June 2001

Superconformal Symmetry in 11D Superspace and the M-Theory Effective Action

S. James Gates, Jr.
Department of Physics
University of Maryland
College Park, MD 20742-4111 USA

ABSTRACT

We establish a theorem about non-trivial 11D supergravity fluctuations that are conformally related to flat superspace geometry. Under the assumption that a theory of conformal 11D supergravity exists, similar in form to that of previously constructed theories in lower dimensions, this theorem demands the appearance of non-vanishing dimension 1/2 torsion tensors in order to accommodate a non-trivial 11D conformal compensator and thus M-theory corrections that break superconformal symmetry. At the complete non-linear level, a presentation of a conventional minimal superspace realization of Weyl symmetry in eleven dimensional superspace is also described. All of our results taken together imply that there exists some realization of conformal symmetry relevant for the M-theory effective action. We thus led to conjecture this is also true for the full and complete M-theory.

PACS: 03.70.+k, 11.30.Rd, 04.65.+e

Keywords: Gauge theories, Supersymmetry, Supergravity.

1 Supported in part by National Science Foundation Grants PHY-98-02551
2gatess@wam.umd.edu
1 Introduction

In a recent note [1], we have resumed studies of a class of problems, the on-shell superspace and perturbative description of higher derivative supergravity, that has been one of several foci of fascination for us since we inaugurated such investigations [2, 3, 4, 5, 6] in the middle eighties. Our proposed method uses on-shell superspace to describe, in a perturbative manner, supergravity actions containing higher-order curvature terms. As first stated in the initial efforts, in order for such constructions to make sense there must be a dimensional parameter with respect to which such a perturbative description is made. One might think that such a parameter is present in all ordinary supergravity systems. After all, the Newton constant is always present in such theories. In fact this is not sufficient. When supergravity theories are written in superspace formulations, it is always possible to perform certain implicit “re-scalings” of component fields in such a way that Newton’s constant is effectively absent. One requires a second dimensional constant and this is supplied in superstring theory by the string tension. Similarly, in discussions of the low-energy M-theory effective action there is also such a constant that we refer to as $\ell_{11}$. We can choose this parameter to possess the units of length or that of an inverse mass. As proposed [2, 3, 4, 5, 6] in our inaugural papers on this topic, the superspace torsion, curvature and field strength can be ‘deformed’ perturbatively as power series expansions in terms of such a dimensional parameter. Although our arguments were made with regard to superstring/heterotic string theories, the same reasoning applies to superstring/M-theory.

Many years ago, we made a conjecture [7] about some of the structure that is required to describe an eleven dimensional supergravity theory whose equations of motion is different from those derived from the standard Cremmer-Julia theory. In fact in 1980, it was pointed out that such modification would in all likelihood require the existence of a dimension 1/2 and spin-1/2 multiplet of currents (which we now refer to as the $F$-tensor). At that time an equation was given for how this multiplet of currents would begin to modify the supergeometry of eleven dimensional supergravity. In a 1996 work [8] we attempted to extend this investigation. However, due to missing terms first noted by Howe [9], this was not a carried out convincingly. In fact, Howe even argued that our proposed dimension 1/2 and spin-1/2 multiplet could not play a role in 11D supergravity/M-theory. This was formalized in a result sometimes called “Howe’s Theorem.” Most recently [1], we have emphasized that Howe’s Theorem cannot be valid unless supergravity/M-theory dynamics admits a scale-invariance. Since the proposed lowest order M-theory correction violates this condition, it is fully

\footnote{Interestingly, these terms were already present in the 1980 work.}
our expectation that the dimension 1/2 and spin-1/2 multiplet of currents must play a critical role.

In the mean time, research has been undertaken based upon “Howe’s Theorem” and investigating what modifications are allowed within its context. This has led to the appearance of additional multiplets of currents known as $X$-tensors to be introduced so as to accommodate the higher derivative M-theory corrections. The appearance of the $X$-tensors does not contradict the appearance of the spin-1/2 $J$-tensor. In fact, in both our 1980 and 1996 discussions of the modified 11D supergeometry we made explicit reference to our expectation that the spin-1/2 $J$-tensor would most likely be only part of an off-shell theory i. e. the $J$-tensor must be accompanied by other tensors to provide a superspace description of the M-theory corrections.

One of the most powerful argument that reveals why the $X$-tensors alone are insufficient to describe an off-shell Poincaré supergravity theory is because they are actually superconformal field strengths$^4$ similar to but distinct from, the superspace Weyl multiplet superfield. We should also note that the appearance of the superconformal $X$-tensors was also foreshadowed in a much overlooked study of conformal symmetry in 4D, $N > 4$ superspace geometry [10]. In this work there appears the statement, “The present analysis shows that conformal symmetry may still be relevant for $N > 4$ supergravity. However, it requires at least a second conformal field strength...” Thus, by this work, the realization that a superconformal superspace field strength not related to the Weyl tensor has appeared in the literature for some time. Although our comments were directed to 4D, $N > 4$ superspace geometry, dimensional oxidation implies that this must be true for 11D superspace geometry also. So the significance of our observation has largely been ignored and has led to much confusion on this subject.

2 Conformal Class of Bosonic Spaces

A special class of infinitesimal fluctuations are those that are conformally related to the flat metric of eleven dimensional superspace. We begin our study of eleven dimensional supergeometry with this class of theories. However, before we do so, we wish to review this class of structures within the non-supersymmetric case followed by considerations in a well-known supersymmetric context.

In a purely bosonic space, the class of graviton fluctuations that are conformally

\footnote{This fact seems not to have been noticed until the recent work in [1].}
related to flat space are defined by

\[ \nabla_a = \partial_a + \psi \partial_a + k_0 (\partial_b \psi) M_a^b, \quad (1) \]

where the infinitesimal scalar field \( \psi \) (an arbitrary function of the spacetime coordinates) is known as the “scale factor,” “scale compensator,” or “conformal compensator.” Torsion and curvature tensors can be defined by calculating the commutator algebra of this derivative.

\[
\begin{align*}
[\nabla_a, \nabla_b] &= t_{ab}{}^c \nabla_c + \frac{1}{2} r_{ab}{}^d M_d^c, \\
t_{ab}{}^c &= (1 + k_0) \left( \partial_{[a} \psi \right) \delta_{b]}^c, \\
r_{ab}{}^{de} &= k_0 \left[ (\partial^d \partial_a \psi) \delta_b^e - (\partial^e \partial_a \psi) \delta_b^d \right].
\end{align*}
\]

(2)

A Riemannian geometry has no torsion tensor and this is accomplished by demanding,

\[ t_{ab}{}^c = 0 \rightarrow k_0 = -1, \quad (3) \]

thus determining the value of the otherwise undetermined parameter \( k_0 \) above. As the vanishing torsion determines the Christoffel connection, we see that fixing \( k_0 = -1 \), is equivalent to choosing the Christoffel connection in (1). Geometries that are conformally related to flat geometry are also compatible with a Riemannian geometry possessing completely vanishing torsion. Stated another way, the vanishing of the torsion places no dynamical restriction on the compensator \( \psi \).

For finite values of the conformal compensator, we can integrate (1) to obtain

\[ \nabla_a = \psi \left[ \partial_a - (\partial_b \ln \psi) M_a^b \right]. \quad (4) \]

3 Minimal Conformal Class of 4D, \( N = 1 \)

Superspace

We can continue this line of deliberation by looking at the case of 4D, \( N = 1 \) superspace. The analog of (1) takes the form

\[
\begin{align*}
\nabla_\alpha &= D_\alpha + \frac{1}{2} \Psi D_\alpha + \ell_0 (D_\beta \Psi) M_\alpha^\beta, \\
\nabla_\theta &= \partial_\theta + \frac{1}{2} \left( \Psi + \overline{\Psi} \right) \partial_\theta + i \ell_1 (D_\alpha \overline{\Psi}) D_\alpha + i \ell_1 (D_\alpha \Psi) D_\alpha \\
&\quad + i \ell_2 \left( [D_\gamma, D_\alpha] \Psi \right) M_\alpha^\gamma - i \ell_2 \left( [D_\alpha, D_\gamma] \overline{\Psi} \right) M_\alpha^\gamma \\
&\quad + \ell_3 (\partial_\gamma \Psi) M_\alpha^\gamma + \overline{\ell}_3 (\partial_\alpha \overline{\Psi}) \overline{M}_\alpha^\gamma, \quad (5)
\end{align*}
\]
In this expression, the constants $\ell_0$, $\ell_1$, $\ell_2$ and $\ell_3$ are the supersymmetric analogs of $k_0$ in (1) and $\Psi$ is a complex scalar and an infinitesimal superfield. It is an exercise to compute all the dimension 1/2 torsion tensors associated with (5) and we find

$$T_{\alpha\beta\gamma} = \frac{1}{2} (\ell_0 + 1) (D_\alpha \Psi) \delta_{\beta\gamma} ,$$

$$T_{\alpha\beta\gamma} = (\ell_1 + \frac{1}{2}) (D_\alpha \overline{\Psi}) \delta_{\beta\gamma} ,$$

$$T_{\alpha\beta\gamma} = \left\{ \frac{1}{2}[D_\alpha((1 - \ell_0)\Psi + \overline{\Psi})] \delta_{\beta\gamma} + [D_\beta(\ell_1 \Psi + \ell_0 \overline{\Psi})] \delta_{\alpha\gamma} \right\} \delta_{\beta\gamma} .$$

The constraints of 4D, $N = 1$ supergravity act much as their analog in (3). The condition

$$\nabla_a = -i [\nabla_\alpha, \nabla_\beta] ,$$

determines the constants $\ell_1$, $\ell_2$ and $\ell_3$ in terms of $\ell_0$. This is equivalent to providing a definition of $E_a$ and $\omega_{\alpha b c}$. The condition

$$T_{\alpha\beta\gamma} = 0 ,$$

determines the constant $\ell_0$ and this is equivalent to providing a definition of $\omega_{\alpha \beta\gamma}$. The form of the fluctuations we are considering also require that $\omega_{\alpha \beta\gamma} = 0$.

Demanding that the first two dimension 1/2 torsions should vanish leads to $\ell_0 = -1$ and $\ell_1 = -\frac{1}{2}$. In principle in order for the last dimension 1/2 torsion to vanish, we are led to two distinct conditions

$$D_\alpha[(1 - \ell_0)\Psi + \overline{\Psi}] = 0 , \quad D_\beta[\ell_0 \Psi + \ell_1 \overline{\Psi}] = 0 ,$$

except for $\ell_0 = -1$ and $\ell_1 = -\frac{1}{2}$ these are proportional to the self-same equation

$$D_\alpha(2\Psi + \overline{\Psi}) = 0 .$$

A solution of this equation introduces the well-known “chiral compensator” superfield, $\varphi$, of 4D, $N = 1$ supergravity [10].

$$\Psi = (2\overline{\varphi} - \varphi) .$$

We see that 4D, $N = 1$ supergravity fluctuations, conformally related to flat superspace, do not require non-vanishing dimension 1/2 torsions if and if those fluctuations are described by a chiral compensator. The vanishing of the dimension 1/2 torsion places no dynamical restriction on the chiral compensator. The results of (3) - (11) for the special values $\ell_0 = -1$ and $\ell_1 = -\frac{1}{2}$ correspond precisely to the infinitesimal limit of the minimal 4D, $N = 1$ prepotential formulation of supergravity with the
added restriction that we are working in a gauge where the conformal prepotential, $U^\mathbf{m}$, has also been set to zero. The property that the full supergravity solution allows us to consistently separate the fluctuations in $\nabla_\mathbf{A}$ involving $U^\mathbf{m}$ from those involving $\varphi$ may be referred to as “separability.” Separability is equivalent to the statement that a Poincaré supergravity multiplet can be thought of as the combination of a Weyl supergravity multiplet and a compensator multiplet. The compensator multiplet contains all of the Goldstone fields required to break the superconformal symmetry group to the super Poincaré group.

4 The Conformal Class of Flat 11D Superspace

We now wish to prove a theorem about the eleven dimensional supersymmetric case. We call this the “11D Torsion–Conformal Compensator Theorem” (11D T–C$^2$ Theorem). Below we will show that 11D supergravity has a behavior that is drastically different from either the purely bosonic theory or 4D, $N = 1$ supergravity discussed above. The formal statement of the 11D T–C$^2$ Theorem is given below.

\begin{quote}
If 11D supergravity is completely separable into a Weyl supermultiplet and a compensator supermultiplet, then the 11D Poincaré supergeometry must possess non-vanishing dimension 1/2 torsion super-tensors.
\end{quote}

We now prove this by explicit construction. The condition of complete separability means that we can set to zero the superfield that contains the Weyl multiplet without affecting the dynamics of the conformal compensator. This condition is true in all superfield supergravity theories presently known.

For the class of eleven dimensional superspaces, we note that the analog of (4) and (5) takes the form

\begin{align}
\nabla_\alpha &= D_\alpha + \frac{1}{2} \Psi D_\alpha + \ell_0 (D_\beta \Psi) (\gamma^{de}_\alpha)_{\alpha}^{\beta} \mathcal{M}_{de} , \\
\nabla_a &= \partial_a + \Psi \partial_a + i \ell_1 (\gamma_a^{\alpha \beta} (D_\alpha \Psi) D_\beta + \ell_2 (\partial_c \Psi) M_a^c \quad (12) \\
&\quad \quad + i \ell_3 (\gamma_a^{de})_{\alpha \beta} (D_\alpha D_\beta \Psi) \mathcal{M}_{de} ,
\end{align}

where $\Psi$ in (12)\footnote{Although we use the same symbol here, this superfield should not be confused with the one that appears in (5).} is a real scalar and an infinitesimal superfield and the $\ell_i$’s are a set of constants (essential like those of the 4D, $N = 1$ theory). Of course the form of $E_\alpha$
is given by
\[ E_\alpha = D_\alpha + \frac{1}{2} \Psi D_\alpha , \quad (13) \]
which is appropriate for a supergeometry that is related by a scale transformation to a “flat” eleven dimensional supergeometry. The constants \( \ell_2 \) and \( \ell_3 \) can be eliminated by imposing the analog of (7) to the 11D theory. This leaves only \( \ell_0 \) and \( \ell_1 \) to be fixed.

Computing the dimension 1/2 torsion tensors associated with (12), we find
\[
T_{\alpha b}^c = (1 + 2 \ell_0) (D_\alpha \Psi) \delta_b^c - (\ell_1 + 2 \ell_0) (\gamma^c \gamma_b)_{\alpha \beta} (D_{\beta} \Psi) \\
T_{\alpha \beta}^\gamma = \frac{1}{2} (D_\alpha \Psi) \delta_\beta^\gamma + \frac{1}{2} \ell_0 (D_\delta \Psi) (\gamma^{[2]}),_{(\alpha}^\delta (\gamma_{2])_{\beta})^\gamma \\
+ \ell_1 (\gamma^c)_{\alpha \beta} (\gamma_c)^\delta \gamma (D_\delta \Psi) \\
= \frac{1}{32} \left[ (32 \ell_1 - 1 - 70 \ell_0) (\gamma^c)_{\alpha \beta} (\gamma_c)^\gamma \delta \\
+ \frac{1}{2} (38 \ell_0 + 1) (\gamma^{[2]}),_{(\alpha}^\delta (\gamma_{2])_{\beta})^\gamma \\
+ \frac{1}{120} (10 \ell_0 - 1) (\gamma^{[5]}),_{(\alpha}^\delta (\gamma_{5])_{\beta})^\gamma \right] (D_\delta \Psi) . \quad (14)
\]

In reaching the second form of the last equation, we used two Fierz identities in order to make clear the full content of the equation. Demanding that the dimension 1/2 torsions vanish leads to five independent conditions
\[
0 = -1 - 70 \ell_0 + 32 \ell_1 , \quad 0 = 1 + 38 \ell_0 , \\
0 = 1 - 10 \ell_0 , \quad 0 = 1 + 2 \ell_0 , \quad 0 = \ell_1 + 2 \ell_0 , \quad (15)
\]
on two constants (\( \ell_0 \) and \( \ell_1 \)). The resulting system is thus overdetermined and inconsistent. Our argument thus far runs exactly parallel to the case of 4D, \( N = 1 \) supergravity.

The only consistent solution for completely vanishing dimension 1/2 torsions in this circumstance is to also impose the extra condition \( D_\alpha \Psi = 0 \). But unlike the 4D, \( N = 1 \) theory discussed above, this has an additional dire consequence,
\[
D_\alpha \Psi = 0 \rightarrow D_\alpha D_\beta \Psi = 0 \rightarrow [D_\alpha , D_\beta ] \Psi = 0 \\
\rightarrow i (\gamma^c)_{\alpha \beta} \partial_c \Psi = 0 \rightarrow \partial_c \Psi = 0 . \quad (16)
\]
This implies that \( \Psi \) must be a constant so that (12) reduces to a trivial constant rescaling of the superframe fields. Thus eleven dimensional superspace geometry is quite unlike its bosonic counterpart. Here we see non-constant supergravity fluctuations that are conformally related to the flat superspace necessarily produce dimension 1/2 and spin-1/2 torsion tensors that are non-vanishing. At most only two of the five
independent structures that occur in the dimension 1/2 torsions can be set to zero, if the theory is to possess a non-trivial conformal compensator. This is also seen to be substantially different from the case of 4D, N = 1. There it was the case that the existence of the chiral compensator still permitted the existence of non-trivial fluctuations that are conformally related flat superspace even though all dimension 1/2 torsions vanish. This is not possible for the 11D case since the notion of a chiral superfield is non-existent for an 11D theory!

Within the class of derivatives defined by (2) we next wish to define one that satisfies the conventional constraints

$$\nabla_a = i \frac{1}{32} (\gamma_a)^{\alpha\beta} [\nabla_\alpha, \nabla_\beta] \ , \ T_{\alpha \beta} = \frac{2}{55} (\gamma_{\alpha \beta}) T_\beta \ d \ = \ 0 \ . \quad (17)$$

The first of these defines E\_a and ω\_abc in terms of E\_α and ω\_abc (c. f. Eq. (3)). The second of these defines the spin-connection ω\_abc in terms of the anholonomy (c. f. Eq. (8)). We can satisfy these condition by choosing the constants as

$$\ell_0 = \frac{1}{10} \ , \ \ell_1 = \frac{1}{4} \ , \ \ell_2 = \frac{1}{5} \ , \ \ell_3 = \frac{1}{100} \ . \quad (18)$$

Utilizing the fluctuations described by (12) and (18), we calculate the complete commutator algebra associated with this supergravity covariant derivative to find

$$T_{\alpha \beta}^c = i (\gamma^c)_{\alpha \beta} \ ,$$

$$T_{\alpha \beta}^\gamma = \frac{3}{32} (\gamma^{de})_{\alpha \gamma} (\gamma_{de})^{\beta \gamma} (D_\delta \Psi) \ ,$$

$$R_{\alpha \beta} \ ^d \ e = - \frac{1}{10} \left[ (\gamma^{de})_{\alpha \gamma} (D_{[\beta}D_{\gamma]} \Psi) + (\gamma^{de})_{\beta \gamma} (D_{[\alpha}D_{\gamma]} \Psi) + \frac{1}{5} (\gamma^b)_{\alpha \beta} (\gamma^c_{de})^{\gamma \delta} (D_\gamma D_\delta \Psi) \right] \ ,$$

$$R_{ab}^c = \frac{3}{5} \left[ 2 (D_a \Psi) \delta_b^c - \frac{3}{4} (\gamma^c \gamma_b)_\alpha (D_\gamma \Psi) \right] \ ,$$

$$R_{ab}^\gamma = \left[ i \frac{1}{5} (D_{[\alpha}D_{\beta]} \Psi) (\gamma^b)_{\gamma \gamma} - i \frac{1}{2 \times 80} (\gamma^{[3]}_{\delta \epsilon}) (D_\delta D_\epsilon \Psi) (\gamma_{[3]} \gamma_b)_\alpha \gamma \right. \ 
- i \frac{1}{2 \times 80} (\gamma^{[3]}_{\delta \epsilon}) (D_\delta D_\epsilon \Psi) (\gamma_b \gamma_{[3]} \alpha) \gamma - \frac{1}{2 \times 80} (\gamma_b \gamma^c)_\alpha \gamma (\partial_\epsilon \Psi) \left. + \frac{7}{32} (\gamma^c \gamma_b)_\alpha \gamma (\partial_\epsilon \Psi) \right] \ ,$$

$$R_{ab} \ ^d \ e = \left[ - i \frac{1}{2 \times 80} (\gamma^b_{de}) (D_{[\alpha}D_{\beta]}D_{[\delta]} \Psi) - \frac{1}{80} (\gamma^c \gamma_b)_{de} (\gamma_{[3]} \gamma_b)_\alpha \gamma \right. \ 
+ \frac{1}{5} (\gamma^c_{de})_{\alpha \gamma} (\partial_b D_\gamma \Psi) + \frac{1}{5} (\partial_d D_\alpha \Psi) \delta_e^c \right] \ ,$$

$$T_{ab}^c = \frac{1}{5} \left[ 6 (\partial_{[a} \Psi) \delta_{[b]}^c + i \frac{1}{5} (\gamma_{ab})_{\alpha \beta} (D_a D_\beta \Psi) \right] \ ,$$

$$T_{ab}^\gamma = - i \frac{1}{4} (\gamma_{[a]})^{\gamma \delta} (\partial_{[b]} D_\delta \Psi) \ ,$$

$$R_{ab} \ ^d \ e = \frac{1}{5} \left[ (\partial_{[a]} \partial^d \Psi) \delta_{[b]}^c - (\partial_{[a]} \partial^d \Psi) \delta_{[b]}^d \right. \ 
+ i \frac{1}{32} (\gamma_{[a]})_{\alpha \beta} (\partial_{[b]} D_\alpha D_\beta \Psi) \left. \right] . \quad (19)$$
These equations emphasize a remarkable fact which is true in all supergeometries. The most general supergeometry that is conformally related to the flat superspace may be viewed as a geometrical description of a scalar superfield. We have long been aware of this fact and have used it previously ([22]) to derive the first off-shell description of 2D, \(N = 4\) supergravity. Within 2D theories, this result is even more powerful. It implies that the supergravity constraints for 2D theories are in one-to-one correspondence with irreducible scalar superfields and thus the supergravity constraints are \textit{totally} determined by the differential equations that define 2D irreducible scalar multiplets and \textit{vice-versa}.

The covariant derivatives in (12) may be “integrated with respect to \(\Psi\)” so that in the case of a finite conformal compensator \(\Psi\) we find

\[
\nabla_\alpha = \Psi^{1/2} \left[ D_\alpha + \frac{1}{10} (D_\gamma \ln \Psi) (\gamma^d e)_{\alpha}^\gamma M_{de} \right], \\
\nabla_a = \Psi \left[ \partial_a + i \frac{1}{4} (\gamma_a)_{\gamma}^\delta (D_\gamma \ln \Psi) D_\delta + \frac{1}{5} (\partial_b \ln \Psi) M_a^b \\
+ i \frac{1}{100} (\gamma_a^a)_{\gamma}^\delta (D_\gamma D_\delta \ln \Psi) M_{de} \\
+ i \frac{27}{800} (\gamma_a^a)_{\gamma}^\delta (D_\gamma \ln \Psi) (D_\delta \ln \Psi) M_{de} \right].
\]

To summarize the main result in this section, we have shown that under the assumption that 11D supergravity is separable into a Weyl multiplet superfield \((H_\alpha^m)\) and a conformal compensator superfield \((\Psi)\), Howe’s Theorem in the limit of vanish Weyl multiplet forbids the appearance of the conformal compensator superfield. This behavior is not congruent with that (e.g. the discussion from (5-11)) found in all previous known cases of off-shell superfield supergravity.

5 Traditional Approach to Weyl Symmetry in 11D Superspace

Although Howe’s 1997 paper [4] has described a \textit{new} formalism to realize the presence of Weyl symmetry for 11D superspace, in fact there is a traditional manner for accomplishing this goal. This traditional approach was initiated by Howe & Tucker [12] and then developed by others [13]. We have even been able to extend this traditional description all the way to 10D, \(N = 1\) superspace [14]. In our recent work [1], we began the process of extending this traditional approach to the 11D theory. As was shown in [1], the newer approach requires that there must exist degrees of freedom in addition to those that reside in Eq. It is thus a \textit{non-minimal} realization of Weyl symmetry in 11D superspace. The traditional approach [12], does not suffer
from this drawback. So it will be the goal of this chapter to extend completely the
traditional formalism for Weyl symmetry in superspace to the 11D case and thereby
establish the realization of Weyl symmetry in a minimal manner within this venue.

In our recent work [1], an analysis of the 11D vielbein degrees of freedom was
performed. This work implies that a solely conventional set of constraints\(^6\) can be
chosen as

\[
i (\gamma_a)^{\alpha \beta} T_{\alpha \beta}^b = 32 \delta_a^b \quad , \quad (\gamma_a)^{\alpha \beta} T_{\alpha \beta}^\gamma = 0 \quad , \quad T_{[de]} - \frac{2}{55} (\gamma_{de})^a_\gamma T_{\gamma b}^b = 0 \quad ,
\]

\[
(\gamma_a)^{\alpha \beta} R_{\alpha \beta}^{de} = 0 \quad , \quad (\gamma_{ab})^{\alpha \beta} T_{\alpha \beta}^b = 0 \quad , \quad (\gamma_{[ab]})^{\alpha \beta} T_{\alpha \beta}^{[c]} = 0 \quad ,
\]

\[
(\gamma_{abcde})^{\alpha \beta} T_{\alpha \beta}^e = 0 \quad , \quad \frac{1}{6!} \epsilon^{abcdef} (\gamma_{abcde})^{\alpha \beta} T_{\alpha \beta}^f = 0 \quad . \tag{21}
\]

The last four constraints determine the degrees of freedom in \(E_{a \mu}^\alpha\) that correspond to
the elements of the coset

\[
\frac{\text{SL}(32, \mathbb{R})}{\text{SO}(1, 10) \otimes \text{SO}(1, 1)} \quad . \tag{22}
\]

The constraints imply that all 11D supergravity fields are contained in two semi-
prepotentials; \(\Psi\) (conformal compensator) and \(H_{a \mu}^m\) (Weyl supermultiplet) and we
will use these constraints in the following.

The last four constraints of our table may be called “coset conventional con-
straints\(^7\)” and in fact their existence is an almost universal feature of superspace
supergravity theories. The easiest way to understand this is to consider torus com-
pactification of the 11D result. So for example, in four dimensional \(N\)-extended
supergravity, these coset constraints take the form

\[
\frac{\text{SL}(4N, C)}{\text{SO}(1, 3) \otimes \text{SO}(1, 1) \otimes \text{U}(N)} \quad . \tag{23}
\]

for \(N \neq 4\) and

\[
\frac{\text{SL}(16, C)}{\text{SO}(1, 3) \otimes \text{SO}(1, 1) \otimes \text{SU}(4)} \quad . \tag{24}
\]

for \(N = 4\). It is perhaps also useful to point out that an example of coset conven-
tional constraints was discussed [14] when we provided the first explicit solution to
constraints for 2D, \(N = 1\) supergravity. In fact, 4D, \(N = 1\) supergravity is the ex-
ception rather than the rule when it comes to coset conventional constraints. While

\(^6\)To our knowledge, this is the first time this particular set of off-shell constraints for 11D
superspace has been suggested. These are different from our previous works (e. g. \[1\]) but
but are convenient for many purposes. The factor of 2/55 in particular leads to ‘nice’ nor-
malizations in many, many subsequent calculations.

\(^7\)The 11D coset conventional constraints are closely related to the constraints discussed in \[15\].
4D, $N = 1$ superspace supergravity theory does not require such constraints, most
generic superspace supergravity theories do require this type of constraint.

A perennial question we often encounter when discussing superspace systems of
constraints is caused by the fourth entry in our table. The question is, “Shouldn’t
$\omega_{abc}$ be determined by the condition that appears in (3)?” The answer to this is that
one can certainly replace the first constraint in the second row by $T_{abc} = 0$. However,
this determines the connection in an “un-improved manner.” Certain improvement
terms (as seen in the 4D, $N = 1$ case) will occur with our choice.

Motivated by the discussion in the last two chapters and by our work in [1, 14], we
can define a minimal realization of a scale transformation law for the 11D superspace
covariant derivative by

$$
\delta_S \nabla_\alpha = \frac{1}{2} L \nabla_\alpha + \frac{1}{10} (\nabla_\gamma L) (\gamma^{bc})_\alpha \gamma \mathcal{M}_{bc} ,
$$

$$
\delta_S \nabla_a = L \nabla_a + i \frac{1}{4} (\gamma^{a\beta}) (\nabla_\alpha L) \nabla_\beta + \frac{1}{5} (\nabla_\gamma L) \mathcal{M}_a ,^c (25)
$$

$$
+ i \frac{1}{100} (\gamma^{a\beta}) (\nabla_\alpha \nabla_\beta L) \mathcal{M}_{bc} .
$$

The importance of these equations is that they permit us to calculate the scale vari-
ations of all superspace torsions and curvatures at the full non-linear level. It should
also be clearly understood that the derivatives $\nabla_A$ that appear in (25) are not re-
stricted to describe fluctuations conformally related to flat superspace as in (12). In
particular the torsions in this chapter are not given solely by the terms in (15). We find

$$
\delta_S T_{abc} = 0 ,
$$

$$
\delta_S T_{ab}^\gamma = \frac{1}{2} L T_{ab}^\gamma - i \frac{1}{4} (\delta^{[a\beta} (\nabla_\gamma L) T_{[\alpha\delta]} T_{\beta\gamma}) + \frac{1}{2} (\nabla_\gamma L) \delta_{\beta\gamma}
$$

$$
+ \frac{1}{20} (\nabla_\delta L) (\gamma^{[a]} (\delta_{\beta]} (\gamma^{\beta])_{\beta\gamma} ,
$$

$$
\delta_S T_{ab}^c = \frac{1}{2} L T_{ab}^c - i \frac{1}{4} (\gamma_b) (\nabla_\gamma L) T_{ab}^c + (\nabla_\gamma L) \delta_{b}^c
$$

$$+ \frac{1}{5} (\nabla_\gamma L) (\gamma_b^c)_{a\gamma} ,
$$

$$
\delta_S T_{ab}^\gamma = L T_{ab}^\gamma - i \frac{1}{4} (\delta^{[a\beta} (\nabla_\gamma L) T_{[\alpha\delta]} T_{\beta\gamma}) - i \frac{1}{2} (\nabla_\delta L) (\gamma^{a\beta}) (\delta_{\gamma}) (\gamma_{\beta\gamma}
$$

$$- \frac{1}{2} (\nabla_\gamma L) \delta_{\gamma\gamma} - \frac{1}{10} (\nabla_\delta L) (\gamma^{[a]} (\delta_{\beta]} (\gamma^{\beta])_{\gamma})
$$

$$+ i \frac{1}{4} (\gamma_b^c) (\nabla_\gamma L) - i \frac{1}{20} (\nabla_\gamma L) (\gamma^{a\beta} (\nabla_\delta L) (\gamma_{\beta\gamma})_{a\gamma} ,
$$

$$
\delta_S T_{ab}^c = L T_{ab}^c + i \frac{1}{4} (\delta^{[a\beta} (\nabla_\gamma L) T_{[\alpha\delta]} T_{\beta\gamma}) + \frac{6}{5} (\nabla_\gamma L) \delta_{b}^c
$$

$$+ i \frac{1}{100} (\gamma_{ab}^c) (\nabla_\gamma L) ,
$$

$$
\delta_S T_{ab}^\gamma = \frac{3}{2} L T_{ab}^\gamma - i \frac{1}{4} (\delta^{[a\beta} (\nabla_\gamma L) T_{[\alpha\delta]} T_{\beta\gamma}) + i \frac{1}{4} (\delta^{[a\beta} (\nabla_\gamma L) T_{\beta\gamma})
$$

$$- i \frac{1}{4} (\gamma_{a}^c) (\nabla_\gamma L) .
$$
In our previous work [1], we pointed out that there is also an alternate possibility of
or alternately we see

This transformation law implies that for the various curvature tensor components.

Although these conformal transformation laws may seem quite complicated when
compared to those of ([2]), an investigation of previous lower dimensional off-shell
supergravity theories will reveal many similarities.

An especially important superfield occurs at dimension 1/2. We denote this superfield
by \( \mathcal{J}_\alpha \) where

\[
\mathcal{J}_\alpha \equiv \frac{4}{3} T_{ab}^\beta \rightarrow \delta_S \mathcal{J}_\alpha = \frac{1}{2} L \mathcal{J}_\alpha + (\nabla_\alpha L) .
\]

This transformation law implies that \( \mathcal{J}_\alpha \) is not a superconformal invariant superfield.

It has been proposed [5] that a rank six and dimension zero field strength denoted
by \( X_3^a \) is critical to obtain a superspace description of the M-theory effective action.
In our previous work [1], we pointed out that there is also an alternate possibility of
allowing another similar tensor, \( X_{2}^a \). These both can be found in

\[
T_{\alpha \beta}^c = \frac{i}{2} \left( \gamma^c \right)_{\alpha \beta} + \frac{1}{2} \left( \gamma^{[2]} \right)_{\alpha \beta} X_{[2]}^c + \frac{i}{120} \left( \gamma^{[5]} \right)_{\alpha \beta} X_{[5]}^c ,
\]

or alternately we see

\[
X_{[ab]}^k \equiv \frac{1}{32} \left( \gamma_{ab} \right)^{\alpha \beta} T_{\alpha \beta}^k ,
\]

\[
X_{[abcde]}^k \equiv \frac{i}{32} \left( \gamma_{abcde} \right)^{\alpha \beta} T_{\alpha \beta}^k .
\]

It is interesting to note what result is obtained from the quantity defined by
\( T_{ab}^b - (3/4) T_{\alpha \beta}^\beta \) under the action of the scale transformation law in (25). In fact this
object is a dimension 1/2 spin-1/2 scale covariant. Starting from the third equation
in (26) we see
\[ \delta S T_{\alpha \beta} = \frac{1}{2} L T_{\alpha \beta} - \frac{i}{4} T_{\alpha \beta}^c (\gamma_c)^{\delta \beta} (\nabla_\delta L) + \frac{1}{2} (\nabla (\alpha L) \delta_\beta)^\beta \\
+ \frac{1}{160} (\nabla_\delta L) (\gamma_{[2]})(\delta_\beta)^\beta \]
\[ = \frac{1}{2} L T_{\alpha \beta} - \frac{i}{4} \left[ i (\gamma^c)_{\alpha \beta} + \frac{1}{2} (\gamma_{[2]})(\alpha \beta)^c \right] (\gamma_c)^{\delta \beta} (\nabla_\delta L) \\
+ \frac{1}{2} (\nabla (\alpha L) \delta_\beta)^\beta + \frac{1}{20} (\nabla_\delta L) (\gamma_{[2]})(\delta_\beta)^\beta \]
\[ = \frac{1}{2} L T_{\alpha \beta} - \frac{i}{4} (\nabla_\alpha L) + \frac{33}{2} (\nabla_\alpha L) - \frac{55}{20} (\nabla_\alpha L) , \]
\[ \rightarrow \delta S \left( \frac{1}{11} T_{\alpha \beta} \right) = \frac{1}{2} L \left( \frac{1}{11} T_{\alpha \beta} \right) + (\nabla_\alpha L) . \]

The reason the terms involving the $X$-tensors have “disappeared” from the final result can be seen from the following considerations.
\[ \begin{align*}
\frac{1}{2} (\gamma_{[2]})_{\alpha \beta} (\gamma_c)^{\delta \beta} X_{[2]}^c &= - \frac{1}{2} (\gamma_{[2]} (\gamma_c)^{\delta \beta} X_{[2]}^c \\
&= - \frac{1}{2} \left[ (\gamma_{[abc]})_\alpha^\delta - 2 \eta_{ca}(\gamma_b)_\alpha^\delta \right] X^{[ab]c} = 0 , \\
i \frac{1}{120} (\gamma_{[5]})_{\alpha \beta} (\gamma_c)^{\delta \beta} X_{[5]}^c &= - i \frac{1}{120} (\gamma_{[5]} (\gamma_c)^{\delta \beta} X_{[5]}^c \\
&= \frac{1}{120} \left[ \frac{1}{120} \epsilon_{abcdef}[5^s] (\gamma_{[5^s]})_\alpha^\delta \\
- i 15 \eta_{ca}(\gamma_{bdef})_\alpha^\delta \right] X^{[abdef]c} = 0 .
\end{align*} \]

The conventional constraints imposed upon the $X$-tensors eliminate their appearance from the final result in (31). So that we see an alternate definition of the $J$-tensor is given by $J_\alpha = \frac{1}{11} T_{\alpha \beta}$. The appearance of this non-scalar covariant field Poincaré supergravity strength should not come as a surprise. At least one non-scale invariant field strength superfield can be verified in every known off-shell superspace formulation that has ever been given.

The reason that this superfield is important is that it allows the definition of a Poincaré superspace spinorial differentiation operation that acts consistently to remain solely within the space of superconformal tensors. To see this, we note that a superconformal tensor denoted by $T^{(w)}_{a_1...a_p b_1...b_q}$ and of weight $w$ can be defined to transform according to
\[ \delta S T^{(w)}_{a_1...a_p b_1...b_q} = w L T^{(w)}_{a_1...a_p b_1...b_q} , \]
under the action of the superspace scale transformation. From this it follows that we
\[
\delta_S(\nabla_\alpha T^{(w)}_{a_1...a_p b_1...b_q}) = \left( w + \frac{1}{2} \right) L \left( \nabla_\alpha T^{(w)}_{a_1...a_p b_1...b_q} \right)
+ w \left( \nabla_\alpha L \right) T^{(w)}_{a_1...a_p b_1...b_q}
+ \frac{1}{10} \left( \nabla_\gamma L \right) (\gamma^{bc})_\alpha^\gamma (M_{bc} T^{(w)}_{a_1...a_p b_1...b_q})
\]
\[
\delta_S(J_\alpha T^{(w)}_{a_1...a_p b_1...b_q}) = \left( w + \frac{1}{2} \right) L \left( J_\alpha T^{(w)}_{a_1...a_p b_1...b_q} \right)
+ \left( \nabla_\alpha L \right) T^{(w)}_{a_1...a_p b_1...b_q}
\]
\[
\delta_S(J_\gamma (\gamma^{bc})_\alpha^\gamma M_{bc} T^{(w)}_{a_1...a_p b_1...b_q}) = \left( w + \frac{1}{2} \right) L J_\gamma (\gamma^{bc})_\alpha^\gamma (M_{bc} T^{(w)}_{a_1...a_p b_1...b_q})
+ \left( \nabla_\gamma L \right) (\gamma^{bc})_\alpha^\gamma (M_{bc} T^{(w)}_{a_1...a_p b_1...b_q})
\]  

These three equations taken together inform us that the quantity defined by
\[
(\hat{\nabla}_\alpha T^{(w)}_{a_1...a_p b_1...b_q}) \equiv \left[ (\nabla_\alpha - w J_\alpha - \frac{1}{10} J_\gamma (\gamma^{bc})_\alpha^\gamma M_{bc} T^{(w)}_{a_1...a_p b_1...b_q}) \right]
\]  
possesses a covariant scale transformation law
\[
\delta_S(\hat{\nabla}_\alpha T^{(w)}_{a_1...a_p b_1...b_q}) = \left( w + \frac{1}{2} \right) L (\hat{\nabla}_\alpha T^{(w)}_{a_1...a_p b_1...b_q})
\]  
with scale weight \((w + \frac{1}{2})\).

Examples of 11D scale-covariant supergravity tensors are provided by the “X-tensors” [15]. According to the first result in (26) both X-tensors are scale-covariant having \(w = 0\). Thus, we see
\[
\delta_S X^{[2]}_c = 0 \quad , \quad \delta_S X^{[5]}_c = 0 \quad , \quad \rightarrow \\
\hat{\nabla}_\alpha X^{[2]}_c = (\nabla_\alpha - \frac{1}{10} J_\gamma (\gamma^{bc})_\alpha^\gamma M_{bc}) X^{[2]}_c \\
\hat{\nabla}_\alpha X^{[5]}_c = (\nabla_\alpha - \frac{1}{10} J_\gamma (\gamma^{bc})_\alpha^\gamma M_{bc}) X^{[5]}_c
\]  
and consequently
\[
\delta_S(\hat{\nabla}_\alpha X^{[2]}_c) = \frac{1}{2} L (\hat{\nabla}_\alpha X^{[2]}_c) \quad , \quad \delta_S(\hat{\nabla}_\alpha X^{[5]}_c) = \frac{1}{2} L (\hat{\nabla}_\alpha X^{[5]}_c)
\]  
and we find the “hatted derivative” of the \(w = 0\) scale-covariant X-tensors are \(w = 1/2\) scale-covariant tensors. All of these results are the expected generalization of the ones found for the 10D, \(N = 1\) superspace [14].

A second especially important superfield occurs at dimension one and is the on-shell 11D supergravity field strength. We denote this superfield by \(W_{abcd}\) and note

\[\text{In their work, Cederwall, Gran, Nielsen and Nilsson only retained the } X^{[5]}_a \text{ tensor.}\]
that it contains the usual supercovariantized Weyl tensor\textsuperscript{9} at second order in its \(\theta\) expansion. Given the superspace scale transformation law of the various torsion components (26), it follows that a superscale-covariant quantity of dimension one and with \(w = 1\) is given by

\[
W_{abcd} \equiv \frac{1}{32} \left[ i (\gamma^e\gamma_{abcd})_\gamma^\alpha T_\alpha e^\gamma - \frac{1}{3} (\gamma_{abcd})^{\alpha\beta} (\nabla_\alpha T_{\beta c}^\ c + \frac{14}{1,815} T_{ac}^\ c T_{\beta d}^\ d) \right] \rightarrow \delta_S W_{abcd} = L W_{abcd} .
\]

This 4-form superfield quite properly may be called the 11D Weyl multiplet superfield. However, as we noted above, the component-level supercovariantized Weyl tensor is \textit{not} the leading component of this superfield. Since this superfield contains the Weyl multiplet, it does not vanish when the 11D supergravity multiplet obeys the equations of motion associated with 11D super Poincaré action. It also thus corresponds to the on-shell field strength of 11D supergravity. Our discussion above can be applied to solve a small puzzle in the work of [15]. Namely this formula necessarily defines the on-shell field strength, containing the physical degrees of freedom of the 11D theory, as it picks out a particular linear combination of the two four-forms described in the work of Ref. [15]\textsuperscript{10}.

The equation in (35) has an important component implication. If we denote the local supersymmetry variation in the putative 11D superconformal theory by \(\hat{\delta}Q\), the corresponding one in the 11D Poincaré theory by \(\delta_Q\), the local scale variation by \(\delta_S\) and the local Lorentz variation by \(\delta_{LL}\), then Eq. (35) implies

\[
\hat{\delta}Q(\epsilon^\alpha) = \delta_Q(\epsilon^\alpha) - \delta_S(\epsilon^\alpha \lambda_\alpha) - \frac{1}{10} \delta_{LL}(\epsilon^\alpha (\gamma^{bc})_\alpha^\beta \lambda_\beta) ,
\]

where \(\epsilon^\alpha(x)\) is the local supersymmetry parameter and \(\lambda_\alpha(x)\) is the \(\theta \to 0\) limit of \(J_\alpha\).

The main results of this section are summarized in the following table.

| Superfield | Conformal | Weight | Engineering dim. |
|------------|-----------|--------|------------------|
| \(W_{abcd}\) | Yes       | 1      | 1                |
| \(J_\alpha\) | No        | 1/2    | 1/2              |
| \(X_{[abdef]}^c\) | Yes | 0      | 0                |
| \(X_{[ab]}^c\) | Yes | 0      | 0                |

Table 1: 11D Supergravity Field Strength Superfields

\textsuperscript{9}We use the non-calligraphic symbol \(W_{abcd}\) for the usual Weyl tensor superfield.

\textsuperscript{10}The definition of \(W_{abcd}\) can change slightly depending on the choice of conventional constraints. Our definition is the one that follows from choosing the constraints as in (21).
6 Weyl 11D Superspace Geometry from Super Poincaré Geometry

The spinorial “hatted” supercovariant derivative introduced in the last chapter allows the construction of supergeometrical objects that possess superconformal symmetry. The spinorial Weyl “hatted” supercovariant derivative given in (33)

\[ \hat{\nabla}_\alpha \equiv \nabla_\alpha - w J_\alpha - \frac{1}{10} J_\gamma (\gamma^\beta)_\alpha \gamma^c M_{bc} \quad , \]

may be used to define a bosonic Weyl “hatted” supercovariant derivative,

\[ \hat{\nabla}_a \equiv \frac{i}{32} (\gamma_a)^{\alpha\beta} [\hat{\nabla}_\alpha , \hat{\nabla}_\beta] \quad , \]

\[ = \nabla_a - i \frac{1}{4} (\gamma_a)^{\alpha\beta} J_\alpha \nabla_\beta - i w \frac{1}{10} (\gamma_a)^{\alpha\beta} [\nabla_\alpha J_\beta] I \]

\[ - i \frac{1}{160} (\gamma_a \gamma_{de})^{\alpha\beta} [(\nabla_\alpha - \frac{13}{5} J_\alpha) J_\beta] M_{de} \quad . \]

From the results in (25), (28) and (36) we find the following variations,

\[ \delta_S[\nabla_a T^{(w)}] = (w + 1) L \nabla_a T^{(w)} + i \frac{1}{4} (\gamma_a)^{\alpha\beta} (\nabla_\alpha L)(\nabla_\beta T^{(w)}) + w (\nabla_a L) T^{(w)} \]

\[ + \frac{1}{5} (\nabla_a L) (M_a d^T)^{\alpha\beta} + i \frac{1}{10} (\gamma_a \gamma_{de})^{\alpha\beta} (\nabla_\alpha \nabla_\beta L)(M_{de} T^{(w)}) \quad , \]

\[ \delta_S[i (\gamma_a)^{\alpha\beta} J_\alpha (\nabla_\beta T^{(w)})] = i (w + 1) L (\gamma_a)^{\alpha\beta} J_\alpha (\nabla_\beta T^{(w)}) \]

\[ + i (\gamma_a)^{\alpha\beta} (\nabla_\alpha L)(\nabla_\beta T^{(w)}) \]

\[ + i w (\gamma_a)^{\alpha\beta} J_\alpha (\nabla_\beta L) T^{(w)} \]

\[ + i \frac{1}{160} (\gamma_a \gamma_{de})^{\alpha\beta} J_\alpha (\nabla_\beta L)(M_{de} T^{(w)}) \quad , \]

\[ \delta_S[i (\gamma_a)^{\alpha\beta} (\nabla_a J_\beta) T^{(w)}] = i (w + 1) L (\gamma_a)^{\alpha\beta} (\nabla_a J_\beta) T^{(w)} + 16 (\nabla_a L) T^{(w)} \]

\[ - i 4 (\gamma_a)^{\alpha\beta} J_\alpha (\nabla_\beta L) T^{(w)} \quad , \]

\[ \delta_S[i (\gamma_d)^{\alpha\beta} (\nabla_a J_\beta)(M_a d^T)^{\alpha\beta}] = i (w + 1) L (\gamma_d)^{\alpha\beta} (\nabla_a J_\beta)(M_a d^T)^{\alpha\beta} \]

\[ - i 4 (\gamma_a)^{\alpha\beta} J_\alpha (\nabla_\beta L)(M_a d^T)^{\alpha\beta} \]

\[ + 16 (\nabla_a L)(M_a d^T)^{\alpha\beta} \quad , \]

\[ \delta_S[i (\gamma_a \gamma_{de})^{\alpha\beta} (\nabla_a J_\beta)(M_{de} T^{(w)})] = i (w + 1) L (\gamma_a \gamma_{de})^{\alpha\beta} (\nabla_a J_\beta)(M_{de} T^{(w)}) \]

\[ + \frac{6}{5} (\gamma_a \gamma_{de})^{\alpha\beta} J_\alpha (\nabla_\beta L)(M_{de} T^{(w)}) \]

\[ + i (\gamma_a \gamma_{de})^{\alpha\beta} (\nabla_a \nabla_\beta L)(M_{de} T^{(w)}) \quad , \]

\[ \delta_S[i (\gamma_a \gamma_{de})^{\alpha\beta} (J_\alpha J_\beta)(M_{de} T^{(w)})] = i (w + 1) L (\gamma_a \gamma_{de})^{\alpha\beta} (J_\alpha J_\beta)(M_{de} T^{(w)}) \]

\[ + i 2 (\gamma_a \gamma_{de})^{\alpha\beta} J_\alpha (\nabla_\beta L)(M_{de} T^{(w)}) \quad , \]
In writing these we have used the short-hand notation \( T^{(w)} \equiv T^{(w)}_{a_1...a_p b_1...b_q} \). The definition of \( \hat{\nabla}_a \) insures that the following relation is satisfied,

\[
\delta_S T^{(w)}_{a_1...a_p b_1...b_q} = w L T^{(w)}_{a_1...a_p b_1...b_q} 
\]

\[
\delta_S (\hat{\nabla}_a T^{(w)}_{a_1...a_p b_1...b_q}) = (w + 1) L \hat{\nabla}_a T^{(w)}_{a_1...a_p b_1...b_q} .
\] (49)

A calculation of the graded commutator algebra of the “hatted” derivatives yield “hatted” torsion and curvature superfields. As well a dilatation field strength superfield \( F_{AB} \) can be defined from the graded commutator,

\[
[\hat{\nabla}_A, \hat{\nabla}_B] \equiv \hat{T}_{AB} \gamma^\gamma \hat{\nabla}_\gamma + \hat{T}_{AB}^c \hat{\nabla}_c + \frac{1}{2} \hat{R}_{ABd}^e M_e d - w F_{AB} I .
\] (50)

Given the results of (41) and (42) we also find

\[
[\hat{\nabla}_A, \hat{\nabla}_B] = \left[ \nabla_A, \nabla_B \right] + \mathcal{K}_{AB} \gamma \nabla_\gamma + \frac{1}{2} \mathcal{K}_{ABd}^e M_e d - w F_{AB} I .
\] (51)

The quantities \( \mathcal{K}_{AB} \), \( \mathcal{K}_{ABd} \) and \( F_{ABd} \) are expressed\(^{11}\) in terms of the Poincaré superspace covariant derivative \( \nabla_A \) and \( J_\alpha \). The two equations immediately above inform us that the super-Weyl covariants \( \hat{T}_{AB} \) and \( \hat{R}_{ABd} \) are related to the corresponding Poincaré objects \( T_{AB} \) and \( R_{ABd} \) via,

\[
\hat{T}_{AB} = T_{AB} + \hat{\mathcal{K}}_{AB} , \quad \hat{R}_{ABd} = R_{ABd} + \hat{\mathcal{K}}_{ABd} .
\] (52)

7 Scale vs. Non-Scale Invariant Equations of Motion

The analysis of the conformal properties of the full non-linear theory given in a previous chapters once again provides an argument against the result known as “Howe’s Theorem” that asserts that there cannot appear a dimension 1/2 and spin-1/2 auxiliary field strength superfield. Thus Howe’s Theorem is equivalent to the imposition of the condition that \( J_\alpha = 0 \) be imposed as a kinematic constraint. Let us now impose this condition in addition to those that appear in (21). The scale transformation properties of the superspace are still described by (23)\(^{12}\). In turn this

\(^{11}\)Although the calculation of the \( \mathcal{K} \)-quantities, \( \hat{\mathcal{K}} \)-quantities and as well \( F_{ABd} \) is straightforward, we will forego giving explicit expressions for all of these.
observation together with a consistency condition applied to (28) leads to the condition \( \nabla_a L = 0 \). This is the full nonlinear extension of (10) and leads to the exact same problem as we found in our pre-potential analysis.

Let us use these insights in a different way. In this very short section, we will simply compare properties of the non-supersymmetric Poincaré gravity equations of motions with two possible supersymmetric extensions.

For the ordinary 11D \( x \)-space covariant derivative, a scale transformation law is given by

\[
\delta_S \nabla_a = \ell \nabla_a - (\nabla_k \ell) \mathcal{M}_a^k ,
\]

where \( \ell(x) \) is a local scale parameter. This implies that the 11D \( x \)-space Riemann curvature tensor transforms as

\[
\delta_S r_{ab}^{\text{de}} = \ell r_{ab}^{\text{de}} - (\nabla[^a \nabla[^d \ell) \delta_\ell]^c] .
\]

The constraint is, of course, that the torsion tensor vanishes \( t_{abc} = 0 \). In the absence of matter, the usual Einstein-Hilbert action leads to the expected equation of motion

\[
\mathcal{E}_{ab} \equiv r_{ac}^{\text{b}c} - \frac{1}{2} \eta_{ab} r_{cd}^{\text{cd}} = 0 ,
\]

i.e. the Einstein tensor vanishes. It is a simple matter to show that this equation of motion is not a scale-invariant condition since,

\[
\delta_S \mathcal{E}_{ab} = 2\ell \mathcal{E}_{ab} - 9 (\nabla_a \nabla_b \ell) + 9 \eta_{ab} (\nabla^c \nabla_c \ell) .
\]

Now in the scheme that was proposed in the works in [1] and [8], the equations of motions are

\[
\mathcal{J}_a = 0 , \ X_{[ab]}^c = 0 , \ X_{[abdef]}^c = 0 ,
\]

Due to the observation in (28), we see that the first of these superfield equations is also not a super scale-invariant condition.

This is to be contrasted with the schemes ([9] and [15]) where the equations of motions are

\[
X_{[ab]}^c = 0 , \ X_{[abdef]}^c = 0 .
\]

but the condition \( \mathcal{J}_a = 0 \) is considered to be a constraint. Thus, in these schemes the equations of motion are super scale covariant. From this point of view, the role of the \( \mathcal{J} \)-tensor is to extend the non-scale covariance of the Poincaré equations of motion in \( x \)-space into the non-scale covariance of Poincaré equations of motion in superspace.
So to accept the validity of Howe’s Theorem\textsuperscript{12} is equivalent to imposing super scale covariance on the equations of motion.

In M-theory, the low-energy effective action is one whose lowest order terms describe 11D Poincaré supergravity. Some of the structure of the next order corrections in an expansion $\ell_{11}$ have been discussed in the literature. None to date lead to scale-invariant terms appearing in the equations of motion. Thus, it is our position that this alone precludes any superspace obeying Howe’s Theorem from being able to describe the M-theory effective action. The non-scale covariance of the M-theory effective action equations of motion demand the appearance of the $J$-tensor.

\section{Summary Discussion}

Under the impetus of the effective M-theory action’s superspace formulation, problems in 11D supergravity are now being faced. This is welcomed activity. However, we are still without truly fundamental insight into off-shell 11D, $N = 1$ supergravity, despite the work underway. It is useful to use 4D, $N = 1$ supergravity to make this point most sharply. We can do this by giving further consideration to the situation of the scale compensator. There is another layer to the structure of the theory. The truly irreducible minimal 4D, $N = 1$ off-shell supergravity compensator satisfies the equation $J_\alpha = 0$. The solution to this in four dimensions implies that $\Psi$ is a linear combination of a chiral superfield and its conjugate. It is the analog of this irreducibility condition that we still lack in the 11D theory because chiral superfields cannot exist in 11D superspace.

Since chirality does not exist in 11D, there are no non-trivial conformal fluctuations of the flat metric to the equation $J_\alpha = 0$.

The supergravity vielbein in (12) with its $2^{32}$ d. f. (d. f. $\equiv$ degrees of freedom) is an off-shell but highly reducible supergravity representation. It requires the imposition of differential equations upon it in order to reach a minimal irreducible off-shell representation. Based on some structures observed in an unusual class of algebras \textsuperscript{21}, we conjecture that such irreducible representation exist. For example we have found some evidence that there may well exist a 32,768 bosonic d. f. and 32,768 fermionic d. f. irreducible representation of $D = 11$ supersymmetry in superspace. However, confirmation of this remains for the future. The problem of classifying 11D superspace irreducible representations remains an important unsolved puzzle.

\textsuperscript{12}The authors of \cite{15} have noted the need to check explicit M-theory currents for consistency with Howe’s theorem.
We again point out for skeptics that non-minimal 4D, \( N = 1 \) superspace geometry is the avatar to set the pattern to be followed by 11D supergravity/M-theory superspace. It is a demonstrable fact that imposing the solely conventional constraints leads to four field strength superfields; \( W_{abcd}, J_\alpha, X_{[ab]}^c \) and \( X_{[abdef]}^c \), one on-shell and three off-shell field strengths. The first of these contains all the usual conformal degrees of freedom, the second allows for the traditional realization of the Weyl symmetry within a Poincaré superspace and the latter two superfield are roughly speaking the analogs of \( G_a \) in the non-minimal 4D, \( N = 1 \) superspace supergravity theory. However, \( X_{[ab]}^c \) and \( X_{[abdef]}^c \) differ from \( G_a \) in the important respect that they are also conformally covariant like \( W_{abcd} \). The superfield \( W_{abcd} \) has conformal weight one while \( X_{[ab]}^c \) and \( X_{[abdef]}^c \) have conformal weight zero. In terms of semi-prepotentials, all the dynamics are contained in two superfields \( H_\alpha^c \) and \( \Psi \). The first of these is the gauge field for the Weyl degrees of freedom and the second is the Goldstone superfield for breaking superconformal symmetry to super Poincaré symmetry.

Some years ago, we also conjectured \cite{23} that the ultimate formulation of covariant string field theory must be one in which the fundamental string field functional should appear as a pre-potential in an as-yet undiscovered geometrical formulation. In the most complete formulation of open string field theory to date \cite{24}, Berkovits has presented a formulation that is hauntingly reminiscent of the 4D, \( N = 1 \) pre-potential formulation of Yang-Mills theory. This is precisely in accord with our conjecture in \cite{23} and encourages our belief in a type of universality of pre-potential formulations in all theories that contain supergravity.

It has been the suggestion of Cederwall, Gran, Nielsen and Nilsson \cite{13} that the quantities \( X_{[ab]}^c \) and \( X_{[abdef]}^c \) are both equations of motion. In the work of \cite{1} we have pointed out that if one can be set equal to zero as a constraint that the choice \( X_{[abdef]}^c = 0 \) leads to a smaller supergravity multiplet (if this is a viable option). It also the case that both of these \( X \)-field strength superfields cannot simultaneously be set to zero as constraints. To do so would force the conformal semi-prepotential \( H_\alpha^c \) to be zero up to a pure gauge transformation.

If 11D supergravity is like (i.e. separable) other prepotential formulation of supergravity theories that possess both a conformal multiplet and a conformal compensator then the latter will necessarily demand non-vanishing torsion at dimension 1/2. The 11D T–C\(^2\) theorem is the formal statement of the most fundamental reason why it has been our position that Howe’s Theorem is specious. Stated another way, if Howe’s Theorem were true and the \( X \)-tensors were the only ones required to completely describe the dynamics of the M-theory 11D effective action, then this effective action
would possess a superconformal symmetry. Not possessing a clear understanding of this has led to a number of confused efforts in the research literature by different research groups working on both 11D and 10D superspace problems [25].

In this present work we have found evidence that in 11D supergravity, the canonical split of the fundamental degrees of freedom into a superconformal pre-potential $H_\alpha{}^c$ and a conformal compensator $\Psi$ is valid and that it is possible to conventionally realize a superconformal symmetry. As 11D supergravity has been proposed as a limit of M-theory, our findings naturally suggest that this canonical split likely carries over to the complete M-theory itself! That is, it appear likely to us that M-theory has a similar split and that it is possible to realize a superconformal symmetry on whatever may play the role of the fundamental degrees of freedom that describe M-theory. Thus, the concept of the pre-potential will likely play an important role in the final formulation of M-theory. We thus conjecture that there exist some formulation of M-theory that possesses all the canonical structures of superspace supergravity; a conformal pre-potential, a conformal compensating pre-potential and a realization of superconformal symmetry.

“Understanding and wisdom only comes to us when our respective muses deign to receive an audience. Knowing this is the beginning of both.”
– Anonymous

Acknowledgment

We wish to acknowledge discussions with M. Cederwall and P. Howe, B. Nilsson and H. Nishino.

Added Note in Proof

After the completion of this work, we received communications from B. Nilsson stating that the issue of superconformal symmetry leads one to “... reconsider Weyl superspace as soon as one puts in explicit expressions for the $X$-tensors.” As well their work requires a computation of the Weyl-connection to verify the validity of ‘Weyl superspace’.

A component level discussion of 11D superconformal symmetry may be found in our final reference.

Appendix : Conventions and Notation

The conventions that we use for 11D superspace have been stated in some detail in our previous work [8]. In particular, we use real (i.e. Majorana) 32-component
spinors for the Grassmann coordinates of superspace. Our $\gamma$-matrices are defined by

$$\{ \gamma^a, \gamma^b \} = 2 \eta^{ab} I ,$$  \hspace{1cm} (A.1)

where the signature of the metric is the “mostly minus one,” i.e. diag. $(+, -, \cdots, -)$. This implies that our gamma matrices satisfy the complex conjugation conditions

$$\begin{align*}
\left[ (\gamma^a)_\alpha^\beta \right]^* &= - (\gamma^a)_\alpha^\beta, \\
\left[ (\gamma^{[2]})_\alpha^\beta \right]^* &= (\gamma^{[2]})_\alpha^\beta, \\
\left[ (\gamma^{[5]})_\alpha^\beta \right]^* &= - (\gamma^{[5]})_\alpha^\beta.
\end{align*}$$

Our ‘spinor metric,’ with which we raise and lower spinor indices, is denoted by $C_{\alpha\beta}$ and satisfies,

$$C_{\alpha\beta} = - C_{\beta\alpha}, \quad [C_{\alpha\beta}]^* = - C_{\alpha\beta} \hspace{1cm} (A.3)$$

The inverse spinor metric $C^{\alpha\beta}$ is defined to satisfy

$$C_{\alpha\beta} C^{\gamma\beta} = \delta_\alpha^\gamma \hspace{1cm} (A.4)$$

We also use superspace conjugation which permits the appearance of appropriate factors of $i$ even within a theory of involving solely real spinors. A complete discussion of superspace conjugation can be found in a recent pedagogical presentation [21] (p. 13).

We define our gamma matrices with multiple numbers of vector indices through the equations

$$\begin{align*}
\gamma_a \gamma_b &= \gamma_{ab} + \eta_{ab} , \\
\gamma_a \gamma_{bc} &= \gamma_{abc} + \eta_{[b|c]} , \\
\gamma_a \gamma_{bcd} &= \gamma_{abcd} + \frac{1}{2} \eta_{[b|c|d]} , \\
\gamma_a \gamma_{bcde} &= \gamma_{abcde} + \frac{1}{6} \eta_{[b|c|d|e]} , \\
\gamma_a \gamma_{bcdef} &= i \frac{1}{120} \epsilon_{bcdef}^{[5]} \gamma^{[5]} + \frac{1}{24} \eta_{[b|c|d|e|f]} , \\
\gamma_{bcdef} \gamma_a &= - i \frac{1}{120} \epsilon_{bcdef}^{[5]} \gamma^{[5]} + \frac{1}{24} \eta_{[b|c|d|e|f]}.
\end{align*}$$

(A.5)
The basic identities for non-vanishing traces over the gamma matrices are

\[
\frac{1}{32} \text{Tr} (\gamma^a \gamma^b) = \delta^a_b, \\
\frac{1}{32} \text{Tr} (\gamma_{a_1 a_2} \gamma^{b_1 b_2}) = - \delta_{[a_1}^{b_1} \delta_{a_2]^{b_2}}, \\
\frac{1}{32} \text{Tr} (\gamma_{a_1 a_2 a_3} \gamma^{b_1 b_2 b_3}) = - \delta_{[a_1}^{b_1} \delta_{a_2]^{b_2} \delta_{a_3]^{b_3}}, \\
\frac{1}{32} \text{Tr} (\gamma_{a_1 a_2 a_3 a_4} \gamma^{b_1 b_2 b_3 b_4}) = \delta_{[a_1}^{b_1} \delta_{a_2]^{b_2} \delta_{a_3]^{b_3} \delta_{a_4]^{b_4}}, \\
\frac{1}{32} \text{Tr} (\gamma_{a_1 \cdots a_{11}}) = i \epsilon_{a_1 \cdots a_{11}}.
\]

Using the spinor metric to lower one spinor index of the quantities in (A.2) we find

\[
[ (\gamma^a)_{\alpha \beta} ]^* = (\gamma^a)_{\alpha \beta}, \quad [ (\gamma^{[2]} )_{\alpha \beta} ]^* = -(\gamma^{[2]} )_{\alpha \beta}, \\
[ (\gamma^{[3]} )_{\alpha \beta} ]^* = (\gamma^{[3]} )_{\alpha \beta}, \quad [ (\gamma^{[4]} )_{\alpha \beta} ]^* = -(\gamma^{[4]} )_{\alpha \beta}, \quad (A.7)
\]

and as well the same equations apply to the matrices with two raised spinor indices. In addition these satisfy the symmetry relations

\[
(\gamma^a)_{\alpha \beta} = (\gamma^a)_{\beta \alpha}, \quad (\gamma^{[2]} )_{\alpha \beta} = (\gamma^{[2]} )_{\beta \alpha}, \quad (\gamma^{[5]} )_{\alpha \beta} = (\gamma^{[5]} )_{\beta \alpha}, \quad (A.8)
\]

\[
(\gamma^{[3]} )_{\alpha \beta} = -(\gamma^{[3]} )_{\beta \alpha}, \quad (\gamma^{[4]} )_{\alpha \beta} = -(\gamma^{[4]} )_{\beta \alpha},
\]

where the same equations apply to the matrices with two raised spinor indices.

Other identities on the 11D gamma matrices include

\[
\gamma^{[P]} \gamma_{[Q]} \gamma^{[P]} = c_{[Q][P]} \gamma^{[Q]}, \quad (A.9)
\]

where the coefficients \(c_{[Q][P]}\) are given in the following table.

| \( [Q] = 1 \) | \( [Q] = 2 \) | \( [Q] = 3 \) | \( [Q] = 4 \) | \( [Q] = 5 \) |
|---|---|---|---|---|
| \( [P] = 1 \) | -9 | -70 | 450 | 2,160 | -5,040 |
| \( [P] = 2 \) | 7 | -38 | -126 | -144 | -5,040 |
| \( [P] = 3 \) | -5 | -14 | -30 | -528 | 1,680 |
| \( [P] = 4 \) | 3 | 2 | 66 | -144 | 1,680 |
| \( [P] = 5 \) | -1 | 10 | -30 | 240 | -1,200 |

For example this table implies

\[
\gamma^{[3]} \gamma^{[2]} \gamma^{[3]} = -126 \gamma^{[2]}, \quad (A.10)
\]
The role of this object is that it allows us to convert the components of a super 7-form to those of a super 4-form. Some useful Fierz-type identities include the following:

\[
0 = (\gamma^a)_{(\alpha\beta)}(\gamma_{ab})_{\gamma\delta}, \\
0 = 5(\gamma^a)_{(\alpha\beta)}(\gamma_a)_{\gamma\delta} + \frac{1}{2}(\gamma_2)_{(\alpha\beta)}(\gamma_{[2]})_{\gamma\delta}, \\
0 = 6(\gamma^a)_{(\alpha\beta)}(\gamma_a)_{\gamma\delta} + \frac{1}{5!}(\gamma_5)_{(\alpha\beta)}(\gamma_{[5]})_{\gamma\delta}, \\
0 = (\gamma^a)_{(\alpha\beta)}(\gamma_{abcede})_{\gamma\delta} - \frac{1}{8}(\gamma_{ab})_{(\alpha\beta)}(\gamma_{[cd]})_{\gamma\delta}, \\
0 = \frac{1}{2}(\gamma^{ab})_{(\alpha\beta)}(\gamma_{abcede})_{\gamma\delta} + 2(\gamma_{[c]})_{(\alpha\beta)}(\gamma_{[de]})_{\gamma\delta}, \\
0 = (\gamma^a)_{(\alpha\beta)}(\gamma_a)_{\gamma\delta} - \frac{1}{2}(\gamma_2)_{(\alpha\beta)}(\gamma_{[2]})_{\gamma\delta} + \frac{1}{5!}(\gamma_5)_{(\alpha\beta)}(\gamma_{[5]})_{\gamma\delta}. \\
\]

(A.11)

Our Lorentz generator is defined to realize

\[
[M_{ab}, (\gamma_a)_{\alpha\beta}] = 0, \\
[M_{ab}, \nabla_a] = \frac{1}{2}(\gamma_{ab})_{\alpha\beta}\nabla_\beta, \\
[M_{ab}, \nabla_c] = \eta_{ca}\nabla_b - \eta_{cb}\nabla_a, \\
[M_{ab}, M_{cd}] = \eta_{ca}M_{bd} - \eta_{cb}M_{ad} - \eta_{da}M_{bc} + \eta_{db}M_{ac}. \\
\]

(A.12)

One notational device not discussed in our previous paper is the definition of the 11D super-epsilon tensor, which we denote by \(\tilde{\epsilon}_{A_1...A_4B_1...B_7}\), and define by

\[
\tilde{\epsilon}_{a_1...a_4}^{b_1...b_7} \equiv \epsilon_{a_1...a_4}^{b_1...b_7}, \\
\tilde{\epsilon}_{a_1a_2,a_4}^{b_1b_2,...b_7} \equiv (\gamma^a_{b_1})_{\alpha_1\beta_1}\epsilon_{a_1...a_4}^{b_1...b_7}, \\
\tilde{\epsilon}_{a_1a_2,a_3a_4}^{b_1b_2b_3b_4} \equiv (\gamma^a_{b_1})_{\alpha_1\beta_1}(\gamma^b_{b_2})_{\alpha_2\beta_2}\epsilon_{a_1...a_4}^{b_1...b_7}, \\
\tilde{\epsilon}_{a_1a_2,a_3a_4}^{b_1b_2b_3b_4b_5b_6b_7} \equiv (\gamma^a_{b_1})_{\alpha_1\beta_1}(\gamma^b_{b_2})_{\alpha_2\beta_2}(\gamma^c_{b_3})_{\alpha_3\beta_3}\epsilon_{a_1...a_4}^{b_1...b_7}. \\
\]

(A.13)

The role of this object is that it allows us to convert the components of a super 7-form into the dual components of a super 4-form via the definitions:

\[
\tilde{X}_{a_1...a_4} \equiv \frac{1}{7!}\tilde{\epsilon}_{a_1...a_4}^{b_1...b_7}X_{b_1...b_7}, \\
\tilde{X}_{a_1a_2,a_3a_4} \equiv \frac{1}{6!}\tilde{\epsilon}_{a_1a_2,a_3a_4}^{b_1b_2...b_7}X_{b_1b_2...b_7}, \\
\tilde{X}_{a_1a_2,a_3a_4} \equiv \frac{1}{3!}\tilde{\epsilon}_{a_1a_2,a_3a_4}^{b_1b_2b_3b_4...b_7}X_{b_1b_2b_3b_4...b_7}, \\
\tilde{X}_{a_1a_2,a_3a_4} \equiv \frac{1}{2!}\tilde{\epsilon}_{a_1a_2,a_3a_4}^{b_1b_2b_3b_4}X_{b_1b_2b_3b_4}, \\
\tilde{X}_{a_1a_2,a_3a_4} \equiv \frac{1}{1!}\tilde{\epsilon}_{a_1a_2,a_3a_4}^{b_1b_2b_3b_4b_5b_6b_7}X_{b_1b_2b_3b_4b_5b_6b_7}. \\
\]

(A.14)

As well, it can be used for the reverse purpose of converting the components of a super 4-form to those of a super 7-form. The super-epsilon tensor concept has proven very useful for 10D theories as we suspect will also be the case for 11D theories.
References

[1] S. J. Gates, Jr. and H. Nishino, “Deliberations on 11D Superspace for the M-Theory Effective Action,” Univ. of MD preprint UMDEPP 00-032, hep-th/0101037 (revised).

[2] S. J. Gates, Jr. and H. Nishino, Phys. Lett. **173B** (1986) 52.

[3] S. J. Gates, Jr. and S. I. Vashakidze, Nucl. Phys. **B291** (1987) 173; S. J. Gates, Jr. and H. Nishino, Nucl. Phys. **B291** (1987) 205.

[4] S. Bellucci and S. J. Gates, Jr., Phys. Lett. **208B** (1988) 456.

[5] S. Bellucci, S. J. Gates, Jr. and D. Depireux, Phys. Lett. **238B** (1990) 315.

[6] S. Bellucci, S. J. Gates, Jr., B. Radak, P. Majumdar and S. Vashakidze, Mod. Phys. Lett. **A21** (1989) 1985.

[7] S. J. Gates, Jr., Phys. Lett. **96B** (1980) 305.

[8] H. Nishino and S. J. Gates, Jr., Phys. Lett. **388B** (1996) 504.

[9] P. Howe, Phys. Lett. **415B** (1997) 149 (hep-th/9707184).

[10] S. J. Gates, Jr. and R. Grimm, Phys. Lett. **133B** (1983) 192.

[11] W. Siegel, “A Derivation of the Supercurrent Superfield,” HUTP-77/A089, Dec 1977; ibid. “The Superfield Supergravity Action,” HUTP-77/A080, Dec 1977; idem. Nucl. Phys. **B142** (1978) 301.

[12] P. Howe and R. Tucker, Phys. Lett. **80B** (1978) 138.

[13] W. Siegel, Phys. Lett. **80B** (1979) 224; S. J. Gates, Jr., in *Supergravity* eds. P. van Nieuwenhuizen and D. Z. Freedman, North-Holland, Amsterdam, 1979, pp. 215-219; idem. Nucl. Phys. **B162** (1980) 79; ibid. Nucl. Phys. **B176** (1980) 397; P. Howe, Phys. Lett. **100B** (1980) 389; idem., Nucl. Phys. **199** (1982) 309.

[14] S. J. Gates, Jr. and H. Nishino, Phys. Lett. **266B** (1991) 14.

[15] M. Cederwall, U. Gran, M. Nielsen and B. Nilsson, “Manifestly Supersymmetric M-Theory,” Göteborg ITP preprint, hep-th/007035.

[16] S. J. Gates, Jr. and H. Nishino, Class. and Quant. Grav. **3** (1986) 391.
[17] M. Brown and S. J. Gates, Jr., Annals of Physics, 122, No. 2 (1979) 443; S. J. Gates, Jr. and W. Siegel, Nucl. Phys. B163 (1980) 519; S. J. Gates, Jr., M. T. Grisaru, M. Roček and W. Siegel, “Superspace or One Thousand and One Lessons in Supersymmetry,” Benjamin Cummings (Addison-Wesley), Reading, MA (1983).

[18] P. Breitenlohner, Phys. Lett. 67B (1977) 49; idem. Nucl. Phys. B124 (1977) 500; W. Siegel, Harvard preprint HUTP-77/A068 (November, 1977) and Harvard preprint HUTP-77/A080 (November, 1977); S. J. Gates, Jr. and J. A. Shapiro, Phys. Rev. D18 (1978) 2768; W. Siegel and S. J. Gates, Jr., Nucl. Phys. B147 (1979) 77; S. J. Gates, Jr. and M. Brown, Nucl. Phys. B165 (1980) 445.

[19] K. Peeters, P. Vanhove and A. Westerberg, “Supersymmetric $R^4$ Actions and Quantum corrections to Superspace Torsion Constraints,” [hep-th/0010182]; idem. “Supersymmetric Higher-derivative Actions in Ten and Eleven Dimensions, the Associated Superalgebras and Their Formulation in Superspace’’, [hep-th/0010167].

[20] S. J. Gates, Jr. and L. Rana, “A Primer on Supersymmetric Quantum Mechanics (I),” preprint UMDEPP 96-38, in preparation.

[21] S. J. Gates, Jr., “Basic Canon in $D = 4, N = 1$ Superfield Theory: Five Primer Lectures,” in Supersymmetry, Supergravity and Supercolliders: TASI 97, ed. J. Bagger, World Scientific, Singapore, 1999, [hep-th/9809064].

[22] S. J. Gates, Jr., L. Lu and R. Oerter, Phys. Lett. 218B (1989) 33.

[23] S. J. Gates, Jr., “Strings, Superstrings and Two-Dimensional Lagrangian Field Theory”, in Functional Integration, Geometry and Strings, the Proceedings of the XXV Winter School of Theoretical Physics in Karpacz, Poland (Feb., 1989) Z.Haba and J.Sobczyk, Birkhauser-Verlag Press (1989) pp. 140-184.

[24] N. Berkovits, “Review of Open Superstring Field Theory,” [hep-th/0105230].

[25] L. Bonora, P. Pasti, and M. Tonin, Phys. Lett. 188B (1987) 335; S. Ferrara, P. Fré and M. Porrati, Ann. Phys. 175 (1987) 112.

[26] E. Bergshoeff and M. de Roo, Phys. Lett. 138B (1984) 67.