TOPOLOGICAL DIAGONALIZATIONS AND HAUSDORFF DIMENSION

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Abstract. The Hausdorff dimension of a product $X \times Y$ can be strictly greater than that of $Y$, even when the Hausdorff dimension of $X$ is zero. But when $X$ is countable, the Hausdorff dimensions of $Y$ and $X \times Y$ are the same. Diagonalizations of covers define a natural hierarchy of properties which are weaker than “being countable” and stronger than “having Hausdorff dimension zero”. Fremlin asked whether it is enough for $X$ to have the strongest property in this hierarchy (namely, being a $\gamma$-set) in order to assure that the Hausdorff dimensions of $Y$ and $X \times Y$ are the same.

We give a negative answer: Assuming the Continuum Hypothesis, there exists a $\gamma$-set $X \subseteq \mathbb{R}$ and a set $Y \subseteq \mathbb{R}$ with Hausdorff dimension zero, such that the Hausdorff dimension of $X + Y$ (a Lipschitz image of $X \times Y$) is maximal, that is, 1. However, we show that for the notion of a strong $\gamma$-set the answer is positive. Some related problems remain open.

1. Introduction

The Hausdorff dimension of a subset of $\mathbb{R}^k$ is a derivative of the notion of Hausdorff measures [4]. However, for our purposes it will be more convenient to use the following equivalent definition. Denote the diameter of a subset $A$ of $\mathbb{R}^k$ by $\text{diam}(A)$. The Hausdorff dimension of a set $X \subseteq \mathbb{R}^k$, $\dim(X)$, is the infimum of all positive $\delta$ such that for each positive $\epsilon$ there exists a cover $\{I_n\}_{n \in \mathbb{N}}$ of $X$ with

$$\sum_{n \in \mathbb{N}} \text{diam}(I_n)^\delta < \epsilon.$$ 

From the many properties of Hausdorff dimension, we will need the following easy ones.

Lemma 1.

1. If $X \subseteq Y \subseteq \mathbb{R}^k$, then $\dim(X) \leq \dim(Y)$.
2. Assume that $X_1, X_2, \ldots$ are subsets of $\mathbb{R}^k$ such that $\dim(X_n) = \delta$ for each $n$. Then $\dim(\bigcup_n X_n) = \delta$.
3. Assume that $X \subseteq \mathbb{R}^k$ and $Y \subseteq \mathbb{R}^m$ is such that there exists a Lipschitz surjection $\phi : X \to Y$. Then $\dim(X) \geq \dim(Y)$.
4. For each $X \subseteq \mathbb{R}^k$ and $Y \subseteq \mathbb{R}^m$, $\dim(X \times Y) \geq \dim(X) + \dim(Y)$.

Equality need not hold in item (4) of the last lemma. In particular, one can construct a set $X$ with Hausdorff dimension zero and a set $Y$ such that $\dim(X \times Y)$ is maximal, that is, 1. However, we show that for the notion of a strong $\gamma$-set the answer is positive. Some related problems remain open.

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and at step \( n \) if for each sequence \( U_n \) times the size of the interval. For each \( n \) (shrunk) copies of \( \lambda \) subinterval of size the unit interval, and at each step removing from the middle of each interval a subinterval of size \( \lambda \) times the size of the interval (So that \( C(1/3) \) is the canonical middle-third Cantor set, which has Hausdorff dimension \( \log 2/\log 3 \)). It is easy to see that if \( \lambda_n \searrow 1 \), then \( \dim(C(\lambda_n)) \searrow 0 \).

Thus, define a special Cantor set \( C(\{\lambda_n\}_{n \in \mathbb{N}}) \) by starting with the unit interval, and at step \( n \) removing from the middle of each interval a subinterval of size \( \lambda_n \) times the size of the interval. For each \( n \), \( C(\{\lambda_n\}_{n \in \mathbb{N}}) \) is contained in a union of \( 2^n \) (shrunk) copies of \( C(\lambda_n) \), and therefore \( \dim(C(\{\lambda_n\}_{n \in \mathbb{N}})) \leq \dim(C(\lambda_n)) \). \( \square \)

As every countable set has strong measure zero, the latter notion can be thought of an “approximation” of countability. In fact, Borel conjectured in [3] that every strong measure zero set is countable, and it turns out that the usual axioms of mathematics (ZFC) are not strong enough to prove or disprove this conjecture: Assuming the Continuum Hypothesis there exists an uncountable strong measure zero set (namely, a Luzin set), but Laver [10] proved that one cannot prove the existence of such an object from the usual axioms of mathematics.

The property of strong measure zero (which depends on the metric) has a natural topological counterpart. A topological space \( X \) has Rothberger’s property \( C'' \) [13] if for each sequence \( \{U_n\}_{n \in \mathbb{N}} \) of covers of \( X \) there is a sequence \( \{V_n\}_{n \in \mathbb{N}} \) such that for each \( n \) \( U_n \in \mathcal{U} \), and \( \{V_n\}_{n \in \mathbb{N}} \) is a cover of \( X \). Using Scheepers’ notation [15], this property is a particular instance of the following selection hypothesis (where \( \mathcal{U} \) and \( \mathcal{V} \) are any collections of covers of \( X \)):

**Proposition 2** (folklore). There exists a perfect set of reals \( X \) with Hausdorff dimension zero.

**Proof.** For \( 0 < \lambda < 1 \), denote by \( C(\lambda) \) the Cantor set obtained by starting with the unit interval, and at each step removing from the middle of each interval a subinterval of size \( \lambda \) times the size of the interval. For each \( \lambda_n \), \( C(\{\lambda_n\}_{n \in \mathbb{N}}) \) is a union of \( 2^n \) (shrunk) copies of \( C(\lambda_n) \), and therefore \( \dim(C(\{\lambda_n\}_{n \in \mathbb{N}})) \leq \dim(C(\lambda_n)) \). \( \square \)

Let \( \mathcal{O} \) denote the collection of all open covers of \( X \). Then the property considered by Rothberger is \( S_1(\mathcal{O}, \mathcal{O}) \). Fremlin and Miller [5] proved that a set \( X \subseteq \mathbb{R}^k \) satisfies \( S_1(\mathcal{O}, \mathcal{O}) \) if, and only if, \( X \) has strong measure zero with respect to each metric which generates the standard topology on \( \mathbb{R}^k \).
But even Rothberger’s property for $X$ is not strong enough to have Equation 1 hold: It is well-known that every Luzin set satisfies Rothberger’s property (and, in particular, has Hausdorff dimension zero).

**Lemma 3.** The mapping $(x, y) \mapsto x + y$ from $\mathbb{R}^2$ to $\mathbb{R}$ is Lipschitz.

*Proof.* Observe that for nonnegative reals $a$ and $b$, $(a - b)^2 \geq 0$ and therefore $a^2 + b^2 \geq 2ab$. Consequently,

$$a + b = \sqrt{a^2 + 2ab + b^2} \leq \sqrt{2(a^2 + b^2)} = \sqrt{2a^2 + 2b^2}.$$ 

Thus, $|(x_1 + y_1) - (x_2 + y_2)| \leq \sqrt{2} \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. □

Assuming the Continuum Hypothesis, there exists a Luzin set $L \subseteq \mathbb{R}$ such that $L + L$, a Lipschitz image of $L \times L$, is equal to $\mathbb{R}$ [9].

We therefore consider some stronger properties. An open cover $U$ of $X$ is an $\omega$-cover of $X$ if each finite subset of $X$ is contained in some member of the cover, but $X$ is not contained in any member of $U$. $U$ is a $\gamma$-cover of $X$ if it is infinite, and each element of $X$ belongs to all but finitely many members of $U$. Let $\Omega$ and $\Gamma$ denote the collections of open $\omega$-covers and $\gamma$-covers of $X$, respectively. Then $\Gamma \subseteq \Omega \subseteq \mathcal{O}$, and these three classes of covers introduce 9 properties of the form $S_1(\Omega, \Gamma)$. If we remove the trivial ones and check for equivalences [9, 20], then it turns out that only six of these properties are really distinct, and only three of them imply Hausdorff dimension zero:

$$S_1(\Omega, \Gamma) \to S_1(\Omega, \Omega) \to S_1(\mathcal{O}, \mathcal{O}).$$

The properties $S_1(\Omega, \Gamma)$ and $S_1(\Omega, \Omega)$ were also studied before. $S_1(\Omega, \Omega)$ was studied by Sakai [14], and $S_1(\Omega, \Gamma)$ was studied by Gerlits and Nagy in [8]: A topological space $X$ is a $\gamma$-set if each $\omega$-cover of $X$ contains a $\gamma$-cover of $X$. Gerlits and Nagy proved that $X$ is a $\gamma$-set if, and only if, $X$ satisfies $S_1(\Omega, \Gamma)$. It is not difficult to see that every countable space is a $\gamma$-set. But this property is not trivial: Assuming the Continuum Hypothesis, there exist uncountable $\gamma$-sets [7].

$S_1(\Omega, \Omega)$ is closed under taking finite powers [9], thus the Luzin set we used to see that Equation 1 need not hold when $X$ satisfies $S_1(\mathcal{O}, \mathcal{O})$ does not rule out that possibility that this Equation holds when $X$ satisfies $S_1(\Omega, \Omega)$. However, in [2] it is shown that assuming the Continuum Hypothesis, there exist Luzin sets $L_0$ and $L_1$ satisfying $S_1(\Omega, \Omega)$, such that $L_0 + L_1 = \mathbb{R}$. Thus, the only remaining candidate for a nontrivial property of $X$ where Equation 1 holds is $S_1(\Omega, \Gamma)$ ($\gamma$-sets). Fremlin (personal communication) asked whether Equation 1 is indeed provable in this case. We give a negative answer, but show that for a yet stricter (but nontrivial) property which was considered in the literature, the answer is positive.

The notion of a strong $\gamma$-set was introduced in [7]. However, we will adopt the following simple characterization from [20] as our formal definition. Assume that $\{U_n\}_{n \in \mathbb{N}}$ is a sequence of collections of covers of a space $X$, and that $\mathcal{Y}$ is a collection of covers of $X$. Define the following selection hypothesis.

$S_1(\{U_n\}_{n \in \mathbb{N}}, \mathcal{Y})$: For each sequence $\{U_n\}_{n \in \mathbb{N}}$ where $U_n \in \mathcal{U}_n$ for each $n$, there is a sequence $\{U_n\}_{n \in \mathbb{N}}$ such that $U_n \in \mathcal{U}_n$ for each $n$, and $\{U_n\}_{n \in \mathbb{N}} \in \mathcal{Y}$.

A cover $U$ of a space $X$ is an $n$-cover if each $n$-element subset of $X$ is contained in some member of $U$. For each $n$ denote by $\mathcal{O}_n$ the collection of all open $n$-covers of a space $X$. Then $X$ is a strong $\gamma$-set if $X$ satisfies $S_1(\{\mathcal{O}_n\}_{n \in \mathbb{N}}, \Gamma)$. 

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In most cases $S_1(\{O_n\}_{n \in \mathbb{N}}, \mathcal{U})$ is equivalent to $S_1(\Omega, \mathcal{U})$ [20], but not in the case $\mathcal{U} = \Gamma$: It is known that for a strong $\gamma$-set $G \subseteq \{0, 1\}^\mathbb{N}$ and each $A \subseteq \{0, 1\}^\mathbb{N}$ of measure zero, $G \oplus A$ has measure zero too [7]; this can be contrasted with Theorem 5 below. In Section 3 we show that Equation 1 is provable in the case that $X$ is a strong $\gamma$-set, establishing another difference between the notions of $\gamma$-sets and strong $\gamma$-sets, and giving a positive answer to Fremlin’s question under a stronger assumption on $X$.

2. The product of a $\gamma$-set and a set of Hausdorff dimension zero

Theorem 4. Assuming the Continuum Hypothesis (or just $p = c$), there exist a $\gamma$-set $X \subseteq \mathbb{R}$ and a set $Y \subseteq \mathbb{R}$ with Hausdorff dimension zero such that the Hausdorff dimension of the algebraic sum

$$X + Y = \{x + y : x \in X, y \in Y\}$$

(a Lipschitz image of $X \times Y$ in $\mathbb{R}$) is 1. In particular, $\dim(X \times Y) \geq 1$.

Our theorem will follow from the following related theorem. This theorem involves the Cantor space $\{0, 1\}^\mathbb{N}$ of infinite binary sequences. The Cantor space is equipped with the product topology and with the product measure.

Theorem 5 (Bartoszyński and Recław [1]). Assume the Continuum Hypothesis (or just $p = c$). Fix an increasing sequence $\{k_n\}_{n \in \mathbb{N}}$ of natural numbers, and for each $n$ define

$$A_n = \{f \in \{0, 1\}^\mathbb{N} : f \mid [k_n, k_{n+1}) \equiv 0\}.$$

If the set

$$A = \bigcap_{n \in \mathbb{N}} \bigcup_{n \geq m} A_n$$

has measure zero, then there exists a $\gamma$-set $G \subseteq \{0, 1\}^\mathbb{N}$ such that the algebraic sum $G \oplus A$ is equal to $\{0, 1\}^\mathbb{N}$ (where where $\oplus$ denotes the modulo 2 coordinatewise addition).

Observe that the assumption in Theorem 5 holds whenever $\sum_n 2^{-(k_{n+1} - k_n)}$ converges.

Lemma 6. There exists an increasing sequence of natural numbers $\{k_n\}_{n \in \mathbb{N}}$ such that $\sum_n 2^{-(k_{n+1} - k_n)}$ converges, and such that the sequence $\{B_n\}_{n \in \mathbb{N}}$ defined by

$$B_n = \left\{ \sum_{i \in \mathbb{N}} \frac{f(i)}{2^{i+1}} : f \in \{-1, 0, 1\}^\mathbb{N} \text{ and } f \mid [k_n, k_{n+1}) \equiv 0 \right\}$$

for each $n$, the set

$$Y = \bigcap_{n \in \omega} \bigcup_{n \geq m} B_n$$

has Hausdorff dimension zero.

Proof. Fix a sequence $p_n$ of positive reals which converges to 0. Let $k_0 = 0$. Given $k_n$ find $k_{n+1}$ satisfying

$$3^{k_n} \cdot \frac{1}{2p_n(k_{n+1} - 2)} \leq \frac{1}{2n}.$$

Clearly, every $B_n$ is contained in a union of $3^{k_n}$ intervals such that each of the intervals has diameter $1/2^{k_{n+1} - 2}$. For each positive $\delta$ and $\epsilon$, choose $m$ such that
Proof. Choose a sequence zero such that \( p_n < \delta \) for all \( n \geq m \). Now, \( Y \) is a subset of \( \bigcup_{n \geq m} B_n \), and

\[
\sum_{n \geq m} 3^{k_n} \left( \frac{1}{2^{k_{n+1}+1}} \right) < \sum_{n \geq m} 3^{k_n} \left( \frac{1}{2^{k_{n+2}+1}} \right)^{p_n} < \sum_{n \geq m} \frac{1}{2^n} < \epsilon.
\]

Thus, the Hausdorff dimension of \( Y \) is zero.

The following lemma concludes the proof of Theorem 4.

Lemma 7. There exists a \( \gamma \)-set \( X \subseteq \mathbb{R} \) and a set \( Y \subseteq \mathbb{R} \) with Hausdorff dimension zero such that \( X + Y = \mathbb{R} \). In particular, \( \dim(X + Y) = 1 \).

Proof. Choose a sequence \( \{k_n\}_{n \in \mathbb{N}} \) and a set \( Y \) as in Lemma 6. Then \( \sum_n 2^{-(k_{n+1} - k_n)} \) converges, and the corresponding set \( A \) defined in Theorem 5 has measure zero. Thus, there exists a \( \gamma \)-set \( I \) such that \( I \cap A = \emptyset \).

Define \( \Phi : \{0,1\}^N \to \mathbb{R} \) by

\[
\Phi(f) = \sum_{i \in \mathbb{N}} f(i) 2^{i+1}.
\]

As \( \Phi \) is continuous, \( X = \Phi[I] \) is a \( \gamma \)-set of reals. Assume that \( z \) is a member of the interval \( [0,1] \), let \( f \in \{0,1\}^N \) be such that \( z = \sum_i f(i) 2^{i+1} \). Then \( f = g + a \) for appropriate \( g \in G \) and \( a \in A \). Define \( h \in \{-1,0,1\}^N \) by \( h(i) = f(i) - g(i) \). For infinitely many \( n \), \( a \upharpoonright [k_n,k_{n+1}] = 0 \) and therefore \( f \upharpoonright [k_n,k_{n+1}] = g \upharpoonright [k_n,k_{n+1}] \), that is, \( h \upharpoonright [k_n,k_{n+1}] = 0 \) for infinitely many \( n \). Thus, \( y = \sum_i h(i) 2^{i+1} \in Y \), and for \( x = \Phi(g) \),

\[
x + y = \sum_{i \in \mathbb{N}} g(i) 2^{i+1} + \sum_{i \in \mathbb{N}} h(i) 2^{i+1} = \sum_{i \in \mathbb{N}} g(i) + h(i) 2^{i+1} \frac{f(i)}{2^{i+1}} = \frac{f(i)}{2^{i+1}} = z.
\]

This shows that \( [0,1] \subseteq X + Y \). Consequently, \( X + (Y + \mathbb{Q}) = (X + Y) + \mathbb{Q} = \mathbb{R} \). Now, observe that \( Y + \mathbb{Q} \) has Hausdorff dimension zero since \( Y \) has.

3. THE PRODUCT OF A STRONG \( \gamma \)-SET AND A SET OF HAUSDORFF DIMENSION ZERO

Theorem 8. Assume that \( X \subseteq \mathbb{R}^k \) is a strong \( \gamma \)-set. Then for each \( Y \subseteq \mathbb{R}^l \), \( \dim(X \times Y) = \dim(Y) \).

Proof. The proof for this is similar to that of Theorem 7 in [7].

It is enough to show that \( \dim(X \times Y) \leq \dim(Y) \).

Lemma 9. Assume that \( Y \subseteq \mathbb{R}^l \) is such that \( \dim(Y) < \delta \). Then for each positive \( \epsilon \) there exists a large cover \( \{I_n\}_{n \in \mathbb{N}} \) of \( Y \) (i.e., such that each \( y \in Y \) is a member of infinitely many sets \( I_n \)) such that \( \sum_n \dim(I_n) < \epsilon \).

Proof. For each \( m \), choose a cover \( \{I^m_n\}_{n \in \mathbb{N}} \) of \( Y \) such that \( \sum_n \dim(I^m_n) < \epsilon/2^m \). Then \( \{I^m_n : m,n \in \mathbb{N}\} \) is a large cover of \( Y \), and \( \sum_{m,n} \dim(I^m_n) < \sum_n \epsilon/2^m = \epsilon \).

Lemma 10. Assume that \( Y \subseteq \mathbb{R}^l \) is such that \( \dim(Y) < \delta \). Then for each sequence \( \{\epsilon_n\}_{n \in \mathbb{N}} \) of positive reals there exists a large cover \( \{A_n\}_{n \in \mathbb{N}} \) of \( Y \) such that for each \( n \), \( A_n \) is a union of finitely many sets, \( I^1_{m_1}, \ldots, I^m_{m_n} \), such that \( \sum_j \dim(I^j_{m_j}) < \epsilon_n \).
Proof. Assume that \( \{\epsilon_n\}_{n \in \mathbb{N}} \) is a sequence of positive reals. By Lemma 9, there exists a large cover \( \{I_n\}_{n \in \mathbb{N}} \) of \( Y \) such that \( \sum_n \text{diam}(I_n)^\delta < \epsilon_1 \). For each \( n \) let \( k_n = \min\{m : \sum_{j \geq m} \text{diam}(I_j)^\delta < \epsilon_n\} \). Take

\[
A_n = \bigcup_{j=k_n}^{k_n+1-1} I_j.
\]

\( \Box \)

Fix \( \delta > \dim(Y) \) and \( \epsilon > 0 \). Choose a sequence \( \{\epsilon_n\}_{n \in \mathbb{N}} \) of positive reals such that \( \sum_n 2n\epsilon_n < \epsilon \), and use Lemma 10 to get the corresponding large cover \( \{A_n\}_{n \in \mathbb{N}} \).

For each \( n \) we define an \( n \)-cover \( U_n \) of \( X \) as follows. Let \( F \) be an \( n \)-element subset of \( X \). For each \( x \in F \), find an open interval \( I_x \) such that \( x \in I_x \) and

\[
\sum_{j=1}^{m_n} \text{diam}(I_x \times I_j^n)^\delta < 2\epsilon_n.
\]

Let \( U_F = \bigcup_{x \in F} I_x \). Set

\[
U_n = \{U_F : F \text{ is an } n \text{-element subset of } X\}.
\]

As \( X \) is a strong \( \gamma \)-set, there exist elements \( U_{F_n} \in U_n \), \( n \in \mathbb{N} \), such that \( \{U_{F_n}\}_{n \in \mathbb{N}} \) is a \( \gamma \)-cover of \( X \). Consequently,

\[
X \times Y \subseteq \bigcup_{n \in \mathbb{N}} (U_{F_n} \times A_n) \subseteq \bigcup_{n \in \mathbb{N}} \bigcup_{x \in F_n} \bigcup_{j=1}^{m_n} I_x \times I_j^n
\]

and

\[
\sum_{n \in \mathbb{N}} \sum_{x \in F_n} \sum_{j=1}^{m_n} \text{diam}(I_x \times I_j^n)^\delta < \sum_{n} n \cdot 2\epsilon_n < \epsilon.
\]

\( \Box \)

4. Open problems

There are ways to strengthen the notion of \( \gamma \)-sets other than moving to strong \( \gamma \)-sets. Let \( B_\Omega \) and \( B_\Gamma \) denote the collections of countable Borel \( \omega \)-covers and \( \gamma \)-covers of \( X \), respectively. As every open \( \omega \)-cover of a set of reals contains a countable \( \omega \)-subcover [9], we have that \( \Omega \subseteq B_\Omega \) and therefore \( S_1(B_\Omega, B_\Gamma) \) implies \( S_1(\Omega, \Gamma) \). The converse is not true [17].

**Problem 11.** Assume that \( X \subseteq \mathbb{R} \) satisfies \( S_1(B_\Omega, B_\Gamma) \). Is it true that for each \( Y \subseteq \mathbb{R} \), \( \dim(X \times Y) = \dim(Y) \)?

We conjecture that assuming the Continuum Hypothesis, the answer to this problem is negative. We therefore introduce the following problem. For infinite sets of natural numbers \( A, B \), we write \( A \subseteq^* B \) if \( A \setminus B \) is finite. Assume that \( \mathcal{F} \) is a family of infinite sets of natural numbers. A set \( P \) is a pseudointersection of \( \mathcal{F} \) if it is infinite, and for each \( B \in \mathcal{F} \), \( A \subseteq^* B \). \( \mathcal{F} \) is centered if each finite subcollection of \( \mathcal{F} \) has a pseudointersection. Let \( \mathfrak{p} \) denote the minimal cardinality of a centered family which does not have a pseudointersection. In [17] it is proved that \( \mathfrak{p} \) is also the minimal cardinality of a set of reals which does not satisfy \( S_1(B_\Omega, B_\Gamma) \).

**Problem 12.** Assume that the cardinality of \( X \) is smaller than \( \mathfrak{p} \). Is it true that for each \( Y \subseteq \mathbb{R} \), \( \dim(X \times Y) = \dim(Y) \)?
Another interesting open problem involves the following notion [18, 19]. A cover $U$ of $X$ is a $\tau$-cover of $X$ if it is a large cover, and for each $x, y \in X$, one of the sets $\{U \in U : x \in U \text{ and } y \notin U\}$ or $\{U \in U : y \in U \text{ and } x \notin U\}$ is finite. Let $T$ denote the collection of open $\tau$-covers of $X$. Then $\Gamma \subseteq T \subseteq \Omega$, therefore $S_1(\{O_n\}_{n \in \mathbb{N}}, T)$ implies $S_1(\{O_n\}_{n \in \mathbb{N}}, T)$.

**Problem 13.** Assume that $X \subseteq \mathbb{R}$ satisfies $S_1(\{O_n\}_{n \in \mathbb{N}}, T)$. Is it true that for each $Y \subseteq \mathbb{R}$, $\dim(X \times Y) = \dim(Y)$?

It is conjectured that $S_1(\{O_n\}_{n \in \mathbb{N}}, T)$ is strictly stronger than $S_1(\Omega, T)$ [20]. If this conjecture is false, then the results in this paper imply a negative answer to Problem 13.

Another type of problems is the following: We have seen that the assumption that $X$ is a $\gamma$-set and $Y$ has Hausdorff dimension zero is not enough in order to prove that $X \times Y$ has Hausdorff dimension zero. We also saw that if $X$ satisfies a stronger property (strong $\gamma$-set), then $\dim(X \times Y) = \dim(Y)$ for all $Y$. Another approach to get a positive answer would be to strengthen the assumption on $Y$ rather than $X$.

If we assume that $Y$ has strong measure zero, then a positive answer follows from a result of Scheepers [16] (see also [21]), asserting that if $X$ is a strong measure zero metric space which also has the Hurewicz property, then for each strong measure zero metric space $Y$, $X \times Y$ has strong measure zero. Indeed, if $X$ is a $\gamma$-set then it has the required properties.

Finally, the following question of Krawczyk remains open.

**Problem 14.** Is it consistent (relative to ZFC) that there are uncountable $\gamma$-sets but for each $\gamma$-set $X$ and each set $Y$, $\dim(X \times Y) = \dim(Y)$?

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