Soliton wave solutions of ion-acoustic waves a
cold plasma with negative ions

Dexu Zhao¹, Dianchen Lu¹, Samir A Salama², Piyaphong Yongphet³,⁴ and
Mostafa MA Khater¹,⁵

Abstract
This article investigates the computational complex wave solutions of the modified Korteweg–de Vries equation combined
with an adverse order of the Korteweg–de Vries model. This model was derived in 2017, where the recursion and inverse
recursion operators are employed to select the integrable merged MKdV with a negative MKdV model. This integrable
property is tested utilizing the Painlevé property. Verosky gave the description and properties of the opposing order
recursion operator. We handle this model by implementing eleven contemporary techniques. We obtain a novel formula of
complex solitary wave solutions for this model. Complex solitary wave solutions describe wave propagation, and it is also
considered more mathematically concise tools to explain more details about the physical properties of models. The main
goals of our paper are a comparison between these methods and introducing a novel modified method. All solutions are
checked for accuracy by putting them back into the model via two different software (Maple 17 and Mathematica 12).

Keywords
merged MKdV with negative MKdV equation, recursion operator, solitary wave solutions, traveling solutions

AMS classification: 02.30.Jr; 02.30.Hq; 02.60.-x

Introduction
The recursion operator is regarded as a fundamental instrument in nonlinear wave theory.¹–³ This operator is based on the
expansion of a power series across fields and the representation of momentum. Calculating the recursion operator using the
Hamiltonian is one of the techniques. The recursion operator has many features, including the following:

1. It allows writing the families of equations integrable by a given spectral issue in a compact design.
2. It supports picking up the wholesome family of the equation.
3. It is accompanied by the Hamiltonian approach of integrable equations.
4. It provides the generating operator for the family of Hamiltonian structures.

For an example of using recursion operator:
This operator is used to the KdV equation to derive all of its families and determine the family’s integrability. It is written as follows

\[ \frac{\partial u(x,t)}{\partial t} + \partial L^* u = 0, \]  

(1.1)
where \([L = \delta^2 + 2 u + 2 \delta^{-1} u \delta]\). Also, the integrability tested by using Hamiltonian structures that has the next formula

\[
[F, H]_n = \int_{-\infty}^{\infty} dx \frac{\delta F}{\delta u(x)} \frac{\delta H}{\delta u^n} \tag{1.2}
\]

where \([n = 0, \pm 1, \pm 2, \ldots]\). As previously stated, there are two types of recursion operators: weak recursion operators and robust recursion operators. This moniker refers to the vulnerable recursion operators that enable iteratively generating the infinite family of Hamiltonian equations. Simultaneously, the strong recursion operator generates an endless family of equations from a given equation.

Recursion operators are also used on the modified KdV equation, and it has the following formula

\[
\varphi(v) = \delta_x^2 + p_3 v^2 - p_3 v_x^2 \varphi(v). \tag{1.3}
\]

The modified KdV equation describes the nonlinear wave propagation in a variety of polarity-symmetric physical formations.\(^5\)\(^7\) Additionally, it defined nonlinear ion-acoustic waves as a collisionless plasma composed of adiabatic warm ions, a weakly relativistic electron beam, and non-isothermal electrons.\(^8\)\(^9\) The negative order equation is denoted by ref.\(^10\)

\[
\varphi(v_t) = -v_t. \tag{1.4}
\]

Now, combining the modified Korteweg–de Vries equation combined with an adverse order of the same models, this model has the nested formula

\[
v_t + p_1 v^2 v_x + v_{xxx} + v_{xxxt} + p_3 v^2 v_x + p_3 v_x \delta^{-1}(v_x) = 0. \tag{1.5}
\]

Differentiating equation (1.5) obtains

\[
\begin{align*}
v_{tt} + p_1 v^2 v_{tt} + v_{xxx} + v_{xxxt} + p_3 v^2 v_{tt} + p_3 v_{tt} v_x - v_{xx} v_t &= 0, \\

\{v_t + p_1 v^2 v_x + v_{xxx} + v_{xxxt} + p_3 v^2 v_x\} &= 0, \quad \text{where,} \quad p_2 = 2p_1. \tag{1.6}
\end{align*}
\]

Numerous approaches are generated by analyzing nonlinear partial differential equations with an integer order, a fractional order, a complementary order, or an adverse order. For a more detailed discussion of these techniques, see: Refs.\(^{12-27}\).

This article combines the modified Korteweg–de Vries equation with an adverse order of the same models to generate novel traveling wave solutions using over eleven recent analytical techniques.\(^{28-39}\)

The remainder of this article is organized as follows. In Application, we applied some recent methods on the combining equation \([\text{Exp}(-\varphi(\theta))]-\text{expansion method, an improved (f'/f) -expansion method, direct algebraic method, the generalized Kudryashov method, an extended tanh-function method, the simplest equation method, an extended Fan-expansion method, Riccati expansion method, an improved F-Expansion method, generalization Sinh-Gorden expansion method, and modified Khater method}]\). In Result and discussion, we represent our obtained solutions and show the convergence between our solutions and those obtained with different methods. In Conclusion, we demonstrate the conclusion of our paper.

Application

Using the next wave transformation \([v(x,t) = \varphi(\theta), \theta = l_2 t + l_1 x]\) transforms the nonlinear partial differential equation form of equation (1.6) to the following nonlinear ordinary differential equation.

\[
l_1^2 (l_1 + l_2) \left( \frac{d^4 \varphi(\theta)}{d \theta^4} - \varphi(\theta) \varphi''(\theta) \right) + p_2 \varphi(\varphi'(\theta))^3 = 0. \tag{2.1}
\]

Evaluating the value of balance between the terms of equation (2.1) by using the next formula of balance rule

\[
\begin{align*}
D \left[ \frac{dx Q(\theta)}{d \theta} \right] &= N + \zeta, \\
D \left[ Q\left( \frac{d^4 Q(\theta)}{d \theta^4} \right) \right] &= N + \zeta + \mathcal{S}(N + \epsilon), \tag{2.2}
\end{align*}
\]
and the following auxiliary equations of the abovementioned computational schemes, respectively

\[
\begin{align*}
G''(\theta) &= -\mu f'(\theta) = l_2 f(\theta)^2 + l_2 f(\theta) + l_1, \quad Q(\theta) = Q(\theta)^2 - Q(\theta), \quad f'(\theta) = d + \phi(\theta)^2, \\
f''(\theta) &= c_2 f'(\theta)^2 + c_1 f(\theta), \quad f'(\theta)^2 = \xi^2 \phi(\theta)^4 + 2 \xi \zeta \phi(\theta) + (\xi^2 + 2 \xi \zeta) \phi(\theta)^2 + 2 \xi \zeta \phi(\theta) + \xi_1, \quad Q'(\theta) = \xi_2 Q(\theta)^2 + \xi_3 Q(\theta) + \xi_4, \quad f'(\theta)^2 + a, \quad f''(\zeta) = \frac{\delta f'(\theta)}{\mu} + \frac{\delta f''(\theta) + \chi}{\ln(k)}, \quad \text{get } N = 1. \quad \text{Thus, the general solutions of equation (2.1) based on the considered computational techniques are given by}
\end{align*}
\]

\[
\begin{align*}
\sum_{j=-N}^{N} a_ie^{-j\phi(\theta)} &= a_{-1}e^{\phi(\theta)} + a_1e^{-\phi(\theta)} + a_0, \\
\sum_{j=1}^{N} \left( a_i \frac{G'(\theta)}{G(\theta)} + b_j \frac{G'(\theta)}{G(\theta)^2} \right)^{i-1} &\left( \sigma \left( \frac{G'(\theta)}{G(\theta)} \right)^2 + 1 \right) + a_0 = a_i \frac{G'(\theta)}{G(\theta)} + a_0 + b_1 \left( \sigma \left( \frac{G'(\theta)}{\mu G(\theta)^2} + 1 \right) \right), \\
\sum_{j=-N}^{N} a_i f(\theta)^j &= \frac{a_{-1}}{f(\theta)} + a_1 f(\theta) + a_0, \\
\sum_{j=0}^{m} a_i Q(\theta)^j &= a_2 Q(\theta)^2 + a_1 Q(\theta) + a_0, \\
\sum_{j=0}^{N} b_j Q(\theta)^j &= a_2 Q(\theta)^2 + a_1 Q(\theta) + a_0, \\
Q(\theta) &= \sum_{j=-N}^{N} a_i \phi(\theta)^j = \frac{a_{-1}}{\phi(\theta)} + a_1 \phi(\theta) + a_0, \\
\sum_{j=0}^{n} a_i f(\theta)^j &= a_1 f(\theta) + a_0, \\
\sum_{j=0}^{N} a_i \phi(\theta)^j &= a_1 \phi(\theta) + a_0, \\
\sum_{j=1}^{N} a_i Q(\theta)^j + \sum_{j=1}^{N} b_j Q(\theta)^{-j} + a_0 &= a_2 Q(\theta) + a_0 + \frac{b_1}{\frac{Q(\theta)}{Q(\theta)^2}}, \\
\sum_{j=-N}^{N} a_i (\mu + \phi(\theta))^j &= \frac{a_{-1}}{\mu + \phi(\theta)} + a_1 (\mu + \phi(\theta)) + a_0, \\
\sum_{j=1}^{N} \cosh^{-1}(W) \left[ B_i \sinh(W) + A_i \cosh(W) \right] + A_0 &= A_1 \cosh(W) + A_0 + B_1 \sinh(W), \\
\sum_{j=1}^{N} a_i k^{j\theta} + \sum_{j=1}^{N} b_j k^{-j\theta} + a_0 &= a_1 k^{\theta} + a_0 + b_1 k^{-\theta},
\end{align*}
\]

(2.3)

where \(a_i, b_i\) are arbitrary constants to be determined later. Handling equation (2.1) with equation (2.3) along its auxiliary equations gives the value of the abovementioned parameters and explicit solutions of equation (1.5) as follows:
**Exp (− φ(δ)) -expansion method**

Employing Exp (− φ(δ)) -expansion method gives the following sets of the abovementioned parameters:

Set I

\[ a_{-1} \to a_1 \sigma, a_0 \to \frac{a_1 \sigma}{2}, l_1 \to \frac{i a_1 \sqrt{p_2}}{2 \sqrt{3}}, \quad \text{where} \quad (p_2 < 0, \ i = \sqrt{-1}). \]

Set II

\[ a_0 \to \frac{a_{-1} \sigma}{2}, a_1 \to 0, l_1 \to \frac{i a_{-1} \sqrt{p_2}}{2 \sqrt{3} \sigma}, \quad \text{where} \quad (p_2 < 0, \ i = \sqrt{-1}). \]

Set III

\[ a_{-1} \to 0, a_0 \to \frac{a_1 \sigma}{2}, l_1 \to \frac{i a_1 \sqrt{p_2}}{2 \sqrt{3}}, \quad \text{where} \quad (p_2 < 0, \ i = \sqrt{-1}). \]

Thus, the solitary wave solutions of equation (1.5) can be formulated in the following formulas:

For \( p^2 - 4\sigma > 0, \ \sigma \neq 0 \), the solutions are given by

\[ v_{1,1}(x,t) = \frac{1}{2} a_1 \left( -\frac{4\sigma}{\rho + \sqrt{p^2 - 4\sigma} \tanh \left( \frac{1}{2} \sqrt{\rho^2 - 4\sigma} \left( \frac{i a_1 \sqrt{p_2}}{2 \sqrt{3}} + l_2 t + \Omega \right) \right)} - \sqrt{\rho^2 - 4\sigma} \tanh \left( \frac{1}{2} \sqrt{\rho^2 - 4\sigma} \left( \frac{i a_1 \sqrt{p_2}}{2 \sqrt{3}} + l_2 t + \Omega \right) \right) \right), \]

\[ v_{1,2}(x,t) = \frac{1}{2} a_1 \left( \left( -\sqrt{\rho^2 - 4\sigma} \coth \left( \frac{1}{2} \sqrt{\rho^2 - 4\sigma} \left( \frac{i a_1 \sqrt{p_2}}{2 \sqrt{3}} + l_2 t + \Omega \right) \right) \right) \right), \]

\[ v_{2,1}(x,t) = -\frac{1}{2\sigma} a_{-1} \sqrt{\rho^2 - 4\sigma} \tanh \left( \frac{1}{2} \sqrt{\rho^2 - 4\sigma} \left( \frac{i a_1 \sqrt{p_2}}{2 \sqrt{3}} + l_2 t + \Omega \right) \right), \]

\[ v_{2,2}(x,t) = -\frac{1}{2\sigma} a_{-1} \sqrt{\rho^2 - 4\sigma} \coth \left( \frac{1}{2} \sqrt{\rho^2 - 4\sigma} \left( \frac{i a_1 \sqrt{p_2}}{2 \sqrt{3}} + l_2 t + \Omega \right) \right), \]

\[ v_{3,1}(x,t) = \frac{1}{2} a_1 \left( \frac{4\sigma}{\rho + \sqrt{p^2 - 4\sigma} \tanh \left( \frac{1}{2} \sqrt{\rho^2 - 4\sigma} \left( \frac{i a_1 \sqrt{p_2}}{2 \sqrt{3}} + l_2 t + \Omega \right) \right)} \right), \]

\[ v_{3,2}(x,t) = \frac{1}{2} a_1 \left( \frac{4\sigma}{\rho + \sqrt{p^2 - 4\sigma} \coth \left( \frac{1}{2} \sqrt{\rho^2 - 4\sigma} \left( \frac{i a_1 \sqrt{p_2}}{2 \sqrt{3}} + l_2 t + \Omega \right) \right)} \right). \]

For \( p^2 - 4\sigma > 0, \ \sigma = 0 \), the solutions are given by

\[ v_{1,3}(x,t) = \frac{1}{2} a_1 \rho \coth \left( \frac{1}{12} \rho \left( i \sqrt{3} a_1 \sqrt{p_2} x + 6l_2 t + 6\Omega \right) \right), \]
\[ v_{\text{II},3}(x,t) = \frac{1}{2} a_1 \rho \coth \left( \frac{1}{12} \rho \left( \frac{3a^3}{i\sqrt{3}a_1 \sqrt{p_2 x} + 6l_2 t + 6\Theta} + 12 - \frac{12\sigma}{i\sqrt{3}a_1 \sqrt{p_2 x} + 6l_2 t + 6\Theta - \rho\sigma} \right) \right). \] (2.11)

For \( \rho^2 - 4\sigma = 0, \sigma \neq 0, \rho \neq 0 \), the solutions are given by

\[ v_{\text{II},4}(x,t) = \frac{2a_1}{\rho^2} \left( \frac{3a^3}{i\sqrt{3}a_1 \sqrt{p_2 x} + 6l_2 t + 6\Theta + 12} - \frac{12\sigma}{i\sqrt{3}a_1 \sqrt{p_2 x} + 6l_2 t + 6\Theta - \rho\sigma} \right), \] (2.12)

\[ v_{\text{II},5}(x,t) = \frac{a_1}{2\rho^2} \left( \frac{8}{\frac{a_1 \sqrt{p_2 x}}{\sqrt{3}} + l_2 t + \Theta} + \frac{\rho^3}{\sigma} - 4\rho \right), \] (2.13)

\[ v_{\text{II},6}(x,t) = \frac{6a_1 \rho}{i\sqrt{3}a_1 \sqrt{p_2 x} + 6l_2 t + 6\Theta + 12}. \] (2.14)

For \( \rho^2 - 4\sigma = 0, \sigma = 0, \rho = 0 \), the solutions are given by

\[ v_{\text{II},5}(x,t) = \frac{a_1}{\frac{a_1 \sqrt{p_2 x}}{\sqrt{3}} + l_2 t + \Theta}, \] (2.15)

\[ v_{\text{II},6}(x,t) = \frac{a_1}{\frac{a_1 \sqrt{p_2 x}}{\sqrt{3}} + l_2 t + \Theta}. \] (2.16)

For \( \rho^2 - 4\sigma < 0, \sigma \neq 0 \), the solutions are given by

\[ v_{\text{II},6}(x,t) = \frac{1}{2} a_1 \left( \sqrt{4\sigma - \rho^2} \tan \left( \frac{1}{2} \sqrt{4\sigma - \rho^2} \left( \frac{ia_1 \sqrt{p_2 x}}{2\sqrt{3}} + l_2 t + \Theta \right) \right) \right. \]

\[ - \frac{4\sigma}{\rho - \sqrt{4\sigma - \rho^2} \tan \left( \frac{1}{2} \sqrt{4\sigma - \rho^2} \left( \frac{ia_1 \sqrt{p_2 x}}{2\sqrt{3}} + l_2 t + \Theta \right) \right)}, \] (2.17)

\[ v_{\text{II},7}(x,t) = \frac{1}{2} a_1 \left( \sqrt{4\sigma - \rho^2} \cot \left( \frac{1}{2} \sqrt{4\sigma - \rho^2} \left( \frac{ia_1 \sqrt{p_2 x}}{2\sqrt{3}} + l_2 t + \Theta \right) \right) \right. \]

\[ - \frac{4\sigma}{\rho - \sqrt{4\sigma - \rho^2} \cot \left( \frac{1}{2} \sqrt{4\sigma - \rho^2} \left( \frac{ia_1 \sqrt{p_2 x}}{2\sqrt{3}} + l_2 t + \Theta \right) \right)}, \] (2.18)

\[ v_{\text{II},4}(x,t) = \frac{a_1 \sqrt{4\sigma - \rho^2}}{2\sigma} \tan \left( \frac{1}{2} \sqrt{4\sigma - \rho^2} \left( \frac{ia_1 \sqrt{p_2 x}}{2\sqrt{3}} + l_2 t + \Theta \right) \right), \] (2.19)

\[ v_{\text{II},5}(x,t) = \frac{a_1 \sqrt{4\sigma - \rho^2}}{2\sigma} \cot \left( \frac{1}{2} \sqrt{4\sigma - \rho^2} \left( \frac{ia_1 \sqrt{p_2 x}}{2\sqrt{3}} + l_2 t + \Theta \right) \right), \] (2.20)

\[ v_{\text{II},6}(x,t) = \frac{1}{2} a_1 \left( \rho - \frac{4\sigma}{\rho - \sqrt{4\sigma - \rho^2} \tan \left( \frac{1}{2} \sqrt{4\sigma - \rho^2} \left( \frac{ia_1 \sqrt{p_2 x}}{2\sqrt{3}} + l_2 t + \Theta \right) \right)} \right), \] (2.21)
\[ v_{III}(x,t) = \frac{1}{2} a_1 \left( \frac{\rho - \frac{4\sigma}{\rho - \sqrt{4\sigma - \rho^2} \cot \left( \frac{1}{2} \sqrt{4\sigma - \rho^2} \left( \frac{m_i \sqrt{\mu}}{2} + l_2 t + \mathcal{U} \right) \right)}}{\rho - \sqrt{4\sigma - \rho^2} \cot \left( \frac{1}{2} \sqrt{4\sigma - \rho^2} \left( \frac{m_i \sqrt{\mu}}{2} + l_2 t + \mathcal{U} \right) \right)} \right). \] (2.22)

**Improved (G'/G) -expansion method**

Employing improved (G'/G) -expansion method gives the following sets of the abovementioned parameters:

**Set I**

\[ a_0 \to 0, \quad a_1 \to 0, \quad p_2 \to -\frac{12 \mu l_1^2}{b_1^2 \sigma} \]

**Set II**

\[ a_0 \to 0, \quad b_1 \to 0, \quad p_2 \to -\frac{12 l_1^2}{a_1} \]

**Set III**

\[ a_0 \to 0, \quad b_1 \to \frac{a_1 \sqrt{\mu}}{\sqrt{\sigma}}, \quad l_1 \to \frac{ia_1 \sqrt{p_2}}{\sqrt{3}} \quad \text{where} \quad (p_2 < 0, \ i = \sqrt{-1}) \]

Thus, the solitary wave solutions of equation (1.5) can be formulated in the following formulas:

For \( \mu > 0 \), the solutions are given by

\[ v_{I,1}(x,t) = b_1 \sqrt{\frac{(C_1^2 + C_2^2) \sigma}{(C_1 \sin(\sqrt{\mu} (l_2 t + l_1 x)) + C_2 \cos(\sqrt{\mu} (l_2 t + l_1 x)))^3}}, \] (2.23)

\[ v_{II,1}(x,t) = \frac{a_1 \sqrt{\mu} (C_1 - C_2 \tan(\sqrt{\mu} (l_2 t + l_1 x)))}{C_1 \tan(\sqrt{\mu} (l_2 t + l_1 x)) + C_2}, \] (2.24)

\[ v_{III,1}(x,t) = a_1 \sqrt{\mu} \left[ \frac{(C_1^2 + C_2^2) \sigma}{\sqrt{\sigma}} \right] \sqrt{\left[ \sqrt{\frac{C_1 \sin(\sqrt{\mu} (l_2 t + \frac{ia_1 \sqrt{p_2}}{\sqrt{3}})) + C_2 \cos(\sqrt{\mu} (l_2 t + \frac{ia_1 \sqrt{p_2}}{\sqrt{3}}))}{C_1 + C_1 \tan\left(\sqrt{\mu} (l_2 t + \frac{ia_1 \sqrt{p_2}}{\sqrt{3}})\right)} \right]^2} \] (2.25)

For \( \mu < 0 \), the solutions are given by

\[ v_{I,2}(x,t) = b_1 \sqrt{\frac{(C_1^2 - C_2^2) \sigma}{(C_1 \cos(\sqrt{\mu} (l_2 t + l_1 x)) + C_2 \sinh(\sqrt{-\mu} (l_2 t + l_1 x)))^3}}, \] (2.26)

\[ v_{II,2}(x,t) = \frac{a_1 \sqrt{-\mu} (C_2 \cos(\sqrt{\mu} (l_2 t + l_1 x)) + C_1 \sinh(\sqrt{-\mu} (l_2 t + l_1 x)))}{C_1 \cos(\sqrt{\mu} (l_2 t + l_1 x)) + C_2 \sinh(\sqrt{-\mu} (l_2 t + l_1 x))}, \] (2.27)
\[ v_{\text{III},2}(x,t) = a_1 \sqrt{\mu} \left( \frac{(C_1^2 - C_2^2)}{\left( C_1 \cos \left( \sqrt{\mu} \left( l_2 t + \frac{ia_1 \sqrt{p_x^2}}{\sqrt{3}} \right) \right) + C_2 \sinh \left( \sqrt{-\mu} \left( l_2 t + \frac{ia_1 \sqrt{p_x^2}}{\sqrt{3}} \right) \right) \right)^2} + \frac{1}{C_1 \sqrt{\mu} + \tan \left( \sqrt{\mu} \left( l_2 t + \frac{ia_1 \sqrt{p_x^2}}{\sqrt{3}} \right) \right)} - \frac{C_1 \sin \left( \sqrt{\mu} \left( l_2 t + \frac{ia_1 \sqrt{p_x^2}}{\sqrt{3}} \right) \right)}{C_1 \cos \left( \sqrt{\mu} \left( l_2 t + \frac{ia_1 \sqrt{p_x^2}}{\sqrt{3}} \right) \right) + C_2 \sinh \left( \sqrt{-\mu} \left( l_2 t + \frac{ia_1 \sqrt{p_x^2}}{\sqrt{3}} \right) \right)} \right) \] (2.28)

**Direct algebraic method**

Employing the direct algebraic method gives the following sets of the abovementioned parameters:

**Set I**

\[ a_0 \rightarrow a_{-1} \frac{h_2}{2h_1}, \quad a_1 \rightarrow a_{-1} \frac{h_2}{h_1}, \quad p_2 \rightarrow -\frac{12h_1^2 \mu}{a_{-1}^2}. \]

**Set II**

\[ a_0 \rightarrow a_{-1} \frac{h_2}{2h_1}, \quad a_1 \rightarrow 0, \quad p_2 \rightarrow -\frac{12h_1^2 \mu}{a_{-1}^2}. \]

**Set III**

\[ a_{-1} \rightarrow 0, \quad a_0 \rightarrow a_{-1} \frac{h_2}{2h_3}, \quad p_2 \rightarrow -\frac{12h_1^2 \mu}{a_{-1}^2}. \]

Thus, the solitary wave solutions of equation (1.5) can be formulated in the following formulas:

For \( h_2 = 0, \ h_1h_3 > 0 \), the solutions are given by

\[ v_{\text{I},1} = \frac{2a_{-1} \sqrt{h_1} h_3}{h_1} \csc \left( 2 \sqrt{h_1} h_3 (\zeta + l_2 t + l_1 x) \right), \] (2.29)

\[ v_{\text{II},1} = \frac{a_{-1} \sqrt{h_1} h_3}{h_1} \cot \left( \sqrt{h_1} h_3 (\zeta + l_2 t + l_1 x) \right), \] (2.30)

\[ v_{\text{II},2} = \frac{a_{-1} \sqrt{h_1} h_3}{h_1} \tan \left( \sqrt{h_1} h_3 (\zeta + l_2 t + l_1 x) \right), \] (2.31)

\[ v_{\text{III},1} = \frac{a_{-1} \sqrt{h_1} h_3}{h_3} \tan \left( \sqrt{h_1} h_3 (\zeta + l_2 t + l_1 x) \right), \] (2.32)

\[ v_{\text{III},2} = \frac{a_{-1} \sqrt{h_1} h_3}{h_3} \cot \left( \sqrt{h_1} h_3 (\zeta + l_2 t + l_1 x) \right). \] (2.33)

For \( h_2 = 0, \ h_1h_3 < 0 \), the solutions are given by

\[ v_{\text{I},2} = -\frac{2a_{-1} \sqrt{h_1} h_3}{h_1} \csch \left( 2 \left( \sqrt{h_1} h_3 (l_2 t + l_1 x) \mp \frac{\log(\zeta)}{2} \right) \right), \] (2.34)
\begin{align*}
\nu_{ll,3} &= \frac{a_{-1}\sqrt{-h_1 h_3}}{h_1} \coth \left( \sqrt{-h_1 h_3} \left( l_2 t + l_1 x \right) + \frac{\log(\xi)}{2} \right), \quad (2.35) \\
\nu_{ll,4} &= \frac{a_{-1}\sqrt{-h_1 h_3}}{h_1} \tanh \left( \sqrt{-h_1 h_3} \left( l_2 t + l_1 x \right) + \frac{\log(\xi)}{2} \right), \quad (2.36) \\
\nu_{ll,5} &= \frac{a_1\sqrt{-h_1 h_3}}{h_3} \tanh \left( \sqrt{-h_1 h_3} \left( l_2 t + l_1 x \right) + \frac{\log(\xi)}{2} \right), \quad (2.37) \\
\nu_{ll,6} &= \frac{a_1\sqrt{-h_1 h_3}}{h_3} \coth \left( \sqrt{-h_1 h_3} \left( l_2 t + l_1 x \right) + \frac{\log(\xi)}{2} \right). \quad (2.38)
\end{align*}

For \( h_1 = 0, h_2 > 0 \), the solutions are given by
\begin{equation}
\nu_{ll,5} = \frac{a_1 h_2}{h_3 \left( e^{h_3 (\zeta + l_2 t + l_1 x)} - 1 \right)} - \frac{a_1 h_3}{2 h_3}. \quad (2.39)
\end{equation}

For \( h_1 = 0, h_2 < 0 \), the solutions are given by
\begin{equation}
\nu_{ll,6} = \frac{a_1}{h_3 \left( e^{h_3 (\zeta + l_2 t + l_1 x)} + 1 \right)} + \frac{a_1 h_2}{2 h_3} - a_1. \quad (2.40)
\end{equation}

For \( 4h_1 h_3 > h_2^2 \), the solutions are given by
\begin{align*}
\nu_{ll,3} &= \frac{a_{-1}\sqrt{4h_1 h_3 - h_2^2}}{2 h_1} \tan \left( \frac{1}{2} \sqrt{4h_1 h_3 - h_2^2} \left( \zeta + l_2 t + l_1 x \right) \right) \\
&\quad - \frac{2a_{-1} h_3}{h_2 - \sqrt{4h_1 h_3 - h_2^2} \tan \left( \frac{1}{2} \sqrt{4h_1 h_3 - h_2^2} \left( \zeta + l_2 t + l_1 x \right) \right) }, \quad (2.41) \\
\nu_{ll,4} &= \frac{a_{-1}\sqrt{4h_1 h_3 - h_2^2}}{2 h_1} \cot \left( \frac{1}{2} \sqrt{4h_1 h_3 - h_2^2} \left( \zeta + l_2 t + l_1 x \right) \right) \\
&\quad - \frac{2a_{-1} h_3}{h_2 - \sqrt{4h_1 h_3 - h_2^2} \cot \left( \frac{1}{2} \sqrt{4h_1 h_3 - h_2^2} \left( \zeta + l_2 t + l_1 x \right) \right) }, \quad (2.42) \\
\nu_{ll,5} &= \frac{a_{-1} h_2}{2 h_1} - \frac{2a_{-1} h_3}{h_2 - \sqrt{4h_1 h_3 - h_2^2} \tan \left( \frac{1}{2} \sqrt{4h_1 h_3 - h_2^2} \left( \zeta + l_2 t + l_1 x \right) \right) }, \quad (2.43) \\
\nu_{ll,6} &= \frac{a_{-1} h_2}{2 h_1} - \frac{2a_{-1} h_3}{h_2 - \sqrt{4h_1 h_3 - h_2^2} \cot \left( \frac{1}{2} \sqrt{4h_1 h_3 - h_2^2} \left( \zeta + l_2 t + l_1 x \right) \right) }, \quad (2.44) \\
\nu_{ll,7} &= \frac{a_1\sqrt{4h_1 h_3 - h_2^2}}{2 h_3} \tan \left( \frac{1}{2} \sqrt{4h_1 h_3 - h_2^2} \left( \zeta + l_2 t + l_1 x \right) \right), \quad (2.45) \\
\nu_{ll,8} &= \frac{a_1\sqrt{4h_1 h_3 - h_2^2}}{2 h_3} \cot \left( \frac{1}{2} \sqrt{4h_1 h_3 - h_2^2} \left( \zeta + l_2 t + l_1 x \right) \right). \quad (2.46)
\end{align*}
Generalized Kudryashov method

Employing the generalized Kudryashov method gives the following sets of the abovementioned parameters:

Set I

\[ a_0 \rightarrow \frac{a_2 b_0}{2 b_1}, \quad a_1 \rightarrow \frac{2 a_2 b_0 - a_2 b_1}{2 b_1}, \quad p_2 \rightarrow -\frac{12 b_1^2 l_1^2}{a_1^2}. \]

Set II

\[ a_0 \rightarrow \frac{a_2 b_0(2 b_0 + b_1)}{2 b_1^2}, \quad a_1 \rightarrow \frac{2 a_2 b_0 - a_2 b_1}{2 b_1}, \quad p_2 \rightarrow -\frac{12 b_1^2 l_1^2}{a_1^2}. \]

Set III

\[ a_0 \rightarrow \frac{a_2}{2}, \quad a_1 \rightarrow -a_2, \quad b_1 \rightarrow -2b_0, \quad p_2 \rightarrow -\frac{48 b_0^2 l_1^2}{a_1^2}. \]

Thus, the solitary wave solutions of equation (1.5) can be formulated in the following formulas

\[ v_1(x,t) = \frac{a_2}{2 b_1} \left( 2 \frac{2}{A e^{2x+2t} + 1} - 1 \right), \quad (2.47) \]

\[ v_2(x,t) = \frac{a_2}{2 b_1} \left( 2 \frac{2 b_1}{A e^{2x+2t} + 1} - 2 b_1 (b_0 + b_1) \right), \quad (2.48) \]

\[ v_3(x,t) = \frac{a_2}{2 b_1} \left( 1 - \frac{2}{A e^{2x+2t} + 1} \right). \quad (2.49) \]

Extended tanh-function method

Employing the extended tanh-function gives the following sets of the abovementioned parameters:

Set I

\[ a_{-1} \rightarrow 0, \quad a_0 \rightarrow 0, \quad p_2 \rightarrow -\frac{12 l_1^2}{a_1^2}. \]

Set II

\[ a_{-1} \rightarrow a_1 d, \quad a_0 \rightarrow 0, \quad p_2 \rightarrow -\frac{12 l_1^2}{a_1^2}. \]

Set III

\[ a_0 \rightarrow 0, \quad a_1 \rightarrow 0, \quad p_2 \rightarrow -\frac{12 d^2 l_1^2}{a_{-1}^2}. \]

Thus, the solitary wave solutions of equation (1.5) can be formulated in the following formulas:

For \( d < 0 \), the solutions have been evaluated as following

\[ v_{1,1}(x,t) = a_1 \sqrt{d} \tan \left( \sqrt{d} (l_2 t + l_1 x) \right), \quad (2.50) \]

\[ v_{1,2}(x,t) = a_1 \left(-\sqrt{d}\right) \cot \left( \sqrt{d} (l_2 t + l_1 x) \right), \quad (2.51) \]

\[ v_{1,1}(x,t) = 2a_1 \sqrt{d} \csc \left( 2\sqrt{d} (l_2 t + l_1 x) \right), \quad (2.52) \]
\[ v_{I,2}(x,t) = -2a_1 \sqrt{d} \csc\left(2\sqrt{d} (l_2 t + l_1 x)\right), \] (2.53)
\[ v_{I,1}(x,t) = \frac{a_1}{\sqrt{d}} \cot\left(\sqrt{d} (l_2 t + l_1 x)\right), \] (2.54)
\[ v_{I,3}(x,t) = \frac{a_1}{\sqrt{d}} \tan\left(\sqrt{d} (l_2 t + l_1 x)\right). \] (2.55)

For \( d > 0 \), the solutions have been evaluated as following
\[ v_{I,3}(x,t) = a_1 \sqrt{d} \tan\left(\sqrt{d} (l_2 t + l_1 x)\right), \] (2.56)
\[ v_{I,4}(x,t) = a_1 (-\sqrt{d}) \cot\left(\sqrt{d} (l_2 t + l_1 x)\right), \] (2.57)
\[ v_{I,3}(x,t) = 2a_1 \sqrt{d} \csc\left(2\sqrt{d} (l_2 t + l_1 x)\right), \] (2.58)
\[ v_{I,4}(x,t) = -2a_1 \sqrt{d} \csc\left(2\sqrt{d} (l_2 t + l_1 x)\right), \] (2.59)
\[ v_{I,3}(x,t) = \frac{a_1}{\sqrt{d}} \cot\left(\sqrt{d} (l_2 t + l_1 x)\right), \] (2.60)
\[ v_{I,4}(x,t) = -\frac{a_1}{\sqrt{d}} \tan\left(\sqrt{d} (l_2 t + l_1 x)\right). \] (2.61)

For \( d = 0 \), the solutions have been evaluated as following
\[ v_{I,5}(x,t) = -\frac{a_1}{l_2 t + l_1 x}, \] (2.62)
\[ v_{I,5}(x,t) = \frac{a_1 (d(l_2 t + l_1 x)^2 + 1)}{l_2 t + l_1 x}. \] (2.63)

**Simplest equation method**

Employing the simplest equation method gives the following sets of the abovementioned parameters:

Set I For \( c_2 = -1 \)
\[ a_0 \rightarrow -\frac{1}{2} a_1 c_1, l_1 \rightarrow \frac{ia_1 \sqrt{p_2}}{2 \sqrt{3}}, \text{ where } (p_2 < 0, \ i = \sqrt{-1}). \]

Set I For \( c_1 = 1, \ c_2 = -1 \)
\[ a_0 \rightarrow -\frac{a_1}{2}, l_1 \rightarrow \frac{ia_1 \sqrt{p_2}}{2 \sqrt{3}}, \text{ where } (p_2 < 0, \ i = \sqrt{-1}). \]

Thus, the solitary wave solutions of equation (1.5) can be formulated in the following formulas
\[ v_{I,1}(x,t) = \frac{1}{2} a_1 c_1 \tanh\left(\frac{1}{2} c_1 \left(\frac{ia_1 \sqrt{p_2 x}}{2 \sqrt{3}} + J + l_2 t\right)\right), \] (2.64)
\[ v_{I,2}(x,t) = \frac{1}{2} a_1 c_1 \tanh\left(\frac{1}{2} c_1 \left(\frac{ia_1 \sqrt{p_2 x}}{2 \sqrt{3}} + J + l_2 t\right)\right), \] (2.65)
Employing the extended Fan-expansion method gives the following sets of the abovementioned parameters

\[ a_0 \to \frac{a_1 \zeta_3}{2 \zeta_3} p_2 \to -\frac{12 \zeta_3^2 \xi^2}{a_1^4}. \]

Thus, the solitary wave solutions of equation (1.5) can be formulated in the following formulas:

For \( \zeta_2^2 - 4 \zeta_3 \zeta_1 > 0, \zeta_2 \zeta_3 \neq 0, \zeta_3 \zeta_1 \neq 0 \), the solutions have been evaluated as following

\[ v_{il}(x,t) = \frac{1}{2} a_1 \tanh \left( \frac{1}{12} \left( 6 \xi t + i \sqrt{3} a_1 \sqrt{p_2} x \right) \right). \]

(2.66)

**Extended fan-expansion method**

Employing the extended Fan-expansion method gives the following sets of the abovementioned parameters

Thus, the solitary wave solutions of equation (1.5) can be formulated in the following formulas:

\[ v_1(x,t) = a_1 \sqrt{\frac{\zeta_2^2 - 4 \zeta_1 \zeta_3}{2 \zeta_3}} \tanh \left( \frac{1}{2} \sqrt{\frac{\zeta_2^2 - 4 \zeta_1 \zeta_3}{2 \zeta_3}} (l_2 t + l_1 x) \right), \]

(2.67)

\[ v_2(x,t) = \frac{a_1 \sqrt{\zeta_2^2 - 4 \zeta_1 \zeta_3}}{2 \zeta_3} \coth \left( \frac{1}{2} \sqrt{\frac{\zeta_2^2 - 4 \zeta_1 \zeta_3}{2 \zeta_3}} (l_2 t + l_1 x) \right), \]

(2.68)

\[ v_3(x,t) = \frac{a_1 \sqrt{\zeta_2^2 - 4 \zeta_1 \zeta_3}}{2 \zeta_3} \left( \tanh \left( \frac{1}{2} \sqrt{\frac{\zeta_2^2 - 4 \zeta_1 \zeta_3}{2 \zeta_3}} (l_2 t + l_1 x) \right) \pm \text{sech} \left( \sqrt{\frac{\zeta_2^2 - 4 \zeta_1 \zeta_3}{2 \zeta_3}} (l_2 t + l_1 x) \right) \right), \]

(2.69)

\[ v_4(x,t) = \frac{a_1 \sqrt{\zeta_2^2 - 4 \zeta_1 \zeta_3}}{2 \zeta_3} \left( \coth \left( \frac{1}{2} \sqrt{\frac{\zeta_2^2 - 4 \zeta_1 \zeta_3}{2 \zeta_3}} (l_2 t + l_1 x) \right) \pm \text{csch} \left( \sqrt{\frac{\zeta_2^2 - 4 \zeta_1 \zeta_3}{2 \zeta_3}} (l_2 t + l_1 x) \right) \right), \]

(2.70)

\[ v_5(x,t) = \frac{a_1 \sqrt{\zeta_2^2 - 4 \zeta_1 \zeta_3}}{4 \zeta_3} \tanh \left( \frac{1}{4} \sqrt{\frac{\zeta_2^2 - 4 \zeta_1 \zeta_3}{2 \zeta_3}} (l_2 t + l_1 x) \right) \left( \coth \left( \frac{1}{2} \sqrt{\frac{\zeta_2^2 - 4 \zeta_1 \zeta_3}{2 \zeta_3}} (l_2 t + l_1 x) \right) + 1 \right), \]

(2.71)

\[ v_6(x,t) = \frac{a_1 \left( \sqrt{\frac{\zeta_2^2 - 4 \zeta_1 \zeta_3}{2 \zeta_3}} (A^2 + B^2) - A \sqrt{\frac{\zeta_2^2 - 4 \zeta_1 \zeta_3}{2 \zeta_3}} \cosh \left( \sqrt{\frac{\zeta_2^2 - 4 \zeta_1 \zeta_3}{2 \zeta_3}} (l_2 t + l_1 x) \right) \right)}{2 \zeta_3 \left( A \text{sinh} \left( \sqrt{\frac{\zeta_2^2 - 4 \zeta_1 \zeta_3}{2 \zeta_3}} (l_2 t + l_1 x) \right) + B \right)}, \]

(2.72)

\[ v_7(x,t) = -\frac{a_1 \left( \sqrt{\frac{\zeta_2^2 - 4 \zeta_1 \zeta_3}{2 \zeta_3}} (B^2 - A^2) + A \sqrt{\frac{\zeta_2^2 - 4 \zeta_1 \zeta_3}{2 \zeta_3}} \sinh \left( \sqrt{\frac{\zeta_2^2 - 4 \zeta_1 \zeta_3}{2 \zeta_3}} (l_2 t + l_1 x) \right) \right)}{2 \zeta_3 \left( A \cosh \left( \sqrt{\frac{\zeta_2^2 - 4 \zeta_1 \zeta_3}{2 \zeta_3}} (l_2 t + l_1 x) \right) + B \right)}, \]

(2.73)

\[ v_8(x,t) = \frac{1}{2} a_1 \left( \frac{\zeta_2}{\zeta_3} - \frac{4 \zeta_1}{\zeta_2 - \sqrt{\frac{\zeta_2^2 - 4 \zeta_1 \zeta_3}{2 \zeta_3}} \tanh \left( \frac{1}{2} \sqrt{\frac{\zeta_2^2 - 4 \zeta_1 \zeta_3}{2 \zeta_3}} (l_2 t + l_1 x) \right) \right), \]

(2.74)

\[ v_9(x,t) = \frac{1}{2} a_1 \left( \frac{\zeta_2}{\zeta_3} - \frac{4 \zeta_1}{\zeta_2 - \sqrt{\frac{\zeta_2^2 - 4 \zeta_1 \zeta_3}{2 \zeta_3}} \coth \left( \frac{1}{2} \sqrt{\frac{\zeta_2^2 - 4 \zeta_1 \zeta_3}{2 \zeta_3}} (l_2 t + l_1 x) \right) \right), \]

(2.75)
\[ v_{10}(x,t) = \frac{1}{2} a_1 \left( \frac{\zeta_2}{\zeta_3} + \left( 4\zeta_1 \cosh \left( \sqrt{\zeta_2^2 - 4\zeta_1^2} (l_{2t} + l_1) \right) \right) \right) \left( \sqrt{\zeta_2^2 - 4\zeta_1^2} \sinh \left( \sqrt{\zeta_2^2 - 4\zeta_1^2} (l_{2t} + l_1) \right) \right) \]
\[ \left( \zeta_2 \cosh \left( \sqrt{\zeta_2^2 - 4\zeta_1^2} (l_{2t} + l_1) \right) \pm i \sqrt{\zeta_2^2 - 4\zeta_1^2} \right) \right) \right), \]
\[ (2.76) \]
\[ v_{11}(x,t) = \frac{1}{2} a_1 \left( \frac{\zeta_2}{\zeta_3} + \left( 4\zeta_1 + \left( \cosh \left( \sqrt{\zeta_2^2 - 4\zeta_1^2} (l_{2t} + l_1) \right) \right) \left( \sqrt{\zeta_2^2 - 4\zeta_1^2} \cosh \left( \sqrt{\zeta_2^2 - 4\zeta_1^2} (l_{2t} + l_1) \right) \right) \right) \right) \right) \right), \]
\[ (2.77) \]
\[ v_{12}(x,t) = \frac{1}{2} a_1 \left( \frac{\zeta_2}{\zeta_3} - 4\zeta_1 \right) \left( \left( \left( \zeta_2 - \sqrt{\zeta_2^2 - 4\zeta_1^2} \cosh \left( 2\sqrt{\zeta_2^2 - 4\zeta_1^2} (l_{2t} + l_1) \right) \right) \right) \cosh \left( \frac{1}{2} \sqrt{\zeta_2^2 - 4\zeta_1^2} \times (l_{2t} + l_1) \right) \right), \]
\[ (2.78) \]
where \( A \) and \( B \) are two non-zero constants satisfying \( B^2 - A^2 > 0 \).
For \( \zeta_2^2 - 4\zeta_1^2 \zeta_3 < 0, \zeta_2 \zeta_3 \neq 0, \zeta_3 \zeta_1 \neq 0 \), the solutions have been evaluated as following
\[ v_{13}(x,t) = \frac{1}{2} a_1 \left( \frac{\zeta_2}{\zeta_3} - \left( \frac{4\zeta_1}{\zeta_2 - \sqrt{\zeta_2^2 - 4\zeta_1^2} \cosh \left( \frac{1}{2} \sqrt{\zeta_2^2 - 4\zeta_1^2} \times (l_{2t} + l_1) \right) \right) \right) \right) \right), \]
\[ (2.79) \]
\[ v_{14}(x,t) = -\frac{a_1 \sqrt{4\zeta_1^2 \zeta_3 - \zeta_2^2 \zeta_3 \cosh \left( \frac{1}{2} \sqrt{4\zeta_1^2 \zeta_3 - \zeta_2^2} \zeta_3 \zeta_1 \times (l_{2t} + l_1) \right) \right) \cosh \left( \frac{1}{2} \sqrt{4\zeta_1^2 \zeta_3 - \zeta_2^2} \zeta_3 \zeta_1 \times (l_{2t} + l_1) \right) \right), \]
\[ (2.80) \]
\[ v_{15}(x,t) = \frac{a_1 \sqrt{\zeta_2^2 - 4\zeta_1^2} \zeta_3}{2\zeta_3} \left( \tan \left( \sqrt{4\zeta_1^2 \zeta_3 - \zeta_2^2} \zeta_3 \zeta_1 \times (l_{2t} + l_1) \right) \right) \pm \sec \left( \sqrt{4\zeta_1^2 \zeta_3 - \zeta_2^2} \zeta_3 \zeta_1 \times (l_{2t} + l_1) \right) \right), \]
\[ (2.81) \]
\[ v_{16}(x,t) = -\frac{a_1 \sqrt{\zeta_2^2 - 4\zeta_1^2} \zeta_3}{2\zeta_3} \left( \cot \left( \sqrt{4\zeta_1^2 \zeta_3 - \zeta_2^2} \zeta_3 \zeta_1 \times (l_{2t} + l_1) \right) \right) \pm \csc \left( \sqrt{4\zeta_1^2 \zeta_3 - \zeta_2^2} \zeta_3 \zeta_1 \times (l_{2t} + l_1) \right) \right), \]
\[ (2.82) \]
\[ v_{17}(x,t) = -\frac{a_1 \sqrt{\zeta_2^2 - 4\zeta_1^2} \zeta_3}{4\zeta_3} \left( \tan \left( \frac{1}{4} \sqrt{4\zeta_1^2 \zeta_3 - \zeta_2^2} \zeta_3 \zeta_1 \times (l_{2t} + l_1) \right) \right) \left( \cot^2 \left( \frac{1}{4} \sqrt{4\zeta_1^2 \zeta_3 - \zeta_2^2} \zeta_3 \zeta_1 \times (l_{2t} + l_1) \right) - 1 \right), \]
\[ (2.83) \]
\[ v_{18}(x,t) = \frac{1}{4} \left( \sqrt{\zeta_2^2 - 4\zeta_1^2} \zeta_3 \right) \left( (A - B)(A + B) - A \sqrt{4\zeta_1^2 \zeta_3 - \zeta_2^2} \cos \left( \sqrt{4\zeta_1^2 \zeta_3 - \zeta_2^2} \zeta_3 \zeta_1 \times (l_{2t} + l_1) \right) \right) \right) \left( \right), \]
\[ (2.84) \]
\[ v_{19}(x,t) = -\frac{a_1 \sqrt{4\zeta_1^2 \zeta_3 - \zeta_2^2} \zeta_3 \sin \left( \sqrt{4\zeta_1^2 \zeta_3 - \zeta_2^2} \zeta_3 \zeta_1 \times (l_{2t} + l_1) \right) \right) - \sqrt{\zeta_2^2 - 4\zeta_1^2} \zeta_3 \left( B^2 - A^2 \right) \right), \]
\[ (2.85) \]
Employing the generalized Riccati expansion method gives the following sets of the abovementioned parameters:

\[ v_{20}(x,t) = \frac{1}{2} a_1 \left( \frac{\zeta_2}{\zeta_3} - \frac{4 \zeta_1}{\zeta_2 + \sqrt{4 \zeta_1 \zeta_3 - \zeta_2^2 \tan \left( \frac{1}{2} \sqrt{4 \zeta_1 \zeta_3 - \zeta_2^2} (l_2 t + l_1 x) \right)} \right), \]  

(2.86)

\[ v_{21}(x,t) = \frac{1}{2} a_1 \left( \frac{\zeta_2}{\zeta_3} + 4 \zeta_1 \sinh \left( \frac{1}{2} \sqrt{4 \zeta_1 \zeta_3 - \zeta_2^2} (l_2 t + l_1 x) \right) \right) \left( \sqrt{4 \zeta_1 \zeta_3 - \zeta_2^2} \cos \left( \frac{1}{2} \sqrt{4 \zeta_1 \zeta_3 - \zeta_2^2} (l_2 t + l_1 x) \right) \right) \left( 4 \zeta_1 \zeta_3 - \zeta_2^2 \tan \left( \frac{1}{2} \sqrt{4 \zeta_1 \zeta_3 - \zeta_2^2} (l_2 t + l_1 x) \right) \right), \]  

(2.87)

\[ v_{22}(x,t) = \frac{1}{2} a_1 \left( \frac{\zeta_2}{\zeta_3} - 4 \zeta_1 \cos \left( \frac{1}{2} \sqrt{4 \zeta_1 \zeta_3 - \zeta_2^2} (l_2 t + l_1 x) \right) \right) \left( \sqrt{4 \zeta_1 \zeta_3 - \zeta_2^2} \sin \left( \frac{1}{2} \sqrt{4 \zeta_1 \zeta_3 - \zeta_2^2} (l_2 t + l_1 x) \right) \right) \left( + \left( \zeta_2 \cos \left( \frac{1}{2} \sqrt{4 \zeta_1 \zeta_3 - \zeta_2^2} (l_2 t + l_1 x) \right) \pm \sqrt{4 \zeta_1 \zeta_3 - \zeta_2^2} \right) \right), \]  

(2.88)

\[ v_{23}(x,t) = \frac{1}{2} a_1 \left( \frac{\zeta_2}{\zeta_3} + 4 \zeta_1 \sin \left( \frac{1}{2} \sqrt{4 \zeta_1 \zeta_3 - \zeta_2^2} (l_2 t + l_1 x) \right) \right) \left( \sqrt{4 \zeta_1 \zeta_3 - \zeta_2^2} \cos \left( \frac{1}{2} \sqrt{4 \zeta_1 \zeta_3 - \zeta_2^2} (l_2 t + l_1 x) \right) \right) \left( \pm \sqrt{4 \zeta_1 \zeta_3 - \zeta_2^2} \right), \]  

(2.89)

\[ v_{24}(x,t) = \frac{1}{2} a_1 \left( \frac{\zeta_2}{\zeta_3} - 4 \zeta_1 \right) \left( \zeta_2 - \sqrt{4 \zeta_1 \zeta_3 - \zeta_2^2} \cos \left( \frac{1}{2} \sqrt{4 \zeta_1 \zeta_3 - \zeta_2^2} (l_2 t + l_1 x) \right) \right) \csc \left( \frac{1}{2} \sqrt{4 \zeta_1 \zeta_3 - \zeta_2^2} \right) \left( (l_2 t + l_1 x) \right), \]  

(2.90)

where \( A \) and \( B \) are two non-zero constants satisfying \( B^2 - A^2 < 0 \).

**Generalized Riccati expansion method**

Employing the generalized Riccati expansion method gives the following sets of the abovementioned parameters:

Set I

\[ a_0 \rightarrow \frac{a_1 \varsigma_2}{2 \varsigma_3}, \quad b_1 \rightarrow \frac{a_1 \varsigma_1}{\varsigma_3}, \quad p_2 \rightarrow - \frac{12 \nu_1 \varsigma_2^3}{a_1}. \]

Set II

\[ a_0 \rightarrow \frac{b_1 \varsigma_2}{2 \varsigma_1}, \quad a_1 \rightarrow 0, \quad p_2 \rightarrow - \frac{12 \nu_1 \varsigma_1^3}{b_1}. \]

Set III

\[ a_0 \rightarrow \frac{a_1 \varsigma_2}{2 \varsigma_3}, \quad b_1 \rightarrow 0, \quad p_2 \rightarrow - \frac{12 \nu_1 \varsigma_3^3}{a_1}. \]

Thus, the solitary wave solutions of equation (1.5) can be formulated in the following formulas:

For \( \varsigma = \varsigma_2^2 - 4 \varsigma_1 \varsigma_3 > 0, \varsigma_2 \varsigma_3 = 0, \varsigma_3 \varsigma_1 = 0 \), the solutions have been evaluated as following

\[ v_{1,1}(x,t) = \frac{1}{2} a_1 \left( \frac{4 \varsigma_1}{\sqrt{\nu_1 \varsigma_2^2} (l_2 t + l_1 x) + \varsigma_2} - \sqrt{\nu_1 \varsigma_1} \tanh \left( \frac{1}{2} \sqrt{\nu_1} (l_2 t + l_1 x) \right) \right) \left( \frac{4 \varsigma_1}{\sqrt{\nu_1 \varsigma_1} (l_2 t + l_1 x) + \varsigma_1} \right), \]  

(2.91)
\begin{align}
\nu_{1,2}(x,t) &= \frac{1}{2} a_1 \left( -\sqrt{\Omega} \coth \left( \frac{1}{2} \sqrt{\Omega} (l_xt + l_1x) \right) \frac{4\varsigma_1}{\sqrt{\Omega} \coth \left( \frac{1}{2} \sqrt{\Omega} (l_xt + l_1x) \right) + \varsigma_2} \right), \\
\nu_{1,3}(x,t) &= \frac{1}{2} a_1 \left( -\sqrt{\Omega} \left( \tanh \left( \frac{1}{2} \sqrt{\Omega} (l_xt + l_1x) \right) + \text{csch} \left( \frac{1}{2} \sqrt{\Omega} (l_xt + l_1x) \right) \right) \right) \\
&- \frac{4\varsigma_1}{\sqrt{\Omega} \left( \tanh \left( \frac{1}{2} \sqrt{\Omega} (l_xt + l_1x) \right) + \text{csch} \left( \frac{1}{2} \sqrt{\Omega} (l_xt + l_1x) \right) \right)} \\
\nu_{1,4}(x,t) &= \frac{1}{2} a_1 \left( -\sqrt{\Omega} \left( \coth \left( \frac{1}{2} \sqrt{\Omega} (l_xt + l_1x) \right) + \text{csch} \left( \frac{1}{2} \sqrt{\Omega} (l_xt + l_1x) \right) \right) \right) \\
&- \frac{4\varsigma_1}{\sqrt{\Omega} \left( \coth \left( \frac{1}{2} \sqrt{\Omega} (l_xt + l_1x) \right) + \text{csch} \left( \frac{1}{2} \sqrt{\Omega} (l_xt + l_1x) \right) \right)} \\
\nu_{1,5}(x,t) &= \frac{1}{4} a_1 \left( -\sqrt{\Omega} \left( \tanh \left( \frac{1}{4} \sqrt{\Omega} (l_xt + l_1x) \right) \left( \coth \left( \frac{1}{4} \sqrt{\Omega} (l_xt + l_1x) \right) + 1 \right) \right) \\
&- \frac{16\varsigma_1}{\sqrt{\Omega} \left( \tanh \left( \frac{1}{4} \sqrt{\Omega} (l_xt + l_1x) \right) + \coth \left( \frac{1}{4} \sqrt{\Omega} (l_xt + l_1x) \right) \right) + 2\varsigma_2} \\
\nu_{1,6}(x,t) &= \frac{1}{2} a_1 \left( \frac{\sqrt{\Omega} (A^2 + B^2) - A \sqrt{\Omega} \coth \left( \frac{1}{2} \sqrt{\Omega} (l_xt + l_1x) \right)}{\varsigma_3 \left( \text{Asinh} \left( \frac{1}{2} \sqrt{\Omega} (l_xt + l_1x) \right) + B \right)} \right) \\
&- \frac{4\varsigma_1}{A \sqrt{\Omega} \coth \left( \frac{1}{2} \sqrt{\Omega} (l_xt + l_1x) \right) - \sqrt{\Omega} \left( A^2 + B^2 \right) + \varsigma_2} \\
\nu_{1,7}(x,t) &= \frac{1}{2} a_1 \left( -\sqrt{\Omega} \left( B^2 - A^2 \right) + A \sqrt{\Omega} \sinh \left( \frac{1}{2} \sqrt{\Omega} (l_xt + l_1x) \right) \right) \\
&- \frac{4\varsigma_1}{\sqrt{\Omega} \left( B^2 - A^2 \right) + A \sqrt{\Omega} \sinh \left( \frac{1}{2} \sqrt{\Omega} (l_xt + l_1x) \right) + \varsigma_2} \\
\nu_{1,8}(x,t) &= \frac{1}{2} a_1 \left( \sqrt{\Omega} \tanh \left( \frac{1}{2} \sqrt{\Omega} (l_xt + l_1x) \right) \right) \\
&- \frac{4\varsigma_1}{\varsigma_2 - \sqrt{\Omega} \tanh \left( \frac{1}{2} \sqrt{\Omega} (l_xt + l_1x) \right)} \\
\nu_{1,9}(x,t) &= \frac{1}{2} a_1 \left( \sqrt{\Omega} \coth \left( \frac{1}{2} \sqrt{\Omega} (l_xt + l_1x) \right) \right) \\
&- \frac{4\varsigma_1}{\sqrt{\Omega} \coth \left( \frac{1}{2} \sqrt{\Omega} (l_xt + l_1x) \right) - \varsigma_2}.
\end{align}
\[ v_{1,10}(x,t) = \frac{1}{2} a_1 \left( \frac{4\varsigma_1 \cosh(\sqrt{3}(l_2 t + l_1 x))}{\sqrt{3} \sinh(\sqrt{3}(l_2 t + l_1 x))} - \left(\varsigma_2 \cosh(\sqrt{3}(l_2 t + l_1 x)) \pm i \sqrt{3}\right) \right) + \frac{1}{\varsigma_3} \frac{\operatorname{sech}(\sqrt{3}(l_2 t + l_1 x))}{\sqrt{3} \tanh(\sqrt{3}(l_2 t + l_1 x))} + \varsigma_2, \] (2.100)

\[ v_{1,11}(x,t) = a_1 \left( \frac{\cosh(\sqrt{3}(l_2 t + l_1 x))}{\sqrt{3} \tanh(\frac{1}{2} \sqrt{3}(l_2 t + l_1 x)) + \varsigma_2} \right) \cdot \frac{2\varsigma_3}{2\varsigma_1} \frac{\cosh(\sqrt{3}(l_2 t + l_1 x))}{\sqrt{3} \tanh(\frac{1}{2} \sqrt{3}(l_2 t + l_1 x)) + \varsigma_2} - \varsigma_2, \] (2.101)

\[ v_{1,12}(x,t) = \frac{1}{2} a_1 \left( \frac{4\varsigma_1}{\sqrt{3} \coth(\sqrt{3}(l_2 t + l_1 x)) + \varsigma_2} \right) \cdot \frac{\varsigma_2}{\varsigma_3} \frac{2\varsigma_3}{\sqrt{3} \coth(\sqrt{3}(l_2 t + l_1 x)) + \varsigma_2} - \varsigma_2 \right), \] (2.102)

\[ v_{1,1}(x,t) = \frac{1}{2} b_1 \left( \frac{\varsigma_2}{\varsigma_1} - \frac{4\varsigma_2}{\sqrt{3} \tanh(\frac{1}{2} \sqrt{3}(l_2 t + l_1 x)) + \varsigma_2} \right), \] (2.103)

\[ v_{1,2}(x,t) = \frac{1}{2} b_1 \left( \frac{\varsigma_2}{\varsigma_1} - \frac{4\varsigma_2}{\sqrt{3} \coth(\frac{1}{2} \sqrt{3}(l_2 t + l_1 x)) + \varsigma_2} \right), \] (2.104)

\[ v_{1,3}(x,t) = \frac{1}{2} b_1 \left( \frac{\varsigma_2}{\varsigma_1} - \frac{4\varsigma_2}{\sqrt{3} \tanh(\frac{1}{2} \sqrt{3}(l_2 t + l_1 x)) + \varsigma_2} \right), \] (2.105)

\[ v_{1,4}(x,t) = \frac{1}{2} b_1 \left( \frac{\varsigma_2}{\varsigma_1} - \frac{4\varsigma_2}{\sqrt{3} \coth(\frac{1}{2} \sqrt{3}(l_2 t + l_1 x)) + \varsigma_2} \right), \] (2.106)

\[ v_{1,5}(x,t) = \frac{1}{2} b_1 \left( \frac{\varsigma_2}{\varsigma_1} - \frac{8\varsigma_3}{\sqrt{3} \left(\tanh(\frac{1}{4} \sqrt{3}(l_2 t + l_1 x)) + \coth(\frac{1}{4} \sqrt{3}(l_2 t + l_1 x))\right) + \varsigma_2} \right), \] (2.107)

\[ v_{1,6}(x,t) = \frac{1}{2} b_1 \left( \frac{\varsigma_2}{\varsigma_1} - \frac{4\varsigma_3}{A \sqrt{3} \cosh(\sqrt{3}(l_2 t + l_1 x)) - \sqrt{3}(A^2 + B^2)} \right), \] (2.108)

\[ v_{1,7}(x,t) = \frac{1}{2} b_1 \left( \frac{\varsigma_2}{\varsigma_1} - \frac{4\varsigma_3}{A \cosh(\sqrt{3}(l_2 t + l_1 x)) + B} \right), \] (2.109)

\[ v_{1,8}(x,t) = \frac{b_1 \sqrt{3}}{2\varsigma_1} \tanh\left(\frac{1}{2} \sqrt{3}(l_2 t + l_1 x)\right), \] (2.110)
\[ \nu_{II,10}(x,t) = \frac{b_I}{2\varsigma_1} \left( \tanh \left( \frac{1}{2} \sqrt{\Theta} (l_2 t + l_1 x) \right) - \varsigma_2 \coth \left( \frac{1}{2} \sqrt{\Theta} (l_2 t + l_1 x) \right) \right), \]

\[ \nu_{II,11}(x,t) = \frac{b_I}{2\varsigma_1} \left( \csc \left( \sqrt{\Theta} (l_2 t + l_1 x) \right) \left( \sqrt{\Theta} \cosh \left( \sqrt{\Theta} (l_2 t + l_1 x) \right) \pm \sqrt{\Theta} \right) \right), \]

\[ \nu_{II,12}(x,t) = \frac{b_I}{4\varsigma_1} \left( \frac{1}{2} \sqrt{\Theta} (l_2 t + l_1 x) \right) \left( \coth^2 \left( \frac{1}{2} \sqrt{\Theta} (l_2 t + l_1 x) \right) + 1 \right), \]

\[ \nu_{III,1}(x,t) = -\frac{a_I}{2\varsigma_3} \left( \frac{1}{2} \sqrt{\Theta} (l_2 t + l_1 x) \right) \left( \coth^2 \left( \frac{1}{2} \sqrt{\Theta} (l_2 t + l_1 x) \right) + 1 \right), \]

\[ \nu_{III,2}(x,t) = -\frac{a_I}{2\varsigma_3} \left( \frac{1}{2} \sqrt{\Theta} (l_2 t + l_1 x) \right) \left( \coth \left( \frac{1}{2} \sqrt{\Theta} (l_2 t + l_1 x) \right) \pm \csc \left( \frac{1}{2} \sqrt{\Theta} (l_2 t + l_1 x) \right) \right), \]

\[ \nu_{III,3}(x,t) = -\frac{a_I}{2\varsigma_3} \left( \frac{1}{4} \sqrt{\Theta} (l_2 t + l_1 x) \right) \left( \coth^2 \left( \frac{1}{4} \sqrt{\Theta} (l_2 t + l_1 x) \right) + 1 \right), \]

\[ \nu_{III,4}(x,t) = \frac{a_I}{2\varsigma_3} \left( \frac{1}{2} \sqrt{\Theta} (l_2 t + l_1 x) \right) \left( a_i \left( (A^2 + B^2) - A \sqrt{\Theta} \cosh \left( \sqrt{\Theta} (l_2 t + l_1 x) \right) \right) \right), \]

\[ \nu_{III,5}(x,t) = \frac{a_I}{2\varsigma_3} \left( \frac{1}{4} \sqrt{\Theta} (l_2 t + l_1 x) \right) \left( \cosh \left( \frac{1}{4} \sqrt{\Theta} (l_2 t + l_1 x) \right) \right), \]

\[ \nu_{III,6}(x,t) = \frac{a_I}{2\varsigma_3} \left( \frac{1}{2} \sqrt{\Theta} (l_2 t + l_1 x) \right) \left( a_i \left( B^2 - A^2 \right) + A \sqrt{\Theta} \sinh \left( \sqrt{\Theta} (l_2 t + l_1 x) \right) \right), \]

\[ \nu_{III,7}(x,t) = \frac{a_I}{2\varsigma_3} \left( \frac{1}{4} \sqrt{\Theta} (l_2 t + l_1 x) \right) \left( a_i \left( (A^2 + B^2) - A \sqrt{\Theta} \cosh \left( \sqrt{\Theta} (l_2 t + l_1 x) \right) \right) \right), \]

\[ \nu_{III,8}(x,t) = \frac{1}{2} a_i \left[ \frac{\varsigma_2}{\varsigma_3} - \frac{4\varsigma_1}{\varsigma_2 - \sqrt{\Theta} \tanh \left( \frac{1}{2} \sqrt{\Theta} (l_2 t + l_1 x) \right)} \right], \]

\[ \nu_{III,9}(x,t) = \frac{1}{2} a_i \left[ \frac{4\varsigma_1}{\sqrt{\Theta} \coth \left( \frac{1}{2} \sqrt{\Theta} (l_2 t + l_1 x) \right)} + \frac{\varsigma_2}{\varsigma_3} \right], \]

\[ \nu_{III,10}(x,t) = \frac{1}{2} a_i \left[ \frac{\varsigma_2}{\varsigma_3} + \frac{4\varsigma_1 \cosh \left( \sqrt{\Theta} (l_2 t + l_1 x) \right)}{\sqrt{\Theta} \sinh \left( \sqrt{\Theta} (l_2 t + l_1 x) \right) - \left( \varsigma_2 \cosh \left( \sqrt{\Theta} (l_2 t + l_1 x) \right) \pm i\sqrt{\Theta} \right)} \right], \]

\[ \nu_{III,11}(x,t) = \frac{1}{2} a_i \left[ \csc \left( \sqrt{\Theta} (l_2 t + l_1 x) \right) \left( \sqrt{\Theta} \cosh \left( \sqrt{\Theta} (l_2 t + l_1 x) \right) \pm \sqrt{\Theta} \right) \right]. \]
where \( A \) and \( B \) are two non-zero real constants satisfying \( B^2 - A^2 > 0 \).

For \( \Omega = \varsigma_2^2 - 4 \varsigma_1 \varsigma_3 < 0 \), \( \varsigma_2 \varsigma_3 \neq 0 \), \( \varsigma_2 \varsigma_1 \neq 0 \), the solutions have been evaluated as following

\[
\begin{align*}
\nu_{1,13}(x,t) &= \frac{1}{2} a_1 \left( -\frac{4 \varsigma_1}{\sqrt{\Omega} \tanh \left( \frac{1}{2} \sqrt{\Omega} (l_2 t + l_1 x) \right) + \varsigma_2} - \frac{\sqrt{\Omega}}{\varsigma_3} \tanh \left( \frac{1}{2} \sqrt{\Omega} (l_2 t + l_1 x) \right) \right), \\
\nu_{1,14}(x,t) &= \frac{1}{2} a_1 \left( -\frac{\sqrt{\Omega}}{\varsigma_3} \coth \left( \frac{1}{2} \sqrt{\Omega} (l_2 t + l_1 x) \right) - \frac{4 \varsigma_1}{\sqrt{\Omega} \coth \left( \frac{1}{2} \sqrt{\Omega} (l_2 t + l_1 x) \right) + \varsigma_2} \right), \\
\nu_{1,15}(x,t) &= \frac{1}{2} a_1 \left( \frac{\sqrt{-\Omega}}{\varsigma_3} \left( \tan \left( \sqrt{-\Omega} (l_2 t + l_1 x) \right) \pm \text{sech} \left( \sqrt{-\Omega} (l_2 t + l_1 x) \right) \right) \right) \\
&+ \frac{4 \varsigma_1}{\sqrt{-\Omega} \left( \tan \left( \sqrt{-\Omega} (l_2 t + l_1 x) \right) \pm \text{sech} \left( \sqrt{-\Omega} (l_2 t + l_1 x) \right) \right) - \varsigma_2}, \\
\nu_{1,16}(x,t) &= \frac{1}{2} a_1 \left( -\frac{\sqrt{-\Omega}}{\varsigma_3} \left( \cot \left( \sqrt{-\Omega} (l_2 t + l_1 x) \right) \pm \csc \left( \sqrt{-\Omega} (l_2 t + l_1 x) \right) \right) \right) \\
&- \frac{4 \varsigma_1}{\sqrt{-\Omega} \left( \cot \left( \sqrt{-\Omega} (l_2 t + l_1 x) \right) \pm \csc \left( \sqrt{-\Omega} (l_2 t + l_1 x) \right) \right) + \varsigma_2} \right) \right), \\
\nu_{1,17}(x,t) &= \frac{1}{4} a_1 \left( -\frac{16 \varsigma_1}{\sqrt{\Omega} \tanh \left( \frac{1}{4} \sqrt{\Omega} (l_2 t + l_1 x) \right) \left( \coth \left( \frac{1}{4} \sqrt{\Omega} (l_2 t + l_1 x) \right) + 1 \right) + 2 \varsigma_2} \right) \\
&- \frac{-\sqrt{\Omega} \tanh \left( \frac{1}{4} \sqrt{\Omega} (l_2 t + l_1 x) \right) \left( \coth \left( \frac{1}{4} \sqrt{\Omega} (l_2 t + l_1 x) \right) + 1 \right) \right), \\
\nu_{1,18}(x,t) &= \frac{a_1}{2} \left( \frac{1}{A \varsigma_3} \csc \left( \sqrt{-\Omega} (l_2 t + l_1 x) \right) \left( \sqrt{A^2 - B^2 - \Omega} - A + \sqrt{-\Omega} \cosh \left( \sqrt{-\Omega} (l_2 t + l_1 x) \right) \right) \right) \\
&- \frac{4 A \varsigma_1}{\left( A - \sqrt{A^2 - B^2 - \Omega} \right) \csc \left( \sqrt{-\Omega} (l_2 t + l_1 x) \right) + A \varsigma_2 + \left( -\sqrt{\Omega} \right) \coth \left( \sqrt{-\Omega} (l_2 t + l_1 x) \right) \right), \\
\nu_{1,19}(x,t) &= \frac{1}{2} a_1 \left( \frac{A \sqrt{\Omega} \sinh \left( \sqrt{\Omega} (l_2 t + l_1 x) \right) - \sqrt{\Omega} \left( B^2 - A^2 \right)}{\varsigma_3 \left( A \cosh \left( \sqrt{\Omega} (l_2 t + l_1 x) \right) + B \right)} \right) \\
&- \frac{4 \varsigma_1}{\sqrt{\Omega} \left( B^2 - A^2 \right) - A \sqrt{\Omega} \sinh \left( \sqrt{\Omega} (l_2 t + l_1 x) \right)} \right) \right), \\
\end{align*}
\]
\[ v_{L20}(x,t) = \frac{1}{2}a_1 \left( \frac{\sqrt{\Omega} \tanh \left( \frac{1}{2} \sqrt{\Omega} (l_2 t + l_1 x) \right)}{\varsigma_3} - \frac{4\varsigma_1}{\varsigma_2 - \sqrt{\Omega} \tanh \left( \frac{1}{2} \sqrt{\Omega} (l_2 t + l_1 x) \right)} \right), \]  
\[ v_{L21}(x,t) = \frac{1}{2}a_1 \left( \frac{\sqrt{\Omega} \coth \left( \frac{1}{2} \sqrt{\Omega} (l_2 t + l_1 x) \right)}{\varsigma_3} + \frac{4\varsigma_1}{\sqrt{\Omega} \coth \left( \frac{1}{2} \sqrt{\Omega} (l_2 t + l_1 x) \right) - \varsigma_2} \right), \]  
\[ v_{L22}(x,t) = \frac{1}{2}a_1 \left( \frac{4\varsigma_1 \cosh \left( \frac{1}{2} \sqrt{\Omega} (l_2 t + l_1 x) \right)}{\sqrt{\Omega} \sinh \left( \frac{1}{2} \sqrt{\Omega} (l_2 t + l_1 x) \right)} \right) + \frac{1}{\varsigma_3} \left( \sech \left( \frac{1}{2} \sqrt{\Omega} (l_2 t + l_1 x) \right) \left( -\varsigma_2 \cosh \left( \frac{1}{2} \sqrt{\Omega} (l_2 t + l_1 x) \right) \pm \sqrt{-\Omega} \right) \right) + \sqrt{\Omega} \tanh \left( \frac{1}{2} \sqrt{\Omega} (l_2 t + l_1 x) + \varsigma_2 \right), \]  
\[ v_{L23}(x,t) = a_1 \left( \frac{1}{2\varsigma_3} \csc \left( \frac{1}{2} \sqrt{\Omega} (l_2 t + l_1 x) \right) \left( \sqrt{-\Omega} \cos \left( \frac{1}{2} \sqrt{\Omega} (l_2 t + l_1 x) \right) \pm \sqrt{\Omega} \right) \right) + \frac{2\varsigma_1}{\csc \left( \frac{1}{2} \sqrt{\Omega} (l_2 t + l_1 x) \right) \left( \sqrt{-\Omega} \cos \left( \frac{1}{2} \sqrt{\Omega} (l_2 t + l_1 x) \right) \pm \sqrt{\Omega} \right) - \varsigma_2}, \]  
\[ v_{L24}(x,t) = \frac{1}{2}a_1 \left( \frac{\sqrt{\Omega} \coth \left( \frac{1}{2} \sqrt{\Omega} (l_2 t + l_1 x) \right)}{\varsigma_3} + \frac{4\varsigma_1}{\sqrt{\Omega} \coth \left( \frac{1}{2} \sqrt{\Omega} (l_2 t + l_1 x) \right) - \varsigma_2} \right), \]  
\[ v_{IL13}(x,t) = \frac{1}{2}b_1 \left( \frac{\varsigma_2}{\varsigma_1} - \frac{4\varsigma_3}{\sqrt{\Omega} \tanh \left( \frac{1}{2} \sqrt{\Omega} (l_2 t + l_1 x) \right) + \varsigma_2} \right), \]  
\[ v_{IL14}(x,t) = \frac{1}{2}b_1 \left( \frac{\varsigma_2}{\varsigma_1} - \frac{4\varsigma_3}{\sqrt{\Omega} \coth \left( \frac{1}{2} \sqrt{\Omega} (l_2 t + l_1 x) \right) + \varsigma_2} \right), \]  
\[ v_{IL15}(x,t) = \frac{1}{2}b_1 \left( \frac{4\varsigma_3}{\sqrt{-\Omega} \left( \tan \left( \frac{1}{2} \sqrt{-\Omega} (l_2 t + l_1 x) \right) \pm \sech \left( \frac{1}{2} \sqrt{-\Omega} (l_2 t + l_1 x) \right) \right) + \varsigma_2 \right), \]  
\[ v_{IL16}(x,t) = \frac{1}{2}b_1 \left( \frac{4\varsigma_3}{\sqrt{-\Omega} \left( \cot \left( \frac{1}{2} \sqrt{-\Omega} (l_2 t + l_1 x) \right) \pm \csc \left( \frac{1}{2} \sqrt{-\Omega} (l_2 t + l_1 x) \right) \right) + \varsigma_2 \right), \]  
\[ v_{IL17}(x,t) = \frac{1}{2}b_1 \left( \frac{8\varsigma_3}{\sqrt{\Omega} \tanh \left( \frac{1}{4} \sqrt{\Omega} (l_2 t + l_1 x) \right) \left( \coth \left( \frac{1}{4} \sqrt{\Omega} (l_2 t + l_1 x) \right) + 1 \right) + 2\varsigma_2} \right), \]  
\[ v_{IL18}(x,t) = \frac{1}{2}b_1 \left( \frac{4\varsigma_3}{\varsigma_1} - 4A\varsigma_3 \right) \left( \left( A - \sqrt{A^2 - B^2 - \Omega} \right) \csc \left( \frac{1}{2} \sqrt{-\Omega} (l_2 t + l_1 x) \right) + A\varsigma_2 + \left( -\sqrt{\Omega} \right) \right) \times \coth \left( \frac{1}{2} \sqrt{\Omega} (l_2 t + l_1 x) \right), \]
\[ v_{II,19}(x,t) = \frac{1}{2} b_1 \left( \frac{\varsigma_2 - \frac{4\varsigma_3}{\sqrt{\Omega}(B^2 - A^2) - A\sqrt{\Omega}\sinh\left(\sqrt{\Omega}(l_2 t + l_1 x)\right)} + \varsigma_2}{A\cosh\left(\sqrt{\Omega}(l_2 t + l_1 x)\right) + B} \right), \] 

(2.145)
Employing the improved F-expansion method gives the following sets of the abovementioned parameters:

\[ v_{III.21}(x,t) = \frac{1}{2} a_1 \left( \frac{4\xi_1}{\sqrt{\mathcal{O}} \coth \left( \frac{1}{2} \sqrt{\mathcal{O}} (l_x t + l_x) \right) - \xi_2} + \frac{\xi_2}{\xi_3} \right), \]  
\[ v_{III.22}(x,t) = \frac{1}{2} a_1 \left( \frac{4\xi_1 \cosh \left( \sqrt{\mathcal{O}} (l_x t + l_x) \right)}{\sqrt{\mathcal{O}} \sinh \left( \sqrt{\mathcal{O}} (l_x t + l_x) \right) - \left( \xi_2 \cosh \left( \sqrt{\mathcal{O}} (l_x t + l_x) \right) + \sqrt{\mathcal{O}} \right)} + \frac{\xi_2}{\xi_3} \right), \]  
\[ v_{III.23}(x,t) = \frac{1}{2} a_1 \left( \frac{4\xi_1}{\csc \left( \sqrt{\mathcal{O}} (l_x t + l_x) \right) \left( \sqrt{\mathcal{O}} \cos \left( \sqrt{\mathcal{O}} (l_x t + l_x) \right) + \sqrt{\mathcal{O}} \right) - \xi_2} + \frac{\xi_2}{\xi_3} \right), \]  
\[ v_{III.24}(x,t) = \frac{1}{2} a_1 \left( \frac{\xi_2 \sqrt{\mathcal{O}} \cosh \left( \sqrt{\mathcal{O}} (l_x t + l_x) \right) - \left( \xi_2^2 - 4\xi_1^2 \xi_3 \right) \sin \left( \sqrt{\mathcal{O}} (l_x t + l_x) \right) \right)}{2\xi_3 \left( \sqrt{\mathcal{O}} \cos \left( \sqrt{\mathcal{O}} (l_x t + l_x) \right) - \xi_2 \sin \left( \sqrt{\mathcal{O}} (l_x t + l_x) \right) \right)} \]  

where \( A \) and \( B \) are two non-zero real constants satisfying \( B^2 - A^2 < 0 \).

For \( \xi_1 = 0, \xi_2 \xi_3 = 0 \), the solutions have been evaluated as following

\[ v_{I.25}(x,t) = \frac{a_1}{2\xi_2} \left( \frac{2}{\xi_3^2 \left( e^{2\xi_1 (l_x t + l_x)} + 1 \right) - 1} - \frac{2\xi_1}{\xi_3^2} \left( i + e^{2\xi_1 (l_x t + l_x)} \right) \right), \]  
\[ v_{I.26}(x,t) = \frac{a_1}{2\xi_2} \left( i \left( e^{2\xi_1 (l_x t + l_x)} - 1 \right) \right) + \frac{a_1}{2\xi_3} \left( \frac{2i}{i + e^{2\xi_1 (l_x t + l_x)}} - 1 \right), \]  
\[ v_{II.25}(x,t) = \frac{b_1}{\xi_2} \left( \frac{- e^{2\xi_1 (l_x t + l_x)}}{i} - 1 \right), \]  
\[ v_{II.26}(x,t) = \frac{b_1}{\xi_2} \left( i \left( - e^{2\xi_1 (l_x t + l_x)} - 1 \right) \right) + \frac{b_1}{\xi_3} \left( \frac{2i}{i + e^{2\xi_1 (l_x t + l_x)}} - 1 \right), \]  
\[ v_{III.25}(x,t) = \frac{a_1}{2\xi_3} \left( \frac{2}{\left( e^{2\xi_1 (l_x t + l_x)} + 1 \right) - 1} \right), \]  
\[ v_{III.26}(x,t) = \frac{a_1}{2\xi_3} \left( \frac{2i}{i + e^{2\xi_1 (l_x t + l_x)}} - 1 \right). \]  

For \( \xi_1 = \xi_2 = 0, \xi_3 = 0 \), the solutions have been evaluated as following

\[ v_{I.27}(x,t) = \frac{1}{2} a_1 \left( \frac{\xi_2 - 2\xi_1 (l_x t + l_x) + \theta}{\xi_3} - \frac{2}{\xi_3 (l_x t + l_x) + \theta} \right), \]  
\[ v_{II.27}(x,t) = \frac{b_1}{\xi_2} \left( \xi_2 - 2\xi_1 (l_x t + l_x) + \theta) \right), \]  
\[ v_{III.27}(x,t) = \frac{1}{2} a_1 \left( \frac{\xi_2}{\xi_3} - \frac{2}{\xi_3 (l_x t + l_x) + \theta} \right). \]  

**Improved F-expansion method**

Employing the improved F-expansion method gives the following sets of the abovementioned parameters:
Set I
\[ a_{-1} \to 0, \ a_0 \to a_1(-\mu), \ l_1 \to \frac{ia_1\sqrt{p_2^2}}{2\sqrt{3}}, \] where \( p_2 < 0, \ i = \sqrt{-1} \).

Set II
\[ a_{-1} \to a_1(\mu^2 + \Omega), \ a_0 \to a_1(-\mu), \ l_1 \to \frac{ia_1\sqrt{p_2^2}}{2\sqrt{3}}, \] where \( p_2 < 0, \ i = \sqrt{-1} \).

Set III
\[ a_0 \to -\frac{a_{-1} \mu}{\mu^2 + \Omega}, \ a_1 \to 0, \ l_1 \to \frac{a_{-1} \sqrt{p_2^2}}{2\sqrt{3} \sqrt{-\mu^2 - 2\mu^2 \Omega - \Omega^2}}, \] where \( p_2 < 0, i = \sqrt{-1} \).

Thus, the solitary wave solutions of equation (1.5) can be formulated in the following formulas:

For \( \Omega < 0 \), the solutions have been evaluated as following

\[ v_{I,1}(x,t) = a_1 \sqrt{\Omega} \tan \left( \sqrt{\Omega} \left( l_2 t + \frac{ia_1 \sqrt{p_2^2}}{2\sqrt{3}} \right) \right), \] (2.172)

\[ v_{I,2}(x,t) = a_1 \left( -\sqrt{\Omega} \right) \cot \left( \sqrt{\Omega} \left( l_2 t + \frac{ia_1 \sqrt{p_2^2}}{2\sqrt{3}} \right) \right), \] (2.173)

\[ v_{II,1}(x,t) = a_1 \left( \frac{\mu^2 + \Omega}{\mu + \sqrt{\Omega} \tan \left( \sqrt{\Omega} \left( l_2 t + \frac{ia_1 \sqrt{p_2^2}}{2\sqrt{3}} \right) \right)} + \sqrt{\Omega} \tan \left( \sqrt{\Omega} \left( l_2 t + \frac{ia_1 \sqrt{p_2^2}}{2\sqrt{3}} \right) \right) \right), \] (2.174)

\[ v_{II,2}(x,t) = a_1 \left( \frac{\mu^2 + \Omega}{\mu - \sqrt{\Omega} \cot \left( \sqrt{\Omega} \left( l_2 t + \frac{ia_1 \sqrt{p_2^2}}{2\sqrt{3}} \right) \right)} - \sqrt{\Omega} \cot \left( \sqrt{\Omega} \left( l_2 t + \frac{ia_1 \sqrt{p_2^2}}{2\sqrt{3}} \right) \right) \right), \] (2.175)

\[ v_{III,1}(x,t) = a_{-1} \left( \frac{1}{\sqrt{\Omega} \tan \left( \sqrt{\Omega} \left( \frac{a_{-1} \sqrt{p_2^2}}{2\sqrt{3}} \right) + l_2 t + \frac{ia_1 \sqrt{p_2^2}}{2\sqrt{3}} \right) \right)} - \frac{\mu}{\mu^2 + \Omega} \right), \] (2.176)

\[ v_{III,2}(x,t) = a_{-1} \left( \frac{1}{\mu - \sqrt{\Omega} \cot \left( \sqrt{\Omega} \left( \frac{a_{-1} \sqrt{p_2^2}}{2\sqrt{3}} \right) + l_2 t + \frac{ia_1 \sqrt{p_2^2}}{2\sqrt{3}} \right) \right)} - \frac{\mu}{\mu^2 + \Omega} \right). \] (2.177)

For \( \Omega > 0 \), the solutions have been evaluated as following

\[ v_{I,3}(x,t) = a_1 \sqrt{\Omega} \tan \left( \sqrt{\Omega} \left( l_2 t + \frac{ia_1 \sqrt{p_2^2}}{2\sqrt{3}} \right) \right), \] (2.178)

\[ v_{I,4}(x,t) = a_1 \left( -\sqrt{\Omega} \right) \cot \left( \sqrt{\Omega} \left( l_2 t + \frac{ia_1 \sqrt{p_2^2}}{2\sqrt{3}} \right) \right), \] (2.179)

\[ v_{II,3}(x,t) = a_1 \left( \frac{\mu^2 + \Omega}{\mu + \sqrt{\Omega} \tan \left( \sqrt{\Omega} \left( l_2 t + \frac{ia_1 \sqrt{p_2^2}}{2\sqrt{3}} \right) \right)} + \sqrt{\Omega} \tan \left( \sqrt{\Omega} \left( l_2 t + \frac{ia_1 \sqrt{p_2^2}}{2\sqrt{3}} \right) \right) \right), \] (2.180)
\[ v_{III,4}(x,t) = a_1 \left( \frac{\mu^2 + \nu}{\mu - \nu \cot \left( \frac{\nu}{\mu} \left( l_2 t + \frac{i a_1 \sqrt{p_2^2}}{2\sqrt{3}} \right) \right)} - \sqrt{\nu} \cot \left( \frac{\nu}{\mu} \left( l_2 t + \frac{i a_1 \sqrt{p_2^2}}{2\sqrt{3}} \right) \right) \right), \tag{2.181} \]

\[ v_{III,3}(x,t) = a_{-1} \left( \frac{1}{\sqrt{\nu} \tan \left( \frac{\nu}{\mu} \left( \frac{a_{-1} \sqrt{p_2^2}}{2\sqrt{3}} \sqrt{\mu^2 - 2\mu^2 \nu^2} + l_2 t \right) \right) + \mu} - \frac{\mu}{\mu^2 + \nu} \right), \tag{2.182} \]

\[ v_{III,4}(x,t) = a_{-1} \left( \frac{1}{\mu - \sqrt{\nu} \cot \left( \frac{\nu}{\mu} \left( \frac{a_{-1} \sqrt{p_2^2}}{2\sqrt{3}} \sqrt{\mu^2 - 2\mu^2 \nu^2} + l_2 t \right) \right)} - \frac{\mu}{\mu^2 + \nu} \right). \tag{2.183} \]

For \( q = 0 \), the solutions have been evaluated as following

\[ v_{I,5}(x,t) = -\frac{a_1}{l_2 t + \frac{ia_1 \sqrt{p_2^2}}{2\sqrt{3}}}, \tag{2.184} \]

\[ v_{II,5}(x,t) = a_1 \left( \frac{\mu^2 + \nu}{\mu - \frac{1}{l_2 t + \frac{ia_1 \sqrt{p_2^2}}{2\sqrt{3}}} - \frac{1}{l_2 t + \frac{ia_1 \sqrt{p_2^2}}{2\sqrt{3}}} \right), \tag{2.185} \]

\[ v_{III,5}(x,t) = a_{-1} \left( \frac{1}{\mu - \frac{a_{-1} \sqrt{p_2^2}}{2\sqrt{3}} \sqrt{\mu^2 - 2\mu^2 \nu^2} + l_2 t} - \frac{1}{\mu} \right). \tag{2.186} \]

**Generalization sinh-Gordon expansion method**

Employing the generalization sinh-Gordon expansion method gives the following sets of the abovementioned parameters:

Case **one** When \( W' = \sinh(W) \), we get the following sets of solutions.

Set I

\[ A_0 \to 0, A_1 \to 0, l_1 \to \frac{ib_1 \sqrt{p_2^2}}{2\sqrt{3}}, \quad \text{where} \quad p_2 < 0, i = \sqrt{-1}. \]

Set II

\[ A_0 \to 0, B_1 \to 0, l_1 \to \frac{ia_1 \sqrt{p_2^2}}{2\sqrt{3}}, \quad \text{where} \quad p_2 < 0, i = \sqrt{-1}. \]

Set III

\[ A_0 \to 0, B_1 \to -A_1, l_1 \to \frac{ia_1 \sqrt{p_2^2}}{\sqrt{3}}, \quad \text{where} \quad p_2 < 0, i = \sqrt{-1}. \]

Thus, the solitary wave solutions of equation (1.5) can be formulated in the following formulas.
Thus, the solitary wave solutions of equation (1.5) can be formulated in the following formulas

\[ v_{l,1}(x,t) = iB_1 \text{sech} \left( l_t + \frac{iB_1 \sqrt{p_2 x}}{2\sqrt{3}} \right), \]  
(2.187)

\[ v_{l,2}(x,t) = iB_1 \text{csch} \left( l_t + \frac{iB_1 \sqrt{p_2 x}}{2\sqrt{3}} \right), \]  
(2.188)

\[ v_{l,1}(x,t) = -A_1 \tanh \left( l_t + \frac{iA_1 \sqrt{p_2 x}}{2\sqrt{3}} \right), \]  
(2.189)

\[ v_{l,2}(x,t) = A_1 \left( -\coth \left( l_t + \frac{iA_1 \sqrt{p_2 x}}{2\sqrt{3}} \right) \right), \]  
(2.190)

\[ v_{l,1}(x,t) = A_1 \left( -\tanh \left( l_t + \frac{iA_1 \sqrt{p_2 x}}{\sqrt{3}} \right) - i \text{sech} \left( l_t + \frac{iA_1 \sqrt{p_2 x}}{\sqrt{3}} \right) \right), \]  
(2.191)

\[ v_{l,2}(x,t) = A_1 \left( -\coth \left( l_t + \frac{iA_1 \sqrt{p_2 x}}{\sqrt{3}} \right) \right) - iA_1 \text{csch} \left( l_t + \frac{iA_1 \sqrt{p_2 x}}{\sqrt{3}} \right). \]  
(2.192)

**Case one** When \( W = \cosh(W) \), we get the following sets of solutions.

**Set A**

\[ A_0 \to 0, A_1 \to 0, l_1 \to \frac{iB_1 \sqrt{p_2}}{2\sqrt{3}}, \] where \( p_2 < 0, i = \sqrt{-1} \).

**Set B**

\[ A_0 \to 0, B_1 \to 0, l_1 \to \frac{iA_1 \sqrt{p_2}}{2\sqrt{3}}, \] where \( p_2 < 0, i = \sqrt{-1} \).

**Set C**

\[ A_0 \to 0, B_1 \to -A_1, l_1 \to \frac{iA_1 \sqrt{p_2}}{\sqrt{3}}, \] where \( p_2 < 0, i = \sqrt{-1} \).

Thus, the solitary wave solutions of equation (1.5) can be formulated in the following formulas

\[ v_{l,1}(x,t) = B_1 \tan \left( l_t + \frac{B_1 \sqrt{p_2 x}}{2\sqrt{3}} \right), \]  
(2.193)

\[ v_{l,2}(x,t) = B_1 \left( -\cot \left( l_t + \frac{iB_1 \sqrt{p_2 x}}{2\sqrt{3}} \right) \right), \]  
(2.194)

\[ v_{l,1}(x,t) = A_1 \sec \left( l_t + \frac{iA_1 \sqrt{p_2 x}}{2\sqrt{3}} \right), \]  
(2.195)

\[ v_{l,2}(x,t) = A_1 \csc \left( l_t + \frac{iA_1 \sqrt{p_2 x}}{2\sqrt{3}} \right), \]  
(2.196)

\[ v_{c,1}(x,t) = A_1 \left( \sec \left( l_t + \frac{iA_1 \sqrt{p_2 x}}{\sqrt{3}} \right) - \tan \left( l_t + \frac{iA_1 \sqrt{p_2 x}}{\sqrt{3}} \right) \right), \]  
(2.197)

\[ v_{c,2}(x,t) = A_1 \cot \left( \frac{1}{6} \left( 3l_t + i\sqrt{3}A_1 \sqrt{p_2 x} \right) \right). \]  
(2.198)
**Modified Khater method**

Employing the modified Khater method gives the following sets of the abovementioned parameters:

Set I

\[
a_0 \rightarrow \frac{a_1 \chi}{2 \delta}, \quad b_1 \rightarrow \frac{a_1 Q}{\delta}, \quad p_2 \rightarrow -\frac{12\delta^2 f_1}{a_1^2}.
\]

Set II

\[
a_0 \rightarrow \frac{a_1 \chi}{2 \delta}, \quad b_1 \rightarrow 0, \quad p_2 \rightarrow -\frac{12\delta^2 f_1}{a_1^2}.
\]

Set III

\[
a_0 \rightarrow \frac{b_1 \chi}{2 \delta}, \quad a_1 \rightarrow 0, \quad p_2 \rightarrow -\frac{12\delta f_1^2}{b_1^2}.
\]

Thus, the solitary wave solutions of equation (1.5) can be formulated in the following formulas:

For \( \chi^2 - 4\delta Q < 0, \delta \neq 0 \), the solutions are given by

\[
v_{1,1}(x,t) = \frac{1}{2} a_1 \left( \frac{\sqrt{4\delta Q - \chi^2}}{\delta} \tan \left( \frac{1}{2} \sqrt{4\delta Q - \chi^2} (l_2 t + l_1 x) \right) - \frac{4 \delta}{\chi - \sqrt{4\delta Q - \chi^2} \tan \left( \frac{1}{2} \sqrt{4\delta Q - \chi^2} (l_2 t + l_1 x) \right)} \right),
\]

(2.199)

\[
v_{1,2}(x,t) = \frac{1}{2} a_1 \left( \frac{\sqrt{4\delta Q - \chi^2}}{\delta} \cot \left( \frac{1}{2} \sqrt{4\delta Q - \chi^2} (l_2 t + l_1 x) \right) - \frac{4 \delta}{\chi - \sqrt{4\delta Q - \chi^2} \cot \left( \frac{1}{2} \sqrt{4\delta Q - \chi^2} (l_2 t + l_1 x) \right)} \right),
\]

(2.200)

\[
v_{1,1}(x,t) = \frac{a_1 \sqrt{4\delta Q - \chi^2}}{2 \delta} \tan \left( \frac{1}{2} \sqrt{4\delta Q - \chi^2} (l_2 t + l_1 x) \right),
\]

(2.201)

\[
v_{1,2}(x,t) = \frac{a_1 \sqrt{4\delta Q - \chi^2}}{2 \delta} \cot \left( \frac{1}{2} \sqrt{4\delta Q - \chi^2} (l_2 t + l_1 x) \right),
\]

(2.202)

\[
v_{1,1}(x,t) = \frac{1}{2} b_1 \left( \frac{\chi}{\delta} - \frac{4 \delta}{\chi - \sqrt{4\delta Q - \chi^2} \tan \left( \frac{1}{2} \sqrt{4\delta Q - \chi^2} (l_2 t + l_1 x) \right)} \right),
\]

(2.203)

\[
v_{1,2}(x,t) = \frac{1}{2} b_1 \left( \frac{\chi}{\delta} - \frac{4 \delta}{\chi - \sqrt{4\delta Q - \chi^2} \cot \left( \frac{1}{2} \sqrt{4\delta Q - \chi^2} (l_2 t + l_1 x) \right)} \right).
\]

(2.204)

For \( \chi^2 - 4\delta Q > 0, \delta \neq 0 \), the solutions are given by
\[ v_{l,3}(x,t) = \frac{1}{2} a_1 \left( -\frac{4 \delta}{\sqrt{x^2 - 4 \delta Q} \tanh \left( \frac{1}{2} \sqrt{x^2 - 4 \delta Q} (l_x t + l_x) \right) + \chi \right) - \frac{\sqrt{x^2 - 4 \delta Q}}{\delta} \tanh \left( \frac{1}{2} \sqrt{x^2 - 4 \delta Q} (l_x t + l_x) \right) \right), \tag{2.205} \]

\[ v_{l,4}(x,t) = \frac{1}{2} a_1 \left( -\frac{4 \delta}{\sqrt{x^2 - 4 \delta Q} \coth \left( \frac{1}{2} \sqrt{x^2 - 4 \delta Q} (l_x t + l_x) \right) + \chi \right) - \frac{\sqrt{x^2 - 4 \delta Q}}{\delta} \coth \left( \frac{1}{2} \sqrt{x^2 - 4 \delta Q} (l_x t + l_x) \right) \right), \tag{2.206} \]

\[ v_{l,3}(x,t) = -\frac{a_1 \sqrt{x^2 - 4 \delta Q}}{2 \delta} \tanh \left( \frac{1}{2} \sqrt{x^2 - 4 \delta Q} (l_x t + l_x) \right), \tag{2.207} \]

\[ v_{l,4}(x,t) = -\frac{a_1 \sqrt{x^2 - 4 \delta Q}}{2 \delta} \coth \left( \frac{1}{2} \sqrt{x^2 - 4 \delta Q} (l_x t + l_x) \right), \tag{2.208} \]

\[ v_{\text{III},3}(x,t) = \frac{1}{2} b_1 \left( \frac{\chi}{\delta} - \frac{4 \delta}{\sqrt{x^2 - 4 \delta Q} \tanh \left( \frac{1}{2} \sqrt{x^2 - 4 \delta Q} (l_x t + l_x) \right) + \chi \right), \tag{2.209} \]

\[ v_{\text{III},4}(x,t) = \frac{1}{2} b_1 \left( \frac{\chi}{\delta} - \frac{4 \delta}{\sqrt{x^2 - 4 \delta Q} \coth \left( \frac{1}{2} \sqrt{x^2 - 4 \delta Q} (l_x t + l_x) \right) + \chi \right). \tag{2.210} \]

For \( \delta Q > 0, \xi \neq 0, \delta \neq 0, \chi = 0 \), the solutions are given by

\[ v_{l,5}(x,t) = \frac{2a_1 \xi}{\sqrt{\delta Q}} \csc \left( 2 \sqrt{\delta Q} (l_x t + l_x) \right), \tag{2.211} \]

\[ v_{l,6}(x,t) = -\frac{2a_1 \xi}{\sqrt{\delta Q}} \csc \left( 2 \sqrt{\delta Q} (l_x t + l_x) \right), \tag{2.212} \]

\[ v_{l,5}(x,t) = \frac{a_1 \xi}{\sqrt{\delta Q}} \tan \left( \sqrt{\delta Q} (l_x t + l_x) \right), \tag{2.213} \]

\[ v_{l,6}(x,t) = -\frac{a_1 \xi}{\sqrt{\delta Q}} \cot \left( \sqrt{\delta Q} (l_x t + l_x) \right), \tag{2.214} \]

\[ v_{\text{III},5}(x,t) = \frac{b_1 \delta}{\sqrt{\delta Q}} \cot \left( \sqrt{\delta Q} (l_x t + l_x) \right), \tag{2.215} \]

\[ v_{\text{III},6}(x,t) = -\frac{b_1 \delta}{\sqrt{\delta Q}} \tan \left( \sqrt{\delta Q} (l_x t + l_x) \right). \tag{2.216} \]

For \( \delta Q < 0, \xi \neq 0, \delta \neq 0, \chi = 0 \), the solutions are given by
\[ v_{I,7}(x,t) = \frac{2a_1 \sqrt{\delta}}{\sqrt{\delta}} \csc \left( 2 \sqrt{\delta} \sqrt{Q} (l_2 t + l_1 x) \right), \quad (2.217) \]

\[ v_{I,8}(x,t) = \frac{2a_1 \sqrt{\delta}}{\sqrt{-\delta Q}} \csc \left( 2 \sqrt{-\delta Q} (l_2 t + l_1 x) \right), \quad (2.218) \]

\[ v_{II,7}(x,t) = \frac{a_1 \sqrt{\delta}}{\sqrt{-\delta Q}} \tanh \left( \sqrt{-\delta Q} (l_2 t + l_1 x) \right), \quad (2.219) \]

\[ v_{II,8}(x,t) = \frac{a_1 \sqrt{\delta}}{\sqrt{-\delta Q}} \coth \left( \sqrt{-\delta Q} (l_2 t + l_1 x) \right), \quad (2.220) \]

\[ v_{III,7}(x,t) = \frac{b_1 \sqrt{\delta}}{\sqrt{Q}} \cot \left( \sqrt{\delta} \sqrt{Q} (l_2 t + l_1 x) \right), \quad (2.221) \]

\[ v_{III,8}(x,t) = \frac{b_1 \sqrt{-\delta Q}}{\sqrt{Q}} \tanh \left( \sqrt{-\delta Q} (l_2 t + l_1 x) \right). \quad (2.222) \]

For \( \chi = 0, \varrho = -\delta \), the solutions are given by

\[ v_{I,5}(x,t) = 2a_1 \csc \left( 2Q (l_2 t + l_1 x) \right), \quad (2.223) \]

\[ v_{II,9}(x,t) = a_1 \coth \left( Q (l_2 t + l_1 x) \right), \quad (2.224) \]

\[ v_{III,9}(x,t) = b_1 \tanh \left( Q (l_2 t + l_1 x) \right). \quad (2.225) \]

For \( \chi = \frac{\delta}{2} = \kappa, \delta = 0 \), the solutions are given by

\[ v_{III,10}(x,t) = \frac{1}{4} b_1 \left( \frac{4}{e^{\left( l_2 t + l_1 x \right)} - 2} + 1 \right). \quad (2.226) \]

For \( \chi = \delta = \kappa, \varrho = 0 \), the solutions are given by

\[ v_{I,10}(x,t) = -\frac{1}{2} a_1 \coth \left( \frac{1}{2} \kappa (l_2 t + l_1 x) \right), \quad (2.227) \]

\[ v_{II,10}(x,t) = -\frac{1}{2} a_1 \coth \left( \frac{1}{2} \kappa (l_2 t + l_1 x) \right). \quad (2.228) \]

For \( \chi = \delta = 0, \varrho \neq 0 \), the solutions are given by

\[ v_{III,11}(x,t) = \frac{b_1}{l_2 t Q + l_1 x Q}. \quad (2.229) \]

For \( \chi = \varrho = 0, \delta \neq 0 \), the solutions are given by

\[ v_{I,11}(x,t) = -\frac{a_1}{\delta l_2 t + \delta l_1 x}. \quad (2.230) \]

\[ v_{II,11}(x,t) = \frac{a_1}{\delta l_2 t + \delta l_1 x}. \quad (2.231) \]

For \( \chi = 0, \varrho = \delta \), the solutions are given by

\[ v_{I,12}(x,t) = 2a_1 \csc \left( 2(C + l_2 t Q + l_1 x Q) \right), \quad (2.232) \]

\[ v_{II,12}(x,t) = a_1 \tan \left( C + l_2 t Q + l_1 x Q \right). \quad (2.233) \]
Figure 1. Breather solitary wave of equation (2.4) in (a) 3D, (b) 2D, (c) contour, (d) spherical, and (d) polar plots.

Figure 2. Kink solitary wave of equation (2.6) in (a) 3D, (b) 2D, (c) contour, (d) spherical, and (d) polar plots.
\[ \nu_{III,12}(x,t) = b_1 \cot(C + C_2 t + C_1 x). \] (2.234)

For \( \delta = 0, \chi \neq 0, \rho \neq 0 \), the solutions are given by
\[ \nu_{III,13}(x,t) = \frac{1}{2} b_1 \chi \left( \frac{1}{Q} - \frac{2}{\chi Q(\chi t + \chi x)} \right). \] (2.235)

For \( \chi^2 - 4\delta \rho = 0 \), the solutions are given by
\[ \nu_{II,13}(x,t) = \frac{a_1}{2\chi} \left( \frac{\chi^3}{\delta(\chi t + \chi x + 2)} - \frac{4\rho}{\chi Q(\chi t + \chi x)} - 2\chi Q \right), \] (2.236)
\[ \nu_{II,13}(x,t) = \frac{a_1}{2\chi} \left( \frac{\chi^3}{\delta} - \frac{8\rho}{\chi t + \chi x} - 4\chi Q \right), \] (2.237)
\[ \nu_{III,14}(x,t) = \frac{b_1 \chi}{\chi t + \chi x + 2\rho}. \] (2.238)

**Result and discussion**

The merging MKdV with a negative MKdV equation was used in this paper to create unique explicit traveling wave solutions using eleven current approaches. Numerous special formulas for exponential, trigonometric, and hyperbolic solutions have been employed to represent the investigated model’s wave solutions. These solutions define the physical...
qualities more straightforwardly and across a considerably broader range, which enables us to apply our obtained responses in the models’ application. Now, and in this portion of our article, we will examine our solutions and their relationships and the techniques themselves to make the primary notion of these approaches evident. From an essential examination of our obtained answers, we can observe that a large number of ways provide solutions. These solutions have two parts; the first part is closed to other solutions that got by another manner, for example, [Equations (2.4), (2.6), (2.8), (2.26), (2.28), (2.34), (2.47), (2.48), (2.50), (2.64), (2.67), (2.71), (2.72), (2.91), (2.103), (2.115), (2.172), (2.173),

Figure 4. Singular solitary wave of equation (2.26) in (a) 3D, (b) 2D, (c) contour, (d) spherical, and (d) polar plots.
Figure 5. Periodic solitary wave of equation (2.28) in (a) 3D, (b) 2D, (c) contour, (d) spherical, and (d) polar plots.

Figure 6. Singular solitary wave of equation (2.34) in (a) 3D, (b) 2D, (c) contour, (d) spherical, and (d) polar plots.
Figure 7. Kink solitary wave of equation (2.36) in (a) 3D, (b) 2D, (c) contour, (d) spherical, and (d) polar plots.

Figure 8. Kink solitary wave of equation (2.47) in (a) 3D, (b) 2D, (c) contour, (d) spherical, and (d) polar plots.
Figure 9. Singular solitary wave of equation (2.48) in (a) 3D, (b) 2D, (c) contour, (d) spherical, and (d) polar plots.

Figure 10. Solitary wave of equation (2.50) in (a) 3D, (b) 2D, (c) contour, (d) spherical, and (d) polar plots.
Figure 11. Kink wave of equation (2.64) in (a) 3D, (b) 2D, (c) contour, (d) spherical, and (d) polar plots.

Figure 12. Kink wave of equation (2.67) in (a) 3D, (b) 2D, (c) contour, (d) spherical, and (d) polar plots.
Figure 13. Singular wave of equation (2.71) in (a) 3D, (b) 2D, (c) contour, (d) spherical, and (d) polar plots.

Figure 14. Kink wave of equation (2.72) in (a) 3D, (b) 2D, (c) contour, (d) spherical, and (d) polar plots.
Figure 15. Breather wave of equation (2.91) in (a) 3D, (b) 2D, (c) contour, (d) spherical, and (d) polar plots.

Figure 16. Kink wave of equation (2.103) in (a) 3D, (b) 2D, (c) contour, (d) spherical, and (d) polar plots.
Figure 17. Solitary wave of equation (2.115) in (a) 3D, (b) 2D, (c) contour, (d) spherical, and (d) polar plots.
Figure 18. Kink wave of equation (2.172) in (a) 3D, (b) 2D, (c) contour, (d) spherical, and (e) polar plots.
Figure 19. Cone solitary wave of equation (2.173) in (a) 3D, (b) 2D, (c) contour, (d) spherical, and (d) polar plots.
Figure 20. Breather wave of equation (2.187) in (a) 3D, (b) 2D, (c) contour, (d) spherical, and (d) polar plots.
Figure 21. Kink solitary wave of equation (2.189) in (a) 3D, (b) 2D, (c) contour, (d) spherical, and (e) polar plots.
Figure 22. Breather solitary wave of equation (2.205) in (a) 3D, (b) 2D, (c) contour, (d) spherical, and (d) polar plots.

Figure 23. Kink solitary wave of equation (2.207) in (a) 3D, (b) 2D, (c) contour, (d) spherical, and (d) polar plots.
(2.187), (2.189), (2.205), (2.207), (2.210)], and so on. However, the second part is new formulas of solutions which are the rest of the solutions. These solutions have been represented in 3D, 2D, contour, spherical, and polar plots to show more physical and dynamical behavior of the investigated model (Figures 1–24) by using some special values of the above shown parameters.

Convergence and divergence of solutions may be traced back to the similarity of techniques in the auxiliary equation, which is often a remarkable instance of the Riccati equation.

**Conclusion**

We utilized eleven techniques to the Merged MKdV with a negative MKdV equation in this research. We obtained several innovative formulas for this model’s sophisticated solitary wave solutions. Complex solitary wave solutions are used to explain the propagation of waves. Additionally, it is considered a more mathematically succinct tool for elaborating on the physical features of models. A representation of complicated solutions simplifies the handling of wave superpositions significantly. The correctness of all solutions found has been verified using the Mathematica software.

**Acknowledgments**

We greatly thank Taif University for providing fund for this work through Taif University Researchers Supporting Project number (TURSP-2020/52), Taif University, Taif, Saudi Arabia.

**Authors’ contribution**

Dexu Zhao and Mostafa Khater have revised the conceptualization, data curation, and methodology. Dianchen Lu and Mostafa Khater have revised data curation, investigation, and software. Samir Salama and Piyaphong Yongphe have revised the physical meaning of the obtained solutions and raised the given graphs resolutions. All authors have read and agreed to the published version of the manuscript.
Declaration of conflicting interests
The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Funding
The author(s) disclosed receipt of the following financial support for the research, authorship, and/or publication of this article: We greatly thank Taif University for providing fund for this work through Taif University Researchers Supporting Project number (TURSP-2020/52), Taif University, Taif, Saudi Arabia.

Availability of data and material
The data that support the findings of this study are available from the corresponding author upon reasonable request.

Code availability
The used code of this study is available from the corresponding author upon reasonable request.

ORCID iD
Mostafa MA Khater https://orcid.org/0000-0001-8466-168X

References
1. Fokas A and Santini P. The recursion operator of the Kadomtsev-Petviashvili equation and the squared eigenfunctions of the Schrödinger operator. Stud Appl Math 1986; 75(2): 179–185.
2. Lou S-y and Chen W-z. Inverse recursion operator of the AKNS hierarchy. Phys Lett A 1993; 179(4–5): 271–274.
3. Bruschi M and Ragnisco O. Recursion operator and Bäcklund transformations for the Ruijsenaars-Toda lattice. Phys Lett A 1988; 129(1): 21–25.
4. Zakharov VE and Konopelchenko B. On the theory of recursion operator. Commun Math Phys 1984; 94(4): 483–509.
5. Nagatani T. Modified KdV equation for jamming transition in the continuum models of traffic. Phys A 1998; 261(3–4): 599–607.
6. Wazwaz A-M. A two-mode modified KdV equation with multiple soliton solutions. Appl Math Lett 2017; 70: 1–6.
7. Wazwaz A-M. Exact soliton and kink solutions for new (3+1)-dimensional nonlinear modified equations of wave propagation. Open Eng 2017; 7(1): 169–174.
8. Chowdury A, Ankiewicz A, and Akhmediev N. Periodic and rational solutions of modified Korteweg-de Vries equation. Eur Phys J D 2016; 70(5): 104.
9. Wazwaz A-M and Xu G-q. An extended modified KdV equation and its Painlevé integrability. Nonlinear Dyn 2016; 86(3): 1455–1460.
10. Hosseini K and Ansari R. New exact solutions of nonlinear conformable time-fractional Boussinesq equations using the modified Kudryashov method. Waves Random Comp Media 2017; 27(4): 628–636.
11. Wazwaz A-M. A new integrable equation combining the modified KdV equation with the negative-order modified KdV equation: multiple soliton solutions and a variety of solitonic solutions. Waves Random Comp Media 2018; 28(3): 533–543.
12. Yue C, Khater MM, Inc M, et al. Abundant analytical solutions of the fractional nonlinear (2+1)-dimensional BLMP equation arising in incompressible fluid. Int J Mod Phys B 2020; 34(09): 2050084.
13. Abdel-Aty A-H, Khater MM, Dutta H, et al. Computational solutions of the HIV-1 infection of CD4+ T-cells fractional mathematical model that causes acquired immunodeficiency syndrome (AIDS) with the effect of antiviral drug therapy. Chaos Solitons & Fractals 2020; 139: 110092.
14. Khater MM, Attia RA, Park C, et al. On the numerical investigation of the interaction in plasma between (high & low) frequency of (langmuir & ion-acoustic) waves. Results Phys 2020; 18: 103317.
15. Ali AT, Khater MM, Attia RA, et al. Abundant numerical and analytical solutions of the generalized formula of Hirota-Satsuma coupled KdV system. Chaos Solitons & Fractals 2020; 131: 109473.
16. Abdel-Aty A-H, Khater MM, Attia RA, et al. On the new explicit solutions of the fractional nonlinear space-time nuclear model. Fractals 2020; 28(8): 2040035.
17. Park C, Khater MM, Attia RA, et al. An explicit plethora of solution for the fractional nonlinear model of the low–pass electrical transmission lines via Atangana–Baleanu derivative operator. Alexandria Eng J 2020; 59(3): 1205–1214.
18. Khater MM, Attia RA, Abdel-Aty A-H, et al. Analytical and semi-analytical ample solutions of the higher-order nonlinear Schrödinger equation with the non-kerr nonlinear term. Results Phys 2020; 16: 103000.
19. Abdel-Aty A-H, Khater MM, Baleanu D, et al. Abundant distinct types of solutions for the nervous biological fractional Fitzhugh–Nagumo equation via three different sorts of schemes. *Adv Diff Equ* 2020; 2020(1): 1–17.

20. Khater MM, Attia RA, Alodhaibi SS, et al. Novel soliton waves of two fluid nonlinear evolutions models in the view of computational scheme. *Int J Mod Phys B* 2020; 34(10): 2050096.

21. Attia RA, Lu D, Ak T, et al. Optical wave solutions of the higher-order nonlinear Schrödinger equation with the non-kerr nonlinear term via modified Khater method. *Mod Phys Lett B* 2020; 34(05): 2050044.

22. Khater MM, Nofal TA, Abu-Zinadah H, et al. Novel computational and accurate numerical solutions of the modified Benjamin–Bona–Mahony (BBM) equation arising in the optical illusions field. *Alexandria Eng J* 2021; 60(1): 1797–1806.

23. Yue C, Khater MM, Attia RA, et al. The plethora of explicit solutions of the fractional KS equation through liquid–gas bubbles mix under the thermodynamic conditions via Atangana–Baleanu derivative operator. *Adv Diff Equ* 2020; 2020(1): 1–12.

24. Khater MM, Alzaidi J, Attia RA, et al. Analytical and numerical solutions for the current and voltage model on an electrical transmission line with time and distance. *Physica Scripta* 2020; 95(5): 055206.

25. Khater MM, Park C, and Lu D. Two effective computational schemes for a prototype of an excitable system. *AIP Adv* 2020; 10(10): 105120.

26. He J-H, Hou W-F, He C-H, et al. Variational approach to fractal solitary waves. *Fractals* 2021; 29(7): 2150199.

27. Liu C. Periodic solution of fractal Phi-4 equation. *Therm Sci* 2021; 25(2): 32–32.

28. Khater MM, Attia RA, Mahmoud EE, et al. On the interaction between (low & high) frequency of (ion-acoustic & langmuir) waves in plasma via some recent computational schemes. *Results Phys* 2020; 19: 103684.

29. Khater MM, Attia RA, and Baleanu D. Abundant new solutions of the transmission of nerve impulses of an excitable system. *The Eur Phys J Plus* 2020; 135(2): 1–12.

30. Khater MM, Attia RA, and Lu D. Computational and numerical simulations for the nonlinear fractional Kolmogorov–Petrovskii–Piskunov (FKPP) equation. *Physica Scripta* 2020; 95(5): 055213.

31. Khater MM, Mohamed MS, Park C, et al. Effective computational schemes for a mathematical model of relativistic electrons arising in the laser thermonuclear fusion. *Results Phys* 2020; 19: 103701.

32. Abdel-Aty A-H, Khater MM, Baleanu D, et al. Oblique explicit wave solutions of the fractional biological population (BP) and equal width (EW) models. *Adv Diff Equ* 2020; 2020(1): 1–17.

33. Khater MM, Lu D, and Hamed Y. Computational simulation for the (1+ 1)-dimensional Ito equation arising quantum mechanics and nonlinear optics. *Results Phys* 2020; 19: 103572.

34. Khater MM, Bekir A, Lu D, et al. Analytical and semi-analytical solutions for time-fractional Cahn–Allen equation. *Math Met Appl Sci* 2021; 44(3): 2681–2691.

35. Rezazadeh H, Souleymanou A, Korkmaz A, et al. New exact solitary waves solutions to the fractional Fokas-Lenells equation via Atangana-Baleanu derivative operator. *Int J Mod Phys B* 2020; 34(31): 2050309.

36. Khater MM, Attia RA, Abdel-Aty A-H, et al. Abundant analytical and numerical solutions of the fractional microbiological densities model in bacteria cell as a result of diffusion mechanisms. *Chaos Solitons & Fractals* 2020; 136: 109824.

37. Khater MM, Ghanbari B, Nisar KS, et al. Novel exact solutions of the fractional Bogoyavlensky–Konopelchenko equation involving the Atangana-Baleanu-Riemann derivative. *Alexandria Eng J* 2020; 59(5): 2957–2967.

38. Khater MM and Baleanu D. On abundant new solutions of two fractional complex models. *Adv Diff Equ* 2020; 2020(1): 1–14.

39. Tian Y and Liu J. Direct algebraic method for solving fractional Fokas equation. *Therm Sci* 2021; 25(3): 2235–2244.

40. He J-H, Hou W-F, Qie N, et al. Hamiltonian-based frequency-amplitude formulation for nonlinear oscillators. *Ser Facta UniversitatisMechanical Eng* 2022; 41(11): 209–215.