CORRIGENDUM: SIMILARITY DEGREE OF FOURIER ALGEBRAS

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Abstract. We address two errors made in our paper [7]. The most significant error is in Theorem 1.1. We repair this error, and show that the main result, Theorem 2.5 of [7], is true. The second error is in one of our examples, Remark 2.4 (iv), and we partially resolve it.

1. On Theorem 1.1 of [7]

1.1. A corrected version of the theorem. We begin with a simple observation whose straightforward proof we omit.

Lemma 1.1. Let $V$ be a an operator space and $V_0$ and $W$ be subspaces with $V_0$ dense in $V$, $V_0 \cap W$ dense in $W$, and $W$ closed in $V$. Then the map

$$v + V_0 \cap W \mapsto v + W : V_0/(V_0 \cap W) \to V/W$$

is a complete isometry. Hence we identify $V_0/(V_0 \cap W)$ as a subspace of $V/W$ and, for any $n \in \mathbb{N}$, the quotient map takes the matricial open unit ball $b_1(M_n \otimes V_0)$ onto $b_1(M_n \otimes [V_0/(V_0 \cap W)]) \cong b_1([M_n \otimes V_0]/[M_n \otimes (V_0 \cap W)])$.

We now recall the notation of [7]. For a Banach algebra and operator space $A, c \geq 1, \tilde{A}_c$ (contained in $B(H_c)$) is the universal operator algebra generated by representations on Hilbert spaces $\pi : A \to B(H)$ with completely bounded norm $\|\pi\|_{cb} \leq c$, and $\iota_c : A \to \tilde{A}_c$ is the canonical embedding. Note that we assume that $\iota_1$ is injective. We say that $A$ satisfies the similarity property for completely bounded homomorphisms if for each completely bounded homomorphism $\pi : A \to B(H)$, there is an invertible $S$ in $B(H)$ for which $\|S\pi(\cdot)S^{-1}\|_{cb} \leq 1$. We also consider the “weighted multiplication” map on the $N$-fold Haagerup tensor product of $A$ with itself, $m_{N,c} : A^{N\otimes h} \to \tilde{A}_c$, given on elementary tensor by

$$m_{N,c}(u_1 \otimes \cdots \otimes u_N) = \frac{1}{c^N} \iota_c(u_1) \cdots \iota_c(u_n) = \frac{1}{c^N} \iota_c(u_1, \ldots, u_N)$$

which is a complete contraction.

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We let $m_N : \mathcal{A}^{\otimes N} \to \mathcal{A}$ be the multiplication map on the $N$-fold algebraic tensor product and $\mathcal{A}^N = m_N(\mathcal{A}^{\otimes N})$. We say that $\mathcal{A}$ is square-dense provided the closure, $\overline{\mathcal{A}^N}$, is all of $\mathcal{A}$. This clearly implies that $\overline{\mathcal{A}^N} = \mathcal{A}$ for all $N \geq 2$.

**Theorem 1.2.** Suppose that $m_{N,1} : \mathcal{A}^{N \otimes k} \to \widetilde{\mathcal{A}}_1$ is a complete surjection, i.e. the induced map $\mu_{N,1} : \mathcal{A}^{N \otimes k} / \ker m_{N,1} \to \widetilde{\mathcal{A}}_1$ is a complete isomorphism which is completely bounded below by $1/K$, and either of the following conditions holds:

(i) $\mathcal{A}$ is square-dense and $\ker m_N$ is dense in $\ker m_{N,1}$; or

(ii) $\mathcal{A}$ satisfies the similarity property for completely bounded homomorphisms. Then any completely bounded homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ admits an invertible operator $S$ in $\mathcal{B}(\mathcal{H})$ for which

\[(1.1) \quad \| S \pi(\cdot) S^{-1} \|_{cb} \leq 1 \quad \text{and} \quad \| S \| \cdot \| S^{-1} \| \leq K \| \pi \|_{cb}^N.\]

In (i) we add to [7, Theorem 1.1] both the assumption of square-density of $\mathcal{A}$, and density of $\ker m_N$ in $\ker m_{N,1}$, and gain the similarity property. In (ii) we assume the similarity property, but gain information about completely bounded similarity degree $d_{cb}(\mathcal{A})$: the smallest $N$ for which (1.1) holds.

In the proof of [7, Theorem 1.1], it is indicated that $\iota_c \circ \iota_1^{-1} : \iota_1(\mathcal{A}) \to \widetilde{\mathcal{A}}_c$ extends to a completely bounded map on $\mathcal{A}_1$. There is a gap in that proof, which we repair, below.

**Proof of Theorem 1.2.** We begin with assumptions of (i). Let $K_N = \ker m_N$ and $\mathcal{K}_{N,1} = \ker m_{N,1}$. The injectivity of $\iota_1$ provides that

$$\mathcal{K}_N = \mathcal{A}^{N \otimes} \cap \mathcal{K}_{N,1}.$$}

In particular, we may use Lemma 1.1 to regard $\mathcal{A}^{N \otimes} / K_N$ as a dense subspace of $\mathcal{A}^{N \otimes k} / K_{N,1}$ with

\[(1.2) \quad \mu_{N,1}(\mathcal{A}^{N \otimes} / K_N) = m_{N,1}(\mathcal{A}^{N \otimes}) = \iota_1 \circ m_N(\mathcal{A}^{\otimes N}) = \iota_1(\mathcal{A}^N).\]

Let $n \in \mathbb{N}$ and $a$ be in the open unit ball $b_1(\mathcal{M}_n \otimes \iota_1(\mathcal{A}^N)) \subseteq \mathcal{B}(\mathcal{M}_n \otimes \widetilde{\mathcal{A}}_1)$. Our assumptions on $\mu_{N,1}$ provide a $T$ in the open $K$-ball $b_K(\mathcal{M}_n \otimes (\mathcal{A}^{N \otimes k} / K_{N,1}))$, for which $\text{id}_n \otimes \mu_{N,1}(T) = a$. Moreover, since $\mu_{N,1}$ is a bijection, (1.2) shows that $T \in \mathcal{M}_n \otimes (\mathcal{A}^{N \otimes} / K_N)$, whence Lemma 1.1 and homogeneity provide a $t \in b_K(\mathcal{M}_n \otimes \mathcal{A}^{N \otimes}) \subseteq b_K(\mathcal{M}_n \otimes \mathcal{A}^{N \otimes k})$ for which $T = t + \mathcal{M}_n \otimes K_N$.

Now we fix $c \geq 1$. We observe that

\[(1.3) \quad \iota_c \circ \iota_1^{-1} \circ m_{N,1}|_{\mathcal{A}^{N \otimes}} = c_N \circ m_{N,c}|_{\mathcal{A}^{N \otimes}} : \mathcal{A}^{N \otimes} \to \widetilde{\mathcal{A}}_c.\]

If $a \in b_1(\mathcal{M}_n \otimes \iota_1(\mathcal{A}^N))$, choose $T$ and $t$ as above, so $a = \mu_{N,1}(T) = m_{N,1}(t)$ and we use (1.3) to see that

$$\| \text{id}_n \otimes \iota_c \circ \iota_1^{-1}(a) \| = \| \text{id}_n \otimes \iota_c \circ \iota_1^{-1} \circ m_{N,1}(t) \| = c_N \| \text{id}_n \otimes m_{N,c}(t) \| \leq Kc^N.$$  

Taking supremum over all such $a$ and all such $n$ yields that $\| \iota_c \circ \iota_1^{-1} \|_{cb} \leq Kc^N$ on $\iota_1(\mathcal{A}^N)$. The assumption of square density of $\mathcal{A}$ provides that $\mathcal{A}^N$ in dense in $\mathcal{A}$, whence $\iota_1(\mathcal{A}^N)$ is dense in $\widetilde{\mathcal{A}}_1$ and thus we find that $\iota_c \circ \iota_1^{-1}$ extends to a map $\iota_{1,c} : \widetilde{\mathcal{A}}_1 \to \mathcal{A}_c$, with $\| \iota_{1,c} \|_{cb} \leq Kc^N$, as desired.

If we assume (ii), then $\iota_c : \mathcal{A} \to \mathcal{A}_c \subseteq \mathcal{B}(\mathcal{H}_c)$ is similar to a complete contraction and hence $\iota_c \circ \iota_1^{-1}$ is automatically completely bounded. Then we can simplify (1.3) above to $\iota_c \circ \iota_1^{-1} \circ m_{N,1} = c_N \circ m_{N,c}$ on $\mathcal{A}^{N \otimes k}$, and gain the same estimate on $\| \iota_{1,c} \|_{cb}$, as above, using only the complete surjectivity of $m_{N,1}$. 
The rest of proof follows as in [7, Theorem 1.1]. \[\square\]

**Remark 1.3.** In the case where \( A \) is unital, and hence square-dense, Theorem 1.2 with assumptions (i) is a partial converse to [8, Theorem 2.5], where Pisier shows that if \( A \) satisfies a certain similarity property, then \( m_{N,1} : \mathcal{A}^N \otimes h \to \tilde{A}^1 \) is a complete surjection. Our result with assumptions (ii) is really the aspect of [8, Theorem 2.5] which begins from the complete surjectivity result to obtain (1.1). We include it merely for context and completeness of presentation.

Regrettably, the essential error of the authors in [7, Theorem 1.1] is also made in [10, Theorem 4.2.9].

1.2. **On applying the corrected theorem to Fourier algebras.** Let \( G \) be a locally compact group and \( A(G) \) its Fourier algebra. We used the flawed [7, Theorem 1.1] to prove our main theorem [7, Theorem 2.5]. We wish to deduce the latter result from our present Theorem 1.2 instead.

In the proof of [7, Theorem 2.5] we successfully showed, for a quasi-small invariant neighbourhood (QSIN) group \( G \), that

\[
m = m_{2,1} : A(G) \otimes h \to \mathcal{C}_0(G) \text{ is a complete quotient map.}
\]

This was achieved by showing that

\[
m^* : M(G) \to \text{VN}(G) \otimes \text{eh} \text{ VN}(G) \text{ is a complete isometry}
\]

It is shown in [7, Proposition 2.1], that completely contractive homomorphisms are exactly the \( \ast \)-homomorphisms for Fourier algebras of QSIN groups, and hence \( \mathcal{C}_0(G) = \tilde{A}(G)_1 \). This gives the first assumption of our present Theorem 1.2. In order to apply this theorem to prove [7, Theorem 2.5], we shall verify the two aspects of condition (i) of Theorem 1.2. To see the first aspect we have that

\[
A(G) \text{ is square-dense thanks to the Tauberian theorem of [3]. The second aspect is more delicate.}
\]

It is observed in [2, Section 3] that \( A(G) \otimes h A(G) \) is a semisimple commutative Banach algebra, and that \( A(G \times G) \cong A(G) \otimes A(G) \) (operator projective tensor product) completely contractively embeds in \( A(G) \otimes h A(G) \) with dense range. Hence this is a regular Banach algebra on its Gelfand spectrum \( G \times G \). Given a closed subset \( E \) of \( G \times G \) we let

\[
I_h(E) = \{ u \in A(G) \otimes h A(G) : u|_E = 0 \}, \quad \text{and} \quad I_h^0(E) = \{ u \in I_h(G) : \text{supp}(u) \cap E = \emptyset \text{ and supp } u \text{ is compact} \}.
\]

We say that \( E \) is a set of spectral synthesis if \( I_h^0(E) = I_h(E) \). In the regular algebra \( A(G) \otimes h A(G) \), this implies that any ideal with vanishing set \( E \) is dense in \( I_h(E) \).

The following is shown as [1, Corollary 4.3], where it was proved in response to a question to M. Alaghmandan by the authors. This result was previously shown for compact groups in [9, Proposition 3.1], and virtually abelian groups in [2, Corollary 5.4]. For convenience of the reader, we provide a proof of this general result, based on [1,3].

**Proposition 1.4.** Let \( G \) be a QSIN group. Then the diagonal \( \Delta = \{(s,s) : s \in G\} \) is a set of spectral synthesis for \( A(G) \otimes h A(G) \).
Corollary 1.5. The multiplication map $m^\ast (\mu) = \int_G \lambda(s) \otimes \lambda(s) \, d\mu(s) = \int_G \lambda(s, s) \, d\mu(s) = \lambda_\Delta(\mu)$

where each integral is understood as a weak* integral with respect to the respective predual.

Now let $I(\Delta)$ and $I^0(\Delta)$ be the ideals in $\Lambda(G \times G)$, defined analogously to $I_h(\Delta)$ and $I^0_h(\Delta)$, above. Since $\Delta$ is a closed subgroup, [14] provides that $\Delta$ is a set of spectral synthesis for $\Lambda(G \times G)$, and further, that $I(\Delta) \perp = VN(\Delta \times G)$. Hence in $VN(G) \otimes^{cb} VN(G) \subseteq VN(G \times G)$ we have that $I^0_h(\Delta) \perp \subseteq I_h(\Delta) \perp = VN(\Delta \times G)$.

Let $T \in I^0_h(\Delta) \perp$. Then by decomposing into a sum of two self-adjoint operators, the Kaplansky density theorem provides a net $(\mu_\alpha)$ from the space of measures $M(G)$ for which

$$T = w^* \lim_\alpha \lambda_\Delta(\mu_\alpha), \text{ with each } \|\lambda_\Delta(\mu_\alpha)\| \leq 2\|T\| \text{ in } VN(G \times G).$$

Combining with (1.5) and (1.7) we thus see that each

$$\|\mu_\alpha\|_{M(G)} = \|m^\ast (\mu_\alpha)\|_{VN(G) \otimes^{cb} VN(G)} \leq \|\lambda_\Delta(\mu_\alpha)\| \leq 2\|T\|.$$ 

Thus, by dropping to subnet, we may suppose that $\mu = w^* \lim_\alpha \mu_\alpha$ exists in $M(G)$. But then $T = \lambda_\Delta(\mu)$ in $VN(G \times G)$, hence, using again (1.7), $T = m^\ast (\mu)$ in $VN(G) \otimes^{cb} VN(G)$. Thus, if $u \in I_h(\Delta)$, then $\langle T, u \rangle = \int_G u(s, s) \, d\mu(s) = 0$, so $T \in I_h(\Delta) \perp$.

In summary, we have shown that $I^0_h(\Delta) \perp \subseteq I_h(\Delta) \perp$, so the bipolar theorem shows that $I_h(E) \subseteq I^0_h(E)$, which establishes the equality of these ideals. □

Corollary 1.5. The multiplication map $m_2 : \Lambda(G) \otimes \Lambda(G) \to \Lambda(G)$ satisfies that $\ker m_2 = \ker m_{2,1}$ in $\Lambda(G) \otimes^h \Lambda(G)$.

Proof. Since $K_2 = \ker m_2$ is an ideal in $\Lambda(G) \otimes \Lambda(G)$, its closure $\overline{K_2}$ is an ideal in $\Lambda(G) \otimes^h \Lambda(G)$. The regularity of $\Lambda(G)$ provides that the vanishing set of $K_2$, hence that of $\overline{K_2}$, is $\Delta$. Thus $\overline{K_2} = I_h(\Delta)$, by Proposition 1.3. But $I_h(\Delta) = \ker m_{2,1}$. □

The combination of [12], [16] and Corollary 1.5 give the assumptions Theorem 1.2 with condition (i) for $\Lambda(G)$ with $G$ a QSIN group. We hence conclude that [7] Theorem 2.5 is true: if $G$ is a QSIN group, then any completely bounded homomorphism $\pi : \Lambda(G) \to B(H)$ admits an invertible $S$ in $B(H)$ for which

$$S \pi(\cdot) S^{-1} \text{ is a } *-\text{homomorphism, and } \|S\| \|S^{-1}\| \leq \|\pi\|_{cb}^2.$$ 

The proofs of results [7] Corollaries 2.9 & 2.10 relied only on [7] Theorem 2.5], as stated, and hence remain valid.
2. ON REMARK 2.4 (IV) OF [7]

This remark is in error, as stated. The failure of the a Γ-space to be amenable, in a sense to which the authors implicitly appeal, does not imply non-existence of invariant means.

We can partially recover this result, with a simple adaptation of an argument in the preprint [5]. We shall use terminology as introduced in [7].

Let \( \Gamma \) be a dense subgroup of \( S = \text{SL}_2(\mathbb{R}) \), treated as a discrete group. Then \( G = \mathbb{R}^2 \rtimes \Gamma \) is not QSIN. Indeed, as noted in the proof of [7, Theorem 2.5], the QSIN condition would provide an asymptotically inner invariant net \((v_\alpha) \subset L^1(G)\) which would further satisfy

\[
v_\alpha \geq 0, \quad \int_G v_\alpha = 1 \quad \text{and} \quad \text{supp} (v_\alpha) \searrow \{e\}.
\]

Hence we may suppose that \((v_\alpha)\) is supported in the open subgroup \( \mathbb{R}^2 \), and can be realized as a net in \( L^1(\mathbb{R}^2) \) which satisfies

\[
\|v_\alpha \circ \sigma - v_\alpha\|_1 \to 0 \quad \text{for} \quad \sigma \in \Gamma.
\]

We note that as \( \mathbb{R}^2 \setminus \{0\} \cong S/P \) where \( P \) is the amenable fixed point subgroup of any non-zero element of \( \mathbb{R}^2 \). Hence we regard \((v_\alpha)\) as a net of means on the left uniformly continuous functions \( \mathcal{LUC}(S/P) \); any cluster point of this net gives a \( \Gamma \)-invariant, hence \( S \)-invariant mean. But this violates [4, §3, 1°].

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