ALGEBRAIC CONDITIONS FOR
CONFORMAL SUPERINTEGRABILITY
IN ARBITRARY DIMENSION

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Abstract. We show that the definition of a second order superintegrable system on a (pseudo-) Riemannian manifold gives rise to a conformally invariant notion of superintegrability. Conformal equivalence is the natural extension of the well-known Stäckel transform, which in turn originates from the classical Maupertuis-Jacobi principle. We extend our recently developed algebraic geometric approach for the classification of second order superintegrable systems in arbitrarily high dimension to conformally superintegrable systems, which are presented via conformal scale choices of second order superintegrable systems defined within a conformal geometry.

For superintegrable systems on constant curvature spaces, we find that the conformal scales of Stäckel equivalent systems arise from eigenfunctions of the Laplacian and that their equivalence is characterised by a conformal density of weight two.

Our approach yields an algebraic equation that governs the classification under conformal equivalence for a prolific class of second order conformally superintegrable systems. This class contains all non-degenerate examples known to date, and is given by a simple algebraic constraint of degree two on a general harmonic cubic form. In this way the yet unsolved classification problem is put into the reach of algebraic geometry and geometric invariant theory. In particular, no obstruction exists in dimension three, and thus the known classification of conformally superintegrable systems is reobtained in the guise of an unrestricted univariate sextic. In higher dimensions, the obstruction is new and has never been revealed by traditional approaches.

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1. Introduction

Transformation groups play an important role in the natural sciences: The Poincaré group, for instance, and its subgroups, are pivotal in special relativity, for Maxwell’s field equations, in

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particle physics, and many other fields. Felix Klein’s Erlangen program has put the concept of transformations at the core of geometry, later generalised by Cartan [Kle00, SC00]. In particular, relaxing the invariance group of the geometry considered, some properties are not preserved any more. Inspired by this idea, the current paper reconsiders second order superintegrable Hamiltonian systems. These have been extensively studied as structures in (pseudo-)Riemannian geometry, but not yet as structures in conformal geometry.

Superintegrable systems, traditionally, are Hamiltonian systems on a (pseudo-)Riemannian geometry that admit a maximal amount of (hidden) symmetry. They are often seminal models in science. Historically, the theory of superintegrability arose from classical (and quantum) mechanics: While it is impossible, even for relatively simple models, to solve Hamiltonian’s or Schrödinger’s equation in exact, closed terms, for superintegrable systems the solution can be found by quadrature, i.e. using algebraic operations and the integration of known functions. Prominent examples of second order superintegrable systems are the Kepler-Coulomb and the Harmonic oscillator models. They have fundamental significance for the understanding of celestial mechanics, atomic orbitals, material science and many other disciplines.

1.1. What Geometry underpins superintegrability? Traditionally, second order superintegrable systems are defined on a (pseudo-)Riemannian manifold \((M,g)\). The suitable symmetry group for these systems is the semi-direct product \(\mathfrak{S} = \text{Diff}(M) \rtimes \text{Aff}(\mathbb{R})\) of diffeomorphisms (coordinate transformations on \(M\)) and the affine group \(\text{Aff}(\mathbb{R}) = \mathbb{R}^* \ltimes \mathbb{R}\), where \(\mathbb{R}^* = \mathbb{R} \setminus \{0\}\), see Section 2. However, these are not the only possible transformations of superintegrable systems. Indeed, conformal geometry manifests itself in the theory of superintegrable systems through Stäckel equivalence or coupling constant metamorphosis; these will be discussed in detail in Section 2.1. Historically, they can be traced back to the 1700s in the form of Maupertuis’ principle [Tsi01, Lag88, Jac51, dM50]. Stäckel transformations are linked to very special conformal transformations, namely those that originate from superintegrable potentials.

Arbitrary conformal transformations, however, do not preserve superintegrability. Instead they lead to conformally superintegrable systems. Although these systems are well studied, their underlying conformal geometry is understood only superficially to date. One purpose of the present paper is to remedy this, and to derive a suitable concept of conformal equivalence on conformally superintegrable systems from Stäckel equivalence.

Given the significance of second order (conformally) superintegrable systems, it is natural to seek a classification. In [KSV19], the authors present an algebraic geometric framework for a classification of second order superintegrable systems in arbitrarily high dimension, and on arbitrary geometries. This framework put earlier attempts by various authors (see below) onto a firm base, yet it is not closed under conformal transformations of superintegrable systems. Instead, conformal transformations lead to the more general concept of conformally superintegrable systems. The current paper develops an algebraic geometric framework for conformally superintegrable systems that is closed under conformal transformations. In particular we obtain: Non-degenerate second order superintegrable systems on conformal geometries are characterised by a structure tensor, i.e. a trace-free and totally symmetric tensor field \(S_{ijk}\). This field is invariant under the geometry’s transformation group, i.e. under the conformal group, and from \(S_{ijk}\) we can reconstruct all superintegrable potentials and integrals of the system.

This will lead to a natural definition of superintegrability on conformal manifolds, whose symmetry group we identify as \(\mathfrak{S} = \text{Conf}(M) \rtimes \mathbb{R}^*\). In this way we naturally incorporate conformal geometry into the theory of superintegrable systems. Somewhat surprisingly, this appears to never have been attempted before, although it sheds considerable light on the geometry underpinning superintegrability and opens the subject for subsequent studies using Cartan geometry, tractor calculus, algebraic geometry, representation theory and geometric invariant theory. In particular, superintegrable systems on (pseudo-)Riemannian geometries can be viewed as specific conformal scale choices of a conformally invariant superintegrable system.
1.2. State of the art. A vast literature exists both on second order conformally superintegrable systems and on Stäckel transformations. Most importantly, second order conformally superintegrable systems are classified completely in dimension 2 [KKM05c, KKM05a]. For conformally flat spaces in dimension 3, at least so called non-degenerate systems are classified [Cap14, KKM06]. The conformal classes of non-degenerate systems are classified in dimensions 2 and 3 [Kre07, Cap14].

Stäckel transform is well understood as an equivalence relation on second order (conformally) superintegrable systems. It has first been introduced, under the name coupling constant metamorphosis, by Hietarinta et al. in [HGDR84], for integrable Hamiltonian systems with potential. The concept of Stäckel transformations has been introduced in [BKM86] by Boyer, Kalnins & Miller, for integrable systems that admit separation of variables. Coupling constant metamorphosis and Stäckel transform are not identical in general, but do coincide for second order (conformal) integrals of the motion. They are therefore equivalent in the context which is of interest here. Details on the interrelation between both concepts can be found in [Pos10]. Higher order integrals are also discussed in [KMP09], where it is proven that in general coupling constant metamorphosis neither preserves the order of the integrals of the motion, nor even their polynomiality in momenta. A multi-parameter generalisation of Stäckel transform exists as well [SB08, BM12, BM17]. Instead of following this direction, we are going to use a geometric approach that does not rely on a specific parametrisation of the space of compatible potentials of a given superintegrable system, but instead is based on the symmetry groups mentioned in Section 1.1 above.

Stäckel transformations can historically be traced back to Maupertuis-Jacobi transformations, which take a Hamiltonian with potential to a potential-free one, see for example [BKF95, Tsi01]. Equivalence classes of superintegrable systems, under Stäckel transforms, have been established for dimension 2 and 3 in [Kre07] and in [KKM06, Cap14], respectively. For these geometries, every conformally superintegrable system can, via a Stäckel transformation using the potential, be transformed into a properly\(^1\) superintegrable system [Cap14].

As mentioned, the existing work on second order conformally superintegrable systems and their conformal equivalence is restricted to dimensions 2 and 3. In dimension 2, a classification exists, but ignores the geometric structure of the classification space [Kre07]. In dimension 3, a classification in terms of representations of the rotation group exists [Cap14, CK14, CKP15]. These latter references are one major inspiration for our work as they highlight the power of the geometric approach, revealing for example a natural algebraic hierarchy of systems related to an inclusion tree of certain algebraic ideals. Unfortunately, there is little hope to apply the methods used in those references to higher dimensions, neither conceptually nor practically, as the equations become ever more extensive with increasing dimension. The current paper develops a new approach, extending and generalising the framework from [KS18, KSV19], which formulates the governing equations for second order properly superintegrable systems in dimension \(n \geq 3\) in a concise form, making the problem manageable for higher dimensions.

For completeness we mention that conformal transformations are not the only possible transformations of superintegrable systems. For instance, Bôcher transformations of certain conformally superintegrable systems are studied in [KMS16, CKP15] and there is some indication that they can be understood as boundaries of orbit closures on the algebraic geometric variety classifying the superintegrable systems [KS18]. Yet another transformation of superintegrable systems is possible if the underlying metrics share the same geodesics up to reparametrisation. Such metrics are called projectively (or geodesically) equivalent. For some examples of superintegrable systems defined on projectively equivalent geometries, see [Val16, KKMW03, Vol20].

To summarise, second order non-degenerate conformally superintegrable systems are to date classified in dimensions two and three, for manifolds that are conformally flat. Higher dimensions are out of the scope of traditional methods, which rely on the correspondence with properly superintegrable systems and on the extensive use of computer algebra. A particular challenge with

\(^1\)For clarity, in this paper we use the adjective “proper” to refer to superintegrable systems, emphasising the distinction to conformally superintegrable systems.
traditional methods is the fast growth of the number of partial differential equations with increasing dimension. In the current paper, we shall overcome this problem and outline how to approach the classification of second order conformally superintegrable systems in arbitrarily high dimension. For the most prolific class of systems we find, somewhat surprisingly, that the underlying structure equations reduce to only a single, algebraic equation of degree 2.

1.3. Special functions and superintegrable systems. Special functions are ubiquitous and powerful tools in science and technology. For instance spherical harmonics appear in the solution of the Schrödinger equation for the hydrogen atom, eventually giving rise to the periodic table of the elements. Superintegrable systems have long been known as a rich source of special functions, such as hypergeometric orthogonal polynomials [KMP07, KMP13], Painlevé transcendents [MPR20, Gra04], Jacobi-Dunkl polynomials [GIVZ13], and exceptional polynomials [PTV12, HMPZ18]. In particular, a striking resemblance has been found between two directed graphs: The first describes degenerations and confluenes of hypergeometric polynomials in the Askey scheme [Ask85, AW85]. The vertices of the other graph represent (conformal classes of) superintegrable systems in dimension 2, and its edges represent orbit degenerations and Böcher contractions. In dimension 3, the generic superintegrable system on the 3-sphere has indeed been related to bivariate Wilson polynomials [KMP11], and interbasis expansions for the isotropic 3D harmonic oscillator are linked to bivariate Krawtchouk polynomials [GVZ14].

A systematic documentation is indispensable for the use of special functions, and also needs to include their properties, interrelations and hierarchies. Unfortunately, such documentation quickly becomes extensive and laborious. The Bateman Manuscript Project, for instance, fills five volumes [Bat53, Bat54] and the steadily growing Mathematical Functions Site [WMT] comprises to date more than 300,000 formulae. In the face of the sheer amount of information, a natural question is whether there is a unified theory of special functions. While a general unified theory might be out of reach, a systematisation of certain classes of special functions is a realistic goal. Ideally, such a framework should explain and organise at least some of the major properties of the functions it comprises. One attempt at such a partial systematisation is the Askey-Wilson scheme [Ask85, AW85], which establishes a hierarchy of hypergeometric orthogonal polynomials.

Conformal superintegrability has also been related to different Laplace equations [KKMP11, KWM16], particularly in the context of the corresponding quantum systems. With respect to special functions we find that second order properly superintegrable systems on constant curvature spaces correspond to eigenfunctions of the Laplace operator, where the eigenvalues are determined by the scalar curvature.

The classification of special functions arising from superintegrable systems is clearly out of the scope of the current paper. Our approach has the potential to explain this correspondence, including higher-dimensional hypergeometric polynomials and higher dimensional generalisations of the Askey-Wilson scheme.

1.4. Classifying second order superintegrability in arbitrarily high dimension. In reference [KSV19] the authors have developed an algebraic geometric framework for the classification of superintegrable systems. This framework generalises previous work in dimension two [KS18] to arbitrarily high dimensions. Older works in the field are [KKM07c] and [KKM07d] for dimensions two and three, respectively. While [KSV19] for the first time provides a framework to classify, in an algebraic geometric way, superintegrable systems in arbitrarily high dimensions, this framework in its original form is not yet closed under conformal transformations. The present paper extends the existing algebraic geometric framework to conformally superintegrable systems on (pseudo-)Riemannian metrics. This new framework is closed under conformal transformations. Second order conformally superintegrable systems will be thoroughly introduced in Section 3.1 and 3.2. Conceptually, these systems are traditionally defined using a Hamiltonian

\[ H = g^{ij}(q)p_ip_j + V(q) \]  

(1.1)
where \( g_{ij} \) denotes the underlying metric and where \( q \) and \( p \) are the canonical position and momenta variables on the manifold. A second order conformally superintegrable system is a Hamiltonian system with a sufficiently high number of functions \( F : T^*M \to \mathbb{R} \),

\[
F = \mathcal{K}^{ij}(q)p_i p_j + W(q),
\]

satisfying

\[
\{ H, F \} = \omega_q(p) H \tag{1.2}
\]

for some 1-form \( \omega = \omega_i dx^i \). In this context, the scalar function \( V \) is called the potential. Functions \( F \) satisfying (1.2) are called (conformal) integrals. The collection of integrals \( F \) forms a linear space \( \mathcal{F} \). On the other hand, if we start from the space \( \mathcal{F} \), we are going to see that the collection \( \mathcal{V} \) of potentials \( V \) compatible with \( \mathcal{F} \) forms a linear space of functions on the base manifold.

Before formulating the main results of the present paper, we would like to draw the reader’s attention to a subtlety regarding the spaces \( \mathcal{F} \) and \( \mathcal{V} \), as different conventions can be found in the literature. In the present paper, we will be working with the maximal spaces \( \mathcal{V} \) and \( \mathcal{F} \) modulo some normalisation detailed below. This allows us to formulate the results in a clean and concise manner. This convention also appears to be the most common one in the relevant literature on Stäckel transforms, as it ensures well-definedness of the conformal equivalence of two superintegrable Hamiltonians. A competing convention regularly found in the literature is to restrict to a linear subspace of \( \mathcal{F} \) with a basis of functionally independent integrals \( \mathcal{F} \) and with a Hamiltonian for which \( V \in \mathcal{V} \) is a specific function. Our results can straightforwardly, but somehow tediously, be adapted to these settings. Such specifications are omitted here as they do not contribute to further understanding the geometry underpinning superintegrability, which is the main objective of the paper.

### 1.5. First Main Result: Conformal superintegrability in higher dimensions

The method carried out in reference [KSV19] facilitates the classification of second order properly superintegrable systems, in particular of so-called abundant systems. Abundantness is going to be introduced thoroughly later, and so here we limit ourselves here to saying that these systems comprise all known non-degenerate second order conformally superintegrable systems. In the present paper, we extend the framework to conformally superintegrable systems. We find that it is closed under conformal transformations and leads to a well-defined concept of superintegrability on conformal geometries arising from conformal equivalence classes of conformally superintegrable systems. For the abundant case in dimensions \( n \geq 3 \), we show in Sections 5 and 6 that such systems are in natural correspondence with (trace-free) cubic forms \( \Psi_{ijk} p^i p^j p^k \) on \( \mathbb{R}^n \) that satisfy the simple algebraic equation

\[
\left( g^{ab} \Psi_{bi[j} \Psi_{k]a} \right)_\circ = 0 \tag{1.3}
\]

where \( g^{ab} \) is an inner product on \( \mathbb{R}^n \) with the same signature as the metrics on the underlying manifold. Here square brackets denote antisymmetrisation over enclosed indices, and the subscript \( \circ \) stands for projection onto the trace-free part.

We show that initial data in the form of a cubic form \( \Psi_{ijk} p^i p^j p^k \) satisfying (1.3) can be extended, locally to a conformal structure tensor \( S_{ijk} \) of an abundant second order superintegrable system. We make this precise by introducing the concept of c-superintegrable systems, i.e. conformal equivalence classes whose underlying geometry is a conformal manifold. In Section 6.4 we derive conformally invariant structural equations for abundant c-superintegrable systems. The equations governing abundant properly superintegrable systems [KSV19] naturally follow from the equations we present here.

Condition (1.3) is conformally invariant, and therefore a suitable foundation for an algebraic geometric classification of second order systems on the level of conformal geometries. Condition (1.3) is also surprisingly simple, and in dimensions \( n \geq 4 \) it encodes new obstructions to conformal superintegrability. These obstructions do not exist in lower dimensions and have not been revealed by classical approaches. It is worthwhile to compare (1.3) to the corresponding equation for the case of proper superintegrability in [KSV19], which is not projected onto the trace-free part.
We also show: Abundant conformally superintegrable systems can only exist on conformally flat geometries. Such systems naturally correspond to solutions of Equation (1.3). The task of classifying equivalence classes of n-dimensional conformally superintegrable systems is therefore equivalent to classifying harmonic cubics in n variables that satisfy (1.3). Note that while a general classification of harmonic cubics under the rotation group is out of sight, a classification under the additional condition (1.3) may well be simple enough to admit a manageable solution.

In dimension n = 3, particularly, (1.3) is trivially satisfied. Thus abundant superintegrable systems in dimension 3 are in 1-to-1 correspondence with harmonic ternary cubic forms, or, equivalently, with univariate sextic polynomials

\[ p(x) = a_6 x^6 + a_5 x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0. \]

The details are discussed in Section 7. It is known that any conformally superintegrable system is Stäckel equivalent to a properly superintegrable systems [Cap14]. In dimension 3, every conformally superintegrable system is even Stäckel equivalent to a properly superintegrable system on a constant curvature space [KKM06, Theorem 4].

1.6. Second Main Result: Superintegrable systems on constant curvature geometries. In our framework, second order conformally superintegrable systems are conformal scale choices of c-superintegrable systems. Thus the conformal structure tensor \( S_{ijk} \) determines a conformally superintegrable system up to the choice of a conformal scale expressed via a function \( \sigma \) that transforms as a weight-1 tensor density. For the prolific class of abundant second order conformally superintegrable systems we find that the conformal scale satisfies a Helmholtz like equation,

\[
\left( R - 4 \frac{n-1}{n-2} \Delta \right) \sigma^{1-\frac{n-2}{2}} = S \sigma^{1-\frac{n-2}{2}},
\]

where \( S = S_{abc} S^{abc} \) is a conformal density obtained from the structure tensor \( S_{ijk} \) mentioned earlier, and where \( R \) is the scalar curvature. The operator on the left hand side of (1.4) is the conformal Laplace operator.

If we restrict to properly superintegrable systems, the conformal structure tensor \( S_{ijk} \) determines a superintegrable system up to provision of a suitable conformal scale. In the present paper, we prove the following: For abundant properly superintegrable systems on manifolds of constant sectional curvature, the conformal scale \( \sigma \) is (the power of) an eigenfunction of the conformal Laplacian for an eigenvalue determined by the curvature \( R \),

\[
\left( \Delta + 2 \frac{n+1}{n-1} R \right) \sigma^{n+2} = 0,
\]

which holds in addition to (1.4), see Theorem 6.13. Note that the operator in (1.5) is not conformally invariant as Equation (1.5) does not describe a property of the conformal class, but of an individual superintegrable system. In particular we find that on the n-sphere the conformal scale function satisfies a Laplace equation with quantum number \( n + 1 \). Theorem 6.16 shows: The generic system on the n-sphere is never conformally equivalent to a superintegrable system on flat space.

Moreover, in Theorem 6.19 we extend the concept of structure functions to c-superintegrable systems: If two abundant second order properly superintegrable systems on constant curvature spaces are conformally equivalent, then their structure functions behave like conformal densities of weight \(-2\). In fact (up to a certain conformally equivariant gauge transformation that leaves the conformally equivariant tensor \( S_{ijk} \) unchanged)

\[
b = B \det(g)^{\frac{1}{n}} \in \mathcal{E}[-2],
\]

computed from the structure function \( B \) and metric \( g \) of any one of the two systems, coincides with the one computed from the structure function and metric of the other system, see Corollary 6.20.

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2. Preliminaries

Before generalising to conformally superintegrable systems, it is instructive to briefly review
properly superintegrable systems. We recall that for clarity, the adjective “proper” is used to refer
to superintegrable systems, whenever a distinction from conformally superintegrable systems is
required. While self-contained, this review only highlights the aspects needed for a later comparison
to conformally superintegrable systems. For a more in-depth review of proper superintegrability,
we refer the interested reader to the literature cited in the introduction. Reference [KKM18], in
addition, provides a synopsis of a large part of the existing literature. The PhD thesis [Cap14]
and the articles [KS18, KSV19] are foundational for the algebraic geometric approach to properly
superintegrable systems, see also [KKM07a, KKM07d].

A Hamiltonian system is a dynamical system characterised by a function $H(p, q)$, referred to
as Hamiltonian. Here, the position and momentum coordinates on the phase space are denoted by
$q = (q_1, \ldots, q_n)$ and $p = (p_1, \ldots, p_n)$, respectively. The evolution of the system is determined by
Hamilton’s equations

$$
\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = +\frac{\partial H}{\partial p}.
$$

(2.1)

An integral, aka first integral or constant of the motion, for the Hamiltonian $H$ is a function $F(p, q)$
on phase space that commutes with $H$ with respect to the canonical Poisson bracket. It is therefore
constant along solutions of (2.1),

$$
\dot{F} = \{F, H\} = \frac{\partial F}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial F}{\partial p} \frac{\partial H}{\partial q} = 0.
$$

(2.2)

Note that, if the underlying manifold is endowed with a (pseudo-)Riemannian metric $g$, the usual
derivatives in this equation may be replaced by covariant derivatives, using the Levi-Civita metric $\nabla^g$, without changing the Poisson bracket. For simplicity we assume the metric to be analytic from
now on.

An integral restricts the trajectory of the system to a hypersurface in phase space. A (properly)
superintegrable system is a Hamiltonian system that possesses the maximal number of $2n - 1$
functionally independent constants of motion $F^{(0)}, \ldots, F^{(2n-2)}$. Its trajectories in phase space are
the (unparametrised) curves given as the intersections of the hypersurfaces $F^{(\alpha)}(p, q) = c^{(\alpha)}$, where
the constants $c^{(\alpha)}$ are determined from the initial conditions. For convenience it is customary to
choose $F^{(0)} = H$ without loss of generality. In particular, we assume the base manifold is endowed
with a (pseudo-)Riemannian metric $g$ and a natural Hamiltonian (1.1),

$$
H = G(q, p) + V(q),
$$

where $G(q, p) = g_q(p, p)$ denotes the kinetic part and $V(q)$ is a smooth scalar function called
potential.
Definition 2.1. A second order superintegrable system is a Hamiltonian together with a linear space $F$ of integrals of the form

$$ F = K(q,p) + W(q) := K^{ij}(q)p_ip_j + W(q), $$

(2.3)

satisfying (2.2). Moreover, $F$ must contain $2n-1$ integrals that are functionally independent.

Note that $\dim(F) \geq 2n-1$. In case of the equality, it is common practice to only specify the $2n-1$ linearly independent generators $F^{(\alpha)}$. We also recall that, for (2.3), Equation (2.2) is a polynomial condition in the momenta with homogeneous components of cubic and linear degree, respectively:

$$ \{K,G\} = 0 $$

(2.4a)

$$ \{K,V\} + \{W,G\} = 0 $$

(2.4b)

Condition (2.4a) is equivalent to the requirement that the components $K^{ij}$ in $K(q,p) = K^{ij}_ip_ip_j$ are components of a Killing tensor field

$$ K_{(ij,k)} = 0. $$

(2.5)

Here, the comma denotes a covariant derivative and round brackets denote symmetrisation in the enclosed indices. Condition (2.4b) can be rewritten in the form

$$ W_{,j} = K^k_j V_k \quad \text{or even} \quad dW = KdV, $$

where by a slight abuse of notation $K$ denotes the endomorphism obtained from $K^{ij}$ using the metric $g$. The integrability condition for $W$ is known as the Bertrand-Darboux condition [Ber57, Dar01],

$$ dKdV = 0. $$

(2.6)

The Bertrand-Darboux Equation (2.6) is the compatibility condition for the potential $V$ and the space of Killing tensors $K_{ij}$. Let us denote the linear space of kinetic parts of the integrals $F \in F$, viewed as endomorphisms, by

$$ \mathcal{K} = \{ K : K(q,p) + W \in F \text{ for some } W \}. $$

Definition 2.2. For a second order superintegrable system with potential $V$, we introduce the spaces

$$ \mathcal{V}^{\text{max}} = \{ V : dKdV = 0 \text{ holds for every } K \in \mathcal{K} \}, $$

$$ \mathcal{K}^{\text{max}} = \{ K : dKdV = 0 \text{ holds for every } V \in \mathcal{V}^{\text{max}} \}. $$

Remark 2.3. A second order superintegrable system is said to be irreducible if the endomorphisms $K^{ij} = g^{ij}K_{ij}$ obtained from its associated Killing tensors $K \in \mathcal{K}$ form an irreducible set [KSV19]. In reference [KSV19], it is shown that for such irreducible systems we can solve (2.6) for all second derivatives of the potential except its Laplacian. Thus the Wilczynski equation is obtained,

$$ V_{,ij} = T_{ij}^k V_{,k} + \frac{1}{n}g_{ij}\Delta V, $$

(2.7)

where $T_{ij}^k$ is a tensor symmetric and trace-free in the first two indices, depending on the values of the Killing tensors $K^{(\alpha)}$ and their derivatives.

The properties of the partial differential equation (2.7) are discussed thoroughly in [KSV19], and similar equations appear in [KKM05b]. The most important fact is that in (2.7) the tensor $T_{ij}^k$ is determined by $\mathcal{K}$ independently from the potential. More precisely, at a point $x_0 \in M$, $T_{ij}^k$ is determined by the values of the Killing tensors $K_{ij}$ and their derivatives in $x_0$.

In the classification theory of second order superintegrable systems, non-degenerate systems have received particular attention, e.g. [KKM18, KSV19, Cap14]. These are the systems satisfying (2.7) for which the dimension of $\mathcal{V}^{\text{max}}$ is maximal, i.e.

$$ \dim(\mathcal{V}^{\text{max}}) = n + 2. $$

Note that for an analytic metric, the Killing tensors are analytic, and thus the structure tensor and the potentials are also analytic.
The integrability conditions of (2.7) are then generically satisfied \([KKM07b, KKM07d, KSV19]\). Resubstituting (2.7) into (2.6) and considering the coefficients of \(\nabla V\) and \(\Delta V\), one furthermore finds
\[
K_{ij,k} = \frac{1}{3} \frac{j+1}{k} T_{ij}^a K_{ak}
\]
(2.8)
for non-degenerate systems. Now consider non-degenerate systems for which the dimension of \(K_{\text{max}}\) is maximal as well, i.e.
\[
\dim(K_{\text{max}}) = \frac{n(n + 1)}{2}.
\]
Such systems are called \textit{abundant} \([KSV19]\). Abundantness implies that the integrability conditions of (2.8) are generically satisfied. Abundant systems should be considered the most important class of second order superintegrable systems, for at least two reasons: First, in dimensions two and three, all non-degenerate systems are abundant \([KKM05b]\). Second, all examples of non-degenerate systems known to date are abundant systems.

2.1. \textbf{Stäckel equivalence}. In the literature, Stäckel transformations are typically introduced by considering a Hamiltonian with a coupling constant. We shall start following this convention, but we are then going to present an alternative and equivalent formulation that is better suited for our purposes. Consider a family of second order superintegrable systems on a (pseudo-)Riemannian manifold \((M, g)\) given by the family of Hamiltonians \(H_\beta = H_0 + V + \beta U\). Here \(H_0 = g(p, p)\) is the \textit{free Hamiltonian}, \(H_0 + V\) is the background Hamiltonian, and \(\beta\) is called the \textit{coupling parameter}. The concept of Stäckel equivalence is based on the following fact, see for example \([KMP09, Pos10, KKM05b, HGDR84, BKM86]\).

\textbf{Lemma 2.4}. Let \(H = H_0 + V + \beta U\) be a family of second order superintegrable Hamiltonians with integrals \(F(\beta)\), for \(V, U \in V_{\text{max}}\). Then the Hamiltonian \(\tilde{H} = \frac{H + 2\nu}{U}\) admits the integral of motion \(\tilde{F}(\eta) = F(\tilde{H})\), parametrised by \(\eta\).

The Hamiltonian \(\tilde{H}\) is called the \textit{Stäckel transform} of \(H\) with conformal factor \(U\). While the lemma is true for either sign, it will often be preferable to work with \(U > 0\) exclusively, in order to preserve the signature of the underlying metric. Note that locally this is always possible by a redefinition of \(\beta\), if needed.

Stäckel transformations are also known under the name \textit{coupling constant metamorphism} since they can be thought of as a transformation exchanging the roles of the coupling parameter and the energy. However, we remark that the two concepts coincide for second order superintegrability, but they are not identical in general \([KMP09, Pos10]\).

Note that in Lemma 2.4, it has been exploited that we may add any constant \(\eta\) to \(H\) without changing the integrals. We could analogously have written down the Stäckel transform of the Hamiltonian \(H = H_0 + V + \beta U + \eta\) admitting the integrals \(F^{(\alpha)}\) \([BKM86, Pos10]\), as
\[
\tilde{H} = U^{-1} H,
\]
(2.9a)
\[
\tilde{F}^{(\alpha)} = F^{(\alpha)} + \frac{1 - W^{(\alpha)}}{U} H.
\]
(2.9b)
We observe that this transformation preserves the kinetic part up to a trace contribution, i.e. up to a term proportional to \(g(p, p)\) with a coefficient that depends on the position only.

One special case deserves explicit mentioning: When \(H = H_0 + \beta U\), we arrive at \(\tilde{H} = \beta H_0 + \beta\). This special case is known as a \textit{Maupertuis-Jacobi transformation} \([Tsi01, Lag88, Jac51, dM50]\). For conciseness, we have restricted ourselves to one coupling parameter in Lemma 2.4. A multi-parameter generalisation of Stäckel transforms exists as well \([SB08, BM12, BM17]\). We are not going to follow this direction further, however, because for the purposes of the present paper a parameter-free viewpoint on Stäckel equivalence is much more appropriate. Equivalently and better suited for the purposes of the present paper, we define Stäckel equivalence as follows.
**Definition 2.5.** Two second order properly superintegrable systems are said to be Stäckel equivalent if their Hamiltonians and integrals satisfy (2.9).

We observe that for Stäckel equivalent Hamiltonians \( H = H_0 + V \) and \( \tilde{H} = \tilde{H}_0 + \tilde{V} \), their underlying metrics are conformally equivalent, i.e. \( \tilde{g}_{ij} = \Omega^2 g_{ij} \), if \( U = \Omega^2 > 0 \). In case of negative sign, \( U < 0 \), the metric’s signature is merely inverted. For the corresponding integrals \( F = K^{ij}p_ip_j + W \) and \( \tilde{F} = \tilde{K}^{ij}p_ip_j + \tilde{W} \), (2.9) implies

\[
\tilde{K} = K + \frac{1 - W}{U} g,
\]

\[
\tilde{W} = W + \frac{1 - W}{U} V.
\]

The trace-free part of the Killing tensors, with raised indices, therefore remain unchanged under Stäckel transformations.

2.2. **Symmetry group.** Consider a non-degenerate second order properly superintegrable system with potential \( V \). The symmetry group of such a system is the semi-direct product \( \mathcal{G} = \text{Diff}(M) \rtimes \text{Aff}(\mathbb{R}) \) of the diffeomorphisms of the manifold \( M \) and the affine group \( \text{Aff}(\mathbb{R}) \simeq \mathbb{R} \rtimes \mathbb{R}^* \). An element \( \Phi = (\phi, (a,b)) \in \mathcal{G} \) transforms a Hamiltonian according to

\[
\Phi(g^{ij}p_ip_j + V) = \phi^*(g)^{ij}p_ip_j + a\phi^*(V) + b,
\]

where \( \phi^* \) is the pullback with \( \phi \). Indeed, the underlying geometry and the space of compatible Killing tensors does not change under \( \mathcal{G} \). Moreover, the structure tensor \( T_{ijk} \) in (2.7) remains unchanged under \( \mathcal{G} \). The structure tensor remains unchanged even\(^3\) under \( \mathcal{G}' = \mathcal{G} \rtimes \mathbb{R}^* \), where \( \Phi' = (\phi, (a,b,c)) \in \mathcal{G} \) transforms a Hamiltonian according to

\[
\Phi'(g^{ij}p_ip_j + V) = c\phi^*(g)^{ij}p_ip_j + a\phi^*(V) + b.
\]

The quotient of the \((n+2)\)-dimensional space \( V^\text{max} \) under the affine group \( \text{Aff}(\mathbb{R}) \) is an \( n \)-dimensional projective space. Finally, note that the Stäckel transformations of two equivalent Hamiltonians are also equivalent.

2.3. **Young projectors.** In order to keep the notation concise, tensor symmetries are described by Young projectors in the following. In doing so, we adhere to the convention used for properly superintegrable systems in [KSV19], which we briefly review in the current section. A comprehensive introduction to representations of symmetric and linear groups is out of scope of the present paper, but can be found in [Ful97, FH00] for instance.

Let \( n > 0 \) be an integer. A partition of \( n \) into a sum of ordered, positive integers can be represented by a **Young frame**, i.e. by non-increasing rows of square boxes, which by convention are left-aligned. For instance, to denote the partition \( 5 = 3 + 1 + 1 \) we may draw the Young frame

\[
\begin{array}{ccc}
\Box & \Box & \Box \\
\Box & \Box \\
\end{array}
\]

Irreducible representations of the permutation group \( S_n \) and the induced Weyl representations of \( \text{GL}(n) \) can also be labelled by Young frames. A Young frame filled with tensor index names is called a **Young tableau**; it explicitly defines a projector onto an irreducible representation. Two simple examples are complete symmetrization,

\[
\begin{array}{c|c|c}
1 & 2 & 3 \\
\end{array}
\]

and antisymmetrization,

\[
\begin{array}{cccc}
1 & 2 & & \\
& 1 & 2 & \\
& & 1 & 3 \\
& & & 1 \\
\end{array}
\]

\(^3\)Note that the action might not be proper.
For example, a 2-tensor $\tau_{ij}$ can be decomposed into its symmetric and antisymmetric parts,

$$\tau_{ij} = \frac{1}{2} \tau_{ij} + \frac{1}{2} \tau_{ji} = \frac{1}{2} \left( \tau_{ij} + \tau_{ji} \right) + \frac{1}{2} \left( \tau_{ij} - \tau_{ji} \right).$$

The symmetric part can be decomposed further, according to irreducible representations of $SL(n)$, into a trace-free and a trace component. The projection onto the completely trace-free part of a tensor is denoted by the sub- or superscript “$\circ$”. For example,

$$\tau_{ij} = \frac{1}{2} \tau_{ij} + \frac{1}{n} g_{ij} \tau^a_a + \frac{1}{2} \tau_{ij} \tau_{ij}. $$

A general Young tableau denotes the composition of its row symmetrisers and column antisymmetrisers, and by convention the antisymmetrisers are applied first. For instance,

$$T_{ijk} = \frac{1}{6} T_{ijk} + \frac{1}{4} T_{ijk} + \frac{1}{4} T_{ikj} + \frac{1}{6} T_{ikj}.$$

One particular 4-index Young tableaux that we make intensive use of is the projector

$$T_{ijkl} = \frac{1}{12} T_{ijkl} + \frac{1}{4(n-1)} T_{ijkl} + \frac{1}{8(n-1)} g_{ik} g_{jl}.$$

where

$$W_{ijkl} = \frac{1}{12} R_{ijkl} \quad \text{is the Weyl curvature},$$

$$\hat{R}_{ij} \quad \text{the trace-free part of the Ricci tensor and } R \quad \text{the scalar curvature.}$$

The Schouten tensor is given by

$$(n - 2)P_{ij} = R_{ij} - \frac{1}{2(n-1)} R g_{ij} = \hat{R}_{ij} + \frac{n-2}{2n(n-1)} R g_{ij}. \quad (2.12)$$

3. Conformal structure tensors

In the present chapter superintegrable systems on (pseudo-)Riemannian manifolds are generalised to \textit{conformally superintegrable systems}. We subsequently reconsider them as systems on conformal geometries. Before we begin, we need to introduce the conformal counterpart of Killing tensors.

\textbf{Definition 3.1.} A (second order) \textit{conformal Killing tensor} is a symmetric tensor field $C_{ij}$ on a (pseudo-)Riemannian manifold satisfying the conformal Killing equation

$$C_{ij,k} = \frac{1}{n+2} \left( 2C_{a,k} + C_{a,a,k} \right), \quad (3.1)$$

where $\omega$ is a 1-form.

The 1-form $\omega$ can be expressed in terms of the conformal Killing tensor. Indeed, contracting (3.1) in $(i, j)$, we find

$$\omega_k = \frac{1}{n+2} \left( 2C_{a,k} + C_{a,a,k} \right). \quad (3.2)$$
Remark 3.2.
(i) Any Killing tensor is also a conformal Killing tensor, with $\omega = 0$. In particular, the metric $g$ is trivially a conformal Killing tensor.
(ii) If $K_{ij}$ is a conformal Killing tensor, any trace modification $C_{ij} = K_{ij} + \lambda g_{ij}$ is also a conformal Killing tensor. If $K_{ij}$ is a proper Killing tensor, $C_{ij}$ is a conformal Killing tensor with $\omega = d\lambda$.

We mention that while for a proper Killing tensor $K_{ij}$, the function $K(p, q)$ is preserved along geodesics, for a conformal Killing tensor $C_{ij}$ the function $C(p, q)$ is preserved along null geodesics.

3.1. Conformally superintegrable systems. The concept of a superintegrable system on a (pseudo-)Riemannian manifolds is generalised as follows.

Definition 3.3.
(i) By a conformally (maximally) superintegrable system, we mean a Hamiltonian system admitting $2n - 1$ functionally independent conformal integrals of the motion $F^{(\alpha)}$,

$$\{F^{(\alpha)}, H\} = \omega^{(\alpha)} H \quad \alpha = 0, 1, \ldots, 2n - 2,$$

with a function $\omega(p, q)$ polynomial in momenta. The Hamiltonian can be assumed to be among the conformal integrals. Thus by convention we set

$$F^{(0)} = H, \quad \omega^{(0)} = 0.$$

(ii) A conformal integral of the motion is second order if it is of the form

$$F^{(\alpha)} = C^{(\alpha)} + V^{(\alpha)},$$

where

$$C^{(\alpha)}(p, q) = \sum_{i=1}^{n} C^{(\alpha)}_{ij}(q) p^i p^j$$

is quadratic in momenta and $V^{(\alpha)} = V^{(\alpha)}(q)$ a function depending only on positions. In this case, the function $\omega$ has to be linear in the momenta $p$.

(iii) A conformally superintegrable system is second order if its conformal integrals $F^{(\alpha)}$ are second order and

$$H = G + V,$$

where

$$G(p, q) = \sum_{i=1}^{n} g_{ij}(q) p^i p^j$$

is given by the (pseudo-)Riemannian metric $g_{ij}(q)$ on the underlying manifold.

(iv) We call $V$ a conformal superintegrable potential if the Hamiltonian (3.5) gives rise to a conformally superintegrable system.

A function $F(q(t), p(t))$ that satisfies (3.3) is constant on the null locus of the Hamiltonian, since

$$F = \{F, H\} = \omega H.$$

By adding a constant $c$ to the Hamiltonian, we achieve that $F$ is constant on shells where the new Hamiltonian is constant and equal to $c$. Since we are concerned exclusively with second order maximally superintegrable systems, we omit the terms “second order” and “maximally” without further mentioning.

Assumption 3.4. From now on, unless otherwise stated, we assume that the quadratic parts correspond to trace-free conformal Killing tensors, except for the Hamiltonian $H = F^{(0)}$. This is no restriction, as the trace-free part of any such conformal Killing tensor is itself a conformal Killing tensor.
For (3.4), condition (3.3) splits into two homogeneous parts with respect to momenta. These parts are cubic respectively linear in $p$

\[
\{C^{(\alpha)}, g\} = 2\omega G \quad \text{(3.6a)}
\]
\[
\{C^{(\alpha)}, V\} + \{V^{(\alpha)}, G\} = 2\omega V \quad \text{(3.6b)}
\]

The condition (3.6a) for $C(p, q) = C_{ij}(q)p^i p^j$ is equivalent to $C_{ij}$ being a conformal Killing tensor.

The components of $\omega(p, q) = \omega^i(q)p_i$ are given by a 1-form, also denoted by $\omega$ in the following. Compare (3.6) to the analogous equations (2.4) for proper superintegrability.

The metric $g$ allows us to identify symmetric forms and endomorphisms by abuse of notation. Interpreting a conformal Killing tensor in this way as an endomorphism on $1$-forms, Equation (3.6b) can be written in the form

\[
dV^{(\alpha)} = C^{(\alpha)}dV + \omega V.
\]

Its integrability condition is the Bertrand-Darboux condition

\[
d(C^{(\alpha)}dV) + V d\omega + dV \wedge \omega = 0 \quad \text{(3.8a)}
\]

which in components reads

\[
\begin{bmatrix}
i \\
j
\end{bmatrix}
\left(C^{(\alpha)}_{m1} V_{jm} + C^{(\alpha)}_{m1j} V_{jm} + \omega^i V_{j} + \omega^i j V\right) = 0.
\]

This is the counterpart to condition (2.6) from proper superintegrability.

By virtue of the Bertrand-Darboux equation (3.8), the potentials $V^{(\alpha)}$ for $\alpha \neq 0$ are eliminated from our equations. As we are going to see in the following, for non-degenerate systems the remaining potential $V = V^{(0)}$ can be eliminated as well, leaving equations on the conformal Killing tensors $C^{(\alpha)}$ alone. In analogy to proper superintegrability we denote by $\mathcal{F}$ the linear space spanned by second order conformal integrals $F^{(\alpha)} = C^{(\alpha)}(p, p) + V^{(\alpha)}$ satisfying (3.3).

**Definition 3.5.** Let $H = g(p, p) + V$ with $\mathcal{F} = \{F^{(\alpha)}\}$. Analogously to Definition 2.2 we introduce first

\[C = \{C : C(p, p) + W \in \mathcal{F} \text{ for some } W\}\]

and then

\[V^{\max} = \{V : (3.8) \text{ holds for every } C \in \mathcal{C}\}\]
\[C^{\max} = \{C : (3.8) \text{ holds for every } V \in V^{\max}\}\].

**Assumption 3.6.** Unless otherwise stated, a conformally superintegrable Hamiltonian will be considered together with all its conformal integrals $F = C_{ij} p^i p^j + W$ where $C \in C^{\max}$ and where $W$ satisfies Equation (3.7), i.e.

\[dW = C dV + \omega V.\]

This assumption is no restriction, and ensures a tidy exposition as we do not need to specify the subspace $\mathcal{C} \subset C^{\max}$ each time.

**3.2. Conformal equivalence.** In analogy to Section 2.2 we obtain a symmetry group of conformally superintegrable systems on (pseudo-)Riemannian manifolds. For the time being, we insist that the metric is preserved (up to coordinate transformations). In contrast to properly superintegrable systems, the affine group does not map potentials to potentials for conformally superintegrable systems, since (1.2) contains the potential $V$ without derivatives. The symmetry group of a conformally superintegrable system on a fixed (pseudo-)Riemannian manifold therefore is $\mathcal{G} = \text{Diff}(M) \rtimes \mathbb{R}^*$ where $\Phi = (\phi, a) \in \mathcal{G}$ acts as

\[\Phi(g^{ij} p^i p_j + V) = \phi^*(g)^{ij} p^i p_j + a \phi^*(V)\]
This is the counterpart of the symmetry group (2.10) for properly superintegrable systems. Analogously to (2.11) in Section 2.2, the symmetry group of the structure tensor $S_{ijk}$ of a conformally superintegrable system is $\mathcal{S}' = \mathcal{S} \times \mathbb{R}^*$.

In the current section, Stäckel equivalence is going to be generalised to conformal equivalence of conformally superintegrable systems on (pseudo-)Riemannian geometries. In a next step, we then define superintegrability on conformal geometries, and we shall see that with this definition, the symmetry group of the structure tensor coincides with the symmetry group of the superintegrable system.

Lemma 2.4 is the basis for the definition of Stäckel equivalence of properly superintegrable Hamiltonians, see Definition 2.5. For conformally superintegrable systems, we have the following generalisation, compare for example [Kre07, KKM05a, KKM06, BKM86].

**Definition 3.7.** Consider two second order conformally superintegrable systems with Hamiltonians $H = g(p, p) + V$ respectively $\tilde{H} = \tilde{g}(p, p) + \tilde{V}$ and conformal integrals $F^{(\alpha)}$ respectively $\tilde{F}^{(\alpha)}$ as in Definition 3.3. We say that the two systems are **conformally equivalent** if the associated Hamiltonians $H$ and $\tilde{H}$ satisfy $\tilde{H} = \Omega^{-2}H$ for some function $\Omega$, and if the trace-free parts of the associated conformal Killing tensors, viewed as $(2,0)$-tensors with upper indices, span the same linear space.

Compare this to Definition 2.5. Stäckel equivalence is a special case of conformal equivalence, insofar as in (2.9b) the conformal factor $\Omega$ is given by a potential $U \in \mathcal{V}^{\text{max}}$. Regarding the Stäckel transformation of the integrals, Equation (2.9b), we remark the following: In [Cap14, Theorem 4.1.8] it is proven that if $H = g(p, p) + V$ is a Hamiltonian with conformal integral $F = C(p, p) + W$, and $U \in \mathcal{V}^{\text{max}}$, then there is a trace correction $\lambda$ such that $\tilde{H} = \frac{H}{\Omega^2}$ admits the proper integral $\tilde{F} = C(p, p) + \lambda g(p, p) - \frac{VW}{\Omega^2}$. This fact provides us with a way to transform a second order conformally superintegrable system into a properly superintegrable system, possibly on a different (pseudo-)Riemannian manifold. We conclude that Stäckel equivalence can indeed be understood as special case of conformal equivalence.

In dimensions 2 and 3 it is even proven that any second order non-degenerate superintegrable system on a conformally flat manifold is Stäckel equivalent to a properly superintegrable system on a manifold of constant curvature; see [KKM05a, Theorem 3] and [KKM06, Theorem 4] respectively. Stäckel classes of 2-dimensional second order systems are studied in [Kre07] using properties of their associated quadratic algebras.

As mentioned earlier, we impose trace-freeness on conformal Killing tensors, except for the metric, which thereby becomes a distinguished conformal Killing tensor. Trace-freeness is motivated, among other reasons, by the conformal transformation rules for conformal Killing tensors. Let us assume that $H, \tilde{H}$ are two conformally equivalent Hamiltonians. Let $F(q) = C_q(p, p) + W(q)$ be a conformal integral for $H$, i.e.

$$\{H, C(p, p) + W\} = \omega(p) H.$$  

Then we compute

$$\{\tilde{H}, F\} = \left\{ \frac{H}{\Omega^2}, C(p, p) + W \right\}$$

$$= \frac{1}{\Omega^2} \left\{ H, C(p, p) + W \right\} - \frac{2}{\Omega^3} \left\{ \Omega, C(p, p) \right\} H$$

$$= \left( \omega(p) - 2C(p, d\Upsilon) \right) \tilde{H}$$

with $\Upsilon = \ln \Omega$. The following proposition is thus obtained, recalling Assumption 3.6, which ensures that the spaces $\mathcal{F}, \tilde{\mathcal{F}}$ in the claim are maximal.

**Proposition 3.8.** Let $g$ and $\tilde{g} = \Omega^2 g$ be a pair of conformally equivalent metrics, $\Omega > 0$.

(i) If $C_{ij}$ is a trace-free conformal Killing tensor for $g$ then $\tilde{C}_{ij} = \Omega^4 C_{ij}$ is a trace-free conformal Killing tensor for $\tilde{g}$. 

(ii) If \((g, V, \mathcal{F})\) and \((\tilde{g}, \tilde{V}, \tilde{\mathcal{F}})\) are conformally equivalent second order conformally superintegrable systems, where for \(F = C(p, p) + W \in \mathcal{F}\) with \(F \not\equiv H\), the coefficients \(C_{ij}\) are the components of a trace-free conformal Killing tensor. Then \(\tilde{\mathcal{F}} = \mathcal{F}\).

Note that \(\tilde{C}_{ij}\) is trace-free with respect to \(\tilde{g}\), but also with respect to \(g\), since \(g\) and \(\tilde{g}\) are conformally equivalent.

Proof. For part (i), we have the equation \(\Box E\) \(C_{ij,k} - g_{ij}\omega_k = 0\) for the conformal Killing tensor \(C\). A straightforward computation then shows that

\[
\begin{align*}
\tilde{C}_{ij,k} - \tilde{g}_{ij}\tilde{\omega}_k &= 0, \\
\tilde{\omega}^i &= \Omega^2 (\omega_i - 2C_{ia}\nabla^a), \\
\end{align*}
\]

where \(\tilde{C}_{ij} := \Omega^4 C_{ij}\), \(\tilde{\omega}^i := \Omega^2 (\omega_i - 2C_{ia}\nabla^a)\), and \(\nabla = \ln \Omega\). Part (ii) then follows immediately from (3.9) in the light of Assumption 3.6. \(\Box\)

Proposition 3.8 yields the transformation rules under conformal equivalence: The Hamiltonian is conformally modified, but the space of integrals is preserved. We can encode this in an efficient manner using weighted tensor densities. A conformal density \(\delta\) of weight \(w\) is a section in \(\mathcal{E}[w] := S^2 T^* M \otimes (\Lambda^w TM)^{-w/n}\) such that \(\phi^* (\delta) = \Omega^w \delta\) if \(\phi \in \text{Conf}(M)\) is a conformal transformation, i.e. \(\phi^* (g) = \Omega^2 g\). Given the data \((g, V, \mathcal{F})\), we observe that \(g \in \mathcal{E}_{(0,2)}[2]\) and \(V \in \mathcal{E}_{[-2]}\) are weighted densities. We are thus able to define conformally invariant densities of weight \(0\), \(g \in \mathcal{E}_{(0,2)}[0]\) and \(V \in \mathcal{E}[0] = C^\infty (M)\), given in local components by

\[
g_{ij} = \frac{g_{ij}}{|\det(g)|^{1/2}},
\]

compare [CG14], and

\[
v = |\det(g)|^{1/2} V,
\]

respectively. One straightforwardly verifies that \(\phi^* (g) = g\) and \(\phi^* (v) = v\). This leads us to the following definition:

**Definition 3.9.** Let \(M\) be a conformal manifold. A second order c-superintegrable system on \(M\) is given by a conformally invariant function \(v\) and a maximal linear space \(\mathcal{F}\) of invariant scalar functions \(F: T^* M \rightarrow \mathbb{R}\) on \(M\) such that

(i) If \(F \in \mathcal{F}\), then \(F = C(q)^{ij} p_i p_j + W(q)\) where \(C^{ij}\) are the components of a trace-free conformal Killing tensor.

(ii) There is a density \(\Omega \in \mathcal{E}[1]\) of weight \(1\) such that \(V = \Omega^2 v\) satisfies the Bertrand-Darboux condition (3.8) for all \(F \in \mathcal{F}\).

Note that the definition is conformally invariant, and for any \(\Omega \in \mathcal{E}[1]\) we have that \((\Omega^{-2} g, \Omega^2 v, \mathcal{F})\) is a conformally superintegrable system.

**Remark 3.10.** Let us consider the symmetry group of c-superintegrable systems. It is given by \(\mathcal{G} = \text{Conf}(M) \rtimes \mathbb{R}^+\), where \(\mathbb{R}^+\) acts as

\[
(v, F) \mapsto (av, F)
\]

It contains, in particular, all diffeomorphisms, and the transformations having a constant conformal factor. We remark that two different conformal factors do not necessarily result in different geometries. For instance, if \(g\) is a Euclidean metric in dimension two, then any conformally equivalent metric \(\Omega^2 g\) with

\[
\left(\frac{\partial \Omega}{\partial x}\right)^2 + \left(\frac{\partial \Omega}{\partial y}\right)^2 = \Omega \frac{\partial^2 \Omega}{\partial x^2} + \Omega \frac{\partial^2 \Omega}{\partial y^2}
\]

will also be Euclidean.
3.3. **Structure tensors of a conformally superintegrable system.** We consider the Bertrand-Darboux equation (3.8). It is a compatibility condition for the potential and the space of conformal Killing tensors associated to a second order superintegrable systems. Our aim is to solve the overdetermined system of linear equations (3.8) for the highest derivatives of the potential $V$. Following the analogous discussion in [KSV19], we use a generalisation of Cramer’s rule.

**Definition 3.11.** On an inner product space, the *Gram Coefficients* $G_k(A)$ of a linear map $A$ are the coefficients of the polynomial

$$\det(1 + tAA^*) = \sum_{k=0}^{\infty} G_k(A)t^k,$$

where $A^*$ is the adjoint of $A$.

Up to sign and order, the $G_k(A)$ are the coefficients of the characteristic polynomial of $AA^*$.

The following proposition provides us with an explicit expression for the structure tensor, which we will then apply to irreducible systems.

**Proposition 3.12.** [DTGVL05] A linear map $A$ on an inner product space has rank $r$ if and only if

$$G_r(A) \neq 0 = G_{r+1}(A).$$

(3.10)

In this case, the system of linear equations

$$Ax = b$$

has a solution $x$ if and only if the augmented matrix $(A|b)$ satisfies

$$G_{r+1}(A|b) = 0.$$

Moreover, the minimal norm solution is given by

$$x = A^\dagger b,$$

where

$$A^\dagger = \frac{1}{G_r(A)} \sum_{k=1}^{r} G_{r-k}(A)(-A^*A)^{k-1}A^*.$$

(3.11)

is the Moore-Penrose inverse of $A$.

We consider the Bertrand-Darboux equation (3.8), when written in local coordinates, as a linear system

$$AX = b_1(dV) + b_0V,$$

(3.12)

where $X$ is a vector that contains the unknown components of the trace-free Hessian

$$\nabla^2_{ij}V = \nabla^2_{ji}V - \frac{1}{n} g_{ij}\Delta V.$$

The coefficient matrix $A$ contains the components of the Killing tensors $C^{(\alpha)}$, $\alpha = 0, 1, \ldots, 2(n-1)$. On the right hand side, $b_1$ is a vector of terms involving the components of the differential $dV$ of $V$, i.e. $b_1(dV)$ comprises the components of the second and third term in the Bertrand-Darboux equation for each $C^{(\alpha)}$. Likewise, $b_0$ is a column vector such that $b_0V$ contains the components of the fourth term in (3.8b). Note that the rank $r$ for (3.12) does not need to be constant. If the conformal Killing tensors are analytic, however, then so are the components of the matrix $A$. Consequently the Gram coefficients $G_k(A)$ are also analytic. Proposition 3.12 ensures that in such case the rank of $A$ is constant on an open and dense subset.

Provided $r$ is constant, we can express the trace-free Hessian of $V$ using the Moore-Penrose inverse, by writing

$$X = A^\dagger b_1(dV) + A^\dagger b_0V.$$

(3.13)

**Definition 3.13.** A conformally superintegrable system on a (pseudo-)Riemannian manifold $M$ has *rank $r$*, if the coefficient matrix $A$ in (3.12) has rank $r$ on an open and dense subset of $M$. 

Remark 3.14. The rank of a conformally superintegrable system is at most
\[ r_{\text{max}} = \frac{n(n+1)}{2} - 1 = \frac{(n-1)(n+2)}{2}. \] (3.14)
As in the case of properly superintegrable systems [KSV19], we can characterise systems of maximal rank in terms of their trace-free conformal Killing tensors. As before, the metric allows us to identify (trace-free) bilinear forms and (trace-free) endomorphisms on the tangent space. This identification will tacitly be made in the following.

Due to Proposition 3.8, any conformally superintegrable system that is conformally equivalent to a maximal rank conformally superintegrable system is itself of maximal rank.

Definition 3.15.

(i) A set of endomorphisms is **irreducible** if they do not have a non-trivial invariant subspace in common.

(ii) A set of endomorphism fields on a (pseudo-)Riemannian manifold \( M \) is called **irreducible**, if they are pointwise irreducible on an open and dense subset of \( M \).

(iii) We call a conformally superintegrable system **irreducible**, if its conformal Killing tensors form an irreducible set.

(iv) We call a c-superintegrable system **irreducible**, if its conformal Killing tensors form an irreducible set.

The next lemma follows analogously to the corresponding statement in the case of properly superintegrable systems [KSV19].

Lemma 3.16. A conformally superintegrable system has maximal rank if and only if it is irreducible.

Irreducibility thus ensures that we can solve (3.12) for \( X \). In particular we find the minimal-norm solution (3.13), to which we may add any element from the kernel of \( A \). For second order conformally superintegrable systems, the Bertrand-Darboux equation (3.8) can therefore be solved assuming irreducibility. We have the requirement that the trace of the Hessian of \( V \) is the Laplacian of \( V \) and thus the potential \( V \) of an irreducible conformally superintegrable system satisfies
\[ V_{ij} = T_{ij} \, ^mV_m + \frac{1}{2} \, g_{ij} \Delta V + \tau_{ij} V \] (3.15)
with a (not necessarily unique) \((2,1)\)-tensor \( T_{ijk} \) and a \((2,0)\)-tensor \( \tau_{ij} \). We refer to (3.15) as the **Wilczynski equation** because our methods are inspired by Wilczynski’s series of papers on the projective differential geometry of surfaces [Wil07, Wil09]. Equations similar to (3.15) appear in [KKM05b] in local coordinates and for dimension three. The tensors \( T \) and \( \tau \) necessarily satisfy the following symmetries
\[ T_{ij} \, ^mV_m + \tau_{ij} V = 0 \] (3.16a)
\[ g^{ij} \, T_{ij} \, ^mV_m + \tau_{ij} V = 0. \] (3.16b)

We call \( T_{ijk} \) the **primary structure tensor** and \( \tau_{ij} \) the **secondary structure tensor** of the conformally superintegrable system. Note that these tensors are not invariant under conformal transformations.

The analog of (3.15) in proper superintegrability is Equation (2.7), which formally coincides with (3.15) for \( \tau_{ij} = 0 \). However, note that (2.7) was obtained from (2.6), where \( K \) is a proper Killing tensor. Instead, the Wilczynski equation (3.15) is obtained from the Bertrand-Darboux equation (3.8), where trace-free conformal Killing tensors appear. In spite of this difference, the following lemma shows that vanishing \( \tau \) indeed follows from proper superintegrability. We are going to see below in Corollary 5.3 that, for non-degenerate second order conformally superintegrable systems, the converse holds as well.

Lemma 3.17. Consider a second order superintegrable system with potential \( V \) and associated proper Killing tensors \( K^{(\alpha)} \). Let \( C^{(\alpha)} = K^{(\alpha)} \) and assume they satisfy the Wilczynski equation (3.15) with \( V \). Then \( \tau_{ij} = 0 \).
Proof. We choose a specific $\alpha$ and suppress the corresponding superscript for conciseness. By hypothesis, there is a function $\lambda$ such that

$$K_{ij} = C_{ij} + \frac{1}{n} \lambda g_{ij}$$

(3.17)

satisfies (2.6).

The proper Killing tensor $K_{ij}$ satisfies the Killing equation $K_{ijk,k} = 0$. Substituting (3.17) into the Killing equation, and then using the conformal Killing equation $K_{ij,k} = 0$, we find $\lambda_k = -n \omega_k$. We conclude that $\omega$ is exact, $d\omega = 0$. It then follows that $\tau_{ij} = 0$, as (3.8) does not have a term involving $V$ without derivative. □

3.4. Non-degenerate and abundant systems. In the previous section, it was shown that for irreducible second order superintegrable systems, the Bertrand-Darboux equation (3.8) can be solved for the second derivatives of $V$ up to the Laplacian $\Delta V$. The Wilczynski equation (3.15) then allows one to express all higher covariant derivatives of $V$ linearly in $V$, $\nabla V$, and $\Delta V$. Hence all higher derivatives of $V$ at some point are determined by the value of $(V, \nabla V, \Delta V)$ at this point, i.e. by $n + 2$ constants. This motivates the following definition.

Definition 3.18. A conformally superintegrable system is called non-degenerate if it satisfies the Wilczynski condition (3.15), and if (3.15) admits an $(n + 2)$-dimensional space of solutions $V$.

Due to (3.16), the structure tensors satisfy the following symmetry conditions for a non-degenerate potential:

$$T_{ji}^m = T_{ij}^m$$
$$\tau_{ij} = \tau_{ji}$$

$$g^{ij} T_{ij}^m = 0$$
$$g^{ij} \tau_{ij} = 0.$$ 

Lemma 3.19. For a non-degenerate conformally superintegrable system the structure tensors $T_{ijk}$ and $\tau_{ij}$ are unique.

Proof. Assume that the Wilczynski condition (3.15) were satisfied for two different pairs of structure tensors, say $T_{ijk}$, $\tau_{ij}$ and $\tilde{T}_{ijk}$, $\tilde{\tau}_{ij}$ respectively. Then, subtracting the corresponding copies of (3.15),

$$0 = (T_{ij}^k - \tilde{T}_{ij}^k) V_k + (\tau_{ij} - \tilde{\tau}_{ij}) V.$$ 

By the hypothesis of non-degeneracy, the coefficients of $V_k$ and $V$ have to vanish independently, i.e. $T_{ij}^k = \tilde{T}_{ij}^k$ and $\tau_{ij} = \tilde{\tau}_{ij}$.

Example 3.20. The isotropic harmonic oscillator is an irreducible system in the sense of Definition 3.15 and has the potentials

$$V(x) = \frac{\omega^2}{2} (x - x_0)^2 + V_0$$

with the $n + 2$ free parameters $\omega^2$, $x_0$ and $V_0$. $V(x)$ solves the Wilczynski equation (3.15). The \textit{isotropic harmonic oscillator} on flat $n$-dimensional space has vanishing structure tensor $T$. It is properly superintegrable and therefore also the structure tensor $\tau_{ij}$ vanishes. Any conformally superintegrable system conformally equivalent to the isotropic harmonic oscillator is characterised by $\tilde{T}_{ijk} = 0$, and we obtain (3.15) in the form

$$V_{,ij} = \frac{nt_i V_j + nt_j V_i - 2 g_{ij} t^m V_m}{(n-1)(n+2)} + \frac{1}{n} g_{ij} \Delta V + \tau_{ij} V.$$ (3.18)

We are going to see in (5.9) that

$$\tau_{ij} = \frac{2}{3} \left( \frac{n}{(n-1)(n+2)} \right)^2 (t_i t_j) - 2 \hat{P}_{ij},$$

where $\hat{P}_{ij}$ is the trace-free part of the Schouten tensor.

\textsuperscript{4}Note that for an analytic metric, the (trace-free) conformal Killing tensors are analytic, and thus the structure tensors and the potentials are also analytic.
Note that non-degeneracy is conformally invariant, and thus can be defined for a c-superintegrable system.

**Definition 3.21.** We call a c-superintegrable system *non-degenerate* if one (and hence all) members of the corresponding equivalence class are non-degenerate in the sense of Definition 3.18.

From now on we restrict to non-degenerate systems that satisfy the Wilczynski equation (3.15). In the present section, our aim is to formulate and study the integrability conditions imposed by (3.15) and the non-degeneracy condition onto the structure tensors of a conformally superintegrable system.

By definition, a superintegrable system on an \( n \)-dimensional (pseudo-)Riemannian manifold has the maximal number of \( 2n - 1 \) functionally independent integrals. However, as for properly superintegrable systems, all known non-degenerate second order systems also admit maximally many linearly independent conformal Killing tensors. In analogy to properly superintegrable systems, we therefore define:

**Definition 3.22.** We call a non-degenerate second order conformal superintegrable system in dimension \( n \) abundant, if the subspace
\[
\hat{C} = \{ \hat{C} : C \in C^{\text{max}} : \text{tr}(C) = 0 \}
\]
has dimension
\[
\dim(\hat{C}) = \frac{n(n+1)}{2} - 1 = \frac{(n-1)(n+2)}{2} = r_{\text{max}}.
\]

By virtue of Proposition 3.8, abundantness is a conformally invariant property and hence a property of a c-superintegrable system. Moreover, it is manifest that if a conformally superintegrable system is abundant, then a properly superintegrable system that is conformally equivalent to it is also abundant in the sense of [KSV19].

Definition 3.22 is going to become natural in Section 5, where we find that it is tantamount with the generic integrability of the prolongation equations for the trace-free conformal Killing tensors arising from a second order conformally superintegrable system. At the end of the current section we shall also give an equivalent characterisation of abundantness relying on a relaxation of linear independence.

We remark that Definition 3.22 generalises the concept of abundantness introduced in Section 2. Abundantness is trivial in dimension \( n = 2 \). In dimension \( n = 3 \), the situation is a bit more involved. It is still true that every non-degenerate second order (properly) superintegrable system on a conformally flat manifold extends from \( 2n - 1 = 5 \) functionally independent integrals to \( \frac{n(n+1)}{2} = 6 \) linearly independent integrals [KKM05b]. This is known as the “(5 \( \Rightarrow \) 6)-Lemma”. Furthermore, every 3-dimensional conformal system is Stäckel equivalent to a proper one with constant curvature [Cap14].

**Remark 3.23.** For the sake of expositional simplicity, we require an abundant system to have \( \frac{n(n+1)}{2} - 1 \) linearly independent trace-free conformal Killing tensors. Functional independence of \( 2n - 1 \) of the arising conformal integrals is not yet required in the definition, but in Lemma 6.8 we prove that systems which do admit \( 2n - 1 \) functionally independent conformal integrals lie dense among abundant systems.

We devote the remainder of this paragraph to an alternative characterisation of abundantness.

**Definition 3.24.** We say that a collection of linearly independent Killing tensors \( K^{(\alpha)}_{ij} \) is conformally linearly independent if
\[
\sum_{\alpha} c_{\alpha} K^{(\alpha)} = f g,
\]
with constants \( c_{\alpha} \) and a function \( f \), implies \( f = 0 \) and \( c_{\alpha} = 0 \) for all \( \alpha \).
The following lemma ensures that a conformally superintegrable system is abundant if the conformal Killing tensors $C^{(\alpha)}$ for $\alpha \in \{1, 2, \ldots, \frac{n(n+1)}{2}\}$ in Definition 3.3 are conformally linearly independent.

**Lemma 3.25.** Let $(K^{(1)}, \ldots, K^{(m)})$ be a collection of Killing tensors and denote the corresponding trace-free conformal Killing tensors by $C^{(\alpha)} = K^{(\alpha)} - \frac{1}{n} \text{tr}(K^{(\alpha)}) g$. Then the tuple $(g, C^{(1)}, \ldots, C^{(m)})$ is linearly independent if any only if $(K^{(1)}, \ldots, K^{(m)})$ is conformally linearly independent.

**Proof.** Assume first that $(g, C^{(1)}, \ldots, C^{(m)})$ is linearly dependent. This means there is a combination

$$\sum c_\alpha C^{(\alpha)} = c_0 g$$

with constants $(c_0, c_1, \ldots, c_m) \neq 0$. This means

$$\sum c_\alpha K^{(\alpha)} = f g$$

for a function $f$ obtained from $c_0$ and the trace terms. This proves one implication. For the other implication, we assume that $(K^{(1)}, \ldots, K^{(m)})$ are conformally linearly dependent. Therefore we have

$$\sum_{\alpha=1}^m c^{(\alpha)} C^{(\alpha)} = \left( f + \frac{1}{n} \sum c^{(\alpha)} \text{tr}(K^{(\alpha)}) \right) g.$$

In this equation the left hand side and the right hand side have to vanish independently, as they are trace-free respectively pure trace. We conclude

$$\sum c_\alpha C^{(\alpha)} = 0, \quad f = -\frac{1}{n} \sum c_\alpha \text{tr}(K^{(\alpha)})$$

Because of the hypothesis of conformal linear dependence, we have $(c_1, \ldots, c_m) \neq 0$. This implies that $(g, C^{(1)}, \ldots, C^{(m)})$ are linearly dependent.

Moreover we observe: If the integrals $F^{(\alpha)} = C^{ij}_{(\alpha)} p_i p_j + W^{(\alpha)} \in \mathcal{F}$ are functionally independent, then their kinetic parts are associated to conformally linearly independent conformal Killing tensors. This follows from Theorem 1 of [KKM05b].

### 3.5 Structure tensors and $c$-superintegrability

We are now going to determine how the structure tensors behave under conformal changes of the superintegrable system.

**Lemma 3.26.** Let $H = g(p,p) + V$ and $\tilde{H} = \Omega^{-2} H$ be a pair of conformally equivalent Hamiltonians, $\Omega > 0$. Assume $\tilde{H}$ gives rise to an irreducible conformally superintegrable system such that the Wilczynski equation (3.15) is satisfied. Then $\tilde{H}$ satisfies (3.15) as well. (We decorate the corresponding objects with a tilde). In particular, the structure tensors $T_{ijk}$ and $\tau_{ij}$ are transformed to, respectively,

$$\tilde{T}_{ij}^\ k = T_{ij}^\ k - 3 \frac{i \ j}{a} \gamma_i g_j^\ k$$

$$\tilde{\tau}_{ij} = \tau_{ij} + 2 T_{ij}^\ k \gamma_k - \frac{i \ j}{a} (\gamma_{ij} + 2 \gamma_i \gamma_j),$$

where $\gamma = \ln \Omega$.

**Proof.** By a straightforward computation, using the Wilczynski equation (3.15) and the product rule, we arrive at

$$\nabla_{ij}^2 \tilde{V} = \frac{1}{n} \tilde{g}_{ij} \Delta \tilde{V} + (T_{ij}^\ k - 3 \ (\gamma_i g_j^\ k + \gamma_j g_i^\ k) - 2 \ n \ g_{ij} \gamma^k) \ \tilde{V}_k$$

$$+ \left( \tau_{ij} + 2 T_{ij}^\ k \gamma_k - \frac{2 \ i \ j}{a} \gamma_i \gamma_j \right) \tilde{V}.$$

The result then simplifies further using $\Omega^{-1} \nabla_{ij}^2 \Omega = \gamma_i \gamma_j + \frac{1}{2} \frac{i \ j}{a} \gamma_{ij}$. According to the Wilczynski equation (3.15), the structure tensor $T_{ijk}$ of a conformally superintegrable system determines the conformally superintegrable potential up to the action of $\mathcal{S} = \text{Diff}(M) \times \mathbb{R}^*$. The following corollary is straightforwardly obtained, but will be fundamental.
Corollary 3.27. Let the hypothesis and notation be as in Lemma 3.26.

(i) Under conformal transformations, the trace
\[ t_i = T^a_{ia} = T^a_{ai} \]
undergoes a translation by \( \Upsilon, i \),
\[ \tilde{t}_i = t_i - \frac{3}{n} (n - 1)(n + 2) \Upsilon, i \].

(ii) Under conformal transformations, the trace
\[ T^a_{ai} \] remains unchanged,
\[ \tilde{T}^a_{ai} = T^a_{ai} \).

(iii) Under conformal transformations, the trace-free part of the primary structure tensor remains unchanged,
\[ \tilde{T}^{k}_{ij} = \tilde{T}^{k}_{ij} \).

Remark 3.28. Although \( \tilde{T}^{k}_{ij} \) is conformally invariant, it is often advantageous to work with the tensor \( \tilde{T}^{k}_{ijk} \) instead, which in the context below turns out to be a totally symmetric tensor field. While \( \tilde{T}^{k}_{ijk} \) is not actually invariant, under conformal transformations it is just multiplied with a power of \( \Omega \),
\[ \tilde{T}^{k}_{ijk} = \Omega^2 g^{ka} \tilde{T}^{a}_{ij} = \Omega^2 \tilde{T}^{k}_{ijk} \), i.e. it has weight 2.

According to [KSV19], \( \tilde{T}^{k}_{ijk} \) is closely related to the Weyl curvature. Under mild assumptions we shall find below that \( \tilde{T}^{k}_{ijk} \) carries enough information to reconstruct the conformal equivalence class of a (conformally) superintegrable system.

In view of properly superintegrable systems, at first sight it might seem that the conformal case would require additional information, in the form of an additional structure tensor \( \tau_{ij} \). We will find, however, that for abundant systems \( \tau_{ij} \) is determined by \( T^{k}_{ijk} \) and the Ricci curvature. We summarise the discussion in this section with the following proposition.

Proposition 3.29. On a (pseudo-)Riemannian manifold \( M \), every non-degenerate irreducible conformally superintegrable system admits tensor fields \( T \) and \( \tau \) with the following properties:

(i) \( T \) is well-defined and smooth on the open and dense subset \( N = \{ G_{r_{\text{max}}}(A) \neq 0 \} \subseteq M \). It is of valence 3, symmetric and trace-free in its first two indices:
\[ T_{jk} = T_{ijk} \quad g^{ij} T_{ijk} = 0 \] (3.23)

Moreover, \( \tau \) is well-defined and smooth on \( N \). It is of valence 2, symmetric and trace-free:
\[ \tau_{ji} = \tau_{ij} \quad g^{ij} \tau_{ij} = 0 \] (3.24)

(ii) The conformally superintegrable potential satisfies the Wilczynski equation (3.15).

(iii) \( T \) and \( \tau \) are uniquely determined by the metric and by the trace-free conformal Killing tensors \( C^{(\alpha)} \) in the conformally superintegrable system, and depend only on the subspace \( C \) spanned by these \( C^{(\alpha)} \), i.e. it is invariant under basis changes on \( C \). Note that such a change of basis is conformally invariant.

(iv) The trace-free part \( T^{k}_{ijk} \) of the \((2,1)\)-tensor field \( T^{k}_{ij} \) is invariant under conformal changes of the conformally superintegrable system.

The components \( T_{ijk} \) of \( T \) are given explicitly in terms of the Killing tensors by the rank-\( r \) Moore-Penrose inverse, where \( r = r_{\text{max}} \) is the maximal rank (3.14).

Proof. The first three assertions follow analogous to the case of properly superintegrable systems, see [KSV19], such that the tensors \( T \) and \( \tau \) are given by \( A^{1} b_1 \) and \( A^{1} b_0 \), respectively, using Equation (3.12). To see (iv), take the trace-free part of (3.20a). \( \square \)

Let us reconsider the aforesaid in the light of Definition 3.9. The corollary below is easily obtained using the transformation rules (3.20) and recalling the following transformation rules for the curvature.
Remark 3.30. The following transformation rules can be found in \cite{CG14, Kul69, Kul70}, for instance. The Weyl tensor is conformally invariant. The trace-free Ricci tensor and the scalar curvature transform, respectively, according to
\[ \tilde{R}_{ij} \to \tilde{R}_{ij} - (n-2)(\tilde{\Upsilon}_{ij} - \tilde{\Upsilon}_i \tilde{\Upsilon}_j), \]
\[ R \to \Omega^{-2} \left( R + 2(n-1) \left( \Delta \tilde{\Upsilon} - \frac{n-2}{2} \tilde{\Upsilon}^a \tilde{\Upsilon}_a \right) \right). \]
Therefore the Schouten tensor transforms according to
\[ P_{ij} \to P_{ij} - \tilde{\Upsilon}_{ij} + \tilde{\Upsilon}_i \tilde{\Upsilon}_j - \frac{1}{2} g_{ij} \tilde{\Upsilon}_c \tilde{\Upsilon}^c. \]

Corollary 3.31. 
(i) Every non-degenerate irreducible c-superintegrable system on a conformal manifold \((M,c = [g])\) admits a well-defined totally symmetric and trace-free tensor field \(S = \tilde{T}\) that is invariant under conformal transformations.
(ii) If this tensor coincides for two non-degenerate irreducible c-superintegrable systems on the same conformal manifold, then these systems are conformally equivalent if also
\[ \tau_{ij} - \frac{2}{3} S_{ijk} \tilde{t}^k - 2 \tilde{P}_{ij} + \frac{1}{3} \left( \tilde{\Gamma}_{ij} \tilde{t}_j \right) \]
coincides for both systems.

4. Conformally superintegrable potentials
Written in local coordinates, the Wilczynski equation (3.15) is a second order partial differential equation (PDE) for the potential \(V\). In Proposition 3.29 we have seen that for irreducible second order superintegrable systems, the coefficients in this PDE are determined by the space of trace-free conformal Killing tensors and the metric.

4.1. Prolongation of a superintegrable potential. The Wilczynski equation (3.15) expresses the trace-free part of the Hessian of the potential \(V\) linearly in the differential \(\nabla V\), the Laplacian \(\Delta V\), and the potential \(V\) itself. The coefficients in this expression are determined by the structure tensors \(T\) and \(\tau\). In the following Proposition, the Wilczynski equation (3.15) is extended by a second equation which expresses the derivatives of \(\Delta V\) linearly in \(\Delta V\), \(\nabla V\) and \(V\). Again, the coefficients are determined by the structure tensors. The system (4.1) below is an extended system of PDEs for (3.15). Such an extended system is called a prolongation of the initial PDE, and it allows one to make use of the powerful theory of parallel linear connections. In particular, by virtue of (4.1) all higher derivatives of \(V\), \(\nabla V\) and \(\Delta V\) are expressed in terms of these.

Proposition 4.1. Equation (3.15) has the prolongation
\[ V_{,ij} = T_{ij}^m V_{,m} + \frac{1}{n} g_{ij} \Delta V + \tau_{ij} V \]
\[ \frac{n-1}{n} (\Delta V)_{,k} = q_k^m V_{,m} + \frac{1}{n} t_k \Delta V + \gamma_k V, \]
with
\[ q_j^m := Q_{aj}^a \]
\[ t_j := T_{aj}^a \]
\[ \gamma_k := \Gamma_{ak}^a \]
where
\[ Q_{ijk}^m := T_{ij}^m g_{,k} + T_{ij}^r T_{rk}^m - R_{ijk}^m + \tau_{ij} g_{,k}^m. \]
\[ \Gamma_{ijk} := \tau_{ij,k} + T_{ij}^a \tau_{ak} \]
Proof. Equation (4.1a) is nothing but the Wilczynski equation (3.15). Substituting it into its covariant derivative, we obtain

\[ V_{ijk} = (T_{ij}^m \cdot k + T_{ij} T_{ik}^m + \tau_{ij} \delta^m_k) V_m + \frac{1}{n} (T_{ijk} \Delta V + g_{ij} (\Delta V)_k + (\tau_{ij} + T_{ij}^m \tau_{mk}) V. \]

Antisymmetrising in \( j \) and \( k \) and applying the Ricci identity gives

\[ R^m_{ijk} V_m = \frac{j}{k} \left[ (T_{ij}^m \cdot k + T_{ij} T_{ik}^m + \tau_{ij} \delta^m_k) V_m + \frac{1}{n} (T_{ijk} \Delta V + g_{ij} (\Delta V)_k + \Gamma_{ijk} V \right]. \]

Solving for the term involving \( (\Delta V)_k \) on the right hand side, we get

\[ \frac{1}{n} \frac{j}{k} g_{ij} (\Delta V)_k = - \frac{j}{k} \left( Q_{ijk} m V_m + \frac{1}{n} T_{ijk} \Delta V + \Gamma_{ijk} V \right). \]

The contraction of this equation in \( i \) and \( j \) now yields (4.1b), since \( T_{ijk} \) and \( Q_{ijk}^m \) are trace-free in \( i \) and \( j \) by definition. \( \square \)

4.2. Integrability conditions for a non-degenerate potential. From the perspective of the equations (4.1), non-degeneracy implies that the corresponding integrability conditions be satisfied generically, independently of the potential. With this condition we finally eliminate the potential equations (4.1), non-degeneracy implies that the corresponding integrability conditions be satisfied

\[ \text{Integrability conditions for a non-degenerate potential.} \]

Proposition 4.2. The Wilczynski equation (3.15) locally has a non-degenerate solution \( V \) if and only if the following integrability conditions hold:

1. \[ \frac{j}{k} \left( T_{ijk} + \frac{1}{n-1} g_{ij} t_k \right) = 0 \]
2. \[ \frac{j}{k} \left( Q_{ijkl} + \frac{1}{n-1} g_{ij} q_{kl} \right) = 0 \]
3. \[ \frac{j}{k} \left( \Gamma_{ijk} + \frac{1}{n-1} g_{ij} \gamma_k \right) = 0 \]
4. \[ \frac{k}{l} \left( q_{,l} + T_{ml} q_{k}^m + \frac{1}{n-1} g_{k} t_l^m + g_{\gamma k} \right) = 0 \]
5. \[ \frac{k}{l} \left( \gamma_{,i} + q_i \tau_{mj} + \frac{1}{n-1} t_i \gamma_j \right) = 0. \]

Proof. The system (4.1) allows one to write all higher derivatives of \( V, \nabla V \) and \( \Delta V \) as linear combinations of \( V, \nabla V \) and \( \Delta V \). Necessary and sufficient integrability conditions are then given by the Ricci identities

\[ \frac{j}{k} V_{ijk} = R^m_{ijk} V_m \quad \frac{k}{l} (\Delta V)_{,kl} = 0, \]

where we substitute (3.15) in the left hand side of the equations. We obtain

\[ \frac{j}{k} \left( Q_{ijk}^m + \frac{1}{n-1} g_{ij} q_k^m \right) V_m + \frac{1}{n} \frac{j}{k} \left( T_{ijk} + \frac{1}{n-1} g_{ij} t_k \right) \Delta V + \frac{j}{k} \left( \Gamma_{ijk} + \frac{1}{n-1} g_{ij} \gamma_k \right) V = 0 \]

and, respectively,

\[ \frac{k}{l} \left( q_{,l} + T_{ml} q_{k}^m + \frac{1}{n-1} g_{k} t_l^m \right) V_n + \frac{1}{n} \frac{k}{l} \left( t_{k,l} + q_{kl} \right) \Delta V + \frac{n}{n} \frac{k}{l} \left( \gamma_{,l} + q_k \tau_{ml} + \frac{1}{n-1} t_k \gamma_l \right) V = 0. \]
For a non-degenerate superintegrable potential the coefficients of $\Delta V$, $\nabla V$ and $V$ must vanish separately. In addition to the stated integrability conditions, this yields the condition

$$\frac{k}{l} \left( t_{k,l} + q_{kl} \right) = 0.$$ \hspace{1cm} (4.6)

The latter is redundant, since it can be obtained from (4.5b) by a contraction over $i$ and $l$. \hfill \square

Note that the equations (4.5) are not invariant under conformal transformations, as they emerge from coefficients of $V$, $\nabla V$ and $\Delta V$, respectively. Still, after a conformal transformation as in Proposition 3.8, the form of (4.5) is the same, but the metric, the curvature and the structure tensors are replaced. We can solve Equation (4.5a) right away, because it is linear and does not involve derivatives.

**Proposition 4.3.** The first integrability condition for a superintegrable potential, Equation (4.5a), has the solution

$$T_{ijk} = S_{ijk} + \frac{1}{6} \left( \bar{t}_i g_{jk} - \frac{1}{n} g_{ij} \bar{t}_k \right),$$ \hspace{1cm} (4.7)

where $S$ is an arbitrary totally symmetric and trace-free tensor. The 1-form $\bar{t}_i$ is given by

$$\bar{t}_i = \frac{n}{(n-1)(n+2)} t_i = \frac{n}{(n-1)(n+2)} T_{ia}^a.$$ \hspace{1cm} (4.8)

Note that $S_{ijk}$ and $\bar{t}_i$ are uniquely determined by $T$.

**Proof.** Let us decompose $T_{ijk}$, which by definition is trace-free and symmetric in $(i,j)$,

$$\begin{align*}
\Box \otimes \Box & \simeq \Box \otimes \Box \simeq \Box \otimes \Box \simeq = S_{ijk} \\
\circ \otimes \circ & \simeq \Box \otimes \Box \simeq \Box \otimes \Box \simeq .
\end{align*}$$

Due to (4.5a), the penultimate component of this decomposition vanishes, and therefore we obtain

$$T_{ijk} = S_{ijk} + \frac{1}{6} \left( \bar{t}_i g_{jk} - \frac{1}{n} g_{ij} \bar{t}_k \right).$$

where $s_k$ and $\xi_k$ are components of some 1-forms. Substituting (4.5a), and taking the trace in $(j,k)$,

$$t_i = T_{ia}^a = \frac{n+2}{3} s_i - \frac{n-1}{4} \xi_i.$$ 

Taking the trace in $(i,j)$ instead,

$$0 = T_{ai}^a = \frac{n+2}{3} s_i + \frac{n-1}{2} \xi_i.$$ 

Solving for $s_i$ and $\xi_i$, we find

$$s_i = \frac{2t_i}{n+2}, \quad \xi_i = -\frac{4t_i}{3(n-1)}.$$ 

Resubstituting into the initial formula for $T_{ijk}$, we arrive at (4.7). \hfill \square

**Corollary 4.4.**

(i) The tensor $q_{ij}$ is symmetric, i.e. $q_{ji} = q_{ij}$.

(ii) The 1-form $\bar{t}_i$ is the derivative of a function $\bar{t}$, i.e. $\bar{t}_i = \bar{t}, i$.

Note that consequently also $t_i = t, i$. Without loss of generality we impose $\bar{t} = \frac{n}{(n-1)(n+2)} t$ on the arbitrary integration constant.

**Proof.** The first statement follows from substituting (4.7) into the definition (4.2a) of $q_{ij}$. The second then follows from (4.6). \hfill \square
Proposition 4.5. The third integrability condition for a superintegrable potential, Equation (4.5c), has the solution
\[ \Gamma_{ijk} = \Sigma_{ijk} + \frac{1}{n} g_{ij} \bar{\gamma}_k, \] (4.9)
where \( \Sigma \) is an arbitrary totally symmetric and trace free tensor and
\[ \bar{\gamma}_i = \frac{n}{(n + 2)(n - 1)} \gamma_i. \]
Note that \( \Sigma \) and \( \gamma \) are uniquely determined by \( \Gamma \).

The proof is the same as that of Proposition 4.3.

4.3. **The conformal scale function.** As we have just seen, the trace \( t_i \) of the primary structure tensor is the differential of a function \( t \). We thus obtain the following transformation rule under conformal changes of the Hamiltonian, which is an immediate consequence of (3.21).

Lemma 4.6. Under a conformal change of the superintegrable system, say \( H \mapsto \Omega^{-2} H, \Omega > 0 \), the function \( t \) transforms as
\[ \bar{t} \mapsto \bar{t} - 3 \Upsilon \quad \text{up to an irrelevant constant}, \] (4.10)
where \( \Upsilon = \ln \Omega \).

Note that the function \( \bar{t} \) is determined by the structure tensor \( T \) only up to an irrelevant constant. The symmetry group of c-superintegrable systems is \( \mathfrak{S} = \text{Conf}(M) \rtimes \mathbb{R}^* \), c.f. Remark 3.10. The second factor of \( \mathfrak{S} \) does not affect \( \bar{t} \). Indeed, we see that if \( t^{\text{new}} - t^{\text{old}} = c \in \mathbb{R} \), then \( \Omega = e^{-\frac{2c}{3}} \) and thus
\[ H^{\text{new}} = e^{\frac{2c}{3}} H^{\text{old}}. \]

Lemma 4.6 above therefore confirms that \( \bar{t} \) behaves like a scale function.

Definition 4.7. The conformal scale function is the density of weight 1 defined by
\[ \sigma = e^{-\frac{1}{3} \bar{t}}. \]

Lemma 4.6 allows us to change \( \bar{t} \) arbitrarily, resulting in a natural gauge freedom of a conformally superintegrable system. There are three natural scale choices, in particular, that are relevant in this paper, each of which has specific features that we can exploit to gain information or to simplify certain computations. Table 1 summarises some properties of the scale choices and the notation we use.

4.3.1. **Standard scale.** This scale choice realises \( \bar{t}_{ij} = 0 \).

Definition 4.8. A conformally superintegrable system with \( \bar{t}_{ij} = 0 \) is said to be in standard scale.

We shall use a specific notation for the metric and the secondary structure tensor when we work in standard scale. Given an arbitrary scale choice, we can apply a conformal transformation with conformal rescaling \( \Omega = e^{\frac{2c}{3}} \). Let \( \Upsilon = \ln \Omega \). The transformed metric of the system in standard scale, then is
\[ \tilde{g}_{ij} = e^{\frac{4}{3} \bar{t}} g_{ij} =: g_{ij}, \] (4.11a)
and the new structure tensors become \( \tilde{T}^{k}_{ij} = S^k_{ij} \) and
\[ \tilde{\tau}_{ij} = \tau_{ij} + \frac{2}{3} S^k_{ij} \bar{\tau}_{k} - \frac{1}{3} \tilde{\tau}_{ij} \left( \frac{1}{3} \tilde{t}_{ij} - \frac{4}{9} \tilde{t}_{i} \tilde{t}_{j} \right) =: \eta_{ij}. \] (4.11b)
For later reference, we mention the Schouten curvature \( \mathcal{P}_{ij} \) of \( g_{ij} \), which is given by
\[ \mathcal{P}_{ij} = P_{ij} - \Upsilon_{ij} + \Upsilon_{,i} \Upsilon_{,j} - \frac{1}{2} g_{ij} \Upsilon_{,c} \Upsilon^{c}, \] (4.11c)
while the Weyl curvature remains unchanged under conformal transformations. Equations (4.11b) and (4.11c) are obtained, respectively, from (3.20b) and (3.26).

Note that the standard scale is not unique, as we may add any constant to $\bar{t}$. For simplicity we usually choose $\bar{t} = 0$. As discussed after Lemma 4.6, however, this only means that the Hamiltonian is multiplied by a constant, and moreover the structure tensor is not changed in the process. From the viewpoint of conformally superintegrable systems however, if we multiply the metric by a constant, this typically changes the underlying metric (unless the transformation is already in $\text{Diff}(M)$). Yet, the space $\mathcal{C}$ of conformal Killing tensors of a conformally superintegrable system remains unaffected by such a change.

The standard scale has two major advantages: On the one hand, it yields very compact equations, facilitating some otherwise tedious computations. On the other hand, the standard scale exposes the invariant data of the problem, which is going to be particularly helpful when we discuss conformal equivalence classes.

**Example 4.9.** The systems VII [5], O and A in Table 3 are in standard scale.

### 4.3.2. Flat scale.

This scale choice only exists for conformally flat metrics.

**Definition 4.10.** A conformally superintegrable system with flat curvature is said to be in **flat scale**.

We find, using (4.11a), that there is a function $\rho : C^{\infty}(M) \to \mathbb{R}$ such that

$$ g_{ij} = e^{2\rho} h_{ij}. \tag{4.12} $$

where $h$ has vanishing curvature. A major advantage of flat scales is that covariant derivatives coincide with partial ones, facilitating concrete computations in local coordinates. Moreover, the existence of a flat scale permits us to express the Ricci curvature in terms of the scalar function $\rho$ using (3.25).

Flat scales are not unique. For example, we can add any constant to $\rho$. According to [Kul70], any conformal change transforming a flat metric into a flat metric is given via $\rho \to \rho - \eta$ where $\eta$ is a function satisfying

$$ [Q(Y, Z) + g(Y, Z)r] X - [Q(X, Z) + g(X, Z)r] Y + g(Y, Z)Q(X) - g(X, Z)Q(Y) = 0 $$

where $Q(X, Y) = (\nabla^2 \eta)(X, Y) - X(\eta)Y(\eta)$ with $g(Q(X), Y) = Q(X, Y)$ and $r = g(d\eta, d\eta)$.

**Example 4.11.** All systems in Table 3 are in flat scale. In particular, note that the systems III [23] and V [32] are conformally equivalent.

### 4.3.3. Proper scale.

A third natural choice is the proper scale, in which the system is properly superintegrable (up to a trace correction of the trace-free conformal Killing tensors). As mentioned earlier, any conformally superintegrable system is conformally equivalent to a properly superintegrable system [Cap14, Theorem 4.1.8]. According to Lemma 3.17 it satisfies $\tau_{ij} = 0$.

**Definition 4.12.** A conformally superintegrable system with $\tau_{ij} = 0$ is said to be in **proper scale**.

Again, the proper scale choice is not unique. Its advantage is that all the known results about properly superintegrable systems can be invoked. Yet it is less useful for gaining insight into the underlying conformal geometry. Nevertheless, from the viewpoint of conformal geometry, proper scale choices have some interesting properties which we explore in Section 6.5 for constant curvature spaces.
Example 4.13. All systems in Table 3 are in proper scale. For an example that is in proper scale, but neither in flat nor standard scale, consider the metric 
\[ g = \left( \frac{z \bar{z} + 4}{z - \bar{z}} \right)^2 + a_1 \left( \frac{z \bar{z} + 4}{z - \bar{z}} \right)^2 + a_2 \left( \frac{z \bar{z} - 4}{z - \bar{z}} \right)^2 + a_3 \]
with parameters \( a_i \in \mathbb{R} \), and satisfies
\[ \tilde{t} = \frac{3}{2} \ln \frac{i (z \bar{z} + 4)^3}{(z - \bar{z})^2} \]

Example 4.14 (Generic system on the 3-sphere). Consider the 3-sphere with metric
\[ g = d\phi^2 + \sin^2(\phi) (d\theta^2 + \sin^2(\theta) d\psi^2) \]
The potential
\[ V = a_1 \cos^2(\phi) + a_2 \sin^2(\phi) \cos^2(\theta) + a_3 \sin^2(\phi) \sin^2(\theta) \cos^2(\psi) + a_4 \sin^2(\phi) \sin^2(\theta) \sin^2(\psi) + a_0 \]
is non-degenerate and defines the so-called generic system on the 3-sphere; in [KKM06] it is labelled VIII. It is in proper scale, but neither in flat nor in standard scale.

Note that the Harmonic Oscillator, see Example (3.20), is simultaneously in standard, flat and proper scale. In Section 6 we find that this is an immediate consequence of \( T_{ijk} = 0 \).

| Objects                  | Standard Scale | Flat scale | Proper scale |
|-------------------------|----------------|------------|--------------|
| Function \( \tilde{t} \) | 0              | \( \rho \) | \( \tilde{t} \) |
| Schouten tensor \( P_{ij} \) | \( \mathcal{P}_{ij} \) | 0           | \( P_{ij} \) |
| Secondary structure tensor \( \tau_{ij} \) | \( \mathcal{h}_{ij} \) | \( \tau_{ij} \) | 0 |
| Metric \( g_{ij} \)     | \( g_{ij} \)   | \( h_{ij} \) | \( g_{ij} \) |

Table 1. Notation and conventions for the three natural scale choices.

5. Conformal Killing tensors in conformally superintegrable systems

5.1. Prolongation equations for trace-free conformal Killing tensors. In Section 4.1 we have discussed a prolongation system for the potential \( V \). Similarly, we can write down a prolongation for an arbitrary trace-free conformal Killing tensor \( C_{ij} \). In general this system can be rather complicated [Wei77], given also the explicit but complicated expressions well known for proper second order Killing tensors in [Wol98, GL19]. However, as is shown in [KSV19], the prolongation system for proper second order Killing tensors in non-degenerate superintegrable systems simplifies considerably. In fact, the prolongation system in this case closes after the first covariant derivative. We observe the same phenomenon with trace-free conformal Killing tensors. We remark that trace-freeness is paramount. Indeed, for conformal Killing tensors with non-vanishing trace, the prolongation system would not be finite.

Theorem 5.1. A trace-free conformal Killing tensor \( C_{ij} \) in a non-degenerate conformally superintegrable system satisfies
\[ C_{ij,k} = \frac{1}{3} \left( \frac{\{ j, i \}}{k} T^m_{ji} g_k^n - \frac{2}{n} g_{ij}(t^m g^n_k - T_k^m n) \right) C_{mn}, \]  
with the primary structure tensor \( T_{ijk} \) given by the Wilczynski equation (3.15). The Bertrand-Darboux condition (3.8) in this situation is equivalent to (5.1) and
\[ \frac{\{ j, k \}}{k} \left( \omega_{j,k} + C^m_{ij} \tau_{km} \right) = 0. \]
Note that from (5.1) we obtain \( \omega_i \) using Formula (3.2). We also remark that (5.1) does not contain the secondary structure tensor \( \tau_{ij} \). Indeed, we shall see that, under the hypothesis of the theorem, the tensor \( \tau_{ij} \) is obtained from \( T_{ijk} \) and the Ricci curvature.

**Remark 5.2.** Equation (5.1) should be compared to the prolongation equation (2.8) for a Killing tensor in a properly superintegrable system. However, here \( K_{ij,k} \) is not trace-free. Therefore, we need to subtract the trace, obtaining

\[
K^a_{a,k} = \frac{2}{3} (t^m g^n_k - T_{k}^{mn}) K_{mn}. \tag{5.3}
\]

Next, verify that

\[
\begin{aligned}
\left( \frac{j}{k} \right)^i_j T_{j}^{mn} g_k^n - \frac{2}{n} g_{ij} (t^m g^n_k - T_{k}^{mn})
\end{aligned}
\]

which, combined with (2.8) and (5.3), yields

\[
C_{ij,k} = \frac{1}{3} \left( \frac{j}{k} \right)^i_j T_{j}^{mn} g_k^n - \frac{2}{n} g_{ij} (t^m g^n_k - T_{k}^{mn}) \right) C_{mn}
\]

where \( C_{ij} = K_{ij} - \frac{1}{n} g_{ij} K^a_a \). Summarizing, we have thus confirmed that the trace-free part \( C_{ij} \) of a properly superintegrable Killing tensor \( K_{ij} \) satisfies (5.1).

**Proof of Theorem 5.1.** We decompose \( C_{ij,k} \) as

\[
C_{ij,k} = \frac{1}{3} \left( \frac{j}{k} \right)^i_j C_{ij,k} + \frac{1}{6} \left( \frac{j}{k} \right)^i_j \omega_{k} g_{ij} \tag{5.4}
\]

The totally symmetric component is given by the conformal Killing equation,

\[
\left( \frac{j}{k} \right)^i_j C_{ij,k} = \left( \frac{j}{k} \right)^i_j (T_{j}^{kl} C_{lk} + g_{ik} \omega_{k})
\]

The hook symmetric component is obtained as follows: Substituting the Wilczynski equation (3.15) into the Bertrand-Darboux equation (3.8) gives

\[
\left( \frac{j}{k} \right)^i_j \left( \left( C_{j}^{m,k} - T_{j}^{m} C_{k}^{l} + \omega_{j} g_{lm} \right) V_{m} + \left( C_{m}^{k} \tau_{km} + \omega_{j,k} \right) V \right) = 0. \tag{5.5}
\]

From non-degeneracy it follows that the coefficients of \( V_{m} \) and \( V \) vanish independently. The coefficient of \( V \) yields (5.2). From the coefficients of \( V_{m} \) we obtain

\[
\left( \frac{j}{k} \right)^i_j \omega_{k} g_{ij} (T_{j}^{kl} C_{lk} + g_{ik} \omega_{k})
\]

Altogether, using (5.4),

\[
C_{ij,k} = \frac{1}{3} \left( \frac{j}{k} \right)^i_j \left( T_{j}^{kl} C_{lk} + g_{ik} \omega_{k} \right) + \frac{1}{6} \left( \frac{j}{k} \right)^i_j \omega_{k} g_{ij}
\]

The trace-freeness of \( C_{ij} \) now implies

\[
\omega_{k} = -\frac{1}{3n} g^{ij} \left( \frac{j}{k} \right)^i_j T_{j}^{mn} C_{mk} = \frac{2}{3n} (T_{k}^{ab} - t^{i} g_{ik}) C_{ab}, \tag{5.6}
\]

which completes the proof. \( \square \)

Equation (5.2) allows us to prove the converse of Lemma 3.17 for non-degenerate systems. Note that there is a natural mapping from the space \( K \) of Killing tensors into the space \( C^0 \) of trace-free conformal Killing tensors,

\[
K \rightarrow C^0, \quad K_{ij} \mapsto C_{ij} = K_{ij} - \frac{1}{n} K^a_a g_{ij}.
\]

This map is not surjective as not every conformal Killing tensor arises from a proper Killing tensor. Its range consists of trace-free conformal Killing tensors whose \( \omega \) from (3.2) is exact, \( \omega = d\lambda \), and thus

\[
\{ C \in C^0 : 2C_{a,k}^{a} + C_{a,k}^{a} = \lambda, k \text{ for some scalar } \lambda \} \rightarrow K, \quad C \mapsto C - \frac{1}{n} \lambda g,
\]
is surjective. It is not injective as we may always add a constant multiple of the metric to a Killing tensor. From (5.2) we infer that \( \omega \) is exact for trace-free conformal Killing tensors that commute with \( \tau \).

**Corollary 5.3.** If \( \tau_{ij} = 0 \) for a non-degenerate second order conformally superintegrable system, then the system is properly superintegrable.

Note that in the proof we do not take functional independence into account yet, but we will account for it in Lemma 6.8. This lemma ensures the existence of sufficiently many functionally independent conformal integrals for almost any potential of a non-degenerate system, which suffices here as we consider the space \( V_{\text{max}} \).

**Proof.** Let \( C_{ij} \) be a trace-free conformal Killing tensor of the conformally superintegrable system. We need to find a function \( \lambda \) such that \( K_{ij} = C_{ij} + \frac{1}{n} g_{ij} \lambda \) is a proper Killing tensor, i.e. it satisfies the Bertrand-Darboux condition (2.6). We proceed in two steps. First we show that \( d \omega = 0 \). Then we prove that this leads to a properly superintegrable system.

For the first step, take the coefficient of \( V \) in (5.5). For a non-degenerate system (5.2) yields

\[
2d\omega_{ij} = \frac{i}{j} \omega_{i,j} = \frac{i}{j} C^a_{ji} \tau_{ia} = \frac{i}{j} K^a_{ij} \tau_{ia}
\]

and therefore \( \omega \) is exact if \( \tau_{ij} = 0 \), i.e. \( \omega = d\lambda \). Let \( K_{ij} = C_{ij} + \frac{1}{n} g_{ij} \lambda \) with this specific function \( \lambda \).

We conclude, using the trace-freeness of \( C_{ij} \),

\[
(d(KdV))_{ij} = \frac{1}{2} \sum_{i,j} (K_{ia,j} V^{a} + K^{a}_{i} V_{aj}) = 0,
\]

due to the conformal Bertrand-Darboux condition (3.8). So \( K_{ij} \) satisfies the proper Bertrand-Darboux equation (2.6). This proves the claim. \( \square \)

To allow for a concise notation, we introduce the shorthand

\[
P_{mn}^{ijk} := \frac{1}{6} m n (T_{jik} - T^{m}_{ji} g_{nk} - 2 n g_{ij} (T_{m}^{n} g^{k} - T^{m}_{k} g^{n})).
\]

Consequently, we have (5.1) in the form

\[
K_{ij,k} = P_{ijk}^{ab} K_{ab}.
\]

Given \( T_{ijk} \), we can compute \( P_{ijk}^{mn} \). The following lemma shows that \( P_{ijk}^{mn} \) contains all the information about \( \tau_{ij} \), i.e. for abundant systems the secondary structure tensor is redundant.

**Lemma 5.4.** In an abundant conformally superintegrable system, the tensor \( \tau_{ij} \) is given by

\[
\tau_{ij} = \frac{2}{n} \left( \Lambda^{a}_{j,a,i} - \Lambda^{a}_{ij} c^{a} + \Lambda^{ab}_{ij} P_{ab,cj} + \Lambda^{ab}_{ij} P_{abcj} \right),
\]

where

\[
\Lambda^{ab}_{ij} = \frac{1}{3n} \Lambda^{ab}_{ijkl} (T_{ijkl} - 2 t^{a} g_{ik}^{b}).
\]

Moreover,

\[
\sum_{i,j} P_{i,j}^{mn} (\Lambda_{mnj} + \Lambda_{ij} P_{abmn}) = 0.
\]

Note that \( \tau_{ij} \) in (5.9) is symmetric and trace-free due to (5.10). Because of Equation (5.9) the superintegrable potential is completely determined by the primary structure tensor \( T_{ij}^{k} \), and this observation can be interpreted as follows: Any conformally superintegrable system corresponds to a properly superintegrable system [Cap14], for which the Wilczynski equation (3.15) holds with \( \tau_{ij} = 0 \). Applying a conformal transformation, due to (3.20b) the tensor \( \tau_{ij} \) can only contain information from the properly superintegrable system (and the conformal factor). Indeed, this is the information appearing on the right hand side of Equation (5.9).
Proof of Lemma 5.4. Equation (5.2) yields the antisymmetric part of $\omega_{i,j}$, 
\[ \frac{\partial}{\partial j} \omega_{i,j} = \frac{\partial}{\partial j} C^{m} \tau_{im}. \] (5.11)
On the other hand we obtain from (5.6), after one differentiation, 
\[ \omega_{i,j} = \left( \Lambda_{i}^{mn} \right)_{j} + \Lambda_{i}^{ab} P_{abj}^{mn} \right) C_{mn} \]
Resubstituting into (5.11),
\[ \frac{\partial}{\partial j} \left( \Lambda_{i}^{mn} \right)_{j} + \Lambda_{i}^{ab} P_{abj}^{mn} + \frac{\partial}{\partial j} \tau_{im} g^{n} \right) C_{mn} = 0. \] (5.12)
Next, using the fact that there are \( \frac{n(n+1)}{2} - 1 = r_{\text{max}} \) linearly independent, trace-free and symmetric $C_{ij}$, we conclude that the symmetrisation of the coefficients of $C_{mn}$ must vanish independently,
\[ \frac{\partial}{\partial j} \left( \Lambda_{imn,j} + \Lambda_{i}^{ab} P_{abjmn} + \tau_{im} g_{n} \right) \right) C_{mn} = 0. \]
Contracting in $(n, j)$ yields (5.9). Contracting (5.12) in $(n, m)$ shows that (5.9) is the only independent trace of (5.12). The trace-free part of (5.12) is (5.10), and this completes the proof. □

Corollary 5.5. In an abundant system, the structure tensors can be recomputed from $P_{ijk}^{mn}$ defined in (5.8). We have
\[ t_{k} = -\frac{3}{n} P_{ab}^{iab} \]
\[ S_{ijk} = \frac{3}{n} \left( P_{ijk}^{a} + \frac{n-1}{n} g_{ij} P_{abk}^{ab} + \frac{n-2}{n} g_{ik} P_{abj}^{ab} \right), \]
which yield $T_{ijk}$, and (5.9), which yields $\tau_{ij}$.
Proof. This follows from $P_{ab}^{iab} = -\frac{n}{3} t_{i},$ and
\[ P_{ijk}^{a} = \frac{n}{3} S_{ijk} + \frac{n-1}{3} g_{ij} t_{k} + \frac{n-2}{3} g_{ik} t_{j}. \]
Together with Lemma 5.4 the claim follows. □

5.2. Integrability conditions in an abundant system. A trace-free conformal Killing tensor in an abundant conformally superintegrable system satisfies the prolongation system (5.1). Due to the condition of abundantness, its integrability condition will only depend on $g, T$ and $\nabla T$. We have already seen that non-degeneracy is the condition for the generic integrability of $V$. Along a similar line, abundantness is then the condition for the generic integrability of $K_{ij}$.

Proposition 5.6. For the trace-free conformal Killing tensor fields in an abundant (conformally) superintegrable system, the integrability condition of (5.1) reads
\[ \frac{R_{m}^{nkl}}{\Gamma_{k}} \left( P_{ijk}^{mn} + P_{ipj}^{pq} P_{pql}^{mn} \right) = \frac{1}{2} \frac{\partial}{\partial j} \frac{\partial}{\partial k} R_{m}^{nkl} g_{j}. \] (5.13)
Note that the integrability conditions (5.13) are not conformally invariant. This is entirely analogous to (4.5), which are not invariant either. However, our further analysis is going to show that we can distill invariant conditions out of (5.13) and, as we shall see, these already imply (4.5).

Proof. Writing
\[ C_{ij,k} = P_{ijk}^{mn} C_{mn}, \] (5.14)
and taking the covariant derivative yields
\[ C_{ij,kl} = P_{ijk}^{mn} C_{mn} + P_{ijk}^{mn} C_{mn,l}. \]
After antisymmetrisation over \((k, l)\) we can eliminate all derivatives of \(C\) by using the Ricci identity

\[
\frac{k}{l} \ C_{ijk,kl} = \frac{i}{j} \ R_{ikl}^m C_{mj}
\]
on the left hand side, and substituting (5.14) for \(C_{mn,l}\) on the right hand side. We obtain

\[
\frac{i}{j} \ R_{ikl}^m g^n j C_{mn} = \frac{k}{l} \left( P_{ijkl}^{mn,i} + P_{ijkl}^{pq} P_{pql}^{mn} \right) C_{mn}.
\]

(5.15)

An abundant conformally superintegrable system has \(\frac{n(n+1)}{2} - 1 = r_{\max}\) linearly independent trace-free conformal Killing tensors \(C\). Since this is exactly the number of independent components of the trace-free symmetric tensor \(C_{mn}\), we can replace \(C_{mn}\) by a symmetrisation in \(m\) and \(n\) as the expression in parentheses in (5.15) is already trace-free in \(m\) and \(n\). We have thus obtained the claim.

\[\square\]

**Lemma 5.7.** For an abundant system, the curvature tensor \(R^i_{ijk}\) satisfies

\[
R^i_{ijk} = \frac{2}{n+2} \ \frac{i}{k} \ \frac{j}{l} \ \left( P_{iaj}^{\ \ la,k} + P_{iaj}^{\ pq} P_{pql}^{\ la} \right).
\]

(5.16)

**Proof.** Contracting (5.13) in \(n\) and \(j\) immediately yields the result. \[\square\]

Lemma 5.7 allows us to express the curvature in terms of the superintegrable structure tensor. Alternatively we can also view it as a curvature obstruction to the structure tensor. In any case, it enables us to (almost) eliminate the curvature from the integrability conditions.

**Lemma 5.8.** An abundant conformally superintegrable system satisfies the curvature independent equation

\[
\frac{k}{l} \left( P_{ijkl}^{\ \ mn,i} + P_{ijkl}^{\ pq} P_{pql}^{\ mn} \right) = \frac{i}{j} \ \frac{m}{n} \ \frac{k}{l} \ \left( P_{iaj}^{\ \ ma,l} + P_{iaj}^{\ pq} P_{pql}^{\ ma} \right) \ \frac{g^n_{j}}{n+1}.
\]

(5.17)

The proof is straightforward.

By a tedious computation, the following is confirmed:

\[\square\]

**Corollary 5.9.** For an abundant system, (5.16) and (5.17) imply (4.5).

Therefore, for abundant systems, the integrability conditions for the potential \(V\), its trace-free conformal Killing tensors \(C_{ij}\) and their respective scalar parts \(W\) are equivalent to (5.9), (5.10), (5.16) and (5.17).

5.3. **Non-linear prolongation equations for the structure tensor.** We have found the prolongation (4.1) for the potential and the prolongation (5.1) for the trace-free conformal Killing tensors in a conformally superintegrable system. Now we show that these imply a third, non-linear prolongation for the structure tensor \(T_{ijk}\), which expresses covariant derivatives of \(T_{ijk}\) polynomially in terms of \(T_{ijk}\) and the Ricci tensor \(R_{ij}\).

**Proposition 5.10.** For an abundant conformally superintegrable potential in dimension \(n \geq 3\), the primary structure tensor \(T_{ijk}\), decomposed according to (4.7) as

\[
T_{ijk} = S_{ijk} + \frac{i}{j} \ g_{jk} - \frac{1}{n} g_{ij} \ t_{ik}
\]
satisfies the following non-linear prolongation:

\[
\nabla_i t_j \ = \ \frac{3}{4(n-2)} \ R_{ij} + \frac{1}{4(n-2)} S_{ij}^{ab} S_{ab} + \frac{1}{3} t_{ij} a + \frac{1}{n} g_{ij} \nabla^a a
\]

(5.18a)

\[
\nabla^a a = \frac{3}{2(n-1)} R + \frac{3}{6(n-1)(n+1)} S_{abc} S_{abc} - \frac{(n-2)}{6} t^a a
\]

(5.18b)

\[
\nabla_i S_{ijk} = \frac{1}{18} \ \frac{i}{j} \ \frac{j}{k} \ \left( S_{il}^a S_{jka} + 3 S_{ijl} t_k + S_{ijk} t_l + \left( \frac{4}{n-2} S_{ij}^{ab} S_{kab} - 3 S_{jka} \right) g_d \right)
\]

(5.18c)
Remark 5.11. Remarkably, the system (5.18) is the same as the one found for properly superintegrable systems in [KSV19], with the curvature term in (5.18a) replaced using (5.9). Here we leave the curvature term, in order to not re-introduce $\tau_{ij}$ into (5.18).

Proof of Proposition 5.10. The proof is analogous to the corresponding result in [KSV19], but we provide more details here. Using (5.17), define

$$E_{ijk}{}^{mn} l = (n+1) \left[ \frac{k}{l} \left( P_{ijk}{}^{mn} l + P_{ijk}{}^{pq} P_{pql}{}^{mn} \right) - \frac{m}{n} \right] \left( P_{iak}{}^{ma} l + P_{pq}{}^{iak} P_{ma}{}^{pql} \right) g^a_j ,$$

where we observe that $E_{ijk}{}^{mn} l$ is symmetric in $(i,j)$ and in $(m,n)$, antisymmetric in $(k,l)$, and pure trace. Its trace components are given by

$$E_{kml}^{(1)} = E_{akml}^a , \quad E_{ijkl}^{(2)} = E_{jia}^{mn} l , \quad E_{ijnl}^{(3)} = E_{ija} a nl .$$

These are not independent, satisfying the relation

$$2(n-2) \left( E_{ijkl}^{(2)} - E_{ijnl}^{(3)} \right) = (n^2 - 2n - 2) E_{iklj}^{(1)} .$$

This equation allows us to express the trace-free part of $\nabla_i S_{ijkl}$ in terms of $T_{ijkl}$ and the metric, yielding the trace-free part of (5.18c). The trace part $\nabla^a S_{ij} a$ is obtained from $E_{ij} a$, which implies

$$\nabla^a S_{ij} a = \frac{2n}{3(n-2)} S^{ik} a S_{jka} - \frac{n}{3} S_{ija} t a - \frac{2}{3(n-2)} g_{ij} S^{abc} S_{abc} .$$

This confirms (5.18c). The two other equations, (5.18a) and (5.18b), are obtained from (5.16) by contraction. Contracting (5.16) in $i$ and $k$, we find (5.18a). Contracting (5.16) further, in $(i,k)$ as well as $(j,l)$, we find (5.18b). □

Remarkably, the system (5.18) is already conformally invariant, which we show in detail Section 6.4. Indeed, in Equations (5.18a) and (5.18b) the terms involving $\bar{t}$ absorb the transformation behaviour of $R_{ij}$ under (3.25).

5.4. The integrability conditions for abundant systems. In addition to the integrability conditions for Killing tensors in abundant systems, we have two more equations, namely (5.9) and (5.10). Note that only (5.9) involves the secondary structure tensor $\tau_{ij}$, as it allows us to express $\tau_{ij}$ in terms of the structure tensor $T_{ijkl}$.

Lemma 5.12.

(i) The non-linear prolongation (5.18) implies Equation (5.10).

(ii) For abundant systems in dimension $n \geq 3$, the equations of the non-linear prolongation (5.18), together with

$$W_{ijkl} = \frac{k}{l} S_{ik} a S_{jia} = 0 ,$$

are equivalent to the integrability condition (5.6).

(iii) With (5.18), Equation (5.9) becomes

$$3(n-2) \left( \frac{1}{2} \tau_{ij} + \bar{P}_{ij} \right) = \left( (n-2)(S_{ija} t a + \bar{t} \bar{t}_j) - S^{ik} a S_{jka} \right) .$$

Proof. For the first part simply resubstitute (5.18) into (5.10). The proof of part (ii), namely of Equation (5.19), is analogous to that of Theorem 5.9 in [KSV19]. Finally, for part (iii), resubstitute (5.18) into Equation (5.9). We obtain (5.20). □

As an immediate consequence of (5.18) and (5.19), we obtain the following obstruction on the geometry underlying an abundant system.

Corollary 5.13. Abundant conformally superintegrable systems can only exist on conformally flat manifolds.
Proof. It follows immediately from (5.19) that a conformally superintegrable system can only exist on a Weyl flat manifold. Therefore, for dimension \( n \geq 4 \), they can exist only on conformally flat manifolds. In dimension 2, any metric is conformally flat. We are therefore left with the case \( n = 3 \). Using standard scale, i.e. \( \tilde{t} = 0 \), Equation (5.18c) yields that

\[
S_i^{ab} S_j^{ab} - \frac{3}{20} S^{abc} S_{abc} g_{ij}
\]

is a Codazzi tensor. The claim then follows from the Weyl-Schouten Theorem. \( \square \)

6. Equivalence classes of abundant superintegrable systems

So far, we have considered conformally superintegrable systems whose underlying geometry is a \((pseudo-)Riemannian manifold\). We now turn towards conformal equivalence classes, i.e. towards c-superintegrable systems on conformal manifolds. For such systems, \( S_{ijk} \) is the conformally invariant structure tensor. According to (5.20) or (5.9), the secondary structure tensor \( \tau_{ij} \) is determined by \( T_{ijk} \) and the Ricci curvature. Table 2 contrasts the setting of properly and conformally superintegrable systems as opposed to c-superintegrable systems.

| Type | Proper superintegrability | Conformal superintegrability | C-superintegrability |
|------|--------------------------|-------------------------------|--------------------|
| Geometry | pseudo-Riem. metric | pseudo-Riem. metric | conformal metric |
| Constants of motion | (proper) integrals | conformal integrals |
| Primary structure tensor | \( T_{ijk} = S_{ijk} + \tilde{t} g_{ijk} \) | \( S_{ijk} = \tilde{T}_{ijk} \) |
| Secondary structure tensor | \( \tau_{ij} = 0 \) given by (5.20) | none (not conformally invariant) |

Table 2. Synopsis of the main objects in proper, conformal and c-superintegrability.

6.1. Obstructions to the integrability of the non-linear prolongation (5.18). Consider the non-linear prolongation (5.18) of PDEs for \( \tilde{t} \) and \( S_{ijk} \). We now investigate the integrability conditions for this system. The prolongation equations (5.18) are non-linear in the components of \( S_{ijk} \) and \( \tilde{t}_i \). Therefore the Ricci conditions,

\[
\begin{align}
\tilde{t} \nabla_m \nabla_i S_{ijk} &= \tilde{t} \tilde{t} \tilde{L}_k R_{\alpha m} S_{\alpha jk} \\
\tilde{t}_k \nabla_j \tilde{t}_i &= R_{\alpha jk} \tilde{t}_i,
\end{align}
\]

are necessary but need not be sufficient for the integrability of (5.18). Sufficiency is guaranteed if not only the Ricci condition but also all of its differential consequences are satisfied in a given point \( x_0 \) \[Gol67\]. We find that the integrability conditions, of which a priori there can be infinitely many, reduce to a single algebraic equation of the form (1.3).

**Theorem 6.1.**

(i) If Equation (5.19) holds, then the integrability conditions (6.1) of (5.18) are satisfied.

(ii) Let \( M \) be a conformally flat, pseudo-Riemannian manifold of dimension \( n \geq 3 \) and let \( x_0 \in M \) be a point on this manifold. Then any solution \( \Psi_{ijk} = S_{ijk}(x_0) \) of (1.3) together with the arbitrary
initial values $\bar{t}(x_0)$ and $\nabla \bar{t}(x_0)$ can be extended, in a neighborhood of $x_0$, to solutions $S_{ijk}(x)$ and $\bar{t}(x)$ of the non-linear prolongation (5.18).

**Proof.** (i) Since the integrability condition cannot depend on $\bar{t}$, we may w.l.o.g. perform a conformal transformation such that the transformed system is in standard scale, $\bar{t} = 0$. As a result, (5.18a) and (5.18b) determine the Ricci curvature tensor in this scale. We shall comment on this after finishing the proof.

First, let us investigate the integrability condition for (5.18c), which by virtue of the aforementioned transformation has turned into

$$\nabla_l S_{ijk} = \frac{1}{18} \epsilon_{ijk} \left( S^a_{il} S^b_{jka} + \frac{4}{n-2} S^a_{j} S^b_{ka} g_{il} \right).$$  \hspace{1cm} (6.2)

Using (5.18a), (5.18b) and Weyl-flatness (5.19), we replace the Riemann curvature in (6.1a) by a quadratic expression in $S$. Due to (5.19) in combination with the non-linear prolongation (5.18) and (5.20), Equation (6.1a) is equivalent to the conformally invariant condition

$$\epsilon_{jk} S_{abc} S_{ija} S_{kbc} = 0,$$

which is confirmed to be an algebraic consequence of (5.19) by way of contracting (5.19) with $S$. We have therefore established (5.19) as the only first order integrability condition of (5.18c). By an analogous computation we then confirm that all Ricci identities of (5.18) are satisfied if (5.19) holds. As explained earlier, however, the first order integrability conditions (6.1) need not be sufficient for the integrability of (5.18). We now proceed to show their sufficiency.

(ii) In order to find sufficient, pointwise integrability criteria, higher order integrability conditions have to be taken into account. Concretely, all differential consequences of (5.19) need to be satisfied in a fixed point $x_0$ in order to allow us to extend $S_{ijk}$ and $\bar{t}$ such that the extensions satisfy (5.18) in a neighborhood of $x_0$. Taking a covariant derivative of (5.19) and replacing derivatives of $S_{ijk}$ by (6.2), we find an algebraic condition on $S_{ijk}$. Using (5.19) again, it can be verified that this second order condition is an algebraic consequence of the first order one, i.e. of (5.19). This concludes the proof. □

If instead of a pseudo-Riemannian manifold we consider a conformal manifold, we obtain the following statement.

**Corollary 6.2.** Consider a flat conformal manifold $(M, c)$ of dimension $n \geq 3$. Then a solution of (1.3) can be extended to a totally symmetric and trace-free tensor field $S_{ijk}(x)$ in a neighborhood of a point $x_0 \in M$ such that there exists $g \in c$ and (6.2) holds where $\nabla$ is the Levi-Civita connection of $g$.

**Proof.** Note that (1.3) is an invariant condition. The statement follows from statement (ii) in the theorem, making use of the fact that there exists a flat conformal scale choice, which removes the curvature from the non-linear prolongation system (5.18). If (1.3) holds in $x_0 \in M$ for this case, the system is integrable in a neighborhood of $x_0$. Note that the specific scale choice is technical and irrelevant for the statement. □

**Corollary 6.3.** In dimension $n = 3$, (5.18) can be integrated for any $S_{ijk}(x_0)$.

**Proof.** Note that (5.19) has Weyl symmetry and thus vanishes trivially in dimension $n = 3$. □

**Remark 6.4.** The corollary is consistent with the existing literature. For dimension 3, [Cap14] finds that conformally superintegrable systems lead to the irreducible representations $\bigcirc$ and $\Box$ of the rotation group. Specifically, these representations are of dimension seven and three. These correspond to our tensors $S_{ijk}$ and $t_i$ (or $t_i$), respectively. Moreover, this reference shows that every element of this 7-dimensional representation of the rotation group corresponds to at least one superintegrable system on a 3-dimensional conformally flat geometry, in line with the statement of Corollary 6.3.
6.2. The conformal factors between standard scale, proper scale and flat scale. In the standard scale $\bar{t} = 0$, Equations (5.18a) and (5.18b) become algebraic conditions on the curvature tensor. In contrast, if a flat scale exists, exactly the curvature terms disappear. In an arbitrary scale choice we have the following formula for the curvature, in terms of the conformally invariant tensor $S_{ijk}$ and the conformal scale function $\sigma$.

**Proposition 6.5.** The Ricci tensor satisfies

$$R_{ij} = -\frac{1}{9} \left( S_{iab} S_{jab} + \frac{n}{3(n+2)} g_{ij} S^{abc} S_{abc} \right) + (n-2) \left( \frac{\nabla_{ij}^2 \sigma}{\sigma} - \frac{4(n-1)}{n(n-2)^2} \frac{\Delta \sigma^{n-2}}{\sigma^{n-2}} g_{ij} \right)$$

(6.3)

where $\sigma = \exp(-\frac{1}{3} \bar{t})$ is the conformal scaling from Definition 4.7.

**Proof.** Solve (5.18a) and (5.18b) for the Ricci tensor. □

Note that on the right hand side of (6.3), the first term is invariant (up to a conformal factor), while the second term vanishes in standard scale. In a flat scale, $R_{ijkl} = 0$, and the left hand side vanishes. The function $\bar{t} = \rho$ arising from the primary structure tensor measures ‘how far’ the flat scale is from the standard scale. According to (4.12), the conformal scale factor between the two scales is

$$\theta := e^{-\frac{1}{3} \rho}$$

(6.4)

(note that now we transform from the scale defined by $h_{ij}$ back to the standard scale with metric $g_{ij}$). From (6.3) we see that the flat conformal scales $\theta$ are determined by $S_{ijk}$. The following lemma makes this explicit.

**Lemma 6.6.** For any conformal class of abundant superintegrable systems there are functions $\theta > 0$ satisfying

$$\nabla_{ij}^2 \theta = -\frac{(S_{iab} S_{jab})_c}{9(n-2)} \theta$$

(6.5a)

$$\Delta \theta^{1-\frac{2}{n}} = \frac{(n-2)(3n+2)}{36(n-1)(n+2)} S^{abc} S_{abc} \theta^{1-\frac{2}{n}}.$$

(6.5b)

where $\nabla^2$ is the flat, trace-free Hessian and $\Delta$ the flat Laplace operator, and where $S_{ijk}$ is the conformal structure tensor in the flat scale.

**Proof.** Let the function $\bar{t} = \rho$ be the trace of the structure tensor after a conformal transformation to a flat scale. Then a conformal transformation with $\Upsilon = -\nabla^2 \rho$ takes us back to the standard scale. Rewriting (5.18a) and (5.18b) in terms of $\theta$, and decomposition of the result into its trace-free and trace parts, confirms the claim. □

Note that, by Corollary 5.13, solutions $\theta$ do exist. However, solutions are not unique. Indeed, a positive constant scalar multiple of $\theta$ is again a solution, and in general more solutions can exist. For instance, in Table 3 the systems III and V are both in flat scale, and they are conformally equivalent, see [Cap14]. Two different solutions $\theta$ represent two different flat conformally superintegrable systems within the same conformal class. They share the same $\bar{T}_{ij} = S_{ij}$ but the traces of $T_{ijk}$ will be different unless the $\bar{t}$ differ only by an additive constant. In order to understand the space of solutions $\theta$ better, let us study (6.5a) further, ignoring the additional constraint (6.5b) for a moment. We find:

**Lemma 6.7.** Equation (6.5a) has the linear prolongation

$$\nabla_{ij}^2 \theta = -\frac{(S_{iab} S_{jab})_c}{9(n-2)} \theta + \frac{1}{n} g_{ij} \Delta \theta$$

(6.6a)

$$(\Delta \theta)_k = \frac{n}{9(n-2)} S^{abc} S_{abc} \theta + \frac{3n+2}{27(n-1)(n-2)} S_{k ab} S_{a cd} S_{bcd} \theta$$

(6.6b)
where $\nabla^2$ is the flat Hessian and $\Delta$ the flat Laplace operator, and where $S_{ijk}$ is the conformal structure tensor in the flat scale. The integrability conditions for (6.6) are equivalent to (5.19).

Proof. Equations (6.6) are obtained in formal analogy to (4.1), where formally $T \equiv 0$. Its integrability conditions are satisfied due to (5.19), as (6.6a) is a special case of (5.18a).

Equations (6.6) are a linear prolongation system for $\theta$ and its coefficients are determined by $S_{ijk}$. Therefore the solutions $\theta$ lie in an (at most) $(n+2)$-dimensional linear space $\mathcal{B}$, determined by the values of $\Delta \theta, \nabla \theta$ and $\theta$ in a fixed point. The additional constraint (6.5b) defines a quadric in $\mathcal{B}$.

6.3. Classifying the conformal classes of conformally superintegrable systems. In the previous section we have found algebraic integrability conditions whose form is the same for any conformally superintegrable system within a class (in the next section we reformulate them as equivariant conditions). As initial data we need to specify $\Psi_{ijk} = S_{ijk}(x_0)$. For a conformally flat geometry we may choose a flat metric, which facilitates determining a solution for $S_{ijk}$. In order to reconstruct an abundant conformally superintegrable system from the initial data $\Psi_{ijk} = S_{ijk}(x_0)$, we recall that an abundant superintegrable system requires a conformally flat metric $g = \sigma^2 h$ where $h$ is the flat metric. Then we can use the following procedure.

(i) Let $\Psi_{ijk} = S_{ijk}(x_0)$ be the initial data given in a point $x_0$. Assume that $\Psi_{ijk}$ solves the algebraic condition (1.3). The $\Psi_{ijk}$ do not depend on $\phi$, and if $\Psi_{ijk}$ are solutions then so are $k \Psi_{ijk}$ for $k \neq 0$.

(ii) We extend the initial data $\Psi_{ijk}$ to a solution in a neighborhood of $x_0$ such that the nonlinear prolongation (5.18) holds. This is possible by virtue of Corollary 6.2. For a concrete computation we should choose some conformal scale, and the flat scale is a reasonable choice. Then we need to specify the initial data $\nabla \rho(x_0)$ and $\rho(x_0)$ in addition to $\Psi_{ijk}$.

(iii) This yields $T_{ijk}$ up to a conformal transformation. Integrating the Wilczynski equation (3.15) for $V$ in the specific scale given by $\ell = \rho$, and computing $v = \epsilon^2 \nabla V$, we find the conformally invariant potential as an $(n + 2)$-parameter family of densities of weight $-2$. This is the space $V^{\text{max}}$. The space $C^{\text{max}}$ of conformal Killing tensors is similarly obtained by integration of (5.1).

Since all integrability conditions are satisfied generically, we find at least $\frac{1}{2} n(n+1)$ many linearly independent conformal integrals. We address their functional independence below in Lemma 6.8.

The procedure just outlined allows one to reconstruct an abundant c-superintegrable system from the given initial data and the knowledge of the underlying conformal metric up to the choice of the potential from $V^{\text{max}}$. We recall Assumption 3.6, but remark that with Lemma 6.8 below, we are able to restrict the space $C^{\text{max}}$ in order to obtain $2n - 1$ functionally independent conformal integrals. Let us reinterpret the aforesaid in the light of classifying c-superintegrable systems. In [KSV19, Theorem 6.4] it is shown that the classification space for irreducible non-degenerate superintegrable systems on a (pseudo-)Riemannian manifold $M$ with analytic metric is a quasi-projective subvariety $\mathcal{U} \subset G_{2n-1}(\mathcal{K}(M))$ in the Grassmannian of $(2n-1)$-dimensional subspaces in the space $\mathcal{K}(M)$ of Killing tensors on $M$. Since any c-superintegrable system admits at least one system in proper scale, it follows that the classification space of irreducible non-degenerate c-superintegrable systems with analytic metric is the quotient $\mathcal{U} = \mathcal{U} / \text{Conf}(M)$. For non-degenerate irreducible conformally superintegrable systems on analytic metrics, the classification space is a fibre bundle over $\mathcal{U}$.

The following lemma was proven for properly superintegrable systems in reference [KSV19]. We adapt it for conformal systems.

Lemma 6.8. Let $C^{(\alpha)}$ be $2n-2$ linearly independent, trace-free conformal Killing tensors satisfying the integrability conditions (5.15) for (5.1), and (4.5) for (4.1). Then, in the linear space $V^{\text{max}}$ of solutions $V$ to Equation (4.1), those $V$ that give rise to functionally dependent integrals are confined to an affine subspace of $V$ with non-empty complement.
Proof. Suppose the integrals (3.4) were functionally dependent. Then there is a function \( \varphi : \mathbb{R}^{2n-2} \to \mathbb{R} \), non-zero in an open subset of its domain, such that
\[
\varphi(F^{(1)}, \ldots, F^{(2n-2)}) = 0
\]
(6.7)
This implies the infinitesimal condition
\[
\sum_{\alpha=1}^{2n-2} \lambda_{(\alpha)} dF^{(\alpha)} = 0,
\]
(6.8)
where
\[
\lambda_{(\alpha)} = \frac{\partial \varphi}{\partial F^{(\alpha)}}(F^{(\alpha)})
dF^{(\alpha)} = \frac{\partial F^{(\alpha)}}{\partial x^k} dx^k + \frac{\partial F^{(\alpha)}}{\partial p^k} dp^k.
\]
By a direct computation we find
\[
\frac{\partial F^{(\alpha)}}{\partial x^k} = C^{(\alpha)}_{ij,k} p^i p^j + V^{(\alpha)}_{,k},
\]
\[
\frac{\partial F^{(\alpha)}}{\partial p^k} = 2C^{(\alpha)}_{jk} p^j.
\]
Separating the components of (6.8) and substituting (3.7), we conclude
\[
\sum_{\alpha} \lambda_{(\alpha)} \left( C^{(\alpha)}_{ij,k} p^i p^j + C^{(\alpha)}_{jk} V^{(\alpha)} + \omega^{(\alpha)}_k V \right) = 0
\]
(6.9a)
\[
\sum_{\alpha} \lambda_{(\alpha)} C^{(\alpha)}_{jk} p^j = 0.
\]
(6.9b)
Invoking (5.1), we obtain
\[
C^{(\alpha)}_{ij,k} p^i p^j = \frac{2}{3} \left( T^a_{ij} C^{(\alpha)}_{ka} - T^a_{kj} C^{(\alpha)}_{ia} \right) p^i p^j - \frac{2}{3n} \left( T^a_{k ab} C^{(\alpha)}_{ab} - t^a C^{(\alpha)}_{ab} \right) p^i p_c.
\]
Multiplying with \( \lambda_{(\alpha)} \) and summing over \( \alpha \), we find, using (6.9b) and the decomposition (4.7),
\[
\sum_{\alpha} \lambda_{(\alpha)} C^{(\alpha)}_{ij,k} p^i p^j = \frac{2}{3} \sum_{\alpha} \lambda_{(\alpha)} \left( S^a_{ij} C^{(\alpha)}_{ka} - \frac{1}{n} t^a g_{ij} C^{(\alpha)}_{ka} - \frac{1}{n} S^a_{k ab} C^{(\alpha)}_{ab} g_{ij} \right) p^i p^j.
\]
Substituting this back into (6.9a), invoking (3.2), and using again the decomposition (4.7), we conclude
\[
C_{ab} \eta^{kab} = 0,
\]
(6.10)
where we use the abbreviations
\[
C_{ab} := \sum_{\alpha} \lambda_{(\alpha)} C^{(\alpha)}_{ab}
\]
and
\[
\eta^{kab} = g^{k\alpha} \left( S^b_{ij} p^i p^j - \frac{1}{n} h^b p^c p_c + \frac{3}{2} V^b - \frac{1}{n} \frac{(n+1)(n-2)}{(n-1)(n+2)} t^b V \right) + \frac{1}{n} S^{kab} (V - p^b p_c).
\]
Note that \( C(x_0) \neq 0 \). Indeed, otherwise the \( C_{ab}^{(\alpha)}(x_0) \) would be linearly dependent,
\[
\sum_{\alpha} k_{(\alpha)} C^{(\alpha)}_{ab}(x_0) = 0, \quad k_{(\alpha)} = \lambda_{(\alpha)}(x_0).
\]
Because of (5.1) the derivatives of \( C_{ab}^{(\alpha)} \) are linearly dependent at \( x_0 \), with the same constants \( k_{(\alpha)} \). Iterated application of (5.1) to higher derivatives shows that the same is true for all higher derivatives. It readily follows that \( \sum_{\alpha} k_{(\alpha)} C^{(\alpha)} = 0 \) everywhere, which contradicts the linear independence of the conformal Killing tensors \( C^{(\alpha)} \).

Now, for \( x_0 \in M \), consider the mapping \( \Xi : T^{\otimes 3}_{x_0} M \to T_{x_0} M \), given by contracting with \( C_{ab} \),
\[
\Xi(\eta^{kab}) = C_{ab} \eta^{kab}.
\]
By virtue of Equation (6.10), we conclude that for any potential \( V \in \mathcal{V}^{\max} \)
\[
\eta^{kab}(x_0) \in \ker \Xi.
\]
Using the linearity of the kernel, we conclude further that
\[
\left[ \frac{3}{2} g^{kb} V^a + \frac{1}{n} \left( S_{kb} - \frac{(n+1)(n-2)}{(n-1)(n+2)} g^{kb} a \right) V \right]_{x_0} \in \ker \Xi.
\]
Choosing \( V(x_0) = 0 \), we obtain that
\[
C_k^a V_a(x_0) = 0
\]
for any choice of \( V_a(x_0) \), contradicting that \( C(x_0) \neq 0 \).

Theorem 6.9. Abundant conformally superintegrable Hamiltonians with their \((n+2)\)-parameter family of potentials, and identified under conformal transformations, are classified by (1.3).

Proof. An abundant conformally superintegrable Hamiltonian with its \((n+2)\)-dimensional space \( V_{\max} \) of all compatible potentials, can be recovered from \( S_{ijk} \) up to a conformal transformation of superintegrable systems and every abundant system satisfies (1.3).

6.4. Invariant formulation of the non-linear prolongation equations. In this section we express the non-linear prolongation equations (5.18) in a conformally invariant way.

Proposition 6.10.
(i) Equation (5.18a) is equivalent to
\[
\tilde{H}_{ij} \sigma = -\frac{1}{9(n-2)} (S_{ab}^{ij} S_{ijab}) \sigma, \tag{6.11}
\]
where \( \sigma = e^{-1/6} \) and where \( \tilde{H} \) is the conformally invariant trace-free Hessian, defined by
\[
\tilde{H}_{ij} = (\nabla_i \sigma - P_{ij}) \sigma.
\]
(ii) Equation (5.18b) is equivalent to
\[
\mathbb{L}^{1-\frac{n}{2}} \sigma\sigma^{1-\frac{n}{2}} = -\frac{2}{9} \frac{3n+2}{n+2} S_{abc} S_{abc} \sigma^{1-\frac{n}{2}}, \tag{6.12}
\]
where \( \mathbb{L} \) denotes the conformal Laplacian,
\[
\mathbb{L} = -4 \frac{n-1}{n-2} \Delta + R.
\]
(iii) Equation (5.18c) is equivalent to
\[
\nabla^i S_{ijk} = \frac{1}{3} \left( \frac{1}{m} \right) \left( S_{ij}^{a} S_{jka} - \frac{4}{n-2} g_{kl} S_{ij}^{ab} S_{jka} \right), \tag{6.13}
\]
where \( \nabla^i \) is the conformally equivariant Weyl connection [Wey18] defined by
\[
\nabla^i \alpha_j = \nabla_i \alpha_j - \frac{m+1}{3} \tilde{t}_i \alpha_j - \frac{1}{3} \tilde{t}_j \alpha_i + \frac{1}{3} \tilde{t}^a \alpha_a g_{ij}, \tag{6.14}
\]
for \( \alpha_j \) of conformal weight \( m \), i.e. \( \alpha_j \rightarrow \Omega^m \alpha_j \) under conformal transformations. Here we have \( m = -\frac{2}{3} \).

Proof. Parts (i) and (ii) are straightforward. For part (iii), apply (6.14) to \( S_{ijk} \),
\[
\nabla^i S_{ijk} = \nabla_i S_{ijk} - \frac{1}{18} \left( \frac{1}{m} \right) \left( 3 S_{ij}^{ab} S_{jka} + S_{ij} \tilde{t}_k - 3 g_{kl} S_{ij} \tilde{t}^a \right).
\]
A direct computation indeed confirms that \( \nabla^i \) is invariant under conformal changes up to multiplication by a factor: The replacement rules are \( g \rightarrow \Omega^2 g \) and \( \tilde{t} \rightarrow \tilde{t} - 3 \ln|\Omega| \), as well as, respectively, \( S_{ijk} \rightarrow \Omega^2 S_{ijk} \) or \( \alpha_i \rightarrow \Omega^m \alpha_i \) with \( m = -\frac{2}{3} \).
Two remarks are in place with regard to the above proposition. First, note that the conformal weights required for the trace-free conformal Hessian and the conformal Laplacian are different, leading to different powers of the conformal scale function. The second remark concerns the conformal invariance of the operators. Note that, under conformal transformations with rescale function $\Omega$, we have

$$\sigma^{1-\frac{n}{2}} \to \Omega^{1-\frac{n}{2}} \sigma^{1-\frac{n}{2}},$$

and the conformal invariance of $\mathbb{L}$ means

$$\mathbb{L} \circ \Omega^{1-\frac{n}{2}} = \Omega^{-1} \circ \mathbb{L},$$

which is consistent as $S^{abc}S_{abc} \to \Omega^{-2}S^{abc}S_{abc}$. Note that in the standard scale, i.e. for $\bar{t} = 0$ resp. $\sigma = 1$, Equation (6.12) is an expression for the scalar curvature in terms of $S_{ijk}$, and (6.11) for the Schouten tensor.

**Remark 6.11.** To determine whether the non-linear system (5.18) is integrable, it is sufficient to know the invariants in (6.11), (6.12) and (6.13) as well as those in (6.6), which are constructed algebraically from $S$. These invariants are

$$A_{ijkl} = \begin{bmatrix} i & j & k & l \end{bmatrix}_o (S^a_i S_{jka}) + \begin{bmatrix} i & j & k & l \end{bmatrix}_o (S^a_i S_{jka}),$$

$$B_k = S^a_k S^d_a S_{bcd},$$

$$\Sigma_{ij} = \begin{bmatrix} i & j \end{bmatrix}_o S^a_i S_{jab}.$$

The last of these invariants has a nice geometric interpretation. First, due to (5.18a), we have

$$Q_{ij} = \frac{1}{9(n-2)} \Sigma_{ij},$$

where the Schouten curvature in standard scale is denoted by $Q_{ij}$. The trace of the Schouten curvature satisfies

$$\Psi^a = \frac{3n+2}{18(n-1)(n+2)} \Sigma^a.$$

The second invariant also has a geometric meaning. It is easy to show that

$$\Psi^{a,k} = -\frac{(3n+2)}{27(n-2)(n-1)} B_k,$$

and therefore $B_k = 0$ characterises the case when the standard scale system has constant scalar curvature. We also observe that $B_k$ and $\Sigma^a$ are not (differentially) independent.

### 6.5. Properly superintegrable systems on constant curvature manifolds.

In reference [KSV19] abundant properly superintegrable systems are studied. These systems satisfy $\tau_{ij} = 0$ due to Lemma 3.17 and thus (5.20) becomes

$$[(n-2)(S_{ij} t^a i + i t_j) - S^a i S_{jab}]_o = 3 \bar{R}_{ij}.$$  \hspace{1cm} (6.16)

In the case of a constant curvature metric, the right hand side of this equation vanishes. In [KSV19] the following equation is then proven.

**Lemma 6.12.** On a manifold of constant sectional curvature, (6.16) together with the non-linear prolongation (5.18) implies

$$S^{abc} S_{abc} - (n-1)(n+2) t^a i_a = 9R.$$  \hspace{1cm} (6.17)

Using the condition found in the lemma, we obtain the following conditions for Stäckel equivalent properly superintegrable systems.

**Theorem 6.13.** If (6.17) holds, then Equation (5.18b) becomes

$$\Delta \sigma^{n+2} = -2 \frac{n+1}{n-1} R \sigma^{n+2}$$  \hspace{1cm} (6.18)

where $\sigma = e^{-\frac{i}{t}}$ as in Definition 4.7.
Note that Equation (6.18) is (1.5).

**Proof.** If (6.17) holds, then $S^{abc}S_{abc}$ can be eliminated from Equation (5.18b), yielding

$$\Delta \bar{t} = \frac{6(n + 1)}{(n - 1)(n + 2)} R + \frac{n + 2}{3} \bar{t}^a \bar{t}^a .$$

In terms of $\sigma = e^{-\frac{1}{3} \bar{t}}$, this rewrites as (6.18). \(\square\)

For constant curvature spaces, Equation (5.18b) thus becomes a Laplace eigenvalue problem, and a power of the scale function $\sigma$ is an eigenfunction of $\Delta$. For a flat manifold, (6.18) merely implies that $\sigma^{n+2}$ is harmonic. On the round sphere $S^n \subset \mathbb{R}^{n+1}$, we have spherical harmonics with the quantum number $\mu = n + 1$ satisfying

$$\mu (\mu - n - 1) := 2 \frac{n + 1}{n - 1} R r^2 = 2 n(n + 1) , \quad (6.19)$$

where the second equality follows from $R = \frac{n(n - 1)}{r^2}$ with $r > 0$ denoting the radius of the sphere.

A close connection between the Helmholtz-Laplace equations and conformal superintegrability has been found in [KMS16, KKMP11]. Such links also appear in the present paper, although in different context: Earlier we have seen that on conformally flat spaces we find a scalar function $\theta^{1 - \frac{2}{n}}$ satisfying the generalised Helmholtz equation (6.5b). Now we have found (6.12), which is a conformally invariant generalised Helmholtz equation. In particular, in the case of proper superintegrability, the $(n + 2)$-nd power of the conformal scale function satisfies the generalised Helmholtz equation (6.18). It is a proper Helmholtz equation in the case of constant scalar curvature.

We now use Equation (6.18) to study conformally equivalent properly superintegrable systems further.

**Proposition 6.14.** Assume we are provided with an abundant second order properly superintegrable system on the sphere with the round metric $g$, which is conformally equivalent to a properly superintegrable system on flat space with the flat metric $h = \Omega^{-2} g$. Then the conformal factor $\sigma$ on the sphere has to satisfy

$$\Omega \left( \Delta \Omega - g(\ln(\sigma^{n+2}), d\Omega) \right) + g(d\Omega, d\Omega) = 0 . \quad (6.20)$$

Note that $\Omega$ is the conformal factor mediating between the standard and the spherical scale, while $\sigma$ mediates between the spherical and the flat scale.

**Proof.** Due to (6.18), a properly superintegrable system on flat space must satisfy the condition $\Delta_{\text{flat}} (\Omega^{-(n+2)} \sigma^{n+2}) = 0$. A direct computation using (6.18) then shows

$$\Delta_{\text{flat}} (\Omega^{-(n+2)} \sigma^{n+2}) = (3n + 2) \sigma^{n+2} \Omega^{-(n+6)} \left[ \Omega \Delta \Omega + \Omega^a \Omega_{,a} - \Omega \Omega^a (\ln \sigma^{n+2})_{,a} \right] ,$$

taking into account that

$$R = -2 \frac{n - 1}{\Omega^2} \left( \Omega \Delta \Omega - \frac{n}{2} \Omega^a \Omega_{,a} \right) .$$

due to (3.25). The desired condition now follows from (6.18) after a conformal transformation via $\Omega$. \(\square\)

The following example employs condition (6.20) to show that there is no conformal transformation that takes the generic system on the $n$-sphere to a properly superintegrable system on flat space. Note that there is always a conformal transformation taking it to a conformally superintegrable system on flat space.

The example thereby generalises a result shown in [KKM06], which addresses the specific case of dimension 3, see also [Cap14]. Note that the proof presented here is a relatively simple exercise,
while with traditional methods the claim, if at all, cannot be obtained for arbitrary dimension in a straightforward fashion.\footnote{We remark that the generic system on the $n$-sphere can be transformed into a proper superintegrable systems on flat space using Bôcher transformations or orbit degenerations [Cap14]. Opposed to conformal transformations, however, these are not equivalence relations on conformally superintegrable systems.}

**Example 6.15** (Generic system on the $n$-sphere). Consider the generic system on the $n$-sphere, with $n \geq 3$. It has already been introduced for dimension 3 in Example 4.14. In arbitrary dimension we have the metric

$$g = \sum_{m=1}^{n} \left( \prod_{k=2}^{m} \sin^{2}(\phi_{k-1}) \right) d\phi_{m}^{2}$$

with angular coordinates $\phi_{1}, \ldots, \phi_{n}$. The superintegrable potential defining the generic system is

$$V = a_{0} + \sum_{m=1}^{n} \left( \frac{a_{m}}{\cos^{2}(\phi_{m}) \prod_{k=2}^{m} \sin^{2}(\phi_{k-1})} \right) + \frac{a_{n+1}}{\prod_{k=1}^{n} \sin^{2}(\phi_{k})}$$

For this system, $\sigma^{n+2}$ satisfies the Laplace eigen-equation with quantum number $n + 1$,

$$\Delta \sigma^{n+2} = -2n(n+1)\sigma^{n+2}.$$  

Solutions of this equation span a vector space of dimension $(n+2)^{2}$ whose basis is given by hyperspherical harmonics [Fry91] or one of the bases in [LR03, LR04]. Concretely, for the generic system, we have

$$\sigma^{n+2} = \prod_{k=1}^{n} \cos(\phi_{k}) \sin^{n-k+1}(\phi_{k}),$$

which does not satisfy (6.20).

Since the generic system on the $n$-sphere does not satisfy (6.20), we have proven the following.

**Theorem 6.16.** The generic system on the $n$-sphere is not conformally equivalent to a properly superintegrable system on flat space.

The next example illustrates further how the presented framework can be invoked for proving statements in arbitrary dimension with ease and in a rigorous manner.

**Example 6.17.** A non-degenerate properly superintegrable system on the $n$-sphere cannot be conformally equivalent to the harmonic oscillator. Indeed, the harmonic oscillator has a vanishing structure tensor, $T_{ijk} = 0$. Therefore the system has to satisfy $S_{ijk} = 0$, because $S_{ijk}$ is conformally invariant. Moreover, being proper, the system on the sphere satisfies (6.11), (6.12) and (6.18), i.e.

$$\tilde{\nabla}^{2} \sigma = 0, \quad \Delta \sigma^{1-2} = 0, \quad \Delta \sigma^{n+2} = -2 \frac{n+1}{n-1} R \sigma^{n+2} \neq 0,$$

where $\tilde{\nabla}^{2}$ and $\Delta$ are the trace-free Hessian and the Laplace-Beltrami operator on the sphere of constant scalar curvature $R \neq 0$. This system does not admit a solution.

We continue our study of pairs of conformally equivalent, properly superintegrable systems on manifolds of constant curvature.

**Definition 6.18.** We say that a $c$-superintegrable system is basic if it contains a member system that is an abundant properly superintegrable system on a manifold of constant curvature.

In reference [KSV19], it is proven that the structure tensor of an abundant second order properly superintegrable system on a constant curvature manifold of dimension $n \geq 3$ satisfies

$$T_{ijk} = \frac{1}{6} \left[ \frac{1}{1} \frac{1}{k} {B}_{ijk} + \frac{1}{1} \frac{1}{j} g_{ij} \left( \frac{2(n+1)}{n(n-1)} R {B}_{ij} \right) \right],$$

where $B$ is a scalar function, called its structure function. The following theorem allows us to extend this definition of the structure function $B$ to any basic $c$-superintegrable system.
Theorem 6.19. Consider two manifolds of constant curvature and with properly superintegrable systems that are conformally equivalent. Denote their metrics by $g$ and $\tilde{g} = \Omega^2 g$. Assume the superintegrable systems have structure functions $B$ and $\tilde{B}$, respectively, c.f. (6.21). Then

$$\tilde{B} = \Omega^2 B \quad \text{modulo gauge transformations.} \quad (6.22)$$

Proof. On a manifold with Hamiltonian $H = g^{ij} p_i p_j + V$, we infer from [KSV19] the following formula for $B_{ijk}$ in an abundant constant-curvature system,

$$B_{ijk} = T_{ijk} + \frac{n+2}{n} g_{ij} \bar{t}_k + \frac{1}{2(n-2)} \bar{i} j k g_{ij} C, k$$

where

$$C = \frac{n-2}{n+2} \Delta B + \frac{2(n+1)}{n(n-1)} R B - (n - 2) \bar{t}$$

and

$$B_{ijk} = \frac{1}{6} i j k \left( B_{ij} + \frac{4R}{n(n-1)} g_{ij} B \right)_{,k}$$

up to an irrelevant constant. By virtue of (3.20), we know the transformation behavior of $S_{ijk}$ and $\bar{t}_i = \bar{t}_i$, 

$$S_{ijk} \to \tilde{S}_{ijk} = \Omega^2 S_{ijk} \quad \text{and} \quad \bar{t}_i \to \bar{t}_i - 3\Omega^{-1} \Omega_i,$$

and therefore that of $B_{ijk}$,

$$\tilde{B}_{ijk} = \Omega^2 (B_{ijk} + \text{trace terms}). \quad (6.24)$$

Secondly, we also know by construction that $S_{ijk} = \frac{1}{i j k} \circ B_{ijk}$, where on the right hand side we recall that comma denotes the covariant derivative. An analogous equation holds for $\tilde{S}_{ijk}$ with a function $\tilde{B}$. Now, let us denote by $\nabla$ and $\bar{\nabla}$ the Levi-Civita connections of $g$ and $\tilde{g}$, respectively. Then, for the third derivatives,

$$\bar{i} j k \circ \bar{\nabla}^3_{ijk} \tilde{B} = \Omega^2 \bar{i} j k \circ \nabla^3_{ijk} B \quad (6.25)$$

because of the invariance of $S_{ijk}$. A straightforward computation verifies that $\tilde{B} = \Omega^2 B$ satisfies (6.25). We have therefore confirmed that $\bar{t} = \bar{t} - 3 \ln |\Omega|$ and $\tilde{B} = \Omega^2 B$ yield the correct structure tensors $\tilde{S}_{ijk}$ and $\bar{t}_i$, for the conformally transformed manifold with Hamiltonian $\tilde{H} = \Omega^{-2} H$. Since the structure functions are unique up to gauge transformations, this concludes the proof. \qed

The theorem allows us to extend the definition of the structure function $B$ as follows: In [KSV19] the structure function $B$ is defined for abundant properly superintegrable systems on constant curvature spaces. We are now able to define a corresponding object for any c-superintegrable system arising from such systems.

Corollary 6.20. Abundant second order properly superintegrable systems on constant curvature spaces in dimension $n \geq 3$ are Stäckel equivalent if and only if their densities $b \in \mathcal{E}[-2]$ given by

$$b = B \det(g)^{\frac{1}{2}}.$$

coincide up to a gauge transformation.

Example 6.21. It is well understood that on 3-dimensional flat space the systems III and V are equivalent. There is no Stäckel equivalent system of this class on the 3-sphere. In common coordinates $(x, z, \bar{z})$, we may write

$$g_{III} = \frac{dx^2 + dz d\bar{z}}{z^2} \quad V_{III} = \omega \bar{z}^2 (4x^2 + z \bar{z}) + a_1 x z^2 + a_2 z^2 + \frac{a_3}{z} + a_4$$

$$g_{V} = dx^2 + dz d\bar{z} \quad V_{V} = \omega (4x^2 + z \bar{z}) + a_1 x + \frac{a_4}{z} + \frac{a_3}{z^3} + a_2$$

for which we find

$$B_{III} = -\frac{3}{2} \ln(z) \bar{z}, \quad B_{V} = -\frac{3}{2} \ln(z) z \bar{z}.$$
The density \( b \) shared by these structure functions is
\[
b = -3 \ln(z) z \bar{z}.
\]
There is no properly superintegrable system conformally equivalent to those obtained from \( g_{III} \) and \( V_{III} \) (or equivalently \( g_V \) and \( V_V \)). In order to confirm this, note that \( Vg \) is invariant, since \( V \) transforms with \( \Omega^{-2} \) and \( g \) transforms with \( \Omega^3 \). Indeed, the metric
\[
\hat{g} = V_{III} g_{III} = V_V g_V
\]
cannot have constant scalar curvature 1 for any constants \( a_1, \ldots, a_4 \) and \( \omega \).

7. Application to dimension three

We have already discussed a few examples in the previous section, along with their behavior under conformal transformations. In the present section, we apply our framework to the 3-dimensional case. Non-degenerate second order conformally superintegrable systems in dimension 3 are classified in [KKM06, Cap14]. Also, it is known that all these systems are abundant [KKM05b]. In [KKM06] it has been established that any non-degenerate second order conformally superintegrable system is Stäckel equivalent to a non-degenerate second order abundant and proper system on a constant curvature geometry. We shall therefore restrict to the study of abundant systems for constant curvature metrics. All non-degenerate 3-dimensional systems are equivalent to these. Recall that in dimension 3, the non-linear condition (5.19) is void. Consequently, no further restriction exists on the tensor \( S_{ijk} \). Hence any trace-free symmetric initial conditions \( \Psi_{ijk} = S_{ijk}(x_0) \) in a point \( x_0 \in M \) can be integrated to a structure tensor of an abundant second order conformally superintegrable system, c.f. Corollary 6.3. Therefore the set of conformal equivalence classes of such systems is parametrised by the seven dimensional space of trace-free symmetric 3-tensors \( \Psi_{ijk} \) or, equivalently, harmonic ternary cubics \( \Psi(p) = \Psi_{ijk} p^i p^j p^k \). This parametrisation is equivariant with respect to the stabiliser subgroup of the point \( x_0 \) in the conformal group, which is isomorphic to \( SO(3) \).

We comment that this agrees with the references [KKMP11, CK14, Cap14]. In [KKMP11] a 10-parameter classification space is mentioned, corresponding to a 10-dimensional representation of \( SO(3) \) in [Cap14, CK14]. This 10-dimensional representation decomposes into two irreducible components of dimension 7 and 3, corresponding to \( S_{ijk} \) and \( \ell_i \) in our framework. Note that the 3-dimensional component is restricted in the references, which corresponds to imposing proper superintegrability here. The 7-dimensional component is realised as the space of binary sextics and it is shown that no restrictions exist on this component. The relation to our framework is given by a known correspondence between harmonic ternary cubics and binary sextics as follows.

The adjoint action of \( SL(2, \mathbb{C}) \) on its Lie algebra \( sl(2, \mathbb{C}) \cong \mathbb{C}^3 \) preserves the Killing form. This defines a group morphism \( SL(2, \mathbb{C}) \rightarrow SO(3, \mathbb{C}) \) with kernel \( \{ \pm 1 \} \) and hence an isomorphism \( SL(2, \mathbb{C})/\mathbb{Z}_2 \rightarrow SO(3, \mathbb{C}) \). The standard action of \( SL(2, \mathbb{C}) \) on \( \mathbb{C}^2 \) induces an \( SL(2, \mathbb{C}) \)-action on \( S^2 \mathbb{C}^2 \) which descends to an \( SO(3, \mathbb{C}) \)-action, because the elements \( \pm 1 \) act trivially. The latter induces an \( SO(3, \mathbb{C}) \)-action on \( S^3 S^2 \mathbb{C}^2 \) which descends to \( S^6 \mathbb{C}^2 \) under total symmetrisation \( S^3 S^2 \mathbb{C}^2 \rightarrow S^6 \mathbb{C}^2 \). Together with the isomorphism \( S^2 \mathbb{C}^2 \cong \mathbb{C}^3 \), we obtain a morphism \( S^3 \mathbb{C}^3 \rightarrow S^6 \mathbb{C}^2 \) of \( SO(3, \mathbb{C}) \)-representations, giving an \( SO(3, \mathbb{C}) \)-equivariant morphism from the 10-dimensional space of ternary cubics to the 7-dimensional space of binary sextics. Its restriction to the 7-dimensional space of harmonic ternary cubics is non-trivial and hence an isomorphism by Schur's lemma. Explicitly, it is given by defining a sextic \( s \) from the cubic \( \Psi(p) \) via
\[
s(z, w) = \Psi(z^2 - w^2, 2zw, z^2 + w^2).
\]

Note that the stabiliser subgroup contains only rotations. The action of translations is not linear and more involved.

7.1. Special case: Simultaneous standard scale and proper scale. Let us confine to constant curvature geometries, since in dimension 3 any non-degenerate system is conformally equivalent to one on constant curvature [KKM06, Theorem 4]. We begin with the very particular situation where
the standard scale choice is a properly superintegrable system, i.e. we have both  \( \bar{t} = 0 \) and  \( \tau_{ij} = 0 \).
In this case, due to (5.20) and (5.18a), the tracefree Ricci tensor satisfies
\[
\bar{R}_{ij} = (S^{ab}_{i} S_{jab})_a = 0 ,
\]
(7.2)
A comparison with [KKM05b] shows that there are only three normal forms on 3-dimensional Euclidean space that satisfy this criterion, c.f. also Table 3. These are the systems labeled VII, A and O in [KKM05b].

7.2. General picture. Table 3 lists the established normal forms for 3-dimensional non-degenerate systems on flat space, see [KKM06, Cap14]. The functions  \( B \) and \( \bar{t} \) are obtained as established in [KSV19], and using Theorem 6.19, as well as Equation (4.10), we may compute the corresponding functions for any 3-dimensional non-degenerate system conformally equivalent to one of the systems in Table 3.
If in Table 3, we take the quotient under conformal equivalence for each example, then the systems III and V are identified and we obtain a list of nine abundant c-superintegrable systems. Due to [Cap14] these are all abundant c-superintegrable systems in dimension 3, up to one exception. Indeed, from [KKM05b, KKM06, Cap14] it follows that there is one equivalence class of non-degenerate 3-dimensional superintegrable system that does not admit a representative properly superintegrable system on flat 3-space. This system is the generic system on the 3-sphere from Example 4.14, see also Theorem 6.16. Its conformally equivariant structure tensor  \( S_{ijk} \) is generated by the structure function (up to gauge freedom)
\[
B = -\frac{3}{2} \sum_k s_k^2 \ln(s_k).
\]

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## Non-degenerate second order superintegrable systems in dimension $n = 3$, for Euclidean geometry $g = \sum_i dx_i^2$

| Example | potential mod superintegrable systems | B mod gauge terms | $-\frac{6}{5}t$ mod const. |
|---------|--------------------------------------|-------------------|-----------------------------|
| **Regular systems** (linked with elliptic separation coordinates) | | | |
| “Generic system” I [2111]/Smarodinsky-Winternitz I | $\sum_{i=1}^n \left( \frac{a_i}{x_i} + \omega x_i^2 \right)$ | $-\frac{3}{2} (x^2 \ln(x) + y^2 \ln(y) + z^2 \ln(z))$ | $\sum_k \ln(x_k)$ |
| System II [221] | $\omega (x^2 + y^2 + z^2) + a_1 \frac{x - iy}{(x + iy)^3} + a_2 \frac{1}{(x + iy)^2} + \frac{a_3}{z^2}$ | $(x^2 + y^2)(\frac{1}{2} \ln(x^2 + y^2) - i \arctan(\frac{y}{x})) + z^2 \ln(z)$ | $\ln(z) + \ln(x^2 + y^2) - 2i \arctan(\frac{z}{x})$ |
| System III [23] | $\omega (x^2 + y^2 + z^2) + \frac{a_1}{(x + iy)^3} + \frac{a_2 z}{(x + iy)^2} + a_3 \frac{x^2 y^2 - 3z^2}{(x + iy)^4}$ | $(y^2 + z^2)(\frac{1}{2} \ln(y^2 + z^2) - i \arctan(\frac{z}{y}))$ | $\frac{3}{2} \ln(x^2 + y^2) - 3i \arctan(\frac{z}{y})$ |
| System V [32] | $\omega (4x^2 + y^2 + z^2) + a_1 x + \frac{a_2}{x^2} + \frac{a_3}{z^2}$ | $y^2 \ln(y) + z^2 \ln(z)$ | $\ln(y) + \ln(z)$ |
| System IV [311]/Smarodinsky-Winternitz II | $\omega (x^2 + y^2 + z^2) + a_1 x + \frac{a_2}{x^2} + \frac{a_3}{z^2}$ | $y^2 \ln(y) + z^2 \ln(z)$ | $\ln(y) + \ln(z)$ |
| System VI [41] | $\omega (z^2 - 2(x - iy)^3 + 4x^2 + 4y^2) + a_1 (2x + 2iy - 3(x - iy)^2) + a_2(x - iy) + \frac{a_3}{z^2}$ | $z^2 \ln(z) + \frac{1}{6} (x - iy)^3$ | $\ln(z)$ |
| System VII [5] | $\omega ((x + iy)^3 + 6(x^2 + y^2 + z^2)) + a_1 ((x + iy)^3 + \frac{5}{4} (x + iy)^2) + \frac{a_2}{16} (5(x + iy)^4 + x^2 + y^2 + z^2 + 6(x + iy)^2 z)$ | $-\frac{1}{3} ((x + iy)^2 + 6z) (x + iy)^2$ | 0 |

| **Exceptional systems** (linked with degenerate separation coordinates) | | | |
| Isotropic oscillator O | $\omega^2 \sum_i x_i^2 + \sum_i \alpha_i x_i$ | 0 | 0 |
| System OO | $\frac{\omega}{2} \left( x^2 + y^2 + \frac{z^2}{4} \right) + a_1 x + a_2 y + \frac{a_3}{z^2}$ | $z^2 \ln(z)$ | $\ln(z)$ |
| System A | $\omega ((x - iy)^3 + 6(x^2 + y^2 + z^2)) + a_1 ((x - iy)^3 + 2(x + iy)) + a_2 (x - iy) + a_3 z$ | $-\frac{1}{18} (x - iy)^3$ | 0 |

Table 3. The properly superintegrable systems in dimension 3 on flat space. The Systems III and V are conformally equivalent under Stäckel transform, see Example 6.21 for details.