Energies of the static solitary wave solutions of the one-dimensional Gross-Pitaevskii equation

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Abstract. We calculated the energies of the static solitary wave solutions of the one-dimensional Gross-Pitaevskii equation with the time-dependent parabolic trap, the time-dependent scattering wave length of s-wave, and the time-dependent external potential describing a gain or loss term. Some written solutions of the equation were used, two of which are based on the experimental results. The solutions satisfy the condition of solitary wave solution since they are localized over the space. By this argument, the energies were obtained by integrating the Hamiltonian density over the space formulated in the classical field theory. To do that, we constructed the appropriate Lagrangian density representing the equation by initially writing the ansatz Lagrangian density and then substituting into the Euler-Lagrange equation. We found that two of the solutions have the same energies and the other one should mathematically have the pure imaginary function describing the gain-loss term to achieve the real energy.

1. Introduction
It has been noted that the Gross-Pitaevskii equation (GPE), which represents the dynamics of a condensate in the Bose-Einstein condensation, can be extended by adding the other external potentials [1-7], which represent the realistic system. The GPE itself, which can be treated as a kind of the nonlinear quantum oscillator [8,9], has been proposed to discuss the characteristics of boson gases at the very low temperature. The detailed explanations of the characteristics of the GPE can be found in Refs. [10-14]. Interestingly, this nonlinear equation has analytically produced the solitary wave solutions although it needs an approximation [15]. In the nonlinear wave theory, the solitary wave solutions can be associated with the particles since they can define a mass. This mass can be considered as the accumulation of the energy of the solitary wave, which is localized over the space. Therefore, by using the classical field theory, the definition of the accumulation of the energy can be calculated [16].

This time, we are interested in the analytical solutions of the solitary waves obtained by Atre et al. [17], which is related to the experimental results attained by Strecker et al. [18]. Strecker et al. have observed the bright solitons of $^7$Li atoms in a stable Bose-Einstein condensate [18]. In their report, the atoms were confined in the one-dimensional optical trapping potential and they observed the presence of repulsive interaction between two solitons. To follow the results of Strecker et al., Atre et al. have extended some time-dependent parameters and another time-dependent external potential describing a gain or loss term into the GPE [17]. They have classified some solutions of the solitary wave by reducing the three-dimensional GPE into the one-dimensional GPE, assuming the appropriate conditions, and then proposing their ansatz solutions.
The purpose of this paper is to calculate the energy of some static solitary wave solutions of the one-dimensional GPE, which have been obtained by Atre et al [17]. These energies are actually obtained by integrating the Hamiltonian density over the space, which has been stated by the classical field theory [16]. In this paper, we follow the idea to obtain the energies after constructing the appropriate Hamiltonian density. We organized the paper as follows: we derive the Euler-Lagrange equation and the Hamiltonian density by minimizing the action function in Sec. 2. In Sec. 3 we initially write our ansatz Lagrangian density and choose the appropriate constants in order to fix our one-dimensional GPE by substituting it into the corresponding Euler-Lagrange equation. Finally, we summarize our discussion in Sec. 4 with conclusions based on our final results.

2. The energy-momentum tensor

Here, we concern the derivation of the Euler-Lagrange equation and the energy-momentum tensor, which are obtained by minimizing the action function and applying the total variation of the field. This time, we initially work in the (1+3) space-time dimension and reduce it in the next section into (1+1) space-time dimension since we only consider for the case of the one-dimensional GPE. First, we start with the Lagrangian density, which is a function of the complex field and its conjugate, and the four-dimensional space-time coordinates explicitly $L = L(\psi, \psi^*, \partial_x \psi, \partial_x \psi^*)$. Here, $v$ runs 0 to 3 and we also define $(x^0 = t, x^1 = x, x^2 = y, x^3 = z)$, where $x_v = (x_0, x_1, x_2, x_3) = (t, -x, -y, -z)$. The field and its conjugate are well-defined in the four-dimensional region $R$ and confined in the boundary of its region $\partial R$, for further discussion one can see a reference book (19). Then, the action for the corresponding Lagrangian density can be written as $S = \int L d^4 x$.

The variation for an action then can be written as

$$\delta S = \int (\delta L + L \delta \psi^* d^4 x),$$

where

$$\delta L = \frac{\partial L}{\partial \psi} \delta \psi + \frac{\partial L}{\partial \psi^*} \delta \psi^* + \frac{\partial L}{\partial (\partial_x \psi)} \delta (\partial_x \psi) + \frac{\partial L}{\partial (\partial_x \psi^*)} \delta (\partial_x \psi^*) + \frac{\partial L}{\partial x^v} \delta x^v.$$  \hspace{1cm} (2)

One can obtain Eq. (1) by applying the transformation of the space-time coordinates and the field as follows

$$x^v = x^v + \delta x^v, \quad \psi'(x^v) = \psi(x^v) + \delta \psi(x^v).$$

Note that the conjugate field has the similar transformation as in Eq. (3) by replacing $\psi \rightarrow \psi^*$. By observing that $\delta (\partial_v \psi) = \partial_v (\delta \psi)$, we obtain

$$\delta S = \int_R \left( \frac{\partial L}{\partial \psi} - \partial_v \left( \frac{\partial L}{\partial (\partial_x \psi)} \right) \right) \delta \psi + \frac{\partial L}{\partial (\partial_x \psi^*)} \delta (\partial_x \psi^*) + \partial_v \left( \frac{\partial L}{\partial (\partial_x \psi^*)} \right) \delta \psi^* + \partial_v \left( \frac{\partial L}{\partial (\partial_x \psi)} \right) \delta \psi + \left( \frac{\partial L}{\partial (\partial_x \psi^*)} \right) \delta \psi^* + L \delta x^v \right) d^4 x.$$  \hspace{1cm} (4)

By observing Eq. (4), we know that the third term is the surface integral and by using the Gauss theorem it can be rewritten as

$$\int_R \partial_v \left( \frac{\partial L}{\partial (\partial_x \psi)} \right) \delta \psi + \left( \frac{\partial L}{\partial (\partial_x \psi^*)} \right) \delta \psi^* + L \delta x^v \right) d^4 x$$

$$= \int_{\partial R} \left( \frac{\partial L}{\partial (\partial_x \psi)} \right) \delta \psi + \left( \frac{\partial L}{\partial (\partial_x \psi^*)} \right) \delta \psi^* + L \delta x^v \right) d \sigma_v.$$  \hspace{1cm} (5)

By minimizing the action in Eq. (1), $\delta S = 0$, and applying the boundary conditions of the region $R$, $\delta \psi = \delta x^v = 0$ on $\partial R$, we obtain that the third term totally vanishes. Following the same boundary conditions, we also obtain two independent Euler-Lagrange equations as

$$\frac{\partial L}{\partial \psi} - \partial_v \left( \frac{\partial L}{\partial (\partial_x \psi)} \right) = 0, \quad \frac{\partial L}{\partial \psi^*} - \partial_v \left( \frac{\partial L}{\partial (\partial_x \psi^*)} \right) = 0.$$  \hspace{1cm} (6)

Following Ryder (19), to obtain the energy-momentum tensor we have to extend the transformation for $\psi$ in Eq. (3) as $\psi'(x^v) = \psi(x^v) + \delta \psi(x^v)$, where the total variation is given by $\Delta \psi = \delta \psi + (\partial_v \psi) \delta x^v$. Note that those transformations also hold for the conjugate field. Then, by employing those transformations and manipulating the right-hand side of Eq. (5), we can rewrite Eq. (5) as

$$\int_{\partial R} \partial_v \left( \frac{\partial L}{\partial (\partial_x \psi)} \right) \delta \psi + \left( \frac{\partial L}{\partial (\partial_x \psi^*)} \right) \delta \psi^* + L \delta x^v \right) d^4 x$$

.$$
\[ \frac{\partial L}{\partial (\partial_\nu \psi)} \Delta \psi + \left( \frac{\partial L}{\partial (\partial_\nu \psi^*}) \right) \Delta \psi^* - T^\nu_{\mu} \delta x^\mu \right] d\sigma_\nu. \] (7)

As directly seen in Eq. (7), it is clear that the total variation does not change the equation of motion represented by the Euler-Lagrange equation since the boundary conditions still hold. Thus, in Eq. (7) we have defined our energy-momentum tensor written as

\[ T^\nu_{\mu} = \frac{\partial L}{\partial (\partial_\nu \psi)} \partial_\mu \psi + \frac{\partial L}{\partial (\partial_\nu \psi^*)} \partial_\mu \psi^* - (\delta^\nu_{\mu}) L. \] (8)

In the field theory, we can cast the components of energy-momentum tensor into the Hamiltonian density and 3-momentum density, respectively

\[ T^0_{\ 0} = \frac{\partial L}{\partial (\partial_0 \psi)} \partial_0 \psi + \frac{\partial L}{\partial (\partial_0 \psi^*)} \partial_0 \psi^* - L, \] (9)

\[ T^0_{\ i} = \frac{\partial L}{\partial (\partial_\psi \psi)} \partial_i \psi + \frac{\partial L}{\partial (\partial_\psi \psi^*)} \partial_i \psi^*, \] (10)

with \( i = 1,2,3 \). In the next section, we will use our previous results to reach our purpose.

3. Results and discussions

In our previous paper [20], it has been discussed the ordinary GPE coupled by a gain or loss term based on the formulation in Ref. [17]. In this section, we consider the other discussions by allowing some quantities to be time dependent. In order to discuss the experimental results obtained by Strecker et al. [18], Atre et al. [17] has suggested that the GPE should be extended by allowing the time-dependent frequency of three-dimensional anisotropic parabolic trap, the time-dependent scattering wave length of s-wave, and adding the gain-loss term represented by an arbitrary time-dependent function. Before discussing more detailed, we rewrite some previous results which have been obtained by Atre et al. [17].

According to them, the equation of motion of a condensate trapped by the anisotropic potential with a gain or loss term was given by (17)

\[ i\hbar \frac{\partial \psi(\vec{r},t)}{\partial t} = \left( -\frac{\hbar^2}{2m} \nabla^2 + V(\vec{r},t) + U|\psi(\vec{r},t)|^2 + i \eta(t) \right) \psi(\vec{r},t), \] (11)

where \( \eta(t) \) is the time-dependent arbitrary function representing a gain-loss term, \( U \) is the time-dependent parameter, which can be written as \( U = 4\pi\hbar^2a(t)/m \), and the trapping potential is given by

\[ V(\vec{r}) = \frac{1}{2} m \omega^2 (x^2 + y^2 + \lambda_z(t) z^2), \] (12)

with \( \lambda_z(t) = \omega_z^2(t)/\omega^2 \). By proposing an ansatz solution given by (20)

\[ \psi(\vec{r},t) = \frac{1}{\sqrt{2\pi a_0 \hbar \omega}} \psi \left( \frac{z}{a_0} \sqrt{\frac{\hbar}{m \omega}}, \sqrt{\gamma(t)} \right) \exp \left( -\frac{\omega_a t}{2a_0^2} \psi \right), \] (13)

they found the one-dimensional GPE in the dimensionless units (17)

\[ i \partial_\tau \psi = -\frac{1}{2} \partial_{xx} \psi + \gamma(t)|\psi|^2 \psi + \frac{1}{2} \lambda_z(t) z^2 \psi + i \frac{\eta(t)}{2} \psi, \] (14)

where it has been defined that \( g(t) = \eta(t)/\hbar \omega \), \( a_0 = \sqrt{\hbar/m \omega} \), \( \gamma(t) = 2a(t)/a_B \), and \( a_B \) is the radius of Bohr. By exploring Eq. (14), they have constructed the general ansatz solution of the solitary waves by assuming that its solution is the multiplication between the amplitude and phase terms. The amplitudes, for the general cases, are different one to the others, but the phases have the same form.

Now, we are ready to construct our Hamiltonian density for the one-dimensional GPE stated in Eq. (14). First, we consider the ansatz Lagrangian density written as

\[ L = A \psi^* \phi + B \psi \phi^* + C |\psi|^2 + D \gamma(t)|\psi|^4 + E \lambda_z(t) z^2 |\psi|^2 + F g(t)|\psi|^2, \] (15)

where \( A, B, C, D, E, \) and \( F \) are constants that should be determined in order to keep Eq. (14), when Eq. (15) is substituted into one of the Euler-Lagrange equations in Eq. (6). Since we obtain two Euler-Lagrange equations, we have to choose the appropriate one, i.e., the conjugate one. Then by taking \( \nu = 0,3 \) into the Euler-Lagrange for the conjugate version in Eq. (6) and inserting Eq. (15) into Eq. (6), we find the complete Lagrangian density

\[ L = \frac{1}{2} (\psi^* \phi^* - \psi^*_t \phi^*_t) - \frac{1}{2} |\psi|^2 - \frac{1}{2} \gamma(t)|\psi|^4 - \frac{1}{2} \lambda_z(t) z^2 |\psi|^2 - \frac{1}{2} g(t)|\psi|^2. \] (16)

Finally, by substituting our complete Lagrangian density in Eq. (16) into Eq. (9), the Hamiltonian density becomes
Here we have replaced the symbol of the Hamiltonian density, as written in Eq. (17). Note that according to the classical field theory, the energy of the solitary wave can be formulated by integrating the Hamiltonian density over the space $E = \int_{-\infty}^{\infty} H(z, t) dz$. Although the Hamiltonian density depends on time, we are interested in calculating the energy of the static solitary wave solution since the solution itself describes the fundamental configuration of the energy which is nontrivial [16]. Before we present our calculation, we have to remind the reader that the phase of the solution can be written as [17].

$$\Phi(z, t) = a(t) + b(t) z - \frac{1}{2} c(t) z^2,$$

where all the above functions have the relations

$$c_t - c^2 (t) = \lambda z(t), \quad b(t) = A_0 \exp \left( \int_0^t c(t') dt' \right), \quad a(t) = a_0 + \frac{a-1}{2} \int_0^t b^2(t') dt'.$$

Now, we have already obtained the complete expressions to calculate some energies of the static solitary wave solutions obtained by Atre et al. [17] for some cases as follows:

### 3.1. Trains of soliton trapped in the potential of confining oscillator

By generating the confining oscillator potential and removing the gain-loss term, Atre et al. [17], based on the experimental results obtained by Strecker et al. [18], have found the moving trains of soliton, which is given by

$$\psi(z, t) = \sqrt{A_0} \sec(\lambda_0 t) \exp (i\Phi(z, t)) \cn \left( \frac{A_0 \sec(\lambda_0 t) [z - \frac{A_0 \sin(\lambda_0 t)}{\tau_0}], m) \right),$$

where $\cn(z, m)$ is one of the Elliptic functions whose several values are listed in the table. By choosing $m = 1$, we find the energy by substituting Eq. (20) into $E = \int_{-\infty}^{\infty} H(z, t) dz$

$$E = b^2(0) \tau_0 + \frac{c^2(0) \pi^2 \tau_0^4}{12 A_0^2} + \frac{1}{12} \frac{\lambda_0 \pi^2 \tau_0^3}{A_0^2} + \frac{A_0^2}{8 \tau_0} + \frac{2}{3} \gamma_0 \tau_0 A_0.$$

### 3.2. The presence of collapse and revival of the condensate

In this case, based on the experiment results obtained by Strecker et al [18], Atre et al. [17] have chosen some time-dependent parameters $g(t)$ and $\gamma(t)$. This leads to a complete solution [17]

$$\psi(z, t) = \sqrt{A_0} \sec(\lambda_0 t) \sec h \left( \frac{A_0 \sec(\lambda_0 t) z}{\tau_0} \right) \exp \left( G(t)/2 \right) \exp \left( i\Phi(z, t) \right),$$

with $G(t)$ obeys the relation $G(t) = \int_0^t g(t') dt'$. By inserting the solution in Eq. (22) into $E = \int_{-\infty}^{\infty} H(z, t) dz$, we find the energy as

$$E = \exp[G(0)] \left( b^2(0) \tau_0 + \frac{c^2(0) \pi^2 \tau_0^3}{12 A_0^2} + \frac{1}{12} \frac{\lambda_0 \pi^2 \tau_0^3}{A_0^2} + \frac{A_0^2}{8 \tau_0} + \frac{2}{3} A_0 \tau_0 \gamma(0) \exp \left( G(0) \right) + i g(0) \tau_0 \right).$$

By observing Eq. (21), we directly conclude that the function $g(t)$ should be pure imaginary since the energy must be real. However, this argument has a consequence. This consequence is that the gain-loss term has no physical meaning, so we cannot study the influence of the term.

### 3.3. Soliton in trapping potential for expulsive case

By choosing the appropriate time-dependent parameter again for $\lambda z(t)$ and $\gamma(t)$, Atre et al. have proposed a complete solution, which is given by [17]

$$\psi(z, t) = \sqrt{A_0} e^{2i t} \sec h \left( \frac{A_0 \sec(\lambda_0 t) z}{1 + e^t \tau_0} \right) \exp \left( i\Phi(z, t) \right).$$
We find the energy of the solution after inserting Eq. (24) into \( E = \int_{-\infty}^{\infty} H(z, t) dz \)
\[
E = b^2(0)\tau_0 + \frac{c^2(0)\pi^2\tau_0^2}{12A_0}\frac{\lambda_0\pi^2\tau_0^3}{6\tau_0} + \frac{2}{3} A_0^2 + \frac{2}{3} \gamma_0\tau_0 A_0. \tag{25}
\]

| Cases                                      | \( \lambda(t) \)         | \( \gamma(t) \)     | \( g(t) \)     |
|--------------------------------------------|---------------------------|----------------------|----------------|
| Trains of soliton trapped in the oscillator potential | \( \lambda_0^2 \)         | \( \gamma_0 \)       | 0              |
| The presence of collapse and revival of the condensate | \( \lambda_0^2 \)         | \( \gamma_0 \exp \left[ -\frac{a_z t^2}{2} - \frac{a_z}{\beta} \cos(\beta t) \right] \) \( \sec(\lambda_0 t) \) | \( a_1(t) - a_z \sin(\beta t) \) |
| Soliton in the trapping potential for expulsive case | \( \frac{3}{2} [1 + \tan(\gamma(t/2))] - 4 \) | \( \gamma_0 \left( \frac{e^{\gamma_0}}{1 + e^{\gamma_0}} \right) \) \( \gamma_0 < 0 \) | 0              |

By observing Table 1, we can identify which case can give the quality of future applications, such as for the nonlinear optics. One can also check the intensity of the wave by calculating the density of wave for each solution given in Eqs. (20), (22), and (24). The energy together with the intensity can give a clear description on how to apply them for the electronic devices.

4. Conclusions
We have presented the calculation of energies of the solitary wave solutions of the one-dimensional GPE based on the formulation proposed by Atre et al. [17], where their ansatz solutions correspond to the experimental results that have reported by Strecker et al. [18]. To make the relation between the solutions and experiment results, Atre et al. determined the appropriate parameters and showed the results by the simulations [17]. They extended the ordinary GPE by proposing some time-dependent parameters. The obtained solutions describe the existence of the solitons since the localized solutions have been shown [17].

Here, the energies for each solution have been obtained by integrating the Hamiltonian density over the space based on the formulation in the classical field theory. In the case of the trains of soliton, we only choose the value \( m = 1 \) since for \( m = 0 \) the energy will be divergent. We also show that for the case of the collapse and revival of condensates, one has to choose a pure imaginary function for describing the gain-loss term. It can be understood since the imaginary function in that term should be vanished to obey the real energy. However, the consequence is that the gain-loss term has no physical meaning. Finally, we also conclude that in the case of the trains soliton and the soliton confined in the trapping potential, those energies are the same, except if we give the other values for \( m \).

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