New Insights on Time-Symmetry in Quantum Mechanics

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The “time-asymmetry” attributed to the standard formulation of Quantum Mechanics (QM) was inherited from a reasonable tendency learned from Classical Mechanics (CM) to predict the future based on initial conditions: once the equations of motion are fixed in CM, then the initial and final conditions are not independent, only one can be fixed arbitrarily. In contrast, as a result of the uncertainty principle, the relationship between initial and final conditions within QM can be one-to-many: two “identical” particles with identical environments can subsequently exhibit different properties under identical measurements. These subsequent identical measurements provide fundamentally new information about the system which could not in principle be obtained from the initial conditions. Although this lack of causal relations seemed to conflict with basic tenets of science, many justified it by arguing “nature is capricious.” This lead to Einstein’s objection “God doesn’t play dice.” Nevertheless, after 100 years of experimental verification, QM has won over Einstein’s objection.

QM’s “time-asymmetry” is the assumption that measurements only have consequences after they are performed, i.e. towards the future. Nevertheless, a positive spin was placed on QM’s non-trivial relationship between initial and final conditions by Aharonov, Bergmann and Lebowitz (ABL) [2] who showed that the new information obtained from measurements was also relevant for the past of every quantum-system and not just the future. This inspired ABL to re-formulate QM in terms of Pre-and-Post-Selected-ensembles. The traditional paradigm for ensembles is to simply prepare systems in a particular state and thereafter subject them to a variety of experiments. These are “pre-selected-only-ensembles.” For pre-and-post-selected-ensembles, we add one more step, a subsequent measurement or post-selection. By collecting only a subset of the outcomes for this later measurement, we see that the “pre-selected-only-ensemble” can be divided into sub-ensembles according to the results of this subsequent “post-selection-measurement.” Because pre-and-post-selected-ensembles are the most refined
quantum ensemble, they are of fundamental importance and subsequently led to the two-vector or Time-Symmetric re-formulation of Quantum Mechanics (TSQM) \[5, 6\]. TSQM provides a complete description of a quantum-system at a given moment by using two-wavefunctions, one evolving from the past towards the future (the one utilized in the standard paradigm) and a second one, evolving from the future towards the past.

While TSQM is a new conceptual point-of-view that has predicted novel, verified effects which seem impossible according to standard QM, TSQM is in fact a re-formulation of QM. Therefore, experiments cannot prove TSQM over QM (or vice-versa). The motivation to pursue such re-formulations, then, depends on their usefulness. The intention of this article is to answer this by discussing how TSQM fulfils several criterion which any reformulation of QM should satisfy in order to be useful and interesting:

- TSQM is consistent with all the predictions made by standard QM (§1),
- TSQM has revealed new features and effects of QM that were missed before (§2),
- TSQM has lead to new mathematics, simplifications in calculations, and stimulated discoveries in other fields (as occurred, e.g., with the Feynman re-formulation of QM) (§3),
- TSQM suggests generalizations of QM that could not be easily articulated in the old language (§4).

A more conservative scientist may choose to utilize all the pragmatic, operational advantages listed above\[^1\], but stick to the standard time-asymmetric QM formalism. Our view is that these new effects form a logical, consistent, and intuitive pattern (in contrast to the traditional interpretation). Therefore, we believe there are deeper reasons which underly TSQMs success in predicting them. One generalization suggested by TSQM (§4.1) addresses the “artificial” separation in theoretical physics between the kinematic and dynamical descriptions \[23\]; another (§4.2) provides a novel solution to the

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\[^1\]While this paper focuses on theoretical issues, we emphasize that many of the novel predictions have been tested in quantum-optics laboratories utilizing Townes’ laser technology \[66\]. In addition, TSQM has suggested a number of innovative new technologies which could be implemented with lasers \[8.1\].
measurement problem. Consequently, we are able to change the meaning of uncertainty from “capriciousness” to exactly what is needed in order that the future can be relevant for the present, without violating causality, thereby providing a new perspective to the question “Why does God play dice?” (§5.1) In other words, TSQM suggests that two “identical” particles are not really identical, but there is no way to find their differences based only on information coming from the past, one must also know the future. We also show how the second generalization involving “destiny” is consistent with free-will (§5.2). Finally, we speculate on the novel perspectives that TSQM can offer for several other themes of this volume, such as emergence.

1 Consistency of Time-symmetric Quantum Mechanics with Standard Quantum Mechanics

We first motivate TSQM with a paradox concerning the relativistic covariance of the state-description in QM.

1.1 Motivation - A Relativistic Paradox

Consider two experimentalists A and B corresponding to two spin-1/2 particles prepared in a superposition with correlated spins ($\hat{\sigma}_A + \hat{\sigma}_B = 0$), i.e. in an EPR state:

$$|\Psi_{EPR}(t = 0)\rangle = \frac{1}{\sqrt{2}} \{|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B\} \tag{1.1}$$

Suppose at some later time $t_2$, the particles separate to a distance $L$ and A measures his spin in the z-direction and obtains the outcome $|\uparrow_z\rangle_A$. According to the usual interpretation, ideal-measurements on either particle will instantly reduce the state from a superposition $|\Psi_{EPR}(t_2 - \varepsilon)\rangle = |\Psi_{EPR}(0)\rangle = \frac{1}{\sqrt{2}} \{|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B\}$ into a direct product $|\Psi(t_2 + \varepsilon)\rangle = |\uparrow_z\rangle_A |\downarrow_z\rangle_B$. I.e. after A performs his measurement at $t = t_2$, then the joint wavefunction collapses so B’s wavefunction also collapses to $|\downarrow_z\rangle_B$ which can be confirmed if B actually performs a measurement. When should B perform this measurement? Consider a “lab” frame-of-reference which is at rest relative to A and B, in which case the collapse is simultaneous (see figure 1.a) as indicated
by the space-time coordinates:

$$A : \begin{pmatrix} ct_2 \\ 0 \end{pmatrix} \quad B : \begin{pmatrix} ct_2 \\ L \end{pmatrix}$$

However, what is simultaneous in one frame-of-reference is not simultaneous in another: e.g. as we change to a rocket frame-of-reference which moves with velocity \( \beta = \frac{v}{c} \) in the \( x \) direction (with the same space-time origin), then the “plane-of-simultaneity” changes (see figure 1b) as can be seen with the new coordinates after a Lorentz-transformation:

$$A : \begin{pmatrix} \gamma & -\beta \gamma \\ -\beta \gamma & \gamma \end{pmatrix} \begin{pmatrix} ct_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma ct_2 \\ -\beta \gamma ct_2 \end{pmatrix}$$

$$B : \begin{pmatrix} \gamma & -\beta \gamma \\ -\beta \gamma & \gamma \end{pmatrix} \begin{pmatrix} ct_2 \\ L \end{pmatrix} = \begin{pmatrix} \gamma ct_2 - \beta \gamma L \\ -\beta \gamma ct_2 + \gamma L \end{pmatrix}$$

In the rocket frame-of-reference, the collapse of the wavefunction of \( B \) happens at \( t_1 = \gamma t_2 - \frac{\beta}{c} \gamma L < t_2 \), i.e. the rocket frame-of-reference notices at \( t_1 < t_2 \), that \( B \) is in the state \( | \downarrow_z \rangle_B \), implying that the joint EPR wavefunction had collapsed at \( t_1 \) or before, so the state of \( A \) should be \( | \uparrow_z \rangle_A \) no later than \( t_1 \). If we transform back to our lab-frame-of-reference:

$$A : \begin{pmatrix} \gamma & \beta \gamma \\ \beta \gamma & \gamma \end{pmatrix} \begin{pmatrix} ct_1 \\ -\beta ct_1 \end{pmatrix} = \begin{pmatrix} \gamma ct_2 - \beta L \\ 0 \end{pmatrix}$$

we see that the particle on \( A \)’s side was in the \( | \uparrow_z \rangle_A \) state even before \( A \) made the measurement at \( t_2 \) (contradicting our notion that \( A \)’s measurement supposedly caused the collapse in the first place.)

In summary, this paradox focuses on “when did the collapse take place?”. In the lab-frame, \( A \)’s measurement occurs first and then \( B \)’s measurement occurs (see figure 1a). However, in a rocket-frame, \( B \)’s measurement occurs first and then \( A \)’s (see figure 1b). The lab-frame believes that \( A \)’s measurement caused the collapse whereas the rocket-frame disagrees and believes that \( B \)’s measurement caused the collapse. While the two different versions give the same statistical results at the level of probabilities, they differ completely on the state-description during the intermediate times and there is nothing in QM to suggest which version is the correct one. A similar arrangement has been probed experimentally producing results consistent with TSQMs hypothesis. This paradox has two possible resolutions:

\(^2\)This paradox can be sharpened in several ways.
Figure 1: Collapse of singlet state in 2-different frames of reference; a) the $t_{\text{lab}} = 0$ hypersurface intersects the $B$ worldline before $B$’s measurement, b) the $t_{\text{rocket}} = 0$ hypersurface intersects the $B$ worldline after $B$’s measurement.

1. Collapse cannot be described covariantly in a relativistic theory at the level of the state-description, only at the level of probabilities. This thereby precludes progress on questions such as “Why God plays dice.”

2. As first pointed out by Bell [48, 47, 9], Lorentz-covariance in the state-description can be preserved in a theory like TSQM [32] (§2.4). In addition, we believe it to be the most fruitful approach to probe deeper quantum realities beyond probabilities, thereby providing insight to questions like “Why God plays dice.”

1.2 The Main Idea

TSQM contemplates measurements which occur at the present time $t$ while the state is known both at $t_{\text{in}} < t$ (past) and at $t_{\text{fin}} > t$ (future). More precisely, we start at $t = t_{\text{in}}$ with a measurement of a nondegenerate operator $\hat{O}_{\text{in}}$. This yields as one potential outcome the state $|\Psi_{\text{in}}\rangle$, i.e. we prepared the “pre-selected” state $|\Psi_{\text{in}}\rangle$. At the later time $t_{\text{fin}}$, we perform another measurement of a nondegenerate operator $\hat{O}_{\text{fin}}$ which yields one possible outcome: the post-selected state $|\Psi_{\text{fin}}\rangle$. At an intermediate time $t \in [t_{\text{in}}, t_{\text{fin}}]$, we measure a non-degenerate observable $A$ (for simplicity), with eigenvectors $\{|a_j\rangle\}$. We wish to determine the conditional probability of $a_j$, given that we have both boundary conditions, $|\Psi_{\text{in}}\rangle$ and $\langle \Psi_{\text{fin}}|$.

3Such an arrangement has long been considered in actual experiments: consider a bubble chamber scattering experiment. The incoming particle, $|\Psi_{\text{in}}\rangle$, interacts with a target and then evolves into various outgoing states, $|\Psi_{\text{fin}}\rangle_1, |\Psi_{\text{fin}}\rangle_2$, etc. Typically, photographs are not taken for every target-interaction, but only for certain ones that were triggered.
we use the time displacement operator: $U_{t_{\text{in}}-t} = \exp\{-iH(t - t_{\text{in}})\}$ where $H$ is the Hamiltonian for the free system. For simplicity, we assume $H$ is time independent and set $\hbar = 1$. The standard theory of collapse states that the system collapses into an eigenstate $|a_j\rangle$ after the measurement at $t$ with an amplitude $\langle a_j | U_{t_{\text{in}}-t} | \Psi_{\text{in}} \rangle$. The amplitude for our series of events is $\alpha_j \equiv \langle \Psi_{\text{fin}} | U_{t_{\text{in}}-t} | a_j \rangle \langle a_j | U_{t_{\text{in}}-t} | \Psi_{\text{in}} \rangle$ which is illustrated in figure C.2a. This means that the conditional probability to measure $a_j$ given $|\Psi_{\text{in}}\rangle$ is pre-selected and $|\Psi_{\text{fin}}\rangle$ will be post-selected is given by the ABL formula [2].

$$Pr(a_j, t|\Psi_{\text{in}}, t_{\text{in}}; \Psi_{\text{fin}}, t_{\text{fin}}) = \frac{\langle \Psi_{\text{fin}} | U_{t_{\text{in}}-t} | a_j \rangle \langle a_j | U_{t_{\text{in}}-t} | \Psi_{\text{fin}} \rangle}{\sum_n \langle \Psi_{\text{fin}} | U_{t_{\text{in}}-t} | a_n \rangle \langle a_n | U_{t_{\text{in}}-t} | \Psi_{\text{fin}} \rangle}$$ (1.5)

As a first step toward understanding the underlying time-symmetry in the ABL formula, we consider the time-reverse of the numerator of eq. 1.6 and figure 2a. First we apply $U_{t_{\text{in}}-t}$ on $\langle \Psi_{\text{fin}} \rangle$ instead of on $\langle a_j \rangle$. We note that $\langle \Psi_{\text{fin}} | U_{t_{\text{in}}-t} = \langle U_{t_{\text{in}}-t}^{\dagger} | \Psi_{\text{fin}} \rangle$ by using the well-known QM symmetry $U_{t_{\text{in}}-t}^{\dagger} = \{e^{-iH(t_{\text{fin}}-t)}\}^{\dagger} = e^{iH(t_{\text{fin}}-t)} = e^{-iH(t-t_{\text{fin}})} = U_{t_{\text{fin}}-t}$. We also apply $U_{t_{\text{in}}-t}$ on $a_j$ instead of on $|\Psi_{\text{fin}}\rangle$ which yields the time-reverse re-formulation of the numerator of eq. 1.6 $\langle U_{t_{\text{fin}}-t}^{\dagger} | \Psi_{\text{fin}} | a_j \rangle \langle a_j | U_{t_{\text{in}}-t} | \Psi_{\text{fin}} \rangle$ as depicted in fig. 2b. Further work is needed to formulate what we mean by the 2-vectors in TSQM. E.g. if we are interested in the probability for possible outcomes of $a_j$ at time $t$, we must consider both $U_{t_{\text{in}}-t} | \Psi_{\text{in}} \rangle$ and $\langle U_{t_{\text{fin}}-t}^{\dagger} \Psi_{\text{fin}} | a_j \rangle$, since these expressions propagate the pre-and-post-selection to the present time $t$ (see the conjunction of both figures 2a and 2b giving 2c which is re-drawn in figure 3.b; these 2-vectors are not just the time-reverse of each other). This represents the basic idea behind the Time-Symmetric re-formulation of this symmetry by subsequently interacting with detectors. In CM, there is (in principle) a one-to-one mapping between incoming states and outgoing states, whereas in QM, it is one-to-many. By selecting a single outcome for the post-selection-measurement, we define the pre-and-post-selected-ensemble that has no classical analog.

ABL is intuitive: $|\langle a_j | U_{t_{\text{in}}-t} | \Psi_{\text{in}} \rangle|^2$ is the probability to obtain $|a_j\rangle$ having started with $|\Psi_{\text{in}}\rangle$. If $|a_j\rangle$ was obtained, then the system collapsed to $|a_j\rangle$ and $|\langle \Psi_{\text{fin}} | U_{t_{\text{in}}-t} | a_j \rangle|^2$ is then the probability to obtain $|\Psi_{\text{fin}}\rangle$. The probability to obtain $|a_j\rangle$ and $|\Psi_{\text{fin}}\rangle$ then is $|\alpha_j|^2$. This is not yet the conditional probability since the post-selection may yield outcomes other than $|\Psi_{\text{fin}}\rangle$. The probability to obtain $|\Psi_{\text{fin}}\rangle$ is $\sum_j |\alpha_j|^2 = |\langle \Psi_{\text{fin}} | \Psi_{\text{fin}} \rangle|^2 < 1$. The question being investigated concerning probabilities of $a_j$ at $t$ assumes we are successful in obtaining the post-selection and therefore requires the denominator in eq. 1.6 $\sum_j |\alpha_j|^2$, which is a re-normalization to obtain a proper probability.
Quantum Mechanics (TSQM)\[^5\]

\[
Pr(a_j, t|\Psi_{\text{in}}, t_{\text{in}}; \Psi_{\text{fin}}, t_{\text{fin}}) = \frac{|\langle U_{t_{\text{fin}}-t}\Psi_{\text{fin}}|a_j\rangle\langle U_{t_{\text{in}}-t}\Psi_{\text{in}}|\rangle^2}{\sum_n |\langle U_{t_{\text{fin}}-t}\Psi_{\text{fin}}|a_n\rangle\langle U_{t_{\text{in}}-t}\Psi_{\text{in}}|\rangle^2} \quad (1.6)
\]

While this mathematical manipulation clearly proves that TSQM is consistent with QM, it yields a very different interpretation. For example, the action of \(U_{t_{\text{fin}}-t}\) on \(\langle \Psi_{\text{fin}}|\) (i.e. \(\langle U_{t_{\text{fin}}-t}\Psi_{\text{fin}}|\)) can be interpreted to mean that the time displacement operator \(U_{t_{\text{fin}}-t}\) sends \(\langle \Psi_{\text{fin}}|\) back in time from the time \(t_{\text{fin}}\) to the present, \(t\). A number of new categories of states (figure 3) are suggested by the TSQM formalism and have proven useful in a wide variety of situations.

In summary, the ABL formulation clarified a number of issues in QM. E.g.: in this formulation, both the probability and the amplitude are symmetric under the exchange of \(\Psi_{\text{in}}\) and \(\Psi_{\text{fin}}\). Therefore, the possibility of wavefunction collapse in QM does not necessarily imply irreversibility of an

\[^5\]We note that because (full) collapses take place at the \(t_{\text{in}}\) and \(t_{\text{fin}}\) measurements, there is no meaning to information coming from \(t > t_{\text{fin}}\) or \(t < t_{\text{in}}\). Therefore, at least in this context, there is no meaning to a “multi-vector” formalism.
1.2.1 Pre-and-post-selection and Spin-1/2

One of the simplest, surprising, example of pre-and-post-selection is to pre-select a spin-1/2 system with $|\Psi_{\text{in}}\rangle = |\hat{\sigma}_x = +1\rangle = |\uparrow_x\rangle$ at time $t_{\text{in}}$. After the pre-selection, spin measurements in the direction perpendicular to $x$ yields complete uncertainty in the result \footnote{E.g. in the $z$-basis the state is $\frac{1}{\sqrt{2}}(|\uparrow_z\rangle + |\downarrow_z\rangle)$ which yields equal probability either spin-up or spin-down in the $z$-direction.}, so if we post-select at time $t_{\text{fin}}$ in the $y$-direction, we obtain $|\Psi_{\text{fin}}\rangle = |\hat{\sigma}_y = +1\rangle = |\uparrow_y\rangle$ half the time. Since the particle is free, the spin is conserved in time and thus for any $t \in [t_{\text{in}}, t_{\text{fin}}]$.
an ideal-measurement of either $\hat{\sigma}_x$ or $\hat{\sigma}_y$, yields $+1$ for this pre-and-post-selection. This by itself, two non-commuting observables known with certainty, is a most surprising property which no pre-selected-only-ensemble could possess.

We now ask a slightly more complicated question about the spin in a direction $\xi = 45^\circ$ relative to the $x-y$ axis. This yields:

$$\hat{\sigma}_\xi = \hat{\sigma}_x \cos 45^\circ + \hat{\sigma}_y \sin 45^\circ = \frac{\hat{\sigma}_x + \hat{\sigma}_y}{\sqrt{2}}$$ \hspace{1cm} (1.7)

From the results $Pr(\hat{\sigma}_x = +1) = 1$ and $Pr(\hat{\sigma}_y = +1) = 1$, one might wonder why we couldn’t insert both values, $\hat{\sigma}_x = +1$ and $\hat{\sigma}_y = +1$ into eq. 1.7 and obtain $\hat{\sigma}_\xi = \frac{1+1}{\sqrt{2}} = 2 \neq 1$. Such a result is incorrect for an ideal-measurement because the eigenvalues of any spin operator, including $\hat{\sigma}_\xi$, must be $\pm 1$. The inconsistency can also be seen by noting $\left(\frac{\sigma_x+\sigma_y}{\sqrt{2}}\right)^2 = \frac{\sigma_x^2 + \sigma_y^2 + \sigma_x \sigma_y + \sigma_y \sigma_x}{2} = \frac{1 + 1 + 0}{2} = 1$. By implementing the above argument, we would expect $\left(\frac{\sigma_x+\sigma_y}{\sqrt{2}}\right)^2 = \left(\frac{1+1}{\sqrt{2}}\right)^2 = 2 \neq 1$. Performing this step of replacing $\hat{\sigma}_x = +1$ and $\hat{\sigma}_y = +1$ in eq. 1.7 can only be done if $\hat{\sigma}_x$ and $\hat{\sigma}_y$ commute, which would allow both values simultaneously to be definite. Although it appears we have reached the end-of-the-line with this argument, nevertheless, it still seems that there should be some sense in which both $Pr(\hat{\sigma}_x = +1) = 1$ and $Pr(\hat{\sigma}_y = +1) = 1$ manifest themselves simultaneously to produce $\hat{\sigma}_\xi = \sqrt{2}$.

### 1.2.2 Pre-and-post-selection and 3-Box-paradox

Another example of a surprising pre-and-post-selection effect is the 3-box-paradox [3] which uses a single quantum particle that is placed in a superposition of 3 closed, separated boxes. The particle is pre-selected to be in the state $|\Psi_{\text{in}}\rangle = 1/\sqrt{3} (|A\rangle + |B\rangle + |C\rangle)$, where $|A\rangle$, $|B\rangle$ and $|C\rangle$ denote the particle localized in boxes $A$, $B$, or $C$, respectively. The particle is post-selected to be in the state $|\Psi_{\text{fin}}\rangle = 1/\sqrt{3} (|A\rangle + |B\rangle - |C\rangle)$. If an ideal-measurement is performed on box $A$ in the intermediate time

\footnote{This is also evident from ABL: the probability to obtain $\hat{\sigma}_x = +1$ at the intermediate time if an ideal-measurement is performed is $Pr(\hat{\sigma}_x = +1) = \frac{1+\cos(\xi)+\sin(\xi)+\cos(\xi)\sin(\xi)}{1+\cos(\xi)\sin(\xi)}$. We see that if $\xi = 0^\circ$ (i.e. $\hat{\sigma}_x$) then the intermediate ideal-measurement will yield $\hat{\sigma}_x = +1$ with certainty and when $\xi = 90^\circ$ (i.e. $\hat{\sigma}_y$), then the intermediate ideal-measurement will again yield $\hat{\sigma}_y = +1$ with certainty. E.g. $\hat{\sigma}_{\text{xyz}}^{45} = \pm 1$ is displayed in figure 9.a}
(e.g. we open the box), then the particle is found in box $A$ with certainty. This is confirmed by the ABL probability for projection in $A$: $Pr(\hat{P}_A) = \frac{|\langle \Psi_{\text{fin}} | \hat{P}_A | \Psi_{\text{in}} \rangle|^2}{|\langle \Psi_{\text{fin}} | \hat{P}_A | \Psi_{\text{in}} \rangle|^2 + |\langle \Psi_{\text{fin}} | \hat{P}_B + \hat{P}_C | \Psi_{\text{in}} \rangle|^2} = 1$. This can also be seen intuitively by contradiction: suppose we do not find the particle in box $|A\rangle$. In that case, since we do not interact with box $|B\rangle$ or $|C\rangle$, we would have to conclude that the state that remains after we didn’t find it in $|A\rangle$ is proportional to $|B\rangle + |C\rangle$. But this is orthogonal to the post-selection (which we know will definitely be obtained). Because this is a contradiction, we conclude that the particle must be found in box $A$. Similarly, the probability to find the particle in box $B$ is 1, i.e. $Pr(\hat{P}_B = 1) = 1$. The “paradox” is, what “sense” can these 2 definite statements be simultaneously true. We cannot detect the distinction with ideal-measurements: e.g. $Pr(\hat{P}_A = 1) = 1$ if only box $A$ is opened, while $Pr(\hat{P}_B = 1) = 1$ if only box $B$ is opened. If ideal-measurements are performed on both box $A$ and box $B$, then obviously the particle will not be found in both boxes, i.e. $\hat{P}_A \hat{P}_B = 0$.

\begin{align*}
|A\rangle & \quad + \quad |B\rangle & \quad - \quad |C\rangle \\
\begin{tikzpicture}
\draw[->] (0,0) -- (1,1) node[midway,above] {\text{time}};
\draw[->] (0,1) -- (1,0) node[midway,right] {\text{ideal measurement of particles in } A};
\draw[->] (0,2) -- (1,1) node[midway,above] {\text{ideal measurement of particles in } B};
\draw[->] (0,1) -- (1,2) node[midway,left] {\text{pre-selection}};
\draw[->] (0,0) -- (1,2) node[midway,right] {\text{post-selection}};
\end{tikzpicture}
\end{align*}

Figure 4: a) pre-selected vector $|\Psi_{\text{in}}\rangle = 1/\sqrt{3} (|A\rangle + |B\rangle + |C\rangle)$ propagates forwards in time from $t_{\text{in}}$ to $t_1$, and post-selected vector $|\Psi_{\text{fin}}\rangle = 1/\sqrt{3} (|A\rangle + |B\rangle - |C\rangle)$ propagating backwards in time from $t_{\text{fin}}$ to $t_2$. b) ideal-measurement of $\hat{P}_A$ at $t_1$ and of $\hat{P}_B$ at $t_2$.

1.2.3 Counterfactuals

There is a widespread tendency to “resolve” these paradoxes by pointing out that there is an element of counter-factual reasoning: the contradictions arise only because inferences are made that do not refer to actual experiments. Had the experiment actually been performed, then standard measurement

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8The mystery is increased by the fact that both $\hat{P}_A$ and $\hat{P}_B$ commute with each other, so one may ask “how is it possible that measurement of one box can disturb the measurement of another?”
theory predicts that the system would have been disrupted so that no paradoxical implications arises. Suppose we applied this to the 3-box-paradox: the resolution then is that there is no meaning to say that the particle is in both boxes without actually **measuring** both boxes during the intermediate time.

We have proven [11, 32] that one shouldn’t be so quick in throwing away counter-factual reasoning; though indeed counter-factual statements have no observational meaning, such reasoning is actually a very good pointer towards interesting physical situations. **Without invoking counter-factual reasoning**, we have shown that the apparently paradoxical reality implied counter-factually has new, **experimentally accessible** consequences. These observable consequences become evident in terms of **weak measurements**, which allow us to test - to some extent - assertions that have been otherwise regarded as counter-factual.

The main argument against counter-factual statements is that if we actually perform ideal-measurements to test them, we disturb the system significantly, and such disturbed conditions hide the counter-factual situation, so no paradox arises. TSQM also provides some novel insights for this “disturbance-based-argument”. E.g., for the spin-1/2 case (§1.2.1), if we verify \( \hat{\sigma}_x \) at \( t = t_1 \) and \( \hat{\sigma}_y \) at \( t = t_2 \), \( t_{\text{in}} < t_1 < t_2 < t_{\text{fin}} \), then \( Pr(\hat{\sigma}_x = +1) = 1 \) and \( Pr(\hat{\sigma}_y = +1) = 1 \) are simultaneously true. But if we switch the order and perform \( \hat{\sigma}_y \) before \( \hat{\sigma}_x \), then \( Pr(\hat{\sigma}_x = +1) = 1 \) and \( Pr(\hat{\sigma}_y = +1) = 1 \) are not simultaneously true, since measuring \( \hat{\sigma}_y \) at time \( t = t_1 \) would not allow the information from the earlier (\( t_{\text{in}} < t \)) pre-selection of \( \hat{\sigma}_x = +1 \) to propagate to the later time (\( t_2 > t_1 > t_{\text{in}} \)) of the \( \hat{\sigma}_x \) measurement. As a consequence, the \( \hat{\sigma}_x \) measurement at time \( t_2 \) would yield both outcomes \( \hat{\sigma}_x = \pm 1 \). So, in general, the finding that \( \hat{\sigma}_x = +1 \) with certainty or \( \hat{\sigma}_y = +1 \) with certainty in the pre-and-post-selected ensemble only held when **one** of these two measurements was performed in the intermediate time, not both. Therefore, we should not expect both \( \hat{\sigma}_x = +1 \) and \( \hat{\sigma}_y = +1 \) when measured simultaneously through \( \hat{\sigma}_{\xi = 45^\circ} \).

For the spin-1/2 case, the ABL-assignment relied on only the pre-**or**-post-selection, while in the 3-box-paradox, the ABL assignment relies on both the pre-**and**-post-selection. However, ABL still only gives an answer for one **The same argument applies in the reverse direction of time. The 4-outcomes are consistent with** \( |\Psi_{\text{in}}\rangle = |\hat{\sigma}_x = +1\rangle \) and \( |\Psi_{\text{fin}}\rangle = |\hat{\sigma}_y = +1\rangle \). Physically, the ideal-measurement of \( \hat{\sigma}_z \) exposes the particle to a magnetic field with a strong gradient in the \( \xi = 45^\circ \) direction, which causes the spin to revolve around this axis in an uncertain fashion.
actual ideal-measurement. What happens if we tried to obtain two answers for the 3-box-paradox? In order to deduce $\hat{P}_A = 1$, we used information from both pre-and-post-selected vectors. When we actually measured $\hat{P}_A$, then this ideal-measurement will limit the “propagation” of the 2-vectors that were relied on to make this determination (see fig. 4b). If we subsequently were to measure $\hat{P}_B$, then the necessary information from both the pre-and-post-selected vectors is no longer available (i.e. information from $t_{in}$ cannot propagate beyond the ideal-measurement of $\hat{P}_A$ at time $t_1$ due to the disturbance caused by the ideal-measurement of $\hat{P}_A$). Thus, even though $\hat{P}_A$ and $\hat{P}_B$ commute, ideal-measurements of one can disturb ideal-measurement of the other.10

Since we have understood the reason why both statements are not simultaneously true as a result of disturbance, we can now see the “sense” in which the definite ABL assignments can be simultaneously relevant. Our main argument is that if one doesn’t perform absolutely precise (ideal) measurements but is willing to accept some finite accuracy, then one can bound the disturbance on the system. For example, according to Heisenberg’s uncertainty relations, a precise measurement of position reduces the uncertainty in position to zero $\Delta x = 0$ but produces an infinite uncertainty in momentum $\Delta p = \infty$. On the other hand, if we measure the position only up to some finite precision $\Delta x = \Delta$ we can limit the disturbance of momentum to a finite amount $\Delta p \geq \hbar/\Delta$. By replacing precise measurements with a bounded-measurement paradigm, counter-factual thought experiments become experimentally accessible. What we often find is that the paradox remains - measurements produce surprising and often strange, but nevertheless consistent structures. With limited-disturbance measurements, there is a sense in which both $Pr(\hat{\sigma}_x = +1) = 1$ and $Pr(\hat{\sigma}_y = +1) = 1$ are simultaneously relevant because measurement of one does not disturb the other. Since measurement of $\hat{\sigma}_z$ also can be understood as a simultaneous measurement of $\hat{\sigma}_x$ and $\hat{\sigma}_y$, we will see that with limited-disturbance measurements, we can simultaneously use both $\hat{\sigma}_x = +1$ and $\hat{\sigma}_y = +1$ to obtain

10This is related to a violation of the product rule. In general, if $|\Psi_1\rangle$ is an eigenvector of $\hat{A}$ with eigenvalue $a$ and $|\Psi_2\rangle$ is an eigenvector of $\hat{B}$ with eigenvalue $b$ and $[\hat{A}, \hat{B}] = 0$, then if $\hat{A}$ and $\hat{B}$ are known only by either pre-selection or post-selection, then the product rule is valid, i.e. $\hat{A}\hat{B} = ab$. However if $\hat{A}$ and $\hat{B}$ are known by both pre-selection and post-selection, then the product rule is not valid, i.e. $\hat{A}\hat{B} \neq ab$, i.e. they can still disturb each other, even though they commute. [31]
\[(\hat{\sigma}_{\xi=45^\circ})_w = \frac{\langle 1_y | \hat{\sigma}_x + \hat{\sigma}_z \| 1_x \rangle}{\sqrt{2} \langle 1_y | 1_x \rangle} = \frac{\langle 1_y | 1 + 1 \| 1_x \rangle}{\sqrt{2} \langle 1_y | 1_x \rangle} = \sqrt{2}.\]

2 TSQM has revealed new features and effects: Weak-Measurements

ABL considered the situation of measurements between two successive ideal-measurements where one transitions from a pre-selected state \(|\Psi_{\text{in}}\rangle\) to a post-selected state \(|\Psi_{\text{fin}}\rangle\). The state of the system at a time \(t \in [t_{\text{in}}, t_{\text{fin}}]\), i.e. after \(t_{\text{in}}\) when the state is \(|\Psi_{\text{in}}\rangle\) and before \(t_{\text{fin}}\) when the state is \(|\Psi_{\text{fin}}\rangle\) is generally disturbed by an intermediate ideal-measurement. A subsequent theoretical development arising out of the ABL work was the introduction of the weak-value of an observable which can be probed by a new type of measurement called the weak-measurement \([5]\). The motivation behind these measurements is to explore the relationship between \(|\Psi_{\text{in}}\rangle\) and \(|\Psi_{\text{fin}}\rangle\) by reducing the disturbance on the system at the intermediate time. This is useful in many ways, e.g. if a weak-measurement of \(\hat{A}\) is performed at the intermediate time \(t \in [t_{\text{in}}, t_{\text{fin}}]\) then, in contrast to the ABL situation, the basic object in the entire interval \(t_{\text{in}} \rightarrow t_{\text{fin}}\) for the purpose of calculating other weak-values for other measurements is the pair of states \(|\Psi_{\text{in}}\rangle\) and \(|\Psi_{\text{fin}}\rangle\).

2.1 Quantum Measurements

Weak-measurements \([5]\) originally grew out of the quantum measurement theory developed by von Neumann \([12]\). First we consider ideal-measurements of observable \(\hat{A}\) by using an interaction Hamiltonian \(H_{\text{int}}\) of the form \(H_{\text{int}} = -\lambda(t)\hat{Q}_{\text{md}}\hat{A}\) where \(\hat{Q}_{\text{md}}\) is an observable of the measuring-device (e.g. the position of the pointer) and \(\lambda(t)\) is a coupling constant which determines the duration and strength of the measurement. For an impulsive measurement we need the coupling to be strong and short and thus take \(\lambda(t) \neq 0\) only for \(t \in (t_0 - \varepsilon, t_0 + \varepsilon)\) and set \(\lambda = \int_{t_0 - \varepsilon}^{t_0 + \varepsilon} \lambda(t)dt\). We may then neglect the time evolution given by \(H_{\text{s}}\) and \(H_{\text{md}}\) in the complete Hamiltonian \(H = H_{\text{s}} + H_{\text{md}} + H_{\text{int}}\). Using the Heisenberg equations-of-motion for the momentum \(\hat{P}_{\text{md}}\) of the

\[\text{[11]}\text{Weak-measurements and their outcome, weak-values, can be derived in all approaches to quantum measurement theory. E.g. the usual projective measurement typically utilized in quantum experiments is a special case of these weak-measurements [50].}\]
measuring-device (conjugate to the position $\hat{Q}_{\text{md}}$), we see that $\hat{P}_{\text{md}}$ evolves according to $\frac{d\hat{P}_{\text{md}}}{dt} = \lambda(t) \hat{A}$. Integrating this, we see that $P_{\text{md}}(T) - P_{\text{md}}(0) = \lambda \hat{A}$. Integrating this, we see that $P_{\text{md}}(T) = P_{\text{md}}(0) + \lambda \hat{A}$.

Under these conditions (e.g. if the measuring-device approaches a delta function in $P_{\text{md}}$), then the disturbance or back-reaction on the system is increased due to a larger $H_{\text{int}}$, the result of the larger $\Delta Q_{\text{md}} (\Delta Q_{\text{md}} \geq \frac{1}{\Delta P_{\text{md}}})$. When $\hat{A}$ is measured in this way, then any operator $\hat{O}$ is disturbed because it evolved according to $\frac{d}{dt} \hat{O} = i\lambda(t)[\hat{A}, \hat{O}] \hat{Q}_{\text{md}}$, and since $\lambda \Delta Q_{\text{md}}$ is not zero, $\hat{O}$ changes in an uncertain way proportional to $\lambda \Delta Q_{\text{md}}$.

In the Schroedinger picture, the time evolution operator for the complete system from $t = t_0 - \varepsilon$ to $t = t_0 + \varepsilon$ is $\exp \{-i \int_{t_0-\varepsilon}^{t_0+\varepsilon} H(t) dt \} = \exp \{-i \lambda \hat{Q}_{\text{md}} \hat{A} \}$. This shifts $P_{\text{md}}$ (see figure 5a). If before the measurement the system was in a superposition of eigenstates of $\hat{A}$, then the measuring-device will also be superposed proportional to the system. This leads to the “quantum measurement problems,” discussed in [26]. A conventional solution to this problem is to argue that because the measuring-device is macroscopic, it cannot be in a superposition, and so it will “collapse” into one of these states and the system will collapse with it.

2.1.1 Weakening the interaction between system and measuring device

Following our intuition we now perform measurements which do not disturb either the pre-or-post-selections. The interaction $H_{\text{int}} = -\lambda(t) \hat{Q}_{\text{md}} \hat{A}$ is weakened by minimizing $\lambda \Delta Q_{\text{md}}$. For simplicity, we consider $\lambda \ll 1$ (assuming without lack of generality that the state of the measuring-device is a Gaussian with spreads $\Delta P_{\text{md}} = \Delta Q_{\text{md}} = 1$). We may then set $e^{-i\lambda \hat{Q}_{\text{md}} \hat{A}} \approx 1 - i \lambda \hat{Q}_{\text{md}} \hat{A}$.

\[ E.g. \text{ in the spin-1/2 example, the conditions for an ideal-measurement } \delta P_{\text{md}} = \lambda \hat{\sigma}_z \gg \Delta P_{\text{md}} \text{ will also necessitate } \Delta Q_{\text{md}} \gg \frac{1}{\lambda \hat{\sigma}_z} \text{ which will thereby create a back-reaction causing a precession in the spin such that } \Delta \Theta \gg 1 \text{ (i.e. more than one revolution), thereby destroying (i.e. making completely uncertain) the information that in the past we had } \hat{\sigma}_x = +1, \text{ and in the future we will have } \hat{\sigma}_y = +1. \]
Figure 5: a) with an ideal or “strong” measurement at \( t \) (characterized e.g. by \( \delta P_{md} = \lambda a_1 \gg \Delta P_{md} \)), then ABL gives the probability to obtain a collapse onto eigenstate \( a_1 \) by propagating \( |\Psi_{\text{fin}}\rangle \) backwards in time from \( t_{\text{fin}} \) to \( t \) and \( |\Psi_{\text{in}}\rangle \) forwards in time from \( t_{\text{in}} \) to \( t \); in addition, the collapse caused by ideal-measurement at \( t \) creates a new boundary condition \( |a_1\rangle \langle a_1| \) at time \( t \in [t_{\text{in}}, t_{\text{fin}}] \); b) if a weak-measurement is performed at \( t \) (characterized e.g. by \( \delta P_{md} = \lambda A_w \ll \Delta P_{md} \)), then the outcome of the weak-measurement, the weak-value, can be calculated by propagating the state \( |\Psi_{\text{fin}}\rangle \) backwards in time from \( t_{\text{fin}} \) to \( t \) and the state \( |\Psi_{\text{in}}\rangle \) forwards in time from \( t_{\text{in}} \) to \( t \); the weak-measurement does not cause a collapse and thus no new boundary condition is created at time \( t \).

and use a theorem \[19\]:

\[
\hat{A}|\Psi\rangle = \langle \hat{A} \rangle |\Psi\rangle + \Delta A|\Psi\perp\rangle,
\]

(2.8)

to show that before the post-selection, the system state is:

\[
e^{-i\lambda \hat{Q}_{md}\hat{A}}|\Psi_{\text{in}}\rangle = (1 - i\lambda \hat{Q}_{md}\langle \hat{A} \rangle)|\Psi_{\text{in}}\rangle = (1 - i\lambda \hat{Q}_{md}\langle \hat{A} \rangle)|\Psi_{\text{in}}\rangle - i\lambda \hat{Q}_{md}\Delta \hat{A}|\Psi_{\text{in}}\perp\rangle
\]

(2.9)

Using the norm of this state \( \| (1 - i\lambda \hat{Q}_{md}\hat{A})|\Psi_{\text{in}}\rangle \|^2 = 1 + \lambda^2 \hat{Q}_{md}^2 \langle \hat{A}^2 \rangle \), the probability to leave \( |\Psi_{\text{in}}\rangle \) un-changed after the measurement is:

\[
\frac{1 + \lambda^2 \hat{Q}_{md}^2 \langle \hat{A}^2 \rangle}{1 + \lambda^2 \hat{Q}_{md}^2 \langle \hat{A}^2 \rangle} \rightarrow 1 \quad (\lambda \rightarrow 0)
\]

(2.10)

while the probability to disturb the state (i.e. to obtain \( |\Psi_{\text{in}}\perp\rangle \)) is:

\[
\frac{\lambda^2 \hat{Q}_{md}^2 \Delta \hat{A}^2}{1 + \lambda^2 \hat{Q}_{md}^2 \langle \hat{A}^2 \rangle} \rightarrow 0 \quad (\lambda \rightarrow 0)
\]

(2.11)

where \( \langle \hat{A} \rangle = \langle \Psi|\hat{A}|\Psi\rangle \), \( |\Psi\rangle \) is any vector in Hilbert space, \( \Delta A^2 = \langle \Psi|(\hat{A} - \langle \hat{A} \rangle)^2|\Psi\rangle \), and \( |\Psi\perp\rangle \) is a state such that \( \langle \Psi|\Psi\perp\rangle = 0 \).
The final state of the measuring-device is now a superposition of many substantially overlapping Gaussians with probability distribution given by

\[ Pr(P_{\text{md}}) = \sum_i \left | \langle a_i | \Psi_{\text{in}} \rangle \right |^2 \exp \left\{ -\frac{(P_{\text{md}} - \lambda a_i)^2}{2 \Delta P_{\text{md}}^2} \right\} \].

This sum is a Gaussian mixture, so it can be approximated by a single Gaussian

\[ \tilde{\Phi}^{\text{fin}}_{\text{md}}(P_{\text{md}}) \approx \langle P_{\text{md}} | e^{-i \lambda \hat{Q}_{\text{md}}(\hat{A})} | \Phi_{\text{in}}^{\text{md}} \rangle \approx \exp \left\{ -\frac{(P_{\text{md}} - \lambda \langle \hat{A} \rangle)^2}{2 \Delta P_{\text{md}}^2} \right\} \]

centered on \( \lambda \langle \hat{A} \rangle \).

### 2.1.2 Information gain without disturbance: safety in numbers

It follows from eq. 2.11 that the probability for a collapse decreases as \( O(\lambda^2) \), but the measuring-device’s shift grows linearly \( O(\lambda) \), so \( \delta P_{\text{md}} = \lambda a_i \). For a sufficiently weak interaction (e.g. \( \lambda \ll 1 \)), the probability for a collapse can be made arbitrarily small, while the measurement still yields information but becomes less precise because the shift in the measuring-device is much smaller than its uncertainty \( \delta P_{\text{md}} \ll \Delta P_{\text{md}} \) (figure 5.b). If we perform this measurement on a single particle, then two non-orthogonal states will be indistinguishable. If this were possible, it would violate unitarity because these states could time evolve into orthogonal states \( |\Psi_1\rangle|\Phi_{\text{in}}^{\text{md}}(1)\rangle \rightarrow |\Psi_1\rangle|\Phi_{\text{in}}^{\text{md}}(1)\rangle \) and \( |\Psi_2\rangle|\Phi_{\text{in}}^{\text{md}}\rangle \rightarrow |\Psi_2\rangle|\Phi_{\text{in}}^{\text{md}}(2)\rangle \), with \( |\Psi_1\rangle|\Phi_{\text{in}}^{\text{md}}(1)\rangle \) orthogonal to \( |\Psi_2\rangle|\Phi_{\text{in}}^{\text{md}}(2)\rangle \). With weakened measurement interactions, this does not happen because the measurement of these two non-orthogonal states causes a smaller shift in the measuring-device than it’s uncertainty. We conclude that the shift \( \delta P_{\text{md}} \) of the measuring-device is a measurement error because \( \tilde{\Phi}^{\text{MD}}_{\text{fin}}(P_{\text{md}}) \approx \langle P_{\text{md}} | e^{-i \lambda \hat{Q}_{\text{md}}(\hat{A})} | \Phi_{\text{in}}^{\text{md}} \rangle \approx \exp \left\{ -\frac{(P_{\text{md}} - \lambda \langle \hat{A} \rangle)^2}{2 \Delta P_{\text{md}}^2} \right\} \) for \( \lambda \ll 1 \) Nevertheless, if a large \( (N \geq N') \) ensemble of particles is used, then the shift of all the measuring-devices \( \delta P_{\text{tot}}^{\text{md}} \approx \lambda \langle \hat{A} \rangle N' = N' \langle \hat{A} \rangle \) becomes distinguishable because of repeated integrations, while the collapse probability still goes to zero. That is, for a large ensemble of particles which are all either \( |\Psi_2\rangle \) or \( |\Psi_1\rangle \), this measurement can distinguish between them even if \( |\Psi_2\rangle \) and \( |\Psi_1\rangle \) are not orthogonal.

Using these observations, we now emphasize that the average of any operator \( \hat{A} \), i.e. \( \langle \hat{A} \rangle \equiv \langle \Psi | \hat{A} | \Psi \rangle \), can be obtained in three distinct cases [33, 85]:

1. **Statistical method with disturbance:** the traditional approach is to perform ideal-measurements of \( \hat{A} \) on each particle, obtaining a variety of different eigenvalues, and then manually calculate the usual statistical average to obtain \( \langle \hat{A} \rangle \).

\footnote{because the scalar product \( \langle \Psi^{(N)}_1 | \Psi^{(N)}_2 \rangle = \cos^n \theta \to 0 \)}
2. **Statistical method without disturbance** as demonstrated by using \( \hat{A}|\Psi\rangle = \langle \hat{A} |\Psi\rangle + \Delta A |\Psi_\perp\rangle \). We can also verify that there was no disturbance: consider the spin-1/2 example ([1.2.1]), pre-selecting an ensemble, \(|\uparrow_x\rangle\), then performing a weakened-measurement of \( \hat{\sigma}_\xi \) and finally a post-selection again in the \( x \)-direction (figure 6). For every post-selection, we will again find \(|\uparrow_x\rangle\) with greater and greater certainty (in the weakness limit), verifying our claim of no disturbance. Each measuring device is centered on \( \langle \uparrow_x | \sigma_\xi | \uparrow_x \rangle = \frac{1}{\sqrt{2}} \) and the whole ensemble can be used to reduce the spread (figure 7c). The weakened interaction for \( \hat{\sigma}_\xi \) means that the inhomogeneity in the magnetic field induces a shift in momentum which is less than the uncertainty \( \delta P_\xi < \Delta P_\xi \) and thus a wave packet corresponding to \( \hat{\sigma}_x + \hat{\sigma}_y \sqrt{2} = 1 \) will be broadly overlapping with the wave packet corresponding to \( \hat{\sigma}_x + \hat{\sigma}_y \sqrt{-1} = -1 \). A particular example is depicted in fig. 7.a. (following [51]) with \( \Phi_{\text{in}}^{\text{md}}(P_{\text{md}}) = (\Delta^2 \pi)^{-1/4} \exp\{-P_{\text{md}}^2/2\Delta^2\} \) and \( \Delta \equiv \Delta P_{\text{md}} \) now parametrizes the “weakness” of the interaction instead of \( \lambda \). In the ideal-measurement regime of \( \Delta << 1 \), the probability distribution of the measuring-device is a sum of 2 distributions centered on eigenvalues \( \pm 1 \), figure 7.a.

\[
Pr(P_{\text{md}}) = \cos^2(\pi/8) e^{-(P_{\text{md}}-1)^2/\Delta^2} + \sin^2(\pi/8) e^{-(P_{\text{md}}+1)^2/\Delta^2} \quad (2.12)
\]

The weak regime occurs when \( \Delta \) is larger than the separation between the eigenvalues of \( \pm 1 \) (i.e. \( \Delta >> 1 \)); e.g. 7b.

3. **Non-statistical method without disturbance** is the case where \( \langle \Psi|A|\Psi\rangle \) is the “eigenvalue” of a single “collective operator,” \( \hat{A}^{(N)} \equiv \frac{1}{N} \sum_{i=1}^{N} \hat{A}_i \) (with \( \hat{A}_i \) the same operator \( \hat{A} \) acting on the \( i \)-th particle). Using this, we are able to obtain information about \( \langle \Psi|A|\Psi\rangle \) without causing disturbance (or a collapse) and without using a statistical approach because any product state \( |\Psi^{(N)}\rangle \) becomes an eigenstate of the operator \( \hat{A}^{(N)} \). To see this, we apply the theorem \( \hat{A}|\Psi\rangle = \langle \hat{A} |\Psi\rangle + \Delta A |\Psi_\perp\rangle \) to \( \hat{A}^{(N)}|\Psi^{(N)}\rangle \), i.e.:

\[
\hat{A}^{(N)}|\Psi^{(N)}\rangle = \frac{1}{N} \left[ N\langle \hat{A} |\Psi^{(N)}\rangle + \Delta A \sum_i |\Psi_{\perp}^{(N)}(i)\rangle \right] \quad (2.13)
\]

where \( \langle \hat{A} \rangle \) is the average for any one particle and the states \( |\Psi_{\perp}^{(N)}(i)\rangle \) are mutually orthogonal and are given by \( |\Psi_{\perp}^{(N)}(i)\rangle = |\Psi_1\rangle_1 |\Psi_2\rangle_2 \ldots |\Psi_i\rangle_i \ldots |\Psi_N\rangle \).
That is, the $i$th state has particle $i$ changed to an orthogonal state and all the other particles remain in the same state. If we further define a normalized state $|\Psi_{\perp}^{(N)}\rangle = \sum_{i} \frac{1}{\sqrt{N}} |\Psi^{(N)}(i)i\rangle$ then the last term of eq. 2.13 is $\frac{\Delta A}{\sqrt{N}} |\Psi^{(N)}\rangle$ and it’s size is $|\frac{\Delta A}{\sqrt{N}} |\Psi^{(N)}\rangle|^2 \propto \frac{1}{N} \rightarrow 0$. Therefore, $|\Psi^{(N)}\rangle$ becomes an eigenstate of $\hat{A}^{(N)}$, with the value $\langle \hat{A} \rangle$ and not even a single particle has been disturbed (as $\hat{N} \rightarrow \infty$).

In the last case, the average for a single particle becomes a robust property over the entire ensemble, so a single experiment is sufficient to determine the average with great precision. There is no longer any need to average over results obtained in multiple experiments.

\[\text{all } |\hat{\sigma}_x = +1\rangle \text{ weakened measurement of } \sigma_\xi = 45^\circ \text{ at time } t \]

\[\text{all } |\hat{\sigma}_x = +1\rangle \text{ at time } t_f \]

\[\text{figure 6: Obtaining the average for an ensemble.}\]

Tradition has dictated that when measurement interactions are limited so there is no disturbance on the system, then no information can be gained. However, we have now shown that when considered as a limiting process, the disturbance goes to zero more quickly than the shift in the measuring-device, which means for a large enough ensemble, information (e.g. the expectation value) can be obtained even though not even a single particle is disturbed. This viewpoint thereby shifts the standard perspective on two fundamental postulates of QM.\[^{15}\]

\[^{15}\text{This is also helpful to understand the quantum to classical transition because typical classical interactions involve these collective observables which do not disturb each other.}\]


Figure 7: “Spin component measurement without post-selection.” Probability distribution of the pointer variable for measurement of $\sigma_\xi$ when the particle is pre-selected in the state $|\uparrow_x\rangle$. (a) Strong measurement, $\Delta = 0.1$. (b) Weak measurement, $\Delta = 10$. (c) Weak-measurement on the ensemble of 5000 particles. The original width of the peak, 10, is reduced to $10/\sqrt{5000} \simeq 0.14$. In the strong measurement (a) the pointer is localized around the eigenvalues $\pm 1$, while in the weak-measurements (b) and (c) the peak is located in the expectation value $\langle |\uparrow_x\rangle |\sigma_\xi|\uparrow_x\rangle = 1/\sqrt{2}$.” From [51]
2.1.3 Adding a post-selection to the weakened interaction: Weak-Values and Weak-Measurements

Having established a new measurement paradigm -information gain without disturbance- it is fruitful to inquire whether this type of measurement reveals new values or properties. With weak-measurements (which involve adding a post-selection to this ordinary -but weakened- von Neumann measurement), the measuring-device registers a new value, the weak-value. As an indication of this, we insert a complete set of states \(\{|\Psi_{\text{fin}}\rangle_j\}\) into the outcome of the weak interaction of §2.1.1 (i.e. the expectation value \(\langle \hat{A} \rangle\)):

\[
\langle \hat{A} \rangle = \langle \Psi_{\text{in}} | \sum_j |\Psi_{\text{fin}}\rangle_j \langle \Psi_{\text{fin}}|_j \hat{A} |\Psi_{\text{in}}\rangle = \sum_j |\langle \Psi_{\text{fin}}|_j \langle \Psi_{\text{fin}}|_j \langle \Psi_{\text{fin}}|_j \hat{A} |\Psi_{\text{in}}\rangle|_j \rangle^2 \frac{|\langle \Psi_{\text{fin}}|_j \hat{A} |\Psi_{\text{in}}\rangle|_j \langle \Psi_{\text{fin}}|_j \Psi_{\text{in}}\rangle}{\langle \Psi_{\text{fin}}|_j \Psi_{\text{in}}\rangle}
\]

\[
(2.14)
\]

If we interpret the states \(|\Psi_{\text{fin}}\rangle_j\) as the outcomes of a final ideal-measurement on the system (i.e. a post-selection) then performing a weak-measurement (e.g. with \(\lambda \Delta Q_{\text{md}} \to 0\)) during the intermediate time \(t \in [t_{in}, t_{\text{fin}}]\), provides the coefficients for \(|\langle \Psi_{\text{fin}}|_j \langle \Psi_{\text{fin}}|_j \hat{A} |\Psi_{\text{in}}\rangle|_j \rangle\) which gives the probabilities \(Pr(j)\) for obtaining a pre-selection of \(|\Psi_{\text{in}}\rangle\) and a post-selection of \(|\Psi_{\text{fin}}\rangle_j\). The intermediate weak-measurement does not disturb these states and the quantity \(A_w(j) \equiv \frac{|\langle \Psi_{\text{fin}}|_j \hat{A} |\Psi_{\text{in}}\rangle|_j \langle \Psi_{\text{fin}}|_j \Psi_{\text{in}}\rangle}{\langle \Psi_{\text{fin}}|_j \Psi_{\text{in}}\rangle}\) is the weak-value of \(\hat{A}\) given a particular final post-selection \(\langle \Psi_{\text{fin}}|_j \rangle\). Thus, from the definition \(\langle \hat{A} \rangle = \sum_j Pr(j) A_w(j)\), one can think of \(\langle \hat{A} \rangle\) for the whole ensemble as being constructed out of sub-ensembles of pre-and-post-selected-states in which the weak-value is multiplied by a probability for a post-selected-state.

The weak-value arises naturally from a weakened measurement with post-selection: taking \(\lambda << 1\), the final state of measuring-device in the momentum representation becomes:

\[
\langle P_{\text{md}}|\langle \Psi_{\text{fin}}|e^{-i\lambda \hat{Q}_{\text{md}} A}|\Psi_{\text{in}}\rangle|\Phi_{\text{MD}}\rangle \approx \langle P_{\text{md}}|\langle \Psi_{\text{fin}}|\{1 + i\lambda \hat{Q}_{\text{md}} A\}|\Psi_{\text{in}}\rangle|\Phi_{\text{MD}}\rangle
\]

\[
\approx \langle P_{\text{md}}|\langle \Psi_{\text{fin}}|_j \langle \Psi_{\text{fin}}|_j \{1 + i\lambda \hat{Q}_{\text{md}} A\}|\Psi_{\text{in}}\rangle|\Phi_{\text{MD}}\rangle
\]

\[
\approx \langle \Psi_{\text{fin}}|_j \langle \Psi_{\text{fin}}|_j \langle P_{\text{md}}|e^{-i\lambda \hat{Q}_{\text{MD}} A_w}|\Phi_{\text{MD}}\rangle
\]

\[
\to \langle \Psi_{\text{fin}}|_j \langle \Psi_{\text{fin}}|_j \exp \left\{-(P_{\text{md}} - \lambda A_w)^2\right\}
\]

\[
(2.15)
\]

where \(A_w = \frac{|\langle \Psi_{\text{fin}}|_j \hat{A} |\Psi_{\text{in}}\rangle|_j \langle \Psi_{\text{fin}}|_j \Psi_{\text{in}}\rangle}{\langle \Psi_{\text{fin}}|_j \Psi_{\text{in}}\rangle}\).
The final state of the measuring-device is almost un-entangled with the system; it is shifted by a very unusual quantity, the weak-value, $A_w$, which is not in general an eigenvalue of $\hat{A}$. We have used such limited disturbance measurements to explore many paradoxes (see, e.g. [11, 32]). A number of experiments have been performed to test the predictions made by weak-measurements and results have proven to be in very good agreement with theoretical predictions [54, 55, 56, 52]. Since eigenvalues or expectation values can be derived from weak-values [14], we believe that the weak-value is indeed of fundamental importance in QM. In addition, the weak-value is the relevant quantity for all generalized weak interactions with an environment, not just measurement interactions. The only requirement being that the 2-vectors, i.e. the pre-and-post-selection, are not significantly disturbed by the environment.

2.2 Fundamentally new features of weak-values

2.2.1 Weak-values and 3-box-paradox

Returning to the 3-box-paradox (§1.2.2), we can calculate the weak-values of the number of particles in each box, e.g.:

$$\langle A \rangle_w = \frac{\langle \Psi_{\text{fin}} | A | \Psi_{\text{in}} \rangle}{\langle \Psi_{\text{fin}} | \Psi_{\text{fin}} \rangle} = \frac{\frac{1}{\sqrt{3}} (\langle A | + \langle B | - \langle C |) \langle A | + \langle B | + \langle C |) + \frac{1}{\sqrt{3}} (\langle A | + \langle B | - \langle C |) \langle A | + \langle B | + \langle C |)}{\frac{1}{\sqrt{3}} (\langle A | + \langle B | - \langle C |) \langle A | + \langle B | + \langle C |)} = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{3}$$

However, we can more easily ascertain the weak-values without calculation due to the following theorems:

**Theorem 1**: The sum of the weak-values is equal to the weak-value of the sum [64]:

$$\text{if } (\hat{P}_A)_w = (\hat{P}_B + \hat{P}_C)_w \text{ then } (\hat{P}_A)_w = (\hat{P}_B)_w + (\hat{P}_C)_w \quad (2.16)$$

**Theorem 2** [65]: If a single ideal-measurement of an observable $\hat{P}_A$ is performed between the pre-and-post-selection, then if the outcome is definite (e.g. $\text{Prob}(\hat{P}_A = 1) = 1$) then the weak-value is equal to this eigenvalue (e.g. $(\hat{P}_A)_w = 1$) [6].

This also provides a direct link to the counterfactual statements (§1.2.3) because all counterfactual statements which claim that something occurs

\[16\] Thereby challenging another fundamental postulate of QM.
with certainty, and which can actually be experimentally verified by separate ideal-measurements, continue to remain true when tested by weak-measurements. However, given that weak-measurements do not disturb each other, all these statements can be measured simultaneously.

Applying Theorem 2 to the 3-box-paradox, we know the following weak-values with certainty:

\[ (\hat{P}_A)_{\text{w}} = 1, \quad (\hat{P}_B)_{\text{w}} = 1, \quad \hat{P}_{\text{total}} = (\hat{P}_A + \hat{P}_B + \hat{P}_C)_{\text{w}} = 1. \] (2.17)

Using theorem 1, we obtain:

\[
(\hat{P}_C)_{\text{w}} = \frac{\langle \Psi_{\text{fin}} | \hat{P}_{\text{total}} - \hat{P}_A - \hat{P}_B | \Psi_{\text{in}} \rangle}{\langle \Psi_{\text{fin}} | \Psi_{\text{in}} \rangle} = (\hat{P}_A + \hat{P}_B + \hat{P}_C)_{\text{w}} - (\hat{P}_A)_{\text{w}} - (\hat{P}_B)_{\text{w}} = -1. \] (2.18)

This surprising theoretical prediction of TSQM has been verified experimentally using photons \[39\]. What interpretation should be given to \((\hat{P}_C)_{\text{w}} = -1\)? Any weak-measurement which is sensitive to the projection operator \(\hat{P}_C\) will register the opposite effect from those cases in which the projection operator is positive, e.g. a weak-measurement of the amount of charge in box \(C\) in the intermediate time will yield a negative charge (assuming it is a positively charged particle). For numerous reasons, we believe the most natural interpretation is: there are \(-1\) particles in box \(C\).

### 2.2.2 How the weak-value of a spin-1/2 can be 100

The weak-value for the spin-1/2 considered in \[1.2.1\] (which was confirmed experimentally for an analogous observable, the polarization \[54\]) is:

\[
(\hat{\sigma}_{\xi=45^\circ})_{\text{w}} = \frac{\langle \uparrow_y | \hat{\sigma}_y | \uparrow_x \rangle}{\langle \uparrow_y | \uparrow_x \rangle} = \frac{\{\langle \uparrow_y | \hat{\sigma}_y \rangle + \{\hat{\sigma}_x | \uparrow_x \rangle \}}{\frac{\sqrt{2}}{\sqrt{2}} \langle \uparrow_y | \uparrow_x \rangle} = \frac{\langle \uparrow_y | 1 + 1 | \uparrow_x \rangle}{\sqrt{2} \langle \uparrow_y | \uparrow_x \rangle} = \sqrt{2} \] (2.19)

Normally, the component of spin \(\hat{\sigma}_{\xi}\) is an eigenvalue, \(\pm 1\), but the weak-value \((\hat{\sigma}_{\xi})_{\text{w}} = \sqrt{2}\) is \(\sqrt{2}\) times bigger, (i.e. lies outside the range of eigenvalues of \(\hat{\sigma} \cdot \mathbf{n}\))\(^{17}\). How do we obtain this? Instead of post-selecting \(\hat{\sigma}_x = 1\) (figure

---

\(^{17}\)Weak-values even further outside the eigenvalue spectrum can be obtained by post-selecting states which are more anti-parallel to the pre-selection: e.g. if we post-select the +1 eigenstate of \((\cos \alpha) \sigma_x + (\sin \alpha) \sigma_z\), then \((\hat{\sigma}_z)_{\text{w}} = \lambda \tan \frac{\alpha}{2}\), yielding arbitrarily large values such as spin-100.
we post-select $\hat{\sigma}_y = 1$ which will be satisfied in one-half the trials (figure 8).

To show this in an actual calculation, we use eq. 2.16 and the post-selected state of the quantum system in the $\sigma_\xi$ basis (\(| \uparrow_\xi \rangle \equiv \cos(\pi/8) | \uparrow_\xi \rangle - \sin(\pi/8) | \downarrow_\xi \rangle\)), the measuring-device probability distribution is:

$$
Pr(P_{md}) = N^2[\cos^2(\pi/8)e^{-(P_{md}-1)^2/\Delta^2} - \sin^2(\pi/8)e^{-(P_{md}+1)^2/\Delta^2}]^2
$$

With a strong or ideal-measurement, $\Delta \ll 1$, the distribution is localized again around the eigenvalues $\pm 1$, as illustrated in figures 9.a and 9.b, similar to what occurred in figure 7.a. What is different, however, is that when the measurement is weakened, i.e. $\Delta$ is made larger, then the distribution changes to one single distribution centered around $\sqrt{2}$, the weak-value, as illustrated in figures 9.c-f, (the width again is reduced with an ensemble 9.f). Using eq. 2.14, we can see that the weak-value is just the pre-and-post-selected sub-ensemble arising from within the pre-selected-only ensembles. That is, 9.f is a sub-ensemble from the full ensemble represented by the expectation value, figure 7.c.

The non-statistical aspect mentioned in case-3 (\(\S\) 2.1.2) can also be explored by changing the problem slightly. Instead of considering an ensemble of spin-1/2 particles, we now consider “particles” which are composed of many ($N$) spin-1/2 particles, and perform a weak-measurement of

\[\text{either } |\hat{\sigma}_y = +1\rangle \text{ or } |\hat{\sigma}_y = -1\rangle \]

weak-measurement of $\sigma_\xi=45^\circ$ at time $t$

all $|\hat{\sigma}_x = +1\rangle$

\begin{align*}
&\text{particle 1} \quad \text{particle 2} \quad \text{particle 3} \quad \text{\ldots} \quad \text{\ldots} \quad \text{\ldots} \quad \text{particle N} \\
&\text{\ldots} \quad \text{\ldots} \quad \text{\ldots} \quad \text{\ldots} \quad \text{\ldots} \quad \text{\ldots} \quad \text{\ldots}
\end{align*}

Figure 8: Statistical weak-measurement ensemble.

\[^{18}\text{If a post-selection does not satisfy } \hat{\sigma}_y = +1, \text{ then that member of the sub-ensemble must be discarded. This highlights a fundamental difference between pre-and-post-selection due to the macroscopic arrow-of-time: in contrast to post-selection, if the pre-selection does not satisfy the criteria, then a subsequent unitary transformation can transform to the proper criteria.}\]
Figure 9: **Measurement on pre-and-post-selected ensemble.** Probability distribution of the pointer variable for measurement of $\sigma_\xi$ when the particle is pre-selected in the state $|\uparrow_x\rangle$ and post-selected in the state $|\uparrow_y\rangle$. The strength of the measurement is parameterized by the width of the distribution $\Delta$. (a) $\Delta = 0.1$; (b) $\Delta = 0.25$; (c) $\Delta = 1$; (d) $\Delta = 3$; (e) $\Delta = 10$. (f) Weak-measurement on the ensemble of 5000 particles; the original width of the peak, $\Delta = 10$, is reduced to $10/\sqrt{5000} \simeq 0.14$. In the strong measurements (a)-(b) the pointer is localized around the eigenvalues $\pm 1$, while in the weak-measurements (d)-(f) the peak of the distribution is located in the weak-value $(\sigma_\xi)_w = \langle \uparrow_y | \sigma_\xi | \uparrow_x \rangle / \langle \uparrow_y | \uparrow_x \rangle = \sqrt{2}$. The outcomes of the weak-measurement on the ensemble of 5000 pre-and-post-selected particles, (f), are clearly outside the range of the eigenvalues, $(-1,1)$. From [51]
the collective observable \( \hat{\sigma}_{\xi}^{(N)} \equiv \frac{1}{N} \sum_{i=1}^{N} \hat{\sigma}_{\xi} \) in the 45\(^{\circ}\) -angle to the \( x-y \) plane. Using \( H_{\text{int}} = -\frac{\lambda(\xi)}{N} \hat{Q}_{\text{md}} \sum_{i=1}^{N} \hat{\sigma}_{\xi} \), a particular pre-selection of \( |\uparrow_x\rangle \) (i.e. \( |\Psi_{\text{in}}^{(N)}\rangle = \prod_{i=1}^{N} |\uparrow_x\rangle \)) and post-selection \( |\uparrow_y\rangle \) (i.e. \( \langle \Psi_{\text{fin}}^{(N)} | = \prod_{k=1}^{N} |\uparrow_y\rangle_k = \prod_{n=1}^{N} \{ \langle z|n + i\langle z|n \} \)), the final state of the measuring-device is:

\[
|\Phi_{\text{fin}}^{\text{MD}}\rangle = \prod_{j=1}^{N} |\uparrow_y\rangle_j \exp \left\{ \frac{\lambda}{N} \hat{Q}_{\text{md}} \sum_{k=1}^{N} \hat{\sigma}_{\xi} \right\} \prod_{i=1}^{N} |\uparrow_x\rangle_i |\Phi_{\text{fin}}^{\text{MD}}\rangle
\]  

(2.21)

Since the spins do not interact with each other, we can calculate one of the products and take the result to the \( N \)th power:

\[
|\Phi_{\text{fin}}^{\text{MD}}\rangle = \prod_{j=1}^{N} |\uparrow_y\rangle_j \exp \left\{ \frac{\lambda}{N} \hat{Q}_{\text{md}} \hat{\sigma}_{\xi} \right\} |\uparrow_x\rangle |\Phi_{\text{fin}}^{\text{MD}}\rangle = \left\{ \langle \uparrow_y | \exp \left\{ \frac{\lambda}{N} \hat{Q}_{\text{md}} \hat{\sigma}_{\xi} \right\} |\uparrow_x\rangle \right\} |\Phi_{\text{fin}}^{\text{MD}}\rangle
\]  

(2.22)

Using the following identity \( \exp \{ i\alpha \hat{\sigma}_{\xi} \} = \cos \alpha + i\hat{\sigma}_{\xi} \sin \alpha \) [57], this becomes:

\[
|\Phi_{\text{fin}}^{\text{MD}}\rangle = \left\{ \langle \uparrow_y | \left[ \cos \frac{\lambda \hat{Q}_{\text{md}}}{N} - i\hat{\sigma}_{\xi} \sin \frac{\lambda \hat{Q}_{\text{md}}}{N} \right] |\uparrow_x\rangle \right\} |\Phi_{\text{fin}}^{\text{MD}}\rangle
\]  

(2.23)

where we have substituted \( \alpha_w \equiv \langle \hat{\sigma}_{\xi} \rangle_w = \frac{\langle \uparrow_y | \hat{\sigma}_{\xi} |\uparrow_x\rangle}{\langle \uparrow_y | \uparrow_x \rangle} \). We consider only the second part (the first bracket, a number, can be neglected since it does not depend on \( \hat{Q} \) and thus can only affect the normalization):

\[
|\Phi_{\text{fin}}^{\text{MD}}\rangle = \left\{ 1 - \frac{\lambda^2 \langle \hat{Q}_{\text{md}} \rangle^2}{N^2} - i\langle \uparrow_y | \hat{\sigma}_{\xi} |\uparrow_x\rangle \right\} |\Phi_{\text{fin}}^{\text{MD}}\rangle \approx e^{i\lambda \alpha_w \hat{Q}_{\text{md}} / N} |\Phi_{\text{fin}}^{\text{MD}}\rangle
\]  

(2.24)

When\(^{19}\) projected onto \( P_{\text{md}} \), i.e. the pointer, we see that the pointer is robustly shifted by the the same weak-value obtained with the previous statistical method, i.e. \( \sqrt{2} \):

\[
\langle \hat{\sigma}_{\xi} \rangle_w = \frac{\prod_{k=1}^{N} \langle \uparrow_y | \hat{\sigma}_{\xi} \rangle_{\uparrow_x} \prod_{j=1}^{N} |\uparrow_x\rangle_j}{\sqrt{2} N \langle \langle y| \uparrow_x \rangle \rangle^N} = \sqrt{2} \pm O \left( \frac{1}{\sqrt{N}} \right).
\]  

(2.25)

\(^{19}\)The last approximation was obtained as \( N \to \infty \), using \( (1 + \frac{a}{N})^N = (1 + \frac{a}{N})^N \approx e^a \).
A single experiment is now sufficient to determine the weak-value with great precision and there is no longer any need to average over results obtained in multiple experiments as we did in the previous section. Therefore, if we repeat the experiment with different measuring-devices, then each measuring-device will show the very same weak-values, up to an insignificant spread of $\frac{1}{\sqrt{N}}$ and the information from both boundary conditions, i.e. $|\Psi_{\text{in}}\rangle = \prod_{i=1}^{N} |\uparrow_x\rangle_i$ and $\langle \Psi_{\text{fin}} | = \prod_{i=1}^{N} \langle \downarrow_y |$, describes the entire interval of time between pre-and-post-selection. Following [51], we consider an example with $N = 20$. The probability distribution of the measuring-device after the post-selection is:

$$\text{prob}(Q_{md}^{(N)}) = \mathcal{N}^2 \left( \sum_{i=1}^{N} (-1)^i (\cos^2(\pi/8))^N - i (\sin^2(\pi/8))^i e^{-\left(\frac{(Q_{md}^{(N)} - (2N-1)^2/2\Delta^2)^2}{2}\right)} \right).$$

(2.26)

and is drawn for different values of $\Delta$ in figure 10. While this result is rare, we have recently shown [33] how any ensemble can yield robust weak-values like this in a way that is not rare and for a much stronger regime of interaction. We have thereby shown that weak-values are a general property of every pre-and-post-selected ensemble.  

2.2.3 Hardy’s Paradox

Another surprising pre-and-post-selection effect is Hardy’s gedanken-experiment which is a variation of interaction-free measurements (IFM) [58], consisting of two “superposed” Mach-Zehnder interferometers (MZI)(figure 11), one with a positron and one with an electron. Consider first a single interferometer, for instance that of the positron (labeled by +). By adjusting the arm lengths, it is possible to arrange specific relative phases in the propagation amplitudes for paths between the beam-splitters $BS1^+$ and $BS2^+$ so that the positron can only emerge towards the detector $C^+$. However, the phase difference can be altered by the presence of an object, for instance in the lower arm, in which case detector $D^+$ may be triggered. In the usual IFM setup, this is illustrated by the dramatic example of a sensitive bomb that absorbs the particle with unit probability and subsequently explodes. In this way, if $D^+$ is triggered, it is then possible to infer the presence of the bomb

\footnote{We have also proposed this as another innovative new laser-technology, e.g. in the amplification of small non-random signals by minimizing uncertainties in determining the weak value and by minimizing sample size. [33]}
Figure 10: "Measurement on a single system. Probability distribution of the pointer variable for the measurement of $A = \left( \sum_{i=1}^{20} (\sigma_i)_z \right) / 20$ when the system of 20 spin-$\frac{1}{2}$ particles is pre-selected in the state $|\Psi_1\rangle = \prod_{i=1}^{20} |\uparrow_x\rangle_i$ and post-selected in the state $|\Psi_2\rangle = \prod_{i=1}^{20} |\uparrow_y\rangle_i$. While in the very strong measurements, $\Delta = 0.01 - 0.05$, the peaks of the distribution located at the eigenvalues, starting from $\Delta = 0.25$ there is essentially a single peak at the location of the weak-value, $A_w = \sqrt{2}$." From [51]
without “touching” it, i.e., to know both that there was a bomb and that the particle went through the path where there was no bomb.

Figure 11: a) counterfactual resolution: $D_0^-$ disturbs the electron and the electron could end up in the $D^-$ detector even if no positron were present in the overlapping arm, b) electron must be on the overlapping path $\hat{N}_O^- = 1$, c) positron also must be on overlapping path $\hat{N}_O^+ = 1$

In the double-MZI, things are arranged so that if each MZI is considered separately, the electron can only be detected at $C^-$ and the positron only at $C^+$. However, because there is now a region where the two particles overlap, there is also the possibility that they will annihilate each other. We assume that this occurs with unit probability if both particles happen to be in this region.

According to QM, the presence of this interference-destroying alternative allows for a situation similar to IFM in which detectors $D^-$ and $D^+$ may click in coincidence (in which case, obviously, there is no annihilation).

Suppose $D^-$ and $D^+$ do click. Trying to “intuitively” understand this situation leads to paradox. For example, we should infer from the clicking of $D^-$ that the positron must have gone through the overlapping arm; otherwise nothing would have disturbed the electron, and the electron couldn’t have ended in $D^-$. Conversely, the same logic can be applied starting from the clicking of $D^+$, in which case we deduce that the electron must have also gone through the overlapping arm. But then they should have annihilated, and couldn’t have reached the detectors. Hence the paradox.

These statements, however, are counter-factual, i.e. we haven’t actually measured the positions. Suppose we actually measured the position of the
electron by inserting a detector $D_O^-$ in the overlapping arm of the electron-MZI. Indeed, the electron is always in the overlapping arm. But, we can no longer infer from a click at $D^-$ that a positron should have traveled through the overlapping arm of the positron MZI in order to disturb the electron (figure 11.a). The paradox disappears.

As we mentioned (§1.2.3), weak-measurements produce only limited disturbance and therefore can be performed simultaneously, allowing us to experimentally test such counter-factual statements. Therefore we would like to test [11, 32] questions such as “Which way does the electron go?”, “Which way does the positron go?”, “Which way does the positron go when the electron goes through the overlapping arm?” etc. In other words, we would like to measure the single-particle “occupation” operators

$$
\hat{N}_{NO}^+ = |NO\rangle_p \langle NO|_p \\
\hat{N}_{NO}^- = |NO\rangle_e \langle NO|_e \\
\hat{N}_{O}^+ = |O\rangle_p \langle O|_p \\
\hat{N}_{O}^- = |O\rangle_e \langle O|_e
$$

which tell us separately about the electron and the positron. We note a most important fact, which is essential in what follows: the weak-value of a product of observables is not equal to the product of their weak-values. Hence, we have to measure the single-particle occupation-numbers independently from the pair occupation-operators:

$$
\hat{N}_{NO,O}^{+,-} = \hat{N}_{NO}^{+} \hat{N}_{O}^{-} \\
\hat{N}_{O,NO}^{+,-} = \hat{N}_{O}^{+} \hat{N}_{NO}^{-} \\
\hat{N}_{NO,NO}^{+,-} = \hat{N}_{NO}^{+} \hat{N}_{NO}^{-}
$$

These tell us about the simultaneous locations of the electron and positron. The results of all our weak-measurements on the above quantities, echo, to some extent, the counter-factual statements, but go far beyond that. They are now true observational statements (and experiments have successfully verified these results [59]). In addition, weak-values obey an intuitive logic of their own which allows us to deduce them directly. While this full-intuition is left to published articles [11, 32], we discuss the essence of the paradox which is defined by three counterfactual statements:

- The electron is always in the overlapping arm.
- The positron is always in the overlapping arm.
- The electron and the positron are never both in the overlapping arms.
To these counterfactual statements correspond the following \textit{observational} facts. In the cases when the electron and positron end up at $D^-$ and $D^+$ respectively and if we perform a single ideal-measurement of:

- $\hat{N}_O^-$, we always find $\hat{N}_O^- = 1$ (figure 11b).
- $\hat{N}_O^+$, we always find $\hat{N}_O^+ = 1$ (figure 11c).
- $\hat{N}_{O,O}^{+, -}$, we always find $\hat{N}_{O,O}^{+, -} = 0$ (figure 12a).

The above statements seem paradoxical but, of course, they are valid only if we perform the measurements separately; they do not hold if the measurements are made simultaneously. However, Theorem 2 says that when measured weakly all these results remain true simultaneously:

$$N_{Ow}^- = 1, \quad N_{Ow}^+ = 1 \quad (2.29)$$

Using theorems 1 and 2, all other weak-values can be trivially deduced:

$$N_{NOw}^- = 0, \quad N_{NOw}^+ = 0 \quad (2.30)$$

$$N_{O, Ow}^{+, -} = 0 \quad (2.31)$$

$$N_{O, NOw}^{+, -} = 1, \quad N_{NO, Ow}^{+, -} = 1 \quad (2.32)$$

$$N_{NO, NOw}^{+, -} = -1. \quad (2.33)$$

What do all these results tell us?

First of all, the single-particle occupation numbers (2.29) are consistent with the intuitive statements that “the positron must have been in the overlapping arm otherwise the electron couldn’t have ended at $D^-$” (figure 11c) and also that “the electron must have been in the overlapping arm otherwise the positron couldn’t have ended at $D^+$” (figure 11b). But then what happened to the fact that they could not be both in the overlapping arms since this will lead to annihilation? QM is consistent with this too - the pair occupation number $N_{O, Ow}^{+, -} = 0$ shows that there are zero electron-positron pairs in the overlapping arms (figure 12a)!

We also feel intuitively that “the positron must have been in the overlapping arm otherwise the electron couldn’t have ended at $D^-$, and furthermore,
the electron must have gone through the non-overlapping arm since there was no annihilation” (figure 12.b). This is confirmed by $N_{O,NO}^+ = 1$. But we also have the statement “the electron must have been in the overlapping arm otherwise the positron couldn’t have ended at $D^-$ and furthermore the positron must have gone through the non-overlapping arm since there was no annihilation”. This is confirmed too, $N_{NO,OW}^+ = 1$. But these two statements together are at odds with the fact that there is in fact just one electron-positron pair in the interferometer. QM solves the paradox in a remarkable way - it tells us that $N_{NO,NOw}^+ = -1$, i.e. that there is also minus one electron-positron pair in the non-overlapping arms which brings the total down to a single pair (figure 12.c)!

Figure 12: a) $\hat{N}_{O,O}^+ = 0$, b) $N_{O,NOw}^+ = 1$, $N_{NO,OW}^+ = 1$, c) $N_{NO,NOw}^+ = -1$

2.3 Contextuality

TSQM and WMs have proven very useful in exploring many unsettled aspects of QM. For example, using TSQM, we have shown that it is possible to assign definite values to observables in a new way in situations involving “contextuality.” Traditionally, contextuality was thought to be a requirement for certain hypothetical modifications of QM. However, using pre-and-post-selection and weak-measurements, we have shown that QM implies contextuality directly [69, 70, 32].

What is contextuality? Bell-Kochen-Specker (BKS) proved that one cannot assign unique answers (i.e. a Hidden-Variable-Theory, HVT) to yes-no questions in such a way that one can think that measurement simply reveals the answer as a pre-existing property that was intrinsic solely to the quantum
system itself. BKS assumed that the specification of the HVT, i.e. $V_\psi(\hat{A})$, should satisfy: $V_\psi(F\{\hat{A}\}) = F\{V_\psi(\hat{A})\}$, i.e. any functional relation of an operator that is a member of a commuting subset of observables must also be satisfied if one substitutes the values for the observables into the functional relations. A consequence of this is satisfaction of the sum and product rules and therefore BKS showed that with any system (of dimension greater than 2) the $2^n$ possible “yes-no” assignments (to the $n$ projection operators representing the yes-no questions) cannot be compatible with the sum and product rules for all orthogonal resolutions of the identity. Thus, a HVT-the hypothetical modification of QM-must be contextual.

In [32] it was first pointed out and extensively discussed and later proven [28], that whenever there is a logical pre-and-post-selection-paradox (as in the 3-box-paradox §1.2.2), then there is a related proof of contextuality. However, the elements in the proof are all counter-factual. TSQM has taught us that by applying theorems 1 and 2, we can for the first time obtain an experimental meaning to the proof.

By way of example, we consider Mermin’s version of BKS with a set of 9 observables. It is intuitive [30] to represent all the “functional relationships between mutually commuting subsets of the observables,” i.e. $V_\psi(F\{\hat{A}\}) = F\{V_\psi(\hat{A})\}$, by drawing them in fig. 13 and arranging them so that all the observables in each row (and column) commute with all the other observables in the same row (or column).

\[
\begin{array}{c|c|c|c}
\hat{\sigma}_x^1 & \hat{\sigma}_y^1 & \hat{\sigma}_z^1 & 1 \\
\hat{\sigma}_y^2 & \hat{\sigma}_z^2 & \hat{\sigma}_z^1\hat{\sigma}_z^2 & 1 \\
\hat{\sigma}_x^1\hat{\sigma}_y^2 & \hat{\sigma}_x^1\hat{\sigma}_y^2 & \hat{\sigma}_x^1\hat{\sigma}_z^2 & 1 \\
1 & 1 & -1 & 1 \\
\end{array}
\]

Figure 13: 4-D BKS example

$V_\psi(F\{\hat{A}\}) = F\{V_\psi(\hat{A})\}$ requires that the value assigned to the product of all three observables in any row or column must obey the same identities that the observables themselves satisfy, i.e. the product of the values assigned to
the observables in each oval yields a result of +1 except in the last column which gives −1. E.g. computing column 3 of fig. 13:

\[
\{\hat{\sigma}_x^1\hat{\sigma}_y^2\{\hat{\sigma}_y^1\hat{\sigma}_y^2\{\hat{\sigma}_z^1\hat{\sigma}_z^2\} = \hat{\sigma}_x^1\hat{\sigma}_y^2\hat{\sigma}_y^1\hat{\sigma}_z^1\hat{\sigma}_x^2\hat{\sigma}_y^2 = \hat{\sigma}_x^1\hat{\sigma}_y^2\hat{\sigma}_y^1\hat{\sigma}_z^1\hat{\sigma}_x^2\hat{\sigma}_y^2 = i\hat{\sigma}_z^1 = i\hat{\sigma}_z^2
\]

Computing the product of the observables in the third row, i.e.:

\[
\{\hat{\sigma}_x^1\hat{\sigma}_y^2\{\hat{\sigma}_z^2\hat{\sigma}_y^1\{\hat{\sigma}_z^1\hat{\sigma}_z^2\} = \hat{\sigma}_x^1\hat{\sigma}_y^2\hat{\sigma}_z^2\hat{\sigma}_y^1\hat{\sigma}_z^1\hat{\sigma}_z^2 = i\hat{\sigma}_x^2\hat{\sigma}_y^1 \hat{\sigma}_z = i\hat{\sigma}_x^2\hat{\sigma}_y^1 \hat{\sigma}_z = -1
\] (2.34)

If the product rule is applied to the value assignments made in the rows, then:

\[
\begin{array}{l}
\underline{\text{row 1}}: V_\psi(\hat{\sigma}_x^1)V_\psi(\hat{\sigma}_y^2)V_\psi(\hat{\sigma}_z^1) = +1 \\
\underline{\text{row 2}}: V_\psi(\hat{\sigma}_x^1)V_\psi(\hat{\sigma}_y^2)V_\psi(\hat{\sigma}_z^2) = +1 \\
\underline{\text{row 3}}: V_\psi(\hat{\sigma}_x^1)V_\psi(\hat{\sigma}_y^2)V_\psi(\hat{\sigma}_z^1) = +1
\end{array}
\] (2.36)

while the column identities require:

\[
\begin{array}{l}
\underline{\text{column 1}}: V_\psi(\hat{\sigma}_x^1)V_\psi(\hat{\sigma}_y^2)V_\psi(\hat{\sigma}_z^1) = +1 \\
\underline{\text{column 2}}: V_\psi(\hat{\sigma}_x^1)V_\psi(\hat{\sigma}_y^2)V_\psi(\hat{\sigma}_z^2) = +1 \\
\underline{\text{column 3}}: V_\psi(\hat{\sigma}_x^1)V_\psi(\hat{\sigma}_y^2)V_\psi(\hat{\sigma}_z^2) = -1
\end{array}
\] (2.37)

However, it is easy to see that the 9 numbers \(V_\psi\) cannot satisfy all 6 constraints because multiplying all 9 observables together gives 2 different results, a +1 when it is done row by row and a −1 when it is done column by column.

\footnote{The value assignments are given by \(V_\psi(\hat{\sigma}_x^1) = \langle \hat{\sigma}_x^1 \otimes I^2 \rangle\), \(V_\psi(\hat{\sigma}_y^2) = \langle I^1 \otimes \hat{\sigma}_y^2 \rangle\)...
\(V_\psi(\hat{\sigma}_z^2) = \langle \hat{\sigma}_z^2 \otimes \hat{\sigma}_z^2 \rangle\).}
column:

\[
\begin{align*}
V_\psi(\hat{\sigma}_x) V_\psi(\hat{\sigma}_x^2) & V_\psi(\hat{\sigma}_y) V_\psi(\hat{\sigma}_y^2) & V_\psi(\hat{\sigma}_z) V_\psi(\hat{\sigma}_z^2) = +1 \\
V_\psi(\hat{\sigma}_x) V_\psi(\hat{\sigma}_x^2) & V_\psi(\hat{\sigma}_y) V_\psi(\hat{\sigma}_y^2) & V_\psi(\hat{\sigma}_z) V_\psi(\hat{\sigma}_z^2) = -1
\end{align*}
\]

There obviously is no consistent solution to eqs. \(2.39\) and \(2.38\) since they contain the same set of numbers, simply ordered differently. Therefore the values assigned to the observables cannot obey the same identities that the observables themselves obey, \(V_\psi(F\{\hat{A}\}) \neq F\{V_\psi(\hat{A})\}\), and an HVT would have to assign values to observables in a way that depended on the choice of which of 2 mutually commuting sets of observables that were also chosen to measure, i.e. the values assigned are contextual. For example, the assignment \(\hat{\sigma}_x^1 \hat{\sigma}_x^2 = \pm 1\) depends on whether we associate \(\hat{\sigma}_x^1 \hat{\sigma}_x^2\) with row-3 or with column-3.

We briefly summarize application of TSQM to the Mermin example \[69, 70, 32\]. A single pre-and-post-selection (figure 14) allows us to assign a definite value to any single observable in figure 13. That by itself is new and surprising.

Moreover, for “contextuality,” we must determine how many of the products of the 9 observables in figure 13 can be ascertained together with certainty. In order to ascertain the products of any 2 pairs, the generalized state is required, an outcome that Mermin describes as “intriguing” \[29\].
The generalized state is defined by [6]: \( \Psi = \sum_i \alpha_i \langle \Psi_i | \Phi_i \rangle \). The outcome for the product of the first two observables in column 3 of figure 13 with the pre-and-post-selection of fig. 14a is \( \sigma_1^x \sigma_2^x \sigma_1^y \sigma_2^y = +1 \). However, if we measure the operators corresponding to the first 2 observables of row 3 in figure 13, i.e. \( \sigma_1^x \sigma_2^y \sigma_2^x \sigma_1^y \), given this particular pre-and-post-selection shown in fig. 14a, then the sequence of measurements interfere with each other (as represented by the slanted ovals in figure 17a). To see this, consider that \( \sigma_1^x \sigma_2^y \sigma_2^x \sigma_1^y \) corresponds to the sequence of measurements represented in figure 16a. While the pre-selection of particle 2 is \( \sigma_2^2 = 1 \) at \( t_{in} \), the next measurement after the pre-selection at \( t_2 \) is for \( \sigma_2^2 \) and only after that a measurement of \( \sigma_2^2 \) is performed at \( t_3 \). Thus, there is no guarantee that the \( \sigma_2^2 \) measurement at \( t_3 \) will give the same value as the pre-selected state of \( \sigma_2^2 = 1 \) or that the \( \sigma_2^2 \) measurement will give the same value as the post-selected state of \( \sigma_2^2 = 1 \). In TSQM, this is due to the disturbance of the 2-vector boundary conditions which is created by the ideal-measurement ([2.1]): the initial pre-selected vector \( \sigma_2^2 = 1 \) from \( t_{in} \) is “destroyed” when the \( \sigma_2^2 \) measurement at time \( t_2 \) is performed and therefore cannot inform the later \( \sigma_2^2 \) measurement at time \( t_3 \). In other words, with the particular pre-and-post-selection given in fig. 14a and 16a, the operator, \( \sigma_1^x \sigma_2^y \sigma_2^x \sigma_1^y \) depends on information from

[22] This correlated state can be created by preparing at \( t_{in} \) a correlated state \( \sum_i \alpha_i \langle \Psi_i | \Phi_i \rangle \) with \( \langle \Phi_i \rangle \) an orthonormal set of states of an ancilla. Then the ancilla is “guarded” so there are no interactions with the ancilla during the time \( (t_{in}, t_{fin}) \). At \( t_{fin} \) we post select on the particle and ancilla the state \( \frac{1}{\sqrt{N}} \sum_i \langle \Phi_i | \Phi_i \rangle \). If we are successful in obtaining this state for the post-selection, then the state of 2 particles is described in the intermediate time by the entangled state (see figure 15). This is yet another example of a useful generalization of QM.

Figure 15: Generalized State: superpositions of 2-vectors.
both the pre-selected vector $\hat{\sigma}_1 = 1, \hat{\sigma}_2 = 1$ and the post-selected vector $\hat{\sigma}_1 = 1, \hat{\sigma}_2 = 1$ in a “diagonal-pre-and-post-selection” sense. We call this diagonal-pre-and-post-selection because a line connecting $\hat{\sigma}_1(t_1)$ with $\hat{\sigma}_2(t_3)$ will be diagonal or will cross the line connecting $\hat{\sigma}_1(t_2)$ with $\hat{\sigma}_2(t_4)$, where $t_{in} < t_1 < t_2 < ... < t_{fin}$, see fig. 17(a). However, $\hat{\sigma}_1 \hat{\sigma}_2$ is assigned different values in different pre-and-post-selections. It is precisely because of this connection between particular pre-and-post-selections and different values for $\hat{\sigma}_1 \hat{\sigma}_2$ that the issue of contextuality arises when we consider products of these observables. In other words, the contextuality here is manifested by the fact that $\hat{\sigma}_1 \hat{\sigma}_2 \hat{\sigma}_1 \hat{\sigma}_2 = -1$ (given the pre-and-post-selection of fig. 16(a)) even though separately $\hat{\sigma}_1 \hat{\sigma}_2 = +1$ and $\hat{\sigma}_2 \hat{\sigma}_1 = +1$. But these 3 outcomes can be measured weakly without contradiction because the product of weak-values

Figure 16: Time sequence of pre-and-post-selection measurements for Mermin example.

Figure 17: a) Measurement of $\hat{\sigma}_1 \hat{\sigma}_2 \hat{\sigma}_2 \hat{\sigma}_1$ is diagonal, b) measurement of $\hat{\sigma}_1 \hat{\sigma}_2 \hat{\sigma}_1 \hat{\sigma}_2$ is diagonal.
is not equal to the weak-value of the product. Therefore, instead of contextuality being an aspect of a hypothetical replacement for QM (the HVT), we have shown that contextuality is directly part of QM [69, 70, 32].

2.4 Nonlocality

Traditionally, it was believed that “contextuality” was very closely related to “kinematic-nonlocality.” Typically, kinematic-nonlocality refers to correlations, such as eq. [1] that violate Bell’s-inequality with the consequence that QM cannot be replaced with a local realistic model. Similarly, contextuality refers to the impossibility of replacing QM with a noncontextual realistic theory. Applying this now to the relativistic-paradox (§[1]), we see that Lorentz covariance in the state-description can be preserved in TSQM [9] because the post-selected vector $\sigma_A^A = +1$ propagates all the way back to the initial preparation of an EPR state, eq. [1] $|\Psi_{EPR}\rangle = \frac{1}{\sqrt{2}} \{ |\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B \}$. E.g. if $A$ changes his mind and measures $\sigma_A^y$ instead of $\sigma_A^z$ or if we consider a different frame-of-reference, then this would change the post-selected vector all the way back to $|\Psi_{EPR}\rangle$. More explicitly, suppose the final post-selected-state is $\langle \Psi_{fin} | = \frac{1}{\sqrt{2}} \{ \langle \uparrow_z | A \{ \langle \downarrow | B + \langle \uparrow | B \} = \frac{1}{\sqrt{2}} \{ \langle \uparrow_z | A \langle \uparrow | B + \langle \uparrow_z | A \langle \downarrow | B \} \}$. The full state-description is the bra-ket combination (which is not just a scalar product):\begin{equation}
\langle \Psi_{fin} || \Psi_{EPR} \rangle = \frac{1}{\sqrt{2}} \{ \langle \uparrow | A \langle \uparrow | B + \langle \uparrow | A \langle \downarrow | B \} \} \frac{1}{\sqrt{2}} \{ \langle \uparrow | A \langle \downarrow | B - \langle \down | A \langle \uparrow | B \} \}
\end{equation}
(2.40)

There is no longer a need to specify a moment when a non-local collapse occurs, thereby removing the relativistic paradox.

Finally, TSQM and weak-measurements also provide insight into a very different kind of non-locality, namely dynamical-nonlocality, e.g. that of the Aharonov-Bohm (AB) effect [1]. We have shown how this novel kind of nonlocality can be measured with weak-measurements [71].
3 TSQM lead to new mathematics, simplifications in calculations, and stimulated discoveries in other fields

TSQM has influenced work in many areas of physics, e.g. in cosmology [22, 81], in black-holes [21, 44], in superluminal tunneling [19, 38], in quantum information [8, 82, 83], etc. We review two examples here.

3.1 Super-oscillations

Superoscillations [60] are functions which oscillate with an arbitrarily high frequency $\alpha$, but which, surprisingly, can be understood as superpositions of low frequencies, $|k| < 1$, seemingly a violation of the Fourier theorem:

$$\sum_{|k|<1} c_k e^{ikx} \to e^{i\alpha x} \quad (3.41)$$

Superoscillations were originally discovered through the study of weak-values. By way of example, consider again eq. 2.23:

$$|\Phi_{MD}^{fin}\rangle = \left\{ \cos \frac{\lambda \hat{Q}_{md}}{N} - i\alpha_w \sin \frac{\lambda \hat{Q}_{md}}{N} \right\} |\Phi_{in}^{MD}\rangle$$

$$= \left\{ \frac{e^{i\lambda \hat{Q}_{md}}}{\sqrt{2}} + \alpha_w - \frac{e^{-i\lambda \hat{Q}_{md}}}{\sqrt{2}} \right\} |\Phi_{in}^{MD}\rangle$$

$$= \left\{ e^{i\lambda \hat{Q}_{md}} \frac{(1 + \alpha_w)}{\sqrt{2}} + e^{-i\lambda \hat{Q}_{md}} \frac{(1 - \alpha_w)}{\sqrt{2}} \right\} |\Phi_{in}^{MD}\rangle \quad (3.42)$$

We already saw how this could be approximated as $e^{i\lambda \alpha_w \hat{Q}_{md}} |\Phi_{in}^{MD}\rangle$ which produced a robust-shift in the measuring-device by the weak-value $\sqrt{2}$. However, we can also view $\psi(x) = \left\{ e^{i\lambda \hat{Q}_{md}} \frac{(1 + \alpha_w)}{\sqrt{2}} + e^{-i\lambda \hat{Q}_{md}} \frac{(1 - \alpha_w)}{\sqrt{2}} \right\}^N$ in a very different way, by performing a binomial expansion:

$$\psi(x) = \sum_{n=0}^{N} \frac{(1+\alpha_w)^n(1-\alpha_w)^{N-n}}{2^N n!(N-n)!} \exp \left\{ \frac{in\lambda \hat{Q}_{md}}{N} \right\} \exp \left\{ -i\lambda \hat{Q}_{md}(N-n) \right\}$$
\[ = \sum_{n=0}^{N} c_n \exp \left\{ \frac{i\lambda Q_{\text{mod}}(2n-N)}{N} \right\} = \sum_{n=0}^{N} c_n \exp \left\{ \frac{i\lambda Q_{\text{mod}}\lambda_n}{N} \right\} \] (3.43)

We see that this wavefunction is a superposition of waves with small wavenumbers \(|k| \leq 1\) (because \(-1 < \frac{(2n-N)}{N} < 1\)). For a small region (which can include several wavelengths \(2\pi/\alpha_w\), depending on how large one chooses \(N\)), \(\psi(x)\) appears to have a very large momentum, since \(\alpha_w\) can be arbitrarily large, i.e. a super-oscillation. Because these regions of superoscillations are created at the expense of having the function grow exponentially in other regions, it would be natural to conclude that the superoscillations would be quickly “over-taken” by tails coming from the exponential regions and would thus be short-lived. However, it has been shown that superoscillations are remarkably robust \([61]\) and can last for a surprisingly long time. This has therefore led to proposed/practical applications of superoscillations to situations which were previously probed by evanescent waves (e.g. in the superresolution of very fine features with lasers). \(^{23}\)

As we mentioned in the introduction, TSQM is a re-formulation of QM, and therefore it must be possible to view the novel effects from the traditional single-vector perspective. This is precisely what super-oscillations teach us. In summary, there are 2 ways to understand weak-values:

- the measuring-device is registering the weak-value as a property of the system as characterized by TSQM
- the weak-value is a result of a complex interference effect in the measuring-device; the system continues to be described with a single-vector pursuant to QM

Oftentimes, calculations are either much simplified or can only be performed by utilizing the first approach (e.g. when the measuring-device is classical) \([32]\).

### 3.2 Quantum Random Walk

Another fundamental discovery arising out of TSQM is the Quantum-Random-Walk \([8]\) which has also stimulated discoveries in other areas of physics (for a

\(^{23}\)In \([68, 84]\), we uncover several new relationships between the physical creation of the high-momenta associated with the superoscillations, eccentric weak-values, and modular variables which have been used to model the dynamical non-locality discussed in \([24, 02, 63, 13]\).
Figure 18: “Demonstration of an approximate equality given by $\sum_{n=0}^{N} c_n f(t - a_n) \approx f(t - \alpha)$. The sum of a function shifted by the 14 values $c_n$ between 0 and 1 and multiplied by the coefficients, yields approximately the same function shifted by the value 10. The dotted line shows $f(t)$; the dashed line shows $f(t - 10)$; and the solid line shows the sum.” From [51]
41

review, see [34]). In the second bullet above, the measuring-device is shifted by the operator \( \hat{\sigma}_\xi^{(N)} \) with it’s \( N + 1 \) eigenvalues equally spaced between \(-1\) and \(+1\) [32]. How can a superposition of small shifts between \(-1\) and \(+1\) give a shift that is arbitrarily far outside \( \pm 1 \)? The answer is that states of the measuring-device interfere constructively for \( \hat{P}_{md}^{(N)} = \alpha_w \) and destructively for all other values of \( \hat{P}_{md}^{(N)} \) such that \( \Phi_{MD}^{MD}(P) \rightarrow \Phi_{MD}^{MD}(P - A_w) \), the essence of quantum-random-walk[8]. If the coefficients for a step to the left or right were probabilities, as would be the case in a classical random walk, then \( N \) steps of step size \( 1 \) could generate an average displacement of \( \sqrt{N} \), but never a distance larger than \( N \). However when the steps are superposed with probability \( \textit{amplitudes} \), as with the quantum-random-walk, and when one considers probability amplitudes that are determined by pre-and-post-selection, then the random walk can produce any displacement. In other words, instead of saying that a “quantum step” is made up of probabilities, we say that a quantum step is a superposition of the amplitude for a step “to the left” and the amplitude for a “step to the right,” then one can superpose small Fourier components and obtain a large shift. This phenomenon is very general: if \( f(t - a_n) \) is a function shifted by small numbers \( a_n \), then a superposition can produce the same function but shifted by a value \( \alpha \) well outside the range of \( a_n \): \( \sum_{n=0}^{N} c_n f(t - a_n) \approx f(t - \alpha) \). The same values of \( a_n \) and \( c_n \) are appropriate for a wide class of functions and this relation can be made arbitrarily precise by increasing the number of terms in the sum, see figure 18. The key to this phenomenon is the extremely rapid oscillations in the coefficients \( c_n \equiv \frac{(1+\alpha_w)^n(1-\alpha_w)^{N-n}}{2^n n!(N-n)!} \) in \( \sum_{n=0}^{N} c_n \exp \{i\lambda \hat{Q}_{md} k_n\} \).

4 TSQM suggests generalizations of QM

4.1 Reformulation of Dynamics: each moment a new universe

We review a generalization of QM suggested by TSQM [31 32] which addresses the “artificial” separation in all areas of theoretical physics between the kinematic and dynamical descriptions. David Gross has predicted [23] that this distinction will be blurred as the understanding of space and time is advanced and indeed we have developed a new way in which these traditionally distinct constructions can be united. We note [31 32] that the description of the time evolution given by QM does not appropriately represent...
multi-time-correlations which are similar to Einstein-Podolsky-Rosen/Bohm entanglement (eq. 1.1) but instead of being between two particles in space, they are correlations for a single particle between two different times. Multi-time-correlations, however, can be represented by using TSQM. As a consequence, the general notion of time in QM is changed from the current conceptual framework which was inherited from CM, i.e.:

1): the universe is viewed as unique, and the objects which inhabit it just change their state in time. In this view, time is “empty,” it just propagates a state forward; the operators of the theory create the time evolution;

to a new conceptual framework in which:

2): each instant corresponds to a new pair of Hilbert spaces, (i.e., each instant is a new degree of freedom; in a sense, a new universe); instead of the operators creating the time evolution as in the previous approach, an entangled state (in time) “creates” the propagation: a whole new set of structures within time is able to “propagate” a quantum state forward in time,

This new approach has a number of useful qualities, e.g.: 1) the dynamics and kinematics can both be represented simultaneously in the same language, a single entangled vector (in many Hilbert spaces), and 2) a new, more fundamental complementarity between dynamics and kinematics is naturally introduced. This approach also leads to a new solution to the measurement problem which we model by uncertain Hamiltonians. Finally, these considerations are also relevant to the problem of the “Now,” which was succinctly expressed by Davies [75] as “why is it ’now’ now?” The kinematic-dynamics generalization [34, 32] suggests a new fourth approach to time besides the traditional “block universe,” “presentism,” and “possibilism” models.

While we leave all details to our other publications [34, 32], in brief, consider a spin-1/2 particle, initially polarized “up” along the $z$ axis, and having the Hamiltonian $H = 0$. In this case the time evolution of the particle is trivial,

$$|\Psi(t)\rangle = constant = |\sigma_z = 1\rangle. \quad (4.44)$$

To see the deficiency in representing multi-time-correlations, we will consider an isomorphism between the correlations for a single particle at multiple instants of time and the correlations between multiple particles at a single
instant of time. Therefore, we ask if we could prepare $N$ spin-1/2 particles such that if we perform measurements on them at some time $t_0$ we would obtain the same information as we would obtain by measuring the state of the original particle at $N$ different time moments, $t_1, t_2...t_N$? Since the state of the original particle at all these moments is $|\sigma_z = 1\rangle$, one would suppose that this task can be accomplished by preparing the $N$ particles each polarized “up” along the $z$ axis, that is eq. (4.45) (see also fig. 19):

$$|\sigma_z = 1\rangle_1 |\sigma_z = 1\rangle_2 ... |\sigma_z = 1\rangle_N$$ (4.45)

Figure 19: $N$ spin-1/2 particles all in the initial or pre-selected state of $\sigma_z = +1$.

But this mapping is not appropriate for many reasons. One reason is that the time evolution (4.44) contains subtle correlations (i.e. multi-time-correlations), which usually are not noticed, and which do not appear in the state (4.45) but which can actually be measured. It is generally believed that since the particle is at every moment in a definite state of the $z$-spin component, the $z$-spin component is the only thing we know with certainty about the particle - all other spin components do not commute with $\sigma_z$ and cannot thus be well-defined. However, there are multi-time variables whose values are known with certainty, given the evolution (4.44). For example, although the $x$ spin component is not well defined when the spin is in the $|\sigma_z = 1\rangle$ state, we know that it is constant in time, since the Hamiltonian is zero. Thus, for example, the two-time observable $\sigma_x(t_4) - \sigma_x(t_2) = 0$ is definite ($t_2 < t_4$). However, there is no state of $N$ spins such that

$$\hat{\sigma}^1_{\hat{n}} = \hat{\sigma}^2_{\hat{n}} = ... = \hat{\sigma}^N_{\hat{n}}$$ (4.46)

for every direction $\hat{n}$ as would be required for all the multi-time-correlations. At best, one may find a two-particle state eq. (1) for which the spins are anti-correlated instead of correlated i.e. $\hat{\sigma}^1_{\hat{n}} = -\hat{\sigma}^2_{\hat{n}}$. However, e.g., for 3 particles, only 2 of them can be completely anti-correlated, thus it cannot be
extended to $N$ particles.

$$\begin{align*}
&\psi_1^1 = \langle \downarrow | 1 \rangle \\
&\psi_2^1 = \langle \uparrow | 1 \rangle \\
&\psi_1^2 = \langle \downarrow | 2 \rangle \\
&\psi_2^2 = \langle \uparrow | 2 \rangle \\
&\Phi_1 = | \uparrow \rangle_2 \\
&\Phi_2 = | \downarrow \rangle_2
\end{align*}$$

Figure 20: particle 1 is correlated to the pre-selected state of particle 2.

Although a state of $N$ spin 1/2 particles with complete correlations among all their spin components as required by eq. (4.46) doesn’t exist in the usual sense, there are pre-and-post-selected states with this property given by TSQM. By way of example (see figure 20), the post-selected state of particle 1 can be completely correlated with the pre-selected state of particle 2 as described by the state $\Phi = \frac{1}{\sqrt{2}} \{ \langle \downarrow | 1 \rangle | \uparrow \rangle_2 - \langle \uparrow | 1 \rangle | \downarrow \rangle_2 \}$. We are now able to preserve the single particle’s multi-time-correlations by simply “stacking” the $N$ spin-1/2 particles “one on top of the other” along the time axis (fig. 21). As a result of the correlations between the pre-and-post-selected states, a verification measurement of $\hat{\sigma}_x (t_4) - \hat{\sigma}_x (t_2)$ (see left part of fig. 21), will yield 0, i.e. perfect multi-time correlations because $\hat{\sigma}_x (t_2, \text{particle } 2) - \hat{\sigma}_x (t_2, \text{particle } 1) = 0$ (see right part of fig. 21). When “stacked” onto the time axis, these correlations act like the identity operator and thus evolve the state forward, handing-off or effectively propagating a state from one moment to the next (although nothing is “really” propagating in this picture).

### 4.2 Destiny states: new solution to measurement problem

Up until now we have limited ourselves to the possibility of 2 boundary conditions which obtain their assignment due to selections made before and after a measurement. It is feasible and even suggestive to consider an extension of QM to include both a wavefunction arriving from the past and a second “destiny” wavefunction coming from the future which are determined by 2 boundary conditions, rather than a measurement and selection. This proposal could solve the issue of the “collapse” of the wavefunction in a new and
more natural way: every time a measurement takes place and the possible measurement outcomes decohere, then the future boundary condition simply selects one out of many possible outcomes \[35, 32\]. It also implies a kind of “teleology” which might prove fruitful in addressing the anthropic and fine-tuning issues\[77\] The possibility of a final boundary condition on the universe could be probed experimentally by searching for “quantum miracles” on a cosmological scale. While a “classical miracle” is a rare event that can be explained by a very unusual initial boundary-condition, “Quantum Miracles” are those events which cannot naturally be explained through any special initial boundary-condition, only through initial-and-final boundary-conditions. By way of example, destiny-post-selection could be used to create the right dark energy or the right negative pressure (etc \[81\]).
5 Discussion of big questions and major unknowns concerning time-symmetry

5.1 Why God Plays Dice

Why does uncertainty seem to play such a fundamental role in QM? First of all, uncertainty is necessary to obtain non-trivial pre-and-post-selections. In addition, this uncertainty is needed in the measuring-device to preserve causality. These two uncertainties work together perfectly \[32\]. Returning to the example discussed in §'s 1.2.1, 1.2.3, 2.2.2, since the weak-measurement result $\sqrt{2}$ was “obtained” at a time arbitrarily earlier than the post-selection time, couldn’t we then ascertain that a future post-selection should produce $\sigma_y = +1$, seemingly in violation of causality? While the weak-value depended on the post-selection, we now show that this cannot violate causality because the uncertainty in the measuring-device forces us to interpret the outcomes of weak-measurements as errors. If this were not true, then the outcome of a weak-measurement would force us to perform a particular post-selection (seemingly in violation of our free-will). In summary, eccentric-weak-values like this cannot be discerned with certainty from the statistics of pre-selected-only-ensembles for two principle reasons:

- **Analyticity of the measuring-device**: Any measurement produces only bounded changes in the pointer variable\[24\] which can produce an “erroneous” value. The disturbance to the wavefunction of the system being measured is bounded only if we prepare the measuring-device in an initial state with $Q$ bounded, i.e. $\tilde{\Phi}_{in}^{MD}(Q)$ has compact support. But this implies that the Fourier transform of $\tilde{\Phi}_{in}^{MD}(Q)$, i.e. $\Phi_{in}^{MD}(P)$ is analytic. Therefore, there is a non-zero probability that the pointer produces “erroneous” values even from the initial state $\Phi_{in}^{MD}(P)$. That is, it must be possible to constructively produce interference in the tails of $\Phi_{in}^{MD}(P)$ in order to reconstruct the initial wavefunction of the measuring-device in the “forbidden” region, i.e. $\Phi_{in}^{MD}(P - \langle A \rangle_w)$ centered around $A_w$, just as occurred with super-oscillations.

- **The probability to obtain the weak-value as an error of the measuring-device is greater than the probability of obtaining**

\[24\]A bounded change in position, for example, occurs in the weak-measurements discussed in \[2\]
an actual weak-value. This follows from the requirement that the uncertainty in $P$ must be of the same order as the maximum separation between the eigenvalues (figure 5b), so that superposition of the measuring-device wavefunction can destructively interfere in the region where the normal spectrum is defined.

Therefore we conclude that the weak-value structure is completely hidden if we are looking at a pre-selected-only system because the measuring-device always hides the weak-value structure. If the spread in $P$ did not hide the components of $A_w$, then we could obtain some information about the choice of the post-selection, which could violate causality. Nevertheless, usually one says that a causal connection between events exists if the existence of a single event is “followed” by many other events, i.e. that there is a one-way correlation. If we consider a number $N$ of weak-measurements during $t \in [t_{in}, t_{fin}]$, then when the correct post-selection is obtained, then this post-selection forces all the weak-measurements to be centered on $A_w^1 = A_w^2 = ... = A_w^N = \frac{\langle \Psi_{fin} | \hat{A} | \Psi_{in} \rangle}{\langle \Psi_{fin} | \Psi_{in} \rangle}$.

Therefore, the one-way correlation between $\langle \Psi_{fin} |$ and $A_w$ is consistent with this “causality” condition. Finally, we have also used these considerations to probe the axiomatic structure of QM [32, 73, 74, 13]. Traditionally, the

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25 As shown in [2.1] and [33], there are several regimes for valid weak-measurement and each rigorously preserves causality: 1) we must have a small system-measuring-device interaction strength, $\lambda$, as compared to strong-measurements in which the accuracy is increased by increasing $\lambda$; 2) a very rare pre-and-post-selection; 3) with robust-weak-measurement approach [33], causality is again preserved because the corrections can only be made using the relative coordinates which can only be obtained after the particles go through the pre-and-post-selections. If one attempted to utilize very eccentric-weak-values, then we note that as the weak-value goes further and further outside the operator spectrum, $|\langle \Psi_{fin} | \Psi_{in} \rangle|^2$ from eq. [2.14] (the probability to see this particular weak-value) becomes smaller and smaller, and therefore the probability of obtaining these weak-values becomes smaller and smaller. As the fluctuation in the system increases, the probability of a rare or eccentric post-selection also increases. An attempt to discern this fluctuation through the use of weak-measurements requires the spread in the measuring-device to be increased. This increases the probability of seeing strange result as an error of the measuring-device. This is a general condition which protects causality: the probability of obtaining the weak-value as an error of the measuring-device must be greater than the probability of post-selection. In other words, restricting $\hat{Q}$ to a finite interval forces $\Phi(P)$ to be analytic which means that $\Phi(P)$ has tails. The tails allow the exponential to be expanded (eq. [2.10] and therefore, the measuring-device will register the weak-value-again without changing the shape of $\Phi(P)$. The existence of these tails means that if the measuring-device registers $\sqrt{2}$, then it is more likely to be an error than a valid weak-value. This prevents any “acausal” indicator of the post-selection process.
uncertainty of QM meant that nature is capricious, i.e. “God playing dice.” A different meaning for uncertainty can be obtained from two axioms: 1) the future is relevant to the present and 2) causality is maintained. In this program, uncertainty is derived as a consequence of the consistency between causality and weak-values; in order to enrich nature with temporal non-locality, and yet preserve cause-effect relations, we must have uncertainty.

5.2 The Problem of Free-Will

The “destiny-generalization” of QM inspired by TSQM (§4.2) posits that what happens in the present is a superposition of effects, with equal contribution from past and future events. At first blush, it appears that perhaps we, at the present, are not free to decide in our own mind what our future steps may be. Nevertheless, we have shown that freedom-of-will and destiny can “peacefully co-exist” in a way consistent with the aphorism “All is foreseen, yet choice is given” [78, 76].

The concept of free-will is mainly that the past may define the future, yet after this future effect takes place, i.e. after it becomes past, then it cannot be changed: we are free from the past, but, in this picture, we are not necessarily free from the future. Therefore, not knowing the future is a crucial requirement for the existence of free-will. In other words, the destiny-vector cannot be used to inform us in the present of the result of our future free choices.

We have also shown that free-will does not necessarily mean that nobody can in principle know what the future will be because any attempt to communicate such knowledge will make the memory system unstable, thereby allowing the freedom to change the future. Suppose there is a person who can see into the future, a prophet. Then while we, at the present are making a decision, and have not yet decided, the prophet knows exactly what this decision will be. At this point, as long as this prophet does not tell us what our decision will be, we are still free to make it, since we know that if the prophet had told us what our decision was going to be, then we would be free to change it and his prophecy would no longer be true. Therefore, the prophet could be accurate as long as he doesn’t tell us our future decision.

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26 This was Bell’s main concern with retrodictive solutions to Bell’s theorem.
I.e., we are still free to make decisions based on nothing but the past and our own mind. Our decisions stand alone and the prophet’s knowledge does not affect our free-will.

From TSQM and the destiny-generalization, we may say that this prophet is the information of future measurements propagating back from the future to affect the results of measurements conducted at the present. Since a measurement of a weak-value is dependent upon a certain type of post-selection (which is only one of a few possible post-selections), but we at the present do not know whether the weak-value measured is due to an experimental error, or due to the post-selection. In addition, because a weak-measurement could be an error, there is nothing that forces us to perform a particular post-selection in the future. Only in the future, when all the measurements are finished and we actually make the post-selection, can we retrospectively conclude whether the eccentric-weak-value shown by the measuring-device was either an error, or a real result due to the concrete post-selection. Again, the conditions for a weak-measurements require a high probability of experimental error.

From this we conclude that our prophet, the post-selected vector coming from the future, does not tell us the information we need to violate our free-will, and we are still free to decide what kind of future measurements to conduct. Therefore, free-will survives.

5.3 Emergence and Origin of Laws

TSQM also provides novel perspectives on several other themes explored in this volume, e.g. on the question of emergence [24]:

- Contextuality (§2.3) suggests that the measuring-device determines the sets of possible micro-states [69, 70, 32].

- A crucial component of contextuality, namely the failure of the product rule, suggests other novel forms of emergence [32]. By way of example, another surprising pre-and-post-selection effect is the ability to separate a system from its properties [32, 13], as suggested by the Cheshire cat story: “Well! I’ve often seen a cat without a grin,” thought Alice; “but a grin without a cat! Its the most curious thing I ever saw in all

\[27\] The weak-value of a product of observables is not equal to the product of their weak-values.
Figure 22: Chesire Cat grin states. From [13]

my life!” [79]. We approximate the cat by a single particle with grin states given by $|\sigma_z = +1\rangle$ (grinning) and $|\sigma_z = -1\rangle$ (frowning). Besides spin, we also specify the particle’s location as either in a box on the left $|\psi_L\rangle$, or a box on the right $|\psi_R\rangle$ (figure 22). Consider the pre-selection: $|\Psi_{in}\rangle = |\psi_L\rangle \{ |\sigma_z = +1\rangle + |\sigma_z = -1\rangle \} + |\psi_R\rangle |\sigma_z = +1\rangle$ and the post-selected state: $|\Psi_{fin}\rangle = \{ |\psi_L\rangle - |\psi_R\rangle \} \{ |\sigma_z = +1\rangle - |\sigma_z = -1\rangle \}$. Using the isomorphism between spin states and boxes, if $N_L(+1)$ is the number of $\sigma_z = +1$ particles in the left box (etc.), then the total number of particles in the left box is: $N_L(+1) + N_L(-1) = 0$. But the magnetic moment in the left box is: $N_L(+1) - N_L(-1) = 2N$. Thus, there are no particles in the left box, yet there is twice the magnetic field there! Alice would say “Curiouser and curiouser”: the particles are all in the right box, but there is no field there, thereby challenging the notion that all properties “sit” on the particle.

- Finally, the “destiny-vector” (§4.2) suggests a form of top-down causality which is stable to fluctuations because post-selections are performed on the entire Universe and by definition no fluctuation exists outside the Universe.

These are examples of emergence with respect to properties. As Barrows and Davies [77] have emphasized, the questions of fine-tuning, the origin of the physical laws, and the anthropic principle are significant outstanding
problems in physics. What novel perspectives can be gleaned from TSQM on these questions? E.g. the dynamics-kinematics generalization (§4.1 [34, 32]) suggests a novel way to think about dynamical laws. One implication is the fact that although we may know the dynamics on a particular time-scale $T$, this doesn’t mean that we know anything about the dynamics on a smaller time-scale: consider a superposition of unitary evolutions (using $e^{-iHT} = \{e^{-iH N} \}^N$):

$$
\int g(\nu)e^{-iH(\nu)t}d\nu \rightarrow \int g(\nu)\{1 + iH(\nu)t\}d\nu \xrightarrow{i \int g(\nu)d\nu = 1} 1 + i \int g(\nu)H(\nu)t d\nu
$$

(5.47)

This theory is the same as the usual theory but with an effective Hamiltonian

$$
H_{eff} = \int g(\nu)H(\nu)t d\nu
$$

(5.48)

The finer grained Hamiltonian can be expressed as a superposition of evolutions $e^{-iH N} = \sum \alpha_n e^{-i\beta_n H N}$, i.e. the Hamiltonian can be represented as a superposition of different laws given by pre-and-post-selection [32].

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[49] Theorem: for every observable $A$ and a normalized state $|\psi\rangle$, we have: $A|\psi\rangle = \langle A|\psi\rangle + \Delta A|\psi_\perp\rangle$ for some state $|\psi_\perp\rangle$ which is orthogonal to $|\psi\rangle$. To prove this, we begin with: $A|\psi\rangle = \langle A|\psi\rangle + A|\psi\rangle - \langle A|\psi\rangle$ now, we set:
\[ |\tilde{\psi}_\perp \rangle = A|\psi \rangle - \langle A|\psi \rangle, \]

so: \[
\langle \tilde{\psi}_\perp |\psi \rangle = (\langle \psi |A - \langle \psi |\langle A \rangle|\psi \rangle = \langle \psi |A|\psi \rangle - \langle A|\psi \rangle = 0
\]

now we set: \[ |\psi_\perp \rangle = b|\tilde{\psi}_\perp \rangle, \]

where \[ |\psi_\perp \rangle \]

is normalized and \( b \) real (note that \( \langle \psi |\psi_\perp \rangle = 0 \)). so: \[ A|\psi \rangle = \langle A|\psi \rangle + b|\psi_\perp \rangle. \]

Now we multiply from the left by \( \langle \psi_\perp | \), and we get: \[
\langle \psi_\perp |A|\psi \rangle = b
\]

and the result: \[ A|\psi \rangle = \langle A|\psi \rangle + \Delta A|\psi_\perp \rangle \]

is proved.

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[57] The identity \( \exp \{ i \alpha \hat{\sigma}_n \} = \cos \alpha + i \hat{\sigma}_n \sin \alpha \) is easily proven using the fact that for any integer \( k \): \[ \sigma_n^{2k} = I \]

and \( \sigma_n^{2k+1} = \sigma_n \)

and now it follows that: \[ e^{i\alpha \sigma_n} = \sum_{k=0}^{\infty} \frac{(i\alpha)^k \sigma_n^k}{k!} = \sum_{k=0}^{\infty} \frac{(i\alpha)^{2k}}{(2k)!} + \sigma_n \sum_{k=0}^{\infty} \frac{(i\alpha)^{2k+1}}{(2k+1)!} \]

\[ e^{i\alpha \sigma_n} = \cos \alpha + i \sigma_n \sin \alpha \]

and the identity is proven.

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[64] Proof: from linearity\[
\frac{\langle \Psi_{\text{fin}} | \hat{P}_B + \hat{P}_C | \Psi_{\text{in}} \rangle}{\langle \Psi_{\text{fin}} | \Psi_{\text{in}} \rangle} = \frac{\langle \Psi_{\text{fin}} | \hat{P}_B | \Psi_{\text{in}} \rangle}{\langle \Psi_{\text{fin}} | \Psi_{\text{in}} \rangle} + \frac{\langle \Psi_{\text{fin}} | \hat{P}_C | \Psi_{\text{in}} \rangle}{\langle \Psi_{\text{fin}} | \Psi_{\text{in}} \rangle}.
\]

[65] Proof: Given that\[\hat{P}_A = \sum_n a_n | \alpha_n \rangle \langle \alpha_n |,\]if an eigenvalue, e.g.\[\hat{P}_A = a_n,\]is obtained with certainty, then for\[n \neq m, \quad \hat{P}_A \equiv | \alpha_m \rangle \langle \alpha_m | = 0\]because the probability to obtain another eigenvalue by ABL is\[\propto \langle \Psi_{\text{fin}} | \alpha_m \rangle \langle \alpha_m | \Psi_{\text{in}} \rangle = 0.\]In this case, the weak-value\[\langle \Psi_{\text{fin}} | \hat{P}_A | \Psi_{\text{in}} \rangle \rangle = \langle | \alpha_m \rangle \langle \alpha_m | \rangle \rangle = \frac{\langle \Psi_{\text{fin}} | \alpha_m \rangle \langle \alpha_m | \Psi_{\text{in}} \rangle}{\langle \Psi_{\text{fin}} | \Psi_{\text{in}} \rangle} = 0.\]In addition,\[\sum_m \frac{\langle \Psi_{\text{fin}} | \alpha_m \rangle \langle \alpha_m | \Psi_{\text{in}} \rangle}{\langle \Psi_{\text{fin}} | \Psi_{\text{in}} \rangle} = 1\]because\[\sum_m | \alpha_m \rangle \langle \alpha_m | = 1.\]But since\[\langle \Psi_{\text{fin}} | \alpha_m \rangle \langle \alpha_m | \Psi_{\text{in}} \rangle = 0\]for\[n \neq m,\]the only term left is\[n.\]Therefore, the weak-value is 1, the same as the ideal value.

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