The Particle-Field Theory and Its Relativistic Generalization

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Abstract

As a serious attempt for constructing a new foundation for describing micro-entities from a causal standpoint, it was explained before in [1, 2, 3] that by unifying the concepts of information, matter and energy, each micro-entity is assumed to be composed of a probability field joined to a particle called a particle-field or PF system.

In this essay, the relativistic generalization of the PF theory has been considered. The equation of motion for the PF system is derived in a form which is Lorentz-invariant. Moreover, based on constitutional similarities to classical equations of motion, a well-defined relativistic time-independent Schrödinger equation is derived, which is one of our main achievements in developing a micro-relativistic physics of PF theory. This relativistic Schrödinger equation is solved for a relativistic micro-particle in one-dimensional box to find its eigenstate and energy spectrum.

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1 Introduction

"Quantum mechanics is certainly imposing. But an inner voice tells me that it is not yet the real thing. The theory says a lot, but does not really bring us closer to the secret of the Old One. I, at any rate, am convinced that He is not playing at dice" [4].

This view on quantum mechanics is now shared by a large number of scientists spanning the entire spectrum of physics, from pure theoretical ones to cutting edge experimenters. There is no inclusive consensus among the physicists and the philosophers of science about the meaning of quantum theory and
the way one is preferred to look at quantum world. There are many weird quantum phenomena that the most important of them are the measurement problem, the EPR paradox [5] and the quantum interference phenomenon described by the famous double-slit experiments. No generally accepted variant of quantum theory has been provided up to now to explain these puzzling phenomena.

To be involved in such an important business, in recent years a new foundation for describing micro-events from a deterministic causal standpoint is formulated, in which a micro-entity is supposed to be an allied particle-field system, instead of composing of a particle and (or) a field (wave) [1, 2, 3]. It has been explained in the first essay of this series that in the microworld, one encounters an unified concept of information, matter and energy [1]. In this new approach, the principles of realism and causality based on the classic-like equations of motion are presumed and the meaning of wave function is explained to explain why its form (according to Born postulate) determines the probability density of finding a particle somewhere in space. One can also see a clear depiction for some weird quantum phenomena such as the tunneling effect, double slit experiment and the so-called EPR thought experiment [2, 3].

From a more fundamental point of view, this theory provides us with a new formulation of quantum phenomena based on a unified concept of information (by which we gain knowledge about the possible locations with in which a particle can be found), matter (characterized by the existence of a particle) and energy (attributed to the whole particle-field). This is somehow similar to the special relativity theory in which the concepts of matter and energy are joined to one concept in the relativistic domain.

So, we suppose that there is a field associated with a particle which together form a unit entity called "particle-field" (PF). The field has a mathematical representation which determines the spatial distribution of the entire system, i.e., it determines the probability of finding the particle within a definite interval of space, when one measures its position. Moreover, both the particle and the field satisfy deterministic equations of motion, but the field has no independent identity without the existence of the particle. We are not able to see an independent particle directly without any intervention. A PF system is indeed an extended notion of particle which its nature differs from a classical particle because the particle shares some of its energy with its surrounding space [1].

The other side of non-classical behavior of a system is that, the world is not only quantized but also it is relativistic. It is a four dimensional universe in which the laws of physics obey the principles of relativity. So, we need to show that this new theory fulfills the requirements of the theory of relativity. For this requirement we introduce a unified concept of spacetime which shows the infinitesimal distance of two separated points based on the defined entities in the PF theory and show that this element is invariant under Lorentz transformation.

Moreover, we obtain the relativistic time-independent Schrödinger equation which is one of the achievements of the relativistic generalization of this theory. Here, we consider a non-relativistic potential and solve the Schrödinger equation for a relativistic particle in a one dimensional-box. Then, we find the energy spectrum and the states of the particle.

This paper is organized as follows: In section 2, we review the basic elements of the new theory for a one-particle one-dimensional microsystem. In section 3, we find the relativistic equation of motion for a PF system and in section 4, we
show that it is invariant under Lorentz transformation. In section 5, using the basic elements of the theory depicted in [1], we will derive relativistic Schrödinger equation for a one-dimensional conservative system, considering both cases of mass-dependent and mass-independent potentials. Then in section 6, we will find the state and the energy of a relativistic particle in a one-dimensional box to show that the result is physically intuitive and consistent. At the end, in section 7, the whole content of our paper is discussed and concluded.

2 Review of Basic Elements

In this section, we give a brief review of basic elements of the PF theory. More details are available in [1]. For a one-dimensional, one-particle microsystem, three physical entities are introduced:

1. A particle with mass \( m \) and position \( x(t) \) whose dynamics is given by the Newton’s second law:
   \[
   m \frac{d^2 x(t)}{dt^2} = f_P,
   \]
   where \( f_P \) is the force defined for the particle. For the conservative forces, the particle possesses a conserved energy \( E_P = V_P + K_P \), where \( K_P = \frac{p_P^2}{2m} \) is the kinetic energy and \( p_P \) is the linear momentum of the particle.

2. Like the particle aspect of the PF system, there is a field denoted by \( X(x(t),t) \) with velocity \( v_F = |\frac{dX}{dx}| = |\dot{X}| \) along the positive direction of \( x \), where
   \[
   \dot{X} = \left( \frac{\partial X}{\partial x} \right) v_P + \left( \frac{\partial X}{\partial t} \right),
   \]
   and \( v_P \) is the velocity of the particle along the same direction. The amplitude of the field has a dimension of length. Similar to the particle, we assume that the field obeys a Newton-like dynamics too in the same direction,
   \[
   m \frac{d\dot{X}}{dt} = f_F,
   \]
   where \( f_F \) is the force the field is subjected to. If the particle is subjected to a conservative force \( f_P \), we shall consider \( X = \chi(x(t)) \). Then, one can show that
   \[
   v_F = |(\frac{d\chi}{dx})|v_P = |\chi'|v_P,
   \]
   and
   \[
   f_F = mv_P^2 \frac{d|\chi'|}{dx} + |\chi'|f_P.
   \]
   From a physical point of view, the field \( X \) merely enfolds the particle. It experiences its own mechanical-like force introduced as \( f_F \) in (3), although the presence of the particle is essential for defining the force of the field. If there is no particle, there will not be any associated field too. The existence of the field depends on the existence of the particle, but the opposite is not true, because \( X \) is a function of the particle’s position, not vice versa.

For a conservative field subjected to the force \( f_F \) in (3), one can define the energy \( E_F = V_F + K_F \) where \( K_F = \frac{1}{2}mv_F^2 = K_P|\chi'|^2 \). The kinetic energy of the field includes the kinetic energy of the particle. Here, one can’t separate the
meaning of $K_F$ from $K_P$.

In the quantum domain, the quantities $E_P$ and $E_F$ are not practically discernible, but the total energy $E = E_P + E_F$ is an observable property. One can write the total energy as:

$$E = V_P + (E_F + \frac{p^2}{2m}),$$

$$= V_P + \frac{p^2}{2m}$$

(6)

where $\frac{p^2}{2m} = (E_F + \frac{p^2}{2m})$, and $V_P$ is the particle’s potential.

3. Neither the particle, nor the field representation alone is adequate for explaining the physical behavior of a microsystem, comprehensively. What really gives us a thorough understanding of the nature of a quantum system is a holistic depiction of both particle and its associated field which we call here a PF system. The kinetic energy of a PF system is proportional to $K_P + K_F$, but its total energy is the same as $E$ in (6). Let us define the kinetic energy of a PF as $K_{PF} = \frac{1}{2}m\dot{q}^2$, where $q$ denotes the position of the PF and $\dot{q}$ is its velocity. Then, it is legitimate to suppose that $K_{PF} \propto K_P + K_F$, or

$$\dot{q}^2 = g_{PF}^2(\dot{x}^2 + |\dot{X}|^2)$$

(7)

where $g_{PF}$ is a proportionality factor and $\dot{x} = v_P$. For many problems, this factor is equal to one, but the non-oneness of its value in general is crucial in some other problems [1, 2].

The above relation can be rewritten as a geometric relation in Euclidean space:

$$dq^2 = g_{PF}^2(dx^2 + |dX|^2)$$

(8)

From this relation, one can obtain the trajectories of a PF system:

$$q(x, t) = g_{PF} \int dx \sqrt{\left(1 + \left|\frac{dX(x, t)}{dx}\right|^2\right)}.$$  

(9)

The relation (9) shows that while we expect the particle to move along the infinitesimal displacement $dx$ in the $x$ direction, the displacement of the whole system is equal to $dq$, not $dx$. The difference here is due to the existence of the associated field which adds a new term, in addition to the direction the particle moves along. Hence, the PF system indeed keeps going through an integrated path determined by the whole action of the particle and its associated field.

Using the relation (9), one can obtain the finite displacement $q$ of a PF system in terms of the particle’s location $x(t)$ and time, when the field $X(x, t)$ is known. Then, if the form of dependence of $x$ to $t$ is also known for a given physical problem, it is possible to write $q$ totally in terms of $t$. For stationary states, however, $q = q(x(t))$ and there is no explicit time-dependency. Therefore, one can see that the time variable could be kept concealed in equations of motions, so that the spatial direction $x$ would be sufficient for illustrating the behavior of $q$.

The dynamics of the PF system can also be described according to a Newtonian equation. So, we have

$$m\frac{d^2q}{dt^2} = f_{PF},$$

(10)
where $f_{PF}$ is the force the PF system is subjected to. Using the relations (1) and (7), one can obtain a relation for $f_{PF}$ in stationary states (i.e., the states in which $X = \chi(x(t))$):

$$f_{PF} = g_{PF} \left[ f_{P}(1 + \chi'^2)^{1/2} + \frac{m\dot{x}^2\chi''}{(1 + \chi'^2)^{1/2}} \right],$$  \hspace{1cm} (11)

Regarding our description of a PF system, one may pose the question that what the differences are between this approach and Bohmian account for a micro-system. In other words, what is the advantage of the PF description instead of a Bohmian one?

Here are some main points:

1. Contrary to Bohmian Mechanics [6, 7], a PF system is not composed of a particle and a wave. Instead, it is a unified system for which the particle and the wave notions are only abstract constructions without real manifestation. A PF system is neither a particle nor a wave, not also a combination of these two entities. It is a totality of both wave and particle notions, so that one can imagine it as a field that enfolds a particle. However, this is only an imagination. We abstract the notions particle and field to describe the PF system more elaborately. So, it seems that particle and field construct the PF system and when the energy of the field approaches zero, the classical particle appears. Yet, in reality, there are no distinct entities such as the particle and the field. Only when the PF system loses its all holistic nature, it reduces to a known classical particle. Thus, it looks like we have two different energies; one for the particle and the other for the field and the latter causes the quantum behavior of the system.

This important feature of a PF system enables one, e.g., to show why the squared modulus of the wave function behaves like a probability density in spatial coordinates. While, in Bohmian theory Born postulate is accepted a priori. There are other important consequences too which are mentioned in the following items.

2. Considering a PF system allows one to explain the origin of the Schrödinger equation, since here we assume that the underlying dynamics of a supposed field is influenced by an oscillatory force which could be approximated to a harmonic one as the first order [1]. Taking into account anharmonic effects, one can obtain non-linear forms of the Schrödinger equation with additional terms which change the amount of energy of the system at a scale smaller than the hyperfine structure. Such new predictions are forbidden in Bohmian approach in which only quantum predictions are reproduced.

3. The PF theory is not in contradiction with Special Relativity in its origin. Since, contrary to Bohmian Mechanics, there is no need to assume faster-than-signaling between the particles in a many-body system. Yet, the theory is an instance of a contextual local model which have holistic features [1, 2]. This helps us to search for a Lorentz-invariant form of equations in PF theory which is the main purpose of this paper. This fact alone is enough to show the importance of this work.

In addition to above points, there are other differences between the Bohmian and the PF approaches. From a fundamental point of view, these two theories explain bizarre quantum phenomena like the measurement problem, tunneling effect and double-slit experiment in completely different directions. The inter-
ested reader can follow the corresponding fashions of explanation in each model in [1, 2, 3] and [6, 7].

3 Relativistic Equation of Motion for the PF System

In this section we are going to find a relativistic form of the relation. (9), to reach a unified concept of spacetime that is invariant under Lorentz transformation.

With the definition of the kinetic energy of the conservative field, \( K_F = K_P \chi'^2 \), we define the relativistic kinetic energy of the stationary field as

\[
K_{rF} = K_{rP} \chi'^2
\]  

(12)

where \( K_{rP} = m_0 c^2 (\gamma_p - 1) \) is the relativistic energy of the particle and \( c \) is the velocity of light. So, one can define the kinetic energy of the PF system as

\[
K_{rPF} = K_{rP} + K_{rF} = m_0 c^2 (\gamma_{PF} - 1).
\]  

(13)

Here

\[
\gamma_{PF} = (1 - \frac{\dot{q}^2}{c^2})^{-\frac{1}{2}}
\]  

(14)

where \( \dot{q}^2 \) is the velocity of the PF system and \( \dot{q} \leq c \). Using the relations (12) – (14), we can express an explicit relation for \( \dot{q} \) as the following:

\[
\dot{q} = c \left( 1 - \frac{1}{[(\gamma_p - 1)(1 + \chi'^2)]^2} \right)^{\frac{1}{2}}.
\]  

(15)

Relation (15) shows clearly that for photons, (for which \( \gamma_p \to \infty \)), \( v_p = \dot{q} = c \).

We know that two infinitesimally separated points \((ct, q)\) and \((ct + \Delta t, q + \Delta q)\) can be connected by a light signal, according to the following relation:

\[
ds^2 = c^2 \Delta t^2 - \Delta q^2.
\]  

(16)

This introduces a unified concept of spacetime in the microworld. In the following, we show that (16), defined as the proper distance, is invariant under Lorentz transformation.

4 Lorentz Invariance of the Proper Distance for a PF System

We can verify whether the relation (16) is invariant under Lorentz transformation in the PF theory, i.e., it has the same form in frames of reference which are moving relative to each other with a constant uniform velocity, \( v_{PF} \). Here \( v_{PF} \) shows the relative velocity of two reference frames corresponding to the PF system. Let \( Q \) and \( Q' \) be reference frames for the coordinate systems \((t, q_x, q_y, q_z)\) and \((t', q'_x, q'_y, q'_z)\), respectively. Without loose of generality, we concern ourselves with the case that the corresponding axes are aligned, with \( q_x \) and \( q'_x \) along the line of the relative motion, so that \( Q' \) has velocity \( v_{PF} \) in
with the particle, we have a field denoted by $\chi_\gamma$ where $t$ coincide at $t = t' = 0$. Hereafter, we refer to this arrangement as the "standard configuration" of a pair of reference frames.

In such a standard configuration, if an event has coordinates $(t, q_x, q_y, q_z)$ in $Q$, then its coordinates in $Q'$ are given by

$$q_x' = \frac{q_x - v_{PF}t}{\sqrt{1 - \frac{v_{PF}^2}{c^2}}}$$

$$q_y' = q_y$$

$$q_z' = q_z$$

$$t' = \frac{t - \frac{v_{PF}}{c^2}q_x}{\sqrt{1 - \frac{v_{PF}^2}{c^2}}}$$

where $v_{PF} = \dot{q}$. To clarify the approach followed here, let us remember the entities explained in section 2. Three physical entities have been defined, a particle with mass $m$, position $x(t)$ and velocity $v_p = \frac{dx}{dt} = \dot{x}$. Associated with the particle, we have a field denoted by $\chi(x)$ with the velocity $v_p = \frac{dx}{dt} = |\chi|v_p$ (here, we are considering the stationary fields). The whole PF-system is characterized by the position $q$ and the velocity $v_{PF} = \frac{dq}{dt} = \dot{q}$.

Using the relation (15), the proper distance defined in (16) is obtained as

$$ds^2 = c^2dt^2 - dq^2 = c^2dt^2[1 - \frac{1}{c^2} \left(\frac{dq}{dt}\right)^2]$$

$$= c^2dt^2 \frac{1}{(\gamma_p - 1)(1 + \chi^2) + 1^2},$$

where $\gamma_p = \frac{1}{\sqrt{1 - \frac{v_{PF}^2}{c^2}}}$ and $\chi' = \frac{dx}{dt}$. It is easy to obtain (18) in the primed frame as

$$ds^2 = c^2\gamma_{PF}^2 \left(\frac{v_{PF}}{c^2} dq' + dt'\right)^2 - \gamma_{PF}^2 (dq' + v_{PF} dt')^2$$

$$= c^2\gamma_{PF}^2 dt'^2 [1 + \frac{v_{PF}}{c^2} \left(\frac{dq'}{dt'}\right)^2 - \gamma_{PF}^2 dt'^2 (v_{PF} + \frac{dq'}{dt'})^2]$$

$$= c^2dt'^2 [1 - \frac{1}{c^2} \left(\frac{dq'}{dt'}\right)^2 - 1] (1 + \gamma_{PF}^2 (\frac{dx}{dt'} - \frac{v_{PF}}{c^2} \frac{dx}{dt'})) + 1^2$$

where we have used the following equations:

$$dt = \gamma_{PF} \left(\frac{v_{PF}}{c^2} dq' + dt'\right),$$

$$dq = \gamma_{PF} (v_{PF} dt' + dq'),$$

$$\chi' = \frac{dx}{dt} = \gamma_{PF} \left(\frac{dx}{dt'} - \frac{v_{PF}}{c^2} \frac{dx}{dt'}\right)$$

$$= \gamma_{PF} \frac{dx}{dt'} (1 - \frac{v_{PF}^2}{c^2})$$

If $\chi \rightarrow 0$, the relation (18) will go to $ds^2 = c^2 dt^2 - dx^2 = c^2dt^2 \gamma_p^2 - v_{PF}^2 = c^2dt^2 (1 - \frac{v_{PF}^2}{c^2})$. 


and $\gamma_{PF}$ is defined in (14). The relation (19) is the infinitesimal interval of two events in the frame $Q'$. It seems that the form of the relations (19) and (18) are different in the two frames $Q$ and $Q'$ which are moving relative to each other with the constant velocity $v_{PF}$.

Now, we try to find the constant $\gamma_{PF}$ so that the relations (19) and (18) have the same form in the two reference frames. It is worth to note that $\gamma_{PF}$ is a function of $v_{P}$, $v'_{P}$, $\frac{d\chi}{dt}$, $\frac{d\chi}{dt}'$, provided that when $\chi \to 0$, we get the following relations between the gamma factors:

$$\gamma_{PF} = \frac{1}{\sqrt{1 - \frac{v'^2_{P}c^2}{c^2}}} = \frac{1 + \frac{v'_{P}c}{c}}{\sqrt{1 - \frac{v'^2_{P}c^2}{c^2}}} \sqrt{1 - \frac{v^2_{P}c^2}{c^2}}$$

(21)

where $v_{P}$ and $v'_{P}$ are the velocities of the particle in the frames $Q$ and $Q'$, respectively.

It will be useful to write the relation (18) as

$$ds^2 = c^2 dt^2 \left[ \gamma_{P}^2 - 2 \chi' - 2 \chi' \left( \frac{\gamma_{P}}{\gamma_{P}^2 - 1} \right)^2 \right]$$

(25)

We see that if the field character $\chi$ vanishes, relation (19) will lead to

$$ds'^2 = c^2 dt'^2 - dx'^2 = c^2 dt'^2 \gamma_{P}^{-2} = c^2 dt'^2 (1 - \frac{v'^2_{P}}{c^2})$$

The value of $\chi$ is small enough to write the relations (18) and (19) as (1).

$$ds^2 = c^2 dt^2 \left[ \gamma_{P}^2 - 2 \chi' - 2 \chi' \left( \frac{\gamma_{P}}{\gamma_{P}^2 - 1} \right)^2 \right]$$

(22)

and the relation (19) as

$$ds^2 = c^2 dt^2 \left[ \gamma_{PF}^2 - 2 \gamma_{PF} \gamma_{P} \chi' - 2 \gamma_{PF} \gamma_{P} \chi' \left( \frac{\gamma_{PF}}{\gamma_{PF}^2 - 1} \right)^2 \right]$$

(23)

where $a$ and $b$ are respectively defined as

$$a = 1 - \frac{v_{P}v'_{P}c}{c^2} \gamma_{P} (1 - \frac{v_{P}v'_{P}c}{c^2})$$

(24a)

$$b = \frac{\gamma_{PF} \gamma_{P} \chi'}{c^2} \left( \frac{d\chi}{dt} \right) - \frac{\gamma_{PF} \gamma_{P} \chi'}{c^2} \left( \frac{d\chi}{dt}' \right)$$

(24b)

The value of $\chi$ is small enough to write the relations (18) and (19) as (1).

$$ds^2 = c^2 dt^2 \gamma_{P}^{-2} [1 - 2 \chi^2 (\frac{\gamma_{P} - 1}{\gamma_{P}}) + ...]$$

(25)

We see that if the field character $\chi$ vanishes, relation (19) will lead to $ds'^2 = c^2 dt'^2 - dx'^2 = c^2 dt'^2 \gamma_{P}^{-2} = c^2 dt'^2 (1 - \frac{v'^2_{P}}{c^2})$
For large values of $\gamma_p \gg 1$ and we can write the relation \((25)\) in a simpler form

$$ds^2 \simeq c^2 dt^2 \gamma_p^{-2}[1 - 2\chi^2 + \ldots]. \quad (27)$$

So, we can find $\gamma_{PF}$, such that the relations \((27)\) and \((26)\) have the same form under Lorentz transformation, i.e.,

$$\gamma_p^{-2}[1 - 2\left(\frac{dX}{dx}\right)^2] = (\gamma_{PF}a)^{-2}[1 - 2\beta^2\gamma_{PF}^2],$$

and

$$\gamma_{PF}^{-2}[1 - 2\left(\frac{dX}{dx}\right)^2] = \gamma_p^{-2}[1 - 2\beta^2\gamma_p^2]. \quad (28)$$

Substituting $a$ and $b$ from the relations \((24a)\) and \((24b)\), one gets

$$\gamma_{PF}^{-2}(1 - \frac{v_p v_{PF}}{c^2})^2[1 - 2\left(\frac{dX}{dx}\right)^2] = \gamma_p^{-2}[1 - 2\gamma_p^2\left(\frac{dX}{dx}\right)^2(1 - \frac{v_p v_{PF}}{c^2})^2]. \quad (29)$$

So, we have:

$$\gamma_{PF}^{-2}[1 - 2\left(\frac{dX}{dx}\right)^2](1 - \frac{v_p v_{PF}}{c^2})^2\gamma_p^{-2} + 2\left(\frac{dX}{dx}\right)^2(1 - \frac{v_p v_{PF}}{c^2})^2 = \gamma_p^{-2}. \quad (30)$$

Then

$$\gamma_{PF}^2 = \frac{\gamma_p^2}{[(1 - 2\left(\frac{dX}{dx}\right)^2)(1 - \frac{v_p v_{PF}}{c^2})^2\gamma_p^{-2} + 2\left(\frac{dX}{dx}\right)^2(1 - \frac{v_p v_{PF}}{c^2})^2]^2}, \quad (31)$$

or

$$\gamma_{PF} = \frac{\gamma_p}{[(1 - 2\left(\frac{dX}{dx}\right)^2)(1 - \frac{v_p v_{PF}}{c^2})^2\gamma_p^{-2} + 2\left(\frac{dX}{dx}\right)^2(1 - \frac{v_p v_{PF}}{c^2})^2]^2}. \quad (32)$$

It is convenient to write the relation \((32)\) as

$$\gamma_{PF} = \frac{\gamma_p^2}{[(1 - \frac{2\left(\frac{dX}{dx}\right)^2)(1 - \frac{v_p v_{PF}}{c^2})^2\gamma_p^{-2} + 2\left(\frac{dX}{dx}\right)^2(1 - \frac{v_p v_{PF}}{c^2})^2]^2} \frac{\gamma_p^2}{[(1 - 2\left(\frac{dX}{dx}\right)^2)(1 - \frac{v_p v_{PF}}{c^2})^2\gamma_p^{-2} + 2\left(\frac{dX}{dx}\right)^2(1 - \frac{v_p v_{PF}}{c^2})^2]^2} \gamma_{PF} \gamma_{PF}' = \frac{\gamma_p^2}{[(1 - \frac{2\left(\frac{dX}{dx}\right)^2)(1 - \frac{v_p v_{PF}}{c^2})^2\gamma_p^{-2} + 2\left(\frac{dX}{dx}\right)^2(1 - \frac{v_p v_{PF}}{c^2})^2]^2} \gamma_{PF}' \gamma_{PF}' = \frac{\gamma_p^2}{[(1 - 2\left(\frac{dX}{dx}\right)^2)(1 - \frac{v_p v_{PF}}{c^2})^2\gamma_p^{-2} + 2\left(\frac{dX}{dx}\right)^2(1 - \frac{v_p v_{PF}}{c^2})^2]^2}, \quad (33)$$

where $v_P = \frac{dX}{dt} = \frac{dx}{dt}$ and $v_P' = \frac{dX}{dt'} = \frac{dX}{dt} \frac{dt'}{dt}$ are the velocities of the field in two reference frames $Q$ and $Q'$, respectively. In the relation \((33)\), when $\chi \to 0$, we obtain $\gamma_p^2 a^2 = \gamma_{PF}^2$, which as we expect, is the exact relationship between the gamma factors in the relation \((21)\). In fact, in this limit, the PF system converts to the classical particle.

From the relation \((33)\), it is obvious that for large values of $v_P$, the energy of the particle has the most contribution to the energy of the PF system. One can also see this fact from Eq. \((33)\). For large values of $v_P$, we find the relation \((21)\) i.e., PF system transforms to the classical particle. So, it seems that if $\gamma_{PF}$ satisfies the relation \((33)\), infinitesimal distance of two separated points, $ds^2$, defined in the PF theory is invariant under Lorentz transformations.
5 Relativistic Generalization of Time Independent Schrödinger Equation

Now, we are going to derive a relativistic Schrödinger equation in a general form. The physical structure of the PF formalism which has constitutional similarities to classical equations of motion permits us to derive a well-defined relativistic Schrödinger equation for stationary fields, regardless of spin variable. The dynamics of a stationary real field in one dimension (denoted by $\chi = \chi(x(t))$) in the relativistic regime can be represented as

$$\frac{d(m_p \chi)}{dt} = f_{rF}, \quad (34)$$

where $f_{rF}$ is the force defined for the field under the relativistic conditions and $m_p$ is the relativistic mass of the particle:

$$m_p = \gamma_p m_0; \quad \gamma_p = \left(1 - \frac{v^2_p}{c^2}\right)^{-\frac{1}{2}}, \quad (35)$$

where $m_0$ is the rest mass, as before. The stationary field $\chi(x(t))$ does not explicitly depend on time. So, one can find out that

$$f_{rF} = f_{rP} \chi' + \gamma_p m_0 v_p^2 \chi'', \quad (36)$$

where $\chi' = \frac{d\chi}{dx}$ and $f_{rP} = m_0 v_p \gamma_p^3$ is the force exerted on the particle. For stationary real fields, there exists an oscillating-like term in the force expression (denoted by the second term in (36), when $\gamma_p \rightarrow 1$) from which the non-relativistic time-independent Schrödinger equation can be resulted [1]. Here, we suppose that the same situation holds true under the relativistic conditions. That is, for stationary real fields, we postulate the following equality as a general rule:

$$-m_p \ddot{\chi} = \gamma_p m_0 v_p^2 \chi'', \quad (37)$$

where $m_p$ was defined in relation (35) and $\ddot{\chi} = k^2 v_p^2$. Here again, we define $k = \frac{p}{\hbar}$, where $p$ is the relativistic de Broglie momentum. From the relation (37), it is immediately concluded that

$$-\hbar^2 \chi'' = p^2 \chi, \quad (38)$$

which has the same form as the non-relativistic Schrödinger equation. To find an appropriate relation for $p^2$ in relation (38), however, one should first note that depending on whether the potential energy of the particle is a function of mass or not, the total energy of the PF system can be respectively written as

$$E = \gamma_p (V_{nrP} + m_0 c^2) + E_{rF}$$

or

$$E = V_{nrP} + \gamma_p m_0 c^2 + E_{rF}$$

$$= V_{nrP} + \gamma m_0 c^2. \quad (40)$$

\[3\text{Sometimes expressed inversely, that is whether the mass is potential-dependent or not [?]}\]
where $V_{nrP}$ is the non relativistic potential energy of the particle and $E_{rF}$ is the relativistic energy of the field. Also, using the relation $p = \gamma m_0 v$, $\gamma$ can be defined as

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} = \sqrt{1 + \frac{v^2}{m_0^2 c^2}} \quad (41)$$

At the nonrelativistic limit, we take $\gamma V_{nrP} \rightarrow V_{nrP}$, $\gamma m_0 c^2 \rightarrow \frac{p^2}{2m_0} + m_0 c^2$ and $E_{rF} \rightarrow E_{nrF}$, so that the total energy in relations (39) or (40) can be expressed as

$$E = V_{nrP} + \frac{p^2}{2m_0} + m_0 c^2 = E_{nr} + m_0 c^2,$$

where $p^2 = p^2_P + 2m_0 E_{nrF}$. From relations (39) and (41), we can deduce that

$$p^2 = \left(1 + \frac{V_{nrP}}{p^2}\right)^{-2} \left[\frac{E^2}{c^2} - m_0^2 c^2 \left(1 + \frac{V_{nrP}}{p^2}\right)^2\right] \quad (42)$$

In a similar manner, from (40) and (41), one obtains

$$p^2 = \frac{1}{c^2}(E - V_{nrP})^2 - m_0^2 c^2 \quad (43)$$

Inserting the relation (42) in (38), we derive relativistic Schrödinger equations for the cases that the potential energy of the particle includes the relativistic mass:

$$-\frac{\hbar^2}{2m_0} \chi'' + \frac{1}{2} m_0 c^2 \chi = \frac{E^2}{2m_0 c^2} \left(1 + \frac{V_{nrP}}{m_0 c^2}\right)^{-2} \chi, \quad (44)$$

and using (43), we get the relativistic Schrödinger equation, when the potential energy of the particle is independent of mass:

$$-\frac{\hbar^2}{2m_0} \chi'' + \frac{1}{2} m_0 c^2 \chi = \frac{1}{2m_0 c^2}(E - V_{nrP})^2 \chi. \quad (45)$$

So, one can solve relations (44) or (45) for different problems. In the following, we consider the problem of a particle in a one-dimensional box to find its eigenstates and its energy spectrum. We solve such equations for two other problems, one-dimensional harmonic oscillator and the relativistic Hydrogen in a separate article.

### 6 Relativistic Micro-Particle in One-Dimensional Box

The simplest system which could be analyzed is a PF system in a one-dimensional box. We consider a particle for which the nonrelativistic potential energy is defined as

$$V_{nrP} = 0 \quad 0 \leq x \leq a \quad (46)$$

$$V_{nrP} = \infty \quad \text{elsewhere.}$$

So the relation (44) can be written as

$$\chi''(x) = -k^2 \chi(x), \quad (47)$$
where
\[ k^2 = \frac{1}{\hbar^2 c^2} (E^2 - m_0^2 c^4). \] (48)

At the boundaries, we have,
\[ \chi(0) = \chi(a) = 0. \]

We solve the relativistic Schrödinger equation (45) to obtain the energy spectrum and the stationary eigenfields, respectively, as,
\[ \left( \frac{E_n}{c} \right)^2 = n^2 \frac{\hbar^2}{4a^2} + m_0^2 c^2. \] (49)

and
\[ \chi_n(x) = A_{rn} \sin \left( \frac{n \pi x}{a} \right); n = 1, 2, 3, \ldots \] (50)

where \( A_{rn} \) is the amplitude of the field which can be found, if the relativistic energy of the stationary field \( E_{rF} \) is known. The trajectories of the PF system can be obtained by integrating (15) over \( t \), but since \( \chi_n'(x) \) is a function of \( x(t) \), the solution is complicated.

For a photonic PF system in one-dimensional box, \( E_n = \frac{n \hbar c}{2a} \). The energies of the particle \( E_{rP} = \gamma_P m_0 c^2 \) and its associated field \( E_{rF} \) are finite but unknown. The de Broglie momentum of the photonic PF system is also sharp around the values \( \pm \frac{n \hbar}{2a} \).

For a free photonic PF system, we have the same relation as (48), but with \( E = pc \) and \( p = \frac{\lambda}{c} \) where \( \lambda \) is the wavelength. The energy of a free photon is not quantized, but it can be still described as a PF system comprised of a particle and its allied field which the latter behaves like a plane wave and the whole system propagates with velocity \( c \). The trajectories of the free photonic PF system are straight lines (in terms of \( t \)), because \( \dot{q} = c \) in (15), regardless of the form of \( \chi \).

7 Discussion

The new formalism of microphenomena introduced in [1] can be used to generalize Quantum Mechanics to include relativistic equations, regardless of the spin notion.

Using the relativistic kinetic energy of the field and the particle denoted by a PF system, we used a more general, equation of motion of a PF system relation (15), to show that an infinitesimally separated point according to (16) is invariant under Lorentz transformation. Considering a standard configuration of a pair reference frames, we found the proper distance in both of them. Since in the PF theory a field accompanies the particle, the relative velocity of two reference frames is a function of the velocity of the particle and its field. So, if \( \gamma_P \) satisfies the relation (18), the proper distance is invariant under Lorentz transformation.

Moreover, considering the classical potential energy, we derived the relativistic Schrödinger equation for the case that the potential energy of the particle includes the relativistic mass and the case that it is independent of it. We solved the relativistic Schrödinger equation to find the energy spectrum and the stationary eigenfields of a relativistic PF system for the one-dimensional box.
It is worth mentioning here that we have not yet defined the spin which is one of the conceptual puzzles in Quantum Mechanics. Although there is a consensus about elementary particles having some quantum mechanical property called spin, the understanding of the physical nature of the spin is still incomplete [8]. Historically, the concept of spin was introduced in order to explain some experimental findings such as the emission spectra of alkali metals and Stern-Gerlach experiments. Though the spin is regarded as a fundamental property of the electron, a universally accepted spin operator for the Dirac theory is still missing [9]. So, we note that the relativistic Schrödinger equations, i.e., the relations (44) and (45), found in the relativistic generalization of the PF theory are completely different with Dirac equation, derived for half spin particles. Also it is not the same as Klein-Gordon equation for spin zero particles, and doesn’t have the problem of negative energies.

Last but not least, it is important mentioning that the causal basis of the PF theory along with its capability to be reformulated on a geometric ground makes it one think over the general relativistic development of this new theory in a rigorous way.

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