Multiple little $q$-Jacobi polynomials

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Abstract

We introduce two kinds of multiple little $q$-Jacobi polynomials $p_{\vec{n}}$ with multi-index
$\vec{n} = (n_1, n_2, \ldots, n_r)$ and degree $|\vec{n}| = n_1 + n_2 + \cdots + n_r$ by imposing orthogonality
conditions with respect to $r$ discrete little $q$-Jacobi measures on the exponential
lattice $\{q^k, k = 0, 1, 2, 3, \ldots\}$, where $0 < q < 1$. We show that these multiple little $q$-
Jacobi polynomials have useful $q$-difference properties, such as a Rodrigues formula
(consisting of a product of $r$ difference operators). Some properties of the zeros of
these polynomials and some asymptotic properties will be given as well.

Key words: $q$-Jacobi polynomials, basis hypergeometric polynomials, multiple
orthogonal polynomials

1 Little $q$-Jacobi polynomials

Little $q$-Jacobi polynomials are orthogonal polynomials on the exponential lattice $\{q^k, k = 0, 1, 2, \ldots\}$, where $0 < q < 1$. In order to express the orthogonality relations, we will use
the $q$-integral

$$\int_0^1 f(x) \, d_q x = (1 - q) \sum_{k=0}^{\infty} q^k f(q^k), \quad (1.1)$$

(see, e.g., [2, §10.1], [5, §1.11]) where $f$ is a function on $[0, 1]$ which is continuous at 0. The orthogonality is given by

$$\int_0^1 p_n(x; \alpha, \beta | q) x^k w(x; \alpha, \beta | q) \, d_q x = 0, \quad k = 0, 1, \ldots, n - 1, \quad (1.2)$$

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where
\[ w(x; a, b|q) = \frac{(qx; q)_{\infty}}{(q^b x; q)_{\infty}} x^\alpha. \] (1.3)

We have used the notation
\[ (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad (a; q)_{\infty} = \prod_{k=0}^{\infty} (1 - aq^k). \]

In order that the \( q \)-integral of \( w \) is finite, we need to impose the restrictions \( \alpha, \beta > -1 \). The orthogonality conditions (1.2) determine the polynomials \( p_n(x; \alpha, \beta|q) \) up to a multiplicative factor. In this paper we will always use monic polynomials and these are uniquely determined by the orthogonality conditions. The \( q \)-binomial theorem
\[ \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} z^n = \frac{(az; q)_{\infty}}{(z; q)_{\infty}}, \quad |z|, |q| < 1, \] (1.4)

(see, e.g., [2, §10.2], [5, §1.3]) implies that
\[ \lim_{q \to 1} w(x; \alpha, \beta|q) = (1 - x)^\beta x^\alpha, \quad 0 < x < 1, \]

so that \( w(x; \alpha, \beta|q) \) is a \( q \)-analog of the beta density on \([0, 1]\), and hence
\[ \lim_{q \to 1} p_n(x; \alpha, \beta|q) = P_n^{(\alpha, \beta)}(x), \]

where \( P_n^{(\alpha, \beta)} \) are the monic Jacobi polynomials on \([0, 1]\). Little \( q \)-Jacobi polynomials appear in representations of quantum \( SU(2) \) [9], [10], and the special case of little \( q \)-Legendre polynomials was used to prove irrationality of a \( q \)-analog of the harmonic series and \( \log 2 \) [14]. Their role in partitions was described in [1]. A detailed list of formulas for the little \( q \)-Jacobi polynomials can be found in [8, §3.12], but note that in that reference the polynomial \( p_n(x; a, b|q) \) is not monic and that \( a = q^\alpha, b = q^\beta \). Useful formulas are the lowering operation
\[ D_q p_n(x; \alpha, \beta|q) = \frac{1 - q^n}{1 - q} p_{n-1}(x; \alpha + 1, \beta + 1|q), \] (1.5)

where \( D_q \) is the \( q \)-difference operator
\[ D_q f(x) = \begin{cases} \frac{f(x) - f(qx)}{(1 - q)x}, & \text{if } x \neq 0, \\ f'(0), & \text{if } x = 0, \end{cases} \] (1.6)

and the raising operation
\[ D_p[w(x; \alpha, \beta|q)p_n(x; \alpha, \beta|q)] = -\frac{1 - q^{n+\alpha+\beta}}{(1 - q)q^{n+\alpha+\beta}} w(x; \alpha - 1, \beta - 1|q)p_{n+1}(x; \alpha - 1, \beta - 1|q), \] (1.7)
where \( p = 1/q \). Repeated application of the raising operator gives the Rodrigues formula

\[
w(x; \alpha, \beta|q) p_n(x; \alpha, \beta|q) = \frac{(-1)^n(1 - q)^n q^{an+n(n-1)}}{(q^{a+\beta+n+1}; q)_n} D^n p w(x; \alpha + n, \beta + n|q). \tag{1.8}
\]

A combination of the raising and the lowering operation gives a second order \( q \)-difference equation. The Rodrigues formula enables us to give an explicit expression as a basic hypergeometric sum:

\[
p_n(x; \alpha, \beta|q) = \frac{x^n q^{n(a+\alpha)}(q^{-n-a}; q)_n}{(q^{n+a+\beta+1}; q)_n} \phi_2 \left( \begin{array}{c} q^{-n}, q^{-n-a}, 1/x \\ q^{\beta+1}, 0 \end{array} \right| q; q),
\]

which by some elementary transformations can also be written as

\[
p_n(x; \alpha, \beta|q) = \frac{q^{(n+a)n}(q^{-n-a}; q)_n}{(q^{n+a+\beta+1}; q)_n} \sum_{k=0}^{n} \frac{(q^{-n}; q)_k(q^{n+a+\beta+1}; q)_k}{(q^{\alpha+1}; q)_k(q; q)_k} q^k x^k.
\tag{1.9}
\]

2 Multiple orthogonal polynomials

Multiple orthogonal polynomials (of type II) are polynomials satisfying orthogonality conditions with respect to \( r \geq 1 \) positive measures \([3][4][11, \S 4.3][15]\). Let \( \mu_1, \mu_2, \ldots, \mu_r \) be \( r \) positive measures on the real line and let \( \vec{n} = (n_1, n_2, \ldots, n_r) \in \mathbb{N}^r \) be a multi-index of length \( |\vec{n}| = n_1 + n_2 + \cdots + n_r \). The corresponding type II multiple orthogonal polynomial \( p_{\vec{n}} \) is a polynomial of degree \( \leq |\vec{n}| \) satisfying the orthogonality relations

\[
\int p_{\vec{n}}(x) x^k d\mu_j(x) = 0, \quad k = 0, 1, \ldots, n_j - 1, \quad j = 1, 2, \ldots, r.
\]

These orthogonality relations give \( |\vec{n}| \) homogeneous equations for the \( |\vec{n}| + 1 \) unknown coefficients of \( p_{\vec{n}} \). We say that \( \vec{n} \) is a normal index if the orthogonality relations determine the polynomial \( p_{\vec{n}} \) up to a multiplicative factor. Multiple orthogonal polynomials of type I (see, e.g., \([3][11, \S 4.3][4][15]\)) will not be considered in this paper. Multiple little \( q \)-Jacobi polynomials are multiple orthogonal polynomials where the measures \( \mu_1, \ldots, \mu_r \) are supported on the exponential lattice \( \{q^k, k = 0, 1, 2, \ldots\} \) and are all of the form \( d\mu_i(x) = w(x; \alpha_i, \beta_i|q) d_q x \), where \( w(x; \alpha, \beta|q) d_q x \) is the orthogonality measure for little \( q \)-Jacobi polynomials. It turns out that in order to have formulas and identities similar to those of the usual little \( q \)-Jacobi polynomials one needs to keep one of the parameters \( \alpha_i \) or \( \beta_i \) fixed and change the other parameters for the \( r \) measures. This gives two kinds of multiple little \( q \)-Jacobi polynomials. Note that these multiple little \( q \)-Jacobi polynomials should not be confused with multivariable little \( q \)-Jacobi polynomials, introduced by Stokman
All the multi-indices will be normal when we impose the condition that $\sup$.

Suppose that $\alpha$.

**Theorem 2.1**

There are $\mu_1$, $\ldots$, $\mu_r$ form a so-called AT-system, which implies that all the multi-indices are normal [11, Theorem 4.3].

There are $r$ raising operations for these multiple orthogonal polynomials.

**Theorem 2.1** Suppose that $\alpha_1, \ldots, \alpha_r, \beta > 0$, with $\alpha_i - \alpha_j \not\in \mathbb{Z}$ whenever $i \neq j$, and put $p = 1/q$, then

$$\mathcal{D}_p[w(x; \alpha_j, \beta|q)p_{\vec{n}}(x; \vec{\alpha}, \beta|q)]$$

$$= \frac{q^{\alpha_{j-1}+\beta+|\vec{n}|}}{(1-q)^{q^{\alpha_{j-1}+|\vec{n}|}} \alpha_{j-1} \beta - 1|q)}w(x; \alpha_j - 1, \beta - 1|q)p_{\vec{n}+\vec{e}_j}(x; \vec{\alpha} - \vec{e}_j, \beta - 1|q), \quad (2.2)$$

for $1 \leq j \leq r$, where $\vec{e}_1 = (1, 0, 0, \ldots, 0), \ldots, \vec{e}_r = (0, \ldots, 0, 0, 1)$ are the standard unit vectors.

Observe that these operations raise one of the indices in the multi-index and lower the parameter $\beta$ and one of the components of $\vec{\alpha}$.

**Proof:** First observe that

$$\mathcal{D}_p[w(x; \alpha_j, \beta|q)p_{\vec{n}}(x; \vec{\alpha}, \beta|q)]$$

$$= w(x; \alpha_j - 1, \beta - 1|q)\frac{(1-q^\beta x)p_{\vec{n}}(x; \vec{\alpha}, \beta|q) - p^{\alpha_j}(1-x)p_{\vec{n}}(px; \vec{\alpha}, \beta|q)}{1-p},$$

so that

$$\mathcal{D}_p[w(x; \alpha_j, \beta|q)p_{\vec{n}}(x; \vec{\alpha}, \beta|q)] = - \frac{1-q^{\alpha_{j-1}+\beta+|\vec{n}|}}{(1-q)^{q^{\alpha_{j-1}+|\vec{n}|} - 1}}w(x; \alpha_j - 1, \beta - 1|q)Q_{|\vec{n}|+1}(x), \quad (2.3)$$

[13]. In [12] the multiple little $q$-Jacobi polynomials of the first kind are used to prove some irrationality results for $\zeta_q(1)$ and $\zeta_q(2)$.

### 2.1 Multiple little $q$-Jacobi polynomials of the first kind

Multiple little $q$-Jacobi polynomials of the first kind $p_{\vec{n}}(x; \vec{\alpha}, \beta|q)$ are monic polynomials of degree $|\vec{n}|$ satisfying the orthogonality relations

$$\int_0^1 p_{\vec{n}}(x; \vec{\alpha}, \beta|q)x^k w(x; \alpha_j, \beta|q) d_q x = 0, \quad k = 0, 1, \ldots, n_j - 1, j = 1, 2, \ldots, r, \quad (2.1)$$

where $\alpha_1, \ldots, \alpha_r, \beta > -1$. Observe that all the measures are orthogonality measures for little $q$-Jacobi polynomials with the same parameter $\beta$ but with different parameters $\alpha_j$.

All the multi-indices will be normal when we impose the condition that $\alpha_i - \alpha_j \not\in \mathbb{Z}$ whenever $i \neq j$, because then all the measures are absolutely continuous with respect to $w(x; 0, \beta|q) d_q x$ and the system of functions

$$x^{\alpha_1}, x^{\alpha_1+1}, \ldots, x^{\alpha_1+n_1-1}, x^{\alpha_2}, x^{\alpha_2+1}, \ldots, x^{\alpha_2+n_2-1}, \ldots, x^{\alpha_r}, x^{\alpha_r+1}, \ldots, x^{\alpha_r+n_r-1}$$

is a Chebyshev system on $(0, 1)$, so that the measures $(\mu_1, \ldots, \mu_r)$ form a so-called AT-system, which implies that all the multi-indices are normal [11, Theorem 4.3].

Observe that these operations raise one of the indices in the multi-index and lower the parameter $\beta$ and one of the components of $\vec{\alpha}$.
where $Q_{|\vec{n}|+1}$ is a monic polynomial of degree $|\vec{n}| + 1$. We will show that this monic polynomial $Q_{|\vec{n}|+1}$ satisfies the multiple orthogonality conditions (2.1) of $p_{\vec{n}+\vec{e}_j}(x; \vec{\alpha} - \vec{e}_j, \beta - 1|q)$ and hence, since all $\alpha_i - \alpha_j \notin \mathbb{Z}$ whenever $i \neq j$, the unicity of the multiple orthogonal polynomials implies that $Q_{|\vec{n}|+1}(x) = p_{\vec{n}+\vec{e}_j}(x; \vec{\alpha} - \vec{e}_j, \beta - 1|q)$. Integration by parts for the $q$-integral is given by the rule

$$
\int_0^1 f(x) \mathcal{D}_q g(x) = -q \int_0^1 g(x) \mathcal{D}_q f(x), \quad \text{if } g(p) = 0.
$$

(2.4)

If we apply this, then

$$
\frac{1 - q^{\alpha_j + \beta + |\vec{n}|}}{(1 - q)q^{\alpha_j + |\vec{n}|-1}} \int_0^1 x^k w(x; \alpha_j - 1, \beta - 1|q) Q_{|\vec{n}|+1}(x) \, dq \, x
$$

$$
= -q \int_0^1 w(x; \alpha_j, \beta|q) p_{\vec{n}}(x; \vec{\alpha}, \beta|q) \mathcal{D}_q x^k \, dq \, x,
$$

and since

$$
\mathcal{D}_q x^k = \begin{cases} 
\frac{1-q^k}{1-q} x^{k-1} & \text{if } k \geq 1, \\
0 & \text{if } k = 0,
\end{cases}
$$

we find that

$$
\int_0^1 x^k w(x; \alpha_j - 1, \beta - 1|q) Q_{|\vec{n}|+1}(x) \, dq \, x = 0, \quad k = 0, 1, \ldots, n_j.
$$

For the other components $\alpha_i$ ($i \neq j$) of $\vec{\alpha}$ we have

$$
\frac{1 - q^{\alpha_i + \beta + |\vec{n}|}}{(1 - q)q^{\alpha_i + |\vec{n}|-1}} \int_0^1 x^k w(x; \alpha_i, \beta - 1|q) Q_{|\vec{n}|+1}(x) \, dq \, x
$$

$$
= \frac{1 - q^{\alpha_i + \beta + |\vec{n}|}}{(1 - q)q^{\alpha_i + |\vec{n}|-1}} \int_0^1 x^{k+\alpha_i - \alpha_j + 1} w(x; \alpha_j - 1, \beta - 1|q) Q_{|\vec{n}|+1}(x) \, dq \, x
$$

$$
= - q \int_0^1 w(x; \alpha_j, \beta|q) p_{\vec{n}}(x; \vec{\alpha}, \beta|q) \mathcal{D}_q x^{k+\alpha_i - \alpha_j + 1} \, dq \, x,
$$

and since $\alpha_i - \alpha_j \notin \mathbb{Z}$ we have

$$
\mathcal{D}_q x^{k+\alpha_i - \alpha_j + 1} = \frac{1 - q^{k+\alpha_i - \alpha_j}}{1-q} x^{k+\alpha_i - \alpha_j},
$$

hence

$$
\int_0^1 x^k w(x; \alpha_i, \beta - 1|q) Q_{|\vec{n}|+1}(x) \, dq \, x = 0, \quad k = 0, 1, \ldots, n_i - 1.
$$

Hence all the orthogonality conditions for $p_{\vec{n}+\vec{e}_j}(x; \vec{\alpha} - \vec{e}_j, \beta - 1|q)$ are indeed satisfied. \qed

As a consequence we find a Rodrigues formula:
Theorem 2.2 The multiple little $q$-Jacobi polynomials of the first kind are given by

$$p_{\tilde{m}}(x; \tilde{\alpha}, \beta|q) = C(\tilde{\alpha}, \tilde{\alpha}, \beta) \frac{(q^{\beta+1}x; q)_{\infty}}{(qx; q)_{\infty}} \prod_{j=1}^{r} (x^{-\alpha_j}D_{p}^{\alpha_j} x^{\alpha_j} + j) \frac{(qx; q)_{\infty}}{(q^{\beta+|\tilde{m}|+1}x; q)_{\infty}}, \quad (2.5)$$

where the product of the difference operators can be taken in any order and

$$C(\tilde{\alpha}, \tilde{\alpha}, \beta) = (-1)^{|\tilde{m}|} \left(1 - q\right)^{|\tilde{m}|} \frac{\sum_{j=1}^{r}(\alpha_j - 1)_{nj} + \sum_{1 \leq j \leq k \leq r} n_j n_k}{\prod_{j=1}^{r} (q^{\alpha_j + \beta + |\tilde{m}| + 1}; q)_{nj}}.$$

Proof: If we apply the raising operator for $\alpha_j$ recursively $n_j$ times, then

$$D_{p}^{\alpha_j}w(x; \alpha_j, \beta|q)p_{\tilde{m}}(x; \alpha_j, \beta|q) = (-1)^{n_j} \frac{(q^{\alpha_j + \beta + |\tilde{m}| - n_j + 1}; q)_{ni}}{(1 - q)^{n_j q^{\alpha_j + |\tilde{m}| - 1})_{nj}} \times w(x; \alpha_j - n_j, \beta - n_j|q)p_{\tilde{m} + n_j \tilde{e}_j}(x; \tilde{\alpha} - n_j \tilde{e}_j, \beta - n_j|q). \quad (2.6)$$

Use this expression with $\tilde{m} = \tilde{0}$ and $j = 1$, then

$$D_{p}^{\alpha_1}w(x; \alpha_1, \beta|q) = (-1)^{n_1} \frac{(q^{\alpha_1 + \beta - n_1 + 1}; q)_{n_1}}{(1 - q)^{n_1 q^{\alpha_1 - 1})n_1}} \times w(x; \alpha_1 - n_1, \beta - n_1|q)p_{n_1 \tilde{e}_1}(x; \tilde{\alpha} - n_1 \tilde{e}_1, \beta - n_1|q).$$

Multiply both sides by $w(x; \alpha_2, \beta - n_1|q)$ and divide by $w(x; \alpha_1 - n_1, \beta - n_1|q)$, then

$$x^{n_1 + \alpha_2 - \alpha_1}D_{p}^{\alpha_1}w(x; \alpha_1, \beta|q) = (-1)^{n_1} \frac{(q^{\alpha_1 + \beta - n_1 + 1}; q)_{n_1}}{(1 - q)^{n_1 q^{\alpha_1 - 1})n_1}} \times w(x; \alpha_2 - n_1, \beta - n_1|q)p_{n_1 \tilde{e}_1}(x; \tilde{\alpha} - n_1 \tilde{e}_1, \beta - n_1|q).$$

Apply (2.6) with $j = 2$, then

$$D_{p}^{\alpha_2}x^{n_1 + \alpha_2 - \alpha_1}D_{p}^{\alpha_1}w(x; \alpha_1, \beta|q) = (-1)^{n_1 + n_2} \frac{(q^{\alpha_1 + \beta - n_1 + 1}; q)_{n_1} (q^{\alpha_2 + \beta - n_2 + 1}; q)_{n_2}}{(1 - q)^{n_1 + n_2 q^{\alpha_1 - 1})n_1 + (\alpha_2 - 1 + n_1)n_2}} \times w(x; \alpha_2 - n_2, \beta - n_1 - n_2|q)p_{n_1 \tilde{e}_1 + n_2 \tilde{e}_2}(x; \tilde{\alpha} - n_1 \tilde{e}_1 - n_2 \tilde{e}_2, \beta - n_1 - n_2|q).$$

Continuing this way we arrive at

$$\left( D_{p}^{\alpha_r}x^{\alpha_r} \right) \left( x^{n_{r-1} - \alpha_{r-1}}D_{p}^{\alpha_{r-1}}x^{\alpha_{r-1}} \right) \ldots \left( x^{\alpha_1}D_{p}^{\alpha_1} \right) w(x; \alpha_1, \beta|q) = \frac{(-1)^{|\tilde{m}|} \prod_{j=1}^{r}(q^{\alpha_j + \beta - n_j + 1}; q)_{nj}}{(1 - q)^{|\tilde{m}|} q^{\sum_{j=1}^{r}(\alpha_j - 1)_{nj} + \sum_{1 \leq j \leq k \leq r} n_j n_k}} w(x; \alpha_r - n_r, \beta - |\tilde{m}||q)p_{\tilde{m}}(x; \tilde{\alpha} - \tilde{\alpha}, \beta - |\tilde{m}|)q).$$

Now replace each $\alpha_j$ by $\alpha_j + n_j$ and $\beta$ by $\beta + |\tilde{m}|$, then the required expression follows. The order in which we took the raising operators is irrelevant. □
We can obtain an explicit expression of the multiple little $q$-Jacobi polynomials of the first kind using this Rodrigues formula. Indeed, if we use the $q$-binomial theorem, then
\[
\frac{\langle qx; q \rangle_\infty}{\langle q^{\beta+|\vec{n}|}+1; x; q \rangle_\infty} = \sum_{k=0}^{\infty} \frac{(q^{-\beta-|\vec{n}|}; q)_k}{(q; q)_k} q^{(\beta+|\vec{n}|+1)k} x^k.
\]

Use this in (2.5), together with
\[
x^{-\alpha} P^{n_\alpha|\vec{n}} \alpha+n+k = \frac{(q^{\alpha+1}; q)_n (q^{\alpha+n+1}; q)_k}{(1-q^n) (q^{\alpha+1}; q)_k} q^{-n(k+n) - (n-1)/2} x^k,
\]
then this gives
\[
p_n(x; \vec{\alpha}, \beta|q) = C(\vec{n}, \vec{\alpha}, \beta) \prod_{j=1}^{r} (q^{\alpha_j+1}; q)_{n_j} q^{-\sum_{j=1}^{r} \alpha_j n_j - \sum_{j=1}^{r} \binom{n_j}{2}}
\]
\[
\frac{(q^{\beta+1}; q)_\infty}{(qx; q)_\infty} \phi_r \left( \begin{array}{c}
q^{-\beta-|\vec{n}|}, q^{\alpha_1+n+1}, \ldots, q^{\alpha_r+n+1} \\
q^{\alpha_1+1}, \ldots, q^{\alpha_r+1}
\end{array} \right) q^{\beta+1} x^r. \quad (2.7)
\]

This explicit expression uses a non-terminating basic hypergeometric series, except when $\beta$ is an integer. Another representation, using only finite sums, can be obtained by using the Rodrigues formula (1.8) $r$ times. For $r = 2$ this gives

**Theorem 2.3** The multiple little $q$-Jacobi polynomials of the first kind (for $r = 2$) are given by
\[
p_{n,m}(x; (\alpha_1, \alpha_2), \beta|q) = \frac{q^{nm+n^2+\alpha_1 n + \alpha_2 m} (q^{-\alpha_1-n}; q)_n (q^{-\alpha_2-m}; q)_m}{(q^{\alpha_1+\beta+n+m+1}; q)_n (q^{\alpha_2+\beta+n+m+1}; q)_m}
\]
\[
\times \sum_{k=0}^{n} \sum_{\ell=0}^{m} \frac{(q^{-n}; q)_\ell (q^{-m}; q)_k (q^{\alpha_2+\beta+m+n+1}; q)_k (q^{\alpha_1+\beta+n+1}; q)_k (q^{\alpha_1+n+1}; q)_k}{(q^{\alpha_2+1}; q)_k (q^{\alpha_1+1}; q)_k} q^{k+l} x^{k+l}
\]
\[
\times \frac{q^{k+\ell} x^{k+\ell}}{q^{kn}(q; q)_k (q; q)_l}. \quad (2.8)
\]

**Proof:** For $r = 2$ the Rodrigues formula (2.5) is
\[
p_{n,m}(x; (\alpha_1, \alpha_2), \beta|q) = \frac{(-1)^{n+m}(1-q)^{n+m} q^{\alpha_1 n + \alpha_2 m - n + m + n^2 + m^2}}{(q^{\alpha_1+\beta+n+m+1}; q)_n (q^{\alpha_2+\beta+n+m+1}; q)_m}
\]
\[
\times \frac{(q^{\beta+1}; q)_\infty}{(qx; q)_\infty} x^{-\alpha_1} P^{n_\alpha|\vec{n}} \alpha+n-\alpha_2 P^{m_\alpha|\vec{m}} \alpha_2+m \frac{(qx; q)_\infty}{(q^{\beta+\alpha+n+1}; q)_\infty}.
\]

Observe that by the Rodrigues formula (1.8) for the little $q$-Jacobi polynomials
\[
\frac{\langle qx; q \rangle_\infty}{\langle q^{\beta+n+1}+1; x; q \rangle_\infty} = \frac{(-1)^{m} (q^{\alpha_2+\beta+n+m+1}; q)_m}{(1-q)^m q^{\alpha_2 m + m^2 - m}} \frac{(qx; q)_\infty}{(q^{\beta+n+1}; q)_\infty} x^{\alpha_2} p_m(x; \alpha_2, \beta + n|q),
\]
and hence
\[ p_{n,m}(x; (\alpha_1, \alpha_2), \beta | q) = \frac{(-1)^n (1 - q)^n q^{\alpha_1 n - n + nm + n^2}}{(q^{\alpha_1 + \beta + n + m + 1}; q)_n} \times \frac{(q^{3 + 1}; q)_\infty}{(q x; q)_\infty} x^{-\alpha_1} \mathcal{D}_{p}^{n x^{\alpha_1 + n}} \frac{(q x; q)_\infty}{(q^{3 + n + 1} x; q)_\infty} p_m(x; \alpha_2, \beta + n | q). \]

Now use the explicit expression (1.9) to find
\[ p_{n,m}(x; (\alpha_1, \alpha_2), \beta | q) = \frac{(-1)^n (1 - q)^n q^{\alpha_1 n + \alpha_2 m - n + nm + n^2 + m^2 (q^{-m - \alpha_2}; q)_m}}{(q^{\alpha_1 + \beta + n + m + 1}; q)_n (q^{\alpha_2 + \beta + n + m + 1}; q)_m} \times \frac{(q^{\beta + 1}; q)_\infty}{(q x; q)_\infty} x^{-\alpha_1} \sum_{k=0}^{m} \frac{(q^{-m}; q)_k (q^{\alpha_2 + \beta + n + m + 1}; q)_k q^k}{(q^{\alpha_2 + 1}; q)_k (q; q)_k} \mathcal{D}_{p}^{n x^{\alpha_1 + n + k}} \frac{(q x; q)_\infty}{(q^{3 + n + 1} x; q)_\infty} p_n(x; \alpha_1 + k, \beta | q), \]

In this expression we recognize
\[ \mathcal{D}_{p}^{n x^{\alpha_1 + n + k}} \frac{(q x; q)_\infty}{(q^{3 + n + 1} x; q)_\infty} = \frac{(-1)^n (q^{\alpha_1 + \beta + k + n + 1}; q)_n}{(1 - q)^n q^{\alpha_1 n + k + n^2 - n}} x^{\alpha_1 + k} \frac{(q x; q)_\infty}{(q^{\beta + 1}; q)_\infty} p_n(x; \alpha_1 + k, \beta | q), \]

hence
\[ p_{n,m}(x; (\alpha_1, \alpha_2), \beta | q) = \frac{q^{\alpha_2 m + nm + m^2 (q^{-m - \alpha_2}; q)_m}}{(q^{\alpha_1 + \beta + n + m + 1}; q)_n (q^{\alpha_2 + \beta + n + m + 1}; q)_m} \times \sum_{k=0}^{m} \frac{(q^{-m}; q)_k (q^{\alpha_2 + \beta + n + m + 1}; q)_k q^k}{(q^{\alpha_2 + 1}; q)_k (q; q)_k q^n} x^k p_n(x; \alpha_1 + k, \beta | q). \]

If we use the explicit expression (1.9) for the little $q$-Jacobi polynomials once more, then after some simplifications we finally arrive at (2.8). \[\square\]

2.2 Multiple little $q$-Jacobi polynomials of the second kind

Multiple little $q$-Jacobi polynomials of the second kind $p_{\vec{n}}(x; \alpha, \vec{\beta} | q)$ are monic polynomials of degree $|\vec{n}|$ satisfying the orthogonality relations
\[ \int_{0}^{1} p_{\vec{n}}(x; \alpha, \vec{\beta} | q)x^k w(x; \alpha, \beta_j | q) \, dx = 0, \quad k = 0, 1, \ldots, n_j - 1, \quad j = 1, 2, \ldots, r, \quad (2.9) \]

where $\alpha, \beta_1, \ldots, \beta_r > -1$. Observe that all the measures are orthogonality measures for little $q$-Jacobi polynomials with the same parameter $\alpha$ but with different parameters $\beta_j$. All the multi-indices will be normal when we impose the condition that $\beta_i - \beta_j \notin \mathbb{Z}$ whenever $i \neq j$, because then all the measures are absolutely continuous with respect to
$(qx; q)_{\infty} w(x; \alpha, 0|q) d_q x$ and the system of functions
\[
\frac{1}{(q^{b_1+1}; q)_{\infty}} x \frac{1}{(q^{b_1+1}; q)_{\infty}} x^{n_1-1} \frac{1}{(q^{b_2+1}; q)_{\infty}} x \frac{1}{(q^{b_2+1}; q)_{\infty}} x^{n_2-1} \cdots \frac{1}{(q^{b_r+1}; q)_{\infty}} x \frac{1}{(q^{b_r+1}; q)_{\infty}} x^{n_r-1}
\]
is a Chebyshev system\(^1\) on $[0, 1]$, so that the vector of measures $(\mu_1, \ldots, \mu_r)$ forms an AT-system, which implies that all the multi-indices are normal [11, Theorem 4.3].

Again there are $r$ raising operations.

**Theorem 2.4** Suppose that $\alpha, \beta_1, \ldots, \beta_r > 0$, with $\beta_i - \beta_j \notin \mathbb{Z}$ when $i \neq j$, and put $p = 1/q$, then
\[
\mathcal{D}_p \left[ w(x; \alpha, \beta_j|q) p_{\bar{n}}(x; \alpha, \bar{\beta}|q) \right] = \frac{q^{\alpha+\beta_j+|\bar{n}|} - 1}{(1 - q) q^{\alpha+|\bar{n}|} - 1} w(x; \alpha - 1, \beta_j - 1|q) p_{\bar{n}+\bar{e}_j}(x; \alpha - 1, \bar{\beta} - \bar{e}_j|q), \quad (2.10)
\]
for $1 \leq j \leq r$, where $\bar{e}_1 = (1, 0, 0, \ldots, 0), \ldots, \bar{e}_r = (0, \ldots, 0, 0, 1)$ are the standard unit vectors.

Observe that these operations raise one of the indices in the multi-index and lower the parameter $\alpha$ and one of the components of $\bar{\beta}$.

**Proof:** Again we see that
\[
\mathcal{D}_p \left[ w(x; \alpha, \beta_j|q) p_{\bar{n}}(x; \alpha, \bar{\beta}|q) \right] = \frac{q^{\alpha+\beta_j+|\bar{n}|} - 1}{(1 - q) q^{\alpha+|\bar{n}|} - 1} w(x; \alpha - 1, \beta_j - 1|q) Q_{|\bar{n}|+1}(x), \quad (2.11)
\]
where $Q_{|\bar{n}|+1}$ is a monic polynomial of degree $|\bar{n}| + 1$. We will show that this monic polynomial $Q_{|\bar{n}|+1}$ satisfies the multiple orthogonality conditions (2.9) of $p_{\bar{n}+\bar{e}_j}(x; \alpha - 1, \bar{\beta} - \bar{e}_j|q)$ and hence, since all $\beta_i - \beta_j \notin \mathbb{Z}$ whenever $i \neq j$, the unicity of the multiple orthogonal polynomials implies that $Q_{|\bar{n}|+1}(x) = p_{\bar{n}+\bar{e}_j}(x; \alpha - 1, \bar{\beta} - \bar{e}_j|q)$. Integration by parts gives
\[
\frac{1 - q^{\alpha+\beta_j+|\bar{n}|}}{(1 - q) q^{\alpha+|\bar{n}|} - 1} \int_0^1 x^k w(x; \alpha - 1, \beta_j - 1|q) Q_{|\bar{n}|+1}(x) d_q x
\]
\[
= -q \int_0^1 w(x; \alpha, \beta_j|q) p_{\bar{n}}(x; \alpha, \bar{\beta}|q) D_q x^k d_q x,
\]
so that
\[
\int_0^1 x^k w(x; \alpha - 1, \beta_j - 1|q) Q_{|\bar{n}|+1}(x) d_q x = 0, \quad k = 0, 1, \ldots, n_j.
\]

\(^1\) The fact that this system is a Chebyshev system is not obvious but is left as an advanced problem for the reader.
For the other components \( \beta_i (i \neq j) \) of \( \vec{\beta} \) we have

\[
1 - \frac{q^{\alpha+\beta_j+|\vec{n}|}}{(1-q)q^{\alpha+|\vec{n}|-1}} \int_0^1 x^k w(x; \alpha - 1, \beta_i|q)Q_{|\vec{n}|+1}(x) \, dq x
\]

\[
= \frac{1 - q^{\alpha+\beta_j+|\vec{n}|}}{(1-q)q^{\alpha+|\vec{n}|-1}} \int_0^1 x^k \frac{(q^{\beta_j}x; q)_{\infty}}{(q^{\beta_i+1}x; q)_{\infty}} w(x; \alpha - 1, \beta_j - 1|q)Q_{|\vec{n}|+1}(x) \, dq x
\]

\[
= - q \int_0^1 w(x; \alpha, \beta_j|q)p_{\vec{n}}(x; \alpha, \vec{\beta}|q)D_q \left( \frac{x^k (q^{\beta_j}x; q)_{\infty}}{(q^{\beta_i+1}x; q)_{\infty}} \right) \, dq x,
\]

and since \( \beta_i - \beta_j \notin \mathbb{Z} \) we have

\[
D_q \left( \frac{x^k (q^{\beta_j}x; q)_{\infty}}{(q^{\beta_i+1}x; q)_{\infty}} \right) = x^{k-1} \frac{(q^{\beta_j+1}x; q)_{\infty}}{(q^{\beta_i+1}x; q)_{\infty}} a_k(x),
\]

where each \( a_k \) is a polynomial of degree exactly 1 and \( a_0(0) = 0 \). Therefore

\[
\int_0^1 x^k w(x; \alpha - 1, \beta_i|q)Q_{|\vec{n}|+1}(x) \, dq x = 0, \quad k = 0, 1, \ldots, n_i - 1.
\]

Hence all the orthogonality conditions for \( p_{\vec{n}+\vec{e}_j}(x; \alpha - 1, \vec{\beta} - \vec{e}_j|q) \) are indeed satisfied. \( \square \)

As a consequence we again find a Rodrigues formula:

**Theorem 2.5** The multiple little \( q \)-Jacobi polynomials of the second kind are given by

\[
p_{\vec{n}}(x; \alpha, \vec{\beta}|q) = C(\vec{n}, \alpha, \vec{\beta}) \frac{1}{(qx; q)_{\infty}^{\alpha+|\vec{n}|}} \prod_{j=1}^r \left( \frac{(q^{\beta_j+1}x; q)_{\infty}D_{p}^{n_j}}{(q^{\beta_j+n_j+1}x; q)_{\infty}} \right) (q x; q)_{\infty} x^{\alpha+|\vec{n}|}, \tag{2.12}
\]

where the product of the difference operators can be taken in any order and

\[
C(\vec{n}, \alpha, \vec{\beta}) = (-1)^{|\vec{n}|} \frac{(1-q)^{|\vec{n}|}q^{(\alpha+|\vec{n}|-1)|\vec{n}|}}{\prod_{j=1}^r (q^{\alpha+\beta_j+|\vec{n}|-1}; q)_{n_j}}.
\]

**Proof:** The proof can be given in a similar way as in the case of little \( q \)-Jacobi polynomials of the first kind by repeated application of the raising operators. Alternatively one can use induction on \( r \). For \( r = 1 \) the Rodrigues formula is the same as (1.8). Suppose that the Rodrigues formula (2.12) holds for \( r - 1 \). Observe that the multiple orthogonal polynomials with multi-index \((n_1, \ldots, n_{r-1})\) for \( r-1 \) measures \((\mu_1, \ldots, \mu_{r-1})\) coincide with the multiple orthogonal polynomials with multi-index \((n_1, n_2, \ldots, n_{r-1}, 0)\) for \( r \) measures \((\mu_1, \ldots, \mu_r)\) for any measure \( \mu_r \). Use the Rodrigues formula for \( r - 1 \) for the polynomial \( p_{\vec{n}-n_r \vec{e}_r}(x; \alpha + n_r, \vec{\beta} + n_r \vec{e}_r|q) \) to find

\[
w(x; \alpha + n_r, \beta_r + n_r|q)p_{\vec{n}-n_r \vec{e}_r}(x; \alpha + n_r, \vec{\beta} + n_r \vec{e}_r|q) = C(\vec{n} - n_r \vec{e}_r, \alpha + n_r, \vec{\beta}) \times \frac{1}{(q^{\beta_r+n_r+1}x; q)_{\infty}^{\alpha+|\vec{n}|}} \prod_{j=1}^{r-1} \left( \frac{(q^{\beta_j+1}x; q)_{\infty}D_{p}^{n_j}}{(q^{\beta_j+n_j+1}x; q)_{\infty}} \right) (q x; q)_{\infty} x^{\alpha+|\vec{n}|}.
\]
Now apply the raising operation (2.10) for $\beta_r$ to this expression $n_r$ times to find the required expression. □

In a similar way as for the first kind multiple little $q$-Jacobi polynomials we can find an explicit formula with finite sums using the Rodrigues formula for little $q$-Jacobi polynomials $r$ times. For $r = 2$ this gives the following:

**Theorem 2.6** The multiple little $q$-Jacobi polynomials of the second kind (for $r = 2$) are explicitly given by

$$p_{n,m}(x; \alpha, (\beta_1, \beta_2)|q) = \frac{q^{\alpha(n+m)+n^2+m^2+nm}(q^{-m-\alpha}; q)_m(q^{-n-\alpha}; q)_n(q^{\alpha+1}; q)_m}{(q^{\alpha+\beta_1+n+m+1}; q)_n(q^{\alpha+\beta_2+n+m+1}; q)_m} \times \sum_{\ell=0}^{n} \sum_{k=0}^{m} \frac{(q^{-n}; q)_\ell(q^{-m}; q)_k(q^{\alpha+n+1}; q)_m(q^{\alpha+1}; q)_m}{(q^{\alpha+1}; q)_{k+\ell}(q^{\alpha+\beta_1+n+1}; q)_k} \times \frac{q^{k+\ell}x^k}{q^{nk}(q; q)_k(q; q)_\ell}.$$ (2.13)

**Proof:** The Rodrigues formula (2.12) for $r = 2$ becomes

$$p_{n,m}(x; \alpha, (\beta_1, \beta_2)|q) = \frac{(-1)^{n+m}(1 - q)^{n+m}q^{(\alpha+n+m-1)(n+m)}}{(q^{\alpha+\beta_1+n+m+1}; q)_n(q^{\alpha+\beta_2+n+m+1}; q)_m} \times x^{-\alpha}(q^{\beta_1+1}; q)_n(q^{\beta_2+1}; q)_m D^\alpha\frac{(q;x)_\infty}{(q^{\beta_1+1}; q)_\infty} D^\beta \frac{(q^{\beta_2+1}; q)_\infty}{(q^{\beta_2+1}; q)_\infty} x^{\alpha+n+m}.$$

The Rodrigues formula (1.8) for little $q$-Jacobi polynomials gives

$$D^\alpha\frac{(q;x)_\infty}{(q^{\beta_1+1}; q)_\infty} D^\beta \frac{(q;x)_\infty}{(q^{\beta_2+1}; q)_\infty} p_m(x; \alpha + n, \beta_2|q),$$

hence

$$p_{n,m}(x; \alpha, (\beta_1, \beta_2)|q) = \frac{(-1)^{n}(1 - q)^n q^{\alpha+n^2+nm-n}}{(q^{\alpha+\beta_1+n+m+1}; q)_n} \times x^{-\alpha}(q^{\beta_1+1}; q)_n q^\alpha \frac{(q;x)_\infty}{(q^{\beta_1+1}; q)_\infty} p_m(x; \alpha + n, \beta_2|q).$$

Now use the explicit expression (1.9) for the little $q$-Jacobi polynomials to find

$$p_{n,m}(x; \alpha, (\beta_1, \beta_2)|q) = \frac{(-1)^{n}(1 - q)^n q^{\alpha(n+m)+n^2+m^2+2nm-n}(q^{-m-n-\alpha}; q)_m}{(q^{\alpha+\beta_1+n+m+1}; q)_n(q^{\alpha+\beta_2+n+m+1}; q)_m} \times \frac{(q^{\beta_1+1}; q)_\infty}{x^\alpha(q;x)_\infty} \sum_{k=0}^{m} \frac{(q^{-m}; q)_k(q^{\alpha+\beta_2+n+m+1}; q)_k q^k}{(q^{n+1}; q)_k(q; q)_k} D^\alpha x^{\alpha+n+k} \frac{(q;x)_\infty}{(q^{\beta_1+n+1}; q)_\infty}.$$
Again we recognize a little $q$-Jacobi polynomial

\[
\mathcal{D}_p^n x^{\alpha+n+k} (qx; q)_\infty \frac{(q^n x; q)_\infty}{(q^{\beta_1+n+1} x; q)_\infty} = (-1)^n (q^{\alpha+\beta_1+k+n+1}; q)_n x^{\alpha+k} (qx; q)_\infty p_n(x; \alpha+k; \beta_1|q),
\]

and if we use the explicit expression (1.9) for this little $q$-Jacobi polynomial, then we find (2.13) after some simplifications. □

3 Zeros

The zeros of the multiple little $q$-Jacobi polynomials (first and second kind) are all real, simple and in the interval $(0,1)$. This is a consequence of the fact that $\mu_1, \ldots, \mu_r$ form an AT-system [11, first Corollary on p. 141]. For the usual orthogonal polynomials with positive orthogonality measure $\mu$ we know that an interval $[c,d]$ for which the orthogonality measure has no mass, i.e., $\mu([c,d]) = 0$, can have at most one zero of each orthogonal polynomial $p_n$. In particular this means that each orthogonal polynomial $p_n$ on the exponential lattice \( \{q^k, k = 0, 1, 2, \ldots \} \) can have at most one zero between two points $q^{k+1}$ and $q^k$ of the lattice. A similar result holds for multiple orthogonal polynomials if we impose some conditions on the measures $\mu_i$.

**Theorem 3.1** Suppose $\mu_1, \ldots, \mu_r$ are positive measures on $[a,b]$ with infinitely many points in their support, which form an AT system, i.e., $\mu_k$ is absolutely continuous with respect to $\mu_1$ for $2 \leq k \leq r$ with

\[
\frac{d\mu_k(x)}{d\mu_1(x)} = w_k(x),
\]

and

\[
1, x, \ldots, x^{n_1-1}, w_2(x), xw_2(x), \ldots, x^{n_2-1}w_2(x), \ldots, w_r(x), xw_r(x), \ldots, x^{n_r-1}w_r(x)
\]

are a Chebyshev system on $[a,b]$ for every multi-index $\vec{n}$. If $[c,d]$ is an interval such that $\mu_1([c,d]) = 0$, then each multiple orthogonal polynomial $p_{\vec{n}}$ has at most one zero in $[c,d]$.

**Proof:** Suppose that $p_{\vec{n}}$ is a multiple orthogonal polynomial with two zeros $x_1$ and $x_2$ in $[c,d]$. We can then write it as $p_{\vec{n}}(x) = (x-x_1)(x-x_2)q_{|[\vec{n}]-2}(x)$, where $q_{|[\vec{n}]-2}$ is a polynomial of degree $|\vec{n}|-2$. Consider a function $A(x) = \sum_{j=1}^r A_j(x)w_j(x)$, where $w_1 = 1$ and each $A_j$ is a polynomial of degree $m_j - 1 \leq n_j - 1$, with $|\vec{m}| = |\vec{n}|-1$. Since we are dealing with a Chebyshev system, there is a unique function $A$ satisfying the interpolation conditions

\[
A(y) = \begin{cases} 
0 & \text{if } y \text{ is a zero of } q_{|[\vec{n}]-2}, \\
1 & \text{if } y = x_1.
\end{cases}
\]
Furthermore $A$ has $|\vec{n}| - 2$ zeros in $[a, b]$ and these are the only sign changes on $[a, b]$. Hence
\[
\int_{a}^{b} p_{\vec{n}}(x)A(x) \, d\mu_{1}(x) = \int_{[a, b]\setminus[c, d]} (x - x_{1})(x - x_{2})q_{|\vec{n}|-2}(x)A(x) \, d\mu_{1}(x) \neq 0,
\]
since the integrand does not change sign on $[a, b] \setminus [c, d]$. On the other hand
\[
\int_{a}^{b} p_{\vec{n}}(x)A(x) \, d\mu_{1}(x) = \sum_{j=1}^{r} \int_{a}^{b} p_{\vec{n}}(x)A_{j}(x) \, d\mu_{j}(x) = 0,
\]
since every term in the sum vanishes because of the orthogonality conditions. This contradiction implies that $p_{\vec{n}}$ can’t have two zeros in $[c, d]$. $\square$

In particular this theorem tells us that the zeros of the multiple little $q$-Jacobi polynomials are always separated by the points $q^{k}$ and that between two points $q^{k+1}$ and $q^{k}$ there can be at most one zero of a multiple little $q$-Jacobi polynomials. Note that the points $q^{k}$ have one accumulation point at 0, hence as a consequence the zeros of the multiple little $q$-Jacobi polynomials (first and second kind) accumulate at the origin.

4 Asymptotic behavior

The asymptotic behavior of little $q$-Jacobi polynomials was given by Ismail and Wilson [7] and an asymptotic expansion was given by Ismail [6]. In this section we give the asymptotic behavior of the multiple little $q$-Jacobi polynomials which extends the result of Ismail and Wilson.

**Theorem 4.1** For the multiple little $q$-Jacobi polynomials of the first kind we have
\[
\lim_{n, m \to \infty} x^{n+m} p_{n,m}(1/x; (\alpha_{1}, \alpha_{2}), \beta|q) = (x; q)_{\infty},
\]
(4.1)

The order in which the limits for $n$ and $m$ are taken is irrelevant.

**Proof:** If we use (2.8) and reverse the order of summation (i.e., change variables $m-k = j$ and $n - \ell = i$), then
\[
\begin{align*}
&x^{n+m} p_{n,m}(1/x; (\alpha_{1}, \alpha_{2}), \beta|q) = \frac{q^{nm+n^{2}+\alpha_{1}n+\alpha_{2}m}(q^{-\alpha_{1}-n}; q)_{n}(q^{-\alpha_{2}-m}; q)_{m}}{(q^{\alpha_{1}+\beta+n+m+1}; q)_{n}(q^{\alpha_{2}+\beta+n+m+1}; q)_{m}} \\
&\times \sum_{i=0}^{n} \sum_{j=0}^{m} (q^{-n}; q)_{n-i}(q^{-m}; q)_{m-j}(q^{\alpha_{1}+\beta+m+n+1}; q)_{m-j}(q^{\alpha_{1}+\beta+n+1}; q)_{m+n-i-j}(q^{\alpha_{1}+1}; q)_{m-j} \\
&\times \frac{q^{m+n-i-j+1}}{q^{|m-j|n}(q; q)_{m-j}(q; q)_{n-i}}.
\end{align*}
\]

Now observe that
\[(q^{-m}; q)_{m-j} = (-1)^{m-j} q^{-\frac{m(m+1)}{2}} \frac{\beta^{m+1} (q; q)_m,}{(q; q)_j},
\]
\[(q^{-m-\alpha}; q)_{m} = (-1)^{m} q^{-m(m+1)/2} q^{-\alpha} (q^{\alpha+1}; q)_{m-m},
\]
\[(q^{c+n}; q)_{m} = \frac{(q^c; q)_{n+m}}{(q^c; q)_n},
\]
therefore we find
\[
x^{n+m} p_{n,m}(1/x; (\alpha_1, \alpha_2), \beta|q) = \frac{(q^{\alpha_2+1}; q)_n(q^{\alpha_1+\beta+1}; q)_{n+m}(q; q)_m(q; q)_n}{(q^{\alpha_1+\beta+1}; q)_{2n+m}(q^{\alpha_2+\beta+1}; q)_{n+2m}}
\]
\[
\times \sum_{i=0}^{n} \sum_{j=0}^{m} \frac{(q^{\alpha_2+\beta+1}; q)_{n+2m-j}(q^{\alpha_1+\beta+1}; q)_{2n+m-i-j}(q^{\alpha_1+1}; q)_{n+m-j}}{(q^{\alpha_1+1}; q)_{m+n-i-j}(q^{\alpha_1+1}; q)_{n-i}(q; q)_{m-j}}
\]
\[
\times (-1)^{i+j} q^2 \frac{q^{nj} x^{i+j}}{(q; q)_i(q; q)_j}.
\]
If we use Lebesgue’s dominated convergence theorem, then we take \(n, m \to \infty\) in each term of the sum. The factor \(q^{nj}\) tends to zero whenever \(j > 0\), hence the only contributions come from \(j = 0\), and we find
\[
\lim_{n,m \to \infty} x^{n+m} p_{n,m}(1/x; (\alpha_1, \alpha_2), \beta|q) = \sum_{i=0}^{\infty} q^2 \frac{(-x)^i}{(q; q)_i}.
\]
The right hand side is the \(q\)-exponential function
\[
E_q(-x) = (x, q)_{\infty},
\]
[5, (II.2) in Appendix II], which gives the required result. \(\square\)

**Theorem 4.2**  For the multiple little \(q\)-Jacobi polynomials of the second kind we have
\[
\lim_{n,m \to \infty} x^{n+m} p_{n,m}(1/x; \alpha, (\beta_1, \beta_2)|q) = (x; q)_{\infty}.
\]
The order in which the limits for \(n\) and \(m\) are taken is irrelevant.

**Proof:** The proof is similar to the case of the first kind multiple little \(q\)-Jacobi polynomials, except that now we use the expression (2.13). \(\square\)

As a consequence (using Hurwitz’ theorem) we see that every zero of \((1/x; q)_{\infty}\), i.e., each number \(q^k\), \(k = 0, 1, 2, \ldots\), is an accumulation point of zeros of the multiple little \(q\)-Jacobi polynomial \(p_{n,m}\) of the first and of the second kind.

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