Modeling finite element for stress state calculation in combined structures

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Abstract. A method for constructing a transition finite element for orthotropic multilayer shells is developed in the article. Based on a three-dimensional finite element of the continuum, the family of the new finite elements is constructed, allowing joining elements of various structures in uniform element model. Calculation of the stabilizer of the plane on durability and stability for various cases of load is taken into account, along with the comparison of results of calculation with the data received in package ANSYS.

Keywords: transition finite element, stress state, orthotropic material, aircraft stabilizer

Introduction

While constructing physical models of multilayer orthotropic shells, various approaches are introduced, based on both different hypotheses for each layer of the shell [1-3], and on uniform hypotheses for all layers of a thin-walled structure [2,4-9]. In the first case, the order of the resolution system depends on the number of layers. In the second case, the order of the system does not depend on the number of layers, which opens up the possibilities for the effective use of FEM in the calculations of orthotropic multilayer shells. This orthotropic multilayer finite element (FE) was obtained based on a three-dimensional isotropic eight-node FE shell [10, 22], which consists discretizing three-dimensional
equations of the theory of elasticity in a curved coordinate system using some shell hypotheses. The application of such approaches to the calculation of shells of medium thickness is discussed in papers [11-18]. The introduction of orthotropy and multilayer into the design scheme of shells is described in publications [17-21].

To calculate combined structures and structures of substantially variable thickness by the finite element method, it is advisable to use both three-dimensional finite elements (FE) [10, 22] and shell ones. Various methods are used to dock them into a single calculation model. In [23], the creation and study of transitional three-dimensional FEs used in solving thermo elastic problems is described, in [24] transitional finite elements are used that connect quadratic three-dimensional elements of a continuous medium and elements of shells of the Timoshenko type into a single calculation model, which are chosen as nodal degrees of freedom with three projections of the displacement vector and two normal rotation angles; in [25], to calculate the stress-strain state of spatial structures, a combined model is used, where these conditions of elastic conjugation with shell elements are taken into account using the penalty method. In [26], the necessary conjugation conditions along the boundary between three-dimensional and shell FEs are realized by introducing the Lagrange multipliers into the original function, the parameters of which are excluded from the number of variable values at the assembly level of the structure. A review of some algorithms for creating transient FEs is given in [27].

The aim of this work is to develop a method to construct a three-dimensional finite element of a continuous medium, which allows modelling combined structures consisting of three-dimensional bodies and shells of medium thickness in a linear formulation. Typically, FE shells of medium thickness are constructed using degrees of freedom defined at the nodes of the median surface and including the rotation angles of a normal fiber. As a rule, FEs with angular degrees of freedom demonstrate good accuracy and are effective in calculating shells of small and medium thicknesses. However, it is very difficult to use them in modelling conjugations of shells with massive three-dimensional bodies, since it is necessary to express the nodal displacements of three-dimensional elements through the angles of rotation of the shell FEs. Therefore, the use of special shell elements, one of which is presented in this paper, allows us to simplify the procedure for joining elements modelling three-dimensional and shell substructures.

In this paper, we introduce approximations of the radius vector, covariant and contravariant basis vectors, metric tensors, displacements, strain and stress tensors. The technology of using the double approximation method for super convergence points and the technique of “lowering the approximation order” of lateral shear strains in a three-dimensional setting are described in detail in [10, 22]. The hypothesis of small compressive stresses is described in approximate form, the “simplified Hooke law”, which relates the compressive stress to the compressive strain, is not used. The matrix of elastic constants for an orthotropic material is determined. The elastic characteristics and the winding angle can be generally different for each layer of the finite element. The assembly relationships of the stiffness matrix for a multilayer orthotropic FE are described.

Several test problems have been solved, and hence the efficiency of the proposed FE is tested and verified with solutions of other authors. The results show the acceptability of this technique to determine the stress-strain state of thin-walled structures. For illustration, the results of a static calculation of the stabilizer of a light-engine aircraft, consisting of multilayer shells, are presented.

1. Kinematic relationships

Within each element, a local coordinate system \( \xi_1, \xi_2, \xi_3 \) is introduced, which translates the curved box into a unit cube. We define a vector function of an arbitrary material point of a thin-walled structure.

\[
\mathbf{r} = x'(\xi_1, \xi_2, \xi_3) \mathbf{\bar{e}},
\]

which would ensure the continuity of itself and its first derivatives, i.e. basis vectors, when crossing interelement boundaries. The projections of the radius vector are approximated as follows
\[ x' (\xi_1, \xi_2, \xi_3) = \sum_{i=1}^{k} x'_i N_i (\xi_1, \xi_2, \xi_3), \]

where \( N_i (\xi_1, \xi_2, \xi_3) \) are the functions of the shape of the FE shell.

To define covariant basis vectors

\[ r_i = \frac{\partial x'}{\partial \xi_i} e_i = r'_i e_i. \]

We write the components of the corresponding basis in the form

\[ r'_i = \sum_{i=1}^{k} x'_i \frac{\partial N_i}{\partial \xi_i}. \]

The covariant basis is calculated by the formulas

\[ r^i = \frac{\partial \xi_i}{\partial \xi'} r_e = r^i e_i. \]

Metric tensors in various bases characterizing the geometry can be represented using the following relations

\[ g_{ij} = r_i r_j = \sum_{m} r'^m r'^m, \quad g^{ij} = r'^i r'^j = \sum_{m} r'^m r'^m. \]

We introduce approximations of the vector \( u \) and the gradient of the displacement vector \( u \),

\[ u = u^m (\xi_1, \xi_2, \xi_3) e_m, \quad u_i = e_i \sum_{m} u^m \frac{\partial N_m}{\partial \xi_i}. \]

The components of the strain tensor are defined as follows

\[ \varepsilon_{ij} = \frac{1}{2} (u_i r_j + r_i u_j) = \sum_{m} \sum_{m} u^m E_{ij}^{mn}, \]

when

\[ E_{ij}^{mn} = \sum_{i=1}^{k} x'^m \left( \frac{\partial N_i}{\partial \xi''_j} + \frac{\partial N_i}{\partial \xi''_i} \frac{\partial N_i}{\partial \xi''_j} \right) \frac{\partial N_i}{\partial \xi''_i} \frac{\partial N_i}{\partial \xi''_j} \frac{\partial N_i}{\partial \xi''_i} \frac{\partial N_i}{\partial \xi''_j}. \]

The technology of lowering the approximation order for transverse shear deformations \( \varepsilon_{ij} \), \( i = 1, 2 \) is described in detail in [10, 22].

Let us describe the technique of introducing a “decrease in the approximation order” for a three-dimensional finite element. Consider the shear strain \( E_{13}^{13}, E_{23}^{23} \)

\[ E_{13}^{13} = \frac{1}{128} X_{13}^m \left[ \xi_1^3 \xi_2^3 \xi_3^3 \left( 1 + \xi_1^3 \xi_2^3 \right) \left( 1 + \xi_1^3 \xi_3^3 \right) \left( 1 + \xi_2^3 \xi_3^3 \right) \right]. \]

The first “reduction of approximation” consists in eliminating the variability of these strains in thickness.

\[ E_{13}^{13} = \frac{1}{128} X_{13}^m \left[ \xi_1^3 \xi_2^3 \xi_3^3 \left( 1 + \xi_1^3 \xi_2^3 \right) \left( 1 + \xi_1^3 \xi_3^3 \right) \left( 1 + \xi_2^3 \xi_3^3 \right) \right]. \]
The second “reduction” is aimed at eliminating variability along coordinates \( \xi^1, \xi^2 \), deformations \( \varepsilon_{13}, \varepsilon_{23} \), respectively.

\[
E_{13} \equiv \frac{1}{128} X_n \left[ \frac{\varepsilon_1^2}{\gamma_1} \left( 1 + \frac{\varepsilon_1^1}{\gamma_1} \right) \left( 1 + \frac{\varepsilon_1^2}{\gamma_1} \right) \right].
\]

\[
E_{23} \equiv \frac{1}{128} X_n \left[ \frac{\varepsilon_2^2}{\gamma_2} \left( 1 + \frac{\varepsilon_2^1}{\gamma_2} \right) \left( 1 + \frac{\varepsilon_2^2}{\gamma_2} \right) \right].
\]

The third modification involves defining strains \( \varepsilon_{13}, \varepsilon_{23} \) as a linear approximation for \( \varepsilon_1^1, \varepsilon_2^2 \) at two points \( \xi = \pm 1 \) and \( \xi = \pm 1 \), respectively. To do this, consider the quadratic function \( g(\xi) \) of the form

\[
g(\xi) = \alpha_1 + \beta_1 \xi + \gamma_1 \xi^2,
\]

where \( \alpha_1, \beta_1, \gamma_1 \) are some coefficients. The linear approximation will have the form

\[
g(\xi) = \frac{g(1) + g(-1)}{2} \xi = \beta_1 + \beta_2 \xi.
\]

By analogy, we write

\[
E_{13} \equiv \frac{1}{128} X_n \left[ \frac{\varepsilon_1^2}{\gamma_1} \left( 1 + \frac{\varepsilon_1^1}{\gamma_1} \right) \left( 1 + \frac{\varepsilon_1^2}{\gamma_1} \right) \right].
\]

\[
E_{23} \equiv \frac{1}{128} X_n \left[ \frac{\varepsilon_2^2}{\gamma_2} \left( 1 + \frac{\varepsilon_2^1}{\gamma_2} \right) \left( 1 + \frac{\varepsilon_2^2}{\gamma_2} \right) \right].
\]

It should be noted that taking lesser degrees in expressions for strains than those presented is not recommended, as this will lead to a reduced rank of the stiffness matrix, or in other words, to additional false rigid displacements.

A certain drawback of the scheme described above is the violation of the variational principle, as a result of which the conditions for the compatibility of deformations for truncated deformations are not satisfied, which leads to nonmonotonic convergence. Nevertheless, the choice of such approximations is expedient and justified from the standpoint of MSFE (moment scheme of finite elements) [16-18].

Hooke’s generalized law is written as follows

\[
\sigma_{ij} = D_{ijmn} \varepsilon_{mn},
\]

We introduce the hypothesis of small compression stresses

\[
\sigma_{33} = 0.
\]

Using relation (1), we write expression (2) in the following form

\[
D_{33mn} \varepsilon_{mn} = 0.
\]

We represent (3) as an expression

\[
D_{3311} \varepsilon_{11} + D_{3312} \varepsilon_{12} + D_{3313} \varepsilon_{13} + D_{3321} \varepsilon_{21} + D_{3322} \varepsilon_{22} + D_{3323} \varepsilon_{23} + D_{3332} \varepsilon_{32} + D_{3333} \varepsilon_{33} = 0
\]

We express from (4) the compression strain

\[
\varepsilon_{33} = -\left( \frac{D_{3311} \varepsilon_{11} + D_{3312} \varepsilon_{12} + D_{3313} \varepsilon_{13} + D_{3321} \varepsilon_{21} + D_{3322} \varepsilon_{22} + D_{3323} \varepsilon_{23} + D_{3332} \varepsilon_{32} + D_{3333} \varepsilon_{33}}{D_{3333}} \right)
\]

Substituting (5) in (1), we get
\[ \sigma^*_{ij} = D_{g11} \varepsilon_{11} + D_{g12} \varepsilon_{12} + D_{g13} \varepsilon_{13} + D_{g21} \varepsilon_{21} + D_{g22} \varepsilon_{22} + \\
+ D_{g23} \varepsilon_{23} + D_{g31} \varepsilon_{31} + D_{g32} \varepsilon_{32} - D_{g33} \left( D_{3331} \varepsilon_{11} + \\
+ D_{3332} \varepsilon_{12} + D_{3333} \varepsilon_{13} + D_{3334} \varepsilon_{21} + D_{3335} \varepsilon_{22} + D_{3336} \varepsilon_{23} + \\
+ D_{3337} \varepsilon_{31} + D_{3338} \varepsilon_{32} \right) / D_{3333} \]

We write (6) in a simpler form

\[ \sigma^*_{ij} = D'_{ij} \varepsilon_{mn} = \left( D_{omm} - \frac{D_{333m} D_{333n}}{D_{3333}} \right) \varepsilon_{mn}, \]

when

\[
D'_{g11} = D_{g11} - \frac{D_{g33} D_{g311}}{D_{3333}}, \quad D'_{g12} = D_{g12} - \frac{D_{g33} D_{g312}}{D_{3333}}, \\
D'_{g13} = D_{g13} - \frac{D_{g33} D_{g313}}{D_{3333}}, \quad D'_{g21} = D_{g21} - \frac{D_{g33} D_{g321}}{D_{3333}}, \\
D'_{g22} = D_{g22} - \frac{D_{g33} D_{g322}}{D_{3333}}, \quad D'_{g23} = D_{g23} - \frac{D_{g33} D_{g323}}{D_{3333}}, \\
D'_{g31} = D_{g31} - \frac{D_{g33} D_{g311}}{D_{3333}}, \quad D'_{g32} = D_{g32} - \frac{D_{g33} D_{g312}}{D_{3333}}, \\
D'_{g33} = D_{g33}. 
\]

The hypothesis of small compressive stresses recorded in this form makes it possible to explicitly introduce physical constants for both isotropic and orthotropic materials into the design scheme.

2. Orthotropy and layering

Among the many methods used to calculate layered composite shells, a special place is occupied by numerical methods, in particular, FEM. The main advantage of such algorithms is the possibility of calculating thin-walled structures of complex geometry with variable mechanical characteristics under the influence of various loads.

The generalized Hooke law for an orthotropic body has the following form:

\[
\varepsilon_{ij} = \frac{1}{E_i} \left( \sigma_{ij} - \mu_{ij} \varepsilon_{22} - \mu_{ij} \varepsilon_{33} \right), \quad \gamma_{23} = 2\varepsilon_{23} = \frac{\sigma_{23}}{G_{23}}, \\
\varepsilon_{22} = \frac{1}{E_2} \left( \sigma_{22} - \mu_{22} \varepsilon_{33} - \mu_{22} \varepsilon_{11} \right), \quad \gamma_{12} = 2\varepsilon_{12} = \frac{\sigma_{12}}{G_{12}}, \\
\varepsilon_{33} = \frac{1}{E_3} \left( \sigma_{33} - \mu_{33} \varepsilon_{11} - \mu_{33} \varepsilon_{22} \right), \quad \gamma_{31} = 2\varepsilon_{31} = \frac{\sigma_{31}}{G_{31}}. 
\]

Then the components of the elasticity matrix for an orthotropic material can be written as follows:
When calculating the stiffness matrix, the Gauss-Legendre quadrature formula of the order \( 2 \times 2 \times N \) where \( N \) is the number of layers used, is applied. Thus, the coordinates of the quadrature points in the direction of the axis \( \zeta \) are determined in the middle of the thickness of each layer. Then the components of the stiffness matrix can be calculated as follows:

\[
D_{ij}^{rs} = g^m g^n D_{mn} E_{rm} E_{sn} \sqrt{g} 2\Delta_k,
\]

when \( r, c = 1, 2, 3, a, b, \ldots \), and \( p = 1, 2 \).

The stress components are calculated at four quadrature points defined in the middle of each layer and then interpolated into nodes. Since a single-layer approximation is used when calculating the stress-strain state of thin-walled structures, acceptable accuracy is already achieved with the number of layers \( N \geq 3 \).

3. Calculation of the stabilizer and elevator of a light-engine aircraft for strength

As an example, the calculation of the stress-strain state of the elevator and stabilizer of a light-engine aircraft is given. The stabilizer and elevator are a complex combined shell structure (Fig. 1), reinforced by root ribs and spars. When sampling the computational domain, both multilayer FE (for modelling the stabilizer and elevator sheathing) and a three-dimensional FE of a continuous medium (for modelling elements of root ribs and spars) are used.
Calculation of the stabilizer and elevator strength is carried out for balancing and maneuvering loads given in SNiPs, the application diagram of which is shown in Figure 2. Only symmetric cases of load application are considered.

The stabilizer mounting diagram for the fuselage is shown in Figure 3 (the elevator is attached to the stabilizer). Based on the symmetry of the design, only half of the stabilizer is shown. Fastening to the fuselage (for half of the stabilizer) is carried out at three points $A$, $B$ and $C$ (Fig. 3). A sliding hinge is realized at a point $A$, an ordinary hinge at a point $B$, and at point $C$ a stabilizer is attached to the keel with a brace.

To verify the program, we compared the solution of the problem with the results obtained in the ANSYS program based on the finite element SHELL181. Table 1 shows a comparison of the reactions $R_x$, $R_y$ and $R_z$ occurring at the points of attachment of the stabilizer to the fuselage at the points $A$ and $B$.

**Table 1.** Reactions (H) at the attachment points and transmitted to the fuselage from the stabilizer for three cases of loading.

| №  | $R_x$ | $R_y$ | $R_z$ |
|----|-------|-------|-------|
|    | progr | ansys | progr | ansys | progr | ansys |
| 1  | A     | -     | -5841 | -5062 | 63,2  | -9,2  |
|    | B     | -524  | -443  | 2164  | 1926  | 26,3  | 0     |
| 2  | A     | -     | -9124 | -8615 | -21   | -16   |
|    | B     | -349  | -337  | 11016 | 10497 | -13   | -3    |
Analysis of the calculation results shows that the difference in the results of the calculation of force factors during the static calculation of a complex structure between those obtained by the developed method and in the ANSYS program does not exceed 15%.

To illustrate, the results of calculation in Figure 4 shows the distribution of the deflection $W$ for half stabilizer constructions (in meters) for loading scheme 2.

**Figure 4.** The picture of the deflections of half the stabilizer design for the loading scheme 2.

Based on the results obtained, it should be noted that the developed numerical method for studying the stress-strain state of orthotropic multilayer shells of complex geometry yields results that are in good agreement with the data obtained in other application software packages, in particular, in the ANSYS program. Therefore, based thereon, similar designs can be calculated and reliable results can be obtained.

**Conclusion** In this paper, we consider a technology for constructing a transitional finite element. The essence of the methodology is to impose shell hypotheses (the hypothesis of small compressive stresses, truncation of transverse shear deformations, “simplified Hooke’s law”, reduced integration, etc.) on quadrature points, which are then interpolated into nodes at the junction of the element with the shell finite element. The nodes in which the transitional finite element is docked with massive three-dimensional elements are exempt from the imposition of shell hypotheses. Thus, smoothing of displacements within the finite element occurs, but does not guarantee smoothing of stresses within a given transitional FE, i.e., it is understood that such an operation should be applied to areas remote from the actual studied places of stress concentration. The criterion for assessing stresses in a given zone is the study of stress values while dividing this zone by various types of finite elements (three-dimensional, shell and transitional).

Thanks to this approach, it is possible to build a family of isotropic and orthotropic transition finite elements. Joining of single-layer and multi-layer finite elements is also possible.

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