Computing Dense and Sparse Subgraphs of Weakly Closed Graphs

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Abstract
A graph $G$ is weakly $\gamma$-closed if every induced subgraph of $G$ contains one vertex $v$ such that for each non-neighbor $u$ of $v$ it holds that $|N(u) \cap N(v)| < \gamma$. The weak closure $\gamma(G)$ of a graph, recently introduced by Fox et al. (SIAM J Comput 49(2):448–464, 2020), is the smallest number such that $G$ is weakly $\gamma$-closed. This graph parameter is never larger than the degeneracy (plus one) and can be significantly smaller. Extending the work of Fox et al. (2020) on clique enumeration, we show that several problems related to finding dense subgraphs, such as the enumeration of bicliques and $s$-plexes, are fixed-parameter tractable with respect to $\gamma(G)$. Moreover, we show that the problem of determining whether a weakly $\gamma$-closed graph $G$ has a subgraph on at least $k$ vertices that belongs to a graph class $\mathcal{G}$ which is closed under taking subgraphs admits a kernel with at most $\gamma k^2$ vertices. Finally, we provide fixed-parameter algorithms for INDEPENDENT DOMINATING SET and DOMINATING CLIQUE when parameterized by $\gamma + k$ where $k$ is the solution size. Furthermore, we

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show that \textsc{Independent Dominating Set} does not admit a polynomial kernel for constant $\gamma$ under standard assumptions.

\textbf{Keywords} Fixed-parameter tractability · $c$-Closure · Degeneracy · Clique relaxations · Bicliques · \textsc{Dominating Set}

\section{Introduction}

In the quest to design efficient algorithms for NP-hard graph problems, a very successful approach is to exploit the sparsity of input graphs: many problems that are assumed to be hard in general graphs turn out to be efficiently solvable in sparse graphs [1, 17, 25, 34, 36, 39, 45]. One popular sparseness measure that has been used for a variety of graph problems is the degeneracy of the input graph $G$, defined as follows. For a vertex $v \in V(G)$, let $\deg_G(v) := |N(v)|$ denote the degree of $v$.

\textbf{Definition 1} A graph $G$ is $d$-degenerate if one of the following two equivalent conditions holds:

- There exists a degeneracy ordering $\delta := (v_1, \ldots, v_n)$ of $G$, that is, an ordering such that $\deg_{G_i}(v_i) \leq d$ where $G_i := G[\{v_i, \ldots, v_n\}]$
- Every induced subgraph $G'$ of $G$ has a vertex $v$ with $\deg_{G'}(v) \leq d$.

The degeneracy of a graph $G$ is the smallest integer $d$ such that $G$ is $d$-degenerate.

Many graph algorithms which exploit the fact that the input graph has bounded degeneracy have been proposed. For example, there is an algorithm that enumerates all maximal cliques of a graph in $O(3^{d/3} \cdot dn)$ time and performs very efficiently on real-world input instances [17]. This algorithm is an FPT-algorithm for the parameter $d$ since the exponential part of the running time depends only on $d$. Further applications of degeneracy include FPT-algorithms for clique relaxations [34, 36] and for biclique enumeration algorithms [16, 27]. Degeneracy can also be used for problems that are \textit{W[1]}-hard for their standard parameterization by solution size. For example, \textsc{Dominating Set} and related problems are \textit{W[1]}-hard with respect to the solution size $k$ but have FPT-algorithms for $d + k$ [1, 24, 42].

In a recent work, Fox et al. [20] proposed exploiting a different property of real-world graphs that is motivated by the triadic closure principle. This principle postulates that people in a social network which have many common friends are likely to be friends themselves. Many real-world social networks give evidence for this postulate as they contain no pair of nonadjacent vertices with many common neighbors. The degree to which a given graph adheres to the triadic closure principle can be expressed in the closure number of $G$, defined as follows.

\textbf{Definition 2} [20] Let $\text{cl}_G(v) := \max_{v' \in V \setminus N[v]} |N(v) \cap N(v')|$ denote the closure number of a vertex $v$ in a graph $G$. A graph $G$ is $c$-closed if $\text{cl}_G(v) < c$ for all $v \in V(G)$. The closure number of a graph $G$ is the smallest integer $c$ such that $G$ is $c$-closed.

Fox et al. [20] showed that a $c$-closed graph has $O(3^{c/3} \cdot n^2)$ maximal cliques. Given that all maximal cliques can be enumerated in $O(\alpha \cdot n^2)$ time, where $\alpha$ is the number of
maximal cliques [8], this bound implies that all maximal cliques of a \( c \)-closed graph can be enumerated in \( O^*(3^{c/3}) \) time.\(^1\) This means that the clique enumeration problem has an FPT-algorithm with respect to the closure number of the input graph. In companion works, we showed that several hard graph problems such as INDEPENDENT SET, DOMINATING SET, INDUCED MATCHING and PARTIAL VERTEX COVER admit polynomial kernels on \( c \)-closed graphs when parameterized by the respective solution size [30, 32]. Recently, FPT-algorithms for further problems related to DOMINATING SET such as PERFECT CODE were obtained by Kanesh et al. [28]. Koana and Nichterlein [33] studied the time complexity of finding and enumerating small induced subgraphs in \( c \)-closed graphs.

Fox et al. [20] suggested a further graph parameter which combines sparseness and triadic closure, the weak closure of a graph.

**Definition 3** [20] A graph \( G \) is weakly \( \gamma \)-closed\(^2\) if one of the following holds:

- There exists a weak closure ordering \( \sigma := (v_1, \ldots, v_n) \) of \( G \), that is, an ordering such that \( \cl_{G_i}(v_i) < \gamma \) for all \( i \in [n] \) where \( G_i := G([v_i, \ldots, v_n]) \).
- Every induced subgraph \( G' \) of \( G \) has a vertex \( v \in V(G') \) such that \( \cl_{G'}(v) < \gamma \).

The weak closure number of a graph \( G \) is the smallest integer \( \gamma \) such that \( G \) is weakly \( \gamma \)-closed.

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\(^1\) The \( O^* \) notation hides polynomial factors in the input size.

\(^2\) To avoid confusion with the closure number \( c \), we denote the weak closure by \( \gamma \) instead of \( c \).

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### Table 1  
A comparison of the number \( n \) of vertices, number \( m \) of edges, the maximum degree \( \Delta \), the closure \( c \), the degeneracy \( d \) and the weak closure \( \gamma \) in social and biological networks

| Instance name          | \( n \) | \( m \) | \( \Delta \) | \( c \) | \( d \) | \( \gamma \) |
|------------------------|--------|--------|------------|------|------|-----------|
| adjnoun-adjacency      | 112    | 425    | 49         | 14   | 6    | 6         |
| arenas-jazz            | 198    | 2742   | 100        | 42   | 29   | 18        |
| ca-netscience          | 379    | 914    | 34         | 5    | 8    | 3         |
| bio-celegans           | 453    | 2025   | 237        | 26   | 10   | 9         |
| bio-diseasome          | 516    | 1188   | 50         | 9    | 10   | 5         |
| soc-wiki-Vote          | 889    | 2914   | 102        | 18   | 9    | 8         |
| arenas-email           | 1133   | 5451   | 71         | 19   | 11   | 8         |
| bio-yeast              | 1458   | 1948   | 56         | 8    | 5    | 4         |
| ca-CSphd               | 1882   | 1740   | 46         | 3    | 2    | 3         |
| soc-hamsterster        | 2426   | 16,630 | 273        | 77   | 24   | 19        |
| ca-GrQc                | 4158   | 13422  | 81         | 43   | 43   | 9         |
| soc-advogato           | 5167   | 39,432 | 807        | 218  | 25   | 21        |
| bio-dmela              | 7393   | 25,569 | 190        | 72   | 11   | 12        |
| ca-HepPh               | 11,204 | 117,619| 491        | 90   | 238  | 54        |
| ca-AstroPh             | 17,903 | 196,972| 504        | 61   | 56   | 30        |
| soc-brightkite         | 56,739 | 212,945| 1134       | 184  | 52   | 49        |
The weak closure number $\gamma$ of a graph $G$ is never larger than $d + 1$ where $d$ is the degeneracy of $G$ and also never larger than the closure number $c$ of $G$. Consequently, fixed-parameter algorithms for $\gamma$ are, in principle, preferable to those for the closure number $c$ or the degeneracy $d$. From an application point of view, the weak closure number is also an excellent parameter in such graphs since it tends to take on very small values in real-world social networks [20] (see also Table 1). Fox et al. [20] showed that a graph has $O(3^{\gamma/3} \cdot n^2)$ maximal cliques which, again using known clique enumeration algorithms, gives an algorithm that enumerates all maximal cliques in $O^*(3^{\gamma/3})$ time.

Very recently, it was shown that Dominating Set is FPT with respect to $\gamma + k$ [38] and that several problems like CONNECTED VERTEX COVER and INDUCED MATCHING admit kernels of size $k^{O(\gamma)}$ [31].

**Our Results**

In a nutshell, we show that when the graph has a small weak closure number, it helps in solving a variety of graph problems that are related to searching for sparse or dense subgraphs or for sparse or dense dominating sets. Our main results for clique relaxations are listed in Table 2; our main results for variants of DOMINATING SET are listed in Table 3.

Our results improve over the state of the art in the following sense: the best known tractability results for these problems employ the degeneracy of the input graph as a parameter and, as discussed above, the weak closure is essentially a smaller parameter. For some problems, we also provide results for the $c$-closure parameter. There are two reasons for this. First, for some problems we obtain better running time bounds for the parameter $c$. Second, we provide some lower bounds for the problems under consideration and, whenever possible, we provide them for the larger closure parameter $c$.

From a practical point of view, the most important results are, in our opinion, the enumeration algorithms for maximal non-induced bicliques and maximal $s$-plexes whose running times grow moderately with $\gamma$. Both algorithms are based on the algorithm to enumerate all maximal cliques in weakly $\gamma$-closed graphs [20]. Independently, Behera et al. [2] obtained similar results for the enumeration of maximal $s$-plexes and further dense subgraphs parameterized by the $c$-closure; it seems that their algorithms for $s$-plex enumeration can be adapted to parameterization by weak closure as well [2].

**Preliminaries**

For $n \in \mathbb{N}$, we denote by $[n]$ the set $\{1, \ldots, n\}$. For a graph $G$, we denote by $V(G)$ and $E(G)$ its vertex set and edge set, respectively. We let $n := |V(G)|$ denote the number of vertices. Let $X \subseteq V(G)$ be a vertex set. We let $G[X]$ denote the subgraph induced by $X$ and $G - X := G[V(G) \setminus X]$ the graph obtained by removing the vertices of $X$. We denote by $N_G(X) := \{y \in V(G) \setminus X \mid xy \in E(G), x \in X\}$ and $N_G[X] := N_G(X) \cup X$, the open and closed neighborhood of $X$, respectively. For all these notations, when $X$ is a singleton $\{x\}$ we may write $x$ instead of $\{x\}$. The maximum degree of $G$ is $\Delta := \max_{v \in V(G)} \deg_G(v)$. The $h_G$-index of a graph $G$ is the largest integer $h$ such that $G$ has at least $h$ vertices of degree at least $h$ [18]. We may drop the subscript $\cdot_G$ when it is clear from context.

Instances $(I, k)$ of a parameterized problem consist of a classical input instance $I$ and a parameter $k \in \mathbb{N}$. A parameterized problem is fixed-parameter tractable if every
Table 2  An overview of our results for clique relaxations. Our algorithms for s-PLEX and NON-INDUCED (k₁, k₂)-BICLIQUE and our $O(2^k n^{s+3})$-time algorithm for s-DEFECTIVE CLIQUE are based on algorithms enumerating all maximal s-plexes (Theorem 2), non-induced bicliques (Theorem 8), and s-defective cliques (Theorem 4), respectively.

| Problem                        | Result                                                | Reference |
|--------------------------------|-------------------------------------------------------|-----------|
| INDDEPENDENT SET               | $O(\gamma k^2)$-vertex kernel                        | Corollary 1 |
| s-PLEX                         | W[1]-hard for $k$ even if $c = 2$                     | Theorem 3  |
|                               | $O(2^k n^{2s+1})$-time algorithm for $s \geq 2$      | Corollary 4 |
| s-Defective Clique             | W[1]-hard for $k$ even if $c = 2$                    | [43]      |
|                               | $O(2^k n^{s+3})$-time algorithm                      | Corollary 5 |
|                               | $2^{O(\sqrt{s} + s \log n)} \cdot n^{O(\sqrt{s})}$-time algorithm | Theorem 6          |
| 2-CLUB                         | NP-hard for $c = 4$                                  | Theorem 7  |
| NON-INDUCED (k₁, k₂)-BICLIQUE  | $O^*(2^\sqrt{s})$-time algorithm                     | Theorem 9  |
| INDUCED (k, k)-BICLIQUE        | $O^*(\gamma n^{O(\gamma)})$-time algorithm          | Theorem 10 |
| INDUCED (k₁, k₂)-BICLIQUE      | $O^*(1.6107\gamma)$-time algorithm if $k_1 \geq 2$  | Theorem 11 |
|                               | NP-hard if $k_1 = 1$ for $c = 3$ and $\gamma = 2$   | Theorem 12 |
|                               | $P$ for $c = 2$                                      | Corollary 8 |
|                               | $P$ for $k_1 = 1$ and $\gamma = 1$                  | Theorem 13 |
|                               | $P$ for $k_1 \geq 2$ and $\gamma \leq k_1 + 1$     | Theorem 13 |
|                               | NP-hard for $k_1 \geq 2$ and $\gamma \geq k_1 + 2$ | Theorem 13 |

Table 3  An overview of our results for variants of DOMINATING SET

| Problem                        | Result                                                | Reference |
|--------------------------------|-------------------------------------------------------|-----------|
| INDEPENDENT DOMINATING SET     | $O^*((\gamma - 1)/2^{k-2})$-time algorithm            | Theorem 14 |
|                               | No $k^{O(1)}$ kernel for $\gamma = 2$                | Theorem 15 |
|                               | No $(k + c)^{O(1)}$ kernel                            | Theorem 15 |
| DOMINATING CLIQUE              | $O^*((\gamma - 1)^{k-1})$-time algorithm             | Theorem 16 |
|                               | NP-hard for $c = 3$                                  | Proposition 5 |
|                               | No $O(k^{c-1-\epsilon})$ kernel                      | Proposition 3 |

instance $(I, k)$ can be solved in $f(k) \cdot |I|^{O(1)}$ time for some computable function $f$. An algorithm with such a running time is an FPT-algorithm. A basic class of parameterized intractability is W[1]: it is widely assumed that W[1]-hard problems do not admit an FPT-algorithm. W[1]-hardness can be shown via a parameterized reduction from a W[1]-hard problem. A parameterized reduction from a parameterized problem $L$ to a parameterized problem $L'$ is an algorithm that maps each instance $(I, k)$ of $L$ in $f(k) \cdot |I|^{O(1)}$ time to an equivalent instance $(I', k')$ of $L'$ such that $k' \leq g(k)$ for some computable function $g$.

A kernelization is a polynomial-time algorithm which transforms every instance $(I, k)$ into an equivalent instance $(I', k')$ such that $|I'| + k' \leq g(k)$ for some computable function $g$. If $g$ is a polynomial function, then we speak of a polynomial kernel. A
problem is fixed-parameter tractable if and only if it admits a kernelization. There are, however, many problems which are fixed-parameter tractable but do not admit a polynomial kernel under standard complexity-theoretic assumptions.

For more details on parameterized complexity, we refer to the standard monographs [12, 15].

2 Sparse Subgraphs

In this section we study problems that are related to finding sparse subgraphs of a given graph. The most fundamental problem in this context is the INDEPENDENT SET problem, where one aims to find a large edgeless subgraph or, in other words, a large set of vertices without edges between them.

**INDEPENDENT SET**

**Input** A graph $G$ and $k \in \mathbb{N}$.

**Question** Is there a vertex set $S \subseteq V(G)$ such that $|S| \geq k$ and the vertices in $S$ are pairwise nonadjacent?

Since INDEPENDENT SET is NP-hard already on graphs with maximum degree 3 [22], there is no hope for FPT-algorithms for parameterization by the closure number $c$ or the weak closure number $\gamma$. Parameterization by $c + k$, however, leads to fixed-parameter tractability: in a companion work [32], we provided an $O(ck^2)$-vertex kernel for INDEPENDENT SET. Here, we strengthen this result by showing that a generalization of INDEPENDENT SET admits a polynomial kernel with respect to the parameter $k + \gamma$.

The problem that we consider is defined as follows. Let $G$ be a graph class. We say that $G$ is **monotone** if $G$ is closed under vertex and edge deletions. That is, if $G \in G$, then for each (not necessarily induced) subgraph $H$ of $G$ we have $H \in G$. The aim is now to find a large induced subgraph belonging to $G$.

**$G$-SUBGRAPH**

**Input** A graph $G$ and $k \in \mathbb{N}$.

**Question** Is there a vertex set $S \subseteq V(G)$ with $|S| \geq k$ such that $G[S] \in G$?

When $G$ is the class of edgeless graphs, then $G$-SUBGRAPH is the same as INDEPENDENT SET.

The kernelization algorithm consists of one reduction rule that works on a weak closure ordering $(v_1, v_2, \ldots, v_n)$ of the input graph $G$. The correctness of the reduction rule hinges on the following observation about the size of common neighborhoods of nonadjacent vertices $v_i$ and $v_j$ when we consider only the vertices with higher index than $v_i$. To state this observation and the data reduction itself, let $G_i := G[V_i]$ for $V_i = \{v_i, v_{i+1}, \ldots, v_n\}$.

**Lemma 1** Let $j \in [n]\{i\}$. If $v_i v_j \notin E(G)$, then $|N_{G_i}(v_i) \cap N_{G_j}(v_j)| < \gamma$.

**Proof** First, assume that $j < i$. Then, we have $|N_{G_j}(v_i) \cap N_{G_j}(v_j)| < \gamma$ by the definition of weak closure orderings. Since $V_i \subseteq V_j$ this implies that $|N_{G_i}(v_i) \cap
$N_G(v_j) < \gamma$. Second, assume that $j > i$. By the definition of weak closure orderings we have $|N_{G_i}(v_i) \cap N_{G_i}(v_j)| < \gamma$. \hfill $\square$

Lemma 1 allows us to show the correctness of the following reduction rule, which removes vertices with many neighbors that have higher index in the weak closure ordering.

**Reduction Rule 1** If $\text{deg}_{G_i}(v_i) \geq \gamma k$, then remove $v_i$.

**Lemma 2** Reduction Rule 1 is correct for monotone graph classes.

**Proof** Let $G':= G - v_i$ for $v_i \in V$ with $\text{deg}_{G_i}(v_i) \geq \gamma k$ be the graph obtained by applying Reduction Rule 1. Clearly, if $G'[S] \in \mathcal{G}$ for some vertex set $S \subseteq V(G')$, then also $G[S] \in \mathcal{G}$.

Hence, it remains to show that if there is a vertex set $S \subseteq V(G)$ of size $k$ such that $G[S] \in \mathcal{G}$, then there is a vertex set $S' \subseteq V(G')$ of size $k$ such that $G'[S'] \in \mathcal{G}$. If $v_i \notin S$, we observe that $G'[S] \in \mathcal{G}$. Thus, in the following we assume that $v_i \in S$. Let $S_i := S \setminus N_G(v_i)$ be the set of vertices in $S$ that are not adjacent to $v_i$. We show that there is some vertex $u \notin S$ that is not adjacent to any vertex of $S_i$. By Lemma 1, any vertex $v_j \in S_i$ has less than $\gamma$ neighbors in $N_{G_i}(v_i)$. Since $\text{deg}_{G_i}(v_i) \geq \gamma k$ and $|S_i| < k$, there exists at least one vertex $u$ in $N_{G_i}(v_i)$ that is not adjacent to any vertex from $S_i$. Consequently, $N_{G_i}(u) \cap S \subseteq N_{G_i}(v_i) \cap S$. Since $\mathcal{G}$ is monotone, we may thus replace $v_i$ in $S$ with $u$: for $S' := (S \setminus \{v_i\}) \cup \{u\}$ we have $G'[S'] \in \mathcal{G}$. \hfill $\square$

**Theorem 1** Let $\mathcal{G}$ be a monotone graph class. Then, $\mathcal{G}$-SUBGRAPH has a kernel with at most $\gamma k^2$ vertices.

**Proof** One can exhaustively apply Reduction Rule 1 in polynomial time. The resulting graph has a weak closure ordering where every vertex $v_i$ has less than $\gamma k$ neighbors in $G_i$. Hence, this graph has degeneracy $d < \gamma k$. Note that any graph $G$ on at least $(d + 1)k$ vertices contains an independent set $S$ of size $k$. Due to the monotonicity of $\mathcal{G}$, $G[S] \in \mathcal{G}$ for an independent set $S$. Thus, returning Yes is correct whenever $|V(G)| \geq \gamma k^2$ and we obtain an equivalent instance with at most $\gamma k^2$ vertices. \hfill $\square$

Since the class of edgeless graphs is monotone, we obtain the following.

**Corollary 1** INDEPENDENT SET has a kernel with at most $\gamma k^2$ vertices.

Theorem 1 also implies kernels for many other problems, including A CYCLIC SUBGRAPH, BIPARTITE SUBGRAPH, PLANAR SUBGRAPH, and BOUNDED DEGREE SUBGRAPH. These problems ask whether the input graph $G$ contains a vertex set $S \subseteq V(G)$ such that $|S| \geq k$ and $G[S]$ is acyclic, bipartite, planar, or has bounded maximum degree, respectively. All of these problems are W[1]-hard in general graphs [29].

**Corollary 2** Each of ACYCLIC SUBGRAPH, BIPARTITE SUBGRAPH, BOUNDED DEGREE SUBGRAPH, and PLANAR SUBGRAPH has a kernel with $\gamma k^2$ vertices.

Moreover, it follows from Theorem 1 that SPARSEST-$k$-SUBGRAPH, the problem of finding an induced subgraph with exactly $k$ vertices and at most $t$ edges, also admits a polynomial kernel in weakly $\gamma$-closed graphs.
Corollary 3  Sparsest-$k$-Subgraph has a kernel with at most $\gamma k^2$ vertices.

This is in sharp contrast to Densest-$k$-Subgraph, where one asks for a set $S \subseteq V(G)$ of exactly $k$ vertices such that $G[S]$ has at least $t$ edges: Densest-$k$-Subgraph is W[1]-hard with respect to $k$ even in 2-closed graphs [43].

3 Clique Relaxations

In this section, we present algorithms for generalizations of the Clique problem. In contrast to the variants of Independent Set considered in Sect. 2, here we only consider parameterization by the weak closure number $\gamma$. Recall, that Fox et al. [20] showed that a graph has $O(3^{\sqrt[3]{\gamma}} \cdot n^2)$ many maximal cliques. Using known clique enumeration algorithms this gives an algorithm that enumerates all maximal cliques in $O^*(3^{\sqrt[3]{\gamma}})$ time.

3.1 $s$-Plex

A clique is a vertex set $S$ such that each vertex $v \in S$ is adjacent to each other vertex in $S$. One way to relax the clique definition is to allow each vertex $v \in S$ to have at most $s$ non-neighbors in $S$. This idea can be formalized as follows.

Definition 4  In a graph $G = (V, E)$ a set $S \subseteq V$ is an $s$-plex if every vertex in $G[S]$ has degree at least $|S| - s$ in $G[S]$.

Observe that cliques are exactly the 1-plexes. Here, we study the task of enumerating maximal $s$-plexes which has received some interest in practice [9, 10], and the problem of finding a sufficiently large $s$-plex, defined as follows.

$s$-PLEX

Input  A graph $G$ and $k \in \mathbb{N}$.

Question  Does $G$ contain an $s$-plex $S$ of size at least $k$?

On the negative side, $s$-PLEX is W[1]-hard when parameterized by $k$ for all $s \in \mathbb{N}$ [29, 35]. On the positive side, a simple algorithm can enumerate all maximal $s$-plexes of a $d$-degenerate graph in $2^d n^s + O(1)$ time [34].

For the problem of enumerating all maximal $s$-plexes, we obtain an FPT-algorithm for the weak closure number.

Theorem 2  For $s \geq 2$, a graph $G$ has $O(2^{s} n^{2s-1})$ maximal $s$-plexes. Moreover, all maximal $s$-plexes of $G$ can be enumerated in $O(2^{s} n^{2s+1})$ time.

Proof  First, we show the bound on the number of maximal $s$-plexes in a weakly $\gamma$-closed graph. Let $v \in V(G)$ be a vertex such that $cl_G(v) < \gamma$ and let $G' := G - v$ be the graph obtained by deleting $v$. Let $S$ and $S'$ be the collections of all maximal $s$-plexes (without duplicates) in $G$ and $G'$, respectively. We show that $|S| \leq |S'| + 2^{s} n^{2s-2}$ and that $S$ can be constructed from $S'$ in $O(|S'| \cdot n + 2^{s} n^{2s})$ time. To obtain the bound we identify the following four types of maximal $s$-plexes in $G$:
Type 1: $S$ does not contain $v$. Then, $S$ is also maximal in $G'$.
Type 2: $S$ contains $v$ and $S \setminus \{v\}$ is maximal in $G'$.
Type 3: $S$ contains $v$, $S \setminus \{v\}$ is not maximal in $G'$, and $S$ contains a non-neighbor of $v$ (that is, $S \setminus N_G(v) \neq \emptyset$).
Type 4: $S$ contains $v$, $S \setminus \{v\}$ is not maximal in $G'$, and $S$ is contained in the neighborhood of $v$, that is, $S \subseteq N_G(v)$.

Clearly, each maximal $s$-plex is of one of these four types. It is easy to see that there are $|S'|$ maximal $s$-plexes of Type 1 and Type 2. Hence, it remains to bound the number of maximal $s$-plexes of Type 3 and Type 4.

Next, we bound the number of maximal $s$-plexes of Type 3. Consider such an $s$-plex $S$. We may partition $S$ into three parts as follows: We first divide $S$ into $S_v := S \cap N_G(v)$ and $\tilde{S}_v := S \setminus N_G(v)$. We divide $S_v$ further into $S_{uv} := S_v \cap N_G(u)$ and $\tilde{S}_{uv} := S_v \setminus N_G(u)$ for some vertex $u \in \tilde{S}_v$. Here, $u$ is any non-neighbor of $v$ to exploit the weak $\gamma$-closure. By the definition of $s$-plexes, $|\tilde{S}_v| < s$ and $|\tilde{S}_{uv}| < s$. Hence, there are at most $n^{2s-2}$ choices for $\tilde{S}_v$ and $\tilde{S}_{uv}$. For $S_{uv}$, there are at most $2^{s-1}$ choices because $S_{uv} \subseteq N_G(v) \cap N_G(u)$ and $|N_G(v) \cap N_G(u)| < cl_G(v) < \gamma$. Overall, there are at most $2^{s-1} n^{2s-2}$ maximal $s$-plexes of Type 3.

It remains to bound the number of maximal $s$-plexes of Type 4. Let $S$ be one of these $s$-plexes. Since $S' := S \setminus \{v\}$ is not maximal in $G'$, there exists a vertex $u \in V(G) \setminus S$ such that $S' \cup \{u\}$ is an $s$-plex in $G'$. If $u \in N_G(v)$, then $S \cup \{u\}$ is also an $s$-plex in $G$, which contradicts the fact that $S$ is maximal in $G$. Hence, we can assume that $u \notin N_G(v)$. Then, $S \setminus N_G(u)$ contains at most $s - 1$ vertices, which in turn implies that there are at most $n^{s-1}$ choices for $S \setminus N_G(u)$. Since $S \subseteq N(v)$ we observe that $S \cap N_G(u) \subseteq N_G(v) \cap N_G(u)$ and $|N_G(v) \cap N_G(u)| \leq cl_G(v) < \gamma$. Thus, we have $2^{s-1}$ choices for $S \cap N_G(u)$. All in all, there are at most $2^{s-1} n^{s}$ maximal $s$-plexes of Type 4.

By the above analysis, we obtain $|S| \leq |S'| + 2^{s-1} n^{2s-2} + 2^{s-1} n^s \leq |S'| + 2^{s-1} n^{2s-2}$. Next, we bound the overall number of maximal $s$-plexes in a graph with $n$ vertices. To this end, let $a_n$ be the number of maximal $s$-plexes in weakly $\gamma$-closed graphs on $n$ vertices. Clearly, $a_1 = 1$. Furthermore, the above analysis showed that $a_n - a_{n-1} = |S| - |S'| \leq 2^{s-1} n^{2s-2}$. Hence, by induction we obtain $a_n = a_1 + \sum_{i=2}^{n} (a_i - a_{i-1}) \leq 2^{s-1} n^{2s-1} + 1$. In other words, a weakly $\gamma$-closed graph on $n$ vertices has at most $2^{s-1} n^{2s-1} + 1$ maximal $s$-plexes.

Second, we bound the overall time needed to enumerate all maximal $s$-plexes. To obtain this bound, we again let $v \in V(G)$ be any vertex such that $cl_G(v) < \gamma$, let $G' := G - v$ be the graph obtained by deleting $v$, and let $S$ and $S'$ be the collections of all (without duplicates) maximal $s$-plexes in $G$ and $G'$. Observe that all maximal $s$-plexes of Type 1 and 2 can be found in $O(|S'| \cdot n)$ time. Furthermore, maximal $s$-plexes of Type 3 and 4 can be enumerated in $O((2^{s-1} n^{2s-2} + 2^{s-1} n^s) \cdot n^2)$ time, because it takes $O(n^2)$ time to verify whether a vertex set is a maximal $s$-plex or not. Finally, we remove duplicates in $O((|S'| + 2^{s-1} n^{2s-2} + 2^{s-1} n^s) \cdot n) = O(|S'| \cdot n + 2^{s-1} n^{2s-1})$ time, using radix sort. Altogether, the algorithm needs $O(|S'| \cdot n + 2^{s-1} n^{2s})$ time to enumerate all maximal $s$-plexes in $G$. Recall that $a_n$ is the number of maximal $s$-plexes in a weakly $\gamma$-closed graph on $n$ vertices. Thus, all maximal $s$-plexes of a weakly $\gamma$-closed
A factor of $n^{2s-2}$ for the number of maximal s-plexes in Theorem 2 is unavoidable: Consider a graph $G$ consisting of two cliques $C_1$ and $C_2$ of equal size. Clearly, $G$ is 1-closed. Each subset of $C_1$ of size exactly $s-1$ and each subset of $C_2$ of size exactly $s-1$ together form a maximal $s$-plex. Hence, there exist 1-closed graphs with $\Omega((n/2)^{2s-2})$ maximal $s$-plexes.

For $s$-Plex, Theorem 2 directly implies the following.

**Corollary 4** For $s \geq 2$, $s$-Plex can be solved in $O(2^{2s} n^{2s+1})$ time.

Next, we show that there is presumably no $f(k) \cdot n^{O(1)}$-time algorithm for $s$-Plex in 2-closed graphs. Moreover, our reduction also shows that $s$-Plex is W[1]-hard for the parameter $k+s+d$.

**Theorem 3** $s$-Plex is W[1]-hard in 2-closed graphs when parameterized by $k+s+d$.

**Proof** We reduce from Clique. An illustration of our construction is shown in Fig. 1. Let $(G, k)$ be an instance of Clique with $k \geq 4$. First, we subdivide each edge $uv$ of $G$ twice. That is, we remove the edge $uv$ and add edges $ux_u^v$, $x_u^v x_v^u$, and $x_v^u$, where $x_u^v$ and $x_v^u$ are two new vertices. Second, for each edge $uv \in E(G)$, we introduce $k-3$ vertices $x_{1uv}, \ldots, x_{k-3uv}$. Let $X_{uv} := \{x_{1uv}, x_{uv}, x_{1uv}, \ldots, x_{k-3uv}\}$ and let $X := \bigcup_{uv \in E(G)} X_{uv}$. We then add edges so that $X_{uv}$ forms a clique. Lastly, we introduce a set $T := \{t_1, \ldots, t_{k-3}\}$ of $k-3$ vertices and add edges between $x_{1uv}^i$ and $t_i$ for each $uv \in E(G)$ and each $i \in [k-3]$. Let $G'$ be the resulting graph.

It is easy to verify that $G'$ is 2-closed. Moreover, $G'$ is $(k-1)$-degenerate: Each vertex $x \in X_{uv}$ is of degree $k-1$ and there is no edge in $G' - X$. We show that $G$ has a

![Fig. 1 Illustration of the construction of Theorem 3.](image-url)

- The graph $G$ of the Clique instance and
- the graph $G'$ of $s$-Plex. Here, the sets $X_{uv}$ and $X_{uv}$ are not drawn. Note that the sets $X_{uv}$ and $X_{uv}$ are cliques.
clique of size $k$ if and only if $G'$ has an $s$-plex of size $k'$, where $k' := 2k - 3 + (k - 1){k \choose 2}$ and $s := k' - (k - 1)$.

Suppose that $G$ has a clique $S$ of size exactly $k$. Let $S' = S \cup T \cup \bigcup_{u, v \in S} X_{uv}$. Observe that $|S'| = k'$. We verify that each vertex in $G'[S']$ has degree at least $k' - s = k - 1$.

- Let $v \in S$. By construction, we have $X_v^u \in N_{G'}(v)$ for each $u \in S \setminus \{v\}$. Since $X_v^u$ is contained in $S'$, $v$ has at least $k - 1$ neighbors in $G'[S']$.
- We have $\deg_{G'[S']}(t^i) \geq {k \choose 2} \geq k - 1$ for each $i \in [k - 3]$, because $t^i$ is adjacent to $x_{uv}^i$ for all $uv \in E(G[S])$.
- Consider $x_{uv}^i$ for $uv \in E(G[S])$. We have $u \in N_{G'}(x_{uv}^i)$ by construction. Moreover, $x_{uv}^i$ is adjacent to all $k - 2$ vertices in $X_{uv} \setminus \{x_{uv}^i\}$. Thus, we have $\deg_{G'[S']}(x_{uv}^i) \geq k - 1$.
- Consider $x_{uv}^i$ for $uv \in E(G[S])$ and $i \in [k - 3]$. We have $t^i \in N_{G'}(x_{uv}^i)$ by construction. Moreover, $x_{uv}^i$ is adjacent to all $k - 2$ vertices in $X_{uv} \setminus \{x_{uv}^i\}$. Thus, we have $\deg_{G'[S']}(x_{uv}^i) \geq k - 1$.

Thus, every vertex has at least $k - 1 = k' - s$ neighbors in $G'[S']$.

Conversely, suppose that $S'$ is an $s$-plex of size exactly $k'$. We start with the following claim.

**Claim 1** If $S'$ contains a vertex $x$ of $X_{uv}$ for some $uv \in E(G)$, then $S'$ also contains all vertices in $N_{G'}(X_{uv})$, that is, $\{u, v\} \cup X_{uv} \cup T \subseteq S'$.

**Proof of Claim** By construction, $\deg_{G'}(x) = k - 1$. Since each vertex in $G'[S']$ has degree $|S'| - s \geq k - 1$ by the definition of $s$-plexes, we have $N_{G'}(X_{uv}) \subseteq S'$. ■

Let $\ell = |S' \cap V(G)|$. We conclude that there are at most ${k \choose 2}$ edges $uv \in E(G)$ with $X_{uv} \cap S' = \emptyset$ since otherwise the above claim would imply that $|S' \cap V(G)| > \ell$. By construction, we have $|X_{uv}| = k - 1$ for each $uv \in E(G)$. Thus, we have

$$|S'| = |S' \cap V(G)| + |T| + |S' \cap X| \leq \ell + k - 3 + (k - 1){k \choose 2}.$$ 

Since $|S'| = k' = 2k - 3 + {k \choose 2}$, we obtain $\ell \geq k$.

By definition, each vertex $v \in S' \cap V(G)$ has at least $|S'| - s \geq k - 1$ neighbors in $G'[S']$. So there are at least $\ell(k - 1)/2$ edges $uv \in E(G)$ such that $S' \cap X_{uv} = \emptyset$. From the above claim we know that $X_{uv} \subseteq S'$ for each $X_{uv}$ with $X_{uv} \cap S' = \emptyset$. Hence, we obtain that

$$|S'| \geq |S' \cap V(G)| + |T| + |S' \cap X| \geq \ell + k - 3 + (k - 1) \cdot \ell(k - 1)/2.$$ 

Since $|S'| = k' = 2k - 3 + (k - 1){k \choose 2}$, we obtain $\ell = k$ and $|S' \cap X| = (k - 1){k \choose 2}$. Since $|S' \cap X| = (k - 1){k \choose 2}$ we conclude that each two vertices in $S' \cap V(G)$ are adjacent. Thus, $S' \cap V(G)$ is a clique of $k$ vertices in $G$ by construction. □
3.2 s-Defective Clique

A clique is a vertex set $S$ such that no vertex pair in $S$ is nonadjacent. One way to relax the clique definition is to allow up to $s$ nonadjacent vertex pairs. This idea can be formalized as follows.

**Definition 5** In a graph $G = (V, E)$ a set $S \subseteq V$ is an $s$-defective clique if $G[S]$ has at least \(|S|/2 - s\) edges.

Note that cliques are exactly the 0-defective cliques. Similar to $s$-plexes, we consider the problems of enumerating all maximal $s$-defective cliques and finding a sufficiently large $s$-defective clique in (weakly) closed graphs. The latter problem can be formalized as follows.

$s$-Defective Clique

**Input** A graph $G$ and $k \in \mathbb{N}$.

**Question** Does $G$ contain an $s$-defective clique $S$ of size at least $k$?

One can show that $s$-DEFECTIVE CLIQUE is W[1]-hard with respect to $k$ even if $c = 2$ by adapting a previous hardness proof for DENSEST-k-SUBGRAPH on 2-degenerate graphs [43, Theorem 20].

First, we study the problem of enumerating all maximal $s$-defective cliques. To obtain an FPT-algorithm for this problem for the weak closure number, we adapt the algorithm of Theorem 2. The only difference to the proof of Theorem 2 is the following: For bounding the number of $s$-plexes of Type 3 the sets $\tilde{S}_v$ and $\tilde{S}_{uv}$ were bounded by $s - 1$ each. Since a maximal $s$-defective clique contains at most $s$ non-edges and $uv \notin E(G)$ we observe that $|\tilde{S}_v \cup \tilde{S}_{uv}| < s$. Hence, there are at most $2^{\gamma - 1}n^{s - 1}$ maximal $s$-defective cliques of Type 3. Thus, we can bound the overall number of maximal $s$-defective cliques by $2^{\gamma}n^{s+1} + 1$. Since the rest of the proof is completely analogous, we omit it.

**Theorem 4** For $s \geq 2$, there are $O(2^\gamma n^{s+1})$ maximal $s$-defective cliques in weakly $\gamma$-closed graphs and they can be enumerated in $O(2^\gamma n^{s+3})$ time.

A factor of $n^{s+1}$ in the number of maximal $s$-defective cliques in Theorem 4 is inevitable due to the following lower bound: Again we consider the graph $G$ consisting of two disjoint cliques $C_1$ and $C_2$, each of size $n/2$. For each clique $C \subseteq C_1$ of size $s$ and each $v \in C_2$, the vertex set $C \cup \{v\}$ is a maximal $s$-defective clique. Thus, $G$ has $\Omega((n/2)^{s+1})$ maximal $s$-defective cliques.

Second, we study $s$-DEFECTIVE CLIQUE, the decision problem of finding a sufficiently large $s$-defective clique. Theorem 4 directly implies the following.

**Corollary 5** $s$-DEFECTIVE CLIQUE can be solved in $O(2^\gamma n^{s+3})$ time.

Next, we present faster algorithms in terms of the dependence on $s$. First, we show that each $s$-defective clique can be covered by $O(\sqrt{s})$ maximal cliques.

**Lemma 3** Let $S$ be an $s$-defective clique for $s \geq 1$. Then, there is a collection $C$ of at most $O(\sqrt{s})$ cliques such that $S \subseteq \bigcup_{C \in C} C$. 

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**Proof** Let $H$ denote the complement graph of $G[S]$. By definition, $H$ has at most $s$ edges. Since a clique becomes an independent set in the complement graph, it suffices to show that there is an $O(\sqrt{s})$-coloring of $H$ (that is, $\chi(H) = O(\sqrt{s})$). Although this is known folklore, we describe its proof for the sake of completeness. Consider an optimal coloring. Then, for each pair of colors, say red and blue, there is at least one edge with one endpoint red and the other blue (otherwise we find a coloring with fewer colors). Recall that $H$ is the complement graph of $G[S]$. Hence, $H$ has at most $s$ edges, we obtain $s \geq (\chi(H))^2$, or equivalently, $\chi(H) \leq \sqrt{2s + \frac{1}{4} + \frac{1}{2}}$. $\square$

Note that a trivial brute-force algorithm can enumerate all (not necessarily maximal) cliques in $O(2^{d}dn)$ time. Lemma 3 says that each $s$-defective clique is covered by at most $O(\sqrt{s})$ cliques. Hence, by a simple brute-force we obtain the following.

**Theorem 5** $s$-DEFECTIVE CLIQUE can be solved in $2^{O(d\sqrt{s})}n^{O(\sqrt{s})}$ time.

We can also use Lemma 3 to obtain an algorithm in terms of the smaller parameter $\gamma$ instead of the degeneracy $d$ without increasing the exponent of $n$.

**Theorem 6** $s$-DEFECTIVE CLIQUE can be solved in $2^{O(\gamma\sqrt{s} + s \log k)}n^{O(\sqrt{s})}$ time.

**Proof** We first enumerate all maximal cliques in $(3^{\gamma/3} \cdot n^{O(1)})$ time [20]. If there is a clique of size at least $k$, then return Yes, since each clique is also an $s$-defective clique. Now, we assume that there is no clique of size at least $k$. By Lemma 3, it suffices to check whether there is an $s$-defective clique of size $k$ in $\bigcup \{\mathcal{C} \in \mathcal{C} \} |\mathcal{C}| = \mathcal{O}(\sqrt{s})$ maximal cliques. Observe that each fixed collection in $\mathcal{C}$ has $O(k\sqrt{s})$ vertices. Let $W_{\mathcal{C}}$ denote the vertex set of $\mathcal{C}$. By applying the algorithm of Corollary 5 to find the largest $s$-defective clique, we can determine in $O(2^{\gamma}(\sqrt{s}k)^{O(s+3)})$ time whether $W_{\mathcal{C}}$ contains an $s$-defective clique of size at least $k$. Since there are $O^{*}(3^{\gamma/3})$ maximal cliques, the overall running time of this algorithm is $(3^{\gamma/3} \cdot n^{O(1)})^{O(\sqrt{s})} \cdot O(2^{\gamma}(\sqrt{s}k)^{O(s+3)}) = 2^{O(\gamma\sqrt{s} + s \log k)}n^{O(\sqrt{s})}$. $\square$

For $c$-closed graphs, we can obtain an algorithm whose running time does not depend on $k$. This is due to the following lemma.

**Lemma 4** Let $S \subseteq V(G)$ be an $s$-defective clique in $G$, in which at least one pair of vertices is nonadjacent. Then, $|S| \leq c + s$.

**Proof** Let $u, v \in S$ be vertices such that $uv \notin E(G)$. We show that $|S'| \leq c + s - 2$ for $S' = S \setminus \{u, v\}$. Since $G$ is $c$-closed, there are at most $c - 1$ vertices in $S'$ adjacent to both $u$ and $v$. Moreover, there are at most $s - 1$ vertices in $S'$ which are nonadjacent to either $u$ or $v$ in $S'$, by the definition of $s$-defective cliques. Thus, we obtain $|S'| \leq (c - 1) + (s - 1) = c + s - 2$. $\square$

From Lemma 4 we directly obtain the following.

**Corollary 6** $s$-DEFECTIVE CLIQUE can be solved in $2^{O(c\sqrt{s} + s \log(c+s))}n^{O(\sqrt{s})}$ time.
3.3 2-Clubs

A clique is a vertex set $S$ such that each vertex in $S$ has distance 1 to each other vertex within $S$. One way to relax the clique definition is to allow greater distances of pairs of vertices within $G[S]$. This idea can be formalized as follows.

**Definition 6** For a graph $G$ and $s \in \mathbb{N}$, a set $S \subseteq V(G)$ is an $s$-club if each pair of vertices in $S$ has distance at most $s$ in $G[S]$.

Note that cliques are exactly the 1-clubs. This definition leads to the following decision problem.

2-Club

**Input** A graph $G$ and $k \in \mathbb{N}$.

**Question** Does $G$ contain a 2-club $S$ of size at least $k$?

It is known that 2-Club parameterized by $k$ admits an FPT-algorithm [7, 44] and that it does not admit a polynomial kernel unless coNP $\subseteq$ NP/poly[44]. Since the largest 2-club containing some vertex $v$ is $N_2[v]$, we observe that the size of a largest 2-club is $\Delta^2 + 1$. This implies fixed-parameter tractability for $\Delta$. In contrast, 2-Club is $\text{W}[1]$-hard with respect to $h$-index and it is NP-hard on 6-degenerate graphs [26]. Since $\gamma \leq d + 1$, this also implies NP-hardness for constant values of $\gamma$. We extent these results, by showing that 2-Club remains NP-hard even on 4-closed graphs.

**Theorem 7** 2-Club remains NP-hard even on 4-closed graphs.

**Proof** We reduce from Cliques.

**Construction** Let $(G, k)$ be an instance of Cliques. We construct an equivalent instance $(G', k')$ of 2-Club such that $G'$ is 4-closed. We set $k' := k \cdot n^2$. For each vertex $w \in V(G)$, we add a clique $K_w := \{w_j \mid j \in \{0, \ldots, n^2 - 1\}\}$ of size $n^2$ to $G'$. We denote the graph constructed so far by $G^0$. Furthermore, let $(e_1, e_2, \ldots, e_m)$ be an arbitrary but fixed ordering of the edges in $E(G)$. We will add edges corresponding to each edge $e_i \in E(G)$ to $G'$. We denote by $G^i$ the graph after we added the gadgets for the edges $e_1$ to $e_i$ to $G^0$. Note that $G^0$ is the graph constructed so far; a disjoint union of cliques, and that $G^m = G'$. The idea for the gadget of edge $e_i = uv$ is as follows: We add a matching between the vertices of the cliques $K_u$ and $K_v$. More precisely, we add the edges $u_i v_{i + \ell_{uv} \mod n^2}$ for each $i \in \{0, \ldots, n^2 - 1\}$ and some fixed integer $\ell_{uv}$. We call $\ell_{uv}$ the *shift* of $uv$. We will assume that $\ell_{uv} + \ell_{vu} = n^2$. To simplify notation, we will assume that the modulo $n^2$ is taken after the addition of a shift. The difficult part lies in choosing $\ell_{uv}$ carefully to obtain a graph with constant closure.

For a vertex pair $(a, b)$ of $G$ by $A_{ab}$ we denote the set of vertices in the cliques $K_a$ and $K_b$ and by $B_{ab}$ the remaining vertices of $V(G')$. Next, we prove the following invariant which is an essential ingredient to show that $G' = G^m$ has constant closure number.

**Invariant** For each $i$, there is a shift $\ell_{uv}$ for the gadget of the $i$th edge $e_i = uv$ such that for each two nonadjacent vertices $x \in K_a$ and $y \in K_b$ for any vertices $a, b \in V(G)$ we have $|N_{G^i}(x) \cap N_{G^i}(y) \cap B_{ab}| \leq 1$. Moreover, we can find $\ell_{uv}$ is polynomial time.
That is, we want to maintain the invariant that two nonadjacent vertices in \( K_a \cup K_b \) have at most one common neighbor in \( B_{ab} \). Recall that \( G^0 \) is a disjoint union of cliques. Thus, the invariant holds for \( G^0 \). In the following, we assume that the invariant holds for the graph \( G^{i-1} \). Recall that the graph \( G^i \) is constructed from \( G^{i-1} \) by adding the matching for the edge \( e_i = uv \). We will show that the invariant can be maintained for \( G^i \). More precisely, we show that we can compute a shift \( \ell_{uv} \in \{0, \ldots, n^2 - 1 \} \) in polynomial time such that adding the edges \( u_j v_{j+\ell_{uv}} \) for each \( j \in \{0, \ldots, n^2 - 1 \} \) to \( G^{i-1} \) does not violate the invariant.

Assume to the contrary that the invariant is violated by two nonadjacent vertices in \( G^i \). Observe that there could be three possibilities on how the invariant could be violated in \( G^i \):

Case 1 Two nonadjacent vertices in \( A_{pq} \) for \( p, q \in V(G) \setminus \{u, v\} \) violate the invariant,
Case 2 two nonadjacent vertices in \( A_{uv} \) violate the invariant, or
Case 3 two nonadjacent vertices in \( A_{wp} \) for \( w \in \{u, v\} \) and \( p \in V(G) \setminus \{u, v\} \) violate the invariant.

In the following, we show that we can choose the shift \( \ell_{uv} \) in such a way to fulfill the invariant also for \( G^i \). We distinguish the three above cases:

Case 1 Let \( x \) and \( y \) be a pair of nonadjacent vertices in \( A_{pq} \) violating the invariant. Note that each edge added to \( G^{i-1} \) to obtain \( G^i \) is of the form \( u_t v_s \). Clearly, \( u_t, v_s \notin A_{pq} \). Hence, we conclude that \( |N_{G^i}(x) \cap N_{G^i}(y) \cap B_{pq}| = |N_{G^{i-1}}(x) \cap N_{G^{i-1}}(y) \cap B_{pq}| \leq 1 \) since the invariant holds for \( G^{i-1} \). Thus, this case is not possible.

Case 2 Let \( x \) and \( y \) be a pair of nonadjacent vertices in \( A_{uv} \) violating the invariant. As in case 1, since only edges with both endpoints in \( A_{uv} \) are added to the graph \( G^i \), we obtain that \( |N_{G^i}(x) \cap N_{G^i}(y) \cap B_{uv}| = |N_{G^{i-1}}(x) \cap N_{G^{i-1}}(y) \cap B_{uv}| \leq 1 \) since the invariant holds for \( G^{i-1} \). Thus, this case is also not possible.

Case 3 Without loss of generality, assume that \( w = u \). Recall that adding a matching between the cliques \( K_u \) and \( K_v \) can increase the number of common neighbors in \( B_{up} \) of two nonadjacent vertices in \( A_{up} \) by at most 1. Thus, two vertices in \( A_{up} \) violating the invariant in \( G^i \) have a common neighbor in some clique \( K_t \) in \( G^{i-1} \). Since only the matchings corresponding to the edges \( u_t, pt \in E(G) \) result in edges between \( K_u \) and \( K_t \) and between \( K_p \) and \( K_t \), the matchings corresponding to the edges \( u_t \) and \( pt \) are already added to \( G^{i-1} \). To obtain \( G^i \) from \( G^{i-1} \) only a matching between \( K_u \) and \( K_v \) is added. Thus, we conclude that the matching corresponding to the edge \( pv \) was already present in \( G^{i-1} \).

For every \( j \in \{0, \ldots, n^2 - 1 \} \), we have \( N(u_j) \cap K_t = \{t_{j+\ell_{ut}}\} \) and \( N(t_{j+\ell_{ut}}) \cap K_p = \{p_{j+\ell_{ut}+\ell_{tp}}\} \). Hence, \( u_j \) and \( p_j \) have a common neighbor in \( K_p \) if and only if \( j' - j \equiv \ell_{ut} + \ell_{tp} \). Similarly, \( u_j \) and \( p_{j'} \) have a common neighbor in \( K_v \) if and only if \( j' - j \equiv \ell_{uv} + \ell_{vp} \). Consequently, the invariant is only violated if \( \ell_{uv} \equiv \ell_{ut} + \ell_{tp} + \ell_{pv} \). Thus, for each \( p \) and \( t \), there is at most one shift violating the invariant, amounting to at most \((n-2)^2 \) forbidden shifts. Since there are \( n^2 \) possible shifts, we conclude that we can choose a shift \( \ell_{uv} \) in a way which does not violate the invariant. Note that this does not only show the existence of a shift maintaining the invariant, the above argument also shows that the shift \( \ell_{uv} \) can be constructed in polynomial time, although no explicit formula for \( \ell_{uv} \) is given here.
Thus, we have shown that the invariant is maintained for each $i$, in particular for $i = m$ and hence for the resulting graph $G'$.

**Bounded Closure** We use the invariant to show that $G'$ is 4-closed. Consider two nonadjacent vertices $x \in K_u$ and $y \in K_v$ in $G'$. Observe that $u \neq v$ since otherwise $xy \in E(G')$. By the invariant, we have $|N_{G'}(x) \cap N_{G'}(y) \cap B_{uv}| \leq 1$. Recall that $A_{uv} = K_u \cup K_v$. Since $x$ has at most one neighbor in $K_v$ and since $y$ has at most one neighbor in $K_u$, we conclude that $x$ and $y$ have at most three common neighbors. Thus, $G'$ is 4-closed.

**Correctness** Suppose that $G$ contains a clique $C$ of size at least $k$. Let $S := \{K_v \mid v \in C\}$. Clearly, $S$ has size $k' = k \cdot n^2$. It remains to show that $S$ is a 2-club. Consider two nonadjacent vertices $x$, $y \in S$. Note that $x \in K_u$ and $y \in K_v$ for $u, v \in C$ such that $u \neq v$ since otherwise $xy \in E(G')$. Since $C$ is a clique, we have $uv \in E(G)$ and thus we added a matching between the cliques $K_u$ and $K_v$. Hence, $x$ has a neighbor $z$ in $K_v$ and thus $x$ and $y$ have distance 2 since $K_v$ is a clique.

Conversely, suppose that $S$ contains a 2-club $S$ of size at least $k' = k \cdot n^2$. Let $T := \{v \mid |K_v \cap S| \geq n + 1\}$. Observe that $|T| \geq k$, since otherwise $|S| \leq |T| \cdot n^2 + (n - |T|) \cdot n \leq kn^2 - (k - 1)n$. In the following, we show that $T$ is a clique in $G$. Assume towards a contradiction that $T$ is not a clique and let $u, v \in T$ such that $uv \notin E(G)$. Let $u^*$ be a vertex in $K_u \cap T$ and let $U := N(u^*) \setminus K_u$. Note that since $uv \notin E(G)$ we have $U \cap K_v = \emptyset$. Furthermore, note that by construction each vertex $y \in K_w$ has at most one neighbor in $K_v$ for any $w, x \in V(G)$ such that $w \neq x$. Thus, $|U| \leq n$. Furthermore, by the same argument we obtain that each vertex in $U$ has at most 1 neighbor in $K_v$. Thus, $u^*$ has distance at most 2 to at most $n$ vertices in $K_v$. This is a contradiction to the fact that $|S \cap K_v| \geq n + 1$ and that $S$ is an 2-club. Hence, $T$ is a clique and thus $G$ contains a clique of size at least $k$. \qed

We leave the complexity of 2-CLUB on 2-closed graphs and 3-closed graphs open. We want to point out that 2-closed graphs of diameter two are also known to be geodetic, that is, each pair of vertices has a unique shortest path between them. Moreover, it is known that every 2-closed graph $G$ of diameter two satisfies one of the following [3]:

- $G$ contains a vertex $v$ such that $N(v) = V(G)$, or
- $G$ is strongly regular, that is, $G$ is regular and for some $\lambda, \mu \in \mathbb{N}$, every two adjacent (nonadjacent) vertices have $\lambda$ ($\mu$, respectively) common neighbors (note that $\mu = 1$ since $G$ is 2-closed), or
- $G$ has exactly two vertex degrees.

To show that 2-CLUB in 2-closed graph is solvable in polynomial time exploiting these three properties might be helpful.

For 2-clubs we only studied the decision variant 2-CLUB in which we ask for an sufficiently large 2-club in $c$-closed graphs. The enumeration of all maximal 2-clubs is not possible in FPT-time even for graphs with constant closure: Observe that in the construction of Theorem 7, $C$ is a maximal clique in the graph $G$ of the CLIQUE instance if and only if $\{K_c \mid c \in C\}$ is a maximal 2-club in the graph $G'$ of the 2-CLUB instance. The number of maximal cliques in an $n$-vertex graph is $3^{n/3}$ [40]. Hence, the above correspondence shows that even a 4-closed graph may have up to $3^{n/3}$ maximal 2-clubs.
4 Bicliques

The counterpart of cliques in bipartite graphs are (non-) induced bicliques. In this section we study the parameterized complexity of enumerating all maximal (non-) induced bicliques and finding a sufficiently large (non-) induced biclique in (weakly) closed graphs.

4.1 Non-induced Biclique

In this subsection, we study problems of finding non-induced maximal bicliques fulfilling certain cardinality constraints. Next, we formally define non-induced bicliques.

**Definition 7** In a graph $G = (V, E)$ two disjoint vertex sets $S \subseteq V$ and $T \subseteq V$ are a non-induced biclique if $st \in E(G)$ for each $s \in S$ and each $t \in T$.

There is an algorithm that enumerates in $O^*(2^d)$ time all maximal pairs of sets $S$ and $T$ such that each vertex of $S$ is adjacent to each vertex of $T$ [16]. We also consider the problem of finding a sufficiently large non-induced biclique.

**Non-Induced** $(k_1, k_2)$-**Biclique**

**Input** A graph $G$ and $k_1, k_2 \in \mathbb{N}$.

**Question** Does $G$ contain a non-induced biclique with vertex sets $S$ and $T$ such that $|S| \geq k_1$ and $|T| \geq k_2$?

Non-Induced $(k_1, k_2)$-BICLIQUE is W[1]-hard with respect to $k_1$ even if $k_1 = k_2$ [37]. We also consider Non-Induced Max-Edge BICLIQUE where we demand that $|S| \cdot |T| \geq k$ instead of putting constraints on the partition sizes. We may assume that $\min(|S|, |T|) \leq \sqrt{k}$. Thus, Non-Induced Max-Edge BICLIQUE can be solved by solving $\sqrt{k}$ instances of Non-Induced $(k_1, k_2)$-BICLIQUE and thus the latter problem can be considered to be more difficult in our setting. Non-Induced Max-Edge BICLIQUE can be solved in $O(k^{2.5}k^{\sqrt{k}}n)$ time by applying the algorithm for Induced Max-Edge BICLIQUE on bipartite graphs [19].

First, we study the parameterized complexity of enumerating all maximal non-induced bicliques in weakly $\gamma$-closed graphs. We need to define carefully, however, what we mean by enumerating bicliques: The algorithm of Eppstein [16] enumerates in $O^*(2^d)$ time all maximal pairs of sets $S$ and $T$ such that each vertex of $S$ is adjacent to each vertex of $T$. For this enumeration problem, an FPT-algorithm for the weak closure is unattainable since any clique of size $n$ is 1-closed and admits $\Theta(2^n)$ bipartitions that need to be enumerated. To circumvent this issue, we view a biclique as a vertex set that can be partitioned into sets $S$ and $T$. Thus, in order to strengthen the parameterization from $d$ to $\gamma$, we go from an explicit listing of bicliques with bipartitions to a compact representation of bicliques as vertex sets and this is indeed necessary. We say that a vertex set $U \subseteq V(G)$ is a non-induced biclique if $G[U]$ contains a biclique as a (not necessarily induced) subgraph. Note that it can be decided in $O(n^2)$ time whether a

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3 Eppstein [16] describes an algorithm with running time $O^*(2^{2a})$ for the graph parameter arboricity $a$ which is linearly bounded in $d$ by the inequality $a \leq d \leq 2a - 1$. It can be shown that this algorithm also has running time $O^*(2^d)$. 

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vertex set $U \subseteq V(G)$ is a non-induced biclique or not, because $U$ is a non-induced biclique if and only if the complement of $G[U]$ has multiple connected components. We adapt the algorithm of Theorem 2 to obtain an $O^*(2^{|V|})$-time algorithm to enumerate all maximal non-induced bicliques.  

Recall that in Theorem 2 we bounded the overall number of maximal s-plexes in a weakly $\gamma$-closed graph $G$ by distinguishing 4 different types of maximal s-plexes if we are provided with the set of maximal s-plexes of $G - v$. As in the proof of Theorem 2, we aim to enumerate all maximal non-induced bicliques in $G$, provided with the collection $S'$ of all non-induced maximal bicliques in $G' := \overline{G} - v$. Again, we define the same four types of non-induced bicliques $S$: Type 1: $S$ does not contain $v$, Type 2: $S$ contains $v$ and $S \setminus \{v\}$ is maximal in $G'$, Type 3: $S$ contains $v$, $S \setminus \{v\}$ is not maximal in $G'$, and $S$ contains a non-neighbor $u$ of $v$, and Type 4: $S$ contains $v$, $S \setminus \{v\}$ is not maximal in $G'$, and $S$ is contained in the neighborhood of $v$, that is, $S \subseteq N_G[v]$. 

First and foremost, all maximal non-induced bicliques of Type 1 and Type 2 can be enumerated from $S'$ in $|S'| \cdot n^2$ time. We claim that there are at most $2^{\gamma-1}n$ maximal non-induced bicliques of Type 3: Let $U$ be such a non-induced biclique with a bipartition $(S, T)$. Without loss of generality, assume that $u, v \in S$. There are at most $n$ choices for $u \in S \setminus N_G[v]$ and there are at most $2^{\gamma-1}$ choices for $T \subseteq N_G(v) \cap N_G(u)$. Since $U$ is a maximal non-induced biclique, we obtain $S = \bigcap_{w \in T} N_G(w)$. Finally, there is only one maximal non-induced biclique of Type 4, namely $N_G[v]$. Thus, we obtain the following theorem.

**Theorem 8** All maximal non-induced bicliques can be enumerated in $O^*(2^{|V|})$ time.

Second, we consider the decision variant of this problem. We show that NON-INDUCED $(k_1, k_2)$-BICLIQUE can be solved in $O^*(2^{|V|})$ time, using this enumeration algorithm.

**Theorem 9** NON-INDUCED $(k_1, k_2)$-BICLIQUE can be solved in $O^*(2^{|V|})$ time.

**Proof** With the algorithm behind Theorem 8 we can enumerate the vertex sets of all maximal non-induced bicliques. This algorithm, however, only returns the vertex set, and not a bipartition of any maximal non-induced biclique. To check whether any of these maximal non-induced bicliques has a bipartition into sets $S$ and $T$ such that $|S| \geq k_1$ and $|T| \geq k_2$, we use the following observation: Let $G'$ denote the complement graph of $G$. Any connected component of $G'$ is either completely contained in $S$ or completely contained in $T$. Now, we can use this observation to define an instance of SUBSET SUM to check whether there exists a valid bipartition. SUBSET SUM is formally defined as follows.

**SUBSET SUM**

**Input** A set $A = \{a_1, \ldots, a_n\}$ of $n$ positive integers and $k_1 \leq k_2 \in \mathbb{N}$.

**Question** Is there a set $B \subseteq A$ such that $k_1 \leq \sum_{b \in B} b \leq k_2$?

A standard dynamic programming algorithm can solve SUBSET SUM in $O(n \cdot \sum_{a \in A} a)$ time. To solve NON-INDUCED $(k_1, k_2)$-BICLIQUE, we construct an instance $(A', k_1', k_2')$ of SUBSET SUM for each maximal non-induced biclique $U$ with $|U| \geq$
\[ k_1 + k_2 \] returned by the algorithm of Theorem 8, where \( k'_1 := k_1, k'_2 := |U| - k_2, \) and \( A' := \{|C_i| : i \in [\ell]\} \) for the connected components \( C_1, \ldots, C_\ell \subseteq V(G) \) of the complement of \( G[U] \). Observe that \((G, k_1, k_2)\) is a Yes-instance if and only if the constructed instance of \textsc{Subset Sum} is a Yes-instance for some maximal non-induced biclique \( U \): note that \( k'_1 \) is a lower bound and \( k'_2 \) is an upper bound for the size of the smaller side of any valid bipartition and any solution \( B \) of the \textsc{Subset Sum} instance corresponds to \( S \), the smaller side of the bipartition of \( U \), and \( A \setminus B \) corresponds to the other part of the bipartition. \( \square \)

Recall that \textsc{Non-Induced Max-Edge Biclique} can be solved by solving \( \sqrt{k} \) instances of \textsc{Non-Induced \((k_1, k_2)\)-Biclique}. Hence, we obtain the following from Theorem 9.

**Corollary 7** \textsc{Non-Induced Max-Edge Biclique} can be solved in \( \mathcal{O}^*(2^\sqrt{k}) \) time.

### 4.2 Induced Biclique

In this subsection, we study problems where one aims to find \textit{induced} maximal bicliques fulfilling certain cardinality constraints. Formally, we consider the following.

**Definition 8** In a graph \( G = (V, E) \) two disjoint vertex sets \( S \subseteq V \) and \( T \subseteq V \) are an \textit{induced biclique} if \( G[S \cup T] \) is isomorphic to a complete bipartite graph, that is, \( st \in E(G) \) for each \( s \in S \) and each \( t \in T \), and \( tt' \not\in E(G) \) for each \( t, t' \in T \).

Gaspers et al. [23] provided an \( \mathcal{O}^*(3^{n/3}) \)-time algorithm to enumerate all maximal induced bicliques. Moreover, all maximal induced bicliques can be enumerated in \( \mathcal{O}^*(3^{(\Delta+d)/3}) \) time [27]. On the negative side, it is impossible to enumerate all maximal induced bicliques in time \( \mathcal{O}^*(f(d+c)) \) for any function \( f \) because a graph may have too many maximal induced bicliques [27]: Consider the graph with a single universal vertex \( u \) and \((n-1)/3\) disjoint triangles. This graph is 3-degenerate and 2-closed, and it has \( 3^{(n-1)/3} \) maximal induced bicliques where one part consists of \( u \).

In addition to the enumeration problem, we also study the following decision problem.

**INDUCED \((k_1, k_2)\)-BICLIQUE**

**Input** A graph \( G \) and \( k_1, k_2 \in \mathbb{N} \) such that \( k_1 \leq k_2 \).

**Question** Does \( G \) contain an induced biclique with vertex sets \( S \) and \( T \) such that \( |S| \geq k_1 \) and \( |T| \geq k_2 \)?

When \( k_1 = k_2 \), we will refer to the problem as \textsc{Induced \((k, k)\)-Biclique}. \textsc{Induced \((k, k)\)-Biclique} is \textsc{W[1]}-hard [12]. We also consider \textsc{Induced Max-Edge Biclique} where we demand that \( |S| \cdot |T| \geq k \) instead of putting constraints on the partition sizes. \textsc{Induced Max-Edge Biclique} is \textsc{NP}-hard [41] and \textsc{W[1]}-hardness with respect to the solution size \( k \) can be shown by a reduction from \textsc{Independent Set} where we attach an universal vertex. As in the non-induced case, \textsc{Induced Max-Edge Biclique} can be solved by solving \( \sqrt{k} \) instances of \textsc{Induced \((k_1, k_2)\)-Biclique}. Thus, positive results for \textsc{Induced \((k_1, k_2)\)-Biclique} transfer to \textsc{Induced Max-Edge Biclique}.

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First, we present an FPT-algorithm for Induced \((k, k)\)-BICLIQUE parameterized by \(\gamma\).

**Theorem 10** \(\text{Induced} \ (k, k)\)-BICLIQUE can be solved in \(O^*(\gamma^{O(\gamma)})\) time.

**Proof** Since a biclique \(K_{\gamma, \gamma}\) is not weakly \(\gamma\)-closed, \((G, k, k)\) is a No-instance if \(k \geq \gamma\). Moreover, \(\text{Induced} \ (k, k)\)-BICLIQUE is trivially solvable in polynomial time when \(k \leq 1\). Hence, we may assume that \(2 \leq k < \gamma\). Let \(\sigma\) be a fixed weak closure ordering of \(G\). Suppose that \((S, T)\) is a solution of \((G, k)\). Furthermore, let \(v \in S \cup T\) be the vertex of \(S \cup T\) that appears in \(\sigma\) before all other vertices of \(S \cup T\). We assume without loss of generality that \(v\) lies in \(S\). Note that there are at most \(n\) choices for \(v\).

Let \(G'\) be the graph obtained by removing all vertices preceding \(v\) in \(\sigma\). Furthermore, let \(v' \in V(G')\setminus\{v\}\) be another vertex which is contained in \(S\). Note that there are at most \(n\) choices for \(v'\). Next, we determine an independent set \(T \subseteq N_{G'}(v) \cap N_{G'}(v')\) of at least \(k\) vertices. Since \(|N_{G'}(v) \cap N_{G'}(v')| < \gamma\), there are at most \(2\gamma^k\) possibilities for \(T\). Now, it remains to find an independent set \(S' \subseteq \bigcap_{u \in T} N_{G'}(u)\) of size at least \(k\) in \(G'\). Recall that INDEPENDENT SET admits a kernel with at most \(\gamma^k\) vertices by Corollary 1, and thus this can be achieved in \(O^*((\gamma k^2)^k)\) time. Since \(k < \gamma\), the overall running time is \(O^*(2^\gamma \gamma^{3\gamma}) = O^*(\gamma^{O(\gamma)})\). \(\square\)

For \(c\)-closed graphs, we show that there is a single-exponential time algorithm when \(k_1 \geq 2\). Our algorithm is based on a reduction to a variant of INDEPENDENT SET called BICOLORED INDEPENDENT SET [11].

**BICOLORED INDEPENDENT SET**

**Input** A graph \(G\), a partition \((V_1, V_2)\) of \(V(G)\), and \(k_1, k_2 \in \mathbb{N}\).

**Question** Is there an independent set \(I \subseteq V(G)\) with \(|I \cap V_1| = k_1\) and \(|I \cap V_2| = k_2\)?

**Theorem 11** \(\text{Induced} \ (k_1, k_2)\)-BICLIQUE with \(k_1 \geq 2\) can be solved in \(O^*(1.611^c)\) time.

**Proof** Let \((G = (V, E), k_1, k_2)\) be an instance of \(\text{Induced} \ (k_1, k_2)\)-BICLIQUE.

Since \(k_1 \geq 2\) any induced biclique with \(k_1\) vertices in one partite set and with \(k_2\) vertices in the other partite set contains at least on cycle on four vertices. For each induced cycle \((u_S, u_T, v_S, v_T)\) on four vertices in \(G\) we search the largest induced biclique containing these four vertices. Now, we construct an instance \((G', V'_1, V'_2, k_1, k_2)\) of BICOLORED INDEPENDENT SET, where

- \(V'_1 := N_G(u_S) \cap N_G(v_S)\),
- \(V'_2 := N_G(u_T) \cap N_G(v_T)\), and
- \(G' := (V'_1 \cup V'_2 \cup E(G[V'_1]) \cup E(G[V'_2]) \cup \{(v'_1 v'_2) \mid v'_1 \in V'_1, v'_2 \in V'_2, v'_1 v'_2 \notin E(G)\})\).

In other words, \(G'\) is constructed from \(G[V'_1 \cup V'_2]\) by flipping the adjacency between \(V'_1\) and \(V'_2\). By the \(c\)-closure of \(G\), there are at most \(2c - 2\) vertices in \(G'\). Since \(v'_1 \in V'_1\) and \(v'_2 \in V'_2\) are adjacent in \(G\) if and only if they are not in \(G'\), there is a \((k_1, k_2)\)-biclique containing \(u_S, u_T, v_S, v_T\) if and only if \((G', V'_1, V'_2, k_1, k_2)\) is a Yes-instance. Since BICOLORED INDEPENDENT SET is \(O^*(1.2691^c)\)-time solvable on \(n\)-vertex graphs [11], we obtain an \(O^*(1.611^c)\)-time algorithm for \(\text{Induced} \ (k_1, k_2)\)-BICLIQUE. \(\square\)
By using a reduction similar to the one in the proof of Theorem 11, and using the algorithm of Gaspers et al. [23] to enumerate all maximal induced bicliques in $O^*(3^{n/3})$ time we obtain the following.

**Proposition 1** All maximal induced bicliques in which each part has at least two vertices can be enumerated in $O^*(3^{2c/3})$ time.

However, even 2-closed graphs may have $\Omega(3^{n/3})$ maximal induced bicliques: Consider the aforementioned graph proposed by Hermelin and Manoussakis [27], which consists of a single universal vertex $u$ and $(n - 1)/3$ disjoint triangles. Observe that this graph is 2-closed and has $3^{(n-1)/3}$ maximal induced bicliques where one part consists of $u$.

In contrast to our positive result for $k_1 \geq 2$ presented in Theorem 11, we prove that INDUCED $(1, k)$-BICLIQUE is NP-hard even on graphs with constant $h$-index, $c$-closure, and weak $\gamma$-closure.

**Theorem 12** INDUCED MAX-EDGE BICLIQUE and INDUCED $(1, k_2)$-BICLIQUE remain NP-hard even on graphs with $h$-index 4, $c$-closure 3, and weak $\gamma$-closure.

**Proof** We first show the NP-hardness for INDUCED MAX-EDGE BICLIQUE. We reduce from INDEPENDENT SET, which is NP-hard even on graphs in which each vertex has degree at most 3 [21]. Recall that in INDEPENDENT SET we are given a graph $G$ and an integer $k$, and ask whether $G$ contains an independent set of size at least $k$. We assume that $k \geq 10$, since otherwise the instance $(G, k)$ can be solved in polynomial time. We construct an instance $(G', k')$ of INDUCED MAX-EDGE BICLIQUE as follows: We begin with a copy of $G$. Then, each edge $uv \in E(G)$ is replaced by a path on four vertices $u, u_v, v_u$, and $v$. Finally, we introduce a new universal vertex $w$ (that is, $N_G(w) = V(G')$) and set $k' := k + |E(G)|$. It is easy to see that $G'$ has $h$-index 4 (because every vertex except $w$ has degree at most 4), is 3-closed and weakly 2-closed. It remains to show that $G$ contains an independent set of size $k$ if and only if $G'$ contains an induced biclique with at least $k' = k + |E(G)|$ edges.

Suppose that $G$ contains an independent set $I$ of size at least $k$. Then, there is an independent set $I'$ of size $k + |E(G)|$ in $G' - w$: Since $I$ is an independent set, for each edge $uv \in E(G)$ we have without loss of generality that $u \notin I$. Let $F := \{u_v \mid uv \in E(G)\}$ be the union of the neighbors of these vertices $u$ not in the independent set in paths on four vertices in $G'$. Then, $I'$ is the disjoint union of $I$ and $F$. Thus, the set $I' \cup \{w\}$ is an induced biclique with at least $k + |E(G)|$ edges in $G'$.

Conversely, suppose that $G'$ contains a biclique $(S, T)$ with at least $k' = k + |E(G)|$ edges. Since each vertex in $G' - w$ has degree at most 3 and $k \geq 10$, we see that vertex $w$ is contained in $(S, T)$. Without loss of generality, assume that $w \in S$. Since $w$ is a universal vertex, we obtain $S = \{w\}$. It follows that $T$ is an independent set of size at least $k + |E(G)|$ in $G'$. We may assume $|T \cap \{u_v, v_u\}| = 1$: For each edge $uv \in E(G)$, the set $T$ contains at most one of $u_v$ and $v_u$. If neither is in $T$, then $(T \setminus \{u\}) \cup \{u_v\}$ is another independent set of size $k'$. Thus, we may assume that $|T \cap \{u_v, v_u\}| = 1$ for every $uv \in E(G)$. No pair of adjacent vertices $u$ and $v$ in $G$ are part of $T$ since otherwise $T$ contains three vertices from a path $(u, u_v, v_u, v)$. Thus, $T \cap V(G)$ is an independent set of size $|T'| - |E(G)| \geq k$. 

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Finally, note that this reduction also shows NP-hardness of Induced \((1, k_2)\)-BICLIQUE (let \(k_2 = k'\)).

Together with Theorem 12, the next theorem paints a full picture of the complexity of Induced \((k_1, k_2)\)-BICLIQUE with respect to the weak closure number.

**Theorem 13** For constant \(k_1 \geq 2\), Induced \((k_1, k_2)\)-BICLIQUE on weakly \(\gamma\)-closed graphs is polynomial-time solvable if \(\gamma \leq k_1 + 1\) and NP-hard otherwise. Moreover, Induced \((1, k_2)\)-BICLIQUE on weakly 1-closed graphs is polynomial-time solvable.

**Proof** We start with the NP-hardness. We adapt the reduction in the proof of Theorem 12: Instead of adding a single universal vertex \(w\), we add \(k_1\) universal vertices (which are pairwise nonadjacent). Note that the graph constructed by our reduction is weakly \((k_1 + 2)\)-closed (consider an ordering in which all the universal vertices appear last).

Our polynomial-time algorithms solve INDEPENDENT SET on weakly 1-closed graphs as a subroutine. We fix a weak closure ordering \(\sigma\). Start with \(I = \emptyset\).

In a first step, we add the last vertex \(v\) of \(\sigma\) to \(I\) and then delete \(N[v]\) from the graph. For the correctness of this step, observe that the neighborhood of \(v\) is a clique. Otherwise, there exists a non-neighbor \(u\) of \(v\) with \(u <_{\sigma} v\) and distance 2 to \(v\). Since \(u\) and \(v\) have at least one common neighbor, we obtain a contradiction to the fact that the graph is weakly 1-closed. Since \(N[v]\) is a clique, there exists a maximum independent set containing \(v\). We repeat this step until the graph is empty.

Next, we give a polynomial-time algorithm for Induced \((1, k_2)\)-BICLIQUE on weakly 1-closed graphs. Without loss of generality, we assume that the input graph \(G\) is connected. Observe that there is a universal vertex \(u\) that is adjacent to every other vertex. Now, observe that there is an induced \((1, k_2)\)-biclique in \(G\) if and only if a maximum independent set of size \(k_2\) in \(G - u\). Since a maximum independent set in a weakly 1-closed graph can be found in polynomial time, we are done.

Finally, we prove the polynomial-time solvability for \(\gamma \leq k_1 + 1\). Observe that if \(\gamma \leq k_1\), then we have a No-instance of Induced \((k_1, k_2)\)-BICLIQUE since an induced \((k_1, k_2)\)-biclique has weak closure \(k_1 + 1\). Hence, in the following we assume that \(\gamma = k_1 + 1\). Now, consider a hypothetical solution \((S, T)\) with \(|S| = k_1\) and \(|T| = k_2\). We can guess which vertices correspond to the smaller side \(S\) in \(O(n^{k_1})\) time. Let \(\sigma\) be a fixed weak closure ordering and let \(X\) be the set of vertices that occur in \(\sigma\) before any vertex in \(S\). Since \(T \cap X\) are common neighbors of \(S\) we observe that \(T \cap X\) has size at most \(\gamma - 1 = k_1\). Hence, in \(O(n^{k_1})\) time, we can guess \(T \cap X\). It remains to find \(T \cap X\). Note that \(T \cap X \subseteq U := \bigcap_{s \in S} N(s)\), that is, \(S \subseteq N(t)\) for every vertex \(t \in T \cap X\). Observe that \(G[U \cap X]\) is weakly 1-closed: In the ordering \(\sigma\), two nonadjacent vertices \(u, u' \in U \cap X\) such that \(u <_{\sigma} u'\) have no common neighbor \(w \in U \cap X\) with \(u <_{\sigma} w\) since \(u\) and \(u'\) have \(S\) as common neighbors which appear after \(u'\) in \(\sigma\), and \(S\) has size \(k_1 = \gamma - 1\). As argued above, we can find a maximum independent set in \(G[U \cap X]\) in polynomial time. Thus, Induced \((k_1, k_2)\)-BICLIQUE can be solved in polynomial time if \(k_1\) is a constant and \(\gamma \leq k_1 + 1\).

To complete the dichotomy with respect to \(c\), we prove that Induced MAX-EDGE BICLIQUE and Induced \((k_1, k_2)\)-BICLIQUE can be solved in polynomial time.
if $c = 2$. Observe that Theorem 11 implies a polynomial-time algorithm for $k_1 \geq 2$ if $c = 2$. Hence, it remains to show that INDUCED $(1, k_2)$-BICLIQUE can be solved in polynomial-time if $c = 2$. For this, is it sufficient to consider diamond-free graphs since each 2-closed graph is diamond-free.

**Proposition 2** INDUCED $(1, k_2)$-BICLIQUE can be solved in polynomial time on diamond-free graphs.

**Proof** Suppose that the input graph $G$ is diamond-free. Then, for each vertex $v \in V(G)$ the graph $G[N(v)]$ is a disjoint union of cliques. Thus, $(G, 1, k_2)$ is a Yes-instance if and only if there is a vertex $v \in V(G)$ such that $G[N(v)]$ has at least $k_2$ connected components.

Now, from Proposition 2 ($k_1 = 1$) and Theorem 11 ($k_1 \geq 2$) we obtain the following.

**Corollary 8** INDUCED $(k_1, k_2)$-BICLIQUE and INDUCED MAX-EDGE BICLIQUE can be solved in polynomial time on 2-closed graphs.

Our results for INDUCED $(k_1, k_2)$-BICLIQUE can be summarized as follows (see also Table 2): If $k_1 = k_2$, then the problem becomes FPT with respect to the weak closure number $\gamma$ (Theorem 9). In the general case, the complexity strongly depends on whether $k_1 \geq 2$ or $k_1 = 1$. If $k_1 \geq 2$, the problem is polynomial-time solvable for $\gamma \leq k_1 + 1$ (Theorem 13), NP-hard for $\gamma \geq k_1 + 2$ (Theorem 13), and FPT for the parameterization by $c$ (Theorem 11). If $k_1 = 1$, then we have a complexity dichotomies in terms of $c$ and $\gamma$: we have a polynomial-time algorithm for $c = 2$ (Corollary 8) and $\gamma = 1$ (Theorem 13) and NP-hardness for $c \geq 3$ (Theorem 12) and $\gamma \geq 2$ (Theorem 12).

### 5 Variants of Dominating Set

In companion work [32], we showed that DOMINATING SET admits a kernel of size $k^{O(c)}$. Recently, Lokshtanov and Surianarayanan showed that DOMINATING SET parameterized by $\gamma + k$ can be solved in $O^*(k^{O(\gamma^2k^3)})$ time [38]. Here, we develop FPT-algorithms for the related INDEPENDENT DOMINATING SET and DOMINATING CLIQUE problems in weakly $\gamma$-closed graphs.

#### 5.1 Independent Dominating Set

We consider the INDEPENDENT DOMINATING SET problem. The task in this problem is to find a small independent set $S$ that dominats all vertices in $G$.

**Definition 9** A vertex set $S \subseteq V(G)$ is a dominating set in $G$ if $S \cap N[v] \neq \emptyset$ for each $v \in V$. Moreover, $S \subseteq V(G)$ is an independent dominating set in $G$ if $S$ is a dominating set and all vertices of $S$ are pairwise nonadjacent.
Algorithm 1 An FPT-Algorithm for INDEPENDENT DOMINATING SET.

1: function SolveIDS(G, k)
2: if \( k = 0 \) and \( V(G) \neq \emptyset \) then return No
3: let \( I := \emptyset \) and \( G' := G \).
4: while \( V(G') \neq \emptyset \) and \( |I| \leq k \) do
5: let \( v \) be a vertex such that \( cl_{G'}(v) \leq \gamma - 1 \).
6: \( I := I \cup \{v\} \) and \( G' := G' - N_G[v] \).
7: if \( |I| \leq k \) then return Yes
8: else
9: \( P := \{v \mid v \text{ is a common neighbor of at least two vertices in } I\} \)
10: for each \( u \in P \) do
11: if SolveIDS\( (G - N_G[u], k - 1) \) returns Yes then
12: return Yes
13: end if
14: end for
15: end if
16: end while
17: return No

INDEPENDENT DOMINATING SET

Input A graph \( G \) and \( k \in \mathbb{N} \).

Question Does \( G \) contain an independent dominating set \( S \subseteq V(G) \) of size at most \( k \)?

INDEPENDENT DOMINATING SET is \( W[2] \)-hard for the parameter \( k \) [15]. There are several fixed-parameter tractability results in restricted graph classes: INDEPENDENT DOMINATING SET has a kernel of \( O(d^2 k^d + 1) \) vertices computable in \( O^*(2^d) \) time [42]. Moreover, when the graph contains no cycles of length 3 or 4, INDEPENDENT DOMINATING SET can be solved in \( O^*(k^{O(k)}) \) time [43].

We present an FPT-algorithm \( \text{SolveIDS} \) (Algorithm 1) with running time \( O^*((\frac{\gamma - 1}{2})^k k^{2k}) \). Note that our algorithm extends the \( O^*(k^{O(k)}) \) time algorithm of Raman and Saurabh [43], because any graph without cycles of length 3 or 4 is 2-closed. Let \( G' \) be a copy of \( G \). Algorithm 1 first greedily computes an independent set \( I \) of \( G \) of size at most \( k + 1 \) by iteratively choosing vertices \( v \) such that \( cl_{G'}(v) \leq \gamma - 1 \) and afterwards removing \( N[v] \) from \( G' \) (Line 6). If \( I \) is inclusion-maximal and of size at most \( k \), then \( I \) constitutes a solution. Otherwise, we find a vertex set \( P \) to branch on. The choice of \( I \) will ensures that \( P \) at most \( (\gamma - 1)(\frac{k + 1}{2}) \) vertices.

Theorem 14 INDEPENDENT DOMINATING SET can be solved in \( O^*((\frac{\gamma - 1}{2})^k k^{2k}) \) time.

Proof We show that the search tree algorithm Algorithm 1 solves any instance \( (G, k) \) of INDEPENDENT DOMINATING SET in the claimed time. First, we prove the correctness of Algorithm 1. Let \( I \) be the independent set of size at most \( k + 1 \) of \( G \) obtained in Lines 4 to 7. Suppose that \( |I| \leq k \). Since \( I \) is a maximal independent set, each vertex \( v \in V(G) \) is either contained in \( I \) or a neighbor of a vertex in \( I \). Hence, \( I \) is an independent dominating set of size at most \( k \) of \( G \). Thus, \( (G, k) \) is a Yes-instance. Now, suppose that \( |I| = k + 1 \). Let \( P \) be the set of vertices in \( G \) which have at least two neighbors in \( I \) (Line 10). Since \( |I| = k + 1 \), the sought solution \( S \) must contain at least one vertex \( u \in P \). If \( u \in S \), then \( S \) does not contain any neighbor of \( u \). Thus, the branching into \( (G - N_G[u], k - 1) \) in Line 13 is correct.

Now, we analyze the running time of Algorithm 1. First, we bound the number of children of any node in the search tree. To do so, we prove that \( |P| \leq (\gamma - 1)(\frac{k + 1}{2}) \).
Let $v_i$ be the $i$th vertex added to $I$ in Line 7 and let $G_i := G - N_G([v_1, \ldots, v_{i-1}])$ for each $i \in [k + 1]$. Observe that $c_{G_i}(v_i) \leq \gamma - 1$ for each $i \in [k]$ since $G$ is weakly $\gamma$-closed. For a vertex $u \in P$, let $v_i \in I$ be the first vertex that $u$ is adjacent to. Then, $u$ is present in the graph $G_i$ and we have $u \in N_{G_i}(v_i)$. Thus, $P \subseteq \bigcup_{i \in [k + 1]} N_{G_i}(v_i)$ and we see that $|P| \leq \sum_{i \in [k + 1]} |N_{G_i}(v_i) \cap P|$. Moreover, we have $|N_{G_i}(v_i) \cap P| \leq \sum_{j \in [i + 1, k + 1]} |N_{G_i}(v_i) \cap N_{G_i}(v_j)|$ for each $i \in [k]$. Since $I$ is an independent set we obtain that $|N_{G_i}(v_i) \cap N_{G_i}(v_j)| \leq \gamma - 1$ for each $j \in [i + 1, k + 1]$. Therefore,

$$|P| \leq \sum_{i < j \in [k + 1]} |N_{G_i}(v_i) \cap N_{G_i}(v_j)| \leq (\gamma - 1)\binom{k + 1}{2}.$$ 

It is easy to see that finding an independent set $I$ in Lines 4 to 7 only requires polynomial time. Hence, the algorithm spends polynomial time in each search tree node. Since each node has at most $(\gamma - 1)\binom{k + 1}{2}$ children in the search tree and its depth is at most $k$, the overall running time of Algorithm 1 is $O^*((\gamma - 1 \cdot \binom{k + 1}{2})^k) = O^*((\frac{\gamma - 1}{2})^k 2^k)$.

A natural next question is whether \textsc{Independent Dominating Set} admits a polynomial kernel in weakly $\gamma$-closed graphs. We answer this question in the negative way, that is, we provide kernel lower bounds for \textsc{Independent Dominating Set} via a cross-composition [4, 5].

An equivalence relation $R$ on $\Sigma^*$ is called a polynomial equivalence relation if the following two conditions hold: (1) There is an algorithm that given two strings $x, y \in \Sigma^*$ decides whether $x$ and $y$ belong to the same equivalence class in $(|x| + |y|)^O(1)$ time, and (2) for any finite set $S \subseteq \Sigma^*$ the equivalence relation $R$ partitions the elements of $S$ into at most $(\max_{x \in S} |x|)^O(1)$ classes.

**Definition 10** Let $L \subseteq \Sigma^*$ be a set and let $Q \subseteq \Sigma^* \times \mathbb{N}$ be a parameterized problem. We say that $L$ cross-composes into $Q$ if there is a polynomial equivalence relation $R$ and an algorithm which, given $2^t$ strings $x_1, x_2, \ldots, x_{2^t}$ belonging to the same equivalence class of $R$, computes an instance $(x^*, k^*) \in \Sigma^* \times \mathbb{N}$ in time polynomial in $\sum_{i=1}^{2^t} |x_i|$ such that:

1. $(x^*, k^*) \in Q$ if and only if $x_i \in L$ for some $i \in [2^t]$, and
2. $k^*$ is bounded by a polynomial in $\max_{i \in [2^t]} |x_i| + t$.

It is known that if an NP-hard problem cross-composes into a parameterized problem, then the parameterized problem does not admit a kernel of polynomial size unless coNP $\subseteq$ NP/poly [4, 5].

**Theorem 15** Unless coNP $\subseteq$ NP/poly, \textsc{Independent Dominating Set} admits

- no kernel of size $(k + c)^O(1)$ and
- no kernel of size $k^{O(1)}$ even if $\gamma = 2$.

**Proof** We provide a cross-composition from \textsc{Independent Dominating Set} on 2-closed graphs. Note that \textsc{Independent Dominating Set} remains NP-hard on 2-closed graphs and hence also on weakly 2-closed graphs. This follows from the fact
that \textsc{Independent Dominating Set} is \textsc{NP}-hard on graphs of girth at least five \cite{6, 46}. In particular, the graph constructed by the cross-composition procedure is weakly 2-closed and $(t + 2)$-closed.

Assume that we are given $2^t$ instances $I_x := (G_x, k)$ of \textsc{Independent Dominating Set} on 2-closed graphs for $x \in [2^t]$. We will describe how to construct an instance $(G^*, k')$ of \textsc{Independent Dominating Set} with weak closure 2 and closure $t + 2$ that meets the requirements as specified in Definition 10. To do so, we write an integer $x \in [2^t]$ in binary encoding $(x_1, \ldots, x_t)$. For $y \in \{0, 1\}$ let $\tilde{y} := 1 - y$. Furthermore, for any string $x := (x_1, \ldots, x_s)$ a string $(x_1, \ldots, x_p)$ for some $p \in [s]$ is a prefix of $x$.

First, we construct the \textit{instance selector} gadget $H_t$. For each string $z \in \{0, 1\}^*$ of length at most $t$, we introduce a vertex $w_z$ to $H_t$. We add an edge $w_zw_y$ whenever $z$ is a prefix of $y$. Furthermore, we add an edge $w_zw_y$ whenever $z$ and $y$ are of the same length and they differ only in the last bit, that is, $z := (z_1, \ldots, z_{s-1}, z_s)$ and $y := (z_1, \ldots, z_{s-1}, \tilde{z}_s)$. This concludes the construction of $H_t$.

To construct $G^*$, start with a disjoint union of $G_x$ for all $x \in [2^t]$ and with $H_t$. We then add an edge from vertex $w_z$ to every vertex in $V(G_x)$ whenever $z$ is a prefix of $x$. Finally, we set $k' := k + t$. As we will show, every independent set of $H_t$ avoids dominating the vertices of $G_x$ for some $x \in [0, 1]^t$ (see Claim 2). Intuitively speaking, this ensures that $G_x$ has an independent dominating set of size at most $k$ whenever $(G', k')$ is a Yes-instance.

Before showing the correctness, we verify that $G^*$ is weakly 2-closed. To this end, we show that every induced subgraph $G^*$ of $G'$ has a vertex $v \in V(G^*)$ such that $\text{cl}_{G^*}(v) < 2$. If $G^*$ does not contain any vertex of $H_t$, then $G^*$ is 2-closed. Otherwise, assume that $G^*$ contains at least one vertex of $H_t$. Let $w_z \in V(G^*)$ be a vertex such that $z := (z_1, \ldots, z_{s-1}, z_s)$ has the shortest length among all vertices $w_y \in V(H_t) \cap V(G^*)$. We show that $w_z$ and any vertex $v \in V(G^*) \setminus N[w_z]$ have at most one common neighbor. We show this claim for $v = w_y$; the proof for $v \in V(G_x)$ with $x \in [2^t]$ is analogous because $N(v) \subseteq N(w_x)$. Since $w_zw_y \notin E(G')$, we observe that $z$ is not a prefix of $y$. By construction, we have

$$N_{G'}(w_z) \cap N_{G'}(w_y) \subseteq Z \cup \{w_{(z_1, \ldots, z_s)}\},$$

Here, $Z := \{w_{z'} | z' \text{ is a prefix of both } y \text{ and } z\}$. Since $z$ has the shortest length among all strings $z'$ such that $w_{z'} \in V(G^*)$, we have $Z \cap V(G^*) = \emptyset$. Hence, we have $|N_{G^*}(w_z) \cap N_{G^*}(w_y)| \leq 1$. We thus have shown that $G'$ is weakly 2-closed.

We then examine the $c$-closure of $G'$. By construction, each vertex has at most $t$ neighbors in $H_t$. Hence, any two nonadjacent vertices of $G'$ have at most $t$ common neighbors in $H_t$. Moreover, since the $2^t$ many instances of \textsc{Independent Dominating Set} are 2-closed and disjoint and any two vertices in $H_t$ having a common neighbor in some $G_x$ are adjacent, we conclude that any two nonadjacent vertices of $G'$ have at most one common neighbor in $\bigcup_{x \in [2^t]} V(G_x)$. Thus, $G'$ is $(t + 2)$-closed.

Next we show that $G'$ contains an independent dominating set of size at most $k'$ if and only if $G_x$ contains an independent dominating set of size at most $k$ for some $x \in [2^t]$.

Assume that $S$ is an independent dominating set of size at most $k$ for the instance $I_x$. Recall that the binary encoding of $x$ is $(x_1, \ldots, x_t)$ where $x_i \in \{0, 1\}$ for each $i$. Springer
Let $I := \{x_1, \ldots, x_t\}$ be the start of the induction. Observe that $w(0)$ and $w(1)$ are adjacent in $G'$ (note that these correspond to the 1-bit strings). Hence, $I'$ can contain at most one of these two vertices. In other words, $w_i \notin I'$ for some $i \in \{0, 1\}$. We set $x_1 := i$. Now, we consider the inductive step. Here we assume that we already constructed the string $(x_1, \ldots, x_s)$ for some $s \in [t - 1]$ and now we aim to construct $x_{s+1}$. Observe that for $y := (x_1, \ldots, x_s, 0)$ and $z := (x_1, \ldots, x_s, 1)$ the vertices $w_y$ and $w_z$ are adjacent in $G'$. Hence, $I'$ can contain at most one of these two vertices. In other words, $w(\text{prefix of } x_s, i) \notin I'$ for some $i \in \{0, 1\}$. We set $x_{s+1} := i$. Now, the claim follows after constructing $x_t$.

Let $I' := S' \cap V(H_t)$. In the following, let $x = (x_1, \ldots, x_t)$ be a string fulfilling the conditions of Claim 2, that is, for any prefix $y$ of $x$, the vertex $w_y$ is not contained in $I'$ and hence also not in $S'$. Furthermore, let $G_x$ be the graph of the INDEPENDENT DOMINATING SET instance corresponding to $x$. Since $w_y \notin S'$ for any prefix $y$ of $x$, we obtain that $S' \cap V(G_x) \neq \emptyset$. Consider the vertex $w_{y_i}$ where $y_i := (x_1, \ldots, x_{i-1}, \tilde{x}_i)$ for some $i \in [t]$. By construction, the vertices $\{w_{y_i} \mid i \in [t]\}$ are pairwise unreachable in $G' - N[S' \cap V(G_x)]$. Hence, $G' - N[S' \cap V(G_x)]$ has at least $t$ connected components. Since $S'$ contains at least one vertex of each connected component, it follows that there is an independent dominating set in $G_x$ of size at most $k' - t = k$. \hfill \Box
Algorithm 2 An algorithm for finding a dominating clique $S$ that contains $v_i$ as the first vertex in the fixed weak closure ordering $\sigma$ of $G$. Initially we have $T := \{v_i\}$.

1: function SolveDC($G, k, T$) \[\triangleright T \subseteq \{v_1, \ldots, v_n\} \text{ and } v_i \in T\]
2: if $k = 0$ and $V(G) \neq N[T]$ then return No
3: if $V(G) = N[T]$ then return Yes
4: Find a vertex $w$ such that $v_i \overline{w} \in E(G)$
5: for each $u \in \bigcap_{x \in T} N(x) \cap N(w) \cap V(G_i)$ do
6: \[\triangleright G_i := G[\{v_1, \ldots, v_n\}]\]
7: if SolveDC($G, k - 1, T \cup \{u\}$) returns Yes then
8: return Yes
9: return No

5.2 An FPT-Algorithm for Dominating Clique

We now consider the DOMINATING CLIQUE problem. The task in this problem is to find a small clique that dominates all vertices.

**Definition 11** A set $S \subseteq V(G)$ is a dominating clique in $G$ if all vertices of $S$ are pairwise adjacent and $S$ is a dominating set.

**DOMINATING CLIQUE**

**Input** A graph $G$ and a parameter $k \in \mathbb{N}$.

**Question** Does $G$ contain a dominating clique of size at most $k$?

It is known that DOMINATING CLIQUE is W[2]-hard with respect to $k$ even on graphs which do not contain a 4-claw (a $K_{1,4}$) as an induced subgraph [14].

Note that there is a straightforward $O^*(d^k)$-time algorithm for DOMINATING CLIQUE on $d$-degenerate graphs: Enumerate all cliques of size at most $k$ and check if any of them dominates all vertices. To see the running time bound, observe that if we may use a degeneracy ordering $(v_1, \ldots, v_n)$ of $G$ and recall that for this ordering $\deg_{G_i}(v_i) \leq d$ where $G_i := G[\{v_1, \ldots, v_n\}]$. By considering all $n$ possibilities for the first vertex of the dominating clique in this ordering, we can enumerate every clique of at most $k$ vertices in $O(n \cdot d^k)$ time. Instead, one may also solve DOMINATING CLIQUE in $O^*(2^d)$ time by enumerating all cliques of $G$. At first glance, an $O^*(2^d)$-time algorithm may sometimes seem preferable to the $O^*(d^k)$-time algorithm. However, a more precise running time bound of the latter algorithm is $O^*(\binom{d}{k})$ which is never larger than $O^*(2^d)$.

In this subsection, we describe an FPT-algorithm for weakly $\gamma$-closed graphs, resulting in an $O^*((\gamma - 1)^k)$-time algorithm. Note that a maximal clique of a weakly $\gamma$-closed graph may be arbitrarily large. Thus, a simple brute-force search on maximal cliques may require $\Omega(n^k)$ time even on graphs with constant weak closure. Moreover, we want to avoid enumerating all maximal cliques since this alone incurs a running time of $\Omega(3^{\gamma/3})$ [20]. Instead, we will use Algorithm 2 for each vertex $v_i$ in a fixed weak closure ordering $\sigma$. The key idea is that we assume that $v_i$ is the first vertex in the dominating clique with respect to $\sigma$. As we shall see in the proof of Theorem 16 this guarantees that for each vertex $w$ which is not adjacent to $v_i$, we may branch into at most $\gamma - 1$ cases to determine a vertex that dominates $w$.

**Theorem 16** DOMINATING CLIQUE can be solved in $O^*((\gamma - 1)^k)$ time.
Proof To solve an instance \((G, k)\) of DOMINATING CLIQUE, we first compute a weak closure ordering \(\sigma\). Afterwards, we invoke SolveDC on input \((G, k - 1, \{v_i\})\) for each vertex \(v_i \in V\). In the call SolveDC\((G, k - 1, \{v_i\})\), we assume that \(v_i\) is the first vertex in the dominating clique \(S\) with respect to the weak closure ordering \(\sigma\).

We first show that SolveDC\((G, k, T)\) is correct in the following sense: it returns \(\text{Yes}\) if and only if there is a dominating clique \(S\) of size at most \(k\) which contains all vertices of \(T\), and vertex \(v_i\) is the first vertex in \(S\) with respect to \(\sigma\) (where \(v_i\) is the minimal vertex of \(T\) with respect to \(\sigma\)). It is easy to see that the terminal condition in Line 2 is correct. Moreover, Line 3 (where we return \(\text{Yes}\) when \(V(G) = N[T]\)) is correct if \(T\) is a clique, we will argue below that this is always the case. Let \(w \notin N(v_i)\) be the vertex computed in Line 4. Since we want to compute a dominating clique \(S\) which contains \(T\), where vertex \(v_i\) is the first vertex in \(S\) with respect to the weak closure ordering \(\sigma\) and since \(v_i, w \notin E(G)\), any dominating set must contain at least one vertex \(u\) of \(N(w) \cap V(G_i)\). Moreover, since we are searching for a dominating clique, we have that \(u\) must also be a common neighbor of all vertices in \(T\), that is, \(u \in (\bigcap_{x \in T} N(x))\). Thus, the branching into \((G, k - 1, T \cup \{u\})\) in Lines 5 and 7 is correct. Since each vertex \(u\) chosen in Line 5 is a common neighbor of all vertices in \(T\), we conclude that \(T\) is a clique and thus Line 3 returns \(\text{Yes}\) if and only if \(G\) contains a dominating clique of size at most \(k\). Furthermore, each vertex \(u\) chosen in Line 5 is contained in \(G_i\). Hence \(v_i <_\sigma u\). In other words, vertex \(v_i\) is the smallest vertex in \(T\) with respect to \(\sigma\).

Let us analyze the time complexity of SolveDC. It is easy to see that Lines 2 to 4 can be performed in polynomial time. Consider the search tree where each node corresponds to an invocation of SolveDC. We show that each node in the search tree has at most \(\gamma - 1\) children. To this end, we bound the size of \(|N(v_i) \cap N(w) \cap V(G_i)|\) which is an upper bound on the number of branches created in Line 5. If \(v_i <_\sigma w\), then \(|N(v_i) \cap N(w) \cap V(G_i)| \leq \gamma - 1\) by Definition 3. Otherwise, if \(w <_\sigma v_i\), then \(v_i\) and \(w\) have at most \(\gamma - 1\) common neighbors in \(\{v' \mid w <_\sigma v'\}\) and thus also in \(V(G_i)\). Hence, each node has at most \(\gamma - 1\) children. Moreover, the depth of the search tree is at most \(k - 1\). Thus, we spend \(O^*(\gamma - 1)^{k-1}\) time for each vertex \(v_i \in V(G)\) and the claimed running time bound follows.

In companion work [32], we showed by a reduction from \(\lambda\)-Hitting Set that DOMINATING SET does not admit kernels of size \(O(k^{c-1-\epsilon})\) under some standard complexity-theoretic assumptions. The idea of this well-known reduction is to construct a split graph in which the universe is the clique and the sets of the set family are the vertices in the independent set. In other words, the split graph is obtained from the incidence graph of the set family by making the universe a clique. We thus directly obtain the following hardness results for DOMINATING CLIQUE from this reduction.

Proposition 3 For \(c \geq 3\), DOMINATING CLIQUE has no kernel of size \(O(k^{c-1-\epsilon})\) unless coNP \(\subseteq NP/poly\).

Proposition 4 Unless the ETH fails, there is no \(n^{o(k)}\)-time algorithm for DOMINATING CLIQUE.

In view of Proposition 4, it is unlikely that the running time \(O^*(\gamma - 1)^k\) of Theorem 16 can be substantially improved: an algorithm running in time \(O^*(\gamma^{o(k)})\) or...
$O^*(e^{o(k)})$ would disprove the ETH. Furthermore, for $\lambda = 2$, when $\lambda$-Hitting Set is the Vertex Cover problem, the reduction shows NP-hardness for constant closure since the independent set vertices in the constructed instance have degree 2.

**Proposition 5** DOMINATING CLIQUE remains NP-hard even on 3-closed graphs.

Thus, both parameters $\gamma$ and $k$ are necessary in Theorem 16: DOMINATING CLIQUE is W[2]-hard with respect to $k$ even on graphs which do not contain a 4-claw as an induced subgraph [14] and NP-hard even for $\gamma = 3$ since $\gamma \leq c$ (Proposition 5).

### 6 Conclusion

We have provided further applications of the weak closure parameter $\gamma$ which was introduced for clique enumeration [20]. Given the algorithmic usefulness of the class of weakly closed graphs, it seems important to further study its properties. For example, it would be nice to obtain a forbidden subgraph characterization. We note that the weakly-1-closed graphs are exactly the graphs that do not contain a $C_4$ or a $P_4$ as an induced subgraph. These graphs are also known as quasi-threshold graphs. Can we obtain a similar characterization for weakly 2-closed graphs?

Further FPT-algorithms for the parameter $\gamma$ would also be very interesting from a theoretical and practical point of view. In particular, obtaining kernelization algorithms for the class of weakly closed graphs is unexplored for many problems. For example, DOMINATING SET has an FPT-algorithm for the parameter $\gamma + k$ [38] but it remains open whether DOMINATING SET admits a polynomial kernel for $k$ if $\gamma$ is a constant. Only for special graph classes, kernels for $\gamma + k$ are known: In companion work [31] we provided almost tight kernels of size $k^{O(\gamma)}$ for split graphs and of size $k^{O(\gamma^2)}$ for graphs with constant clique size. In contrast, for the larger parameters degeneracy and closure almost tight upper and lower bounds are known: DOMINATING SET admits a kernel with $O(k^{(d+1)^2})$ vertices [42], and a kernel of size $O(k^{(d-3)(d-1)-\varepsilon})$ is unlikely [13]. Similarly, DOMINATING SET admits a kernel with $k^{O(c)}$ vertices [32] and a kernel of size $O(k^{c-1-\varepsilon})$ is unlikely [32]. Observe in this context that, for INDEPENDENT DOMINATING SET, we showed that a kernel of size $k^{g(\gamma)}$ is unlikely. Such a kernel may, however, still be achievable for DOMINATING CLIQUE.

Also, some questions about clique relaxations in (weakly) closed graphs remain open: In s-CLUB we ask for a vertex set $S$ of size at least $k$ in a graph $G$ which is an $s$-club. Recall that in an $s$-club $S$ each pair of vertices in $S$ has distance at most $s$ in $G[S]$ (see also Definition 6). We showed that 2-CLUB is NP-hard even in 4-closed graphs (Theorem 7). It is open, whether 2-CLUB is also NP-hard 2-closed or 3-closed graphs. Finally, the complexity of s-CLUB for $s \geq 3$ on graphs with constant closure remains open.

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