A survey on M. B. Levin’s proofs for the exact lower discrepancy bounds of special sequences and point sets

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Abstract

The goal of this overview article is to give a tangible presentation of the breakthrough works in discrepancy theory [3, 5] by M. B. Levin. These works provide proofs for the exact lower discrepancy bounds of Halton’s sequence and a certain class of \((t, s)\)-sequences. Our survey aims at highlighting the major ideas of the proofs and we discuss further implications of the employed methods. Moreover, we derive extensions of Levin’s results.

1 Introduction and statement of main results

In [3] and [5] M. B. Levin proved optimal lower discrepancy bounds for certain shifted \((t, m, s)\)-nets and for the \(s\)-dimensional Halton sequence. The main ideas of these proofs are also basis for later, even deeper works of Levin on this topic, see [4, 6]. However, these papers will not be discussed in our survey. In [3] and [5] Levin showed the subsequent Theorems 1 and 2, which we will state below in a simplified version. We start with fixing the notation for basic quantities and concepts, which will be needed for the formulation of Levin’s results and of our extensions.

Let \((x_n)_{n \in \mathbb{N}}\) be an infinite sequence in the \(s\)-dimensional unit cube \([0, 1)^s\),

\[
y = (y^{(1)}, \ldots, y^{(s)}),
\]

and

\[
[0, y) = [0, y^{(1)}) \times \ldots \times [0, y^{(s)}) \subseteq [0, 1)^s.
\]

We call \(\Delta(\cdot, (x_n)_{n=1}^N) : [0, 1]^s \to \mathbb{R},\)

\[
\Delta(y, (x_n)_{n=1}^N) = \sum_{n=1}^N (\chi_{[0,y]}(x_n) - y^{(1)} \cdots y^{(s)}),
\]

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the discrepancy function of the sequence \((x_n)_{n \in \mathbb{N}}\). We define the star-discrepancy of an \(N\)-point set \((x_n)_{n=1}^{N}\) as

\[
D^*\left((x_n)_{n=1}^{N}\right) = \sup_{y \in (0,1)^s} \left| \frac{1}{N} \Delta(y, (x_n)_{n=1}^{N}) \right|.
\]

Further, we need the definition of a \((t, m, s)\)-net in base \(b\) introduced by H. Niederreiter \[2\] and the so-called \(d\)-admissibility property of nets.

**Definition 1.** For integers \(b \geq 2, s \geq 1, m \text{ and } t, \) with \(0 \leq t \leq m, \) a \((t, m, s)\)-net in base \(b\) is defined as a set of points \(\mathcal{P} = \{x_0, \ldots, x_{b^m-1}\} \) in \([0,1)^s\), which satisfies the condition that every interval with volume \(b^{-m+t}\) of the form \(J = \prod_{i=1}^{s} \left[ \frac{a_i}{b^i}, \frac{a_i+1}{b^i} \right), \) with \(d_i \in \mathbb{N}_0, a_i \in \{0, 1, \ldots, b^d_i-1\}, \) for \(i = 1, \ldots, s, \) contains exactly \(b^t\) points of \(\mathcal{P}. \) We will call these intervals \(J\) elementary intervals.

**Definition 2.** For \(x = \sum_{i \geq 1} \frac{x_i}{b^i}, \) where \(x_i \in \{0, 1, \ldots, b-1\}\) and \(m \in \mathbb{N}, \) the truncation is defined as

\[
[x]_m = \sum_{i=1}^{m} \frac{x_i}{b^i}.
\]

For \(x = (x^{(1)}, \ldots, x^{(s)})\) the truncation is defined as \([x]_m = ([x^{(1)}]_m, \ldots, [x^{(s)}]_m). \) Moreover, we define \([x]_0 := 0.\)

Keep in mind that for an arbitrary number \(x \in \mathbb{R}, \) \([x]\) denotes the integer part of \(x.\) For the next definition recall the concept of the digital shift. For a point \(x = \sum_{i \geq 1} \frac{x_i}{b^i}\) and a shift \(\sigma = \sum_{i \geq 1} \frac{\sigma_i}{b^i}\) we have that

\[
x \oplus \sigma := \sum_{i \geq 1} \frac{y_i}{b^i}; \quad \text{where} \quad y_i \equiv x_i + \sigma_i \mod b
\]

and analogously

\[
x \ominus \sigma := \sum_{i \geq 1} \frac{y_i}{b^i}; \quad \text{where} \quad y_i \equiv x_i - \sigma_i \mod b.
\]

For \(x = (x^{(1)}, \ldots, x^{(s)})\) and \(\sigma = (\sigma^{(1)}, \ldots, \sigma^{(s)})\) the \(b\)-adic digitally shifted point is defined by \(x \oplus \sigma = (x^{(1)} \oplus \sigma^{(1)}, \ldots, x^{(s)} \oplus \sigma^{(s)})\). Analogously we define \(x \ominus \sigma.\)

**Definition 3.** For \(x = \sum_{i \geq 1} \frac{x_i}{b^i}, \) where \(x_i = 0\) for \(i = 1, \ldots, k\) and \(x_{k+1} \neq 0, \) the absolute valuation of \(x\) is defined as

\[
\|x\|_b = \frac{1}{b^{k+1}}.
\]

For \(x = (x^{(1)}, \ldots, x^{(s)})\) the absolute valuation is defined as \(\|x\|_b := \prod_{i=1}^{s} \|x^{(i)}\|_b.\)

With this definition we can introduce point sets with a special property which is essential for the further considerations of this chapter.
Definition 4. For an integer $d$, we say that a point set $P = \{\mathbf{x}_0, \ldots, \mathbf{x}_{b^m-1}\}$ in $[0,1)^s$ is $d$-admissible in base $b$ if
\[
\min_{0 \leq k < n < b^m} \|\mathbf{x}_n \ominus \mathbf{x}_k\|_b > \frac{1}{b^{m+d}}.
\]

We remind the definition of the Halton sequence in bases $b_1, \ldots, b_s$, where $s \geq 1$. Throughout this survey all occurring bases $b_1, \ldots, b_s$, are assumed to be pairwise coprime integers.

Definition 5. Let $b_1, \ldots, b_s$, $b_i \geq 2$ ($i = 1, \ldots, s$), for some dimension $s \geq 1$, be integers. Then the $s$-dimensional Halton sequence in bases $b_1, \ldots, b_s$, denoted by $(H_s(n))_{n \in \mathbb{N}_0}$, is defined as
\[
H_s(n) := (\phi_{b_1}(n), \ldots, \phi_{b_s}(n)), \quad n = 0, 1, \ldots,
\]
where $\phi_{b_i}$ denotes the radical inverse function in base $b_i$, i.e., the function $\phi_{b_i} : \mathbb{N}_0 \to [0,1)$, defined as
\[
\phi_{b_i}(n) := \sum_{j=0}^{\infty} n_j b_i^{-j-1},
\]
where $n = n_0 + n_1 b_i + n_2 b_i^2 + \ldots$, with $n_0, n_1, n_2, \ldots \in \{0, 1, \ldots, b_i-1\}$.

It is well known in discrepancy theory that the Halton sequence (requiring that the underlying bases are pairwise coprime) is a low discrepancy sequence, i.e., the star-discrepancy is of order $O\left(\frac{(\log N)^s}{N}\right)$ (see, e.g., [1]). Succeeding in showing that the discrepancy of the Halton sequence satisfies $D^*(H_s(n))_{n=1}^N \geq c_s\frac{(\log N)^s}{N}$, for infinitely many $N$, with a constant $c_s > 0$, would prove that this order is exact.

For $(t, m, s)$-nets in base $b$, denoted by $P$, we know that their discrepancy always satisfies $D^*(P) \leq c_{s,b} b^t \left(\frac{(\log N)^s-1}{N}\right)$. We will show that the order $O\left(\frac{(\log N)^s-1}{N}\right)$ is exact for certain $(t, m, s)$-nets.

Now, we can state Levin’s main results from [3] and [5] (in a simplified form).

Theorem 1. Let $s \geq 2, d \geq 1, m \geq 9(d+t)(s-1)^2$ and let $(\mathbf{x}_n)_{0 \leq n < b^m}$ be a $d$-admissible $(t, m, s)$-net in base $b$. Then, we can provide an explicitly given $\mathbf{w}$ such that
\[
b^m D^*((\mathbf{x}_n \oplus \mathbf{w})_{0 \leq n < b^m}) \geq \frac{(4(d+t)(s-1)^2)^{s+1}}{b^d m^{s-1}}.
\]

In particular, we have
\[
D^*((\mathbf{x}_n \oplus \mathbf{w})_{0 \leq n < N}) \geq c_{s,d} \frac{(\log N)^s-1}{N},
\]
with a constant $c_{s,d} > 0$ and $N = b^m$. 

Theorem 2. Put $B = b_1 \cdots b_s$, $s \geq 2$ and $m_0 = [2B \log_2 B] + 2$, then the estimate for the star-discrepancy of the Halton sequence

$$\sup_{1 \leq N \leq 2^{m_0}} ND^*((H_s(n))_{n=1}^N) \geq m^*(8B)^{-1},$$

is valid for $m \geq B$. In particular, there exists some constant $c_s > 0$, such that

$$D^*((H_s(n))_{n=1}^N) \geq c_s \left( \frac{\log N}{N} \right)^s,$$

for infinitely many $N \in \mathbb{N}$.

The implied constant $c_s$ also depends on the bases but not on $N$.

The aim of this paper is two-fold. **First**, we will give an easier and simpler access to the ideas of Levin. To this end, we are eager to give a clear and illustrative re-proof of Theorems 1 and 2. We use absolutely the same ideas as Levin, but focus on a clearer presentation. To achieve this goal, we restrict the re-proof of Theorem 1 to the two-dimensional case and carry out the steps in detail. For this case of course, the exact lower discrepancy bound follows (for an arbitrary $w$) by the general lower bound for the discrepancy of two-dimensional point sets by W. M. Schmidt [7]. For simplicity we will also restrict ourselves to base $b = 2$. Moreover, we focus on the optimal quality parameter $t = 0$ and for ease of presentation we formulate and prove the result for $m \equiv 0 \mod 4$. We also state the result without the shift and require a certain condition on $x_0$ instead. (The ideas for the proof in the general case are the same as in this special version.) This gives Theorem 3:

**Theorem 3.** Let $(x_n)_{0 \leq n < 2^m}$ be a $(0, m, 2)$-net in base 2 with $m \geq 4$, $m \equiv 0 \mod 4$ and $x_0 = \gamma = (\gamma^{(1)}, \gamma^{(2)})$,

$$\gamma^{(1)} = 1 \cdot 2^{-2} + 1 \cdot 2^{-4} + \cdots + 1 \cdot 2^{-m/2},$$
$$\gamma^{(2)} = 1 \cdot 2^{-m/2+2} + 1 \cdot 2^{-m/2+4} + \cdots + 1 \cdot 2^{-m}.$$  

Then it holds for the interval $J_\gamma = [0, \gamma^{(1)}) \times [0, \gamma^{(2)})$ that

$$\frac{1}{N} \Delta(\gamma, (x_n)_{0 \leq n < 2^m}) \leq -\frac{1}{4} \frac{1}{2^{m+2}m},$$

and consequently

$$D^*((x_n)_{0 \leq n < N}) \geq \frac{1}{16 \log 2} \frac{\log N}{N},$$

with $N = 2^m$.

The **second aim** is to give a - in a certain sense - quantitative extension of Theorems 1 and 2. We will show:

**Theorem 4.** Let $m \geq 2s^*(s - 1)^s$. Then, there is a set $\Gamma \subseteq [0, 1)^s$, $s \geq 2$, with the following properties:
For all $x \in [0,1)^s$ there exists a $\gamma \in \Gamma$ with
\[ \|x - \gamma\| < b\sqrt{s} \frac{1}{L^{1/2}}. \]
Here, $\| \cdot \|$ denotes the Euclidean norm.

- If $P = \{x_0, \ldots, x_{b^m-1}\}$ is a $(0, m, s)$-net in base $b$, and if $x_i \in \Gamma$ for some $i \in \{0, \ldots, b^m-1\}$, then, with $N = b^m$,
\[ D^*(P) \geq \frac{(b-1)^s(2s-3)^{s-1}}{b^s(4s^2(s-1)^2 \log b)^{s-1}} \frac{(\log N)^{s-1}}{N}. \]

**Theorem 5.** There are constants $c_1$ and $c_2 > 0$, such that for infinitely many $N$ there exists a set $\Lambda_N \subseteq [0,1)^2$ with the following properties:

- We have $\lambda_2(\Lambda_N) \geq c_1$, where $\lambda_2$ denotes the 2-dimensional Lebesgue measure.
- For all $x \in \Lambda_N$ there exists a $y \in [0,1)^2$ with $\|x - y\| < \sqrt{\frac{8}{N^2}}$ and
\[ \left| \Delta(y, (H_2(n))_{n=1}^N) \right| \geq c_2(\log N)^2. \]

**Remark 1.** An analogous result can be obtained for arbitrary dimensions. For sake of simplicity our considerations will be restricted to the two-dimensional case. The basic ideas become better visible in this case and can be adopted to higher dimensions in a straightforward manner.

The remainder of this paper is organised as follows: In Chapter 2, we will discuss the $d$-admissibility property in more detail. Of course, the proof of Theorem 3 will be the major part of this chapter. We relax some of the conditions of Theorem 3 in Chapter 3 and derive a more general result (Theorem 4). In Chapter 4, we will prove Theorem 2 in detail. Chapter 5 will be solely dedicated to the proof of Theorem 5.

## 2 Remarks on admissibility of nets and Re-proof of Theorem 3

Before stating the proof of Theorem 3, we discuss the $d$-admissibility property for $(0, m, s)$-nets, since in this theorem we restrict ourselves to the quality parameter $t = 0$.

**Lemma 2.1.** A point set $P = \{x_0, \ldots, x_{b^m-1}\}$ in $[0,1)^s$ is $s$-admissible if and only if $P$ is a $(0, m, s)$-net in base $b$. Moreover, $P$ cannot be $d$-admissible for $d < s$. 


Proof. Let $\mathcal{P}$ be a $(0, m, s)$-net in base $b$. First, we show that

$$\frac{1}{b^{m+s-1}} \geq \min_{0 \leq k < n < b^m} \|x_n \ominus x_k\|_b,$$

by taking special elementary intervals into account. Since $\mathcal{P}$ is a $(0, m, s)$-net, we know by definition that every elementary interval of order $m$ in base $b$, i.e., every elementary interval with volume $\frac{1}{b^m}$, contains exactly one point of $\mathcal{P}$. Therefore, this is also true for intervals of the form

$$\left[\frac{k}{b^m}, \frac{k+1}{b^m}\right) \times [0, 1)^{s-1}, \quad k \in \{0, \ldots, b^m - 1\}.$$

Now let $x = (x^{(1)}, \ldots, x^{(s)})$ be the unique point of $\mathcal{P}$ for which it holds that $x^{(1)} \in \left[0, \frac{1}{b^m}\right)$. Moreover, let $y = (y^{(1)}, \ldots, y^{(s)})$ be the point of $\mathcal{P}$ such that $y^{(1)} \in \left[\frac{b-1}{b^m}, \frac{b}{b^m}\right)$. This is equivalent to

$$0 \leq x^{(1)} < \frac{1}{b^m}, \quad \frac{b-1}{b^m} \leq y^{(1)} < \frac{1}{b^{m-1}}.$$

Therefore, we know that $x^{(1)}$ and $y^{(1)}$ can be written as

$$x^{(1)} = \frac{\alpha_1}{b^{m+1}} + \frac{\alpha_2}{b^{m+2}} + \cdots,$$

$$y^{(1)} = \frac{b-1}{b^m} + \frac{\beta_1}{b^{m+1}} + \frac{\beta_2}{b^{m+2}} + \cdots,$$

where $\alpha_i, \beta_i \in \{0, 1, \ldots, b - 1\}$ for $i \geq 1$. Thus, $\|y^{(1)} \ominus x^{(1)}\|_b = \frac{1}{b^m}$. Moreover, for $x^{(i)}$ and $y^{(i)}$, $i = 2, \ldots, s$, it holds that $\|y^{(i)} \ominus x^{(i)}\|_b \leq \frac{1}{b^{m-1}}$. Therefore, it follows, that

$$\|y \ominus x\|_b \leq \frac{1}{b^{m+s-1}}.$$

If we can prove that $\min_{0 \leq k < n < b^m} \|x_n \ominus x_k\|_b > \frac{1}{b^{m+s-1}}$, then the first implication of the assertion immediately follows. Suppose that there exist points $x = (x^{(1)}, \ldots, x^{(s)}), x \in \mathcal{P}$ and $y = (y^{(1)}, \ldots, y^{(s)}), y \in \mathcal{P}$ such that $\|y \ominus x\|_b \leq \frac{1}{b^{m+s-1}}$. Then, there exist integers $l^{(1)}, \ldots, l^{(s-1)}$ such that

$$\|y^{(i)} \ominus x^{(i)}\|_b \leq \frac{1}{b^{l^{(i)}}}, \quad \text{for } i = 1, \ldots, s-1,$$

and

$$\|y^{(s)} \ominus x^{(s)}\|_b \leq \frac{1}{b^{m+s-\sum_{i=1}^{s-1} l^{(i)} - 1}}.$$

This implies that the first $l^{(i)} - 1$ digits of the $b$-adic expansion of $x^{(i)}$ and $y^{(i)}$, $i = 1, \ldots, s-1$ are identical. Also, the first $m + s - \sum_{i=1}^{s-1} l^{(i)} - 1$ digits of the $b$-adic expansion of
$x^{(s)}$ and $y^{(s)}$ are identical. Consequently, $x$ and $y$ are contained in an elementary interval of volume $\frac{1}{b^m}$. This contradicts our assumption that $\mathcal{P}$ is a $(0, m, s)$-net.

Let now $\mathcal{P}$ be an arbitrary $b^m$-point set in $[0, 1]^s$ which is not a $(0, m, s)$-net. Then there exists an elementary interval $J_1 \subseteq [0, 1]^s$ of volume $1/b^m$ which contains no point of $\mathcal{P}$ or at least two points of $\mathcal{P}$. In the second case it immediately follows (by the same considerations as above) that $\mathcal{P}$ is not $s$-admissible. Consider now the first case: We can partition $[0, 1]^s$ into $b^m$ elementary intervals $J_i$ of the same shape as $J_1$. Since $J_1$ contains no point of $\mathcal{P}$ there exists at least one $i$ such that $J_i$ contains at least two points, and this again contradicts the $s$-admissibility.

**Remark 2.** Note, that it might happen that a $(1, m, s)$-net in base $b$ is non-admissible for any integer $d$. To see this, just take $b$ copies of a $(0, m - 1, s)$-net in base $b$. This gives an example of a $(1, m, s)$-net in base $b$ which is not $d$-admissible for any $d \in \mathbb{N}$.

These preliminary considerations put us in the position to prove Theorem 3. In Chapter 3 we give the proof for a more general result in the general case. Note, that for $(t, m, s)$-nets with nonzero quality parameter the $d$-admissibility condition has to be required additionally. The idea underlying the proof of the theorem in the general case is exactly the same.

**Proof of Theorem 3:** 
Note that by Lemma 2.1 $(x_n)_{0 \leq n < 2^m}$ is 2-admissible. To begin with, we want to find a suitable partition of the interval $J_\gamma$. Let therefore $r = (r_1, r_2) \in \mathbb{N}^2$. For

$$r_1 = 2j_1 \quad \text{and} \quad r_2 = m/2 + 2j_2,$$

with $j_1, j_2 \in \{1, ..., m/4\}$ it holds that

$$\gamma^{(1)} = \sum_{r_1} \frac{1}{2r_1} \quad \text{and} \quad \gamma^{(2)} = \sum_{r_2} \frac{1}{2r_2}.$$

Now define the set $A$ which contains all combinations of the indices $r_1$ and $r_2$, i.e.,

$$A = \{(r_1, r_2) \mid r_1 = 2j_1, \ r_2 = m/2 + 2j_2, \ j_1, j_2 \in \{1, ..., m/4\}\}.$$

The partition of $J_\gamma$ is then given by

$$J_{r, \gamma} = \left[\left[\gamma^{(1)}\right]_{r_1 - 1}, \left[\gamma^{(1)}\right]_{r_1 - 1} + \frac{1}{2r_1}\right] \times \left[\left[\gamma^{(2)}\right]_{r_2 - 1}, \left[\gamma^{(2)}\right]_{r_2 - 1} + \frac{1}{2r_2}\right],$$

for $(r_1, r_2) \in A$. Furthermore, let

$$A_1 = \{r \in A \mid r_1 + r_2 \leq m\},$$

$$A_2 = \{r \in A \mid r_1 + r_2 = m + 1\},$$

$$A_3 = \{r \in A \mid r_1 + r_2 \geq m + 2\},$$

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such that $A = A_1 \cup A_2 \cup A_3$. The intervals $J_{\gamma, r}$ are elementary intervals in base 2 with volume $\frac{1}{2^{r_1+r_2}}$, i.e., of order $r_1 + r_2$. Moreover, all $J_{\gamma, r}$ are disjoint and therefore, we obtain with

$$A(r) := \sum_{n=0}^{2m-1} \chi_{J_{\gamma, r}}(x_n)$$

$$\frac{1}{N} \Delta(\gamma, (x_n)_{0 \leq n < 2^m}) = \sum_{r \in A} \left( \frac{A(r)}{2^m} - \lambda_2(J_{\gamma, r}) \right)$$

$$= \sum_{r \in A_1} \left( \frac{A(r)}{2^m} - \lambda_2(J_{\gamma, r}) \right) + \sum_{r \in A_2} \left( \frac{A(r)}{2^m} - \lambda_2(J_{\gamma, r}) \right) + \sum_{r \in A_3} \left( \frac{A(r)}{2^m} - \lambda_2(J_{\gamma, r}) \right)$$

$$=: \Delta_1(\gamma) + \Delta_2(\gamma) + \Delta_3(\gamma).$$

Consider $\Delta_1$. Since $(x_n)_{0 \leq n < 2^m}$ is a $(0, m, 2)$-net, it is fair with respect to all elementary intervals of order $\leq m$. For $r \in A_1$ it holds that $r_1 + r_2 \leq m$ and therefore

$$\Delta_1(\gamma) = \sum_{r \in A_1} \frac{A(r)}{2^m} - \lambda_2(J_{\gamma, r}) = 0.$$

Consider $\Delta_2$. From the condition that $r \in A_2 \subseteq A$ we get that

$$r_1 = 2j_1 \quad \text{and} \quad r_2 = m/2 + 2j_2,$$

where $j_1, j_2 \in \{1, \ldots, m/4\}$. It follows that

$$r_1 + r_2 = m + 2(j_1 + j_2 - m/4).$$

Since $j_1 + j_2 - m/4 \in \mathbb{Z}$ we know that $2(j_1 + j_2 - m/4) \neq 1$ which is a contradiction to the assumption that $r_1 + r_2 = m + 1$ for all $r \in A_2$. Therefore, $A_2 = \emptyset$ and $\Delta_2 = 0$.

Consider $\Delta_3$. As a first step we want to show that $J_{\gamma, r}$ with $r_1 + r_2 \geq m + 2$ cannot contain any point of $(x_n)_{0 \leq n < 2^m}$ and we will do that by deriving a contradiction.

Suppose there exists $x_k \in J_{\gamma, r}$ for some $k < 2^m$ and some $r \in A_3$. Then we know for the first coordinate

$$[\gamma(1)]_{r_1-1} \leq x_k^{(1)} < [\gamma(1)]_{r_1-1} + \frac{1}{2^{r_1}},$$

which is equivalent to

$$\frac{1}{2^2} + \frac{1}{2^4} + \ldots + \frac{1}{2^{r_2-2}} \leq \frac{x_k^{(1)}}{2} + \frac{x_k^{(1)}}{2^{r_1-1}} + \frac{x_k^{(1)}}{2^{r_1}} + \ldots < \frac{1}{2^2} + \frac{1}{2^4} + \ldots + \frac{1}{2^{r_2-2}} + \frac{1}{2^{r_1}}.$$

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Therefore, it has to hold that \(x_{k,2}^{(1)} = x_{k,4}^{(1)} = \ldots = x_{k,r_1-2}^{(1)} = 1\) and \(x_{k,1}^{(1)} = x_{k,3}^{(1)} = \ldots = x_{k,r_1-1}^{(1)} = 0\). An analogous procedure can be done for the second coordinate. Hence,

\[
\begin{align*}
\gamma^{(1)}_{r_1-1} &= x_{k,r_1-1}^{(1)} \quad \text{and} \quad \gamma^{(2)}_{r_2-1} = x_{k,r_2-1}^{(2)}. \\
\end{align*}
\]

Combining (1) and the assumption that \(x_0 = \gamma\) leads to

\[
\left[ (x_k \ominus x_0)_{r_1-1}^{(1)} \right] = 0 \quad \text{and} \quad \left[ (x_k \ominus x_0)_{r_2-1}^{(2)} \right] = 0.
\]

Thus, we get \(\|x_k^{(i)} \ominus x_0^{(i)}\|_2 \leq \frac{1}{2^r}\). Since \(r \in A_3\), i.e., \(r_1 + r_2 \geq m + 2\), it follows that

\[
\|x_k \ominus x_0\|_2 \leq \frac{1}{2^{r_1+r_2}} \leq \frac{1}{2^{m+2}}.
\]

This is a contradiction to the assumption that \((x_n)_{0 \leq n < 2^m}\) is a 2-admissible \((0, m, 2)\)-net in base 2. Hence, \(A(r) = 0\) for all \(r \in A_3\) and

\[
\Delta_3(\gamma) = \sum_{r \in A_3} \left( \frac{A(r)}{2^m} - \lambda_2(J_r, \gamma) \right) \\
= -\sum_{r \in A_3} \frac{1}{2^{r_1+r_2}} \\
\leq -\sum_{r \in A_3} \frac{1}{2^{m+2}} \\
= -|A_4| \frac{1}{2^{m+2}}
\]

with

\[
A_4 = \{ r \in A_3 | r_1 + r_2 = m + 2 \}.
\]

It is easy to see that

\[
|A_4| = \frac{m}{4}
\]

for \(m \geq 4\) and \(m \equiv 0 \mod 4\), and so we finally get

\[
\frac{1}{N} \Delta(\gamma, (x_n)_{0 \leq n < 2^m}) = \Delta_3(\gamma) \\
\leq -\frac{1}{2^{m+2}} |A_4| \\
= -\frac{1}{4} \frac{1}{2^{m+2}} m.
\]

\(\square\)
3 Proof of Theorem 4

The first aim of this section is to focus on the assumption of Theorem 3 that there exists a point \( x_0 \in P \) such that \( x_0 = \gamma \) (of course the condition \( x_0 = \gamma \) can be replaced by \( x_n = \gamma \) for any \( n \in \{0, \ldots, 2^m - 1\} \)). This restriction on the point set is weakened by showing that there are many possible choices for \( \gamma \) such that the proof of Theorem 3 can still be performed in an analogous way. In fact, it turns out that \( \gamma \) only has to fulfill some simple properties as the following lemma shows:

**Lemma 3.1.** Let \( (x_n)_{0 \leq n < b^m} \) be a \((0,m,s)\)-net in base \( b \). Let \( x_0 \in \prod_{j=1}^{s} [\gamma^{(j)}, \gamma^{(j)} + \frac{1}{b^{\max(R_j)}}) \),

\[
\gamma^{(j)} = \sum_{r \in R_j} \frac{a_r^{(j)}}{b^r},
\]

\( a_r^{(j)} \in \{1,2,\ldots,b-1\} \) and \( R_j \subseteq \{1,2,\ldots,m\} \) for \( j = 1,\ldots,s \). Here the \( R_j \) are arbitrary, but for \( r = (r_1,r_2,\ldots,r_s) \in R_1 \times R_2 \times \ldots \times R_s \), the following constraints need to be satisfied:

- \( |\{r| m + 1 \leq \sum_{j=1}^{s} r_j < m + s\}| \leq \frac{m^{s-1}}{s} \),
- \( |\{r| \sum_{j=1}^{s} r_j = m + \alpha\}| \geq \frac{m^{s-1}}{\beta} \),

for some constant \( \beta > 0 \), some integer \( \alpha \geq s \) and for \( \delta > \frac{b^{\alpha} (b^s - 1)}{b^{s-1}} \). Then, it holds for the interval \( J_\gamma = \prod_{j=1}^{s} [0, \gamma^{(j)}) \) that

\[
\frac{1}{N} \Delta(\gamma, (x_n)_{0 \leq n < b^m}) \leq -\frac{m^{s-1}}{b^m} \left( -\frac{(b-1)^s b^{s-1} - 1}{\delta} + \frac{(b-1)^s 1}{\beta} \right),
\]

where \( \left( -\frac{(b-1)^s b^{s-1} - 1}{\delta} + \frac{(b-1)^s 1}{\beta} \right) > 0 \).

**Proof.** Let \( A = \{r| r_j \in R_j, j = 1,\ldots,s\} \) be the set of indices which can be split into three disjoint subsets

\[
A_1 = \{r \in A| \sum_{j=1}^{s} r_j \leq m\},
\]

\[
A_2 = \{r \in A| m + 1 \leq \sum_{j=1}^{s} r_j < m + s\},
\]

\[
A_3 = \{r \in A| \sum_{j=1}^{s} r_j \geq m + s\}.
\]

Further let

\[
A_4 = \{r \in A| \sum_{j=1}^{s} r_j = m + \alpha\}.
\]
A partition of the interval \( J \) is given by the subintervals

\[ J_{r, \gamma, g} = \prod_{j=1}^{s} \left[ \gamma(j)_{r_j-1} + \frac{g_j}{b^{r_j}}, \gamma(j)_{r_j-1} + \frac{g_j + 1}{b^{r_j}} \right] \]

where \( g = (g_1, ..., g_s) \) with \( g_j \in \{0, 1, ..., a_{r_j} - 1\} \). The intervals \( J_{r, \gamma, g} \) are disjoint elementary intervals of order \( \sum_{j=1}^{s} r_j \) in base \( b \). We define

\[ \mathcal{A}(r, g) := \sum_{n=0}^{b^{m-1}} \chi_{J_{r, \gamma, g}}(x_n). \]

Then, it is possible to split the estimation of the discrepancy function into three parts corresponding to the sets \( A_1, A_2 \) and \( A_3 \),

\[
\frac{1}{N} \Delta(\gamma, (x_n)_{0 \leq n < b^m}) = \sum_{r \in A_1} \frac{\mathcal{A}(r, g)}{b^m} - \lambda_s(J_{r, \gamma, g}) + \sum_{r \in A_2} \frac{\mathcal{A}(r, g)}{b^m} - \lambda_s(J_{r, \gamma, g}) + \sum_{r \in A_3} \frac{\mathcal{A}(r, g)}{b^m} - \lambda_s(J_{r, \gamma, g}) = \Delta_1 + \Delta_2 + \Delta_3.
\]

It follows by the net property and the fact that \( J_{r, \gamma, g} \) are elementary intervals that

\[
\Delta_1 = \sum_{r \in A_1} \left( \frac{\mathcal{A}(r, g)}{b^m} - \lambda_s(J_{r, \gamma, g}) \right) = 0.
\]

Since \( J_{r, \gamma, g}, r \in A_2 \), are elementary intervals of order greater or equal to \( m + 1 \), they either contain one point of the \((0, m, s)\)-net or they are empty. Let us consider these two cases:

1. \( \exists x_k \in J_{r, \gamma, g} \). Then it holds that

\[
\frac{1}{b^m} - \frac{1}{b^{m+1}} \leq \frac{\mathcal{A}(r, g)}{b^m} - \lambda_s(J_{r, \gamma, g}) = \frac{1}{b^m} - \frac{1}{b^{\sum_{j=1}^{s} r_j}} \leq \frac{1}{b^m} - \frac{1}{b^{m+s-1}}.
\]

2. \( \nexists x_k \in J_{r, \gamma, g} \). In this case it holds that

\[
-\frac{1}{b^{m+1}} \leq \frac{\mathcal{A}(r, g)}{b^m} - \lambda_s(J_{r, \gamma, g}) = -\frac{1}{b^{\sum_{j=1}^{s} r_j}} \leq -\frac{1}{b^{m+s-1}}.
\]
Then, by the assumptions on \( A_2 \) we obtain the estimate

\[
- \frac{1}{b^{m+1}} \frac{m^{s-1}}{\delta} (b - 1)^s \leq \Delta_2 \leq \left( \frac{1}{b^{m}} - \frac{1}{b^{m+s-1}} \right) \frac{m^{s-1}}{\delta} (b - 1)^s.
\]

Now, consider \( \Delta_3 \). The first step is again to show that \( J_{r,\gamma,g} \) with \( r \in A_3 \) and for all associated \( g \), cannot contain any point of a \((0, m, s)\)-net which has an element \( x_0 \in \prod_{j=1}^{s}[\gamma^{(j)}(\gamma^{(j)}) + \frac{1}{b^{\max(R_j)}}] \). The condition that \( x_0 \) is contained in this set, is equivalent to

\[
[\gamma^{(j)}]_{r_j} = [x^{(j)}_0]_{r_j}, \quad \text{for} \ j = 1, \ldots, s.
\]  

(2)

Suppose there exists \( x_k \in J_{r,\gamma,g} \) for some \( k < b^m \), some \( r \in A_3 \) and some \( g \). It then follows that

\[
[\gamma^{(j)}]_{r_j-1} = [x^{(j)}_k]_{r_j-1}, \quad \text{for} \ j = 1, \ldots, s.
\]

Therefore,

\[
\| x_k \ominus x_0 \|_b \leq \frac{1}{b^{\sum_{j=1}^{s} r_j}} \leq \frac{1}{b^{m+s}}.
\]

This is a contradiction to the assumption that \( x_k \) and \( x_0 \) are elements of a \((0, m, s)\)-net in base \( b \) because from Lemma 2.1 we know that \( \min_{x,y \in P} \| x \ominus y \|_b = \frac{1}{b^{m+s-1}} \). Hence, all \( J_{r,\gamma,g} \) where \( r \in A_3 \) are empty. Using the fact that \( |A_4| \geq \frac{m^{s-1}}{\beta} \), we then get

\[
\Delta_3 = \sum_{r \in A_3} \left( \frac{A(r,g)}{b^{m}} - \lambda_s(J_{r,\gamma,g}) \right)
\]

\[
= - \sum_{r \in A_3} \frac{1}{b^{\sum_{j=1}^{s} r_j}}
\]

\[
\leq - \sum_{r \in A_4} \frac{1}{b^{m+\alpha}}
\]

\[
\leq - \frac{m^{s-1}}{\beta} (b - 1)^s \frac{1}{b^{m+\alpha}}.
\]

Finally, we get the estimate

\[
\frac{1}{N} \Delta(\gamma, (x_n)_{0 \leq n < b^m}) = \Delta_1 + \Delta_2 + \Delta_3
\]

\[
\leq \left( \frac{1}{b^{m}} - \frac{1}{b^{m+s-1}} \right) \frac{m^{s-1}}{\delta} (b - 1)^s - \frac{m^{s-1}}{\beta} \frac{1}{b^{m+\alpha}}
\]

\[
= - \frac{m^{s-1}}{b^{m}} \left( \frac{(b - 1)^s}{\delta} b^{s-1} - 1 \right) + \frac{(b - 1)^s}{\beta} \frac{1}{b^{\alpha}} < 0
\]

for \( \delta > \frac{b^{\alpha}(b-1)^s}{b^{s-1}} \). \( \square \)
Subsequently, we now derive Theorem 4, which in some sense describes how dense possible choices of $\gamma$ are in $[0, 1)^s$.

**Proof of Theorem 4:**
Let $\Gamma$ be defined as the set, which contains all points of the form $\gamma = (\sum_{i=1}^{r_1} \frac{1}{b^{r_1}}, \ldots, \sum_{r_s} \frac{1}{b^{r_s}})$, where $r_i \in R_i \subseteq \{1, 2, \ldots, m\}$ for $i = 1, \ldots, s$ and the sets $R_i$ fulfill the following conditions:

- $|\{(r_1, \ldots, r_s)\mid m + 1 \leq \sum_{i=1}^{s} r_i < m + s\}| = 0$,
- $|\{(r_1, \ldots, r_s)\mid \sum_{i=1}^{s} r_i = m + s\}| \geq \frac{m^{s-1}(2s-3)^{s-1}}{(4s^2(s-1)^2)^{s-1}}$.

Consider now the $b$-adic digit expansion of some $x = (x^{(1)}, \ldots, x^{(n)}) \in [0, 1)^s$,

$x^{(i)} = \sum_{s_i \in S_i} a_{s_i} b^{s_i},$

where $S_i \subseteq \mathbb{N}$ is the set of indices for which we have $a_{s_i} \in \{1, 2, \ldots, b-1\}$ for $i = 1, \ldots, s$. Now we have to construct a point $\gamma$ with the following properties:

$$\|x - \gamma\| < b^{s} \frac{1}{b^{m/(m-1)}},$$

$$\gamma \in \Gamma,$$  

where $\Gamma$ is defined as above.

Let $\gamma = (\gamma^{(1)}, \ldots, \gamma^{(s)})$,

$$\gamma^{(i)} = \sum_{r_i \in R_i} a_{r_i} b^{r_i},$$

where

$$R_i = \{s_i \in S_i\mid s_i \leq k\} \cup T_i,$$

where $k := \left\lfloor \frac{m}{2(s-1)s} \right\rfloor$, and where $t_i \in T_i$ has the form

$$t_i = \left\lfloor \frac{m}{2s(s-1)} \right\rfloor + sj_i$$

for $i = 1, \ldots, s-1$ and $t_s \in T_s$ has the form

$$t_s = m - (s-1) \left( \left\lfloor \frac{m}{2s(s-1)} \right\rfloor + s\bar{m} \right) + sj_s.$$

Here, $j_1, \ldots, j_s \in \{1, \ldots, \bar{m}\}$ with

$$\bar{m} = \left\lfloor \frac{m(2s-3)}{2s^2(s-1)} \right\rfloor.$$

Moreover, we choose $a_{r_i} = a_{s_i}$ for all $r_i \in \{s_i \in S_i\mid s_i \leq k\}$ and otherwise, $a_{r_i} = 1$. By the choice of $S_i$ it then holds that $[x^{(i)}]_k = [\gamma^{(i)}]_k$ for all $i = 1, \ldots, s$. This implies that $x$ and $\gamma$ are contained in the same square elementary interval of order $sk$, i.e.,

$$x, \gamma \in \prod_{i=1}^{s} \left( \frac{A_i}{b^k} - \frac{A_i + 1}{b^k} \right).$$
for some $A_i \in \{0, 1, ..., b^k - 1\}$. Therefore, it holds that

$$
||x - \gamma|| < \sqrt{s} \frac{1}{b^k} \leq b\sqrt{s} \frac{1}{b^{2(s-1)}}.
$$

Hence, (3) is shown. It remains to check, whether the condition on $\gamma$, mentioned at the beginning of the proof, are satisfied, i.e., if $\gamma \in \Gamma$. Obviously, $R_i \subseteq \{1, 2, ..., m\}$ for all $i = 1, ..., s$.

To begin with, observe that for any $r_i \in R_i$, where $i = 1, ..., s - 1$, and for any $s_s \in S_s, s_s \leq k$ we have that

$$
\sum_{i=1}^{s-1} r_i + s_s \leq (s - 1) \left[ \frac{m}{2s(s-1)} \right] + \bar{m}s + k
$$

$$
\leq (s - 1) \left[ \frac{m}{2s(s-1)} \right] + m(2s - 3) \frac{s}{2s^2(s-1)} + \bar{m}s \leq m.
$$

Additionally, for any $s_1 \in S_1, s_1 \leq k$ and $r_i \in R_i$, where $i = 2, ..., s$ it holds that

$$
s_1 + \sum_{i=2}^{s} r_i \leq k + (s - 1) \left( \left[ \frac{m}{2s(s-1)} \right] + \bar{m}s \right) + \bar{m}s
$$

$$
\leq s \frac{m}{2s(s-1)} + (s - 1)s \frac{m(2s - 3)}{2s^2(s-1)} + s \frac{m(2s - 3)}{2s^2(s-1)} = m.
$$

Hence, we can conclude that

$$
\left| \{(r_1, ..., r_s) | \sum_{i=1}^{s} r_i > m, r_i \in R_i \} \right| = \left| \{(t_1, ..., t_s) | \sum_{i=1}^{s} t_i > m, t_i \in T_i \} \right|
$$

Therefore, let us consider $t_i \in T_i$ for $i = 1, ..., s$. We have that

$$
\sum_{i=1}^{s} t_i = m + s(j_1 + ... + j_s - (s - 1)\bar{m}) \neq m + s,
$$

because of the fact that $\bar{m} \in \mathbb{Z}$. It follows that

$$
\left| \{(r_1, ..., r_s) | m + 1 \leq \sum_{i=1}^{s} r_i < m + s \} \right| = 0.
$$

For the case $t_1 + ... + t_s = m + s$ it holds that

$$
j_s = 1 + (s - 1)\bar{m} - j_1 - ... - j_{s-1}.
$$

This implies that the following inequality must be fulfilled:

$$
1 \leq 1 + (s - 1)\bar{m} - j_1 - ... - j_{s-1} \leq \bar{m}.
$$
Obviously, the left inequality holds for any choice of $j_1, \ldots, j_{s-1}$. For the right inequality consider the case that $j_1 = \ldots = j_{s-1}$. Then we can conclude that it has to hold

$$j_1 \geq \left\lceil \frac{(s - 2)\bar{m}}{s - 1} \right\rceil + 1.$$ 

Hence, we obtain

$$\left|\{(r_1, \ldots, r_s) \mid \sum_{i=1}^{s} r_i = m + s\}\right| = \left|\{(t_1, \ldots, t_s) \mid \sum_{i=1}^{s} t_i = m + s\}\right|$$

$$= \left(\bar{m} - \left\lceil \frac{(s - 2)\bar{m}}{s - 1} \right\rceil\right)^{s-1}$$

$$\geq \left\lceil \frac{\bar{m}}{s - 1} \right\rceil^{s-1}$$

$$\geq m^{s-1}(2s - 3)^{s-1}$$

$$\geq \frac{m^{s-1}(2s - 3)^{s-1}}{(4s^2(s - 1)^2)^{s-1}}$$

by using the estimate

$$\left\lceil \frac{\bar{m}}{s - 1} \right\rceil = \left\lceil \frac{m(2s-3)}{2s^2(s-1)} \right\rceil \geq \frac{m(2s-3)}{4s^2(s-1)^2} \quad \text{for } m \geq \frac{2s^2(s-1)^2}{2s - 3}.$$ 

Thus, also (4) is shown. Now we finish the proof of Theorem 4. It remains to show the second item. Let $P = \{x_0, \ldots, x_{b^m-1}\}$ be a $(0, m, s)$-net in base $b$ for which some element $x_i$ belongs to the set $\Gamma$. Therefore, the conditions of Lemma 3.1 are satisfied with $\alpha = s, \beta = \left(\frac{4s^2(s-1)^2}{(2s-3)^{s-1}}\right)^{s-1}$ and for any $\delta > \frac{b^{(s-1)(4s^2(s-1)^2)^{s-1}}}{(2s-3)^{s-1}}$. By considering the limit $\delta \to \infty$ we obtain

$$\frac{1}{N}\Delta(\gamma, (x_n)_{0 \leq n < b^m}) \leq -\frac{m^{s-1}(b - 1)^s(2s - 3)^{s-1}}{b^m \cdot 4s^2(s - 1)^2},$$

and the assertion follows with $N = b^m$. \hfill \Box

## 4 Re-proof of Theorem 2

In the interest of clear presentation, the proof of Theorem 2 will be split into several auxiliary lemmas. The necessity of the following two results should be motivated. In a later step, we will define a special axes-parallel box $[0, y]$ and partition this multi-dimensional interval into several disjoint axes-parallel boxes (see, equation (5)). Lemma 4.1 and Lemma 4.2 show under which condition on $n$ a sequence element $H_s(n)$ of the Halton sequence is contained in one of these disjoint intervals.
Lemma 4.1. Define $x_i := \sum_{j=1}^{\infty} x_{i,j} b_i^{-j}$, $x_{i,j} \in \{0, 1, \ldots, b_i - 1\}$, and its truncation $[x_i]_r := \sum_{j=1}^{r} x_{i,j} b_i^{-j}$, for $i = 1, \ldots, s$, $r = 1, 2, \ldots$. Then, we have

$$\phi_b(n) \in [[x_i]_r, [x_i]_r + b_i^{-r}) \iff n \equiv \hat{x}_{i,r} \mod b_i^r, \text{ where } \hat{x}_{i,r} = \sum_{j=1}^{r} x_{i,j} b_i^{j-1}.$$ 

Proof. The result follows immediately from the definition of the Halton sequence. \hfill \Box

Lemma 4.2. For a vector $r = (r_1, \ldots, r_s)$ of positive integers, let $B_r := \prod_{i=1}^{s} b_i^{r_i}$, and the integer $M_{i,r}$, be defined such that $M_{i,r}(B_r b_i^{-r_i}) \equiv 1 \mod b_i^{r_i}$, then we have

$$\phi_b(n) \in [[x_i]_r, [x_i]_r + b_i^{-r_i}), \text{ for } i = 1, \ldots, s \iff n \equiv \hat{x}_{r} \mod B_r,$$

with $\hat{x}_{r} = \sum_{i=1}^{s} M_{i,r} B_r b_i^{-\tau_i} \hat{x}_{i,r_i}$. 

Proof. This follows immediately from Lemma 4.1 and the Chinese remainder theorem. \hfill \Box

In order to obtain further information about the discrepancy function $\Delta(\cdot, (H_s(n))_{n=1}^{N})$ of the Halton sequence, we will investigate this function for a special setting of the interval $[0, y)$ and thereby exploit the information gained by the previous lemmas. Accordingly, let $y_i$, $i = 1, \ldots, s$, be defined as

$$y_i := \sum_{j=1}^{m} b_i^{-j\tau_i}, \text{ with } \tau_i = \min\{1 \leq k < B^{(i)}|b_i^k \equiv 1 \mod B^{(i)}\},$$

where $m \in \mathbb{N}$, $m \geq B$ and $B^{(i)} = \frac{B}{b_i}$. If we consider, for instance, the two-dimensional Halton sequence in bases $b_1 = 2$ and $b_2 = 3$, we obtain $\tau_1 = 2$ and $\tau_2 = 1$.

Having gathered these tools, we put $[0, y) = [0, y^{(1)}) \times \ldots \times [0, y^{(s)}) \subset [0,1)^s$. The pertinence of introducing the integers $\tau_i$ will be revealed at a later step in Lemma 4.5. For a further analysis concerning $[0, y)$, it turns out to be beneficial to consider a disjoint partitioning of this interval. To achieve the goal of a disjoint decomposition, a truncation of the one-dimensional interval borders $y_i$, of the form $[y_i]_{\tau_i k_i} = \sum_{j=1}^{k_i} b_i^{-j\tau_i}$, $k_i \geq 1$, $i = 1, \ldots, s$, is taken into account. Collecting the integers $k_i$ in a vector $k = (k_1, \ldots, k_s)$ we arrive at

$$[0, y) = \bigcup_{1 \leq k_1, \ldots, k_s \leq m} P_k, \text{ with } P_k := \prod_{i=1}^{s} [[y_i]_{\tau_i k_i} - b_i^{-\tau_i k_i}, [y_i]_{\tau_i k_i}). \tag{5}$$

We apply Lemma 4.2 to the interval $P_k$ and obtain:

Lemma 4.3. An element $H_s(n)$ of the Halton sequence is contained in $P_k$ if and only if

$$n \equiv \sum_{i=1}^{s} M_{i,\tau \cdot k} B_{\tau \cdot k} b_i^{-\tau_i k_i} \hat{y}_{i,\tau_i (k_i - 1)} \mod B_{\tau \cdot k}, \text{ where } \hat{y}_{i,\tau_i k_i} := \sum_{j=1}^{k_i} b_i^{-\tau_i k_i - 1}. \tag{6}$$

Note, that $\tau = (\tau_1, \ldots, \tau_s)$ and the product $\tau \cdot k$ denotes the vector $(\tau_1 k_1, \ldots, \tau_s k_s)$.
A slight reformulation of relation (6) is required. Although, by the previous lemma, we have found a criterion for a sequence element to be contained in \( P_k \), key steps of the proof of Theorem 2 will be based on a congruence of the form

\[
\sum_{i=1}^{s} -M_{i,k}B_{\tau_k}b_i^{-1} \mod B_{\tau_k}, \quad A_k \in [0, B_{\tau_k}).
\]

This form is obtained as follows: We have

\[
\sum_{i=1}^{s} M_{i,k}b_i^{-1}\tilde{y}_{i,k}(k_i-1)
\]

\[
= \sum_{i=1}^{s} M_{i,k}b_i^{-1}\tilde{y}_{i,k} - \sum_{i=1}^{s} M_{i,k}b_i^{-1}
\]

\[
\equiv \sum_{i=1}^{s} M_{i,k}B_{\tau_k}b_i^{-1}\tilde{y}_{i,k}(m+1) - \sum_{i=1}^{s} M_{i,k}b_i^{-1}
\]

\[
\equiv: \tilde{y}_m + A_k \mod B_{\tau_k}.
\]

Here \( \tilde{y}_m \) is chosen such that \( \tilde{y}_m \in [0, B_{\tau_k(m+1)}) \). The first of the congruences above follows by elementary computations. We summarize:

\[
H_s(n) \in P_k \iff n \equiv \tilde{y}_m + A_k \mod B_{\tau_k}.
\]

Note that the multiplication \( \tau(m+1) \) has to be understood componentwise, i.e., we have \( \tau(m+1) = (\tau_1(m+1), \ldots, \tau_s(m+1)) \).

Employing the information received from Lemma 4.3, the equality

\[
\sum_{n=N_1B_{\tau_k}}^{(N_1+1)B_{\tau_k}-1} (\chi_{P_k}(H_s(n)) - B_{\tau_k}^{-1}) = 0,
\]

holds for any integer \( N_1 \geq 0 \), since amongst \( B_{\tau_k} \) consecutive integers the congruence of relation (6) has exactly one solution. Moreover, for an integer \( N_2 \in [0, B_{\tau_k}), \) we have

\[
\sum_{n=\tilde{y}_m+N_1B_{\tau_k}+N_2-1}^{\tilde{y}_m+N_1B_{\tau_k}+N_2-1} (\chi_{P_k}(H_s(n)) - B_{\tau_k}^{-1}) = \sum_{n\in[\tilde{y}_m,\tilde{y}_m+N_2]} (\chi_{P_k}(H_s(n)) - B_{\tau_k}^{-1}).
\]

(7)

Recalling that

\[
H_s(n) \in P_k \iff n \equiv \tilde{y}_m + A_k \mod B_{\tau_k} \iff \\
\exists \ l \in \mathbb{Z}, \text{ such that } n = lB_{\tau_k} + \tilde{y}_m + A_k, \quad \in [0, B_{\tau_k})
\]
the characteristic function in the sum (7) only has a nonzero contribution for \( n = \tilde{y}_m + A_k \), i.e., \( l = 0 \), since for all other values of \( l \), \( n \) does not belong to the interval \([\tilde{y}_m, \tilde{y}_m + N_2] \). Hence, these arguments enable to restate (7) by the expression

\[
\sum_{n \in [\tilde{y}_m, \tilde{y}_m + N_2], n = \tilde{y}_m + A_k} 1 - N_2 B_{\tau, k}^{-1} = \begin{cases} 
1 - N_2 B_{\tau, k}^{-1}, & 0 \leq A_k < N_2, \\
-N_2 B_{\tau, k}^{-1}, & \text{else,}
\end{cases} = \chi(0, N_2)(A_k) - N_2 B_{\tau, k}^{-1}.
\]

So far, we have constructed a special interval \([0, y]\), partitioned this box into subintervals and derived criteria to verify if some sequence element \( H_s(n) \) is contained in a fixed box \( P_k \). To make the star-discrepancy of the Halton sequence sufficiently large, we additionally have to construct infinitely many values for \( N \), which are bad in the sense that they yield (in combination with the special interval \([0, y]\)) a large discrepancy. The decisive idea is to show the existence of such \( N \), rather to give an explicit construction. This consideration is realised by taking a quantity \( \alpha_m \) into account, which represents the average of the discrepancy function, evaluated for the sequence elements \((H_s(n))_{n=\tilde{y}_m}^{\tilde{y}_m+N-1}\) for several different values of \( N \). Succeeding in showing that \(|\alpha_m| \geq c_s m^s \), with \( c_s > 0 \), would allow to conclude Theorem 2.

**Lemma 4.4.** Let

\[
\alpha_m := \frac{1}{B_{\tau, m}} \sum_{N=1}^{B_{\tau, m}} \Delta(y, (H_s(n))_{n=\tilde{y}_m}^{\tilde{y}_m+N-1}),
\]

then

\[
\alpha_m = \sum_{1 \leq k_1, \ldots, k_s \leq m} \left( \frac{1}{2} - \frac{A_k}{B_{\tau, k}} - \frac{1}{2 B_{\tau, k}} \right). \tag{8}
\]

**Proof.** We have

\[
\alpha_m = \frac{1}{B_{\tau, m}} \sum_{N=1}^{B_{\tau, m}} \Delta(y, (H_s(n))_{n=\tilde{y}_m}^{\tilde{y}_m+N-1})
= \sum_{1 \leq k_1, \ldots, k_s \leq m} \frac{1}{B_{\tau, m}} \sum_{N=1}^{B_{\tau, m}} \sum_{n=\tilde{y}_m}^{\tilde{y}_m+N-1} (\chi_{B_k}(H_s(n)) - B_{\tau, k}^{-1})
\]

The summands \( \alpha_{m,k} \) can be reformulated in the following way:

\[
\alpha_{m,k} = \frac{1}{B_{\tau, m}} \sum_{N=1}^{B_{\tau, m}} \sum_{n=\tilde{y}_m}^{\tilde{y}_m+N-1} (\chi_{B_k}(H_s(n)) - B_{\tau, k}^{-1})
= \frac{1}{B_{\tau, m}} \sum_{N_1=0}^{B_{\tau, m}/B_{\tau, k}-1} \sum_{N_2=1}^{B_{\tau, k}} \left( \sum_{n=\tilde{y}_m}^{\tilde{y}_m+N_1+B_{\tau, k}-1} (\chi_{B_k}(H_s(n)) - B_{\tau, k}^{-1}) \right)
= 0
\]
\[
\begin{align*}
&\hat{y}_m + N_1 B_{\tau,k} + N_2 - 1 \\
&\quad + \sum_{n=\hat{y}_m + N_1 B_{\tau,k}}^{\hat{y}_m + N_1 B_{\tau,k} + N_2 - 1} (\chi_{P_k}(H_s(n)) - B_{\tau,k}^{-1}) \\
&= \chi_{[0,N_2]}(A_k) - N_2 B_{\tau,k}^{-1} \\
&= \frac{1}{B_{\tau,m}} \sum_{N_1=0}^{B_{\tau,m}/B_{\tau,k} - 1} \sum_{N_2=1}^{B_{\tau,k}} (\chi_{[0,N_2]}(A_k) - N_2 B_{\tau,k}^{-1}) \\
&= \frac{1}{B_{\tau,k}} \left( \sum_{N_2=1}^{B_{\tau,k}} \chi_{[0,N_2]}(A_k) - \sum_{N_2=1}^{B_{\tau,k}} N_2 B_{\tau,k}^{-1} \right).
\end{align*}
\]

By virtue of the fact that \( A_k \in [0, B_{\tau,k}) \) the first sum of (9) is not vanishing and simplifies to \( B_{\tau,k} - A_k \). We therefore arrive at

\[
\alpha_{m,k} = \frac{1}{2} - \frac{A_k}{B_{\tau,k}} - \frac{1}{2 B_{\tau,k}},
\]

and consequently

\[
\alpha_m = \sum_{1 \leq k_1, \ldots, k_s \leq m} \left( \frac{1}{2} - \frac{A_k}{B_{\tau,k}} - \frac{1}{2 B_{\tau,k}} \right).
\]

**Lemma 4.5.** Let \( \alpha_m \) be defined as in the previous lemma. Then we have

\[
|\alpha_m| \geq c_s m^s, \text{ with } c_s > 0.
\]

**Proof.** For simplicity reasons, we will prove this lemma only for the two-dimensional Halton sequence in bases \( b_1 = 2 \) and \( b_2 = 3 \). The general case works analogously with a bit more technical effort. To estimate the absolute value of \( \alpha_m \) from below, we investigate the three occurring sums in (8) separately. We have

\[
\sum_{1 \leq k_1, k_2 \leq m} \frac{1}{2} = \frac{m^2}{2}.
\]

The definition of \( A_k \) gives

\[
\frac{A_k}{B_{\tau,k}} \equiv -\frac{2}{i=1} M_i B_{\tau,k} b_i^{-1} \mod 1,
\]

and therefore it is necessary to examine the expression \( M_i B_{\tau,k} b_i^{-1} \mod 1 \) in detail. According to the choice of the integer \( M_i B_{\tau,k} \) and \( \tau \), we obtain in our special case:

\[
M_{1,\tau,k} 3^{k_2} \equiv 1 \mod 2^{2k_1},
\]

hence

\[
M_{1,\tau,k} 3^{k_2} \equiv 1 \mod 2
\]

and consequently

\[
M_{1,\tau,k} \equiv 1 \mod 2.
\]

Further

\[
M_{2,\tau,k} 3^{k_1} \equiv 1 \mod 3^{k_2},
\]
hence
\[ M_{2, \tau \cdot k} 2^{2k_1} \equiv 1 \mod 3 \]
and consequently
\[ M_{2, \tau \cdot k} \equiv 1 \mod 3. \]
Combining this result with (10) yields
\[ \frac{A_k}{B_{\tau \cdot k}} \equiv -\frac{1}{b_1} - \frac{1}{b_2} = -\frac{1}{2} - \frac{1}{3} \mod 1 = 1 - \frac{1}{2} - \frac{1}{3} = \frac{1}{6}. \]
Summing up the reformulated addends of equation (8), gives
\[ |\alpha_m| = \left| m^2 \left( \frac{1}{2} - \frac{1}{6} \right) - \sum_{1 \leq k_1, k_2 \leq m} \frac{1}{2B_{\tau \cdot k}} \right| \geq c_2 m^2, \text{ with } c_2 > 0, \]
and \( m \) sufficiently large. \( \square \)

This estimate gives us the necessary tools to conclude Theorem 2.

Proof of Theorem 2:
From the definition of \( \alpha_m \) (see formulation of Lemma 4.4) and from Lemma 4.5 we conclude that for every \( m \) there is an \( N \) with \( 1 \leq N \leq B_{\tau m} \) such that
\[ \left| \Delta(y, (H_s(n))_{n=\tilde{y}_m+N-1}) \right| \geq c_s m^s. \]
Hence,
\[ \left| \Delta(y, (H_s(n))_{n=0}^{\tilde{y}_m-1}) \right| \geq \frac{c_s}{2} m^s \quad \text{and} \quad \left| \Delta(y, (H_s(n))_{n=0}^{\tilde{y}_m+N-1}) \right| \geq \frac{c_s}{2} m^s. \]
Assume, the second estimate holds (the other case is treated analogously) and set \( N_m := \tilde{y}_m + N \), i.e.,
\[ \left| \Delta(y, (H_s(n))_{n=0}^{N_m-1}) \right| \geq \frac{c_s}{2} m^s. \]
Now note that
\[ N_m = \tilde{y}_m + N \leq B_{\tau(m+1)} + B_{\tau m} \leq B^{3m(\tau_1+\ldots+\tau_s)}, \]
i.e.:
\[ m \geq \frac{\log N_m}{\log B^{3(\tau_1+\ldots+\tau_s)}}, \]
and therefore
\[ \left| \Delta(y, (H_s(n))_{n=0}^{N_m-1}) \right| \geq \frac{c_s}{2(\log B^{3(\tau_1+\ldots+\tau_s)})^s} (\log N_m)^s. \]
It can easily be argued that we can obtain infinitely many such \( N_m \), with this property and the result follows. \( \square \)
5 Proof of Theorem 5

The investigations of the current section are restricted to the two-dimensional Halton sequence in bases $b_1 = 2$ and $b_2 = 3$. In the following, we survey possible options to modify the intervals $[0, y^{(1)}]$ and $[0, y^{(2)}]$, and discuss whether these changes still allow to derive the estimate $|\alpha_m| \geq c_2 m^2$ or not. A way to obtain further possible values for $y^{(1)}$ or $y^{(2)}$ would be to remove some addends of the specification of $y^{(1)}$ or $y^{(2)}$, i.e., to consider for example

$$y^{(1)} = \sum_{j=1, j \neq l}^m 2^{-j \tau_1}, \text{ or } y^{(2)} = \sum_{j=1, j \neq l}^m 3^{-j \tau_2}, \text{ with } l \in \mathbb{N} \text{ and } 1 \leq l \leq m.$$ 

Recalling equation (8), the choice of the modified box $[0, y^{(1)}) \times [0, y^{(2)})$ would have the consequence that (8) amounts to

$$\alpha_m = \sum_{1 \leq k_1, k_2 \leq m} \left( \frac{1}{2} - \frac{A_k}{B_{\tau \cdot k}} - \frac{1}{2B_{\tau \cdot k}} \right).$$

Note, that all previous steps of the proof of Theorem 2 can easily be adapted to this modified choice of the axes-parallel box. Since $k_1$ only takes on $(m - 1)$ different values, we get

$$\alpha_m = \frac{1}{3} m(m - 1) - \sum_{1 \leq k_1, k_2 \leq m} \frac{1}{2B_{\tau \cdot k}}$$

and therefore we are still in the position to derive a lower bound for $|\alpha_m|$ of the form $c_2 m^2$. The next corollary focuses on the questions of how many addends can be removed from the representation of $y^{(1)}$ (or $y^{(2)}$).

**Corollary 5.1.** Let $\epsilon > 0$ and fix an $m > \hat{c}_2(\epsilon)$, with a sufficiently large constant $\hat{c}_2(\epsilon)$. If we remove at most $m(1 - \epsilon)$ addends from the representation of $y^{(1)}$ ($y^{(2)}$), while $y^{(2)}$ ($y^{(1)}$) remains unchanged, then we still have $|\alpha_m| \geq c_2(\epsilon) m^2$, with $c_2(\epsilon) > 0$.

Up to now we have only modified $y^{(1)}$ ($y^{(2)}$) and kept $y^{(2)}$ ($y^{(1)}$) unchanged. If we remove addends from the representation of $y^{(1)}$ and from the one of $y^{(2)}$, we obtain the following corollary.

**Corollary 5.2.** Let $\epsilon > 0$ and fix an $m > \hat{c}_3(\epsilon)$, with a sufficiently large constant $\hat{c}_3(\epsilon)$. If we remove at most $m(1 - \epsilon)$ addends from the representation of $y^{(1)}$ and $y^{(2)}$ then we still have $|\alpha_m| \geq c_3(\epsilon) m^2$, with $c_3(\epsilon) > 0$.

Based on these preliminary considerations, we will derive the following lemma, which states, that there are, in some sense, many feasible choices for the interval borders $y^{(1)}$ and $y^{(2)}$. 

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Lemma 5.1. Let \( m \) be sufficiently large (as in Corollary 5.2). Then, there is a set \( \Upsilon \subseteq [0,1)^2 \) with the following property: For all \( \mathbf{x} \in [0,1)^2 \) there exists a \( \mathbf{y} \in \Upsilon \) with

\[
\| \mathbf{x} - \mathbf{y} \| < \sqrt{\frac{8}{2m/2}}.
\]

Furthermore, for such a \( \mathbf{y} \), we have \( |\alpha_m| \geq c_2 m^2 \), with some constant \( c_2 > 0 \).

Proof. Let \( y^{(1)} = 0.010101\ldots01 \) in base 2, and \( y^{(2)} = 0.11\ldots1 \) in base 3, the original choice of the interval borders of the two-dimensional box \([0,y^{(1)}) \times [0,y^{(2)})\). We now consider modified interval borders of the form \( \tilde{y}^{(1)} = 0.a_1\ldots a_i0101\ldots01 \), with \( a_1,\ldots,a_i \in \{0,1\} \) and \( \tilde{y}^{(2)} = 0.b_1\ldots b_i11\ldots11 \), with \( b_1,\ldots,b_i \in \{0,1,2\} \). The question is of course, how large \( l_1 = l_1(m) \) and \( l_2 = l_2(m) \) can be chosen for a given \( m \), such that we still have \( |\alpha_m| \geq c_2 m^2 \) for this modified choice of the interval. The set \( \Upsilon \) is then defined as the set of all feasible choices of \((\tilde{y}^{(1)}, \tilde{y}^{(2)})\). Let \( \tilde{k}_1^{(i)} \) and \( \tilde{k}_1^{(i-1)} \leq l_1/2 \) be integers, for which \( a_{2\tilde{k}_1^{(i)}} = a_{2\tilde{k}_1^{(i-1)}} = 1 \). If one of the digits \( a_{2\tilde{k}_1^{(i-1)}+1},\ldots,a_{2\tilde{k}_1^{(i-1)}} \) is one, we split an interval of the form

\[
[[\tilde{y}^{(1)}]_{2\tilde{k}_1^{(i-1)}}, [\tilde{y}^{(1)}]_{2\tilde{k}_1^{(i)}})
\]

into the two disjoint intervals

\[
[[\tilde{y}^{(1)}]_{2\tilde{k}_1^{(i-1)}}, [\tilde{y}^{(1)}]_{2\tilde{k}_1^{(i)}} - 2^{-2\tilde{k}_1^{(i)}}) \land [[\tilde{y}^{(1)}]_{2\tilde{k}_1^{(i)}}, [\tilde{y}^{(1)}]_{2\tilde{k}_1^{(i)}}).
\]

Now, let \( \tilde{k}_2^{(i)} \leq l_2 \), be an integer, for which \( b_{\tilde{k}_2^{(i)}} = 2 \). Then, we split an interval of the form

\[
[[\tilde{y}^{(2)}]_{\tilde{k}_2^{(i)}}, [\tilde{y}^{(2)}]_{\tilde{k}_2^{(i)}})
\]

into the two disjoint intervals

\[
[[\tilde{y}^{(2)}]_{\tilde{k}_2^{(i)}}, [\tilde{y}^{(2)}]_{\tilde{k}_2^{(i)}} - 2 \cdot 3^{-\tilde{k}_2^{(i)}}, [\tilde{y}^{(2)}]_{\tilde{k}_2^{(i)}}, [\tilde{y}^{(2)}]_{\tilde{k}_2^{(i)}} \land [[\tilde{y}^{(2)}]_{\tilde{k}_2^{(i)}}, [\tilde{y}^{(2)}]_{\tilde{k}_2^{(i)}} - 2 \cdot 3^{-\tilde{k}_2^{(i)}}, [\tilde{y}^{(2)}]_{\tilde{k}_2^{(i)}}, [\tilde{y}^{(2)}]_{\tilde{k}_2^{(i)}}).
\]

We investigate the influence of this additional interval on the quantity \( \alpha_m \). Therefore, we consider the average of the discrepancy function for the interval

\[
J_1 = [[[\tilde{y}^{(1)}]_{2\tilde{k}_1^{(i-1)}}, [\tilde{y}^{(1)}]_{2\tilde{k}_1^{(i)}} - 2^{-2\tilde{k}_1^{(i)}}, [\tilde{y}^{(2)}]_{\tilde{k}_2^{(i)}}, [\tilde{y}^{(2)}]_{\tilde{k}_2^{(i)}} \times [0, \tilde{y}^{(2)}),
\]

i.e., we study:

\[
\tilde{c}_m^{(1)} = \frac{1}{B_{\tau m}} \sum_{N=1}^{B_{\tau m}} \left( \sum_{n=\tilde{y}_m} \sum_{n=\tilde{y}_m} \chi_{J_1}(H_2(n)) - N \lambda_2(J_1) \right).
\]
\[
\frac{1}{B_{\text{tr}}} \sum_{N=1}^{B_{\text{tr}}} \left( \sum_{n=g_n}^{g_{n+1} - 1} \chi_{J_1}(H_s(n)) \right) - \frac{B_{\text{tr}} + 1}{2} \left( \sum_{j=2k_l^{(l-1)} + 1}^{2k_l^{(l)} - 1} a_j \left( \sum_{i=1}^{l_2} \frac{b_l}{3^i} + \sum_{i=l_2+1}^{m} \frac{1}{3^i} \right) \right)
\]
\[
\geq \frac{1}{B_{\text{tr}}} \sum_{N=1}^{B_{\text{tr}}} \left( \sum_{j=2k_l^{(l-1)} + 1}^{2k_l^{(l)} - 1} \sum_{i=1}^{l_2} a_j b_l \left[ \frac{N}{2j3^i} \right] + \sum_{j=2k_l^{(l-1)} + 1}^{2k_l^{(l)} - 1} \sum_{i=l_2+1}^{m} a_j \left[ \frac{N}{2j3^i} \right] \right)
\]
\[
- \frac{B_{\text{tr}} + 1}{2} \left( \sum_{j=2k_l^{(l-1)} + 1}^{2k_l^{(l)} - 1} \sum_{i=1}^{l_2} a_j \left( \sum_{i=1}^{l_2} \frac{b_l}{3^i} + \sum_{i=l_2+1}^{m} \frac{1}{3^i} \right) - \sum_{i=1}^{l_2} b_l \right) \sum_{j=2k_l^{(l-1)} + 1}^{2k_l^{(l)} - 1} a_j
\]
\[
\geq (m - l_2) \sum_{j=2k_l^{(l-1)} + 1}^{2k_l^{(l-1)} - 1} a_j
\]
\[
\geq -2m \sum_{j=2k_l^{(l-1)} + 1}^{2k_l^{(l-1)} - 1} a_j.
\]

We get an analogue upper bound for \(\tilde{\alpha}_m^{(1)}\), by estimating \(\sum_{n=g_n}^{g_{n+1} - 1} \chi_{J_1}(H_s(n))\) with the expression
\[
\sum_{j=2k_l^{(l-1)} + 1}^{2k_l^{(l)} - 1} \sum_{i=1}^{l_2} a_j b_l \left( \left\lfloor \frac{N}{2j3^i} \right\rfloor + 1 \right) + \sum_{j=2k_l^{(l-1)} + 1}^{2k_l^{(l-1)} - 1} \sum_{i=l_2+1}^{m} a_j \left( \left\lfloor \frac{N}{2j3^i} \right\rfloor + 1 \right).
\]

To sum up, we get:
\[
\left| \tilde{\alpha}_m^{(1)} \right| \leq 2m \sum_{j=2k_l^{(l-1)} + 1}^{2k_l^{(l-1)} - 1} a_j.
\]
In total, all intervals of this form yield therefore a contribution of at most $l_1m$.

Studying the average of the discrepancy function for an interval of the form

$$J_2 = [0, \tilde{y}^{(1)}] \times \left[[\tilde{y}^{(2)}]_{k_2^{(1)}}, 3^{-k_2^{(1)}}, [\tilde{y}^{(2)}]_{k_2^{(2)}}\right),$$

we get, analogously to above, an additional contribution to $\alpha_m$ of at most $l_2m$. In total, we thus have, an contribution of the magnitude

$$m(l_1 + l_2).$$

Therefore, if $l_1 + l_2 < m$, we still can derive an estimate of the form $|\alpha_m| \geq c_2m^2$ for the modified box $[0, \tilde{y}^{(1)}] \times [0, \tilde{y}^{(2)}]$. Let now $m$ be given and $x = (x_1, x_2) \in [0, 1)^2$, arbitrary but fixed, where

$$x_1 = \sum_{i \geq 1}^\infty a_i \frac{2^i}{2^{i+1}}, \quad a_i \in \{0, 1\} \quad \text{and} \quad x_2 = \sum_{i \geq 1}^\infty b_i \frac{3^i}{3^{i+1}}, \quad b_i \in \{0, 1, 2\}.$$ 

Due to above considerations, we can find $y \in \Upsilon$, which satisfies

$$\|x - y\| < \sqrt{\left(\frac{1}{2^{\lfloor \frac{m}{2}\rfloor - 1}}\right)^2 + \left(\frac{2}{3^{\lfloor \frac{m}{2}\rfloor - 1}}\right)^2} < \sqrt{8 \frac{1}{2m/2}},$$

and also allows to derive $|\alpha_m| \geq c_2m^2$.

Based on the previous lemma, we are in the position to prove Theorem 5, which gives a lower bound for the discrepancy for a specific $N$ and not just for the average.

**Proof of Theorem 5:**

Fix an $m$, which satisfies the condition of Lemma 5.1 and recall $N_m = N + \tilde{y}_m$, as in the proof of Theorem 2. Consider now squares $Q_i \subseteq [0, 1)^2$ of side length $\frac{2\sqrt{3}}{3^{m/2}}$. Due to Lemma 5.1, we know that each such square contains elements of the set $\Upsilon$ (defined as in Lemma 5.1). We partition $[0, 1)^2$ into $\frac{2^m}{3^2}$ such squares $Q_i$. Choose, for each $Q_i$, $y_i \in Q_i \cap \Upsilon$. For some fixed $y_i$, we have

$$|\alpha_m(y_i)| \geq c_2m^2. \quad (11)$$

Let $c_2 > 0$ be small enough, such that this estimate holds for all other choices $y_j \in Q_j \neq Q_i$ as well.

Note, that we always have $|\alpha_m| \leq cm^2$ for a fixed constant $c > 0$, since

$$D^*(H_2(n)) \leq c \frac{(\log N)^2}{N}, \quad \text{for all} \quad N.$$ 

Now, we claim that the number of $N$s with $1 \leq N \leq B_{\tau m}$ and

$$\left|\Delta(y_i, (H_2(n)))_{n=1}^{N_m}\right| < \frac{c_2}{2}m^2$$

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is at most $\kappa B_{\tau m}$, with $\kappa := \frac{c - c_2}{c - c_2/2}$.

Suppose the number of $N$ with $1 \leq N \leq B_{\tau m}$ and

$$\left| \Delta(y_i, (H_2(n))_{n=1}^{N_m}) \right| < \frac{C_2}{2} m^2$$

would be larger than $\kappa B_{\tau m}$. Then, we would have

$$|\alpha_m(y_i)B_{\tau m}| < \kappa B_{\tau m} \frac{C_2}{2} m^2 + (1 - \kappa)B_{\tau m}cm^2 = c_2 B_{\tau m}m^2,$$

which is a contradiction to inequality (11).

Therefore, the number of $N$ with $1 \leq N \leq B_{\tau m}$ and

$$\left| \Delta(y_i, (H_2(n))_{n=1}^{N_m}) \right| \geq \frac{C_2}{2} m^2$$

is at least $(1 - \kappa)B_{\tau m} = \frac{c_2}{c - c_2} B_{\tau m}$.

To sum up, we have $\frac{2^m}{32}$ squares $Q_i$, and for each of them, we have identified $(1 - \kappa)B_{\tau m}$ distinct values for $N$, $1 \leq N \leq B_{\tau m}$, which give a sufficiently large discrepancy. Thus, in total we have identified $\frac{2^m}{32}(1 - \kappa)B_{\tau m}$ many $N$ and this implies that at least one of those $N$ is identified at least $\frac{2^m}{32}(1 - \kappa)$-times. Let $N_0$ be an $N$ with this certain multiplicity. Further, this means that there exist at least $\frac{2^m}{32}(1 - \kappa)$ distinct $y_i \in \cup_i Q_i \cap \Upsilon$, such that

$$\left| \Delta(y_i, (H_2(n))_{n=1}^{N^{(0)}_m}) \right| \geq \frac{C_2}{2} m^2,$$

where $N^{(0)}_m := N_0 + \tilde{y}_m$. Note, that the union of all squares $Q_i$ containing the $y_i$, with this property, forms the set $\Lambda_{N_0}$ and therefore $\lambda_2(\Lambda_N) \geq 1 - \kappa$. It remains to verify, that for all $x \in \Lambda_{N_0}$ there exists a $y \in [0, 1)^2$ having a distance less than $\sqrt{\frac{8}{N^{(0)}_m}}$. Since $1 \leq N_0 \leq B_{\tau m}$, the claim immediately follows by Lemma 5.1 and the estimate $\tilde{y}_m + B_{\tau m} < 2^m$.

**Remark 3.** We note, that the considerations of this section can also be adopted to an arbitrary dimension $s > 2$. For ease of notation, we have only presented them in the two-dimensional case for the bases $b_1 = 2$ and $b_2 = 3$.

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