Electronic glasses

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1. – Introduction

For more than 50 years our understanding of interacting electronic systems has been essentially based on the Fermi liquid theory. In this way of viewing electrons, the electron is essentially described by a (nearly free) plane wave excitation, with a well defined momentum and energy (see e.g. [1]). Most of the phenomena in solids can thus be understood in this wave description, with the crucial addition of the Pauli principle which embeds the antisymmetric nature of the total wavefunction of the system. The Pauli principle blocks states too deep in energy and ensures the existence of a Fermi surface. Such an approach has been extremely successful to explain quite remarkable phenomena such as superconductivity. It has been crucial also to interpret the subtle effects of disorder in solids. Indeed although the naive scattering of electrons on impurities can be understood in a semi-classical approach, more subtle effects like Anderson localization rests on the quantum nature of the particles and can be interpreted as wave interferences. Using both scaling theories and sophisticated field theoretical techniques, it is now known that free electrons are localized by disorder in one and two dimensions, whereas a mobility edge exists in three dimensions. [2, 3, 4, 5, 6, 7]

However when the interactions become strong this “wave” starting point can be a poor description of the system. Indeed as was described a long time ago by Wigner [8], if the Coulomb interactions dominate the wave functions of each particle becomes localized in space, instead of being a plane wave. The electrons form a crystal (Wigner crystal (WC)). In such a crystal phase our intuition based on the wave description of electrons fails. The wavefunction of the electrons is localized so the electrons become discernable particles by their position, and the Pauli principle plays a much smaller role.
Quantum fluctuations are still there however since the vibration modes of such crystal are still quantized. One is thus facing a novel phase of quantum systems with quite fascinating properties to explore. For example, for the pure system if the interactions are reduced the quantum fluctuations will increase and one can expect the crystal to melt. What is the nature of this melting and to which phase it melts is of course a crucial, and still an open question. Whether this is simply a Fermi liquid in which the particle have recovered their “plane wave” nature or whether some intermediate phase exists is yet to be understood. Needless to say one can also expect quite interesting effects of the disorder in such systems. In the crystal phase the disorder effects cannot be viewed as scattering of waves but rather as the pinning of this crystal on the random potential due to impurities. It is thus interesting to know how this pictures of a pinned crystal connects with the standard description of interacting disordered systems, adding interactions on the top of disordered free electrons [9, 10, 11]. From a theoretical point of view this is quite a challenging question. Most of the theoretical approaches used for free electrons either fail or become much more complicated when interactions are included which makes it more difficult to obtain unambiguous answers. Perturbative calculations or renormalization group calculations can be made for weak interactions. Unfortunately they scale to strong coupling, which leaves the question of the large scale/low energy physics still open. The crystal description thus provides and alternative way to study this quite complicated problem. Even in the case of the crystal the competition of repulsion between particles, that wants the crystal to be nicely ordered, and the impurities that want the particles of the crystal at random position is a quite difficult problem. As a result, physics has been mostly computed in the limit of very strong disorder where particles are pinned individually [12]. A summary of the situation is presented in Fig. 1.

In addition to this pure theoretical interest, there is also a direct experimental relevance. The progress of nanotechnology have made it possible to produce such quantum crystals. There are various situations where one can reach Wigner crystallization. In order to enhance the interactions effects compared to the kinetic energy it is necessary to reduce the electron density (kinetic energy is killed faster than interactions at low densities). Thus semiconductors where one can reach very low density of carriers have been prime candidates to observe Wigner crystallization. Two dimensional electron gas (2DEG) where the density can be controlled by an external gate are good candidates. Because the crystallization requires quite low densities, one can help it by killing the kinetic energy of the electrons by application of a very strong magnetic field. In that case the electrons describe cyclotron orbits and their kinetic energy is quenched helping the interactions. This allows to reach Wigner crystallization for higher densities than in the absence of a field. Thus the first signs of Wigner crystallization were observed in systems where the fractional quantum hall effect was seen, but at higher magnetic fields.

Some experimental data in these systems is shown in Fig. 2. This data prompts for an immediate question: in all these systems there is no direct evidence of the crystal structure. Some attempts have been made to image such systems but one still cannot resolve individual crystal sites [16]. This is at variance with classical crystals such as the
Fig. 1. – (top) combined effects of disorder $D$ and interactions (denoted loosely $U$) in $d = 3$. Whether an intermediate phase exists between the crystal phase and a standard Fermi liquid phase is an open question. (bot) same thing in two dimensions. In $d = 2$ it is unclear whether a metallic phase exists, or whether a true crystalline phase can exist. Nevertheless, for strong interactions a description in terms of a crystal is a much better starting point than a plane wave description of the electrons. Although on these pictures the phase diagram is written in terms of the interactions for simplicity, usually the density is varied or the kinetic energy is killed by application of a strong magnetic field in order to reach the crystal phase.
vortex lattice where good imaging techniques exists (see e.g. [17]), and thus where it is possible to directly “see” the crystalline order. Here one has really to infer the existence of the crystal from circumstantial evidence, i.e. from transport measurements (mostly) or some other indirect measurements (acoustic wave absorption, compressibility etc.). In order to know whether the transport experiments can be considered as a proof or not of the existence of the Wigner crystal is it thus specially important to have a reliable theory that allows to compute the transport properties. Such a task is far from being trivial given the complexity of the problem.
Quite recently systems with even lower densities could be achieved. In these systems a “metal-insulator” transition was observed. There is considerable debate whether the metallic phase really exists [18], but in my sense a more interesting question is whether or not the insulating phase (which is the low density phase) is a pinned Wigner crystal. Resistivity data is shown in Fig. 3. Finally and although I will not insist on these systems in these notes, Luttinger liquids can be viewed also as an electronic crystal. A detailed review of the properties of such systems and their connection with disordered electronic crystals can be found in [20].

There is thus considerable interest in understanding the properties of these electronic crystals, which is the subject I will present in these lectures. It is clear that I cannot expect to cover such a subject in these few pages, so I will just give the basic ideas and some references. From the theoretical side, the concepts and methods used are quite different from the methods used usually to tackle disordered electrons, and I will introduce some of them. I will discuss the minimal model needed to describe an electronic glass in Sec. 2. I will present some basic concepts for disordered crystals in Sec. 3 and discuss also some incorrect preconceived ideas on the physics of such systems. In order to give a more precise solution I will present two more sophisticated theoretical tools in Sec. 4: the replica method and a functional renormalization group. Some of the results for equilibrium transport will be examined in Sec. 5. Finally some conclusions and perspectives can be found in Sec. 6.

In complement and connection with the present notes there are three other recent
set of notes that present other aspects of disordered elastic systems and where further references to other reviews or original papers can be found. The notes \[17\] deal with classical crystals such as vortex lattices. \[20\] is a rather complete review on quantum problems. It also presents other techniques and discusses in details one dimensional systems. The present notes are complementary with this review (there is some overlap) in the sense that I will much more insist here on two dimensional electronic systems. \[21\] deals with the specific case of Wigner crystal, and discusses in much more detail the issues of compressibility and out of equilibrium dynamics that I will only briefly mention in these notes.

2. – Basic Description

2.1. Elastic description. – Starting from the full electronic Hamiltonian (fermions with interactions and disorder) is a near impossible task. In the crystal phase some simplifications can be made since the particles are now discernable by their position. This allows for a minimal phenomenological model to describe such a crystal \[22, 23, 24, 21\]: one assumes that the particles are characterized by an equilibrium position \(R^0_i\) and a displacement \(u_i\) relative to this equilibrium position. In order to define uniquely the displacement one should not have topological defects such as dislocations in the crystal. I will come back to this point in Sec. 5.2. From the original quantum problem one has to define the “particles” of the crystal. If the wavefunction is localized enough then one can indeed ignore the exchange between the various sites and thus define “particles” that have a size given by the extension of the wavefunction as indicated in Fig. 4. Of course the density fixes the lattice spacing \(a\). These two lengthscales (size of particle, lattice spacing) are independent and should be kept. One then has to deal with the vibration modes of this crystal. As usual one considers the interaction between particles

\[
H = \frac{1}{2} \sum_{i \neq j} V(R_i - R_j) = \frac{1}{2} \int d\mathbf{r} d\mathbf{r}' V(\mathbf{r} - \mathbf{r}') \rho(\mathbf{r}) \rho(\mathbf{r}')
\]

and expands to second order in the displacements. If the potential \(V\) is short range this gives the standard elasticity. Here one has to be more careful since \(V\) is the (long range) Coulomb interaction between the electrons. One has thus to distinguish the two modes presented in Fig. 5. The compression mode changes the smeared density at one point of the crystal, creating an excess or a deficit charge. Such charges interact through the long range part of the Coulomb interaction. Since the density is essentially \(\nabla u\) (I will be more precise later), this mode gives a contribution \(V(q)q^2 u^*_q u_q\) in the energy, where \(V(q)\) is the Fourier transform of the Coulomb interaction. Since \(V(r) \sim 1/r\), \(V(q) \sim 1/q\) in two dimensions. The compression mode thus has an elasticity going as \(q^2\). On the other hand the shear mode does not change the density (averaged over a couple of unit cells), and thus the long range part of the interaction does not couple to this mode. The coupling comes from the short range part (or if one prefers \(V(q \sim 1/a)\)). The shear mode has thus still an elasticity going as \(q^2\). Being more precise one can define the longitudinal
Fig. 4. – The three length characterizing the Wigner crystal. The size $l_c$ of the “particles” in the crystal (at low temperature it is essentially given by the extension of the wavefunction around the equilibrium position, at large temperatures it is controlled by the thermal fluctuations and is the Lindemann length), $a$ the lattice spacing is controlled by the density of particles, and the disorder is correlated over a length $r_f$. The inset shows the triangular structure of the Wigner crystal. Particles are labeled by an equilibrium position $R_i^0$ and a displacement $u_i$. (From [24])

and transverse displacements for a mode $q$ by

\begin{equation}
\vec{u}(q) = \vec{q} \frac{u_L(q)}{q} + (\vec{q} \wedge \vec{z})u_T(q)
\end{equation}

Fig. 5. – (left) Bulk compression mode. This mode changes the smeared density and is thus coupled to the long range nature of the interaction. (right) shear mode. This mode leaves the smeared density invariant and feels only the short range part of the interaction.
This leads to the action (see [24] for more details):

\[
S = \frac{1}{2} \int_{\mathbf{q}} \frac{1}{\beta} \sum_{\omega_n} u_L(q, \omega_n)(\rho_m \omega_n^2 + c_L(q)) u_L(-q, -\omega_n) + u_T(q, \omega_n)(\rho_m \omega_n^2 + c_T(q)) u_T(-q, -\omega_n) \\
+ \rho_m \omega_n \omega_n [u_L(q, \omega_n) u_T(-q, -\omega_n) - u_T(q, \omega_n) u_L(-q, -\omega_n)]
\]

\[
\int_{\mathbf{q}} \text{denote the integration over the Brillouin zone } \int_{BZ} d^2 \mathbf{q} (2\pi)^2 \text{ and the } \omega_n \text{ are the standard Matsubara frequencies. The } \omega_n^2 \text{ terms are coming from the } P^2/2m \text{ term in the Hamiltonian, and are the ones embedding the quantum nature of the crystal. For a quantum crystal it is impossible to have the particles at rest since both its position and position would be determined simultaneously. Thus even at } T = 0 \text{ quantum fluctuations exists. The third term in (3) comes from the Lorentz force, that exists if the particles are submitted to a magnetic field. The action corresponding to this force is simply the work of the force } \mathbf{F}_L \cdot \mathbf{u} \text{ where } \mathbf{F}_L = q \tau \mathbf{u} \wedge \mathbf{B}. \rho_m \simeq \frac{m}{\pi a^2} \text{ is the mass density and } \omega_c = eB/m \text{ the cyclotron frequency.}
\]

The precise coefficients \(C_{L,T}(q)\) are the elastic coefficients for the longitudinal and transverse modes respectively. These coefficients can be obtained from an expansion of the coulomb correlation energy of the WC in terms of the displacements [25, 26]. As I discussed \(C_L(q) \sim |q|\) and \(C_T(q) \sim q^2\). Keeping the lowest terms of the expansion one has

\[
C_L(q) = d|q| - bq^2 \\
C_T(q) = cq^2
\]

where \(c,d,b\) are positive constants.

2.2. Consequences for pure system. – For a pure system the consequences of the quadratic Hamiltonian (3) are easy to carry out. The eigenmodes of the system are obtained by doing the analytic continuation \(i\omega_n \rightarrow \omega + i\delta\). In the absence of magnetic field the longitudinal and transverse modes are decoupled. The longitudinal one has a dispersion

\[
\rho_m \omega_L^2 = C_L(q) \sim d|q|
\]

This is the standard plasmon dispersion in \(d = 2\). Because of the Coulomb repulsion it is extremely hard to excite this mode. The system is very rigid (much more than a normal elastic system) since it does not like to have its local density changed. This is not the case of the transverse mode which keeps a phonon-like dispersion

\[
\rho_m \omega_T^2 = C_T(q) \sim cq^2
\]

In the presence of a large magnetic field the two modes are mixed. It is easy to obtain the new eigenmodes by diagonalizing the two by two matrix in (3). This gives the modes...
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Table I. – q dependence of the eigenmodes in a Wigner crystal in the absence of magnetic field or for a very strong field.

| Mode    | zero field | High magnetic field |
|---------|------------|---------------------|
| ω−(q)  | ∝ q² (trans.) | ∝ \(2^{3/2}\omega_c\) |
| ω+(q)  | ∝ q (long.) | \(\sim \omega_c\) |

The “plasmon” mode is now pushed to the cyclotron frequency \(\omega_c\), while there is a low energy mode with a \(q^{3/2}\) dispersion. This low energy mode is the mode that was probed in the sound absorption experiment shown in Fig. 2.

For the crystal, the current is simply given by \(J = e\rho_0 \partial_t u\), making thus the conductivity very simple to compute since it is essentially the correlator of the displacements (up to a factor \(\omega^2\)). The conductivity is thus given by

\[
\sigma_{\alpha\beta}(\omega) = -(e\rho_0)^2 i(\omega + i\delta) \langle u_\alpha(q = 0, \omega_n) u_\beta(q = 0, \omega_n) \rangle_{\omega_n \rightarrow \omega + i\delta}
\]

In the absence of magnetic field the conductivity is simply

\[
\sigma_{xx}(\omega) = \frac{e^2 \rho_0}{m} \frac{i}{\omega + i\delta}
\]

The real part is thus just a \(\delta\)-function at zero frequency. This traduces the fact that the crystal can slide freely in response to an external force (the electric field). The pure quantum crystal is thus a perfect conductor. In the presence of a finite magnetic field, the conductivity becomes

\[
\sigma_{xx}(\omega) = \sigma_{yy}(\omega) = \frac{e^2 \rho_0}{m} \frac{-i(\omega + i\delta)}{-(\omega + i\delta)^2 + \omega_c^2}
\]

Thus the \(\delta\)-function peak is pushed at the finite frequency \(\omega_c\). The d.c. conductivity is zero. This is due to the fact that in presence of a magnetic field the electrons describe cyclotron orbits, and do not drift in response to a static electric field. Paradoxically the system is then an insulator. On the other hand the Hall conductivity is given by

\[
\sigma_{xy}(\omega) = \frac{e^2 \rho_0}{m} \frac{\omega_c}{-(\omega + i\delta)^2 + \omega_c^2}
\]

This gives for the Hall resistivity

\[
\rho_{xy}(\omega) = \frac{\sigma_{xy}}{\sigma_{xx}^2 + \sigma_{xy}^2} = \frac{m\omega_c}{e^2 \rho_0} = \frac{B}{e\rho_0}
\]

which quite remarkably is the classical Hall resistivity.
Of course these results are for the pure system only and the crucial question, in order to make contact with the experiments, is to determine how the disorder changes the above results.

2.3. Disorder. – The coupling of the crystal to the disorder writes

\[ S_{\text{dis}} = \int d\tau \int dr V(r) \rho(r, \tau) \]  

(12)

The potential \( V \) represents the impurities. In these systems there are usually two main sources of disorder. One is provided by donors/acceptors far from the 2DEG. They interact with the 2DEG via the long range Coulomb interaction (see e.g. [27]). Since they are far from the 2DEG this produces a smooth potential. Another source of disorder is provided by interface roughness [28]. This type of disorder has a wavelength which is comparable or shorter than the lattice spacing of the crystal, and is most likely the main source of pinning.

In the limit where the impurities are weak and the pinning is collective, the disorder can be modelled by a gaussian potential, (see [22, 17] for a more in depth discussion of this limit)

\[ V(r)V(r') = \Delta_{r_f}(r - r') \]  

(13)

\( \Delta_{r_f} \) is a delta-like function of range \( r_f \) which is the characteristic correlation length of the disorder potential (see Figure 4). The density reads

\[ \rho(r) = \sum_i \delta(r - R_0^i - u_i) \]  

(14)

where \( \delta \) is a \( \delta \)-like function of range \( l_c \) (see Figure 4) and \( u_i \equiv u(R_0^i) \). Since the disorder can vary at a lengthscale \( r_f \) \textit{a priori} shorter or comparable to the lattice spacing \( a \), the continuum limit \( u_i \to u(r) \), valid in the elastic limit \( |u_i - u_{i+1}| \ll a \) should be taken with care in the disorder term. This can be done using the decomposition of the density in terms of its Fourier components [29, 30]

\[ \rho(r) \simeq \rho_0 - \rho_0 \nabla \cdot u + \rho_0 \sum_{K \neq 0} e^{iK \cdot (r - u(r))} \]  

(15)

where \( \rho_0 \) is the average density and \( K \) are the reciprocal lattice vectors of the perfect crystal. This is a decomposition of the density in Fourier harmonics determined by the periodicity of the underlying perfect crystal as shown on Fig. 6. Let me examine the various terms. The first term is just the average density. The second one is the textbook expression of the density in terms of the displacements. It corresponds to a density averaged at a scale of a few lattice spacing (smeared density). This part of the density thus traduces local changes of the density due to compression modes similar to the ones in Fig. 5. The other terms describes change of the density at a lengthscale comparable or
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Fig. 6. – Various harmonics of the density. If one is only interesting in variations of the density at lengthscales large compared to the lattice spacing one has the standard “elastic” expression of the density in terms of the displacements. In a crystal however on has to consider variations of density at lengthscales smaller than the lattice spacing and higher harmonics are needed [29, 30].

\[ \rho(x) = \rho_0 - \rho_0 \nabla u(x) + \rho_0 \sum_K e^{iK(x-u(x))} \]

+ smaller than the lattice spacing. The sum over the reciprocal lattice vectors \( K \) obviously reproduces the \( \delta \) function peaks of the density (14). The finite range of \( \delta \) is recovered [30] by restricting the sum over \( K \) to momentum of order \( K_{\text{max}} \sim \pi/l_c \). This expression of the density is quite useful since it allows to express the density at all scales in term of the slowly varying function \( u(r) \). The price to pay is the more complicated form of the operator that embeds in its very form the information about the periodicity of the crystal and the size of the particles. This is a quite useful trick, initially introduced for Luttinger liquids [31]. One keeps only the low energy-long wavelength part of the displacement field, since this field has a well defined continuum limit, and thus a quite simple continuum Hamiltonian (elastic Hamiltonian in this case). But it is still possible using this field that varies at a lengthscale large compared to the lattice spacing (this is the essence of the elastic limit) to describe phenomena that vary at a much shorter lengthscale, provided that one uses the proper form for the operator in terms of this continuum field. This expression of the density allows to write quite simply the coupling to disorder. Using the representation of density (15) it is easy to show [22, 20] that a Luttinger liquid is indeed very similar to the quantum crystals that are described here.

Using the expression of the density (15) one sees that different Fourier component of the disorder play different roles. The Fourier components close to \( q \sim 0 \) couple to \( \nabla u \). These terms play little role (see [20] for more details). In our case because the compression modes are quite stiff thanks to the Coulomb repulsion this is even more so. The main effect of the disorder comes from the other Fourier modes, i.e. the ones that are close to \( q \sim K \) where \( K \) is one of the vectors of reciprocal lattice of the crystal. The effect of the disorder can thus be written as

\[ S_{\text{dis}} \sim \sum_{K \neq 0} \int dr V(r) \rho_0 e^{iK(r-u(r))} \]
The quantum crystal can also be viewed as a periodic array of lines that get pinned by columnar disorder, i.e., by disorder that is independent of one of the coordinates $\tau$ which is the imaginary time for the quantum problem. The lines are the space-time trajectories of the particles. Although the action can be computed with the imaginary time $\tau$, to obtain the physical response for the quantum problem an analytic continuation is needed. This analytical continuation has an impact on many physical quantities, so the analogy between the classical system and the quantum one should be exerted with care.

The coupling to the displacements is highly nonlinear which of course makes this problem very difficult to solve. The action (12) and (3) is the minimal model needed to describe a disordered quantum crystal.

For a quantum problem, the disorder is time independent. Thus if one looks at the space-time trajectories of the particles, as shown in Fig. 7, one can view the system as a classical system of lines in $d + 1$ dimensions $[22, 20]$. These lines are pinned by columnar defects, i.e., defects independent of one of the coordinates ($\tau$ in this case). The problem of lines pinned by columnar defects has been studied extensively in the context of vortex lattices $[32, 33]$, where in that case the defects are produced by heavy ions irradiation. Thus some features of the two systems are indeed similar and one can gain considerable insight by thinking in term of the classical system of lines. There is however an important difference. In the classical system $\tau$ is simply one spatial direction, one is thus looking at the static of such systems. The dynamics of classical systems is a different problem, where an additional time $t$ needs to be introduced (see e.g. $[17, 34]$). For the quantum problem, $\tau$ is the imaginary time of the problem. Thus $\tau$ dependence of the quantities is describing the equilibrium dynamics of the quantum problem. In order to get the proper quantum dynamics however one should make the analytical continuation $\tau \rightarrow it$ (or more precisely $i\omega_n \rightarrow \omega + i\delta$). This analytical continuation, although seemingly innocent has severe consequences and changes many behaviors. Though although some analogy between the static (but $z = \tau$ dependent quantities such as e.g. the tilt of the lines) behavior of classical crystals with columnar defects and the dynamics behavior of quantum crystals can be made, one should exert great care because of the analytic continuation needed for the quantum problems, which ultimately leads to some differences in physics.
3. Basic description and preconceived ideas

3.1. Relevance of disorder and basic length scales. In order to define the basic properties of such systems let me consider first what happens in the ground state of the problem. Since the disorder is time independent, one can consider all displacements as time independent to get the ground state (it corresponds to the mode $\omega_n = 0$). The problem is thus equivalent to a classical system in $d$ dimensions. The time integration is just providing a factor $\beta$ in front of the action and the problem is equivalent to determine the partition function of a system

$$Z = \int \mathcal{D}u[r] e^{-\beta H_{cl}}$$

where

$$H_{cl} = \frac{c^2}{2} \int d^d r (\nabla u)^2 + \rho_0 \int V(r) \rho(r)$$

I have forgotten for simplicity in the above formula the difference between longitudinal and transverse modes and simply rewritten everything with a standard elasticity (this corresponds in fact to the transverse mode with an elastic constant $C_T(q) \sim q^2$). It is thus enough to examine a classical crystal, in presence of point like defects (for more details see [17, 34]).

In order to understand the basic physics of such problem a simple (but groundbreaking!) scaling argument was put forward by Larkin [35]. To know whether the disorder is relevant and destroys the perfect crystalline order, let me assume that there exists a characteristic lengthscale $R_a$ for which the relative displacements are of the order of the lattice spacing $u(R_a) - u(0) \sim a$. If the displacements vary of order $a$ over the lengthscale $R_a$ the cost in elastic energy from (18) is

$$\frac{c^2}{2} R_a^{d-2} a^2$$

by simple scaling analysis. Thus in the absence of disorder minimizing the energy would lead to $R_a = \infty$ and thus to a perfect crystal. In presence of the disorder the fact that displacements can adjust to take advantage of the pinning center on a volume of size $R_a^d$ allow to gain some energy. Since $V$ is random the energy gained by adapting to the random potential is the square root of the potential over the volume $R_a^d$, thus one gains an energy from (18)

$$-V R_a^{d/2} \rho_0$$

Thus minimizing (19) plus (20) shows that below four dimensions the disorder is always relevant and leads to a finite lengthscale

$$R_a \sim a \left( \frac{c^2 \rho_0}{V^2 \rho_0} \right)^{1/(4-d)}$$
at which the displacements are of order $a$. The conclusion is thus that even an arbitrarily weak disorder destroys the perfect positional order below four spatial dimensions, and thus no disordered crystal can exist for $d \leq 4$. This is an astonishing result, which has been rediscovered in other context (for charge density waves $R_a$ is known as Fukuyama-Lee \cite{36} length and for random field Ising model this is the Imry-Ma length \cite{37}). Of course it immediately prompt the question of what is the resulting phase of elastic system plus disorder?

Since solving the full problem is tough another important step was made by Larkin \cite{35,38}. For small displacements he realized that (18) could be expanded in powers of $u$ leading to the simpler disorder term

\begin{equation}
H_{\text{larkin}} = \int d^d r f(r) u(r)
\end{equation}

where $f(r)$ is some random force acting on the vortices. Because the coupling to disorder is now linear in the displacements the Larkin Hamiltonian is exactly solvable. Taking a local random force $f(r)f(r') = \Delta \delta(r-r')$ gives for the relative displacements correlation function and structure factor

\begin{equation}
B(r) = \langle [u(r) - u(0)] \rangle = B_{\text{thermal}}(r) + \frac{\Delta}{c^2} r^{4-d}
\end{equation}

where $\langle \rangle$ denotes the thermodynamics average and $\overline{\cdots}$ denotes the disorder average. $B_{\text{thermal}}$ is the same correlation function in the absence of disorder due to thermal fluctuations (which remain bounded in $d > 2$ and are thus negligeable at large distance compared to the disorder term). Thus the solution of the Larkin model confirms the scaling analysis: (i) displacements do grow unboundedly and thus perfect positional of the crystal is lost; (ii) the lengthscale at which the displacements are of the order of the lattice is the similar to the one given by the scaling analysis. In addition the Larkin model tells us how fast the positional order is destroyed: the displacements grow as power law thus one can expect the positional order to be destroyed exponentially fast. However these two conclusions should be taken with a serious grain of salt. Indeed the Larkin model is an expansion in powers of $u$, and thus cannot be valid at large distance (since the displacements grow unboundedly the expansion has to break down at some lengthscale). What is this characteristic lengthscale? A naive expectation is that the Larkin model cease to be valid when the displacements are of order $a$ i.e. at $r = R_a$. In fact this is too naive as was understood by Larkin and Ovchinikov. To understand why, in a transparent way, let me use the expression for the density (15) This immediately allows to reproduce the Larkin model by expanding the disorder term (18)

\begin{equation}
\rho_0 \int d^d r \sum_{K} e^{iK(r-u(r))} V(r)
\end{equation}

in powers of $u$. Clearly the expansion is valid as long as $K_{\text{max}} u \ll 1$ This will thus be valid up to a lengthscale $R_c$ such that $u(R_c)$ is of the order of the size of the particles $l_c$. 
Note that this lengthscale is different (and quite generally smaller) than the lengthscale $R_a$ at which the displacements are of the order of the lattice spacing $a$. The Larkin model cease to be valid way before the displacements become of the order of $a$ and thus cannot be used to deduce the behavior of the positional order at large length scale. In addition it is easy to check that because the coupling to disorder is linear in the Larkin model, this model does not exhibit any pinning, since the model is invariant by changing $u(r) \rightarrow u(r) + C$ where $C$ is an arbitrary constant (remember that the total of the random forces has to be zero $\int dr f(r) = 0$). Thus sliding by an arbitrary amount does not cost any energy and any addition of an external force leads to a sliding of the vortex lattice. Thus the lengthscale at which this model breaks down is precisely the lengthscale at which pinning appears [38]. The lengthscale $R_c$ is thus the lengthscale above which various chunks of the system are collectively pinned by the disorder. A simple scaling analysis on the energy gained when putting an external force

\begin{equation}
H = \int d^d r F_{\text{ext}} u(r)
\end{equation}

allows to determine the critical force needed to unpin the lattice. Assuming that the critical force needed to unpin the lattice is when the energy gained by moving due to the external force is equal to the balance of elastic energy and disorder $\frac{1}{2} \epsilon_c^2 R_c^{d-2}$, one obtains for the critical force needed to move the lattice

\begin{equation}
F_c \propto \frac{\epsilon_c}{R_c^2}
\end{equation}

This is the famous Larkin-Ovchinnikov relation which allows to relate a dynamical quantity (the critical force at $T=0$ needed to unpin the lattice) to purely static lengthscales, here the Larkin-Ovchinnikov length at which the displacements are of the order of the size of the particle. Let me insist again that this lengthscale controlling pinning is quite different from the one $R_a$ at which displacements are of the order of the lattice spacing at that controls the properties of the positional order.

These two lengthscales $R_c$ and $R_a$ are thus the crucial ones in controlling the properties of a pinned classical crystal, and thus will of course be crucial for our quantum crystals. For the quantum crystals this leaves us with two main questions: (i) what is the conductivity of the system, and which length controls it; (ii) what happens beyond the length $R_a$.

3.2. Conventional wisdom and dynamics. – Up to know we have been on relatively firm ground, but answering the above questions, even for a classical crystal is a tough cookie. Given the difficulty in obtaining the answers, instead of a solution people have been happy for a long time with a kind of consensus on how the system “should” behave, based on physical intuition and some incomplete, or in some cases incorrect proofs. I will give in this section a summary of these preconceived ideas on disordered crystals, since they were at the root of most of the physical solutions given for the dynamics.
Fig. 8. – The (incorrect) physical image that was the commonly accepted view of what a disordered elastic system would look like. The crystal would be broken into crystallites of size $R_a$ by the disorder. Dislocations would be generated by the disorder at the same lengthscale.

Fortunately recent progress have made it possible to truly analyze these systems. As usual the true behavior turned out to be quite different from what was naively believed. I will examine these solutions in the next section. But for the moment let me go along and describe the (incorrect) picture of a disordered crystal, that was used for a long time.

The first point is to naturally assume that because there exists a lengthscale $R_a$ for which displacements are of the order of the lattice spacing, positional order is destroyed exponentially fast beyond this lengthscale. This behavior would be in agreement with the too naive extrapolation of the Larkin model to arbitrary length scales and is in agreement with exact solutions of interface problems in random environments (so called random manifold problems) and solutions in one spatial dimension. It was thus quite naturally assumed that an algebraic growth of displacements was the correct physical solution of the problem, and thus that the positional order would be destroyed exponentially beyond the length $R_a$. This led to an image of the disordered vortex lattice that consisted of a crystal “broken” into crystallites of size $R_a$ due to disorder. To reinforce this image (incorrect) “proofs” were given [39] to show that due to disorder dislocations would be generated at the lengthscale $R_a$ (even at $T = 0$) further breaking the crystal apart and leaving no hope of keeping positional order beyond $R_a$. A summary of this (incorrect) physical image is shown on Fig. 8.

If one assumes that all positional order is lost, this allows to solve for the dynamics of the problem. In that case one can consider that each crystallite is pinned independently. When one tries to move it it sees a pinning potential that one can approximate by a harmonic well. The shape of the well depends on the size of the crystallite. If the system is submitted to an electric field at frequency $\omega$, each crystallite will respond as a particle in a well. Crudely speaking the minimal frequency one can excite is (using $\omega = cq$ and $q_{\text{min}} \sim 1/R_a$) is

$$\omega_{\text{pin}} \propto \frac{c}{R_a}$$

(27)

The perfect conductivity of the pure crystal, i.e. the $\delta$-function peak that existed at
Fig. 9. – In an image where the system is broken into crystallites of size $\rho_a$, one can imagine that each crystallite is experiencing a pinning force and is in an energy well. There is thus a natural pinning frequency, $\omega_p$ which is controlled by $\rho_a$. The conductivity has thus a peak at this frequency. Coupling between crystallites and distribution of pinning forces leads to a broadening of this pinning peak, very often chosen as lorentzian for lack of a better function.

Zero frequency is now pushed to the finite frequency $\omega_{pin}$, as shown in Fig. 9. If there is a finite magnetic field the peak at $\omega_c$ is essentially unchanged and there is a second peak that appears at $\omega_{pin}$. Of course the approximation of independent crystallites is not exactly true and the coupling between the crystallites as well as the distribution of pinning strength leads to a broadening of the peaks. The broadening is hard to determine precisely since one does not know exactly the static solution, so quite often a lorentzian broadening is assumed. This physical image inspired from pioneering theories [36] used for charge density waves allows to compute, using some approximations, the optical conductivity (see e.g. [20] for further details on that point). This gives specific predictions, in particular the density dependence of the pinning frequency would be from the above mentioned approximations $\omega_p \propto n^{1/2}$.

Measuring the optical conductivity is thus a very powerful way to check for the presence of a pinned Wigner crystal. An example of data is shown in Fig. 10. One clearly observes a pinning peak in the optical conductivity. However there are many problems. First the broadening is far from being simply lorentzian as is obvious when simply looking at the data. A much more serious problem comes from the density dependence of the pinning frequency that can be extracted from the above data and is shown in Fig. 11. When one compares the experiments, with these theoretical predictions the agreement is qualitatively wrong. The predicted result would be totally opposite to the data which shows a decrease of the pinning frequency with the density. Such important problems when one tries to compare with the data could cast serious doubts on the interpretation of the
Fig. 10. – Optical conductivity for various densities for a 2DEG under strong magnetic field. The peak at a characteristic frequency (pinning frequency) is an expected characteristics of a pinned crystalline structure. The density and magnetic field dependence of the pinning peak provide a stringent test for the existence of a pinned Wigner crystal phase (from [40]).

Fig. 11. – Variation of the pinning frequency with the density for a 2DEG under a strong magnetic field. The pinning frequency decreases with the density in all systems (different symbols), in contradiction with the naive calculations based on the physical image shown in Fig. 8. Such calculations would lead to $\omega_p \propto n^{1/2}$, in qualitative disagreement with experiments. (From [40])
insulating phase in terms of a Wigner crystal and quite naturally other interpretations for this phase have been proposed [41].

In fact, the reason of this discrepancy is that the phenomenological image of nearly independent crystallites is simply plain wrong. In order to be able to compute the transport properties we thus need sharper tools to solve the problem of pinned quantum crystals. These are the tools I now present.

4. – Methods

Let us see which tools of our theoretical toolbox one can use to tackle a disordered problem. If we put a random term in our Hamiltonian, then every observable depends on the specific realization of the random potential. What is needed is then to take an average of these observables with respect of the probability distribution of the disorder to get the average response of the system. This is of course a theoretical trick. A real experimental system has usually only one realization of the disorder, and the self-averaging comes from the fact that the system is large enough so that little pieces of the system more or less behave as independent sub-systems. Solving the problem for a given specific realization of the disorder and averaging afterwards is of course a totally impossible program. In addition of being impossible it is in general stupid, since a given realization of the disorder breaks many symmetries of the system. Since a given realization of the disorder is not invariant by translation, all correlation functions depend now on both coordinates $x$ and $x'$ at which they are computed and not simply on the difference $x - x'$ as in a translationally invariant system. On the other hand the averaged correlation function is invariant by translation so it is much simpler. Of course averages can be done order by order in a perturbation theory, but if one wants to go beyond perturbation one wants a method to average over disorder from the start. Unfortunately many techniques, that were useful for the non interacting systems (such as supersymmetry [7]) fail when interactions are included. I present here a quite general method that still works and is known as the replica method. The other useful method is the dynamical method (so called Keldysh technique) but it is of a more complex use. For a recent review on dynamical methods see [42]. This method is specially important if one wants to do out of equilibrium dynamics in quantum systems since this is the only one that can deal with this question.

Let me assume that we want to compute the average value of some observable $O$ for a system with a random potential $V$. The average value can be written as a functional integral over the configurations of the system as

$$\langle O \rangle_V = \frac{\int D\phi O(\phi)e^{-S_V(\phi)}}{\int D\phi e^{-S_V(\phi)}}$$

(28)

where $S_V(\phi)$ is the action of the system for a given realization of the random potential $V$. Of course $\langle O \rangle_V$ depends on $V$ itself, so we have to average over all realizations of $V$. If we assume that the disorder has a probability distribution $p(V)$ the average over
disorder is

\[ \langle O \rangle = \frac{\int D V p(V) \langle O \rangle_V}{\int D V p(V)} \]

In general \( S_V(\phi) \) is linear in \( V \), something like (this is indeed the case in the disordered crystal problem)

\[ S_V = S_0(\phi) + \int dx d\tau V(x) A(\phi(x, \tau)) \]

Note that for a quantum problem the disorder is time independent. For the disorder one takes in general a gaussian disorder. This is very often justified by the central limit theorem. For example using a distribution

\[ p(V) = e^{-\frac{1}{2D} \int dx V(x)^2} = e^{-\frac{1}{2D} \sum_q V_q^* V_q} \]

corresponds to the average

\[ \overline{V(x)V(x')} = D\delta(x - x') \]

With the distribution (31) it would be very easy to perform the average (29) if it were not for the denominator in (28). Indeed in the absence of such denominator one has

\[ e^{\frac{1}{2D} \int dx V(x)^2} \int D V e^{-\int dx\int d\tau V(x) A(\phi(x, \tau))} = \]

\[ e^{\frac{D}{2} \int dx \int d\tau \int d\tau' A(\phi(x, \tau)) A(\phi(x, \tau'))} \]

One would end up with an effective action where the disorder has been eliminated and has given after average an interaction term so the action would be

\[ S_{\text{eff}} = S_0(\phi) - \frac{D}{2} \int dx \int d\tau \int d\tau' A(\phi(x, \tau)) A(\phi(x, \tau')) \]

we could then treat this problem using our favorite method since it would not be more complicated than the type of problems that we already encountered in this book.

The idea of the replica method [43] is thus to get rid of the denominator and to transform it into a numerator (see the lecture notes by I. V. Lerner in this volume, for the use of replicas for calculation of the partition function). As with any really great method the idea is very simple. One can rewrite

\[ \frac{1}{\int D \phi e^{-S_V(\phi)}} = \left[ \int D \phi e^{-S_V(\phi)} \right]^{n-1} \]
with \( n = 0 \). If we forget \( n = 0 \) for a moment and consider \( n \) as a positive integer \( n = 2, 3, 4, \ldots \), then

\[
\left[ \int D\phi e^{-S_V(\phi)} \right]^{n-1} = \left[ \int D\phi_2 e^{-S_V(\phi_2)} \right] \cdots \left[ \int D\phi_n e^{-S_V(\phi_n)} \right]
\]

where we have introduced the fields \( \phi_2, \phi_3, \ldots \). The denominator can thus be rewritten as the product of \( n-1 \) copies. The average (28) can thus be rewritten

\[
\langle O \rangle_V = \left[ \int D\phi_1 O(\phi_1) e^{-S_V(\phi_1)} \right] \left[ \int D\phi_2 e^{-S_V(\phi_2)} \right] \cdots \left[ \int D\phi_n e^{-S_V(\phi_n)} \right]
\]

\[
= \int D\phi_1 D\phi_2 \cdots D\phi_n O(\phi_1) e^{-\sum_{a=1}^n S_V(\phi_a)}
\]

(37)

There is no denominator anymore. The price to pay is the introduction of \( n \) copies of the system. Of course one would only recover (28) if one can take the limit \( n \to 0 \) at the end. Before averaging over disorder in (37) all replicas (copies) are independent. Since there is no denominator in (37) one can do the average over disorder in the manner described above. One thus finds

\[
\langle O \rangle = \int D\phi_1 D\phi_2 \cdots D\phi_n O(\phi_1) e^{-S_{\text{eff}}}
\]

where the effective action is now

\[
S_{\text{eff}} = \sum_{a=1}^n S_0(\phi_a) - \frac{D}{2} \sum_{a,b=1}^n \int dx \int d\tau \int d\tau' A(\phi_a(x, \tau)) A(\phi_b(x, \tau'))
\]

(39)

This is nearly the same form except that now one has \( n \) fields and the interaction couples all fields together. We have thus traded a disordered system with only one field for a clean interacting problem of \( n \) coupled fields. Of course the second one is more complicated due to the presence of the \( n \) fields, but as explained before we should be more equipped to tackle it. One important difficulty is of course that we should obtain a good enough solution for any \( n \) to be able to make the analytic continuation to \( n \to 0 \) at the end, since it is only in this limit that one recovers the disordered solution. Taking this limit is far from being obvious and contains hidden difficulties that I will briefly allude to later in this section.

If we use this method we end up with an action \( S = S_{\text{el}} + S_{\text{dis}} \) where \( S_{\text{el}} \) is the elastic part of the action and the disorder part is

\[
S_{\text{dis}} = -\rho_0^2 \Delta \sum_{a,b=1}^n \sum_{K} \int dx d\tau d\tau' \cos(K(u_a(r, \tau) - u_b(r, \tau')))
\]

(40)
The term (40) is of course extremely complicated to solve. To tackle such a term two paths are available. The first one rests on the hope that the properties of the system are dominated by small fluctuations around the minimum of one of the cosine. One would thus hope that expanding the cosine is relatively accurate. In fact things are more complicated because of the presence of replicas. The second possible way is to tackle the problem as a standard critical problem and try to find renormalization equations for the disorder term. Here again the situation is more complicated than one would naively think. I will examine these two methods in the following.

### 4.1. Variational method

The first method is to try to reduce this problem to the “best” quadratic problem approximating \( S_{\text{el}} + S_{\text{dis}} \). The most brutal way would be to simply expand the cosine, but as is well known for sine-Gordon problems this is very often a poor approximation. Here is would be a complete catastrophe for reasons that will become clear later. An improved way is to perform a self consistent calculation. The simplest way to implement such a scheme it through Feynman’s variational approach [44]. The method has been extended to deal with replicas in [45]. Let me define a trial action \( S_0 \). Then the free energy \( F \) of the system verifies

\[
F \leq S_{\text{var}} = F_0 + \langle S - S_0 \rangle_0
\]

where \( \langle \rangle_0 \) denotes an average taken with the action \( S_0 \). If \( S_0 \) depends on some variational parameter one can hope that the “best” way (i.e. the one giving a physics as close as possible to the one of the true action \( S \)) of choosing this parameter is to minimize \( S_{\text{var}} \).

Let us take for \( S_0 \) the most general quadratic action

\[
S_0 = \frac{1}{\beta \Omega} \sum_{ab} \sum_{q, \omega_n} G^{-1}_{ab}(q, \omega_n) u_a^*(q, \omega_n) u_b(q, \omega_n)
\]

The choice of a quadratic action for \( S_0 \) has two motivations: (i) quadratic actions are the only ones with which one is able to compute something; (ii) on a physical basis one can expect the properties of the system to be dominated by the small oscillations around one of the minima of the cosine. We will have to come back to this assumption. In doing so one implicitly neglects excitations that can take the system from one minimum of the cosine to the other. For the sine-Gordon model these excitations are known as solitons [46]. For the pure problem this method is equivalent to the well known self consistent harmonic approximation, and gives excellent results. The disordered case is more involved [22, 20]. In particular special case should be paid to the mode \( \omega_n = 0 \) since the disorder is time independent. I will only sketch the main lines here. The full Green’s functions are here the variational parameters. They are determined by

\[
\frac{\partial S_{\text{var}}}{\partial G_{ab}(q, \omega_n)} = 0
\]
Fig. 12. – (left) Replica symmetric structure. The matrix $G_{ab}$ has one value on the diagonal $\tilde{G}$ and one value away from the diagonal $G$. (right) Replica symmetry breaking structure. The example of a two step RSB is shown. The matrix $G_{ab}$ is a hierarchical matrix. There is one value $\tilde{G}$ on the diagonal and several values (here three) away from the diagonal. In the limit $n \to 0$ the value away from the diagonal becomes a function $G(u)$ labelled by an index $u \in [0, 1]$, giving its position in the hierarchical structure. In this example the function $G(u)$ takes three values with two breakpoints (hence the name two step RSB).

Using (41) leads to nasty integral equations, roughly of the form

$$G_{ab}^{-1}(q, \omega_n) = (\omega_n^2 + q^2)\delta_{ab} + \int d\tau [1 - \cos(\omega_n \tau)] e^{-\frac{1}{\beta} \sum_{q, \nu_n} G_{ab}(q, \nu_n) \cos(\nu_n \tau)} \ldots$$

One has to remember that the limit $n \to 0$ should be taken at the end so $G_{ab}$ is in fact a $0 \times 0$ matrix. In order to invert this matrix to solve the equations (44) one needs to know what is the structure of such matrices. In the original action, $S_{\text{cl}}$ is purely diagonal in replicas, whereas $S_{\text{dis}}$ couples each replica with another and is thus independent of the value of the indices $a, b$. This suggests that a good structure for $G_{ab}$ is to take one (for each value of $q$ and $\omega_n$ of course) value on the diagonal, and one unique value away from the diagonal

$$G_{ab} = [\tilde{G} - G]\delta_{ab} + G$$

Such a matrix structure is shown in Fig. 12. This is the natural choice, and is called replica symmetric since it respects the permutation symmetry between the replicas. The matrix depends in fact on only two numbers: the value on the diagonal $\tilde{G}$ and the off diagonal value $G$. It is easy to see that such matrices can be inverted and have a well defined limit when $n \to 0$. Calling $\tilde{G}^{-1}$ and $G^{-1}$ the diagonal and off diagonal value of the matrix $G_{ab}^{-1}$ and the connected part $G_c = \sum_b G_{ab}$ one has the inversion formulas (in the limit $n \to 0$)

$$G_c = \frac{1}{G_c^{-1}} \quad , \quad G = -\frac{G^{-1}}{(G_c^{-1})^2}$$
This allows to close the integral equations (44) and in principle to solve them. Unfortunately, it is easy to see that (45) cannot be right. This can be seen if one brutally expand the cosine: This indeed corresponds to a RS structure and gives “directly” $S_0$ (this approximation corresponds to the variational approach if one assumes the RS structure and that the disorder $\Delta$ is large). In that case only the $\omega_n = 0$ frequency appears due to the double integration on $\tau$ and $\tau'$. Since the action is quadratic this means that the disorder would only affect the mode $\omega_n = 0$, and thus would be unable to change the current. This is obviously completely opposite to the idea of pinning, so such an approximation misses all this physics. The answer to this paradox resides in the assumption of the replica symmetric structure for the matrix $G_{ab}$. Indeed as is now well known for spin glasses, such a saddle point might be unstable [47]. In that case one should look for another structure for $0 \times 0$ matrices. This structure was discovered by Parisi, and breaks the replica symmetry [48]. The matrix is now a hierarchical matrix. There is not a single element off diagonal but many values defined in a hierarchical way. I will not enter in the structure of the theory, and refer the reader to [49, 20] for a discussion. A pictorial representation of the matrix is shown in Fig. 12. Such matrices can also be inverted in the $n \to 0$ limit. This structure is called replica symmetry breaking. In this limit the off diagonal element becomes a function $G(v)$ of a continuous variable varying between zero and one. If $G(v)$ is a continuous function one speaks of continuous RSB, whereas if $G(v)$ has step(s) one talks about $n$-step RSB ($n$ being the number of steps). I will not dwell too much on the physical interpretation and refer the reader to [20]. As is well known for spin glasses, the occurrence of RSB signals glassiness, and the presence of metastable states. Since there is one matrix $G_{ab}(q, \omega_n)$ for each value of $q$ and $\omega_n$, the replica structure depends a priori on $q$ and $\omega_n$. For quantum problems since the disorder is time independent one can show [22] that one has off diagonal replica terms only for the mode $\omega_n = 0$. The structure is replica symmetric for $q > 1/R_c$ and RSB for $q < 1/R_c$. This confirms the physical picture that pinning and metastability appear above the Larkin length $R_c$.

4.2. Functional renormalization group. – RSB might look like magic(1), so let us try to solve our disordered problem by a more conventional renormalization approach. Doing it on quantum problems is tedious so I will explain the method on a classical system. One simply takes $\hbar = 0$ in the quantum action which removes all time dependence in time of $u(r, \tau)$. The system is thus described by the classical action

\begin{equation}
S = \frac{c}{2T} \sum_a \int d^4 r (\nabla u_a(r))^2 - \frac{\Delta}{T^2} \sum_{ab} \int d^4 r \cos(K(u_a(r) - u_b(r)))
\end{equation}

where I have retained only a single harmonic in the disorder term for simplicity. To implement an RG procedure one would decompose $u_a(r)$ into fast and slow modes. If for example $\Lambda$ is a cutoff in $q$ space and $\Lambda'$ a smaller cutoff, one defines ($\Omega$ is the volume of

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
Column 1 & Column 2 & Column 3 \\
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Row 1 & Row 2 & Row 3 \\
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\end{tabular}
\caption{Table caption}
\end{table}

(1) This is metaphysical question, since the only point is whether a method works or not!
the system)

\[ u_a(r) = u^{<}(r) + u^{>}(r) \]

\[ = \frac{1}{\Omega} \sum_{q<\Lambda'} u_a(q)e^{iqr} + \frac{1}{\Omega} \sum_{\Lambda'<q<\Lambda} u_a(q)e^{iqr} \]

(48)

One then integrates over the fast modes ($\Lambda' < q < \Lambda$) to get an effective action with new coupling constants for the slow modes. Since the cutoff has changed it is then convenient to rescale space by $q = q'\Lambda'/\Lambda$ or $r = r'\Lambda'/\Lambda$, such that the new space variable $q'$ has again a cutoff $\Lambda$. One can then iterate this procedure. Even without performing the integration over the fast modes let me look at the bare dimensions of the operators (tree level). Let us assume that $u$ scales as $u \sim L^{\zeta}$. One sees directly that periodic systems are peculiar. Because the disorder term is a cosine, with a periodicity which corresponds to the lattice spacing, this periodicity cannot be changed in the renormalization. This imposes $\zeta = 0$ [29, 30]. Let me parametrize the cutoff by $\Lambda(l) = \Lambda_0 e^{-l}$, where $\Lambda_0$ is the initial cutoff. In order to keep the elastic part of the action invariant, one should thus have

\[
\frac{1}{T(l + dl)} = \frac{1}{T(l)} \left( \frac{\Lambda(l)}{\Lambda(l + dl)} \right)^{d-2}
\]

(49)

the $d-2$ comes from the integration $d^dr$ and the $\nabla^2$. This gives the scaling dimension for the temperature

\[
\frac{dT(l)}{dl} = T(l)(2 - d)
\]

(50)

We thus see that for $d > 2$ the temperature scales to zero. This is in agreement with the fact that the competition between disorder and elasticity is an energy problem, where the temperature plays little role. Let us look at the renormalization of the disorder term. Performing the same rescaling of $d^dr$ and using (49) for the temperature gives

\[
\Delta(l + dl) = \Delta(l) \left( \frac{\Lambda(l + dl)}{\Lambda(l)} \right)^{d-4}
\]

(51)

giving the equation

\[
\frac{d\Delta(l)}{dl} = \Delta(l)(4 - d)
\]

(52)

We thus see that the disorder is relevant below $d = 4$. Note that around $d = 4$ the temperature is indeed irrelevant. One could thus naively think that this problem is quite similar to the standard $4 - \epsilon$ renormalization in critical phenomena. There is however an important difference. Let us compare the present problem with a standard
(non disordered problem) in the standard $\phi^4$ theory. Let me denote the action for this problem

$$\frac{c}{2T} \int d^d r (\nabla \phi)^2 + \frac{\lambda}{T} \int d^d r \phi^4 + \frac{\mu}{T} \int d^d r \phi^6$$

(53)

In that case and contrarily to the disorder case both the elastic term and the “potential” terms have the same power of the temperature. This is of course the signature that a normal phase transition results from a competition energy-entropy. The extra factor of $1/T$ in front of the cosine in (47) is the way for the replicated action to indicate that before averaging one had to compare the two energies of the elastic term and of the disorder term. For (53) one absorbs the factor $e^{-(d-2)dl}$ coming from the spatial rescaling in the rescaling of $u$ and one leaves the temperature invariant. This traduces the fact that a problem at $T = T_c$ stays at $T_c$. The rescaling for $u$ thus becomes

$$\phi^2(l + dl) = \phi^2(l)e^{(d-2)dl}$$

(54)

This gives for the renormalization of the coupling constants

$$\frac{d\lambda}{dl} = \lambda(4 - d)$$
$$\frac{d\mu}{dl} = \mu(6 - 2d)$$

(55)

Thus all terms in the potential that have high (i.e. larger than 4) powers of $u$ are irrelevant. Thus regardless of the precise form of the potential, one can always expand in power series and the only important term is the $\phi^4$ term. The situation is totally different for the disordered system [50]. In that case $u$ does not scale. It means that all terms in a power series expansion of the potential have the same scaling dimension and are equally relevant around four dimensions. Thus instead of having a single RG equation for a single parameter $\lambda$ one has now an infinity of (a priori coupled) RG equations for all the coefficients of the terms in the power series expansion of the potential. In other words for the disorder problem one has now to renormalize the whole function corresponding to the potential term. This means that one has to start with a potential term

$$S_{\text{dis}} = -\frac{1}{T^2} \sum_{a,b} \int d^d r \Delta(u_a(r) - u_b(r))$$

(56)

where $\Delta(z)$ is a function whose initial ($l = 0$) value is $\Delta_{l=0}(z) = \cos(Kz)$ but which will change form in the course of the renormalization. Because one has to keep the whole function under renormalization this procedure is known as functional renormalization group. I will not derive the equations to the next order (a good description of the derivation can be found in an appendix of [51]) but just give the result (for a classical
crystal)] [29, 30]

\[
\frac{d\Delta(z)}{dl} = \epsilon \Delta(z) + \frac{1}{2} \Delta''(z)^2 - \Delta''(z = 0)\Delta''(z)
\]

(57)

Needless to say this is much more complicated to solve than if one had a single parameter. But in addition there is an important feature of the solution. Because we eliminate modes \( u_a \) becomes smoother and smoother thus in a sense \( u_a - u_b \) becomes smaller and smaller under renormalization. Thus if the function \( \Delta \) stayed analytic one could expand it in powers of \( u \) to get

\[
\Delta(u_a - u_b) = \Delta(0) + \frac{1}{2} \Delta''(0)(u_a - u_b)^2 + \frac{1}{4!} \Delta^{(4)}(0)(u_a - u_b)^4 + \ldots
\]

(58)

and one would be asymptotically dominated by the \((u_a - u_b)^2\) term. This would give back a RS solution, and in fact exactly Larkin’s result. Since we know that this is incorrect beyond a certain lengthscale, something has to go wrong with the above expansion. The only way to escape this is if \( \Delta^{(4)}(0) \) becomes very large (in fact infinite) so that the above expansion breaks down and that higher power continue to play a role. Thus to get the glassy physics the function \( \Delta(z) \) should become nonanalytic beyond a certain lengthscale. One can check that this is indeed the case. The equation for \( \Delta^{(4)}(0) \) is from

\[
\frac{d\Delta^{(4)}(0)}{dl} = \epsilon \Delta^{(4)}(0) + 3[\Delta^{(4)}(0)]^2
\]

(59)

which indeed explodes at the lengthscale

\[
\epsilon^l = \left( \frac{\epsilon}{3\Delta_0} \right)^{1/z}
\]

(60)

This corresponds indeed to the Larkin scale (compare with (21)). Beyond this lengthscale, although the function itself remains finite, it develop a non-analyticity. If one looks at the second derivative of \( \Delta \) it develops a cusp, as shown on Fig. 13. I will stop there on the FRG method since this would take us way beyond the level of these notes. For more on the FRG and further references see [49, 34, 52, 53]

5. – Bragg glass and disordered Wigner crystal

5.1. Elastic model and experiments. – Let me now examine what are the results for the disordered Wigner crystal. I will not reproduce the calculations here that can be found in [22, 20] for quantum crystals in general and in [23, 24] for the specific case of the Wigner crystal.

Using the variational method on (3) and (12) one finds that the most stable solution is an RSB one. This is suggestive that the pinned system indeed has glassy properties due to the pinning. I will thus use loosely the term Wigner glass in the following. Second,
Fig. 13. – In the FRG approach one has to renormalize the full correlator of the disorder. One starts with an analytic function (left). For crystals this function is essentially a cosine. After a certain lengthscale, corresponding to the Larkin length, the function develops a non-analyticity (cusp) at the origin (right). This non-analyticity is the signature in this method of the glassy properties of the system.

since we know have from the displacement-displacement propagator, one can compute the conductivity. An example of the results is shown in Fig. 14. At first sight the agreement with the experimental data is much better (see [24] for a full discussion). Note that the

Fig. 14. – An example of the conductivity of the Wigner glass as computed by the variational approach. Note the non lorentzian broadening of the peak and the magnetic field dependence of the pinning frequency (from [24]).
broadening is non lorentzian in agreement with the data. More importantly one finds a decrease of the pinning frequency with the density as $\omega_p \propto n^{-3/2}$, as well as a good magnetic field dependence of the pinning frequency. This very good agreement gives a good confirmation that the insulating phase in the 2DEG under strong magnetic field is indeed a Wigner crystal collectively pinned by impurities.

What is the reason of this difference with the naive answers explained in Sec. 3.2? An inspection of the variational solution [23, 24] shows that in the full solution the pinning frequency is not controlled by $R_a$ as in the naive answer but by $R_c$ (in fact the formula is more complicated). This is a very satisfactory and physical result. Indeed as we discussed, for a classical crystal $R_c$ is the length that controls the pinning force. Although it is of course impossible to directly extrapolate it is quite reasonable to obtain that the pinning frequency, that is another manifestation of pinning, is also controlled by this lengthscale. This difference is crucial since, as in the Wigner crystal $l_c$ and $a$ are quite different $R_c$ and $R_a$ corresponds to quite different lengthscales and have in general quite different dependence in the various parameters. Since $R_c$ depends on the size of the particle $l_c$, this gives, for example for the case of strong magnetic field for which $l_c$ is just the cyclotron orbit, additional magnetic field dependence to the pinning frequency. This is to be contrasted with charge density waves for which $l_c \sim a$ due to the nearly sinusoidal density modulation and thus $R_c \sim R_a$. Thus borrowing directly the solutions that have been developed [36] for this case is dangerous and gives part of the physics incorrectly.

I refer the reader to [23, 24], for other physical properties such as the Hall conductivity. I will just mention here the compressibility. Defining the compressibility for a charged system is a delicate question since one has to maintain the neutrality of the system. Naively one relates the compressibility to the density-density correlation function by

$$\kappa = \lim_{q \to 0} \langle \rho(q, \omega_n = 0) \rho(-q, \omega_n = 0) \rangle$$

Let me look first at the pure system. The compressibility is simply (only the longitudinal mode plays a role)

$$\kappa(q) = \lim_{q \to 0} \frac{q^2}{c_L(q)}$$

If only short range interactions are present in the system the longitudinal mode is a phonon-like mode $c_L(q) \propto q^2$ and the one recovers a finite compressibility. On the other hand if one has long range Coulomb interactions $c_L(q) \propto |q|$ and the compressibility becomes zero. This is simply due to the fact that (61) measures the density response to a change of chemical potential while keeping the neutralizing background unchanged. A charged system thus does not remain neutral, hence the infinite compressibility. One has thus to define the compressibility more precisely. Many derivations of the compressibility use instead directly a derivative of the free energy with respect to the number of particles. The free energy can be computed for a neutral system for an arbitrary number of particles.
which solves the above-mentioned problem. Unfortunately very often the calculation is only possible in some sort of approximate way such as an Hartree-Fock approximation. Here again the link with the direct measurements of the compressibility is not clear. Using such procedures, so called “negative” compressibilities are found for some range of the interactions, for interacting electrons. Similarly, experiments measure such negative “compressibilities” [54, 16] (see [18] for further references and discussion on this question).

In order to make the physics of such negative compressibility more transparent, I discuss now a very simple way to compute it [21]. It is in fact a direct calculation of the quantity that is actually measured to determine the “compressibility”, i.e. the capacitance of a system made by with the 2DEG [54]. For simplicity I take here a capacitor formed of two identical systems, as shown in Fig. 15. Taking one system and one metallic plate would not change the results in an essential way. The Hamiltonian of the system is thus

\[ (63) H_1^0 + H_2^0 + \sum_{\alpha, \beta=1, 2} \int_{r, r'} \frac{1}{2} V(r-r') [\rho_\alpha (r) - \rho_0] [\rho_\beta (r') - \rho_0] + \frac{\mu}{2} \int [\rho_1 (r) - \rho_2 (r)] \]

If one assumes that the system is neutral in the absence of \( \mu \), then the charge on one plate when a potential \( \mu \) is applied is

\[ (64) \langle \rho_1 \rangle = \frac{\mu}{2} (\langle \rho_1 \rho_1 \rangle - \langle \rho_1 \rho_2 \rangle) \]

in linear response. (64) give directly the capacitance \( \langle \rho_1 \rangle / \mu \).

One can use the general formula (64) to compute the capacitance for the Wigner crystal. One substitutes in (3) the density decomposition (15). The \( \nabla u \) terms give directly the contribution of the long range part of the Coulomb interaction

\[ (65) H_{\text{long-range}} = \frac{1}{2} \rho_0^2 \sum_q \sum_{\alpha, \beta=1, 2} [V_{\alpha \beta} (q) u_\alpha^2 (q) u_\beta (-q)] \]
Since $V_{11}(q) \sim 1/q$, (65) gives obviously the part proportional to $q$ in the elastic coefficients for an isolated plane. The higher harmonics give the regular part (i.e. the part proportional to $q^2$ in the elastic coefficients. Such a way to determine the coefficient is equivalent the calculation of the coefficients in [25].

Taking a pure system the Hamiltonian becomes (only the $\omega_n = 0$ term of the longitudinal part needs to be computed to have the compressibility)

$$H = \begin{pmatrix} u_L^1(q) \\ u_L^2(q) \end{pmatrix} = \begin{pmatrix} c_{SR}^L(q) + \rho_0^2 q^2 V_{11}(q) & \rho_0^2 q^2 V_{12}(q) \\ \rho_0^2 q^2 V_{12}(q) & c_{SR}^L(q) + \rho_0^2 q^2 V_{11}(q) \end{pmatrix} \begin{pmatrix} u_L^1(-q) \\ u_L^2(-q) \end{pmatrix}$$

where $c_{SR}^L(q)$ is the “short range” part of the elastic coefficients. Using (64) and the expression of the density for small $q$ from (15) $\rho_L(q) = -\rho_0 q u_L(q)$ one gets for the capacitance $C = C_{\text{geom}} + \frac{1}{C_{\text{el}}}$, where $C_{\text{geom}} = 1/(4\pi d)$ is the standard geometrical capacitance and

$$\frac{1}{C_{\text{el}}} = \lim_{q \to 0} \frac{2c_{SR}^L(q)}{\rho_0^2 q^2}$$

(again the factor of 2 comes from the fact that here I took two identical plates). The electronic capacitance corresponds to the propagator where only the short range part of the elastic coefficients is kept. Using [25] $c_{SR}^L(q) = -\omega_0^2 (0.18...) (aq)^2$, where $\omega_0 = \frac{4\pi e^2}{\sqrt{3} m a}$, one finds for the Wigner crystal a “negative” compressibility. The fact that a system of discrete charges can lead to such effects has been noted before for classical Wigner crystals (see e.g. [55] and references therein).

The disorder can be considered easily. The variational method discussed in Sec. 5 gives straightforwardly the density-density correlation function which is simply related to the $\langle uu \rangle$ correlation. Since it is a gaussian approximation one can easily use the capacitance method shown above. The capacitance is thus given by (67) but where one should use the propagator in the presence of disorder. It is given by (within the variational approximation used [23, 24]) by:

$$\langle uu \rangle = \frac{1}{\rho m \omega_n^2 + c_L(q) + \Sigma(1 - \delta_{n,0}) + I(\omega_n)}$$

where $\Sigma$ and $I(\omega_n)$ are respectively a constant and a function related to the disorder verifying $I(0) = 0$. At $\omega_n = 0$ (68) leads to a compressibility in identical to the one of the pure system, and thus also “negative”. More on this point can be found in [21].

5'2. Defects. – We thus see that our elastic description of an electronic glass is a very good starting point to extract the physical properties since we have now reliable analytic methods to tackle such problems. However, in view of the commonly accepted point of view that disorder would induce topological defects in a crystal (as explained in Sec. 3'2) it is important to ascertain the validity of the elastic description itself. In fact the creation of defects by disorder has been grossly overestimated and the elastic
Fig. 16. – In $d = 3$ a classical crystal plus disorder is in a Bragg glass state (for weak disorder). Although the disorder destroys the perfect positional order of the crystal, it does it smoothly. The topological order of the crystal is preserved and there are no unpaired defects such as dislocations generated by the disorder. The system is thus nearly as ordered as a perfect crystal. In particular the relative displacements of two neighbors remain small, and thus such a system is totally described by an elastic theory.

The theory is in fact much more stable to the presence of topological defects than initially anticipated following the naive picture of Sec. 3.2.

In $d = 3$ it is now known that below a certain threshold of disorder no topological defects can be induced by the disorder. The disordered elastic system is in a Bragg glass state [30] with a quasi long range positional order, much more ordered than the incorrect image of Fig. 8 suggests (see e.g. [17, 34] for a discussion and references on this point). The distance between unpaired defects is thus infinite in this case. Though although the crystal is distorted by the disorder and the perfect positional order is lost, the deformation is smooth and the topological structure of the crystal is preserved. The displacements of two neighbors remain small even if large distortions can accumulate at large lengthscales. The elastic description is thus perfectly accurate at all lengthscales in that case. An image of the Bragg glass is indicated in Fig. 16.

In $d = 2$ the situation is marginal, and defects do appear in the ground state, but at distance $\xi_D$ much larger (for weak disorder) than the lengthscale $R_a$ and not at that lengthscale [30, 56], as shown in Fig. 17. Since the ratio $\xi_D/R_a$ can become arbitrarily large for weak disorder, it means that all important effects of disorder (they occur at the lengthscales $R_c$ and $R_a$) can be reliably described by an elastic theory. Since the pinning peak is controlled by the length $R_c \ll R_a \ll \xi_D$ it means that dislocations will spoil the results of the elastic theory for the optical conductivity only well below the pinning peak, as shown on Fig. 18. This implies that the theory is a reliable tool to compute the characteristics of the peak and above, and thus most of the a.c. transport. Thus optical conductivity measurements are an accurate way of probing the Wigner glass nature of the system, since one has a reliable theory to compute the characteristics of the pinning peak in a Wigner glass phase. This allows for a detailed testing, as I showed above for the case of the 2DEG under strong magnetic field. For the systems in the absence
Fig. 17. – In $d = 2$ for a classical crystal, defects are generated by disorder at large lengthscales, so an elastic description is not valid beyond the distance $\xi_d$ between these topological defects. However for weak disorder $\xi_D$ is much larger than the lengthscale $R_a$ (right). This is in marked contrast with the incorrect naive picture of Fig. 8 (left), where both length would be of the same order. It means that an elastic theory can reliably be used to describe the effects of disorder below $\xi_D$ (from [56]).

of a magnetic field shown in Fig. 3, a measurement of the optical conductivity could thus provide an unambiguous answer on the presence of a Wigner glass in the insulating phase. Such an experiment remains to be performed.

On the contrary this shows that to know d.c. conductivity it is necessary to take topological defects of the crystal into account (vacancies, interstitials, dislocations etc.).

Fig. 18. – (a) If dislocations occurred at scale $R_a$ and the pinning frequency was controlled by $R_a$, as was naively believed, the elastic theory is incapable of giving any reliable information on the pinning peak. It would be necessary to include dislocations from the start. (b) As was discussed in the text, dislocations occur in fact at $\xi_D \gg R_a$ and the pinning peak depends on $R_c \ll R_a$. Thus the pinning peak is given quantitatively by a purely elastic theory. It is necessary to take into account topological defects such as dislocations only at much lower frequencies, and in particular if one wants to obtain reliable results for the d.c. transport (from [24]).
This makes the calculation of the d.c. conductivity much more difficult since at the moment there is no theory able to take into account topological defects, elasticity and disorder at the same time. It is thus impossible to make specific predictions specific to the Wigner crystal and that could be quantitatively checked. The only case where such a calculation of d.c. conductivity can be done reliably is the case of one dimension (Luttinger liquids) [57].

6. – Further steps and unsolved questions

In these short notes I have discussed some of the physics of quantum particles in the presence of strong repulsion. In that case the physics changes drastically from the standard “wave” description of indiscernable particles. The particles localize to form a quantum crystal. Examples of such systems are the Wigner crystal and Luttinger liquids. In such a quantum crystal, quantum effects are still important but particles become essentially discernable by their position, making the problem much simpler to tackle. The effects of disorder in such a disordered crystal are also quite different than on noninteracting electrons, and can be viewed as the pinning of the quantum crystal on impurities, in a similar way than for a classical crystal.

I have briefly described the methods used to tackle such disordered elastic systems. Using them the a.c. transport and thermodynamic quantities such as the compressibility can be reliably computed. A pinned quantum crystal exhibit glassy properties, in a way similar to the Bragg glass for classical systems (quantum Bragg glass). There is good agreement with the predicted a.c. transport and the observed behavior of a 2DEG under a strong magnetic field, making a strong case for a Wigner crystal phase in such systems. The compressibility is found to be negative both for a pure Wigner crystal and in the presence of disorder, and detailed comparison with experiments on that point would be clearly fruitful. One important point is that the elastic description is an excellent starting point to obtain the a.c. properties of such systems. Indeed, although in $d = 2$ topological defects in the crystal (dislocations etc.) are generated by disorder, the distance at which unpaired defect appear is very large compared to the scale at which the effects of disorder manifests themselves. This makes the a.c. transport a probe of choice to decide on the presence of a Wigner glass phase, since direct imaging techniques are not yet at the point where the crystalline structure can be seen.

The d.c. transport is a much more difficult question because of the need to explicitly take the presence of topological defects into account. There are however some answers that can be obtained when the crystal is set in motion at finite velocity. In that case the defects become less important, and the properties are again crucially dependent on the periodicity of the system (Moving glass phase). As was shown in [58, 59], periodic moving periodic systems have a quite specific dynamics. Indeed due to the existence of the periodicity in the direction transverse to the direction of motion, the motion cannot average completely over the disorder. The moving system is thus submitted to a random potential, which leads to a channel like motion. The very existence of such channels has an important consequence if one tries to make the system move in the transverse direction.
Indeed although the particles do move along the channels, the channels themselves are pinned. This means that even above the longitudinal threshold field, if one tries to apply a force in the transverse direction a transverse critical force still exists. For the two dimensional electronic system, putting a magnetic field is a simple way of applying a transverse force. If the lattice is sliding at velocity \( v \), it is submitted to a Lorentz transverse force \( F_L = evB \). The existence of the transverse critical force \( F_{tr} \) thus implies that the channel structure should not slide as long as \( F_L < F_{tr} \). There will thus be no hall voltage generated. On the other hand when \( F_L > F_{tr} \) the channel structure should slide and a Hall voltage exists. The periodicity of the crystalline structure thus implies that one needs a finite longitudinal current before a Hall voltage exists. Experiments [60] have shown that a finite longitudinal current is clearly needed to develop a Hall voltage, in good agreement with the existence of a transverse threshold. The existence of such an effect is another direct probe of the crystalline (existence of a transverse periodicity) nature of the phase.

We see that there are thus efficient, if not easy, ways via transport to check for the presence of a pinned crystalline phase (electronic glass). Most of the experiments discussed in the preceding sections were for a 2DEG under a strong magnetic field. It would of course be extremely interesting to use the same techniques to probe the 2DEG in the absence of the magnetic field, and analyze the experiments in the line of what was discussed above to check for the existence of a Wigner crystal in these systems. Among the interesting possible experiments one can note:

- Measurement of the optical conductivity. In particular the density dependence of the pinning peak can be directly checked against the theoretical predictions of the pinned Wigner crystal.

- I did not dwell too much of the non-linear transport and on the existence of a threshold field (as shown in Fig. 2), but a threshold field is also a characteristic of a pinned crystal. Since it is controlled by the Larkin length (26) it has to be related to the pinning frequency. So a comparison between the threshold field in the d.c. transport and the pinning frequency is another probe of the pinned Wigner crystal.

- The Hall tension vs the longitudinal current (i.e. the measure of the transverse pinning force), as I briefly discussed here.

- Although not discussed in these notes, noise measurements are also a good way to probe the periodic nature of the systems (see e.g. [61] and references therein).

The electronic glasses are thus fascinating objects to play with, both theoretically and experimentally. Although we have some answers there are clearly many more unsolved questions than answers at the moment. Here are some possible directions to explore in the future in this field:

- What is the d.c. transport? To solve this question one needs first to understand the behavior of topological defects in a quantum crystal. Even for classical crystals
this question is open. At the moment there is only an answer for Luttinger liquids 
\((d = 1)\) where topological defects do not play a role [20, 57].

- In connection with the d.c. transport, what is the out of equilibrium dynamics of 
such systems. What is the \(I - V\) characteristics. This is quite general question for 
 PINNED structures. For lack of space I will not discuss the question of the moving 
crystals here but one can find more details and further references in [17, 34] for 
classical crystals and in [21] for the Wigner crystal.

- Since the system is a glass, he should show some of the properties that have been 
observed other types of glasses. In particular glasses show aging [62], i.e. the result 
of a measurement depends on the time at which the measurement is done (the 
origin of time is fixed by the time at which the glassy phase has been formed), 
since a glass relax very slowly towards its ground state. The question of quantum 
aging is in infancy [63] and the experimental consequences for electronic glasses 
have hardly been worked out.

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REFERENCES

[1] P. Nozieres, Theory of Interacting Fermi systems (W. A. Benjamin, New York, 1961).
[2] V. L. Berezinskii, Sov. Phys. JETP 38, 620 (1974).
[3] A. A. Abrikosov and J. A. Rhyzkin, Adv. Phys. 27, 147 (1978).
[4] E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan, Phys. Rev. 
Lett. 42, 673 (1979).
[5] F. Wegner, Z. Phys. B 35, 207 (1979).
[6] K. B. Efetov, A. I. Larkin, and D. E. Khmel’nitskii, Sov. Phys. JETP 52, 568 (1980).
[7] K. B. Efetov, Adv. Phys. 32, 53 (1983).
[8] E. Wigner, Phys. Rev. 46, 1002 (1934).
[9] B. L. Altshuler and A. G. Aronov, in Electron-electron interactions in disordered systems, 
edited by A. L. Efros and M. Pollak (North-Holland, Amsterdam, 1985).
[10] A. M. Finkelstein, Z. Phys. B 56, 189 (1984).
[11] P. A. Lee and T. V. Ramakrishnan, Rev. Mod. Phys. 57, 287 (1985).
[12] B. I. Shklovskii and A. L. Efros, Sov. Phys. JETP 54, 218 (1981).
[13] E. Y. Andrei and al., Phys. Rev. Lett. 60, 2765 (1988).
[14] R. L. Willett et al., Phys. Rev. B 38, R7881 (1989).
[15] F. I. B. Williams and et al, Phys. Rev. Lett. 66, 3285 (1991).
[16] S. Ilani, A. Yacoby, D. Mahalu, and H. Shtrikman, Phys. Rev. Lett. 84, 3133 (2000).
[17] T. Giamarchi and S. Bhattacharyya, in High Magnetic Fields: Applications in Condensed Matter Physics and Spectroscopy, edited by C. Berthier et al. (Springer-Verlag, Berlin, 2002), p. 314, cond-mat/0111052.
[18] E. Abrahams, S. V. Kravchenko, and M. P. Sarachik, Rev. Mod. Phys. 73, 251 (2001).
[19] S. V. Kravchenko et al., Phys. Rev. B 51, 7038 (1995).
[20] T. Giamarchi and E. Orignac, in Theoretical Methods for Strongly Correlated Electrons, CRM Series in Mathematical Physics, edited by D. Sénéchal et al. (Springer, New York, 2003), cond-mat/0005220.
[21] T. Giamarchi, in Strongly correlated fermions and bosons in low dimensional disordered systems, edited by I. V. Lerner et al. (Kluver, Dordrecht, 2002), cond-mat/0205099.
[22] T. Giamarchi and P. Le Doussal, Phys. Rev. B 53, 15206 (1996).
[23] R. Chitra, T. Giamarchi, and P. Le Doussal, Phys. Rev. Lett. 80, 3827 (1998).
[24] R. Chitra, T. Giamarchi, and P. Le Doussal, Phys. Rev. B 65, 035312 (2001).
[25] L. Bonsall and A. A. Maradudin, Phys. Rev. B 15, 1959 (1977).
[26] K. Maki and X. Zotos, Phys. Rev. B 28, 4349 (1983).
[27] I. M. Rusin, S. Marianer, and B. I. Shklovskii, Phys. Rev. B 46, 3999 (1992).
[28] P. I. B. Williams, 2001, private communication.
[29] T. Giamarchi and P. Le Doussal, Phys. Rev. Lett. 72, 1530 (1994).
[30] T. Giamarchi and P. Le Doussal, Phys. Rev. B 52, 1242 (1995).
[31] F. D. M. Haldane, Phys. Rev. Lett. 47, 1840 (1981).
[32] D. R. Nelson and V. M. Vinokur, Phys. Rev. B 48, 13060 (1993).
[33] G. Blatter et al., Rev. Mod. Phys. 66, 1125 (1994).
[34] T. Nattermann and S. Scheidl, Adv. Phys. 49, 607 (2000).
[35] A. I. Larkin, Sov. Phys. JETP 31, 784 (1970).
[36] H. Fukuyama and P. A. Lee, Phys. Rev. B 17, 535 (1978).
[37] Y. Imry and S. K. Ma, Phys. Rev. Lett. 35, 1399 (1975).
[38] A. I. Larkin and Y. N. Ovchinnikov, J. Low Temp. Phys 34, 409 (1979).
[39] D. S. Fisher, M. P. A. Fisher, and D. A. Huse, Phys. Rev. B 43, 130 (1990).
[40] C. C. Li et al., Phys. Rev. B 61, 10905 (2000).
[41] S. C. Zhang, S. Kivelson, and D. H. Lee, Phys. Rev. Lett. 69, 1252 (1992).
[42] A. Kamenev, in Strongly correlated Fermions and Bosons in low dimensional disordered systems, edited by I. V. Lerner et al. (Kluwer Academic Publishers, Dordrecht, 2002), p. 313.
[43] S. F. Edwards and P. W. Anderson, J. Phys. F 5, 965 (1975).
[44] R. P. Feynman, Statistical Mechanics (Benjamin Reading, MA, 1972).
[45] M. Mézard and G. Parisi, J. de Phys. I 1, 809 (1991).
[46] R. Rajaraman, Solitons and Instantons: An Introduction to solitons and Instantons in Quantum Field Theory (North Holland, Amserdam, 1982).
[47] J. R. L. de Almeida and D. J. Thouless, J. Phys. A 11, 983 (1978).
[48] M. Mézard, G. Parisi, and M. A. Virasoro, Spin Glass Theory and beyond (World Scientific, Singapore, 1987).
[49] T. Giamarchi and P. Le Doussal, in Spin Glasses and Random fields, edited by A. P. Young (World Scientific, Singapore, 1998), p. 321, cond-mat/9705096.
[50] D. S. Fisher, Phys. Rev. B 31, 7233 (1985).
[51] L. Balents and D. S. Fisher, Phys. Rev. B 48, 5959 (1993).
[52] P. Le Doussal, K. J. Wiese, and P. Chauve, Phys. Rev. B 66, 174201 (2002).
[53] K. J. Wiese, Acta Physica Slovaca 52, 341 (2002), cond-mat/0205116.
[54] J. P. Eisenstein, L. N. Pfeiffer, and K. W. West, Phys. Rev. B 73, 535 (1978).
[55] T. T. Nguyen, A. Y. Grosberg, and B. I. Shklovskii, 2001, cond-mat/0101103.
[56] P. Le Doussal and T. Giamarchi, Physica C 331, 233 (2000).
[57] T. Nattermann, T. Giamarchi, and P. Le Doussal, preprint, cond-mat/0303233., 2003.
[58] T. Giamarchi and P. Le Doussal, Phys. Rev. Lett. 76, 3408 (1996).
[59] P. Le Doussal and T. Giamarchi, Phys. Rev. B 57, 11356 (1998).
[60] F. Perruchot et al., Physica B 284, 1984 (2000).
[61] Y. Togawa et al., Phys. Rev. Lett. 85, 3716 (2000).
[62] L. F. Cugliandolo, J. Kurchan, J. P. Bouchaud and M. Mezard, in Spin Glasses and Random fields, edited by A. P. Young (World Scientific, Singapore, 1998).
[63] L. F. Cugliandolo et al., Phys. Rev. B 66, 014444 (2002).