The Fundamental Theorem of Barzilai does not hold

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Abstract

The extensively cited work of Barzilai, J. (1997): Deriving weights from pairwise comparison matrices, published in *Journal of the Operational Research Society*, 48(12), 1226-1232., derives the geometric mean method from two simple axioms. This note reveals that the central result of the paper called the Fundamental Theorem by the author does not hold as there exists at least one further method satisfying both requirements.

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1 Notations

Denote by $\mathbb{R}_{+}^{n\times n}$ the set of positive (with all elements greater than zero) matrices of size $n \times n$.

**Definition 1.** *Pairwise comparison matrix*: Matrix $A = [a_{ij}] \in \mathbb{R}_{+}^{n\times n}$ is a pairwise comparison matrix if $a_{ji} = 1/a_{ij}$ for all $1 \leq i, j \leq n$.

**Definition 2.** *Consistency*: A pairwise comparison matrix $A = [a_{ij}] \in \mathbb{R}_{+}^{n\times n}$ is consistent if $a_{ik} = a_{ij}a_{jk}$ for all $1 \leq i, j, k \leq n$.

Denote by $\mathbb{R}_{+}^{n}$ the set of positive (with all elements greater than zero) vectors of size $n$.

**Definition 3.** *Weight vector*: Vector $w = [w_{i}] \in \mathbb{R}_{+}^{n}$ is a weight vector if $\sum_{i=1}^{n} w_{i} = 1$.

Denote by $\mathcal{A}_{+}^{n\times n}$ the set of pairwise comparison matrices of size $n \times n$.

Denote by $\mathcal{R}_{+}^{n}$ the set of weight vectors of size $n$.

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Definition 4. Weighting method: Mapping \( f : A^{n \times n} \to R^n \) is a weighting method.

\( f_i(A) \) stands for the weight of alternative \( i \) in the pairwise comparison matrix \( A \in A^{n \times n} \) according to the weighting method \( f : A^{n \times n} \to R^n \).

We discuss two particular weighting methods.

Definition 5. Geometric mean method: The geometric mean method is the mapping \( A \to w^{GM}(A) \) such that the weight vector \( w^{GM}(A) \) is given by

\[
    w^{GM}_i(A) = \frac{\prod_{j=1}^n a_{ij}^{1/n}}{\sum_{k=1}^n \prod_{j=1}^n a_{kj}^{1/n}}, \quad (1)
\]

that is, it can be obtained by normalising the vector of row geometric means.

Definition 6. Column mean method: The column mean method is the mapping \( A \to w^{CM}(A) \) such that the weight vector \( w^{CM}(A) \) is given by

\[
    w^{CM}_i(A) = \frac{\sum_{j=1}^n a_{ij}/n}{\sum_{k=1}^n \sum_{j=1}^n a_{kj}/n}, \quad (2)
\]

that is, it can be obtained by normalising the vector of row arithmetic means.

2 The axioms

Barzilai (1997) requires the following two conditions for a weighting method to be an acceptable solution.

Axiom 1. Correctness: Let \( A \in A^{n \times n} \) be a consistent pairwise comparison matrix. Weighting method \( f : A^{n \times n} \to R^n \) is correct if \( f_i(A)/f_j(A) = a_{ij} \) for all \( 1 \leq i, j \leq n \).

Axiom 2. Independence of relative measurements among other alternatives: Let \( A \in A^{n \times n} \) be a pairwise comparison matrix. Weighting method \( f : A^{n \times n} \to R^n \) is independent of relative measurements among other alternatives if for an arbitrarily fixed \( 1 \leq k \leq n \), \( f_k(A) \) does not depend on \( a_{ij} \) for all \( i, j \neq i \).

3 The result

Proposition 1. The geometric mean and the column mean methods meet correctness and independence of relative measurements among other alternatives.

Proof. It immediately follows from formulas (1) and (2), respectively. \( \square \)

Corollary 1. Barzilai (1997, Theorem 1) states that there is exactly one weighting method satisfying Axioms 1 and 2, namely, the geometric mean. This result does not hold as presented by our Proposition 1.

Remark 1. Since Barzilai (1997) gives a proof to his Fundamental Theorem, it should contain a mistake: the author incorrectly assumes that if a weighting method \( f \) satisfies both axioms, furthermore, \( A = [a_{ij}] \in A^{n \times n} \) and \( B = [b_{ij}] \in A^{n \times n} \) are pairwise comparison matrices where \( b_{ij} = a_{kj}/a_{ki} \) (consequently, \( b_{ik} = a_{ik} \) and \( b_{kj} = a_{kj} \)), hence \( B \) is consistent, then \( f_k(B) = w_k^{GM}(B) \) implies \( f \) being the geometric mean because any other weighting method satisfying Axiom 1 (for example, the column mean method) coincides with the geometric mean on the set of consistent pairwise comparison matrices.
The column mean method shows poor performance in simulations since it sums up the columns which are not in commensurate units (Zahedi, 1986; Choo and Wedley, 2004; Bajwa et al., 2008). However, it is not meaningless as having three desirable properties in addition to Axioms 1 and 2: it preserves ranks strongly, as well as satisfies smoothness and comparison order invariance (Bajwa et al., 2008).

4 Summary

Barzilai (1997, Theorem 1) provides a characterisation for the geometric mean with using only two axioms, but this result is proved to be incorrect. On the other hand, there exists a powerful axiomatisation of the geometric mean in Csató (2019), which exchanges Axiom 2 to a property called invariance to $\alpha$-transformation on a triad. Csató (2018) gives another characterisation for the ranking induced by this method based on the famous theorem of Aczél and Saaty (1983) concerning the aggregation of pairwise comparison matrices.

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