Complete Hierarchy of Relaxation for Constrained Signomial Positivity

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Abstract

In this article, we prove that the Sums-of-AM/GM Exponential (SAGE) relaxation for verifying signomial positivity over a constrained set is complete, with a compactness assumption. The high-level structure of the proof is as follows. We first apply variable change to convert a set of rational exponents to polynomial equations. In addition, we make the observation that linear constraints of the variables may also be converted to polynomial equations after a variable change. Note that any convex set may be expressed as a set of linear constraints. Further, we use redundant constraints to find reduction to Positivstellensatz. We rely on Positivstellensatz results from algebraic geometry to obtain a decomposition of positive polynomials. Lastly, we explicitly show that the decomposition is of a form certifiable by SAGE.

1 Introduction

A signomial function is one of the form \( f(x) = \sum_{j=1}^{l} c_j \exp(a_j^\top x) \), where \( c_j \in \mathbb{R}, a_j \in \mathbb{R}^n \) for \( j = 1 \ldots l \) are fixed. Optimization of such function subject to signomial inequality and equality is called signomial programming (SP). SPs are non-convex in general and are NP-Hard in special cases [3].

Geometric programming (GP) constitutes a subclass of SPs in which the objective function to be minimized is a *posynomial*, with \( c_j \geq 0 \) for \( j = 1 \ldots l \), subject to upper constraints on posynomials. GPs have wide applications in many areas such as control in communication systems [3], circuit design [14], approximations to the matrix permanent [10], and the computation of capacities of point-to-point communication channels [4]. However, the modeling power of SPs on arbitrary signomials are useful in many additional applications in chemical engineering [6], aeronautics [15], and communications network optimization [12].

In this article, we are concerned with a broader class of problems, in which one minimizes an arbitrary signomial over any arbitrary convex set \( C \) characterized by a set of constraints. Such optimization problem may be reduced to the verification of signomial positivity over a constrained set.

2 Outline and Contribution

We first describe the problem of interest. In particular, we describe the relaxation for verifying signomial positivity based on a certification of signomial positivity with at most one negative term. While the certificate has been previously considered, we independently arrived an equivalent formulation in the constrained

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set and will present additional results. We also show how such verification of positivity can be used for constrained signomial optimization. The verification is a relaxation because although it ensures that a signomial is positive over a constrained set, not all such signomials may be verified as so.

The main contribution of the article is to describe the hierarchy of relaxation and a completeness theorem. While the verification is a relaxation, it has a hierarchy such that with increasing computational complexity, increasing subset of signomials that are positive over a constrained set may be verified as so. The completeness theorem shows that any signomial positive over a constrained set can be certified under the same framework, at finite level on the hierarchy.

3 Background

The problem of interest is verifying the positivity of a signomial over a convex set. That is, given a signomial function $f(x)$ and a convex set $C$, whether

$$\inf \{ f(x) : x \in C \} > 0 \tag{1}$$

In 2016, Chandrasekaran and Shah proposed the Sums-of-AM/GM Exponential (SAGE) certificates of signomial positivity, which provided a new convex relaxation framework for signomial programs akin to sum-of-square (SOS) methods for polynomial optimization \cite{2}. This work concerns the case where $C = \mathbb{R}^n$ and provides an efficiently computable certificate for a signomial with at most one negative term. The certificate is based on the AM/GM inequality and is exact, and the relaxation for an arbitrary signomial is based on finding a decomposition of a signomial such that each term has at most one negative term and is positive. The relaxation was shown to have a hierarchy that is \textit{complete}. That is, any positive signomial may be certified as such at some finite (although not bounded or known) level of the relaxation hierarchy.

In 2019, Murray, Chandrasekaran and Wierman extended the Sums-of-AM/GM Exponential (SAGE) relaxation for signomial positivity in the constrained case \cite{9}. We have independently produced the same results, and will describe it in the following sections. However, while the previous work discusses a hierarchy, it does not extend the completeness theorem to the constrained case.

More broadly, the notion of hierarchy of relaxation and completeness is well known in the SOS method for polynomial optimization \cite{5, 7}. The hierarchy and completeness results proved in this article are analogous to the ones in polynomial optimization, and the Positivstellensatz used in this article is the constrained analogue of Reznick’s Positivstellensatz supporting the SOS hierarchy \cite{11}.

4 Notations

Given a matrix defined by a finite collection of vectors $A = [a^{(1)}, a^{(2)}, \ldots, a^{(l)}] \in \mathbb{R}^{n \times l}$, let $\tilde{A}^{(i)} = [a^{(1)} - a^{(i)}, \ldots, a^{(i)} - a^{(i)}, \ldots, a^{(l)} - a^{(i)}] \in \mathbb{R}^{n \times l}$, and let $\tilde{A}^{(i)}_{\backslash i}$ be a matrix having $i$th column removed from $\tilde{A}^{(i)}$. i.e. $[a^{(1)} - a^{(i)}, \ldots, a^{(i-1)} - a^{(i)}, a^{(i+1)} - a^{(i)}, \ldots, a^{(l)} - a^{(i)}]$. Given $c \in \mathbb{R}^n$, use $c_{\backslash i}$ to denote a vector with the $i$th entry removed from $c$.

5 Constrained-AGE Certificate

We want to certify the positivity of signomial with at most one negative term over a constrained set. Recall that for any function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$, its conjugate is defined as:

$$f^*(y) = \sup_x x^\top y - f(x). \tag{2}$$

With the Fenchel conjugate we recall the celebrated Fenchel-Rockafellar duality theorem.
Theorem 1 (e.g., [1, Theorem 3.3.5]) Let \( f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\} \) and \( g : \mathbb{R}^l \to \mathbb{R} \cup \{\infty\} \) be two functions, \( A : \mathbb{R}^n \to \mathbb{R}^l \) be a linear map, then
\[
\inf_x f(x) + g(Ax) \geq -\inf_y f^*(A^T y) + g^*(-y).
\] (3)

If, furthermore, \( f \) and \( g \) are lower semicontinuous convex and (for instance) \( 0 \in \text{int} (\text{dom}(g) - \text{Adom}(f)) \), then equality holds and the second infimum is attained if finite.

Of interest here is the special case where \( f(x) = \iota_C(x) = \begin{cases} 0, & \text{if } x \in C \\ \infty, & \text{otherwise} \end{cases} \) \( (4) \) and its conjugate is the support function of \( C \):
\[
f^*(z) = \sigma_C(z) = \sup_{x \in C} x^T z.
\] (5)

The goal is to provide a convex certificate of the following AM/GM-signal inequality:
\[
\forall x \in C, \ c_0 \exp(a_0^T x) + \sum_{j=1}^l c_j \exp(a_j^T x) \geq 0? \tag{6}
\]

Here \( c_j \geq 0 \) for \( j = 1, \ldots, m \) but we allow \( c_0 \) to be negative (otherwise the answer would be trivially yes). The important observation made in [2] is that we can divide both sides of (6) by \( \exp(a_0^T x) \) and arrive at the equivalent problem:
\[
\inf_{x \in C} \sum_{j=1}^l c_j \exp[(a_j - a_0)^T x] \geq -c_0? \tag{7}
\]

Unlike the left-hand side of (6), the left-hand side of (7) is a convex problem (assuming \( C \) is a convex set).

Let \( \tilde{A}_{(0)} = [a_1 - a_0, \ldots, a_l - a_0]^T \in \mathbb{R}^{l \times n}, g(y) = \sum_{j=1}^l c_j \exp(y_j) \), and \( f(x) = \iota_C(x) \), then applying the Fenchel-Rockafellar duality we have (note that \( \dim(g) = \mathbb{R}^l \) hence the condition in Theorem 1 holds):
\[
\inf_{y \in \mathbb{R}_+^l} \sigma_C(-\tilde{A}_{(0)}^T y) + \text{KL}(y\|\mathbf{e}c) \leq c_0? \tag{8}
\]

Here we have used the fact that for the function \( g(y) = \sum_{j=1}^l c_j \exp(y_j) \),
\[
g^*(z) = \sup_y \sum_{j=1}^l z_j y_j - c_j \exp(y_j) = \sum_{j=1}^m z_j \ln \frac{z_j}{c_j e} =: \text{KL}(z\|\mathbf{e}c), \tag{9}
\]
where \( e = \exp(1) \) is the Euler constant. Note that the infimum in (8) is actually attained if the infimum is finite. Thus, w.l.o.g., we can rewrite (8):
\[
\exists y \in \mathbb{R}_+^l \text{ such that } \sigma_C(-\tilde{A}^T y) + \text{KL}(y\|\mathbf{e}c) \leq c_0? \tag{10}
\]

We remark that the condition in (8) remains to be sufficient to guarantee (7), for any set \( C \), while it is also necessary if \( C \) is closed convex. On the other hand, the left-hand side of (8) is always a convex problem hence can be efficiently computed.

More generally, define the following.
Definition 1 Given a matrix defined by a finite collection of vectors $A = [a^{(1)}, a^{(2)}, \ldots, a^{(l)}] \in \mathbb{R}^{n \times l}$ and convex set $C \subseteq \mathbb{R}^n$ consider signomials defined by $A$ with a most one negative coefficient occurring at the $i$th index that are positive over a convex set $C$. i.e. $f(x) = \sum_{j=1}^{l} c^j \exp\{a^{(j)} \cdot x\}$, $c_{\setminus i} \geq 0$. Then such signomials are defined as:

$$AGE(A, C, i) = \{ f \mid f = \sum_{j=1}^{l} c^j \exp\{a^{(j)} \cdot x\} \geq 0 \ \forall x \in C, \ c_{\setminus i} \geq 0 \}$$

(11)

$$= \{ f \mid f = \sum_{j=1}^{l} c^j \exp\{a^{(j)} \cdot x\}, \ c_{\setminus i} \geq 0$$

$$\exists y \in \mathbb{R}^l_+ \text{ such that } \sigma_C(-({\tilde{A}}^{(i)}_{\setminus i})^\top y) + KL(y\|c_{\setminus i}) \leq c_i \}$$

(13)

We also define the coefficients of such signomials

Definition 2 Given a matrix defined by a finite collection of vectors $A = [a^{(1)}, a^{(2)}, \ldots, a^{(l)}] \in \mathbb{R}^{n \times l}$ and convex set $C \subseteq \mathbb{R}^n$, the set of coefficients defining $AGE(A, C, i)$ are described as follows:

$$C_{AGE}(A, C, i) = \{ c \mid f(x) = \sum_{j=1}^{l} c^j \exp\{a^{(j)} \cdot x\}, \ c_{\setminus i} \geq 0 \}$$

(14)

$$= \{ c \mid f(x) = \sum_{j=1}^{l} c^j \exp\{(a^{(j)} - a^{(i)}) \cdot x\} \geq 0 \ \forall x \in C, \ c_{\setminus i} \geq 0 \}$$

(15)

$$= \{ c \mid \exists y \in \mathbb{R}^l_+ \text{ such that } \sigma_C(-({\tilde{A}}^{(i)}_{\setminus i})^\top y) + KL(y\|c_{\setminus i}) \leq c_i \}$$

(16)

We observe that $C_{AGE}(A, C, i)$ is a closed convex cone. Using the above two definitions, we may define summation of such signomials.

Lemma 1 $AGE(A, C, i)$ and $C_{AGE}(A, C, i)$ are both closed under closure

Proof: Clear from the definition of positivity over a convex set and positivity of coefficients on all but ith term, both are closed under addition.

6 Constrained-SAGE Relaxation

We may use the above certificate to provide relaxation for signomial positivity over a constrained set. The key is to find a decomposition of a signomial such that each part has at most one negative term and is positive over a constrained set (i.e. Sums-of-AGE).

Definition 3 Given a matrix defined by a finite collection of vectors $A = [a^{(1)}, a^{(2)}, \ldots, a^{(l)}] \in \mathbb{R}^{n \times l}$ and convex set $C \subseteq \mathbb{R}^n$, signomials defined by $A$ that can be decomposed into $AGE(A, C, i)$ are defined as:

$$SAGE(A, C) = \{ f \mid f = \sum_{j=1}^{l} f_i, f_i \in AGE(A, C, i) \}$$

(17)

From the definitions the following are clear:

Remark 1 $\forall i \ AGE(A, C, i) \subseteq SAGE(A, C)$

Remark 2 $SAGE(A, C) \subseteq \{ f : f = \sum_{j=1}^{l} c^j \exp\{a^{(j)} \cdot x\} \geq 0 \ \forall x \in C \}$
We also define the coefficients of such signomials.

**Definition 4** Given a matrix defined by a finite collection of vectors \( \mathbf{A} = [\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(l)}] \in \mathbb{R}^{n \times l} \) and convex set \( C \subseteq \mathbb{R}^n \), the set of coefficients defining \( \text{SAGE}(\mathbf{A}, C) \) are described as follows:

\[
\text{C}_{\text{SAGE}}(\mathbf{A}, C) = \left\{ c \in \mathbb{R}^l \mid \sum_{j=1}^l c_j \exp\{\mathbf{a}^{(j)\top} \mathbf{x}\} \in \text{SAGE}(\mathbf{A}, C, i) \right\}
\]

(18)

\[
= \left\{ c \in \mathbb{R}^l \mid \exists \mathbf{c}^{(j)} \in \mathbb{R}^l, j = 1, \ldots, l \text{ s.t. } c = \sum_{j=1}^l c^{(j)} \right\}
\]

(19)

\[
\left( \begin{array}{c}
\mathbf{c}_{(j)}^{(j)} \\
\mathbf{c}_{(j)}^{(j)}
\end{array} \right) \in \text{C}_{\text{AGE}}(\tilde{\mathbf{A}}^{(i)}, C, i)
\]

(20)

\[
\text{Lemma 2} \quad \text{SAGE}(\mathbf{A}, C) \quad \text{and} \quad \text{C}_{\text{SAGE}}(\mathbf{A}, C) \quad \text{are both closed under closure}
\]

**Proof:** Follows from additive closure of \( \text{AGE}(\mathbf{A}, C, i) \) and \( \text{C}_{\text{AGE}}(\mathbf{A}, C, i) \) as in Lemma 1.

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## 7 Convex Relaxation for Constrained Signomial Optimization

We discuss how the above definitions can be used for convex relaxation of constrained signomial optimization.

Given a matrix defined by a finite collection of vectors \( \mathbf{A} = [\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(l)}] \in \mathbb{R}^{n \times l} \), and a convex set \( C \), let \( f(\mathbf{x}) := \sum_{j=1}^l c_j \exp\{\mathbf{a}^{(j)\top} \mathbf{x}\} \). WLOG assume that \( \mathbf{a}^{(1)} = 0 \). Consider the following problem:

\[
f^* = \inf_{\mathbf{x} \in C} f(\mathbf{x})
\]

(21)

The minimization problem can be reformulated as a certification of positivity.

\[
f^* = \inf_{\mathbf{x} \in C} f(\mathbf{x}) = \sup_\gamma \gamma \text{ s.t. } f(\mathbf{x}) - \gamma \geq 0 \ \forall \mathbf{x} \in C
\]

(22)

(23)

### 7.1 Inner Approximation

However, the problem is in general intractable, as certifying signomial positivity is intractable. The definition of \( \text{SAGE}(\mathbf{A}, C) \) provides a tractable lower bound. Consider the following problem and its solution.

**Definition 5** Let \( f_{\text{SAGE}} = \sup_\gamma \gamma \text{ s.t. } f(\mathbf{x}) - \gamma \in \text{SAGE}(\mathbf{A}, C) \). Recall Remark 2. Then it is clear that \( f_{\text{SAGE}} \leq f^* \).

A more concrete formulation of the problem is below:

\[
f_{\text{SAGE}} = \sup_\gamma \gamma \text{ s.t. } f(\mathbf{x}) - \gamma \in \text{SAGE}(\mathbf{A}, C)
\]

(24)

\[
= \sup_\gamma \gamma \text{ s.t. } \mathbf{c}_f - \gamma \mathbf{e}^{(1)} \in \text{C}_{\text{SAGE}}(\mathbf{A}, C)
\]

(25)

where \( \mathbf{e}^{(1)} \in \mathbb{R}^n \text{ s.t. } \mathbf{e}_1^{(1)} = 1, \mathbf{e}_i^{(1)} = 0 \ \forall i \neq 1 \), and \( \text{C}_{\text{SAGE}}(\mathbf{A}, C) \) is characterized by convex constraints in section 6.

The above can be considered a relaxation through an inner approximation of signomial positivity over a constrained set. That is, we rely on the the property that \( \text{SAGE}(\mathbf{A}, C) \subseteq \{ f : f = \sum_{j=1}^l c_j \exp\{\mathbf{a}^{(j)\top} \mathbf{x}\}^\top \mathbf{x} \geq 0 \ \forall \mathbf{x} \in C \} \). It is an approximation because the LHS is a subset of the RHS, but not necessarily equal to it.
8 Hierarchy of Relaxation

Given a matrix defined by a finite collection of vectors $A = [a^{(1)}, a^{(2)}, \ldots, a^{(l)}] \in \mathbb{R}^{n \times l}$, define the following notation.

**Definition 6** For some $p \in \mathbb{Z}_{++}$

$$E_p(A) = \{ a \in \mathbb{R}^n : a = \sum_{i=1}^{l} w_i a^{(i)} \text{ such that } w_i \in \mathbb{Z} \forall i, \text{ and } \sum_{i} w_i \leq p \}$$

Note that $E_1(A) = A$. Also it is clear that $|E_p(A)| = l^p$. Using the above set of exponentials, we may define a quantity as follows.

**Definition 7** Let $f_{SAGE}^{(p)} = \sup_{\gamma} \gamma \text{ s.t. } f(x) - \gamma \in SAGE(E_p(A), C)$

The significance of the quantity is expressed in the following theorem.

**Theorem 2** For some $p \in \mathbb{Z}_{++}$, and any $f(x), f_{SAGE}^{(p)}(x) \leq f_{SAGE}^{(p+1)}(x) \leq f^*$.

That is, convex relaxation of the constrained optimization problem has a hierarchy that becomes non-decreasingly accurate. Before proving the theorem, we prove the following two theorems for better understanding of the set $SAGE(E_p(A), C)$.

**Theorem 3** For any $p \in \mathbb{Z}_{++}$, if $f(x) \in SAGE(E_p(A), C)$, then $(\sum_{j=1}^{l} \exp(a^{(j)\top} x))f(x) \in SAGE(E_{p+1}(A), C)$.

**Proof of Theorem 3** Let $E_p(A) = \{ d_k \}_{k=1}^{l}$. Suppose $f(x) \in SAGE(E_p(A), C)$. Then $f(x) = \sum_{i=1}^{l} f_i(x)$ where $f_i(x) \in AGE(E_p(A), C, i)$. By definition, this implies $\forall i, f_i(x) = \sum_{k=1}^{l} c_{k}^{(i)} \exp(d_{k}^{(i)\top} x) \geq 0 \forall x \in C$ and $c_{k}^{(i)} \geq 0$.

The conditions hold for $(\exp(a^{(j)\top} x))f_i(x) = \sum_{k=1}^{l} c_{k}^{(i)} \exp((a^{(j)} + d_{k})^{\top} x) \forall i, j$ as well. So

$$\forall i, j (\exp(a^{(j)\top} x))f_i(x) \in AGE(E_{p+1}(A), C, i) \subseteq SAGE(E_{p+1}(A), C)$$

Finally, by additive closure property (lemma 2 of $SAGE(E_{p+1}(A), C)$, we have $(\sum_{j=1}^{l} \exp(a^{(j)\top} x))f(x) \in SAGE(E_{p+1}(A), C)$

**Theorem 4** For any $p \in \mathbb{Z}_{++}$, $SAGE(E_p(A), C) \subseteq \{ f : f(x) \geq 0 \forall x \in C \}$

**Proof of Theorem 4** Note that multiplication of a posynomial does not change the positivity of a signomial over a convex set. In particular, for any $p \in \mathbb{Z}_{++}$, $f(x) \geq 0 \forall x \in C$ if and only if $(\sum_{j=1}^{l} \exp(a^{(j)\top} x))p f(x) \geq 0 \forall x \in C$. Thus $f(x) \in SAGE(E_{p+1}(A), C) \implies f(x) \geq 0 \forall x \in C$.

**Proof of Theorem 2** Let $f_{SAGE}^{(p)}(x) = \gamma$. Then

$$\sum_{j=1}^{l} \exp(a^{(j)\top} x))^p (f(x) - \gamma) \in SAGE(E_p(A), C)$$

$$\implies \sum_{j=1}^{l} \exp(a^{(j)\top} x))^p+1 (f(x) - \gamma) \in SAGE(E_{p+1}(A), C) \text{ by Theorem 3}$$

$$\implies f_{SAGE}^{(p+1)}(x) \geq \gamma = f_{SAGE}^{(p)}(x)$$
And we have the first inequality. Let $f_{SAGE}^{(p+1)}(x) = \gamma$. Then

$$\left(\sum_{j=1}^{l} \exp\{a^{(j)\top} x\}\right)^{p+1} (f(x) - \gamma) \in SAGE(E_{p+1}(A), C)$$

$$\implies \left(\sum_{j=1}^{l} \exp\{a^{(j)\top} x\}\right)^{p+1} (f(x) - \gamma) \geq 0 \forall x \in C \quad \text{by Theorem 4}$$

$$\implies f(x) - \gamma \geq 0 \forall x \in C \implies f^* \geq \gamma = f_{SAGE}^{(p+1)}(x)$$

And we have the second inequality.

\section{A Completeness Theorem}

\textbf{Theorem 5} Let $\{a^{(j)}\}_{j=1}^{l} \subset \mathbb{Q}^n$ be a collection of rational vectors. Let $H$ be a set of (possibly infinite) halfspaces defined by rationals. $\forall h \in H, h = \{x : w^{(h)\top} x \leq t^{(h)}\}$ where $w^{(h)} \in \mathbb{Q}^n$ and $t^{(h)} \in \mathbb{Q}$. Let $S_H = \bigcap_{h \in H} h$. Assume that $S_H$ is compact. Let $A = [a^{(1)} \ldots a^{(l)}] \in \mathbb{Q}^{l \times n}$

Consider a signomial $f(x) : \mathbb{R}^n \to \mathbb{R}$ defined by such collection of vectors. $f(x) = \sum_{j=1}^{l} c_j \exp\{a^{(j)\top} x\}$. Suppose that the signomial is positive over $S_H$. There exists some $r \in \mathbb{Z}_+$ s.t. $(\sum_{j=1}^{l} \exp\{a^{(j)\top} x\})^r f(x) \in SAGE(E_{p+1}(A), S_H)$.

We also note the following classic result from convex analysis [13].

\textbf{Theorem 6} Any closed convex set $C$ may be expressed as the intersection of (possibly infinite) halfspaces.

Here, we restrict the halfspaces to be defined by rationals.

\section{Proof of Completeness Theorem}

The proof structure is as follows. First, we show that the halfspace constraints and rational exponents can be converted into polynomial equations after a change of variable. Then, we make modifications to the polynomials so that they are homogeneous, and add redundant constraints so that the its extension from positive orthant to the non-negative orthant does not increase the feasible region. The goal of variable change operation is to reduce signomial positivity over a convex set to positivity over the intersection semi-algebraic set and the non-negative orthant. Then, we apply Positivstellensatz result from algebraic geometry to decompose the positive polynomial into sum of homogeneous polynomials [11]. Lastly, we show that the decomposition is certifiable as SAGE.

Without loss of generality, we may make the following assumptions about the collection of vectors $\{a^{(j)}\}_{j=1}^{l} \subset \mathbb{Q}^n$.

(a) the first $n$ vectors $\{a^{(j)}\}_{j=1}^{n}$ are linearly independent

(b) $a^{(n+1)} = 0$

The assumptions on the exponents are in fact not restrictive. To satisfy the first condition, we may select a set of linearly independent vectors as the first $n$. The proof is easily generalized to the case when the span of the vectors has dimension less than $n$. The second condition is not restrictive either, since we may insert a zero vector into the set of exponents. However, it is a variable required for satisfying certain conditions in the proof.
10.1 Variable Change

In this section we prove the following.

**Theorem 7** Consider a signomial function \( f(x) = \sum_{j=1}^{l} c_j \exp\{a^{(j)^T} x\} \) and a set of halfspaces \( H \) satisfying conditions in Theorem 5. Then, there exists a set \( T_{++}^{(A, H)} \) at the intersection of the positive orthant and the a set of polynomial equations, such that \( \inf_{x \in S_H} f(x) > 0 \iff \inf_{y \in T_{++}^{(A, H)}} c^T y > 0 \)

10.1.1 Exponents Defined by Rational Vectors

Consider set of exponents \( \{\exp\{a^{(j)^T} x\}\}_{j=1}^{l} \) defined by rational vectors satisfying conditions in Theorem 5.

Apply a change of variable by letting \( y_j = \exp\{a^{(j)^T} x\} \). First, since the first \( n \) vectors are linearly independent, they span \( \mathbb{R}^n \). Thus the set of first \( n \) exponents \( \{\exp\{a^{(i)^T} x\}\}_{i=1}^{n} = \{y_j\}_{j=1}^{n} \) are free, and may take any value. Next, since \( a_{n+1} = 0 \), \( y_{n+1} = \exp\{a^{(n+1)^T} x\} = 1 \). The rest of the vectors may be expressed as linear combinations of the first \( n \) vectors. Moreover, the linear combinations are defined by rationals since the exponents are rationals. Thus, they are constrained with respect to the first \( n \) vectors. For \( a^{(j)} \) with \( j \geq n + 2 \), \( a^{(j)} = \sum_{i=n+1}^{l} w^{(j)}_i a^{(i)} \). Then:

\[
y_j = \exp\{a^{(j)^T} x\} = \exp\{\sum_{i=1}^{n+1} w^{(j)}_i a^{(i)} x\} = \exp\{\prod_{i=1}^{n+1} (\exp\{a^{(i)} x\})^{w^{(j)}_i}\} = \prod_{i=1}^{n+1} y^{w^{(j)}_i} = \prod_{i=1}^{n} y^{w^{(j)}_i} = \prod_{i=1}^{n+1} y^{w^{(j)}_i} = \prod_{i=1}^{n} y^{w^{(j)}_i}
\]

The last step is from the fact that \( a_n = 0 \). \( w^{(j)}_i \geq 0 \ \forall i \) and since \( w^{(j)}_i \)'s are rationals, we may raise both sides by the smallest common denominator to clear the fractions. For example, \( y_j = y_1^x y_2^y \iff y_j = y_1^x y_2 \).

We may apply such operation to \( y_j \) for all \( j \geq n + 2 \). Note that the operation is only valid in the positive orthant. Thus, with the assumptions in Theorem 5 change of variable has converted a set of rational exponents to polynomial equations as follows.

\[
y_j = \exp\{a^{(j)^T} x\} \ \forall j = 1...l
\]
\[
\iff y_j = \prod_{i=1}^{n} y^{w^{(j)}_i} \ \forall j = n+2...l
\]
\[
\iff y^{\lambda^{(j)}_i} = \prod_{i=1}^{n} y^{\lambda^{(j)}_i} \ \forall j = n+2...l
\]
\[
\iff y^{\lambda^{(j)}_i} - \prod_{i=1}^{n} y^{\lambda^{(j)}_i} = 0 \ \forall j = n+2...l
\]

Where \( \lambda^{(j)} \)'s are obtained from the procedure as above.
10.1.2 Halfspace Defined by Rationals

We first consider a single rational halfspace constraint on \( x \in \mathbb{R}^n \). Let \( h = \{ x : w^T x \leq t \} \). \( w \in \mathbb{Q}^n \) and \( t \in \mathbb{Q} \).

We note the following known theorem in linear algebra [8]:

**Theorem 8** Let \( P \subset \mathbb{R}^n \) be a polyhedron and let \( T : \mathbb{R}^n \rightarrow \mathbb{R}^p \) be a linear transformation. Then \( T(P) \subset \mathbb{R}^p \) be a polyhedron. Further, if \( P \) is a rational polyhedron and \( T \) is a rational linear transformation (that is, the matrix of \( T \) is rational), then \( T(P) \) is a rational polyhedron.

Thus, given the rational halfspace constraint on \( x \), we may find a rational polyhedron constraint on \( Ax \). \( w^T x \leq t \iff B(Ax) \leq d \) for some \( B \in \mathbb{Q}^{p \times l} \) and \( d \in \mathbb{Q}^p \). The dimension \( p \) is arbitrary, but is finite by the above theorem. We apply a series of elementary arithmetic operations as below.

\[
\begin{align*}
y & = \exp\{Ax\} : w^T x \leq t \\
\iff & y = \exp\{Ax\} : B(Ax) \leq d \\
\iff & \log y = Ax : B(Ax) \leq d \\
\iff & B(\log y) \leq d \\
\iff & \sum_i B_{ki}(\log y_i) \leq d_k \quad \forall k = 1 \ldots p \\
\iff & \sum_i \log y_i^{B_{ki}} \leq d_k \quad \forall k = 1 \ldots p \\
\iff & \log \prod_i y_i^{B_{ki}} \leq d_k \quad \forall k = 1 \ldots p \\
\iff & \prod_{B_{ki} > 0} y_i^{B_{ki}} \leq \prod_{B_{ki} < 0} y_i^{-B_{ki}} \exp\{d_k\} \quad \forall k = 1 \ldots p
\end{align*}
\]

The last step moves exponents with negative terms by multiplication on both sides. For example; \( y_1^2 y_2^{-3} \leq 1 \iff y_1 \leq y_2^3 \). Again, since \( B \in \mathbb{Q}^{k \times l} \) has rational entries, we may raise both sides by a common denominator to clear fraction.

\[
\begin{align*}
\prod_{B_{ki} > 0} y_i^{B_{ki}} & \leq \prod_{B_{ki} < 0} y_i^{-B_{ki}} \exp\{d_k\} \quad \forall k = 1 \ldots p \\
\iff & \prod_{B_{ki} > 0} y_i^{m(k)B_{ki}} \leq \prod_{B_{ki} < 0} y_i^{-m(k)B_{ki}} \exp\{m(k)d_k\} \quad \forall k = 1 \ldots p \\
\iff & \prod_{B_{ki} < 0} y_i^{-m(k)B_{ki}} \exp\{m(k)d_k\} - \prod_{B_{ki} > 0} y_i^{m(k)B_{ki}} \geq 0 \quad \forall k = 1 \ldots p
\end{align*}
\]

Where \( m(k) \) is the common denominator of the fractions.

10.1.3 Intersection of Halfspaces Defined by Rationals

It is easy to extend the above to intersection of (possibly infinite) halfspaces. In the above, single halfspace constraint has generated a finite set of polynomial equations. Given a set of halfspaces, we simply take the intersection of the polynomial equations generated from them. For each \( k \), Let \( \gamma_i = m(k)B_{ki} \in \mathbb{Z}^l \) and \( c^{(k)} = \exp\{m(k)d_k\} \in \mathbb{R}_+ \). We may write as below.

\[
c^{(k)} \prod_{\gamma_i < 0} y_i^{-\gamma_i} - \prod_{\gamma_i > 0} y_i^{\gamma_i} \geq 0 \quad \forall k \in K
\]

\( K \) is a set of (possibly infinite) indices, each corresponding to some polynomial generated from the rational halfspaces.
10.1.4 Positivity of Signomial to Positivity of Polynomial

In the previous sections, we have performed a change of variable transforming rational exponents over the intersection of rational halfspaces to a vector in the positive orthant constrained by polynomial equations. Now we consider the optimization in the new variables. We define a set of feasible values of $y$ as below.

$$T_{++}^{(A,H)} = \{ y = \exp(\{A\mathbf{x}\}) \in \mathbb{R}^n_{++} : \mathbf{x} \in S_H \}$$

$$= \{ y \in \mathbb{R}^n_{++} : y_{n+1} = 1, \quad y_j^{\lambda_j} - \prod_i^n y_i^{\lambda_i(j)} = 0 \quad \forall j = n + 2...l$$

$$\quad c^{(k)} \prod_i y_i^{-\gamma_i(k)} - \prod_i y_i^{\gamma_i(k)} \geq 0 \quad \forall k \in K \}$$

And the optimization problem can be recasted as follows

$$\inf_{x \in S_H} f(x) = \inf_{x \in S_H} \sum_j c_j \exp(\{a^{(j)T}\mathbf{x}\})$$

$$= \inf_{y \in T_{++}^{(A,H)}} c^T y$$

We recall Theorem 7 which can be written as follows. Based on the variable change from section 10.1, we have:

$$\inf_{x \in S_H} f(x) > 0 \iff \inf_{y \in T_{++}^{(A,H)}} c^T y > 0$$

10.2 Positivity to Positivestellensatz

So far we have reduced the positivity of a signomial over a convex set to the positivity of a polynomial over the intersection of the positive orthant and a set defined by (possibly infinite) polynomial equations. Now we note the key Positivestellensatz results [11].

**Theorem 9** Let be $m \in \mathbb{Z}_{++}$ and $f_0,...,f_m \in \mathbb{R}[\mathbf{x}]$ be homogeneous polynomials on $\mathbb{R}^n$ such that $f_0(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathbb{R}^n_+ \cap \bigcap_{i=1}^m f_i^{-1}(\mathbb{R}_+) \setminus \{0\}$ and $f_1(\mathbf{x}) = 1$. Then for some $r \in \mathbb{Z}_+$, there exists homogeneous polynomials $g_1,...,g_m \in \mathbb{R}[\mathbf{x}]$ such that all of their coefficients are nonnegative and $(\sum_{i=1}^n x_i)^r f_0(\mathbf{x}) = \sum_{i=1}^m f_i(\mathbf{x}) g_i(\mathbf{x})$

The condition of this Positivestellensatz requires positivity of a polynomial over the intersection of nonnegative orthant and a semialgebraic set. $T_{++}^{(A,H)}$ is not such set. There are three conditions required by the Positivestellensatz theorem that $T_{++}^{(A,H)}$ does not satisfy.

1. $T_{++}^{(A,H)}$ does not include the faces of the nonnegative orthant.
2. $T_{++}^{(A,H)}$ is defined by polynomials that are possibly non-homogeneous.
3. $T_{++}^{(A,H)}$ is defined by possibly infinite polynomials.

The goal of this section is to show the following:

**Theorem 10** There exists a set $T_{++}^{(A,H)}'$ defined as the intersection of the nonnegative orthant and a semialgebraic set s.t.

$$\inf_{y \in T_{++}^{(A,H)}} c^T y > 0 \implies \inf_{y \in T_{++}^{(A,H)} \setminus \{0\}} c^T y > 0$$

Note that implication only needs to be true in one direction. In other words, $T_{++}^{(A,H)}'$ satisfy the conditions for Positivestellensatz, positivity over $T_{++}^{(A,H)}$ implies positivity over $T_{++}^{(A,H)}'$. In the next three sections, we show how modifications on $T_{++}^{(A,H)}$ to satisfy the three conditions, while positivity condition remains true.
10.2.1 Inclusion of Points on the Faces of Nonnegative Orthant

First consider the following set.

\[ T_{++}^{(A,H)} = \{ y \in \mathbb{R}^n_+ : y^{A(x)^{(j)}} - \prod_{i} y_i^{A(x)^{(j)}} = 0 \quad \forall j = n + 2 \ldots \} \]

\[ c^{(k)} \prod_{i} y_i^{-\gamma(H)^{(k)}} - \prod_{i} y_i^{\gamma(H)^{(k)}} \geq 0 \quad \forall k \in K \}

which extends the definition of \( T_{++}^{(A,H)} \) to nonnegative orthant.

A sufficient condition for \( \inf_{y \in T_{++}^{(A,H)}} c^T y < \inf_{y \in T_{++}^{(A,H)}} c^T y \) is that \( \text{cl}(T_{++}^{(A,H)}) \supseteq T_{++}^{(A,H)} \). However, this is not true in general for any \( A \) and \( H \).

**Proposition 1** There exists \( A \) and \( H \) satisfying conditions in Theorem 5 such that \( \text{cl}(T_{++}^{(A,H)}) \supseteq T_{++}^{(A,H)} \).

**Proof of Proposition** We prove this by an example. Let \( A = [a_1, a_2, a_3, a_4] \) where \( a_1 = [1, 0]^T, a_2 = [0, 1]^T, a_3 = [0, 0]^T, a_4 = [0, 1, 0]^T \). Let \( H = \{ [x] \in \mathbb{R}^3 : -2x_1 + x_2 \leq 0 \}, [x] \in \mathbb{R}^3 : x_1 - 2x_2 \leq 0 \}. \) It is easy to check that they satisfy the conditions in Theorem 5.

Let \( y = \exp\{Ax\} \). We have

\[ y_1 = \exp\{x_1\} \]
\[ y_2 = \exp\{x_2\} \]
\[ y_3 = 0 \]
\[ y_4^{10} = y_1 y_2 \]

Also

\[ x \in S_H \iff -2x_1 + x_2 \leq 0 \text{ and } x_1 - 2x_2 \leq 0 \]
\[ -2 \log y_1 + \log y_2 \leq 0 \text{ and } \log y_1 - 2 \log y_2 \leq 0 \]
\[ \log(y_1^2 y_2) \leq 0 \text{ and } \log(y_1 y_2^2) \leq 0 \]
\[ y_1^2 y_2 \leq 1 \text{ and } y_1 y_2^2 \leq 1 \]
\[ y_2 \leq y_1^2 \text{ and } y_1 \leq y_2^2 \]
\[ y_2^2 - y_2 \geq 0 \text{ and } y_2 - y_1 \geq 0 \]

Therefore \( T_{++}^{(A,H)} = [y \in \mathbb{R}^4_+ : y_4^{10} = y_1 y_2, y_1^2 - y_2 \geq 0, y_2^2 - y_1 \geq 0] \) and \( T_{++}^{(A,H)} = [y \in \mathbb{R}^4_+ : y_4^{10} = y_1 y_2, y_1^2 - y_2 \geq 0, y_2^2 - y_1 \geq 0] \). Consider \( y' = [0, 0, 1, 0]^T \). It is easy to check that that \( y' \in T_{++}^{(A,H)} \) but \( y' \notin \text{cl}(T_{++}^{(A,H)}) \).

We may add redundant constraint to \( H \) so that the the closure of \( T_{++}^{(A,H)} \) includes \( T_{++}^{(A,H)} \). We formalize this below.

**Lemma 3** Given any \( A \in \mathbb{Q}^{1 \times n} \) and \( H \) as defined in Theorem 5 there exists a set of polynomial equations \( \Upsilon \) such that \( \text{cl}(T_{++}^{(A,H)}) \supseteq T_{++}^{(A,H)} \cap \Upsilon = T_{++}^{(A,H)} \).

**Proof of Lemma** Since \( S_H \) is compact, \( v_1 \leq x \leq v_2 \) for some vectors \( v_1 \in \mathbb{R}^n, v_2 \in \mathbb{R}^n \). \( y_i = \exp\{A_i x\} \) so \( \min(\exp\{A_i v_1\}, \exp\{A_i v_2\}) \leq y_i \leq \max(\exp\{A_i v_1\}, \exp\{A_i v_2\}) \).
Let \( l_i = \min(\exp(\mathbf{A}, \mathbf{v}_1), \exp(\mathbf{A}, \mathbf{v}_2)) \) and \( u_i = \max(\exp(\mathbf{A}, \mathbf{v}_1), \exp(\mathbf{A}, \mathbf{v}_2)) \). Letting
\[
\Upsilon = \{ y : u_i - y_i \geq 0, y_i - l_i \geq 0 \quad \forall i = 1 \ldots l \}
\]
Since \( y \notin \Upsilon \) if \( y_i = 0 \) for any \( i \), \( \text{cl}(\mathcal{T}_+^{(A,H)}) \supseteq \mathcal{T}_0^{(A,H)} \cap \Upsilon \). By the definitions of \( l_i \) and \( u_i \), \( \mathcal{T}_+^{(A,H)} \cap \Upsilon = \mathcal{T}_+^{(A,H)} \).

Note that the first condition implies \( \inf_{y \in \mathcal{T}_+^{(A,H)}} c^\top y > 0 \implies \inf_{y \in \mathcal{T}_+^{(A,H)} \cap \Upsilon} c^\top y > 0 \). The significance of the second condition will be apparent later.

### 10.2.2 Positivity over Non-negative Orthant of Homogeneous Polynomials

We made the observation that polynomials in the definition of \( \mathcal{T}_+^{(A,H)} \cap \Upsilon \) are not homogeneous. Recall the definition
\[
\mathcal{T}_+^{(A,H)} \cap \Upsilon = \{ y \in \mathbb{R}^n_+ : y_{n+1} = 1 \}
\]
\[
y_j^{\lambda_j} - \prod_{i} y_i^{\lambda_i} = 0 \quad \forall j = n + 2 \ldots \end{array}
\]
\[
\alpha^{(k)} \prod_{i \gamma_i^{(k)} < 0} y_i^{-\gamma_i^{(k)}} - \prod_{i \gamma_i^{(k)} > 0} y_i^{\gamma_i^{(k)}} \geq 0, \gamma^{(k)} \in \mathbb{Z}^n \quad \forall k \in K
\]
\[
u_i - y_i \geq 0, y_i - l_i \geq 0 \quad \forall i = 1 \ldots n
\]

Where in the last expression we have written the terms abstractly. Each \( p_1^{(j)}(y), p_2^{(j)}(y), q_1^{(k)}(y), q_2^{(k)}(y) \) are monomials. We modify the polynomials to be homogeneous by making the following transformation. Let \( [x]_+ = \max(x, 0) \)
\[
\tilde{\mathcal{T}}_+^{(A,H)} \cap \bar{\Upsilon} = \{ y \in \mathbb{R}^n_+ : y_{n+1}^{[\deg(p_2^{(j)}) - \deg(p_1^{(j)})]} \begin{array}{c}
p_1^{(j)}(y) - y_{n+1}^{[\deg(p_1^{(j)}) - \deg(p_2^{(j)})]} p_2^{(j)}(y) = 0 \quad \forall j = n + 2 \ldots \end{array}
\]
\[
y_{n+1}^{[\deg(q_2^{(k)}) - \deg(q_1^{(k)})]} q_1^{(k)}(y) - y_{n+1}^{[\deg(q_1^{(k)}) - \deg(q_2^{(k)})]} q_2^{(k)}(y) \geq 0 \quad \forall k \in K
\]
\[
u_i y_{n+1} - y_i \geq 0, y_i - l_i y_{n+1} \geq 0 \quad \forall i = 1 \ldots l
\]

Notice that we have removed the condition \( y_{n+1} = 1 \). Now we claim the following lemma.

**Lemma 4** Consider a signomial function \( f(\mathbf{x}) = \sum_{j=1}^l c_j \exp(\mathbf{a}^{(j)}_+ \mathbf{x}) \) and a set of halfspaces \( H \) satisfying conditions in Theorem 3. Let \( \mathcal{T}_+^{(A,H)} \) and \( \tilde{\mathcal{T}}_+^{(A,H)} \cap \bar{\Upsilon} \) be defined as above. Then \( \inf_{y \in \mathcal{T}_+^{(A,H)} \cap \Upsilon} c^\top y > 0 \implies \inf_{y \in \tilde{\mathcal{T}}_+^{(A,H)} \cap \bar{\Upsilon}} c^\top y > 0 \)

**Proof of Lemma 4** Consider some \( y \in (\tilde{\mathcal{T}}_+^{(A,H)} \cap \bar{\Upsilon}) \setminus 0 \). First we have that \( y_{n+1} \neq 0 \) for otherwise \( y = 0 \) by the constraints \( u_i y_{n+1} - y_i \geq 0 \quad \forall i \).
Let \( \tilde{y} = y / y_{n+1} \). Since \( \tilde{T}_+^{(A,H)} \cap \tilde{\Upsilon} \) is a semialgebraic set defined by homogeneous polynomials, it is closed under positive scaling thus \( \tilde{y} \in \tilde{T}_+^{(A,H)} \cap \tilde{\Upsilon} \). Since \( y_{n+1} = 1 \), the conditions for \( \tilde{T}_+^{(A,H)} \cap \tilde{\Upsilon} \) reduces to conditions for \( T_+^{(A,H)} \cap \Upsilon \), and thus \( \tilde{y} \in T_+^{(A,H)} \cap \Upsilon \). By assumption \( c^\top \tilde{y} > 0 \), and so \( c^\top y = c^\top (y_{n+1} \tilde{y}) > 0 \).

**10.2.3 Positivity Over the Non-negative Orthant of Finite Homogeneous Polynomials**

We claim the following.

**Lemma 5** Consider a signomial function \( f(x) = \sum_{j=1}^l c_j \exp\{a(j)^\top x\} \) and a set of halfspaces \( H \) satisfying conditions in Theorem 5. \( \tilde{T}_+^{(A,H)} \cap \tilde{\Upsilon} \) be defined as in Section 10.2.3. Then there exists a set \( T^{(A,H)}' \) defined at the intersection of non-negative orthant and semialgebraic set, defined by finite polynomials such that 
\[
\inf_{y \in \tilde{T}_+^{(A,H)} \cap \tilde{\Upsilon} \setminus 0} c^\top y > 0 \implies \inf_{y \in T^{(A,H)}'} c^\top y > 0
\]

Now consider the following theorem. It is adapted from [11].

**Theorem 11** Consider a set of homogeneous polynomials \( \{f_0\} \cup \{f_i \mid i \in I\} \subseteq \mathbb{R}[x] \) with infinite cardinality. If \( f_0(x) > 0 \) for all \( x \in \mathbb{R}_+^n \), if \( \bigcap_{i \in I} f_i^{-1}(\mathbb{R}_+) \setminus \{0\} \), there exists a subset \( J \subseteq I \) of finite cardinality such that \( f_0(x) > 0 \) for all \( x \in \mathbb{R}_+^n \cap \bigcap_{i \in J} f_i^{-1}(\mathbb{R}_+) \setminus \{0\} \).

The proof of Lemma 5 as well. It is involved and is thus left in the appendix.

**Proof of Theorem 10** In Theorem 11 let \( \tilde{T}_+^{(A,H)} \cap \tilde{\Upsilon} = \mathbb{R}_+^n \cap \bigcap_{i \in I} f_i^{-1}(\mathbb{R}_+) \) and \( T^{(A,H)}' = \mathbb{R}_+^n \cap \bigcap_{i \in J} f_i^{-1}(\mathbb{R}_+) \). The desired result follows. Note that this is a non-constructive proof, but the finite polynomials are a subset of ones defining \( \tilde{T}_+^{(A,H)} \cap \tilde{\Upsilon} \).

Before ending this section, also consider the following lemma.

**Lemma 6** Given any \( A \in \mathbb{Q}^{n \times n} \) and \( H \) as defined in Theorem 5, let \( x \in \mathbb{R}_+^n \) and let \( y = \exp\{Ax\} \). Then \( x \in S_H \implies y \in (T^{(A,H)})' \setminus 0 \).

**Proof of Lemma 6** From Section 10.1.4 it is clear that \( x \in S_H \implies y \in T^{(A,H)} \). By Lemma 3 \( \tilde{T}_+^{(A,H)} = T_+^{(A,H)} \cap \Upsilon \) so \( y \in T_+^{(A,H)} \cap \Upsilon \). Also, \( x \in S_H \) implies \( y_{n+1} = \exp\{a_{n+1}x\} = 1 \). By construction of \( \tilde{T}_+^{(A,H)} \) \( \cap \tilde{\Upsilon} \) in Section 10.2.3 if \( y \in T_+^{(A,H)} \cap \Upsilon \) and \( y_{n+1} = 1 \), then \( y \in \tilde{T}_+^{(A,H)} \cap \tilde{\Upsilon} \). Thus \( y \in \tilde{T}_+^{(A,H)} \cap \tilde{\Upsilon} \). \( T^{(A,H)}' \supseteq \tilde{T}_+^{(A,H)} \cap \tilde{\Upsilon} \) by construction of choosing a subset of polynomials, so \( y \in T^{(A,H)}' \). Lastly, \( y = \exp Ax \neq 0 \). The desired result is obtained.

To summarize Section 10.2.

**Proof of Theorem 10**

\[
\inf_{x \in S_H} f(x) > 0 \implies \inf_{y \in T_+^{(A,H)}} c^\top y > 0 \quad \text{by Theorem 7}
\]
\[
\implies \inf_{y \in (T_+^{(A,H)} \cap \Upsilon) \setminus 0} c^\top y > 0 \quad \text{by Lemma 5}
\]
\[
\implies \inf_{y \in (T_+^{(A,H)} \cap \tilde{\Upsilon}) \setminus 0} c^\top y > 0 \quad \text{by Lemma 4}
\]
\[
\implies \inf_{y \in (T_+^{(A,H)} \cap \tilde{\Upsilon}) \setminus 0} c^\top y > 0 \quad \text{by Lemma 5}
\]

Also:

\[
x \in S_H \implies y \in (T^{(A,H)})' \setminus 0
\]
10.3 Positivestellensatz to SAGE decomposition

From the previous section, we have that

\[ \inf_{x \in S_H} f(x) > 0 \implies \inf_{y \in T^{(A,H)' \setminus 0}} c^\top y > 0 \]

Observe that all homogeneous polynomials defining \( T^{(A,H)'} \) are of the form \( m_1(y) - m_2(y) \); the difference of two monomials. Thus without loss of generality, \( T^{(A,H)'} = \{ y : m_1^{(j)}(y) - m_2^{(j)}(y) \geq 0 \; j = 1 \ldots m \} \).

Applying Theorem 9, for some \( r \in \mathbb{Z} \), there exists homogeneous posynomials \( g_1(y) \ldots g_m(y) \) s.t.

\[ (\sum_{i=1}^{n} y_i)^r c^\top y = \sum_{j=1}^{m} g_j(y)(m_1^{(j)}(y) - m_2^{(j)}(y)) \]

**Theorem 12** Consider a signomial \( f(x) = \sum_{i=1}^{J} c_j \exp(a_j^\top x) \) and a set of halfspaces \( H \) satisfying conditions in Theorem 4 and \( T^{(A,H)'} \) as constructed in the previous sections. Then the RHS of Positivestellensatz, after change of variable, is a conditional-SAGE.

**Proof of Theorem 12** The condition is that \( \inf_{y \in T^{(A,H)'} \setminus 0} c^\top y > 0 \). A key observation is that the LHS is a homogeneous polynomial. Thus without loss of generality, for all \( j \), \( \deg(g_j(y)) + \deg(m_1^{(j)}(y)) = r + 1 \). Moreover, for each \( j \), \( g_j(y) = \sum_{k}^{l(j)} h_k^{(j)}(y) \) where \( h_k^{(j)} \) is a monomial of one term. Then

\[ (\sum_{i=1}^{n} y_i)^r c^\top y = \sum_{j=1}^{m} g_j(y)(m_1^{(j)}(y) - m_2^{(j)}(y)) \]

\[ \quad = \sum_{j=1}^{m} \sum_{k=1}^{l(j)} h_k^{(j)}(y)(m_1^{(j)}(y) - m_2^{(j)}(y)) \]

Now:

\[ x \in S_H \implies y = \exp A x \in T^{(A,H)'} \]  by Lemma 7

\[ \implies m_1^{(j)}(y) - m_2^{(j)}(y) \geq 0 \; \forall j \; \text{by definition of } T^{(A,H)'} \]

\[ \implies h_k^{(j)}(y)(m_1^{(j)}(y) - m_2^{(j)}(y)) \geq 0 \; \forall j, k \; \text{since } h_k^{(j)}(y) \geq 0 \]

\[ \implies h_k^{(j)}(\exp A x)(m_1^{(j)}(\exp A x) - m_2^{(j)}(\exp A x)) \geq 0 \; \forall j, k \]

Let \( l_k^{(j)}(x) = h_k^{(j)}(\exp A x)(m_1^{(j)}(\exp A x) - m_2^{(j)}(\exp A x)) \; \forall j, k \). One may verify that it is a signomial in the exponential form. Make the following observations

- \( l_k^{(j)}(x) \) has one negative term.

- Since \( \deg(g_j(y)) + \deg(m_1^{(j)}(y)) = r + 1 \), the exponential of \( l_k^{(j)}(x) \) are one of \( E_{p+1}(A) \)

Since it is also non-negative over \( x \in S_H, l_k^{(j)}(x) \in SAGE(E_{r+1}(A), S_H) \). By Lemma 2, \( SAGE(E_{r+1}(A), S_H) \) is closed under addition, so \( \sum_{j=1}^{m} \sum_{k=1}^{l(j)} l_k^{(j)}(x) \in SAGE(E_{r+1}(A), S_H) \).
10.4 Discussion

To summarize Section 10, we have

\[ \inf_{x \in S_H} f(x) > 0 \implies \inf_{y \in T(A,H)} c^\top y > 0 \text{ by Theorem 7} \]

\[ \implies \inf_{y \in (T(A,H) \setminus 0)} c^\top y > 0 \text{ by Theorem 10} \]

\[ \implies (\exp(Ax)^\top 1)^\top c^\top (\exp Ax) \in SAGE(E_r+1(A), S_H) \]

for some \( r \in \mathbb{Z}_{++} \) by Theorem 12.

Notice that the compactness assumption allows the construction of redundant constraints in Lemma 3. The redundant constraints are used by two arguments in the proof. First, they ensure that the intersection of the semi-algebraic set and the nonnegative orthant is the same as the intersection of the semi-algebraic and the positive orthant, without restricting the semi-algebraic set (somewhat counterintuitively), as shown in Lemma 3. Second, they allow reducing the positivity over semi-algebraic set to the positivity over semi-algebraic set defined by homogeneous polynomials, as shown in Lemma 4.

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Appendix

Proof of Theorem 11 Consider the set $\Omega = \{x \in \mathbb{R}_+^n : \|x\|_2 = 1\}$, which is compact. $f_0(x) > 0$ for all $x \in \mathbb{R}_+^n \cap \bigcap_{i \in J} f_i^{-1}(\{0\}) \setminus \{0\}$ if $f_0(x) > 0$ for all $x \in S \cap \mathbb{R}_+^n \cap \bigcap_{i \in J} f_i^{-1}(\{0\})$ since $f_0(x)$ is homogeneous and its positivity is invariant to the scale of $x$. Make the following observations:

1. Without loss of generality assume that $\deg(f_i(x)) \leq 1 \forall i \in I$. This implies that $\max_{x \in \Omega}\{\|\nabla f_i(x)\|\} \leq 1 \forall i \in I$.

2. By mean value theorem, for any $x, y \in \Omega$, and any $i \in I$, there exists $\alpha \in [0, 1]$ s.t. $f_i(x) - f_i(y) = (x - y)^\top \nabla f_i(\alpha x)$. Thus

   \[
   \|f_i(x) - f_i(y)\|_2 = \|\nabla f_i(\alpha x)\|_2 \\ \leq \|x - y\|_2 \|\nabla f_i(\alpha x)\|_2 \\ \leq \|x - y\|_2 \max_{x \in \mathcal{S}} \{\|\nabla f_i(x)\|\} \\ \leq \|x - y\|_2 \forall i \in I
   \]

   This implies $\forall i \in I$, $f_i(x)$ is a continuous function.

3. $\forall i \in I$, $\|f_i(x)\|_2 = \|f_i(x) - f_i(0)\|_2 \leq \|x\|_2 \leq 1 \forall x \in \Omega$.

Now define the following compact sets:

- $\Omega_0 = \Omega \cap f_0^{-1}(-\mathbb{R}_+)$
- $\Omega = \Omega \cap f_i^{-1}(\mathbb{R}_+)$
- $\Omega = \Omega \cap \bigcap_{i \in J} f_i^{-1}(\mathbb{R}_+) \setminus \bigcup_{i \in J} f_i^{-1}(\{0\}) \forall J \subseteq I$

Observe that $f_0(x) > 0$ for all $x \in \mathbb{R}_+^n \cap \bigcap_{i \in I} f_i^{-1}(\mathbb{R}_+) \setminus \{0\}$ iff $\Omega_0 \cap \Omega_I = 0$. The goal is to show that $\exists J \subseteq I$ $\Omega_0 \cap \Omega_J = 0$, which implies then $f_0(x) > 0$ for all $x \in \mathbb{R}_+^n \cap \bigcap_{i \in J} f_i^{-1}(\mathbb{R}_+) \setminus \{0\}$.

Consider the function: $\xi(x) = \sup\{-f_i(x) \mid i \in I\} \forall x \in \Omega$. By observation (2), $\xi(x)$ is a supremum of continuous function and is thus continuous. $\Omega_0 \cap \Omega_I = 0$ by assumption, so range of $\xi(x) > 0 \forall x \in \Omega_0$. Moreover, by observation (3), $\xi(x) \in (0, 1] \forall x \in \Omega_0$. Let $\epsilon = \min_{x \in \Omega_0} \xi(x)$. Since $\Omega_0$ is compact, and $\xi(x)$ is continuous, by the extreme value theorem, the min is attained in $\Omega_0$. So $\epsilon \in (0, 1]$.

Consider the following two claims. (1) any $x \in \Omega_0$, there exists some $i \in I$ s.t. $-f_i(x) \geq \frac{2}{3} \xi(x) \geq \frac{2}{3} \epsilon > 0$.

(2) for any $y \in \Omega$ s.t. $\|x - y\|_2 \leq \frac{1}{3} \epsilon$, $f_i(y) \leq f_i(x) + \|x - y\|_2 \leq \frac{2}{3} \epsilon + \frac{1}{3} \epsilon < 0$. So $y \notin \Omega_I$. Now consider the following procedure. Suppose $z_i$ is chosen at $t_i$th iteration of while loop. First, $\exists i \in I$ to add to $J$, by claim

Algorithm 1: Finding $J \subseteq I$ s.t. $\Omega_0 \cap \Omega_J = 0$

Let $J = \emptyset$

while $\exists z \in \Omega_0 \cap \Omega_J$ do

| If for some $i \in I$, $f_i(z) \leq -\frac{2}{3} \epsilon$, then $J = J \cup \{i\}$ |

return $J$

(1) Since $f_i(z_t) \leq -\frac{2}{3} \epsilon < 0$, by claim (2), for any $z_{t+1}$ s.t. $\|z_{t+1} - z_t\|_2 \leq \frac{1}{3} \epsilon$, $z_{t+1} \notin \Omega_i \implies z_{t+1} \notin \Omega_{J \cup I}$. Thus in each iteration, $z_t$ has a distance of at least $\frac{1}{3} \epsilon$ from the previous ones. Since $\Omega_0$ is a compact set, the algorithm terminates in finite time, which implies $\Omega_0 \cap \Omega_J = 0$. The desired $J \subseteq I$ is obtained.

[15] K. Yun and C. Xi. Second-order method of generalized geometric programming for spatial frame optimization. Computer Methods in Applied Mechanics and Engineering, 141:117–123, 1997.
10.5 Constrained AGE Certificate for Common Convex Sets

In Section 5, we derived the condition of positivity for signomial with at most one negative term over a constrained set.

\[ \exists y \in \mathbb{R}_+^t \text{ such that } \sigma_C(-\tilde{A}^T y) + \text{KL}(y\|e c) \leq c_0? \] (26)

Here we show more concrete conditions for frequently occurring convex sets.

10.5.1 Unconstrained

In this degenerate case we have \( C = \mathbb{R}^n \) hence

\[ \sigma_C = \iota(0). \] (27)

This is the case considered by [2], and the condition in (10) can be simplified as:

\[ \exists y \in \mathbb{R}_+^t \text{ such that } A^T y = 0, \text{KL}(y\|e c) \leq c_0? \] (28)

10.5.2 Box constraint

Let \( C = \{ x : l \leq x \leq u \} \) be a box constraint. Then, clearly

\[ \sigma_C(y) = \iota(y) + (u - l)^T y_+, \] (29)

where \( y_+ = \max\{y, 0\} \) is the component-wise positive part of \( y \). Hence, (10) can be simplified as:

\[ \exists (y, z) \in \mathbb{R}_+^t \times \mathbb{R}_+^k \text{ such that } z + A^T y \geq 0, -1^T A^T y + (u - l)^T z + \text{KL}(y\|e c) \leq c_0? \] (30)

10.5.3 Linear constraint

Let \( C = \{ x : Wx \leq r \} \) be a linear constraint, where \( W : \mathbb{R}^n \rightarrow \mathbb{R}^k \) is a linear map. Then, the support function

\[ \sigma_C(y) = \begin{cases} \lambda^T r, & \text{if } y = W^T \lambda \text{ for some } \lambda \geq 0 \\ \infty, & \text{otherwise} \end{cases}, \] (32)

and (10) can be simplified as:

\[ \exists (y, \lambda) \in \mathbb{R}_+^t \times \mathbb{R}_+^k \text{ such that } A^T y + W^T \lambda = 0, \lambda^T r + \text{KL}(y\|e c) \leq c_0? \] (33)

If, instead, some inequality constraints are actually equalities, then we need only drop the corresponding nonnegativity constraint on \( \lambda \).

We can also treat the box constraint as a special case of linear constraints, although a direct treatment as above seems more transparent.

10.5.4 Norm ball constraint

Let \( C = \{ x : \|x\| \leq \gamma \} \), where \( \| \cdot \| \) is some norm (more generally, a closed gauge function) on \( \mathbb{R}^n \) and \( \gamma > 0 \) is a constant. Then,

\[ \sigma_C(y) = \gamma\|y\|_0, \] (34)

where \( \| \cdot \|_0 \) is the dual norm (polar) of \( \| \cdot \| \). We can thus simplify (10) as:

\[ \exists y \in \mathbb{R}_+^m \text{ such that } \gamma\|A^T y\|_0 + \text{KL}(y\|e c) \leq c_0? \] (35)
10.5.5 Quadratic constraint

Let $C = \{x : x^\top Q x + q^\top x + p \leq 0\}$ be a convex quadratic constraint (i.e., $Q \succeq 0$). May also be able to consider intersection of two nonconvex quadratics, use S-lemma.

10.5.6 Posynomial constraint

Let $C = \{x : \sum_{j=1}^k d_j \exp(b_j^\top x) \leq 1\}$ where $d \geq 0$. We can compute the support function using Lagrangian duality:

$$
\sigma_C(z) = \sup_{x \in \mathbb{R}^n} x^\top z \text{ s.t. } \sum_{j=1}^k d_j \exp(b_j^\top x) \leq 1 \tag{36}
$$

$$
= \sup_{x \in \mathbb{R}^n, t \in \mathbb{R}_+^k} x^\top z \text{ s.t. } \sum_{j=1}^k d_j t_j \leq 1, \exp(b_j^\top x) \leq t_j \tag{37}
$$

$$
= \sup_{x \in \mathbb{R}^n, t \in \mathbb{R}_+^k} x^\top z \text{ s.t. } \sum_{j=1}^k d_j t_j \leq 1, b_j^\top x \leq \ln t_j \tag{38}
$$

(Slater’s condition)

$$
\min_{(\lambda)\geq 0} \sup_{x \in \mathbb{R}^n, t \in \mathbb{R}_+^k} x^\top z + \lambda (1 - d^\top t) - v^\top (Bx - \ln t) \tag{39}
$$

$$
= \min_{(\lambda)\geq 0} \lambda + KL(v\|\lambda\delta) \text{ s.t. } B^\top v = z, \tag{40}
$$

where we introduce the matrix $B = [b_1, \ldots, b_k]^\top \in \mathbb{R}^{k \times n}$.

Thus, we can again simplify (10) as:

$$
\exists (y, v, \lambda) \in \mathbb{R}_+^m \times \mathbb{R}_+^k \times \mathbb{R}_+, \text{ such that } B^\top v + A^\top y = 0 \text{ and } \lambda + KL(v\|\lambda\delta) + KL(y\|\epsilon) \leq c_0? \tag{41}
$$

$$
\exists (y, v, \lambda) \in \mathbb{R}_+^m \times \mathbb{R}_+, \text{ such that } A^\top z = 0 \text{ and } \lambda + KL(z\|\epsilon \delta + \lambda\Delta) \leq c_0? \tag{42}
$$

We remark that if $B = A$, then we can further simplify (11) by a change of variable: $z = y + v$. Indeed, one can verify that

$$
KL(z\|\alpha + \beta) = \min_{(y, v) : y + v = z} KL(v\|\alpha) + KL(y\|\beta), \tag{43}
$$

where the minimum is attained at $y = \frac{\alpha}{\alpha + \beta} z$. Applying (44) element-wise we can reduce (11) to the following equivalent condition:

$$
\exists (z, \lambda) \in \mathbb{R}_+^m \times \mathbb{R}_+, \text{ such that } A^\top z = 0 \text{ and } \lambda + KL(z\|\epsilon \delta + \lambda\Delta) \leq c_0? \tag{44}
$$

A similar simplification holds when $A$ and $B$ partially overlap.

10.5.7 Spectrahedron constraint

Let $C = \{x : \sum_{j=1}^k x_j S_j \succeq 0\}$ be a spectrahedron constraint. Using Lagrangian duality we have

$$
\sigma_C(y) = \begin{cases} 0, & \text{if } \exists \Lambda \succeq 0, \text{ such that } \forall j, \text{tr} (S_j \Lambda) + y_j = 0, \\ \infty, & \text{otherwise} \end{cases} \tag{45}
$$

Thus, we can reformulate (10) as:

$$
\exists y \in \mathbb{R}_+^l, \Lambda \succeq 0, \text{ such that } \forall j, \text{tr} (S_j \Lambda) - (a_j - a_0)^\top y = 0, KL(y\|\epsilon) \leq c_0. \tag{46}
$$
10.5.8 Sublevel set constraint

Let \( C = \{ x : h(x) \leq t \} \) be the (sub)level set of a (closed) convex function \( h \). Then, according to [13],

\[
\sigma_C(y) = \inf_{\lambda \geq 0} \lambda h^*(y/\lambda) + \lambda t.
\]

(47)

Thus, (10) can be reformulated as:

\[
\exists (y, \lambda) \in \mathbb{R}^l_+ \times \mathbb{R}_+ \text{ such that } \lambda h^*(-A^Ty/\lambda) + \lambda t + KL(y\|ec) \leq c_0?
\]

(48)

If we take \( h(y) = \|y\| \) and \( t = 1 \) then \( h^* = \iota_{\|\cdot\|} \leq 1 \) and we recover the norm ball constraint. On the other hand, if \( h \) is positive homogeneous and \( t = 0 \), then \( C \) is a convex cone \( K \), in which case we can further simplify (48) as:

\[
\exists y \in \mathbb{R}^l_+ \text{ such that } A^Ty \in K^*, KL(y\|ec) \leq c_0?
\]

(49)

Here \( K^* \) is the dual cone of \( K \).

10.5.9 Epigraphic constraint

Let \( C = \text{epi}(h) := \{ (x, t) : h(x) \leq t \} \)

\[
\sigma_C \left( \begin{pmatrix} y \\ -s \end{pmatrix} \right) = \begin{cases} sh^*(y/s), & \text{if } s \geq 0 \\ \infty, & \text{otherwise} \end{cases}.
\]

(50)

Thus, (10) can be reformulated as:

\[
\exists (y, \lambda) \in \mathbb{R}^{l+1}_+ \text{ such that } (a_m - a_0)^\top y = \lambda, \lambda h^*(-\tilde{A}^Ty/\lambda) + KL(y\|ec) \leq c_0?
\]

(51)

Here we use \( \tilde{A} \) to denote the submatrix of \( A \) with the last column removed.

In particular, if \( h(x) = \|x\| \) is a norm, then \( C \) is a special cone constraint and we can simplify (51) further as:

\[
\exists y \in \mathbb{R}^l_+ \text{ such that } \|\tilde{A}^Ty\| \leq (a_m - a_0)^\top y, KL(y\|ec) \leq c_0?
\]

(52)

10.5.10 Intersection of constraints

We recall the following classical result:

**Theorem 13** Let \( C_1, \ldots, C_k \) be closed convex sets in \( \mathbb{R}^n \) that have a point in common in their relative interiors, then

\[
\sigma_{C_1 \cap C_2 \cap \cdots \cap C_k}(z) = \min_{z_1, \ldots, z_k} \sum_{i=1}^k \sigma_{C_i}(z_i), \text{ s.t. } z = \sum_{i=1}^k z_i.
\]

(53)

Theorem 13 allows us to combine the previous results in a straightforward manner. Indeed, for \( C = \cap_i C_i \), the condition (10) can be reformulated as, thanks to Theorem 13

\[
\exists y \in \mathbb{R}^l_+, (z_1, \ldots, z_k) \in (\mathbb{R}^n)^k \text{ such that } A^Ty + \sum_{i=1}^k z_i = 0 \text{ and } \sum_{i=1}^k \sigma_{C_i}(z_i) + KL(y\|ec) \leq c_0?
\]

(54)

For example, by intersecting the linear constraints in Section 10.5.3 we get an equivalent condition for a polyhedron \( C \). Similarly, by intersecting the posynomial constraints in Section 10.5.6 we get an equivalent condition for a convex set \( C \) cut by many posynomials. We omit other details.