Membrany corrections
to the string anti-string potential
in M5-brane theory

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We study the potential between a string and an anti-string source in M5-theory by using the adS/CFT duality conjecture. We find that the next to leading order corrections in a saddle point approximation renormalize the classical result.
1. Introduction

One of the outstanding problems in M-theory is a better understanding of the world volume theory of the M five-brane. In the recent past there have been various publications discussing that issue from different points of view. A small sample of references is given in [1], [2], [3] and [4]. In the present paper we will use two descriptions of the M-theory five-brane. What we will call a “perturbative” description is the picture that the five-brane is formed by defects in eleven dimensional Minkowski space which arise due to open membranes ending on flat 5+1 dimensional hypersurfaces. From this perspective the world volume theory has longitudinal and transversal degrees of freedom. Since we will not enter a quantitative discussion relying on the perturbative description the qualitative picture will be sufficient for us. (Nevertheless it should be interesting to study the presented configuration from an effective field theoretic approach.)

The dual description which we will call “non-perturbative” is based on Maldacena’s conjecture [5] (further elaborated in [6],[7]), where the five-brane theory is given by M-theory on $ads_7 \times S^4$. In the context of this paper we will read the M of M-theory as an abbreviation for membrane. The precise statement is that M-theory on the space with the metric

$$ds^2 = l_p^2 R^2 \left[ U^2 dx_\parallel^2 + 4 \frac{dU^2}{U^2} + d\Omega_4^2 \right]$$

is equivalent to the world volume theory of $N$ M5-branes sitting on top of each other. The eleven dimensional Planck length has to be taken to zero in the end. (Since it drops out of all our final results we will put it formally to one from now on.) In difference to [5] we have rescaled the world-volume coordinates of the five-brane $x_\parallel \rightarrow R^{3/2} x_\parallel$. The radius $R$ (in Planck units) is related to the number of five-branes by the relation

$$R = (\pi N)^{1/3}.$$  

The supergravity solution [1] is reliable for a large number of five-branes $N$.

We want to apply this duality to the computation of a potential energy density between two straight string sources in the M5-brane theory. In the next section we will do this by a saddle point approximation. (This has already been discussed in [3].) In the following sections we will study corrections to this result due to membrane fluctuations.
2. The Background

In the present paper we take as a “perturbative” definition of the M5-brane theory the picture that the degrees of freedom on the world volume of the M5-brane are described by membranes ending on them. The term “perturbative” means here that the embedding space is 11 dimensional Minkowski space. (This is in analogy to the perturbative definition of Yang-Mills theory on D-branes.) Especially we are interested in a situation where we have one M5 brane separated by a very large distance from a bunch of $N$ M5 branes. In addition we span two straight Membranes between the five-branes such that they end in two parallel strings with distance $L$ on the M5-branes (Fig. 1).

![Diagram of M5-branes and membranes](image)

**Fig. 1:** “perturbative” picture: the embedding space is flat

On the world volume theory of the $N$ M5-branes this corresponds to a pair of a string anti-string. The $L$ dependent part of the energy density of the two membranes corresponds to the potential energy density of the string anti-string pair in the M5-brane-theory. Since in the perturbative picture the gravitational interaction in the bulk is neglected the force between the two membranes is solely carried by M5 world volume fields. The $L$ independent part of the energy density arises due to the self energy of the two membranes. It should be proportional to the separation distance of the single M5-brane from the $N$ M5-branes. In the M5 field theoretic description longitudinal and transversal modes couple to the

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1 “Anti” refers to the opposite orientation from the five dimensional point of view.
string sources. Exchanges of longitudinal quanta will lead to the \( L \)-dependent potential whereas the transversal quanta result in an \( L \)-independent contribution which diverges when the single M5 is taken infinitely far away from the \( N \) M5-branes. Here, we are in the strange situation that we do not know how to compute this potential energy density in the “perturbative” regime but we do know how to do it non-perturbatively.

The non-perturbative dual of the above configuration is given by a membrane living in \( \text{adS}_7 \times S^4 \) with the boundary condition that it ends in two parallel strings separated by a distance \( L \) at the boundary of \( \text{AdS}_7 \times S^4 \) (Fig. 2).

Since the fermionic zero-modes of the membrane background in \( \text{adS}_7 \times S^4 \) are zero for our problem we can obtain the configuration of fig. 2 by minimizing the world volume of the membrane which is the Nambu-Goto action

\[
S = \frac{1}{2\pi} \int d^3\sigma \sqrt{-\det (G_{\mu\nu} \partial_a X^{\mu} \partial_b X^{\nu})},
\]

(3)

where \( a = (\tau, \sigma, \phi) \) labels the world volume coordinates of the membrane and \( G_{\mu\nu} \) is the metric of the embedding space (II). We chose a static gauge

\[
X^0 = \tau, \quad X^1 = \sigma, \quad X^2 = \phi.
\]

(4)

Further we take the ansatz \( U = U(\sigma) \) and the rest of the embedding coordinates is constant. Like in the string case one can employ the explicit \( \sigma \) independence of the Lagrangian to reduce the equations of motion to a first order differential equation

\[
\partial_\sigma U = \pm \frac{U^2}{2U_0^3} \sqrt{U^6 - U_0^6},
\]

(5)
where $U_0$ is an integration constant which we will relate to the string anti-string distance $L$. (In the following we will restrict ourself to positive values of $\sigma$ and chose the upper sign in (5).) Eq. (5) can be integrated to give $X^1 = \sigma$ as a function of $U$

$$X^1 = \frac{2}{U_0} \sqrt{\frac{U}{U_0}} \int_1^{U_{\text{max}}} \frac{dy}{\sqrt{y^6 - 1}}.$$  

(6)

The boundary condition $X^1 (U = \infty) = \frac{L}{2}$ leads to the relation

$$U_0 = \frac{2}{3L} B \left( \frac{2}{3}, \frac{1}{2} \right),$$  

(7)

where $B$ denotes Euler’s Beta-function. In order to obtain the energy density we integrate the Lagrange density in (3) over $\sigma$ and substitute $\sigma = \sigma (U)$.

$$\varepsilon = R^3 \lim_{U_{\text{max}} \to \infty} \frac{2}{\pi U_0^2} \int_1^{U_{\text{max}}} \frac{dy}{\sqrt{y^6 - 1}}.$$  

(8)

where we have introduced an upper cut-off for the $U$ integration. Now, we split the energy-density into self-energy contribution and a potential energy density

$$\varepsilon = \varepsilon_{\text{self}} + \varepsilon_{\text{pot}},$$  

(9)

with

$$\varepsilon_{\text{self}} = R^3 \lim_{U_{\text{max}} \to \infty} \frac{1}{\pi U_0^2} \int_1^{U_{\text{max}}} \frac{dy}{\sqrt{y^6 - 1}} \left( 2y^6 + 1 \right)$$  

$$= R^3 \frac{U_{\text{max}}^2}{\pi} + \ldots.$$  

(10)

where the dots stand for terms vanishing in the limit $U_{\text{max}} \to \infty$. The potential energy density comes out to be

$$\varepsilon_{\text{pot}} = -R^3 \frac{U_0^2}{\pi} \int_1^{\infty} \frac{dy}{y^2 \sqrt{y^6 - 1}}$$  

$$= -\frac{2R^3}{27\pi} B \left( \frac{2}{3}, \frac{1}{2} \right)^3 \frac{1}{L^2}.$$  

(11)

A few remarks are in order. By rescaling $x_\parallel \to R^{3/2} x_\parallel$ we had changed the world volume of the M5-brane. To undo this we should divide the energy densities by $R^3$ but at the
same time also replace $L \rightarrow R^{-3/2}L$ \footnote{These rescalings may look strange at the first sight but are correct. The energy density is measured with respect to a volume which is $R^3$ times smaller than the original one (including the time) and our $L$ has to be expressed in terms of the original one which is $R^{3/2}$ times bigger.}. This removes the $R$ dependence of the self energy contribution (10) but leaves the potential energy (11) unchanged. Upon compactifying $X^2$ and $\phi$ on a circle (double dimensional reduction) one obtains a Coulomb law in a $4 + 1$ dimensional theory - a result which has been used already in \cite{[9]}. Even though its derivation is given in \cite{[8]} we decided to present it in some detail since in the rest of the paper we will study fluctuations around this background membrane.

3. Fluctuations

The result of the previous section is valid for large $R$ where the supergravity background (geometry) is reliable as well as the saddle point approximation is good. In \cite{[10]} it was argued that there are no corrections to the geometry due to finite $N$. Therefore, we will focus on corrections resulting from fluctuations around the background membrane. In order to do so we have also to include the fermionic fluctuations. A $\kappa$-symmetric action for the membrane on $adS_7 \times S^4$ can be found in \cite{[11]}. After rescaling their fermionic coordinates $\theta \rightarrow \sqrt{R}\theta$ the only $R$ dependence of the action appears as an overall factor of $R^3$. Therefore the loop expansion gives a power series in $1/R^3$. We will be interested in the next to leading order ($R^0$) contribution to the potential energy density (11). To this end, we need to background field expand the membrane action to second order in fluctuations. Since the background in the $S^4$ direction and in fermionic directions is trivial the bosonic fluctuations in $adS_7$ direction, in $S^4$ direction and in fermionic directions decouple and we can discuss their actions separately. In order to obtain translation invariant functional measures we use the normal coordinate expansion developed in \cite{[12]}. There fluctuations are parameterized by tangent vectors $\xi^a$ (with $a$ being a Lorentz index) to geodesics connecting the background with its fluctuation. It is useful to take the world volume metric to be the full induced metric because it saves one from dealing with constraints. With this remarks the calculation should be straightforward and we will not enter into its details but just present the results.
3.1. Fluctuations in $\text{adS}_7$ direction

The part of the action second order in fluctuations $\xi^a$ ($a = 0, \ldots, 6$) is

$$S_{\text{adS}}^{(2)} = \frac{1}{4\pi} \int d^3\sigma \sqrt{-h} \left[ h^{ij} \left( \sum_{a=3}^5 \partial_i \xi^a \partial_j \xi^a + \partial_i \xi_\perp \partial_j \xi_\perp \right) + \frac{3}{4} \sum_{a=3}^5 (\xi^a)^2 + \frac{3}{4} \left(1 - \frac{2U_0^6}{U^6}\right)(\xi_\perp)^2 \right],$$

where the metric $h_{ij}$ is (up to a factor of $R^2$) the induced background metric

$$ds^2 = -U^2 d\tau^2 + \frac{U_0^8}{U^6} d\sigma^2 + U^2 d\phi^2,$$

and we have redefined

$$\xi^\parallel = \frac{U_0^3}{U^3} \xi^1 + \frac{\sqrt{U_0^6 - U^6}}{U^3} \xi^6,$$

$$\xi_\perp = -\frac{\sqrt{U_0^6 - U^6}}{U^3} \xi^1 + \frac{U_0^3}{U^3} \xi^6.$$

Note that the Jacobian of this redefinition is one. The new fields $\xi^\parallel, \xi_\perp$ are fluctuations which lie in the one-six plane and parameterize fluctuations parallel respectively perpendicular to the background membrane.

We observe that (12) degenerates since it does not depend on $\xi^0, \xi^2, \text{ and } \xi^\parallel$. This originates from the freedom of performing world volume diffeomorphisms. We remove the degeneracy by gauge fixing

$$\xi^0 = \xi^2 = \xi^\parallel = 0.$$

3.2. Fluctuations in $S^4$ direction

Since the background is trivial in the $S^4$ direction we obtain a very simple action quadratic in $\xi^a$ ($a = 7, \ldots, 10$)

$$S_{S^4}^{(2)} = \frac{1}{4\pi} \int d^3\sigma \sqrt{-h} h^{ij} \sum_{a=7}^{10} \partial_i \xi^a \partial_j \xi^a.$$

3.3. Fluctuations in fermionic directions

In order to obtain the part of the action quadratic in fermionic fluctuations we need to take the part of the membrane action [11] bilinear in fermions and plug in there our background for the bosons. Then one obtains a result containing only $\Gamma^a$ ($a = 0, \ldots, 6$),
where $\Gamma^a$ denotes an eleven dimensional Gamma-matrix. Now, one can write $\Gamma^a = \gamma^a \otimes \gamma^5'$ where $\gamma^a$ is a gamma-matrix of the seven-dimensional tangent space of $adS_7$ and $\gamma^5'$ belongs to the tangent space of $S^4$. We split our 32-component spinor into two sixteen component spinors $\theta^1, \theta^2$ according to their eigenvalue with respect to $\gamma^5'$,

$$\gamma^5' \theta^1 = \theta^1 \quad , \quad \gamma^5' \theta^2 = -\theta^2. \quad (17)$$

Further we should fix $\kappa$-symmetry. A $\kappa$-fixed action of the membrane on $adS_7 \times S^4$ is discussed in [13], for our purpose we find a different gauge fixing convenient however. First, define (cf (14))

$$\gamma^\parallel = \frac{U_3^3}{U_3} \gamma^1 + \frac{\sqrt{U^6 - U'^6}}{U^3} \gamma^6. \quad (18)$$

With this we choose as a $\kappa$-fixing condition

$$\left(1 + \gamma^0^\parallel^2\right) \theta^1 = 0$$

$$\left(1 - \gamma^0^\parallel^2\right) \theta^2 = 0. \quad (19)$$

Further we will need the dreibeine and spin-connections belonging to [13] (numbers denote Lorentz-indices),

$$e_0^0 = e_2^2 = U \quad , \quad e_1^1 = \frac{U^4}{U_0^3} \quad , \quad \omega^0_1 = \omega^2_1 = \frac{\sqrt{U^6 - U'^6}}{2U^2}, \quad (20)$$

and all other components are zero. With some algebra and employing (19) one can write the equations of motion for the fermionic fluctuations as follows

$$\rho^a e^i_a \left( \partial_i + \frac{1}{4} \omega^bc_i \rho_b \rho_c + A_i \right) \theta^1 = -\frac{3}{4} \theta^1, \quad (21)$$

where we have defined $\rho$-matrices satisfying a 2 + 1 dimensional Clifford algebra

$$\rho^0 = \gamma^0 \quad , \quad \rho^1 = \gamma^0^2 \quad , \quad \rho^2 = \gamma^2. \quad (22)$$

The field

$$A_\sigma = \frac{3U}{4} \gamma^{16} \quad (23)$$

appears as a background value of a gauge field belonging to local rotations in the one-six plane (the $\rho$’s commute with $A$). For $\theta^2$ we obtain the same equation (21) but with $\rho^1$ replaced by $-\rho^1$. So, the condition (19) allows us to write the equations of motion for the
fermionic fluctuations in a covariant three dimensional form where the target space spinors ‘metamorphosed’ into world-volume spinors.

Multiplying the kinetic operator from (21) with its adjoint gives

$$-\Delta - \frac{R^{(3)}}{4} + \frac{9}{16},$$

where $R^{(3)}$ is the scalar curvature computed from (13)

$$R^{(3)} = \frac{3}{2} \frac{U^6 + U_0^6}{U^6}$$

and $\Delta$ is the Laplacian including spin- and gauge connections.

3.4. Adding up the fluctuations

From (12) (16) and (24) we obtain for the one loop effective action

$$S_{1-loop}^{1-eff} = \frac{1}{2} \log \det^3 \left(-\Delta_0 + \frac{3}{4}\right)$$

$$+ \frac{1}{2} \log \frac{\det (-\Delta_0 + \frac{9}{4} - R^{(3)}) \det^4 (-\Delta_0)}{\det (-\Delta - \frac{1}{4}R^{(3)} + \frac{9}{16})},$$

where $\Delta_0$ is the Laplacian with respect to (13) acting on scalars. The power in the fermionic determinant (in the denominator) results from 32 real fermionic components which have been reduced to 16 by $\kappa$-fixing. Since $\Delta$ is an eight by eight matrix and we have squared the fermionic operators we arrive at the expression (26). Unfortunately we are not able to evaluate the full expression (26). However, we can extract the uv-divergent contributions. The formulas we are going to use can be found e.g. in [14]. In 2 + 1 dimensions there are two potentially divergent contributions to the effective action. For an operator of the form $-\Delta + E$ there is a cubic divergence

$$a_0 = \frac{1}{\Lambda^3} (4\pi)^{-\frac{3}{2}} \text{tr} 1$$

where the trace is taken over all fields and includes an integration with the covariant measure. Since in our case the Laplacian is dimensionless ($U$ has mass dimension one) the short distance cut-off $\Lambda$ is dimensionless as well. (When taking the limit of vanishing Planck length the short distance cut-off in Planck units is held fixed.) From (26) we see that we do not encounter cubic divergences. The linear divergence is

$$a_2 = \frac{(4\pi)^{-\frac{3}{2}}}{6\Lambda} \text{tr} \left(6E + R^{(3)}\right).$$
We have\(^3\)
\[ \text{tr} E = \int d^3 \sigma \sqrt{-h} R^{(3)} \] (29)
and hence the divergent contribution to the effective energy density is
\[ \varepsilon^{\text{div}} = \frac{1}{\Lambda} (4\pi)^{-\frac{3}{2}} \int d\sigma \sqrt{-h} R^{(3)} = \varepsilon^{\text{div}}_{\text{self}} + \varepsilon^{\text{div}}_{\text{pot}}. \] (30)

For the divergent contribution to the self energy we find
\[ \varepsilon^{\text{div}}_{\text{self}} = \frac{3}{4\Lambda} (4\pi)^{-\frac{3}{2}} \int d\sigma \frac{2U^6 + U_0^6}{U_0^3} = \frac{3}{2\Lambda} (4\pi)^{-\frac{3}{2}} U_{\text{max}}^2 + \ldots, \] (31)
where the dots stand again for terms vanishing when \( U_{\text{max}} \) is taken to infinity. The potential energy density receives the following divergent contribution
\[ \varepsilon^{\text{div}}_{\text{pot}} = \frac{3}{4\Lambda} (4\pi)^{-\frac{3}{2}} \int d\sigma U_0^3 = \frac{1}{9\Lambda} (4\pi)^{-\frac{3}{2}} B \left( \frac{2}{3}, \frac{1}{2} \right) \frac{3}{L^2}. \] (32)
So, both the self energy and the potential energy are renormalized.

So far we did not mention boundary contributions which appear in theorem 4.1 of [14]. We take Dirichlet boundary conditions (the configuration on the M5 brane is not allowed to fluctuate). By fermion-boson matching one realizes that \( a_1 \) and the boundary contribution to \( a_2 \) vanish but \( a_3 \) does not. This leads to the following additional part in the energy density
\[ \frac{\log \Lambda}{16\pi} \sqrt{-h} \left. R^{(3)} \right|_{U = U_{\text{max}}}, \] (33)
where \( h_{ij}^{(2)} \) is obtained from \([13]\) by fixing \( U = U_{\text{max}} \) (\( \sigma = \text{constant} \)). In the limit \( U_{\text{max}} \to \infty \) one observes that \([33]\) gives an additional contribution to the self energy density \( \varepsilon^{\text{div}}_{\text{self}} \).

To summarize, we have linearly divergent contributions to the self energy density \([31]\) and to the potential energy density \([32]\), in addition there is a logarithmically divergence in the self energy density \([33]\). Since the self energy density is infinite in our set up from the beginning their renormalization does not look like a real problem. (It can be absorbed in a redefinition of the infinite \( U \)-integration cut off \( U_{\text{max}} \).) The part which is difficult to interpret is the linear divergence \([32]\). It does not introduce an additional scale since \( \Lambda \) is dimensionless. However, our calculation seems to imply that in the \( AdS_7 \times S^4 \) case the Maldacena conjecture needs to be supplemented by the information to what value the UV cut off (in Planck units) has to be taken in the near horizon limit. We do not know to which data of the M5 brane theory this information belongs.

\(^3\) We take care of the halfs in front of the logarithms in \([26]\) by restricting the \( \sigma \) integration on the region between zero and \( L/2 \).
4. Conclusions

Starting from the Maldacena conjecture and assuming that M-theory is described by membranes we computed the potential energy density between a string and an anti-string source in the M5-brane theory. We found that after double dimensional reduction the potential energy follows a 4 + 1 dimensional Coulomb law. Then we discussed corrections to this result due to membrane fluctuations to the next to leading order. We proposed a way of world volume diffeomorphism and $\kappa$-symmetry fixing which seems suitable for the given problem. Finally, we found that the next to leading order renormalizes the classical result.

Analogous techniques can be applied to the Wilson loop computation based on the duality between $\mathcal{N} = 4$ supersymmetric Yang-Mills theory and string theory on $\text{adS}_5 \times S^5$. A publication dealing with that case is in preparation [15]. But even without going through the details of the calculation one can guess what to expect. According to [16] divergent contributions to the partition function of the string on $\text{adS}_5 \times S^5$ result at most in a constant contribution to the dilaton beta-function. Hence, a potential divergence will arise with a factor $\int \sqrt{-h}R^{(2)}$ where $h$ is now the induced metric resulting from the background discussed in [8]. Computing this integral one finds that it only contains a self energy contribution (in the case of $D3$ branes one can associate this to a $W$ mass via the Higgs mechanism).

It should be interesting to extend the presented discussion to the finite temperature case. There it may be possible to deduce in certain limits more than just the divergent contributions to the partition function (cf [17]).

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