Gravitational lensing by gravitational waves

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Abstract

Gravitational lensing by gravitational wave is considered. We notice that although final and initial direction of photons coincide, displacement between final and initial trajectories occurs. This displacement is calculated analytically for the plane gravitational wave pulse. Estimations for observations are discussed.

1 Introduction

According to general relativity, any gravitational field can change trajectory of photons or, in other words, deflect light rays. Hence the gravitational field may act as a gravitational lens.

Gravitational lensing by gravitational waves in different cases was considered by many authors (see [1], [2], [3] and references therein). It was found that the deflection angle vanishes for any localized gravitational wave packet because of transversality of gravitational waves [3]. Thus if the photon passes through finite gravitational wave pulse its deflection due to this wave is equal to zero. Nevertheless we notice that the displacement between trajectories of the photon before and after passing the wave may occur.

In this work we confirm analytically vanishing of deflection angle for plane wave pulses. However, we have found that the gravitational wave (GW) changes the photon propagation in another way, simply shifting its whole trajectory after passing through the GW (see figs 1,2). This displacement is found analytically for the photon passing through the plane GW. On the basis of this result we obtain an approximate formula for the estimation of observational effects.
2 The deflection angle

The wave vector of the photon $k^i$ (tangent to a trajectory) by definition is equal to $k^i = \frac{dx^i}{d\lambda}$, where $\lambda$ is a parameter changing along the trajectory [4]. The geodesic equation is written as $Dk^i = 0$ or $Dk_i = 0$, where $D$ denotes a covariant derivative. It is more convenient to use the second expression. After some transformation

$$\frac{dk_i}{d\lambda} = \Gamma_{k,il}^k k^l,$$  

we obtain the equation of motion for the photon:

$$\ddot{x}_i = \frac{1}{2} \dot{x}_i \dot{x}_j \frac{\partial h_{kj}}{\partial x^i},$$  

where dot denotes derivative with respect to parameter $\lambda$.

Let us consider the gravitational wave in a flat space with a metric $g_{ik} = \eta_{ik} + h_{ik}$, where $\eta_{ik}$ is a flat metric $(-1, 1, 1, 1)$ and $h_{ik}$ is a small perturbation (gravitational wave). In this approximation one can integrate equation (3), calculating right-hand side of equation with unperturbed trajectory of the photon. Performing integration, we obtain the expression for the deflection angle (compare with [3]):

$$\hat{\alpha}_i = \frac{k_i(+\infty) - k_i(-\infty)}{k} = \int_{-\infty}^{+\infty} \frac{1}{2} k^k k^l \frac{\partial h_{kj}}{\partial x^i} d\lambda$$

where $h_{ik}$ is calculated along the straight line trajectory and $k^i = const$ along unperturbed trajectory.

Consider the photon moving along $z$-axis. Its unperturbed trajectory is $z = z_0 + ct$, and we can use the coordinate $z$ as the parameter $\lambda$. Then the wave vector is $k^i = (1, 0, 0, 1)$, $k = 1$. When the photon passes through the finite wave packet, we denote the $z$-coordinate of the input of the photon into the wave front as $z_1$ and the $z$-coordinate of the output from the wave front as $z_2$ ($z_1 < z_2$). Hence we have the expression for the deflection angle in the form (compare with [2]):

$$\hat{\alpha}_i = \frac{1}{2} \int_{z_1}^{z_2} \frac{\partial}{\partial x^3} (h_{00} + 2h_{03} + h_{33}) ~dz .$$

Let us calculate photon deflection by the plane GW pulse. Let us consider a light ray propagating under the angle $\varphi = -(\pi - \theta)$ relative to the direction of the plane GW packet propagation (see Fig.1). Let us define for convenience two reference systems $K$ and $K'$. The first one is connected with direction of the light ray: the photon moves along $z$-axis in a positive direction in the reference frame $K$. The second one is connected with the direction of propagation of the gravitational wave.
pulse. The gravitational wave packet moves along \( z' \)-axis in a positive direction in the reference frame \( K' \) (see Fig.1). The systems are at the rest relative to each other and their origins of coordinates coincide. At the initial time \( t = 0 \) the photon is situated at \( z_0 < 0 \) \( (x_0 = y_0 = 0) \), the wave vector of the photon is \( k^t = (k^0, 0, 0, k^z) = (1, 0, 0, 1), k = k^z \). The form of the wave pulse is sinusoidal (the top part of the sinusoid, with the phase changing from 0 to \( \pi \), and with zero perturbations on the boundaries):

\[
h_{ik}' \propto \sin \xi', \quad \xi' = 0...\pi, \quad \xi' = \omega t - k'_g z', \quad k'_g = \omega/c,
\]

(6)

where \( \omega \) and \( k'_g \) are the frequency and the wave vector of gravitational wave in \( K' \) correspondingly. The pulse width \( \delta \) (in space) is \( \delta = c\pi/\omega \).

It is convenient to use non-dimensional variables for time \( \tilde{t} = t/t_0 \), \( t_0 = 1/\omega \) and distances \( \tilde{x} = x/x_0 \), \( x_0 = c/\omega \). Hereafter we omit tildes for simplicity. In non-dimensional variables the equation of motion (3) looks the same, and the gravitational wave form is written as \( \sin(t - z') \).

The right side of (5) includes components \( h_{00}, h_{03}, h_{33} \), which are components of gravitational perturbation in the reference system \( K \). Gravitational wave moves along the axis \( Oz' \) in the reference system \( K' \), therefore the GW has non-zero components \( h'_{11}, h'_{12}, h'_{21}, h'_{22} \) only.

The reference system \( K \) transforms into the system \( K' \) by rotation by the angle \( \varphi = -(\pi - \theta) \) around the axis \( x \) (see Fig.1). Hence we have:

\[
h_{00} = h_{03} = 0, \quad h_{33} = \sin^2 \varphi h'_{22}.
\]

(7)

Writing \( h'_{22} \) as \( h'_{22} = h \sin(t - z') \), where \( h \) is the amplitude of wave, we obtain:

\[
h_{33} = h \sin^2 \theta \sin(t + z \cos \theta - y \sin \theta).
\]

(8)

Taking into account that the straight line ray has the trajectory \( z = z_0 + t \), one can find the points of intersection of the photon and the wave front. The point of the input is \( z_1 \), the point of the output is \( z_2 \), the point of the perturbation maximum is \( z_m \) (in the reference system \( K \) the gravitational wave moves in the negative direction of the axis \( z \), therefore we have \( z_1 < z_m < z_2 \)):

\[
z_1 = \frac{z_0}{1 + \cos \theta}, \quad z_2 = \frac{\pi + z_0}{1 + \cos \theta}, \quad z_m = \frac{\pi/2 + z_0}{1 + \cos \theta}.
\]

(9)

Because of the symmetry, the deflection may happen only in the plane \( (zy) \): \( \alpha_y \). Let us define \( F_y(z) \) as

\[
F_y(z) = \frac{1}{2} h \sin^2 \theta \left[ \frac{\partial}{\partial y} \left( \sin(z - z_0 + z \cos \theta - y \sin \theta) \right) \right] \bigg|_{y=0} = \frac{\partial \varphi_y}{\partial y},
\]

\[
\varphi_y = \frac{1}{2} h \sin^2 \theta \sin(z - z_0 + z \cos \theta - y \sin \theta),
\]

\( \varphi_y = 0 \) outside the GW pulse.
Then the deflection angle in the first part of the way within the wave is:

\[ \alpha_1 = \int_{z_1}^{z_m} F_y(z)dz = -\frac{1}{2} h \frac{\sin^3 \theta}{1 + \cos \theta} = -\frac{1}{2} h (1 - \cos \theta) \sin \theta . \]  

(11)

The deflection angle in the second part of the way within the wave is:

\[ \alpha_2 = \int_{z_m}^{z_2} F_y(z)dz = \frac{1}{2} h \frac{\sin^3 \theta}{1 + \cos \theta} = \frac{1}{2} h (1 - \cos \theta) \sin \theta . \]  

(12)

And the total deflection angle is:

\[ \hat{\alpha} = \int_{z_1}^{z_2} F_y(z)dz = 0 . \]  

(13)

The top part of the sinusoid is symmetrical relative to the vertical axis. We also have considered a non-symmetrical plane waveform and have obtained numerically that vanishing of the deflection angle occurs in this case too. We also checked vanishing for different velocities of the photon (in the medium, where photon has velocity \(< c\)).

### 3 The displacement

Let us find a displacement analytically for the plane gravitational wave pulse. To find the displacement we need to integrate farther the equation of motion \[\text{(3)}\], what gives:

\[ y(z) = \int_{z_1}^{z} \left[ \int_{z_1}^{z'} F_y(z'')dz'' \right] dz' , \quad y(z_1) = 0 . \]  

(14)

We obtain for the trajectory \( y(z) \) of the photon within the wave \((z_1 < z < z_2)\):

\[ y(z) = \frac{1}{2} h \frac{(-1 + \cos(z - z_0 + z \cos \theta)) \sin^3 \theta}{(1 + \cos \theta)^2} , \quad y(z_1) = 0 . \]  

(15)

We see that the photon trajectory has a sinusoid form within wave. Using \[\text{(13)}\], we see that the total displacement along the axis \( y \) does not vanish, and it is equal to:

\[ \Delta y = y(z_2) - y(z_1) = -h \frac{\sin^3 \theta}{(1 + \cos \theta)^2} = -h \frac{1 - \cos \theta}{1 + \cos \theta} \sin \theta . \]  

(16)

Thus although initial and final directions of photon coincide, and the deflection angle vanishes, the displacement in the trajectory occurs. This displacement is absent in case of \( \theta = 0 \) (the photon and the gravitational wave directions are parallel) and reaches its maximum in
the case at $\theta = \pi/2$ (the photon and the gravitational wave directions are orthogonal). It is clear that this displacement will be equal to zero if we consider the whole sinusoid with the top and the bottom parts, because the displacement due to the top part of the sinusoid will be cancelled by the displacement due to the bottom part. Therefore this displacement takes place mainly in the case of isolated wave pulses, which have a form similar to the top part of the sinusoid or when it has the non-symmetrical top and bottom parts of wave profile, and may, in principal, vanish for the periodic wave of a long duration. The wave pulses may be produced, for example, during stellar collapse (see [5], paper contains many figures with waveforms) or during formation of large scale structure of the Universe (see [6]).

We calculate displacement in the non-dimensional variables. In dimensional variables we have:

$$\Delta y = -h \frac{\delta}{\pi} \frac{\sin^3 \theta}{(1 + \cos \theta)^2}.$$  \hfill (17)

Let us estimate observational effects caused by this displacement.

4 The observational effects of displacement

Directions of photons passing through the gravitational wave packet does not change, therefore any focusing of rays does not occur in this case. Thus the displacement in trajectories does not lead to any magnification effect. But the displacement leads to change of the angular position of object for distant observer. The change of the angular position due to passing of the light ray through the gravitational wave pulse $\Delta \alpha_d$ is as (see fig.3)

$$\Delta \alpha_d = \frac{\Delta y}{D_s} \simeq \frac{h \delta}{D_s},$$  \hfill (18)

where $h$ is the amplitude of the GW pulse, $\delta$ is its thickness and $D_s$ is a distance between the source and the observer.

Let us estimate the change of the angular position for the GW pulses produced during formation of large scale structure of the Universe in dark matter (see [6]). For estimates we put $h = 10^{-11}$, $\delta = Mpc$, $D_s = 100Mpc$, then we obtain

$$\Delta \alpha_d \simeq 2 \cdot 10^{-8} \, \text{arcsec}.$$  \hfill (19)

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Figure 1: Lensing of the photon by the plane GW pulse. The initial state. The photon moves along $z$-axis in the reference system $K$. Gravitational wave packet moves along $z'$-axis in the reference system $K'$. The reference system $K$ transforms into the system $K'$ by rotation by the angle $\varphi = -(\pi - \theta)$ around the axis $x$ (positive rotation is anticlockwise).
Figure 2: Passing of the photon through the GW. The positions of the wave packet at the time of the photon input and the photon output are shown by full line and dashed line correspondingly.
Figure 3: The observational effect of the displacement in trajectory of the photon.
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