We prove that the field equations of general relativity and other metric theories can be derived from the conservation of energy-momentum without using the assumption of least action principle. We show a new procedure for perturbative derivation of symmetric pseudo-energy momentum for the gravity field.

1 Introduction

The principle of least action is traditionally considered as essential for deriving the field equations from an action. The field equations are then enforced for each symmetry of the action, in order to obtain a conserved Noether current. There are cases where this logical order can be changed, as in the electromagnetic theory where we get Maxwell equations from the demand of gauge symmetry and conservation of energy momentum, without the need to assume the principle of least action.

Recently it has been shown [1] that it is possible to obtain the field equations in a number of classical models, under the assumption of some symmetries and conservation of energy-momentum, without the use of principle of least action. The motivation for this alternative derivation is both conceptual and practical. While the principle of least action is a very effective technical tool for deriving the field equations, it can not be directly verified by experiment, and in general it’s physical meaning is vague and difficult to comprehend. In contrast, conservation of a charge, like energy-momentum, is a concept for which we have a strong intuition, and in addition it can be checked experimentally.

In this paper we prove that the field equations of any metric gravitational theory can be derived under the assumptions of symmetry and conservation of energy momentum. Beside the zero covariance divergence of the matter energy...
momentum tensor, we assume the conservation of total energy-momentum. The space integral of the energy momentum pseudo-tensor is the energy momentum vector, which is a well defined and a measurable quantity. Assuming 4 conservation equations of the divergence free pseudo-tensor, i.e., conservation of 4 energy-momentum vector components up to Lorentz transformation, we deduce the 10 metric field equations, without using the principle of least action.1

2 Method

Starting from the action

\[ S = \int (\mathcal{L}_G + \mathcal{L}_m) d^4x \]  \hspace{1cm} (2.1)

where \( \mathcal{L}_G \) and \( \mathcal{L}_m \) are the gravity and matter Lagrangian densities. We develop the Lagrangian density formally as a power series in the deviation from the Minkowski metric

\[ \mathcal{L} = \sum_{n=2}^{\infty} \mathcal{L}^{(n)}(\eta^\mu{}^{\nu}, h_{\mu\nu}, h_{\mu\nu,\rho}, \phi) \]  \hspace{1cm} (2.2)

where \( h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu} \) and \( \phi \) stands for matter fields. The index \( (n) \) represents the number of times the field \( h_{\mu\nu} \) or its derivatives \( h_{\mu\nu,\rho} \) appears in that indexed addend. We identify the symmetric total energy momentum pseudo-tensor for the system by implementing the definition of the Hilbert energy-momentum to a system with dynamical metric fields, and we use formally an exterior field for metric operations. We substitute back \( g_{\mu\nu} - \eta_{\mu\nu} \) for \( h_{\mu\nu} \). We then replace all the constant matrices \( \eta^{\mu\nu} \) with an exterior metric field \( \hat{g}^{\mu\nu} \), i.e. a non dynamical tensor field that is used for raising indices and to define a covariant derivative. We also replace \( \eta_{\mu\nu} \) with \( \hat{g}_{\mu\nu} \) where \( \hat{g}_{\mu\nu} \) is defined by \( \hat{g}_{\mu\nu}\hat{g}^{\nu\rho} = \delta_{\nu}^{\rho} \). Correspondingly, we replace partial derivatives \( \partial_{\rho} \) with a covariant derivative with respect to our new metric field \( \hat{\nabla}_{\rho} \) and the integration measure \( d^4x \) with a scalar density \( \sqrt{\hat{g}}d^4x \). The modified action is

\[ \hat{S} = \int \sum_{n=2}^{\infty} \hat{\mathcal{L}}^{(n)}(\hat{g}^{\mu\nu}, g_{\mu\nu} - \hat{g}_{\mu\nu}, \hat{\nabla}_{\rho} g_{\mu\nu}, \phi) \]  \hspace{1cm} (2.3)

where

\[ \hat{\mathcal{L}}^{(n)} \equiv \mathcal{L}(\hat{g}^{\mu\nu}, g_{\mu\nu} - \hat{g}_{\mu\nu}, \hat{\nabla}_{\rho} g_{\mu\nu}, \phi) \]  \hspace{1cm} (2.4)

The total energy momentum pseudo-tensor for the original system is derived by variation of the modified action \( \hat{S} \) with respect to the exterior metric and then substitution the Minkowski metric.

1In this context it is appropriate to refer to [2] where an equivalence of field equations and conservation laws in general relativity, is proven by other methods.

2It should be emphasized that while we start from an action, We do not assume the principle of least action. In the following, we don’t get the field equations by variation of the action with respect to the metric, because we don’t assume that such a variation is necessarily zero.
\[ \tau_{\mu\nu} = \hat{T}_{\mu\nu} \bigg|_{g_{\mu\nu} = \eta_{\mu\nu}} \quad (2.5) \]

where \( \hat{T}_{\mu\nu} \equiv \frac{2}{\sqrt{\hat{g}}} \frac{\delta \hat{S}}{\delta \hat{g}_{\mu\nu}} \). We emphasize that the tensor \( \hat{T}_{\mu\nu} \) corresponds to the variation of the whole system with respect to some background hat metric, that is, we put the graviton field of the usual metric and standard matter field on the same level and couple them both to the same external hat metric. After substitution of the Minkowski metric as the external metric in \( \hat{T}_{\mu\nu} \) we get the total energy momentum pseudo-tensor \( \tau_{\mu\nu} \) of the original system, so \( \tau_{\mu\nu} = T_{\mu\nu} + t_{\mu\nu} \) where \( T_{\mu\nu} \) is the usual matter energy momentum tensor, and \( t_{\mu\nu} \) is an energy momentum pseudo-tensor for the gravity field. The total energy-momentum pseudo-tensor does not have a local meaning as it may be set to zero at any chosen spacetime point with some coordinate transformation, but its global conservation is well defined \[3\]. There are a number of nonequivalent forms for the energy pseudo-tensor \[4, 5, 6\], but with the desired flat boundary conditions, their 3-space integration is finite and equal \[7\] and transforms as a vector under all coordinate transformations which leave the Minkowski metric at infinity unchanged \[4\]. The total energy can also be defined for non-flat boundary conditions \[8\].

We now start an iterative process. In the first (linear) approximation we take the terms that are quadratic in the deviations of the metric from the Minkowski matrix.

\[ \hat{S} \cong \hat{S}^L = \int \hat{L}^{(2)} \sqrt{\hat{g}} d^4 x \quad (2.6) \]

The modified linear action, a scalar, is invariant under the diffeomorphism \( x^\mu \to x^\mu + \xi^\mu (x^\nu) \) where \( \xi^\mu \) are arbitrary functions in the domain of integration and vanish on the border

\[ \delta \xi \hat{S}^L = \int \left( \frac{\delta \hat{L}^{(2)}}{\delta g_{\mu\nu}} \delta \xi g_{\mu\nu} + \frac{\delta \hat{L}^{(2)}}{\delta \phi} \delta \xi \phi \right) \sqrt{\hat{g}} d^4 x = 0 \quad (2.7) \]

Using \( \delta \xi g_{\mu\nu} = \hat{\nabla}^\nu \xi^\mu + \hat{\nabla}^\mu \xi^\nu \) and integration by parts of the first addend we get

\[ \delta \xi \hat{S}^L = \int \left( \hat{\nabla}^\nu \hat{T}_{\mu\nu} \xi^\mu + \frac{\delta \hat{L}^{(2)}}{\delta g_{\mu\nu}} \delta \xi g_{\mu\nu} + \frac{\delta \hat{L}^{(2)}}{\delta \phi} \delta \xi \phi \right) \sqrt{\hat{g}} d^4 x = 0 \quad (2.8) \]

We take \( \hat{g}^{\mu\nu} = \eta^{\mu\nu} \). We then assume conservation of the total energy momentum

\[ \hat{\nabla}^\nu \hat{T}_{\mu\nu} \bigg|_{g_{\mu\nu} = \eta_{\mu\nu}} = \tau_{\mu\nu} = 0 \quad (2.9) \]

and the matter field equations \( \frac{\delta \xi}{\delta \xi} = 0 \) that we get from the invariance of the matter action \( \delta \xi S_m = 0 \) and local conservation of the matter energy momentum \( \nabla^\nu T_{\mu\nu} = 0 \) \[11\].
We are then left with the equation
\[ \int \frac{\delta L^{(2)}}{\delta g_{\mu\nu}} \delta g_{\mu\nu} d^4x = 0 \] (2.10)

In general, the variation of the field \( g_{\mu\nu} \) under this coordinate transformation is
\[ \delta \xi g_{\mu\nu} = -g_{\mu\lambda} \xi^\lambda_{,\nu} - g_{\lambda\nu} \xi^\lambda_{,\mu} - g_{\mu\nu,\lambda} \xi^\lambda \] (2.11)

But with the linear approximation we can omit terms with \( h_{\mu\nu} \), and we are left with a variation of the form \( \delta \xi g_{\mu\nu} = -\eta_{\mu\lambda} \xi^\lambda_{,\nu} - \eta_{\lambda\nu} \xi^\lambda_{,\mu} \). If the ten component of symmetric matrix array of equations
\[ -\eta_{\mu\lambda} \xi^\lambda_{,\nu} - \eta_{\lambda\nu} \xi^\lambda_{,\mu} = F_{\mu\nu} \] (2.12)
can be solved for any symmetric \( F_{\mu\nu} \) then we have 10 arbitrary variations which are enough to deduce, with eq. (2.10) that the functional derivative must be zero, that is:
\[ \frac{\delta L^{(2)}}{\delta g_{\mu\nu}} = 0 \] (2.13)

which are the metric field equations in the linear approximation.

We now prove that we can always find a solution \( \xi^\mu \) for eq. (2.12) with any symmetric matrix \( F_{\mu\nu} \). On the left side of this array there are only first derivatives and we treat the equation as a linear algebraic equations, which may be solved smoothly at every spacetime point \( x^\mu_0 \). This matrix has 10 different components with arbitrary dependence on 4 coordinates. We have (4 components)\(^4\) (4 coordinates)=16 first derivatives of the four vector \( \xi^\mu \). These are not independent functions, but are restricted by integrability conditions, namely that the second derivatives be symmetric. There are 16 second derivatives for each of the 4 components of the vector \( \xi^\mu \). They define (4 components)\(^4\) (6 different symmetric relations) = 24 different equations of the form \( \xi^\mu_{,\nu,\rho} = \xi^\mu_{,\rho,\nu} \) with \( \rho \neq \nu \). Each of the second derivative relations restricts one coordinate dependence of some first derivative function, e.g., the equation \( \xi^\mu_{,2,1} = \xi^\mu_{,1,2} \) restricts the dependence of \( \xi^\mu_2 \) on \( x^1 \) or the dependence of \( \xi^\mu_1 \) on \( x^2 \). Thus there are all 4 coordinates restriction of (24 one coordinate restrictions)/(4 coordinates)=6 first derivative functions. Having started with 16 first derivatives, this leaves us with exactly (16 first derivatives)-(6 restricted)=10 arbitrary functions we need.

After we got the metric field equations in the first approximation, we assume that the metric field equations are derived to the n-th approximation:

**Assumption 1.** for the n-th approximation we have
\[ \frac{\delta \sum_{k=2}^{n+1} L^{(k)}}{\delta g_{\mu\nu}} = 0 \]
We then show that the $n+1$-th approximation of the field equations can be derived. Go back to eq. (2.3) and take the next approximation. Now

$$\hat{S} \approx \hat{S}^{(n+1)} = \int \sum_{k=2}^{n+2} \hat{\mathcal{L}}^{(k)} \sqrt{g} d^4 x$$

(2.14)

All the addends in the modified action are manifestly scalars, as opposed to the original action which is a scalar only in the full infinite sum, i.e. not perturbed form. So, as before, the modified action is a scalar and is invariant under the coordinate transformation. Employing again the conservation of total energy momentum (2.3) and matter field equations we get

$$\int \delta \left( \sum_{k=2}^{n+2} \mathcal{L}^{(k)} \right) \frac{\delta \xi_{\mu \nu}}{\delta g_{\mu \nu}} d^4 x = 0$$

(2.15)

In this approximation we omit terms in which the deviation $h_{\mu \nu}$ or its derivatives appear $(n+2)$ times, so with (2.11) the above equation can take the explicit form of

$$\int \left( \frac{\delta \left( \sum_{k=2}^{n+2} \mathcal{L}^{(k)} \right)}{\delta g_{\mu \nu}} (-\eta_{\mu \lambda} \xi^{\lambda}_{\nu} - \eta_{\lambda \nu} \xi^{\lambda}_{\mu}) + \frac{\delta \left( \sum_{k=2}^{n+1} \mathcal{L}^{(k)} \right)}{\delta g_{\mu \nu}} (-h_{\mu \lambda} \xi^{\lambda}_{\nu} - h_{\lambda \nu} \xi^{\lambda}_{\mu} - h_{\mu \nu, \lambda} \xi^{\lambda}) \right) d^4 x = 0$$

(2.16)

From our assumption which is exact in the $n$-th approximation, we deduce that

$$\frac{\delta \sum_{k=2}^{n+1} \mathcal{L}^{(k)}}{\delta g_{\mu \nu}} = O(h^{n+1})$$

The expression in the second brackets in (2.16) is of first order so the total second addend in (2.16) is of $(n+2)$ order, and we are left with

$$\int \left( \frac{\delta \left( \sum_{k=2}^{n+2} \mathcal{L}^{(k)} \right)}{\delta g_{\mu \nu}} (-\eta_{\mu \lambda} \xi^{\lambda}_{\nu} - \eta_{\lambda \nu} \xi^{\lambda}_{\mu}) d^4 x = 0$$

(2.17)

Using the above arguments that have been stated for (2.12, 2.13), we deduce that

$$\frac{\delta \left( \sum_{k=2}^{n+2} \mathcal{L}^{(k)} \right)}{\delta g_{\mu \nu}} = 0$$

(2.18)

that is, the metric field equations in the $n+1$ approximation. The metric equations for general relativity and for other metric theories can thus derived inductively from a Lagrangian without the principle of least action.
3 Summary

The hypothesis that equations of motion can be deduced from the conservation of energy momentum laws without the principle of least action, is proven for general metric gravitational theory. We deduce it from the assumption of total energy momentum and local matter energy conservation. The demonstrated procedure for perturbative derivation of a (Hilbert like) pseudo-energy momentum tensor for system with dynamical metric fields, can be justified by our results, and by comparing them with similar results which have been derived by other methods [2].

The method demonstrated in this paper may be useful for other gauge theories. The field variation under the gauge symmetry transformation, in first order, does not depend on the fields themselves, and from algebraic arguments one can deduce the field equations. In a higher order of approximation, the contribution for the variation in the action from terms in the Lie derivative which depends on the fields, is of higher order, and can be neglected. Thus, with the same algebraic arguments we can deduce the fields equation to any order of approximation.

The question whether symmetries and their corresponding charge conservation are sufficient for derivation of all field equations in general, is a matter of further research.

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