THE GELFAND–SHILOV TYPE ESTIMATE
FOR GREEN’S FUNCTION
OF THE BOUNDED SOLUTIONS PROBLEM

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Abstract. An analogue of the Gelfand–Shilov estimate of the matrix exponential is proved for Green’s function of the problem of bounded solutions of the ordinary differential equation \( x'(t) - Ax(t) = f(t) \).

Introduction

In [11, p. 68, formula (13)], it was established the following statement. Let the eigenvalues of an \( N \times N \)-matrix \( A \) lie in the half-plane \( \text{Re} \lambda < -\gamma \). Then the matrix exponential satisfies the estimate

\[
\|e^{At}\| \leq e^{-\gamma t} \sum_{j=0}^{N-1} c_j t^j, \quad t > 0,
\]

where the coefficients \( c_j \geq 0 \) depend only on \( \|A\| \) (see Corollary 12 for details). In particular, it easily follows from this estimate that \( \lim_{t \to +\infty} e^{(\gamma - \epsilon)t} \|e^{At}\| = 0 \) for any \( \epsilon > 0 \) uniformly for any bounded family of matrices \( A \). Applications of estimates of \( \|e^{At}\| \) can be found in [6, 11, 12].

In this paper, we prove a similar estimate for Green’s function for the problem of bounded on the axis solutions of the differential equation

\[
x'(t) - Ax(t) = f(t).
\]

The proof is similar to that of [11] and uses some constructions from [17].

In Sections 1 and 2, preliminaries are collected. In Section 3, we recall the definition of Green’s function and some its properties and describe its representation in the form of the Newton interpolating polynomial. In Section 4, we prove our estimate (Theorem 11).

1. The Newton interpolating polynomial

Let \( \mu_1, \mu_2, \ldots, \mu_N \) be given complex numbers (some of them may coincide with others) called points of interpolation. Let a complex-valued function \( f \) be defined and analytic in a neighbourhood \( U \) of these points. Divided differences of the function \( f \) with respect to
the points \( \mu_1, \mu_2, \ldots, \mu_N \) are defined (see, e.g., [1]) by the recurrent relations
\[
\begin{align*}
f[\mu_i] &= f(\mu_i), \\
f[\mu_i, \mu_{i+1}] &= \frac{f(\mu_{i+1}) - f(\mu_i)}{\mu_{i+1} - \mu_i}, \\
f[\mu_i, \ldots, \mu_{i+m}] &= \frac{f(\mu_{i+1}, \ldots, \mu_{i+m}) - f(\mu_1, \ldots, \mu_{i+m-1})}{\mu_{i+m} - \mu_i}.
\end{align*}
\]
In these formulas, if the denominator vanishes, then the quotient is understood as the derivative with respect to the corresponding argument of the previous divided difference.

**Proposition 1** ([1, ch. 1, formula (54)]). Let the function \( f \) be analytic in a neighbourhood of the points of interpolation \( \mu_1, \mu_2, \ldots, \mu_N \). Then
\[
f[\mu_1, \ldots, \mu_N] = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{\Omega(z)} dz,
\]
where the contour \( \Gamma \) encloses all the points of interpolation and
\[
\Omega(z) = \prod_{k=1}^{N} (z - \mu_k).
\]

**Proof.** The statement follows from Proposition 1. \( \square \)

**Proposition 2** ([1, ch. 1, formula (48)]). Let the points of interpolation \( \mu_j \) be distinct. Then
\[
f[\mu_1, \ldots, \mu_N] = \sum_{j=1}^{N} \frac{f(\mu_j)}{\prod_{k=1}^{N}_{k \neq j} (\mu_j - \mu_k)}.
\]

**Proposition 3** ([1, ch. 1, formula (49)]). Let the domain \( U \) of \( f \) contain the convex hull of the set \( \{\mu_1, \ldots, \mu_N\} \). Then the following estimate holds:
\[
|f[\mu_1, \ldots, \mu_N]| \leq \frac{1}{(N - 1)!} \max_{\lambda \in \text{ch} \{\mu_1, \ldots, \mu_N\}} |f^{(N-1)}(\lambda)|,
\]
where \( \text{ch} \{\mu_1, \ldots, \mu_N\} \) means the convex hull of the set \( \{\mu_1, \ldots, \mu_N\} \).

The set \( \lambda_1, \ldots, \lambda_M \in \mathbb{C} \) of interpolation points together with the set \( n_1, \ldots, n_M \in \mathbb{N} \) of their multiplicities is called multiple interpolation data. We set \( N = n_1 + \cdots + n_M \).

Let \( U \subseteq \mathbb{C} \) be an open neighbourhood of the set \( \lambda_1, \ldots, \lambda_M \) of the points of interpolation and \( f : U \to \mathbb{C} \) be an analytic function. An interpolating polynomial of \( f \) that corresponds to the multiple interpolation data is a polynomial \( p \) of degree \( N - 1 \) satisfying the equalities
\[
p^{(j)}(\lambda_k) = g^{(j)}(\lambda_k), \quad k = 1, \ldots, M; \quad j = 0, 1, \ldots, n_k - 1.
\]

**Proposition 4** ([1, p. 20]). For any analytic function \( f \), the interpolating polynomial exists and is unique. Let \( \mu_1, \ldots, \mu_N \) be the points of multiple interpolation data \( \lambda_1, \ldots, \lambda_M \), listed in an arbitrary order and repeated as many times as their multiplicities \( n_1, \ldots, n_M \). Then the interpolating polynomial possesses the representation
\[
p(z) = f[\mu_1] + f[\mu_1, \mu_2](z - \mu_1) + f[\mu_1, \mu_2, \mu_3](z - \mu_1)(z - \mu_2) + f[\mu_1, \mu_2, \mu_3, \mu_4](z - \mu_1)(z - \mu_2)(z - \mu_3) + \cdots
\]
\[
+ f[\mu_1, \mu_2, \ldots, \mu_N](z - \mu_1)(z - \mu_2) \cdots (z - \mu_{N-1}). \tag{1}
\]
Representation (1) is called [1, 3] the interpolating polynomial in the Newton form or shortly the Newton interpolating polynomial with respect to the points \( \mu_1, \mu_2, \ldots, \mu_N \).
2. Matrix functions

Let $A$ be a complex $N \times N$-matrix. Let $1$ be the identity matrix. The polynomial

$$p_A(\lambda) = \det(\lambda 1 - A)$$

is called the characteristic polynomial of the matrix $A$. Let $\lambda_1, \ldots, \lambda_M$ be the complete set of the roots of the characteristic polynomial $p_A$, and $n_1, \ldots, n_M$ be their multiplicities; thus $n_1 + \cdots + n_M = N$. It is well known that $\lambda_1, \ldots, \lambda_M$ are eigenvalues of $A$. The numbers $n_k$ are called (algebraic) multiplicities of the eigenvalues $\lambda_k$. The set $\sigma(A) = \{\lambda_1, \ldots, \lambda_M\}$ is called the spectrum of $A$.

Let $U \subseteq \mathbb{C}$ be an open set that contains the spectrum $\sigma(A)$. Let $f : U \to \mathbb{C}$ be an analytic function. The function $f$ of the matrix $A$ is defined $[\text{7}, \text{p. 17}], [\text{8}, \text{ch. VII}], [\text{14}, \text{ch. V, } \S \text{1}]$ by the formula

$$f(A) = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) (\lambda 1 - A)^{-1} d\lambda,$$

where the contour $\Gamma$ surrounds the spectrum $\sigma(A)$.

**Proposition 5** ([\text{14}, Theorem 5.2.5]). The mapping $f \mapsto f(A)$ preserves algebraic operations, i.e.,

$$(f + g)(A) = f(A) + g(A),$$

$$(\alpha f)(A) = \alpha f(A),$$

$$(fg)(A) = f(A)g(A),$$

where $f + g$, $\alpha f$ and $fg$ are defined pointwise.

**Proposition 6** (see, e.g., [\text{3}, Proposition 2.3]). Let $p$ be an interpolating polynomial of $f$ that corresponds to the points $\lambda_1, \ldots, \lambda_M$ of the spectrum of the matrix $A$ counted according to their multiplicities $n_1, \ldots, n_M$. Then

$$p(A) = f(A).$$

**Remark 1.** Proposition 6 remains valid if one assumes that $n_1, \ldots, n_M$ are the maximal sizes of the corresponding Jordan blocks. This assumption decreases the degree $N - 1$ of the interpolating polynomial.

3. Green’s function

In this Section, we recall the definition and some properties of Green’s function. Let $A$ be a complex $N \times N$-matrix. We consider the differential equation

$$x'(t) = Ax(t) + f(t), \quad t \in \mathbb{R}. \quad (2)$$

We are interested in bounded solutions problem, i.e. seeking bounded solutions $x : \mathbb{R} \to \mathbb{C}^N$ under the assumption that the free term $f : \mathbb{R} \to \mathbb{C}^N$ is a bounded function. The bounded solutions problem has its origin in the work of Perron [\text{23}]. Its different modifications can be found in [\text{4, 5, 6, 13, 16, 21, 22, 24, 26}]; see also references therein.
Suppose that \( \sigma(A) \) does not intersect the imaginary axis. In this case the functions

\[
\exp^+\lambda(t) = \begin{cases} 
  e^{\lambda t}, & \text{if } \Re \lambda < 0, \\
  0, & \text{if } \Re \lambda > 0,
\end{cases}
\]

\[
\exp^-\lambda(t) = \begin{cases} 
  0, & \text{if } \Re \lambda < 0, \\
  e^{\lambda t}, & \text{if } \Re \lambda > 0,
\end{cases}
\]

\[
g_t(\lambda) = \begin{cases} 
  -\exp^-\lambda(t), & \text{if } t < 0, \\
  \exp^+\lambda(t), & \text{if } t > 0
\end{cases}
\]

are analytic in the neighbourhood \( \mathbb{C} \setminus i\mathbb{R} \) of the spectrum \( \sigma(A) \). We set

\[
G(t) = g_t(A), \quad t \neq 0.
\]

(3)

The function \( G \) is called Green’s function of the bounded solutions problem for equation (2).

The main property of Green’s function is described in the following theorem.

**Theorem 7** ([7, Theorem 4.1, p. 81]). Equation (2) has a unique bounded on \( \mathbb{R} \) continuously differentiable solution \( x \) for any bounded continuous function \( f \) if and only if the spectrum \( \sigma(A) \) does not intersect the imaginary axis. This solution possesses the representation

\[
x(t) = \int_{-\infty}^{\infty} G(t-s)f(s)\,ds,
\]

where \( G \) is Green’s function (3) of equation (2).

Below we assume that \( A \) is a fixed complex \( N \times N \)-matrix and its spectrum does not intersect the imaginary axis. We denote by \( \mu_1, \ldots, \mu_k \) the roots of the characteristic polynomial that lie in the open right half-plane \( \Re \mu > 0 \) counted according to their multiplicities; and we denote by \( \nu_1, \ldots, \nu_m \) the roots of the characteristic polynomial that lie in the open left half-plane \( \Re \nu < 0 \) counted according to their multiplicities. Thus, \( k + m = N \). We denote by \( \gamma_-, \gamma_+ > 0 \) real numbers such that

\[
\Re \mu_i \geq \gamma_+ \quad \text{for } 1 \leq i \leq k, \\
\Re \nu_j \leq -\gamma_- \quad \text{for } 1 \leq j \leq m.
\]

(4)

**Proposition 8** ([17]). Let an analytic function \( f \) be identically zero in the open right half-plane \( \Re \mu > 0 \) (an example of such a function is the function \( \exp^+_\mu \)). Then

\[
f[\mu_1, \ldots, \mu_k; \nu_1, \ldots, \nu_m] = \tilde{f}[\nu_1, \ldots, \nu_m],
\]

where

\[
\tilde{f}(z) = \frac{f(z)}{\prod_{i=1}^{k}(z - \mu_i)}.
\]

**Proof.** Suppose that all multiplicities equal 1. By Proposition 4 we have

\[
f[\mu_1, \ldots, \mu_k; \nu_1, \ldots, \nu_m] = \sum_{q=1}^{m} \frac{f(\nu_q)}{\prod_{i=1}^{k}(\nu_q - \mu_i) \prod_{j=1}^{m}(\nu_q - \nu_j)} = \sum_{q=1}^{m} \frac{f(\nu_q)}{\prod_{i=1}^{k}(\nu_q - \mu_i) \prod_{j=1}^{m}(\nu_q - \nu_j)}
\]

\[
= \tilde{f}[\nu_1, \ldots, \nu_m].
\]
From Proposition 6 it easily follows that divided differences continuously depend on their arguments. Hence, the case of multiple points of interpolation is obtained by a passage to the limit.

Theorem 9 ([7]). Let us arrange the roots of the characteristic polynomial in the following order:

\[ \mu_1, \ldots, \mu_k; \nu_1, \ldots, \nu_m. \]  

Then the Newton interpolating polynomial \( p^+_t \) of the function \( \exp^+_t \) takes the form

\[ p^+_t(z) = (z - \mu_1) \ldots (z - \mu_k) q^+_t(z), \]

where

\[ q^+_t(z) = \exp^+_t[\nu_1] + \cdots + \exp^+_t[\nu_1, \ldots, \nu_m](z - \nu_1) \ldots (z - \nu_{m-1}) \]

is the interpolating polynomial of the function

\[ \exp^+_t(z) = \frac{\exp^+_t(z)}{\prod_{i=1}^k (z - \mu_i)} \]

with respect to the points \( \nu_1, \ldots, \nu_m \). The interpolating polynomial \( p^-_t \) of the function \( \exp^-_t \) can be represented in the form

\[ p^-_t(z) = (z - \nu_1) \ldots (z - \nu_m) q^-_t(z), \]

where

\[ q^-_t(z) = \exp^-_t[\mu_1] + \cdots + \exp^-_t[\mu_1, \ldots, \mu_k](z - \mu_1) \ldots (z - \mu_{k-1}) \]

is the interpolating polynomial of the function

\[ \exp^-_t(z) = \frac{\exp^-_t(z)}{\prod_{i=1}^m (z - \nu_i)} \]

with respect to the points \( \mu_1, \ldots, \mu_k \).

Proof. We observe that \( \exp^+_t(\mu_i) = 0, i = 1, \ldots, k \). Therefore

\[ \exp^+_t[\mu_1] = \cdots = \exp^+_t[\mu_1, \ldots, \mu_k] = 0. \]

Now from Proposition 4 it follows that

\[ p^+_t(z) = \exp^+_t[\mu_1, \ldots, \mu_k; \nu_1](z - \mu_1) \ldots (z - \mu_k) + \]

\[ + \exp^+_t[\mu_1, \ldots, \mu_k; \nu_1, \ldots, \nu_m](z - \mu_1) \ldots (z - \mu_k)(z - \nu_1) \ldots (z - \nu_{m-1}). \]

It remains to apply Proposition 8. \( \square \)

4. The estimate

In this Section we prove an estimate of Green’s function. As a potential application of this estimate, we note that knowing an estimate of the function \( t \mapsto \| G(t) \| \) is an important information in the freezing method for equations with slowly varying coefficients [4, § 10.2], [13, § 7.4], [20, ch. 10, § 3], [1, 3, 11, 13, 23, 24, 28]. See also references therein.

Lemma 10. Let \( \gamma^-, \gamma^+ > 0 \), \( \Re z \leq -\gamma^- \), and \( \Re \mu_j \geq \gamma^+ \) for \( j = 1, \ldots, k \). Then for \( k \geq 1 \) we have

\[ \left| \frac{d^l}{dz^l} \frac{e^{zt}}{\prod_{j=1}^k (z - \mu_j)} \right| \leq e^{-\gamma^- t} \sum_{i=0}^l t^{l-i} \binom{l}{i} \frac{(k + i - 1)!}{(k - 1)!} \frac{1}{\gamma^{k+i}}, \quad t > 0, \]

(7)
where $\gamma = \gamma^+ + \gamma^-$. But for $k = 0$

$$\left| \frac{d^l}{dz^l} e^{zt} \right| \leq e^{-\gamma^- t} t^l, \quad t > 0. \tag{8}$$

Remark 2. Formula (8) becomes a special case of (9) if one sets $(-1)! = 1$ and $(i-1)! = 0$ for $i = 1, 2, \ldots$.

Proof. By the general Leibniz product differentiation rule [1] ch. 1, § 3, Proposition 2 we have the identity

$$\left[ \frac{e^{zt}}{\prod_{j=1}^k (z - \mu_j)} \right]^{(l)} = e^{zt} \sum_{i=0}^l \binom{l}{i} t^{m-i} \left[ \frac{1}{\prod_{j=1}^k (z - \mu_j)} \right]^{(i)}.$$

In order to complete the proof, it is enough to show that

$$\left| \left[ \frac{1}{\prod_{j=1}^k (z - \mu_j)} \right]^{(i)} \right| \leq (k + i - 1)! \frac{1}{(k-1)!} \frac{1}{\gamma^{k+i}}.$$

We recall that among the numbers $\mu_j$ there may be repeating ones.

We note that by the product differentiation rule [1] ch. 1, § 1, Proposition 3], the derivative $\left[ \frac{1}{\prod_{j=1}^k (z - \mu_j)} \right]'$ is the sum of $k$ summands of the form $\frac{-1}{\prod_{j=1}^{k+1} (z - \mu_j)}$, where $\mu_j^{(1)}$ are the old numbers $\mu_j$, but one of them is repeated twice. In the course of the next differentiation, each term $\frac{-1}{\prod_{j=1}^{k+1} (z - \mu_j)}$ turns into $k+1$ terms of the form $\frac{-1}{\prod_{j=1}^{k+2} (z - \mu_j)}$, and the entire first derivative is transformed into $k(k+1)$ terms of the form $\frac{-1}{\prod_{j=1}^{k+2} (z - \mu_j)}$, where $\mu_j^{(2)}$ are some numbers satisfying the condition $\text{Re} \mu_j^{(2)} \geq \gamma^+$. The third derivative $\left[ \frac{1}{\prod_{j=1}^k (z - \mu_j)} \right]^{(3)}$ consists of $k(k+1)(k+2)$ summands of the form $\frac{-1}{\prod_{j=1}^{k+3} (z - \mu_j)}$. And so on.

Each term of the $i$-th derivative is less in absolute value than or equal to $\frac{1}{\gamma^{i+1}}$, and the total number of terms is $k(k+1)(k+2)(k+i-1) = \frac{(k+i-1)!}{(k-1)!}$.

Let us fix a norm in $\mathbb{C}^N$. We define the norm of an $N \times N$-matrix $A$ as the norm of the linear operator acting in $\mathbb{C}^N$ induced by $A$.

**Theorem 11.** Let assumption (8) be satisfied. We set $\gamma = \gamma^+ + \gamma^-$. Then Green’s function satisfies the estimates

$$\|G(t)\| \leq e^{-\gamma^- t} \sum_{j=0}^{m-1} \frac{t^j}{(j-i)!} \sum_{i=0}^j \binom{k+i-1}{k-1} \frac{(2\|A\|)^{k+j}}{\gamma^{k+i}}, \quad t > 0, \tag{9}$$

$$\|G(t)\| \leq e^{\gamma^+ t} \sum_{j=0}^{k-1} \frac{t^j}{(j-i)!} \sum_{i=0}^j \binom{m+i-1}{m-1} \frac{(2\|A\|)^{m+j}}{\gamma^{m+i}}, \quad t < 0. \tag{10}$$

Proof. We consider the case $t > 0$. We represent the Newton interpolating polynomial $p_i^+$ of the function $\exp_t^+$ in the form (8). By Propositions [3] and [3] we have

$$G(t) = \exp_t^+ (A) = p_i^+(A) = (A - \mu_1 1) \cdots (A - \mu_k 1) q_i^+(A), \quad t > 0,$$

where

$$q_i^+(A) = \exp_t^+[\nu_1] 1 + \cdots + \exp_t^+[\nu_1, \ldots, \nu_m](A - \nu_1 1) \cdots (A - \nu_{m-1} 1). \tag{11}$$
Clearly, $\|A - \mu_i 1\| \leq 2\|A\|$. Therefore
\[
\|G(t)\| = \|p_t(A)\| \leq (2\|A\|)^k \|q^+_t(A)\|. \tag{12}
\]
From representation (14), Proposition 3, Theorem 1 and Lemma 4 we have
\[
\|q^+_t(A)\| \leq \sum_{j=0}^{m-1} \left| \exp_{\frac{t}{j!}}[\nu_1, \ldots, \nu_{j+1}] \right| (2\|A\|)^j
\]
\[
\leq \sum_{j=0}^{m-1} \frac{1}{j!} \max_{\lambda \in \mathbb{C}(\nu_1, \ldots, \nu_{j+1})} \left| \exp_{\frac{t}{j!}}(\lambda) \right| (2\|A\|)^j
\]
\[
\leq e^{-\gamma t} \sum_{j=0}^{m-1} \frac{1}{j!} \binom{j}{i} \frac{(k+i-1)!}{(k-1)!} (2\|A\|)^j \frac{t^j}{\gamma^{k+i}}
\]
\[
\leq e^{-\gamma t} \sum_{j=0}^{m-1} \frac{1}{(j-i)!} \binom{j-i}{k-1} (2\|A\|)^j \frac{t^j}{\gamma^{k+i}}.
\]
Taking (12) into account we arrive at
\[
\|G(t)\| \leq e^{-\gamma t} \sum_{j=0}^{m-1} \frac{1}{(j-i)!} \binom{j-i}{k-1} (2\|A\|)^j \frac{t^j}{\gamma^{k+i}}, \quad t > 0.
\]
Formula (17) is proved in a similar way. \qed

Example 1. For $N = 6$ and $k = 3$ estimate (3) has the form
\[
\|G(t)\| \leq e^{-\gamma t} \left( 6\|A\|^5 + 6\|A\|^5 t + 2\|A\|^5 t^2 + 3\|A\|^4 t + 2\|A\|^4 t + \|A\|^3 \right), \quad t > 0.
\]

Corollary 12 (3, p. 131, Lemma 10.2.1, 14, p. 68, formula (13)). Let the eigenvalues of the matrix $A$ lie in the half-plane $\Re \lambda < -\gamma^-$, where $\gamma^- > 0$. Then
\[
\|e^{At}\| \leq e^{-\gamma^- t} \sum_{j=0}^{N-1} \frac{j!}{(2t\|A\|)^j}, \quad t > 0.
\]

Proof. The proof is similar to that of Theorem 11. The proof can also be obtained as a special case of (17) if we take into account Remark 4. \qed

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