Knotting of algebraic curves in complex surfaces

Sergey Finashin

Abstract. For any \( d \geq 5 \), I constructed infinitely many pairwise smoothly non-equivalent surfaces \( F \subset \mathbb{C}P^2 \) homeomorphic to a non-singular algebraic curve of degree \( d \), realizing the same homology class as such a curve and having abelian fundamental group \( \pi_1(\mathbb{C}P^2 \setminus F) \).

It is a special case of a more general theorem, which concerns for instance those algebraic curves, \( A \), in a simply connected algebraic surface, \( X \), which admit irreducible degenerations to a curve \( A_0 \), with a unique singularity of the type \( X_9 \), and such that \( A \cdot A > 16 \).

1. Introduction

Theorem 1.1. For any \( d \geq 5 \) there exist infinitely many smooth oriented closed surfaces \( F \subset \mathbb{C}P^2 \) representing class \( d \in H_2(\mathbb{C}P^2) = \mathbb{Z} \), having genus(\( F) = \frac{1}{2}(d - 1)(d - 2) \) and \( \pi_1(\mathbb{C}P^2 \setminus F) \cong \mathbb{Z}/d \), such that the pairs \((\mathbb{C}P^2, F)\) are pairwise smoothly non-equivalent. Moreover, \( d\)-fold cyclic coverings over \( \mathbb{C}P^2 \) branched along \( F \) differ by their Seiberg-Witten invariants and thus are non-diffeomorphic.

This theorem, which answers an old question (cf. [6], Problem 4.110), is proved in [2] for even \( d \geq 6 \). In this paper I added the proof for odd \( d \) and generalized Theorem 1.1 (see below Theorem 1.6). Sections 2-3 and the Appendix reproduce the content of [2] whereas Section 5 extends the results from there.

Remark 1.1. Note that the surfaces that I construct are not symplectic. Some speculation referring to Gromov’s theorem suggests that any symplectic surface in \( \mathbb{C}P^2 \) may be isotopic to an algebraic curve. As far as I know, at the moment it is proved only for degrees \( d \leq 4 \).

The knotting construction used to obtain surfaces \( F \) is a relative of the rim-surgery defined in [5]. An alternative way to achieve Theorem 1.1 is to use the tangle-surgery of Viro introduced in [3]. For technical reasons I prefer to use the rim-surgery in this paper, and give below an idea about the other approach just because it inspired this paper.

1.1. The idea that inspired my construction

Any kind of a surgery on a codimension two submanifold, \( F \), in some fixed \( n \)-manifold \( X \) gives rise to some \( n \)-dimensional surgery on the double covering \( Y \rightarrow X \) branched along \( F \). Vice versa, considering a surgery on \( Y \), one can try to perform it equivariantly with
respect to the covering transformation, which results in some surgery on a pair \((X, F)\). Sometimes \(X\) is preserved, and only \(F\) as an embedded submanifold is modified by this surgery. I call such an ambient surgery on \(F\) in \(X\) the folding of the corresponding surgery on \(Y\).

For example, if \(Y\) is a complex surface defined over \(\mathbb{R}\), and \(X = Y/\text{conj}\) is the quotient by the complex conjugation \(\text{conj}: Y \to Y\), then the projection \(p: Y \to X\) is a double covering branched along \(F = \text{Fix}(\text{conj})\) (the real locus of \(Y\)). Algebraic transformations (say, a blow-up, or a logarithmic transform) can be applied to \(Y\) in the real category. It turns out (at least in the examples known to the author) that the quotient \(X = Y/\text{conj}\) is not changed if a transformation is irreducible over \(\mathbb{C}\), i.e., if it does not contain a pair of \(\text{conj}\)-symmetric transformations localized outside the real part \(F\).

Say, the folding of a blow-up at a real point of \(Y\) is a real blow-up of \(F\), that is an ambient connected sum \((X, F) \# (S^4, \mathbb{R}P^2)\), because \(\mathbb{CP}^2/\text{conj} \cong S^4\). Viro observed [3] that the folding of a logarithmic transform is a certain tangle-surgery on \(F\). This yields "exotic knottings" of \(F = \#_10 \mathbb{R}P^2\) in \(S^4 = Y/\text{conj}\), where \(Y = E(1) = \mathbb{CP}^2#_9 \mathbb{CP}^2\) is a rational elliptic surface, being modified by logarithmic transforms (which produce Dolgachev surfaces defined over \(\mathbb{R}\)).

The same construction applied to a K3 surface, \(Y = E(2)\), instead of \(E(1)\), gives "exotic knottings" of \(F\) in \(X = Y/\text{conj}\). For a suitable choice of the real structure in \(Y\), the quotient \(X\) is diffeomorphic to \(\mathbb{CP}^2\) and \(F\) becomes a sextic in \(X\), so the surgery gives examples for \(d = 6\) in Theorem 1.1. Viro’s tangle surgery can be applied, in general, along any null-framed annulus membrane on a surface in a four-manifold, which gives in the covering space a logarithmic transform. Suitable membranes on algebraic curves in \(\mathbb{CP}^2\) are described in what follows.

It turned out that the Fintushel-Stern’s surgery on \(Y\) admits also a folding, i.e., can be made equivariantly, with the quotient \(X\) being preserved, provided the knot that we use is a double knot, i.e., \(K \# K\). This folding is just what I call below “an annulus rim surgery”.

1.2. An annulus rim-surgery

Our surgery, like the Viro tangle surgery, requires a suitable annulus membrane and produces a new surface via knotting an old one along such a membrane. By an annulus membrane for a smooth surface \(F\) in a 4-manifold \(X\) I mean a smoothly embedded surface \(M \subset X, M \cong S^1 \times I\), with \(M \cap F = \partial M\) and such that \(M\) comes to \(F\) normally along \(\partial M\). Assume that such a membrane has framing 0, or equivalently, admits a diffeomorphism of its regular neighborhood \(\phi: U \to S^1 \times D^3\) mapping \(U \cap F\) onto \(S^1 \times f\), where \(f = I \sqcup I \subset D^3\) is a disjoint union of two segments, which are unknotted and unlinked in \(D^3\), that is to say that a union of \(f\) with a pair of arcs on a sphere \(\partial D^3\) bounds a trivially embedded band, \(b \subset D^3, b \cong I \times I\), so that \(f = I \times (\partial I) \subset b\) (see Figure 1). The annulus \(M\) can be viewed as \(S^1 \times \{ \frac{1}{2} \} \times I\) in \(S^1 \times b \subset S^1 \times D^3 \cong U\).

If \(X\) and \(F\) are oriented, then \(f\) inherits an orientation as a transverse intersection, \(f = F \cap D^3\), and we may choose a band \(b\) so that the orientation of \(f\) is induced from
some orientation of \( b \). It is convenient to view \( f = I \sqcup I \) as is shown on Figure 1, so that the segments of \( f \) are parallel and oppositely oriented, with \( b \) being a thin band between them. Such a presentation is always possible if we allow a modification of \( \phi \), since one of the segments of \( f \) may be turned around by a diffeomorphism of \( D^3 \to D^3 \) leaving the other segment fixed.

Given a knot \( K \subset S^3 \), we construct a new smooth surface, \( F_{K,\phi} \), obtained from \( F \) by tying a pair of segments \( I \sqcup I \) along \( K \) inside \( D^3 \), as is shown on Figure 1. More precisely, we consider a band \( b_K \subset D^3 \) obtained from \( b \) by knotting along \( K \) and let \( f_K \) denote the pair of arcs bounding \( b_K \) inside \( D^3 \). We assume that the framing of \( b_K \) is chosen the same as the framing of \( b \), or equivalently, that the inclusion homomorphisms from \( H_1(\partial D^3 \setminus (\partial f)) \to H_1(B^3 \setminus f) \) and to \( H_1(B^3 \setminus f_K) \) have the same kernel. Then \( F_{K,\phi} \) is obtained from \( F \) by replacing \( S^1 \times f \subset S^1 \times D^3 \) \( \cong U \) with \( S^1 \times f_K \). It is obvious that \( F_{K,\phi} \) is homeomorphic to \( F \) and realizes the same homology class in \( H_2(X) \).

![Figure 1. Knotting of a band \( b_K \)](image)

The above construction is called in what follows an annulus rim-surgery, since it looks like the rim-surgery of Fintushel and Stern \[5\], except that we tie two strands simultaneously, rather then one. Recall that the usual rim-surgery is applied in \[5\] to surfaces \( F \subset X \) which are primitively embedded, that is \( \pi_1(X \setminus F) = 0 \), which is not the case for the algebraic curves in \( \mathbb{CP}^2 \) of degree \( > 1 \). The primitivity condition is required to preserve the fundamental group of \( X \setminus F \) throughout the knotting. An annulus rim-surgery may preserve a non-trivial group \( \pi_1(X \setminus F) \), if we require commutativity of \( \pi_1(X \setminus (F \cup M)) \), instead of primitivity of the embedding.

**Proposition 1.2.** Assume that \( X \) is a simply connected closed 4-manifold, \( F \subset X \) is an oriented closed surface with an annulus-membrane \( M \) of index 0, \( \phi : U \to S^1 \times D^3 \) is a trivialization like described above and \( K \subset S^3 \) is any knot. Assume furthermore that \( F \setminus \partial M \) is connected and the group \( \pi_1(X \setminus (F \cup M)) \) is abelian. Then the group \( \pi_1(X \setminus F_{K,\phi}) \) is cyclic and isomorphic to \( \pi_1(X \setminus F) \).
1.3. Maximal nest curves

To prove Theorem 1.1, I apply an annulus rim-surgery inside \( X = \mathbb{CP}^2 \) letting \( F = \mathbb{CA} \) be the complex point set of a suitable non-singular real algebraic curve, containing an annulus, \( M \), among the connected components of \( \mathbb{RP}^2 \setminus \mathbb{RA} \), where \( \mathbb{RA} = \mathbb{CA} \cap \mathbb{RP}^2 \) is the real locus of the curve.

One may take, for instance, a real algebraic curve \( \mathbb{CA} \) of degree \( d \), with a maximal nest real scheme. Such a curve for \( d = 2k \) is constructed by a small real perturbation of a union of \( k \) real conics, whose real parts (ellipses) are ordered by inclusion in \( \mathbb{RP}^2 \). For \( d = 2k + 1 \), we add to such conics a real line not intersecting the conics in \( \mathbb{RP}^2 \) and then perturb the unions. The real part, \( \mathbb{RA} \), of our non-singular curve contains \( k \) components, \( O_1, \ldots, O_k \), called ovals (just deformed ellipses). We order the ovals so that \( O_i \) lies inside \( O_{i+1} \) and denote by \( R_i \) the annulus-component of \( \mathbb{RP}^2 \setminus \mathbb{RA} \) between \( O_i \) and \( O_{i+1} \) for \( i = 1, \ldots, k - 1 \). \( R_0 \) is a topological disk bounded from outside by \( O_1 \), and \( R_k \) is the component bounded from inside by \( O_k \).

The closures, \( \text{Cl}(R_i) \), for \( i = 1, \ldots, k - 1 \) are obviously 0-framed annulus-membranes on \( \mathbb{CA} \). For simplicity, let us choose \( M = \text{Cl}(R_1) \).

**Proposition 1.3.** The assumptions of Proposition 1.2 hold if we put \( X = \mathbb{CP}^2 \), let \( F = \mathbb{CA} \) be a maximal nest real algebraic curve of degree \( d \geq 5 \) and choose \( M = \text{Cl}(R_1) \).

1.4. Proof of Theorem 1.1 for even \( d \)

Assuming that the class \([F] \in H_2(X; \mathbb{Z}/2)\) vanishes, one can consider a double covering \( p : Y \to X \) branched along \( F \); such a covering is unique if we require in addition that \( H_1(X; \mathbb{Z}/2) = 0 \). Similarly, we consider the double coverings \( Y(K, \phi) \to X \) branched along \( F_{K, \phi} \). To prove non-equivalence of pairs \((\mathbb{CP}^2, F_{K, \phi})\) for some family of knots \( K \), it is enough to show that \( Y(K, \phi) \) are not pairwise diffeomorphic. To show it, I use that \( Y(K, \phi) \) is diffeomorphic to the 4-manifolds \( Y_{K \# K} \) obtained from \( Y \) by a surgery introduced in [4] (I call it FS-surgery).

**Proposition 1.4.** The above \( Y(K, \phi) \) is diffeomorphic to a 4-manifold obtained from \( Y \) by the FS-surgery along the torus \( T = p^{-1}(M) \) via the knot \( K \# K \subset S^3 \).

To distinguish the diffeomorphism types of \( Y_{K \# K} \) one can use the formula of Fintushel and Stern [4] for SW-invariants of a 4-manifold \( Y \) after FS-surgery along a torus \( T \subset Y \). Recall that this formula can be applied if the SW-invariants of \( Y \) are well-defined and a torus \( T \), realizing a non-trivial class \([T] \in H_2(Y)\), is c-embedded (the latter means that \( T \) lies as a non-singular fiber in a cusp-neighborhood in \( Y \), cf. [4]). Being an algebraic surface of genus \( \geq 1 \), the double plane \( Y \) has well-defined SW-invariants. The conditions on \( T \) are also satisfied.

**Proposition 1.5.** Assume that \( X, F \), and \( M \) are like in Proposition 1.2, \([F] \in H_2(X; \mathbb{Z}/2)\) vanishes and \( p : Y \to X \) is like above. Then the torus \( T = p^{-1}(M) \) is primitively embedded in \( Y \) and therefore \([T] \in H_2(Y)\) is an infinite order class. If, moreover, \( X, F \), and \( M \) are chosen like in Proposition 1.3, then \( T \subset Y \) is c-embedded.
Recall that the product formula [4]
\[ \text{SW}_{Y_K} = \text{SW}_Y \cdot \Delta_K(t), \quad \text{where} \quad t = \exp(2[T]) \]
expresses the Seiberg-Witten invariants (combined in a single polynomial) of the manifold \( Y_K \), obtained by an FS-surgery, in terms of the Seiberg-Witten invariants of \( Y \) and the Alexander polynomial, \( \Delta_K(t) \), of \( K \).

This formula implies that the basic classes of \( Y_K \) can be expressed as \( \pm \beta + 2n[T] \), where \( \pm \beta \in H_2(Y) \) are the basic classes of \( Y \) and \( |n| \leq \deg(\Delta_K(t)) \), are the degrees of the non-vanishing monomials in \( \Delta_K(t) \). So, if \([T]\) has infinite order, then the manifolds \( Y(K, \varphi) \) differ from each other by their SW-invariants, and moreover, by the numbers of their basic classes, for an infinite family of knots \( K \), since the number of the basic classes is determined by the number of the terms in \( \Delta_K = (\Delta_K)^2 \) (one can take any family of knots with Alexander polynomials of distinct degrees).

### 1.5. A generalization

More generally, one can produce “fake algebraic curves” under the following conditions.

**Theorem 1.6.** Assume that \( F \) is a non-singular connected curve in a simply connected complex surface \( X \), which admits a deformation degenerating \( F \) into an irreducible curve \( F_0 \subset X \), with a singularity of the type \( X_9 \), such that the fundamental group \( \pi_1(X \setminus F_0) \) is abelian. Then there exists an infinite family of surfaces \( F_K, \varphi \subset X \) homeomorphic to \( F \) and realizing the same homology class as \( F \), having the same fundamental group of the complement, but with the smoothly non-equivalent pairs \( (X, F_K, \varphi) \).

I remind that \( X_9 \)-singularity is a point where 4 non-singular branches meet pairwise transversally. Nori’s theorem [7] gives conditions under which \( \pi_1(X \setminus F_0) \) must be abelian. For instance, it is so if \( A_0 \) has no other singularities except \( X_9 \) and \( A \circ A > 16 \).

**Remark 1.2.** The claim of Theorem 1.6 holds also if \( F_0 \) has a more complicated than \( X_9 \) singularity, provided the group \( \pi_1(X \setminus F_0) \) is abelian.

### 2. Commutativity of the fundamental group throughout the knotting

**Lemma 2.1.** The assumptions of Proposition 1.2 imply that \( \pi_1(X \setminus (F \cup M)) = \pi_1(X \setminus F) \) is cyclic with a generator presented by a loop around \( F \).

**Proof.** The Alexander duality in \( X \) combined with the exact cohomology sequence of a pair \( (X, F \cup M) \) gives
\[
H_1(X \setminus (F \cup M)) \cong H^1(X, F \cup M) = H^2(F \cup M) / i^*H^2(X)
\]
where \( i : F \cup M \to X \) is the inclusion map. If \( F \) is oriented and \( F \setminus \partial M \) is connected, then the Mayer-Vietoris Theorem yields \( H^2(F \cup M) \cong H^2(F) \cong \mathbb{Z} \), and thus \( H_1(X \setminus (F \cup M)) \cong H_1(X \setminus F) \) is cyclic with a generator presented by a loop around \( F \). The same property holds for the fundamental groups of \( X \setminus (F \cup M) \) and \( X \setminus F \), since they are abelian by the assumption of Proposition 1.2. \( \square \)
Proof of Proposition 1.2. Put $X_0 = \text{Cl}(X \smallsetminus U)$. Then $\partial X_0 = \partial U \cong S^1 \times S^2$ and $\partial U \smallsetminus F$ is a deformational retract of $U \smallsetminus (F \cup M)$, so
$$\pi_1(X_0 \smallsetminus F) = \pi_1(X \smallsetminus (F \cup M))$$
Since this group is cyclic and is generated by a loop around $F$, the inclusion homomorphism $h : \pi_1(\partial U \smallsetminus F) \to \pi_1(X_0 \smallsetminus F)$ is epimorphic and thus $\pi_1(X_0 \smallsetminus F) = \pi_1(\partial U \smallsetminus F)/k$, where $k$ is the kernel of $h$.

Applying the Van Kampen theorem to the triad $(X_0 \smallsetminus F, U \smallsetminus F, K, \phi)$, we conclude that
$$\pi_1(X \smallsetminus F_{K,\phi}) \cong \pi_1(X_0 \smallsetminus F)/j(k)$$
where $j : \pi_1(\partial U \smallsetminus F) \to \pi_1(U \smallsetminus F_{K,\phi})$ is the inclusion homomorphism. Furthermore, in the splitting
$$\pi_1(U \smallsetminus F_{K,\phi}) \cong \pi_1(S^1 \times (D^3 \smallsetminus f_K)) \cong \mathbb{Z} \times \pi_1(D^3 \smallsetminus f_K)$$
factorization by $j(k)$ kills the first factor $\mathbb{Z}$ and adds some relations to $\pi_1(D^3 \smallsetminus f_K)$, one of which affects $\pi_1(D^3 \smallsetminus f_K)$ as if we attach a 2-cell along a loop, $m_b$, going once around the band $b_K$ (to see it, note that factorization by $k$ leaves only one generator of $\pi_1(\partial D^3 \smallsetminus f_K) = \pi_1(S^2 \smallsetminus \{4\text{pts}\})$). Attaching such a 2-cell effects to $\pi_1$ as connecting together a pair of the endpoints of $f_K$, which transforms $f_K$ into an arc (see Figure 2). This arc is unknotted and thus factorization by $j(k)$ makes $\pi_1(D^3 \smallsetminus f_K)$ cyclic and leaves $\pi_1(X \smallsetminus F_{K,\phi})$ isomorphic to $\pi_1(X_0 \smallsetminus F) \cong \pi_1(X \smallsetminus (F \cup M)) \cong \pi_1(X \smallsetminus F)$.

![Figure 2. Gluing a 2-cell along $m_b$ effects as transforming $f_K$ into an unknotted arc](image)

Proof of Proposition 1.3. All the assumptions of Proposition 1.2 except the last two are obviously satisfied. It is well known that $\mathcal{C}A \smallsetminus RA$ splits for a maximal nest curve $\mathcal{C}A$ into a pair of connected components permuted by the complex conjugation, and thus, $\mathcal{C}A \smallsetminus \partial M$ is connected, provided $\partial M \subset RA$, which is the case for $d \geq 5$. So, it is only left to check that the group $\pi_1(CP^2 \smallsetminus (\mathcal{C}A \cup M))$ is abelian.

There are several ways to check it. For instance, one can refer to my old work [1] containing computation of the homotopy type of $CP^2 \smallsetminus (\mathcal{C}A \cup RP^2)$ and, in particular, of its fundamental group (see also §4 in [3]). This computation concerns a real curve $\mathcal{C}A \subset CP^2$ if it is an $L$-curve, i.e., $\mathcal{C}A$ can be obtained by a non-singular perturbation
from a curve $C_A = CL_1 \cup \ldots \cup CL_d$ splitting into $d$ real lines, $CL_i$, in a generic position. The maximal nest curves, $CA \subset \mathbb{CP}^2$, can be easily constructed as $L$-curves, and the result of [1] gives a presentation $\pi = \pi_1(\mathbb{CP}^2 \setminus (CA \cup \mathbb{RP}^2)) = \langle a, b | a^d b^d = 1 \rangle$, where $a$, $b$ are represented by loops around the two connected components of $CA \setminus RA$. More specifically, a basis point and these loops can be taken on the conic $C = \{x^2 + y^2 + z^2 = 0\} \subset \mathbb{CP}^2$, which have the real point set empty. The group $\pi_1(\mathbb{CP}^2 \setminus (CA \cup M))$ is obtained from $\pi$ by adding the relations corresponding to puncturing the components $R_i$, $0 \leq i \leq k$, $i \neq 1$, of $\mathbb{RP}^2 \setminus RA$ (here $d = 2k$ or $d = 2k + 1$). Such a relation (as we puncture $R_i$) is $a^{d-i} b^i = b^d - ia^i = 1$, see [1], or §4 in [3]. A pair of the relations for $i = 2$ and $i = 3$ implies that $a = b$.

The arguments from [1] and [3] relevant to the above calculation are briefly summarized in the Appendix.

**Remark 2.1.** It follows from the proof above that $\pi_1(\mathbb{CP}^2 \setminus (CA \cup M))$ is not abelian and $CA \setminus \partial M$ is not connected for a maximal nest quartic, $CA$.

### 3. The double surgery in the double covering

**Proof of Proposition 1.4.** The proof is based on the following two observations. First, we notice that $Y(K, \phi)$ is obtained from $Y$ by a pair of FS-surgeries along the tori parallel to $T$, then we notice that such pair of surgeries is equivalent to a single FS-surgery along $T$. The both observations are corollaries of Lemma 2.1 in [5], so, I have to recall first the construction from [4], [5].

An FS-surgery [4] on a 4-manifold $X$ along a torus $T \subset X$, with the self-intersection $T \circ T = 0$, via a knot $K \subset S^3$ is defined as a fiber sum $X \#_{T=1 \times m_{K}} S^1 \times M_{K}$, that is an amalgamated connected sum of $X$ and $S^1 \times M_{K}$ along the tori $T$ and $S^1 \times m_{K} \subset S^1 \times M_{K}$. Here $M_{K}$ is a 3-manifold obtained by the 0-surgery along $K$ in $S^3$, and $m_{K}$ denotes a meridian of $K$ (which may be seen both in $S^3$ and in $M_{K}$). Such a fiber sum operation can be viewed as a direct product of $S^1$ and the corresponding 3-dimensional operation, which I call $S^1$-fiber sum.

More precisely, $S^1$-fiber sum $X \#_{K=1 \times L} Y$ of oriented 3-manifolds $X$ and $Y$ along oriented framed knots $K \subset X$ and $L \subset Y$ is the manifold obtained by gluing the complements $Cl(X \setminus N(K))$ and $Cl(Y \setminus N(L))$ of tubular neighborhoods, $N(K)$, $N(L)$, of $K$ and $L$ via a diffeomorphism $f : \partial N(K) \to \partial N(L)$ which identifies the longitudes of $K$ with the longitudes of $L$ preserving their orientations, and the meridians of $K$ with the meridians of $L$ reversing the orientations. As it is shown in Lemma 2.1 of [5], tying a knot $K$ in an arc in $D^3$ can be interpreted as a fiber sum $D^3 \#_{m=m_{K}} M_{K}$, where $m$ is a meridian around this arc. The meridians $m$ and $m_{K}$ are endowed here with the 0-framings (0-framing of a meridian makes sense as a meridian lies in a small 3-disc). To understand this observation, it is useful to view an $S^1$-fiber sum with $M_{K}$ as surgering a tubular neighborhood, $N(m)$, of $m$ and replacing it by the complement, $S^3 \setminus N(K)$ of a tubular neighborhood, $N(K)$, of $K$, so that the longitudes of $m$ are glued to the meridians of $K$ and the meridians of $m$ to the longitudes of $K$. The framing of an arc in $D^3$ is preserved under such a fiber
sum, so tying a knot in the band $b \subset D^3$ is equivalent to taking an $S^1$-fiber sum with $M_K$ along a meridian $m_b$ around $b$.

The double covering over $D^3$ branched along $f$ is a solid torus, $N \cong S^1 \times D^2$, and the pull back of $m_b$ splits into a pair of circles, $m_1, m_2 \subset N$, parallel to $m = S^1 \times \{0\}$. Therefore, $Y(K, \phi)$ is obtained from $Y$ by performing FS-surgery twice, along the tori

$$T_i = S^1 \times m_i \subset p^{-1}(U) \cong S^1 \times N, \quad i = 1, 2$$

The following Lemma implies that this gives the same result as a single FS-surgery along $T = p^{-1}(M)$ via the knot $K \# K$.

**Lemma 3.1.** For any pair of knots, $K_1, K_2$, the manifold

$$M_{K_1 \# m_{K_1} = m_1} \# m_{m_2 = m_{K_2}} M_{K_2}$$

obtained by taking an $S^1$-fiber sum twice, is diffeomorphic to $N \# m = m_K M_K$, for $K = K_1 \# K_2$, via a diffeomorphism identical on $\partial N$.

**Proof.** A solid torus $N$ can be viewed as the complement $N = S^3 - N'$ of an open tubular neighborhood $N'$ of an unknot, so that $m, m_1, m_2$ represent meridians of this unknot. Taking a fiber sum of $S^3$ with $M_{K_1}$ along $m_i = m_{K_1}$ is equivalent to knotting $N'$ in $S^3$ via $K_i$. So, performing $S^1$-fiber sum twice, along $m_1$ and $m_2$, we obtain the same result as after taking fiber sum along $m$ once, via $K = K_1 \# K_2$. □

**Remark 3.1.** The above additivity property can be equivalently stated as

$$M_{K_1 \# m_{K_1} = m_{K_2}} M_{K_2} \cong M_{K_1 \# K_2}$$

**Proof of Proposition 1.5.** Lemma 2.1 implies that, in the assumptions of Proposition 1.2, $\pi_1(Y \setminus (F \cup T))$ is a cyclic group with a generator represented by a loop around $F$. Thus, $\pi_1(Y \setminus T) = 0$ and, by the Alexander duality, $H_3(Y, T) = H^1(Y \setminus T) = 0$, which implies that $[T] \in H_2(Y)$ has infinite order.

To check that $T$ is c-embedded it is enough to observe that there exists a pair of vanishing cycles on $T$, or more precisely, a pair of $D^2$-membranes, $D_1, D_2 \subset Y$, on $T$, having $(-1)$-framing and intersecting at a unique point $x \in T$, so that $[\partial D_1], [\partial D_2]$ form a basis of $H_1(T)$. In the setting of Proposition 1.3, $Y \to \mathbb{C}P^2$ is a double covering branched along a maximal nest curve $\mathcal{CA}$ and $T$ is a connected component of the real part of $Y$ (with respect to a certain real structure on $Y$ lifted from $\mathbb{C}P^2$). Two nodal degenerations of $\mathcal{CA}$ shown on the top part of Figure 3 give nodal degenerations of the double covering $Y$.

In the first of the degenerations of $\mathcal{CA}$, a node appears as an oval $O_1$ is collapsed into a point. In the second degeneration a crossing-like node can be seen as the fusion point of the ovals $O_1$ and $O_2$. Existence of such degenerations for our explicitly constructed curve $\mathcal{CA}$ is known and trivial. Another simple observation (which is obvious for quartics and thus follows for any maximal nest curve of a higher degree) is that our pair of nodal degenerations can be united into one cuspidal degeneration. This means in particular that the two vanishing cycles in $Y$ intersect transversally at a single point.
Furthermore, our complex vanishing cycles in $Y$ can be chosen conj-invariant. Being a $(−2)$-sphere, each of such complex cycles is divided by its real pair into a pair of $(−1)$-discs. Choosing one disc from each pair, we obtain $D_1$ and $D_2$ that we need.

It is easy to view these $(−2)$-spheres and the $(−1)$-disks explicitly. First, note that $R_0$ is a $(−1)$-membrane on $CA$ and $p^{-1}(R_0)$ is the first of the conj-symmetric vanishing cycles. The $(−1)$-disk $D_1$ is any of its halves. Furthermore, there is another $(−1)$-disk membrane, $Q$ on $CA$ corresponding to the second nodal degeneration. It can be chosen conj-invariant and then is split by $Q \cap \mathbb{R}P^2$ into semi-discs $Q = Q_1 \cup Q_2$ permuted by conj. $Q_i$ is bounded by the arcs $Q \cap \mathbb{R}P^2$ and $Q_i \cap CA$. The disk $D_2$ is any of the discs $p^{-1}(Q_i)$.

![Diagram showing nodal degenerations of curves and tori](https://example.com/diagram.png)

**Figure 3.** Nodal degenerations of $\mathbb{R}A$ providing $(−1)$-framed $D^2$-membranes on $T f_K$ into an unknotted arc

### 4. The case of $d$-fold branched covering

Consider as before a maximal nest curve, $CA \subset \mathbb{CP}^2$, of degree $d \geq 2$, and $CA_{K,\phi}$ obtained from $CA$ via an annulus rim-surgery along $R_1$, but now let us denote by $p : Y \to \mathbb{CP}^2$ and $Y(K, \phi) \to \mathbb{CP}^2$ the $d$-fold coverings branched along $CA$ and $CA_{K,\phi}$ respectively. Consider a $d$-fold covering $N \to D^3$ branched along $f$. The pull-back of $m_b$ consists of $d$ circles, $m_1, \ldots, m_d$, which are cyclically ordered. Using a homeomorphism $(D^3, f) \cong (D^2 \times [0, 1], \{z_1, z_2\} \times [0, 1])$, where $\{z_1, z_2\} \subset \text{Int}(D^2)$, we present $N$ as $F \times [0, 1]$, where $F$ is a sphere with $d$ holes. The circles $m_i$ go around these holes. An annulus rim-surgery in $\mathbb{CP}^2$ along $m_b \times S^1 \subset D^3 \times S^1$, is covered by $d$ copies of FS-surgery along the tori $T_i = m_i \times S^1 \subset N \times S^1$.

The following observation implies that the Fintushel-Stern formula for Seiberg-Witten invariants can be applied in this setting.
Proposition 4.1. Each of the tori $T_i$ is primitively $c$-embedded in the complement of the others.

Proof. A pair of $(-1)$-disc membranes, $D_1^i, D_2^i$, on each of $T_i$ is constructed like in the proof of Proposition 1.5. Namely, $p^{-1}(R_0)$ consists of $d$ disks which yield the disks $D_1^i$, that are glued along $\{\text{pt}\} \times S^1 \subset m_i \times S^1$.

Furthermore, $p^{-1}(Q_1)$ splits also into $d$ disks, $Q_1^1, \ldots, Q_d^1$. Let us choose their orientations induced from a fixed orientation of $Q_1$ and cyclically order in accord with the ordering of $T_i$, then the unions $Q_1^i \cup (-Q_1^{i+1})$ provide the required disks $D_2^i$, which are glued along $m_i \times \{\text{pt}\}$. More precisely, $D_2^i$ are the parts of the components of $p^{-1}(R_0)$ bounded by the intersections of the components with the tori $T_i$, whereas $D_1^i$ are obtained from $Q_1^i \cup (-Q_1^{i+1})$ by a small shift making them membranes on $T_i$.

Next, we observe that there exists only one linear dependence relation between the classes $[T_i] \in H_2(Y)$.

Proposition 4.2. The inclusion map $H_2(\bigcup T_i) \to H_2(Y)$ has kernel $\mathbb{Z}$ generated by the relation $\sum_{i=1}^d [T_i] = 0$. Here $T_i$ are oriented uniformly in accord with some fixed orientation of $m_i \times S^1$.

Proof. It is enough to show that $\pi_1(Y \setminus (N \times S^1)) = 0$, since it implies that $H_3(Y, N \times S^1) \cong H^1(Y \setminus (N \times S^1)) = 0$ and thus the inclusion map $H_2(N \times S^1) \to H_2(Y)$ is monomorphic. The first inclusion map in the composition $H_2(\bigcup T_i) \to H_2(N \times S^1) \to H_2(Y)$ that we analyze, is just $H_1(\partial F) \otimes H_1(S^1) \to H_1(F) \otimes H_1(S^1)$, and has kernel $H_2(F, \partial F) \otimes H_1(S^1) \cong \mathbb{Z}$, as stated in the Proposition.

Now note that $p^{-1}(R_1)$ is a deformational retract (spine) of $N \times S^1$, so it is enough to check the triviality of $\pi_1(Y \setminus (p^{-1}(R_1)))$. This triviality follows from that $\pi_1(CP^2 \setminus (CA \cup R_1))$ is $\mathbb{Z}/d$, with a generator represented by a loop around $CA$ (say, by the computation in [1] reproduced in the Appendix), and thus $\pi_1(Y \setminus p^{-1}(CA \cup R_1)) = 0$.

Proposition 4.2 together with the Fintushel-Stern formula [4] guarantees that the Seiberg-Witten invariants of $Y(K, \phi)$ are distinct for some sequence of knots $K$ with increasing degrees of $\Delta_K(t)$.

Proof of Theorem 1.6 The case of a primitive class $[F] \in H_2(X)$ is considered in [5]. More precisely, the assumptions in Theorem 1.1 in [5] are satisfied because our condition on the fundamental group yields that $\pi_1(X \setminus F)$ is abelian and thus trivial, existence of an irreducible deformation of $F$ implies that $F \circ F \geq 0$, and $X_0$-degeneration guarantees that $F$ is not a rational curve.

If $[F]$ is divisible by $d \geq 2$, then we consider a $d$-fold covering, $p : Y \to X$, branched along $F$ and perform an annulus rim-surgery on $F$ along a membrane $M$ defined as follows. Consider a local topological model of the singularity $X_0$, defined in $\mathbb{C}^2$ by the equation $(x^2 + y^2)(x^2 + 2y^2) = 0$, and a model of its perturbation, $(x^2 + y^2 - 4\varepsilon)(x^2 + 2y^2 - \varepsilon) = \delta$, where $\varepsilon, \delta \in \mathbb{R}$, $0 \ll \delta \ll \varepsilon \ll 1$. The real locus of a perturbed singularity contains a pair of ovals which bound together in $\mathbb{R}^2$ an annulus that we take as $M$. 

10
The assumptions of Theorem 1.6 imply those of Proposition 1.2. Namely, irreducibility of $F_0$ implies that $F \setminus \partial M$ is connected and commutativity of $\pi_1(X \setminus F_0)$ implies commutativity of $\pi_1(X \setminus (F \cup M))$ via Van Kampen theorem. Moreover, the singularity $X_0$ provides the topological picture that was used in the above proof of Theorem 1.1, in the case of $d$-fold covering. Namely, $X_0$ yields the both $(-1)$-disk membranes that were used to show that the Fintushel and Stern formula can be applied to $Y$.

Remark 4.1. Note that to apply the formula [5] it is not required that $b_2^+(Y) > 1$. Nevertheless, it is so, because $b_2^+(Y) \geq d$, which can be proved by observing $d$ linearly independent pairwise orthogonal classes in $H_2(Y)$, having non-negative squares. One of these classes is $[F]$, and the other $(d-1)$ come from $p^{-1}(M)$, due to Proposition 4.2 (each of these $(d-1)$ classes has self-intersection 0).

5. Appendix: The topology of $\mathbb{CP}^2 \setminus (\mathbb{RP}^2 \cup \mathbb{CA})$ for $L$-curves $\mathbb{CA}$

Let $\mathbb{CA}_0 = \mathbb{CL}_1 \cup \cdots \cup \mathbb{CL}_d \subset \mathbb{CP}^2$ denote the complex point set of a real curve of degree $d$ splitting into $d$ lines, $\mathbb{CL}_i$. Put $\hat{V} = C \cap \mathbb{CA}_0$, where $C$ is the conic from the proof of Proposition 1.3. Our first observation is that $C \setminus \hat{V}$ is a deformational retract of $\mathbb{CP}^2 \setminus (\mathbb{RP}^2 \cup \mathbb{CA}_0)$, and moreover, the latter complement is homeomorphic to $(C \setminus \hat{V}) \times \text{Int}(D^2)$. To see it, it suffices to note that $\mathbb{CP}^2 \setminus \mathbb{RP}^2$ is fibered over $C$ with a 2-disc fiber, each fiber being a real semi-line, that is a connected component of $\mathbb{CL} \setminus \mathbb{RL}$ for some real line $\mathbb{CL} \subset \mathbb{CP}^2$, where $\mathbb{RL} = \mathbb{CL} \cap \mathbb{RP}^2$. This fiberings maps a semi-line into its intersection point with $C$.

It is convenient to view the quotient $C / \text{conj}$ of the conic $C$ by the complex conjugation as the projective plane, $\mathbb{RP}^2$, dual to $\mathbb{RP}^2 \subset \mathbb{CP}^2$, since each real line, $\mathbb{CL}$, intersects $C$ in a pair of conjugated points. If we let $V = \{l_1, \ldots, l_d\} \subset \mathbb{RP}^2$ denote the set of points $l_i$ dual to the lines $RL_i \subset \mathbb{RP}^2$, then $\hat{V} = q^{-1}(V)$, where $q : C \to C / \text{conj}$ is the quotient map.

The information about a perturbation of $\mathbb{CA}_0$ is encoded in a genetic graph of a perturbation, $\Gamma \subset \mathbb{RP}^2$. The graph $\Gamma$ is a complete graph with the vertex set $V$, whose edges are line segments. Note that there exist two topologically distinct perturbations of a real node of $\mathbb{CA}_0$ at $p_{ij} = RL_i \cap RL_j$, as well as there exist two line segments in $\mathbb{RP}^2$ connecting the vertices $l_i, l_j \in V$. Let $\mathbb{RA}$ denotes a real curve obtained from $\mathbb{RA}_0$ by a sufficiently small perturbation. Then the edge of $\Gamma$ connecting $l_i$ and $l_j$ contains the points dual to those lines passing through $p_{ij}$ which do not intersect $\mathbb{RA}$ locally, in a small neighborhood of $p_{ij}$.

The complement $\mathbb{CP}^2 \setminus (\mathbb{CA} \cup \mathbb{RP}^2)$ turns out to be homotopy equivalent to a 2-complex obtained from $C \setminus \hat{V}$ by adding 2-cells glued along a figure-eight shaped loops along the edges of $\hat{\Gamma} = q^{-1}(\Gamma) \subset C$. Such 2-cells identify pairwise certain generators of $\pi_1(C \setminus \hat{V})$ “along the edges” of $\hat{\Gamma}$ (cf. [3] for details). This easily implies that the group $\pi_1(\mathbb{CP}^2 \setminus (\mathbb{CA} \cup \mathbb{RP}^2))$ is generated by a pair of elements, $a$ and $b$, represented by a pair of loops in $C \setminus \hat{V}$ around a pair of conjugated vertices of $\hat{V}$.

For example, for a maximal nest curve, the graph $\Gamma$ is contained in an affine part of $\mathbb{RP}^2$, i.e., has no common points with some line in $\mathbb{RP}^2$, namely, with a line dual to a point inside the inner oval of the nest. Therefore, the graph $\hat{\Gamma}$ splits into two connected components separated by a big circle in $C$. A loop around any vertex of $\hat{V}$ from one of these components represents $a$, ...
and a loop around a vertex from the other component represents $b$. It is trivial to observe also the relation $a^d b^d = 1$ (which is indeed a unique relation in the case of maximal nest curves).

As we puncture $\mathbb{R}P^2$ at a point $x \in \mathbb{R}P^2 \setminus \mathbb{R}A_0$, we attach a 2-cell to $C \setminus \tilde{V}$ along the big circle $S_x \subset C$ dual to $x$. If $x$ moves across a line $RL$, then $S_x$ moves across the pair of points $q^{-1}(l_i)$. Since a small perturbation and puncturing are located at distinct points of $\mathbb{C}P^2$ and can be done independently, it is not difficult to see that if we choose $x \in R_i$ (in the case of a maximal nest curve $C \Gamma$), then the big circle $S_x$ cuts $C$ into the hemispheres, one of which contains $i$ vertices from one component of $\tilde{\Gamma}$ and $d - i$ vertices from the other component. This gives relations $a^i b^{d-i} = a^{d-i} b^i = 1$.

References

[1] S. Finashin, Topology of the Complement of a Real Algebraic Curve in $\mathbb{C}P^2$, Zap. Nauch. Sem. LOMI 122 (1982), 137–145
[2] S. Finashin, Knotting of algebraic curves in $\mathbb{C}P^2$, math.GT/9907108, to appear in “Topology”
[3] S. Finashin, M. Kreck, O. Viro, Non-diffeomorphic but homeomorphic knottings of surfaces in the 4-sphere Spr. Lecture Notes in Math. 1346 (1988), 157–198
[4] R. Fintushel, R. Stern, Knots, Links and 4-manifolds Invent. Math. 134 (1988), 2, 363–400
[5] R. Fintushel, R. Stern, Surfaces in 4-manifolds Math. Res. Lett., 4 (1997), 907–914
[6] R. Kirby, Problems in Low-dimensional Topology, 1996
[7] M. V. Nori, Zariski conjecture and related problems Ann. Sci. Ec. Norm. Sup., 4 Ser., 16 (1983), 305–344

 democracy

Middle East Technical University, Ankara 06531 Turkey
E-mail address: serge@metu.edu.tr

12