Improved WKB analysis of Slow-Roll Inflation

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We extend the WKB method for the computation of cosmological perturbations during inflation beyond leading order and provide the power spectra of scalar and tensor perturbations to second order in the slow-roll parameters. Our method does not require that the slow-roll parameters be constant. Although leading and next-to-leading results in the slow-roll parameters depend on the approximation technique used in the computation, we find that the inflationary theoretical predictions obtained may reach the accuracy required by planned observations. In two technical appendices, we compare our techniques and results with previous findings.

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I. INTRODUCTION

One of the main topics in theoretical cosmology is the computation of the spectra of perturbations during inflation (for an introduction and review see, e.g. Refs. [1]), which are then to be compared with the data for the Cosmic Microwave Background Radiation (CMBR) obtained in present and future experiments [2, 3]. It is in general impossible to obtain such spectra analytically, and therefore approximate methods have been proposed, amongst which stands out the slow-roll approximation of Ref. [4]. This amounted to introducing a suitable hierarchy of parameters, whose time-dependence was frozen to first order, and corresponding spectra and spectral indices were then determined uniquely.

The task of extending standard slow-roll results beyond first order is however highly non-trivial. As was noted in Ref. [4], one expects that the time-dependence of the parameters that describe the inflationary Universe could then appear in the order of the Bessel functions which describe the perturbations. Since this is, in general, inconsistent, other ways have been devised in order to by-pass this issue, such as the Green’s function method of Refs. [6, 7], and the “uniform” approximation developed in Refs. [8, 9, 10].

Our approach to this problem is to make use of an improvement of the WKB approximation (first applied to cosmological perturbations in Ref. [11]) introduced in Ref. [12]. We followed the method of Ref. [13] in order to illustrate an improved (uniform) leading order approximation and then introduced two expansions: the “adiabatic” expansion of Ref. [12] and a perturbative Green’s function expansion. In the present paper, we shall use our methods (in particular the adiabatic expansion of Ref. [12]) in conjunction with the slow-roll approximation [4, 14, 15] in order to obtain higher order terms in the slow-roll parameters. Our method will not require that the slow-roll parameters be constant and extends the results illustrated in Ref. [15]. Let us emphasize here the distinction between the slow-roll and WKB formalisms. The latter (also referred to as “semiclassical” or adiabatic expansion) is used to obtain approximate solutions to the general cosmological perturbation equations, whereas the former corresponds to an approximate solution for the homogeneous mode in which the inflaton kinetic term is neglected and one has a hierarchy of horizon flow functions (see Section III). Thus, they correspond to diverse, and independent, approximation schemes, even if, as is obvious, the validity (or not) of the slow-roll approximation will influence the form of the approximate WKB solution.

In Section III we briefly recall the main definitions and standard slow-roll results. Our results to leading WKB and second slow-roll order, and next-to-leading WKB and first slow-roll order from Ref. [12] are respectively reviewed in detail in Sections IV and V. New results for the next-to-leading WKB and second slow-roll order are given in Section VI. We finally comment on our work in Section VII and compare with other results in the literature in Appendices A and B.

II. BASICS

Let us begin by recalling the Robertson-Walker metric in conformal time $\eta$,

$$ds^2 = a^2(\eta) \left[ -d\eta^2 + \mathbf{d}x \cdot \mathbf{d}x \right],$$

(1)

where $a$ is the scale factor of the Universe. Scalar (density) and tensor (gravitational wave) fluctuations are given on this background, respectively by $\mu = \mu_5 = \mu Q$ ($Q$ is the Mukhanov variable [16]) and $\mu = \mu_T = \mu h$ ($h$ is the amplitude of the two polarizations of gravitational waves [17, 18]), where the functions $\mu$ must satisfy the
one-dimensional Schrödinger-like equation
\[ \left[ \frac{d^2}{d\eta^2} + \Omega^2(k, \eta) \right] \mu = 0 , \tag{2} \]
together with the initial condition
\[ \lim_{\eta \to +\infty} \mu(k, \eta) \approx \frac{e^{-i k \eta}}{\sqrt{2} k} . \tag{3} \]
In the above $k$ is the wave-number, $H \equiv a'/a^2$ is the Hubble parameter, and
\[ \Omega^2(k, \eta) \equiv k^2 - \frac{z''}{z} , \tag{4} \]
where $z = z_0 \equiv a^2 \phi'/H$ for scalar and $z = z_T \equiv a$ for tensor perturbations ($\phi$ is the homogenous inflaton and primes denote derivatives with respect to $\eta$). The dimensionless power spectra (PS henceforth) of scalar and tensor fluctuations are then given by
\[ P_\zeta \equiv \frac{k^3}{2 \pi^2} \left| \frac{\mu}{\mu_S} \right|^2 , \quad P_h \equiv \frac{4 k^3}{\pi^2} \left| \frac{\mu}{\mu_T} \right|^2 \tag{5a} \]
and the spectral indices and runnings
\[ n_S - 1 \equiv \frac{d \ln P_\zeta}{d \ln k} , \quad n_T \equiv \frac{d \ln P_h}{d \ln k} \tag{5b} \]
\[ \alpha_S \equiv \frac{d^2 \ln P_\zeta}{(d \ln k)^2} , \quad \alpha_T \equiv \frac{d^2 \ln P_h}{(d \ln k)^2} . \tag{5c} \]
We also define the tensor-to-scalar ratio
\[ R \equiv \frac{P_h}{P_\zeta} . \tag{5d} \]
The evolution of the Universe is usually described by means of a set of flow equations \[11, 19\]. The zero horizon flow function is defined by
\[ \epsilon_0 \equiv \frac{H(N_i)}{H(N)} , \tag{6} \]
where $N$ is the number of e-folds, $N \equiv \ln(a/a_i)$ [where $a_i = a(\eta_i)$] after the arbitrary initial time $\eta_i$. The hierarchy of horizon flow functions (HFF henceforth) is then defined according to
\[ \epsilon_{i+1} \equiv \frac{d \ln |\epsilon_i|}{d N} , \quad i \geq 0 \tag{7} \]
and inflation takes place for $\epsilon_1 < 1$, (see Appendix A for a comparison with different conventions). We finally recall that the frequencies $\Omega$ can now be expressed in terms of $\epsilon_1$, $\epsilon_2$, and $\epsilon_3$ only, as
\[ \frac{k^2 - \Omega_T^2}{a^2 H^2} = 2 - \epsilon_1 \tag{8} \]
\[ \frac{k^2 - \Omega_S^2}{a^2 H^2} = 2 - \epsilon_1 - \frac{3}{2} \epsilon_2 - \frac{1}{2} \epsilon_3 \epsilon_2 + \frac{1}{4} \epsilon_2^2 + \frac{1}{2} \epsilon_2 \epsilon_3 . \]
In general, $\Omega^2(k, \eta)$ vanishes for a certain time $\eta = \eta_0(k)$ (the classical turning point).

A. WKB Formalism

Let us briefly summarize the essential formulae from our Ref. \[12\]. First of all, we apply the Langer transformation
\[ x \equiv \ln \left( \frac{k}{H a} \right) , \quad \chi \equiv (1 - \epsilon_1)^{1/2} e^{-x/2} \mu \tag{9} \]
so that Eq. (2) becomes
\[ \left[ \frac{d^2}{dx^2} + \omega^2(x) \right] \chi(x) = 0 , \tag{10} \]
where for scalar and tensor perturbations one now has
\[ \omega_S^2(x) = \frac{e^{2x}}{(1 - \epsilon_1)^2} - \frac{1}{4} \left( \frac{3 - \epsilon_1}{1 - \epsilon_1} \right)^2 - \frac{(3 - 2 \epsilon_1) \epsilon_2}{2 (1 - \epsilon_1)^2} - \frac{(1 - 2 \epsilon_1) \epsilon_2 \epsilon_3}{2 (1 - \epsilon_1)^3} - \frac{(1 - 4 \epsilon_1) \epsilon_2^2}{4 (1 - \epsilon_1)^4} \tag{11} \]
\[ \omega_T^2(x) = \frac{e^{2x}}{(1 - \epsilon_1)^2} - \frac{1}{4} \left( \frac{3 - \epsilon_1}{1 - \epsilon_1} \right)^2 + \frac{\epsilon_1 \epsilon_2}{2 (1 - \epsilon_1)^2} + \frac{(2 + \epsilon_1) \epsilon_2^2}{4 (1 - \epsilon_1)^4} . \]
and the corresponding turning points are mapped into $x = x_0(\eta_0, k)$. We then denote quantities evaluated in the region to the right (left) of the turning point $x_0$ with
the subscript I (II). For example,

\[ \omega_I(x) = \sqrt{\omega^2(x)} \quad \omega_{II}(x) = \sqrt{-\omega^2(x)} \]  

(12)

and

\[ \xi_I(x) = \int_x^{x_0} \omega_I(y) \, dy \quad \xi_{II}(x) = \int_x^{x_0} \omega_{II}(y) \, dy . \]  

(13)

The next-to-leading WKB solutions of Eq. (10), in the adiabatic expansion and for one linear turning point, is then given by a combination of Bessel functions and their derivatives as [12, 13]

\[ U_I^{(\pm)}(x) = [1 + \phi_{I(1)}(x)] \, u_I^{(\pm)}(x) + \gamma_{I(1)}(x) \, u_I^{(\mp)'}(x) \]  

(14)

\[ U_{II}^{(\pm)}(x) = [1 + \phi_{II(1)}(x)] \, u_{II}^{(\pm)}(x) + \gamma_{II(1)}(x) \, u_{II}^{(\mp)'}(x) , \]

where

\[ u_I^{(\pm)}(x) = \sqrt{\frac{\xi_I(x)}{\omega_I(x)}} J_{\pm \frac{1}{2}}[\xi_I(x)] \]  

(15)

\[ u_{II}^{(\pm)}(x) = \sqrt{\frac{\xi_{II}(x)}{\omega_{II}(x)}} I_{\pm \frac{1}{2}}[\xi_{II}(x)] . \]

The above expressions satisfy an equation of motion

\[ \left[ \frac{d^2}{dx^2} + \omega^2(x) - \sigma(x) \right] \chi(x) = 0 , \]  

(16)

where the residual term is defined by

\[ \sigma(x) \equiv \frac{3}{4} \left( \frac{\omega'}{\omega} \right)^2 - \frac{\omega''}{2 \omega} - \frac{5}{36} \frac{\omega^2}{\xi^2} . \]  

(17)

As we showed in Ref. [12], such a quantity is expected to be small for the cases of interest, that is, in the subhorizon limit

\[ \frac{k}{a \, H} = e^x \to +\infty , \]  

(18)

in the superhorizon limit

\[ \frac{k}{a \, H} = e^x \to 0 , \]  

(19)

and also at the turning points (zeros) \( x = x_0 \) of the frequencies. In Eq. (17), we omitted the subscripts I and II for brevity and primes hereafter will denote derivatives with respect to \( x \).

The next-to-leading WKB PS in the adiabatic expansion are given by

\[ \mathcal{P}_S = \frac{H^2}{\pi \epsilon_I m_{Pl}^2} \left( \frac{k}{a \, H} \right)^3 \frac{e^{2 \xi_{II,S}} (1 + g_{I(1)S}^{AD})}{(1 - \epsilon_I) \omega_{II,S}} \]  

(20a)

\[ \mathcal{P}_h = \frac{16}{\pi} \frac{H^2}{m_{Pl}^2} \left( \frac{k}{a \, H} \right)^3 \frac{e^{2 \xi_{II,T}} (1 + g_{I(1)T}^{AD})}{(1 - \epsilon_I) \omega_{II,T}} . \]  

The spectral indices are also given by

\[ n_S - 1 = 3 + \frac{2 \, d \xi_{II,S}}{d \ln k} + \frac{d g_{I(1)S}^{AD}}{d \ln k} \]  

(20b)

\[ n_T = 3 + \frac{2 \, d \xi_{II,T}}{d \ln k} + \frac{d g_{I(1)T}^{AD}}{d \ln k} , \]

and their runnings by

\[ \alpha_S = 2 \frac{d^2 \xi_{II,S}}{(d \ln k)^2} + \frac{d^2 g_{I(1)S}^{AD}}{(d \ln k)^2} \]  

(20c)

\[ \alpha_T = 2 \frac{d^2 \xi_{II,T}}{(d \ln k)^2} + \frac{d^2 g_{I(1)T}^{AD}}{(d \ln k)^2} . \]

Finally, the tensor-to-scalar ratio takes the form

\[ R = 16 \epsilon_1 e^{2 \xi_{II,T}} \left( 1 + g_{I(1)T}^{AD} \right) \omega_{II,S} \]  

(20d)

where all quantities are evaluated in the super-horizon limit \( x \ll x_0 \) and we used the results

\[ g_{I(1)S,T}^{AD}(x) = 2 \left\{ \phi_{I(1)}(x) - \gamma_{I(1)}(x) \left[ \omega_{II}(x) + \frac{\omega_{I(1)}'(x)}{2 \omega_{II}(x)} \right] \right. \]

(21a)

\[ + \frac{1}{2} \gamma_{II}(x_i) - \phi_{I(1)}(x_i) \} \]  

\[ \phi_{I(1)}(x) = -\frac{1}{2} \int_x^{x_0} \left[ \gamma_{II}'(y) + \sigma(y) \gamma_{I(1)}(y) \right] \, dy \]  

(21b)

\[ \gamma_{I(1)}(x) = \frac{1}{2 \omega(x)} \int_x^{x_0} \frac{\sigma(y)}{\omega(y)} \, dy , \]

where it is understood that the integration must be performed from \( x_0 \) to \( x \) in region I and from \( x \) to \( x_0 \) in region II, and the indices S and T refer to the use of the corresponding frequencies. Let us note that if we take the above formulae with \( g_{I(1)S,T}^{AD}(x) \equiv 0 \), we obtain the leading WKB results given in Ref. [11] for the PS, spectral indices and \( \alpha \)-runnings.

\[ \text{B. Slow-Roll Results} \]

Let us briefly recall the known results obtained from the slow-roll approximation [4, 11, 14]. To first order in the slow-roll parameters [4], we have the PS
where $C \equiv \gamma_e + \ln 2 - 2 \approx -0.7296$, $\gamma_e$ being the Euler-Mascheroni constant, and all quantities are evaluated at the Hubble crossing $\eta_e$, that is the moment of time when $k_* = (aH)(N_e) \equiv (aH)_e$. (The number $k_*$ is usually called “pivot scale”). From Eqs. (22a), we can also obtain the spectral indices and $\alpha$-runnings,

$$n_S - 1 = -2 \epsilon_1 - \epsilon_2 , \quad n_T = -2 \epsilon_1 \quad (22b)$$

$$\alpha_S = \alpha_T = 0 , \quad (22c)$$

on using respectively Eqs. (16a) and (16d). From Eq. (16d), the tensor-to-scalar ratio becomes

$$R = 16 \epsilon_1 \left[ 1 + C \epsilon_2 + \epsilon_2 \ln \left( \frac{k}{k_*} \right) \right] . \quad (22d)$$

III. LEADING WKB AND SECOND SLOW-ROLL ORDER

We now consider the possibility of obtaining consistent second order results in the parameters $\epsilon_i$’s from the leading WKB approximation.

We first set $\eta^{AD}(x) \equiv 0$ in Eqs. (20a), (20b), (20c) and (20d). We then find it convenient to re-express all relevant quantities in terms of the conformal time $\eta$. For this purpose, we employ the relation

$$-k \eta = \frac{k}{aH (1 - \epsilon_1)} \left[ 1 + \epsilon_1 \epsilon_2 + O(\epsilon_i^2) \right] \quad (23a)$$

$$= \frac{k}{(1 - \epsilon_1)} \left[ 1 + \epsilon_1 \epsilon_2 + O(\epsilon_i^2) \right] . \quad (23b)$$

Now consider the possibility of obtaining consistent second order results in the parameters $\epsilon_i$’s from

$$\omega_s^2(\eta) = k^2 \eta^2 (1 - 2 \epsilon_1 \epsilon_2) - \left( \frac{9}{4} + 3 \epsilon_1 + 4 \epsilon_1^2 + \frac{3}{2} \epsilon_2 + 2 \epsilon_1 \epsilon_2 + \frac{1}{4} \epsilon_2^2 + \frac{1}{2} \epsilon_2 \epsilon_3 \right)$$

$$\omega_T^2(\eta) = k^2 \eta^2 (1 - 2 \epsilon_1 \epsilon_2) - \left( \frac{9}{4} + 3 \epsilon_1 + 4 \epsilon_1^2 - \frac{1}{2} \epsilon_1 \epsilon_2 \right) , \quad (24)$$

and the PS become

$$\mathcal{P}_\zeta = \frac{H^2}{\pi \epsilon_1 \epsilon_2 m_p^2} (-k \eta)^3 \frac{2}{3} \left[ 1 - \frac{8}{3} \epsilon_{1,1} f - \frac{1}{3} \epsilon_{2,1} f + \frac{19}{9} \epsilon_{1,1}^2 f - \frac{19}{9} \epsilon_{1,1} \epsilon_{1,2,1} f + \frac{1}{9} \epsilon_{2,1} f - \frac{1}{9} \epsilon_{2,1} \epsilon_{3,1} f \right] e^{2\xi_{II,V},S}$$

$$\mathcal{P}_h = \frac{16 H^2}{\pi m_p^2} (-k \eta)^3 \frac{2}{3} \left[ 1 - \frac{8}{3} \epsilon_{1,1} f + \frac{19}{9} \epsilon_{1,1}^2 f - \frac{26}{9} \epsilon_{1,1} \epsilon_{1,2,1} f \right] e^{2\xi_{II,V},T} , \quad (25)$$

From now on, the subscript $f$ will denote that the given quantity is evaluated in the super-horizon limit.

We must now compute the arguments of the exponentials in the above expressions, which are in general of the form

$$\xi_{II}(\eta_0, \eta; k) = \int_{\eta_0}^{\eta} \sqrt{A^2(\eta) - k^2 \eta^2} \frac{d\eta}{\eta} , \quad (26)$$
where the function $A^2(\eta)$ contains the HFF, but does not depend on $k$, and we have used

$$\text{d} x = \frac{\text{d} \eta}{\eta} \left[ 1 + \epsilon_1(\eta) \epsilon_2(\eta) + \mathcal{O}(\epsilon_3^3) \right].$$  \hspace{1cm} (27)$$

Let us also note that, at the turning point $\eta = \eta_0$, one has

$$A(\eta_0) = -k \eta_0.$$  \hspace{1cm} (28)$$

It is now clear that, in order to obtain consistent results to second order in the slow-roll parameters, we must consider the time-dependence of the $\epsilon_i$’s, and the function $A^2(\eta)$ may not be approximated by a constant. This does not allow us to perform the integral unless the scale factor $a = a(\eta)$ is given explicitly or, as we shall see, some further approximation is employed.

In Ref. [15], we proposed a procedure which will now be described in detail. Let us start with the general exact relation

$$\int_{\eta_1}^{\eta_2} \sqrt{A^2(\eta) - k^2 \eta^2} \frac{\text{d} \eta}{\eta} = \sqrt{A^2(\eta) - k^2 \eta^2} \bigg|_{\eta_1}^{\eta_2} + \frac{A(\eta)}{2} \ln \left( \frac{A(\eta) - \sqrt{A^2(\eta) - k^2 \eta^2}}{A(\eta) + \sqrt{A^2(\eta) - k^2 \eta^2}} \right) \bigg|_{\eta_1}^{\eta_2} - \int_{\eta_1}^{\eta_2} \ln \left( \frac{A(\eta) - \sqrt{A^2(\eta) - k^2 \eta^2}}{A(\eta) + \sqrt{A^2(\eta) - k^2 \eta^2}} \right) \frac{A^2(\eta)}{4} \text{d} \eta$$  \hspace{1cm} (29)$$

which holds for every function $A^2(\eta)$. We can derive this relation by initially considering the integral in Eq. (29) with $A$ constant (i.e. independent of $\eta$, which is just power-law inflation). In this case, the integration can be performed and yields the first and second term in the r.h.s. of Eq. (29) with the function $A$ independent of $\eta$, whereas the third term vanishes identically. If we then reinstate the $\eta$-dependence of $A$ in the two non-vanishing terms and differentiate them with respect to $\eta_2$, we obtain the original integrand in Eq. (29) plus another term, originating from $A^2(\eta)$. We then obtain the result (29) by (formally) integrating the latter term and subtracting it from the previous two.

One can also repeat the above procedure for the new integral in the r.h.s. of Eq. (29). To do this, we define

$$[A^2(\eta)]' = \frac{1}{\eta} B(\eta),$$  \hspace{1cm} (30)$$

and, after several simplifications, we can write the general relation as

$$-\int_{\eta_1}^{\eta_2} \ln \left( \frac{A(\eta) - \sqrt{A^2(\eta) - k^2 \eta^2}}{A(\eta) + \sqrt{A^2(\eta) - k^2 \eta^2}} \right) \frac{A^2(\eta)}{4} \text{d} \eta$$  

$$- \int_{\eta_1}^{\eta_2} \ln \left( \frac{A(\eta) - \sqrt{A^2(\eta) - k^2 \eta^2}}{A(\eta) + \sqrt{A^2(\eta) - k^2 \eta^2}} \right) \frac{B^2(\eta)}{8 A^3(\eta)} \text{d} \eta,$$

with

$$Y(\eta) = \ln \left( \frac{-k \eta}{2 A(\eta)} \right) \ln \left( \frac{A(\eta) - \sqrt{A^2(\eta) - k^2 \eta^2}}{A(\eta) + \sqrt{A^2(\eta) - k^2 \eta^2}} \right)$$

$$+ \text{Li}_2 \left( \frac{A(\eta) - \sqrt{A^2(\eta) - k^2 \eta^2}}{2 A(\eta)} \right)$$

$$- \text{Li}_2 \left( \frac{A(\eta) + \sqrt{A^2(\eta) - k^2 \eta^2}}{2 A(\eta)} \right),$$  \hspace{1cm} (31)$$

and $\text{Li}_2(z)$ is the dilogarithm function (see, e.g. Ref. [20]),

$$\text{Li}_2(z) \equiv \sum_{k=1}^{\infty} \frac{z^k}{k^2} = -\int_0^z \frac{\ln(1 - z')}{z'} \text{d} z'.$$  \hspace{1cm} (32)$$

On putting together all the pieces, we finally obtain

$$\int_{\eta_1}^{\eta_2} \sqrt{A^2(\eta) - k^2 \eta^2} \frac{\text{d} \eta}{\eta} = \sqrt{A^2(\eta) - k^2 \eta^2} \bigg|_{\eta_1}^{\eta_2} + Y(\eta) \left( \frac{B^2(\eta)}{16 \eta A^3(\eta)} + \frac{B'(\eta)}{8 A(\eta)} \right) \bigg|_{\eta_1}^{\eta_2} - \int_{\eta_1}^{\eta_2} \ln \left( \frac{A(\eta) - \sqrt{A^2(\eta) - k^2 \eta^2}}{A(\eta) + \sqrt{A^2(\eta) - k^2 \eta^2}} \right) \frac{B^2(\eta)}{8 \eta A^3(\eta)} \text{d} \eta.$$  \hspace{1cm} (33)$$

For the cases of interest to us [i.e. for the frequencies (24)], the functions $A$ for scalar and tensor pertur-
so that
\[ B_S = 3 \epsilon_1 \epsilon_2 + \frac{3}{2} \epsilon_2 \epsilon_3 , \quad B_T = 3 \epsilon_1 \epsilon_2 . \tag{35b} \]
For the results to second order in the HFF, we can neglect

the last two integrals in Eq. (33), since \( B'(\eta) = O(\epsilon_i^3) \) and \( B^2(\eta) = O(\epsilon_i^4) \). Thus our approximation consists in neglecting such terms and not in assuming the HFF be constant (or any specific functional form for them). We can finally write \( \xi_{II,S}(\eta_0, \eta_i; k) \) and \( \xi_{II,T}(\eta_0, \eta_i; k) \) as

\[
\xi_{II,S} \simeq -\frac{3}{2} + \frac{3 \ln 3}{2} + \ln 3 \epsilon_1, + \ln 3 \epsilon_2, + \left( \ln 3 + \frac{1}{3} \right) \epsilon_1, + \frac{1}{12} \epsilon_2, + \left( \frac{11 \ln 3}{6} - \frac{\ln^2 3}{24} + \frac{\pi^2}{6} + \frac{1}{12} \right) \epsilon_2, \epsilon_3, \\
+ \left[ \frac{3}{2} - \epsilon_1, - \frac{1}{2} \epsilon_2, - \epsilon_1, \epsilon_2 + \left( \ln 3 - \frac{11}{6} \right) \epsilon_1, + \left( \frac{1}{2} - \frac{1}{6} \right) \epsilon_2, \epsilon_3 \right] \ln (-k \eta) \\
- \frac{1}{2} \left( \epsilon_1, \epsilon_2 + \frac{1}{2} \epsilon_2, \epsilon_3 \right) \ln^2 (-k \eta) \tag{36a} \]

\[
\xi_{II,T} \simeq -\frac{3}{2} + \frac{3 \ln 3}{2} + \ln 3 \epsilon_1, + \left( \ln 3 + \frac{1}{3} \right) \epsilon_1, + \left( \frac{4 \ln 3}{3} - \frac{\ln^2 3}{2} + \frac{\pi^2}{6} \right) \epsilon_1, \epsilon_2, \\
+ \left[ \frac{3}{2} - \epsilon_1, - \epsilon_1^2, + \left( \ln 3 - \frac{4}{3} \right) \epsilon_1, \epsilon_2 \right] \ln (-k \eta) - \frac{1}{2} \epsilon_1, \epsilon_2 \ln^2 (-k \eta) , \tag{36b} \]

where the arguments in \( \xi_{II} \) have been omitted. On inserting the above quantities in Eqs. (23), we can calculate the PS.

In order to compare the PS from Eqs. (24), (26a) and (36a), with the slow-roll expressions in Eqs. (31a), we need a relation between \( H(N_f) \) and \( H(N_*) \), \( \epsilon_1(N_f) \) and \( \epsilon_1(N_*) \), and so on, to second order in the HFF (hereafter, quantities without a subscript will be evaluated at the Hubble crossing \( N_* \) corresponding to the pivot scale \( k_0 \)). We expand the parameters \( \epsilon_i \) to first order in \( \Delta N \equiv N_f - N_* \) in the numerators and to second order in the denominator of the scalar spectrum,

\[
\frac{\epsilon_i(N_f)}{\epsilon_i} \simeq 1 + \epsilon_{i+1} \Delta N + \frac{1}{2} \left( \epsilon_{i+1} + \epsilon_{i+1} \epsilon_{i+2} \right) \Delta N^2 , \tag{37} \]

and, analogously, we find

\[
\frac{H^2(N_f)}{H^2} \simeq 1 - 2 \epsilon_1 \Delta N - \left( \epsilon_1 \epsilon_2 - 2 \epsilon_1^2 \right) \Delta N^2 . \tag{38} \]

We can eliminate \( \eta_i \) in the logarithms by expressing it in terms of \( 1/(a H)_i \),

\[
\ln (-k \eta_i) \simeq \ln \left[ \frac{k (1 + \epsilon_1, \epsilon_2)}{(a H)_i} \right] \tag{39} \]

where in the first equality we used Eq. (23a), in the second one the definition of the pivot scale, and, in the last relation, Eq. (37). If we now use the above expressions we obtain the following PS, where the superscript (2) stands for second slow-roll order and the subscript WKB for leading adiabatic order,
\[ P_{\zeta,\text{WKB}}^{(2)} = \frac{H^2}{\pi \epsilon_1 m_{p}^2} A_{\text{WKB}} \left\{ 1 - 2(D_{\text{WKB}} + 1) \epsilon_1 - D_{\text{WKB}} \epsilon_2 + \left( 2D_{\text{WKB}}^2 + 2D_{\text{WKB}} - \frac{1}{9} \right) \epsilon_1^2 \right. \\
+ \left( D_{\text{WKB}}^2 - D_{\text{WKB}} + \frac{\pi^2}{12} - \frac{20}{9} \right) \epsilon_1 \epsilon_2 + \left( \frac{1}{2} D_{\text{WKB}}^2 + \frac{2}{9} \right) \epsilon_2^2 \left. + \left( -\frac{1}{2} D_{\text{WKB}}^2 + \frac{\pi^2}{24} - \frac{1}{18} \right) \epsilon_2 \epsilon_3 \right) \\
+ \left[ -2 \epsilon_1 - \epsilon_2 + 2(2D_{\text{WKB}} + 1) \epsilon_2^2 + (2D_{\text{WKB}} - 1) \epsilon_1 \epsilon_2 + D_{\text{WKB}} \epsilon_2^2 - D_{\text{WKB}} \epsilon_2 \epsilon_3 \right] \ln \left( \frac{k}{k_*} \right) \\
+ \frac{1}{2} \left( 4 \epsilon_1^2 + 2 \epsilon_1 \epsilon_2 + \epsilon_2^2 - \epsilon_2 \epsilon_3 \right) \ln^2 \left( \frac{k}{k_*} \right) \right\} \\
\] (40a)

\[ P_{h,\text{WKB}}^{(2)} = \frac{16 H^2}{\pi m_{p}^2} A_{\text{WKB}} \left\{ 1 - 2(D_{\text{WKB}} + 1) \epsilon_1 + \left( 2D_{\text{WKB}}^2 + 2D_{\text{WKB}} - \frac{1}{9} \right) \epsilon_1^2 \right. \\
+ \left( -D_{\text{WKB}}^2 - 2D_{\text{WKB}} + \frac{\pi^2}{12} - \frac{19}{9} \right) \epsilon_1 \epsilon_2 + \left[ -2 \epsilon_1 + 2(2D_{\text{WKB}} + 1) \epsilon_2^2 - 2(D_{\text{WKB}} + 1) \epsilon_1 \epsilon_2 \right] \ln \left( \frac{k}{k_*} \right) \\
+ \frac{1}{2} \left( 4 \epsilon_1^2 - 2 \epsilon_1 \epsilon_2 \right) \ln^2 \left( \frac{k}{k_*} \right) \right\} . \\
\] (40b)

In the above, we have defined \( D_{\text{WKB}} \equiv \frac{1}{3} \ln(3) \approx -0.7653, \) which approximates the coefficient \( C \) in Eqs. (22a) with an error of about 5%, and the factor \( A_{\text{WKB}} \equiv 18/e^3 \approx 0.896 \) which gives an error of about 10% on the amplitudes. One of the main results of this paper, already reported in Ref. [12], is that the PS to second order in the slow-roll parameters do not depend on \( \Delta N, \) once the laborious dependence on the Hubble crossing has been worked out: this fact is in complete agreement with the constancy of the growing modes of \( \zeta \) and \( h \) on large scales. The spectral indices, from Eqs. (20b), and their runnings, from Eqs. (20c), are analogously given by

\[ n_{S,\text{WKB}}^{(2)} - 1 = -2 \epsilon_1 - \epsilon_2 - 2 \epsilon_2^2 - (2D_{\text{WKB}} + 3) \epsilon_1 \epsilon_2 - D_{\text{WKB}} \epsilon_2 \epsilon_3 - 2 \epsilon_1 \epsilon_2 \ln \left( \frac{k}{k_*} \right) - \epsilon_2 \epsilon_3 \ln \left( \frac{k}{k_*} \right) \\
\] (40b)

\[ n_{T,\text{WKB}}^{(2)} = -2 \epsilon_1 - 2 \epsilon_1^2 - 2(D_{\text{WKB}} + 1) \epsilon_1 \epsilon_2 - 2 \epsilon_1 \epsilon_2 \ln \left( \frac{k}{k_*} \right) \] (40c)

\[ \alpha_{S,\text{WKB}}^{(2)} = -2 \epsilon_1 \epsilon_2 - \epsilon_2 \epsilon_3 , \quad \alpha_{T,\text{WKB}}^{(2)} = -2 \epsilon_1 \epsilon_2 . \] (40c)

From Eq. (20d) the tensor-to-scalar ratio becomes

\[ R_{\text{WKB}}^{(2)} = 16 \epsilon_1 \left[ 1 + D_{\text{WKB}} \epsilon_2 + \left( D_{\text{WKB}} + \frac{1}{9} \right) \epsilon_1 \epsilon_2 + \left( \frac{1}{2} D_{\text{WKB}}^2 - \frac{2}{9} \right) \epsilon_2^2 + \left( \frac{1}{2} D_{\text{WKB}}^2 - \frac{\pi^2}{24} + \frac{1}{18} \right) \epsilon_2 \epsilon_3 \right. \\
\left. + (\epsilon_2 + \epsilon_1 \epsilon_2 + D_{\text{WKB}} \epsilon_2^2 + D_{\text{WKB}} \epsilon_2 \epsilon_3) \ln \left( \frac{k}{k_*} \right) + \frac{1}{2} (\epsilon_2^2 + \epsilon_2 \epsilon_3) \ln^2 \left( \frac{k}{k_*} \right) \right] . \] (40d)
IV. NEXT-TO-LEADING WKB AND FIRST SLOW-ROLL ORDER

To first order in the HFF, the function $A$ can be taken as constant. In this case, let us then introduce the more convenient notation

$$\xi_{\pm}(x_0, x; k) = \pm \int_{x_0}^{x} \sqrt{\pm \omega^2(y)} \, dy$$

$$\simeq \pm \int_{\eta}^{\eta_0} \sqrt{\pm \left( k^2 \tau^2 - A^2 \right)} \frac{d\tau}{\tau}$$

$$= \pm A \left[ \text{atan}_{\pm}(\Theta_{\pm}) - \Theta_{\pm} \right],$$

where we have used $d\xi = d\tau/\tau$, and

$$A_{\delta}^2(\epsilon_i) = \frac{9}{4} + 3\epsilon_1 + \frac{3}{2} \epsilon_2$$

(42a)

$$A_{\gamma}^2(\epsilon_i) = \frac{9}{4} + 3\epsilon_1$$

where $\epsilon_1$ and $\epsilon_2$ are constant as well. It is also useful to define

$$\Theta_{\pm} \equiv \sqrt{\pm \left( 1 - \frac{\eta^2}{\eta_0^2} \right)}$$

(43a)

$$\text{atan}_+(x) \equiv \arctanh(x)$$

(43b)

$$\text{atan}_-(x) \equiv \arctan(x),$$

where the plus (minus) sign corresponds to region II (I).

With this notation, the expressions for $\gamma(1)(\eta)$ and $\phi(1)(\eta)$ in Eqs. (21b) can be explicitly written as

$$\gamma_{\pm}(1)(\eta) = \frac{1}{A^2} \left\{ -\frac{5 + 3 \Theta_{\pm}^2}{24 \Theta_{\pm}^4} \pm \frac{5}{72 \Theta_{\pm} \left[ \text{atan}_{\pm}(\Theta_{\pm}) - \Theta_{\pm} \right]} \right\}$$

$$\phi_{\pm}(1)(\eta) = \frac{1}{A^2} \left\{ \frac{23}{3150} \frac{505 + 654 \Theta_{\pm}^2 + 153 \Theta_{\pm}^4}{1152 \Theta_{\pm}^6} - \frac{5}{10368 \Theta_{\pm}^4 \left[ \text{atan}_{\pm}(\Theta_{\pm}) - \Theta_{\pm} \right]^2} \right\}.$$  

(44)

The limits of interest are then given by

$$\lim_{\eta \to -\infty} \gamma^{-}(1)(\eta) = 0$$

(45a)

$$\lim_{\eta \to -\infty} \phi^{-}(1)(\eta) = \frac{23}{3150} \frac{A^2}{A^2}$$

(45b)

$$\lim_{\eta \to 0^-} \gamma^{+}(1)(\eta) = -\frac{1}{12 A^2}$$

(45c)

$$\lim_{\eta \to 0^-} \phi^{+}(1)(\eta) = \frac{181}{16800 A^2},$$

(45d)

where we have used

$$\lim_{\eta \to -\infty} \Theta^{-}(\eta) = \infty$$

(45e)

$$\lim_{\eta \to 0^-} \Theta^{+}(\eta) = 1,$$

(45f)

and

$$\lim_{\eta \to 0^-} \left( \frac{\omega_\Pi + \omega'_\Pi}{2 \omega_\Pi} \right) = A.$$  

(45g)

We can now use the values in Eqs. (42a) to calculate the correction $g_{\text{AD}}^{(1)}$ in the super-horizon limit ($\eta \to 0^-$) for scalar and tensor perturbations, obtaining

$$g_{\text{AD}}^{(1)S} = \frac{37}{324} - \frac{19}{243} \left( \epsilon_1 + \frac{1}{2} \epsilon_2 \right)$$

(46)

$$g_{\text{AD}}^{(1)T} = \frac{37}{324} - \frac{19}{243} \epsilon_1.$$  

(46)

Finally, from Eqs. (20a) and (46), we can write the expressions for the scalar and tensor PS, with the superscript (1) for first slow-roll order and the subscript WKB* for next-to-leading adiabatic order, as
\[ P_{\xi,\text{WKB}}^{(1)} = \frac{H^2}{\pi \epsilon_1 m_{\text{Pl}}^2} A_{\text{WKB}} \left[ 1 - 2 \left( D_{\text{WKB}} + 1 \right) \epsilon_1 - D_{\text{WKB}} \epsilon_2 - (2 \epsilon_1 + \epsilon_2) \ln \left( \frac{k}{k_*} \right) \right] \]  

\[ P_{h,\text{WKB}}^{(1)} = \frac{16 H^2}{\pi m_{\text{Pl}}^2} A_{\text{WKB}} \left[ 1 - 2 \left( D_{\text{WKB}} + 1 \right) \epsilon_1 - 2 \epsilon_1 \ln \left( \frac{k}{k_*} \right) \right], \]  

where \( D_{\text{WKB}} \equiv \frac{7}{18} - \ln 3 \approx -0.7302 \), which approximates the coefficient \( C \) in Eqs. \((22a)\) with an error of about 0.08%, and the factor \( A_{\text{WKB}} \equiv 361/18 e^3 \approx 0.999 \), which gives an error of about 0.1% on the amplitudes. We also obtain the standard slow-roll spectral indices and \( \alpha \)-runnings,

\[ n_{S,\text{WKB}}^{(1)} - 1 = -2 \epsilon_1 - \epsilon_2, \quad n_{T,\text{WKB}}^{(1)} = -2 \epsilon_1 \]  

on respectively using Eqs. \((20)\) and \((20b)\). From Eq. \((20d)\) the tensor-to-scalar ratio becomes

\[ R_{\text{WKB}}^{(1)} = 16 \epsilon_1 \left[ 1 + D_{\text{WKB}} \epsilon_2 + \epsilon_2 \ln \left( \frac{k}{k_*} \right) \right]. \]  

V. NEXT-TO-LEADING WKB AND SECOND SLOW-ROLL ORDER

In order to give the results to next-to-leading WKB order and second slow-roll order, we should evaluate the corrections \( g_{(1)}^{AD} \) for scalar and tensor perturbations to second order in the \( \epsilon_i \)'s. This, in turn, would require the computation of the functions \( \phi_{\pm(1)} \) and \( \gamma_{\pm(1)} \) in Eqs. \((21c)\), whose analytical calculation is extremely difficult. We shall therefore try to give the expressions for the corrections by following a heuristic (and faster) method.

Let us start from the first order slow-roll expressions given in Eqs. \((40)\) and add all possible second order terms in the HFF. Some of such terms can then be (partially) fixed by making reasonable requirements, which will be explained below, so that the corrections to the PS read

\begin{align*}
1 + g_{(1)}^{AD} &= \frac{361}{324} \left\{ 1 - \frac{4}{57} \left[ \epsilon_{1,f} + \frac{1}{2} \epsilon_{2,f} \right] - \frac{8}{361} \epsilon_{1,f}^2 + \left[ \frac{4}{57} \left( b_S - D_{\text{WKB}} \right) - \frac{262}{3249} \right] \epsilon_{1,f} \epsilon_{2,f} + \frac{13}{1083} \epsilon_{2,f}^2 \\
&\quad + \left[ \frac{2}{57} \left( d_S - D_{\text{WKB}} \right) - \frac{2}{171} \right] \epsilon_{2,f} \epsilon_{3,f} - \left( \frac{4}{57} \epsilon_{1,f} \epsilon_{2,f} + \frac{2}{57} \epsilon_{2,f} \epsilon_{3,f} \right) \ln \left( -k \eta_1 \right) \right\} \\
1 + g_{(1)}^{AD} &= \frac{361}{324} \left\{ 1 - \frac{4}{57} \epsilon_{1,f} - \frac{8}{361} \epsilon_{1,f}^2 + \left[ \frac{4}{57} \left( b_T - D_{\text{WKB}} \right) - \frac{16}{171} \right] \epsilon_{1,f} \epsilon_{2,f} - \frac{4}{57} \epsilon_{1,f} \epsilon_{2,f} \ln \left( -k \eta_1 \right) \right\}. 
\end{align*}

First of all, we have factorized the number \( 361/324 \) so as to recover the standard slow-roll amplitudes \((22a)\) with a very good accuracy. As already mentioned, the first order terms are the same as those in the results \((40)\) of Section \(\text{IV}\) evaluated however in the super-horizon limit.

The form of the coefficients multiplying the second order monomials in the HFF have been partially determined by using the expressions given in Section \(\text{IV}\) \[\text{Eq. (45a)-\(48g\)}\] with Eqs. \((45a)-\(48g\)) replacing Eqs. \((42a)\). By this procedure, one expects to obtain results correct up to derivatives of \( A^2 \), which can just contain mixed terms to second order. The numerical coefficients in front of \( \epsilon_{1,f}^2 \) and \( \epsilon_{2,f}^2 \) are thus uniquely determined, whereas the coefficients multiplying the mixed terms \( \epsilon_{1,f} \epsilon_{2,f} \) and \( \epsilon_{2,f} \epsilon_{3,f} \) remain partially ambiguous. The last number in each square bracket arises directly from this procedure, whereas the other terms are chosen so as to match the “standard” dependence on \( D_{\text{WKB}} \) in the PS, with \( b_{S,T} \) and \( d_S \) left undetermined. Finally, we have also required that the PS do not depend on \( \Delta N \), consistently with the constancy in time of the growing modes for \( h \) and \( \zeta \). This requirement fixes the coefficients in front of the logarithms in Eqs. \((48)\), uniquely \((22a)\), and leads to spectral indices which do not depend on \( b_{S,T} \) and \( d_S \).

Proceeding then as in Section \(\text{III}\) from the above corrections we obtain the expressions for the scalar and tensor PS,
\[ \mathcal{P}_{\zeta, \text{WKB}}^{(2)} = \frac{H^2}{\pi \epsilon_1 m_{\text{Pl}}^2} A_{\text{WKB}} \left\{ 1 - 2(D_{\text{WKB}} + 1) \epsilon_1 - D_{\text{WKB}} \epsilon_2 + \left( 2 D_{\text{WKB}}^2 + 2 D_{\text{WKB}} - \frac{71}{1083} \right) \epsilon_1^2 \right. \\
+ \left( D_{\text{WKB}}^2 - D_{\text{WKB}} + \frac{\pi^2}{12} + b_S \frac{4}{57} - \frac{2384}{1083} \right) \epsilon_1 \epsilon_2 + \left( \frac{1}{2} D_{\text{WKB}}^2 + \frac{253}{1083} \right) \epsilon_2^2 \\
+ \left( -\frac{1}{2} D_{\text{WKB}}^2 + \frac{\pi^2}{24} + d_S \frac{2}{57} - \frac{49}{722} \right) \epsilon_2 \epsilon_3 \\
+ \left[ -2 \epsilon_1 - 2 + (2 D_{\text{WKB}} + 1) \epsilon_1^2 + (2 D_{\text{WKB}} - 1) \epsilon_1 \epsilon_2 + D_{\text{WKB}} \epsilon_2^2 - D_{\text{WKB}} \epsilon_2 \epsilon_3 \right] \ln \left( \frac{k}{k_*} \right) \\
+ \frac{1}{2} \left( 4 \epsilon_1^2 + 2 \epsilon_1 \epsilon_2 + \epsilon_2^2 - \epsilon_2 \epsilon_3 \right) \ln^2 \left( \frac{k}{k_*} \right) \right\} (49a) \]

\[ \mathcal{P}_{h, \text{WKB}}^{(2)} = \frac{16 H^2}{\pi m_{\text{Pl}}^2} A_{\text{WKB}} \left\{ 1 - 2(D_{\text{WKB}} + 1) \epsilon_1 + \left( 2 D_{\text{WKB}}^2 + 2 D_{\text{WKB}} - \frac{71}{1083} \right) \epsilon_1^2 \\
+ \left( -D_{\text{WKB}}^2 - 2 D_{\text{WKB}} + \frac{\pi^2}{12} + b_T \frac{4}{57} - \frac{771}{361} \right) \epsilon_1 \epsilon_2 \\
+ \left[ -2 \epsilon_1 + 2(2 D_{\text{WKB}} + 1) \epsilon_1^2 - 2(D_{\text{WKB}} + 1) \epsilon_1 \epsilon_2 \right] \ln \left( \frac{k}{k_*} \right) + \frac{1}{2} \left( 4 \epsilon_1^2 - 2 \epsilon_1 \epsilon_2 \right) \ln^2 \left( \frac{k}{k_*} \right) \right\} . \]

We also obtain the spectral indices \[ n_{S, \text{WKB}}^{(2)} = 2 \epsilon_1 - 2 \epsilon_2 - 2 \epsilon_1^2 - (2D_{\text{WKB}} + 3) \epsilon_1 \epsilon_2 - D_{\text{WKB}} \epsilon_2 \epsilon_3 - 2 \epsilon_1 \epsilon_2 \ln \left( \frac{k}{k_*} \right) - \epsilon_2 \epsilon_3 \ln \left( \frac{k}{k_*} \right) (49b) \]

\[ n_{T, \text{WKB}}^{(2)} = -2 \epsilon_1 - 2 \epsilon_2^2 - 2(D_{\text{WKB}} + 1) \epsilon_1 \epsilon_2 - 2 \epsilon_1 \epsilon_2 \ln \left( \frac{k}{k_*} \right) (49c) \]

\[ \alpha_{S, \text{WKB}}^{(2)} = -2 \epsilon_1 \epsilon_2 - \epsilon_2 \epsilon_3, \quad \alpha_{T, \text{WKB}}^{(2)} = -2 \epsilon_1 \epsilon_2, \quad (49c) \]

and the tensor-to-scalar ratio becomes

\[ R_{\text{WKB}}^{(2)} = 16 \epsilon_1 \left\{ 1 + D_{\text{WKB}} \epsilon_2 + \left[ D_{\text{WKB}} + (b_T - b_S) \frac{4}{57} + \frac{71}{1083} \right] \epsilon_1 \epsilon_2 \right. \\
+ \left( \frac{1}{2} D_{\text{WKB}}^2 - \frac{253}{1083} \right) \epsilon_2^2 + \left( \frac{1}{2} D_{\text{WKB}}^2 - \frac{\pi^2}{24} - d_S \frac{2}{57} + \frac{49}{722} \right) \epsilon_2 \epsilon_3 \right. \\
\left. + \left( \epsilon_2 + \epsilon_1 \epsilon_2 + D_{\text{WKB}} \epsilon_2^2 + D_{\text{WKB}} \epsilon_2 \epsilon_3 \right) \ln \left( \frac{k}{k_*} \right) + \frac{1}{2} \left( \epsilon_2^2 + \epsilon_2 \epsilon_3 \right) \ln^2 \left( \frac{k}{k_*} \right) \right\} . \]

Let us end this Section with a few remarks. The undetermined coefficients \( b_{S, T} \) and \( d_S \) still appear in the PS (and their ratio \( R \)), but not in the spectral indices and runnings, which are therefore uniquely specified by our (heuristic) procedure. In general, one may expect that the complete adiabatic corrections to second order contain HFF calculated both in the super-horizon limit and at the turning points (zeros) of the frequencies. However, this is not relevant in the present context, since the possible numerical coefficients that would multiply such terms can be eventually re-absorbed in the definitions of \( b_{S, T} \) and \( d_S \). This could only be an issue (in particular for the spectral indices) if we considered third order terms (since it would then become important where the second order terms are evaluated).

VI. CONCLUSIONS

We have shown that the improved WKB treatment of cosmological perturbations agrees with the standard slow-roll approximation \([4]\) to within 0.1%, finally resolving the issue of a 10% error in the prediction of the amplitudes to lowest order which was raised in Ref. \([11]\).
The next issues are the inflationary predictions to second order in the slow-roll parameters. After the results on the running of the power spectra \[21\], second order results have been obtained using the Green’s function method with the massless solution in a de Sitter spacetime \[3, 7\]. In a previous manuscript \[13\], and in full detail here, we have employed the leading WKB approximation to obtain the scalar and tensor power spectra to second order in the slow-roll parameters. The key technical point is to use Eq. (34), which allows one to express the power spectra on large scales as a function of the Hubble crossing quantities, but with no explicit dependence on \(\Delta N\). As one of our main results, we find that the polynomial structure of the power spectra in the \(\epsilon_i\’s\) obtained from the WKB method agrees with the one arising from the Green’s function approach of Refs. \[3, 7\].

In Section \[V\] as a further development of Ref. \[13\], we employ the next-to-leading WKB approximation to second order in the \(\epsilon_i\’s\) and, by requiring that the power spectra do not explicitly depend on \(\Delta N\) (a property which instead was derived both in Sections \[III\] and \[IV\], and using the expressions found in Section \[V\], we obtain unique expressions for the spectral indices and runnings. Our findings show that different ways of approximating cosmological perturbations show up at the leading order in the amplitude, but in the next-to-leading order in the derivatives of the power-spectra with respect to wave number \(k\). The accuracy in the theoretical predictions on spectral indices to second order in the slow-roll parameters, evaluated at \(k = k_s\), is now striking: the Green’s function method coefficient \(C \approx -0.7296\) is replaced by \(D_{\text{WKB}} \approx -0.7302\) in the WKB method, leading to a precision of 1 part on 1000 in the predictions of the coefficients of \(O(\epsilon_i^2)\) terms in the spectral indexes.

\[\epsilon_V = \epsilon_1 \left[ 1 + \frac{\epsilon_2}{2(3 - \epsilon_1)} \right]^2 \sim \epsilon_1, \quad \eta_V = \frac{6\epsilon_1 - \frac{3}{2} \epsilon_2 - 2\epsilon_1^2 + \frac{5}{2} \epsilon_1 \epsilon_2 - \frac{3}{2} \epsilon_2^2 - \frac{7}{2} \epsilon_2 \epsilon_3}{3 - \epsilon_1} \sim 2\epsilon_1 - \frac{1}{2} \epsilon_2,\]

\[\xi_V^2 = \frac{2(\epsilon_1 - 6 - \epsilon_2) [8\epsilon_1^4 - 6\epsilon_1^3 (4 + 3\epsilon_2) + \epsilon_1 \epsilon_2 (18 + 6\epsilon_2 + 7\epsilon_3) - \epsilon_2 \epsilon_3 (3 + \epsilon_2 + \epsilon_3 + \epsilon_4)]}{4 (3 - \epsilon_1)^2}\]

\[\sim 4\epsilon_1^3 - 3\epsilon_1 \epsilon_2 + \frac{1}{2} \epsilon_2 \epsilon_3,\]

where we have shown both the exact formulas and approximate relations to the order of interest.

We complete this review with the first three exact connection formulae between the PSR and HSR parameters \[22\]

\[\frac{\epsilon_V}{\epsilon_H} = \frac{(3 - \eta_H)}{(3 - \epsilon_H)^2}, \quad \eta_V = \frac{3(\epsilon_H + \eta_H) - \eta_H^2 - \xi_H^2}{3 - \epsilon_H},\]

\[\xi_V^2 = \frac{9(3\epsilon_H \eta_H + \epsilon_H^2 - \epsilon_H \eta_H^2)}{(3 - \epsilon_H)^2} - 3(4\eta_H \xi_H^2 - \eta_H^2 \xi_H^2 + \sigma_H^2) + \eta_H \sigma_H^2,\]

and so on with \(\sigma_H\) the next HSR parameter.

**APPENDIX A: SLOW-ROLL VS HORIZON FLOW**

In this Appendix, we compare the HFF we used in the text with some other (equivalent) hierarchies.

In Table \[I\] we give the relations between the slow-roll parameters defined by some other authors and the HFF defined in Eq. (7).

| J. Martin et al. \[14]\textsuperscript{a} | E.D. Stewart et al. \[4, 6]\textsuperscript{b} | S. Habib et al. \[8\] | S. Habib et al. \[9, 10\] |
|----------------|----------------|----------------|----------------|
| \(\epsilon = \epsilon_1\) | \(\epsilon = \epsilon_1\) | \(\delta = \frac{1}{2} \epsilon_2 - \epsilon_1\) | \(\xi_2 = \frac{1}{2} \epsilon_2 \epsilon_3\) |
| \(\delta_1 = \frac{1}{2} \epsilon_2 - \epsilon_1\) | \(\delta_2 = \frac{-3}{2} \epsilon_1 \epsilon_2 + \frac{3}{2} \epsilon_2^2 + \frac{1}{2} \epsilon_2 \epsilon_3\) | \(\delta_1 = \frac{1}{2} \epsilon_2 - \epsilon_1\) | \(\delta_2 = \frac{-3}{2} \epsilon_1 \epsilon_2 + \frac{3}{2} \epsilon_2^2 + \frac{1}{2} \epsilon_2 \epsilon_3\) |

\textsuperscript{a} In Ref. \[14\] \(\xi\) is used instead of \(\xi_2\). \textsuperscript{b} In Ref. \[4\] \(\epsilon_1, \delta, \text{and} \dot{\phi}/H \dot{\phi}\) are used instead of \(\epsilon, \delta_1, \text{and} \delta_2\) respectively.

We also compare the Hubble-slow-roll (HSR) and the potential-slow-roll (PSR) parameters (see Ref. \[22\]) with the HFF. We recall that some authors redefine \(\xi_V\) and \(\xi_H\) as \(\xi_V^\prime\) and \(\xi_H^\prime\) and write only \(\epsilon, \eta\) and \(\xi\), which can be confusing. The relations between the \(\epsilon_i\’s\) and HSR parameters are

\[\epsilon_H = \epsilon_1, \quad \eta_H = \epsilon_1 - \frac{1}{2} \epsilon_2,\]

\[\xi_H^2 = \epsilon_1^2 - \frac{3}{2} \epsilon_1 \epsilon_2 + \frac{1}{2} \epsilon_2 \epsilon_3,\]

and for the PSR parameters we have

**APPENDIX B: OTHER RESULTS IN THE LITERATURE**

We use Table \[II\] and Eqs. (A1), to compare the results for spectral indices - and for PS and runnings when available - in terms of the HFF in Tables \[I\] and \[II\].
The change of variables Eq. (9) of Ref. [8] can be obtained from our Eq. (2) by in different hierarchies. Papers which give the same results, although expressed sor perturbations given in Ref. [7], as calculated with the scaling in \( \ln (k) \).

\[ \zeta = \frac{H^2}{8\pi m^2} \left( 1 - 2(C + 1) \epsilon_1 - C \epsilon_2 + \left( 2C^2 + 2C + \frac{\pi^2}{5} - 5 \right) \epsilon_1^2 \right. \]

\[ \left. + \left( C^2 + 2 + \frac{\pi^2}{3} - 7 \right) \epsilon_1 \epsilon_2 + \left( \frac{1}{2} C^2 + \frac{\pi^2}{2} - 1 \right) \epsilon_2^2 + \left( -\frac{1}{2} C^2 + \frac{\pi^2}{2} \right) \epsilon_2 \epsilon_3 \right. \]

\[ \left. + \left[ -2 \epsilon_1 - 2 \epsilon_2 + 2(2C + 1) \epsilon_1^2 + (2C - 1) \epsilon_1 \epsilon_2 + C \epsilon_2^2 - C \epsilon_2 \epsilon_3 \right] \ln \left( \frac{k}{k_*} \right) + \frac{1}{2} \left( 4 \epsilon_1^2 + 2 \epsilon_1 \epsilon_2 + \epsilon_2^2 - \epsilon_2 \epsilon_3 \right) \ln^2 \left( \frac{k}{k_*} \right) \right) \]

\[ \mathcal{P}_h = \frac{16H^2}{m^2} \left( 1 - 2(C + 1) \epsilon_1 + \left( 2C^2 + 2C + \frac{\pi^2}{5} - 5 \right) \epsilon_1^2 + \left( -C^2 - 2C + \frac{\pi^2}{2} - 2 \right) \epsilon_1 \epsilon_2 \right. \]

\[ \left. + \left[ -2 \epsilon_1 + 2 \epsilon_2 + (2C + 1) \epsilon_1^2 - 2(C + 1) \epsilon_1 \epsilon_2 \right] \ln \left( \frac{k}{k_*} \right) + \frac{1}{2} \left( 4 \epsilon_1^2 - 2 \epsilon_1 \epsilon_2 \right) \ln^2 \left( \frac{k}{k_*} \right) \right) \]

\[ n_S - 1 = -2 \epsilon_1 - \epsilon_2 - 2 \epsilon_1^2 - (2C + 3) \epsilon_1 \epsilon_2 - C \epsilon_2^2, \quad n_T = -2 \epsilon_1 - 2 \epsilon_1^2 - 2(C + 1) \epsilon_1 \epsilon_2 \]

\[ \alpha_S = -2 \epsilon_1 \epsilon_2 - \epsilon_2 \epsilon_3, \quad \alpha_T = -2 \epsilon_1 \epsilon_2 \]

\[ R = 16 \epsilon_1 \left[ 1 + C \epsilon_2 + \left( -\frac{\pi^2}{5} + 5 \right) \epsilon_1 \epsilon_2 + \left( \frac{1}{2} C^2 - \frac{\pi^2}{5} + 1 \right) \epsilon_2^2 + \left( \frac{1}{2} C^2 - \frac{\pi^2}{5} \right) \epsilon_2 \epsilon_3 \right] \]

In Table II we summarize the results for scalar and tensor perturbations given in Ref. [7], as calculated with the method proposed in Ref. [6]. We also refer to some other papers which give the same results, although expressed in different hierarchies.

Let us then compare with the results obtained by the uniform approximation [9]. The integrand in Eq. (9) of Ref. [8] can be obtained from our Eq. (7) by the change of variables \( x = \ln (-k \eta) \), \( u = e^{-x/2} \mu \), which of course differs from our Eq. (9). Therefore, the integrands in our Eq. (13) and Eq. (9) of Ref. [8] differ by \( O(\epsilon_1^4) \) terms, once both are expressed in conformal time. We also note that in Refs. [3], [9], the authors just give expressions for the spectral indices which depend on \( k \) (i.e., without referring to a pivot scale \( k = k_* \)). Further, it is not easy to see that they obtain the correct scaling in \( \ln (k/k_*) \), which yields the \( \alpha \)-runnings, and the \( \epsilon_i \)’s are evaluated at the classical turning points for the frequencies.

In Table III we display the results for scalar and tensor perturbations given in Ref. [10]. As it was properly stated in Ref. [10], the results for scalar and tensor spectral indices given in Ref. [3], [6] were not fully expanded to second order in the slow-roll parameters.

**TABLE II:** Results obtained with the Green’s function method [6] suggested in Ref. [3]: \( C \equiv \ln 2 + \gamma_E - 2 \simeq -0.7296 \) with \( \gamma_E \) the Euler-Mascheroni constant. We display the original results for spectral indices, \( \alpha \)-runnings and tensor-to-scalar ratio evaluated at the pivot scale \( k_* \).

| \( \epsilon_1 \)’s are evaluated at the classical turning points for the frequencies. |

**TABLE III:** Results obtained with the uniform approximation.

| \( n_S - 1 = -2 \epsilon_1 - \epsilon_2 - 2 \epsilon_1^2 - (2C + 3) \epsilon_1 \epsilon_2 - C \epsilon_2^2, \quad n_T = -2 \epsilon_1 - 2 \epsilon_1^2 - 2(C + 1) \epsilon_1 \epsilon_2 \) |

| S. Habib et al. [10] |

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For the same reason (i.e. to avoid the appearance of $\Delta N$), we cannot consider terms such as $\mathcal{O}(\epsilon_i^2) \ln^2(-k \eta)$ in Eqs. [13].