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Published in:
Astrophysical Journal

DOI:
10.3847/1538-4357/ac56dd

Publication date:
2022

Document version
Publisher's PDF, also known as Version of record

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Citation for published version (APA):
Gressel, O., & Pessah, M. E. (2022). Finite-time Response of Dynamo Mean-field Effects in Magnetorotational Turbulence. Astrophysical Journal, 928(2), [118]. https://doi.org/10.3847/1538-4357/ac56dd
Finite-time Response of Dynamo Mean-field Effects in Magnetorotational Turbulence

Oliver Gressel\textsuperscript{1,2,} and Martin E. Pessah\textsuperscript{2} \textsuperscript{\textsuperscript{\textregistered}}

\textsuperscript{1}Leibniz-Institut für Astrophysik Potsdam (AIP), An der Sternwarte 16, D-14482, Potsdam, Germany; ogressel@aip.de
\textsuperscript{2}Niels Bohr International Academy, The Niels Bohr Institute, Blegdamsvej 17, DK-2100, Copenhagen Ø, Denmark

Received 2021 December 16; revised 2022 February 17; accepted 2022 February 17; published 2022 March 31

Abstract

Accretion disk turbulence along with its effect on large-scale magnetic fields plays an important role in understanding disk evolution in general, and the launching of astrophysical jets in particular. Motivated by enabling a comprehensive subgrid description for global long-term simulations of accretion disks, we aim to further characterize the transport coefficients emerging in local simulations of magnetorotational disk turbulence. For the current investigation, we leverage a time-dependent version of the test-field method, which is sensitive to the turbulent electromotive force (EMF) generated as a response to a set of pulsating background fields. We obtain Fourier spectra of the transport coefficients as a function of oscillation frequency. These are well approximated by a simple response function, describing a finite-time buildup of the EMF as a result of a time-variable mean magnetic field. For intermediate timescales (i.e., slightly above the orbital frequency), we observe a significant phase lag of the EMF compared to the causing field. Augmented with our previous result on a nonlocal closure relation in space, and incorporated into a suitable mean-field description that we briefly sketch out here, the new framework will allow us to drop the restrictive assumption of scale separation.

Unified Astronomy Thesaurus concepts: Magnetic fields (994); Magnetohydrodynamical simulations (1966); Magnetohydrodynamics (1964)

1. Introduction

Thirty years after its ultimate discovery by Balbus & Hawley (1991), the magnetorotational instability (MRI) is practically synonymous with accretion disk turbulence and is believed to be the key to understanding the structure and evolution of disks ranging from circumplanetary, to circumstellar (including those around black holes or neutron stars, as well as the innermost and outer reaches of protoplanetary ones), and all the way to active galactic nuclei.

Enhanced transport coefficients—stemming from correlated fluctuations in the MRI turbulence—have a profound impact on disk evolution. This can happen either directly, when disk accretion is enabled by angular momentum exchange through enhanced viscosity, or indirectly via a magnetocentrifugal disk outflow. Notably, the latter scenario requires large-scale ordered poloidal fields that are either created in situ by a disk dynamo (e.g., von Rekowski et al. 2003; Stepanovs et al. 2014), or—when an inherited large-scale field is invoked—are at least affected by enhanced field dissipation as a result of eddy diffusivity. Recent attempts of incorporating subgrid-scale physics into jet-launching simulations (e.g., Bucciantini & Del Zanna 2013; Dyda et al. 2018; Fednt & Gaßmann 2018; Mattia & Fednt 2020a, 2020b; Vourellis & Fednt 2021) illustrate the need for comprehensive parameterizations that are ideally based on first-principles, resolved MRI simulations.

Depending on (i) the level of inherited/accumulated net-vertical magnetic flux, and (ii) the relevance of the vertical disk structure, the MRI relies to a varying degree on the presence of an intrinsic dynamo of some sort to become a self-sustained mechanism for powering disk accretion (see Rincon 2019, for an excellent review on this subject). A pronounced shortcoming of nonstratified box simulations is that they are very sensitive to the vertical aspect ratio (see Shi et al. 2016; Walker & Boldyrev 2017).

Notably, when including vertical stratification, both local-box (going back to Brandenburg et al. 1995) and global (as recent as Dhang et al. 2020) fully nonlinear MRI simulations alike robustly develop near-periodic cycles in the (horizontally/azimuthally averaged) mean magnetic field with a characteristic propagation away from the disk midplane—providing a natural explanation to sustaining the MRI via a large-scale dynamo action (e.g., Brandenburg 2005, 2008; Blackman 2010).

The morphology of this so-called butterfly diagram—the hallmark of the mean-field dynamo—was previously found to depend somewhat on the amount of net-vertical magnetic flux (see, e.g., Gressel & Pessah 2015; Salvesen et al. 2016). We here nevertheless focus on the limit of negligible net-vertical magnetic flux, in which a way is the crucial test for providing a robust accretion engine from MRI turbulence. Another important issue raised pertains to the onset of convective turnover (Bodo et al. 2012; Gressel 2013; Hirose et al. 2014), which was found to drastically affect the regularity of the magnetic-field cycles (see the discussion in Coleman et al. 2017). As with the net-vertical magnetic flux, we take a rather conservative stance and focus our investigation on the isothermal case, avoiding the complications associated with arguably more realistic thermodynamic representations.

A central question that remains unanswered is whether the dynamo wave can be reconciled with a conventional $\alpha\Omega$ dynamo (i.e., driven via the interplay of helical turbulence with differential rotation), and/or whether its dynamics are enforced by the near-exact conservation of magnetic helicity at a high magnetic Reynolds number (e.g., Vishniac 2009; Gressel 2010; Oishi & Mac Low 2011). While the cycle period as a function of shear rate can nicely be explained using the dispersion relation of a near-critical $\alpha\Omega$ dynamo (Gressel & Pessah 2015), the propagation direction away from the midplane is still not...
well understood, possibly requiring a magnetic buoyancy contribution near the midplane (Brandenburg 1998).

Both the spatial nonlocality (see Brandenburg & Sokoloff 2002) of the dynamo closure relation, the noninstantaneous aspects (i.e., so-called “memory effects”; e.g., Hubbard & Brandenburg 2009), as well as their combined effect (see, e.g., Rheinhardt & Brandenburg 2012) have been demonstrated to influence the characteristics of the dynamo cycle. Another comprehensive example of how finite-time effects can influence dynamo-generated fields has been presented by Chamandy et al. (2013a, 2013b) in the context of galactic magnetic fields. To complement our previous investigation of the scale dependence of turbulence (Gressel & Pessah 2015, Section 3.4), we here investigate the potential role of finite-time effects in the mean-field closure relation.

Our paper is organized in the following manner: Section 2 briefly describes the numerical simulations and introduces the newly adopted noninstantaneous closure relation to the mean-field induction equation, as well as how it can be captured using the test-field (TF) method. We present the results obtained from a fiducial MREI shearing box simulation in Section 3, and we discuss how these findings may be exploited in the future, in Section 4.

\section{Methods}

As in previous work, we solve the equations of isothermal, ideal magnetohydrodynamics (MHD) in a local shearing box (e.g., Gressel & Ziegler 2007) frame of reference. Lacking explicit dissipation, the purist may call this an “implicit” large-eddy simulation (iLES). For practical purposes, we will nevertheless refer to this as a direct numerical simulation (DNS). For brevity, we here only briefly recapitulate the essential properties of our numerical approach, and refer the reader to Sections 2.1 et seq. of Gressel & Pessah (2015) for a more in-depth discussion, motivating our particular choices.

\subsection{Brief Specification of the Direct Simulations}

We here use local Cartesian coordinates, \((x, y, z)\), but refer to some tensor coefficients in cylindrical components, \((r, \phi, z)\), for an easier comparison with global models. Differential rotation is expressed via the parameter \(q \equiv \mathrm{dln} \Omega / \mathrm{dln} r = -3/2\) for a Keplerian rotation curve, and we use the “orbital advection” scheme of Stone & Gardner (2010) to treat the background shear flow, \(\mathbf{v}_k \equiv q \Omega x \hat{y}\), with the benefit of a position-independent truncation error. The equations expressed in the local Eulerian velocity, \(\mathbf{v}\), are

\[\begin{align*}
\frac{\partial \mathbf{v}}{\partial t} + \nabla \cdot (\mathbf{v} \mathbf{v}) &= 0, \\
\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v} + \mathbf{p}^* \mathbf{B} - \mathbf{BB}) &= -2 \rho \Omega \hat{z} \times \mathbf{v} - \rho \nabla \Phi, \\
\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) &= 0,
\end{align*}\]

(1)

with the total pressure \(\mathbf{p}^* \equiv \mathbf{p} + \mathbf{B}^2/2\), and the combined (i.e., tidal plus gravitational) effective potential

\[\Phi(x, z) = q \Omega^2 x^2 + \frac{1}{2} \Omega^2 z^2,\]

(2)
defined in the locally corotating frame of reference at fixed angular frequency \(\Omega \equiv \Omega \hat{z}\). Horizontal boundary conditions are shear-periodic (see Gressel & Ziegler 2007), for details), and we apply standard outflow conditions in the vertical direction. We chose an intermediate box size of \(L_x \times L_y \times L_z = H \times \pi H \times 6H\) with a linear resolution of \(\approx 32/H\) in all space dimensions—amounting to \(32 \times 100 \times 192\) cells in the radial (\(x\)), azimuthal (\(\phi\)), and vertical (\(z\)) coordinate directions, respectively. The initial plasma parameters in the disk midplane are \(\beta_p = 800\) and \(\beta_p = 2.2 \times 10^5\) for the zero-net-flux contribution, and the additional net-vertical field, respectively. As previously, we (i) include an artificial mass diffusion term (see Gressel et al. 2011) to circumvent undue time-step constraints resulting from low-density regions in the upper disk corona, and (ii) replenish the mass lost via outflow through the vertical domain boundary to obtain an overall steady-state disk structure.

\subsection{The Noninstantaneous Closure Relation}

Adopting the well-established framework of mean-field MHD (Krause & Rädler 1980), we seek a parameterization for the turbulent electromotive force (EMF), \(\mathbf{E} \equiv \mathbf{v} \times \mathbf{B}^\ast\) with fluctuating magnetic and velocity fields defined as \(\mathbf{B}^\ast \equiv \mathbf{B} - \overline{\mathbf{B}(z)}\) and \(\mathbf{v}^\ast \equiv \mathbf{v} - \overline{\mathbf{v}(z)}\), respectively.

Here, and in the following, the overbar implies geometric averaging over horizontal slabs. This is the natural choice for the adopted box geometry and trivially satisfies the Reynolds rules required for a consistent mean-field description. By virtue of its definition, the EMF captures correlations in fluctuating velocity and magnetic fields, whose nonzero mean appears as a source term on the right-hand side of the (one-dimensional) mean-field induction equation

\[\frac{\partial \mathbf{B}(z)}{\partial t} - \nabla \times (\mathbf{v}(z) \times \overline{\mathbf{B}(z)}) = \nabla \times \overline{\mathbf{E}(z)}.\]

(3)

By construction, this equation describes the long-term evolution of the (comparatively slowly changing) mean magnetic field under the effect of the underlying turbulence. To make progress over a direct simulation approach, the EMF is then typically expanded into a linear functional of the mean magnetic field and its gradients as

\[\mathbf{E}(z, t) = \alpha_{ij}(z, t) \overline{\mathbf{B}_i(z, t)} - \eta_{ij}(z, t) \epsilon_{jkl} \partial_k \overline{\mathbf{B}_l(z, t)},\]

(4)

where \(\epsilon_{jkl}\) is the Levi-Civita tensor, and where the indices \(i, j, l\) label the coordinates \(x, y\) and contraction over repeated indices is understood. Note that it is unnecessary to include the radial and azimuthal gradients in our case, which is due to the homogeneity of the turbulence in any given horizontal plane.

Under steady-state conditions, the second-rank tensors, \(\alpha_{ij}(z)\) and \(\eta_{ij}(z)\), become time-independent and are expected to capture the statistical properties of the chaotic flow (see Krause & Rädler 1980). If the system at hand is sufficiently anisotropic (e.g., due to rotation) and inhomogeneous (e.g., due to vertical gravity/stratification), \(\alpha_{ij}(z)\)—as well as the off-diagonal elements of \(\eta_{ij}(z)\)—is expected to be nonvanishing. Together, the tensors encapsulate the emergence of the mean EMF as a response to imposing an external mean magnetic field—or, in general, to the presence of a self-consistently

\footnotetext{As we will be using the fluctuating velocity \(\mathbf{u} \equiv \mathbf{v} - \mathbf{v}_k\) in some places, we note that, because \(\mathbf{v}_k(x)\) vanishes when averaging, \(\mathbf{v} \equiv \mathbf{u} \hat{z}\), trivially.}
evolving mean field. The purpose of the present paper is to elucidate a possible finite-time character of this response.

Typically, the turbulent closure coefficients are thought to connect \( \mathcal{E}(z, t) \) to the mean magnetic field, \( \mathcal{B}(z, t) \), and its curl \( \varepsilon_{ij} \partial_iz \hat{B}(z, t) \), in a local and instantaneous fashion.\(^4\) However, in contrast to this instantaneous characterization of the closure relation, the power-law nature of the turbulent cascade suggests that the spacetime domain of dependence of \( \mathcal{E}(z, t) \) is indeed finite—implying so-called “memory effects” (Hubbard & Brandenburg 2009), that is, a delayed (i.e., out-of-phase) response to an applied mean field.\(^5\) Under the assumption of statistically stationary turbulence, a simple noninstantaneous closure relation (also see Gressel & Elstner 2020) can be formulated as a convolution integral in time of the form

\[
\mathcal{E}(z, t) = \int [\hat{\alpha}_y(z, t') \mathcal{B}(z, t-t')] - \tilde{\eta}_y(z, t') \varepsilon_{ij} \partial_iz \hat{B}(z, t-t')] \, dt'.
\]  

(5)

In the local box geometry, the integral kernels \( \hat{\alpha}_y(z, t') \) and \( \tilde{\eta}_y(z, t') \) are functions of the vertical coordinate, \( z \), only. Moreover, in its Fourier-space representation, the above relation can be expressed as a simple multiplication (see Hubbard & Brandenburg 2009, Appendix A) with the Fourier transform, \( \hat{\alpha}_y \), of the kernel. That is, (dropping the explicit \( z \)-dependence) we write

\[
\hat{\mathcal{E}}(\omega) = \hat{\alpha}_y(\omega) \hat{\mathcal{B}}(\omega) - \tilde{\eta}_y(\omega) i k_z \varepsilon_{ij} \hat{B}(\omega),
\]  

(6)

attributing a spectral flavor to the mean-field coefficients. While all of these quantities are in general complex functions, we can obtain real values (of, e.g., \( \mathcal{E} \)) in physical space and time by adding up the contributions from positive and negative frequencies.

Complementing the result of Gressel & Pessah (2015) on the nonlocal, scale-dependent character of the mean-field effects in magnetorotational turbulence, we here aim to obtain frequency-dependent closure coefficients, corresponding to convolution kernels in the time domain. The frequency dependence can very naturally be obtained via the TF method (Schrimmer et al. 2005, 2007) employing oscillating test fields as briefly outlined in the next section.

2.3. The Spectral Test-field Method

The defining advantage of the TF method, compared with other methods of inference, is that it relies on analytically prescribed “test fields” that can be chosen to span a nondegenerate basis for determining all tensor coefficients in an unambiguous manner. This differs from direct inversion methods (see, e.g., discussion in Bendre et al. 2020) that are founded on the (potentially degenerate) mean fields, \( \mathcal{B}(z, t) \), developing in the DNS. To invert Equation (6), and solve for the tensorial closure coefficients, \( \hat{\alpha}_y(\omega) \) and \( \tilde{\eta}_y(\omega) \), we apply the flavor of the method where the TFs, \( \mathcal{B}(\omega) \), are quadruplets of trigonometric functions (also see, e.g., Brandenburg 2005; Brandenburg et al. 2008; Sur et al. 2008):

\[
\begin{align*}
\mathcal{B}_{(0)} &= \cos(\chi \omega) \cos(k_z z) \hat{x}, \\
\mathcal{B}_{(1)} &= \cos(\chi \omega) \sin(k_z z) \hat{x}, \\
\mathcal{B}_{(2)} &= \cos(\chi \omega) \cos(k_z z) \hat{y}, \\
\mathcal{B}_{(3)} &= \cos(\chi \omega) \sin(k_z z) \hat{y}.
\end{align*}
\]  

(7)

For the purpose of determining the time response, we here focus on a fixed vertical scale \( k_z = k_z^{\text{TF}} = \frac{2\pi}{L_z} \), with \( L_z = 6 \) as the vertical size of the box. This makes us sensitive to the coefficients representative of the largest scales available, and we refer the interested reader to Section 3.4 of Gressel & Pessah (2015) for a complementary discussion about the scale dependence (also see Rheinhardt & Brandenburg 2012, for the general case of full spatiotemporal dependence). Having specified \( k_z^{\text{TF}} \), we moreover use 11 spectral modes \( \omega = \omega^{\text{TF}} = 1/32, 1/16, 1/8, ... , 16, 32 \times 2\pi/P_0 \), centered around \( P_0 = 1/4 \) orbit for the temporal domain. We have arrived at this sampling interval by a combination of educated guessing and trial and error, and have found a posteriori that the relevant dynamic range appears to be covered.

In total, we are hence solving \( 4 \times 11 = 44 \) additional induction equations, one for each of the TF fluctuations, \( \mathcal{B}^{(i)}(r, t) \), alongside the DNS. In terms of the fluctuating velocity, \( \mathbf{u} \), these are

\[
\partial_t \mathcal{B}^i = \nabla \times [\mathbf{u}^i \times \mathcal{B}^i + (\mathbf{u}^j + \mathbf{v}_k) \times \mathcal{B}^j - \mathbf{u}^i \times \mathcal{B}^j + \mathbf{u}^j \times \mathcal{B}^i],
\]  

(8)

Importantly, these equations are passive in that they do not influence the evolving magnetic fields in the original DNS. Since we are dealing with MRI turbulence—driven via an underlying genuinely magnetic instability—they likely are preexisting magnetic fluctuations, \( \mathcal{B}^0 \), that are statistically independent from the developing (horizontal) mean fields. Such fluctuations, if correlated with the turbulent velocity, may result in an additional EMF, namely \( \mathcal{E}^i = \mathbf{u}^i \times \mathcal{B}^0 \) —which does, however, not enter our parameterization. To obtain the coefficients, we evaluate the corresponding mean EMF \( \mathcal{E}^{(\omega)} = \mathbf{u}^i \times \mathcal{B}^{(\omega)} \) for each of the quadruplets from Equation (7). A formal solution to Equation (6) is then obtained as

\[
\begin{align*}
\hat{\alpha}_y(k_z, \omega) &= e^{i\omega t} \begin{pmatrix} \cos(k_z z) \\
-\sin(k_z z) \end{pmatrix}, \\
\hat{\eta}_y(k_z, \omega) &= e^{i\omega t} \begin{pmatrix} \sin(k_z z) \\
\cos(k_z z) \end{pmatrix},
\end{align*}
\]  

(9)

where the tensors \( \tilde{\eta}_y(k_z, \omega) \) (i.e., with respect to the current) and \( \hat{\alpha}_y(k_z, \omega) \) (i.e., with respect to the field gradients) are simply related via (see Hubbard & Brandenburg 2009)

\[
\hat{\eta}_y = \varepsilon_{jkz} \hat{\alpha}_y.
\]  

(10)

In order to arrive at a statistical sound basis and eliminate stochastic fluctuations, Equation (9) can simply be accumulated over time as needed. This appears to be particularly relevant for slowly oscillating TFs, motivating our preference for a long simulation time over grid resolution. Note that in the time-dependent case, that is, for \( \omega^{\text{TF}} \neq 0 \), complex coefficients can arise, reflecting a frequency-dependent phase shift of the resulting EMF with respect to the originating oscillating TF. In practical terms, we replace the complex factor \( e^{i\omega t} \) in Equation (9) by either \( \cos(\omega t) \) or \( \sin(\omega t) \), in order to project the real/imaginary parts of the coefficients, respectively. In the Appendix of Gressel & Elstner (2020), we have benchmarked

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\(^4\) This local relation formally demands a “scale separation” between \( \mathcal{B} \), on one hand, and \( \mathcal{B}^{(i)} \) on the other hand (so that the slowly varying mean field can be pulled out of the integral describing the time evolution of the EMF).

\(^5\) Note that, while we often speak of “imposing” or “applying” mean fields, and the EMF as a “response” (using the language of signal processing), these words can easily be replaced by “preexisting” or “emerging” to better capture the spontaneous character of the chaotic turbulent flow.
the implementation of the described spectral TF method, using the simple test case of helical forcing in the strictly kinematic limit.

3. Results

We present results from a single generic shearing box simulation of the MRI in the presence of a weak (i.e., midplane β_p = 2.2 \times 10^7) net-vertical field, with L_c = ±3 H and outflow boundary conditions and a moderate resolution of 32 grid cells per pressure scale height. The simulation quickly reaches a steady state with a dimensionless accretion stress (i.e., Reynolds and Maxwell) of about 0.01, and we evolve the simulation for 555 orbits to obtain decent statistics.

In Figure 1, we plot vertical profiles of the dynamo α effect, where we show real (upper panel/red tones) and imaginary parts (lower panel/blue tones) separately. The parametric curves represent the variation with the imposed oscillation frequency, ω^{FR}, of the TF inhomogeneity.6

Looking at the raw array of curves by eye makes it rather cumbersome to grasp anything but the most fundamental trends in the data. As a remedy, and to illustrate the basic features of the spectral response, we resort to sampling point values at the spectral response, we resort to sampling point values at the

amplitude, which is about a factor of 2 higher, these broadly match the characteristics of α_{pol}(z).

Unlike for the case of supernova-driven turbulence in the multiphase interstellar medium (Gressel & Elstner 2020)—where the off-diagonal elements were found to be antisymmetric and distinct from the diagonal elements—the diagonal and off-diagonal components of the α tensor here (see Figures 1 and 2, respectively) show a rather similar time response. The imaginary part displays a broad peak around ω = 2. At the same time, the real part has a moderate overshoot around ω = 1, before it reaches the asymptotic value for slowly varying mean fields.

The general shape of the response can be understood by visualizing the overall character of (rotating) turbulence. Let us briefly recall what the α effect entails. It describes the emergence of a mean turbulent EMF as the direct consequence7 of the presence of a large-scale magnetic field. In the limit of high frequencies, this “presence” obviously loses its coherent character and magnetic fluctuations created by the \( \mathbf{u} \times \mathbf{B} \) term in Equation (8) will tend to become uncorrelated with the velocity and their contribution to the EMF will consequently average out to zero; this is reflected in the vanishing amplitudes toward high frequencies.

Conversely, at low frequencies, we simply approach the previously reported (Gressel & Pessah 2015) amplitudes. In particular, the imaginary part of the effect also tends to zero in this limit, so that the α effect becomes a real number, implying that there is no longer a phase difference. The interesting regime falls in the region of intermediate frequencies that roughly correspond to the eddy turnover time and/or rotational frequency of the turbulence. Here the finite-time character of the relation between the imposed large-scale field \( \mathbf{B}(t) \) (as a “cause”) and the turbulent \( \mathbf{E}(t) \) (as a “response”) becomes most obvious. In particular, owing to the noninstantaneous buildup

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6 Note that these curves have been spatially filtered using a truncated series expansion into Legendre polynomials (up to order \( l = 12 \)), and this serves the purpose to extract a meaningful magnitude.

7 That is, in a “linear” (or, leading-order) sense.
The real part of the assumed mean field matches the turnover of eddies that are affected by the Coriolis force.

In comparison with the \( \alpha \) tensor, the diffusion coefficient, \( \eta_T \), (shown in Figure 3), shows a somewhat reduced coherence time. This broadly matches the expectation that the mixing aspect of the chaotic flow field depends somewhat less on the buildup of correlated motions and is hence preserved further into the limit of high frequencies. Moreover, the real part of \( \eta_T(\omega) \) remains strictly monotonic around \( \omega = 1 \), hinting at a reduced influence of the orbital timescale on the mere random diffusion of field.

### 3.1. Characteristic Response Function

Pertaining to the noninstantaneous closure relation, and for the case of simple helically forced turbulence, Hubbard & Brandenburg (2009) have demonstrated a frequency dependence of the form of an oscillating decay,

\[
\alpha(t) \propto \Theta(t) e^{-t/\tau_c} \cos(\omega_0 t),
\]

where \( \Theta(t) \) simply denotes the Heaviside step function, enforcing causality by suppressing dependence on future times. Translated into Fourier space, the spectral shape function becomes

\[
\alpha(\omega) = A_0 \frac{1 - i \omega \tau_c}{(1 - i \omega \tau_c)^2 + (\omega_0 \tau_c)^2},
\]

with independent coefficients \( A_0, \tau_c, \) and \( \omega_0 \)—and with corresponding expressions for the other two coefficients of interest. For the purpose of curve fitting the frequency response, we write Equation (12) separated into real and imaginary parts as

\[
\Re = A_0 \frac{1 + (\omega^2 + \omega_0^2) \tau_c^2}{4 \omega^2 \tau_c^2 + (1 - (\omega^2 - \omega_0^2) \tau_c^2)^2},
\]

\[
\Im = A_0 \frac{1 + (\omega^2 - \omega_0^2) \tau_c^2}{4 \omega^2 \tau_c^2 + (1 - (\omega^2 - \omega_0^2) \tau_c^2)^2} \omega \tau_c,
\]

Table 1

|                  | \( A_0 \) | \( \tau_c \) | \( \omega_0 \) | \( \omega_0 \tau_c \) |
|------------------|---------|-----------|-------------|--------------|
| dynamo \( \alpha_\phi \) | 0.88    | 3.92      | 0.20        | 0.78         |
| off-diagonal \( \alpha_\text{sym} \) | 1.86    | 4.06      | 0.20        | 0.80         |
| diffusivity \( \eta_f \) | 0.56    | 3.05      | 0.17        | 0.52         |

The presented results clearly display the finite-time character of the mean-field dynamo effect emerging in stratified zero-net-flux MRI turbulence. The approximate functional form presented in the preceding section will enable us to incorporate the effects into a more comprehensive mean-field description of the evolution of large-scale magnetic fields in accretion disks.

4. Discussion and Conclusions

The purpose of curve fitting the frequency response, we write Equation (12) separated into real and imaginary parts as

\[
\Re = A_0 \frac{1 + (\omega^2 + \omega_0^2) \tau_c^2}{4 \omega^2 \tau_c^2 + (1 - (\omega^2 - \omega_0^2) \tau_c^2)^2},
\]

\[
\Im = A_0 \frac{1 + (\omega^2 - \omega_0^2) \tau_c^2}{4 \omega^2 \tau_c^2 + (1 - (\omega^2 - \omega_0^2) \tau_c^2)^2} \omega \tau_c,
\]

with separate sets of fit parameters \( A_0, \tau_c, \) and \( \omega_0 \) for the three coefficients \( \alpha_\phi(\omega), \alpha_\text{sym}(\omega), \) and \( \eta_f(\omega), \) respectively. We note that—while we express the complex dependence in terms of two separate functional shapes via Equations (13) and (14)—we fit the real and imaginary branches simultaneously in practical terms.

The real parts (solid lines) and imaginary parts (dashed lines) of the fitted curves are overlaid in the insets of Figures 1–3, and represent the data rather well. We, moreover, report best-fit values for coefficients (sampled at \( z = 2.67 \)) in Table 1, where we also provide the dimensionless number, \( \omega_0 \tau_c \), which we naively expect to be on the order of unity.

As a consequence of the vertical stratification in density, one may conjecture that the largest eddy size depends on the height in the disk. This in turn should be reflected in the \( \tau_c \) and \( \omega_0 \) fit coefficients that we determine. To test this assumption, we plot the two numbers in the upper and lower panels of Figure 4, respectively. It can again be seen that the dynamo \( \alpha \) term and the off-diagonal elements are very similar, and at the same time, distinct from the diffusive coefficient. While \( \tau_c \) remains fairly constant for the latter, the former two show a pronounced increase (by a factor of 3) in the turbulent correlation time toward the disk midplane. In contrast to this, their oscillatory parameter, \( \omega_0 \), remains fairly even in that limit but instead shows a moderate peak around \( z = 1.75 \) \( H \). As consistent with the monotonic profile seen in Figure 3, the \( \omega_0 \) parameter is reduced in the \( \eta_f \) coefficient, and even drops to quite small values toward the disk midplane.

What precisely causes the observed trends is unclear at this point, and it is important to keep in mind that MRI turbulence is critically affected by magnetic forces so that intuition from hydrodynamic turbulence may be of limited value. Irrespective of this, comparing mean-field models with and without variation in \( \tau_c \) will allow us to establish whether the seen variations do have an impact on the appearance of the butterfly diagram.
Figure 4. Height dependence of the two fit parameters from Equation (12) describing the temporal behavior of the response. Circles indicate the values shown in Table 1 above. The fit amplitude, $A_0$, simply follows the original curves from the previous figures and is omitted here.

As a first step in that direction, one may neglect the $\omega_0$ contribution, related to oscillatory behavior at intermediate frequencies. In this case, Equation (12) simplifies to

$$\bar{\alpha}(\omega) = A_0^{(\alpha)} \frac{1}{1 - i \omega \tau_c^{(\alpha)}},$$

(15)

with a corresponding expression for $\tau_c$. With the further approximation $\tau_c^{(\alpha)} = \tau_c^{(0)}$ (see Table 1 for judging to what degree this is justified), the characteristic time, $\tau_c$, enters as a relaxation time and implies a time-dependent (i.e., noninstantaneous) EMF response. In contrast to the algebraic relation from Equation (4), this needs to be modeled as an extra partial differential equation (also see Rheinhardt & Brandenburg 2012) of the form

$$\left(1 + \tau_c \frac{\partial}{\partial t} - l_c^2 \frac{\partial^2}{\partial z^2}\right) \mathcal{E}(z, t) = ..., $$

(17)

where the $l_c$ appears as a smoothing term.

This, however, still neglects the possible advection (i.e., via a term $\mathbf{v} \cdot \nabla \mathcal{E}$) of the EMF with the disk outflow, $\mathbf{v}$, as well as potential effects related to $\nabla \cdot \mathbf{B} = \partial_x B_x + \partial_y B_y + \partial_z B_z$. These may act to (de)compress the EMF—similar to the $(\nabla \times (\mathbf{v} \times \mathbf{B})$ term in the induction equation itself. Moreover, if one were to restore the effect related to $\omega_0$, one would obtain a wavelike second-order time derivative of the EMF on the left-hand side, as well as time derivative terms related to $\mathbf{B}$ appearing on the right-hand side (M. Rheinhardt, private communication). A straightforward Crank–Nicolson discretization of Equation (17) has already been implemented into a simple dynamo code. In view of the mentioned complications, we however defer a detailed mean-field treatment along these lines to a later point in time.

Coming back to the inference of the dynamo coefficients from the DNS, an often-mentioned shortcoming of the TF method pertains to the absence of magnetic fluctuations stemming directly from the simulation. Operating in the so-called “quasi-kinematic” realm (also see discussion in Gressel & Pessah 2015), the QKTFM is agnostic to the presence of a possible $\mathcal{E}_B$, and the velocity $\mathbf{v}(r, t)$ is the only manifest link in Equation (8) to the physical evolution traced by the DNS. While this may indeed be seen as a shortcoming of the current approach, we highlight that it merely implies that the detected mean-field effects likely are not exhaustive, but simply restricted to the chosen ansatz. A promising avenue to accounting for these additional contributions has first been laid out by Rheinhardt & Brandenburg (2010) for a simplified set of equations (i.e., lacking the pressure and self-advection terms in the momentum equation). More recently, a workable solution has been found also for the complete set of MHD equations (Käpylää et al. 2021). It appears natural to test this approach for MRI turbulence as well.

We thank Tobias Heinemann for useful discussions, and Matthias Rheinhardt and Kandaswamy Subramanian for comments on a draft version. This work used the NIRVANA code version 3.3, developed by Udo Ziegler at the Leibniz-Institut für Astrophysik Potsdam (AIP). All computations were performed on the Steno node at the Danish Center for Supercomputing (DCSC). M.E.P. gratefully acknowledges support from the Independent Research Fund Denmark (DFF) via grant no. DFF 8021-00400B.

ORCID iDs

Oliver Gressel @ https://orcid.org/0000-0002-5398-9225
Martin E. Pessah @ https://orcid.org/0000-0001-8716-3563

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