Ornstein-Uhlenbeck semi-groups on stratified groups

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Abstract

We consider, in the setting of stratified groups $G$, two analogues of the Ornstein-Uhlenbeck semi-group, namely Markovian diffusion semi-groups acting on $L^q(p(\gamma)d\gamma)$, whose invariant density $p$ is a heat kernel at time 1 on $G$.

The first one is symmetric on $L^2(pd\gamma)$, its generator is $\sum_{i=1}^n X_i^*X_i$, where $(X_i)_{i=1}^n$ is a basis of the first layer of the Lie algebra of $G$.

The second one, denoted by $T_t = e^{-tN}, t > 0$, is non symmetric on $L^2(pd\gamma)$ and the formal real part of $N$ is $\sum_{i=1}^n X_i^*X_i$. The operators $e^{-tN}$ are compact on $L^q(pd\gamma), 1 < q < \infty$. The spectrum of $N$ on this space is the set of integers $\mathbb{N}$ if polynomials are dense in $L^2(p(\gamma)d\gamma)$, i.e if $G$ has at most 4 layers; and we determine in this case its eigenspaces. When $G$ is step 2, we give another description of these eigenspaces, very similar to the classical definition of "Hermite polynomials" by their generating function.

Keywords: stratified groups, sub Laplacian, heat kernel measure, Ornstein-Uhlenbeck semi-groups.

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1 Introduction and notation

Let $G$ be a stratified Lie group equipped with its (biinvariant) Haar measure $dg$ and dilations $(\delta_t)_{t\geq 0}$. Let $Q$ be the homogeneous dimension of $G$. We denote by $\mathcal{D}(G)$ the space of $C^\infty$ compactly supported functions on $G$, by $\mathcal{S}(G)$ the space of Schwartz functions, by $\mathcal{S}'(G)$ its dual, and $L^q(\varphi dg) = L^q(G, \varphi dg)$ for a measurable non negative function $\varphi$. 

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As usual, elements $Z$ of the Lie algebra $G$ are identified with left invariant vector fields by

$$(Zf)(g) = \frac{d}{dt} |_{t=0} f(g \exp tZ).$$

Let $L$ be a subLaplacian on $G$, i.e. an operator on $\mathcal{S}(G)$ defined by

$$L = - \sum_{1}^{n} X_{i}^{2} \tag{1}$$

where $(X_{i})_{1 \leq i \leq n}$ is a linear basis of the first layer of $G$. Obviously $L$ commutes with left translations and satisfies

$$\delta_{t^{-1}}L\delta_{t} = t^{2}L, \ t > 0. \tag{2}$$

The following facts are well known, see e.g. [FS, propositions 1.68, 1.70, 1.74]: $-\frac{L}{2}$ generates a strongly continuous semi-group $e^{-\frac{t}{2}L}$ of convolution operators which are contractions on $L^{q}(dg), \ 1 \leq q \leq \infty$. The kernel $p_{t}$ of $e^{-\frac{L}{2}}$ is a positive function such that $p_{t}(g) = p_{t}(g^{-1})$, it lies in $\mathcal{S}(G)$ and has norm one in $L^{1}(dg)$. Denoting $p_{1} = p$,

$$p_{t}(g) = t^{-\frac{q}{2}}p \circ \delta_{\frac{1}{\sqrt{t}}}g. \tag{3}$$

Equivalently, for $f \in L^{q}(dg)$,

$$e^{-\frac{t}{2}L}(f)(\gamma) = f * p_{t}(\gamma) = \int_{G} f(\gamma g^{-1})p_{t}(g)dg = \int_{G} f(\gamma \delta_{\sqrt{t}}g^{-1})p(g)dg. \tag{3}$$

The aim of this paper is to generalize the Ornstein-Uhlenbeck semi-group in the setting of stratified groups, namely to consider Markovian semi-groups acting on $L^{q}(p(\gamma)d\gamma), \ 1 \leq q \leq \infty$, for which $p(\gamma)d\gamma$ is an invariant measure, whose generators are related to the first layer gradient

$$\nabla = (X_{1}, ..., X_{n}).$$

The classical Ornstein-Uhlenbeck semi-group is defined on $\mathcal{S}(\mathbb{R}^{n})$ by Mehler formula

$$e^{-tN_{0}}(f)(x) = \int_{\mathbb{R}^{n}} f(e^{-t}x + \sqrt{1-e^{-2tx}}y)p(y)dy, \ t \geq 0,$$
where the gaussian density \( p(y) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}|y|^2} \) is the kernel of \( e^{-\Delta} \), and \( \Delta \) is the (positive) Laplacian on \( \mathbb{R}^n \). The O-U semi-group is contracting on \( L^q(\mathbb{R}^n, p\,dx) \), \( 1 \leq q \leq \infty \), compact if \( 1 < q < \infty \), but not compact on \( L^1(\mathbb{R}, p\,dx) \) [D, theorem 4.3.5], and \( p \) is an invariant measure. The generator \(-N_0\) satisfies

\[
N_0 = \sum_{j=1}^{n} (\frac{\partial}{\partial x_j})^* \frac{\partial}{\partial x_j} = \Delta - \sum_{j=1}^{n} \frac{\partial p}{\partial x_j} \frac{\partial}{\partial x_j} = \Delta + \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} = \Delta + A
\]

where \((\frac{\partial}{\partial x_j})^*\) denotes the adjoint on \( L^2(\mathbb{R}^n, p\,dx) \) and \( A \) is the generator of dilations on \( \mathbb{R}^n \). On \( L^q(\mathbb{R}^n, p\,dx) \), \( 1 < q < \infty \), the spectrum of \( N_0 \) is \( \mathbb{N} \), and the Hermite polynomials on \( \mathbb{R}^n \) form an orthogonal basis of eigenvectors of \( e^{-tN_0} \) in \( L^2(\mathbb{R}^n, p\,dx) \).

The generator \( N_0 \) has a fruitful generalization in (commutative or non commutative) analysis on deformed or \( q \)-Fock spaces, namely the number operator \( N \), i.e. the second differential quantization of identity. A substitute of Mehler formula holds and \((e^{-tN})_{t>0}\) is the compression of a one parameter group of unitary dilations, see e.g. [LP2].

Our motivation in this paper is to exploit Mehler formula in another direction: in the setting of stratified groups Mehler formula still defines a semi-group \((e^{-tN})_{t>0}\) and we study which properties of the classical O-U semi-group remain valid. We also hope that this semi-group might throw some light on properties of the heat density \( p \).

**Results and organization of the paper**

In section 2 we recall some properties of the self-adjoint semi-group on \( L^2(pd\gamma) \) whose generator is \(-\nabla^*\nabla = -\sum_{i=1}^{n} X_i^*X_i \), \( X_i^* \) being the formal adjoint of \( X_i \) with respect to \( L^2(pd\gamma) \). We give in passing a simple proof of the known Poincaré inequality in \( L^2(pd\gamma) \).

In the main section 3 we consider another generalization, the Mehler semi-group, which is defined for \( t \geq 0 \) by (theorem 3)

\[
T_t(f)(\gamma) = \int_{G} f(\delta_{e^{-t}\gamma}\delta_{\sqrt{e^{-t}g}})p(g)dg = e^{-tN}(f)(\gamma).
\]

Some properties are described in 3.2, in particular \( pd\gamma \) is an invariant measure. This semi-group is not selfadjoint on \( L^2(pd\gamma) \), but formally the real part of its generator \(-N\) is \(-\nabla^*\nabla \) and \( N = L + A \) where \( A \) is the generator of the group \((\delta_{e^t})_{t\in\mathbb{R}}\) of dilations, studied in 3.3.
We show in 3.4 that every $T_t$, $t > 0$, is compact on $L^q(p\gamma)$, $1 < q < \infty$, (proposition 6), with common spectrum $e^{-tN}$ on the closed subspace spanned by polynomials (theorem 7), which coincides with the whole space only if the number of layers of $\mathcal{G}$ is $\leq 4$ (proposition 8). We describe the eigenspaces in this case.

In 3.5 we give another description of these eigenspaces if $G$ is step two, similar to the usual definition of one variable Hermite polynomials by their generating function.

**More notation**

We denote $G = V_1 \oplus \ldots \oplus V_k$, where $V_1, \ldots, V_k$ are the layers of the Lie algebra $\mathcal{G}$ of $G$, $V_k = \mathcal{Z}$ being the central layer, so that [FS, p. 5]

$$[V_j, V_h] \subset V_{j+h}, \ [V_1, V_k] = V_{h+1}, 1 \leq h < k$$

The homogeneous dimension of $G$ is

$$Q = \sum_{j=1}^{k} j \dim V_j.$$ 

Generic elements of the layers are denoted respectively by $X, Y, \ldots, U$, and respective basis of the layers are denoted by $(X_1, \ldots, X_n)$, $(Y_1, \ldots, Y_m)$, $\ldots$, $(U_1, \ldots, U_k)$. Such a basis is also denoted by $(Z_j)_{1 \leq j \leq N}$. We denote accordingly

$$g = \exp(\sum x_iX_i + \sum y_iY_i + \ldots + \sum u_iU_i) = \exp(X + Y + \ldots + U)$$

$$= (x, y, \ldots, u) = \exp(\sum_{j=1}^{N} z_jZ_j) = (z_j)_{j=1}^{N},$$

since the mapping $(z_j)_{j=1}^{N} \rightarrow g$ is a diffeomorphism: $\mathbb{R}^N \rightarrow G$.

We denote by $\mathcal{P}$ the space of polynomials on $G$, as defined in [FS, chapter I-C] for the fixed basis $(Z_j)_{j=1}^{N}$: they are polynomials w.r. to the coordinates $z_j, 1 \leq j \leq N$.

The dilation $\delta_t$, $t \geq 0$, are defined on $\mathcal{G}$ and $G$ by

$$\delta_t(X + Y + \ldots + U) = tX + t^2Y + \ldots + t^kU, \ \delta_t(\exp Z) = \exp \delta_t(Z), \ Z \in \mathcal{G}.$$ 

For a function $f$ on $G$, 

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[FS]: [The reference should be provided here.]
\[ \delta_t(f) = f \circ \delta_t. \]

The generator \( A \) of the one parameter group \((\delta_s)_{s \in \mathbb{R}}\) of dilations on \( G \) satisfies: for \( f \in \mathcal{S}(G) \) and \( s > 0 \)

\[
\frac{d}{dt} |_{t=1} f \circ \delta_t = A(f) = -tt^A \frac{d}{dt} t^{-A}(f) = -t \delta_t \frac{d}{dt} (f \circ \delta_t). \tag{4}
\]

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## 2 The semi-group \( e^{-t\nabla^*\nabla} \) on \( L^2(pdg) \)

This semi-group has already been introduced in [BHT], under a probabilistic point of view, in connection with some Markov processes on Lie groups. We use instead an analytic point of view as in [O]. We consider this semi-group firstly because it is a natural generalization of the classical O-U semi-group, secondly because its generator \( \nabla^*\nabla \) is the real part of the generator \( N \) we shall study in part 3, see theorem 3.

### 2.1 Definition and some properties

We consider the (closed) accretive sesquilinear form

\[
a(f, h) = \int_G (\nabla f, \nabla h) pdg = \int_G \sum_{i=1}^{n} X_i f \overline{X_i h} pdg
\]

whose (dense) domain in \( L^2(pdg) \) is the Hilbert space

\[
H^1(p) = \{ f \in L^2(pdg) \mid X_i f \in L^2(pdg), 1 \leq i \leq n \}
\]

equipped with the norm \( \| f \|_{H^1(p)}^2 = \| f \|_{L^2(p)}^2 + \| \nabla f \|_{L^2(p)}^2 \); this form is continuous on \( H^1(p) \times H^1(p) \).

Hence [O] proposition 1.51, theorem 1.53] it defines an operator, which we denote by \( \nabla^*\nabla \), such that \( -\nabla^*\nabla \) is the generator of a strongly continuous semi-group of contractions on \( L^2(pdg) \); moreover this semi-group is holomorphic on the sector \( \Sigma_{\frac{\pi}{2}} = \{ \arg z < \frac{\pi}{2}, z \neq 0 \} \), and \( e^{-z\nabla^*\nabla} \) is a contraction on \( L^2(pdg) \) for \( z \in \Sigma_{\frac{\pi}{2}} \). Obviously, on \( \mathcal{S}(G) \),
\[ \nabla^* \nabla = \sum_{i=1}^{n} X_i^* X_i = L - \sum_{i=1}^{n} \frac{X_i p}{p} X_i = L - B. \quad (5) \]

Since \( X_i \) is a derivation, the chain rule holds, hence \( X_i(f^+ - f^-) = (X_i f)_{f > 0} \) by the same proof as for usual derivations on \( \mathbb{R}^N \) \cite[proposition 4.4]{O}, and \( a(f^+, f^-) = 0 \); since the form \( a \) also preserves real valued functions, the semi-group \( e^{-t \nabla^* \nabla} \) is positivity preserving \cite[theorem 2.6]{O}. Since \( e^{-t \nabla^* \nabla}(1) = 1 \), the semi-group is thus contracting on \( L^\infty(pdg) \). Since moreover \( \nabla^* \nabla \) is self-adjoint, \( e^{-t \nabla^* \nabla} \) is measure preserving, i.e.

\[ \int_G e^{-t \nabla^* \nabla}(f) pdg = \int_G f pdg, \quad t > 0, \]

so it extends as a contraction semi-group on \( L^1(pdg) \) hence on \( L^q(pdg), 1 < q < \infty \) by interpolation.

2.2 Poincaré inequality in \( L^2(pdg) \)

Poincaré inequality \cite[theorem 4.2]{DM} means that the spectrum of \( \nabla^* \nabla \) on \( L^2(pdg) \) lies in \( \{0\} \cup [C^{-1}, \infty] \): there exists \( C > 0 \) such that, for \( f \in \mathcal{S}(G) \),

\[ \left\| f - \int_G f \right\|_{L^2(pdg)}^2 \leq C \int_G |\nabla f|^2 pdg = C \int_G f (\nabla^* \nabla f) pdg. \quad (6) \]

(6) follows from the inequality (used for \( q = 2 \)) \cite[theorem 4.1]{DM}

\[ |\nabla(e^{-tL} f)|^q \leq C_q e^{-tL} (|\nabla f|^q), \quad 1 < q < \infty, \quad (7) \]

which B. Driver and T. Melcher proved, first for \( H_1 \), then for nilpotent groups \( G \) (see T. Melcher’s thesis), using Malliavin calculus. See also \cite{BHT} for some extensions.

We shall show in proposition \ref{prop1} that (7) also follows easily from gaussian estimates of \( p \) and \( \nabla p \).

Using the explicit formula for the Carnot-Caratheodory distance, H.Q. Li \cite[corollary 1.2]{Li} obtained (7) for \( q = 1 \), on the 3-dimensional Heisenberg group \( G = H_1 \). As well known \cite[théorème 5.4.7]{A}, this implies Log-Sobolev inequality for the measure \( pdg \) on \( H_1 \) and (6). Another proof of this Log-Sobolev inequality for \( H_1 \), hence for \( H_k \), is given in \cite[theorem 7.3]{HZ}.
Proposition 1 \[ DM \] Let \( G \) be a stratified group. Then \([7]\) and Poincaré inequality \([a]\) hold true.

Proof: By \([DM]\) theorem 4.2, proposition 2.6, lemma 2.3 it is enough to prove \([7]\) for \( t = \frac{1}{2} \), at \( \gamma = 0 \). Hence, it is enough to prove, for an element \( X \) of the basis of \( V_1 \), and \( f \in S(G) \),

\[
\left| X(e^{-\frac{1}{2} L} f)(0) \right| = |X(f * p)(0)| = \left| \int_G (\tilde{X} f)(g)p(g)dg \right| \leq C_{q,X} \| \nabla f \|_{L^q(pdg)};
\]

here \([FS]\) p. 22 and proposition 1.29

\[
(\tilde{X} f)(g) = \frac{d}{dt} |_{t=0} f((\exp tX)g), \quad \tilde{X} = X + \sum_{j>n} Q_{X,j} Z_j
\]

where \((Z_j)_{j=1}^N\) is a basis of \( G \) respecting the layers and \( Q_{X,j} \) is a polynomial (with homogeneous degree \( h - 1 \) if \( Z_j \in V_h, 2 \leq h \leq k \)).

Since \([V_1, V_{h-1}] = V_h, 2 \leq h \leq k \), we may choose \( Z_j \in V_h \) such that \( Z_j = [Y, A] \), where \( Y \) is an element of the basis of \( V_1 \) and \( A \in V_{h-1} \). Then

\[
\left| \int_G Z_j f(g) Q_{X,j}(g)p(g)dg \right| \leq \left| \int_G Y f A(Q_{X,j}p)dg \right| + \left| \int_G A f Y(Q_{X,j}p)dg \right|.
\]

Iterating for \( A \in V_1 + ... + V_{k-1} \) and so on, \( \left| \int_G (\tilde{X} f)(g)p(g)dg \right| \) is finally less than a finite number (which does not depend on \( f \)) of terms \( \left| \int_G Y f Z(Qp)dg \right| \) where \( Y \) is an element of the basis of \( V_1, Z \in G \), and \( Q \) is a polynomial. Each of these terms can be estimated by

\[
\left| \int_G Y f Z(Qp)dg \right| \leq \| \nabla f \|_{L^q(pdg)} \left( \| ZQ \|_{L^q(pdg)} + \| Q \frac{Zp}{p} \|_{L^q(pdg)} \right)
\]

where \( \frac{1}{q} + \frac{1}{q'} = 1 \). Then \( \| ZQ \|_{L^q(pdg)} \) is finite since \( ZQ \) is a polynomial and \( p \in S(G) \). The main point is that \( \| Q \frac{Zp}{p} \|_{L^q(pdg)} \) is finite. Indeed, denoting \( d(g) = d(0, g) \) where \( d \) is the Carnot-Caratheodory distance on \( G \), one uses \([CSV]\) theorem IV.4.2 and Comments on chapter IV] for \( 0 < \varepsilon < 1 \),

\[
C_\varepsilon e^{-\frac{1}{1 - 2\varepsilon} d^2(g)} \leq p(g) \leq K_\varepsilon e^{-\frac{1}{1 - 2\varepsilon} d^2(g)}.
\]

and, for \( Z \in G \),

\[
(Zp)(g) \leq K_{\varepsilon,Z} e^{-\frac{1}{1 - 2\varepsilon} d^2(g)}.
\]

Hence \( Q \frac{Zp}{p} \) lies in \( L^r(pdg) \), \( 1 \leq r < \infty \), which ends the proof. \( \blacksquare \)
3 Definition and properties of the Mehler semigroup

3.1 Preliminaries

The next proposition extends a classical property of independent Gaussian variables and will imply the semi-group property of our family of operators.

**Proposition 2** Let $\gamma, g$ be independent $G$-valued random variables with law $\mathcal{P}_d \mathcal{G}$. Then the r.v.

$$\delta \cos \theta \gamma \delta \sin \theta g, \ 0 \leq \theta \leq \frac{\pi}{2}$$

has the same law, i.e. for any bounded borelian function $f$ on $G$,

$$\int_{G^2} f(\delta \cos \theta \gamma \delta \sin \theta g)p(\gamma)p(g)d\gamma dg = \int_G f(g)p(g)dg.$$ 

More generally, if $g_1, \ldots, g_n$ are $G$-valued i.i.d r.v. with law $\mathcal{P}_d \mathcal{G}$ and

$$\sum_{1 \leq j \leq n} a_j^2 = 1, \ (a_j \geq 0),$$

the law of \(\prod_{j=1}^{n} \delta a_j g_j\) is $\mathcal{P}_d \mathcal{G}$.

**Proof:** By two changes of variables, denoting $C = \sin \theta \cos \theta$,

$$\int_{G^2} f(\delta \cos \theta \gamma \delta \sin \theta g)p(g)p(\gamma)d\gamma dg = \frac{1}{CQ} \int_{G^2} f(\gamma' g')p(\delta \cos \frac{\gamma'}{C^2} \gamma')p(\delta \sin \frac{\gamma'}{C^2} g')d\gamma' dg'$$

$$= \frac{1}{CQ} \int_{G^2} f(g)p(\delta \cos \frac{\gamma'}{C^2} \gamma')p(\delta \sin \frac{\gamma'}{C^2} (\gamma'^{-1} g'))d\gamma' dg$$

$$= \int_G f(g)(p_{\cos^2 \theta} * p_{\sin^2 \theta})(g)dg$$

$$= \int_G f(g)p(g)dg.$$ 

The second assertion follows by iteration.

**Remark 1:** A central limit theorem for i.i.d centered random variables with values in a stratified group $G$ and law $\mu$ with order 2 moments is proved in [CR, theorem 3.1]. The density $p$ of the limit law is the kernel at time 1 of a diffusion semi-group whose generator satisfies (2).

**Remark 2:** If $X, Y$ are i.i.d standard Gaussian vectors with values in $\mathbb{R}^n$, the couple $(X \cos \theta + Y \sin \theta, \frac{\partial}{\partial \theta} (X \cos \theta + Y \sin \theta))$ has the same joint law as
(X, Y). This fact implies, in the O-U case, that \(\cos^{N_0} \theta\) is the compression of the isometry \(R_\theta\) of \(L^2(\mathbb{R}^n \times \mathbb{R}^n, p(x)p(y)dx\,dy)\) defined by

\[
R_\theta(F)(x, y) = F(x \cos \theta + y \sin \theta, -x \sin \theta + y \cos \theta)
\]

and \((R_\theta)_{\theta \in \mathbb{R}}\) is a one parameter group preserving the measure \(p(x)p(y)dx\,dy\). This point of view was exploited e.g. in [P, theorem 2.2] in order to get a concentration inequality for the gaussian measure.

In the stratified setting we were not able to exhibit explicit unitary dilations for the Mehler operators \(T_t\) defined below.

### 3.2 The Mehler semi-group

We now define the Mehler semi-group on \(L^q(G, pd\gamma)\).

**Theorem 3** Let \(L,\) defined by (4), be a sub-Laplacian on a stratified group \(G,\) and let \(p\) be the kernel of \(e^{-\frac{L}{2}}\).

a) The family of operators \((T_t)_{t \geq 0}\) defined on \(\mathcal{S}(G)\) by

\[
T_t(f)(\gamma) = \int_G f(\delta_{e^{-t}\gamma} \delta_{\sqrt{1-e^{-2t}}} g)p(g)dg = e^{-\frac{t}{4}(1-e^{-2t})}(f)(\delta_{e^{-t}\gamma})
\]

(10)
is a semi-group whose generator \(-N\) is defined on \(\mathcal{S}(G)\) by

\[
N = L + A.
\]

(11)
b) The probability measure \(pd\gamma\) is invariant by \((T_t)_{t \geq 0}\) i.e.

\[
\int_G T_t(f)(\gamma)p(\gamma)d\gamma = \int_G f(\gamma)p(\gamma)d\gamma
\]

(12)

and, for \(f \in \mathcal{S}(G),\) \(\int_G (Nf)p\,dg = 0.\)

c) \((T_t)_{t \geq 0}\) extends as a Markovian semi-group of contractions on \(L^q(G, pd\gamma), 1 \leq q \leq \infty,\) strongly continuous if \(q \neq \infty.\)

d) If \(f \in L^q(pd\gamma), 1 \leq q < \infty,\)

\[
\left\| T_t(f) - \int_G f pd\gamma \right\|_{L^q(pd\gamma)} \to_{t \to \infty} 0.
\]
c) \((T_t)_{t>0}\) is not self-adjoint on \(L^2(G, \mu)\) as soon as \(G\) is not abelian. Formally \(\nabla^* \nabla\) is the real part of \(N\), i.e., for \(f, h \in \mathcal{S}(G)\),

\[
\langle Nf, h \rangle_{L^2(\mu)} = \langle (\nabla^* \nabla + iC) f, h \rangle_{L^2(\mu)}
\]

where \(C\) is a non zero first order differential operator satisfying \(\langle Cf, h \rangle = \langle f, Ch \rangle\). In particular, for \(f \in \mathcal{S}(G)\),

\[
\Re \int_G (Nf)fpd\gamma = \int_G |\nabla f|^2 pd\gamma = \int_G (\nabla^* \nabla f)fpd\gamma.
\]

If moreover \(f\) is real valued, the left integral is real.

By the change of notation \(e^{-t} = \cos \theta, \ < \theta < \pi/2\), (10) can be rewritten as

\[
\cos^N \theta(f)(\gamma) = \int_G f(\delta_{\cos\theta\gamma\delta_{\sin\theta}}g)p(g)dg = \delta_{\cos\theta} \circ e^{-\frac{1}{2} \sin^2 \theta \mu}(f)(\gamma).
\]  

(13)

Proof: a) Let \(\varphi(g') = T_t(f)(g')\); we compute

\[
T_s(\varphi)(\gamma) = \int_G \varphi(\delta_{e^{-s\gamma}} \delta_{\sqrt{1-e^{-2s}}}h)p(h)dh
\]

\[
= \int_G \int_{G^2} f(\delta_{e^{-s\gamma}} \delta_{\sqrt{1-e^{-2s}}}h \delta_{\sqrt{1-e^{-2s}}}g)p(g)p(h)dgdh
\]

\[
= \int_G f(\delta_{e^{-(s+t)\gamma}} \delta_{\sqrt{1-e^{-2(s+t)}}}k)p(k)dk = T_{s+t}(f)(\gamma)
\]

where the third equality comes from proposition \(\mathbb{2}\) applied to \((h, g)\).

By the chain rule applied to (10),

\[
Nf = -\frac{d}{dt} \bigg|_{t=0} T_t(f) = Lf + A(f).
\]

b) Proposition \(\mathbb{2}\) gives (12). Differentiating (12) at \(t = 0\) for \(f \in \mathcal{S}(G)\) implies

\[
\int_G (Nf)pd\gamma = 0.
\]

Another proof will be given in Remark 3.
c) $T_t$ is contracting both on $L^1(G, pd\gamma)$, since it is positivity and measure preserving, and on $L^\infty(G, pd\gamma)$, since it is positivity preserving and $T_t(1) = 1$. Hence $T_t$ is contracting on $L^q(G, pd\gamma)$, $1 \leq q \leq \infty$ by interpolation.

Since $\mathcal{D}(G)$ is norm dense in $L^q(G)$, it is norm dense in $L^q(pd\gamma)$, $1 \leq q < \infty$ : indeed, if $F \in L^q(pd\gamma)$ $(\frac{1}{q} + \frac{1}{q'} = 1)$ and $\int_G fFpd\gamma = 0$ for every $f \in \mathcal{D}(G)$, then $Fp \in L^q(G)$ hence $Fp = 0$ $d\gamma$ a.s.. Writing $e^{-t} = \cos \theta$, one has, for $f \in \mathcal{D}(G)$,

$$\|T_t(f) - f\|^q_{L^q(pd\gamma)} = \left\| \int_G [f(\delta_{\cos \theta}g\delta_{\sin \theta}g) - f(\gamma)]p(g)dg \right\|^q_{L^q(pd\gamma)} \leq \int_{G^2} |f(\delta_{\cos \theta}g\delta_{\sin \theta}g) - f(\gamma)|^q p(\gamma)p(g)d\gamma dg,$$

which converges to 0 as $\theta \to 0$ by the dominated convergence theorem. Since $T_t$ is contracting, the strong continuity on $L^q(pd\gamma)$ follows by density.

d) Similarly, if $f$ is bounded and continuous on $G$,

$$f(\delta_{e^{-t}\gamma} \delta_{\sqrt{1-e^{-2t}}}g) \to_{t \to \infty} f(g);$$

by dominated convergence theorem $T_t(f) \to_{t \to \infty} \int_G f(g)p(g)dg$ pointwise and in the norm of $L^q(pd\gamma)$. The claim follows by density.

e) By (11), (5) and lemma 4 below, for $f \in \mathcal{S}(G)$,

$$(N - \nabla^*\nabla)f = A(f) + \sum_{1 \leq j \leq n} \frac{X_j p}{p} X_j f = \sum_{1 \leq j \leq N} b_j Z_j f$$

where the functions $b_j$ are not all zero if $j > n = \text{dim} V_1$. Hence for $h \in \mathcal{S}(G)$,

$$\int_G (N - \nabla^*\nabla)(f) \overline{h} pdg = - \int_G f \left[ \sum_{1 \leq j \leq N} b_j(g)(Z_j \overline{h})p + \overline{h} Z_j(b_j p) \right] dg.$$ 

By b), the left hand side is zero for $h = 1$, hence $\sum_{1 \leq j \leq N} Z_j(b_j p) = 0$. Since $T_t$ preserves real valued functions, so does $N$, hence

$$\int_G (N - \nabla^*\nabla)(f) \overline{h} pdg = - \int_G f(N - \nabla^*\nabla)(h) pdg = - \int_G f(N - \nabla^*\nabla)(h) pdg,$$

which proves $(iC)^* = -iC$, where $iC = N - \nabla^*\nabla = A + B$. The remaining assertions are obvious.
Remark 3: We now give another instructive proof of \( \int_G (Nf)pdg = 0, f \in \mathcal{S}(G) \), hence of (12). We claim that, for \( f, h \in \mathcal{S}(G) \),
\[
\int_G (Nf)hdg = \int_G f [L(h) - Qh + \frac{d}{ds} \bigg|_{s=1} h \circ \delta_s^1]dg = \int_G f (L - QId - A)(h)dg.
\]
Indeed, \( N = L + A, L \) is formally selfadjoint on \( L^2(dg) \) and the claim follows by differentiating at \( s = 1 \) the right hand side of
\[
\int_G f(\delta_s^1 \gamma)h(\gamma)d\gamma = s^{-Q} \int_G f(\gamma')h(\delta_s^1 \gamma)d\gamma'.
\]
By (4) and [LP, lemma 2], \( p \) may be precisely defined as the unique solution in \( L^1(G) \), satisfying \( \int_G p(g)dg = 1 \), of
\[
(L - QId - A)(p) = Lp - Qp + s\delta_s^1 \frac{d}{ds}(p \circ \delta_s^1) = 0.\]

Remark 4: As already mentioned in section 2.2, Log-Sobolev inequality for \( pd\gamma \) is known for \( H_k \). It is equivalent both to hypercontractivity of \( e^{-tN} \) and to hypercontractivity of \( e^{-t\nabla^* \nabla} \) on \( H_k \), since \( p \) is an invariant measure for these markovian semigroups and \( N, \nabla^* \nabla \) are diffusion operators [A, theorem 2.8.2].

3.3 The generator of dilations

We may identify \( G \) with a group of finite matrices [V, theorem 3.6.6]. The derivation formula for an exponential of a matrix valued function, see e.g. [H, theorem 69], applied to a smooth function \( Z(s): \mathbb{R} \to \mathcal{G} \), where \( \mathcal{G} \) has \( k \) layers, gives
\[
\frac{d}{ds} \exp Z(s) = \lim_{h \to 0} \frac{\exp Z(s + h) - \exp Z(s)}{h} = \exp(Z(s)) V(Z(s)),
\]
where
\[ V(Z(s)) = (d \exp)_Z(Z'(s)) = Z'(s) + \sum_{l=1}^{k-1} \frac{(-1)^l}{(l+1)!} (AdZ(s))^l(Z'(s)). \] (15)

Hence

\[ \exp Z(s + h) = \exp Z(s) \exp h[V(Z(s)) + o(1)], \]

which entails for \( f \in C^\infty(G) \)

\[ \frac{d}{ds} f(\exp Z(s)) = V(Z(s))(f)(\exp Z(s)). \] (16)

**Lemma 4** Let \( A \) be the generator of the group of dilations \( (\delta_e^t)_{t \in \mathbb{R}} \). Then

\[ A(f)(g) = \sum_{1 \leq j \leq N} a_j(g) Z_j f(g) \]

where the functions \( a_j \) are polynomials w.r. to the coordinates of \( g \), and are not all zero for \( j > n = \dim V_1 \).

Proof: Assume that \( G \) has \( k \) layers, \( k \geq 2 \). Let

\[ \delta_s g = \exp(sX + s^2Y + \ldots + s^kU) = \exp Z(s). \]

By (16) \( A = V(Z(1)) \). Noting that \( Z' - Z \in V_2 + \ldots + V_k \), we get \( (AdZ(1))^l(Z'(1)) \in V_3 + \ldots + V_k, l \geq 1 \). So \( V(Z(1)) - (X + 2Y) \) lies in \( V_3 + \ldots + V_k \).

**Notation:** We denote by \( P_n \) the (finite dimensional) space of homogeneous polynomials on \( G \) with homogeneous degree \( n, n \in \mathbb{N} \), i.e. satisfying

\[ \delta_s(P) = s^n P, \ P \in P_n; \] (17)
equivalently, \( P_n \) is the eigensubspace of \( A \) on \( P \) associated to \( n \). The finite dimensional subspaces \( B_n = P_0 + \ldots + P_n \) are stable under \( L \) and dilations, hence under \( e^{-\frac{iL}{\hbar}} \) and \( \cos^N \theta \) by (10), these operators being naturally extended on \( \mathcal{S}'(G) \). In particular \( e^{-\frac{iL}{\hbar}} \) is well defined on \( B_n \) and is the inverse of \( e^{-\frac{iL}{\hbar}} \), which is thus one to one on every \( B_n \) hence on \( P = \cup_{n \geq 0} B_n \).

The next lemma is the key for the computation of the spectrum of \( \cos^N \theta \). It will be exploited again in section 3.5.
Lemma 5

a) The generator $A$ of dilations on $G$ satisfies $[L, A] = 2L$ on $\mathcal{C}^\infty(G)$.

b) $e^{-\frac{L}{2}} \circ \cos^N \theta = \delta_{\cos \theta} e^{-\frac{L}{2}}$ on $S'(G)$.

c) The set of polynomials $e^{\frac{L}{2}}(P_n)$ is a space of eigenvectors of $\cos^N \theta$ associated to the eigenvalue $\cos^n \theta, n \geq 0$.

Proof: a) We rewrite (2) as

$$Le^{tA} = e^{2t} e^{tA} L, \quad t \in \mathbb{R},$$

and a) follows by differentiating at $t = 0$.

b) By (3), on $S(G)$, hence on $S'(G)$, for $t > 0$,

$$e^{-\frac{L^2}{2} L} = \delta_1 \circ e^{-\frac{L}{2}} \circ \delta_t. \quad (18)$$

Hence, on $S'(G)$, by (10) and (18) applied to $t = \cos \theta$,

$$e^{-\frac{L}{2}} \circ \cos^N \theta = e^{-\frac{L}{2}} \circ \delta_{\cos \theta} \circ e^{-\frac{\sin^2 \theta}{2} L} = \delta_{\cos \theta} \circ e^{-\frac{L}{2}}.$$

c) Since $e^{-\frac{L}{2}}$ is invertible on $\mathcal{P}$, and $\mathcal{P}$ is stable under $\cos^N \theta$, b) implies on $\mathcal{P}$

$$\cos^N \theta \circ e^{\frac{L}{2}} = e^{\frac{L}{2}} \circ \delta_{\cos \theta}.$$

Applying this to $\mathcal{P}_n$ proves the result.

3.4 Compacity and spectrum of $\cos^N \theta$ on $L^q(pd\gamma)$

Proposition 6 Let $\cos^N \theta$ be defined by (13). Then

a) $\cos^N \theta$ is a Hilbert-Schmidt operator on $L^2(pd\gamma)$.

b) $\cos^N \theta$ is compact on $L^q(pd\gamma), 1 < q < \infty$; its non zero eigenvalues and corresponding eigenspaces are the same on $L^2(pd\gamma)$ and $L^q(pd\gamma)$. In particular its spectrum $\sigma(\cos^N \theta)$ does not depend on $q$ and

$$\sigma(\cos^N \theta) = (\cos \theta)^{\sigma(N)} \cup \{0\}.$$

Actually, $\cos^N \theta$ is a trace class operator on $L^2(pd\gamma)$ by a) and the semi-group property of $(e^{-tN})_{t>0}$.

Proof: a) We must show that the kernel of $\cos^N \theta$ lies in $L^2(G \times G, pd\gamma \otimes pdg)$. For fixed $\gamma$ and $\theta, 0 < \theta < \frac{\pi}{2}$,
\[
\int_G f(\delta_{\cos \theta} \delta_{\sin \theta}) p(g) dg = \frac{1}{\sin \beta} \int_G f(z) p(\delta_{\cos \theta} \gamma^{-1} \delta_{\sin \theta}) dz,
\]
so we must prove the convergence of the integral
\[
I(\theta) = \int_{C^2} p^2(\delta_{\cos \theta} \gamma^{-1} \delta_{\sin \theta}) \frac{p(\gamma)}{p(z)} dz d\gamma.
\]
By the gaussian estimates \([8]\)
\[
\frac{C_\varepsilon}{K_\varepsilon} p^2(\delta_{\cos \theta} \gamma^{-1} \delta_{\sin \theta}) \frac{p(\gamma)}{p(z)} \leq \exp \left( \frac{d^2(z)}{2-2\varepsilon} - \frac{d^2(\gamma)}{2+2\varepsilon} - \frac{d^2(\delta_{\cos \theta} \gamma^{-1} \delta_{\sin \theta})}{1+\varepsilon} \right) = \exp \beta.
\]
The Carnot distance \(d\) satisfies
\[
d(g) \leq d(\gamma^{-1}g) + d(\gamma) \text{ and } d(\delta_t g) = td(g).
\]
Hence
\[
(1+\varepsilon) \beta \leq \frac{d^2(z)}{2(1-\varepsilon)^2} - \frac{d^2(\gamma)}{2} - \left( \frac{1}{\sin \theta} d(z) - \frac{\cos \theta}{\sin \theta} d(\gamma) \right)^2
\]
\[
\leq d^2(z) \left( \frac{1}{2} - \frac{1-\cos \theta}{\sin^2 \theta} \right) + d^2(\gamma) \left( \frac{\cos \theta - \cos^2 \theta}{\sin^2 \theta} - \frac{1}{2} \right).
\]
Since \(\frac{1-\cos \theta}{\sin^2 \theta} > \frac{1}{2}\) on \([0, \frac{\pi}{2}]\), the coefficient of \(d^2(\gamma)\) is strictly negative, and so is the coefficient of \(d^2(z)\) for small enough \(\varepsilon > 0\). Hence, for some \(c, C > 0\),
\[
I(\theta) \leq C \int \int_{C^2} e^{-c(d^2(z)+d^2(\gamma))} dz d\gamma = C \left( \int_G e^{-cd^2(z)} dz \right)^2.
\]
By the left hand side of \([8]\), for small \(\varepsilon\),
\[
C_\varepsilon \int_G e^{-cd^2(z)} dz \leq \int_G p^{2c(1-\varepsilon)}(z) dz,
\]
and the last integral is finite since \(p \in S(G)\). This proves a).

b) By interpolation, since \(\cos^N \theta\) is compact on \(L^2(p(g) dg)\) and bounded on \(L^\infty(pdg)\) and \(L^1(pdg)\), it is compact on \(L^q(pdg), 1 < q < \infty\), with the same spectrum and the same eigenspaces associated to non zero eigenvalues \([D]\) theorems 1.6.1 and 1.6.2).

By the compacity on \(L^q(pdg)\), the set of these eigenvalues is \(\{ \cos^\lambda \theta \mid \lambda \in \sigma_q(N) \}\) where \(\sigma_q(N)\) denotes the spectrum of \(N\) on \(L^q(pdg)\) \([L]\) chap. 34.5, theorem 13]. Hence \(\sigma_q(N) = \sigma_2(N)\) is discrete and lies in \(\{ \lambda \in \mathbb{C} \mid \Re \lambda \geq 0 \}\) since \(\cos^N \theta\) is contracting on \(L^2(pdg)\) (or since \(\Re \langle Nf, f \rangle \geq 0\)).
Theorem 7  Let $G$ be a step $k$ stratified group.

1) If $k \leq 4$
   a) the spectrum of $\cos^N \theta$ on $L^2(pdg)$ is $\sigma(\cos^N \theta) = (\cos \theta)^N \cup \{0\}$ and $\sigma(N) = \mathbb{N}$.
   b) the corresponding eigenspaces $E_n, n \geq 0$, (which are not pairwise orthogonal in $L^2(pdg)$) are
      
      $$E_n = e^{\frac{i}{2}L}(P_n).$$

2) If $k > 4$, assertions a) b) remain true for the restriction of $\cos^N \theta$ to the closed subspace $L^2_P(pdg)$ spanned by polynomials.

If $k = 1$ polynomials in $E_n$ are the Hermite polynomials with degree $n$.
Proof: 1) follows from 2) and proposition 8 below.
2) We first define $E_n$ by $E_n = e^{\frac{i}{2}L}(P_n)$. By lemma 5 $E_n$ lies in the eigenspace of $\cos^N \theta$ associated to the eigenvalue $\cos^n \theta$. By proposition 6 $\cos^N \theta$ is compact on $L^2_P(pdg)$. The claim then follows from the following facts:

Let $T : E \to E$ be a compact operator on an infinite dimensional Banach space $E$; let $\Lambda$ be a set of eigenvalues of $T$ and let $E_\lambda, \lambda \in \Lambda$, be eigensubspaces whose union is total in $E$. Then

a) the spectrum of $T$ is $\Lambda \cup \{0\}$
   b) for $\lambda \in \Lambda, E_\lambda$ is the whole eigenspace associated to $\lambda$.

Indeed, assume that $T$ has an eigenvalue $\lambda_0 \notin \Lambda$. Then $T - \lambda_0 I$ has a closed range with non zero finite codimension (see e.g. [L, chap. 21.1, theorems 3, 4]). But this range contains the linear span of the $E_\lambda$’s, $\lambda \in \Lambda$, hence is the whole of $E$. This is a contradiction, which proves a).

Let $\lambda_0 \in \Lambda$; since $E_{\lambda_0}$ is stable under $T$, $T$ acts on the quotient space $E/E_{\lambda_0}$ and is still compact. The $E_\lambda$’s, $\lambda \in \Lambda \setminus \{\lambda_0\}$ span a dense subspace of $E/E_{\lambda_0}$. Applying a) to $E/E_{\lambda_0}, \lambda_0$ cannot belong to the spectrum of $T$ on the quotient space, which proves b).

The proof of the next proposition is essentially due to W. Hebisch (private communication).

Proposition 8  Let $G$ be a stratified group. Then the polynomials are dense in $L^2(pdg)$ if and only if $G$ is step $k$ with $k \leq 4$.

Proof: 1) We recall that polynomials are dense in $L^2(\mathbb{R}, e^{-c|x|^\alpha} dx)$ if and only if $\alpha \geq \frac{1}{2}$: obviously, this does not depend on $c$ and is equivalent to the density of polynomials in $L^2(\mathbb{R}^+, e^{-x^\alpha} dx)$. If $0 < \alpha < \frac{1}{2}$, [PS, Part III, problem 153] produces a non zero bounded function $g_\alpha$ which is orthogonal to polynomials.
in $L^2(\mathbb{R}^+, e^{-\cos(\alpha x)x^\alpha} dx)$. If $\alpha \geq \frac{1}{2}$, the result follows from the trick of [Ham, p 197-198]. Indeed, if $\psi \in L^2(\mathbb{R}^+, e^{-x^\alpha} dx)$ and $\alpha \geq \frac{1}{2}$, the function

$$F(z) = \int_{\mathbb{R}^+} \psi(x)e^{\sqrt{\pi}z}e^{-x^\alpha} dx = \int_{\mathbb{R}^+} \psi(y^2)e^{\sqrt{\pi}z}e^{-y^{2\alpha}} y dy$$

is bounded and holomorphic on $\{\Re z < \beta\}$ for some $\beta > 0$, by Cauchy-Schwarz inequality.

Expanding $z \to e^{\sqrt{\pi}z}$ in power series, one gets $F(-z) = -F(z)$ if $\psi$ is orthogonal to polynomials in $L^2(\mathbb{R}^+, e^{-x^\alpha} dx)$. Thus $F$ extends as a bounded entire function, which must be zero by Liouville theorem since $F(0) = 0$. Hence the Fourier transform of $y \to \psi(y^2)e^{-y^{2\alpha}} y$ is zero, i.e. $\psi = 0$ a.s..

2) We identify $g = \exp Z \in G$ with the coordinates $(x, y, \ldots, w)$ of $Z$ w.r. to a basis respecting the layers and denote

$$\eta(g) = \sum_{i \leq l} |x_i|^2 + \sum_{i \leq m} |y_i|^2 + \ldots + \sum_{i \leq r} |w_i|^2.$$ 

Obviously $\eta(\delta g) = s^2 \eta(g)$, in particular $\eta(g) = d^2(g)\eta(\delta_{\frac{1}{|z|}}g)$, $d$ denoting the Carnot distance. Since $\eta$ is strictly positive and bounded on the $d$-unit sphere of $G$, there exist constants $c', C' > 0$ such that

$$c'\eta(g) \leq d^2(g) \leq C'\eta(g).$$

By (8) there exist constants $c, C > 0$ such that the following embeddings

$$L^2(e^{-C\eta(g)} dg) \rightarrow L^2(pdg) \rightarrow L^2(e^{-C\eta(g)} dg)$$

are continuous, with dense ranges since $\mathcal{D}(G)$ is dense in the three spaces.

3) The algebraic tensor product

$$\mathcal{E} = \otimes_{i \leq l} L^2(e^{-C\eta^2} dx_i) \otimes \ldots \otimes_{i \leq p} L^2(e^{-C|w_i|^2} dw_i),$$

is dense in $L^2(e^{-C\eta(g)} dg)$. For $k \leq 4$, one variable polynomials are dense in every factor of $\mathcal{E}$ by step 1), hence polynomials are dense in $L^2(e^{-C\eta(g)} dg)$ and in $L^2(pdg)$.

Let $k \geq 5$. By 1) there exists a non zero function $g \in L^2(e^{-c|w_r|^2} dw_r)$ which is orthogonal to polynomials w.r. to $w_r$. Then $1 \otimes \ldots \otimes 1 \otimes g \in L^2(e^{-c\eta(g)} dg)$ is orthogonal to all polynomials, so polynomials are neither dense in $L^2(e^{-c\eta(g)} dg)$, nor in $L^2(pdg)$. ■
3.5 Generating functions of polynomial eigenvectors of \( N \)

The usual Hermite polynomials on \( \mathbb{R} \), denoted by \( H_n, n \in \mathbb{N} \), are the eigenvectors of the Ornstein-Uhlenbeck operator \( N_0 \), and have the generating function

\[
e^{ixt + \frac{1}{2}t^2} = \sum_{n \geq 0} \frac{(it)^n}{n!} H_n(x) = e^{\frac{1}{2}t^2} e^{\Delta (e^{ixt})} = e^{\frac{1}{2}t^2} \circ \delta_t (e^{ix})\]

noting that \( x \to e^{ix} \) is a bounded eigenvector of \( \Delta \). In particular

\[
i^n H_n(x) = \left. \frac{d^n}{dt^n} \right|_{t=0} e^{\frac{1}{2}t^2} \circ \delta_t (e^{ix}).
\]

We shall verify (proposition 11) that a similar formula gives polynomial eigenvectors of \( N \). When \( G \) is step two, these vectors are total in \( L^q(pdg), 1 \leq q < \infty \), see theorem 12 below. More precisely we give in 3.5.1 a technical lemma producing eigenvectors of \( N \) out of eigenvectors of \( L \). In 3.5.3 we use this lemma when \( \varphi \) is both an eigenvector of \( L \) and a coefficient function of a representation of \( G \) (proposition 11). We shall first gather in 3.5.2 well known facts about these functions.

3.5.1 Candidates for generating functions of eigenvectors of \( N \)

In the next lemma 9 we state technical assumptions ensuring the validity of the computation of some eigenvectors of \( N \). Using lemma 5 b), the point is to define "\( e^{\frac{1}{2}t^2} \varphi \)" for suitable functions \( \varphi : \) in lemma 5 c), we choose \( \varphi \in \mathcal{P} \), here we choose eigenvectors of \( L \).

**Lemma 9** Let \( G \) be a stratified group and let \( \varphi \in \mathcal{S}'(G) \cap \mathcal{C}^\infty(G) \) be an eigenvector of \( L \) such that \( L \varphi = \lambda \varphi \). We assume that, for \( n \geq 1 \),

\[
(i) \left. \frac{d^n}{dt^n} \right|_{t=0} \int_G \delta_t(\varphi)(\gamma g^{-1})p(g)dg = \int_G \left. \frac{d^n}{dt^n} \right|_{t=0} \delta_t(\varphi)(\gamma g^{-1})p(g)dg \\
(ii) \left. \frac{d^n}{dt^n} \right|_{t=0} \delta_t(\varphi) \text{ is a polynomial on } G.
\]

Let

\[
f_t = e^{\frac{1}{2}t^2} \delta_t(\varphi), \ t > 0; \ h_n = \left. \frac{d^n}{dt^n} \right|_{t=0} f_t.
\]

Then \( h_n \) is a polynomial on \( G \) and

\[
\cos^n \theta(h_n) = \cos^n \theta h_n.
\]
Proof: Since $\phi \in C^\infty(G)$, $t \to f_t$ is $C^\infty$ on $\mathbb{R}^+$. By (2) $L \circ \delta_t(\phi) = t^2 \lambda \delta_t(\phi)$, so that $\delta_t(\phi) = e^{-\frac{\lambda t}{2}} f_t$. By lemma 5 b)

$$e^{-\frac{\lambda}{2} \cos^N \theta} f_t = \delta_{\cos \theta} e^{-\frac{\lambda}{2} f_t} = \delta_{\cos \theta} \delta_t(\phi) = \delta_{t \cos \theta}(\phi) = e^{-\frac{\lambda}{2} f_t \cos \theta}. \quad (19)$$

We claim that

$$\frac{d^n}{dt^n} \bigg|_{t=0} e^{-\frac{\lambda}{2} \cos^N \theta} f_t = \frac{d^n}{dt^n} \bigg|_{t=0} e^{-\frac{\lambda}{2} \cos^N \theta} f_t = \frac{d^n}{dt^n} \bigg|_{t=0} e^{-\frac{\lambda}{2} \cos^N \theta} h_n. \quad (20)$$

In particular, applying (20) with $\theta = 0$, $\frac{d^n}{dt^n} \bigg|_{t=0} e^{-\frac{\lambda}{2} f_t} = e^{-\frac{\lambda}{2} h_n}$.
Hence, by (20) and (19),

$$e^{-\frac{\lambda}{2} \cos^N \theta} \cos \theta f_t \cos \theta = \delta_{t \cos \theta} \delta_t(\phi) = \delta_{t \cos \theta}(\phi) = e^{-\frac{\lambda}{2} f_t \cos \theta} \cos \theta. \quad (21)$$

By Leibnitz rule, it is enough to prove the claim for $\delta_t(\phi)$ instead of $f_t$. By lemma 5 b) we may replace $e^{-\frac{\lambda}{2} \cos^N \theta}$ in the claim by $\delta_{\cos \theta} e^{-\frac{\lambda}{2} f_t}$. The claim now follows from assumption (i).

By Leibnitz rule and assumption (ii), $h_n$ is a polynomial. So is $\cos^N \theta h_n$ and the result follows from (21) since $e^{-\frac{\lambda}{2} f_t}$ is one to one on $\mathcal{P}$.

**Remark 5:** $\phi$ and $\phi \circ \delta_\beta$, $\beta > 0$, give colinear $h_n$'s, since

$$\frac{d^n}{dt^n} \bigg|_{t=0} e^{\frac{\lambda}{2} \cos^N \theta} \delta_{\cos \theta} \delta_t(\phi) = \beta^n \frac{d^n}{dt^n} \bigg|_{t=0} e^{\frac{\lambda}{2} \cos^N \theta} \delta_{\cos \theta} \delta_t(\phi) = \beta^n h_n. \quad (22)$$

### 3.5.2 A total set of eigenvectors of $L$ in $L^q(pd\xi)$, $1 \leq q < \infty$.

Let $\Pi : G \to B(L^2(\mathbb{R}^k, d\xi))$ be a non trivial unitary irreducible representation of $G$. By definition, $F \in L^2(\mathbb{R}^k)$ is a $C^\infty$ vector for $\Pi$ if the vector valued function: $g \to \Pi(g)(F)$ is $C^\infty$ on $G$. We still denote by $\Pi$ the associated differential representation, defined for a $C^\infty$ vector $F$ and $X \in G$ by

$$X \Pi(g)(F) = \frac{d}{dt} \bigg|_{t=0} \Pi(g \exp tX)(F) = \Pi(g) \Pi(X)(F), \quad g \in G, \quad (22)$$

and $\Pi(X^m) = \Pi(X)^m$, see e.g. [CG: p.227]; by definition, $\Pi(X^m)(F)$ still lies in $L^2(\mathbb{R}^k)$ and is still a $C^\infty$ vector for $\Pi$.
$\Pi$ extends as a representation of the convolution algebra $M(G)$ by
\[ \Pi(\mu) = \int_G \Pi(g) d\mu(g). \]

In particular \((\Pi(p_t dg))_{t \geq 0}\) is a semigroup of operators on \(L^2(\mathbb{R}^k)\), whose generator is \(-\Pi(L)\). Indeed, for a \(C^\infty\) vector \(F\), by (22),

\[
- \frac{d}{dt} \int_G \Pi(g)(F)p_t(g) dg = \int_G \Pi(g)(F)(Lp_t)(g) dg = \int_G L \circ \Pi(g)(F)p_t(g) dg
\]

\[
= \int_G \Pi(g) \circ \Pi(L)(F)p_t(g) dg \rightarrow_{t \to 0^+} \Pi(L)(F).
\]

Since \(p \in \mathcal{S}(G)\), \(\Pi(pdg) = e^{-\frac{1}{2} \Pi(L)}\) is a trace class operator [CG, theorem 4.2.1]; in particular its non zero eigenvalues are \(\{e^{-\frac{1}{2} \lambda}, \lambda \in \sigma_2(\Pi(L))\}\), where \(\lambda\) runs through the eigenvalues of \(\Pi(L)\) on \(L^2(\mathbb{R}^k)\). Moreover, for \(F \in L^2(\mathbb{R}^k)\), the function \(\Pi(pdg)(F)\) is a \(C^\infty\) vector for \(\Pi\) [CG, theorem A.2.7 p. 241].

Let \(U\) be a set of non trivial unitary irreducible representations of \(G\) whose equivalence classes support the Plancherel measure for \(G\). By Kirillov theory, there exists an integer \(k\), which does not depend on \(\Pi \in U\), such that \(\Pi : G \to B(L^2(\mathbb{R}^k))\), see more details in 3.5.4 below.

**Proposition 10** Let \(G\) be a stratified group and let \(\mathcal{F}\) be the set of coefficient functions

\[ \mathcal{F} = \{ \varphi_{\Pi, \mu, \mu'} = \langle \Pi(\cdot)(F_{\mu}), F_{\mu'} \rangle \mid \Pi \in \mathcal{U}, F_{\mu}, F_{\mu'} \in \mathcal{B}_\Pi \} \subset L^\infty(dg) \]

where \(\mathcal{B}_\Pi\) is an orthogonal basis of \(L^2(\mathbb{R}^k)\) chosen among eigenvectors of \(e^{-\frac{1}{2} \Pi(L)}\). Then \(\mathcal{F}\), which lies in \(C^\infty(G)\), is a set of eigenvectors of \(L\) which is total in \(L^q(p(g) dg), 1 \leq q < \infty\).

For fixed \(\Pi, \mu\) the functions \(\{ \varphi_{\Pi, \mu, \mu'} \mid F_{\mu'} \in \mathcal{B}_\Pi \}\) are independent and belong to the same eigenspace of \(L\).

Proof: a) For every non trivial unitary irreducible representation \(\Pi\) of \(G\), since \(\Pi(pdg)(F_{\mu}) = e^{-\frac{1}{2} \Pi(L)}(F_{\mu}) = e^{-\frac{1}{2} \lambda_{\mu}} F_{\mu}\), \(F_{\mu}\) is a \(C^\infty\) vector for \(\Pi\), hence \(\varphi_{\Pi, \mu, \mu'} \in C^\infty(G)\); \(\varphi_{\Pi, \mu, \mu'}\) is an eigenvector of \(L\) with eigenvalue \(\lambda_{\mu}\) by (22).

Since \(\Pi\) is irreducible, the closed invariant subspace

\[ \{ F \in L^2(\mathbb{R}^k) \mid \forall g \in G \langle \Pi(g)(F_{\mu}), F \rangle = 0 \} \]
is reduced to \( \{0\} \), which implies the independence of the \( \varphi^{\Pi,\mu,\mu'} \)'s. (In the Heisenberg case, see [1] p. 19, 51).

b) Let \( \psi \in L^q(pdg), \frac{1}{q} + \frac{1}{q'} = 1 \), be orthogonal to \( \mathcal{F} \), i.e. for \( \Pi \in \mathcal{U} \),

\[
0 = \int_{G} \langle \Pi(g)(F_\mu), F_{\mu'} \rangle \psi(g)p(g)dg = \left( \int_{G} \Pi(g)\psi(g)p(g)dg \right) \langle F_\mu, F_{\mu'} \rangle.
\]

Equivalently \( \Pi(\psi p) = \widehat{\psi p}(\Pi) = 0 \) for \( \Pi \in \mathcal{U} \). Then Plancherel formula for \( G \) (see e.g. [CG, theorem 4.3.10]) implies that \( \psi p = 0 \) a.s.. Indeed, this is clear if \( \psi p \in L^2(dg) \), in particular if \( q' \geq 2 \). In general, \( \psi p \in L^1(dg), \| (\psi p) * p_t - \psi p \|_{L^1(dg)} \to 0 \) and \( (\psi p) * p_t \in L^2(dg) \); moreover \( (\psi p) * p_t = 0 \) a.s. since, for every \( \Pi \in \mathcal{U} \),

\[
\Pi((\psi p) * p_t) = \Pi(\psi p)\Pi(p_t) = 0. \blacksquare
\]

### 3.5.3 Polynomial eigenvectors of \( N \) built from coefficients of representations

We now consider the functions \( e^{\frac{i}{2} t \lambda_\mu} \varphi^{\Pi,\mu,\mu'} \circ \delta_t \) as generating functions of polynomial eigenvectors of \( N \).

**Proposition 11** Let \( \varphi^{\Pi,\mu,\mu'} = \langle \Pi(.) (F_\mu), F_{\mu'} \rangle \in \mathcal{F} \) be as in proposition [10]. For \( n \geq 1 \), let

\[
h_n^{\Pi,\mu,\mu'} = \frac{d^n}{dt^n} \bigg|_{t=0} e^{\frac{i}{2} t \lambda_\mu} \varphi^{\Pi,\mu,\mu'} \circ \delta_t.
\]

Then \( h_n^{\Pi,\mu,\mu'} \) is a polynomial eigenvector of \( \cos N \theta \) with eigenvalue \( \cos^n \theta \).

**Proof:** By proposition [10] and lemma [9] it is enough to prove assumptions (i) and (ii) in lemma [9]. We claim the existence of a polynomial \( \psi_n, n \geq 1 \), which does not depend on \( t \), such that, for \( 0 \leq t \leq 1 \) and \( n \geq 0 \),

\[
\left| \frac{d^n}{dt^n} \varphi^{\Pi,\mu,\mu'} \circ \delta_t \right| \leq \psi_n.
\]

Since \( g \to \psi_n(\gamma g^{-1}) \) is still a polynomial, it lies in \( L^1(pdg) \) for every \( \gamma \in G \), and this will prove assumption (i). We now verify the claim.

**Case 1:** The computation of derivatives being easier if \( G \) is step two, we first consider this setting.
By Schur lemma, the restriction of $\Pi$ to the center $\exp Z$ of $G$ is given by a character $u \rightarrow e^{i\langle l, u \rangle}$ where $l$ is some linear form on $Z$, see e.g. [CG, p. 184]. If $g = (x, u)$ and $X = \sum_{j=1}^{n} x_j X_j \in V_1$,

$$\varphi^{\Pi,\mu,\mu'}(\delta_t g) = e^{it^2 \langle l, u \rangle} \langle \Pi(\exp tX)(F_\mu), F_{\mu'} \rangle = e^{it^2 \langle l, u \rangle} \Phi_t^{\Pi,\mu,\mu'}(x)$$

and, by (22),

$$\frac{d^m}{dt^m} \Phi_t^{\Pi,\mu,\mu'}(x) = \langle \Pi(\exp tX)(X)^m(F_\mu), F_{\mu'} \rangle.$$  

Since $\Pi(X)^m(F_\mu)$ lies in $L^2(\mathbb{R}^k, d\xi)$, $\langle \Pi(X)^m(F_\mu), F_{\mu'} \rangle$ and $\|\Pi(X)^m(F_\mu)\|_{L^2(d\xi)}$ are polynomials w.r. to $x$, $\frac{d^m}{dt^m} |_{t=0} \delta_t(\varphi^{\Pi,\mu,\mu'})$ is a polynomial w.r. to $x, u$, and $|_{t=0} \frac{d^m}{dt^m} e^{it^2 \langle l, u \rangle} \Phi_t^{\Pi,\mu,\mu'}(x)$ is, for $0 \leq t \leq 1$, less than a polynomial $\psi_n$ which does not depend on $t$. This proves (i) and (ii) in this case.

**General case:** As in (14) and (15), for $g = \exp Z = \exp(X + Y + .. + U)$ and $t > 0$, since $V(\Pi(\delta_t Z)) = \Pi(V(\delta_t Z))$,

$$\frac{d}{dt}\varphi^{\Pi,\mu,\mu'}(\delta_t g) = \frac{d}{dt} \langle \exp \Pi(\delta_t Z)(F_\mu), F_{\mu'} \rangle = \langle \Pi(V(\delta_t Z))(F_\mu), \exp -\Pi(\delta_t Z)(F_{\mu'}) \rangle.$$  

At $t = 0$ this reduces to the polynomial $\langle \Pi(X)(F_\mu), F_{\mu'} \rangle$. Since $\Pi(V(\delta_t Z)$ has polynomial coefficients w.r. to $t$ and the coordinates of $g$, so does $\|\Pi(V(\delta_t Z))(F_\mu)\|_{L^2(d\xi)}$. Hence there is a polynomial $\psi_1$ w.r. to the coordinates of $g$ such that

$$\sup_{0 \leq t \leq 1} \|\Pi(V(\delta_t Z))(F_\mu)\|_{L^2(d\xi)} \leq \psi_1.$$  

This proves the claim for $n = 1$. Clearly this can be iterated for upper derivatives, which proves (i) and (ii).■

**3.5.4 The step two setting: generalized Hermite polynomials**

In this case, the key facts are the extension of the explicit functions $\varphi^{\Pi,\mu,\mu'} \in \mathcal{F}$ as entire functions on the complexification of $G$ and the explicit expression of $p$. Theorem 12 gives another proof of theorem 7 a) in this setting, with another description of the eigenspaces of $N$ by generating functions.

**Theorem 12** Let $G$ be a step two stratified group. Then
a) every $\varphi^{\Pi,\mu,\mu'} \in \mathcal{F}$ lies in the closed subspace of $L^q(pdg), 1 \leq q < \infty$, spanned by constants and the polynomials $\{h_n^{\Pi,\mu,\mu'}, n \geq 1\}$ defined in proposition 17.

b) The set of generalized Hermite polynomials

$$\bigcup_{\varphi^{\Pi,\mu,\mu'} \in \mathcal{F}} \{h_n^{\Pi,\mu,\mu'}, n \geq 1\}$$

together with the constants is a set of eigenvectors of $N$ which is total in $L^q(pdg), 1 \leq q < \infty$.

c) For fixed $n \geq 1$, $\bigcup_{\varphi^{\Pi,\mu,\mu'} \in \mathcal{F}} \{h_n^{\Pi,\mu,\mu'}\}$ spans the eigenspace of $N$ associated to $n$ in $L^q(pdg), 1 < q < \infty$.

In contrast, if $G$ has more than 4 layers, assertion b) is false by proposition 8, hence a) is false for some $\varphi^{\Pi,\mu,\mu'} \in \mathcal{F}$, by proposition 10. If $G$ has 3 or 4 layers, we do not know if the conclusions of theorem 12 hold true.

Proof of theorem 12: a) implies b) by propositions 10 and 11.

b) implies c) as recalled in the proof of theorem 7.

a) The proof is given in three steps. In step 1 we state two standard sufficient conditions ensuring statement a); in step 2 we verify these conditions when $G$ is a Heisenberg group; in step 3 we show how the general step 2 case mimicks the Heisenberg case.

Step 1: Let $\varphi^{\Pi,\mu,\mu'} \in \mathcal{F}$ and assume that

(i) for every $g \in G$, the function $t \to \varphi^{\Pi,\mu,\mu'}(\delta_t g)$ extends as a holomorphic function $z \to \varphi_z^{\Pi,\mu,\mu'}(g)$ on $\mathbb{C}$.

(ii) for some connected neighborhood $\Omega$ of the real axis, for every compact $K \subset \Omega$, there exists $w_K \in L^q(pdg), 1 \leq q < \infty$, such that

$$\left| \varphi_z^{\Pi,\mu,\mu'} \right| \leq w_K, \ z \in K.$$ 

We claim that $\varphi^{\Pi,\mu,\mu'} = \varphi$ then lies in the closed subspace of $L^q(pdg)$ spanned by $h_n^{\Pi,\mu,\mu'}, n \geq 1$. Indeed, let $\psi \in L^q(pdg), \frac{1}{q} + \frac{1}{q'} = 1$, and let

$$m(t) = \int_G \varphi(\delta_t g)\psi(g)p(g)dg.$$ 

By the assumptions, $m$ extends as a holomorphic function on $\Omega$ and

$$\frac{d^n}{dz^n}m = \int_G \left(\frac{d^n}{dz^n}\varphi_z\right)\psi p dg, \ m \geq 0.$$
By proposition 10, $L(\varphi) = \lambda \varphi$ for some $\lambda = \lambda_\mu$. Hence $t \to e^{\frac{1}{2} t^2 \lambda} m(t)$ also extends as a holomorphic function on $\Omega$ and

$$
\frac{d^n}{dz^n} \big|_{z=0} e^{\frac{1}{2} t^2 \lambda} m = \int_G \left[ \frac{d^n}{dz^n} \big|_{z=0} e^{\frac{1}{2} t^2 \lambda} \varphi \right] \psi pdg = \int_G h_n^{\Pi,\mu,\mu'} \psi pdg, \quad n \geq 0.
$$

If $\psi$ is orthogonal to $\{ h_n^{\Pi,\mu,\mu'}, n \geq 0 \}$, these derivatives are zero, hence $e^{\frac{1}{2} t^2 \lambda} m$ is zero on $\Omega$. In particular $m(1) = 0$, i.e. $\psi$ is orthogonal to $\varphi$, which proves the claim.

**Step 2: The Heisenberg groups $\mathbb{H}_k$**

A basis of the first layer of the Lie algebra is $X_1, Y_1, \ldots, X_k, Y_k$ where $[X_j, Y_j] = -4U$, $U$ spans the center, and the other commutators are zero. By the Campbell-Hausdorff formula,

$$
g = \exp(\sum_{j=1}^{k} x_j X_j + y_j Y_j + u U) = \exp uU \prod_{j=1}^{k} \exp(-2x_jy_jU) \exp y_j Y_j \exp x_j X_j.
$$

We first consider the Schrödinger (unitary irreducible) representation $\Pi_S : \mathbb{H}_k \to B(L^2(\mathbb{R}^k))$, defined on the Lie algebra by

$$
\Pi_S(X_j) = \frac{\partial}{\partial \xi_j}, \quad \Pi_S(Y_j) = i\xi_j, \quad \Pi_S(U) = -\frac{1}{4} \left[ \frac{\partial}{\partial \xi_j}, i\xi_j \right] = -\frac{i}{4} I.
$$

For $F \in L^2(\mathbb{R}^k)$, this implies

$$
\Pi_S(g)(F)(\xi) = e^{-i\xi/4} e^{\frac{1}{2} \sum_{j=1}^{k} x_j y_j} e^{i \sum_{j=1}^{k} y_j \xi_j} F(\xi + x), \quad (24)
$$

and

$$
\Pi_S(L) = H = \sum_{j=1}^{k} \left( -\frac{\partial^2}{\partial \xi_j^2} + \xi_j^2 \right)
$$

is the harmonic oscillator. If $k = 1$, an o.n. basis of eigenvectors of $H$ in $L^2(\mathbb{R})$ is the sequence of Hermite functions $F_\mu, \mu \in \mathbb{N}$. The so called special Hermite functions [1] p. 18-19 are, for $\mu, \mu' \in \mathbb{N}$ and $\varepsilon_{\mu,\mu'} = sgn(\mu' - \mu),

$$
\langle \Pi_S(x, y, 0)(F_\mu), F_{\mu'} \rangle = \Phi_{\mu,\mu'}(x, y) = \int_{\mathbb{R}} e^{iy\xi} F_\mu(\xi + \frac{x}{2}) F_{\mu'}(\xi - \frac{x}{2}) d\xi
$$

$$
= r_{\mu,\mu'}(x^2 + y^2) e^{-\frac{1}{2}(x^2 + y^2)} (x + i \varepsilon_{\mu,\mu'} y)^{|\mu - \mu'|}, \quad (25)
$$

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where \( r_{\mu,\mu'} = r_{\mu',\mu} \) is a one variable polynomial with real coefficients.

An o.n basis of eigenvectors of \( H \) in \( L^2(\mathbb{R}^k) \) is the sequence \( \left( \prod_{j=1}^k F_{\mu_j}(\xi_j) \right)_{\mu \in \mathbb{N}^k} \), which gives, for \( \mu, \mu' \in \mathbb{N}^k \) and \( g = (x, y, u) \),

\[
\varphi^{\Pi_S,\mu,\mu'}(g) = \langle \Pi(g)(F_\mu), F_{\mu'} \rangle = e^{-i\frac{\mu}{2}} \prod_{j=1}^k \Phi_{\mu_j,\mu'_j}(x_j, y_j).
\]

By (25) the function \( z \to \varphi^{\Pi_S,\mu,\mu'}(zx, zy, z^2u) \) is holomorphic on \( \mathbb{C} \). Let \( Ra,\delta = \{ \alpha + i\beta \mid |\alpha| < a, |\beta| < \delta \} \subset \mathbb{C} \).

For some constant \( C_{a,\delta} \), and \( z \in \overline{Ra,\delta} \),

\[
\left| \varphi^{\Pi_S,\mu,\mu'}(zx, zy, z^2u) \right| \leq C_{a,\delta} e^{\frac{a\delta}{2}|u|} \prod_{j=1}^k e^{\delta^2(x_j^2+y_j^2)}.
\]

We now look for conditions on \( a, \delta \) ensuring that the right hand side lies in \( L^q(pd\mu) \). We recall [Hu] that

\[
p(x, y, u) = \int_{\mathbb{R}} e^{i\lambda u} Q(x, y, \lambda) d\lambda = c_k \int_{\mathbb{R}} e^{i\lambda u} \prod_{j=1}^k \frac{2\lambda}{sh2\lambda} e^{-\frac{\lambda}{sh^2}(x_j^2+y_j^2)} d\lambda.
\]

Noting that \( Q(x, y, \lambda) = \prod_{j=1}^k Q_1(x_j, y_j, \lambda) \) is even w.r. to \( \lambda \), we get, for \( q \geq 1 \),

\[
\frac{1}{2} \int_{\mathbb{R}} e^{2a\delta|u|} p(x, y, u) du \leq \int_{\mathbb{R}} ch(q^2a\delta u)p(x, y, u) du = Q(x, y, i\alpha q^2/2).
\]

We need the convergence of

\[
\int_{\mathbb{R}^{2k}} \prod_{j=1}^k e^{q\delta^2(x_j^2+y_j^2)} Q_1(x_j, y_j, i\alpha q^2/2) dx_j dy_j = c \prod_{j=1}^k \int_{\mathbb{R}^2} e^{\left(q\delta^2 - \frac{x^2+y^2}{4\alpha^2\delta^2}\right)}(x_j^2+y_j^2) dx_j dy_j,
\]

which holds for \( qa\delta \leq \frac{q}{4} \) and \( a > 2\delta \). Thus, taking \( a = N \in \mathbb{N} \), \( \varphi^{\Pi_S,\mu,\mu'} \) satisfies the assumptions of step 1 on
Ω = \bigcup_{N \geq 2} R_N \frac{1}{\eta_N}.

Plancherel formula for \( \mathbb{H}_k \) (see e.g. [T, Theorem 1.3.1] or [CG, p.154]) involves the representations

\[ \rho_h(x, y, u) = e^{-\frac{i}{4} hu} \Pi_S(x, hy, 0). \]

By the Stone-Von Neumann theorem [T, theorem 1.2.1] every irreducible unitary representation \( \Pi \) of \( \mathbb{H}_k \) satisfying \( \Pi(0, 0, u) = e^{-\frac{i}{4} hu} \) for a real \( h \neq 0 \) is unitarily equivalent to \( \rho_h \). Hence \( \rho_{\beta^2} \) (resp. \( \rho_{-\beta^2} \)) is unitarily equivalent to \( \Pi_S \circ \delta_\beta \), \( \beta > 0 \), where \( \sigma \) is the automorphism of \( \mathbb{H}_k \) defined by \( \sigma(x, y, u) = (x, -y, -u) \).

Since \( \Pi_S(L) = \Pi_S \circ \sigma(L) \), we get \( \varphi^{\Pi_S \circ \sigma, \mu, \mu'} = \varphi^{\Pi_S, \mu, \mu'} \circ \sigma = \varphi^{\Pi_S, \mu, \mu'} \), hence

\[ F = \left\{ \varphi^{\Pi_S, \mu, \mu'} \circ \delta_\beta, \varphi^{\Pi_S, \mu, \mu'} \circ \delta_\beta, \beta > 0, \mu, \mu' \in \mathbb{N}^k \right\}. \]

The conditions of step 1 are satisfied by \( \varphi^{\Pi_S, \mu, \mu'} \circ \delta_\beta \), replacing \( R_{a, \delta} \) by \( R_{\beta a, \beta \delta} \), which ends the proof of theorem 12 for \( \mathbb{H}_k \). Taking remark 5 into account, the set \( \cup_{\mu, \mu', n} \{ h^{\Pi_S, \mu, \mu'}, h^{\Pi_S, \mu, \mu'}_n \} \) is total in \( L^2(\mathbb{H}_k, pdg) \).

**Step 3.** We first recall some more facts on representations and compute the set \( F \) for step 2 stratified groups. We shall follow Cygan’s scheme [Cy].

Let \( t \in \mathcal{G}^* \). Among the Lie subalgebras \( \mathcal{M} \subset \mathcal{G} \) satisfying \( \{ l, [X, Y] \} = 0 \) for every \( X, Y \in \mathcal{M} \), some have minimal codimension \( m_t \) and are denoted by \( \mathcal{M}_t \). Then the map

\[ Z \in \mathcal{M}_t \rightarrow e^{i(l, Z)} \]

is a representation of the subgroup \( \exp \mathcal{M}_t \) and induces an irreducible unitary representation of \( G \) as follows [CG, theorems 1.3.3, 2.2.1 and p 41] : One chooses independent vectors \( (X_j)_{i=1}^{m_t} \) such that \( \mathcal{G} = \mathcal{M}_t + \text{span}\{(X_j)_{i=1}^{m_t}\} \). For \( (g, \xi) \in G \times \mathbb{R}^{m_t} \) there exist \( (\xi', M) \in \mathbb{R}^{m_t} \times \mathcal{M}_t \) such that

\[ \exp\left( \sum_{i=1}^{m_t} \xi_i X_i \right). g = \exp M . \exp\left( \sum_{i=1}^{m_t} \xi_i' X_i \right). \]

Then, for \( F \in L^2(\mathbb{R}^{m_t}) \),

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\[ \Pi_l(g)(F)(\xi) = e^{i(l,M)} F(\xi'). \]  

The set of \( \mathcal{C}^\infty \) vectors for \( \Pi_l \) is \( \mathcal{S}(\mathbb{R}^k) \) [CG corollary 4.1.2]. Every irreducible unitary representation of \( G \) is equivalent to a representation constructed in this way; different \( \mathcal{M}_l, \mathcal{M}'_l \) and different \( l, l' \) in the same coadjoint orbit induce equivalent representations [CG theorems 2.2.2, 2.2.3, 2.2.4].

By Kirillov theory there is an integer \( k \) and a set \( \mathcal{U}_0 \subset G^* \) of "generic" orbits with maximal dimension \( 2k \), such that \( m_l = k \) for \( l \in \mathcal{U}_0 \). The Plancherel measure is supported by \( \mathcal{U}_0 \) [CG, theorem 4.3.10].

We now compute such a \( \Pi_l \) when \( G \) is step 2. Let \( U_1, \ldots, U_d \) be a basis of the central layer \( \mathcal{Z} \) and let \( \chi_1, \ldots, \chi_n \) be a basis of the first layer \( \mathcal{V}_1 \) of \( G \). Let \( A_\lambda \) be the \( n \times n \) matrix with coefficients \( \langle \lambda, [\chi_j, \chi_h] \rangle \).

By Campbell-Hausdorff formula, for \( Y \in \mathcal{G}, X \in \mathcal{V}_1, U \in \mathcal{Z}, g = \exp(X + U) \),

\[ \exp Adg(Y) = g \exp Y g^{-1} = e^{[X,Y]} \exp Y = \exp(Y + [X,Y]), \]

hence the coadjoint orbit of \( l \), i.e. \( \{ l \circ Adg, g \in \mathcal{G} \} \subset \mathcal{G}^* \), is \( l + \text{range } A_\lambda \).

We now assume that \( l \) lies in \( \mathcal{U}_0 \), so that the range of \( A_\lambda \) has dimension \( 2k \). There exists an orthogonal matrix \( \Omega_\lambda \) such that

\[ A_\lambda = \Omega_\lambda A'_\lambda \Omega_\lambda^* \]

where \( A'_\lambda \) is block diagonal, the non zero blocks having the form

\[ \nu_j(\lambda) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \nu_j(\lambda) > 0. \]  

The new basis of \( \mathcal{V}_1 \) (defined by the columns of \( \Omega_\lambda \)) is denoted by \( X_1, Y_1, \ldots, X_k, Y_k, S_1, \ldots, S_{n-2k} \), so that

\[ \langle \lambda, [X_j, X_h] \rangle = 0 = \langle \lambda, [Y_j, Y_h] \rangle, \quad \langle \lambda, [X_j, Y_h] \rangle = \nu_j(\lambda) \delta_{jh}, \quad 1 \leq j, h \leq k. \]  

We denote \( t = \Omega_\lambda(x, y, s) \in \mathbb{R}^n \), where

\[ \sum_{j=1}^{n} t_j \chi_j = \sum_{j=1}^{k} x_j X_j + y_j Y_j + \sum_{h=1}^{n-2k} s_h S_h = X + Y + S \in \mathcal{V}_1. \]
Choosing $\mathcal{M}_l = \mathbb{Z} + \text{span}\{Y_j, S_h\}_{1 \leq j \leq k, 1 \leq h \leq n - 2k}$, let us compute $\Pi_l$. By definition $\Pi_l(\exp u_j U_j) = e^{i u_j \lambda_j}$. For $g = \exp(X + Z)$ and $Z = Y + S$,

$$\exp(\sum_{j=1}^{k} \xi_j X_j) g = \exp[\sum_{j=1}^{k} \xi_j X_j, X + Z] g \exp(\sum_{j=1}^{k} \xi_j X_j)$$

$$= \exp(\sum_{j=1}^{k} \xi_j X_j, X + Z + \frac{1}{2}[X, Z]) \exp Z \exp(\sum_{j=1}^{k} \xi_j X_j)$$

$$= \exp M \exp(\sum_{j=1}^{k} (\xi_j + x_j) X_j).$$

Hence, by (27) and (29), for $F \in L^2(\mathbb{R}^k)$,

$$\Pi_l(g)(F)(\xi) = e^{i(l, M)} F(\xi + x) = e^{i \sum_{j=1}^{k} \nu_j \xi_j + x_j} e^{i(l, Y + S)} F(\xi + x). \quad (30)$$

Since we may replace $l$ by $l'$ in the orbit of $l$, we may suppose $\langle l, Y_j \rangle = 0$, $1 \leq j \leq k$. In particular, by (30),

$$\Pi_l(X_j) = \frac{\partial}{\partial \xi_j}, \quad \Pi_l(Y_j) = i \nu_j \xi_j, \quad 1 \leq j \leq k; \quad \Pi_l(S_h) = i \langle l, S_h \rangle I, \quad 1 \leq h \leq n - 2k.$$

Since $\Omega_\lambda$ is orthogonal, 

$$-L = \sum_{j=1}^{k} (X_j^2 + Y_j^2) + \sum_{h=1}^{n-2k} S_h^2,$$

which entails

$$\Pi_l(L) = \sum_{j=1}^{k} -\frac{\partial^2}{\partial \xi_j^2} + \nu_j^2 \xi_j^2 + \sum_{h=1}^{n-2k} \langle l, S_h \rangle^2 I.$$

A basis of eigenvectors of $\Pi_l(L)$ is thus

$$\left( \prod_{j=1}^{k} F_{\mu_j}(\sqrt{\nu_j} \xi_j) \right)_{\mu \in \mathbb{N}^k}.$$

By (30) and (25), for $g = (x, y, s, u)$,

$$\varphi^{\Pi_l, \mu, \mu'}(g) = e^{i(\lambda, u)} e^{i \sum_{h=1}^{n-2k} s_h \langle l, S_h \rangle} \prod_{j=1}^{k} \frac{1}{\sqrt{\nu_j}} \Phi_{\mu_j, \mu'_j}(\sqrt{\nu_j} x_j, \sqrt{\nu_j} y_j).$$

Hence, for $z \in R_{a, \delta}$ and some constant $C_{a, \delta}$, with $t = \Omega_\lambda(x, y, s)$,
\[ \left| \varphi_{\Pi, \mu, \mu'} (zt, z^2 u) \right| \leq C_{a, \delta} e^{2a\delta|\langle \lambda, u \rangle|} e^{\delta \sum_{h=1}^{n-2k} \left| s_{h, \langle l, S_h \rangle} \right|} \prod_{j=1}^{k} \frac{1}{\sqrt{v_j}} e^{\delta^2 v_j (x_j^2 + y_j^2)} \]

\[ = e^{2a\delta|\langle \lambda, u \rangle|} w_{a, \delta, t}(x, y, s). \]

By [Cy] corollary 5.5 the heat kernel \( p(t, u) \) is the Fourier transform of \( CQ(t, \lambda) \) w.r. to the central variables, where

\[ Q(t, \lambda) = \prod_{h=1}^{n-2k} e^{-\frac{1}{2} s_h^2} \prod_{j=1}^{k} Q_1(x_j, y_j, \frac{\nu_j}{4}) = Q(t, -\lambda). \]

Again, we need the convergence of

\[ \int_{\mathbb{R}^n} w_{a, \delta, t}^g (x, y, s) \prod_{h=1}^{n-2k} e^{-\frac{1}{2} s_h^2} \prod_{j=1}^{k} Q_1(x_j, y_j, \frac{iqa\delta \nu_j}{2}) dx dy ds, \]

which holds if \( qa\delta \max \nu_j \leq \frac{\pi}{4} \) and \( a > 2\delta \). This ends the proof of theorem [12].

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