Spectral properties of stochastic resonance in quantum transport

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We investigate theoretically and experimentally stochastic resonance in a quantum dot coupled to electron source and drain via time-dependent tunnel barriers. A central finding is a transition visible in the current noise spectrum as a bifurcation of a dip originally at zero frequency. The transition occurs close to the stochastic resonance working point and relates to quantized pumping. For the evaluation of power spectra from measured waiting times, we generalize a result from renewal theory to the ac driven case. Moreover, we develop a master equation method to obtain phase-averaged current noise spectra for driven quantum transport.

Stochastic resonance (SR) is a counter intuitive phenomenon by which the output signal of a device improves due to the action of external noise [1, 2]. Typically it emerges as an interplay of periodic driving, nonlinearities, and noise-induced activation. The paradigmatic example consists of two states separated by an energetic barrier, where an external oscillating force causes periodic transitions considered as signal. When the force is rather weak, noise may help to cross the barrier and, thus, improves the signal. For very strong noise, however, the output inherits too much randomness and degrades. This reflects a prominent feature of SR, namely an optimal working point at an intermediate noise level. A further characteristic property is that SR predominantly occurs when the driving frequency roughly matches one half the intrinsic decay rate of the system, \( f = \frac{\Gamma_0}{2} \) [2]. SR has been suggested as the mechanism behind very different phenomena ranging from the periodic recurrence of ice ages to biological signal processing [3]. Many of these ideas have been realized experimentally in the classical regime, while the quantum regime has been explored mainly theoretically [4–6].

Recently in an experiment with a biased quantum dot with time-dependent tunnel rates, SR has been extended to the realm of quantum transport with the zero-frequency noise of the current as a measure for the signal quality [7]. It turned out that the current noise indeed assumes its minimum when the driving frequency obeys the mentioned SR condition. However, as only zero-frequency properties of the experimental data were evaluated, the question arises whether additional information can be extracted from the full power spectrum of the current fluctuations.

With this letter, we demonstrate that the current noise spectrum provides relevant insight to the SR mechanism in quantum transport. We develop a method for computing the frequency-dependent Fano factor [8–10] for ac driven transport making use of the time evolution of conditional cumulants [11] and compare the results with experimental data from an ac-driven quantum dot similar to Ref. [7]. For the data analysis, we generalize the relation between waiting times and the power spectrum of a spike train known from renewal theory [12, 13] to the ac-driven case. Finally, we discuss possible applications for quantized charge pumping and current standards such as those of Refs. [14, 15].

**Experimental setup and model.**—The experiments have been performed on a Schottky gate defined quantum dot based on the two-dimensional electron gas of a GaAs/AlGaAs heterostructure as shown in Fig. 1(a). An adjacent quantum point contact acts as charge monitor. By applying sufficiently negative voltages to the center gates, the visible gap in the center is electrostatically closed and the two paths are galvanically isolated. From the upper side, the quantum dot is confined by two tunnel barrier gates and a plunger gate, which are used to manipulate the tunneling rates and the energy levels of the dot.

![Fig. 1](image_url)

**FIG. 1.** (a) Strongly biased quantum dot with periodically time-dependent tunnel couplings. It can be charged by electrons entering from the source (green arrow) and discharged when they leave to the drain (magenta). The quantum point contact measures the dot occupation. (b) Corresponding theoretical model. (c) Time-derivative of \( I_{qpc} \). The sign of the spikes reflects the change of the dot occupation.
quantum dot. All measurements have been performed at 1.5 K.

The quantum dot is tunnel coupled to biased leads, where voltages are applied to the plunger gate and the tunnel barrier gates such that its lowest level, hosting up to one electron, lies in the center of the bias window. Care has been taken to tune the dot to a symmetric coupling to source and drain, such that the tunnel rates from the source and to the drain are equal. The ac components of the gate voltages let the dot level oscillate as sketched in Fig. 1(b), and the tunnel rates becomes time-dependent [7, 14].

We model this situation by the transition rates

$$
\Gamma_{L/R}(t) = \Gamma_0 \exp[\pm \alpha_{L/R} A \cos(\Omega t)],
$$

with the driving amplitude $A$ and a period $T = 2\pi/\Omega \equiv 1/f$. The leverage factors $\alpha_{L/R}$ as well as the intrinsic decay rate $\Gamma_0$ are adjusted such that the $\Gamma_{L/R}(t)$ match the rates in the experiment. Transport phenomena in this open system can be described by a master equation of the form $\dot{\rho} = \mathcal{L}(t)\rho$, where $\rho$ is the reduced density operator of the central conductor with the $T$-periodic Liouvillian $\mathcal{L}(t) = \mathcal{L}(t + T)$, see the Supplemental Material [10].

Current measurements are affected by the displacement current of the fluctuating charge configurations [21, 22]. In our case, their origin is the stochastic charging and discharging of the quantum dot described by the jump operators, $\mathcal{J}_{L/R}$ and $\mathcal{J}_Q^\pm = \mathcal{J}_L^\pm - \mathcal{J}_R^\pm$ [9, 15], where $L$ and $R$ denote source and drain, respectively, while $Q$ refers to the dot occupation. As a consequence, current measurements in the leads of a mesoscopic two-terminal device provide the so-called total (or Ramo-Shockley) current $I = -\kappa_L I_L + \kappa_R I_R$, where $I_L$ and $I_R$ are the particle currents at the interfaces. $\kappa_L$ and $\kappa_R = 1 - \kappa_L$ are normalized gate capacitances which we assume time independent and symmetric, $\kappa_L = \kappa_R = 1/2$. While this distinction is irrelevant for the average current and the zero-frequency noise [23], it is quite important for the current noise spectrum [9, 24]. A measurement of the dot occupation via the charge monitor provides the full information about all currents. Nevertheless, we focus on the total current, because it turns out that its noise spectrum is most significantly affected by SR.

The relation between the noise of the total current $S_{\text{tot}}$ and that of $I_L, I_R$, and $Q$ is readily obtained from charge conservation, $N_L + N_R + Q = \text{const}$. Hence, the corresponding current correlation function obeys $S_{\text{tot}} = \kappa_L S_L + \kappa_R S_R - \kappa_L \kappa_R S_Q$, which holds in the time domain as well as in the frequency domain [9, 24].

**Frequency-dependent Fano factor.**—We consider the particle current as the change of the electron number, given by the stochastic variable $j = \dot{n}$, in a region which may be a lead or the quantum dot (or any other compound of coupled quantum dots). Its symmetrized auto correlation function $S(t, t') = \frac{1}{2}(\langle \Delta j(t) \Delta j(t') \rangle + \langle \Delta j(t') \Delta j(t) \rangle)$ in the stationary limit, must obey the discrete time-translation invariance of the Liouvillian, namely $S(t, t') = S(t + T, t' + T)$. By introducing the time difference $\tau = t - t'$, one sees that $S(t, t - \tau)$ is invariant under $t \rightarrow t + T$, i.e., for constant $\tau$ it is $T$-periodic in $t$ [23]. This implies that time-averages over a driving period are equivalent to averages over the phase of the driving [23].

Hence, we define the phase-averaged correlation function $\bar{S}(\tau) \equiv \frac{1}{2}(S(t + \tau, t) + \bar{S}(t + \tau, t))$, where the second equality follows readily from simultaneous translation of all times. The corresponding phase-averaged spectral density $\bar{S}(\omega)$ is normalized to the average current $\bar{I}$ to yield as dimensionless noise spectrum the frequency-dependent Fano factor $F(\omega) = \bar{S}(\omega)/\bar{I}$, which is our main quantity of interest.

To compute $\bar{S}(\omega)$, we establish its relation to the conditional second moment of the electron number in the lead as $M_2(t|t') = \langle \Delta n^2(t) \rangle_{t'}$, where the subscript $t'$ denotes the reference time from which on we consider the fluctuations. Owing to $\dot{n}(t) = j(t)$, one finds [10]

$$
M_2(t|t_0) = \int_{t_0}^{t} dt'' \int_{t_0}^{t} dt' S(t'', t'),
$$

whose time derivative is the conditional second current cumulant $c_2(t|t_0) = 2 \int_{t_0}^{t} dt' S(t, t')$. Via the substitution $t' \rightarrow t' - \tau$, a subsequent phase average, and Fourier transformation, we obtain the generalized MacDonald formula [15]

$$
\bar{S}(\omega) = \omega \int_0^\infty d\tau \sin(\omega \tau) \int_0^{\tau} dt c_2(t + \tau|t).
$$

The remaining tasks are the computation of the time evolution of the conditional current cumulant $c_2$ for sufficiently many values of $t$ (or initial phases) and a numerical $t$-integration.

For this purpose, we employ a recent propagation method for the full-counting statistics [11]. It is based on a generalized master equation that consists of the Liouvillian and the jump operator of the considered current together with a counting variable [26]. Taylor expansion in the counting variable provides the usual master equation $\dot{\rho} = \mathcal{L}_0 \rho$ and the time dependent current expectation value, $I_\nu(t) = \text{tr}(\mathcal{J}_\nu^+ - \mathcal{J}_\nu^-)\rho$ for $\nu = L, R, Q$. The next order yields the second cumulant, $c_2(t|t') = \text{tr}(\mathcal{J}_\nu^+ - \mathcal{J}_\nu^-)\rho(t) + 2 \text{tr}(\mathcal{J}_\nu^+ - \mathcal{J}_\nu^-)X_1(t)$, where the auxiliary operator $X_1$ obeys the equation of motion $X_1(t) = \mathcal{L}_1 X_1 + [\mathcal{J}_\nu^+ - \mathcal{J}_\nu^- - I(t)]\rho(t)$. Since we are interested in stationary correlations, the boundary condition at time $t'$ is that (i) the density operator $\rho(t')$ must no longer contain transients and (ii) $X_1(t') = 0$. For details of the derivation, see the Supplemental Material [10].

**Power spectrum of the measured current.**—In the experiment, the charge monitor provides the times at which electrons tunnel from or to the dot, such that we can consider the current as a spike train as shown in Fig. 1(c). The experimental data at hand are the times between two subsequent tunnel events. In the absence of the driving, the distribution function of these times relates to the power spectrum of the spike train [12, 13, 27, 28]. In the Supplemental Material [19], we generalize this relation to
Moreover, for adiabatic driving, the symmetry gets lost due to the constant Fano factor $F$ in a symmetric quantum dot has white noise character.

First mention that in the absence of driving, the current and the noise properties are primarily manifest in the total current. The magnitude and the position of the noise reduction is analyzed in Fig. 3. Panels (a) and (b) show how the minimum of $F(\omega)$, as observed in Fig. 2(a), changes with the driving frequency $f$. The data confirm that the minimum of the Fano factor in the adiabatic limit, i.e. low driving frequencies, assumes values considerably larger than the standard value 1/2, as discussed above. Upon increasing the driving frequency, the minimum becomes lower until at an amplitude-dependent value $f_{\text{min}}$, it starts to increase again. Figure 3(b) shows a nice agreement between theory and experiment for the development of the minimum of the Fano factor for an amplitude $A = 10$ meV considering a certain noise in the experimental data. In particular, the data clearly confirm the frequency independence of the minimum beyond the SR point. Next we consider the location of the minimum in the spectrum, $\omega_{\text{min}}$, depicted in Fig. 3(c), where the transition from $\omega_{\text{min}} = 0$ to a finite value corresponds to the splitting of the dip as observed in Fig. 2(a). This happens at a frequency $f^*$ which increases with increasing driving amplitude. In the inset of Fig. 3(c) the detailed dependence on the driving amplitude is plotted. The growth of $f^*$ with the driving amplitude reminds one to the shift of the working point which has also been observed for the “usual” SR in closed systems [2].

Figure 3(d) depicts the minimum of the Fano factor as function of the driving amplitude for three driving frequencies (beyond the SR). A strong reduction of the Fano factor is clearly observed, as also seen in the experimental data (marked by circles) for a driving frequency $A = 10$ meV and various driving frequencies. The colored lines mark experimental data, while the black lines are computed with the master equation approach. The other parameters are $\alpha_L = 0.09$, $\alpha_R = 0.065$, and $\Gamma_0 = 1.675$ kHz.

The periodically time-dependent case and show that the probability distributions of the waiting times between a tunnel event and its $(\ell + 1)$st successor, $w_\ell(\tau)$, which oscillate with the driving frequency [29] and which we sample from experimental data.

For $\varphi$ we only know that it is $T$-periodic and has zero mean [16]. Without the driving, it vanishes such that Eq. (4) recovers a result from renewal theory [12] [13]. To determine $\varphi$, we notice that for large time difference $\tau$, the tunnel events are uncorrelated and, thus, $S$ vanishes. Therefore, $\varphi$ can be identified as the long-time oscillations of $w(\tau)$. Accordingly in the frequency domain we use the fact that finite-time Fourier transform converts long-time oscillations to poles of first order, while $S(\omega)$ is expected to be a smooth function. Therefore, poles in the Fourier transformed of $w(\tau)$ can be attributed to $\varphi$. They can be determined by fitting.

**SR signatures in the Fano factor.**—In Ref. [7], the existence of SR in quantum transport has been demonstrated with the zero-frequency Fano factor as a noise measure. However, a complete picture of the noise must include its full spectral properties. As a reference, let us first mention that in the absence of driving, the current in a symmetric quantum dot has white noise characterized by the constant Fano factor $F(\omega) = 1/2$ [30-32]. Moreover, for adiabatic driving, the symmetry gets lost such that most of the time the zero-frequency noise is enhanced [33]. Since we are interested in SR, we consider much larger frequencies of the order $\Gamma_0$. Figure 2(a) shows noise spectra of the total current for various nonadiabatic driving frequencies. For the relatively low frequency $f = 0.4$ kHz, the zero frequency noise is already smaller than the standard value 1/2 expected for the undriven dot. In addition, in the vicinity of $\omega = 0$, however, $F(\omega) > 1/2$. With increasing frequency $f$, we witness the dip in the noise spectrum at $\omega = 0$ becoming deeper and broader. Close to the SR condition $f \approx \Gamma_0/2$, the dip evolves into a double dip located at the driving frequency $\pm f$ ($f = 2$ kHz and 4 kHz), which underlines the importance of considering the whole noise spectrum. While the zero-frequency noise insinuates disappearance of the SR effect—and becomes almost insensitive to it at $f = 4$ kHz—the frequency-dependent analysis reveals that the noise suppression remains, but occurs in the spectrum at finite frequency. In contrast, the current noise at source and drain depends only weakly on the driving, as can be seen in Figs. 2(b) and 2(c). Only when the driving frequency exceeds the SR frequency, i.e., for $f \geq \Gamma_0/2$, the Fano factor of the source and drain currents develop small dips at $\omega \approx \pm 2\pi f$. Interestingly, the noise of the dot current [Fig. 2(d)] and, thus, that of the dot occupation are practically independent of the driving. This emphasizes that for transport SR, the noise properties are primarily manifest in the total current.
of 8 kHz. At strong driving the reduction is enhanced for frequencies closer to the SR condition, see the curve for 2 kHz in comparison with the curve at 8 kHz.

To investigate the amplitude dependence in more detail (Fig. 3) shows the frequency dependent Fano factor $F(\omega)$ at three different applied amplitudes for a driving frequency much larger than $f^*$. The curves exhibit the typical double-dip structure discussed above. With an increasing amplitude, the shape of the Fano factor starts to deviate from the Lorentzian obtained for weak driving. Moreover, for $A = 30$ meV, we witness that the impact of non-linearities is visible as a tiny additional dip at $\omega/2\pi \approx 3f$. In the experimental data the additional dip is less clear, because the large driving amplitude makes it increasingly difficult to determine with sufficient precision the poles stemming from the last term in Eq. 1 of Ref. 10. The increased broadening of the dips with increasing amplitude is, by contrast, even more expressed in the experimental data.

Recent interest in controlled single-electron tunneling stems from the challenge of building current standards that transport an integer number of electrons per cycle. Let us therefore discuss the frequency-dependent Fano factor in this context. Figure 4(a) shows the average (electric) current as a function of the driving amplitude and frequency. In the same figure also the current $I = ef$ (black line) and the above discussed frequencies $f_{\text{min}}$ (red dashed line) and $f^*$ (blue dashed line) are shown as a function of the driving amplitude. For amplitudes smaller than $A \approx 25$ meV, the three lines more or less overlap, i.e. there is no clear difference between $f_{\text{min}}$ and $f^*$ and they mark the quantized current with $I = ef$. For larger amplitudes a plateau-like structure with current $I \approx ef$ is observed [white region in Fig. 4(b)] as expected from the experiments investigating single-electron pumping for current standards. The line on which $I = ef$ is fulfilled exactly lies in the middle of this plateau and between transition frequency $f^*$ and the frequency $f_{\text{min}}$ at which the Fano factor is minimal. With increasing amplitude, both the width of the plateau and the difference between $f_{\text{min}}$ and $f^*$ become larger. Accordingly, for large frequencies, very low Fano factors require larger amplitudes, see Fig. 3(d). Such a plateau widening with increasing amplitude was also observed in the pumping experiments investigating current standards, see e.g. Refs. 35, 36. Interestingly, the two frequencies $f_{\text{min}}$ and $f^*$ mark the borders of the plateau. In this way an analysis of the frequency dependent Fano factor can help to optimize the pumping conditions for the current standard.

**Conclusions.**—We have analyzed experimentally and theoretically the frequency dependent current noise in a transport SR experiment. The most noticeable effect is visible in the power spectrum of the total current as a splitting of a dip at zero frequency to a double dip located at the driving frequency. For small amplitudes, the transition between these two qualitatively different regimes occurs when the SR condition is met. With increasing amplitudes, the transition frequency shifts towards larger values. Our results show the relation between transport SR and quantized electron pumping used...
for current standards.

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SUPPLEMENTAL MATERIAL

Appendix A: Master equation and jump operators

We consider a quantum dot that for energetic reasons can be occupied with at most one electron, such that the dynamics is restricted to the spin states $|\uparrow\rangle$, $|\downarrow\rangle$, and the empty state $|0\rangle$ with probabilities $P_{\uparrow}$, $P_{\downarrow}$, and $P_{0}$. For weak tunnel coupling and large bias, electrons can enter only from the left lead (source) and will leave to the right lead (drain). In the absence of magnetic fields and spin effects, this situation is captured by the Markovian master equation

$$\frac{d}{dt} \begin{pmatrix} P_{\uparrow} \\ P_{\downarrow} \\ P_{0} \end{pmatrix} = \begin{pmatrix} -\Gamma_{R} & 0 & \Gamma_{L}/2 \\ 0 & -\Gamma_{R} & \Gamma_{L}/2 \\ \Gamma_{R} & \Gamma_{R} & -\Gamma_{L} \end{pmatrix} \begin{pmatrix} P_{\uparrow} \\ P_{\downarrow} \\ P_{0} \end{pmatrix},$$

where the factor $1/2$ for the source-dot tunneling is introduced for the ease of notation. In our case, the rates $\Gamma_{L/R}$ are periodically time-dependent owing to an ac gate voltage applied to the tunnel barriers. Introducing the probability $P_{1} = P_{\uparrow} + P_{\downarrow}$ for the occupation with any spin projection, we obtain a master equation $\dot{\rho} = \mathcal{L}\rho$ for $\rho = (P_{1}, P_{0})$ with the Liouvillian

$$\mathcal{L} = \begin{pmatrix} -\Gamma_{R} & \Gamma_{L} \\ \Gamma_{R} & -\Gamma_{L} \end{pmatrix}.$$  \tag{A2}

Using the convention that an upper index $+$ refers to currents flowing from the central conductor of one of the leads, the corresponding jump operators read [26]

$$\mathcal{J}_{L}^{-} = \begin{pmatrix} 0 & \Gamma_{L} \\ 0 & 0 \end{pmatrix}, \quad \mathcal{J}_{R}^{+} = \begin{pmatrix} 0 & 0 \\ \Gamma_{R} & 0 \end{pmatrix},$$

while for uni-directional transport, $\mathcal{J}_{L}^{+} = \mathcal{J}_{R} = 0$.

Appendix B: Frequency-dependent Fano factor

We consider a Markovian master equation $\dot{\rho} = \mathcal{L}(t)\rho$ for the reduced density operator of the conductor with a periodically time dependent Liouvillian $\mathcal{L}(t) = \mathcal{L}(t + T)$. The time-dependence may affect the conductor itself as well as the leads or the conductor-lead couplings. Our goal is a propagation method for the computation of the stationary symmetric current-current correlation function

$$S(t, t') = \frac{1}{2} \langle [\Delta j(t), \Delta j(t')]_{+} \rangle,$$  \tag{B1}

where $j$ is defined as the time derivative of the particle number $n$ in a given region such as the leads or the central conductor, while $\Delta j = j - \langle j \rangle$.

An alternative characterization of current fluctuations is the full counting statistics of the transported particles with the conditional moments

$$M_{k}(t|t') = \langle n^{k}(t) \rangle_{t'},$$  \tag{B2}

with the boundary condition $M_{k}(t'|t') = 0$ for all $k > 0$. Both concepts are related by the MacDonald formula [17] which for time-independent problems establishes a connection between the current correlation function \(\text{(B1)}\) and the variance of the number of transported particles. In a first step, we generalize this formula to the time-dependent case and subsequently use the result as a basis for computing the phase-averaged current noise spectrum.

1. MacDonald formula for time-dependent transport

The moments $M_{k}$ of a probability distribution contain the same information as the corresponding cumulants (or irreducible moments) [18]. For a Markovian transport process, the latter eventually grow linear in time, which motivates the definition of current cumulants as their time derivatives [20]. Obviously, the first current cumulant, i.e., the rate by which the number of transported particles grows is the current, while the second current cumulant is the time derivative of the variance, $c_{2}(t|t') = \frac{1}{n} \mathcal{M}_{2}(t|t') - \mathcal{M}_{2}^{2}(t|t')$. Upon noticing that $\dot{n} = j$, the definitions of $S$ and $c_{2}$ directly provide the relation

$$c_{2}(t|t') = 2 \int_{t}^{t'} dt'' S(t, t'').$$  \tag{B3}

For time-independent systems, two-time expectation values such as $S(t, t')$ are called stationary if they are homogeneous in time, i.e., if $S(t, t') = S(t + \tau, t' + \tau)$ for any time translation $\tau$. Setting $\tau = -t'$ yields the known fact that stationary correlation functions depend only on time differences, $S(t, t') = S(t - t')$. Then Eq. \(\text{(B3)}\) in Fourier representation reads $c_{2}(\omega) = 2iS(\omega)/\omega$, which allows one to compute $S(\omega)$ directly from the time evolution of $c_{2}$ [17].

For periodically driven systems, time translation invariance is granted only for time shifts by multiples of
the driving period $T$. Hence, stationarity corresponds to
the weaker relation
\[ S(t, t') = S(t + T, t' + T). \] (B4)

Nevertheless, one can define a noise measure that depends on only a single time argument, namely the phase-averaged correlation function \[ \overline{S} \], which is equivalent to the average over one driving period when keeping the time difference $\tau = t - t'$ constant. Hence, we can characterize the current fluctuations by the function
\[ \tilde{S}(\tau) = \int_{\text{period}} dt S(t + \tau, t). \] (B5)

As the integrand in this expression is periodic in $t$ and by definition symmetric in its time arguments, one can easily see that $\tilde{S}(\tau) = \tilde{S}(-\tau)$. To find a relation between the conditional current cumulant and the correlation function, we substitute in Eq. \[ \text{(B3)} \] the primed times by the time differences $t = t'$ and $t'' = t - t''$ and introduce the time shift $t \to t + \tau$ to obtain
\[ c_2(t + \tau|t) = 2 \int_0^\tau d\tau' S(t + \tau, t + \tau - \tau'). \] (B6)

According to Eq. \[ \text{(B4)} \], the right-hand side of this relation is $T$-periodic in $t$, and hence the left-hand side as well. Therefore, we can perform an average over the driving period and after taking the derivative with respect to $\tau$, we obtain
\[ \tilde{c}_2(\tau) = \frac{1}{2} \frac{d}{d\tau} c_2(\tau) \] (B7)
with the time-averaged second current cumulant \[ \tilde{c}_2(\tau) = \int_{\text{period}} dt c_2(t + \tau|t). \] (B8)

We will see below that $c_2$ can be computed by numerical propagation of a generalized master equation.

In a final step, we use the symmetry $\tilde{S}(\tau) = \tilde{S}(-\tau)$ to write $\tilde{S}(\omega)$ as a Fourier cosine transformed and integrate by parts to obtain
\[ \tilde{S}(\omega) = \omega \int_0^\infty d\tau \sin(\omega \tau) \tilde{c}_2(\tau), \] (B9)
which represents the requested generalization of the MacDonald formula for periodically driven conductors. It relates the phase-averaged noise spectrum $\tilde{S}(\omega)$ to the conditional second cumulant. For time-independent problems, $c_2(t + \tau|t) = c_2(\tau)$, such that the $t$-integration in Eq. \[ \text{(B5)} \] becomes trivial and one recovers the classic MacDonald formula \[ \text{(L7)}. \]

Let us remark that the zero-frequency limit of Eq. \[ \text{(B9)} \] obeys the relation \[ \lim_{\omega \to 0} \tilde{S}(\omega) = \lim_{\tau \to \infty} c_2(t + \tau|t), \] (B10)
which is independent of $t$. For periodically driven conductors, this may be proven upon noticing that the right-hand side of Eq. \[ \text{(B3)} \] is the zero-frequency limit of a Fourier integral \[ \text{[24]} \] or via the long-time theorem of Laplace transformation.

2. Propagation method for cumulants

The time evolution of the current cumulants for a time-dependent transport problem can be computed with the hierarchical scheme derived in Ref. \[ \text{[11]} \]. Here we briefly sketch the underlying ideas and adapt them to our needs.

We consider the number of electrons $n$ in one of the leads as the main observable. To keep track of this degree of freedom even after tracing it out, we introduce a counting variable $\chi$ and define the moment generating function $\overline{\mathcal{Z}}(\chi) = \langle e^{i\chi n} \rangle$. For its computation, we employ a generalized reduced density operator $\overline{R}(\chi)$ constructed such that $\text{tr} \overline{R}(\chi) = \overline{\mathcal{Z}}(\chi)$. It obeys the master equation (we omit obvious time arguments) \[ \text{[20]} \]
\[ \frac{d}{dt} \overline{R}(\chi) = [\mathcal{L} + \mathcal{J}(\chi)] \overline{R}(\chi), \] (B11)
\[ \mathcal{J}(\chi) = (e^{i\chi} - 1) \mathcal{J}^+ + (e^{-i\chi} - 1) \mathcal{J}^-, \] (B12)
where the jump operators $\mathcal{J}^\pm = \mathcal{J}^\pm_{L/R}$ describe incoherent electron tunneling from the conductor to the respective lead and back.

The associated current cumulants are the derivatives of the generating function $\phi(\chi) = \frac{d}{d\chi} \ln \overline{\mathcal{Z}}(\chi)$ and read $c_k = (\partial/\partial i\chi)^k \phi(\chi)|_{\chi = 0} \text{[20]}$. Defining $X(\chi) = \overline{R}(\chi)/\overline{\mathcal{Z}}(\chi)$, Taylor expansion of the generalized master equation \[ \text{[B11]}, \phi(\chi), \text{and } X(\chi) \text{ yields the iteration } \text{[11]} \]
\[ c_k = \sum_{k' = 0}^{k - 1} \binom{k}{k'} \text{tr } \mathcal{J}^{(k-k')} X_{k'}, \] (B13)
\[ \dot{X}_k = \mathcal{L} X_k + \sum_{k' = 0}^{k - 1} \binom{k}{k'} \{ \mathcal{J}^{(k-k')} - c_{k-k'} \} X_{k'}, \] (B14)
where
\[ \mathcal{J}^{(k)} = \mathcal{J}^+ + (-1)^k \mathcal{J}^- \] (B15)
while $c_k$ and $X_k$ are the Taylor coefficients of $\phi$ and $X$, respectively. Notice that for $k = 0$, Eq. \[ \text{(B14)} \] is the usual master equation for the reduced density operator $X_0$. For time-independent problems, this scheme is equivalent to the iteration derived in Refs. \[ \text{[19, 20]} \]. Truncating Eqs. \[ \text{[B13] and [B14]} \] at second order provides the time-dependent current expectation value and the corresponding second cumulant.

The initial condition of the differential equation \[ \text{[B14]} \] deserves some attention. Let us recall that we are interested in the stationary conditional moments, which means that at initial time $t_0$ of the integration, (i) all moments and cumulants must vanish and (ii) transients
of the density operator must have decayed already. Condition (i) is granted when \( \chi(t) \) at \( t_0 \) is \( \chi \)-independent. Thus, all its Taylor coefficients vanish with the exception of \( \chi_0 \). For \( \chi_0 \), condition (ii) means that it must be the steady-state density operator at \( t_0 \). \( X_0(t_0) = \rho_{\text{ss}}(t_0) \) It is computed by starting the propagation at some time \( t_0 - t_{\text{trans}} \), where \( t_{\text{trans}} \) is the timescale on which transients decay.

3. Ramo-Shockley theorem

Currents in the leads are affected by the displacement current of the electric field of the fluctuating charge distribution. While this is usually irrelevant for the average current, it may play a significant role for the current noise at finite frequency [24]. Therefore, it is here important to notice that the measured current in the leads (also referred to as the total current) flowing from the left to the right lead (with electron numbers \( N_{L/R} \)) is a weighted average of the currents at both contacts. Using the sign convention \( I_{L/R} = N_{L/R} \), it is given by

\[
I_{\text{tot}} = -\kappa_L I_L + \kappa_R I_R, \tag{B16}
\]

where the weights \( \kappa_L \) and \( \kappa_R \) reflect the capacitances at the contacts normalized such that \( \kappa_L + \kappa_R = 1 \). We assume that the driving does not affect these capacitances.

As the total charge is conserved, the number of electrons on the conductor changes as \( \dot{Q} = -I_L - I_R \). Then the correlation \( S_{\text{tot}} \) of the total current becomes

\[
S_{\text{tot}} = \kappa_L S_L + \kappa_R S_R - \kappa_L \kappa_R S_Q, \tag{B17}
\]

which holds in the time domain and in the frequency domain, as well as for the phase average in Eq. (B5). While \( S_L \) and \( S_R \) can be computed as described above, the direct computation of \( S_Q \) needs more care, because generally, one cannot simply add the jump operators of the left and the right lead as insinuated by Eq. (B16).

To derive the jump operators for the total current and for the displacement current, we employ ideas of Ref. [9]. We start from the integrated form of Eq. (B16) and charge conservation, i.e. \( N_{\text{tot}} = -\kappa_L N_L + \kappa_R N_R \) and \( Q = -N_L - N_R \), respectively, where \( N_{\text{tot}} \) is the number of electrons transported with the total current. Then we write the exponent in the expression for the moment generating function \( Z(\chi_L, \chi_R) \), in terms of \( N \) and \( Q \) as

\[
\chi_L N_L + \chi_R N_R = ( -\kappa_R \chi_L - \kappa_L \chi_R ) Q + ( \chi_R - \chi_L ) N_{\text{tot}},
\]

which lets us conclude that the counting variables for \( N_{\text{tot}} \) and \( Q \) read \( \chi_{\text{tot}} = \chi_R - \chi_L \) and \( \chi_Q = -\kappa_R \chi_L - \kappa_L \chi_R \), respectively. In turn,

\[
\begin{align*}
\chi_L &= -\kappa_L \chi_{\text{tot}} - \chi_Q, \\
\chi_R &= \kappa_R \chi_{\text{tot}} - \chi_Q. \tag{B18}
\end{align*}
\]

Now the corresponding jump operators follow simply by differentiation of the Liouvillian with respect to the associated counting variable using the chain rule [9].

\[
\begin{align*}
\mathcal{J}^{(k)}_{\text{tot}} &= \partial_{\chi_{\text{tot}}} \mathcal{L}(\chi_L, \chi_R)|_{0} = (-\kappa_L)^k \mathcal{J}^{(k)}_L \rho_{\text{tot}}, \\
\mathcal{J}^{(k)}_Q &= \partial_{\chi_Q} \mathcal{L}(\chi_L, \chi_R)|_{0} = (-1)^k \mathcal{J}^{(k)}_L + (-1)^k \mathcal{J}^{(k)}_R. \tag{B19}
\end{align*}
\]

In the final expression no mixed derivatives occur, because the contribution of each lead enters as a separate term. Interestingly, the first-order jump operators, \( \mathcal{J}^{(1)}_{\text{tot}} \) and \( \mathcal{J}^{(1)}_Q \), are as naively expected, while in particular the higher orders of \( \mathcal{J}^{(k)}_{\text{tot}} \) contain powers of the contact capacities. Notice that a possibly different sign convention for the currents merely leads to a global prefactor \( (-1)^k \), which is irrelevant for derivatives of even order and, thus, for the current correlation function.

4. Numerical scheme

For the practical computation, we start from the Liouvillian for the conductor, identify the jump terms and attribute them counting variables to find \( \mathcal{J}_{L/R} \) and, thus, \( \mathcal{J}_{\text{tot}} \) and \( \mathcal{J}_Q \). Then we use the iteration in Eqs. (B13) and (B14) to obtain for each current the corresponding conditional second moment \( \bar{c}_2(t + \tau | t) \) as function of \( \tau \) for different initial times \( t \). Averaging over \( t \) yields \( \bar{c}_2(\tau) \), which we finally insert into Eq. (B9). The remaining Fourier integral is conveniently evaluated via discrete Fourier transformation. Since the zero-frequency contribution of the resulting spectrum may be rather sensitive to numerical noise, we determine it from the long-time limit in Eq. (B10).

Appendix C: Current noise from waiting-time distributions

We consider the current as a spike train

\[ j(t) = \sum_k \delta(t - s_k) \tag{C1} \]

with tunnel events at times \( s_k \), which in the experiment can be obtained by monitoring the dot occupation. However, the \( s_k \) may not be known with sufficient precision to exclude effects of possible long-time phase drifts of the external driving. Therefore, we will base the evaluation on the differences between subsequent events, \( t_k = s_k - s_{k-1} \), which also provide the times between events with \( \ell \) events in between. In practice, each data set consists of the order \( 10^6 \) events, while for \( \ell \) values up to several \( \sim 100 \) are sufficient. Our goal is now to obtain from these data the phase-averaged power spectral density \( S(\tau) \) of the spike train. For this purpose, we generalize a relation
between the power spectrum and the waiting time statistics known from renewal theory \cite{12, 13} to periodically time-dependent systems.

The power spectrum of a (time-independent) renewal processes can be computed from the waiting time distribution between two subsequent events, \( w_0(\tau) \). The derivation makes use of the fact that the waiting time distributions \( w_\ell(\tau) \) of the \((\ell + 1)\)st successor of an event, \( w_\ell(\tau) \) for \( \ell > 0 \) can be computed simply by an \( \ell \)-fold auto convolution of \( w_0 \) \cite{12, 13}. However, this relation holds true only in the time-independent case, while we will sample the \( w_\ell(\tau) \) from our experimental data. In the following, we demonstrate that in periodically time-dependent cases, knowledge of the \( w_\ell \) allows one to compute the phase-averaged power spectrum \( S(\omega) \).

Let us start by noticing that for the spike train in Eq. (C1), any contribution to the expectation value \( \langle j(t + \tau) j(t) \rangle \) originates from spikes at times \( t + \tau \) and \( t + \tau + \tau \), where for the moment we assume \( \tau > 0 \). Therefore, this expectation value must be proportional to the joint probability density for spikes at times \( t \) and \( t + \tau \), i.e.,

\[
\langle j(t + \tau) j(t) \rangle = \lambda p(t + \tau, t), \tag{C2}
\]

where \( p(t', t) dt' dt \) is the probability for an event in the time interval \([t, t + dt]\) and a further one in the different interval \([t', t' + dt']\). The proportionality factor \( \lambda \) will be determined later. Subtracting from the left-hand side the product of the expectation values \( \gamma(t) = \langle j(t) \rangle \) and \( \gamma(t + \tau) \) obviously yields the auto correlation of the spike train,

\[
S(t + \tau, t) = \langle j(t + \tau) j(t) \rangle - \gamma(t + \tau) \gamma(t). \tag{C3}
\]

Owing to the periodic time-dependence of \( \gamma(t) \), the \( t \)-average of the last term is not only the square of the mean spike rate, \( \bar{\gamma} = \langle \gamma(t) \rangle \), but in addition contains a \( T \)-periodic term \( \varphi(\tau) \) with zero mean. Thus, the phase-averaged auto correlation becomes

\[
\bar{S}(\tau) = \lambda p(t + \tau, t)^2 - \bar{\gamma}^2 - \varphi(\tau). \tag{C4}
\]

Next we relate the right-hand side of Eq. (C4) to the waiting-time distributions. To this end, we notice that \( w_\ell(\tau) \) is the \( \ell \)-averaged probability density for finding a spike at time \( t + \tau \) conditioned to the occurrence of a spike at time \( t \) and \( \ell \) spikes in between. Thus,

\[
w_\ell(\tau) = \langle p(t + \tau | t) \rangle_\ell = \langle p(t + \tau) p(t) \rangle_\ell, \tag{C5}
\]

where \( \langle x(t) \rangle_\ell = \frac{1}{\ell} \int_0^T dt x(t) p(t) = \bar{x(t)} p(t) \) denotes the expectation value of a quantity \( x \) that occurs with probability \( p(t) dt \) in the time interval \([t, t + dt]\). The overbar, by contrast, refers to the simple average by time integration over one driving period as in Eq. (B5), i.e. the phase-average. Bayes theorem now tells us that \( p(t + \tau | t) p(t) = p(t + \tau, t) \), which is the joint probability density for events at time \( t \) and \( t + \tau \) with \( \ell \) events

\[
\text{in between. Summation over } \ell \text{ removes the latter restriction leading to } \sum_\ell p(t + \tau, t) = p(t + \tau, t). \text{ Hence, the right-hand side of Eq. (C5) can be identified as the term required in Eq. (C4), namely}
\]

\[
p(t + \tau, t) = \sum_\ell w_\ell(\tau) \equiv w(\tau). \tag{C6}
\]

A typical example of \( w(\tau) \) for our experiment is shown in Fig. S1.

As \( w(\tau) \) is a probability density of events, despite some conditions at an earlier time, in the long-time limit, at least on average, it must be equal to the mean event rate \( \bar{\gamma} \). This implies that the \( \tau \)-averaged long-time limit of

\[
\text{FIG. S2. Data shown in Fig. 2 of the main text, but without removing the contribution of the poles caused by } \varphi(\omega) \text{ at multiples of the driving frequencies.}
\]
Eq. (C4) becomes $0 = \lambda^2 - \bar{\gamma}^2$ [recall that $\varphi(\tau)$ has zero mean], which provides the missing proportionality factor $\lambda = \bar{\gamma}$.

So far, we have assumed $\tau > 0$. To obtain a relation valid for any $\tau$, we recall that $S(t, t') = S(t', t)$ is symmetric by definition and at $t = t'$ must have a $\delta$-peak that reflects shot noise, $\gamma(t)\delta(t - t')$ [12, 27, 30]. Hence, we find that the auto correlation of the spike train and the sum of the waiting time distributions relate as

$$\bar{S}(\tau) + \bar{\gamma}^2 = \bar{\gamma}\delta(\tau) + \bar{\gamma}w(|\tau|) - \varphi(|\tau|).$$  \hspace{1cm} (C7)

A final Fourier transformation provides the power spectral density

$$\bar{S}(\omega) = \bar{\gamma} + 2\bar{\gamma} \int_0^\infty d\tau \cos(\omega \tau)w(\tau) - \sum_k \varphi_k(\omega - k\Omega)$$  \hspace{1cm} (C8)

with some irrelevant coefficients $\varphi_k$, where $\varphi_0$ absorbs the term $\bar{\gamma}^2$. This relation forms the basis of our evaluation of the experimental data.

For a finite set of sampled data, $w(\tau)$ can be determined only within a finite time window. Then instead of $\delta$-peaks, the discrete Fourier transform of $\varphi$ leads to poles of the type $e^{(\omega - k\Omega)\tau_{\text{max}}}/(\omega - k\Omega)$, as can be seen in Fig. [SC2] Their contribution can be determined by analyzing the long-time behavior of $w(\tau)$ which is $T$-periodic and equal to $\varphi(\tau)$. Alternatively, one may compute the discrete Fourier transformation of $w(\tau)$ and determine the emerging poles in $\bar{S}(\omega) + \varphi(\omega)$ by fitting the result in small regions around all relevant resonances $k\Omega$ to functions $A_k/(\omega - B_k)$ and subtract the result to obtain $\bar{S}(\omega)$. Owing to the discrete representation of the spectrum and to numerical noise, it is advantageous to treat $B_k$ as a fit parameter, despite that it is known to match $k\Omega$. For our data, this approach worked rather reliably.

Finally, we have to relate the spike trains with the currents discussed in the main text. To this end, we consider the spike trains $j_L, j_R$, and $\bar{j}$ for the events at the source, at the drain, and at both contacts, respectively. We evaluate each as described above to obtain the power spectra $S_L, S_R, \bar{S}$. As $\bar{j} = j_L + j_R$, this also provides the cross correlation $\langle j_L, j_R \rangle = \bar{S} - S_L - S_R$. The latter allows us to obtain the power spectra of the displacement current and the total (or Ramo-Shockley) current

$$S_Q = 2(S_L + S_R) - \bar{S},$$  \hspace{1cm} (C9)

$$S_{\text{tot}} = \kappa_L \kappa_R (\kappa_L - \kappa_R) S_L + \kappa_R (\kappa_R - \kappa_L) S_R,$$  \hspace{1cm} (C10)

respectively, where one has to pay attention to the sign convention. For the present symmetric case, $\kappa_L = \kappa_R = 1/2$, Eq. (C10) becomes $S_{\text{tot}} = \bar{S}/4$. Since the mean spike rate for $\bar{j}$ is twice the electron current from source to drain, the Fano factor of the total current is $F = \bar{F}/2$.

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[1] P. Jung and P. Hänggi, Phys. Rev. A 44, 8032 (1991)
[2] L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, Rev. Mod. Phys. 70, 223 (1998).
[3] F. Moss, L. M. Ward, and W. G. Sannita, Clin. Neurophysiol. 115, 267 (2004).
[4] M. Grifoni and P. Hänggi, Phys. Rev. Lett. 76, 1611 (1996).
[5] T. Wellens and A. Buchleitner, Phys. Rev. Lett. 84, 5118 (2000).
[6] C. K. Lee, L. C. Kwek, and J. Cao, Phys. Rev. A 84, 062113 (2011).
[7] T. Wagner, P. Talkner, J. C. Bayer, E. P. Rupergamigablo, P. Hänggi, and R. J. Haug, Nature Phys. 15, 330 (2019).
[8] C. Emary, D. Marcos, R. Aguado, and T. Brandes, Phys. Rev. B 76, 161404(R) (2007).
[9] D. Marcos, C. Emary, T. Brandes, and R. Aguado, New J. Phys. 12, 123009 (2010).
[10] R. Hussein, J. Gómez-García, and S. Kohler, Phys. Rev. B 90, 155424 (2014).
[11] M. Benito, M. Niklas, and S. Kohler, Phys. Rev. B 94, 105433 (2016).
[12] D. R. Cox, Renewal Theory (Methuen & Co., London, 1962).
[13] W. Gerstner, W. M. Kistler, R. Naud, and L. Paninski, Neural Dynamics (Cambridge University Press, 2014).
[14] B. Kaestner and V. Kashcheyevs, Rep. Prog. Phys. 78, 103901 (2015).
[15] S. Platonov, B. Kästner, H. W. Schumacher, S. Kohler, and S. Ludwig, Phys. Rev. Lett. 115, 106801 (2015).
[16] See Supplemental Material, which includes Refs. [17–20], for a derivation of the formalism and the relation between $\bar{S}(\omega)$ and the waiting time distributions.
[17] D. K. C. MacDonald, Rep. Prog. Phys. 12, 56 (1949).
[18] H. Risken, The Fokker-Planck Equation, 2nd ed., Springer Series in Synergetics, Vol. 18 (Springer, Berlin, 1989).
[19] C. Flindt, T. Novotný, A. Braggio, M. Sassetti, and A.-P. Jauho, Phys. Rev. B 82, 155407 (2010).
[20] C. Flindt, T. Novotný, A. Braggio, and A.-P. Jauho, Phys. Rev. B 79, 155407 (2008).
[21] S. Ramo, Proc. I. R. E. 27, 584 (1939).
[22] W. Shockley, J. Appl. Phys. 9, 635 (1938).
[23] S. Kohler, J. Lehmann, and P. Hänggi, Phys. Rep. 406, 379 (2005).
[24] Y. M. Blanter and M. Büttiker, Phys. Rep. 336, 1 (2000).
[25] P. Jung and P. Hänggi, Phys. Rev. A 41, 2977 (1990).
[26] D. A. Bagrets and Y. V. Nazarov, Phys. Rev. B 71, 055316 (2005).
[27] N. Ubbelohde, C. Fricke, C. Flindt, F. Hohls, and R. J.
Haug, Nature Comm. 3, 612 (2012).

[33] R.-P. Riwar, J. Splettstoesser, and J. König, Phys. Rev. B 87, 195407 (2013).

[34] As the experimental spectra are noisy, we have convoluted them with a Gaussian of width $\sigma = 100$ Hz before reading off the minima.

[35] M. D. Blumenthal, B. Kaestner, L. Li, S. Giblin, T. J. B. M. Janssen, M. Pepper, D. Anderson, G. Jones, and D. A. Ritchie, Nature Phys. 3, 343 (2007).

[36] B. Kaestner, V. Kashcheyevs, G. Hein, K. Pierz, U. Siegner, and H. W. Schumacher, Appl. Phys. Lett. 92, 192106 (2008).