CORES AND OPTIMAL FUZZY COMMUNICATION STRUCTURES OF FUZZY GAMES

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Abstract. In real game problems not all players can cooperate directly, games with communication structures introduced by Myerson in 1977 can deal with these problems quite well. More recently, this concept has been introduced into fuzzy games. In this paper, games on (fuzzy) communication structures were studied. We proved that if a coalitional game has a nonempty core, then the game restricted on an n-person connected graph also has a nonempty core. Further, the fuzzy game restricted on the n-person connected graph also has a nonempty core. Moreover, we proved the above two cores are identical and the core of the coalitional game is included in them. In addition, optimal fuzzy communication structures of fuzzy games were studied. We showed that the optimal communication structures do exist and proposed three allocating methods. In the end, a full illustrating example was given.

1. Introductions. There are two main solution concepts in cooperative games. Core is one of the major set-valued solutions while Shapley value is the uppermost one-point solution. The two definitions are simple and straightforward and can be characterized by several axioms [3, 5, 8, 13, 14]. They have been extensively studied since their inception and these two solutions are applied in fuzzy games immediately with the emergence of the fuzzy game concept [1].

In classic cooperative games there is an implicit assumption that any two players can communicate directly and freely. Actually that may not be the case. Some players may lack necessary resources or trust to establish links. For example, buyers and sellers need middlemen; both sides in a conflict need a third party for mediation. Myerson [4, 10] introduced ‘communication structure’. A communication structure is an undirected graph whose vertices are the players in a game. The Myerson value of a game on a communication structure is a generalized Shapley value. If a communication structure becomes a complete graph, the Myerson value is just the Shapley value.

The concept of fuzzy cooperative game is due to Aubin [1]. He pointed out that under certain circumstance some players may not fully participate in a coalition, but to a certain extent. He introduced the core and the value of games with fuzzy coalitions by extending the corresponding definitions in classic cooperative games.

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More recently, the concept of communication structure was introduced into fuzzy games by Jiménez-Losada et al. [6, 7], they proposed a general framework in order to define fuzzy Myerson values.

It is conceivable that a generalized core can be defined in games on communication structures. And, the generalized core should be consistent with the classic (fuzzy) core if the communication structure becomes a complete graph. We have shown that this is possible. In most solutions of cooperative games, coalitions or fuzzy coalitions are seen as bargaining powers in dividing the worth of the grand coalition. Whether (fuzzy) coalitions can be formed is not considered seriously. If resources are divisible, forming several fuzzy coalitions may be better even if the grand coalition is efficient. The second point of this paper focuses on optimal fuzzy communication structures of fuzzy games and discusses the value allocation problem under the optimal structure.

2. Preliminaries. Let $N$ be a nonempty finite set of agents who consider different cooperation possibilities. Each subset $S$ of $N$ is referred as a coalition. The set $N$ is called the grand coalition and $\emptyset$ is called the empty coalition. We denote the collection of coalitions, i.e. the set of all subsets of $N$ by $2^N$.

A cooperative game in characteristic function form is a pair $(N,v)$ consisting of the player set $N$ and the characteristic function $v: 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$.

Often we identify the game $(N,v)$ with its characteristic function $v$ and call $v(S)$ the value of coalition $S$. In this paper a cooperative game in characteristic function form is abbreviated as a cooperative game. The set of all cooperative games is denoted by $G^N$.

A fuzzy coalition is a subset of $N$, which is identified with a function from $N$ to $[0, 1]$. Then for a fuzzy coalition $S$ and player $i$, $s(i)$, or $s_i$ indicates the membership grade of $i$ in $S$, i.e., the rate of the ith player’s participation in $S$. The class of all fuzzy subsets of $N$ is denoted by $[0, 1]^N$.

A fuzzy cooperative game is a pair $(N,v)$ consisting of the player set $N$ and the characteristic function $v: [0, 1]^N \to \mathbb{R}$ with $v(\emptyset) = 0$. The set of all fuzzy cooperative games is denoted by $FG^N$.

A game $v \in FG^N$ is superadditive if $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \in [0, 1]^N$ with $S \cap T = \emptyset$. According to von Neumann et al. [2,16], for any coalition $S$, $v(S)$ is regarded as the least profit which can be achieved by its members when $S$ formed. It follows that any game $v$ is superadditive either it is crisp or fuzzy.

Definition 2.1. A payoff vector $x \in \mathbb{R}^n$ is an imputation for the game $v \in G^N$ if it is efficient and individually rational, i.e.

a) $\sum_{i \in N} x_i = v(N)$;

b) $x_i \geq v(i)$ for all $i \in N$.

Individual rationality means that any player’s payoff should not be less than the income under solo effort. Efficiency means that the worth of the grand coalition must be divided completely.

Definition 2.2. The core of a game $v \in G^N$ is the subset of imputations which are stable against any possible deviation by coalitions, i.e.

$$c(v) = \left\{ x \in \mathbb{R}^N \left| \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \text{ for all } S \in 2^N \right. \right\}$$
Definition 2.3. The core of a fuzzy game $v \in FG^N$ is the subset of imputations which are stable against any possible deviation by fuzzy coalitions, i.e.

$$c(v) = \left\{ x \in \mathbb{R}^N \middle| \sum_{i \in S} x_i = v(N), \sum_{i \in S} s_i x_i \geq v(S) \text{ for all } S \subseteq [0, 1]^N \right\}$$

Obviously, Definition 2.2 is a special case of Definition 2.3.

Definition 2.4. \( \forall S \in [0, 1]^N \), let \( Q(S) = \{ S(i)|S(i) > 0, i \in N \} \), and let \( q(s) \) be the cardinality of \( Q(S) \). We write the elements of \( Q(S) \) in the creasing order as \( h_1 < \cdots < h_{q(S)} \). Then a game \( v \in FG^N \) is said to be a fuzzy game ‘with Choquet integral form’ if and only if the following holds:

$$v(S) = \sum_{h=1}^{q(S)} v(S_{h_i}) \cdot (h_l - h_{l-1})$$

for all \( S \in [0, 1]^N \), where \( h_0 = 0 \).

In a fuzzy game with Choquet integral form, a fuzzy coalition is decomposed into several fuzzy coalitions in each of which players share a same participation rate. Consider a three-player game with Choquet integral form. Suppose \( S = (0.3, 0.6, 0.4) \), \( S \) is decomposed into a grand coalition in which three players share a same participation rate 0.3, a coalition of player II and III with participation rate (0.4–0.3) and a solo coalition of player II with participation rate (0.6–0.4). \( v(S) = 0.3v(\{1, 2, 3\}) + 0.1v(\{2, 3\}) + 0.2v(\{2\}) \). More about fuzzy games with Choquet integral form, one can refer to Tsurumi et al. [11,15].

A communication structure is a graph vertices of which correspond to the players in the game. Let \( g^N \) denote the complete graph formed by all members of \( N \). Let \( gR \) be the set of all graphs over \( N \), i.e., \( gR = \{ g|g \subseteq g^N \} \). Given \( g \in gR, S \subseteq N \), then a partition is formed over \( S \) by \( g \). We denote it by \( S/g = \{ \{i|i\text{ and }j\text{ are linked in }S\text{ by }g\}|j \in S \} \). Obviously, \( S \) is connected in \( g \) if and only if \( S = S/g \). The set of all connected coalitions in \( g \) is denoted by \( 2^N/g \).

Definition 2.5. A fair allocation rule of a cooperative game \((N, v)\) is a function satisfy the following equations:

\( a) \sum_{i \in S} \phi_i (v, g) = v(S), \forall g \in gR, \forall S \in N/g \)

\( b) \phi_i (v, g) - \varphi_i (v, g - \{i, j\}) = \varphi_j (v, g) - \phi_j (v, g - \{i, j\}) \)

\( \forall g \in gR, \forall S \in N/g, \forall \{i, j\} \in g \)

Condition \( a) \) requires the worth of coalition \( S \) must be divided completely; condition \( b) \) requires the two endpoints of a side should get equal payoffs from this side. Myerson [1] proved that there exists a unique fair allocation rule for a given cooperative game \((N, v)\). Particularly, there is \( \varphi_i (v, g^S) = \varphi_i (v, S) \) for all \( S \subseteq N \). Therefore, the fair allocation rule can be regarded as a generalization of the Shapley function and the communication structure is a generalized cooperative form. The set of all games on cooperative structures is denoted by \( CSG^N \).

3. Cores of games on communication structures. As shown above, there exists a generalized Shapley function for games on communication structures. In this section we will propose a generalized core for games on communication structures and discuss the relationship between the two cores.

For a game on a communication structure, the value of coalition \( S \) does not necessarily be \( v(S) \) but depends on how it is split by the graph \( g \). Let \( v/g(S) \) denotes
the value of coalition $S$ in games with communication structure then $v/g(S) = \sum_{R \in S/g} v(R)$.

**Example 3.1.** Consider a four-player game cooperating on the above graph:

Then $v/g(12) = v(12), v/g(14) = v(1) + v(4), v/g(124) = v(12) + v(4)$ since Player 4 is not connected with either Player 1 or Player 2. Since $g$ is connected over $N, v/g(N) = v(N)$.

Using this notation, the fair allocation rule, or Myerson value for a given cooperative game $(N,v)$, $\phi_i(v, g)$ is equal to $\phi_i(v g)$, the Shapley value of game $(N,v/g)$.

**Definition 3.2.** the core of a game on a communication structure $v/g \in GCS^N$ is a set of payoff vectors satisfying $\sum_{i \in N} x_i = v/g(N)$ and $\sum_{i \in S} x_i \geq v/g(S)$ for all $S \subseteq N$. The core of a game on a communication structure is denoted by $c(v/g)$.

**Theorem 3.3.** If the core of a game $v$ is nonempty, then this game on a communication structure $v/g$ has a nonempty core as long as $g$ is connected over $N$ and $c(v) \subseteq c(v/g)$.

**Proof.** Let $x \in c(v)$, since $g$ is connected over $N$, there is $\sum_{i \in N} x_i = v/g(N) = v(N)$. Now consider any coalition $S$, if $S$ is connected by $g$, then there is $\sum_{i \in S} x_i \geq v(S) = v/g(S)$. If $S$ is not connected then $S$ is split into several connected blocks by $g$. For any $R \in S/g$, the inequality $\sum_{i \in R} x_i \geq v(R)$ holds. Then there is $v/g(S) = \sum_{R \in S/g} v(R) \leq \sum_{i \in S} x_i$. That is, $x \in c(v/g)$.

Please note that if $g$ is not connected over $N$, then the conclusion may not hold.

For a game on a communication structure $v/g$, we consider the following linear programming problem:

\[
\begin{align*}
\text{Min}_x(N) & \\
\text{s.t. } x(S) & \geq v/g(S) \text{ for all } S \subseteq N, S \neq \emptyset.
\end{align*}
\]

Clearly, $v/g$ has a nonempty core if and if the programming optimal value is $v/g(N)$. If $g$ is connected over $N$, then the programming problem can be simplified to the following form:

\[
\begin{align*}
\text{Min}_x(N) & \\
\text{s.t. } x(S) & \geq v(S) \text{ for all } S \in 2^N/g, S \neq \emptyset. \quad (1)
\end{align*}
\]

The characteristic vector $e^S$ of $S$ is a member of $\mathbb{R}^N$ where $e^S_i = 1$, if $i \in S$; $e^S_i = 0$, otherwise. Using the foregoing notion the dual program of (1) is:

\[
\begin{align*}
\text{Max} & \sum_S \delta_S v(S) \\
\text{s.t. } \sum_S \delta_S e^S & = e^N \\
& \text{where } \delta_S \geq 0, S \in 2^N/g \text{ and } S \neq \emptyset. \quad (2)
\end{align*}
\]

As both programming problems are feasible, game $v/g$ has a nonempty core if and only if $v(N) \geq \sum_{S \in 2^N/g} \delta_S v(S)$ for all feasible vectors $(\delta_S)_{S \in 2^N/g}$ of (2). This assertion leads to Theorem 3.5 that is in exactly the same form as the weak form of Bondareva-Shapley Theorem.
Definition 3.4. A collection $B \subseteq 2^N$, $\emptyset \notin B$ is called balanced if there exist $\delta_S > 0, S \in B$ such that $\sum_{S \in B} \delta_S e^S = e^N$ where $(\delta_S)_{S \in B}$ is called a system of balancing weights.

Theorem 3.5. A necessary and sufficient condition that the core of a game $(N,v/g)$ is not empty is that for each balanced collection $B \subseteq 2^N/g$ and each system $(\delta_S)_{S \in B}$ of balancing weights, the inequalities

$$v(N) \geq \sum_{S \in B} \delta_S v(S).$$

hold.

Clearly, the set of feasible vectors of (2) is a convex polytope. Then, the core of a game $(N,v/g)$ is not empty if and only if $v(N) \geq \sum_{S \in B} \delta_S v(S)$ for every extreme point of (2). If $g$ is an n-dimension complete graph, then (2) becomes (3).

$$\begin{align*}
\max \sum_S \delta_S v(S) \\
\text{s.t.} \quad \sum_S \delta_Se^S = e^N \\
\text{where } \delta_S \geq 0, S \in 2^N \text{ and } S \neq \emptyset.
\end{align*}$$

A balanced collection is called minimal balanced if it does not contain a proper balanced subcollection. It’s already been proved that a collection is minimal balanced if and only if its system of balancing weights $(\delta_S)_{S \in B}$ is an extreme point of (3). See Peleg [12]. By the same manipulation, one can proof that a collection $B \subseteq 2^N/g$ is minimal balanced if and only if its system of balancing weights $(\delta_S)_{S \in B}$ is an extreme point of (2).

Thus, the above discussion leads to Theorem 3.6 almost the same as the sharp form of Bondareva-Shapley Theorem.

Theorem 3.6. A necessary and sufficient condition that the core of a game $(N,v/g)$ is not empty is that for each minimal balanced collection $B \subseteq 2^N/g$,

$$v(N) \geq \sum_{S \in B} \delta_S v(S).$$

where $(\delta_S)_{S \in B}$ is the system of balancing weights for $B$.

4. Cores of games on fuzzy communication structures. Following Myerson, Jiménez-Losada et al. [7] introduced the concept of fuzzy communication structures. Let $L$ be the set of bilateral communications among the players in $N$.

Definition 4.1. A fuzzy communication structure for the game $v$ is an undirected fuzzy graph over $N$, this is a pair $\gamma = (\tau,\rho)$ where $\tau \in [0,1]^N$ denotes the fuzzy set of vertices and $\rho \in [0,1]^L$ denotes the fuzzy set of links satisfying $\rho(i,j) \leq \tau(i) \land \tau(j)$ for all $\{i,j\} \in L$. The set of fuzzy communication structures over $N$ is denoted by $FCS^N$. The set of all fuzzy games on fuzzy communication structures is denoted by $FCSG^N$.

Let $\gamma = (\tau,\rho)$ be a fuzzy graph. The number $\tau(i)$ is interpreted as the real level of involvement of player $i \in N$ in the game $v$. The number $\rho(i,j)$ represents the maximal level to which the link $\{i,j\}$ can be used.
Then the fair allocation rule is naturally extended as fuzzy fair allocation rule. After characterizing the fuzzy Myerson values, they discussed two well known models, i.e., proportional model and Choquet model on fuzzy graphs in detail [6,9].

In Choquet model on fuzzy graphs, the fuzzy graph is decomposed into a number of simple smaller fuzzy graphs, participants and links in each smaller fuzzy graph share a common activity level. In this case, the value of the fuzzy graph is the sum of the value of each simple graph multiplied by its activity level. Given a communication structure \( g \), there exists a unique maximum fuzzy graph in the links set of which satisfying \( \rho(i, j) = \tau(i) \land \tau(j) \) or \( \rho(i, j) = 0 \) if there is no link between \( i \) and \( j \) in \( g \) corresponding to a fuzzy coalition.

**Example 4.2.** Consider the fuzzy graph \( \gamma \) in Figure 1. \( \gamma \) is decomposed into three simple graphs. In the first graph the activity level of all vertices (participants) and links is 0.4; the other two graphs can be interpreted in the same manner. And, the worth of the fuzzy graph \( \gamma, v(\gamma) \) is by definition equal to the total worth of the following three simple graphs. Thus, fuzzy games on fuzzy graphs are transformed into crisp games on ordinary graphs. A player’s fuzzy Myerson value is the sum of his (her) fuzzy Myerson values in the three simple fuzzy graphs.

With the concept of fuzzy communication structure we can introduce a core concept extended from Definition 3.2.

**Definition 4.3.** the core of a game on a fuzzy communication structure \( v/g \in FCSG^N \) is a set of payoff vectors satisfying \( \sum_{i \in N} x_i = v(g)(N) \) and \( \sum_{i \in S} \tau_i x_i \geq v(g)(S) \) for all \( S \subseteq N \), where \( \tau_i \) is the \( i \)th player’s activity level in \( S \).

Just as \( v(S) \) already been mentioned above, here \( v(g)(S) \) is the value of a (fuzzy) coalition \( S \) on graph \( g \). Note that \( v(g)(S) \) is not necessarily equal to \( v(S) \). The core of a game on fuzzy a communication structure is denoted by \( c(v/g) \).

Definition 4.3 can be explained in the same manner: the worth of the grand coalition must be divided and no sub fuzzy graph can refuse it.

**Theorem 4.4.** If the core of a cooperative game on a communication structure \( v/g \) is nonempty, then the game on fuzzy communication structures \( v/g \) also has a nonempty core. And there is \( c(v/g) = c(v/g) \).

**Proof.** Obviously, there is \( c(v/g) \subseteq c(v/g) \), so we only need to show \( c(v/g) \subseteq c(v/g) \). Consider a fuzzy graph \( v/g \subseteq g, v \) can be decomposed into a number of small fuzzy graphs. Let \( \{g_1, g_2, \ldots, g_r\} \) denote the set of the small graphs and \( \{p_1, p_2, \ldots, p_r\} \) denote the set of activity levels of the corresponding graphs. The vertices set of graph \( g_i \) is denoted by \( S_i \). Let \( x \in c(v(g)) \), then there is

\[
v(g)(\phi) = \sum_i v(g)(p_i g_i) = \sum_i p_i v(g)(g_i) \leq \sum_i (p_i \sum_{j \in S_i} x_j) = \sum_j p_j x_j.
\]
And clearly, there is
\[ \sum_{i \in N} x_i = v/g(N) = v/fg(N). \]
That is, \( x \in c(v/fg) \). That ends the proof. \( \square \)

Note that \( v/g(N) \) does not necessarily equal to \( v(N) \) unless \( g \) is connected over \( N \).
By theorem 3.3 and 4.4, there always is \( c(v) \subseteq c(v/g) = c(v/fg) \).

5. **Optimal fuzzy communication structures of fuzzy games and allocating methods.** In a classic crisp cooperative game, if the game is supperadditive, then the grand coalition \( N \) is surely the best one. When we talk about fuzzy games, we mean the bargaining power of a fuzzy coalition \( S \). Can \( S \) get more from the total worth? Whether \( S \) can really form has not been taken seriously. If fuzzy coalitions allowed, players can earn more even if the game is supperadditive.

**Example 5.1.** Consider a four-player crisp cooperative game such as \( v(S) = 1, 3, 5, 6 \) when \( |S| = 1, 2, 3, 4 \). Apparently, this game is supperadditive. In the grand coalition, each player gets \( 6/4 = 1.5 \). Now assume players can partly participate in several coalitions simultaneously, then players will form four three-player coalitions and each player joins three of them. That is,
\[ 1/3(123), 1/3(124), 1/3(134), 1/3(234). \]
In this coalition structure each player gets \( 5/3 \). All players now are better than before.

Zhan and Zhang [17] discussed the above question and drew a general conclusion. Now we turn to games on communication structures. Likewise, if fuzzy communication structures allowed, player may also form certain fuzzy communication structures. Considering all fuzzy communication structures is neither possible nor necessary. Note that the worth of a non-connected graph is equal to the total worth of its connected sub graphs and fuzzy graphs are decomposed into simple fuzzy graphs each of which is equivalent to an ordinary graph multiplied by a constant. Hence only ordinary connected graphs need to be taken into consideration. The remaining question is to determine the activity level, or, participating rate of each of such graphs.

Assume that players in a game \((N,v)\) are cooperating on a graph \( g \) that is connected over \( N \). Let \( S \subseteq N \) be a crisp coalition connected in \( g \), and \( x_S \cdot S \) denotes the fuzzy coalition, or called fuzzy graph, formed by all players in \( S \) with a same participating rate \( x_S \). Note that the total participating rate of a player in all fuzzy graphs equals 1. So we can get the linear programming as follows:
\[
\begin{align*}
\max & \sum_{S \subseteq 2^N/g} v(x_S \cdot S) \\
\text{s.t.} & \sum_{i \in S} x_S = 1 & i \in N \\
& x_S \geq 0 \\
\end{align*}
\]
(4)
By careful examination one can find formulas (2) and (4) are actually the same. (2) is the dual of the linear programming describing core while (4) comes to the same conclusion from a different perspective. That is, if the game \( v/g \) has a nonempty
core, then the players will cooperate on the grand graph $g$. If $v/g$ isn’t balanced, then the players will find cooperating on several small graphs more profitable.

**Example 5.2** (continuing 5.1). Now the four players are cooperating on the following graph. Other conditions are same as above. What fuzzy communication structures will they form?

![Diagram](image)

According to (4), there is the following linear programming.

$$\max \sum_{S \in 2^n / g} v(x_S \cdot S)$$

s.t. \[
\begin{align*}
  x_A + x_{AB} + x_{AC} + x_{AD} + x_{ABC} + x_{ACD} + x_{ABD} + x_N &= 1 \\
  x_B + x_{AB} + x_{BC} + x_{ABC} + x_{ABD} + x_N &= 1 \\
  x_C + x_{AC} + x_{BC} + x_{ABC} + x_{ACD} + x_N &= 1 \\
  x_D + x_{AD} + x_{ABD} + x_{ACD} + x_N &= 1 \\
  x_S &\geq 0
\end{align*}
\]

Solving the problem, $x_{BC} = x_{ABD} = x_{ACD} = 0.5$, other variables equal zero. They will form the following three fuzzy communication structures instead of the big graph $g$. In this instance players can achieve 6.5.

![Diagram](image)

Then there comes a natural question: how to divide the total worth reasonably? This is critical because it will determine whether the optimal fuzzy communication structures can stay stable. At least every player should not become worse in this situation than before. If the worth of each fuzzy graph in the optimal communication structures is only divided in itself, then we can’t make sure that every player will become better.

One feasible way is a four-step method. First allocate $v(N)$ by fair allocation rule, each player gets his (her) Myerson value. Next allocate the increment to fuzzy communication structures in proportion to their values. Then allocating the worth distributed in the second step in each fuzzy communication structure by fair allocation rule again. Lastly, calculate players’ final payoffs.

The second method is much simpler. Taking the value of the optimal communication structures as the value of the graph $g$, allocating the value by fair allocation rule, each player gets his (her) bigger Myerson value.

The third method is somewhat complex. We turn to consider whether there exist an appropriate core concept for the problem. We need check Example 5.1 and 5.2 together. If $v(N)$ increases to 6.5, the core of the game $(N, v)$ is still empty. But according to Formula 2 and Theorem 3.5, the core of the game $(N, v/g)$ is not
empty, and, by Theorem 4.4, the core of the game \((N, vfg)\) is also not empty this moment. Can the core be taken as an allocation method?

Since the grand coalition can earn more by forming fuzzy communication structures, of course other coalitions also can do so. That is, any crisp or fuzzy coalition \(S\) (or called ordinary graph or fuzzy graph) may earn more by forming fuzzy communication structures in its members. Modifying the linear programming (4) slightly we can get the new programming that describes the optimal fuzzy communication structures in any graph either it is ordinary or fuzzy.

\[
\max \sum_{T \subseteq \text{Supp } \gamma, T \in 2^N / g} v(x_T \cdot T)
\]

\[
\text{s.t. } \sum_{i \in T} x_T = \tau(i) \ \forall i \in \tau
\]

\[
x_T \geq 0
\]

(5)

In (5) \(\gamma\) is a (fuzzy) graph and \(\text{Supp } \gamma\) is a ordinary graph whose vertices and links are same with those of \(\gamma T\) is a ordinary connected sub graph of \(\text{Supp } \gamma\), i.e., a crisp coalition connected in \(\text{Supp } \gamma\).

Consider a game \((N, v^* )\) cooperating on graph \(g\) (and its sub fuzzy graphs) where \(v^*(\gamma) = \max \sum_{T \subseteq \text{Supp } \gamma, T \in 2^N / g} v(x_T \cdot T)\). If the game \(v^*/fg\) has a nonempty core then it can be taken as an allocating method for optimal fuzzy communication structures.

**Theorem 5.3.** the core of the game \(v^*/fg\) is nonempty.

**Proof.** We first introduce a game \((N, v')\) cooperating on graph \(g\) where \(v'(N) = v^*(N)\) and \(v'(S) = v(S)\) for any \(S \neq N\). According to Formula 2 and Theorem 3.5, the core of the game \((N, v'/g)\) is nonempty. Consider \(\gamma \subseteq g\), assume that \(v^*(\gamma)\) is realized by fuzzy communication structures \(\{t_1T_1, t_2T_2, \ldots, t_lT_l\}\). Then the optimal fuzzy communication structure of \(T_j\) is \(T_j\) itself, that is, \(v^*(T_j) = v(T_j)\). Let \(x \in c(v'/g)\), then

\[
\sum_{i \in T_j} x_i \geq v^*(T_j) \Rightarrow \sum_{j=1}^l t_j \left( \sum_{i \in T_j} x_i \right) \geq \sum_{j=1}^l t_j v^*(T_j) = v^*(\gamma),
\]

while

\[
\sum_{j=1}^l t_j \left( \sum_{i \in T_j} x_i \right) = \sum_{i \in \gamma} \tau_i x_i,
\]

so we get

\[
\sum_{i \in \gamma} \tau_i x_i \geq v^*(\gamma).
\]

No graph can reject \(c(v'/g)\), that is, \(c(v'/g) \subseteq c(v^*/fg)\). Obviously, there is \(c(v^*/fg) \subseteq c(v'/g)\). Hence we have \(c(v^*/fg) = c(v'/g)\). That ends the proof.

6. A full illustrating example. Consider a four-player crisp cooperative game such as \(v(S) = 1, 3, 5, 7\) when \(|S| = 1, 2, 3, 4\).
If they are cooperating on the complete graph, it’s quite easy to find the core $c(v)$.

$$c(v) = \{(x_1, x_2, x_3, x_4) | x_i \geq 5/3 \text{ and } \sum x_i = 7\}$$

If there is no direct link between Player B, D and Player C, D, the core $(v/g)$ is described by the following inequalities:

$$\begin{cases}
    x_1 + x_2 \geq 3, & x_1 + x_3 \geq 3, & x_1 + x_4 \geq 3, & x_2 + x_3 \geq 3 \\
    x_1 + x_2 + x_3 \geq 5, & x_1 + x_2 + x_4 \geq 5, & x_1 + x_3 + x_4 \geq 5 \\
    x_1 + x_2 + x_3 + x_4 = 7
\end{cases}$$

It’s straight to see that $c(v) \subseteq c(v/g)$. And, $(3, 1.5, 1.5, 1) \in c(v/g)$, but $(3, 1.5, 1.5, 1) \notin c(v)$.

Now we turn back to Example 5.2, suppose $v(N) = 6$, we’ve already known the optimal cooperative structures, i.e., $x_{BC} = x_{DAB} = x_{DAC} = 0$. The optimal value is 6.5. Now we’ll calculate the specific results of the three distribution methods mentioned in the above section.

**Method I** the four-step method.

Firstly, calculate each player’s Myerson value. That has already been given in Example 5.2.

$$\varphi_A(v,g) = 46/24, \varphi_B(v,g) = \varphi_C(v,g) = 34/24, \varphi_D(v,g) = 30/24.$$  

Secondly, allocate the increment (6.5–6) to fuzzy communication structures in proportion to their values. Note that

$$v(0.5 \ast BC) = 3/2, v(0.5 \ast ABD) = v(0.5 \ast ACD) = 5/2,$$

hence the three fuzzy communication structures get 3/26, 5/26, 5/26 respectively.

Thirdly, allocate the worth distributed in the former step in each fuzzy communication structure by fair allocation rule again.

In the fuzzy communication structure 0.5*BC, respectively, Player B, C get

$$3/52, 3/52.$$  

In the fuzzy communication structure 0.5*ABD, respectively, Player A, B, D get

$$1/13, 3/52, 3/52.$$  

In the fuzzy communication structure 0.5*ACD, respectively, Player A, C, D get

$$1/13, 3/52, 3/52.$$  

Lastly, calculate each player’s payoff in total. Player A’s final payoff is 323/156 ($46/24 + 1/13 + 1/13$). Similarly, we can get $B, C,$ and $D$’s final payoffs. They are 239/156, 239/156 and 213/156.
Method II: the bigger Myerson value method.

Although players are now cooperating on three small fuzzy communication structures, we are still pretending they are cooperating on the grand graph. And now $v(N)$ is not 6, but 6.5. Note that players' final payoffs don't increase proportionally. By fair allocation rule,

$$\varphi_A(v, g) = \frac{49}{24}, \varphi_B(v, g) = \varphi_C(v, g) = \frac{37}{24}, \varphi_D(v, g) = \frac{33}{24}.$$  

Method III: the core method.

According to Theorem 5.3, $c(v*/fg)$ is described by the following inequalities:

$$\begin{align*}
&x_i \geq 1 \\
&x_1 + x_2 \geq 3, \ x_1 + x_3 \geq 3, \ x_1 + x_4 \geq 3, \ x_2 + x_3 \geq 3 \\
&x_1 + x_2 + x_3 \geq 5, \ x_1 + x_2 + x_4 \geq 5, \ x_1 + x_3 + x_4 \geq 5 \\
&x_1 + x_2 + x_3 + x_4 = 6.5
\end{align*}$$

After some calculation, we get the core $c(v*/fg)$. That is,

$$c(v*/fg) = (2 + x, 1.515, 1.5 - x),$$

where $x \in [0, 0.5]$.

Now we can easily verify that the solutions of the former two methods are not belonging to the core. To make everyone better than before, there must be

$$(2 + x, 1.515, 1.5 - x) \geq (46/24, 34/24, 34/24, 30/24),$$

i.e., $x \in [0, 0.25]$.

7. Conclusions and discussions. In this paper cores of games on (fuzzy) communication structures were introduced. We got very similar results as those in common coalitional games. The relationship of cores of games on (fuzzy) communication structures and cores of common coalitional games was discussed. Just as in common coalitional games, players in games on communication structures can improve their situation by forming several small fuzzy communication structures. Three allocation methods were shown and an illustrating example was given in the end to elaborate the methods. However, if graphs are not connected, we have not come up with a general conclusion. And, in the theory of endogenous coalition, forming coalitions and distributing payoffs are dealt with in a same framework. In this paper communication structures are still given, not endogenous. Further study on the dynamic relation between communication structures evolving and distributing payoffs is needed.

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