Structure of critical perturbations in a horizontal layer of melted water with the prescribed heat flux at the boundaries

V A Sharifulin\textsuperscript{1} and T P Lyubimova\textsuperscript{2}

\textsuperscript{1} Perm National Research Polytechnic University, 29, Komsomolsky ave., Perm, 614990, Russia
\textsuperscript{2} Institute of Continuous Media Mechanics UB RAS, 1, Korolev St., Perm, 614013, Russia

E-mail: vadsharif@bk.ru

Abstract. The paper deals with the investigation of critical perturbations structure in a horizontal layer of fluid with temperature inversion of density. The lower boundary of the layer is rigid and upper boundary is free and undeformable. The constant vertical heat flux is imposed at both boundaries. The thermocapillarity, evaporation and radiation are neglected. The temperature dependence of the density is assumed to be quadratic. The coordinate of the density inversion point which shows the position of the plane of maximal density in the fluid layer in the conductive state is used for the description of density inversion effect. The influence of the location of density inversion point on structure of critical perturbations is studied. It is shown that, depending on the location of the inversion point, velocity profile of longwave critical perturbations can have two-floor or three-floor structure. Stability to finite-wavelength perturbations is studied for the entire range of their existence. The asymptotic formulas for critical values of the Rayleigh number and wave number are obtained, the structure of finite-wavelength critical perturbations is determined.

1. Introduction

Non-monotonic, i.e. non-linear dependence of thermal expansion of water was mentioned in the Book [1] by G. Ostroumov on thermal convection. Veronis G. in [2] suggested to approximate this non-linear temperature dependence of water density by simple quadratic formula. In 1963 it was shown independently by Eklund H. [3] that in normal conditions in the range between zero and eight degrees the specific volume of water is well described by quadratic dependence on temperature. Independently of [2,3], the quadratic temperature dependence of density was obtained by Goren S.L. in [4]. In modern notations it has the form:

$$\rho = \rho_0(1 - \alpha(T - T_0))$$

where $\alpha = 0.8 \cdot 10^{-3} (C)^{-2}$, $T_0 = 4.00C$.

The onset of convection in a horizontal fluid layer heated from below in the presence of density inversion described by (1) was studied for the first time in [2]. In [2] the case of isothermal boundaries was considered and the subcritical excitation of convection was detected. The mathematical equivalency of the conductive state stability problem for a fluid layer with rigid isothermal boundaries in the presence of density inversion and the Couette flow between rotating cylinders was noted in this
paper. In the latter case, the ratio of angular velocities of inner and outer cylinders plays the same role as the ratio of the height of stably stratified part of the layer to its full thickness. A simple algebraic formula relating the Taylor and Rayleigh numbers was derived.

In [5] the important cases of fixed temperature and fixed heat flux (directed vertically from the bottom to the top, see figure 1) at the boundaries were firstly investigated; however, an error was made in deriving the linearized equations. Correct formulation of the problem of the conductive state stability for the case of rigid lower and free and undeformable upper boundary was given in [6]. The dimensionless inversion point coordinate $z_i$ was used for the description of density inversion effect. The parameter $z_i$ has simple physical meaning: this parameter describes the position of a maximum density plane within a fluid layer in the conductive state.

At $z_i < 0$ the density stratification in layer is stable throughout the entire layer and the conductive state is absolutely stable; at $z_i > 1$ (the dimensionless thickness of the layer equals to 1) stratification is unstable throughout the entire fluid layer and at $0 < z_i < 1$ the density stratification is unstable only in a part of the layer, namely, at $0 < z < z_i$.

The thermal conditions corresponding to a fixed heat flux are favorable for the development of longwave perturbations which, in the case of linear temperature dependence of the density (the Boussinesq fluid) are the most dangerous [7]. In [6] it was shown that differ from the Boussinesq fluid, in the case of fluid with density inversion this is not always the case. The longwave perturbations remain the most dangerous if potentially unstable stratification exists throughout the entire fluid layer, or, at least, if the layer with stable stratification is thin. However, when the layer with unstable stratification is thin, the longwave perturbations are strongly stabilized and the finite-wavelength perturbations become the most dangerous.

In this paper we investigate the structure of both longwave and finite-wavelength critical perturbations.

![Figure 1. Horizontal layer with fixed heat flux at the boundaries.](image)

2. Method of solution

In conditions of stationary heat flux $A$ from bottom to top possible motionless equilibrium state. Let us consider normal monotonic perturbations of this state. In [6] it is shown that the form of normal critical perturbations and the values of critical Rayleigh number for fixed values of wave number $k$ and parameter $z_i$ can be obtained from the solution of the boundary-value problem:

$$\begin{align*}
\Delta^2 W + Ra(z - z_i)k^2 \phi &= 0, \\
\Delta \phi + W &= 0, \\
z = 0: \phi' = W = W' &= 0, \\
z = 1: \phi' = W = W'' &= 0.
\end{align*}$$

(2)
Here $W$ and $\theta$ – dimensionless amplitudes of vertical velocity and temperature perturbations, $Ra = 2\alpha g A^2 h^3 / \chi^2$ - Rayleigh number. Analytical method for the long wave limit $k \to 0$ and numerical method for finite $k$ are described in [6]. In the present work the form of critical perturbations is defined from the solution of the problem (2), such study was not performed earlier.

3. Results

Longwave instability occurs at $z_i = 5/9 \approx 0.55$ [6]. In the range $0.61 < z_i < \infty$ the longwave perturbations are more dangerous than perturbations with finite wavelength. The influence of $z_i$ is presented in figures 2, 3 (the boundary between unstably and stably stratified layers is marked by the dotted line). The stable layer in these figures is large enough and comparable in its size with the unstable one.

![Figure 2](image1.png)  
*Figure 2.* At the moment of the onset of longwave instability there are 3 layers with different form of critical perturbations. For this value of $z_i$ the perturbations with finite wavelength are more dangerous. The profile of horizontal velocity $X_i$ of neutral longwave perturbations at $z_i = 0.57$ is presented. Velocity values are normalized by the value of Rayleigh number. The dotted line describes the upper boundary of the layer with unstable stratification.

![Figure 3](image2.png)  
*Figure 3.* Profile of horizontal velocity $X_i$ of neutral longwave perturbations for $z_i = 0.6104$. For this value of $z_i$ the perturbations with finite wavelength are not the most dangerous. The dotted line marks the upper boundary of the layer with unstable stratification.
Surprisingly, the perturbations presented in figure 3 has the velocity values on the free surface close to zero. Maximal velocity of fluid is replaced deep in unstable layer.

The growth of the thickness of unstable layer significantly changes the form of the profile of critical perturbations. The maximum value of velocity is reached now on the free surface (see figure 4 for $z_i = 0.8$). Further growth of $z_i$ does not change significantly the form of critical longwave perturbations of the velocity.

Figure 4. Profile of horizontal velocity $X_1$ of neutral longwave perturbations for $z_i = 0.8$.

For this value of $z_i$ the perturbations with finite wavelength are not the most dangerous. The dotted line marks the upper boundary of the layer with unstable stratification.

If the thickness of unstably stratified layer is less than 0.61 the finite-wavelength perturbations become more dangerous than the longwave ones. The neutral curves obtained by the solution of linear stability problem are presented in figure 5 for three values of $z_i$.

Figure 5. Neutral curves for three values of $z_i$. 
Figure 6. Neutral curves of finite-wavelength instability for $z_i = 1/2$ (dashed line) and $z_i = 1/3$ (solid line).

Figure 7. The conductive state stability boundaries to the longwave (dashed line) and finite-wavelength (solid line) perturbations.

Figure 8. Streamlines of critical finite-wavelength perturbations for $z_i = 1/3$. Dotted line marks upper boundary of layer with unstable stratification. One can see that cells penetrate deeply in stably stratified layer of fluid.

We introduce the modified Rayleigh number $Ra'$ and the modified wave number $k'$, determined using the thickness of unstably stratified layer, i.e. $z_i$. They are related to the Rayleigh number determined by the thickness of the layer as

$$Ra' = z_i^3 Ra, \quad k' = k / z_i.$$  \hspace{1cm} (3)

The calculations show that in these variables the neutral curves are almost independent for $z_i \leq 1/3$. This means that the instability boundary with respect to finite-wavelength perturbations (separating the regions I and II in figure 8) for $z_i \leq 1/3$ can be described by the asymptotic formula:
\[ Ra_{cr} = 591/\varepsilon^3 \]  

(4)

Conclusions

The structure of both longwave and finite-wavelength critical perturbations has been investigated для всех значений координаты точки инверсии \( 0 < \varepsilon < \infty \).

In the case when unstable stratification occupies the whole layer or its major part (0.61 < \varepsilon < \infty), the critical longwave perturbations have simple two-floor structure, beside at \( \varepsilon = 0.61 \) the velocity of the perturbations at the free surface is close to zero. In the case 0.55 < \varepsilon < 0.61 when the longwave perturbations are less dangerous than the wavelength perturbations the longwave perturbations have exotic three-floor structure.

Finite-wavelength perturbations at \( \varepsilon < 0.55 \) have two-floor structure and occupy the whole layer. The asymptotic formula is obtained for the dependence of critical Rayleigh number on \( \varepsilon \) from the analysis of the obtained results.

Acknowledgments

The work was supported by the Ministry of Education and Science of Russian Federation (project No. 3.6990.2017/BCh).

References

[1] Ostroumov G A 1958 Free convection under the conditions of the internal problem. NASA Tech. Memo No. 1407
[2] Veronis G 1963 Astrophys. J. 137 641
[3] Eklund H 1963 Science 142 1457-8
[4] Goren S 1966 Chemical Engineering Science 21 515
[5] Merker G, Waas P, Straub J and Grigull U 1976 Warme und Stoffubertrag 9 99
[6] Lyubimov D, Lyubimova T and Sharifulin V 2012 Fluid Dynamics 47 448
[7] Gershuni G Z and Zhukhovitskii E M 1972 Convective Stability of Incompressible Fluids (Moscow: Nauka) [in Russian]