Generic uniqueness of saddle point for two-person zero-sum differential games

1 Introduction

In the 1950s, Isaacs [1] initiated the study of two-person zero-sum differential games. Later in the 1960s and 1970s, Berkovitz [2], Elliott-Kalton [3], Fleming [4], and Friedman [5] also made contributions. Two-person zero-sum differential games were investigated extensively in the literature as they are widely used in many fields, such as biology, finance, and engineering, and also play a key role in the research of general differential games. Ramaswamy and Shaiju [6] proved convergence theorems for the approximate value functions by Yosida type approximations and constructed approximate saddle-point strategies within the sense of feedback in Hilbert Space. Berkovitz [7] defined differential games of fixed duration and showed that games of fixed duration that satisfy Isaacs condition have saddle point. Ghosh and Shaiju [8] proved the existence of saddle point equilibrium for two-player zero-sum differential games in Hilbert space. Ammar et al. [9] derived sufficient and necessary conditions for an open-loop saddle point of rough continuous differential games for two-person zero-sum rough interval continuous differential games. In particular, Sun [10] derived a sufficient condition of the existence of an open-loop saddle point for two-person zero-sum stochastic linear quadratic differential games in 2021. We refer the reader to [11,12] and references therein.

It is worth noting that uniqueness is important in both practice and theory, especially in mathematical problems including two-person zero-sum differential games. However, how many problems have a unique solution? In fact, most mathematical problems cannot guarantee the uniqueness of the solution. So, we have to settle for the second thing: generic uniqueness (see Remark 3.1).

Regarding the generic uniqueness, many results have been investigated. Kenderov [13] studied the solutions of optimization problems and obtained an important result: most optimization problems have a unique solution. Ribarska and Kenderov [14] in their work proved that most two-person zero-sum continuous games have a unique solution in the sense of Baire's category. Tan et al. [15] studied the saddle point for general functions and derived the generic uniqueness of saddle points by the set-valued analysis method. Yu et al. [16] considered the generic uniqueness of equilibrium points for general equilibrium problems.
On the other hand, Yu et al. [17] presented the existence and stability of optimal control problems using set-valued analysis theory in 2014 and showed that most of the optimal control problems are generic stable. After that, Deng and Wei [18, 19] proved that generic stability result of optimal control problems governed by semi-linear evolution equation and nonlinear optimal control problems with 1-mean equilibrium controls, respectively. In 2020, the generic stability of Nash equilibria is investigated by Yu and Peng in their work [20] on noncooperative differential games in the sense of Baire’s category.

To the best of our knowledge, there is no published result for the generic uniqueness of saddle point for two-person sum-person differential games. The purpose of this paper is to study such problems. We point out that the main idea of the present paper comes from the works of Kenderov [13], Ribarska and Kenderov [14], and Yu et al. [15, 20].

The remainder of this paper is organized as follows. The next section is devoted to formulating the game model, collecting some basic preliminary, and stating some properties of a saddle point. In Section 3, we formulate a space of problem and introduce a set-valued mapping. We then state some continuous dependence of state trajectory and cost functional and present some main results in this paper. Finally, some conclusions are given in Section 4.

2 Model and preliminaries

We begin with classical differential games governed by ordinary equations. Let $R^n$ and $R^q$ be Euclidean space, $U \subset R^n$ and $V \subset R^q$ be bounded closed and convex set. Let $T > 0$, for initial state $x_0 \in R^n$, consider the following control systems:

$$
\begin{align*}
\dot{X}(t) &= f(t, X(t), u(t), v(t)), \quad t \in [0, T], \\
X(0) &= x_0,
\end{align*}
$$

where $f : [0, T] \times R^n \times U \times V \rightarrow R^n$ is a given map. $X(\cdot)$ is called the state trajectory, $u(\cdot)$ and $v(\cdot)$ are control functions valued in $U$ and $V$, respectively. We denote

$$
\begin{align*}
U[t, s] &= \{ u : [t, s] \rightarrow U \mid u(\cdot) \text{ is continuous} \}, \\
V[t, s] &= \{ v : [t, s] \rightarrow V \mid v(\cdot) \text{ is continuous} \}.
\end{align*}
$$

Under some mild conditions, for initial pair $(0, x_0)$ and any $(u(\cdot), v(\cdot)) \in U[0, T] \times V[0, T]$, control system (1) admits a unique solution.

**Remark 2.1.** It is obvious that $X(\cdot)$, which is the solution of control system (1), depends on $f$, $u$, and $v$. Thus, let $X(\cdot) \equiv X_{u,v}(\cdot)$. See the below section for more description with respect to continuous dependence.

We now introduce the following cost functionals which measures the performance of the control $u(\cdot)$ and $v(\cdot)$.

$$
J_i(u(\cdot), v(\cdot)) = \int_0^T \varphi_i(t, X(t), u(t), v(t))dt + \psi_i(X(T)), \quad i = 1, 2,
$$

for some given maps $\varphi_i : [0, T] \times R^n \times U \times V \rightarrow R$ and $\psi_i : R^n \rightarrow R$ $(i = 1, 2)$. The following two-person differential games is posed.

**Problem (DG).** For a given initial pair $(0, x_0)$, Player 1 finds a control $\bar{u}(\cdot) \in U[0, T]$ and Player 2 finds a control $\bar{v}(\cdot) \in V[0, T]$ such that

$$
\begin{align*}
J_1(\bar{u}(\cdot), \bar{v}(\cdot)) &= \inf_{u(\cdot) \in U[0,T]} J_1(u(\cdot), \bar{v}(\cdot)), \\
J_2(\bar{u}(\cdot), \bar{v}(\cdot)) &= \inf_{v(\cdot) \in V[0,T]} J_2(\bar{u}(\cdot), v(\cdot)).
\end{align*}
$$
Any $(\bar{u}(), \bar{v}()) \in \mathcal{U}[0, T] \times \mathcal{V}[0, T]$ satisfying (4) is called an open-loop Nash equilibrium control.

Now, we let cost functionals (3) satisfies
\[
\begin{align*}
\phi(t, X(t), u, v) + \psi(t, X(t), u, v) &= 0, \\
\psi(X(T)) &= 0,
\end{align*}
\]
where $\phi_t(t, X(t), u(t), v(t)) = h(t, X(t)) + Wu(t) + Zv(t)$ $(i = 1, 2)$, and $W, Z$ are constant positive definite matrix. $h : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ is the given mapping. Then, one has
\[
J(u(\cdot), v(\cdot)) + J_2(u(\cdot), v(\cdot)) = 0.
\]
In this case, Problem(DG) is a two-person zero-sum differential game. For convenience, we call it Problem(ZDG). Define
\[
\begin{align*}
\phi(t, X(t), u, v) &= -\phi_2(t, X(t), u, v), \\
\psi(X(T)) &= -\psi_2(X(T)),
\end{align*}
\]
and
\[
J(u(\cdot), v(\cdot)) = J_1(u(\cdot), v(\cdot)) = -J_2(u(\cdot), v(\cdot)).
\]
This yields that
\[
J(\bar{u}(), \bar{v}()) = \inf_{u(\cdot) \in \mathcal{U}[0, T]} J(u(\cdot), \bar{v}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[0, T]} J_1(u(\cdot), v(\cdot)) = J_1(\bar{u}(\cdot), \bar{v}(\cdot)).
\]
and
\[
J(\bar{u}(), \bar{v}()) = \inf_{v(\cdot) \in \mathcal{V}[0, T]} J(\bar{u}(\cdot), v(\cdot)) = -\inf_{v(\cdot) \in \mathcal{V}[0, T]} J_2(\bar{u}(\cdot), v(\cdot)) = -J_2(\bar{u}(\cdot), \bar{v}(\cdot)).
\]

Remark 2.2. In this paper, our objective is to investigate generic uniqueness of Problem(ZDG) against the perturbation of the right-hand side function of control system. To this end, we assume that cost functional is linear with regard to $u(\cdot)$ and $v(\cdot)$, which does not impact our main idea.

Definition 2.1. Let initial pair $(0, x_0)$ be fixed. A control pair $(\bar{u}(\cdot), \bar{v}(\cdot)) \in \mathcal{U}[0, T] \times \mathcal{V}[0, T]$ is called an open-loop saddle point of Problem(ZDG), if for any $(u(\cdot), v(\cdot)) \in \mathcal{U}[0, T] \times \mathcal{V}[0, T]$, it satisfies
\[
J(\bar{u}(\cdot), v(\cdot)) \leq J(\bar{u}(\cdot), \bar{v}(\cdot)) \leq J(u(\cdot), \bar{v}(\cdot)).
\]
In this paper, $\| \cdot \|$ represents a Euclidean norm.

We make the following assumptions.

[F] The map $f : [0, T] \times \mathbb{R}^n \times U \times V \to \mathbb{R}^n$ is measured in $t$ and continuous with respect to $u$ and $v$. There exist constant $L > 0$ and $\phi_1(\cdot) \in L^p([0, T]; \mathbb{R})$ $(p \geq 1)$ such that
\[
\begin{align*}
\| f(t, x, u, v) - f(t, y, u, v) \| &\leq L\| x - y \|, \\
\| f(t, 0, u, v) \| &\leq \phi(t),
\end{align*}
\]
\[
\forall (t, x, u, v) \in [0, T] \times \mathbb{R}^n \times U \times V.
\]

[H1] The maps $\phi : \mathbb{R}^n \to \mathbb{R}$ and $\psi : [0, T] \times \mathbb{R}^n \times U \times V \to \mathbb{R}$ are continuous in $(t, x, u, v) \in [0, T] \times \mathbb{R}^n \times U \times V$. There exists constant $K > 0$ such that
\[
\phi(t, x, u, v), \psi(x) \geq -K, \quad \forall (t, x, u, v) \in [0, T] \times \mathbb{R}^n \times U \times V.
\]

[H2] For $0 \leq t \leq T$, the map $\epsilon(t, \cdot) : \mathbb{R}^n \to 2^{\mathbb{R}^n \times \mathbb{R}^n}$ has Cesari properties, i.e.,
\[
\bigcap_{\delta > 0} \epsilon_{\delta, \epsilon}(t, O_{\delta}(x)) = \epsilon(t, x),
\]
for all $x \in \mathbb{R}^n$, where $O_{\delta}(x)$ is a $\delta$-neighborhood of $x \in \mathbb{R}^n$, and for any $(t, x) \in [0, T] \times \mathbb{R}^n$. 

\[
\varepsilon(t, x) = \begin{cases} 
(z^0, z) \in \mathbb{R} \times \mathbb{R}^n & \text{if } z^0 \geq \varphi(t, x, u, v), \\
& z = f(t, x, u, v), \\
& (u, v) \in U \times V,
\end{cases}
\] (6)

[1] The following condition holds for any \((t, x) \in [0, T] \times \mathbb{R}^n,\)
\[
\inf_{u \in U} \sup_{v \in V} \langle p, f(t, x, u, v) \rangle + \varphi(t, x, u, v) = \sup_{v \in V} \inf_{u \in U} \langle p, f(t, x, u, v) \rangle + \varphi(t, x, u, v), \quad \forall p \in \mathbb{R}^n.
\]

**Remark 2.3.** Under the assumptions [F], [I], and [H1]–[H2], Problem(ZDG) admits open-loop saddle point (see [6–8] and references therein).

Next, we state some property on saddle point.

**Property 2.1.** Let \((\bar{u}(\cdot), \bar{v}(\cdot)) \in \mathcal{U}[0, T] \times \mathcal{V}[0, T].\) Then \((\bar{u}(\cdot), \bar{v}(\cdot))\) is a saddle point of Problem(ZDG) if and only if (for short, iff)
\[
\inf_{u(\cdot) \in \mathcal{U}[0, T]} \sup_{v(\cdot) \in \mathcal{V}[0, T]} J(u(\cdot), v(\cdot)) = \sup_{v(\cdot) \in \mathcal{V}[0, T]} \inf_{u(\cdot) \in \mathcal{U}[0, T]} J(u(\cdot), v(\cdot)).
\] (7)

**Proof.** Let \((\bar{u}(\cdot), \bar{v}(\cdot))\) be a saddle point, then for any \(u(\cdot) \in \mathcal{U}[0, T]\), we have \(J(\bar{u}(\cdot), \bar{v}(\cdot)) \leq J(u(\cdot), \bar{v}(\cdot)).\) This implies that \(J(\bar{u}(\cdot), \bar{v}(\cdot)) \leq \sup_{v(\cdot) \in \mathcal{V}[0, T]} \inf_{u(\cdot) \in \mathcal{U}[0, T]} J(u(\cdot), v(\cdot)),\) which results in
\[
J(\bar{u}(\cdot), \bar{v}(\cdot)) \leq \inf_{u(\cdot) \in \mathcal{U}[0, T]} \sup_{v(\cdot) \in \mathcal{V}[0, T]} J(u(\cdot), v(\cdot)).
\]

Similarly, we can prove that
\[
J(\bar{u}(\cdot), \bar{v}(\cdot)) \geq \sup_{v(\cdot) \in \mathcal{V}[0, T]} \inf_{u(\cdot) \in \mathcal{U}[0, T]} J(u(\cdot), v(\cdot)).
\]
From the above, (7) holds.

Conversely, let \(\omega = J(\bar{u}(\cdot), \bar{v}(\cdot)),\) that is \(\omega = \inf_{u(\cdot) \in \mathcal{U}[0, T]} \sup_{v(\cdot) \in \mathcal{V}[0, T]} J(u(\cdot), v(\cdot)) = \sup_{v(\cdot) \in \mathcal{V}[0, T]} \inf_{u(\cdot) \in \mathcal{U}[0, T]} J(u(\cdot), v(\cdot)).\) Then for any \(u(\cdot) \in \mathcal{U}[0, T]\) and \(v(\cdot) \in \mathcal{V}[0, T],\) we have
\[
\omega = \inf_{u(\cdot) \in \mathcal{U}[0, T]} J(u(\cdot), \bar{v}(\cdot)) = \sup_{v(\cdot) \in \mathcal{V}[0, T]} J(\bar{u}(\cdot), v(\cdot)).
\]
So,
\[
J(\bar{u}(\cdot), v(\cdot)) \leq \omega \leq J(u(\cdot), \bar{v}(\cdot)),
\]
i.e.,
\[
J(\bar{u}(\cdot), v(\cdot)) \leq J(\bar{u}(\cdot), \bar{v}(\cdot)) \leq J(u(\cdot), \bar{v}(\cdot)).
\]
This completes the proof. □

**Property 2.2.** Let \((\bar{u}_1(\cdot), \bar{v}_1(\cdot)), (\bar{u}_2(\cdot), \bar{v}_2(\cdot)) \in \mathcal{U}[0, T] \times \mathcal{V}[0, T]\) be saddle point of Problem(ZDG). Then \((\bar{u}_1(\cdot), \bar{v}_2(\cdot)), (\bar{u}_2(\cdot), \bar{v}_1(\cdot)) \in \mathcal{U}[0, T] \times \mathcal{V}[0, T]\) are also saddle point and
\[
J(\bar{u}_1(\cdot), \bar{v}_1(\cdot)) = J(\bar{u}_2(\cdot), \bar{v}_1(\cdot)) = J(\bar{u}_2(\cdot), \bar{v}_2(\cdot)) = J(\bar{u}_1(\cdot), \bar{v}_2(\cdot)).
\] (8)

**Proof.** Since \((\bar{u}_1(\cdot), \bar{v}_1(\cdot)), (\bar{u}_2(\cdot), \bar{v}_2(\cdot))\) are saddle points, then for any \((u(\cdot), v(\cdot)) \in \mathcal{U}[0, T] \times \mathcal{V}[0, T],\) we have
\[
J(u(\cdot), \bar{v}_1(\cdot)) \geq J(\bar{u}_1(\cdot), \bar{v}_1(\cdot)) \geq J(\bar{u}_1(\cdot), v(\cdot)).
\] (9)
\[
J(u(\cdot), \bar{v}_2(\cdot)) \geq J(\bar{u}_2(\cdot), \bar{v}_2(\cdot)) \geq J(\bar{u}_2(\cdot), v(\cdot)).
\] (10)
We denote \(u(\cdot) = \bar{u}_2(\cdot), v(\cdot) = \bar{v}_2(\cdot),\) and \(u(\cdot) = \bar{u}_1(\cdot), v(\cdot) = \bar{v}_1(\cdot)\) in (9) and (10), respectively.
\[ J(\tilde{u}(\cdot), \tilde{v}(\cdot)) \geq J(\tilde{u}_1(\cdot), \tilde{v}_1(\cdot)) \geq J(\tilde{u}_2(\cdot), \tilde{v}_2(\cdot)), \]
\[ J(\tilde{u}_1(\cdot), \tilde{v}_1(\cdot)) \geq J(\tilde{u}_2(\cdot), \tilde{v}_2(\cdot)) \geq J(\tilde{u}_3(\cdot), \tilde{v}_3(\cdot)). \]

(11)

It follows from (11) that
\[ J(\tilde{u}_3(\cdot), \tilde{v}_3(\cdot)) = J(\tilde{u}_2(\cdot), \tilde{v}_2(\cdot)) = J(\tilde{u}_1(\cdot), \tilde{v}_1(\cdot)). \]

Namely, (8) holds. From (8) and (11), we obtain that
\[ J(u(\cdot), v(\cdot)) \geq J(u_1(\cdot), v_1(\cdot)) \geq J(u_2(\cdot), v_2(\cdot)), \]
\[ J(u_1(\cdot), v_1(\cdot)) \geq J(u_2(\cdot), v_2(\cdot)) \geq J(u_3(\cdot), v_3(\cdot)), \]
for any \((u(\cdot), v(\cdot)) \in \mathcal{U}[0, T] \times \mathcal{V}[0, T] \). This completes the proof. □

3 Generic uniqueness

To investigate the generic uniqueness of open-loop saddle point for Problem (ZDG), we construct the following model. Let
\[ \Omega = \{ f \mid f \text{ satisfy } [F] \}. \]

(12)

We denote the following set of open-loop saddle points of Problem (ZDG).
\[ E(f) = \{ (\tilde{u}(\cdot), \tilde{v}(\cdot)) \in \mathcal{U}[0, T] \times \mathcal{V}[0, T] \mid (\tilde{u}(\cdot), \tilde{v}(\cdot)) \text{ is open-loop saddle point of Problem (ZDG), for any } f \in \Omega \}. \]

Then, the correspondence \( f \to E(f) \) yields a set-valued mapping \( E : \Omega \to 2^{U \times V} \). We shall study the generic uniqueness of \( E(f) \). The associated metric \( d : \Omega \times \Omega \to R \) is defined by
\[ d(f, g) = \sup_{(t,x,u,v) \in [0,T] \times R^2, x, u, v} \| f(t,x,u,v) - g(t,x,u,v) \|, \quad \forall f, g \in \Omega. \]

Then, one can easily prove that \((\Omega, d)\) is a complete metric space.

Next, we recall a series of definitions on set-valued mapping from [21] to study the generic uniqueness of Problem (ZDG).

Let \( U \times V \) be a metric space. A set-valued mapping \( E : \Gamma \to 2^{U \times V} \) is called:
1. upper semi-continuous at \( f \in \Omega \) iff for each open set \( O \) in \( U \times V \) with \( E(f) \subseteq O \) (respectively, \( O \cap E(f) = \emptyset \)), there exists \( \delta > 0 \) such that \( E(g) \subseteq O \) (respectively, \( O \cap E(g) = \emptyset \)) for any \( g \in \Omega \) with \( \rho(f, g) < \delta \); (2) continuous at \( f \in \Omega \) iff \( E \) is both upper and lower semi-continuous at \( f \); (3) an usc mapping with compact values iff \( E \) is upper semi-continuous and \( E(f) \) is nonempty compact for each \( f \in \Omega \); and (4) closed iff \( \text{Graph}(E) \) is closed, where \( \text{Graph}(E) = \{(f, u, v) \in \Omega \times U \times V \mid (u, v) \in E(f)\} \) is the graph of \( \Omega \). Also recall that a subset \( Q \subset \Omega \) is called a residual set iff it contains countably many intersections of open and dense subsets of \( \Omega \). If \( \Omega \) is a complete space, any residual subset of \( \Omega \) must be dense in \( \Omega \) and it is a second category set.

Lemma 3.1. [22] Let set-valued mapping \( E : \Omega \to 2^{U \times V} \) be closed and \( U \times V \) be compact, then \( E \) is upper semi-continuous at each \( f \in \Omega \).

Lemma 3.2. [23] Let \( \Omega \) be a complete metric space, \( U \times V \) be a metric space, and \( E : \Omega \to 2^U \) be an usc mapping with compact. Then there exists a dense residual subset \( Q \) of \( \Omega \) such that \( E \) is lower semi-continuous at every point in \( Q \).

Remark 3.1. Let \( Q \subset \Omega \) be a dense residual set, if for any \( \beta \in Q \), a certain property \( P \) depending on \( \beta \) holds. Then \( P \) is called generic property on \( \Omega \). Since \( Q \) is a second category, we may say that the property \( P \) holds for most of the points of \( \Omega \) in the sense of Baire’s category.

In what follows, inspired by the literature [18] and [20], we give some basic property about continuous dependence for state trajectory.
Property 3.1. Let \( \{ f_k \} \subset \Omega \) with \( f_k \to f \in \Omega \). For any \((u_k(\cdot), v_k(\cdot)) \in \mathcal{U}[0, T] \times \mathcal{V}[0, T] \) with \((u(\cdot), v(\cdot)) \to (\bar{u}(\cdot), \bar{v}(\cdot)) \in \mathcal{U}[0, T] \times \mathcal{V}[0, T] \), one has \( X_{k}^{u_k, v_k}(\cdot) \to X_{u, v}(\cdot) \) as \( k \to \infty \).

Proof. For any \( t \in [0, T] \), according to control system (1), we have

\[
\begin{align*}
X_k(t) &= x_0 + \int_0^T f_k(t, X_k(t), u_k(t), v_k(t)) \, dt, \\
X(t) &= x_0 + \int_0^T f(t, X(t), u(t), v(t)) \, dt.
\end{align*}
\]

Since \( f_k \to f \), for any \( \varepsilon > 0 \), there exists \( N_1 > 0 \) such that for any \( k > N_1 \), \( d(f_k, f) < \frac{\varepsilon}{3T} \). \( X(t) \) is continuous at \( [0, T] \), then there exists constant \( a_1 > 0 \) such that \( \max_{t \in [0, T]} \| X(t) \| \leq a_1 \). \( U \subset \mathbb{R}^p \) and \( V \subset \mathbb{R}^q \) are bounded closed and convex set. That is, \( U \) and \( V \) are also compact. Because \( u(\cdot) \) and \( v(\cdot) \) are continuous in \( [0, T] \), there exist constants \( a_2 > 0 \) and \( a_3 > 0 \) such that \( \max_{t \in [0, T]} \| u(t) \| \leq a_2 \) and \( \max_{t \in [0, T]} \| v(t) \| \leq a_3 \). Thus, \( f \) is uniformly continuous on the set

\[
\Sigma = [0, T] \times \{ X \in \mathbb{R}^n \mid \| X(t) \| \leq a_1 \} \times \{ u \in U \mid \| u(t) \| \leq a_2 \} \times \{ v \in V \mid \| v(t) \| \leq a_3 \}.
\]

Owing to \((u_k(\cdot), v_k(\cdot)) \to (\bar{u}(\cdot), \bar{v}(\cdot))\), there exists constant \( N_2 > 0 \) such that for any \( t \in [0, T] \), when \( k \geq N_2 \), one has

\[
\| f(t, X(t), u_k(t), v_k(t)) - f(t, X(t), u(t), v(t)) \| < \frac{\varepsilon}{3T}.
\]

There exists constant \( N_3 > 0 \) such that for any \( t \in [0, T] \), when \( k \geq N_3 \), one has

\[
\| f(t, X(t), u(t), v_k(t)) - f(t, X(t), u(t), v(t)) \| < \frac{\varepsilon}{3T}.
\]

Therefore, choose \( N = \max\{N_0, N_2, N_3\} \) such that for any \( t \in [0, T] \), when \( k \geq N \), one has

\[
\| X_k(t) - X(t) \| \leq \int_0^T \| f_k(t, X_k(t), u_k(t), v_k(t)) - f(t, X(t), u(t), v(t)) \| \, dt
\]
\[
\leq \| f_k(t, X_k(t), u_k(t), v_k(t)) - f(t, X_k(t), u_k(t), v_k(t)) \| \, dt
\]
\[
+ \int_0^T \| f(t, X_k(t), u_k(t), v_k(t)) - f(t, X(t), u_k(t), v_k(t)) \| \, dt
\]
\[
+ \int_0^T \| f(t, X(t), u_k(t), v_k(t)) - f(t, X(t), u(t), v_k(t)) \| \, dt
\]
\[
+ \int_0^T \| f(t, X(t), u(t), v_k(t)) - f(t, X(t), u(t), v(t)) \| \, dt
\]
\[
\leq \int_0^T \frac{\varepsilon}{3T} \, dt + \int_0^T \| X_k(t) - X(t) \| \, dt + \int_0^T \frac{\varepsilon}{3T} \, dt + \int_0^T \frac{\varepsilon}{3T} \, dt
\]
\[
\leq \varepsilon L \int_0^T \| X_k(t) - X(t) \| \, dt.
\]

Thanks to Gronwall's inequality, we have

\[
\| X_k - X \| \leq e^{LT} \varepsilon.
\]

From the arbitrary of \( \varepsilon > 0 \), it yields \( X_{k}^{u_k, v_k}(\cdot) \to X_{u, v}(\cdot) \). \( \Box \)
From Property 3.1, the following result is easily obtained.

**Corollary 3.1.** Let \( \{ f_k \} \subset \Omega \) with \( f_k \to f \in \Omega \).

1. For any \( u_k(\cdot) \in \mathcal{U}[0, T] \) with \( u_k(\cdot) \to \bar{u}(\cdot) \in \mathcal{U}[0, T] \). Then for any \( \nu(\cdot) \in \mathcal{V}[0, T] \), \( X_{\nu, u_k}(\cdot) \to X_{\nu, \bar{u}}(\cdot) \) as \( k \to \infty \).

2. For any \( v_k(\cdot) \in \mathcal{V}[0, T] \) with \( v_k(\cdot) \to \nu(\cdot) \in \mathcal{V}[0, T] \). Then for any \( u(\cdot) \in \mathcal{U}[0, T] \), \( X_{\nu, u_k}(\cdot) \to X_{\nu, \bar{u}}(\cdot) \) as \( k \to \infty \).

3. For any \( (u(\cdot), \nu(\cdot)) \in \mathcal{U}[0, T] \times \mathcal{V}[0, T] \). Then \( X_{\nu, u}(\cdot) \to X_{\nu, \bar{u}}(\cdot) \) as \( k \to \infty \).

**Corollary 3.2.** Let \( f \in \Omega \).

1. For any \( (u_k(\cdot), \nu_k(\cdot)) \in \mathcal{U}[0, T] \times \mathcal{V}[0, T] \) with \( (u_k(\cdot), \nu_k(\cdot)) \to (\bar{u}(\cdot), \bar{\nu}(\cdot)) \in \mathcal{U}[0, T] \times \mathcal{V}[0, T] \). Then \( J_f(u_k(\cdot), \nu_k(\cdot)) \to J_f(\bar{u}(\cdot), \bar{\nu}(\cdot)) \) as \( k \to \infty \).

2. For any \( u_k(\cdot) \in \mathcal{U}[0, T] \) with \( u_k(\cdot) \to \bar{u}(\cdot) \in \mathcal{U}[0, T] \). Then for any \( \nu(\cdot) \in \mathcal{V}[0, T] \), \( J_f(u_k(\cdot), \nu(\cdot)) \to J_f(\bar{u}(\cdot), \nu(\cdot)) \) as \( k \to \infty \).

3. For any \( \nu_k(\cdot) \in \mathcal{V}[0, T] \) with \( \nu_k(\cdot) \to \nu(\cdot) \in \mathcal{V}[0, T] \). Then for any \( u(\cdot) \in \mathcal{U}[0, T] \), \( J_f(u(\cdot), \nu_k(\cdot)) \to J_f(u(\cdot), \nu(\cdot)) \) as \( k \to \infty \).

Now, we present the main results in this paper.

**Theorem 3.1.** Set-valued mapping \( E : \Omega \to 2^{\mathcal{U} \times \mathcal{V}} \) is an usc mapping with compact.

**Proof.** Since \( U \subset \mathbb{R}^p \) and \( V \subset \mathbb{R}^q \) are bounded closed and convex set, then \( U \times V \subset \mathbb{R}^{p+q} \) is also bounded closed and convex set, i.e., \( U \times V \) is compact and convex set. By Lemma 3.2, it suffices to show that the graph of \( E \) is closed, where \( \text{Graph}(E) = \{(f, u, v) \in \Omega \times U \times V \mid (u, v) \in E(f)\} \). Suppose that \( \{f_k\} \subset \Omega \) with \( f_k \to f \in \Omega \), for any \( (u_k(\cdot), \nu_k(\cdot)) \in (E(f_k)) \) with \( (u_k(\cdot), \nu_k(\cdot)) \to (\bar{u}(\cdot), \bar{\nu}(\cdot)) \). Let us show that \((\bar{u}(\cdot), \bar{\nu}(\cdot)) \in E(f)\).

By \((u_k(\cdot), \nu_k(\cdot)) \in (E(f_k))\), then for any \((u, v) \in (U \times V) \), we have

\[
J_{f_k}(u_k(\cdot), \nu_k(\cdot)) \geq J_f(u_k(\cdot), \nu(\cdot)) \geq J_f(u(\cdot), \nu(\cdot)).
\]

Since \((u_k(\cdot), \nu_k(\cdot)) \to (\bar{u}(\cdot), \bar{\nu}(\cdot))\), and \( f_k \to f \), by Property 3.1 and its Corollaries, we obtain that

\[
J_{f_k}(u_k(\cdot), \nu_k(\cdot)) \to J_f(u(\cdot), \nu(\cdot)),
\]

\[
J_{f_k}(u_k(\cdot), \nu(\cdot)) \to J_f(\bar{u}(\cdot), \bar{\nu}(\cdot)), \quad \text{as} \quad k \to \infty.
\]

Therefore, for any \((u(\cdot), \nu(\cdot)) \in (U \times V) \), it results in

\[
J_f(u(\cdot), \nu(\cdot)) \geq J_f(\bar{u}(\cdot), \bar{\nu}(\cdot)) \geq J_f(\bar{u}(\cdot), \nu(\cdot)),
\]

which yields \((\bar{u}(\cdot), \bar{\nu}(\cdot)) \in E(f)\). This completes the proof.

**Theorem 3.2.** There exists a dense residual subset \( Q \) of \( \Omega \) such that for any \( \omega \in Q \), \( E(\omega) \) is a singleton set.

**Proof.** Since \( U \times V \) is compact and \((\Omega, d)\) is a complete metric space, according to Theorem 3.1, set-valued mapping \( E \) is an usc mapping with compact. By using Lemma 3.2, there exists a dense residual subset \( Q \) such that for any \( \omega \in Q \), \( E(\omega) \) is lower semi-continuous at \( \omega \), which implies \( E \) is continuous at \( \omega \).

Assume that \( E(\omega) \) is not a singleton set for some \( \omega \in Q \). Then there exists \((u_1, v_1) \neq (u_2, v_2) \in E(\omega)\), and \((u_1, v_1) \neq (u_2, v_2) \). Without loss of generality, let \( u_1 \neq u_2 \). By separation theorem of convex set, there exists continuous linear functional \( \eta \) in \( E \) such that \( \eta(u_1) \neq \eta(u_2) \), let \( g : U \to \mathbb{R} \) be defined by

\[
g(u) = \frac{\eta(u) - \eta(u_1)}{\eta(u_1) - \eta(u_2)}, \quad \text{for any} \quad u \in U.
\]

Then \( g(u_1) = 1, g(u_2) = 0 \), and \( g \) is continuous and bounded in \( U \). Take \((u, v) \in U \times V \), for any \( \varepsilon > 0 \), define a function \( \omega_\varepsilon(u, v) = \omega(u, v) - \varepsilon g(u) \). It is easy to prove that \( \omega_\varepsilon \in \Omega \) and \( \omega_\varepsilon \to \omega \) as \( \varepsilon \to 0 \).
Let $G = \{ u \in U \mid g(u) > \frac{1}{2} \} \times V$, then $G \subset U \times V$ is an open set. Since $g(u_i) = 1, (u_i, v_i) \in G, G \cap E(\omega) \neq \emptyset$. Since set-valued mapping $E$ is lower semi-continuous, thus, when $\varepsilon > 0$ is very small, we have $G \cap E(f_\varepsilon) \neq \emptyset$.

Take $(\bar{u}, \bar{v}) \in G \cap E(\omega)$, that is, $(\bar{u}, \bar{v}) \in E(\omega)$ and $g(\bar{u}) > \frac{1}{2}$.

$$V_\varepsilon = \inf_{u \in U} \sup_{v \in V} \omega(u, v) \geq \inf_{u \in U} \sup_{v \in V} \omega(\bar{u}, v) = \omega(\bar{u}, v) = \sup_{v \in V} \omega(\bar{u}, v)$$

$$\geq \sup_{v \in V} [\omega(\bar{u}, v) - \varepsilon g(\bar{u})] = \sup_{v \in V} [\omega(\bar{u}, v) - \varepsilon g(\bar{u})]$$

$$> \inf_{u \in U} \sup_{v \in V} \omega(u, v) = \frac{\varepsilon}{2} = \omega - \frac{\varepsilon}{2}$$

where $\omega = \inf_{u \in U} \sup_{v \in V} \omega(u, v)$.

On the other hand, since $g(u_0) = 0$ and $(u_1, v_1), (u_2, v_2) \in E(\omega)$, by Property 2.2, $(u_2, v_1) \in E(\omega)$.

$$\omega = \inf_{u \in U} \sup_{v \in V} \omega(u, v) \geq \inf_{u \in U} \sup_{v \in V} \omega(\bar{u}, v) = \sup_{v \in V} \omega(\bar{u}, v)$$

$$= \sup_{v \in V} [\omega(\bar{u}, v) - \varepsilon g(\bar{u})] = \sup_{v \in V} [\omega(\bar{u}, v) - \varepsilon g(\bar{u})]$$

$$\geq \inf_{u \in U} \sup_{v \in V} \omega(\bar{u}, v) = V_\varepsilon,$$

which is a contradiction with $V_\varepsilon > \omega - \frac{\varepsilon}{2}$. Thus, the proof is complete. 

\[\square\]

4 Conclusion

By constructing a complete metric space, based on the theory of set-valued mappings, this paper investigates the generic uniqueness of saddle point with respect to the right-hand side functions of the control system for two-person zero-sum differential games within the class of open-loop. That is, most of the two-person zero-sum differential games have unique saddle point in the sense of Baire’s category. However, it is great that our cost functional is linear with respect to control functions $u(\cdot)$ and $v(\cdot)$. We will investigate the corresponding stability for a general cost functional in the future.

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding information: This work was supported by the Natural Science Foundation of China (Grant No. 12061021).

Author contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of interest: Authors state no conflict of interest.

Data availability statement: Data sharing is not applicable to this paper as no data sets were generated or analyzed during the current study.

References

[1] R. Isaacs, Differential Games, Wiley, New York, 1965.

[2] L. D. Berkovitz, A variational approach to differential games, Advances in Game Theory, Princeton University Press, Princeton, 1964.
[3] R. J. Elliott and N. J. Kalton, *The Existence of Value in Differential Games*, American Mathematical Society, Providence, 1972.

[4] W. H. Fleming, *The convergence problem for differential games*, J. Math. Anal. Appl. 3 (1961), no. 1, 102–116.

[5] A. Friedman, *Differential Games*, Wiley-Interscience, New York, 1971.

[6] M. Ramaswamy and A. J. Shaiju, *Construction of approximate saddle-point strategies for differential games in a Hilbert spaces*, J. Optim. Theory Appl. 141 (2009), 349–370, DOI: https://doi.org/10.1007/s10957-008-9478-z.

[7] L. D. Berkovitz, *The existence of value and saddle point in games of fixed duration*, SIAM J. Control Optim. 23 (1985), no. 2, 172–196, DOI: https://doi.org/10.1137/0323015.

[8] M. K. Ghosh and A. J. Shaiju, *Existence of value and saddle point infinite-dimensional differential games*, J. Optim. Theory Appl. 121 (2004), no. 2, 301–325, DOI: https://doi.org/10.1023/B:JOTA.0000037407.15482.72.

[9] E. S. Amar, M. G. Brikaa, and E. A. Rehim, *A study on two-person zero-sum rough interval continuous differential games*, OPSEARCH 56 (2019), 689–716, DOI: https://doi.org/10.1007/s12597-019-00383-2.

[10] J. Sun, *Two-person zero-sum stochastic linear-quadratic differential games*, SIAM J. Control Optim. 59 (2021), no. 3, 1804–1829, DOI: https://doi.org/10.1137/20M1340368.

[11] P. Bernhard, *Linear quadratic, two-person, zero-sum differential games: Necessary and sufficient conditions*, J. Optim. Theory Appl. 27 (1979), no. 1, 51–69, DOI: https://doi.org/10.1007/BF00933325.

[12] J. Yong, *Differential Games: A Concise Introduction*, World Scientific, New York, 2014.

[13] P. S. Kenderov, *Most of the optimization problems has unique solutions*, In: B. Brosowski, F. Deutsch, (eds), Parametric Optimization and Approximation, International Series of Numerical Mathematics, vol. 72, Birkhäuser, Basel, 1984, pp. 203–216, DOI: https://doi.org/10.1007/978-3-0348-6253-0_13.

[14] N. Ribarska and P. S. Kenderov, *Most of the two person zero-sum games have unique solutions*, In: Workshop/ Miniconference on Functional Analysis and Optimization, Australian National University, Mathematical Sciences Institute, Australia, 1988, pp. 73–82.

[15] K. K. Tan, J. Yu, and X. Yuan, *The uniqueness of saddle points*, Bull. Pol. Acad. Sci. Math. 43 (1995), 119–129.

[16] J. Yu, D. Peng, and S. Xiang, *Generic uniqueness of equilibrium points*, Nonlinear Anal. 74 (2011), 6326–6332, DOI: https://doi.org/10.1016/j.na.2011.06.011.

[17] J. Yu, D. Peng, D. Xu, and Y. Zhou, *Existence and stability analysis of optimal control*, Optimal Control Appl. Methods 35 (2014), no. 6, 721–729, DOI: https://doi.org/10.1002/oca.2096.

[18] H. Deng and W. Wei, *Existence and stability analysis for nonlinear optimal control problems with 1-mean equilibrium controls*, J. Ind. Manag. Optim. 11 (2015), no. 4, 1409–1422, DOI: https://doi.org/10.3934/jimo.2015.11.1409.

[19] H. Deng and W. Wei, *Stability analysis for optimal control problems governed by semilinear evolution equation*, Adv. Differential Equations 2015, 103, DOI: https://doi.org/10.1186/s13662-015-0443-5.

[20] J. Yu and D. Peng, *Generic stability of Nash equilibrium for noncooperative differential games*, Oper. Res. Lett. 48 (2020), no. 2, 157–162, DOI: https://doi.org/10.1016/j.orl.2020.02.001.

[21] D. Peng, J. Yu, and N. Xiu, *Generic uniqueness theorems with some applications*, J. Global Optim. 56 (2013), 713–725, DOI: https://doi.org/10.1007/s10898-012-9903-6.

[22] M. K. Fort, *Essential and nonessential fixed points*, Amer. J. Math. 72 (1950), no. 2, 315–322, DOI: https://doi.org/10.2307/2372035.

[23] M. K. Fort, *Points of continuity of semicontinuous functions*, Publ. Math. Debrecen 2 (1951), 100–102.