Deterministic Non-cooperative Binding in Two-Dimensional Tile Assembly Systems Must Have Ultimately Periodic Paths

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Abstract

We consider non-cooperative binding, so-called ‘temperature 1’, in deterministic or directed (called here confluent) tile self-assembly systems in two dimensions and show a necessary and sufficient condition for such system to have an ultimately periodic assembly path. We prove that an infinite maximal assembly has an ultimately periodic assembly path if and only if it contains an infinite assembly path that does not intersect a periodic path in the $\mathbb{Z}^2$ grid. Moreover we show that every infinite assembly must satisfy this condition, and therefore, contains an ultimately periodic path. This result is obtained through a superposition and a combination of two paths that produce a new path with desired properties, a technique that we call co-grow of two paths.

The paper is an updated and improved version of the first part of Durand-Lose et al. [2019].

Keywords

Tile assembly system; Directed (confluent) system; Non-cooperation; Ultimately periodic.

1 Introduction

The abstract tile self-assembly model (aTAM) was introduced by Winfree in 1998 [Winfree, 1998] as a theoretical model that describes DX DNA self-assembly processes. The DX molecule can be designed with four sticky ends such that their assembly forms a 2D surface area [Winfree et al., 1998] as if tiled with square tiles. Hence, motivated by Wang tiles, the abstract tile assembly model is based on square tiles with colored edges (‘glues’, simulating the DNA sticky ends). Starting from a seed assembly (or a seed tile) the assembly grows through matching

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glue attachments of tiles. Unlike Wang tiles, the glues have strength and when the matching
glues are strong enough, the tiles can attach to the growing structure, although there may be
mismatched glues on other sides of the tile. It was observed that two or three weaker matching
.glues can achieve bonding of the tile similar to the bonding with a higher strength single glue.
The notion of glue strength is captured in the model through a parameter called ‘temperature’.
If all tiles have uniform strength sticky ends allowing attachment, then it is said that the model
describes ‘temperature 1’ bonding. When there are two or more strengths on the sticky ends,
the temperature can be higher than one. In temperature 2, for example, a tile can attach to the
growing assembly either by matching of a single glue of strength 2, or, at least two weaker tile
.glues of strength one are matched. The latter case is called ‘cooperative’ binding. Such bond
cooperation is not needed (although can appear) when tiles have uniform strength on their sticky
.ends, or when the model runs at ‘temperature 1’, or non-cooperative binding.

There are several experimental assemblies of DNA-based tile arrays that show that aTAM
can carry out computation. These include a binary counter using four DX-based tiles [Evans,
2015], binary addition by TX molecules [LaBean et al., 2000], Sierpinski triangle as a pattern
.on a substrate [Rothemund et al., 2004, Fujibayashi et al., 2008], transducer simulations by TX
.molecules [Chakraborty et al., 2012], and the most recent one where different combinations of
input tiles achieve a variety of computations [Woods et al., 2019].

In his thesis [Winfree, 1998], Winfree showed that the abstract tile assembly model at tem-
perature 2 can assemble (simulate) a trace of computation of any Turing machine, thereby proving
that the model has universal computational power. At the same time it was conjectured that
temperature 1 systems have strictly lower computational power. In [Rothemund and Winfree,
2000], it was also observed that temperature 2 aTAM can assemble certain structures, such as
.squares, with much smaller number of tile types compared with temperature 1 systems, indicating
a possible difference in the computational power of the two models.

The theoretical proofs for universal computational power of aTAM in temperature 2, as well
as many other observations for structure assemblies rely on so-called local determinism in the
assembly [Soloveichik and Winfree, 2007]; in other words, for every two producible assemblies
there is a larger assembly that contains both as sub-assemblies. This property is sometimes
called ‘directed assembly’ or ‘determinism of the system’ [Patitz, 2014]. In order to avoid the
ambiguity and being guided by similar notions in other systems, here this property is also called
confluence.

The standing question about the computational power of systems with non-cooperative bind-
ing (temperature 1 systems) has initiated several studies of these systems. It has been ob-
observed that even small modifications of the model can provide universal computational power.
For example, by considering three-dimensional tile assemblies that can add another layer of
tiles (essentially having one array above the other) it was shown that confluent (directed) non-
cooperative assembly system has universal computational power [Cook et al., 2011]. It was also
observed that by allowing one tile with a repelling glue (glue with strength $-1$), the con-
fluent non-cooperative system becomes computationally universal [Patitz et al., 2011]. On the
other side, if one allows certain tiles to be be added to the system in stages, then again, the
system gains universal computing power [Behsaz et al., 2012]. It was also observed that if tiles
are equipped with signals that activate glues stepwise, the system can simulate any tempera-
ture 2 system [Karpenko, 2015], that is, including the one that provides intrinsic universality of
temperature 2 systems [Doty et al., 2012].

Infinite ribbon construction or snake tilings in non-cooperative (temperature 1) binding sys-
tems were also given attention [Adleman et al., 2002, 2009, Brijder and Hoogeboom, 2009, Kari, 2002]. It was observed that non-determinism, or non-confluence, also adds power to the system. First it was shown that it is undecidable whether one can obtain an infinite ribbon (snake tiling) with a given non-deterministic system [Adleman et al., 2009]. This was achieved with simulating special type of Wang tiles and a space filling curve. In this case, the notion of a ‘directed’ system implies that the design of the tiles is accompanied with arrows that guide the direction of the assembly rather than the system being deterministic (or confluent). On the other side, one can use the snake tiles, and the space filling curve, to obtain a non-confluent system that can generate recognizable picture languages [Brijder and Hoogeboom, 2009]. And because recognizable picture languages contain the rectangular shapes that can be obtained from Wang tiles, which are known to have universal computational power [Wang, 1975], we have that non-determinism, i.e., non-confluence of the system together with a pre-defined condition on acceptable assembly provide universal computing power.

The limitation of a confluent (deterministic) non-cooperative (temperature 1) binding was first observed through so-called ‘pumpable’ paths [Doty et al., 2011]. An infinite path is pumpable if there is a segment of the path that can extend into ultimately periodic within the assembled structure. By assuming that a system can have every sufficiently long path ‘pumpable’, it was observed that only a limited number of structures can be constructed with this system. In particular, in this case the finite assembly is either a ‘grid’ or a ‘finite set of combs’. This implies that the maximal assembly covers a semi-linear subset of the integer lattice. It was also proved that confluent non-cooperative binding cannot simulate a trace of bounded Turing machine computation whose halting appears on the boundary of the computation [Meunier and Woods, 2017]. The paper also shows that such a system cannot be intrinsically universal, that is, there is no temperature-1 confluent aTAM that can simulate any other such a system.

A pumping lemma for temperature 1 confluent aTAM appears in Meunier et al. [2020a], a sketch of the proof was accepted to STOC 2020 [Meunier et al. 2020b]. The authors prove that if an assembly path starting from the seed is long enough, then there is a shield structure beyond which there is a path that can be pumped and assembled. This observation covers a large part of the long assembly path cases, but not all. For example the ‘teeth’ of the combs start away from the seed. The authors provide a bound for the size of the path to ensure the existence of a shield structure and thus obtain a pumpable path. This assembly either exists in the maximal assembly or is ‘fragile’ (for a non-deterministic system, the growth can be blocked by a previously assembled path). In confluent system, no path can be fragile, so that long enough paths starting at the seed are pumpable, hence the final assembly has ultimately periodic paths. In a new paper [Meunier and Regnault, 2021] it was proven that the final structure of a confluent temperature-1 is decidable.

This paper supplements and reinforces the result in [Meunier et al., 2020b,a] by providing a necessary and sufficient condition for a confluent (directed) non-cooperative tile assembly system to have an ultimately periodic assembly path. This is also an alternative proof of the result in [Meunier et al., 2020b]. We show that such a path exists in an infinite maximal assembly if and only if there is an infinite assembly path that does not intersect an ultimately periodic path in the two-dimensional grid of $\mathbb{Z}^2$ (Lem. 13). With this observation we prove that a maximal infinite assembly must contain an ultimately periodic assembly path in the confluent aTAM at temperature 1. Hence this is another proof of the main result in [Meunier et al., 2020b] using different tools such as ‘free-paths’ (Sect. 2.1) and superposition of free-paths called ‘co-grow’ (Sect. 2.3) that are interesting on their own.
For the existence of ultimately periodic assembly paths we consider finite portions of paths, which we call off-the-wall paths, that are bounded by a line in $\mathbb{Z}^2$. These paths are used to show that the necessary and sufficient condition for existence of an ultimately periodic assembly path always holds (Th. [21]). Two main notions are used in the proofs: (a) left and right regions in the plane separated by a bi-infinite path and (b) superposition of two paths, which we call co-grow to obtain a new path that takes the ‘right-most’ way of the two, similarly to the ‘right-priority’ path in [Meunier et al., 2020b, Meunier and Woods, 2017], except in our case co-grow is used as a function that produces a new ‘right-most’ path of two that are in the input. The co-grow of two paths is possible if they are in a well defined region that has ‘no obstructions’ which we call non-causal. Because the system is confluent, the intersection of the two paths during co-grow is always at a vertex that can be associated with only one tile type.

2 Definitions

The set of integers from $a$ to $b$ is denoted $[a, b]$ ($a$ and $b$ can be infinite). The two dimensional integer lattice $\mathbb{Z}^2$ is considered as a two dimensional grid, a periodic graph whose vertices are the elements of $\mathbb{Z}^2$ and two vertices $x$ and $y$ are connected by an edge if $||x - y|| = 1$. A path in $\mathbb{Z}^2$ is a simple path without repetition of the vertices and edges, it can be finite or (bi-)infinite. A cycle is a simple path whose first and last vertex are the same. The set of vertices visited by a path $\pi$ is the domain of $\pi$ and is denoted $\text{dom}(\pi)$. We denote with $\pi_i$ the $i$th vertex visited by $\pi$ and for $a \leq b$ we denote with $\pi_{[a,b]}$ the segment $\pi_a \pi_{a+1} \cdots \pi_b$ of a path $\pi$. We allow for one, or both $a$ and $b$ to be infinite. The origin of $\mathbb{Z}^2$ is $o (= (0, 0))$. The intersection of two paths is the set of vertices visited by both paths.

The set of unit vectors $D = \{e=(1, 0), n=(0, 1), s=(0, -1), w=(-1, 0)\}$ is called the set of directions. The vectors $e, n, s,$ and $w$ correspond to the east, north, south and west directions, respectively.

A graph $H$ that is a sub-graph of a graph $G$ is denoted by $H \subseteq G$. We consider paths as graphs and the same notation is used for paths and subpaths. Let $G, H$ and $H'$ be graphs such that $H \subseteq G$ and $H' \subseteq G$, $H \sqcup H'$ denotes the subgraph that is union of $H$ and $H'$ in $G$.

The set of finite (resp. forward infinite, backward infinite, bi-infinite) sequences of elements, words, over alphabet $D = \{w, e, n, s, w\}$ is denoted $D^*$ (resp. $D^\omega$, $D^\omega$, $D^\omega D$ and $D^\omega D^\omega$ is denoted $D^2\omega$). The union of $D^*$, $D^\omega$, $D^\omega D$ and $D^\omega D^\omega$ is denoted $D^2$. The empty sequence is $\epsilon$. We consider $d \in D$ as a symbol in the alphabet $D$ and a unit vector in $\mathbb{Z}^2$.

2.1 Free Paths and Paths in $\mathbb{Z}^2$

We say that two paths $\pi$ and $\pi'$ in $\mathbb{Z}^2$ are equivalent if $\pi'$ is a translation of $\pi$ in $\mathbb{Z}^2$. The equivalence class of $\pi$, denoted $\langle \pi \rangle$, is called a free path associated with $\pi$. The equivalence class is uniquely determined by a sequence of unit vectors $\kappa$, a word over $D$ (i.e., an element of $D^2$) such that $\kappa_i = d$ if and only if $\pi_{i+1} = \pi_i + d$. We intermittently use the notion ‘free path’ and a word notation $\kappa$ to represent both $\langle \pi \rangle$ and a word in $D^2\omega$. The null free path is $\epsilon$ and $\langle \epsilon \rangle$ is the set of vertices in $\mathbb{Z}^2$. If $m_1$ is not forward infinite and $m_2$ is not backward infinite, then $m_1 m_2$ designates their concatenation and represents a free path only if it corresponds to a path. If $m$ is a finite free path, then $m^\omega = m m m \cdots$ is its infinite (forward) repetition, $\omega m = \cdots m m m$ is its infinite backward repetition and $\omega m^\omega = \cdots m m m \cdots$ is its bi-infinite repetition. The set of cyclic
rotations of a finite free path \( m = d_1 \ldots d_k \) is \( \text{rot}(m) = \{d_i d_{i+2} \ldots d_k d_1 d_2 \ldots d_{i-1} | 1 \leq i \leq k \} \).

For any finite free path \( m = d_1 \ldots d_k \in D^* \), the associated displacement vector, \( \overrightarrow{m} \) (of \( \mathbb{Z}^2 \)) is defined as the sum of its elements \( \overrightarrow{m} = d_1 + \ldots + d_k \). Two finite free paths are collinear if their associated displacement vectors are collinear. There are infinitely many free paths associated to a given displacement vector of \( \mathbb{Z}^2 \) and they are all mutually collinear.

For a path \( \pi \) in \( \mathbb{Z}^2 \) and \( A \in \text{dom}(\pi) \) we also use a notation \( \pi = b.A.f \) where \( b \) and \( f \) are free paths such that \( bf = \langle \pi \rangle \). We say that the free path \( bf \) is grounded at vertex \( A \) and that \( \pi \) is the resulting path. That is, \( b.A.f \) is an instance of a free path \( bf \) such that the end vertex of the sub-path corresponding to \( b \) is \( A \), which is the first vertex to the subpath corresponding to \( f \). If any of these free paths, \( b \) or \( f \), is null, the notation simplifies to \( b.A \), \( A.f \) or just \( A \).

Extending this notation, a path can also be denoted as a sequence of paths and free-paths; i.e., 
\[
q = f_1,\pi_1, f_2, \pi_2, \ldots
\]
where \( f_i \)'s are free paths and \( \pi_i \)'s are paths. In this case \( q \) is the unique path instance in the equivalence class \( f_1 \langle \pi_1 \rangle f_2 \langle \pi_2 \rangle \cdots \) ‘grounded’ by the vertices in the domains of \( \pi_1, \pi_2, \ldots \) (if it exists as a path).

If \( A \) is a vertex of \( \mathbb{Z}^2 \) and \( \overrightarrow{v} \) is a vector in \( \mathbb{Z}^2 \), then \( A + \overrightarrow{v} \) is the vertex \( A \) translated by \( \overrightarrow{v} \). For any path, \( bm.A = b.(A - \overrightarrow{m}).m\) where \( b \in D^* \cup \varnothing D \) and \( m \in D^* \). The reverse of a free path \( m = d_1d_2 \ldots d_k \) is \( \overrightarrow{m} = d_kd_{k-1} \ldots d_1 \) where \( \overrightarrow{c} = \overrightarrow{w}, \overrightarrow{n} = \overrightarrow{s}, \overrightarrow{s} = \overrightarrow{n}, \) and \( \overrightarrow{w} = \overrightarrow{e} \). In particular \( \overrightarrow{m} \) traverses the free path \( m \) in reverse.

A free path \( \pi = mp^\omega \) for \( m, p \in D^* \) and \( p \neq \varepsilon \) is called an ultimately periodic free path, the prefix \( m \) is called the transient part of \( \pi \) and \( p^\omega \) is the periodic part of \( \pi \). Similarly, the path \( \pi = A.mp^\omega \) is an ultimately periodic path, \( A.m \) is the transient part of \( \pi \) and \( (A + \overrightarrow{m}).p^\omega \) is the periodic part of \( \pi \). A periodic (free) path is a (free) path whose transient path is \( \varepsilon \).

Let \( \pi \) be a path and \( \pi_i \) be a legal vertex of \( \pi \). The notation \( \pi_p \) (resp. \( \pi_r \)) corresponds to the subpath of \( \pi \) starting at vertex \( \pi_i \) till the end of \( \pi \), with \( \pi_i \) included (resp. excluded). The notation \( \pi_q \) (resp. \( \pi_s \)) corresponds to the subpath of \( \pi \) up to vertex \( \pi_i \), with \( \pi_i \) included (resp. excluded). For example, for a bi-infinite path \( \pi \), \( \pi_p = \pi_{[i,+\infty]} \), \( \pi_r = \pi_{[i+1, +\infty]} \), \( \pi_q = \pi_{[-\infty, i]} \), and \( \pi_s = \pi_{[-\infty, i-1]} \).

### 2.2 Regions

A region, \( R \), is a connected subgraph of \( \mathbb{Z}^2 \). For a vertex \( v \in \mathbb{Z}^2 \), the neighborhood of \( v \), \( \mathcal{N}(v) \), is the subgraph of \( \mathbb{Z}^2 \) induced by the nine vertices at \( x \)-distance and \( y \)-distance at most 1 from \( v \). A boundary vertex for a region \( R \) is a vertex \( A \in R \) such that \( \mathcal{N}(A) \not\subseteq R \). The boundary of \( R \), denoted \( \partial R \), is the subgraph of \( R \) induced by the sets of its boundary vertices. The interior of \( R \), denoted \( \overset{\circ}{R} \), is the complement (with complement taken out of vertices and edges) of \( \partial R \) in \( R \), that is \( \overset{\circ}{R} = R \setminus \partial R \).

Let \( \pi \) be a cycle or a bi-infinite path in \( \mathbb{Z}^2 \). Then, by Jordan curve theorem, \( \pi \) defines two regions in the plane whose intersection is \( \pi \) itself. We distinguish these two regions as ‘left’ and ‘right’ as described below.

Since all paths are in dimension 2, a path \( \pi \) can be considered oriented by orienting the edges from \( \pi_i \) to \( \pi_{i+1} \) so that its left side can be defined. For \( d,d' \in D \) we say that \( d' \) is to the left of \( d \) or \( d \) is to the right of \( d' \) if \( (d,d') \in \{(\textbf{n},\textbf{w}), (\textbf{w},\textbf{s}), (\textbf{s},\textbf{e}), (\textbf{e},\textbf{n})\} \). A vertex \( A \) in \( \mathbb{Z}^2 \) is directly to the left of \( \pi \) if it does not belong to \( \pi \) and there are \( i \in \mathbb{N} \) and a direction \( d' \) to the left of \( d \) such that \( \pi_i + d = \pi_{i+1} \) and \( \pi_{i+1} + d' = A \), or \( \pi_{i-1} + d = \pi_i \) and \( \pi_i + d' = A \). In this case the edge \((A,\pi_i)\) is directly to the left of \( \pi \). Similarly we define a vertex, and an edge directly to the right of \( \pi \).
The **left region** of a path $\pi$ is the subgraph of $\mathbb{Z}^2$ consisting of vertices $A$ that are either in $\text{dom}(\pi)$, or there is a path $A.m$ for some free path $m$ that ends at $\pi$ with an edge directly to the left of $\pi$ and does not intersect with $\pi$ in any other vertex. The **right region** of $\pi$ is defined similarly. Because $\pi$ is bi-infinite or a cycle, the left and right regions are well defined, and their intersection is $\pi$.

A path $\pi$ is **inside a region** $R$ if $\pi \subseteq R$. It is **strictly inside** $R$ if $\pi \subseteq \hat{R}$.

Let $\vec{v} = (p, q)$ $(p, q \in \mathbb{Z})$ be a vector and consider the line $\ell$ defined with $o + s\vec{v}$ for $s \in \mathbb{R}$ in the Euclidean plane $\mathbb{R}^2$. Then, we also define the right and the left regions of $\ell$ such that $A = (a, b) \in \mathbb{Z}^2$ is in the **left region of** $\ell$ if the dot product $A \cdot \vec{v}_\perp$ (where $\vec{v}_\perp = (-q, p)$) is positive or null. Similarly, $A$ is in the **right region of** $\ell$ if $A \cdot \vec{v}_\perp \leq 0$. For an arbitrary line $\ell'$ that is a translation of $\ell$ defined with $B + s\vec{v}$ for $B \in \mathbb{Z}^2$, $s \in \mathbb{R}$, the vertex $A \in \mathbb{Z}^2$ is in the **right region of** $\ell'$ if $A - B$ is in the right region of $\ell$. The **left region of** $\ell'$ is defined analogously. In particular, the points of $\mathbb{Z}^2$ that belong to $\ell$ (or $\ell'$) belong to both, the left and the right region of $\ell$ (or $\ell'$). Observe that if $R$ is the right region of $\ell$ (or $\ell'$), then $\partial R$ is a bi-infinite periodic path because $\vec{v}$ has integer coordinates. Similarly the boundary of the left region is a bi-infinite periodic path (generally different from the boundary of the right region).

Let $A \in \mathbb{Z}^2$ be in the left region of $B + s\vec{v}$. The **ribbon** between points $A$ and $B$ in $\mathbb{Z}^2$, $A \neq B$ in direction $\vec{v}$ is the intersection of the left region of $B + s\vec{v}$ and the right region of $A + s\vec{v}$ and is denoted $(A, B, \vec{v})$. Directly from the definition it follows that if $C$ is a vertex in the ribbon $(A, B, \vec{v})$ then all points of the integer lattice $\mathbb{Z}^2$ that lie on the line $C + s\vec{v}$ are also in the ribbon.

**Lemma 1** (double implies periodic). Let $m$ be a non null finite free path. If $m^2 = mm$ is a free path then so is $\omega m\omega$.

**Proof.** Let $m = d_1d_2 \cdots d_i$ be a free path such that $m^2$ is a free path. We denote $m_1 = 0$ and $m_r = d_1 + \cdots + d_{i-1}$. Because $m$ is non null and is a free path, it follows that $m \neq 0$. Suppose that $k$ is such that $o.m^k$ is a path, but $o.m^{k+1}$ is not a path but a walk.

Let $A$ and $B$ be such that all vertices of the walk $o.m^{k+1}$ are included in the ribbon $(A, B, \vec{m})$ and $||A - B||$ is minimal. If $A = B$, because the ribbon contains the path $o.m^k$ (connected subgraph), then $\vec{m}$ is either horizontal, or vertical, in which case $o.m^{k+1} = d \cdots d$ for some direction $d \in D$, and the lemma holds. Hence we assume $0 < ||A - B||$. Because by definition all vertices of the walk $o.m^{k+1}$ are included in the ribbon, and by the minimality condition, the vertices of $o.m^{k+1}$ are either boundary vertices for the ribbon or they are strictly inside the region defined by the ribbon.

By the minimality of $||A - B|| > 0$, there are integers $i$ and $j$ with $\vec{m}_i = d_1 + \cdots + d_{i-1}$ and a sub-path $\vec{m}_i.m_{[i,j]}$ of $o.m$ that splits the ribbon in two parts, that is, $\vec{m}_i.d_i$ is at one boundary of the ribbon (say the boundary of the right region of $A + s\vec{m}$), and $\vec{m}_j.d_j$ is on the other boundary of the ribbon (the boundary of the left region of $B + s\vec{m}$). If $i < j$, we can choose $i$ and $j$ such that none of the vertices $\vec{m}_r.d_r$ are on the lines $A + s\vec{m}$ and $B + s\vec{m}$ for $i < r < j$. Any bi-infinite path within the ribbon passing from one part of the ribbon to the other must intersect with a vertex from $\vec{m}_i.m_{[i,j]}$. Moreover, for every integer $l$, the vertices of $(l\vec{m}).m$ are included in the ribbon and $(l\vec{m} + \vec{m}_i).m_{[i,j]}$ splits the ribbon in two parts.

Being $o.m^k$ a path, but $o.m^{k+1}$ a walk and not a path, $(km).m$ must intersect $o.m^k$. This means that it either intersects the tail of $(k-1)\vec{m}.m$ or it must cross the sub-path $(k-1)\vec{m} + \vec{m}_i).m_{[i,j]}$. Hence it has to intersect $(k-1)\vec{m}.m$, implying that $(k-1)\vec{m}.m^2$ is not a path, but a walk. This is in contradiction with our assumption that $m^2$ is a free path. \(\square\)
2.3 Co-grow

We describe a method of superimposing two free paths to form a new free path that is in some sense the ‘rightmost’ portion of both. The idea is similar to taking the “right-priority” in [Meunier and Woods 2017]. Here we define a general method to take two bi-infinite free paths that intersect and obtain a forward finite, or infinite free path starting at one of the intersection points that coincides with at least one of the paths and lies within the right regions of both paths. We call this combined new path as ‘co-grow’ of both. The co-grow is a free path and becomes a path once we ground it at a starting vertex.

Definition 2 (co-grow). Let \(bf, b'f'\) be free bi-infinite paths (where \(b, b'\) are backward infinite and \(f, f'\) are forward infinite) such that \(f\) and \(f'\) start with the same direction. Let \(R\) (resp. \(R'\)) be the right region of \(b.0.f\) (resp. \(b'.0.f'\)). The (right) co-grow of the free paths \(b, f, b'\) and \(f'\), denoted \(\hat{f} = \text{coGrow}(b, f, b', f')\) is a forward, possibly infinite, maximal free path \(\hat{f}\) corresponding to the path \(0.\hat{f}\) defined inductively as follows:

- \(0.\hat{f}_1 = o.d\) where the direction \(d = f_1 = f'_1\) is the common initial direction of \(f\) and \(f'\);
- if \(0.\hat{f}_1 \ldots \hat{f}_i\) is defined and it is a path from \(o\) to \(A\), then \(\hat{f}_{i+1} = d\) is defined if
  - \(d\) is the rightmost direction with respect to \(\hat{f}_i\), such that \(A.d \subseteq o.f\) or \(A.d \subseteq o.f'\);
  - \(0.\hat{f}_1 \ldots \hat{f}_{i+1} = o.\hat{f}_1 \ldots \hat{f}_i.d\) is a path that is a subgraph of \(R \cap R'\).

The notation of paths and regions in co-grow remain fixed for the rest of the current section. Paths, regions and expected free path \(\hat{f}\) are illustrated in Fig.1. The symbol \(\odot\) is used to indicate that start of the co-grow. The free paths \(b\) and \(b'\) do not take part in the construction of \(\hat{f}\) but are used to define the two right regions \(R\) and \(R'\) producing the boundaries \(b.0.f\) and \(b'.0.f'\) that limit the extension of the co-grow (see Fig.1a). If paths, instead of free paths, are used as arguments for co-grow, then the co-grow is assumed to be taken with the associated free paths.

The ‘growing direction’ of the co-grow at any vertex coincides with the direction of at least one of the paths \(o.f\) and \(o.f'\) at the same location, and if they intersect at that vertex it always takes the ‘rightmost’ path of the two paths at the current vertex. A way to obtain the path \(\hat{f} = \text{coGrow}(b, f, b', f')\) is to start with \(e\) from the origin and follow the direction of both paths \(f\) and \(f'\) until one of the paths takes a direction different from the other. At that point one follows the path that takes the rightmost direction, until the point when both paths intersect. In a sense, in between any two intersections of paths \(o.f\) and \(o.f'\) one follows the path that is strictly to the right of the other.

The co-grow produces a ‘maximal’ free path \(\hat{f}\) in the boundary of the region \(R \cap R'\) that contains the origin. Observe that \(R \cap R'\) may be finite or not be connected (see figures 1b and 1c), but the path \(\hat{f}\) (in green) that is a co-grow of two paths (red and blue), lies on the boundary of the connected component that contains the starting vertex. In the case of Fig.1b the region that is bounded by the co-grow is finite and \(\hat{f}\) is a finite path (not bounding the region completely). In the case of Fig.1c the co-grow is possibly infinite. A component in \(R \cap R'\) could be connected only through a path (isthmus in graph theory). The rightmost extension in the definition is important as illustrated in Fig.2a where the condition prevents \(\hat{f}\) from taking an isthmus between \(f\) and \(f'\) at point \(A\) and get trapped in a finite region (dotted path). Note that there may be an isthmus between \(f\) and \(b'\) as shown in Fig.2b but this condition may
Figure 1: co-grow. (a) The dashed (green) line indicates the path \( \hat{f} = \text{coGrow}(b, f, b', f') \) that is a co-grow of \( f \) (in blue) and \( f' \) (in red). Examples where the intersection of the right region of the red path and the blue path is (b) finite and (c) not connected. However, the co-grow of the two paths (in green) bounds the connected component of the region intersection that contains the starting vertex. In case of (b) \( \hat{f} \) is finite, not necessarily bounding the whole component. It cannot be extended because there is no rightmost direction that belongs to \( f \) or \( f' \) that is also in \( R \cap R' \). In case of (c) the region bounded by \( \hat{f} \) is potentially infinite, and so may be \( \hat{f} \).

not necessarily enforce finiteness of \( \hat{f} \) because \( b' \) is part of the right region \( R' \), as well as \( f \) is a part of \( R \) (in comparison with Fig. 1b where \( f \) crosses into the complement of \( R' \)). The situation when the co-grow produces an infinite free path is of our interest; therefore we have the following lemma.

**Lemma 3.** Let \( b f, b' f' \) be bi-infinite free paths such that \( f \) and \( f' \) start in the same direction, and let \( \hat{f} = \text{coGrow}(b, f, b', f') \). Let \( R \) (resp. \( R' \)) be the right region of \( b. o. f \), (resp. \( b'. o. f' \)). If there is a forward infinite path in \( R \cap R' \) starting at \( o \), then \( \hat{f} \) is infinite.

**Proof.** Let \( K \) be the connected component of \( R \cap R' \) that contains \( o \). Assume that there is a forward infinite path in \( R \cap R' \) starting at \( o \). The connected component \( K \) then must be infinite.

Figure 2: Isthmus and the construction of \( \hat{f} \).
We say that a path $q$ crosses into some region $S$ over some path $q'$ when there are two indices $i \leq j$ such that: $q_{[j,i]} \subseteq q'$ and one of $q_{i-1}$ and $q_{i+1}$ belongs to the interior of $S$, while the other does not belong to $S$. If $q_i$ intersects $q'$ and one of $q_{i+1}$ or $q_{i-1}$ is in the interior of $S$, then $q_i$ is called the crossing point into $S$ over $q'$.

Since $K$ contains the origin and is infinite, $b'.o$ cannot cross into $K$ over $o.f$. Otherwise there is a cycle between the crossing point into $K$, following $b'$ to $o$, then following $f$ ending at the crossing point into $K$. This cycle contains the origin and would bound $K$ inside a finite region, but since $K$ is infinite, this is a contradiction. Similarly, $b.o$ cannot cross into $K$ over $o.f'$.

We denote $o.f$ by $\pi$, $o.f'$ by $\pi'$ and $o.f^{new}$ by $\hat{\pi}$ where $\hat{\pi}$ is the co-grow of $b$, $f$, $b'$, and $f'$. Because every vertex in $\hat{\pi}$ belongs to at least one of $\pi$ or $\pi'$, let $\{j_n\}_n$ and $\{i_n\}_n$ be the respective indexes of vertices of $\pi$ and $\pi'$ intersecting $\hat{\pi}$ listed in the order they are encountered. We prove that at least one of these sequences is infinite by showing that they must be increasing sequences and the rightmost direction of co-grow always exists. By Def. 2 $\hat{\pi}$ has at least one edge because $f$ and $f'$ start with the same direction and so $j_0 = i_0 = 0$ and $j_1 = i_1 = 1$.

Inductively, for $k \geq 1$, suppose $\hat{\pi}_1 ... \hat{\pi}_k = \hat{\pi}_{[0,k]}$ is defined intersecting $\pi$ at vertices with indexes $j_0,j_1,...,j_k$ (on $\pi$), and intersecting $\pi'$ at indexes $i_0,i_1,...,i_k$ (on $\pi'$). Assume further that if $\pi_{m} = \pi_{j_s}$ and $\pi_{m'} = \pi_{j_{s+1}}$, then $m < m'$ implies $j_n < j_{n+1}$, i.e., $\hat{\pi}_{[0,k]}$ meets vertices of $\pi$ with increasing indexes, that is, $\hat{\pi}$ always follows the same direction as $f$ when using edges from $f$. Suppose the same holds for $\pi'$ and $f'$.

Because $f$ and $f'$ are forward infinite, at least one of the edges of $f$ or $f'$ is rightmost starting at $\hat{\pi}_k$, and the co-grow $\hat{\pi}$ could extend with that edge. The only way that co-grow’s $(k + 1)$-st edge would not exist is when the rightmost edge exits the region $R \cap R'$ (the second condition of the inductive step in Def. 2). This happens when taking the next rightmost direction crosses one of $b.o$ or $b'o$, which, as observed above, cannot happen.

Assume that rightmost direction from $\hat{\pi}_k$ is a direction using $f$ (the case when the co-grow uses $f'$ is similar) with $\hat{\pi}_k = \pi_{j_s}$ and $\hat{\pi}_{k+1} = \pi_{j_{s+1}}$. The last direction of $\hat{\pi}$ joining $\hat{\pi}_{k-1}$ and $\hat{\pi}_k$ either follows $f$ or not. Case 1: if $\hat{\pi}_{k-1} = \pi_{j_{s-1}}$, and $\hat{\pi}_k = \pi_{j_s}$, that is, the last direction reaching $\hat{\pi}_k$ taken by $\hat{\pi}$ is following $f$, then $\pi_{j_{s-1}} \pi_{j_s} \pi_{j_{s+1}}$ is a two edge sub-path of $\pi$. By inductive hypothesis $j_{s-1} < j_s$ implies that $j_{s-1} + 1 = j_s$, so it must be $j_s + 1 = j_{s+1}$, i.e., $j_s < j_{s+1}$.

![Figure 3: Impossible orientation for $f$ (in blue) when the last direction taken by $\hat{\pi}$ coincides with $f'$ (in green).](image)

Case 2: suppose $\hat{\pi}_{k-1} = \pi'_{j_{s-1}}$, and $\hat{\pi}_k = \pi'_{j_s}$, that is, the last direction reaching $\hat{\pi}_k$ taken by $\hat{\pi}$ is following $f'$. Because $\hat{\pi}_k \hat{\pi}_{k+1}$ is an edge in $\pi$, we have $\pi'_{j_s}$ and either $j_s - 1 = j_{s+1}$
either contains \( o \) or not. The two possible cases are depicted in Figs. 3a and 3b. In the former case (Fig. 3a), the cycle \( \pi'_{l'j'} \odot f_{l'j'} \) would bound a finite region that must include the forward infinite \( \pi_{j,j} \). In the latter case (Fig. 3b), the same cycle bounds a finite region containing \( o \), so the backward infinite path \( b.o \) must cross either \( o.f \) or \( o.f' \) which, as observed above, cannot happen. Therefore, it must be that \( j_i + 1 = j_{i+1} \), and \( \hat{f} \) follows the same direction as \( f \).

Because the co-grow extends at every step, and \( \{j_n\}_n \) and \( \{i_n\}_n \) are increasing sequences, at least one of the sequence of intersection indexes \( \{j_n\}_n \), or \( \{i_n\}_n \), is infinite, and hence, \( \hat{f} \) is infinite.

\[\square\]

### 2.4 Tile Assembly System

Let \( \Sigma \) be an a finite set called an alphabet whose elements are symbols that we will also call glues. A tile type is a map \( t : D \rightarrow \Sigma \). We use the notation \( t_d \) for the value (glue) of \( t \) in direction \( d \).

A (temperature 1) tile assembly system (TAS) is a pair \( T = (T, \sigma) \) where \( T \) is a finite set of tile types and \( \sigma \) is a tile type (not necessarily in \( T \)) called the seed. In order to separate the seed from the tiles used in the assembly, we assume that \( \sigma \notin T \).

For a TAS with set of tile types \( T \) and seed \( \sigma \), an assembly over \( T \) is a partial map \( \alpha : \mathbb{Z}^2 \rightarrow T \cup \{\sigma\} \) where \( \alpha^{-1}(\sigma) \) is empty or a singleton. The domain of \( \alpha \) is the set of points of \( \mathbb{Z}^2 \) for which \( \alpha \) is defined, and is denoted \( \text{dom}(\alpha) \). The binding graph of \( \alpha \) is a subgraph of the lattice \( \mathbb{Z}^2 \) with vertices \( \text{dom}(\alpha) \) such that for \( A, A' \in \text{dom}(\alpha) \) there is an edge with endpoints \( A \) and \( A' \) if and only if \( A + d = A' \) for some direction unit vector \( d \) and \( \alpha(A)_d = \alpha(A')_d \). The assembly \( \alpha \) is stable if its binding graph is connected. An assembly \( \alpha \) is producible in \( T \) if it is stable and \( \alpha(0,0) \) is the seed. The seed appears in \( \alpha \) only at the origin.

Note that although neighboring vertices \( A, A' \in \mathbb{Z}^2 \) with \( A + d = A' \) may be in the domain of \( \alpha \) it may happen that \( \alpha(A)_d \neq \alpha(A')_d \). In this case the tiles \( \alpha(A) \) and \( \alpha(A') \) mismatch in direction \( d \), and the binding graph of \( \alpha \) has no edge between \( A \) and \( A' \).

A producible assembly \( \alpha \) over \( T \) is said to be an assembly path if its binding graph has as a subgraph a path \( \pi \) with \( \pi_0 = (0,0) \) that visits all vertices of \( \alpha \)'s binding graph, i.e., \( \text{dom}(\pi) = \text{dom}(\alpha) \). In other words \( \alpha \) is an assembly path if its binding graph has a spanning tree that is a path in \( \mathbb{Z}^2 \) starting at the origin. In this case we write \( \alpha = \alpha_{\pi} \). We note that for an assembly path a path \( \pi \) such that \( \alpha = \alpha_{\pi} \) may not be unique. For example, for an assembly path \( \alpha \) that has as a binding graph the unit square, we have two paths \( \pi_1 = \text{o.enw} \) and \( \pi_2 = \text{o.nes} \) such that \( \alpha = \alpha_{\pi_1} = \alpha_{\pi_2} \). We extend the notions for ultimately periodic paths to ultimately periodic assembly paths. We say that an assembly path \( \alpha \) is ultimately periodic if there is a path \( \pi \) with \( \alpha = \alpha_{\pi} \) and there are free paths \( m \) and \( p = p_1 \cdots p_k \) such that for all \( i \geq 0 \), and all \( s = 1, \ldots, k \), \( \alpha(m + \vec{p} + p_1 \cdots p_i) = \alpha(m + p_1 \cdots p_i) \).

We introduce the following property for tile assembly systems that we show holds for all confluent systems and helps to characterize the assemblies obtained in these systems.

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\(^1\)The seed can also be taken to be an assembly larger than a singleton tile (e.g. [Doty et al., 2012]).
Definition 4. Two assemblies $\alpha$ and $\alpha'$ are compatible if for all $v \in \text{dom}(\alpha) \cap \text{dom}(\alpha')$ we have that $\alpha(v) = \alpha'(v)$.

Definition 5 (confluent or directed). A tile assembly system $\mathcal{T}$ is confluent if every two producible assemblies $\alpha$ and $\alpha'$ are compatible.

An assembly $\beta$ is maximal for a system $\mathcal{T}$ if for any other assembly $\alpha$ satisfying $\text{dom}(\beta) \subseteq \text{dom}(\alpha)$ we have that $\text{dom}(\alpha) = \text{dom}(\beta)$. In a confluent system, any two assembly paths can ‘coexist’ within a larger assembly because all intersections of their domains are mapped to the same tiles by both paths.

Lemma 6. [Doty et al., 2011] If $\mathcal{T}$ is confluent then there is a unique maximal producible stable assembly $\alpha_{\text{max}}$ such that for every other stable assembly $\alpha$, $\text{dom}(\alpha) \subseteq \text{dom}(\alpha_{\text{max}})$.

Notation. In the rest of the paper we assume that $\mathcal{T}$ is a fixed confluent tile assembly system that produces an infinite stable maximal assembly denoted $\alpha_{\text{max}}$. We introduce several straightforward lemmas that are used later in the text.

Lemma 7. In a confluent system if for an assembly path $\alpha$ there is a forward infinite ultimately periodic path $\pi$ such that $\alpha = \alpha_{\pi}$, then $\alpha$ is ultimately periodic.

Proof. Suppose $\alpha = \alpha_{\pi}$ and $\pi = o.m.p^o$ is ultimately periodic. Observe that because $\mathcal{T}$ is finite, there are $i < j$ such that $\alpha_{\pi}(\vec{m} + i\vec{p}) = \alpha_{\pi}(\vec{m} + j\vec{p})$. Let $q$ be any prefix of $p^{-i}$. Because of confluence of $\mathcal{T}$, it must be that $\alpha_{\pi}(\vec{m} + i\vec{p} + \vec{q}) = \alpha_{\pi}(\vec{m} + j\vec{p} + \vec{q})$. Hence, $\alpha$ is ultimately periodic since for all $k \geq 0$, we have $\alpha(\vec{m}' + kp' + \vec{q}) = \alpha(\vec{m}' + \vec{q})$ where $\vec{m}' = mp'$ and $p' = p^{-i}$. \hfill $\square$

Example 8. The tile set in figs. 4a and 4b generates only one maximal assembly, the one depicted in Fig. 4c. If the seed had a glue $c$ to the north side, then the system would not have been confluent because both tiles A and C could assemble north of $\sigma$ and the maximal assembly would not be unique.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example}
\caption{Example of $\mathcal{T}$, the east, west sides of tiles C and D as well as north of B have glues that do not match any other glues, hence are not indicated.}
\end{figure}

For assembly paths $\{\alpha_i\}_{i \in I}$ we say that $\beta = \bigcup_{i \in I} \alpha_i$ if $\beta$ has a binding graph that is union of the paths that are binding graphs for $\alpha_i$ for all $i \in I$, and $\beta|_{\text{dom}(\alpha_i)} = \alpha_i$. Note this is well defined because all paths are pairwise compatible.
Lemma 9. A stable assembly in a confluent TAS $\mathcal{T}$ is a union of assembly paths.

Proof. It follows directly from the fact that the binding graph of a stable assembly is connected and $\mathcal{T}$ is confluent. □

Corollary 10. The maximal assembly of $\mathcal{T}$, is $\alpha_{\text{max}} = \bigcup_{\alpha \in \Pi} \alpha$ where $\Pi$ denotes the set of all assembly paths.

In the rest of the paper, to simplify notations, $\alpha_{\text{max}}$ is often used to refer to its binding graph. In particular, $\pi \subseteq \alpha_{\text{max}}$ denotes that $\pi$ is a subgraph of the binding graph of $\alpha_{\text{max}}$.

Non-causal. A vertex $B$ in $\mathbb{Z}^2$ is non-causal for vertex $A$ in $\mathbb{Z}^2$ if either $A$ does not belong to $\alpha_{\text{max}}$ or there is a path in $\alpha_{\text{max}}$ from $o$ to $A$ that does not contain $B$. The set of non-causal vertices for $A$ in $\text{dom}(\alpha_{\text{max}})$ is

$$\text{non}_\text{causal}(A) = \{B \in \mathbb{Z}^2 \mid \exists m \in D^*, o.m \subseteq \alpha_{\text{max}} \text{ with } o + \overrightarrow{m} = A \text{ and } B \notin \text{dom}(o.m)\} \cup \{A\}$$

The point $A$ itself is included in the set in order to simplify later expressions. By the definition, if $A \notin \text{dom}(\alpha_{\text{max}})$ then $\text{non}_\text{causal}(A)$ is whole $\mathbb{Z}^2$.

An assembly $\alpha$ that has a path as a binding graph may not necessarily be part of an assembly path, i.e., may not necessarily be part of a producible assembly. If $\alpha$ starts with a tile $t_0$, it may be the case that every assembly path (that starts from the seed, i.e., the origin) that reaches $t_0$ also forms an obstacle for generating $\alpha$ starting at $t_0$. The non-causal set of vertices for a given point in $x \in \mathbb{Z}^2$ identifies the region that is free from those obstacles; in other words, $B$ is in the non-causal region for $A \in \text{dom}(\alpha_{\text{max}})$ if one can reach $A$ within $\alpha_{\text{max}}$ without passing through $B$. In particular, we have the lemma.

Lemma 11. Let $A \in \text{dom}(\alpha_{\text{max}})$. Suppose $p$ is a free path, and $C.p$ is a path such that there is an assembly $\alpha$ (not necessarily producible) whose binding graph has $C.p$ as a subgraph and $\text{dom}(\alpha) = \text{dom}(C.p)$ with a start tile $a(C) = \alpha_{\text{max}}(A)$. If $\text{dom}(A.p) \subseteq \text{non}_\text{causal}(A)$ then $A.p \subseteq \alpha_{\text{max}}$.

Proof. Inductively, for $p = e$ we have $A.p = A \subseteq \alpha_{\text{max}}$. Suppose there is an assembly $\alpha$ with binding graph spanned by the path $C.p' = C.pd$ such that $\text{dom}(A.pd) \subseteq \text{non}_\text{causal}(A)$ and $A.p \subseteq \alpha_{\text{max}}$. Let $B = (A + \overrightarrow{p}).d$ be the end vertex of $A.d$, and hence it is in $\text{non}_\text{causal}(A)$. Let $o.q$ be an assembly path in $\alpha_{\text{max}}$ such that $B \notin \text{dom}(o.q)$ and $o + \overrightarrow{q} = A$. Note that $q$ may use vertices in $\text{dom}(A.p)$ although all of the vertices along $A.p$ are in $\text{non}_\text{causal}(A)$. Suppose $p_1p_2 = p$ is such that $p_1$ is the longest prefix of $p$ where $A + \overrightarrow{p_1} \in \text{dom}(o.q)$ (the suffix $p_2$ could be empty). If $p_1 = e$ then trivially $o.qpd \subseteq \alpha_{\text{max}}$. Let $q'$ be the prefix of $q$ such that $A + \overrightarrow{p_1} = o + \overrightarrow{q'}$. Then, $o.q'p_2d$ is a subgraph of a binding graph of a producible assembly because $\mathcal{T}$ is confluent, $o.q'p_2$ is a path in $\alpha_{\text{max}}$, and $p_2d$ is a suffix of an assembly path that exists. Therefore, the edge between $A + \overrightarrow{p}$ and $B$ exists, i.e., $A.pd \subseteq \alpha_{\text{max}}$. □

The above lemma shows that if $A, C \in \text{dom}(\alpha_{\text{max}})$ and $\alpha_{\text{max}}(A) = \alpha_{\text{max}}(C)$, for every path $p$, if $C.p \subseteq \alpha_{\text{max}}$ then $A.p \subseteq \alpha_{\text{max}}$ as soon as $\text{dom}(A.p) \subseteq \text{non}_\text{causal}(A)$. In particular, for an assembly path $\pi$, for any positive index $i$, $\pi_i$, and $\pi$, are both in $\text{non}_\text{causal}(\pi_i)$. Moreover, the site in $\mathbb{Z}^2$ that is not in $\text{non}_\text{causal}(\pi_i)$ must be in $\pi_{i'}$ (and $\pi_{i''}$).

The lemma below shows that $\hat{f} = \text{coGrow}(b, f, b', f')$ allows to extend paths within $\alpha_{\text{max}}$ in a compatible way since $\mathcal{T}$ is confluent. We use the following setup.
Let $bqf$ and $b'q'f'$ be two bi-infinite free paths with $b, b'$ being backward infinite, $q$ and $q'$ finite and starting with the same direction and $f$ and $f'$ forward infinite. Let $\hat{q}\hat{f} = \text{coGrow}(b, qf, b', q'f')$ be defined and $\hat{q}$ is the maximal portion of the co-growth that consists of segments of $q$ or $q'$ only. Let $A, A' \in \mathbb{Z}^2$ and consider the right regions $R, R'$ of $b.A.qf$, and $b'.A'.q'f'$, respectively. We further suppose that $A.q \subseteq \alpha_{\text{max}}$ and $A'.q' \subseteq \alpha_{\text{max}}$ and $\alpha_{\text{max}}(A) = \alpha_{\text{max}}(A')$.

**Lemma 12** (co-Grow compatibility). If $R \subseteq \text{non}_\text{causal}(A)$ and $R' \subseteq \text{non}_\text{causal}(A')$ then both paths $A.q$ and $A'.\hat{q}$ are subgraphs of $\alpha_{\text{max}}$.

**Proof.** Because $A.q$ is in $R \cap R'$, the hypotheses imply that $\text{dom}(A.q)$ is in $\text{non}_\text{causal}(A)$ and $\text{dom}(A'.\hat{q})$ in $\text{non}_\text{causal}(A')$, hence by Lem. 11 the paths $A.q$ and $A'.\hat{q}$ are subgraphs of $\alpha_{\text{max}}$. Furthermore, whenever the paths $q$ and $q'$ intersect, because of the confluence of the $T$, the tiles in $\alpha_{\text{max}}$ corresponding to these intersections must coincide. Hence $A.q$ and $A'.\hat{q}$ are binding graphs of the same assembly in $T$; just one is a translation of the other. \qed

### 3 $\alpha_{\text{max}}$ has an ultimately periodic assembly path

In this section we show that an infinite $\alpha_{\text{max}}$ in a confluent tiling assembly system must contain an ultimately periodic path. There are two cases:

1. $\alpha_{\text{max}}$ has an infinite assembly path having an empty intersection with an ultimately periodic path in the grid $\mathbb{Z}^2$, and
2. all infinite assembly paths in $\alpha_{\text{max}}$ intersect all ultimately periodic paths in $\mathbb{Z}^2$ infinitely often.

We prove that the first case is equivalent with $\alpha_{\text{max}}$ having an ultimately periodic assembly path, and the second case is impossible. We start with Lem. 13, saying that the existence of an ultimately periodic path in $\alpha_{\text{max}}$ is equivalent to case (1).

Then, for a given $\delta > 0$, we consider a finite segment $\pi_{[\epsilon, r]}$ (called ‘off-the-wall’) of an assembly path $\pi$ such that all intersections of $\pi_{[\epsilon, r]}$ with the $x$-axis ($y = 0$) lie within a finite segment of the $x$-axis $y = 0$ of length $\delta$. We consider the area ‘above’ $y = 0$, bounded by the $x$-axis and off-the-wall path $\pi_{[\epsilon, r]}$. We observe (Lem. 19) that if, for a given $\delta$, the set of such areas above $y = 0$ in $\alpha_{\text{max}}$ is not bounded, then the property 2 of Lem. 13 holds and $\alpha_{\text{max}}$ satisfies case (1).

Finally, in Th. 21 considering case (2), we start with the assumption that any infinite assembly path in $\alpha_{\text{max}}$ intersects the $x$-axis infinitely often. This provides infinitely many off-the-wall assembly paths such that their height cannot be bounded. We show that there is $\delta$ and a set of off-the-wall paths above $\delta$, that are sufficiently high such that it is possible to use co-grow to prove that the set of areas for this $\delta$ cannot be bounded. This proves the condition of Lem. 19 and consequently Lem. 13 holds, contradicting possibility of case (2).

**Lemma 13.** The following two properties are equivalent:

1. $\alpha_{\text{max}}$ contains an ultimately periodic assembly path; and
2. there is a point $A$ in $\mathbb{Z}^2$, an infinite periodic free path $p^o$ and an infinite path $o.\pi \subseteq \alpha_{\text{max}}$ with $\text{dom}(A,p^o) \cap \text{dom}(o.\pi) = \emptyset$.

**Proof.** Property 1 implies Property 2 by taking the ultimately periodic path as $\pi$, the same period $p$ and choosing a point $A$ sufficiently away from the seed. It remains to prove that Property 2 implies Property 1.
First, we consider the case that \( p \) extends eastwards (i.e., \( \overrightarrow{p} \) has a positive first component) and that the infinite path \( \pi \) infinitely extends eastwards north of \( A.p^\omega \) (i.e., for all large enough \( x_0 \), the vertical line \( x = x_0 \) intersects both \( \pi \) and \( A.p^\omega \), and the intersection with \( \pi \) has a larger \( y \) value than the intersection with \( A.p^\omega \)). By replacing, if necessary, \( A \) by \( A + k\overrightarrow{p} \) for some positive \( k \), we also assume that the vertical line passing through \( A \) intersects \( \pi \). Let \( a \) be the least positive integer such that \( A + a\overrightarrow{n} \) is \( \pi_i \) for some \( i \). The situation is depicted in Fig. 5a.

The path \( \tau = \omega \overrightarrow{p}.A.n^\omega .n_\tau \) is bi-infinite. By hypothesis on \( A.p^\omega \) and definition of \( \pi_i \), the finite path \( \pi_{[0,1]} \) does not cross the infinite path \( \tau \). So, \( \pi_{[0,1]} \) is in the interior of either the right, or the left region of \( \tau \). This means that the right region of \( \tau \) is included in \( \text{non-causal}(\pi_i) \) if and only if \( \overrightarrow{o} \) belongs to the left region.

If \( \overrightarrow{o} \) is in the right region of \( \tau \), that is \( \overrightarrow{o} \) is not included in \( \text{non-causal}(\pi_i) \), then \( \pi_{[0,1]} \) is inside the region as illustrated in Fig. 5b. In this case, let \( A' = A + k\overrightarrow{p} \) for some positive \( k \) and let \( \pi_{i'} \) be the intersection of \( \pi \) and \( A'.n^\omega \). By choosing \( k \) large enough, this intersection cannot be on the finite path \( \pi_{[0,1]} \) and, thus, \( i < i' \). By construction \( \pi_i \) is in the left region of \( \omega \overrightarrow{p}.A'.n^\omega .n_\pi \). Since \( i < i' \) then the whole path \( \pi_{[0,1]} \) has to be in this left region and so is \( \overrightarrow{o} \). Thus, the right region of \( \omega \overrightarrow{p}.A'.n^\omega .n_\pi \) is included in \( \text{non-causal}(\pi_i) \). From now on, we can assume w.l.o.g. that \( A \) and \( \pi_i \) are such that the right region of \( \tau \) is included in \( \text{non-causal}(\pi_i) \).

Let \( \Pi \) be the set of forward infinite paths in \( \sigma_{\max} \) that start at \( \pi_i \), are in the right region of \( \tau \), and do not intersect \( A.p^\omega \). The set \( \Pi \) is not empty because it contains \( \pi_i \). Let \( \pi' \) be the path in \( \Pi \) such that, except for \( \pi' \), there is no other infinite path in \( \Pi \) that is inside the right region \( \Pi' \) of \( \omega \overrightarrow{p}.A.n^\omega .n_\pi \). This path does exist because \( \mathbb{Z}^2 \) is discrete and it can be inductively defined from \( \pi_i \) as follows. Starting with \( \overrightarrow{n}, \pi_i \), let \( d \) be the rightmost direction such that there is an infinite path in \( \Pi \) with prefix \( \pi_i, d \). We set \( \pi_0' = \pi_i \) and \( \pi_1' = \pi_i + d \). Given \( k \geq 0 \) and a path \( \pi_0' \ldots \pi_k' \), let \( d \) be the rightmost direction with respect to \( \pi_{k-1}' \pi_k' \) such that there is an infinite path in \( \Pi \) starting with \( \pi_{k}'d \) and does not intersect \( A.p^\omega \). Then, we set \( \pi_{k+1}' = \pi_k' + d \). We denote the right region of \( \omega \overrightarrow{p}.A.n^\omega .n_\pi \) by \( \Pi' \). By construction \( \pi' \) is the only path in \( \Pi' \) that starts at \( \pi_i \) and does not intersect \( A.p^\omega \).

We consider an infinite set of vertices \( \pi_i' \) \((i = 1, 2, \ldots)\) on \( \pi' \) in the following way. Let \( k \) be minimal such that \( (A + k\overrightarrow{n})p^\omega \cap \pi' \neq \emptyset \) (see Fig. 6). If this intersection is infinite \( \{ \pi_1', \pi_2', \ldots \} \) then these vertices are the desired set. If the intersection is finite, say \( \{ \pi_{j_1}', \pi_{j_2}', \ldots \} \) we set \( k_i = k \) and \( A' = A + k_i\overrightarrow{n} \). Let \( \pi_{j_i}' = A' + n\overrightarrow{p} + p' \) for some proper prefix \( p' \) of \( p \). Then, we set

![Figure 5: Situation for Lem.13](image-url)
$A_1 = A' + (n+1)\overrightarrow{p} = \pi_j + \overrightarrow{q_1}$, where $q_1$ is a suffix of $p$ such that $p'q_1 = p$. Consider the minimal $k_2$ such that $(A_1 + k_2\overrightarrow{n}).\overrightarrow{p}' \cap \overrightarrow{\pi}'_j \neq \emptyset$. If this intersection is infinite, we append the infinite sequence of vertices to $\{\overrightarrow{\pi}'_{j_1}, \overrightarrow{\pi}'_{j_2}, \ldots, \overrightarrow{\pi}'_{j_h}\}$ to obtain the desired set. Otherwise, let $\{\overrightarrow{\pi}'_{j_{s+1}}, \overrightarrow{\pi}'_{j_{s+2}}, \ldots, \overrightarrow{\pi}'_{j_h}\}$ be the set of intersection vertices. We repeat the process by setting $A_2 = \pi_j + \overrightarrow{q_2}$, where $q_2$ is a suffix of $p$ such that $A_2, \overrightarrow{p}'$ is a subpath of $(A_1 + k_1\overrightarrow{n}).\overrightarrow{p}'$, and take $k_3$ to be minimal such that $(A_2 + k_3\overrightarrow{n}).\overrightarrow{p}' \cap \overrightarrow{\pi}'_j \neq \emptyset$, etc.

Because $\overrightarrow{\pi}'$ does not intersect $A.\overrightarrow{p}'$, by construction, for each $\pi_j$ ($i = 1, 2, \ldots$) there is a suffix of $p$ such that $\pi_j, q_i\overrightarrow{p}'$ is a subpath of $(A + k\overrightarrow{n}).\overrightarrow{p}'$ for some $k$ and moreover, $\overrightarrow{p}'\overrightarrow{\pi}'_j$ is a bi-infinite path entirely in region $R^*$.

All this is exemplified in Fig. 6. The forward infinite path $A.\overrightarrow{p}'$ is shifted by $k\overrightarrow{n}$ to intersect $\overrightarrow{\pi}'$ (and there is no intersection for a lesser $k$). There is only one intersection, $\overrightarrow{\pi}'_j$ and $q_1$ is the free path from $\overrightarrow{\pi}'_j$ to $A_1$. The process is restarted from $A_1$ instead of $A$. This leads to two intersection points $\overrightarrow{\pi}'_{j_2}$ and $\overrightarrow{\pi}'_{j_3}$ with respective free paths $q_2$ and $q_3$. The process then restarts from $A_2$.

![Diagram showing extraction of special points of $\overrightarrow{\pi}'$.](image)

Since there are infinitely many such $\overrightarrow{\pi}'_j$, there must be two distinct indices $j_1, j_2$ ($j_1 < j_2$) such that $\alpha_{\max}(\overrightarrow{\pi}'_{j_1}) = \alpha_{\max}(\overrightarrow{\pi}'_{j_2})$, $\overrightarrow{\pi}'$ follow $\overrightarrow{\pi}'_{j_1}$ and $\overrightarrow{\pi}'_{j_2}$ in the same direction and $q_{j_1} = q_{j_2} (= q)$. Since $\overrightarrow{p}'\overrightarrow{\pi}'_{j_1}$ and $\overrightarrow{p}'\overrightarrow{\pi}'_{j_2}$ are bi-infinite paths, the forward free path $f$ generated by $\text{coGrow}(\overrightarrow{p}'\overrightarrow{\pi}'_{j_1}, \overrightarrow{p}'\overrightarrow{\pi}'_{j_2})$ is infinite since the right regions of both $\overrightarrow{p}'\overrightarrow{\pi}'_{j_1}$ and $\overrightarrow{p}'\overrightarrow{\pi}'_{j_2}$ contain the backward infinite path $\overrightarrow{p}'\overrightarrow{\pi}'_{j_1}$ (Lem. 3).

Moreover, the right region of $\overrightarrow{p}'\overrightarrow{\pi}'_{j_1}$ is included in $R^*$ and hence is in non-causal($\pi_{j_1}$) because the $\omega$ is outside this region. Then, by Lem. 12 the path $\overrightarrow{\pi}'_{j_1}.f$ belongs to $\alpha_{\max}$.

By definition of $\overrightarrow{\pi}'$, $\overrightarrow{\pi}'$ is the only path in $\Pi$ that is inside region $R^*$, and because $\overrightarrow{\pi}'_{[j_1]}f$ is in $\Pi$, it must be included in the left region of $\overrightarrow{p}'\overrightarrow{A}.\overrightarrow{n}.\overrightarrow{\pi}'_{j_1}$. By co-grow, $\overrightarrow{\pi}'_{j_1}.f$ also has to be in the right region $R^*$ which includes the right regions of $\overrightarrow{p}'\overrightarrow{\pi}'_{j_1}$ and $\overrightarrow{p}'\overrightarrow{\pi}'_{j_2}$. Therefore $\overrightarrow{\pi}'_{j_1}.f$ has to be on the boundary of $R^*$ and hence $\overrightarrow{\pi}' = \overrightarrow{\pi}'_{[j_1]}f$. The same is true for the index $j_2$; so $\overrightarrow{\pi}' = \overrightarrow{\pi}'_{[1,j_1]}f = \overrightarrow{\pi}'_{[1,j_2]}f$. Since $j_1 < j_2$, $\overrightarrow{\pi}'$ has to be ultimately periodic and thus, by Lem. 7 there is an ultimately periodic assembly path in $\alpha_{\max}$ with domain $\overrightarrow{\pi}'$. 

Figure 6: Extraction of special points of $\overrightarrow{\pi}'$. 

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If \( p \) and \( \pi \) do not extend eastwards, the TAS \( \mathcal{T} \) can be rotated until they do so. A TAS \( \mathcal{T} \) is rotated (by 90° clockwards) by rotating its seed and all its tile types. A free path is rotated by replacing \( n \) by \( e \), \( e \) by \( s \) ... TAS \( \mathcal{T} \) can be rotated (by 90° clockwards) by rotating its seed and all its tile types. A free path is rotated by replacing \( n \) by \( e \), \( e \) by \( s \)... TAS \( \mathcal{T} \) is rotated (by 90° clockwards) by rotating its seed and all its tile types. A free path is rotated by replacing \( n \) by \( e \), \( e \) by \( s \)...

Observe that the path \( \pi \) can be quite complicated, as depicted in Fig. 7. In the figure \( p = e^6n \) and the unshaded (white) region is the right region of \( \omega_{A, n^\omega} A. p^\omega \). This region belongs to \( \text{non}_{-}\text{causal}(A + an) \). An example of co-grow is displayed with dashed lines showing that \( \pi_{[i, +\infty]} \) cannot be the path \( \pi \) defined above.

Note that the above lemma does not provide a construction of the ultimately periodic path \( \pi' \) but only shows its existence. In [Durand-Lose et al., 2019] we show an algorithmic way how to use Lem.13 to provide a finite description of \( \alpha_{max} \) that provides ultimately periodic paths comprising \( \alpha_{max} \).

**Corollary 14.** Let \( \pi \) be an infinite path in \( \alpha_{max} \), \( mp^\omega \) be a free path and \( i \) be a positive number such that \( \text{dom}(o, \pi) \cap \text{dom}(\pi, mp^\omega) = \{ \pi_i \} \). Then, one of the three following possibilities appears:

1. \( \pi \) is ultimately periodic,
2. there is an ultimately periodic path in \( \alpha_{max} \) intersecting \( \text{dom}(\pi) \) on an infinite set, or
3. there is an ultimately periodic path in \( \alpha_{max} \) strictly inside the right region of \( mp^\omega, \pi_{[i, \infty]} \).

**Proof.** The proof follows directly from the proof of Lem.13. If the first two conditions are not satisfied, then the path \( \pi' \) constructed in the proof of Lem.13 satisfies the third condition. \( \Box \)
**Definition 15** (off-the-wall path). Let $\delta$ a positive integer. A finite path $\pi = \pi_0 \cdots \pi_r$ is (rightwards) $\delta$-off-the-wall if there exists a positive integer $\ell$ less than $r$ such that:

1. $\pi$ is an assembly path in $\alpha_{\text{max}}$.
2. there is $x_0 \in \mathbb{Z}$ satisfying $\pi_\ell = (x_0, 0)$ and $\pi_r = (x_0 + \delta, 0)$,
3. $\text{dom}(\pi) \cap \text{dom}(e^{\ell} \pi_r) = \text{dom}(\pi) \cap \text{dom}(e^{r} \pi_o)$, and
4. and $o$ is in the right region of the bi-infinite path $\omega e^{\ell} \pi_r e^r$.

\[ \text{(1)} \]

The leftwards $\delta$-off-the-wall is defined in a symmetric way by swapping east and west. We use simply the phrase $\delta$-off-the-wall to denote rightwards $\delta$-off-the-wall, unless otherwise stated.

The positive number $\delta$ is called the width of the off-the-wall path. The wall is the subgraph of $\mathbb{Z}^2$: $\omega e^0 o e^\ell$. The segment $\pi_{[\ell, r]}$ of $\pi$ is called above-the-wall part of $\pi$ (even though some of its portions might be under ‘the wall’ as illustrated in Fig. 8). We call $\pi_\ell$ the left end, and $\pi_r$ the right end of the off-the-wall path.

**Definition 16** (height, surface and area above the wall). Let $\pi$ be any $\delta$-off-the-wall path. Its **height** is the maximal $y$-coordinate it reaches above the wall. The **surface above** (the wall) is the intersection of the left region of $\omega e^0 o e^\ell$ with the right region of $\omega e^{\ell} \pi_{[\ell, r]} e^r$. The **area above** is the area of the surface above the wall. If $\pi$ is an assembly path, the **wall valuation** of an off-the-wall path is the pair of tile types at $\pi_\ell$ and $\pi_r$.

The notions of off-the-wall path, the height above, and surface above are illustrated in Fig. 8 where the surface above is shaded. The area above is the area of the shaded portion. The wall valuation corresponds to the pair of green dotted tiles.

**Figure 8:** $\delta$-off-the-wall path (the surface above is shaded).

The following lemma allows to combine and extend off-the-wall paths.

**Lemma 17** (Off-the-wall combination). Let $\pi = \pi_0 \cdots \pi_r$ and $\pi' = \pi'_0 \cdots \pi'_r$ be two $\delta$-off-the-wall assembly paths with the same wall valuation. There exists a free path $g$ such that: the path $\pi_{[0, \ell]} g$ (resp. $\pi'_{[0, \ell']} g$) is a $\delta$-off-the-wall path with the same wall valuation and the area of its surface above the wall contains the area of the surface above the wall of $\pi$ (resp. $\pi'$).
Proof. The portions above the wall of \( \pi \) and \( \pi' \) are \( \pi_{[\ell, r]} \) and \( \pi'_{[\ell', r']} \) respectively. Since they have the same wall valuations, the tile type at their extreme intersections with the \( x \)-axis are identical, but possibly shifted horizontally. By the definition of the off-the-wall path, everything in the left region of, including, \( \alpha \cdot \mathbf{e} \cdot \pi_{[\ell, r]} \cdot \mathbf{e}^{\omega} \) (resp. in the left region of \( \alpha' \cdot \mathbf{e} \cdot \pi'_{[\ell', r']} \cdot \mathbf{e}^{\omega} \)) belongs to \( \text{non\_causal}(\pi) \) (resp., \( \text{non\_causal}(\pi'_{[\ell', r']}) \)). Let \( p \) and \( p' \) be the free paths corresponding to \( \pi_{[\ell, r]} \) and \( \pi'_{[\ell', r']} \) respectively. We left-co-grow the bi-infinite paths \( \alpha \cdot \mathbf{e} \cdot \mathbf{e}^{\omega} \) and \( \alpha' \cdot \mathbf{p} \cdot \mathbf{e}^{\omega} \).

Let \( f = 1\text{-coGrow} (\alpha \cdot \mathbf{e}, \mathbf{p} \cdot \mathbf{e}^{\omega}, \alpha' \cdot \mathbf{p}' \cdot \mathbf{e}^{\omega}) \). The infinite path \( \mathbf{e} \cdot \mathbf{o} \) is in both left regions of \( \alpha \cdot \mathbf{e} \cdot \mathbf{o} \cdot \mathbf{p} \cdot \mathbf{e}^{\omega} \) and \( \alpha' \cdot \mathbf{e} \cdot \mathbf{p}' \cdot \mathbf{e}^{\omega} \); so \( f \) is infinite by the left-co-grow version of Lem. 3. By construction of co-grow, \( \mathbf{e} \cdot \mathbf{f} \) is equal to \( \mathbf{e} \cdot \mathbf{g} \cdot \mathbf{e}^{\omega} \) where \( \mathbf{g} \) is a prefix of \( f \) with displacement \( \mathbf{g} = ( \delta, 0 ) \) where \( \delta \) is the common width of the off-the-wall paths \( \pi \) and \( \pi' \). By Lem. 12 both \( \pi_{[0, r]} \cdot \mathbf{g} \) and \( \pi'_{[0, r']} \cdot \mathbf{g} \) are paths in \( \alpha_{\text{max}} \). Because the co-grow free path \( \mathbf{g} \) takes the leftmost segment of the two paths, the area of the surface above \( \pi \) (and resp. \( \pi' \)) is included in the area of the surface above \( \pi_{[0, r]} \cdot \mathbf{g} \) (resp. \( \pi'_{[0, r']} \cdot \mathbf{g} \)). The constructed off-the-wall paths have the same width \( \delta \) and wall valuation. \( \square \)

Above lemma allows to generate paths in \( \alpha_{\text{max}} \) that are compatible with respect to their wall valuations while increasing the areas of the surfaces above of the original paths.

Corollary 18. If the set of areas above \( \delta \)-off-the-wall paths with the same wall valuation is bounded then there is a unique \( \delta \)-off-the-wall path \( \hat{\pi} \) with the same wall valuation whose area above the wall is maximal. Moreover, the height of \( \hat{\pi} \) is maximal among all paths with the same wall valuation.

Proof. Consider the set of all \( \delta \)-off-the-wall paths with identical wall valuation. Suppose that this set is finite. If this set has only one path, then that path is \( \hat{\pi} \). Consider two off-the-wall paths \( \pi_{[\ell, r]} \) and \( \pi'_{[\ell', r']} \) in this set with a maximal area above. By Lem. 17, there is a free path \( \mathbf{g} \) such that both \( \pi_{[0, r]} \cdot \mathbf{g} \) and \( \pi'_{[0, r']} \cdot \mathbf{g} \) are off-the-wall paths with the same wall valuation whose area above the wall contains the areas above both \( \pi \) and \( \pi' \). Since \( \pi_{[\ell, r]} \) and \( \pi'_{[\ell', r']} \) have maximal areas above the wall, it must be that \( \pi_{[\ell, r]} \cdot \mathbf{g} = \pi_{[\ell, r]} \) and \( \pi'_{[\ell', r']} \cdot \mathbf{g} = \pi'_{[\ell', r']} \). We observe that \( \pi_{[\ell, r]} \cdot \mathbf{g} \) (resp. \( \pi'_{[\ell', r']} \cdot \mathbf{g} \)) must also have a maximal height because its area above the wall contains all areas above the wall for the set of all \( \delta \)-off-the-wall paths with the same wall valuation. \( \square \)

Lemma 19. If the set of areas above \( \delta \)-off-the-wall paths in \( \alpha_{\text{max}} \) with a given wall valuation is not bounded, then \( \alpha_{\text{max}} \) contains an ultimately periodic assembly path.

Proof. We consider only the off-the-wall paths that correspond to the width \( \delta \) and have the same wall valuation. By the lemma hypothesis, for each \( k \) there is a \( \delta \)-off-the-wall path \( \pi^k \) such that its area above the wall is greater than \( k \). Let \( \hat{\pi}_0 = \pi^0 \) and for \( k \geq 0 \) let \( \hat{\pi}^{k+1} \) be obtained from \( \hat{\pi}^k \) and \( \pi^{k+1} \) by the combination lemma (Lem. 17).

All the \( \hat{\pi}^k \) pass through \( \pi^0 \), and \( \hat{\pi}_0^0 \). By Lem. 17 they are all \( \delta \)-off-the-wall with the same wall valuation and belong to \( \alpha_{\text{max}} \) (i.e., are assembly paths).

Consider the the subgraph \( G \) of \( \alpha_{\text{max}} \) that consists of the union of all off-the-wall segments of all \( \hat{\pi}^k \). This subgraph \( G \) must be a subgraph of \( \alpha_{\text{max}} \) (because all co-grow \( \delta \)-off-the-wall paths are subgraphs of \( \alpha_{\text{max}} \)) and it is infinite because the areas of the surfaces above the co-grows are not bounded. The graph \( G \) is also connected because it is a union of co-grow of \( \hat{\pi}^k \). Moreover, \( G \) has no intersections with \(( \hat{\pi}_r + (1, 0)), e^{\omega} \) because all off-the-wall paths that comprise \( G \) intersect the \( x \)-axis between \( \hat{\pi}_r \) and \( \hat{\pi}_r \).
By König’s lemma there is an infinite path \( \pi’ \) in \( G \) starting at \( u_0^0 \). Because of the co-grow constructions, this infinite \( \pi’ \) does not intersect \( \pi_0^0 \). Therefore \( \pi = \pi_0^0 \cdot \pi’ \) is an infinite assembly path in \( \alpha_{\text{max}} \). Then, \((\pi_0^0 + (1,0)).e^\omega\) is a periodic path in \( \mathbb{Z}^2 \) that has no intersection with the infinite assembly path \( \pi \). By Lem.\( \tau^0 \) \( \alpha_{\text{max}} \) contains an ultimately periodic assembly path. \qed

In the following we observe that the property\( \tau^0 \) in Lem.\( \tau^0 \) is always satisfied in a confluent system \( \mathcal{T} \) as soon as there exists an infinite path. To conclude this, we show that there are always off-the-wall paths with the same wall valuation and unbounded set of areas above; therefore satisfying Lem.\( \tau^0 \). We concentrate on paths in \( \alpha_{\text{max}} \) intersecting the \( x \)-axis an infinite number of times. If there exists an infinite path in \( \alpha_{\text{max}} \) that does not intersect \( o.e^\omega \) (or \( o.w^\omega \)) an infinite number of times, then by Lem.\( \tau^0 \) there is an ultimately periodic assembly path in \( \alpha_{\text{max}} \). We start by observing that the heights of the off-the-wall paths that intersect both sides of the \( x \)-axis (that is, \( o.e^\omega \) and \( o.w^\omega \)), infinite number of times is unbounded.

**Lemma 20.** If there is an infinite assembly path \( \pi \) intersecting both \( o.e^\omega \) and \( o.w^\omega \) an infinite number of times then, up to a vertical symmetry, there exist infinitely many off-the-wall paths and the set of heights of these paths is unbounded.

**Proof.** Let \( W \) be the sets of indices \( \ell’ \) of \( \pi \) such that \( \pi_{\ell'} \in o.w^\omega \) and all the indices (on \( \pi \)) of vertices in the intersection of \( \pi \) and \( \pi_{\ell’}.w^\omega \) are greater than \( \ell’ \). Let \( E \) be defined similarly on the east direction. Both \( E \) and \( W \) are infinite because \( \pi \) intersects the \( x \)-axis an infinite number of times on both sides. There are infinitely many pairs \((w, e)\) such that: \( w \in W, e \in E, w < e, \left[ w + 1, e - 1 \right] \cap (E \cup W) = \emptyset \). An example is depicted in Fig.\( \tau^0 \) where the indices 240 and 300 form such a pair, while \((130, 300)\) is not because \( 240 \in [129, 299] \cap (E \cup W) \).

![Figure 9: Points in E and W.](image)

One of the left, or the right region of the bi-infinite path \( ^o.e.\pi_{[w,e]}e^\omega \) must contain the entire path \( \pi_{[0,w-1]} \). If it is in the right region, then \( \pi_{[0,e]} \) is an off-the-wall path, otherwise, \( \pi_{[0,e]} \) is an off-the-wall path for the system that is vertically symmetric to \( \mathcal{T} \). In this way we obtain infinitely many off-the-wall paths for \( \mathcal{T} \) (or for its vertically symmetric one).

Consider two such pairs \((w, e)\) and \((w’, e’)\). It must be \( w < e < w’ < e’ \) because the intervals \([w + 1, e - 1]\) and \([w’, e’]\) cannot intersect. By definition of \( E \) and \( W \), \( \pi_{\ell’} \) is further to the east than \( \pi_{w} \) and \( \pi_{w’} \) is further to the west than \( \pi_{w} \). Then, \( \pi_{[0,e]} \) must have a greater height (goes ‘above’) than the height of \( \pi_{[0,e]} \). As pairs of indices \((w, e)\) increase, the off-the-wall paths have to pass one ‘above’ the other with an increasing height. Therefore the set of heights and the set of areas above for these paths are not bounded. \qed
The points of interest defined below are used in the proof of the main theorem of this section.

A point of interest of an off-the-wall path \( \pi \) is any point that is west-most on any horizontal line above the wall, that is, a point \( \pi_k \) is a point of interest if \( \pi \) does not intersect \((\pi_k + \mathbf{w}), \mathbf{w}^0\).

In particular, \( \pi_{k_e} \) and \( \pi_{k_f} \) do not intersect \( \pi_k, \mathbf{w}^0 \). There is a point of interest on every horizontal line above the wall up to the height of \( \pi \), and distinct points of interest are on a distinct height above the wall. We point out that points of interest defined above have similar flavor as the notion of ‘visible glues’ used in [Meunier and Woods, 2017, Meunier et al., 2020b,a], except, we are not concerned with the glues of the tiles but the vertices where they appear. Figure 10 shows points of interest \( \pi_{k_1} \) and \( \pi_{k_2} \). The label "no past, no future" indicates that \( \pi \) does not intersect the horizontal lines to the west.

We will use the following lemma in the main theorem.

**Theorem 21.** A confluent tiling system \( \mathcal{T} \) either has a finite \( \alpha_{\text{max}} \) or \( \alpha_{\text{max}} \) has an ultimately periodic assembly path.

**Proof.** Suppose \( \alpha_{\text{max}} \) is infinite. If property of Lem. 13 hold, then \( \alpha_{\text{max}} \) contains an ultimately periodic path. So assume that the properties of Lem. 13 do not hold. Hence, every infinite path in \( \alpha_{\text{max}} \) is intersecting \( \mathbf{o.e}^0 \) and \( \mathbf{o.w}^0 \) infinitely number of times. Moreover, the conditions of Lem. 19 do not hold, because otherwise, as seen in the proof, the conditions of Lem. 13 hold, and there is an ultimately periodic assembly path in \( \alpha_{\text{max}} \). Thus, for every width \( \delta \) and every wall valuation, the set of areas above the wall of \( \delta \)-off-the-wall paths with the same valuation is bounded. On the other side, by Lem. 20, there are off-the-wall paths with heights larger than \( h \) for every \( h \).

Consider \( h \) large enough such that any off-the-wall path \( \pi' \) with height larger than \( h \) must have two distinct points of interest such that the tile types at these vertices in \( \alpha_{\text{max}} \) are the same, and the edges in \( \pi' \) incident to these vertices are in the same directions. Because \( h \) can be arbitrarily large and \( \mathcal{T} \) is finite, such \( h \), off-the-wall path \( \pi' \) of height \( h \), and points of interest exist. Consider an off-the-wall path \( \pi \) that has maximal area above the wall with the same wall valuation as the path \( \pi' \) of height \( h \). Then, \( \pi \) is of height at least \( h \) (by Cor. 18) and hence, it has points of interest \( \pi_{k_1} \) and \( \pi_{k_2} \) (with \( k_1 < k_2 \)) on \( \pi \) such that \( \alpha_{\text{max}}(\pi_{k_1}) = \alpha_{\text{max}}(\pi_{k_2}) \) and the edges \( \pi_{k_1}, \pi_{k_1+1} \) and \( \pi_{k_2}, \pi_{k_2+1} \) are in the same direction.

We show that either there is an ultimately periodic assembly path in \( \alpha_{\text{max}} \) with the same prefix as \( \pi \), or there is an off-the-wall path with same wall valuation as \( \pi \) and a larger surface area above the wall (hence contradicting the maximality of the area bounded by \( \pi \), implying that Lem. 19 must hold, and hence there is an ultimately periodic assembly path). Let \( m \) and \( p \) be free paths that correspond to \( \pi_{[0,k_1]} \) and \( \pi_{[k_1,k_2]} \) respectively. Then, \( \pi = \mathbf{o.m.p.} \pi_{k_2} \). This is illustrated in Fig. 10.

Consider the paths \( f_2 = \pi_{[k_2,r]} \mathbf{e}^0 \) and \( f_1 = \mathbf{p} f_2 \). The free paths \( f_1 \) and \( f_2 \) differ because there is at least one \( n \) in \( p \). Let \( f \) be the \( \ell \)-coGrow \( \langle \mathbf{e}, f_1, \mathbf{e}, f_2 \rangle \). Since the infinite path \( \mathbf{e.o} \) is in each of the left regions of \( \mathbf{e.o.f_1} \) and \( \mathbf{e.o.f_2} \), by Lem. 3, \( f \) is finite, and \( f \) must end with \( \mathbf{e}^0 \). Let \( f = g \mathbf{e}^0 \) where \( g \) is the segment generated by the co-grow of both segments \( \pi_{[k_1,r]} \) and \( \pi_{[k_2,r]} \). Because the left regions of \( \mathbf{e.}\pi_{k_1}, f_1 \) and \( \mathbf{e.}\pi_{k_2}, f_2 \) are subsets of non-causal(\( \pi_{k_1} \)) and non-causal(\( \pi_{k_2} \)) respectively, the free path \( f \) can start from both \( \pi_{k_1} \) and \( \pi_{k_2} \) and both \( \pi_{k_1} \mathbf{g} \) and \( \pi_{k_2} \mathbf{g} \) belong to \( \alpha_{\text{max}} \) by Lem. 12.

The infinite free path \( f \) is either \( f_1 \), or \( f_2 \), or \( g \) is a (leftmost) combination of \( p, \pi_{k_2} \), and \( \pi_{k_2}^* \), taking segments of both paths. In the last case when \( g \) is a leftmost combination of both \( p, \pi_{k_2} \), and \( \pi_{k_2}^* \), taking segments of both paths, we consider the assembly paths \( \pi_{k_1} \mathbf{g} \) and \( \pi_{k_2} \mathbf{g} \). At

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Figure 10: Two points of interest \( \pi_{k_1} \) and \( \pi_{k_2} \) with the same tile in \( \alpha_{\text{max}} \) and the same direction following \( \pi \).

At least one of these assemblies goes strictly to the left of \( \pi \), that is, disconnects from \( \pi \), and then reconnects with it before the end of \( g \) (otherwise \( g \) is either prefix of \( f_1 \) or \( f_2 \). As depicted in Fig. 11a, this forms an off-the-wall path that has strictly larger area above the wall than \( \pi \) without changing the wall valuation. This is not possible because \( \pi \) is supposed to have a maximal area above the wall.

![Diagram](image)

(a) area above the wall enlargement (dotted area) when \( g \) is a combination of \( \pi_{[k_1,r]} \) and \( \pi_{[k_2,r]} \)

(b) impossible \( f = f_2 \)

(c) infinitely periodic in \( \alpha_{\text{max}} \) when \( f = f_1 \)

Figure 11: Co-grow cases for \( f \).

It is not possible that \( g = \pi_{[k_2,r]} \), i.e., \( f \) is \( f_2 \), because \( \pi_{k_1}, g \omega = \pi_{k_1}, f = \pi_{k_1}, f_2 \) (as illustrated in Fig. 11b) would intersect \( \pi_{k_2}, f_2 \) being the same segment of \( \pi \) that starts from a point, \( \pi_{k_1} \), shifted south. But then \( \pi_{k_1}, f \) cannot pass north of a north shifted version of itself, and \( \pi_{k_1}, f_2 \) must intersect \( \pi_{k_2}, f_2 \). This implies that \( g \) is a (leftmost) combination of \( p, \pi_{k_2} \), and \( \pi_{k_2} \), taking segments of both paths, which, as seen above, produces a path with larger area above the wall than \( \pi \) and contradicts our choice of \( \pi \).

The only case left to consider is \( f = f_1 \), i.e., \( g = \{ \pi_{[k_1,r]} \} \). In this case \( \pi_{k_2}, f \) is a shift of \( \pi_{k_1}, f_1 \) northwise and there are no vertices of \( \pi_{k_2}, f_2 \) strictly to the left of \( \pi_{k_2}, f = \pi_{k_1}, f_1 \). Thus, \( p \) is a prefix of \( g \) and \( \pi_{k_2}, p \) is in \( \alpha_{\text{max}} \). Since \( \pi_{k_1} + p = \pi_{k_2}, \pi_{k_1}, p^2 \) is also in \( \alpha_{\text{max}} \) and, by Lem. 1, \( p \) can form a forward infinite periodic path \( p^\omega \). Let \( A = \pi_{k_1} \) and \( B = \pi_{k_2} \); then \( A.pf_2 = A.f_1 = A.f \), and also \( A.p \) ends at \( B \). We have that \( B.f_2 \) is in the right region of \( ^\omega e.p.B.pf_2 = ^\omega e.A.pp f_2 \)
because \( pf_2 = f \) is the left co-grow of \( f_1 \) and \( f_2 \). Similarly, \( B.pf_2 \) is in the right region of \( ^a e_2.B.ppf_2 = ^a e_1.A.pppf_2 \) and hence \( B.f_2 \) is in the right region of \( ^a e_1.A.pppf_2 \). Inductively, we have that \( B.f_2 \) is in the right region of \( ^a e_1.A.p^\omega \), i.e., \( A.p^\omega \) is in the left region of \( ^a e_1.A.f \). Since \( \pi_{[0,k_1-1]} \) is in the interior of the right region of \( ^a e_1.A.f \), \( \text{dom}(A.p^\omega) = \text{dom}(\pi_{k_1},p^\omega) \subseteq \text{non}_\text{causal}(\pi_{k_1}) \) and therefore \( \pi_{k_1}.p^\omega \) is a subgraph in \( \alpha_{\text{max}} \), as depicted in Fig. 11c and there is an ultimately periodic assembly path in \( \alpha_{\text{max}} \).

4 Conclusion

We showed that for every confluent (deterministic) temperature 1 system \( \mathcal{T} \), if the maximal assembly is infinite, then \( \alpha_{\text{max}} \) contains an ultimately periodic assembly path. In our proof we used notions of ‘left’ and ‘right’ regions of a bi-infinite path. Being in two dimensions, the Jordan curve theorem ensures that a bi-infinite path divides the plane in two regions, and hence \( \mathbb{Z}^2 \) is divided in two regions by a bi-infinite path. In our case, the regions are connected subgraphs of \( \mathbb{Z}^2 \) and both regions contain the path itself. The notions of left and right of a path are also used in [Meunier and Woods, 2017] [Meunier et al., 2020b], except in our case the paths are bi-infinite and are not necessarily part of \( \alpha_{\text{max}} \). The other tool developed for the proof is the co-grow of two paths, which is a function that produces a free path that is a superposition of the two paths by taking the rightmost turns of the two paths (a similar notion of ‘right-priority’ was used in [Meunier and Woods, 2017] [Meunier et al., 2020b]). This tool provides a way to identify, or construct, an ultimately periodic path in \( \alpha_{\text{max}} \). In order to co-grow two paths we relied on the confluence of the system.

The co-grow can be applied in any confluent two-dimensional system. It may be of interest to extend this notion to systems that are not necessarily confluent, nor at temperature 1, or even in higher dimensions. We believe that the presence of ultimately periodic assembly paths is not decidable in non-confluent systems, following the result of undecidability of growing infinite ribbons [Adleman et al., 2009].

Our result is a first step toward identifying the two basic structures that an infinite \( \alpha_{\text{max}} \) is conjectured to have: a grid, or a finite union of combs [Doyt et al., 2011]. These two structures were identified as main structures of \( \alpha_{\text{max}} \) in confluent systems if the pumpability of paths is assumed [Doyt et al., 2011]. We do not show the necessary pumping lemma, but we believe that the main result of Th. 21, i.e., the merely existence of an ultimately periodic path, and the condition 2. of Lem. 13 provides a tool for characterizing \( \alpha_{\text{max}} \) through a design of a finite automaton-like structure (called ‘quipu’) [Durand-Lose et al., 2019]. Quipu is a specific automaton associated with a given \( \alpha_{\text{max}} \) that contains a cycle for every ultimately periodic path. The finiteness of the quipu and the fact that it is constructible would imply that a temperature 1 system \( \mathcal{T} \) cannot have universal computing power.

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# Table of symbols

| Symbol | Definition | Page |
|--------|------------|------|
| $\pi_i$ | Path up to index $i$, included. It correspond to $\pi_{[0,i]}$ | 1 |
| $\pi^i_j$ | Path up to index $i$, excluded. It correspond to $\pi_{[0,j-1]}$ | 1 |
| $\pi_r$ | Path from index $i$, included. It correspond to $\pi_{[i,\infty]}$ | 1 |
| $\pi^r_i$ | Path from index $i$, excluded. It correspond to $\pi_{[i+1,\infty]}$ | 1 |
| $\mathbb{Z}$ | Set of all integers | 1 |
| $\pi$ | Some path in $\mathbb{Z}^2$ | 4 |
| dom | Domain of a path, i.e. visited vertices | 4 |
| o | The origin of $\mathbb{Z}^2$, $=(0,0)$ | 4 |
| $D$ | The set of directions, $\{e, n, s, w\}$ | 4 |
| $e$ | Vector for East, $=(1,0)$ | 4 |
| n | Vector for North, $=(0,1)$ | 4 |
| s | Vector for South, $=(0,-1)$ | 4 |
| w | Vector for West, $=(-1,0)$ | 4 |
| $\sqsubseteq$ | Sub-graph inclusion (vertices and edges) | 4 |
| $\sqcup$ | Union of sub-graphs (vertices and edges) | 4 |
| $\epsilon$ | Empty word / empty free path | 4 |
| $\langle \pi \rangle$ | Free path associated to $\pi$ | 4 |
| rot | Rotations of a finite free path | 5 |
| $\partial$ | Boundary of a region/subgraph of $\mathbb{Z}^2$ | 5 |
| $\mathbb{N}$ | Set of all natural numbers | 5 |
| $\text{coGrow}$ | (right) co-growth of a free path inside 2 regions | 7 |
| $\ominus$ | Co-growth starting tile | 7 |
| l–$\text{coGrow}$ | (left) Co-growth of a free path inside 2 regions | 7 |
| $\Sigma$ | Set of glues | 10 |
| $\mathcal{T}$ | Tile Assembly System | 10 |
| $T$ | Set of tile types | 10 |
| $\sigma$ | The seed tile type | 10 |
| $\alpha$ | Some assembly of $\mathcal{T}$ | 10 |
| $\alpha_{\text{max}}$ | The unique maximal assembly of $\mathcal{T}$ | 11 |
| $\beta$ | Some other assembly of $\mathcal{T}$ | 11 |
| non-causal | Non-causal vertices for a vertex in $\mathbb{Z}^2$ | 12 |
| $\tau$ | Some other path in $\mathbb{Z}^2$ | 14 |