The fractional Brownian motion and the halo mass function

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ABSTRACT

The fractional Brownian motion with index \( \alpha \) is introduced to construct the fractional excursion set model. A new mass function with single parameter \( \alpha \) is derived within the formalism, of which the Press-Schechter mass function (PS) is a special case when \( \alpha = 1/2 \). Although the new mass function is computed assuming spherical collapse, comparison with the Sheth-Tormen fitting function (ST) shows that the new mass function of \( \alpha \approx 0.435 \) agrees with ST remarkably well in high mass regime, while predicts more small mass halos than the ST but less than the PS. The index \( \alpha \) is the Hurst exponent, which exact value in context of structure formation is modulated by properties of the smoothing window function and the shape of power spectrum. It is conjectured that halo merging rate and merging history in the fractional set theory might be imprinted with the interplay between halos at small scales and their large scale environment. And the mass function in high mass regime can be a good tool to detect the non-Gaussianity of the initial density fluctuation.

Key words: cosmology: theory – large scale structure of the Universe – galaxies: halos – methods: analytical

1 INTRODUCTION

Halo models are widely applied in the campaign of cosmological parameter estimation in precision from cosmic large scale structures as well as in the expeditions of understanding structure formation. The mass function is a fundamental ingredient of halo models. The most famous analytical formula of mass function, the Press-Schechter mass function (hereafter PS) was derived by Press & Schechter (1974) based on the spherical collapse model. The PS function can be alternatively derived with the random walk or the excursion set formalism (Bond et al. 1991). By smoothing the linear density field on different scales with a sharp k-space filter, the density fluctuation within the characteristic scale could be regarded as a random walk against the variance of the smoothed field at this scale. Consequently the whole theory of random walk can be grafted to model the density contrast field. The elegant theory provides a concise analytical framework to study various processes in cosmic structure formation, and is embraced with great interests by the community. For instance, Lacey & Cole (1993) explicitly calculated the merger rate, halo formation time, and relevant properties of galaxy clusters; Sheth & Tormen (2002) adopted the excursion set theory with moving barrier to study ellipsoidal collapse of halos; Zhang & Hui (2006) solved the excursion set theory with moving barrier of arbitrary shape and discussed the HII bubble size during reionization; and voids phenomenon is explored within the framework by Furlanetto & Pirani (2006).

The success of the random walk formalism in cosmology is prominent, but the primary product of the excursion set theory, the PS mass function, is a poor description to simulations at all epochs (Reed et al. 2006). The common practice is to parameterize the PS function, and then fit the function to simulations to pin down free parameters (e.g. Sheth & Tormen 1999, Jenkins et al. 2001, Reed et al. 2002, Warren et al. 2006). Many functional forms have been proposed by various authors to account for different effects of ellipsoidal collapse (Sheth & Tormen 2002), angular momentum (Del Popolo 2006a) and the index of power spectrum (Reed et al. 2000; Betancort-Rijo & Montero-Dorta 2003). Betancort-Rijo & Montero-Dorta (2006) claims that the “all-mass-at-center” problem shall be properly formulated to obtain the correct mass function in high mass regime. Lee (2006) assumes there is a break in the hierarchical merging process and obtains much shallower mass function in low mass regime.

In this report, we construct a fractional excursion set theory by replacing the conventional random walk with the fractional Brownian motion of index \( \alpha \). The standard excursion set theory is simply a special case of the the new theory. The difference between the normal random walk and the fractional random walk lies in that the latter takes
the correlation between walking steps into account. A new mass function is derived with the fractional excursion set theory, which contains one parameter \( \alpha \) in connection with the correlation of steps of the random walk. Although the new mass function is derived with the boundary condition of a single fixed absorbing barrier, i.e. in spherical collapse scenario, it is in good agreement with the Sheth-Tormen formula \( \text{[Sheth & Tormen 1999, hereafter ST]} \) with \( \alpha \approx 0.435 \) in high mass regime, while has more small mass halos than ST and less small mass halos than PS.

The layout of this paper is that at first we recite the excursion set theory briefly in Section 2, then in Section 3 we introduce the fractional Brownian motion to develop the fractional excursion set theory and subsequently derive a new mass function, and the last section is of discussion.

2 THE EXCURRENT SET THEORY

The initial density fluctuation \( \delta = \rho/\bar{\rho} - 1 \ll 1 \) in early universe is Gaussian and evolves linearly. If the density contrast in a region exceeds a critical value \( \delta_c \), the mass in that region will collapse and be virialised in future to form a halo.

As pointed out by \( \text{[Bond et al. 1991]} \), at an arbitrary point in the universe, the density contrast smoothed with a window function \( W_M(R) \) of characteristic scale \( R \) is a function of the underlying total mass \( M(R) \sim \bar{\rho} R^3 \) included by the smoothing window, the \( M \) effectively represents the scale \( R \). The variation of the smoothed density contrast \( \delta(M) \) forms a trajectory in the plane of \( \delta(M) - M \). The collapsing condition \( \delta_c \) is turned into an absorbing barrier over the trajectory, at the largest \( M \) where \( \delta(M) \) firstly crosses the barrier, the trajectory will be absorbed, i.e. an object will form. The task to find how many objects will form in mass range \( (M, M + dM) \) is converted to the problem of tracing the fraction of trajectories passing through the barrier.

A quantity used to represent the smoothing scale in stead of the mass \( M \) is the variance of the smoothed field

\[
S(M) = \sigma^2(M) = \langle |\delta_M|^2 \rangle = \sum_k |\delta_k|^2 \hat{W}_M(k),
\]

where \( \delta_k \) and \( \hat{W}_M(k) \) are the Fourier transform of \( \delta \) and the window function \( W_M(r) \) respectively. The smoothed density fluctuation can be written as

\[
\delta(S) = \delta(M) = \sum_k \delta_k \hat{W}_M(k),
\]

which actually tells us that \( \delta(S) \) is the sum of \( \delta_k \) weighted by the window function \( W_M(k) \). If the smoothing scale \( R \) is sufficiently large, \( S \) and \( \delta(S) \) will be zero. Once we decrease the smoothing scale \( R \), since the window \( W \) is a function of \( R \), the weighting to Fourier modes of \( \delta \) will change. Naturally the feature of \( \delta \) (S) trajectories depends on the weighting pattern of Fourier modes, i.e. properties of the window function (see examples in \( \text{[Bond et al. 1991]} \).

If the window function is sharp in k-space (a top-hat function spanning from \( k = 0 \) to \( k \sim 1/R \)), the increment \( \delta(S + dS) - \delta(S) \) of a step from \( S \) to \( S + dS \) comes from a new set of Fourier modes in a thin shell of \( (k, k + dk) \). Phases of \( \delta_k \) are uniformly distributed in \( [0, 2\pi] \), the sum \( \sum_{k}^{k+dk} \delta_k \) is a random Gaussian variable and uncorrelated with previous increments \( \text{[Bond et al. 1991, Lacey & Cole 1993]} \). This is exactly a Brownian random walk. If we define \( Q(\delta, S) \) as the number density of trajectories at \( S \) within \( (\delta, \delta + d\delta) \), the Brownian random walk satisfies a simple diffusion equation

\[
\frac{\partial Q}{\partial S} = \frac{1}{2} \frac{\partial^2 Q}{\partial \delta^2}, \quad (3)
\]

and \( S = 0, \delta(S) = 0 \). In absence of barrier, we have solution

\[
Q(\delta, S) = \frac{1}{\sqrt{2\pi S}} \exp \left( -\frac{\delta^2}{2S} \right). \quad (4)
\]

According to \( \text{[Chandrasekhar 1943]} \), a trajectory \( \delta(S) \) reaches the barrier \( \delta_c \) at \( S \) has equal probability to walk above or below the barrier, therefore the solution of Eq. (4) with an absorbing barrier boundary condition is

\[
Q(\delta, S, \delta_c) = \frac{1}{\sqrt{2\pi S}} \left[ e^{-\frac{\delta^2/2S}{2}} - e^{-(\delta - \delta_c)^2/2S} \right]. \quad (5)
\]

The probability of a trajectory absorbed by the barrier \( \delta_c \) must equal to the reduction of trajectories survived below the barrier in interval \( (S, S + dS) \),

\[
f_S(S, \delta_c) dS = -\frac{\partial}{\partial S} \int_{-\infty}^{\delta_c} Q d\delta . \quad (6)
\]

Substituting Eq. (6) and (5) into the above equation gives

\[
f_S(S, \delta_c) dS = \frac{\delta_c}{\sqrt{2\pi}} S^{-3/2} \exp \left( -\frac{\delta^2}{2S} \right) dS , \quad (7)
\]

which is the fraction of mass associated with halos in the range of \( S \) and consequently \( M \). So the comoving number density of halos of mass at epoch \( z \) is simply

\[
\frac{dn}{dM} dM = \frac{\bar{\rho}}{M^2} f_S \left( \frac{dS}{dM} \right) dM = \frac{\bar{\rho}}{M^2} f_{PS}(\sigma) \left| \frac{d\ln \sigma}{d \ln M} \right| dM \quad (8)
\]

where

\[
f_{PS}(\sigma) = \sqrt{\frac{2}{\pi \sigma}} \exp \left( -\frac{\sigma^2}{2\sigma^2} \right) . \quad (9)
\]

This is the well-known Press-Schechter mass function.

3 THE FRACTIONAL EXCURSION SET THEORY

3.1 motivation

It is clear that the validity of the Brownian random motion prescription to the trajectory of \( \delta(S) \) is guaranteed by the sharp k-space filtering. Lack of correlation between the new increment with any previous steps delimits the Markov nature of the Brownian motion. In context of structure formation, it means that the formation of halos at small scales is not correlated with the density fluctuation smoothed at large scales, henceforth halo formation is completely independent of environment.

If we choose a different smoothing window function such as a Gaussian or a top-hat in real space, \( \delta(S + dS) \) contains the same set of \( \delta_k \) as \( \delta(S) \) though in the summation each Fourier mode is weighted differently by the window function. In this circumstance \( \delta(S) \) is apparently correlated with earlier steps, which can not be described by the Brownian random walk formalism any longer. In general there is no
The fractional Brownian motion (FBM) is designed to score.

3.2 the fractional Brownian motion

The FBM is a generalization of the normal Brownian random walk introduced by Mandelbrot & van Ness (1968). FBM, though not well-known in astronomy community, has been widely used to model geometry and growth of many types of rough surfaces in nature like mountain terrain, clouds, percolation and diffusion-limited aggregation. Interestingly it finds its application in financial market (c.f. clouds, percolation and diffusion-limited aggregation). Inter-

FBM, though not well-known in astronomy community, has

Thus the solution under boundary condition of a fixed ab-

formally, with index $\alpha$ ($0 < \alpha < 1$), a FBM is defined as a random process $X(t)$ on some probability space such that:

(i) with probability 1, $X(t)$ is continuous and $X(0) = 0$;
(ii) for any $t \geq 0$ and $h > 0$, the increment $X(t+h) - X(t)$ follows a normal distribution with mean zero and variance $h^{2\alpha}$, so that

$$P(X(t+h) - X(t) \leq x) = \frac{h^{-\alpha}}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2h^{2\alpha}} du . \quad (10)$$

If $\alpha = 1/2$, FBM backs to the normal Brownian motion (c.f. Fedele 1988).

The index $\alpha$ is named the Hurst exponent, which is used originally in the rescaled range analysis (R/S analysis) to portray scaling behaviors of time series. It has strong connection with the fractal dimensions of time series or spatial structures, but the exact relation is case dependent (Meakin 1988). Here, $\alpha$ tells us how strongly correlated the step incresemnt is with previous steps. The trajectory of a FBM with smaller index $\alpha$ is more noisy than that of a FBM with higher index, so sometimes $\alpha$ is called the roughness exponent.

It is very interesting that the FBM has infinitely long-

run correlations. For instance, the past increments $X(0) - X(-t)$ are correlated with future increments $X(t) - X(0)$: as $X(0) = 0$, the correlation function of the “past” and “future” is

$$C(t) = \langle X(-t)X(t) \rangle / \langle X^2(t) \rangle = 2^{2\alpha - 1} - 1 \quad (11)$$

which is invariant with the “time” $t$ and only vanishes when $\alpha = 1/2$! This is an impressive feature of FBM, which leads us to classify FBM into two types:

(i) persistence FBM with $\alpha > 1/2$, which means that an increasing trend in past will result in an increasing trend in future for arbitrary large $t$, i.e. a positive feedback process;
(ii) anti-persistence FBM with $\alpha < 1/2$, which refers to an increasing trend in past will lead to a decreasing trend in future, i.e. a negative feedback.

It might help understanding characteristics of FBM to know the generation methods of FBM. To simulate a 1-dimensional FBM, the simplest method is

$$X(t) = \frac{G(t) + \sum_{s=t-n}^{t-1} (t-s)^{\alpha - 1/2} G(s)}{\Gamma(\alpha + 1/2)} , \quad (12)$$

in which $G(t)$ and $G(s)$ are uncorrelated random numbers extracted from a normal distribution with zero mean and unity variance, $n$ is a practical cut-off number which shall be as large as possible.

To generate a $(d + 1)$-dimensional surface of FBM by Fourier transformation, in the first instance we place a grid in Fourier space, and fill the grid with complex numbers $\delta(k)$ with Gaussian distributed amplitudes and random phases. Spatial correlation is introduced by

$$\delta'(k) = k^{-(d+1/2)} \delta(k) . \quad (13)$$

Then Fourier transformation of the random field $\delta'(k)$ will give a self-affine surface modelled by FBM.

3.3 the fractional excursion set theory

The number density of trajectories $Q_\alpha(\delta, S)$ of fractional Brownian motion with index $\alpha$ obeys the diffusion equation (c.f. Latza 2001).

$$\frac{\partial Q_\alpha}{\partial S} = \alpha S^{2\alpha - 1} \frac{\partial^2 Q_\alpha}{\partial \delta^2} , \quad (14)$$

which has solution in absence of barrier

$$Q_\alpha(\delta, S) = S^{-\alpha} \frac{\Gamma(2\alpha)}{\sqrt{2\pi}} \exp \left( -\frac{\delta^2}{2S^{2\alpha}} \right) . \quad (15)$$

Apprantly the distribution of $\delta(S)$ at $S$ is still a Gaussian, the argument of Chandrasekhar (1943) shall be valid, thus the solution under boundary condition of a fixed absorbing barrier $\delta_c$ is

$$Q_\alpha(S, \delta, \delta_c) = S^{-\alpha} \sqrt{\frac{2\pi}{\sqrt{2\pi}}} \left[ \exp \left( -\delta^2/2S^{2\alpha} \right) - \exp \left( -(\delta - \delta_c)^2/2S^{2\alpha} \right) \right] . \quad (16)$$

After a straightforward and tedious calculation, the halo mass function is

$$\frac{dn}{dM} dM = \frac{\bar{\rho}}{M^2} f_\alpha(\sigma) \frac{dln \sigma}{dln M} dM \quad (17)$$

with the kernel

$$f_\alpha(\sigma) = \frac{4\alpha}{\sqrt{2\pi}} \frac{\delta_c}{\sigma^{2\alpha}} \exp \left( -\frac{\delta_c^2}{2\sigma^{2\alpha}} \right) . \quad (18)$$

It is easy see that it is the PS function when $\alpha = 1/2$. For comparison, we reproduce the kernel of the Sheth-Tormen mass function here,

$$f_{ST}(\sigma) = A \sqrt{\frac{2\alpha}{\pi}} \left[ 1 + \left( \frac{\alpha^2}{a^2\delta_c^2} \right)^{1/2} \right] \frac{\delta_c}{\sigma} \exp \left( -\frac{a^2\delta_c^2}{2\sigma^2} \right) . \quad (19)$$
where \( A = 0.3222, \alpha = 0.707 \) and \( p = 0.3 \).

\( f_\alpha \) of different Hurst exponent \( \alpha \) in comparison with PS and ST formulas are displayed in Fig. 1. The mass function from persistence FBM is very different with that of anti-persistence FBM. It appears that the ST function is in good agreement with our new mass function of \( \alpha \approx 0.435 \) in high mass regime. The arrow indicates the region where all curves cross the line \( f/f_{PS} = 1 \).

\( f_\alpha \) is very sensitive to the value of \( \alpha \). In present days' simulations, we have to leave it to future to tell which mass function is better in small mass regime. A quick check indicates that \( f_\alpha \) is obtained from fitting to simulations and has good accuracy in large mass scale environment, an immediate conclusion is that trajectories \( \delta(S) \) in our universe is actually anti-persistent.

In small mass regime, the dependence on \( \alpha \) of \( f_\alpha \) is relatively weak. If \( \alpha < 0.5 \) the new mass function predicts less number of halos than the PS formula by \( \sim 10 - 20\% \), but up to \( \sim 30\% \) more than what ST function gives. It is very difficult and unreliable to resolve halos with mass lower than \( 10^9M_\odot \) in present days' simulations, we have to leave it to future to tell which mass function is better in small mass regime. A quick check indicates that \( f_\alpha \) has very different shape with ST formula at \( \ln \sigma^{-1} < \sim 0.3 \), we can only achieve good fit to \( f_{ST} \) in range of \( -0.5 < \ln \sigma^{-1} < 0.3 \) with \( \alpha \approx 0.35 \).

\section{Discussion}

The fractional Brownian motion of index \( \alpha \) is introduced to construct the fractional excursion set theory. The new mass function computed with the theory is analytical and simple, of which the PS mass function is only a special case of \( \alpha = 1/2 \). Comparison with the ST function nurtured by N-body simulations demonstrates the excellent performance of the new mass function.

In Fig. 1 it is observed that high mass halo abundance is very sensitive to the value of \( \alpha \), the high mass halo abundance observed can be potentially a very powerful tool to detect the non-Gaussianity of the initial density fluctuation field: non-Gaussianity will change the correlation between walking steps of \( \delta(S) \) and therefore modify the \( \alpha \) effectively.

The success of applying FBM formalism to model structure formation is attributed to the inclusion of the correlation between density fluctuations at different scales. The correlation strength characterized by the Hurst exponent \( \alpha \) could be resulted from properties of window function and the intrinsic correlation of the cosmic density field. We know that a non-sharp filtering in k-space will induce correlation \cite{Bond}, but are unclear how \( \alpha \) changes with features of the window function. More of interests is the relation between \( \alpha \) and the power spectrum of density field. The generation method Eq. 13 provides some clues, however there is the complication that the scaling of trajectory \( \delta(S) \) is founded relative to the variance \( \sigma^2 \), not the physical scale \( R \). Numerical experiments with scale free simulations shall be able to improve our understanding effects of window function and power spectrum on \( \alpha \).

In this work only the mass function is computed. In principle, the fractional excursion set theory may have many applications, for example, those works of Lacey & Cole \cite{Lacey}, Mo & White \cite{Mo} and Zhang & Hui \cite{Zhang} can all be revisited with FBM. Since \( \alpha \) denotes the correlation of \( \delta \) at different scales, the subsequently calculated halo merger rate and merger history is marked with the stamp of large scale environment on halo formation at small scales. We might be able to explain the halo clustering dependence on halo formation history and environment \cite{Gao, Wechsler}.

The new halo mass function is obtained assuming spherical collapse. To improve the accuracy of the model, ellipsoidal collapse has to be taken into account. The poor performance of Eq. 13 of \( \alpha \approx 0.435 \) in low mass regime (see Fig. 1) is very likely due to our simplification of adopting the spherical collapse model. Essentially to calibrate the ellipsoidal collapse, we replace the fixed barrier \( \delta \) with a moving barrier \( B(S) \) as in Sheth & Tormen \cite{Sheth} and Del Popolo \cite{Del}, and then solve the diffusion equation with the new boundary condition. Technique details and comparison with simulations will be presented elsewhere (Fosalba & Pan, in preparation).

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\section{References}

Betancort-Rijo J. E., Montero-Dorta A. D., 2006, ApJ in press, astro-ph/0605608

Bond J. R., Cole S., Efstathiou G., Kaiser N., 1991, ApJ, 379, 440

Chandrasekhar S., 1943, Reviews of Modern Physics, 15, 1
fractional excursion set theory

Del Popolo A., 2006a, AJ, 131, 2367
Del Popolo A., 2006b, A&A, 448, 439
Feder J., 1988, Fractals. Plenum Press, New York
Furlanetto S. R., Piran T., 2006, MNRAS, 366, 467
Gao L., Springel V., White S. D. M., 2005, MNRAS, 363, L66
Harker G., Cole S., Helly J., Frenk C., Jenkins A., 2006, MNRAS, 367, 1039
Jenkins A., Frenk C. S., White S. D. M., Colberg J. M., Cole S., Evrard A. E., Couchman H. M. P., Yoshida N., 2001, MNRAS, 321, 372
Lacey C., Cole S., 1993, MNRAS, 262, 627
Lee J., 2006, submitted to ApJ, astro-ph/0605697
Lutz E., 2001, Phys. Rev. E, 64, 051106
Mandelbrot B. B., van Ness J. W., 1968, SIAM Review, 10, 422
Meakin P., 1998, Fractals, scaling and growth far from equilibrium. Cambridge University Press, Cambridge
Mo H. J., White S. D. M., 1996, MNRAS, 282, 347
Press W. H., Schechter P., 1974, ApJ, 187, 425
Reed D., Bower R., Frenk C., Jenkins A., Theuns T., 2006, submitted to MNRAS, astro-ph/0607150
Reed D., Gardner J., Quinn T., Stadel J., Fardal M., Lake G., Governato F., 2003, MNRAS, 346, 565
Sheth R. K., Tormen G., 1999, MNRAS, 308, 119
Sheth R. K., Tormen G., 2002, MNRAS, 329, 61
Sheth R. K., Tormen G., 2004, MNRAS, 350, 1385
Warren M. S., Abazajian K., Holz D. E., Teodoro L., 2006, ApJ, 646, 881
Wechsler R. H., Zentner A. R., Bullock J. S., Kravtsov A. V., Allgood B., 2006, ApJ in press, astro-ph/0512416
Zhang J., Hui L., 2006, ApJ, 641, 641