Quantum speed limits and optimal Hamiltonians for driven systems in mixed states

O Andersson and H Heydari

Department of Physics, Stockholm University, SE-10691 Stockholm, Sweden
E-mail: olehandersson@gmail.com and hoshang@fysik.su.se

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Abstract

Inequalities of Mandelstam–Tamm (MT) and Margolus–Levitin (ML) type provide lower bounds on the time that it takes for a quantum system to evolve from one state into another. Knowledge of such bounds, called quantum speed limits, is of utmost importance in virtually all areas of physics, where determination of the minimum time required for a quantum process is of interest. Most MT and ML inequalities found in the literature have been derived from growth estimates for the Bures length, which is a statistical distance measure. In this paper we derive such inequalities by differential geometric methods, and we compare the quantum speed limits obtained with those involving the Bures length. We also characterize the Hamiltonians which optimize the evolution time for generic finite-level quantum systems.

Keywords: quantum speed limit, optimal Hamiltonian, symplectic reduction, mixed state, Bures length
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1. Introduction

The fundamental problem of determining the minimum time required to perform a quantum process and the dual problem of designing time-optimal Hamiltonians have recently attracted much attention because of their significance in several modern applications of quantum mechanics. These include quantum metrology [1–3], quantum computation and information [4–6], and optimal control theory [7–11]. Limits on the minimal evolution time also play a role in cosmology [4], and quantum thermodynamics [12].

In this paper, we use the terminology introduced by Margolus and Levitin [13] and call lower bounds on the time that it takes for a quantum system to evolve from one state into another quantum speed limits. More specifically, we refer to lower bounds that involve the
energy uncertainty as Mandelstam–Tamm (MT) quantum speed limits. Bhattacharyya [14] was one of the first to derive an MT quantum speed limit (see also [15]). Assuming Mandelstam and Tamm’s uncertainty relation, he showed through accurate estimates of the rate of change of the ‘quantum non-decay probability’ that it takes at least the time $\pi \hbar /2\Delta E$ for a system to evolve between two orthogonal pure states. A few years later, Anandan and Aharonov [16] confirmed Bhattacharyya’s result. But more importantly, they revealed the geometric nature of the MT quantum speed limits, showing that $\frac{1}{\hbar}$ times the path integral of the energy uncertainty of a unitarily evolving pure state equals the Fubini–Study length of the curve traced out by the state.

Few evolution time estimates for quantum systems in mixed states have been derived by differential geometric methods, despite the differential geometric approach in [16]. Until now, most MT quantum speed limits for mixed states have been obtained from estimates of the growth of the Bures length [17], which is a statistical distance measure. Soon after the publication of [16], Uhlmann [18] showed that the time that it takes for a system to evolve from one mixed state into another is bounded from below by $\frac{1}{\hbar}$ times the fraction of the Bures length between the states and the energy uncertainty of the system. Uhlmann’s result has been verified in several publications, e.g. [19–23].

In this paper we develop Anandan and Aharonov’s approach. We use symplectic reduction to construct a principal fiber bundle over a general space of isospectral (i.e. unitarily equivalent) mixed quantum states. The bundle, which generalizes the Hopf bundle for pure states, gives rise in a canonical way to a Riemannian metric and a symplectic form on the space of isospectral mixed states. Using these we then derive an MT quantum speed limit for unitarily driven quantum systems, which proves to be sharper than the Uhlmann limit.

The speed of a quantum evolution depends also on the system’s energy resources. Margolus and Levitin [13] showed that the time that it takes for a non-driven system to unitarily evolve between two orthogonal pure states is bounded from below by a factor that is inversely proportional to the energy of the system. Generalizing Margolus and Levitin’s result to systems in mixed states has proven to be quite difficult. Giovannetti et al [21, 22] have derived an ‘implicit’ Margolus–Levitin (ML) quantum speed limit for non-driven systems, and Deffner and Lutz [23] used the methods put forward by Jones and Kok [19] to derive an estimate of ML type for driven systems. The present paper also contains a generalization of Margolus and Levitin’s quantum speed limit to driven systems in mixed states. The speed limit is different from that of Deffner and Lutz, and it reduces to a greater speed limit than theirs for systems with time-independent Hamiltonians.

The paper is organized as follows. In section 2, we set up the geometric framework, and introduce most of the notation that we use in the rest of the paper. In sections 3 and 4 we derive an MT quantum speed limit and an ML quantum speed limit, respectively, for the time that it takes to unitarily run a quantum system from one mixed state into another. Section 5 contains a characterization of the Hamiltonians that optimize evolution time for finite-level systems in generic mixed states, as well as an example of a system for which the MT quantum speed limit derived in section 3 is greater than the corresponding limit involving the Bures length. The paper ends with a conclusion.

1.1. Conventions

Evolving quantum states will be represented by curves of density operators. The curves are assumed to be defined on an unspecified interval $0 \leq t \leq \tau$, and the final time $\tau$ will be referred to as the evolution time. Compositions of linear mappings will be written as concatenations. By a ‘function’ we mean a real-valued smooth function, and by a ‘functional’ we mean a
2. Reduced purification bundles

This paper concerns quantum systems in mixed states which evolve unitarily. The systems will be modeled on a Hilbert space $\mathcal{H}$ and their states will be represented by density operators. We write $\mathcal{D}(\mathcal{H})$ for the space of density operators on $\mathcal{H}$, and $\mathcal{D}_k(\mathcal{H})$ for the space of density operators on $\mathcal{H}$ which have finite rank at most $k$.

A quantum state is called pure if it can be represented by a single unit vector. In quantum information theory and geometric quantum mechanics it is common to make use of the fact that mixed states can be considered as reduced pure states [24, 25]. In this paper, we will use

\[ P(\sigma) = \sum_{j=1}^I p_j \Pi_j, \quad \Pi_j = \sum_{i=m_1+\ldots+m_{j-1}+1} |i\rangle \langle i|, \]

where the $p_j$ are the density operator’s different positive eigenvalues, listed in descending order, and the $m_j$ are the eigenvalues’ multiplicities. Throughout the rest of this paper we fix such a spectrum $\sigma$ and write $\mathcal{D}(\sigma)$ for the corresponding $\mathcal{U}(\mathcal{H})$-orbit in $\mathcal{D}_k(\mathcal{H})$. Furthermore, we fix an orthonormal computational basis $\{ |1\rangle, |2\rangle, \ldots, |k\rangle \}$ in $\mathcal{K}$ and we define a spectral characteristic Hermitian operator $P(\sigma)$ on $\mathcal{K}$ by

\[ L_U(\psi) = U \psi, \quad R_V(\psi) = \psi V. \]

We write $u(\mathcal{H})$ and $u(\mathcal{K})$ for the Lie algebras of $\mathcal{U}(\mathcal{H})$ and $\mathcal{U}(\mathcal{K})$, and $X_\xi$ and $\tilde{\eta}$ for the fundamental vector fields corresponding to $\xi$ in $u(\mathcal{H})$ and $\eta$ in $u(\mathcal{K})$:

\[ X_\xi(\psi) = \frac{d}{dt} [L_{\exp(t\xi)}(\psi)]_{t=0} = \xi \psi, \quad \tilde{\eta}(\psi) = \frac{d}{dt} [R_{\exp(t\eta)}(\psi)]_{t=0} = \psi \eta. \]

Every functional on $u(\mathcal{K})$ has the form $\mu_{\Lambda}(\xi) = i \hbar \text{Tr}(A\xi)$ for some Hermitian operator $A$ on $\mathcal{K}$. Let $u(\mathcal{K})^*$ be the space of all functionals on $u(\mathcal{K})$ and define $J : \mathcal{L}(\mathcal{K}, \mathcal{H}) \rightarrow u(\mathcal{K})^*$ by

\[ J(\psi) = \mu_{\psi^* \psi}. \]
Theorem 2.1. $J$ is a coadjoint-equivariant momentum map for the Hamiltonian $\mathcal{U}(K)$-action on $L(K, \mathcal{H})$, and $\mu_{P(\sigma)}$ is a regular value of $J$ whose isotropy group acts freely and properly on $J^{-1}(\mu_{P(\sigma)})$.

Now let $S(\sigma)$ be the set of $\psi$ in $L(K, \mathcal{H})$ satisfying $\psi^\dagger \psi = P(\sigma)$, and define

$$\pi : S(\sigma) \rightarrow D(\sigma), \quad \pi(\psi) = \psi \psi^\dagger. \quad (6)$$

The map $\pi$ is a principal fiber bundle with gauge group $U(\sigma)$ consisting of all unitary operators on $K$ which commute with $P(\sigma)$. In fact, $S(\sigma) = J^{-1}(\mu_{P(\sigma)})$, $U(\sigma)$ is the isotropy group of $\mu_{P(\sigma)}$, and $\pi$ is canonically isomorphic to the reduced space submersion $J^{-1}(\mu_{P(\sigma)}) \rightarrow J^{-1}(\mu_{P(\sigma)})/U(\sigma)$; see [26]. The action of $U(\sigma)$ on $S(\sigma)$ is induced by the right action of $U(K)$ on $L(K, \mathcal{H})$. We write $u(\sigma)$ for the Lie algebra of $U(\sigma)$. This algebra consists of all anti-Hermitian operators on $K$ which commute with $P(\sigma)$. It follows from the Marsden–Weinstein–Meyer symplectic reduction theorem [27, 28] that $D(\sigma)$ admits a symplectic form which is pulled back to $\Omega|_{S(\sigma)}$ by $\pi$. We will not need the full strength of this fact, only that $S(\sigma)$ is preserved by the left action by $U(\mathcal{H})$. This, in turn, implies that solutions to Schrödinger equations which extend from elements in $S(\sigma)$ remain in $S(\sigma)$.

2.2. Riemannian structure and the mechanical connection

The metric $G$ restricts to a gauge-invariant metric on $S(\sigma)$. We define the vertical and horizontal bundles over $S(\sigma)$ to be the subbundles $V_S(\sigma) = \ker d\pi$ and $H_S(\sigma) = V_S(\sigma)^\perp$ of the tangent bundle $T_S(\sigma)$; see figure 1. Here $d\pi$ is the differential of $\pi$ and $^\perp$ denotes the orthogonal complement with respect to $G$. Vectors in $V_S(\sigma)$ and $H_S(\sigma)$ are called vertical and horizontal, respectively. We equip $D(\sigma)$ with the unique metric $g$ which makes $\pi$ a Riemannian submersion. Thus, $g$ is such that the restriction of $d\pi$ to every fiber of $H_S(\sigma)$ is an isometry.

Example 1. If $\sigma = (1; 1)$, the operators in $D(\sigma)$ represent pure states. In fact, $D(\sigma)$ is the projective space over $\mathcal{H}$, $S(\sigma)$ is the unit sphere in $\mathcal{H}$, $\pi$ is the Hopf bundle, and $g$ is the Fubini–Study metric [16, 29, 30].

Remark 1. For a general spectrum, $S(\sigma)$ is diffeomorphic to the Stiefel manifold of $k$-frames in $\mathcal{H}$; see [29, 30]. However, $g$ is different from the Riemannian metric induced by the standard bi-invariant metric on $U(\mathcal{H})$. 

![Figure 1. Illustration of the bundle $\pi$ and the decomposition of each tangent space of $S(\sigma)$ into a vertical and a horizontal subspace.](image)
The fundamental vector fields of the gauge group action on $S(\sigma)$ yield canonical isomorphisms between $u(\sigma)$ and the fibers in $V S(\sigma)$. Furthermore, $H S(\sigma)$ is the kernel bundle of the gauge-invariant mechanical connection $A_\psi = I^{-1} J_\psi$, where

$$I : S(\sigma) \times u(\sigma) \to u(\sigma)^*, \quad I_\psi \xi(\eta) = G(\hat{\xi}(\psi), \hat{\eta}(\psi)),$$

$$J : T S(\sigma) \to u(\sigma)^*, \quad J_\psi (X)(\xi) = G(X, \hat{\xi}(\psi))$$

are the locked inertia tensor and metric momentum map, respectively. The inertia tensor is of constant bi-invariant type since $I_\psi$ is an adjoint-invariant form on $u(\sigma)$ which is independent of $\psi$. Thus all $I_\psi$ define the same metric on $u(\sigma)$, namely

$$\xi \cdot \eta = -\frac{1}{2} \text{Tr}((\xi \eta + \eta \xi) P(\sigma)).$$

This metric can be used to derive the following explicit formula for the mechanical connection:

$$A_\psi (X) = \sum_{j=1}^l \Pi_j \psi^\dagger X \Pi_j P(\sigma)^{-1}.$$

For details, consult [26]. The next proposition will be important in section 5.

**Proposition 2.2.** Geodesics in $S(\sigma)$ have conserved metric momenta. Therefore, a geodesic in $S(\sigma)$ which is initially horizontal remains horizontal.

**Proof.** Let $\psi = \psi(t)$ be a curve in $S(\sigma)$ and $\xi$ be any element in $u(\sigma)$. Then

$$\frac{d}{dt} J_\psi (\dot{\psi})(\xi) = \frac{1}{2} \frac{d}{dt} \text{Tr}((\psi^\dagger \dot{\psi} - \dot{\psi}^\dagger \psi) \xi) = \frac{1}{2} \text{Tr}((\nabla_t \psi^\dagger \psi - \psi^\dagger \nabla_t \psi) \xi).$$

Thus $J_\psi (\dot{\psi})$ is constant if $\psi$ is a geodesic. \qed

**Example 2.** The proof of proposition 2.2 also shows that solutions in $S(\sigma)$ to Schrödinger equations with time-independent Hamiltonians have conserved metric momenta. For if $\psi$ satisfies $i\hbar \dot{\psi} = \hat{H} \psi$, where $\hat{H}$ is a time-independent Hamiltonian, then

$$\frac{1}{2} \text{Tr}((\nabla_t \psi^\dagger \psi - \psi^\dagger \nabla_t \psi) \xi) = -\frac{1}{2\hbar^2} \text{Tr}((\psi^\dagger \hat{H}^2 \psi - \psi^\dagger \hat{H} \hat{H}^* \psi) \xi) = 0.$$

In section 5 we restrict our study to the case where $\hat{H}$ is finite dimensional and the density operators in $D(\sigma)$ are invertible. Under these conditions, we characterize the Hamiltonians which transport elements of $S(\sigma)$ along horizontal geodesics.

### 3. An inequality of Mandelstam–Tamm type

Anandan and Aharonov [16] showed that the distance between two pure quantum states equals the length of that evolution curve connecting the two states which has the least average fluctuation in energy. In this section we generalize Aharonov and Anandan’s result to evolutions of quantum systems in mixed states. To be precise, we show that $1/\hbar$ times the path integral of the energy uncertainty of an evolving mixed state is bounded from below by the length of the curve traced out by the density operator of the state, and we show that every curve of isospectral density operators is generated by a Hamiltonian for which $1/\hbar$ times the uncertainty path integral equals the curve’s length. These observations give rise to an MT evolution time estimate, which we compare with previously established MT estimates involving the Bures length.
3.1. Parallel and perpendicular Hamiltonians

The average energy function $H$ of a Hamiltonian $\hat{H}$ on $\mathcal{H}$ is defined by $H(\rho) = \text{Tr}(\hat{H}\rho)$. We write $X_H$ for the Hamiltonian vector field of $H$, $X_H(\rho) = [\rho, \hat{H}]_{\text{i}h}$. This field has a distinguished gauge-invariant lift $X_{\hat{H}}$ to $S(\sigma)$, $X_{\hat{H}}(\psi) = \hat{H}\psi/\text{i}h$. We say that $\hat{H}$ is parallel at a density operator $\rho$ if $X_{\hat{H}}(\psi)$ is horizontal at some, and hence every, $\psi$ in the fiber over $\rho$. Furthermore, we say that $\hat{H}$ parallel transports $\rho$ if the solution curve to the initial value Schrödinger equation

$$\psi = X_{\hat{H}}(\psi), \quad \psi(0) \in \pi^{-1}(\rho),$$

is horizontal. We remind the reader that for every curve $\rho(t)$ in $\mathcal{D}(\sigma)$ and every initial value $\psi_0$ in the fiber over $\rho(0)$, there exists a unique horizontal curve $\psi(t)$ in $\mathcal{S}(\sigma)$ that extends from $\psi_0$ and is projected onto $\rho$; e.g. see [29, p 69, proposition 3.1]. Furthermore, $\psi(t)$ is the solution to (13) for some, possibly time-dependent, Hamiltonian because $U(\hat{H})$ acts transitively on $\mathcal{S}(\sigma)$.

The locked inertia tensor can be used to measure the deviation from parallelism: given a Hamiltonian $\hat{H}$ we define a $u(\sigma)$-valued field $\xi_{\hat{H}}$ on $\mathcal{D}(\sigma)$ by $\pi^*\xi_{\hat{H}} = A \circ X_{\hat{H}}$. Then $\xi_{\hat{H}} \cdot \xi_{\hat{H}}$ equals the square of the norm of the vertical part of $X_{\hat{H}}$. (Recall that $\cdot$ is the metric on $u(\sigma)$ given by (9).) The field $\xi_{\hat{H}}$ is intrinsic to the quantum system, and contains complete information about the expectation values of $\hat{H}$; cf (15) below.

The opposite of parallelism we call perpendicularity. Thus, $\hat{H}$ is perpendicular at $\rho$ if $X_{\hat{H}}$ is vertical along the fiber over $\rho$, or equivalently, if $X_{\hat{H}}(\psi) = \psi \xi_{\hat{H}}(\rho)$ for every lift $\psi$ of $\rho$. In this case $X_{\hat{H}}(\rho) = 0$. Note also that $\hat{H}$ is perpendicular at $\rho$ provided that $\rho$ represents a mixture of eigenstates of $\hat{H}$.

The precision to which the value of a Hamiltonian $\hat{H}$ can be known is quantified by its uncertainty function $\Delta H(\rho) = \sqrt{\text{Tr}(\hat{H}^2\rho) - \text{tr}(\hat{H}\rho)^2}$. Let $\xi_{\hat{H}}^\perp$ be the projection of $\xi_{\hat{H}}$ on the orthogonal complement of the unit vector $-i1$ in $u(\sigma)$.

**Theorem 3.1.** The Hamiltonian vector field of $H$ satisfies

$$h^2 g(X_H, X_H) = \Delta H^2 - \xi_{\hat{H}}^\perp \cdot \xi_{\hat{H}}^\perp. \quad (14)$$

In particular, $h^2 g(X_H(\rho), X_H(\rho)) = \Delta H(\rho)^2$ if $\hat{H}$ is parallel at $\rho$.

**Proof.** Let $\psi$ be a purification of $\rho$. Then

$$\text{Tr}(\hat{H}\rho) = i\hbar \text{Tr}(A_\psi (X_{\hat{H}}(\psi)) P(\sigma)) = i\hbar \text{Tr}(\xi_{\hat{H}}(\rho) P(\sigma)) = h(-i1) \cdot \xi_{\hat{H}}(\rho), \quad (15)$$

$$\text{Tr}(\hat{H}^2\rho) = h^2 G(X_{\hat{H}}(\psi), X_{\hat{H}}(\psi)) = h^2 g(X_{\hat{H}}(\rho), X_{\hat{H}}(\rho)) + h^2 \xi_{\hat{H}}(\rho) \cdot \xi_{\hat{H}}(\rho). \quad (16)$$

It follows that

$$\Delta H^2 = h^2 (g(X_H, X_H) + \xi_{\hat{H}} \cdot \xi_{\hat{H}}) - H^2 = h^2 (g(X_H, X_H) + \xi_{\hat{H}}^\perp \cdot \xi_{\hat{H}}^\perp). \quad (17)$$

In particular, $h^2 g(X_H(\rho), X_H(\rho)) = \Delta H(\rho)^2$ if $\xi_{\hat{H}}(\rho) = 0$. □

**Example 3.** For pure states, the vertical bundle is one-dimensional. Therefore $\xi_{\hat{H}}^\perp = 0$. It follows that $h^2 g(X_H, X_H) = \Delta H^2$. This is consistent with the observations made in [16].

**Example 4.** There is a canonical procedure for creating a parallel Hamiltonian from a given one. Suppose that $\rho = \rho(t)$ is a solution to the von Neumann equation with Hamiltonian $\hat{H}$. Let $\psi = \psi(t)$ be any solution to the Schrödinger equation on $\mathcal{S}(\sigma)$ with Hamiltonian $\hat{H}$, and $\hat{H}_t$ be any Hamiltonian on $\mathcal{H}$ which is such that $\hat{H}_t(t) \psi(t) = \hat{H}(t) \psi(t) - i\hbar \psi(t) \xi_{\hat{H}}(\rho(t))$. 
Figure 2. A lift \( \psi \) of an evolution curve \( \rho \), and the shift of \( \psi \) into a horizontal curve \( \psi \| \).

(This uniquely defines \( \hat{H}_\| (t) \) on the image of \( \psi (t) \).) Then \( \hat{H}_\| \) parallel transports \( \rho (0) \) along \( \rho \) with the same speed as \( \hat{H} \) because \( \xi_{\hat{H}_\|} (\rho) = 0 \) and \( [\hat{H}_\|, \rho] = [\hat{H}, \rho] \). Indeed, the solution to the Schrödinger equation of \( \hat{H}_\| \) which extends from \( \psi (0) \) is the gauge shift of \( \psi \) into a horizontal curve

\[
\psi_{\|} (t) = \psi (t) \exp + \left( - \int_0^t A_\psi (\dot{\psi}) \, dt \right);
\]

see figure 2. Here \( \exp + \) is the positive time-ordered exponential.

3.2. A Mandelstam–Tamm quantum speed limit

The geodesic distance between two density operators with common spectrum \( \sigma \) is defined as the infimum of the lengths of all curves in \( \mathcal{D} (\sigma) \) that connect them. There is at least one curve whose length equals the distance, and all such curves are geodesics. Moreover, horizontal lifting of curves is length preserving because \( \pi \) is a Riemannian submersion, and a curve in \( \mathcal{D} (\sigma) \) is a geodesic if and only if one (and hence all) of its horizontal lifts is a geodesic in \( \mathcal{S} (\sigma) \); see [31]. The next theorem generalizes the main result of [16].

**Theorem 3.2.** The distance between two isospectral density operators \( \rho_0 \) and \( \rho_1 \) is

\[
\text{Dist}(\rho_0, \rho_1) = \inf \frac{1}{\hbar} \int_0^1 \Delta H (\rho) \, dt,
\]

where the infimum is taken over all Hamiltonians \( \hat{H} \) for which the boundary value von Neumann equation

\[
\dot{\rho} = X_H (\rho), \quad \rho (0) = \rho_0, \quad \rho (\tau) = \rho_1,
\]

is solvable.

**Proof.** The length of a curve \( \rho = \rho (t) \) in \( \mathcal{D} (\sigma) \) is

\[
\text{Length}[\rho] = \int_0^\tau \sqrt{g (\rho, \dot{\rho})} \, dt.
\]
Theorem 3.1 tells us that if $\rho$ is the integral curve of $X_H$ for some Hamiltonian $\hat{H}$, then the length of $\rho$ is a lower bound for the energy dispersion integral along $\rho$:

$$\text{Length}(\rho) \leq \frac{1}{\hbar} \int_0^\tau \Delta H(\rho) \, dt. \quad (22)$$

There is a Hamiltonian $\hat{H}$ that generates a horizontal lift of $\rho$ because the unitary group of $H$ acts transitively on $S(\sigma)$. For such a Hamiltonian we have equality in (22) by theorem 3.1. Moreover, if $\rho$ is a shortest geodesic, then

$$\text{Dist}(\rho_0, \rho_1) = \frac{1}{\hbar} \int_0^\tau \Delta H(\rho) \, dt. \quad (23)$$

This proves (19). □

Aharonov and Bohm’s [32] interpretation of the classic MT time–energy uncertainty relation gives rise to a limit on the speed of dynamical evolution [14]. For systems prepared in pure states it implies that the time that it takes for a state to evolve to an orthogonal state is bounded from below by $\pi \hbar/2$ times the inverse of the average energy uncertainty of the system. Uhlmann [18] showed that the same inequality holds for mixed states when orthogonality is replaced by full distinguishability, which means that the fidelity of the states vanishes [33, 34]. To be precise, Uhlmann showed that if $\rho$ is a solution to (20), then the Bures length between the initial and final state,

$$\mathcal{L}_B(\rho_0, \rho_1) = \arccos \sqrt{F(\rho_0, \rho_1)}, \quad F(\rho_0, \rho_1) = \langle \sqrt{\rho_0} \rho_1 \sqrt{\rho_0} \rangle^2, \quad (24)$$

is bounded from above by the energy dispersion integral:

$$\mathcal{L}_B(\rho_0, \rho_1) \leq \frac{1}{\hbar} \int_0^\tau \Delta H(\rho) \, dt. \quad (25)$$

Consequently, the evolution time is bounded from below by $\hbar/\Delta E$ times the Bures length:

$$\tau \geq \frac{\hbar}{\Delta E} \mathcal{L}_B(\rho_0, \rho_1), \quad \Delta E = \frac{1}{\tau} \int_0^\tau \Delta H(\rho) \, dt. \quad (26)$$

If we combine theorem 3.2 with the findings of Uhlmann, we see that

$$\tau \geq \frac{\hbar}{\Delta E} \text{Dist}(\rho_0, \rho_1) \geq \frac{\hbar}{\Delta E} \mathcal{L}_B(\rho_0, \rho_1). \quad (27)$$

In section 5.2 we will construct examples of density operators for which the second inequality in (27) is strict. Thus the quantum speed limit given by the middle term in (27) is sometimes greater than the quantum speed limit derived by Uhlmann. However, for fully distinguishable states they are the same:

**Proposition 3.3.** If $\rho_0$ and $\rho_1$ in $S(\sigma)$ are fully distinguishable, then

$$\text{Dist}(\rho_0, \rho_1) = \mathcal{L}_B(\rho_0, \rho_1) = \pi/2. \quad (28)$$

**Lemma 3.4.** Purifications of fully distinguishable mixed states have orthogonal supports, and hence they are Hilbert–Schmidt orthogonal.

**Proof.** Let $\psi_0$ and $\psi_1$ be purifications of $\rho_0$ and $\rho_1$, and assume that $\rho_0$ and $\rho_1$ represent two fully distinguishable mixed states. Then $\rho_0$ and $\rho_1$ have orthogonal supports (see [33] and [34, theorem 1]), and the same is true for $\psi_0$ and $\psi_1$ because the support of $\psi_0$ equals the support of $\rho_0$, and likewise for $\psi_1$ and $\rho_1$. A compact way to express this is $\psi_0^\dagger \psi_1 = 0$. □

**Proof of proposition 3.3.** Let $\psi_0$ in $\pi^{-1}(\rho_0)$ and $\psi_1$ in $\pi^{-1}(\rho_1)$ be such that the $G|_{S(\sigma)}$-geodesic distance between them equals $\text{Dist}(\rho_0, \rho_1)$. If we consider $\psi_0$ and $\psi_1$ as elements
in $S(K, H)$, then they are a distance of $\pi/2$ apart. In fact, $\psi(t) = \cos(t)\psi_0 + \sin(t)\psi_1$, 0 ≤ t ≤ $\pi/2$, is a length minimizing unit speed curve from $\psi_0$ to $\psi_1$. Consequently, Dist$(\rho_0, \rho_1) \geq \pi/2$. However, direct computations yield $\psi^*\psi = P(\sigma)$ and $\psi^*\psi = 0$. Thus $\psi$ is a horizontal curve in $S(\sigma)$. We conclude that Dist$(\rho_0, \rho_1) = \pi/2$. □

4. An inequality of Margolus–Levitin type

By theorem 3.1, the MT quantum speed limit derived in the previous section has a geometric origin. This limit does not depend on the energy of the system because two Hamiltonians with different energies may have the same uncertainties and solution spaces for their von Neumann equations:

Proposition 4.1. Suppose $\hat{H}$ is a Hamiltonian on $H$. Let $E = E(t)$ be any function and define $\hat{K} = \hat{H} - E1$. Then $\Delta K = \Delta H, X_K = X_H$, and $\xi^\perp_K = \xi^\perp_H$. But $K = H - E$.

However, there is also a dynamical quantum speed limit which does depend on the energy of the evolving system. Margolus and Levitin [13] showed that, when the evolution is governed by a time-independent Hamiltonian $\hat{H}$, the time that it takes for a system to evolve from one pure state into an orthogonal one is never less than $\pi \hbar/2(H - E_0)$, where $E_0$ is the ground state energy. The same is true for an evolution between fully distinguishable mixed states because according to lemma 3.4, their purifications are orthogonal. Next, we show that a similar inequality holds for a driven quantum system when $\hat{H}(s)$ and $\hat{H}(t)$ commute for 0 ≤ s, t ≤ $\tau$. To this end, we construct a time-averaged Hamiltonian $\tilde{H}$ with the same eigenspaces as $\hat{H}$ but whose eigenvalues are averages of the ground state energy shifted eigenvalues of $\hat{H}$. Thus, if $\hat{H}(t) = \sum_n E_n(t)\langle n(t)\rangle\langle n(t)\rangle$ is a continuously varying family of instantaneous spectral decompositions of $\hat{H}$, where $\{|n(t)\rangle\}$ is an orthonormal eigenframe for $\hat{H}(t)$, then define

$$\tilde{H}(t) = \sum_n E_n(t)\langle n(t)\rangle\langle n(t)\rangle, \quad \tilde{E}_n(t) = \frac{1}{t} \int_0^t (E_n(t) - E_0(t)) \, dt. \quad (29)$$

Then we have the following generalization of the Margolus and Levitin estimate.

Theorem 4.2. Suppose $\rho_0$ and $\rho_1$ are the initial and final states, respectively, of an evolution $\rho$ governed by a Hamiltonian $\hat{H}$ such that $\hat{H}(t)$ and $\hat{H}(t)$ commute for 0 ≤ s, t ≤ $\tau$. If $\rho_0$ and $\rho_1$ are fully distinguishable, then

$$\tau \geq \frac{\pi \hbar}{2\tilde{E}} \quad \tilde{E} = \text{Tr}(\tilde{H}(\tau)\rho_0) = \text{Tr}(\tilde{H}(\tau)\rho_1). \quad (30)$$

Proof. Suppose $\psi_0$ is a purification of $\rho_0$, and let $\psi = \psi(t)$ be the solution to the Schrödinger equation with Hamiltonian $\hat{H} - E_01$ which extends from $\psi_0$. Then $\psi$ is a lift of $\rho$. We set $\psi_1 = \psi(\tau)$. Like Margolus and Levitin, we use the inequality $\cos x \geq 2(x + \sin x)/\pi$ for $x \geq 0$ to estimate the real part of the inner product of $\psi_0$ and $\psi_1$:

$$\text{Re} \text{Tr} \psi_0^*\psi_1 = \sum_n \text{Re}\langle n(\tau)|\psi_1\psi_0^*|n(\tau)\rangle$$

$$= \sum_n \langle n(\tau)|\rho_0|n(\tau)\rangle \cos\left(\frac{\tau}{\hbar}\tilde{E}_n(\tau)\right)$$

$$\geq \sum_n \langle n(\tau)|\rho_0|n(\tau)\rangle \left(1 - \frac{2}{\pi}\left(\frac{\tau}{\hbar}\tilde{E}_n(\tau) + \sin\left(\frac{\tau}{\hbar}\tilde{E}_n(\tau)\right)\right)\right)$$

$$= 1 - \frac{2\tau}{\pi\hbar} \sum_n \langle n(\tau)|\rho_0|n(\tau)\rangle\tilde{E}_n(\tau) + \frac{2}{\pi} \text{Im} \text{Tr} \psi_0^*\psi_1. \quad (31)$$
Moreover,
\[ \text{Tr}(\bar{H}(\tau)\rho_0) = \sum_n \langle n(\tau)|\rho_0|n(\tau)\rangle \bar{E}_n(\tau) = \text{Tr}(\bar{H}(\tau)\rho_1) \] (32)
because \( \langle n(\tau)|\rho_0|n(\tau)\rangle = \langle n(\tau)|\rho_1|n(\tau)\rangle \) for every \( n \). Equations (31) and (32) yield
\[ \text{Re}(\langle \psi_0|\psi_1 \rangle) \geq 1 - \frac{2\bar{E}}{\pi^2} - \frac{2}{\pi} \text{Im}(\langle \psi_0|\psi_1 \rangle), \quad \bar{E} = \text{Tr}(\bar{H}(\tau)\rho_0) = \text{Tr}(\bar{H}(\tau)\rho_1). \] (33)
Now, by lemma 3.4, \( \psi_0 \) and \( \psi_1 \) are orthogonal if \( \rho_0 \) and \( \rho_1 \) are fully distinguishable. Then,
\[ \tau \geq \frac{\pi\hbar}{2\bar{E}}. \] (34)

4.1. The Margolus–Levitin quantum speed limit

Margolus and Levitin’s estimate has been generalized to arbitrary pairs of isospectral mixed states [21]. Recently, Deffner and Lutz [23] proved that if \( \rho_0 \) and \( \rho_1 \) are the initial and final states of a solution curve \( \rho \) to a von Neumann equation with Hamiltonian \( \hat{H} \) such that \( \hat{H}(s) \) and \( \hat{H}(t) \) commute for \( 0 \leq s, t \leq \tau \), then
\[ \tau \geq \frac{4\hbar}{\pi^2(H - E_0)} L_B(\rho_0, \rho_1)^2, \quad \langle H - E_0 \rangle = \frac{1}{\tau} \int_0^\tau (\text{Tr}(\hat{H}\rho) - E_0) \, dt. \] (35)
We show how a minor modification of the proof of theorem 4.2 gives rise to a similar estimate, which actually provides a greater lower bound on the evolution time of systems with time-independent Hamiltonians than (35).

Theorem 4.3. Suppose \( \rho_0 \) and \( \rho_1 \) are the initial and final states, respectively, of an evolution governed by a Hamiltonian \( \hat{H} \) such that \( \hat{H}(s) \) and \( \hat{H}(t) \) commute for \( 0 \leq s, t \leq \tau \). Let \( \beta \approx 0.724 \) be such that \( 1 - \beta x \) is a line tangent to \( \cos x \); see figure 3(a). Then
\[ \tau \geq \frac{4\hbar}{\beta^2\pi^2 E} L_B(\rho_0, \rho_1)^2, \quad \bar{E} = \text{Tr}(\bar{H}(\tau)\rho_0) = \text{Tr}(\bar{H}(\tau)\rho_1). \] (36)

Remark 2. \( \bar{E} = \langle H - E_0 \rangle = H - E_0 \) if \( \hat{H} \) is time independent.
where \( \exp^{-} \) vanishes along \( \rho \).

Inspired by theorem 3.1, we call a Hamiltonian \( H \) optimal if and only if \( H - H^1 \) is parallel and optimal.

Proof. Let \( \psi_0 \) and \( \psi_1 \) be as in the proof of theorem 4.2. Since \( \cos x \geq 1 - \beta x \) for \( x \geq 0 \) (see figure 3(a)), we have that

\[
|\Tr \psi_0^0 \psi_1| \geq \Re \Tr \psi_0^0 \psi_1 \geq \sum_n (n(\tau) | \rho_0 \rangle \langle n(\tau) |) (1 - \beta \tau \overline{E}_n(\tau)/\hbar) = 1 - \beta \tau \overline{E}/\hbar,
\]

and since \( 1 - x \geq 4 \arccos^2 x/\pi^2 \) for \( 0 \leq x \leq 1 \) (see figure 3(b)),

\[
\tau \geq \frac{\hbar}{\beta E} (1 - |\Tr \psi_0^0 \psi_1|) \geq \frac{4\hbar}{\beta \pi^2 E} \arccos^2 |\Tr \psi_0^0 \psi_1| \geq \frac{4\hbar}{\beta \pi^2 E} \overline{L}_B(\rho_0, \rho_1)^2.
\]

5. Optimal Hamiltonians

Generically, the number of independent kets in a mixed state of a finite-level quantum system equals the dimension of the Hilbert space. Such mixed states are represented by invertible density operators. From now on we assume that \( \mathcal{H} \) has finite dimension \( n \), and that the density operators in \( D(\sigma) \) are invertible. Then we can put \( K = \mathcal{H} \) and express all density operators as matrices with respect to the computational basis.

5.1. Optimal Hamiltonians for states represented by invertible density operators

Inspired by theorem 3.1, we call a Hamiltonian \( \hat{H} \) optimal for a density operator \( \rho_0 \) if the solution \( \rho \) to the von Neumann equation of \( \hat{H} \) with initial state \( \rho_0 \) is a geodesic and \( \xi_H^1 \) vanishes along \( \rho \). For optimal Hamiltonians we have that \( \tau = h \text{Dist}(\rho_0, \rho_1)/\Delta E \) —at least if the distance between \( \rho_0 \) and \( \rho_1 = \rho(\tau) \) is smaller than the injectivity radius of \( D(\sigma) \).

This follows directly from theorem 3.1. In this section we characterize the optimal Hamiltonians. Recall that a curve in \( D(\sigma) \) is a geodesic if and only if its horizontal lifts are geodesics. Recall also that all the horizontal lifts of a given curve in \( D(\sigma) \) satisfy the same Schrödinger equation. According to the next proposition (and proposition 4.1), we need only characterize those optimal Hamiltonians that are parallel.

Proposition 5.1. A Hamiltonian \( \hat{H} \) is optimal if and only if \( \hat{H} - H^1 \) is parallel and optimal.

Proof. The solution spaces of the von Neumann equations of \( \hat{H} \) and \( \hat{H}_|| = \hat{H} - H^1 \) are identical, and \( \xi_{\hat{H}_||} = \xi_H^1 \) by (15).

With each curve \( \xi = \xi(t) \) in \( u(\mathcal{H}) \) we associate the Hamiltonian

\[
\hat{H}_\xi(t) = i\hbar \exp\left(\int_0^t \xi(t) \, dt\right) \exp_+ \left( -\int_0^t \xi(t) \, dt\right),
\]

where \( \exp_- \) and \( \exp_+ \) denote the negative and positive time-ordered exponentials. The Schrödinger equation with Hamiltonian \( \hat{H}_\xi \) and initial value \( \psi_0 \) is

\[
\dot{i} \hbar \psi = \hat{H}_\xi \psi, \quad \psi(0) = \psi_0.
\]

We describe conditions for \( \xi \) that, when satisfied, make \( \hat{H}_\xi \) transport \( \psi_0 \), and hence every purification of \( \rho_0 = \psi_0 \psi_0^\dagger \), along a horizontal geodesic. Recall that proposition 2.2 guarantees that a geodesic in \( \mathcal{S}(\sigma) \) remains horizontal if it is initially horizontal.

The left action by \( U(\mathcal{H}) \) on \( \mathcal{S}(\sigma) \) is free and transitive. Therefore, the fundamental vector fields of this action define an isomorphism \( \xi \mapsto X_\xi(\psi_0) \) from \( u(\mathcal{H}) \) to the tangent space of \( \mathcal{S}(\sigma) \) at \( \psi_0 \). We equip \( u(\mathcal{H}) \) with the metric \( \xi \star \eta \) that makes this isomorphism an isometry:

\[
\xi \star \eta = -\frac{1}{2} \Tr((\xi \eta + \eta \xi) \rho_0).
\]
Furthermore, we write $\Lambda_\xi$ for the left invariant vector field on $S(\sigma)$ which coincides with $X_\xi$ at $\psi_0$. Thus if $\psi = U\psi_0$, then

$$\Lambda_\xi(\psi) = \Lambda_\xi(L_U(\psi_0)) = dL_U(X_\xi(\psi_0)) = U\xi\psi_0 = U\xi U^\dagger \psi.$$ (42)

With each curve $\psi = \psi(t)$ in $S(\sigma)$, we can associate a curve $\xi = \xi(t)$ in $u(H)$ by declaring $\dot{\psi} = \Lambda_\xi(\psi)$, and $\psi$ solves (40) if it extends from $\psi_0$:

**Proposition 5.2.** A curve extending from $\psi_0$ is an integral curve of $\Lambda_\xi$ if and only if it satisfies the Schrödinger equation (40).

**Proof.** Suppose $\psi = \psi(t)$ is the integral curve of $\Lambda_\xi$ that extends from $\psi_0$. There is a unique curve of unitaries $U = U(t)$ such that $\psi = U\psi_0$. By (42) and the fact that $\psi_0$ is invertible, this curve satisfies $\dot{U} = U\xi$ and $U(0) = I$. Thus $U(t) = \exp\left(\int_0^t \xi \, dt\right)$. Now,

$$i\hbar \dot{\psi} = i\hbar U\psi_0 = i\hbar U\xi \psi_0 = i\hbar U\xi U^\dagger \psi_0 = i\hbar U\xi U^\dagger \psi = \dot{\psi} = \dot{\psi}.$$ (43)

The opposite implication follows from the uniqueness of solutions to (40). $\square$

A curve $\xi = \xi(t)$ in $u(H)$ satisfies the *Euler–Arnold equation* if $\dot{\xi} = \text{ad}_{\xi}^\dagger \eta$, where $\text{ad}_{\xi}^\dagger \eta$ is the unique element in $u(H)$ such that $\text{ad}_{\xi}^\dagger \eta \cdot \eta = \eta \cdot [\xi, \eta]$ for every $\eta$ in $u(H)$. According to the next proposition, integral curves of $\Lambda_\xi$ are geodesics if and only if $\xi$ satisfies the Euler–Arnold equation:

**Proposition 5.3.** Suppose $\xi = \xi(t)$ is a curve in $u(H)$, and let $\psi = \psi(t)$ be an integral curve of $\Lambda_\xi$. Then $\psi$ is a geodesic if and only if $\xi$ satisfies the Euler–Arnold equation.

**Proof.** The covariant derivative of the velocity field of $\psi$ is $\nabla_t \dot{\psi} = \Lambda_\xi(\psi) + \nabla_A \Lambda_\xi(\psi)$. By the Kozul formula [29, p 160, proposition 2.3],

$$2G(\nabla_A \Lambda_\xi, \Lambda_\eta) = \Lambda_\xi G(\Lambda_\xi, \Lambda_\eta) + \Lambda_\xi G(\Lambda_\eta, \Lambda_\xi) - \Lambda_\eta G(\Lambda_\xi, \Lambda_\xi)$$

$$-G(\Lambda_\xi, [\Lambda_\xi, \Lambda_\eta]) + G(\Lambda_\xi, [\Lambda_\eta, \Lambda_\xi]) + G(\Lambda_\eta, [\Lambda_\xi, \Lambda_\xi])$$

$$= -\xi \cdot [\eta, \xi] + \xi \cdot [\xi, \eta]$$

$$= -2 \text{ad}_{\xi}^\dagger \eta \cdot \eta$$

$$= -2G(\text{ad}_{\xi}^\dagger \eta, \Lambda_\eta)$$ (44)

for every $\eta$ in $u(H)$. Thus $\nabla_t \dot{\psi} = \Lambda_\xi - \text{ad}_{\xi}^\dagger (\psi)$. $\square$

The following theorem, which follows from propositions 2.2, 5.2, and 5.3, summarizes the conditions under which the Hamiltonian $H_\xi$ transports $\psi_0$ along a horizontal geodesic:

**Theorem 5.4.** Every curve in $S(\sigma)$ that extends from $\psi_0$ is the solution to (40) for some curve $\xi$. Moreover, the solution to (40) is a horizontal geodesic if and only if $\xi$ satisfies the Euler–Arnold equation, and the fundamental vector field of $\xi(0)$ is horizontal at $\psi_0$.

### 5.2. Density operators with two distinct eigenvalues, and almost pure qubit systems

A geodesic orbit space is a Riemannian homogeneous space in which each geodesic is an orbit of a one-parameter subgroup of its isometry group. If $\sigma$ contains precisely two different, possibly degenerate, eigenvalues, then $D(\sigma)$ is a geodesic orbit space because every geodesic is then generated by a time-independent Hamiltonian. We verify this for the geodesics extending from a density operator which is diagonal with respect to the computational basis. The general result then follows from the fact that the conjugation action of $U(H)$ on $D(\sigma)$ is transitive and by isometries.
Assume that \( \sigma = (p_1, p_2; m_1, m_2) \), where \( m_1 + m_2 = n \). Let \( \xi \) be an anti-Hermitian operator on \( \mathcal{H} \) such that \( X_\xi \) is horizontal along the fiber over \( \rho_0 = P(\sigma) \), where \( P(\sigma) \) is the density operator defined in (2). Further, let \( \eta \) be any anti-Hermitian operator on \( \mathcal{H} \), and express \( \xi \) and \( \eta \) as matrices with respect to the computational basis:

\[
\xi = \begin{bmatrix} 0 & \xi_{12}^\dagger \\ -\xi_{12} & 0 \end{bmatrix}, \quad \eta = \begin{bmatrix} \eta_{11} & \eta_{12} \\ -\eta_{12} & \eta_{22} \end{bmatrix}.
\] (45)

Here, \( \xi_{12} \) and \( \eta_{12} \) have dimensions \( m_1 \times m_2 \), and \( \eta_{11} \) and \( \eta_{22} \) have dimensions \( m_1 \times m_1 \) and \( m_2 \times m_2 \), respectively. Now \( \text{ad}_\xi \eta = 0 \) because

\[
\xi * [\xi, \eta] = \frac{i}{2} \left( p_1 \text{Tr} \left[ \xi_{12}\xi_{12}^\dagger, \eta_{11} \right] + p_2 \text{Tr} \left[ \xi_{12}^\dagger\xi_{12}, \eta_{22} \right] \right) = 0,
\] (46)

as commutators of matrices have vanishing trace. This in turn implies that every curve \( \xi \) which satisfies the conditions in theorem 5.4 is stationary, and hence that \( \mathcal{H}_\xi \) is time independent. Next we use this observation to produce density operators representing mixed qubit states for which the second inequality in (27) is strict.

Assume that \( \dim \mathcal{H} = 2 \). Two independent qubits are represented by the computational basis vectors \( |1 \rangle \) and \( |2 \rangle \). Consider an ensemble of qubits prepared such that the proportion of qubits in state \( |j \rangle \) is \( p_j \), where \( p_1 > p_2 > 0 \). The initial state of the ensemble is represented by the density operator \( \rho_0 = \text{diag}(p_1, p_2) \). Chose \( \psi_0 = \text{diag}(\sqrt{p_1}, \sqrt{p_2}) \) in the fiber over \( \rho_0 \), and let \( \xi \) be an arbitrary anti-Hermitian operator on \( \mathcal{H} \) such that \( X_\xi (\psi_0) \) is horizontal:

\[
\xi = \begin{bmatrix} 0 & ae^{i\theta} \\ -ae^{-i\theta} & 0 \end{bmatrix}, \quad a > 0.
\] (47)

The solution to the Schrödinger equation of \( \mathcal{H}_\xi = i\hbar \xi \) which extends from \( \psi_0 \) is

\[
\psi(t) = \begin{bmatrix} \sqrt{p_1} \cos at & -\sqrt{p_2}e^{i\theta} \sin at \\ -\sqrt{p_2}e^{-i\theta} \sin at & \sqrt{p_1} \cos at \end{bmatrix}.
\] (48)

This curve is a horizontal geodesic, and its projection is a geodesic extending from \( \rho_0 \):

\[
\rho(t) = \begin{bmatrix} p_1 \cos^2 at + p_2 \sin^2 at & e^{i\theta} (p_2 - p_1) \cos at \sin at \\ e^{-i\theta} (p_2 - p_1) \cos at \sin at & p_1 \sin^2 at + p_2 \cos^2 at \end{bmatrix}.
\] (49)

Set \( \rho_1 = \rho(\tau) \), let \( d > 0 \) be the (spectrum-dependent) injectivity radius of \( \mathcal{D}(\sigma) \), and assume that \( 0 < \tau < d/a \). Then \( \rho \) is a shortest geodesic between \( \rho_0 \) and \( \rho_1 \), and

\[
\text{Dist}(\rho_0, \rho_1) = \text{Length}[\rho] = \frac{1}{\hbar} \int_0^\tau \Delta H_\xi(\rho) \, dt = a\tau.
\] (50)

Next, we will argue that the Bures length between \( \rho_0 \) and \( \rho_1 \) is strictly less than \( a\tau \), and hence that there exist states for which the second inequality in (27) is strict.

By [25, p 225, equation (9)], the fidelity of \( \rho_0 \) and \( \rho_1 \) is

\[
F(\rho_0, \rho_1) = \text{Tr}(\rho_0 \rho_1) + 2\sqrt{\det \rho_0 \det \rho_1} = (p_1 - p_2)^2 \cos^2(\alpha\tau) + 4p_1p_2.
\] (51)

Therefore, the Bures length between \( \rho_0 \) and \( \rho_1 \) is

\[
L_B(\rho_0, \rho_1) = \arccos \sqrt{(p_1 - p_2)^2 \cos^2(\alpha\tau) + 4p_1p_2}.
\] (52)

The difference \( a\tau - \arccos \sqrt{(p_1 - p_2)^2 \cos^2(\alpha\tau) + 4p_1p_2} \) is a positive function of \( \tau > 0 \). (In figure 4 we have plotted the difference for three different spectra, for when \( 0 < \alpha \tau < \pi \).)

Note, however, that \( d \) might be smaller than \( \pi \). Consequently, \( \text{Dist}(\rho_0, \rho_1) > L_B(\rho_0, \rho_1) \).
6. Conclusion

Quantum speed limits are fundamental lower bounds on the time required for a quantum system to evolve from one state into another. In this paper, we have derived a sharp MT quantum speed limit, by differential geometric methods, and we have characterized the Hamiltonians that optimize the evolution time for finite-level quantum systems in generic mixed states. The paper also contains a quantum speed limit of ML type, which, under certain circumstances, such as that where the Hamiltonian is time independent, is sharper than those known previously.

Quantum speed limits for open quantum systems are also available [35–37]. It is the intention of the authors to develop differential geometric methods by which one can derive quantum speed limits also for open systems.

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