JORDAN DERIVATIONS OF SOME EXTENSION ALGEBRAS

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Abstract. In this paper, we mainly study Jordan derivations of dual extension algebras and those of generalized one-point extension algebras. It is shown that every Jordan derivation of dual extension algebras is a derivation. As applications, we obtain that every Jordan generalized derivation and every generalized Jordan derivation on dual extension algebras are both generalized derivations. For generalized one-point extension algebras, it is proved that under certain conditions, each Jordan derivation of them is the sum of a derivation and an anti derivation.

1. Introduction

Let us begin with some definitions. Let \( R \) be a commutative ring with identity, \( \mathcal{A} \) be a unital algebra over \( R \) and \( \mathcal{Z}(\mathcal{A}) \) be the center of \( \mathcal{A} \). We denote the Jordan product by \( a \circ b = ab + ba \) for all \( a, b \in \mathcal{A} \). Recall that an \( R \)-linear mapping \( \Theta \) from \( \mathcal{A} \) into itself is called a derivation if

\[
\Theta(ab) = \Theta(a)b + a\Theta(b)
\]

for all \( a, b \in \mathcal{A} \), an anti-derivation if

\[
\Theta(ab) = \Theta(b)a + b\Theta(a)
\]

for all \( a, b \in \mathcal{A} \), and a Jordan derivation if

\[
\Theta(a \circ b) = \Theta(a) \circ b + a \circ \Theta(b).
\]

Every derivation is obviously a Jordan derivation. The converse statement is in general not true. Moreover, in the 2-torsion free case the definition of a Jordan derivation is equivalent to

\[
\Theta(x^2) = \Theta(x)x + x\Theta(x)
\]

for all \( x \in \mathcal{A} \). Those Jordan derivations which are not derivations are said to be proper.

There has been an increasing interest in the study of Jordan derivations of various algebras since last decades. The standard problem is to find out whether a Jordan derivation degenerate to a derivation. Jacobson and Rickart [14] proved that every Jordan derivation of a full matrix algebra over a 2-torsion free unital ring is a derivation by relating the problem to the decomposition of Jordan homomorphisms. In [22], Herstein showed that every Jordan derivation from a 2-torsion free prime ring into itself is also a derivation. Zhang and Yu [30] obtained that every Jordan

\[
\begin{align*}
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\end{align*}
\]

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derivation on a triangular algebra with faithful assumption is a derivation. This result was extended to the higher case by Xiao and Wei [27]. They obtained that any Jordan higher derivation on a triangular algebra is a higher derivation. The aforementioned results have been extended to different rings and algebras in various directions, see [5, 6, 22, 23, 30] and the references therein.

Path algebras of quivers come up naturally in the study of tensor algebras of bimodules over semisimple algebras. It is well known that any finite dimensional basic $K$-algebra is given by a quiver with relations when $K$ is an algebraically closed field. In [10], Guo and Li studied the Lie algebra of differential operators on a path algebra $K\Gamma$ and related this Lie algebra to the algebraic and combinatorial properties of the path algebra $K\Gamma$. In [18], the current authors studied Lie derivations and Jordan derivations of a class of path algebras of quivers without oriented cycles, which can be viewed as one-point extensions. It is proved in this case that each Lie derivation is of the standard form and each Jordan derivation is a derivation. Moreover, the standard decomposition of a Lie derivation is unique.

For path algebras of finite quivers without oriented cycles, Xi [24] constructed their dual extension algebras to study quasi-hereditary algebras. This construction were further refined in details in [7, 9, 25] by Deng and Xi. A more general construction, the twisted doubles, were studied in [8, 15, 26]. It turns out that dual extension algebras inherit some nice properties of the representation theory aspect from given algebras. On the other hand, in [19], the current authors proved that all Lie derivations of the dual extension algebra are of the standard form. Then it is natural to ask whether all Jordan derivations of dual extension algebras are derivations. We will give a positive answer in this paper. More precisely, one of the main results of this paper is as follows:

**Theorem.** Let $K$ be a field with $\text{char }\neq 2$. Let $(\Gamma, \rho)$ be a finite quiver without oriented cycles. Then each Jordan derivation on the dual extension algebra of path algebra $K(\Gamma, \rho)$ is a derivation.

It should be remarked that each associative algebra with non trivial idempotents is isomorphic to a generalized matrix algebra. The form of Jordan derivations on generalized matrix algebras has been characterized by current authors in [20]. We proved that under certain conditions, each Jordan derivation is the sum of a derivation and an anti-derivation. An example of proper Jordan derivation was also given there. To find a proper Jordan derivation is not an easy task in general. Recently Bencovič [4] introduced the so-called singular Jordan derivations which are usually anti-derivations. He gave a sufficient condition for a Jordan derivation on a unital algebra with a nontrivial idempotent to be the sum of a derivation and a singular Jordan derivation. Our result on Jordan derivations of dual extension algebras implies that neither the conditions in [20] nor those in [4] are necessary. Of course, we want to give other examples to illustrate this fact. The so-called generalized one-point extension algebras introduced in [19] just provide us another class of examples. We prove that under certain conditions, each Jordan derivation on a generalized one-point extension algebra is the sum of a derivation and an anti-derivation.
The paper is organized as follows. After a quick review of some needed preliminaries on path algebras and generalized matrix algebras in Section 2, we investigate Jordan derivations of dual extension algebras in Section 3. Jordan generalized derivations and generalized Jordan derivations are also considered. Then in Section 4, we study Jordan derivations of generalized one-point extension algebras. An interesting example will also be given there.

2. Path algebras and generalized matrix algebras

In this section, we give a quick review of path algebras of quivers and generalized matrix algebras. For more details, we refer the reader to [1] and [27].

2.1. Path algebras. Recall that a finite quiver \( \Gamma = (\Gamma_0, \Gamma_1) \) is an oriented graph with the set of vertices \( \Gamma_0 \) and the set of arrows between vertices \( \Gamma_1 \) being both finite. For an arrow \( \alpha \), we write \( s(\alpha) = i \) and \( e(\alpha) = j \) if it is from the vertex \( i \) to the vertex \( j \). A sink is a vertex without arrows beginning at it and a source is a vertex without arrows ending at it. A nontrivial path in \( \Gamma \) is an ordered sequence of arrows \( p = \alpha_n \cdots \alpha_1 \) such that \( e(\alpha_m) = s(\alpha_{m+1}) \) for each \( 1 \leq m < n \). Define \( s(p) = s(\alpha_1) \) and \( e(p) = e(\alpha_n) \). The length of \( p \) is defined to be \( n \). A trivial path is the symbol \( e_i \) for each \( i \in \Gamma_0 \). In this case, we set \( s(e_i) = e(e_i) = i \). The length of a trivial path is defined to be zero. A nontrivial path \( p \) is called an oriented cycle if \( s(p) = e(p) \). Let us denote the set of all paths by \( P \).

Let \( K \) be a field and \( \Gamma \) a quiver. Then the path algebra \( K\Gamma \) is the \( K \)-algebra generated by the paths in \( \Gamma \) and the product of two paths \( x = \alpha_n \cdots \alpha_1 \) and \( y = \beta_t \cdots \beta_1 \) is defined by

\[
xy = \begin{cases} 
\alpha_n \cdots \alpha_1 \beta_t \cdots \beta_1, & e(y) = s(x) \\
0, & \text{otherwise.}
\end{cases}
\]

Clearly, \( K\Gamma \) is an associative algebra with the identity \( 1 = \sum_{i \in \Gamma_0} e_i \), where \( e_i (i \in \Gamma_0) \) are pairwise orthogonal primitive idempotents of \( K\Gamma \).

A relation \( \sigma \) on a quiver \( \Gamma \) over a field \( K \) is a \( K \)-linear combination of paths

\[
\sigma = \sum_{i=1}^{n} k_i p_i,
\]

where \( k_i \in K \) and

\[
e(p_1) = \cdots = e(p_n), \quad s(p_1) = \cdots = s(p_n).
\]

Moreover, the number of arrows in each path is assumed to be at least 2. Let \( \rho \) be a set of relations on \( \Gamma \) over \( K \). The pair \( (\Gamma, \rho) \) is called a quiver with relations over \( K \). Denote by \( \langle \rho \rangle \) the ideal of \( K\Gamma \) generated by the set of relations \( \rho \). The \( K \)-algebra \( K(\Gamma, \rho) = K\Gamma/\langle \rho \rangle \) is always associated with \( (\Gamma, \rho) \). For arbitrary element \( x \in K\Gamma \), write by \( \overline{x} \) the corresponding element in \( K(\Gamma, \rho) \). We often write \( \overline{x} \) as \( x \) if there is no confusion caused.

2.2. Generalized matrix algebras. The definition of generalized matrix algebras is given by a Morita context. Let \( \mathcal{R} \) be a commutative ring with identity. A Morita context consists of two \( \mathcal{R} \)-algebras \( A \) and \( B \), two bimodules \( AM_B \) and \( BN_A \),...
and two bimodule homomorphisms called the pairings $\Phi_{MN} : M \otimes N \rightarrow A$ and $\Psi_{NM} : N \otimes M \rightarrow B$ satisfying the following commutative diagrams:

\[
\begin{array}{ccc}
M \otimes N \otimes M & \xrightarrow{\Phi_{MN} \otimes I_M} & A \otimes M \\
\downarrow I_M \otimes \Psi_{NM} & & \downarrow I_M \otimes \Phi_{MN} \\
M \otimes B & \xrightarrow{=} & M
\end{array}
\quad \begin{array}{ccc}
N \otimes M \otimes N & \xrightarrow{\Psi_{NM} \otimes I_N} & B \otimes N \\
\downarrow I_N \otimes \Phi_{MN} & & \downarrow I_N \otimes \Psi_{NM} \\
N \otimes A & \xrightarrow{=} & N
\end{array}
\]

Let us write this Morita context as $(A, B, M_B, A_M, \Phi_{MN}, \Psi_{NM})$. If $(A, B, M_B, B_N, A_M, \Phi_{MN}, \Psi_{NM})$ is a Morita context, then the set

\[
\begin{bmatrix}
A & M \\
N & B
\end{bmatrix}
\]

form an $R$-algebra under matrix-like addition and matrix-like multiplication. There is no constraint condition concerning the bimodules $M$ and $N$. Of course, they can be equal to zeros. Such an $R$-algebra is called a \textit{generalized matrix algebra} of order 2 and is usually denoted by $\mathcal{G} = [A \ M \ N \ B]$ or $\mathcal{G} = (A, M, N, B)$. The structure and properties of linear mappings on generalized matrix algebras have been investigated in our systemic works [17, 18, 20, 21, 27].

Any unital $R$-algebra $A$ with nontrivial idempotents is isomorphic to a generalized matrix algebra as follows:

\[
\mathcal{G} = \begin{bmatrix}
e A e & e A (1 - e) \\
(1 - e) A e & (1 - e) A (1 - e)
\end{bmatrix}
= \left\{ \begin{bmatrix}
e a e & e c (1 - e) \\
(1 - e) d e & (1 - e) b (1 - e)
\end{bmatrix} \middle| a, b, c, d \in A \right\},
\]

where $e$ is a nontrivial idempotent in $A$.

3. JORDAN DERIVATIONS OF DUAL EXTENSION

Let us first recall the definition of dual extension algebras introduced in [24]. Let $\Lambda = K(\Gamma, \rho)$, where $\Gamma$ is a finite quiver. Let $\Gamma^*$ be a quiver whose vertex set is $\Gamma_0$ and

\[
\Gamma_1^* = \{ \alpha^* : i \rightarrow j \mid \alpha : j \rightarrow i \text{ is an arrow in } \Gamma_1 \}.
\]

Let $p = \alpha_n \cdots \alpha_1$ be a path in $\Gamma$. Write the path $\alpha_n^* \cdots \alpha_1^*$ in $\Gamma^*$ by $p^*$. Define $\mathcal{D}(\Lambda)$ to be the path algebra of the quiver $(\Gamma_0, \Gamma_1 \cup \Gamma_1^*)$ with relations

\[
\rho \cup \rho^* \cup \{ \alpha \beta^* \mid \alpha, \beta \in \Gamma_1 \}.
\]

If $\Gamma$ has no oriented cycles, then $\mathcal{D}(\Lambda)$ is called the \textit{dual extension} of $\Lambda$. A more general definition of dual extension algebras was given in [25] to study global dimensions of dual extension algebras. We omit the details here because it will not be involved in this paper. Clearly, if $|\Gamma_0| = 1$, then the algebra is trivial. Let us assume that $|\Gamma_0| \geq 2$ from now on. Then $\mathcal{D}(\Lambda)$ is isomorphic to a generalized matrix algebra $\mathcal{G} = [A \ M \ N \ B]$. According to the construction of dual extension, it is easy to verify that the pairings $\Phi_{MN} = 0$ and $\Psi_{NM} \neq 0$. If $M \neq 0$, then $N \neq 0$. 
Moreover, it is helpful to point out that $M$ need not to be faithful as left $A$-module or as right $B$-module. Some examples were given in [19].

We always assume, without specially mentioned, that every algebra and every bimodule considered is 2-torsion free. Let us recall some indispensable descriptions about derivations and Jordan derivations of generalized matrix algebras. For details, we refer the reader to [17] and [20].

**Lemma 3.1.** [17] Proposition 4.2] An additive map $\Theta$ from $G$ into itself is a derivation if and only if it has the form

$$\Theta \left( \begin{bmatrix} a & m \\ n & b \end{bmatrix} \right) = \left[ \begin{array}{cc} \delta_1(a) - mn_0 - m_0n & \overline{a}m_0 - m_0b + \tau_2(m) \\ \overline{n}_0a - bm_0 + \nu_3(n) & \nu_3m + nm_0 + \mu_4(b) \end{array} \right],$$  \hspace{1cm} (\star 1)

$$\forall \left[ \begin{array}{cc} a & m \\ n & b \end{array} \right] \in G,$$

where $m_0 \in M, n_0 \in N$ and

$$\delta_1 : A \rightarrow A, \quad \tau_2 : M \rightarrow M, \quad \nu_3 : N \rightarrow N, \quad \mu_4 : B \rightarrow B$$

are all $R$-linear mappings satisfying the following conditions:

1. $\delta_1$ is a derivation of $A$ with $\delta_1(mn) = \tau_2(m)n + m\tau_3(n);$$
2. $\mu_4$ is a derivation of $B$ with $\mu_4(nm) = n\tau_2(m) + \nu_3(n)n;$$
3. $\tau_2(am) = a\tau_2(m) + \delta_1(a)m$ and $\tau_2(mb) = \tau_2(m)b + m\mu_4(b);$$
4. $\nu_3(aa) = \nu_3(a)n + n\delta_1(a)$ and $\nu_3(bb) = b\nu_3(n) + \mu_4(b)n.$

**Lemma 3.2.** [20] Proposition 4.2] An additive map $\Theta$ from $G$ into itself is a Jordan derivation if and only if it is of the form

$$\Theta \left( \begin{bmatrix} a & m \\ n & b \end{bmatrix} \right) = \left[ \begin{array}{cc} \delta_1(a) - mn_0 - m_0n & \overline{a}m_0 - m_0b + \tau_2(m) + \tau_3(n) \\ \overline{n}_0a - bm_0 + \nu_3(m) + \nu_3(n) & \nu_3m + nm_0 + \mu_4(b) \end{array} \right],$$  \hspace{1cm} (\star 2)

$$\forall \left[ \begin{array}{cc} a & m \\ n & b \end{array} \right] \in G,$$

where $m_0 \in M, n_0 \in N$ and

$$\delta_1 : A \rightarrow A, \quad \tau_2 : M \rightarrow M, \quad \tau_3 : N \rightarrow M, \quad \nu_2 : M \rightarrow N, \quad \nu_3 : N \rightarrow N, \quad \mu_4 : B \rightarrow B$$

are all $R$-linear mappings satisfying the following conditions:

1. $\delta_1$ is a Jordan derivation on $A$ and $\delta_1(mn) = \tau_2(m)n + m\tau_3(n);$$
2. $\mu_4$ is a Jordan derivation on $B$ and $\mu_4(nm) = n\tau_2(m) + \nu_3(n)n;$$
3. $\tau_2(am) = a\tau_2(m) + \delta_1(a)m$ and $\tau_2(mb) = \tau_2(m)b + m\mu_4(b);$$
4. $\nu_3(aa) = \nu_3(a)n + n\delta_1(a)$ and $\nu_3(bb) = b\nu_3(n) + \mu_4(b)n;$$
5. $\nu_2(na) = \nu_2(a)n, \tau_3(bn) = \tau_3(n)b, \tau_3(n)n = 0, \tau_3(n)n = 0;$$
6. $\nu_2(am) = \nu_2(a)m, \nu_2(mb) = b\nu_2(m), \mu_2(mb) = 0, \nu_2(m)m = 0.$

Let $\Lambda = K(\Gamma, \rho)$, where $\Gamma$ is a finite connected quiver without oriented cycles, and let $\mathcal{D}(\Lambda)$ be the dual extension algebra. Assume that $i \in \Gamma_0$ is a source and $\mathcal{D}(\Lambda) \simeq \mathcal{G} = [A N \ M B]$, where $B \simeq e_i \mathcal{D}(\Lambda) e_i$. Thus an arbitrary Jordan derivation on $\mathcal{D}(\Lambda)$ can be characterized by the methods of generalized matrix algebras as follows.
Lemma 3.3. Let $\Theta$ be a Jordan derivation of $\mathcal{D}(\Lambda)$. Then $\Theta$ is of the form
\[
\Theta\left(\begin{array}{ccc}
a & m \\
n & b
\end{array}\right) = \left(\begin{array}{ccc}
d_1(a) & a m_0 - m_0 b + \tau_2(m) \\
n_0 a - b n_0 + \nu_3(n) & n_0 m + n m_0 + \mu_4(b)
\end{array}\right),
\]
where $m_0 \in M, n_0 \in N$ and
\[
d_1 : A \to A,
\tau_2 : M \to M,
\nu_3 : N \to N,
\mu_4 : B \to B
\]
are all $\mathcal{R}$-linear mappings satisfying the following conditions:

1. $\delta_1$ is a Jordan derivation on $A$;
2. $\mu_4$ is a derivation on $B$ and $\mu_4(nm) = n \tau_2(m) + \nu_3(n)m$;
3. $\tau_2(am) = a \tau_2(m) + \delta_1(a)m$ and $\tau_2(mb) = \tau_2(m)b + m \mu_4(b)$;
4. $\nu_3(bn) = b \nu_3(n) + \mu_4(b)n$ and $\nu_3(na) = \nu_3(n)a + n \delta_1(a)$.

Proof. Let $\Theta$ be a Jordan derivation of $\mathcal{D}(\Lambda)$ with the form $(\star 2)$. We first prove that $\tau_3 = 0$, $\nu_2 = 0$. Let $\alpha \in N$ be an arbitrary arrow. Then $e(\alpha) = i$. Assume that $s(\alpha) = j$, where $j \in \Gamma_0$. In view of condition (5) of Lemma 3.2, we know that
\[
\tau_3(\alpha) = \tau_3(\alpha e_j) = e_j \tau_3(\alpha).
\]
This implies that if $\tau_3(\alpha) \neq 0$, then $\alpha \tau_3(\alpha) \neq 0$. However, $\alpha \tau_3(\alpha) \neq 0$ is impossible by condition (5) of Lemma 3.2. Thus $\tau_3(\alpha) = 0$ for all $\alpha \in N$. Note that all path $p \in N$ with length more than 1 is of the form $\alpha p'$, where $\alpha$ is an arrow ending at $i$. Then $\tau_3(p) = \tau_3(\alpha p') = p' \tau_3(\alpha) = 0$. This completes the proof of $\tau_3 = 0$. It is proved similarly that $\nu_2 = 0$. Furthermore, we have from $B$ is commutative that every Jordan derivation of $B$ is a derivation. Finally, the fact $\Phi_{MN} = 0$ leads to $mn_0 = m_0 n = 0$ for all $m \in M$ and $n \in N$. \hfill $\square$

Now we are in a position to describe Jordan derivations of a dual extension algebra.

Theorem 3.4. Let $\Gamma$ be a finite connected quiver without oriented cycles and $\Lambda = K(\Gamma, \rho)$. Let $\mathcal{D}(\Lambda)$ be the dual extension algebra of $\Lambda$. Then each Jordan derivation of $\mathcal{D}(\Lambda)$ is a derivation.

Proof. If the algebra $\mathcal{D}(\Lambda)$ is trivial, then the theorem clearly holds. Suppose that $\Gamma_0 \geq 2$ and that $i \in \Gamma_0$ is a source. Let $\Theta$ be a Jordan derivation on $\mathcal{D}(\Lambda)$. Let us denote by $(\Gamma', \rho')$ the quiver obtained by removing the vertex $i$ and the relations starting at $i$ and write $\Lambda' = K(\Gamma', \rho')$. It follows from Lemma 3.3 that each Jordan derivation on $\mathcal{D}(\Lambda)$ is a derivation if each Jordan derivation on $\mathcal{D}(\Lambda')$ is a derivation. Thus it is sufficient to determine whether every Jordan derivation on $\mathcal{D}(\Lambda')$ is a derivation. We continuously repeat this process and ultimately arrive at the algebra $K$ after finite times, since $\Gamma_0$ is a finite set. Clearly, every Jordan derivation on $K$ is a derivation. This completes the proof. \hfill $\square$

Remark 3.5. In [20], the current authors and L. W. Wyk proved that for a generalized matrix algebra $G = (A, M, N, B)$ with $M$ being faithful as left $A$-module and as right $B$-module, if the pairings $\Phi_{MN} = 0$ and $\Psi_{NM} = 0$, then each Jordan derivation of $G$ is the sum of a derivation and an anti-derivation. Theorem 3.4 implies that neither the faithful condition nor the pairings being both zero is necessary. Benković and Širovnik [1] introduced the so-called singular Jordan derivations and
showed that under certain condition, each Jordan derivation of a generalized matrix algebra is the sum of a derivation and a singular Jordan derivation. Theorem 3.4 also implies that Bencovič’s condition is not necessary.

At the end of this section, let us characterize Jordan generalized derivations and generalized Jordan derivations of dual extension algebras. Let \( R \) be a commutative ring with identity, \( A \) a unital algebra over \( R \) and \( M \) an \((A,A)\)-bimodule. Recall that a linear mapping \( f: A \to A \) is called a Jordan generalized derivation if there exists a linear mapping \( d: A \to A \) such that
\[
f(x \circ y) = f(x) \circ y + x \circ d(y)
\]
for all \( x, y \in A \), where \( d \) is called an associated linear mapping of \( f \). A linear mapping \( f: A \to A \) is called a generalized Jordan derivation if there exists a linear mapping \( d: A \to A \) such that
\[
f(xy) = f(x)y + xy + xd(y) + yd(x)
\]
for all \( x, y \in A \). A linear mapping \( f: A \to A \) is called a generalized derivation if
\[
f(xy) = f(x)y + xd(y) \quad \text{for all} \quad x, y \in A.
\]

For generalized Jordan derivations on dual extension algebras, the following result is a simple corollary of [3, Lemma 4.1] and Theorem 3.4.

**Corollary 3.6.** Every generalized Jordan derivation on a dual extension algebra \( \mathcal{D}(\Lambda) \) is a generalized derivation.

In order to deal with Jordan generalized derivations, we need the following two lemmas obtained in [16] by the first author and Benkovič.

**Lemma 3.7.** [16, Proposition 2.1] Let \( f: A \to M \) be a generalized derivation with an associated linear mapping \( d \). Then \( d \) is a derivation and \( f(1) = f(1)x + d(x) \) for all \( x \in A \).

**Lemma 3.8.** [16, Theorem 2.2] Let \( f: A \to M \) be a Jordan generalized derivation with an associated linear mapping \( d \). The following statements are equivalent:

1. Element \( f(1) \) belongs to the center of \( M \).
2. The mapping \( d \) is a Jordan derivation and \( f(x) = f(1)x + d(x) \) for all \( x \in A \).

Let \( f \) be a Jordan generalized derivation on a dual extension algebra \( \mathcal{D}(\Lambda) \). It follows from Lemma 3.7, Lemma 3.8 and Theorem 3.4 that if \( f(1) \in Z(\mathcal{D}(\Lambda)) \), then every Jordan generalized derivation is a generalized derivation. In order to prove that \( f(1) \in Z(\mathcal{D}(\Lambda)) \), the following lemma obtained in [16] will play an important role.

**Lemma 3.9.** [16, Lemma 2.4] Let \( f: A \to M \) be a Jordan generalized derivation with an associated linear mapping \( d: A \to M \). Then the following holds:

1. \([x,y], f(1) = 0 \) for all \( x, y \in A \);
2. \( f(1) = ef(1)e + (1-e)f(1)(1-e) \) for any idempotent \( e \in A \).

**Lemma 3.10.** Let \( f \) be a Jordan generalized derivation on a dual extension algebra \( \mathcal{D}(\Lambda) \). Then \( f(1) \in Z(\mathcal{D}(\Lambda)) \).

**Proof.** It is not difficult to see that \( e_i(1-e_i) = (1-e_i)e_i = 0 \) for all \( i \in \Gamma_0 \). Applying Lemma 3.9 (2) yields that
\[
e_i f(1) = e_i f(1)e_i = f(1)e_i.
\]

Moreover, let \( p \) be a nontrivial path with \( s(p) = r, e(p) = t \) and \( s \neq t \). Then \( p = [p, e_r] \). Combining this fact with Lemma 3.9 (2) gives that

\[
f(1)p = pf(1). \tag{3.2}
\]

Furthermore, we claim that \( e_if(1)e_j = 0 \) for arbitrary \( i, j \in \Gamma_0 \) with \( i \neq j \). In fact, Lemma 3.9 (2) implies that

\[
f(1) = (e_i + e_j)f(1)(e_i + e_j) + (1 - e_i - e_j)f(1)(1 - e_i - e_j). \tag{3.3}
\]

Left multiplication of (3.3) by \( e_i \) leads to

\[
e_i f(1) = e_i f(1)e_i + e_i f(1)e_j. \tag{3.4}
\]

Note that \( e_i f(1) = e_if(1)e_i \). Then (3.4) forces \( e_i f(1)e_j = 0 \). This shows that the paths \( p \) with \( s(p) \neq e(p) \) do not appear in the expansion of \( f(1) \). Hence for all nontrivial paths \( p \) with \( s(p) = e(p) \),

\[
f(1)p = pf(1). \tag{3.5}
\]

Combining (3.1), (3.2) with (3.5) yields that \( f(1) \in \mathcal{Z}(\mathcal{P}(\Lambda)) \).

Applying Theorem 3.4, Lemma 3.7 and Lemma 3.10 yields the following result.

**Proposition 3.11.** Every Jordan generalized derivation on a dual extension algebra \( \mathcal{P}(\Lambda) \) is a generalized derivation.

### 4. Jordan Derivations of Generalized One-Point Extensions

We introduced the notion of generalized one-point extension algebras in \([19]\) and studied Lie derivations on them. In this section, we will investigate Jordan derivations of generalized one-point algebras. Let us first recall the definition.

Let \( \Gamma = (\Gamma_0, \Gamma_1) \) be a finite quiver without oriented cycles and \( |\Gamma_0| \geq 2 \). Let \( \Gamma^\ast \) be a quiver whose vertex set is \( \Gamma_0 \) and

\[
\Gamma^\ast_1 = \{ \alpha^\ast : i \to j \mid \alpha : j \to i \text{ is an arrow in } \Gamma_1 \}.
\]

For a path \( p = \alpha_n \cdots \alpha_1 \) in \( \Gamma \), write the path \( \alpha_n^\ast \cdots \alpha_1^\ast \) in \( \Gamma^\ast \) by \( p^\ast \). Given a set \( \rho \) of relations, denote by \( \Lambda = K(\Gamma, \rho) \). Define the generalized one-point extension algebra \( E(\Lambda) \) to be the path algebra of the quiver \( (\Gamma_0, \Gamma_1 \cup \Gamma^\ast_1) \) with relations

\[
\rho \cup \rho^\ast \cup \{ \alpha \beta^\ast \mid \alpha, \beta \in \Gamma_1 \} \cup \{ \alpha^\ast \beta \mid \alpha, \beta \in \Gamma_1 \}.
\]

It is helpful to point out that if we choose a suitable idempotent, then neither \( M \) nor \( N \) need not to be faithful. Let us illustrate an example here.

**Example 4.1.** Let \( \Gamma \) be a quiver as follows

\[
\begin{array}{c}
1 \\
\alpha \\
2 \\
\beta \\
3 \\
\gamma
\end{array}
\]

and let \( \Lambda = KT \). The generalized one-point extension algebra \( E(\Lambda) \) has a basis

\[
\{ e_1, e_2, e_3, e_4, \alpha, \beta, \gamma, \alpha^\ast, \beta^\ast, \gamma^\ast, \beta \alpha, \alpha^\ast \beta^\ast \}.
\]

Taking the nontrivial idempotent to be \( e_1 + e_2 \), then \( E(\Lambda) \) is isomorphic to a generalized matrix algebra \( \mathcal{G} = \begin{bmatrix} A & M \\ N & B \end{bmatrix} \), where \( A \) has a basis \( \{ e_1, e_2, \alpha, \alpha^\ast \} \), \( B \) has a basis \( \{ e_3, e_4, \gamma, \gamma^\ast \} \), \( M \) has a basis \( \{ \alpha^\ast \beta^\ast, \beta^\ast \} \) and \( N \) has a basis \( \{ \beta, \beta \alpha \} \). It is easy to check that \( \alpha \in \text{Ann}(AM) \) and \( \gamma \in \text{Ann}(MB) \), that is, \( M \) is neither faithful
as left $A$-module nor as right $B$-module. Similarly, we obtain $\gamma \in \text{Ann}(N_B)$ and $\alpha \in \text{Ann}(N_A)$ that is, $N$ is neither faithful as left $B$-module nor as right $A$-module.

Moreover, in \cite{Benkovic} Benkovič proved that for a generalized matrix algebra $G = \left[ \begin{array}{c} A \\ M \\ N \\ B \end{array} \right]$, if

1. $aM = 0$ and $Na = 0$ imply that $a = 0$;
2. $Mb = 0$ and $bN = 0$ imply that $b = 0$,
then every Jordan derivation on $G$ is the sum of a derivation and an anti-derivation.

Clearly, our example does not satisfy Benkovič’s conditions.

The aim of this section is to prove that under certain conditions, each Jordan derivation of $E(\Lambda)$ is the sum of a derivation and an anti-derivation. Let us first characterize anti-derivations of generalized one-point extension algebras.

**Lemma 4.2.** Let $\Gamma$ be a finite quiver without oriented cycles and $\Lambda = K(\Gamma, \rho)$. Let $\Theta$ be an anti-derivation on $E(\Lambda)$ and $\alpha \in \Gamma_1 \cup \Gamma_1^*$ with $s(\alpha) = r$ and $e(\alpha) = t$.

Then

1. $\Theta(e_r) = \sum_{s(p) = i, \text{ or } e(p) = i} k_{e_i}^p$;
2. $\Theta(\alpha) = \sum_{s(p) = t, e(p) = r} k_{\alpha}^p$.

Moreover, $\Theta(p) = 0$ for all path $p$ with length more than one. If there exists a nontrival path $\beta$ such that $\beta\alpha \neq 0$ or $\alpha\beta \neq 0$, then $\Theta(\alpha) = 0$.

**Proof.** (1) Suppose that

\[
\Theta(e_r) = \sum_{i \in \Gamma_0} k_i e_i + \sum_{s(p) \neq e(p)} k_{e_p}^p.
\]

It follows from the fact $e_r^2 = e_i$ that

\[
\Theta(e_r) = \Theta(e_r)e_r + e_r\Theta(e_r).
\]

Combining (4.1) with (4.2) gives that $k_r = 0$. If there exists $j \in \Gamma_0$ with $i \neq j$ such that $k_j \neq 0$, then the coefficient of $e_j$ in the expansion of $\Theta(e_r)e_j$ is $k_j$. On the other hand, since $e_j$ does not appear in the expansion of $e_r\Theta(e_j)$, we conclude that $e_j$ does not appear in the expansion of $e_r\Theta(e_j)$ too. This implies that $\Theta(e_j e_r) \neq 0$, which is impossible.

(2) Let $\Theta$ be an anti-derivation on $E(\Lambda)$ and let $\alpha \in \Gamma_1$ with $s(\alpha) = r$ and $e(\alpha) = t$. Suppose that

\[
\Theta(\alpha) = \sum_{i \in \Gamma_0} k_i e_i + \sum_{s(p) \neq e(p)} k_{\alpha e_p}^p.
\]

Then on one hand,

\[
\Theta(\alpha) = \Theta(e_t \alpha) = \Theta(\alpha)e_t + \alpha \Theta(e_t).
\]

Substituting (4.3) into (4.4) gives that

\[
\sum_{i \in \Gamma_0} k_i e_i + \sum_{s(p) \neq e(p)} k_{\alpha e_p}^p = (\sum_{i \in \Gamma_0} k_i e_i + \sum_{s(p) \neq e(p)} k_{\alpha e_p}^p)e_t + \alpha \Theta(e_t)
\]

\[
= k_t e_t + \sum_{s(p) = t} k_{\alpha e_p}^p + \alpha \Theta(e_t).
\]
On the other hand,\[\Theta(\alpha) = \Theta(\alpha e_r) = e_r \Theta(\alpha) + \Theta(e_r)\alpha.\] (4.5)

Substituting (4.3) into (4.5) yields that
\[\sum_{i \in \Gamma_0} k_i e_i + \sum_{s(p) \neq e(p)} k_p^a p = e_r \sum_{i \in \Gamma_0} k_i e_i + \sum_{s(p) \neq e(p)} k_p^a p + \Theta(e_r)\alpha\]
\[= k_r e_r + \sum_{e(p) = r} k_p^a p + \Theta(e_r)\alpha.\]

By the above equalities it follows that

\[\sum_{i \in \Gamma_0} k_i e_i + \sum_{s(p) \neq e(p)} k_p^a p = k_r e_r + \sum_{e(p) = r} k_p^a p + \Theta(e_r)\alpha.\]

This implies that \(k_i = 0\) for all \(i \in \Gamma_0\) and the coefficients of all paths \(p\) with \(s(p) \neq t\) or \(e(p) \neq r\) in the expansion of \(\Theta(\alpha)\) are zero, that is,

\[\Theta(\alpha) = \sum_{s(p) = t, e(p) = r} k_p^a p.\] (4.6)

If there exists a non trivial path \(\beta\) such that \(\beta \alpha \neq 0\), then \(\alpha \beta^* = 0\). However, \(\Theta(\alpha \beta^*) = \Theta(\beta^*) \alpha + \beta^* \Theta(\alpha)\). If \(\Theta(\alpha) \neq 0\), then by (4.5) we know that

\[\beta^* \Theta(\alpha) = \sum_{s(p) = t, e(p) = r} k_p^a \beta p \neq 0,\]

and hence \(\Theta(\alpha \beta^*) \neq 0\), which is a contradiction. This forces that \(\Theta(\alpha) = 0\). Similarly, we can show that if \(\alpha \beta = 0\), then \(\Theta(\alpha) = 0\).

Since \(\Gamma\) is a quiver without oriented cycles, we can take a source \(i\) in \(\Gamma\). Let \(e_i\) be the corresponding idempotent in \(E(\Lambda)\). Then \(E(\Lambda)\) is isomorphic to a generalized matrix algebra \(G = [A M B] \) with \(A \simeq E(\Lambda')\), where the quiver \(\Gamma'\) of \(\Lambda'\) is obtained by removing the vertex \(i\) and the relations starting at \(i\). Moreover, we have from the construction of \(E(\Lambda)\) that the bilinear pairings are both zero. In this case, the form (\#1) of any Jordan derivation of \(E(\Lambda)\) becomes as follows:

**Lemma 4.3.** Let \(\Lambda = K \Gamma\) and \(E(\Lambda)\) be the generalized one-point extension. Then an arbitrary Jordan derivation \(\Theta\) on \(E(\Lambda)\) is of the form

\[\Theta\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \delta_1(a) & am_0 - m_0 b + \tau_2(m) + \tau_3(n) \\ n_0 a - bm_0 + \nu_2(m) + \nu_3(n) & 0 \end{bmatrix},\] (\#1)

\(\forall \begin{bmatrix} a & m \\ n & b \end{bmatrix} \in G,\)

where \(m_0 \in M, n_0 \in N\) and

\(\delta_1 : A \rightarrow A, \ \tau_2 : M \rightarrow M, \ \tau_3 : N \rightarrow M, \ \nu_2 : M \rightarrow N, \ \nu_3 : N \rightarrow N,\)
are all $\mathcal{R}$-linear mappings satisfying the following conditions:

1. $\delta_1$ is a Jordan derivation on $A$.
2. $\tau_2(am) = a\tau_2(m) + \delta_1(a)m$ and $\tau_2(mb) = \tau_2(m)b$;
3. $\nu_3(bn) = \nu_3(n)a + n\delta_1(a)$;
4. $\tau_3(na) = \tau_3(n)$, $\tau_3(bn) = \tau_3(n)b$;
5. $\nu_2(am) = \nu_2(m)a$, $\nu_2(mb) = \nu_2(m)b$.

Proof. We only need to prove $\mu_4 = 0$. But this is clear because $\mu_4$ is a Jordan derivation on $B = K$. \qed

In [20], the form of an arbitrary anti-derivation on a generalized matrix algebra $G = [A \phi B]$ has been characterized under the condition that $M$ being faithful as left $A$-module and also as right $B$-module. If we remove the faithful assumption on $M$, the form of an anti-derivation on $G$ is as follows:

Lemma 4.4. An additive mapping $\Theta$ is an anti-derivation of $G$ if and only if $\Theta$ has the form

$$\Theta \left( \begin{array}{cc} a & m \\ n & b \end{array} \right) = \left[ \begin{array}{cc} \delta_1(a) & am_0 - m_0b + \tau_3(n) \\ n_0a - bn_0 + \nu_2(m) & \mu_4(b) \end{array} \right], (\ast 2)$$

where $m_0 \in M, n_0 \in N$ satisfying for all $a, a' \in A, b, b' \in B, m \in M$ and $n \in N$

1. $[a, a']m_0 = 0, m_0[b, b'] = 0, n_0[a, a'] = 0, [b, b']n_0 = 0$;
2. $mn_0 = 0, nmn = 0, n_0m = 0$;

and $\delta_1 : A \rightarrow A, \tau_3 : N \rightarrow M, \nu_2 : M \rightarrow N, \mu_4 : B \rightarrow B$

are $\mathcal{R}$-linear mappings satisfying for all $a \in A, b \in B, m, m' \in M$ and $n, n' \in N$

3. $\delta_1$ is an anti-derivation on $A$ and $\delta_1(mn) = 0, \delta_1(a)m = 0, n\delta_1(a) = 0$;
4. $\mu_4$ is an anti-derivation on $B$ and $\mu_4(mn) = 0, m\mu_4(b) = 0, \mu_4(b)n = 0$;
5. $\tau_3(na) = \tau_3(n)$, $\tau_3(bn) = \tau_3(n)b$, $n\tau_3(n') = 0$, $\tau_3(n)n' = 0$;
6. $\nu_2(am) = \nu_2(a), \nu_2(mb) = \nu_2(m), m\nu_2(m') = 0, \nu_2(m)m' = 0$.

Proof. It can be proved similarly as that of [20 Proposition 3.6]. \qed

As a consequence of Lemma 4.3 and Lemma 4.4 we have

Proposition 4.5. Let $\Theta$ be a Jordan derivation on a generalized one-point extension algebra $E(\Lambda) \simeq [A \phi B]$. If there exists an anti-derivation $f$ on $A$ with $\text{Im}(f) \subset \text{Ann}(A \phi M)$ such that $\delta_1 - f$ is a derivation of $A$, then $\Theta$ is the sum of a derivation and an anti-derivation.

We are now in a position to state the main result of this section.

Theorem 4.6. Let $\Gamma$ be a finite quiver without oriented cycles and $\Lambda = K(\Gamma, \rho)$. If there is no path $p$ with length more than one, then every Jordan derivation on the generalized one point extension algebra $E(\Lambda)$ is the sum of a derivation and an anti-derivation.
Proof. Let $\Theta$ be a Jordan derivation on $E(\Lambda)$. Then by Lemma 4.2, it is of the form (1). We claim that if each Jordan derivation on $A$ is the sum of a derivation and an anti-derivation, then so is $E(\Lambda)$. In fact, assume that $\delta = d + f$, where $d$ is a derivation of $A$ and $f$ is an anti-derivation of $A$. By Lemma 4.2, we know that all $e_i$ do not appear in $f(a)$ for $a \in A$. Note that the length of each path is not more than one. This implies that $f(a)m = 0$ for all $a \in A$ and $m \in M$. Similarly, we can show that $nf(a) = 0$ for all $a \in A$ and $n \in N$. Define a linear mapping $f'$ on $E(\Lambda)$ by

$$f' \left( \begin{bmatrix} a & m \\ n & b \end{bmatrix} \right) = \begin{bmatrix} f(a) & \tau_3(n) \\ \nu_2(m) & 0 \end{bmatrix}, \forall \begin{bmatrix} a & m \\ n & b \end{bmatrix} \in E(\Lambda)$$

Then Lemma 4.2 and Lemma 4.3 give that $f'$ is a anti-derivation of $E(\Lambda)$. Furthermore, the linear mapping

$$D \left( \begin{bmatrix} a & m \\ n & b \end{bmatrix} \right) = \begin{bmatrix} d(a) & am_0 - m_0b + \tau_2(m) \\ n_0a - b_0n + \nu_3(n) & 0 \end{bmatrix}$$

is a derivation of $E(\Lambda)$. This completes the proof of our claim. Repeating this process, we arrive at the algebra $K$, on which every Jordan derivation is zero. This completes the proof. 

Finally, we illustrate an example which satisfies the condition of Theorem 4.6

**Example 4.7.** Let $\Gamma$ be a quiver as follows

$$
\begin{array}{c}
 1 & \alpha & 2 \\
\end{array}
\begin{array}{c}
 3 & \beta \\
\end{array}
$$

and let $\Lambda = K\Gamma$. Then the generalized one point extension algebra $E(\Lambda)$ has a basis

$$\{e_1, e_2, e_3, \alpha, \beta, \alpha^*, \beta^*\}.$$

Define a linear mapping on $E(\Lambda)$ by

$$\Theta(e_1) = \Theta(e_2) = \Theta(e_3) = 0, \quad \Theta(\alpha) = \alpha^*, \quad \Theta(\alpha^*) = \alpha,$n

$$\Theta(\beta) = \beta + \beta^*, \quad \Theta(\beta^*) = \beta - \beta^*.$$

Then a direct computation shows that $\Theta$ is a proper Jordan derivation on $E(\Lambda)$. On the other hand, we can also define two linear mappings $\Theta_1$ and $\Theta_2$ by

$$\Theta_1(e_1) = \Theta_1(e_2) = \Theta_1(e_3) = 0, \quad \Theta_1(\alpha) = 0, \quad \Theta_1(\alpha^*) = 0,$n

$$\Theta_1(\beta) = \beta, \quad \Theta_1(\beta^*) = -\beta^*.$$

and

$$\Theta_2(e_1) = \Theta_2(e_2) = \Theta_2(e_3) = 0, \quad \Theta_2(\alpha) = \alpha^*, \quad \Theta_2(\alpha^*) = \alpha,$n

$$\Theta_2(\beta) = \beta^*, \quad \Theta_2(\beta^*) = \beta.$$

It is easy to see that $\Theta_1$ is a derivation on $E(\Lambda)$ and $\Theta_2$ is an anti-derivation on $E(\Lambda)$. Therefore, $\Theta$ is the sum of the derivation $\Theta_1$ and the anti-derivation $\Theta_2$.

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