Singular Welschinger invariants

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Abstract

We suggest an invariant way to enumerate nodal and nodal-cuspidal real deformations of real plane curve singularities. The key idea is to assign Welschinger signs to the counted deformations. Our invariants can be viewed as a local version of Welschinger invariants enumerating real plane rational curves.

Introduction

Gromov-Witten invariants of the plane can be identified with the degrees of Severi varieties, which parameterize irreducible plane curves of given degree and genus. As a local version, one can consider a versal deformation of an isolated plane curve singularity \((C, z) \subset \mathbb{C}^2\) with base \(B(C, z) \simeq (\mathbb{C}^n, 0)\), and the following strata in \(B(C, z)\):

\[ EG_{C,z}^i, \quad 1 \leq i \leq \delta(C, z), \quad (1) \]

parameterizing deformations with the total \(\delta\)-invariant greater or equal to \(i\);

\[ EC_{C,z}^k, \quad 0 \leq k \leq \kappa(C, z) - 2\delta(C, z), \quad (2) \]

parameterizing deformations with the total \(\delta\)-invariant equal to \(\delta(C, z)\) and the total \(\kappa\)-invariant equal to \(2\delta(C, z) + k\) (a necessary information on \(\delta\)-and \(\kappa\)-invariants can be found in [7] or [9, Section 3.4]). Note also that \(EC_{C,z}^0 = EG_{C,z}^{\delta(C,z)}\).

The strata (1) are called *Severi loci*; among them, \(D_{C,z} := EG_{C,z}^1\) is the discriminant hypersurface in \(B(C, z)\), and \(EG_{C,z} := EG_{C,z}^{\delta(C,z)}\) is the so-called *equigeneric locus*. We call the strata (2) *generalized equiclassical loci*, and

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among them $EC_{C,z} := EC_{C,z}^{\kappa(C,z)-2\delta(C,z)}$ is the so-called equiclassical locus. The incidence relations are as follows:

$$EG_{C,z}^i \subseteq EG_{C,z}^{i+1} \subseteq EC_{C,z}^k \subseteq EC_{C,z}^{k+1}$$

for all $1 \leq i < \delta(C,z)$ and $1 \leq k < \kappa(C,z) - 2\delta(C,z)$. All these loci are pure-dimensional germs of complex spaces (cf. [14, 15]).

A natural problem is to compute the multiplicities of $EG_{C,z}^i, EC_{C,z}^k$ for all $i, k \geq 1$. This problem was solved for the equigeneric stratum $EG_{C,z}$ in [8]. In the particular case of an irreducible germ with one Puiseux pair, i.e., topologically equivalent to $x^p + y^q = 0$, $2 \leq p < q$, gcd$(p,q) = 1$, one has (see [3, Proposition 4.3] and [8, Section G])

$$\text{mult } EG_{C,z} = \frac{1}{p+q} \binom{p+q}{p}.$$ 

The multiplicities of all Severi loci $EG_{C,z}^i$ were expressed in [15] in terms of the Euler characteristics of Hilbert schemes of points on curve germs representing a given singularity. The multiplicities of the equiclassical loci $EC_{C,z}^k$ are not known except for the case of the smoothness mentioned in [6, Theorems 2 and 27].

The multiplicity admits an enumerative interpretation: it can be regarded as the number of intersection points of a locus $V \subset B(C,z)$ with a generic affine subspace $L \subset B(C,z)$ of the complementary dimension (equal to codim$V$) chosen to be transversal to the tangent cone $T_0V$.

The goal of this note is to define real multiplicities of the Severi loci and of the generalized equiclassical loci. Let the singularity $(C,z)$ be real. Then the Severi loci and the generalized equiclassical loci are defined over the reals. Thus, given such a locus $V$, we count real intersection points of $V$ with a generic real affine subspace $L \subset B(C,z)$ of the complementary dimension. Our main result is that, in certain cases, the count of real intersection points of $V$ and $L$ equipped with Welschinger-type signs is invariant, i.e., does not depend on the choice of $L$. We were motivated by [11, Lemma 15], which, in fact, states the existence of a Welschinger type invariant for the equigeneric stratum $EG_{C,z}$. In this note, we go further and prove the

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1 We understand the multiplicity of a point of an algebraic variety embedded into an affine space as the intersection number at this point with a generic smooth germ of the complementary dimension (cf. [13, Chapter 5, Definition 5.9]).

2 Under the real object we always understand a complex object invariant with respect to the complex conjugation.
existence of similar Welschinger type invariants for $E_{\theta}^{\delta(C,z)}$ (see Proposition 3.2 in Section 3) and for $E_{\theta}^{1} = D_{C,z} \subset B(C,z)$ (see Proposition 3.3 in Section 3) as well as for all the loci $E_{\theta}^{k}$ (see Proposition 4.1 in Section 4).

We remark that a similar enumeration of real plane rational curves with at least one cusp is not invariant, i.e., depends on the choice of point constraints (cf. [17]).

As an example, we perform computations for singularities of type $A_{n}$ (see Section 5).

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1 Singular Welschinger numbers

We shortly recall definitions and basic properties of objects of our interest. Details can be found in [7] and [9, Chapter II].

Let $(C,z)$ be the germ of a plane complex analytic curve $C$ at its isolated singular point $z = (0,0) \in \mathbb{C}^2$, which is given by an analytic equation $f(x,y) = 0$, $f \in \mathbb{C}\{x,y\}$. We shortly call it singularity. The Milnor ball $D(C,z) \subset \mathbb{C}^2$ is a closed ball centered at $z$ such that $C \cap D(C,z)$ is closed and smooth outside $z$ with the boundary $\partial(C \cap D(C,z)) \subset \partial D(C,z)$, and the intersection of $C$ with any 3-sphere in $D(C,z)$ centered at $z$ is transversal. Pick integer $N > 0$ and consider the small neighborhood $B(C,z)$ of 0 in the space (which is a $\mathbb{C}$-algebra) $R(C,z) := \mathbb{C}\{x,y\}/((f) + m_{z}^N)$, where $m_{z} \subset \mathbb{C}\{x,y\}$ is the maximal ideal. We can suppose that, for any $\varphi \in B(C,z)$, the curve $C_{\varphi} := \{f + \varphi = 0\} \cap D(C,z)$ has only isolated singularities in $D(C,z)$, is smooth along $\partial D(C,z)$, and intersects the sphere $\partial D(C,z)$ transversally. It is well-known that the deformation $\pi : C \to B(C,z)$ of $(C,z)$, where $\pi^{-1}(\varphi) = C_{\varphi}$, is versal for $N > 0$ sufficiently large (cf. [2] Page 165 or [7] Section 3). The space $B(C,z)$ contains the equigeneric stratum $E_{\theta}^{\delta(C,z)} \subset B(C,z)$, formed by $\varphi \in B(C,z)$ such that $C_{\varphi}$ has the total $\delta$-invariant equal to $\delta(C,z)$ (the maximal possible value), the equiclassical locus $E_{\theta}^{\kappa(C,z)} \subset E_{\theta}^{\delta(C,z)} \subset B(C,z)$, formed by $\varphi \in E_{\theta}^{\delta(C,z)}$ such that $C_{\varphi}$ has the total $\kappa$-invariant equal to $\kappa(C,z)$ (also the maximal possible value), and
the discriminant

\[ D_{C,z} = \{ \varphi \in B(C, z) : C_\varphi \text{ is singular} \} . \]

The following statement summarizes some known facts on the above strata (see [7, Theorems 1.1, 1.3, 4.15, 4.17, 5.5, Corollary 5.13] and [6, Theorems 2 and 27]).

**Lemma 1.1.** (1) The stratum \( EG_{C,z} \) is irreducible of codimension \( \delta(C, z) \); it is smooth iff all irreducible components of \( (C, z) \) are smooth; in general, the normalization of \( EG_{C,z} \) is smooth and projects one-to-one onto \( EG_{C,z} \). The tangent cone \( T_0 EG_{C,z} \) is the linear space \( J_{C,z}^{\text{cond}}/m_z^N \) of codimension \( \delta(C, z) \), where \( J_{C,z}^{\text{cond}} \subset \mathbb{C}\{x, y}\}/f \) is the conductor ideal. Furthermore, \( EG_{C,z} \) contains an open dense subset \( EG_{C,z}^* \) that parameterizes the curves \( C_\varphi \) having \( \delta(C, z) \) nodes as their only singularities.

(2) The stratum \( EC_{C,z} \) is irreducible of codimension \( \kappa(C, z) - \delta(C, z) \); it is smooth iff each local branch of \( (C, z) \) either is smooth, or has topological type \( x^m + y^{m+1} = 0 \) with \( m \geq 2 \); in general, the normalization of \( EC_{C,z} \) is smooth and projects one-to-one onto \( EC_{C,z} \). The tangent cone \( T_0 EC_{C,z} \) is the linear space \( J_{C,z}^{\text{ec}}/m_z^N \) of codimension \( \kappa(C, z) - \delta(C, z) \), where \( J_{C,z}^{\text{ec}} \subset \mathbb{C}\{x, y}\}/f \) is the equiclassical ideal. Furthermore, the stratum \( EC_{C,z} \) contains an open dense subset \( EC_{C,z}^* \) that parameterizes the curves \( C_\varphi \) having \( 3\delta(C, z) - \kappa(C, z) \) nodes and \( \kappa(C, z) - 2\delta(C, z) \) ordinary cusps as their only singularities.

(3) The discriminant \( D_{C,z} \) is an irreducible hypersurface with the tangent cone \( T_0 D_{C,z} = m_z/((f)+m_z^N) \). Furthermore, an open dense subset \( D_{C,z}^* \subset D_{C,z} \) parameterizes the curves \( C_\varphi \) having one node and no other singularities.

In the same way one can establish similar properties of the Severi loci (1) and generalized equiclassical loci (2).

**Lemma 1.2.** (1) Each Severi locus \( EG_{C,z}^i \) is a (possibly reducible) germ of a complex space of pure codimension \( i \). A generic element of each component of \( EG_{C,z}^i \) is a curve with \( i \) nodes as its only singularities.

(2) Each generalized equiclassical locus \( EC_{C,z}^k \) is a (possibly reducible) germ of a complex space of pure codimension \( \delta(C, z) + k \). A generic element of each component of \( EC_{C,z}^k \) is a curve with \( \delta(C, z) - k \) nodes and \( k \) ordinary cusps as its only singularities.

It is well-known that \( \text{mult} D_{C,z} = \mu(C, z) \) (the Milnor number), \( \text{mult} EG_{C,z} \) has been computed in [8] as the Euler characteristic of an appropriate compactified Jacobian.
Now we switch to the real setting. We call the complex space $V$ real if it is invariant under the (natural) action of the complex conjugation and denote by $\mathbb{R}V$ its real point set. Suppose that $(C, z)$ is real.

**Definition 1.3.** Let $V \subset B(C, z)$ be an equivariant union of irreducible components of either a Severi locus $EG_{C, z}^i$, $1 \leq i \leq \delta(C, z)$, or a generalized equiclassical locus $EC_{C, z}^k$, $1 \leq k \leq \kappa(C, z) - 2\delta(C, z)$, and let $\widehat{T}_0V$ be a linear subspace of $R(C, z)$ of dimension $\dim V$. Assume that $L_0 \subset R(C, z)$ is a real linear subspace of dimension $\dim L_0 = \text{codim}_{B(C, z)} V$, which meets $\widehat{T}_0V$ only at the origin, and let $U(L_0)$ be a neighborhood of the origin such that $L_0 \cap V \cap U(L_0) = \{0\}$. For a sufficiently close to $L_0$ real affine space $L \subset R(C, z)$ of dimension $\dim L = \dim L_0$, intersecting $V \cap U(L_0)$ along $V^*$ and with total multiplicity $\text{mult}_V$, we set

$$W(C, z, V, L) = \sum_{\varphi \in L \cap \mathbb{R}V \cap U(L_0)} w(\varphi),$$

where $w(\varphi) = (-1)^{s(\varphi) + ic(\varphi)}$, with $s(\varphi)$ being the number of real elliptic node of $C_\varphi$ and $ic(\varphi)$ the number of pairs of complex conjugate cusps of $C_\varphi$. In case of $V = EG_{C, z}$ or $EC_{C, z}$, we write $W^{eg}(C, z, L)$ or $W^{ec}(C, z, L)$, respectively.

In what follows we examine the dependence on $L$ and prove some invariance statements.

## 2 Singular Welschinger invariant $W^{eg}(C, z)$

The following statement is a consequence of [11, Lemma 15]. We provide a proof, since in a similar manner we treat other instances of the invariance.

**Proposition 2.1.** Given a real singularity $(C, z)$, the number $W^{eg}(C, z, L)$ does not depend on the choice of $L$.

**Proof.** Let $L'_0, L''_0 \subset R(C, z)$ be two real linear subspaces of dimension $\delta(C, z)$ transversally intersecting $T_0EG_{C, z}$ at the origin, and let $L', L'' \subset R(C, z)$ be real affine subspaces of dimension $\delta(C, z)$, which are sufficiently close to $L'_0, L''_0$, respectively, in the sense of Definition [3]. We can connect the pairs $(L'_0, L')$ and $(L''_0, L'')$ by a generic smooth path $\{L_0(t), L(t)\}_{t \in [0, 1]}$ consisting of real linear subspaces $L_0(t)$ of $R(C, z)$ of dimension $\delta(C, z)$, which are transversal to $T_0EG_{C, z}$, and real affine subspaces $L(t)$ of dimension $\delta(C, z)$ sufficiently close to $L_0(t)$ in the sense of Definition [3].

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3A real node is called elliptic if it is equivariantly isomorphic to $x^2 + y^2 = 0$.  


follows from Lemma [11,1] (1) that, for all $t \in [0, 1]$, the space $L(t)$ intersects $EG_{C,z}$ transversally at each element of $L(t) \cap EG_{C,z}$. Furthermore, all but finitely many spaces $L(t)$ intersect $EG_{C,z}$ along $EG^*_{C,z}$, transversally at each intersection point. The remaining finite subset $F \subset (0, 1)$ is such that, for any $\hat{t} \in F$, the intersection $L(\hat{t}) \cap EG_{C,z}$ consists of elements of $EG^*_{C,z}$ and one real element $\varphi$ belonging to a codimension one substratum of $EG_{C,z}$. The classification of these codimension one substrata is known (see, for instance [7, Theorem 1.4]): an element $\varphi$ of such a substratum is as follows:

(n1) either $C_{\varphi}$ has an ordinary cusp $A_2$ and $\delta(C, z) - 1$ nodes,

(n2) or $C_{\varphi}$ has a tacnode $A_3$ and $\delta(C, z) - 2$ nodes,

(n3) or $C_{\varphi}$ has a triple point $D_4$ and $\delta(C, z) - 3$ nodes.

In cases (n2) and (n3), the stratum $EG_{C,z}$ is smooth at $\varphi$ (cf. Lemma [11,1]), and the deformation of $C_{\varphi}$ under the variation of $L(t)$ induces independent equivariant deformations of all (smooth) local branches of $C_{\varphi}$ at the non-nodal singular point. Thus, the crossing of these strata does not affect $W^{\text{reg}}(C, z, L(t))$.

In case (n1), the germ of $B(C, z)$ at $\varphi$ can be represented as $B(A_2) \times B(A_1)^{\delta(C,z)-1} \times (\mathbb{C}^{n-\delta(C,z)-1}, 0)$ (cf. [9, Proposition I.1.14 and Theorem I.1.15] and [11, Lemma 13]), where $n = \dim B(C, z)$, $B(A_2) \simeq (\mathbb{C}^2, 0)$ is a miniversal deformation base of an ordinary cusp, which we without loss of generality can identify with the base of the deformation $\{y^2 - x^3 - \alpha x - \beta : \alpha, \beta \in (\mathbb{C}^2, 0)\}$, and $B(A_1) \simeq (\mathbb{C}, 0)$ stands for the versal deformation of an ordinary node. Here

$$(EG_{C,z}, b) = EG(A_2) \times EG(A_1)^{\delta(C,z)-1} \times (\mathbb{C}^{n-\delta(C,z)-1}, 0),$$

where $n = \dim B(C, z)$ and

$$EG(A_2) = \left\{ \frac{\alpha^3}{27} - \frac{\beta^2}{4} \right\} \subset B(A_2), \quad EG(A_1) = \{0\} \subset B(A_1),$$

$$T_bEG_{C,z} = EG(A_2) \times \{0\}^{\delta(C,z)-1} \times \mathbb{C}^{n-\delta(C,z)-1}.$$
not intersect \( EG(A_2) \) in real points, or it intersects \( EG(A_2) \) in two real points \((\alpha_1, \beta_1), (\alpha_2, \beta_2)\) with \( \beta_1 < 0 < \beta_2 \), where the former point corresponds to a real curve with a hyperbolic node in a neighborhood of the cusp, while the latter one - to a real curve with an elliptic node. Hence, the Welschinger signs of these intersections of \( L^1(t) \) with \( EG(A_2) \) cancel out, which confirms the constancy of \( W^{eq}(C, z, L(t)) \), \( |t - \hat{t}| < \eta \), in the considered wall-crossing.

We mention also two more useful properties of the invariant \( W^{eq}(C, z) \).

Lemma 2.2. (1) The number \( W^{eq}(C, z) \) is an invariant of a real equisingular deformation class. That is, if \( (C_t, z)_{t \in [0, 1]} \) is an equisingular family of real singularities, then \( W^{eq}(C_0, z) = W^{eq}(C_1, z) \).

(2) Let \( (C, z) = \bigcup_i (C_i, z) \) be the decomposition of a real singularity \( (C, z) \) into irreducible over \( \mathbb{R} \) components. Then \( W^{eq}(C, z) = \prod_i W^{eq}(C_i, z) \).

Proof. (1) It is sufficient to verify the local constancy of \( W^{eq}(C, z) \) in real equisingular deformations. Recall that the equisingular stratum \( ES_{C, z} \subset B(C, z) \) is a smooth subvariety germ. Furthermore, for \( N > \mu(C, z) + 1 \), the germ of \( B(C, z) \) at any point \( \psi \in ES_{C, z} \) is a versal deformation of of the singularity \( C_\psi \). Then the equality \( W^{eq}_{C, z} = W^{eq}_{C_\psi} \) follows by the argument in the proof of Proposition 2.1.

(2) The second statement of the lemma follows from the fact that a equigeneric deformation of \( (C, z) \) induces independent equigeneric deformations of the components \( (C_i, z) \) and vice versa (see [16, Theorem 1, page 73], [5, Corollary 3.3.1], and also [9, Theorem II.2.56]), and from the fact that the deformed components \( (C_i, z) \) and \( C_j, z \), \( i \neq j \), can intersect only in hyperbolic real nodes and in complex conjugate nodes.

3 Singular Welschinger invariants associated with \( EG^δ_{C, z} \) and \( D_{C, z} \)

The key ingredient of the proof of Proposition 2.1 is that the tangent cone to the equigeneric stratum \( EG_{C, z} \) is a linear space of dimension equal to \( \dim EG_{C, z} \). We intend to establish a similar statement for \( EG^δ_{C, z} \).

Recall the following fact used in the sequel: By [14, Theorem 1.1] the closure of each irreducible component of \( EG^δ_{C, z} \) contains \( EG_{C, z} \), and a

4“Equisingular” means “preserving the (complex) topological type”.

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generic element of such a component can be obtained by smoothing a node of an element of \( EG^*_{C,z} \).

**Lemma 3.1.** For the following real substrata \( V \subset EG^\delta(C,z) - 1 \), the tangent cones \( \overline{T}_0 \mathbb{R}V \) are linear subspaces of \( \mathbb{R}R(C,z) \) of (real) codimension \( \delta(C,z) - 1 = \operatorname{codim} \mathbb{R}EG^\delta(C,z) - 1 \):

(i) \( (C, z) \) contains a real singular local branch \( (C', z) \), and \( V \subset EG^\delta(C,z) - 1 \) is the union of those irreducible components of \( EG^\delta(C,z) - 1 \), which contain nodal curves obtained from the curves \( C_\varphi, \varphi \in \mathbb{E}G^*_{C,z} \), by smoothing out a real node on the component of \( C_\varphi \) corresponding to the local branch \( (C', z) \);

(ii) \( (C, z) \) contains a pair of complex conjugate local branches \( (C', z), (C'', z) \), and \( V \subset EG^\delta(C,z) - 1 \) is the union of those irreducible components of \( EG^\delta(C,z) - 1 \), which contain nodal curves obtained from the curves \( C_\varphi, \varphi \in \mathbb{E}G^*_{C,z} \), by smoothing out a real intersection point on the components of \( C_\varphi \) corresponding to the local branches \( (C', z), (C'', z) \).

**Proof.** (i) Notice, first, that \( V \) can be identified with \( EG^\delta(C',z) - 1 \times EG^\delta(C'',z) - 1 \), where \( (C'', z) \) is the union of the local branches of \( (C, z) \) different from \( (C', z) \). Hence, we can simply assume that \( (C, z) \) is irreducible.

If \( C_\varphi, \varphi \in EG^\delta(C,z) - 1 \), has precisely \( \delta(C, z) - 1 \) nodes as its only singularities, then the tangent space \( T_\varphi EG^\delta(C,z) - 1 \) can be identified with the space of elements \( \psi \in R(C,z) \) vanishing at the nodes of \( C_\varphi \). It has codimension \( \delta(C,z) - 1 \), and we have the following bound for the intersection:

\[
(C_\psi \cdot C_\varphi) \geq 2\delta(C,z) - 2, \quad \psi \in T_\varphi EG^\delta(C,z) - 1.
\]

Hence, any limit of the tangent spaces \( T_\varphi EG^\delta(C,z) - 1 \) as \( \varphi \to 0 \) is contained in the linear space

\[
\{ \psi \in R(C,z) : \operatorname{ord} \psi \mid_{(C,z)} \geq 2\delta(C,z) - 2 \}
\]

of codimension at most \( \delta(C,z) - 1 \). By [1 Proposition 5.8.6] we have

\[
\{ \psi \in R(C,z) : \operatorname{ord} \psi \mid_{(C,z)} \geq 2\delta(C,z) - 1 \}
\]

\[
= \{ \psi \in R(C,z) : \operatorname{ord} \psi \mid_{(C,z)} \geq 2\delta(C,z) \} = J^\operatorname{cond}_{C,z}/m^N_z.
\]
Hence
\[
\text{codim}\{\psi \in R(C, z) : \text{ord } \psi_{|(C, z)} \geq 2\delta(C, z) - 2\} \\
\geq \text{codim } J^\text{cond}_{C, z} / m_z^N - 1 = \delta(C, z) - 1,
\]
and we are done.

(ii) As in the preceding case, we can assume that \((C, z) = (C', z) \cup (C'', z)\).

The above argument yields that the limits of the tangent spaces \(T_\varphi \mathbb{R}V\) as \(\varphi \in \mathbb{R}V^*\) tends to 0, are contained in the linear subspace
\[
\{\psi \in \mathbb{R}R(C, z) : \text{ord } \psi_{|(C', z)} = \text{ord } \psi_{|(C'', z)} \geq 2\delta(C', z) + (C' \cdot C'')_z - 1\}
\]
which then must be of (real) codimension at most \(\delta(C, z) - 1\). So, it remains to show that the latter codimension equals exactly \(\delta(C, z) - 1\), and we will prove that the complex codimension of the space
\[
\{\psi \in R(C, z) : \text{ord } \psi_{|(C', z)} = \text{ord } \psi_{|(C'', z)} \geq 2\delta(C', z) + (C' \cdot C'')_z - 1\}
\]
is at least \(\delta(C, z) - 1\). Namely, we just impose an extra linear condition and show that the resulting space
\[
\Lambda = \{\psi \in R(C, z) : \text{ord } \psi_{|(C', z)} \geq 2\delta(C', z) + (C' \cdot C'')_z \\
\text{ord } \psi_{|(C'', z)} \geq 2\delta(C'', z) + (C' \cdot C'')_z - 1\}
\]
has codimension \(\geq \delta(C, z)\). Write \(f = f'f''\), where \(f' = 0\) and \(f'' = 0\) are equations of \((C', z)\), \((C'', z)\), respectively. By the Noether’s theorem in the form of [10, Theorem II.2.1.26], any \(\psi \in \Lambda\) can be represented as \(\psi = af' + bf''\), where \(a, b \in R(C, z)\) and
\[
\text{ord } a_{|(C'', z)} \geq 2\delta(C'', z) - 1, \quad \text{ord } b_{|(C', z)} \geq 2\delta(C', z).
\]
Again by [4, Proposition 5.8.6], the former inequality yields
\[
\text{ord } a_{|(C'', z)} \geq 2\delta(C'', z),
\]
which finally implies that \(\Lambda \subset J^\text{cond}_{C, z} / m_z^N\), and hence \(\text{codim } \Lambda \geq \delta(C, z)\).}

**Proposition 3.2.** Let \(V \subset EC_{C, z}^{\delta(C, z) - 1}\) satisfy the hypotheses of one of the cases in Lemma 3.1. Then \(W(C, z, V, L)\) does not depend on the choice of the real affine space \(L\) as in Definition 1.3.
Proof. We closely follow the argument in the proof of Proposition 2.1. The classification of codimension one substrata of $V$ contains the cases (n1)-(n3) as in the proof of proposition 2.1 and one additional case:

(n4) the substratum is $EG_{C,z}$ (i.e., its generic element $\varphi$ has $\delta(C,z)$ nodes).

The analysis of the cases (a)-(c) literally coincides with that in the proof of Proposition 2.1. In case (d), the germ of $RV$ at $\varphi$ consists of $k$ pairwise transversal smooth real germs of codimension $\delta(C,z)-1$ in $\mathbb{R}R(C,z)$, where $k$ is the number of such real nodes $p$ of the curve $C_\varphi$ that the smoothing of $p$ yields an element of $RV$ (depending on $V$ as defined in Lemma 3.1). For any smooth germ $M$ in this union, the intersection of $L(t) \cap M$, $0 < |t - \hat{t}| < \eta$, yields a curve $C_\psi$ whose Welschinger sign depends only on the real nodes of $C_\varphi$ different from $p$, and hence does not depend on $t$. \hfill \Box

By Lemma 1.1(3), the tangent cone $T_0 D_{C,z}$ is a hyperplane. As in the preceding case, this yields

Proposition 3.3. Given a real singularity $(C, z)$, the number $W_{\text{discr}}(C,z,L) := W(C,z,D_{C,z},L)$ does not depend on the choice of a real line $L$.

The proof literally follows the argument in the proof of Propositions 2.1 and 3.2.

4 Singular Welschinger invariants associated with $EC^k_{C,z}$

We start with the equiclassical stratum $EC_{C,z}$, which is the most interesting.

Proposition 4.1. (1) Given a real singularity $(C, z)$, the number $W^{\text{ec}}(C, z, L)$ does not depend on the choice of $L$.

(2) The number $W^{\text{ec}}(C, z)$ is an invariant of a real equisingular deformation class. That is, if $(C_t, z)_{t \in [0,1]}$ is an equisingular family of real singularities, then $W^{\text{ec}}(C_0, z) = W^{\text{ec}}(C_1, z)$.

(3) Let $(C, z) = \bigcup_i (C_i, z)$ be the decomposition of a real singularity $(C, z)$ into irreducible over $\mathbb{R}$ components. Then $W^{\text{ec}}(C, z) = \prod_i W^{\text{ec}}(C_i, z)$.

Proof. Again the proof follows the argument in the proof of Proposition 2.1. So, we accept the initial setting and the notations in the proof of Proposition 2.1. Then we study the wall-crossings that correspond to codimension one substrata in $EC_{C,z}$. If $\varphi \in EC_{C,z}$ is a general element of a codimension one substratum, then
(n1') either $C_{\varphi}$ has $3\delta(C, z) - \kappa(C, z) - 1$ nodes and $\kappa(C, z) - 2\delta(C, z) + 1$ cusps,

(n2') or $C_{\varphi}$ has $3\delta(C, z) - \kappa(C, z) - 2$ nodes, $\kappa(C, z) - 2\delta(C, z)$ cusps, and one tacnode $A_3$,

(n3') or $C_{\varphi}$ has $3\delta(C, z) - \kappa(C, z) - 3$ nodes, $\kappa(C, z) - 2\delta(C, z)$ cusps, and one triple point $D_4$,

(c1') or $C_{\varphi}$ has $3\delta(C, z) - \kappa(C, z) - 1$ nodes, $\kappa(C, z) - 2\delta(C, z) - 1$ cusps, and one singularity $A_4$,

(c2') or $C_{\varphi}$ has $3\delta(C, z) - \kappa(C, z) - 2$ nodes, $\kappa(C, z) - 2\delta(C, z) - 1$ cusps, and one singularity $D_5$,

(c3') or $C_{\varphi}$ has $3\delta(C, z) - \kappa(C, z) - 1$ nodes, $\kappa(C, z) - 2\delta(C, z) - 2$ cusps, and one singularity $E_6$.

First, we notice that the wall-crossings of types (n1'), (n2'), (n3') are completely similar to the wall-crossing (n1), (n2), (n3), respectively, considered in the proof of Proposition 2.1, since they involve only the nodal part of the singularities of degenerating elements of $\mathbb{R}EC_{C, z}^*$. Hence, the constancy of $W_{\text{ec}}(C, z, L(t))$, $|t - t^*| < \eta$, follows in the same way.

Next we explain why (c1'), (c2'), (c3') are the only codimension one strata of $EC_{C, z}$ that involve cusps of the degenerating elements of $\mathbb{R}EC_{C, z}^*$. To this end, we show that, any other collection of singularities of $C_{\varphi}$ can be deformed into $3\delta(C, z) - \kappa(C, z)$ nodes and $\kappa(C, z) - 2\delta(C, z)$ cusps in two successive non-equisingular deformations. By our assumption, at least one of the non-nodal-cuspidal singularities of $C_{\varphi}$ must contain a singular local branch. Thus,

- if $C_{\varphi}$ has at least two non-nodal-cuspidal singularity, we, first, deform one such singularity into nodes and cusps (along its equiclassical deformation), then all other singularities;

- if the non-nodal-cuspidal singularity of $C_{\varphi}$ has at least three local branches (one of which denoted $P$ is singular), we, first, shift away a branch, different from $P$, then equiclassically deform the obtained curve into a nodal-cuspidal one;

- if the non-nodal-cuspidal singularity of $C_{\varphi}$ has two singular branches $P_1, P_2$, we, first, shift $P_2$ so that $P_2$ remains centered at a smooth point of $P_1$, then equiclassically deform the obtained curve into a nodal-cuspidal one;
• if the non-nodal-cuspidal singularity of $C_\varphi$ has two branches, $P_1$ smooth and $P_2$ singular, which is different from an ordinary cusp, then we, first, equiclassically deform the local branch $P_2$ into nodes and (necessarily appearing) cusps, while keeping one cusp centered on $P_1$, then deform the obtained triple singularity into nodes and one cusp;

• if the non-nodal-cuspidal singularity of $C_\varphi$ has two branches, $P_1$ smooth and $P_2$ singular of type $A_2$, which is tangent to $P_1$, then we, first, rotate $P_1$ so that it becomes transversal to $P_2$, then deform the obtained singularity $D_5$ into two nodes and one cusp;

• if the non-nodal-cuspidal singularity of $C_\varphi$ is unibranch either of multiplicity $m \geq 3$ and not of the topological type $y^m + x^{m+1} = 0$, or of multiplicity 2 and not of type $A_4$, then we, first, equigenerically deform this singularity into some nodes and a singularity of topological type $y^m + x^{m+1} = 0$, if $m \geq 3$, or a singularity $A_4$, if $m = 2$ (this can be done by the blow-up construction as in the proof of [1], see also [12, Section 2.1]), then equiclassically deform the obtained curve into a nodal-cuspidal one;

• if the non-nodal-cuspidal singularity of $C_\varphi$ is of the topological type $y^m + x^{m+1} = 0$, $m \geq 4$, then the codimension of the its equisingular stratum in a versal deformation base equals $\frac{m^2 + 3m}{2} - 3$, while the codimension of the equiclassical stratum equal

$$\kappa(\{y^m + x^{m+1} = 0\}) = \frac{m^2 + m}{2} - 1$$

$$= \left(\frac{m^2 + 3m}{2} - 3\right) - (m - 2) \leq \left(\frac{m^2 + 3m}{2} - 3\right) - 2.$$

Now we analyze the wall-crossings of type (c1'), (c2'), and (c3') as described above.

In case (c1'), the miniversal unfolding of an $A_4$ singularity $y^2 = x^5$ is given by the family $y^2 = x^5 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$ with the base $B = \{(a_0, \ldots, a_3) \in (\mathbb{C}^4, 0)\}$, while the equiclassical locus $EC \subset B$ is a curve given by $y^2 = (x - 2t)^3(x + 3t)^2$, $t \in (\mathbb{C}, 0)$. This curve has an ordinary cusp at the origin. The natural projection of the germ of $B(C, z)$ at $C_\varphi$ onto $B$ takes the affine spaces $L(t)$, $|t - t^*| < \eta$, to real three-dimensional affine spaces transversal to the tangent line to $EC$ at the origin. Similarly to the case (n1) in the proof of Proposition [21], in the considered bifurcation, two real intersections with $EC$, one corresponding to a curve with a cusp.
and a hyperbolic node and the other corresponding to a curve with a cusp
and an elliptic node, turns in the wall-crossing into two complex conjugate
intersections, and hence the constancy of \( W_{\text{ec}}(C, z, L(t)), |t - t^*| < \eta, \) follows.

In case \((c2')\), the equiclassical locus in a miniversal deformation base of a
singularity \( D_5 \) given, say, by \( x(y^2 - x^3) = 0 \) is smooth and can be described by
a family \( (x - t)(y^2 - x^3) = 0 \). So, in the considered wall-crossing a real curve
with a cusp and two hyperbolic nodes turns into a curve with a cusp and
two complex conjugate nodes, and hence the constancy of \( W_{\text{ec}}(C, z, L(t)), |t - t^*| < \eta, \) follows.

In case \((c3')\), the equiclassical locus in a miniversal deformation base
of a singularity \( E_6 \) is smooth (cf. \[6, \text{Theorem 27}\]) and one-dimensional.
It is not difficult to show that one half branch of \( \mathbb{R}EC(E_6) \) parameterizes
curves with two real cusps and one hyperbolic node, while the other half
branch parameterizes curves with two complex conjugate cusps and one
elliptic node. Thus, the constancy of \( W_{\text{ec}}(C, z, L(t)), |t - t^*| < \eta, \) follows. \( \square \)

The other loci \( EC^k_{C, z}, 1 \leq k < \kappa(C, z) - 2\delta(C, z), \) may be reducible.
Assume that \((C, z) = (C_1, z) \cup \ldots \cup (C_s, z)\) is the splitting into irreducible
(over \( \mathbb{C} \)) components. Given a partition \( k = (k_1, \ldots, k_s) \) such that
\[
k_1 + \ldots + k_s = k, \quad 0 \leq k_i \leq \kappa(C_i, z) - 2\delta(C_i, z), \quad i = 1, \ldots, s, \tag{3}
\]
we define the substratum \( EC^k_{C, z} \subset EC^k_{C, z}, \) which is the union of those
irreducible components of \( EC^k_{C, z} \) whose generic elements \( \varphi \) are such that
\( C_{i, \varphi} = C_{1, \varphi} \cup \ldots \cup C_{s, \varphi} \) with \( C_{i, \varphi} \in EC^k_{C_i, z}, i = 1, \ldots, s. \)

**Lemma 4.2.** In the above notation, the tangent cone \( \mathcal{T}_0 EC^k_{C, z} \) is a linear
subspace of \( R(C, z) \) of codimension \( k + \delta(C, z) = \text{codim} EC^k_{C, z}. \)

**Proof.** It is sufficient to treat the case of an irreducible singularity \((C, z)\).
Let \( \varphi \) be a generic element of a component of \( EC^k_{C, z}. \) The tangent space
\( T_{\varphi} EC^k_{C, z} \) at \( \varphi \) can be identified with the space
\[
\{ \psi \in R(C, z) : \psi(\text{Sing}(C_{\varphi})) = 0, \quad \text{ord } \psi \big|_P \geq 3 \text{ for each cuspidal local branch } P \},
\]

and hence the limit of each sequence of tangent spaces \( T_{\varphi} EC^k_{C, z} \) as \( \varphi \to 0 \)
is contained in the linear space
\[
\{ \psi \in R(C, z) : \text{ord } \psi \big|_{C, z} \geq 2\delta(C, z) + k \}.
\]
It remains to notice that
\[
\text{codim}\{\psi \in R(C, z) : \text{ord}_{C,z}\psi \geq 2\delta(C, z) + k\} = \delta(C, z) + k .
\]
The latter follows, for instance, from [4, Propositions 5.8.6 and 5.8.7].

As a corollary we obtain

**Proposition 4.3.** Given a real singularity \((C, z)\) splitting into irreducible (over \(\mathbb{C}\)) irreducible components \((C_i, z), i = 1, ..., s\), and a sequence \(k = (k_1, ..., k_s)\) satisfying (3) and an extra condition \(k_i = k_j\) as long as \((C_i, z)\) and \((C_j, z)\) are complex conjugate, the locus \(EC_{C,z}^k\) is real, and the number \(W(C, z, EC_{C,z}^k, L)\) does not depend on the choice of \(L\).

The proof literally coincides with the proof of Proposition 4.1.

**5 Example: singularities of type \(A_n\)**

A complex singularity of type \(A_n\) is analytically isomorphic to the canonical one \(\{y^2 - x^{n+1} = 0\} \subset (\mathbb{C}^2, 0)\), and its miniversal deformation can be chosen to be

\[
\left\{ y^2 - x^{n+1} - \sum_{i=0}^{n-1} a_i x^i = 0 \right\}_{a_0, ..., a_{n-1} \in (\mathbb{C}, 0)}
\]

with the base \(B(A_n) = \{(a_0, ..., a_{n-1}) \in (\mathbb{C}^n, 0)\}\).

**Lemma 5.1.** (1) For any \(n \geq 1\), and \(1 \leq i \leq \delta(A_n) = \left\lceil \frac{n+1}{2} \right\rceil\),

\[
\hat{T}_0 EG^i_{A_n} = \{a_0 = ... = a_{i-1} = 0\} \subset B(A_n)
\]

the linear subspace of codimension \(i = \text{codim } EG^i_{A_n}\).

(2) If \(n\) is odd, then \(EC_{A_n} = EG_{A_n}\). If \(n\) is even, then

\[
\hat{T}_0 EC_{A_n} = \{a_0 = ... = a_k = 0\} \subset B(A_n)
\]

**Proof.** Let \((C, z)\) be a canonical singularity of type \(A_n\). The tangent space to \(EG^i_{C,z}\) at a generic element \(\varphi\) consists of \(\psi \in B(C, z)\) such that \(C_{\psi}\) passes through all \(i\) nodes of \(C_{\varphi}\), and hence, \((C_{\psi} \cdot C_{\varphi})_{D(C,z)} \geq 2i\). It follows that the limit of any sequence of these tangent spaces as \(\varphi \to 0\) is contained in the linear space \(\{\psi \in B(C, z) : (C_{\psi} \cdot C)_z \geq 2i\}\), which one can easily identify with the space in the right-hand side of (4). So, the first claim of the lemma follows for the dimension reason. The same argument settles the second claim.

\(\square\)
Proposition 5.2. For any $n \geq 1$ and $k \geq 1$, we have
\[
\text{mult } E_{A_n}^i = \binom{n+1-i}{i}, \quad \text{for all } i = 1, ..., \delta(A_n) = \left\lceil \frac{n+1}{2} \right\rceil,
\]
and \[
\text{mult } E(C_{2k}) = k.
\]

Remark 5.3. The multiplicities mult $E_{A_n}^i$ were computed in [15, Section 5, page 540]. Here we provide another, more explicit computation, which will be used below for computing singular Welschinger invariants.

Proof. (1) If $n+1 = 2i$, then $E_{A_n}^i = EC(A_n) = EC(A_n)$ is smooth; hence, the multiplicity equals 1. Thus, suppose that $n+1 > 2i$. By Lemma 5.1(1), the question on mult $E_{A_n}^i$ reduces to the following one: How many polynomials $P(x)$ of degree $\leq i - 1$ satisfy the condition
\[
x^{n+1} + x^i + P(x) = Q(x)^2 R(x),
\]
where $Q, R$ are monic polynomials of degree $i$, $n+1 - 2i$, respectively?

Combining relation (5) with its derivative, we obtain
\[(n+1-i)x^i + ((n+1)P - xP') = ((n+1)QR - 2xQ'R - xQR') Q,
\]
which immediately yields
\[(n+1)QR - 2xQ'R - xQR' = n+1 - i.
\]
Substituting
\[Q(x) = x^i + \sum_{j=1}^{i} \alpha_j x^{i-j}, \quad R(x) = x^{n+1-2i} + \sum_{j=1}^{n+1-2i} \beta_j x^{n+1-2i-j}
\]
into (6), we obtain that the terms of the top degree $n+1-i$ cancel out, while the coefficients of $x^m$, $m = 0, ..., n-i$, yield the system of equations
\[
\begin{cases}
2\alpha_1 + \beta_1 = 0, \\
2j\alpha_j + j\beta_j + \sum_{0<m<j} c_{jm} \alpha_{j-m} \beta_m = 0, \quad j = 2, ..., n-i, \\
(n+1)\alpha_i \beta_{n+1-2i} = n+1 - i,
\end{cases}
\]
where we assume $\alpha_j = 0$ as $j > i$ and $\beta_j = 0$ as $j > n+1-2i$.

Suppose that $\deg Q = i \geq \deg R = n+1-2i$. From the $(n+1-2i)$ first equations in (7) we express $\beta_j$ as a polynomial in $\alpha_1, ..., \alpha_i$ of homogeneity degree $j$, while $\alpha_m$ has weight $m$, for all $j = 1, ..., n+1-2i$. Substituting
these expressions into the other equations, we obtain a system of \(i\) equations in \(\alpha_1, \ldots, \alpha_i\) of homogeneity degrees \(n + 2 - 2i, \ldots, n + 1 - i\), respectively. Thus, (cf. the computation in [8 Section G, Example 1]) the number of solutions (counted with multiplicities) appears to be

\[
\frac{(n + 2 - 2i) \cdot \ldots \cdot (n + 1 - i)}{i!} = \binom{n + 1 - i}{i}
\]
as required. In the same way we treat the case when \(\deg Q = i \leq \deg R = n + 1 - 2i\).

(2) For \(n = 2k\), by Lemma 5.1(2), the question on mult \(EC(A_{2k})\) reduces to the following one: How many polynomials \(P(x)\) of degree \(k\) satisfy the condition

\[
x^{2k+1} + x^{k+1} + P(x) = Q(x)^2(x + \beta)^3,
\]
where \(Q(x)\) is a monic polynomial of degree \(k - 1\)?

The preceding argument subsequently gives an equation

\[(2k + 1)(x + \beta)Q - 3xQ - 2Q'(x + \beta) = k\]
with \(Q(x) = x^{k-1} + \sum_{j=1}^{k-1} \alpha_j x^{k-1-j}\), which develops into the system

\[
\begin{align*}
2\alpha_1 + 3\beta &= 0, \\
(2j + 2)\alpha_j + (2j + 3)\alpha_{j-1} \beta &= 0, & j = 2, \ldots, k - 1, \\
(2k + 1)\alpha_{k-1} \beta &= k + 1,
\end{align*}
\]

admitting a simplification of the form

\[
\alpha_j = \nu_j \beta^j, \quad j = 1, \ldots, k - 1, \quad (2k + 1)\nu_{k-1} \beta^k = k + 1
\]
with some \(\nu_1, \ldots, \nu_{k-1} \in \mathbb{Q}\). So, we finally obtain \(k\) solutions as required. \(\square\)

Now we pass to the real setting. The complex singularity of type \(A_n\) has a unique real form \(y^2 = x^{2k+1}\) if \(n = 2k\), and has two real forms \(y^2 = x^{2k}\) and \(y^2 = -x^{2k}\) (denoted by \(A_{2k-1}^h\) and \(A_{2k-1}^e\), respectively) if \(n = 2k - 1\).

**Lemma 5.4.** (1) For all \(k \geq 1\) and \(i = 1, \ldots, k\), there exist singular Welschinger invariants

\[
W\left(A_{2k-1}^h, EG_{A_{2k-1}}^i\right), \ W\left(A_{2k-1}^e, EG_{A_{2k-1}}^i\right), \text{ and } W\left(A_{2k}, EG_{A_{2k-1}}^i\right). \ (10)
\]

(2) Furthermore,

\[
W^{eg}(A_{2k-1}^e) = (-1)^k, \quad W^{eg}(A_{2k-1}^h) = 1
\]
\[ W^{eg}(A_{2k}) = \begin{cases} 0, & k \equiv 1 \mod 2, \\ 1, & k \equiv 0 \mod 2, \end{cases} \]

\[ W^{ec}(A_{2k}) = \begin{cases} 0, & k \equiv 0 \mod 2, \\ 1, & k \equiv 1 \mod 2. \end{cases} \]

**Proof.** The existence of the invariants \((10)\) follows from Lemma \(5.1\) and the argument used in the proof of Propositions \(2.1\) and \(3.2\).

Since \(\text{mult} \ EG(A_{2k-1}) = 1\), we have \(W^{eg} = \pm 1\) for \(A^{h}_{2k-1}\) and \(A^{e}_{2k-1}\). More precisely, an equigeneric nodal deformation of \(A^{h}_{2k-1}\) has the form \(y^2 - Q(x)^2 = 0, \deg Q = k\), and hence it has only hyperbolic real nodes, i.e., \(W^{eg}(A^{h}_{2k-1}) = 1\), while an equigeneric nodal deformation of \(A^{e}_{2k-1}\) has the form \(y^2 + Q(x)^2 = 0, \deg Q = k\), and hence it has only elliptic real nodes, whose number is of the same parity as \(k\), i.e., \(W^{eg}(A^{e}_{2k-1}) = (-1)^k\).

Consider singularities \(A_{2k}\). For \(EG(A_{2k}) = EG^k_{A_{2k}}\), system \((7)\) takes the form

\[
\begin{aligned}
2\alpha_1 + \beta_1 &= 0, \\
2j\alpha_j + (2j - 1)\alpha_{j-1}\beta_1 &= 0, \quad j = 2, \ldots, k, \\
(2k + 1)\alpha_k\beta_1 &= k + 1,
\end{aligned}
\]

which yields

\[
\alpha_j = \lambda_j\beta_1^j, \quad (-1)^j\lambda_j > 0, \quad j = 1, \ldots, k, \quad \lambda_k\beta_1^{k+1} = \frac{k + 1}{2k + 1}.
\]

So, if \(k\) is odd, we have no real solutions, and hence \(W^{eg}(A_{2k}) = 0\). If \(k\) is even, than we have a unique real solution such that \(\beta_1 > 0\) and \((-1)^j\alpha_j > 0\). That is, \(Q(x)\) has only positive real roots (if any), and hence the curve \(y^2 - (x + \beta_1)Q(x)^2 = 0\) has only hyperbolic real nodes, i.e., \(W^{eg}(A_{2k}) = 1\).

In the same manner we analyze system \((9)\) and obtain the values of \(W^{ec}(A_{2k})\) as stated in the lemma. \(\square\)

**Remark 5.5.** (1) The problem of computation of the invariants \(W^{eg}\) and \(W^{ec}\) for arbitrary real singularities (even for quasihomogeneous singularities) remains widely open. A possible relation to enumerative invariants of (global) plane algebraic curves could be a key to this problem.

(2) The values of \(W^{eg}\) and \(W^{ec}\) for \(A_n\)-singularities are 0 or \(\pm 1\). The same can be showed for other simple singularities. Is it true for an arbitrary real singularity?
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