Improved Examples of Non-Termination for Ruppert’s Algorithm

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Abstract

Improving the best known examples, two planar straight-line graphs which cause the non-termination of Ruppert’s algorithm for a minimum angle threshold as low as $\alpha \geq 29.06^\circ$ are given.

Introduction

Given a planar straight-line graph (PSLG), Ruppert’s algorithm [8] produces a conforming Delaunay triangulation satisfying a minimum angle bound. The standard analysis [8, 10] demonstrates that when the input contains no angles smaller than 60$^\circ$ Ruppert’s algorithm produces a size-optimal mesh (up to a constant factor) for any minimum angle bound $\alpha \leq 20.7^\circ$. A more detailed analysis can slightly improve this restriction to $\alpha \leq 22.2^\circ$ for non-acute input [6] and an additional (very mild) assumption further improves the guarantee to 26.5$^\circ$ [3].

In practice, Ruppert’s algorithm succeeds for substantially larger minimum angle bounds than those guaranteed by the theory. Ruppert observed that the minimum angle reaches 30$^\circ$ during typical runs of the algorithm [8]. In a number of experiments Shewchuk found the algorithm to terminate for $\alpha \leq 33.8^\circ$ [9]. Certain modifications of the vertex insertion procedure further improve the algorithm suggesting that this constraint can be improved to possibly 40$^\circ$ or more [1].

Pav demonstrated an example of non-termination of Ruppert’s algorithm on a simple non-acute PSLG for any $\alpha > 30^\circ$ [4]. This example combined with the fact that the analysis of Ruppert’s algorithm on point sets breaks down at 30$^\circ$ led to a natural conjecture that Ruppert’s algorithm terminates and produces a well-graded mesh for all $\alpha < 30^\circ$. While some examples corroborated this idea [6], a recent example shows non-termination for $\alpha$ as low as $\approx 29.51^\circ$ [7].

We give two examples that improve upon those given in [7]. The first example produces non-termination of Ruppert’s algorithm for $\alpha \geq 29.10^\circ$ and has a minimum input angle of about 87$^\circ$. The second example gives a slight improvement, $\alpha \geq 29.06^\circ$ but requires a 60$^\circ$ input angle.

First, some notation is defined. The line-segment between endpoints $v_1$ and $v_2$ is denoted $\overrightarrow{v_1v_2}$ and the triangle with vertices $v_1$, $v_2$, and $v_3$ is $\triangle v_1 v_2 v_3$. Let $\angle v_0 v_1 v_2$ be the angle at vertex $v_0$ between line-segments $\overrightarrow{v_0v_1}$ and $\overrightarrow{v_0v_2}$.

Example 1

This example PSLG consists of five input vertices and three adjacent input segments. By carefully constructing the input (as described below) Ruppert’s algorithm inserts a circumcenter, followed by three consecutive circumcenters that yield to midpoints of the segments. The result is a configuration that is similar to the input but exactly half the size and thus Ruppert’s algorithm can repeat this cycle indefinitely. Depicted in

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The following construction describes how to place the input vertices \( v_0, \ldots, v_4 \) to achieve non-termination for the smallest possible angle threshold. We begin by fixing \( \gamma_1 \) which serves as the smallest angle for a number of skinny triangles in the example. Then the placement of the vertices is determined by the following steps; recall Figure 1.

1. Begin with a segment between two vertices \( v_0 \) and \( v_1 \).
2. Let \( m_3 \) be the midpoint of \( v_0 v_1 \).
3. Place \( v_4 \) such that \( \angle v_4 v_1 v_0 = 90^\circ \) and \( \angle v_1 v_4 v_0 = \gamma_1 \).
4. Let \( c_1 \) denote the circumcenter of \( \triangle v_0 v_1 v_4 \).
5. Select \( v_2 \) such that the circumcenter of \( \triangle v_0 v_2 c_1 \), denoted \( c_2 \), lies on the diametral ball of \( v_0 v_2 \).
6. Let \( m_1 \) denote the midpoint of \( v_0 v_2 \).
7. Select \( m_2 \) so that the circumcenter of \( \triangle v_0 v_1 m_2 \), denoted \( c_4 \), lies on the diametral ball of \( v_0 v_1 \) and the opposite side of \( v_0 v_1 \) as \( c_1 \).
8. Define \( v_3 \) so that \( m_2 \) is the midpoint of \( v_0 v_3 \).
9. Let \( c_3 \) denote the circumcenter of \( \triangle v_0 v_3 m_1 \).
When $\alpha > \gamma_1$, the construction ensures that triangles $\Delta v_0v_1v_4$ and (following the insertion of $c_1$) $\Delta v_0v_2c_1$ are split by Ruppert’s algorithm. Moreover, if $m_2$ is inserted, then $\Delta v_0v_1m_2$ is skinny. Also, by construction, if/when $c_2$ and $c_4$ are inserted, they encroach $\overline{v_0v_2}$ and $\overline{v_0v_4}$, respectively, resulting in the insertion of $m_1$ and $m_3$.

For non-termination to occur, $\Delta v_0v_3m_1$ must also be skinny and then its circumcenter $c_3$ must encroach $\overline{v_0v_3}$. When $\gamma_1 \in [25, 30]$, $c_3$ always encroaches $\overline{v_0v_3}$ and this encroachment is not ‘sharp’ unlike the encroachment of $\overline{v_0v_4}$ and $\overline{v_0v_2}$ which is constructed to occur on the boundary of the diametral ball.

Letting $\gamma_2 := \angle v_0v_3m_1$, non-termination occurs for $\alpha > \max(\gamma_1, \gamma_2)$. Thus, the best example will minimize $\max(\gamma_1, \gamma_2)$. Since $\gamma_2$ is a function of $\gamma_1$ (i.e., given $\gamma_1$, performing the construction yields $\gamma_2$), this optimization need only be performed over a one-parameter family. Numerically we find that $\gamma_1 = \gamma_2 \approx 29.10^\circ$ produces the smallest required threshold.

Remarks

- The PSLG input constructed is slightly acute: $\angle v_2v_0v_3 \approx 87.3^\circ$. Restricting the construction to ensure that all input angles are larger than $90^\circ$ yields an example of non-termination for $\alpha > 30^\circ$, i.e., no better than Pav’s original example.

- There appears to be some ‘slack’ in the construction since $c_3$ lies well inside the diametral ball of $\overline{v_0v_3}$. However, there is no perturbation of $v_2$ or $v_3$ that yields a valid encroachment sequence and improves on the needed angle threshold.

- Removing symmetry is an important part of the construction. The original example [6] contained four similar configurations between adjacent segments and the first improvement [7] broke the symmetry leaving two similar constructions. This second improved example goes one step further by eliminating all symmetry.

- The requirement $\angle v_0v_1v_4 = 90^\circ$ is essential to ensure that $c_1$ lies in the correct location for subsequent iterations.

Example 2

As noted in the previous remark, there is little flexibility in the position of the unconnected vertex $v_4$. If segment $\overline{v_0v_4}$ is included in the input, some flexibility is gained since the midpoint (denoted $m_0$) creates a similar configuration for the next cycle of the algorithm even if $\angle v_0v_1v_4 \neq 90^\circ$. Essentially the same sequence of vertex insertions is caused by Ruppert’s algorithm except now $v_1$ encroaches segment $\overline{v_0v_4}$; see Figure 2. Then, $c_2$, $m_1$, etc. are inserted in a similar fashion where $m_0$ replaces $c_1$ in the sequence. The new construction and requirements on $\alpha$ are now described.

Construction

The construction is nearly identical to Example 1. In addition to adding the segment $\overline{v_0v_4}$ to the input, the placement of $v_4$ is slightly different. Steps 3 and 4 of the construction in Example 1 are replaced with the following steps.

3*. Place $v_4$ such that $\angle v_4v_0v_1 = 60^\circ$ and $\angle v_1v_4v_0 = \gamma_1$.

4*. Let $m_0$ denote the midpoint of segment $\overline{v_0v_4}$.

The later steps of the construction in Example 1 are all identical with a single exception: midpoint $m_0$ replaces circumcenter $c_1$ in all cases.

As in the first example, $\gamma_1$ can be selected to match the smallest angle $\gamma_2$ of the final skinny triangle (recall $\gamma_2 = \angle v_0v_3m_1$). The result is an input for which Ruppert’s algorithm does not terminate for all $\alpha \gtrapprox 29.06^\circ$. 
Figure 2: Example 2. The black vertices and segments comprise the input. Light blue vertices are circum-centers of skinny triangles which are rejected for encroaching segments. The red vertices are midpoints of encroached segments which are inserted by the algorithm. (left) The sequence of vertices considered for insertion by Ruppert’s algorithm with the relevant skinny triangles and encroached diametral circles. (right) Labels for each of the vertices.

Remarks

- Example 2 more clearly identifies the challenge in utilizing the ‘slack’ in positioning c₃ to improve the example. Such modification seeks to increase ∠v₂v₀v₃ at the expense of ∠v₁v₀v₄ which would violate our restriction that input angles are larger than 60°.

- Relaxing the 60° restriction on input angles to 51° (which corresponds to the best known counterexample [4]) would improve our result to α ≥ 28.46°. Allowing a single 45° input angle (the value at which trivial alternating midpoint insertion can occur) pushes the bound to α ≥ 28.00°.

Conclusion

Examples of the non-termination of Ruppert’s algorithm serve an important role in the development of a sharp analysis of the algorithm. The PSLG inputs we have constructed improve the best known examples of non-termination due to the size of the minimum angle threshold selected.

If a refined analysis of Ruppert’s algorithm is going to yield guarantees that more closely resemble its behavior in practice, these counterexamples demonstrate why additional mild assumptions on the input or the algorithm must be considered. Non-acute input (rather than admitting 60° input angles), groomed input so that adjacent segments have equal length, restrictions on queue ordering, and non-circumcenter Steiner vertices are all possible candidates which have been used to improve Delaunay refinement in theory and/or practice; e.g., [3, 11, 5, 2, 6]. Without explicitly utilizing any of these modifications, any extension of the analysis must be limited by the α ≈ 29.06° example.

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