Coded Computation Against Distributed Straggling Channel Decoders in the Cloud for Gaussian Uplink Channels

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Abstract—The uplink of a Cloud Radio Access Network (C-RAN) architecture is studied, where decoding at the cloud takes place at distributed decoding processors. To mitigate the impact of straggling decoders in the cloud, the cloud re-encodes the received frames via a linear code before distributing them to the decoding processors. Focusing on Gaussian channels, and assuming the use of lattice codes at the users, in this paper the maximum user rate is derived such that all the servers can reliably recover the linear combinations of the messages corresponding to the employed linear code at the cloud. Furthermore, two analytical upper bounds on the frame error rate (FER) as a function of the decoding latency are developed, in order to quantify the performance of the cloud’s linear code in terms of the tradeoff between FER and decoding latency at the cloud.

I. INTRODUCTION

A Cloud Radio Access Network (C-RAN) architecture can leverage network function virtualization (NFV) in order to implement baseband functionalities on commercial off-the-shelf (COTS) hardware, such as general purpose servers. An important challenge of this solution is to ensure a prescribed latency performance despite the variability of the servers’ runtimes [1].

The problem of straggling processors, that is, processors lagging behind in the execution of a certain function, has been widely studied in the context of distributed computing [2]. [1] demonstrates the effectiveness of decomposing tasks in parallel runnable small jobs over a distributed computing architecture in terms of latency while avoiding overhead.

For distributed computing, it has been recently shown in [3], [4] that parallel processing can be improved by carrying out linear precoding of the data prior to processing, as long as the function to be computed is linear. The key idea is that, by employing a proper linear block code over fractions of size $1/K$ of the original data, a function may be completed as soon as a number of $K$ or more processors have finalized their operation, irrespective of their identity.

The NFV-based C-RAN model considered in this paper is illustrated by Fig. 1. The packets sent by a user in the uplink are received by the remote radio head (RRH) through an additive white Gaussian noise (AWGN) channel and forwarded to a cloud over a RRH-to-cloud link. Decoding is carried out on a distributed architecture consisting of COTS servers $1, \ldots, N$.

We investigate the use of linear coding on the received packets as a means to improve over parallel processing in order to mitigate the impact of straggling decoders at the cloud. The idea was first studied in [5], [6] where the packets are received by the RRH via a binary symmetric channel (BSC). In this paper, we tackle the problem of extending the design and analysis to Gaussian channels.

With Gaussian channels, the model at hand is similar to the compute-and-forward (C&F) problem [7] emerging in Gaussian relay networks. In this problem, the relays attempt to linearly combine received packets as a means to improve over parallel processing in order to mitigate the impact of straggling decoders at the cloud. The main difference is that in the C&F transmitted signals are mixed by the channel, while in our model linear combining is applied at the cloud. Accordingly, in the NFV scenarios, the linearly combined received packets contain an accumulated noise term (i.e., $\tilde{y}_i = \sum_{j=1}^{K} a_{ij} (x_j + z_j)$), while this is not the case in C&F setting (i.e., $\tilde{y}_i = \sum_{j=1}^{K} a_{ij} x_j + z_j$).

The accumulated noise terms (i.e., $\sum_{j=1}^{K} a_{ij} z_j$) affect the functions of the servers in terms of the following two aspects. First, noise powers are accumulated, which leads to a variation on the decoding error probability of each individual server compared to the C&F problem. Second, the common terms in $\sum_{j=1}^{K} a_{ij} z_j$ make the noise terms seen by the servers in general dependent.

To account for the first aspect, we derive the computation rate that guarantees correct decoding for each server in Sec. III. As for the second aspect, we analyze the dependency among the servers by using the dependency graph of the linear NFV code as introduced in [5]. Then, we derive two analytical upper bounds on the frame error rate (FER) as a function of the decoding latency. The bounds on FER depend on the properties of both the channel coding adopted by the user and the linear NFV code applied at the cloud.

Notation: Let $+$, $\sum$ and $\oplus$, $\bigoplus$ denote addition and summation over reals and finite fields, respectively. Let $\| h \| \triangleq$
\[
\sqrt{\sum_{i=1}^{N} |h_i|^2}
\]

denote the norm of a vector \( h \). \([K]\) denotes the set \( \{1, 2, \cdots, L\} \). All logarithms are of base two. Let \( \log^+(x) \triangleq \max(\log(x), 0) \). \([F]\) denote cardinality of \( \mathcal{F} \).

## II. PROBLEM STATEMENT

### A. System Model

As illustrated in Fig. 1, we focus on the uplink of a C-RAN system with a multi-server cloud decoder connected to an RRH via a dedicated fronthaul link. As detailed next, the model follows reference [5], but it considers the more realistic AWGN channel for the user-RRH link, requiring a redesign of the operation at the cloud.

The user encodes a file \( u \) of length \( L \) over a finite field \( \mathbb{F}_p \) for uplink transmission, where \( p > 0 \) is a prime in \( \mathbb{Z} \). Each symbol is drawn independently and uniformly over the finite field. Before encoding, the file is divided into \( K \) blocks \( u_1, u_2, \ldots, u_K \) of equal length \( k \triangleq L/K \) symbols. The user’s encoder, \( \mathcal{E} : \mathbb{F}_p^k \to \mathbb{R}^n \), then maps each length-\( k \) block to a length-\( n \) real valued codeword, \( x_j = \mathcal{E}(u_j) \). The encoder is subject to the power constraint \( \mathbb{E}([|x_j|^2]) \leq nP \). The transmission rate \( R \) of the user is the length of its message normalized by the number of channel uses, i.e., \( R = k/n \log p \).

At the output of the user-RRH AWGN channel, the length-\( n \) received packet for the \( j \)-th block at the RRH is given as

\[
y_j = x_j + z_j,
\]

where \( z_j \) is a vector of i.i.d. Gaussian random variables with zero-mean and variance \( N_0 \). For convenience, we define the signal-to-noise ratio (SNR) as \( \text{SNR} \triangleq P/N_0 \). The \( K \) packets \((y_1, y_2, \ldots, y_K)\) are transmitted by the RRH to the cloud over a fronthaul link. Decoding is carried out at the cloud.

To this end, the cloud consists of \( N \) available servers, namely, Server 1, \ldots, \( N \), and a master server, i.e., Server 0. Each server can decode a packet within a random time \( T_i = T_{1,i} + T_{2,i} \), where times \( \{T_{1,1}, \ldots, T_{N,N}\} \) are mutually independent. Time \( T_{1,i} \) accounts the unavailability of the processor, and is independent of the workload, while \( T_{2,i} \) models the execution runtime and it grows as the size \( n \) of the packet. The variable \( T_{1,i} \) follows an exponential distribution with mean \( 1/\mu_1 \), while \( T_{2,i} \) is a shifted exponential with shift equal to \( a \geq 0 \) and average equal to \( a + 1/\mu_2 \times n \) so that \( 1/\mu_2 \) is the time required for an input symbol. The probability that a given set of \( l \) out of \( N \) servers has finished decoding by time \( t \) is given as \( \Pr(t, l) = F(t)^l (1 - F(t))^{N-l} \), where \( F(t) \) is the cumulative distribution function of \( T_i \).

In order to mitigate the effect of decoding straggling, we adapt the NFV coding scheme in [5] to the AWGN channel. NFV coding operates as follows. The \( K \) packets are first linearly encoded by Server 0 into \( N \geq K \) coded blocks of the same length \( n \), as depicted in Fig. 1. The reason for this partitioning is that each block is forwarded to a different server in the cloud for decoding. For linear coding, consider an \((N, K)\) linear code \( C_c \) with \( K \times N \) generator matrix \( G_c \in g(\mathbb{F}_{p'})^{N \times K} \), where \( p' > 0 \) is a prime and \( g(\cdot) \) is the natural map from \( \mathbb{F}_p \) to the integers \( \{0, 1, 2, \ldots, p' - 1\} \). Note that the prime \( p' \) may be different from the prime \( p \) used to define the user code. Accordingly, the encoded packets are obtained as

\[
\tilde{Y} = YG_c,
\]

where \( Y = [y_1, y_2, \ldots, y_K] \) is a \( n \times K \) matrix, and \( Y = [\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_N] \) is a \( n \times N \) matrix. From (1), the encoded packet \( \tilde{y}_i \) can be written as

\[
\tilde{y}_i = \sum_{j=1}^{K} y_j g_{c,ji} = \sum_{j=1}^{K} x_j g_{c,ji} + \sum_{j=1}^{K} z_j g_{c,ji},
\]

where \( g_{c,ji} \) is the \((j,i)\) entry of matrix \( G_c \).

Each server \( i \in [N] \) aims at decoding a linear combination of the messages

\[
\hat{u}_i = \hat{c}_i \hat{G}_c u_j,
\]

where \( \hat{g}_{c,ji} = g^{-1}([g_{c,ji}] \bmod p) \) are coefficients taking values in \( \mathbb{F}_p \). To this end, Server \( i \) is equipped with a decoder, \( D_i : \mathbb{R}^n \to \mathbb{F}_p \), that maps the observed output \( \hat{y}_i \) to an estimate \( \hat{u}_i = D_i(\hat{y}_i) \) of the equation \( \hat{u}_i \).

Let \( d_{\text{min}} \) be the minimum distance of the NFV code \( C_c \). Server 0 is able to decode the message \( u_i \) or equivalently the \( K \) packets \( u_j \) for \( j \in [K] \), as soon as \( N - d_{\text{min}} + 1 \) servers
have decoded successfully. The output $\hat{u}_i(t)$ at the $i$th Server at time $t$ is $\hat{u}_i(t) = \tilde{u}_i$, if $T_i \leq t$; and $\hat{u}_i(t) = \emptyset$, otherwise. The output $\hat{u}(t)$ of the decoder at Server $0$ at time $t$ is a function of $\hat{u}_i(t)$ for $i \in [N]$. The frame error rate (FER) at time $t$ is defined as

$$P_{\text{FER}}(t) = \Pr(\hat{u}(t) \neq u).$$

(5)

III. Analytical Bounds on the FER

In this section we study the trade-off between the decoding latency and the decoding error probability, by deriving an upper bound on the FER $P_{\text{FER}}(t)$ in (5).

Each Server $i$ with $i \in [N]$ outputs the correct equation $\tilde{u}_i$ by time $t$ if: (i) the server completes decoding at time $t$, and (ii) the decoder can correctly decode despite the noise caused by the AWGN channel. We define the indicator variables $C_i(t) = 1 \{T_i \leq t\}$ and $D_i(t) = 1 \{\tilde{u}_i = \hat{u}_i\}$, which equal $1$ if the above two events occur, respectively, and zero otherwise. Recalling that an error occurs at time $t$ if the number of servers that have successfully decoded by time $t$ is smaller than $N - d_{\text{min}} + 1$. With these definitions, the FER is given by

$$P_{\text{FER}}(t) = \Pr\left(\sum_{i=1}^{N} C_i(t) D_i(t) \leq N - d_{\text{min}}\right).$$

(6)

The variables $C_i(t)$ are independent Bernoulli random variables across the servers $i \in [N]$, due to the independence among the decoding times $\{T_i\}_{i \in [N]}$. However, the variables $D_i(t)$ are dependent Bernoulli random variables, since there may exist common terms among the noise terms $\sum_{j=1}^{R} z_j g_{c,ji}$ in (3) at the decoders. The dependency of variables $D_i(t)$ is accounted for when deriving the upper bound on the FER shown in Sec. III-B.

In order to compute an upper bound on the FER, we first evaluate the computation rate, which gives the maximum rate for each Server $i$ to decode the desired equation $\tilde{u}_i$ with average probability of error approaching zero. Based on this auxiliary result, we then employ the error exponent given in [8, Theorems 8-11] to characterize the upper bounds on the decoding error probability of each Server $i$ under a given coefficient vector $g_{c,i}$ and a given SNR. Finally, we give two upper bounds on the FER by taking account the combined impact from the dependence of $D_i(t)$ and the accumulated noise.

A. Computation Rate

In order to allow servers to decode the desired equations in a manner similar to C&F, we assume that the user adopts a nested lattice code. In this subsection, we derive conditions on the NFV code that enable the servers to decode the desired equations.

To proceed, the following definitions are useful. An $n$-dimensional lattice is a discrete subgroup of $\mathbb{R}^n$ which can be described by

$$\Lambda = \{ \lambda = Bz : z \in \mathbb{Z}^n \},$$

(7)

where $B$ is the full rank generator matrix. The Voronoi region $\mathcal{V}$ of a lattice $\Lambda$ is

$$\mathcal{V} \triangleq \{ z : Q_\Lambda(z) = 0 \},$$

(8)

where $Q_\Lambda(z) \triangleq \min_{\lambda \in \Lambda} \| z - \lambda \|$. Let $\text{Vol}(\mathcal{V})$ denote the volume of $\mathcal{V}$ and $\text{Vol}(\mathcal{V}) = |\det(B)|$. The second moment of a lattice $\Lambda$ is defined as

$$\sigma^2 \Lambda \triangleq \frac{1}{n\text{Vol}(\mathcal{V})} \int_{\mathcal{V}} \|z\|^2 \, dz,$$

(9)

and the normalized second moment (NSM) is defined as

$$G(\Lambda) \triangleq \frac{\sigma^2 \Lambda}{(\text{Vol}(\mathcal{V}))^{2/n}}.$$  

(10)

A lattice $\Lambda$ is said to be nested in a lattice $\Lambda_f$ if $\Lambda \subseteq \Lambda_f$. Refer $\Lambda_f$ as the fine lattice and $\Lambda$ as the coarse lattice.

The following theorem provides a condition on the transmission rate $R$ that guarantees reliable decoding of given equations at the servers.

Theorem 1. For a given NFV code matrix $G_c$ and $n$ large enough, there exists a nested lattice code $\Lambda \subseteq \Lambda_f$ with rate $R$, such that for all coefficient vectors $g_{c,1}, g_{c,2}, \ldots, g_{c,N} \in g(\mathbb{F}_p)^K$, any Server $i \in [N]$ can recover the linear combination of messages $\tilde{u}_i$ given in (4) with average probability of error $\epsilon$ as long as the inequality

$$R < \min_{i : g_{c,i} \neq 0} \frac{1}{2} \log^+ \left( \frac{P}{\| g_{c,i}\|^2 N_0 (\alpha_i^2 + \text{SNR} (\alpha_i - 1)^2)} \right)$$

holds for some choice of parameters $\alpha_1, \ldots, \alpha_N \in \mathbb{R}^n$.

Proof: See Appendix A.

Based on Theorem 1, we define the computation rate for each Server $i$ as

$$R^* (g_{c,i}) = \max_{\alpha_i \in \mathbb{R}} \frac{1}{2} \log^+ \left( \frac{P}{\| g_{c,i}\|^2 N_0 (\alpha_i^2 + \text{SNR} (\alpha_i - 1)^2)} \right).$$

(12)

By Theorem 1 this is the rate that guarantees correct decoding at Server $i$.

Theorem 2. The computation rate (12) is uniquely maximized by choosing $\alpha_i$ to be the minimum mean square error (MMSE) coefficient $\alpha_{\text{MMSE}} = \frac{\text{SNR}}{1 + \text{SNR}}$ which results in a computation rate of

$$R^* (g_{c,i}) = \frac{1}{2} \log^+ \left( \frac{1 + \text{SNR}}{\| g_{c,i}\|^2} \right).$$

(13)

Proof: See Appendix B.

Remark 3. The computation rate from Theorem 2 is zero if the coefficient vector $g_{c,i}$ satisfies $\| g_{c,i}\|^2 \geq 1 + \text{SNR}$.
B. Upper Bounds on the FER

In order to analyze the FER, we need to first evaluate the decoding error probability for each Server $i$, for $i \in [N]$, as a function of the vector $g_{c,i}$ defined by the NFV code.

To this end, define the gap to the computation rate as
\[
\Delta = \frac{1}{2} \log^+ \left( 1 + \frac{1}{\|g_{c,i}\|^2} \right) - R, \tag{14}
\]
and let $\mu \triangleq 2^{2\Delta}$. Assuming maximum likelihood (ML) decoding, an upper bound on the decoding error probability is given by $P_e^{ML}(g_{c,i})$ \cite{5} Theorems 8-11, where

\[
P_e^{ML}(g_{c,i}) \triangleq \begin{cases} 
\frac{e^{-nE_r(\mu)} e^{1/2\mu}}{2\pi n^e}, & \mu > 2 \\
\frac{e^{-nE_r(\mu)} e^{1/2\mu}}{8\pi n^e}, & \mu = 2 \\
\frac{e^{-nE_r(\mu)} (e/n)^{1/2}}{2(\mu-1)}, & 2 > \mu > 1, \\
0, & \mu \leq 1.
\end{cases} \tag{15}
\]

where $a \triangleq b$ indicates that $\frac{a}{b} \to 1$, and $E_r(\cdot)$ is the Poltyrev random coding exponent defined as \cite{9}

\[
E_r(\mu) = \begin{cases} 
\frac{1}{2} \left[ \ln (\mu) + \ln (e/4) \right], & \mu \geq 2 \\
\frac{1}{2} \left[ \mu - 1 - \ln (\mu) \right], & 2 \geq \mu \geq 1 \\
0, & \mu \leq 1.
\end{cases} \tag{16}
\]

Based on the bound (15), we now provide an upper bound on the FER by leveraging the approach introduced in \cite{5}. Accordingly, we use the notion of the dependence graph and its chromatic number for the NFV code to characterize the dependence of the correct decoding indications $D_i$.

The dependence graph $G(G_c) = (V, \mathcal{E})$ comprises a set $V$ of $N$ vertices and a set $\mathcal{E} \subseteq V \times V$ of edges, where the edge $(i,j) \in \mathcal{E}$ is included if both the $i$th and $j$th columns of $G_c$ have at least a non-zero term in the same row. Each vertex of $G(G_c)$ represents a decoding server, and an edge indicates that the noise terms in $G_c$ for the two servers are correlated. The chromatic number $\chi(G_c)$ of $G(G_c)$ is the smallest number of colors needed to color the vertices of $G(G_c)$, such that no two adjacent vertices share the same color. We then give a large deviation bound (LDB) on the FER.

**Theorem 4.** \cite{5} *Theorem 1*] Let $P_e^{min} = \min\{P_e^{ML}(g_{c,i})\}_{i=1}^N$ according to (15). Then, for all $t \geq a - \frac{1}{\mu} \ln \left( \frac{d_{min} \sum_{i=1}^N P_e^{ML}(g_{c,i})}{N - \sum_{i=1}^N P_e^{ML}(g_{c,i})} \right)$, the FER is upper bounded as

\[
P_e^{FER}(t) \leq \exp \left( -\frac{S(t)}{b(\varphi)} \right) \varphi \left( \frac{4b(\varphi)(NF(t) - F(t) \sum_{i=1}^N P_e^{ML}(g_{c,i}) - N + d_{min})}{5S(t)} \right), \tag{17}
\]

where $b(t) \triangleq F(t)(1 - P_e^{min})$, $S(t) \triangleq \sum_{i=1}^N F(t)(1 - P_e^{ML}(g_{c,i}))(1 - F(t)(1 - P_e^{ML}(g_{c,i})))$, and $\varphi(x) \triangleq (1 + x) \ln (1 + x) - x$.

This upper bound captures the dependency of the FER caused by the NFV code, and also the error probability $P_e^{ML}(g_{c,i})$ depending on both the channel code and the NFV code. The following gives a union bound (UB) that is tighter and valid for all times $t$.

**Theorem 5.** \cite{5} *Theorem 2*] For any subset $A \subseteq [N]$, define $P_e^{min}(A) \triangleq \min\{P_e^{ML}(g_{c,i})\}_{i \in A}$ and $P_e^A \triangleq \sum_{i \in A} P_e^{ML}(g_{c,i})$, and let $G_A$ be the $K \times |A|$, submatrix of $G_c$, with column indices in the subset $A$. Then, the FER is upper bounded by

\[
P_e^{FER}(t) \leq 1 - \sum_{l=N-d_{min}+1}^N \Pr(l,t) \sum_{A \subseteq [N]:|A|=l} \left( 1 - \exp \left( -\frac{S_A}{b_A \chi(G_A)} \varphi \left( \frac{4b_A(1 - P_e^A)}{5S_A} \right) \right) \right),
\]

where $S_A(t) \triangleq \sum_{i \in A} P_e^{ML}(g_{c,i})(1 - P_e^{ML}(g_{c,i}))$ and $b_A \triangleq 1 - P_e^{min}(A)$.

IV. NUMERICAL RESULTS

In this section, we provide some numerical results to obtain insights into the performance of NFV codes based on the FER bounds presented in the previous section, in terms of the trade-offs between decoding latency and FER. We employ a frame length of $L = 504$ and $N = 8$ servers. The user code is selected to be binary (i.e., $p = 2$) with rate $R = 0.5$.

We set $\mu_1 = 50$, $\mu_2 = 10$, and $a = 1$. Unless stated, otherwise, we have $p' = p = 2$. Furthermore, we leave the performance comparison with simulated results based on specific user lattice codes to future work (See \cite{5} for the case of binary symmetric channels).

We compare the performance of the following solutions:

(i) Single-server (SS) decoding, where there is a single server...
in the cloud that decodes the entire frame \((K = 1)\), so that we have \(n = 1008\) and \(\mathcal{X}(G_c) = d_{\text{min}} = 1\); (ii) Repetition coding (RPT), where the entire frame is duplicated at all servers, so that we have \(n = 1008\) and \(\mathcal{X}(G_c) = d_{\text{min}} = 8\); (iii) Parallel processing (PRL), where the frame is divided into \(K = N\) disjoint parts processed by different servers in parallel, and hence we have \(n = 126\) and \(\mathcal{X}(G_c) = d_{\text{min}} = 2\); and (iv) Single parity check code (SPC), with \(K = 7\), where one server decodes a sum of all other \(K\) received packets, and hence we have \(n = 144\) and \(\mathcal{X}(G_c) = d_{\text{min}} = 2\); and (v) an NFV code \(C\), with generator matrix \(G_c\) defined in [5 Eq. (8)] which is characterized by \(K = 4\), \(n = 252\) and \(\mathcal{X}(G_c) = d_{\text{min}} = 3\).

In order to elaborate on the optimal computation rate in Theorem 2, Figure 2 shows the LDB and UB for three parallel coding schemes with generator matrices \(G_c = I^{N \times N}\), \(3G_c\), and \(5G_c\). Note that all these parallel codes have the same minimum Hamming distance \(d_{\text{min}} = 1\) and the same chromatic number \(\chi(G_c) = 1\), since the positions of all the non-zeros elements are the same. However, they take entries from different field sizes, e.g., \(p' = 2, 5, 7\). Figure 2 confirms the main result in Theorem 2 that, under the same SNR, the NFV codes with larger norms on the column vectors of the generator matrix entail a larger equivalent noise for the server to decode the message equations, causing a larger error floor, and accordingly, a worse trade-off between latency and FER.

Figure 3 shows both LDB and UB for all the five schemes under SNR = 7 dB. As first observation, Figure 3 confirms that UB is tighter than the LDB, and we note that leveraging multiple servers for decoding yields a better trade-off between latency and FER.

Figure 3 shows that, according to the derived upper bounds, the NFV code \(C\) provides the smallest FER for a sufficiently small latency level, improving over all schemes including parallel processing. The latter scheme is in fact very sensitive to the unavailability of the servers, requiring all servers to completely decoding, and hence it needs a longer latency in order to achieve a low FER. As for the SPC scheme, although it has an extra parity-check server as compared to parallel processing, its performance is limited by the large equivalent noise determined by its coding matrix. We emphasize that these conclusions are drawn based solely on the derived upper bound, but simulation results for practical codes are expected to show a similar behavior (see [12]).

V. CONCLUSION

In this work, we have extended the idea of coding to improve the robustness of uplink channel decoding in the cloud over AWGN channels. Explicit calculations on the computation rate are provided to quantify the impact on the accumulated noise terms caused by linear coding over the received packets. Taking account the dependency among servers and the equivalent noise for each server, we have derived upper bounds on the FER depending on both the channel coding and the NFV coding, and evaluate the trade-offs between FER and decoding latency under various coding schemes. As future work, we mention here the optimized design of NFV codes as a function of the field size.

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APPENDIX A

The user’s encoder \(E\) maps its finite field message vector \(u_t\) to a lattice point \(t_j \in A_f \cap V\), using the function \(\phi\) from
Lemma 5], i.e., \( t_j = \phi(u_j) \). In order to recover \( \tilde{u}_i \), each server needs to decode the lattice equation

\[
\mathbf{v}_i = \left[ \sum_{j=1}^{K} t_j g_{c,ji} \right] \mod \Lambda
\]

of the lattice points \( t_j \) for \( j \in [K] \).

Dither vectors \( d_j \) are generated independently by a uniform distribution over the Voronoi region \( \mathcal{V} \) of the coarse lattice \( \Lambda \). All dither vectors are available at the servers. The user transmits

\[
x_j = [t_j - d_j] \mod \Lambda.
\]

By [7, Lemma 7], the vector \( x_j \) is uniform over \( \mathcal{V} \), so we have the equality \( \mathbb{E}[\|x_j\|^2] = nP \), where the expectation is over all dithers. Furthermore, it is argued in [7] that there exist fixed dithers that meet the power constraint \( \|x_j\|^2 \leq nP \).

The input of server \( i \in [N] \) is given by (3). Each server computes

\[
s_i = \alpha_i \tilde{v}_i + \sum_{j=1}^{K} d_j g_{c,ji}.
\]

Let \( Q_f \) denote the lattice quantizer for the fine lattice \( \Lambda_f \). To obtain an estimation of the lattice equation \( \mathbf{v}_i \), this vector is quantized onto \( \Lambda_f \) modulo the coarse lattice \( \Lambda \).

\[
\hat{v}_i = [Q_f(s_i)] \mod \Lambda
\]

The following sequence of qualities shows that \( [s_i] \mod \Lambda \) is equivalent to \( \mathbf{v}_i \) with some added noise terms.

\[
\begin{align*}
[s_i] & \mod \Lambda \\
&= \left[ \sum_{j=1}^{K} g_{c,ji} ([t_j - d_j] \mod \Lambda + d_j) + \sum_{j=1}^{K} g_{c,ji} ((\alpha_i - 1)x_j + \alpha_i z_j) \right] \mod \Lambda \\
&= \left[ \sum_{j=1}^{K} g_{c,ji} t_j + \sum_{j=1}^{K} g_{c,ji} ((\alpha_i - 1)x_j + \alpha_i z_j) \right] \mod \Lambda \\
&= \left[ \mathbf{v}_i + \sum_{j=1}^{K} g_{c,ji} ((\alpha_i - 1)x_j + \alpha_i z_j) \right] \mod \Lambda.
\end{align*}
\]

By [7, Lemma 7], the pair \( (\mathbf{v}_i, \hat{v}_i) \) has the same joint distribution as the pair \( (\mathbf{v}_i, \tilde{v}_i) \), where \( \tilde{v}_i \) is defined as

\[
\tilde{v}_i \triangleq [Q_f(\mathbf{v}_i + z_{eq,i})] \mod \Lambda,
\]

where

\[
z_{eq,i} \triangleq \sum_{j=1}^{K} g_{c,ji} ((\alpha_i - 1)x_j + \alpha_i z_j),
\]

and \( x_j \) is drawn independently and uniformly distributed over \( \mathcal{V} \). By [7, Lemma 8], the density of \( z_{eq,i} \) can be upper bounded by an i.i.d. zero-mean Gaussian vector \( \mathbf{z}_i^* \) whose variance \( \sigma_{eq,i}^2 \) approaches

\[
N_{eq,i} = \|g_{c,i}\|^2 N_0 \left( \alpha_i^2 + \text{SNR} (\alpha_i - 1)^2 \right),
\]

as \( n \to \infty \).

The probability of error \( \Pr(\hat{v}_i \neq \mathbf{v}_i) \) is thus equal to the probability that the equivalent noise leaves the Voronoi region surrounding the codeword, \( \Pr(\mathbf{z}_{eq,i} \notin \mathcal{V}) \). Also, we design the fine lattice such that \( \Lambda_f \) satisfies AWGN-goodness [9], which requires that \( \epsilon_i = \Pr(\mathbf{z}_i^* \notin \mathcal{V}) \) goes to zero exponentially in \( n \) as long as the volume-to-noise ratio is such that

\[
\mu(\Lambda_f, \epsilon_i) \triangleq \frac{\text{Vol}(\mathcal{V}_f)^{2/n}}{\sigma_{eq,i}^2} > 2\pi e.
\]

Under this condition, \( \epsilon_i = \Pr(\mathbf{z}_{eq,i} \notin \mathcal{V}) \) also goes to zero exponentially in \( n \). By the union bound, the average probability of error \( \epsilon \) is upper bounded by \( \epsilon \leq \sum_{i=1}^{N} N_{eq,i} \). To ensure that \( \epsilon_i \) goes to zero for all desired equations, \( \mathcal{V}_f \) must satisfy (26) for all servers with \( g_{c,ji} \neq 0 \). We set \( \mathcal{V}_f \) small enough, we can approach the computation rate as close as we desired.

As a result, the servers can make estimates \( \hat{v}_i \) of lattice equations \( \mathbf{v}_i \) with coefficient vectors \( g_{c,1}, g_{c,2}, \ldots, g_{c,N} \in g(\mathbb{F}_p)^K \) such that \( \Pr(\hat{v}_i \neq \mathbf{v}_i) < \epsilon \) for \( \epsilon > 0 \) and large \( n \) enough as long as

\[
R < \min_{g_{c,ji} \neq 0} \frac{1}{2} \log^+ \left( \frac{P}{N_{eq,i} \epsilon} - \log (1 + \delta) \right).
\]

Therefore, by choosing \( \delta \) small enough, we can approach the computation rate as close as we desired.
Let \( f(\alpha_i) \) denote the denominator of the computation rate \([12]\). Since it is quadratic in \( \alpha_i \), it can be uniquely minimized by setting its first derivative to zero.

\[
f(\alpha_i) = \alpha_i^2 + \text{SNR} (\alpha_i - 1)^2
\]

\[
\frac{df}{d\alpha_i} = 2\alpha_i + 2\text{SNR} (\alpha_i - 1) = 0
\]

\( \alpha_{\text{MMSE}} = \frac{\text{SNR}}{1 + \text{SNR}} \) \hspace{1cm} (32)

We plug \( \alpha_{\text{MMSE}} \) back into \( f(\alpha_i) \) and substituting this into \( \log^+ \left( \frac{P}{\|g_{c,i}\|^2 N_0 f(\alpha_i)} \right) \) yields the desired computation rate.