Meaning of Non-Extensive Entropies in Micro-Macro Duality

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Abstract. The purpose of this presentation is to clarify the universal meaning of generalized non-extensive entropies and/or statistics based on Tsallis entropy from the viewpoint of “Micro-Macro duality”. What plays here the most important roles is the notions of scales and scaling transformations which will be seen to be related closely to the main themes of this workshop, generalized non-extensive entropies and/or statistics and the information-geometrical structures of the classifying spaces.

1. Non-additive Statistics & Micro-Macro Duality
Since the importance of non-additive statistics does not seem to be fully recognized yet in the science community, I originally intended to clarify in my talk the universal meaning of non-additive statistics based on Tsallis entropy [1, 2], from my own viewpoint of “Micro-Macro duality” [3]. After listening to many excellent lectures in this workshop given by the famous experts in the field including Prof. Tsallis himself, however, I have realized that it is redundant to (re-)interpret its essence from some general viewpoints. I have, therefore, decided to shift the focus of my talk into a challenge to reformulate the Tsallis entropy as a solution to an inverse problem, which is one of the typical questions arising from the context of Micro-Macro duality. When I gave my talk at the workshop in July 2009, however, my understanding was not sufficient at the most important points in the problem of specifying the context for the Tsallis entropy and its relative version to be uniquely determined. While it is not completely solved yet, there has been good progress after the workshop, which will be reported at the end of this paper.

In this context the points to be discussed are as follows:

1) The essence of “statistics” in general found in the relations between “unknown” & “known” regions in the world:
While such identifications as unknown ≃ microscopic & known ≃ macroscopic need not be fixed (as opposite connections possible), we can gain a deep insight from analyzing their mutual relations. The general scheme based on “Micro-Macro duality” seems to be well suited for this purpose [⇒ Sec.2 & 4].

2) In this context, scales and scale transformations play essential roles in moving from one side to another:
This is also one of the most important aspects treated in “Micro-Macro duality” [⇒ Sec.5].

3) The distinction and mutual relations between Micro and Macro, and between quantum and classical, can be systematically formulated and treated in the above scheme, via the method to construct a composite system to unify both extreme ends into one [⇒ Sec.3]. This machinery
seems to be useful for understanding the relations between usual statistics with extensivity and non-extensive one.

4) Important roles of “inverse problem” [⇒ Sec.4 & 5]: because of the bi-directionality inherent in duality, many of seemingly one-directional processes can be inverted, as seen below. I shall try to put Tsallis entropy in the context.

2. Micro-Macro Duality & “Duheme-Quine Paradox”
To discuss the above problems, we explain briefly the essence of Micro-Macro duality. It provides a general framework for treating unknown systems and phenomena on the basis of “dualities” between unknown and/or invisible worlds to be described (symbolically called “Micro”) and known and/or visible ones to serve as reference frames (symbolically “Macro”) in the description:

Basic ingredients of our quadrality scheme [4] for description, consist of:

- **Dyn** = dynamics (= time development) of object system,
- **Alg** = algebra of observables to characterize the object system,
- **Rep.’s & States** = states and representations of **Alg** to describe configurations of the system via measured values,
- **Spec** = classifying space to parametrize the classified structures of the object system and phenomena.

1) Basic duality relation of [Alg ⇔ Reps] is due to the “duality between objects and their attributes”:

    [an object = family of attributes] ⇔ [an attribute= objects sharing that attribute = a set (in mathematics)]

2) The above duality relations themselves change chronologically, in contrast to invariant quantities

   ⇒ Duality of [Dyn ⇔ Spec] is based on the “duality between changes and invariants”

3) “5W1H” nature inherent in all the theoretical descriptions of phenomena:

   **Spacetime localization**
   = “when” & “where”

   **Algebra** =
   “what” (& “who”)

   **Dynamics**
   “how”

   **States** = “how”

   “why”
With suitable re-interpretations, many examples of the above scheme can be found in the basic structures of physical (and statistical) theories.

A No-Go theorem, “Duhem-Quine paradox”, however, prevents us from determining uniquely a theory to reproduce phenomenological data, owing to inevitable finiteness in number and resolutions of measured quantities.

implies This universal dilemma is overcome in our quadrality scheme in such a way that the very origin of the difficulty leads to a solution of the problem! by the necessity to restrict the aspects to be focused and the degrees of exactitude, a certain context is always (pre-)chosen, to within which a theoretical explanation can be unificied by “Micro-Macro duality” as a context-dependent matching condition [4] between the phenomena to be described and the theory to describe; its mathematical formulation entitles “Macro” as standard reference system satisfying “universality”.

3. Quantum-Classical Composite System

To explain the above, we define the notion of a sector = (thermodynamic) pure phase, parametrized by an order parameter:

Mathematically, a sector \( \pi \) of algebra \( \mathcal{A} \) of observables, as a minimal unit of representations classified by quasi-equivalence (= unitary equivalence up to multiplicities) and characterized by trivial centre \( \pi(\mathcal{A})'' \cap \pi(\mathcal{A})' =: 3_\pi(\mathcal{A}) = \mathbb{C} \) (i.e., central observables commuting with all other observables are only scalars \( \in \mathbb{C} \)), where the symbol \( B' \) for any subset \( B \) of \( \mathcal{A} \) means the commutant \( B' := \{ T ; TB = BT \text{ for } \forall B \in B \} \) of \( B \). It can be shown that any two factor representations \( \pi_1 \) and \( \pi_2 \) are either mutually quasi-equivalent or disjoint in the sense that any intertwiner \( T \) between \( \pi_1 \) and \( \pi_2 \), \( T \pi_1(A) = \pi_2(A)T \) for \( \forall A \in \mathcal{A} \), is vanishing: \( T = 0 \).

Remark: In quantum mechanics (QM) with finite degrees of freedom, we have only one sector(!)...(*), as all the pure states/representations are unitarily equivalent according to Dirac theory of unitary transformations based on Stone-von Neumann uniqueness theorem. This means that the usual QM with finite degrees of freedom is destined, from the beginning, to fail in explaining the mutual relations between quantum and classical levels in nature.

The case opposite to this one-sector (= pure phase) situation is a mixed phase \( \pi = \bigoplus \pi_\gamma \) consisting of many sectors \( \pi_\gamma \) (= macroscopically different configurations of \( \gamma \in Sp(3_\pi(\mathcal{A})) \)) “Micro” parametrized by order parameters as spectrum \( \gamma \in Sp(3_\pi(\mathcal{A})) \) of the non-trivial centre \( 3_\pi(\mathcal{A}) \) (= Macro), which is simultaneously diagonalized owing to its commutativity.

Thus, we encounter in this situation a Quantum-Classical Composite System consisting of both intra-sectorial quantum aspects within each sector and inter-sectorial classical ones extending over different sectors:

| inter-sector | \( \gamma_1 \) | \( \gamma_2 \) | visible = classical | \( \gamma_N \) | \( \cdots \) |
| intra-sector | \( \pi_\gamma \) | \( \pi_\gamma \) | \( \pi_\gamma \) | \( \pi_\gamma \) | \( \cdots \) |
| inside sectors | ↑ | ↑ | ↑ | ↑ | ↑ |
| invisible quantum | \( \pi_{\gamma_1} \) | \( \pi_{\gamma_2} \) | \( \pi_{\gamma} \) | \( \pi_{\gamma_N} \) | \( \cdots \) |
| \( \pi_{\gamma_1} \) | \( \pi_{\gamma_2} \) | \( \pi_{\gamma} \) | \( \pi_{\gamma_N} \) | \( \cdots \) |
| \( \pi_{\gamma_1} \) | \( \pi_{\gamma_2} \) | \( \pi_{\gamma} \) | \( \pi_{\gamma_N} \) | \( \cdots \) |
| \( \pi_{\gamma_1} \) | \( \pi_{\gamma_2} \) | \( \pi_{\gamma} \) | \( \pi_{\gamma_N} \) | \( \cdots \) |
| \( \pi_{\gamma_1} \) | \( \pi_{\gamma_2} \) | \( \pi_{\gamma} \) | \( \pi_{\gamma_N} \) | \( \cdots \) |

It is well known that Heisenberg had once complained of the absence of a universal constant with dimensionality of length in quantum theory in relation with the ultraviolet divergences in
quantum field theory. Owing to this absence we cannot specify a fixed boundary (i.e., the so-called “Heisenberg cut”) between quantum and classical, but, once a constant of this sort were present, such an absolute limitation would have been implied that the physical world smaller than certain length scale can never be visible owing to the absolute boundary between visible classical and invisible quantum worlds. In the above formulation based on the notion of sectors, we can, fortunately enough, benefit from two kinds of freedoms, one to specify a temporary and conditional boundary between classical and quantum regimes and the other to shift this boundary according to the relative configurations of length scales of factors relevant to the observed situation. Such a “conditional” boundary can clearly be defined mathematically in a specified context of restricted degrees of accuracy, as follows. Actually, this is in sharp contrast to the traditional approach where the absence (*) of inter-sectorial classical level in QM inevitably forces us to treat separately either purely quantum or purely classical situations, neglecting mutual connections and/or transitions between them! This lack of theoretical formulation always needs be made up by heuristic arguments, which may often be wrong as shown below.

By the above definition of sectors, any observables \( A \) of a quantum-classical composite system in a mixed phase \( \pi = \bigoplus_{\gamma \in S_p(3 \gamma(A))} \pi_\gamma \) with many disjoint sectors \( \pi_\gamma \) have the following block-diagonal structure:

\[
\pi(A) = \begin{pmatrix}
\pi_{\gamma_1}(A) & 0 & \cdots & 0 \\
0 & \pi_{\gamma_2}(A) & 0 & \vdots \\
\vdots & 0 & \pi_{\gamma_3}(A) & 0 \\
0 & \cdots & 0 & \ddots
\end{pmatrix},
\]

where all the off-diagonal terms can be shown to be 0 according to the mutual disjointness of \( \pi_\gamma \)'s. Therefore, any expectation value \( \langle \psi | A \pi \rangle \) in a vector state \( \psi = c_1 \psi_1 \oplus c_1 \psi_2 \oplus \cdots \in \mathcal{H}_\pi \) reduces automatically to a mixed state:

\[
\langle \psi | \pi(A) | \psi \rangle = |c_1|^2 \langle \psi_1 | \pi_{\gamma_1}(A) \psi_1 \rangle + |c_2|^2 \langle \psi_2 | \pi_{\gamma_2}(A) \psi_2 \rangle + \cdots = Tr_{\rho_\psi} \pi(A)
\]

with a density operator \( \rho_\psi \) given by:

\[
\rho_\psi = |c_1|^2 |\psi_1\rangle \langle \psi_1| + |c_2|^2 |\psi_2\rangle \langle \psi_2| + \cdots.
\]

Thus, such a common belief in QM is wrong that any vector state given as a superposition \( c_1 \psi_1 + c_2 \psi_2 + \cdots \) is a pure state showing quantum interference effects. From this viewpoint, the famous paradox of Schrödinger’s cat is merely an ill-posed question, based on the level confusions about quantum-classical boundaries. Namely, because of the absence of such a physical observable \( A \) as \( \langle \psi_{\text{dead}} | A \psi_{\text{alive}} \rangle \neq 0 \), the actual transition from the cat’s being alive to dead can take place, not at the micro-level of the Geiger counter, but by macroscopic accumulation of infinitely many microscopic processes! This last point can be understood by such quantum-classical correspondence that classical Macro level consisting of order parameters to describe inter-sectorial structure emerges from microscopic levels through condensation of infinite number of quanta. (As a matter of course, the presence or absence of microscopic observables triggering macroscopic state changes depends highly on the situations and/or aspects in consideration, like the case, for instance, of the visual eyesight controlled by photo-chemical reactions involving the rhodopsin molecules at the retiniae.)

Thus, it is important to understand properly the roles played by a systematic methodology to construct a unified system combining extreme cases into one, where one of the most important advantages of the scheme based on “Micro-Macro duality” can be found: What we attain in a “unified” treatment is not only a simple unification of two (or more) poles but the implementation of bi-directional transitions among them, inherent in the very notion of duality.
4. Fourier-Galois Duality & Bi-directionality ⇒ “Inverse Problem”

The most familiar sort of duality is well known in Fourier duality, which allows one to go back and forth between an abstract group $G$ and its group dual $\hat{G}$ or category $\text{Rep} G$ of (certain class of) representations of $G$. This can be extended to the Fourier-Galois duality as a duality between [a dynamical system consisting of an algebra $\mathcal{F} \rtimes G$ acted on by a group $G$] and [the fixed-point subalgebra $\mathcal{F}^G \rtimes \hat{G}$ with a co-action $\hat{\tau}$ of $G$ (= action of $\hat{G}$ or $\text{Rep} G$)] i.e.,

$$[\mathcal{F} \rtimes G \simeq \mathcal{F}^G] \rtimes \hat{G}$$

Its basic structure fits well to our quadrality scheme:

$$\text{Spec:} \ G \overset{\rtimes}{\longrightarrow} \mathcal{F}^G, \ Rep.$$  

A concrete and important example of this kind of duality can be found in Doplicher-Haag-Roberts (DHR) superselection theory with unbroken symmetry [5, 6]: in the direction $\leftarrow$ from Macro to Micro, the fixed-point subalgebra $A = \mathcal{F}^G$ consisting of $G$-invariant observables can recover the group $G$ of symmetry and the whole dynamical system $\mathcal{F} \rtimes G$, respectively, as the Galois group $G = \text{Gal}(\mathcal{F}/A)$ and the Galois extension $\mathcal{F} = A \bowtie \hat{G}$ of $A$ (= coefficients of the “equation”), when $A$ is combined with phenomenological data on its sector structure parametrized by $\hat{G}$ (= solutions of certain “equation”); this is just the core of Fourier-Galois duality. Its essence lies in the recovery of the whole system through the re-mobilization by $\hat{G}$ of $G$-invariant fixed-points $\mathcal{F}^G$, analogous to $G$ recovered from $\hat{G}$ by its inverse Fourier transform $\sim G$.

What is more important is the guarantee of universality and of uniqueness of the recovered results, which is due to the matching condition [4] to restrict the focus of our interest to the aspects of $G$-dynamical systems, in a consistent way with “Duheme-Quine Paradox”! In the case of the above DHR theory, such a matching condition can be found in the form of DHR selection criterion [5] to pick up all the states $\omega$ carrying localizable charges whose deviations from the vacuum $\omega_0$ are restricted to finitely extended spacetime regions $\mathcal{O}$: $\pi_\omega |_{\mathcal{O}} \equiv \pi_{\omega_0} |_{\mathcal{O}}$, where $\pi_\omega$ and $\pi_{\omega_0}$ are the GNS representation corresponding to $\omega$ and its restriction to the local subalgebras in the causal complement $\mathcal{O}'$ of $\mathcal{O}$. While this theory originally applied only to the unbroken internal symmetries, it has been extended to the situations with spontaneously [4] and explicitly broken symmetries [7] and also to a general scheme for quantum measurements where measurement processes involving transitions from Micro to Macro is shown to be dual to the reconstruction of the original quantum system from measured data [3] in the opposite direction, from Macro to Micro. These examples of duality suggest the possibility to consider such problems as:

- solving equations to obtain solutions ⇐ reconstructing equations from solutions,
- duality between deduction (from Micro theory to Macro data) ⇐ induction (from Macro data to Micro theory), etc., etc.

In the search for a new structure to incorporate the old and standard one as a special case, one usually attempts trial-and-error searches in a heuristic way which seems to be unavoidable. How and to which extent can this be made systematic by the method for solving an “inverse problem”? 
5. Tsallis Entropy & Scale Transformations

What I have so far elaborated on is the merits of examining the mutual relations among different levels of nature on the basis of Micro-Macro duality. Along this line, it would be interesting to examine the “inverse problem” to trace back a path specified by a sequence $S_q$ of Tsallis entropy from the common knowledge of its limit point $S = \lim_{q \to 1} S_q$ as Shannon entropy:

$$S_q = \sum_{i=1}^{k} p_i^q - 1$$

$$S = \sum_{i=1}^{k} p_i \log p_i$$

Together with the corresponding quantum version, the relative entropy or the Kullback-Leibler information, as a quantity naturally related with the entropy, is given on the bottom line for two density matrices $\rho$ and $\sigma$, in terms of which classical as well as quantum systems (with finite degrees of freedom) can be treated on the same footing. In contrast to the axiomatic-type discussions on non-extensive entropies [2], the focus point here is the mutual relation between the standard Shannon entropy $S$ and the non-extensive Tsallis entropy $S_q$ (as well as their relative versions), and the more important problem is in which context the paths $q \to 1$ absorbed into a fixed point $S$ at $q = 1$ can be reversed uniquely into the family of Tsallis entropies $S_q$ parametrized by $q$. We expect that, when this problem is solved, it will give a clear-cut answer to the question, under which condition and in what sense one is necessarily forced to use generalized statistics with Tsallis entropy.

To prepare a setting up for this consideration, we need to clarify what is the variable dual to the parameter $q$ so as to specify the context to validate a duality structure relevant to the problem. For this purpose, we recall the crucial roles played by scale transformations in moving from one scale region to another. We can formulate a scheme where scale invariance can be treated as a broken symmetry accompanied by the so-called power-law behaviours as the polynomial corrections to the vanishing exponent $m \to 0$:

$$\text{Spec} \quad \text{Micro} \quad \text{Macro} \quad \text{unclear} \quad \text{Micro-Macro boundary}$$

$$\exp(-m(-)) \quad \sqrt{\ } \quad m \to 0 \quad \sqrt{\ } \quad \text{power-law corrections}$$

$$\sqrt{\ } \quad N \to 0 \quad \frac{1}{x} \quad \text{inversion}$$

In [7] the (inverse) temperature $\beta$ is shown to appear as order parameter associated with this symmetry breaking. In view of (non-commutative) $L^p$-theory formulated in terms of the
modular operators $\Delta_\beta^{1/p}$ which can roughly be viewed as the Boltzmann factor $\exp(-\beta H/p) = \exp(-H(k_BpT)^{-1})$, this broken scale invariance can be related to the change of $p$ appearing in $L^p$-$L^{p'}$ duality (with $1/p + 1/p' = 1$), and the latter is also related with the information geometry and $\alpha$-divergence through such identification [7] as $1/p = (1-\alpha)/2 = q$ and $1/p' = (1+\alpha)/2 = 1-q$, or equivalently, $\alpha := 1/p - 1/p'(\neq \pm 1)$, $1/p + 1/p' = 1$. In this thermodynamic as well as interpolation-theoretical context, the meaning of Tsallis relative entropy can be naturally understood as the analytic continuation of the standard Shannon type relative entropy $S(\rho)$ given on the real-time axis $\mathbb{R}$ into the so-called KMS strip $D_\beta := \{t + i\tau; t \in \mathbb{R}, 0 \leq \tau \leq \beta\}$ with imaginary time $i\tau$ where any thermal equilibrium states $\omega_\beta$ at the inverse temperature $\beta$ are characterized mathematically by the KMS condition [8]. The formula for an $\alpha$-divergence [9] is given in [7] for non-commutative $L^p$-spaces associated with an arbitrary von Neumann algebra $\mathcal{M}$ [10] by

$$D^{(\alpha)}(T_1||T_2) = pp' \left[ \frac{||T_1||^p_p}{p} + \frac{||T_2||^{p'}_{p'}}{p'} - \text{Re}[T_1, T_2]_{\phi_0} \right],$$

where $[T_1, T_2]_{\phi_0}$ is the pairing between $L^p$ and $L^{p'}$ w.r.t. a faithful normal semifinite weight $\phi_0$ (or, a faithful normal state such as $\omega_\beta$) for $T_1 = u_1 \Delta_\beta^{1/p}, \chi \in L^p(\mathcal{M}, \phi_0)$, $T_2 = u_2 \Delta_\beta^{1/p'}, \chi \in L^{p'}(\mathcal{M}, \phi_0)$, with $u_i$: partial isometries and $\Delta_{\beta, \phi_0}$: relative modular operator from $\phi_0$ to $\phi_i \in \mathcal{M}_i$. This can easily be adapted to the above familiar case of density matrix states $\rho$ and $\sigma$ (typically identified with Gibbs states $\omega_\beta$) by $T_1 = u_1 \Delta_\beta^{1/p}, \chi \rightarrow \rho^{1/p}, T_2 = u_2 \Delta_\beta^{1/p'}, \chi \rightarrow \sigma^{1/p'}, ||T_1||^p_p \rightarrow \phi_0(T_1)||^p_p = Tr((\rho_1||p)) = Tr\rho = 1, ||T_2||^{p'}_{p'} \rightarrow Tr\sigma = 1, Re[T_1, T_2]_{\phi_0} \rightarrow Tr(\rho^{1/p}\sigma^{1/q}) = Tr(\rho^{1/q})$, and hence,

$$D^{(\alpha)}(\rho^{1/p}||\sigma^{1/q}) = pp' \left[ \frac{1}{p} + \frac{1}{p'} - Tr(\rho^{1/p}\sigma^{1/q}) \right] = \frac{1}{q} S_q(\rho||\sigma).$$

If we rewrite the expressions above by $D^{(\alpha)}(\rho^{1/p}||\sigma^{1/q}) \rightarrow D^{(\alpha)}(\rho, \sigma), S_q(\rho||\sigma) \rightarrow K_q(\rho||\sigma)$, we can easily see the agreement between the formulæ here and those in [11]. Note that the GNS-representation space for density matrix states is given by the Hilbert space $\mathcal{H}_{HS}$ consisting of Hilbert-Schmidt class operators with the square root $\rho^{1/2}$ of $\rho$ playing the role of the GNS cyclic vector $[\Omega_\rho]$. While the discussion here is given in terms of density matrices, this is simply for the sake of familiarity and the general essence can be easily extended to the systems with infinite degrees of freedom described by non-type I von Neumann algebras.

Now a relative entropy functional involving observables $A$ and $B$ can be introduced by generalizing Uhlmann’s formulæ [12] into the following form:

$$D^{(\alpha)}(\rho||\sigma)(A, B) = \frac{1}{q(1-q)} \left[ Tr(A^\ast \rho B) - Tr(A^\ast \rho^q B \sigma^{1-q}) \right]$$

$$= \frac{1}{q} \frac{\langle \Omega_\sigma| A^\ast (\Delta_{\rho, \sigma} - \Delta_{\rho, \sigma}^q) B \Omega_\sigma \rangle}{1-q};$$

$$S(\rho||\sigma)(A, B) = \frac{1}{q(1-q)} \left( \langle \Omega_\sigma| A^\ast (\Delta_{\rho, \sigma} - \Delta_{\rho, \sigma}^q) B \Omega_\sigma \rangle \right)$$

$$= \langle \Omega_\sigma| A^\ast \Delta_{\rho, \sigma} \log \Delta_{\rho, \sigma} B \Omega_\sigma \rangle = -\langle \Omega_\rho| B \log(\Delta_{(\rho, \sigma)} A^\ast \Omega_\rho) \rangle.$$

While $Tr(\sigma(-)) = \langle \Omega_\sigma| (-) \Omega_\sigma \rangle = \omega_\sigma(-)$ expresses the procedure to take an expectation value, the appearance of $\rho^q$ or the relative modular operator $\Delta_{\rho, \sigma}$ indicate the roles played by the
so-called “escort distribution”. In view of the appearance of the relative modular operators in various operator-convex functions, it would be convenient to unify all the relevant formulae in the form of the f-divergences or quasi-(relative) entropies [13]. In view of the close relationship among α-divergences, canonical correlations = Bogoliubov–Kubo-Mori inner products (BKM products), transport coefficients given in terms of BKM products [14] and the entropy production [15], the consideration of this sort would be useful not only for various problems related with power-law behaviours and also conformal symmetry encountered in many transient areas in nature, but also for statistical physical problems in the usual situations with valid additivity.

Along the above line of thought, the case of entropy can be treated formally by considering the equality $\rho = \Delta_{\rho,Tr}$ for a density matrix $\rho$ and the corresponding relative modular operator between $\omega_\rho$ and the trace $Tr$ (as a normal faithful semi-finite weight corresponding to $\sigma = 1$ ), which involves, however, the wrong sign due to the lack of normalization for the trace functional:

$$S_q(\rho; A, B) = -\frac{Tr(A^*\rho B) - Tr(A^*\rho^n B)}{1 - q} \bigg|_{A=B\rightarrow 1} - \frac{Tr\rho^n - 1}{q - 1} = S_q(\rho) \bigg|_{q \rightarrow 1} S(\rho).$$

In any case, the above arguments show again that the origin of Tsallis entropy $S_q(\rho)$ can be found mathematically in the $L^p$-interpolation, whose physical essence lies in the analytic continuation of correlation functions into the imaginary time axis for describing the dissipative effects of thermal fluctuations. In short, $S_q(\rho)$ is just a dilation from the Shannon type entropy $S(\rho)$ living in the real world with real time $t$ with vanishing imaginary time $\tau = 1/p = 0$ (corresponding to $L^\infty$) into the dissipative domain $\tau \neq 0$ in the KMS strip $D_\beta$.

Concerning the direction $\leftrightarrow$ itself opposite to the standard arrow $\rightarrow$, the context to be focused on is now clear in relation with the interpolation theory and its physical implications involving inverse temperature or imaginary time $\beta$. To solve this inverse problem in a literal sense, however, we need to proceed one step further. In this connection, it would be helpful to recall some more ingredients typically used in the context of q-deformed mathematics, among which the notion of Jackson derivative seems to be most relevant, as is suggested by Mr. H. Saigo: in the above discussions we see that the controlling parameter $1/p = q$ moves around in the imaginary time interval $[0, 1] \ni 1/p$ of the KMS strip $D_\beta$, the former of which is compact (in the form of $1/p$ but may not be so in the form of $p \in [1, +\infty)$). It is clear that the controlling parameter for solving the inverse problem is dual to this, running over a discrete set, on which the Jackson derivative may act. In this way, many results obtained in the fields involving Tsallis entropy and related objects are expected to shed new lights on the above well-established traditional areas, while the effects acting in the opposite direction may also be beneficial for both sides. Indeed, this kind of bi-directionality is to be emphasized again as the core essence of the approach based on Micro-Macro duality!

Acknowledgments

Let me express my sincere thanks to Profs. H. Suyari, T. Wada and A. Ohara for inviting me to this inspiring workshop and for the continuation of fruitful discussions; I benefited so much from the lectures by and the discussions with Prof. Tsallis, the pioneer of this field, Prof. Topsoe, Prof. Naudts and Prof. Amari, and others. Last but not least, many discussions with Mr. H. Saigo, including his suggestion on the Jackson derivative are gratefully acknowledged.
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