Efficient Quantum Transforms

Peter Høyer*
Odense University†

February 11, 1997

Abstract

Quantum mechanics requires the operation of quantum computers to be unitary, and thus makes it important to have general techniques for developing fast quantum algorithms for computing unitary transforms. A quantum routine for computing a generalized Kronecker product is given. Applications include re-development of the networks for computing the Walsh-Hadamard and the quantum Fourier transform. New networks for two wavelet transforms are given. Quantum computation of Fourier transforms for non-Abelian groups is defined. A slightly relaxed definition is shown to simplify the analysis and the networks that computes the transforms. Efficient networks for computing such transforms for a class of metacyclic groups are introduced. A novel network for computing a Fourier transform for a group used in quantum error-correction is also given.

1 Introduction

The quantum computational version of the discrete Fourier transform is without doubt the most important transform developed for quantum computing so far. It is in the heart of all quantum computational issues discussed until now. All main quantum algorithms, including Shor’s celebrated factoring algorithm [29] and Grover’s searching algorithm [21], use it as a subroutine. All known relativized separation results for quantum computation are based on quantum algorithms that use the discrete Fourier transform [3, 4, 30, 3]. Fundamental

*Supported in part by the ESPRIT Long Term Research Programme of the EU under project number 20244 (ALCOM-IT). Current address: Dépt. IRO, Université de Montréal. Email: hoyer@IRO.UMontreal.CA.
†Department of Mathematics and Computer Science, Odense University, Campusvej 55, DK–5230 Odense M, Denmark. Email: u2pi@imada.ou.dk.
concepts of quantum error-correction rely on it; see for example [9, 10, 18, 31, 32]. It is, in conclusion, the most important single routine for obtaining an insight in the previous work done in quantum computing. Still, this seemingly simple transform is not yet fully understood.

When referring to the discrete Fourier transform, one often does not refer to a single transform, but rather to a family of transforms. For any positive integer \( n \) and any \( n \)-dimensional complex vector space \( V_n \), one defines a discrete Fourier transform \( F_n \) (see for example [33]). More generally, given \( r \) positive integers \( \{n_i\}_{i=1}^r \), and \( r \) complex vector spaces \( \{V_i\}_{i=1}^r \), where \( V_i \) is of dimension \( n_i \), one defines a discrete Fourier transform, denoted \( F_{n_1} \otimes \cdots \otimes F_{n_r} \), for the tensor product space \( V_1 \otimes \cdots \otimes V_r \). (See Section 4 for details.)

Earlier, the quantum versions of the discrete Fourier transforms were defined, but efficient quantum networks where only known for few of them. Currently, efficient networks implementing \( F_n \) exactly have been found for all smooth integers \( n \) [11, 12, 16, 20, 29], where \( n \) is considered smooth if all its prime factors are less than \( \log^c(n) \) for some constant \( c \) [29]. Furthermore, any discrete Fourier transform can be efficiently approximated to any degree of accuracy by some quantum circuit [7, 24].

Common for the networks discussed above are that their description has been taken from the point of view that the transforms were to be implemented by quantum networks. In this paper, we discuss quantum computation not as opposed to classical (perhaps parallelized) computation, but more as a variant. We believe that the fundamental object is the unitary transform, which then can be considered a quantum or a classical (reversible) algorithm. With this point of view, the problem of finding an efficient algorithm implementing a given unitary transform \( U \), reduces to the problem of factorizing \( U \) into a small number of “sparse” unitary transforms such that those sparse transforms should be known to be efficiently implementable. As an example of this, we show that the quantum networks implementing the quantum versions of the discrete Fourier transforms can be very easily derived from the mathematical descriptions of their classical counterparts.

More generally, we consider a new tool for finding quantum networks implementing any given unitary transform \( U \). We show that if \( U \) can be expressed as a certain generalized Kronecker product (defined below) then, given efficient quantum networks implementing each factor in this expression, we also have an efficient quantum network implementing \( U \). The expressive power of the generalized Kronecker product includes several new transforms. Among these are the two wavelet transforms: the Haar transform [22] and Daubechies’ \( D^4 \) transform [14], and we are thus able to devise new quantum networks that compute these transforms.

There exists a group theoretical interpretation of the discrete Fourier transforms which establishes a bijective correspondence between this family of transforms and the set of finite Abelian groups. To further demonstrate the power of the generalized Kronecker products, we use them to give a simple re-development of the quantum Fourier transforms for Abelian
groups. More interestingly, the Fourier transforms can be generalized to arbitrary finite non-Abelian groups (see for example [25] for an introduction), and we give a definition of what it means that a quantum network computes such transforms. Moreover, we give a slightly relaxed definition where we only compute a Fourier transform up to phase factors. Classically, the idea of relating a Fourier transform for a group to one of its subgroups has proven to be very useful [25], and we show that this carries over to quantum computers. We apply these ideas to give new networks for quantum computing Fourier transforms for the quaternionic group and for a class of metacyclic groups.

Since Shor demonstrated that quantum error-correction is possible by given an explicit nine-bit code [28], several new classes of quantum codes have been developed. See for example [9, 10, 18, 31, 32] for some of the many results. Many of these are stabilizer codes, which are subgroups of a certain non-Abelian group $E_n$ (defined in Section 7). For that group, we also give a simple and efficient network for computing a Fourier transform, again using the framework of generalized Kronecker products.

Independently of this work, Robert Beals has found quantum networks implementing Fourier transforms for the symmetric groups [3]. A challenging open question related to that result is whether it can be used to find a polynomial quantum circuit solving the famous graph isomorphism problem.

2 Generalized Kronecker products

All matrices throughout this paper are finite. Matrices are denoted by bold capital letters and tuples of matrices by calligraphic letters. Indices of tuples and row and column indices of matrices and vectors are numbered starting from zero; the $(i, j)$–th element of $A$ is referred to as $a_{ij}$. A single integer as subscript on a unitary matrix denotes its dimension, e.g., we let $I_q$ denote the $(q \times q)$ identity matrix. The transpose of $A$ is denoted by $A^t$. Recall that a square matrix is unitary if it is invertible and its inverse is the complex conjugate of its transpose. The complex conjugate of a number $c$ is denoted by $\overline{c}$.

**Definition 1** Let $A$ be a $(p \times q)$ matrix and $C$ a $(k \times l)$ matrix. The left and right Kronecker product of $A$ and $C$ are the $(pk \times ql)$ matrices

$$
\begin{bmatrix}
A_{c00} & A_{c01} & \ldots & A_{c0,l-1} \\
A_{c10} & A_{c11} & \ldots & A_{c1,l-1} \\
\vdots & \vdots & \ddots & \vdots \\
A_{ck-1,0} & A_{ck-1,1} & \ldots & A_{ck-1,l-1}
\end{bmatrix}
$$

and

$$
\begin{bmatrix}
a_{00}C & a_{01}C & \ldots & a_{0,q-1}C \\
a_{10}C & a_{11}C & \ldots & a_{1,q-1}C \\
\vdots & \vdots & \ddots & \vdots \\
a_{p-1,0}C & a_{p-1,1}C & \ldots & a_{p-1,q-1}C
\end{bmatrix},
$$

respectively.
We denote the left Kronecker product by $A \otimes_L C$ and the right Kronecker product by $A \otimes_R C$. When some property holds for both definitions, we use $A \otimes C$. Note that the Kronecker product is a binary matrix operator as opposed to the tensor product which is a binary operator defined for algebraic structures like modules. The Kronecker product can be generalized in different ways; see for example [17] and [26]. In this paper, we use (an even further generalized version of) the generalized Kronecker product discussed in [17], defined as follows.

**Definition 2** Given two tuples of matrices, a $k$–tuple $A = (A^i)_{i=0}^{k-1}$ of $(p \times q)$ matrices and a $q$–tuple $C = (C^i)_{i=0}^{q-1}$ of $(k \times l)$ matrices, the generalized right Kronecker product is the $(pk \times ql)$ matrix $D = A \otimes_R C$ where

$$d_{ij} = d_{uk+v,xl+y} = a^{u}_{ux} c^{x}_{vy}$$

with $0 \leq u < p$, $0 \leq v < k$, $0 \leq x < q$, and $0 \leq y < l$. □

The generalized right Kronecker product can be found from the standard right Kronecker product by, for each sub-matrix $a_{ux}C$ in Definition 1 substituting it with the following sub-matrix

$$
\begin{bmatrix}
  a^{0}_{ux}c^{x}_{00} & a^{0}_{ux}c^{x}_{01} & \cdots & a^{0}_{ux}c^{x}_{0,l-1} \\
  a^{1}_{ux}c^{x}_{10} & a^{1}_{ux}c^{x}_{11} & \cdots & a^{1}_{ux}c^{x}_{1,l-1} \\
  \vdots & \vdots & \ddots & \vdots \\
  a^{k-1}_{ux}c^{x}_{k-1,0} & a^{k-1}_{ux}c^{x}_{k-1,1} & \cdots & a^{k-1}_{ux}c^{x}_{k-1,l-1}
\end{bmatrix}
$$

The generalized left Kronecker product is the $(pk \times ql)$ matrix $D = A \otimes_L C$ where the $(i,j)$–th entry holds the value

$$d_{ij} = d_{up+v,xq+y} = a^{u}_{vy} c^{y}_{ux}$$

with $0 \leq u < k$, $0 \leq v < p$, $0 \leq x < l$, and $0 \leq y < q$.

As for standard Kronecker products, we let $A \otimes C$ denote either of the two definitions. If the matrices $A^i = A$ are all identical, and also $C^i = C$, the generalized Kronecker product $A \otimes C$ reduces to the standard Kronecker product $A \otimes C$. Denote by $A \otimes C$ the generalized Kronecker product of a $k$–tuple $A$ of $(p \times q)$ matrices, and a $q$–tuple $C$ of identical $(k \times l)$ matrices $C$. Denote $A \otimes C$ similarly.
Example 3

\[
\left( \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \otimes_R \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}
\]

\[
\left( \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \otimes_L \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}.
\]

To analyze generalized Kronecker products we need the shuffle permutation matrix of dimension \((mn \times mn)\), denoted \(\Pi_{mn}\) as shorthand for \(\Pi_{(m,n)}\), defined by

\[
\pi_{rs} = \pi_{dn + e, d'm + e'} = \delta_{de}^d\delta_{e'e}
\]

where \(0 \leq d, e' < m\); \(0 \leq d', e < n\), and \(\delta_{xy}\) denotes the Kronecker delta function which is zero if \(x \neq y\), and one otherwise. It is unitary and satisfies \(\Pi_{mn}^{-1} = \Pi_{mn}' = \Pi_{nm}\).

Given two tuples of matrices, \(k\)-tuple \(\mathcal{A} = (A^i)_{i=0}^{k-1}\) of \((p \times r)\) matrices and \(k\)-tuple \(\mathcal{C} = (C^i)_{i=0}^{k-1}\) of \((r \times q)\) matrices, let \(\mathcal{AC}\) denote the \(k\)-tuple where the \(i\)-th entry is the \((p \times q)\) matrix \(A^iC^i\), \(0 \leq i < k\). For any \(k\)-tuple \(\mathcal{A}\) of matrices, let \(\text{Diag}(\mathcal{A})\) denote the direct sum \(\bigoplus_{i=0}^{k-1} A_i\) of the matrices \(A^0, \ldots, A^{k-1}\). The generalized Kronecker products satisfy the following important Diagonalization Theorem of [17].

**Theorem 4** [Diagonalization Theorem] Let \(\mathcal{A} = (A^i)_{i=0}^{k-1}\) be a \(k\)-tuple of \((p \times q)\) matrices and \(\mathcal{C} = (C^i)_{i=0}^{q-1}\) a \(q\)-tuple of \((k \times l)\) matrices. Then

\[
\mathcal{A} \otimes_R \mathcal{C} = \left( \Pi_{pk} \text{Diag}(\mathcal{A}) \Pi_{kq} \right) \times \text{Diag}(\mathcal{C}) \tag{1}
\]

\[
\mathcal{A} \otimes_L \mathcal{C} = \text{Diag}(\mathcal{A}) \times \left( \Pi_{kq} \text{Diag}(\mathcal{C}) \Pi_{ql} \right). \tag{2}
\]
**Corollary 5** Let $A = (A^i)_{i=0}^{k-1}$ be a $k$–tuple of $(p \times q)$ matrices and $C = (C^i)_{i=0}^{q-1}$ a $q$–tuple of $(k \times l)$ matrices. Then

\[ A \otimes_R C = \Pi_{pk} \left( A \otimes_L C \right) \Pi_{ql} \]

\[ A \otimes_L C = \Pi_{kp} \left( A \otimes_R C \right) \Pi_{ql}. \]

Until now, we have not assumed anything about the dimension of the involved matrices. In the next theorem, we assume that the matrices involved are square matrices. The theorem is easily proven from the Diagonalization Theorem. For any $k$–tuple $A = (A^i)_{i=0}^{k-1}$ of invertible matrices, let $A^{-1}$ denote the $k$–tuple where the $i$–th entry equals the inverse of $A^i$, $0 \leq i < k$.

**Corollary 6** Let $A, C$ be $m$–tuples of $(n \times n)$ matrices, and $D, E$ be $n$–tuples of $(m \times m)$ matrices. Then

\[ AC \otimes DE = \left( A \otimes I_m \right) \times \left( C \otimes D \right) \times \left( I_n \otimes E \right). \] (3)

Furthermore, if the matrices in the tuples $A$ and $C$ are invertible, then

\[ \left( A \otimes_R C \right)^{-1} = \Pi_{nm} \left( C^{-1} \otimes_R A^{-1} \right) \Pi_{mn} = C^{-1} \otimes_L A^{-1} \]

\[ \left( A \otimes_L C \right)^{-1} = \Pi_{nm} \left( C^{-1} \otimes_L A^{-1} \right) \Pi_{mn} = C^{-1} \otimes_R A^{-1} \]

If $A$ and $C$ are unitary, then so is $A \otimes C$.

### 3 Quantum routines

In this section, we give a method for constructing a quantum network for computing any given generalized Kronecker product. A primary application of this method is as a tool to find a quantum network of a given unitary matrix. As two examples, we use it to develop quantum networks for computing two wavelet transforms, the Haar transform and Daubechies’ $D^4$ transform.

As our quantum computing model, we adopt the now widely used quantum gates arrays [1, 5]. Let $\tau : |u, v, v⟩ \mapsto |u, v \oplus v⟩$ denote the two-bit exclusive-or operation, and $U$ the set of all one-bit unitary operations. Following [1], by a **basic operation** we mean either a $U$ operation or the $\tau$ operation. The collection of basic operations is universal for quantum networks in the sense that any finite quantum network can be approximated with arbitrary precision by a quantum network $Q$ consisting only of gates implementing such operations [13, 36, 1].

Define the one-bit unitary operations

\[ X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad Y = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad W = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \]
Given a unitary matrix $C$, let $\Lambda((j, x), (k, C))$ denote the transform where we apply $C$ on the $k$–th register if and only if the $j$–th register equals $x$. Given an $n$–tuple $C = (C_i)_{i=0}^{n-1}$ of unitary matrices, let $\Lambda((j, i), (k, C_i))$ denote $\Lambda((j, n-1), (k, C^{n-1})) \cdots \Lambda((j, 0), (k, C^0))$. Given a $k$–th root of unity, say $\omega$, let $\Phi(\omega)$ denote the unitary transform given by $|u\rangle|v\rangle \mapsto \omega^{uv}|u\rangle|v\rangle$.

If the first register holds a value from $\mathbb{Z}_n$, and the second holds a value from $\mathbb{Z}_m$, then $\Phi(\omega) = \Phi(n,m)(\omega)$ can be implemented in $\Theta(\lceil \log n \rceil \lceil \log m \rceil)$ basic operations [11, 12, 24].

Quantum shuffle transform. For every $m > 1$, let the operation $DM_m$ perform the unitary transform $|k\rangle|0\rangle \mapsto |k \text{ div } m\rangle|k \text{ mod } m\rangle$. Let SWAP denote the unitary transform $|u\rangle|v\rangle \mapsto |v\rangle|u\rangle$. Then $\Pi_{mn}$ can be implemented on a quantum computer by one application of $DM_m$, one swap operation, and one application of $DM_n^{-1}$,

$$\Pi_{mn} \equiv DM_n^{-1} \text{ SWAP } DM_m.$$  

Quantum direct sum. Let $C$ be an $n$–tuple of $(m \times m)$ unitary matrices. It is not difficult to see that $\text{Diag}(C)$ can be implemented as follows,

$$\text{Diag}(C) \equiv DM_m^{-1} \Lambda((1, i), (2, C^i))_i \text{ DM}_m.$$  

Quantum Kronecker product. Let $A$ be an $m$–tuple of $(n \times n)$ unitary matrices and $C$ an $n$–tuple of $(m \times m)$ unitary matrices. By the Diagonalization Theorem, the generalized Kronecker product can be applied by applying two direct sums and two shuffle transforms. Removing cancelling terms we get

$$A \otimes_R C \equiv DM_m^{-1} \Lambda((2, i), (1, A^i))_i \Lambda((1, i), (2, C^i))_i \text{ DM}_m \quad (4)$$

$$A \otimes_L C \equiv DM_n^{-1} \Lambda((1, i), (2, A^i))_i \Lambda((2, i), (1, C^i))_i \text{ DM}_n. \quad (5)$$

Thus, an application of a generalized right Kronecker product can be divided up into the following four steps: in the first step, we apply $DM_m$. In the second step, we apply the controlled $C^i$ transforms on the second register, and in the third step, the controlled $A^i$ transforms on the first register. Finally, in the last step, we apply $DM_m^{-1}$ to the result.

Example 7 Let $A$ be a 4–tuple of $(2 \times 2)$ unitary matrices, and $C$ a 2–tuple of $(4 \times 4)$ unitary matrices. The generalized Kronecker product $A \otimes_R C$ can be implemented by a quantum network as follows.

![Quantum Network Diagram](https://example.com/quantum_network.png)
The dots and the circles represent control-bits: if the values with the dots are one, and if the values with the circles are zero, the transform is applied, otherwise it is the identity map. The least (most) significant bit is denoted by LSB (MSB). Note that most of the transforms are orthogonal, and thus the gates commute. Note also that, following the ideas of Griffiths and Niu in [19], a semi-classical generalized Kronecker product transform can be defined.

3.1 Quantum wavelet transforms

A main application of the Diagonalization Theorem is as a tool to find quantum networks computing large unitary matrices. Suppose we have factorized a unitary matrix \( U \) via a generalized Kronecker product into a product of some simpler matrices. Then, if we have quantum networks for computing these simpler transforms, we also have a quantum network for computing \( U \) by applying the methods developed above. We give two examples of this technique; in both cases implementing a wavelet transform.

Example 8 The Haar wavelet transform [22], \( H \), can be defined using the generalized Kronecker product as follows.

\[
H_2 = W \\
H_{2^{n+1}} = \prod_{2^{n}} \times \left( (H_{2^{n}}, I_{2^{n}}) \otimes_R W \right), \quad n = 1, 2, \ldots
\]

Applying the decomposition of the generalized right Kronecker product given in Equation (4), we immediately obtain an efficient quantum circuit for computing the Haar transform. Let \( S_{2^{n+1}} \) denote the bit-shift transform given by \(|b_n \ldots b_0) \mapsto |b_0 b_n \ldots b_1)\). This transform efficiently implements \( \prod_{2^{n}} \). For \( n = 3 \), the quantum circuit defined by Equation (6) is given below. Here the two \( S \) transforms are the bit-shift transforms of the appropriate dimensions.

By Equation (3), we can rewrite the recursive definition in Equation (6) as

\[
H_{2^{n+1}} = \prod_{2^{n}} \times \left( (H_{2^{n}}, I_{2^{n}}) \otimes_R I_2 \right) \times \left( I_{2^{n}} \otimes_R W \right).
\]

We refer to the right-most factor in this factorization as the scaling matrix of dimension \((2^{n+1} \times 2^{n+1})\) for the Haar wavelet transform. In general, given any family of unitary matrices \( \{D_{2^i}\}_{i \geq i_0} \), define a family of unitary transforms \( \{U_{2^i}\}_{i \geq i_0} \) as follows,

\[
U_{2^0} = D_{2^0} \\
U_{2^{n+i_0}} = \prod_{2^{n+i_0-1}} \times \left( (U_{2^{n+i_0-1}}, I_{2^{n+i_0-1}}) \otimes_R I_2 \right) \times D_{2^{n+i_0}}, \quad n = 1, 2, \ldots
\]
We refer to the $D_i$ matrices as a family of scaling matrices, and the family of $U_i$ matrices as a wavelet transform. Suppose that we have a family of efficient quantum networks for computing a given family of scaling matrices. Then, as in Example 8, we also have efficient quantum networks for computing the associated wavelet transform. The next example gives a factorization of the scaling matrices used in Daubechies’ $D^4$ wavelet transform [14].

**Example 9** Let $m \geq 4$ be an even integer, and let

$$k_{0/1} = \frac{3 \pm \sqrt{3}}{4\sqrt{2}} \quad \text{and} \quad k_{2/3} = \frac{1 \mp \sqrt{3}}{4\sqrt{2}}.$$ 

Daubechies’ $D^4_m$ scaling matrix [14] of dimension $(m \times m)$ is the matrix with

$$d_{ij} = \begin{cases} k_{j-i+x} & \text{i is even} \\ (-1)^j k_{2+i-j-x} & \text{i is odd,} \end{cases}$$

where

$$x = \begin{cases} 4 & \text{if } i \geq m - 2 \text{ and } j < 2 \\ 0 & \text{otherwise,} \end{cases}$$

and $k_l = 0$ if $l < 0$ or if $l > 3$.

Let $P_m$ be the $(m \times m)$ permutation matrix which subtracts two if the input is odd, i.e., $p_{ij} = 1$ if $i = j$ and $i$ is even, or if $i + 2 \equiv j \pmod{m}$ and $i$ is odd. Let $C_0$ and $C_1$ denote the two one-bit unitary operations,

$$C_0 = 2 \begin{pmatrix} k_3 & -k_2 \\ k_2 & k_3 \end{pmatrix} \quad \text{and} \quad C_1 = \frac{1}{2} \begin{pmatrix} k_0/k_3 & 1 \\ 1 & k_1/k_2 \end{pmatrix}.$$ 

The scaling matrix $D^4_m$ can then be factorized using two Kronecker products

$$D^4_m = (I_{m/2} \otimes_R C_1) \times P_m \times (I_{m/2} \otimes_R C_0). \quad (7)$$

Set $n = \lceil \log m \rceil$. The permutation transform $P_m$ can be implemented in $\Theta(n)$ basic operations [34]. Each of the other two factors on the right hand side of Equation (7) can be implemented in one basic operation. Thus, $D^4_m$ can be implemented in $\Theta(n)$ basic operations. We remark that we have not been able to find this factorization of $D^4_m$ elsewhere in the literature—despite the fact that using it compared to straightforward use of $D^4_m$ saves $m$ additions in the classical case. Furthermore, note that $(I_{m/2} \otimes_R C_1) \times (I_{m/2} \otimes_R C_0) = I_{m/2} \otimes_R W$, which is the scaling matrix used in the Haar transform. \qed
4 Group representations and quantum Fourier transforms

In the rest of this paper, $G$ will denote a finite group, written multiplicative with identity $e$, and $\eta$ the order of $G$. Let $\mathbb{C}G$ denote the complex group algebra of $G$. Let $\mathcal{B}_{\text{time}}$ denote the standard basis of $\mathbb{C}G$, that is, $\{g_1, \ldots, g_\eta\}$, and let $(u, v) = \sum_{g \in G} u(g)v(g)$ denote the natural inner product in $\mathbb{C}G$. Let $\text{GL}_d(\mathbb{C})$ denote the multiplicative group of $(d \times d)$ invertible matrices with complex entries. We start by reviewing some basic facts from the theory of linear representations of finite groups. For a general introduction to group representation theory, see for example [13] or [27].

A complex matrix representation $\rho$ of $G$ is a group-homomorphism $\rho : G \rightarrow \text{GL}_d(\mathbb{C})$. The dimension $d = d_\rho$ is called the degree or dimension of the representation $\rho$. Two representations, $\rho_1$ and $\rho_2$, of degree $d$ are equivalent if there exists an invertible matrix $A \in \text{GL}_d(\mathbb{C})$ such that $\rho_2(g) = A^{-1}\rho_1(g)A$ for all $g \in G$. A representation $\rho : G \rightarrow \text{GL}_d(\mathbb{C})$ is irreducible if there is no non-trivial subspace of $\mathbb{C}^d$ which is invariant under $\rho(g)$ for all $g \in G$, and it is unitary if $\rho(g)$ is unitary for all $g \in G$. For every representation there exists an equivalent unitary representation. Up to equivalence, there are only a finite number of irreducible representations, say $\nu$, of $G$. This number equals the number of distinct conjugate classes of $G$.

Let $\mathcal{R} = \{\rho^1, \ldots, \rho^\nu\}$ be a complete set of inequivalent, irreducible and unitary representations of $G$ with $d_i$ equal to the degree of $\rho^i$. For any representation $\rho \in \mathcal{R}$, the vector $\rho_{kl} \in \mathbb{C}G$ defined by considering the $(k, l)$–th entry of $\rho(g)$ for each $g \in G$ is called a matrix coefficient of $\mathcal{R}$. The inner product of two matrix coefficients of $\mathcal{R}$ is non-zero if and only if they are equal. For each matrix coefficient $\rho_{kl}$, let $b_{\rho,k,l}$ denote the normalized matrix coefficient, and let $\mathcal{B}_{\text{freq}} = \{b_{\rho,k,l}\}$ denote the set of orthonormalized matrix coefficients. Since one can show that the degrees $d_i$ of the representations $\rho_i \in \mathcal{R}$ satisfy the relation $\sum_{i=1}^{\nu} d_i^2 = \eta$, it follows that $\mathcal{B}_{\text{freq}}$ is an orthonormal basis of the vector space $\mathbb{C}G$.

The linear operator $F_G$ on $\mathbb{C}G$ which maps a vector $v \in \mathbb{C}G$ given in the standard basis $\mathcal{B}_{\text{time}}$ to its representation $\hat{v} \in \mathbb{C}G$ in basis $\mathcal{B}_{\text{freq}}$ is called the Fourier transform for $\mathbb{C}G$ on $\mathcal{R}$. Each entry of $\hat{v}$, denoted $\hat{v}(\rho_{kl})$ or just $\hat{\rho}_{kl}$, is called a Fourier coefficient of $v$ (on $\mathcal{R}$).

In the recent years, many new exciting results have been found concerning the computation of Fourier transforms for finite groups on classical computers—see [25] for a nice survey. In this paper, we consider the computation of Fourier transforms on quantum computers. Since quantum mechanics requires the operation of the computer to be unitary, our definition of a Fourier transform given above is slightly more strict than the most common used definitions for the classical case.
We now define what it means that a quantum circuit computes a Fourier transform. Let $F_G$ be a Fourier transform for $\mathbb{C}G$ on $\mathcal{R}$. Let $E_{\text{time}} : B_{\text{time}} \to \mathbb{Z}_\eta$ and $E_{\text{freq}} : B_{\text{freq}} \to \mathbb{Z}_\eta$ be two bijections. These functions induce an ordering on $B_{\text{time}}$ and $B_{\text{freq}}$, respectively. Let $E : B_{\text{time}} \cup B_{\text{freq}} \to \mathbb{Z}_\eta$ denote the extension of $E_{\text{time}}$ and $E_{\text{freq}}$. We say that $E$ is an encoding for the linear transform $F_G$. With respect to $E$, $F_G$ can be viewed as a matrix $F_G$ in $\text{GL}_\eta(\mathbb{C})$. This matrix is unitary by construction, and thus there exists a quantum circuit computing it \[15\]. We say that the circuit computes $F_G$ with respect to $E$.

Given a $k$–tuple of complex numbers of unit norm, $(\phi_i)_{i=1}^k$, let $\phi = \text{diag}(\phi_i) \in \text{GL}_k(\mathbb{C})$ denote the unitary diagonal matrix with $\phi_i$ at the $i$–th diagonal entry. Let $F_G$ be a Fourier transform for $\mathbb{C}G$ on $\mathcal{R}$, $E$ an encoding for $F_G$, and $F_G$ the resulting unitary matrix. We say that a quantum circuit computes $F_G$ up to phase factors (with respect to $E$) if there exists a unitary diagonal matrix $\phi \in \text{GL}_\eta(\mathbb{C})$ such that the circuit computes $F_G^\phi = \phi F_G$. Given a network that computes $F_G$ up to phase factors, we can obtain a quantum circuit for computing $F_G$ exactly by applying first $F_G^\phi$ and then the unitary transform $\phi^{-1} = \phi^*$. Note the dependencies of the set $\mathcal{R}$ and the encoding $E$ of $B_{\text{time}}$ and $B_{\text{freq}}$ in the above definitions. A Fourier transform for $\mathbb{C}G$ is defined only with respect to $\mathcal{R}$. A quantum circuit computing the Fourier transform is, in addition, defined with respect to an encoding $E$ of the basis-elements in $B_{\text{time}}$ and $B_{\text{freq}}$.

The quantum computation time of $F_G$ with respect to $\mathcal{R}$ and the encoding $E$ is defined as the minimum number of basic operations in any quantum circuit computing $F_G$, and it is denoted by $\text{QT}(G)(\mathcal{R}, E)$. The quantum computation time of a Fourier transform for $\mathbb{C}G$, denoted $\text{QT}(G)$, is defined as the minimum of $\text{QT}(G)(\mathcal{R}, E)$ over all possible choices of $\mathcal{R}$ and $E$.

In the rest of this paper, $\mathcal{R}$ denotes a complete set of inequivalent, irreducible and unitary representations of $G$.

5 Quantum Fourier transforms for cyclic groups

As an introductionary example, consider the problem of quantum computing the discrete Fourier transform. We start by developing an efficient quantum routine for computing the discrete Fourier transform using the generalized Kronecker product discussed in Section 2. Then, we review a group theoretical interpretation of the transform which relates it to the cyclic groups.
The discrete Fourier transform for a quantum computer is defined as follows. For any positive integer \( n \), let

\[
F_n|x\rangle = \frac{1}{\sqrt{n}} \sum_{y=0}^{n-1} \omega_n^{xy} |y\rangle,
\]

for each \( x = 0, \ldots, n-1 \), where \( \omega_n = \exp(2\pi\sqrt{-1}/n) \) is the principal \( n \)-th root of unity. The unitary Fourier transform \( F_{nm} \) can be defined from \( F_n \) and \( F_m \) using a generalized Kronecker product

\[
F_{nm} = \Pi_{nm} \times (F_n \otimes L I_m) \times \left( (D_{nm}^{s})_{s=0}^{m-1} \otimes L I_m \right) \times (I_n \otimes L F_m)
\]

where \( D_{nm}^{s} = \text{diag}(\omega^{si}) \) for \( 0 \leq s < m \) and \( \omega = \omega_{nm} \). Equation (9) is referred to as a \textit{radix–}\( n \)-\textit{splitting} in [33], where a proof of the identity can be found.

Equation (9) gives an efficient quantum routine for computing \( F_{nm} \) from \( F_n \), \( F_m \), and \( D_{nm}^{s} \). Interestingly, the resulting routine obtained this way is the same as the one found by Cleve [11] using a direct method. The transform \( (D_{nm}^{s})_{s} \otimes L I_m \) is a special application of the \( \Phi \) transform defined in Section 3 and is thus easily applied. For powers of 2, the computation of \( F_{2^n} \) uses \( \Theta(n^2) \) basic operations [12].

We now review the well-known group theoretical correspondence to the discrete Fourier transform. Let \( G = \mathbb{Z}_n \) be the cyclic group of order \( n \). For Abelian groups all irreducible representations are one-dimensional, and hence equivalent representations are equal. There are \( n \) distinct representations, \( \mathcal{R} = \{\zeta^0, \ldots, \zeta^{n-1}\} \), given by

\[
\zeta^i(j) = [\omega_n^j] \quad \text{for every} \quad j \in \mathbb{Z}_n.
\]

The collection of normalized matrix coefficients are \( \mathcal{B}_{\text{freq}} = \{b_{\zeta^0,1,1}, \ldots, b_{\zeta^{n-1},1,1}\} \) where \( (b_{\zeta^i,1,1,j}) = \frac{1}{\sqrt{n}} \omega_n^{ij} \) for all \( b_{\zeta^i,1,1} \in \mathcal{B}_{\text{freq}} \) and \( j \in \mathcal{B}_{\text{time}} \). Hence, for all \( b_{\zeta^i,1,1} \in \mathcal{B}_{\text{freq}} \) we have \( b_{\zeta^i,1,1} = \frac{1}{\sqrt{n}} \sum_{j \in \mathcal{B}_{\text{time}}} \omega_n^{ij} j \). Thus, by choosing the encoding \( E \) given by \( E_{\text{time}}(j) = j \) and \( E_{\text{freq}}(b_{\zeta^i,1,1}) = i \), respectively, the quantum circuit defined by Equation (8) is seen to compute the Fourier transform for the cyclic group \( \mathbb{Z}_n \) with respect to \( E \). We remark that it is possible also to give a group theoretical interpretation of the decomposition given by Equation (4); see for example [25] for details.

### 5.1 Direct product groups

Suppose we are given quantum networks (as black-boxes) for quantum computing Fourier transforms for the group algebras \( \mathbb{C}G_1 \) and \( \mathbb{C}G_2 \). Consider the problem of computing a Fourier transform for the direct product group algebra \( \mathbb{C}G = \mathbb{C}(G_1 \times G_2) \). Classically, this
problem has a very simple solution. In this section, we show that this carries over in the
quantum circuit model.

Let $G_1$ and $G_2$ be finite groups of order $\eta_1$ and $\eta_2$, respectively. Our first task is to establish
a specific isomorphism $\varphi$ between the algebras $\mathbb{C}G_1 \times \mathbb{C}G_2$ and $\mathbb{C}(G_1 \times G_2)$. Let $B^i_{\text{time}}$ denote
the standard basis of $\mathbb{C}G_i$, $i = 1, 2$, and let $B_{\text{time}} = \{(g_i, g_2) : g_i \in G_i, \ i = 1, 2\}$ denote the
standard basis of $\mathbb{C}(G_1 \times G_2)$. Let $\mathbb{C}G_1 \otimes \mathbb{C}G_2$ denote the tensor product algebra of $\mathbb{C}G_1$
and $\mathbb{C}G_2$, and $\varphi : \mathbb{C}G_1 \otimes \mathbb{C}G_2 \to \mathbb{C}(G_1 \times G_2)$ the natural algebra isomorphism defined by

$$\varphi(g_1 \otimes g_2) = (g_1, g_2) \quad (g_1 \otimes g_2 \in B^1_{\text{time}} \otimes B^2_{\text{time}}).$$

With these definitions, we can write

$$B_{\text{time}} = \varphi \left( B^1_{\text{time}} \otimes B^2_{\text{time}} \right). \quad (10)$$

Let $R_i$ be a complete set of inequivalent, irreducible and unitary representations of $G_i$, 
$i = 1, 2$. We need the following lemma from representation theory.

**Lemma 10** Let $G_i$ and $R_i$ be given as above, $i = 1, 2$. Then

$$\mathcal{R} = \mathcal{R}_1 \otimes_R \mathcal{R}_2 = \{\rho_1 \otimes_R \rho_2 : \rho_i \in \mathcal{R}_i, \ i = 1, 2\}$$

is a complete set of inequivalent, irreducible and unitary representations of $G = G_1 \times G_2$.

Let $B^i_{\text{freq}}$ denote the set of orthonormalized matrix coefficients of $\mathcal{R}_i$, $i = 1, 2$, and $B_{\text{freq}}$ the
set of orthonormalized matrix coefficients of $\mathcal{R}$, where $\mathcal{R}$ is given as in Lemma 10. By that
lemma, it follows by straightforward algebra that

$$B_{\text{freq}} = \varphi \left( B^1_{\text{freq}} \otimes B^2_{\text{freq}} \right). \quad (11)$$

Having established the above isomorphism, we now state the main result from representation
theory to be used in this section. Essentially, it reduces the problem of computing a Fourier transform for $\mathbb{C}(G_1 \times G_2)$ to those of computing Fourier transforms for $\mathbb{C}G_1$ and $\mathbb{C}G_2$.

**Theorem 11** Let $F_1$ and $F_2$ be Fourier transforms for $\mathbb{C}G_1$ and $\mathbb{C}G_2$ on $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively. Define the linear transform $F'_G$ over $\mathbb{C}G_1 \otimes \mathbb{C}G_2$ by

$$F'_G(g_1 \otimes g_2) = F_1(g_1) \otimes F_2(g_2).$$

Then $F_G = \varphi F'_G \varphi^{-1}$ is a Fourier transform for $\mathbb{C}G = \mathbb{C}(G_1 \times G_2)$ on $\mathcal{R} = \mathcal{R}_1 \otimes_R \mathcal{R}_2$. 

13
This reduction is, however, only as abstract computations over vector spaces. To give a concrete quantum circuit for computing the Fourier transform for the product group, we also need to consider the choices of bases for the involved transforms in the reduction. Let $E_i$ be an encoding for $F_i$ and $\mathbf{F}_i$ the corresponding matrix representation of $F_i$, $i = 1, 2$. In ket-notation, the transform $\mathbf{F}' G$ reads

$$|g_1\rangle|g_2\rangle \rightarrow (\mathbf{F}_1|g_1\rangle)(\mathbf{F}_2|g_2\rangle)$$

for all $g_1 \otimes g_2 \in \mathcal{B}_{\text{time}}^1 \otimes \mathcal{B}_{\text{time}}^2$.

Define the bijections $E_{\text{time}}: \mathcal{B}_{\text{time}} \rightarrow \mathbb{Z}_{\eta_1 \eta_2}$ and $E_{\text{freq}}: \mathcal{B}_{\text{freq}} \rightarrow \mathbb{Z}_{\eta_1 \eta_2}$ by

$$E_{\text{time}}(\varphi(g_1 \otimes g_2)) = \eta_2 E_1(g_1) + E_2(g_2)$$

$$E_{\text{freq}}(\varphi(b_1 \otimes b_2)) = \eta_2 E_1(b_1) + E_2(b_2).$$

Let $E$ denote the extension of $E_{\text{time}}$ and $E_{\text{freq}}$. With respect to the encoding $E$, $F_G$ as defined in Theorem 11 has the matrix representation

$$\mathbf{F}_G = \mathbf{F}_1 \otimes_R \mathbf{F}_2.$$

Applying $\mathbf{F}_G$ is thus done by applying $\mathbf{F}_1$ on the most significant bits, and $\mathbf{F}_2$ on the least significant bits. We have shown

**Theorem 12** Let $G_1$ and $G_2$ be groups of order $\eta_1$ and $\eta_2$, respectively. Suppose we have quantum networks $\mathbf{F}_1$ and $\mathbf{F}_2$ for computing Fourier transforms for $\mathbb{C}G_1$ and $\mathbb{C}G_2$ with respect to the encodings $E_1$ and $E_2$, respectively. Then the following quantum circuit computes a Fourier transform for $\mathbb{C}(G_1 \times G_2)$ with respect to the encoding $E$ as defined in Equation (12).

As an application of this, consider the Walsh-Hadamard transform $W$, defined as follows. For any positive integer $n$, let

$$W_{2^n}|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} (-1)^{\sum_{i=0}^{n-1} x_i y_i}|y\rangle,$$

for each $x = 0, \ldots, 2^n - 1$, where $x = x_{n-1} \ldots x_0$ and $y = y_{n-1} \ldots y_0$. This unitary transform can also be defined using the standard Kronecker product as follows

$$W_2 = W, \quad W_{2^{n+1}} = W \otimes_R W_{2^n}, \quad n = 1, 2, \ldots$$

(13)
Appealing to the generalized Kronecker product routine in Section 3, we immediately obtain the well-known method for computing the Walsh-Hadamard transform $W_{2^n}$ on a quantum computer [16]: apply the transform $W$ on each of the $n$ qubits.

It is easy to check that $W$ is the Fourier transform for the cyclic group $\mathbb{Z}_2$ of order two. Thus, by Theorem 12, we have the well-known fact that the Walsh-Hadamard transform coincides with the Fourier transform for the Abelian group $\mathbb{Z}_n^2$. This transform has been extensively used in quantum algorithms, for example by Deutsch and Jozsa [16], Simon [30], and Grover [21, 8]. One of its advantages is that it can be computed in only $\Theta(n)$ basic operations [16].

Now one might ask if a similar statement holds for subgroups in general. That is, if $H \leq G$ is a subgroup and we encode elements of $G$ using two registers, the first for coset representatives, the second for elements from $H$, to what extent do the gates then need to involve both registers? We consider this question in the next section, and give an answer to it for some classes of non-Abelian groups.

In this section, we have carefully distinguished between isomorphic vector spaces in order to prove Theorem 12. In the following, we will relax slightly upon this to avoid cumbersome notation. Let $U$ and $V$ be any two inner product spaces of dimension $m$ and $n$, respectively, with orthonormalized bases $\{u_1, \ldots, u_m\}$ and $\{v_1, \ldots, v_n\}$, respectively. Then the tensor product $U \otimes V$ and the vector space spanned by $\{(u_i, v_j) : 1 \leq i \leq m, 1 \leq j \leq n\}$ are isomorphic under the natural isomorphism $\varphi$ given by $\varphi(u_i \otimes v_j) = (u_i, v_j)$. When appropriate, we will not distinguish between $u_i \otimes v_j$ and $(u_i, v_j)$ in the rest of this paper. Note that the set $\{u_i \otimes v_j\}$ is an orthonormalized basis for $U \otimes V$.

6 Adapted representations

In the previous section, we related a quantum Fourier transform for $G = G_1 \times G_2$ to quantum Fourier transforms for $G_1$ and $G_2$. In the classical case, relating a Fourier transform of a group to a Fourier transform to one of its subgroup has shown to be very useful; see for example [23] and the references therein. The main ideas in this approach are factorization of the group elements, and the use of an adapted set of representations. For example, in Section 5.1, we used the factorization $(g_1, g_2) = (g_1, e_2) \cdot (e_1, g_2)$, where $e_i$ denotes the identity of $G_i$, $i = 1, 2$.

For any subgroup $H \leq G$ and any representation $\rho$ of $G$, let $\rho \downarrow H$ denote the representation of $H$ obtained by restricting $\rho$ to $H$. The representation $\rho \downarrow H$ is unitary but not necessarily irreducible. Recall that we in this paper assume that all representations are irreducible and unitary.
Definition 13 Let $H \leq G$ be a subgroup, and $\mathcal{R}$ be a complete set of representations of $G$. Then $\mathcal{R}$ is called $H$–adapted if there is a complete set $\mathcal{R}^H$ of representations of $H$ such that the set of restricted representations $(\mathcal{R} \downarrow H) = \{\rho \downarrow H : \rho \in \mathcal{R}\}$ is a set of matrix direct sums of the representations in $\mathcal{R}^H$. □

The set $\mathcal{R}$ is said to be adapted to a chain of subgroups if it is adapted to each subgroup in the chain. Adapted representations always exist.

Let $H \leq G$ be a subgroup of order $m$, and $T$ a left transversal for $H$ in $G$. Let $\mathcal{R}^H$ be a complete set of representations of $H$, and let $\mathcal{R}$ be a complete set of representations of $G$ that is $H$–adapted relative to $\mathcal{R}^H$. Let $\mathcal{B}_{freq}^H$ and $\mathcal{B}_{freq}$ denote the collections of normalized matrix coefficients for $H$ and $G$, respectively. Let $\rho \in \mathcal{R}$ be a representation of degree $d$. The matrix coefficient $\rho_{kl} \in \mathbb{C}G$ can be written as a linear sum of the basis-elements $\mathcal{B}_{time}$:

$$\rho_{kl} = \sum_{g \in G} \rho_{kl}(g) = \sum_{t \in T} \sum_{h \in H} \sum_{i=1}^{d} \rho_{ki}(t) \rho_{il}(h) th = \sum_{t \in T} \sum_{i=1}^{d} \rho_{ki}(t) \left( \sum_{h \in H} \rho_{il}(h) th \right).$$

Since $\mathcal{R}$ is $H$–adapted by assumption, $\rho$ is a matrix direct sum of representations in $\mathcal{R}^H$. Therefore, either $\rho_{il}(h) = 0$ for all $h \in H$, or there exist $\rho' \in \mathcal{R}^H$ of degree $d'$ and $1 \leq i', l' \leq d'$ such that $\rho_{il}(h) = \rho'_{il}(h)$ for all $h \in H$. In the former case, let $\rho'_{i'l'}$ (and $b_{\rho', i', l'}$) denote the zero vector in $\mathbb{C}H$. Then we have

$$\rho_{kl} = \sum_{i=1}^{d} \sum_{t \in T} \rho_{ki}(t) \left( \sum_{h \in H} \rho'_{i'l'}(h) th \right),$$

so

$$b_{\rho, k, l} = \sum_{i=1}^{d} \sum_{t \in T} b_{\rho, k, i}(t) \left( \sum_{h \in H} \rho'_{i'l'}(h) th \right) = \sum_{i=1}^{d} \sum_{t \in T} \sqrt{\frac{m}{d'}} b_{\rho, k, i}(t) \left( \sum_{h \in H} b_{\rho', i', l'}(h) th \right).$$

(14)

A Fourier transform, $F_G$, is a change of basis in $\mathbb{C}G$ from the standard basis to a basis of normalized matrix coefficients. Let $F_H$ be the Fourier transform for $\mathbb{C}H$ on $\mathcal{R}^H$. For obtaining an $H$–adapted method for computing $F_G$, consider the complex vector space spanned by the basis

$$T \otimes \mathcal{B}_{time}^H = \{ t \otimes h : t \in T, h \in \mathcal{B}_{time}^H \}.$$ 

This vector space is clearly isomorphic to $\mathbb{C}G$ under the natural map $\varphi : \langle T \otimes \mathcal{B}_{time}^H \rangle \to \langle \mathcal{B}_{time} \rangle$ given by $\varphi(t \otimes h) = th$. Here, and in the rest of this paper, $\langle \cdot \rangle$ means span($\cdot$). Another basis is

$$\mathcal{B}_{temp} = T \otimes \mathcal{B}_{freq}^H = \{ t \otimes b_{\rho', i', l'} : t \in T, b_{\rho', i', l'} \in \mathcal{B}_{freq}^H \},$$
and using \( \varphi \), Equation (14) reads

\[
b_{\rho,k,l} = \sum_{t \in T} \sum_{i=1}^d \sqrt{m} b_{\rho,k,i}(t) \varphi(t \otimes b_{\rho',i',l}).
\]

Let \( V : \langle B_{\text{freq}} \rangle \rightarrow \langle B_{\text{temp}} \rangle \) denote the transform

\[
V : b_{\rho,k,l} \mapsto \sum_{t \in T} \sum_{i=1}^d \sqrt{m} b_{\rho,k,i}(t) t \otimes b_{\rho',i',l}.
\]

By construction of \( V \),

\[
(I \otimes F_H) \circ \varphi^{-1} = V \circ F_G
\]

as illustrated in the following commutative diagram

\[
\begin{array}{ccc}
\langle T \otimes B_{\text{time}}^H \rangle & \xrightarrow{\varphi} & \langle B_{\text{time}} \rangle \\
I \otimes F_H & \downarrow & \downarrow F_G \\
\langle T \otimes B_{\text{freq}}^H \rangle & \xleftarrow{V} & \langle B_{\text{freq}} \rangle
\end{array}
\]

Since \( \varphi \) is an isomorphism, and \( F_H \) and \( F_G \) are unitary, \( V \) is invertible. Let \( U : \langle B_{\text{temp}} \rangle \rightarrow \langle B_{\text{freq}} \rangle \) denote the inverse of \( V \), that is,

\[
U : \sum_{t \in T} \sum_{i=1}^d \sqrt{m} b_{\rho,k,i}(t) t \otimes b_{\rho',i',l} \mapsto b_{\rho,k,l}
\]

which maps a vector \( \tilde{v} \in \langle B_{\text{temp}} \rangle \) given relative to basis \( B_{\text{temp}} \) to its representation \( \hat{v} \in \mathbb{C}G \) relative to basis \( B_{\text{freq}} \). Hence, we have factorized the Fourier transform \( F_G \) into a product of three unitary transforms,

\[
F_G = U \circ (I \otimes F_H) \circ \varphi^{-1}.
\]

A quantum implementation of the adapted method for computing a Fourier transform can be obtained as follows. Given a vector \( v \in \mathbb{C}G \), let \( v_t \in \mathbb{C}G \) denote the vector which is non-zero only on the coset \( tH \), on which it is given by \( v_t(h) = v(th) \) for all \( h \in H \). Initially, we hold the superposition \( v = \sum_{g \in B_{\text{time}}} v(g)|g\rangle \) and we want to compute the superposition \( \hat{v} = \sum_{b_i \in B_{\text{freq}}} \hat{v}(b_i)|b_i\rangle \). The quantum routine consists of three steps. In the first step, we apply \( \varphi^{-1} \), computing

\[
v = \sum_{g \in B_{\text{time}}} v(g)|g\rangle \mapsto \sum_{t \in T} \sum_{h \in B_{\text{time}}^H} v(th)|t\rangle|h\rangle = \sum_{t \in T} |t\rangle \left( \sum_{h \in B_{\text{time}}^H} v_t(h)|h\rangle \right).
\]
Then, we apply a quantum Fourier transform $F_H$ with respect to $R_H$ to the second register, producing

$$\sum_{t \in T} |t \rangle \left( \sum_{b'_i \in B_{freq}} \hat{v}_t(b'_i) |b'_i \rangle \right) = \sum_{t \in T} \sum_{b'_i \in B_{freq}} \hat{v}_t(b'_i) |t \rangle |b'_i \rangle = \tilde{v}.$$  

Finally, in the third step, we apply the linear transform $U$ given by Equation (15), producing

$$\sum_{b_i \in B_{freq}} \hat{v}(b_i) |b_i \rangle = \hat{v}.$$  

The transform $U$ is unitary since the first two steps (that is, $(I \otimes F_H) \circ \varphi^{-1}$) are unitary and the composition of the three steps (that is, $F_G$) is unitary. In the following sections, we apply the technique just described to develop quantum Fourier transforms for some non-Abelian groups.

### 6.1 The quaternionic groups

The quaternionic group $Q_n$ of order $4n$ is the group

$$Q_n = \langle r, c : r^{2n} = c^4 = 1, cr = r^{2n-1}c, c^2 = r^n \rangle.$$  

For simplicity, we consider only the case that $n$ is even. The case when $n$ is odd is very similar, and in fact slightly simpler. When $n$ is even, $Q_n$ has a complete set $R$ consisting of four one-dimensional and $n - 1$ two-dimensional representations

$$\rho^1 \equiv 1 \quad \rho^2(r) = \rho^2(-c) = 1 \quad \rho^3(-r) = \rho^3(-c) = 1 \quad \sigma^i(r) = \begin{bmatrix} \omega^i & 0 \\ 0 & \omega^{-i} \end{bmatrix} \quad \sigma^i(c) = \begin{bmatrix} 0 & (-1)^i \\ 1 & 0 \end{bmatrix}$$

where $1 \leq i < n$ and $\omega = \omega_{2n}$.

The group has a cyclic subgroup $H$ generated by $r$ of index two. Let $T = \{e, c\}$ be a left transversal for $H$ in $Q_n$, and write $Q_n = TH$. Let $R^H$ denote the complete set of one-dimensional representations of $H$ given in Section 4. The set of restricted representations of $R$ is

$$(R \downarrow H) = \{\zeta^0, \zeta^n\} \cup \{\zeta^l \oplus \zeta^{2n-l} : 1 \leq l < n\},$$

so $R$ is $H$–adapted. Let $B_{time}, B_{freq}, B^H_{time}$, and $B^H_{freq}$ be defined as in Section 4. Let

$$B_{temp} = T \otimes B^H_{freq} = \{t \otimes b_{\zeta^l, 1, 1} : t \in T, b_{\zeta^l, 1, 1} \in B^H_{freq}\}.$$  

A main part of the development of subgroup-adapted Fourier transforms is the determination and implementation of the transform $U$ defined by Equation (13). For this purpose, consider the matrix coefficient $\sigma^{i}_{11} \in \mathbb{C}Q_n$. By writing

$$\sigma^i_{11} = \sum_{t \in T} \sum_{h \in H} \sigma^i_{11}(th) th = \sum_{x \in \mathbb{Z}_{2n}} \overline{\omega}^x r^x,$$
we have that
\[ b_{\sigma^*,1,1} = \frac{1}{\sqrt{2n}} \sum_{x\in\mathbb{Z}_{2n}} \varphi(x) = \varphi(e \otimes \zeta^i). \]

Each of the other Fourier coefficients can similarly be written as a linear sum of the basis-elements \( B_{\text{temp}} \),
\[
\begin{align*}
  b_{\sigma^*,1,1} &= \varphi(e \otimes \zeta^i) \\
  b_{\sigma^*,1,2} &= (-1)^i \varphi(c \otimes \zeta^{2n-i}) \\
  b_{\sigma^*,2,1} &= \varphi(c \otimes \zeta^i) \\
  b_{\sigma^*,2,2} &= \varphi(c \otimes \zeta^{2n-i})
\end{align*}
\]
and
\[
\begin{align*}
  b_{\rho^*,1,1} &= \frac{1}{\sqrt{2}} (\varphi(e \otimes \zeta^0) + \varphi(c \otimes \zeta^0)) \\
  b_{\rho^*,1,1} &= \frac{1}{\sqrt{2}} (\varphi(e \otimes \zeta^0) - \varphi(c \otimes \zeta^0)) \\
  b_{\rho^*,1,1} &= \frac{1}{\sqrt{2}} (\varphi(e \otimes \zeta^n) + \varphi(c \otimes \zeta^n)) \\
  b_{\rho^*,1,1} &= \frac{1}{\sqrt{2}} (\varphi(e \otimes \zeta^n) - \varphi(c \otimes \zeta^n))
\end{align*}
\]
where \( i = 1, \ldots, n-1 \).

Equation (17) defines the transform \( U : \langle B_{\text{temp}} \rangle \rightarrow \langle B_{\text{freq}} \rangle \) appearing in the factorization \( F_G = U \circ (I \otimes F_H) \circ \varphi^{-1} \). What remains in order to obtain a concrete circuit computing the Fourier transform \( F_G \), is an encoding of the bases for the transforms. With respect to the encoding
\[
E^H_{\text{time}}(r^k) = k \quad E^H_{\text{freq}}(\zeta^i) = i,
\]
the Fourier transform \( F_H \) for \( \mathbb{C}H \) defined by Equation (8) has the matrix representation \( F_H \).

Let therefore the encoding of \( \mathcal{B}_{\text{time}} \) be given by \( E_{\text{time}}(c^i r^k) = 2nj + k \), and the encoding of \( B_{\text{temp}} \) be given by \( E_{\text{temp}}(c^i r^k) = 2nj + i \). With respect to this encoding, the transform \( (I \otimes F_H) \circ \varphi^{-1} \) has the matrix representation \( I_2 \otimes_R F_H \).

Computing the \( U \) transform with respect to \( E_{\text{temp}} \) is very simple
\[
|j\rangle\langle i| \mapsto \begin{cases} 
  \frac{1}{\sqrt{2}} (|0\rangle + (-1)^j |1\rangle) & \text{if } i = 0 \text{ or } i = n \\
  (-1)^j |j\rangle\langle i| & \text{if } i > n \text{ and } i \text{ is odd} \\
  |j\rangle\langle i| & \text{otherwise}.
\end{cases}
\]

So, given a network computing the Fourier transform \( F = F_{2n} \) for the cyclic group of order \( 2n \), a network computing the Fourier transform for the quaternionic group \( Q_n \) can be constructed as follows.

```
MSB ─── W ─── Z ─── MSB
|      |     |      |
|      |     |      |
|      |     |      |
|      |     |      |
| F    |     |      |
|      |     |      |
|      |     |      |
|      |     |      |
LSB ───     ───     ─── LSB
```

Note that \( Z \) and \( W \), both defined in Section 3, operate on distinct states and thus commute. If we, on the circuit given above, remove the \( Z \) gate, then we have a circuit that computes a Fourier transform for both the dihedral and the semidihedral group of order \( 4n \). In the following, we show that this is no coincidence.
6.2 Metacyclic Groups

In this section, we give a general quantum circuit for computing a Fourier transform for a class of metacyclic groups. A group is called metacyclic if it contains a cyclic normal subgroup $H$ so that the quotient group $G/H$ is also cyclic. Let $G = \{b^j a^i : 0 \leq j < q, 0 \leq i < m\}$ be a metacyclic group where

$$b^{-1}ab = a^r, \quad b^q = a^s, \quad a^m = 1$$

and with $(m, r) = 1$, $m|s(r - 1)$, and $q$ prime. Let $d = (r - 1, m)$. The group has a cyclic subgroup $H$ generated by $a$ of index $q$. Let $T = \{b^j : 0 \leq j < q\}$ be a left transversal for $H$ in $G$, and write $G = TH$. Let $E_{\text{time}} : \mathcal{B}_{\text{time}} \to \mathbb{Z}_{qm}$ be the encoding of $\mathcal{B}_{\text{time}}$ given by

$$b^i a^j \mapsto mj + i.$$ 

**Theorem 14** The following network computes a Fourier transform for $\mathbb{C}G$ up to phase factors with respect to $E_{\text{time}}$. Here $\omega = \omega_{qd}^s$.

![Diagram](image)

The phase factors involved in the theorem depend on the actual group structure. Before proving the theorem, we consider the representations of the group. The group $G$ has $qd$ one-dimensional representations, $\{\rho^{ij}\}_{i,j=0}^{i=d-1,j=q-1}$, each given by

$$\rho^{ij}(a) = \omega^i_d \quad \rho^{ij}(b) = \omega^i_d \omega^j_s.$$ 

Let $\mathcal{R}^H$ be the complete set of representations of $H$ given in Section 3. For every $\zeta^i \in \mathcal{R}^H$ define the induced representation $\tilde{\zeta}^i : G \to \text{GL}_q(\mathbb{C})$ by

$$a \mapsto \begin{bmatrix} \omega_m^i & \cdots & \omega_m^{frq-1} \\ & \cdots & \\ \omega_m^{frq-1} & \cdots & \omega_m^i \end{bmatrix} \quad b \mapsto \begin{bmatrix} 1 & \cdots & \omega_s^m \\ \cdots & \ddots & \ \end{bmatrix}.$$ 

The group $G$ has an $H$–adapted set of representations $\mathcal{R}$ consisting of the $qd$ one-dimensional and $(m - d)/q$ $q$–dimensional representations. The $q$–dimensional representations in $\mathcal{R}$ are all induced representations [13].
The matrix coefficient \( \rho^{ij} \in \mathbb{C}G \) can be written as a linear sum of the basis-elements \( B_{\text{temp}} = T \otimes B_{\text{freq}} \):

\[
\rho^{ij} = \sum_{g \in G} \rho^{ij}(g) g
\]

\[
= \sum_{k \in \mathbb{Z}_q} \sum_{x \in \mathbb{Z}_m} \rho^{ij}(b^k a^x) b^k a^x
\]

\[
= \sum_{k \in \mathbb{Z}_q} \rho^{ij}(b^k) \sum_{x \in \mathbb{Z}_m} \rho^{ij}(a^x) b^k a^x
\]

\[
= \sum_{k \in \mathbb{Z}_q} \omega^j_k \left( \omega^{ik}_q \sum_{x \in \mathbb{Z}_m} \omega^{xim/d}_m b^k a^x \right)
\]

\[
= \sqrt{m} \sum_{k \in \mathbb{Z}_q} \omega^j_k \left( \omega^{ik}_q \omega_{\zeta^{im/d},1,1} \right)
\]

so, by definition of \( U : \langle B_{\text{temp}} \rangle \to \langle B_{\text{freq}} \rangle \) as given in Section 6,

\[
U^{-1} : b_{\rho^{ij},1,1} \mapsto \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{Z}_q} \omega^j_k \left( \omega^{ik}_q b^k \otimes b_{\zeta^{im/d},1,1} \right).
\]  (18)

We refer to a matrix coefficient of an induced representation as an **induced** matrix coefficient.

Any induced matrix coefficient \( \bar{\zeta}_{kl} \in \mathbb{C}G \) is non-zero on exactly one coset of \( H \). For example, \( \bar{\zeta}_{kl} = \bar{\zeta}_{31} \) is non-zero on the coset \( b^2 H \). In general, the matrix coefficient \( \bar{\zeta}_{kl} \in \mathbb{C}G \) can be written as a linear sum of the basis-elements \( B_{\text{temp}} \) as follows.

\[
\bar{\zeta}_{kl} = \sum_{g \in G} \bar{\zeta}_{kl}(g) g
\]

\[
= \sum_{t \in T} \sum_{h \in H} \bar{\zeta}_{kl}(th) th
\]

\[
= \sum_{t \in T} \bar{\zeta}_{kl}(t) \sum_{h \in H} \bar{\zeta}_{kl}(h) th
\]

\[
= \bar{\zeta}_{kl}(b^k) \sum_{h \in H} \bar{\zeta}_{kl}(h) b^k h
\]

\[
= \bar{\zeta}_{kl}(b^k) \sum_{x \in \mathbb{Z}_m} \omega^j_k b^k a^x
\]

\[
= \sqrt{m} \phi \varphi(b^k \otimes b_{\zeta^{im/d},1,1}).
\]

Here, \( \phi = \bar{\zeta}_{kl}(b^k) \) is some \( m \)-th root of unity. So

\[
U^{-1} : b_{\bar{\zeta}_{kl},1,1} \mapsto \phi b^k \otimes b_{\zeta^{im/d},1,1}.
\]  (19)

that is, \( b_{\bar{\zeta}_{kl},1,1} \in B_{\text{freq}} \) is mapped by \( U^{-1} \) to one of the basis-elements \( B_{\text{temp}} \) up to a phase-factor. To find an expression for \( U \) instead of \( U^{-1} \), we need Lemma 16 which easily follows from the following lemma for which a proof can be found, for example, in [13, Lemma (47.8)].
Lemma 15 The induced representation $\tilde{\zeta}^i$ is reducible if and only if there exists a $j$, $1 \leq j \leq q - 1$, such that $ir^j \equiv i \pmod{m}$.

Lemma 16 Let $\tilde{\zeta}_{kl}^i$ be any induced matrix coefficient. If $\tilde{\zeta}^i$ is irreducible, then $ir^l$ is not a multiple of $m/d$.

Proof To prove the contrapositive, suppose that $ir^l \equiv 0 \pmod{m/d}$. Since $(r, m) = 1$, $i \equiv 0 \pmod{m/d}$, so $id \equiv 0 \pmod{m}$. Since $d$ divides $r - 1$, we have $ir \equiv i \pmod{m}$ and the statement follows from Lemma 15. □

Lemma 17 The transform $U : \langle B_{\text{temp}} \rangle \to \langle B_{\text{freq}} \rangle$ is given by

$U(b^k \otimes b_{\zeta^{x,1,1}}) = \begin{cases} \phi b_i & \text{if } x \text{ is not a multiple of } m/d \\ \frac{1}{\sqrt{q}} \omega_{qd}^{sjk} \sum_{j \in \mathbb{Z}_q} \omega_q^{jk} b_{\rho^{j,1,1}} & \text{if } x = jm/d. \end{cases}$

Here, $\phi$ is some $m$-th root of unity and $b_i \in B_{\text{freq}}$, both depending on the value of $k$ and $x$.

Proof Write $B_{\text{temp}}$ as a disjoint union of two sets, $B_{\text{temp}}^1$ and $B_{\text{temp}}^2$, where $B_{\text{temp}}^1 = T \otimes \{b_{\zeta^{x,1,1}} : x \text{ is a multiple of } m/d\}$. Write similarly $B_{\text{freq}}$ as a disjoint union of two sets, $B_{\text{freq}}^1$ and $B_{\text{freq}}^2$, where $B_{\text{freq}}^1 = \{b_{\rho^{j,1,1}} : 0 \leq i < d, 0 \leq j < q\}$. We first show that

$\langle U^{-1}(B_{\text{freq}}^2) \rangle = \langle B_{\text{temp}}^2 \rangle$ (20)

by a simple counting argument. For each of the $q(m - d)$ elements $b_{\zeta^{x,k,l}} \in B_{\text{freq}}^2$, we have that $\zeta^i \in \mathcal{R}$ is irreducible. By Lemma 16, $ir^j$ is not a multiple of $m/d$, and therefore $U^{-1}(b_{\zeta^{x,k,l}}) \in \langle B_{\text{temp}}^2 \rangle$. Since $B_{\text{freq}}^2$ and $B_{\text{temp}}^2$ has the same cardinality, and since $U$ is unitary, Equation (20) follows. By Equation (19), the first case in the lemma follows.

By the unitarity of $U$, we also have that

$\langle U^{-1}(B_{\text{freq}}^1) \rangle = \langle B_{\text{temp}}^1 \rangle$.

The action of $U^{-1}$ on $B_{\text{freq}}^1$ is given by Equation (18), and the action of its inverse (that is, of $U$) on $B_{\text{temp}}^1$ is given by the second case in the lemma. □

Let $U_1 : \langle B_{\text{temp}} \rangle \to \langle B_{\text{freq}} \rangle$ denote the unitary transform which acts on $B_{\text{temp}}^1$ as $U$, and which on $B_{\text{temp}}^2$ is given by $b^k \otimes b_{\zeta^{x,k,l}} \mapsto b_{\zeta^{x,k,l}}$. Since we are only interested in a quantum
The network that computes a Fourier transform for \( \mathbb{C}G \) up to phase factors, by Lemma 17, it suffices to implement \( U_1 \) instead of \( U \). In conclusion, we have shown that the transform \( \mathcal{F}_G = U_1 \circ (I \otimes F_m) \circ \varphi^{-1} \) is the Fourier transform for \( \mathbb{C}G \) on \( \mathbb{R} \) up to phase factors. Here, \( F_m = F_H \) is the Fourier transform for \( \mathbb{C}H \) defined in Section 5.

We now consider the implementation of \( \mathcal{F}_G \). The encoding \( \mathcal{E}_{\text{time}} : \mathcal{B}_{\text{time}} \rightarrow \mathbb{Z}_{qm} \) is given above. Let \( \mathcal{E}_{\text{freq}} : \mathcal{B}_{\text{freq}} \rightarrow \mathbb{Z}_{qm} \) be given by \( b^j \otimes \zeta^i \mapsto mj + i \). With respect to \( \mathcal{E}_{\text{time}} \) and \( \mathcal{E}_{\text{freq}} \), the transform \( \mathcal{H} \) is implemented by \( I_q \otimes_R F_m \).

With respect to \( \mathcal{E}_{\text{time}} \) and \( \mathcal{E}_{\text{freq}} \), the transform \( U_1 \) can be represented by

\[
|km + im/d + x\rangle \mapsto \begin{cases} 
|km + im/d + x\rangle & \text{if } 1 < x < m/d \\
\omega_q^{sk} \frac{1}{\sqrt{q}} \sum_{j \in \mathbb{Z}_q} \omega_q^{jk} |jm + im/d + x\rangle & \text{if } x = 0.
\end{cases}
\]

Here, \( k \in \mathbb{Z}_q, i \in \mathbb{Z}_d, \) and \( x \in \mathbb{Z}_{m/d} \).

Written as a generalized Kronecker product, this is

\[
(F_q \otimes_R I_d) \times \Phi_{qd}(\omega_q^s), I_{qd}, \ldots, I_{qd}) \otimes_R I_{m/d}.
\]

Thus, with respect to \( \mathcal{E}_{\text{time}} \) and \( \mathcal{E}_{\text{freq}} \), \( \mathcal{F}_G \) is computed up to phase factors by a quantum circuit implementing

\[
\mathcal{F}_G^\phi = \left( (F_q \otimes_R I_d) \times \Phi_{qd}(\omega_q^s), I_{qd}, \ldots, I_{qd} \right) \otimes_R I_{m/d} \times \left( I_q \otimes_R F_m \right).
\]

Theorem 14 follows.

### 7 Fourier transforms related to error-correction

In this section, we give a quantum circuit for computing a Fourier transform for a certain subgroup \( E_n \) of the orthogonal group \( O(2^n) = \{ A \in \text{GL}_{2^n}(\mathbb{C}) : AA^t = I \} \). The group \( E_n \) was used independently by Gottesman [18] and Calderbank et. al. [9] to give a group theoretical framework for studying quantum error-correcting codes.

For all \( i = 1, \ldots, n \), define

\[
X_i = I_{2^{i-1}} \otimes_R X \otimes_R I_{2^{n-i}} \]
\[
Z_i = I_{2^{i-1}} \otimes_R Z \otimes_R I_{2^{n-i}} \]
\[
Y_i = I_{2^{i-1}} \otimes_R Y \otimes_R I_{2^{n-i}} \]
where $X, Z,$ and $Y$ are given as in Section 3. The group $E_n$ is the group generated by these $3n$ unitary matrices. Its order is $2 \cdot 4^n$. Every element squares to either $I$ or $-I$, and two elements either commute or anti-commute. When $n = 0$, $E_n = \{[\pm 1]\}$ is a cyclic group of order two, and if $n = 1$, $E_n$ is isomorphic to $D_4$. For larger $n$, $E_n$ is isomorphic to $D_4^n/K_n$ where $K_n$ is a normal subgroup isomorphic to $Z_2^{n-1}$. Given $a, c \in Z_2^n$, $a = (a_1, \ldots, a_n)$ and $c = (c_1, \ldots, c_n)$, let $X(a)$ and $Z(c)$, respectively, denote the elements $\prod_{i=1}^n X_{i}^{a_i}$ and $\prod_{i=1}^n Z_{i}^{c_i}$, respectively. Then every element $g$ of $E_n$ can be written uniquely in the form
\[ g = (-I)^{X(a)}Z(c) \tag{21} \]
where $\lambda \in Z_2$, and $a, c \in Z_2^n$. We denote $g$ by the 3–tuple $(\lambda, a, c)$. By rewriting Equation (21), $g$ can be written as a right Kronecker product
\[ g = (\lambda, a, c) = ((-I_2)^{X_{a_1}Z_{c_1}}) \otimes_R (X_{a_2}Z_{c_2}) \otimes_R \cdots \otimes_R (X_{a_n}Z_{c_n}) \tag{22} \]
For $n \geq 1$, let $H \leq E_n$ be the subgroup $\{ (\lambda, a, c) \in E_n : a_n = c_n = 0 \}$ of index 4, and identify $E_{n-1}$ with $H$ in $E_n$. Write $E_n = T E_{n-1}$ where $T = \{ X_{n}^{a_n}Z_{c_n} : a_n, c_n \in Z_2 \}$ is a left transversal for $E_{n-1}$ in $E_n$. The group $E_n$ has a complete set $R_{(n)}$ of $1 + 2^{2n}$ inequivalent, irreducible and unitary representations, all but one of dimension one (except for $n = 0$ where both representations, denoted $(0)\rho$ and $(0)\sigma$, are one-dimensional). The $2^{2n}$ one-dimensional representations $\{ (n)\rho^{xz} \}_{x, z \in Z_2^n}$ are given by
\[ (n)\rho^{xz}(g) = (n)\rho^{xz}((\lambda, a, c)) = (-1)^{x \cdot a + z \cdot c}. \]
The last representation, $(n)\sigma$, has dimension $2^n$ and is the group itself. From Equation (22), we have the following recursive expression for the $(kk, ll)$--th entry of the element $g = (\lambda, aa_n, cc_n) \in E_n$, $a, c, k, l \in Z_2^{n-1}$,
\[ (n)\sigma_{kk, ll}((\lambda, aa_n, cc_n)) = (-1)^{l_n c_n} \delta_{d_n a_n} (n-1)\sigma_{kl}((\lambda, a, c)) \]
where $d_n = k_n \oplus l_n \in Z_2$. Hence, $R_{(n)}$ is $E_{n-1}$–adapted relative to $R_{(n-1)}$. We use the concept of adapted representations to find a Fourier transform for $\mathbb{C}E_n$. Let the bases $B_{\text{temp}}$, $B_{\text{freq}}$, $B_{\text{time}}$, and $B_{\text{temp}} = T \otimes B_{\text{freq}}$ be given as in Section 4. Let $\varphi : \langle T \otimes B_{\text{temp}} \rangle \to \langle B_{\text{time}} \rangle$ denote the natural isomorphism defined in Section 4.
The matrix coefficients of $\mathcal{R}_n$ can be written as linear sums of the basis-elements $\mathcal{B}_{\text{temp}}$

\[
(n)\rho^{x_n z_n} = \sum_{\lambda \in \mathbb{Z}_2} \sum_{a, c \in \mathbb{Z}_2} (n)\rho^{x_n z_n} \phi((\lambda, a, c)) (\lambda, a, c)
\]

\[
= \sum_{a_n \in \mathbb{Z}_2} \sum_{c_n \in \mathbb{Z}_2} (-1)^{a_n x_n + c_n z_n} \varphi(X_n^{a_n} Z_n^{c_n} \otimes (n-1)\rho^{x z})
\]

\[
(n)\sigma_{kkll} = \sum_{\lambda \in \mathbb{Z}_2} \sum_{a, c \in \mathbb{Z}_2} (n)\sigma_{kkll} \phi((\lambda, a, c)) (\lambda, a, c)
\]

\[
= \sum_{c_n \in \mathbb{Z}_2} (-1)^{c_n l_n} \left( \sum_{a_n \in \mathbb{Z}_2} \sum_{a, c \in \mathbb{Z}_2}^{n-1} (n-1)\sigma_{kl} ((\lambda, a, c)) (\lambda, a d_n, c c_n) \right)
\]

\[
= \sum_{c_n \in \mathbb{Z}_2} (-1)^{c_n l_n} \varphi(X_n^{a_n} Z_n^{c_n} \otimes (n-1)\sigma_{kl})
\]

where $x, z, k, l \in \mathbb{Z}_2^{n-1}$ and $d_n = k_n \oplus l_n \in \mathbb{Z}_2$. Hence

\[
\begin{align*}
 b_{(n)\rho^{x_n z_n}, 1, 1} &= \frac{1}{2} \sum_{a_n \in \mathbb{Z}_2} \sum_{c_n \in \mathbb{Z}_2} (-1)^{a_n x_n + c_n z_n} \varphi(X_n^{a_n} Z_n^{c_n} \otimes b_{(n-1)\rho^{x z}, 1, 1}) \\
 b_{(n)\sigma_{kkll}, ll} &= \frac{1}{\sqrt{2}} \sum_{c_n \in \mathbb{Z}_2} (-1)^{c_n l_n} \varphi(X_n^{a_n} Z_n^{c_n} \otimes b_{(n-1)\sigma_{kl}})
\end{align*}
\]

Equation (23) seems to have the form of two $W$ transforms for the one-dimensional representations $\rho$, and a single $W$ transform for the $\sigma$ representation. With respect to an appropriate encoding, this is indeed the case. Choose the encoding $E^{(n)} : E_n \rightarrow \mathbb{Z}_2^{2n+1}, n \geq 0$,

\[
E^{(0)}_{\text{time}}((\lambda, \epsilon, c)) = \lambda
\]

\[
E^{(n)}_{\text{time}}((\lambda, a a_n, c c_n)) = E^{(n-1)}_{\text{time}}((\lambda, a, c)) a_n c_n
\]

\[
E^{(n)}_{\text{temp}}(X_n^{a_n} Z_n^{c_n} \otimes (n-1)\sigma_{kl}) = E^{(n-1)}_{\text{freq}} ((n-1)\sigma_{kl}) a_n c_n
\]

\[
E^{(0)}_{\text{freq}}((0)\rho) = 0
\]

\[
E^{(0)}_{\text{freq}}((0)\sigma) = 1
\]

\[
E^{(n)}_{\text{freq}}((n)\rho^{x_n z_n}) = E^{(n-1)}_{\text{freq}} ((n-1)\rho^{x_z}) x_n z_n
\]

\[
E^{(n)}_{\text{freq}}((n)\sigma_{kkll}) = E^{(n-1)}_{\text{freq}} ((n-1)\sigma_{kl}) a_n l_n \quad \text{(where } a_n = k_n \oplus l_n)\]

On the right hand side of the expressions, think of the images of the encoding as binary strings with standard string concatenation.
With respect to this encoding, the transform \( U : \langle \mathcal{B}_{\text{temp}} \rangle \rightarrow \langle \mathcal{B}_{\text{freq}} \rangle \), for \( n \geq 1 \), can be represented by

\[
|\lambda s a_n c_n \rangle \mapsto \begin{cases}
\frac{1}{2} \sum_{x_n \in \mathbb{Z}_2} \sum_{z_n \in \mathbb{Z}_2} (-1)^{a_n x_n + c_n z_n} |\lambda s a_n z_n \rangle & \text{if } \lambda = 0 \\
\frac{1}{\sqrt{2}} \sum_{l_n \in \mathbb{Z}_2} (-1)^{c_n l_n} |\lambda s a_n l_n \rangle & \text{if } \lambda = 1
\end{cases}
\]

where \( \lambda \in \mathbb{Z}_2, s \in \mathbb{Z}_2^{2n-2} \) and \( a_n, c_n \in \mathbb{Z}_2 \). As a generalized Kronecker product, this reads

\[
\left( I_2 \otimes_R (I_{2^{2n-2}} \otimes_R \mathbf{W}, I_{2^{2n-1}}) \right) \otimes_R \mathbf{W}.
\]

(24)

For \( n \geq 1 \), let \( \mathbf{E} \) be a quantum circuit computing the Fourier transform for \( \mathbb{C}E_{n-1} \) on \( \mathcal{R}_{(n-1)} \) with respect to the above encoding. Then, by Equation (24), the following network computes the Fourier transform for \( \mathbb{C}E_n \) on \( \mathcal{R}_{(n)} \), also with respect to the above encoding.

For \( n = 0 \), the one-bit network consisting only of the \( \mathbf{W} \) transform computes the Fourier transform. Thus, expanding this recursively defined network given above, we have

**Theorem 18** The following network computes a Fourier transform for \( \mathbb{C}E_n \).

8 Conclusion

The problem of finding efficient quantum algorithms computing a given unitary transform can be formulated as a purely matrix factorization problem. Let \( \mathcal{U} \) be a set of basic unitary matrices. Given a unitary matrix \( U \) of dimension \( (n \times n) \), can \( U \) be factorized into a product of basic unitary matrices such that the number of components in this product is polynomial bounded in \( \log(n) \)? Previously, the only operations considered allowed in this product have
been the basic binary matrix operations: multiplication and standard Kronecker product. In this paper, we have shown that allowing a generalization of the latter, efficient networks can still be obtained.

This generalized operation has several advantages. First of all, it gives a new tool when searching for factorizations of unitary matrices. The two new quantum networks given in Section 3.1 implementing the wavelet transforms were found this way. Secondly, it gives a nice compact mathematical description of more complex transforms. Thirdly, it directly gives quantum networks for computing unitary transforms which already were known to be expressible by generalized Kronecker products. This is for example the case for the Fourier transforms for the finite Abelian groups.

In this paper, we have also discussed the issue of computing Fourier transforms for finite non-Abelian groups. We have given a definition of such computations on quantum computers, and especially we have given a slightly relaxed definition where we only compute a Fourier transform up to phase factors. Using this latter definition, we have devised a quantum network computing a Fourier transform for a class of meta-cyclic groups—even without completely knowing the group structure. This relaxed definition is in particular useful if the computation are to be followed by a measurement [19], as for example in the algorithms of Deutsch and Jozsa [16], Simon [30], Shor [29], and Boneh and Lipton [7].

We have also given a simple quantum circuit computing a Fourier transform for a certain group [18, 1] used in quantum error-correcting. Together with Beals’ proposal of a quantum network for the symmetric group [4], this emphasizes a challenging question which has only been partly discussed in this paper. Namely, which applications are there for these new transforms? Clearly, one can define quantum versions of the classical applications, but are there any other applications? For example, a crucial insight in Shor’s algorithm [29] was the possibility of using the quantum version of the discrete Fourier transform to find the index of an unknown subgroup in a cyclic group. No efficient classical counterpart of this idea is known. Is this phenomena present for non-Abelian groups, too?

Acknowledgments

I am very grateful to Joan Boyar and Gilles Brassard for many valuable discussions and for their interest in this work. I am also grateful to André Berthiaume for interesting discussions on generalized Kronecker products, and to Hans J. Munkholm and René Depont Christensen for helpful discussions on representation theory. This work was completed at the Laboratoire d’informatique théorique et quantique at Université de Montréal, and I would like to thank the faculty and the students, especially Alain Tapp, for their hospitality.
References

[1] Adriano Barenco, Charles H. Bennett, Richard Cleve, David P. DiVincenzo, Norman Margolus, Peter W. Shor, Tycho Sleator, John Smolin, and Harald Weinfurter. Elementary gates for quantum computation. *Physical Review A*, 52:3457 – 3467, 1995.

[2] Robert Beals. Quantum computation of Fourier transforms over symmetric groups. In *Proc. 29th Annual ACM Symposium on Theory of Computing*, 1997. To appear.

[3] Charles H. Bennett, Ethan Bernstein, Gilles Brassard, and Umesh Vazirani. Strengths and weaknesses of quantum computing. *SIAM Journal on Computing*, 1997. To appear.

[4] Ethan Bernstein and Umesh Vazirani. Quantum complexity theory. *SIAM Journal on Computing*, 1997. To appear.

[5] André Berthiaume. Quantum computation. In *Complexity Theory Retrospective II*. Springer-Verlag, 1997. To appear.

[6] André Berthiaume and Gilles Brassard. The quantum challenge to structural complexity theory. *Journal of Modern Optics*, 41:2521 – 2535, 1994.

[7] Dan Boneh and Richard J. Lipton. Quantum cryptoanalysis of hidden linear functions (extended abstract). In *Proc. Advances in Cryptology—Crypto’95*, volume 963 of *Lecture Notes on Computer Science*, pages 424 – 437, 1995.

[8] Michel Boyer, Gilles Brassard, Peter Høyer, and Alain Tapp. Tight bounds on quantum searching. In *Proc. 4th Workshop on Physics and Computation*, pages 36 – 43, 1996.

[9] A. Robert Calderbank, Eric M. Rains, Peter W. Shor, and Neil J. A. Sloane. Quantum error correction and orthogonal geometry. *Physical Review Letters*, 1997. To appear.

[10] A. Robert Calderbank and Peter W. Shor. Good quantum error-correcting codes exist. *Physical Review A*, 54:1098 – 1106, 1996.

[11] Richard Cleve. A note on computing Fourier transformation by quantum programs. Dept. of Computer Science, University of Calgary, 1994. Unpublished.

[12] Don Coppersmith. An approximate Fourier transform useful in quantum factoring. Technical Report RC 19642, IBM T. J. Watson Research Center, 1994.

[13] Charles W. Curtis and Irving Reiner. *Representation Theory of Finite Groups and Associative Algebras*. Pure and Applied Mathematics. Interscience Publishers, 1962.
[14] Ingrid Daubechies. Orthonormal bases of compactly supported wavelets. *Communications on Pure and Applied Mathematics*, XLI(7):909–996, 1988.

[15] David Deutsch. Quantum computational networks. *Proceedings of the Royal Society of London*, A425:73–90, 1989.

[16] David Deutsch and Richard Jozsa. Rapid solution of problems by quantum computation. *Proceedings of the Royal Society of London*, A439:553–558, 1992.

[17] Bernard J. Fino and V. Ralph Algazi. A unified treatment of discrete fast unitary transforms. *SIAM Journal on Computing*, 6(4):700–717, 1977.

[18] Daniel Gottesman. Class of quantum error-correcting codes saturating the quantum Hamming bound. *Physical Review A*, 54(3):1862–1868, 1996.

[19] Robert B. Griffiths and Chi-Sheng Niu. Semiclassical Fourier transform for quantum computation. *Physical Review Letters*, 76:3228–3231, 1996.

[20] Dima Yu. Grigoriev. Testing the shift-equivalence of polynomials using quantum machines. In *Proc. International Symposium on Symbolic and Algebraic Computation*, pages 49–54, 1996.

[21] Lov K. Grover. A fast quantum mechanical algorithm for database search. In *Proc. 28th Annual ACM Symposium on Theory of Computing*, pages 212–219, 1996.

[22] Alfréd Haar. Zur theorie der orthogonalen funktionensysteme. *Mathematische Annalen*, LXIX:331–371, 1910.

[23] M. J. Hadamard. Résolution d’une question relative aux déterminants. *Bulletin des Sciences Mathématiques*, XVII:240–246, 1893.

[24] Alexey Yu. Kitaev. Quantum measurements and the Abelian stabilizer problem. L. D. Landau Institute for Theoretical Physics, Moscow, 1995. Unpublished.

[25] David K. Maslen and Daniel N. Rockmore. Generalized FFTs—A survey of some recent results. In *Proc. DIMACS Workshop in Groups and Computation—II*, 1995.

[26] Phillip A. Regalia and Sanjit K. Mitra. Kronecker products, unitary matrices and signal processing applications. *SIAM Review*, 31(4):586–613, 1989.

[27] Jean-Pierre Serre. *Linear Representations of Finite Groups*, volume 42 of *Graduate texts in mathematics*. Springer-Verlag, 1977.

[28] Peter W. Shor. Scheme for reducing decoherence in quantum computer memory. *Physical Review A*, 52:2493–2496, 1995.
[29] Peter W. Shor. Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer. *SIAM Journal on Computing*, 1997. To appear.

[30] Daniel R. Simon. On the power of quantum computation. In *Proc. 35th Annual Symposium on Foundations of Computer Science*, pages 116–123, 1994.

[31] Andrew Steane. Multiple particle interference and quantum error correction. *Proceedings of the Royal Society of London*, A452:2551, 1996.

[32] Andrew Steane. Simple quantum error-correcting codes. *Physical Review A*, 54:4741–4751, 1996.

[33] Charles Van Loan. *Computational Frameworks for the Fast Fourier Transform*, volume 10 of *Frontiers in Applied Mathematics*. Society for Industrial and Applied Mathematics, 1992.

[34] Vlatko Vedral, Adriano Barenco, and Artur Ekert. Quantum networks for elementary arithmetic operations. *Physical Review A*, 54:147–153, 1996.

[35] M. Joseph Leonard Walsh. A closed set of normal orthogonal functions. *American Journal of Mathematics*, XLV:5–24, 1923.

[36] Andrew Chi-Chih Yao. Quantum circuit complexity. In *Proc. 34th Annual Symposium on Foundations of Computer Science*, pages 352–361, 1993.