Coreset Construction via Randomized Matrix Multiplication

Jiasen Yang∗  Agniva Chowdhury†  Petros Drineas‡

Abstract

Coresets are small sets of points that approximate the properties of a larger point-set. For example, given a compact set \( S \subseteq \mathbb{R}^d \), a coreset could be defined as a (weighted) subset of \( S \) that approximates the sum of squared distances from \( S \) to every linear subspace of \( \mathbb{R}^d \). As such, coresets can be used as a proxy to the full dataset and provide an important technique to speed up algorithms for solving problems including principal component analysis, latent semantic indexing, etc. In this paper, we provide a structural result that connects the construction of such coresets to approximating matrix products. This structural result implies a simple, randomized algorithm that constructs coresets whose sizes are independent of the number and dimensionality of the input points. The expected size of the resulting coresets yields an improvement over the state-of-the-art deterministic approach. Finally, we evaluate the proposed randomized algorithm on synthetic and real data, and demonstrate its effective performance relative to its deterministic counterpart.

1 Introduction

In computational geometry, coresets are small sets of points that approximate the shape and properties of a larger point-set. Many natural optimization problems have coresets that approximate the optimal solution to within relative-error accuracy and which can be found quickly, in linear or near-linear time on the input size. The size of those coresets is typically bounded by a function of the accuracy parameter and is independent of the input size.

In such settings, the existence of coresets for a particular optimization problem allows the design and analysis of (near) linear-time approximation schemes, based on the idea of finding a coreset and then applying an exact optimization algorithm to it. If the size of the coreset is bounded and independent of the cardinality of the original point-set, the running time of the resulting approximation scheme will also be independent of the cardinality of the original point-set. Assuming that the coreset construction is efficient, such approximation schemes could lead to significant gains in running times with a small, provably bounded, loss in accuracy [1].

To make our discussion more precise, let the matrix \( A \in \mathbb{R}^{n \times d} \) represent a collection of \( n \) objects described with respect to \( d \) features, or, equivalently, \( n \) points in \( \mathbb{R}^d \). In this setting, a coreset would correspond to a subset of rows of \( A \) (perhaps weighted by non-negative weights) that would suffice to approximate the solution to an optimization problem (e.g., regression problems with respect to various \( \ell_p \) norms [16, 17, 9, 10]) up to relative error \( \varepsilon \). Ideally, the size of the coreset would depend only on \( 1/\varepsilon \) and be independent of \( n \) and hopefully even \( d \), namely the number and dimensionality of the input points. The running time to construct the coreset would ideally be proportional to the number of non-zero entries in \( A \) (the sparsity of \( A \)), perhaps up to polylogarithmic factors. Then,
the resulting coreset (the subset of rows of $A$) could be used to solve, say, a regression problem on just the coreset while guaranteeing a relative-error $(1+\varepsilon)$-approximation to the regression problem induced by the full matrix $A$. 

In this work, we follow the lines of [21] (which builds upon [20]) and focus on the following definition of a coreset. Given a matrix $A$ representing $n$ points in $\mathbb{R}^d$, we seek a coreset consisting of a small number of rows of $A$ such that the sum of squared distances from any given $k$-dimensional (linear) subspace to the rows of $A$ is approximately the same as the sum of squared (weighted) distances to the rows of the coreset. The following definition formalizes this concept.$^4$

**Definition 1** (Property 1 in [21]). Let $A \in \mathbb{R}^{n \times d}$, $k \in \{1,2,\ldots,d-1\}$, and $0 < \varepsilon \leq 1$. For a diagonal matrix $W \in \mathbb{R}^{n \times n}$ with non-negative entries, the matrix $C = WA$ is a $(k, \varepsilon)$-coreset of $A$ if for every matrix $X \in \mathbb{R}^{d \times (d-k)}$ such that $X^T X = I$, we have

$$
(1 - \varepsilon)\|AX\|_F^2 \leq \|WAX\|_F^2 \leq (1 + \varepsilon)\|AX\|_F^2. 
$$

(1)

We note that the problem in Definition $\S$ is of paramount importance in machine learning. This particular notion of $(k, \varepsilon)$-coresets is intimately related to dimensionality reduction techniques such as principal components analysis (PCA) and latent semantic indexing (LSI). We refer the reader to [21] for further discussion motivating Definition $\S$ Indeed, [21] was the first paper to explicitly address the long-open research question regarding the existence and efficient construction of a $(k, \varepsilon)$-coreset that is independent of the input size, and to articulate its connections to PCA and LSI.

### 1.1 Our Contributions

Our contributions are three-fold. First, we provide a structural result (Theorem $\S$) that connects the construction of $(k, \varepsilon)$-coresets to approximating matrix products. To the best of our knowledge, this structural result is novel and is much simpler than its predecessors (cf. Section $\S$). Importantly, it can leverage any rank-$k$ approximation to the input matrix $A$, instead of working with only the best rank-$k$ approximation to $A$ as computed by the singular value decomposition (SVD). Second, the structural result almost immediately implies a simple, randomized algorithm which constructs a coreset of size (in expectation) $O(k \ln(k)/\varepsilon^2)$ that satisfies the conditions of Definition $\S$ with constant probability (Theorem $\S$). The algorithm utilizes a Bernoulli-sampling analog of well-known results on randomized matrix multiplication that date back to [12, 13]. We present a self-contained description of this Bernoulli-sampling version of randomized matrix multiplication in Appendix $\S$ to the best of our knowledge, many of the proofs therein have not explicitly appeared in prior work and are of independent interest. We note that the proposed randomized algorithm (Algorithm $\S$) depends not only on (approximations to) the leverage scores $[25, 15]$ of $A$, but also on the norms of the rows of the residual matrix (cf. Eq. $\S$). To the best of our knowledge, this is a novel feature of our approach, which indicates that sampling rows proportional to their leverage scores alone is not sufficient to guarantee the structural conditions required by Theorem $\S$. Third, we present a thorough empirical evaluation of our approach on synthetic and real data matrices, demonstrating that randomized algorithms are competitive to (and often outperform) their deterministic counterparts, even when an approximate sampling distribution is used.

Formally, the following two theorems capture the theoretical substance of our contributions. First, let the input matrix $A$ be decomposed as $A = ZZ^T A + E$, where $Z \in \mathbb{R}^{n \times k}$ and $Z^T Z = I$

$^4$Our definition deviates slightly from Property 1 in [21]. First, we square the Frobenius norms to be consistent with the actual result shown in the proof of Theorem 1 in Section 3.1 of [21]. Second, and more importantly, we omit the second condition from Property 1 of [21]. According to [18], this condition, as stated, is erroneous and should be omitted.
(which implies that $Z^TE = 0$). For example, it suffices to take $Z$ to be the matrix consisting of the top $k$ left singular vectors of $A$. In this case, $ZZ^TA$ forms the best rank-$k$ approximation of $A$ under the Frobenius norm. More generally, we can take $Z$ to be any approximation to the top $k$ left singular vectors of $A$, as long as we have control on the approximation error:

$$\|E\|_F \leq (1 + \varepsilon_0)\|A - A_k\|_F$$

for some $\varepsilon_0 \geq 0$, where $A_k$ is the best rank $k$ approximation to $A$ as computed by the SVD. This strategy was pioneered by [22] [4] [5], which also presented algorithms for constructing such a matrix $Z$ in time $O(ndk/\varepsilon_0)$. Using the projection method of [7], the $nd$ factor in the running time can be reduced to $\text{nnz}(A)$, the sparsity of the input matrix $A$ (see [30] for details).

**Theorem 1.** Given a matrix $A \in \mathbb{R}^{n \times d}$, let

$$A = ZZ^TA + E$$

be a decomposition of $A$ for some $Z \in \mathbb{R}^{n \times k}$ such that $Z^TZ = I$ and

$$\|E\|_F \leq (1 + \varepsilon_0)\|A - A_k\|_F$$

for some $\varepsilon_0 \geq 0$, where $A_k$ is the best rank $k$ approximation to $A$ as computed by the SVD. Assume that for some constants $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \geq 0$, there exists a diagonal matrix $W \in \mathbb{R}^{n \times n}$ with non-negative diagonal entries such that the conditions

$$\|Z^TW^TZW - I\|_2 \leq \varepsilon_1$$

$$\|W^2 - \|E\|^2_F\|_F \leq \varepsilon_2\|E\|^2_F$$

$$\|E^TW^TWE - E^TE\|_F \leq \varepsilon_3\|E\|^2_F$$

$$\|E^TW^TWZ\|_F \leq \varepsilon_4\|E\|_F$$

are all satisfied. Then, for any $X \in \mathbb{R}^{d \times (d-k)}$ satisfying $X^TX = I$, we have

$$\left|\|WAX\|^2_F - \|AX\|^2_F\right| \leq \left[\varepsilon_1 + \sqrt{2(\varepsilon_2^2 + k\varepsilon_3^2)(1 + \varepsilon_0)^2 + \varepsilon_4(1 + \varepsilon_0)}\right]\|AX\|^2_F. \tag{8}$$

The above theorem provides structural conditions that are sufficient to guarantee the existence of a $(k, \varepsilon)$-coreset whose size equals the number of non-zero entries in the diagonal of $W$. The following theorem provides a particularly simple construction that fulfills all the required structural conditions.

**Theorem 2.** Given a matrix $A \in \mathbb{R}^{n \times d}$, there exists a randomized algorithm (Algorithm [1]) that constructs a random diagonal matrix $W \in \mathbb{R}^{n \times n}$ with at most

$$c = O\left(\frac{k \ln k}{\varepsilon^2}\right) \tag{9}$$

non-zero entries in its diagonal (in expectation), such that with probability at least $1 - \delta$, Eqs. (4), (5), (6), (7) are all satisfied with $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ set to $O(\varepsilon)$ for some $0 < \varepsilon \leq 1$.

Adjusting constants, combining with Theorem [1] and noting the existence of various algorithms to construct decompositions of the form $A = ZZ^TA + E$ that guarantee $\varepsilon_0 = O(\varepsilon)$, the above result implies that Algorithm [1] constructs a $(k, \varepsilon)$-coreset of size (in expectation) at most $c = O(k \ln k/\varepsilon^2)$. The running time of Algorithm [1] is dominated by the time it takes to compute the decomposition $A = ZZ^TA + E$, since the remaining steps of the algorithm can be easily implemented in one sequential pass over the computed matrices $Z$ and $E$. 

---

3
1.2 Related Work

Recent work on various coreset constructions for machine learning problems include [19] [20] [3] [21]. The work most closely related to ours is [21], which provided the first deterministic construction of a $(k, \varepsilon)$-coreset with size $O(k^2/\varepsilon^2)$ that is independent of both the number of input points $n$ and their dimensionality $d$. In this paper, we provide a key insight on coreset constructions, connecting them to the well-studied problem of (randomized or deterministic) approximate matrix multiplication [12] [13] [23] [15]. Our approach is very different from previous approaches that are largely inspired by geometric intuition. Our structural result (Theorem 1) is somewhat reminiscent of similar results in [27] (e.g., the structural results in Chapter 3), which also appeared later in [8]. However, our structural result is much simpler (perhaps because the objective is slightly different) and, to the best of our knowledge, novel. In terms of coreset size, we note that our algorithm has a failure probability (as opposed to the deterministic approach of [21]), but results in a provably smaller coreset size (in expectation) that is almost optimal (up to a logarithmic factor). To the best of our knowledge, achieving coresets of size $O(k\ln k/\varepsilon^2)$ cannot be accomplished by sampling with respect to the leverage scores only (i.e., using only the first term in Eq. (21)); using probabilities that depend on the residual matrix seems necessary. Finally and interestingly, we note that the coreset construction algorithms of [19] [3] [24] also involve importance sampling with respect to a mixture of probability distributions, although the specific form of their mixing components as well as their algorithmic intuitions are quite different from our proposed approach.

2 Preliminaries

We write the compact SVD of $A \in \mathbb{R}^{n \times d}$ as $A = U_{n \times r} \Sigma_{r \times r} V_{r \times d}^T$, where $r = \text{rank}(A)$, $U^T U = V^T V = I$ and $\Sigma = \text{diag}\{\sigma_1, \ldots, \sigma_r\}$ consists of the non-zero singular values of $A$ sorted in non-increasing order, $\sigma_1 \geq \cdots \geq \sigma_r > 0$. Denote $\Sigma_k = \text{diag}\{\sigma_1, \ldots, \sigma_k\} \in \mathbb{R}^{k \times k}$ and let $U_k \in \mathbb{R}^{n \times k}$ and $V_k \in \mathbb{R}^{d \times k}$ be the top $k$ left and right singular vectors of $A$, respectively. Similarly, denote $\Sigma_{k,\perp} = \text{diag}\{\sigma_{k+1}, \ldots, \sigma_r\} \in \mathbb{R}^{(r-k) \times (r-k)}$ and let $U_{k,\perp} \in \mathbb{R}^{n \times (r-k)}$ and $V_{k,\perp} \in \mathbb{R}^{d \times (r-k)}$ be the bottom $r-k$ left and right singular vectors of $A$. Writing $A_k = U_k \Sigma_k V_k^T$ and $A_{k,\perp} = U_{k,\perp} \Sigma_{k,\perp} V_{k,\perp}^T$, we have that $A = A_k + A_{k,\perp}$ and $A_k^T A_{k,\perp} = 0$. It is well-known that $A_k$ gives the best rank-$k$ approximation to $A$ under the Frobenius (or any unitarily invariant) norm, i.e., $A_k = \text{argmin}_{B \in \mathbb{R}^{n \times d}, \text{rank}(B)=k} \|A - B\|_F$.

Next, we review von Neumann’s trace inequality, an important tool in our analysis. Recall that the trace of a square matrix $A$, denoted $\text{tr}(A)$, is the sum of its diagonal entries.

**Proposition 3** (von Neumann’s trace inequality [20]). For any matrices $A, B \in \mathbb{R}^{n \times n}$ with singular values $\{\sigma_i(A)\}_{i=1}^n$ and $\{\sigma_i(B)\}_{i=1}^n$ respectively,

$$|\text{tr}(AB)| \leq \sum_{i=1}^n \sigma_i(A) \sigma_i(B)$$

where equality is achieved when $A$ and $B$ are simultaneously diagonalizable.

3 Proof of the Structural Result (Theorem 1)

In this section, we outline the proof of our structural result (Theorem 1). First, recall that we are working with a decomposition of the input matrix $A$, namely $A = ZZ^T A + E$, where $Z \in \mathbb{R}^{n \times k}$ and
Writing $\tilde{A} \triangleq ZZ^T A$, we have $A = \tilde{A} + E$, and note that $\tilde{A}^T E = A^T ZZ^T E = 0$. Using this decomposition, we can rewrite the left-hand-side of Eq. (8) as follows:

$$\|WAX\|_F^2 - \|AX\|_F^2 = \left| \text{tr} \left( X^T \tilde{A}^T W^T WAX - X^T \tilde{A}^T AX \right) \right|$$

$$\leq \left| \text{tr} \left( X^T \tilde{A}^T W^T WAX - X^T \tilde{A}^T AX \right) \right| + \left| \text{tr} \left( X^T E^T W^T WE - X^T E^T EX \right) \right|$$

$$+ \left| \text{tr} \left( X^T E^T W^T WEX - X^T E^T AX \right) \right| + \left| \text{tr} \left( X^T E^T W^T WEX - X^T E^T EX \right) \right|$$

$$= \left| \text{tr} \left( X^T \tilde{A}^T W^T WAX - X^T \tilde{A}^T AX \right) \right| + \left| \text{tr} \left( X^T E^T W^T WEX - X^T E^T AX \right) \right|$$

$$+ 2 \left| \text{tr} \left( X^T \tilde{A}^T W^T WEX \right) \right| .$$

We now proceed to bound the terms $\Delta_1$, $\Delta_2$, and $\Delta_3$ separately.

**Bounding $\Delta_1$.** For the first term in Eq. (10), we have

$$\Delta_1 = \left| \text{tr} \left( X^T \tilde{A}^T W^T WAX - X^T \tilde{A}^T AX \right) \right|$$

$$= \left| \text{tr} \left( X^T A^T ZZ^T W^T WZAX - X^T A^T ZZ^T ZZ^T AX \right) \right|$$

$$= \left| \text{tr} \left( X^T A^T Z (Z^T W^T WZ - I) Z^T AX \right) \right| = \left| \text{tr} \left( X^T A^T Z E^T Z^T AX \right) \right| ,$$

where $E_1 \triangleq Z^T W^T WZ - I$. Eq. (4) implies that $\|E_1\|_2 \leq \varepsilon_1$. Therefore, applying von Neumann’s trace inequality (Proposition 3), we have

$$\left| \text{tr} \left( X^T A^T Z E^T Z^T AX \right) \right| \leq \sum_i \sigma_i(E_1) \sigma_i(Z^T AX (Z^T AX)^T) = \sum_i \sigma_i(E_1) \sigma_i^2(Z^T AX)$$

$$\leq \|E_1\|_2 \sum_{i=1}^k \sigma_i^2(Z^T AX) = \varepsilon_1 \|Z^T AX\|_F^2 = \varepsilon_1 \|ZZ^T AX\|_F^2 ,$$

$$= \varepsilon_1 \|AX\|_F^2 ,$$

where the last inequality in Eq. (12) follows from matrix Pythagoras’ theorem, $\|AX\|_F^2 = \|\tilde{A}X\|_F^2 + \|EX\|_F^2$ since $\tilde{A}^T E = 0$. Finally, combining Eq. (11) and Eq. (12), we have shown that

$$\Delta_1 = \left| \text{tr} \left( X^T \tilde{A}^T W^T WAX - X^T \tilde{A}^T AX \right) \right| \leq \varepsilon_1 \|AX\|_F^2 .$$

**Bounding $\Delta_2$.** For the second term in Eq. (10), we have

$$\Delta_2 = \left| \text{tr} \left( X^T E^T W^T WEX - X^T E^T EX \right) \right|$$

$$= \left| \text{tr} \left( X^T (E^T W^T WE - E^T E) X \right) \right| = \left| \text{tr} \left( X^T E_2 X \right) \right| = \left| \text{tr} \left( E_2 XX^T \right) \right| ,$$

which we will bound separately.
where \( E_2 \triangleq E^T W^T W E - E^T E \). If we let \( X_\perp \) be an \( d \times k \) matrix with \( k \) orthonormal columns in \( \mathcal{R}(X)_\perp \) (the orthogonal component of the column space of \( X \)), then \( XX^T + X_\perp X_\perp^T = I \), and

\[
\Delta_2^2 = \text{tr}^2 \left( E_2 (I - X_\perp X_\perp^T) \right) = \left( \text{tr} (E_2) - \text{tr} \left( E_2 X_\perp X_\perp^T \right) \right)^2 \leq 2 \left( \text{tr}^2 (E_2) + \text{tr}^2 \left( E_2 X_\perp X_\perp^T \right) \right).
\]

(14)

Next, applying Proposition 3 and the Cauchy-Schwarz inequality, consecutively to the second term of the right-hand side in Eq. (14) yields

\[
\text{tr}^2 \left( E_2 X_\perp X_\perp^T \right) \leq \left( \sum_i \sigma_i (E_2) \sigma_i (X_\perp X_\perp^T) \right)^2 \leq \left( \sum_i \sigma_i^2 (E_2) \right) \left( \sum_j \sigma_j^2 (X_\perp X_\perp^T) \right)
= \text{tr} \left( E_2^T E_2 \right) \text{tr} \left( X_\perp X_\perp^T \right) = k \|E_2\|_F^2.
\]

(15)

Combining Eqs. (14), (15) and applying Eqs. (5), (6), (3), we have

\[
\Delta_2^2 \leq 2 \left( \text{tr}^2 (E_2) + \text{tr}^2 \left( E_2 X_\perp X_\perp^T \right) \right) \leq 2 \left( \text{tr}^2 (E_2) + k \|E_2\|_F^2 \right) \leq 2 (\varepsilon_2^2 + k \varepsilon_3^2) \|E\|_F^4.
\]

Finally, since \( A_k \) is the best rank-\( k \) approximation to \( A \), and \( AX_\perp X_\perp^T \) has rank at most \( k \), we have

\[
\|A - A_k\|_F^2 \leq \|A - AX_\perp X_\perp^T\|_F^2 = \|A(I - X_\perp X_\perp^T)\|_F^2 = \|AXX^T\|_F^2 = \|AX\|_F^2,
\]

(16)

and thus

\[
\Delta_2 \leq \sqrt{2 (\varepsilon_2^2 + k \varepsilon_3^2) (1 + \varepsilon_0)^2 \|AX\|_F^2}.
\]

(17)

**Bounding \( \Delta_3 \).** For the third term in Eq. (10), we have

\[
\Delta_3 = \left| \text{tr} \left( X^T E^T W^T W \tilde{A} X \right) \right| = \left| \text{tr} \left( E^T W^T W Z Z^T A X X^T \right) \right| = \left| \text{tr} \left( E_3 Z^T A X X^T \right) \right|
= \left| \text{tr} \left( E_3 Z^T A X X^T \right) \right|,
\]

(18)

where \( E_3 \triangleq E^T W^T W Z \). Now, applying Proposition 3 and the Cauchy-Schwarz inequality to the right-hand-side of Eq. (18) yields

\[
\left| \text{tr} \left( E_3 Z^T A X X^T \right) \right| \leq \sum_i \sigma_i (E_3) \sigma_i (Z^T A X X^T) \leq \sqrt{ \left( \sum_i \sigma_i^2 (E_3) \right) \left( \sum_j \sigma_j^2 (Z^T A X X^T) \right)}
= \sqrt{\|E_3\|_F^2 \|Z^T A X X^T\|_F^2} = \|E_3\|_F \|Z Z^T A X X^T\|_F
= \|E_3\|_F \|	ilde{A} X X^T\|_F = \|E_3\|_F \|	ilde{A} X\|_F = \|E_3\|_F \|	ilde{A} \|_F \leq \|E_3\|_F \|A X\|_F = \|E_3\|_F \|A X\|_F,
\]

(19)

where the last inequality follows from the matrix Pythagoras’ theorem, \( \|A X\|_F^2 = \|	ilde{A} X\|_F^2 + \|EX\|_F^2 \), since \( \tilde{A}^T E = 0 \).

Now, combining Eqs. (18), (19) and applying Eqs. (3) and (16), we have

\[
\Delta_3 \leq \|E_3\|_F \|A X\|_F \leq \varepsilon_4 (1 + \varepsilon_0) \|A_{k,\perp}\|_F \|A X\|_F \leq \varepsilon_4 (1 + \varepsilon_0) \|A X\|_F^2.
\]

(20)
Final step. Putting together Eqs. (13), (17), and (20), we have shown that for any $X \in \mathbb{R}^{d \times (d-k)}$ satisfying $X^TX = I$,
\[
\|WXAX\|_F^2 - \|AX\|_F^2 \leq \Delta_1 + \Delta_2 + \Delta_3
\]
\[
\leq \left[ \varepsilon_1 + \sqrt{2(\varepsilon_2^2 + k\varepsilon_3^2)} (1 + \varepsilon_0)^2 + \varepsilon_4 (1 + \varepsilon_0) \right] \|AX\|_F^2,
\]
which concludes the proof of Theorem 1.

4 A Randomized Algorithm for Coreset Construction

We can leverage the structural result of Theorem 1 to design a provably accurate randomized algorithm for coreset construction (Algorithm 1). Given the decomposition $A = ZZ^TA + E$, the algorithm samples each row of $A$ with probability $p_i = \min\{c\widetilde{p}_i, 1\}$, $i = 1, \ldots, n$, where
\[
\widetilde{p}_i \triangleq \frac{1}{2} \frac{\|Z_{i*}\|_2^2}{k} + \frac{1}{2} \frac{\|E_{i*}\|_2^2}{\|E\|_F^2}
\]
(21)
satisfies $\sum_{i=1}^n \widetilde{p}_i = 1$. Algorithm 1 produces a diagonal sampling-and-rescaling matrix $W \in \mathbb{R}^{n \times n}$ which contains $c = \sum_{i=1}^n p_i$ non-zero entries in expectation.

**Algorithm 1: Randomized Algorithm for Coreset Construction**

**Input:** Matrix $A \in \mathbb{R}^{n \times d}$ and the expected coreset size $c$.

**Output:** Diagonal matrix $W \in \mathbb{R}^{n \times n}$ with at most $c$ non-zero entries in expectation.

1. Compute the decomposition $A = ZZ^TA + E$, where $Z \in \mathbb{R}^{n \times k}$ satisfies $Z^TZ = I$, and $\|E\|_F \leq (1 + \varepsilon_0) \|A_{k,\perp}\|_F$ for some $\varepsilon_0 \geq 0$.
2. for $i = 1, \ldots, n$ do
3. \[ \widetilde{p}_i \leftarrow \frac{1}{2} \frac{\|Z_{i*}\|_2^2}{k} + \frac{1}{2} \frac{\|E_{i*}\|_2^2}{\|E\|_F} \]
4. \[ p_i \leftarrow \min\{c\widetilde{p}_i, 1\} \]
5. \[ W_{ii} \leftarrow \begin{cases} \frac{1}{\sqrt{p_i}} & \text{with probability } p_i \\ 0 & \text{with probability } 1 - p_i \end{cases} \]
6. return $W$

By Theorem 1 to prove that the random diagonal matrix $W$ constructed in Algorithm 1 constitutes a $(k, \varepsilon)$-coreset of $A$, it suffices to show that $W$ satisfies Eqs. (4), (5), (6), (7) simultaneously (up to a constant failure probability). We now restate a more precise version of Theorem 2.

**Theorem 4.** Given a matrix $A \in \mathbb{R}^{n \times d}$, let $W \in \mathbb{R}^{n \times n}$ be the random diagonal matrix constructed in Algorithm 1. For any constants $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \delta \geq 0$, if the expected coreset size
\[
c \geq \max \left\{ 4 \left( 1 + \frac{\varepsilon_1}{3} \right) \frac{k \ln(8k/\delta)}{\varepsilon_2^2}, \frac{8}{\delta} \min\{\varepsilon_2, \varepsilon_3\}^2, \frac{8k}{\delta \varepsilon_4^2} \right\},
\]
(22)
then Eqs. (4), (5), (6), (7) hold together with probability at least $1 - \delta$.

\footnote{We can also prove a variant of Theorem 4 that holds with probability $1 - \delta$, while the coreset size grows as a function of $\ln(1/\delta)$. However, the sampling probabilities $\{p_i\}_{i=1}^n$ there are somewhat more involved, leading to a lengthier analysis, and we omit the discussion here.}
Prior to presenting the proof of Theorem 4, we first discuss how it can be combined with Theorem 1. Set $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon$, $\varepsilon_4 = \varepsilon / \sqrt{k}$ and assume that the algorithm used to decompose $A$ as $A = ZZ^T A + E$ satisfies $\|E\|_F \leq (1 + \varepsilon)\|A_{k, \perp}\|_F$ (see our discussion in Section 1.1). Substituting in Eq. (22) and using $\varepsilon \leq 1$, we see that setting

$$c = O \left( \frac{k \ln k}{\varepsilon^2} \right)$$

suffices to obtain (from Theorem 1):

$$\left| \|WAX\|_F^2 - \|AX\|_F^2 \right| \leq 11\varepsilon \|AX\|_F^2.$$  

The above bound holds with constant probability; adjusting constants yield the desired bound described in Section 1.1. Clearly, the (expected) size of the coreset is at most $c$, which is equal to the number of non-zero entries in the diagonal of $W$.

**Proof of Theorem 4.** A complete description of our results for randomized matrix multiplication under the Bernoulli-sampling scheme can be found in Appendix A; the following proof shall make references to several results stated therein.

To prove Eq. (4), notice that the sampling probabilities $p_i = \min\{c\tilde{p}_i, 1\}$, $i = 1, \ldots, n$ satisfy

$$\tilde{p}_i \geq \frac{1}{2} \frac{\|Z_i^*\|^2}{k}, \forall i = 1, \ldots, n \quad \text{and} \quad \sum_{i=1}^n \tilde{p}_i = 1.$$

Applying Lemma 9, if the number of sampled rows $c \geq 4 \left( 1 + \frac{\varepsilon_3}{\varepsilon_2} \right) \frac{k \ln(n/k)}{\varepsilon_1^2}$, then

$$\mathbb{P} \left( \|Z^TW^TW - I\|_2 \geq \varepsilon_1 \right) \leq \frac{\delta}{4}.$$  

To prove Eqs. (5) and (6), notice that

$$\tilde{p}_i \geq \frac{1}{2} \frac{\|E_i^*\|^2}{\|E\|^2_F}, \forall i = 1, \ldots, n \quad \text{and} \quad \sum_{i=1}^n \tilde{p}_i = 1.$$

Applying Corollary 7, we have

$$\mathbb{E} \left[ \|E^TW^TW - E^TE\|^2_F \right] \leq \frac{2}{c} \|E\|_F^4,$$

$$\mathbb{E} \left[ (\|WE\|_F^2 - \|E\|_F^2)^2 \right] \leq \frac{2}{c} \|E\|_F^4.$$

By Markov’s inequality, if the number of sampled rows $c \geq \frac{8}{\delta \min\{\varepsilon_2, \varepsilon_3\}^2}$, then

$$\mathbb{P} \left( \|WE\|_F^2 - \|E\|^2_F \geq \varepsilon_2 \|E\|^2_F \right) \leq \frac{\delta}{4},$$  

$$\mathbb{P} \left( \|E^TW^TW - E^TE\|_F \geq \varepsilon_3 \|E\|^2_F \right) \leq \frac{\delta}{4}.$$  

To prove Eq. (7), applying Lemma 6 and noticing that $E^TZ = 0$, we have

$$\mathbb{E} \left[ \|E^TW^TWZ\|^2_F \right] = \sum_{i=1}^n \frac{\|E_i^*\|^2 \|Z_i^*\|^2}{p_i} - \sum_{i=1}^n \|E_i^*\|^2 \|Z_i^*\|^2.$$  

8
Since $p_i = \min\{c \tilde{p}_i, 1\}, i = 1, \ldots, n$, where
\[
\tilde{p}_i \geq \frac{1}{2} \frac{\|Z_{is}\|^2}{k}, \forall i = 1, \ldots, n \quad \text{and} \quad \sum_{i=1}^{n} \tilde{p}_i = 1,
\]
we have
\[
E \left[ \|E^T W^T W Z\|_F^2 \right] = \sum_{\{i: \tilde{p}_i \leq c/2\}} \left( \frac{\|E_{is}\|^2}{c \tilde{p}_i} \right) \|Z_{is}\|^2 + \sum_{\{i: \tilde{p}_i > c/2\}} \left( \frac{\|E_{is}\|^2}{c \tilde{p}_i} \right) \|Z_{is}\|^2 - \sum_{i=1}^{n} \left( \frac{\|E_{is}\|^2}{c \tilde{p}_i} \right) \|Z_{is}\|^2 \leq \frac{2k}{c} \|E\|_F^2.
\]
By Markov’s inequality, if the number of sampled rows $c \geq \frac{8k}{\delta \varepsilon^2_4}$, then
\[
P \left( \|E^T W^T W Z\|_F \geq \varepsilon d \|E\|_F \right) \leq \frac{\delta}{4}.
\] (26)

Finally, applying the union bound to Eqs. (23), (24), (25), (26), we observe that if the number of sampled rows
\[
c \geq \max \left\{ 4 \left(1 + \frac{1}{3} \varepsilon_1 \right) \frac{k \ln(8k/\delta)}{\varepsilon_2^2}, \frac{8}{\delta (\min\{\varepsilon_2, \varepsilon_3\})^2}, \frac{8k}{\delta \varepsilon^2_4} \right\},
\]
then Eqs. (4), (5), (6), (7) hold together with probability at least $1 - \delta$. This concludes the proof of Theorem 4.

5 Experimental Evaluation

**Synthetic data.** We generate synthetic datasets using the procedure implemented by [21]. First, we generate a sparse matrix $A \in \mathbb{R}^{n \times d}$ of density $10^{-6}$ whose non-zero entries are drawn from a $\text{Uniform}(0, 1)$ distribution. Next, we generate a dense matrix $H \in \mathbb{R}^{n \times s}$ ($s \leq d$) where each entry is drawn from $\text{Uniform}(0, 1)$. Then, we randomly select $s$ columns of $A$, and replace them with the columns of $H$. In our experiments, we set $n = 5000, d = 1000, s = 500$.

**Real data.** We also perform experiments on two real data collections:

**Document-term data:** These data come from the TechTC-300 collection of the Open Directory Project (ODP) [11]. We utilize a subset of 10 matrices from this collection where PCA has been found to consistently separate the two document classes [6]. Each matrix contains a pair of categories from the ODP and consists of around 150 documents (rows); each document is described with respect to 10,000–20,000 words (columns). The matrix entries take on real values in $[-1, 1]$, and the matrices are sparse, each with density $(\text{nnz}(A)/n \cdot d)$ around 2%.

**Population genetics data:** These data come from the Population Reference Sample (POPRES) [28]. In particular, we use the European samples from the POPRES dataset to form 22 matrices, one for each autosomal chromosome. Each matrix contains 1,387 columns and a varying number of rows that is equal to the number of single nucleotide polymorphisms (SNPs, well-known biallelic loci of genetic variation across the human genome) in the respective chromosome. The rows of each matrix were mean-centered as a preprocessing step. The resulting matrices contain real-valued entries in $[-2, 2]$, and the density of each matrix is around 95%. It has been shown that PCA on these matrices separates the various European populations in the sample [28, 11].

Summary statistics for the real data matrices can be found in Appendix B.

9
Coreset construction methods. We compare the performance of the proposed randomized coreset construction algorithm (Algorithm 1 (Coreset-R)) with the deterministic coreset construction algorithm of [21] (Coreset-D), as well as the following random sampling methods:

Uniform random sampling: Select $c$ rows of the input matrix $A \in \mathbb{R}^{n \times d}$ uniformly at random, and form the diagonal matrix $W \in \mathbb{R}^{n \times n}$, where $W_{ii} = 1$ if the $i$-th row is selected and $W_{ii} = 0$ otherwise. Then, scale the weights by setting $W \leftarrow \alpha W$, where $\alpha = \|A\|_F/\|WA\|_F$.

Weighted random sampling: Given an input matrix $A \in \mathbb{R}^{n \times d}$, compute $U_k$, the top $k$ left singular vectors. Sample $c$ rows of $A$ with probability proportional to their leverage scores:

$$p_i = \frac{\| (U_k)_{i\ast} \|_2^2}{\sum_{i=1}^n \| (U_k)_{i\ast} \|_2^2}, \quad i = 1, \ldots, n,$$

and form the diagonal matrix $W \in \mathbb{R}^{n \times n}$, where $W_{ii} = 1$ if the $i$-th row is sampled and $W_{ii} = 0$ otherwise. Then, scale the weights by setting $W \leftarrow \alpha W$, where $\alpha = \|A\|_F/\|WA\|_F$.

For $k \in \{10, 20, 50\}$, given the coreset $C = WA$ computed using each method, we measure the average approximation error (relative to the best rank-$k$ approximation computed by an exact SVD) per data point as in [21]:

$$\text{err}(A, C) \triangleq \frac{1}{n} \cdot \frac{\| A - AQ_k Q_k^T \|_F^2 - \| A - AV_k V_k^T \|_F^2}{\| A - AV_k V_k^T \|_F^2},$$

where $V_k$ are the top $k$ right singular vectors of $A$ and $Q_k$ are the top $k$ right singular vectors of $C$. For the randomized methods, we perform 10 repeated trials and plot the means and standard errors.

Results. For the synthetic dataset, Figure 1 shows that the compared methods performed very similarly. For the real datasets, Figure 2 shows the results for one document-term data matrix and one population genetics data matrix; the complete results for all the matrices can be found in Appendix B. We observe that the proposed randomized algorithm (Coreset-R) attains consistently lower approximation error when the coreset size $c$ is large and the subspace dimension $k$ is small. Similar observations can be made for the results on the other data matrices.

For Coreset-D, Coreset-R, and weighted random sampling, an initial call to MATLAB’s svds function is required to compute the top $k$ singular vectors up to a certain tolerance $\text{tol}$. We experiment with setting $\text{tol} \in \{10^{-4}, 10^{-2}, 10^{-1}\}$ to examine the effects of approximate SVD on the quality

---

3For Coreset-D and the uniform/weighted random sampling methods, we utilize the publicly available implementation of [21] at [http://people.csail.mit.edu/mikhail/NIPS2016/].
of the constructed coresets, but do not notice any appreciable qualitative difference in the results. Finally, we note that the running times for Coreset-D, Coreset-R, and weighted random sampling are all comparable, since their common overhead is the initial call to MATLAB’s `svds` function.

6 Conclusion

Building upon the coreset definition of [21], we proved a novel and simple structural result connecting coreset construction with approximating matrix multiplication. Using tools from (randomized) matrix multiplication, we proposed a simple, randomized algorithm that constructs coresets whose size is independent of the number and dimensionality of the input points. We showed that the expected size of the resulting $(k, \varepsilon)$-coreset is $O(k \ln k / \varepsilon^2)$, which improves upon the deterministic approach of [21]. We conducted a thorough evaluation of the proposed randomized algorithm on synthetic and real data, and demonstrated that it performs comparably or superior to its deterministic counterpart.
References

[1] Pankaj K Agarwal, Sariel Har-Peled, and Kasturi R Varadarajan. Geometric approximation via coresets. *Combinatorial and Computational Geometry*, 52:1–30, 2005.

[2] Jason Altschuler, Aditya Bhaskara, Gang Fu, Vahab Mirrokni, Afshin Rostamizadeh, and Morteza Zadimoghaddam. Greedy column subset selection: New bounds and distributed algorithms. In *Proceedings of the 33rd International Conference on International Conference on Machine Learning*, volume 48 of *Proceedings of Machine Learning Research*, pages 2539–2548. PMLR, 20–22 Jun 2016.

[3] Olivier Bachem, Mario Lucic, and Andreas Krause. Coresets for nonparametric estimation - the case of DP-means. In *Proceedings of the 32nd International Conference on International Conference on Machine Learning*, volume 37, pages 209–217, 2015.

[4] Christos Boutsidis, Petros Drineas, and Malik Magdon-Ismail. Near-optimal column-based matrix reconstruction. In *The 52nd Annual IEEE Symposium on Foundations of Computer Science*, 2011.

[5] Christos Boutsidis, Petros Drineas, and Malik Magdon-Ismail. Near-optimal column-based matrix reconstruction. *The SIAM Journal on Computing*, 43(2):687–717, 2014.

[6] Christos Boutsidis, Michael W. Mahoney, and Petros Drineas. Unsupervised feature selection for principal components analysis. In *Proceedings of the 14th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, pages 61–69, 2008.

[7] Kenneth L. Clarkson and David P. Woodruff. Low rank approximation and regression in input sparsity time. In *Proceedings of the 45th annual ACM symposium on Theory of Computing*, pages 81–90, 2013.

[8] Michael B. Cohen, Sam Elder, Cameron Musco, Christopher Musco, and Madalina Persu. Dimensionality reduction for k-means clustering and low rank approximation. *arXiv:1410.6801*, 2015.

[9] Anirban Dasgupta, Petros Drineas, Boulos Harb, Ravi Kumar, and Michael W. Mahoney. Sampling algorithms and coresets for \( \ell_p \) regression. In *Proceedings of the 19th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 932–941, 2008.

[10] Anirban Dasgupta, Petros Drineas, Boulos Harb, Ravi Kumar, and Michael W. Mahoney. Sampling algorithms and coresets for \( \ell_p \) regression. *The SIAM Journal on Computing*, (38):2060–2078, 2009.

[11] Dmitry Davidov, Evgeniy Gabrilovich, and Shaul Markovitch. Parameterized generation of labeled datasets for text categorization based on a hierarchical directory. In *Proceedings of the 27th Annual International ACM SIGIR Conference on Research and Development in Information Retrieval*, pages 250–257, 2004.

[12] Petros Drineas and Ravi Kannan. Fast Monte-Carlo algorithms for approximate matrix multiplication. In *Proceedings of the 42nd IEEE Symposium on Foundations of Computer Science*, pages 452–459, 2001.

[13] Petros Drineas, Ravi Kannan, and Michael W. Mahoney. Fast Monte Carlo algorithms for matrices I: Approximating matrix multiplication. *The SIAM Journal on Computing*, 36:132–157, 2006.
[14] Petros Drineas, Jamey Lewis, and Peristera Paschou. Inferring geographic coordinates of origin for Europeans using small panels of ancestry informative markers. *PLoS ONE*, 5(8):e11892, 2010.

[15] Petros Drineas and Michael W. Mahoney. RandNLA: Randomized numerical linear algebra. *Communications of the ACM*, 59:80–90, 2016.

[16] Petros Drineas, Michael W. Mahoney, and S. Muthukrishnan. Sampling algorithms for $\ell_2$ regression and applications. In *Proceedings of the 17th Annual ACM-SIAM Symposium on Discrete Algorithm*, pages 1127–1136, 2006.

[17] Petros Drineas, Michael W Mahoney, S Muthukrishnan, and Tamás Sarlós. Faster least squares approximation. *Numerische Mathematik*, 117:219–249, 2011.

[18] Dan Feldman. Personal communication. 2017.

[19] Dan Feldman, Matthew Faulkner, and Andreas Krause. Scalable training of mixture models via coresets. In *Advances in Neural Information Processing Systems 24*, 2011.

[20] Dan Feldman, Melanie Schmidt, and Christian Sohler. Turning Big data into tiny data: Constant-size coresets for $k$-means, PCA and projective clustering. In *Proceedings of the 24th Annual ACM-SIAM Symposium on Discrete algorithms*, pages 1434–1453, 2013.

[21] Dan Feldman, Mikhail Volkov, and Daniela Rus. Dimensionality reduction of massive sparse datasets using coresets. In *Advances in Neural Information Processing Systems 29*, 2016.

[22] Nathan Halko, Per-Gunnar Martinsson, and Joel A. Tropp. Finding structure with randomness: Probabilistic algorithms for constructing approximate matrix decompositions. *SIAM Review*, 53(2):217–288, 2011.

[23] John T. Holodnak and Ilse C. F. Ipsen. Randomized approximation of the Gram matrix: Exact computation and probabilistic bounds. *The SIAM Journal on Matrix Analysis and Applications*, 36(1):110–137, 2015.

[24] Mario Lucic, Olivier Bachem, and Andreas Krause. Strong coresets for hard and soft Bregman clustering with applications to exponential family mixtures. In *Proceedings of the 19th International Conference on Artificial Intelligence and Statistics*, 2016.

[25] Michael W. Mahoney. Randomized algorithms for matrices and data. *Foundations and Trends in Machine Learning*, 3(2):123–224, 2011.

[26] Leon Mirsky. A trace inequality of John von Neumann. *Monatshefte für Mathematik*, 79(4):303–306, 1975.

[27] Cameron N Musco. *Dimensionality Reduction for k-Means Clustering*. M.Sc. Thesis, MIT, 2015.

[28] John Novembre, Toby Johnson, Katarzyna Bryc, Zoltán Kutalik, Adam R. Boyko, Adam Auton, Amit Indap, Karen S. King, Sven Bergmann, Matthew R. Nelson, Matthew Stephens, and Carlos D. Bustamante. Genes mirror geography within Europe. *Nature*, 456(7218):98–101, 2008.

[29] Joel A. Tropp. An introduction to matrix concentration inequalities. *Foundations and Trends in Machine Learning*, 8(1-2):1–230, 2015.

[30] David P. Woodruff. Sketching as a tool for numerical linear algebra. *Foundations and Trends in Theoretical Computer Science*, 10(1-2), 2014.
Appendix A    Randomized Matrix Multiplication

We study randomized matrix multiplication via Bernoulli sampling. To the best of our knowledge, many of the following results have not appeared in prior literature, therefore we gather and prove all necessary results here for convenient reference. To be consistent with the notation used in the literature on randomized matrix multiplication [12, 13, 15, 23], we write the dimensions of the input matrix as $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$ instead of $A \in \mathbb{R}^{n \times d}$ as in the main text.

Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$, recall that their product $AB$ can be computed via

$$AB = \sum_{k=1}^{n} A_{sk}B_{ks}$$

where $A_{sk}$ denotes the $k$-th column of $A$ and $B_{ks}$ the $k$-th row of $B$.

We shall approximate the product $AB$ by sampling a subset of columns of $A$ without replacement and the corresponding rows of $B$ to form the matrices $C$ and $R$, respectively. Formally, let the random variables $Z_k \sim \text{Ber}(p_k)$, $k = 1, 2, \ldots, n$ denote whether the $k$-th column of $A$ (or equivalently, the $k$-th row of $B$) is sampled. Define the diagonal sampling-and-rescaling matrix as

$$W_{n \times n} \triangleq \text{diag} \left\{ \frac{Z_1}{\sqrt{p_1}}, \frac{Z_2}{\sqrt{p_2}}, \ldots, \frac{Z_n}{\sqrt{p_n}} \right\} \quad (28)$$

and form the matrices $C_{m \times n} = AW$ and $R_{n \times p} = WB$. Note that the sampling probabilities $\{p_k\}_{k=1}^{n}$ do not have to sum to one; in fact, the expected number of sampled rows/columns is equal to $c = \sum_{k=1}^{n} p_k$. Moreover, for any $k$ such that $Z_k = 0$, the column $C_{sk}$ and row $R_{ks}$ will be all-zeros; in practice they can be dropped without affecting the product $CR$. The sampling algorithm is shown in Algorithm 2.

**Algorithm 2: Randomized Matrix Multiplication via Bernoulli Sampling**

**Input:** Matrices $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$, the expected number of sampled rows/columns $c \geq 0$, and a set of sampling probabilities $\{\tilde{p}_k\}_{k=1}^{n}$ that satisfy $\tilde{p}_k \geq 0$, $\forall k$ and $\sum_{k=1}^{n} \tilde{p}_k = 1$.

**Output:** Diagonal matrix $W \in \mathbb{R}^{n \times n}$ with at most $c$ non-zero entries in expectation.

1. for $k = 1, \ldots, n$ do
2. \hspace{1em} $p_k \leftarrow \min\{c\tilde{p}_k, 1\}$
3. \hspace{1em} $W_{kk} \leftarrow \begin{cases} \frac{1}{\sqrt{p_k}} & \text{with probability } p_k \\ 0 & \text{with probability } 1 - p_k \end{cases}$
4. return $W$

We begin by showing that for any set of sampling probabilities $\{p_k\}_{k=1}^{n}$, the resulting product $CR$ is an unbiased estimate of $AB$, and compute the variance of each entry.

**Lemma 5.** Given matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$, let $C_{m \times n} = AW$ and $R_{n \times p} = WB$, where $W$ is defined in Eq. (28), we have that

$$\mathbb{E}[(CR)_{ij}] = (AB)_{ij}$$

$$\text{Var}((CR)_{ij}) = \sum_{k=1}^{n} \frac{A_{ik}^2B_{kj}^2}{p_k} - \sum_{k=1}^{n} A_{ik}^2B_{kj}^2$$

$$\text{Cov}((CR)_{i_1j_1}, (CR)_{i_2j_2}) = \sum_{k=1}^{n} \left( \frac{1 - p_k}{p_k} \right) A_{i_1k}A_{i_2k}B_{kj_1}B_{kj_2}.$$
for any indices \( i, j, i_1, j_1, i_2, j_2 \in \{1, \ldots, n\} \).

**Proof.** For any \( i, j \in \{1, \ldots, n\} \), we have that
\[
\mathbb{E} [(\mathbf{CR})_{ij}] = \mathbb{E} \left[ \sum_{k=1}^{n} \mathbf{A}_{ik} \mathbf{W}^{2}_{kk} \mathbf{B}_{kj} \right] = \mathbb{E} \left[ \sum_{k=1}^{n} \mathbf{A}_{ik} \left( \frac{Z_{k}^{2}}{p_{k}} \right) \mathbf{B}_{kj} \right] = \sum_{k=1}^{n} \left( \mathbf{A}_{ik} \mathbf{B}_{kj} \right) \mathbb{E} \left[ Z_{k}^{2} \right]
\]
\[
= \sum_{k=1}^{n} \mathbf{A}_{ik} \mathbf{B}_{kj} = (\mathbf{AB})_{ij},
\]

since \( Z_{k}^{2} \overset{d}{=} Z_{k} \sim \text{Ber}(p) \) and thus \( \mathbb{E} [Z_{k}^{2}] = \mathbb{E} [Z_{k}] = p \), where \( \overset{d}{=} \) denotes equality in distribution. By the independence of the \( Z_{k} \)'s, and noting that \( \text{Var} (Z_{k}) = p_{k}(1 - p_{k}) \), we have that
\[
\text{Var} ((\mathbf{CR})_{ij}) = \text{Var} \left( \sum_{k=1}^{n} \mathbf{A}_{ik} \left( \frac{Z_{k}^{2}}{p_{k}} \right) \mathbf{B}_{kj} \right)
\]
\[
= \sum_{k=1}^{n} \left( \frac{\mathbf{A}_{ik} \mathbf{B}_{kj}^{2}}{p_{k}} \right)^{2} \text{Var} (Z_{k}^{2}) = \sum_{k=1}^{n} \left( \frac{1 - p_{k}}{p_{k}} \right) \mathbf{A}_{ik}^{2} \mathbf{B}_{kj}^{2}
\]
\[
= \sum_{k=1}^{n} \frac{\mathbf{A}_{ik}^{2} \mathbf{B}_{kj}^{2}}{p_{k}} - \sum_{k=1}^{n} \mathbf{A}_{ik}^{2} \mathbf{B}_{kj}^{2}.
\]

Finally, for any pairs of indices \((i_1, j_1)\) and \((i_2, j_2)\), we have that
\[
\text{Cov} ((\mathbf{CR})_{i_1j_1}, (\mathbf{CR})_{i_2j_2}) = \text{Cov} \left( \sum_{k_1=1}^{n} \mathbf{A}_{i_1k_1} \left( \frac{Z_{k_1}^{2}}{p_{k_1}} \right) \mathbf{B}_{k_1j_1}, \sum_{k_2=1}^{n} \mathbf{A}_{i_2k_2} \left( \frac{Z_{k_2}^{2}}{p_{k_2}} \right) \mathbf{B}_{k_2j_2} \right)
\]
\[
= \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \left( \frac{\mathbf{A}_{i_1k_1} \mathbf{A}_{i_2k_2} \mathbf{B}_{k_1j_1} \mathbf{B}_{k_2j_2}}{p_{k_1}p_{k_2}} \right) \text{Cov} (Z_{k_1}^{2}, Z_{k_2}^{2})
\]
\[
= \sum_{k=1}^{n} \left( \frac{\mathbf{A}_{i_1k} \mathbf{A}_{i_2k} \mathbf{B}_{kj_1} \mathbf{B}_{kj_2}}{p_{k}^{2}} \right) \text{Var} (Z_{k}^{2})
\]
\[
= \sum_{k=1}^{n} \left( \frac{1 - p_{k}}{p_{k}} \right) \mathbf{A}_{i_1k} \mathbf{A}_{i_2k} \mathbf{B}_{kj_1} \mathbf{B}_{kj_2}.
\]

This concludes the proof. \( \square \)

Next, we consider various notions of approximation error.

**Lemma 6.** Under the setting of Lemma 5, the approximation error
\[
\mathbb{E} \left[ \| \mathbf{AB} - \mathbf{CR} \|_{F}^{2} \right] = \sum_{k=1}^{n} \frac{\| \mathbf{A}_{*k} \|_{2}^{2} \| \mathbf{B}_{*k} \|_{2}^{2}}{p_{k}} - \sum_{k=1}^{n} \frac{\| \mathbf{A}_{*k} \|_{2}^{2} \| \mathbf{B}_{*k} \|_{2}^{2}}{p_{k}}.
\]

Furthermore, if \( m = p \), the approximation error
\[
\mathbb{E} \left[ (\text{tr} (\mathbf{AB} - \mathbf{CR}))^{2} \right] = \sum_{k=1}^{n} \frac{[(\mathbf{BA})_{kk}]^{2}}{p_{k}} - \sum_{k=1}^{n} [(\mathbf{BA})_{kk}]^{2}.
\]
Proof. Using Lemma 5, we have that

\[
\mathbb{E} \left[ \| AB - CR \|_F^2 \right] = \sum_{i=1}^{m} \sum_{j=1}^{p} \mathbb{E} \left[ ((AB)_{ij} - (CR)_{ij})^2 \right] = \sum_{i=1}^{m} \sum_{j=1}^{p} \text{Var} ((CR)_{ij}) \\
= \sum_{i=1}^{m} \sum_{j=1}^{p} \left[ \frac{n A_{ik}^2 B_{kj}^2}{p_k} - \frac{n A_{ik}^2 B_{kj}^2}{p_k} \right] \\
= \sum_{k=1}^{n} \left( \frac{1}{p_k} - 1 \right) \left( \sum_{i=1}^{m} A_{ik}^2 \right) \left( \sum_{j=1}^{p} B_{kj}^2 \right) \\
= \sum_{k=1}^{n} \frac{\| A_{ik} \|_2^2 \| B_{kj} \|_2^2}{p_k} - \sum_{k=1}^{n} \| A_{ik} \|_2^2 \| B_{kj} \|_2^2.
\]

Furthermore, if \( m = p \), we have that

\[
\mathbb{E} \left[ \text{tr} (AB - CR)^2 \right] = \mathbb{E} \left[ \left( \sum_{i=1}^{m} (AB - CR)_{ii} \right)^2 \right] \\
= \mathbb{E} \left[ \left( \sum_{i=1}^{m} (AB - CR)_{ii} \right) \left( \sum_{j=1}^{p} (AB - CR)_{jj} \right) \right] \\
= \sum_{i=1}^{m} \sum_{j=1}^{m} \mathbb{E} [(AB - CR)_{ii} (AB - CR)_{jj}] = \sum_{i=1}^{m} \sum_{j=1}^{p} \text{Cov} ((CR)_{ii}, (CR)_{jj}) \\
= \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{n} \left( 1 - \frac{p_k}{p_k} \right) A_{ik} A_{jk} B_{ki} B_{kj} \text{ (by Lemma 5)} \\
= \sum_{k=1}^{n} \left( 1 - \frac{p_k}{p_k} \right) \left( \sum_{i=1}^{m} A_{ik} B_{ki} \right) \left( \sum_{j=1}^{m} A_{jk} B_{kj} \right) \\
= \sum_{k=1}^{n} \left( 1 - \frac{p_k}{p_k} \right) [BA]_{kk}^2 = \sum_{k=1}^{n} \frac{[BA]_{kk}^2}{p_k} - \sum_{k=1}^{n} [BA]_{kk}^2.
\]

This concludes the proof.

We now apply Lemma 6 to obtain bounds on the approximation errors for coreset sampling.

Corollary 7. Given \( A \in \mathbb{R}^{m \times n} \), let \( C_{m \times n} = WA \), where

\[ W_{m \times m} \triangleq \text{diag} \left\{ \frac{Z_1}{\sqrt{p_1}}, \frac{Z_2}{\sqrt{p_2}}, \ldots, \frac{Z_m}{\sqrt{p_m}} \right\} \]

and \( Z_i \overset{\text{iid}}{\sim} \text{Ber}(p_i), i = 1, 2, \ldots, m \). If we set the sampling probabilities to be \( p_i \triangleq \min\{c \bar{p}_i, 1\} \), where

\[ \bar{p}_i \geq \beta \frac{\| A_{is} \|_2^2}{\| A \|_F^2}, \forall i = 1, \ldots, m \quad \text{and} \quad \sum_{i=1}^{m} \bar{p}_i = 1 \]

for some constant \( \beta \in (0, 1] \), then

\[
\mathbb{E} \left[ \| A^T A - C^T C \|_F^2 \right] = \mathbb{E} \left[ (\| A \|_F^2 - \| C \|_F^2)^2 \right] = \sum_{i=1}^{m} \frac{\| A_{is} \|_2^2}{p_i} - \sum_{i=1}^{m} \| A_{is} \|_2^4 \leq \frac{\| A \|_F^4}{c \beta}.
\]
Proof. Substituting $A$ by $A^T$ and $B$ by $A$ in Lemma 6 we have

$$
\mathbb{E} \left[ \|A^T A - C^T C\|_F^2 \right] = \sum_{i=1}^m \frac{\|A_{ix}\|^4}{p_i} - \sum_{i=1}^m \|A_{ix}\|^4
$$

and

$$
\mathbb{E} \left[ (\|A\|_F^2 - \|C\|_F^2)^2 \right] = \sum_{i=1}^m \frac{(AAT)_{ii}^2}{p_i} - \sum_{i=1}^m [(AAT)_{ii}]^2 = \sum_{i=1}^m \frac{\|A_{ix}\|^4}{p_i} - \sum_{i=1}^m \|A_{ix}\|^4.
$$

Notice that

$$
\sum_{i=1}^m \frac{\|A_{ix}\|^4}{p_i} - \sum_{i=1}^m \|A_{ix}\|^4 = \sum_{\{i: \tilde{p}_i \leq \gamma/\epsilon\}} \frac{\|A_{ix}\|^4}{\epsilon c\tilde{p}_i} + \sum_{\{i: \tilde{p}_i > \gamma/\epsilon\}} \|A_{ix}\|^4 - \sum_{i=1}^m \|A_{ix}\|^4
$$

and thus

$$
\mathbb{E} \left[ \|A^T A - C^T C\|_F^2 \right] = \mathbb{E} \left[ (\|A\|_F^2 - \|C\|_F^2)^2 \right] = \sum_{i=1}^m \frac{\|A_{ix}\|^2}{p_i} - \sum_{i=1}^m \|A_{ix}\|^2 \leq \frac{\|A\|_F^4}{c\beta},
$$

which concludes the proof. \qed

Next, we derive concentration bounds on the approximation error of Bernoulli sampling. We shall employ the following matrix Bernstein inequality:

**Proposition 8.** [29, Theorem 1.6.2] Let $X_1, \ldots, X_n \in \mathbb{R}^{d_1 \times d_2}$ be independent, centered random matrices, and assume that each one is uniformly bounded: $\mathbb{E} [S_k] = 0$ and $\|X_k\|_2 \leq L_1$ for each $k = 1, \ldots, n$. Let $L_2 \triangleq \max \left\{ \left\| \sum_{k=1}^n \mathbb{E} [X_k^T X_k] \right\|_2 , \left\| \sum_{k=1}^n \mathbb{E} [X_k X_k^T] \right\|_2 \right\}$. Then, for all $t \geq 0$,

$$
P \left( \left\| \sum_{k=1}^n X_k \right\|_2 \geq t \right) \leq (d_1 + d_2) \exp \left\{ -\frac{t^2/2}{L_2 + L_1 \epsilon t/3} \right\}.
$$

The following result is similar in spirit to Theorem 4.2 of [23], but their analysis was performed under the paradigm of repeated sampling with replacement.

**Lemma 9.** Let $A \in \mathbb{R}^{m \times n}$ and let $C = WA$, where

$$
W_{m \times m} \triangleq \text{diag} \left\{ \frac{Z_1}{\sqrt{p_1}}, \frac{Z_2}{\sqrt{p_2}}, \ldots, \frac{Z_m}{\sqrt{p_m}} \right\}
$$

and $Z_i \overset{\text{iid}}{\sim} \text{Ber}(p_i)$, $i = 1, 2, \ldots, m$. Let the sampling probabilities be $p_i = \min \{ c\tilde{p}_i, 1 \}$, $i = 1, \ldots, m$, where $\{\tilde{p}_i\}_{i=1}^m$ satisfy

$$
\tilde{p}_i \geq \beta \frac{\|A_{ix}\|^2}{\|A\|_F^2}, \quad \forall i = 1, \ldots, m \quad \text{and} \quad \sum_{i=1}^m \tilde{p}_i = 1
$$

for some constant $\beta \in (0, 1]$. Given $0 < \delta < 1$ and $0 < \epsilon \leq 1$, if the number of sampled rows is at least

$$
c \geq 2 \left( 1 + \frac{\epsilon}{3} \right) \frac{\|A\|_F^2 \ln(2n/\delta)}{\beta \|A\|_F^2 \epsilon^2}
$$

then with probability at least $1 - \delta$,

$$
\|C^T C - A^T A\|_2 \leq \epsilon \|A^T A\|_2.
$$
Therefore, applying Proposition 8, we have

\[
C^T C - A^T A = \sum_{i=1}^{m} \left( \frac{Z_i}{p_i} - 1 \right) (A_{i*})^T A_{i*} = \sum_{i=1}^{m} X_i.
\]

To apply Proposition 8, we need to find upper-bounds \( L_1 \) and \( L_2 \) such that \( \max_i \|X_i\|_2 \leq L_1 \) and \( \|\sum_{i=1}^{m} \mathbb{E} [X_i^T X_i]\|_2 \leq L_2 \). Observe that for any \( i \) such that \( p_i = 1 \), we also have \( Z_i = 1 \) almost surely, and thus \( X_i \equiv 0 \). Thus, for both \( \max_i \|X_i\|_2 \) and \( \|\sum_{i=1}^{m} \mathbb{E} [X_i^T X_i]\|_2 \) we only need to restrict attention to \( i \)'s such that \( \bar{p}_i < 1/c \). Notice that

\[
\max_i \|X_i\|_2 = \max_i \left\{ \left| \frac{Z_i}{c p_i} - 1 \right| \|A_{i*}\|_2^2 \right\} \leq \max_i \left\{ \max \left( \frac{\|A\|_F^2}{c \beta \|A_{i*}\|_2^2} - 1, 1 \right) \|A_{i*}\|_2^2 \right\}
\]

\[
= \max \left( \frac{\|A\|_F^2}{c \beta} - \min_i \|A_{i*}\|_2^2, \max_i \|A_{i*}\|_2^2 \right)
\]

\[
\leq \max \left( \frac{\|A\|_F^2}{c \beta}, \max_i \|A_{i*}\|_2^2 \right) \leq \frac{\|A\|_F^2}{c \beta},
\]

where the last inequality holds because \( \bar{p}_i < 1/c \) implies that \( \|A_{i*}\|_2^2 \leq \frac{\|A\|_F^2}{c \beta} \). Furthermore,

\[
\left\| \sum_{i=1}^{m} \mathbb{E} [X_i^T X_i]\right\|_2 = \left\| \sum_{i=1}^{m} \mathbb{E} \left[ \left( \frac{Z_i}{p_i} - 1 \right)^2 \right] (A_{i*})^T A_{i*} (A_{i*})^T A_{i*} \right\|_2
\]

\[
= \left\| \sum_{i=1}^{m} \left( \frac{1}{p_i} - 1 \right) A^T e_i \|A_{i*}\|_2^2 e_i^T A \right\|_2
\]

\[
\leq \|A\|_2^2 \sum_{i=1}^{m} \|A_{i*}\|_2^2 \left( \frac{1}{p_i} - 1 \right) e_i e_i^T \right\|_2 = \|A\|_2^2 \max_i \left\{ \|A_{i*}\|_2^2 \left( \frac{1}{p_i} - 1 \right) \right\}
\]

\[
\leq \|A\|_2^2 \max_i \left\{ \frac{\|A\|_F^2}{c \beta} - \|A_{i*}\|_2^2 \right\} = \frac{\|A\|_2^2 \|A\|_F^2}{c \beta} - \min_i \|A_{i*}\|_2^2
\]

\[
\leq \frac{\|A\|_2^2 \|A\|_F^2}{c \beta}.
\]

Therefore, applying Proposition 8 we have

\[
\mathbb{P} \left( \left\| \sum_{i=1}^{m} X_i \right\|_2 \geq \varepsilon \middle| \|A^T A\|_2 \geq \varepsilon \right) \leq 2 n \exp \left\{ - \frac{c \beta \|A\|_2^2 \varepsilon^2/2}{\|A\|_F^2 - 1 + \varepsilon/3} \right\}.
\]

Finally, setting

\[
\delta = 2 n \exp \left\{ - \frac{c \beta \|A\|_2^2 \varepsilon^2/2}{\|A\|_F^2 - 1 + \varepsilon/3} \right\}
\]

and solving for \( c \) yields

\[
c = 2 \left( 1 + \frac{\varepsilon}{3} \right) \frac{\|A\|_F^2 \ln(2n/\delta)}{\beta \|A\|_2^2 \varepsilon^2},
\]

which concludes the proof. \( \Box \)
Appendix B  Experiment Results

The summary statistics for the document-term and population genetics data matrices are shown in Tables 1 and 2 respectively.

| Dataset   | # Rows \((n)\) | # Columns \((d)\) | Density  |
|-----------|----------------|-------------------|----------|
| 10567_11346 | 139            | 17,204            | 0.0198   |
| 10567_12121 | 138            | 13,600            | 0.0197   |
| 11346_22294 | 125            | 16,259            | 0.0215   |
| 11498_14517 | 125            | 17,149            | 0.0269   |
| 14517_186330 | 130         | 19,764            | 0.0175   |
| 20186_22294 | 130            | 14,401            | 0.0207   |
| 22294_25575  | 127            | 11,608            | 0.0185   |
| 332386_61792 | 159          | 17,869            | 0.0171   |
| 61792_814096 | 159          | 18,052            | 0.0172   |
| 85489_90753  | 154            | 16,604            | 0.0168   |

| Dataset   | # Rows \((n)\) | # Columns \((d)\) | Density  |
|-----------|----------------|-------------------|----------|
| Chr1      | 36,641         | 1,387             | 0.9451   |
| Chr2      | 37,778         | 1,387             | 0.9455   |
| Chr3      | 30,917         | 1,387             | 0.9522   |
| Chr4      | 29,346         | 1,387             | 0.9443   |
| Chr5      | 29,225         | 1,387             | 0.9493   |
| Chr6      | 28,713         | 1,387             | 0.9540   |
| Chr7      | 23,525         | 1,387             | 0.9513   |
| Chr8      | 25,140         | 1,387             | 0.9473   |
| Chr9      | 20,912         | 1,387             | 0.9551   |
| Chr10     | 26,017         | 1,387             | 0.9502   |
| Chr11     | 23,963         | 1,387             | 0.9484   |
| Chr12     | 22,724         | 1,387             | 0.9476   |
| Chr13     | 17,596         | 1,387             | 0.9490   |
| Chr14     | 14,287         | 1,387             | 0.9479   |
| Chr15     | 13,103         | 1,387             | 0.9468   |
| Chr16     | 14,096         | 1,387             | 0.9440   |
| Chr17     | 10,331         | 1,387             | 0.9529   |
| Chr18     | 13,556         | 1,387             | 0.9459   |
| Chr19     | 5,848          | 1,387             | 0.9501   |
| Chr20     | 11,372         | 1,387             | 0.9531   |
| Chr21     | 6,487          | 1,387             | 0.9519   |
| Chr22     | 5,635          | 1,387             | 0.9449   |
Figures 3 and 4 show the complete experiment results for all document-term data matrices, while Figures 6–10 show the complete experiment results for all population genetics data matrices.

Figure 3: Relative approximation errors on document-term data.
Figure 4: Relative approximation errors on document-term data (continued).
Figure 5: Relative approximation errors on document-term data (continued).

Figure 6: Relative approximation errors on population genetics data.
Figure 7: Relative approximation errors on population genetics data (continued).
Figure 8: Relative approximation errors on population genetics data (continued).
Figure 9: Relative approximation errors on population genetics data (continued).
Figure 10: Relative approximation errors on population genetics data (continued).
Figure 11: Relative approximation errors on population genetics data (continued).