Intersection Type Distributors

Federico Olimpieri
LIPN, Université Sorbonne Paris Nord, France
Email: olimpieri@lipn.univ-paris13.fr

Abstract—We study a family of distributors-induced bicategorical models of \( \lambda \)-calculus, proving that they can be syntactically presented via intersection type systems. We first introduce a class of 2-monads whose algebras are monoidal categories modelling resource management. We lift these monads to distributors and define a parametric Kleisli bicategory, giving a sufficient condition for its cartesian closure. In this framework we define a proof-relevant semantics: the interpretation of a term associates to it the set of its typing derivations in appropriate systems. We prove that our model characterize solvability, adapting reducibility techniques to our setting. We conclude by describing two examples of our construction.

I. INTRODUCTION

1) A Logical Approach to Resources: The notion of resource is very important in Computer Science. A resource can be copied or deleted, and these two basic operations affect the behavior of programs. Hence, a mathematical approach to the notion of resource is naturally required, as it can clarify the understanding of how programs behave. A well-known resource-sensitive mathematical framework is linear logic, introduced by Girard [30] in the 80s. The decomposition of the intuitionistic arrow

\[
A \Rightarrow B = ! A \twoheadrightarrow B
\]

expresses the general non-linear behaviour of programs. The \(!\) construction says that we are allowed to copy or delete the input as many times as needed. Linear logic is thus immediately connected to quantitative aspects of computation.

2) Resources via Types: A few years before Girard’s introduction of linear logic, Coppo and Dezani [14] proposed intersection types, a type-theoretic framework sensitive to the fact that a \( \lambda \)-term can be typed in several ways. In order to define an intersection type system, they add another constructor to the syntax: \( a \cap b \). Then typability with an intersection type is equivalent to being typable with both types \( a \) and \( b \). This kind of type disciplines proved to be very useful to characterize fundamental notions of normalization in \( \lambda \)-calculus (e.g., head-normalization, \( \beta \)-normalization, strong normalization) [39, 3, 8]. Moreover, if the intersection type \( a \cap b \) is non-idempotent [29, 9], i.e., \( a \cap a \neq a \), the considered type system is resource sensitive. In that case, the arrow type

\[
a_1 \cap \cdots \cap a_k \Rightarrow a
\]

encodes the exact number of times that the program needs its input during computation. The resource awareness of non-idempotent intersection has been used to prove normalization and standardization results by combinatorial means [8], to study infinitary computation [57] and to express the execution time of programs and proof-nets [9, 11, 10]. The non-idempotent intersection type system \( R \), is also strongly connected to the Taylor expansion of \( \lambda \)-terms [19, 9]. Thus, resource sensitive intersection corresponds also to linear approximation. Another important feature of intersection type systems is that they determine a class of filter models for pure \( \lambda \)-calculus [13]. The correspondence between intersection types and Engeler-like models is also well-known [35]. Hence intersection types are both syntactic and semantic objects.

3) A Categorical Approach: The semantic side of intersection types is connected also to categorical semantics. A simple and informative categorical model for \( \lambda \)-calculus is the relational model (MRel)\(^1\). Objects of MRel are sets, while morphisms are multirelations \( f \subseteq M_f(A) \times B \), where \( M_f(A) \) is the free commutative monoid over \( A \). This model arises from the linear logic decomposition. It is well-known that this relational semantics corresponds to the non-idempotent intersection type system \( R \) [9]. This correspondence says that the categorical interpretation of a \( \lambda \)-term can be presented in a concrete way, as a form of typing assignment. In particular, the intersection type constructor \( \cap \) corresponds to the product in the free commutative monoid construction that gives the interpretation of the linear logic exponential connective. This fact suggests the possibility to model, in all generality, intersection types via monads. With some relevant modifications, one can also achieve in this way an idempotent intersection [17, 16].

4) Lifting to Bicategories: The idea of a bidimensional semantics for \( \lambda \)-calculus was first presented by Seely [53] and further studied in [33]. The passage from 2-category to bicategories consists in a weakening of the structure. In particular, associativity and identity laws for horizontal composition are now only up to coherent isomorphisms. In this setting, there is a natural generalization of the category of relations: the bicategory of distributors (Dist). A relation \( f \subseteq A \times B \) is the same as its characteristic function \( \chi_f : A \times B \to \{0, 1\} \). In particular, the former function naturally induces a functor from \( A \times B \), taken as discrete category, to the 2 elements category. It is then natural to relax the hypothesis and consider functors of the shape \( F : B^a \times A \to Set \) where \( A \) and \( B \) are arbitrary small categories. These functors are called distributors\(^2\). Cattani and Winskel [12] proposed a distributor-
induced semantics of concurrency. In particular, they also gave a distributor model of linear logic, generalizing Scott’s domains. In subsequent papers, Fiore, Gambino, Hyland and Winskel [26, 22] introduced the bicategory of generalized species of structures (Esp), a rich framework encompassing both multirelations and Joyal’s combinatorial species [37]. They also proved that Esp is cartesian closed and, hence, a bicategorical model for λ-calculus.

Inspired by their result, Tsukada, Asada and Ong [54, 55] showed that the generalized species semantics of λ-calculus has a syntactic counterpart in the rigid Taylor expansion of λ-terms. At the same time, building on [48, 34], Mazza, Pellissier and Vial [46] presented a higher categorical approach to intersection types and linear approximation, rooted in the framework of multicategories and discrete distributors.

This bicategorical setting has several advantages. First, one can model term rewriting in a categorical way, via 2-cells between terms denotations. Second, as shown in [46], 2-dimensional categorical constructions can be useful to reason parametrically on syntax, sparing a lot of time that normally is lost in checking special cases. Third, distributors are an example of categorification: the set-theoretic notion of relations is replaced by a category-theoretic one. It turns out that this replacement makes explicit relevant information that was hidden in the non-categorified setting.

5) Our Contribution: Building on [22, 21, 28, 54, 44], we define a family of distributor-induced denotational semantics of λ-calculus. These bicategories of distributors are Kleisli bicategories for an appropriate collection of pseudomonads, the resource monads. These are 2-monads over categories, whose algebras are some special kind of strict monoidal categories. We can sum up the results of the paper in a procedural way:

(i) Take a resource monad S and apply the construction of [21] to obtain a pseudomonad S̃ (Section II-E).
(ii) Consider the Kleisli bicategory of S̃, S-Dist. Its opposite bicategory (S-Dist)op = S-CatSym, the bicategory of S-symmetric sequences,3 is cartesian closed if the algebras of S are symmetric strict monoidal categories (Section II-F).
(iii) Consider the λ-calculus semantics induced by S-CatSym (Section III). Following the construction presented in Section III-A, get the parametric category of types DA and intersection type system EA generated by a small category A of atomic types and the resource monad S.
(iv) By the results of Section III-A2, the considered type system is a proof relevant denotational semantics for λ-calculus. The distributor that interprets a λ-term M, its intersection type distributor, is defined in the following way:

\[
\left[\left[ M \right]\right]_{A}(\Delta, a) = \begin{cases} \tilde{\pi} & \Delta \vdash M : a \\
\end{cases}
\]

where \(\tilde{\pi}\) is an equivalence class of typing derivations, \(a\) is a type and \(\Delta\) is a type context.

The equivalence relation is induced by composition in the appropriate bicategory S-CatSyn. The equivalence is crucial, since it forces the preservation under reduction not only of typability, but of the amount of classes of typing derivations. We remark that this refines and improves the standard relational semantics, where the denotation of a term is just a test of typability and do not give any information about derivations. As in [46], our construction gives rise to four intersection type systems, linear, affine, relevant and cartesian ones. The structure of the resource monad S gives the kind of intersection connective. For example, the 2-monad for symmetric strict monoidal categories determines a non-idempotent (linear) intersection. By contrast, the 2-monad for cartesian categories determines an intersection that admits duplication and erasing of resources.

Moreover, our model internalizes subtyping in a categorical framework: the preorder relation \(a \leq b\) between intersection types is replaced by an arrow \(f : a \rightarrow b\) in an appropriate category of types. The intuition behind it is that \(f\) is now a witness of subtyping. The construction of morphisms between types naturally generalizes the standard subtyping rules, as expected.

The strength of our approach is twofold. First, we are able to give a concrete presentation of a relevant class of quite abstract and esoteric semantics for λ-calculus. Second, this presentation determines a parametric theory of intersection types. In particular, our theory can account for proof-relevance, subtyping and denotational semantics.

6) Discussion of Related Work:

(i) Our approach is independent from [24, 25, 52]. Fiore and Saville presented a bicategorical extension of simply typed λ-calculus that corresponds to the appropriate type theory for cartesian closed bicategories. Our intersection type systems can be seen as an approximation theory for simple types and arise by making explicit the structure of a special class of bicategorical models.

(ii) We vastly generalize the results of [32], where a categorification of non-idempotent intersection type is presented. Our parametric construction over resource monads determines a categorification of linear (non-idempotent), affine, relevant and cartesian (idempotent) ones. Then [32] becomes a special case of our method (Section V)4.

(iii) In [46] a parametric 2-categorical construction of intersection type systems is presented. Intersection type systems are seen as special kind of fibrations. This contribution can be seen as a “syntactic categorification” of intersection types. Indeed, while the construction of Mazza et al. is an elegant and very general approach to intersection type disciplines, that also allows to prove normalization theorems in a modular way, it does not pro-

3A parametric generalisation of the bicategory of categorical symmetric sequences introduced in [28] and biequivalent to generalized species.

4The only sensible difference is that while in [32] an untyped call-by-push value calculus [41, 31, 18] is considered, in the present setting we chose pure λ-calculus. Our choice is only instrumental to avoid additional technicalities.
vide a type-theoretic denotational semantics\textsuperscript{5}. Moreover, their work is limited to the discrete case, i.e., they do not consider subtyping. We shall see that what is needed to obtain both denotational semantics and subtyping is highly non-trivial.

(iv) Our work is closely related to the rigid Taylor expansion semantics [54]. However, Tsukada et al. contribution is restricted syntactically to \( \eta \)-long simply typed terms and semantically to generalized species over groupoids. The generalization of their approach to the whole simply typed and untyped \( \lambda \)-calculus and to the parametric bicategory \( S \text{-CatSym} \) is, again, highly non-trivial and is one of the goals of our work\textsuperscript{6}.

7) Outline: Section II introduces some categorical background. The main goal is to define a family of Kleisli bicategories of distributors, associated with the lifting of a bicategory of distributors, associated with the lifting of a bicategory \( S \text{-CatSym} \) to the parametric bicategory \( S \text{-CatSym} \) is, again, highly non-trivial and is one of the goals of our work.\textsuperscript{6}

8) Notations: Given a category \( C \) we write \( C^o \) for its full sub-2-category of small categories. Given (bi)categories \( A, B \) we denote as \( \coprod_{i=1}^{n} A_i \) their product. Given (bi)categories \( A_1, \ldots, A_n \) we denote as \( \bigoplus_{i=1}^{n} A_i \) their coproduct. Given categories \( A, B \), we use either \( [A, B] \) or \( \text{Cat}(A, B) \) to denote their functor category. We denote the initial category as \( 0 \). We use linear logic notations for the general notions of cartesian product, terminal object, etc.

II. CATEGORICAL BACKGROUND

We suppose that the reader is familiar with the basics of bicategory theory, for which we refer to [5].

A. Integers and Lists

We consider the category \( \Omega_f \) where object are finite ordinals \( [n] = \{1, \ldots, n\} \), for \( n \in \mathbb{N} \), and morphisms are functions. The category \( \Omega_f \) is symmetric strict monoidal, with tensor product given by addition: \( [n] \oplus [m] = [n + m] \).

\textsuperscript{5}Given \( M \rightarrow \beta \mathbb{N} \), the type theoretic structure associated to \( M \) is not, in general, isomorphic to the one of \( \mathbb{N} \) [45][pp. 65-66].

\textsuperscript{6}However, in the present paper we consider intersection types instead of terms approximations. In [49, Chapter 4] is shown that a naive generalization of Tsukada, Asada and Ong’s approach fails. The general notion of approximation needs to take into account the subtyping information given by typing derivations.

A. Integers and Lists

Let \( k_1, \ldots, k_n \) be natural numbers and \( \alpha : [m] \rightarrow [n] \) we define \( \bar{\alpha} : \bigoplus_{i=1}^{n} k_i \rightarrow \bigoplus_{i=1}^{m} k_i \) as follows:

\[
\bar{\alpha}(\sum_{j=1}^{m} k_{\alpha(j)} + p) = \sum_{i=1}^{\alpha(l)-1} k_i + p
\]

with \( l \in [m] \), and \( 1 \leq p \leq k_{\alpha(i)} \). If we apply the former construction to bijections, we get the symmetries of the tensor product.

From \( \Omega_f \) we can build categories of indexed families of objects over finite ordinals. Let \( \{a_1, \ldots, a_k\} \) be a list of elements of \( A \). We write \( \text{len}(\bar{a}) \) for its length. We denote lists as \( \bar{a}, \bar{b}, \bar{c} \ldots \) Given a list \( \bar{a} = \langle a_1, \ldots, a_k \rangle \) and a function \( \alpha : [k] \rightarrow [k'] \) we define the right action of \( \alpha \) on \( \bar{a} \) as \( \bar{a}\{\alpha\} = \langle a_{\alpha(1)}, \ldots, a_{\alpha(k)} \rangle \). Given a category \( A \), we define the category \( \Omega_f A \) of lists of \( A \), as follows:

1) \( \text{Obj}(\Omega_f A) = \{\{a_1, \ldots, a_n\} \mid a_i \in A\} \).
2) \( \Omega_f A(a_1, \ldots, a_n), (b_1, \ldots, b_m) = \{\langle \alpha, f_1, \ldots, f_m \rangle \mid \alpha : [m] \rightarrow [n] \text{ and } f_i : a_{\alpha(i)} \rightarrow b_i\} \).
3) For \( \langle \bar{\alpha}, \bar{f} \rangle : \bar{a} \rightarrow \bar{b} \) and \( \langle \beta, \bar{g} \rangle : \bar{b} \rightarrow \bar{c} \), composition is given by

\[
\langle \beta, \bar{g} \rangle \circ \langle \alpha, \bar{f} \rangle = \langle \alpha \circ \beta, \bar{g} \circ \bar{f}\rangle
\]

The category \( \Omega_f A \) is cartesian monoidal, with products given by lists concatenation. For \( \langle \bar{a}_1, \ldots, \bar{a}_n \rangle \) and \( \alpha : [m] \rightarrow [n] \) with \( \text{len}(\bar{a}_i) = k_i \) we define

\[
\alpha^* : \bigoplus_{i=1}^{n} \bar{a}_i \rightarrow \bigoplus_{j=1}^{m} \bar{a}_{\alpha(j)}
\]

as \( \alpha^* = \langle \bar{\alpha}, \bigoplus_{i=1}^{n} \bar{a}_{\alpha(i)} \rangle \). The former construction encompasses all arrows that are compositions of structural morphisms, i.e., of symmetries, terminal arrows, projections and diagonals.

B. Coend calculus

Virtually everything that follows is rooted in the notion of coend.

**Definition 1.** Let \( F : C^o \times C \rightarrow D \) be a functor. A co wedge for \( F \) is an object \( T \in D \) together with a family of morphisms \( w_c : F(c, c) \rightarrow T \) such that the following diagram commutes

\[
\begin{array}{ccc}
F(e', c) & \xrightarrow{F(f, 1)} & F(c, c) \\
\downarrow{F(1, f)} & & \downarrow{w_c} \\
F(e', e') & \xrightarrow{w_{e'}} & T
\end{array}
\]

for \( f : c \rightarrow e' \).

A coend is then an universal co wedge. We denote the coend of \( F \) as \( \int_{c \in C} F(c, c) \). Clearly a coend is a kind of colimit,
precisely a coequalizer. The integral notation is justified by the formal calculus connected with this notion\(^7\).

C. Presheaves

For a small category \( A \) define \( PA = [A^\circ, \text{Set}] \), the category of presheaves of \( A \) and natural transformations. If \( A \) is monoidal, for \( X, Y \in PA \), we define the Day convolution tensor product [15] pointwise

\[
(X \otimes Y)(a) = \int^{a_1, a_2 \in A} X(a_1) \times Y(a_2) \times A(a, a_1 \otimes a_2).
\]

It is well-known and crucial that \( PA \) is the free cocompletion of \( A \). This derives directly from the Yoneda embedding and what is called the density theorem, i.e., that presheaves are canonical colimits of representables. The freeness condition is then satisfied by the left Kan extension:

\[
\begin{array}{ccc}
A & \xrightarrow{Y_A} & PA \\
\downarrow F & & \downarrow L_Y(F) \\
B & \xleftarrow{L_Y(F)} & PB
\end{array}
\]

Where \( B \) is a cocomplete category, \( Y_A \) is the Yoneda embedding and \( F \) functor.

D. Distributors

We now define the bicategory of distributors.

- **0-cells** are small categories \( A, B, C \ldots \);
- **1 cells** \( F : A \to B \) are functors \( F : B^\circ \times A \to \text{Set} \).
- By the cartesian closed structure of the 2-category of categories, functors and natural transformations we have the following correspondence:

\[
F : B^\circ \times A \to \text{Set} \\
F^A : A \to PB
\]

Hence we will switch from one to the other presentation of distributors when convenient.

- **2-cells** \( \alpha : F \Rightarrow G \) are natural transformations.
- For fixed 0-cells \( A \) and \( B \), 1-cells and 2-cells organize themselves as a category \( \text{Dist}(A, B) \). Composition \( \alpha \circ \beta \) in \( \text{Dist}(A, B) \) is called vertical composition.
- For \( A \in \text{Dist} \), the identity \( 1_A : A \to A \) is defined as the Yoneda embedding \( 1_A(a, a') = A(a, a') \).
- For 1-cells \( F : A \to B \) and \( G : B \to C \) the horizontal composition is given by

\[
(G \circ F)(a, c) = \int^{b \in B} G(c, b) \times F(b, a).
\]

associative and identities are only up to canonical isomorphism. For this reason \( \text{Dist} \) is a bicategory [5].

- There is a symmetric monoidal structure on \( \text{Dist} \) given by the cartesian product of categories: \( A \otimes B = A \times B \). The bicategory of distributors is compact closed and orthogonality is given by taking the opposite category \( A^\perp = A^\circ \). The linear exponential object is then defined as \( A \to B = A^\circ \times B \).
- For \( A, B \in \text{ob}(\text{Dist}) \) there is a zero distributor \( \emptyset_{A,B} \in \text{Dist}(A, B) \) such that for all \( \langle b, a \rangle \in B \times A \), \( \emptyset_{A,B}(b, a) = \emptyset \).

Given a functor \( F : A \to B \) we can define distributors \( \hat{F} : A \to B, \tilde{F} : B \to A \) as \( \hat{F}(b, a) = B(b, F(a)) \) and \( \tilde{F} = B(F(a), b)^8 \).

E. Pseudomonads and Algebras

For a proper introduction to two-dimensional monad theory we refer to [4].

**Definition 2.** Let \( C \) be a 2-category. A 2-monad over \( C \) is a triple \( (T, \eta, \mu) \) where \( T \) is a 2-endofunctor on \( C \) and \( \eta : 1 \to T \) and \( \mu : T^2 \to T \) are 2-natural transformations satisfying the usual monadic commutative diagrams. A pseudomonad over \( C \) is the same as a 2-monad but the commutation of diagrams is only up to coherent isomorphisms.

Given a 2-monad \( \langle S : C \to C, \eta, \mu \rangle \) we can build the category of lax algebras of \( S \), \( S\text{-LAlg}_C \) as follows:

- An object of \( S\text{-LAlg}_C \) is given by an object \( A \in C \), called the underlying object, a 1-cell \( h_A : SA \to A \) called the structure map and 2-cells \( t_1, t_2 \):

\[
\begin{array}{ccc}
SSA & \xrightarrow{h_A} & SA \\
\downarrow h_{SA} & & \downarrow h_A \\
SA & \xrightarrow{t_1} & A \\
\downarrow h_A & & \downarrow h_{SA} \\
A & \xrightarrow{t_2} & A
\end{array}
\]

The 2-cells need to verify 2 additional coherence conditions [43]. We denote lax algebras by \( \hat{A}, \hat{B}, \ldots \). If the 2-cells \( t_1, t_2 \) are isos, \( \hat{A} \) is called a pseudoalgebra. If they are identities, \( \hat{A} \) is a strict algebra.

- For lax algebras \( \hat{A}, \hat{B} \) a 1-cell or morphism \( \varphi : \hat{A} \to \hat{B} \) is a morphism \( F : A \to B \) together with an invertible 2-cell

\[
\begin{array}{ccc}
SA & \xrightarrow{SF} & SB \\
\downarrow h_A & & \downarrow h_B \\
A & \xrightarrow{F} & B
\end{array}
\]

required to satisfy two coherence conditions [4][p.3]. If \( \zeta \) is an isomorphism, then the morphism is called a pseudomorphism. If \( \zeta \) is the identity, then the morphism is called a strict morphism.

- The category of lax-algebras can be also equipped with a 2-dimensional structure [4].

We denote the 2-categories of pseudoalgebras and strict algebras as respectively \( S\text{-PAlg}_C \) and \( S\text{-Alg}_C \), in both cases the 1-cell considered are pseudomorphisms. Clearly we have that \( S\text{-Alg}_C \) is a full 2-subcategory of \( S\text{-PAlg}_C \).

\(^7\)For a proper introduction to coend calculus see [42].

\(^8\)The two distributors are adjoint 1-cells in the bicategory \( \text{Dist} \).
1) **Resource Monads:** We present a list of 2-monads over $\text{CAT}$, the 2-category of locally small categories, functors and natural transformations. We follow the spirit of [44]. We call these monads *resource monads*. The intuition is that each of these monadic constructions gives a particular notion of resource management.

We start by giving a canonical presentation of some free monoidal constructions. We assume that the reader is familiar with monoidal categories. We explicitly denote an arbitrary (symmetric) monoidal category as $\mathbb{A} = \langle C, \otimes, 1, \alpha, \lambda, \rho, \sigma \rangle$, where $C$ is its underlying category, $\otimes$ its tensor, $1$ its unit, $\alpha$ is the associator, $\lambda, \rho$ the unitalisers and $\sigma$ the symmetry.

**Definition 3.** A *semicartesian monoidal category* is a symmetric monoidal category $\mathbb{A} = \langle C, \otimes, 1, \alpha, \lambda, \rho, \sigma \rangle$ such that the unit is a terminal object. We write then $e_a : a \to 1$ for the terminal morphism.

**Definition 4.** A *relevant monoidal category* is a symmetric monoidal category $\mathbb{A} = \langle C, \otimes, 1, \alpha, \lambda, \rho, \sigma \rangle$ equipped with a natural transformation $e_a : a \to a \otimes a$, called the *diagonal*, which has to satisfy additional coherence conditions.

A monoidal category that is both semicartesian and relevant is a cartesian category.

**Proposition II.1.** For $A \in \text{Cat}$ and $\vec{a}, \vec{b} \in \mathbb{O}_A$ with $n = l(\vec{a}), m = l(\vec{b})$ we define

$$\mathbb{O}_A^{*} (\vec{a}, \vec{b}) = \sum_{\alpha : [m] \to [n]} \prod_{i \in [m]} A(a_{\alpha(i)}, b_i)$$

for $\alpha : [m] \to [n]$ being restricted either to general functions, bijections, surjections, injections or identities. The following holds:

1) If $\alpha$ is restricted to identities, then $\mathbb{O}_A^{*} (\vec{a}, \vec{b})$ is the homset of the free strict monoidal category on $A$.
2) If $\alpha$ is restricted to bijections, then $\mathbb{O}_A^{*} (\vec{a}, \vec{b})$ is the homset of the free symmetric strict monoidal category on $A$.
3) If $\alpha$ is restricted to injections, then $\mathbb{O}_A^{*} (\vec{a}, \vec{b})$ is homset of the free semicartesian strict monoidal category on $A$.
4) If $\alpha$ is restricted to surjections, then $\mathbb{O}_A^{*} (\vec{a}, \vec{b})$ is the homset of free relevant strict monoidal category on $A$.
5) If $\alpha$ is a general function then $\mathbb{O}_A^{*} (\vec{a}, \vec{b})$ is the homset of the free cartesian monoidal strict category on $A$.

**Proof.** The proof exploits the fact that each $\mathbb{O}_A^{*} (\vec{a}, \vec{b})$ defines a subcategory of $\mathbb{O}_A$. The unit $\eta_A : A \to \mathbb{O}_A^{*}$ is given by the singleton embedding $a \mapsto \langle a \rangle$. $\square$

The resource monads are then the following 2-monads.

1) The *strict monoidal resource monad*: the 2-monad over $\text{CAT}$ that sends a category $A$ to its free strict monoidal completion;
2) The *linear resource monad*: the 2-monad over $\text{CAT}$ that sends a category $A$ to its free symmetric strict monoidal completion;
3) The *semicartesian resource monad*: the 2-monad over $\text{CAT}$ that sends a category $A$ to the free semicartesian strict monoidal category on $A$;
4) The *relevant resource monad*: the 2-monad over $\text{CAT}$ that sends a category $A$ to the free relevant strict monoidal category on $A$;
5) The *cartesian resource monad*: the 2-monad over $\text{CAT}$ that sends a category $A$ to its free cartesian strict monoidal completion.

For $S$ resource monad, we call the *tensor product of $S$* the tensor product on $SA$. We call $S$-monoidal functor a functor that preserves the structure on the nose. We denote as $\mathbb{O}_S^f$ the full subcategory of $\mathbb{O}_f$ where morphisms depend on the structure of $S$ (via Proposition II.1).

**Proposition II.2.** Let $A, B \in \text{Cat}$ and $S$ be a resource monad.

If the tensor product of $S$ is symmetric, then we have $S(A \sqcup B) \simeq SA \times SB$.

We can extend the former proposition to finite products and coproducts of categories $S(A_1 \sqcup \cdots \sqcup A_n) \simeq SA_1 \times \cdots \times SA_n$.

We denote the two components of the former equivalence as respectively $\mu_0 : S(A_1 \sqcup \cdots \sqcup A_n) \to SA_1 \times \cdots \times SA_n$ and $\mu_1 : SA_1 \times \cdots \times SA_n \to S(A_1 \sqcup \cdots \sqcup A_n)$.

2) **The 2-monadic Lifting:** In [21], a method to extend 2-monads over $\text{Cat}$ to pseudomonads over $\text{Dist}$ is introduced. The construction is based on the intuition that the bicategory of distributors is the Kleisli bicategory for a suitable pseudomonad of presheaf on the 2-category $\text{Cat}$. Indeed, this idea is very natural: a distributor is just a functor $F : A \to \mathsf{PSh}(A)$.

However, this is not strictly possible, since for a small category $A, PA$ is not small any more. In [21] the notion of relative pseudomonad is defined, in order to deal with this problem.

**Definition 5 (Relative pseudomonad).** Let $J : C \to D$ be a pseudofunctor between 2-categories. A *relative pseudomonad* $T$ over $J$ is the collection of the following data:

- for $A \in C$, an object $TA \in D$;
- for $A, B \in C$, a pseudofunctor $(-)_A : \mathbb{D}(JA, TB) \to \mathbb{D}(TA, TB)$;
- for $A \in C$, a morphism $i_A : JA \to TA$;
- for $f : A \to B$ and $g : B \to C$ a family of invertible two-cells $\mu_{f,g} : (g^* \circ f)^* \cong g^* \circ f^*$;
- for $f : JA \to TB$ a family of invertible two cells $\eta_f : f \cong f^* \circ i_2$;
- a family of invertible two cells $i_A^* \cong 1_{TA}$.

This data has also to satisfy two coherence conditions [21].

Relative pseudomonads are equipped with an appropriate notion of Kleisli bicategory [21, Theorem 4.1].

Given a relative pseudomonad $T$ over $J : C \to D$ and a 2-monad $S$ over $C$, we can define a notion of *lifting* of $T$ to pseudogebras of $S$ [21, Definition 6.2]. The idea is that the lifting, denoted $T$, determines a relative pseudomonad over the lifted pseudofunctor $J : S-\text{Alg}_{QC} \to S-\text{PsAlg}_D$.

$^9$In order to obtain this lifting one has to add the condition that the 2-monad $S$ restricts along $J$ [21].
Lemma II.3 ([21, Example 4.2]). Distributors are the Kleisli bicategory for the relative pseudomonad of presheaves $P$ over the inclusion functor $j : \text{Cat} \to \text{CAT}$.

Proposition II.4 ([21, Theorem 6.3]). Let $S$ be a 2-monad over $C$. If a relative pseudomonad $T$ over $j : C \to D$ lifts to pseudofunctors of $S$, then $S$ can be extended to a pseudomonad on $\text{Kl}(T)$.

In our case, $T$ will be the relative pseudomonads of presheaves $P$, $\text{Kl}(P)$ the bicategory of distributors and $S$ an arbitrary resource monad.

Theorem II.5. Let $S$ be a resource monad. The relative pseudomonad $P$ lifts to the pseudofunctors of $S$.

Proof. For the monoidal strict monad, the symmetric monoidal strict monad and the semi-cartesian strict monad, the result was already proved in [21]. The cartesian resource monad is actually a direct corollary of the lifting of the 2-monad for finite products, again proved in [21]. All cases derives from a straightforward adaptation of Kelly’s result on the universal property of the Day convolution [36]. We prove the result for the relevant resource monad and the cartesian resource monad. In order to prove the case of symmetric strict monoidal categories with diagonals, we need to check three conditions:

1) The considered monoidal structure lifts to presheaves.
2) The Yoneda embedding preserves the considered structure.
3) Let $A$ be $S$-monoidal category, $B$ be a $S$-monoidally complete category and $F : A \to B$ be a strong monoidal $S$-functor. Cocontinuous functors $F : P\text{A} \to B$ preserve the relevant structure.

The three condition are verified exploiting the fact that a presheaf is a canonical colimit of representables. □

F. The Bicategory $S\text{-CatSym}$

From now on we restrict ourselves to resource monad that have a symmetric tensor product\textsuperscript{10}. Thanks to Theorem II.5 and Proposition II.4, given a resource monad $S$, we obtain a (relative) pseudomonad $S$ over distributors. We denote as $S\text{-Dist}$ the Kleisli bicategory for this pseudomonad. We define the bicategory of $S\text{-categorical symmetric sequences}$, as $S\text{-CatSym} = S\text{-Dist}^{op}$. It is useful to give an explicit definition of the relevant structure of $S\text{-CatSym}$. When we write $SA^{n}$ (resp. $SA^{n}$) we always mean $(SA)^{n}$ (resp. $(SA)^{n}$).

1) $\text{ob}(S\text{-CatSym}) = \text{ob}(\text{Cat})$.
2) For $A, B \in S\text{-Dist}$, we have $S\text{-CatSym}(A, B) = S\text{-Dist}(B, A) = \text{Dist}(B, SA)$.
3) The identity is defined as

$$1_A(\bar{a}, a) = SA(\bar{a}, \langle a \rangle).$$

\textsuperscript{10}This is crucial since the Seely equivalence is needed in order to establish the cartesian closure.

4) For $F : A \leadsto B$ and $G : B \leadsto C$ $S$-categorical symmetric sequences, composition is given by considering $F$ and $G$ as $S$-distributors:

$$(G \circ F)(\bar{a}, c) = \int_{\vec{b} \in SB} G(\vec{b}, c) \times F^\circ(\bar{a}, \vec{b})$$

where

$$F^\circ(\bar{a}, \vec{b}) = \int_{\vec{a}_1, \ldots, \vec{a}_n} \prod_{i=1}^{\text{len}(\vec{b})} F(\vec{a}_i, b_i) \times SA(\bigoplus_{i=1}^{\text{len}(\vec{b})} \vec{a}_i, \vec{a}).$$

5) $S\text{-CatSym}$ is cartesian. The cartesian product is the disjoint union $A \times B = A \cup B$ and the projections are defined as follows:

$$\pi_{1, 2}(\vec{c}, a) = (S(A \cup B)(\vec{c}, \langle \iota_1(a) \rangle)).$$

The terminal object is the empty category.

6) The bicategory $S\text{-CatSym}$ is cartesian closed, with exponential object $A \to B = SA^{\circ} \times B$.

Indeed, exploiting the Seely equivalence we get the following chain of equivalences:

$$S\text{-CatSym}(A \times B, C) = \text{Dist}(C, S(A \cup B)) = \text{CAT}(S(A \cup B)^{\circ} \times C, \text{Set}) \simeq \text{CAT}(S(A^{\circ} \times (SB^{\circ} \times C), \text{Set}) = S\text{-Dist}(SB^{\circ} \times C, A) = S\text{-CatSym}(A, SB^{\circ} \times C).$$

III. MODELS FOR PURE $\lambda$-CALCULUS

We build a family of non-extensional bicategorical models for pure $\lambda$-calculus. These models will then be syntactically presented as appropriate categories of intersection types.

Definition 6. Let $A$ be a small category. We define by induction a family of small categories as follows:

$$D_0 = A \quad D_{n+1} = (SD_n^{\circ} \times D_n) \cup A$$

We define by induction on $n \in \mathbb{N}$ a sequence of inclusions

$$\iota_n : D_n \hookrightarrow D_{n+1} : \iota_0 = \iota_A \quad \iota_{n+1} = (S(\iota_n)^{\circ} \times \iota_n) \cup 1_A$$

Then we set $D_A = \lim_{n \in \mathbb{N}} D_n$.

The category $D_A$ is the filtered colimit for the diagram $(D_n \hookrightarrow D_{n+1})_{n \in \mathbb{N}}$.

This definition is actually a special case of the standard free-algebra construction for an (unpointed) endofunctor [38]. In our case the endofunctor is $S(-)^{\circ} \times (-) : \text{Cat} \to \text{Cat}$ and the free algebra is $\langle D_A, \iota : SD_A^{\circ} \times D_A \to D_A \rangle$, where $\iota$ is a canonical embedding. This determines a retraction $D_A \Rightarrow D_A \triangleleft D_A$ in the bicategory $S\text{-CatSym}$. The retraction pair is given by

$$i : \langle S(\iota)^{\circ} \times D_A \rangle \to D_A \quad (\vec{a}, a) \mapsto SDA(\iota(\vec{a}), \langle a \rangle)$$

$$j : SDA^{\circ} \times (SD_A^{\circ} \times D_A) \to \text{Set} \quad \langle \vec{a}, \langle a \rangle \rangle \mapsto SDA(\bar{a}, \langle \iota(\bar{a}, a) \rangle)$$
Types:
\[ a := o \in \text{ob}(A) \mid (a_1, \ldots, a_k) \Rightarrow a \]

Morphisms:
\[ f : o \to o' \]
\[ f : \alpha \Rightarrow \beta \]
\[ f : \alpha \Rightarrow \beta' \]
\[ \alpha \in \mathcal{D}(\mathcal{K}, [k', k]) \]
\[ \forall \alpha, \beta, \beta' : \mathcal{D}(\mathcal{K}, [k', k]) \]
\[ f_1 : a_{\alpha(1)} \to \alpha_1' \cdots \quad f_k : a_{\alpha(k)} \to \alpha_k' \]
\[ \langle \alpha, f_1, \ldots, f_k \rangle : \langle a_1, \ldots, a_k \rangle \Rightarrow \langle a_1', \ldots, a_k' \rangle \]

(a) Category of Types \( D_A \).

Derivations:
\[ f_1 : \tilde{a}_1 \Rightarrow \langle \rangle, \ldots, f : \tilde{a}_i \Rightarrow \langle a \rangle, \ldots, f_n : \tilde{a}_n \Rightarrow \langle \rangle \]
\[ x_1 : \tilde{a}_1, \ldots, x_i : \tilde{a}_i, \ldots, x_n : \tilde{a}_n \vdash x_i : a \]
\[ \Delta, x : \tilde{a} \vdash M : a \]
\[ \Delta, x : \tilde{a} \vdash \lambda x.M : a \]
\[ \Delta, x : \tilde{a} \vdash \lambda x.M : a \]
\[ \Delta, x : \tilde{a} \vdash \lambda x.M : a \]
\[ \Delta, x : \tilde{a} \vdash \lambda x.M : a \]

(b) Parametric Intersection Type System \( E_A^S \).

Figure 1: Type Theoretic Presentation of the Semantics.

\[ \llbracket x \rrbracket(\Delta, a) = SD^n(\Delta, \langle \rangle, \ldots, \langle a \rangle, \ldots, \langle \rangle) \]
\[ \llbracket \lambda x.M \rrbracket(\Delta, a) = \begin{cases} \llbracket M \rrbracket(\Delta, a) & \text{if } a = \iota(\tilde{a}', a') \\ \emptyset & \text{otherwise} \end{cases} \]
\[ \llbracket MN \rrbracket(\Delta, a) = \int_{\tilde{a} = (a_1, \ldots, a_k) \in SD} \int_{\Gamma_0, \ldots, \Gamma_k \in SD^n} \llbracket M \rrbracket(\Delta, a) \times SD^n(\Delta, (\bigotimes_{i=0}^{k} \Gamma_i)) \]

Figure 2: Denotation of \( \lambda \)-terms.

\[ \left( f_1 : \tilde{a}_1 \Rightarrow \langle \rangle, \ldots, f_i : \tilde{a}_i \Rightarrow \langle a \rangle, \ldots, f_n : \tilde{a}_n \Rightarrow \langle \rangle \right) \eta \]
\[ x_1 : \tilde{a}_1, \ldots, x_i : \tilde{a}_i, \ldots, x_n : \tilde{a}_n \vdash x_i : a \]
\[ \Delta, x : \tilde{a} \vdash M : a \]
\[ \Delta \vdash \lambda x.M : a \]
\[ \Delta \vdash \lambda x.M : a \]
\[ \Delta \vdash \lambda x.M : a \]
\[ \Delta \vdash \lambda x.M : a \]
\[ \Delta \vdash \lambda x.M : a \]
\[ \Delta \vdash \lambda x.M : a \]

Where \( \tilde{a} = (a_1, \ldots, a_k) \) and \( \eta = (g_1, \ldots, g_n) : \Delta' \Rightarrow \Delta \).

Figure 3: Right action on derivations.

Hence, \( D_A \) is a (weak) reflexive object\(^\text{11}\). If we set \( (a_1, \ldots, a_k) \Rightarrow a := \iota((a_1, \ldots, a_k), a) \), we can give a completely type-theoretic presentation of the category \( D_A \) as in Figure 1a.

We fix a countable set of variables \( x, y, z, \ldots \in \mathcal{V} \). The set of \( \lambda \)-terms is defined by induction in the usual way:
\[ M, N \in \Lambda ::= x \mid \lambda x.M \mid MN \]
Terms are considered up to renaming of bound variables. As usual, we assume that application associates to the left. We denote the capture-free parallel substitution of variables as \( M \{ N_1, \ldots, N_n / x_1, \ldots, x_n \} \). Given a term \( M \), a list of terms \( N = \langle N_1, \ldots, N_n \rangle \) and a list of variables \( \bar{x} = \langle x_1, \ldots, x_m \rangle \) we set \( MN = MN_1 \cdots N_n \lambda \bar{x}.M = \lambda x_1 \cdots \lambda x_m.M \).

The interpretation of \( \lambda \)-terms in the bicategory \( S \text{-CatSym} \) is given by induction, following the standard categorical definition ([2, Section 4.6]). We fix a small category \( A \) and a constant type \( D \) such that \( D = D \Rightarrow D \).

1) On types:
\[ \llbracket \top \rrbracket = D_A \quad \llbracket \Pi = D_1, \ldots, D_n \rrbracket = \llbracket D_1 \rrbracket \& \cdots \& \llbracket D_n \rrbracket \]
2) On terms:
\[ \llbracket x : D_1, \ldots, x_n : D \vdash x : D \rrbracket = \pi_{x_1, \ldots, x_n} \]

\(^\text{11}\)Weak in this case means that the retraction condition is satisfied only up to canonical invertible 2-cell.

\(^\text{12}\)It is worth noting that we do not require for this equation to be semantically satisfied, i.e. we consider non-extensional models.
\[ \Gamma \vdash \lambda x. M : D \] = \{ \eta : \Delta \vdash M : D \} \\
\[ \Gamma \vdash P Q : D \] = ev_{D,D} \circ (j \circ \Gamma \vdash P : D, \Gamma \vdash Q : D) .
\]
Where \((i,j)\) is an appropriate retraction pair. Given \(\Gamma = x_1 : D_1, \ldots, x_n : D_n\) we set \(\text{supp}(\Gamma) = (x_1, \ldots, x_n)\).

1) Denotations of Terms: The category \(D_A\) is a non-extensional model for pure \(\lambda\)-calculus. We will denote, with a small abuse of language, \(SD_A\) as \(SD\) and \(D_A\) as \(D\). We now want to make explicit the idea that the semantics induced by this category is an intersection type system. In order to do so, we are going to define a parallel semantics, that we call the denotation of a \(\lambda\)-term. The intuition is that the denotation is the type-theoretic presentation, up to isomorphism, of the categorical semantics.

We call intersection type contexts, or contexts for short, the objects of \(SD^n\). Since \(SD\) is monoidal, the category \(SD^n\) admits a tensor product, that we denote as \(\otimes\), defined as follows: for \(\Gamma = \langle a_1, \ldots, a_n \rangle\), we set \(\Gamma \otimes = \langle b_1, \ldots, b_n \rangle\). This tensor product inherits all the structure from \(\otimes\), i.e., if \(\otimes\) is symmetric (resp., semicartesian, relevant, cartesian) then also \(\otimes\) is so.

We define the denotation of a \(\lambda\)-term by induction in Figure 2. We have that \(\llbracket M \rrbracket_x : D \rightarrow SD^n\).

The denotation of an application is defined via the Day convolution as follows. Consider the functor \(F : SD^n \times SD \times (SD^n)^o \rightarrow D \rightarrow \text{Set}\)

\[ (\bar{a}, \bar{b}) = (b_1, \ldots, b_k) \otimes D \rightarrow \text{Set} \]

\[ \langle [M]_x(-, \bar{a} \Rightarrow a) \otimes \bigotimes_{i=1}^k [N]_x(-, b_i) \rangle (\Delta) \]

\[ \llbracket MN \rrbracket_x (\Delta, a) = \int_{\bar{a} \in SD} F(\bar{a}, \bar{a}, \Delta, a) \]

The action on morphism is given by the universal property of the coend construction.

The denotation of a term is isomorphic to its bicategorical interpretation via the Seely equivalence:

**Theorem III.1.** Let \(M \in \Delta, \bar{x} \supseteq FV(M)\) and \(\Gamma \vdash M : D\) such that \(\text{supp}(\Gamma) = \bar{x}\). We have a natural isomorphism \(\llbracket M \rrbracket_x \cong \llbracket M \rrbracket_{\bar{x}} \circ \Delta_1 \circ \text{Diag} \cdot \llbracket \Gamma \vdash M : D \rrbracket\).

**Proof.** By induction on the structure of \(M\) via lengthy but straightforward coend manipulations.

A. The Denotation as an Intersection Type System

We now give a type-theoretic description of the denotation of a \(\lambda\)-term. We define the intersection type system \(E_{\Delta}^M\), where types and morphisms live in the category \(D_A\) (Figure 1a).

Thanks to this type theoretic description, we can present the denotation’s action on morphism as right and left actions on typing derivations:

\[ \pi : \Delta \vdash M : a \]

\[ \llbracket f \rrbracket \pi (\eta) \]

\[ \Delta' \vdash M : a' \]

with \(f : a \rightarrow a'\) and \(\eta : \Delta' \rightarrow \Delta\). The actions are inductively defined in Figures 3 and 4. By an easy inspection of the definitions, we get \(\llbracket \pi (\eta) \rrbracket (\theta) = \pi (\eta \circ \theta), \llbracket (f \pi) (\eta) \rrbracket = \llbracket f \rrbracket \pi (\eta) \rrbracket\) and \(g (\llbracket f \rrbracket (\pi)) = [g \circ f] (\pi)\).

We observe that in the variable rule of our system (Figure 1b) the morphisms \(f_j : \bar{a}_j \Rightarrow \emptyset\) for \(j \neq i \in [n]\), are unique, by the structure of resource monads. In particular, if \(S\) is irrelevant (cartesian or semicartesian resource monad), \(f_j\) is the the terminal morphism \(\top_{\bar{a}_j} : \bar{a}_j \Rightarrow \emptyset\). Otherwise (linear or relevant resource monad) \(f_j\) is the identity \(1_{\emptyset} : \emptyset \Rightarrow \emptyset\).

1) Congruence on Typing Derivations: The definition of denotation of an application \(MN\) depends on the notion of coend. In the Set enriched setting this notion boils down to an appropriate quotient sum of sets. Hence, if we want to give a syntactic presentation of the denotation via the intersection type system \(E_{\Delta}^M\), we shall need to translate the quotient in the setting of typing derivations.

We set \(\hat{\pi}\) as the equivalence class of \(\pi\) for the smallest congruence generated by the rules of Figure 5. By an easy inspection of the definitions, we get that if \(\pi \sim \pi'\) then \(\pi (\theta) \sim \pi (\theta)\) and \([g \pi] \sim [g \pi']\).

**Definition 7.** Let \(\bar{x} \supseteq f v(M)\) and \(\text{len}(\bar{x}) = n\). We now define the \(S\)-intersection type distributor of \(M\), \(T_D(M)_{\bar{x}} : D \rightarrow SD^n\), as follows:

1) on objects

\[ T_D(M)_{\bar{x}}(\Delta, a) = \left\{ \begin{array}{c} \hat{\pi} : \\
\Delta' \vdash M : a' \end{array} \right\} \]

2) on morphisms

\[ T_D(M, f, \eta) : T_D(M)_{\bar{x}}(\Delta, a) \rightarrow T_D(M)_{\bar{x}}(\Delta', a') \]

\[ \hat{\pi} \Rightarrow [f \pi (\eta)] \]

**Theorem III.2.** Let \(M \in \Delta\). We have a natural isomorphism

\[ \llbracket M \rrbracket_{\bar{x}} \cong T_D(M)_{\bar{x}} \]

**Proof.** By induction on the structure of \(M\). The only non-trivial part is showing that in the application case, the distributor \(T_D(M)_{\bar{x}}\) can be described as a coend.

2) Typing Derivations under Reduction: In this section we will prove that \(\llbracket [M]_{\bar{x}}(\Delta, a) \cong [N]_{\bar{x}}(\Delta, a)\) when \(M \rightarrow_{\beta} N\), refining the standard subject reduction and expansion for intersection types. Indeed, we recall that, by Theorem III.1, \(\llbracket M \rrbracket_{\bar{x}} \cong T_D(M)_{\bar{x}}\) hence, if we prove that \(\llbracket M \rrbracket_{\bar{x}}(\Delta, a) \cong [N]_{\bar{x}}(\Delta, a)\) when \(M \rightarrow_{\beta} N\), in particular we have \(T_D(M)_{\bar{x}} \cong T_D(N)_{\bar{x}}\). This means that we have a natural bijection between the set of equivalence classes of typing derivations with conclusion \(\Delta \vdash M : a\) and the set of equivalence classes of typing derivations with conclusion \(\Delta' \vdash N : a\), that is what we called a proof relevant denotational semantics.

Let \(M, N \in \Delta, f v(M) \setminus \{x\} \cup f v(N) \subseteq \bar{x}\) and \(x \notin \bar{x}\). We set \(\text{Sub}_{M, \bar{x}, N}(\Delta, a) = \int_{\bar{a} \in SD} \int_{\bar{\Gamma} \in SD^n} \llbracket [M]_{\bar{x}}(\bar{\Gamma}(0), a) \times \)

\[ \llbracket \bar{\Gamma} \rrbracket_{\bar{x}}(\bar{\Gamma}(0) + \langle \bar{a} \rangle, a) \times \]

\[ \Delta' \vdash M : a' \]

The actions are inductively defined in Figures 3 and 4. By an easy inspection of the definitions, we get \(\llbracket \pi (\eta) \rrbracket (\theta) = \pi (\eta \circ \theta), \llbracket (f \pi) (\eta) \rrbracket = \llbracket f \rrbracket \pi (\eta) \rrbracket\) and \(g (\llbracket f \rrbracket (\pi)) = [g \circ f] (\pi)\).

We observe that in the variable rule of our system (Figure 1b) the morphisms \(f_j : \bar{a}_j \Rightarrow \emptyset\) for \(j \neq i \in [n]\), are unique, by the structure of resource monads. In particular, if \(S\) is irrelevant (cartesian or semicartesian resource monad), \(f_j\) is the the terminal morphism \(\top_{\bar{a}_j} : \bar{a}_j \Rightarrow \emptyset\). Otherwise (linear or relevant resource monad) \(f_j\) is the identity \(1_{\emptyset} : \emptyset \Rightarrow \emptyset\).
Proof.\\[\begin{align*}
\Gamma_0 \vdash M : \bar{a} &\Rightarrow a, \\
\Gamma_i \vdash M : a_j &\Rightarrow \eta : \Delta \Rightarrow \otimes_{j=0}^k \Gamma_j,
\end{align*}\]

where \(\bar{a} = \langle a_1, \ldots, a_k \rangle\).

Figure 4: Left action on derivations.

If \(H(M) = M\) we say that \(M\) is a head-normal form.

Lemma IV.1. Let \(M \in \Lambda\) be a head-normal form. Then \(\llbracket M \rrbracket_x \neq \emptyset_{D_A, SD^\alpha(x)}\).

Proof. We have that \(M = \lambda x_1 \ldots \lambda x_m.xQ_1 \ldots Q_n\). We prove it for \(xQ_1 \ldots Q_n\), choosing as list of variables \(\bar{y} = \bar{x} \cup \{x_1, \ldots, x_m\} = \{y_1, \ldots, y_k\}\) where \(k = m + \text{len}(\bar{x})\), the extension being immediate.

Let \(b = \langle \rangle \Rightarrow \cdots \Rightarrow \langle \rangle \Rightarrow a\). It is enough to take the following typing derivation \(\pi\):

\[
\begin{array}{c}
y_1 : \langle \rangle, \ldots, x : \langle b \rangle, \ldots, y_k : \langle \rangle \vdash x : \langle \rangle \Rightarrow \cdots \Rightarrow \langle \rangle \Rightarrow a \\
y_1 : \langle \rangle, \ldots, x : \langle b \rangle, \ldots, y_k : \langle \rangle \vdash xQ_1 \cdots Q_n : a
\end{array}
\]

Then \(TD(M)_{\bar{y}}(\langle \langle \rangle, \ldots, \langle b \rangle, \ldots, \langle \rangle \rangle, a)\) is non-empty for all types \(a \in D\). Then we apply Theorem III.1 and conclude.

Corollary IV.2. Let \(M \in \Lambda\). If \(M\) is head-normalizable then \(\llbracket M \rrbracket_x \neq \emptyset_{D_A, SD^\alpha(x)}\).

Proof. Corollary of the former lemma and Theorem III.4.

We are now ready to present our reducibility argument. For a set \(X \subseteq \Lambda\) we say that \(X\) is saturated if \(M \{N/x\} N_1 \cdots N_n \in X\) implies \((\lambda x.M) N_1 \cdots N_n \in X\). Given \(X_1, X_2 \subseteq \Lambda\), we write \(X_1 \Rightarrow X_2\) if \(\{M \in \Lambda\mid X_1 \subseteq M \Rightarrow X_2\}\).

Given a small category \(A\) an interpretation is a functor \(I : A \rightarrow (\langle \mathcal{V} \rangle, \subseteq)\), where \(\langle \mathcal{V} \rangle^* = \{X \subseteq \Lambda \mid X\) is saturated \}. Given \(\delta \in D_A \cup SD_A\) we define the set of realizers of \(\delta\) by induction as follows:

\[
\llbracket [a]_{I} = I(\langle \rangle) \quad \llbracket [\langle \rangle]_{I} = \Lambda \\
\llbracket [\langle a_0, \ldots, a_k \rangle]_{I} = \bigotimes_{i=0}^{k}[a_i]_{I} \\
\llbracket [\bar{a} \Rightarrow \bar{a}]_{I} = \llbracket \bar{a} \rrbracket_{I} \Rightarrow \llbracket \bar{a} \rrbracket_{I}
\]

By construction we have that \(\llbracket \delta \rrbracket_{I}\) is saturated.

Lemma IV.3. Let \(\delta, \delta' \in D_A \cup SD_A\). If \(f : \delta \Rightarrow \delta'\) then \(\llbracket [\delta]_{I} \subseteq \llbracket [\delta']_{I}\).

Definitions and lemmas without proofs have been omitted for space reasons.

IV. HEAD-NORMALIZATION

In this section we present a parametric head-normalization theorem for our systems, adapting the reducibility argument of [9, 39] to our setting. The construction of the argument is classical, but there is a technical improvement to be made in order to lift it to a category-theoretic perspective.

We remark that our argument can be adapted also to characterize weak and strong normalization, as shown in [49]. Given a \(\lambda\)-term \(M\), we denote as \(H(M)\) its head-reduct\(^{14}\):

\[
H(M) = \begin{cases} 
M &\text{if } M = \lambda \bar{x}. M N \\
\lambda \bar{x}. M \{N/x\} \bar{N} &\text{if } M = \lambda \bar{x}. (\lambda x.M) N \bar{N}.
\end{cases}
\]

\(^{14}\)Reading the integral as an existential quantifier and the product as a conjunction, this analogy should be quite evident.

\(^{15}\)In the definition we use a well-known characterization of \(\lambda\)-terms, see [39].
\[ \pi_0 \vdash \Gamma_0 \vdash M : b \Rightarrow a \]
\[ \Delta \vdash MN : a \]
\[ \pi_0(\theta) \]
\[ \Gamma_0 \vdash M : \bar{a} \Rightarrow a \]
\[ \Delta \vdash MN : a \]
\[ \Gamma_0 \vdash N : a_i \]
\[ \eta : \Delta \rightarrow \bigotimes_{j=0}^{k} \Gamma_j \]
\[ \Delta \vdash MN : a \]
\[ \Gamma_0 \vdash M : \bar{a} \Rightarrow a \]
\[ \Delta \vdash MN : a \]
\[ \Gamma_0 \vdash N : a_i \]
\[ \eta : \Delta \rightarrow \bigotimes_{j=0}^{k} \theta_j \]

Where \( \alpha, f_1, \ldots, f_{k'} \) : \( a = a_1, \ldots, a_k \) \( \Rightarrow b = b_1, \ldots, b_{k'} \) and \( \theta : \Gamma_i \rightarrow \Gamma'_i \).

Figure 5: Congruence on typing derivations.

Proof. By induction on the structure of \( \delta' \).

Lemma IV.4. Let \( M, N_1, \ldots, N_n \in \Lambda \) and \( I \) be an interpretation. If \( x_1 : a_1, \ldots, x_n : a_n \vdash M : a \) and \( N_i \in \mathcal{H}N_0 \) then \( M \{ N_1, \ldots, N_n / x_1, \ldots, x_n \} \in \mathcal{H}N_1 \).

Proof. By induction on the structure of \( M \), applying Lemma IV.3.

We define \( \mathcal{H}N = \{ M \in \Lambda \mid \text{The head-reduction of } M \text{ ends} \} \) and \( \mathcal{H}N_0 = \{ x : x_1, \ldots, x_n \in \Lambda \} \). We remark that \( V \subseteq \mathcal{H}N_0 \).

Lemma IV.5. \( \mathcal{H}N \) is saturated.

We set \( I_{\mathcal{H}N} : A \rightarrow (\mu \Lambda)^*, \subseteq \) to be the functor such that \( \forall a \in \Lambda, I_{\mathcal{H}N}(a) = \mathcal{H}N, \) the action on morphisms being the trivial one. We define in the same way \( I_N \) and \( I_{SN} \).

Lemma IV.6. For all \( a \in D_A \) we have that \( \mathcal{H}N_0 \subseteq [a]_{I_{\mathcal{H}N}} \subseteq \mathcal{H}N \).

Lemma IV.7. Let \( M \in \Lambda \). If \( M \) is typable in the system \( E^S_A \) then the head-reduction of \( M \) ends.

Proof. Direct consequence of Lemma IV.6 and Lemma IV.4.

Theorem IV.8. Let \( M \in \Lambda \). The following statements are equivalent:

1) \( \llbracket M \rrbracket \not\subseteq \emptyset_{D,S,D^{\text{linear}}(x)} \).
2) The head-reduction of \( M \) ends.
3) \( M \) is head-normalizable.

Proof. \( (1) \Rightarrow (2) \) Corollary of Theorems III.2 and Lemma IV.7. \( (2) \Rightarrow (3) \) immediate by definition. \( (3) \Rightarrow (1) \) by Theorem III.1 and Corollary IV.2.

V. WORKED OUT EXAMPLES

We present two concrete constructions of the distributor-induced denotational semantics that we introduced in the previous sections. We choose the examples of the linear resource monad (symmetric monoidal strict completion) and of the cartesian one (cartesian strict completion). Those two examples are particularly relevant since they correspond to the categorification of the two best known intersection type systems: the linear logic induced Gardener-De Carvalho System \( \mathcal{R} \) [29, 9] and the original Coppo-Dezani System \( D\Omega \) [14]. The first one is non-idempotent, the second one is idempotent. In our setting, the idempotency issue is replaced by an operational one: which operations do we allow on intersections?

A. Example 1: Linear Resources

The content of this subsection corresponds to a Call-by-Name version of [32]. We present the non-idempotent intersection type system \( \mathcal{R} \). That system has a categorical counterpart in the linear logic induced relational model for pure \( \lambda \)-calculus [9]. The intersection type is given by multisets. In our case, we achieve a non-idempotent and commutative (up to isos) intersection type system applying our construction in the special case where the resource monad \( S \) is the 2-monad for symmetric strict monoidal categories. The corresponding intersection type system is system \( R_A \) in Figure 6.

In the linear case, we can prove the head-normalization theorem in a combinatory way. We set \( \llbracket M \rrbracket^{RA}_A \) to be the denotation of \( M \) in the case where \( S \) is the linear resource monad. We define the size \( s(\pi) \) of a typing derivation \( \pi \) as the number of application rules that appear in it.

By an easy inspection of the definitions, we have that the size is stable under actions and under congruence: if \( \pi, \pi' \in R_A \) and \( \pi \sim \pi' \) then \( s(\pi) = s(\pi') \).

Let \( (\rho) : \llbracket M \rrbracket^{RA}_A (\Delta, a) \cong T_D(\Delta) \llbracket M \rrbracket^{RA}_A (\Delta, a) \) be the isomorphism given by Theorem III-A. Then for \( \alpha \in \llbracket M \rrbracket^{RA}_A (\Delta, a) \) we set \( s(\alpha) = s(\rho_{\Delta,a}(\alpha)) \). Given \( \alpha = (\alpha_1, \ldots, \alpha_k) \) with \( \alpha_i \in \llbracket M \rrbracket^{RA}_A \), we set \( s(\alpha) = \sum_{i=1}^{k} s(\alpha_i) \).

Lemma V.1. Let \( M, N \) be two \( \lambda \)-terms, \( \bar{x} \supseteq \text{fv}(M) \cup \text{fv}(N) \) with \( x \notin \bar{x} \) and \( \text{sub}_{\Delta,a}^{M,x,N,\bar{x}} : \text{sub}_{\bar{x}}^{M,x,N}(\Delta, a) \cong \llbracket M \{ N/x \} \rrbracket^{RA}_A (\Delta, a) \) be the natural isomorphism given by Theorem III.1. For all \( \alpha = (\pi, \bar{\psi}) \in \text{sub}_{\Delta,a}^{M,x,N,\bar{x}}(\Delta, a) \), we have

\[ s\left( \text{sub}_{\Delta,a}^{M,x,N,\bar{x}}(\alpha) \right) = s(\pi) + s(\bar{\psi}) \]

Theorem V.2. Let \( M, N \in \Lambda \). We have a natural isomorphism

\[ \varphi_{\Delta,a} : \llbracket H(M) \rrbracket^{RA}_A (\Delta, a) \cong \llbracket H(N) \rrbracket^{RA}_A (\Delta, a) \]
such that for $\alpha \in [M]^A_{e}(\Delta, a), s(\varphi_{\Delta, a}(\alpha)) \leq s(\alpha)$.

**Proof.** Direct corollary of the former lemma. $\square$

**Theorem V.3.** Let $M \in \Delta$. If $[M]^A_{e} \neq \emptyset_{D, SD^{\text{sym}}(e)}$ (the head reduction of $M$ ends).

**Proof.** We have that, for $\varphi : [M]^A_{e} \cong [H(M)]^A_{e}$. If $[M]^A_{e} \neq \emptyset_{D, SD^{\text{sym}}(e)}$ then $H(M)^A_{e} \neq \emptyset_{D, SD^{\text{sym}}(e)}$. We consider $\alpha \in [M]^A_{e}(\Delta, a)$ for some $(\Delta, a) \in SD^{\text{sym}}(e) \times D$. Then, by the former theorem, $s(\varphi_{\Delta, a}(\alpha)) < s(\alpha)$. Then we can apply the IH and conclude. $\square$

**Example V.4.** We provide a simple example of reduction of typing derivations to ease the understanding of the congruence’s role in establishing the natural isomorphisms. Consider $M = (\lambda x.x)y$. We type it with the following typing derivations:

$$\begin{align*}
\pi_1 &= \frac{h \circ f : a \to b}{x : (b) \vdash x : b} \quad \frac{g : c \to a}{y : (c) \vdash y : a} \quad \frac{\lambda x.x : \langle a \rangle \Rightarrow b}{y : (c) \vdash \langle \lambda x.x \rangle y : b} \quad \frac{h \circ f' : d \to b}{x : (b) \vdash x : b} \quad \frac{g' : c \to d}{y : (c) \vdash y : d} \quad \frac{\lambda x.x : \langle d \rangle \Rightarrow b}{y : (c) \vdash \langle \lambda x.x \rangle y : b}
\end{align*}$$

We have that $\pi_1 \sim \pi_2$. Indeed, by the first rule of Figure 5:

$$\begin{align*}
\pi_1 &= \frac{h : b \to b}{x : (b) \vdash x : b} \quad \frac{f \circ g : c \to b}{y : (c) \vdash y : b} \quad \frac{\lambda x.x : \langle b \rangle \Rightarrow b}{y : (c) \vdash \langle \lambda x.x \rangle y : b} \\
\pi_2 &= \frac{h : b \to b}{x : (b) \vdash x : b} \quad \frac{f' \circ g' : c \to b}{y : (c) \vdash y : b} \quad \frac{\lambda x.x : \langle b \rangle \Rightarrow b}{y : (c) \vdash \langle \lambda x.x \rangle y : b}
\end{align*}$$

and by the hypothesis that $f \circ g = f' \circ g'$ we can conclude by transitivity. In particular, this means that the quotient identify all couple of morphisms leading to the same composition. Now, we have that $M \to y$. Consider the following typing derivation of $y$:

$$\begin{align*}
\pi_3 &= \frac{h \circ (f \circ g) : c \to b}{y : (c) \vdash y : b}
\end{align*}$$

**Example V.5.** We provide some example of typing derivations in system $C_A$, giving also some intuition for what concerns the congruence on typing derivations.

1) Let us type the term $M = (\lambda x.(xx))x$. Let $b = \langle a \rangle \Rightarrow \langle a \rangle \Rightarrow a$. Consider the following typing derivation $\pi$:

$$\begin{align*}
\pi_1 &= \frac{h : b \to b}{x : (b) \vdash x : b} \quad \frac{f \circ g : c \to b}{y : (c) \vdash y : b} \quad \frac{\lambda x.x : \langle b \rangle \Rightarrow b}{y : (c) \vdash \langle \lambda x.x \rangle y : b} \\
\pi_2 &= \frac{h : b \to b}{x : (b) \vdash x : b} \quad \frac{f' \circ g' : c \to b}{y : (c) \vdash y : b} \quad \frac{\lambda x.x : \langle b \rangle \Rightarrow b}{y : (c) \vdash \langle \lambda x.x \rangle y : b}
\end{align*}$$

Now consider the following typing derivation $\rho$:

$$\begin{align*}
\pi &= \frac{\lambda x.(xx)x : \langle b, a \rangle \Rightarrow a}{N : b \vdash N : a}
\end{align*}$$

2) Let us type the term $M = (\lambda x.(xx)x)N : a$. Consider the following typing derivation $\pi'$:

$$\begin{align*}
\pi_1 &= \frac{h : b \to b}{x : (b) \vdash x : b} \quad \frac{f \circ g : c \to b}{y : (c) \vdash y : b} \quad \frac{\lambda x.x : \langle b \rangle \Rightarrow b}{y : (c) \vdash \langle \lambda x.x \rangle y : b} \\
\pi_2 &= \frac{h : b \to b}{x : (b) \vdash x : b} \quad \frac{f' \circ g' : c \to b}{y : (c) \vdash y : b} \quad \frac{\lambda x.x : \langle b \rangle \Rightarrow b}{y : (c) \vdash \langle \lambda x.x \rangle y : b}
\end{align*}$$

Now consider the following typing derivation $\rho'$:

$$\begin{align*}
\pi' &= \frac{\lambda x.(xx)x : \langle b, a, a \rangle \Rightarrow a}{N : b \vdash N : a}
\end{align*}$$

**B. Example 2: Cartesian Resources**

We now focus on the type theoretic semantics induced by the cartesian resource monad. In this framework, a resource can be copied and deleted at wish.

When $SA$ is cartesian, the Day convolution on $PSA$ is isomorphic to the cartesian product. Hence, we have the following natural isomorphism\(^{16}\)

$$\begin{align*}
G \circ F(\bar{a}, c) &\cong \int_{\langle b_1, \ldots, b_k \rangle \in SD} G(\bar{b}, c) \times \prod_{i \in [n]} F(\bar{a}, b_i)
\end{align*}$$

For $F, G : D \rightsquigarrow D$ in $S$-CatSyn. This Kleisli bicategory is known as the bicategory of cartesian distributors [23]. By straightforward coend manipulations, we derive the type system $C_A$ described in Figure 6. Actions on typing derivations are defined in the straightforward way. The equivalence on typing derivation in this case is generated only by the rule of Figure 7, since now the coend on contexts disappeared. It is worth noting that the cartesian category $SD_A$ admits all the basic axioms imposed on the preorder over idempotent intersection types [1]. This means that our construction generalizes the standard subtyping relation, as expected. However, the two conditions

$$\begin{align*}
\pi_{1,2} : a_1 \oplus a_2 \to a_3 \\
c_{\emptyset} : a \to a \oplus a
\end{align*}$$

do not determine an idempotency $a \oplus a \cong a$. In our categorified setting, idempotency is replaced by the possibility to perform two operations on resources: copying and deleting.
We have that non-extensional models for pure Kleisli bicategories to be cartesian closed. We then defined a family of Kleisli bicategories of distributors, parametric framework for a general theory of intersection types. We de...

Where \( \vec{a} = \langle a_1, \ldots, a_k \rangle \).

We have that \( \pi = [c^* \Rightarrow 1] \pi' \) where \( c^* = 1_{(b)} \oplus c_{(a)} \). If we consider then the following derivation \( \rho' \):

\[
\begin{align*}
\pi' & : \\
\vdash \lambda x.(xx)x : (b, a, a) \Rightarrow a & \quad \vdash N : b \quad \vdash N : a \quad \vdash N : a \\
\vdash (\lambda x.(xx)x)N : a
\end{align*}
\]

We have that \( \rho \sim \rho' \) by the rule of congruence of Figure 7.

VI. CONCLUSIONS

1) Results: Bringing together several independent results and perspectives, we gave a consistent argument in favour of considering the bicategory of distributors as an appropriate framework for a general theory of intersection types. We defined a family of Kleisli bicategories of distributors, parametric over a resource monad. We gave a sufficient condition for these Kleisli bicategories to be cartesian closed. We then defined non-extensional models for pure \( \lambda \)-calculus. We showed how each resource monad is equivalent to a particular intersection type construction. Each model that we presented can be seen as an appropriate category of types. From this category of types we defined an intersection type system and, consequently, a proof relevant denotational semantics. We then proved that these semantics are coherent with respect to solvability.

2) Perspectives: The flexibility of our approach opens a considerable amount of possible future investigations. From an abstract standpoint, it is tempting to go even a bit further in the direction of [46] and identify our construction of intersection type distributors with an interpretation morphism between the symmetric 2-operad of \( \lambda \)-terms and the bicategory \( \text{Dist} \). This identification makes intuitively sense because of the strict connection between Kleisli bicategories of distributors and multicategories [21]. We leave all these speculations to future work.

We believe that an appropriate presentation of the results of [49, Chapter 4] about the relationship between intersection type distributors and \( \lambda \)-terms rigid approximants [50, 54] would be of great interest. In particular, this would make explicit the relationship between our semantic approach and the one of [54, 55].

The study of the relationship between our distributors induced semantics and Melliès template games semantics [47] would shed some light on the connection between intersection types, term approximants and game semantics. The natural starting point for this investigation would be a particular special case of the general construction of [47]: the bicategory of spans over groupoids. What happens in that framework with the exponential modality is particularly interesting and could give a homotopic flavour to our semantic perspective.

Another possibility is the investigation of extensional collapse, in the sense of [17]. We believe that connecting the approaches of the present work and of [27] we could reach an understanding of the semantic link between non-idempotent intersection types and idempotent ones in the bicategorical setting of distributors.

An extension of our approach to probabilistic computation, algebraic \( \lambda \)-calculus [56] would also be of great interest. This would be somehow related to [55], [40] and [6].

Finally, another interesting question arises in the context of Multiplicative Exponential Linear Logic (MELL). Since the notion of experiment [20] can be thought as the proof-net version of typing derivations, a possible extension of this work to that setting could give relevant information about the experiments reduction [11].
REFERENCES

[1] Fabio Alessi, Franco Barbanera, and Mariangiola Dezani-Ciancaglini. “Intersection types and lambda models”. In: *Theoretical Computer Science* 355.2 (2006). Logic, Language, Information and Computation, pp. 108–126. DOI: https://doi.org/10.1016/j.tcs.2006.01.004.

[2] Roberto M. Amadio and Pierre-Louis Curien. *Domains and Lambda-calculi*. New York, NY, USA: Cambridge University Press, 1998. ISBN: 0-521-62277-8.

[3] Alexis Bernadet and Stéphane Jean Lengrand. “Non-idempotent intersection types and strong normalisation”. In: *Logical Methods in Computer Science* Volume 9, Issue 4 (2013). DOI: 10.2168/LMCS-9(4:3)2013.

[4] R. Blackwell, G. Maxwell Kelly, and A. John Power. “Two-dimensional monad theory”. In: *Journal of Pure and Applied Algebra* 59.1 (1989), pp. 1–41. ISSN: 0022-4049. DOI: https://doi.org/10.1016/0022-4049(89)90160-6.

[5] Francis Borceux. *Handbook of Categorical Algebra*. Vol. 1. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1994. DOI: 10.1017/CBO9780511525858.

[6] Flavien Breuvart and Ugo Dal Lago. “On Intersection Types and Probabilistic Lambda Calculi”. In: *Proceedings of the 20th International Symposium on Principles and Practice of Declarative Programming, PPDP 2018*, Frankfurt am Main, Germany, September 03-05, 2018, 8:1–8:13. DOI: 10.1145/3236950.3236968.

[7] Flavien Breuvart, Giulio Manzonetto, and Domenico Ruoppolo. “Relational Graph Models at Work”. In: *Logical Methods in Computer Science* Volume 14, Issue 3 (2018). DOI: 10.23638/LMCS-14(3:2)2018.

[8] Antonio Bucciarelli, Delia Kesner, and Daniel Ventura. “Non-idempotent intersection types for the Lambda-Calculus”. In: *Logic Journal of the IGPL* 25.4 (2017), pp. 431–464. DOI: 10.1093/jigpal/jzx018.

[9] Daniel de Carvalho. “Sémantique de la logique lineaire et temps de calcul”. In: PhD thesis, Aix-Marseille Université, 2007.

[10] Daniel de Carvalho and Lorenzo Tortora de Falco. “A semantic account of strong normalization in linear logic”. In: *Inf. Comput.* 248 (2016), pp. 104–129. DOI: 10.1016/j.ic.2015.12.010.

[11] Daniel de Carvalho, Michele Pagani, and Lorenzo Tortora de Falco. “A semantic measure of the execution time in linear logic”. In: *Theoretical Computer Science* 412.20 (2011), pp. 1884–1902. ISSN: 0304-3975. DOI: https://doi.org/10.1016/j.tcs.2010.12.017.

[12] Gian Luca Cattani and Glynn Winskel. “Profunctors, open maps and bisimulation”. In: *Mathematical Structures in Computer Science* 15.3 (2005), pp. 553–614. DOI: 10.1017/S0960129505004718.

[13] M. Coppo, M. Dezani-Ciancaglini, F. Honsell, and G. Longo. “Extended Type Structures and Filter Lambda Models”. In: *Logic Colloquium ‘82*. Vol. 112. Studies in Logic and the Foundations of Mathematics. Elsevier, 1984, pp. 241–262. DOI: https://doi.org/10.1016/S0049-237X(08)71819-9.

[14] Mario Coppo and Mariangiola Dezani-Ciancaglini. “A new type-assignment for lambda terms”. In: *Archiv für Mathematische Logik und Grundlagenforschung*. 1978, pp. 139–156.

[15] Brian Day. “On closed categories of functors”. In: *Reports of the Midwest Category Seminar IV*. Berlin, Heidelberg: Springer Berlin Heidelberg, 1970, pp. 1–38. ISBN: 978-3-540-36292-0.

[16] Thomas Ehrhard. “Call-By-Push-Value from a Linear Logic Point of View”. In: *Programming Languages and Systems*. Ed. by Peter Thiemann. Berlin, Heidelberg: Springer Berlin Heidelberg, 2016, pp. 202–228. ISBN: 978-3-662-49498-1.

[17] Thomas Ehrhard. “The Scott model of linear logic is the extensional collapse of its relational model”. In: *Theoretical Computer Science* 424 (2012), pp. 20-45. ISSN: 0304-3975. DOI: https://doi.org/10.1016/j.tcs.2011.11.027.

[18] Thomas Ehrhard and Giulio Guerrieri. “The bang calculus: An untyped lambda-calculus generalizing Call-By-Name and Call-By-Value”. In: *Proceedings of the 18th International Symposium on Principles and Practice of Declarative Programming, PPDP 2016*. Association for Computing Machinery, 2016, pp. 174–187. DOI: 10.1145/2967973.2968608.

[19] Thomas Ehrhard and Laurent Regnier. “Uniformity and the Taylor Expansion of ordinary λ-terms”. In: *Theoretical Computer Science* 403.2-3 (2008). DOI: 10.1016/j.tcs.2008.06.001.

[20] Lorenzo Tortora de Falco. “Obsessional Experiments For Linear Logic Proof-Nets”. In: *Mathematical Structures in Computer Science* 13.6 (2003), pp. 799–855. DOI: 10.1017/S0960129503003967.

[21] Marcelo Fiore, Nicola Gambino, Martin Hyland, and Glynn Winskel. “Relative pseudomonads, Kleisi bicategories, and substitution monoidal structures”. In: *Selecta Mathematica* 24.3 (Nov. 2017), pp. 2791–2830. ISSN: 1420-9020. DOI: 10.1007/s00029-017-0361-3.

[22] Marcelo Fiore, Nicola Gambino, Martin Hyland, and Glynn Winskel. “The cartesian closed bicategory of generalised species of structures”. In: *Journal of the London Mathematical Society* (2008). DOI: 10.1112/jlms/dfm096.

[23] Marcelo Fiore and André Joyal. “Theory of Parad-Toposes”. Talk at the Category Theory Conference. 2015.

[24] Marcelo Fiore and Philip Saville. “A type theory for cartesian closed bicategories (Extended Abstract)”. In: *34th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2019*, Vancouver, BC, Canada, June 24-27, 2019, 2019, pp. 1–13. DOI: 10.1109/LICS.2019.8785708.
[25] Marcelo Fiore and Philip Saville. “Coherence and Normalisation-by-Evaluation for Bicategorical Cartesian Closed Structure”. In: *Proceedings of the 35th Annual ACM/IEEE Symposium on Logic in Computer Science*. LICS ’20. Saarbrücken, Germany: Association for Computing Machinery, 2020, pp. 425–439. DOI: 10.1145/3373718.3394769.

[26] Marcelo P. Fiore. “Mathematical Models of Computational and Combinatorial Structures”. In: *Foundations of Software Science and Combinatorial Structures*. Ed. by Vladimiro Sassone. Berlin, Heidelberg: Springer Berlin Heidelberg, 2005, pp. 25–46. ISBN: 978-3-540-31982-5.

[27] Zeinab Galal. “A Profunctorial Scott Semantics”. In: *5th International Conference on Formal Structures for Computation and Deduction (FSCD 2020)*. Ed. by Zena M. Ariola. Vol. 167. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2020, 16:1–16:18. ISBN: 978-3-95977-155-9. DOI: 10.4230/LIPIcs.FSCD.2020.16.

[28] Nicola Gambino and André Joyal. “On operads, bimodules and analytic functors”. In: *Memoirs of the American Mathematical Society* 249.1184 (Sept. 2017). ISSN: 1947-6221. DOI: 10.1090/memo/1184.

[29] Philippa Gardner. “Discovering needed reductions using type theory”. In: *Theoretical Aspects of Computer Software*. Berlin, Heidelberg: Springer Berlin Heidelberg, 1994, pp. 555–574. ISBN: 978-3-540-48383-0.

[30] Jean-Yves Girard. “Linear logic”. In: *Theoretical Computer Science* 50.1 (1987), pp. 1–101. DOI: https://doi.org/10.1016/0304-3975(87)90045-4.

[31] Giulio Guerrieri and Giulio Manzonetto. “The Bang Calculus and the Two Girard’s Translations”. In: *Proceedings Joint International Workshop on Linearity & Trends in Linear Logic and Applications, Linearity-TLLA@FLoC 2018*, Oxford, UK, 7-8 July 2018, pp. 15–30. DOI: 10.4204/EPTCS.292.2.

[32] Giulio Guerrieri and Federico Olimpieri. “Categorifying Non-Idempotent Intersection Types”. In: *29th EACSL Annual Conference on Computer Science Logic (CSL 2021)*. Ed. by Christel Baier and Jean Goubault-Larrecq. Vol. 183. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2021, 25:1–25:24. ISBN: 978-3-95977-175-7. DOI: 10.4230/LIPIcs.CSL.2021.25.

[33] Tom Hirschowitz. “Cartesian closed 2-categories and permutation equivalence in higher-order rewriting”. In: *Log. Methods Comput. Sci.* 9.3 (2013). DOI: 10.2168/LMCS-9(3:10)2013.

[34] Martin Hyland. “Classical lambda calculus in modern dress”. In: *Mathematical Structures in Computer Science* 27.5 (2017), pp. 762–781. DOI: 10.1017/S0960129515000377.

[35] Martin Hyland, Misao Nagayama, John Power, and Giuseppe Rosolini. “A Category Theoretic Formulation for Engeler-style Models of the Untyped $\lambda$-Calculus”. In: *Electronic Notes in Theoretical Computer Science* 161 (2006). Proceedings of the Third Irish Conference on the Mathematical Foundations of Computer Science and Information Technology (MCSFIT 2004), pp. 43–57. DOI: https://doi.org/10.1016/j.entcs.2006.04.024.

[36] Geun Bin IM and G.Maxwell Kelly. “A universal property of the convolution monoidal structure”. In: *Journal of Pure and Applied Algebra* 43.1 (1986), pp. 75–88. DOI: https://doi.org/10.1016/0022-4049(86)90005-8.

[37] André Joyal. “Foncteurs analytiques et espèces de structures”. In: *Combinatoire énumérative*. Berlin, Heidelberg: Springer Berlin Heidelberg, 1986, pp. 126–159.

[38] G. Maxwell Kelly. “A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on”. In: *Bulletin of the Australian Mathematical Society* 22.1 (1980), pp. 1–83. DOI: 10.1017/S0004972700006353.

[39] Jean-Louis Krivine. “Lambda-calculus, types and models”. In: *Ellis Horwood series in computers and their applications*. 1993.

[40] Jim Laird, Giulio Manzonetto, Guy McCusker, and Michele Pagani. “Weighted Relational Models of Typed Lambda-Calculi”. In: *Proceedings of the 2013 28th Annual ACM/IEEE Symposium on Logic in Computer Science*. LICS ’13. USA: IEEE Computer Society, 2013. DOI: 10.1109/LICS.2013.36.

[41] Paul Blain Levy. “Call-by-Push-Value: A Subsuming Paradigm”. In: *Proceedings of the 4th International Conference on Typed Lambda Calculi and Applications*, TLCA ’99. Berlin, Heidelberg: Springer-Verlag, 1999, pp. 228–242. ISBN: 3540657630.

[42] Fosco Loregian. *Coend calculus*. 2015. arXiv: 1501.02503 [math.CT].

[43] Fernando Lucatelli Nunes. “On lifting of biadjoints and lax algebras”. In: *Categories and General Algebraic Structures with Applications* 9.1 (2018), pp. 29–58. ISSN: 2345-5853.

[44] Dan Marsden and Maaike Zwart. “Quantitative Foundations for Resource Theories”. In: *27th EACSL Annual Conference on Computer Science Logic, CSL 2018, September 4-7, 2018, Birmingham, UK*. 2018, 32:1–32:17. DOI: 10.4230/LIPIcs.CSL.2018.32.

[45] Damiano Mazza. “Polyadic Approximations in Logic and Computation”. Habilitation thesis. Université Paris 13, 2017.

[46] Damiano Mazza, Luc Pellissier, and Pierre Vial. “Polyadic approximations, fibrations and intersection types”. In: 2018. DOI: 10.1145/3158094.

[47] P. Mellière. “Template games and differential linear logic”. In: *2019 34th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*. 2019, pp. 1–13. DOI: 10.1109/LICS.2019.8785830.

[48] Paul-André Mellière and Noam Zeilberger. “Functors Are Type Refinement Systems”. In: *SIGPLAN Not.* 50.1
[49] Federico Olimpieri. “Intersection Types and Resource Calculi in the Denotational Semantics of Lambda-Calculus”. PhD thesis. Aix-Marseille Université, 2020.

[50] Federico Olimpieri and Lionel Vaux Auclair. On the Taylor expansion of λ-terms and the groupoid structure of their rigid approximants. 2020. arXiv: 2008.02665 [cs.LO].

[51] C.-H. Luke Ong. “Quantitative semantics of the lambda calculus: Some generalisations of the relational model”. In: 32nd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2017, Reykjavik, Iceland, June 20-23, 2017. IEEE Computer Society, 2017, pp. 1–12. DOI: 10.1109/LICS.2017.8005064.

[52] Philip Saville. Cartesian closed bicategories: type theory and coherence. 2020. arXiv: 2007.00624 [math.CT].

[53] R. A. G. Seely. “Modelling Computations: A 2-Categorical Framework”. In: LICS. 1987.

[54] Takeshi Tsukada, Kazuyuki Asada, and C.-H. Luke Ong. “Generalised Species of Rigid Resource Terms”. In: Proceedings of the 32nd Annual ACM/IEEE Symposium on Logic in Computer Science. LICS 2017. 2017. DOI: 10.1109/LICS.2017.8005093.

[55] Takeshi Tsukada, Kazuyuki Asada, and C.-H. Luke Ong. “Species, Profunctors and Taylor Expansion Weighted by SMCC: A Unified Framework for Modelling Nondeterministic, Probabilistic and Quantum Programs”. In: Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science. LICS ’18. Oxford, United Kingdom, 2018, pp. 889–898. ISBN: 978-1-4503-5583-4. DOI: 10.1145/3209108.3209157.

[56] Lionel Vaux. “The algebraic lambda calculus”. In: Mathematical Structures in Computer Science 19.5 (2009), pp. 1029–1059. DOI: 10.1017/S0960129509900089.

[57] Pierre Vial. “Infinitary intersection types as sequences: A new answer to Klop’s problem”. In: 2017 32nd Annual ACM/IEEE Symposium on Logic in Computer Science (LICS). 2017, pp. 1–12. DOI: 10.1109/LICS.2017.8005103.