On the explicit upper and lower bounds for the number of zeros of the Selberg class

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Abstract

In this paper we prove explicit upper and lower bounds for the error term in the Riemann-von Mangoldt type formula for the number of zeros inside the critical strip. Furthermore, we also give examples of the bounds.

1 Introduction

The Selberg class $S$, defined by Selberg [13], consists of functions $L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ which satisfy the following conditions:

1. Ramanujan hypothesis: For any $\epsilon > 0$ we have $|a(n)| \ll n^\epsilon$.

2. Analytic continuation: There is an integer $k \geq 0$ such that $(s-1)^k L(s)$ is an entire function of finite order.

3. Functional equation: There exists a positive integer $f$ and a real number $Q$ and for integer $j \in [1, f]$ there are positive real numbers $\lambda_j$ and complex numbers $\omega$, $\mu_j$, $d_L = 2 \sum_{j=1}^{f} \lambda_j$, $\lambda = \prod_{j=1}^{f} \lambda_j^{2\lambda_j}$ where $|\omega| = 1$ and $\Re(\mu_j) \geq 0$ which satisfy

$$\Lambda_L(s) = \omega \Lambda_L(1 - s)$$

where

$$\Lambda_L(s) = L(s)Q^s \prod_{j=1}^{f} \Gamma(\lambda_j s + \mu_j).$$

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4. Euler product: We have

\[ L(s) = \prod_p L_p(s), \]

where

\[ L_p(s) = \exp \left( \sum_{l=1}^{\infty} b(p^l) p^l s \right) \]

with some coefficients \( b(p^l) \) satisfying \( b(p^l) \ll p^{\theta} \) for some \( \theta < \frac{1}{2} \).

We denote \( s = \sigma + it \) where \( \sigma \) and \( t \) are real numbers. We also choose the principal branch of the logarithm \( \log L(s) \) on the real axis as \( \sigma \to \infty \). For other points we use the analytic continuation of the logarithm.

The zeros \( s = -\frac{l + it}{\lambda} \) where \( l = 0, 1, 2, \ldots \) and \( j \in [1, f] \) of the function \( L(s) \) are called trivial zeros. We notice that for all trivial zeros it holds that \( \sigma \leq 0 \). Since we suppose that the function has an Euler product, we know that \( L(s) \) does not have zeros for which \( \sigma > 1 \) and \( a(1) = 1 \). The zeros which lie in the strip \( 0 \leq \sigma \leq 1 \) are called non-trivial zeros. The function may have a trivial and a non-trivial zero at the same point. By \([5]\) if \( dL = 0 \) then \( L(s) \equiv 1 \) and if \( dL > 0 \) then \( dL \geq 1 \). Since the number of zeros for \( L(s) \equiv 1 \), we concentrate to the cases \( dL \geq 1 \).

Riemann-von Mangoldt formula \([22, 23]\) describes the number of the zeros of the Riemann zeta function inside the certain strip. According to this formula, the number of the zeros \( \rho \) for which \( 0 \leq \Im(\rho) \leq T \) for a positive real number \( T \) is

\[ \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T). \]

A. Selberg \([14]\) and A. Fujii \([10]\) proved similar formulas for the zeros of the \( L \)-functions inside the critical strip for height \( T \) and \( [T, T + H] \) respectively. J. Steuding \([17]\) and L. Smaljović \([16]\) estimated the number of the zeros of the Selberg class functions. The number of the zeros \( \rho \) of the function \( L \in S \), for which \( 0 \leq \Im(\rho) \leq T \) or \( -T \leq \Im(\rho) \leq 0 \), is

\[ \frac{dL}{2\pi} T \log \frac{T}{e} + \frac{T}{2\pi} \log (\lambda Q^2) + O(\log T). \]

In this paper we are interested in the non-trivial zeros for which \( T_0 < \Im(\rho) \leq T \) or \( -T \leq \Im(\rho) < T_0 \) when \( T > T_0 \) is large enough. We prove that

\[ \left| N_L^\pm(T_0, T) - \frac{dL}{2\pi} T \log \frac{T}{e} - \frac{T}{2\pi} \log (\lambda Q^2) \right| < c_{L,1} \log T + c_{L,2}(T_0) + \frac{c_{L,3}(T_0)}{T}, \]

where \( N_L^\pm(T_0, T) \) denotes the number of the non-trivial zeros of the function \( L \in S \) for \( T_0 < \Im(\rho) \leq T \) or \( -T \leq \Im(\rho) < T_0 \) when \( T > T_0 \) is large enough. The terms \( c_{L,\lambda}(T_0) \) are real numbers which depend on the function \( L \) and the
number \( T_0 \) and the real number \( c_{L,1} \) depends only on the function \( L \). This formula involves no unknown, undefined constants in the error term. This is proved in Theorem 5.2. In 1916 R. J. Backlund [1] proved similar formula for the Riemann zeta function. E. Carneiro and R. Finder [4], [8] proved an explicit bound for the function 
\[ S_1(t, \pi) = \frac{1}{\pi} \int_1^\infty \log |L(\sigma + it, \pi)| d\sigma, \]
where \( L(s, \pi) \) is in a subset of the \( L \)-functions, assuming generalized Riemann hypothesis. There are also explicit upper bounds for the number of the zeros for the Riemann zeta-function assuming Riemann hypothesis [2] and along the critical line [20] and \( L \)-functions [3]. G. França and A. LeClair [9] proved an exact equation for the \( n \)th zero of the \( L \)-functions on the critical line. There are also explicit results for Dirichlet \( L \)-functions and Dedekind zeta-functions, see for example the papers from K. S. McCurley [12] and T. S. Trudgian [21].

In the Section 2 we prove lemmas which are used in the other sections. We also prove a formula which describes the sum of the real parts of the zeros inside certain strip. The sum depends on the four integrals of the function \( L(s) \). The main result follows from the estimates of the integrals. In the Sections 3 and 4 we estimate the integrals. In the Section 5 we combine the results of the Sections 2, 3 and 4 and prove the main result. We follow the proofs of the article [17] and the chapters 6 and 7 of the book [18]. In the Section 6 we give examples of the main result.

2 Preliminary results

In this section we prove some preliminary results which are needed to prove the main theorem. The results are used in the Sections 3, 4 and 5.

2.1 Basic theory of the function \( L(s) \)

To shorten our notation we define

\[ v = \sum_{j=1}^{f} \lambda_j \log \lambda_j, \quad u = \sum_{j=1}^{f} \left( \tilde{\mu}_j - \frac{1}{2} \right) \log \lambda_j \]

and \( \mu = 4 \sum_{j=1}^{f} \left( \frac{1}{2} - \mu_j \right) \).

By the Ramanujan hypothesis (\( \epsilon = 1 \)) there is a constant \( a_1 \) such that for all \( n \) we have \( |a(n)| \leq a_1 n \). Note that \( a_1 \geq 1 \) since we have assumed that \( a(n) = 1 \). Let \( a \) be a real number for which \( a > 2 \) and

\[ \sum_{n=2}^{\infty} \frac{a_1}{n^a} < \frac{1}{2} \]

and let \( b < -3 \) be a negative real number which has the following property:

\[ \sum_{n=2}^{\infty} \frac{a_1}{n^{-b-1}} < 1. \]
In the chapter 7.1 of the book [18] it is proved that:

Lemma 2.1. Let $T_0$ and $T > T_0$ be a positive real numbers. Let number $\rho$ denote the zero of the function $L \in S$. Then

$$2\pi \sum_{T_0 < \Im(\rho) \leq T} (\Re(\rho) - b) = \int_{T_0}^{T} \log |L(b + it)| dt - \int_{T_0}^{T} \log |L(a + it)| dt$$

$$- \int_{b}^{a} \arg L(\sigma + iT_0) d\sigma + \int_{b}^{a} \arg L(\sigma + iT) d\sigma$$

$$:= I_1(T_0, T, b) + I_2(T_0, T, a) - I_3(T_0, a, b) + I_3(T, a, b).$$

The goal is to estimate the integrals and get the main result by using these estimates. By the functional equation we have

$$L(s) = \Delta L(s) \overline{L(1-s)},$$

where

$$\Delta L(s) = \omega Q^{1-2s} \prod_{j=1}^{f} \frac{\Gamma(\lambda_j(1-s) + \bar{\mu}_j)}{\Gamma(\lambda_j s + \mu_j)}.$$  \hfill (1)

We use this formula to estimate the term $\log |\Delta L(s)|$. We need this when we estimate the terms $I_1(T_0, T, b) - I_1(T_0, T, b + 1)$, $I_3(T_0, a, b)$ and $I_3(T, a, b)$. To do this we define that $B_n$ is $n$th Bernoulli number and we need the following lemma from T. J. Stieltjes [19, paragraph 9]:

Lemma 2.2. Let $z$ be a complex number such that $|\arg(z)| < \pi$. Then for $N = 1, 2, \ldots$

$$\log \Gamma(z) = z \log z - z + \frac{1}{2} \log \frac{2\pi}{z} + \sum_{n=1}^{N-1} \frac{B_{2n}}{2n(2n-1)z^{2n-1}} + W_N(z),$$

where

$$|W_N(z)| \leq \frac{B_{2N}}{2N(2N-1)|z|^{2N-1}} \sec^{2N} \left(\frac{|\arg z|}{2}\right)$$

is a holomorphic function.

Let

$$V_j(s) = \left(-\lambda_j s + \lambda_j + \bar{\mu}_j - \frac{1}{2}\right) \log \left(1 + \frac{\lambda_j + \bar{\mu}_j}{-\lambda_j s}\right) - (\lambda_j + \bar{\mu}_j)$$

$$- \left(\lambda_j s + \mu_j - \frac{1}{2}\right) \log \left(1 + \frac{\mu_j}{\lambda_j s}\right) + \mu_j + W(-\lambda_j s) - W(\lambda_j s),$$

where $|W(z)| \leq \frac{B_2}{2|z|} \sec^2 \left(\frac{|\arg z|}{2}\right)$ is a holomorphic function if $|\arg(z)| < \pi$. Let also $V(s) = \sum_{j=1}^{f} V_j(s)$. Using the previous lemma and simplifying expressions we get
Lemma 2.3. Let $|\arg (\lambda_j (1-s) + \bar{\mu}_j)| < \pi$, $|\arg (\lambda_j s + \mu_j)| < \pi$ and $t > 0$. Then

$$
\log |\Delta_L(s)| = \left( \frac{1}{2} - \sigma \right) \left( d_L \log t + \log (\lambda Q^2) \right) + d_L \sigma \\
+ \Re \left( \log \left( 1 - \frac{\sigma_i t}{2} \right) \left( d_L \left( \frac{1}{2} - s \right) + \frac{3(\mu)i}{2} \right) + V(s) \right)
$$

Proof. It is enough to estimate the product of the $\Gamma$-functions of the formula (1). First we do some basic calculation and then we use Lemma 2.2. After short computations we have

$$
\log (-\lambda_j s) = \log (\lambda_j) - \frac{\pi i}{2} + \log t + \log \left( 1 - \frac{\sigma_i t}{2} \right)
$$

and

$$
\log (\lambda_j s) = \log (\lambda_j) + \frac{\pi i}{2} + \log t + \log \left( 1 - \frac{\sigma_i t}{2} \right).
$$

We apply Lemma 2.2 for $N = 1$ and using the previous formulas, we get

$$
\log \left( \prod_{j=1}^{f} \frac{\Gamma (\lambda_j (1-s) + \bar{\mu}_j)}{\Gamma (\lambda_j s + \mu_j)} \right)
= \sum_{j=1}^{f} \left( \left( \lambda_j - \lambda_j s + \bar{\mu}_j - \frac{1}{2} \right) \left( \log (\lambda_j) - \frac{\pi i}{2} + \log t + \log \left( 1 - \frac{\sigma_i t}{2} \right) \right) + \lambda_j s \right)
- \sum_{j=1}^{f} \left( \left( \lambda_j s + \mu_j - \frac{1}{2} \right) \left( \log (\lambda_j) + \frac{\pi i}{2} + \log t + \log \left( 1 - \frac{\sigma_i t}{2} \right) \right) - \lambda_j s \right)
+ V(s)
= \log t \left( d_L \left( \frac{1}{2} - s \right) + \frac{3(\mu)i}{2} \right) + 2 \Re (u)i + d_L s + 2v \left( \frac{1}{2} - s \right)
- \frac{\pi i}{4} \left( d_L - \Re (\mu) \right) + \log \left( 1 - \frac{\sigma_i t}{2} \right) \left( d_L \left( \frac{1}{2} - s \right) + \frac{3(\mu)i}{2} \right) + V(s).
$$

The claim follows from the previous computations and the definition (1) of the function $\Delta_L(s)$.

2.2 The estimate of the function $V_j(s)$

The formula of the function $\log |\Delta_L(s)|$ in Lemma 2.3 contains the term $V(s)$. Thus we want also estimate the term $V(s)$. Since $V(s) = \sum_{j=1}^{f} V_j(s)$, it is sufficient to estimate the terms $V_j(s)$. Before doing this we prove a lemma. The estimate of the term $V_j(s)$ follows from this lemma.
Lemma 2.4. If \( t \geq \max \frac{2|\lambda_j + \mu_j|}{\lambda_j}, \) \( t \geq \max \frac{2|\mu_j|}{\lambda_j} \) and \( \sigma \) is a constant then for all \( j \)

\[
|W(-\lambda_j s)| + |W(\lambda_j s)| < \frac{B_2}{2|\lambda_j t|} \left( 2 + \sec^2 \left( \frac{\arg \left( \lambda_j \left( -|\sigma| + \max_j \frac{2|\lambda_j + \mu_j|}{\lambda_j}, \frac{2|\mu_j|}{\lambda_j} \right) i \right)}{2} \right).
\]

Proof. We assume that \( \sigma < 0 \) since we can prove the case \( \sigma \geq 0 \) similarly. By the definition of the function \( W(z) \) we have

\[
|W(z)| \leq \frac{B_2}{2|z|} \sec^2 \left( \frac{\arg z}{2} \right)
\]

if \( |\arg z| < \pi \). Also, by the assumptions for the number \( t \) we have \( |\arg(\pm \lambda_j s)| < \pi \). The goal is to estimate the functions \( \sec^2 \left( \frac{\arg(-\lambda_j s)}{2} \right) \) and \( \sec^2 \left( \frac{\arg(\lambda_j s)}{2} \right) \). We do the estimate by using basic properties of the secant function.

First we estimate the arguments of the complex numbers \(-\lambda_j s\) and \(\lambda_j s\). Since \( t \geq \max \frac{2|\lambda_j + \mu_j|}{\lambda_j}, \) \( t \geq \max \frac{2|\mu_j|}{\lambda_j} \) and \( \sigma < 0 \), the argument of the complex numbers \(-\lambda_j s\) is in the interval \((-\frac{\pi}{2}, 0)\). Thus \( \frac{\arg(-\lambda_j s)}{2} \in (-\frac{\pi}{4}, 0) \).

Similarly we have

\[
\frac{\arg(\lambda_j s)}{2} \in \left( \frac{\pi}{4}, \frac{\pi}{2} \right) \subset \left( \frac{\pi}{4}, \frac{\pi}{2} \right).
\]

Next we use the estimates of the arguments. Because the function \( \sec^2(z) \) is an increasing function for \( z \in \left[ 0, \frac{\pi}{2} \right) \) and an even function we have

\[
\sec^2 \left( \frac{\arg(-\lambda_j s)}{2} \right) < \sec^2 \left( -\frac{\pi}{4} \right) = 2
\]

and

\[
\sec^2 \left( \frac{\arg(\lambda_j s)}{2} \right) \leq \sec^2 \left( \frac{\arg \left( \lambda_j \left( \sigma + \max_j \frac{2|\lambda_j + \mu_j|}{\lambda_j}, \frac{2|\mu_j|}{\lambda_j} \right) i \right)}{2} \right).
\]

The claim follows from the previous equations. \( \square \)

Now we estimate the term \( V_j(s) \).
Lemma 2.5. If $\sigma$ is a constant and $t$ is as in Lemma 2.4 then for all $j$

$$|V_j(s)| < \frac{2}{|\lambda_j s|} \left( \frac{(\lambda_j + \bar{\mu}_j)^2}{\lambda_j t} + 2 \left| \frac{(\lambda_j + \bar{\mu}_j)(\lambda_j + \mu_j - \frac{1}{2})}{\lambda_j t} \right| + \left| \frac{\mu_j^2}{\lambda_j t} \right| + 2 \left| \frac{\mu_j(\mu_j - \frac{1}{2})}{\lambda_j t} \right| \right) \left( \arg \left( \lambda_j (1 + |\sigma| + \max_j \left\{ \left\{ \frac{2\lambda_j + \mu_j}{\lambda_j}, \frac{2\mu_j}{\lambda_j} \right\} i \right\}) \right) \right).$$

Proof. By the definition of the function $V_j(s)$ and the series expansion we have

$$V_j(s) = \left( \frac{\lambda_j + \bar{\mu}_j}{\lambda_j s} \right)^2 \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n} \left( \frac{\lambda_j + \bar{\mu}_j}{-\lambda_j s} \right)^{n-2}$$

$$+ \left( \frac{\lambda_j + \bar{\mu}_j - 1}{2} \right) \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left( \frac{\lambda_j + \bar{\mu}_j}{-\lambda_j s} \right)^{n-2} - \left( \frac{\mu_j - 1}{2} \right) \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left( \frac{\mu_j}{\lambda_j s} \right)^{n-2} - \left( \frac{\mu_j - 1}{2} \right) \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left( \frac{\mu_j}{\lambda_j s} \right)^{n-2} W(-\lambda_j s) - W(\lambda_j s).$$

Since $t \geq \max_j \left\{ \frac{2\lambda_j + \mu_j}{\lambda_j} \right\}$ and $t \geq \max_j \left\{ \frac{2\mu_j}{\lambda_j} \right\}$, we have $\left| \frac{\lambda_j + \bar{\mu}_j}{\lambda_j s} \right| \leq \frac{1}{2}$ and $\left| \frac{\mu_j}{\lambda_j s} \right| \leq \frac{1}{2}$ for all $j$. The claim follows from these estimates and Lemma 2.4.

3 The difference $I_1(T_0, T, b) - I_1(T_0, T, b + 1)$

In this section we estimate the integral

$$I_1(T_0, T, b) - I_1(T_0, T, b + 1) = \int_{T_0}^{T} \log |L(b + it)| - \log |L(b + 1 + it)| dt.$$

3.1 Preliminaries for the difference $I_1(T_0, T, b) - I_1(T_0, T, b + 1)$

In this section we prove preliminary results which are used to estimate the term $I_1(T_0, T, b) - I_1(T_0, T, b + 1)$. The first one of these describes the properties of the logarithm.
Lemma 3.1. Let $z$ be a complex number. Then

(a) if $|z| < \frac{1}{2}$ we have \[|\log(1 + z)| < 2|z|,\]

(b) if $z$ is a real number we have \[|\log(1 - zi)| < 7|z|.

Proof. (a) Assume that $|z| < \frac{1}{2}$. By the series expansion of the logarithm we have

\[|\log(1 + z)| = \left| \sum_{n=1}^{\infty} \frac{(-1)^{n+1}z^n}{n} \right| \leq \sum_{n=1}^{\infty} \frac{|z|^n}{n}.

Since $|z| < \frac{1}{2}$ and $n \geq 1$ in the sum, we have

\[
\sum_{n=1}^{\infty} \frac{|z|^n}{n} < \sum_{n=1}^{\infty} |z| \frac{1}{2^n-1} = 2|z|.
\]

(b) Assume that $z$ is a real number. If $|z| < \frac{1}{2}$ then by the previous case we have $|\log(1 - zi)| < 2|z|$. Let $|z| \geq \frac{1}{2}$. Now we have

\[
|\log(1 - zi)|^2 = (\log |1 - zi|)^2 + (\arg(1 - zi))^2 < 5|z|^2 + 4\pi^2|z|^2.
\]

Thus

\[|\log(1 - zi)| < \sqrt{5 + 4\pi^2}|z| < 7|z|.
\]

We use the previous lemma and basic properties of the absolute value to obtain the following two inequalities.

Lemma 3.2. If $\sigma < -3$ and $t > 0$ then

\[
\left| \Re \left( \log \left( 1 - \frac{\sigma i}{t} \right) \left( d_L \left( \frac{1}{2} - \sigma \right) + \frac{\Im(\mu)i}{2} \right) \right. \right. \\
- \log \left( 1 - \frac{(\sigma + 1)i}{t} \right) \left( d_L \left( -\frac{1}{2} - \sigma \right) + \frac{\Im(\mu)i}{2} \right) \left. \right) \right| \\
< \frac{2}{t} \left| -d_L \sigma + \frac{\Im(\mu)i}{2} \right| - \frac{7d_L}{2t}(2\sigma + 1).
\]
Proof. We have
\[
\left| \Re \left( \log \left( 1 - \frac{\sigma i}{t} \right) \left( d_L \left( \frac{1}{2} - \sigma \right) + \frac{\Im(\mu)i}{2} \right) \right) \right| \leq \left| \left( -d_L \sigma + \frac{\Im(\mu)i}{2} \right) \log \left( 1 + \frac{i}{t - (\sigma + 1)i} \right) \right| \tag{2}
\]
\[
+ \left| \log \left( 1 - \frac{\sigma i}{t} \right) \frac{d_L}{2} \right| + \left| \log \left( 1 - \frac{(\sigma + 1)i}{t} \right) \frac{d_L}{2} \right|.
\]

We want to estimate the previous terms. First we estimate the factor \(\log \left( \frac{1 + i}{t - (\sigma + 1)i} \right)\). Since \(\sigma < -3\), we have
\[
\left| \frac{i}{t - (\sigma + 1)i} \right|^2 = \frac{1}{t^2 + (\sigma + 1)^2} < \frac{1}{4}.
\]

Thus \(\left| \frac{i}{t - (\sigma + 1)i} \right| < \frac{1}{4}\) and by Lemma 3.1 we get
\[
\left| \left( -d_L \sigma + \frac{\Im(\mu)i}{2} \right) \log \left( 1 + \frac{i}{t - (\sigma + 1)i} \right) \right| < 2 \left| -d_L \sigma + \frac{\Im(\mu)i}{2} \right| \left| \frac{1}{t - (\sigma + 1)i} \right|. \tag{3}
\]

Further, by Lemma 3.1 we have
\[
\left| \log \left( 1 - \frac{\sigma i}{t} \right) \frac{d_L}{2} \right| < -7 \frac{d_L \sigma}{2t}, \tag{4}
\]
and
\[
\left| \log \left( 1 - \frac{(\sigma + 1)i}{t} \right) \frac{d_L}{2} \right| < -7 \frac{d_L (\sigma + 1)}{2t}. \tag{5}
\]

The claim follows from the formulas (2), (3), (4) and (5).

Lemma 3.3. If \(|\sigma| \geq 1\) and \(t > 0\) then
\[
\left| \Re \left( -d_L - d_L \log \left( 1 - \frac{\sigma i}{t} \right) it + d_L \log \left( 1 - \frac{(\sigma + 1)i}{t} \right) it \right) \right| < d_L \left( \frac{3(\sigma^2 + \sigma)}{t^2} + \frac{2}{t} \right).
\]

Proof. We can calculate
\[
\Re \left( -d_L - d_L \log \left( 1 - \frac{\sigma i}{t} \right) it + d_L \log \left( 1 - \frac{(\sigma + 1)i}{t} \right) it \right)
\]
\[
= -d_L \left( 1 + t \arg \left( 1 - \frac{i}{t - \sigma i} \right) \right).
\]
To obtain the absolute value of the term $-d_L \left(1 + t \arg \left(1 - \frac{i}{t - \sigma i}\right)\right)$ we calculate the upper and lower bounds of this expression. We have $\arg(1 - \frac{i}{t - \sigma i}) = \arctan \left(\frac{-t}{t^2 + \sigma^2 + \sigma}\right)$. Thus $\arg(1 - \frac{i}{t - \sigma i}) \leq 0$ and

$$-d_L \left(1 + t \arg \left(1 - \frac{i}{t - \sigma i}\right)\right) < -d_L \left(1 + \frac{-t^2}{t^2 + \sigma^2 + \sigma}\right) \leq -d_L (1 - 1) = 0.$$  

Next we compute the lower bound of the term $-d_L \left(1 + t \arg \left(1 - \frac{i}{t - \sigma i}\right)\right)$.

By \[15\] we have $\arctan(x) \leq \frac{3x}{1 + 2\sqrt{1 + x^2}}$ for $x \leq 0$. Thus

$$1 + t \arctan \left(\frac{-t}{t^2 + \sigma^2 + \sigma}\right) \leq 1 - \frac{3t^2}{(t^2 + \sigma^2 + \sigma) \left(1 + 2\sqrt{1 + \left(\frac{-t}{t^2 + \sigma^2 + \sigma}\right)^2}\right)}.$$  

The right hand side is

$$= \frac{t^2 + \sigma^2 + \sigma + 2\sqrt{(t^2 + \sigma^2 + \sigma)^2 + t^2} - 3t^2}{t^2 + \sigma^2 + \sigma + 2\sqrt{(t^2 + \sigma^2 + \sigma)^2 + t^2}}.$$  

Since

$$t^2 + \sigma^2 + \sigma + 2\sqrt{(t^2 + \sigma^2 + \sigma)^2 + t^2} - 3t^2 < 3(\sigma^2 + \sigma) + 2t$$

and

$$t^2 + \sigma^2 + \sigma + 2\sqrt{(t^2 + \sigma^2 + \sigma)^2 + t^2} > t^2,$$

by \[5\] and \[4\] we have

$$1 + t \arctan \left(\frac{-t}{t^2 + \sigma^2 + \sigma}\right) < \frac{3(\sigma^2 + \sigma)}{t^2} + \frac{2}{t},$$

as required.

Next we estimate two integrals. The estimates are used in the next section.

**Theorem 3.4.** Let $T_0$ and $T > T_0$ be positive real numbers. Then

$$\left|\int_{T_0}^{T} \log |\mathcal{L}(1 - b + it)|\right| < \frac{\pi^2}{3\log 2} \quad \text{and} \quad \left|\int_{T_0}^{T} \log |\mathcal{L}(-b + it)|\right| < \frac{\pi^2}{3\log 2}.$$
Proof. We prove the claim only for the integral $|\int_{T_0}^{T} \log |L(-b + it)|$ since the other case can be proved similarly. First we look at the sum $L(-b + it) - 1 = \sum_{n=2}^{\infty} \frac{a(n)}{n^{b+it}}$. By the Ramanujan hypothesis and the assumptions for the numbers $a_1$ and $b$

$$\left| \sum_{n=2}^{\infty} \frac{a(n)}{n^{b+it}} \right| \leq \sum_{n=2}^{\infty} \frac{a_1}{n^{b-1}} < 1.$$ 

Thus we can use the Taylor series expansion of the logarithm of $L(-b+it) = 1 + \sum_{n=2}^{\infty} \frac{a(n)}{n^{b+it}}$ and get

$$\log |L(-b + it)| = \Re \left( \sum_{l=1}^{\infty} \frac{(-1)^l}{l} \sum_{n_1=2}^{\infty} \cdots \sum_{n_l=2}^{\infty} \frac{a(n_1) \cdots a(n_l)}{(n_1 \cdots n_l)^{-b+it}} \int_{T_0}^{T} \frac{dt}{(n_1 \cdots n_l)^{it}} \right).$$

We have

$$\left| \int_{T_0}^{T} \log |L(-b + it)| \right| = \left| \Re \left( \sum_{l=1}^{\infty} \frac{(-1)^l}{l} \sum_{n_1=2}^{\infty} \cdots \sum_{n_l=2}^{\infty} \frac{a(n_1) \cdots a(n_l)}{(n_1 \cdots n_l)^{-b}} \int_{T_0}^{T} \frac{dt}{(n_1 \cdots n_l)^{it}} \right) \right|$$

By the Ramanujan hypothesis

$$\Re \left( \sum_{l=1}^{\infty} \frac{(-1)^l}{l} \sum_{n_1=2}^{\infty} \cdots \sum_{n_l=2}^{\infty} \frac{a(n_1) \cdots a(n_l)}{(n_1 \cdots n_l)^{-b}} \int_{T_0}^{T} \frac{dt}{(n_1 \cdots n_l)^{it}} \right) \leq \sum_{l=1}^{\infty} \frac{1}{l} \sum_{n_1=2}^{\infty} \cdots \sum_{n_l=2}^{\infty} \frac{a_l}{(n_1 \cdots n_l)^{-b-1}} \left| \int_{T_0}^{T} \frac{dt}{(n_1 \cdots n_l)^{it}} \right|.$$ 

For $n \geq 2^l$ we have

$$\left| \int_{T_0}^{T} \frac{dt}{n^{it}} \right| = \left| \frac{i}{\log n} (e^{-iT \log n} - e^{-iT_0 \log n}) \right| \leq \frac{2}{l \log 2}.$$

Thus

$$\sum_{l=1}^{\infty} \sum_{n_1=2}^{\infty} \cdots \sum_{n_l=2}^{\infty} \frac{a_l}{(n_1 \cdots n_l)^{-b-1}} \left| \int_{T_0}^{T} \frac{dt}{(n_1 \cdots n_l)^{it}} \right| \leq \sum_{l=1}^{\infty} \frac{2}{l^2 \log 2} \left( \sum_{n=2}^{\infty} \frac{a_1}{n^{b-1}} \right)^l.$$ 

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Since we assume that \( \sum_{n=2}^{\infty} \frac{a_n}{n^{b-1}} < 1 \), we have
\[
\sum_{l=1}^{\infty} \frac{2}{l^2 \log 2} \left( \sum_{n=2}^{\infty} \frac{a_n}{n^{b-1}} \right)^l < \sum_{l=1}^{\infty} \frac{2}{l^2 \log 2} = \frac{\pi^2}{3 \log 2}.
\]

3.2 The estimate of the difference \( I_1(T_0, T, b) - I_1(T_0, T, b + 1) \)

In this section we estimate the term
\[
\left| I_1(T_0, T, b) - I_1(T_0, T, b + 1) - \int_{T_0}^{T} d\mathcal{L} \log t + \log(\lambda Q^2) dt \right|
\]
which is used to get the main result. We obtain the estimate by using the results which we have obtained in Sections 2 and 3.1. We want to simplify the notation and thus we define for real numbers \( T_0 \) and \( T \)
\[
S_j(T_0, T, b) = \frac{2}{\lambda_j} \log \frac{T}{T_0} \left( |\lambda_j + \bar{\mu}_j|^2 + 2 \left| (\lambda_j + \bar{\mu}_j) \left( \lambda_j + \bar{\mu}_j - \frac{1}{2} \right) \right| + |\mu_j|^2 + 2 |\mu_j \left( \mu_j - \frac{1}{2} \right)| \right)
+ \frac{|B_2|}{4} \left( 2 + \sec^2 \left( \frac{\arg \left( \lambda_j \left( -b + \max_j \left\{ \frac{2|\lambda_j + \bar{\mu}_j|}{\lambda_j}, \frac{2|\mu_j|}{\lambda_j} \right\} i \right) \right) }{2} \right)
+ \frac{|B_2|}{4} \left( 2 + \sec^2 \left( \frac{\arg \left( \lambda_j \left( -b - 1 + \max_j \left\{ \frac{2|\lambda_j + \bar{\mu}_j|}{\lambda_j}, \frac{2|\mu_j|}{\lambda_j} \right\} i \right) \right) }{2} \right)
\]
\[
S(T_0, T, b) = \sum_{j=1}^{f} S_j(T_0, T, b)
\]
and
\[
R_1(T_0, T, b) = \log \frac{T}{T_0} \left( -7\frac{d\mathcal{L}}{2} (2b + 1) + 2 \left| -d\mathcal{L} b + \frac{3\mu i}{2} \right| + 2d\mathcal{L} \right)
+ \frac{3d\mathcal{L} (b^2 + b)}{T_0} + S(T_0, T, b).
\]

Theorem 3.5. If \( T_0 \geq \max_j \left\{ \frac{2|\lambda_j + \bar{\mu}_j|}{\lambda_j} \right\} \), \( T_0 > \max_j \left\{ \frac{2|\mu_j|}{\lambda_j} \right\} \) and \( T > T_0 \) then
\[
\left| I_1(T_0, T, b) - I_1(T_0, T, b + 1) - \int_{T_0}^{T} d\mathcal{L} \log t + \log(\lambda Q^2) dt \right|
< R_1(T_0, T, b) + \frac{2\pi^2}{3 \log 2}
\]
Proof. By the definition of the integral $I_1(T_0, T, b)$ and since $L(s) = \Delta_L(s) \bar{L}(1-s)$, we have

$$I_1(T_0, T, b) - I_1(T_0, T, b + 1)$$

$$= \int_{T_0}^T \log |\Delta_L(b + it)| - \log |\Delta_L(b + 1 + it)| dt$$

$$+ \int_{T_0}^T \log |\mathcal{L}(1 - b + it)| - \log |\mathcal{L}(-b + it)| dt.$$  \hfill (9)

We will first estimate the terms and the claim follows when we sum the estimates.

First we look at the part $\log |\Delta_L(b + it)| - \log |\Delta_L(b + 1 + it)|$ and its integral. Since $T_0 > \max \left\{ \frac{2|\mu_j|}{\lambda_j} \right\}$, for $t \geq T_0$ it holds that

$$\begin{cases}
|\arg(\lambda_j(1 - b - it) + \overline{\mu}_j)| < \pi, \\
|\arg(\lambda_j(b + it) + \mu_j)| < \pi, \\
|\arg(\lambda_j(-b - it) + \overline{\mu}_j)| < \pi \text{ and} \\
|\arg(\lambda_j(b + 1 + it) + \mu_j)| < \pi
\end{cases}$$

Thus by Lemma 2.3 we have

$$\log |\Delta_L(b + it)| - \log |\Delta_L(b + 1 + it)|$$

$$= (d_L \log t + \log(\lambda Q^2)) - d_L + \Re[V(b + it) - V(b + 1 + it)]$$

$$+ \Re \left( \log \left(1 - \frac{b+1}{t} \right) \left(d_L \left(\frac{1}{2} - b - it\right) + \frac{3(\mu)i}{2} \right) - \log \left(1 - \frac{b}{t} \right) \left(d_L \left(-\frac{1}{2} - b - it\right) + \frac{3(\mu)i}{2} \right) \right).$$

Now we estimate the last difference between the last term and the term $d_L$. By Lemma 3.2 and Lemma 3.3 we have

$$\left| \Re \left( \log \left(1 - \frac{b}{t} \right) \left(d_L \left(\frac{1}{2} - b - it\right) + \frac{3(\mu)i}{2} \right) - \log \left(1 - \frac{b + 1}{t} \right) \left(d_L \left(-\frac{1}{2} - b - it\right) + \frac{3(\mu)i}{2} \right) \right) \right|$$

$$\leq \frac{2}{t} \left| d_L b + \frac{3(\mu)i}{2} \right| - t d_L \left(2b + 1\right) + d_L \left(\frac{3b^2 + b}{t^2} + \frac{2}{t} \right).$$
We can integrate this and get

\[
\int_{T_0}^{T} \frac{2}{t} \left| -\frac{3(\mu)i}{2} - \frac{7d_L}{2t}(2b + 1) + \frac{3(b^2 + b)}{t^2} + \frac{2}{t} \right| dt < \log \frac{T}{T_0} \left( -\frac{7d_L}{2}(2b + 1) + 2 \left| -\frac{3(\mu)i}{2} + 2d_L \right| + 3d_L(b^2 + b) \right). 
\]

Further, we can estimate the integral of the term \( \Re(V(b + it) - V(b + 1 + it)) \).

We remember that \( V(s) = \sum_{j=1}^{J} V_j(s) \). By Lemma 2.4,

\[
|V_j(b + it)| < 2 \left| \frac{(\lambda_j + \bar{\mu}_j)^2}{\lambda_j t} \right| + 4 \left| \frac{\lambda_j + \bar{\mu}_j}{\lambda_j t} \right| + 2 \left| \frac{\mu_j}{\lambda_j t} \right| + 4 \left| \frac{\mu_j(\mu_j - \bar{\mu}_j)}{\lambda_j t} \right| 
+ \frac{|B_2|}{2|\lambda_j t|} \left( 2 + \sec^2 \left( \arg \left( \frac{\lambda_j(-b + \max_j \left\{ \frac{2|\lambda_j + \bar{\mu}_j|}{\lambda_j}, \frac{2|\mu_j|}{\lambda_j} \right\})}{2} \right) \right) 
+ \frac{|B_2|}{2|\lambda_j t|} \left( 2 + \sec^2 \left( \arg \left( \frac{\lambda_j(-b - 1 + \max_j \left\{ \frac{2|\lambda_j + \bar{\mu}_j|}{\lambda_j}, \frac{2|\mu_j|}{\lambda_j} \right\})}{2} \right) \right) \right) 
\]

We can sum these terms, integrate and get

\[
\left| \int_{T_0}^{T} \Re(V(b + it) - V(b + 1 + it)) dt \right| < S(T_0, T, b). 
\]

By Theorem 3.4,

\[
\left| \int_{T_0}^{T} \log |\mathcal{L}(1 - b + it)| - \log |\mathcal{L}(-b + it)| dt \right| < \frac{2\pi^2}{31 \log 2}. 
\]

The claim follows immediately when we sum the estimates (9), (10), (11) and (12) together.

**4 Integral \( I_3(T, a, b) \)**

In this section we estimate the integral \( I_3(T, a, b) = \int_{b}^{a} \arg \mathcal{L}(\sigma + iT) d\sigma \).

**4.1 Preliminaries for the integral \( I_3(T, a, b) \)**

In this section we collect some preliminary results which are needed in the estimates of the integral \( I_3(T, a, b) \). Our first goal is to estimate the function \( \mathcal{L}(s) \), and to do this, we apply the Phragmén-Lindelöf principle for a strip. We also need the following lemma.
Lemma 4.1. If \( t \geq 1 \) then
\[
\Re \left( \log \left( 1 + \frac{2i}{t} \right) \left( d_{\mathcal{L}} \left( \frac{5}{2} - it \right) + \Im(\mu)i \right) \right) < \frac{5\sqrt{5} + 4}{2} d_{\mathcal{L}} + |\Im(\mu)|.
\]

Proof. Since \( t \geq 1 \), we get
\[
\Re \left( \log \left( 1 + \frac{2i}{t} \right) \left( d_{\mathcal{L}} \left( \frac{5}{2} - it \right) + \Im(\mu)i \right) \right) = \log \left| 1 + \frac{5}{2} d_{\mathcal{L}} + \arg \left( 1 + \frac{2i}{t} \right) d_{\mathcal{L}} t - \arg \left( 1 + \frac{2i}{t} \right) \frac{\Im(\mu)}{2} \right|
\]
\[
< \frac{5}{2} \sqrt{1^2 + 2^2 d_{\mathcal{L}} + \frac{2}{t} d_{\mathcal{L}} t + \frac{2}{t} \Im(\mu)} < \frac{5\sqrt{5} + 4}{2} d_{\mathcal{L}} + |\Im(\mu)|.
\]

We denote \( R = a - b \). Note that \( R > 0 \) and \( a - 2R < 0 \). Let
\[
M_1 = \max \left\{ 3^k \frac{a_1 \pi^2}{6}, 3^k \frac{a_1 \pi^2}{6}, |\lambda Q|^2 \right\} \exp \left( \frac{5\sqrt{5}}{2} d_{\mathcal{L}} + |\Im(\mu)| + \frac{1}{|\lambda_j(T - 2R)|} \right) \times
\]
\[
\sum_{j=1}^{f} \left( |\lambda_j + \bar{\mu}_j|^2 + 2 \left( \lambda_j + \bar{\mu}_j \right) \left( \lambda_j + \bar{\mu}_j - \frac{1}{2} \right) \right) + \left( |\mu_j|^2 + 2 \left| \mu_j \left( \mu_j - \frac{1}{2} \right) \right| \right)
\]
\[
+ \frac{|B_2|}{2} \left( 2 + \sec^2 \left( \arg \left( \lambda_j(2 + \max_{j} \left\{ \frac{2|\lambda_j + \bar{\mu}_j|}{\lambda_j}, \frac{2|\mu_j|}{\lambda_j} \right\}) \right) \right) \right) \right\}.
\]

Let also \( k \) be defined as in the condition 2 for Selberg class. By using the Phragmén-Lindelöf principle for a strip we get the following theorem.

Theorem 4.2. Let
\[
T \geq \begin{cases} 
2R + 1, \\
2R + \max_{j} \left\{ \frac{2|\lambda_j + \bar{\mu}_j|}{\lambda_j} \right\}, \\
2R + \frac{1}{2^{2\pi - 1}}, \text{ if } k > 0
\end{cases}
\]
$T > 2R + \max_j \left\{ \frac{2|\mu_j|}{\lambda_j} \right\}$ and $t \in [T - 2R, T + 2R]$. Then $|\mathcal{L}(s)| \leq \frac{a_1 \pi^2}{6}$, if $\sigma \geq 3$,

$$|\mathcal{L}(s)| \leq \frac{t^{d_{\mathcal{L}}(\frac{1}{2} - \sigma)} a_1 \pi^2}{6} |\lambda Q^2|^{\frac{1}{2} - \sigma} \times$$

$$\times \exp \left( d_{\mathcal{L}} \sigma + \Re \left( \log \left( 1 - \frac{\sigma i}{t} \right) \left( d_{\mathcal{L}} \left( \frac{1}{2} - s \right) + \frac{\Im(\mu)i}{2} \right) + V(s) \right) \right),$$

if $\sigma \leq -2$ and $|\mathcal{L}(s)| < 2^{\frac{1}{5}} \frac{d_{\mathcal{L}} + 1}{6} M_1 t^{\frac{1}{5} d_{\mathcal{L}} (3 - \sigma)}$ if $-2 \leq \sigma \leq 3$.

**Proof.** We prove the claim in the different parts depending on the value of the real number $\sigma$. Assume $\sigma \geq 3$. Then

$$|\mathcal{L}(s)| = \left| \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \right| \leq \sum_{n=1}^{\infty} \frac{a_1}{n^2} = \frac{a_1 \pi^2}{6}.$$ 

Next we look at the case $\sigma \leq -2$. We have $\mathcal{L}(s) = \Delta_{\mathcal{L}}(s) \overline{\mathcal{L}(1 - s)}$ and by the previous case $|\overline{\mathcal{L}(1 - s)}| \leq \frac{a_1 \pi^2}{6}$. Since $t > \max \left\{ \frac{2|\mu_j|}{\lambda_j} \right\}$, it holds that $|\arg(\lambda_j (1 - s) + \mu_j)| < \pi$, $|\arg(\lambda_j s + \mu_j)| < \pi$ and $t > 0$. Thus by Lemma 2.3 for $\sigma \leq -2$ we have

$$|\mathcal{L}(s)| \leq t^{d_{\mathcal{L}}(\frac{1}{2} - \sigma)} \frac{a_1 \pi^2}{6} |\lambda Q^2|^{\frac{1}{2} - \sigma} \times$$

$$\times \exp \left( d_{\mathcal{L}} \sigma + \Re \left( \log \left( 1 - \frac{\sigma i}{t} \right) \left( d_{\mathcal{L}} \left( \frac{1}{2} - s \right) + \frac{\Im(\mu)i}{2} \right) + V(s) \right) \right).$$

Now we look at the case $-2 \leq \sigma \leq 3$. First we look at the function $(s - 1)^k \mathcal{L}(s)$. Since $(s - 1)^k \mathcal{L}(s)$ is an analytic function of the finite order, we can apply the Phragmén-Lindelöf principle for a strip [11 Theorem 5.53]. By the case $\sigma \geq 3$ and since $t > 0$, we have

$$|\mathcal{L}(3 + it)| \leq \frac{a_1 \pi^2}{6} = \frac{a_1 \pi^2}{6} (1 + t)^0.$$ 

Further, by the case $\sigma \leq -2$ and Lemma 4.4 we have

$$|\mathcal{L}(-2 + it)| \leq t^{d_{\mathcal{L}}(\frac{1}{2} + 2)} \frac{a_1 \pi^2}{6} |\lambda Q^2|^{\frac{1}{2} + 2} \exp \left( -2d_{\mathcal{L}} \right.$$

$$+ \Re \left( \log \left( 1 + \frac{2i}{t} \right) \left( d_{\mathcal{L}} \left( \frac{1}{2} + 2 - it \right) + \frac{\Im(\mu)i}{2} \right) + V(-2 + it) \right)$$

$$< (1 + t)^{\frac{1}{2} d_{\mathcal{L}} \frac{a_1 \pi^2}{6} |\lambda Q^2|^\frac{1}{2} \exp \left( \frac{5\sqrt{5}}{2} d_{\mathcal{L}} + |\Im(\mu)| + \sup_{t \in [T - 2R, T + 2R]} |V(-2 + it)| \right).$$
Also, from Lemma 2.5 it follows that

\[
\sup_{t \in [T-2R, T+2R]} |V(-2 + it)|
\]
\[
< \frac{1}{|\lambda_j(T - 2R)|} \sum_{j=1}^{f} \left| \lambda_j + \bar{\mu}_j \right|^2
\]
\[
+ 2 \left| (\lambda_j + \bar{\mu}_j) \left( \lambda_j + \bar{\mu}_j - \frac{1}{2} \right) \right| + |\mu_j|^2 + 2 \left| \mu_j \left( \mu_j - \frac{1}{2} \right) \right|
\]
\[
+ \frac{|B_2|}{2} \left( 2 + \sec^2 \left( \frac{\arg \left( \lambda_j (-2 + \max_j \left( \frac{2(\lambda_j + \bar{\mu}_j)}{\lambda_j}, \frac{2|\mu_j|}{\lambda_j} \right) \right))}{2} \right) \right).
\]

(13)

Let \( l(x) = -\frac{x}{2} + \frac{x}{3} \). We estimate the function \((s - 1)^k \mathcal{L}(s)\) in two different cases; \( k = 0 \) and \( k > 0 \).

First we look at the case \( k = 0 \). This means that \((s - 1)^k \mathcal{L}(s) = \mathcal{L}(s)\). Let

\[
M_{\sigma_1} = a_1 \frac{\pi^2}{6} |\lambda Q^2|^{\frac{3}{2}} \exp \left( \frac{5\sqrt{5}}{2} d_L + |3(\mu)| + \sup_{t \in [T-2R, T+2R]} |V(-2 + it)| \right),
\]

\( \text{and } M_{\sigma_2} = a_1 \frac{\pi^2}{6} \). By the Phragmén-Lindelöf principle for a strip for \(-2 \leq \sigma \leq 3\) and the inequality (13) we have

\[
|\mathcal{L}(s)| \leq M_{\sigma_1}^{k(\sigma)} M_{\sigma_2}^{1-k(\sigma)} (1 + t)^{l(\sigma)\frac{5}{2}d_L} < M_1 (1 + t)^{\frac{5}{2}d_L (3-\sigma)}.
\]

(14)

Now we look at the case \( k > 0 \). Since \( k \) is an integer, we have \( k \geq 1 \). By triangle inequality for \( \sigma \in [-2, 3] \) we have

\[
|s - 1|^k \leq (|\sigma - 1| + t)^k \leq 3^k \left( 1 + \frac{t}{3} \right)^k < 3^k (1 + t)^k.
\]

(15)

This inequality is used when we apply the Phragmén-Lindelöf principle for a strip. Let

\[
M_{\sigma_1} = 3^k a_1 \frac{\pi^2}{6} |\lambda Q^2|^{\frac{3}{2}} \exp \left( \frac{5\sqrt{5}}{2} d_L + |3(\mu)| + \sup_{t \in [T-2R, T+2R]} |V(-2 + it)| \right)
\]

\( \text{and } M_{\sigma_2} = 3^k a_1 \frac{\pi^2}{6} \). By the Phragmén-Lindelöf principle for a strip in \(-2 \leq \sigma \leq 3\), the inequality (13) and the inequality (15) we have

\[
|s - 1|^k |\mathcal{L}(s)| \leq M_{\sigma_1}^{k(\sigma)} M_{\sigma_2}^{1-k(\sigma)} (1 + t)^{l(\sigma)\frac{5}{2}d_L + k(1-l(\sigma))} < M_1 (1 + t)^{\frac{5}{2}d_L (3-\sigma) + k}.
\]
Thus $|\mathcal{L}(s)| < M_1 \left( \frac{1 + t}{|s - 1|} \right)^k (1 + t)^\frac{k}{2} d\mathcal{L}(3-\sigma)$. Since by assumptions $t \in [T - 2R, T + 2R]$, we have $t \geq \frac{1}{2\pi - 1}$. Thus

$$\frac{M_1}{(s - 1)^k} (1 + t)^\frac{k}{2} d\mathcal{L}(3-\sigma) \leq 2M_1 (1 + t)^\frac{k}{2} d\mathcal{L}(3-\sigma). \quad (16)$$

By the inequalities (13) and (10) we have

$$|\mathcal{L}(s)| < M_1 (1 + t)^\frac{k}{2} d\mathcal{L}(3-\sigma) < 2M_1 (1 + t)^\frac{k}{2} d\mathcal{L}(3-\sigma)$$

if $k = 0$ and

$$|\mathcal{L}(s)| < 2M_1 (1 + t)^\frac{k}{2} d\mathcal{L}(3-\sigma)$$

if $k > 0$. Further, for $\sigma \in [-2, 3]$ and $t \geq 1$ we have

$$(1 + t)^\frac{k}{2} d\mathcal{L}(3-\sigma) \leq (2t)^\frac{k}{2} d\mathcal{L}(3-\sigma) \leq 2^k d\mathcal{L}(3-\sigma).$$

Thus

$$|\mathcal{L}(s)| < 2^k d\mathcal{L} + 1 M_1 t^\frac{k}{2} d\mathcal{L}(3-\sigma)$$

for all $k$.

The following property is also useful in estimating the integral $I_3(T, a, b)$.

**Lemma 4.3.** Let $\sigma \in [a - 2R, a + 2R]$, $T > 2R$ and $t \in [T - 2R, T + 2R]$. Then

$$\Re \left( \log \left( 1 - \frac{\sigma i}{t} \right) \left( \frac{1}{2} - s \right) + \frac{\Im(\mu) i}{2} \right)$$

$$< \left| 1 - \frac{(a + 2R)i}{T - 2R} \right| d\mathcal{L} \left( \frac{1}{2} - a + 2R \right) + d\mathcal{L}(a + 2R) + \frac{a + 2R}{T - 2R} \left| \frac{\Im(\mu)}{2} \right|.$$ 

**Proof.** We have

$$\Re \left( \log \left( 1 - \frac{\sigma i}{t} \right) \left( \frac{1}{2} - s \right) + \frac{\Im(\mu) i}{2} \right)$$

$$= \log \left| 1 - \frac{\sigma i}{t} \right| d\mathcal{L} \left( \frac{1}{2} - a \right) + \arg \left( 1 - \frac{\sigma i}{t} \right) \left( d\mathcal{L} t - \frac{\Im(\mu)}{2} \right). \quad (17)$$

By the assumptions for the numbers $\sigma$ and $t$ we have $|\sigma| \leq a + 2R$ and $t \geq T - 2R > 0$. Thus

$$\log \left| 1 - \frac{\sigma i}{t} \right| d\mathcal{L} \left( \frac{1}{2} - a \right) < \left| 1 - \frac{(a + 2R)i}{T - 2R} \right| d\mathcal{L} \left( \frac{1}{2} - a + 2R \right) \quad (18)$$

and

$$\arg \left( 1 - \frac{\sigma i}{t} \right) \left( d\mathcal{L} t - \frac{\Im(\mu)}{2} \right) < d\mathcal{L}(a + 2R) + \frac{a + 2R}{T - 2R} \left| \frac{\Im(\mu)}{2} \right|. \quad (19)$$

The claim follows immediately from the formulas (17), (18) and (19). \qed
4.2 The estimate of the integral $I_3(T, a, b)$

In this section we estimate the integral $I_3(T, a, b)$. Since the estimate contains the term $V(s)$, where $\sigma$ and $t$ lie in specific intervals, we also need estimate the term $V(s)$ on these intervals. We want to shorten our notation and thus we define the following terms: Let $T$ be a real number,

$$V^*(T) = \frac{1}{T-2R} \sum_{j=1}^{f} \frac{1}{\lambda_j} \left( |\lambda_j + \bar{\mu}_j|^2 + 2 |\lambda_j + \bar{\mu}_j - \frac{1}{2}| + |\mu_j|^2 + 2 |\mu_j - \frac{1}{2}| \right)$$

and

$$R_2(T) = \frac{1}{\log 2} \left( d_L \left( \frac{1}{2} - a + 2R \right) \log(2T) \log \frac{\log 2^2}{3(\mu)} + \right.$$}

$$+ \max \left\{ \log |\lambda Q^2|, \left( \frac{1}{2} - a + 2R \right) \log |\lambda Q^2| \right\} - 2d_L$$

$$+ \log \left( \frac{5}{2} d_L + 1 \right) \log 2 + k \log 3 + \max \left\{ 0, \frac{5}{2} \log |\lambda Q^2| + \frac{5\sqrt{5}}{2} d_L + |\Im(\mu)| \right\}$$

$$\left(20\right)$$

**Lemma 4.4.** Assume that $T$ satisfies the same conditions as in Theorem 4.2. Let

$$g(z) = \frac{1}{2} \left( \mathcal{L}(z + iT) + \overline{\mathcal{L}(z + iT)} \right).$$

Then

$$\left| \frac{1}{2\pi \log 2} \int_0^{2\pi} \log |g(a + 2Re^{i\theta})|d\theta \right| < R_2(T)$$

if $\int_0^{2\pi} \log |g(a + 2Re^{i\theta})|d\theta \geq 0$.

**Proof.** We assume that $\int_0^{2\pi} \log |g(a + 2Re^{i\theta})|d\theta \geq 0$. First we estimate the term $|g(a + 2Re^{i\theta})|$. To do this we estimate the functions $\mathcal{L}(z + iT)$ and
$\mathcal{L}(z+iT)$ by Theorem 4.2 and Lemma 4.3. First we define the function $M(T)$. Let
\begin{equation}
M(T) = \frac{\alpha_1 \pi^2}{6} \max \left\{ |\lambda Q^2|^\frac{1}{2} \exp \left( \frac{1}{2} - a + 2R \right) \right\}
\end{equation}
\begin{equation}
+ d_{\mathcal{L}}(a + 2R) + \frac{a + 2R}{T - 2R} \left| \frac{3(\mu)}{2} \right|, |\lambda Q^2|^\frac{1}{2} \exp \left( - \frac{2d_{\mathcal{L}}}{2} - a + 2R \right) + \frac{a + 2R}{T - 2R} \left| \frac{3(\mu)}{2} \right|, \right\}
\end{equation}
\begin{equation}
2^\frac{5}{2}d_{\mathcal{L}} + 13 & 2^\frac{5}{2}d_{\mathcal{L}} + 13 |\lambda Q^2|^\frac{1}{2} \exp \left( \frac{5\sqrt{5}}{2} \right)
\end{equation}
\begin{equation}
\left| \lambda Q^2 \right| \exp \left( \frac{5\sqrt{5}}{2} \right)
\end{equation}
\begin{equation}
\left| \lambda Q^2 \right| \exp \left( \frac{5\sqrt{5}}{2} \right)
\end{equation}
\begin{equation}
\left| \lambda Q^2 \right| \exp \left( \frac{5\sqrt{5}}{2} \right)
\end{equation}
Since the estimates of the functions $\mathcal{L}(z+iT)$ and $\mathcal{L}(z+iT)$ contain restrictions of the imaginary and real parts, we estimate them. We have
\begin{equation}
\Re(a + 2Re^{\pm i\theta} + iT) = a + 2R \cos(\theta) \in [a - 2R, a + 2R].
\end{equation}
Further, we have
\begin{equation}
|\Re(a + 2Re^{\pm i\theta} + iT)| = |2R \sin(\pm \theta) + T| \in [T - 2R, T + 2R].
\end{equation}
Note that $[\pm 3] \subset [a - 2R, a + 2R]$. Further, note that by Lemma 2.5
\begin{equation}
\sup_{\sigma \in [a - 2R, a + 2R], \epsilon \in [T - 2R, T + 2R]} |V(s)| < V^*(T).
\end{equation}
Thus by Theorem 4.2 and Lemma 4.3
\begin{equation}
|\mathcal{L}(a + 2Re^{\pm i\theta} + iT)| < (2R + T)^d_{\mathcal{L}}(2 - a + 2R) M(T) \exp(V^*(T)).
\end{equation}
The same estimate holds also for $\mathcal{L}(\bar{z} + iT)$. Thus
\begin{equation}
\left| g(a + 2Re^{i\theta}) \right| < (2R + T)^d_{\mathcal{L}}(1 - a + 2R) M(T) \exp(V^*(T)). \quad (21)
\end{equation}
We have estimated the term $|g(a + 2Re^{i\theta})|$. Next we estimate the term $\frac{1}{2\pi \log 2} \int_0^{2\pi} \log |g(a + 2Re^{i\theta})| d\theta$. Let
\begin{equation}
\chi_{\log |g| \geq 0} = \begin{cases} 1, & \text{if } \log |g| \geq 0 \\ 0, & \text{otherwise} \end{cases}
\end{equation}
Now, since $\int_0^{2\pi} \log |g(a + 2Re^{i\theta})| d\theta \geq 0$, we have
\begin{equation}
\left| \int_0^{2\pi} \log |g(a + 2Re^{i\theta})| d\theta \right| \leq \int_0^{2\pi} \chi_{\log |g| \geq 0} \log |g(a + 2Re^{i\theta})| d\theta.
\end{equation}
Since for $\log |g(a+2Re^{i\theta})| \geq 0$ it holds that $|g(a+2Re^{i\theta})| \geq 1$, it is enough to know the upper bound of the term $|g(a+2Re^{i\theta})|$. Further, by the inequality (21) we have

$$
\int_0^{2\pi} \chi_{|g|\geq 0} \log |g(a+2Re^{i\theta})| d\theta \\
< \int_0^{2\pi} \left| \log |(2R + T)^{d_L(i(\frac{1}{2}-a)+2R)} M(T) \exp(V^*(T))| \right| d\theta.
$$

The last step is to integrate the terms. By the definition of the functions $M(T)$ and $V^*(T)$ we have

$$
\left| \log |(2R + T)^{d_L(i(\frac{1}{2}-a)+2R)} M(T) \exp(V^*(T))| \right| \\
\leq d_L \left( \frac{1}{2} - a + 2R \right) \log(2T) + \log \frac{a_1 \pi^2}{6} + \max \left\{ \max \left\{ \frac{5}{2} \log |\lambda Q^2|, \left( \frac{1}{2} - a + 2R \right) \log |\lambda Q^2| \right\} - 2d_L + \left| 1 - \frac{(a + 2R)i}{T - 2R} \right| d_L \left( \frac{1}{2} - a + 2R \right) \right. \\
+ d_L(a + 2R) + \frac{a + 2R}{T - 2R} \frac{\Im(\mu)}{2} \left. \cdot \left( \frac{5}{2} d_L + 1 \right) \log 2 + k \log 3 \right. \\
+ \max \left\{ 0, \frac{5}{2} \log |\lambda Q^2| + \frac{5\sqrt{5}}{2} d_L + |\Im(\mu)| \left\} \right\} + V^*(T).
$$

Thus we have

$$
\left| \frac{1}{2\pi \log 2} \int_0^{2\pi} \log |g(a+2Re^{i\theta})| d\theta \right| \\
< \frac{1}{\log 2} \left( d_L \left( \frac{1}{2} - a + 2R \right) \log(2T) + \log \frac{a_1 \pi^2}{6} + V^*(T) \right. \\
+ \max \left\{ \max \left\{ \frac{5}{2} \log |\lambda Q^2|, \left( \frac{1}{2} - a + 2R \right) \log |\lambda Q^2| \right\} - 2d_L \right. \\
+ \left. \left| 1 - \frac{(a + 2R)i}{T - 2R} \right| d_L \left( \frac{1}{2} - a + 2R \right) + d_L(a + 2R) + \frac{a + 2R}{T - 2R} \frac{\Im(\mu)}{2}, \right. \\
\left. \left( \frac{5}{2} d_L + 1 \right) \log 2 + k \log 3 + \max \left\{ 0, \frac{5}{2} \log |\lambda Q^2| + \frac{5\sqrt{5}}{2} d_L + |\Im(\mu)| \left\} \right\} \right) \right) \\
= R_2(T).
$$

We define a new function $n(r)$ and estimate it. The estimate is used to estimate the integral $I_3(T, a, b)$. 21
Theorem 4.5. Assume that \( T \) satisfies the same conditions as in Theorem 4.2. Let \( n(r) \) be the number of the zeros of the function \( g(z) \) in \( |z - a| \leq r \), where \( g(z) \) is as in Lemma 4.4. Then \( n(R) < R_2(T) + 1 \).

Proof. First we estimate the value of the function \( n(R) \) with the function \( g(z) \) and its integral. Then we estimate the previous terms and obtain \( n(R) < R_2(T) + 1 \).

Since \( \Im(z + iT) > 0 \) and \( \Im(\bar{z} + iT) > 0 \) for \( |z - a| < T \), the functions \( L(z + iT) \) and \( \overline{L(\bar{z} + iT)} \) are analytic in the disc \( |z - a| < T \). Thus the function \( g(z) \) is analytic in the disc \( |z - a| < T \). We have

\[
\int_0^{2R} \frac{n(r)}{r} dr \geq \int_R^{2R} \frac{n(R)}{r} dr = n(R) \log(2).
\]

By Jensen’s formula

\[
\int_0^{2R} \frac{n(r)}{r} dr = \frac{1}{2\pi} \int_0^{2\pi} \log |g(a + 2Re^{i\theta})| d\theta - \log |g(a)|.
\]

Thus

\[
n(R) \leq \frac{1}{2\pi \log 2} \int_0^{2\pi} \log |g(a + 2Re^{i\theta})| d\theta - \frac{\log |g(a)|}{\log 2}. \tag{22}
\]

From the previous formula and Lemma 4.4 we see that it is enough to estimate the terms

\[
\frac{1}{2\pi \log 2} \int_0^{2\pi} \log |g(a + 2Re^{i\theta})| d\theta
\]

if \( \int_0^{2\pi} \log |g(a + 2Re^{i\theta})| d\theta < 0 \) and \( \frac{\log |g(a)|}{\log 2} \). By the assumptions for the number \( a \) we have

\[
\Re(L(a + iT)) = \Re \left( 1 + \sum_{n=2}^{\infty} \frac{a(n)}{n^{a+iT}} \right) \in \left[ \frac{1}{2}, \frac{3}{2} \right].
\]

Since \( g(a) = \Re(L(a + iT)) \), we have \( |\log |g(a)|| \leq \log 2 \). Thus

\[
\left| \frac{\log |g(a)|}{\log 2} \right| \leq 1. \tag{23}
\]

Next we look at the case \( \int_0^{2\pi} \log |g(a + 2Re^{i\theta})| d\theta < 0 \). By the definition of the function \( n(R) \) we have \( n(R) \geq 0 \). Also by (22) and (23)

\[
-1 \leq - \left| \frac{\log |g(a)|}{\log 2} \right| + n(R) \leq \frac{1}{2\pi \log 2} \int_0^{2\pi} \log |g(a + 2Re^{i\theta})| d\theta.
\]

Thus

\[
\left| \frac{1}{2\pi \log 2} \int_0^{2\pi} \log |g(a + 2Re^{i\theta})| d\theta \right| \leq 1. \tag{24}
\]
if \( \int_0^{2\pi} \log |g(a + 2Re^{i\theta})|d\theta < 0 \). By (22), (23), (24) and Lemma 4.4 we get the result
\[ n(R) < R_2(T) + 1. \]

In the following theorem we estimate the integral \( I_3(T, a, b) = \int_b^a \mathcal{L}(\sigma + iT)d\sigma \) using the previous theorem.

**Theorem 4.6.** If \( T \) satisfies the same conditions as in Theorem 4.2, then
\[ |I_3(T, a, b)| < \pi R (R_2(T) + 2). \]

**Proof.** Assume that the function \( \Re(\mathcal{L}(\sigma + iT)) \) has \( N \) zeros for \( b \leq \sigma \leq a \). Now the sign of the \( \Re(\mathcal{L}(\sigma + iT)) \) changes at most \( N + 1 \) times in the interval \( \sigma \in [b, a] \). We can divide the interval \([b, a]\) to \( N + 1 \) parts where \( \Re(\mathcal{L}(\sigma + iT)) \) is of constant sign. When we sum the maximum absolute values of the argument in each of these intervals, we get
\[ |\arg(\mathcal{L}(\sigma + iT))| \leq \pi (N + 1). \]

Let \( n(R) \) and \( g(z) \) be as in Theorem 4.2. Since \( (b, a) \subseteq \{z : \Re z \leq R\} \) and \( g(\sigma) = \Re(\mathcal{L}(\sigma + iT)) \), we have \( N \leq n(R) \). Thus
\[ \pi (N + 1) \leq \pi (n(R) + 1). \]

By Theorem 4.5 we have \( \pi (n(R) + 1) < \pi (R_2(T) + 2) \). It follows that
\[ |I_3(T, a, b)| \leq \int_b^a |\arg(\mathcal{L}(\sigma + iT))|d\sigma < \pi R (R_2(T) + 2). \]

**Remark 4.7.** Since \( (b + 1, a) \subset (b, a) \) it also holds that \( |I_3(T, a, b + 1)| < \pi R (R_2(T) + 2) \).

## 5 Main result

In this section we prove the explicit version of the Riemann-von Mangoldt type formula for the functions of the set \( S \). Let \( \mathcal{N}_+^\mathcal{L}(T_0, T) \) and \( \mathcal{N}_-^\mathcal{L}(T_0, T) \) be the number of the non-trivial zeros \( \rho \) of the function \( \mathcal{L}(s) \) with \( T_0 < \Im(\rho) \leq T \) and \( -T \leq \Im(\rho) < -T_0 \) respectively. First we combine the results from the Sections 2, 3 and 4 to estimate the functions \( \mathcal{N}_+^\mathcal{L}(T_0, T) \) and \( \mathcal{N}_-^\mathcal{L}(T_0, T) \). We remember that \( a_1 \) is a constant such that for all \( n \) we have \( |a(n)| \leq a_1 n \). Also, \( a \) is a real number for which \( a > 2 \) and
\[ \sum_{n=2}^{\infty} \frac{a_1}{n^a} < \frac{1}{2}. \]
and $b < -3$ is a negative real number which has the following property:

$$\sum_{n=2}^{\infty} \frac{a_1}{n^{b-1}} < 1.$$ 

We have defined that $R = a - b$. The constants $d_L$, $\lambda$, $Q$, $f$, $\mu_j$ and $\lambda_j$ depend on the function $L$ and are defined at the beginning of the Section 1. The function $R_1(T_0, T, b)$ is defined in the formula (8) of the Section 3.2 and the function $R_2(t)$ is defined in the formula (20) of the Section 4.2. Let

$$R_L(T_0, T)$$

$$= \frac{d_L}{2\pi} T_0 \log \frac{T_0}{e} + \frac{T_0}{2\pi} |\log(\lambda Q^2)| + \frac{R_1(T_0, T, b)}{2\pi} + \frac{\pi}{3 \log 2}$$

$$+ \left( R - \frac{1}{2} \right) \left( R_2(T_0) + R_2(T) + 4 \right) + f \cdot \left( \begin{array}{c} (b + 1) \max_j \{\lambda_j\} + \min_j \{\mathcal{R}(\mu_j)\} \\ \min_j \{\mathcal{R}(\mu_j)\} - \min_j \{\mathcal{R}(\mu_j)\} + \max_j \{\mathcal{R}(\mu_j)\} \end{array} \right).$$

First we prove a useful lemma and then we use it to prove the main result.

**Lemma 5.1.** Assume that $T_0$ satisfies the same conditions as $T$ in Theorem 4.2 and $T > T_0$ is a real number. Then

$$\left| \mathcal{N}_L^+(T_0, T) - \frac{d_L}{2\pi} T \log \frac{T}{e} - \frac{T}{2\pi} \log(\lambda Q^2) \right| < R_L(T_0, T).$$

**Proof.** Since the functions $L(\bar{s})$ and $\overline{L(\bar{s})} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ have the same zeros, we need to prove the claim only for the function $\mathcal{N}_L^+(T_0, T)$. By Lemma ?? for the zeros $\rho$ of the function $L$ we have

$$2\pi \sum_{T_0 < \Im(\rho) \leq T \atop \Re(\rho) > b} (\Re(\rho) - b) = I_1(T_0, T, b) - I_2(T_0, T, a) - I_3(T_0, a, b) + I_3(T, a, b).$$

We can subtract the formula containing $b + 1$ from the formula containing $b$ and get

$$2\pi \mathcal{N}_L^+(T_0, T) + 2\pi \sum_{T_0 < \Im(\rho) \leq T \atop 0 > \Re(\rho) > b + 1} 1 + 2\pi \sum_{T_0 < \Im(\rho) \leq T \atop b + 1 \geq \Re(\rho) > b} (\Re(\rho) - b)$$

$$= I_1(T_0, T, b) - I_1(T_0, T, b + 1) - I_3(T_0, a, b)$$

$$+ I_3(T_0, a, b + 1) + I_3(T, a, b) - I_3(T, a, b + 1).$$

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By Theorems 3.5, 4.6 and Remark 4.7 we have
\[
\left| N^T_\mathcal{L} (T_0, T) - \frac{d_\mathcal{L}}{2\pi} T \log \frac{T}{e} - \frac{T}{2\pi} \log(\lambda Q^2) \right|
\leq \frac{d_\mathcal{L}}{2\pi} T_0 \log \frac{T_0}{e} + \frac{T_0}{2\pi} |\log(\lambda Q^2)| + \frac{R_1(T_0, T, b)}{2\pi} + \frac{\pi}{3\log 2}
\]
\[
+ \left( R - \frac{1}{2} \right) (R_2(T_0) + R_2(T) + 4) + \left\{ \sum_{T_0 < \Im(\rho) \leq T \atop 0 > \Re(\rho) > b+1} 1 + \sum_{T_0 < \Im(\rho) \leq T \atop b+1 \geq \Re(\rho) > b} (\Re(\rho) - b) \right\}.
\]
Since all the trivial zeros are of the form \( s = \frac{l+\mu_l}{\lambda_j} \), where \( l = 0, 1, 2, \ldots \) and \( j \in [1, f] \), we have
\[
\left| \sum_{T_0 < \Im(\rho) \leq T \atop 0 > \Re(\rho) > b+1} 1 + \sum_{T_0 < \Im(\rho) \leq T \atop b+1 \geq \Re(\rho) > b} (\Re(\rho) - b) \right|
\leq f \cdot \left( (b + 1) \max_j \{ \lambda_j \} + \min_j \{ \Re(\mu_j) \} \right)
\leq b \max_j \{ \lambda_j \} + (b + 1) \min_j \{ \lambda_j \} - \min_j \{ \Re(\mu_j) \} + \max_j \{ \Re(\mu_j) \} \right).
\]
The claim follows from the inequalities (25) and (26).

Next we prove the main result by estimating the term \( R_\mathcal{L}(T_0, T) \). We want that it holds that
\[
|R_\mathcal{L}(T_0, T)| \leq c_{\mathcal{L}, 1} \log T + c_{\mathcal{L}, 2}(T_0) + \frac{c_{\mathcal{L}, 3}(T_0)}{T},
\]
where the terms \( c_{\mathcal{L}, j}(T_0) \) are real numbers which depend on the function \( \mathcal{L} \) and the number \( T_0 \) and the real number \( c_{\mathcal{L}, 1} \) depends only on the function \( \mathcal{L} \). To shorten our notation we define that
\[
h_{\mathcal{L}, 1} = (1 - \alpha) \left( \max \left\{ \frac{5}{2} \log |\lambda Q^2|, \left( \frac{1}{2} - a + 2R \right) \log |\lambda Q^2| \right\} + d_\mathcal{L} \left( -\frac{3}{2} + 4R \right) \right)
+ \alpha \left( \frac{5}{2} d_\mathcal{L} + 1 \right) \log 2 + k \log 3 + \max \left\{ 0, \frac{5}{2} \log |\lambda Q^2| + \frac{5\sqrt{3}}{2} d_\mathcal{L} + |\Im(\mu)| \right\}
\]
and
\[
h_{\mathcal{L}, 2} = (1 - \alpha) \left( d_\mathcal{L} \left( \frac{1}{2} - a + 2R \right) (a + 2R) + (a + 2R) \left| \Im(\mu) \right| \right)
\]
where the number \( \alpha \in \{0, 1\} \). If the sum \( h_{\mathcal{L}, 1} + \frac{h_{\mathcal{L}, 2}}{2\alpha + 1} \) is bigger for \( \alpha = 0 \) than \( \alpha = 1 \) then \( \alpha = 0 \). Otherwise \( \alpha = 1 \). Using this notation we obtain the main result:
Theorem 5.2. Suppose that $T_0$ and $T$ satisfy the same conditions as in Lemma 5.1. Then we have

$$|R_L(T_0, T)| \leq c_{\mathcal{L}, 1}(T_0) \log T + c_{\mathcal{L}, 2}(T_0) + \frac{c_{\mathcal{L}, 3}(T_0)}{T},$$

where

$$c_{\mathcal{L}, 1} = \frac{1}{2\pi} \left( -\frac{7d_L}{2}(2b + 1) + 2 \left| -d_L b + \frac{\Im(\mu)i}{2} \right| + 2d_L + S(1, e, b) \right)$$

$$+ \frac{1}{\log 2} \left( R - \frac{1}{2} \right) d_L \left( \frac{1}{2} - a + 2R \right),$$

$$c_{\mathcal{L}, 2}(T_0) = \frac{d_L T_0 \log T_0}{2\pi} \left[ \log(\lambda Q^2) \right] + \frac{\pi}{3\log 2} + 4R - 2 + \frac{3d_L(b^2 + b)}{2\pi T_0} + f \cdot \left( \left( b + 1 \right) \max_j \{ \lambda_j \} + \min_j \{ \Re(\mu_j) \} \right)$$

$$- b \max_j \{ \lambda_j \} + (b + 1) \min_j \{ \lambda_j \} - \min_j \{ \Re(\mu_j) \} + \max_j \{ \Re(\mu_j) \}$$

$$+ \frac{1}{2\pi} R_1(T_0, 1, b) + \left( R - \frac{1}{2} \right) \left( R_2(T_0) + d_L \left( \frac{1}{2} - a + 2R \right) \right)$$

$$+ \frac{1}{\log 2} \left( R - \frac{1}{2} \right) \left( \log \frac{a_1 \pi^2}{6} + h_{\mathcal{L}, 1} \right)$$

and

$$c_{\mathcal{L}, 3}(T_0) = \frac{1}{\log 2} \left( R - \frac{1}{2} \right) \frac{T_0}{T_0 - 2R} \left( V^*(2R + 1) + h_{\mathcal{L}, 2} \right).$$

Proof. Since $1 - \frac{(a + 2R)i}{T_0 - 2R} \leq 1 + \frac{a + 2R}{T_0 - 2R}$ and $1 - \frac{T_0}{T_0 - 2R} \leq \frac{T_0}{T_0 - 2R} T$, the claim follows from the definition of the term $R_L(T_0, T)$.\]

Using the main result we can prove a useful corollary. If we know the number of up to height $T_0$, we can also estimate the number of zeros up to height $T$. Let $\mathcal{N}_L^+(t)$ and $\mathcal{N}_L^-(t)$ denote the number of the non-trivial zeros of the function $L$ for which $0 \leq \Im(\rho) \leq t$ and $-t \leq \Im(\rho) \leq 0$ respectively. We also notice that by [IS] we have that $\mathcal{N}_L^+(T) \sim \frac{d_L}{2\pi} T \log T$ and thus the numbers $\mathcal{N}_L^+(T_0)$ are finite. Using these properties we obtain the following corollary:

Corollary 5.3. Suppose that $T_0$ and $T$ satisfy the same conditions as in Lemma 5.1. Since for all positive real numbers $c$ it holds that $c \leq c \log T$, by Theorem 5.2 we get

$$|R_L(T_0, T) + \mathcal{N}_L^+(T_0)| \leq c_{\mathcal{L}, 1} \log T + C_{\mathcal{L}, 2}(T_0) + \frac{c_{\mathcal{L}, 3}(T_0)}{T},$$

for example, when $C_{\mathcal{L}, 2}(T_0) = c_{\mathcal{L}, 2}(T_0) + \max\{\mathcal{N}_L^+(T_0), \mathcal{N}_L^-(T_0)\}$.\]

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Furthermore, we also would like to note one interesting and useful result. For the number of zeros in the interval \((T, 2T]\) we obtain a formula with the error term where coefficients of the terms \(\log T, 1\) and \(\frac{1}{T}\) don’t depend on the number \(T\).

**Remark 5.4.** Using similar methods as in the proof of Lemma 5.1 and Theorem 5.2 we can prove that

\[
\left| N_{\ell}(T, 2T) - \frac{d_{\ell}}{2\pi} T \log \frac{4T}{e} - \frac{T}{2\pi} \log(\lambda Q^2) \right| \leq c_1 \log T + c_2 + \frac{c_3}{T},
\]

where

\[
c_1 = \frac{d_{\ell}}{\log 2} (2R - 1) \left( \frac{1}{2} - a + 2R \right),
\]

\[
c_2 = \frac{\log 2}{2\pi} \left( -\frac{7d_{\ell}}{2} (2b + 1) + 2 \left| -d_{\ell} b + \frac{\Im(\mu_i)}{2} \right| + 2d_{\ell} \right) + \frac{S(1, 2, b)}{2\pi} + \frac{2\pi}{3\log 2}
\]

\[
+ 4R - 2 + (2R - 1) \left( \frac{3d_{\ell}}{2} \left( \frac{1}{2} - a + 2R \right) + \frac{1}{\log 2} \left( \log \frac{a_1\pi^2}{6} + h_{\ell, 1} \right) \right)
\]

and

\[
c_3 = \frac{3d_{\ell} (b^2 + b) R}{4\pi} + \frac{3T_0}{2(T_0 - 2R) \log 2} \left( R - \frac{1}{2} \right) \left( V^*(2R + 1) + h_{\ell, 2} \right).
\]

**6 Example: \(L\)-function associated with a holomorphic newform**

In this section we give examples of the values of the terms \(c_{\ell, 1}, c_{\ell, j}(T_0)\) and \(c_j\) which are defined in Section 5. Since we have estimated these terms for a general set which contains \(L\)-functions other than the Riemann zeta function and Dirichlet \(L\)-functions, the estimates of the term \(c_{\ell, 1}\) and \(c_{\ell, j}(T_0)\) for these functions are not as strong as previous estimates, see [1] and [21].

Let \(g\) be a newform of even weight \(\kappa\) for some congruence subgroup

\[
\Gamma_0(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL_2(\mathbb{Z}) : C \equiv 0 \mod N \right\},
\]

where \(N\) is a positive integer and \(SL_2(\mathbb{Z})\) is a set of \(2 \times 2\) matrices with integer entries and which determinant is 1. We also assume that for \(z \in \mathbb{H}\) the function \(g\) has a Fourier expansion

\[
g(z) = \sum_{n=1}^{\infty} c(n) \exp(2\pi i n z).
\]

We define

\[
\mathcal{L}(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s},
\]

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where \( a(n) = c(n)n^{\frac{1 + \kappa}{2}} \). The function \( \mathcal{L}(s) \) satisfies the equation

\[
\Lambda \mathcal{L}(s) = \mathcal{L}(s) \left( \frac{\sqrt{N}}{2\pi} \right)^s \Gamma \left( s + \frac{\kappa - 1}{2} \right),
\]

where

\[
\Lambda \mathcal{L}(s) = i^\kappa \Lambda \mathcal{L}(1 - s).
\]

Hence, we can choose

\[
f = 1, \quad Q = \frac{\sqrt{N}}{2\pi}, \quad \lambda_j = 1, \quad \omega = i^\kappa \quad \text{and} \quad \mu_j = \frac{\kappa - 1}{2}.
\]

Thus we also have

\[
d \mathcal{L} = 2, \quad \lambda = 1 \quad \text{and} \quad \mu = 4 - 2\kappa.
\]

We also have \( k = 0 \). By Deligne \([6, 7]\) \(|a(n)| \leq 1 \) and we can choose that

\[
a_1 = 1, \quad a_2 = 3 \quad \text{and} \quad b_1 = -4.
\]

By Theorem 5.2 we have

\[
T_0 \geq 15 + \kappa \quad \text{and} \quad c_{\mathcal{L}, 1} = 299 \log 2 + \frac{15 + \kappa}{2\pi} \left| \log \frac{N}{4\pi^2} \right| + \frac{\pi}{3 \log 2} + \frac{353}{2} + \frac{1}{12} \left( \sec^2 \left( \frac{\arg (3 + (k + 1)i)}{2} \right) + \sec^2 \left( \frac{\arg (3 + (k + 1)i)}{2} \right) \right),
\]

\[
c_{\mathcal{L}, 2}(15 + \kappa) = \frac{1}{\pi} (15 + \kappa) \left( \frac{15 + \kappa}{e} + \frac{15 + \kappa}{2\pi} \left| \log \frac{N}{4\pi^2} \right| + \frac{3}{3 \log 2} + \frac{353}{2} \right)
\]

\[
+ \frac{36}{\pi (15 + \kappa)} + \left| \frac{\kappa - 7}{2} \right| + \frac{72}{2\pi (15 + \kappa)}
\]

\[
- \frac{\log (15 + \kappa)}{2\pi} \left( \frac{9\kappa^2 - 6\kappa + 217}{3} \right)
\]

\[
+ \frac{1}{12} \left( \sec^2 \left( \frac{\arg (4 + (k + 1)i)}{2} \right) + \sec^2 \left( \frac{\arg (3 + (k + 1)i)}{2} \right) \right)
\]

\[
+ \frac{13}{12(1 + \kappa) \log 2} \left( \frac{9\kappa^2 - 6\kappa + 10}{2} \sec^2 \left( \frac{\arg (-17 + (k + 1)i)}{2} \right) \right)
\]

\[
+ \frac{299}{2 \log 2} \left( \log (30 + 2\kappa) + \left| 1 - \frac{17i}{\kappa + 1} \right| \right)
\]

\[
+ \frac{13}{2 \log 2} \left( 2 \log \frac{\pi^2}{6} + 2 \max \left\{ \frac{5}{2} \log \frac{N}{4\pi^2}, \frac{23}{2} \log \frac{N}{4\pi^2} \right\} + 83 \right)
\]

and

\[
c_{\mathcal{L}, 3}(15 + \kappa) = \frac{13}{12 \log 2} \left( 15 + \kappa \right) \left( \frac{9\kappa^2 - 6\kappa + 2356}{2} + \frac{1}{2} \sec^2 \left( \frac{\arg (-17 + (k + 1)i)}{2} \right) \right).
\]
Furthermore, by Remark 5.4 we also have

\[ c_1 = \frac{299}{\log 2}, \]

\[ c_2 = \frac{2\pi}{3\log 2} + 923 + \frac{\log 2}{6\pi} \left( 9\kappa^2 - 6\kappa + 217 \right) + \frac{1}{4} \left( \sec^2 \left( \frac{\arg (4 + (\kappa + 1)i)}{2} \right) + \sec^2 \left( \frac{\arg (3 + (\kappa + 1)i)}{2} \right) \right) \]
\[ + \frac{13}{\log 2} \left( \log \frac{\pi^2}{6} + \max \left\{ \frac{5}{2} \log \frac{N}{4\pi^2}, \frac{23}{2} \log \frac{N}{4\pi^2} \right\} + 53 \right) \]

and

\[ c_3 = \frac{18}{\pi} + \frac{13(15 + \kappa)(17 + 3\kappa)}{4(1 + \kappa)(8 + \kappa) \log 2} \left( \frac{1}{6(1 + \kappa)} \left( 9\kappa^2 - 6\kappa + 10 \right) + \frac{1}{2} \sec^2 \left( \frac{\arg (17 + (\kappa + 1)i)}{2} \right) \right) + 391. \]

We can see different values of the ceiling function of the numbers \( c_{\mathcal{L},1}, c_{\mathcal{L},j}(T_0) \) and \( c_j \) from Table II.
Table 1: Different values of the terms $c_L, c_L(T_0)$ and $c_j$

| $N$ | $\kappa$ | $T_0$ | $c_L,1$ | $c_L,2(T_0)$ | $c_L,3(T_0)$ | $c_1$ | $c_2$ | $c_3$ |
|-----|------|-----|------|----------------|----------------|-----|-----|-----|
| 1   | 12   | 27  | 293  | 1945           | 11637          | 432 | 1811| 10506 |
| 1   | 34   | 49  | 769  | 415            | 27478          | 432 | 2141| 8183  |
| 1   | 36   | 51  | 835  | 172            | 29742          | 432 | 2187| 8133  |
| 1   | 38   | 53  | 905  | -91            | 32127          | 432 | 2235| 8092  |
| 1   | 40   | 55  | 979  | -374           | 34631          | 432 | 2286| 8060  |
| 1   | 50   | 65  | 1405 | -2087          | 48918          | 432 | 2582| 7983  |
| 2   | 8    | 23  | 256  | 2112           | 11554          | 432 | 1817| 12323 |
| 2   | 10   | 25  | 272  | 2040           | 11375          | 432 | 1829| 11247 |
| 11  | 2    | 17  | 229  | 2941           | 21661          | 432 | 1879| 24239 |
| 11  | 10   | 25  | 272  | 2113           | 11375          | 432 | 1909| 11247 |
| 11  | 12   | 27  | 293  | 2047           | 11637          | 432 | 1923| 10506 |
| 11  | 36   | 51  | 835  | 265            | 29742          | 432 | 2299| 8133  |
| 11  | 38   | 53  | 905  | 1              | 32127          | 432 | 2347| 8092  |
| 11  | 40   | 55  | 979  | -282           | 34631          | 432 | 2399| 8060  |
| 21  | 6    | 21  | 243  | 2314           | 12460          | 432 | 1919| 14017 |
| 21  | 8    | 23  | 256  | 2214           | 11554          | 432 | 1928| 12323 |
| 40  | 2    | 17  | 229  | 3000           | 21661          | 432 | 1942| 24239 |
| 40  | 6    | 21  | 243  | 2345           | 12460          | 432 | 1951| 14017 |
| 40  | 36   | 51  | 835  | 317            | 29742          | 432 | 2362| 8133  |
| 40  | 38   | 53  | 905  | 53             | 32127          | 432 | 2410| 8092  |
| 63  | 36   | 51  | 835  | 419            | 29742          | 432 | 2460| 8133  |
| 63  | 38   | 53  | 905  | 155            | 32127          | 432 | 2508| 8092  |
| 64  | 36   | 51  | 835  | 422            | 29742          | 432 | 2463| 8133  |
| 64  | 38   | 53  | 905  | 159            | 32127          | 432 | 2512| 8092  |
| 64  | 40   | 55  | 979  | -125           | 34631          | 432 | 2563| 8060  |
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