A Damping of the de Haas-van Alphen Oscillations in the superconducting state

K. P. Duncan and B. L. Gyorffy

H. H. Wills Physics Laboratory, University of Bristol, Tyndall Avenue, Bristol BS8 1TL, UK.

Abstract. – Deploying a recently developed semiclassical theory of quasiparticles in the superconducting state we study the de Haas-van Alphen effect. We find that the oscillations have the same frequency as in the normal state but their amplitude is reduced. We find an analytic formulae for this damping which is due to tunnelling between semiclassical quasiparticle orbits comprising both particle-like and hole-like segments. The quantitative predictions of the theory are consistent with the available data.

The revival of interest in studying the de Haas-van Alphen (dHvA) effect in the superconducting state is driven by the hope that this would provide new k-vector dependent information about the superconducting gap \( \Delta(k) \). Evidently this would be of particular importance in connection with anisotropic superconductors where \( \Delta(k) \) can have lines of zero’s on the Fermi surface. Unfortunately at this stage there is no concensus concerning the mechanism of how the experimentally observed oscillations of the diamagnetic response of a Type II superconductor come about. Using our recently developed very general semiclassical theory of quasiparticles in the superconducting state, in what follows we develop a semiclassical picture of Landau like orbits of quasiparticles suggested by the simple model calculation of Miller and Gyorffy. Clearly the long term aim of a semiclassical theory is to provide an analogue of the Lifshitz-Kosevich formulae for superconductors. Hopefully such a formulae would allow the interpretation of experiments in terms of the Fermi surface and the variation of \( \Delta(k) \) on the Fermi surface. At this stage we only deal with conventional superconductors with the usual s-wave pairing. As it happens this is the only case for which reliable data already exists. Within the limits of a number of simplifying assumptions the above semiclassical theory provides for Bohr-Sommerfeld like quantisation rules for quasiparticles. In particular it allows for the analogue of magnetic breakdown which involves tunnelling between distinct semiclassical orbits. We show that there are orbits which enclose areas that are precisely the same size as the Landau orbits in the normal state, but involve such tunnelling. As will be seen the consequence of these tunnelling events is a damping factor in the Lifshitz-Kosevich formulae in agreement with experiments.

The theory we wish to use seeks the semiclassical spectrum of the following Bogoliubov-de
The effective classical mechanics, Hamilton-Jacobi equations for the quasiparticles, corresponding to (1), is described by the following effective Hamiltonians \[16\]:

\[
\begin{pmatrix}
\frac{1}{2m} (\mathbf{p} + e\mathbf{A}(\mathbf{r}))^2 + V(\mathbf{r}) - \epsilon_F
& |\Delta(\mathbf{r})|e^{i\phi(\mathbf{r})} \\
|\Delta(\mathbf{r})|e^{-i\phi(\mathbf{r})} & -\frac{1}{2m} (\mathbf{p} - e\mathbf{A}(\mathbf{r}))^2 - V(\mathbf{r}) + \epsilon_F
\end{pmatrix}
\begin{pmatrix}
u_\lambda(\mathbf{r}) \\
v_\lambda(\mathbf{r})
\end{pmatrix}
= E_\lambda
\begin{pmatrix}
u_\lambda(\mathbf{r}) \\
v_\lambda(\mathbf{r})
\end{pmatrix},
\]

where \(u_\lambda(\mathbf{r})\) and \(v_\lambda(\mathbf{r})\) are the probability amplitudes that an elementary excitation is a quasiparticle and quasihole respectively, \(|\Delta(\mathbf{r})|e^{i\phi(\mathbf{r})}\) is the order parameter and the other symbols have the conventional meaning. For Type II superconductors in large magnetic fields \(\Delta(\mathbf{r})\) takes the form of the Abrikosov flux lattice \[17\], comprising a periodic array of vortices. The effective classical mechanics, Hamilton-Jacobi equations for the quasiparticles, corresponding to (1), is described by the following effective Hamiltonians \[16\]:

\[
E^\alpha(\mathbf{p}, \mathbf{r}) = \mathbf{p} \cdot \mathbf{v}_s(\mathbf{r}) + \alpha \sqrt{\left(\frac{p^2}{2m} + \frac{1}{2} m v_s^2(\mathbf{r}) + V(\mathbf{r}) - \epsilon_F\right)^2 + |\Delta(\mathbf{r})|^2},
\]

where \(\alpha = \pm\) and \(m \mathbf{v}_s(\mathbf{r}) = \frac{1}{2} \hbar \nabla \phi(\mathbf{r}) + e \mathbf{A}(\mathbf{r})\) is the superfluid velocity. On a constant energy surface these determine, implicitly, the functions \(p^\alpha(\mathbf{r})\) to be used in the Bohr-Sommerfeld quantisation rule. To simplify matters we take the crystal potential \(V(\mathbf{r})\) to be a constant. It is clear from the single vortex solution \[16\], that the detailed shape of \(|\Delta(\mathbf{r})|\) is not essential. This fact will be used below. By contrast the rôle of the line of phase singularities \[16\], which runs along the vortex core, needs more careful consideration. The topologically non-trivial behaviour of the phase gradient (\(\oint \nabla \phi \cdot d\mathbf{r} \neq 0\) for any path enclosing lines of phase singularities) causes the phase gradient, \(\nabla \phi\), to cancel out (on the average) the increase in \(\mathbf{A}(\mathbf{r})\) across the cell. Consequently the Abrikosov form for \(\mathbf{v}_s(\mathbf{r})\) is periodic. The Hamiltonian, (2), then describes quasiparticles which correspond to spinor Bloch waves \[18\]. On the other hand the topologically non-trivial behaviour of \(\Delta(\mathbf{r})\) arises from the superposition of topologically trivial solutions to the Ginzburg-Landau theory, so long as the coefficients are suitably determined from minimization of the free energy for the nonlinear problem \[17\]. We can therefore consider the solution of the BdG equations for a topologically trivial \(\Delta(\mathbf{r}) = |\Delta(y)| \exp(ik_x x)\), with \(|\Delta(y)| = \Delta_0 \exp\left(-\frac{1}{2}(y - \frac{\hbar k_x}{2eB})^2/(2\pi)^{-1} d^2\right)\) and \(k_x = 2\pi n/d\) (\(d\) the flux lattice cell size, \(n\) an integer). A full Bloch wave solution to equation (3) can then be constructed by forming an appropriate superposition and determining the coefficients from self-consistency. The single-valuedness of each solution ensures the resulting superposition is also single-valued, and the former requires the quantisation of a Bohr-Sommerfeld integral for quasiparticles with the \(\Delta(\mathbf{r})\) given above.

Taking this approach, and working in the Landau gauge, we rewrite (3) as

\[
E^\alpha(\mathbf{p}, \mathbf{r}) = P_x v_{s,x}(y) + \alpha \sqrt{\left(\frac{p^2}{2m} + \frac{1}{2} mv_{s,x}(y) - \epsilon_F\right)^2 + |\Delta(y)|^2},
\]

where \(mv_{s,x}(y) = \hbar k_x/2 + eA_x(y)\) and we have approximated \(\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r})\) by a constant (averaged) \(\mathbf{B}\)-field. The quasiparticle dynamics described by (3) are integrable and we can deploy the full semiclassical machinery developed in \[16\]. This we now do.

The momentum branches defined by the Hamilton-Jacobi equations \(E^\alpha(\mathbf{p}, \mathbf{r}) = E^\alpha\) are

\[
p^\pm(\mathbf{y}) = \sqrt{p^2_F - P_x^2 - \frac{m^2}{4} \omega_c^2 - \frac{\hbar k_x}{2eB} - y^2} \pm \frac{1}{2} m \sqrt{\left(E^\alpha + P_x \omega_c (y - \frac{\hbar k_x}{2eB})\right)^2 - |\Delta(y)|^2},
\]

We see immediately that \(\frac{\hbar k_x}{2eB}\) behaves like an orbit centre, though restricted to \(\frac{\hbar k_x}{2eB} = nd\) (\(n\) integer). Less obviously, \(P_x\), also behaves like an orbit centre (see below). It is natural
Fig. 1 – Phase space orbits defined by $p_\alpha^\pm$ given in (4) with $|\Delta(y)| = |\Delta|$. The solid line is for a particle-like excitation, the dashed line is for a hole-like one. Arrows indicate the direction of the velocity. (a) $P_x = 0$. The turning points along the $y$-axis are given by $p_\alpha^\pm = 0$. (b) $P_x \neq 0$ and small. The orbits are shifted in opposite directions. The vertical dotted lines indicate the position of the Andreev-like turning points, $y_+ (P_x)$ and $y_- (P_x)$, see equation (6). (c) As $P_x$ increases further the Andreev-like turning point $y_+$, for which $p_\alpha^+ \neq 0$, is reached before the normal turning points given by $p_\alpha^\pm = 0$. MAS takes place (see text). (d) $P_x$ increases even further. Now MAS also occurs at $y_-$ creating a second particle-hole orbit. The two orbits are separated in real space by $\delta y = 2|\Delta|/P_x\omega_c$.

This is the Landau level spectrum, shifted into two square root singularities (Fig. 2), which was studied previously by Miller and Győrffy [8] and Miyake [9]. As it turns out this theory is over simplified but nevertheless it contains the basic physics. The picture is one of Landau levels which march across the gap given in (6) giving the same frequency as in the normal state, but with an extra damping, due to $|\Delta|$ by $R_{sc} = aK_1(\alpha)$, with $a = 2\pi \ell |\Delta|^{1/2} \omega_c$, $(\ell$ integer), where $K_1$ is the Bessel function for imaginary argument. Whilst $R_{sc} (\alpha)$ fits the experimental data it predicts too much damping for realistic values of $|\Delta|$. However, as was pointed out to restrict it to one flux cell, i.e., $-d/2 \leq P_x/eB \leq +d/2$. Then, since $P_x$ lies within a cell, and $\hbar k_x/2eB$ represents translations by lattice cell vectors, the pair combined allow us to centre orbits anywhere in the sample. As we will see, different placements of orbits within a cell lead to different $E_\lambda$, i.e., degeneracy within a cell is lifted. On the other hand translations of such an orbit by lattice vectors (different choices of $k_x$) leave $E_\lambda = E_\lambda (P_x)$ invariant. One further simplification replaces $|\Delta(y)|$ by $|\Delta(y)| = |\Delta|$ in the region of interest. Outside this region we expect to set $|\Delta(y)| = 0$ but, since from (6), any choice in the range $0 \leq |\Delta(y)| \leq |\Delta|$ for $y > \hbar k_x/2eB > d$ makes little difference, we can for simplicity still take $|\Delta(y)| = |\Delta|$. Taking all these observations into account the constant energy orbits corresponding to these relations are the same as those studied by Burmistrov and Dubovskii [13].
in ref. [3] $P_z \neq 0$ changes the spectra drastically because, as is clear from [3] ‘Landau level’ energies depend upon $P_z$. The rest of this paper deals with these complications.

Thus we consider the orbits and spectrum for $P_z \neq 0$. Figure 3(b), (c) and (d) demonstrate the marked change in the structure of the phase space orbits as $P_z$ increases from zero (figure 3(a)). To understand the behaviour of the branches of $p_y^{\pm, \alpha}$, it is helpful to view the orbit structure as resulting from the competition between two different types of turning point. The usual turning points, $p_y^{\pm, \alpha} = 0$, are due to the harmonic oscillator confining potential, $\frac{1}{2}m_\omega c^2 y^2$, $(\gamma = y - \hbar k_x/2eB)$ in 3, and are those seen in figure 3(a) and (b). However the presence of the second square root in 4, $\sqrt{(E + P_x \omega, \gamma)^2 - |\Delta|^2}$, which we denote by $\epsilon(\gamma)$ provides an alternative reflection mechanism for which $p_y^{\pm, \alpha} \neq 0$. If $\epsilon(\gamma)$ becomes zero before the normal turning points are reached the momentum becomes complex and the wave function will correspondingly exhibit evanescent decay. Classically the excitation undergoes reflection. These new turning points given by $\epsilon(\gamma) = 0$ are

$$y_\pm = \pm |\Delta| - E \over P_x \omega_c. \tag{6}$$

This reflection process is analogous to Andreev scattering [19]. Andreev scattering is due to scattering from inhomogeneities in $|\Delta(\mathbf{r})|$ Thus $\sqrt{E^2 - |\Delta(\mathbf{r})|^2} \to 0$ as $|\Delta(\mathbf{r})| \to E$ from below. The reflection process we consider has $|\Delta| = \text{constant}$, but due to the non-zero vector potential the extra term, $P_x \omega_c \gamma$, can still cause $\epsilon(\gamma) \to 0$. (It is for this reason that the detailed shape of $|\Delta(y)|$ is not essential.) To emphasise the similarity of this process with Andreev reflection, and the role of the vector potential we shall call the scattering mechanism Magnetic Andreev Scattering (MAS), and $y_\pm$ Andreev-like turning points.

The appearance of MAS when $P_z$ is varied (Fig. 3(c) and (d)) can then be explained as follows. From 3 we see that as $P_z \to 0$, $y_\pm \to -\infty (E \geq |\Delta|)$. When $P_z$ becomes non-zero the Andreev-like turning points move in from $-\infty$ towards the normal turning points. We also see, as stated above, that $P_z$ acts essentially like an orbit centre, the particle-like and hole-like orbits (see below) being pulled in opposite directions due to their opposite charge (Fig. 3(b)). The orbits first intersect when $p_y^{\pm}(y_+) = p_y^{-}(y_+)$ i.e., when the Andreev-like turning point reaches the normal turning point. For a more physical picture of the nature of these turning points remember that the spinor (the eigenfunction to (4)) being transported along a given trajectory has both particle, $|u|$, and hole, $|v|$, amplitudes and thus carries an effective charge $e^* = \epsilon(|u(\mathbf{r})|^2 - |v(\mathbf{r})|^2)$. Evidently $e^*(\mathbf{r})$ is a function of position and in particular changes from $e^* > 0$ along $p_y^{+}$ (Fig. 3(c)), through zero (at the turning point) to $e^* < 0$ along $p_y^{-}$. So the effective charge of the excitation changes sign at $y_+$ and correspondingly the direction of circulation in the magnetic field is reversed. A particle-like excitation is reflected as a hole-like excitation, and a given orbit has both particle-like and hole-like segments.

Having discussed the appearance and interpretation of the MAS orbits we now turn to their quantisation. The spectrum obtained from semiclassical quantisation is shown in Fig. 3. Firstly note that at $P_z = 0$ there are Landau levels corresponding to 3 which are excluded from the gap. As we increase $P_z$ we reach the situation shown in Fig. 3(c). The sensitive dependence of the turning point $y_+(P_z)$ upon $P_z$ results in a spectrum which is highly $P_z$ dependent. Increasing $P_z$ further results in the situation shown in Fig. 3(d). The dramatic new feature of the spectrum for $P_z \neq 0$ is the existence of states inside the gap. These states are new features of the theory which were not there in the $P_z = 0$ case. However despite the existence of these states it is clear that they cannot on their own account for the experimental facts. These states have considerably different phase space (and hence momentum space) area, so that the dHvA frequency for each individual orbit, which is related to this area, will
Fig. 2 – Landau level spectrum defined by (5). The BCS density of states is included to emphasise how it is broken up in a magnetic field into Landau levels, which are nonetheless pushed apart by the gap, $|\Delta|$. 

Fig. 3 – Semiclassical spectrum obtained by quantising the orbits in Fig. 1. Region A corresponds to quantising the orbits in Fig. 1(a) and (b), whilst region B and C correspond to quantising the one or two scattering orbits in Fig. 1(c) and (d). The spectrum first becomes gapless when $P_x = p_{\min} \approx |\Delta(B)|/v_F$. 

be very different from that in the normal state. To explain the experiments we in fact require one further bit of physics. That is a phenomenon analogous to ‘Magnetic Breakdown’, [21,22] which we shall now discuss.

We observed that the separation of the two orbits in Fig. 1(d) is given by $\delta y = y_+ - y_- = 2|\Delta|/P_x \omega_c$ and it thus inversely proportional to both $P_x$ and the $B$-field (through $\omega_c = eB/m$). For a fixed $B$, as $P_x$ increases this separation in real space rapidly shrinks and quasiparticle tunnelling becomes a viable option. Purely on physical grounds we see that tunnelling between orbit segments (Fig. 3) can reproduce an orbit in phase space with approximately the normal area. To quantise explicitly the tunnelling orbit the Bohr-Sommerfeld integral can be represented as a complex contour integral. The Riemann surface for the problem is somewhat similar to that of Fortin et al [20]. In our case the MAS orbits live on two different sheets of the Riemann surface but the tunnelling orbit remains on only one sheet. Consequently the integral is equal to the residue at $\infty$, on this sheet, and we find 

$$E_n(p_z) = \hbar \omega_c \left( n + \frac{1}{2} \right) + \frac{p_z^2}{2m} - \epsilon_F. \quad (7)$$

The spectrum is $P_x$-independent and the tunnelling orbits have exactly the normal state frequency. However the tunnelling coefficient, which depends upon the imaginary part (Im) of the momentum, is 

$$T(P_x) = e^{-\frac{1}{\hbar} \int_{y_-(P_x)}^{y_+(P_x)} dy' \text{Im} p_+(y')}, \quad (8)$$

and clearly depends upon $P_x$ and $B$. For fixed $P_x$, $\delta y \propto 1/B \to 0$ as $B$ increases, so that the possibility of tunnelling is enhanced. We can therefore view the tunnelling as ‘Magnetic
Breakdown of MAS orbits'. Magnetic Breakdown in the normal state is a well studied phenomena [20]. The oscillatory magnetisation formulae is modified due to the tunnelling. To account for this we must include the factor \( T^2 k \) (every revolution of the orbit we pick up 2 tunnelling coefficients, and in general we have \( k \) revolutions). The \( P_x \) dependence of (8) requires the \( \sum P_x T^2 k \) to be carried out when calculating the magnetisation. (An additional \( \sum k \) yields the usual degeneracy factor in the formulae for the magnetisation.) Observing that \( P_{x,\text{max}} \ll p_{F}^2 \), we find \( \bar{h} - \frac{1}{2} \int dy' \text{Im} p_{y}^\dagger (y') \approx \frac{1}{2} \frac{|\Delta|^2}{k_{F}} \frac{1}{v_F} P_x \), where \( v_F \) is the Fermi velocity. Using \( |\Delta(B)| = \Delta(0)(1 - B/B_c)^{1/2} \to 0 \) as \( B \to B_c \) we have the important observation that \( T(P_x) \to 1 \) for all \( P_x \) as \( B \to B_c \). Thus the additional damping vanishes as \( |\Delta| \to 0 \) as is to be expected. Note, tunnelling is not a small correction, it is zeroth order. The normal state is recovered for \( T(P_x) \to 1 \).

The main contribution to \( \sum P_x T^{2k}(P_x) \) is obtained for the largest value of \( P_x = eBd/2 \) and is given by

\[
T^{2k}(P_x = eBd/2) = e^{-2\frac{f}{\Lambda} \frac{\Delta^2}{k_{F}}} \frac{1}{v_F},
\]

(9)

where \( \Lambda = (2eBh)^{-1/2} \). Interestingly this result is very similar to that of Wasserman and Springford [12]. More generally, the extra damping factor in the superconducting state is found to be

\[
R^{sc}(B) = \frac{1}{p_d} \int_{p_{\min}(B)}^{p_d(B)} dP_x e^{-2\frac{f}{\Lambda} \frac{\Delta^2}{k_{F}}} \int_{y_{-}(P_x,B)}^{y_{+}(P_x,B)} dy' \text{Im} p_{y}^\dagger (y',P_x),
\]

(10)

where \( p_d = eBd/2 \), and \( p_{\min}(B) \equiv |\Delta(B)|/v_F \) is the smallest value of \( P_x \) for which the spectrum is gapless (see Fig. 3). Although the above results have been derived for a free electron model with short range attraction, constant (averaged) pairing potential (|\Delta|) and uniform magnetic field \( B \), provisionally the formulae for the damping factor given in (10) can be compared with that deduced from experiments. We replace \( v_F \) by its orbitally averaged...
quantity, and use \( \langle 1/v_F \rangle = m_b/\sqrt{2\hbar eF} \) with \( m_b = 0.9m_e \), \( F = 1560\text{T} \), and \( |\Delta| = 4.4 \text{ meV} \) to compare (10) with the data for \( V_3\text{Si} \) in figure 5. Even at this early stage we have a satisfactory fit to the data. Clearly, \( |\Delta| \) in the above analysis is an effective quantity whose value cannot be expected to agree with the zero field \( |\Delta| \) of 2.6 meV (see for example ref. [2]). A more detailed analysis of the existing data as well as a more complete presentation of the theory will be given elsewhere (1).

To summarise we developed a semiclassical theory in the superconducting state. We consider new semiclassical orbits involving ‘magnetic breakdown’ of MAS orbits. These new orbits, when quantised, have precisely the normal state frequency. The tunnelling involved gives rise to an extra damping in the superconducting state in agreement with experiments in such classic superconductors as \( V_3\text{Si} \). When generalised to include period crystal potential, a more complete description of the flux lattice, and anisotropic pairing [5], the above theory will also be suitable for analysing experimental results on exotic superconductors whenever these become available.

\* \* \*

We would like to thank Steve Hayden for supplying us with the data for \( V_3\text{Si} \) in Fig. 5. This work was supported by EPSRC, grant No. GR/M53844.

REFERENCES

[1] Onuki Y., Umehara I., Ebihara T., Nagai N., and Takita T., J. Phys. Soc. Jpn., 61 (1992) 9691
[2] Janssen T. J. B., Haworth C., Hayden S. M., Meeson P., Springford M. and Wasserman A., Phys. Rev. B, 57 (1998) 11 689
[3] Goll G., Heinecke M., Jansen A. G. M., Joss W., Nguyen L., Steep E., Winzer K. and Wyder P., Phys. Rev. B, 53 (1996) 8871
[4] Graebner J. E. and Robbins M., Phys. Rev. Lett., 36 (1976) 422
[5] Mineev V. P. and Samokhin K. V., Introduction to Unconventional Superconductivity (Gordon and Breach Science Publications) 1999
[6] Gor’kov L. P. and Schrieffer J. R., Phys. Rev. Lett., 80 (1998) 3360
[7] Maniv T., Zhuravlev V., Vagner I. D. and Wyder P., Rev. Mod. Phys., 73 (2001) 867
[8] Miller P. and Györyffy B. L., J. Phys. Condens. Matter, 7 (1995) 5579-5606
[9] Miyake K., Physica B, 186-188 (1993) 115-117
[10] Dukan S. and Tešanović Z., Phys. Rev. Lett., 74 (1995) 2311
[11] Norman M. R., MacDonald A. H., and Hiroshi Akera., Phys. Rev. B, 51 (1995) 5927
[12] Wasserman A. and Springford M., Adv. in Phys., 45 (1996) 471
[13] Burmistrov S. N., and Dubovskii L. B., Czechoslovak Journal of Physics, 46 (1996) 895
[14] Bruun G. M., Nikos Nicopoulos V. and Johnson N. F., Phys. Rev. B, 56 (1997) 809-825
[15] Vavilov M. G. and Mineev V. P., Sov. Phys. JETP., 85 (1997) 1024
[16] Duncan K. P. and Györyffy B. L., cond-mat/0112160, to appear in Ann. Phys.,
[17] Abrikosov A. A., Fundamentals of The Theory of Metals (North-Holland) 1988
[18] Franz M. and Tešanović Z., Phys. Rev. Lett., 84 (2000) 554
[19] Andreev A. F., Sov. Phys. JETP., 19 (1964) 128-1238
[20] Fortin J. Y., Bellissard J., Guismag M. and Ziman T., Phys. Rev. B, 57 (1998) 1484
[21] Duncan K. P., The semiclassical theory of the de Haas-van Alphen oscillations in type-II superconductors (Ph.D thesis, University of Bristol) 1999
[22] Duncan K. P. and Györyffy B. L., Bulletin APS, 44 (1999) 1063

(1) work in preparation