Thermodynamical properties of topological Born-Infeld-dilaton black holes

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We examine the \((n+1)\)-dimensional \((n \geq 3)\) action in which gravity is coupled to the Born-Infeld nonlinear electrodynamic and a dilaton field. We construct a new \((n+1)\)-dimensional analytic solution of this theory in the presence of Liouville-type dilaton potentials. These solutions which describe charged topological dilaton black holes with nonlinear electrodynamics, have unusual asymptotics. They are neither asymptotically flat nor (anti)-de Sitter. The event horizons of these black holes can be an \((n-1)\)-dimensional positive, zero or negative constant curvature hypersurface. We also analyze thermodynamics and stability of these solutions and disclose the effect of the dilaton and Born-Infeld fields on the thermal stability in the canonical ensemble.

I. INTRODUCTION

The pioneering theory of the non-linear electromagnetic field was proposed by Born and Infeld in 1934 for the purpose of solving various problems of divergence appearing in the Maxwell theory \[1\]. Although it became less popular with the introduction of QED, in recent years, the Born-Infeld action has been occurring repeatedly with the development of superstring theory, where the dynamics of D-branes is governed by the Born-Infeld action \[2, 3\]. It has been shown that charged black hole solutions in Einstein-Born-Infeld gravity are less singular in comparison with the Reissner-Nordström solution. In other words, there is no Reissner-Nordström-type divergence term \(q^2/r^2\) in the metric near the singularity while it exist only a Schwarzschild-type term \(m/r\) \[4, 5\]. In the absence of a dilaton field, exact solutions of Einstein-Born-Infeld theory with/without cosmological constant have been constructed by many authors \[6, 7, 8, 9, 10, 11, 12, 13\]. In the scalar-tensor theories of gravity, black hole solutions coupled to a Born-Infeld nonlinear electrodynamics have been studied in \[14\]. The Born-Infeld action coupled to a dilaton field, appears in the low energy limit of open superstring theory \[2\]. Although one can consistently truncate such models, the presence of the dilaton field cannot be ignored if one consider coupling of the gravity to other gauge fields, and therefore one remains with Einstein-Born-Infeld gravity in the presence of a dilaton field. Many attempts have been done to construct solutions of Einstein-Born-Infeld-dilaton

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The appearance of the dilaton field changes the asymptotic behavior of the solutions to be neither asymptotically flat nor (anti)-de Sitter [(A)dS]. The motivation for studying non-asymptotically flat nor (A)dS solutions of Einstein gravity comes from the fact that, these kind of solutions can shed some light on the possible extensions of AdS/CFT correspondence. Indeed, it has been speculated that the linear dilaton spacetimes, which arise as near-horizon limits of dilatonic black holes, might exhibit holography [21]. Black hole spacetimes which are neither asymptotically flat nor (A)dS have been explored widely in the literature (see e.g. [22, 23, 24, 25, 26, 27, 28, 29, 30]).

On the other hand, it is a general belief that in four dimensions the topology of the event horizon of an asymptotically flat stationary black hole is uniquely determined to be the two-sphere $S^2$ [31, 32]. Hawking’s theorem requires the integrated Ricci scalar curvature with respect to the induced metric on the event horizon to be positive [31]. This condition applied to two-dimensional manifolds determines uniquely the topology. The “topological censorship theorem” of Friedmann, Schleich and Witt is another indication of the impossibility of non-spherical horizons [33, 34]. However, when the asymptotic flatness and the four dimensional spacetime are given up, there are no fundamental reasons to forbid the existence of static or stationary black holes with nontrivial topologies. It was confirmed that black holes in higher dimensions bring rich physics in comparison with the four dimensions. For instance, for five-dimensional asymptotically flat stationary black holes, in addition to the known $S^3$ topology of event horizons, stationary black hole solutions with event horizons of $S^2 \times S^1$ topology (black rings) have been constructed [36]. It has been shown that for asymptotically AdS spacetime, in the four-dimensional Einstein-Maxwell theory, there exist black hole solutions whose event horizons may have zero or negative constant curvature and their topologies are no longer the two-sphere $S^2$. The properties of these black holes are quite different from those of black holes with usual spherical topology horizon, due to the different topological structures of the event horizons. Besides, the black hole thermodynamics is drastically affected by the topology of the event horizon. It was argued that the Hawking-Page phase transition [37] for the Schwarzschild-AdS black holes does not occur for locally AdS black holes whose horizons have vanishing or negative constant curvature, and they are thermally stable [38]. The studies on the topological black holes have been carried out extensively in many aspects (see e.g. [39, 40, 41, 42, 43, 44, 45, 46, 47]).

In this paper, we would like to explore thermodynamical properties of the topological Born-Infeld-dilaton black holes in higher dimensional spacetimes in the presence of Liouville-type potentials for the dilaton field. The motivation for studying higher dimensional solutions of Einstein
gravity originates from superstring theory, which is a promising candidate for the unified theory of
everything. As the superstring theory can be consistently formulated only in 10-dimensions, the
existence of extra dimensions should be regarded as the prediction of the theory. Although for a
while it was thought that the extra spatial dimensions would be of the order of the Planck scale,
making a geometric description unreliable, but it has recently been realized that there is a way
to make the extra dimensions relatively large and still be unobservable. This is if we live on a
three dimensional surface (brane) in a higher dimensional spacetime (bulk). In such a scenario, all
gravitational objects such as black holes are higher dimensional. Indeed, the large extra dimension
scenarios open up new exciting possibilities to relate the properties of higher dimensional black
holes to the observable world by direct probing of TeV-size mini-black holes at future high energy
colliders [48]. Besides, it was argued that through Hawking radiation of higher dimensional black
holes, it is possible to detect these extra dimensions [49]. In the light of all mentioned above, it
becomes obvious that further study of black hole solutions in higher dimensional gravity is of great
importance.

This paper is organized as follows: In section II we construct a new class of \((n+1)\)-dimensional
topological black hole solutions in EBIId theory with two liouville type potentials and general
dilaton coupling constant, and investigate their properties. In section III we obtain the conserved
and thermodynamic quantities of the \((n+1)\)-dimensional topological black holes and verify that
these quantities satisfy the first law of black hole thermodynamics. In section IV we perform a
stability analysis and show that the dilaton creates an unstable phase for the solutions. The last
section is devoted to summary and conclusions.

II. TOPOLOGICAL DILATON BLACK HOLES IN BORN-INFELD THEORY

We examine the \((n+1)\)-dimensional \((n \geq 3)\) action in which gravity is coupled to dilaton and
Born-Infeld fields
\[
S = \frac{1}{16\pi} \int d^{n+1}x \sqrt{-g} \left( \mathcal{R} - \frac{4}{n-1} (\nabla \Phi)^2 - V(\Phi) + L(F, \Phi) \right),
\]
(1)
where \(\mathcal{R}\) is the Ricci scalar curvature, \(\Phi\) is the dilaton field and \(V(\Phi)\) is a potential for \(\Phi\). The
Born-Infeld \(L(F, \Phi)\) part of the action is given by
\[
L(F, \Phi) = 4\beta^2 e^{4\alpha \Phi/(n-1)} \left( 1 - \sqrt{1 + \frac{e^{-8\alpha \Phi/(n-1)} F^2}{2\beta^2}} \right).
\]
(2)
Here, $\alpha$ is a constant determining the strength of coupling of the scalar and electromagnetic fields, $F^2 = F_{\mu\nu}F^{\mu\nu}$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor, and $A_\mu$ is the electromagnetic vector potential. $\beta$ is the Born-Infeld parameter with the dimension of mass. In the limit $\beta \to \infty$, $L(F, \Phi)$ reduces to the standard Maxwell field coupled to a dilaton field

$$L(F, \Phi) = -e^{-4\alpha\Phi/(n-1)} F^2.$$  

On the other hand, $L(F, \Phi) \to 0$ as $\beta \to 0$. It is convenient to set

$$L(F, \Phi) = 4\beta^2 e^{4\alpha\Phi/(n-1)} \mathcal{L}(Y),$$

where

$$\mathcal{L}(Y) = 1 - \sqrt{1 + Y},$$

$$Y = \frac{e^{-8\alpha\Phi/(n-1)} F^2}{2\beta^2}. \quad (6)$$

The equations of motion can be obtained by varying the action (1) with respect to the gravitational field $g_{\mu\nu}$, the dilaton field $\Phi$ and the gauge field $A_\mu$ which yields the following field equations

$$\mathcal{R}_{\mu\nu} = \frac{4}{n-1} \left( \partial_\mu \Phi \partial_\nu \Phi + \frac{1}{4} g_{\mu\nu} V(\Phi) \right) - 4 e^{-4\alpha\Phi/(n-1)} \partial_\nu \mathcal{L}(Y) F_{\mu\eta} F^{\eta \nu}$$

$$+ \frac{4\beta^2}{n-1} e^{4\alpha\Phi/(n-1)} [2 Y \partial_\nu \mathcal{L}(Y) - \mathcal{L}(Y)] g_{\mu\nu}, \quad (7)$$

$$\nabla^2 \Phi = \frac{n-1}{8} \frac{\partial V}{\partial \Phi} + 2\alpha \beta^2 e^{4\alpha\Phi/(n-1)} [2 Y \partial_\nu \mathcal{L}(Y) - \mathcal{L}(Y)],$$

$$\partial_\mu \left( \sqrt{-g} e^{-4\alpha\Phi/(n-1)} \partial_\nu \mathcal{L}(Y) F^{\mu\nu} \right) = 0. \quad (9)$$

In particular, in the case of the linear electrodynamics with $\mathcal{L}(Y) = -\frac{1}{2} Y$, the system of equations (7)-(9) reduce to the well-known equations of Einstein-Maxwell-dilaton gravity [22].

We would like to find topological solutions of the above field equations. The most general such metric can be written in the form

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 R(r) h_{ij} dx^i dx^j, \quad (10)$$

where $f(r)$ and $R(r)$ are functions of $r$ which should be determined, and $h_{ij}$ is a function of coordinates $x_i$ which spanned an $(n - 1)$-dimensional hypersurface with constant scalar curvature $(n-1)(n-2)k$. Here $k$ is a constant and characterizes the hypersurface. Without loss of generality, one can take $k = 0, 1, -1$, such that the black hole horizon or cosmological horizon in (10) can
be a zero (flat), positive (elliptic) or negative (hyperbolic) constant curvature hypersurface. The electromagnetic field equation (9) can be integrated immediately to give

\[ F_{tr} = \frac{\beta q e^{4\alpha \Phi/(n-1)}}{\sqrt{\beta^2 (rR)^{2n-2} + q^2}}, \]  

(11)

where \( q \) is an integration constant related to the electric charge of the black hole. Defining the electric charge via

\[ Q = \frac{q \omega_{n-1}}{4\pi}, \]  

(12)

where \( \omega_{n-1} \) represents the volume of constant curvature hypersurface described by \( h_{ij}dx^i dx^j \). It is worthwhile to note that the electric field is finite at \( r = 0 \). This is expected in Born-Infeld theories. Meanwhile it is interesting to consider three limits of (11). First, for large \( \beta \) (where the Born-Infeld action reduces to the Maxwell case) we have

\[ F_{tr} = \frac{qe^{4\alpha \Phi/(n-1)}}{(rR)^{n-1}} \]  

as presented in [22]. On the other hand, if \( \beta \to 0 \) we get \( F_{tr} = 0 \). Finally, in the absence of the dilaton field (\( \alpha = 0 \)), it reduces to the case of \((n+1)\)-dimensional Einstein-Born-Infeld theory [11]

\[ F_{tr} = \frac{\beta q}{\sqrt{\beta^2 r^{2n-2} + q^2}}. \]  

(13)

Our aim here is to construct exact, \((n+1)\)-dimensional topological solutions of the EBId gravity with an arbitrary dilaton coupling parameter \( \alpha \). The case in which we find topological solutions of physical interest is to take the dilaton potential of the form

\[ V(\Phi) = 2\Lambda_0 e^{2\zeta_0 \Phi} + 2\Lambda e^{2\zeta \Phi}, \]  

(14)

where \( \Lambda_0, \Lambda, \zeta_0 \) and \( \zeta \) are constants. This kind of potential was previously investigated by a number of authors both in the context of Friedmann-Robertson-Walker (FRW) scalar field cosmologies [50] and dilaton black holes (see e.g. [18, 19, 22, 30, 47]). In order to solve the system of equations (7) and (8) for three unknown functions \( f(r) \), \( R(r) \) and \( \Phi(r) \), we make the ansatz

\[ R(r) = e^{2\alpha \Phi/(n-1)}. \]  

(15)

Using (15), the electromagnetic field (11) and the metric (10), one can show that equations (7) and (8) have solutions of the form

\[
\begin{align*}
f(r) &= \frac{k(n-2)\left(\alpha^2 + 1\right)^2 b^{-2\gamma}}{(\alpha^2 - 1)(n + \alpha^2 - 2)} r^{2\gamma} - \frac{m}{r^{(n-1)(1-\gamma)-1}} + 2 \frac{\left(\Lambda - 2\beta^2\right)\left(\alpha^2 + 1\right)^2 b^{2\gamma}}{(n-1)(\alpha^2 - n)} r^{2-2\gamma} \\
&\quad - \frac{4\beta^2(\alpha^2 + 1)b^{2\gamma}}{n-1} r^{(n-1)(\gamma-1)+1} \int r^{(n+1)(1-\gamma)-2} \sqrt{1 + \eta dr},
\end{align*}
\]  

(16)
\[ \Phi(r) = \frac{(n-1)\alpha}{2(1+\alpha^2)} \ln \left( \frac{b}{r} \right), \]  

where \( b \) is an arbitrary constant, \( \gamma = \alpha^2/(\alpha^2 + 1) \), and
\[
\eta = \frac{q^2 b^{2\gamma(1-n)}}{\beta^2 r^{2(n-1)(1-\gamma)}}. \tag{18}
\]

In the above expression, \( m \) appears as an integration constant and is related to the ADM (Arnowitt-Deser-Misner) mass of the black hole. According to the definition of mass due to Abbott and Deser [47], the mass of the solution (16) is [47]
\[
M = \frac{b^{(n-1)\gamma}(n-1)\omega_{n-1}}{16\pi(\alpha^2 + 1)} m. \tag{19}
\]

In order to fully satisfy the system of equations, we must have
\[
\zeta_0 = \frac{2}{\alpha(n-1)}, \quad \zeta = \frac{2\alpha}{n-1}, \quad \Lambda_0 = \frac{k(n-1)(n-2)\alpha^2}{2b^2(\alpha^2 - 1)}. \tag{20}
\]

Notice that here \( \Lambda \) is a free parameter which plays the role of the cosmological constant. For later convenience, we redefine it as \( \Lambda = -n(n-1)/2l^2 \), where \( l \) is a constant with dimension of length. The integral can be done in terms of hypergeometric function and can be written in a compact form as
\[
f(r) = -\frac{k(n-2)(\alpha^2 + 1)^2 b^{-2\gamma}}{(\alpha^2 - 1)(n + \alpha^2 - 2)} r^{2\gamma} \left( \frac{m}{r^{(n-1)(1-\gamma)-1}} + \frac{2\Lambda (\alpha^2 + 1)^2 b^{2\gamma}}{(n-1)(\alpha^2 - n)} r^{2(1-\gamma)} \right) \]
\[-4\beta^2(\alpha^2 + 1)^2 b^{2\gamma} r^{2(1-\gamma)} \frac{(n-1)(\alpha^2 - n)}{(n-1)(\alpha^2 - n)} \times \left( 1 - 2F_1 \left( \left[ -\frac{1}{2}, \frac{n-2}{2n-2} \right], \frac{\alpha^2 + n-2}{2n-2}, -\eta \right) \right). \tag{21}
\]

One may note that as \( \beta \rightarrow \infty \) these solutions reduce to the (n+1)-dimensional topological dilaton black hole solutions given in [47]
\[
f(r) = -\frac{k(n-2)(\alpha^2 + 1)^2 b^{-2\gamma}}{(\alpha^2 - 1)(\alpha^2 + n - 2)} r^{2\gamma} \left( \frac{m}{r^{(n-1)(1-\gamma)-1}} + \frac{2\Lambda (\alpha^2 + 1)^2 b^{2\gamma}}{(n-1)(\alpha^2 - n)} r^{2(1-\gamma)} \right) \]
\[+ \frac{2q^2(\alpha^2 + 1)^2 b^{-2n-2}}{(n-1)(\alpha^2 + n - 2)} r^{2(n-2)(\gamma-1)}. \tag{22}
\]

In the absence of a nontrivial dilaton (\( \alpha = \gamma = 0 \)), the solution (21) reduces to
\[
f(r) = k - \frac{m}{r^{n-2}} + \frac{r^2}{l^2} + \frac{4\beta^2 r^2}{n(n-1)} \times \left( 1 - 2F_1 \left( \left[ -\frac{1}{2}, \frac{n}{2n-2} \right], \frac{n-2}{2n-2}, -\frac{q^2}{\beta^2 r^{2n-2}} \right) \right). \tag{23}
\]

which describes an (n+1)-dimensional asymptotically (A)dS topological Born-Infeld black hole with a positive, zero or negative constant curvature hypersurface [12]. Using the fact that \( 2F_1(a,b,c,z) \) has a convergent series expansion for \( |z| < 1 \), we can find the behavior of the metric for large \( r \).
FIG. 1: The function \( f(r) \) versus \( r \) for \( \alpha = 0.7, m = 2, \beta = 1, n = 4 \) and \( q = 1 \). \( k = -1 \) (bold line), \( k = 0 \) (continuous line), and \( k = 1 \) (dashed line).

This is given by

\[
f(r) = \frac{k (n - 2) (\alpha^2 + 1)^2 b^{-2\gamma}}{(\alpha^2 - 1) (n + \alpha^2 - 2)} \left( \frac{2q^2(\alpha^2 + 1)^2 b^{-2(\gamma - 1)}}{(n - 1)(\alpha^2 + n - 2)r^{2(n - 2)(1 - \gamma)}} - \frac{2\Lambda (\alpha^2 + 1)^2 b^{2\gamma}}{(n - 1)(\alpha^2 - n)} r^{2(1 - \gamma)} \right).
\]

The last term in the right hand side of the above expression is the leading Born-Infeld correction to the topological black hole with dilaton field \[47\]. Note that for \( \alpha = \gamma = 0 \), the above expression reduces to

\[
f(r) = k - \frac{m}{r^{n-2}} + \frac{r^2}{l^2} + \frac{2q^2}{(n - 1)(n - 2)r^{2(n - 2)}} - \frac{q^4}{2\beta^2(n - 1)(3n - 4)r^{2(2n - 3)}}.
\]

which has the form of the \((n + 1)\)-dimensional topological black hole in \((A)dS\) spacetime in the limit \( \beta \to \infty \) (see e.g. \[41, 42\]).

**Physical Properties of the Solutions**

In order to study the physical properties of the solutions, we first look for the curvature singularities. In the presence of a dilaton field, the Kretschmann scalar \( R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa} \) diverges at \( r = 0 \), it is finite for \( r \neq 0 \) and goes to zero as \( r \to \infty \).Thus, there is an essential singularity located at \( r = 0 \). The spacetime is neither asymptotically flat nor \((A)dS\). It is notable to mention that

in the \( k = \pm 1 \) cases these solutions do not exist for the string case where \( \alpha = 1 \). As one can see from Eq. (21), the solution is ill-defined for \( \alpha = \sqrt{n} \). The cases with \( \alpha < \sqrt{n} \) and \( \alpha > \sqrt{n} \) should be considered separately. In the first case where \( \alpha < \sqrt{n} \), there exist a cosmological horizon for \( \Lambda > 0 \), while there is no cosmological horizons if \( \Lambda < 0 \). Indeed, where \( \alpha < \sqrt{n} \) and \( \Lambda < 0 \),
FIG. 2: The function $f(r)$ versus $r$ for $m = 2$, $\beta = 1$, $q = 1$, $n = 4$ and $k = 0$. $\alpha = 0.6$ (bold line), $\alpha = 0.75$ (continuous line), and $\alpha = 0.85$ (dashed line).

FIG. 3: The function $f(r)$ versus $r$ for $m = 2$, $\alpha = 0.6$, $q = 1$, $n = 4$ and $k = 0$. $\beta = 1$ (bold line), $\beta = 2$ (continuous line), and $\beta = 15$ (dashed line).

FIG. 4: The function $m(r_h)$ versus $r_h$ for $\beta = 1$, $q = 1$, $n = 4$ and $k = 0$. $\alpha = 0$ (bold line), $\alpha = 0.8$ (continuous line), and $\alpha = 1.2$ (dashed line).
the spacetimes associated with the solution (21) exhibit a variety of possible casual structures depending on the values of the metric parameters (see Figs. 1-3). For simplicity in these figures, we kept fixed \( l = b = 1 \). These figures show that our solutions can represent topological black hole, with two event horizons, an extreme topological black hole or a naked singularity provided the parameters of the solutions are chosen suitably. In the second case where \( \alpha > \sqrt{n} \), the spacetime has a cosmological horizon. One can obtain the casual structure by finding the roots of \( f(r) = 0 \). Unfortunately, because of the nature of the exponent in (21), it is not possible to find analytically the location of the horizons. To have further understanding on the nature of the horizons, as an example for \( k = 0 \), we plot in Figs. 4 and 5 the mass parameter \( m \) as a function of the horizon radius for different values of dilaton coupling constant \( \alpha \) and charge parameter \( q \). Again, we have fixed \( l = b = 1 \), for simplicity. It is easy to show that the mass parameter of the black hole can be expressed in terms of the horizon radius \( r_h \) as

\[
m(r_h) = -\frac{k(n-2)(\alpha^2+1)^2b^{-2\gamma}}{(\alpha^2-1)(n+\alpha^2-2)r_h^{n-2+\gamma(3-n)}} + \frac{2\Lambda(\alpha^2+1)^2b^{2\gamma}n(1-\gamma)-\gamma}{(n-1)(\alpha^2-n)r_h^n} \times \left(1 - 2F_1\left(\left[-\frac{1}{2}, \frac{\alpha^2}{2n-2}\right], \left[\frac{\alpha^2+n-2}{2n-2}\right], -\eta\right)\right),
\]

These figures show that for a given value of \( \alpha \), the number of horizons depend on the choice of the value of the mass parameter \( m \). We see that, up to a certain value of the mass parameter \( m \), there are two horizons, and as we decrease the \( m \) further, the two horizons meet. In this case we get an extremal black hole with mass \( m_{\text{ext}} \) (see the next section). Figure 4 shows that with increasing \( \alpha \), the \( m_{\text{ext}} \) also increases. It is worth noting that in the limit \( r_h \to 0 \) we have a nonzero value for the mass parameter \( m \). This is in contrast to the Schwarzschild black holes in which the mass parameter goes to zero as \( r_h \to 0 \). As we have shown in figure 5, this is due to the effect of

FIG. 5: The function \( m(r_h) \) versus \( r_h \) for \( \beta = 1, \alpha = 0.8, n = 4 \) and \( k = 0 \). \( q = 0 \) (bold line), \( q = 1 \) (continuous line), and \( q = 1.5 \) (dashed line).
the charge parameter \( q \) and the nature of the Born-Infeld field, and in the case \( q = 0 \), the mass parameter \( m \) goes to zero as \( r_h \to 0 \). In summary, the metric of Eqs. (10) and (21) can represent a charged topological dilaton black hole with inner and outer event horizons located at \( r_- \) and \( r_+ \), provided \( m > m_{\text{ext}} \), an extreme topological black hole in the case of \( m = m_{\text{ext}} \), and a naked singularity if \( m < m_{\text{ext}} \).

III. THERMODYNAMICS OF TOPOLOGICAL DILATON BLACK HOLE

We now would like to study the thermodynamical properties of the solutions we have just found. The temperature of the black hole can be obtained by continuing the metric to its Euclidean sector via \( t = -i\tau \) and requiring the absence of conical singularity at the horizon. It is a matter of calculation to show that

\[
T_+ = \frac{\kappa}{2\pi} = \frac{f'(r_+)}{4\pi},
\]

where \( \kappa \) is the surface gravity. The temperature is then

\[
T_+ = -\frac{(\alpha^2 + 1)b^{2\gamma}r_+^{1-2\gamma}}{2\pi(n-1)} \left( \frac{k(n-2)(n-1)b^{-4\gamma}}{2(\alpha^2 - 1)} \right) r_+^{4\gamma - 2} + \Lambda - 2\beta^2 (1 - \sqrt{1 + \eta_+})
\]

\[
= -\frac{k(n-2)(\alpha^2 + 1)b^{-2\gamma}}{2\pi(\alpha^2 + n - 2)} r_+^{2\gamma - 1} \left( \frac{n - \alpha^2}{4\pi(\alpha^2 + 1)} r_+^{(n-1)(\gamma-1)} - \frac{q^2(\alpha^2 + 1)b^{2(2-n)\gamma}}{\pi(\alpha^2 + n - 2)} r_+^{2(2-n)(1-\gamma)-1} \right)
\]

\[
\times 2F_1 \left( \left[ \frac{1}{2}, \frac{n + \alpha^2 - 2}{2n - 2} \right], \left[ \frac{3n + \alpha^2 - 4}{2n - 2} \right], -\eta_+ \right).
\]

where \( \eta_+ = \eta(r = r_+) \). There is also an extreme value for the mass parameter in which the temperature of the event horizon of black hole is zero. It is a matter of calculation to show that

\[
m_{\text{ext}} = \frac{2k(n-2)(\alpha^2 + 1)b^{-2\gamma}}{(n-\alpha^2)(\alpha^2 + n - 2)} r_+^{(2-n)(\gamma-1)+\gamma} + \frac{4q^2(\alpha^2 + 1)b^{2(2-n)\gamma}}{(n-\alpha^2)(\alpha^2 + n - 2)} r_+^{(3-n)(1-\gamma)-1}
\]

\[
\times 2F_1 \left( \left[ \frac{1}{2}, \frac{n + \alpha^2 - 2}{2n - 2} \right], \left[ \frac{3n + \alpha^2 - 4}{2n - 2} \right], -\eta_+ \right).
\]

The entropy of the topological black hole typically satisfies the so called area law of the entropy which states that the entropy of the black hole is a quarter of the event horizon area \[52\]. This near universal law applies to almost all kinds of black holes, including dilaton black holes, in Einstein gravity \[53\]. It is a matter of calculation to show that the entropy of the topological black hole is

\[
S = \frac{k(n-1)\gamma \omega_{n-1} r_+^{(n-1)(1-\gamma)}}{4}.
\]

The electric potential \( U \), measured at infinity with respect to the horizon, is defined by

\[
U = A_\mu \chi^\mu \left|_{r\to\infty} - A_\mu \chi^\mu \right|_{r=r_+},
\]
where $\chi = \partial_t$ is the null generator of the horizon. One can easily show that the gauge potential $A_t$ corresponding to the electromagnetic field \([11]\) can be written as
\[
A_t = \frac{q b^{(3-n)\gamma}}{\Upsilon r^+} 2F_1 \left( \left[ 1, \frac{\alpha^2 + n - 2}{2n - 2} \right], \left[ \frac{\alpha^2 + 3n - 4}{2n - 4} \right], -\eta \right),
\]
where $\Upsilon = (n - 3)(1 - \gamma) + 1$. Therefore, the electric potential may be obtained as
\[
U = \frac{q b^{(3-n)\gamma}}{\Upsilon r^+} 2F_1 \left( \left[ 1, \frac{\alpha^2 + n - 2}{2n - 2} \right], \left[ \frac{\alpha^2 + 3n - 4}{2n - 4} \right], -\eta^+ \right).
\]

In figures 6 and 7 we have shown the behavior of the electric potential $U$ as a function of horizon radius. As one can see from these figures, $U$ is finite even for $r^+ = 0$. Then, we consider the first law of thermodynamics for the topological black hole. For this purpose, we first obtain the mass $M$ as a function of extensive quantities $S$ and $Q$. Using the expression for the charge, the mass and the entropy given in Eqs. \([12]\), \([19]\) and \([30]\) and the fact that $f(r^+) = 0$, one can obtain a
FIG. 8: $\left(\frac{\partial^2 M}{\partial S^2}\right)_Q$ versus $\alpha$ for $r_+ = 0.8$, $\beta = 1$, $n = 5$, and $k = 0$. $q = 0.5$ (bold line), $q = 1$ (continuous line), and $q = 1.5$ (dashed line).

Smarr-type formula as

$$M(S, Q) = -\frac{k(n-1)(n-2)(\alpha^2 + 1)b^{-\alpha^2}}{16\pi(\alpha^2 - 1)(\alpha^2 + n - 2)}(4S)^{(\alpha^2+n-2)/(n-1)} + \frac{\Lambda(\alpha^2 + 1)b^\alpha}{8\pi(\alpha^2 - n)}(4S)^{(n-\alpha^2)/(n-1)}$$

$$-\frac{\beta^2(\alpha^2 + 1)b^{\alpha^2}}{4\pi(\alpha^2 - n)}(4S)^{(n-\alpha^2)/(n-1)} \times \left(1 - \frac{1}{2}\right)^{-\frac{n}{2n-2}} \left[\alpha^2 + \frac{n-2}{2n-2}\right], \left[-\frac{\pi^2Q^2}{\beta^2S^2}\right] \right)$$

(34)

Then we can regard the parameters $S$ and $Q$ as a complete set of extensive parameters for the mass $M(S, Q)$ and define the intensive parameters conjugate to $S$ and $Q$. These quantities are the temperature and the electric potential

$$T = \left(\frac{\partial M}{\partial S}\right)_Q, \quad U = \left(\frac{\partial M}{\partial Q}\right)_S.$$  

(35)

Now that we have all the relevant thermodynamic quantities, we can easily verify the first law of black hole thermodynamics. We find that

$$dM =TdS + UdQ,$$

(36)

is satisfied. In the next section we will explore the thermal stability of the solutions.

**IV. STABILITY IN THE CANONICAL ENSEMBLE**

Finally, we investigate the thermal stability of the topological black hole solutions of Einstein-Born-Infeld-dilaton gravity in the canonical ensemble. In the canonical ensemble, the charge is a fixed parameter, and therefore the positivity of the heat capacity or $\left(\frac{\partial T}{\partial S}\right)_Q = \left(\frac{\partial^2 M}{\partial S^2}\right)_Q$ is sufficient to ensure the local stability of the system. In order to investigate the effects of the dilaton
FIG. 9: \( (\partial^2 M / \partial S^2)_Q \) versus \( \alpha \) for \( r_+ = 0.8, \beta = 1, n = 3, \) and \( k = -1 \). \( q = 0.5 \) (bold line), \( q = 1 \) (continuous line), and \( q = 1.5 \) (dashed line).

FIG. 10: \( (\partial^2 M / \partial S^2)_Q \) versus \( \alpha \) for \( r_+ = 0.8, n = 4, q = 1 \) and \( k = -1 \). \( \beta = 0.2 \) (bold line), \( \beta = 2 \) (continuous line) and \( \beta = 20 \) (dashed line).

FIG. 11: \( (\partial^2 M / \partial S^2)_Q \) versus \( \alpha \) for \( r_+ = 0.8, \beta = 1, q = 1 \) and \( n = 4 \). \( k = -1 \) (bold line), \( k = 0 \) (continuous line), and \( k = 1 \) (dashed line).
FIG. 12: $(\partial^2 M/\partial S^2)_Q$ versus $q$ for $r_+ = 0.8$, $\beta = 2$, $n = 5$ and $k = -1$. $\alpha = 0.8$ (bold line), $\alpha = 1.2$ (continuous line), and $\alpha = \sqrt{2}$ (dashed line).

FIG. 13: $(\partial^2 M/\partial S^2)_Q$ versus $q$ for $r_+ = 0.8$, $\beta = 1$, $n = 5$ and $\alpha = \sqrt{2}$. $k = -1$ (bold line), $k = 0$ (continuous line), and $k = 1$ (dashed line).

FIG. 14: $(\partial^2 M/\partial S^2)_Q$ versus $\beta$ for $r_+ = 0.8$, $q = 1$, $n = 5$ and $k = -1$. $\alpha = 0.6$ (bold line), $\alpha = 1.1$ (continuous line), and $\alpha = 1.4$ (dashed line).
FIG. 15: $(\partial^2 M/\partial S^2)_Q$ versus $\beta$ for $r_+ = 0.8$, $q = 1$, $\alpha = \sqrt{2}$ and $n = 5$. $k = -1$ (bold line), $k = 0$ (continuous line), and $k = 1$ (dashed line).

coupling constant $\alpha$, the electric charge parameter $q$, and the Born-Infeld parameter $\beta$ on the stability of the solutions, we plot $(\partial^2 M/\partial S^2)_Q$ versus $\alpha$, $q$ and $\beta$. As one can see from figures 8-11, the topological black hole solutions are stable independent of the values of the charge parameter $q$, the curvature constant parameter $k$, and the Born-Infeld parameter $\beta$, in any dimensions if $\alpha < \alpha_{\text{max}}$, while for $\alpha > \alpha_{\text{max}}$ the system has an unstable phase. That is the dilaton field makes the solution unstable. These figures also show that $\alpha_{\text{max}}$ increases with increasing $q$ and $\beta$ (see Figs. 8-11), while it decreases with increasing $k$ (see Fig. 11). Again we kept fixed $l = b = 1$ in these figures. Note that as in the case of black hole solutions in Einstein-Maxwell theory, the solutions are unstable for small values of electric charge (see Figs. 12 and 13). That is the electric charge increases the stable phase of the system. Besides, as the non-linearity of the electromagnetic field increases ($\beta$ decreases), the stable phase of the solution decreases. That is the non-linearity of the electromagnetic field makes the solutions more unstable. This fact may be seen in figures 14 and 15 which show that the solutions are unstable for very highly nonlinear field (small value of the Born-Infeld parameter).

V. SUMMARY AND CONCLUSIONS

The construction and analysis of topological black holes in AdS space is a subject of much recent attention. This is primarily due to their relevance for the AdS/CFT correspondence. In particular, they allow us to study the dual conformal field theory on spaces of the form $S^1 \times M^{d-2}$, where $M^{d-2}$ is an Einstein space of positive, zero, or negative curvature [38, 54]. Apart the motivation comes from AdS/CFT side, the Born-Infeld lagrangian coupled to a dilaton field appears very
frequently in string theory. Therefore it is of great importance to investigate various properties of the topological black hole solutions in this theory. In this paper, we constructed a new class of \((n+1)\)-dimensional charged topological black hole solutions of Einstein-Born-Infeld-dilaton action. In contrast to the topological black holes in the Einstein-Maxwell theory, which are asymptotically AdS, the topological dilaton black holes we found here, are neither asymptotically flat nor (A)dS. Indeed, the Liouville-type potentials (the negative effective cosmological constant) play a crucial role in the existence of these black hole solutions, as the negative cosmological constant does in the Einstein-Maxwell theory. In the \(k = \pm 1\) cases, these solutions do not exist for the string case where \(\alpha = 1\). They are also ill-defined for \(\alpha = \sqrt{n}\). In the absence of a dilaton field \((\alpha = 0 = \gamma)\), our solutions reduce to the \((n+1)\)-dimensional topological Born-Infeld black hole solutions presented in [12], while in the limit \(\beta \to \infty\) they reduce to the topological black holes in Einstein-Maxwell-dilaton gravity [47]. We computed the conserved and thermodynamic quantities of the solutions and verified that these quantities satisfy the first law of black hole thermodynamics. Finally, we analyzed the thermal stability of these black holes and disclosed the effect of the dilaton and Born-Infeld fields on the stability of the solutions.

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