LOWER BOUNDS FOR THE CENTERED HARDY-LITTLEWOOD
MAXIMAL OPERATOR ON THE REAL LINE

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Abstract. Let $1 < p < \infty$. We prove that there exists an $\varepsilon_p > 0$ such that for each $f \in L^p(\mathbb{R})$, the centered Hardy-Littlewood maximal operator $M$ on $\mathbb{R}$ satisfies the lower bound $\|Mf\|_{L^p(\mathbb{R})} \geq (1 + \varepsilon_p)\|f\|_{L^p(\mathbb{R})}$.

1. Introduction

Given a locally integrable real-valued function $f$ on $\mathbb{R}^d$ define its uncentered maximal function $M_{u}f(x)$ as

$$M_{u}f(x) = \sup_{B \ni x} \frac{1}{|B|} \int_{B} |f(y)| dy,$$

where the supremum is taken over all balls $B \in \mathbb{R}^d$ containing the point $x$; here $|B|$ denotes the $d$-dimensional Lebesgue measure of the ball $B$. The usefulness of this and other maximal functions comes from the fact that they are larger than the original function $f$, but not much larger, and usually improve regularity. Since $M_{u}f$ is often used as a close upper bound for $f$, it is interesting to know precisely how much larger $M_{u}f$ is, and the same question can be asked about other maximal operators.

It is well known that $M_{u}f(x) \geq f(x)$ a.e. On the other hand, since an average does not exceed a supremum, $\|M_{u}f\|_{L^{\infty}(\mathbb{R}^d)} = \|f\|_{L^{\infty}(\mathbb{R}^d)}$. It is shown in [7] that $M_{u}$ has no nonconstant fixed points. In [6] A. Lerner studied whether given any $1 < p < \infty$, there is a constant $\varepsilon_{p,d} > 0$ such that

$$\|M_{u}f\|_{L^p(\mathbb{R}^d)} \geq (1 + \varepsilon_{p,d})\|f\|_{L^p(\mathbb{R}^d)} \quad \text{for all } f \in L^p(\mathbb{R}^d).$$

We note that lack of existence of nonconstant fixed points does not imply (1). Using Riesz’s sunrise lemma, Lerner proved for the real line that

$$\|M_{u}f\|_{L^p(\mathbb{R})} \geq \left( \frac{p}{p-1} \right)^{1/p} \|f\|_{L^p(\mathbb{R})}.$$

A proof of inequality (1) for every dimension $d \geq 1$ and every $1 < p < \infty$ was obtained in [3]. Inequality (1) has been shown to be true for other maximal functions, say, maximal functions defined taking the supremum over shifts and dilates of a fixed centrally symmetric...
convex body, maximal functions defined over \(\lambda\)-dense family of sets, almost centered maximal functions (see [3]) and dyadic maximal functions [8].

For the centered maximal function

\[ Mf(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)|\,dy, \]

Lerner’s inequality

\[ \|Mf\|_{L^p(\mathbb{R}^d)} \geq (1 + \varepsilon_{p,d})\|f\|_{L^p(\mathbb{R}^d)} \quad \text{for all } f \in L^p(\mathbb{R}^d), \]

need not hold. First of all, it was shown in [5] that \(M\) has a nonconstant fixed point \(f \in L^p(\mathbb{R}^d)\) if and only if \(d \geq 3\) and \(p > d/(d - 2)\). But, as was noted before, the lack of nonconstant fixed points does not imply (2). In this context, Ivanisvili and Zbarsky (cf. [4]) noted that (2) is valid for any \(d\) when \(p \equiv p_d\) is sufficiently close to 1.

The main result in [4] proves for \(d = 1\) and every \(1 < p < 2\) that (2) is true, in the form

\[ \|Mf\|_{L^p(\mathbb{R})} \geq \left( \frac{p}{2(p-1)} \right)^{1/p} \|f\|_{L^p(\mathbb{R})}. \]

They also proved that inequality (2) holds for \(d = 1\) and \(1 < p < \infty\), if we restrict \(f\) to the class of indicator functions or unimodal functions. Besides, they conjectured (see [4, p. 343]) that (2) is valid for \(d = 1\) and \(1 < p < \infty\) without restrictions on the functions.

In this paper we give an affirmative answer to their conjecture, proving the following

**Theorem 1.1.** Let \(1 < p < \infty\). Then there exists an \(\varepsilon_p > 0\) such that

\[ \|Mf\|_{L^p(\mathbb{R})} \geq (1 + \varepsilon_p)\|f\|_{L^p(\mathbb{R})} \quad \text{for any } f \in L^p(\mathbb{R}). \]

Furthermore, if \(A_p\) is the best constant for the strong \((p, p)\) inequality satisfied by the centered maximal operator on the real line, and \(\gamma_n\) is as in Definition [2.4], then for every \(n \geq 1\) we can select

\[ (1 + \varepsilon_p)^p = 1 + \left( \frac{A_p - 1}{A^n_p - 1} \right)^p \left( \frac{\gamma_n p}{(p-1)} \right)^{1/p} - 1 \]

Let us note that this expression is strictly larger that 1 if we suitably choose \(n\), taking into account that \(\gamma_n \uparrow 1\) (see Remark 2.5).

Our approach consists of extending the methods in [4] and using the following inequality (see Lemma 2.7 below) for any locally integrable function in \(\mathbb{R}\):

\[ M^n f \geq \gamma_n M_L f, \]

where \(M_L\) denotes the left maximal operator and \(M^n\) denotes the iteration of the centered maximal operator \(n\) times. This inequality extends the trivial inequality \(Mf \geq M_L f/2\).

Using (4), we prove

**Theorem 1.2.** Let \(n \in \mathbb{N}\) and \(f \in L^p(\mathbb{R})\). Then,

\[ \|M^n f\|_p \geq \left( \frac{\gamma_n p}{(p-1)} \right)^{1/p} \|f\|_p. \]
Since $\gamma_1 = 1/2$, this result is an extension of (3).

Let us remark that simultaneously and independently, Zbarsky [9] has proved (2) for $d = 1$ and $d = 2$ and the centered maximal operator associated to centrally symmetric convex bodies. This extends Theorem 1.1 but without an explicit expression for the lower constant $\varepsilon_p$.

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2. Definitions and lemmas

Definition 2.1. For all $n \in \mathbb{N} \cup \{0\}$, define the following functions $g_n : [-1/2, \infty) \rightarrow [0, 1]$. Let $g_0$ be the null function and for $n \geq 1$, set

$$g_n(t) := \frac{1 + \int_0^{1+2t} g_{n-1}(u) du}{2(1+t)}, \quad t \geq -\frac{1}{2}.$$  

In the next lemma we give an explicit formula for the functions $g_n$.

Lemma 2.2. Let $\{g_n\}_{n=0}^{\infty}$ be the functions from Definition 2.1. Then,

1. $0 \leq g_n(t) \leq 1$ for all $n \in \{0\} \cup \mathbb{N}$ and all $t \geq -1/2$.
2. For all $n \geq 0$ and $t \geq -1/2$, we have

$$g_n(t) = \frac{\log(2 + 2t)}{1 + t} \sum_{j=1}^{n} \frac{\log^j(2^j(1+t))}{2^j(j-1)!}.$$  

3. For all $t \geq -1/2$, we have $\lim_{n \to \infty} g_n(t) = 1$.

Proof. Part 1 of the lemma follows by simple induction in $n$. To prove part 2, for each $n \in \{0\} \cup \mathbb{N}$, we define $h_n(t) := g_{n+1}(t) - g_n(t)$. Since $g_0(t) = 0$, it holds that

$$g_n(t) = \sum_{j=0}^{n-1} h_j(t).$$

Let us note that $h_0(t) = g_1(t) - g_0(t) = g_1(t) = 1/(2 + 2t) > 0$. Besides, by (5),

$$h_n(t) = \int_0^{1+2t} h_{n-1}(u) du, \quad n \geq 1, \ t \geq -1/2.$$  

Now we set for each $n \geq 0$,

$$c_n(t) = \frac{\log(2 + 2t)}{1 + t} \frac{1}{2^{n+1}n!} \log^{n-1}(2^{n+1}(1+t)), \quad t \geq -1/2,$$

where $c_0(-1/2)$ is defined by continuity, i.e., $c_0(-1/2) = \lim_{t \to -1/2^+} c_0(t) = \lim_{t \to -1/2^+} 1/(2 + 2t) = 1$. Then we have that $h_0(t) = c_0(t)$ for all $t \in [-1/2, \infty)$. Moreover, it is a calculus exercise to check that (6) also holds with $c_n$ and $c_{n-1}$ instead of $h_n$ and $h_{n-1}$, for every $n \geq 1$. As a consequence, we have that $c_n(t) = h_n(t)$ for all $n \geq 0$ and all $t \in [-1/2, \infty)$. Thus, part 2 of the lemma holds.
Finally we will prove part 3 of the lemma. By part 2, we have that
\[
\lim_{n \to \infty} g_n(t) = \frac{\log(2 + 2t)}{1 + t} \sum_{j=1}^{\infty} \frac{\log^j(2^j(1 + t))}{2^j(j - 1)!}, \quad t \geq -1/2.
\]

As a consequence of Lagrange expansion \([1, p.206, eq.6.24]\), we can obtain
\[
e^{xy} = \sum_{k=0}^{\infty} x(x + k)z^{k-1} \left(\frac{ye^{-yz}}{k!}\right)^k, \quad x, y, z \in \mathbb{R}.
\]
This equation with \(y = 1, x = \log(2 + 2t)\) and \(z = \log 2\) implies
\[
2 + 2t = \sum_{k=0}^{\infty} \log(2 + 2t)(\log(2 + 2t) + k \log 2)^{k-1} \frac{2^{-k}}{k!}.
\]
From this equation and (7), part 3 of the lemma follows. \(\square\)

**Remark 2.3.** Part 3 of the lemma could also be proved by suitably bounding the functions \(g_n\), using an argument inspired in \([4, p.4-5]\) to obtain:
\[
1 \geq g_n(t) \geq 1 - \left(\frac{\sqrt{8}}{3}\right)^n \sqrt{1 + t}, \quad n \in \mathbb{N}, t \geq 0.
\]

**Definition 2.4.** For each \(n \in \mathbb{N}\), let us denote by \(\gamma_n := g_n(0)\), where \(g_n\) are the functions from Definition 2.1.

**Remark 2.5.** It follows from the previous definition and Lemma 2.2 that
\[
\gamma_n = \frac{1}{2} \sum_{j=1}^{n} \frac{j^{-2}}{(j - 1)!} \left(\frac{\log 2}{2}\right)^{j-1}, \quad n \in \mathbb{N}.
\]
Besides, \(\gamma_1 = 1/2\) and \(\gamma_n\) increases to 1 when \(n \to \infty\).

**Definition 2.6.** Let \(f : \mathbb{R} \to \mathbb{R}\) be a locally integrable function. We define the left maximal function \(M_L f\) as
\[
M_L f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f(u)|du, \quad x \in \mathbb{R}.
\]
It is easy to see that \(Mf \geq M_L f/2\). In the next lemma we extend this inequality to the iterated centered maximal operator, defined via \(M^1 f := Mf\), and for \(n \geq 2\), \(M^n f := M(M^{n-1} f)\).

**Lemma 2.7.** Let \(\{\gamma_n\}_{n=1}^{\infty}\) be the sequence from Definition 2.4. Then, for all \(n \in \mathbb{N}\) and all \(f \in L^1_{loc}(\mathbb{R})\), \(M^n f \geq \gamma_n M_L f\).

**Proof.** Let us assume that \(f \geq 0\). Fix \(x \in \mathbb{R}\) and \(h > 0\). Define \(F(x, h) := \frac{1}{h} \int_{x-h}^{x} f(t)dt\). Now, using an inductive process in \(n \in \mathbb{N}\), we will prove that for all \(y \geq x\),
\[
M^n f(y) \geq F(x, h) g_n \left(\frac{y - x}{h}\right),
\]
where \( g_n \) comes from Definition \([2.1]\) Indeed, for \( n = 1 \) and every \( y \geq x \), we have
\[
Mf(y) \geq \frac{1}{2(y-x+h)} \int_{x-h}^{2y-x+h} f(t)dt \geq \frac{hF(x,h)}{2(y-x+h)} = \frac{F(x,h)}{2(1 + \frac{y-x}{h})} = F(x,h)g_1\left(\frac{y-x}{h}\right).
\]
Hence, by induction hypothesis, for all \( n \geq 2 \) and all \( y \geq x \),
\[
M^n f(y) \geq \frac{1}{2(y-x+h)} \int_{x-h}^{2y-x+h} M^{n-1} f(t)dt \geq \frac{hF(x,h) + F(x,h) \int_x^{2y-x+h} g_{n-1}\left(\frac{1-x}{h}\right)dt}{2(y-x+h)} = \frac{hF(x,h) + \int_0^{1+2y-x/h} g_{n-1}(z)dz}{2(y-x+h)} = \frac{F(x,h) + \int_0^{1+2y-x/h} g_{n-1}(z)dz}{2(1 + \frac{y-x}{h})} = F(x,h)g_n\left(\frac{y-x}{h}\right),
\]
so (9) is proved. As a consequence, \( M^n f(x) \geq F(x,h)g_n(0) = F(x,h)\gamma_n \), and taking the supremum over \( h > 0 \) we obtain
\[
M^n f(x) \geq M_L f(x) \cdot \gamma_n, \quad n \in \mathbb{N}.
\]
\[\square\]

**Remark 2.8.** It is known that for every \( f \in L^p(\mathbb{R}) \), we have \( \|M_L f\|_p \geq \left(\frac{p}{p-1}\right)^{1/p} \|f\|_p \) (see \([2 \ p.93, 2.1.11(a)]\) and integrate). This inequality, together with the previous lemma, leads immediately to
\[
\|M^n f\|_p \geq \gamma_n \|M_L f\|_p \geq \gamma_n \left(\frac{p}{p-1}\right)^{1/p} \|f\|_p.
\]
This inequality is enough to prove Theorem \([1.1] \) but with a smaller \( \varepsilon_p \). Indeed, inequality (10) will be improved in Theorem \([1.2] \) by the use of the following lemma, which is an extension of \([4 \ Lemma 3]\) and uses the same arguments. We include it here for the reader’s convenience.

**Lemma 2.9.** Let \( 0 < \lambda < \infty \) and \( n \in \mathbb{N} \). For every locally integrable function \( f \geq 0 \) defined on the real line, it holds that
\[
|\{M^n f > \lambda\}| \geq \frac{\gamma_n}{\lambda} \int_{\{f > \lambda\}} f.
\]

**Proof.** Since \( M^n f \geq f \) almost everywhere and, by Lemma \([2.7] \) \( M^n f \geq \gamma_n M_L f \), we have (with the exception of a null set) that
\[
\{M^n f > \lambda\} \supseteq \{f > \lambda\} \cup \{M_L f > \frac{\lambda}{\gamma_n}\}.
\]
Then, we separate this into two disjoint sets, take Lebesgue measure, apply \( M_L f \geq f \) a.e. and using Riesz’s rising sun lemma \([2 \ p.93] \) we obtain,
\[
|\{M^n f > \lambda\}| \geq |\{f > \lambda\} \setminus \{M_L f > \frac{\lambda}{\gamma_n}\}| + |\{M_L f > \frac{\lambda}{\gamma_n}\}| \geq
\]
\[
\geq \frac{\gamma_n}{\lambda} \int_{\{f > \lambda\} \setminus \{M_L f > \frac{\lambda}{\gamma_n}\}} f + \frac{\gamma_n}{\lambda} \int_{\{M_L f > \frac{\lambda}{\gamma_n}\}} f \geq \frac{\gamma_n}{\lambda} \int_{\{f > \lambda\} \cup \{M_L f > \frac{\lambda}{\gamma_n}\}} f \geq \frac{\gamma_n}{\lambda} \int_{\{f > \lambda\}} f. \]

\[
\□ \quad 3. \text{Proofs of the theorems}
\]

To prove the theorems one just has to use the previous lemmas and some arguments from [4]. We include here the proofs for the reader’s convenience.

**Proof of Theorem 1.2.** Without loss of generality we assume that \( f \geq 0 \). By Lemma 2.9 we have:

\[
|\{M^n f > \lambda\}| \geq \frac{\gamma_n}{\lambda} \int_{\mathbb{R}} f(x) \chi_{(\lambda, \infty)}(f(x)) dx.
\]

We multiply both sides of the previous inequality by \( p \lambda^{p-1} \) and integrate:

\[
\int_{\mathbb{R}} (M^n f)(x) dx \geq \int_{0}^{\infty} \gamma_n p \lambda^{p-2} \int_{\mathbb{R}} f(x) \chi_{(\lambda, \infty)}(f(x)) dxd\lambda = \gamma_n p \int_{\mathbb{R}} f(x) \int_{0}^{f(x)} \lambda^{p-2} d\lambda dx = \frac{\gamma_n p}{(p-1)} \int_{\mathbb{R}} f(x)^p dx.
\]

**Proof of Theorem 1.1.** First, we have that

\[
(11) \quad \|Mf\|_p^p \geq \|f\|_p^p + \|Mf - f\|_p^p.
\]

Besides, if we denote by \( A_p > 1 \) the best constant for the strong \((p, p)\) inequality satisfied by \( M \), it holds that

\[
(12) \quad \|M_n f - f\|_p \leq \sum_{i=1}^{n} M_i f - M_i^{i-1} f\|_p \leq \sum_{i=1}^{n} A_i^{i-1} \|M f - f\|_p = \frac{A_p^n - 1}{A_p - 1} \|M f - f\|_p.
\]

Furthermore, by Theorem 1.2

\[
\left( \frac{\gamma_n p}{(p-1)} \right)^{1/p} \|f\|_p \leq \|M^n f\|_p \leq \|M^n f - f\|_p + \|f\|_p.
\]

Then

\[
(13) \quad \left[ \left( \frac{\gamma_n p}{(p-1)} \right)^{1/p} - 1 \right] \|f\|_p \leq \|M^n f - f\|_p.
\]

Now, putting (11), (12) and (13) together we get

\[
\|Mf\|_p^p \geq \|f\|_p^p + \left( \frac{A_p - 1}{A_p^n - 1} \right)^p \|M^n f - f\|_p^p \geq \|f\|_p^p + \left( \frac{A_p - 1}{A_p^n - 1} \right)^p \left[ \left( \frac{\gamma_n p}{(p-1)} \right)^{1/p} - 1 \right]^p \|f\|_p^p =
\]
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\[
\|f\|_p^p \left\{ 1 + \left( \frac{\frac{1}{p} - 1}{\frac{1}{p} - 1} \right) \left[ \left( \frac{\gamma_n}{(p-1)} \right)^{1/p} - 1 \right]^p \right\}.
\]

Let us note that, by Remark 2.5, for \( n \) big enough, \( \gamma_n/(p-1) > 1 \).

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