TAKENS’ LAST PROBLEM AND EXISTENCE OF NON-TRIVIAL WANDERING DOMAINS

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ABSTRACT. In this paper, we give an answer to a version of the open problem of Takens in [35] which is related to historic behavior of dynamical systems. To obtain the answer, we show the existence of non-trivial wandering domains near a homoclinic tangency, which is conjectured by Colli and Vargas [6, §2]. Concretely speaking, it is proved that any Newhouse open set in the $C^r$ topology of two-dimensional diffeomorphisms with $r \geq 2$ is contained in the closure of the set of diffeomorphisms which have non-trivial wandering domains whose forward orbits have historic behavior. Moreover, this result implies an answer in the $C^r$ category to one of the open problems of van Strien [33] which is concerned with wandering domains for Hénon family.

1. Introduction

1.1. Historic behavior and wandering domain. Consider a dynamical system with a compact state space $X$, given by a continuous map $\varphi : X \to X$. We say that the orbit \( \{ x, \varphi(x), \varphi^2(x), \ldots \} \) of $x \in X$ has historic behavior if the partial average

$$\frac{1}{m + 1} \sum_{i=0}^{m} \delta_{\varphi^i(x)}$$

does not converge as $m \to \infty$ in the weak topology, where $\delta_{\varphi^i(x)}$ is the Dirac measure on $X$ supported at $\varphi^i(x)$. The terminology of historic behavior was given by Ruelle in [31]. The following is the last open problem presented by Floris Takens (see [35]).

Takens’ Last Problem. Whether are there persistent classes of smooth dynamical systems such that the set of initial states which give rise to orbits with historic behavior has positive Lebesgue measure?

The first example of historic behavior was given in [15], where it is shown that the logistic family contains elements for which almost all orbits have historic behavior. This was extended to generic full families of unimodal maps, see [8]. While Takens showed in [36] that Bowen’s 2-dimensional flow with an attracting heteroclinic loop has a set of positive Lebesgue measure consisting of initial points of orbits with historic behavior, but this example is not persistent. Moreover he presented an idea using Dowker’s result [10] to show that the doubling map on the circle persistently has orbits with historic behavior, for which the collection of initial points is a residual subset on the circle, see [35].

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In this paper, we obtain an answer to Takens’ Last Problem for non-hyperbolic diffeomorphisms having homoclinic tangencies by a different way from the previous works. To solve the problem we use an open set, called a \textit{non-trivial wandering domain}, whose images do not intersect each other but are wandering around non-attracting invariant sets. Wandering domains were studied in advance of Smale’s perspective on hyperbolic dynamical systems from the end of 60s to the beginning of 70s. In fact, Bohl \cite{bohl1916} in 1916 and Denjoy \cite{denjoy1932} in 1932 constructed examples of $C^1$ diffeomorphisms on the circle with any irrational rotation number which have wandering domains whose $\omega$-limit set is a Cantor set. However, it can not be extend to any $C^2$ as well as $C^1$ diffeomorphism whose derivative is a function of bounded variation, see in \cite{kuusi1955}. Subsequently, similar phenomena for high dimensional diffeomorphisms were studied by several authors, for example \cite{guckenheimer1973, keller1975, keller1973, keller1974}. Same as Denjoy’s example, their results were not observed in diffeomorphisms with higher differentiability. Also, for unimodal as well as multimodal maps on an interval or a circle, the main difficulty in the their classification in real analytic category was to show the absence of wandering domains, which were developed by many dynamicists \cite{guckenheimer1973, lecalvez1991, katok1980, bui1998, kaloshin1998}, see the survey of van Strien \cite{vansrien1998}.

On the other hand, a wide variety of investigations derived from Smale’s perspective yielded abundant developments, and provided a focal point for us to explore beyond hyperbolic phenomena. Thus, we here focus entirely on one of non-hyperbolic phenomena called homoclinic tangencies, which were pioneered by Newhouse, Palis, Takens and others. It is somewhat surprising that homoclinic tangencies and wandering domains were not studied together until E. Colli and E. Vargas presented an essential example in \cite{colli2017}. We furthermore discuss these two themes in more general situation to solve Takens’ Last Problem.

To state our results precisely we need to introduce some definitions. Let $M$ be a closed two-dimensional $C^\infty$ manifold and $\text{Diff}^r(M)$, $r \geq 2$, the set of $C^r$ diffeomorphisms on $M$ endowed with $C^r$ topology. We say that $f \in \text{Diff}^r(M)$ has a \textit{homoclinic tangency} of a saddle periodic point $p$ if the stable manifold $W^s(p, f)$ and unstable manifold $W^u(p, f)$ have a non-empty and non-transversal intersection. Newhouse showed in \cite{newhouse1978} that, for any $C^2$ neighborhood of $f$ having a homoclinic tangency, there is an open set $\mathcal{N} \subset \text{Diff}^2(M)$ whose closure $\text{Cl}(\mathcal{N})$ contains $f$ such that any element in $\mathcal{N}$ is approximated by a diffeomorphism with a homoclinic tangency associated with the continuation of $p$, and moreover the element has a $C^2$-persistent tangency associated with some hyperbolic basic sets. The open set $\mathcal{N}$ is called a \textit{Newhouse open set}. In $\mathcal{N}$, various non-hyperbolic phenomena were observed but still far from being completely understood. For example, Newhouse first showed in \cite{newhouse1978} that generic elements of $\mathcal{N}$ have infinitely many sinks or sources. Moreover, $\mathcal{N}$ contains a dense subset of diffeomorphisms exhibiting infinitely many Hénon-like attractors or repellers \cite{nicol1997}. Kaloshin proved in \cite{kaloshin2017} that the number of periodic points for diffeomorphisms in a residual subset of $\mathcal{N}$ grows super-exponentially. As applications to the (original) Hénon family, there exist open subsets of its parameter space each element of which is an Hénon map contained in $\mathcal{N}$ \cite{nicol1997}, and admit homoclinic classes without SRB measure and strange attractors with SRB measure simultaneously \cite{nicol1998}. In \cite{newhouse1978}, we also presented a subfamily of Hénon maps which have generically unfolding cubic homoclinic tangency. Consequentially, the subfamily exhibits persistent anti-monotonic tangencies and cubic polynomial-like strange attractors.
In this paper, we will show that any two-dimensional $C^r$ diffeomorphism belonging to any Newhouse open set is characterized by some wandering domains. In [8], de Melo and van Strien defined that, for a circle diffeomorphism $f$, an open interval in $S^1$ is a non-trivial wandering domain if

- $D, f(D), f^2(D), \ldots$ are pairwise disjoint;
- the $\omega$-limit set of $D$ is not equal to a single periodic orbit,

where the former condition expresses the property of wandering, and the latter non-triviality. Note that the wandering domain of Denjoy’s example satisfies these properties. However, it might not be enough for the case of dimension greater than one, since one has a trivial two-dimensional example: any open set in the basin of an attracting circle for a $C^2$-diffeomorphism which has an irrational rotation, see in Section 2 of [6]. Thus, we need to define non-triviality for wandering domains in two-dimensional dynamics so as to exclude trivial examples. For this purpose, we observe trivial examples of domains $D$ for two-dimensional diffeomorphisms $f$ which satisfy

\[(1.1) \quad f^i(D) \cap f^j(D) = \emptyset\]

for any $i, j \in \mathbb{Z}$ with $i \neq j$, as follows.

1. If $D$ is a sufficiently small domain contained in the basin of a periodic sink for $f$, then $D$ satisfies the condition (1.1). Even if $D$ is contained in the immediate basin of a dissipative saddle-node periodic point, it satisfies the condition (1.1). For each case, the diameter of $f^n(D)$ converges to zero as $n \to \infty$.

2. Consider a classical structurally stable diffeomorphism $f$ on $S^2$ for which the chain recurrent set is comprised of Smale’s affine horseshoe $\Lambda = \bigcap_{i \in \mathbb{Z}} f^i(S)$ and a sink $p_0$ and a source $p_\infty$ in the complement of $S$, where $S$ is the unit square.

   2a. Same as in (1), if $D$ belongs to the basin of $p_0$, then $D$ satisfies the condition (1.1) and the diameter of $f^n(D)$ converges to zero as $n \to \infty$.

   2b. On the other hand, if $D$ is the interior of any vertical component $V$ of $S \setminus f(S)$, then $D$ is a domain with the condition (1.1) whose $\omega$-limit set is equal to $\Lambda \cup \{p_0\}$, and the diameter of $f^n(D)$ converges to some positive constant as $n \to \infty$.

Such trivial examples are not the target for our study. Colli and Vargas described some “trivial class” for wandering domains in some immediate basin of a weak attractor in [6]. Nevertheless, if one follows it, the above 2b) is categorized into the “non-trivial class” since the $\omega$-limit set in 2b) is not dynamically connected which is one of their hypotheses for weak attractors. (However, it does not give any damage to their results in [6] and their ideas are essential for the present paper.) So, we here give a simpler description of non-triviality for wandering domains which is also a natural extension of that of de Melo and van Strien. A subset $\Lambda$ of a surface $M$ is called a basic set if it is a compact hyperbolic and locally maximal invariant set which is transitive and contains a dense subset of periodic orbits. The basic set $\Lambda$ is non-trivial if it is not just a periodic orbit. In this paper, a nonempty connected open set $D$ in $M$ is called a non-trivial wandering domain for $f \in \text{Diff}^r(M)$ if

- $f^i(D) \cap f^j(D) = \emptyset$ for any $i, j \in \mathbb{Z}$ with $i \neq j$;
- there is a non-attracting and non-trivial basic set $\Lambda$ such that, for any $x \in D$, the $\omega$-limit set $\omega(x)$ contains $\Lambda$.
A wandering domain $D$ is called **contracting** if the diameter of $f^n(D)$ converges to zero as $n \to \infty$. Note that the contracting wandering domain detected in Colli-Vargas [6] is non-trivial also in the sense of our definition as above.

### 1.2. Main results.

Colli and Vargas presented a simple non-hyperbolic model with a wandering domain, which will be revisited in the next section. Their wandering domain is detected for a two-dimensional diffeomorphism which has an affine horseshoe whose stable and unstable manifolds have persistent tangencies with countable many but arbitrarily small perturbations. They moreover expressed a strong belief in the existence of non-trivial wandering domains in more general situations, see the end of [6, Section 2].

We now state the main result of this paper where the conjecture of Colli and Vargas is true in $C^r$ topology and a solution to Takens’ Last Problem will be obtained using some non-trivial wandering domains.

**Theorem A.** For any closed surface $M$ and any integer $r$ with $2 \leq r < \infty$, any Newhouse open set of $\text{Diff}^r(M)$ is contained in the closure of the set of diffeomorphisms $f$ of $\text{Diff}^r(M)$ having contracting non-trivial wandering domains $D$ such that, for any $x \in D$, the forward orbit of $x$ under $f$ has historic behavior.

Note that any wandering domain is an open set, and hence in particular $D$ has positive Lebesgue measure, which is the condition required in Takens’ Last Problem.

Next we consider the Hénon family $f_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$f_{a,b}(x,y) = (1-ax^2 + y, bx),$$

where $a, b$ are real parameters. This family will play a significant role in the renormalization near the homoclinic tangency. As $b = 0$, the dynamics of $f_{a,0}$ is perfectly controlled by the family of quadratic maps $\varphi_a(x) = 1 - ax^2$. It is known that there is a parameter value of $a$ such that there is a $C^1$ unimodal map which is semi-conjugated to $\varphi_a$ and has a non-trivial wandering interval [13]. Moreover, under the sufficient differentiability, some large class of multimodal maps including the quadratic maps cannot have non-trivial wandering intervals, see [34]. However, as $b \neq 0$, it has not been known whether or not $f_{a,b}$ has non-trivial wandering domains, which is one of open problems in [33] [23]. We can answer such a problem in the $C^r$ category with $2 \leq r < \infty$ as in the following corollary of Theorem A together with the fact that there exists $(a, b)$ arbitrarily close to $(2, 0)$ such that $f_{a,b}$ has a quadratic homoclinic tangency for some saddle fixed point which unfolds generically with respect to $a$, see for example in [18] [20].

**Corollary B.** There is an open set $O$ of the parameter space of Hénon family with $\text{Cl}(O) \ni (2,0)$ such that, for every $(a,b) \in O$, $f_{a,b}$ is $C^r$-approximated by diffeomorphisms which have contracting non-trivial wandering domains.

In the end of this section we recall that non-trivial wandering phenomena are observable in circle homeomorphisms in the $C^\infty$ category by Hall [12] but not in the $C^\omega$ category by Yoccoz [37], which is the answer to one of problems by Poincaré [28]. Note that every discussion of the present paper unfolds in the $C^r$ category with $2 \leq r < \infty$, but some tools would not be applied directly to discussion in the $C^\infty$ as well as $C^\omega$ category. See Remark 7.7(2). Thus, the open problem for
wandering domains of the Hénon family by van Strien et al. is unsolved yet in $C^\infty$ and $C^\omega$ categories. So it is worth recalling the following:

**Question.** Does there exist a parameter value $(a, b)$ for the Hénon family (1.2) such that $f_{a,b}$ has a non-trivial wandering domain?

Note that Astong et al. [1] study the existence of wandering Fatou components for polynomial skew-product maps and present an example which admits a wandering Fatou component intersecting $\mathbb{R}^2$. However, it does not contain the Hénon family.

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2. Motivation and outline of the paper

We have endeavored to make self-contained expositions bringing together all topics related to our results in the subsequent sections. Before that, we give the motivated example presented by Colli and Vargas, and an outline of our paper.

2.1. Colli-Vargas model revisited. As mentioned in the preceding subsection, our study is much inspired by the work of Colli-Vargas [6]. Their ingredients containing key results called Linking Lemma, Linear Growth Lemma, Critical Chain Lemma and Rectangles Lemma are all described for a model having a thick affine horseshoe with persistent homoclinic tangencies. Note that this simple model not technical but essential in their discussions. So we will extend their ingredients and develop the theory to more general situations in this paper. Still it may be useful here to recall the simple model and ideas by Colli-Vargas to understand our discussion. But the reader may skip this subsection, if he/she knows the result of Colli-Vargas.

Let $D$ be a disk on $\mathbb{R}^2$ defined as $D = D_0 \cup D_+ \cup D_-$ where $D_0 = [0,1]^2$ and $D_\pm$ are two hemi-disks satisfying $D_\pm \cap D_0 = [-1,1] \times \{1\}$, respectively. We consider Smale’s horseshoe diffeomorphism $g$ satisfying

- $g(D_0) \subset D_0$ and $g(D_+) \subset \text{Int}(D_0)$ and $g(D_+) \cap D_+ = \emptyset$;
- $g|_{D_-}$ is a contraction;
- $g|_{S_\pm}$ is an affine map in neighborhoods of the horizontal strips $S_\pm = [-1,1] \times [\pm 2^{-1} - \sigma^{-1}, \pm 2^{-1} + \sigma^{-1}]$ for some $\sigma > 2$ defined by
  \[
g|_{S_\pm}(x,y) = \left( \mp \lambda x \pm 2^{-1}, \mp \sigma(y \mp 2^{-1}) \right),\]
  where $\lambda$ is a constant satisfying $0 < \lambda < \sigma^{-1}$;
- $g(S_- \cup S_+) \subset D_-$ and $g(S_0) \subset D_+$, where $S_-, S_0, S_+$ are the horizontal components of $D_0 \setminus (S_- \cup S_+)$.

From the definition, $g$ has the affine horseshoe $\Lambda = \bigcap_{i \in \mathbb{Z}} g^i(D_0)$ with two fixed points in $S_\pm \cap \Lambda$, respectively. Denote by $p = (-x_0, -y_0)$ a unique fixed point of $g$ which lies in the third quadrant. See Figure 2.1(a).

By the local perturbation $\Psi_\mu$ given in [27, Section 5.1], one obtains a one-parameter family $g_\mu := \Psi_\mu \circ g$, $\mu \in [-1,1]$ which has the following properties:

- as $\mu = -1$, $g_{-1} = g;$
as $\mu$ close to 0, for any $(x, y)$ in a small neighborhood $U$ of $(-x_0, 0)$,
\[
g_2^\mu(x, y) = (\alpha y, -y_0 + \mu - \beta y^2 - \gamma(x + 1))
\]
for some constants $\alpha, \beta, \gamma > 0$.

Note that, the affine horseshoe $A$ is not affected by the perturbation. That is, for every $\mu \in [-1, 1]$, $g_\mu$ has the same affine horseshoe $A$ which is equal to the product of two Cantor sets $K^u$ and $K^s$ in $[-1, 1]$. Moreover, as $\mu = 0$, the first homoclinic tangency of $W^s(p, g_\mu)$ and $W^u(p, g_\mu)$ unfolds generically. Let $F_{loc}^u$ and $F_{loc}^s$ be local foliations compatible to $W_{loc}^u(A) = K^s \times [-y_0, y_0]$ and $W_{loc}^s(A) = [-x_0, x_0] \times K^u$, respectively. From the definition of $g_\mu$, all tangencies between $F_{loc}^u$ and $g_\mu(F_{loc}^s)$ are contained in the vertical line $\{x = 0\}$. Thus, the intersection of $F_{loc}^u$ with the vertical line is a part of $K^u$, while the intersection of $g_\mu(F_{loc}^s)$ with the same line is a part of $\tilde{K}^u = -y_0 + \mu - \gamma(K^u + 1)$. Under the open condition $(1 - 2\lambda)(\sigma - 2) < \lambda$, one has
\[
\tau(K^s)\tau(\tilde{K}^u) > 1,
\]
where $\tau$ represents the thickness of the corresponding Cantor set. See Section 3 for the definition of the thickness of Cantor sets. Thus, from Gap lemma 2.16, for any small $\mu > 0$, $g_\mu^2(W_{loc}^u(A))$ and $W_{loc}^s(A)$ have a $C^2$-persistent homoclinic tangency contained the vertical line $\{x = 0\}$. So far this is a standard setting for wild hyperbolic sets.

To obtain the Colli-Vargas model from $g_\mu$, we first observe bridges and gaps for the Cantor sets. Let $H_-$ and $H_+$ be the two horizontal components of $g(R) \cap R$, where $R = [-x_0, x_0] \times [-y_0, y_0]$. For a $j \in \mathbb{N}$, one writes $z_j := \pm i$ if $g^j(x, y) \in H_\pm$, respectively. For any $k \in \mathbb{N}$, the strips $\mathbb{B}_k^u$ and $\mathbb{B}_k^s$ are respectively defined as
\[
\mathbb{B}_k^u = \mathbb{B}^u(k; z_1 \ldots z_k) = \{(x, y) \in D_0 : g^{-j}(x, y) \in H_{z_j}, j = 1, \ldots, k\}; \quad \mathbb{B}_k^s = \mathbb{B}^s(k; z_1 \ldots z_k) = \{(x, y) \in D_0 : g^{j-1}(x, y) \in H_{z_j}, j = 1, \ldots, k\}.
\]
Moreover, the maximal horizontal strip inside $\mathbb{B}^u(k_1 \ldots z_k)$ and lying between $\mathbb{B}^u(k+1; z_1 \ldots z_k)$ is denoted by $G_k^u = G^u(k; z_1 \ldots z_k)$, while the maximal vertical strip inside $\mathbb{B}^s(k_1 \ldots z_k)$ and lying between $\mathbb{B}^s(k+1; z_1 \ldots z_k)$ is denoted by $G_k^s = G^s(k; z_1 \ldots z_k)$. Let $\pi^u$ be the orthogonal projection to the horizontal segment $\{y = -y_0\} \cap W^u(p)$ and $\pi^s$ the orthogonal projection to the vertical segment $\{x = -x_0\} \cap W^u(p)$. Using the projections, we have stable bridge and stable gap for the Cantor set $K^s$ defined as
\[
B^s_k := \pi^u(\mathbb{B}^s(k; z_1 \ldots z_k)), \quad G^s_k := \pi^u(G^s(k; z_1 \ldots z_k)),
\]
and unstable ones for $K^u$ defined as
\[
B^u_k := \pi^u(\mathbb{B}^u(k; z_1 \ldots z_k)), \quad G^u_k := \pi^u(G^u(k; z_1 \ldots z_k)),
\]
where $k$ is called the generation and $(z_1 \ldots z_k)$ is called the itinerary for the corresponding bridges and gaps. Moreover, since both $\mathbb{B}_k^u$ and $G_k^s$ intersect with $U$ if $k$ is sufficiently large, one has the images of $\mathbb{B}_k^u \cap U$ and $G_k^s \cap U$ by $g_\mu^2$ to the line $\{x = 0\}$ defined as
\[
B_k^u(\mu) = -y_0 + \mu - \gamma(\pi^u(B_k^s) + 1), \quad G_k^u(\mu) = -y_0 + \mu - \gamma(\pi^u(G_k^s) + 1).
\]
We can write $B_k^s(\mu) = B_k^s(\mu) + G_k^s(\mu) = G_k^s(\mu)$, where $B_k^s := B_k^s(0)$ and $G_k^s := G_k^s(0)$. 
Figure 2.1. Linking Lemma and Linear Growth Lemma in \[6\] ensure the existence of a sequence of pairs of unstable and stable bridges which are uniformly linked. To be precise, there exist a constant \(\xi_0 > 0\) depending on \(\tau(K^u)\), a positive value \(\mu_0\) arbitrarily close to 0, integers \(N_u, N_s > 1\) and sequences \(\{\tilde{B}_u^k\}\) of unstable bridges and \(\{\tilde{B}_s^k\}\) of stable bridges such that, for every \(k \geq 1\),

- the pair of \(\tilde{B}_u^k + \mu_0\) is \(\xi_0\)-linked and \(\lambda\)-proportional, i.e.,
  \[\tilde{B}_u^k \cap (\tilde{B}_s^k + \mu_0) \geq \xi_0 \min\{|\tilde{B}_u^k|, |\tilde{B}_s^k|\}, \lambda|\tilde{B}_u^k| < |\tilde{B}_s^k| \leq |\tilde{B}_u^k|;\]
- \(\tilde{n}_k < \tilde{n}_{k+1} \leq \tilde{n}_k + N_u\) and \(\tilde{m}_k < \tilde{m}_{k+1} \leq \tilde{m}_k + N_s\), where \(\tilde{n}_k\) and \(\tilde{m}_k\) are the generations of \(\tilde{B}_u^k\) and \(\tilde{B}_s^k\), respectively.

Using such a sequence of uniformly linking pairs, Colli and Vargas detected an appropriate local perturbation \(h_t\) depending on a perturbation sequence (2.2)
\[
(t_1, t_2, \ldots, t_k, \ldots)
\]
to realize the claim in Critical Chain Lemma of \[6\]: for any integer \(T \gg N_s\), there exist sequences of real numbers \(t_k\) with \(t_1 = 0\) and \(|t_k| < \lambda^T\), \(u\)-bridges \(B_{u_nk}^k \subset \tilde{B}_u^k\), and \(s\)-bridges \(B_{s_nk}^k \subset \tilde{B}_s^k + \mu_0\) with the generation \(n_k = O(k^2)\), such that the center of \(t_{k+1} + B_{u_nk}^k\) coincides with the center of \(B_{u_nk}^k\). See Figure 2.1-(b). Note that for each \(k \geq 1\), \(B_{u_nk}^k\) contains the gap \(G_{n_k}^u\) whose center and itinerary are equal to those of \(B_{u_nk}^k\). It follows that, if we take
\[
f = g_{\mu_0} \circ h_t,
\]
then one has a sequence \(\{R_k\}\) of mutually disjoint rectangles which satisfies the following properties for each \(k \geq 1\):
- \(R_k \subset G_{n_k}^u\) where \(G_{n_k}^u\) is the unstable gap satisfying
  \[G_{n_k}^u = -y_0 + \mu_0 - \gamma(\pi^s(G_{n_k}^u) + 1) + t_k;\]
the center of \( R_k \) is the center \((0, y_k)\) of \( \mathbb{G}^u_{n_k} \); the width of \( R_k \) equals the width of \( \mathbb{G}^u_{n_k} \cap V_0 \) where \( V_0 := R \setminus (g(H_-) \cup g(H_+)) \), and its height equals \( 2\sigma_{n_k} \);

- let \( F_k := f^{n_k+2}|_{R_k} \); then
  \[
  F_k(x, y_k + y) = (\pm \alpha \sigma^{n_k} y, y_{k+1} - \beta \sigma^{2n_k} y^2 \pm \gamma \lambda^{n_k} x),
  \]
  in particular \( F_k(0, y_k) = (0, y_{k+1}) \) which is the center of \( \mathbb{G}^u_{n_k+1} \).

Finally, in Rectangles Lemma of [6], they presented a sequence \( \{ R^*_k \} \) of sub-rectangles \( R^*_k \subset R_k \) with

\[
F_k(\operatorname{Int}(R^*_k)) \subset \operatorname{Int}(R^*_{k+1}).
\]

It implies that \( R^*_1 \) is a wandering domain for \( f \), which is non-trivial in the sense that the union of the \( \omega \)-limit sets of every point of \( R^*_1 \) is contained in the union of \( \Lambda \) and the set of homoclinic tangencies between the stable and unstable manifolds of \( \Lambda \). In addition to this result, they elucidated several stochastic properties for the orbits of the wandering domain, see [6, Section 9]. This ends the outline of the Colli-Vargas model.

### 2.2. Outline of the proof of Theorem A

Our paper contains two heavy substances. One is a generalization of discussions involving several lemmas for linking properties by Colli-Vargas, and the other is how essential difficulties brought about this generalization are overcome by our several ideas with arduous tasks. These substances cannot be divided exactly into different parts, which are convolutedly dependent on each other. However, a scenario of our discussions which is simplified as much as possible will help the reader to understand the whole of the paper.

#### 2.2.1. Standard setting for general situations

For the beginning, it could not be better than that assumptions are minimized. So we start our discussions not for some convenient models, but rather for any two-dimensional diffeomorphism \( f \) which has a homoclinic tangency for a saddle fixed point, say \( p \). See Figure 2.2.

For such a situation, it naturally reminds us of the renormalization scheme near the homoclinic tangency by Palis-Takens [27], see Theorem 3.1. In fact, we will take much advantage of the scheme as follows.

In this scheme, we have two main hyperbolic basic sets. One is a horseshoe \( \Lambda \) which is associated with a transverse homoclinic intersection of \( p \) but not affine in general. The other is a hyperbolic basic set, denoted by \( \Gamma_m \), which is created by an Hénon-like return map \( \varphi_m \) of (3.1) in the renormalization near the homoclinic tangency, where \( m \) is the period of some periodic point for \( \varphi_m \). Those ingredients and their cyclic interconnection by way of persistent heteroclinic tangencies are precisely described in Sections 3.

#### 2.2.2. Main Cantor sets and bridges

From the basic sets \( \Lambda \) and \( \Gamma_m \) one can obtain several dynamically defined Cantor sets, among which the following three are especially important in this paper:

\[
K^s_\Lambda := \pi_{F^u_\text{loc}(\Lambda)}(\Lambda), \quad K^u_\Lambda := \pi_{F^u_\text{loc}(\Lambda)}(\Lambda), \quad K^u_m := \pi_m(\Gamma_m),
\]

where \( \pi_{F^u_\text{loc}(\Lambda)} \), respectively \( \pi_{F^s_\text{loc}(\Lambda)} \), is the projection on \( W^u_{\text{loc}}(p) \), respectively \( W^s_{\text{loc}}(p) \), along the leaves of an unstable foliation \( F^u_\text{loc}(\Lambda) \), respectively stable foliation \( F^s_\text{loc}(\Lambda) \), and \( \pi_m \) is the projection on an arc \( \ell \subset W^u_{\text{loc}}(P) \) along the leaves of a stable foliation \( F^s_\text{loc}(\Gamma_m) \) compatible with \( W^s_{\text{loc}}(\Gamma_m) \). Here \( P \) stands for the
saddle fixed point for $\varphi_n$ as illustrated in Figure 2.2 which is not contained in $\Gamma_m$. See (4.2) and (4.9) respectively. Moreover, one has the sequences of $s, u$-bridges related to the three Cantor sets, respectively denoted by

$$\{B_s^k\}, \{B_u^k\}, \{A_u^k\},$$

where the former two are defined in Subsection 4.2, see Figure 4.1, and the latter is in Subsections 4.3-5.1, see Figure 4.2. In Section 4, we give descriptions of bounded distortion properties of these bridges.

2.2.3. Generalized uniformly linking properties for $\{B_s^k(\Delta)\}$ and $\{A_u^k(\Delta)\}$. In the case of the Colli-Vargas model as described above, non-trivial wandering domains $\text{Int}R_k$ were detected in intersections between stable gaps $G_{s_k}$ and unstable gaps $G_{u_k}$ associated with affine Cantor sets $K_s$ and $K_u$ satisfying (2.1). On the other hand, we cannot directly unfold the same story into our case, because there is no promise in general such that the product of thickness of $K_s$ and $K_u$ is larger than one.

However, a bypass of this problem is already given in the renormalization scheme, see Theorem 3.1. In fact, since the thickness of the third Cantor set $K_u$ has an arbitrarily large value by taking $m$ large enough, one can obtain a $C^2$-persistent heteroclinic tangency associated with $W^u(A_g)$ and $W^s(\Gamma_{m,g})$, where $\Lambda_g$ and $\Gamma_{m,g}$ are the continuations of $\Lambda$ and $\Gamma_m$, respectively, see (S-3) in Section 3. For simplicity, we denote such continuations by the same notations as $\Lambda$ and $\Gamma_m$, respectively, in this outline. Therefore, we can discuss linking properties between the images of continuations for bridges $\{B_s^k\}$ and $\{A_u^k\}$ on the arc $L$ of tangencies between...
we show the existence of uniformly linking subsequence of \( \{ s_k \} \) for any \( k \). Actually, these linking situations will be found by the projected images (5.3) on \( L \).

To obtain linked pair of the continuations for bridges, we have to add the first perturbation in \( f^{-1}(B_{\delta_0}) \) where \( B_{\delta_0} \) is a small \( \delta_0 \)-disk which meets the inverse image \( \tilde{L} \) of \( L \), see Figure 5.5. Actually, this perturbation is the horizontal \( \delta \)-shift with \( |\delta| \ll \delta_0 \), and hence the perturbed map is given as \( f_{\delta}(x) = f(x) + (\delta, 0) \) for any \( x \in f^{-1}(B_{\delta_0}/2) \). Using this perturbation, we present Lemma 5.2 which is a generalization of Linking Lemma of Colli-Vargas. Moreover, in Lemma 6.1 we show the existence of uniformly linking subsequence of \( \{ A^u_k(\Delta) \} \) and \( \{ B^u_k(\Delta) \} \) where \( \Delta = \sum_{k=1}^{\infty} \delta_k \), see Lemma 6.2. This is a generalization of Linear Growth Lemma of Colli-Vargas. The proofs of the results are supported by Lemma 4.3 and Lemma 4.5 in which bounded distortion properties are presented for \( s \)-bridges \( \{ B^u_k \} \) and \( u \)-bridges \( \{ A^u_k \} \), respectively.

2.2.4. Critical chains in \( \{ B^u_k(\Delta) \} \) and \( \{ B^s_k(\Delta) \} \). Note that the uniformly linking subsequence of \( \{ B^u_k(\Delta) \} \) and \( \{ A^s_k(\Delta) \} \) are constructed on the arc \( L \) including heteroclinic persistent tangencies of \( W^s(\Lambda) \) and \( W^u(\Gamma_m) \). At this stage, since we use only one way from \( \Lambda \) to \( \Gamma_m \), even if one takes any domain constructed from the linking subsequence by some perturbations, there is no certification that the orbit of domain comes back to and wanders around \( \Lambda \) non-trivially. However, we can construct a return route by using the fact that \( \Gamma_m \) and \( \Lambda \) are homoclinically related to each other, see the condition (S \( e \)) of Section 2. It follows that the stable foliation \( \mathcal{F}^s(\Lambda) \) and some gap of \( K_{m,L} \) have a transverse intersection. Hence, one obtains some gap \( \tilde{G}^u_{L,k+1} \) of \( A^u_{L,k+1} \) which contains some \( u \)-bridge \( B^u_{L,k+1} := \pi_{\tilde{A}^e_k}(B^u_{k+1}) \), see Figure 7.1 where \( \pi_{\tilde{A}^e_k} : B(0) \to L \) is the projection defined as (5.2) and \( B^u_{k+1} \) is the \( u \)-bridge whose itinerary satisfies (7.1). Moreover, by taking the itinerary of the \( s \)-bridge \( B^s_{k+1} \) as in (7.1), one can obtain linking situations which is desired in Lemma 7.1 (Critical Chain Lemma): there is an interval \( J^s_{k+1} \) such that \( t \in J^s_{k+1} \) if and only if \( B^s_{L,k+1} + t \cap B^s_{L,k+1} \neq \emptyset \), where \( B^s_{L,k+1} \) is the image of the \( s \)-bridge \( B^s_{L,k+1} \) of \( K^s_k \) by the projection \( \pi^s : S \to L \) of (5.4). Note that the itineraries given in \( (7.1) \) will be crucial to control the orbit of any point in the wandering domain obtained in the later sections.

2.2.5. Multidirectional perturbations and critical chains of rectangles. We may consider the inverse image \( L \) of \( L \) which is contained in the neighborhood of \( \Lambda \), as shown in Figure 7.3. Lemma 7.2 implies that, for all sufficiently large \( k \), there exists almost horizontal line \( L_k \) such that \( L_k \) meets \( L \) transversely at a single point \( x_k \). Moreover we take a point \( \tilde{x}_k = f^{N_k} \circ f^{i_k} \circ f^{N_0}(x_k) \) in \( \tilde{L}_k \), and define a sequence \( \{ \tilde{x}_k \} \) with \( \tilde{x}_k = f^{i_k^2+k}(\tilde{x}_k) \), where \( k \) is the integer of (7.2) with \( \lim_{k \to \infty} (k)/k^2 = 1 \). For \( \{ x_k \} \) and \( \{ \tilde{x}_k \} \), one has the sequence

\[
\mathbf{t} = (t_2, t_3, \ldots, t_k, \ldots),
\]

where each \( t_k \) is the vector given by \( f^{N_k}(\tilde{x}_k) + t_{k+1} = x_{k+1} \), see Figure 7.6 which is the second perturbation corresponding to (2.2) of Colli-Vargas. Note that each entry vector of the perturbation is not necessary in the same direction. For \( \mathbf{t} \), we now define the diffeomorphism \( f^\mathbf{t}_L \) by

\[
f^\mathbf{t}_L := f \circ \psi^\mathbf{t}_L
\]
so that \( f_k(x_k) = x_{k+1} \), where \( \psi_k \) is the \( C^r \)-map defined as (7.13). Note that Lemma 7.6 claims that, if \( T \) is sufficiently large, then \( \psi_k \) is arbitrarily \( C^r \)-close to the identity and hence \( f_k \) is a \( C^r \)-diffeomorphism arbitrarily \( C^r \)-close to \( f \).

2.2.6. Non-trivial wandering domains. Around each \( x_k = (x_k, y_k) \), we define a rectangle as

\[
R_k = [x_k - b_k^{1/2}, x_k + b_k^{1/2}] \times [y_k - b_k, y_k + b_k],
\]

where \( b_k \) is the positive number given in (8.2), and show in Lemma 8.3 (Rectangle Lemma) that there is an integer \( k_0 > 0 \) such that, for any \( k \geq k_0 \), \( g_k(R_k) \subset \text{Int}(R_{k+1}) \), where \( g_k \) is given as (8.6). That is, the interior of \( D_k \) is a wandering domain. It follows immediately by our construction of \( D \) that the wandering domain is contracting. Moreover, the non-triviality of the wandering domain is implied by taking the words \( \nu_{k+1} \in \{1, 2\}^k \) in itineraries of (7.1) suitably. See Proposition 8.4.

2.2.7. Historic behavior. Note that the first perturbation of \( f \) in Subsection 5.2 does not depend on the sequence \( z = \{z_k\}_{k=1}^\infty \) such that each entry \( z_k \) is either \( z_0 \) or \( z_0 + 1 \) but the second perturbation in Subsection 7.2 does. The diffeomorphism \( f_n \) obtained in Proposition 8.4 and the wandering domain \( D = \text{Int}R_{k_0} \) also depend on \( z \). Thus, in the proof of Theorem A we express the dependence by \( f_{n, z} \) and \( D_z = \text{Int}R_{k_0, z} \). From the setting of (7.1), the itinerary of any orbit starting from the non-trivial wandering domain \( D_z \) contains \( \frac{1}{2^{k_z}} \) and \( 2^{k_z} \) with a sufficiently large \( k_z \), and the remaining part of the itinerary is corresponding to at most order \( k \) iterations of \( f_{n, z} \). Using this, we can show that for any \( x \in D_z \) there is the subsequence \( \{\mu_x(\hat{m}_k)\}_{k=k_0}^\infty \) of partial averages

\[
\mu_x(\hat{m}_k) = \frac{1}{\hat{m}_k + 1} \sum_{i=0}^{\hat{m}_k} \delta_{f_{n, z}^i(x)}
\]

which tends to the distinct two probability measures:

\[
\nu_0 = \frac{1}{z_0 + 1} (z_0 \delta_p + \delta_{\bar{p}}), \quad \nu_1 = \frac{1}{z_0 + 2} (z_0 + 1) \delta_p + \delta_{\bar{p}}
\]

as \( k \to \infty \), where \( p \) and \( \bar{p} \) are different saddle fixed points in the horseshoe \( \Lambda \). It implies that the orbit of any point in the non-trivial wandering domain \( D_z \) has historic behavior. This finishes the outline of the proof of Theorem A.

3. Preparations

In this section, we present standard notions concerning planer homoclinic bifurcations introduced by Palis-Takens [27], which are given in forms adaptable to our discussions.

Fix an integer \( r \) with \( 2 \leq r < \infty \) and suppose that \( f \) is a diffeomorphism on a closed surface \( M \) with a saddle periodic point \( p \) whose stable manifold \( W^s(p) \) and unstable manifold \( W^u(p) \) have a tangency. For simplicity, replacing \( f^{\text{Per}(p)} \) by \( f \) (where \( \text{Per}(p) \) stands for the period of \( p \) and adding a small \( C^r \)-perturbation to \( f \), we may assume that \( p \) is a saddle fixed point of \( f \) with \( |\det(Df)_p| \neq 1 \). Moreover, one can suppose that \( p \) is dissipative, i.e. \( |\det(Df)_p| < 1 \), if necessary considering \( f^{-1} \) instead of \( f \). By a small perturbation near the homoclinic tangency \( q \), one can also suppose that the tangency is quadratic. Moreover, as in [27], Section 6.5], performing several arbitrarily small perturbations, we obtain a diffeomorphism which has both a transverse homoclinic intersection and a homoclinic tangency of
the continuation of $p$ simultaneously. Using the transverse homoclinic intersection, one obtains a basic set containing the continuation of $p$ called a horseshoe, i.e. a compact invariant hyperbolic set which is transitive and contains a dense subset of periodic orbits and such that the restriction of the diffeomorphism on the set is conjugate to the 2-shift. In what follows, if no confusion can arise, we will denote a $C^\infty$ diffeomorphism which is arbitrarily $C^r$ close to $f$ again by $f$. So we will work under the assumption that $f$ is a diffeomorphism of $C^\infty$-class and return to a diffeomorphism of $C^r$-class with $3 \leq r < \infty$ at the stage of (7.15) in Subsection 7.2.

In accordance with the discussion as above, we may suppose that $f$ has

(S-i) a horseshoe $\Lambda$ containing a dissipative saddle fixed point $p$;
(S-ii) a non-degenerate homoclinic tangency $q$ of $p$.

Moreover, if a $C^\infty$ diffeomorphism $C^r$-close to $f$ satisfies an open and dense Sternberg condition [32] concerning the eigenvalues at the continuation of $p$, then $f$ is $C^r$-linearizable in a neighborhood $U$ of the continuation. So, one can suppose that the above $f$ has

(S-iii) a $C^r$-linearizing coordinate in a neighborhood of $p$ such that $f(x, y) = (\lambda x, \sigma y)$ with $0 < \lambda < 1 < \sigma$ and $\lambda \sigma < 1$ (we replace $f$ by $f^2$ if $\lambda$ or $\sigma$ is negative).

Note that one might proceed without (S-iii) by using techniques in Gonchenko et al. [11], but (S-iii) is more appropriate here from a standpoint of simple descriptions.

Let $\{f_\mu\}_{\mu \in \mathbb{R}}$ be a one-parameter family of $C^\infty$ diffeomorphisms on a two-dimensional manifold $M$ with $f_0 = f$ and such that the homoclinic quadratic tangency $q$ of $p$ unfolds generically at $\mu = 0$. The renormalization of return maps near the tangency provides a better description as follows.

**Theorem 3.1** (Renormalization Theorem, Palis-Takens [27]). There exists an integer $N_\star > 0$ such that, for any sufficiently large integer $n > 0$, there are a $C^r$ reparameterization $\Theta_n : \mathbb{R} \to \mathbb{R}$ and a $\bar{\mu}$-dependent $C^r$ coordinate change $\Phi_n : \mathbb{R}^2 \to M$ satisfying the followings:

- $\Theta_n'(\bar{\mu}) > 0$;
- for any $(\bar{\mu}, \bar{x}, \bar{y}) \in \mathbb{R} \times \mathbb{R}^2$, $(\Theta_n(\bar{\mu}), \Phi_n(\bar{x}, \bar{y}))$ converges to $(0, q)$ as $n \to \infty$;
- for any $\bar{\mu} \in \mathbb{R}$, the diffeomorphisms $\varphi_n$ on $\mathbb{R}^2$ defined by

$$\varphi_n : (\bar{x}, \bar{y}) \mapsto \Phi_n^{-1} \circ f_{\mu_n}^{N_\star + n} \circ \Phi_n(\bar{x}, \bar{y})$$

converge to

$$\varphi_{\mu, \nu} : (\bar{x}, \bar{y}) \mapsto (\bar{y}, \bar{y}^2 + \bar{\mu})$$

as $n \to \infty$ in the $C^r$ topology, where $\mu_n := \Theta_n(\bar{\mu})$.

**Proof.** See Palis-Takens [27], §3.4, Theorem 1] for the proof. \hfill \Box

Here the integer $N_\star$ is taken so that $f^{-N_\star}(q)$ is a point in $W^u_{\text{loc}}(p)$ sufficiently near $p$. In short this lemma ensures that the return maps $f_{\mu_n}^{N_\star + n}$ near the homoclinic tangency can be approximated by diffeomorphisms so called Hénon-like maps which are close to the quadratic endomorphism $\Phi_{\mu, \nu}$.

For simplicity, we consider the map

$$\varphi_{\mu, \nu}(x, y) = (y, \mu + \nu x + y^2),$$
which is equivalent to the original Hénon map given in [12] via appropriate parameter and coordinate changes, see [18]. More general Hénon-like families are obtained by adding small higher-order terms to the above form. From [27, §6.3, Proposition 1], for a given $m \geq 3$, there exist a neighborhood $U(-2,0)$ of the point $(-2,0)$ in the parameter space and continuous maps $P, Q_m$ and $\Gamma_m$ which map $(\mu, \nu) \in U(-2,0)$ to the fixed point $P_{\mu,\nu}$, the periodic point $Q_{m,\mu,\nu}$, and the non-trivial invariant set $\Gamma_{m,\mu,\nu}$ for $\varphi_{\mu,\nu}$, respectively, and furthermore satisfy the following properties:

- $P_{-2,0} = (2, 2)$ is a saddle fixed point and $Q_{m,-2,0}$ is a saddle periodic point of period $m$ both of which are contained in a parabolic arc which is convex downward between $P_{-2,0}$ and $P_{2,0} := (-2, 2)$, see Figure 4.3.
- For any $(\mu, \nu) \in U(-2,0)$ with $\nu \neq 0$, $\Gamma_{m,\mu,\nu}$ is a basic set containing the orbit of $Q_{m,\mu,\nu}$.

More detailed information on these ingredients will be given in the next section. Note that the same properties hold for any Hénon-like map which is sufficiently close to $\varphi_{-2,0}$.

By the above results, there exists a positive integer $n(m) > 0$ such that, for any $n \geq n(m)$, $f_{\nu_n}$ is a diffeomorphism arbitrarily $C^r$ close to the original $f$ and satisfying not only (S-i)-(S-iii) but also

(S-iv) the restriction $\varphi_n$ of $f_{\mu_n}^{N_\nu+n}$ near the tangency $q$ is a return map $C^r$-approximated by an Hénon-like map and has the continuation $P_{\bar{\mu}}$ of the saddle fixed point $P$, the continuation $\Gamma_{m,\bar{\mu}}$ of the basic set $\Gamma_m$ which contains the continuation $Q_{m,\bar{\mu}}$ of the saddle periodic point $Q_m$ of period $m$, where $\bar{\mu} = \Theta^{-1}_n(\mu_n)$.

We here recall two important relations on a pair of basic sets. We say that disjoint basic sets $\Lambda$ and $\Gamma$ are homoclinically related if both $W^u(\Lambda) \cap W^s(\Gamma)$ and $W^s(\Lambda) \cap W^u(\Gamma)$ contain non-trivial transverse intersections. Basic sets $\Lambda$ and $\Gamma$ for $f$ have a $C^2$-robust tangency if there exists a $C^2$ neighborhood $U(f)$ of $f$ satisfying the following condition: for every $g \in U(f)$, either $W^u(\Lambda_g) \cap W^s(\Gamma_g)$ or $W^s(\Lambda_g) \cap W^u(\Gamma_g)$ contains a tangency, where $\Lambda_g$ and $\Gamma_g$ are the continuations of $\Lambda$ and $\Gamma$, respectively.

By [27, Section 6.4], we may also suppose that

(S-v) the continuation $\Lambda_n := \Lambda(f_{\mu_n})$ of the horseshoe $\Lambda$ in (S-i) and the basic set $\Gamma_{m,n} := \Gamma_m(\varphi_n)$ in (S-iv) are homoclinically related, and they have a $C^2$-robust tangency. To be more precise, $W^u(\Lambda_g) \cap W^s(\Gamma_{m,n})$ contains a tangency $a$ for any diffeomorphism $g$ $C^2$-near $f$, see Figure 3.1.

Now we recall the construction of the return map $\varphi_n$ by Palis-Takens. There exists a small rectangle $D_n$ near $q$ such that $\Gamma_{m,n} = \bigcap_{k=-\infty}^{\infty} f_{\mu_n}^{k(N_\nu+n)}(D_n)$. According to [27, Section 6.4], there is a transverse intersection point $b$ of $W^u(p)$ and $W^s(P)$ such that

(S-vi) the sub-arc $\alpha^u$ in $W^u(p)$ connecting $p$ with $b$ is disjoint from the union $X_n = D_n \cup f_{\mu_n}(D_n) \cup \cdots \cup f_{\mu_n}^{N_\nu+n}(D_n)$, and

(S-vii) $f_{\mu_n}^i(D_n) \cap f_{\mu_n}^j(D_n) = \emptyset$ for any $i, j \in \{1, 2, \ldots, N_\nu + n\}$ with $i \neq j$, see Figure 3.2.

Note that the condition (S-vii) does not necessarily hold for $i, j \notin \{1, 2, \ldots, N_\nu + n\}$. In fact, for any integer $N$ sufficiently larger than $N_\nu$, $f_{\mu_n}^N(\alpha^u)$ meets all leaves of
Figure 3.1. \( W^u_{loc}(\Gamma_m) \) transversely, which is suggested in Figure 3.1 and hence in particular \( f^n_{\mu_n}(\alpha^u) \cap D_n \neq \emptyset \).

A nonempty compact subset \( K \) of an interval \( I \) is called a Cantor set if \( K \) has neither interior points nor isolated points. A gap of the Cantor set \( K \) is the closure of a connected component of \( I \setminus K \). Let \( G \) be a gap and \( p \) a boundary point of \( G \). A closed interval \( B \subset I \) is called the bridge at \( p \) if \( B \) is maximal among all closed intervals \( B' \) in \( I \) with \( G \cap B' = \{ p \} \) and such that \( B' \) does not intersect any gap whose length is at least that of \( G \). The thickness for the Cantor set \( K \) at \( p \) is defined by \( \tau(K, p) = |B|/|G| \), where \( B \) and \( G \) are a bridge and a gap satisfying \( G \cap B = \{ p \} \). The thickness \( \tau(K) \) of \( K \) is the infimum over these \( \tau(K, p) \) for all boundary points \( p \) of gaps of \( K \). Two Cantor sets \( K_1 \) and \( K_2 \) are said to be linked if neither \( K_1 \) is contained in the interior of any gap of \( K_2 \) nor \( K_2 \) is contained in the interior of any gap of \( K_1 \). Gap Lemma (see [26, §4], [27, §4.2]) shows that, for any linked Cantor sets \( K_1 \) and \( K_2 \) with \( \tau(K_1) \tau(K_2) > 1 \), \( K_1 \cap K_2 \neq \emptyset \) holds.
We say that a bridge $B$ of $K$ is adjacent to a gap $G$ if $B \cap G \neq \emptyset$ and $\text{Int} B \cap G = \emptyset$. If two bridges $B, B'$ are adjacent to a common gap $G$, then $G$ is called the connecting gap for $B$ and $B'$ and denoted by $\text{Gap}(B, B')$.

4. Bounded distortions

4.1. Classical bounded distortion lemma. A Cantor set $K$ in an interval $I$ is said to be dynamically defined if the following conditions hold: there are mutually disjoint closed sub-intervals $B_1, B_2, \ldots, B_r \subset I$ and a differentiable map $\Psi$ defined in a neighborhood $U$ of $B_1 \sqcup \cdots \sqcup B_r$ in $I$ such that

- $\Psi$ is uniformly hyperbolic on $K$, that is, there are constants $C, \sigma > 0$ and $\delta > 0$ such that $|((\Psi^n)'(x))| \geq C \sigma^n$ for every $x \in K$ and $n \geq 1$, and
- $\{B_1, \ldots, B_r\}$ is a Markov partition satisfying

$$K = \bigcap_{n \in \mathbb{N}} \Psi^{-n}(B_1 \sqcup \cdots \sqcup B_r).$$

The next classical result, called Bounded Distortion Lemma, will play an important role in this paper.

**Lemma 4.1** (Palis-Takens [27]). Let $K$ be a dynamically defined Cantor set as above associated with a uniformly hyperbolic map $\Psi$ and $a_0$ the minimum positive integer with $C \sigma^{a_0} > 1$. If $\Psi$ satisfies the $C^{1+\alpha}$ Hölder condition for some $0 < \alpha \leq 1$, then, for every $\delta > 0$, there exists a constant $c(\delta) > 0$ satisfying

$$e^{-c(\delta)} \leq |((\Psi^{n_0})'(q))| |((\Psi^{n_0})'(\tilde{q}))|^{-1} \leq e^{c(\delta)}$$

for any $q, \tilde{q} \in K$ and integer $n \geq 1$ such that (i) $|\Psi^{n_0}(q) - \Psi^{n_0}(\tilde{q})| \leq \delta$; (ii) the interval between $\Psi^i(q)$ and $\Psi^i(\tilde{q})$ is contained in $B_1 \sqcup \cdots \sqcup B_r$ for all $0 \leq i \leq (n - 1)a_0$. Moreover, $c(\delta)$ is of order $\delta^\alpha$. In particular, $c(\delta)$ converges to zero as $\delta \to 0$.

**Proof.** Since $\Psi$ is uniformly hyperbolic on $K$, $|((\Psi^n)'(x))| \geq C \sigma^n$ for any $x \in K$ and $n \geq 1$. Then there exists a constant $\gamma > 1$ and an open neighborhood $V$ of $K$ in $U$ such that $|((\Psi^n)'(x))| \geq \gamma$ for any $x \in V$ and $n \geq 1$. For all sufficiently small $\delta > 0$, any points $q, \tilde{q}$ of $K$ satisfying the conditions (i) and (ii) bound an interval contained in $V$. Hence, one can apply [27] §4.1, Theorem 1 directly to the expanding map $\Psi^{a_0}$, and obtain a constant

$$c(\delta) = \tilde{C} \delta^\alpha \frac{(C \sigma^{a_0})^{\alpha}}{1 - (C \sigma^{a_0})^{-\alpha}}$$

satisfying the condition (4.1), where $\tilde{C}$ is a positive constant independent of $\delta$. This completes the proof. \hfill $\square$

4.2. Horseshoes and $s$-bridges. Let us recall a simple example of a dynamically defined Cantor set and give the bounded distortion property for the set.

Consider a two-dimensional $C^r$ diffeomorphism $f$ admitting a horseshoe $\Lambda$ as in (S2) of Section 3, which contains a saddle fixed point $p$. Let $\mathcal{F}^u_{\text{loc}}(\Lambda)$ and $\mathcal{F}^s_{\text{loc}}(\Lambda)$ be local unstable and stable foliations on $S = [0, 2] \times [0, 2]$ compatible with $W^u_{\text{loc}}(\Lambda)$ and $W^s_{\text{loc}}(\Lambda)$ respectively. Here we may assume that

- $W^u_{\text{loc}}(p) = [0, 2] \times \{0\}$, $W^s_{\text{loc}}(p) = \{0\} \times [0, 2]$,
- $[0, 2] \times \{2\}$ is a leaf of $\mathcal{F}^s_{\text{loc}}(\Lambda)$ disjoint from $W^s_{\text{loc}}(\Lambda)$ and $\{2\} \times [0, 2]$ is a leaf of $\mathcal{F}^u_{\text{loc}}(\Lambda)$ disjoint from $W^u_{\text{loc}}(\Lambda)$. 

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Let $\pi_{F^s_{\text{loc}}(\Lambda)} : S \to W^s_{\text{loc}}(p)$ be the projection along the leaves of $F^s_{\text{loc}}(\Lambda)$, and $\pi_{F^u_{\text{loc}}(\Lambda)} : S \to W^u_{\text{loc}}(p)$ the projection along the leaves of $F^u_{\text{loc}}(\Lambda)$. Consider the Cantor sets

$$K^s_\Lambda := \pi_{F^s_{\text{loc}}(\Lambda)}(\Lambda), \quad K^u_\Lambda := \pi_{F^u_{\text{loc}}(\Lambda)}(\Lambda),$$

associated with $\Lambda$ dynamically defined by $\Psi_s := \pi_{F^s_{\text{loc}}(\Lambda)} \circ f^{-1}$ and $\Psi_u := \pi_{F^u_{\text{loc}}(\Lambda)} \circ f$, respectively. Since $f$ is a $C^r$ map and $\pi_{F^s_{\text{loc}}}, \pi_{F^u_{\text{loc}}}$ are $C^{1+\alpha}$ maps with $0 < \alpha < 1$ (see [27, §4.1]), it follows that both $\Psi_s$ and $\Psi_u$ are of $C^{1+\alpha}$ class. Note that both $\Psi_s$ and $\Psi_u$ are expanding maps.

**Remark 4.2.** In this paper, we use three local ‘stable’ foliation on $S$, one of which is the above $F^s_{\text{loc}}(\Lambda)$. The other two are $F^s$ in Subsection 5.1 and $G^s_{\text{loc}}(\Lambda)$ in Subsection 7.2.

Let $B^s(0)$ (respectively $B^u(0)$) be the smallest interval in $W^s_{\text{loc}}(\Lambda)$ (respectively $W^u_{\text{loc}}(\Lambda)$) containing $K^s$ (respectively $K^u$).

One may suppose that the curve $W^s_{\text{loc}}(p)$ (respectively $W^u_{\text{loc}}(p)$) is a closed interval $B^s(0)$ (respectively $B^u(0)$) containing $K^s_\Lambda$ (respectively $K^u_\Lambda$). We give here descriptions associated only with $K^s_\Lambda$. Similar arguments work also in the case of $K^u_\Lambda$. There exists a Markov partition of $K^s_\Lambda$ in $B^s(0)$ which consists of two components, and denote one of them by $B^s(1; 1)$ and the other by $B^s(1; 2)$. If necessary replacing the original $f$ by $f^n$ with a large integer $n$ and $\Lambda$ by a smaller $f^n$-invariant set, we may suppose that

$$|B^s(1; 1)|, |B^s(1; 2)| < |B^s(0)|/2,$$

where $| \cdot |$ stands for the length of the corresponding interval. For each integer $k \geq 1$ and $w_i \in \{1, 2\}$ $(i = 1, \ldots, k)$, we define the interval $B^s(k; w_1 \ldots w_k)$, called an $s$-bridge of generation $k$, as

$$B^s(k; w_1 \ldots w_k) = \{x \in I : \Psi_s^{i-1}(x) \in B^s(1; w_i), i = 1, \ldots, k\},$$

where the sequence $(w_1 w_2 \ldots w_k)$ is the itinerary for the $s$-bridge. If one writes $w = (w_1 w_2 \ldots w_k)$, then $w^{-1}$ stands for the reverse sequence $(w_k w_{k-1} \ldots w_1)$. The $u$-bridges $B^u(k; w_1 \ldots w_k)$ of generation $k$ associated with $K^u_\Lambda$ can be defined similarly by using $\Psi_u$. For our convenience, we regard that $B^s(0)$ and $B^u(0)$ are bridges of generation $0$ with empty itinerary.

For any integer $k \geq 0$, let $B^s_k$ be the collection of all $B^s(k; w_1 \ldots w_k)$, see Figure 4.1. Note that $B^s_k$ consists of mutually disjoint $2^k$ $s$-bridges. The union $B^s = \bigcup_{k=0}^{\infty} B^s_k$ is the set of all $u$-bridges of $K^s_\Lambda$. The set $B^u = \bigcup_{k=0}^{\infty} B^u_k$ of all $u$-bridges of $K^u_\Lambda$ is defined similarly.

![Figure 4.1](image)

**Figure 4.1.** For any $w_1, w_2 \in \{1, 2\}$, $\Psi_s(B^s(2; w_1 w_2)) = B^s(1; w_2)$. 

Note that $B^s_k \cup B^u_k$ generation 0 with empty itinerary.
Lemma 4.3 (Bounded distortions for $s$-bridges). The exist a non-negative integer $k_0$ and constants $r_{s+} \geq r_{s-} > 2$ satisfying the following conditions. For any integer $k \geq k_0$, let $B^s_k$ and $B^s_{k+1}$ be $s$-bridges for $K^A_k$ of generation $k$ and $k+1$ such that the first $k$ entries in the itinerary of $B^s_{k+1}$ are identical to the entries in the itinerary of $B^s_k$, that is,

$$B^s_k := B^s(k; w_1 \ldots w_k), \quad B^s_{k+1} := B^s(k + 1; w_1 \ldots w_kw_{k+1}).$$

Then the inequality

$$r_{s-} \leq |B^s_k||B^s_{k+1}|^{-1} \leq r_{s+}$$

holds.

Proof. By the mean-value theorem, for any bridges $B^s_k$ and $B^s_{k+1}$ as above, there are points $q \in B^s_k$, $\bar{q} \in B^s_{k+1}$ and $s$-bridges $B^s_{k_0} = B^s(k_0; w_{k-k_0} \ldots w_k)$, $B^s_{k_0+1} = B^s(k_0 + 1; w_kw_{k+1})$, such that

$$|B^s_{k_0}| = |B^s_k||(\Psi^{k-k_0}_s)'(\bar{q})|, \quad |B^s_{k_0+1}| = |B^s_{k+1}||(\Psi^{k-k_0}_s)'(q)|.$$

Thus,

$$|B^s_k||B^s_{k+1}|^{-1} = ([B^s_k][B^s_{k+1}]^{-1})||((\Psi^{k-k_0}_s)'(q))||(\Psi^{k-k_0}_s)'(\bar{q})|^{-1}.$$ (4.4)

Since $\Psi_s$ is expanding, one can apply Lemma 4.1 to $\Psi_s$ by setting $a_0 = 1$. We set $\delta(k_0) = \max\{|B^s(k_0; w_1 \ldots w_{k_0})|; \; w_1 \in \{1, 2\} \; (i = 1, \ldots, k_0)\}$. Since $\Psi_s$ is of $C^{1+\alpha}$ class, it follows from Lemma 4.1 that there exists a constant $c = c(\delta(k_0)) > 0$ of order $\delta(k_0)^\alpha$ satisfying

$$e^{-c} \leq ||(\Psi^{k-k_0}_s)'(q))||(\Psi^{k-k_0}_s)'(\bar{q})|^{-1} \leq e^c.$$ (4.5)

For any $B^s_{k_0} = B^s(k_0; w_{k-k_0+1} \ldots w_k)$ and $i = 1, 2$, we write

$$B^s_{k_0+1}(i) = B^s(k_0 + 1; w_kw_{k+1}).$$

if $w_{k+1} = i$. By (4.3), both

$$q_{s-} = \min\{|B^s_{k_0}||B^s_{k_0+1}(i)|^{-1}; \; w_j \in \{1, 2\} \; (j = k - k_0 + 1, \ldots, k)\}, \quad i \in \{1, 2\}$$

$$q_{s+} = \max\{|B^s_{k_0}||B^s_{k_0+1}(i)|^{-1}; \; w_j \in \{1, 2\} \; (j = k - k_0 + 1, \ldots, k)\}, \quad i \in \{1, 2\}$$

are greater than 2. Since $c(\delta(k_0)) \to 0$ as $k_0 \to \infty$, one can take $k_0$ large enough so that $r_{s-} := e^{-c(\delta(k_0))}q_{s-}$ is greater than 2. By (4.4) and (4.5), $r_{s+} := e^{c(\delta(k_0))}q_{s+}$ and $r_{s-}$ satisfy our desired inequality.

4.3. Quadratic maps and $u$-bridges. We recall another Cantor set for one-dimensional maps fundamental properties of which are succeeded by Hénon-like maps in Section 4.4.

Let $F_\mu$ be the family of one-dimensional quadratic maps on $\mathbb{R}$ defined as

$$F_\mu(x) = x^2 + \mu,$$

where $\mu$ is a real parameter. For any integer $m \geq 3$, $F_\mu$ has a periodic orbit of period $m$ if $\mu$ is sufficiently close to $-2$. Denote by $q_1$ and $q_2$, respectively, the minimum and maximum points of the $m$-periodic orbit, which satisfies $F_\mu(q_1) = q_2$. We denote by $A^u(0)$ the interval $[q_1, q_2]$, and define the mutually disjoint $m - 1$ intervals as

$$A^u(1; z_1) := \begin{cases} F_\mu^{-1}(A^u(0)) \cap \{x < 0\} & \text{if } z_1 = 1; \\ F_\mu^{-z_1+1}(F_\mu^{-1}(A^u(0)) \cap \{x < 0\}) \cap \{x > 0\} & \text{if } z_1 = 2, \ldots, m-1. \end{cases}$$
Moreover, for every integer \( k \geq 2 \), we inductively define the \((m - 1)^k\) intervals \( A^u(k; z_1 z_2 \ldots z_k) \) as

\[
A^u(k; z_1 z_2 \ldots z_k) := A^u(k - 1; z_1 \ldots z_{k-1}) \cap F^{-z_1}_\mu (A^u(k - 1; z_2 \ldots z_k)),
\]

where each entry \( z_i \) is an element of \( \{1, 2, \ldots, m - 1\} \). See Figure 4.2 for the case of \( m = 4 \). The interval \( A^u(k; z_1 z_2 \ldots z_k) \) is called a \( u \)-bridge for \( F_\mu \) of generation \( k \), and the sequence \( (z_1 \ldots z_k) \) is the itinerary of the \( u \)-bridge. A unique point of \( \partial A^u(k; z_1 z_2 \ldots z_k) \cap \partial A^u(k + 1; z_1 z_2 \ldots z_k) \) (resp. \( \partial A^u(k; z_1 z_2 \ldots z_k) \cap \partial A^u(k + 1; z_1 z_2 \ldots z_k m - 1) \)) is called the leading point (resp. bottom point) of \( A^u(k; z_1 z_2 \ldots z_k) \). Consider the Cantor set

\[
K^u_m(\mu) := A^u(0) \cap \bigcup_{k=1}^{\infty} \bigcup_{(z_1, \ldots, z_k) \in \{1, \ldots, m - 1\}^k} A^u(k; z_1 \ldots z_k)
\]

dynamically defined by \( F_\mu \) and associated with the \( m \)-periodic orbit. For our convenience, we regard that \( A^u(0) \) is a \( u \)-bridge of generation 0 with empty itinerary. The leading gap of \( A^u(k; z_1 \ldots z_k) \) is the gap in \( A^u(k; z_1 \ldots z_k) \) bounded by \( A^u(k + 1; z_1 \ldots z_k) \) and \( A^u(k + 1; z_1 \ldots z_k m - 1) \). For each integer \( k \geq 0 \), let \( A^u_k \) be the collection of all \( k \)-bridges \( A^u(k; z_1 \ldots z_k) \), see Figure 4.2.

Figure 4.2. For any \( z \in \{1, 2, 3\} \), \( F_\mu(A^u(2; 3z)) = A^u(2; 2z) \), \( F_\mu(A^u(2; 2z)) = A^u(2; 1z) \), and \( F_\mu(A^u(2; 1z)) = A^u(1; z) \). The red dots represent the leading points of \( u \)-bridges of generation 0 and 1. \( G^u_1 \) is the leading gap of \( A^u(1; 1) \).

**Remark 4.4.** As \( \mu = -2 \), \( F_{-2} \) is topologically conjugate to the tent map \( T : x \mapsto |2x - 1| + 1 \) via \( g : x \mapsto 2 - 4 \sin^2(\pi/2) x \). It implies that, if \( \mu \) is contained in a small neighborhood \( I \) of \( -2 \) and \( m \geq 3 \), \( K^u_m(\mu) \) is a uniformly hyperbolic set for \( F_\mu \), see [26] §6.2.

**Lemma 4.5** (Bounded distortions for \( u \)-bridges). For any integer \( m \geq 3 \), there exist \( \eta(m) > 0 \) and an integer \( \kappa(m) \geq 1 \) such that, for any \( k \geq \kappa(m) \) and \( \mu \in (-2 - \eta(m), -2 + \eta(m)) \), the following conditions (1)-(3) hold, where \( A^u_k \) and \( A^u_{k+1} \) are \( u \)-bridges for \( K^u_m(\mu) \) of generation \( k \) and \( k + 1 \) respectively such that the first \( k \)
entries in the itinerary of \( A^u_{k+1} \) are the same as the entries in the itinerary of \( A^u_k \), that is,
\[
A^u_k = A^u(k; z_1 \ldots z_k), \quad A^u_{k+1} = A^u(k+1; z_1 \ldots z_k z_{k+1}).
\]

(1) If \( z_{k+1} = j \in \{1, \ldots, m-1\} \), then
\[
3 \cdot 2^{j-2} \leq |A^u_k| |A^u_{k+1}|^{-1} \leq 5 \cdot 2^{j-2}.
\]

In particular, we have
\[
\frac{3}{2} \leq |A^u_k| |A^u_{k+1}|^{-1} \leq 5 \cdot 2^{m-3}
\]
for any \( z_{k+1} \in \{1, \ldots, m-1\} \). Moreover, \( |A^u_k| |A^u_{k+1}|^{-1} \leq 5 \) if \( z_{k+1} \) is either 1 or 2.

(2) Let \( I^u_k \) be the minimum sub-interval of \( A^u_k \) containing \( A^u_{k+1} \) and the bottom point of \( A^u_k \). Then
\[
|A^u_{k+1}| |I^u_k|^{-1} \geq \frac{1}{3}.
\]

(3) Suppose that \( z_{k+1} \leq m-2 \) and \( \tilde{A}^u_{k+1} = A^u(k+1; z_1 \ldots z_k z_{w+k+1} + 1) \). Let \( G^u_{k+1} \) be the connecting gap for \( A^u_{k+1} \) and \( A^u_h \), i.e., \( G^u_{k+1} = \text{Gap}(A^u_{k+1}, A^u_h) \).

Then
\[
|\tilde{A}^u_{k+1}| |A^u_{k+1}|^{-1} \geq \frac{1}{3} \quad \text{and} \quad |G^u_{k+1}| |A^u_{k+1}|^{-1} \geq \frac{1}{2^{m+1}}.
\]

Proof. There are constants \( C > 0 \) and \( \sigma > 1 \) such that \( |(F^u_{\mu}')(x)| \geq C \sigma^k \) for any \( x \in K^u_{\mu}(\mu) \) and \( k \geq 1 \). Let \( k_0 = k_0(\mu, m) \) be the minimum non-negative integer with \( C \sigma^{k_0} > 1 \). As in the proof of Lemma 4.1, we have a constant \( \gamma > 1 \) and an open neighborhood \( V \) of \( K^u_{\mu}(\mu) \) such that \( |(F^u_{\mu}')(x)| \geq \gamma \) for any \( x \in V \) and \( n \geq 1 \).

Take a positive integer \( \kappa \). For any integer \( k \geq k_0 + \kappa \), there exist integers \( n \geq 0 \) and \( h \in \{\kappa, \ldots, \kappa + k_0 - 1\} \) such that \( F^{n_{k_0}}(A^u_k) = A^u_h, F^{n_{k_0}}(A^u_{k+1}) = A^u_{h+1} \), where
\[
A^u_k = A^u(h; z_{k-1} \ldots z_k), \quad A^u_{k+1} = A^u(h+1; z_{k-1} \ldots z_k z_{k+1}).
\]

By the mean-value theorem, there are \( q \in A^u_k \) and \( \tilde{q} \in A^u_{k+1} \) such that
\[
|A^u_k| = |A^u_k||(F^{n_{k_0}}_{\mu})'(q)|, \quad |A^u_{k+1}| = |A^u_{k+1}||(F^{n_{k_0}}_{\mu})'(\tilde{q})|.
\]

Let \( \delta(k) \) be the maximum width of elements in \( A(k) \). Applying Lemma 4.1 to \( F^{n_{k_0}}_{\mu} \), there exists a constant \( \tilde{c} = \tilde{c}(\delta(k)) \) of order \( \delta(k)^{\alpha} \) independent of \( n \) such that
\[
e^{\tilde{c}} \leq |(F^{n_{k_0}}_{\mu})'(q)| ||(F^{n_{k_0}}_{\mu})'(\tilde{q})|^{-1} \leq e^{\tilde{\epsilon}}.
\]

Thus we have
\[
e^{\tilde{c}} r_{u-}(m, j) \leq |A^u_k| |A^u_{k+1}|^{-1} \leq e^{\tilde{c}} r_{u+}(m, j),
\]
where
\[
r_{u-}(m, j) = \min\{ |A^u_k| |A^u_{k+1}|^{-1} ; h \in \{\kappa, \ldots, \kappa + k_0 - 1\}, \ z_{k-h+1}, \ldots, z_k \in \{1, \ldots, m-1\}, \ z_{k+1} = j \};
\]
\[
r_{u+}(m, j) = \max\{ |A^u_k| |A^u_{k+1}|^{-1} ; h \in \{\kappa, \ldots, \kappa + k_0 - 1\}, \ z_{k-h+1}, \ldots, z_k \in \{1, \ldots, m-1\}, \ z_{k+1} = j \}.
\]

One can suppose that \( \delta(k) \) is arbitrarily small by taking \( \kappa \) large enough, and hence \( \tilde{c}(\delta(k)) \) is arbitrarily close to 0.
First we consider the case of $\mu = -2$. Set $A^u_{k'} = g^{-1}(A^u_k)$ for the conjugation map $g$ given in Remark 4.4. By [27] §6.2, $A^u(0) = [2\delta, 1 - \delta]$ and $A^u(1; j) = \left[\frac{1}{2^j} + \frac{1}{2^j}, \frac{1}{2^j} + \frac{1}{2^j} - \frac{1}{2^j} - \frac{1}{2^j}\right]$ for $j = 1, \ldots, m - 1$. $A^u(1; 2) = \left[\frac{1}{2^j} + \frac{1}{2^j}, \frac{1}{2^j} - \frac{1}{2^j} - \frac{1}{2^j}\right]$, where $\delta = \frac{1}{2^{m-1}}$. Thus we have $|A^u(0)| = 1 - 3\delta$ and $|A^u(1; j)| = \frac{1}{2^j}(1 - 3\delta)$. This implies that

$|A^u(0)||A^u(1; j)|^{-1} = 2^j$.

Since the tent map $T$ is a piecewise linear map with $|DT_x| = 2$ for any $x \neq \frac{1}{2}$, $|A^u_{k'}||A^u_{k'+1}|^{-1} = 2^j$ for any $k \in \{\kappa, \ldots, \kappa + k_0 - 1\}$, $z_{k-k_0+1}, \ldots, z\in \{1, \ldots, m-1\}$ and $z_{k+1} = j$. The width of the interval $A^u_{k'}$ can be arbitrarily small if we take $\kappa$ sufficiently large. Since the conjugation map $g$ is almost affine on such a short interval, one can suppose that $r_u(m, j) > \frac{3}{4} \cdot 2^j = 3 \cdot 2^{j-2}$ and $r_u(m, j) < \frac{3}{4} 2^j = 5 \cdot 2^{j-2}$ for $\mu = -2$. Since $F^u_2$ uniformly converges to $F^u_2$ on $[-3, 3]$, as $\mu \to -2$ for $t = 1, \ldots, k_0$, there exist $\eta(m) > 0$ and an integer $\kappa(m) \geq 1$ such that $e^{-(k)} r_u(m, j) > 3 \cdot 2^{j-2}$ and $e^{-(k)} r_u(m, j) < 5 \cdot 2^{j-2}$ if $k \geq \kappa(m)$ and $\mu \in (2 - \eta(m), -2 + \eta(m))$. These show (3).

3 Suppose that $\mu = -2$. Then $A^u(0) = [2\delta, 1 - \delta]$ and $A^u(1; l) = \left[\frac{1}{2^j} + \frac{1}{2^j}, \frac{1}{2^j} + \frac{1}{2^j} - \frac{1}{2^j} - \frac{1}{2^j}\right]$ for $l = 1, \ldots, m-1$. Let $I^u_l$ be the minimum sub-interval of $A^u(0)$ containing $A^u(1; l)$ and $2\delta$, that is, $I^u_l = [2\delta, \frac{1}{2^j} - \frac{1}{2^j} - \frac{1}{2^j}\delta]$. Since $|A^u(1; l)| = \frac{1}{2} (1 - 3\delta)$ and $|I^u_l| = (\frac{1}{2^j} - \frac{1}{2^j} - \frac{1}{2^j} - \frac{1}{2^j}\delta) - 2\delta \leq \frac{1}{2^j}(1 - 3\delta)$, $|A^u(1; l)|/|I^u_l|^{-1} \geq \frac{1}{2}$. By using the argument as in (1), one can show that

$|A^u_{k+1}||I^u_k|^{-1} \geq \frac{1}{3}$

if necessary retaking $\kappa(m)$ by a larger integer and $\mu(m)$ by a smaller positive number. This shows (3).

3 The proof is quite similar to that of (2). Since $|A^u(1; l)| = \frac{1}{2^j}(1 - 3\delta)$ and $|A^u(1; l + 1)| = \frac{1}{2^j}(1 - 3\delta)$ for any $l \in \{1, \ldots, m-2\}$, we have $|A^u(1; l+1)|/|A^u(1; l)|^{-1} = \frac{1}{2}$. The connecting gap $G_l^u$ for $A^u(1; l)$ and $A^u(1; l + 1)$ has the length $|G_l^u| = \frac{1}{2^j} \delta$. Thus $|A^u(1; l)|/|G_l^u|^{-1} \leq (1 - 3\delta)\delta^{-1} = \frac{1}{2^j}$. Then one can take $\kappa(m)$ and $\eta(m)$ again so that the inequalities of (3) hold. This completes the proof.

As $\mu$ is close to $-2$, $K^u_m(\mu)$ is contained in the interior $\operatorname{Int}(I)$ of $I = [-3, 3]$. In such a situation, observe that, for every $p \in K^u_m(\mu)$, there exist $k \geq 0$, $A^u \in A^u_k$ and the closure $G$ a component of $I \setminus \bigsqcup_{A^u \in A^u_k} A^u$ such that $A^u$ and $G$ are the bridge and gap satisfying $A^u \cap G = \{p\}$. Note that $K^u_m(\mu)$ depends on the initially given $m$-periodic orbit containing $q_1$ and $q_2$. This fact together with Remark 4.4 implies the following:

Remark 4.6. $\tau(K^u_m(\mu))$ can be arbitrarily large if we take $m$ sufficiently large, see [27] §6.2.

4.4. Translation into Hénon-like maps. Hénon map introduced in Section 3 is written as

$\varphi_{\mu, \nu}(x, y) = (y, \nu x + F_\mu(y))$,

where $\nu$ is a real parameter and $F_\mu$ is the quadratic map of (4.6). Fortunately all the properties given in Section 3.3 are inherited by Hénon maps with $\nu \approx 0$.

Using the Cantor set $K^u_m(\mu)$ for $F_\mu$ with $\mu \approx -2$, one can define the subset $\{(x, F_\mu(x)) ; x \in K^u_m(\mu)\}$ of $\mathbb{R}^2$, which is a Cantor set on the parabolic curve.
\[ \text{Im}(\varphi_{\mu,0}) = \{(x, \mu + x^2); -\infty < x < \infty\}. \] For simplicity, we denote the Cantor set \( \varphi_{\mu,0}(K_m^u(\mu)) \) in \( \text{Im}(\varphi_{\mu,0}) \) again by \( K_m^u(\mu) \).

Let \( U(-2,0) \) be a small neighborhood of \((-2,0)\) in the parameter space, and let \( P_{\mu,\nu}, Q_{m;\mu,\nu}, \Gamma_{m;\mu,\nu} \) be respectively the fixed point, periodic point and basic set for \( \varphi_{\mu,\nu} \) with \( (\mu, \nu) \in U(-2,0) \) defined in Section 3. Now we consider a continuous map \( \tilde{P}: U(-2,0) \to \mathbb{R}^2 \) with \( \tilde{P}_{\mu,\nu} := \tilde{P}(\mu, \nu) \in W^u(P_{\mu,\nu}) \) and \( \tilde{P}_{-2,0} = (-2, 2) \), see Figure 4.3 For any \( (\mu, \nu) \in U(-2,0) \), let \( \tilde{l}_{\mu,\nu}^u \) be the arc in \( W^u(P_{\mu,\nu}) \) connecting \( P_{\mu,\nu} \) with \( \tilde{P}_{\mu,\nu} \). By the stable manifold theorem (for example see [30], Chapter 5, Theorem 10.1U)), \( \tilde{l}_{\mu,\nu}^u \) \( C^r \)-converges to \( \tilde{l}_{-2,0}^u \) as \( (\mu, \nu) \to (-2, 0) \), see Figure 4.3.

\[ \partial l_1 \cup \partial l_2 = \partial h_1 \cup \partial h_2 \] and such that the union \( l_1 \cup h_1 \cup l_2 \cup h_2 \) is a simple closed curve in \( \mathbb{R}^2 \) bounding a compact region \( E_{\mu,\nu} \) which contains the basic set \( \Gamma_{m;\mu,\nu} \)

\[ \text{Figure 4.3.} \]

For any \( (\mu, \nu) \approx (-2, 0) \) with \( \nu \neq 0 \), consider arcs \( h_i \) \((i = 1, 2)\) in \( W^u_{\text{loc}}(\varphi_{\mu,\nu}^1(Q_{m;\mu,\nu})) \) as illustrated in Figure 4.4 Let \( l_j \) \((j = 1, 2)\) be parabolic curves in \( \mathbb{R}^2 \) with

\[ \text{Figure 4.4. (a) The case of } \nu > 0. \text{ (b) The case of } \nu < 0. \]
and such that \( \ell^u_{\mu,\nu} = \hat{\ell}^u_{\mu,\nu} \cap E_{\mu,\nu} \) is an arc connecting \( h_1 \) with \( h_2 \). Consider a local stable foliation \( F^s_{\mu,\nu} = F^s_{\text{loc}}(\Gamma_{m;\mu,\nu}) \) on \( E_{\mu,\nu} \) compatible with \( W^s_{\text{loc}}(\Gamma_{m;\mu,\nu}) \), that is,

- each component of \( W^u_{\text{loc}}(\Gamma_{m;\mu,\nu}) \cap E_{\mu,\nu} \) is a leaf of \( F^s_{\mu,\nu} \);
- \( \ell^u_{\mu,\nu} \) crosses \( F^s_{\mu,\nu} \) exactly, that is, each leaf of \( F^s_{\mu,\nu} \) intersects \( \ell^u_{\mu,\nu} \) transversely in a single point and any point of \( \ell^u_{\mu,\nu} \) is passed through by a leaf of \( F^s_{\mu,\nu} \);
- leaves of \( F^s_{\mu,\nu} \) are \( C^3 \) curves such that themselves, their directions, and their curvatures vary \( C^1 \) with respect to any transverse direction and \((\mu, \nu)\).

See [17, Lemma 4.1] and [18, §2.3] for details. Let \( \pi^u_{\mu,\nu} : E_{\mu,\nu} \to \ell^u_{\mu,\nu} \) be the projection along the leaves of \( F^s_{\mu,\nu} \). Define

\[
K^u_{m;\mu,\nu} := \pi^s_{\mu,\nu}(\Gamma_{m;\mu,\nu}),
\]

which is a Cantor set dynamically defined by \( \pi^u_{\mu,\nu} \circ \varphi_{\mu,\nu} \). Here we note that the set \( K^u_{m;\mu,\nu} \) does not depend on the choice of the local stable foliation \( F^s_{\mu,\nu} \) on \( E_{\mu,\nu} \) compatible with \( W^s_{\text{loc}}(\Gamma_{m;\mu,\nu}) \).

If \((\mu, \nu)\) is close to \((-2,0)\), then one can define the presentation involved with bridges and gaps for the Cantor set \( K^u_{m;\mu,\nu} \) in a manner quite similar to that for \( K^u_m \). Then Lemma 4.5 and Remark 4.6 are translated as follows if necessary replacing \( \mathcal{U}(-2,0) \) by a smaller neighborhood of \((-2,0)\):

**Remark 4.7.** (1) For every \((\mu, \nu)\in\mathcal{U}(-2,0)\), the bounded distortion property in Lemma 4.5 holds for \( K^u_{m;\mu,\nu} \).

(2) The thickness \( \tau(K^u_{m;\mu,\nu}) \) of \( K^u_{m;\mu,\nu} \) converges to \( \tau(K^u_{m;\mu,0}) \) as \( \nu \to 0 \), and hence it can have an arbitrarily large value if we take \( m \) large enough and \((\mu, \nu)\in\mathcal{U}(-2,0) \) sufficiently close to \((-2,0)\), see [27 §6.3, Proposition 1].

Note that the return map

\[
\varphi_n := \Phi_n^{-1} \circ f_{\mu_n}^{N_n+n} \circ \Phi_n(x, y)
\]

of Theorem 3.1 is arbitrarily \( C^r \)-close to \( \varphi_{-2,0} \) if \( n \) is sufficiently large. Locally identifying the coordinate on \( \mathbb{R}^2 \) with that on a small neighborhood \( U(q) \) of \( q \) in \( M \), we may set

\[
\varphi_n = f_{\mu_n}^{N_n+n}.
\]

The \( \varphi_n \) is an Hénon-like map with the saddle fixed point \( P(\varphi_n) \) and the basic set \( \Gamma_m = \Gamma_m(\varphi_n) \) corresponding to \( P_{\mu,\nu} \) and \( \Gamma_{m;\mu,\nu} \). Let \( E \) be a compact region corresponding to \( E_{\mu,\nu} \) and \( \pi_m : E \to \ell^u(\varphi_n) \) the projection along the leaves of a stable foliation \( F^s_{\text{loc}}(\Gamma_m) \) on \( E \) compatible with \( W^s_{\text{loc}}(\Gamma_m) \), where \( \ell^u(\varphi_n) \) is the curve in \( W^u(P(\varphi_n)) \cap E \) corresponding to \( l^u(\varphi_{\mu,\nu}) \). Then one obtains the dynamically defined Cantor set

\[
K^u_m := \pi_m(\Gamma_m).
\]

**Remark 4.8.** The distortion and thickness for \( K^u_m \) have the same properties as those in Remarks 4.7 if \( \varphi_n \) is sufficiently close to \( \varphi_{-2,0} \).

5. **Linking property for bridges**

Recall that \( \{f_{\mu}\}_{\mu \in \mathbb{R}} \) is the one-parameter family of two-dimensional \( C^r \) diffeomorphisms given in Section 3. In this section, we will present Linking Lemma, which is crucial in the proof of Theorem A.
5.1. **Heteroclinic tangencies.** Let \( h : [0,1] \times [0,1] \to M \) be an embedding. Set \( R = h([0,1] \times [0,1]) \), \( zR = h([0,1] \times \{0,1\}) \) and \( R = h([0,1] \times [0,1]) \). Note that \( zR \cup bR = \partial R \). We say that the pair \((R, zR)\) (for short \( R \)) is a strip with the edge \( zR \). Similarly, the pair \((R, bR)\) is a strip with the edge \( bR \). A strip \((R', zR')\) is called a sub-strip of \((R, zR)\) if \( R' \subset R \) and each component of \( zR \) contains a component of \( zR' \).

One can take a coordinate neighborhood of the fixed point \( p \) in \( M \) such that \( p = (0,0), f^N_0(q) = (0,1) \) and \( f_\mu \) is linear on \( S = [0,2] \times [0,2] \) and satisfying the condition \( (S_{3.1}) \) in Section 3, where \( N_\ast \) is the positive integer given in Theorem 3.1. Consider the foliation \( \mathcal{F}_S \) on \( S \) consisting of horizontal leaves, that is, any leaf of \( \mathcal{F}_S \) has form \([0,2] \times \{y\}\) for some \( 0 \leq y \leq 2 \). Though \( \mathcal{F}_S \) is not in general an \( f_\mu \)-invariant foliation, the \( f_\mu^{-n}\)-image of the restriction of \( \mathcal{F}_S \) on \([0,2^{\lambda_n}] \times [0,2] \) is a sub-lamination of \( \mathcal{F}_S \) for any positive integer \( n \).

For any \( \mu \) near \(-2\), let \( \mu_n = \Theta_n(\mu) \) be the parameter value used in 3.2. We set \( f_{\mu_n} = f_\mu \) for short throughout the remainder of this subsection. Take an integer integer \( m \geq 3 \). From the condition \( (S_{3.1}) \), for any sufficiently large integer \( n \), we have the horseshoe \( \Lambda := \Lambda(f_n) \) containing \( p \) for \( f_n \) the basic set \( \Gamma_m := \Gamma_m(\varphi_n) \) for the return map \( \varphi_n := f_n^{N_\ast+n} \) defined as \((4.8)\) near the homoclinic tangency \( q \). Recall that \( \Gamma_m \) contains the saddle fixed point \( \bar{P} := P(\varphi_n) \) and the \( m \)-periodic orbit \( Q_m^{(i)} := \varphi_n^i(Q_m) \) for \( i = 0, 1, \ldots, m - 1 \). Let \( E \) be the compact region and \( \ell^u(\varphi_n) \) the arc given in Subsection 4.4. Again by the condition \( (S_{4.4}) \), there is an integer \( N_1 > 0 \) such that each leaf of \( f_n^{-N_1}(\mathcal{F}_S) \) meets leaves of \( W_{\text{loc}}(\Gamma_m) \) transversely in \( E \). See Figure 5.1 for the case of \( m = 4 \). For any integer \( i \geq 0 \), let \( A^u(i) \) be the set of \( u \)-bridges \( A^u(i, z) \) in \( \ell^u(\varphi_n) \) of generation \( i \) with \( A^u(0) = \ell^u(\varphi_n) \) with respect to the Cantor set \( K_n^\ast \) of \((4.9)\), and let \( A^u = \bigcup_{i=0}^{\infty} A^u(i) \). Note that the itinerary \( z = (z_1, \ldots, z_l) \) of \( A^u(i, z) \) is an element of \( \{1, \ldots, m - 1\}^i \). Let \( G^u(0) \) be the leading gap of \( A^u(0) \). The closure \( \mathbb{C}^u(0) \) of the component of \( E \setminus W_{\text{loc}}^u(\Gamma_m) \) containing \( G^u(0) \) is a strip with \( W_{\text{loc}}^u(\Gamma_m) \cap \mathbb{C}^u(0) = b\mathbb{C}^u(0) \). For any \( A^u \in A^u \), the strip \( \mathbb{G}(A^u) \) containing the leading gap \( G(A^u) \) of \( A^u \) is defined similarly, see Figure 5.2.
For the square $S = [0, 2] \times [0, 2]$, set $\delta S = [0, 2] \times \{0, 2\}$ and $\beta S = \{0, 2\} \times [0, 2]$. For our choice of $N_1$, there exists a sub-strip $(S(0), \delta S(0))$ of $(S, \delta S)$ such that $\beta S(0)$ consists of two arcs in $S$ meeting $F_S$ transversely and $(f^{-N_1}_n(S(0)), f^{-N_1}_n(\beta S(0)))$ is a sub-strip of the strip $((\mathcal{G}^u(0), \mathcal{G}^u(0)))$. If the generation of a $u$-bridge $A^u$ is $i$, then there exists a unique integer $\bar{i}$ with

\[(5.1) \quad i \leq \bar{i} \leq (m-1)i\]

such that $(\varphi^\bar{i}_n(\mathcal{G}(A^u)), \varphi^\bar{i}_n(\beta \mathcal{G}(A^u)))$ is a sub-strip of $(\mathcal{G}^u(0), \beta \mathcal{G}^u(0))$. Then

\[
\ell^u(\varphi_n) = \pi^m \circ \varphi^\bar{i}_n(A^u)
\]

holds. There exists a sub-strip $(S(A^u), \delta S(A^u))$ of $(S(0), \delta S(0))$ such that $f^{-N_1}_n(S(A^u)) = \varphi^\bar{i}_n(\mathcal{G}(A^u)) \cap f^{-N_1}_n(S(0))$. Then we have a sub-strip $(E(A^u), \delta E(A^u))$ of $(\mathcal{G}^u(0), \beta \mathcal{G}(A^u))$ with

\[
(\varphi^\bar{i}_n(E(A^u)), \varphi^\bar{i}_n(\delta E(A^u))) = (f^{-N_1}_n(S(A^u)), f^{-N_1}_n(\beta S(A^u))).
\]

The strip $E(A^u)$ has the foliation $F_{A^u}$ induced from $F_S$ via $\varphi^{-\bar{i}} \circ f^{-N_1}$. One can retake the local unstable foliation $F^u_{\text{loc}}(\Gamma_m)$ on $E$ compatible with $W^u_{\text{loc}}(\Gamma_m)$ and extending $\bigcup_{A^u \in A} F_{A^u}$. The reason why we choose the unstable foliation with $\varphi^{-\bar{i}} \circ f^{-N_1}(F^u_{\text{loc}}|_{S(A^u)})$ will be explained in the proof of Lemma 8.3.

Recall that $F^u_{\text{loc}}(\Lambda)$ is a local unstable foliation on $S$ compatible with $W^u_{\text{loc}}(\Lambda)$ and containing $\{2\} \times [0, 2]$ as a leaf. From the conditions (S[V], S[VI]) of Section 3 there are integers $N_0 > 0$, $N_2 > N_1$, and a $C^1$-arc $L$ in $U(q)$ meeting $f^{-N_0}_n(F^u_{\text{loc}}(\Gamma_m))$ exactly and such that $L' = L \cap f^{-N_2}_n(F^u_{\text{loc}}(\Lambda))$ is a sub-arc of $L$ each element of which is a quadratic tangency of leaves of $f^{-N_0}_n(F^u_{\text{loc}}(\Gamma_m))$ and $f^{-N_2}_n(F^u_{\text{loc}}(\Lambda))$. See Figure 5.3. Let $\pi^u : E \rightarrow L$ be the projection along the leaves of $f^{-N_0}_n(F^u_{\text{loc}}(\Gamma_m))$.

For the sub-strip $S(A^u)$ of $S(0)$ given as above, let $\pi_{S,A^u} : S(A^u) \rightarrow L$ be the projection defined by $\pi_{S,A^u} = \pi^u \circ \varphi^{-\bar{i}} \circ f^{-N_1}_n|_{S(A^u)}$. Let

\[(5.2) \quad \pi_{A^u} : B^u(0) \rightarrow L
\]

be the composition $\pi_{S,A^u} \circ \psi_{A^u}$, where $\psi_{A^u} : B^u(0) \rightarrow S(A^u)$ is a diffeomorphism from $B^u(0) = \{0\} \times [0, 2]$ onto a component of $\beta S(A^u)$ along the leaves of $F_S$. Let $K^u_{\Lambda,L}$, $K^u_{m,L}$ be the Cantor sets on $L$ defined as

\[(5.3) \quad K^u_{\Lambda,L} = \pi^u(K^u_{\Lambda}) \quad \text{and} \quad K^u_{m,L} = \pi^u(K^u_{m}),
\]
where

$$\pi^s : S \rightarrow L$$

is the projection along the leaves of \( f^{N_2}(F^u_{loc}(\Lambda)) \). For any \( B^s \in B^s \), let \( B^s_L := \pi^s(B^s) \) is a bridge of \( K^s_{t,L} \). Similarly, for any \( A^u \in A^u \), let \( A^u_L := \pi^s(A^u) \) is a bridge of \( K^u_{m,L} \).

### 5.2. Encountering of \( s \)-bridges and \( u \)-bridges, I.

Here we will study the heteroclinical connection between \( s \)-bridges \( B^s \) of \( K^s_{t,L} \) and \( u \)-bridges \( A^u \) of \( K^u_{m,L} \) in \( L \).

Let \( U(\varphi_{-2,0}) \) be a sufficiently small \( C^r \)-neighborhood of \( \varphi_{-2,0} \). For any sufficiently large \( n \), the return map \( \varphi_ := \varphi_n \) of (4.8) is contained in \( U(\varphi_{-2,0}) \). Theorem 3.1 together with Remark 4.8 assures that, if we take the integer \( m \) sufficiently large, then the Cantor set \( K^u_m \) satisfies

$$\tau(K^u_m) > \max \{ r^2_{s+}, \tau(K^s_t)^{-1}, 3^n \}.$$  

Here \( n_+(m) \) is a positive integer with \( \lim_{m \to \infty} n_+(m) = \infty \) such that \( f_{\mu n_+} \) is arbitrarily \( C^r \)-close to \( f \). Write \( \tau := \tau(K^u_m) \) for short and let

$$\xi_0 := \left( \frac{1}{r_{s+}} - \frac{2^{1/2}}{2^{1/2}} \right) r_{s-}.$$  

By (5.6) and Lemma 4.3 we have \( 0 < \xi_0 < 1 \).

By Theorem 3.1 one can choose \( \mu_n = \Theta_{n, (\hat{\mu})} \neq 0 \) so that the Cantor sets \( K^s_{t,L} \) and \( K^u_{m,L} \) on \( f_{\mu n} \) are linked in \( L \). We denote the \( f_{\mu n} \) again by \( f \). Note that the leaves of \( f^{N_0}(F^s_{loc}(\Gamma_m)) \) are almost horizontal in \( U(q) \). Then the vertical line \( L_0 := \{ x = 1 \} \cap U(q) \) meets the leaves of \( f^{N_0}(F^s_{loc}(\Gamma_m)) \) almost orthogonally, see Figure 5.4. We consider a new \( C^{1+\alpha} \)-coordinate on a small neighborhood \( U(L) \) of \( L \) in \( U(q) \) such that each horizontal line is a leaf of \( f^{N_0}(F^s_{loc}(\Gamma_m)) \) and each vertical line is contained in a vertical line with respect to the original coordinate on \( U(q) \).
A parametrization of \( L \). The \( C^1 \)-arc \( L \) has the parametrization naturally induced from the \( C^{1+\alpha} \)-coordinate on \( U(L) \). However, we need another parameterization on \( L \) suitable to our argument. For the sub-arc \( L' = L \cap f^{N_2}(F_{loc}^u(A)) \), \( L = f^{-N_2}(L') \) is a \( C^1 \)-arc in \( S \) connecting the components of \( \flat S \). See Figure 7.3. Since \( L \) meets \( f^{N_2}(\{0\} \times [0, 2]) \) transversely, \( L' \) also meets \( \{0\} \times [0, 2] \) transversely. Thus there exists a small \( \delta > 0 \) such that \( \tilde{L}_\delta = f^{-N_2}(\tilde{L}_\delta) \) is a sub-arc of \( \tilde{L} \) meeting vertical lines in \( [0, \delta] \times [0, 2] \) transversely. We suppose that the parameter value of \( x \in \tilde{L}_\delta := f^{-N_2}(\tilde{L}_\delta) \) is \( t \) if the first coordinate of \( f^{-N_2}(x) \) is \( t \) with respect to the orthogonal coordinate on \( S \) and \( L \) has a \( C^1 \)-parametrization extending that on \( \tilde{L}_\delta \). For any subset \( A \) of \( L \) and any constant \( c \), \( A + c \) is the subset of \( L \) obtained from \( A \) by the \( c \)-shift along \( L \) if it is well defined. For any interval \( I \) of \( L \), \( |I| \) denotes the diameter of \( I \) with respect to the parametrization on \( L \). See Figure 5.4.

We need another coordinate on \( U(L) \), which is a moving \( C^3 \)-coordinate \( C^{1+\alpha} \)-depending on a given point \( x_0 \) of \( U(L) \). As seen in Subsection 4.4, each leaf of \( f^{-N_0}(F_{loc}^u(\Gamma_m)) \) is a \( C^3 \)-curve. In particular, the horizontal line \( L_{x_0}^{hor} \) with respect to the \( C^{1+\alpha} \)-coordinate passing through \( x_0 \) is a \( C^3 \)-curve. Thus one can define a \( C^3 \)-coordinate on \( U(L) \) with the origin \( x_0 \) and each vertical line of which is equal to an original vertical line and each horizontal line of which is obtained by a parallel translation of \( L_{x_0}^{hor} \) to the vertical direction. Such a \( C^3 \)-coordinate will be used in Remark 8.2.

For given bridges \( B^s_L \) of \( K^{s}_{\Lambda, L} \) and \( A^u_L \) of \( K^{uc}_{m, L} \), we say that the pair \( (B^s_L, A^u_L) \) is linked if

- \( \text{Int}(B^s_L \cap A^u_L) \neq \emptyset \), and
- \( B^s_L \) is not contained in a gap of \( A^u_L \cap K^{uc}_{m, L} \) and \( A^u_L \) is not contained in a gap of \( B^s_L \cap K^{s}_{\Lambda, L} \).

Moreover the pair is called \( \xi \)-linked for a constant \( 0 < \xi \leq 1 \) if

\[ |B^s_L \cap A^u_L| \geq \xi \min\{|B^s_L|, |A^u_L|\}. \]

The pair \( (B^s_L, A^u_L) \) is \( \gamma \)-proportional for a constant \( \gamma \) with \( 0 < \gamma < 1 \) if

\[ |A^u_L| \geq |B^s_L| \geq \gamma |A^u_L|, \]

and the pair is just proportional if \( |A^u_L| \geq |B^s_L| \).

\[ \text{Figure 5.4.} \]
Remark 5.1. Let $B^s, A^u$ be the bridges of $K^s_\lambda$ and $K^u_m$ respectively. Since $\pi^s$ and $\pi^u\circ \pi_m$ are $C^1$-maps, $\pi^s|_{B^s} : B^s \to B^s_L$ and $\pi^u|_{A^u} : A^u \to A^u_L$ are almost affine if $|B^s|$ and $|A^u|$ are sufficiently small. Thus one can suppose the following conditions without loss of generality.

(i) For bridges $B^s_{L,k}, B^s_{L,k+1}$ of bridges of $K^s_{m,L}$ with sufficiently large generations $k, k+1$ and $B^u_{L,k} \supset B^u_{L,k+1}$, the conclusion of Lemma 4.3 holds, that is,

\[ r_{s-} \leq |B^s_{L,k}|^{-1} \leq r_{s+} \]

if necessary modifying $r_{s-}$ and $r_{s+}$ slightly.

(ii) For bridges $A^u_{L,k}, A^u_{L,k+1}$ of bridges of $K^u_{m,L}$ with sufficiently large generations $k, k+1$ and $A^u_{L,k} \supset A^u_{L,k+1}$ and the gaps $G^u_{L,k+1}$, the interval $I^u_k$ corresponding to those in Lemma 4.5 the conclusions (1)–(3) of Lemma 4.5 hold.

(iii) From the definition of the thickness, the restricted Cantors sets satisfy $\tau(B^s \cap K^s_\lambda) \geq \tau(K^s_\lambda)$ and $\tau(A^u \cap K^u_m) \geq \tau(K^u_m)$. Thus, for any $0 < \varepsilon < 1$ and bridges $B^s, A^u$ with sufficiently large generation, $\tau(B^s \cap K^s_{m,L}) \geq (1-\varepsilon)\tau(B^s \cap K^s_\lambda)$ and $\tau(A^u \cap K^u_{m,L}) \geq (1-\varepsilon)\tau(A^u \cap K^u_m)$ hold. From (5.6), one can suppose that

\[ \tau(B^s_L \cap K^s_{m,L}) \tau(A^u_L \cap K^u_{m,L}) > 1. \]

See Subsections 4.1 and 4.2 of [19] for similar arguments.

**The first perturbation of $f$.** Consider the gap strip $G^u(0) = \pi^{-1}_{\mathcal{F}^u_{loc}(\Lambda)}(G^u(0))$ in $S$, where $G^u(0) = \text{Gap}(B^u(1;1), B^u(1;2))$ and $\pi_{\mathcal{F}^u_{loc}(\Lambda)} : S \to \{0\} \times [0,2]$ is the projection along the leaves of $\mathcal{F}^u_{loc}(\Lambda)$. Recall that $L'$ is the sub-arc of $L$ given in Subsection 5.1 meeting $f^{-N_2}(\mathcal{F}^u_{loc}(\Lambda))$ exactly (see Figure 5.3) and $L = f^{-N_2}(L')$ is a $C^1$-arc in the strip $G^u(0)$ disjoint from $\mathfrak{s}G^u(0)$ and meeting $\mathcal{F}^u_{loc}(\Lambda)$ exactly. See Figure 7.3. For a sufficiently small $\delta_0 > 0$, let $B_{h_0}$ and $B_{\delta_0}$ be the disks in $M$ centered at the left edge of $L$ of radius $\delta_0/2$ and $\delta_0$ respectively such that $B_{\delta_0} \cap S \subset G^u(0)$ and $f^{-1}(B_{\delta_0}) \cap X_n = \emptyset$, where $X_n$ is the union of rectangles used in Section 3 to define the basic set $\Gamma_m$ satisfying the conditions (S.3) and (S.7). See Figure 3.2. From the disjointness, any perturbation of $f$ supported on $f^{-1}(B_{\delta_0})$ does not affect the invariant set $\Gamma_m$ and hence the local stable foliation $\mathcal{F}^s_{loc}(\Gamma_m)$ on $E$. We parametrize $L \cap B_{\delta_0}$ by the first coordinates of elements. Then $f^N_{L_{\mathcal{F}^u_{loc}}(\Gamma_m) : L \cap B_{\delta_0} \to L$ is a parameter-preserving map, where $L$ has the parametrization defined as in Subsection 5.2.

For any $\delta$ with $|\delta|$ sufficiently smaller than $\delta_0$, consider the perturbation of $f$ supported on $f^{-1}(B_{\delta_0})$ such that the restriction of the perturbed map $\tilde{f}_\delta$ on $f^{-1}(B_{\delta_0/2})$ is the horizontal $\delta$-shift. Strictly, this means that $f_\delta(x) = f(x) + (\delta,0)$ for any $x \in f^{-1}(B_{\delta_0/2})$. One can construct $f_\delta$ so as to $C^\infty$-converge to $f$ as $\delta \to 0$. In particular, any $f_\delta$ can be supposed to satisfy the conditions (5.6), (5.8) and (5.9).

Note that the perturbation deforms the arc $L$. Let $\tilde{L}(\delta)$ be the arc consisting of tangencies between $\mathcal{F}^u_{loc}(\Lambda; \delta) := f_\delta(\mathcal{F}^u_{loc}(\Lambda))|_{S \cap B_{\delta_0}} \cup \mathcal{F}^u_{loc}(\Lambda)|_{S \cap B_{\delta_0}}$ and $f^{-(N_0+N_2)}(\mathcal{F}^u_{loc}(\Gamma_m))$. See Figure 5.5. From the Implicit Function Theorem, the arcs $\tilde{L}(\delta) C^1$-depend on $\delta$ and $\tilde{L}(0) = L$. Let $B^s(\delta)$ be the bridge and $K^s_\lambda(\delta)$ the Cantor set in $\tilde{L}(\delta)$ which are projected respectively onto the bridge $B^s$ and the Cantor $K^s_\lambda$ in $W^s_{loc}(p)$ along the leaves of $\mathcal{F}^u_{loc}(\Lambda; \delta)$. Let $L(\delta)$ be an arc in $U(L)$ containing $f^{N_2}(\tilde{L}(\delta))$ and crossing $f^{-N_2}(\mathcal{F}^u_{loc}(\Gamma_m))$ exactly, and let $\pi^u : E \to L(\delta)$ be the projection along the leaves
δ for some –(2) and satisfying the following properties of Lemma 4.5 and Remark 5.1 hold. Then, for any \( ε \) of \( \tilde{K} \) bridge of \( B \) Moreover, suppose that the generations of \( y \) \( y \) satisfy (5.10). Let \( \pi_\delta: \tilde{L} \to \tilde{L}(\delta) \) be the projection along the leaves of \( \mathcal{F}_\text{loc}^u(\Lambda; \delta) \), and let \( \pi^y_s: \tilde{L} \to \tilde{L}(\delta) \) be the projection along the leaves of \( f^{-(N_0+N_2)}(\mathcal{F}_\text{loc}^u(\Gamma_m)) \). From our construction of \( f_\delta \), \( B^u(\delta) = \pi^y_\delta(B^u(0)) \) for any \( s \)-bridge \( B^u(0) \) in \( \tilde{L} \cap B_{\delta_0/2} \). Similarly, \( \pi_s^\delta(A^u(0)) = A^u(\delta) \) for any \( u \)-bridge \( A^u(0) \) in \( \tilde{L} \cap B_{\delta_0/2} \).

For \( x, x' \) in \( B^u(0) \), let \( x_\delta, x_\delta' \in B^u(\delta) \) be the \( \pi^\delta_\delta \)-images of \( x, x' \), and similarly \( y_\delta, y_\delta' \in A^u(\delta) \) the \( \pi^\delta_\delta \)-images of \( y, y' \) in \( A^u(0) \). Since both \( \pi^\delta_\delta \) and \( \pi^\delta_\delta \) \( C^1 \)-converge to the identity of \( \tilde{L} \) as \( \delta \to 0 \), there exists a constant \( C > 0 \) independent of the bridges and satisfying

\[
\begin{align*}
(1 - C|\delta|)||x - x'|| &\leq ||x_\delta - x_\delta'|| \leq (1 + C|\delta|)||x - x'||, \\
(1 - C|\delta|)||y - y'|| &\leq ||y_\delta - y_\delta'|| \leq (1 + C|\delta|)||y - y'||, \\
(1 - C|\delta|)||x + (\delta, 0) - y|| &\leq ||x_\delta - y_\delta|| \leq (1 + C|\delta|)||x + (\delta, 0) - y|| + o(\delta),
\end{align*}
\]

where the terms \( \pm o(\delta) \) in the third inequalities are derived form the fact that any leaves of the \( C^{1+\varepsilon} \) foliations \( \mathcal{F}_\text{loc}^u(\Lambda; \delta) \) and \( f^{-(N_0+N_2)}(\mathcal{F}_\text{loc}^u(\Gamma_m)) \) passing through the same point of \( \tilde{L}(\delta) \) are tangent to each other at the point.

**Lemma 5.2 (Linking Lemma).** Suppose that the dynamically defined Cantor sets \( K_\Lambda^u \) and \( K_m^u \) satisfy \( \tilde{L}(\delta) \). Let \( B^u(0) \) be an \( s \)-bridge of \( K^u_\Lambda(0) \) and \( A^u(0) \) a \( u \)-bridge of \( K^u_m(0) \) contained in \( B_{\delta_0/2} \cap \tilde{L} \) such that the pair \( (B^u(0), A^u(0)) \) is linked. Moreover, suppose that the generations of \( B^u \) and \( A^u \) are so large that the conditions of Lemma 4.5 and Remark 5.1 hold. Then, for any \( \varepsilon \) with \( 0 < \varepsilon \leq |B^u(0) \cap A^u(0)| \) and sufficiently smaller than \( \delta_0 \), there exist bridges \( B^u_1, \tilde{B}^u_1, A^u_1, \tilde{A}^u_1 \) and gaps \( G^u, G^u \) in \( \tilde{L} \) with

\[
B^u_1, \tilde{B}^u_1 \subset B^u(0), \quad G^u = \text{Gap}(B^u_1, \tilde{B}^u_1), \quad A^u_1, \tilde{A}^u_1 \subset A^u(0), \quad G^u = \text{Gap}(A^u_1, \tilde{A}^u_1)
\]

and satisfying the following properties \( 1 \),(2) for any \( \nu \) with \( |\nu| < \varepsilon \) and \( 3 \),(4) \) for some \( \delta \) with \( |\delta| < \varepsilon \):

1. \( r_{s+1}^{-1}|r|^{-5/4} \leq |B^u_1(\nu)| < r_{s+1}^{-1}|r|^{-3/4} \varepsilon, \quad r_{s+1}^{-1}|r|^{-5/4} \leq |\tilde{B}^u_1(\nu)| < r_{s+1}^{-1}|r|^{-3/4} \varepsilon; \)
(2) both \((B^1_1(v), A^1_1(v))\) and \((\tilde{B}^1_1(v), \tilde{A}^1_1(v))\) are \(\tau^{-1}\)-proportional;

(3) both \((B^2_2(\delta), A^2_2(\delta))\) and \((\tilde{B}^2_2(\delta), \tilde{A}^2_2(\delta))\) are \(\xi_0\)-linked pairs, where \(\xi_0\) is the constants given in \((5.7)\);

(4) \(G^u(\delta)\) and \(G^s(\delta)\) have a common middle point.

Note that \(\xi_0\) of \((5.7)\) is a universal constant independent of the choices of \(B^*, A^u\) or \(\varepsilon\). Here we set \(B^1_1, B^1_1, A^u, \tilde{A}^u\) for simplicity instead of \(B^1_1(0), \tilde{B}^1_1(0), A^1_1(0), \tilde{A}^1_1(0)\) respectively.

The following lemma is essential in the proof of Lemma \(5.2\).

**Lemma 5.3.** Let \((B^*(0), A^*(0))\) be the linked pair given in Proposition \(5.2\). For any \(\varepsilon\) with \(0 < \varepsilon \leq |B^*(0) \cap A^*(0)|\) and sufficiently smaller than \(\delta_0\), there exist an interval \(J_1\) with \(J_1 \subset (-\varepsilon, \varepsilon)\), sub-bridges \(\tilde{B}^1_1 \subset B^*(0)\) and \(\tilde{A}^1_1 \subset A^*(0)\) satisfying the following conditions.

1. \(\tau^{-5/4} \leq |\tilde{B}^1_1(\nu)| < \tau^{-3/4} \varepsilon\) for any \(\nu\) with \(|\nu| \leq \varepsilon\);
2. \(\tau^{-1/2} |\tilde{A}^1_1(\nu)| \leq |\tilde{B}^1_1(\nu)| < \tau^{-1/4} |\tilde{A}^1_1(\nu)|\) for any \(\nu\) with \(|\nu| \leq \varepsilon\);
3. \(\tilde{B}^1_1(\nu) \cap \tilde{A}^1_1(\nu) \neq \emptyset\) if and only if \(\nu \in J_1\).

**Proof.**

1. First we consider the case of \(\nu = 0\). Since the pair \((B^*(0), A^*(0))\) is linked and \(\tau(B^*(0) \cap K_A^*(0)) \tau(A^*(0) \cap K_m^*(0)) > 1\) by \((5.9)\), it follows from Gap Lemma that \((B^*(0) \cap K_A^*(0)) \cap (A^*(0) \cap K_m^*(0))\) contains a point, say \(a_0\). Take an \(s\)-bridge \(B^*(i)\) with \(B^*(i) \ni a_0\) and \(|B^*(i)| < \tau^{-3/4}\varepsilon\) where \(i\) represents the generation of \(B^*(i)\). If \(|B^*(i)| \geq \tau^{-5/3}\varepsilon\), then we set \(\tilde{B}^1_1 = B^*(i)\). Otherwise, consider the \(s\)-bridge \(B^*(i-1)\) with \(B^*(i-1) \ni a_0\). Since \(r^2_{s+1} < \tau\) by \((5.6)\), we have from Lemma \(4.3\) that

\[
|B^*(i-1)| \leq r_{s+1}|B^*(i)| \leq r_{s+1}\tau^{-5/4}\varepsilon < \tau^{-3/4}\varepsilon \quad \text{and} \quad |B^*(i-1)| \geq 2|B^*(i)|.
\]

If \(|B^*(i-1)| \geq \tau^{-5/4}\varepsilon\), then we set \(\tilde{B}^1_1 = B^*(i-1)\). Otherwise, we repeat the same process until we get the \(s\)-bridge containing \(a_0\) and satisfying the inequality of \((1)\). We adopt the bridge as \(\tilde{B}^1_1\). This shows \((1)\) for the case of \(\nu = 0\). From \((5.10)\), we know that it is not hard to generalize this result to the case of \(|\nu| \leq \varepsilon\).

2. First we consider the case of \(\nu = 0\). We mean by \(A^*(i)\) that the generation of the \(u\)-bridge is \(i\). Suppose that \(A^u = A^u(j)\). First we show that there exists a sub-bridge \(A^u_0\) of \(A^u\) with

\[
\frac{\varepsilon}{3\tau^{1/4}} \leq |A^u_0| \leq \frac{\varepsilon}{3}.
\]

and contained in a closed sub-arc of \(A^u\) of width \(\varepsilon/3\) and containing \(a_0\). Let \(A^u(j + 1)\) be the sub-bridge of \(A^u\) containing \(a_0\). If \(A^u(j + 1) \geq \varepsilon/3\), then we repeat the argument using \(A^u(j + 1)\) instead of \(A^u\). So it suffices to consider the case of \(A^u(j + 1) < \varepsilon/3\). Suppose that \(I\) is a sub-arc of \(A^u\) with \(|I| = \varepsilon/3\) and containing \(a_0\) as a boundary point. If \(A^u(j + 1) \geq \varepsilon/3\tau^{1/4}\), then one can set \(A^u_0 = A^u(j + 1)\). Otherwise, consider the maximum sub-arc \(I'\) of \(A^u\) with \(a_1\) as a boundary point and containing \(I\), where \(a_1\) is the the boundary point of \(I\) other than \(a_0\), see Figure \(5.6\). Let \(A^u_1(j + 1)\) be the sub-bridge of \(A^u\) closest to \(a_0\) among all \(u\)-bridges not contained in \(I'\). By Lemma \(1.5\) \((2)\) (together with Remark \(5.1\) \((ii)\) for strictly), \(|A^u_1(j + 1)| \geq |I'|/3 \geq |I|/3 = \varepsilon/9\). By Lemma \(4.5\) \((3)\) and \((5.6)\), the \(u\)-bridge \(A^u_2(j + 1)\) closest to \(a_1\) among all \(u\)-bridges contained in \(I'\) satisfies \(|A^u_2(j + 1)| \geq |A^u_1(j + 1)|/3 > \varepsilon/27 > \varepsilon/3\tau^{1/4}\). Thus \(A^u_0 := A^u_2(j + 1)\) satisfies \((5.11)\).
Consider the sequence of $u$-bridges

$$A^n_0 = A^n(k) \supseteq A^n(k+1) \supseteq \cdots \supseteq A^n(k+i) \supseteq A^n(k+i+1) \supseteq \cdots$$

such that, for any integer $i \geq 0$, the bottom point of $A^n(k+i+1)$ is equal to the leading point of $A^n(k+i)$. By (5.11),

$$|A^n_0| \geq \frac{\varepsilon}{3^{1/4}} = \tau^{1/4} \frac{\varepsilon}{3^{1/4}} > \tau^{1/4} \frac{\varepsilon}{4^{1/4}} \geq \tau^{1/4} |\hat{B}_1^n|.$$

Thus there exists $i \geq 0$ such that $|A^n(k+i+1)| \leq \tau^{1/4} |\hat{B}_1^n| < |A^n(k+i)|$. By Lemma 4.5 (1),

$$|\hat{B}_1^n| \geq \tau^{-1/4} |A^n(k+i+1)| \geq \tau^{-1/4} \frac{|A^n(k+i)|}{\varepsilon} > \tau^{-1/2} |A^n(k+i)|.$$

Here we used the inequality $\tau > 3^8 > 5^4$ derived from (5.6). Thus $\hat{A}^n_i := A^n(k+i)$ satisfies the inequality of (2) for $\nu = 0$. Again by using (5.10), one can generalize this result to the case of $|\nu| \leq \varepsilon$.

(3) Let $\nu$ be any number with $|\nu| \leq \varepsilon$. Since $\hat{A}^n_{\nu} \subset I$, $\hat{A}^n_{\nu}(\nu)$ is contained in $I(\nu)$. By (1), $|\hat{B}_1^n(\nu)| < \varepsilon/3$. Let $x_B$ be the first coordinate of the left edge $x_B$ of $\hat{B}_1^n$ and $x_I$ that of the right edge $x_I$ of $I$. Since $|\hat{B}_1^n| < \varepsilon/3$ and $I$ is an arc of length $\varepsilon/3$ with $I \cap \hat{B}_1^n \neq \emptyset$, $x_B - x_I > -2\varepsilon/3$. Let $x_B(\nu), x_I(\nu)$ be the first coordinates of the points in $\hat{L}(\nu)$ corresponding to $x_B, x_I$ respectively. By (5.10),

$$x_B(\nu) - x_I(\nu) > \nu - \frac{2\varepsilon}{3}(1 + C\nu) + o(\nu)$$

for any $0 \leq \nu \leq \varepsilon$. This implies that, if

$$\nu > \frac{2\varepsilon}{3 - 2\varepsilon C - 3o(1)_{\nu \to 0}},$$

then $\hat{B}_1^n(\nu)$ lies in the right component of $\hat{L}(\nu) \setminus I(\nu)$. One can choose $\varepsilon > 0$ so small that the right hand side of the preceding inequality is smaller than $\varepsilon$. Similarly, if $\nu < 2\varepsilon/(3 - 2\varepsilon C - 3o(1)_{\nu \to 0})$, then $\hat{B}_1^n(\nu)$ lies in the left component of $\hat{L}(\nu) \setminus I(\nu)$. Thus the interval $J_1$ satisfying the condition (3) is contained in $(-\varepsilon, \varepsilon)$. This completes the proof.

Proof of (1) and (2) of Lemma 5.2. To show (1), we will present a procedure how to define our desired sub-bridges and gaps. Suppose that the generations of $\hat{A}^n_i(\nu)$ and $\hat{B}_1^n(\nu)$ given in Lemma 5.3 are $k$ and $l$, respectively. Let $A^n_i, A^n_l$ be sub-bridges of $\hat{A}^n_i$ of generation $k+1$ with the connecting gap $G^u$ and such that one of $A^n_i$
and $A_1^u$ contains the leading point of $A_1^u$, that is, $G^u$ is the leading gap of $\hat{A}_1^u$. Let $B_1^u$ and $\hat{B}_1^u$ be sub-bridges of $\hat{B}_1^u$ of generation $l+1$ with the connecting gap $G^u$. We may assume that $B_1^u$ and $A_1^u$ lie in the left sides of $G_2^u$ and $G_1^u$ respectively if necessary exchanging notations. By Lemmas 4.3 and 5.3-(1), for any $\nu$ with $|\nu| < \varepsilon$, 

$$\tau - r^{1-5/4}/\varepsilon \leq |B_1^u(\nu)| < \tau - r^{1-3/4}/\varepsilon.$$ 

The inequality concerning $|\hat{B}_1^u(\nu)|$ is proved in the same manner. This shows (1).

(2) By Lemmas 4.3, 4.5 and 5.3-(2), for any $\nu$ with $|\nu| < \varepsilon$, 

$$|A_1^u(\nu)| \geq |\hat{A}_1^u(\nu)| \geq \frac{\tau^{1/4}|\hat{B}_1^u(\nu)|}{5} \geq \frac{\tau^{1/4}r_1|B_1^u(\nu)|}{5} \geq |B_1^u(\nu)|,$$

$$|B_1^u(\nu)| \geq r_1^{-1}|\hat{B}_1^u(\nu)| \geq r_1^{-1}|A_1^u(\nu)| \geq r_1^{-1}|G_1^u(\nu)| \geq \tau^{-1}|A_1^u(\nu)|.$$ 

This shows that $(B_1^u(\nu), A_1^u(\nu))$ is $\tau^{-1}$-proportional. The $\tau^{-1}$-proportionality of $(\hat{B}_1^u(\nu), \hat{A}_1^u(\nu))$ is shown quite similarly. This proves (2). \hfill \Box

We need the following inequality in the proof of (3).

$$(5.12) \quad \tau^{-1/2} \geq \frac{|G^u(\nu)|}{|B_1^u(\nu)|} (|\nu| < \varepsilon).$$ 

In fact, by Lemma 5.3-(2), $|\hat{B}_1^u(\nu)| \geq \tau^{-1/2}|\hat{A}_1^u(\nu)| \geq \tau^{-1/2}|A_1^u(\nu)|$. From the definition of thickness, $\tau \leq |A_1^u(\nu)|/|G^u(\nu)|$. It follows that 

$$\tau^{-1}/|A_1^u(\nu)| > \tau^{-1/2}|G^u(\nu)| = \tau^{-1/2}|G^u(\nu)|.$$ 

Hence (5.12) holds.

Proof of (3) and (4) of Lemma 5.3: Since $G^s(\delta) \subseteq \hat{B}_1^u(\delta)$ and $G^u(\delta) \subseteq \hat{A}_1^u(\delta)$, there is a $\delta \in J_1$ such that the middle point of $G^s(\delta)$ is equal to that of $G^u(\delta)$. Again by Lemmas 4.3-(2) and 5.3-(2), $|A_1^u(\delta)| \geq |\hat{A}_1^u(\delta)|/5 \geq \tau^{1/4}|\hat{B}_1^u(\delta)/5 > |\hat{B}_1^u(\delta)|$. Similarly $|\hat{A}_1^u(\delta)| \geq |\hat{B}_1^u(\delta)|$. Thus we have $\text{Int} A_1^u(\delta) \supset \hat{B}_1^u(\delta)$ and $|A_1^u(\delta)| \geq |\hat{B}_1^u(\delta)|$. By Lemma 4.3, $|B_1^u(\delta)| \geq r_1^{-1}|\hat{B}_1^u(\delta)| \geq r_1^{-1}|G^u(\delta)| \geq |G^u(\delta)|$. This implies that $B_1^u(\delta)$ is not contained in $G^u(\delta)$.

To show that the pair $(B_1^u(\delta), A_1^u(\delta))$ is $\xi_0$-linked, we need to consider the two cases of (a) $G^s(\delta) \subseteq G^u(\delta)$ and (b) $G^s(\delta) \supset G^u(\delta)$, see Figure 5.7.

$\text{Figure 5.7.}$

First we consider the case (a). One of the boundary points of $B_1^u(\delta)$ is contained in $A_1^u(\delta)$ and the other is contained in $G^u(\delta)$. It follows that $B_1^u(\delta) \cap A_1^u(\delta) \neq \emptyset$. $B_1^u(\delta)$ is not contained in a gap of $A_1^u(\delta) \cap K_1^u(\delta)$ and $A_1^u(\delta)$ is not contained in a
gap of $B_1^\ast(\delta) \cap K_3^\ast(\delta)$. This implies that $(B_1^\ast(\delta), A_1^\ast(\delta))$ is a linked pair. By Lemmas 5.3 and 4.3

$$|B_1^\ast(\delta) \cap A_1^\ast(\delta)| = |B_1^\ast(\delta)| + \frac{|G^\ast(\delta)|}{2} - \frac{|G^\ast(\delta)|}{2} > |B_1^\ast(\delta)| - \frac{|G^\ast(\delta)|}{2} \geq \frac{\tilde{B}_1^\ast(\delta)}{r_s} - \frac{|G^\ast(\delta)|}{2}.$$ 

Since $\min\{|B_1^\ast(\delta)|, |A_1^\ast(\delta)|\} = |B_1^\ast(\delta)|$, we have by \((5.7)\)

$$\frac{|B_1^\ast(\delta) \cap A_1^\ast(\delta)|}{\min\{|B_1^\ast(\delta)|, |A_1^\ast(\delta)|\}} = \frac{|B_1^\ast(\delta)|}{|B_1^\ast(\delta)|} \geq \left(\frac{|\tilde{B}_1^\ast(\delta)|}{r_s} - \frac{|G^\ast(\delta)|}{2}\right) \frac{r_s}{|B_1^\ast(\delta)|} \geq \left(\frac{1}{r_s} - \frac{1}{2\tau^{1/2}}\right) r_s = \xi.$$

In the case (b), it is immediately seen that $B_1^\ast(\delta) \cap A_1^\ast(\delta) \neq \emptyset$ and $A_1^\ast(\delta)$ is not contained in a gap of $B_1^\ast(\delta) \cap K_3^\ast(\delta)$. Moreover, we will show that $B_1^\ast(\delta)$ is not contained in a gap of $A_1^\ast(\delta) \cap K_3^\ast(\delta)$ by contradiction. Suppose that there would exist a gap $G_1^\ast(\delta)$ of $A_1^\ast(\delta) \cap K_3^\ast(\delta)$ with $G_1^\ast(\delta) \supset B_1^\ast(\delta)$. This implies that there is a $u$-bridge $A_u^\ast(\delta)$ which is adjacent to $G_1^\ast(\delta)$ and contained in $G^\ast(\delta)$. Thus,

$$\frac{|A_u^\ast(\delta)|}{|G^\ast(\delta)|} \geq \frac{|B_1^\ast(\delta)|}{|G^\ast(\delta)|} > 1.$$ 

On the other hand, we have from \((5.9)\)

$$\frac{|A_u^\ast(\delta)|}{|G^\ast(\delta)|} \geq \frac{|B_1^\ast(\delta)|}{|G^\ast(\delta)|} > \tau(B^\ast(\delta) \cap K_3^\ast(\delta)) \tau(A^\ast(\delta) \cap K_3^\ast(\delta)) > 1.$$ 

This is a contradiction. Hence, we conclude that $B_1^\ast(\delta)$ is not contained in a gap of $A_1^\ast(\delta) \cap K_3^\ast(\delta)$. Since $B_1^\ast(\delta) \subseteq A_1^\ast(\delta)$, one has

$$\frac{|B_1^\ast(\delta) \cap A_1^\ast(\delta)|}{\min\{|B_1^\ast(\delta)|, |A_1^\ast(\delta)|\}} = \frac{|B_1^\ast(\delta)|}{|B_1^\ast(\delta)|} = 1 > \xi.$$

This completes the proof of \((3)\). \hfill \Box

6. LINEAR GROWTH PROPERTY OF LINKED PAIRS

As in the preceding section, any $s$-bridge $B^\ast(\delta)$ here means a bridge with respect to $K^\ast(\delta)$ and any $u$-bridge $A^\ast(\delta)$ means a bridge with respect to $K^\ast(\delta)$ for any $\delta$ with $|\delta| < \epsilon$. The main result of this section is as follows:

**Lemma 6.1** (Linear Growth Lemma). Let $\xi_0$ be the constant of \((5.7)\) and let $(B^\ast(0), A^\ast(0))$ be a linked pair in $\hat{L} \cap B_{b_0/2}$. For any $0 < \epsilon < |B^\ast(0) \cap A^\ast(0)|$, there exist a constant $\Delta$ with $|\Delta| < \epsilon \tau^{-3/4}/2$, collections of sub-bridges $\{B^\ast_k\}_{k \geq 1}$ of $B^\ast(0)$ and $\{A^\ast_k\}_{k \geq 1}$ of $A^\ast(0)$, positive integers $N_s$ and $N_u$ independent of $\epsilon$ which satisfy the following (1)–(3) for every $k \geq 1$.

1. $(B^\ast_k(\Delta), A^\ast_k(\Delta))$ is a proportional $\xi_0/2$-linked pair.
2. For the union

$$I_k = A^\ast_k(\Delta) \cup B^\ast_k(\Delta),$$

which is an arc in $\hat{L}(\Delta)$, there exists a positive constant $\alpha_0$ independent of $k$ such that, for any integer $l > k$, the $\alpha_0|A^\ast_l(\Delta)|$-neighborhood of $I_k$ in $L(\Delta)$ is disjoint from the $\alpha_0|A^\ast_l(\Delta)|$-neighborhood of $I_l$ in $\hat{L}(\Delta)$. 


(3) If $n_k$ and $i_k$ are generations of $B^*_k$ and $A^*_k$, respectively, then
\[ n_k < n_{k+1} \leq n_k + N_s, \quad i_k < i_{k+1} \leq i_k + N_u. \]

Lemma 6.1 follows immediately from the next technical lemma.

**Lemma 6.2.** Under the assumptions same as in Lemma 6.1 there exist sequences
\[ \{n_k\}_{k \geq 1}, \{i_k\}_{k \geq 1} \] of positive integers,
\[ \{\delta_k\}_{k \geq 1} \] of real numbers with
\[ |\delta_k| \leq 2^{-1}\xi_0 \tau^{-3/4}\varepsilon r_{s-}^{-k}; \]
\[ \{B^*_k\}_{k \geq 1}, \{\overline{B}^*_k\}_{k \geq 1} \] of $s$-bridges of generation $n_k$ with $B^*_k, \overline{B}^*_k \subset \overline{B}^*_{k-1}, \overline{B}^*_0 = B^*(0)$ which have the connecting gaps $G^*_k = \text{Gap}(B^*_k, \overline{B}^*_k);$
\[ \{A^*_k\}_{k \geq 1}, \{\overline{A}^*_k\}_{k \geq 1} \] of $u$-bridges of generation $i_k$ with $A^*_k, \overline{A}^*_k \subset \overline{A}^*_{k-1}, \overline{A}^*_0 = A^*(0)$ which have the connecting gaps $G^*_k = \text{Gap}(A^*_k, \overline{A}^*_k);$

satisfying the following (1)–(3) for each $k \geq 1$.

1. For any $t = 1, \ldots, k$ and the positive number $\xi_{k-t}$ defined as (6.2), both $(B^*_t(\Delta_k), A^*_t(\Delta_k))$ and $(\overline{B}^*_t(\Delta_k), \overline{A}^*_t(\Delta_k))$ are proportional $\xi_{k-t}$-linked pairs, where $\Delta_k = \delta_1 + \cdots + \delta_k.$

2. $G^*_k(\Delta_k)$ and $G^*_k(\Delta_k)$ have a common middle point.

3. There exist integers $1 \leq \tilde{n}_s, N_u < \infty$ independent of $k$ such that
\[ n_k < n_{k+1} \leq n_k + N_s, \quad i_k < i_{k+1} \leq i_k + N_u. \]

Moreover, $\Delta := \sum_{k=1}^\infty \delta_k$ is an absolutely convergent series with $\Delta_\ast := \sum_{k=1}^\infty |\delta_k| < \varepsilon \tau^{-3/4}/2.$

Let $\{n_k\}_{k=1}^\infty$ be the strictly increasing sequence of generations given in Lemma 6.2 and $n_0 = 0.$ For any integer $k \geq 1,$ let
\[ \xi_k := \xi_0 \left(1 - \frac{1}{2} \sum_{i=1}^k r_{s-}^{-\tilde{n}_i}\right), \]
where $\{\tilde{n}_i\}_{i=1}^\infty$ is the sequence defined by
\[ \tilde{n}_i = \inf\{n_{i+l} - n_i; l = 0, 1, 2, \ldots\}. \]
Since $n_{i+1} \geq n_i + 1,$ we have $\tilde{n}_i \geq i$ for any $i \geq 1.$ The inequality $r_{s-} > 2$ of Lemma 4.3 implies
\[ \xi_k \geq \xi_0 \left(1 - \frac{1}{2} \sum_{i=1}^k r_{s-}^{-\tilde{n}_i}\right) \geq \xi_0 \left(1 - \frac{1}{2} r_{s-}^{-1}\right) = \xi_0 \left(1 - \frac{1}{2}\right) = \frac{\xi_0}{2} > 0. \]

Proof of Lemma 6.2: Applying Lemma 5.2 to the linked pair $(B^*(0), A^*(0)),$ we obtain a constant $\delta_1$ with $|\delta_1| < \varepsilon,$ sub-bridges $B^*_1, \overline{B}^*_1$ of $B^*(0)$ and $A^*_1, \overline{A}^*_1$ of $A^*(0)$ such that $(B^*_1(\delta_1), A^*_1(\delta_1))$ and $(\overline{B}^*_1(\delta_1), \overline{A}^*_1(\delta_1))$ are proportional $\xi_0$-linked pairs and the connecting gaps $G^*_1(\delta_1)$ and $G^*_1(\delta_1)$ have a common middle point. Let $a$ be the constant defined as
\[ a = \max \left\{2, \frac{4\xi_0(1 + C\varepsilon)}{r_{s+}(1 - r_{s-}^{-1})(1 - 2r_{s-}^{-1})}\right\}, \]
which will be used later to prove Lemma 6.1.(2). Again applying Lemma 5.2 to the linked pair $(B^*_1(\delta_1), A^*_1(\delta_1))$ for $\varepsilon_1 := \xi_0 B^*_1/2ar_{s+}$ instead of $\varepsilon,$ we obtain a
constant $\delta_2$ with $|\delta_2| \leq \varepsilon_1$, sub-bridges $B^*_{\delta_1}(\delta_1), \tilde{B}^*_{\delta_1}(\delta_1)$ of $\tilde{B}^*_{\delta_1}(\delta_1)$ and $A^*_{\delta_1}(\delta_1), \tilde{A}^*_{\delta_1}(\delta_1)$ of $\tilde{A}^*_{\delta_1}(\delta_1)$ such that $(B^*_{\delta_1}(\Delta_2), A^*_{\delta_1}(\Delta_2))$ and $(\tilde{B}^*_{\delta_1}(\Delta_2), \tilde{A}^*_{\delta_1}(\Delta_2))$ are proportional $\xi_0$-linked pairs and the connecting gaps $G^*_{\delta_1}(\Delta_2)$ and $G^*_{\delta_1}(\Delta_2)$ have a common middle point. Similarly, for any $k \geq 2$, there exist a constant $\delta_k$ with $|\delta_k| \leq \varepsilon_{k-1} := \frac{\varepsilon_0}{|\Delta_{k-1}| / 2ar_{s^+}}$, sub-bridges $B^*_k(\Delta_{k-1}), \tilde{B}^*_k(\Delta_{k-1})$ of $\tilde{B}^*_k(\Delta_{k-1})$ and $A^*_k(\Delta_{k-1}), \tilde{A}^*_k(\Delta_{k-1})$ of $\tilde{A}^*_k(\Delta_{k-1})$ such that $(B^*_k(\Delta_k), A^*_k(\Delta_k))$ and $(\tilde{B}^*_k(\Delta_k), \tilde{A}^*_k(\Delta_k))$ are proportional $\xi_0$-linked pairs and the connecting gaps $G^*_k(\Delta_k)$ and $G^*_k(\Delta_k)$ have a common middle point. This shows $[2]$.

Now we will show that, for any $k \geq 1$, $(B^*_k(\Delta_k), A^*_k(\Delta_k))$ is a proportional $\xi_{k-1}$-linked pair for each $t = 1, \ldots, k$. Suppose that the assertion holds until the $k$-th step and consider the $(k+1)$-st step. When $t = k+1$, the proof is already done. So we may suppose that $t \leq k$. Since $(B^*_k(\Delta_k), A^*_k(\Delta_k))$ is a proportional $\xi_{k-1}$-linked pair,

$$|B^*_k(\Delta_k) \cap A^*_k(\Delta_k)| \geq \xi_{k-1}|B^*_k(\Delta_k)|.$$ 

By this inequality together with $(5.10)$,

$$|B^*_k(\Delta_{k+1}) \cap A^*_k(\Delta_{k+1})| \geq (1 - C|\delta_{k+1}|) \left( |B^*_k(\Delta_k) \cap A^*_k(\Delta_k)| - |\delta_{k+1}| \right) - o(\delta_{k+1})$$

$$\geq (1 - C|\delta_{k+1}|) \left( \xi_{k-1}|B^*_k(\Delta_k)| - |\delta_{k+1}| \right) - o(\delta_{k+1})$$

$$\geq (1 - C|\delta_{k+1}|) \left( \xi_{k-1}|B^*_k(\Delta_k) - |\delta_{k+1}| \right) - o(\delta_{k+1})$$

$$= \xi_{k-1}|B^*_k(\Delta_k)| - \left( 1 + \frac{2C\xi_{k-1}|B^*_k(\Delta_k)|}{1 + C|\delta_{k+1}|} + o(1) \right) |\delta_{k+1}|.$$

Since $|B^*_k(\Delta_{k+1})| < \varepsilon$ by Lemma 5.2 and $\xi_{k-1} < 1$ by $(6.2)$, one can choose $\varepsilon > 0$ so that the contribution of the last parenthesis is smaller than two. Then

$$|B^*_k(\Delta_{k+1}) \cap A^*_k(\Delta_{k+1})| \geq \xi_{k-1}|B^*_k(\Delta_k)| - 2|\delta_{k+1}|$$

$$= (\xi_{k-1} - 2|\delta_{k+1}|) |B^*_k(\Delta_{k+1})|.$$ 

Let $n_k$ be the generation of $\tilde{B}^*_k$. By Lemma 4.3, $|\tilde{B}^*_k(\Delta_{k+1})| \leq r_s^{-\left(n_k-n_{k-1}\right)}|B^*_k(\Delta_{k+1})|$ and $|\tilde{B}^*_k(\Delta_{k+1})| \leq r_s+|B^*_k(\Delta_{k+1})|$. Since $a \geq 2$ by $(6.4)$,

$$2|\delta_{k+1}| |B^*_k(\Delta_{k+1})|^{-1} \leq \frac{2\varepsilon_0}{2ar_{s^+}} |B^*_k(\Delta_{k+1})|^{-1}$$

$$\leq \frac{\varepsilon_0 r_s^{-\left(n_k-n_{k-1}\right)}}{ar_{s^+}} |\tilde{B}^*_k(\Delta_{k+1})|^{-1}$$

$$\leq \frac{\varepsilon_0 r_s^{-\left(n_k-n_{k-1}\right)}}{2}.$$ 

Then

$$\xi_{k-1} - 2|\delta_{k+1}| |B^*_k(\Delta_{k+1})|^{-1} \geq \xi_0 \left( 1 - \frac{1}{2} \sum_{i=1}^{k-t} r_s^{-\tilde{n_i}} \right) - \xi_0 r_s^{-\left(n_k-n_{k-1}\right)}$$

$$= \xi_0 \left( 1 - \frac{1}{2} \left( \sum_{i=1}^{k-t} r_s^{-\tilde{n_i}} + r_s^{-\left(n_k-n_{k-1}\right)} \right) \right)$$

$$\geq \xi_0 \left( 1 - \frac{1}{2} \left( \sum_{i=1}^{k-t} r_s^{-\tilde{n_i}} + r_s^{-n_{k+1-t-1}} \right) \right) = \xi_{k+1-t}.$$
Since $\xi_{k+1} - t \geq \xi_0/2$ by (6.3), it follows that $(B^*_k(\Delta_{k+1}), A^*_k(\Delta_{k+1}))$ is a $\xi_0/2$-linked pair. This shows (1).

By Lemma 5.2-4, the length of $\tilde{B}^*_k$ is evaluated as follows:

(6.5) $|\tilde{B}^*_k| \geq \frac{r_s^{-1}\epsilon^{-5/4}\xi_k}{2r_{s+}} \geq \frac{r_s^{-1}\epsilon^{-5/4}\xi_0}{2r_{s+}} = 2^{-1}\frac{r_s^{-2}\tau^{-5/4}\xi_0}{2r_{s+}}|\tilde{B}^*_k|.$

Since the generation of $\tilde{B}^*_k$ is $n_k$, by Lemma 4.3,

$$r_s^{-(n_{k+1} - n_k)} \geq |\tilde{B}^*_k|^{-1}.$$ 

This implies that

$$n_{k+1} - n_k \leq \log\left(\frac{2^{-1}r_s^{-2}\tau^{-5/4}\xi_0}{\log(r_s^{-1})}\right).$$

Thus, the maximum integer $N_s$ not greater than the right hand side of this inequality satisfies $n_{k+1} \leq n_k + N_s$ for any $k \geq 1$. It follows from (6.5) and the proportionality condition in Lemma 5.2-(2) that

$$|\tilde{A}^*_k| \geq |\tilde{B}^*_k| \geq 2^{-1}\frac{r_s^{-2}\tau^{-5/4}\xi_0}{2r_{s+}} = 2^{-1}\frac{r_s^{-2}\tau^{-5/4}\xi_0}{2r_{s+}}|\tilde{A}^*_k|.$$ 

We suppose that the generation of $A^*_k$ is $i_k$. Since $|\tilde{A}^*_k| |\tilde{A}^*_k|^{-1} \leq (\frac{2}{5})^{i_{k+1} - i_k}$ by Lemma 4.5, one has a positive integer $u_k$ independent of $k$ and satisfying

$$i_{k+1} - i_k \leq N_s \leq \frac{\log\left(\frac{2^{-1}r_s^{-2}\tau^{-5/4}\xi_0}{\log(\frac{2}{5})}\right)}{1.}

$$

for any $k \geq 1$.

Since $a \geq 1$, $n_{k-1} - n_1 \geq (k - 1) - 1 = k - 2$ and $0 < \xi_0 < 1$, it follows from Lemmas 4.3 and 5.2 that

$$|\delta_k| \leq \frac{\xi_0}{2ar_{s+}} \leq \frac{\xi_0}{2r_{s+}} \frac{r_s^{-k-2}(r_s^{-3/4}\xi)}{r_s^{-n_k+1} + n_1(r_s^{-1} - \tau^{-3/4}\xi)} \leq \frac{\xi_0}{2r_{s+}} \frac{r_s^{-k-2}(r_s^{-3/4}\xi)}{r_s^{-n_k+1} + n_1(r_s^{-1} - \tau^{-3/4}\xi)}.$$ 

This shows that

$$\sum_{k=1}^{\infty} |\delta_k| < \frac{\epsilon}{2} \frac{r_s^{-3/4}}{\tau^{-3/4}} \frac{1}{r_s - 1} < \frac{\epsilon}{2} \tau^{-3/4}.$$ 

In particular, $\Delta = \sum_{k=1}^{\infty} \delta_k$ is an absolutely convergent series with $\Delta_s = \sum_{k=1}^{\infty} |\delta_k| < \epsilon \tau^{-3/4}/2$. This shows (1) and completes the proof.

The proof of (1) of Lemma 6.1 is obtained immediately from Lemma 6.2. So it remains to prove (2).

Proof of (2) of Lemma 6.1. Since $A_k^*(\Delta) \cap B^*_k(\Delta) \neq \emptyset$, $I_k = A_k^*(\Delta) \cup B^*_k(\Delta)$ is an arc in $\tilde{L}(\Delta)$. The union $B^*_k(\Delta) = B^*_k(\Delta) \cup G^*_k(\Delta) \cup \tilde{B}^*_k(\Delta)$ is the smallest $s$-bridge containing $B^*_k(\Delta)$ and $\tilde{B}^*_k(\Delta)$. By Lemma 4.3, $|B_k^*(\Delta)|, |\tilde{B}_k^*(\Delta)| \leq r_s^{-1}|B_k^*(\Delta)|$. It follows that

$$|G_k^*(\Delta)| = |\tilde{B}_k^*(\Delta)| - (|B_k^*(\Delta)| + |\tilde{B}_k^*(\Delta)|) \geq |\tilde{B}_k^*(\Delta)|(1 - 2r_s^{-1}) \geq |\tilde{B}_k^*(\Delta)|(1 - 2r_s^{-1}).$$
Similarly we have \(|G_k^s(\Delta)| \geq |B_k^s(\Delta)| (1 - 2r_s^{-1})\). As in the proof of Lemma 6.2 for any integer \(l \geq k + 1\),

\[
|\delta_l| < \frac{\xi_0 |\tilde{B}_{k-1}^s(\Delta)|}{2ar_{s+}} \leq \frac{(1 + C\varepsilon)\xi_0 |\tilde{B}_k^s(\Delta)|}{2ar_{s+}} \leq \frac{(1 + C\varepsilon)\xi_0}{2ar_{s+}} r_{s-}^{-n_{l-1}+n_{k}}|\tilde{B}_k^s(\Delta)|.
\]

Thus, by (6.4), we have

\[
(6.6) \quad \sum_{l=k+1}^{\infty} |\delta_l| \leq (1 + C\varepsilon)\xi_0 |\tilde{B}_k^s(\Delta)| \leq \frac{|G_k^s(\Delta)|}{8}.
\]

Define the constant \(\alpha_0\) as

\[
\alpha_0 = \max \left\{ \frac{1 - 2r_s^{-1}}{8r}, \frac{1}{2m^{r+3}} \right\}.
\]

Then we have

\[
(6.7) \quad \alpha_0 |A_k^u(\Delta)| \leq \alpha_0 |B_k^s(\Delta)| \leq \alpha_0 \frac{\tau}{1 - 2r_s^{-1}} |G_k^s(\Delta)| = \frac{|G_k^s(\Delta)|}{8}.
\]

Since \(|A_k^u(\Delta)| \leq 2^{m+1} |G_k^u(\Delta)|\) by Lemma 4.3-(3),

\[
(6.8) \quad \alpha_0 |A_k^u(\Delta)| \leq \alpha_0 2^{m+1} |G_k^u(\Delta)| \leq \frac{|G_k^u(\Delta)|}{4}.
\]

We have chosen \(\Delta_k\) so that \(G_k^s(\Delta_k)\) and \(G_k^u(\Delta_k)\) have the common middle point \(x_k\), see Figure 6.1. By (5.10) and (6.6), for any \(x \in B_k^s(\Delta_k)\) and all sufficiently large \(k\),

\[
\|\tilde{\pi}_k^u(x) - \tilde{\pi}_k^u(x_k)\| \geq (1 - C|\Delta - \Delta_k|) \|x - x_k\| - |\Delta - \Delta_k| - o(|\Delta - \Delta_k|)
\]

\[
\geq (1 - C|\Delta - \Delta_k|) G_k^2(\Delta) - |\Delta - \Delta_k| - o(|\Delta - \Delta_k|)
\]

\[
\geq \frac{|G_k^u(\Delta)|}{4},
\]

where \(\tilde{\pi}_k^u : \tilde{L}(\Delta_k) \rightarrow \tilde{L}(\Delta)\) is the composition of the shift map \(x \mapsto x + (\Delta - \Delta_k, 0)\) followed by the projection along the leaves of \(F^u_{loc}(\Lambda; \Delta)\), while \(\tilde{\pi}_k^u : \tilde{L}(\Delta_k) \rightarrow \tilde{L}(\Delta)\).
\( \tilde{L}(\Delta) \) is the projection along the leaves of \( f^{-(N_0+N_2)}(F^u_\text{loc}(\Gamma_m)) \). Thus, by \( (6.7) \), \( \mathcal{N}(B^u_{k+1}(\Delta);\alpha_0|A^u_k|) \) does not contain \( \tilde{\pi}^u_k(x_k) \), where \( \mathcal{N}(I, \eta) \) denotes the \( \eta \)-neighborhood of \( J \) in \( \tilde{L}(\Delta) \) for \( \eta > 0 \) and a compact subset \( J \) of \( \tilde{L}(\Delta) \). Since \( B_{k+1}(\Delta) \) is contained in \( \tilde{B}_k(\Delta) \) and \( |A^u_{k+1}(\Delta)| \leq |A^u_k(\Delta)| \), one can show similarly that \( \mathcal{N}(B^u_{k+1}(\Delta);\alpha_0|A^u_k(\Delta)|) \) does not contain \( \tilde{\pi}^u_k(x_k) \). By \( (6.9) \), \( \mathcal{N}(A^u_k(\Delta);\alpha_0|A^u_k(\Delta)|) \) also does not contain \( \tilde{\pi}^u_k(x_k) \). Since \( A^u_{k+1}(\Delta) \subset A^u_k(\Delta) \) and \( B^u_{k+1}(\Delta) \subset B^u_k(\Delta) \), we have similarly \( \mathcal{N}(A^u_k(\Delta);\alpha_0|A^u_k(\Delta)|) \notin \tilde{\pi}^u_k(x_k) \). This shows that

\[
(6.9) \quad \mathcal{N}(I_k;\alpha_0|A^u_k(\Delta)|) \cap \mathcal{N}(I_{k+1};\alpha_0|A^u_{k+1}(\Delta)|) = \emptyset.
\]

Since \( A^u_{k+1}(\Delta) \subset \tilde{A}^u_k(\Delta) \) and \( \tilde{B}^u_{k+1}(\Delta) \subset \tilde{B}^u_k(\Delta) \), one can also prove that

\[
\mathcal{N}(I_k;\alpha_0|A^u_k(\Delta)|) \cap \mathcal{N}(\tilde{I}_{k+1};\alpha_0|A^u_{k+1}(\Delta)|) = \emptyset,
\]

where \( \tilde{I}_{k+1} = \tilde{A}^u_{k+1}(\Delta) \cup \tilde{B}^u_{k+1}(\Delta) \). From this fact together with \( I_{k+2} \subset \tilde{I}_{k+1} \), it follows that

\[
(6.10) \quad \mathcal{N}(I_k;\alpha_0|A^u_k(\Delta)|) \cap \mathcal{N}(I_{k+2};\alpha_0|A^u_{k+2}(\Delta)|) = \emptyset.
\]

The assertion \( (2) \) is completed by applying an argument similar to that from \( (6.9) \) to \( (6.10) \) repeatedly.

### 7. Critical chains

Since \( |\Delta| < \varepsilon \tau^{-3/4} \), by Lemma \( 6.1 \), we may suppose that the perturbed diffeomorphism \( f_\Delta \) given in Subsection \( 5.2 \) is arbitrarily \( C^r \)-close to the original diffeomorphism \( f \). So we reset the notations and write \( f_\Delta = f, \tilde{L}(\Delta) = \tilde{L}, L(\Delta) = L, F^u_\text{loc}(L;\Delta) = F^u_\text{loc}(L) \) and so on. Since \( \tilde{L} \) and \( L \) are parametrized so that \( f^{N_1}_{\tilde{L}} : \tilde{L} \to L \) is parameter-preserving, Linear Growth Lemma (Lemma \( 6.1 \)) holds for the bridges \( B^r_{L,k} = f^{N_2}(B^r_k(\Delta)) \) and \( A^u_{L,k} = f^{N_2}(A^u_k(\Delta)) \) with respect to the Cantor set \( K^r_{L,L} = f^{N_2}(K^r_{L,L}(\Delta)) \) and \( K^u_{L,L} = f^{N_2}(K^u_{L,L}(\Delta)) \) in \( L \).

#### 7.1. Encountering of \( s \)-bridges and \( u \)-bridges, II.

In Subsection \( 5.2 \) we have studied the heteroclinical connection between \( s \)-bridges \( B^r(\Delta) \) of \( K^r_{L,L}(\Delta) \) and \( u \)-bridges \( A^u(\Delta) \) of \( K^u_{L,L}(\Delta) \) in \( \tilde{L} \). To construct non-trivial wandering domains, we also need to study the homoclinical connection between \( s \)-bridges \( B^r_{L} \) of \( K^r_{L,L} \) and \( u \)-bridges \( B^u_{L} \) of \( K^u_{L,L} \) in \( L \).

We write \( 1^{(n)} := \underbrace{1, \ldots, 1}_n \) and \( 2^{(n)} := \underbrace{2, \ldots, 2}_n \) and prove the following key lemma.

Consider a positive integer \( z_0 \) independent of \( k \) and satisfying the conditions \( (8.4) \) and \( (8.5) \) which are given later. Let \( \{z_k\}_{k=1}^{\infty} \) be any sequence of integers such that each entry \( z_k \) is either \( z_0 \) or \( z_0 + 1 \).

**Lemma 7.1** (Critical Chain Lemma). Let \( (B^r_k(\Delta), A^u_k(\Delta)) \) (\( k = 0, 1, 2, \ldots \)) be the sequence of the proportional \( \xi_0/2 \)-linked pairs of bridges for \( (K^r_{L,L}(\Delta), K^u_{L,L}(\Delta)) \) given in Lemma \( 6.1 \). Then there exists a constant \( T_0 > 0 \) such that, for any \( T \geq T_0 \) and integers \( k \geq 1 \), there are

- a \( u \)-bridge \( \hat{A}^u_{L,k} \) of \( K^u_{L,L} \) with \( \hat{A}^u_{L,k} \subset A^u_{L,k} \) and a \( u \)-bridge \( B^u_{k} \) of \( K^u_{L,L} \) such that \( B^u_{k} := \pi_{\hat{A}^u_k}(B^u_k) \) is contained in the leading gap of \( \hat{A}^u_{L,k} \), where \( \pi_{\hat{A}^u_k} : B^u_k(0) \to L \) is the projection defined as \( (5.2) \);
- an \( s \)-bridge \( \hat{B}^s_{L,k+1} \) of \( K^s_{L,L} \) with \( \hat{B}^s_{L,k+1} \subset B^s_{L,k+1} \);
Suppose that $\hat{B}^u_{L,k+1}$ is of generation $\hat{n}_{k+1}$ and has the itinerary $\hat{w}_{k+1}$. Let $B^u_k$ and $B^u_{k+1}$ be the bridges of $K^u_A$ and $K^u_A$ with the itineraries $1^{(z_k k^2 k^2)} 2^{(k^2)} k_{+1} [\hat{w}_{k+1}]^{-1}$ and $\hat{w}_{k+1} [\hat{w}_{k+1}]^{-1}$ respectively, that is,

$$B^u_k = B^u(z_k k^2 + \langle k \rangle; 1^{(z_k k^2 k^2)} 2^{(k^2)} k_{+1} [\hat{w}_{k+1}]^{-1}),$$

$$B^u_{k+1} = B^u(z_k k^2 + \langle k \rangle; \hat{w}_{k+1} [\hat{w}_{k+1}]^{-1}) 2^{(k^2)} 1^{(z_k k^2)}),$$

where

$$\langle k \rangle = \hat{n}_{k+1} + k^2 + k$$

and $\hat{w}_{k+1}$ is an arbitrarily chosen element of $\{1, 2\}^k$, that is, $\hat{w}_{k+1}$ is a non-specified itinerary of length $k$.

(2) Let $B^u_{L,k+1} := \pi^u(B^u_{k+1})$, where $\pi^u : S \to L$ is the projection of $[5.4]$. Then there exists an interval $J_{k+1} \subset (-\varepsilon_0 r_{-1} - T(k+1), \varepsilon_0 r_{-1} - T(k+1))$ such that $(B^u_{L,k+1} + t) \cap B^u_{L,k+1} \neq \emptyset$ if and only if $t \in J_{k+1}$.

(3) Let $G(\hat{A}^u_k)$ be the strip of the leading gap $G(\hat{A}^u_k)$ with $\pi^u(\hat{A}^u_k) = \hat{A}^u_{L,k}$ defined as in Subsection 5.4, see Figure 5.2. Suppose that $\hat{i}_k$ is the minimum positive integer satisfying $f^{\hat{i}_k}(G(\hat{A}^u_k)) \subset G^u(0)$ and having $N_s + n_s$ as a divisor, where $N_s$ and $n_s$ are the integers given in Theorem 5.1 and Subsection 5.4 respectively. In other words, $\hat{i}_k = \hat{i}_k / (N_s + n_s)$ is the minimum integer with $(\varphi_k(G(\hat{A}^u_k)), \varphi_k(G(\hat{A}^u_k))) \subset (G^u(0), G^u(0))$. Then the inequality $\hat{i}_k + \hat{n}_{k+1} < C_1 T_k + C_2$ holds.

See Figure 7.1 for the situation of Lemma 7.1 where $G^u_{L,k+1} = \pi^u(G(\hat{A}^u_{k+1}))$. Figure 7.2 illustrates the transition from $B^u_k$ to $B^u_{L,k}$ via $f^{-N_0} \circ f^{-\hat{i}_k} \circ f^{-N_1}$ schematically.

**Proof of Lemma 7.1** Let $T_0$ be the constant defined as

$$T_0 = \frac{N_s \log(r_{s+1})}{\log(r_{s-})}.$$
By Lemmas 4.3 and 6.1-(2),

\[ |B_{s,L,k}^u \cap A_{u,L,k}^u| \geq \xi_0 |B_{s,0}^u(\Delta)| \geq \frac{\xi_0 |B_0^u(\Delta)|}{2} r_{s+} - N_{s,k} = \varepsilon_0 r_{s+} - T_{0,k} \]

for any integer \( k \geq 1 \), where \( \varepsilon_0 = \xi_0 |B_0^u(\Delta)|/2 \). Here we fix an integer \( T \geq T_0 \). By Lemma 5.3, there exist an interval \( J_k \) with \( J_k \subset ( -\varepsilon_0 r_{s-} - T_k, \varepsilon_0 r_{s-} - T_k ) \), sub-bridges \( \hat{B}_{s,L,k}^u \subset B_{s,L,k}^u \) and \( \hat{A}_{u,L,k}^u \subset A_{u,L,k}^u \) satisfying the following conditions:

(a) \( \tau^{5/4} \varepsilon_0 r_{s-} - T_k \leq |\hat{B}_{s,L,k}^u| < \tau^{-3/4} \varepsilon_0 r_{s-} - T_k \);

(b) \( \tau^{-1/4} |\hat{A}_{u,L,k}^u| \leq |\hat{B}_{s,L,k}^u| < \tau^{-1/4} |\hat{A}_{u,L,k}^u| \);

(c) \( (\hat{B}_{s,L,k}^u + t) \cap \hat{A}_{u,L,k}^u \neq \emptyset \) if and only if \( t \in J_k \).

Suppose that \( \hat{A}_{u,L,k}^u \) is of generation \( i_k > 0 \). By (5.1), we have the integer \( \tilde{i}_k \) with \( \varphi_{s}^u(\hat{G}(A_{u,k}^u)), \varphi_{s}^u(\hat{G}(A_{u}^u)) \subset (\hat{G}(0), \hat{G}(0)) \) and \( i_k \leq \tilde{i}_k \leq (m - 1) i_k \). We set \( \tilde{i}_k = \tilde{i}_k(N_s + i_s) \). By the same procedure as in (6), there is a sub-interval \( J_{k+1}^{*} \) of \( J_{k+1} \) such that \( (B_{s,L,k+1}^u + t) \cap B_{s,L,k}^u \neq \emptyset \) if and only if \( t \in J_{k+1}^{*} \). This completes the proofs of (1) and (2).

Now we will show (3). By Lemma 4.3 one has

\[ r_{s+}^{-\tilde{n}_k} |B_{0}^u(\Delta)| \leq |\hat{B}_{k}^u(\Delta)| \leq r_{s-}^{-\tilde{n}_k} |B_{0}^u(\Delta)|, \]

and from (5)

\[ \tau^{-5/4} \varepsilon_0 r_{s-} - T_k \leq r_{s-}^{-\tilde{n}_k} |B_{0}^u(\Delta)|. \]

Hence, for every \( l \geq 0 \),

\[ \tilde{n}_{k+l} \leq T(k + l) + \frac{\log(\tau^{5/4} \varepsilon_0^{-1} |B_{0}^u(\Delta)|)}{\log r_{s-}}. \]
By (7.3) and (7.4), one can get positive constants $C$.

By using the inequalities as above, we have

$$
\log \left( \frac{3}{2} \right) \hat{i}_k \leq \log(r_{s-}TK) + \log(\tau \varepsilon_0^{-1}|A_{L,0}^u|) + \hat{n}_k \log(r_{s+}/r_{s-}) \\
\leq \log(r_{s-}TK) + \log(\tau \varepsilon_0^{-1}|A_{L,0}^u|) + \left( TK + \frac{\log(\tau^{5/4} \varepsilon_0^{-1}|B_0^u(\Delta)|)}{\log(r_{s-})} \right) \log(r_{s+}/r_{s-}) \\
= \log(r_{s-}TK) + C_3,
$$

where $C_3 = \log(\tau \varepsilon_0^{-1}|A_{L,0}^u|) + \frac{\log(\tau^{5/4} \varepsilon_0^{-1}|B_0^u(\Delta)|)}{\log(r_{s-})} \log(r_{s+}/r_{s-})$. Hence

$$
(7.4) \quad \hat{i}_k \leq (m-1)i_k(N_* + n_*) \leq \frac{\log(r_{s+}^{(N_*+n_*)(m-1)})}{\log(\frac{3}{2})} TK + C_3(N_* + n_*)(m-1) \frac{\log(\hat{i}_k)}{\log(\frac{3}{2})}.
$$

By (7.3) and (7.4), one can get positive constants $C_1$ and $C_2$ satisfying (3).

7.2. The second perturbation of $f$. In Subsection 5.2 we perturbed $f$ by performing the ‘$\Delta$-sliding’ of $f$ along the arc $\hat{L}$. The second perturbation presented here is ‘switchback slidings’ in neighborhood of the brides $B^\pm_\alpha(\Delta)$ ($k = 1, 2, \ldots$) which are done individually by using bump functions with mutually disjoint supports in $S$.

Consider bridges $B^s$ in $W^s_{\text{loc}}(p) = [0, 2] \times \{0\}$ and $B^u$ in $W^u_{\text{loc}}(p) = \{0\} \times [0, 2]$ associated with the Cantor sets $K^s_\alpha$ and $K^u_\alpha$ of (4.2), respectively. Set $B^s = (\pi_{\mathcal{F}^s_{\text{loc}}(\Lambda)})^{-1}(B^s)$ and $B^u = (\pi_{\mathcal{F}^u_{\text{loc}}(\Lambda)})^{-1}(B^u)$, where $\pi_{\mathcal{F}^s_{\text{loc}}(\Lambda)}$ and $\pi_{\mathcal{F}^u_{\text{loc}}(\Lambda)}$ are the projections given in Subsection 4.2. Note that $B^s$ is a sub-strip of $(S, S_S)$ and $B^u$ is a sub-strip of $(S, \mathfrak{S})$. We call that $B_{\alpha}^u(s)$ is the bridge strip for $B^u_{\alpha}(s)$. From our definition of $B^s_{\alpha}$ and $B^u_{\alpha}$ in (7.1), $f^{z_k+1}_{\kappa}(k)(B^s_{\alpha}) = B^u_{\alpha}$, see Figure 7.3. The gap strip $G^u(\alpha)$ for a gap $G^u(\alpha)$ of $K^u_{\alpha}$ is defined similarly.

Recall that the bridges $B^s(\Delta)$, $A^u(\Delta)$ in Lemma 6.1 are taken in $\hat{L} \cap B_{lo/2}$ so that our perturbation of the diffeomorphisms $f$ does not affect the invariant set $\Gamma_m$ and local stable foliation $\mathcal{F}^s_{\text{loc}}(\Gamma_m)$ on $E$.

Now we need to choose them more carefully. Let $D_{\mu_\alpha}$ be the rectangle used in Section 3 to define the basic set $\Gamma_m$, which satisfies the conditions (S-vi) and (S-vii). The dynamics of $\varphi = \varphi_{\mu}$ on $D_{\mu_\alpha}$ is determined by that of $f$ on $X_{\mu} = D_{\mu_\alpha} \cup f(D_{\mu_\alpha}) \cup \cdots \cup f^{N_*+n_*}(D_{\mu_\alpha})$, where $N_*$ and $n_*$ are the integers given in Theorem 3.1 and Subsection 5.2, respectively. So our aim is accomplished by perturbing $f$ in the complement of a neighborhood of $X_{\mu}$. By (S-vi) and Theorem 5.1 one can choose the parameter $\mu_\alpha = \Theta_{\mu}(\mu_\alpha)$ and bridges $B^s(\Delta)$ of $K_{\alpha}(\Delta)$ and $A^u(\Delta)$ of $K_{\alpha}(\Delta)$ so that $f = f_{\mu}$ satisfies the conditions of Lemma 5.2 and the following extra conditions for a small constant $\alpha_1 > 0$.

(B-i) For the sub-arc $I = B^s(\Delta) \cup A^u(\Delta)$ of $\hat{L}$, the $\alpha_1|I|$-neighborhood $\hat{I}$ of the strip $I = \pi_{\mathcal{F}^u_{\text{loc}}(\Lambda)}^{-1} \circ \pi_{\mathcal{F}^s_{\text{loc}}(\Lambda)}(I)$ in $S$ is disjoint from $X_{\mu}$, where $\pi_{\mathcal{F}^s_{\text{loc}}(\Lambda)} : S \to W^u_{\text{loc}}(\rho)$ is the projection along leaves of $\mathcal{F}^s_{\text{loc}}(\Lambda)$.

(B-ii) For the sub-bridges $B^s_{\alpha}(\Delta)$ of $B^s(\Delta)$ and $A^u_{\alpha}(\Delta)$ of $A^u(\Delta)$ given in Lemma 6.1 and the sub-arc $I_k = B^s_{\alpha}(\Delta) \cup A^u_{\alpha}(\Delta)$ of (6.1), the strips $\hat{I}_k = \pi_{\mathcal{F}^u_{\text{loc}}(\Lambda)}^{-1}(\hat{I}_k)$
In fact, the assertion (B-i) is guaranteed by choosing the parameter \( \bar{\mu} \) and the initial linked pair \( B^* (0) \) and \( A^* (0) \) of Lemma 5.2 suitably so that the sub-arc \( \hat{I}_k \) is contained in an arbitrarily small neighborhood of the left component of \( bS \) in \( S \), which is represented by the shaded region in Figure 3.2. We refer to Subsection 6.5 in [27] (and also Subsection 5.3 in [19]) for such an argument. The assertion (B-ii) holds if we take \( \alpha_1 \) sufficiently small comparing with \( \alpha_0 \) of Lemma 6.1-(2).

We will define an auxiliary stable foliation on \( S \) for Lemma 7.2 below. Consider a \( C^1 \)-foliation \( \mathcal{G}^s (0) \) on \( \mathcal{G}^u (0) \) such that \( \hat{L} \) and any components of \( \mathcal{G}^u (0) \) are leaves of \( \mathcal{G}^s (0) \) and each leaf meets \( \mathcal{L}_{\text{loc}} (\Lambda) \) exactly. Then \( \mathcal{G}^s (0) \) is uniquely extended to a local stable \( C^1 \)-foliation \( \mathcal{G}^s_{\text{loc}} (\Lambda) \) on \( S \) compatible with \( \mathcal{W}^s_{\text{loc}} (\Lambda) \).

Let \( \hat{A}^u_{L,k} \) and \( B^u_{L,k} \) be the bridges of \( K^u_{m,L} \) and \( K^u_{\Lambda,L} \) given in Lemma 7.1, respectively. Note that \( \hat{A}^u_{L,k} \) contains \( B^u_{L,k} \), see Figure 7.1. Consider the curves \( \hat{L}_k \) in \( S \) and \( L_k \) in \( U (q) \) defined by

\[
\hat{L}_k = f^{- (z_k k^2 + (k))} (\mathcal{B}^s_{k+1} \cap \hat{L}), \quad L_k = f^{-N_0} \circ f^{-i_k} \circ f^{-N_1} (\hat{L}_k \cap S (\hat{A}^u_k)),
\]

see Figure 7.3, where \( S (\hat{A}^u_k) \) is the strip in \( S \) associated with \( \hat{A}^u_k \), see also Figure 5.2. Note that \( \hat{L}_k \) is a leaf of the foliation \( \mathcal{G}^s_{\text{loc}} (\Lambda) \).

![Figure 7.3.](image)

Suppose that \( l \) is a compact \( C^1 \)-curve in \( S \) such that any line tangent to \( l \) is not vertical. For any \( a \in l \), let \( \text{slope}_{a} (l) \geq 0 \) be the (absolute) slope of \( l \) at \( a \) and define the maximum slope \( \text{slope} (l) \) of \( l \) by \( \max \{ \text{slope}_{a} (l) : a \in l \} \). The slope is defined similarly for compact \( C^1 \)-curves in \( U (L) \) which do not have vertical tangent lines, where \( U (L) \) is supposed to have the \( C^{1+\alpha} \)-coordinate introduced in Subsection 5.2.

**Lemma 7.2.** There exists a constant \( \alpha > 1 \) satisfying the following inequalities for any integer \( k > 0 \) and any leaf \( l \) of \( \mathcal{G}^s_{\text{loc}} (\Lambda) \) contained in \( \mathcal{B}^u_k \).

\[
\text{slope} (l) < \alpha (\sigma^{-1} \lambda)^{z_k k^2}, \quad \text{slope} (L_k) < \alpha k (\sigma^{-1} \lambda)^{z_k k^2},
\]

In particular, \( \text{slope} (L_k) < \alpha (\sigma^{-1} \lambda)^{z_k k^2} \).
Proof. Since the left component \( \{0\} \times [0, 2] \) of \( \mathcal{S} \) is a leaf of \( \mathcal{F}_{\text{loc}}^\nu(\Lambda) \), any leaf of \( \mathcal{G}_{\text{loc}}^\nu(\Lambda) \) meets \( \{0\} \times [0, 2] \) transversely. If necessary replacing \( \delta \) by a smaller positive number, we may assume that, for any leaf \( l \) of \( \mathcal{G}_{\text{loc}}^\nu(\Lambda) \), the restriction \( l_{[0,\delta]} = l \cap ([0,\delta] \times [0,2]) \) has no vertical tangent line. From the compactness of \( \mathcal{G}_{\text{loc}}^\nu(\Lambda) \), we have a constant \( \alpha_1 > 0 \) independent of \( l \in \mathcal{G}_{\text{loc}}^\nu(\Lambda) \) such that \( \text{slope}(l_{[0,\delta]}) < \alpha_1 \). We denote the left edge of \( \tilde{l} \) by \( e(l) \). Let \( n_0 \) be the smallest integer with \( n_0 \geq \log \delta / \log \lambda \). For any \( n \geq n_0 \), the component \( l_n \) of \( f^{-n}(l_{[0,\delta]}) \cap \mathcal{S} \) containing \( f^{-n}(e(l)) \) is a leaf of \( \mathcal{G}_{\text{loc}}^\nu(\Lambda) \). Since \( f^n \) is the linear map \((\sigma^n x, \lambda^n y)\) on a small neighborhood of \( l_n \) in \( \mathcal{S} \), \( \text{slope}(l_n) < \alpha_1(\sigma^{-1}\lambda)^n \).

Now we suppose that \( l \) is any leaf of \( \mathcal{G}_{\text{loc}}^\nu(\Lambda) \) contained in \( \mathbb{H}^u_k \). It follows from the form \((7.1)\) of \( B_k^u \) that, if \( z_k k^2 > z_0 k^2 > n_0 \), then \( f^z k^2 \) is the linear map \((\sigma^z k^2 x, \lambda^z k^2 y)\) on a small neighborhood of \( l \) with \( f^z k^2(l') \subset l'_{[0,\delta]} \) for some leaf \( l' \) of \( \mathcal{G}_{\text{loc}}^\nu(\Lambda) \). This shows that \( \text{slope}(l) < \alpha_1(\sigma^{-1}\lambda) z_k k^2 \). By replacing \( \alpha_1 \) by a larger constant \( \alpha > 1 \) if necessary, one can suppose that \( \text{slope}(l) < \alpha(\sigma^{-1}\lambda) z_k k^2 \) for any \( k > 0 \). Thus the first inequality holds.

Since \( M \) is compact, there exists a constant \( \gamma > 1 \) satisfying

\[
\gamma^{-1} \leq \|Df^{-1}_x(v)\| < \gamma
\]

for any point \( x \) of \( M \) and any unit tangent vector \( v \in T_x(M) \). By Lemma \((7.1)\), there exists a constant \( H > 0 \) with \( N_0 + \tilde{\gamma} k + N_1 < H k \) for any \( k > 0 \). Note that \( f^{-N_0} \circ f^{-\tilde{\gamma} k} \circ f^{-N_1} \) maps \( \tilde{L}_k \cap S(\tilde{A}_k) \) onto \( L_k \) and the horizontal segment passing through any point of \( L_k \cap S(\tilde{A}_k) \) to the horizontal segment passing through a point of \( \tilde{L}_k \). Since moreover \( \text{slope}(L_k) < \alpha(\sigma^{-1}\lambda) z_k k^2 \),

\[
\text{slope}(L_k) < \alpha \gamma^{2H}(\sigma^{-1}\lambda) z_k k^2 < (\alpha \gamma^{2H})^k(\sigma^{-1}\lambda)^k z_k k^2.
\]

This implies the second inequality by denoting \( \alpha \gamma^{2H} \) newly by \( \alpha \). \( \square \)

In particular, Lemma \((7.2)\) implies that, for all sufficiently large \( k \), \( L_k \) is almost horizontal and hence \( L_k \) meets \( L \) transversely at a single point \( x_k = (x_k, y_k) \). Note that \( \tilde{x}_k = f^{N_k} \circ f^{\tilde{\gamma} k} \circ f^{N_0}(x_k) \) is a point of \( \tilde{L}_k \). There exists a neighborhood \( N_0 \) of \( \tilde{L} \) in \( \mathcal{G}^u(0) \) consisting of leaves of \( \mathcal{G}^u(0) \). In particular, \( N_0 \) is a sub-strip of \( (\mathcal{G}^u(0), \mathcal{F}^u(0)) \) and the components \( l_{1,0}, l_{2,0} \) of \( \mathcal{N}_0 \) meet \( \mathcal{F}_{\text{loc}}^\nu(\Lambda) \) exactly. We set

\[
(7.6) \quad \text{dist}(\mathcal{N}_0, \tilde{L}) = \zeta > 0.
\]

Let \( \mathcal{N}_k \) be the component of \( f^{-z_k k^2 + (k)}(\mathcal{N}_0) \cap \mathcal{S} \) containing \( \tilde{L}_k \). Since \( \mathcal{N}_0 \subset \mathcal{G}^u(0) \), \( \mathcal{N}_k \) is contained in some gap strip in \( \mathbb{H}^u_k \).

For sequences \( \{u_k\}, \{v_k\} \) of positive numbers, \( u_k \asymp v_k \) means that there exist constants \( 0 < c_1 < c_2 \) independent of \( k \) such that \( c_1 \leq \frac{u_k}{v_k} \leq c_2 \) holds for all \( k \).

Lemma 7.3. There exists a constant \( 0 < \nu < 1 \) such that, for all sufficiently large \( k \), \( [0, 2] \times [\bar{y}_k - \nu k^2 \sigma^{-z_k k^2}, \bar{y}_k + \nu k^2 \sigma^{-z_k k^2}] \) is contained in \( \mathcal{N}_k \), where \( \bar{y}_k \) is the \( y \)-entry of \( x_k = (\bar{x}_k, \bar{y}_k) \), see Figure \((7.2)\).

Proof. By \((7.5)\) and \((7.6)\), we have

\[
\text{dist}(\mathcal{N}_k, \tilde{L}_k) \geq \gamma^{-(k)} \sigma^{-z_k k^2} \zeta.
\]
By Lemma 7.1-(3), \( \tilde{n}_{k+1} \approx k \) and hence \( \langle k \rangle = \tilde{n}_{k+1} + k^2 + k \approx k^2 \). Thus one can take a constant \( 0 < \nu < 1 \) satisfying
\[
\text{dist}(\#N_k, \tilde{L}_k) \geq 2\nu k^2 - \frac{\sigma - z_k^2}{k^2}
\]
for any \( k > 0 \). Since the components \( l_{1,k}, l_{2,k} \) of \( \#N_k \) are leaves of \( G^s_{\text{loc}}(\Lambda) \), slope\((l_{i,k}) < \alpha(\sigma^{-1} \lambda)^{z_k^2} \) for \( i = 1, 2 \) by Lemma 7.2. Here we suppose that the integer \( z_0 \) satisfies
\[
(7.7) \quad \lambda z_0 < \nu.
\]
We will see later that the condition (7.7) is implied by the condition (8.5). Since
\[
\text{slope}(l_{i,k}) \leq \frac{\alpha(\nu^{-1} k^2 \lambda z_0^2)}{2} \to 0 \quad (k \to \infty)
\]
for \( i = 1, 2 \), it follows that
\[
[0, 2] \times [\tilde{y}_k - \nu k^2 \sigma - z_k^2, \tilde{y}_k + \nu k^2 \sigma - z_k^2] \subset N_k
\]
for all sufficiently large \( k \).

Consider the map \( h_k \) defined by
\[
(7.8) \quad h_k := f^{-z_k^2 + \langle k \rangle} \circ f^N_1 \circ \hat{f}^N_0 \circ f^{-N_0}
\]
and the sequence \( \{\hat{x}_k\} \) with \( \hat{x}_k = f^{-z_k^2 + \langle k \rangle}(\tilde{x}_k) = h_k(x_k) \). Let \( u_k = (u_k, v_k) \) be the vector with
\[
(7.9) \quad \hat{x}_k + u_k = f^{-N_2}(x_{k+1}).
\]
Figure 7.5 illustrates the transition of base points schematically.

**Lemma 7.4.** There exists a constant \( \beta > 0 \) independent of \( k \) such that \( \|u_k\| \leq \beta r_{s-T}^{-k} \), where \( T \) is a number not smaller than the constant \( T_0 \) given in Lemma 7.1.

**Proof.** Since \( f^{N_2}(\hat{x}_k) \in B_{L,k+1}^s \) and \( x_{k+1} \in B_{k+1}^s \), it follows from Lemma 7.1-(2) that there exists a vector \( t_{k+1} = (s_{k+1}, t_{k+1}) \) with \( |t_{k+1}| < 2^{-1} \xi_0 \) that \( f^{-N_2}(\hat{x}_k) + t_{k+1} = x_{k+1} \). Since \( N_2 \) is independent of \( k \), \( \|u_k\| \approx \|t_{k+1}\| \approx |t_{k+1}| \). Thus we have a constant \( \beta > 0 \) satisfying \( \|u_k\| = \|x_{k+1} - f^{-N_2}(\hat{x}_k)\| \leq \beta r_{s-T}^{-k} \) for any \( k \geq 1 \). \( \square \)
 Throughout the remainder of this paper, we suppose that $r$ is an integer with $3 \leq r < \infty$. Note that $\pi_{F_{\mu}^s(\Lambda)}$, $\pi_{F_{\mu}^u(\Lambda)}$ are $C^{1+\alpha}$-functions but not necessary of $C^r$-class. So we need suitable substitutes for them. Let $\varpi_s : S \to [0, 2]$ and $\varpi_u : S \to [0, 2]$ be $C^r$-maps arbitrarily $C^1$-close to $\pi_{F_s^s(\Lambda)}$ and $\pi_{F_u^s(\Lambda)}$ respectively. Here, we do not require any $f$-invariance property for $\varpi_s$ or $\varpi_u$. Since $N_0$ is contained in $\mathcal{G}^{u}(0) \setminus \mathcal{G}^{u}(0)$, there exist a $d > 0$ and sub-intervals $H$, $\widetilde{H}$ of $G^{u}(0)$ such that $G^{u}(0) = [\min H - 2d|H|, \max H + 2d|H|]$, $\widetilde{H} = [\min H - d|H|, \max H + d|H|]$ and $\pi_{F_{\mu}^s(\Lambda)}(H)$ contains $N_0$. Then one can choose the $C^r$-map $\varpi^s$ so that $H = (\varpi_s)^{-1}(H)$ and $\widetilde{H} = (\varpi^s)^{-1}(\widetilde{H})$ are strips in $S$ with $N_0 \subset H \subset \widetilde{H} \subset \mathcal{G}^{u}(0)$.

We may also assume that each $\tilde{I}_k = (\varpi_u)^{-1}(\tilde{I}_k)$ ($k = 1, 2, \ldots$) is arbitrarily $C^1$-close to the strip $\tilde{I}_k$ defined in (B-ii) so that $\tilde{I}_k$ are mutually disjoint strips in $S$ with $\bigcup_{k=1}^{\infty} \tilde{I}_k \subset \tilde{I}$. See Figure 7.6.

Now we define the bump functions $\theta_k$ and $\tilde{\theta}$ supported on $\tilde{I}_k$ and $\tilde{H}$ respectively. First, consider a non-decreasing $C^\infty$ function on $\mathbb{R}$ with

$$s(x) = \begin{cases} 
0 & \text{if } x \leq -1; \\
1 & \text{if } x \geq 0.
\end{cases}$$

For $\rho > 0$ and the interval $[a, b]$, let $S_{\rho, [a, b]}$ be the non-negative $C^\infty$ function on $\mathbb{R}$ defined as

$$S_{\rho, [a, b]}(x) := s\left(\frac{x - a}{\rho(b - a)}\right) + s\left(\frac{b - x}{\rho(b - a)}\right) - 1.$$ 

The support of $S_{\rho, [a, b]}$ is $[a - \rho(b - a), b + \rho(b - a)]$ and the height on $[a, b]$ is identical to 1. Since $S_{\rho, a}$ is symmetric with respect to $x = \frac{a + b}{2}$, one has

\begin{equation}
(7.10) \quad \|S_{\rho, [a, b]}\|_r \leq \frac{1}{(\rho(b - a))} \|s\|_r,
\end{equation}
where \( \| \cdot \|_r \) is the norm given by the derivative until order \( r \). Then our desired bump functions are given by

\[
\theta_k := S_{\alpha_k/2, \pi(t_k)}, \quad \tilde{\varrho} := S_{d, H},
\]

where \( \pi(t_k) \) is the \( \pi_{\infty}(\Lambda) \)-image of the arc \( I_k \) in \( \mathcal{L} \) defined as \( (6.1) \).

**Lemma 7.5.** Let \( \varepsilon > 0 \) and \( T \) any real number with

\[
T > N_u r \log(5 \cdot 2^{m-3}) / \log(r_{s_-}),
\]

where \( N_u \) is the integer given in Lemma \( 6.2-3 \). Then there exists a constant \( C_T > 0 \) satisfying

\[
\lim_{T \to +\infty} C_T = 0 \quad \text{and} \quad \sum_{k=1}^{\infty} \| u_k \|_{r_{s_-}} < C_T.
\]

**Proof.** By Lemmas \( 4.5-1 \) and \( 6.2-3 \), we have

\[
|A^u_{L,k}| \geq (5 \cdot 2^{m-3})^{-i_k}|A^u_{L,0}| \quad \text{and} \quad i_k \leq N_u k + i_0.
\]

Then

\[
\frac{1}{|A^u_{L,k}|} < \frac{1}{|A^u_{L,0}|} \left( \frac{5 \cdot 2^{m-3} r}{|A^u_{L,0}|} \right)^k < \left( \frac{5 \cdot 2^{m-3} r}{N_u k} \right)^k.
\]

By Lemma \( 7.4 \), one has

\[
\sum_{k=1}^{\infty} \| u_k \|_{r_{s_-}} < \left( \frac{5 \cdot 2^{m-3} r}{N_u k} \right)^k \sum_{k=1}^{\infty} \left( \frac{5 \cdot 2^{m-3} r}{N_u k} \right)^k.
\]

Since \( T > N_u r \log(5 \cdot 2^{m-3}) / \log(r_{s_-}) \), the right-hand side of the inequality is equal to

\[
C_T := \left( \frac{5 \cdot 2^{m-3} r}{N_u k} \right)^k (r_{s_-} - (5 \cdot 2^{m-3} N_u r)).
\]

Since \( r_{s_-} \geq 2 \), \( \lim_{T \to +\infty} C_T = 0. \)

The sequence \( \mathbf{t} = (t_2, t_3, \ldots, t_k, \ldots) \) of vectors with \( f_{N_s}(\mathbf{x}_k) + t_{k+1} = x_{k+1} \) is called the perturbation sequence. Note that \( \mathbf{t} \) depends on the constant \( T \) given in Lemma \( 7.1 \) and each entry \( t_k = \mathbf{t}_k(T) \) converges to the zero vector as \( T \to \infty \).

Using the bump functions \( \{ \theta_k \}_{k \geq 1}, \tilde{\varrho} \) and the vectors \( u_k = (u_k, v_k) \) of \( (7.9) \), we define the sequence of the \( C^r \)-perturbation maps \( \psi_{\mathbf{t},a} : M \to M \) \( (a = 1, 2, \ldots) \) supported on the disjoint union \( \bigcup_{k=1}^{a} Y_k \subset S \) as

\[
\psi_{\mathbf{t},a}(x, y) := \left( x + \sum_{k=1}^{a} u_k \vartheta_k(x, y), y + \sum_{k=1}^{a} v_k \vartheta_k(x, y) \right),
\]

where

\[
\vartheta_k(x, y) = \theta_k(\varpi^u(x, y)) \tilde{\varrho}(\varpi^u(x, y)) \quad \text{and} \quad Y_k = \tilde{I}_k \cap \tilde{H}.
\]

Each \( Y_k \) is a rectangle as illustrated in Figure \( 7.6 \).

**Lemma 7.6.** For any \( \mathbf{t} = \mathbf{t}(T) \), the sequence \( \{ \psi_{\mathbf{t},a} \}_{a=1}^{\infty} \) \( C^r \)-converges to the \( C^r \)-map \( \psi_{\hat{\mathbf{t}}} \) with

\[
\psi_{\hat{\mathbf{t}}}(x, y) := \left( x + \sum_{k=1}^{\infty} u_k \vartheta_k(x, y), y + \sum_{k=1}^{\infty} v_k \vartheta_k(x, y) \right).
\]
for $(x, y) \in S$. Moreover, $\psi_L$ are $C^r$-diffeomorphisms on $M$ for all sufficiently large $T$ which $C^r$-converge to the identity as $T \to \infty$.

**Proof.** Recall that $I_{k+1} = A_{k+1}^u(\Delta) \cup B_{k+1}^s(\Delta)$ is the arc of $(6.1)$ in $L$. Since $N_2$ is independent of $k$, it follows from $|A_{k+1}^u(\Delta)| \geq |B_{k+1}^s(\Delta)|$ that

$$|I_{k+1}| = |A_{k+1}^u(\Delta) \cup B_{k+1}^s(\Delta)| \geq |A_{k+1}^u(\Delta)| \geq |A_{k+1}^u(\Delta)|.$$

By this fact together with (7.10), for any integers $a, b$ with $1 \leq a < b$,

$$\|\psi_{L,b} - \psi_{L,a}\| \leq C\|S_{b,2}\| r \sum_{k=a+1}^{b} \frac{\|u_k\|}{|A_{L,k+1}^u|},$$

where $C$ is a constant independent of $t$. By Lemma 7.5, $\{\psi_{L,a}\}_{a=1}^{\infty}$ is a Cauchy sequence in the space $(\text{Map}^r(M), \| \cdot \|_r)$ of $C^r$-maps on $M$, which is a complete metric space. Thus $\psi_{L,a}$ $C^r$-converges to the $C^r$-map $\psi_L$ with (7.13) as $a \to \infty$. By (7.14),

$$\|\psi_L - \text{id}_M\| \leq C\|S_{b,2}\| r \sum_{k=1}^{\infty} \frac{\|u_k\|}{|A_{L,k+1}^u|}.$$ 

Again from Lemma 7.5, we know that the map $\psi_L$ $C^r$-converges to the identity as $T \to \infty$. Since the identity is a diffeomorphism, $\psi_L$ is also a diffeomorphism for all sufficiently large $T$. This completes the proof. □

From the definition (7.11) of $\theta_k$, $\vec{\theta}$ and the form (7.13) of $\psi_L$, we know that the support $\text{supp}(\psi_L)$ of $\psi_L$ is contained in the disjoint union $\bigcup_{k=1}^{\infty} Y_k$. We now define the $C^r$-map $f_L$ by

$$(7.15) \quad f_L := f \circ \psi_L,$$

for the perturbation sequence $t = (t_2, t_3, \ldots, t_k, \ldots)$. Since $\text{supp}(\psi_L)$ is contained in a small region in $S$ sufficiently close to the left component of $bS$, $\text{supp}(\psi_L)$ is disjoint from $\bigcup_{k=1}^{\infty} f^i(\text{supp}(\psi_L))$, see Figure 3.2. It follows that $f_L$ is equal to $f$ on $\bigcup_{k=1}^{\infty} f^i(\text{supp}(\psi_L))$. Hence, by (7.9) and (7.13),

$$(7.16) \quad f_L^{N_2}(\bar{x}_k) = f_L^{N_2-1} \circ (f \circ \psi_L)(\bar{x}_k) = f^{N_2-1} \circ f(\bar{x}_k + u_k) = x_{k+1},$$

see Figure 7.6. By Lemma 7.6, one can suppose that $f_L$ is a $C^r$-diffeomorphism arbitrarily $C^r$-close to $f$ if we take $T$ sufficiently large.

**Remark 7.7.** (1) Since $\bigcup_{k=1}^{\infty} Y_k \subset \bar{L}$ by (B-1), the support $\text{supp}(\psi_L)$ is disjoint from $X_n$. Thus both the invariant set $\Gamma_m$ and local stable foliation $\mathcal{F}_{\text{loc}}^s(\Gamma_m)$ for the perturbed diffeomorphism $f_L$ are the same as the originals.

(2) It is possible to rearrange our construction so that both $f$ and $\psi_{L,a}$ are of $C^\infty$-class. However, even in the case, Lemma 7.6 still asserts that $\psi_L$ is of $C^r$-class with $2 \leq r < \infty$ but not necessarily of $C^\infty$-class. In fact, though $\psi_L$ is a $C^\infty$-map on the neighborhood $Y_k$ of any $\bar{x}_k$, the authors do not know that it holds also at the limit point of $\{\bar{x}_k\}$. So it may be impossible to suppose that the composition $f_L := f \circ \psi_L$ of (7.15) is a $C^\infty$-diffeomorphism. A similar situation would occur for the diffeomorphism presented by Colli-Vargas [6] Theorem 1].
8. Detection of Wandering Domains with Historic Behavior

8.1. Preliminary. From now on, we write \( f_t = f \) for short. Recall that \( \sigma \) is the unstable eigenvalue of the derivative \( Df_p \) of \( f \) at \( p \) given in (S-iii). Define the constant \( \omega \) by

\[
\omega = \max\{\nu^{-1}, \sup\{\|Df(x)\| : x \in S\}, \sup\{(Df(x))^{-1}\| : x \in S\}\},
\]

where \( \nu \) is the constant given in Lemma 7.3. In this section, the constant

\[
b_k := \varepsilon(5 \cdot 2^{m-3} - p_k \omega - q_k \sigma - r_k)
\]

plays an important role, where \(5 \cdot 2^{m-3} \) is the number presented in Lemma 4.5-(1) and

\[
p_k = \sum_{i=0}^{\hat{t}_{k+1}} \frac{k+i}{2^i}, \quad q_k = \sum_{i=0}^{(k+i)/2}, \quad r_k = \sum_{i=0}^{z_k(k+i)^2} \frac{k+i}{2^i}.
\]

The factor \( \varepsilon \) in (8.2) is a positive constant independent of \( k \) which will be fixed later. Remember that \( z_k \) is either \( z_0 \) or \( z_0 + 1 \) for the integer \( z_0 \) given in Subsection 7.1 and \( (k+i) = \hat{n}_{k+1+i} + (k+i)^2 + k + i \) by (7.2).

**Lemma 8.1.** (1) For any \( \eta > 0 \), suppose that \( z_0 \) satisfies (8.4). Then there exists an integer \( k_* > 0 \) such that, for any \( k \geq k_* \),

\[
b_k^4 \leq \varepsilon^{-\frac{1}{2}}(5 \cdot 2^{m-3} - \frac{1}{2} \hat{t}_{k+1} \omega - \frac{3}{2} \sigma(k+1) + \frac{3}{4} \sigma z_k k^2 b_{k+1},
\]

\[
(k) + \frac{3}{4} \hat{t}_{k+1} < 4k^2.
\]

(2) For any integer \( k > 0 \),

\[
b_{k+1} = \varepsilon^{-1}(5 \cdot 2^{m-3} \hat{t}_k \omega^{2(k)} \sigma^{2z_k k^2 b_k^2}.
\]
This shows the first inequality of (1).

It is immediate from (8.3) that $p_{k+1} - p_k = -\frac{1}{2} + \frac{3}{2}p_{k+1}$ and $q_{k+1} - \frac{q_k}{2} = -\frac{(k+1)}{2} + \frac{3}{4}q_{k+1}$.

Since $f(x) = \frac{4}{2 - (1 + \eta_1)^2} - \frac{1}{1 + \eta_1} - 2$ is a monotone increasing function on $0 \leq x < \sqrt{2} - 1$ with $f(0) = 1$ and $\lim_{x \to \sqrt{2} - 1} f(x) = \infty$, for any $\eta > 0$, there exists a unique $0 < \eta_1 < \sqrt{2} - 1$ with

\[
\frac{4}{2 - (1 + \eta_1)^2} - \frac{1}{1 + \eta_1} - 2 = 1 + \eta.
\]

Since $\lim_{k \to \infty} \frac{(k+1)^2}{k^2} = 1$, there exists an integer $k_* > 0$ such that $(k + 1)^2 \leq (1 + \eta_1)k^2$ for any $k \geq k_*$. We choose the integer $z_0$ so as to satisfy

(8.4) $z_0 + 1 \leq (1 + \eta_1)z_0$.

Then $z_{k+1}(k + 1)^2 \leq (z_0 + 1)(k + 1)^2 \leq (1 + \eta_1)z_k k^2$ holds. Repeating a similar argument, one can have $z_{k+1+i}(k + 1 + i)^2 \leq z_k(1 + \eta_1)^2(k+1)^2$ for any $k \geq k_*$ and $i \geq 0$. Since $0 < \frac{1 + \eta_1}{2} < 1$,

\[
r_{k+1} \leq \left( \sum_{i=0}^{\infty} \frac{(1 + \eta_1)^{2(i+1)}}{2^i} \right) z_k k^2 = \left( \frac{4}{2 - (1 + \eta_1)^2} - 2 \right) z_k k^2.
\]

On the other hand, since $r_k \geq \sum_{i=0}^{\infty} \frac{(k+i)^2}{2^i} = \frac{2}{1 + \eta_1} z_k k^2$,

\[
r_{k+1} - \frac{r_k}{2} \leq \left( \frac{4}{2 - (1 + \eta_1)^2} - 2 \right) z_k k^2 = (1 + \eta)z_k k^2.
\]

This shows the first inequality of (1).

Since $\tilde{n}_{k+1} \gg k$ by Lemma [1, 3], one can choose $k_*$ so that $\langle k \rangle = \tilde{n}_{k+1} + k^2 + k \leq 2k^2$ for any $k \geq k_*$. Since

\[
q_{k+1} - \sum_{i=0}^{\infty} \frac{k^2}{2^i} = \sum_{i=0}^{\infty} \frac{(k+i)^2}{2^i} = \sum_{i=1}^{\infty} \frac{k^2}{2^i} = k,
\]

we may assume that $q_{k+1} < 2\sum_{i=0}^{\infty} \frac{k^2}{2^i} = 4k^2$ for any $k \geq k_*$. It follows that

\[
\frac{\langle k \rangle}{2} + \frac{3}{4} q_{k+1} < k^2 + \frac{3}{4} \cdot 4k^2 = 4k^2.
\]

This shows the second inequality of (1).

(2) The equality of (2) is derived immediately from (8.2) together with

\[
2p_k - p_{k+1} = \tilde{\lambda}k, \quad 2q_k - q_{k+1} = 2\langle k \rangle, \quad 2r_k - r_{k+1} = 2z_k k^2.
\]

This completes the proof. □

Since $z_0$ is required to satisfy (8.4), we need to choose $z_0$ sufficiently large according as $\eta > 0$ is taken small. By (8.4) in Section 3, $\lambda \sigma < 1$. We take and fix a sufficiently small $\eta > 0$ with $\lambda \sigma^{1+\eta} < 1$. So one can choose $z_0$ so as to satisfy

(8.5) $\omega^4(\lambda \sigma^{1+\eta})^{z_0} < 1$. 

holds. Since \( n^{-1} \leq \omega \), it is not hard to show that the condition \( 8.5 \) implies \( 7.7 \). We will see later that the superscript ‘4’ of \( \omega \) corresponds to the coefficient ‘4’ of \( k^2 \) in the second inequality of \( 1 \).

8.2. Critical chain of rectangles. For every \( k \geq 1 \), let \( x_k = (x_k, y_k) \) be the intersection point of \( L_k \) and \( L \) given in Subsection 7.1. We consider the rectangle

\[
R_k = [x_k - b_k^2, x_k + b_k^2] \times [y_k - b_k, y_k + b_k]
\]
centered at \( x_k \) with respect to the \( C^{1+\alpha} \)-coordinate on \( U(L) \) defined in Subsection 5.2, see Figure 7.3. The absolute slope of the diagonals of \( R_k \) is \( b_k^2 \). By (8.2), there exists a constant \( 0 < \gamma < 1 \) such that \( b_k^2 > \gamma k^{-k^2} \). By Lemma 7.2 and (8.5),

\[
\frac{\text{slope}(L_k)}{b_k^2} < \frac{\alpha k^2}{\gamma k^{2} - \omega^{-k^2} \sigma^{-z k^2}} = (\alpha \gamma^{-1}) k^2 \rightarrow +0 \quad (k \rightarrow \infty).
\]

Thus one can suppose that \( L_k \cap R_k \) is an arc in \( R_k \) passing through \( x_k \) and well approximating the horizontal line \( y = y_k \) if \( k \geq k_* \). In particular, the arc connects the components of the edge of \( \tilde{\mathbb{R}}_k \). The strip \( \tilde{\mathbb{R}}_k \) is divided by \( L_k \) into the two strips \( \mathbb{B}_k \). Similarly, \( R_k \) is divided by \( L_k \) into the two strips \( R_k \). See Figure 7.3. By Lemma 4.5 (1), one can have a constant \( C > 0 \) satisfying

\[
\text{dist}(f^{N_k} \circ \hat{f}^k \circ f^{N_0} l_k^\pm, \tilde{L}_k) \leq C(5 \cdot 2^{m-3}) b_k^2,
\]

where \( l_k^\pm = \mathbb{B} \cap R_k \). By (8.1) and (8.2),

\[
(5 \cdot 2^{m-3}) b_k^2 \leq (5 \cdot 2^{m-3}) b_k^2 \leq C \varepsilon \nu k^2 \sigma^{-z k^2}.
\]

It follows from Lemma 7.3 that \( f^{N_k} \circ \hat{f}^k \circ f^{N_0} (R_k^\pm) \subset \mathbb{B}_k \cap N_k \) holds for any \( k \geq k_* \), and \( k \) sufficiently small. In particular, this implies that \( h_k (R_k) \subset \mathbb{B}_k^s \cap \tilde{\mathbb{R}} \), where \( h_k \) is the diffeomorphism defined as (7.8). See Figures 7.3 and 7.6.

Define a new \( C^{1+\alpha} \)-coordinate on the strip \( \tilde{\mathbb{R}}_k \) such that each vertical line is contained in a leaf of \( F_\text{loc}^u (\Lambda) \) and each horizontal line is horizontal also with respect to the original coordinate on \( S \). Set \( \tilde{x}_k = (\tilde{x}_k, \tilde{y}_k) \) on the new coordinate. For any integer \( k > 0 \), consider the composition

(8.6)

\[
g_k := f^{N_2} \circ f^{s + (k)} \circ f^{N_1} \circ \hat{f}^k \circ f^{N_0} = f^{N_2} \circ h_k.
\]

By (7.16), \( f^{N_2} (\tilde{x}_k, \tilde{y}_k) = x_{k+1} = (x_{k+1}, y_{k+1}) \). Since \( f^{N_2} (F_\text{loc}^u (\Lambda)) \) and \( f^{-N_0} (F_\text{loc}^s (\Gamma_m)) \) have a quadratic tangency at \( (x_{k+1}, y_{k+1}) \), \( f^{N_2} \) is represented as

(8.7)

\[
(x_{k+1} + a_1 \tilde{x} + a_2 \tilde{y} + H_1 (\tilde{x}, \tilde{y}), y_{k+1} + \alpha_1 \tilde{x} + \alpha_2 \tilde{y} + H_2 (\tilde{x}, \tilde{y}))
\]

for any \( (\tilde{x}, \tilde{y}) \) near \( (0, 0) \), where \( \alpha_2 \neq 0 \) and \( H_1, H_2 \) are the higher-order terms of \( f^{N_2} \) satisfying

\[
H_1 (\tilde{x}, \tilde{y}) = O(|\tilde{x}|^2 + |\tilde{y}|^2 + |\tilde{y}|^2), \quad H_2 (\tilde{x}, \tilde{y}) = O(|\tilde{x}|^2 + |\tilde{y}|^2 + |\tilde{y}|^3).
\]

Note that the coefficients of the Taylor expansions (8.7) depend only on the values of the partial derivatives of \( f \) at \( \tilde{x}_k \), and the higher order terms depend only on the values of them at \( \tilde{x}_k + \theta \tilde{x} \) for some \( 0 < \theta < 1 \). Since \( f^{N_2} \) is a \( C^r \)-diffeomorphism
and \( \{ \tilde{x}_k \}_{k=1}^\infty \) is contained in the compact set \( \tilde{H} \), there exist positive constants \( c, C \) independent of \( k \) and satisfying
\[
|a_{1,k}| < C, \quad |a_{2,k}| < C, \quad |a_{1,k}| < C, \quad c < |a_{2,k}| < C,
\]
\[
|H_{1,k}(\tilde{x}, \tilde{y})| \leq C(|\tilde{x}|^2 + |\tilde{x}\tilde{y}| + |\tilde{y}|^2), \quad |H_{2,k}(\tilde{x}, \tilde{y})| \leq C(|\tilde{x}|^2 + |\tilde{x}\tilde{y}|^2 + |\tilde{y}|^3)
\]
for any \( k > 0 \) and any \( (\tilde{x}, \tilde{y}) \) near \( (0,0) \).

**Remark 8.2.** Strictly we need \( C^3 \)-coordinates on \( U(L) \) and \( \tilde{H} \) for considering the Taylor expansion of \( f^{N_2} \) near \( \tilde{x}_k \). For any \( x \in U(L) \), we consider the \( C^3 \)-coordinate on \( U(L) \) defined in Subsection 8.2, which \( C^{1+\alpha} \)-depends on the origin \( x \in U(L) \) and such that the two axes coincide with the horizontal and vertical lines \( L_{\tilde{x}} \), \( L_{\tilde{y}} \) passing through \( x \) with respect to the original \( C^{1+\alpha} \)-coordinate on \( U(L) \). Let
\[
\pi_x^{hor(1+\alpha)} : U(L) \rightarrow L_{\tilde{x}}, \quad \pi_x^{vert(1+\alpha)} : U(L) \rightarrow L_{\tilde{y}}
\]
be the projections along the horizontal and vertical lines on the \( C^{1+\alpha} \)-coordinate.

The projections \( \pi_x^{hor(3)}, \pi_x^{vert(3)} \) on the \( C^3 \)-coordinate are defined similarly. Let \( \gamma \) be any smooth arc in \( U(L) \) and \( |\gamma|_{1+\alpha}, |\gamma|_3 \) the lengths of \( \gamma \) with respect to the \( C^{1+\alpha} \) and \( C^3 \)-coordinates respectively. Then it is not hard to see that \( |\gamma|_{1+\alpha} \approx |\gamma|_3 \). Moreover, if the arc \( \gamma \) contains \( x \), then both \( |\pi_x^{hor(1+\alpha)}(\gamma)| \approx |\pi_x^{hor(3)}(\gamma)| \) and \( |\pi_x^{vert(1+\alpha)}(\gamma)| \approx |\pi_x^{vert(3)}(\gamma)| \) hold on \( L_{\tilde{x}} \) and \( L_{\tilde{y}} \), respectively. Similarly, for any \( x \in \tilde{H} \), one can define a \( C^{3} \)-coordinate on \( \tilde{H} \) \( C^{1+\alpha} \) depending on \( \tilde{x} \) such that the origin is \( \tilde{x} \) and the two axes coincide with the horizontal and vertical lines passing through \( \tilde{x} \) with respect to the original \( C^{1+\alpha} \)-coordinate on \( \tilde{H} \). This justifies our argument involved with Taylor expansions.

For sequences \( \{u_k(\varepsilon)\}, \{v_k(\varepsilon)\} \) of positive numbers, \( u_k(\varepsilon) \prec v_k(\varepsilon) \) means that there exists a positive constant \( a \) independent of \( k, \varepsilon \) and satisfying \( u_k(\varepsilon) < av_k(\varepsilon) \) for all \( k \) and \( \varepsilon > 0 \).

The main result of this section is as follows:

**Lemma 8.3 (Rectangle Lemma).** There exists an integer \( k_0 \geq k_* \) such that, for any \( k \geq k_0 \), \( g_k(R_k) \subset \text{Int } R_{k+1} \).

**Proof.** First we show that the \( g_k \)-image of the center vertical segment \( J_k = J_{k+1} \cup J_{k-1} \) of \( R_k \) is contained in \( \frac{1}{2}R_{k+1} = [x_{k+1} - \frac{1}{2}b_{k+1}^2, x_{k+1} + \frac{1}{2}b_{k+1}^2] \times [y_{k+1} - \frac{1}{2}b_{k+1}^2, y_{k+1} + \frac{1}{2}b_{k+1}^2] \), where \( J_{k+1} = \{ x_k \} \times [y_k, y_k + b_k] \) and \( J_{k-1} = \{ x_k \} \times [y_k - b_k, y_k] \), see Figure 8.1. We set \( J_k = h_k(J_{k+1}) \). Note that, by \( 8.2, b_k \) and hence \( J_{k,\pm} \) depend on \( \varepsilon \). Since \( f^{z_k k^2} \) coincides with the linear map \( (x, y) \mapsto (\lambda^{z_k k^2} x, \sigma^{z_k k^2} y) \) on \( f^{N_1} f^{i_k} f^{N_0} (R_k) \) and \( N_0, N_1 \) are independent of \( k \),
\[
|J_k| \prec (5 \cdot 2^{m-3})^{\frac{1}{3}} \lambda^{z_k k^2} \sigma^{z_k k^2} b_k.
\]
Since \( g_k(J_{k+1}) = f^{N_2}(J_{k+1}) \) and \( g_k(J_{k-1}) \) contains the base point \( x_{k+1} \), by Lemma 8.4 (2) and 8.7 together with Remark 8.2, we have
\[
|\pi_x^{vert(1+\alpha)}(g_k(J_{k+1}))| \prec (5 \cdot 2^{m-3})^{\frac{1}{3}} \lambda^{z_k k^2} \sigma^{z_k k^2} b_k = \varepsilon^{\frac{1}{2}} b_{k+1}^2 \quad \text{and}
\]
\[
|\pi_x^{hor(1+\alpha)}(g_k(J_{k+1}))| \prec (5 \cdot 2^{m-3})^{\frac{1}{3}} \lambda^{z_k k^2} \sigma^{z_k k^2} b_k^2 = \varepsilon b_{k+1}.
\]
One can choose \( \varepsilon > 0 \) so that
\[
|\pi_x^{vert(1+\alpha)}(g_k(J_{k+1}))| \leq \frac{1}{2} b_{k+1}^2 \quad \text{and} \quad |\pi_x^{hor(1+\alpha)}(g_k(J_{k+1}))| \leq \frac{1}{2} b_{k+1}.
\]
It follows that \( g_k(J_{k+}) \subset \frac{1}{2} R_{k+1} \). Similarly we have \( g_k(J_{k-}) \subset \frac{1}{2} R_{k+1} \) and hence \( g_k(J_k) \subset \frac{1}{2} R_{k+1} \).

The horizontal segment \( Z_{k,y} = [x_k - b_k^{\frac{3}{2}}, x_k + b_k^{\frac{3}{2}}] \times \{y\} \) of \( R_k \) with \( y_k - b_k \leq y \leq y_k + b_k \) is the union of \( Z_{k,y+} = [x_k, x_k + b_k^{\frac{3}{2}}] \times \{y\} \) and \( Z_{k,y-} = [x_k - b_k^{\frac{3}{2}}, x_k] \times \{y\} \). Note that the intersection \( Z_{k,y} \cap J_k \) consists of the single point \((x_k, y)\). By our construction of the unstable foliation \( \mathcal{F}^u_{\text{loc}}(\Gamma_m) \) in Subsection 5.1, the image \( f^{N_1} \circ f^{k} \circ f^{N_0}(Z_{k,y+}) \) is a strictly horizontal segment in \( S \) with}

\[
|f^{N_1} \circ f^{k} \circ f^{N_0}(Z_{k,y+})| < (5 \cdot 2^{m-3})^{i_k} b_k^{\frac{3}{2}}.
\]

Since \( f^{z_k k^2} \) is the linear map \((\sigma z_k k^2, x, \lambda z_k k^2, y)\) in a small neighborhood of \( f^{N_1} \circ f^{k} \circ f^{N_0}(Z_{k,y+}) \) in \( S \), the curve \( \tilde{Z}_{k,y+} = h_k(Z_{k,y+}) \) satisfies}

\[
|\tilde{Z}_{k,y+}| < (5 \cdot 2^{m-3})^{i_k} \tilde{\omega}(k) \lambda z_k k^2 b_k^{\frac{3}{2}}.
\]

By Lemma 8.1 (1),}

\[
|g_k(Z_{k,y+})| < (5 \cdot 2^{m-3})^{i_k} \tilde{\omega}(k) \lambda z_k k^2 b_k^{\frac{3}{2}}
\]

\[
\leq \varepsilon^{-\frac{1}{2}} (5 \cdot 2^{m-3})^{i_k} \frac{\tilde{\omega}}{\hat{\omega}} \frac{1}{2} p_{k+1} \omega \frac{1}{3} + \frac{3}{4} q_{k+1} (\lambda \sigma^{1+n}) z_k k^2 b_{k+1}
\]

\[
< \varepsilon^{-\frac{1}{2}} (5 \cdot 2^{m-3})^{i_k} \frac{\tilde{\omega}}{\hat{\omega}} \frac{1}{2} p_{k+1} \omega 4^{k^2} (\lambda \sigma^{1+n}) z_k k^2 b_{k+1}.
\]

By Lemma 7.1 (3) and (8.3), \( \frac{\tilde{\omega}}{\hat{\omega}} + \frac{3}{4} p_{k+1} \approx k \). Since \( \omega^4 (\lambda \sigma^{1+n}) z_k < 1 \) by (8.5), there exists an integer \( k_0 \geq k \) such that, for any \( k \geq k_0 \), \( |g_k(Z_{k,y+})| < \frac{1}{2} b_{k+1} \). Since \( g_k(Z_{k,y+}) \cap g_k(J_k) \neq \emptyset \) and \( g_k(J_k) \subset \frac{1}{2} R_{k+1} \), \( g_k(Z_{k,y+}) \) is contained in \( \text{Int} R_{k+1} \). Similarly \( g_k(Z_{k,y-}) \) is contained in \( \text{Int} R_{k+1} \), and hence \( g_k(Z_{k,y}) \subset \text{Int} R_{k+1} \) for all \( y \in [y_k - b_k, y_k + b_k] \). It follows that \( g_k(R_k) \subset \text{Int} R_{k+1} \). This completes the proof. \( \square \)
Proposition 8.4. Let \( f \) be a \( C^r \) diffeomorphism on \( M \) contained in a Newhouse open set \( \mathcal{O} \) of \( \text{Diff}^r(M) \). Then there exist \( C^r \)-diffeomorphisms \( f_n \) on \( M \) which admit non-trivial wandering domains and \( C^r \)-converge to \( f \).

Proof. Let \( f_k \) be the diffeomorphism given in (7.15). By Lemmas 7.6 and 8.3 there exists a perturbation sequence \( t_n = (t_{n,2}, t_{n,3}, \ldots) \) such that \( f_n := f_k \circ t_n \) \( C^r \)-converge to \( f \). Then the interior \( D \) of the rectangle \( R_{k_0} \) given in Lemma 8.3 is a wandering domain of \( f_n \). If not, there would exist an \( a \in \mathbb{N} \) such that \( f_n^a(D) \cap D \neq \emptyset \). Take an integer \( k \) with \( k \geq k_0 \) and \( z_kk^2 > a \). By (7.1), there exists an integer \( q \geq 1 \) such that

\[
 f_n^{q+k^2}(D) \subset B^n((k); \frac{qk^2}{2} \sum_{k+1}^{\infty} |\hat{\omega}_{k+1}|^{-1}) \subset B^n(1; 2),
\]

see Figure 4.1. Moreover we have

\[
 f_n^{q+k^2-a}(D) \subset B^n((k) + a; \frac{1}{2}) \sum_{k+1}^{\infty} |\hat{\omega}_{k+1}|^{-1}) \subset B^n(1; 1).
\]

Hence the intersection \( f_n^{q+k^2-a}(f_n^a(D) \cap D) = f_n^{q+k^2}(D) \cap f_n^{q+k^2-a}(D) \) is empty. This contradicts that \( f_n^a(D) \cap D \neq \emptyset \). Thus \( D \) is a wandering domain of \( f_n \). It is immediate from our construction of \( D \) that \( \lim_{j \to \infty} \text{diam} f_n^j(D) = 0 \). Moreover, for a fixed \( x_0 \in D \), one can suppose that the \( \omega \)-limit set \( \omega(x_0) \) contains \( \Lambda \) by taking the words \( u_{k+1} \in \{1, 2\}^k \) of (7.1) suitably. Since \( \lim_{j \to \infty} \text{diam} f_n^j(D) = 0 \), we also have \( \omega(x) \supset \Lambda \) for any \( x \in D \). This shows that \( D \) is a non-trivial wandering domain of \( f_n \).

Note that the first perturbation of \( f \) in Subsection 5.2 does not depend on the sequence \( z = \{z_k\}_{k=1}^{\infty} \) such that each entry \( z_k \) is either \( 0 \) or \( 1 \) but the second perturbation in Subsection 7.2 does. The wandering domain \( D = \text{Int}R_{k_0} \) also depends on \( z \). We express the dependence by \( f_{n,z} \) and \( D_z = \text{Int}R_{k_0,z} \). On the other hand, since the support of the second perturbation is fully contained in \( G^n(0) \) and disjoint from \( G^n(0), B^n \cap B^u \) is independent of the sequence \( z \) for any bridges \( B^s \) of \( K_\Lambda^s \) and \( B^u \) of \( K_\Lambda^u \).

Remark 8.5. For any integer \( j \geq k_0 \), consider the integer \( m_j \) defined by

\[
 m_j = N_2 + z_j2^j + j + N_1 + \hat{a}_j + N_0
\]

= \( N_2 + z_j2^j + \hat{\omega}_{j+1} + 2^j + j + N_1 + \hat{a}_j + N_0 \),

and \( \hat{\omega}_k = \sum_{j=k_0}^k m_j \) if \( k \geq k_0 \). By (8.6), \( g_j = f^{m_j} \). Our wandering domain \( D_z \) satisfies \( f_{n,z}^j(D_z) \subset \text{Int}R_{k,z} \) for any integer \( k \geq k_0 \) and \( f_{n,z}^j(D_z) \) stays in \( S \) for \( j \) with \( \hat{\omega}_k \leq j < \hat{\omega}_k - N_2 \), where \( \hat{\omega}_k = \hat{\omega}_{k+1} + N_0 + \hat{a}_k + N_1 \). See Figure 7.3

Fix a sufficiently large positive integer \( l \) and suppose that \( k > \max\{k_0, l\} \). From the form of \( B^u_k \) of (7.1) and the fact that \( f_{n,z}^j(D_z) \subset B^u_k \), \( f_{n,z}^j(D_z) \) is contained in \( B^u(l; 1^{(l)}) \cap \hat{B}^s(l; 1^{(l)}) \) for any \( j \) with \( \hat{\omega}_k + l \leq j < \hat{\omega}_k + z_kk^2 - l \) and in \( B^u(l; 2^{(l)}) \cap \hat{B}^s(l; 2^{(l)}) \) for any \( j \) with \( \hat{\omega}_k + z_kk^2 + l \leq j < \hat{\omega}_k + z_kk^2 + k^2 - l \). Since by Lemma 7.4(3) \( N_2 + \hat{a}_{j+1} + j + \hat{a}_j + N_0 > j \), it follows that

\[
 \lim_{k \to \infty} \frac{z_kk^2 - 2l}{\hat{\omega}_k - \hat{\omega}_{k-1}} = \lim_{k \to \infty} \frac{z_kk^2 + k^2 - 4l}{\hat{\omega}_k} = 1.
\]

Thus, for any integer \( l > 0 \) and almost all \( j > 0 \), we have

\[
 f_{n,z}^j(D_z) \subset (B^u(l; 1^{(l)}) \cap \hat{B}^s(l; 1^{(l)})) \cup (B^u(l; 2^{(l)}) \cap \hat{B}^s(l; 2^{(l)})).
\]
Here ‘almost all $j$’ means that, if $d_l(a)$ is the number of integers $j$ ($0 < j \leq a$) satisfying $[8,8]$ for $a > 0$, then $\lim_{a \to \infty} d_l(a)/a = 1$ holds.

Now we are ready to prove our main theorem.

Proof of Theorem A. Let $f_{n,z}$ be the diffeomorphism and $D_z$ the wandering domain of $f_{n,z}$ as above. We will show that the sequence $z = \{z_k\}_{k=1}^\infty$ can be chosen so that, for any $x \in D_z$, the forward orbit $\{x, f_{n,z}(x), f_{n,z}(x), \ldots\}$ has historic behavior.

Note that the horseshoe $\Lambda$ has two fixed points $\hat{p}$ with

$$\{p\} = \bigcap_{i=1}^{\infty} \mathbb{B}^u(t; 1^{(i)}) \cap \mathbb{B}^s(t; 1^{(i)})$$

and $\{\hat{p}\} = \bigcap_{i=1}^{\infty} \mathbb{B}^u(t; 2^{(i)}) \cap \mathbb{B}^s(t; 2^{(i)})$.

Consider the space $\mathcal{P}(M)$ of probability measures on $M$ with the weak topology. For any $x \in D_z = \text{Int}R_{k_0,z}$ and any non-negative integer $m$, the element $\mu_x(m)$ of $\mathcal{P}(M)$ is defined as

$$\mu_x(m) = \frac{1}{m+1} \sum_{i=0}^{m} \delta_{f_{n,x}^i(z)}.$$ We are concerned with the subsequence $\{\mu_x(\hat{m}_k)\}_{k=k_0}^\infty$ of $\{\mu_x(m)\}$. Let $\nu_0$ and $\nu_1$ be the elements of $\mathcal{P}(M)$ defined as

$$\nu_0 = \frac{1}{z_0 + 1} \left( z_0 \delta_p + \delta_{\hat{p}} \right)$$

and $U_0, U_1$ arbitrarily small neighborhoods of $\nu_0$ and $\nu_1$ in $\mathcal{P}(M)$ with $U_0 \cap U_1 = \emptyset$ respectively.

Let $k_1$ be an integer sufficiently larger than $k_0$. Then one can suppose that the integer $l < k_1$ in $[8,8]$ is also large enough. If we take the entries of $z$ so that $z_k = z_0$ for $k = 1, \ldots, k_1$, then $\mu_x(\hat{m}_{k_1})$ is contained in $U_0$.

For any integer $m' > m$, let $\mu_x(m', m)$ be the measure on $M$ defined as $\mu_x(m', m) = \frac{1}{m'+1} \sum_{i=m+1}^{m'} \delta_{f_{n,x}^i(z)}$. If $k > k_1$, then

$$\mu_x(\hat{m}_k) = \frac{\hat{m}_{k_1} + 1}{\hat{m}_k + 1} \mu_x(\hat{m}_{k_1}) + \mu_x(\hat{m}_k, \hat{m}_{k_1}).$$

Thus the contribution of the first term goes to zero as $k \to \infty$. It follows that, if $k_2$ is sufficiently larger than $k_1$, then for any sequence $z = \{z_j\}_{j=1}^\infty$ with $z_j = z_0$ ($j = 1, \ldots, k_1$) and $z_j = z_0 + 1$ ($j = k_1 + 1, \ldots, k_2$), $\mu_x(\hat{m}_{k_2})$ is contained in $U_1$.

Repeating a similar argument, we have a monotone increasing sequence $\{k_a\}_{a=1}^\infty$ such that $\{\mu_x(\hat{m}_{k_a}, z)\}_{a=1}^\infty$ has two subsequences one of which converges to $\nu_0$ and the other to $\nu_1$, where $z = (z_j)_{j=1}^\infty$ is the sequence with

$$\begin{align*}
  z_j &= z_0 & \text{for } j = 1, \ldots, k_1, k_{2a+1}, \ldots, k_{2a+1} (a = 1, 2, \ldots) \\
  z_j &= z_0 + 1 & \text{for } j = k_{2a-1} + 1, \ldots, k_{2a} (a = 1, 2, \ldots).
\end{align*}$$

In particular, this implies that the limit of $\mu_x(m)$ does not exist. It follows that, for any $x \in D_z$, the forward orbit of $x$ under $f_{n,z}$ is historic. This completes the proof of our main theorem. \qed
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