OF COMMUTATORS AND JACOBIANS

TUOMAS P. HYTÖNEN

Dedicated to Professor Fulvio Ricci

Abstract. I discuss the prescribed Jacobian equation

\[ J_u = \det \nabla u = f \]

for an unknown vector-function \( u \), and the connection of this problem to the boundedness of commutators of multiplication operators with singular integrals in general, and with the Beurling operator in particular. A conjecture of T. Iwaniec regarding the solvability for general datum \( f \in L^p(\mathbb{R}^d) \) remains open, but recent partial results in this direction will be presented. These are based on a complete characterisation of the \( L^p \)-to-\( L^q \) boundedness of commutators, where the regime of exponents \( p > q \), unexplored until recently, plays a key role. These results have been proved in general dimension \( d \geq 2 \) elsewhere, but I will here present a simplified approach to the important special case \( d = 2 \), using a framework suggested by S. Lindberg.

Contents

1. The prescribed Jacobian problem
2. Functional analysis behind the results
3. Complex reformulation and connection to commutators for \( d = 2 \)
4. The commutator theorem
5. The classical implications
6. The new case \( p > q \)
References

1. The prescribed Jacobian problem

Given a vector-valued function \( u = (u_j)_{j=1}^d \in \dot{W}^{1,pd}(\mathbb{R}^d)^d \) in the homogeneous Sobolev space

\[ \dot{W}^{1,pd}(\mathbb{R}^d) = \{ v \in L^1_{\text{loc}}(\mathbb{R}^d) : \partial_i v \in L^{pd}(\mathbb{R}^d) \ \forall i \} , \]

it is clear that its Jacobian determinant—a linear combination of \( d \)-fold products of the various \( \partial_i u_j \)—satisfies \( J_u := \det \nabla u := \det(\partial_i u_j)_{i,j=1}^d \in L^p(\mathbb{R}^d) \).

Our starting point is the reverse question: Given \( f \in L^p(\mathbb{R}^d) \), is there \( u \in \dot{W}^{1,pd}(\mathbb{R}^d)^d \) such that \( J_u = f \)? This is a nonlinear PDE, known as the “prescribed Jacobian equation”. It has been mostly studied for smooth functions \( f \) on bounded domains \( \Omega \) \cite{1,2}, in which case there are significant geometric applications (e.g. \cite{1}). In the global \( L^p \) case that we discuss, there is:

\[2010 \text{ Mathematics Subject Classification.} 42B20, 42B25, 42B37, 35F20, 47B47. \]

\[ \text{Key words and phrases.} \ \text{Commutator, Beurling transform, Jacobian determinant.} \]

The author is supported by the Academy of Finland via project Nos. 307333 (Centre of Excellence in Analysis and Dynamics Research) and 314829 (Frontiers of singular integrals).
1.1. Conjecture ([9]). For $p \in (1, \infty)$, there exists a continuous $E : L^p(\mathbb{R}^d) \to W^{1,p_d}(\mathbb{R}^d)^d$ such that $J \circ E = I$.

As suggested in [9], such an $E$ could be interpreted as a “fundamental solution of the Jacobian equation”.

The case $p = 1$ had already been addressed a little earlier. In this case, a simple integration by parts confirms that

$$u \in W^{1,d}(\mathbb{R}^d)^d \Rightarrow \int Ju = 0 \Rightarrow J(W^{1,d}(\mathbb{R}^d)^d) \subseteq L^1(\mathbb{R}^d).$$

A somewhat more careful argument yields:

1.2. Theorem ([2]). For $u \in W^{1,d}(\mathbb{R}^d)^d$, $d \geq 2$, we have

$$\|Ju\|_{H^1(\mathbb{R}^d)} \lesssim \|u\|_{W^{1,d}(\mathbb{R}^d)^d}^d$$

where $H^1(\mathbb{R}^d)$ denotes the Hardy space.

Again in the reverse direction, [2] asked: Given $f \in H^1(\mathbb{R}^d)$, is there $u \in W^{1,d}(\mathbb{R}^d)^d$ such that $Ju = f$? As a partial positive evidence, they proved:

1.3. Theorem ([2]). For every $f \in H^1(\mathbb{R}^d)$, there are $u^i \in W^{1,d}(\mathbb{R}^d)^d$ and $\alpha_i \geq 0$ such that

$$f = \sum_{i=1}^{\infty} \alpha_i J(u^i), \quad \|u^i\|_{W^{1,d}(\mathbb{R}^d)^d} \leq 1, \quad \sum_{i=1}^{\infty} \alpha_i \lesssim \|f\|_{H^1(\mathbb{R}^d)}.$$

What about the (perhaps more usual) non-homogeneous Sobolev space

$$W^{1,p}(\mathbb{R}^d) := \{v \in L^p(\mathbb{R}^d) : \nabla v \in L^p(\mathbb{R}^d)^d\},$$

$$W^{1,p}(\mathbb{R}^d) := \{v \in L^1_{loc}(\mathbb{R}^d) : \nabla v \in L^p(\mathbb{R}^d)^d\}.$$

Given $f \in L^p(\mathbb{R}^d)$ (resp. $H^1(\mathbb{R}^d)$ if $p = 1$), could we even hope to find $u \in W^{1,p_d}(\mathbb{R}^d)^d$ with $Ju = f$? It was only fairly recently that this was shown to fail, and in fact quite miserably:

1.4. Theorem ([10]). The set

$$\left\{ \sum_{i=1}^{\infty} \alpha_i J(u^i) : \|u^i\|_{W^{1,p_d}(\mathbb{R}^d)^d} \leq 1, \sum_{i=1}^{\infty} \alpha_i < \infty \right\},$$

which obviously contains the image $JW^{1,p_d}(\mathbb{R}^d)^d$, has first category in $L^p(\mathbb{R}^d)$ if $p \in (1, \infty)$ resp. in $H^1(\mathbb{R}^d)$ if $p = 1$.

Very roughly speaking, the reason for this negative result is the incompatibility of scaling in $W^{1,p_d}(\mathbb{R}^d)^d$ on the one hand, and in $L^p(\mathbb{R}^d)$ if $p \in (1, \infty)$ resp. in $H^1(\mathbb{R}^d)$ if $p = 1$ on the other hand, but the precise argument is more delicate.

2. Functional analysis behind the results

Both the existence (in Theorem 1.3) and the non-existence (in Theorem 1.4) of the representation $f = \sum \alpha_i J(u^i)$ are based on the following functional analytic lemma from [2] and its elaboration from [10]:

2.1. Lemma ([2]). Let $V \subset X$ be a symmetric bounded subset of a Banach space $X$. Then the following are equivalent:

$$\sum_{i=1}^{\infty} \alpha_i J(u^i) \in V,$$
(1) Every $x \in X$ can be written as $x = \sum_{k=1}^{\infty} \alpha_k v_k$, where $v_k \in V$, $\alpha_k \geq 0$ and \( \sum_{k=1}^{\infty} \alpha_k < \infty \).

(2) $V$ is norming for $X^*$, i.e., \( \|\lambda\|_{X^*} \approx \sup_{v \in V} |\langle \lambda, v \rangle| \quad \forall \lambda \in X^* \).

2.2. Lemma ([10]). Both (1) either holds for all $x \in X$, or in a subset of first category.

For the mentioned theorems, these lemmas are applied with the symmetric set $V = J(B)$, where $B$ = unit ball of $W^{1,p}(\mathbb{R}^d)$ or $W^{1,p}(\mathbb{R}^d)$, which is a bounded subset of the Banach space $X = L^p(\mathbb{R}^d)$ or $X = H^1(\mathbb{R}^d)$. Via the equivalent condition (2), the well-known dual spaces $X^* = L^p(\mathbb{R}^d)$ or $X^* = \text{BMO}(\mathbb{R}^d)$ enter the considerations.

In order to obtain Theorem 1.3, [2] proved that

2.3. Proposition ([2]). Let $d \geq 2$. For every $b \in \text{BMO}(\mathbb{R}^d)$, we have

\[ \|b\|_{\text{BMO}(\mathbb{R}^d)} \approx \sup \left\{ \left| \int bJ(u) \right| : \|\nabla u\|_d \leq 1 \right\}. \]

The analogous result for $p \in (1, \infty)$ read as follows:

2.4. Theorem ([5]). Let $d \geq 2$ and $p \in (1, \infty)$. For every $f \in L^p(\mathbb{R}^d)$, there are $u^i \in W^{1,dp}(\mathbb{R}^d)$ and $\alpha_i \geq 0$ such that

\[ f = \sum_{i=1}^{\infty} \alpha_i J(u^i), \quad \|u^i\|_{W^{1,dp}(\mathbb{R}^d)} \leq 1, \quad \sum_{i=1}^{\infty} \alpha_i \lesssim \|f\|_{L^p(\mathbb{R}^d)}. \]

2.5. Proposition ([5]). Let $d \geq 2$ and $p \in (1, \infty)$. For every $b \in L^p(\mathbb{R}^d)$, we have

\[ \|b\|_{L^p(\mathbb{R}^d)} \approx \sup \left\{ \left| \int bJ(u) \right| : \|\nabla u\|_{dp} \leq 1 \right\}. \]

3. Complex reformulation and connection to commutators for $d = 2$

The various results formulated above are valid, as stated, in all dimensions $d \geq 2$, and their proofs in this generality can be found in the quoted references. We now restrict ourselves to dimension $d = 2$ in order to discuss an alternative complex-variable approach that is available in this situation, as suggested by [10].

For $u = (u_1, u_2) \in \dot{W}^{1,2p}(\mathbb{R}^2; \mathbb{R}^2)$, let us denote

\[ h := u_1 + iu_2 \in \dot{W}^{1,2p}(\mathbb{C}; \mathbb{C}), \quad \partial := \frac{1}{2}(\partial_1 - i\partial_2), \quad \bar{\partial} := \frac{1}{2}(\partial_1 + i\partial_2). \]

Then, after some algebra, we find that

\[ Ju = \det \begin{pmatrix} \partial_1 u_1 & \partial_2 u_1 \\ \partial_1 u_2 & \partial_2 u_2 \end{pmatrix} = |\partial h|^2 - |\bar{\partial} h|^2 = |S(v)|^2 - |v|^2, \]

where $v := \bar{\partial} h \in L^{2p}(\mathbb{C})$ is in isomorphic correspondence with $h \in \dot{W}^{1,2p}(\mathbb{C}; \mathbb{C})$, and $S$ is the (Ahlfors–)Beurling (or 2D Hilbert) transform

\[ S v(z) = -\frac{1}{\pi} \text{p.v.} \int_{\mathbb{C}} \frac{v(y) \, dy_1 \, dy_2}{(z - y)^2}, \]

which satisfied the fundamental relation $S \circ \bar{\partial} = \partial$ and maps $S : L^p(\mathbb{C}) \rightarrow L^p(\mathbb{C})$ bijectively and isometrically for $p = 2$ and isomorphically for all $p \in (1, \infty)$. 


Let us now see how Propositions 2.3 and 2.5 are connected to commutators when $d = 2$. By the reformulations just discussed, we have
\[
\sup \left\{ \left| \int bJ(u) \right| : \|u\|_{W^{1,2p} (\mathbb{R}^2)} \leq 1 \right\} \approx \sup \left\{ \left| \int b(|Sv|^2 - |v|^2) \right| : \|v\|_{L^{2p}(\mathbb{C})} \leq 1 \right\}
\]
denoting $v = \bar{\partial}(u_1 + iv_2)$. We claim that the right side can be further written as
\[
\approx \sup \left\{ \left| \int b(S\bar{S}w - v\bar{w}) \right| : \|v\|_{L^{2p}(\mathbb{C})}, \|w\|_{L^{2p}(\mathbb{C})} \leq 1 \right\}.
\] (3.1)
In fact, “$\leq$” is obvious, while “$\approx$” follows from the elementary polarisation identity
\[
a\bar{b} = \frac{1}{4} \sum_{\varepsilon = \pm 1, \pm i} \varepsilon |a + \varepsilon b|^2, \quad a, b \in \mathbb{C},
\]
applied pointwise to both $(a, b) = (Sv, Sw)$ and $(a, b) = (v, w)$, which implies that
\[
Sv\bar{S}w - v\bar{w} = \frac{1}{4} \sum_{\varepsilon = \pm 1, \pm i} \varepsilon |Sv - \varepsilon Sw|^2 - \frac{1}{4} \sum_{\varepsilon = \pm 1, \pm i} \varepsilon |v - \varepsilon w|^2
\]
\[
= \frac{1}{4} \sum_{\varepsilon = \pm 1, \pm i} \varepsilon \left( |S(v - \varepsilon w)|^2 - |v - \varepsilon w|^2 \right),
\]
where $\|v - \varepsilon w\|_{2p} \leq \|v\|_{2p} + \|w\|_{2p} \leq 2$ if $\|v\|_{2p}, \|w\|_{2p} \leq 1$.

Denoting $g := \bar{S}w$, we have $\overline{g} = Sw$ and hence $S^*g = S^*Sw = w$, where we denoted by $S^*$ the conjugate-linear adjoint of $S$ and used the fact that $S^*S$ is the identity. With this substitution, $g \in L^{2p}(\mathbb{C})$ and $w \in L^{2p}(\mathbb{C})$ are in isomorphic correspondence, and we have
\[
\text{(3.1)} \approx \sup \left\{ \left| \int b(Sv \cdot g - v\bar{S}w) \right| : \|v\|_{L^{2p}(\mathbb{C})}, \|g\|_{L^{2p}(\mathbb{C})} \leq 1 \right\}
\]
Finally, using the duality $\int fS^*\psi = \int Sf \cdot \overline{\psi}$ with $f = b\psi$ and $\psi = \overline{g}$, we have
\[
\int b(Sv \cdot g - v\bar{S}w) = \int (b \cdot Sv \cdot g - S(bv) \cdot \overline{g}) = \int g \cdot [b, S]v,
\] (3.2)
where we finally introduced the commutator
\[
[b, S]v = bSv - S(bv).
\]

Now the supremum of (the absolute value of) (3.2) over $\|g\|_{2p} \leq 1$ is the dual norm $\|[b, S]v\|_{(2p)'}$, and the supremum of this over $\|v\|_{2p} \leq 1$ is the operator norm
\[
\|[b, S]\|_{L^{2p}(\mathbb{C}) \to L^{(2p)'}(\mathbb{C})}.
\]
Summarising the discussion, we have proved:

3.3. Lemma. Let $p \in [1, \infty)$. Then
\[
\sup \left\{ \left| \int bJ(u) \right| : \|u\|_{W^{1,2p}(\mathbb{R}^2)} \leq 1 \right\} \approx \|[b, S]\|_{L^{2p}(\mathbb{C}) \to L^{(2p)'}(\mathbb{C})}.
\]

Thus Propositions 2.3 and 2.5 for $d = 2$, are reduced to understanding the norm of the Beurling commutator $[b, S] : L^{2p}(\mathbb{C}) \to L^{(2p)'}(\mathbb{C})$. When $p = 1$, we have $2p = (2p)' = 2$, and we are talking about $L^2$-boundedness of commutators, which is a well-studied topic since the pioneering work of [3]. When $p \in (1, \infty)$, we have $2p > 2 > (2p)'$, and we are talking about the boundedness of commutators between different $L^p$ spaces. This, too, has been well studied in the case that the target
space exponent is larger (cf. [7]), but we are now precisely in the complementary regime. In this case, the result was only achieved very recently.

4. The commutator theorem

Complementing various classical results starting with [3], the following result was recently completed in [5]:

4.1. Theorem. Let \( T = S \) with \( d = 2 \), or more generally, let \( T \) be any “uniformly non-degenerate” Calderón–Zygmund operator on \( \mathbb{R}^d \), \( d \geq 1 \). Let \( 1 < p, q < \infty \) and \( b \in L^1_{\text{loc}}(\mathbb{R}^d) \). Then

\[
[b, T] : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)
\]

if and only if

1. \( p = q \) and \( b \in \text{BMO} \) [3], or
2. \( p < q \leq p^* \), where \( p^* := \left( \frac{1}{p} - \frac{1}{2} \right)_+ \), and \( b \in C^{0, \alpha} \) with \( \alpha = d(\frac{1}{p} - \frac{1}{2}) \), or
3. \( q > p^* \) and \( b \) is constant (this and the previous case are due to [7]), or
4. \( p > q \) and \( b = a + c \), where \( c \) is constant and \( a \in L^{r'} \) for \( \frac{1}{r'} = \frac{1}{q} - \frac{1}{2} \) [5].

Aside from the new regime of exponents \( p > q \), another novelty of [5] (also when \( p \leq q \)) is the validity of the “only if” implication under the fairly general “uniform non-degeneracy” assumption on \( T \). Recall that [3] proved this direction only for the Riesz transforms, and [7, 11] for “smooth enough” kernels, which has been gradually relaxed in subsequent contributions.

The usual Calderón–Zygmund size condition requires the upper bound

\[
|K(x, y)| \leq \frac{cK}{|x - y|^d},
\]

on the kernel \( K \) of \( T \). “Uniform non-degeneracy” means that we have a matching lower bound essentially over all positions and length-scales, more precisely: For every \( y \in \mathbb{R}^d \) and \( r > 0 \), there is \( x \) such that \( |x - y| \approx r \) and

\[
|K(x, y)| \geq \frac{c_0}{|x - y|^d}.
\]

This is manifestly the case for the Beurling operator, whose kernel \( K(x, y) = -\pi^{-1}/(x - y)^2 \) satisfies both bounds with an equality.

More generally, Theorem 4.1 holds for both

1. two-variable kernels \( K(x, y) \) (with very little continuity), and
2. rough homogeneous kernels

\[
K(x, y) = K(x - y) = \frac{\Omega((x - y)/|x - y|)}{|x - y|^d}
\]

as soon as \( \Omega \) is not identically zero; this was conjectured by [9], who came very close for \( p = q \).

We refer the reader to [5] for the proof of Theorem 4.1 in the stated generality; below we give a much simpler argument in the particular case of the Beurling operator \( T = S \), which is relevant for the two-dimensional Jacobian problem, as discussed above.

Indeed, for \( d = 2 \), Theorems 1.3 and 2.4 are direct corollaries of Theorem 4.1 (via the earlier discussion). For \( d > 2 \), they are not direct consequences of Theorem 4.1 itself, but they can nevertheless be proved by adapting the ideas of the proof of Theorem 4.1 see again [5] for details.
5. The classical implications

We begin with a brief discussion of the “if” implications in Theorem 4.1.

1. The case $p = q$ and $b \in \text{BMO}$ is the only non-trivial “if” statement in Theorem 4.1. There are many excellent discussions of this bound (including two entirely different proofs already in [3]), so we skip it here.

2. If $p < q$ and $b \in C^{0,\alpha}$, we only need the size bound $|K(x, y)| \lesssim |x - y|^{-d}$ to see that

$$
|b, T| f(x) | = \left| \int (b(x) - b(y)) K(x, y) f(y) \, dy \right| \\
\leq \int |b(x) - b(y)| |K(x, y)||f(y)| \, dy \\
\lesssim \int |x - y|^{\alpha} |x - y|^{-d} |f(y)| \, dy.
$$

This is a fractional integral with well-known $L^p \to L^q$ bounds!

3. If $b = c = \text{constant}$, then $|b, T| = 0$ is trivially bounded.

4. If $p > q$ and $b \in L^r$, we use the boundedness of $T : L^p \to L^p$ and $T : L^{q'} \to L^q$ together with Hölder’s inequality

$$
\|bf\|_q \leq \|b\|_r \|f\|_p, \quad \frac{1}{q} = \frac{1}{r} + \frac{1}{p}
$$

to see that both $bT$ and $Tb$ individually are $L^p \to L^q$ bounded.

We then turn to the “only if” part, starting with the beautiful classical argument of [3] for $p = q$. Given a function $b \in L^p_{\text{loc}}(\mathbb{C})$ and a ball (disc) $B = B(z, r) \subset \mathbb{C}$, we can pick an auxiliary function $\sigma$ with $|\sigma(x)| = 1_B(x)$ so that

$$
\int_B |b(x) - \langle b \rangle_B| \, dx = \int_B (b(x) - \langle b \rangle_B) \sigma(x) \, dx,
$$

$$
= \frac{1}{|B|} \int_B \int_B (b(x) - b(y)) \sigma(x) \, dx \, dy \\
= \int_B \int_B b(x) - b(y) \frac{(x - z)^2 - 2(x - z)(y - z) + (y - z)^2}{\pi r^2} \sigma(x) \, dx \, dy \\
= \sum_{i=1}^3 \int g_i(x) \left( \int \frac{b(x) - b(y)}{(x-y)^2} f_i(y) \, dy \right) \, dx = \sum_{i=1}^3 \int g_i[b, S] f_i,
$$

for suitable functions $f_i, g_i$ with $|f_i(x)| + |g_i(x)| \lesssim 1_B(x)$, whose explicit formulae can be easily deduced from above. Thus

$$
\int_B |b - \langle b \rangle_B| \leq \sum_{i=1}^3 \|b, S\|_{L^p \to L^p} \|f_i\|_p \|g_i\|_p' \lesssim \|b, S\|_{L^p \to L^p} \|B\|^{1/p} |B|^{1/p'}.
$$

Dividing by $|B|^{1/p'} |B|^{1/p'} = |B|$ and taking the supremum over all $B$ proves that

$$
\|b\|_{\text{BMO}} \lesssim \|b, S\|_{L^p \to L^p}.
$$

With a simple modification of the previous display observed by [7], we also find that

$$
\int_B |b - \langle b \rangle_B| \leq \sum_{i=1}^3 \|b, S\|_{L^p \to L^p} \|f_i\|_p \|g_i\|_{p'} \lesssim \|b, S\|_{L^p \to L^p} \|B\|^{1/p} |B|^{1/q'},
$$

for suitable functions $f_i, g_i$ with $|f_i(x)| + |g_i(x)| \lesssim 1_B(x)$, whose explicit formulae can be easily deduced from above.
where

$$|B|^{1/p+1/q'} = |B|^{(1/p-1/q)+1} \approx |B| \cdot r_B^{d(1/p-1/q)} = |B| \cdot r_B^n.$$ 

Thus

$$\int_B |b - \langle b \rangle_B| \lesssim r_B^n,$$

which a well-known characterisation of $b \in C^{0,\alpha}$. For $\alpha > 1$, this space has nothing but the constant functions, completing the sketch of the proof of all the classical “only if” statements of Theorem 4.1.

6. The new case $p > q$

We finally discuss the proof of the “only if” implication of Theorem 4.1 in the case $p > q$ that was only recently discovered in [5]. The above estimate

$$\int_B |b - \langle b \rangle_B| \lesssim |B|^{1/p+1/q'} = |B|^{(1/p-1/q)+1} = |B|^{-1/r+1} = |B|^{1/r'}$$

is still true but seems to be useless in this range. How do we even check that a given function is in $L^r + \text{constants}$?

A convenient tool is as follows:

6.1. Lemma ([5], Lemma 3.6). If we have the following bound uniformly for cubes $Q \subset \mathbb{R}^d$:

$$\|b - \langle b \rangle_Q\|_{L^r(Q)} \leq C,$$

then there is a constant $c (= \lim_{Q \to \mathbb{R}^d} \langle b \rangle_Q)$ such that

$$\|b - c\|_{L^r(\mathbb{R}^d)} \leq C.$$

To estimate the local $L^r$ norm, the following result is useful. Depending on one’s background, one may like to call it an iterated Calderón–Zygmund or atomic decomposition; one can also view it as a toy version of an influential formula of [8], featuring merely measurable functions in place $L^1(Q_0)$, the median of $b$ in place of the mean $\langle b \rangle_{Q_0}$, etc. “Sparse bounds” of this type have been extensively used in the last few years; the version below is very elementary compared to several recent highlights, but quite sufficient for the present purposes.

6.2. Lemma. Given a cube $Q_0 \subset \mathbb{R}^d$ and $b \in L^1(Q_0)$, there is a sparse collection $\mathcal{S}$ of the family $\mathcal{D}(Q_0)$ of dyadic subcubes of $Q_0$ such that

$$1_{Q_0}(x)|b(x) - \langle b \rangle_{Q_0}| \lesssim \sum_{Q \in \mathcal{S}} 1_Q(x) \int_Q |b - \langle b \rangle_Q|.$$

A collection of cubes $\mathcal{S}$ is called sparse (or almost disjoint) if there are pairwise disjoint major subsets $E(Q) \subset Q$ for each $Q \in \mathcal{S}$, meaning that

$$E(Q) \cap E(Q') = \emptyset \quad (\forall Q \neq Q'), \quad |E(Q)| \geq \frac{1}{2}|Q|.$$

For $L^p$ estimates, sparse is almost as good as disjoint; namely,

$$\left\| \sum_{Q \in \mathcal{S}} \lambda_Q 1_Q \right\|_p \approx \left( \sum_{Q \in \mathcal{S}} \lambda_Q^p |Q| \right)^{1/p}, \quad \forall \lambda_Q \geq 0,$$

where equality would hold for a disjoint collection.
With these tools at hand, we are ready to prove that \([b, S] : L^p \to L^q\) for 1 < q < p < \(\infty\) only if \(b = a + c\), where \(a \in L^r\) with \(\frac{1}{r} = \frac{1}{q} + \frac{1}{p}\) and \(c\) is constant.

For any cube \(Q_0 \subset \mathbb{R}^d\), we estimate

\[
\|b - \langle b \rangle_{Q_0}\|_{L^r(Q_0)} \lesssim \left\| \sum_{Q \in \mathcal{Q}} 1_Q \int_Q |b - \langle b \rangle_Q| \right\|_{L^r(Q_0)} \quad \text{(by Lemma 6.2)}
\]

\[
\approx \left( \sum_{Q \in \mathcal{Q}} |Q| \left( \int_Q |b - \langle b \rangle_Q| \right)^r \right)^{1/r} \quad \text{(by (6.3))}
\]

\[
= \sum_{Q \in \mathcal{Q}} |Q| \lambda_Q \int_Q |b - \langle b \rangle_Q| = \sum_{Q \in \mathcal{Q}} \lambda_Q \int_Q |b - \langle b \rangle_Q|,
\]

with a suitable dualising sequence \(\lambda_Q\) such that

\[
\sum_{Q \in \mathcal{Q}} |Q| \cdot \lambda_Q' = 1. \tag{6.4}
\]

By the same considerations as in Section 5 in the case of just one ball \(B\), for each of the cubes \(Q \in \mathcal{Q}\) above we find functions \(f_Q^i, g_Q^i\) with

\[
|f_Q^i| + |g_Q^i| \lesssim 1_Q \tag{6.5}
\]

such that

\[
\int_Q |b - \langle b \rangle_Q| = \sum_{i=1}^3 \int g_Q^i [b, S] f_Q^i.
\]

Summarising the discussion so far, we have

\[
\|b - \langle b \rangle_{Q_0}\|_{L^r(Q_0)} \lesssim \sum_{i=1}^3 \sum_{Q \in \mathcal{Q}} \lambda_Q \int g_Q^i [b, S] f_Q^i, \tag{6.6}
\]

where the coefficient \(\lambda_Q\) and the functions \(f_Q^i, g_Q^i\) satisfy (6.4) and (6.5).

We now enter independent random signs \(\varepsilon_Q\) on some probability space, and denote by \(\mathbb{E}\) the expectation. (For the Jacobian theorem in \(d > 2\): we need to use random \(d\)th roots of unity at the analogous step, see [5].) With the basic orthogonality \(\mathbb{E}(\varepsilon_Q \varepsilon_{Q'}) = \delta_{Q,Q'}\) and Hölder’s inequality after observing that

\[
\frac{1}{r} = \frac{1}{q} + \frac{1}{p} \Rightarrow \frac{r'}{q'} = \frac{1}{q'} + \frac{1}{p'} \Rightarrow 1 = \frac{r'}{q'} + \frac{r'}{p'},
\]

we have

\[
\text{RHS of (6.6)} = \sum_{i=1}^3 \mathbb{E} \int \left( \sum_{Q \in \mathcal{Q}} \varepsilon_Q \lambda_Q^{r'/q'} g_Q^i \right) [b, S] \left( \sum_{Q' \in \mathcal{Q}} \varepsilon_Q' \lambda_Q'^{r'/p'} f_Q'^{r'} \right)
\]

\[
\lesssim \|[b, S]\|_{L^p \to L^q} \left\| \sum_{Q \in \mathcal{Q}} \lambda_Q^{r'/q'} 1_Q \right\|_{q'} \left\| \sum_{Q \in \mathcal{Q}} \lambda_Q^{r'/p'} 1_Q \right\|_p \quad \text{(by (6.3))}
\]

\[
\lesssim \|[b, S]\|_{L^p \to L^q} \left( \sum_{Q \in \mathcal{Q}} \lambda_Q^{r'/q'} |Q| \right)^{1/q'} \left( \sum_{Q \in \mathcal{Q}} \lambda_Q^{r'/p'} |Q| \right)^{1/p} \quad \text{(by (6.3))}
\]

\[
= \|[b, S]\|_{L^p \to L^q} \quad \text{(by (6.4))}.
\]

This shows that

\[
\|b - \langle b \rangle_{Q_0}\|_{L^r(Q_0)} \lesssim \|[b, S]\|_{L^p \to L^q}
\]
for every cube $Q_0$, and hence
\[
\|b - c\|_{L^r(C)} \lesssim \|[b, S]\|_{L^p \to L^q}
\]
for some constant $c$ by Lemma 6.1. If we a priori know that $b \in L^r(C)$ (as in Proposition 2.5), then necessarily $c = 0$, and we obtain the desired quantitative bound for $\|b\|_{L^r(C)}$.

References

[1] A. Avila. On the regularization of conservative maps. Acta Math., 205(1):5–18, 2010.
[2] R. Coifman, P.-L. Lions, Y. Meyer, and S. Semmes. Compensated compactness and Hardy spaces. J. Math. Pures Appl. (9), 72(3):247–286, 1993.
[3] R. R. Coifman, R. Rochberg, and G. Weiss. Factorization theorems for Hardy spaces in several variables. Ann. of Math. (2), 103(3):611–635, 1976.
[4] B. Dacorogna and J. Moser. On a partial differential equation involving the Jacobian determinant. Ann. Inst. H. Poincaré Anal. Non Linéaire, 7(1):1–26, 1990.
[5] T. P. Hytönen. The $L^p$-to-$L^q$ boundedness of commutators with applications to the Jacobian operator. Preprint, arXiv:1804.11167, 2018.
[6] T. Iwaniec. Nonlinear commutators and Jacobians. J. Fourier Anal. Appl., 3:775–796, 1997.
[7] S. Janson. Mean oscillation and commutators of singular integral operators. Ark. Mat., 16(2):263–270, 1978.
[8] A. K. Lerner. A pointwise estimate for the local sharp maximal function with applications to singular integrals. Bull. Lond. Math. Soc., 42(5):843–856, 2010.
[9] A. K. Lerner, S. Ombrosi, and I. P. Rivera-Ríos. Commutators of singular integrals revisited. Bull. Lond. Math. Soc., 51(1):107–119, 2019.
[10] S. Lindberg. On the Hardy space theory of compensated compactness quantities. Arch. Ration. Mech. Anal., 224(2):709–742, 2017.
[11] A. Uchiyama. On the compactness of operators of Hankel type. Tôhoku Math. J. (2), 30(1):163–171, 1978.
[12] D. Ye. Prescribing the Jacobian determinant in Sobolev spaces. Ann. Inst. H. Poincaré Anal. Non Linéaire, 11(3):275–296, 1994.