Updating the Inverse of a Matrix When Removing the $i^{th}$ Row and Column with an Application to Disease Modeling

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Abstract

The Sherman-Woodbury-Morrison (SWM) formula gives an explicit formula for the inverse perturbation of a matrix in terms of the inverse of the original matrix and the perturbation. This formula is useful for numerical applications. We have produced similar results, giving an expression for the inverse of a matrix when the $i^{th}$ row and column are removed. However, our expression involves taking a limit, which inhibits use in similar applications as the SWM formula. However, using our expression to find an analytical result on the spectral radius of a special product of two matrices leads to an application. In particular, we find a way to compute the fundamental reproductive ratio of a relapsing disease being spread by a vector among two species of host that undergo a different number of relapses.

1 Introduction

The process of computing the inverse of matrix after altering it is known as updating the inverse of a matrix [2]. The most famous example of this process are the Sherman-Morrisson-Woodbury formula which gives a closed form expression for the inverse of a perturbation of a matrix in terms of its original inverse. In this paper we will be representing the inverse of a matrix when the $i^{th}$ row and column are removed as a limit involving the original inverse. While such methods have numerical applications [2], our method, since it contains a limit, is going to have a more analytical usage. In particular, after some introduction, we will show how this result can be used to compute the fundamental reproductive ratio for a model involving a relapsing disease being spread among two host species by a vector.

2 Main Results

We begin by investigating the determinant of a square matrix $A$ as a diagonal element tends to $\infty$. Note that $A_{[i,j]}$ represents the matrix formed by removing the $i^{th}$ row and $j^{th}$ column, and we will denote a particular element of a matrix with a lower case letter corresponding to the matrix e.g. $a_{mn}$ is the $m,n$th element of $A$. We will also occasionally use $(B)_{ij}$ to represent the $i,j$th element of $B$.

Lemma 1. Let $A$ be an $n \times n$ matrix and suppose that $A_{[i,i]}$ is nonsingular. Then $\lim_{a_{ii} \to \infty} \det A = \pm \infty$.

Proof. By Proposition 2.7.5 of [1]:

$$\det A = \sum_{k=1}^{n} (-1)^{i+k} a_{ik} \det(A_{[i,k]}) = a_{ii} \det A_{[i,i]} + \sum_{k \neq i} (-1)^{i+k} a_{ik} \det(A_{[i,k]})$$

The last sum does not involve $a_{ii}$ and thus has a fixed value as $a_{ii} \to \infty$. Since $A_{[i,j]}$ is nonsingular it has a nonzero determinant, and thus the leading term of the previous sum goes to $\pm \infty$ depending on the sign of $\det A_{[i,i]}$.

As a result of this lemma we can see that there is a sufficiently large value of $a_{ii}$ that makes $A$ invertible, and the matrix remains invertible for all further values. Hence, the hypothesis of $A$ being invertible is not needed in the next result, which tells us how to construct the inverse of $A_{[i,j]}$ from $A^{-1}$:
Theorem 1. If $A_{[i,j]}$ is nonsingular then

$$(A_{[i,j]})^{-1} = \lim_{a_{ii} \to \infty} (A^{-1})_{[i,j]}$$

Furthermore,

$$\lim_{a_{ii} \to \infty} (A^{-1})_{ik} = \lim_{a_{ii} \to \infty} (A^{-1})_{ki} = 0$$

Proof. We will need to consider this proof in four cases. The proof technique in each case is the same, though the indexing in each is different. Throughout let $B^{jk} = (b_{pq}) = A_{[j,k]}$ for $1 \leq p, q \leq n - 1$. We will repeatedly use Corollary 2.7.6 of [1] which is a formula for the $ij$ element of the inverse of a matrix. Also, the use of a “…” denotes terms of a sum that do not involve $a_{ii}$.

**Case 1.** Assume that $1 \leq j, k < i$. Then, on the one hand

$$\lim_{a_{ii} \to \infty} (A^{-1})_{kj} = \lim_{a_{ii} \to \infty} \frac{(-1)^{k+j} \det B^{jk}}{\det A}$$

after expanding $\det B^{jk}$ along its $i - 1$ row. Now we want to identify the term that has $a_{ii}$ in it. Note that since $j, k < i$ we have that $a_{ii} = b_{i-1,i-1}$. So then we let $l = i - 1$ and we have

$$\lim_{a_{ii} \to \infty} (A^{-1})_{kj} = \lim_{a_{ii} \to \infty} \frac{(-1)^{k+j}(-1)^{2i-2}a_{ii} \det B^{jk}_{[i-1,i-1]} + \ldots}{\det A_{[i,i]}}$$

On the other hand

$$((A_{[i,j]})^{-1})_{kj} = \frac{(-1)^{k+j} \det(A_{[i,j]})_{[j,k]}}{\det A_{[i,i]}}$$

Since $j, k < i$ we have that $(A_{[i,j]})_{[j,k]} = (A_{[j,k]})_{[i-1,i-1]} = B^{jk}_{[i-1,i-1]}$ so that

$$((A_{[i,j]})^{-1})_{kj} = \frac{(-1)^{k+j} \det B^{jk}_{[i-1,i-1]}}{\det A_{[i,i]}}$$

Thus when $j, k < i$ we have

$$((A_{[i,j]})^{-1})_{kj} = \lim_{a_{ii} \to \infty} (A^{-1})_{kj}$$

**Case 2.** $n - 1 \geq k, j \geq i$. On the one hand

$$\lim_{a_{ii} \to \infty} (A^{-1})_{k+1,j+1} = \lim_{a_{ii} \to \infty} \frac{(-1)^{k+j+2} \det B^{j+1,k+1}}{\det A}$$

after expanding $\det B^{j+1,k+1}$ along its $i$th row. We have that $a_{ii} = b_{ii}$. So then we let $l = i$ and we have

$$\lim_{a_{ii} \to \infty} (A^{-1})_{kj} = \lim_{a_{ii} \to \infty} \frac{(-1)^{k+j}(-1)^{2i}a_{ii} \det B^{j+1,k+1}_{[i,i]} + \ldots}{\det A_{[i,i]}}$$

$$= \frac{(-1)^{k+j} \det B^{j+1,k+1}_{[i,i]}}{\det A_{[i,i]}}$$
Thus when \( j, k \geq i \) we have that \( (A_{i,i})^{-1}[j,k] = (A_{j+1,k+1})_{i,i} = B_{i,i}^{j+1,k+1} \) so that
\[
((A_{i,i})^{-1})_{k,j} = \frac{(-1)^{k+j} \det(A_{i,i})_{j,k}}{\det A_{i,i}}
\]

Since \( j, k \geq i \) we have
\[
((A_{i,i})^{-1})_{k,j} = \frac{(-1)^{k+j} \det B_{i,i}^{j+1,k+1}}{\det A_{i,i}}
\]

Thus when \( j, k \geq i \) we have
\[
((A_{i,i})^{-1})_{k,j} = \lim_{a_{ii} \to \infty} (A^{-1})_{k+1,j+1}
\]

**Case 3.** \( k = i, j < i \). On the one hand
\[
\lim_{a_{ii} \to \infty} (A^{-1})_{i+1,j} = \lim_{a_{ii} \to \infty} \frac{(-1)^{i+j+1} \det B_{i,i}^{j+1}}{\det A_{i,i}}
\]
\[
= \lim_{a_{ii} \to \infty} \frac{(-1)^{i+j+1} \sum_{l=1}^{n-1} (-1)^{l-1} a_{li} \det B_{[i-1,l]}^{j+1}}{a_{ii} \det A_{[i,i]} + \ldots}
\]

after expanding \( \det B_{i,i}^{j+1} \) along its \( i-th \) row. Since \( j < i \) we have that \( a_{ii} = b_{i-1,i} \). So then we let \( l = i \) and we have
\[
\lim_{a_{ii} \to \infty} (A^{-1})_{i+1,j} = \frac{(-1)^{i+j} \det B_{i,i}^{j+1}}{\det A_{[i,i]}}
\]
\[
= \frac{(-1)^{i+j} \det B_{i,i}^{j+1}}{\det A_{[i,i]}}
\]

On the other hand
\[
((A_{i,i})^{-1})_{i,j} = \frac{(-1)^{i+j} \det(A_{i,i})_{j,i}}{\det A_{[i,i]}}
\]
Since \( j < i \) we have that \( (A_{i,i})_{j,i} = (A_{j,i+1})_{i-1,i} = B_{[i-1,i]}^{j+1} \) so that
\[
((A_{i,i})^{-1})_{i,j} = \frac{(-1)^{i+j} \det B_{[i-1,i]}^{j+1}}{\det A_{[i,i]}}
\]

Thus when \( j < i \) we have
\[
((A_{i,i})^{-1})_{i,j} = \lim_{a_{ii} \to \infty} (A^{-1})_{i+1,j}
\]

**Case 4.** \( k < i, j = i \). On the one hand
\[
\lim_{a_{ii} \to \infty} (A^{-1})_{k,i+1} = \lim_{a_{ii} \to \infty} \frac{(-1)^{i+k+1} \det B_{i,i}^{i+1,k}}{\det A_{i,i}}
\]
\[
= \lim_{a_{ii} \to \infty} \frac{(-1)^{i+k+1} \sum_{l=1}^{n-1} (-1)^{l-1} a_{li} \det B_{[i-1,l]}^{i+1,k}}{a_{ii} \det A_{[i,i]} + \ldots}
\]

after expanding \( \det B_{i,i}^{i+1,k} \) along its \( i \)th row. Since \( k < i \) we have that \( a_{ii} = b_{i-1,i} \). So then we let \( l = i - 1 \) and we have
\[
\lim_{a_{ii} \to \infty} (A^{-1})_{k,i+1} = \frac{(-1)^{i+k} \det B_{[i,i-1]}^{i+1,k}}{\det A_{[i,i]}}
\]
On the other hand
\[
((A_{[i,i]}^{-1})^{-1})_{ki} = \frac{(-1)^{i+k} \det(A_{[i,i]}^{-1})_{[i,k]}}{\det A_{[i,i]}}
\]
Since \(k < i\) we have that \((A_{[i,i]}^{-1})_{[i,k]} = (A_{[i+1,i]}^{-1})_{[i,i-1]} = B_{[i,i-1]}^{i+1,k}\) so that
\[
((A_{[i,i]}^{-1})^{-1})_{ki} = \frac{(-1)^{i+k} \det B_{[i,i-1]}^{i+1,k}}{\det A_{[i,i]}}
\]
Thus when \(k < i\) we have
\[
((A_{[i,i]}^{-1})^{-1})_{ki} = \lim_{a_{ii} \to \infty} (A^{-1})_{k,i+1}
\]
The combination of these four cases gives the first result.

For the second result, we again use Corollary 2.7.6 of [5] to get that
\[
\lim_{a_{ii} \to \infty} (A^{-1})_{ik} = \lim_{a_{ii} \to \infty} \frac{(-1)^{k+i} \det B_{ki}^{i}}{\det A}
\]
\(B_{ki}^{i}\) does not contain \(a_{ii}\), and thus \(\det B_{ki}^{i}\) remains constant for all values of \(a_{ii}\), and by Lemma [5] we have that \(\det A \to \pm \infty\). As a result
\[
\lim_{a_{ii} \to \infty} \frac{(-1)^{k+i} \det B_{ki}^{i}}{\det A} = 0
\]
The result for \((A^{-1})_{ki}\) is done in exactly the same way. \(\Box\)

Lemma [5] and Theorem [5] will allow us to prove a result about the spectral radius of a special product of matrices.

**Corollary 1.** Suppose that \(V_{[i,i]}\) is nonsingular. Then
\[
\lim_{v_{ii} \to \infty} \rho(FV^{-1}) = \rho(F[I_{i,i}]^{-1}(V_{[i,i]}^{-1})^{-1})
\]

**Proof.** As before, because \(V_{[i,i]}\) is nonsingular \(V\) must be nonsingular for sufficiently large \(v_{ii}\). Since eigenvalues are continuous with respect to the entries of a matrix, and the absolute value and maximum of a set of continuous functions is continuous, we have that
\[
\lim_{v_{ii} \to \infty} \rho(FV^{-1}) = \rho(F \lim_{v_{ii} \to \infty} V^{-1})
\]
Let
\[
(V_{[i,i]}^{-1})^{-1} = \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix}
\]
where \(V_1 \in \mathbb{R}^{(i-1) \times (i-1)}, V_2 \in \mathbb{R}^{(i-1) \times (n-i)} \), \(V_3 \in \mathbb{R}^{(n-i) \times (i-1)}\) and \(V_4 \in \mathbb{R}^{(n-i) \times (n-i)}\). Then Lemma [5] says that
\[
\lim_{v_{ii} \to \infty} V^{-1} = \begin{pmatrix} 0_{(i-1) \times 1} & V_2 \\ 0_{(n-i) \times 1} & V_4 \end{pmatrix}
\]
Let
\[
F = \begin{pmatrix} F_1 & f_{(i-1) \times 1}^{(1)} & F_2 \\ f_{1 \times (i-1)}^{(2)} & f_{i,i} & f_{1 \times (n-i)}^{(3)} \\ F_3 & f_{(n-i) \times 1}^{(4)} & F_4 \end{pmatrix}
\]
where \(F_1 \in \mathbb{R}^{(i-1) \times (i-1)}, F_2 \in \mathbb{R}^{(i-1) \times (n-i)} \), \(F_3 \in \mathbb{R}^{(n-i) \times (i-1)}\) and \(F_4 \in \mathbb{R}^{(n-i) \times (n-i)}\). This gives that
\[
F \lim_{v_{ii} \to \infty} V^{-1} = \begin{pmatrix} F_1 V_1 + F_2 V_3 & 0_{(i-1) \times 1} & F_1 V_2 + F_2 V_4 \\ f_{1 \times (i-1)}^{(2)} V_1 + f_{1 \times (n-i)}^{(3)} V_3 & F_3 V_1 + F_4 V_3 & f_{1 \times (i-1)}^{(2)} V_2 + f_{1 \times (n-i)}^{(3)} V_4 \\ 0_{(n-i) \times 1} & F_3 V_2 + F_4 V_4 \end{pmatrix}
\]
We wish to compute the spectral radius of this matrix, so we set up the eigenvalue problem

\[
\det(F \lim_{v_i \to \infty} V^{-1} - \lambda I_n)
\]

\[
= \det\left( F_{\lim}(V^{-1}) = \begin{pmatrix}
F_1 V_1 + F_2 V_3 - \lambda I_{i-1} & 0_{(i-1) \times 1} & F_1 V_2 + F_2 V_4 \\
F_2 (2) V_1 + f_{3 \times (n-1)} V_3 & -\lambda & f_{2 \times (n-1)} V_2 + f_{3 \times (n-1)} V_4 \\
F_3 V_1 + F_4 V_3 & 0_{(n-i) \times 1} & F_3 V_2 + F_4 V_4 - \lambda I_{n-i}
\end{pmatrix}
\right)
\]

\[
= -\lambda \det\left( F_1 V_1 + F_2 V_3 - \lambda I_{i-1} & F_1 V_2 + F_2 V_4 \\
F_3 V_1 + F_4 V_3 & F_3 V_2 + F_4 V_4 - \lambda I_{n-i}
\right)
\]

So the spectrum of \( F \lim_{v_i \to \infty} V^{-1} \) is 0 unioned with the spectrum of \( F_{[i,j]}(V_{[i,j]})^{-1} \). Thus the spectral radius of \( F \lim_{v_i \to \infty} V^{-1} \) is the maximum of the eigenvalues of \( F_{[i,j]}(V_{[i,j]})^{-1} \). That is

\[
\rho(F \lim_{v_i \to \infty} V^{-1}) = \rho(F_{[i,j]}(V_{[i,j]})^{-1})
\]

which gives the result. \( \square \)

3 Applications

To give an application we must first have a brief description of compartmental disease models, following their development in [2]. Suppose a population can be separated into \( n \) homogeneous compartments and the number of members in each compartment will be represented by the vector \( \mathbf{x} \in \mathbb{R}^n \) where the first \( m \) compartments represent infected states while the remaining \( n - m \) compartments are uninfected states. It is natural to insist that \( \mathbf{x} \geq 0 \) (inequality is taken componentwise) since we are dealing with populations. Let \( X_s = \{ \mathbf{x} \geq 0 : x_i = 0, i = 1, \ldots, m \} \) be the set of disease free states. Let \( J_i(\mathbf{x}) \) be the number of new infections in compartment \( i \) (autonomy is assumed). \( V_i^+(\mathbf{x}) \) is the rate of transfer of individuals into compartment \( i \) and \( V_i^-(\mathbf{x}) \) is the rate of transfer out of compartment \( i \). Assume that these functions are at least twice continuously differentiable. The disease transmission model can be written as

\[
\begin{align*}
\dot{x}_i &= f_i(\mathbf{x}) = J_i(\mathbf{x}) + V_i^+(\mathbf{x}) - V_i^-(\mathbf{x}) \\
&= i = 1, \ldots, n
\end{align*}
\]

Let \( V_i(\mathbf{x}) = V_i^-(\mathbf{x}) - V_i^+(\mathbf{x}) \). Suppose that \( \mathbf{x}_0 \in X_s \) is also a fixed point of (1) then we call \( \mathbf{x}_0 \) a disease free equilibrium (DFE). Let \( J \) be the vector valued function with the \( J_i \) as components, and \( V \) similarly defined. Given five conditions (A1-A5 of [3]) the Jacobians of \( J \) and \( V \) must take the form

\[
D\mathbf{F}(\mathbf{x}_0) = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix} \text{ and } D\mathbf{V}(\mathbf{x}_0) = \begin{pmatrix} V & 0 \\ J_1 & J_2 \end{pmatrix}
\]

Where \( F \) and \( V \) are \( m \times m \). Furthermore, under these conditions \( V \) is nonsingular, which allows us to define the fundamental reproductive ratio:

\[
R_0 = \rho(FV^{-1})
\]

where \( \rho \) is the spectral radius. Informally, we can think of \( R_0 \) as being the average number of new infections produced by a single infected individual [3]. With this interpretation in mind, it makes Theorem 2 of [3] expected: For \( R_0 < 1 \) the DFE \( \mathbf{x}_0 \) is stable and for \( R_0 > 1 \) \( \mathbf{x}_0 \) is unstable.

Computing \( R_0 \) can be a difficult and tedious process, particularly when dealing with systems with large numbers of compartments. Recent work has computed \( R_0 \) for vector-borne diseases which relapse an arbitrary number of times. For a full description of these types of models see [4]. In particular, the fundamental reproductive ratio was computed for two kinds of systems:

- One host species undergoing \( j - 1 \) relapses with one vector species spreading the disease.
- Two host species each undergoing \( j - 1 \) relapses with one vector species spreading the disease.
In the first case, we will call the system uncoupled and in the second we will call it a coupled system. The equations describing the dynamics will not be reproduced here but can be found in [4].

For notation, suppose that the $i$th species is the only species in the system (an uncoupled system) and let $R_{0,i,j}$, $i = 1, 2$, be the reproductive ratio when the hosts undergo $j - 1$ relapses and thus have $j$ infected compartments. Let $F_{j,k}$ and $V_{j,k}$ be the Jacobians for the coupled system when the first host species undergoes $j - 1$ relapses and the second undergoes $k - 1$ relapses. Lastly, let $R_{0}^{k}$ be the reproductive ratio for the coupled system where the first species undergoes $j - 1$ relapses and the second species undergoes $k - 1$ relapses. We can write the reproductive ratio for the uncoupled systems in terms of the parameters for the model

$$R_{0,i,j} = f \left[ \frac{c_{i}c_{j}S_{0}}{\mu i} \sum_{k=1}^{j} \prod_{l=1}^{k} \frac{\alpha_{i,l-1}}{\alpha_{i,l} + \mu_{i,l}} \right] = \lim_{v_{jj} \to \infty} \rho \left( F_{j-1,j} \right)^{-1} \left( V_{j-1,j} \right)^{-1} = \lim_{v_{jj} \to \infty} \rho \left( F_{j,j} \right)^{-1} = \lim_{\alpha_{1,j} \to \infty} R_{0}^{j}$$

(2)

The coupled and uncoupled systems can then be related:

$$R_{0}^{j} = \sqrt{\left( R_{0,1,j} \right)^{2} + \left( R_{0,2,j} \right)^{2}} \quad \text{(3)}$$

A full description of the parameters is found in [4], but the relevant portion for our work here will be to note that the first host species leaves the $j$th infected compartment at a rate $\alpha_{i,j}$ and that $\alpha_{i,j} \to \infty$ implies that $(V_{j,k})_{jj} \to \infty$. The average amount of time spent in the $j$th infected compartment is $\frac{1}{\alpha_{1,j}}$, and thus as $\alpha_{1,j} \to \infty$ the average time spent in that compartment will go to zero. This gives us an intuition for the idea that removing a compartment from a system can be achieved through taking a limit.

It is easily observed that from Equations (6)-(8) of [4] that the system with the $j$th infected compartment removed from the relapses of the first species is related to the whole system through the Jacobians:

$$F_{j-1,j} = (F_{j,j})_{[j:j]} \quad \text{and} \quad V_{j-1,j} = (V_{j,j})_{[j:j]}$$

We can now apply the results of Corollary 1 to find that

$$R_{0}^{j-1} = \rho \left( F_{j-1,j}(V_{j-1,j})^{-1} \right) = \rho \left( (F_{j,j})_{[j:j]}(V_{j,j})_{[j:j]}^{-1} \right) = \lim_{\alpha_{1,j} \to \infty} \rho \left( F_{j,j} \right)^{-1} \left( V_{j,j} \right)^{-1} = \lim_{\alpha_{1,j} \to \infty} R_{0}^{j}$$

Using (3) we find that

$$R_{0}^{j-1} = \sqrt{\lim_{\alpha_{1,j} \to \infty} \left( R_{0,1,j} \right)^{2} + \left( R_{0,2,j} \right)^{2}}$$

Apply (2) and observe that

$$\lim_{\alpha_{1,j} \to \infty} \left( R_{0,1,j} \right)^{2} = R_{0,1,j-1}$$

Thus

$$R_{0}^{j-1} = \sqrt{\left( R_{0,1,j-1} \right)^{2} + \left( R_{0,2,j} \right)^{2}}$$

We can repeat this process, making the same observations and applying Corollary 1. Thus we can say that when the first species undergoes $k - 1 < j - 1$ relapses we get

$$R_{0}^{k} = \sqrt{\left( R_{0,1,k} \right)^{2} + \left( R_{0,2,j} \right)^{2}}$$

4 Discussion

We have related the inverse of a matrix when the $i$th row and column are removed to the inverse of the original matrix through a limit. Such updating results generally have numerical uses, but in our case the existence of a limit is a complication. Even the use of this method for giving and approximation to the updated matrix is impractical and inefficient, since it requires computation of the inverse of a larger matrix before the limit is taken. However, we demonstrated an analytical application that allowed us to extend results to coupled systems of host and vectors in the spread of a relapsing disease. In particular, we were able to quantify how the two species undergoing a different number of relapses affects the fundamental reproductive ratio for the disease.
References

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