PULLBACK ATTRACTORS VIA QUASI-STABILITY FOR NON-AUTONOMOUS LATTICE DYNAMICAL SYSTEMS

Radoslaw Czaja
Institute of Mathematics
University of Silesia in Katowice
Bankowa 14, 40-007 Katowice, Poland

In memory of María José Garrido Atienza

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Abstract. In this paper we study long-time behavior of first-order non-autonomous lattice dynamical systems in square summable space of double-sided sequences using the cooperation between the discretized diffusion operator and the discretized reaction term. We obtain existence of a pullback global attractor and construct pullback exponential attractor applying the introduced notion of quasi-stability of the corresponding evolution process.

1. Introduction. We consider non-autonomous first-order lattice systems of the form

\[
\begin{cases}
u u'_i + [2u_i - u_{i+1} - u_{i-1}] = f_i(t, u_i) + g_i(t), & i \in \mathbb{Z}, \quad t > s, \\
u u_i(s) = u^0_i, & i \in \mathbb{Z},
\end{cases}
\]

where \(\nu > 0\), \(f_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) and \(g_i : \mathbb{R} \to \mathbb{R}\) are given functions. Our aim is to study the asymptotic behavior of solutions to these systems in terms of pullback global and exponential attractors for the evolution processes generated by (1). It will be achieved by a suitable extension of I. Chueshov’s concept of quasi-stability of a semigroup from [12] to the case of evolution processes.

Systems of infinite number of ordinary differential equations as in (1) are spatial discretizations of one-dimensional reaction-diffusion equations on the real line and were investigated under various assumptions on the right-hand side and in different spaces of solutions. Mostly these systems were studied in the autonomous case, see e.g. [17, 19, 6] for systems generating semigroups in the weighted spaces of summable double-sided sequences or e.g. [4, 25, 26, 20, 28, 1, 10] in the classical spaces of summable double-sided sequences. The non-autonomous case was investigated e.g. in [24, 2, 27, 18, 3] in both types of spaces. In [24, 2, 3] the authors proved existence of uniform global and exponential attractors under rather restrictive assumptions.

In our setting specific properties of the right-hand side of the system in (1) interplay with the properties of the bounded linear operator on the left-hand side to guarantee the existence of pullback global and exponential attractors for a suitable universe of attracted families of bounded subsets of the space \(\ell^2\) of square summable double-sided real sequences. This cooperation was first observed for lattice
dynamical systems in [10] in the autonomous case (see also [11]). Here we follow these ideas in the context of time dependent nonlinearities by developing a concept of quasi-stability of an evolution process.

Our approach is more general than the one-dimensional case of the problem considered in \( \ell^2 \) in [3] and earlier in [24, 2], where \( g(t) = (g_i(t)) \in \ell^2 \), \( t \in \mathbb{R} \), was almost periodic and

\[
f_i(t, \sigma) = -a_\ast \sigma + h_i(t, \sigma)
\]

with \( a_\ast > 0 \) and \( h_i \) satisfying in particular

\[
h_i(t, \sigma) \leq 0, \ t, \sigma \in \mathbb{R}, \ i \in \mathbb{Z}.
\]

These assumptions come from a prototype of time independent nonlinearities \( f_i \) considered in [2, Section 5] of the form

\[
f_i(t, \sigma) = -\sum_{j=0}^{J} a_{ji} \sigma^{2j+1}
\]

with \( a_{0i} \in [a_\ast, a^\ast] \) and \( a_{ji} \in [0, a^\ast], \ j = 1, \ldots, J \), for some \( a^\ast > a_\ast > 0 \).

The existence of \( a_\ast > 0 \) and the condition (3) are especially restrictive and help establishing asymptotic compactness of the corresponding process. Here instead we distinguish a sequence \( m = (m_i) \in \ell^\infty \) such that the Schrödinger operator \( A - m \) corresponding to the linear operator \( A \) in the left-hand side of (1) (see (7)) is positive definite in \( \ell^2 \). It turns out that not only a constant sequence \( m_i = -a_\ast, \ i \in \mathbb{Z} \), is admissible, but also sequences with infinitely many positive coordinates. Then we make the positive definiteness of \( A - m \) cooperate with the discrete reaction term on the right-hand side of (1) through the structural condition

\[
f_i(t, \sigma) \leq m_i \sigma^2 + d_i(t) |\sigma|, \ i \in \mathbb{Z}, \ t, \sigma \in \mathbb{R},
\]

where \( d(t) = (d_i(t)) \) and \( g(t) = (g_i(t)) \) belong to a common class of functions \( \mathcal{G}_{s_0} \) defined in Definition 2.3. In particular for functions of the form (2), (3) the condition (5) is trivially satisfied with \( m_i \equiv -a_\ast \) and \( d(t) \equiv 0 \), which highly simplifies the argument and excludes many nonlinearities. Moreover, the flexibility of (5) admits functions \( f_i \) with all positive coefficients in the linear part of \( f_i(t, \sigma) - m_i \sigma \) as can be seen for the functions considered in Proposition B.1.

The contents of the paper are as follows. In Section 2 we introduce the setting of our problem reviewing the properties of the linear operator in (1), including the positive definiteness of the corresponding Schrödinger operator and specifying assumptions on the right-hand side of (1). In Proposition 2.7 we show that they guarantee that (1) generates an evolution process \( \{U(t, s) : t \geq s\} \) on \( \ell^2 \).

In Section 3 we consider a universe \( \mathcal{D}_u \) of families of subsets of \( \ell^2 \) uniformly bounded in the past (see Definition 2.2) and prove in Proposition 3.1 that the process is pullback dissipative with a positively invariant pullback \( \mathcal{D}_u \)—absorbing family \( \mathcal{B} \) of closed bounded subsets of \( \ell^2 \). Moreover, we establish pullback \( \mathcal{B} \)—asymptotic compactness of the process by B. Wang’s idea of uniform smallness of tails of pullback orbits adapted to our setting. This leads to the main result of this section, Theorem 3.5, in which we prove existence of the pullback global \( \mathcal{D}_u \cup \{\mathcal{B}\} \)—attractor \( \mathcal{A} = \{A(t) : t \in \mathbb{R}\} \) for the process \( \{U(t, s) : t \geq s\} \) being a minimal, invariant family of compact sets attracting all families from the universe \( \mathcal{D}_u \cup \{\mathcal{B}\} \) with its sections being \( \omega \)—limit sets for the family \( \mathcal{B} \).

Since we are going to prove existence of a pullback exponential \( \mathcal{D}_u \)—attractor in Section 4 and indirectly estimate from above fractal dimension of the pullback
global \( \mathcal{D}_u \)-attractor, we impose further conditions on the right-hand side of (1) in Section 4. They allow us to obtain the estimates of solutions in Proposition 4.2, which in turn imply the quasi-stability of the evolution process - see (52) and (53).

This is a useful tool to construct a pullback exponential attractor as general Theorems A.4 and A.5 show. Having already proved existence of the pullback global attractor, we apply the first one to obtain in our main Theorem 4.4 a positively invariant family \( \mathcal{M} = \{ M(t) : t \in \mathbb{R} \} \) of bounded subsets of \( \ell^2 \), which exponentially pullback attracts all families from \( \mathcal{D}_u \) at a common rate. Moreover, its sections \( M(t) \) have finite fractal dimension and

\[
M(t) = A(t) \cup E(t) \subset B(t),
\]

where \( E(t) \) is a countable subset of \( B(t) \). In Corollary 4.5 we show that Theorem 4.4 applies to the class of nonlinearities from Proposition B.1 and in this case fractal dimension of sections of the pullback exponential attractor \( \mathcal{M} \) is uniformly bounded.

We also observe in Corollary 4.6 that for nonlinearities as in (4) the sections of pullback global and exponential attractors are in fact zero-dimensional.

For the sake of completeness of the presentation in Appendix A we show how quasi-stability of an evolution process (conditions (59) and (60)) helps to construct a pullback exponential attractor and obtain an estimate of the fractal dimension of the pullback global attractor. In Appendix B we give a specific example of the nonlinear part \( f_i \) in (1) to which our results apply.

2. Setting of the problem and existence of evolution process.

We use the Banach spaces of double-sided real sequences

\[
\ell^p = \{ u = (u_i) : \sum_{i \in \mathbb{Z}} |u_i|^p < \infty \}, \quad p \in [1, \infty), \quad \ell^\infty = \{ u = (u_i) : \sup_{i \in \mathbb{Z}} |u_i| < \infty \}
\]

endowed with the norms

\[
\| u \|_p = \left( \sum_{i \in \mathbb{Z}} |u_i|^p \right)^{1/p} \quad \text{and} \quad \| u \|_\infty = \sup_{i \in \mathbb{Z}} |u_i|,
\]

respectively. Our base space \( \ell^2 \) is a Hilbert space with the respective inner product and norm

\[
(u, v) = \sum_{i \in \mathbb{Z}} u_i v_i, \quad \| u \|_2 = \sqrt{(u, u)} =: \| u \|, \quad u = (u_i), \quad v = (v_i) \in \ell^2.
\]

Note that \( \| u \|_\infty \leq \| u \|_p \) for \( u \in \ell^p \) and \( \| uv \|_p \leq \| u \|_p \| v \|_\infty \) for \( u \in \ell^p \) and \( v \in \ell^\infty \). Moreover, we have

\[
\| u \|_p \leq \| u \|_2, \quad u \in \ell^2, \quad p \in [2, \infty].
\]

We also use isometries \( \tau, \tau^* \in \mathcal{L}(\ell^2) \) given by

\[
(\tau u)_i = u_{i+1} \quad \text{and} \quad (\tau^* u)_i = u_{i-1} \quad \text{for} \quad u = (u_i) \in \ell^2,
\]

and operators \( A, Q, Q^* \in \mathcal{L}(\ell^2) \) defined by

\[
A = \nu (2 \text{id} - \tau - \tau^*), \quad Q = \sqrt{\nu} (\tau - \text{id}), \quad Q^* = \sqrt{\nu} (\tau^* - \text{id}).
\]
Lemma 2.1. The operators in (7) satisfy \( A = QQ^\ast = Q^\ast Q \) and
\[
\|Q\|_{L^2(\ell^2)} \leq 2\sqrt{\nu}, \quad \|Q^\ast\|_{L^2(\ell^2)} \leq 2\sqrt{\nu}, \quad \|A\|_{L^2(\ell^2)} \leq 4\nu. \tag{8}
\]
Moreover, we have \((Qu,v) = (u,Q^\ast v)\) for \(u,v \in \ell^2\) and \(A\) is a self-adjoint non-negative operator
\[
(Au,u) = (QQ^\ast u,u) = \|Q^\ast u\|^2 = \|Qu\|^2 \geq 0, \quad u \in \ell^2, \tag{9}
\]
with \(0\) belonging to its continuous spectrum.

One of the main features of our approach will be the positive definiteness of the discretized Schrödinger operator \(A - m\) with \(m = (m_i) \in \ell^\infty\), which was investigated in [10, Proposition 5.1 and Theorem A.3]. Therefore we assume that \(m = (m_i) \in \ell^\infty\) is such that for some \(\alpha_0 > 0\)
\[
(Au,u) - (mu,u) \geq \alpha_0 \|u\|^2, \quad u \in \ell^2. \tag{10}
\]
Examples of such sequences were presented in [10, Remark 5.2], including sequences with infinitely many positive terms. For instance, for arbitrarily chosen real number \(r > 0\) and integers \(0 \leq k < n\) a suitable non-positive sequence with period \(n\) is
\[
m_i = \begin{cases} 
-r, & i = ln, \ldots, k + ln, \\
0, & i = k + 1 + ln, \ldots, n - 1 + ln
\end{cases} \quad \text{for all } l \in \mathbb{Z}.
\]
Moreover, for an arbitrary \(N \in \mathbb{N}\) one can also substitute \(m_i\) at positions \(|i| \leq N\) by zeros. Such a sequence can be further modified by adding any \(\ell^\infty\) sequence with norm less than \(\alpha_0\) in order to preserve positive definiteness of the discretized Schrödinger operator.

Note that for nonlinearities considered in (2) we can simply take \(m_i = -a_* < 0, \ i \in \mathbb{Z}\), and then (9) yields (10) with \(\alpha_0 = a_*\).

Next we introduce a universe of families, which will be attracted by pullback attractors. Pullback attraction of families from a given universe is one of possible approaches for studying asymptotics of non-autonomous problems (compare the discussion in [22]).

Definition 2.2. We define the universe \(\mathcal{D}_u\) of families \(\mathcal{D} = \{D(t) : t \in \mathbb{R}\}\) of nonempty bounded subsets of \(\ell^2\), which are uniformly bounded in the past, i.e., for any \(D \in \mathcal{D}_u\) there exists \(R_D > 0\) such that
\[
\sup_{t \leq 0} \sup_{u \in D(t)} \|u\| \leq R_D. \tag{11}
\]

Note that \(\mathcal{D}_u\) contains in particular all singleton families consisting of a nonempty bounded subset of \(\ell^2\).

The interplay between the discrete diffusion and discrete reaction terms will be reflected in the assumptions on the right-hand side of (1). First, we introduce a class of admissible perturbations \(g_i\) in (1).

Definition 2.3. Let \(0 < \gamma_0 < \alpha_0\). We say that a function \(g(t) = (g_i(t)) \in \ell^2, \ t \in \mathbb{R}\), belongs to class \(\mathfrak{g}_{\gamma_0}\) if \(g : \mathbb{R} \to \ell^2\) is continuous, there exists \(C_g \geq 0\) such that
\[
\int_{-\infty}^{t} \|g(\tau)\|^2 e^{-2\gamma_0(t-\tau)} d\tau \leq C_g, \quad t \in \mathbb{R}, \tag{12}
\]
and for any \(\varepsilon > 0\) there exists \(i_\varepsilon \in \mathbb{N}\) such that
\[
\sum_{|i| \geq i_\varepsilon} g_i^2(t) \leq \varepsilon^2, \quad t \in \mathbb{R}. \tag{13}
\]
Note that in particular (12) holds if \( g \in C(\mathbb{R}, \ell^2) \) is bounded, whereas (13) is satisfied if \( \{ g(t) : t \in \mathbb{R} \} \) is precompact in \( \ell^2 \). The simplest functions belonging to \( \mathcal{G}_{\gamma_0} \) are products of a sequence from \( \ell^2 \) and a continuous bounded real function.

We assume that
\[
g(t) = (g_i(t)), \ t \in \mathbb{R}, \ \text{belongs to} \ \mathcal{G}_{\gamma_0}, \tag{14}
\]
and the functions \( f_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \ i \in \mathbb{Z} \), which form the nonlinear part are continuous and continuously differentiable with respect to the second variable,
\[
f_i(t, 0) = 0, \ t \in \mathbb{R}, \ i \in \mathbb{Z}, \tag{15}
\]
and there exists a continuous function \( P : \mathbb{R} \times (0, \infty) \to (0, \infty) \), which satisfies
\[
\sup_{i \in \mathbb{Z}} \sup_{|\sigma| \leq r} |\frac{\partial f_i}{\partial \sigma}(t, \sigma)| = P(t, r), \ t \in \mathbb{R}, \ r > 0. \tag{16}
\]
Then
\[
F(t, u) = (f_i(t, u_i)), \ t \in \mathbb{R}, \ u = (u_i) \in \ell^2,
\]
defines a function \( F : \mathbb{R} \times \ell^2 \to \ell^2 \), which is Lipschitz continuous with respect to the second variable on bounded subsets of \( \ell^2 \), since by the Mean Value Theorem, (15) and (16) we obtain
\[
\|F(t, u)\|^2 = \sum_{i \in \mathbb{Z}} |\frac{\partial f_i}{\partial \sigma}(t, \theta_i u_i)|^2 |u_i|^2 \leq P^2(t, \|u\|_{\infty}) \|u\|^2, \ t \in \mathbb{R}, \ u \in \ell^2
\]
and, given a ball \( B = \{ u \in \ell^2 : \|u\| \leq r \} \), we have for \( t \in \mathbb{R} \) and \( u, v \in B \)
\[
\|F(t, u) - F(t, v)\|^2 = \sum_{i \in \mathbb{Z}} |\frac{\partial f_i}{\partial \sigma}(t, \theta_i u_i + (1 - \theta_i)v_i)|^2 |u_i - v_i|^2 \leq P^2(t, r) \|u - v\|^2.
\tag{17}
\]
In order to guarantee continuity of \( F : \mathbb{R} \times \ell^2 \to \ell^2 \), owing to (17) it is enough to assume that
\[
\mathbb{R} \ni t \mapsto F(t, u) \in \ell^2 \text{ is continuous for each } u \in \ell^2. \tag{18}
\]

**Remark 2.4.** There are various situations, which imply (18). For example, if for any \( T > 0 \) and \( r > 0 \) there exists \( C_{T, r} \geq 0 \) such that
\[
\sup_{(t, \sigma) \in [-T, T] \times [-r, r]} \sum_{i \in \mathbb{Z}} |\frac{\partial f_i}{\partial \sigma}(t, \sigma)|^2 \leq C_{T, r},
\]
then the Mean Value Theorem implies (18).

Following the idea of [3, Lemma 3.2], condition (18) also holds if for each \( t_0 \in \mathbb{R} \) and \( r > 0 \) there exists \( g \geq 1 \) and a positive sequence \( b = (b_i) \in \ell^2 \) such that
\[
\sup_{i \in \mathbb{Z}} \sup_{|\sigma| \leq r} \frac{|f_i(t, \sigma) - f_i(t_0, \sigma)|}{|\sigma|^g + b_i} \to 0 \text{ as } t \to t_0.
\]

Finally, for the purpose of global solvability and dissipativity we also assume that for \( m = (m_i) \in \ell^\infty \) from (10)
\[
f_i(t, \sigma) \sigma \leq m_i \sigma^2 + d_i(t) |\sigma|, \ i \in \mathbb{Z}, \ t, \sigma \in \mathbb{R}, \text{ for some } d(t) = (d_i(t)) \in \mathcal{G}_{\gamma_0}. \tag{19}
\]
Note that (19) and (15) imply that \( d_i(t) \geq 0 \) for \( i \in \mathbb{Z} \) and \( t \in \mathbb{R} \). We point out that assumption (19) is another advantage of our approach that broadens the class of investigated nonlinear lattice differential equations in comparison with a common simple condition found in the literature of the form (2), (3) (cp. e.g. [3, (1), (25)]).
We view (1) as an abstract Cauchy problem

\[
\begin{align*}
&u' + Au = F(t, u) + g(t), \ t > s, \\
u(s) = u^0 \in \ell^2.
\end{align*}
\]  

(20)

**Proposition 2.5.** Let A be as in (7), \( g = (g_1(\cdot)) \in C(\mathbb{R}, \ell^2) \) and let (15), (16), (18) hold. Then for any \( s \in \mathbb{R} \) and \( u^0 \in \ell^2 \) the Cauchy problem (20) has a unique solution \( u \in C^1([s, t_{\text{max}}(u^0)], \ell^2) \) defined on the maximal interval of existence \([s, t_{\text{max}}(u^0)]\) and

\[
\lim_{t \to t_{\text{max}}(u^0)^-} \|u(t; s, u^0)\| = \infty \quad \text{unless} \quad t_{\text{max}}(u^0) = \infty.
\]

**Proof.** The right-hand side \( F + g: \mathbb{R} \times \ell^2 \to \ell^2 \) is continuous due to (16), (18) and locally Lipschitz continuous with respect to the second variable by (17). Hence the local solutions of (20) exist by the classical Picard theorem.

We show next that all solutions of (20) in \( \ell^2 \) exist in fact globally in time, that is, \( t_{\text{max}}(u^0) = \infty \) for any \( s \in \mathbb{R} \) and \( u^0 \in \ell^2 \).

**Lemma 2.6.** Under the assumptions of Proposition 2.5, if \( m = (m_i) \in \ell^\infty \) and \( \alpha_0 > 0 \) satisfy (10) and (5) holds with some \( d = (d_i(\cdot)) \in C(\mathbb{R}, \ell^2) \), then for any \( 0 < \gamma_0 < \alpha_0 \), \( s \in \mathbb{R} \) and \( u^0 \in \ell^2 \) we have for \( t \in [s, t_{\text{max}}(u^0)] \)

\[
\|u(t; s, u^0)\|^2 \leq \|u^0\|^2 e^{-2\gamma_0 (t-s)} + \frac{1}{\alpha_0 - \gamma_0} \int_s^t (\|d(\tau)\|^2 + \|g(\tau)\|^2) e^{-2\gamma_0 (t-\tau)} d\tau.
\]  

(21)

Hence we obtain \( t_{\text{max}}(u^0) = \infty \).

**Proof.** We take the inner product of the first equation in (20) with \( u \) in \( \ell^2 \) and get

\[
\frac{1}{2} \frac{d}{dt} \|u\|^2 + (Au, u) = (F(t, u) + g(t), u).
\]

Using (5) we get by the Cauchy inequality

\[
(F(t, u) + g(t), u) \leq (mu, u) + (\alpha_0 - \gamma_0) \|u\|^2 + \frac{1}{2(\alpha_0 - \gamma_0)}(\|d(t)\|^2 + \|g(t)\|^2), \ t > s,
\]

which by (10) yields

\[
\frac{d}{dt} \|u\|^2 + 2\gamma_0 \|u\|^2 \leq \frac{1}{\alpha_0 - \gamma_0} (\|d(t)\|^2 + \|g(t)\|^2).
\]

Then the Gronwall inequality implies (21).

**Proposition 2.7.** If \( A \) from (7) satisfies (10) with some \( m = (m_i) \in \ell^\infty \) and \( \alpha_0 > 0 \), \( g = (g_1(\cdot)) \in C(\mathbb{R}, \ell^2) \), \( f_i \) are such that (15), (16), (18) hold and (5) is satisfied with some \( d = (d_i(\cdot)) \in C(\mathbb{R}, \ell^2) \), then the problem (1) generates on \( \ell^2 \) an evolution process \( \{U(t, s): t \geq s\} \) of global solutions of (20) defined by

\[
U(t, s)u^0 = u(t; s, u^0), \ t \geq s, \ u^0 \in \ell^2.
\]

In the next sections we are going to investigate the asymptotic behavior of this process by proving existence of pullback attractors attracting families \( D \) from the universe \( \mathcal{D}_u \) introduced in Definition 2.2.
3. Pullback global attractor. In this section we consider (20) under the assumptions of Proposition 2.7, which guarantee that (20) generates an evolution process \( \{U(t,s) \colon t \geq s \} \) on \( \ell^2 \). Moreover, we assume about \( d = (d_i(\cdot)) \in C(\mathbb{R}, \ell^2) \) and \( g = (g_i(\cdot)) \in C(\mathbb{R}, \ell^2) \) that there exists \( 0 < \gamma_0 < \alpha_0 \) such that for each \( t \in \mathbb{R} \) we have

\[
\int_{-\infty}^{t} (\|d(\tau)\|^2 + \|g(\tau)\|^2) e^{2\gamma_0 \tau} d\tau < \infty. \tag{22}
\]

We now show the existence of a positively invariant pullback absorbing family for the universe \( \mathcal{D}_u \) (cp. [21, Lemma 3.3]).

**Proposition 3.1.** Let the assumptions of Proposition 2.7 hold together with (22). Then for any \( t \in \mathbb{R} \) and \( \mathcal{D} = \{D(t) \colon t \in \mathbb{R} \} \in \mathcal{D}_u \) there exists \( r_{D,t} \geq 0 \) such that \( t \mapsto r_{D,t} \) is nondecreasing and we have

\[
U(t,t-r)D(t-r) \subset B(t), \quad r \geq r_{D,t},
\]

where \( B = \{B(t) \colon t \in \mathbb{R} \} \) with

\[
B(t) = \mathbb{R}^2 \setminus (0,R(t)) = \{u \in \ell^2 \colon \|u\| \leq R(t)\}, \quad t \in \mathbb{R},
\]

and

\[
R^2(t) = 1 + \frac{1}{\alpha_0 - \gamma_0} \int_{-\infty}^{t} (\|d(\tau)\|^2 + \|g(\tau)\|^2) e^{2\gamma_0 \tau} d\tau, \quad t \in \mathbb{R}. \tag{23}
\]

Moreover, we have

\[
U(t,t-r)B(t-r) \subset B(t), \quad t \in \mathbb{R}, \quad r > 0, \tag{24}
\]

and

\[
\limsup_{r \to \infty} \frac{1}{r} \ln^+ (\text{diam}^2 (B(t-r))) \leq \gamma_0, \quad t \in \mathbb{R}. \tag{25}
\]

Denoting \( L_{t,s} = \exp \int_{s}^{t} P(\tau,R(\tau)) d\tau \) we have

\[
\|U(t,s)u^0 - U(t,s)v^0\| \leq L_{t,s} \|u^0 - v^0\|, \quad t \geq s, \quad u^0, v^0 \in B(s). \tag{26}
\]

**Proof.** By (11) for any \( \mathcal{D} = \{D(t) \colon t \in \mathbb{R} \} \in \mathcal{D}_u \) there exists \( R_{\mathcal{D}} > 0 \) such that \( \|u_0\| \leq R_{\mathcal{D}} \) for any \( u_0 \in D(s) \) with \( s \leq 0 \). Let \( r_{\mathcal{D},0} \geq 0 \) be such that \( e^{-2\gamma_0 r} R^2_{\mathcal{D}} \leq 1 \) for \( r \geq r_{\mathcal{D},0} \). Then for any \( r \geq \max \{t,r_{\mathcal{D},0}\} = r_{\mathcal{D},t} \) and \( u^0 \in D(t-r) \) using (22) we get from (21)

\[
\|U(t,t-r)u^0\|^2 \leq 1 + \frac{1}{\alpha_0 - \gamma_0} e^{-2\gamma_0 r} \int_{-\infty}^{t} (\|d(\tau)\|^2 + \|g(\tau)\|^2) e^{2\gamma_0 \tau} d\tau = R^2(t).
\]

Moreover, for \( t \in \mathbb{R}, \quad r > 0 \) and \( u^0 \in B(t-r) \) we have

\[
\|U(t,t-r)u^0\|^2 \leq e^{-2\gamma_0 r} + \frac{1}{\alpha_0 - \gamma_0} e^{-2\gamma_0 r} \int_{-\infty}^{t-r} (\|d(\tau)\|^2 + \|g(\tau)\|^2) e^{2\gamma_0 \tau} d\tau
\]

\[
+ \frac{1}{\alpha_0 - \gamma_0} e^{-2\gamma_0 r} \int_{t-r}^{t} (\|d(\tau)\|^2 + \|g(\tau)\|^2) e^{2\gamma_0 \tau} d\tau \leq R^2(t),
\]

which gives (24).

Let \( t \in \mathbb{R} \) and note that for large \( r \) we have

\[
\ln^+ (\text{diam}^2 (B(t-r))) \leq \ln \left( 2 \sqrt{1 + \frac{e^{-2\gamma_0 (t-r)}}{\alpha_0 - \gamma_0} \int_{-\infty}^{t} (\|d(\tau)\|^2 + \|g(\tau)\|^2) e^{2\gamma_0 \tau} d\tau} \right)
\]

\[
= \ln 2 + \frac{1}{2} \ln \left( e^{2\gamma_0 (t-r)} + \frac{1}{\alpha_0 - \gamma_0} \int_{-\infty}^{t} (\|d(\tau)\|^2 + \|g(\tau)\|^2) e^{2\gamma_0 \tau} d\tau \right) - \gamma_0 (t-r),
\]
Recall that (see [10, (4.5)])
\[
\eta
\]
Using (5) and the facts that \( \theta \) of the pullback global attractor in Theorem 3.5. Back to Bixiang Wang (see [23] or [24, Lemma 4.2]), will help to establish pullback Proposition 3.1 can be uniformly made as small as desired. This idea, which goes back to Bixiang Wang (see [23] or [24, Lemma 4.2]), will help to establish pullback B—asymptotic compactness of the process and in consequence prove the existence of the pullback global attractor in Theorem 3.5.

To this end, we will also require that for any \( \varepsilon > 0 \) there exists \( i_\varepsilon \in \mathbb{N} \) such that
\[
\sum_{|i| > i_\varepsilon} (d_i^2(t) + g_i^2(t)) \leq \varepsilon^2, \quad t \in \mathbb{R}. \quad (27)
\]

**Lemma 3.2.** Let the assumptions of Proposition 2.7 hold together with (22) and (27). Then for any \( \varepsilon > 0 \) there exists \( T_\varepsilon > 0 \) and \( k_\varepsilon \in \mathbb{N} \) such that for any \( t \in \mathbb{R} \), \( r \geq T_\varepsilon \), and \( u_0 \in B(t - r) \) we have
\[
\sum_{|i| > k_\varepsilon} (U(t, t - r)u_0)^2_i \leq \varepsilon^2 R^2(t). \quad (28)
\]

**Proof.** We consider standard cut-off sequences \( \eta_k \in \ell^\infty \), \( k \in \mathbb{N} \), where \( (\eta_k)_i = \theta \left( \frac{|i|}{k} \right), \quad i \in \mathbb{Z} \), with a \( C^1 \) function \( \theta: [0, \infty) \to [0, 1] \) such that \( \theta(s) = 0 \) for \( s \in [0, 1] \) and \( \theta(s) = 1 \) for \( s \geq 2 \) and \( \theta'(s) \leq 2 \) for \( s \geq 0 \) (cp. [10, Section 4]). We take the inner product in \( \ell^2 \) of the first equation in (20) with \( \eta_k^2 u \) to get
\[
\frac{1}{2} \frac{d}{dt} \|\eta_k u\|^2 + (Au, \eta_k^2 u) = (F(t, u) + g(t), \eta_k^2 u). \quad (29)
\]
Recall that (see [10, (4.5)])
\[
(Au, \eta_k^2 u) \geq (A(\eta_k u), \eta_k u) - \frac{20\nu}{k} \|u\|^2, \quad k \in \mathbb{N}, \quad u \in \ell^2. \quad (30)
\]
Using (5) and the facts that \( \eta_k^2 \leq \eta_k \) and \( (\eta_k)_i = 0 \) for \( |i| \leq k \) we get
\[
(F(t, u) + g(t), \eta_k^2 u) \leq (m\eta_k u, \eta_k u) + \sum_{|i| > k} (d_i(t) + |g_i(t)|)(\eta_k)_i |u_i|. \]

Let \( \beta_0 \in (\gamma_0, \alpha_0) \). Applying the Cauchy inequality to the last term yields
\[
(F(t, u) + g(t), \eta_k^2 u) \leq (m\eta_k u, \eta_k u) + (\alpha_0 - \beta_0) \|\eta_k u\|^2 + \frac{1}{2(\alpha_0 - \beta_0)} \sum_{|i| > k} (d_i^2(t) + g_i^2(t)). \quad (31)
\]
Combining (10), (29), (30) and (31), we obtain for any \( k \in \mathbb{N} \)
\[
\frac{d}{dt} \|\eta_k u\|^2 + 2\beta_0 \|\eta_k u\|^2 \leq \frac{40\nu}{k} \|u\|^2 + \frac{1}{\alpha_0 - \beta_0} \sum_{|i| > k} (d_i^2(t) + g_i^2(t)). \quad (32)
\]
We fix $\varepsilon > 0$ and choose $T_\varepsilon > 0$ such that
\[ e^{2(\gamma_0 - \beta_0)r} \leq \frac{\varepsilon^2}{2}, \quad r \geq T_\varepsilon \]
and, due to (27), $l_\varepsilon \in \mathbb{N}$ such that for $k \geq l_\varepsilon$

\[ \frac{40\nu}{k} \leq \frac{\beta_0 - \gamma_0}{2} \varepsilon^2 \quad \text{and} \quad \sum_{|i| > k} (d_i^2(t) + g_i^2(t)) \leq (\alpha_0 - \beta_0)\frac{\beta_0}{2} \varepsilon^2, \quad t \in \mathbb{R}. \]

Let $t \in \mathbb{R}$, $r \geq T_\varepsilon$ and $u_0 \in B(t - r)$. From (32) and (24) we obtain for $k \geq l_\varepsilon$

\[ \frac{d}{dt} \|\eta_k u\|^2 + 2\beta_0 \|\eta_k u\|^2 \leq \frac{\beta_0 - \gamma_0}{2} \varepsilon^2 R^2(\tau) + \frac{\beta_0}{2} \varepsilon^2, \quad \tau > t - r. \]

We multiply (33) by $e^{2\beta_0 r}$ and integrate from $t - r$ to $t$ and get for $k \geq l_\varepsilon$

\[ \|\eta_k u(t)\|^2 \leq R^2(t - r)e^{-2\beta_0 r} + \frac{\varepsilon^2}{2}(\beta_0 - \gamma_0) \int_{t - r}^{t} R^2(\tau)e^{-2\beta_0(t - \tau)}d\tau + \frac{\varepsilon^2}{4}. \]

Since
\[ R^2(\tau) \leq e^{2\gamma_0(t - \tau)} R^2(t) \quad \text{for} \quad \tau \in [t - r, t], \]
and $R(t) \geq 1$, we get for $k \geq l_\varepsilon$

\[ \|\eta_k u(t)\|^2 \leq \frac{\varepsilon^2}{2} R^2(t) + \frac{\varepsilon^2}{4} R^2(t) + \frac{\varepsilon^2}{4} R^2(t) = \varepsilon^2 R^2(t). \]

Since $(\eta_i)_i = 1$ for $|i| \geq 2k$, we obtain (28) with $k_\varepsilon = 2l_\varepsilon$. \qed

We recall the definition of a pullback global $\mathcal{D}$–attractor (see e.g. [8, Definition 2.46]).

**Definition 3.3.** Let $\{U(t, s) : t \geq s\}$ be a process on a metric space $(V, d)$ with a given universe $\mathcal{D}$ of families $\mathcal{D} = \{D(t) : t \in \mathbb{R}\}$ of nonempty subsets of $V$. By a pullback global $\mathcal{D}$–attractor for the process $\{U(t, s) : t \geq s\}$ we call a family $A = \{A(t) : t \in \mathbb{R}\}$ of nonempty compact subsets of $V$ such that

(i) $A$ is invariant under the process, that is, $U(t, s)A(s) = A(t)$ for $t \geq s$,

(ii) $A$ is pullback attracting all families from $\mathcal{D}$, i.e., for any $\mathcal{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}$ and $t \in \mathbb{R}$ we have

\[ \text{dist}^V(U(t, t - r)D(t - r), A(t)) = \sup_{x \in D(t - r)} \inf_{y \in A(t)} d(U(t, t - r)x, y) \to 0 \quad \text{as} \quad r \to \infty, \]

(iii) the family is minimal in the sense that if another family $\{C(t) : t \in \mathbb{R}\}$ of nonempty closed subsets of $V$ pullback attracts all families from $\mathcal{D}$, then $A(t) \subset C(t)$ for $t \in \mathbb{R}$.

For the completeness of the presentation we also quote a general result on the existence of pullback global $\mathcal{D}$–attractors from [14, Theorem 2.16, Corollary 2.17]; see also [5, Theorem 7] and [8, Theorem 2.50] for similar results.

**Theorem 3.4.** Let $\{U(t, s) : t \geq s\}$ be a process on a complete metric space $(V, d)$ with a given universe $\mathcal{D}$ of families $\mathcal{D} = \{D(t) : t \in \mathbb{R}\}$ of nonempty subsets of $V$. Let $\mathcal{B} = \{B(t) : t \in \mathbb{R}\} \in \mathcal{D}$ be a pullback $\mathcal{D}$–absorbing family of nonempty subsets of $V$, that is, for any $\mathcal{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}$ and $t \in \mathbb{R}$ there exists $r_{\mathcal{D}, t} \geq 0$ such that

\[ U(t, t - r)D(t - r) \subset B(t), \quad r \geq r_{\mathcal{D}, t}. \]
If the process is pullback $B$–asymptotically closed and pullback $B$–asymptotically compact, then there exists a pullback global $\mathcal{D}$-attractor $A = \{A(t) : t \in \mathbb{R}\}$ for the process given by

$$A(t) = \text{cl}_V \bigcup_{D \in \mathcal{D}} \omega^V(D, t) = \omega^V(B, t) \subset \text{cl}_V B(t), \ t \in \mathbb{R},$$

where

$$\omega^V(D, t) = \bigcap_{r \leq t} \text{cl}_V \bigcup_{s \leq r} U(t, s)D(s), \ t \in \mathbb{R},$$

denote sections of the pullback $\omega$–limit family for $D \in \mathcal{D}$.

We apply Theorem 3.4 to prove the existence of a pullback global $\mathcal{D}_u \cup \{B\}$–attractor for the process from Proposition 2.7.

**Theorem 3.5.** Under the assumptions of Proposition 2.7 and (22), (27), there exists a pullback global $\mathcal{D}_u \cup \{B\}$–attractor $A = \{A(t) : t \in \mathbb{R}\}$ for the process $\{U(t, s) : t \geq s\}$ on $\ell^2$ generated by (1), which is given by

$$A(t) = \text{cl}_{\ell^2} \bigcup_{D \in \mathcal{D}_u \cup \{B\}} \omega^{\ell^2}(D, t) = \omega^{\ell^2}(B, t) \subset B(t), \ t \in \mathbb{R}.$$

**Proof.** In view of Theorem 3.4 and Proposition 3.1 we need to check pullback $B$–asymptotic closedness and pullback $B$–asymptotic compactness of the process. The first notion means that for any $t > s$ and any sequences $r_n \geq 0$, $r_n \to \infty$ and $x_n \in B(s - r_n)$

$$\text{if } U(s, s - r_n)x_n \to v \text{ and } U(t, s - r_n)x_n \to w, \text{ then } U(t, s)v = w.$$ 

In our case it is an immediate consequence of (26), (24) and the fact that $B(t)$, $t \in \mathbb{R}$, are closed subsets of $\ell^2$.

To show that the process is also pullback $B$–asymptotically compact, we should verify that for any $t \in \mathbb{R}$ and any sequences $r_n \geq 0$, $r_n \to \infty$, and $x_n \in B(t - r_n)$ the sequence $U(t, t - r_n)x_n$ contains a convergent subsequence.

We fix $t \in \mathbb{R}$ and note that the sequence $U(t, t - r_n)x_n$ is bounded as contained in $B(t)$. By the reflexivity of $\ell^2$ there exists $y \in \ell^2$ such that some subsequence $U(t, t - r_{n_l})x_{n_l}$ converges weakly to $y$ in $\ell^2$. We are going to show that in fact the convergence is strong. To this end, we fix $\varepsilon > 0$ and using Lemma 3.2 let $T_\varepsilon > 0$ and $k_\varepsilon^1 \in \mathbb{N}$ be such that

$$\sum_{|i| \geq k_\varepsilon^1} (U(t, t - r)u^{0\{i\}}_i)^2 \leq \frac{\varepsilon^2}{8} R^2(t), \ r \geq T_\varepsilon, \ u^{0} \in B(t - r).$$

For large $l \in \mathbb{N}$ we have $r_{n_l} \geq T_\varepsilon$ and $U(t - T_\varepsilon, t - r_{n_l})x_{n_l} \in B(t - T_\varepsilon)$, so there exists $l_\varepsilon^1 \in \mathbb{N}$ such that

$$\sum_{|i| \geq k_\varepsilon^2} (U(t, t - r_{n_l})x_{n_l})^2 \leq \frac{\varepsilon^2}{8} R^2(t), \ l \geq l_\varepsilon^1.$$

Since $y \in \ell^2$, there exists $k_\varepsilon^2 \geq k_\varepsilon^1$ such that $\sum_{|i| > k_\varepsilon^2} y^2_i \leq \frac{\varepsilon^2}{8}$. By the weak convergence there exists $l_\varepsilon^2 \geq l_\varepsilon^1$ such that

$$\sum_{|i| \leq k_\varepsilon^2} (U(t, t - r_{n_l})x_{n_l})_i - y_i)^2 \leq \frac{\varepsilon^2}{2}, \ l \geq l_\varepsilon^2.$$
Thus for \( l \geq l^2 \) we have
\[
\|U(t, t - r_n)x_{n_1} - y\|^2 \leq \frac{\varepsilon^2}{2} + 2 \sum_{|i| > k^2} \left( (U(t, t - r_n)x_{n_1})^2 + y_i^2 \right) \leq \varepsilon^2 R^2(t).
\]

The existence of the pullback global \( \mathcal{D}_u \cup \{ B \} \)-attractor is thus a consequence of Theorem 3.4.

4. Pullback exponential attractors. We further study the asymptotic behavior of the problem (1) under the assumptions formulated in Section 2, i.e. the linear operator \( A \) from (7) satisfies (10), \( g \) is as in (14) and \( f_i \) fulfill (15), (16), (18) and (19). It follows, in particular, that \( d \) and \( g \) belong to \( \mathcal{G}_{\gamma_0} \), so there exists \( C_{d,g} > 0 \) such that
\[
\int_{-\infty}^{t} (\|d(\tau)\|^2 + \|g(\tau)\|^2)e^{-2\gamma(t-\tau)}d\tau \leq C_{d,g}, \quad t \in \mathbb{R}.
\]

Therefore, \( R(t) \) in (23) is bounded above by \( R_0 \) defined by \( R_0^2 = 1 + \frac{C_{d,g}}{\alpha_0 \gamma_0} \). Hence, the balls \( B(t) \) forming the pullback absorbing family \( B \) from Proposition 3.1 are subsets of the ball
\[
B_0 = \{ u \in \ell^2 : \|u\| \leq R_0 \}.
\]
This means that \( B \in \mathcal{D}_u \) and by Theorem 3.5 the process \{\( U(t,s) : t \geq s \)\} has the pullback global \( \mathcal{D}_u \)-attractor \( \mathcal{A} \).

Since we are going to prove the existence of a pullback exponential \( \mathcal{D}_u \)-attractor and indirectly estimate fractal dimension of the pullback global \( \mathcal{D}_u \)-attractor \( \mathcal{A} \), in this section we additionally assume that
\[
\sup_{|\sigma| \leq r} \sup_{i \in \mathbb{R}} \sup_{t \in \mathbb{R}} \left| \frac{\partial f_i}{\partial \sigma}(t, \sigma) - \frac{\partial f_i}{\partial \sigma}(t, 0) \right| \leq p(r), \quad r > 0,
\]
for some function \( p : (0, \infty) \to (0, \infty) \) satisfying
\[
p(r) \to 0 \text{ as } r \to 0^+
\]
and
\[
\frac{\partial f_i}{\partial \sigma}(t, 0) \leq q_i, \quad t \in \mathbb{R}, \quad i \in \mathbb{Z},
\]
for some sequence \( q = (q_i) \).

Lemma 4.1. For each \( \varepsilon > 0 \) there exists \( j_\varepsilon \in \mathbb{N} \) such that
\[
\sup_{|\sigma| \geq j_\varepsilon} \sup_{i \in \mathbb{R}} \left( \frac{\partial f_i}{\partial \sigma}(t, 0) - m_i \right) \leq \varepsilon.
\]

Proof. We define
\[
\tilde{f}_i(t, \sigma) = f_i(t, \sigma) - \frac{\partial f_i}{\partial \sigma}(t, 0)\sigma, \quad t \in \mathbb{R}, \quad \sigma \in \mathbb{R}, \quad i \in \mathbb{Z}.
\]
Then we have \( \tilde{f}_i(t, 0) = f_i(t, 0) = 0 \) and \( \frac{\partial \tilde{f}_i}{\partial \sigma}(t, \sigma) = \frac{\partial f_i}{\partial \sigma}(t, \sigma) - \frac{\partial f_i}{\partial \sigma}(t, 0) \). For any \( \sigma \neq 0 \) by (19) and the Mean Value Theorem we get
\[
\frac{\partial f_i}{\partial \sigma}(t, 0) - m_i = \frac{f_i(t, \sigma)\sigma - m_i\sigma^2}{\sigma^2} - \frac{\tilde{f}_i(t, \sigma) - \tilde{f}_i(t, 0)}{\sigma} \leq \frac{d_i(t)}{\sigma} - \left( \frac{\partial f_i}{\partial \sigma}(t, \theta_i\sigma) - \frac{\partial f_i}{\partial \sigma}(t, 0) \right)
\]
(39)
with some $\theta_i \in [0, 1]$. For a fixed $\varepsilon > 0$ by (35), (36) let $0 < \delta < \frac{\varepsilon}{2}$ be such that
\[
\left| \frac{\partial f_i}{\partial \sigma}(t, \sigma) - \frac{\partial f_i}{\partial \sigma}(t, 0) \right| \leq \frac{\varepsilon}{2} \quad \text{for} \quad |\sigma| < \delta, \ t \in \mathbb{R}, \ i \in \mathbb{Z}.
\]
Since $d \in \mathfrak{G}_{\gamma_0}$ (see (13)), there exists $j_\varepsilon \in \mathbb{N}$ such that $\sum_{|i| \geq j_\varepsilon} d_i^2(t) \leq \delta^4$ for all $t \in \mathbb{R}$.
In particular, we get $d_i(t) \leq \delta^2$ for $|i| \geq j_\varepsilon$ and $t \in \mathbb{R}$. Plugging $\sigma = \frac{\delta}{2}$ into (39) we obtain
\[
\left| \frac{\partial f_i}{\partial \sigma}(t, 0) - m_i \right| \leq 2\delta + \frac{\varepsilon}{2} < \varepsilon, \ |i| \geq j_\varepsilon, \ t \in \mathbb{R},
\]
which ends the proof. $\square$

In the proposition below we establish the main ingredients of quasi-stability of the evolution process $\{U(t, s) : t \geq s\}$ (see conditions (59) and (60) in Appendix A and compare [12, Definition 3.4.1] for the notion in the autonomous case).

**Proposition 4.2.** There exist constants $T > 0, N \in \mathbb{N}, \eta \in (0, e^{-\gamma_0T})$ and $C_0, C_1 > 0$ such that for any $t \in \mathbb{R}$ and any $w^0, v^0 \in B(t - T)$ for the function
\[
z_t(\tau) = U(\tau + t - T, t - T)v^0 - U(\tau + t - T, t - T)w^0, \ \tau \in [0, T],
\]
the following estimates hold:
\[
sup_{\tau \in [0, T]} \|z_t(\tau)\| \leq e^{C_0 T} \|z_t(0)\|, \quad (40)
\]
\[
\|z_t(T)\|^2 \leq \eta^2 \|z_t(0)\|^2 + C_1 \int_0^T \sum_{|i| \leq N} (z_t(\tau))^2 d\tau. \quad (41)
\]
\[
sup_{\tau \in [0, T]} \|z_t'(\tau)\| \leq (4\nu + \sup_{\tau \in [0, T]} P(\tau + t - T, R_0)) e^{C_0 T} \|z_t(0)\|. \quad (42)
\]

**Proof.** We fix $\beta_0 \in (\gamma_0, \alpha_0)$, where $\alpha_0 > 0$ is taken from (10) and $\gamma_0 < \alpha_0$ is used to define the class $\mathfrak{G}_{\gamma_0}$ in Definition 2.3. We set again $\bar{f}_i$ as in (38). By (35), (36) there exists $\delta > 0$ such that
\[
\left| \frac{\partial \bar{f}_i}{\partial \sigma}(t, \sigma) \right| \leq \frac{\alpha_0 - \beta_0}{2}, \ t \in \mathbb{R}, \ i \in \mathbb{Z}, \ |\sigma| \leq \delta. \quad (43)
\]
By Lemma 4.1 let $k_0 \in \mathbb{N}$ be such that
\[
\sup_{t \in \mathbb{R}} \left( \frac{\partial \bar{f}_i}{\partial \sigma}(t, 0) - m_i \right) \leq \frac{\alpha_0 - \beta_0}{2}, \ |i| > k_0. \quad (44)
\]
Since $R(t) \leq R_0$ for $t \in \mathbb{R}$, we choose $T_\delta > 0$ and $k_\delta \in \mathbb{N}$ from Lemma 3.2 and set $N = \max\{k_0, k_\delta\} \in \mathbb{N}$ to have
\[
(U(t, t - r)x^0)_i \leq \delta, \ |i| > N, \ t \in \mathbb{R}, \ r \geq T_\delta, \ x^0 \in B(t - r).
\]
Let $s \in \mathbb{R}$ and $w^0, v^0 \in B(s)$. Denoting $u(\tau) = U(\tau, s)v^0$ and $v(\tau) = U(\tau, s)v^0$, we have
\[
|u_i(\tau)| \leq \delta \quad \text{and} \quad |v_i(\tau)| \leq \delta \quad \text{for} \ \tau \geq s + T_\delta, \ |i| > N. \quad (45)
\]
We consider $w(\tau) = u(\tau) - v(\tau), \ \tau \geq s$. Combining (20), (44) and (37) we get
\[
\frac{1}{2} \frac{d}{d\tau} \|w\|^2 + (Aw, w) - (mw, w) = \sum_{i \in \mathbb{Z}} (f_i(\tau, u_i) - f_i(\tau, v_i))w_i - \sum_{i \in \mathbb{Z}} m_i w_i^2
\]
\[
= \sum_{i \in \mathbb{Z}} \left( \frac{\partial \bar{f}_i}{\partial \sigma}(\tau, 0) - m_i \right) w_i^2 + \sum_{i \in \mathbb{Z}} (\bar{f}_i(\tau, u_i) - \bar{f}_i(\tau, v_i))w_i
\]
\[ \leq \max_{|i| \leq N} (q_i - m_i) \sum_{|i| \leq N} w_i^2 + \frac{\alpha_0 - \beta_0}{2} \|w\|^2 + \sum_{i \in \mathbb{Z}} (\bar{f}_i(\tau, u_i) - \bar{f}_i(\tau, v_i))w_i, \tau > s. \]

Thus using (10) and the Mean Value Theorem we obtain with some \( \theta_i \in [0, 1] \)
\[ \frac{1}{2} \frac{d}{d\tau} \|w\|^2 + \frac{\alpha_0}{2} \|w\|^2 \leq \max_{|i| \leq N} (q_i - m_i) \sum_{|i| \leq N} w_i^2 - \frac{\beta_0}{2} \|w\|^2 + \sum_{i \in \mathbb{Z}} \left| \frac{\partial \bar{f}_i}{\partial \sigma} (\tau, \theta_iu_i + (1 - \theta_i)v_i) \right| w_i^2, \tau > s. \]  

(46)

Since \( u(\tau), v(\tau) \in B(\tau) \subset B_0 = \{ x^0 \in \ell^2 : \|x^0\| \leq R_0 \} \), we first apply (35) to (46) and get
\[ \frac{d}{d\tau} \|w\|^2 \leq 2C_0 \|w\|^2, \tau > s, \]
with \( C_0 = \max_{|i| \leq N} |q_i - m_i| + p(R_0) \), which then yields
\[ \|w(\tau)\| \leq e^{C_0(\tau-s)} \|w(s)\|, \tau \geq s. \]

(47)

We now apply (35), (43) and (45) to (46) and get
\[ \frac{d}{d\tau} \|w\|^2 + 2\beta_0 \|w\|^2 \leq C_1 \sum_{|i| \leq N} w_i^2, \tau \geq s + T_\delta. \]

(48)

with \( C_1 = 2 \left( \max_{|i| \leq N} |q_i - m_i| + p(R_0) \right) \). Integrating (48) on \([s + T_\delta, t]\) and using (47) we obtain
\[ \|w(t)\|^2 \leq \|w(s)\|^2 e^{2(C_0T_\delta - \beta_0(t-s-T_\delta))} + C_1 \int_{s + T_\delta}^t \sum_{|i| \leq N} w_i^2(\tau)d\tau, \tau \geq s + T_\delta. \]

(49)

We define
\[ T = \left( \frac{C_0 + \beta_0}{\beta_0 - \gamma_0} + 1 \right) T_\delta > T_\delta, \]

(50)

and fix \( t \in \mathbb{R} \). Note that (40) follows directly from (47). Choosing \( s = t - T \) in (49) we get
\[ \|w(t)\|^2 \leq \eta^2 \|w(t-T)\|^2 + C_1 \int_{t-T}^t \sum_{|i| \leq N} w_i^2(\tau)d\tau \]

(51)

with \( \eta = e^{C_0T_\delta - \beta_0(T-T_\delta)} > 0 \). Note that \( \eta < e^{-\gamma_0T} \) by (50) and (51) implies (41).

From (20) using (8), (17) and (47) we obtain
\[ \|w'(\tau)\| \leq \|Aw(\tau)\| + \|F(\tau, u) - F(\tau, v)\| \leq (4\nu + P(\tau, R_0)) \|w(\tau)\| \]
\[ \leq (4\nu + P(\tau, R_0)) e^{C_0(\tau-s)} \|w(s)\|, \tau \geq s, \]
which gives (42). \( \square \)

We rewrite the results of Proposition 4.2 in the form of the quasi-stability of the process. We consider \( Z = C^1([0, T]; \ell^2) \) with the norm
\[ \|z\|_Z = \sup_{\tau \in [0, T]} \|z(\tau)\| + \sup_{\tau \in [0, T]} \|z'(\tau)\|, z \in Z, \]
and with \( \mu = \sqrt{C_1} \) we also define a seminorm on \( Z \)
\[
\mathfrak{n}_Z(z) = \mu \left( \int_0^T \sum_{|i| \leq N} (z_i(\tau))^2 d\tau \right)^{\frac{1}{2}}, \quad z \in Z.
\]

Note that the seminorm \( \mathfrak{n}_Z \) is compact on \( Z \) by the Arzelà-Ascoli Theorem.

We define operators \( K(t) : B(t - T) \to Z, \ t \in \mathbb{R}, \) by
\[
K(t)u^0 = U(\cdot + t - T, t - T)u^0 \in Z, \ t \in \mathbb{R}, \ u_0 \in B(t - T).
\]
Then for \( u^0, v^0 \in B(t - T) \) (41) implies
\[
\|U(t, t - T)u^0 - U(t, t - T)v^0\| \leq \eta \|u^0 - v^0\| + \mathfrak{n}_Z(K(t)u^0 - K(t)v^0),
\]
whereas (40) and (42) give
\[
\|K(t)u^0 - K(t)v^0\|_Z \leq \kappa(t) \|u^0 - v^0\|
\]
with
\[
\kappa(t) = (4\nu + 1 + \sup_{\tau \in [0, T]} P(\tau + t - T, R_0)) e^{C_0T}.
\]

Remark 4.3. If we knew in advance that the function \( P \) in (16) was time independent as e.g. for the nonlinear part considered in Proposition B.1 (cp. also [3, (27)]), then \( \kappa \) in (54) would be constant \( \kappa(t) \equiv \kappa \) and in consequence condition (55) below, which goes back to Caraballo and Sonner (see [7, (2)]), would be trivially satisfied with \( H_\ell = \ln m_Z(2\kappa\delta^{-1}) \). Hence the estimate of the fractal dimension of the pullback exponential \( \mathcal{D}_u \)-attractor and, in consequence of the pullback global \( \mathcal{D}_u \)-attractor from Theorem 3.5, would be time independent.

We now apply Theorem A.4, summarize our results and obtain the existence of a pullback exponential \( \mathcal{D}_u \)-attractor.

Theorem 4.4. We consider (1) under the assumptions of Section 2 and (35), (36) and (37). Then the process \( \{U(t, s) : t \geq s\} \) on \( \ell^2 \) from Proposition 2.7 is pullback dissipative for the universe \( \mathcal{D}_u \) from Definition 2.2 with a pullback absorbing family \( \mathcal{B} = \{B(t) : t \in \mathbb{R}\} \) given in Proposition 3.1 of subsets of the ball \( B_0 \) defined in (34). The process has a pullback global \( \mathcal{D}_u \)-attractor \( \mathcal{A} = \{A(t) : t \in \mathbb{R}\} \) by Theorem 3.5 with
\[
A(t) = \omega^{\mathcal{B}}(B(t) \subset B(t), \ t \in \mathbb{R}.
\]
The process is quasi-stable on \( \mathcal{B} \) so that (52) and (53) hold in \( Z = C^1([0,T];\ell^2) \) with some \( T > 0, \eta \in (0, e^{-\gamma_0 T}) \) and a function \( \kappa \) defined in (54).

We also assume that for some \( \delta \in (0, e^{-\gamma_0 T} - \eta) \)
\[
H_\ell = \limsup_{k \to \infty} \frac{1}{k} \sum_{j=0}^{k-1} \ln m_Z(2\kappa(t - jT)\delta^{-1}) < \infty, \ t \in \mathbb{R},
\]
where \( m_Z(2\kappa(s)\delta^{-1}) \) denotes the maximal number of \( z_j \) in the closed unit ball \( B^Z(0,1) \) in \( Z \) having the property that \( \mathfrak{n}_Z(z_j - z_l) \geq \frac{\delta}{2\kappa(s)} \) for \( j \neq l \).

Then the process has a pullback exponential \( \mathcal{D}_u \)-attractor \( \mathcal{M} = \{M(t) : t \in \mathbb{R}\} \) in \( \ell^2 \) such that
(a) \( M(t) \) is a nonempty compact subset of \( B(t) \) for \( t \in \mathbb{R}, \)
(b) \( U(t, s)M(s) \subset M(t), \ t \geq s, \)
Moreover, such that
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Moreover, the process has a pullback exponential attractor
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Moreover, the process has a pullback exponential attractor
\[
M(t) = A(t) \cup E(t) = \text{cl}_\ell E(t) \subset B(t), \ t \in \mathbb{R},
\]
where \( E = \{ E(t) : t \in \mathbb{R} \} \) is a family of nonempty countable subsets of \( \ell^2 \).

We apply this result in a particular case of nonlinearities considered in Proposition B.1.

**Corollary 4.5.** The problem (1) with \( f_i \) from Proposition B.1 and \( g = (g_i) \) satisfying (14) generates a quasi-stable evolution process \( \{ U(t, s) : t \geq s \} \) on \( \ell^2 \) with a pullback \( D_u \)-absorbing family \( B = \{ B(t) : t \in \mathbb{R} \} \in D_u \) for the universe \( D_u \) from Definition 2.2.

The process has a pullback global \( D_u \)-attractor \( \mathcal{A} = \{ A(t) : t \in \mathbb{R} \} \) with
\[
A(t) = \omega^\ell (B, t) \subset B(t), \ t \in \mathbb{R}.
\]
Moreover, the process has a pullback exponential \( D_u \)-attractor \( \mathcal{M} = \{ M(t) : t \in \mathbb{R} \} \) in \( \ell^2 \), which exponentially pullback attracts all families from \( D_u \) and has uniformly bounded fractal dimension in \( \ell^2 \) of its sections
\[
\dim_f^\ell (M(t)) \leq \chi, \ t \in \mathbb{R},
\]
where \( \chi \) is expressed in terms of \( \gamma_0 \) from (14), constants \( T, \eta \) and \( \kappa \) and the compact seminorm \( n_Z \) on \( Z = C^1([0, T]; \ell^2) \) from the quasi-stability of the process.

Furthermore, \( \mathcal{M} \) has the following structure
\[
M(t) = A(t) \cup E(t) = \text{cl}_\ell E(t) \subset B(t), \ t \in \mathbb{R},
\]
where \( E = \{ E(t) : t \in \mathbb{R} \} \) is a family of nonempty countable subsets of \( \ell^2 \).

We also note that for functions \( f_i \) from (4) the pullback attractors have zero-dimensional sections, since condition (56) below then holds with \( m_i = -a_i, < 0, \ i \in \mathbb{Z} \).

**Corollary 4.6.** Under the assumptions (10), (14), (15), (16) and (18), if
\[
\frac{\partial f_i}{\partial \sigma}(t, \sigma) \leq m_i, \ i \in \mathbb{Z}, \ t, \sigma \in \mathbb{R},
\]
where \( m = (m_i) \in \ell^\infty \) satisfies (10), then there exists a pullback exponential \( D_u \)-attractor \( \mathcal{M} = \{ M(t) : t \in \mathbb{R} \} \) in \( \ell^2 \) for the process generated by (1) on \( \ell^2 \) such that
\[
\dim_f^\ell (A(t)) = \dim_f^\ell (M(t)) = 0, \ t \in \mathbb{R},
\]
where \( \mathcal{A} = \{ A(t) : t \in \mathbb{R} \} \) is the pullback global \( D_u \)-attractor in \( \ell^2 \) from Theorem 3.5.
Proof. First note that (19) holds with $d_i(t) \equiv 0$, since by (15) and (56) we have for some $\theta_i \in [0, 1]$

$$f_i(t, \sigma) \sigma = \left( f_i(t, \sigma) - f_i(t, 0) \right) \sigma = \frac{\partial f_i}{\partial \sigma} (t, \theta_i \sigma) \sigma^2 \leq m_i \sigma^2; \quad i \in \mathbb{Z}, \quad t, \sigma \in \mathbb{R}.$$ 

We consider the difference $w = u - v$ of solutions of (20) starting from $u^0, v^0 \in B(s)$. Using (56) we obtain from (20) for some $\tilde{\theta}_i \in [0, 1]$

$$\frac{1}{2} \frac{d}{dt} \|w\|^2 + (Aw, w) = \sum_{i \in \mathbb{Z}} \frac{\partial f_i}{\partial \sigma} (t, \tilde{\theta}_i u_i + (1 - \tilde{\theta}_i) v_i) w_i^2 \leq (mw, w).$$ 

From (10) we get

$$\frac{d}{dt} \|w\|^2 + 2\alpha_0 \|w\|^2 \leq 0, \quad t > s,$$

which gives

$$\|U(t, s)u^0 - U(t, s)v^0\| \leq \|u^0 - v^0\| e^{-\alpha_0(t-s)}, \quad t \geq s.$$ 

Then for any fixed $T > 0$ we have with $\eta = e^{-\alpha_0 T} \in (0, e^{-\gamma_0 T})$

$$\|U(t, t-T) u^0 - U(t, t-T) v^0\| \leq \eta \|u^0 - v^0\|, \quad u^0, v^0 \in B(t-T).$$ 

Taking $Z = \{0\}$ and $K(t)u^0 = 0$ for $t \in \mathbb{R}, u^0 \in B(t-T)$, we see that (59) and (60) hold with any $\kappa > 0$. Then by Theorem A.5 for any $\delta \in (0, e^{-\gamma_0 T} - \eta)$ there exists a pullback exponential $\mathcal{D}_u$–attractor $M = \{M(t) : t \in \mathbb{R}\}$ in $\ell^2$ such that

$$\text{dim}_{\ell^2}^f (M(t)) \leq \frac{\ln m_Z (2\kappa \delta^{-1})}{T \xi_T} = 0, \quad t \in \mathbb{R},$$

where $\xi_T = -\frac{1}{T} (\ln (\eta + \delta) + \gamma_0 T) > 0$, since $m_Z (2\kappa \delta^{-1}) = 1$. Thus the pullback global $\mathcal{D}_u$–attractor $A = \{\omega^z(\mathcal{B}, t) : t \in \mathbb{R}\}$ from Theorem 3.5 also has zero-dimensional sections, since $A(t) \subset M(t)$ and $\text{dim}_{\ell^2}^f (A(t)) \leq \text{dim}_{\ell^2}^f (M(t))$ for $t \in \mathbb{R}$. 

\qed

Appendix A. Construction of pullback exponential attractors. For the sake of completeness of the presentation we formulate here the main tools to construct a pullback exponential attractor via quasi-stability of the process, a notion adapted from its autonomous counterpart introduced by I. Chueshov (see [12, Definition 3.4.1]).

Recall that a nonempty set $W$ with a function $\rho: W \times W \to [0, \infty)$, which is symmetric, satisfies the triangle inequality and $\rho(x, x) = 0$ for $x \in W$, is called a pseudometric space. A nonempty subset $A$ of $(W, \rho)$ is precompact in $W$ if each sequence of elements of $A$ contains a Cauchy subsequence with respect to $\rho$. Equivalently this means that $A$ is totally bounded, i.e., for any $\varepsilon > 0$ there is a finite cover of $A$ by open $\varepsilon$–balls centered at points from $A$. The minimal number of such balls is denoted by $N^W_\rho (A, \varepsilon)$. By $\tilde{N}^W_\rho (A, \varepsilon)$ we denote the minimal number of subsets of $A$ with diameter no larger than $2\varepsilon$ necessary to cover $A$. Observe that

$$N^W_\rho (A, 3\varepsilon) \leq \tilde{N}^W_\rho (A, \varepsilon) \leq N^W_\rho (A, \varepsilon).$$

Moreover, given $\varepsilon > 0$, we say that a subset $U$ of $W$ is $\varepsilon$–distinguishable in $(W, \rho)$ if

$$\rho(x, y) \geq \varepsilon, \quad x, y \in U, \quad x \neq y.$$ 

Then a nonempty set $A$ is precompact in $(W, \rho)$ if and only if for any $\varepsilon > 0$ the cardinality of each $\varepsilon$–distinguishable subset of $A$ is finite and these cardinalities
are bounded by a finite number. In this case, we denote by $m^W_\rho(A,\varepsilon)$ the maximal cardinality of an $\varepsilon$-distinguishable subset of $A$. We omit the pseudometric in these numbers if it follows from the context.

We are going to construct precompact subsets $A$ of a metric space $(V,d)$ and estimate their fractal dimension defined by

$$\dim^V(A) = \limsup_{\varepsilon \to 0^+} \log_\varepsilon \mathcal{N}^V(A,\varepsilon),$$

where in the role of $\mathcal{N}^V$ one can equivalently take $N^V$ or $\hat{N}^V$.

For this purpose we will use the following fundamental result - for the proof see [13, p. 25].

**Lemma A.1.** Let $A$ be a nonempty subset of a metric space $(V,d)$ and assume that there is a pseudometric $\rho$ on $A$ such that $(A,\rho)$ is a precompact pseudometric space. Suppose that for a map $S: A \to V$ there exists $\eta \geq 0$ such that

$$d(S(x),S(y)) \leq \eta d(x,y) + \rho(x,y), \quad x,y \in A.$$ 

If $A$ can be covered by a finite number of subsets of $A$ with diameter in $(V,d)$ no larger than $2\varepsilon$ for some $\varepsilon > 0$, then for any $\delta > 0$ we have

$$\hat{N}^V(S(A),(\eta + \delta)\varepsilon) \leq \hat{N}^V(A,\varepsilon)e^{C_\rho(A,\varepsilon,\delta\varepsilon)},$$

where

$$C_\rho(A,\varepsilon,\mu) := \sup\{\ln m^A_\rho(F,\mu) : \emptyset \neq F \subset A, \text{ diam}^V(F) \leq 2\varepsilon\} \leq \ln m^A_\rho(A,\mu). \tag{58}$$

We now formulate the standing assumptions of this section.

Let $\{U(t,s) : t \geq s\}$ be a process on a metric space $(V,d)$, that is, a family of operators $U(t,s) : V \to V$, $t \geq s$, $t,s \in \mathbb{R}$, satisfying

$$U(t,s)U(s,r) = U(t,r), \quad t \geq s \geq r, \quad \text{and} \quad U(t,t) = \text{id}, \quad t,s,r \in \mathbb{R},$$

where id denotes the identity operator on $V$. We distinguish a universe $\mathcal{D}$ of families $\mathcal{D} = \{D(t) : t \in \mathbb{R}\}$ of nonempty subsets of $V$ and let $\mathcal{B} = \{B(t) : t \in \mathbb{R}\}$ denote a pullback $\mathcal{D}$-absorbing family of nonempty bounded subsets of $V$ such that

$(H_1)$ $\mathcal{B}$ is positively invariant under the process, i.e.,

$$U(t,s)B(s) \subset B(t), \quad t \geq s,$$

$(H_2)$ there exists $\gamma_0 \geq 0$ such that for any $t \in \mathbb{R}$,

$$\limsup_{r \to \infty} \frac{1}{r} \ln^+(\text{diam}^V(B(t-r))) \leq \gamma_0,$$

$(H_3)$ for every family $\mathcal{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}$ and $t \in \mathbb{R}$ there exists $r_{\mathcal{D},t} \geq 0$ such that

$$U(t,t-r)D(t-r) \subset B(t), \quad r \geq r_{\mathcal{D},t},$$

and, additionally, the function $\mathbb{R} \ni t \mapsto r_{\mathcal{D},t} \in [0,\infty)$ is non-decreasing for every $\mathcal{D} \in \mathcal{D}$.

Thus, defining $r_{\mathcal{B},t} \equiv 0$ if necessary, for any $\mathcal{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D} \cup \{\mathcal{B}\}$ we have

$$U(s,s-r)D(s-r) \subset B(s), \quad s \leq t, \quad r \geq r_{\mathcal{D},t}.$$
Assume that the process is quasi-stable on $B$ in the sense that there exist constants $T > 0$, $\eta \in [0, e^{-\kappa T})$, a function $\kappa : \mathbb{R} \to (0, \infty)$ and maps $K(t) : B(t - T) \to Z$, $t \in \mathbb{R}$, into some auxiliary normed space $Z$ such that for any $t \in \mathbb{R}$,

$$\|K(t)x - K(t)y\|_Z \leq \kappa(t)d(x, y), \quad x, y \in B(t - T),$$

(59)

$$d(U(t, t - T)x, U(t, t - T)y) \leq \eta d(x,y) + n_Z(K(t)x - K(t)y), \quad x, y \in B(t - T),$$

(60)

hold, where $n_Z$ is a compact seminorm on $Z$.

Recall that $n_Z$ is a seminorm on $Z$ if

$$n_Z(x + y) \leq n_Z(x) + n_Z(y), \quad n_Z(\lambda x) = |\lambda| n_Z(x), \quad x, y \in Z, \quad \lambda \in \mathbb{R},$$

and its compactness means that for any bounded sequence $z_n \in Z$ there exists a Cauchy subsequence $z_{n_k}$ with respect to $n_Z$, that is, $n_Z(z_{n_k} - z_{n_l}) \to 0$ as $j, l \to \infty$.

First we show that the quasi-stability of the process implies the covering estimate (62), which is crucial to estimate from above the fractal dimension of constructed pullback exponential attractor (see Proposition A.3 below).

**Proposition A.2.** Setting

$$R_t = \max\{\text{diam}^V(B(t)), 1\}, \quad t \in \mathbb{R},$$

(61)

we have for any $\delta > 0$ and any $t \in \mathbb{R}$

$$N^V(U(t, t - kT)B(t - kT), \frac{3}{2}(\eta + \delta)^k R_{t-kT}) \leq \prod_{j=0}^{k-1} m_Z(2\kappa(t - jT)\delta^{-1}), \quad k \in \mathbb{N},$$

(62)

where $m_Z(2\kappa(s)\delta^{-1})$ denotes the maximal number of $z_j$ in the closed unit ball $B^Z(0, 1)$ in $Z$ having the property that $n_Z(z_j - z_l) \geq \frac{\delta}{2\kappa(s)}$ for $j \neq l$.

**Proof.** Given $t \in \mathbb{R}$, note that $B(t - T)$ is a precompact pseudometric space with

$$\rho_t(x, y) = n_Z(K(t)x - K(t)y), \quad x, y \in B(t - T),$$

since $n_Z$ is compact and (59) holds. For $\delta > 0$ we estimate from above the quantity

$$\zeta_{\rho_t}(B(t - T), \delta) = \sup_{\varepsilon > 0} C_{\rho_t}(B(t - T), \varepsilon, \delta\varepsilon),$$

where $C_{\rho_t}(\cdot, \cdot, \cdot)$ is defined in (58), in order to apply Lemma A.1.

We fix $\varepsilon > 0$ and $\emptyset \neq F \subset B(t - T)$ with $\text{diam}^V(F) \leq 2\varepsilon$. Setting $m_F = m_{\rho_t(t - T)}(F, \delta\varepsilon)$, let $\{x_1, \ldots, x_{m_F}\}$ be the maximal $\delta\varepsilon$-distinguishable subset of $F$ in $(B(t - T), \rho_t)$. We define points $z_j = K(t)x_j \in Z, j = 1, \ldots, m_F$, and see that

$$n_Z(z_j - z_l) \geq \delta\varepsilon \quad \text{for} \quad 1 \leq j, l \leq m_F, j \neq l.$$ 

Also, due to (59), we obtain

$$\|z_j - z_l\|_Z \leq \kappa(t) \text{diam}^V(F) \leq 2\varepsilon \kappa(t), \quad 1 \leq j, l \leq m_F.$$ 

We choose now an arbitrary point $z_j$, denoting it by $z_0$, and note that

$$\frac{1}{2\varepsilon \kappa(t)}(z_j - z_0) \in \overline{B^Z}(0, 1), \quad 1 \leq j \leq m_F,$$

and

$$n_Z\left(\frac{1}{2\varepsilon \kappa(t)}(z_j - z_0) - \frac{1}{2\varepsilon \kappa(t)}(z_l - z_0)\right) \geq \frac{\delta}{2\varepsilon \kappa(t)} \quad \text{for} \quad 1 \leq j, l \leq m_F, j \neq l.$$
By the compactness of \( n_Z \), we see that \( \overline{B}^Z(0,1) \) is a precompact subset of \((Z, \zeta)\) with the pseudometric \( \zeta(x,y) = n_Z(x-y) \) for \( x, y \in Z \). Thus \( m_F \) is bounded from above by \( m_\zeta \left( \overline{B}^Z(0,1), \frac{\delta}{2e^{n_\zeta(t)}} \right) = m_Z \left( 2\kappa(t)\delta^{-1} \right) \), which leads to the estimate

\[
\mathcal{S}_{\rho_1}(B(t-T), \delta) \leq \ln m_Z \left( 2\kappa(t)\delta^{-1} \right).
\]

For any nonempty \( A \subset B(t-T) \) and \( \varepsilon > 0 \) we thus have

\[
C_{\rho_1}(A, \varepsilon, \delta \varepsilon) \leq \mathcal{S}_{\rho_1}(B(t-T), \delta) \leq \ln m_Z \left( 2\kappa(t)\delta^{-1} \right).
\]

We apply Lemma A.1 with \( S = U(t, t-T) \), \( A = B(t-T) \) and \( \varepsilon = \frac{1}{2} R_{t-T} \) to get

\[
\tilde{N}^V(U(t, t-T)B(t-T), \frac{1}{2}(\eta + \delta)R_{t-T}) \leq m_Z \left( 2\kappa(t)\delta^{-1} \right), \quad t \in \mathbb{R},
\]

since \( \tilde{N}^V(B(t-T), \frac{1}{2} R_{t-T}) = 1 \).

Using Lemma A.1 again, we obtain by induction

\[
\tilde{N}^V(U(t, t-kT)B(t-kT), \frac{1}{2}(\eta + \delta)kR_{t-kT}) \leq \prod_{j=0}^{k-1} m_Z \left( 2\kappa(t-jT)\delta^{-1} \right), \quad k \in \mathbb{N}, \quad t \in \mathbb{R}.
\]

Then (62) follows from (57).

We use (62) to construct finite-dimensional family of precompact subsets of \( V \), which attracts \( B \) and families from \( \mathcal{D} \) exponentially fast.

**Proposition A.3.** Define \( R_t \) as in (61) and let

\[
N^V(U(t, t-kT)B(t-kT), aq^k R_{t-kT}) \leq b \prod_{j=0}^{k-1} h_{t-jT}, \quad k \in \mathbb{N}, \quad t \in \mathbb{R}, \quad (63)
\]

hold with some \( T > 0, q \in (0, e^{-\gamma T}), a, b > 0 \) and \( h_t \in [1, \infty) \) for \( t \in \mathbb{R} \). If

\[
H_t = \limsup_{k \to \infty} \frac{1}{k} \sum_{j=0}^{k-1} \ln(h_{t-jT}) < \infty, \quad t \in \mathbb{R},
\]

then there exists a family of sets \( \mathcal{E} = \{E(t) : t \in \mathbb{R}\} \) such that

(i) \( E(t) \subset B(t) \) and \( E(t) \) is precompact in \( V \) for \( t \in \mathbb{R} \),

(ii) \( U(t, t-T)E(t-T) \subset E(t) \) for \( t \in \mathbb{R} \),

(iii) \( E(t) \) is countable, \( E(t) = \bigcup_{k \in \mathbb{N}} Q_k(t) \) with each \( Q_k(t) \subset U(t, t-kT)B(t-kT) \)

and finite and

\[
\dim^V(E(t)) \leq \frac{H_t}{\ln(qe^{-\gamma T})}, \quad t \in \mathbb{R}, \quad (65)
\]

(iv) for any \( \mathcal{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D} \cup \{B\} \) and \( t \in \mathbb{R} \) we have

\[
\lim_{r \to \infty} e^{\xi r} \text{dist}^V(U(t, t-r)D(t-r), E(t)) = 0,
\]

with any \( \xi \in (0, \xi_T) \), where \( \xi_T = -\frac{1}{T} \ln(qe^{-\gamma T}) > 0 \).

**Proof.** Since the construction is based on the ideas from [16], [9], [15] and [7], we only present the main steps of the proof in our setting. We denote by \( W_k(t), t \in \mathbb{R}, \quad k \in \mathbb{N} \), the centers of balls from the cover existing by (63) such that

\[
(w_1) \quad W_k(t) \subset U(t, t-kT)B(t-kT) \subset B(t),
\]

\[
(w_2) \quad \#W_k(t) \leq b \prod_{j=0}^{k-1} h_{t-jT},
\]

and

\[
(w_3) \quad W_k(t) \subset U(t, t-kT)B(t-kT) \subset B(t).
\]
\((w_3)\) \(U(t, t - kT)B(t - kT) \subset \bigcup_{w \in W_k(t)} B^V(u, aq^k R_{t-kT}).\)

We define \(Q_1(t) = W_1(t), t \in \mathbb{R}, \) and set

\[ Q_k(t) := W_k(t) \cup U(t, t-T)Q_{k-1}(t-T), t \in \mathbb{R}, k > 1. \]

Then using \((w_1)-(w_3)\) it follows that the sets \(Q_k(t)\) satisfy for all \(t \in \mathbb{R} \) and \(k \in \mathbb{N}\)

\((q_1)\) \(U(t, t-T)Q_{k-1}(t-T) \subset Q_{k+1}(t), \quad Q_k(t) \subset U(t, t-kT)B(t-kT) \subset B(t),\)

\((q_2)\) \(Q_k(t) = \bigcup_{l=0}^{k-1} U(t, t-lT)W_{k-l}(t-lT), \quad \#Q_k(t) \leq b \sum_{m=0}^{\infty} \prod_{j=m}^{k-1} h_{t-jT}.\)

We now define

\[ E(t) = \bigcup_{k \in \mathbb{N}} Q_k(t), t \in \mathbb{R}. \]

Fix \(t \in \mathbb{R} \) and observe that

\[ E(t) = \bigcup_{k \in \mathbb{N}} \bigcup_{l=0}^{k-1} U(t, t-lT)W_{k-l}(t-lT) = \bigcup_{l=0}^{\infty} \bigcup_{m=1}^{\infty} U(t, t-lT)W_m(t-lT). \]

By \((q_1)\) the set \(E(t)\) is a nonempty subset of \(B(t)\) and we have \(U(t, t-T)E(t-T) \subset E(t).\) Note that by \((q_1)\) and \((H_1)\) we get \(Q_1(t) \subset U(t, t-kT)B(t-kT)\) for any \(l \geq k.\) Consequently, we obtain

\[ E(t) \subset \bigcup_{l=1}^{k} Q_l(t) \cup U(t, t-kT)B(t-kT), k \in \mathbb{N}. \]

Let \(\gamma_0 < \gamma < \frac{1}{T} \ln \frac{1}{q}.\) By \((H_2)\) there exists \(k_0 = k_0(\gamma, t) \in \mathbb{N}\) such that \(R_{t-kT} \leq e^{\gamma_{kT}}\) for \(k \geq k_0.\) Note that the sequence \(k \mapsto a(qe^{\gamma T})^k\) is strictly decreasing to 0, since we have \(ge^{\gamma T} < 1.\) Using \((w_2), (w_3)\) and \((q_2)\) we estimate

\[ N^V(E(t), a(qe^{\gamma T})^k) \leq \# \left( \bigcup_{l=1}^{k} Q_l(t) \right) + \#W_k(t) \leq 2bk^2 \prod_{j=0}^{k-1} h_{t-jT}, \quad k \geq k_0. \quad (67)\]

Thus the set \(E(t)\) is precompact in \(V.\) Consider any sequence \(\varepsilon_n > 0, n \in \mathbb{N},\) convergent to 0 and choose \(k_n \in \mathbb{N}, n \in \mathbb{N}\) such that

\[ k_n \geq k_0 \quad \text{and} \quad a(qe^{\gamma T})^{k_n} \leq \varepsilon_n < a(qe^{\gamma T})^{k_n-1} < 1 \quad \text{for large} \quad n. \]

Due to (67) and (64) we get

\[ \limsup_{n \to \infty} \log \frac{1}{\varepsilon_n} N^V(E(t), \varepsilon_n) \leq \frac{1}{-\ln(qe^{\gamma T})} \limsup_{n \to \infty} \frac{1}{k_n} \sum_{j=0}^{k_n-1} \ln(h_{t-jT}) \leq \frac{H_t}{\ln(\frac{1}{qe^{\gamma T})}} \]

and passing to the limit with \(\gamma\) to \(\gamma_0\) we obtain (65).

By \((w_3)\) we have

\[ \text{dist}^V(U(t, t-kT)B(t-kT), E(t)) \leq aq^k R_{t-kT}, \quad t \in \mathbb{R}, \quad k \in \mathbb{N}. \]

For a fixed \(0 < \xi < \xi_T = -\frac{T}{T} \ln(qe^{\gamma_{kT}})\) we choose \(\gamma_0 < \gamma < \frac{1}{T} \ln(\frac{1}{qe^{\gamma_{kT}}})\) and by \((H_2)\) we see that \(R_{t-kT} \leq e^{\gamma_{kT}}\) for large \(k.\) Then we get

\[ e^{Tt}aq^k R_{t-kT} \leq ae^{(\xi T + \gamma T + \ln q)k} \to 0 \quad \text{as} \quad k \to \infty, \]

and in consequence for any \(0 < \xi < \xi_T\) and \(t \in \mathbb{R}\)

\[ e^{Tt} \text{dist}^V(U(t, t-kT)B(t-kT), E(t)) \to 0 \quad \text{as} \quad k \to \infty. \]
Then (66) follows from \((H_1)\) and \((H_3)\). \(\square\)

In the first result on the existence of pullback exponential attractors we do not assume the completeness of the space \(V\) nor the closedness of \(B(t), t \in \mathbb{R}\), but in advance we require knowledge that there exists a pullback global \(\mathcal{D} \cup \{B\}\)–attractor \(\mathcal{A}\), which is contained in \(B\).

**Theorem A.4.** Assume that a process \(\{U(t,s) : t \geq s\}\) on a metric space \((V,d)\) satisfies \((H_1), (H_2), (H_3), (59), (60)\) with \(T > 0\) and \(\eta \in [0,e^{-\gamma_0 T})\) with \(B(t), t \in \mathbb{R}\), being bounded subsets of \(V\). Assume also that for some \(\delta \in (0,e^{-\gamma_0 T} - \eta)\) we have

\[
H_t = \limsup_{k \to \infty} \frac{1}{k} \sum_{j=0}^{k-1} \ln m_Z \left( 2\kappa(t-jT)\delta^{-1} \right) < \infty, \quad t \in \mathbb{R}.
\]

(68)

If there exists a pullback global \(\mathcal{D} \cup \{B\}\)–attractor \(\mathcal{A} = \{A(t) : t \in \mathbb{R}\}\), which is contained in \(B\), i.e., \(A(t) \subset B(t)\) for \(t \in \mathbb{R}\), then there exists a \(T\)–weak pullback exponential \(\mathcal{D} \cup \{B\}\)–attractor \(\mathcal{M} = \{M(t) : t \in \mathbb{R}\}\) satisfying properties:

(a) \(M(t)\) is a nonempty compact subset of \(B(t)\) for \(t \in \mathbb{R}\),

(b) \(U(t,t-T)M(t-T) \subset M(t), t \in \mathbb{R}\),

(c) for any \(0 < \xi < \xi_T = -\frac{1}{T} (\ln (\eta + \delta) + \gamma_0 T)\) and \(D \in \mathcal{D} \cup \{B\}\)

\[
\lim_{r \to \infty} e^{\xi r} \text{dist}^V(U(t,t-r)D(t-r), M(t)) = 0, \quad t \in \mathbb{R},
\]

(d) the fractal dimension of \(M(t)\) is finite with a given bound, i.e.,

\[
\dim_f^V (M(t)) \leq \frac{H_t}{T\xi_T}, \quad t \in \mathbb{R}.
\]

(69)

Moreover,

\[
M(t) = A(t) \cup E(t) = \operatorname{cl}_V E(t) \subset B(t), \quad t \in \mathbb{R},
\]

(70)

where each \(E(t), t \in \mathbb{R}\), is a nonempty countable subset of \(B(t)\).

If the operator \(U(t,s)\) is Lipschitz continuous on \(B(s)\) with constant \(L_{t,s} > 0\) for \(t \in \mathbb{R}\) and \(s \in [t-T,t]\), i.e.,

\[
d(U(t,s)x, U(t,s)y) \leq L_{t,s} d(x,y), \quad x, y \in B(s), \quad t \in \mathbb{R}, \quad s \in [t-T,t],
\]

(71)

then the family \(\mathcal{M} = \{M(t) : t \in \mathbb{R}\}\) can be corrected to satisfy

\[
U(t,s)M(s) \subset M(t), \quad t \geq s,
\]

and become a pullback exponential \(\mathcal{D} \cup \{B\}\)–attractor with the same rate of convergence \(\xi\) and a suitably modified bound of its fractal dimension

\[
\dim_f^V (M(t)) \leq \frac{H_{nT}}{T\xi_T}, \quad t \in [nT, (n+1)T), \quad n \in \mathbb{Z}.
\]

(72)

Moreover, the structure (70) is preserved.

**Proof.** Let \(R_t\) be as in (61). Since by assumption \(A(t) \subset B(t), t \in \mathbb{R}\), and \(\mathcal{A}\) is invariant, Proposition A.2 implies that

\[
N^V(A(t), 3(\eta + \delta)^k R_{t-kT}) \leq \prod_{j=0}^{k-1} m_Z \left( 2\kappa(t-jT)\delta^{-1} \right), \quad k \in \mathbb{N}, \quad t \in \mathbb{R}.
\]

Repeating the same argument, which led to (65), we obtain

\[
\dim_f^V (A(t)) \leq \frac{H_t}{T\xi_T}, \quad t \in \mathbb{R}.
\]

(73)
By Propositions A.2 and A.3 we construct a family \( E = \{ E(t): t \in \mathbb{R} \} \) satisfying the properties (i)-(iv) of Proposition A.3 with \( q = q + \delta \) and \( H_t, t \in \mathbb{R} \), given in (68).

We define
\[
M(t) = A(t) \cup E(t), \quad t \in \mathbb{R}.
\]
The family \( \mathcal{M} = \{ M(t): t \in \mathbb{R} \} \) consists of nonempty subsets of \( V \) and \( M(t) \subset B(t) \) for \( t \in \mathbb{R} \). Moreover, we have
\[
U(t, t - T)M(t - T) = A(t) \cup U(t, t - T)E(t - T) \subset A(t) \cup E(t) = M(t), \quad t \in \mathbb{R}.
\]
By Proposition A.3 (iv) for any \( \mathcal{D} = \{ D(t): t \in \mathbb{R} \} \in \mathcal{D} \cup \{ \mathcal{B} \} \) we have
\[
\lim_{r \to \infty} e^{\xi r} \dist^V(U(t, t - r)D(t - r), M(t)) = 0, \quad t \in \mathbb{R},
\]
with any \( \xi \in (0, \xi_T) \). Moreover, from (73) and (65) we get (69). Each \( M(t) \) is precompact in \( V \), but we show now that in fact this set is compact.

To this end, we fix \( t \in \mathbb{R} \) and take a sequence \( x_k \in M(t) \). If infinitely many of its elements belong to \( A(t) \), then by the compactness of \( A(t) \), there exists a convergent subsequence to an element of \( A(t) \subset M(t) \). If infinitely many of its elements belong to \( E(t) \), say \( x_{k_i} \in E(t) \), then there exists a sequence \( p_i \in \mathbb{N} \) such that \( x_{k_i} \in Q_{p_i}(t) \), where \( Q_{p_i}(t) \) are taken from Proposition A.3 (iii). If \( p_0 = \sup\{ p_i: i \in \mathbb{N} \} < \infty \), the sequence \( x_{k_i} \) is contained in the finite set \( \bigcup_{j=1}^{p_0} Q_j(t) \) and consequently has a convergent subsequence to an element of this set, hence to an element of \( E(t) \subset M(t) \). Otherwise, if \( \{ p_i: i \in \mathbb{N} \} = \infty \), there exists a subsequence \( p_{i_j} \) such that \( p_{i_j} \to \infty \). Since \( x_{k_{i_j}} \in Q_{p_{i_j}}(t) \subset U(t, t - p_{i_j}T)B(t - p_{i_j}T) \), the family \( \mathcal{A} \) attracts \( \mathcal{B} \) and \( A(t) \) is compact, it follows that the sequence has a convergent subsequence to some element of \( A(t) \subset M(t) \).

We are left to show that \( A(t) \cup E(t) = \cl^V E(t) \). Note that \( \{ \cl^V E(t): t \in \mathbb{R} \} \) is a family of closed sets, which attracts families from \( \mathcal{D} \cup \{ \mathcal{B} \} \). Hence by the minimality of the pullback global \( \mathcal{D} \cup \{ \mathcal{B} \} - \)attractor, we have \( A(t) \subset \cl^V E(t) \). This implies that \( A(t) \cup E(t) \subset \cl^V E(t) \). For the converse inclusion, let \( x \in \cl^V E(t) \) and \( x_k \in E(t) \) be a sequence such that \( x_k \to x \). Then, reasoning as above, either \( x \in E(t) \) or \( x \in A(t) \).

Last claims of the theorem follow by setting \( \widetilde{\mathcal{M}} = \{ \widetilde{M}(t): t \in \mathbb{R} \} \) with
\[
\widetilde{M}(t) = U(t, nT)M(nT), \quad t \in [nT, (n + 1)T), \quad n \in \mathbb{Z}.
\]
By \((H_1)\) we get \( \widetilde{M}(t) \subset B(t), t \in \mathbb{R} \), while (71) implies that \( U(t, nT) \) is Lipschitz continuous on \( M(nT) \), which yields compactness of \( \widetilde{M}(t) \) and
\[
\dim^V(\widetilde{M}(t)) \leq \dim^V(M(nT)) \leq \frac{H_n T}{\xi_T}, \quad t \in [nT, (n + 1)T), \quad n \in \mathbb{Z}.
\]
Moreover, for any \( \mathcal{D} = \{ D(t): t \in \mathbb{R} \} \in \mathcal{D} \cup \{ \mathcal{B} \} \) and \( t \in [nT, (n + 1)T) \) we have
\[
U(nT, t - r)D(t - r) \subset B(nT) \quad \text{for } r \geq t - nT + r_{D,nT}
\]
and
\[
e^{\xi r} \dist^V(U(t, t - r)D(t - r), \widetilde{M}(t))
\]
\[
\leq L_{t,nT} e^{\xi r} \dist^V(U(nT, nT - (r + nT - t))D(nT - (r + nT - t)), M(nT)),
\]
which shows that \( \widetilde{M} \) pullback attracts each \( \mathcal{D} \in \mathcal{D} \cup \{ \mathcal{B} \} \) exponentially with rate \( \xi \).
For the positive invariance of $\tilde{\mathcal{M}}$, let $t \geq s$ and $n \geq m$, $m, n \in \mathbb{Z}$, be such that $t = nT + t_1$ and $s = mT + s_1$ with $t_1, s_1 \in [0, T)$. We have

$$U(t, s)\tilde{\mathcal{M}}(s) = U(t, nT)U(nT, mT)M(mT) \subset U(t, nT)M(nT) = \tilde{\mathcal{M}}(t),$$

which shows that $\tilde{\mathcal{M}}$ is a pullback exponential $\mathcal{D} \cup \{\mathcal{B}\}$–attractor contained in $\mathcal{B}$.

Moreover, for $t \in [nT, (n + 1)T)$ we have

$$\tilde{\mathcal{M}}(t) = U(t, nT)A(nT) \cup U(t, nT)E(nT) = A(t) \cup \tilde{E}(t)$$

with $\tilde{E}(t) = U(t, nT)E(nT)$ being a countable subset of $B(t)$. Then $\tilde{\mathcal{M}}(t) = \text{cl}_V \tilde{E}(t)$, since

$$U(t, nT) \text{cl}_V E(nT) = \text{cl}_V U(t, nT)E(nT)$$

by the continuity of $U(t, nT)$ on $M(nT) = \text{cl}_V E(nT) \subset B(nT)$.

The existence of pullback exponential attractors without knowing a priori that a pullback global attractor exists is guaranteed by the following result.

**Theorem A.5.** Assume that a process $\{U(t, s) : t \geq s\}$ on a complete metric space $(V, d)$ satisfies $(H_1)$, $(H_2)$, $(H_3)$, (59), (60) with $T > 0$ and $\eta \in [0, e^{-\gamma nT})$ with $B(t), t \in \mathbb{R}$, being bounded closed subsets of $V$.

If (68) holds for some $\delta \in (0, e^{-\gamma nT} - \eta)$, then there exists a $T$–weak pullback exponential $\mathcal{D} \cup \{\mathcal{B}\}$–attractor $\mathcal{M} = \{M(t) : t \in \mathbb{R}\}$ satisfying properties (a)-(d) of Theorem A.4. Moreover, we have

$$M(t) = \text{cl}_V E(t) \subset B(t), \ t \in \mathbb{R},$$

(74)

where each $E(t), t \in \mathbb{R},$ is a nonempty countable subset of $B(t)$.

If (71) holds, then $\mathcal{M}$ can be considered as a pullback exponential $\mathcal{D} \cup \{\mathcal{B}\}$–attractor contained in $\mathcal{B}$ with the same rate of convergence $\xi$ and the bound of the fractal dimension of $M(t)$ specified in (72). Moreover, the structure (74) is preserved.

**Proof.** First note that (59) and (60) yield the following Lipschitz condition

$$d(U(t, t - T)x, U(t, t - T)y) \leq L_{t, t - T} d(x, y), \ x, y \in B(t - T), \ t \in \mathbb{R}.$$ (75)

Indeed, on the contrary suppose that for any $l \in \mathbb{N}$ there are $x_l, y_l \in B(t - T)$ such that

$$ld(x_l, y_l) < d(U(t, t - T)x_l, U(t, t - T)y_l).$$

Thus $x_l \neq y_l$ and by (60)

$$l < \eta + n_Z(z_l), \ l \in \mathbb{N},$$

(76)

with $z_l = \frac{K(t)x_l - K(t)y_l}{d(x_l, y_l)}$. Since by (59) we have $\|z_l\|_Z \leq \kappa(t), l \in \mathbb{N}$, it follows from the compactness of $n_Z$ that there exists a Cauchy subsequence $z_l$ with respect to $n_Z$. In particular, the sequence $n_Z(z_l)$ is bounded and we get a contradiction with (76).

We set $M(t) = \text{cl}_V E(t), t \in \mathbb{R}$, where $\{E(t) : t \in \mathbb{R}\}$ is a family constructed in Proposition A.3 using Proposition A.2. Then the assertions are straightforward with (b) following from the continuity of $U(t, t - T)$ on the closed set $B(t - T)$ given in (75).

If (71) holds, then the family $\mathcal{M}$ can be corrected to become a pullback exponential $\mathcal{D} \cup \{\mathcal{B}\}$–attractor following that step in the proof of Theorem A.4. \qed
Remark A.6. Note the following particular situations when (68) is satisfied. If \( \kappa: \mathbb{R} \to (0, \infty) \) is non-decreasing, then (68) holds for any \( \delta \in (0, e^{-\gamma_0 T} - \eta) \) with
\[
H_t \leq \ln m_Z \left( 2\kappa(t)\delta^{-1} \right), \quad t \in \mathbb{R}.
\]
If \( \kappa(t) \equiv \kappa \) is constant, then (68) holds for any \( \delta \in (0, e^{-\gamma_0 T} - \eta) \) with \( H_t \equiv \ln m_Z \left( 2\kappa\delta^{-1} \right) \).

Appendix B. Example of admissible nonlinearities. In this section we give an example of the nonlinear part \( f_i \) in (1), which satisfies assumptions of Theorem 4.4. We consider
\[
f_i(t, \sigma) = m_i(\sigma + a_0(t) + a_1 b_1(t) \sigma |\sigma|^{\rho_1 - 1} + \cdots + a_{J_i} b_{J_i}(t) \sigma |\sigma|^{\rho_J - 1}), \quad t, \sigma \in \mathbb{R}, \quad i \in \mathbb{Z},
\]
where \( m = (m_i) \in l^\infty \) is such that the operator \( A - m \) is positive definite on \( l^2 \) (see Section 2), so that (10) holds, whereas \( 1 = \rho_0 < \rho_1 < \cdots < \rho_J, \quad a_j = (a_{J_i}) \in l^\infty, \) \( 0 \leq j \leq J, \) \( b_j : \mathbb{R} \to \mathbb{R} \) are continuous bounded real functions for \( 1 \leq j \leq J \) and
\[
a_j, b_j(t) < 0, \quad i \in \mathbb{Z}, \quad t \in \mathbb{R} \quad \text{and} \quad \inf_{t \in \mathbb{R}} |b_j(t)| > 0.
\]
Moreover, we assume that
\[
\sum_{i \in \mathbb{Z}} |a_{J_i}|^{2\rho_j - \rho_j} |a_{J_i}|^{-\rho_j} < \infty, \quad 0 \leq j < J.
\]
Note that we do not impose any sign conditions for \( a_j \) and \( b_j \) for \( 0 \leq j < J \).

Proposition B.1. Functions \( f_i: \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad i \in \mathbb{Z}, \) defined by (77)-(79) satisfy conditions (15), (16) with time independent function \( P \), (18), (19), (35), (36) and (37).

Proof. Condition (15) is straightforward. Setting \( b_0(t) \equiv 1, \) using Young’s inequality and omitting time in the notation we get for any \( 0 \leq j < J \) and \( i \in \mathbb{Z} \)
\[
|a_j b_j| |\sigma|^{\rho_j} + 1 \leq \varepsilon_j \rho_j |a_j b_j| |\sigma|^{\rho_j + 1} + |a_j b_j|^{\rho_j} |a_j b_j|^{-\rho_j} \gamma_j |\sigma| \left( \frac{\rho_j - \rho_j}{\rho_j} \right) \varepsilon_j^{-\rho_j},
\]
with any \( \varepsilon_j > 0 \). Taking \( \varepsilon_j = \frac{\rho_j}{j^{\rho_j}}, \) \( 0 \leq j < J \), by (77), (78) we see that the inequality in (19) holds with
\[
d_i(t) = \sum_{j=0}^{J-1} \left( 1 - \frac{\rho_j}{\rho_j} + \frac{\rho_j}{\rho_j} \right) |a_j b_j(t)|^{\rho_j} |a_j b_j|^{-\rho_j} \gamma_j (\eta_j(t) - \eta_j(s)), \quad t \in \mathbb{R}.
\]
Then for some numbers \( \theta_j > 0 \) and some continuous functions \( \eta_j: \mathbb{R} \to \mathbb{R} \) we have
\[
d_i(t) - d_i(s) = \sum_{j=0}^{J-1} \theta_j |a_{J_i}|^{\rho_j} |a_{J_i}|^{-\rho_j} \gamma_j (\eta_j(t) - \eta_j(s)),
\]
which by (79) gives
\[
|d(t) - d(s)|^2 \leq J \sum_{j=0}^{J-1} \theta_j^2 (\eta_j(t) - \eta_j(s))^2 \sum_{i \in \mathbb{Z}} |a_{J_i}|^{2\rho_j} |a_{J_i}|^{-2\rho_j} \gamma_j^2
\]
and shows that \( d: \mathbb{R} \to l^2 \) is continuous.
Moreover, by (78) and the boundedness of functions $b_j$ we obtain with some $c_j \geq 0$

$$d_j^2(t) \leq \sum_{j=0}^{J-1} c_j |a_j|^\frac{2j}{j+\gamma_0} |a_{j+1}|^{\frac{2j}{j+\gamma_0}} , \ t \in \mathbb{R}.$$ 

Furthermore, by (79) we have

$$\|d(t)\|^2 \leq \sum_{j=0}^{J-1} c_j \sum_{i \in \mathbb{Z}} |a_{ji}|^{\frac{2j}{j+\gamma_0}} |a_{ji+1}|^{\frac{2j}{j+\gamma_0}} = C^2, \ t \in \mathbb{R},$$

which shows that $d \in \mathfrak{S}_{\gamma_0}$ for any $0 < \gamma_0 < \alpha_0$ and (19) is satisfied.

Setting $B_j = \sup_{t \in \mathbb{R}} |b_j(t)|$, $1 \leq j \leq J$, we have

$$\left| \frac{\partial f_i}{\partial \sigma} (t, \sigma) \right| \leq \|m\|_\infty + \|a_0\|_\infty + \sum_{j=1}^{J} \rho_j \|a_j\|_\infty |B_j|^{|\rho_j| - 1}, \ t, \sigma \in \mathbb{R},$$

which implies (16) with time independent function $P = P(r)$.

To verify (18) let $u \in \ell^2$ be given. By (6) we then have for $t, s \in \mathbb{R}$

$$\sum_{i \in \mathbb{Z}} |f_i(t, u_i) - f_i(s, u_i)|^2 \leq \sum_{j=1}^{J} (b_j(t) - b_j(s))^2 \|a_j\|_\infty^2 \|u\|^{2\rho_j},$$

which implies (18) by the continuity of each $b_j$.

Furthermore, (35) and (36) are satisfied, since

$$\left| \frac{\partial f_i}{\partial \sigma} (t, \sigma) - \frac{\partial f_i}{\partial \sigma} (t, 0) \right| \leq \sum_{j=1}^{J} \rho_j \|a_j\|_\infty B_j r^{|\rho_j| - 1} = p(r), \ i \in \mathbb{Z}, \ t \in \mathbb{R}, \ |\sigma| \leq r.$$

Finally, we see that $\frac{\partial f_i}{\partial \sigma} (t, 0) = m_i + a_{0i}$ for $i \in \mathbb{Z}$ and $t \in \mathbb{R}$, so (37) holds as well.

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E-mail address: radoslaw.czaja@us.edu.pl