EXISTENCE AND NON-EXISTENCE RESULTS FOR VARIATIONAL HIGHER ORDER ELLIPTIC SYSTEMS

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Abstract. Let \( \alpha \in \mathbb{N}, \alpha \geq 1 \) and \((-\Delta)^\alpha = -\Delta((-\Delta)^{\alpha-1})\) be the polyharmonic operator. We prove existence and non-existence results for the following Hamiltonian systems of polyharmonic equations under Dirichlet boundary conditions

\[
\begin{align*}
(-\Delta)^\alpha u &= H_v(u, v) \quad \text{in } \Omega \subset \mathbb{R}^N \\
(-\Delta)^\alpha v &= H_u(u, v) \\
\frac{\partial u}{\partial r} &= 0, \ r = 0, \ldots, \alpha - 1, \ \text{on } \partial \Omega \\
\frac{\partial v}{\partial r} &= 0, \ r = 0, \ldots, \alpha - 1, \ \text{on } \partial \Omega
\end{align*}
\]

where \( \Omega \) is a sufficiently smooth bounded domain, \( N > 2\alpha, \nu \) is the outward pointing normal to \( \partial \Omega \) and the Hamiltonian \( H \in C^1(\mathbb{R}^2; \mathbb{R}) \) satisfies suitable growth assumptions.

1. Introduction. Let us briefly recall some well-known facts in order to better contextualize our results. Consider the so-called Lane-Emden system:

\[
\begin{align*}
-\Delta u &= |v|^{q-1} v \quad \text{in } \Omega \subset \mathbb{R}^N \\
-\Delta v &= |u|^{p-1} u \\
u &= v = 0 \quad \text{on } \partial \Omega
\end{align*}
\]

where \( N > 2, p, q > 0 \) and \( \Omega \) is a sufficiently smooth bounded domain. System 1 exhibits an Hamiltonian structure, namely it can be written in the form

\[
\begin{align*}
L_1 u &= H_v(u, v) \\
L_2 v &= H_u(u, v)
\end{align*}
\]

where \( L_1, L_2 \) are uniformly elliptic operators and \( H \in C^1(\mathbb{R}^2; \mathbb{R}) \). Moreover, weak solutions to 1 are critical points of the functional

\[
I(u, v) = \int_{\Omega} \nabla u \nabla v \, dx - \frac{1}{p+1} \int_\Omega |u|^{p+1} \, dx - \frac{1}{q+1} \int_\Omega |v|^{q+1} \, dx.
\]

Notice that the quadratic part of 2 turns out to be strongly indefinite, namely it is neither bounded from above nor from below on infinite dimensional spaces (just take pairs \((u, u), (u, -u)\)). This prevents us from applying classical variational results, such as the Mountain Pass Theorem. Moreover, the classical Nehari manifold turns

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out to be of limited success in the study of existence of ground state solutions. This is a deep difference with the scalar case when \( u = v \) and \( p = q \), namely

\[
\begin{aligned}
-\Delta u &= |u|^{p-1} u \quad \text{in } \Omega \subset \mathbb{R}^N \\
u &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\]

whose related functional is given by

\[
I(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \frac{1}{p+1} \int_{\Omega} |u|^{p+1} \, dx,
\]

as here the quadratic part is positive definite. We refer to [10] for an extensive survey about existence and non-existence results for \( 3 \); in particular, we recall that the value \( p = \frac{N+2}{N-2} \) plays the role of critical threshold between existence and non-existence of solutions to \( 3 \) in starshaped domains. As it was first noted by Mitidieri [20], a threshold can be found also for \( 1 \); in this case, it turns out to be a curve, more precisely the following hyperbola

\[
\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}.
\]

If \((p, q)\) lies below \( 5 \), then there exists a nontrivial solution, whereas non-existence on starshaped domains is proved on or above \( 5 \), see the survey papers [4, 26] and the references therein on this topic. In particular, we recall [12], where it is proved that there exists a nontrivial solution to \( 1 \) if \( pq > 1 \) and

\[
\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N}, \quad \max\{(N-4)p, (N-4)q\} < N+4.
\]

This is achieved by means of the Linking Theorem due to Benci and Rabinowitz [3] in the context of fractional order Sobolev spaces. Let us mention that the limiting case when \( N = 2 \) is considered in [25], see also [7, 8].

One may wonder what happens in the case in which polyharmonic operators are taken into account, namely \((-\Delta)\alpha u = -\Delta ((-\Delta)^{\alpha-1}) \), \( \alpha \in \mathbb{N}, \alpha \geq 1 \), in place of the Laplace operator. Actually, these operators appear in many different contexts, such as in the modeling of classical elasticity problems (in particular suspension bridges [13]), as well as Micro Electro-Mechanical Systems (MEMS), see [6] and references therein. Let us first consider the case in which \( \Omega = \mathbb{R}^N \), namely

\[
\begin{aligned}
(-\Delta)^\alpha u &= |v|^{q-1} v \quad \text{in } \mathbb{R}^N \\
(-\Delta)^\alpha v &= |u|^{p-1} u \quad \text{in } \mathbb{R}^N.
\end{aligned}
\]

System \( 6 \) has been studied in [18, 5]: here existence and non-existence results for radially symmetric positive solutions are proved, more precisely the hyperbola

\[
\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2\alpha}{N}.
\]

This turns out to be the substitute for \( 5 \) in the higher order case. As for the non radial case, only partial results are known, see [33, 19, 2] and [21] for non-existence results for supersolutions to \( 6 \). Actually, we point out that even the case \( \alpha = 1 \) has not been yet completely understood, see [23, 31] and references therein.
The purpose of the present paper is to study existence and non-existence results for the following system

\[
\begin{aligned}
(-\Delta)^\alpha u &= H(u,v) \quad \text{in } \Omega \subset \mathbb{R}^N \\
(-\Delta)^\alpha v &= H(u,v) \\
\frac{\partial^r u}{\partial \nu^r} &= 0, \quad r = 0, \ldots, \alpha - 1, \text{ on } \partial \Omega \\
\frac{\partial^r v}{\partial \nu^r} &= 0, \quad r = 0, \ldots, \alpha - 1, \text{ on } \partial \Omega
\end{aligned}
\] (8)

where \(\Omega\) is a \(C^{2,\gamma}\) bounded domain, \(\gamma \in (0,1)\), \(N > 2\alpha\), \(\nu\) is the outward pointing normal to \(\partial \Omega\), \(\alpha \in \mathbb{N}\), \(\alpha \geq 1\) and \(H: \mathbb{R}^2 \to \mathbb{R}\) is a \(C^1\) function.

We consider Dirichlet boundary conditions, which, differently from the so-called Navier boundary conditions, namely

\[
\begin{aligned}
\Delta^r u &= 0, \quad r = 0, \ldots, \alpha - 1, \text{ on } \partial \Omega \\
\Delta^r v &= 0, \quad r = 0, \ldots, \alpha - 1, \text{ on } \partial \Omega
\end{aligned}
\]

do not allow to decoupling 8 into a system of \(2\alpha\) equations. Furthermore, no maximum principle in fairly general bounded domains holds. We refer to [14, 29] for existence results for higher order Lane-Emden systems with Navier boundary conditions and to [18] for the proof of non-existence of positive solutions above the critical curve 7 on star-shaped domains.

Let us first focus on the existence of solutions to 8. In [9] the authors prove a critical point theorem (Theorem 2.3 within) which in particular yields a nontrivial critical point of the functional

\[
J(u) = \frac{q}{q+1} \int_\Omega |\Delta^\alpha u|^\frac{q+1}{q+1} \,dx - \frac{1}{p+1} \int_\Omega |u|^{p+1} \,dx
\]

on the space \(E = W^{2\alpha, \frac{q+1}{q}}(\Omega) \cap W^{\alpha, \frac{q+1}{q}}_0(\Omega)\), provided \((p,q)\) is below the critical hyperbola 7 and \(pq \neq 1\). One can show that, in the case

\[
H(u,v) = \frac{1}{p+1} |u|^{p+1} + \frac{1}{q+1} |v|^{q+1},
\]

given a nontrivial critical point \(u\) of 9, then

\[
(u, ((-\Delta)^\alpha)^{-1}(|u|^{p-1}u))
\]

is a nontrivial weak solution to 8 with

\[
H(u,v) = \frac{|u|^{p+1}}{p+1} + \frac{|v|^{q+1}}{q+1},
\]

see also [27]. However, this method, which is originally due to Lions [17], seems not to be suitable to consider more general nonlinearities than power-like, nor to deal with non variational contexts such as

\[
\begin{aligned}
(-\Delta)^\alpha u &= |v|^{q-1} v \quad \text{in } \Omega \subset \mathbb{R}^N \\
(-\Delta)^\beta v &= |u|^{p-1} u \\
\frac{\partial^r u}{\partial \nu^r} &= 0, \quad r = 0, \ldots, \alpha - 1, \text{ on } \partial \Omega \\
\frac{\partial^r v}{\partial \nu^r} &= 0, \quad r = 0, \ldots, \beta - 1, \text{ on } \partial \Omega
\end{aligned}
\] (10)

with \(\alpha \neq \beta\). In [28] we prove existence results for 10 on a ball by means of degree theory combined with moving planes methods. Here, we take into account the variational case \(\alpha = \beta\) allowing for more general nonlinearities than power-like. Moreover, we deal with a larger class of bounded domains with respect to [28].
However, in order to get the variational setting, a technical (and rather unnatural) assumption has to be imposed, more precisely we require that the exponents $p, q$ satisfy

$$\max\{(N - 4\alpha)p, (N - 4\alpha)q\} < N + 4\alpha.$$  

Notice that this turns out to hold true automatically in the case $p, q > 1$, whereas it is a restrictive assumption if we deal with $pq > 1$.

**Theorem 1.1.** Let $N > 2\alpha$ and let $p, q$ be such that $pq > 1$ and

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N - 2\alpha}{N}, \quad \max\{(N - 4\alpha)p, (N - 4\alpha)q\} < N + 4\alpha.$$  

Let $H: \mathbb{R}^2 \rightarrow \mathbb{R}$ be of class $C^1$ such that

(H1) $H(u, v) \geq 0$ for all $(u, v) \in \mathbb{R}^2$

(H2) There exists $R > 0$ such that, if $(u, v) \in \mathbb{R}^2$ satisfies $|(u, v)| \geq R$, then

$$\frac{1}{p+1} H_u(u, v) \cdot u + \frac{1}{q+1} H_v(u, v) \cdot v > H(u, v) > 0$$

(H3) There exist $r > 0$ and $a > 0$ such that, if $(u, v) \in \mathbb{R}^2$ satisfies $|(u, v)| \leq r$, then

$$H(u, v) \leq a(|u|^{p+1} + |v|^{q+1})$$

(H4) There exists $b > 0$ such that

$$|H_u(u, v)| \leq b(|u|^p + |v|^{p/(q+1)} + 1)$$

$$|H_v(u, v)| \leq b(|v|^q + |u|^{q/(p+1)} + 1).$$

Then, there exists a nontrivial solution

$$(u, v) \in W^{2\alpha, \frac{p+1}{q}}(\Omega) \cap W_0^{\alpha, \frac{p+1}{q}}(\Omega) \times W^{2\alpha, \frac{p+1}{q}}(\Omega) \cap W_0^{\alpha, \frac{p+1}{q}}(\Omega)$$

to problem 8.

**Remark 1.** It follows from (H2) that there exist $c_1, c_2 > 0$ such that

$$H(u, v) \geq c_1(|u|^{p+1} + |v|^{q+1}) - c_2,$$

see [11, Lemma 1.1]. Moreover, by (H4) there exist $a_1, a_2 > 0$ such that

$$H(u, v) \leq a_1(|u|^{p+1} + |v|^{q+1}) + a_2,$$

see [12, p.105].

**Corollary 1.** Let $N > 2\alpha$ and let $p, q$ be such that $pq > 1$ and 11 holds. Then there exists a non trivial solution to

$$\begin{cases}
(-\Delta)^\alpha u = |v|^{q-1} v & \text{in } \Omega \subset \mathbb{R}^N \\
(-\Delta)^{\alpha} v = |u|^{p-1} u & \\
\frac{\partial^r u}{\partial r} = 0, r = 0, \ldots, \alpha - 1, \text{ on } \partial \Omega \\
\frac{\partial^r v}{\partial r} = 0, r = 0, \ldots, \alpha - 1, \text{ on } \partial \Omega.
\end{cases}$$

**Remark 2.** Notice that with standard bootstrap arguments one can infer regularity of solutions to 8 under some additional regularity conditions, precisely in the case $H \in C^2(\mathbb{R}^2)$ (we refer to [4, Lemma 5.16] for the case $\alpha = 1$, see also [30, Theorem 1]). Let us consider the model case 14 and let us take a nontrivial solution

$$(u, v) \in W^{2\alpha, \frac{p+1}{q}}(\Omega) \cap W_0^{\alpha, \frac{p+1}{q}}(\Omega) \times W^{2\alpha, \frac{p+1}{q}}(\Omega) \cap W_0^{\alpha, \frac{p+1}{q}}(\Omega).$$
Figure 1. The light grey region represents values of $p,q$ for which we prove existence of solutions to $14$, whereas the dark grey region is the domain of non-existence given by Corollary 2. The curve $l_1$ is $\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2\alpha}{N}$, whereas $l_2$ is given by $pq = 1$.

Assume $\frac{(p+1)}{p}, \frac{(q+1)}{q} < \frac{N}{2\alpha}$. By Sobolev embeddings we know that $u \in L^{\frac{N-2\alpha}{(q+1)}}(\Omega)$, and as a consequence $u^p \in L^{\frac{N}{(N-2\alpha q-2\alpha p)}}(\Omega)$. By elliptic regularity, see [13, Theorem 2.20], we conclude that $v \in W^{2\alpha,\frac{N}{Nq-2\alpha q-2\alpha p}}(\Omega)$. Since $(p,q)$ is subcritical,

$$\frac{N(q+1)}{Nq-2\alpha q-2\alpha p} > \frac{p+1}{p},$$

thus we have gained some summability of $v$, and similarly for $u$. Repeating a finite number of times this argument, we have $u, v \in W^{2\alpha,r}(\Omega)$ for some $r > \frac{N}{2\alpha}$. Hence, again by Sobolev embedding, $u^p, v^d \in C^{0,\gamma}(\Omega)$ for some $\gamma > 0$. We now apply [13, Theorem 2.19] to get $u, v \in C^{2\alpha,\gamma}(\Omega)$, namely $(u,v)$ is classical. Notice that if $(p+1)/p > N/(2\alpha)$ then $v \in C^{0,\gamma}(\Omega)$ by Sobolev embeddings, hence $(u,v)$ is classical due to [13, Theorem 2.19]. Finally, if $(p+1)/p = N/(2\alpha)$, then $v \in L^{r}(\Omega)$ for any $r$ and the conclusion follows again by [13, Theorem 2.20] and [13, Theorem 2.19]. The same argument applies for $H \in C^2(\mathbb{R}^2)$ satisfying the hypotheses of Theorem 1.1.

The proof of Theorem 1.1 (see Section 2 below) relies on variational methods, extending to the higher order case ideas from [12, 14], where the case $\alpha = 1$ is considered.

In the second part of the paper (Section 3), we establish a non-existence result for 8, by exploiting a Pucci-Serrin type identity [24]. More precisely, we prove the following
Theorem 1.2. Let $\Omega$ be a ball in $\mathbb{R}^N$ and $N > 2\alpha$. Assume that there exists $a \in \mathbb{R}$ such that

$$NH(u, v) - auH_u(u, v) - (N - 2\alpha - a)vH_v(u, v) \leq 0.$$ 

Then, no classical positive solutions to 8 do exist.

As by-product of Theorem 1.2 we have the following

Corollary 2. Let $p, q > 0$ be such that

$$\frac{1}{p+1} + \frac{1}{q+1} \leq \frac{N-2\alpha}{N}. \quad (15)$$

Then, no positive classical solutions to

$$\begin{cases}
(-\Delta)\alpha u = |v|^{q-1} v & \text{in } B_1 \subset \mathbb{R}^N \\
(-\Delta)\alpha v = |u|^{p-1} u & \text{in } B_1 \\
\frac{\partial u}{\partial \nu} = 0, r = 0, \ldots, \alpha - 1, & \text{on } \partial B_1 \\
\frac{\partial v}{\partial \nu} = 0, r = 0, \ldots, \alpha - 1, & \text{on } \partial B_1
\end{cases}$$

do exist, where we set $B_1$ as the the unitary open ball of $\mathbb{R}^N$ centered at the origin.

Notice that both in Theorem 1.2 and Corollary 2 we deal only with the unitary ball $B_1$. This is necessary as we exploit the Hopf lemma for higher order operators, which is not available in a general starshaped domain, see [13] for a comprehensive discussion on the possibility of extending the Hopf Lemma and maximum principles to the higher order context.

2. Existence results: Proof of Theorem 1.1. We first recall some preliminary results about the polyharmonic operator and we present the variational setting. In what follows, we denote by $W^{k,p}_0(\Omega)$ the closure of $C_0^\infty(\Omega)$ in the Sobolev space $W^{k,p}(\Omega) = \{ u \in L^p(\Omega) \text{ s.t. } \forall \alpha, \sum |\alpha| \leq k, \exists D^\alpha u \in L^p(\Omega) \}$ endowed with the norm:

$$\|u\|_{W^{k,p}(\Omega)} = \left( \sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^p dx \right)^{1/p},$$

where $D^\alpha u$ is the weak derivative of $u$ of order $\alpha$. Moreover, $H^k(\Omega) := W^{k,2}(\Omega)$ and $H^k_0(\Omega) := W^{k,2}_0(\Omega)$. In what follows, we will always omit the domain, being clear from the context.

2.1. Preliminaries. Set

$$I(u, v) = \begin{cases}
\int_\Omega \Delta^\alpha u \Delta^\alpha v - \int_\Omega H(u, v) & \text{for even } \alpha \\
\int_\Omega \nabla(\Delta^{\alpha/2} u) \nabla(\Delta^{\alpha/2} v) - \int_\Omega H(u, v) & \text{for odd } \alpha.
\end{cases} \quad (16)$$

A direct approach to 8 would be to consider $16$ as the energy functional. However, in analogy to the case $\alpha = 1$ (see [12, 14]), it turns out to be convenient to work with fractional order Sobolev spaces and to distribute fractionally the derivatives in the quadratic term of $I$, as we will discuss in the sequel.

Let us recall some well-known facts about spectral properties of the polyharmonic operator: the next Lemma follows by applying the spectral theorem for compact and self-adjoint operators and [13, Theorem 2.20].
Lemma 2.1. Let $\alpha \in \mathbb{N}$, $\alpha \geq 1$, $N > 2\alpha$ and $\Omega$ be a smooth bounded domain. There exists an orthonormal basis of $L^2(\Omega)$ composed of eigenfunctions of the operator $(-\Delta)^\alpha$ subject to Dirichlet boundary conditions. These eigenfunctions are in $L^s(\Omega)$ for any $s \geq 1$ and correspond to a diverging sequence of positive eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots$.

Remark 3. Notice that, differently from the Navier case, no maximum principle for

$$\begin{cases}
(-\Delta)^\alpha u = f & \text{in } \Omega \\
\partial_r^\alpha u, \quad r = 0, \ldots, \alpha - 1 \quad & \text{on } \partial \Omega
\end{cases}$$

on a general bounded domain holds, except for the ball and small deformations of the ball, see [13, Sections 3.1, 5.1, 5.2 and Chapter 6]. In particular, we cannot conclude in general that the first eigenfunction of the polyharmonic operator is positive. However, this assumption is not required in the proof we give below and this allows us to deal with general smooth bounded domains.

The fractional Sobolev space $E^s$ is defined as the real interpolation space

$$E^s := [L^2, H^{2\alpha} \cap H_0^\alpha]_{s/2\alpha}$$

with $0 < s < 2\alpha$ real, which in terms of Fourier series is given explicitly by

$$E^s = \left\{ u = \sum_{n=1}^\infty u_n \Phi_n \in L^2 \mid \sum_{n=1}^\infty \lambda_n^{s/\alpha} u_n^2 < \infty \right\},$$

where $\Phi_n$ is an orthonormal basis for $L^2$ of eigenfunctions of $(-\Delta)^\alpha$ with Dirichlet boundary conditions corresponding to eigenvalues $\lambda_n$, and $u_n = \langle u, \Phi_n \rangle$, see [15]. Define for $u \in E^s$

$$A^s u = A^s \left( \sum_{n=1}^\infty u_n \Phi_n \right) = \sum_{n=1}^\infty \lambda_n^{s/\alpha} u_n \Phi_n.$$

Notice that in the case $s = 2\alpha$, there holds $A^s = (-\Delta)^\alpha$. We stress that the space $E^s$ endowed with the scalar product

$$\langle u, v \rangle_{E^s} := \int_\Omega A^s u A^s v$$

is Hilbert. Indeed

$$\|u\|_{E^s} = \left( \int_\Omega |A^s u|^2 \right)^{1/2} = \|A^s u\|_2$$

is a norm, since $\|A^s u\|_2 = 0$ implies $u = 0$ due to the Poincaré inequality

$$\|A^s u\|_2^2 \geq \lambda_1^{s/\alpha} \|u\|_2^2$$

and the space $(E^s, \| \cdot \|_{E^s})$ is Banach: if $u_n$ is a Cauchy sequence in $E^s$, then $A^s u_n$ is a Cauchy sequence in $L^2$, therefore there exists $v \in L^2$ such that $A^s u_n \to v$ in $L^2$; however,

$$v = \sum_{k=1}^\infty v_k \Phi_k = \sum_{k=1}^\infty \lambda_k^{s/\alpha} w_k \Phi_k \quad \text{where} \quad w_k = \frac{v_k}{\lambda_k^{s/(2\alpha)}},$$

namely $v = A^s u$ with $u = \sum_{k=1}^\infty w_k \Phi_k$. Hence, $A^s u_n \to A^s u$ in $L^2$ and $u_n \to u$ in $E^s$. 
Let us define $E = E^s \times E^{2\alpha-s}$, which can be decomposed as $E = E^+ \oplus E^-$, where

$$E^+ = \{(u, A^{-2\alpha+2s}u), u \in E^s \}$$

$$E^- = \{(u, -A^{-2\alpha+2s}u), u \in E^s \}$$

are orthogonal subspaces of $E$. Indeed if $z = (u, v) \in E$ one has

$$z = z^+ + z^-,$$

where

$$z^+ = (u^+, v^+) = \left(\frac{u + A^{2\alpha-2s}v}{2}, \frac{v + A^{-2\alpha+2s}u}{2}\right) \in E^+$$

$$z^- = (u^-, v^-) = \left(\frac{u - A^{2\alpha-2s}v}{2}, \frac{v - A^{-2\alpha+2s}u}{2}\right) \in E^-.$$

Note that this is well defined by 13. The functional 18 may be written also as

$$I(u, v) = \frac{1}{2} \langle L(u, v), (u, v) \rangle_E - J(u, v),$$

where

$$J(u, v) = \int_{\Omega} H(u, v) \, dx$$

by direct computation. Therefore, $E^s \hookrightarrow L^{p+1}$ if $\frac{1}{p+1} \geq \frac{1}{2} - \frac{s}{4N}$ and the embedding is compact provided the strict inequality holds (see [22], see also [16, Theorem 5.1] and [1, Sections 7.22, 7.23]). As a consequence, if one assumes 11, then there exist suitable $s, t \in (0, 2\alpha)$ such that $E^s \times E^t \hookrightarrow L^{p+1} \times L^{q+1}$ compactly and $s + t = 2\alpha$. Indeed, 11 implies:

(i) $N \left(\frac{1}{2} - \frac{1}{p+1}\right) < N \left(\frac{1}{q+1} - \frac{N-4\alpha}{2N}\right)$;

(ii) $N \left(\frac{1}{2} - \frac{1}{p+1}\right) < 2\alpha$;

(iii) $N \left(\frac{1}{q+1} - \frac{N-4\alpha}{2N}\right) > 0$,

and we can fix $s \in (0, 2\alpha)$ such that $N \left(\frac{1}{2} - \frac{1}{p+1}\right) < s < N \left(\frac{1}{q+1} - \frac{N-4\alpha}{2N}\right)$. Notice that here we used the technical assumption

$$\max\{(N-4\alpha)p, (N-4\alpha)q\} < N + 4\alpha$$

in order to have $s \in (0, 2\alpha)$, hence it can not be removed, as it allows us to set the variational setting.

In this context a natural choice for the energy functional related to 8 turns out to be the following:

$$I(u, v) = \int_{\Omega} A^s u A^{2\alpha-s} v \, dx - \int_{\Omega} H(u, v) \, dx. \tag{18}$$

Note that this is well defined by 13. The functional 18 may be written also as

$$I(u, v) = \frac{1}{2} \langle L(u, v), (u, v) \rangle_E - J(u, v),$$

where

$$J(u, v) = \int_{\Omega} H(u, v) \, dx \tag{19}$$
and $L: E \to E$ is given by
\[
\frac{1}{2} \langle L(u, v), (u, v) \rangle_E = \int_{\Omega} A^s u A^{2\alpha-s} v \, dx,
\]

namely
\[
L(u, v) = (A^{2\alpha-2s} v, A^{-2\alpha+2s} u).
\]

**Remark 4.** Note that $J' : E \to E'$ is compact. Indeed, the inclusion $E \hookrightarrow L^{p+1} \times L^{q+1}$ is compact (thus $(L^{p+1} \times L^{q+1})' \hookrightarrow E'$ is compact as well), whereas $J' : L^{p+1} \times L^{q+1} \to (L^{p+1} \times L^{q+1})'$ maps bounded sets into bounded sets, since by (H4) one has

\[
|J'(u, v)(\varphi, \psi)| \leq C(||u||_p ||\varphi||_p + ||v||_q ||\varphi||_p ||\varphi||_p + ||\varphi||_p + ||\varphi||_p + ||v||_q ||\varphi||_q + ||u||_p ||\varphi||_q + ||\psi||_q + ||\psi||_q).
\]

As a consequence,
\[
J' : E \hookrightarrow L^{p+1} \times L^{q+1} \to (L^{p+1} \times L^{q+1})' \hookrightarrow E'
\]
is compact.

Moreover, $L$ is bounded (actually, $||L(u, v)||_E = ||(u, v)||_E$, hence $||L|| = 1$), linear and symmetric, namely self-adjoint since $E$ is Hilbert. Finally, $L$ is invariant on $E^+$, since
\[
L(u, A^{-2\alpha+2s} u) = (A^{2\alpha-2s} A^{-2\alpha+2s} u, A^{-2\alpha+2s} u) = (u, A^{-2\alpha+2s} u), u \in E^s.
\]

Next we prove that critical points of $I$ are weak solutions to 8. This will be done by extending Theorem 1.2 in [12].

**Proposition 1.** Let $(u, v) \in E$ be a critical point of $I$ and $p, q$ satisfy 11. Then $u \in W^{2\alpha, \frac{q+1}{N}} \cap W_0^{\alpha, \frac{q+1}{N}}, v \in W^{2\alpha, \frac{q+1}{N}} \cap W_0^{\alpha, \frac{q+1}{N}}$ and $(u, v)$ is a weak solution to 8.

**Proof.** Since $(u, v)$ is a critical point of $I$, one has
\[
I'(u, v)(\varphi, \psi) = 0
\]
and in particular for $\varphi = 0$ and any $\psi \in E^{2\alpha-s}$ in 20 one has
\[
\int_{\Omega} A^s u A^{2\alpha-s} \psi \, dx = \int_{\Omega} H_v(u, v) \psi \, dx.
\]

If $\psi \in H^{2\alpha} \cap H_0^{\alpha}$ then
\[
\int_{\Omega} A^s u A^{2\alpha-s} \psi \, dx = \int_{\Omega} u A^{2\alpha} \psi \, dx = \int_{\Omega} u(-\Delta)^\alpha \psi \, dx.
\]

Since $v \in L^{q+1}$ and $u \in L^{p+1}$, by (H4) and the Minkowski inequality one has $H_v(u, v) \in L^{\frac{q+1}{q}}$ and hence by elliptic regularity (see [13, Theorem 2.20]) there exists $w \in W^{2\alpha, \frac{q+1}{\sqrt{N}}} \cap W_0^{\alpha, \frac{q+1}{\sqrt{N}}}$ such that
\[
\int_{\Omega} (-\Delta)^\alpha w \psi = \int_{\Omega} H_v(u, v) \psi.
\]

Furthermore, by 11
\[
\frac{1}{2} \geq \frac{q}{q+1} - \frac{2\alpha}{N}.
and by the Sobolev embedding theorem we get $w \in L^2$. Thus
\[ \int_{\Omega} H_v(u,v) \psi \, dx = \int_{\Omega} (-\Delta)^a w \psi \, dx = \int_{\Omega} w (-\Delta)^a \psi \, dx. \] (24)
Hence, by combining 21, 22, 24 one has:
\[ \int_{\Omega} (u - w)(-\Delta)^a \psi \, dx = 0 \]
for any $\psi \in H^{2a} \cap H^\alpha_0$, so that $u = w$. Finally, by 23
\[ \int_{\Omega} (-\Delta)^a u \psi = \int_{\Omega} H_v(u,v) \psi. \]
A similar argument applies to $v$, therefore $(u,v)$ satisfies the regularity conditions in the statement and it is a weak solution to 8.

In view of Proposition 1, the proof of Theorem 1.1 will be straightforward once we show that the hypotheses of a critical point theorem due to Felmer [11], which we recall next, are satisfied in our situation.

**Definition 2.2.** Let $X$ be a Banach space and $I \in C^1(X)$. We say that $\{u_n\} \subset X$ is a Palais-Smale sequence for $I$ if $|I(u_n)| \leq C$ uniformly in $n$ and $\|I(u_n)\| \to 0$ as $n \to \infty$. If any Palais-Smale sequence has a strongly convergent subsequence, then we say that $I$ satisfies the Palais-Smale condition.

**Remark 5.** In the sequel, the Palais-Smale condition will be denoted by (PS).
(C_2) \ h(z,t) = 0, \text{ for all } z \in \partial Q
(C_3) \ h(z,0) = z \text{ for any } z \in \partial Q.

Then the critical level is given by

\[ I(z_0) = \inf_{h \in C^1} \sup_{z \in Q} I(h(z,1)). \]

2.2. Palais Smale condition. Let us recall the following preliminary lemma, see for instance [32, Proposition 2.2].

**Lemma 2.3.** Let \( X \) be a Banach space and \( I \in C^1(X) \) such that:

(i) any Palais-Smale sequence is bounded;

(ii) \( I'(z) = S(z) + K(z) \) where \( S: X \to X' \) is a homeomorphism and \( K: X \to X' \) is a compact map.

Then \( I \) satisfies (PS).

**Proof.** Let \( z_n \) be a Palais-Smale sequence; by hypotheses, it is bounded and \( S(z_n) + K(z_n) = I'(z_n) \to 0 \). Take \( w_n = K(z_n) \). By compactness, there exists a subsequence \( w_{n_k} \) such that \( w_{n_k} \to w \) for some \( w \). Hence

\[ z_{n_k} = S^{-1}(I'(z_{n_k}) - w_{n_k}) \to S^{-1}(-w), \]

thus \( z_{n_k} \) is a strongly convergent subsequence of \( z_n \) and \( I \) satisfies (PS). \( \square \)

**Proposition 3.** Assume condition 11 and let \( pq > 1 \). Then the functional 18 satisfies the (PS) condition.

**Proof.** Let \( (u_n, v_n) \) be a Palais-Smale sequence for \( I \) and assume \( \| (u_n, v_n) \|_E \geq R \), with \( R \) as in (H2). Let us prove that it is bounded. Since the operator norm of \( I'(u_n, v_n) \) satisfies \( \| I'(u_n, v_n) \| \to 0 \), one has for any choice of test functions \( \varphi, \psi \)

\[ |I'(u_n, v_n)(\varphi, \psi)| \leq \varepsilon_n \| (\varphi, \psi) \|_E, \quad (25) \]

with \( \varepsilon_n \to 0 \) as \( n \to \infty \). Moreover,

\[ I'(u, v)(\varphi, \psi) = \int_{\Omega} (A^p u A^{2\alpha-s} \varphi + A^q v A^{2\alpha-s} \psi - H_u(u, v) \varphi - H_v(u, v) \psi) \, dx, \]

thus by (H2)

\[ 0 < \frac{pq - 1}{p + q + 2} \int_{\Omega} H(u_n, v_n) \, dx \]

\[ \leq I(u_n, v_n) - I'(u_n, v_n) \left( \frac{q + 1}{p + q + 2} u_n, \frac{p + 1}{p + q + 2} v_n \right) \]

\[ \leq C_0 (1 + \varepsilon_n \| (u_n, v_n) \|_E). \]

By 12 there exist constants \( c_1, c_2 > 0 \) such that

\[ H(u, v) \geq c_1 (|u|^{p+1} + |v|^{q+1}) - c_2, \]

thus

\[ C_1 (1 + \varepsilon_n \| (u_n, v_n) \|_E) \geq \| u_n \|_{p+1}^{p+1} + \| v_n \|_{q+1}^{q+1}. \quad (26) \]

However, by 25 with \( \psi = 0 \),

\[ \left| \int_{\Omega} A^p \varphi A^{2\alpha-s} v_n \, dx \right| \leq \int_{\Omega} |H_u(u_n, v_n) \varphi| \, dx + \varepsilon_n \| \varphi \|_E. \quad (27) \]
and by Hölder inequality and (H4)

\[
\int_\Omega |H_u(u_n,v_n)| \, |\varphi| \, dx \leq C_3(\|u_n\|_{p+1}^p \|\varphi\|_{E^*}^p + \|v_n\|_{q+1}^{p(q+1)/(p+1)} \|\varphi\|_{E^*} + \|\varphi\|_{E^*}).
\]

(28)

Moreover, one can apply the Riesz Lemma (E* is a Hilbert space) to the functional

\[ T_{v_n} : E^s \to \mathbb{R}, T_{v_n}(\varphi) := \langle \varphi, A^{2\alpha-s}v_n \rangle_{E^*} = \int_\Omega A^s \varphi A^{2\alpha-s}v_n \] to obtain:

\[ \|v_n\|_{E^{2\alpha-s}} = \|A^{2\alpha-s}v_n\|_{E^*} = \|T_{v_n}\| = \sup_{\|\varphi\|_{E^*} = 1} \left| \int_\Omega A^s \varphi A^{2\alpha-s}v_n \, dx \right|. \]

Combining 27 and 28, one has

\[ \|v_n\|_{E^{2\alpha-s}} \leq C_4(\|u_n\|_{p+1}^p + \|v_n\|_{q+1}^{p(q+1)/(p+1)} + 1). \]

Analogously

\[ \|u_n\|_{E^s} \leq C_5(\|v_n\|_{q+1}^q + \|u_n\|_{p+1}^{q(p+1)/(q+1)} + 1), \]

hence by 26

\[ \|(u_n,v_n)\|_{E} \leq C_6(1 + \varepsilon_n (\|(u_n,v_n)\|_{E})) \]

and \((u_n,v_n)\) turns out to be bounded.

Next we apply Lemma 2.3. Indeed,

\[ I'(u,v) = A'(u,v) - J'(u,v), \]

where \(A(u,v) = \int_\Omega A^s u A^{2\alpha-s}v \, dx\) and \(J' : E \to E'\) is an homeomorphism, whereas \(J^*\) is compact, as pointed out in Remark 4.

2.3. Linking geometry. In the sequel, set \(z = (u,v) \in E\) and \(I(z) = \frac{1}{2} \langle Lz, z \rangle_H - J(z)\) as defined in 18.

Proposition 4. Assume condition 11 holds and let \(pq > 1\). Then, there exist two linear, bounded, invertible operators \(B_1, B_2 : E \to E\) such that, given \(\tau \geq 0\), then \(B_\tau = P_2 B_1^{-1} e^\tau B_2 : E^\tau \to E^{\tau}\) is invertible, where \(P_2\) is the projection of \(E\) onto \(E^-\).

Moreover, let \(e^+ = (e^+_1, e^+_2) \in E^+\) with \(\|e^+\|_{E^*} = 1\) and \(e^{-}_1 \in E^s\) eigenfunction of \((-\Delta)^\alpha\) with associated eigenvalue \(\lambda > 0\). Then, there exist constants \(\rho > 0\), \(R_1 > \rho/\|B_1^{-1} B_2 e^+\|_{E^*}\) and \(R_2 > \rho\) such that, setting

\[ S := \{ B_1 z^+ : z^+ \in E^+, \|z^+\|_{E} = \rho \} \]

and

\[ Q := \{ B_2 (t e^+ + z^-) : 0 \leq t \leq R_1, z^- \in E^-, \|z^-\|_{E} \leq R_2 \}, \]

the following conditions hold true:

\((G1)\) \(I(z) \geq \sigma > 0\) on \(S\)

\((G2)\) \(I(z) \leq 0\) on \(\partial Q\).

Proof. Define

\[ B_1(u,v) = (\rho^{\mu-1} u, \rho^{\nu-1} v) \]

and

\[ B_2(u,v) = (R_1^{\mu-1} u, R_1^{\nu-1} v), \]

where \(\rho\) and \(R_1\) will be chosen in the sequel and \(\mu, \nu \geq 1\) satisfy

\[ \frac{1}{p + 1} \leq \frac{\mu}{\mu + \nu}, \quad \frac{1}{q + 1} \leq \frac{\nu}{\mu + \nu}. \]

(29)
Hence $\hat{B}_r$ is invertible, indeed $\hat{B}_r z^- = mz^-$ with $m > 0$ constant if one assumes that $R_1 > 1$ and $\rho < 1$ (see [12, Proposition 3.1]).

Note that with our choice of $B_1, B_2$ one has:

$$S = \{ (\rho^{\mu-1} u^+, \rho^{\nu-1} v^+) : \|(u^+, v^+)\|_E = \rho, \ z^+ = (u^+ + v^+) \in E^+ \} ;$$

$$Q = \{ t(R_1^{\mu-1} e_1^+, R_1^{\nu-1} e_2^+) + (R_1^{\mu-1} u^+ - R_1^{\nu-1} v^-) : 0 \leq t \leq R_1, \ z^- = (u^-, v^-) \in E^-, \|(u^-, v^-)\|_E \leq R_2 \}.$$

Proof of (G1). For any $(\rho^{\mu-1} u^+, \rho^{\nu-1} v^+) \in S$ one has by (H3) and $\rho$ small enough

$$I(\rho^{\mu-1} u^+, \rho^{\nu-1} v^+) \geq \rho^{\mu+\nu-2} \int_\Omega A^s u^+ A^{2\alpha-s} v^+ \, dx - a\rho^{(\mu-1)(p+1)} \int_\Omega |u^+|^{p+1} \, dx$$

$$\quad - a\rho^{(\nu-1)(q+1)} \int_\Omega |v^+|^{q+1} \, dx,$$

hence recalling 17 and the continuous embedding $E \hookrightarrow L^{p+1} \times L^{q+1},$

$$I(\rho^{\mu-1} u^+, \rho^{\nu-1} v^+) \geq \frac{1}{2} \rho^{\mu+\nu-2} \|[z^+]_E^2 - b_1 \rho^{(\mu-1)(p+1)} \|[z^+]_E^{p+1}$$

$$\quad - b_2 \rho^{(\nu-1)(q+1)} \|[z^-]_E^{q+1},$$

for suitable constants $b_1, b_2$. Considering $\|[z^+]_E = \rho,$

$$I(\rho^{\mu-1} u^+, \rho^{\nu-1} v^+) \geq \frac{1}{2} \rho^{\mu+\nu} - b_1 \rho^{(p+1)} - b_2 \rho^{(q+1)}$$

and by 29 this quantity is positive for $\rho$ small enough.

Proof of (G2). Let us split the boundary into three parts as follows:

- $Q \cap \{ t = 0 \}$. By direct computation, $I(z) \leq 0.$ Indeed,

$$I(R_1^{\mu-1} u^-, R_1^{\nu-1} v^-) \leq R_1^{\mu+\nu-2} \int_\Omega A^s u^- A^{2\alpha-s} v^-$$

$$\quad = - R_1^{\mu+\nu-2} \int_\Omega |A^s u^-|^2 \leq 0.$$

- $Q \cap \{ t = R_1 \}$. Fix $R_2 > 0$ arbitrary and choose

$$z^- = (u^-, v^-) = - A^{-2\alpha+2s} u^- \in E^-$$

such that $\|(z^-)\|_E \leq R_2,$ thus

$$z = t(R_1^{\mu-1} e_1^+, R_1^{\nu-1} e_2^+) + (R_1^{\mu-1} u^-, R_1^{\nu-1} v^-) = (u, v) \in Q.$$

We can write $u^- = re_1^+ + w$ where $w \in E^s$ is orthogonal to $e_1^+$ in $L^2$ and $r \in \mathbb{R}.$

Suppose $r \geq 0.$ One has

$$(r + t) \int_\Omega |e_1^+|^2 = \int_\Omega (te_1^+ + u^-) e_1^+ \leq \|te_1^+ + u^-\|_{p+1} \|e_1^+\|_{(p+1)/p}$$

and

$$(r + t) \leq C_1 \|te_1^+ + u^-\|_{p+1}.$$

By 12

$$J(z) \geq c_1 R_1^{(p+1)(\mu-1)} \int_\Omega |te_1^+ + u^-|^{p+1}$$

$$\quad + c_1 R_1^{(q+1)(\nu-1)} \int_\Omega |te_1^+ + v^-|^{q+1} - c_2,$$
thus
\[ J(z) \geq C_2 R_1^{(p+1)(\mu-1)}(r+t)^{p+1} - c_2 \]
and
\[ J(z) \geq C_2 R_1^{(p+1)(\mu-1)}t^{p+1} - c_2. \]

Similarly, if \( r \leq 0 \), since
\[ e_2^+ = A^{-2\alpha+2s}e_1^+ = \lambda^{\frac{2\alpha+2s}{2\alpha}}e_1^+ \]
and \( v^- = -A^{-2\alpha+2s}u^- \), we get
\[
\langle v^-, e_1^+ \rangle = \langle -A^{-2\alpha+2s}u^-, e_1^+ \rangle = \langle -A^{-2\alpha+2s}(re_1^+ + w), e_1^+ \rangle
\]= \( -r\lambda^{\frac{2\alpha+2s}{2\alpha}} \int_{\Omega} |e_1^+|^2 - \langle w, A^{-2\alpha+2s}e_1^+ \rangle = -r\lambda^{\frac{2\alpha+2s}{2\alpha}} \int_{\Omega} |e_1^+|^2, \)
hence
\[
\lambda^{\frac{2\alpha+2s}{2\alpha}}(r + t) \int_{\Omega} |e_1^+|^2 = \int_{\Omega} (te_2^+ + v^-)e_1^+
\leq \|te_2^+ + v^-\|_{q+1} \|e_1^+\|_{(q+1)/q}
\]
and
\[
\lambda^{\frac{2\alpha+2s}{2\alpha}}(r + t) \leq C_3 \|te_2^+ + v^-\|_{q+1}. \]
As a consequence,
\[ J(z) \geq C_4 R_1^{(q+1)(\nu-1)}t^{q+1} - c_2. \]
Concluding, we have that either
\[ J(z) \geq C_2 R_1^{(p+1)(\mu-1)}t^{p+1} - c_2 \]
or
\[ J(z) \geq C_4 R_1^{(q+1)(\nu-1)}t^{q+1} - c_2. \]
Thus by 17 either
\[ I(z) \leq R_1^{\mu+\nu-2} \left( \frac{t^2}{2} - R_1^{\mu+\nu-2} \frac{1}{2} \right) \|z^-\|^2_E - C_2 R_1^{(p+1)(\mu-1)}t^{p+1} + c_2 \] (30)
or
\[ I(z) \leq R_1^{\mu+\nu-2} \left( \frac{t^2}{2} - R_1^{\mu+\nu-2} \frac{1}{2} \right) \|z^-\|^2_E - C_4 R_1^{(q+1)(\nu-1)}t^{q+1} + c_2. \] (31)

Therefore, by 29 one can choose \( t = R_1 \) such that the right-hand sides of both 30 and 31 are negative.

\* Choose \( R_2 \) such that the quantities in 30 and 31 are negative for any \( t \leq R_1 \).

Therefore, \( I \) is negative on \( \partial Q \) taking \( R_1, R_2 \) sufficiently large and the proof is complete.

**Remark 7.** Note that if \( \mu = \nu = 1 \), then conditions 29 imply \( p,q > 1 \). The possibility of choosing different values for \( \mu \) and \( \nu \) allows to deal with \( p,q \) not necessarily both bigger than 1, namely such that \( pq > 1 \).

**Remark 8.** Note that in the proof we have used both the fact that there exists a strictly positive eigenvalue \( \lambda \) and that the eigenfunctions of \((-\Delta)^{s}\) are in \( L^p \) for any \( s \), and these properties hold true by Lemma 2.1.
2.4. **Proof of Theorem 1.1.** Apply Proposition 2 to the situation where $H = E$, $H_1 = E^+$, $H_2 = E^-$, $I$ as in 18. Indeed, $L$ and $J$ satisfies the hypotheses of Proposition 2 due to Remark 4, $I$ satisfies (PS) by Proposition 3 and one can choose constants $\rho, R_1$ and $R_2$ and operators $B_1$ and $B_2$ such that the hypotheses on $S$ and $Q$ are satisfied, as shown in Proposition 4. Thus, one finds a critical point $(u, v)$ of $I$ such that $I(u, v) > 0$, and by Proposition 1 $(u, v)$ is a solution to 8, nontrivial since $I(0, 0) = 0$.

3. **Non-existence results: Proof of Theorem 1.2 and Corollary 2.** Let $(u, v)$ be a positive classical solution to 8 and $\alpha$ be even. The following Pohozaev type identity holds for any $a, b \in \mathbb{R}$, see Section 5 in [24]:

$$
\int_{\partial \Omega} \Delta^{\alpha/2} u \Delta^{\alpha/2} v (x \cdot \nu) = \int_\Omega (NH(u, v) - auH_u(u, v) - (N - a - b - 2\alpha)\Delta^{\alpha/2} u \Delta^{\alpha/2} v - bvH_v(u, v)).
$$

(32)

Similarly, for odd $\alpha$ one has

$$
\int_{\partial \Omega} \nabla \Delta^{(\alpha-1)/2} u \nabla \Delta^{(\alpha-1)/2} v (x \cdot \nu) = \int_\Omega (NH(u, v) - auH_u(u, v) - (N - a - b - 2\alpha)\nabla \Delta^{(\alpha-1)/2} u \nabla \Delta^{(\alpha-1)/2} v - bvH_v(u, v)).
$$

In the sequel, we consider the case of even $\alpha$, the odd case being similar. Theorem 1.2 follows immediately by applying the above Pohozaev type identities to problem 8. Indeed, the Green function of problem

$$
\begin{cases}
(-\Delta)^\alpha u = f & \text{in } B_1 \\
\frac{\partial u}{\partial \nu} = 0, r = 0, \ldots, \alpha - 1, & \text{on } \partial B_1
\end{cases}
$$

is positive; therefore, see [13, Theorem 5.7], if $(u, v)$ is a positive classical solution to 8, then $\Delta^{\alpha/2} u, \Delta^{\alpha/2} v > 0$ on $\partial \Omega$. Hence, by choosing $b = N - 2\alpha - a$, one has

$$
0 < \int_{\partial \Omega} \Delta^{\alpha/2} u \Delta^{\alpha/2} v (x \cdot \nu) = \int_\Omega (NH(u, v) - auH_u(u, v) - (N - 2\alpha - a)vH_v(u, v)) \leq 0,
$$

which is a contradiction. As for Corollary 2 it is enough to choose $a = \frac{N}{p+1}$. Thus, condition

$$
NH(u, v) - auH_u(u, v) - (N - 2\alpha - a)vH_v(u, v) \leq 0
$$

reads as follows

$$
\left(\frac{N}{q + 1} - N + 2\alpha + \frac{N}{p + 1}\right) \int_\Omega |v|^{q+1} \leq 0,
$$

which is equivalent to 15.

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