Sound search in a denotational semantics for first order logic

C.F.M. Vermeulen
February 1, 2008

Abstract

In this paper we adapt the definitions and results from Apt and Vermeulen in [4] to include important ideas about search and choice into the system. We give motivating examples. Then we set up denotational semantics for first order logic along the lines of Apt [1] and Apt and Vermeulen [4]. The semantic universe includes states that consist of two components: a substitution, which can be seen as the computed answer; and a constraint satisfaction problem, which can be seen as the residue of the original problem, yet to be handled by constraint programming. In the set up the interaction between these components is regulated by an operator called infer. In this paper we regard infer as an operator on sets of states to enable us to analyze ideas about search among states and choice between states.

The precise adaptations of definitions and results are able to deal with the examples and we show that given several reasonable conditions, the new definitions ensure soundness of the system with respect to the standard interpretation of first order logic. In this way the ‘reasonable conditions’ can be read as conditions for sound search. We indicate briefly how to investigate efficiency of search in future research.

1 Introduction

The motivating examples for this paper are those examples in constraint programming where a constraint satisfaction problem (CSP) can be handled by distinguishing cases. Let’s look at two such examples: first a rather trivial one, mainly for illustrative purposes; then a more realistic one that is taken from the current literature on constraint programming. All notation will be properly explained later on, but we already employ some of it in the presentation of the examples.

First consider the constraint satisfaction problem $x^2 = 1$ in a situation in which none of the values of the variables have been computed yet. We write this:

$C_0 = \langle x^2 = 1 ; \epsilon \rangle$
where $\epsilon$ is the empty substitution. We may be able to feed this problem to a constraint propagation tool that transfers it into the equivalent form:

$$C_1 = \langle x = 1 \lor x = -1 ; \epsilon \rangle.$$  

In [4] Apt and Vermeulen show how to formalize such a step. The preservation of equivalence in the transition from $C_0$ to $C_1$ is vital. It is essential for the soundness result in [4]. Such a soundness result is the way to check that the computation steps in the system are ‘all right’ according to first order logic.

Now we have a CSP $C_1$ that is disjunctive. So, it makes sense to distinguish two cases:

$$C_2 = \langle x = 1 ; \epsilon \rangle \text{ and } C_3 = \langle x = -1 ; \epsilon \rangle,$$

and proceed by distinguishing these two cases and compute the value of the variable $x$ in each case:

$$C_4 = \langle \top ; \{x/1\} \rangle \text{ and } C_5 = \langle \top ; \{x/-1\} \rangle.$$  

But by splitting things up we lose equivalence: neither $C_2$ nor $C_3$ are equivalent to the original $C_0$. And this loss of equivalence frustrates the soundness result from [4].

Another interesting example is the search for a suitable value for a variable $x$ in a domain $D_x$. Such a search could be organized by following some way of ranking the values $a \in D_x$. Discussion of variable ranking can be found, for example, in Milano and Van Hoeve [10]. They employ the ranking for distinguishing among the values $D_x$ for $x$ a subset of promising values, called $D_{\text{good}} \subset D_x$. Then the search for a solution to a CSP can be speeded up by splitting up the original CSP, first considering the good values in $D_{\text{good}}$ and considering the less promising part of $D_x$ later. Such tricks can result in a notable speed up for several tasks in constraint programming. For example, it is clear that an (in-)consistency check for a CSP, can benefit from such a distinction of likely and un-likely values. But the distinction does not satisfy the equivalence condition from [4]. Hence there is no general result to guarantee the soundness of the search strategy.

So, the situation is that we would like to be able to analyze search strategies and case distinction with the level of generality that is achieved by Apt and Vermeulen in [4]. But in the semantics of [4] methods for search and distinction of cases cannot readily be modeled, as it relies on the preservation of equivalence in the transition from one constraint satisfaction problem to the next. Therefore we propose a different notion of preservation of equivalence here, called pointwise equivalence, and show how with this new notion of equivalence the system in [4] can be adapted to suit the analysis of search strategies and case distinctions. In particular, we will show how this adaptation can be made without losing the crucial soundness properties of the semantics, so that the intended connection with the standard interpretation of first order logic remains intact.

## 2 Constraint propagation in first order logic

We repeat the facts from [4] and the notation introduced there.
Let’s assume that an algebra $\mathcal{J}$ is given over which we want to perform computations. This can be for example: the standard algebra for the language of arithmetic, in case we want to find solutions to equations or systems of equations; the algebra of terms over a first order language, in case we want to compute unifiers of terms; etc.

In each case the basic ingredient of the semantic universe will be the set of states, $\text{states}$. States come in two kinds. First we have an error state, which remains unanalyzed. All other states consist of two components: one component is a constraint satisfaction problem $\mathcal{C}$, the other a substitution $\theta$. Such a state is then written $\langle \mathcal{C}; \theta \rangle$. As always, a substitution $\theta$ is a mapping from variables to terms. It assigns a term $x\theta$ to each variable $x$, but there are only finitely many variables for which $x \neq x\theta$. These variables form $\text{dom}(\theta)$, the domain of $\theta$. The application of a substitution $\theta$ to a term $t$, written $t\theta$, is defined as usual. We denote the empty substitution by $\epsilon$. There is another convenient notation concerning substitutions: we write $\hat{\theta}$ for the conjunction $\bigwedge \{x = x\theta : x \in \text{dom}(\theta)\}$.

However our notion of substitution deviates from the usual notion of substitution in that the terms we assign to a variable are always partially evaluated in the intended algebra $\mathcal{I}$. This trick was developed and motivated by Apt in [1]. For example, if $\mathcal{I}$ is the standard algebra for the language of arithmetic, and we find that $x = 5$, then the substitution will set $\{x/5\}$, i.e. we assign the integer 5 to $x$ rather than the term 5. This strategy of evaluating as much as possible is then extended systematically. So, if $x\theta = 4$ and $y\theta = z$, then $(x + y)\theta = 4 + z$.

We can only compute the value for $x + y$ partially. But, if $x\theta = 4$ and $y\theta = 5$, then $(x + y)\theta = 9$.

Now we can already compute the value of $x + y \in \mathcal{I}$ completely. We refer to Apt’s [1] for more details on this trick for partial evaluation. Its main advantage is that we can now already use some special properties of the algebra $\mathcal{I}$ during the computation.

For the second example mentioned above, where $\mathcal{I}$ is an algebra of terms, the trick for partial evaluation does not make a difference: the partially evaluated substitutions and the usual notion of substitution give exactly the same results. So, we find the standard notion of substitution of logic programming as a special case.

A constraint satisfaction problem (CSP) $\mathcal{C}$, simply is a finite set of formulas of first order logic. In many applications there are extra requirements on the syntactic form of a CSP, but for now we keep things as general as possible. $\bot$ is a special formula which is always false. We also write $\mathcal{C}$ for $\bigwedge \mathcal{C}$, the conjunction over the formulas in $\mathcal{C}$. For a set of states $S$ we write $\bigvee S$ for the disjunction of all the formulas $\mathcal{C} \land \theta$ (for $\langle \mathcal{C}; \theta \rangle \in S$).

Throughout the paper we try to limit the number of brackets and braces as much as possible. In particular, for a finite set $\{A_1, \ldots, A_n\}$ we often write $A_1, \ldots, A_n$. Also, we write $\text{infer}(\mathcal{C}; \theta)$ instead of $\text{infer}(\langle \mathcal{C}; \theta \rangle)$, etc.
For the treatment of local variables we introduce a mapping $\text{DROP}_{u}$ (for each variable $u$). First we define $\text{DROP}_{u}$, a mapping on substitutions:

**Definition 2.1**

$$
\begin{align*}
    u\text{DROP}_{u}(\theta) &= u \\
    x\text{DROP}_{u}(\theta) &= x\theta \text{ for all other variables } x
\end{align*}
$$

So, $\text{dom}(\text{DROP}_{u}(\theta)) = \text{dom}(\theta) - \{u\}$.

We write $\mathcal{C}(u)$ for the part of $\mathcal{C}$ in which $u$ really occurs. Then we can define $\text{DROP}_{u}$, a mapping on states.

**Definition 2.2**

$$
\begin{align*}
    \text{DROP}_{u}\langle \mathcal{C}; \eta \rangle &= \langle \mathcal{C}; \text{DROP}_{u}(\eta) \rangle \quad \text{if } \mathcal{C}(u) = \emptyset \\
    \text{DROP}_{u}\langle \mathcal{C}; \eta \rangle &= \langle \exists u \ (u = u\eta \land \mathbf{y} = \mathbf{y}\eta \land \mathcal{C}(u)), \mathcal{C} - \mathcal{C}(u); \text{DROP}_{u}(\eta) \rangle \quad \text{if } \mathcal{C}(u) \neq \emptyset \\
    \text{DROP}_{u}\text{ERROR} &= \text{ERROR}
\end{align*}
$$

*Here $\mathbf{y}$ denotes the sequence of variables $y_1, \ldots, y_n$ such that $u \in y_i\eta$.*

This shows that $\text{DROP}_{u}$ removes $u$ from the domain of the assignment $\theta$ and existentially quantifies the occurrences of $u$ in the CSP $\mathcal{C}$.

In the definition of the denotational semantics we meet a parameter called $\text{infer}$. This is the crucial parameter in our story. $\text{infer}$ maps sets of states to sets of states. It can be instantiated to cover all kinds of constraint propagation (cf. Apt and Vermeulen [4]). Important examples to keep in mind are: the case where $\mathcal{I}$ is a term algebra and the constraint propagation tool performs unification; the case where $\mathcal{I}$ is the standard algebra of arithmetic and the constraint propagation tool can compute answers for certain types of equations very efficiently. In the first case we could find, for example, that:

$$
\langle \top; \{x/z, y/z\} \rangle \in \text{infer}(\{f(x) = f(y), \epsilon\}).
$$

I.e., the $\text{infer}$ operation computes the unifying substitution $\{x/z, y/z\}$. In the second example we may have a constraint propagation mechanism that solves certain quadratic equations over the integers and find that

$$
\langle x = 1 \lor x = -1; \epsilon \rangle \in \text{infer}(\{x^2 = 1, \epsilon\}).
$$

This type of constraint propagation could already be covered by Apt and Vermeulen in [4]. Here we also incorporate the analysis of search strategies within sets of states and the generation of subproblems from states. Then it can actually happen that we find:
So, infer will be able to come up with two essentially distinct computed answers. This was not allowed in [4].

Below we will discuss natural conditions on the infer operator that guarantee that the computations performed by infer respect first order logic. But for the definition we do not have to worry about these conditions yet.

We can now present our denotational semantics for first order logic in which the infer mapping is a parameter, as explained above. By having the infer parameter we obtain general results, that apply uniformly to various forms of constraint store management, search and case distinction.

We define the mapping $[\phi] : \text{pow}(\text{states}) \rightarrow \text{pow}(\text{states})$, using postfix notation. The definition is presented pointwise, for singleton sets $\{\langle C; \theta \rangle \}$. Then the general case is fixed by the equation $S[\phi] = \bigcup \{ \langle C; \theta \rangle[\phi] : \langle C; \theta \rangle \in S \}$. A final bit of notation is $\text{cons}^+(S)$, which is the subset of $S$ that contains exactly those states that are not inconsistent. So: $\text{cons}^+(S) = \{ \text{error} : \text{error} \in S \} \cup \{ \langle C; \theta \rangle \in S : \not\models \neg(C \land \hat{\theta}) \}$. We will use $\text{cons}(S)$ for $\text{cons}^+(S) - \{ \text{error} \}$.

\begin{definition}
\begin{align*}
\langle C; \theta \rangle[A] & = \text{infer}(C, A; \theta) \quad \text{for an atomic formula } A \\
\langle C; \theta \rangle[\phi_1 \lor \phi_2] & = \langle C; \theta \rangle[\phi_1] \cup \langle C; \theta \rangle[\phi_2] \\
\langle C; \theta \rangle[\phi_1 \land \phi_2] & = \langle (\langle C; \theta \rangle[\phi_1]) \rangle[\phi_2] \\
\langle C; \theta \rangle[\neg \phi] & = \begin{cases} 
\text{infer}(C; \neg \phi; \theta) & \text{if } \text{cons}^+(\langle C; \theta \rangle[\phi]) = \emptyset \\
\emptyset & \text{if } \langle C'; \theta' \rangle \in \text{cons}((\langle C; \theta \rangle[\phi]) \text{ for some } \langle C'; \theta' \rangle \text{ equivalent to } \langle C; \theta \rangle \\
\text{infer}(C; \neg \phi; \theta) & \text{otherwise}
\end{cases} \\
\langle C; \theta \rangle[\exists x \phi] & = \bigcup_{\sigma} \{ \text{infer } \text{drop}_u(\sigma) \} \text{ where, for some fresh } u, \\
& \sigma \text{ ranges over } \text{cons}^+(\langle C; \theta \rangle[\phi[x/u]]) \\
\text{error}[\phi] & = \{ \text{error} \} \text{ for all } \phi
\end{align*}
\end{definition}

The definition relies heavily on the notation that was introduced before. But it is still quite easy to see what goes on. The atomic formulas are handled by means of the infer mapping. Then, disjunction is interpreted as nondeterministic choice, and conjunction as sequential composition. For existential quantification we use the drop$_u$ mapping (for a fresh variable $u$). The error clause says that there is no recovery from error. In the case for negation, three contingencies are present: first, the case where $\phi$ is inconsistent ($\text{cons}^+(\langle C; \theta \rangle[\phi]) = \emptyset$). Then we continue with the input state $\langle C; \theta \rangle$. Secondly, the case where $\phi$ is already
true in (a state equivalent to) the input state. Then we conclude that \( \neg \phi \) yields inconsistency, i.e., we get \( \emptyset \). Finally, we add \( \neg \phi \) to the constraint store \( C \) if it is impossible at this point to reach a decision about the status of \( \neg \phi \).

Here we have made the choice to use \( \emptyset \) for inconsistency or falsehood. Strictly speaking there is something arbitrary about this choice. Any set \( S \) such that \( \text{CONS}^+(S) = \emptyset \) would have done equally well. We will see this also later on, when we discuss the formulation of the soundness theorem and the inconsistency condition (3) on \( \text{infer} \).

Next we show that the denotational semantics with the \( \text{infer} \) parameter is sound. This amounts to two things: 1. successful computations of \( \phi \) result in states in which \( \phi \) holds; 2. if no successful computation of \( \phi \) exists, \( \phi \) is false in the initial state. So, the soundness result show that the denotational semantics respects the standard semantics for first order logic.

### 3 Conditions on propagation and search

In \[4\] we formulated natural conditions on the instantiations of the \( \text{infer} \) operation. The effect of these conditions was that we could prove the soundness of the semantics, i.e., we could show that for each setting of the \( \text{infer} \) mapping that satisfies the conditions, we get a denotational semantics for first order logic that respects the standard interpretation of first order logic. As in \[4\] we use \( \text{infer} \) to analyze constraint propagation, the conditions can be seen as conditions for sound propagation.

The conditions from \[4\] are as follows:\(^1\)

1. If \( \langle C', \theta \rangle \in \text{infer}(\langle C, \theta \rangle) \), then \( C \land \hat{\theta} \models C' \land \hat{\theta}' \) (equivalence)

2. If \( \langle C', \theta' \rangle \in \text{infer}(\langle C, \theta \rangle) \), then also \( \langle C'_v, \theta'_v \rangle \in \text{infer}(\langle C, \theta \rangle) \), where \( \langle C'_v, \theta'_v \rangle \) is obtained from \( \langle C', \theta' \rangle \) by systematically replacing all occurrences of \( u \) by \( v \) for a variable \( u \) that is fresh w.r.t. \( \langle C, \theta \rangle \) and a variable \( v \) that is fresh w.r.t. both \( \langle C, \theta \rangle \) and \( \langle C'_v, \theta'_v \rangle \) (alphabetic variation)

3. If \( \text{infer}(\langle C, \theta \rangle) = \emptyset \), then \( C \theta \models \bot \) (inconsistency)

4. \( \text{infer}(\text{ERROR}) = \{ \text{ERROR} \} \). (error)

Condition (1) is an equivalence condition: it insists on preservation of logical equivalence by the operation \( \text{infer} \). Condition (2) is awkward to read, but it turns out to be the appropriate way of saying that \( \text{infer} \) should not depend on specific choices of fresh variables. Condition (3) insists on the preservation of inconsistency by \( \text{infer} \) and condition (4) is about the propagation of \( \text{ERROR} \).

In this paper we have lifted the \( \text{infer} \) operation to an operation on sets of states. So, we also have to lift these conditions in an appropriate way. Fortunately, for

\[^1\) \( \models \) is shorthand for both \( \phi \models \psi \) and \( \psi \models \phi \).

\[^2\) Please recall the remark about using \( \emptyset \) as the one-and-only inconsistency indicator. If we do not follow this policy, we can weaken the condition in (3) to: \( \text{CONS}^+(\text{infer}(\langle C, \theta \rangle)) = \emptyset \).
conditions (2)-(4) we do not have to change anything. But we do need to adapt the equivalence condition (1), as we have seen that it is not satisfied by the examples of case distinctions from section [1] that we want to analyze. The obvious way to adapt the equivalence condition on infer is perhaps:

**Definition 3.1 Set Equivalence:** \( \bigvee \text{infer}(S) = \models \bigvee S \)

where \( \phi = \models \psi \) stands for: \( \phi \models \psi \) and \( \psi \models \phi \) and \( \bigvee S \) is notation for: \( \bigvee \{ C \land \hat{\theta} : \langle C, \theta \rangle \in S \} \).

This condition allows us to split states in an infer step, as required in the examples. It allows for

\[
\text{infer}\left(\langle x = 1 \lor x = -1; \epsilon \rangle\right) = \{\langle x = 1; \epsilon \rangle, \langle x = -1; \epsilon \rangle\},
\]

as required. But it also allows us to re-group states in a confusing way. For example, if \( \langle x = 1; \epsilon \rangle, \langle x = -1; \epsilon \rangle \in S \), the set equivalence condition allows us to re-group this and have \( \langle x = 1 \lor x = -1; \epsilon \rangle \in \text{infer}(S) \). If we consider an example with three options, \( \langle x = 1 \lor x = 2 \lor x = 3; \epsilon \rangle \) for instance, we can see even more confusing forms of grouping and re-grouping. More generally, this condition gives us no control over the origin of states in \( \text{infer}(S) \). It does not tell us where a particular state \( \langle C; \theta \rangle \in \text{infer}(S) \) is coming from.

This is not the sort of search we are trying to cover and we will see that it messes things up in the soundness proof as well. Instead, we opt for pointwise equivalence:

**Definition 3.2 Pointwise Equivalence:** \( \bigvee \text{infer}(\langle C; \theta \rangle) = \models C \land \hat{\theta} \) for each \( S \) and each \( \langle C; \theta \rangle \in S \)

Now we can relate individual members of \( \text{infer}(S) \) to ancestors in \( S \). And the pointwise equivalence condition makes sure that each state in \( S \) is equivalent to the set of its descendants in \( \text{infer}(S) \). The definitions allow for one state to have several ancestors, but each of these ancestors has to be able to account for its descendants by itself. This way re-grouping as in the example above is no longer allowed. This is entirely compatible with our motivation and it will make the soundness proof run smoothly.

The following property relates the two conditions on infer:

**Definition 3.3 Continuity:** \( \text{infer}(S) = \bigcup \{ \text{infer}(\langle C; \theta \rangle) : \langle C; \theta \rangle \in S \} \)

**Proposition 3.1** Assume that \( \text{infer} \) satisfies the continuity condition. Then set equivalence and pointwise equivalence are equivalent.

**Proof:** Note that the proposition is about the equivalence of two equivalence conditions. First we check that set equivalence implies pointwise equivalence.

- We apply set equivalence to the one element set \( \{ \langle C; \theta \rangle \} \) for \( \langle C; \theta \rangle \in S \). This gives: \( \bigvee \text{infer}(\langle C; \theta \rangle) = \models C \land \hat{\theta} \), as required. (Note that we do not need continuity.)
Next we assume pointwise equivalence and check set equivalence. This is an exercise in handling disjunctions in propositional logic.

- Consider $\bigvee S$. This is a disjunction of formulas of form $C \land \hat{\theta}$ (for $\langle C; \theta \rangle \in S$.) So, to establish $\bigvee S \models \bigvee \text{infer}S$, it suffices to check that $C \land \hat{\theta} \models \bigvee \text{infer}S$, for each $\langle C; \theta \rangle \in S$. From pointwise equivalence we readily obtain: $C \land \hat{\theta} \models \bigvee \text{infer}(C; \theta)$. Now, by continuity we get: $\bigvee \text{infer}(C; \theta) \models \bigvee \text{infer}S$. Jointly this gives: $C \land \hat{\theta} \models \bigvee \text{infer}S$, as required.

- Next consider $\bigvee \text{infer}S$. We need to establish: $\bigvee \text{infer}S \models \bigvee S$. By continuity we know that $\bigvee \text{infer}S$ is the disjunction of all the $\bigvee \text{infer}(C; \theta)$ (for $\langle C; \theta \rangle \in S$). So, it suffices to check that each of these smaller disjunctions entails $\bigvee S$. Pointwise equivalence ensures that: $\bigvee \text{infer}(C; \theta) \models C \land \hat{\theta}$ and hence: $\bigvee \text{infer}(C; \theta) \models \bigvee S$ follows simply by propositional logic.

Below we will always assume pointwise equivalence for $\text{infer}$. Note that the formulation in this paper makes condition (3) a consequence of equivalence. Below we will only refer to condition (3) separately if this adds anything to the readability of the proofs.

4 Sound propagation and search

We start by stating the soundness claim and the preservation lemma in the new setting:

**Theorem 1 (Soundness)** Let $S$, $\langle C, \theta \rangle \in S$, $\phi$ be given. Then we have:

1. $\bigvee S[\phi] \models_\mathfrak{I} \phi$
2. If $\text{cons}^+(S[\phi]) = \emptyset$, then $\bigvee S \models_\mathfrak{I} \neg \phi$.

Here $\emptyset$ is a sign of inconsistency or falsehood: we have run out of options and reached the empty set. We have chosen to use $\emptyset$ as the specific set of states to indicate falsehood in the semantics. But, as was already pointed out before, any set of inconsistent states would have done equally well.

Note that (i) and (ii) have an equivalent pointwise formulation:

(i) for each $\langle C; \theta \rangle \in S[\phi]$: $C \land \hat{\theta} \models_\mathfrak{I} \phi$;

(ii) if $S[\phi] = \emptyset$, then for each $\langle C; \theta \rangle \in S$: $C \land \hat{\theta} \models_\mathfrak{I} \neg \phi$.

**Lemma 1 (Preservation)**

1. If $C \land \hat{\theta} \models_\mathfrak{I} \phi_1$ and $\langle C'; \theta' \rangle \in \langle C; \theta \rangle[\phi_2]$, then $C' \land \hat{\theta'} \models_\mathfrak{I} \phi_1$ (validity)

\footnote{In this section we insist on mentioning $\mathfrak{I}$ all the time to remind us that we are looking at the choice of values from $\mathfrak{I}$.}
ii. If \( C, \hat{\theta} \) and \( \langle \phi_1 \land \phi_2 \rangle \) are consistent (in \( \mathcal{I} \)) and there is a consistent state \( \langle C'; \theta' \rangle \in \langle C; \theta \rangle[\phi_2] \), then there is a state \( \langle C''; \theta'' \rangle \in \langle C; \theta \rangle[\phi_2] \), such that \( C'', \hat{\theta}'' \) and \( \langle \phi_1 \land \phi_2 \rangle \) are consistent (in \( \mathcal{I} \)).

The first part of the lemma insists that the computation of \( \phi_2 \) preserves the validity of \( \phi_1 \) and that the second part of the lemma insists that the computation of \( \phi_2 \) preserves the consistency of \( \phi_1 \) (with \( \phi_2 \) in a suitable state). We see that in the second part we are allowed to make a switch from \( \langle C'; \theta' \rangle \) to \( \langle C''; \theta'' \rangle \). (In the proof this option is only used in the cases for disjunction.)

The proof of the theorem is a simultaneous induction on the construction of \( \phi \). Simultaneity is required for the negation case. In the proof we need the preservation lemma crucially in the case for conjunction. The proof of the lemma itself is again a simultaneous induction, this time on the construction of \( \phi_2 \).

Both proofs follow the corresponding proofs in [4]. So, here we feel free to restrict attention to the atomic cases—they show how pointwise equivalence works—and the conjunction cases—they show the crucial use of the preservation lemma.\(^4\)

**Proof:** [Soundness]

**atoms**

In case \( \phi \) is an atomic formula \( A \) say:

\[
S[A] = \bigcup \{ \langle C; \theta \rangle[A] : \langle C; \theta \rangle \in S \} = \bigcup \{ \text{infer}(\langle C \cup \{ A \}; \theta \rangle) : \langle C; \theta \rangle \in S \}.
\]

i. Consider \( \langle C'; \theta' \rangle \in \text{infer}(\langle C \cup \{ A \}; \theta \rangle) \). Now: \( \langle C \land A \rangle \land \hat{\theta} \models \mathcal{I} A \). So, by pointwise equivalence also: \( C' \land \hat{\theta}' \models \mathcal{I} A \). So, \( \sqrt{S[A]} \models \mathcal{I} A \), as required.

ii. Suppose \( \text{infer}(\langle C \cup \{ A \}; \theta \rangle) \) contains only inconsistent states (for all \( \langle C; \theta \rangle \in S \)). By pointwise equivalence we may conclude \( C \land A \land \hat{\theta} \models \mathcal{I} \perp \) for all \( \langle C; \theta \rangle \in S \). Hence \( C \land \hat{\theta} \models \mathcal{I} \neg A \) for all \( \langle C; \theta \rangle \in S \). From this we conclude:

\( \sqrt{S} \models \mathcal{I} \neg A \), as required.

Note how pointwise equivalence ensures that \( \langle C'; \theta' \rangle \) has an ancestor. Here this can only be one state: \( \langle C \cup \{ A \}; \theta \rangle \).

**conjunction**

In case \( \phi \) is a conjunction, \( \phi_1 \land \phi_2 \) say, \( \langle C''; \theta'' \rangle \in \langle C; \theta \rangle[\phi] \) iff \( \langle C''; \theta'' \rangle \in \langle C'; \theta' \rangle[\phi_2] \) for some \( \langle C'; \theta' \rangle \in \langle C; \theta \rangle[\phi_1] \).

i. Let \( \langle C; \theta \rangle \in S \). Then the induction hypothesis gives: \( C'' \land \hat{\theta}'' \models \mathcal{I} \phi_2 \) and \( C' \land \theta' \models \mathcal{I} \phi_1 \). By persistence (i) we may conclude: \( C'' \land \hat{\theta}'' \models \mathcal{I} \phi_1 \). So, first order logic now gives: \( C'' \land \theta'' \models \mathcal{I} \phi_1 \land \phi_2 \) for each \( \langle C''; \theta'' \rangle \in S[\phi] \). Hence \( \sqrt{S[\phi]} \models \mathcal{I} \phi \), as required.

\(^4\)There is minor divergence from the formulation in [4]. We work with \( C \land \hat{\theta} \) instead of \( C \theta \). This facilitates the proofs marginally, as the reader familiar with [4] can check for himself.
ii. Now we know that $S[\phi_1 \land \phi_2]$ only contains inconsistent states. So, we have: if $\langle C'; \theta' \rangle \in [\phi_1]$ is consistent, then $\langle C'; \theta' \rangle[\phi_2]$ only contains inconsistent states. From this we may conclude by induction hypothesis that: for each consistent $\langle C'; \theta' \rangle \in S[\phi_1]$, $C' \land \theta' \models \neg \phi_2 (\otimes)$.

Now assume that for some $\theta$: $\models I (C \land \hat{\theta} \land \phi_1 \land \phi_2)[\theta]$ and that we have a consistent $\langle C'; \theta' \rangle \in S[\phi_1]$. Then preservation (ii) tells us that the consistency is preserved, i.e., there is a state $\langle C''; \theta'' \rangle \in S[\phi_1]$ and values $[\theta'']$: such that: $\models I (C'' \land \hat{\theta}'' \land \phi_1 \land \phi_2)[\theta'']$. But this contradicts (\otimes). So, for no $\theta$: $\models I (C \land \hat{\theta} \land \phi_1 \land \phi_2)[\theta]$. Hence $C \land \hat{\theta} \models \neg (\phi_1 \land \phi_2)$ (for all $\langle C'; \theta' \rangle \in S)$, which is as required.

\[\Box\]

**Proof:** [Preservation]

**atoms**

In the atomic case $\phi_2 = A$ for some atom $A$ and $\langle C'; \theta' \rangle \in infer(C \cup \{A\}; \theta)$.

i. We know that: $C \land \hat{\theta} \models \phi_1$. So, also: $(C \land A) \land \hat{\theta} \models \phi_1$.

By pointwise equivalence this gives: $C' \land \hat{\theta'} \models \phi_1$, as required.

ii. The assumption gives us values $[\theta]$:

$\models I (C \land \hat{\theta} \land (\phi_1 \land A))[\theta]$.

It is harmless to add a copy of $A$ to get: $\models I (C \land A \land \hat{\theta} \land (\phi_1 \land A))[\theta]$.

Now pointwise equivalence ensures that there are $[\theta']$ such that:

$\models I (C' \land \hat{\theta'} \land (\phi_1 \land A))[\theta']$.

**conjunction**

In this case $\phi_2 = (\psi_1 \land \psi_2)$ and $\langle C''; \theta'' \rangle \in (C'; \theta')[\psi_2]$ for some $\langle C'; \theta' \rangle \in (C'; \theta'][\psi_1]$.

i. By induction hypothesis (for $\phi_1$ and $\psi_1$): $C' \land \hat{\theta'} \models \phi_1$.

By a second application of the induction hypothesis (to $\phi_1$ and $\psi_2$):

$C'' \land \hat{\theta''} \models \phi_1$.

ii. By assumption: $\models I (C \land \hat{\theta} \land (\phi_1 \land (\psi_1 \land \psi_2)))[\theta]$.

So: $\models I (C \land \hat{\theta} \land (\phi_1 \land \psi_1))[\theta]$.

By induction hypothesis we get: $\models I (C'' \land \hat{\theta''} \land (\phi_1 \land \psi_1))[\theta']$ (for suitable $\langle C''; \theta'' \rangle$).

Next the induction hypothesis (for $\phi_1 \land \psi_1$ and $\psi_2$) provides:

$\models I (C''' \land \hat{\theta'''} \land ((\phi_1 \land \psi_1) \land \psi_2))[\theta''']$ (for suitable $\langle C'''; \theta''' \rangle$, as required.
This establishes that all settings of the infer-parameter that satisfy the conditions discussed in section 3 result in sound semantics for first order logic: all the instantiations of infer only produce outcomes of $S[\phi]$ that satisfy $\phi$ and only report false if $\phi$ is false in $S$. In [4] we show how a large number of forms of constraint propagation are sound instances of the infer-parameter. It is clear that also lots of search tricks can be modeled as settings of the infer-parameters. (See section 5 for the discussion of our motivating examples.) If these settings of infer obey the pointwise equivalence condition, they will lead to a sound instantiation of the semantics. So, we can now also read the conditions in section 3 as conditions for sound search.

5 Looking back and ahead

Let’s go back to the motivating examples from section 1. There we used the examples to illustrate the use of all kinds of ‘disjunctive splits’ of states into substates to model search and subproblem selection. Now it is clear that such disjunctive splits leads to definitions of the infer parameter that obey the new, pointwise equivalence condition. For example, if $\{x = 1 \lor x = -1; \epsilon\}$, then $\{(x = 1; \epsilon), (x = -1; \epsilon)\} \subseteq \text{infer}S$ is consistent with the conditions on infer that we propose. Similarly, $S = \{x \in D_x; \epsilon\}$ and $D_x = D_{\text{good}} \cup D_{\text{bad}}$, then $\{(x \in D_{\text{good}}; \epsilon), (x \in D_{\text{bad}} \epsilon)\} \subseteq \text{infer}S$ satisfies pointwise equivalence. So, the adaptation of the definitions pays off: the system proposed in [4] has now been extended to include the investigation of such search strategies in a sound way.

This means that we now have an extremely rich system:

- the denotational semantics for first order logic that we present gives natural computational readings for the logic connectives (following [1]);

- it allows for the investigation of a wide variety of forms of constraint propagation (following [4]);

- and now it also includes the option of analyzing search routines (as suggested in [10]).

All these ingredients are combined in one system in such a way that soundness with respect to the standard interpretation of first order logic is preserved. Hence we can regard the conditions on infer that we have presented as conditions for sound search in constraint programming.

The soundness theorem shows how attractive the combination of ingredients proposed is for establishing general results. In [4] (and here in section 5) it was shown how different forms of constraint propagation can be seen as instantiations of the infer parameter. Hence these forms of constraint programming can be covered all at once, by proving one theorem only. Here we extend the level of generality to several ideas about search. Of the two examples of search in
this paper, the first example has didactic merits only: it certainly is not a ‘hot issue’ in the current literature. But this first example indicates in a convincing way how other sorts of search tricks also fall within the scope of our proposal. In particular the search tricks based on ranking of values of variables from \[10\]. Such search tricks certainly are a real issue in the current literature on constraint programming. And we can cover them in the proposed analysis.

We have given a soundness theorem as an example of general results. Soundness is a natural requirement on search techniques: we do not want to lose anything as we are searching. And the way in which we translate this soundness claim into the format of \[1\] is extremely natural. But we do not yet give an equally natural way of translating other hot issues concerning search, such as efficiency claims about search tricks, into the format. This a clearly an interesting task for future research. As a starting point for such investigations we see \[14\]. There the axiomatization and decidability of the denotational semantics in Apt’s \[1\] is discussed in a way that allows us to estimate upperbounds for the complexity of the semantics. If these results are combined with conditions on the complexity of the \textit{infer} parameter, this should allow the analysis of efficient combinations of computation, constraint propagation and search.

\section*{References}

[1] K.R. Apt. A denotational semantics for first-order logic. In Proc. of the computational logic conference (CL2000), Lecture Notes in Artificial Intelligence 1861, pages 53–69. Springer Verlag, 2000.

[2] K.R. Apt, J. Brunekreef, V. Partington, and A. Schaefer. Alma-0: An imperative language that supports declarative programming. ACM Toplas, 20(5):1014–1066, 1998.

[3] K. R. Apt and A. Schaefer. The Alma project, or how first-order logic can help us in imperative programming. In E.-R. Olderog and B. Steffen, editors, Correct System Design, Lecture Notes in Computer Science 1710, pages 89–113, 1999.

[4] K. R. Apt and C. Vermeulen. First-order Logic as a Constraint Programming Language. In A. Voronkov and M. Baaz, editors, Logic for Programming, Artificial Intelligence and Reasoning, Lecture Notes in Artificial Intelligence 2514, pages 19–35, 2002.

[5] F.S. de Boer, A. Di Pierro, and C. Palamidessi. Nondeterminism and infinite computations in constraint programming. Theoretical Computer Science, 151(1):37–78, 1995.

[6] F.S. de Boer, M. Gabbrielli, E. Marchiori, and C. Palamidessi. Proving concurrent constraint programs correct. In ACM Transactions on Programming Languages and Systems, volume 19(5), pages 685–725, 1997.
[7] F. Fages, P. Ruet, and S. Soliman. Linear concurrent constraint programming: Operational and phase semantics. *Information and Computation*, 165(1):14–41, 2001.

[8] J. Jaffar and J.M. Maher. Constraint logic programming: a survey. *Journal of Logic Programming*, 19/20, 1994.

[9] J. Jaffar, J.M. Maher, K. Marriott, and P. Stuckey. The semantics of constraint logic programs. *Journal of Logic Programming*, 37(1):1–46, 1998.

[10] M. Milano and W.J. van Hoeve. Reduced cost-based ranking for generating promising subproblems. In P. van Henteryck, editor, *Proceedings of CP’02*, Lecture Notes in Computer Science 2470, Springer Verlag, pages 1–16, 2002.

[11] C. Palamidessi, F.S. de Boer, and A. Di Pierro. An algebraic perspective of constraint logic programming. *Journal of Logic and Computation*, 7, 1997.

[12] V. A. Saraswat, M. Rinard, and P. Panangaden. Semantic foundations of concurrent constraint programming. In *Conference Record of the Eighteenth Annual ACM Symposium on Principles of Programming Languages*, pages 333–352, Orlando, Florida, 1991.

[13] V. A. Saraswat. *Concurrent Constraint Programming*. MIT Press, 1993.

[14] C. Vermeulen. Decidability and Axiomatization of a Denotational Semantics for First Order Logic. *Journal for the Theory and Practice of Logic Programming*, to appear.