Remnant masses, spins and recoils from the merger of
generic black hole binaries

Carlos O Lousto, Manuela Campanelli, Yosef Zlochower and
Hiroyuki Nakano

Center for Computational Relativity and Gravitation, and School of Mathematical Sciences,
Rochester Institute of Technology, 85 Lomb Memorial Drive, Rochester, NY 14623, USA

E-mail: colsma@rit.edu (Carlos Lousto)

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Abstract
We obtain empirical formulae for the final remnant black hole mass, spin, and
recoil velocity from merging black hole binaries (BHBs) with arbitrary mass
ratios and spins. Our formulae are based on the mass ratio and spin dependence
of the post-Newtonian expressions for the instantaneous radiated energy, linear
momentum, and angular momentum, as well as the ISCO binding energy
and angular momentum. The relative weight between the different terms is
fixed by amplitude parameters chosen through a least-squares fit of recently
available fully nonlinear numerical simulations. These formulae can be used
for statistical studies of \( N \)-body simulations of galaxy cores and clusters, and
the cosmological growth of supermassive black holes. As an example, we
use these formulae to obtain a universal spin magnitude distribution of merged
black holes and recoil velocity distributions for dry and hot/cold wet mergers.
We also revisit the long-term orbital precession and resonances and discuss how
they affect spin distributions before the merging regime.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Black holes at the centers of galaxies and globular clusters significantly impact the dynamical
evolution of these astronomical structures. Of particular importance to the dynamics are
the black hole (BH) mass, spin, and location (if off-center), properties that can significantly
change following a BH merger. When two galaxies merge, an event that is expected to occur
several times during a galaxy’s evolution, the supermassive BH at their centers forms a black
hole binary (BHB) that eventually inspirals and merges. Similarly, intermediate-mass BHs
in globular clusters can form tight BHBs that inspiral and merge. Consequently, accurate
models for the mass, spin, and gravitational recoil of the merger remnants of BHBs are of great astrophysical interest. However, these models require accurate simulations of merging BHs, a problem in the highly nonlinear regime, that only recently became feasible due to breakthroughs in numerical relativity [1–3].

The first attempts at modeling the remnant BH of BHB mergers using fully nonlinear simulations utilized the ‘Lazarus method’ [4], which combined short-term numerical simulations of BHBs, just prior to merger, with perturbative calculations. With the advent of the ‘moving punctures’ [2, 3] and generalized harmonic [1] approaches, it became possible to accurately model merging BHBs from inspiral through merger and ringdown using fully nonlinear numerical simulations. As a result of these breakthroughs, NR groups from around the world have been able to develop heuristic models for the properties of remnant BHs as a function of the orbital and intrinsic BH parameters of the binary (at least for part of the parameter space).

The initial attempts at modeling the properties of the remnant BH focused on the mass and spin using ad hoc interpolation formulae. In [5–7] we studied equal-mass, spinning BHBs, where the individual BH spins were aligned and counter-aligned with the orbital angular momentum, using fully nonlinear numerical calculations. We found a simple quadratic polynomial relating the final mass and spin of the remnant with the spins of the individual BHs. This scenario was later revisited in [8, 9], and in [10] the authors generalized the formula for the remnant spin (by assuming that the angular momentum is only radiated along the orbital axis, and neglecting the energy loss) in order to model arbitrary BH configurations (these assumptions were relaxed in a follow-up paper [11]). A generic formulae for the final spin was proposed in [12] based on simulations with aligned and non-aligned spins. A more comprehensive approach, using a generic Taylor expansion consistent with the physical symmetries of the problem, and with the parameters chosen by a least-squares fit to many simulations, was developed in [13]. All of these models used low-order polynomial interpolation functions to predict the remnant mass and spin as a function of the individual BH masses and spins. On the other hand, in [14], a different approach, based on approximate analytic models for the merger, was used. Here the authors extended the particle limit approximation for the radiated mass and angular momentum to the comparable-mass regime, ignoring effects of post-ISCO (innermost stable circular orbit) gravitational radiation. This approach was further improved in [15] by taking binding energies into account. All of these approaches show a certain degree of agreement with the remnant masses and spins obtained in a few dozen fully nonlinear numerical simulations available, but significant uncertainties concerning accuracy outside this range of the parameter space remain. Here we propose a set of formulae that incorporate the benefits of both approaches and regimes in a unified way using analytic techniques to develop empirical models with free parameters determined by numerical results.

Due to its significant impact on astrophysics, the modeling of the remnant recoil followed an independent path, particularly since the discovery [16, 17] that the spins of black holes play a crucial role in producing recoils of up to 4000 km s\(^{-1}\). The realization that the merger of BHBs can produce recoil velocities that allow the remnant to escape from major galaxies leads to numerous theoretical and observational efforts to find traces of this phenomenon. Several studies made predictions of specific observational features of recoiling supermassive black holes in the cores of galaxies in the electromagnetic spectrum [18–24]. Notably, there began to appear observations indicating the possibility of detection of such effects [25–27], and although alternative explanations are possible [28–30], there is still the exciting possibility that these observations can lead to the first confirmation of a prediction of general relativity in the highly dynamical, strong-field regime.
This paper is organized as follows. In section 2 we describe our empirical formula for the remnant gravitational recoil and provide the leading coefficients for this formula. In section 3 we describe our formula for the final remnant mass, while in section 4 we describe the formula for the final remnant spins. We provide fits to the constants in the remnant mass and spin formula in section 5. In section 6 we revisit the gravitational alignment and antialignment mechanisms for long-term inspiral orbits, and discuss the consequences and applications of our formulae in section 7.

2. Remnant recoil velocities

In our approach to the recoil problem [16, 17], we used post-Newtonian (PN) theory as a guide to model the recoil dependence on the physical parameters of the progenitor BHB (see equations (3.31) in [31]) while arguing that only full numerical simulations can produce the correct amplitude of the effect (see equation (3) below). For example, in the instantaneous radiated linear momentum, there are terms of the form

$$\frac{d\vec{P}}{dt} = \cdots + \eta^2 \frac{1}{1 + q} [\vec{F}(\vec{r}, \vec{v}) \cdot (\vec{a}_2 - q\vec{a}_1)] \hat{L},$$

(1)

where $\hat{L}$ is the unit vector pointing along the instantaneous orbital angular momentum, $\vec{F}(\vec{r}, \vec{v})$ is a vector in the orbital plane that is only a function of the orbital position and its time derivative, $q = m_1/m_2 \leq 1$ is the mass ratio, $\eta = q/(1+q)^2$ is the symmetric mass ratio, and $\vec{a}_i = \vec{S}_i/m_i^2$ is the intrinsic spin on the black hole $i$. We incorporate this term by adding a term

$$\vec{V}_{\text{recoil}}(q, \vec{\alpha}) = \cdots + \left( K \frac{\eta^2}{(1 + q)} \left[ |\alpha_2^\perp - q\alpha_1^\perp| \cos(\Theta_1/Delta_1 - \Theta_1) \right] \right) \hat{L}$$

(2)

to our fitting formula for the recoil velocity (see equation (3) below), where the fitting constants $K$ and $\Theta_0$ approximate the net effect of the dynamics of this term during the last few $M$ of the rapid plunge (where most of the recoil is generated) and $\Theta_\Delta$ is the angle that $\vec{\Delta} = M^2(\vec{a}_2 - q\vec{a}_1)/(1 + q)$, where $M = m_1 + m_2$, makes with the infall direction at merger. Our heuristic formula describing the recoil velocity of BHB remnants was theoretically verified in several ways. In [17] we confirmed the sinusoidal dependence ($\cos(\Theta_1/Delta_1$ in equation (3)) of the recoil on the direction of the in-plane spin for the so-called superkick configurations, a result that was reproduced in [32] for binaries with different initial separations, while in [33] the authors verified the decomposition of the spin dependence of the recoil into spin components perpendicular and parallel to the orbital plane. Similarly, in [34] the authors determined that the quadratic-in-spin corrections to the in-plane recoil velocity are less than 5% of the total recoil. Recently in [35] we confirmed the leading $\eta^2$ (where $\eta$ is symmetric mass ratio) dependence of the large recoils out of the orbital plane (see also [36]).

Here we augment our original empirical formula with subleading terms, higher order in the mass ratio, and include a new term linear in the total spin, motivated by higher order post-Newtonian computations [37]:

$$\vec{V}_{\text{recoil}}(q, \vec{\alpha}) = v_m \hat{e}_1 + v_\perp (\cos \xi \hat{e}_1 + \sin \xi \hat{e}_2) + v_{\parallel} \hat{n}_{\|},$$

$$v_m = A \frac{\eta^2(1 - q)}{(1 + q)} [1 + B\eta],$$

$$v_\perp = H \frac{\eta^2}{(1 + q)} \left[ (1 + B_H\eta)(\alpha_2^\perp - q\alpha_1^\perp) + H_S \frac{(1 - q)}{(1 + q)^2}(\alpha_2^\parallel + q^2\alpha_1^\parallel) \right].$$

3
\begin{align*}
v_{\parallel} &= K \frac{\eta^2}{(1 + q)} \left[ (1 + B_K \eta) \left| \alpha_2^\perp - q \alpha_1^\parallel \right| \cos(\Theta_\Delta - \Theta_0) \\
&+ K_S \frac{(1 - q)}{(1 + q)^2} \left| \alpha_2^\perp + q^2 \alpha_1^\parallel \right| \cos(\Theta_S - \Theta_1) \right],
\end{align*}

where the indices \( \perp \) and \( \parallel \) refer to perpendicular and parallel to the orbital angular momentum respectively and \( \hat{n}_{\parallel} = \hat{L} \). \( \hat{e}_1, \hat{e}_2 \) are orthogonal unit vectors in the orbital plane, and \( \xi \) measures the angle between the unequal mass and spin contribution to the recoil velocity in the orbital plane. The new constants \( H_S \) and \( K_S \) can be determined from new generic BHB simulations as the data become available. The angles \( \Theta_\Delta \) and \( \Theta_S \) are the angles between the in-plane component of \( \Delta = M(S_2/m_2 - S_1/m_1) \) or \( \vec{S} = \vec{S}_1 + \vec{S}_2 \) and the infall direction at merger. Phases \( \Theta_0 \) and \( \Theta_1 \) depend on the initial separation of the holes for quasicircular orbits. A crucial observation is that the dominant contribution to the recoil is generated near the time of formation of the common horizon of the merging black holes (see, for instance, figure 6 in [38]). Formula (3) above describing the recoil applies at this moment (or averaged coefficients of formation of the common horizon of the merging black holes (see, for instance, figure 6 in [38]). Formula (3) above describing the recoil applies at this moment (or averaged coefficients of formation of the common horizon of the merging black holes (see, for instance, figure 6 in [38]).

The most recent estimates for the above parameters can be found in [35] and references therein. The current best estimates are \( A = 1.2 \times 10^4 \) km s\(^{-1}\), \( B = -0.93 \), \( H = (6.9 \pm 0.5) \times 10^3 \) km s\(^{-1}\), \( K = (6.0 \pm 0.1) \times 10^4 \) km s\(^{-1}\), and \( \xi \sim 145^\circ \). Note that we can use the data from [35] to obtain \( K = (6.072 \pm 0.065) \times 10^4 \) km s\(^{-1}\) if we assume that \( B_K \) and \( K_S \) are negligible. Finally, if we fit the data to find \( K \) and \( K_S \) simultaneously, we obtain \( K = (6.20 \pm 0.12) \times 10^4 \) km s\(^{-1}\) and \( K_S = -0.056 \pm 0.041 \), where we made the additional assumption that \( \Theta_0 = \Theta_1 \) (since \( \vec{S} = \Delta \) for these runs). An attempt to fit \( K \), \( K_S \), \( B_K \) simultaneously does not produce robust results with currently available data (one of the reasons for this is that different values of \( K \) and \( B_K \) produce very similar predicted recoil velocities over the range \( 0.1 \leq q \leq 1 \)). Note that the values for the dominant \( K \) term are reasonably insensitive to the different choices for the fits, while finding the subleading terms require additional runs and higher accuracy.

The above equation (3) contains all the expected linear terms in the spin, and includes ten fitting parameters. Based on the works [37], one could add quadratic terms, and this will be published elsewhere by the authors.

From a practical point of view, for statistical simulations of BHB mergers, where the directions of the spins and infall are not known, one should take a uniform distribution for the in-plane components of \( \hat{e}_1 \) and \( \hat{e}_2 \) over all possible angles, define \( \Theta_S \) and \( \Theta_\Delta \) with respect to a fixed arbitrary in-plane vector (say \( \hat{n} \)), and take \( \Theta_0 = 0 \). The resulting distribution of recoil velocities will be independent of the choice of the arbitrary in-plane vector (but will depend weakly on \( \Theta_1 \)). If we ignore the subleading \( K_S \) correction, then \( \Theta_1 \) will not enter the recoil calculation. Its effects can be incorporated by including the \( K_S \) term and averaging over all possible values of \( \Theta_1 \).

3. Remnant mass

Motivated by the success of the empirical formula for the recoil, we propose a new empirical formula for the total radiated energy based on the post-Newtonian equations for the instantaneous radiated energy (see equations (3.2) in [31], and for the quadratic terms in the spin, see equation (5.4) in [37]):
\[
\delta M/M = \eta E_{\text{ISCO}} + E_2\eta^2 + E_3\eta^3 \\
+ \frac{\eta^2}{(1 + q)^2} \left\{ E_S (\alpha^2_2 + q^2 \alpha^2_1) + E_\Delta (1 - q) (\alpha^2_2 - q \alpha^1_1) + E_A [\tilde{a}_2 + q \tilde{a}_1]^2 \right. \\
+ E_B [\alpha^1_2 + q \alpha^1_1]^2 (\cos^2(\Theta_+ - \Theta_2) + E_C) + E_D [\tilde{a}_2 - q \tilde{a}_1]^2 \\
+ E_E [\alpha^2_2 - q \alpha^2_1]^2 (\cos^2(\Theta_- - \Theta_3) + E_F) \right\},
\]

where \(\Theta_\pm\) are the angles that \(\tilde{\Delta}_\pm = M (\tilde{S}_2/m_2 \pm \tilde{S}_1/m_1)\) make with the radial direction during the final plunge and merger (for comparable-mass BHs, a sizable fraction of the radiation is emitted during this final plunge, see for instance figure 6 in [38]). Phases \(\Theta_{2,3}\) are the parameters that give the angle of maximum radiation for these terms, and depend on the initial separation and parameters of the binary at the beginning of the numerical simulation.

In addition to the terms arising from the instantaneous radiated energy, which gives 12 fitting parameters, we also included terms associated with the secular loss of energy in the inspiral period from essentially infinite separation down to the plunge. In order to model this contribution, we adopted the effective one body form [39] supplemented by \(\eta^2\) effects from self-force calculations [40] and 2PN effects of the spins (see equation (4.6) in [31]) to obtain

\[
\tilde{E}_{\text{ISCO}} \approx (1 - \sqrt{8}/3) + 0.103 \, 803 \eta \\
+ \frac{1}{36\sqrt{3}(1 + q)^2} \left\{ q(1 + 2q)\alpha^1_1 + (2 + q)\alpha^2_2 \right\} \\
- \frac{5}{324 \sqrt{2}(1 + q)^2} \left[ \alpha^2_2 - 3(\alpha^2_2)^2 - 2q(\tilde{a}_1 \cdot \tilde{a}_2 - 3\alpha^1_1\alpha^2_2) + q^2(\tilde{a}_2^2 - 3(\alpha^2_2)^2) \right] \\
+ O(\alpha^3).
\]

The above expression includes only quadratic-in-spin terms for compactness; hence, it is expected to produce reliable results for intrinsic spin magnitudes \(\alpha_i < 0.8\) (because the binding energy is a very steep function of \(\alpha\) for \(\alpha > 0.8\) and the quadratic expressions above are no longer appropriate). Note that we used the full expressions from [39] to obtain our fitting parameters. Here we fit the leading-order parameters using available data, and as new data become available, we expect to be able to fit the remaining parameters.

4. Remnant spin

In an analogous way, we propose an empirical formula for the final remnant spin based on the post-Newtonian equations for the radiated angular momentum and the angular momentum of a circular binary at close separations (see equations (3.28) and (4.7) in [31]):

\[
\tilde{a}_{\text{final}} = (1 - \delta M/M)^{-2} \left\{ \eta \tilde{J}_{\text{ISCO}} + (J_2\eta^2 + J_3\eta^3)\hat{n}_1 \right. \\
+ \frac{\eta^2}{(1 + q)^2} \left\{ (J_A (\alpha^2_2 + q^2 \alpha^2_1) + J_B (1 - q) (\alpha^2_2 - q \alpha^1_1))\hat{n}_1 \right. \\
+ (1 - q)|\alpha^1_2 - q \alpha^1_1|\sqrt{J_D \cos[2(\Theta_- - \Theta_3)]} + J_M \tilde{\hat{h}}_\perp \\
+ |\alpha^2_2 + q^2 \alpha^2_1|\sqrt{J_2 \cos[2(\Theta_+ - \Theta_2)]} + J_M \tilde{\hat{h}}_\perp \right\}.
\]

Note that, even at linear order, there are important contributions of generic spinning black holes producing radiation in directions off the orbital axis that do not vanish in the equal-mass or zero-total-spin cases. The above formula can be augmented by quadratic-in-the-spins terms.
[37] of a form similar to the terms added to the radiated energy formula (4). However, these terms are less significant for modeling the final spin (see, for instance, figure 21 of [7]).

Again, we use the effective one body resummation form [39], supplemented with the effects from self-force calculations [40], and the 2PN effects of the spins (see equation (4.7) in [31]) to obtain

$$J_{\text{ISCO}} \approx \left\{ \begin{array} {l}
2\sqrt{3} - 1.5255862\eta - \frac{1}{9\sqrt{2}(1 + q)^2}\left[q(7 + 8q)a_1^\parallel + (8 + 7q)a_2^\parallel\right] \\
+ \frac{2}{9\sqrt{3}(1 + q)^2}\left[\alpha^2 - 3(a_2^\parallel)^2 - 2q(\alpha_1 \cdot \alpha_2 - 3a_1^\parallel a_2^\parallel) + q^2(\alpha_1^2 - 3(a_1^\parallel)^2)\right]^\parallel \\
- \frac{1}{9\sqrt{2}(1 + q)^2}\left[q(1 + 4q)\alpha_1 + (4 + q)\alpha_2\right] + \frac{1}{\eta}(\alpha_2 + q^2\alpha_1) + O(\alpha^3). \end{array} \right.$$  

(7)

This expression represents a quadratic expansion in the spin dependence; hence, we expect to produce reliable results for intrinsic spin magnitudes $\alpha_i < 0.8$ (hence $\alpha_{\text{final}} < 0.9$).

5. Determination of fitting parameters

Here we show how results from current full numerical simulations can be used to determine the fitting constants in the equations for the final remnant mass and spins of a BHB merger. This procedure can be repeated and extended as we have access to new runs and can also help in designing new simulations to optimally determine all fitting constants and better cover the seven-dimensional physical parameter space of BHBs. We used Mathematica’s LinearRegression and NonLinearRegress functions to find the fitting parameters and estimate the errors in the parameters. Our method for finding the fitting parameters involved a least-squares fit to simulations whose symmetries caused most terms to vanish. Then, after fixing the parameters we found in earlier fits, we fit to simulations with less symmetry to obtain other parameters. For example, we first find $E_2$ and $E_3$, and then using these values, fit additional data to obtain $E_5$, etc.

Energy radiated. For the non-spinning case, we fit the data from eight simulations found in [41, 42] (see also [43]). Here we fit $E_{\text{Rad}}$ versus $\eta$, where $E_{\text{Rad}}$ is the total radiated energy for a given configuration minus the binding energy of the initial configuration (where the binding energy is negative). We calculate the binding energy using the 3PN accurate expressions given in [44]. A fit of the resulting data gives $E_2 = 0.341 \pm 0.014$ and $E_3 = 0.522 \pm 0.062$. In order to estimate $E_5, E_\Delta, E_A$, and $E_D$, we use the remnant masses from 13 simulations for spinning BHBs with spins aligned with the orbital angular momentum given in [45, 46] (see also [5]), and find $E_5 = 0.673 \pm 0.035, E_\Delta = -0.36 \pm 0.37, E_A = -0.014 \pm 0.021$, and $E_D = 0.26 \pm 0.44$. These large uncertainties in the fitting parameters are due to the effect of correcting the binding energy in these simulations. Finally, fits from the final remnant masses from five simulations [17] yield $E_6 = 0.09594 \pm 0.00045$ and fits from five equal-mass configurations in [35] yield $E_6 = 0.045 \pm 0.010$. An accurate fit to $E_D$ is not possible with the configurations available in [35]. Note that our fits for $E_5$, $E_A$, and $E_D$ are consistent with the parameters set to zero. This is due to the fact that the errors introduced in renormalizing the data are of the same order as the effects of these subleading terms.

Angular momentum radiated. For the non-spinning case, we fit the data from eight simulations in [41, 42], and find $J_2 = -2.81 \pm 0.11$ and $J_3 = 1.69 \pm 0.51$. A fit to $J_A$ and $J_B$ from 13
simulations in [45, 46] yields $J_A = -2.97 \pm 0.26$ and $J_B = -1.73 \pm 0.80$. However, we determined that the uncertainty in $J_A$ and $J_B$ is actually closer to 1.0 by considering fits to the independent datasets in [45].

From the combined fit, we find that $2.42\% < \delta M / M < 9.45\%$ and $0.34 < \alpha_{\text{final}} < 0.92$ for the equal-mass, aligned spin scenario, in the region where the fit is valid ($|\alpha_{\text{final}}| < 0.9$).

Finally, we note that much of the errors in the fitting parameters are due to differences in the normalizations between the various runs. Some authors choose to normalize their simulations such that $m_1 + m_2 = 1$, which approximates a binary that inspiraled from infinity with an initial mass of 1, while others choose to normalize their simulations such that the initial ADM mass is 1. In this latter case, we attempted to renormalize the results using the 3PN expression for the binding energy. However, the errors introduced by renormalizing data, or assuming that the ADM mass at infinite separation is 1, introduce uncertainties in our fitting parameters. This affects both $\delta M / M$ directly and $\vec{\alpha}_{\text{final}}$ indirectly through $\delta M / M$. Ideally we would use a set of simulations with the same normalization and all starting from the same initial orbital frequency.

From a practical point of view, for statistical simulations of BHB mergers, where the infall direction and the directions of the spins at merger are not known, one should take a uniform distribution for the in-plane components of $\vec{\alpha}_1$ and $\vec{\alpha}_2$ over all possible angles, define the angles $\Theta_0, \Theta_1, \Theta_2$ (note $\Theta_0 = \Theta_\alpha$) with respect to a fixed arbitrary in-plane vector (say $\hat{x}$), and take a uniform distribution for the unknown angles $\Theta_{1,3,5}$. The angles $\Theta_{0,2,4}$ can be set to zero since the final distributions will be independent of this choice (the distribution will only be a weak function of the relative angles $\Theta_0 - \Theta_1$, etc.). The resulting distributions will be independent of the choice of the arbitrary in-plane vector (but will depend weakly on $\Theta_{1,3,5}$). However, the angles $\Theta_{1,3,5}$ only appear in subleading expressions and the uncertainties in the final distributions of the spins, masses, and recoils are not significant for astrophysical applications.

6. Inspiral phase

One of the important applications of our formulae is the study of the statistical distributions of the final mass, spin, and recoil of the merged remnant black hole, given an initial distribution of individual spins and mass ratios. These types of studies have been performed recently, see for instance [47], assuming initial random distribution of individual spin directions and magnitudes, as well as mass ratios. This choice was supported by a post-Newtonian analysis [47] which concluded that, in the (dry) inspiral phase (prior to the final merger that is modeled here) there is no strong alignment of the spins with the orbital angular momentum, as there would be if, for instance, the system accreted a large amount of gas (wet mergers).

The simulations in [47] indicated that gravitational radiation induced precession of the orbital plane during the inspiral phase leads to a small bias of the spins toward counter-alignment to the orbital angular momentum. These results were the product of integrating 3.5PN equations of motion from separations of $r = 50M$ down to the merger regime at $r = 5M$. As was pointed out in [48–50], the orbital averaged PN equations of motion indicate that, on longer time scales, there are resonances that might affect the distributions of spin directions by the time of merger. Since these studies are complementary to those presented in [47], we will investigate this issue analytically at a lower PN order than in the numerical studies of [47], while retaining the radiation reaction effects on the orbital plane for consistency with the integration of the PN equations of motion in the Hamiltonian formalism.
In terms of the notation and approach of [47], we consider
\[
(\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}_1) = \frac{\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}_1}{|\hat{\mathbf{L}}||\hat{\mathbf{S}}_1|} + \frac{\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}_1|\hat{\mathbf{L}}|}{|\hat{\mathbf{L}}|^2|\hat{\mathbf{S}}_1|} - \frac{\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}_1|\hat{\mathbf{S}}_1|}{|\hat{\mathbf{L}}|^2|\hat{\mathbf{S}}_1|^2},
\]
where we can set $|\hat{\mathbf{S}}_1| = 0$ at this order of approximation. In [47], $\hat{\mathbf{S}}_1$ and the conservative part of $\hat{\mathbf{L}}$ terms did not contribute due to the nature of the statistical studies performed in that paper. Hence, we focus on the dissipative part
\[
(\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}_1)_{\text{dis}} = \frac{\hat{\mathbf{L}}_{\text{dis}} \cdot \hat{\mathbf{S}}_1}{|\hat{\mathbf{L}}||\hat{\mathbf{S}}_1|} - \frac{\hat{\mathbf{L}}_{\text{dis}} \cdot \hat{\mathbf{L}}_{\text{dis}}}{|\hat{\mathbf{L}}|^2|\hat{\mathbf{S}}_1|^2}.
\]
With the PN techniques described [47], we find
\[
(\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}_1)_{\text{dis}} = -\frac{8}{15} \frac{v_{\omega}^{11}}{M} \frac{q}{|\hat{\mathbf{a}}_1|} \left[ q (61q + 48)(\hat{\mathbf{P}} \cdot \hat{\mathbf{a}}_1)^2 + (61 + 48q)(\hat{\mathbf{P}} \cdot \hat{\mathbf{a}}_1)(\hat{\mathbf{P}} \cdot \hat{\mathbf{a}}_2) \right]
\]
(10)
\[
(\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}_2)_{\text{dis}} = -\frac{8}{15} \frac{v_{\omega}^{11}}{M} \frac{q}{|\hat{\mathbf{a}}_2|} \left[ q (61q + 48)(\hat{\mathbf{P}} \cdot \hat{\mathbf{a}}_1)(\hat{\mathbf{P}} \cdot \hat{\mathbf{a}}_2) + (61 + 48q)(\hat{\mathbf{P}} \cdot \hat{\mathbf{a}}_2)^2 \right].
\]
Note that these expressions are negative-definite when averaged over spin directions with only the squared terms $(\hat{\mathbf{P}} \cdot \hat{\mathbf{a}}_i)^2$ contributing. By integrating them over time, we obtain similar results to the expression for $(\hat{\mathbf{L}} \cdot \hat{\mathbf{S}})_{\text{dis}}$ in equation (18) of [47], that lead us to the conclusion that distributions of spins show some bias toward counter-alignment with respect to the orbital angular momentum. Note that the instantaneous counter-alignment mechanism acts at every radius, with increasing strength for small separations, where the orbital velocity $v_{\omega}$ is large.

To investigate the small mass ratio limit, i.e. $q \to 0$, we compute the time integral (roughly speaking, multiply by $1/q$) of equation (11), for instance. We can then see that if the larger black hole’s spin, $\hat{\mathbf{S}}_2$, is initially randomly distributed in the limit $q \to 0$, it ends up with some counter-alignment. On the other hand, the smaller black hole’s spin, $\hat{\mathbf{S}}_1$, would remain randomly oriented as seen from the vanishing of the right-hand side of equation (10).

Note that the above equations do not use orbital averages, since the effect is particularly strong in the latest part of the inspiral, when averages are not a good approximation. In the alternative regime, when the inspiral motion is very slow, resonance orbits have been found using orbit averaged descriptions [48–51]. These resonance orbits lead to alignment or antialignment of spins if one starts from an initial aligned or antialigned large hole and allow random orientations for the less massive one. Note that if both spins are allowed to be chosen at random initially, as we assumed in our computations, then the resulting evolution still leads to randomly distributed spins.

The resonance mechanism is complementary to the mechanism we studied in [47]. The former takes place on very long time scales compared to precession, while the later mechanism is quadratic in the spins (as seen in equations (10) and (11), hence higher order). In order to quantify which of them is the predominant mechanism long-term numerical integration of the (non-averaged) equations of motion is required.

7. Discussion

In this paper, we provided a framework to describe the bulk properties of the remnant of a BHB merger. Our framework is based on PN scaling and fitting the results of full
Table 1. The following parameters give the current best estimates for the constants in equations (4) and (6). These parameters were used to generate the spin-magnitude distribution in figure 1.

| Parameter | Value       |
|-----------|-------------|
| $E_2$     | 0.341       |
| $E_3$     | 0.522       |
| $E_\Delta$| −0.3689     |
| $E_A$     | −0.0136     |
| $E_B$     | 0.045       |
| $E_C$     | 0           |
| $E_D$     | 0.2611      |
| $E_E$     | 0.0959      |
| $J_2$     | −2.81       |
| $J_3$     | 1.69        |
| $J_A$     | −2.9667     |
| $J_B$     | −1.7296     |

numerical simulations. The new formulae are physically motivated, incorporate the correct mass ratio dependence, and account for the radiation of angular momentum both parallel and perpendicular to the orbital angular momentum. These formulae have a symmetric dependence on the mass ratio and spins, while still including the correct particle limit. We also extended the successful recoil formula (3) by adding nonleading terms that include all the linear dependence in the spins, as well as higher mass ratio powers.

Unlike in the formula for the remnant recoil case, the energy lost by the binary during the inspiral phase is a non-trivial fraction of the total radiated energy (and is, in fact, the dominant contribution in the small mass ratio limit). Thus, we included both the instantaneous radiative terms in (4) and the binding energy at the ISCO equation (5) (to take into account the secular loss of energy from very large distances down to the merger and plunge regime). Similarly, in order to model the final remnant spin, we need to take into account both the angular momentum of the system near the ISCO, see equation (7), and the subsequent loss of angular momentum in the final plunge (which is particularly important in comparable-mass mergers).

Using the fitted coefficients in the above formulae, we find that for equal-mass, non-spinning binaries, the net energy radiated is 5% of the total mass and the final spin is $\alpha \approx 0.69$, both in good agreement with the most accurate full numerical runs [52]. For maximally spinning BHBs with spin aligned and counter-aligned, we estimate that quadratic corrections lead to radiated energies between 10% and 3% respectively. As for the magnitude of the remnant spin, the linear estimates are between 0.97 and 0.41 respectively, with quadratic corrections slightly reducing those values. These results show that the cosmic censorship hypothesis is obeyed (i.e. no naked singularities are formed) and are in good agreement with earlier estimates [7].

The set of formulae (4) and (6) with the fitting constants determined as in section 5 can be used to describe the final stage of BHB mergers in theoretical, $N$-body, statistical studies in astrophysics and cosmology [47, 53–64] by choosing a distribution of the initial intrinsic physical parameters of the binaries $(q, \vec{S}_1, \vec{S}_2)$ and mapping them to the final distribution of recoil velocities, spins and masses after the mergers. As an example of such an application of the above formulae, we calculate the expected distribution of spin magnitudes of astrophysical supermassive and intermediate-mass BHs (which are expected to have undergone several mergers). To do this, we first consider a set of $10^6$ binaries with uniform distributions of the mass ratio (from 0 to 1), uniform orientations of the spin directions, and uniform distributions of spin magnitudes. We then use our formulae (see table 1 for the values of the constants that we used) to predict the spin-magnitude distribution of the merger remnants and repeat the calculation, again with uniform distributions in the mass ratio and spin directions, but with this new spin-magnitude distribution (see also [12, 64]). The resulting spin distribution, after each subsequent set of mergers, approaches a fixed distribution. The spin distribution that results after ten generations of mergers is shown in figure 1. The final results are insensitive to the initial distribution and quickly converge, in a few generations of mergers, to the displayed curve, which represents a universal distribution of the intrinsic spin magnitudes (with a
Figure 1. The spin magnitude distribution for dry mergers. We plot the distributions of spins \( \alpha = S/m^2 \) of the final remnant after many mergers. This distribution does not change significantly following additional mergers and peaks at \( \alpha \approx 0.73 \). We also display the distributions representing wet mergers for hot and cold accretion disks. They are highly peaked distributions at around \( \alpha \approx 0.88 \) and \( \alpha \approx 0.9 \) respectively.

Figure 2. The recoil magnitude distribution for dry mergers displaying a tail extending beyond 1000 km s\(^{-1}\). We also display the distributions representing recoils for wet mergers for hot and cold accretion disks. The cold disk leads to a recoil velocity distribution highly peaked at around \( v \approx 80 \text{ km s}^{-1} \), while the hot accretion disk extends the magnitude of the recoil to several hundred km s\(^{-1}\).

maximum near 0.7 and mean in the range (0.5, 0.8)) of the remnant BHs of dry BHB mergers (when neglecting accretion). In order to provide a simple analytical model for this distribution, we fit it to the Kumaraswamy functional form

\[
 f(x; a, b) = abx^{a-1}(1 - x^b)^{b-1},
\]

and find \( a = 5.91 \pm 0.04, b = 5.33 \pm 0.07 \). We choose this functional form because it allows for a skewed distribution (and fits the results better than a beta distribution); however, the fit underestimates the probability for producing small spins.

We have also considered the effect of wet mergers on the final spin and recoil velocities. A first account of accretion effects during the long inspiral phase of binary black holes has been given in [66] using the smoothed particle hydrodynamics approximation (SPH). To evaluate
the accretion effects on the statistical distributions, according to [66], we have considered distributions that at merger have $0.3 \leq \alpha_i \leq 0.9$ and orientations within $10^\circ$ and $30^\circ$ for cold and hot accretion disks respectively. We have also assumed a flat distribution in mass ratios in the region $0 \leq q \leq 1$. The results for the final spin distributions are displayed in figure 1 and show the dramatic change in the spin distributions due to accretion. Note that this accretion effect on spin will be less important on black holes with masses larger than $10^7 M_\odot$ [67].

The same statistical analysis can be made with the magnitude of the recoil velocity of the remnant final black hole when we consider a set of $10^6$ binaries with uniform distributions of mass ratios in the range $0 \leq q \leq 1$. For dry mergers, we consider uniform orientations of spin directions, and uniform distributions of spin magnitudes. We evaluate equation (3) each time and obtain the distribution with the extended tail beyond $1000 \text{km s}^{-1}$ in figure 2. The other two distributions correspond to the wet mergers with $0.3 \leq \alpha_i \leq 0.9$ and orientations within $10^\circ$ and $30^\circ$ for cold and hot accretion disks respectively according to [66]. The results show a tighter distribution around low recoil velocities for cold than for hot accretion disks around the merging black holes.

Finally, another use of the remnant formulae can be found in modeling waveforms in the intermediate and small mass ratio limits using the techniques of [68] by providing an accurate a priori estimation for the background black hole mass and spin.

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