LIPS-thermalization of a relativistic gas

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It is argued that two-particle collisions of relativistic particles "at a distance", irrespective of their position in the configuration space, generate uniform distribution of particles in Lorentz invariant phase space.

In a recent paper by Mereš et al. [1] a new numerical procedure for generation of relativistic particles with the given total energy-momentum is proposed. The procedure makes use of two-particle collisions between particles considered in momentum space only, with no definite positions in configuration space, and seems to produce uniform distribution of particles in Lorentz invariant phase space (LIPS) when applied repeatedly to a system with non-uniformly distributed particles at the beginning. The paper includes a sketch of an argument explaining why it is so. In this note a detailed exposition of the argument is given.

In the procedure developed in [1], collisions are linked into sequences with two collisions per particle and the pairing of particles in each sequence is governed by a single parameter whose value is chosen at random. While this can be advantageous in numerical calculations, it is technically simpler and physically more instructive to consider a gas of particles colliding completely at random. Our aim is to show that the equilibrium distribution of particles in such a gas is uniform in LIPS, so that the distribution necessarily becomes uniform asymptotically, for the number of collisions approaching infinity. If this is the case, the procedure with linked collisions, which uses essentially the same mechanism, will yield an asymptotically uniform distribution, too.

Consider a gas of $N$ nonidentical relativistic particles in which there are perpetually going on elastic 2-particle collisions with a given angular distribution of the outgoing particles in cms. The particles are free between collisions and collide "at a distance", irrespective of their actual position in the configuration space. To describe the evolution of the gas we divide it into steps with one collision per step, and pick the pair of particles entering the collision in any given step at random. Then we choose the directions in which the particles are coming out of the collision in accord with their angular distribution in cms. Both choices are obviously independent of the choices made in previous steps, as

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well as of the sequence number of the actual step. Thus, the evolution of the gas is a homogeneous Markov process.

There is a vast body of literature on Markov processes, including monographs devoted entirely to them (see, for example, [2]) and books on probability theory where their properties are discussed in detail. For a newcomer, the best choice is perhaps to read an exposition on the subject in some lecture notes available online, as are those cited in [3].

A homogeneous Markov process is described by the transfer matrix $P_{ij}$, whose element $(ij)$ is the probability that the system passes in one step from the state $j$ to the state $i$. Denote the probability that the system will be found in the $n$th step in the state $i$ by $p_n(i)$. The probabilities in two subsequent steps are related by the formula

$$p_{n+1}(i) = \sum_j P_{ij} p_n(j).$$

In statistical mechanics, one calls the states $i, j, \ldots$ "microstates", and the states defined by the probability distribution $p_n(i)$ "ensemble states" or "macrostates". Equilibrium is a macrostate into which the system gets after infinitely many steps from the initial macrostate with an arbitrary probability distribution; in particular, an arbitrary microstate (which can be regarded as a macrostate with one probability equal to 1 and all others equal to 0). Thus, for the probability distribution in equilibrium we have

$$p(i) = \sum_j P_{ij} p_0(j), \quad \mathcal{P} = \lim_{n \to \infty} P^n. \quad (1)$$

On the other hand, equilibrium can be defined also as a macrostate whose probability distribution stays constant during the evolution of the system. Thus, the probability distribution in equilibrium should satisfy

$$p(i) = \sum_j P_{ij} p(j). \quad (2)$$

In the first lecture notes cited in ref. [3], the interplay between equations (1) and (2) is demonstrated on a simple, but no less illuminating, example of two-level system.

According to equation (2), the vector $p$ is an eigenvector of the matrix $P$ with the eigenvalue 1. One can easily see that there necessarily exists some vector with this property, the reason being that the matrix $P$ obeys the normalization condition

$$\sum_i P_{ij} = 1. \quad (3)$$
Indeed, if we sum both sides of equation (2) over \( i \) and use equation (3), we arrive at identity; thus, the determinant of the system is zero and the system has at least one nontrivial solution (modulo rescaling). However, we would also like to know that the solution is unique and nonnegative, and that it can be expressed in the form (1). It turns out that all this is indeed the case provided the matrix \( P \) is \textit{irreducible}; that is, either the matrix itself or some finite power of it is positive (has all entries positive). In less formal terms, the solution has desirable properties if we are able to pass from any state to any other state in a finite number of steps, so that we do not eventually become trapped in some subspace of the state space. The proof is based on a well-known theorem from the theory of nonnegative matrices, called \textit{Perron-Frobenius theorem}. More details can be found in [4].

The microstates of the gas are numbered by the \( N \)-tuples of particle momenta \((p_1, p_2, \ldots, p_N)\). Thus, there are continuously infinitely many of them. However, we can pass to the \textit{discretized theory}, dividing the momentum space into elementary cells of finite size. This is actually done in numerical calculations. In the discretized theory, the number of microstates is countable, and if we take into account that the probability distribution we are interested in includes a \( \delta \)-function on total 4-momentum (which is replaced by \( 1/\{4\text{-dimensional volume of the order (size of the cell)}^4 \} \times \text{Kronecker delta after discretization} \)), the number of microstates becomes finite. As a result, previous considerations can be applied without modifications to a gas of colliding particles as a special case.

The basic building block of the theory is the expression for the distribution of outgoing particles in a two-particle collision. Let us compute this distribution in the laboratory frame in case it is isotropic in cms. The quantities referring to cms will be denoted by the index 0. The probability that the particle 1 comes out of the collision in cms in the solid angle \( d\Omega \) is

\[
dP = \frac{d\Omega_10}{4\pi}.
\]

Denote the magnitude of the momenta of ingoing particles in cms by \( \pi_0 \), the momenta of outgoing particles in cms by \((p_{10}, p_{20})\), the total energy of ingoing particles in cms by \( \epsilon_0 \), and the energies of outgoing particles in cms by \( (E_{10}, E_{20}) \). The energies are defined in terms of momenta as \( E_{10} = \sqrt{p_{10}^2 + m_1^2} \) and \( E_{20} = \sqrt{p_{20}^2 + m_2^2} \), where \( m_1 \) and \( m_2 \) are the masses of the colliding particles. The expression for \( dP \) can be rewritten as

\[
dP = \frac{1}{4\pi p_{10}^2} \delta(p_{10} - \pi_0) d^3 p_{10} = \frac{1}{4\pi p_{10}^2} \delta(p_{10} - \pi_0) \delta(p_{10} + p_{20}) d^3 p_{10} d^3 p_{20},
\]

and if we use

\[
\delta(E_{10} + E_{20} - \epsilon_0) = \frac{1}{p_{10}(1/E_{10} + 1/E_{20})} \delta(p_{10} - \pi_0),
\]

3
we find

\[ dP = \frac{1/E_{10} + 1/E_{20}}{4\pi p_{10}} \delta(E_{10} + E_{20} - \epsilon_0)\delta(p_{10} + p_{20})d^3p_{10}d^3p_{20}. \]

The product of \( \delta \)-functions can be written as \( \delta^4(p_{10} + p_{20} - q_0) \), where \( p_{10}^\mu = (E_{10}, p_{10}) \) and \( p_{20}^\mu = (E_{20}, p_{20}) \) are the 4-momenta of outgoing particles in cms and \( q_0^\mu = (\epsilon_0, 0) \) is the total 4-momentum of ingoing particles in cms. From the conservation laws we also have \( 1/E_{10} + 1/E_{20} = \epsilon_0/(E_{10}E_{20}) \) and \( p_{10} = \pi_0 \), hence

\[ dP = \frac{\epsilon_0}{4\pi \pi_0} \delta^4(p_{10} + p_{20} - q_0)\frac{d^3p_{10}d^3p_{20}}{E_{10}E_{20}}. \]  

(4)

The 4-dimensional \( \delta \)-function as well as the element of LIPS by which it is multiplied are relativistic invariants, so that they can be both carried over into the laboratory frame simply by wiping off the index 0. The factor in front of the \( \delta \)-function must be rewritten as an invariant, too, which means that we must express \( \pi_0 \) in terms of \( \epsilon_0 \) and subsequently replace \( \epsilon_0 \) by \( \sqrt{q^2} \) (invariant mass). By squaring the equation \( \epsilon_0 = E_{10} + E_{20} \), separating the term \( 2E_{10}E_{20} \) and squaring the resulting equation again we obtain

\[ \epsilon_0^4 - 2\epsilon_0^2(E_{10}^2 + E_{20}^2) + (E_{10}^2 - E_{20}^2)^2 = 0, \]

and if we insert here the expressions for \( E_{10} \) and \( E_{20} \) with \( p_{10} = p_{20} = \pi_0 \) and solve for \( \pi_0 \), we find

\[ \frac{\pi_0}{\epsilon_0} = \frac{1}{2} \left( 1 - \frac{2M^2}{\epsilon_0^2} + \frac{\mu^4}{\epsilon_0^4} \right)^{1/2}, \]

where \( M^2 = m_1^2 + m_2^2 \) and \( \mu^2 = m_1^2 - m_2^2 \). Inserting this in equation (4) and passing from cms to laboratory frame we finally obtain

\[ dP = \frac{1}{2\pi} \left( 1 - \frac{2M^2}{q^2} + \frac{\mu^4}{q^4} \right)^{-1/2} \delta^4(p_1 + p_2 - q)\frac{d^3p_1d^3p_2}{E_1E_2}. \]  

(5)

The factor in front of the \( \delta \)-function in the expression for \( dP \) serves as a normalization factor, making the total probability equal to 1. The correct normalization of \( dP \) is guaranteed by the way the expression has been derived, and can be easily verified in the nonrelativistic limit.

If the distribution of outgoing particles in cms is anisotropic, we must insert a factor \( f(\cos \theta_{10}) \) into the expression for \( dP \), where \( \theta_{10} \) is the angle between the initial and final directions of motion of the particle 1 in cms and \( f \) is a nonnegative function satisfying

\[ \int_{-1}^{1} f(x)dx = 2. \]
The argument of $f$ can be expressed in terms of $q_1^2$ and $q_1 \cdot p_1$, where $q_1$ and $p_1$ are the initial and final 4-momenta of the particle 1. In particular, for $m_1 = m_2 = m$ we have

$$\cos \theta_{10} = \frac{q_1^2 - 4q_1 \cdot p_1}{q_1^2 - 4m^2}.$$ 

Consider now a gas of $N$ particles with momenta $(q_1, q_2, \ldots, q_N)$, and suppose that at the given moment there occurs a two-particle collision, with all pairs of particles colliding with equal probability. We are interested in the probability of finding the gas after the collision in an infinitesimal neighborhood of the point $(p_1, p_2, \ldots, p_N)$ of the $N$-particle momentum space. This can be written as a sum of contributions of all pairs of particles,

$$dP = \frac{1}{N} \sum_{a<b} dP_{ab} \prod_{c \neq a,b} \delta(p_c - q_c) d^3p_c,$$  

where $N = N(N - 1)/2$ is the number of pairs and $dP_{ab}$ is the probability that after the particles $a$ and $b$ have collided, their momenta will be found in an infinitesimal neighborhood of the point $(p_a, p_b)$ of the two-particle momentum space. Suppose for simplicity that the distribution of outgoing particles in 2-particle collisions is isotropic in cms. Then the probability $dP_{ab}$ is given by equation (5) rewritten for particles $a$ and $b,$

$$dP_{ab} = \frac{1}{2\pi} \left(1 - \frac{2M_{ab}^2}{q_{ab}^2} + \frac{\mu_{ab}^4}{q_{ab}^4}\right)^{-1/2} \delta^4(p_{ab} - q_{ab}) \frac{d^3p_ad^3p_b}{E_aE_b},$$ 

where $p_{ab} = p_a + p_b$, $q_{ab} = q_a + q_b$, $M_{ab}^2 = m_a^2 + m_b^2$ and $\mu_{ab}^2 = m_a^2 - m_b^2$. Taking into account the $\delta$-functions in equation (6), we can rewrite the $\delta$-function in equation (7) as $\delta^4(p - q)$, where $q^\mu$ and $p^\mu$ are the total 4-momenta of in- and outgoing particles. We can also transform the noninvariant 3-dimensional $\delta$-functions in (6) into invariant ones by multiplying them by the energies of the ingoing particles $\epsilon_c$. In this way we arrive at the formula

$$dP = F \delta^4(p - q) \prod \frac{d^3p_c}{E_c},$$ 

with the distribution function $F$ defined as

$$F = \frac{1}{N} \sum_{a<b} \frac{1}{2\pi} \left(1 - \frac{2M_{ab}^2}{q_{ab}^2} + \frac{\mu_{ab}^4}{q_{ab}^4}\right)^{-1/2} \prod_{c \neq a,b} \epsilon_c \delta(p_c - q_c).$$ 

The probability distribution $dP$, regarded as a function of initial and final momenta of the particles, is the transfer matrix for our problem. It is continuous, but can be discretized by the procedure mentioned earlier.
The gas in equilibrium is described by the probability distribution
\[ dp = f\delta^4(p - P)\prod d^3p_c \frac{E_c}{E_c}, \]  
(10)
where \( p^\mu \) is the total 4-momentum of the gas, \( P^\mu \) is the value assigned to \( p^\mu \) and \( f \) is the distribution function in equilibrium depending on the momenta \( (p_1, p_2, \ldots, p_N) \) of the particles forming the gas. Denote the \( N \)-tuple of momenta \( (p_1, p_2, \ldots, p_N) \) by \( \hat{p} \). For the probability distribution \( dp \) we have the continuous version of equation (2),
\[ dp(\hat{p}) = \int dP(\hat{p}, \hat{q}) dp(\hat{q}), \]
where we integrate over the \( N \)-tuple of momenta \( \hat{q} \). By inserting here from equations (6) and (10) we obtain
\[ f(\hat{p})\delta^4(p - P) = \int F(\hat{p}, \hat{q})\delta^4(p - q)f(\hat{q})\delta^4(q - P)\prod d^3q_c \frac{E_c}{E_c}, \]
and if we replace \( q^\mu \) in the argument of the second \( \delta \)-function on the right hand side by \( p^\mu \), using the fact that the value of \( q^\mu \) is fixed by the first \( \delta \)-function, we find
\[ f(\hat{p}) = \int F(\hat{p}, \hat{q})f(\hat{q})\delta^4(p - q)\prod d^3q_c \frac{E_c}{E_c}. \]

(11)

After having obtained an integral equation for the distribution function \( f \), our next goal is to prove that it is solved by constant \( f \). Thus, we want to show that \( F \) satisfies the equation
\[ \int F(\hat{p}, \hat{q})\delta^4(p - q)\prod d^3q_c \frac{E_c}{E_c} = 1. \]

(12)
Note that a similar equation with the integration over the \( N \)-tuple of momenta \( \hat{p} \),
\[ \int F(\hat{p}, \hat{q})\delta^4(p - q)\prod d^3p_c \frac{E_c}{E_c} = 1, \]
is just the normalization condition for \( F \). In fact, it is a continuous version of the normalization “by columns" introduced in equation (3). Our claim is that the transfer matrix we have constructed is normalized also "by rows". (Strictly speaking, the matrices in the two normalization conditions are not the same, since they have different energies in the denominator. [The elementary volumes referring to different momenta do not matter, since they can be always chosen to have a fixed size throughout the momentum space.] However, we can view both matrices as different representations of a single matrix.)

To prove equation (12), we first transform it into an equation for two-particle probability distribution. For that purpose, we pick an arbitrary term \( (a, b) \) in the sum over the pairs of particles in
(9), rewrite the energy-momentum conservation law so that only the 4-momenta of particles \(a\) and \(b\) remain in it, and eliminate the 3-dimensional \(\delta\)-functions by integrating over all \(q\)'s other than \((q_a, q_b)\). In this way we obtain

\[
\int \frac{1}{2\pi} \left( 1 - \frac{2M^2}{q_{12}^2} + \frac{\mu^4}{q_{12}^4} \right)^{-1/2} \delta^4(p_{12} - q_{12}) \frac{d^3q_1 d^3q_2}{\epsilon_1 \epsilon_2} = 1. \tag{13}
\]

This can be viewed as normalization of the two-particle transfer matrix “by columns”. To show that the equation is satisfied, it suffices to replace \(q_{12}^2\) in the expression in front of the \(\delta\)-function by \(p_{12}^2\), rename \((p_1, p_2)\) to \((q_1, q_2)\) and vice versa, and notice that the resulting equation is just the normalization condition for the probability \(dP\) defined in equation (5). Thus, the normalization of the two-particle transfer matrix “by columns” follows immediately from its normalization “by rows”.

Previous reasoning generalizes trivially to anisotropic collisions. In this case, the integral in equation (13) contains an additional factor \(f(\cos \theta_{10})\), but the equation is still satisfied because the in- and outgoing momenta enter the expression for \(\theta_{10}\) symmetrically. This is seen in the formula for \(\cos \theta_{10}\) cited before, and can be regarded as a consequence of the trivial fact that the angles by which the vector \(p_{10}\) is deflected from the vector \(q_{10}\) and vice versa are identical. Thus, equation (12) is valid no matter what the distribution of outgoing particles in cms.

We have shown that the condition of equilibrium is solved by the probability distribution that is uniform in LIPS,

\[
dp = C_P \delta^4(p - P) \prod \frac{d^3p_c}{E_c}, \tag{14}
\]

where \(C_P\) is a normalization constant depending only on the 4-momentum of the gas \(P^\mu\). Thus, the particles forming the gas, if in equilibrium, can be distributed uniformly in LIPS. However, we would like to be sure that they are distributed uniformly in LIPS. In the discretized theory this is guaranteed by the Perron-Frobenius theorem, provided the gas can pass from any initial state to any final state after a finite number of two-particle collisions. In fact, the final state does not need to be varied, since the desired transition for any initial and final state can be combined from one direct and one inverse transition to some reference state. We will not attempt to prove that the interpolating sequence of collisions exists in general, but we will provide an illustrative example showing that such sequence exists for a pair of states that are in a sense maximally distant from each other, and that its length is of order \(N\).

Consider a many-particle gas with the total energy \(E\) and zero total momentum, and suppose for simplicity that the particles are massless. We are interested in the transitions into the reference state...
that is maximally randomized, with the energy of all particles equal to $\epsilon_r = E/N$ and the momenta filling uniformly a sphere in the momentum space. If the gas is to evolve into this state, the least favorable case seems to be if it is in the state with a certain particle $A$ having energy $\mathcal{E}_0 = E/2$ and momentum oriented in one direction, and all remaining particles having energy $\epsilon = \mathcal{E}_0/(N-1) \doteq \epsilon_r/2$ and momenta oriented in the opposite direction (fig. 1 left). Consider a sequence of $N-1$ collisions in

\[ \mathcal{E} \left\{ \vdots \right\} \mathcal{E}_0 \quad \mathcal{E}' \]

![Fig. 1: Randomization of a state with half the total energy carried by a single particle](image)

which the particle $A$ pairs subsequently with all remaining particles, knocking them alternately to one and the other side in a fixed plane and setting their energy to $\epsilon_r$. The energy-momentum conservation laws written for the $\nu$th collision are (fig. 1 right)

\[ \mathcal{E}' + \epsilon_r = \mathcal{E} + \epsilon, \quad \mathcal{E}' \cos \alpha + \epsilon_r \cos \beta = \mathcal{E} - \epsilon \cos \theta, \quad \mathcal{E}' \sin \alpha + \epsilon_r \sin \beta = \epsilon \sin \theta, \quad (15) \]

where $\mathcal{E}$ and $\mathcal{E}'$ is the initial and final energy of the particle $A$, $\alpha$ is the scattering angle of the particle $A$, $\beta$ is the angle between the initial momentum of the particle $A$ and the final momentum of the particle $B$ that is colliding with $A$, and $\theta$ is the cumulative scattering angle of the particle $A$ (the angle between the initial momentum of $A$ in the $\nu$th collision and the momentum $A$ had before the first collision). In all scatterings but a couple of last ones the energy $\mathcal{E}$ is much greater than both $\epsilon$ and $\epsilon'$ (the final energy of the particle $B$) $\doteq 2\epsilon$. This causes $\alpha$ to be small and $\beta$ to be close in absolute value to $\pi/2$. If we choose the first particle $B$ to be declined from its original direction to the right of the particle $A$, $\beta$ will be of the form $\beta = \mp \pi + \delta$, where $\delta$ is small and the upper and lower sign correspond to odd and even collisions respectively. (In the figure, an odd collision is depicted.)

As for the third angle appearing in the problem, for a sequence of collisions under consideration, with an altering direction of motion of particles $B$ after collision, $\theta$ is always less in absolute value than the actual $\alpha$, hence it is small, too.

We want to solve equations (15) in the leading order in the small parameter $\epsilon/\mathcal{E}$. This is a simple exercise, however, at one point we must be cautious: we can put $\epsilon_r = 2\epsilon$ almost everywhere, but in the term $\mathcal{E}' \cos \alpha$ in the second equation (15), in which $\epsilon_r$ appears through $\mathcal{E}'$, we must use the exact
formula \( \epsilon_r = 2(1 - 1/N)\epsilon \). From the first equation (15) we obtain \( \mathcal{E}' = \mathcal{E} - \epsilon \), which yields

\[
\mathcal{E}' = \mathcal{E}_0 - \nu \epsilon = \mu \epsilon,
\]

where \( \mu \) is the number of collisions which remain till the end. For our purposes we can put \( \mu = N - \nu \).

(To obtain correct values of \( \mathcal{E}' \) at the last stage of the process, we must pass to \( \mu = N - 1 - \nu \) and account for the cumulative effect of the small correction to \( \epsilon_r \).) From the remaining two equations we find

\[
\alpha = \pm \frac{2}{\mu}, \quad \delta = \pm \left( \frac{1}{\mu} - \frac{1}{N} \right).
\]

To compute \( \theta \), we must sum \( \alpha \)'s for all collisions up to that with the sequence number \( \nu - 1 \). For odd collisions it holds

\[
\theta_{odd} = \frac{2}{N-1} - \frac{2}{N-2} + \ldots + \frac{2}{\mu} - \frac{2}{\mu + 1} = -2 \left[ \frac{1}{(N-1)(N-2)} \right. + \\
+ \ldots + \left. \frac{1}{\mu(\mu+1)} \right] \approx - \int_{\mu}^{N-1} \frac{dx}{x^2} = - \frac{1}{\mu} + \frac{1}{N-1} \approx - \frac{1}{\mu} + \frac{1}{N},
\]

and for even collisions we have

\[
\theta_{even} = - \frac{1}{\mu + 1} + \frac{1}{N} + \frac{2}{\mu + 1} = \frac{1}{\mu + 1} + \frac{1}{N} \approx \frac{1}{\mu} + \frac{1}{N}.
\]

The momentum acquired by the particle \( B \) after it collided with the particle \( A \) is deflected from the direction perpendicular to the original momentum of the particle \( A \) by the angle \( \Delta = \delta + \theta \). After inserting here for \( \delta \) and \( \theta \) we obtain

\[
\Delta_{odd} = 0, \quad \Delta_{even} = \frac{2}{N}.
\]

The approximate theory developed here holds for a majority of the particles of the gas. After the collisions with the particle \( A \) are completed, these particles split into two equally large groups, one containing particles knocked to one side of the particle \( A \) and the other containing particles knocked to the other side. From equation (18) we can see that the particles of the first group move perpendicularly to the original direction of motion of the particle \( A \), while the particles of the second group are slightly deflected to that side to which the particle \( A \) was heading at the beginning. A small third group consists of particles that collided last and move in various directions in such a way that their total momentum balances the momentum arising from the deflection of the particles of the second group.
Collisions with the particle $A$ represent just the first step of a three-step procedure that brings the gas to the reference state. The next step is to modify the direction of motion of the particles so that they combine into pairs with exactly opposite momenta; or in the discretized theory, with momenta that lie in exactly opposite cells of the momentum space. If the size of the cells $\Delta p$ is greater than $\Delta p_0 = \Delta_{\text{even}} \epsilon_r \approx 4\epsilon / N$, the task is restricted to the particles of the third group and can be presumably completed in a number of steps of order of the number of particles belonging to that group, which is negligible in comparison to $N$. In the opposite case we must take care of the first two groups of particles, too. We must slightly rotate, say, the direction of motion of the particles of the second group so that it will become opposite to the direction of motion of the particles of the first group. This is easily accomplished with one collision per particle; we just have to think of it in advance and pick a slightly different final state in the collisions of this group of particles with the particle $A$. A straightforward computation yields that the final energy $\epsilon'$ of the particles of the second group must differ from the energy $\epsilon_r$ by the quantity

$$\Delta \epsilon' = \frac{4 \epsilon \cos \phi}{N(1 + \sin \phi + \cos \phi)},$$

(19)

where $\phi$ is the angle by which the momentum of the particle of the third group, paired with the given particle of the second group, is deflected from the original momentum of the particle $A$. Note that the expression for $\Delta \epsilon'$ diverges for $\phi = \pi$, therefore the procedure does not work with particles that have not collided with the particle $A$ yet. To find how many collisions are necessary, notice that we must straighten the momenta of half the number of particles, and need one collision for one straightening (perhaps a little more for the particles of the third group, but their number is negligible anyway). Thus, the number of collisions needed at this stage is $N/2$.

The third and final step of the randomization procedure is to distribute the particles uniformly on a sphere in the momentum space with the radius $\epsilon_r$. This is done simply by colliding particles with opposite momenta. The number of collisions is again $N/2$, which makes the total number, in the less favorable case when the cells in the momentum space have a size less than $\Delta p_0$, equal to $2N$.

An isolated gas whose particles collide with each other wanders the energy hypersurface of the $N$-particle momentum space. If the hypersurface consists of domains that are interconnected but not connected with each other, in the sense that points of one domain cannot be reached from another domain, there can exist a macrostate that stays unchanged once it was established, but the gas will not evolve towards it from other macrostates. The question is whether the uniform distribution in LIPS defined in equation (14) does not represent such a “virtual” equilibrium state. The example with
randomization of the gas with one highly energetic particle disputes this possibility since it shows that
even faraway points of the momentum space can be accessible to each other.

For a given homogeneous Markov process, one can calculate the equilibrium probability distribution numerically by starting with an arbitrary state $i_0$, booking the state after every $\mu \gg 1$ steps and computing the relative frequency of every state after $\nu \gg \mu$ steps. The probability distribution $p_{\text{app}}(i)$ obtained in this way is arguably a good approximation to $p(i)$. To see that, consider the first sequence of $\mu$ steps. The probability distribution $p_\mu(i)$ at the end of the sequence is given by a cut-off version of equation (1) with $p_0(i) = \delta_{i_{i0}},$

$$p_\mu(i) = \sum_j P^{(\mu)}_{ij} p_\mu(j) = P^{(\mu)}_{i_{i0}}, \quad P^{(\mu)} = P^{\mu}.$$  

For large enough $\mu$ the information about the initial state is effectively wiped out during the sequence, so that the effect of the cut-off is small and $p_\mu(i)$ is close to $p(i)$. Thus, we could find $p_{\text{app}}(i)$ simply by calculating $p_\mu(i)$ numerically; that is, by repeating the sequence with the same initial state sufficiently many times and computing the relative frequencies of the final states. However, if we picked by chance an initial state for which the system would wander through marginal regions of the state space for a long time (this can presumably happen for initial states with small probability), the resulting probability distribution would be distorted. To be on the safe side, we can use the procedure outlined at the beginning, with the final state of one sequence serving as the initial state of another sequence. This can be described by the formula

$$p_{\text{app}}(i) = \sum_j P^{(\mu)}_{ij} p_{\text{app}}(j).$$

The same formula is valid, as a trivial consequence of equation (2), for $p(i)$. Nevertheless, $p_{\text{app}}(i)$ is still an approximation, since the number of events used in its computation is finite. For a relativistic gas, numerical calculations following this recipe support our claim that the equilibrium distribution of particles is uniform in LIPS [5].

To complete the discussion, consider a gas with linked collisions introduced in [1]. Suppose the two-particle collisions come in sequences consisting of $N$ collisions, with the sequence number of colliding particles, regarded as a periodic variable with the period $N$, raised by one in each consecutive collision. Furthermore, suppose that the difference between the sequence numbers $d$, which stays constant throughout each sequence, assumes all values 1, 2, ..., $N-1$ with the same probability. The collisions fall into three classes (fig. 2): collisions No. 1 to $D = \min(d, N - d)$ in which both incoming particles are in the initial state (class $\alpha$), collisions No. $D + 1$ to $N - D$ (absent for $N$ even and $D = N/2$)
in which one incoming particle is in the initial state and the other incoming particle is in the intermediate state (class $\beta$), and collisions No. $N - D + 1$ to $N$ in which both incoming particle are in the intermediate state (class $\gamma$). The two-particle probability distributions in the respective classes are $dP^{(\alpha)} = dP(qq, ll)$, $dP^{(\beta)} = dP(ql, lp)$ and $dP^{(\gamma)} = dP(ll, pp)$, where $q$, $l$ and $p$ are initial, intermediate and final momenta respectively. The $N$-particle probability distribution for one sequence of collisions is

$$dP = \frac{1}{N-1} \sum dP_d,$$

where the contribution of collisions with the given shift in the sequence number can be written as a product of $N$ two-particle probability distributions integrated over the intermediate momenta,

$$dP_d = \int dP^{(\alpha)}_{1,D+1} \cdots dP^{(\beta)}_{D+1,2D+1} \cdots dP^{(\gamma)}_{N-D+1,1} \cdots$$

The $N$-particle distribution is again of the form (8), and is again normalized “by columns” thanks to the fact that the two-particle distribution is normalized “by columns”. (Equation (12) for the distribution function reduces again to the identity (13).) Thus, the linking of collisions does not affect our conclusion that the uniform distribution in LIPS obeys the condition of equilibrium. The integrations in (21) remove the better part of the $\delta$-functions in the two-particle probability distributions, leaving us with a much greater freedom in choosing the final momenta than we had in the case with independent collisions. Because of that it could be expected that in a many-particle gas the particles will need only $O(1)$ sequences of collisions to get from any state to any other state. This, if true, would make the total number of intermediate collisions of order $N$, just as we have established in the example with randomization. However, to prove that seems to be no less difficult for linked collisions than for independent ones. On the other hand, if the particles could indeed pass from any state to any other state within $O(N)$ independent collisions, they could of course do that also within $O(N)$ sequences of collisions. As a result, if the true (not “virtual”) equilibrium distribution is uniform in a gas with independent collisions, it is necessarily uniform also in a gas with linked collisions.

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