Conventional BCS, Unconventional BCS, and Non-BCS Hidden Dineutron Phases in Neutron Matter

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The nature of pairing correlations in neutron matter is re-examined. Working within the conventional approximation in which the nn pairing interaction is provided by a realistic bare nn potential fitted to scattering data, it is demonstrated that the standard BCS theory fails in regions of neutron number density where the pairing constant λ, depending crucially on density, has a non-BCS negative sign. We are led to propose a non-BCS scenario for pairing phenomena in neutron matter that involves the formation of a hidden dineutron state. In low-density neutron matter where the pairing constant has the standard BCS sign, two phases organized by pairing correlations are possible and compete energetically: a conventional BCS phase and a dineutron phase. In dense neutron matter, where λ changes sign, only the dineutron phase survives and exists until the critical density for termination of pairing correlations is reached at approximately twice the neutron density in heavy atomic nuclei.

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I. INTRODUCTION

Shortly after Bardeen, Cooper, and Schrieffer (BCS) introduced a theory of superconductivity in 1957, A. B. Migdal raised the possibility that the matter inside neutron stars may be superfluid. Since that time, hundreds of papers have been published to elucidate the properties of neutron matter and other nuclear systems implied by nucleonic pairing, within the framework of BCS theory. In the generic zero-temperature Lifshitz phase diagram of a homogeneous 3D Fermi system subject to pairing correlations, the conventional BCS phase lies in the weak-coupling domain of small positive pairing constant λ. Specifically, this dimensionless coupling parameter is defined by $\lambda = -V_F N(0)$, where $V_F = V(p_F, p_F)$ is the diagonal matrix element of the pairing interaction and $N(0) = p_F M^*/\pi^2$ is the density of single-particle states, both evaluated at the Fermi surface. (The Fermi momentum is given by $p_F = (3\pi^2 \rho)^{1/3}$ in terms of the particle density $\rho$, while $M^*$ stands for the effective mass.) The occurrence of the BCS phase in this domain is attributed to the enhancement of pairing correlations stemming from the logarithmic divergence of the propagator of a pair of opposite-spin quasiparticles as their total momentum $P$ approaches zero. This enhancement leads to the formation of a condensate of Cooper pairs with $P = 0$, which entails violation of global U(1) phase rotation symmetry, and is responsible for the superfluidity of the BCS phase. A crucial feature of this phenomenon is the presence of a gap $\Delta(p)$ in the spectrum $E(p)$ of single-particle excitations. In the relevant region of

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PREAMBLE

This contribution is dedicated with deep respect and admiration to Spartak Timofeevich Belyaev on the occasion of his 90th birthday. Three generations of physicists across the globe have taken inspiration from his prodigious achievements in theoretical nuclear physics and quantum many-body theory, as well as the understanding of pairing phenomena in nuclear and current theoretical physics and is central to a microscopic source of surprising and intriguing revelations about the microworld.
the Lifshitz phase diagram, the value of the BCS gap \( \Delta_0 = \Omega (p = p_F, T = 0) \) and critical temperature \( T_c \), above which the BCS gap closes and BCS superfluidity is terminated, turn out to be exponentially small:

\[
\Delta_0 = \Omega_D e^{-2/\lambda}, \quad T_c = 0.57\Delta_0,
\]

where \( \Omega_D \) is the BCS cutoff factor.

BCS theory reigned for several decades as the most successful theory in condensed-matter physics, both fundamentally and quantitatively. However, its limitations became apparent after the discovery of a family of high-temperature superconductors in the late 1980’s. Failure of the theory was conclusively established with the revelation of the so-called pseudogap phase in experimental studies of putatively normal phases of high-\( T_c \) superconductors by means of angular-resolved-photoemission spectroscopy (ARPES). In such a phase, there still exists a gap in the single-particle spectrum, even though the superconductivity is already terminated. BCS theory, a bedrock of our understanding of the phenomena of superfluidity and superconditivity in which termination of these phenomena and closure of the energy gap are inseparable, is manifestly inappropriate when we attempt to describe the pseudogap phase.

A plethora of scenarios have been offered in explanation of such challenging behavior of high-\( T_c \) superconductors. Their discussion is well beyond the scope of the present article, in which we choose to highlight a scenario associated with the original model of in-medium pairing correlations explored by Shafroth, Butler, and Blatt in the years leading up to the breakthrough made by BCS. This scenario envisions the formation of bound pairs in real three-dimensional space. Such a process becomes feasible in the strong-coupling limit when the pair radius turns out to be smaller than the mean interparticle distance, while the pair binding energy \( E \), playing the role of a gap in the spectrum of single-particle excitations above \( T_c \), exceeds the Fermi energy \( E_F \).

It follows that the pairing phase thus envisioned should involve the phenomenon of Bose-Einstein (BE) condensation. The most fully developed treatment of this phenomenon in solid-state physics, known as the theory of bipolaronic superconductivity, is the pioneering work of the late A. S. Alexandrov and his coauthors. To honor his contribution, we call this phase of matter the Shafroth-Butler-Blatt-Alexandrov (SBBBA) phase. The scenario of bipolaronic superconductivity is based on the polaron concept as set forth by Landau in 1933. Conventional polarons, having spin 1/2, result from interactions between electrons and optical phonons, their mass \( M_p \) appearing to be much larger than the electron mass \( M_e \). It is the mass \( M_p \) that enters the criterion for creation of a bound state of two polarons, the so-called bipolaron, and this criterion is met even if the attraction between polarons is moderate. In the description of superconductivity as a BE condensation of bound electron pairs, an idea already advanced by London in 1938, the interplay between bound pairs and the continuum of two-particle states is treated theoretically within the concept of quasichemical equilibrium, in analogy to thermodynamics of ordinary chemical reactions as presented in textbooks.

As \( T \) increases, the density of the superfluid Bose-Einstein condensate of real-space pairs declines and eventually vanishes, terminating superfluidity. The critical temperature \( T_{cBE} \) for destruction of bipolaronic superconductivity is not exponentially small as in Eq. (1), instead showing qualitative agreement with the behavior observed in high-\( T_c \) superconductors. Since SBBA theory attributes the property of superfluidity to the bosonic system of bound pairs, there should be no jump of the specific heat \( C(T) \) at \( T = T_{cBE} \), in contrast to this distinctive signature of BCS pairing at the associated critical temperature. Furthermore, it is easily verified that in SBBA theory the density of unbound fermions is proportional to \( e^{-E(T)/T} \). Hence their contribution can be safely ignored when \( E(T_c) \gg T_c \), and the ARPES data then give evidence for the persistence of the gap \( \Delta(T) \propto E(T) \) in the spectrum of single-particle excitations above \( T_c \).

The crisis faced by the BCS description in dealing with strongly correlated electron systems of solids, happening after 50 years of serenity, calls equally for a reassessment of the theory of nuclear pairing correlations, since nuclear systems, including atomic nuclei and neutron matter, are also composed of strongly correlated fermions.

In neutron matter, there exists a potential nuclear analog of the bipolaron, the in-medium dineutron. Like the bipolaron, the dineutron is non-existent in vacuum. However, analogously to the bipolaron situation, the presence of the background medium might promote the formation of bound dineutron pairs. Highly relevant to this possibility is a distinctive feature of neutron-neutron scattering, namely a narrow resonance lying at the tiny energy of 0.067 MeV, which implies that neutrons attract each other much more effectively by intrinsic nuclear forces than do two electrons by means of phonon exchange.

The dineutron state exhibits itself as a pole in the Cooper channel of the in-medium \( nn \) scattering amplitude, quite unrelated to the Fermi surface and existing until the density \( \rho \) reaches a critical value \( \rho_i \simeq 2\rho_0 \simeq 0.16 \text{ fm}^{-3} \), where \( \rho_0 \) denotes the neutron density in heavy nuclei. Also, the standard BCS approach is legitimate only at low densities \( \rho \leq 0.4\rho_0 \). At larger densities, the pairing constant \( \lambda \), being critically dependent on \( \rho \), changes sign (see Fig. 1), and application of the BCS theory becomes questionable.

Notwithstanding the subtle complexity and inherent richness of fermionic pairing, studies of the implications of pairing correlations over the last half century have been uniquely pursued within standard BCS theory. This article represents a first step toward understanding the details of the interplay between BCS and dineutron
pairing correlations in neutron matter.

II. THE STATE OF THE ART

In the decades following the landmark BCS paper, theorists achieved many successes in quantitative treatment of the pairing interaction between electrons in solids, as derived from electron-phonon exchange between electrons with energies near the Fermi surface. By contrast, the pairing problem in more strongly correlated Fermi systems such as neutron matter and liquid $^3$He continues to present serious challenges to quantitative, \textit{ab initio} microscopic description, in spite of numerous efforts in this direction.$^{24,25}$

Here we will deal with fundamental aspects of this more difficult class of systems that have previously escaped recognition, namely the possibility of a non-BCS pair-organized phase of neutron matter based on dineutron formation. In doing so, we will focus on qualitative rather than quantitative issues. Within this limited objective, it is reasonable to adopt the conventional approximation in which the block of Feynman diagrams irreducible in the Cooper channel is replaced by a vacuum $nn$ interaction potential $V$ of phenomenologically motivated form, fitted to two-nucleon scattering data. The Reid soft core (RSC) potential$^{24}$ is one such interaction that remained popular among nuclear theorists over an extended period. Highly refined potential models of the same type are in current use; prominent among these are the Argonne $V_{14}$ and $V_{18}$ potentials.$^{25}$

As seen from Fig. 1, the RSC interaction has the interesting property that the pairing constant $\lambda$, which has a positive sign in BCS theory, remains \textit{negative} for all values of $p_F$, due the strong inner repulsion present in this potential model. For the currently popular $nn$ potential models mentioned above, notably $V_{14}$ and $V_{18}$, $\lambda$ is seen to have the conventional positive sign at small $p_F$ values. However, this coupling parameter again shows a strong negative excursion as the density increases beyond $p_F \approx 0.8$ fm$^{-1}$. Of course, for a solution of the BCS gap equation to exist, the pairing interaction $V$ must take negative values for some range of particle separations $r$ in \textit{coordinate space}, corresponding to attraction between quasiparticles, and the RSC potential and other more realistic bare $nn$ interaction models certainly do meet this requirement. Thus it is no surprise that solutions of the gap equation for the RSC potential do exist, yielding a substantial maximum of the gap value $\Delta_0$ close to 3 MeV at $p_F \approx 0.85$ fm$^{-1}$ (e.g., see Ref. 26).

Consequent to this behavior, solutions of the BCS gap equation obtained for realistic $nn$ potentials exhibit a striking feature relative to the conventional BCS scenario. According to Eq. (1), the gap value $\Delta_0$ should increase rapidly and monotonically with increasing particle density, since the density of states $N(0)$ entering this formula is proportional to $p_F$. In the case of neutron matter described by the class of $nn$ potentials studied, $\Delta_0$ increases with $p_F$ up to a maximum around 0.85 fm$^{-1}$, then falls off and eventually closes at the critical density $\rho_c = p_F^3/(3\pi^2)$ corresponding to a Fermi momentum $p_F \approx 1.74$ fm$^{-1}$. (Numerical values are cited for the RCS potential; very similar results are obtained for the more modern potential models.) This feature can be ascribed to the occurrence of a bifurcation point in the BCS gap equation when the pairing interaction is constructed from a realistic $nn$ potential.

III. TWO TYPES OF PAIRING INSTABILITY OF THE NORMAL STATE IN NEUTRON MATTER

We shall use the term “conventional BCS solutions” to designate solutions existing in the case $\lambda > 0$ that behave in accordance with Eq. (1) when $\lambda$ tends to 0. The pairing solutions obtained for the RSC potential in Ref. 26, and in numerous independent calculations for the Argonne potentials, must then be identified as \textit{unconventional} solutions of the BCS gap equation, since the associated pairing constants $\lambda$ have the “wrong” (i.e., negative) sign over an extensive density range. In this connection, it is significant that in the density regime relevant to our discussion, the BCS gap function $\Delta(p)$ found for such potentials practically coincides with that of the dineutron solution of the Schrödinger equation in momentum space, provided that the neutron mass is only slightly enhanced (by $\simeq 5\%$) so as to admit a bound dineutron pair. This fact suggests the presence of a hidden dineutron state that is responsible for elements of the unorthodox behavior of the gap amplitude $\Delta_0(\rho)$ in superfluid nuclear matter.

To clarify the situation it is expedient to trace the location of the Cooper singlet-channel pole of the zerotemperature scattering amplitude $\Gamma(P = 0, \omega)$ in the
normal state of the system. This can be done based on the Bethe-Salpeter (BS) equation for the corresponding vertex part $T_{\alpha\beta}(p, \omega) = T(p, \omega) \Gamma_{\alpha\beta}$, where $\alpha, \beta$ are spin indices, $p$ is the momentum of the incoming quasiparticle (with its target having momentum $-p$), $\omega$ is the total two-particle energy measured from $2\mu$, and $\mu$ is the chemical potential. In our treatment, the required equation reads

$$T(p, \omega) = -\int V_0(p, p_1) L(p_1, \omega) T(p_1, \omega) dv_1$$  \hspace{1cm} (2)

in terms of the zeroth harmonic $V_0(p, p_1)$ of the interaction potential $V$ (which replaces a block of diagrams irreducible in the particle-particle channel), the particle-particle propagator $L(p, \omega)$ of the normal ground state, given by

$$L(p, \omega) = -(1 - 2n(p))/\omega - 2\epsilon(p) - i\delta\text{sgn}(p - p_F),$$  \hspace{1cm} (3)

with $\epsilon(p)$ the single-particle spectrum (chosen to coincide with the bare spectrum $\epsilon(p) = p^2/2M - \mu$ and $dv$ the volume element of 3D momentum space.

The central problem encountered in the analysis of Eq. (2) is to decompose the two-particle energy measured from $2\mu$, $\mu$, and $V$ into a separable part and a remainder, so as to derive a linear integral equation for the momentum dependence of the solution in which the presence of this singularity is immaterial. Once this linear equation is solved, we are left with a nonlinear equation for a gap amplitude or other related quantity, whose analysis and numerical solution are far simpler and more accurate than in direct treatment of the original nonlinear equation. As will be seen, this approach also proves advantageous in the forthcoming disclosure of a hidden dineutron phase in neutron matter subject to pairing correlations.

The zeroth harmonic $V_0$ entering Eq. (2) is decomposed as follows

$$V_0(p_1, p_2) = V_F \phi(p_1) \phi(p_2) + R(p_1, p_2),$$  \hspace{1cm} (4)

where $\phi(p) = V_0(p, p_F)/V_F$ with $V_F = V_0(p_F, p_F)$ and hence $\phi(p_F) = 1$. This decomposition is designed to yield the property

$$R(p, p_F) = R(p_F, p) = 0.$$  \hspace{1cm} (5)

Upon insertion of Eq. (4) into Eq. (2) followed by some algebra, we are led to a set of two coupled equations. The first of these,

$$D(p, \omega) \equiv \phi(p) - \int R(p, p_1) L(p_1, \omega) D(p_1, \omega) dv_1,$$  \hspace{1cm} (6)

is an equation for the shape factor $D(p) \equiv T(p)/T(p_F)$, which is almost unaffected by the Cooper singularity because the remainder $R(p, p_1)$ vanishes identically when either argument is on the Fermi surface. The second equation,

$$-1/V_F = \phi(p) L(p, \omega) D(p, \omega) dv,$$  \hspace{1cm} (7)

determines the location of the pole itself.

In the standard BCS situation with the Debye frequency $\Omega_D \ll \epsilon_F$, the remainder $R$ is suppressed. Analytically continuing Eq. (7) into the complex $\omega$ plane, we can therefore employ the first approximation $D^{(1)}(p) = 1$ to find

$$\frac{1}{\lambda} = 0.5 \left( \ln \frac{\Omega_D}{\omega} + i \frac{\pi}{2} \right),$$  \hspace{1cm} (8)

where $\lambda = -N(0)V_F$ as before. This equation has the solution $\omega = i\Omega$, where the real number $\Omega$ is found from the BCS equation

$$\frac{1}{\lambda} = 0.5 \ln \frac{\Omega_D}{\Omega},$$  \hspace{1cm} (9)

implying that one is dealing with the standard Cooper instability, which is eliminated through formation of the Cooper condensate. In the RSC case with $\lambda < 0$, the approximation $D^{(1)}(p) = 1$ fails: the dispersion equation (8) has no solutions at all, at variance with numerical results obtained from an iterative procedure or different methods of solving the standard BCS gap equation.

Pursuant to the point, let us address the case of small $\omega \to 0$ and recast Eq. (7) in the form

$$-1/V_F = I_{11}(\omega) + \int \phi(p) L(p, \omega) \eta(p, \omega) dv,$$  \hspace{1cm} (10)

wherein

$$I_{11}(\omega) = \int \phi^2(p) L(p, \omega) dv = 0.5 N(0) \left( \ln \frac{\epsilon_c}{\omega} + \frac{\pi}{2} \right),$$  \hspace{1cm} (11)

with a cut-off energy $\epsilon_c$, while the function $\eta(p, \omega) = D(p, \omega) - \phi(p)$, determined at arbitrary $\omega$ and $\rho$, obeys the equation

$$\eta(p, \omega) = -\int R(p, p_1; \rho) L(p_1, \omega) (\phi(p_1) + \eta(p_1, \omega)) dv_1.$$  \hspace{1cm} (12)

Since the neighborhood of the Fermi surface contributes divergently only to the first integral on the right side of Eq. (10), it would seem that nontrivial solutions with small $\Omega$ simply do not exist when $V_F > 0$. However, this is not the case. Such solutions do in fact emerge in the vicinity of a critical density $\rho_\sigma$ at which the second term on the right side of Eq. (10) diverges as well due to the divergence of the function $\eta(p, 0)$ at the bifurcation point. Thus, the singular terms conspire to cancel each other. It is known from the theory of
integral equations that the solution of an inhomogeneous linear integral equation such as (12) does indeed diverge at a critical density \( \rho_0 \) where the lowest eigenvalue \( \sigma_0 \) of the kernel \( \mathcal{R}(p, p_1, \rho) L(p, 0) \), determined from the equation

\[
\zeta_0(p) = -\sigma_0 \int \mathcal{R}(p, p_1, \rho) L(p_1, 0) \zeta_0(p_1) dv_1, \tag{13}
\]

is equal to unity.

The structure of the diverging component of \( \eta(p, \omega) \) is readily accessible by standard operations. For consider that the function \( \eta(p, \omega) \) may be expanded in a basis formed by the eigenfunctions \( \zeta_\alpha(p) \) of the above kernel. Extracting the main term proportional to \( \zeta_0(p) \) explicitly, we may write

\[
\eta(p, \omega) = \eta_0(p) \zeta_0(p) \theta(p), \tag{14}
\]

where the remainder \( \theta(p) \) vanishes at the Fermi surface like \( \zeta_0(p) \) and \( \eta(p) \). Inserting this formula into Eq. (12) and gathering all terms explicitly containing the factor \( \eta_0(p) \) on the left side of the equation, we may arrive at

\[
\eta_0(\omega) \left( \zeta_0(p) + \int \mathcal{R}(p, p_1) L(p_1, \omega) \zeta_0(p_1) dv_1 \right) = Y(p, \omega), \tag{15}
\]

where

\[
Y(p, \omega) = -\theta(p) - \int \mathcal{R}(p, p_1) L(p_1, \omega) (\phi(p_1) + \theta(p_1)) dv_1. \tag{16}
\]

With the aid of Eq. (13), the left side of Eq. (15) is recast in the form

\[
\eta_0(\omega) \left( \frac{\kappa}{\sigma_0} \zeta_0(p) + \int \mathcal{R}(p, p_1) \delta L(p_1, \omega) \zeta_0(p_1) dv_1 \right) = Y(p, \omega), \tag{17}
\]

where \( \delta L(p, \omega) = L(p, \omega) - L(p, 0) \). Here we have also introduced the effective stiffness coefficient

\[
\kappa = \sigma_0 - 1, \tag{18}
\]

which is central to the problem under discussion.

At the next step, we multiply both sides of Eq. (17) by the product \( \zeta_0(p) L(p, 0) \) and integrate over the momentum \( p \). Eliminating the operator \( \mathcal{R} \) in the same way as before, we obtain

\[
\eta_0(\omega) \left( \frac{\kappa}{\sigma_0} + \frac{1}{\sigma_0} \int \mathcal{R}(p, p_1) \delta L(p_1, \omega) \zeta_0(p_1) dv_1 \right) = Y(p, \omega), \tag{19}
\]

where the factor \( B \) is given by

\[
B(\omega) = -(I_{00})^{-1} \int \zeta_0(p) \delta L(p, \omega) \zeta_0(p) dv, \tag{20}
\]

while

\[
I_{00} = \int \zeta_0(p) L(p, 0) \zeta_0(p) dv, \tag{21}
\]

\[
I_{10}(\omega) = \sigma_0 \int \zeta_0(p) L(p, 0) Y(p, \omega) dv. \tag{21}
\]

Upon substituting the explicit form of the function \( Y(p, \omega) \) into the last of these integrals, it is found that the terms in the remainder \( \theta \) practically cancel each other. We are left with

\[
I_{10}(\omega) \simeq I_{10} = \int \zeta_0(p) L(p, 0) \phi(p) dv, \tag{22}
\]

and therefore arrive at

\[
\eta_0(\omega) = \frac{I_{10}}{\sigma_0}. \tag{23}
\]

Since \( B(\omega = 0) \) vanishes, we may then infer that the coefficient \( \eta_0(\omega = 0) \) given by Eq. (23), and hence the function \( \eta(p, 0) \), do in fact diverge at the critical density \( \rho_0 \) where \( \kappa(p) \) vanishes.

At the final step, Eqs. (14) and (23) are inserted into Eq. (11). After deleting insignificant contributions from the regular term \( \theta(p) \), we arrive at the required dispersion equation, whose analytical continuation to the complex \( \omega \) plane has the form

\[
0.5 \left( \ln \frac{\varepsilon_c}{\omega} + i \pi/2 \right) = \frac{1}{\lambda} - \frac{\nu^2}{\kappa + B(\omega)}, \tag{24}
\]

where we have employed formula (11) and introduced the notation \( \nu^2 = I_{10}^2/(I_{00} N(0)) \). Setting \( \omega = i \Omega \), Eq. (24) becomes

\[
0.5 \ln \varepsilon_c + \frac{\nu^2}{\Omega} = \frac{1}{\lambda} - \frac{\kappa + B \Omega^2}{\kappa + B \Omega^2} \ln(e_c/\Omega), \tag{25}
\]

with \( B = (\partial^2 B(\Omega)/\partial \Omega^2)_0/\ln(e_c/\Omega) > 0 \). The sign of \( B \) is readily established upon replacing \( \omega \rightarrow i \Omega \) in Eq. (3).

Let us now characterize the solutions of this equation in different quadrants of the Lifshitz plane \((\lambda, \kappa)\), while acknowledging that for any realistic \( nn \) interaction potential these parameters are constrained by one another. In the major part of the first quadrant \((\lambda > 0, \kappa > 0)\), the magnitude of the term proportional to \( \nu^2 \) is suppressed due to the poor overlap between the functions \( \phi(p) \) and \( \zeta_0(p) \) entering the integral \( I_{10} \). Consequently, the role of this term reduces to a renormalization of \( \lambda \), leaving us with the single BCS solution (11).

As already indicated, the function \( \kappa(p) \) becomes negative at \( \rho < \rho_0 \), triggering the onset of the dineutron state. Furthermore, in the quadrant \((\lambda > 0, \kappa < 0)\), which is relevant to the Argonne case at densities \( \rho \) below about 0.4\( \rho_0 \), Eq. (20) has two different solutions. It is instructive to trace the trajectories of both the roots \( \Omega_{1,2} \) versus \( \lambda \). In the limit \( \lambda \rightarrow 0 \), the left root closest to the origin behaves as \( \Omega_1 \propto e^{-2/\lambda} \), with \( \Omega_2(0) = 0 \). It should therefore be identified with the BCS-like root. In the limit addressed, the other root \( \Omega_2 \) occurs close to \( \sqrt{\kappa/B} \). This root is thus definitely of non-BCS nature. As \( \lambda \) increases, both roots move away from the origin. It must be stressed that in our analysis both of the parameters \( \lambda \) and \( \kappa \) are supposed to be small to ensure the smallness of the roots; in this respect the analysis is self-consistent and implies...
that the inequality $\Omega_1(\lambda) < \Omega_2(\lambda)$ holds as the roots evolve. This inequality implies that the non-BCS scenario ensures a shorter relaxation time for the rearrangement of the normal state than the standard BCS one does that obviates the latter scenario.

In the third quadrant ($\lambda < 0, \kappa < 0$), the right non-BCS root no longer exists, but the left one survives. This root may be treated as an unconventional BCS root in the following sense: It is true in a significant domain of the quadrant, the BCS-like behavior ot $\Omega \propto e^{-2/\lambda_{\text{eff}}}$ applies, with $1/\lambda_{\text{eff}} = \nu^2/|\kappa| - 1/|\lambda|$. However, such behavior is completely rearranged near the critical point $\lambda = 0$, where it becomes non-exponential: $\Omega(\lambda \to 0) \to \sqrt{|\kappa|/B \ln(\epsilon_c B/|\kappa|)}$.

IV. COMPETITION BETWEEN BCS AND IN-MEDIUM DINEUTRON CORRELATIONS AT FINITE TEMPERATURE

In this section, we examine the temperature evolution of the two different types of pairing correlations revealed in the quadrant ($\lambda > 0, \kappa < 0$). We focus in each case on a possible phase transition that occurs at the critical temperature for termination of pairing correlations of the given type, as determined from the Thouless criterion. This criterion takes the form of a linear integral equation

$$T(p, T) = -\int V_0(p, p_1) \frac{\tanh(\epsilon(p_1)/2T)}{2\epsilon(p_1)} T(p_1, T) dp_1$$

(26)

analogous to Eq. 4 explored in Sec. III.

To proceed further it is instructive to analyze this equation with the aid of the decomposition procedure introduced in Sec. III. With details relegated to the Appendix, we proceed immediately to the final result

$$0.5 \ln(\epsilon_c/T) = 1/\lambda - \frac{\nu^2}{\kappa(p) + \gamma(T)}$$

(27)

where

$$\gamma(T) = \frac{1}{\hbar_0} \int \zeta_0(p) \left( \frac{\tanh(\epsilon(p)/2T)}{2\epsilon(p)} - \frac{1}{2|\epsilon(p)|} \right) \zeta_0(p) dp.$$  

(28)

This expression can be recast as

$$\gamma(T) \propto \int_0^\infty e^{-T T_0} \frac{d\epsilon}{e^{\epsilon/T} + 1} = \gamma T^2.$$  

(29)

The graphical solution of Eq. 27 is shown in Fig. 2. The two sides of this equation are plotted versus the temperature. The logarithmic curve is seen to cross both branches of the hyperbolic curve, and hence Eq. 27 does possess two roots $T_c$ and $T^*$. In the limit $\lambda \to 0$, we naturally obtain the BCS solution $T_c \propto e^{-1/\lambda}$. Even so, in the region $\lambda > 0, \kappa < 0$ there exists another, non-BCS root of Eq. 27 written as

$$0.5 \ln(\epsilon_c/T) = 1/\lambda - \frac{\nu^2}{|\kappa(p)| - \gamma T^2},$$

(30)

namely $T^*(\lambda \to 0) = \sqrt{|\kappa(p)|/\gamma} > T_c$ that has no exponential smallness. This result informs us that in the region $\lambda > 0, \kappa < 0$, the BCS solution loses the competition with the in-medium dineutron solution not only in the interval $T_c < T < T^*$ where the BCS solution does not exist but also at $T < T_c$ where BCS gain in energy, being exponentially small, ranks below the dineutron one in the value.

In the quadrant $\lambda < 0, \kappa < 0$, the right non-BCS root is seen to disappear, and there remains a single bizarre solution which, at $|\kappa| < \nu^2|\lambda|$, behaves in harmony with the BCS-like formula $\Omega \propto e^{-2/\lambda_{\text{eff}}}$, in which $1/\lambda_{\text{eff}} = \nu^2/|\kappa| - 1/|\lambda|$. Otherwise, however, this solution exhibits non-BCS behavior with $\Omega(T^*) \simeq \sqrt{|\kappa|/\gamma}$.

V. PHASE TRANSITION BETWEEN BCS AND PSEUDOGAP STATES IN STRONGLY CORRELATED FERMI SYSTEMS AT FINITE TEMPERATURE

In principle, the above results, derived employing the standard Fermi-liquid spectrum $\epsilon(p) \propto (p - p_F)$, need not hold in general, and most especially when the system is subject to very strong correlations in the particle-hole
channel. Such correlations often give rise to a so-called quantum critical point (QCP) where the effective mass $M^*$ diverges. This behavior triggers a rearrangement of the Fermi surface. The same is true at finite $T$ as well, up to some critical temperature $T_M$ where this root disappears. The function $\epsilon(p)$ then becomes positive definite, implying that the Fermi surface collapses at this temperature, marking the onset of classical physics. Correspondingly, the left side of Eq. (25) ceases to be logarithmically divergent, and the customary BCS solution, existing at small $\lambda$, disappears. In conventional Fermi liquids having effective masses $M^*$ not so different from the bare mass $M$, the collapse of the Fermi surface occurring at $T_M \simeq \epsilon_F^0$ has no impact on BCS correlations, since BCS superconductivity has already been terminated well before the collapse takes place.

However, in strongly correlated Fermi systems lying on the edge of stability of the Landau state, the form of the spectrum $\epsilon(p)$ can be completely different than in FL theory. This is illustrated in Fig. 3 which presents results of numerical calculations of the single-particle spectrum $\epsilon(p)$, as determined within a model described in Ref. 34. The Landau interaction function $f(q)$ of this model has the dimensionless form

$$f(q)N(0) = \frac{\alpha}{q^2 + \beta^2 p_F^2} \tag{32}$$

with $\beta = 0.3$ and different choices for $\alpha$. The bandwidth $W = |\epsilon(p = 0)|$ is seen to shrink dramatically as the interaction strength approaches a critical value, at which the stability of the conventional Landau state with $n(p) = \theta(p_F - p)$ is lost and the Fermi surface becomes multi-connected. In the vicinity of the critical point, the single root $p_F(T)$ of Eq. (31) at which the spectrum $\epsilon(p, T)$ changes sign is found to approach zero rapidly as the temperature $T$ is raised. This behavior contrasts sharply with that seen in the conventional FL case, where this root moves extremely slowly: $p_F(T) = p_F(0) + O(T^2/(\epsilon_F^0)^2)$.

Suppose now that there exist in nature homogeneous Fermi systems that are so strongly correlated that all quasiparticles go into the fermion condensate (FC), consisting of the totality of single-particle states belonging to a completely flat spectrum. (See Refs. 35–37 for comprehensive reviews of this phenomenon and its implications, as well as Ref. 34.) Illustrative results from numerical calculations are presented in Figs. 4 and Fig. 5 which display single-particle spectra $\epsilon(p)$ evaluated for different temperatures based on the same form (32) for the quasiparticle interaction, but with different input parameters ($\beta = 0.07$ and $\alpha = 1.5$ and 4.2, respectively). The coupling constant $\alpha = 1.5$ is so large that all the
Thus, such an super-strongly correlated Fermi system has no Fermi surface at all. In this statement, the notation \( \alpha \) demonstrates the behavior of the spectra within the same temperature, we find \( \epsilon(p = 0) \approx -0.003\epsilon_F^0 \), whereas at \( T_M \approx 0.1\epsilon_F^0 \), this quantity changes sign, so that at \( T > T_M \) roots of Eq. (34) no longer exist. Fig. 4 demonstrates the behavior of the spectra within the same model \( (\beta) \), but at \( \alpha = 4.2 \). These results inform us that at \( \alpha = 1.5 \) the spectrum \( \epsilon(p) \) assumes negative values only at \( T < 0.05\epsilon_F^0 \), and only over a small range at small \( p \), while at \( \alpha = 4.2 \) the function \( \epsilon(p) \) becomes positive independently of both momentum \( p \) and temperature \( T \). Thus, such an super-strongly correlated Fermi system has no Fermi surface at all. Evidently, if in this case the Thouless equation \( (\beta) \) has a nontrivial solution, it must be attributed to the pseudogap phase.

An examination of the spectra \( \epsilon(p) \) drawn in Figs. 4 and 5 allows one to infer that beyond \( T_M \), the behavior of the integral \( I_{11}(T) \) evolves from the conventional BCS logarithmic character, in which both solutions survive, to a behavior almost independent of \( T \).

In the latter case, we are left with a single solution, which is the non-BCS solution provided \( L < 1/\lambda \) or a BCS solution provided \( L > (1/\lambda + \nu^2/|\kappa|) \); otherwise, nontrivial solutions of the dispersion equation \( (\gamma) \) do not exist at all. In this statement, the notation \( L \) is employed for a posited value of the integral \( I_{11} \) in the case of non-Fermi-liquid behavior of the single-particle spectrum \( \epsilon(p) \). Referring to Fig. 2 the three situations just identified correspond to \( L \) lying (i) below the lower horizontal line (non-BCS), (ii) above the upper horizontal line (BCS), and (iii) between the two horizontal lines (no nontrivial solution).

VI. DISCUSSION AND CONCLUSIONS

For over fifty years, theoretical consideration of pairing correlations in neutron matter has been carried out uniquely within BCS theory. No work based on some other conception of pairing correlations has appeared hitherto, despite the salient fact that in a significant density domain, the pairing constant \( \lambda \) has the “wrong” (i.e., negative) sign with respect to BCS theory. This article is the first to treat on equal footing the BCS pairing correlations giving rise to the Cooper condensate and the non-BCS pairing correlations that induce in-medium dineutron formation.

Let us summarize the findings of our present analysis of the pairing instabilities of the normal state of neutron matter, which may serve as a guide to resolution of the remaining qualitative and quantitative issues that have been exposed. The arguments offered and results obtained demonstrate that contrary to common belief, the Lifshitz phase diagrams of neutron matter and other comparable many-fermion systems that are subject to pairing correlations are characterized by two dimensionless parameters, associated with two different pairing scenarios that operate in different density regions. The first of these, \( \lambda = -V(p_F, p_F)N(0) \), represents the diagonal matrix element of the pairing interaction evaluated on the Fermi surface. This parameter, associated with Cooper pairing of quasiparticles whose energies lie very near the Fermi surface, is relevant in the case \( \lambda > 0 \).

The second parameter characterizing the Lifshitz phase diagram, denoted by \( \kappa \), has scant relation to the Fermi surface. The sign of this parameter is indicative of the possibility of dineutron formation in the medium. If \( \kappa \) is positive, the dineutron correlations interfere with the BCS mechanism so as to suppress conventional pairing correlations. If \( \kappa \) is negative, as expected to apply in neutron matter over some density range below \( \rho_l \approx 2\rho_0 \), the system is able to undergo a dineutron phase transition analogous to the formation of bipolarons in solid state physics. Correspondingly, as we have seen, two different phases organized by pairing correlations a conventional BCS phase with a critical temperature \( T_c \) for termination of the Cooper condensate, and an in-medium dineutron phase.

In the customary situation where the neutron spectrum has the Fermi-liquid form \( \epsilon(p) = pF(p - p_F)/M^* \) with \( M^* \approx M \), the critical temperature \( T^* \) for dissolution of dineutron correlations exceeds the temperature \( T_c \) beyond which the BCS scenario becomes scenario irrelevant. Indeed, near \( T_c \) the BCS gain in energy is extremely small, being proportional to \( (T_c - T)^2 \), while the corresponding shift in energy due to dineutron
correlations contains no such small factor. As for the quadrant \((\lambda < 0, \kappa < 0)\), only a hybrid phase survives, which, at \(|\kappa| \ll \sqrt[2]{|\lambda|}\), exhibits a BCS-like behavior with \(\lambda_{\text{eff}} > 0\), while it becomes a non-BCS phase in the limit \(\lambda \to 0\). Even such a restricted analysis as we have performed demonstrates that there is little or no room for conventional BCS correlations in neutron matter.

In some respects, the situation created by these revelations of the nature of pairing correlations in neutron matter is reminiscent of that which arose for the low-lying \(2^+\)-collective oscillations of atomic nuclei at the dawn of the age of the standard nuclear paradigm. At that time, commonly employed models with simple effective nucleon-nucleon interactions of the quadrupole-quadrupole type entailed a description of these excitations as quanta of zero sound of the collective frequencies was achieved, the posited nature of these excitations as quanta of zero sound of the bulk nucleus, a volume effect. While agreement of the theoretical results with the experimental data on collective oscillations has been established, the nature of the \(2^+\) levels turned out to be incorrect. Indeed, later developments, corroborated by experimental data on inelastic scattering of high-energy electrons, have established that these oscillations belong instead to the Goldstone surface mode associated with loss of translational invariance\(^{23-24}\). In the nuclear pairing problem, model calculations with effective forces having positive pairing constant \(\lambda_{\text{eff}}\) have gained widespread acceptance as well, implying the presence of a Cooper condensate. In our article we argue that this scenario has flaws. Alas, as yet there is no a distinct experimental method for the measurement of the transition density in the Cooper channel. Furthermore, even there were, significant difficulties would be encountered in the interpretation of corresponding experimental data because, unlike the case of low-lying collective excitations of atomic nuclei, there is no a blatant contradiction between the BCS and non-BCS transition densities.

Therefore we adopt a different strategy, based on comparison of the energy shifts \(\delta E_0\) associated with the onset of the pairing correlations in the BCS and intermediate dineutron scenarios where the corresponding phase transition belongs to a family of second-order phase transitions, whose properties are properly explained within the theory of second-order phase transitions. In light of this situation, the analogy with the problem of the spontaneous quadrupole deformations of atomic nuclei is helpful. The theory of nuclear deformations provides the formula

\[
\delta E_0 = C\beta^2 + D\beta^4 + \ldots ,
\]

containing the deformation parameter \(\beta\), the stiffness coefficient \(C\), and the coefficient \(D\) (presumed positive), which is responsible for repulsive interactions between the collective quadrupole modes. On the disordered side of the phase transition, the stiffness \(C\) has a positive value, so that \(\beta = 0\). Beyond the phase transition point, \(C\) changes sign, triggering the emergence of a new phase with the deformation parameter \(\beta^2 = -C/2D\) and the shift in ground state energy \(\delta E_0 = -C^2/4D\). Definitely, near the new equilibrium point where, as seen, \(\delta E_0(\beta) \propto C_\beta(\beta - \beta_0)^2\), the stiffness coefficient \(C_\beta\) turns out to be positive. This is similar to what happens for the first term of the expansion of the thermodynamic potential in the theory of second-order phase transitions.

It is quite significant that formulas analogous to Eq. \(^{23}\) appear not only in the theory of nuclear deformation but also in the Landau theory of second-order phase transitions, in the theory of pion condensation\(^{24}\) in the self-consistent theory of low-energy nuclear phenomena\(^{24}\) and in many refined versions of mean-field theory (see e.g. 44,45). In all these theories, the stiffness coefficient \(C\) is expressed unambiguously in terms of the inverse response function \(\chi^{-1}\) by means of the formula relating \(\chi\) to the variation of the ground state energy. This relation, written symbolically as \(\delta E_0 = (1/2)\chi\delta\rho_0 \delta\rho = (1/2)\chi^{-1}\delta\rho_0 \delta\rho\), is to be evaluated on the disordered side of the transition and then applied on the ordered side. (By definition, \(\chi = \delta\rho_0 / \delta\rho\), where \(\delta\rho\) is the density variation produced by a weak static external field \(\delta\rho_0\).)

In the nuclear pairing problem, the corresponding response function \(\chi_C\) should be evaluated in the Cooper channel, implying that \((\chi_C)_{\alpha\beta} = \chi_C(\tau_2)_{\alpha\beta}\); This can be done with the aid of the same decomposition strategy\(^{1}\) deployed earlier in this text. Omitting intervening mathematical steps, we give the final result

\[
\chi_C(p) = \chi_C(p)T(p) \propto \kappa^{-1}L(p, 0)\zeta_0(p). \quad (34)
\]

Accordingly, the inverse response function \(\chi_C^{-1}\), and hence the stiffness coefficient entering as the first term of the expansion of the energy shift \(\delta E_0\) of the pairing problem, turn out to be proportional to the critical quantity \(\kappa(\rho)\), which becomes negative on the ordered side of the dineutron phase transition. Other than assuming it to be positive, we do not consider here the second term \(D\) of the corresponding expansion, which is proportional to the dineutron scattering amplitude. In this case, the energy shift \(\delta E_0\) increases linearly with \(\kappa^2(\rho)\) as one moves farther from the point of the dineutron phase transition. If the assumption \(D > 0\) fails, the expansion should, as usual, be supplemented by successive terms. At any rate, the shift \(\delta E_0\) is not exponentially small.

Contrariwise, within the established framework of BCS theory, the BCS energy shift near the critical density \(\rho_t\) is of course exponentially small\(^{25}\):

\[
\delta E_0^{\text{BCS}}(T = 0) \propto \Delta_0^2 \propto e^{-a\rho_0 / \kappa(\rho)}, \quad (35)
\]

with \(a\) as a numerical factor. Conclusively, the dineutron effect wins the energetic competition for ascendancy.

Our analysis has been restricted to the vicinity of the critical points, where presumably the second-order phase transition scenario we have employed is applicable. In obtaining all our results, we have proceeded from the
assumption, commonly adopted in work on the nuclear pairing problem, that the pairing interaction is given by the in-vacuum or “bare” interaction potential \( V \). This assumption is clearly of limited validity, because medium effects can significantly alter the pairing interaction from its vacuum counterpart. Unfortunately, the associated “polarization corrections” to the pairing interaction still await proper investigation. In this connection it is worth noting that Pankratov et al. have shown that the gap values in atomic nuclei are overestimated by a factor two when such renormalization effects are ignored. One of explanations of this discrepancy is associated with a complete suppression of pairing correlations in the nuclear interior that renders the pairing correlations the surface phenomenon. This situation highlights the timeliness of the message of S. T. quoted in the Preamble, confirming the wisdom spoken by a great Russian poet Fyodor Tyutchev more than 100 years ago: “Нам не дано предугадать, как слово нами отозвется...”

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Appendix

In this appendix we recast the Thouless equation into a form similar to that obtained in the analysis of the instabilities of the normal state. To begin, the pairing interaction \( V_0 \) is decomposed as

\[
V_0(p_1, p_2) = V_F \phi(p_1)\phi(p_2) + \mathcal{R}(p_1, p_2),
\]

where \( \phi(p) = V_0(p_F, p)_F/V_F \), with \( V_F = V_0(p_F, p_F) \). Upon inserting Eq. (36) into Eq. (26) and performing some algebra, Eq. (26) is transformed into a set of two equivalent equations, the first of which is given by

\[
\chi(p) = \phi(p) - \int \mathcal{R}(p, p_1) \frac{\tanh(\epsilon(p_1)/2T^*)}{2\epsilon(p_1)} \chi(p_1) d\nu,
\]

while the second takes the form

\[
-\frac{1}{V_F} = \int \phi(p) \frac{\tanh(\epsilon(p)/2T^*)}{2\epsilon(p)} \chi(p) d\nu.
\]

After introducing the difference \( \eta(p) = \chi(p) - \phi(p) \) and taking several algebraic steps one arrives at

\[
-\frac{1}{V_F} = I_{11}(T^*) + \int \phi(p) \frac{\tanh(\epsilon(p)/2T^*)}{2\epsilon(p)} \eta(p) d\nu, \quad (39)
\]

where

\[
I_{11}(T) = \int \phi(p) \frac{\tanh(\epsilon(p)/2T)}{2\epsilon(p)} \phi(p) d\nu \approx 0.5N(0) \ln(\epsilon_c/T), \quad (40)
\]

while the function \( \eta(p) \) obeys equation

\[
\eta(p, T) = -\int \mathcal{R}(p, p_1) \frac{\tanh(\epsilon(p_1)/2\epsilon(p_1))}{2\epsilon(p_1)} \left( \phi(p_1) + \eta(p_1, T) \right) d\nu_1. \quad (41)
\]

We observe that at the bifurcation point \( T^* = 0 \), the first term on the right side of Eq. (41) diverges logarithmically, so that a solution \( T^* = 0 \) exists only if the second term also diverges at this point. To confirm that the latter is the case, we expand the function \( \eta(p) \) in a basis formed by the eigenfunctions \( \zeta_\alpha(p) \). Extracting the main term proportional to \( \zeta_0(p) \) explicitly, we write

\[
\eta(p, T) = \eta_0(T) \zeta_0(p) + \vartheta(p), \quad (42)
\]

where, as before, the eigenfunction \( \zeta_0(p) \) obeys the equation

\[
\zeta_0(p) = -\sigma_0 \int \mathcal{R}(p, p_1) \frac{1}{2|\epsilon(p_1)|} \zeta_0(p_1) d\nu_1, \quad (43)
\]

while the remainder \( \vartheta(p) \) vanishes at the Fermi surface like \( \zeta_0(p) \) and \( \eta(p) \). Upon inserting this expansion into Eq. (42) and collecting all terms containing the factor \( \eta_0(T) \) on the left side of Eq. (41), one obtains

\[
\eta_0(T) \left( \zeta_0(p) + \int \mathcal{R}(p, p_1) \frac{\tanh(\epsilon(p_1)/2\epsilon(p_1))}{2\epsilon(p_1)} \zeta_0(p_1) d\nu_1 \right) = Z(p), \quad (44)
\]

where

\[
Z(p) = -\vartheta(p) - \int \mathcal{R}(p, p_1) \frac{\tanh(\epsilon(p_1)/2\epsilon(p_1))}{2\epsilon(p_1)} \left( \phi(p_1) + \theta(p_1) \right) d\nu_1. \quad (45)
\]

The left side of Eq. (44) is conveniently rewritten with the aid of Eq. (43) to yield

\[
\eta_0(T) \left( \frac{\kappa}{\sigma_0} \zeta_0(p) + \int \mathcal{R}(p, p_1) \mathcal{D}(p_1, T) \zeta_0(p_1) d\nu_1 \right) = Z(p), \quad (46)
\]

where

\[
\mathcal{D}(p, T) = \frac{\tanh(\epsilon(p)/2T)}{2\epsilon(p)} - \frac{1}{2|\epsilon(p)|}. \quad (47)
\]

Next, both sides of this equation are multiplied by the product \( \zeta_0(p)/(2|\epsilon(p)|) \), the momentum integration is

\[
\int \zeta_0(p)/(2|\epsilon(p)|) \mathcal{D}(p, T) \zeta_0(p) d\nu = \int \zeta_0(p)/(2|\epsilon(p)|) \left( \frac{\tanh(\epsilon(p)/2T)}{2\epsilon(p)} - \frac{1}{2|\epsilon(p)|} \right) \zeta_0(p) d\nu,
\]

resulting in

\[
\int \zeta_0(p)/(2|\epsilon(p)|) \left( \frac{\tanh(\epsilon(p)/2T)}{2\epsilon(p)} - \frac{1}{2|\epsilon(p)|} \right) \zeta_0(p) d\nu = \int \zeta_0(p)/(2|\epsilon(p)|) \mathcal{D}(p, T) \zeta_0(p) d\nu = \int \zeta_0(p)/(2|\epsilon(p)|) Z(p) d\nu.
\]

This finally yields

\[
\int \zeta_0(p)/(2|\epsilon(p)|) Z(p) d\nu = 0,
\]

which means that

\[
\int \zeta_0(p)/(2|\epsilon(p)|) \mathcal{D}(p, T) \zeta_0(p) d\nu = \int \zeta_0(p)/(2|\epsilon(p)|) Z(p) d\nu = 0.
\]
performed. Eliminating the operator $\mathcal{R}$ with the aid of Eq. (43), we arrive at

$$\eta_0(T) = (\kappa + \gamma(T))^{-1} I_{10}/I_{00},$$

(48)

where $I_{00} > 0$ is given by Eq. (21) of Sec. III, and

$$\gamma(T) = -\frac{I_{10}}{I_{00}} \int \zeta_0(p) \frac{\text{tanh}(\epsilon(p)/2T)}{2|\epsilon(p)|} \zeta_0(p) d\upsilon < 0,$$

while

$$I_{10} = \sigma_0 \int \zeta_0(p) \frac{1}{2|\epsilon(p)|} Z(p) d\upsilon.$$  

(50)

To demonstrate that the sign of $\gamma(T)$ is indeed positive, we rewrite the integrand of Eq. (49) according to

$$\gamma(T) \propto \int \zeta_0^2(p) \frac{1 - \text{tanh}(|\epsilon(p)|/2T)}{2|\epsilon(p)|} d\upsilon.$$  

(51)

Since $\zeta_0(p)$ vanishes at the Fermi surface like $\epsilon(p)$, we immediately conclude that $\gamma(T) = \gamma T^2$. Considering now the integral $I_{10}$, the explicit form of $Z(p)$ is inserted into the integrand of Eq. (50) and it is verified that the terms involving the remainder $\theta$ practically cancel each other. Accordingly, we may take

$$I_{10} = \int \zeta_0(p) \frac{1}{2|\epsilon(p)|} \phi(p) d\upsilon.$$  

(52)

Since $\gamma(T)$ vanishes at $T \to 0$, the coefficient $\eta_0(0)$ does in fact diverge together with the function $\eta(p)$ itself at the critical point where $\kappa$ changes sign.

Next we substitute Eq. (48) into the dispersion equation (39) to find

$$0.5 \ln(\epsilon_c/T) = 1/\lambda - \frac{\mu^2}{\kappa(\rho_0 + \gamma T^2)},$$

(53)

with the understanding that all minor corrections are included in the effective pairing constant denoted $\lambda$, as before. Near the critical density $\rho_c$, the parameter $\kappa$, being negative, behaves as $-1/(\partial \kappa/\partial \rho)_c (\rho_c - \rho)$. The term $1/\lambda$ in Eq. (53) can then be omitted, and we are left with

$$T_c(\rho) \propto e^{-\alpha \rho_c/(\rho_c - \rho)},$$

(54)

where $\alpha$ is a numerical constant. On the other hand, Eq. (53) has the the second, non-exponential root

$$T^* \approx \sqrt{|\kappa|/\gamma}$$

(55)

corresponding to the dineutron phase. We conclude that $T^* > T_c$, and consequently that the BCS solution loses the competition with the dineutron solution.

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47 “We cannot know further ways of our word — how it’ll be drifted . . .”