Determination of the working area and singularity zones of the 3-RRR robot based on the non-uniform coverings method

L A Rybak1, E V Gaponenko1, D I Malyshev1 and L Behera2

1Belgorod State Technological University named after V. G. Shukhov, 46 Kostyukov Str., Belgorod, 308012, Russia
2Indian Institute of Technology Kanpur, Kanpur 208016, India

E-mail: rl_bgtu@intbel.ru

Abstract. The paper considers the optimization algorithms applied to define the working area of a flat 3-RRR-mechanism. The system of inequalities considering the acceptable range of an argument of driving angles arcsine was used to define the working area. The condition of equality of Jacobian matrixes to zero for the first and second type of singularities was used to define special positions. The paper presents the results of modeling for eight combinations of rotation angles of driving pairs. The received results can be used to choose the geometrical parameters of a 3-RRR mechanism setting the boundaries of the working area through the technological process and to plan the movement trajectory of an output link to avoid uncontrollable mobility or jamming of the mechanism in special positions.

1. Introduction

The definition of the robot’s working area is an important task of its design to perform technological and other operations. The volume of the working area shall ensure maneuverability of the robot. However, when parallel robots are used their working space is limited due to their geometry or other characteristics. It is important to increase the working area by applying new perspective designs and technology solutions.

Such scientists as V.A. Glazunov, C. Gosselin, J. P. Merlet made a major contribution to the development of the theory of parallel robots in terms of structural and kinematic analysis [1-3]. Besides, parallel robots are characterized by special positions, which significantly limit their working space and which shall be considered in their design. The methods of analysis of the working space are considered in detail by Evtushenko Yu.G. and Posypkin M.A. [4-6]. This particular study brings these methods to a sharper focus and applies them to study the working area of a flat 3-RRR robot based on the approximation of the system of nonlinear equations. In previous works these methods were applied for other types of parallel robots based on the approximation of the system of inequalities and nonlinear equations [7-9].

2. Algorithm to define the working area and zones of singularity of a 3-RRR mechanism

Let us consider a flat 3-RRR robot. The mechanism includes three chains each containing three rotary kinematic pairs \( A_i, B_i, C_i \) (\( i = 1, 2, 3 \)) (Figure 1). The rotation axes of all pairs are parallel to each other and perpendicular to the plane in which the mechanism is moving. The rotary pairs are fixed on a base and their position is set by coordinates \( x_i, y_i \) in motionless rectangular system of coordinates. The
position of an output link of the mechanism is set by the position of point D and is described by coordinates \( x_0 \) and \( y_0 \) and by the rotation angle \( \varphi \) of this link in relation to some initial position. The output link is moved due to rotation of driving (input) pairs \( A_i \). The rotation angles of these pairs represent the generalized coordinates for this mechanism. \( R \) and \( r \) – radiuses of circles circumscribing triangles \( A_1A_2A_3 \) and \( C_1C_2C_3 \) respectively.

![Figure 1. Scheme of a flat 3-RRR mechanism](image)

The general view of constraint equation for the given mechanism [10]:

\[
(x + l_{3,i} \cos(\gamma_i + \varphi) - x_i - l_{1,i} \cos \theta_i)^2 + (y + l_{3,i} \sin(\gamma_i + \varphi) - y_i - l_{1,i} \sin \theta_i)^2 - l_{2,i}^2 = 0,
\]

where \( i = 1, 2, 3 \) – number of kinematic chains of the mechanism.

Let us express the formulas to calculate angles \( \theta_i \) from equation (1):

\[
\theta_{i,1} = \arcsin \left( \frac{l_{2,i}^2 - l_{1,i}^2 - [b_{i,\cos} - b_{i,\sin}]}{\sqrt{[a_{i,\sin}]^2 + [a_{i,\cos}]^2}} \right) - \varphi,
\]

\[
\theta_{i,2} = \pi - \arcsin \left( \frac{l_{2,i}^2 - l_{1,i}^2 - [b_{i,\cos} - b_{i,\sin}]}{\sqrt{[a_{i,\sin}]^2 + [a_{i,\cos}]^2}} \right) - \varphi,
\]

where

\[
[a_{i,\cos}] = 2 \left[ x_i l_{1,i} - x l_{1,i} - l_{1,i} l_{3,i} \cos(\gamma_i + \varphi) \right],
\]

\[
[ b_{i,\cos} ] = \begin{bmatrix}
 x^2 + 2 x_l l_{3,i} \cos(\gamma_i + \varphi) - 2 x_l l_{3,i}^2 \cos^2(\gamma_i + \varphi) - \\
 -2 x_i l_{3,i} \cos(\gamma_i + \varphi) + x_l^2
\end{bmatrix},
\]

\[
[a_{i,\sin}] = 2 \left[ y_i l_{1,i} - y l_{1,i} - l_{1,i} l_{3,i} \sin(\gamma_i + \varphi) \right],
\]

\[
[b_{i,\sin}] = \begin{bmatrix}
 y^2 + 2 y l_{3,i} \sin(\gamma_i + \varphi) - 2 y y_l l_{3,i}^2 \sin^2(\gamma_i + \varphi) - \\
 -2 y y_l l_{3,i} \sin(\gamma_i + \varphi) + y_l^2
\end{bmatrix},
\]

2
\[
\begin{align*}
\sin \phi &= \frac{[a_i, \cos]}{\sqrt{[a_i, \sin]^2 + [a_i, \cos]^2}}, \\
\cos \phi &= \frac{[a_i, \sin]}{\sqrt{[a_i, \sin]^2 + [a_i, \cos]^2}}.
\end{align*}
\]

The inverse problem regarding the position of this mechanism is solved by the calculation of angles \( \theta_i \) for each kinematic chain corresponding to the position of an output link set by coordinates \( x, y, \varphi \) through formula (2).

Thus, generally for any nonsingular position there are two solutions of the inverse problem on positions. For the mechanism with three kinematic chains the number of solutions and the corresponding configurations of intermediate links will equal eight.

Let us indicate this in (2) \[
l_2^2 - l_1^2 - [b_i, \cos] - [b_i, \sin] = \sin. \]

This function is an arcsine argument, therefore the condition \( 1 \leq \sin \leq 1 \), which was chosen as a criterion of the working area, shall be satisfied. Considering the condition, let us generate the system of inequalities:

\[
\begin{cases}
\sin - 1 \leq 0, \\
-1 - \sin \leq 0, \\
\sin - 1 \leq 0, \\
-1 - \sin \leq 0, \\
\sin - 1 \leq 0, \\
-1 - \sin \leq 0.
\end{cases}
\]

(3)

It is also necessary to exclude special positions (singularity points) of the mechanism from the working space. According to [1], singularities of three types are typical for the 3-RRR mechanism: special positions of the first type, when \( \det(J_B) = 0 \), special positions of the second type, when \( \det(J_A) = 0 \) and special positions of the third type, which combine the properties of special positions of the first and second types.

Let us consider special positions of the first and second types

\[
J_B = \begin{pmatrix}
\frac{\partial F_1}{\partial \theta_1} & 0 & 0 \\
0 & \frac{\partial F_2}{\partial \theta_2} & 0 \\
0 & 0 & \frac{\partial F_3}{\partial \theta_3}
\end{pmatrix},
\]

where \( \frac{\partial F_i}{\partial \theta_j} = 2l_1 \sin \theta_i(x - x_{Al} + l_{3,i} \cos(\varphi + \gamma_i) - l_1 \cos \theta_i) - 2l_1 \cos \theta_i(y - y_{Al} + l_{3,i} \sin(\varphi + \gamma_i) - l_1 \sin \theta_i). \)

\[
J_A = \begin{pmatrix}
\frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial \varphi} \\
\frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial \varphi} \\
\frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial \varphi}
\end{pmatrix},
\]

where \( \frac{\partial F_i}{\partial x} = 2x - 2x_{Al} + 2l_{3,i} \cos(\varphi + \gamma_i) - 2l_1 \cos \theta_i, \frac{\partial F_i}{\partial y} = 2y - 2y_{Al} + 2l_{3,i} \sin(\varphi + \gamma_i) - 2l_1 \sin \theta_i, \frac{\partial F_i}{\partial \varphi} = 2l_{3,i}(y - y_{Al} + l_{3,i} \sin(\varphi + \gamma_i) - l_{1,i} \sin \theta_i) \cos(\varphi + \gamma_i) - 2l_{3,i}(x - x_{Al} + l_{3,i} \cos(\varphi + \gamma_i) - l_{1,i} \cos \theta_i) \sin(\varphi + \gamma_i). \)

(2019) 012057 doi:10.1088/1742-6596/1353/1/012057
Let us define the working area of the mechanism taking into account singularities of the first type.
The algorithm includes two parts: definition of the working area and definition of special positions for
eight combinations of angles \( \theta_{i1} \) and \( \theta_{i2} \).

To define the working area it is necessary to approximate the system of inequalities (3). Let us write
the system of inequalities in general terms:

\[
\begin{align*}
&g_1(x) \leq 0, \\
&\ldots \\
&g_m(x) \leq 0 \\
&a_i \leq x_i \leq b_i, i = 1, \ldots, n.
\end{align*}
\]

(4)

Initial box \( Q \), which contains all solutions \( X \), is defined by interval restrictions \( a_i \leq x_i \leq b_i, i = 1, \ldots, n \). Let us consider any box \( B \). Let \( m_1(B) = \max_{i=1,m} \min_{x \in B} g_i(x) \) and \( M_1(B) = \max_{i=1,m} \max_{x \in B} g_i(x) \). If \( m_0(B) > 0 \), then \( B \) does not contain solutions of the system (3). If \( M_0(B) \leq 0 \), then each point of box \( B \) represents an admissible solution. Hence, it can be added to a covering as an internal box. In other
cases it is split into two smaller boxes if its diameter is bigger than the set approximation accuracy \( \delta \).

For approximation of the equation \( \det(J) = 0 \), the second part of an algorithm is used, which
unlike the first one is repeated eight times for all combinations of angles \( \theta_{i1} \) and \( \theta_{i2} \). Thus, \( m_2(B) = \max_{i=1,m} \min_{x \in B} \det(J) \) and \( M_2(B) = \min_{i=1,m} \max_{x \in B} \det(J) \).

If \( m_2(B) > 0 \) or \( M_2(B) < 0 \), then \( B \) does not contain solutions to the equation \( \det(J) = 0 \). In
other cases it is split into two smaller boxes if its diameter is bigger than the set approximation accuracy \( \delta \). The solution will be the set of remaining boxes with the diameter less than the set approximation
accuracy.

The algorithm functions as follows:
1. On the first step of an algorithm the list of internal approximation \( \mathbb{P} \) is empty, list \( \mathbb{P} \) consists only
of one box \( Q \), which reliably includes the range of movement of point \( C \) \( \{x_{A1} - 2l_1 - r, x_{A2} + 2l_1 + \}
\)r \) along axis \( x \), \( \{y_{A1} - 2l_1 - r, y_{A3} + 2l_1 + r\} \) along axis \( y \) and angle \( \varphi \) of the rotation platform
\( [-\frac{\pi}{2}, \frac{\pi}{2}] \).
2. Let us take box \( B \) from list \( \mathbb{P} \).
3. Let us define \( m_1(B) \) and \( M_1(B) \).
4. If \( m_1(B) > 0 \), then \( B \) is excluded.
5. If \( M_1(B) \leq 0 \) or \( d(B) < \delta \), then \( B \) is added to list \( \mathbb{P} \).
6. In other cases \( B \) is divided into two equal boxes along an edge with the biggest length. These boxes
are included into the end of list \( \mathbb{P} \).
7. If list \( \mathbb{P} \) becomes empty, there is a transition to step 8, in other cases steps 2-7 are repeated.
8. Box \( Q \), which reliably includes the range of movement of point \( C \) \( \{x_{A1} - 2l_1 - r, x_{A2} + 2l_1 + \}
\)r \) along axis \( x \), \( \{y_{A1} - 2l_1 - r, y_{A3} + 2l_1 + r\} \) along axis \( y \) and angle \( \varphi \) of the rotation platform
\( [-\frac{\pi}{2}, \frac{\pi}{2}] \) is included into list \( \mathbb{P} \).
9. Let us take box \( B \) from list \( \mathbb{P} \).
10. Let us define \( m_2(B) \) and \( M_2(B) \).
11. If \( m_2(B) > 0 \) or \( M_2(B) < 0 \), then \( B \) is excluded.
12. If \( d(B) < \delta \), then \( B \) is added to list \( \mathbb{P} \).
13. In other cases \( B \) is split into two equal boxes along an edge with the biggest length. These boxes
are included into the end of list \( \mathbb{P} \).
14. If list \( \mathbb{P} \) becomes empty, there is a transition to step 15, in other cases steps 9-14 shall be repeated.
15. Steps 8-15 are repeated for eight combinations of angles \( \theta_{i1} \) and \( \theta_{i2} \).

The algorithm to define the working area taking into account singularities of the second type, for
which \( \det(J) = 0 \) is similar to the above algorithm, however \( m_2(B) = \max_{i=1,m} \min_{x \in B} d(J) \) and
\( M_2(B) = \min_{i=1,m} \max_{x \in B} d(J) \).
To define the working space and special positions of a 3-RRR mechanism let us set its geometrical parameters considering that $\gamma_1 = 7\pi/6$, $\gamma_2 = 11\pi/6$, $\gamma_3 = \pi/2$, $l_{2,j} = l_{1,j}$: $R = 500$ mm, $r = 100$ mm, $l_1 = 350$ mm. Figures 2 and 3 show the results of modeling for eight combinations of angles $\theta_{i,1}$ and $\theta_{i,2}$. Since there are eight potential solutions of the inverse problem of kinematics of a 3-RRR mechanism, each solution applies different configurations of the manipulator $\Delta_j$ ($j = 1, 2, ..., 8$), namely, $\Delta_1 = [+,-,+]$, $\Delta_2 = [-,+,-]$, $\Delta_3 = [+,-,+]$, $\Delta_4 = [+,-,+]$, $\Delta_5 = [+,-,+]$, $\Delta_6 = [-,+,-]$, $\Delta_7 = [-,+,-]$, $\Delta_8 = [-,+,-]$. The time to calculate the approximation accuracy $\delta = 4$ mm and dimensions of a grid for functions 128x128 on a personal computer depending on the type of singularity and the rotation angle made from 37 to 49 seconds.

Figure 2. Results of modeling for singularities of the first type: a) $\varphi = -20^\circ$, b) $\varphi = 0^\circ$, c) $\varphi = 20^\circ$, d) $\varphi = 40^\circ$. 
Figure 3. Results of modeling for singularities of the second type: a) $\varphi = -20^\circ$, b) $\varphi = 0^\circ$, c) $\varphi = 20^\circ$, d) $\varphi = 40^\circ$.

3. Conclusion
The designed algorithms proved their efficiency. The system of inequalities considering the acceptable range of an argument of driving angles arcsine was used to define the working area. The condition of equality of Jacobian matrixes to zero for the first and second type of singularities was used to define special positions. The received results can be used to choose the geometrical parameters of a 3-RRR mechanism setting the boundaries of the working area through the technological process and to plan the movement trajectory of an output link to avoid uncontrollable mobility or jamming of the mechanism in special positions.

4. Acknowledgments
The study is performed under the financial support of the Russian Foundation for Basic Research within the grant No. 18-57-45014 IND _a of the Committee of Science of the Ministry of Education and Science of the Republic of Kazakhstan No. AP05133190.

References
[1] Gosselin C M and Angeles J 1990 IEEE Transactions on Robotics and Automation 6(3) 281–290
[2] Merlet J-P 2007 Springer Handbook of Robotics eds B Siciliano and O Khatib (Berlin, Heidelberg: Springer) pp 269-285
[3] Aleshin A K, Glazunov V A, Rashoyan G V and Shai O 2016 J. of Machinery Manufacture and Reliability 45(4) 291-296
[4] Evtushenko Y 1971 Computational Mathematics and Mathematical Physics [in Russian – Vychislitelnaja matematika i matematicheskaja fizika] 11(6) 1390–1403
[5] Evtushenko Y, Posypkin M, Rybak L and Turkin A 2018 J. Global Optimization 7 129-145
[6] Evtushenko Y and Posypkin M 2013 Computational Mathematics and Mathematical Physics [in Russian – Vychislitelnaja matematika i matematicheskaja fizika] 53(2) 144-157
[7] Malyshev D, Posypkin M, Rybak L and Usov A 2018 Int. J. of Open Information Technologies 6(7) 15-20
[8] Posypkin M 2019 J. Phys.: Conf. Ser. 1163 012050
[9] Virabyan L G, Khalapyan S Y and Kuzmina V S 2018 Bulletin of BSTU named after V.G. Shukhov 9 106–113
[10] Laryushkin P A and Epanchintseva D S 2015 Engineering Bulletin 9 12-21