WEINGARTEN FLOWS IN RIEMANNIAN MANIFOLDS

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Abstract. Given orientable Riemannian manifolds \( M^n \) and \( \mathbb{M}^{n+1} \), we study flows \( F_t : M^n \to \mathbb{M}^{n+1} \), called Weingarten flows, in which the hypersurfaces \( F_t(M) \) evolve in the direction of their normal vectors with speed given by a function \( W \) of their principal curvatures, called a Weingarten function, which is homogeneous, monotonic increasing with respect to any of its variables, and positive on the positive cone. We obtain existence results for flows with isoparametric initial data, in which the hypersurfaces \( F_t(M) \) are all parallel, and \( \mathbb{M}^{n+1} \) is either a simply connected space form or a rank-one symmetric space of noncompact type. We prove that the avoidance principle holds for Weingarten flows defined by odd Weingarten functions, and also that such flows are embedding preserving.

1. Introduction

Given an open set \( \Gamma \subset \mathbb{R}^n \) containing \( \Gamma_+ := \{(k_1, \ldots , k_n) : k_i > 0\} \), we say that \( W = W(k_1, \ldots , k_n) \in C^\infty(\Gamma) \) is a Weingarten function if it is symmetric, homogeneous, monotonic increasing with respect to any of its variables, and positive on \( \Gamma_+ \). For a hypersurface \( f : M^n \to \mathbb{M}^{n+1} \) (\( M^n \) and \( \mathbb{M}^{n+1} \) are arbitrary orientable Riemannian manifolds), denote by \( k_1, \ldots , k_n \) its principal curvature functions. Assuming that \( (k_1(p), \ldots , k_n(p)) \in \Gamma \) for all \( p \in M \), we define the Weingarten function \( W_f \) of \( f \) associated to \( W \) as

\[
W_f(p) := W(k_1(p), \ldots , k_n(p)), \quad p \in M.
\]

If \( W_f \) is constant on \( M \), we say that \( f \) is a \( W \)-hypersurface.

The higher order mean curvatures \( H_r \), \( 1 \leq r \leq n \), and the squared norm of the second fundamental form \( \|A\|^2 \) are distinguished examples of Weingarten functions. They are defined as

\[
H_r = \sum_{i_1 < \cdots < i_r} k_{i_1} \cdots k_{i_r}, \quad \text{and} \quad \|A\|^2 = \sum_{i=1}^n k_i^2.
\]

In this paper, we shall consider the problem of finding a one-parameter family of smooth oriented immersions \( F(\cdot , t) : M^n \to \mathbb{M}^{n+1} \), \( t \in [0, T) \), which, for a given Weingarten function \( W \in C^\infty(\Gamma) \), satisfy the evolution equation:

\[
\begin{align*}
\frac{\partial F}{\partial t}(p, t) &= W(p, t)N(p, t) \\
F(p, 0) &= f(p),
\end{align*}
\]
where $N(p,t)$ is the unit normal to the hypersurface $F_t := F(.,t)$, and $W(.,t) = W_{F_t}$ is the Weingarten function of $F_t$ associated to $W$. We shall call such a family of immersions a Weingarten flow (or a $W$-flow, in order to specify the function $W$) in $\mathbb{M}^{n+1}$ with initial data $f$.

Huisken and Polden [8] have established existence of short time solutions to (1). Here, we shall seek solutions such that the immersions $F_t : M^n \to \mathbb{M}^{n+1}$ are all parallel to the initial data $f$, that is,

$$F_t(p) = \exp_{f(p)}(\varphi(t)N(p)), \quad (p,t) \in M \times [0,T),$$

where $\exp$ stands for the exponential map of $\mathbb{M}^{n+1}$, $\varphi \in C^\infty[0,T)$ satisfies $\varphi(0) = 0$, and $N$ is the unit normal to $f$. We call $F_t$ a parallel $\varphi$-flow, and choose

$$N(p,t) = d\exp_{f(p)}(\varphi(t)N(p))N(p)$$

as the unit normal field of $F_t$.

As we shall see, if a parallel $\varphi$-flow is a solution to (1), then each $F_t : M^n \to \mathbb{M}^{n+1}$ is necessarily a $W$-hypersurface. This fact leads us to consider isoparametric hypersurfaces of $\mathbb{M}^{n+1}$, i.e., those on which the principal curvatures are constant functions. In this context, the simply connected space forms $Q^+_n$ of constant sectional curvature $\epsilon \in \{0, 1, -1\}$, as well as the rank-one symmetric spaces of noncompact type (i.e., the hyperbolic spaces $\mathbb{H}^n_\epsilon$), are natural sources of parallel $W$-flows, since these spaces have many families of isoparametric hypersurfaces (see Section 2.1 for details).

Our first main result, as stated below, concerns parallel Weingarten flows in space forms of non positive curvature.

**Theorem 1.** For $\epsilon \in \{0, -1\}$, let $f : M^n \to Q^+_n$ be a complete non totally geodesic isoparametric hypersurface of $Q^+_n$, and let $W \in C^\infty(\Gamma)$ be a Weingarten function. Then, there exists a parallel $\varphi$-flow $F_t$ defined on a maximal interval $[0,T)$, $T < +\infty$, which is a solution to (1), and has the following properties, according to the isoparametric type of $f(M)$:

i) If $f(M) \subset \mathbb{H}^{n+1}_\epsilon$ is a horosphere, then $T = +\infty$ and $\{F_t(M), t \in (0, +\infty)\}$ is a family of horospheres in $\mathbb{H}^{n+1}_\epsilon$ which foliates the open horoball bounded by $f(M)$.

ii) If $f(M) \subset \mathbb{H}^{n+1}_\epsilon$ is an equidistant hypersurface to a totally geodesic hyperplane $\Pi \subset \mathbb{H}^{n+1}_\epsilon$, then $T = +\infty$ and $F_t(M) \to \Pi$ as $t \to +\infty$.

iii) If $f(M) \subset \mathbb{H}^{n+1}_\epsilon$ is either a geodesic sphere or a generalized cylinder, then $T < +\infty$, $\varphi(T)$ is the focal distance of $f(M)$, and $F_t(M)$ collapses into the focal set of $f(M)$ at $t = T$.

We also consider $W$-flows in $\mathbb{S}^{n+1}$ from its isoparametric hypersurfaces, obtaining then the following result (see Section 2.1 for definitions).

**Theorem 2.** Let $\mathcal{F} = \{f_\tau : M \to \mathbb{S}^{n+1}; \tau \in (0, \pi/\epsilon)\}$ be a family of positively oriented isoparametric hypersurfaces of $\mathbb{S}^{n+1}$, and let $W \in C^\infty(\Gamma)$ be a Weingarten function such that $W_{f_\tau}$ is well defined for all $f_\tau \in \mathcal{F}$. Given $\tau_0 \in (0, \pi/\epsilon)$, assume that $W_{f_{\tau_0}} > 0$ (resp. $W_{f_{\tau_0}} < 0$), and that the function $\tau \mapsto W_{f_{\tau_0+\epsilon}}$ (resp. $\tau \mapsto W_{f_{\tau_0-\epsilon}}$) is increasing (resp. decreasing) on $[0, \tau_0]$ (resp. on $[0, \pi/\epsilon - \tau_0]$). Under these conditions, the maximal parallel $\varphi$-flow solution $F_t = f_{\tau_0 - \varphi(t)}$ to (1) with initial data $F_0 = f_{\tau_0}$ collapses into the focal set $\mathcal{F}_+$ (resp. $\mathcal{F}_-$) at $t = \varphi^{-1}(\tau_0)$ (resp. $t = \varphi^{-1}(\tau_0 - \pi/\epsilon)$).
It should be mentioned that Theorems 1 and 2 constitute extensions of the main results of [5], where the authors considered mean curvature flows by parallel hypersurfaces in $\mathbb{Q}_{n+1}$.

By considering isoparametric hypersurfaces of the hyperbolic spaces $\mathbb{H}^n_m$, we obtain the following result.

**Theorem 3.** Let $f : M^n \to \mathbb{H}^n_{m_1}$ be either a horosphere or a geodesic sphere of $\mathbb{H}^n_{m_1}$, and let $W \in C^\infty(\Gamma)$ be a Weingarten function. Then, there exists a parallel $\varphi$-flow $F_t$ defined on a maximal interval $[0,T)$, $T < +\infty$, which is a solution to (1). In addition, the following hold:

i) If $f(M)$ is a horosphere, then $T = +\infty$ and $\{F_t(M), t \in (0, +\infty)\}$ is a family of horospheres in $\mathbb{H}^n_{m_1}$ which foliates the open horoball bounded by $f(M)$.

ii) If $f(M)$ is a geodesic sphere, then $T < +\infty$, $\varphi(T)$ is the radius of $f(M)$, and $F_t(M)$ collapses into the center of $f(M)$ at $t = T$.

An important property shared by many kinds of flows in Euclidean space is the avoidance principle, which essentially says that two flows with disjoint initial data remain disjoint until one of them collapses. Here, by means of a result by R. Hamilton, we establish an avoidance principle for $W$-flows $F_t : M^n \to \overline{M}^{n+1}$ whose Weingarten function $W \in C^\infty(\Gamma)$ is odd. Setting $k := (k_1, \ldots, k_n)$, this means that $W$ admits an extension to $-\Gamma := \{-k ; k \in \Gamma\}$ which satisfies $W(-k) = -W(k)$. As one can easily check, for $r$ odd, the mean curvatures $H_r$ are all odd functions.

**Theorem 4** (avoidance principle). Let $M^n_1$, $M^n_2$, and $\overline{M}^{n+1}$ be complete connected Riemannian manifolds, being $M^n_2$ compact. Assume that $W \in C^\infty(\Gamma)$ is an odd Weingarten function, and that

$$F^i : M^n_i \times [0, T) \to \Omega \subset \overline{M}^{n+1}, \ i = 1, 2,$$

are $W$-flows, where $\Omega$ is a strongly convex open set of $\overline{M}^{n+1}$. Under these conditions, we have that the function

$$D(t) := \text{dist}^2(F^1_t(M_1), F^2_t(M_2)), \ t \in [0, T),$$

is no decreasing. In particular, if $F^1_t(M_1)$ and $F^2_t(M_2)$ are disjoint, then $F^1_t(M_1)$ and $F^2_t(M_2)$ are disjoint for all $t \in [0, T)$.

As a consequence of the avoidance principle, if $\overline{M}^{n+1}$ is either a space form $\mathbb{Q}_{n+1}$ or a hyperbolic space $\mathbb{H}^n_m$, then a $W$-flow $F_t : M^n \to \overline{M}^{n+1}$ of a compact manifold $M$ collapses in a finite time $T$, provided that $W$ is odd, or $F_t$ is an embedding for all $t \in [0, T)$ (see Corollary 5 in Section 3).

In our final result, we show that Weingarten flows defined by odd Weingarten functions preserve embeddedness.

**Theorem 5.** Let $\overline{M}^{n+1}$ be a complete connected Riemannian manifold. Assume that $W \in C^\infty(\Gamma)$ is an odd Weingarten function, and that

$$F : M^n \times [0, T) \to \overline{M}^{n+1}$$

is a $W$-flow of a compact connected Riemannian manifold $M$. Under these conditions, if the initial data $F_0$ is an embedding, then $F_t$ is an embedding for all $t \in [0, T)$. 
The paper is organized as follows. In Section 2, we establish general facts on W-flows by parallel hypersurfaces, and present the proofs of Theorems 1–3. We also apply these results to determine the collapsing time of some W-flows in $\mathbb{Q}^{n+1}$ and $\mathbb{H}^n_\theta$ as well. In Section 3, we provide the proofs of Theorems 4 and 5.

2. W-Flows by Parallel Hypersurfaces

The following result gives us a way of obtaining Weingarten flows by parallel hypersurfaces. An interesting property of such a flow is that its hypersurfaces are all W-hypersurfaces.

**Proposition 1.** Given a Weingarten function $W \in C^\infty(\Gamma)$, let $F_t$ be a parallel $\varphi$-flow as in (2), and assume that $W_{F_t}$ is well defined for all $t \in [0, T)$. Then, $F_t$ is a solution to (1) with initial data $f = F_0$ if and only if the function $\varphi$ satisfies

$$\varphi'(t) = W(p, t) \quad \forall (p, t) \in M \times [0, T).$$

If so, $F_t : M^n \to \overline{M}^{n+1}$ is a W-hypersurface for all $t \in [0, T)$.

**Proof.** From (2), we have that

$$\frac{\partial F}{\partial t}(p, t) = d \exp_p(\varphi(t)N(p))\varphi'(t)N(p) = \varphi'(t)N(p, t).$$

This, together with (3), gives that $F_t = F(\cdot, t)$ satisfies (1) if and only if $\varphi$ satisfies (4). In particular, if this equality holds, the Weingarten function $W_{F_t}$ is constant on $M$ (possibly depending on $t$), that is, $F_t$ is a W-hypersurface of $\overline{M}$. □

As an immediate consequence of Proposition 1, we have:

**Corollary 1.** Given a Weingarten function $W \in C^\infty(\Gamma)$, let us suppose that

$$\mathcal{F} := \{f_\tau : M^n \to \overline{M}^{n+1} ; \tau \in (-\delta, \delta)\}$$

is a family of parallel W-hypersurfaces of $\overline{M}$ defined by $f_\tau(p) = \exp_p(\tau N(p))$, where $N$ is the unit normal to $f = f_0$. Then, writing $W(\tau) = W_{f_\tau}$, we have that the solution $\tau = \varphi(t)$ of the initial value problem

$$\begin{cases}
\tau' = W(\tau) \\
\tau(0) = 0
\end{cases}$$

determines a parallel $\varphi$-flow solution to (1).

As we pointed out in the introduction, the fact that hypersurfaces of parallel W-flows are W-hypersurfaces suggests the consideration of isoparametric hypersurfaces. Recall that a one-parameter family $f_\tau : M^n \to \overline{M}^{n+1}$ of parallel hypersurfaces is called isoparametric if, for each $\tau$, any principal curvature function $k_i$ of $f_\tau$ is constant on $M$ (possibly depending on $i$ and $\tau$). In this case, each hypersurface $f_\tau$ is also called isoparametric.

Given a Weingarten function $W \in C^\infty(\Gamma)$, it is clear that any isoparametric hypersurface $f_\tau : M^n \to \overline{M}^{n+1}$ is a W-hypersurface, provided that $W_f$ is well defined. Therefore, in view of Corollary 1, we have the following result.

**Corollary 2.** Suppose that $f : M^n \to \overline{M}^{n+1}$ is an isoparametric hypersurface. Then, for any Weingarten function $W \in C^\infty(\Gamma)$ for which $W_f$ is well defined, there exists a unique solution to (1) by parallel hypersurfaces with initial data $f$. 
2.1. **Parallel W-flows in space forms.** Let us apply the results so far obtained to study $W$-flows in the simply connected space forms $Q^{n+1}_\varepsilon$. In view of Corollary 2, we shall consider the isoparametric hypersurfaces of these spaces. (For details and proofs on this subject we refer to [3, 4].)

For $\varepsilon \leq 0$, the complete isoparametric hypersurfaces of $Q^{n+1}_\varepsilon$ are totally classified. They are:

i) The totally geodesic hyperplanes $Q^n_\varepsilon \subset Q^{n+1}_\varepsilon$.

ii) The geodesic spheres.

iii) The generalized cylinders $Q^{n-k}_\varepsilon \times S^k$, where $Q^{n-k}_\varepsilon$ is a totally geodesic hypersurface of dimension $n-k < n$, and $S^k$ is the $k$-dimensional geodesic sphere of $Q^{n+1}_\varepsilon$.

iv) The horospheres of $H^{n+1}$.

v) The equidistant hypersurfaces to totally geodesic hyperplanes of $H^{n+1}$.

In fact, for $\varepsilon \leq 0$, any isoparametric hypersurface of $Q^{n+1}_\varepsilon$ is necessarily an open set of one of the complete hypersurfaces listed above.

We point out that, in the cases (ii) and (iii), the isoparametric hypersurfaces have focal points. More specifically, any geodesic sphere has a unique focal point, which is its center, and the focal set of a generalized cylinder $Q^{n-k}_\varepsilon \times S^k$ is the totally geodesic submanifold $Q^{n-k}_\varepsilon$. In such cases, we shall take the focal distance as the parameter for a family of isoparametric hypersurfaces, that is, if $F = \{f_\tau : M^n \to Q^{n+1}_{\varepsilon} ; \tau \in I \subset \mathbb{R}\}$ is such a family, then $\tau$ is the distance from $f_\tau(M)$ to its focal set. For instance, if $F$ is a family of concentric geodesic spheres, then $M = S^n$, $I = [0, +\infty)$, and $\tau > 0$ is the radius of $f_\tau(S^n)$.

We also observe that all of the above isoparametric hypersurfaces are connected, orientable, properly embedded and, except for case (i), strictly convex. We shall consider on them the orientation which makes all their principal curvatures positive.

**Proof of Theorem 1.** Firstly, let us write

$$F = \{f_\tau : M^n \to Q^{n+1}_{\varepsilon} ; \tau \in I \subset \mathbb{R}\}$$

for the isoparametric family of complete hypersurfaces of $Q^{n+1}_{\varepsilon}$ (defined in a maximal interval $I \subset \mathbb{R}$) such that $f = f_{\tau_0}$, $\tau_0 \in I$. From the convexity of the hypersurfaces $f_\tau$, and from the positivity of $W$ on $\Gamma_+$, we have that $W_{f_{\tau}} \geq 0$ for all $\tau \in I$.

If $F$ is a family of horospheres, then $I = \mathbb{R}$ and, for any $\tau \in \mathbb{R}$, all principal curvatures of $f_\tau$ are equal to 1, which implies that $W_{f_{\tau}} = W$ is a positive constant independent of $\tau$. Hence, by Corollary 1, the function $\varphi(t) = Wt$, $t \in [0, +\infty)$, determines a $\varphi$-flow $F_t$ which is a solution to (1) with initial data $f_{\tau_0}$. Namely,

$$F_t = f_{Wt+\tau_0}, \; t \in [0, +\infty).$$

Clearly, for all $t > 0$, $F_t(M)$ is a horosphere of $\mathbb{H}^{n+1}$ contained in the open horoball bounded by $f(M)$. This proves (i).

Assume now that $F$ is a family of equidistant hypersurfaces to a totally geodesic hyperplane $\Pi \subset \mathbb{H}^{n+1}$. In this case, $I = \mathbb{R}$ and the parameter $\tau > 0$ is the distance
from \(f_\epsilon(M)\) to \(\Pi\). We can assume, without loss of generality, that \(\tau_0 > 0\). Let \(\varphi : [0, T) \to \mathbb{R}\) be the solution of (6) defined in a maximal interval \([0, T)\). Then,

\[
F_t = f_{\tau_0 - \varphi(t)}, \quad t \in [0, T),
\]

is a solution to (1) satisfying \(F_0 = f_{\tau_0}\). Assume, by contradiction, that \(T < +\infty\). If \(\varphi(T) = \tau_0\), then the flow \(F_t\) can be extended beyond \(T\) just by setting \(F_t = f_0\) for \(t \geq T\) (since, by the homogeneity of \(W\), \(W(0, 0, \ldots, 0) = 0\), contradicting the maximality of \(T\). Analogously, if \(\varphi(T) < \tau_0\), we have that \(F_T = f_{\tau_0 - \varphi(T)}\) is well defined, so that we can extend the flow \(F_t\) beyond \(T\) — again a contradiction. Therefore, \(T = +\infty\).

If \(\varphi(t_0) = \tau_0\) for some \(t_0 \in (0, +\infty)\), then \(F_t(M) = \Pi\) for all \(t \geq t_0\). Hence, we can assume that \(\varphi\) is bounded above by \(\tau_0\). In this case, since \(\phi_t(M)\) moves towards \(\Pi\), the principal curvatures of \(F_t\) are positive decreasing functions of \(t\). This, together with the monotonicity property of \(W\), gives that the function \(W(\varphi(t)) = W_{\tau_0 - \varphi(t)}\) decreases as \(t \to +\infty\). Since \(\varphi'(t) = W(\varphi(t))\), we conclude that \(\varphi''(t) < 0\), that is, \(\varphi\) is positive, increasing, concave and bounded on \([0, +\infty)\). These properties clearly imply that \(\varphi'(t) \to 0\) as \(t \to +\infty\). Therefore,

\[
W_{\tau_0} = 0 = \lim_{t \to +\infty} \varphi'(t) = \lim_{t \to +\infty} W(\varphi(t)) = \lim_{t \to +\infty} W_{\tau_0 - \varphi(t)},
\]

which yields \(\lim_{t \to +\infty} \varphi(t) = \tau_0\). Consequently, \(F_t \to f_0\) as \(t \to +\infty\), which shows assertion (ii).

Finally, let us suppose that \(\mathcal{F}\) is a family of concentric geodesic spheres of \(\mathbb{S}^{n+1}\) (the argument for generalized cylinders is analogous). In this setting, let \(\varphi : [0, T) \to \mathbb{R}\) be the solution of (6), so that \(F_t = f_{\tau_0 - \varphi(t)}\) is the solution to (1) satisfying \(F_0 = f_{\tau_0}\). Since \(\phi_t(M)\) flows towards the center of the spheres \(f_t(M)\), we have that \(\varphi(t) < \tau_0\) for all \(t \in [0, T)\), and also that \(W(\varphi(t)) = \varphi'(t)\) is a positive increasing function of \(t\). Thus, \(\varphi'' > 0\), that is, \(\varphi\) is bounded, increasing and strictly convex, which clearly implies that \(T < +\infty\). Besides, we must have \(\varphi(T) = \tau_0\). Otherwise, arguing as in the preceding paragraph, we derive a contradiction by extending \(\varphi\) beyond \(T\). This completes the proof of (iii), and so of the theorem. \(\square\)

Let us consider now the isoparametric hypersurfaces of \(\mathbb{S}^{n+1}\). A well known result asserts that any such hypersurface has at most \(g\) distinct principal curvatures, where \(g \in \{1, 2, 3, 4, 6\}\). The case \(g = 1\), for instance, correspond to the geodesic spheres of \(\mathbb{S}^{n+1}\). Rather than using the classification theorems for isoparametric hypersurfaces of \(\mathbb{S}^{n+1}\), we shall consider their characterization as level sets of homogeneous polynomials, as done by Münzner [11] (see also [3]).

To be more clear, let \(f : M \to \mathbb{S}^{n+1}\) be an isoparametric hypersurface with \(g\) distinct principal curvatures. Münzner’s result asserts that \(f(M)\) is the intersection of \(\mathbb{S}^{n+1}\) with a level set \(P^{-1}(c), c \in (-1, 1)\), of a homogeneous polynomial function \(P : \mathbb{R}^{n+2} \to \mathbb{R}\) of degree \(g\). Distinct level sets of \(P\) are necessarily parallel in \(\mathbb{S}^{n+1}\), and the focal set of this parallel family has precisely two connected components, which are the intersections of \(\mathbb{S}^{n+1}\) with \(P^{-1}(-1)\) and \(P^{-1}(1)\), respectively. In addition, given \(p \in M\), if we write \(\gamma : (0, \pi/g) \to \mathbb{S}^{n+1}\) for the normalized geodesic from \(P^{-1}(1)\) to \(P^{-1}(-1)\) which is orthogonal to \(f\) at \(p = \gamma(\tau)\), and set the positive orientation \(N(p) = -\gamma'(\tau)\) for \(f\), its \(g\) distinct principal curvatures are given by

\[
k_i = \cot \left( \frac{\tau + (i - 1)\pi}{g} \right), \quad 1 \leq i \leq g.
\]
In this setting, \( \tau \in (0, \pi/g) \) is the focal distance from \( f(M) \) to \( P^{-1}(1) \). Also, all principal curvatures of \( f \) increase as \( \tau \) decreases to 0, and decrease as \( \tau \) increases to \( \pi/g \). We shall denote the multiplicity of \( k_i \) by \( m_i \).

Summarizing, we have that any isoparametric hypersurface of \( S^{n+1} \) with \( g \) distinct principal curvatures is an element of a family

\[
\mathcal{F} = \{ f_\tau : M \to S^{n+1} ; \tau \in (0, \pi/g) \}
\]
of isoparametric hypersurfaces such that \( f_\tau(M) \) is at a distance \( \tau \) from the focal component \( \mathcal{F}_+ := P^{-1}(1) \).

Proof of Theorem 2. Suppose that, for some \( \tau_0 \in (0, \pi/g) \), \( W_{f_\tau_0} > 0 \). In this case, if the function \( \tau \mapsto W_{f_{\tau_0-\tau}} \) is increasing on \([0, \tau_0)\), the \( \varphi \)-flow

\[
F_t = f_{\tau_0-\varphi(t)}, \quad \varphi(0) = 0, \quad \varphi'(t) = W_{f_{\tau_0-\varphi(t)}},
\]

moves towards \( \mathcal{F}_+ \) with increasing velocity. Hence, arguing as in the proof of Theorem 1-(iii), we conclude that \( F_t(M) \) collapses into \( \mathcal{F}_+ \) at \( t = \varphi^{-1}(\tau_0) \).

Analogously, if \( W_{f_{\tau_0}} < 0 \), and the function \( \tau \mapsto W_{f_{\tau_0+\tau}} \) is decreasing on the interval \([0, \pi/g - \tau_0)\), then \( F_t(M) \) collapses into the focal component \( \mathcal{F}_- := P^{-1}(-1) \) at \( t = \varphi^{-1}(\tau_0 - \pi/g) \).

Let us see now that Theorem 2 applies when \( W \) is either the higher order mean curvature \( H_r \) or the squared norm of the second fundamental form \( ||A||^2 \).

Let \( \mathcal{F} \) be as in Theorem 2. Then, in any open interval \((0, \tau_0)\), \( 0 < \tau_0 < \pi/g \), we have that \( k_1^r = \cot \tau \) is unbounded, whereas \( k_i^r = \cot(\tau + (i-1)\pi/g), \quad i = 2, \ldots, g \), is bounded. Assuming that the multiplicity \( m_i \) of \( k_i^r \) (which is the same for all \( \tau \)) satisfies \( m_i \geq r \), where \( r \in \{1, \ldots, n-1\} \), the \( r \)-th mean curvature \( H_r(\tau) \) of \( f_\tau \) is given by

\[
H_r(\tau) = \left( \frac{m_1}{r} \right) \cot^r \tau + \sum_{i=0}^{r-1} \mu_i(\tau) \cot^i \tau,
\]

where the functions \( \mu_i \) are all bounded in \((0, \tau_0)\). In particular, if \( \tau_0 \) is sufficiently small, \( H_r(\tau_0) > 0 \), and the function \( \tau \in [0, \tau_0) \mapsto H_r(\tau_0 - \tau) \) is increasing. In the same manner, if \( r \) is odd, \( m_g \geq r \), and \( \tau_0 \) is sufficiently close to \( \pi/g \), then \( H_r(\tau_0) < 0 \), and the function \( \tau \in [0, \pi/g - \tau_0) \mapsto H_r(\tau_0 + \tau) \) is decreasing. Thus, we have the following

**Corollary 3.** Theorem 2 applies to the Weingarten function \( W = H_r, 1 \leq r \leq n-1 \). More precisely, given \( \tau_0 \in (0, \pi/g) \), if \( m_1 \geq r \) (resp. \( m_g \geq r \) if \( r \) odd), and \( H_r(\tau_0) > 0 \) (resp. \( H_r(\tau_0) < 0 \)), the maximal parallel \( \varphi \)-flow solution \( F_t = f_{\tau_0-\varphi(t)} \) to \( H_r \)-flow with initial data \( F_0 = f_{\tau_0} \) collapses into the focal set \( \mathcal{F}_+ \) (resp. \( \mathcal{F}_- \)) at \( t = \varphi^{-1}(\tau_0) \) (resp. \( t = \varphi^{-1}(\tau_0 - \pi/g) \)).

Theorem 2 also applies to the norm of the second fundamental form \( ||A||^2 \), since \( ||A||^2(\tau_0 - \tau) \) is clearly increasing on \([0, \tau_0)\) for all sufficiently small \( \tau_0 \in (0, \pi/g) \).

**Corollary 4.** Let \( \mathcal{F} \) be as in Theorem 2. Given a sufficiently small \( \tau_0 \in (0, \pi/g) \), the maximal parallel \( \varphi \)-flow solution \( F_t = f_{\tau_0-\varphi(t)} \) to \( ||A||^2 \)-flow with initial data \( F_0 = f_{\tau_0} \) collapses into the focal set \( \mathcal{F}_+ \) at \( t = \varphi^{-1}(\tau_0) \).

Next, we apply the results of this section to determine the collapsing time of some parallel \( W \)-flows in \( \mathbb{Q}^{n+1} \). For that, we shall consider the trigonometric functions \( \cos \) and \( \sin \), as defined in Table 1. The functions \( \tan, \cot, \) and \( \sec \) are defined accordingly, that is, \( \tan_e = \sin_e/\cos_e, \cot_e = \cos_e/\sin_e, \) and \( \sec_e = 1/\cos_e \).
Example 1 (parallel $\|A\|^2$-flow in $Q^n_{\epsilon}$ with spherical initial data). Let

$$f : S^n \to Q^n_{\epsilon}$$

be a (totally umbilical) strictly convex geodesic sphere of radius $R > 0$ and principal curvature $k = \cot \epsilon R$. Set

$$R_{\epsilon} := \begin{cases} \pi/2 & \text{if } \epsilon = 1 \\ +\infty & \text{if } \epsilon \neq 1 \end{cases}$$

and let

$$\mathcal{F} = \{ f_\tau : S^n \to Q^n_{\epsilon} ; \tau \in (0, R_{\epsilon}) \}$$

be the family of parallel geodesic spheres of $Q^n_{\epsilon}$ such that $f_{R_{\epsilon}} = f$. By Theorems 1 and 2, for $W = \|A\|^2$, the flow

$$F_t := f_{R - \varphi(t)} \quad t \in [0, T),$$

where $\varphi$ satisfies

$$\varphi'(t) = W f_{R - \varphi(t)} = n \cot^2 \epsilon (R - \varphi(t)), \quad \varphi(0) = 0,$$

is a solution to (1) which collapses into the center of $f(S^n)$ at time $T = \varphi^{-1}(R)$. Separating variables in (8), we obtain the equation

$$\tan^2 \epsilon (R - \varphi) d\varphi = n dt,$$

which yields

$$(R - \varphi(t))^3 = R^3 - 3nt \quad (\text{for } \epsilon = 0),$$

(9)

$$\tan_{\epsilon} (R - \varphi(t)) + \varphi(t) = \frac{1}{k} - cnt \quad (\text{for } \epsilon = \pm 1).$$

Hence, by making $t = T = \varphi^{-1}(R)$ in (9), one concludes that the collapsing time $T$ for the $\|A\|^2$-flow $F_t$ with initial data $f$ is

$$T = \begin{cases} \frac{R^3}{3n} & \text{(for } \epsilon = 0) \\ \frac{\epsilon(1-kR)}{kn} & \text{(for } \epsilon = \pm 1) \end{cases}$$

Example 2 (parallel $H_r$-flow in $Q^n_{\epsilon}$ with spherical initial data). Let $f$ and $\mathcal{F}$ be as in the preceding example. For the $H_r$-flow, the differential equation for $\varphi$ is

$$\varphi'(t) = W f_{R - \varphi(t)} = \left( \begin{array}{c} n \\ r \end{array} \right) \cot \epsilon (R - \varphi(t)), \quad \varphi(0) = 0,$$

which separates as

$$\tan \epsilon (R - \varphi) d\varphi = \left( \begin{array}{c} n \\ r \end{array} \right) dt.$$

For $\epsilon = 0$, the solution $\varphi$ is given implicitly by

$$\frac{(R - \varphi(t))^{r+1}}{r+1} = \frac{R^{r+1}}{r+1} - \left( \begin{array}{c} n \\ r \end{array} \right) t,$$
which yields
\[ T = \left( \frac{n}{r} \right)^{-1} \frac{R^{r+1}}{r+1} \]
for the collapsing time of \( F_t \).

For \( \epsilon = \pm 1 \), integration on the left hand side of (12) is recurrent. In Table 2, we list the solutions \( \varphi \) and corresponding collapsing times for \( r = 1, 2 \).

| \( r \) | \( \varphi \) | \( T \) |
|------|----------|--------|
| 1    | \( \cos_r(R - \varphi(t)) = e^{n1(r)} \cos \tau \) | \( \frac{\tau}{n} \log(1/\cos \tau) \) |
| 2    | \( \tan_r(R - \varphi(t)) + \varphi(t) = \frac{1}{1} - \epsilon \frac{n2(n-1)}{2} t \) | \( \frac{2(1-k\tau)}{kn(n-1)} \) |

Table 2. Function \( \varphi \) and collapsing time \( T \) for spherical parallel \( H_r \)-flows.

**Example 3** (parallel \( K \)-flow in \( S^{n+1} \) with non spherical initial data). Consider an isoparametric family \( \mathcal{F} \) of hypersurfaces \( f_\tau : M^n \to S^{n+1}, \tau \in (0, \pi/2) \), with two distinct principal curvatures
\[ k_1 = \cot \tau \quad \text{and} \quad k_2 = \cot(\tau + \pi/2) = -\tan \tau, \]
whose multiplicities are \( m_1 \) and \( m_2 \), respectively. By a result due to Cartan, \( M \) is homeomorphic to the product \( S^{m_1} \times S^{m_2} \), and the focal components \( \mathcal{F}_- \) and \( \mathcal{F}_+ \) are isometric to the standard spheres \( S^{m_1} \) and \( S^{m_2} \), respectively. Assuming \( m_1 \) even, we have that the Gaussian curvature \( K(\tau) \) of \( f_\tau \) is
\[ K(\tau) = \cot^{m_1}(\tau) \tan^{m_2}(\tau), \]
which is clearly a positive function on \((0, \pi/2)\).

If \( m_1 = m_2 \), then \( K = 1 \) for all \( \tau \in (0, \pi/2) \). In this case, given \( \tau_0 \in (0, \pi/2) \), the flow \( F_t = f_{\tau_0-t} \) is a solution to \( K \)-flow with initial data \( f_{\tau_0} \) and collapsing time \( T = \tau_0 \).

If \( m_1 > m_2 \), the function \( K(\tau_0 - \tau) = \cot^{m_1-m_2}(\tau_0 - \tau) \) is increasing in \([0, \tau_0]\). So, considering the solution \( \varphi \) of \( \varphi' = K(\tau) \) such that \( \varphi(0) = 0 \), we have from Theorem 2 that the flow \( F_t = f_{\tau_0-\varphi(t)} \) collapses into \( \mathcal{F}_+ \) at \( T = \varphi^{-1}(\tau_0) \). Besides, setting \( m = m_1 - m_2 \), we obtain the function \( \varphi \) and the collapsing time \( T \) by integrating \( \tan^m(\tau_0 - \varphi) \) with respect to \( \varphi \), as in the preceding example. For instance, if \( m = 1 \), the implicit equation for \( \varphi \) and collapsing time \( T \) are
\[ \cos(\tau_0 - \varphi(t)) = e^t \cos \tau_0 \quad \text{and} \quad T = \log(\sec \tau_0). \]

An analogous reasoning applies if \( m_2 \) is odd and \( m_1 < m_2 \), in which case \( F_t \) collapses into \( \mathcal{F}_- \) at \( T = \varphi^{-1}(\tau_0 - \pi/2) \).

**2.2 Parallel \( W \)-flows in rank-one symmetric spaces.** Let us consider now the rank-one symmetric spaces of non compact type, which are precisely the hyperbolic spaces described through the four normed division algebras: \( \mathbb{R} \) (real numbers), \( \mathbb{C} \) (complex numbers), \( \mathbb{H} \) (quaternions) and \( \mathbb{O} \) (octonions). They are denoted by \( \mathbb{H}^{a,b}_\mathbb{R}, \mathbb{H}^{a,b}_\mathbb{C}, \mathbb{H}^{2a}_\mathbb{R} \) and \( \mathbb{H}^{4a}_\mathbb{O} \) and called real hyperbolic space, complex hyperbolic space, quaternionic hyperbolic space and Cayley hyperbolic plane, respectively. The real hyperbolic space is \( \mathbb{H}^{n+1}_\mathbb{R} \), i.e., the simply connected space form of constant sectional curvature \(-1\). We shall adopt the unified notation \( \mathbb{H}^{m}_\mathbb{R} \) for these hyperbolic spaces,
where \( m = 2 \) for \( \mathbb{F} = \mathbb{O} \). The real dimension of \( \mathbb{H}_{m}^{p} \) is \( n + 1 = m \dim \mathbb{F} \). In particular, \( \mathbb{H}_{3}^{3} \) has dimension \( n + 1 = 16 \).

We add that any hyperbolic space \( \mathbb{H}_{m}^{p} \) is a Hadamard manifold with negative bounded sectional curvature everywhere and, more importantly, their geodesic spheres and horospheres are all isoparametric and strictly convex (see [1, 4] for details and proofs).

**Proof of Theorem 3.** Suppose that \( f(M) \) is a geodesic sphere and set

\[ \mathcal{F} = \{ f_\tau : S^n \to \mathbb{H}_{m}^{p} ; \tau \in (0, +\infty) \} \]

for the isoparametric family of geodesic spheres of \( \mathbb{H}_{m}^{p} \) such that \( f = f_{\tau_0} \) for some \( \tau_0 \in (0, +\infty) \). (Recall that the parameter \( \tau \) is the radius of \( f_\tau(S^n) \).)

The principal curvatures \( k^\tau_i \) of \( f_\tau \) with respect to the inward orientation are

\[ k^\tau_1 = \coth(\tau) \text{ with multiplicity } q \]
\[ k^\tau_2 = \frac{1}{2} \coth(\tau/2) \text{ with multiplicity } n - q, \]

where \( q = n \) for \( \mathbb{H}_{n+1}^{2} \), \( q = 1 \) for \( \mathbb{H}_{m}^{p} \), \( q = 3 \) for \( \mathbb{H}_{m}^{3} \), and \( q = 7 \) for \( \mathbb{H}_{d}^{3} \) (see, e.g., [3, pgs. 353, 543] and [9]). In particular, we have

\[ \lim_{\tau \to 0} k^\tau_i = +\infty, \quad i = 1, 2. \]

From the above considerations (and the monotonicity property of W), just as in the real case, we conclude that the parallel \( \varphi \)-flow with initial data \( f = f_{\tau_0} \) collapses to its center at \( T = \varphi^{-1}(\tau_0) \).

As we pointed out, the horospheres of any hyperbolic space \( \mathbb{H}_{m}^{p} \) are isoparametric. In fact, as in the real case, they foliate \( \mathbb{H}_{m}^{p} \) and have all the same constant principal curvatures (cf. the proposition on page 88 of [1]). So, any horosphere of \( \mathbb{H}_{m}^{p} \) moves indefinitely with constant speed under any \( W \)-flow. \( \square \)

In the next example, we calculate the collapsing time of a geodesic sphere of \( \mathbb{H}_{m}^{p} \) moving under \( H \)-flow.

**Example 4** (parallel \( H \)-flow in \( \mathbb{H}_{m}^{p} \) with spherical initial data). Let \( \mathcal{F} \) be as in (13). Then, the mean curvature \( H(\tau) \) of \( f_\tau \) is

\[ H(\tau) = q \coth(\tau) + \frac{n - q}{2} \coth(\tau/2). \]

Given \( R \in (0, +\infty) \), by Corollary 1 and Theorem 3, the flow

\[ F_t := f_{R - \varphi(t)}, \quad t \in [0, T), \]

where \( \varphi \) satisfies

\[ \begin{align*}
\varphi(t) &= H(R - \varphi(t)) = q \coth(R - \varphi(t)) + \frac{n - q}{2} \coth((R - \varphi(t))/2) \\
\varphi(0) &= 0,
\end{align*} \tag{15} \]

is a solution to (1) with initial data \( f_R \) which collapses into the center of \( f_R(S^n) \) at time \( T = \varphi^{-1}(R) \).

From (15), we obtain the equation

\[ \frac{d\varphi}{q \coth(R - \varphi) + ((n - q)/2) \coth((R - \varphi)/2)} = dt. \]
Setting \( x = e^{R-\varphi} \) and integrating the resulting rational function \( f(x)/g(x) \) by means of the identities

\[
\begin{align*}
\int \frac{dx}{ax^2 + bx + c} &= \frac{1}{2a} \log |ax^2 + bx + c| - \frac{b}{2a} \int \frac{dx}{ax^2 + bx + c} + C, \\
\int \frac{dx}{x(ax^2 + bx + c)} &= \frac{1}{2c} \log \left| \frac{x^2}{ax^2 + bx + c} \right| - \frac{b}{2c} \int \frac{dx}{ax^2 + bx + c} + C,
\end{align*}
\]

we conclude that the solution \( \varphi \) of (15) is given implicitly by

\[
\log \left( \frac{e^{R-\varphi(t)}}{a(e^2R - \varphi(t)) + 1} \right)^{1/a} = t + C(R),
\]

where \( a = (n+q)/2 \), \( b = n - q \), and

\[
C(R) = \log \left( \frac{e^R}{a(e^{2R} + 1) + be^R} \right)^{1/a}.
\]

Therefore, the collapsing time \( T = \varphi^{-1}(R) \) is

\[
T = \log \left( \frac{a(e^{2R} + 1) + be^R}{2ne^R} \right)^{\frac{1}{a}}.
\]

3. Avoidance Principle for Weingarten Flows

In this section, we prove Theorem 4, which constitutes an avoidance principle for Weingarten flows whose corresponding Weingarten functions are odd, as we mentioned in the introduction. The fundamental property of such a \( W \)-flow \( F_t : M^n \to \overline{M}^{n+1} \) is that it is invariant under change of orientation. Indeed, given \((p, t) \in M \times [0, T)\), writing \( k_i = k_i(p, t) \) and \( N = N(p, t) \), one has

\[
W(-k_1, \ldots, -k_n)(-N) = -W(k_1, \ldots, k_n)(-N) = W(k_1, \ldots, k_n)N = \frac{\partial F}{\partial t}(p, t).
\]

Along the proof of Theorem 4, we shall consider graphs over tangent spaces of hypersurfaces, as described below.

Let \( f : M^n \to \overline{M}^{n+1} \) be an oriented hypersurface. Fix \( p \in M \), and let \( U \subset T_p M \) be an open neighborhood of the zero vector of the tangent space of \( M \) at \( p \). Given a function \( \phi \in C^\infty(U) \), we call the set (assuming it is well defined)

\[
\Sigma_\phi := \{ \exp_{f(p)}(v + \phi(v)N(p)) \in \overline{M}; v \in U \}
\]

the graph of \( \phi \) on \( U \). Here, \( \exp \) denotes the exponential map of \( \overline{M}^{n+1} \).

Clearly, \( \Sigma_\phi \) is an orientable hypersurface of \( \overline{M}^{n+1} \). Moreover, it is a well known fact that, if the zero vector \( 0 \in U \) is a critical point of \( \phi \), then the Hessian of \( \phi \) at \( 0 \) coincides with the second fundamental form of \( \Sigma_\phi \) at \( \overline{p} = \exp_{f(p)}(\phi(0)N(p)) \in \Sigma_\phi \). In this case, we consider in \( \Sigma_\phi \) the orientation such that the unit normal to \( \Sigma_\phi \) at \( \overline{p} \) is (cf. Theorem 3 in [2], pg. 198):

\[
N_\phi(\overline{p}) = d\exp_{f(p)}(\phi(0)N(p))N(p).
\]

Notice that, if \( \gamma : [0, L] \to \overline{M}^{n+1} \) is the normalized geodesic from \( f(p) \) to \( \overline{p} \) satisfying \( \gamma'(0) = N(p) \), then \( N_\phi(\overline{p}) = \gamma'(L) \).
The following elementary result, which will be useful to us, compares principal curvatures of graphs whose corresponding functions have a common critical point. We adopt the convention of ordering the principal curvatures as $k_1 \leq \cdots \leq k_n$.

**Lemma 1.** With the above notation, assume that $\phi, \mu \in C^\infty(U)$ satisfy $\mu \geq \phi$ on $U$, and that the null vector $0 \in U$ is a minimum of $\mu - \phi$. Then, any principal curvature of $\Sigma_\mu$ at $\bar{p} = \exp_{f(p)}(\mu(0)N(p))$ is greater than, or equal to, the corresponding principal curvature of $\Sigma_\phi$ at $\bar{q} = \exp_{f(p)}(\phi(0)N(p))$.

**Proof.** Since $0$ is a minimum of $\mu - \phi$, we have that the Hessian of $\mu - \phi$ at $0$ is positive semi-definite, which implies that the same is true for the operator $A_\mu - A_\phi$, where $A_\mu$ and $A_\phi$ are the shape operators of $\Sigma_\mu$ at $\bar{q}$ and $\Sigma_\phi$ at $\bar{p}$, respectively. However, a standard result in Linear Algebra (see theorem on page 130 in [6]) asserts the following: If $A$ is self-adjoint and $B$ is positive semi-definite, then the eigenvalues of $A$ do not exceed the corresponding ones of $A + B$. Hence, setting $A = A_\phi$ and $B = A_\mu - A_\phi$, the lemma follows. \qed

The next result, due to R. Hamilton [7] (see also [10]), will play a fundamental role in the sequel.

**Lemma 2** (Hamilton’s trick). Let $u: M \times [0, T) \to \mathbb{R}$ be a $C^1$ function with the following property: For each $t_0 \in [0, T)$, there exist $\delta > 0$ and a compact subset $\Omega \subset M - \partial M$ such that, for any $t \in (t_0 - \delta, t_0 + \delta)$, the minimum

$$u_{\min}(t) := \min_{p \in M} u(p, t)$$

is attained (at least) at one point of $\Omega$. Then, the function $u_{\min}$ is locally Lipschitz in $(0, T)$ and, for each $t \in (0, T)$ where it is differentiable, one has

$$u_{\min}'(t) = \frac{\partial u}{\partial t}(p_0, t),$$

where $p_0 \in M - \partial M$ is any interior point at which $u(\cdot, t)$ attains its minimum.

**Proof of Theorem 4.** Since $M_2$ is compact, for each $t \in (0, T)$, there exists a pair $(p_1, p_2) \in M_1 \times M_2$ (possibly depending on $t$) such that

$$(17) \quad D(t) = \text{dist}^2(F_t^1(p_1), F_t^2(p_2)).$$

In addition, $\text{dist}^2$ is smooth on $\Omega$, which implies that the function

$$u(p, q, t) := \text{dist}^2(F_t^1(p), F_t^2(q)), \quad (p, q, t) \in M_1 \times M_2 \times [0, T),$$

is smooth as well. Thus, Hamilton’s Trick applies and gives that $D(t) = u_{\min}(t)$ is locally Lipschitz, so that $D$ is differentiable almost everywhere (by Rademacher Theorem). Also, at a differentiable point $t_0$, the following equality holds

$$(18) \quad D'(t_0) = \frac{\partial u}{\partial t}(p_1, p_2, t_0),$$

where $(p_1, p_2) \in M_1 \times M_2$ is any pair at which $u(\cdot, t_0)$ attains its minimum. So, it suffices to prove that $D'(t_0) \geq 0$. This is certainly true if $D(t_0) = 0$ (since $D$ is nonnegative), so that we can assume $D(t_0) \neq 0$.

In the above setting, the minimizing normalised geodesic $\gamma_{t_0}: [0, L] \to \mathbb{M}$ joining the points $\bar{p}_1 := F_{t_0}^1(p_1)$ and $\bar{p}_2 := F_{t_0}^2(p_2)$ is orthogonal to both $F_{t_0}^1(M_1)$ (at $\bar{p}_1 = \gamma_{t_0}(0)$) and $F_{t_0}^2(M_2)$ (at $\bar{p}_2 = \gamma_{t_0}(L)$).
Let us denote by $\Pi$ the tangent space of $F^1_{t_0}(M_1)$ at $\bar{p}_1$. It is easily checked that, for $i = 1, 2$, there exists an open neighborhood $U$ of 0 in $\Pi$ such that, for all $t$ sufficiently close to $t_0$, there is a suitable neighborhood of $\bar{p}_i$ in $\overline{M}$, $F^i_t(M_1)$ is a graph of a function $\phi^i_t \in C^\infty(U)$. In particular, we have $\phi^2_t > \phi^1_t$ on $U$. Also, since $\bar{p}_i = \phi^i_{t_0}(0)$, we have that $0 \in U$ is a minimum of $\phi^2_{t_0} - \phi^1_{t_0}$ on $U$.

By (16), we can assume that $N^1_{t_0}(p_1) = \gamma'(0)$ and $N^2_{t_0}(p_2) = \gamma'(L)$. In this case, from Lemma 1, no principal curvature of $F^1_{t_0}(M_1)$ at $p_1$ exceeds the corresponding one of $F^2_{t_0}(M_2)$ at $p_2$. Therefore, from the monotonicity property of the Weingarten function $W$, the following inequality holds:

$$W_{F^1_{t_0}}(p_1) \leq W_{F^2_{t_0}}(p_2).$$

Now, observe that the gradient of the squared distance function of $\overline{M}$ at the point $(\bar{p}_1, \bar{p}_2) \in \overline{M} \times \overline{M}$ is the vector $\nabla \operatorname{dist}^2(\bar{p}_1, \bar{p}_2) = 2 \operatorname{dist}(\bar{p}_1, \bar{p}_2)(-\gamma^1_{t_0}(0), \gamma^2_{t_0}(L))$. So,

$$\nabla \operatorname{dist}^2(\bar{p}_1, \bar{p}_2) = 2 \operatorname{dist}(\bar{p}_1, \bar{p}_2)(-N^1_{t_0}(p_1), N^2_{t_0}(p_2)) \in T_{\bar{p}_1} \overline{M} \times T_{\bar{p}_2} \overline{M}. \quad (20)$$

Putting together identities (17)–(20), and considering the fact that $F^1_t$ and $F^2_t$ are both $W$-flows, we have

$$D'(t_0) = \frac{\partial}{\partial t} \operatorname{dist}^2(F^1_t(p_1, t), F^2_t(p_2, t))|t=t_0 = \left\langle \nabla \operatorname{dist}^2(\bar{p}_1, \bar{p}_2), \left(\frac{\partial F^1_t}{\partial t}(p_1, t_0), \frac{\partial F^2_t}{\partial t}(p_2, t_0)\right)\right\rangle_{\overline{M} \times \overline{M}}$$

$$= 2 \operatorname{dist}(\bar{p}_1, \bar{p}_2)(-W_{F^1_{t_0}}(p_1) + W_{F^2_{t_0}}(p_2)) \geq 0,$$

as we wished to prove. \qed

In the above proof, the hypothesis of $W$ being odd allowed us to choose the orientation of the hypersurfaces $F^i_t$ at $p_i$, $i = 1, 2$, in such a way that their unit normals at these points would coincide with $\gamma'(0)$ and $\gamma'(L)$. In this manner, we could apply Lemma 1 and then obtain the fundamental inequality (19). From this, we conclude that we can drop the assumption on $W$ being odd in the statement of the avoidance principle, as long as we have assured that the orientations of $F^i_t$ follow this pattern.

For instance, suppose that $F_t : M^n \to \overline{M}^{n+1}$, $t \in [0, T)$, is a $W$-flow, where $M^n$ is compact and $\overline{M}^{n+1}$ is either a space form $\mathbb{Q}^{n+1}$ or a hyperbolic space $\mathbb{H}^{n+1}_0$. Assume that $F_0(M)$ is contained in an open totally convex ball $B_R \subset \overline{M}^{n+1}$, whose boundary $\partial B_R$ is a strictly convex geodesic sphere of $\overline{M}^{n+1}$ (i.e., $0 < R < \pi/2$ for $\overline{M}^{n+1} = \mathbb{S}^{n+1}$). In this setting, considering the parallel flow $P_t : \mathbb{S}^n \to \overline{M}^{n+1}$ with initial data $P_0(\mathbb{S}^n) = \partial B_R$ and inward orientation, and assuming that $F_t$ is an embedding with the inward orientation for all $t \in [0, T)$, we have that the normals at the points minimizing the distance between $F_t(M)$ and $P_t(M)$ coincide with the tangent vectors to the minimizing geodesic joining them, as in the above case. Thus, the avoidance principle holds. In particular, by the results of the preceding section, $F_t$ has a finite collapsing time which is at most equal to that of $P_t$.

Summarizing, we have the following result.

**Corollary 5.** Let $\overline{M}^{n+1}$ be either a space form $\mathbb{Q}^{n+1}$ or a hyperbolic space $\mathbb{H}^{n+1}_0$. Given a Weingarten function $W \in C^\infty(\Gamma)$, assume that $F : M^n \times [0, T) \to \overline{M}^{n+1}$ is a $W$-flow of a compact Riemannian manifold $M$ such that $F_0(M)$ is contained in
an open totally convex ball $B_R \subset \mathcal{M}^{n+1}$, whose boundary $\partial B_R$ is a strictly convex geodesic sphere of $\mathcal{M}^{n+1}$. Assume further that one of the following holds:

- $W$ is odd.
- $F_t$ is an embedding with the inward orientation for all $t \in [0,T)$.

Under these conditions, denoting by $P : S^n \times [0, T_R) \rightarrow \mathcal{M}^{n+1}$ the parallel $W$-flow with inward orientation, collapsing time $T_R$, and initial data $P_0(S^n) = \partial B_R, \forall t \in [0, T)$. Consequently, the inequalities $T \leq T_R < \infty$ hold.

Proof of Theorem 5. Since $F_0$ is an embedding, for any sufficiently small $t > 0$, $F_t$ is also an embedding. Let us suppose, by contradiction, that there exists a first time $t_0 > 0$ such that $F_{t_0}$ is not an embedding. In this way,

$$\Omega := \{(p, q) \in M \times M; p \neq q, F_{t_0}(p) = F_{t_0}(q)\}$$

is a nonempty compact set of $M \times M$ which is disjoint from the diagonal $D$ of $M \times M$. Thus, there is an open set $U \subset M \times M$ such that $D \subset U$ and $\Omega$ is disjoint from the closure of $U$ in $M \times M$.

Now, observing that $V := (M \times M) - U$ is compact in $M \times M$, define the function

$$D(t) = \min_{(p, q) \in V} \text{dist}^2 (F_t(p), F_t(q)), \ t \in [0, t_0].$$

Since, for a sufficiently small $t$, $F_t$ is an embedding, for such a $t$ we have $D(t) > 0$. However, proceeding just as in the proof of Theorem 4, we conclude that $D$ is nondecreasing, which contradicts the fact that $D(t_0) = 0$. \(\square\)

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References

[1] Berndt, J., Tricerri, F., Vanhecke, L.: Generalized Heisenberg groups and Damek-Ricci harmonic spaces. Lecture Notes in Mathematics 1598. Springer Verlag (1995).
[2] Bishop, R., Crittenden, R.: Geometry of submanifolds. Academic Press (1964).
[3] Cecil, T., Ryan, P.: Geometry of hypersurfaces. Springer Verlag (2015).
[4] Domínguez-Vázquez, M.: An introduction to isoparametric foliations. Preprint (2018) (available at: http://xtsunxet.usc.es/miguel/teaching/jae2018.html).
[5] dos Reis, H., Tenenblat, K.: The mean curvature flow by parallel hypersurfaces. Proc. Amer. Math. Soc. 146, 4867–4878 (2018).
[6] Gelfand, I.: Lectures on linear algebra, Interscience Publishers, New York–London (1961).
[7] Hamilton, R.: Four-manifolds with positive curvature operator. J. Differential Geom. 24, 153–179 (1986).
[8] Huisken, G., Polden, A.: Geometric evolution equations for hypersurfaces, Calculus of variations and geometric evolution problems (Cetraro, 1996), Lecture Notes in Math., vol. 1713, Springer, Berlin, 1999, pp. 45–84.
[9] Kim, S., Nikolayevsky, Y., Park, J.: Einstein hypersurfaces of the Cayley projective plane. Differential Geom. Appl. 69, 1–6 (2020).
[10] Mantegazza, C.: Lecture notes on mean curvature flow, Birkhäuser (2011).
[11] Münzner, H. F.: Isoparametrische Hyperflächen in Sphären (German), Math. Ann. 251 (1980), no. 1, 57–71.