A basis for the Birman-Wenzl algebra

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Abstract

This paper provides an explicit isomorphism between the Birman-Wenzl algebra $BW_n$, constructed by J. Birman and H. Wenzl, and the Kauffman algebra $MT_n$, subsequently constructed by H.R. Morton and P. Traczyk. The Birman-Wenzl algebra is defined algebraically using generators and relations while the Kauffman algebra has a geometric formulation in terms of tangles. The isomorphism is obtained by constructing an explicit basis in $BW_n$, analogous to a basis previously constructed for $MT_n$ using ‘Brauer connectors’. The geometric isotopy arguments used for $MT_n$ are systematically replaced by algebraic versions using the Birman-Wenzl relations. This not only gives a direct way of determining the dimension of the Birman-Wenzl algebra, but also clarifies the role played by the ring of coefficients, $\Lambda$, and its specialisations.

Foreword

This paper is a very lightly edited version of an article originally written in 1989 but never fully completed. It was intended to be a joint paper with A.J. Wassermann. He had planned to write a final section to make use of the ability to change at will between the Birman-Wenzl algebra, as given by generators and relations, and the geometric framework of the tangles, so as to look in more detail at the representation theory.

One goal of our original approach was to make sure that specialisations of the coefficient ring could be handled confidently, and that the translations to and from the tangle context were on a sound footing.

Subsequently others have made progress in this way, in works such as [6], but we had a number of requests for our earlier account, and so I put it into this more accessible form on the Liverpool knot theory pages in 2000.

In order to have a more permanent place for it I have now put it on ArXiv. I had hoped to make this a joint submission, but I have been unable
to re-establish contact with Wassermann to get his formal agreement. The present paper is largely the result of our joint discussions, although I take responsibility for the eventual content and exposition.

1 Introduction

In recent years there has been considerable interest in deformations of the classical ‘centraliser algebras’ of Schur, Weyl and Brauer. These play an important role in several areas, including exactly solvable models in statistical mechanics, quantum groups, von Neumann algebras and knot theory. It has long been recognized that these links are more than tenuous and if properly exploited lead to fruitful interchanges between the different disciplines. The first and most spectacular instance of this was of course Vaughan Jones’ pioneering work on subfactors, which led to his discovery of new link invariants. Subsequently these invariants were understood in terms of solutions of the quantum Yang-Baxter equation and vertex models. The central thread running through all these topics is the quantum group obtained by deforming the universal enveloping algebra of the unitary group. One has also to deform the centraliser algebras, which amounts to replacing the group algebra of the symmetric group by the Hecke algebra (of type \(A\)).

After these discoveries, somewhat curiously history took a reverse turn. Kauffman discovered new link invariants, based on a purely skein theoretic characterisation of Jones’ original invariants. It was natural to ask whether Kauffman’s invariants could be obtained by algebraic means. This led Birman and Wenzl to introduce a deformation of an abstract algebra first introduced by R. Brauer. An alternative knot-theoretic approach to deforming Brauer’s algebra was later given in terms of tangles by Morton-Traczyk and by Kauffman himself. The original algebra of Brauer bore the same relation to the orthogonal group as the group algebra of the symmetric group did to the unitary group.

By exploiting the new insights provided by Vaughan Jones’ work on subfactors (in particular his ‘basic construction’), Wenzl was able to acquire a fuller understanding of Brauer’s algebra, and resolve some old questions on semisimplicity raised by Brauer and Weyl. Subsequent studies have shown that the Birman-Wenzl algebra does indeed provide the correct analogue of the Hecke algebra for the quantum group corresponding to the orthogonal group. Most recently Wenzl has been able to construct new examples of subfactors using these algebras as a substitute for the Hecke algebras.

In this paper we provide an explicit isomorphism between the Birman-
Wenzl algebra $BW_n$, constructed by J. Birman and H. Wenzl in [1], and
the Kauffman algebra $MT_n$, subsequently constructed by the author and P. Traczyk in [11]. The Birman-Wenzl algebra is defined algebraically using
generators and relations while the Kauffman algebra has a geometric formulation in terms of tangles. We obtain this isomorphism by constructing an
explicit basis in $BW_n$, analogous to a basis previously constructed for $MT_n$
using ‘Brauer connectors’. The geometric isotopy arguments used in [11]
are systematically replaced by algebraic versions using the Birman-Wenzl
relations. This not only gives a direct way of determining the dimension of
the Birman-Wenzl algebra, but also clarifies the role played by the ring of
coefficients, $\Lambda$, an integral domain.

In fact, in [1] the authors prefer to consider the algebra $BW_n \otimes_\Lambda k$,
where $k$ is the field of fractions of $\Lambda$. This enables them to imitate V.
Jones’ basic construction and thus determine the structure of the algebra.
At a crucial point in proof of their main result (theorem 3.7) they need
to use a specialisation of $\Lambda$. Since the algebra is defined by generators
and relations over $\Lambda$, any such specialisation automatically extends to $BW_n$
although not necessarily to the algebra $BW_n \otimes_\Lambda k$. This difficulty can be
overcome by observing that the existence of a basis implies that $BW_n$ is
free as a module over $\Lambda$. The arguments presented in [2] p.55 to prove that
the Hecke algebra is generically semisimple may then be adapted to prove
the same result for $BW_n$, i.e. the specialisations of $BW_n$ are semisimple for
a Zariski open subset of the parameter space Spec($\Lambda$). Wenzl has carried
out a more detailed analysis, based on Jones’ basic construction, in order to
determine precisely when the algebras fail to be semisimple.

This paper is divided into six sections. In section 2 we review the def-
definitions of the algebras to be studied, with some historical comments. In
section 3 we use the basic solution of the Yang-Baxter equation for the or-
thogonal group together with a simple skein-theoretic argument to provide
a short self-contained definition of Kauffman’s two-variable link invariant.
We also briefly discuss the duality between the quantum orthogonal group
and the Birman-Wenzl algebra. In section 4 we give an inductive defini-
tion of a basis for the Birman-Wenzl algebra and outline the more formal
aspects of the proof. The inductive procedure relies on a natural filtration
analogous to the one extensively used by Hanlon and Wales in their studies
of Brauer’s algebra [7]. Effectively the proof that the proposed basis is a
spanning set is achieved by a double induction, which from the point of view
of tangles depends both on the number of strings and then on the number of
‘through’ strings. The remaining two sections are devoted to various stages
of the inductive argument showing that the natural surjective maps from
the Birman-Wenzl algebras to the tangle algebras are isomorphisms. In section 5 we treat the case in which there are no ‘through’ strings: a complete understanding of this case is crucial for the subsequent reasoning since it allows us to use geometry in place of algebra in a controlled way. Finally in section 6 we perform the main step of the induction.

2 Three algebras

2.1 The Birman-Wenzl algebra

We start by recalling the definition of the Birman-Wenzl algebra. We have made a slight change by the introduction of some minus signs, in accordance with Kauffman’s ‘Dubrovnik’ version of his link invariant. As explained in [11] and below, this makes it much easier to see the connection with Brauer’s centraliser algebras.

Let \( \Lambda \) be the quotient ring \( \mathbb{Z}[\lambda^\pm 1, z, \delta]/\langle \lambda^{-1} - \lambda - z(\delta - 1) \rangle \). Thus \( \Lambda \) (or more accurately its complexification) is the coordinate ring of the irreducible quasiprojective variety defined by \( \lambda \neq 0, \lambda^{-1} - \lambda = z(\delta - 1) \) in \( \mathbb{A}^3 \).

Definition. The Birman-Wenzl algebra \( BW_n \) is the quotient of the free algebra over \( \Lambda \) with generators \( g_1^\pm 1, g_2^\pm 1, \ldots, g_n^\pm 1 \) and \( e_1, e_2, \ldots, e_{n-1} \) modulo the ideal generated by the relations:

1. (Kauffman skein relation) \( g_i - g_i^{-1} = z(1 - e_i) \).
2. (Idempotent relation) \( e_i^2 = \delta e_i \).
3. (Braid relations) \( g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \) and \( g_i g_j = g_j g_i \) if \( |i - j| > 1 \).
4. (Tangle relations) \( e_i e_{i\pm 1} e_i = e_i \) and \( g_i g_i e_{i\pm 1} e_i = e_{i\pm 1} e_i \).
5. (Delooping relations) \( g_i e_i = e_i g_i = \lambda e_i \) and \( e_i g_i e_i = \lambda^{-1} e_i \).

Remark. If \( z \) is taken to be invertible then the idempotent relation follows from the delooping and skein relations.

In Birman and Wenzl’s original version several of their relations could be omitted without loss, given invertibility of \( z \). They use \( v \) in place of \( \lambda \) in the coefficient ring.

The presentation given here is intended to be sufficiently symmetric to allow for easy comparison with the tangle algebra, while maintaining the coefficient ring \( \Lambda \) as in [11].
2.2 Kauffman’s tangle algebra

**Definition.** An \((m,n)\)-tangle is a piece of knot diagram in a rectangle \(R\) in the plane, consisting of arcs and closed curves, so that the end points of the arcs consist of \(m\) points at the top of the rectangle and \(n\) points at the bottom, in some standard position.

An example of a \((4,2)\)-tangle is shown in figure 2.1.

![Figure 2.1](image)

**Definition.** Two tangles are *ambient isotopic* if they are related by a sequence of Reidemeister’s moves I, II and III, (see figure 2.2), together with isotopies of \(R\) fixing its boundary.

![Figure 2.2](image)

They are *regularly isotopic* if Reidemeister move I is not used.

**Notation.** Write \(U^m_n\) for the set of \((m,n)\)-tangles up to regular isotopy.

The set \(U^m_n\) admits an *associative multiplication*, defined by placing representative tangles one below the other.

A well-known subset \(B_n\) consists of geometric braids, in this context represented by tangles (necessarily without closed components) where the height coordinate in \(R\) increases monotonically on each component. It can be shown that \(B_n\) is the full group of units in \(U^m_n\) under the multiplication.
The closure, $\hat{T}$, of an $(n,n)$-tangle $T$, is defined, by analogy with the closure of a braid, to be the link diagram (or $(0,0)$-tangle) given from $T$ by joining the points on the top of $R$ to those on the bottom by arcs lying outside $R$ with no further crossings.

We define a closure map $\varepsilon : \mathcal{U}_n \to \mathcal{U}_0$, by $\varepsilon(T) = \hat{T}$.

From $\mathcal{U}_n$ we construct the algebra $M_T$, which we call Kauffman’s tangle algebra, as an algebra over a ring $\Lambda$, as in [11]. We shall take $\Lambda$ to be the ring

$$\Lambda = \mathbb{Z}[\lambda^{\pm 1}, z, \delta]/<\lambda^{-1} - \lambda = z(\delta - 1)>.$$

Then $\Lambda$ is isomorphic to a subring of $\mathbb{Z}[\lambda^{\pm 1}, z^{\pm 1}]$, by taking $\delta = 1 + (\lambda^{-1} - \lambda)/z$. It admits a homomorphism $e : \Lambda \to \mathbb{Z}[\delta]$ with $e(z) = 0$, $e(\lambda) = 1$ and $e(\delta) = \delta$. The main aim of this paper is to show that the Birman-Wenzl algebra $BW_n$ is isomorphic to $M_T$, on specialisation of coefficients.

Certain features of $M_T$, for example its dimension, and its relation to Brauer’s algebra [3], appear here very simply, using the homomorphism $e$ and the Dubrovnik invariant $D$. These features of $BW_n$, not proved directly in the original approach, then follow at once.

**Definition.** Kauffman’s tangle algebra, $M_T$, is the $\Lambda$-module, constructed from $\Lambda[\mathcal{U}_n]$ by factoring out three sets of relations:

$$T^+ - T^- = z(T^0 - T^\infty),$$

(1)

where $T^\pm, T^0, T^\infty$ are represented by tangles differing only as in figure 2.3,

![Figure 2.3](image)

$$T^\text{right} = \lambda^{-1}T, \quad T^\text{left} = \lambda T,$$

(2)

where $T^\text{right}$ and $T^\text{left}$ are given from $T$ by adding a left or right hand curl as in figure 2.4,

![Figure 2.4](image)
\[ T \boxplus O = \delta T, \]  

where \( T \boxplus O \) is the union of \( T \) and a circle having no crossings with \( T \) or itself.

**Proposition 2.1** Composition of tangles induces a \( \Lambda \)-bilinear multiplication on \( MT_n \) making \( MT_n \) an algebra over \( \Lambda \).

**Proof**: The relations (1)-(3) carry down under the multiplication in \( \Lambda[\mathcal{U}_n] \).

\[ \square \]

**Proposition 2.2** The map \( \varepsilon \) induces a \( \Lambda \)-linear map \( \varepsilon : MT_n \rightarrow MT_0 \).

We now give the homomorphism \( \varphi : BW_n \rightarrow MT_n \) which provided the intuition behind Birman and Wenzl’s description of \( BW_n \).

**Definition.** Write \( G_i, E_i \) respectively for the tangles in \( \mathcal{U}_n \) illustrated in figure 2.5. Use the same letters for the elements represented by these tangles in \( MT_n \), called \( s_i, h_i \) in \( \Pi \).

\[ \begin{array}{c}
G_i = \begin{array}{c}
\text{i}
\text{i+1}
\end{array}
\begin{array}{c}
\text{j}\\
\text{k}
\end{array}
\begin{array}{c}
\text{l}\\
\text{m}
\end{array}
\end{array}
\quad
\begin{array}{c}
E_i = \begin{array}{c}
\text{i}
\text{i+1}
\end{array}
\begin{array}{c}
\text{j}\\
\text{k}
\end{array}
\begin{array}{c}
\text{l}\\
\text{m}
\end{array}
\end{array}
\]

Figure 2.5

Then

\[ G_i - G_i^{-1} = z(1 - E_i) \]

in \( MT_n \), from relation (1) applied to the only crossing in \( G_i \).

Similarly, relation (2) shows that

\[ G_iE_i = E_iG_i = \lambda^{-1}E_i \]

\[ G_i^{-1}E_i = E_iG_i^{-1} = \lambda E_i \]

and relation (3) that

\[ E_i^2 = \delta E_i. \]

**Theorem 2.3** A homomorphism \( \varphi : BW_n \rightarrow MT_n \) may be defined by

\( \varphi(g_i) = G_i, \ \varphi(e_i) = E_i. \)
Proof: The relations in $BW_n$ are respected. We have already noted that the skein relation and delooping relations are satisfied by $E_i, G_i$ in $MT_n$. The other relations hold even at the level of the tangle semigroup $U_n^n$. 

Our goal is to prove that $\varphi$ is an isomorphism for all $n$. In this section we find explicit spanning sets for $MT_n$, and show that $\varphi$ is surjective.

In section 4 we give the proof from [11] that the chosen spanning sets are a free basis for $MT_n$, using the existence of Kauffman’s invariant.

The proof that $\varphi$ is injective will subsequently be built up in stages, with the recurring pattern of taking spanning sets for selected subspaces of $BW_n$ and proving that they map to independent sets in $MT_n$.

To save later effort we note here some symmetry of $BW_n$, which carries over by $\varphi$ to two natural operations in $MT_n$.

**Definition.** (1) Write $\rho_n : BW_n \to BW_n$ for the automorphism defined by

\[ \rho_n(g_i) = g_{n-i}, \rho_n(e_i) = e_{n-i}. \]

(2) Write $\alpha : BW_n \to BW_n$ for the reversing anti-automorphism defined by

\[ \alpha(g_i) = g_i, \alpha(e_i) = e_i. \]

**Remark.** The symmetry of the relations in $BW_n$ ensures that $\rho_n, \alpha$ are well-defined.

**Proposition 2.4** There is an automorphism $\rho_n$ of $MT_n$, and an antiautomorphism $\alpha$, with $\varphi \circ \alpha = \alpha \circ \varphi$ and $\varphi \circ \rho_n = \rho_n \circ \varphi$.

**Proof:** Write $\rho_n, \alpha : U_n^n \to U_n^n$ for the natural symmetries given by rotating a tangle $T$ through $\pi$ about one of the two axes shown in figure 2.6.

![Figure 2.6](image)

Clearly $\alpha(G_i) = G_i, \rho_n(G_i) = G_{n-i}$, and similarly for $E_i$. The skein relations are preserved by $\rho_n$ and $\alpha$ so that they induce $\rho_n, \alpha : MT_n \to MT_n$. Since $\rho_n(ST) = \rho_n(S)\rho_n(T)$ and $\alpha(ST) = \alpha(T)\alpha(S)$ these are respectively an automorphism and an antiautomorphism, satisfying the stated relations on the generators of $BW_n$. \qed
We now continue with the proof that $MT_n$ has a finite spanning set, and at
the same time we develop the notation to relate these algebras readily with
Brauer’s centraliser algebras.

2.3 Connectors and Brauer’s algebras

An $(n,n)$-tangle $T$ consists of $n$ arcs and a number, $|T|$, of closed curves. If
each arc joins a point at the top to a point at the bottom then the tangle
determines a permutation in $S_n$.

**Definition.** For a general tangle we extend the idea of a permutation to
that of an $n$-connector, defined to be a pairing of $2n$ points into $n$ pairs.

The set $C_n$ of $n$-connectors has $\binom{2n}{n}$ elements, the product of the
first $n$ odd integers.

Take the set of $2n$ points to be the end points of $(n,n)$-tangles.
The arcs of any $T \in \mathcal{U}_n^n$ pair these end points to give a connector, which we write as
$\text{conn}(T) \in C_n$.

**Remark.** (Brauer’s algebra) Brauer \[3\] uses $C_n$ as the basis for an algebra
over $\mathbb{Z}[\delta]$, (writing $n$ in place of $\delta$ and $f$ in place of $n$). He divides the $2n$
points to be connected into two subsets $t_1,\ldots,t_n$ and $b_1,\ldots,b_n$, arranged
along the top and bottom of a rectangle, and views a connector $c$ as a set of
$n$ intervals with these $2n$ points as endpoints, which join the points paired
by $c$. Two connectors $c_1$ and $c_2$ are composed by placing one rectangle above
the other, giving $n$ arcs whose endpoints are the new top and bottom points,
together with some number $r \geq 0$ of closed curves.

Brauer sets $c_1c_2 = \delta^r d$, where $d$ is the connector defined by the new
arcs. This defines an associative multiplication on $\mathbb{Z}[\delta][C_n] = A_n$ making it
an algebra over $\mathbb{Z}[\delta]$, called **Brauer’s algebra**.

Having divided the $2n$ points in this way there is a natural embedding
$S_n \subset C_n$.

We can modify the map $\text{conn} : \mathcal{U}_n^n \to C_n$ to give a multiplicative homomorphism $c : \mathcal{U}_n^n \to A_n$, which extends to $c : MT_n \to A_n$ as follows.

For $T \in \mathcal{U}_n^n$ set $c(T) = \delta^{|T|} \text{conn}(T) \in A_n$. This can be extended to $c : \Lambda[\mathcal{U}_n^n] \to A_n$ by setting $c(\Sigma \lambda_i T_i) = \Sigma e(\lambda_i)c(T_i)$, using the ring homomorphism $e : \Lambda \to \mathbb{Z}[\delta]$.

**Theorem 2.5** There is an induced homomorphism $c : MT_n \to A_n$.

**Proof:** The relations (1)-(3) defining $MT_n$ are respected. \qed
Remark. We show later that $A_n$ is isomorphic to the $\mathbb{Z}[\delta]$ algebra $MT_n \otimes_{\Lambda} \mathbb{Z}[\delta]$ given from $MT_n$ by replacing the coefficients $\Lambda$ with $\mathbb{Z}[\delta]$, using the homomorphism $e$.

The existence of $c : MT_n \to A_n$ can be viewed as the consequence of specialising the coefficients so that the relations no longer distinguish under-from over-crossings. Then tangles pass to their projections, retaining only the information of their connectors. The crucial technical feature here is that we can specialise $\Lambda$ so as to retain $\delta$, while fixing $\lambda = 1$ and $z = 0$. Complications arise if we try to do this while working in the ring $\mathbb{Z}[\lambda^{\pm 1}, z^{\pm 1}]$.

Definition. Given a tangle $T$, choose a sequence of base-points, consisting firstly of one end point of each arc, and then one point on each closed component. Say that $T$ is totally descending (with this choice of base points) if on traversing all the strands of $T$, starting from the base point of each component in order, each crossing is first met as an overcrossing.

Remark. We shall assume that for each connector $c \in C_n$ a choice of ordering of base-points for the arcs has been made, and we use this same choice for all tangles $T$ with $c = \text{conn}(T)$. Note that there are $n! \cdot 2^n$ potentially different choices possible for each connector. The precise choice is not material, and we shall have occasion to vary the choice in the course of later proofs. The result will be simply to alter the choice of linear basis in $MT_n$.

An example of a totally descending $(3, 3)$-tangle is shown in figure 2.7, with base-points numbered according to a choice of order.

![Figure 2.7](image)

Theorem 2.6 $MT_n$ is spanned by totally descending tangles.

Proof: Let $T$ be a tangle representing an element of $MT_n$. Choose base points for $T$ according to the choice for $\text{conn}(T)$. Traverse the arcs of $T$ in order. At the first non-descending crossing use relation (1) with $T = T_\pm$. Note that $\text{conn}(T_+) = \text{conn}(T_-)$, so that $T_\pm$, resulting from $T$ with the crossing switched, has fewer non-descending crossings. Then $T$ is a linear combination of three tangles, two with fewer crossings and one with fewer non-descending crossings. The theorem follows by induction, firstly on the number of crossings, then on the number of non-descending crossings. \qed
Corollary 2.7 $MT_n$ is spanned by totally descending tangles without closed components.

Proof: If $T$ is totally descending, with $r$ closed components, then these components are unknotted curves stacked below the arcs of $T$. The tangle can then be altered by regular isotopy so that the unknotted components lie well away from the arcs. By using Reidemeister move I as well they can be changed to have no self-crossings. Then by (2) and (3), $T = \lambda^k \delta^r T'$ in $MT_n$, where $T'$ consists simply of the arcs of $T$. \qed

Remark. This result holds as stated for $n = 0$, provided that we admit the ‘empty tangle’ as an element of $U_0^0$. In any event $MT_0$ is spanned by a single element.

Corollary 2.8 $MT_0$ is cyclic.

Theorem 2.9 Let $S$ and $T$ be totally descending $(n,n)$-tangles, without closed components, such that $\text{conn}(S) = \text{conn}(T)$. Then $S$ and $T$ are ambient isotopic, and so $S = \lambda^k T$ in $MT_n$, for some $k$.

Proof: Number the arcs of $S$ and $T$ according to the order of their base points. Since $\text{conn}(S) = \text{conn}(T)$, the $i$th arc in each tangle joins the same pair of end points. The arcs can be arranged to lie in disjoint levels 1 to $n$ above the plane of $R$, since arc $i$ lies above arc $j$ at every crossing when $i < j$. Each individual arc is unknotted, because the tangle is descending, so it can be changed by ambient isotopy to an arc without self-crossings in its level. The resulting tangles are then ambient isotopic by level-preserving isotopy. \qed

Remark. If the arcs of $S$ and $T$ have no self-crossings initially then $S$ and $T$ are regularly isotopic.

Remark. (Construction) For each connector $c \in C_n$, choose an order for the arcs. With this order construct a totally descending tangle with connector $c$ such that any two arcs cross at most once. (Start for example from a diagram of the connector in which any two arcs cross at most once, and make it descending, by choosing the sense of each crossing according to the order of the arcs.) The element $T_c \in U_n^n$ represented by this tangle then depends only on $c$ and the chosen order, by Theorem 2.9.

Remark. For $c \in S_n$ and a natural choice of order the resulting tangles $T_c$ have been studied, [4, 5], under the name ‘positive permutation braids’. 

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They can be represented by a braid in $B_n$ with positive crossings and permutation $c$ in which any two strings cross at most once.

These braids have also been used in [9, 10], to give easily handled generators for the Hecke algebra $H_n$.

**Theorem 2.10** $MT_n$ is spanned, for every choice of order, by the finite set $\{T_c\}, c \in C_n$.

**Proof**: By theorem [2.6] and its corollary, $MT_n$ is spanned by tangles which are ambient isotopic to $T_c$, for various $c$. By use of relation (2), any tangle ambient isotopic to $T_c$ represents $\lambda^k T_c$ in $M_n$, for some $k$. \qed

**Remark.** The number of crossings in a totally descending tangle $T_c$ depends on the connector $c$, not on the order of arcs used. It is simply the number of pairs of arcs which cross in $c$, as dictated by whether or not their endpoints interlock on the boundary rectangle.

Clearly any tangle with $k$ crossings can always be written in $MT_n$ as a linear combination of totally descending tangles with at most $k$ crossings, by induction on $k$, using the procedure of theorem [2.6]. It follows that if $T'_c$, $T_c$ are totally descending tangles with the same connector $c$, arising from different choices of the order of arcs then

$$T'_c = T_c + \sum_d \lambda_d T_d,$$

where $d$ runs over connectors with fewer crossings than $c$.

We finish this section by proving:

**Theorem 2.11** The map $\varphi : BW_n \to MT_n$ is surjective.

**Proof**: We must show that $MT_n$ is generated by $E_i, G_i, 1 \leq i \leq n - 1$. It is enough to show that each totally descending tangle $T_c$ is a monomial in $\{E_i\}$ and $\{G_i^{\pm 1}\}$.

Assuming that the connector $c$ pairs $r$ points at the top with $r$ at the bottom, and connects the remaining $2k = n - r$ points as $k$ pairs, leaving $2k$ points at the bottom connected as $k$ pairs.

We can then draw the tangle $T_c$ (for any order of the arcs) so that there are $r$ arcs running monotonically from top to bottom, $k$ arcs running with a single local minimum from top to top, and $k$ arcs from bottom to bottom with a single local maximum. We can further assume, since the arcs never cross twice, that all the local minima on the top arcs are higher up than the
local maxima, so that there are only \( r \) arcs passing through the middle part of the rectangle.

Now pair arbitrarily the local maxima and minima, and isotop the tangle so that each local minimum moves down to lie directly above its corresponding maximum. We can now decompose the tangle level by level into a composite of simple tangles in each of which there are \( n \) strings all running vertically, except for one pair, which either cross simply, giving \( G_i^{\pm1} \), or form a paired minimum and maximum, giving \( E_i \). An example is shown in figure 2.8.

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{figure2.8.png}
\end{array}
\end{array}
\]

Figure 2.8

**Remark.** It is useful to regard the tangle \( T_c \) with \( r \) through strings as a composite of an \((n,r)\)-tangle and an \((r,n)\)-tangle, and it suggests that a counterpart of \((n,r)\)-tangles might helpfully be studied in relation to \( BW_n \).

## 3 Kauffman’s link polynomial

In this section we discuss Kauffman’s Dubrovnik invariant of links, and its relation to the solutions of the Yang-Baxter equation for the orthogonal group.

Kauffman’s polynomial, in its Dubrovnik form, is a non-zero function \( D : U_0^0 \rightarrow \Lambda \), i.e. a function on knot diagrams which is unaltered by regular isotopy.

This function \( D \) has the basic properties:

1. \( D(K^+) - D(K^-) = z(D(K^0) - D(K^\infty)) \) (skein relation)
   where the diagrams \( K^\pm \), \( K^0 \) and \( K^\infty \) differ only as in figure 2.3, and

2. \( D(K_{\text{left}}) = \lambda D(K), \quad D(K_{\text{right}}) = \lambda^{-1} D(K) \),
   where \( K_{\text{left}} \) and \( K_{\text{right}} \) are given from \( K \) as in figure 2.4.

   It also satisfies

3. \( D(K \amalg O) = \delta D(K) \),
   where \( K \amalg O \) is the union of \( K \) and a circle having no crossings with \( K \) or with itself, and \( \delta \in \Lambda \) satisfies \( \lambda^{-1} - \lambda = z(\delta - 1) \).
Proposition 3.1 Kauffman’s invariant exists if and only if the cyclic module $MT_0$ is free.

Proof: We have shown already that $MT_0$ is cyclic, so $MT_0$ is free if and only if there is a non-zero $\Lambda$-homomorphism $\varphi : MT_0 \to \Lambda$.

If $MT_0$ is free then we may define $D$ on any diagram $K$ by $D(K) = \varphi(K)$. Conversely, if $D$ satisfies (1)-(3) then it defines a non-zero $\Lambda$-homomorphism $D : MT_0 \to \Lambda$. □

Remark. (Uniqueness of Kauffman’s invariant) It follows simply from section 2 that Kauffman’s invariant is unique, because $MT_0$ is cyclic. It is determined uniquely by its value on $O$, the diagram of the unknot without any crossings. $D$ was originally normalised so that $D(O) = 1$. It now appears more natural to assign the value 1 to the ‘empty knot’, so that $D(O) = \delta$.

Kauffman’s original proof of the existence of $D$, [8], requires a considerable amount of combinatorial argument to show that the elements of $\Lambda$ reached by different routes from a given diagram $K$ are independent of any intermediate choices.

We note here an alternative existence proof, using the Yang-Baxter orthogonal invariants.

Proposition 3.2 There exists a regular isotopy invariant of knot diagrams in $\mathbb{Z}[s^{\pm 1}]$ which satisfies relations (1)-(3) with $z = s - s^{-1}$, $\lambda = s^{2n-1}$, $\delta = 1 + (\lambda - \lambda^{-1})/z$, and takes the value 1 on the empty knot.

Proof (Turaev): The invariant is constructed from the $q$-analogue of the fundamental representation of the Lie algebra of $SO(2n)$.

For each $n$ we have a ring homomorphism $e_n : \Lambda \to \mathbb{Z}[s^{\pm 1}]$ defined by $e_n(\lambda) = s^{2n-1}$, $e_n(z) = s - s^{-1}$. Turaev’s invariant then defines a map $\varphi_n : MT_0 \to \mathbb{Z}[s^{\pm 1}]$ with $\varphi_n(aK) = e_n(a)\varphi_n(K)$ for $a \in \Lambda$.

Proposition 3.3 $MT_0$ is a free $\Lambda$-module.

Proof: Suppose not. Then there exists $a \in \Lambda$, $a \neq 0$ such that $aK = 0$, where $K$ is the empty diagram. Now $\varphi_n(K) = 1$ so $0 = \varphi_n(aK) = e_n(a)$ for all $n$. This is impossible, since for any given $a \neq 0$ there exists $n$ with $e_n(a) \neq 0$.

This proves the existence of $D$, given Turaev’s invariants. In principle $D(K)$ could be calculated explicitly for a given link diagram $K$ from knowledge of the invariants $\varphi_n(K)$ for sufficiently many $n$, as follows:
Proof: We know that any element $a$ of $\Lambda$ can be written as a polynomial in $\lambda^\pm 1$, $z$ and $\delta$. Now $z\delta = \lambda^{-1} - \lambda + z$, so $z^k a$ can be rewritten as a polynomial in $\lambda^\pm 1$ and $z$ alone, for large enough $k$.

A simple induction, as in theorem 2.6, shows that $z|\mathcal{D}(K) \in \Lambda$ can always be written as a polynomial in $\lambda^\pm 1$ and $z$; say

$$z|\mathcal{D}(K) = \sum_{r=m}^{M} \lambda^r P_r(z)$$

$$= \sum_{r=m}^{M} \lambda^r Q_r(s),$$

where $Q_r(s) = P_r(s - s^{-1})$.

It is then enough to find $Q_r(s)$, $m \leq r \leq M$.

Now for each $n$,

$$\sum_{r=m}^{M} s^r (2n-1) Q_r(s) = e_n(z|\mathcal{D}(K))$$

$$= (s - s^{-1})|\mathcal{D}(K) \varphi_n(K).$$

Write $V$ for the $k \times k$ Vandermonde matrix with entries

$$s^{(2n-1)r}, \ 1 \leq n \leq k, \ m \leq r \leq M, \ \text{with} \ k = M - m + 1.$$ 

Then

$$V \left(\begin{array}{c} Q_m \\ Q_{m+1} \\ \vdots \\ Q_M \end{array}\right) = (s - s^{-1})|\mathcal{D}(K) \left(\begin{array}{c} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_k \end{array}\right).$$

Since $V$ is invertible, we have $Q_m, \ldots, Q_M$, and hence $\mathcal{D}(K)$ in terms of $\varphi_1, \ldots, \varphi_k$. \qed

In order to make these calculations explicitly we need bounds for $m$ and $M$, in terms of $K$. It is certainly sufficient to note that $|m|, M \leq |K| + c(K)$, where $c(K)$ is the number of crossings in the diagram, although these bounds may turn out to be quite generous.

4 A basis for the tangle algebra

In this section we set out the induction to be used in proving that the algebra $BW_n$ defined by generators and relations is isomorphic to the Kauffman algebra, defined by tangles. We start by reviewing the position for $MT_n$. 

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The algebra $MT_n$ is shown in [11] to be free over $\Lambda$, of the same dimension, $|C_n|$, as Brauer’s algebra $A_n$. The proof, which we give here, is an easy consequence of the existence, however established, of Kauffman’s Dubrovnik invariant $D : MT_0 \to \Lambda$. We make use of the homomorphism $e : \Lambda \to \mathbb{Z}[\delta]$.

**Proposition 4.1** $e(D(K)) = \delta^{|K|}$.

**Proof**: It follows from condition (1) that $e(D(K))$ is unaltered when any crossing in a diagram is switched, and from (2) that it is unaltered by Reidemeister move I. Now any diagram can be changed to any other with the same number of components by a sequence of crossing switches and Reidemeister moves, so $e(D(K)) = e(D(K'))$, where $K'$ is the disjoint union of $|K|$ simple closed curves, giving the result by (3).

**Theorem 4.2** Any set of tangles $\{T_c\}$, $c \in C_n$, without closed components, having $c = \text{conn}(T_c)$ and spanning $MT_n$ forms a free $\Lambda$-basis for $MT_n$.

**Proof**: Define a bilinear map $b : MT_n \times MT_n \to \Lambda$ by $b(S, T) = D(e(ST))$.

Write $A$ for the $|C_n| \times |C_n|$ matrix with entries $a_{cd} = b(T_c, T_d)$.

Suppose that $\Sigma \lambda_i T_i = 0$, $\lambda_i \in \Lambda$. We want to show that $\lambda_i = 0$ for all $i$. For each $c \in C_n$ replace the $c$th column of $A$ by the linear combination of the columns of $A$ with coefficients $\lambda_i$. The new matrix then has determinant $\lambda_c \det A$ and a zero column. The required result follows by proving that $\det A \neq 0$, since $\Lambda$ has no zero-divisors.

Now $e(T_c T_d) \in MT_0$ is represented by a link with $r$ components, say. Each component contains at least one arc from $T_c$ and one from $T_d$, so $r \leq n$. When $r = n$ each component must have exactly one arc from each, so that the connector $d$ is the ‘mirror image’ of $c$, given by interchanging the roles of the top and bottom points. Set $\overline{c} = d$ in this case, so that we have $r = n$ if and only if $d = \overline{c}$.

Now apply the homomorphism $e$ to the entries in $A$. Then, by proposition 4.1, $e(a_{cd}) = \delta^r$, $r \leq n$, and $r = n$ if and only if $d = \overline{c}$. The matrix $e(A)$ has then one entry $\delta^n$ in each row and column, so $e(\det A) = \det(e(A)) \in \mathbb{Z}[\delta]$ has a non-zero coefficient for $\delta^n$. Thus $e(\det A) \neq 0$, so $\det A \neq 0$. This shows that $MT_n$ is a deformation of Brauer’s algebra $A_n$, in the following sense.

**Theorem 4.3** There is an isomorphism of $\mathbb{Z}[\delta]$-algebras induced by $c$ between $MT_n \otimes_{\Lambda} \mathbb{Z}[\delta]$ and $A_n$.  

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Proof: The map $c : MT_n \to A_n$, defined in section 2, factors through a $\mathbb{Z}[\delta]$-homomorphism $MT_n \otimes_\Lambda \mathbb{Z}[\delta] \to A_n$. Since $MT_n \otimes_\Lambda \mathbb{Z}[\delta]$ is spanned over $\mathbb{Z}[\delta]$ by \{\(T_c\)\} which maps onto a basis of $A_n$ of the same cardinality, this set must be a $\mathbb{Z}[\delta]$-basis in the specialisation, and the map is hence an isomorphism.

Corollary 4.4 (to theorem 4.2) Any set of tangles with distinct connectors forms an independent set in $MT_n$.

Proof: We have shown that $c : MT_n \to A_n$ carries a free $\Lambda$-basis to a free $\mathbb{Z}[\delta]$-basis. It follows, using determinantal criteria for independence as in theorem 4.2, that $k$ elements of $MT_n$ whose images are independent in $A_n$ must themselves be independent.

We shall prove by induction on $n$ that the homomorphism $\varphi : BW_n \to MT_n$ is an isomorphism. In the course of the proof we shall construct explicit bases $\varphi^{-1}\{T_c\}$ in $BW_n$. As part of the induction we shall use natural filtrations $BW_n^{(r)}$ and $MT_n^{(r)}$ by 2-sided ideals, analogous to the filtration of $A_n$ used by Hanlon and Wales, [7]. In the case of $MT_n$ this filtration arises from the geometric viewpoint, as in [11], when we consider tangles of rank $\leq r$.

Definition. A tangle $T \in U_n^n$ has rank $\leq r$ if it is the composite $T = AB$ of an $(n,r)$ and an $(r,n)$ tangle.

Remark. Then $\text{conn}(T)$ has at most $r$ arcs connecting top to bottom. However this is not sufficient for $T$ to have rank $r$. For example, the tangle $T$ in figure 4.1 has rank 2, although $\text{conn}(T)$ has no connecting arcs from top to bottom.

![Figure 4.1](image)

Write $MT_n^{(r)}$ for the subspace of $MT_n$ spanned by tangles of rank $\leq r$. Clearly $MT_n^{(r)}$ is a 2-sided ideal, with

$$MT_n = MT_n^{(n)} \supset MT_n^{(n-2)} \supset \ldots.$$
Proposition 4.5 \( MT_{n}^{(r)} \) is generated, as an ideal, by the element \( E_{1}E_{3}...E_{2k-1} \), where \( 2k = n - r \).

Proof: For \( r > 0 \) we can write the identity tangle in \( U_{r} \) as
\[
I = C(E_{1}E_{3}...E_{2k-1})D,
\]
where \( C \) is an \((r,n)\) tangle and \( D \) is an \((n,r)\) tangle, as in figure 4.2.

Then any tangle \( T = AB \) of rank \( \leq r \) can be written as \( T = AC(E_{1}E_{3}...E_{2k-1})BD \) with \( AC, BD \in U_{r}^{n} \).

The case \( r = 0 \) can be handled similarly, by first writing a tangle \( T \) of rank 0 as \( T = AE_{1}B \) where \( A \) is an \((n,2)\) tangle and \( B \) is a \((2,n)\) tangle. \( \square \)

Definition. For \( r = n - 2k \) write \( BW_{n}^{(r)} \) for the 2-sided ideal of \( BW_{n} \) generated by \( e_{1}e_{3}...e_{2k-1} \).

Then
\[
BW_{n} = BW_{n}^{(n)} \supset BW_{n}^{(n-2)} \supset \ldots.
\]
Clearly \( \varphi : BW_{n} \rightarrow MTT_{n} \) restricts to \( \varphi : BW_{n}^{(r)} \rightarrow MTT_{n}^{(r)} \).

Our main result, that \( \varphi \) is an isomorphism, follows from

Theorem 4.6 \( \varphi : BW_{n}^{(r)} \rightarrow MTT_{n}^{(r)} \) is injective for all \( n,r \).

Proof: The detailed lemmas needed appear in later sections. The scheme of the proof follows here.

For fixed \( n \) we prove the result for \( r = 0, 1 \) in section 5 from the injectivity of \( \varphi \) on \( BW_{n-1} (= BW_{n-1}^{(n-1)}) \) using induction on \( n \).

The proof then continues by induction on \( r \).

For this induction we construct a linear subspace \( V_{n}^{(r)} \subset BW_{n}^{(r)} \), complementing \( BW_{n}^{(r-2)} \). The induction step for injectivity of \( \varphi \) follows by establishing:
(1) $V_n^{(r)} + BW_n^{(r-2)}$ is a 2-sided ideal in $BW_n^{(r)}$,
(2) $\phi | V_n^{(r)} \rightarrow MT_n$ is injective,
(3) $e_1 e_3 \ldots e_{2k-1} \in V_n^{(r)}$.

In the construction, given later in this section, we exhibit an explicit spanning set for $V_n^{(r)}$ whose image in $MT_n$ is an independent set of totally descending tangles. This establishes property (2).

Property (3) is immediate from the construction, and property (1) is proved in section 6. \qed

To describe certain elements in $BW_n$ we now draw on Artin’s braid group.

The braid group on $n$ strings, defined by geometric braids, (particular types of $(n,n)$ tangles), is known to have the presentation with generators $\sigma_i$, $i \leq n$ and relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i - j| > 1, \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}.$$

There is then a homomorphism $\psi : B_n \rightarrow BW_n$ defined by $\sigma_i \mapsto g_i$. Any two monomials in $BW_n$ in $g_i^{\pm 1}$ which arise from the same geometric braid $\beta$ will then be equal, and we shall use $\beta$ to picture the element $\psi(\beta)$. We shall also refer to monomials in $g^{\pm 1}$ as braids in $BW_n$.

There is an antihomomorphism $\text{perm} : B_n \rightarrow S_n$ defined by $\text{perm}(\sigma_i) = \tau_i = (i \ i + 1)$. With our convention of composition of geometric braids, the strings in a braid $\beta$ then join the point $i$ at the top to the point $\pi(i)$ at the bottom, with $\pi = \text{perm}(\beta)$.

Among the elements of $B_n$ we shall use particularly the positive permutation braids and, as special cases, the Lorenz braids.

**Definition.** A braid in $B_n$ in which all crossings are positive and every pair of strings crosses at most once is called a positive permutation braid.

**Theorem 4.7** A positive permutation braid $\beta$ is determined by the permutation $\pi = \text{perm}(\beta)$ induced by its strings.

**Proof:** Such braids are examples of ‘totally descending tangles’, as defined in section 2, in which the arcs of the connector all join top to bottom and are ordered by the order of their initial points. \qed

We shall write $\beta_\pi$ for the positive permutation braid with permutation $\pi = \text{perm}(\beta_\pi)$, whose strings join the points $i$ at the top with $\pi(i)$ at the bottom. The element $b_\pi = \psi(\beta_\pi) \in BW_n$, which we shall also call a positive permutation braid, can be conveniently referred to by the permutation $\pi$, rather than choosing one of the many ways of writing it as a
monomial in $g_i$. For example, the permutation $\pi = (14)(23) \in S_4$ gives $b_{\pi} = g_1 g_2 g_3 g_1 g_4 = g_2 g_1 g_2 g_4 g_1 = \ldots$

**Definition.** A Lorenz braid of type $(\ell, r)$ is a braid $\beta_\pi$ where $\pi \in S_n$, $n = \ell + r$, does not permute the first $\ell$ ‘left-hand’ strings, or the last $r$ ‘right-hand’ strings among themselves.

For fixed $(\ell, r)$ there are $\binom{n}{r}$ Lorenz braids, as a Lorenz permutation $\pi$ is determined simply by the free choice of endpoints for the right-hand strings.

Note that $\pi$ is an $(\ell, r)$ Lorenz permutation if and only if $\pi(i) < \pi(j)$ for $1 \leq i < j \leq \ell$ and for $\ell + 1 \leq i < j \leq n$. An example of a $(3, 4)$ Lorenz braid is shown in figure 4.3.

Where $\pi^{-1}$ is a Lorenz permutation the braid $\beta_\pi = \alpha(\beta_{\pi^{-1}})$ can be viewed as a Lorenz braid $\beta_{\pi^{-1}}$ turned upside down. Call $\alpha(\beta_{\pi^{-1}})$ a reverse Lorenz braid. Note that $(\beta_{\pi})^{-1}$ is not the same braid as $\beta_{\pi^{-1}}$ but has all the crossings switched.

**Definition.** For each $n$ and $r = n - 2k$ write $V_n^{(r)}$ for the linear subspace of $BW_n^{(r)}$ spanned by elements $b_{\pi}w_{2k}b_{\tau}b_{\mu}$, where $\pi, \mu$ are $(2k, r)$ Lorenz permutations, $\tau$ is a permutation of the last $r$ strings only, and $w_{2k} \in BW_{2k}^{(0)}$.

**Proposition 4.8** Given that $\varphi|BW_{2k}^{(0)}$ is injective for $n \geq 2k$ then $\varphi|V_n^{(r)} \rightarrow MT_n$ is injective.

**Proof:** We know that $MT_{2k}^{(0)}$ is spanned by $|C_k|^2$ totally descending tangles, one for each $k$-connector of rank 0, and that $\varphi|BW_{2k}^{(0)} \rightarrow MT_{2k}^{(0)}$ is surjective. By hypothesis we can choose a spanning set of $|C_k|^2$ elements for $BW_{2k}^{(0)}$ with this set of tangles as image.

Then $V_n^{(r)}$ is spanned by the $\binom{n}{r}^2|C_k|^2 r!$ elements $b_{\pi}w_{2k}b_{\tau}b_{\mu}$, where $\pi^{-1}, \mu$ are drawn independently from $(2k, r)$ Lorenz permutations, $\tau$ from permutations in $S_r$ and $w_{2k}$ from the spanning set for $BW_{2k}^{(0)}$.

The elements $\varphi(b_{\pi}w_{2k}b_{\tau}b_{\mu})$ are represented by tangles in $MT_n^{(r)}$ each with exactly $r$ through strings, and all having different connectors. A typical
such tangle with $k = 2, \ r = 4$ is illustrated in figure 4.4. It follows by the corollary to theorem 4.2 that these tangles are independent in $MT_n$, and hence that $\varphi|V_n^{(r)}$ is injective.

This establishes property (2) of theorem 4.6 under its induction hypothesis.

From theorem 4.6 we eventually build a basis for $BW_n$ as a union of spanning sets for each $V_n^{(r)}$. The image of this basis in $MT_n$ can be represented by a set of tangles each with a different connector, and each totally descending, for some ordering of the arcs.

We note that this gives a complicated check that the dimension of $BW_n$ is

$$|C_n| = \sum_{k=0}^{[n/2]} \binom{n}{r}^2 |C_k|^2 r!,$$

where we write $r = n - 2k$.

5 Generators and relations for the tangle algebra: the base for induction

In this section we prove injectivity of $\varphi$ on $BW_n^{(0)}$ or $BW_n^{(1)}$, depending on the parity of $n$, given injectivity of $\varphi|BW_{n-1}$. The corresponding sets of tangles in $MT_n$ are those with at most one through string.

We start with some results in $BW_n$ which use only the regular isotopy relations.

**Definition.** The shift map $S : MT_n \to MT_{n+1}$ is a homomorphism defined on an $n$-tangle $T$ as shown in figure 5.1.

\[
S(T) = \begin{array}{c}
T
\end{array}
\]
Thus \( S(G_i) = G_{i+1}, \ S(E_i) = E_{i+1}. \)

It is clear, from the behaviour on tangles, as shown in figure 5.2, that \( WAM = AMS(W) \) for \( W \in MT_m \), where \( AM = GMG_{m-1} \ldots G_1. \)

![Figure 5.2](image)

We can define a shift map with similar properties in \( BW \) as follows.

**Definition.** The shift map \( S : BW_n \to BW_{n+1} \) is defined as a homomorphism by

\[
S(g_i) = g_{i+1}, \ S(e_i) = e_{i+1},
\]

extended linearly.

It is simply necessary to check that the relations are respected by \( S. \)

**Proposition 5.1** The homomorphism \( S \) satisfies

\[
wa_m = a_mS(w), \ wb_m = b_mS(w)
\]

for any \( w \in BW_m \), where

\[
a_m = gmgm−1 \ldots g1, \ b_m = g−1m g−1m−1 \ldots g−11.
\]

**Proof:** When \( w = g_i^{±1} \) or \( w = e_i \) the result is an immediate consequence of the relations, and it follows for monomials \( w \) by induction on their length. \( \square \)

We now define \( F_k \in MT_n, \ 2k \leq n, \) to be the element represented by the tangle shown in figure 5.3.

![Figure 5.3](image)

The following equations in \( MT_n \) are clear from inspection of representative tangles.
Proposition 5.2  For all $i < k$

1. $G_i F_k = G_{2k-i} F_k$,
2. $E_i F_k = E_{2k-i} F_k$,
3. $F_k G_i = F_k G_{2k-i}$,
4. $F_k E_i = F_k E_{2k-i}$.

An example of equation (1) is illustrated in figure 5.4, with $i = 1$ and $k = 3$.

Again it is clear from inspection of the tangles, as shown in figure 5.5, that

$$F_k = \alpha (A_{2k-2} F_{k-1} E_{2k-1} A_{2k-2}).$$

By analogy we define $f_k \in BW_n$ inductively, setting $f_0 = \text{identity}$, and

$$f_k = \alpha (a_{2k-2}) f_{k-1} e_{2k-1}^2 a_{2k-2}.$$

We then have $F_k = \varphi (f_k)$, and $\alpha (f_k) = f_k$, since $f_{k-1}$ and $e_{2k-1}$ commute.

Remark. While it is clear that $\rho_{2k}(F_k) = F_k$ in $MT_n$, it is difficult to prove directly from the definition and relations in $BW_n$ that $\rho_{2k}(f_k) = f_k$.

Proposition 5.3  $BW_n^{(n-2k)} \subset BW_n$ is the 2-sided ideal generated by $f_k$.

Proof: By definition $BW_n^{(n-2k)}$ is the 2-sided ideal generated by $e_1 e_3 \ldots e_{2k-1}$. By induction on $k$ we can write $f_k = \alpha (b_k) e_1 e_3 \ldots e_{2k-1} b_k$ for some invertible element $b_k \in BW_{2k}$, in fact $b_k$ can be chosen to be a braid. □

We now make use of the relations in $BW_n$ to prove the analogous results to proposition 5.2. 23
Proposition 5.4 For all \( i < k \)

1. \( g_i f_k = g_{2k-i} f_k \),
2. \( e_i f_k = e_{2k-i} f_k \),
3. \( f_k g_i = f_k g_{2k-i} \),
4. \( f_k e_i = f_k e_{2k-i} \).

The same results hold with \( \rho_{2k}(f_k) \) in place of \( f_k \).

Proof: Cases (3) and (4) follow from (1) and (2) by applying \( \alpha \). Applying \( \rho_{2k} \) gives the results for \( \rho_{2k}(f_k) \). The result is immediate for \( k = 1 \). For \( i > 1 \) the result follows from proposition 5.1 by induction on \( k \).

For example, in case (1),
\[
    g_i f_k = \alpha(a_{2k-2}) f_{k-1} e_{2k-1} a_{2k-2} \\
    = \alpha(a_{2k-2}) g_{k-1} f_{k-1} e_{2k-1} a_{2k-2}, \text{ by applying } \alpha \text{ to 5.1} \\
    = \alpha(a_{2k-2}) g_{2k-i-1} f_{k-1} e_{2k-1} a_{2k-2}, \text{ by induction} \\
    = g_{2k-i} \alpha(a_{2k-2}) f_{k-1} e_{2k-1} a_{2k-2}, \text{ } (i \geq 2) \\
    = g_{2k-i} f_k.
\]

To prove 5.4 when \( i = 1 \) we set \( h_j = \alpha(a_j) \alpha(a_{j-2}) e_{j+1} e_{j-1} \).

Since \( f_k = h_{2k-2} f_{k-2} a_{2k-4} a_{2k-2} \), the result for cases (1) and (2) will follow by showing that

1. \( g_1 h_j = g_{j+1} h_j \)
2. \( e_1 h_j = e_{j+1} h_j \)

for all \( j \).

We prove (1') and (2') by induction on \( j \), starting with \( j = 2 \). For \( j = 2 \) we have
\[
    h_2 = g_1 g_2 e_1 e_3 = e_2 e_1 e_3 = e_2 e_3 e_1 = g_3 g_2 e_3 e_1.
\]
Then \( g_3 h_2 = g_3 g_1 g_2 e_1 e_3 = g_1 h_2 \) and \( e_1 h_2 = e_1 e_2 e_1 e_3 = e_1 e_3 = e_3 h_2 \).

For the induction step, use the braid relations to write
\[
    \alpha(a_j) \alpha(a_{j-2}) = g_2 g_1 S(\alpha(a_{j-1}) \alpha(a_{j-3})).
\]
(Compare the two braids illustrated in figure 5.6.)
Then $h_j = g_2 g_1 S(h_{j-1})$. So

$g_1 h_j = g_1 g_2 g_1 S(h_{j-1}) = g_2 g_1 S(g_1 h_{j-1})$,

by induction on $j$,

$= g_2 g_1 S(g_j h_{j-1}) = g_{j+1} h_j$, for $j > 2$.

Similarly $e_1 h_j = e_{j+1} h_j$, using the relation in $BW_n$ that $e_1 g_2 g_1 = g_2 g_1 e_2$. □

**Lemma 5.5** Suppose that $\varphi : BW_{m+1} \rightarrow MT_{m+1}$ is injective. Then $BW_{m+1} e_m = BW_m e_m$.

**Proof :** By hypothesis it is enough to prove the corresponding result

$$MT_{m+1} E_m = MT_m E_m.$$  

For an $(m+1,m+1)$ tangle $T$ define $\varepsilon_m(T)$ to be the $(m,m)$ tangle shown in figure 5.7.

![Figure 5.7](image)

Using the standard interpretation of $\varepsilon_m(T)$ as an $(m+1,m+1)$ tangle it is clear that $\varepsilon_m(T) E_m = TE_m$. Extend the definition of $\varepsilon_m$ to linear combinations of tangles to define a linear map $\varepsilon_m : MT_{m+1} \rightarrow MT_m$, (the relations are respected). Then any element $XE_m$ with $X \in MT_{m+1}$ can be rewritten as $XE_m = \varepsilon_m(X) E_m \in MT_m E_m$. □

**Corollary 5.6** Under the same conditions, $BW_{m+1} e_1 = S(BW_m) e_1$.

**Proof :** Apply the automorphism $\rho_{m+1}$. □

**Proposition 5.7** Suppose that $\varphi : BW_{n-1} \rightarrow MT_{n-1}$ is injective. Then

1. $BW_{2k} f_k = BW_k f_k$, for all $k$ with $2k \leq n$,
2. $BW_{2k+1} S(f_k) = BW_{k+1} S(f_k)$, for all $k$ with $2k + 1 \leq n$.  

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Proof:

(1) The case \( k = 1 \) is immediate, since \( g_1^{\pm 1}e_1 \) and \( e_1e_1 \) are multiples of \( e_1 \).

For \( k \geq 2 \) we have \( k + 1 \leq n - 1 \) so that \( BW_{k+1}e_k = BW_ke_k \) by lemma 5.5. It is enough to show that

\[
\begin{align*}
g_iBW_kf_k & \subset BW_kf_k \\
e_iBW_kf_k & \subset BW_kf_k, \text{ for all } i < 2k.
\end{align*}
\]

This is immediate for \( i < k \). For \( i > k \) it follows from 5.4 and the fact that \( BW_k \) then commutes with \( e_i \) and \( g_i \).

Write \( f_k = e_kr_k \) for some \( r_k \in BW_{2k} \), by induction on \( k \). The remaining cases with \( i = k \) follow by noting that

\[
\begin{align*}
g_kBW_ke_k & \subset BW_{k+1}e_k = BW_ke_k \\
e_kBW_ke_k & \subset BW_{k+1}e_k = BW_ke_k.
\end{align*}
\]

(2) The case \( k = 1 \) will be proved directly.

For \( k \geq 2 \) we have \( k + 2 \leq n - 1 \) so that \( BW_{k+2}e_{k+1} = BW_{k+1}e_{k+1} \) from lemma 5.5. We must show that

\[
\begin{align*}
g_iBW_{k+1}S(f_k) & \subset BW_{k+1}S(f_k), \\
e_iBW_{k+1}S(f_k) & \subset BW_{k+1}S(f_k), \text{ for } i \leq 2k.
\end{align*}
\]

This is immediate for \( i < k + 1 \). For \( i > k + 1 \) it follows from proposition 5.4 since \( BW_{k+1} \) commutes with \( g_i \) and \( e_i \). The remaining cases follow as in (1), since \( S(f_k) = e_{k+1}S(r_k) \).

We finish the proof of (2) by showing that \( BW_3e_2 = BW_2e_2 \). Now \( BW_2e_2 \) is spanned by \( e_2, e_1e_2 \) and \( g_1e_2 \), so we must show that products of these elements with \( g_2 \) or \( e_2 \) on the left still lie in \( BW_2e_2 \). It is a matter of a quick check from the relations in \( BW_3 \), to see that \( e_2^2 = \delta e_2, e_2e_1e_2 = e_2, e_2g_1e_2 = \lambda^{-1}e_2, g_2e_1e_2 = \lambda e_2, 2ge_1e_2 = g_1^{-1}e_2 \) and \( g_2g_1e_2 = e_1e_2 \). \( \square \)

**Corollary 5.8** Suppose that \( \varphi|BW_{n-1} \) is injective. Then the ideals generated by \( f_k \) in \( BW_n \), with \( k = [n/2] \), can be written as:

(1) \( BW_{2k}^{(0)} = BW_kf_kBW_k \) when \( n = 2k \), and

(2) \( BW_{2k+1}^{(1)} = BW_{k+1}S(f_k)BW_{k+1} \) when \( n = 2k + 1 \).

Proof (1):

\[
\begin{align*}
BW_{2k}^{(0)} & = BW_{2k}f_kBW_{2k} \\
& = BW_kf_kBW_{2k} \text{ by } 5.7 \\
& = BW_kf_kBW_k \text{ applying } \alpha \text{ to } 5.7
\end{align*}
\]
Proof (2): The ideal \( BW_{2k+1}^{(1)} \) generated by \( f_k \) is equally generated by \( S(f_k) = a_{2k}^{-1} f_k a_{2k} \) so the result follows using [5.7] (2) exactly as in (1). \( \Box \)

We complete this section by showing the injectivity of \( \varphi \) on the 2-sided ideals generated by \( f_k \) in \( BW_n, \ k = [n/2] \), given injectivity on \( BW_{n-1} \).

**Theorem 5.9** Suppose that \( \varphi|BW_{n-1} \to MT_{n-1} \) is injective. Then \( \varphi|BW_{2k}^{(0)} \to MT_{2k}^{(0)} \) is injective, when \( n = 2k \), and \( \varphi|BW_{2k+1}^{(1)} \to MT_{2k+1}^{(1)} \) is injective, when \( n = 2k + 1 \).

**Proof:** In the case \( n = 2k \) we know that \( \varphi|BW_k \) is an isomorphism to \( MT_k \). We may then choose elements \( t_c \in BW_k, c \in C_k \), spanning \( BW_k \), with \( \varphi(t_c) \) represented by a totally descending tangle \( T_c \) say, having connector \( c \).

By corollary [5.8] we have \( BW_{2k}^{(0)} = BW_k f_k BW_k \). This is spanned by \( |C_k|^2 \) elements \( t_c f_k t_d, c, d \in C_k \). It is enough to prove that the images of these elements are independent in \( MT_{2k} \).

Now these images are represented by the tangles \( T_c F_k T_d \). Different pairs of connectors \( (c, d) \) give tangles \( T_c F_k T_d \) with different connectors in \( C_{2k} \), since the tangles consist of a top and a bottom half, each with \( k \) arcs, affected independently by the connectors \( c \) and \( d \). The tangles then represent independent elements in \( MT_{2k} \), by corollary [4.4].

Similarly when \( n = 2k + 1 \) we know that \( \varphi|BW_{k+1} \) is an isomorphism to \( MT_{k+1} \). We may then choose spanning elements \( t_c \in BW_{k+1}, c \in C_k \), with \( \varphi(t_c) \) represented by a totally descending tangle \( T_c \) say, having connector \( c \). Again, by corollary [5.8] we have a spanning set \( \{ t_c S(f_k) t_d, c, d \in C_{k+1} \} \) with \( |C_{k+1}|^2 \) elements, for the ideal \( BW_{2k+1}^{(1)} \).

The images of these elements are represented by tangles \( T_c S(F_k) T_d \). Once more we can see that different pairs of connectors \( (c, d) \) give tangles with different connectors in \( C_{2k+1} \) because all but one of the arcs stays either in the top or in the bottom of the tangle. This guarantees independence in \( MT_{2k+1} \), as before. \( \Box \)

**Remark.** We could in fact show that the composite tangles used in this proof are themselves totally descending, for some suitable ordering of their arcs.

We continue in the next section to examine \( BW_n^{(r)} \) for larger \( r \) having established here the start of our induction on \( r \). Note that we could prove similarly that \( BW_{2k+r}^{(r)} = BW_{k+r} S^r(f_k) BW_{k+r} \) and find a spanning set of \( |C_{k+r}|^2 \) elements. However, a similar attempt to prove the (false) result for
that these are independent would fail, because some different pairs of
connectors in $C_{k+r}$ can yield the same connector in $C_{2k+r}$.

6 Isomorphism between Kauffman’s tangle algebras and the Birman-Wenzl algebras

We finish the proof of injectivity of $\varphi : BW_n \to MT_n$ by proving the remaining induction step, namely that if $\varphi|BW_{n-1}$ is injective, and $\varphi|BW_{n-r-2}$ is injective then $\varphi|BW_{n-r}$ is injective. We do this by finding a complementary subspace $V_n^{(r)}$ to $BW_n^{(r-2)}$ in $BW_n^{(r)}$ on which $\varphi$ is injective.

We recall the definition of $V_n^{(r)}$ given in section 4 as the subspace spanned by 
\[ \{ b_\pi b_\tau BW_{2k} f_k \} \]
where $\alpha(b_\pi)$ is a $(2k, r)$ Lorenz braid and $b_\tau$ is a positive permutation braid in $S_{2k}(B_r)$. Following the scheme of proof in theorem 4.6 we already know, by induction on $n$, that $\varphi|V_n^{(r)}$ is injective.

It remains to show that $V_n^{(r)} + BW_n^{(r-2)} = BW_n^{(r)}$. Since $BW_n^{(r)}$ is the 2-sided ideal generated by $f_k$, and $f_k \in V_n^{(r)}$ we need only show that $V_n^{(r)} + BW_n^{(r-2)}$ is a 2-sided ideal. Now $\alpha(V_n^{(r)}) = V_n^{(r)}$, since the elements $b_\tau$ in $S_{2k}(BW_r)$ commute with $BW_{2k}$. Hence it is enough to show that $V_n^{(r)} + BW_n^{(r-2)}$ is a left ideal.

**Proposition 6.1** Let $n = r + 2k$ and let $X_n^{(r)}$ be the subspace spanned by the set \{ $b_\pi b_\tau BW_{2k} f_k$ \}, where $\alpha(b_\pi)$ is a $(2k, r)$ Lorenz braid and $b_\tau$ is a positive permutation braid in $S_{2k}(B_r)$. Suppose also that $\varphi|BW_{n-1}$ is injective and that $r \geq 2$. Then
\[ L_n^{(r)} = X_n^{(r)} + BW_n^{(r-2)} \]
is a left ideal.

**Corollary 6.2** $V_n^{(r)} + BW_n^{(r-2)}$ is a left ideal, under the hypotheses of proposition 6.1, and hence theorem 4.6 is established.

**Proof :** Since $L_n^{(r)}$ is a left ideal, by 6.1 it follows that $V_n^{(r)} + BW_n^{(r-2)}$ is a left ideal, by noting that $BW_{2k}^{(0)} = BW_{2k} f_k BW_{2k}$. □

The proof of proposition 6.1 occupies the rest of this section. The principal ingredient is an analysis of the elements $g b_\pi$ and $e b_\pi$ for positive permutation braids $b_\pi$. The following two lemmas are a consequence primarily of the braid relations.
Lemma 6.3 Let \( \rho \) be any permutation, and let \( \rho_1 \) be the permutation \( \rho \circ (i \ i + 1) \). Then the positive permutation braid \( b_{\rho_1} \) satisfies the equation

\[
\begin{align*}
    b_{\rho_1} &= g_i b_\rho & \text{if } \rho(i) < \rho(i + 1), \\
    b_{\rho} &= g_i b_{\rho_1} & \text{if } \rho(i) > \rho(i + 1).
\end{align*}
\]

Proof: If \( \rho(i) < \rho(i + 1) \) then each pair of strings in the braid \( g_i b_\rho \) crosses at most once, so it is a positive permutation braid. Its permutation is \( \rho_1 \), so \( g_i b_\rho = b_{\rho_1} \).

If \( \rho(i) > \rho(i + 1) \) then \( \rho_1(i) < \rho_1(i + 1) \) and the same argument holds with \( \rho_1 \) in place of \( \rho \). \( \square \)

Corollary 6.4 Any positive permutation braid \( b_\rho \) can be written as the product of a word in \( \{g_i\} \), \( i \neq \ell \), and an \((\ell, r)\) Lorenz braid.

Proof: By induction on the length of \( b_\rho \), using Lemma 6.3 to write \( b_\rho = g_i b_{\rho_1} \) for some \( i \neq \ell \) if \( b_\rho \) is not already an \((\ell, r)\) Lorenz braid. \( \square \)

Lemma 6.5 Let \( \rho \) be any permutation with \( \rho(i + 1) = \rho(i) + 1 \). Then \( g_i b_\rho = b_\rho g_{\rho(i)} \) and \( e_i b_\rho = b_\rho e_{\rho(i)} \).

Proof: This can be viewed as allowing us to pass a simple crossing along two parallel strings from top to bottom of a braid. By the hypothesis on \( \rho \), both \( g_i b_\rho \) and \( b_\rho g_{\rho(i)} \) are positive permutation braids, and both have the same permutation. Hence they are equal, using only the braid relations, by the fundamental theorem on positive permutation braids. It follows that \( g_i^{-1} b_\rho = b_\rho g_{\rho(i)}^{-1} \) and hence, by the skein relation, that \( z e_i b_\rho = z b_\rho e_{\rho(i)} \).

The lemma follows, if we assume that \( z \) is invertible in \( \Lambda \). Without inverting \( z \) the result follows by induction on the length of \( b_\rho \), together with the relation \( e_i g_i + g_i e_i = g_i + g_i e_i + 1 \) and its reverse in \( BW_n \). For we can write \( b_\rho = g_j b_{\rho_1} \) for some \( j \). Then \( j \neq i \), since the strings \( i \) and \( i + 1 \) do not cross under \( \rho \).

If \( j = i + 1 \) then \( \rho(i + 2) < \rho(i + 1) = \rho(i) + 1 \), so \( \rho(i + 2) < \rho(i) \). We can then, by Lemma 6.3, write \( b_\rho = g_i + g_i b_{\rho_2} \), and then \( e_i b_\rho = g_{i + 1} g_{i + 1} e_{i + 1} b_{\rho_2} \). Now \( \rho_2(i + 2) = \rho(i + 1) = \rho_2(i + 1) + 1 \) and \( b_{\rho_2} \) is shorter than \( b_\rho \), so that we can use induction.

A similar argument can be used when \( j = i - 1 \), while otherwise \(|i - j| > 2\), and \( e_i g_j = g_j e_i \), giving an immediate inductive proof. \( \square \)

Lemma 6.6 \( X_n^{(r)} S^{2k}(BW_r) \subset L_n^{(r)} \).

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Proof: \( X_n^{(r)} e_j \subset BW_n^{(r-2)} \subset L_n^{(r)} \) for \( j > 2k \), since \( f_k e_j \subset BW_n^{(r-2)} \) for \( j > 2k \).

Let \( b_\tau \) be any positive permutation braid in \( S^{2k}B_\tau \) and let \( j > 2k \). Then by 6.3, either

\[
 b_\tau g_j = b_\tau,
\]

or

\[
 b_\tau g_j = b_\tau g_j^2 = b_\tau + zb_\tau - zb_\tau e_j.
\]

Hence \( xg_j \in L_n^{(r)} \) for any spanning element \( x = b_\tau w_{2k} f_k b_\tau \in X_n^{(r)} \).

Thus \( X_n^{(r)} g_j \subset L_n^{(r)} \) for \( j > 2k \). \( \square \)

We now continue the proof of 6.1 to show that \( L_n^{(r)} \) is a left ideal. Lemma 6.6 shows in particular that \( L_n^{(r)} b_\tau \subset L_n^{(r)} \) for \( b_\tau \in S^{2k}(B_r) \). It is enough to show that \( e_i x, g_i x \in L_n^{(r)} \) for each \( x = b_\tau w_{2k} f_k \in X_n^{(r)} \) and each \( i \), where \( \alpha(b_\tau) \) is a \((2k, r)\) Lorenz braid and \( w_{2k} \in BW_{2k} \).

Suppose then that \( x \) and \( i \) are given. We may further suppose that \( \pi(i + 1) > \pi(i) \), otherwise \( b_\tau = g_i b_{\pi_1} \) with \( \pi_1(i + 1) > \pi_1(i) \). We then need only prove that \( e_i x' \in L_n^{(r)} \) where \( x' = b_{\pi_1} w_{2k} f_k \), since \( g_i x = g_i^2 x' = x' + zg_i x' - zg_i e_i x' = x' + xx - \lambda z e_i x' \) and \( e_i x = e_i g_i x' = \lambda e_i x' \) from the skein and delooping relations.

Since \( \pi \) is a reverse \((2k, r)\) Lorenz permutation then \( \pi(i + 1) = \pi(i) + 1 \) if either \( \pi(i + 1) \leq 2k \) or \( \pi(i) > 2k \). By 6.3, \( e_i x = b_\tau e_{\pi(i)} w_{2k} f_k \) in either case.

This lies in \( X_n^{(r)} \) if \( \pi(i) < 2k \) and in \( BW_n^{(r-2)} \) if \( \pi(i) > 2k \), and similarly \( g_i x \in L_n^{(r)} \). It remains to deal with \( e_i x \) and \( g_i x \) when \( \pi(i + 1) > 2k \) and \( \pi(i) \leq 2k \). In this case \( g_i b_\tau \) is a reverse \((2k, r)\) Lorenz braid, by 6.3, so that \( g_i x \in X_n^{(r)} \) and we are left to consider \( e_i x \).

Given \( \pi \) and \( i \), let \( \rho \) be the permutation given by

\[
\rho(j) = \begin{cases} 
2k & j = \pi(i), \\
\pi(i) < j \leq 2k & j + 1, \\
2k + 1 & j = \pi(i + 1), \\
\pi(i + 1) & j > \pi(i + 1), \\
\pi(i + 1) & j > \pi(i + 1)
\end{cases}
\]

Now \( \rho \) only makes pairs of strings cross which have not already been made to cross by the reverse Lorenz braid \( b_\pi \), so that \( b_\pi b_\rho \) is also a positive permutation braid. Then \( b_\pi b_\rho = b_{\pi_1} \), where \( \pi_1 = \rho \circ \pi \). Now \( \rho \) permutes the first \( 2k \) strings and the last \( r \) strings among themselves, moving \( \pi(i) \) to \( 2k \) and \( \pi(i + 1) \) to \( 2k + 1 \), so \( x = b_{\pi_1}(b_\rho)^{-1} w_{2k} f_k \) with \( b_\rho \in BW_{2k} S^{2k}(B_r) \). Note that \( \pi_1^{-1}(i) < \pi_1^{-1}(j) \) for \( i < j < 2k - 1 \).
It is enough, by lemma 6.6, to show that \( e\cdot x' \in L^{(r)}_n \), for \( x' = b_{\pi_1}w_2f_k \). Now \( \pi_1(i+1) = 2k + 1 = \pi_1(i) + 1 \) so, by 6.3, \( e\cdot x' = b_{\pi_1}e_{2k}w_2f_k \). This does not finish the proof, since the element \( e_{2k} \) is stuck between \( BW_{2k} \) and \( S^{2k}(BW_r) \) and we have to use our inductive knowledge of \( w_2f_k \in BW_{2k}^{(0)} \) to free it.

**Lemma 6.7** Suppose that \( \varphi|BW_{2k}^{(0)} \) is injective. Then every element in \( BW_{2k}f_k \) is a linear combination of elements in the sets

\[
g_mg_{m+1} \cdots g_{2k-2}e_{2k-1}BW_{2k}f_k, m = 1, \ldots, 2k - 2, \text{ and } e_{2k-1}BW_{2k}f_k.
\]

**Lemma 6.8** For each \( m = 1, \ldots, 2k - 2 \) and each positive permutation braid \( b_\rho \) with \( \rho^{-1}(i) < \rho^{-1}(j) \) for \( i < j \leq 2k-1 \) and \( \rho^{-1}(2k+1) = \rho^{-1}(2k)+1 \) we have

\[
b_\rho e_{2k}g_mg_{m+1} \cdots g_{2k-2}e_{2k-1} = b_{\rho'}e_{2k-1}
\]

for some positive permutation braid \( b_{\rho'} \).

Proposition 6.1 then follows from 6.7 and 6.8 since we can write the element \( e\cdot x' \) as a linear combination of elements of the form \( b_{\rho'}BW_{2k}f_k \). All of these lie in \( L^{(r)}_n \), since any positive permutation braid \( b_{\rho'} \) can be written as the product \( b_\pi b_{\rho''} \) of a reverse \((2k,r)\) Lorenz braid \( b_\pi \) with a positive braid which does not involve the generator \( g_{2k} \), by the corollary to lemma 6.3 applied to the reverse braids.

**Proof of lemma 6.7** By hypothesis, \( \varphi \) gives an isomorphism from \( BW_{2k}f_k \subset BW_{2k}^{(0)} \) to \( MT_{2k}F_k \). Now every element of \( MT_{2k}F_k \) can be written as a linear combination of totally descending tangles \( T_c \), where the connectors \( c \) join points of the top to the top in some way, and join the bottom points as for \( F_k \). We may choose the order of strings for each connector \( c \) as we wish, so let us assume that in each tangle \( T_c \) the string whose end point is at position \( 2k \) on the top lies above all the others. By isotopy of the strings we may then write each of these tangles \( T_c \) as

\[
G_mG_{m+1} \cdots G_{2k-2}E_{2k-1}TF_k
\]

for some \( m = 1, \ldots, 2k - 2 \) and some \( T \in MT_{2k} \) as illustrated in figure 6.1. The isomorphism \( \varphi \) then gives a spanning set for \( BW_{2k}f_k \) as stated. \( \Box \)
Proof of lemma 6.5: We have
\[ b_\rho e_{2k} g_m g_{m+1} \cdots g_{2k-2} e_{2k-1} = b_\rho g_m g_{m+1} \cdots g_{2k-2} e_{2k} e_{2k-1}. \]

Now the reverse braid \( g_{2k-2} \cdots g_{m+1} g_m \alpha(b_\rho) \) is a positive permutation braid, \( b_{\rho_1} \), say, since \( g_{2k-2} \cdots g_m \) is a positive permutation braid on the first \( 2k-1 \) strings only, while \( \alpha(b_\rho) = b_{\rho^{-1}} \) does not make these strings cross. Now \( \rho_1(2k + 1) = \rho_1(2k) + 1 \), so either \( g_{2k} g_{2k-1} b_{\rho_1} \) or \( g_{2k}^{-1} g_{2k-1}^{-1} b_{\rho_1} \) is a positive permutation braid, \( b_{\rho_2} \), say, depending on whether \( \rho_1(2k - 1) < \rho_1(2k) \) or \( \rho_1(2k - 1) > \rho_1(2k) \), by [6.3].

We can write \( e_{2k-1} e_{2k} = e_{2k-1} g_{2k} g_{2k-1} = e_{2k-1} g_{2k}^{-1} g_{2k}^{-1} \) by the relations in \( BW_n \). Then \( e_{2k-1} e_{2k} b_{\rho_1} = e_{2k-1} b_{\rho_2} \). Apply the reversing map to give
\[
\begin{align*}
  b_\rho e_{2k} g_m g_{m+1} \cdots g_{2k-2} e_{2k-1} &= \alpha(e_{2k-1} e_{2k} b_{\rho_1}) \\
  &= \alpha(e_{2k-1} b_{\rho_2}) \\
  &= b_\rho e_{2k-1},
\end{align*}
\]

where \( \rho'' = \rho_2^{-1} \).

This concludes the proof of proposition 6.1 and the inductive proof of theorem 4.6. We have now established that \( \varphi \) is an isomorphism from \( BW_n \) to \( MT_n \) for all \( n \), so that we are able to use tangle based arguments in dealing with the algebra \( BW_n \). We have established its dimension over \( \Lambda \) and also the geometric description of the natural chain of ideals generated by the elements \( f_k \), so we can also study the composition series of this chain by using the corresponding ideals in \( MT_n \) generated by \( F_k \).

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