On Horizontal Recurrent Finsler Connections

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Abstract. In this paper we adopt the pullback approach to global Finsler geometry. We investigate horizontally recurrent Finsler connections. We prove that for each scalar \( (\pi) \)1-form \( A \), there exists a unique horizontally recurrent Finsler connection whose \( h \)-recurrence form is \( A \). This result generalizes the existence and uniqueness theorem of Cartan connection. We then study some properties of a special kind of horizontally recurrent Finsler connection, which we call special HRF-connection.

Keywords: Finsler manifold; Cartan connection; horizontal recurrent Finsler connection, \( h \)-isotropic; \( P \)-symmetric.

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1. Introduction

The theory of connections is an important field of research of differential geometry. It was initially developed to solve pure geometrical problems. The most important linear connections in Finsler geometry have been studied by many authors, locally (see for example [1, 5, 6, 8, 9]) and globally ([2, 3, 4, 14, 16]). In [14, 16, 17], we have established new proofs of global versions of the existence and uniqueness theorems for the fundamental linear connections on the pullback bundle of a Finsler manifold.

In the present paper we investigate a certain type of Finsler connections that generalize Cartan connection; these connections are called horizontally recurrent Finsler (HRF-) connections. We still adopt the pullback formalism to global Finsler geometry. We prove that for any given scalar \( (\pi) \)1-form, there exists a unique HRF-connection whose \( h \)-recurrence form is \( A \). We display the associated spray and the associated nonlinear connection. We then study a special kind of such connections which we call special HRF-connection. The results of this paper globalize and generalize some results of [10, 11].
2. Notation and Preliminaries

In this section, we give a brief account of some basic concepts of the pullback approach to intrinsic Finsler geometry necessary for this work. For more details, we refer to [9, 12, 16, 17]. We shall use the notations of [16].

In what follows, we denote by \( \pi : TM \to M \) the slit tangent bundle of \( M \), \( \mathfrak{X}(TM) \) the algebra of \( C^\infty \) functions on \( TM \), \( \mathfrak{X}(\pi(M)) \) the \( \mathfrak{X}(TM) \)-module of differentiable sections of the pullback bundle \( \pi^{-1}(TM) \). The elements of \( \mathfrak{X}(\pi(M)) \) will be called \( \pi \)-vector fields and will be denoted by barred letters \( \overline{X} \). The tensor fields on \( \pi^{-1}(TM) \) will be called \( \pi \)-tensor fields. The fundamental \( \pi \)-vector field is the \( \pi \)-vector field \( \overline{\eta} \) defined by \( \overline{\eta}(u) = (u, u) \) for all \( u \in TM \).

We have the following short exact sequence of vector bundles

\[
0 \to \pi^{-1}(TM) \xrightarrow{\gamma} T(\pi^{-1}(TM)) \xrightarrow{\rho} \pi^{-1}(TM) \to 0,
\]

with the well known definitions of the bundle morphisms \( \rho \) and \( \gamma \). The vector space \( V_u(TM) = \{ X \in T_u(TM) : d\pi(X) = 0 \} \) is the vertical space to \( M \) at \( u \in TM \).

Let \( D \) be a linear connection on the pullback bundle \( \pi^{-1}(TM) \). We associate with \( D \) the map \( K : TTM \to \pi^{-1}(TM) : X \mapsto D_X \overline{\eta} \), called the connection map of \( D \). The vector space \( H_u(TM) = \{ X \in T_u(TM) : K(X) = 0 \} \) is called the horizontal space to \( M \) at \( u \). The connection \( D \) is said to be regular if

\[
T_u(TM) = V_u(TM) \oplus H_u(TM) \quad \forall u \in TM.
\]

If \( M \) is endowed with a regular connection, then the vector bundle maps \( \gamma, \rho|_{H(TM)} \) and \( K|_{V(TM)} \) are vector bundle isomorphisms. The map \( \beta := (\rho|_{H(TM)})^{-1} \) will be called the horizontal map of the connection \( D \).

The horizontal ((h)h-) and mixed ((h)hv-) torsion tensors of \( D \), denoted by \( Q \) and \( T \) respectively, are defined by

\[
Q(\overline{X}, \overline{Y}) = T(\gamma \overline{X}, \beta \overline{Y}), \quad T(\overline{X}, \overline{Y}) = T(\overline{\gamma X}, \overline{\beta Y}) \quad \forall \overline{X}, \overline{Y} \in \mathfrak{X}(\pi(M)),
\]

where \( T \) is the (classical) torsion tensor field of \( D \).

The horizontal (h-), mixed (hv-) and vertical (v-) curvature tensors of \( D \), denoted by \( R, P \) and \( S \) respectively, are defined by

\[
R(\overline{X}, \overline{Y}) \overline{Z} = K(\beta \overline{X}, \beta \overline{Y}) \overline{Z}, \quad P(\overline{X}, \overline{Y}) \overline{Z} = K(\beta \overline{X}, \gamma \overline{Y}) \overline{Z}, \quad S(\overline{X}, \overline{Y}) \overline{Z} = K(\gamma \overline{X}, \gamma \overline{Y}) \overline{Z},
\]

where \( K \) is the (classical) curvature tensor field of \( D \).

The contracted curvature tensors of \( D \), denoted by \( \hat{R}, \hat{P} \) and \( \hat{S} \) (known also as the (v)h-, (v)hv- and (v)v-torsion tensors, respectively), are defined by

\[
\hat{R}(\overline{X}, \overline{Y}) = R(\overline{X}, \overline{Y}) \overline{\eta}, \quad \hat{P}(\overline{X}, \overline{Y}) = P(\overline{X}, \overline{Y}) \overline{\eta}, \quad \hat{S}(\overline{X}, \overline{Y}) = S(\overline{X}, \overline{Y}) \overline{\eta}.
\]

Let \((M, L)\) be a Finsler manifold and \( g \) the Finsler metric defined by \( L \). Let \( \nabla \) and \( D^0 \) be the Cartan and Berwald connections associated with \((M, L)\). We quote the following results from [14].

**Proposition 2.1.** The Berwald connection \( D^0 \) is explicitly expressed in terms of Cartan connection \( \nabla \) in the form:

\begin{enumerate}[(a)]
\item \( D^0_{\gamma \overline{X}} \overline{Y} = \nabla_{\gamma \overline{X}} \overline{Y} - T(\overline{X}, \overline{Y}) = \rho[\gamma \overline{X}, \beta \overline{Y}], \)
\end{enumerate}
\[ D^\circ_{\beta X}Y = \nabla_{\beta X}Y + \hat{P}(X, Y) = K[\beta X, \gamma Y]. \]

**Proposition 2.2.** The Berwald connection \( D^\circ \) has the properties:

(a) \( (D^\circ_{\gamma X} g)(Y, Z) = 2T(X, Y, Z), \)

(b) \( (D^\circ_{\beta X} g)(Y, Z) = -2g(\hat{P}(X, Y), Z), \)

where \( T(X, Y, Z) := g(T(X, Y), Z). \)

We terminate this section by some concepts and results concerning the Klein-Grifone approach to intrinsic Finsler geometry. For more details, we refer to [3, 4, 7, 13].

A semi-spray is a vector field \( X \) on \( TM, C^\infty \) on \( TM, C^1 \) on \( TM \), such that \( JX = \mathcal{C} \), where \( J := \gamma \circ \rho \) and \( \mathcal{C} := \gamma \eta \). A semispray \( X \) which is homogeneous of degree 2 in the directional argument (\( [\mathcal{C}, X] = X \)) is called a spray.

**Proposition 2.3.** [7] Let \((M, L)\) be a Finsler manifold. The vector field \( G \) on \( TM \) defined by \( i_G \Omega = -dE \) is a spray, where \( E := \frac{1}{2}L^2 \) is the energy function and \( \Omega := \ddbar{J}E \). Such a spray is called the canonical spray.

A nonlinear connection on \( M \) is a vector 1-form \( \Gamma \) on \( TM, C^\infty \) on \( TM, C^0 \) on \( TM \), such that \( J\Gamma = J \) and \( \Gamma J = -J \). The horizontal and vertical projectors associated with \( \Gamma \) are defined by \( h := \frac{1}{2}(I + \Gamma) \) and \( v := \frac{1}{2}(I - \Gamma) \), respectively. The torsion and curvature of \( \Gamma \) are defined by \( t := \frac{1}{2}[J, \Gamma] \) and \( R := -\frac{1}{2}[h, h] \), respectively. A nonlinear connection \( \Gamma \) is homogenous if \( [\mathcal{C}, \Gamma] = 0 \). It is conservative if \( dhE = 0 \).

**Theorem 2.4.** [4] On a Finsler manifold \((M, L)\), there exists a unique conservative homogenous nonlinear connection with zero torsion. It is given by:

\[ \Gamma = [J, G], \]

where \( G \) is the canonical spray.

This nonlinear connection is called the canonical or the Barthel connection associated with \((M, L)\).

3. Horizontal recurrent Finsler connection

In this section, we prove that for each \( \pi \)-form \( A \) there exists a unique horizontally recurrent Finsler connection whose \( h \)-recurrence form is \( A \). Throughout, we use the notions and results of [13].

**Definition 3.1.** Let \( D \) be a regular connection on \( \pi^{-1}(TM) \) with horizontal map \( \beta \). The semispray \( S := \beta \eta \) will be called the semispray associated with \( D \). The nonlinear connection \( \Gamma_D := 2\beta \circ \rho - I \) will be called the nonlinear connection associated with \( D \).

**Lemma 3.2.** Let \( D \) be a regular connection on \( \pi^{-1}(TM) \) whose connection map is \( K \) and whose horizontal map is \( \beta \). The \( (h)\text{hv-torsion} \) \( T \) of \( D \) has the property that \( T(X, \eta) = 0 \) if and only if the vector form \( \Gamma := \beta \circ \rho - \gamma \circ K \) is a nonlinear connection on \( M \). In this case \( \Gamma \) coincides with the nonlinear connection associated with \( D \): \( \Gamma = \Gamma_D = 2\beta \circ \rho - I \) and, consequently, \( h_{\Gamma} = h_D = \beta \circ \rho, \ v_{\Gamma} = v_D = \gamma \circ K \).
Definition 3.3. Let the pullback bundle $\pi^{-1}(TM)$ be equipped with a metric tensor $g$. A regular connection $D$ on $\pi^{-1}(TM)$ is said to be horizontally recurrent if there exists a scalar 1-form $A$ on $\pi^{-1}(TM)$ such that

$$D_{\bar{\beta}} g = A(X) g \quad \forall X \in \mathfrak{X}(\pi(M)),$$

where $\bar{\beta}$ is the horizontal map of $D$. The scalar form $A$ is called the h-recurrence form of $D$.

Let $(M, L)$ be a Finsler manifold. Let $\nabla$ be the Cartan connection associated with $(M, L)$. We denote by $K, \beta, T$ the connection map, the horizontal map and the (h)hv-torsion of $\nabla$, respectively. We also denote by $R, P, S$ the h-, hv- and v-curvature tensors of $\nabla$, respectively. Now, we announce the main result of the present section.

Theorem 3.4. Let $(M, L)$ be a Finsler manifold and $g$ the Finsler metric defined by $L$. For each scalar ($\pi$)1-form $A$, there exists a unique regular connection $D$ on $\pi^{-1}(TM)$ such that

1. $D$ is horizontally recurrent with h-recurrence form $A$: $(Dg) \circ \bar{\beta} = A \otimes g$,

2. the metric $g$ is $D$-vertically parallel: $D_{\gamma} X g = 0$,

3. the (h)h-torsion $Q$ of $D$ vanishes: $Q = 0$,

4. the (h)hv-torsion $T$ of $D$ satisfies $g(T(X, Y), Z) = g(T(X, Z), Y)$.

Such a connection is called the horizontally recurrent Finsler (HRF-) connection with h-recurrence form $A$.

Proof. First we prove the uniqueness. Since $D$ is a regular connection, then, by Definition 3.1, its horizontal (vertical) projector is given by $\bar{h} := \beta \circ \rho (\bar{v} := I - \bar{\beta} \circ \rho)$. On the other hand, from axioms (C2) and (C4), taking into account Lemma 4 of [14], we deduce that $T(X, \eta) = 0$ for all $X \in \mathfrak{X}(\pi(M))$. Consequently, using Lemma 3.2, it follows that $K = \gamma^{-1}$ on $V(TM)$ and the associated nonlinear connection $\Gamma_{D}$ is given by $\Gamma_{D} = \bar{\beta} \circ \rho - \gamma \circ \bar{K}$, where $\bar{K}$ is the connection map of $D$. Moreover, the horizontal and vertical projectors of $\Gamma_{D}$ are given by $\bar{h} = \bar{\beta} \circ \rho$ and $\bar{v} = \gamma \circ \bar{K}$, respectively.

In view of axioms (C2) and (C4), one can show that, for all $X, Y, Z \in \mathfrak{X}(\pi(M))$,

$$2g(D_{\gamma} X Y, Z) = \gamma X \cdot g(Y, Z) - g(Y, T(X, Z)) + g(Z, T(X, Y)) + g(Y, \rho[\beta Z, \gamma X]) + g(Z, \rho[\gamma X, \beta Y]).$$

(3.1)

As the difference between $\bar{\beta} X$ and $\beta X$ is vertical, one can set

$$\bar{\beta} X = \beta X + \gamma X_t,$$

(3.2)

for some $X_t \in \mathfrak{X}(\pi(M))$. Since $J^2 = 0, [J, J] = 0$ and $\rho \circ J = 0$, we get

$$\rho[\gamma X, \beta Y] = \rho[\gamma X, \beta Y].$$

(3.3)

Making use of (3.3) and Theorem 4(a) of [14], Equation (3.1) implies that

$$D_{\gamma} X Y = \nabla_{\gamma} X Y.$$

(3.4)
Similarly, using axioms (C1) and (C3), we obtain, for all \( \bar{X}, \bar{Y}, \bar{Z} \in \mathfrak{X}(\pi(M)) \),
\[
2g(\mathcal{D}_{\beta \bar{X}} \bar{Y}, \bar{Z}) = \beta \bar{X} \cdot g(\bar{Y}, \bar{Z}) + \beta \bar{Y} \cdot g(\bar{Z}, \bar{X}) - \beta \bar{Z} \cdot g(\bar{X}, \bar{Y}) \\
- g(\bar{X}, \rho[\beta \bar{Y}, \beta \bar{Z}]) - g(\bar{Y}, \rho[\beta \bar{Z}, \beta \bar{X}]) + g(\bar{Z}, \rho[\beta \bar{X}, \beta \bar{Y}]) \\
- A(\bar{X})g(\bar{Y}, \bar{Z}) - A(\bar{Y})g(\bar{Z}, \bar{X}) + A(\bar{Z})g(\bar{X}, \bar{Y}).
\] (3.5)

From (3.2), it is easy to show that
\[
\rho[\beta \bar{X}, \beta \bar{Y}] = \rho[\beta \bar{X}, \beta \bar{Y}] + \rho[\beta \bar{X}, \gamma Y_t] + \rho[\gamma \bar{X}_t, \beta \bar{Y}].
\]

Using the above relation, Proposition 2.1, Proposition 2.2 and Theorem 4(b) of [14], (3.5) becomes
\[
2g(\mathcal{D}_{\beta \bar{X}} \bar{Y}, \bar{Z}) = 2g(\nabla_{\beta \bar{X}} \bar{Y}, \bar{Z}) - A(\bar{X})g(\bar{Y}, \bar{Z}) - A(\bar{Y})g(\bar{Z}, \bar{X}) \\
+ A(\bar{Z})g(\bar{X}, \bar{Y}) + 2g(D^o_{\gamma \bar{X}_t} \bar{Y}, \bar{Z}) + (D^o_{\gamma \bar{X}_t} g)(\bar{Y}, \bar{Z}) \\
+ (D^o_{\gamma \bar{Y}, \bar{Z}}) - (D^o_{\gamma \bar{Z}, \gamma \bar{X}_t})(\bar{X}, \bar{Y}) \\
= 2g(\nabla_{\beta \bar{X}} \bar{Y}, \bar{Z}) - A(\bar{X})g(\bar{Y}, \bar{Z}) - A(\bar{Y})g(\bar{Z}, \bar{X}) \\
+ A(\bar{Z})g(\bar{X}, \bar{Y}) + 2g(D^o_{\gamma \bar{X}_t} \bar{Y}, \bar{Z}) + 2T(\bar{X}_t, \bar{Y}, \bar{Z}) \\
+ 2T(\bar{Y}_t, \bar{Z}, \bar{X}) - 2T(\bar{Z}_t, \bar{X}, \bar{Y}).
\] (3.6)

Setting \( \bar{X} = \bar{Y} = \bar{\eta} \) in (3.6), noting that \( \bar{K} \circ \bar{\beta} = \bar{K} \circ \beta = 0 \) and \( i_{\bar{\eta}}T = 0 \), we get
\[
2\bar{\eta} = 2A(\bar{\eta})\bar{\eta} - L^2\bar{\eta},
\]
where \( g(\bar{\eta}, \bar{X}) := A(\bar{X}) \). From which, by setting again \( \bar{Y} = \bar{\eta} \) into Equation (3.6), we obtain
\[
\beta X - \beta X = \gamma X_t = \frac{1}{2}\{A(\bar{X})\gamma \bar{\eta} + A(\bar{\eta})\gamma X - g(\bar{X}, \bar{\eta}) \gamma \bar{\eta} + L^2 \gamma T(\bar{\eta}, \bar{X})\}.
\] (3.7)

From which, one can show that
\[
2T(\bar{X}_t, \bar{Y}, \bar{Z}) = A(\bar{\eta})T(\bar{X}, \bar{Y}, \bar{Z}) - g(\bar{X}, \bar{\eta})T(\bar{\eta}, \bar{Y}, \bar{Z}) + L^2 T(T(\bar{\eta}, \bar{X}, \bar{Y}, 2T(\bar{X}_t, \bar{Y}, \bar{Z}), (3.8)
\]
\[
2D^o_{\gamma \bar{X}_t} \bar{Y} = A(\bar{X})D^o_{\gamma \bar{X}} \bar{Y} + A(\bar{\eta})D^o_{\gamma \bar{X}} \bar{Y} - g(\bar{X}, \bar{\eta})D^o_{\gamma \bar{X}} \bar{Y} + L^2 D^o_{\gamma T(\bar{X}_t, \bar{Y})}.
\] (3.9)

Now, using (3.7), (3.8) and (3.9), Equation (3.6) reduces to
\[
\mathcal{D}_{\beta \bar{X}} \bar{Y} = \nabla_{\beta \bar{X}} \bar{Y} + 2\{A(\bar{\eta})T(\bar{X}, \bar{Y}) - A(\bar{X})\bar{Y} - A(\bar{Y})X + \bar{\pi}g(\bar{X}, \bar{Y}) \\
+ A(\bar{X})D^o_{\gamma \bar{X}} \bar{Y} + A(\bar{\eta})D^o_{\gamma \bar{X}} \bar{Y} - g(\bar{X}, \bar{\eta})D^o_{\gamma \bar{X}} \bar{Y} + L^2 D^o_{\gamma T(\bar{X}_t, \bar{Y})} \\
- L\bar{\eta}(\bar{X})T(\bar{\eta}, \bar{Y}) - L\bar{\eta}(\bar{Y})T(\bar{\eta}, \bar{X}) + T(\bar{\eta}, \bar{X}, \bar{Y}) \bar{\eta} \\
+ L^2[T(T(\bar{\eta}, \bar{X}, \bar{Y}) + T(T(\bar{\eta}, \bar{Y}, \bar{X}) - T(T(\bar{X}_t, \bar{X}), \bar{\eta})].
\] (3.10)

Consequently, from (3.4), (3.10) and taking into account (3.7), the full expression of \( \mathcal{D}_{\gamma \bar{X}} \bar{Y} \) is given by:
\[
\mathcal{D}_{\gamma \bar{X}} \bar{Y} = \nabla_{\gamma \bar{X}} \bar{Y} + \frac{1}{2}(g(\rho X, \bar{Y})\bar{\pi} - A(\rho X)\bar{Y} - A(\bar{Y})\rho X - L\bar{\eta}(\bar{Y})T(\bar{\eta}, \rho X) \\
+ T(\bar{\eta}, \rho X, \bar{Y}) \bar{\eta} + L^2[T(T(\bar{\eta}, \bar{Y}), \rho X) - T(T(\rho X, \bar{Y}), \bar{\eta})].
\] (3.11)
Hence $\overline{D}_X Y$ is uniquely determined by the right-hand side of (3.11).

Now, we prove the existence of $\overline{D}$. For a given 1-scalar form $A$ on $\pi^{-1}(TM)$, we define $\overline{D}$ by the requirement that (3.11) holds, or equivalently, (3.4), (3.10) and (3.7) hold. Then, using the properties of Cartan connection $\nabla$ and the results of [14], it is not difficult to show that the connection $\overline{D}$ satisfies the conditions (C1)-(C4).

**Remark 3.5.** If in the above theorem we take $A = 0$, the connection $\overline{D}$ reduces to the Cartan connection $\nabla$. Consequently, the above theorem generalizes the existence and uniqueness theorem of Cartan connection [14].

**Corollary 3.6.** The HRF-connection $\overline{D}$ and the Cartan connection $\nabla$ are related by

$$\overline{D}_X Y = \nabla_X Y + N(\rho X, Y);$$

where

$$N(\rho X, Y) := \frac{1}{2}\{g(\rho X, Y)\pi - A(\rho X)Y - A(Y)\rho X - L\ell(Y)T(\pi, \rho X) + T(\pi, \rho X, Y)\eta + L^2[T(T(\pi, Y), \rho X) - T(T(\rho X, Y), \pi)]\}.$$  

In view of Theorem 3.4, we have

**Proposition 3.7.**

(a) The spray $\overline{G}$ associated with $\overline{D}$ is related to the canonical spray $G$ by:

$$\overline{G} = G + A(\eta)C - \frac{1}{2}L^2\gamma\pi.$$  

(b) The nonlinear connection $\overline{\Gamma}$ associated with $\overline{D}$ is related to the Barthel connection $\Gamma$ by:

$$\overline{\Gamma}(X) = \Gamma(X) + \{A(\rho X)C + A(\eta)JX - g(\rho X, \eta)\gamma\pi + L^2\gamma T(\pi, \rho X)\}.$$  

**Proposition 3.8.** Let $\overline{S}$, $\overline{P}$ and $\overline{R}$ be the v-, hv- and h-curvatures of HRF-connection $\overline{D}$, then we have

(a) $\overline{S}(X, Y)Z = S(X, Y)Z$.

(b) $\overline{P}(X, Y)Z = P(X, Y)Z + (\nabla_{X^\eta}N)(X, Z) + N(T(Y, X), Z) + \frac{1}{2}\{A(\eta)S(X, Y)Z - L\ell(X)S(\pi, Y)Z + L^2S(T(\pi, X), Y)Z\}.$

(c) $\overline{R}(X, Y)Z = R(X, Y)Z + P(X, Y_t)Z - P(Y_t, X)Z + S(X_t, Y_t)Z$,

$$+ \{\nabla_{Y_t}N\}(X, Z) + N(Y, N(X, Z)) + N(T(Y_t, X), Z)\},$$

where $Y_t$ is given by (3.7).

$^1\overline{u}_{X,Y} \{B(X,Y)\} := B(X,Y) - B(Y,X).$
4 Special HRF-connection

In this section, we investigate a special horizontally recurrent Finsler connection for which the h-recurrence form $A$ is taken to be $\ell := L^{-1}i_\eta g$. In what follows $\overline{\mathcal{D}}$ will denote the HRF-connection whose h-recurrence form is $\ell$, and will be called the special HRF-connection. In this case $\overline{a} = L^{-1}\eta$.

The following two lemmas are useful for subsequent use.

Lemma 4.1. The nonlinear connection $\overline{\Gamma}$ associated with the special HRF-connection $\overline{\mathcal{D}}$ is given by $\overline{\Gamma} = \Gamma + LJ$. Consequently, $\overline{a} = a = L^{-1}\eta$, $\overline{\ell}(\eta) = L$, and $i_\eta T = 0$.

Proof. The proof follows from Proposition 3.11 and the identities $\overline{a} = L^{-1}\eta$, $\overline{\ell}(\eta) = L$, and $i_\eta T = 0$.

Lemma 4.2. For the special HRF-connection $\overline{\mathcal{D}}$, we have

(a) $N(\overline{X}, \overline{Y}) = \frac{1}{2}\{L^{-1}g(\overline{X}, \overline{Y})\overline{\eta} - \ell(\overline{X})\overline{Y} - \ell(\overline{Y})\overline{X}\}$.

(b) $N(\overline{X}, \overline{\eta}) = -\frac{1}{2}L\overline{X}$.

(c) $(\nabla_{\overline{X}} N)(\overline{Y}, \overline{Z}) = 0$.

(d) $(\nabla_{\overline{Y}} N)(\overline{Y}, \overline{Z}) = \frac{1}{2}\{g(\overline{Y}, \overline{Z})\phi(\overline{X}) - h(\overline{X}, \overline{Y})\overline{Z} - h(\overline{X}, \overline{Z})\overline{Y}\}$.

(e) $(\nabla_{\overline{\gamma}} N)(\overline{Y}, \overline{Z}) = \frac{1}{2}\{g(\overline{Y}, \overline{Z})\phi(\overline{X}) - h(\overline{X}, \overline{Y})\overline{Z} - h(\overline{X}, \overline{Z})\overline{Y}\}$.

where $\phi(\overline{X}) := \overline{X} - L^{-1}\ell(\overline{X})\overline{\eta}$ and $h(\overline{X}, \overline{Y}) = g(\phi(\overline{X}), \overline{Y})$ is the angular metric.

Proof. The proof follows from Corollary 3.6 and fact that $T$ is symmetric and indicatory, taking the identities $\nabla g = 0$, $\nabla_{\overline{X}} L = 0$, $\nabla_{\overline{\gamma}} \ell(\overline{X}) = 0$, $\nabla_{\overline{X}} L = \ell(\overline{X})$ and $(\nabla_{\overline{\gamma}} \ell)(\overline{Y}) = L^{-1}h(\overline{X}, \overline{Y})$ into account.

Theorem 4.3. The (v)hv-torsion tensor $\hat{\mathcal{P}}$ of the special HRF-connection $\overline{\mathcal{D}}$ never vanishes.

Proof. From Proposition 3.15, taking into account Lemma 4.1, Lemma 4.2 and the fact that $T$ and $S$ are indicatory, one can show that

$$\mathcal{P}(\overline{X}, Y, Z) = P(\overline{X}, Y, Z) + \frac{1}{2}\{LS(\overline{X}, Y, Z) + T(\overline{X}, Y, Z)\overline{\eta} - \ell(\overline{Z})T(\overline{X}, \overline{Y})\}$$

$$- \frac{1}{2L}\{h(\overline{Y}, \overline{X})X + h(\overline{Y}, X)\overline{Z} - g(\overline{X}, \overline{Z})\phi(\overline{Y})\}.$$  

From which

$$\hat{\mathcal{P}}(\overline{X}, Y) = \hat{P}(\overline{X}, Y) - \frac{1}{2}\{LT(\overline{X}, \overline{Y}) - \ell(\overline{X})\overline{Y} + L^{-1}g(\overline{X}, \overline{Y})\overline{\eta}\}. \quad (4.1)$$

Now, assume the contrary, that is $\hat{\mathcal{P}} = 0$. Hence, setting $\overline{X} = \overline{\eta}$ in (4.1), using the fact that $\hat{P}$ and $T$ are indicatory, we obtain

$$\frac{1}{2}L \phi(\overline{Y}) = 0 \quad \forall \overline{Y} \in \mathfrak{x}(\pi(M)).$$

Consequently, $(n - 1)L = 0$ (since $Tr \phi = n - 1$), which is a contradiction.  \qed
Corollary 4.4. In view of Equation (4.1), we have

(a) \( \hat{P}(X, Y) - \hat{P}(Y, X) = \frac{1}{2} \{ \ell(X)Y - \ell(Y)X \} \).

(b) \( g(\hat{P}(X, Y), \eta) = -\frac{1}{2} \h(\hat{X}, \hat{Y}) \).

Proposition 4.5. The h-curvature tensor \( \hat{R} \) of the special HRF-connection \( \overline{D} \) has the form

\[
\hat{R}(X, Y)Z = R(X, Y)Z + \frac{1}{4} L^2 S(X, Y)Z + \frac{1}{2} L \{ P(X, Y)Z - P(Y, X)Z \} \\
+ \frac{1}{4} \{ g(X, Z)Y - g(Y, Z)X \}.
\]

Proof. The proof follows from Proposition 3.8, Lemma 4.1 and Lemma 4.2. \( \square \)

Definition 4.6. [15] A Finsler manifold \((M, L)\), with \( \dim M \geq 3 \), is said to be:

(a) h-isotropic with a scalar \( k_0 \) if the h-curvature tensor \( R \) of Cartan connection has the form \( R(X, Y)Z = k_0 \{ g(X, Z)Y - g(Y, Z)X \} \).

(b) of constant curvature \( k \) if the (v)h-torsion tensor \( \hat{R} \) of Cartan connection satisfies the relation \( \hat{R}(\eta, X) = kL^2 \phi(X) \).

Definition 4.7. [15] A Finsler manifold \((M, L)\) is said to be P-symmetric if the mixed curvature tensor \( P \) of Cartan connection satisfies \( P(X, Y)Z = P(Y, X)Z \), for all \( X, Y, Z \in \mathfrak{X}(\pi(M)) \).

Theorem 4.8. Let \((M, L)\) be an h-isotropic Finsler manifold with scalar \((-\frac{1}{4})\). The h-curvature tensor \( \hat{R} \) of the special HRF-connection \( \overline{D} \) vanishes if and only if the v-curvature tensor \( S \) of Cartan connection has property that \( \nabla_{\eta} S = \frac{L}{2} S \).

Proof. The proof follows from Proposition 4.5, Lemma 4.1 and the fact that \( \nabla_{\eta} S = \frac{L}{2} \phi(X) \). \( \square \)

Theorem 4.9. Let \((M, L)\) be a P-symmetric h-isotropic Finsler manifold with scalar \((-\frac{1}{4})\). The h-curvature tensor \( \hat{R} \) of the special HRF-connection \( \overline{D} \) vanishes if and only if the v-curvature tensor \( S \) vanishes.

Proof. The proof follows from Proposition 4.5 and Relation (4.2). \( \square \)

Theorem 4.10. If the (v)h-torsion tensor \( \hat{R} \) of \( \overline{D} \) vanishes, then \((M, L)\) is of constant curvature.

Proof. From proposition 4.5 we obtain

\[
\hat{R}(X, Y) = \hat{R}(X, Y) + \frac{L}{4} \{ \ell(X)Y - \ell(Y)X \}.
\]

Now, if the (v)h-torsion tensor \( \hat{R} \) of \( \overline{D} \) vanishes, the above relation implies that

\[
\hat{R}(\eta, Y) = -\frac{1}{4} L^2 \phi(Y).
\]

Hence, by Definition 4.1(b), the result follows. \( \square \)
References

[1] D. Bao, S. S. Chern, and Z. Shen, An introduction to Riemann-Finsler geometry, Springer-Verlag, Berlin, 2000.

[2] P. Dazord, Propriétés globales des géodésiques des espaces de Finsler, Thèse d’Etat, (575) Publ. Dept. Math. Lyon, 1969.

[3] J. Grifone, Structure presque-tangente et connexions, I, Ann. Inst. Fourier, Grenoble, 22, 1 (1972), 287-334.

[4] J. Grifone, Structure presque-tangente et connexions, II, Ann. Inst. Fourier, Grenoble, 22, 3 (1972), 291-338.

[5] M. Hashiguchi, On determinations of Finsler connections by deflection tensor fields, Rep. Fac. Sci. Kagoshima Univ., 2 (1969), 29-39

[6] S. Hojo, On the determination of generalized Cartan connections and fundamental functions of Finsler spaces, Tensor, N. S., 35 (1981) 333-344.

[7] J. Klein and A. Voutier, Formes extérieures génératrices de sprays, Ann. Inst. Fourier, Grenoble, 18, 1 (1968), 241-260.

[8] M. Matsumoto, Finsler connections with many torsions, Tensor, N. S., 17 (1966), 217-226.

[9] R. Miron and M. Anastasiei, The geometry of Lagrange spaces: Theory and applications, Kluwer Acad. Publ., 59, 1994.

[10] B. N. Prasad and L. Srivastava, On hv-recurrent Finsler connection, Indian J. Pure Appl. Math., 20 (1989), 790-798.

[11] B. N. Prasad, H. S. Shukla and D. D. Singh, On recurrent Finsler connections with deflection and torsion, Publ. Math. Debrecen, 35 (1988), 77-84.

[12] A. Soleiman, Recurrent Finsler manifolds under projective change, Int. J. Geom. Meth. Mod. Phys., 13 (2016), 1650126 (10 pages).

[13] Nabil L. Youssef, Sur les tenseurs de courbure de la connexion de Berwald et ses distributions de nullité, Tensor, N. S., 36 (1982), 275-280.

[14] Nabil L. Youssef, S. H. Abed and A. Soleiman, Cartan and Berwald connections in the pullback formalism, Algebras, Groups and Geometries, 25 (2008), 363-384. arXiv: 0707.1320 [math. DG].

[15] Nabil L. Youssef, S. H. Abed and A. Soleiman, A global approach to the theory of special Finsler manifolds, J. Math. Kyoto Univ., 48 (2008), 857-893. arXiv: 0704.0053 [math. DG].
[16] Nabil L. Youssef, S. H. Abed and A. Soleiman, *A global approach to the theory of connections in Finsler geometry*, Tensor, N. S., 71 (2009), 187-208. arXiv: 0801.3220 [math.DG].

[17] Nabil L. Youssef, S. H. Abed and A. Soleiman, *Geometric objects associated with the fundamental connections in Finsler geometry*, J. Egypt. Math. Soc., 18 (2010), 67-90. arXiv: 0805.2489 [math.DG].