Large deflection of a cantilever beam under transverse loading. A modification of linear theory.

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Abstract. The paper provides a modification method for a classical linear theory of strength of materials in case of a cantilever under transverse loading. The modification uses a condition of saving arc-length of a cantilever. A sagging deflection of the modified linear theory has accuracy comparable with exact solutions in elliptic integrals and functions. The modification expand loading interval of applicability of the linear theory significantly. A comparison with an exact solution was made. We also considered a cantilever under combined loading: concentrated transverse load and uniformly distributed load (an own weight of a cantilever). A comparison with experimental data confirms accuracy of the proposed modification method.

1. Introduction
There are two approaches of solving bending beams problems: small and large deflection. The small deflection approach is of use in strength of materials theory and uses linearized equations. It is believed that an interval of applicability of this approach is limited by the ratio of 3-4% maximum deflection to the length of a beam [1]. The small deflection approach is sufficient for most engineering and construction problems, because deflections are restricted in these problems.

However, many modern problems of science and industry use long elastic rods that are subject to large deflections. For example, these problems include: development of deployable rods construction of small satellites [2], robotic flexible manipulators [3], polymer actuators [4], lamina emergent mechanism and compliant mechanisms [5]. Linearized equations can’t be used for solving of these problems. Solutions for these problems have high complexity and use elliptic integrals and functions. Numeric methods can be used, but it has another difficulties. Therefore development of analytic solutions for different cases of loading is important. It is especially important to obtain simple analytical formulas for practical use.

In the paper we consider a case of thin elastic cantilever under transverse concentrated loading \( P \) on the free end of a cantilever (Figure 1).

![Figure 1. A cantilever under transverse concentrated loading on the free end](image-url)
To solve a bending problem is required to solve Euler-Bernoulli equation (1) in angular or Cartesian coordinates:

\[
\frac{d\theta}{dx} = \frac{d^2 y}{dx^2} [1 + \left(\frac{dy}{dx}\right)^2]^{3/2} = \frac{M(x)}{EJ},
\]

where \(M(x)\) – a bending moment, \(EJ\) – bending stiffness, \(\theta\) – an angle of the tangent at a point as it moves along coordinate \(s\) (the distance measured along the curve itself).

Solving of equation (1) with different one-point and two-point conditions lead up to equations in elliptic integrals and functions. For example, the problem was solved at 1999 in [6]. In this work eigenvalues problem and obtained characteristic critical forces for bending modes was considered. This solution is a special case of a general solution obtained in [7]. The above solutions use elliptic integrals 1\(^{\text{st}}\) and 2\(^{\text{nd}}\) kinds. In work [8] solution for transverse loading with the use elliptic integrals of 1\(^{\text{st}}\) and 3\(^{\text{rd}}\) kind was obtained by using one-point conditions.

2. Linear theory

Linearized Euler-Bernoulli equation (2) describes bending of a beam in classical strength of materials theory [1]:

\[
\frac{d^2 y}{dx^2} = \frac{M(x)}{EJ}
\]

In case of transverse loading on the free end of a cantilever \(M(x) = Px\). After integrating of (2) with initial conditions \(y(L) = 0, y'(L) = 0\) a formula (3) for a bending shape is obtained. The formula in dimensionless coordinates shown below:

\[
\eta = \eta(\xi, \mu) = \frac{\mu}{3} (2 - 3 \xi + \xi^3),
\]

where \(\eta = y/L\) and \(\xi = x/L\) – dimensionless the ordinate and the abscisse respectively, \(\mu = \frac{PL^2}{2EJ}\) a dimensionless parameter of load, \(P\) – an acting force, \(L\) – a length of a cantilever. The parameter can be written in Euler force units \(\lambda\): \(\mu = \frac{\pi^2}{12} (\sim 1.23 \lambda)\). A sagging deflection in the linear theory in the considered case of bending is determined at the point of intersection of the ordinate axis by the bending form (4). It corresponds to the condition of saving a projection of the length of a cantilever on the abscisse axis:

\[
f = f(\mu) = \eta(0, \mu) = \frac{2}{3} \mu.
\]

It necessary to note that this expression can be obtained with a power series expansion of the sagging deflection of the exact solution [6] with respect to the loading parameter [9].

3. A modification of the linear theory

This paper presents the modification of the linear theory for sagging deflection under transverse loading. The modification uses a condition of saving a cantilever arc-length which equals one in dimensionless coordinates. The modification method keeps track of the maximum deflection point on the free end of a cantilever \(\xi_k\). In this way a sagging deflection formula can be found from formula below:

\[
f = \eta(\xi_k, \mu).
\]
where $\xi_k$ calculated with solving equation obtained from of one (that is the length of a cantilever) to an arc-length integral with limits of integration from $\xi_k$ to an abscisse point of the fixed end [10]:

$$\int_{\xi_k}^1 \left\{1 + \left[\frac{d\eta(\xi, \mu)}{d\xi}\right]^2\right\}^{1/2} d\xi = 1.$$  \hspace{1cm} (6)

Root of equation (6) was obtained through numerical computations by the bisection method.

Graphs of sagging deflection of the exact solution [6], the linear theory and the modified linear there was plotted for a comparison in loading interval $0 < \mu < 5$ (Figure 2). Maximum deflection $f$ is on the ordinate axis, loading parameter $\mu$ is on abscissa axis. Also, deviation graph of the modified linear theory from the exact solution was plotted (Figure 3).

![Figure 2. Sagging deflection plots for a cantilever under transverse loading](image)

![Figure 3. Deviation graph of the modified linear theory from the exact solution](image)

We can see on Figure 2 that all solution match in interval $0 < \mu < 0.05$, which corresponds to the range of applicability of the linear theory. The modified linear theory is in good agreement with the exact solution in elliptic integrals over the whole range of load $0 < \mu < 5$. According Figure 3 deviation of the modified linear theory from the exact solution is no more than 6% over the interval $0 < \mu < 5$. These preliminary results are interesting and deserves further investigation because this interval of loads corresponds to the large deflection bending of a cantilever. We can say that the modification expand loading interval of applicability of the linear theory significantly [10].

4. Experimental verification

A comparison with experimental data from [11] was made for experimental verification. In the above paper a demonstration experiment of thin elastic steel cantilever under transverse loading on the free end and cantilever’s own weight was made. The length of the beam is $L = 0.40$ m and it has a uniform rectangular cross-section of width $b = 0.025$ m and height $h = 0.0004$ m. The weight of the beam and the value of the load uniformly distributed over its entire length are $W = 0.3032$ N and $w = W/L = 0.758$ N/m, respectively [11].

From equation (2), by analogy with the previous case, we obtain the expression of the cantilever under combined load: transverse loading on the free end and uniformly distributed load over the whole length. In this case, the bending moment:
\[ M(x) = Px + \frac{1}{2}wx^2. \]

By using initial conditions \( y(L) = 0, \ y'(L) = 0 \) obtain:

\[ \eta = \frac{L^2}{24EJ} [W\xi^4 + 4P\xi^3 - (12P + 4W)\xi + 8P + 3W]. \quad (7) \]

In the linear theory, the expression for the sagging deflection is given by expression (7) with \( \xi = 0 \):

\[ f = \eta(0) = \frac{L^2}{24EJ} (8P + 3W). \quad (8) \]

After differentiate of (7) and substitution to (6) we obtain the following equation for \( \xi_k \) parameter:

\[ \int_{\xi_k}^{1} \left\{ 1 + \frac{L^4}{36(EJ)^2} \left[ 3P(\xi^2 - 1) + W(\xi^3 - 1) \right]^2 \right\}^{1/2} \, d\xi = 1. \quad (9) \]

The combined load sagging deflection plots for the linear theory, the modified linear theory in comparison with experimental data shown on Figure 4. Also, we add the sagging deflection of the exact solution for transversal load for clarity.

![Figure 4. Sagging deflections for case of combined load.](image)

As we can see on Figure 4, the sagging deflection of the linear theory is in agree with experimental data only on the initial part of the curve corresponding to small parameters of load. The sagging deflection of the modified linear theory is fully consistent with experimental data for the entire load range \( 0 < \mu < 1.75 \). For the exact solution it is clear that the shape has the same form as the experimental data.
5. Conclusions
The classical linear theory uses a condition of saving a projection of the length of a cantilever on the abscissa axis. It works well for small deflections. But in the case of large deflection the end of a beam significantly deviates from an equilibrium position and it is impossible to use this approach. It is possible to use a condition of arc-length saving.

In this work the modification method for the linear theory in case of a cantilever under transverse loading was suggested. The sagging deflection of the modified linear theory was validated with a comparison with experimental data. In the experimental comparison we considered case of combined load: transverse loading on the free end and uniformly distributed loading (own weight of a cantilever). The suggested method significantly expands loading range of applicability of the linear theory. A sagging deflection of the modified linear theory has accuracy comparable with exact solutions in elliptic integrals and functions.

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