Decay to bound states of a soliton in a well

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November 21, 2018

Abstract

The decay of a soliton in a trapped state inside a well is shown numerically. Bound states of a kink in an attractive well, both centered and off center are found. Their stability is studied. Unstable soliton solutions inside a repulsive barrier are also found.

PACS 03.40.Kf, 73.40.Gk, 23.60.+e
1 Decay of a soliton in a well

Solitons, originally observed by Scott Russel in a channel near Edinburgh around a century and a half ago and termed by him ‘translational waves’, remained as a curiosity for a long time. The past half century has witnessed the revival of the topic both theoretically and experimentally. Solitons are now generated in nonlinear transmission lines, optical fibers (bright and dark solitons), Josephson junctions, crystals, etc. [1]

Almost a decade ago Kivshar et al. [2] noted that a soliton interacting with an impurity modelled as an external potential or a space dependent mass term, generates a wealth of phenomena ranging from repulsion by an attractive impurity and trapping of a positive kinetic energy soliton in an attractive well. The mechanism of trapping consists in a soliton arriving at the site of an attractive potential and oscillating inside it for an infinite time without the possibility of reemerging from it. These features were demonstrated for a spike-like impurity, $\delta$ function, and later generalized to the case of a finite size potential. [3] (see also ref. [4, 5])

Trapping can be understood in terms of a few degrees of freedom for the soliton [2], namely, the center of the soliton, a shape mode excitation and an impurity mode. [3] These characteristics of soliton scattering from an attractive well are generic and apply also to Sine-Gordon solitons and perhaps even other solitons, not researched until now.

The introduction of a potential is intended to mock-up the behavior of impurities existing in the path of the soliton as in large Josephson junctions where Sine-Gordon solitons are generated. Potentials are real obstacles from which a soliton scatters.

If a soliton is trapped in a well, the oscillations of the soliton are a source of radiation that leads to the decay to lower and lower energy states, until the soliton ends up in a bound state.

The existence of these bound states is here demonstrated for kinks. Sim-
ilar states occur in sine-Gordon solitons. These findings are relevant not only for fluxons, but for any kind of soliton propagating in an inhomogeneous medium that can be represented in terms of a potential. The equivalence of an inhomogeneous medium and an impurity potential is demonstrated in ref.[6].

We here proceed to show that a soliton in a well does indeed decay, its energy being radiated away. We will then find the final states after the kinetic energy is radiated, which are true bound states, similar to those found in quantum mechanics. We will later investigate the stability of the states including also the unstable static solutions of a soliton inside a repulsive barrier.

The kink lagrangian with a potential impurity is[2, 3]

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} \Lambda \left( \phi^2 - \frac{m^2}{\lambda} \right)^2$$

(1)

Here

$$\Lambda = \lambda + V(x)$$

(2)

$\lambda$ being a constant, and $V(x)$ the impurity potential[3]

$$V(x) = h \cosh^{-2} \left( a (x - x_c) \right)$$

(3)

Where $a = \frac{a}{w}$, $w$ being the approximate width of the potential, and we have chosen a bell-shaped potential for the sake of exemplification.

Independently of the choice of parameters it is found that trapped states decay. There are many initial configurations (initial impinging velocities) of the soliton far away from the potential that lead to trapping[2, 3]. After the soliton is trapped inside the well, it oscillates back and forth. Its kinetic energy is still positive.

If the soliton behaved as a pointlike object, it would emerge from the well unscathed. However, due to its extended nature[3] it may be trapped.
The nonlinear wave equation that determines the solitons has also small amplitude solutions, the mesons of the theory\[8\].

These excitations are generated by the oscillations of the soliton in the well, analogously to an oscillating electric dipole.

The waves are emitted in both directions around the well location and drain the kinetic energy of the soliton.

Figure 1 shows the amplitude of the oscillation of the soliton, namely, the value of the field at the center of the well, as a function of time. Here we used $m = 1$, $\lambda = 1$, $\hbar = -3$, $a = 2$, $x_1 = 3$. The soliton impinges from the left. The initial location of the center of the soliton is chosen to be far enough from the well at $x = -3$, with an initial speed $v = 0.025$. The soliton is trapped and immediately starts radiating its energy.

In order to visualize the decay and emission of radiation we extended the x-axis to $-140 \leq x \leq 140$ with a grid of $dx = 0.1$. This coordinate span allows for radiation to propagate for a long distance away from the trapping zone without being reflected.

After a certain time, and due to the finite extent of the x-axis, radiation reflects back from the boundaries and reaches the soliton. The soliton subsequently absorbs the radiation and its amplitude starts to increase. The time taken for radiation to return to the soliton is the travel time for the fastest ‘mesons’ of the theory.

The dispersion relation for the radiated mesons can be extracted from the expansion of the scalar field around the soliton solution. Using $\lambda = m = 1$ we find $\omega^2 = k^2 + 2$. The velocity of the mesons is bounded by

$$u_{\text{max}} = \left(\frac{\omega}{k}\right)_{\text{max}} = 1.$$  

This is clearly observed in figure 1. The reabsorption of radiation starts after the first mesons arrive back from the boundaries to the well. The distance between the well and the boundary is 140, therefore

$$t_{\text{absorption}} = \frac{280}{u_{\text{max}}} = 280.$$  

The frequency of the oscillations of the soliton in a trapped state may be estimated analytically. Using an expansion of the potential in eq. (3)
Figure 1: Amplitude of the oscillation of the soliton in a trapped state as a function of time. Soliton parameters: $m = 1$, $\lambda = 1$, impurity parameters: $h = -3$, $a = 2$, $x_c = 3$. 
around the bottom of the well $V(x) \approx -V_0 + \epsilon y^2$, $y = x - x_c$ and an ansatz appropriate for small oscillations of the soliton around the center of the well $\phi \approx (y + \delta y^2/2) \sin(\omega(t - t_0))$ we find $\omega^2 = 2\mu$.

With $\mu \approx \pm \sqrt{\frac{9}{5}\epsilon + \frac{9}{10}(V_0 - 1)}$. (The positive solution has to be chosen)

The formula compares reasonably well with the leading frequency of oscillation of the soliton inferred from a Fourier analysis of the amplitude of the field at the center of the well. However, the fluctuation of the soliton in the well is anharmonic.

Another way to observe the decay of a trapped state to a bound state consists of adding a dissipative force of the form $\gamma \partial \phi / \partial t$. This force cannot be derived from a Hamiltonian, but, it can arise from the interaction to an environment.

Inserting this term in the soliton equation of motion yields the results depicted in figure 2, where we took the same set of parameters as those of the radiation run of figure 1, but with a friction coefficient $\gamma = .1$.

Attenuation is the dominant effect in this case. The soliton loses its energy by dissipation instead of radiating it. Other choices of parameters may lead to a mixture of both processes. It is, however, evident that trapped states will eventually become bound states.
Figure 2: Same as figure 1 but including attenuation. Friction coefficient $\gamma = .1$. 
2  *Bound states of a soliton in an attractive well*

In the previous section, it was shown that solitons radiate their energy when trapped. Hence, there should exist static bound state solutions of the soliton in the well.

These bound states exist not only for a soliton centered with the well, but also for a soliton located off-center. The former are produced after the soliton radiates its kinetic energy, while the latter seem more difficult to realize. This phenomenon has no counterpart in the classical behavior of particles. Only the bottom of the well is the point where the particles can remain motionless. The extended character of the soliton is playing a major role in generating such unexpected solutions.

It is clear that these off-center solutions are true bound states, because their energy is smaller than the free soliton mass. However, any small perturbation of the soliton will make it drift to the center of the well. The off-center solutions are unstable.

We found the bound state solutions, by integrating the static equations of motion starting from the center of the soliton. There appears to be only a single bound state for each choice of well depth and width, even for large well depths.

Figure 3 shows a bound state centered with the well, for a well depths $h = -1$, (solid line), $h = -5$ (dashed line) and a width parameter $w = 5$. Figure 4 shows a solution for an off-center soliton for a well that is located at a position $x = 5$ and the same parameters. We took wells much wider than the extent of the soliton for the graphs. For narrower wells, we bring the calculated masses only in a table below.

The deeper the well, the more dramatic the distortion of the soliton from its free shape. When the depth of the well exceeds, in absolute value, the mass parameter entering the soliton equation of motion, a topological solution with the right asymptotics is obtained by having a vanishing soliton at the location
Figure 3: Bound state soliton solution in a well with parameters $h=-1$ and $w=5$, full line and $h=-5$ $w=5$, dashed line.
of the well both for centered and off-center wells as evidenced in figure 4. For this reason, the numerical solutions are extremely sensitive to the slope of the soliton at the well.

Table 1 shows the masses of the soliton for several choices of depths and widths. These have to be compared with the free soliton mass of $M=0.9428$. The soliton energy may even become negative for deep wells. A narrow and thin well barely changes the soliton energy. The deeper and wider the well, the stronger the effect, as seen from the last five entries in the table. This feature resembles bound state solutions of the Schrödinger equation.
Table 1: Total energy of a soliton in a well

| well depth | width parameter $w$ | well center | soliton energy |
|------------|---------------------|-------------|----------------|
| -1.        | 5                   | 0           | 0.633          |
|            |                     | 1           | 0.7178         |
|            |                     | 3           | 0.9308         |
|            |                     | 5           | 0.9425         |
| -5         | 1                   | 0           | 0.533          |
|            |                     | 1           | 0.676          |
|            |                     | 3           | 0.9312         |
|            |                     | 5           | 0.9425         |
| -5         | 5                   | 0           | -0.771         |
|            |                     | 1           | -0.6834        |
|            |                     | 2           | -0.4638        |
|            |                     | 4           | 0.025          |
|            |                     | 5           | 0.1177         |
Let us now consider the stability of the soliton solutions for the centered case. We do not have analytical expressions for the soliton, therefore we proceed by considering the behavior of the soliton around its center.

In a small region around the bottom of the well, we can write the soliton solution in terms of a new mass parameter \( \Lambda = 1 + V(x) \), for \( x \approx 0 \), as \( \phi = \tanh \left( \sqrt{\frac{\Lambda}{2}} (x - x_0) \right) \), where \( x_0 \) is the location of the soliton that we deliberately move away by a small amount from \( x = 0 \). In the above ansatz we have used the mass and width parameters as previously \( m = 1, \lambda = 1 \). For \(-V(0) > 1\) we use the \( \tan \) solution instead of the \( \tanh \) one. With such an ansatz in the soliton equation of motion we find the equation for the center of the soliton

\[
\ddot{x}_0 = \frac{\pm 2}{\Lambda(x_0)} \, V'(x_0) \tag{4}
\]

where, the upper sign corresponds to a deep well for which the depth is greater than the mass parameter, \( \Lambda(x = 0) < 0 \), and vice versa. The dots represent time derivatives, the prime space derivative, and velocities are unitless. From equation 4, it is seen that an attractive well, the soliton will be dragged back to the center of the well. The soliton solutions at the center of the well are stable. Using similar techniques, we can show that the off-center solitons are unstable. In this case, a suitable ansatz for the soliton around the center of the well for distances not too far from the center of the soliton, is a quadratic function of the form \( \phi = A \left( 1 - \frac{A}{2}(x - x_0)^2 \right) \) with \( A \), a constant. We obtain the same equation 4 but with a reverse sign. The soliton becomes unstable. It is clear that small amplitude excursions of the soliton from its bound state do not kick it out of the well. However, large amplitude oscillations (maybe produced by radiation) are able to do so. This is essentially the time reversed version of the trapping mechanism.

Finally let us note that there exist static soliton solutions for a repulsive well, with a soliton located at the center of the well. Due to the convexity of
Figure 5: Soliton solution inside a barrier with parameters $h=1$ and $a=5$

the barrier around its maximum, the soliton is unstable here also. Figure 5 shows one such solution for parameters $h=1$ $w=5$.

In summary, if a soliton is trapped in an attractive well it will decay to a stable bound state. It remains for experiments to find the right conditions for the process to happen.

Acknowledgements

This work was supported in part by the Department of Energy under grant DE-FG03-93ER40773 and by the National Science Foundation under grant PHY-9413872.
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