Enhanced and unenhanced dampings of the Kolmogorov flow

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Abstract

The Kolmogorov flow represents the stationary sinusoidal solution \((\sin y, 0)\) to a two-dimensional spatially periodic Navier-Stokes system, driven by an external force. This system admits the additional non-stationary solution \((\sin y, 0) + e^{-\nu t}(\sin y, 0)\), which tends exponentially to the Kolmogorov flow at the minimum decay rate determined by the viscosity \(\nu\). Enhanced damping or enhanced dissipation of the problem is obtained by presenting higher decay rate for the difference between a solution and the non-stationary basic solution. Moreover, for the understanding of the metastability problem in an explicit manner, a variety of exact solutions are presented to show enhanced and unenhanced dampings.

Keywords: Kolmogorov flow, two-dimensional Navier-Stokes equations, metastability, enhanced damping, enhanced dissipation

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1. Introduction

Consider a two-dimensional periodically driven flow with viscosity \(\nu > 0\) governed by the Navier-Stokes equation system

\[
\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p = \nu (a \sin y, 0), \quad \nabla \cdot u = 0
\]

(1.1)

in the flat torus \(T_\alpha = [0, 2\pi/\alpha] \times [0, 2\pi)\) for \(\alpha > 0\) and \(a \geq 0\). The pressure \(p\) and velocity \(u\) are subject to the spatially periodic condition with respect
2π/α period in x and 2π period in y. The vorticity formulation of (1.1) is expressed as
\[
\partial_t \omega + u \cdot \nabla \omega - \nu \Delta \omega = -\nu a \cos y
\]
or
\[
\partial_t \omega + J(\Delta^{-1} \omega, \omega) - \nu \Delta \omega = -\nu a \cos y \tag{1.2}
\]
with the Jacobian \( J(\phi, \varphi) = \partial_x \phi \partial_y \varphi - \partial_y \phi \partial_x \varphi \). The velocity \( u \), vorticity \( \omega \) and stream function \( \psi \) of the flow are subject to the identities
\[
\omega = -\Delta \psi \quad \text{and} \quad u = (\partial_y \psi, -\partial_x \psi).
\]

This problem was introduced by Kolmogorov in 1959 in a seminar \[1\], where he presented the basic stationary flow
\[
u = (\sin y, 0) \quad \text{or} \quad \omega = -\cos y, \tag{1.3}
\]
which is known as the Kolmogorov flow, and encouraged turbulence study from the instability of the flow.

When \( \alpha = 1 \), the linear stability of the Kolmogorov flow for \( \nu > 0 \) was given by Mishalkin and Sinai \[14\] and its nonlinear global stability was obtained by Marchioro \[13\]. When \( 0 < \alpha < 1 \), Iudovich \[11\] showed the instability of the Kolmogorov flow leading to the occurrence of secondary stationary flows. For some \( \alpha \in (0, 1) \), the existence of Hopf bifurcation was obtained by Chen and Price \[6\]. Moreover, if \( \frac{\sqrt{3}}{4} \leq \alpha < \frac{\sqrt{3}}{2} \) and (1.2) is associated with a horizontal a free-slip boundary condition, perturbations around the Kolmogorov flow give rise to Hopf bifurcation (see Chen and Price \[7\]) into oscillatory flows. Nonlinear interaction arises from the coexistence of multiple secondary oscillatory flows from Hopf bifurcation and develops into chaotic flows (see Chen and Price \[8\]). Turbulent Kolmogorov flow was also studied by Chandler and Kerswell \[3\].

In the present study, we are interested in stability problem of the Kolmogorov flow. Especially, for the unforced case \( a = 0 \), the \( L_2 \) inner product of (1.2) with \( \omega \) produces the \( L_2 \) energy estimate
\[
\frac{1}{2} \frac{d}{dt} \|\omega(t)\|^2_{L_2} + \nu \|\nabla \omega(t)\|^2_{L_2} = -\int_{T_\alpha} J(\Delta^{-1} \omega, \omega) \omega dx dy = 0.
\]
This yields the damping of the enstrophy

\[ \| \omega(t) \|_{L_2} \leq e^{-\nu t} \| \omega(0) \|_{L_2}, \quad \text{for} \quad \alpha > 0, \quad (1.4) \]

at the minimum decay rate \( \nu \). However, this damping may be enhanced for some class of initial data with respect to small \( \nu \) or large Reynolds number \((Re = 1/\nu)\). The enhanced damping problem was raised by Beck and Wayne \[2\]. They considered the equation

\[ \partial_t \omega = \nu \Delta \omega - e^{-\nu t} \sin y \partial_x (1 + \Delta^{-1}) \omega, \quad (1.5) \]

which is linearized from the Navier-Stokes equation \(1.2\) with \( a = 0 \) around its exact solution

\[ \omega = -e^{-\nu t} \cos y. \quad (1.6) \]

When the non-local operator \( \Delta^{-1} \) is not involved, they obtained the existence of positive constants \( C \) and \( M \) satisfying the estimate

\[ \| \omega \|_{L_2} \leq C e^{-M \sqrt{\nu} t} \| \omega(0) \|_{L_2}, \quad 0 < t \leq \frac{\tau}{\nu} \quad (1.7) \]

for given \( \tau > 0 \), provided \( \nu \) is small. The estimate \(1.7\) was recently obtained by Wei and Zhang \[16\] and Wei et al. \[17\] for the complete linear equation \(1.5\). When \( a = 1 \), Ibrahim et al. \[10\] considered the equation

\[ \partial_t \omega = \nu \Delta \omega - \sin y \partial_x (1 + \Delta^{-1}) \omega, \quad (1.8) \]

linearization of \(1.2\) around \(1.3\), and obtained the damping estimate \(1.7\).

The Kolmogorov flow moves in a horizontal direction along its streamlines \( y = \text{constants} \), and is called a parallel flow. For the stability and instability of a nonparallel vortex flow moving in multiple directions, we refer to the work of author \[4, 5\].

The present study is motivated by the work of Lin and Xu \[12\], where they noticed that linearized the Navier-Stokes flow \(1.5\) is close to the linearized Euler flow

\[ \partial_t \omega = -\sin y \partial_x (1 + \Delta^{-1}) \omega \quad (1.9) \]

for \( t \) moderate and \( \nu \) sufficiently small. Thus the low frequency modes leading to unenhanced damping in the sense of \[12\] can be controlled by the linearized
Euler flow due to a RAGE theorem \cite{9} and the nonlinear convection term can be bounded by small initial data. They \cite{12} considered the enhanced damping property of the linear equation (1.5) and the nonlinear equation (1.2) with $a = 0$ as

$$\left\| P_{\neq 0} \omega \left( \frac{T}{\nu} \right) \right\|_{L^2} < \delta \| P_{\neq 0} \omega(0) \|_{L^2}$$

for any $\tau, \delta > 0$, provided $\nu$ is small. Here $P_{\neq 0}$ is the projection operator

$$P_{\neq 0} \omega = \omega - P_0 \omega \quad \text{with} \quad P_0 \omega = \frac{\alpha}{2\pi} \int_0^{2\pi/\alpha} \omega \, dx. \quad (1.10)$$

Actually, equation (1.2) has the additional solution

$$u = (a \sin y + e^{-\nu t} \sin y, 0) \quad \text{or} \quad \omega = -a \cos y - e^{-\nu t} \cos y. \quad (1.11)$$

Linearizing the Navier-Stokes equation (1.2) around (1.11), we have

$$\partial_t \omega = \nu \Delta \omega - a \sin y \partial_x(1 + \Delta^{-1}) \omega - e^{-\nu t} \sin y \partial_x(1 + \Delta^{-1}) \omega. \quad (1.12)$$

The present study is twofold. Firstly, we extend the analysis of Lin and Xu \cite{12} to the forced Navier-Stokes equation (1.2). Secondly, to the understanding of the problem in a different direction, we present exact solutions showing enhanced and unenhanced damping properties different to that of \cite{12} in an explicit manner.

An enhanced damping result reads as follows.

**Theorem 1.1.** For any $\alpha > 1$, $\tau > 0$, $\delta > 0$ and $a \in \{0, 1\}$, then the following assertions hold true.

(i) Any solution $\omega$ to the linear equation (1.12) initiated from $\omega(0) \in L^2(\mathbb{T}_\alpha)$ with $P_{\neq 0} \omega(0) = \omega(0)$ satisfies

$$\left\| \omega \left( \frac{T}{\nu} \right) \right\|_{L^2} \leq \delta \| \omega(0) \|_{L^2},$$

provided that $\nu > 0$ is sufficiently small.

(ii) For any solution $\omega$ to the nonlinear equation (1.2) presented in the perturbed form

$$\omega = -a \cos y - e^{-\nu t} \cos y + \omega', \quad (1.13)$$
the perturbed flow $\omega'$ satisfies
\[ \left\| P_{\neq 0}\omega'\left(\frac{T}{\nu}\right)\right\|_{L^2} \leq \delta \left\| P_{\neq 0}\omega'(0)\right\|_{L^2}, \tag{1.14} \]
provided that $\omega'(0) \in L_2(\mathbb{T}_a)$,
\[ \|\omega'(0)\|_{L^2} \leq \nu d \tag{1.15} \]
for $\nu$ and $d$ sufficiently small.

When $a = 1$, this result is on the stability of Kolmogorov flow. Hence metastability rather than enhanced damping may be more suitable to address the present study.

When $a = 0$, this is comparable with an enhanced damping result given by Lin and Xu \cite{12} in the following.

**Theorem 1.2.** (\cite{12}, Theorem 1.2) Consider the nonlinear Navier-Stokes equation (1.1) $(a = 0)$ on $\mathbb{T}_a$ with $\alpha \geq 1$. Denote $P_K$ to be the projection of $L_2(\mathbb{T}_a)$ to the subspace of Kolmogorov flows $W_K = \text{span}\{\cos y, \sin y\}$. Then,

(i) (Rectangular torus) Suppose $\alpha > 1$. There exists $d > 0$, such that for any $\tau > 0$ and $\delta > 0$, if $\nu$ is small enough, then for any solution $\omega(t)$ to (1.2) with initial vorticity $\omega(0) \in L_2(\mathbb{T}_a)$ satisfying
\[ \|P_{\neq 0}\omega(0)\|_{L^2} \leq \nu d, \tag{1.16} \]
we have
\[ \left\| P_{\neq 0}\omega\left(\frac{T}{\nu}\right)\right\|_{L^2} < \delta \left\| P_{\neq 0}\omega(0)\right\|_{L^2}. \tag{1.17} \]

(ii) (Square torus) Suppose $\alpha = 1$. There exist $d > 0$, such that: for any $M > 0$, $\tau > 0$ and $\delta > 0$, if $\nu$ is small enough, then for any solution $\omega(t)$ to (1.2) with initial data $\omega(0) \in L_2(\mathbb{T}_a)$ satisfying (1.16), either
\[ \max_{0 \leq t \leq \tau} \|P_a\omega(t)\|_{L^2} \geq M \|P_{\neq 0}\omega(0)\|_{L^2} \tag{1.18} \]
or
\[ \inf_{0 \leq t \leq \tau} \|(I - P_a)P_{\neq 0}\omega(t)\|_{L^2} < \delta \left\| P_{\neq 0}\omega(0)\right\|_{L^2} \tag{1.19} \]
must hold true. Here $P_a$ is the orthogonal projection mapping $L_2$ onto the space $\text{span}\{\sin x, \cos x\}$.
Remark 1.1. It should be noted that Theorem 1.2 or [12, Theorem 1.2] contradicts to the following fact:

For \( \tau = M = 1 \), \( \delta = e^{-a^2 - 1} \) and any constant \( d > 0 \), the function

\[
\omega = dv \frac{\sqrt{\alpha}}{\sqrt{2\pi}} e^{-(a^2 + 1)\nu t} \sin(\alpha x + y), \quad \alpha \geq 1,
\]

solves the Navier-Stokes equation (1.2) \((a = 0)\) in \( T_\alpha \) and has the following properties:

\[
\| (I - P_K)\omega(0) \|_{L^2} = \| \omega(0) \|_{L^2} = dv, \quad \alpha \geq 1,
\]
\[
P_{\neq 0}\omega = \omega, \quad \alpha \geq 1,
\]
\[
P_a\omega = 0, \quad \alpha = 1,
\]
\[
\| P_{\neq 0}\omega(t) \|_{L^2} \geq \delta \| P_{\neq 0}\omega(0) \|_{L^2}, \quad t \leq \frac{\tau}{\nu}, \alpha > 1,
\]
\[
\| (I - P_a)P_{\neq 0}\omega(t) \|_{L^2} \geq \delta \| P_{\neq 0}\omega(0) \|_{L^2}, \quad t \leq \frac{\tau}{\nu}, \alpha = 1.
\]

Therefore, the initial condition (1.16) holds true, but none of the conclusion estimates (1.17)-(1.19) are valid, no matter how small the viscosity \( \nu \) is.

For a solution \( \omega \) to the Navier-Stokes equation (1.2) with \( a = 0 \) in the perturbed form

\[
\omega(t) = -e^{-\nu t} \cos y + \omega'(t),
\]

we see that \( P_{\neq 0}e^{-\nu t} \cos y = 0 \) and \( (I - P_K)e^{-\nu t} \cos y = 0 \). Hence there holds the identity

\[
P_{\neq 0}\omega(t) = P_{\neq 0}\omega'(t).
\]

Thus one might consider \( \omega' \) and \( \omega \) to be the same in the enhanced damping analysis. However it should be noted that the linear problem (1.12) is invariant in the subspace \((I - P_K)L^2(T_\alpha)\), but this invariance cannot be extended to the nonlinear problem (1.2). Therefore the smallness assumption on \((I - P_K)\omega(0)\) is not enough to produce the desired enhanced damping estimates such as (1.17).

More specifically, for \( \omega \) representing the exact solution (1.20), we see that

\[
\omega(t) = -e^{-\nu t} \cos y + \omega'(t)
\]
with
\[
\omega'(t) = \omega(t) + e^{-\nu t} \cos y = d\nu \frac{\sqrt{\alpha}}{\sqrt{2\pi}} e^{-(\alpha^2+1)\nu t} \sin(\alpha x + y) + e^{-\nu t} \cos y.
\]
and hence
\[
\| P_{\neq 0} \omega(t) \|_{L^2} = \| P_{\neq 0} \omega'(t) \|_{L^2} = e^{-(\alpha^2+1)\nu t} d\nu,
\]
but the initial functions \( \omega'(0) \) and \((I - P_K)\omega(0)\) are quite different as shown in the following.

\[
\| \omega'(0) \|^2_{L^2} = \| d\nu \frac{\sqrt{\alpha}}{\sqrt{2\pi}} \sin(\alpha x + y) \|^2_{L^2} + \| \cos y \|^2_{L^2} = d^2 \nu^2 + \frac{2\pi^2}{\alpha} \geq \frac{2\pi^2}{\alpha},
\]
which is larger than the lower bound \( \frac{2\pi^2}{\alpha} \) for any viscosity \( \nu > 0 \). In contrast, we see that
\[
\| (I - P_K)\omega(0) \|_{L^2} = \| (I - P_K)\omega'(0) \|_{L^2} = \| d\nu \frac{\sqrt{\alpha}}{\sqrt{2\pi}} \sin(\alpha x + y) \|_{L^2} = \nu d.
\]

Hence to exclude the exact solution (1.20) and to ensure the validity of Theorem 1.2, Assumption (1.16) should be replaced by (1.15), or
\[
\| \omega(0) + e^{-\nu t} \cos y \|_{L^2} \leq \nu d.
\]

The enhanced damping property given by Theorem 1.1 is determined by the estimates (1.14) and (1.15), which are not satisfied by (1.20). For compensating this shortness, a variety of exact solutions of (1.2) will be provided for showing their explicit enhanced damping different to that presented in Theorem 1.1.

Theorem 1.1 involves only the rectangular case \( \alpha > 1 \), as the limit case \( \alpha = 1 \) can be treated in a similar manner.

This paper is organized as follows. Theorem 1.1 is to be proved in Section 2. Exact solutions will be discussed in Section 3.
2. Proof of Theorem 1.1

The proof is developed from Lin and Xu [12] together with Constantin et al. [9]. Let $C$ be a generic constant and the condition $\alpha > 1$ is assumed throughout this section.

For convenience of notation, we always assume that $L_2(\mathbb{T}_\alpha)$ denotes the $L_2$ space of all mean zero functions:

$$L_2(\mathbb{T}_\alpha) = \left\{ \omega \mid \int_{\mathbb{T}_\alpha} \omega \, dx \, dy = 0, \|\omega\|_{L_2}^2 = \int_{\mathbb{T}_\alpha} |\omega|^2 \, dx \, dy < \infty \right\}.$$

Recalling the operator $P_{\neq 0}$ from (1.10), we define the $L_2$ subspaces $X = P_{\neq 0}L_2(\mathbb{T}_\alpha)$, that is,

$$X = \left\{ \omega \in L_2(\mathbb{T}_\alpha) \mid P_{\neq 0}\omega = \omega \right\}.$$

Note that $(1+\Delta^{-1})W_K = \{0\}$. It is convenient to use the equivalent $L_2$-norm

$$\|\omega\|_X = \|(1+\Delta^{-1})^{1/2}\omega\|_{L_2} \quad (2.1)$$

in the $L_2$ subspace $(1+\Delta^{-1})L_2(\mathbb{T}_\alpha)$.

2.1. Preliminary estimates

As given by Lin and Xu [12], the enhanced damping lies on a modified RAGE theorem [9] with respect to the linearized Euler equation

$$\partial_t \omega = -(a + 1) \sin y \partial_x (1 + \Delta^{-1})\omega \quad (2.2)$$

in $X$, by avoiding unenhanced damping from low frequency eigenfunctions of the Laplacian operator $-\Delta$. We adopt the complete eigenvalues

$$\alpha^2 \leq \lambda_1 \leq \lambda_2 \leq ...$$

and the corresponding eigenfunctions $e_1, e_2, ...$ of the operator $-\Delta$ in $X$. Let $P_N$ be the $L_2$ projection mapping $X$ onto the subspace span$\{e_1, ..., e_N\}$.

**Lemma 2.1.** ([12, Lemma 2.3] and [9, Lemma 3.2]) Let $K \subset S \equiv \{\varphi \in X \mid \|\varphi\|_{L_2} = 1\}$ be a compact set of $X$. For any $N, \kappa > 0$, there exists $T_c(N, \kappa, K)$ such that for all $T \geq T_c$ and $\omega^0(0)/\|\omega^0(0)\|_{L_2} \in K$,

$$\frac{1}{T} \int_0^T \|P_N\omega^0(t)\|_X^2 \, dt \leq \kappa \|\omega^0(0)\|_X^2, \quad (2.3)$$

where $\omega^0(t)$ is the solution of the linearized Euler equation (2.2) initiated from $\omega^0(0)$.
Rewriting the Navier-Stokes equation (1.2) perturbed around the basic flow \(-a \cos y - e^{-\nu t} \cos y\), we have

\[
\partial_t \omega^\nu - \nu \Delta \omega^\nu = -a \sin y \partial_x (1 + \Delta^{-1}) \omega^\nu - e^{-\nu t} \sin y \partial_x (1 + \Delta^{-1}) \omega^\nu \\
- \sigma J(\Delta^{-1} \omega^\nu, \omega^\nu),
\]

(2.4)

with the constant \(\sigma = 0, 1\). This equation is the linear (1.12) for \(\sigma = 0\) and is the nonlinear perturbed form of (1.2) for \(\sigma = 1\).

The proof of Theorem 1.1 is essentially based on the fundamental \(L^2\) energy estimate of Kolmogorov problem (2.4) given by Iudovich [11, Eq. (2.14)] expressed as

\[
\frac{d}{dt} \int_{\mathbb{T}^2} \omega^\nu(t)(1 + \Delta^{-1}) \omega^\nu(t) dx dy - 2\nu \int_{\mathbb{T}^2} \Delta \omega^\nu(t)(1 + \Delta^{-1}) \omega^\nu(t) dx dy = 0,
\]

(2.5)

or

\[
\frac{d}{dt} \|\omega^\nu(t)\|_X^2 + 2\nu \|\nabla \omega^\nu(t)\|_X^2 = 0.
\]

(2.6)

This is imply due to the identities

\[
\int_{\mathbb{T}^2} (a + e^{-\nu t}) \sin y \partial_x (1 + \Delta^{-1}) \omega^\nu(1 + \Delta^{-1}) \omega^\nu dx dy = 0
\]

and

\[
\int_{\mathbb{T}^2} J(\Delta^{-1} \omega^\nu, \omega^\nu)(1 + \Delta^{-1}) \omega^\nu dx dy = 0,
\]

(2.7)

after integration by parts.

When \(a = 1\), the energy estimate (2.5) gives rise to the steady-state bifurcation analysis of [11] for \(0 < \alpha < 1\) and the Hopf bifurcation analysis of [7] for \(\frac{\sqrt{3}}{4} \leq \alpha < \frac{\sqrt{3}}{2}\).

As a consequence of (2.6), we have an \(L^2\) energy estimate of \(P_{\neq 0} \omega^\nu\) in the following.

**Lemma 2.2.** Let \(\omega^\nu\) be a solution of (2.4) with \(\omega^\nu(0) \in X\). Assume that \(\|\omega^\nu(0)\|_X \leq \nu d\) for a small constant \(d > 0\) when \(\sigma = 1\). Then we have

\[
\frac{d}{dt} \|P_{\neq 0} \omega^\nu(t)\|_X^2 + \frac{15}{8} \nu \|\nabla P_{\neq 0} \omega^\nu(t)\|_X^2 \leq 0.
\]

(2.8)
When $a = 0$, similar estimate has been shown in [12, Lemma 3.2]. For completion of analysis, we sketch a proof developed from [12].

**Proof.** Applying the operator $P_0$ to (2.4), we have

\[
\partial_t P_0 \omega^\nu(t) - \nu \Delta P_0 \omega^\nu(t) = - \sigma P_0 J(\Delta^{-1} P_{\neq 0} \omega^\nu(t) + \Delta^{-1} P_0 \omega^\nu(t) + P_0 \omega^\nu(t))
\]

\[\quad = - \sigma P_0 J(\Delta^{-1} P_{\neq 0} \omega^\nu(t), P_{\neq 0} \omega^\nu(t)). \quad (2.9)\]

Taking the $L_2$ inner product of (2.9) with $2(1 + \Delta^{-1}) P_0 \omega^\nu(t)$, we have

\[
\frac{d}{dt} \| P_0 \omega^\nu(t) \|_X^2 + 2\nu \| \nabla P_0 \omega^\nu(t) \|_X^2
\]

\[
= -2\sigma \int_{\Omega} P_0 J(\Delta^{-1} P_{\neq 0} \omega^\nu(t), P_{\neq 0} \omega^\nu(t))(1 + \Delta^{-1}) P_0 \omega^\nu(t) \, dxdy
\]

\[
\geq -2\sigma \| \nabla \Delta^{-1} P_{\neq 0} \omega^\nu(t) \|_{L_\infty} \| \nabla P_{\neq 0} \omega^\nu(t) \|_{L_2} \| (1 + \Delta^{-1}) P_0 \omega^\nu(t) \|_{L_2}
\]

\[
\geq -2\sigma C \| \nabla P_{\neq 0} \omega^\nu(t) \|_X^2 \| \omega^\nu(t) \|_X
\]

\[
\geq -2\sigma C \| \nabla P_{\neq 0} \omega^\nu(t) \|_X^2 \| \omega^\nu(0) \|_X, \quad (2.10)
\]

after the use of the Sobolev imbedding theorem and (2.6). Here and in what follows, $C$ is a generic constant independent of $\nu$.

On the other hand, rewriting (2.6) as

\[
\frac{d}{dt} \| P_{\neq 0} \omega^\nu(t) \|_X^2
\]

\[
= -2\nu \| \nabla P_{\neq 0} \omega^\nu(t) \|_X^2 - \frac{d}{dt} \| P_0 \omega^\nu(t) \|_X^2 - 2\nu \| \nabla P_0 \omega^\nu(t) \|_X^2 \quad (2.11)
\]

and taking (2.10) into account, we have

\[
\frac{d}{dt} \| P_{\neq 0} \omega^\nu(t) \|_X^2 \leq -2(\nu - \sigma C \| \omega^\nu(0) \|_X) \| \nabla P_{\neq 0} \omega^\nu(t) \|_X^2
\]

\[
\leq -2\nu(1 - \sigma Cd) \| \nabla P_{\neq 0} \omega^\nu(t) \|_X^2 \leq -\frac{15}{8} \nu \| \nabla P_{\neq 0} \omega^\nu(t) \|_X^2,
\]

as $d$ is small so that $2(1 - Cd) > \frac{15}{8}$ when $\sigma = 1$. \hfill \Box

The enhanced damping is to be confirmed when a solution to (2.4) is close to that of the linearized Euler equation (2.2) in the following sense.
Lemma 2.3. For $\tau, \delta, d$ and $\nu$ given in Theorem 1.1, let $\omega^\nu$ be a solution of (2.4) with $\omega^\nu(0) \in L_2(T_\alpha)$ so that $\|\omega^\nu(0)\|_{L_2} < \nu d$ for $\sigma = 1$ and let $\omega^0$ be a solution of (2.2) with $\omega^0(0) \in X$ and $\|\nabla \omega^0(0)\|_X < \infty$. If the estimate

$$\|P_{\neq 0}\omega^\nu(t) - \omega^0(t)\|_X^2 \leq \|P_{\neq 0}\omega^\nu(0) - \omega^0(0)\|_X^2 + C_1\nu(1+t^3)\|\nabla \omega^0(0)\|_X^2$$

(2.12)

holds true for a constant $C_1$ independent of $\nu$, then

$$\left\| P_{\neq 0}\omega^\nu \left( \frac{T}{\nu^2} \right) \right\|_{L_2} < \delta \left\| P_{\neq 0}\omega^\nu(0) \right\|_{L_2}.$$  (2.13)

For the unforced case $a = 0$, similar result has been presented in [12, pp. 1824-1825] originated from [9, Proof of Theorem 1.4]. Following [9], we adopt a compact set $K \subset S$ required by Lemma 2.1.

Proof. For the given $\delta$ and $\tau > 0$, we choose $N$ large enough so that

$$e^{-\lambda_N\tau} < \frac{\alpha^2 - 1}{\alpha^2} \delta^2.$$  (2.14)

In order to use Lemma 2.1, we set the compact set

$$K = \left\{ \phi \in X | \|\nabla \phi\|_X^2 \leq \lambda_N, \|\phi\|_X = 1 \right\}.$$

Let $t_1 = T_c(N, \frac{1}{10}, K)$ from Lemma 2.1 and $C_1$ from (2.12). Choose a small viscosity $\nu_0$ so that

$$\nu_0 C_1(1 + t_1^3) < \frac{1}{10\lambda_N} \quad \text{and} \quad \nu_0 < \frac{\tau}{3t_1}.$$  (2.15)

For any $0 < \nu < \nu_0$, if

$$\lambda_N \left\| P_{\neq 0}\omega^\nu(t) \right\|_X^2 \leq \left\| \nabla P_{\neq 0}\omega^\nu(t) \right\|_X^2$$

for all $t \in [0, \frac{T}{\nu^2}]$, it follows from (2.8) and (2.14) that

$$\left\| P_{\neq 0}\omega^\nu \left( \frac{T}{\nu^2} \right) \right\|_X^2 \leq e^{-\lambda_N\tau} \left\| P_{\neq 0}\omega^\nu(0) \right\|_X^2 < \delta^2 \left\| P_{\neq 0}\omega^\nu(0) \right\|_{L_2}^2$$

(2.16)

and we are done.

Otherwise, there exit $t_0 \in [0, \frac{T}{\nu^2})$ being the first time in the interval such that

$$\left\| \nabla P_{\neq 0}\omega^\nu(t_0) \right\|_X^2 \leq \lambda_N \left\| P_{\neq 0}\omega^\nu(t_0) \right\|_X^2.$$  (2.17)
Now we take the solution \( \omega^0(t_0 + t) \) of (2.2) for \( t \in [0, t_1] \) initiated from

\[
\omega^0(t_0) \equiv P_{\neq 0} \omega^\nu(t_0) \in \{ \omega \in X | \| \nabla \omega \|_X < \infty \}.
\] (2.18)

It follows from (2.12), (2.15), (2.17) and (2.18) that, for \( t \in [0, t_1] \),

\[
\| P_{\neq 0} \omega^\nu(t_0 + t) - \omega^0(t_0 + t) \|^2_X \leq \nu_0 C_1 (1 + t_1^3) \| \nabla \omega^0(t_0) \|^2_X \leq \frac{1}{10} \| \omega^0(t_0) \|^2_X.
\] (2.19)

Since \( \omega^0(t_0)/\| \omega^0(t_0) \|_X \in K \), by the definition of \( t_1 \) and Lemma 2.1, we have

\[
\frac{1}{t_1} \int_0^{t_1} \| P_N^{-1} \omega^0(t_0 + t) \|^2_X dt \leq \frac{1}{10} \| \omega^0(t_0) \|^2_X.
\] (2.20)

By (2.20) and the conservation property (2.6) with \( \nu = 0 \) for the linearized Euler equation (2.2), we have

\[
\frac{1}{t_1} \int_0^{t_1} \| (I - P_N) \omega^0(t_0 + t) \|^2_X dt = \| \omega^0(t_0 + t) \|^2_X - \frac{1}{t_1} \int_0^{t_1} \| P_N \omega^0(t_0 + t) \|^2_X dt
\]

\[
= \| \omega^0(t_0) \|^2_X - \frac{1}{t_1} \int_0^{t_1} \| P_N \omega^0(t_0 + t) \|^2_X dt
\]

\[
\geq \frac{9}{10} \| \omega^0(t_0) \|^2_X.
\]

This together with (2.19) implies

\[
\frac{1}{t_1} \int_0^{t_1} \| (I - P_N) P_{\neq 0} \omega^\nu(t_0 + t) \|^2_X dt
\]

\[
\geq \frac{1}{t_1} \int_0^{t_1} \| (I - P_N) \omega^0(t_0 + t) \|^2_X dt - \frac{1}{t_1} \int_0^{t_1} \| P_{\neq 0} \omega^\nu(t_0 + t) - \omega^0(t_0 + t) \|^2_X dt
\]

\[
\geq \frac{4}{5} \| \omega^0(t_0) \|^2_X.
\]

Hence, we have

\[
\int_0^{t_1} \| \nabla P_{\neq 0} \omega^\nu(t_0 + t) \|^2_X dt \geq \lambda_N \int_0^{t_1} \| (I - P_N) P_{\neq 0} \omega^\nu(t_0 + t) \|^2_X
\]

\[
\geq \frac{4 \lambda_N t_1}{5} \| \omega^0(t_0) \|^2_X.
\]
By (2.28) and the previous inequality, we have
\[
\|P_{\neq 0} \omega(t_0 + t_1)\|_X^2 \leq \|P_{\neq 0} \omega(t_0)\|_X^2 - \frac{15}{8} \nu \int_0^{t_1} \|\nabla P_{\neq 0} \omega(t_0 + t)\|_X^2 dt
\]
\[
\leq (1 - \frac{3}{2} \lambda_N \nu t_1) \|P_{\neq 0} \omega(t_0)\|_X^2
\]
\[
\leq e^{-\frac{3}{2} \lambda_N \nu t_1} \|P_{\neq 0} \omega(t_0)\|_X^2.
\] (2.21)

Moreover, for any interval \([\tau_0, \tau_1]\) such that
\[
\|\nabla P_{\neq 0} \omega(t)\|_X^2 \geq \lambda_N \|P_{\neq 0} \omega(t)\|_X^2, \quad t \in [\tau_0, \tau_1],
\] (2.22)
we have by (2.8) that
\[
\|P_{\neq 0} \omega(\tau_1)\|_X^2 \leq e^{-\lambda_N \nu (\tau_1 - \tau_0)} \|P_{\neq 0} \omega(\tau_0)\|_X^2.
\] (2.23)

Hence, the combination of all decay estimates from (2.21) and (2.23), and the use of \(t_1 < \frac{\tau_1}{3\nu}\) from (2.15) imply the existence of \(t_2 \in \left[\frac{2\tau_1}{3\nu}, \frac{\tau_1}{\nu}\right]\) such that
\[
\|P_{\neq 0} \omega(t_2)\|_X^2 \leq e^{-\frac{4}{3} \lambda_N \nu t_2} \|P_{\neq 0} \omega(0)\|_X^2 \leq e^{-\lambda_N \tau} \|P_{\neq 0} \omega(0)\|_X^2.
\] (2.24)

Therefore, we have
\[
\left\|P_{\neq 0} \omega\left(\frac{\tau_1}{\nu}\right)\right\|_{L^2}^2 \leq \frac{\alpha^2}{\alpha^2 - 1} \left\|P_{\neq 0} \omega\left(\frac{\tau_1}{\nu}\right)\right\|_X^2
\]
\[
\leq \frac{\alpha^2}{\alpha^2 - 1} \left\|P_{\neq 0} \omega(t_2)\right\|_X^2, \quad \text{by (2.8)},
\]
\[
\leq \frac{\alpha^2}{\alpha^2 - 1} e^{-\lambda_N \tau} \left\|P_{\neq 0} \omega(0)\right\|_X^2, \quad \text{by (2.24)},
\]
\[
\leq \delta^2 \left\|P_{\neq 0} \omega(0)\right\|_X^2, \quad \text{by (2.14)},
\]
\[
\leq \delta^2 \|P_{\neq 0} \omega(0)\|_{L^2}^2.
\]

This completes the proof of Lemma 2.3.

\[\square\]

2.2. Proof of Theorem 1.1

By Lemma 2.3, it suffices to prove the validity of (2.12) for the solutions \(\omega\) of (2.4) and \(\omega^0\) of (2.2) when \(\omega(0) \in X\) and \(\omega^0(0) \in X^1\).

Let \(\omega = P_{\neq 0} \omega - \omega^0\). Thus the difference \(\omega\) is subject to the equation
\[
\partial_t \omega = \nu DP_{\neq 0} \omega - a \sin y \partial_x (1 + \Delta^{-1}) \omega - e^{-\nu t} \sin y \partial_x (1 + \Delta^{-1}) \omega
\]
\[
- (e^{-\nu t} - 1) \sin y \partial_x (1 + \Delta^{-1}) \omega^0 - \sigma P_{\neq 0} J(\Delta^{-1} \omega, \omega^\nu).
\]
Taking $L^2$ inner product of the previous equation with $(1 + \Delta^{-1})\omega$ and integrating by parts, we have

$$\frac{1}{2} \frac{d}{dt} \|\omega\|_X^2 + \nu \|\nabla P_{\neq 0} \omega^\nu\|_X^2 = -\nu(\Delta P_{\neq 0} \omega^\nu, (1 + \Delta^{-1})\omega^0 - (e^{-\nu t} - 1)(\sin y \partial_x (1 + \Delta^{-1})\omega^0, (1 + \Delta^{-1})P_{\neq 0} \omega^\nu) - \sigma(P_{\neq 0} J(\Delta^{-1} \omega^\nu, \omega^\nu), (1 + \Delta^{-1})\omega). \quad (2.25)$$

According to integration by parts, the first two linear terms on the right-hand side of (2.25) are bounded by

$$\nu \|\nabla P_{\neq 0} \omega^\nu\|_X \|\nabla \omega^0\|_X + \nu t \|\nabla (1 + \Delta^{-1})\omega^0\|_L^2 \|1 + \Delta^{-1})P_{\neq 0} \omega^\nu\|_L^2 \leq \nu \|\nabla P_{\neq 0} \omega^\nu\|_X \|\nabla \omega^0\|_X + \nu t \|\omega^0\|_X \|P_{\neq 0} \omega^\nu\|_X. \quad (2.26)$$

For the nonlinear problem $\sigma = 1$, we adopt the orthogonal decomposition

$$\omega^\nu = P_{\neq 0} \omega^\nu + P_0 \omega^\nu.$$

Integrating by parts and employing Hölder inequality, Cauchy inequality and Sobolev imbedding, we estimate the third term on the right-hand side of (2.25) as

$$-(P_{\neq 0} J(\Delta^{-1} \omega^\nu, \omega^\nu), (1 + \Delta^{-1})\omega)\] = -(P_{\neq 0} J(\Delta^{-1} P_0 \omega^\nu, P_0 \omega^\nu) + P_{\neq 0} J(\Delta^{-1} P_{\neq 0} \omega^\nu, P_0 \omega^\nu), (1 + \Delta^{-1})\omega)\]$$

$$\leq C \|\Delta^{-1} \omega^\nu\|_{L^\infty} \|\nabla P_{\neq 0} \omega^\nu\|_{L^2} + \|\Delta^{-1} P_{\neq 0} \omega^\nu\|_{L^\infty} \|P_0 \omega^\nu\|_{L^2}$$

$$\leq C \|\nabla P_{\neq 0} \omega^\nu\|_{L^2} \|\omega^\nu\|_{L^2} \|\nabla \omega^\nu\|_{L^2} \leq C \|\nabla P_{\neq 0} \omega^\nu\|_{L^2}^2 \|\omega^\nu\|_{L^2} + C \|\omega^\nu\|_{L^2} \|\nabla \omega^\nu\|_{L^2}^2 \quad (2.27)$$
Collecting terms of (2.25)-(2.27) and employing Cauchy inequality, we have

$$
\frac{d}{dt} \| \omega(t) \|_X^2 \leq -2\nu \| \nabla P_{\neq 0} \omega^{\nu}(t) \|_X^2 + 2\nu \| \nabla P_{\neq 0} \omega^{\nu}(t) \|_X \| \nabla \omega^0(t) \|_X \\
+ 2\nu t \| \nabla P_{\neq 0} \omega^{\nu}(t) \|_X \| \omega^0(t) \|_X + C\sigma \| \nabla P_{\neq 0} \omega^{\nu}(t) \|_X \| \omega^{\nu}(t) \|_{L^2} \\
+ C\sigma \| \nabla P_{\neq 0} \omega^{\nu}(t) \|_X \| \omega^{\nu}(t) \|_{L^2} \| \nabla \omega^0(t) \|_X \\
\leq -\nu \| \nabla P_{\neq 0} \omega^{\nu}(t) \|_X^2 + 2\nu \| \nabla \omega^0(t) \|_X^2 + 2\nu t^2 \| \omega^0(t) \|_X^2 \\
+ C\sigma \| \nabla P_{\neq 0} \omega^{\nu}(t) \|_X \| \omega^{\nu}(t) \|_{L^2} + C\sigma \| \omega^{\nu}(t) \|_{L^2} \| \nabla \omega^0(t) \|_X^2.$$

(2.28)

Therefore, it remains to provide a uniform estimate of \( \| \omega^{\nu}(t) \|_{L^2} \) upper bounded by \( \| \omega^{\nu}(0) \|_{L^2} \). To do so, we note that the operator \( 1 + \Delta^{-1} \) is positive on the space \( (I - P_K) L^2(T_\alpha) \) due to \( \alpha > 1 \), and employ the \( L^2 \) estimate (2.6) to obtain

$$
\min\left\{ \frac{\alpha^2 - 1}{\alpha^2}, \frac{3}{4} \right\} \| (I - P_K) \omega^{\nu}(t) \|_{L^2} \leq \| \omega^{\nu}(t) \|_X^2 \leq \| \omega^{\nu}(0) \|_X^2 \leq \| \omega^{\nu}(0) \|_{L^2}^2.
$$

(2.29)

Moreover, integrating by parts, we see that

$$
\int_{T_\alpha} J(\Delta^{-1} P_K \omega^{\nu}, \omega^{\nu}) P_K \omega^{\nu} dxdy = \int_{T_\alpha} \left( \partial_y (\omega^{\nu} \partial_x \Delta^{-1} P_K \omega^{\nu}) - \partial_x (\omega^{\nu} \partial_y \Delta^{-1} P_K \omega^{\nu}) \right) P_K \omega^{\nu} dxdy = 0.
$$

Hence, taking the inner product of (2.4) with \( P_K \omega^{\nu} \) and integrating by parts, we have

$$
\frac{d}{dt} \| P_K \omega^{\nu}(t) \|_{L^2}^2 + 2\nu \| P_K \omega^{\nu}(t) \|_{L^2}^2 \\
= -2\nu (J(\Delta^{-1} (I - P_K) \omega^{\nu}(t), (I - P_K) \omega^{\nu}(t)), (I - P_K) \omega^{\nu}(t)) \\
\leq 2\nu \| \Delta^{-1} (I - P_K) \omega^{\nu}(t) \|_{L^2} \| (I - P_K) \omega^{\nu}(t) \|_{L^2} ||\nabla P_K \omega^{\nu}(t) ||_{L^\infty} \\
\leq C \| (I - P_K) \omega^{\nu}(t) \|_{L^2}^2 || P_K \omega^{\nu}(t) ||_{L^2} \\
\leq \nu \| P_K \omega^{\nu}(t) \|_{L^2}^2 + C \frac{1}{\nu} \| \omega^{\nu}(0) \|_{L^2}^2.
$$

(2.30)

after the use of (2.29). Integrating (2.30) and combining the resultant equa-
tion with (2.29), we have
\[
\|\omega''(t)\|^2_{L^2} \leq \left(1 + \frac{\alpha^2}{\alpha^2 - 1} + \frac{4}{3}\right)\|\omega''(0)\|^2_{L^2} + C \frac{1}{\nu} \int_0^t e^{-\nu(t-s)}\|\omega''(0)\|^4_{L^2} ds
\]
\[
\leq C\left(\|\omega''(0)\|^2_{L^2} + \frac{1}{\nu^2}\|\omega''(0)\|^4_{L^2}\right) \leq C\nu^2 d^2
\]  
(2.31)
due to the smallness assumption of \(\|\omega''(0)\|_{L^2}\).

Therefore, by (2.31), the conservation of \(\|\omega_0(t)\|_X\) from (2.6) with \(\nu = 0\), and the estimate \([12, \text{Lemma 2.3}]\)
\[
\|\nabla\omega^0(t)\|_X \leq C(1 + t)\|\nabla\omega^0(0)\|_X,
\]  
equation (2.28) can be estimated as
\[
\frac{d}{dt}\|\omega(t)\|^2_X \leq -\nu\|\nabla P_{\neq 0}\omega''(t)\|^2_X + 2\nu\|\nabla\omega^0(t)\|^2_X + 2\nu t^2\|\omega^0(t)\|^2_X
\]
\[
+ C\sigma\|\nabla P_{\neq 0}\omega''(t)\|^2_X\|\omega''(t)\|_{L^2} + C\sigma\|\omega''(t)\|_{L^2}\|\nabla\omega^0(t)\|^2_X
\]
\[
\leq -\nu\|\nabla P_{\neq 0}\omega''(t)\|^2_X + \nu C(1 + t)^2\|\nabla\omega^0(0)\|^2_X + 2\nu t^2\|\omega^0(0)\|^2_X
\]
\[
+ C\sigma d\|\nabla P_{\neq 0}\omega''(t)\|^2_X + C\sigma d(1 + t)^2\|\nabla\omega^0(0)\|^2_X
\]
\[
\leq \nu C(1 + t^2)\|\nabla\omega^0(0)\|^2_X
\]  
(2.32)
for \(C\) independent of \(\nu\) and for \(d\) small so that \(Cd \leq 1\). Hence, we obtain the validity of (2.12), after the integration of the previous equation.

The proof of Theorem 1.1 is complete.

3. Enhanced and unenhanced dampings of exact solutions

In this section, we seek exact solutions, which damps in a manner not discussed in Theorem 1.1.

Let \(\omega\) solve the Navier-Stokes equation (1.2) with \(\alpha > 0\) and \(a \in \{0, 1\}\). If \(\omega\) satisfies the equation
\[
J(\Delta^{-1}\omega, \omega) = 0,
\]  
(3.1)
which is valid when \(\omega\) is a Laplacian eigenfunction expressed as
\[
\Delta \omega = \lambda \omega.
\]  
(3.2)
Hence (1.2) becomes the linear heat equation
\[
\partial_t \omega = \nu \Delta \omega - \nu a \cos y,
\]  
(3.3)
and therefore \(\omega\) is in the following form
\[
\omega = -a \cos y + e^{\nu \Delta t}\left(\omega(0) + a \cos y\right) = -a \cos y + ae^{\nu t} \cos y + e^{\nu \Delta t} \omega(0).
\]
3.1. Exact solutions of (1.2) with $a = 0, 1$

For any function $\phi \in P_0L_2(\mathbb{T}_\alpha)$ or $P \neq 0 \phi = 0$, $\phi$ is a function independent of $x$. That is,

$$\phi = \sum_{n \geq 1} (a_n \cos ny + b_n \sin ny) \in L_2(\mathbb{T}_\alpha)$$

with constants $a_n$ and $b_n$. Therefore (1.2) admits the exact solution

$$\omega = -a \cos y + e^{-\nu t} a \cos y + e^{\nu \Delta t} \phi$$

$$= -a \cos y + e^{-\nu t} a \cos y + \sum_{n \geq 1} e^{-n^2 \nu t} (a_n \cos ny + b_n \sin ny), \quad (3.4)$$

since $\omega$ is independent of $x$ and hence (3.1) holds true.

When $\alpha < 1$, Kolmogorov flow losses stability, but the solutions (3.4) and (3.6) remain in its nonlinear stable manifold.

The solution (3.4) is a parallel flow moving in horizontal lines $y =$ constants, and is in the kernel of the projection operator $P \neq 0$. Thus the solution (3.4) is absent in the metastability estimate in Theorem 1.1 due to the energy estimate (2.8). However, equation (3.4) shows that the stability can be enhanced if $\omega$ involves high frequency modes only.

For nonparallel flows, we consider (1.2) in the extended flat torus domain

$$[0, 2\pi/\alpha) \times [0, 2\pi n)$$

with $\alpha^2 + \frac{1}{n^2} = 1$ and $n \geq 1$. \quad (3.5)

We have the unenhanced damping solution

$$\omega = -\cos y + e^{-\nu t} \cos y + e^{-\nu t} \left( c_1 \sin y + c_2 \cos y + c_3 \sin \alpha x \sin \frac{y}{n} \right)$$

$$+ e^{-\nu t} \left( c_4 \cos \alpha x \cos \frac{y}{n} + c_5 \sin \alpha x \cos \frac{y}{n} + c_6 \cos \alpha x \sin \frac{y}{n} \right) \quad (3.6)$$

for any constants $c_i$, since (3.2) and hence (3.1) hold true. For displaying purpose, we choose a function from (3.6) as

$$\omega = -\cos y + e^{-\nu t} \cos y + e^{-\nu t} \sin \frac{\sqrt{3}}{2} x \sin \frac{1}{2} y$$

$$\quad (3.7)$$

in the torus domain $[0, \frac{4\pi}{\sqrt{3}}) \times [0, 4\pi)$. This solution transforms initially from the four vortices state $\sin \frac{\sqrt{3}}{2} x \sin \frac{1}{2} y$ to the horizontal parallel flow $-\cos y$ (see Figure 1 for the transition). This transition exhibits inverse cascade of two-dimensional fluid motion by enlarging two diagonal vortices, while other two vortices are compressed and then destroyed into pieces.
3.2. More exact solutions of (1.2) with $a = 0$

For the case $a = 0$, the basic stationary solution $-a \cos y$ becomes zero. We thus have more freedom to construct exact solutions.

To display further solutions in $T_\alpha$ with $\alpha > 0$, we choose the following
functions

$$\omega_1 = \sum_{k \geq 1} e^{-\nu(n^2 + m^2)k^2 t}(a_k \cos(k\alpha x + k\alpha y) + b_k \sin(k\alpha x + k\alpha y)) \in L^2(T_\alpha),$$

(3.8)

$$\omega_2 = e^{-\nu(n^2 + m^2)t}\left(c_1 \sin n\alpha x \sin m\alpha y + c_2 \cos n\alpha x \sin m\alpha y \right.$$

$$+ c_3 \sin n\alpha x \cos m\alpha y + c_4 \cos n\alpha x \cos m\alpha y\big),$$

(3.9)

$$\omega_3 = \omega_2 + e^{-\nu j^2 t}(c_5 \sin j\alpha x + c_6 \cos j\alpha x), \text{ when } \alpha^2 n^2 + m^2 = j^2,$$

(3.10)

$$\omega_4 = \omega_2 + e^{-\nu j^2 t}(c_5 \sin i\alpha x + c_6 \cos j\alpha x)$$

$$+ e^{-\nu j^2 t}(c_7 \sin j\alpha y + c_8 \cos j\alpha y), \text{ when } \alpha^2 n^2 + m^2 = \alpha^2 i^2 = j^2,$$

(3.11)

for any given positive integers $n, m, i, j$ and reals $a_k, b_k, c_l$.

It is readily seen that $\omega_2, \omega_3$ and $\omega_4$ solve (1.2) due to the validity of (3.2) and hence (3.1). To show $\omega_1$ satisfying (3.1) for $t \geq 0$, we use the notation $\phi_k = \cos(k\alpha x + k\alpha y)$ and $\varphi_k = \sin(k\alpha x + k\alpha y)$ to obtain

$$J(\Delta^{-1}\omega_1, \omega_1)$$

$$= -\sum_{k, k' \geq 1} e^{-\nu(k^2 + k'^2)(n^2 + m^2)t} \frac{kn\alpha(-a_k \varphi_k + b_k \phi_k) mk'(-a_k \varphi_{k'} + b_k' \phi_{k'})}{n^2 \alpha^2 k^2 + m^2 k'^2}$$

$$+ \sum_{k, k' \geq 1} e^{-\nu(k^2 + k'^2)(n^2 + m^2)t} \frac{km (-a_k \varphi_k + b_k \phi_k)n\alpha k'(-a_k \varphi_{k'} + b_k' \phi_{k'})}{n^2 \alpha^2 k^2 + m^2 k'^2} = 0.$$

Therefore, $\omega_1$ solves (1.2) and is a parallel flow or a bar flow moving along the straight streamlines $n\alpha x + m\alpha y = \text{constants}$.

For example, we take a function from (3.8) as

$$\omega = e^{-4(\alpha^2 + 1)\nu t}\sin(2\alpha x + 2y) + e^{-16(\alpha^2 + 1)\nu t}\cos(4\alpha x + 4y)$$

(3.12)

for $\alpha = \sqrt{6}/2$. 

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Figure 2: Vorticity contours of solutions (3.12) in (a)-(b), (3.13) in (c) and (3.14) in (d) for $\nu = 0.01$.

For nonparallel flows, we choose a solution from (3.10) with $\alpha = \sqrt{5}$ expressed as

$$\omega = e^{-(\alpha^2+4)\nu t} \left( \sin(\alpha x) \sin(2y) + 0.3 \cos(\alpha x) \cos(2y) \right) + 0.4 e^{-9\nu t} \sin(3y),$$

and adopt a solution from (3.11) with $\alpha = 1$ in the following form

$$\omega = e^{-(16\alpha^2+9)\nu t} \left( 0.4 \sin(4\alpha x) \sin(3y) + 0.5 \cos(4\alpha x) \cos(3y) \right)$$

$$+ e^{-25\nu t} \left( \sin(5y) + 0.3 \sin(5x) \right).$$

Vortex contours of enhanced damping flows (3.12), (3.13) and (3.14) are displayed in Figure 2. The solutions (3.9)-(3.11) and their special forms (3.13)-(3.14) can be written as

$$\omega(t) = e^{-(a^2 n^2 + m^2)\nu t}\omega(0),$$

(3.15)
which decays with $t$. The vorticity contours of $\omega(t)$ on the horizontal planes $z = \text{constants}$ are the same with those of $\omega(0)$ on the horizontal planes $z = e^{-(\alpha^2 n^2 + m^2)t}$ constants. Therefore, in Figure 2 (c)-(d), we only provide vorticity contours at the initial state $t = 0$, as the flow patterns remain quasi-stationary when $t$ grows. On the other hand, when $\nu$ is sufficiently small and $t$ is moderate, the exact solutions evolve in a quasi-stationary manner and are close to their initial states, which solve the stationary Euler equation. Solutions (3.8)-(3.11) also exhibit quasi-stationary states such as bar states (parallel flows), dipole states and quadrupole states discussed by Yin et al. [18].

It should be noted that the solution $e^{-(\alpha^2 n^2 + m^2)t} \sin(\alpha nx) \sin(my)$ is known as Taylor flow given by Taylor [15].

**Remark 3.1.** If the fluid domain is the horizontal channel $\mathbb{R} \times [0, 2\pi]$ and the fluid motion is additionally driven by the up moving boundary in the following sense

$$u|_{y=2\pi} = (2\pi, 0), \quad u|_{y=0} = (0, 0),$$

the Navier-Stokes system (1.1) refers to a forced Couette flow problem and has the exact parallel solution

$$u = \left( y + a \sin y + \sum_{n \geq 1} e^{-\nu n^2 t} a_n \sin ny, 0 \right)$$

for any coefficients $a_n$ so that $\sum_{n \geq 1} n^2 a_n^2 < \infty$.

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