Disordered Green’s Functions for Random Resistor Networks

Sayak Bhattacharjee$^{1,*}$ and Kabir Ramola$^{2,†}$

$^1$Department of Physics, Indian Institute of Technology Kanpur, Kanpur 208016, India
$^2$Tata Institute of Fundamental Research, Hyderabad 500107, India

(Dated: October 28, 2022)

We analyze random resistor networks through a study of disordered lattice Green’s functions in arbitrary dimensions. We develop a systematic disorder perturbation expansion to describe the weak disorder regime of such a system. We use this formulation to compute ensemble averaged nodal voltages and bond currents in a hierarchical fashion. We verify the validity of this expansion with direct numerical simulations of a square lattice with resistances at each bond chosen from an exponential distribution. Additionally, we construct a formalism to recursively obtain the exact Green’s functions for finitely many disordered bonds. We provide explicit expressions for lattices with up to four disordered bonds, which can be used to predict nodal voltage distributions for arbitrarily large disorder strengths. Finally, we introduce a novel order parameter that measures the overlap between the bond current and the optimal path (the path of least resistance), for a given resistance configuration, which helps to characterize the weak and strong disorder regimes of the system.

I. INTRODUCTION

The conductivity of disordered random media is a problem of fundamental interest in a variety of condensed matter contexts, including in transport measurements [1, 2] and critical phenomena [3, 4]. Simple tractable models that are often used for modeling such phenomena are electrical networks, most often constituting resistor elements, studied either in the steady-state or as a dynamical system. Recent interest in electric network models has also arisen as synthetic experimental test-beds for topological quantum matter [5–7]. In order to model natural systems, microscopic disorder in such model systems is an important ingredient. Such situations often require a subtle understanding of the properties of the disordered lattice Green’s function (which is the inverse of the lattice Laplacian), and therefore an investigation into techniques that can be used to compute Green’s functions for disordered electrical networks represents a fundamental direction of theoretical as well as experimental relevance.

Within this context, random resistor networks are a popular paradigm for modeling transport in disordered media such as semiconductors [1, 8–15], but also in directed polymers [16–18], and porous rocks [19, 20]. Formally defined, a random resistor network is a network of resistors such that the resistances are sampled from a probability distribution, and the disorder strength may be controlled by a tunable parameter [21–23]. Studies of such networks can be approached through a multitude of ways, most primarily by percolation theory [2, 8, 12, 21, 24–28], but also through random walks [29, 30], and optimization theory [16, 31–33].

In this work, we consider a random resistor network in which resistances are sampled from an exponentially wide distribution [13, 22, 34]. This model is also termed as the hopping percolation model [2, 15, 25] and can be motivated from physical considerations: the conductance between sites in disordered media is proportional to $\exp(-r_{ij}/r_0 - E_{ij}/k_BT)$ where $r_0$ is a length scale for the decay of the wavefunction of the grains, $r_{ij}$ is the distance and $E_{ij}$ is the energy difference between two sites $i$ and $j$. Thus, an exponential disorder of the form $\exp(ax_{ij})$ is natural: the strength $a$ represents an energy and/or length scale of the disordered system corresponding to the random variable $x_{ij}$ [15].

In exponentially disordered networks, past studies have identified two disorder regimes—a weak disorder regime when $L \gg a^\nu$ and a strong disorder regime for $L \ll a^\nu$, where $\nu$ is the percolation connectedness exponent ($\nu = 4/3$ in two dimensions) [2, 15, 25, 34, 35]. The strong disorder regime is characterized by optimal behavior: the current distribution collapses to a self-similar fractal optimal path [33, 34] (see Fig. 2 for a visualization), whose critical exponents ($\delta_{opt} = 1.22$ in two dimensions) have been numerically computed [16]. In the weak disorder regime, the current distribution is delocalized throughout the lattice and the optimal path (defined as the path of least resistance) is shown to be self-affine with critical exponents belonging to the universality class of directed polymers [32, 33]. This crossover from self-affine to self-similar is a characteristic of wide exponential disorder; studies for Gaussian and uniform distributions give self-affine optimal paths across disorder strength [36]. Optimal paths have, of course, been understood as equivalent to domain walls in spin systems as well, where the impurities of the system help pin the wall to energetically favourable sites in the system [37–39]. While critical exponents of the disorder regimes have been explored in depth, a scalable analytical toolbox to analyse such disordered networks has not yet been developed.

In this article, we fill this gap by demonstrating the use of two analytic techniques to compute the disordered lattice Green’s functions of the system. First, we con-
Next, we develop an exact formulation using a dyadic perturbation of the lattice Green’s function, that can in principle yield exact results for arbitrary disorder. The dyadic bond formulation enables us to provide exact formulæ for the Green’s function for a single broken bond in the system [41]. We extend such a formulation to an arbitrary number of bonds with disorder and are able to give an analytically tractable expression for lattices with small number of disordered bonds. In particular, we provide explicit formulæ for lattices with up to four disordered bonds.

We also perform numerical simulations that corroborate our theoretical results, as well as help us probe the different disorder regimes of the system. We compute mean nodal voltages with one, two and three disordered bonds in the system, and study their fluctuations. The fluctuations are shown to peak at a critical disorder strength and our numerics match perfectly with the analytical predictions from exact formulæ for the disordered Green’s functions. We also provide a novel order parameter which we term bond current fidelity. This is defined as the overlap between the current distribution at a particular disorder strength and the optimal path for a particular resistance configuration. We demonstrate clear signatures of the weak and strong disorder regimes in this order parameter, and its scalings are shown to be in line with previously known critical exponents and our analytical predictions using the Green’s function formalism.

The rest of the paper is organized as follows. In Sec. II we introduce the random resistor model and its lattice Laplacian formulation. We discuss the disorder perturbation expansion for the Green’s function and compute the nodal voltages perturbatively in Sec. III. In Sec. IV, we introduce the exact formulation to obtain Green’s functions for arbitrarily many bonds with disorder in the lattice. This work is supplemented by numerical techniques for nodal voltages and discussion of an order parameter in Sec. V. Finally, we discuss and conclude the work in Sec. VI, and present directions for future research.

FIG. 1. The lattice convention used in this paper. Each lattice site \( r_i \) is represented by a \( L^d \) dimensional column vector \(|s_i\rangle\), that form an orthonormal basis. The basic lattice translation vectors are denoted by \( \{\Delta_k\} \equiv \{\pm \hat{x}, \pm \hat{y}, \ldots, \pm \hat{z}\} \) with \( \{\Delta_k^+\} \equiv \{\hat{x}, \hat{y}, \ldots, \hat{z}\} \) denoting the positive translation vectors. The bonds in the lattice may be denoted by bond vectors \(|b_m\rangle \equiv |s_i\rangle - |s_j\rangle\) where the sites \( i \) and \( j \) are connected by a bond on the lattice.

II. RANDOM RESISTOR NETWORK

In this work, we consider a \( d \)-dimensional hypercubic lattice of linear dimension \( L \) with periodic boundary conditions, and resistors placed at each bond. There are \( L^d \) sites and \( dL^d = (N_b) \) bonds on such a \( d \)-dimensional torus. While we consider a hypercubic lattice, our formulation can be easily adapted to other lattices. Each lattice site \( r_i \equiv (x_i, y_i, \ldots, z_i) \) is denoted by the single site vector \(|s_i\rangle \) where \( 1 \leq i \leq L^d \). For the formulation developed in this paper, we find it convenient to use bra-ket notation to denote the total degrees of freedom on the lattice. We define a basis set of \( L^d \) dimensional vectors \(|s_i\rangle\) that denote the sites, so that the basis vector \(|s_i\rangle\) (denoted equivalently by site index \( i \)) is the \( i^{th} \) unit vector. Thus, \( \langle s_j | s_i \rangle = \delta_{ij} \) \((1 \leq i, j \leq L^d)\) and the site vectors form a complete orthonormal set, that is, \( \sum_i \langle s_i | s_i \rangle = 1 \). A site vector subscript on a ket denotes the corresponding entry of the column vector, for example, \(|s_i|_{j^*}\) is the \( j^{th} \) entry of this column vector. The basic lattice translation vectors are denoted by \( \{\Delta_k\} \equiv \{\pm \hat{x}, \pm \hat{y}, \ldots, \pm \hat{z}\} \) with \( \{\Delta_k^+\} \equiv \{\hat{x}, \hat{y}, \ldots, \hat{z}\} \) denoting the positive translation vectors and \( k \equiv x, y, \ldots, z \) is the axis label. The bonds in the lattice may be denoted by bond vectors

\[
|b_m\rangle := |s_i\rangle - |s_j\rangle
\]  

where the sites \( i \) and \( j \) are connected by a bond on the lattice, that is, \( r_j = r_i + \Delta_k^+ \) and the bond index \( m \) \((1 \leq m \leq N_b)\) is equivalent to the tuple \((r_i, \Delta_k^+)\) uniquely marking the bond (see Fig. 1). Note that to define the bonds, it suffices to restrict to the positive translation vectors to avoid double counting of the bonds. For brevity of notation, in the rest of this article, we denote
an arbitrary site by a variable \( \vec{r} \) and a translation vector by \( \Delta \) (or \( \Delta^+ \)), that is, without the subscript.

In the network, each of the bonds between two sites \( \vec{r}_i \) and \( \vec{r}_j \) has a resistance of magnitude \( R_{\vec{r}_i, \vec{r}_j} \). Using the voltages and total current at each site (with the convention that the total current entering the node is positive), we may define a nodal voltage vector \( |V\rangle \) and a nodal current vector \( |I\rangle \). The nodal voltages in this problem are the relevant degrees of freedom. As motivated in the introduction, we study the response of the system under increasing disorder strength. Two configurations are logical: we may either (a) fix the voltages at specific lattice sites or (b) fix the currents at specific lattice sites. We choose the latter in this study, motivated by the desire to develop a theory with the nodal voltages being the degrees of freedom, and also study the convergence to optimal path in the strong disorder regime. We thus set the nodal current configuration to be given by

\[
|I\rangle_{\vec{r}} := \delta_{\vec{r}, \vec{r}_{\text{in}}} - \delta_{\vec{r}, \vec{r}_{\text{out}}},
\]

where the current source node is \( \vec{r}_{\text{in}} \) and the current sink node is \( \vec{r}_{\text{out}} \). We have, without loss of generality, set the (fixed) input current magnitude to 1 unit. While the theory holds for any arbitrary current source-sink configuration, in this particular work, we present simulations for \( \vec{r}_{\text{in}} = (- (L - 1)/2, 0) \) and \( \vec{r}_{\text{out}} = (0, 0) \) on the square lattice in two dimensions, so that the current enters at the midpoint of the left boundary and exits at the center node of the lattice (see Fig. 2 for reference). Note that this source-sink configuration also forces the system size \( L \) to be an odd integer.

**Lattice Laplacian Formulation**

Assuming all circuit elements to be Ohmic, we can apply Kirchoff’s circuit law, given by

\[
|I\rangle_{\vec{r}} = \sum_{\Delta} \frac{|V\rangle_{\vec{r}} - |V\rangle_{\vec{r} + \Delta}}{R_{\vec{r}, \vec{r} + \Delta}}.
\]

We study random resistor networks where the bond resistances are perturbed from a mean resistance \( R_0 \), which without loss of generality we can set equal to 1 unit. We can recast Eq. (3) in the following linear algebraic form

\[
L |V\rangle + |I\rangle = 0,
\]

where \( L \) is the lattice Laplacian (or conductance matrix) [49]. Explicitly, the Laplacian is given by

\[
(L)_{ij} := \begin{cases} 
- \sum_{\Delta} (R_{\vec{r}_i, \vec{r}_j + \Delta})^{-1} & \text{if } i = j \\
(R_{\vec{r}_i, \vec{r}_j})^{-1} & \text{if } \vec{r}_j = \vec{r}_i + \Delta \\
0, & \text{otherwise.}
\end{cases}
\]

Observe that when all resistances are equal to \( R_0 \), \( L \) reduces to the usual circulant form of the lattice Laplacian of a \( d \)-dimensional torus, as expected. We mention here that the basic object of our study is the lattice Green’s function given by

\[
G := L^{-1},
\]

which provides all the system properties. We must be careful to note that, the Laplacian is a non-invertible matrix, and hence must be inverted by projecting out the zero mode, as shown in Eq. (18).

For the bond currents, we define the bond current vectors \( \{|J_{\Delta^+}\rangle\} \) corresponding to the \( d \) bonds along \( \Delta^+ \) assigned to each site. The complete \( N_b \times 1 \) bond current vector is denoted by \( |J\rangle = |J_{\Delta^+}\rangle |J_{\Delta^+}\rangle \cdots |J_{\Delta^+}\rangle \), where \( \langle \rangle \) separates the \( d \) blocks of the ket. Ohm’s law at every bond can be represented as

\[
|J_{\Delta^+}\rangle_{\vec{r}} = \frac{|V\rangle_{\vec{r}} - |V\rangle_{\vec{r} + \Delta^+}}{R_{\vec{r}, \vec{r} + \Delta^+}}.
\]

Again, like before, this may be recast into the following linear algebraic form

\[
D_{\Delta^+} |V\rangle + |J_{\Delta^+}\rangle = 0,
\]
where the difference matrices are given explicitly by

\[(D_{\Delta+})_{ij} := (R_{\vec{r}_i,\vec{r}_i+\Delta})^{-1} \begin{cases} -1 & \text{if } i = j \\ 1 & \text{if } \vec{r}_i = \vec{r}_i + \Delta \\ 0 & \text{otherwise.} \end{cases} \]  

(9)

Again, notice that when all the resistances are equal to \(R_0\), this generalized difference matrix becomes the usual difference operator for a \(d\)-dimensional torus.

As motivated in the introduction, we intend to study the crossover from the regimes of weak to strong disorder, which is brought about by exponential disorder \([34]\).

Thus, we set

\[R_{\vec{r}_i,\vec{r}_j} := e^{ax_{\vec{r}_i,\vec{r}_j}}, \quad (10)\]

where \(x_{\vec{r}_i,\vec{r}_j} \in (0,1)\) is a uniformly distributed random variable and \(a\) controls the strength of the disorder. The limit \(a \to 0\) yields a lattice with zero disorder, while \(a \to \infty\) provides the strong disorder limit. For ease of analytical calculations, we find it convenient to introduce the quenched scalar variables \(\{\zeta_\vec{r}_i,\vec{r}_j\}\) which represent the disorder in the bond resistances. The explicit relationship considered is given by

\[R_{\vec{r}_i,\vec{r}_j} := (1 - \zeta_{\vec{r}_i,\vec{r}_j})^{-1}. \quad (11)\]

Then, from Eqs. (10) and (11), one can find that the distribution of the quenched variables \(\zeta\) is given by (for \(a \geq 0\))

\[f(\zeta) = a^{-1}(1 - \zeta)^{-1} \quad \text{for } 0 < \zeta < 1 - e^{-a}. \quad (12)\]

This also implies that the resistances obey an inverse probability distribution, that is, \(f(R) = 1/(aR)\) for \(1 < R \leq e^a\). The moments of the disorder \(\zeta\) can also be computed exactly; in particular, the first three moments are given by \(\langle \zeta \rangle = (-1 + a + e^{-a})/a\), \(\langle \zeta^2 \rangle = (-3 + 2a + 4e^{-a} - e^{-2a})/(2a)\), and \(\langle \zeta^3 \rangle = (-11 + 6a + 18e^{-a} - 9e^{-2a} + 2e^{-3a})/(6a)\) where \(\langle \cdot \rangle\) denotes a disorder ensemble average.

III. DISORDER PERTURBATION EXPANSION

For the weak disorder regime, that is, a small deviation from the perfect lattice, we can compute the degrees of freedom accurately using a perturbation expansion in the disorder. We control such a perturbation by a tuning parameter \(\lambda\), such that \(0 < \lambda < 1\) and \(\lambda = 0\) corresponds to the zero disorder state. Therefore, we associate the tuning parameter to the disorder \(\zeta\) so that the resistances are redefined as \(R_{\vec{r}_i,\vec{r}_j} \equiv (1 - \lambda \zeta_{\vec{r}_i,\vec{r}_j})^{-1}\). We consider linear perturbations on the Laplacian and difference operators as follows

\[L := L^{(0)} + \lambda L^{(1)}, \quad (13a)\]

\[D_{\Delta+} := D_{\Delta+}^{(0)} + \lambda D_{\Delta+}^{(1)}. \quad (13b)\]

A linear order perturbation in these operators of the network is complete, and should induce perturbation expansions (in \(\lambda\)) up to all higher orders for the system variables. Thus, we assume

\[|V\rangle := |V\rangle^{(0)} + \lambda |V\rangle^{(1)} + \lambda^2 |V\rangle^{(2)} + O(\lambda^3), \quad (14a)\]

\[|J_{\Delta+}\rangle := |J_{\Delta+}\rangle^{(0)} + \lambda |J_{\Delta+}\rangle^{(1)} + \lambda^2 |J_{\Delta+}\rangle^{(2)} + O(\lambda^3) \quad (14b)\]

where the superscript denotes the order of the expansion; naturally, the \((0)\) index denotes the values of the quantities in the zero disorder state.

Applying the constraint that the above equations must obey Ohm’s and Kirchoff’s laws at each order of \(\lambda\), we obtain a hierarchical scheme to explicitly determine the higher order corrections to each of the above quantities. We first use Eqs. (3) and (4) to determine the corrections to the Laplacian matrices and the nodal voltages. Using Eq. (11) in Eq. (3), we have the following exact relationship between the currents and the nodal voltages, given a realization of the disorder

\[|I\rangle_{\vec{r}} = \sum_{\Delta} (|V\rangle_{\vec{r}} - |V\rangle_{\vec{r}+\Delta}) \left(1 - \lambda \zeta_{\vec{r},\vec{r}+\Delta}\right). \quad (15)\]

Next, comparing this with Eq. (4), we notice that

\[(L^{(1)})_{ij} = \begin{cases} \sum_{\Delta} \zeta_{\vec{r}_i,\vec{r}_j+\Delta} & \text{if } i = j \\ -\zeta_{\vec{r}_i,\vec{r}_j} & \text{if } \vec{r}_j = \vec{r}_i + \Delta \\ 0 & \text{otherwise.} \end{cases} \quad (16)\]

Now putting Eqs. (14a) and (14a) in Eq. (4), and equating the terms at each order of \(\lambda\), we obtain the explicit form of the voltages expansion in Eq. (14a). The contribution of each order is obtained in a hierarchical fashion using the contribution of the previous order, and we obtain

\[|V\rangle = G^{(0)}[1 + \lambda L^{(1)} G^{(0)} + \lambda^2 (L^{(1)} G^{(0)})^2 + O(\lambda^3)] |I\rangle, \quad (17)\]

where \(1\) denotes the identity matrix. Here, \(G^{(0)}\) is the lattice Green’s function of the lattice Laplacian \(L^{(0)}\). The singularity of \(L^{(0)}\) necessitates the use of the restricted inverse to be the Green’s function, which is obtained by projecting out the zero eigenvector \(|0\rangle\) of the Laplacian. Specifically, the Green’s function and the Laplacian are related by

\[L^{(0)} G^{(0)} = G^{(0)} L^{(0)} = -(1 - |0\rangle \langle 0|). \quad (18)\]

Since the voltages in the system are equivalent up to an arbitrary constant, the restricted inverse can be used without concern. The complete disordered Green’s function \(G\) satisfies the Dyson equation, given by

\[G = G^{(0)} + G^{(0)} L^{(1)} G. \quad (19)\]
Thus, the disordered Laplacian \( L^{(1)} \) is the so-called self-energy of the system. It is important to note that as the perturbation expansion in Eq. (13a) terminates at order 1, the above equation, is in principle exact at all orders. However, inverting such an equation is in general hard, and therefore is usually solved in a perturbative manner, as we proceed to do. Note that if we set \( \lambda = 1 \), and write \( |\lambda\rangle = G |I\rangle \), where \( G \) is the Green’s function for the disordered system, then we recover the Dyson series given by

\[
G = G^{(0)} + G^{(0)} L^{(1)} G^{(0)} + G^{(0)} L^{(1)} G^{(0)} L^{(1)} G^{(0)} + \ldots \quad (20)
\]

The \( n\)th term of the Dyson series gives us the \( n\)th order correction to the Green’s function, as is evident from the form we obtained in Eq. (17). Computing the nodal voltages is thus simply a task of computing the terms of the Dyson series order by order, which is trivial by the explicit knowledge of the disordered Laplacian as given in Eq. (16).

One can also compute the corrections to the difference matrices and the bond currents. Observe that Eq. (7) can be rewritten as follows

\[
|J_{\Delta^+}\rangle = (|V\rangle |\bar{r}\rangle - |V\rangle |r + \Delta^+\rangle)(1 - \lambda |\xi_{r_\Delta^+} \rangle),
\]

and comparing this with Eq. (8), we obtain

\[
(D^{(1)}_{\Delta^+})_{ij} = \xi_{r_i, r_i + \Delta^+} \begin{cases} 
1 & \text{if } i = j \\
-1 & \text{if } j = r_i + \Delta^+ \\
0 & \text{otherwise.}
\end{cases}
\]

Now putting Eqs. (13) and (14b) in Eq. (8), and again computing in a hierarchical fashion, we obtain the explicit form of the bond current expansion in Eq. (14b) as follows

\[
|J_{\Delta^+}\rangle = -\left[ D^{(0)}_{\Delta^+} G^{(0)} + \lambda \left( D^{(0)}_{\Delta^+} G^{(0)} L^{(1)} G^{(0)} + \right. \right.
\]

\[
D^{(1)}_{\Delta^+} G^{(0)} + \lambda^2 \left( D^{(0)}_{\Delta^+} G^{(0)} L^{(1)} G^{(0)} L^{(1)} G^{(0)} + \right. \right.
\]

\[
D^{(1)}_{\Delta^+} G^{(0)} L^{(1)} G^{(0)} \right] |I\rangle.
\]

The perturbation expansion developed above should lead to an exact answer for the voltage (resp. current) at a given site (resp. bond) for any small value of the disorder. The small value of the disorder is controlled by the scale after which the perturbation expansion is divergent, and this disorder scale is estimated in Sec. IV. In the convergent cases, one only needs to consider the first few terms in the expansion, which rapidly decay at large orders. We illustrate the exact convergence of such an expansion scheme in Fig. 3, where we show that the match between our theoretical predictions and numerical results at orders one and two in \( \lambda \). An excellent match is found between the theoretical and numerical results, as evident from the nodal voltage distributions.

We mention that a similar disorder perturbation expansion has been attempted by Luck and Derrida for dynamical quantities [50], however, our method, is explicitly for steady state (though it may be generalized to dynamical systems) but also is carried out in real space to use the Dyson series as the perturbation expansion solution for system observables.

**Ensemble averages**

The perturbation expansion formalism developed above provides the nodal voltages and bond currents for a particular resistance configuration in the lattice. From experimental considerations, it is more helpful to understand the behaviour of the system in an averaged fashion;
we thus compute the ensemble averages of the nodal voltages and bond currents as a function of the moments of

the quenched disorder \( \zeta \). The averaging is performed over the disorder ensemble with each instance of the ensemble representing a resistance configuration generated from an independent sampling of the disorder distribution. We first compute the ensemble averages of the nodal voltages. Observing the Dyson series (Eq. (20)), we realize that this problem reduces to calculating the ensemble averages of matrix products of alternating disordered Laplacians and perfect lattice Green’s function, that is, quantities of the kind \( (L^{(1)}G^{(0)}L^{(1)} \ldots G^{(0)}L^{(1)})_r \), with \( 2r - 1 \) matrices for the \( r \)th order contribution. Clearly, from inspection of the disordered Laplacian (see Eq. (16)), we notice that

\[
\langle L^{(1)} \rangle = -\langle \zeta \rangle L^{(0)},
\]

where \( \langle \zeta \rangle \equiv \langle \zeta^3 \rangle - 3\langle \zeta^2 \rangle + 2\langle \zeta \rangle^3 \). Notice, the first order correction (Eq. (25a)) reproduces the result we predicted simply by inspection of the matrices as well.

The ensemble averages for the bond currents can be computed in a similar fashion. The perfect and the perturbed lattice difference matrix can also be written in a dyadic representation as follows

\[
D^{(0)}_{\Delta^+} = -\sum_{m=1}^{N_s} |s_i(m)\rangle \langle b_m|,
\]

\[
D^{(1)}_{\Delta^+} = \sum_{m=1}^{N_s} \zeta_m |s_i(m)\rangle \langle b_m|.
\]

where \( |s_i(m)\rangle \) is the site vector associated with the site \( \vec{r}_i \) as well as the bond \( |b_m\rangle \) such that \( m \equiv (\vec{r}_i, \Delta^+) \). As evident from Eq. (23), the relevant quantities to calculate in this case are again matrix products of alternating perfect lattice Green’s function and perturbed Laplacian, except that the first matrix is the disordered difference matrix, that is, quantities of the kind \( (D^{(1)}_\Delta G^{(0)}L^{(1)} \ldots G^{(0)}L^{(1)})_r \).

Owing to the similar structure of the dyadic representation, it is clear that the disorder averaged quantities are exactly the ones computed in Eqs. (25), with any dyad \( |b_m\rangle \langle b_{m'}| \) replaced by \( |s_i(m)\rangle \langle s_i(m')| \), the initial \( L^{(1)} \) replaced by \( D^{(1)}_{\Delta^+} \) in the matrix products, and \( L^{(0)} \) replaced by \( D^{(0)}_{\Delta^+} \).

Given the computed averaged quantities in Eqs. (25), it is straightforward to compute the disorder averaged cumulants of the nodal voltages or bond currents. For example, using the Dyson series for the voltages and series expansion for the current in Eq. (17) and (23) we obtain explicit series expansions for the first moment up to second order

\[
\langle V \rangle = \langle G^{(0)} + \lambda G^{(0)}L^{(1)}G^{(0)} \rangle + \lambda^2 G^{(0)}L^{(1)}G^{(0)}L^{(1)}G^{(0)} + O(\lambda^3) |I|,
\]

\[
\langle J_{\Delta^+} \rangle = -\left[ D^{(0)}_{\Delta^+}G^{(0)} + \lambda \left( D^{(0)}_{\Delta^+}G^{(0)}L^{(1)}G^{(0)} \right) + \langle D^{(1)}_{\Delta^+}G^{(0)} \rangle + \lambda^2 \left( D^{(0)}_{\Delta^+}G^{(0)}L^{(1)}G^{(0)}L^{(1)}G^{(0)} + \langle D^{(1)}_{\Delta^+}G^{(0)}L^{(1)}G^{(0)} \rangle + O(\lambda^3) |I|,\right.
\]

and the relevant averages have been provided in Eq. (25). It is also useful to notice that these disorder averages are general for any arbitrary disorder distribution. Higher \( (r^{th}) \) order corrections are also straightforward to calculate—they would involve \( r \)-sums and the cluster averaged \( r \)th moments of the disorder variables. A convenient way to compute these correlation functions systematically may be via a diagrammatic expansion, and can be, formulated, in principle, for this setup as well. A typical route to diagrammatics is through a cumulant expansion in Fourier space [51–53], which can also be attempted within our formulation, however, we do not pur-
of disorder, while \( G \) denotes the lattice Green’s function for the system with matrices given by we employ is to use the Sherman-Morrison formula for Green’s function (with disorder in all bonds) in the disordered bonds in a hierarchical fashion with

\[
L^{[n]} = L^{[0]} + \langle \tilde{b}_n \rangle \langle \tilde{b}_n \rangle, \quad (31)
\]

which should also be evident from Eq. (24b). Here, in line with the notation for the Green’s functions, a square bracket subscript on the lattice Laplacian denotes the network with finitely many disordered bonds. Similarly, adding a second disordered bond to the lattice modifies the Laplacian matrix of a system with an additional disordered bond, so that \( L^{[2]} = L^{[1]} + \langle \tilde{b}_2 \rangle \langle \tilde{b}_2 \rangle \). Therefore, we can create an entire disordered lattice with \( n \) disordered bonds in a hierarchical fashion with

\[
L^{[n]} = L^{[n-1]} + \langle \tilde{b}_n \rangle \langle \tilde{b}_n \rangle. \quad (32)
\]

It is clear that each level of this hierarchy of adding disordered bonds is amenable to an exact inversion via the Sherman Morrison formula in Eq. (29), as enabled by the disordered bonds entering as dyadic additions to the perfect lattice Laplacian. We therefore obtain the following recursive relation relation between the Green’s functions \( G^{[n]} \) and \( G^{[n+1]} \) for a system with \( n \) and \( n+1 \) bond impurities respectively

\[
G^{[n+1]} = G^{[n]} + \frac{\langle \tilde{b}_{n+1} \rangle \langle \tilde{b}_{n+1} \rangle}{1 - \langle \tilde{b}_{n+1} \rangle \langle G^{[n]} \rangle \langle \tilde{b}_{n+1} \rangle}. \quad (33)
\]

Eq. (33) represents a central result of our study, with the rest of this section devoted to methods that can yield exact results for these recursion relations. For a small number of disordered bonds these may be computed directly. However, we demonstrate a generalized formalism that enables us to calculate the disordered Green’s functions for an arbitrary number of bonds with disorder in an analytically tractable manner.

First, to calculate the Green’s function with a single disordered bond, we simply apply the inversion formula in Eq. (29) for the Laplacian in Eq. (31), which is the \( n = 0 \) case of the recursion in Eq. (33). We thus obtain the Green’s function for a single bond with disorder

\[
G^{[1]} = G^{[0]} + \left( \frac{1}{\beta_1} \right) G^{[0]} \langle \tilde{b}_1 \rangle \langle \tilde{b}_1 \rangle G^{[0]}, \quad (34)
\]

where we have defined

\[
\beta_1 := 1 - \tilde{\mathcal{G}}^{0}, \quad (1 \leq i \leq n) \quad (35a)
\]

\[
\tilde{\mathcal{G}}^{0}_{pq} := \langle \tilde{b}_p \rangle \langle G^{[0]} \rangle \langle \tilde{b}_q \rangle \quad (1 \leq p, q \leq n) \quad (35b)
\]

We term \( \tilde{\mathcal{G}}^{0} \) as the disordered dressed Green’s function (as a generalization of the definition in Section III). As a
limiting case, one can consider the bond percolation limit, where we set the disorder variable $\zeta_1 \rightarrow 1$ in Eq. (34) and recover the single broken bond Green’s function expression derived in [41, 55], as expected.

Having obtained the Green’s function for the system with a single disordered bond, one may obtain the Green’s function for two disordered bonds, by using Eq. (34) in the recursion (Eq. (33)). On doing so, we obtain the following formula

$$G^{[2]} = G^{[0]} + \left( \frac{\tilde{\beta}_2}{\beta_1 \beta_2 - (G^{[0]})^2} \right) G^{[0]} \langle \tilde{b}_1 | \tilde{b}_1 \rangle G^{[0]} + \left( \frac{\tilde{\beta}_1}{\beta_1 \beta_2 - (G^{[0]})^2} \right) G^{[0]} \langle \tilde{b}_2 | \tilde{b}_2 \rangle G^{[0]} + \left( \frac{\tilde{\beta}_0}{\beta_1 \beta_2 - (G^{[0]})^2} \right) \left[ G^{[0]} \langle \tilde{b}_1 | \tilde{b}_1 \rangle G^{[0]} + G^{[0]} \langle \tilde{b}_2 | \tilde{b}_2 \rangle G^{[0]} \right] .$$

(36)

Clearly, a pattern is evident for these formulae. The difference between the disordered and perfect lattice Green’s function is simply the sum of bilinears of the kind $G^{[0]} \langle \tilde{b}_i | \tilde{b}_j \rangle G^{[0]}$ for $1 \leq i, j \leq n$ with disordered coefficients that are functions of the disordered dressed Green’s function $\tilde{G}^{[i]}$. Computing these formulae iteratively using the recursion, however, is hard and below we propose an efficient manner in determining the Green’s functions.

Thus, we now account for an arbitrary number of bonds with disorder. We posit the following sum of bilinears as the disordered lattice Green’s function with $n$ disordered bonds

$$G^{[n]} = G^{[0]} + \sum_{i,j} c_{ij}^{[n]} G^{[0]} \langle \tilde{b}_i | \tilde{b}_j \rangle G^{[0]} .$$

(37)

It is straightforward to show that this solves the recursion relation in Eq. (33), through mathematical induction. Remarkably, the problem of evaluating the disordered Green’s function has now reduced to computing the coefficients $c_{ij}^{[n]}$. By symmetry, $c_{ij}^{[n]} = c_{ji}^{[n]}$ and thus, we need to only compute $n(n+1)/2$ coefficients for $n$ disordered bonds. Multiplying by $(G^{[n]})^{-1} = \left( (G^{[0]})^{-1} - \sum_i \langle \tilde{b}_i | \tilde{b}_i \rangle \right)$ on both sides, and using the linear independence of the $\langle \tilde{b}_i | \tilde{b}_j \rangle$ dyads, we obtain the following relation

$$\tilde{\beta}_i c_{ij}^{[n]} - \sum_{k \neq i} G^{[0]}_{ik} c_{kj}^{[n]} = \delta_{ij} \quad (1 \leq i, j, k \leq n).$$

(38)

Eq. (38) gives us a linear algebraic equation from which we can determine the coefficients $c_{ij}^{[n]}$. The equivalent matrix equation can be formulated in terms of a Kronecker product $\otimes$ as follows

$$\left( \tilde{G}^{[n]} \otimes I_n \right) C^{[n]} = U^{[n]} ,$$

(39)

where $C^{[n]} = (c_{11}^{[n]}, c_{12}^{[n]}, \ldots, c_{nn}^{[n]})^T$ is the $n^2 \times 1$ coefficient vector and $U^{[n]}$ is the constant vector with $n$ unit vector blocks, that is $U^{[n]} \equiv [e_1^{[n]} e_2^{[n]} \ldots e_n^{[n]}]$ (where $e_j^{[n]}$ denotes the $n$ dimensional $j$th unit vector). The dressed Green’s function matrix $\tilde{G}^{[n]}$ is given explicitly by

$$\tilde{G}^{[n]} = \left( \begin{array}{cccc} \tilde{\beta}_1 & -G_{12}^{[0]} & -G_{13}^{[0]} & \cdots & -G_{1n}^{[0]} \\ -G_{21}^{[0]} & \tilde{\beta}_2 & -G_{23}^{[0]} & \cdots & -G_{2n}^{[0]} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -G_{n1}^{[0]} & -G_{n2}^{[0]} & -G_{n3}^{[0]} & \cdots & \tilde{\beta}_n \end{array} \right) .$$

(40)

The block structure of Eq. (39) allows us to solve the matrix equation in a reduced fashion from $n^2 \times n^2$ to $n \times n$ matrix, which gives us a significant computational advantage. Observe that it suffices to solve $n$ matrix equations of the following kind

$$\tilde{G}^{[n]} C_j^{[n]} = e_j^{[n]} ,$$

(41)

where $C_j^{[n]} = (c_{1j}^{[n]}, c_{2j}^{[n]}, \ldots, c_{nj}^{[n]})^T$ is the $n \times 1$ reduced coefficient vector. Since the right-hand side of the matrix equation is simply the unit vector, it is convenient to solve the above systems of equations using Cramer’s rule, which gives us the following neat result

$$c_{ij}^{[n]} = \frac{\det(\tilde{G}^{[n]}(i, j))}{\det(\tilde{G}^{[n]})} ,$$

(42)

where, $\tilde{G}^{[n]}(i, j)$ is the coefficient matrix with the $i$th column replaced with $e_j^{[n]}$. This simplifies to a particularly provocative form for the diagonal terms as below

$$c_{ii}^{[n]} = \frac{\det(\tilde{G}^{[n-1]}(i, i))}{\det(\tilde{G}^{[n]})} .$$

(43)

Here, $G^{[n]}$ is the coefficient matrix in Eq. (41). Thus, the diagonal terms of the coefficients are simply the ratios of the determinants of the dressed Green’s function matrix for $n$ and $n-1$ disordered bonds. This is a remarkable result and suggests that the disordered Green’s function is intrinsically determined by the properties of the determinant of a quantity encoding the lattice structure by means of the dressed Green’s function matrix defined in Eq. (40). Properties of the coefficients of this matrix should help decipher the response of the system as the disorder strength is increased.

We now enlist the coefficients for the disordered Green’s functions for bond impurities of one, two and three bonds respectively. We denote the numerator and denominator of these coefficients by $N[c_{ij}^{[k]}]$ and $D[c_{ij}^{[k]}]$ respectively, with

$$c_{ij}^{[k]} = \frac{N[c_{ij}^{[k]}]}{D[c_{ij}^{[k]}]} .$$

(44)
Note that due to Eq. (42), there is only one (superscript) index in the coefficient for the denominator since it is equal for all coefficients for a given \( n \). Then, for one disordered bond, we have

\[
N[c_1] = 1; \quad D[c_1] = \hat{\beta}_1.
\] (45)

for two bond impurities, we have

\[
N[c_2] = \hat{\beta}_2; \quad N[c_2] = \hat{\beta}_1; \quad N[c_2] = \hat{g}_{12}^0,
\] (46a)

\[
D[c_2] = \hat{\beta}_1 \hat{\beta}_2 - (\hat{g}_{12}^0)^2,
\] (46b)

and for three bond impurities, we have

\[
N[c_3] = \hat{\beta}_2 \hat{\beta}_3 - (\hat{g}_{23}^0)^2,
\] (47a)

\[
N[c_3] = \hat{\beta}_1 \hat{\beta}_3 - (\hat{g}_{13}^0)^2,
\] (47b)

\[
N[c_3] = \hat{\beta}_1 \hat{\beta}_2 - (\hat{g}_{12}^0)^2,
\] (47c)

\[
N[c_3] = \hat{\beta}_1 \hat{g}_{12}^0 + \hat{g}_{13}^0 \hat{g}_{23}^0,
\] (47d)

\[
N[c_3] = \hat{\beta}_1 \hat{g}_{13}^0 + \hat{g}_{12}^0 \hat{g}_{23}^0,
\] (47e)

\[
N[c_3] = \hat{\beta}_1 \hat{g}_{23}^0 + \hat{g}_{12}^0 \hat{g}_{31}^0,
\] (47f)

\[
D[c_3] = \hat{\beta}_1 \hat{\beta}_2 \hat{\beta}_3 - \hat{\beta}_1 (\hat{g}_{23}^0)^2 - \hat{\beta}_2 (\hat{g}_{13}^0)^2 - \hat{\beta}_3 (\hat{g}_{12}^0)^2 - 2 \hat{g}_{12}^0 \hat{g}_{23}^0 \hat{g}_{31}^0.
\] (47g)

This procedure may be extended to arbitrarily many number of bonds and the coefficients can be obtained as before. Observe that the coefficients in Eqs. (45) and (46) are the same as those computed in Eqs. (34) and (36) by directly solving the recursion. Coefficients for four disordered bonds are enlisted in Appendix A.

**Connection with perturbation theory**

The Green’s functions computed in this section are exact for arbitrary disorder. To stay consistent, they must match with the predictions from the perturbation theory in the weak disorder regime. This is indeed the case, and we demonstrate here how one can reproduce the perturbation expansion results by simply computing the series expansions of the Green’s functions in the disorder variable \( \xi \) in the small disorder strength limit. As an illustration, we consider the Green’s function for one disordered bond in Eq. (34) and computing a series expansion (in \( \xi_1 \)) of the denominator (under the assumption that \( \xi_1 \ll (\hat{g}_{11}^0)^{-1} \)), we obtain the following expansion,

\[
G[i] = G[0] + \xi_1 G[0] |b_i\rangle \langle b_i| + \xi_1^2 G[1]^2 G[0] |b_i\rangle \langle b_i| G[0] + \xi_1^3 G[1]^2 G[0] |b_i\rangle \langle b_i| G[0] + \ldots.
\] (48)

Now, recall that the perturbation theory predicts the Dyson series (Eq. (20)) for the Green’s function. Considering that for one bond disorder, the correction to the Laplacian is simply \( L^{(1)} = \xi_1 \langle b_i | b_i \rangle \) (as seen from Eq. (24b)), on using this in the Dyson series we see that we exactly reproduce Eq. (48). This clearly shows how our dyadic bond formulation is consistent with the perturbation theory developed before. This exercise can also be performed for higher number of disordered bonds with the same result.

An alternate benefit of this exercise is that it provides an estimate for the disorder strength up to which the perturbation theory converges. Indeed, we find that as long as \( \xi \ll (\hat{g}_{ij}^0)^{-1} \), or equivalently in terms of the disorder strength, \( a \cdot x_{r_i} r_j \ll \ln \left( 1/(1 - (\hat{g}_{ij}^0)^{-1}) \right) (0 \leq x_{r_i} r_j \leq 1) \) (here, \( r_i \) and \( r_j \) can be identical or nearest-neighbour sites on the lattice), the perturbation theory is expected to provide reasonable results to match with experiments.

Calculating the system variables from these exact Green’s functions is straightforward—\( |V| = G^n |I| \), and \( |J^{\Delta^+}_\pm \rangle = D^{(0)} |V| \). Further, just as we did in the perturbation theory, these disordered Green’s functions can be averaged analytically to obtain the ensemble averaged nodal voltages and bond currents. We expand on this in greater detail in subsection V A.

**V. NUMERICAL RESULTS**

Finally, we test the validity of our theoretical results through numerical simulation of an \( L \times L \) square lattice in two dimensions with exponential disorder. In subsection VA, we first numerically compute the mean nodal voltages and its fluctuations at a generic lattice site for one, two, and three disordered bonds in the lattice. These results are then matched with theoretical predictions from the hierarchical dyadic bond disorder formalism, which show an exact match for the entire disorder range for the weak as well as strong regimes. In subsection VB, we propose a novel order parameter, termed bond current fidelity, which measures the overlap between current distributions at arbitrary and infinite disorder. We provide finite-size scaling estimates of this order parameter in the weak and strong disorder regime, albeit for small lattice sizes.

**A. Nodal voltage fluctuations**

We first numerically compute the nodal voltages and fluctuations at an observation site \( \vec{r}_{o} \) in the lattice. The nodal voltages \( \{ |V|_{\vec{r}_{o}} \} \) are calculated using Kirchoff’s law (Eq. (4)) while the fluctuations are given simply as an average over disorder realizations \( \text{Var}[|V|_{\vec{r}_{o}}] = \langle |V|_{\vec{r}_{o}}^2 \rangle - \langle |V|_{\vec{r}_{o}} \rangle^2 \). The choice of the observation site \( \vec{r}_{o} \) is arbitrary, as our formalism predicts the values at all lattice sites. We notice and expect a similar behaviour at any such lattice site in the bulk and thus it suffices to probe the behaviour at an single arbitrary lattice site. For illustrative purposes, our simulations are performed on a 5 × 5 square lattice with one, two and three disor-
FIG. 4. Plot of the mean nodal voltages $\langle |V\rangle_{\vec{r}_{0}} \rangle$ in subfigures \{(a), (b), (c)\} and their fluctuations $\text{Var} [|V\rangle_{\vec{r}_{0}}]$ in subfigures \{(d), (e), (f)\} for one, two and three disordered bonds in the lattice respectively. The plots display an exact match between the theory (red line) and numerical simulations (blue circles). The insets display a schematic of the $5 \times 5$ random resistor network used in the computation, with the red circles denoting the locations of the source and sink, whereas the green, orange and purple bonds depicting the disordered bonds $\{\vec{b}_{1}, \vec{b}_{2}, \vec{b}_{3}\}$ respectively, and the blue circle denotes the observation point $\vec{r}_{0}$. The averaging is performed over 1000 realizations of the bond disorder. The analytic predictions were computed using Eqs. (49) and (50), with the exact formulae for the $n$-dimensional integrals therein given in Appendix B.

ordered bonds. The locations of the disordered bonds and the observation site are depicted in the schematic in the insets of Fig. 4(a), (c) and (e).

For the cases with one, two and three disordered bonds, one can also compute analytically the quantities of interest using the recursive dyadic bond disorder formulation. It is easy to compute the moments of the nodal voltages using Eq. (37). The mean nodal voltage at site $\vec{r}_{0}$ is given by

$$\langle |V\rangle_{\vec{r}_{0}} \rangle = |V\rangle_{\vec{r}_{0}}^{(0)} + \sum_{i,j}^{n} \mathcal{T}_{ij}^{[n]}(a) \langle s_i | \mathbf{G}^{[0]} | b_i \rangle \langle b_j | \mathbf{G}^{[0]} | I \rangle ,$$

where $\mathcal{T}_{ij}^{[n]}(a)$ is a disorder averaged coefficient obtained by computing an $n$-dimensional integral over the disorder distributions (see Appendix B for explicit formulae). These integrals may be computed analytically for simple cases (and numerically otherwise) and the computed nodal voltages are matched with the numerics (cf. Fig. 4(a), (c) and (e)). We can also compute the fluctuations by a somewhat more tedious calculation. For this, we must compute the second moment of the nodal voltage at site $\vec{r}_{0}$, which may be written as

$$\langle (|V\rangle_{\vec{r}_{0}})^{2} \rangle = \langle |V\rangle_{\vec{r}_{0}}^{(0)} \rangle^{2} + 2 |V\rangle_{\vec{r}_{0}}^{(0)} \sum_{p,q}^{n} \mathcal{T}_{pq}^{[n]} \langle s_o | \mathbf{G}^{[0]} | b_p \rangle \langle b_q | \mathbf{G}^{[0]} | I \rangle + \sum_{p,q,r,s}^{n} \mathcal{J}_{pqrs}^{[n]} \langle s_o | \mathbf{G}^{[0]} | b_p \rangle \langle s_q | \mathbf{G}^{[0]} | b_r \rangle \langle b_s | \mathbf{G}^{[0]} | I \rangle \langle b_t | \mathbf{G}^{[0]} | I \rangle ,$$

(50)

where $\mathcal{T}_{pq}^{[n]}(a)$ and $\mathcal{J}_{pqrs}^{[n]}(a)$ are disorder averaged coefficients. $\mathcal{T}_{pq}^{[n]}(a)$ is the $n$-dimensional integral described above, while $\mathcal{J}_{pqrs}^{[n]}(a)$ is an $n$-dimensional integral connecting four bonds (again, exact formulae are provided in Appendix B). Clearly, the fluctuations are given simply by the four-point term (the third term in Eq. (50)) with a disorder averaged coefficient given by $\mathcal{J}_{pqrs}^{[n]}(a) = \mathcal{J}_{pqrs}^{[n]}(a)$. We show the match of the voltage fluctuations with the simulations in Fig. 4(b), (d) and (f). From Fig. 4, we notice that the nodal voltages increase with increasing disorder strength $a$, and then saturate to a constant value. The saturation is due to the fact that in the strong disorder limit, the current distribution collapses to an optimal path, and hence small changes in the disorder strength does not alter the voltage configuration of the network substantively. The fluctuations, on the other hand, depict a very interesting behaviour—
they peak at a critical disorder value and then decrease as we approach the strong disorder limit. This peak in the fluctuations is representative of the crossover between the weak and strong disorder regimes in the system, for the chosen configuration of small number of impurities. The fluctuations in the nodal voltages thus may be considered as a useful order parameter for the system, at least, for systems with a small number of bonds with disorder.

Although, we have provided theoretical expressions and numerical results for a small lattice size, the case of an infinite lattice size with one, two and three disordered bonds does not require greater effort, as the only change is the replacement of the Green’s function elements for the infinite lattice. Analytic expressions for the same are easy to compute through known recursion relations for the perfect infinite lattice Green’s function [49, 56].

### B. Bond current fidelity

To investigate the behaviour of the system in the weak and strong disorder regimes, we construct a novel order parameter in terms of the bond current observables. Previous studies of the crossover between weak and strong disorder regimes have focused on microscopic observables such as the distribution of tracer path lengths [33], which requires collecting large statistics over many samples, or through the resistance measurements before and after introduction of a perturbation at the bond with a maximal current [2]. In this study, we propose an alternate order parameter computed using a macroscopic observable—the bond currents, thus making the study accessible to experiments. The advantage of this parameter is that it solely depends on the steady state current distribution of a single resistance configuration, and thus does not require, in principle, large averaging, or perturbations to the circuit. Knowledge of the resistance configuration, from an experimental point-of-view requires the simultaneous measurement of the nodal voltages and bond currents, which appears to be accessible within reasonable arrangements.

In order to quantify whether a given system is in a weak or strong disordered phase, it is useful to look at how channelized the flow of the current is. Each resistance configuration has a corresponding optimal path, that is, the path of least resistance between the source and the sink in the circuit. For each disorder strength, the optimal path can be computed by simply computing the path of least total weight in the graph between the source and sink, which is implemented efficiently by Dijkstra’s algorithm [57]. In the perfect lattice, the optimal path is the straight line connecting the source and the sink. As the disorder is increased, in the weak regime, the optimal path fluctuates about this straight line, with its statistics depicting a self-affine behaviour [16]. As the system enters the strong disorder regime, the optimal path changes behaviour—displaying self-similarity. We recover these signatures from the bond current order parameter described in detail below.

In the strong disorder limit, the current distribution collapses completely to the optimal path. Therefore, a relevant order parameter is the deviation of the current distribution in the system from the optimal path. To define the order parameter, it is useful to define the optimal path in terms of bond vectors. Similar to the definition of the bond current vector in Sec. II, we construct $d$ optimal current vectors $\{J_\Delta^a\}$ along $d$ directions and a complete optimal current vector $J^\text{opt}(a) \equiv \begin{pmatrix} J^\text{opt}_\Delta \end{pmatrix}$. We define $J^\text{opt}_\Delta(a)$ as follows

$$J^\text{opt}_\Delta(a) := \begin{cases} 1 & \text{if } |b_m^a| \in \text{optimal path} \\ -1 & \text{if } |b_m^a| \in \text{optimal path} \\ 0 & \text{otherwise} \end{cases}. \quad (51)$$

This construction is consistent with the current definitions and accounts for possible overhangs in the optimal path. Note that the length of the optimal path is given by the 1-norm of the optimal current vector. We then define the bond current fidelity $F$ to be given by

$$F_{\{R\}}(a) \equiv \langle J^\text{opt}(a)|J(a)\rangle, \quad (52)$$

where the subscript $\{R\}$ indicates that the fidelity is calculated for a fixed instance of the random resistance configuration, with the only variable being the disorder strength that is varied. We conjecture that this order parameter $F$ should measure the behaviour of the system succinctly with respect to the behaviour of the optimal path, and one should notice a significant shift in the profile from the self-affine to self-similar profiles of the optimal path in the weak and strong disorder regimes respectively. Since this parameter is an overlap between the actual current distribution and the optimal path at each disorder strength, we term it the bond current fidelity of the system. Similar fidelity parameters have been found useful in a variety of contexts, including in information geometric contexts for probing quantum phase transitions [58].

In Fig. 5, we show numerical simulations of the averaged bond current fidelity for six different system sizes $L = 5, 11, 15, 21, 25$ and $31$ given a fixed disorder configuration. The disorder strength is varied from the weak to the strong disorder regimes, that is, $0.1 \leq a \leq 100$. We observe interesting signatures of the weak and strong disorder regimes in this order parameter. In the weak disorder regime ($L \gg a^{1/3}$) we observe a saturated behaviour of the fidelity, which is very close to the value of the fidelity in the perfect network, with negligible fluctuations. For our largest system size ($L = 31$), we find that weak disorder is given at $a \ll 13$, while strong disorder is given at $a \gg 13$. This is consistent with the profiles observed in Fig. 5. We can also compute the saturation value of the weak disorder limit explicitly using
The averaged bond fidelity in the strong disorder regime is scaled by $L^{1/d_{\text{opt}}}$ where $d_{\text{opt}} = 1.22$. Finally, an additional curious observation is a systematic decrease in the bond current fidelity in the crossover regime between the weak and strong disorders, which, in fact, is also the global minimum of the fidelity for all system sizes. This appears to be a feature of the crossover regime which demands further investigation.

VI. DISCUSSION AND CONCLUSION

In this paper, we have demonstrated a framework to determine the disordered Green’s functions for random resistor networks. We developed two alternate formulations, a perturbative framework that can be used to compute system properties in the weak disordered regime, and an exact (dyadic) framework that can be used to compute quantities at arbitrarily large disorder strengths for a small number of impurities. We demonstrated the equivalence of the two frameworks in the weak disorder regime. In addition, we obtained exact Green’s functions for arbitrary disorder, with analytically tractable expressions for lattices with small number of disordered bonds. We also demonstrated an exact match between numerical and analytical predictions for nodal voltage fluctuations in lattices with a small numbers of impurities (one, two and three). We have also explicitly demonstrated the generalization of the dyadic framework for the case with a large number of disordered bonds in the system, however, in this case the computations become harder to perform. Finally, we proposed a novel order parameter, named the bond current fidelity, which measures the deviation of the currents from the optimal path given a disorder configuration. We found that this order parameter is able to distinguish between the weak and strong disorder regimes of the system, and finite size scaling estimates are consistent with behaviour obtained in previous studies.

There are several directions for further investigation that our study suggests. In this study, we have focused on fundamental observables relevant to experiments, i.e. nodal voltages and bond currents, and our theoretical predictions for the ensemble averages would be interesting to match with steady state experimental observations. As random resistor networks with exponential disorder are paradigmatic systems in condensed matter and statistical physics [2, 14, 22, 25, 34], our results have interesting implications for the characteristics of systems with wide disorder distributions. In particular, it would be very interesting if our dyadic bond formulation can be systematically computed in a manner that sheds more light on the optimal path exponent ($\approx 1.22$ in two dimensions). In this endeavour, it appears that the statistics of the determinant in Eq. (43) are of crucial importance, and a systematic study of the growth of these determinants could lead to an understanding of the behaviour of optimal paths in such systems.

Further, the results presented in this paper are easily generalizable to any underlying disorder, not just expo-
ential, and it would be interesting to study how the nature of the underlying disorder changes the behaviour studied here. In addition, it should be possible to use the scheme of constructing a disordered Green’s function as described in Sec. IV, to construct any arbitrary electrical circuit, such as a bipartite lattice with two types of bonds, a context frequently relevant in condensed matter systems. There are also intriguing connections to be made to the eigenvalues of disordered Laplacians (which could represent Hessian matrices) through the study of their resolvents, which are fundamental quantities of interest in stability studies of random media. Finally, although we have studied the steady state properties of such networks, it would be very interesting to probe dynamical questions through the methods developed in this paper, including investigations of the differences between quenched and annealed disorder in the approach to steady state of such disordered systems.

Acknowledgments

We acknowledge useful discussions with Debankur Das, Roshan Maharana, Soham Mukhopadhyay and Shamashis Sengupta. This project was funded by intramural funds at TIFR Hyderabad from the Department of Atomic Energy (DAE), Government of India.

Appendix A: Exact Green’s functions for four disordered bonds

Here, we present exact disordered Green’s functions for four disordered bonds. For this case we have the following quartic polynomial in $\hat{\beta}$ as the denominator of the coefficients:

$$\hat{\beta}_1 \hat{\beta}_2 \hat{\beta}_3 \hat{\beta}_4 - \hat{\beta}_3 \hat{\beta}_4 (\hat{G}_{12})^2 + \hat{\beta}_2 \hat{\beta}_4 (\hat{G}_{13})^2 + \hat{\beta}_2 \hat{\beta}_3 (\hat{G}_{14})^2 + \hat{\beta}_1 \hat{\beta}_4 (\hat{G}_{23})^2 + \hat{\beta}_1 \hat{\beta}_3 (\hat{G}_{24})^2 + \hat{\beta}_1 \hat{\beta}_2 (\hat{G}_{34})^2 - 2(\hat{\beta}_1 \hat{\beta}_4 \hat{G}_{12} \hat{G}_{13} \hat{G}_{14} + \hat{\beta}_1 \hat{\beta}_3 \hat{G}_{12} \hat{G}_{13} \hat{G}_{24} + \hat{\beta}_1 \hat{\beta}_2 \hat{G}_{12} \hat{G}_{14} \hat{G}_{23}) + (\hat{G}_{12})^2 (\hat{G}_{13})^2 + (\hat{G}_{12})^2 (\hat{G}_{14})^2 + (\hat{G}_{13})^2 (\hat{G}_{24})^2 + (\hat{G}_{14})^2 (\hat{G}_{23})^2 - 2(\hat{G}_{12} \hat{G}_{13} \hat{G}_{24} \hat{G}_{34} + \hat{G}_{12} \hat{G}_{13} \hat{G}_{23} \hat{G}_{24} + \hat{G}_{12} \hat{G}_{14} \hat{G}_{23} \hat{G}_{24} + \hat{G}_{12} \hat{G}_{14} \hat{G}_{24} \hat{G}_{34})$$

We also compute the numerators of the coefficients, which are cubic polynomials in $\hat{\beta}$. Thus, we have

$$\mathcal{N}[c_1] = \hat{\beta}_2 \hat{\beta}_3 \hat{\beta}_4 - \hat{\beta}_2 (\hat{G}_{23})^2 \hat{\beta}_4 (\hat{G}_{24})^2 - \hat{\beta}_3 (\hat{G}_{12})^2 - \hat{\beta}_4 (\hat{G}_{13})^2 - \hat{\beta}_2 \hat{\beta}_3 (\hat{G}_{14})^2 - \hat{\beta}_1 \hat{\beta}_4 (\hat{G}_{23})^2 - \hat{\beta}_1 \hat{\beta}_3 (\hat{G}_{24})^2 - \hat{\beta}_1 \hat{\beta}_2 (\hat{G}_{34})^2$$

$$\mathcal{N}[c_2] = \hat{\beta}_1 \hat{\beta}_3 \hat{\beta}_4 - \hat{\beta}_1 \hat{\beta}_3 (\hat{G}_{14})^2 - \hat{\beta}_1 \hat{\beta}_2 \hat{\beta}_4 (\hat{G}_{13})^2 - \hat{\beta}_1 \hat{\beta}_3 \hat{\beta}_4 (\hat{G}_{23})^2 - \hat{\beta}_3 \hat{\beta}_4 (\hat{G}_{12})^2$$

Appendix B: Disorder averaged coefficients for nodal voltages

We have the following integrals for the disorder averaged coefficients used to compute the nodal voltages—for one and two disordered bonds.

$$I_{11}[1] = \int \frac{\zeta}{\hat{\beta}_1} f(\zeta) d\zeta$$

$$I_{11}[2] = \int \frac{\zeta}{\hat{\beta}_1 \hat{\beta}_2} \frac{\hat{\beta}_2}{\hat{G}_{12}} f(\zeta) f(\zeta') d\zeta d\zeta'$$

$$I_{22}[2] = \int \frac{\zeta}{\hat{\beta}_1 \hat{\beta}_2} \frac{\hat{\beta}_1}{\hat{G}_{12}} f(\zeta) f(\zeta') d\zeta d\zeta'$$

$$I_{12}[2] = \int \frac{\sqrt{\zeta}}{\hat{\beta}_1 \hat{\beta}_2} \frac{\hat{\beta}_2}{\hat{G}_{12}} f(\zeta) f(\zeta') d\zeta d\zeta'$$

where each of the integrals are integrated over the limits 0 to 1 and $e^{-a}$. We also have $I_{21}[2] = I_{12}[2]$ by symmetry. The disorder averaged coefficients for three bonds can also be
written similarly and are not provided here explicitly in view of conciseness.

Finally, we also provide the four-point disorder average coefficients useful for calculating the nodal fluctuations—again, for one and two disordered bonds.

\[
\mathcal{J}_{1111}^{[1]} = \int \frac{\zeta_1^2}{(\beta_1)^2} f(\zeta_1) \, d\zeta_1 \quad \text{(B2a)}
\]

\[
\mathcal{J}_{1111}^{[2]} = \int \frac{\zeta_1^2}{(\beta_1 \beta_2 - (\bar{G}_{12})^2)^2} f(\zeta_1) f(\zeta_2) \, d\zeta_1 \, d\zeta_2 \quad \text{(B2b)}
\]

\[
\mathcal{J}_{2222}^{[2]} = \int \frac{\zeta_1^2}{(\beta_1 \beta_2 = (\bar{G}_{12})^2)^2} f(\zeta_1) f(\zeta_2) \, d\zeta_1 \, d\zeta_2 \quad \text{(B2c)}
\]

\[
\mathcal{J}_{1212}^{[2]} = \int \frac{\zeta_1 \zeta_2}{(\beta_1 \beta_2 - (\bar{G}_{12})^2)^2} f(\zeta_1) f(\zeta_2) \, d\zeta_1 \, d\zeta_2 \quad \text{(B2d)}
\]

\[
\mathcal{J}_{1122}^{[2]} = \int \frac{\zeta_1^2}{(\beta_1 \beta_2 = (\bar{G}_{12})^2)^2} f(\zeta_1) f(\zeta_2) \, d\zeta_1 \, d\zeta_2 \quad \text{(B2e)}
\]

\[
\mathcal{J}_{1111}^{[2]} = \int \sqrt{\zeta_1^2 \zeta_2^2} \frac{\bar{G}_{12} \bar{G}_{12}^*}{(\beta_1 \beta_2 - (\bar{G}_{12})^2)^2} f(\zeta_1) f(\zeta_2) \, d\zeta_1 \, d\zeta_2 \quad \text{(B2f)}
\]

\[
\mathcal{J}_{2212}^{[2]} = \int \sqrt{\zeta_1^2 \zeta_2^2} \frac{\bar{G}_{12} \bar{G}_{12}^*}{(\beta_1 \beta_2 - (\bar{G}_{12})^2)^2} f(\zeta_1) f(\zeta_2) \, d\zeta_1 \, d\zeta_2 \quad \text{(B2g)}
\]

where the integrals are again over the limits 0 to 1 – e^{-\alpha}. The rest of the disorder averaged coefficients in \( J_{pqrs}^{[n]} \) (\( n = 1, 2 \)) are all equal to the entries in the above set by symmetry. The coefficients for higher \( n \) can be constructed similarly.

[1] V. V. Cheianov, V. I. Fal’ko, B. L. Altshuler, and I. L. Aleiner, Random resistor network model of minimal conductivity in graphene, Phys. Rev. Lett. 99, 176801 (2007).
[2] Y. M. Strelinski, R. Berkovits, A. Frydman, and S. Havlin, Percolation transition in a two-dimensional system of ni granular ferromagnets, Phys. Rev. E 69, 065105 (2004).
[3] J. L. Cardy, Electron localisation in disordered systems and classical solutions in ginzburg-landau field theory, Journal of Physics C: Solid State Physics 11, L321 (1978).
[4] M. J. Stephen, Mean-field theory and critical exponents for a random resistor network, Phys. Rev. B 17, 4444 (1978).
[5] T. Helbig, T. Hofmann, C. H. Lee, R. Thomale, S. Imhof, L. W. Molenkamp, and T. Kiessling, Band structure engineering and reconstruction in electric circuit networks, Phys. Rev. B 99, 161114 (2019).
[6] C. H. Lee, S. Imhof, C. Berger, F. Bayer, J. Brehm, L. W. Molenkamp, T. Kiessling, and R. Thomale, Topolectrical circuits, Communications Physics 1, 39 (2018).
[7] V. V. Albert, L. I. Glazman, and L. Jiang, Topological properties of linear circuit lattices, Phys. Rev. Lett. 114, 173902 (2015).
[8] S. Kirkpatrick, Percolation and conduction, Rev. Mod. Phys. 45, 574 (1973).
[9] M. B. Isichenko, Percolation, statistical topography, and transport in random media, Rev. Mod. Phys. 64, 961 (1992).
[10] Z. Ball, H. M. Phillips, D. L. Callahan, and R. Sauerbrey, Percolative metal-insulator transition in excimer laser ir-radiated polyimide, Phys. Rev. Lett. 73, 2099 (1994).
[11] V. Ambegaokar, B. I. Halperin, and J. S. Langer, Hop-
[55] M. Q. Owaidat, R. S. Hijjawi, and J. M. Khalifeh, Substitutional single resistor in an infinite square lattice application to lattice green’s function, Modern Physics Letters B 24, 2057 (2010).

[56] S. Katsura and S. Inawashiro, Lattice green’s functions for the rectangular and the square lattices at arbitrary points, Journal of Mathematical Physics 12, 1622 (1971).

[57] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, Introduction to Algorithms, 2nd ed. (The MIT Press, 2001).

[58] S.-J. Gu, Fidelity approach to quantum phase transitions, International Journal of Modern Physics B 24, 4371 (2010).