Some Hilbert spaces related with the Dirichlet space

1 Introduction

Consider the Dirichlet space \( \mathcal{D} \) on the unit disc \( \{ z \in \mathbb{C} : |z| < 1 \} \) of the complex plane. It can be defined as the Reproducing Kernel Hilbert Space (RKHS) having kernel

\[
k_z(w, z) = \frac{1}{\overline{z}w} \log \frac{1}{1 - \overline{z}w} = \sum_{n=0}^{\infty} \frac{(\overline{z}w)^n}{n+1}.
\]

We are interested in the spaces \( \mathcal{D}_d \) having kernel \( k^d \), with \( d \in \mathbb{N} \). \( \mathcal{D}_d \) can be thought of in terms of function spaces on polydiscs, following ideas of Aronszajn [4]. To explain this point of view, note that the tensor \( d \)-power \( \mathcal{D} \otimes_d \) of the Dirichlet space has reproducing kernel \( k^d_1(z_1, \ldots, z_d; w_1, \ldots, w_d) = \prod_{j=1}^{d} k(z_j, w_j) \). Hence, the space of restrictions of functions in \( \mathcal{D} \otimes_d \) to the diagonal \( z_1 = \cdots = z_d \) has the reproducing kernel \( k^d \), and therefore coincides with \( \mathcal{D}_d \).

We will provide several equivalent norms for the spaces \( \mathcal{D}_d \) and their dual spaces in Theorem 1.1. Then we will discuss the properties of these spaces. More precisely, we will investigate:

- \( \mathcal{D}_d \) and its dual space \( \mathcal{H}S_d \) in connection with Hankel operators of Hilbert-Schmidt class on the Dirichlet space \( \mathcal{D} \);
- the complete Nevanlinna-Pick property for \( \mathcal{D}_d \);
- the Carleson measures for these spaces.

Concerning the first item, the connection with Hilbert-Schmidt Hankel operators served as our original motivation for studying the spaces \( \mathcal{D}_d \).
Note that the spaces \( D_d \) live infinitely close to \( D \) in the scale of weighted Dirichlet spaces \( \tilde{D}_s \), defined by the norms
\[
\| \varphi \|_{\tilde{D}_s} = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \left| \varphi(e^{it}) \right|^2 dt + \int_{|z|<1} |\varphi'(z)|^2 (1-|z|^2)^s dA(z), \quad 0 \leq s < 1,
\]
where \( \frac{dA(z)}{\pi} \) is normalized area measure on the unit disc.

**Notation:** We use multiindex notation. If \( n = (n_1, \ldots, n_d) \) belongs to \( \mathbb{N}^d \), then \( |n| = n_1 + \cdots + n_d \). We write \( A \approx B \) if \( A \) and \( B \) are quantities that depend on a certain family of variables, and there exist independent constants \( 0 < c < C \) such that \( cA \leq B \leq CA \).

### Equivalent norms for the spaces \( D_d \) and their dual spaces \( HS_d \)

**Theorem 1.1.** Let \( d \) be a positive integer and let
\[
a_d(k) = \sum_{|\nu|=k} \frac{1}{(n_1 + 1) \cdots (n_d + 1)}.
\]
Then the norm of a function \( \varphi(z) = \sum_{k=0}^{\infty} \widehat{\varphi}(k) z^k \) in \( D_d \) is
\[
\| \varphi \|_{D_d} = \left( \sum_{k=0}^{\infty} a_d(k)^{-1} |\widehat{\varphi}(k)|^2 \right)^{1/2} \approx [\varphi]_d.
\]
where
\[
[\varphi]_d = \left( \sum_{k=0}^{\infty} \frac{k + 1}{\log^{d-1}(k+2)} |\widehat{\varphi}(k)|^2 \right)^{1/2}.
\]
An equivalent Hilbert norm \( ||\varphi||_{D_d} \approx [\varphi]_d \) for \( \varphi \) in terms of the values of \( \varphi \) is given by
\[
||\varphi||_{D_d} = |\varphi(0)|^2 + \left( \int_{\mathbb{D}} |\varphi'(z)|^2 \frac{1}{\log^{d-1}\left(\frac{1}{1-|z|^2}\right)} \frac{dA(z)}{\pi} \right)^{1/2}.
\]
Define now the holomorphic space \( HS_d \) by the norm:
\[
\| \psi \|_{HS_d} = \left( \sum_{k=0}^{\infty} (k+1)^2 a_d(k) |\widehat{\varphi}(k)|^2 \right)^{1/2}.
\]
Then, \( HS_d = (D_d)^* \) is the dual space of \( D_d \) under the duality pairing of \( D \). Moreover,
\[
\| \psi \|_{HS_d} \equiv [\psi]_{HS_d} := \left( \sum_{k=0}^{\infty} (k+1) \log^{d-1}(k+2) |\widehat{\varphi}(k)|^2 \right)^{1/2} \approx
\]
\[
||\psi||_{HS_d} := |\psi(0)|^2 + \int_{\mathbb{D}} |\psi'(z)|^2 \log^{d-1}\left(\frac{1}{1-|z|^2}\right) \frac{dA(z)}{\pi} \right)^{1/2}.
\]
Furthermore, the norm can be written as
\[
\| \psi \|_{HS_d}^2 = \sum_{(n_1, \ldots, n_d)} |\langle e_{n_1} \cdots e_{n_d}, \psi \rangle_d|^2,
\]
where \( \{e_n\}_{n=0}^{\infty} \) is the canonical orthonormal basis of \( D \), \( e_n(z) = \frac{z^n}{\sqrt{n+1}} \).
The remainder of this section is devoted to the proof of Theorem 1.1. The expression for $\|\varphi\|_{D_d}$ in (1) follows by expanding $(kz)^d$ as a power series. The equivalence $\|\varphi\|_{D_d} \approx [\varphi]_d$, as well as $\|\varphi\|_{HS_d} \approx [\varphi]_{HS_d}$, are consequences of the following lemma. We denote by $c, C$ positive constants which are allowed to depend on $d$ only, whose precise value can change from line to line.

**Lemma 1.2.** For each $d \in \mathbb{N}$ there are constants $c, C > 0$ such that for all $k \geq 0$ we have

$$c a_d(k) \leq \frac{\log^{d-1}(k + 2)}{k + 1} \leq C a_d(k).$$

Consequently, if $t \in (0, 1)$, then

$$c \left( \frac{1}{t} \log \frac{1}{1-t} \right)^d \leq \sum_{k=0}^{\infty} \frac{\log^{d-1}(k + 2)}{k + 1} \log t^k \leq C \left( \frac{1}{t} \log \frac{1}{1-t} \right)^d.$$

**Proof of Lemma 1.2.** We will prove the Lemma by induction on $d \in \mathbb{N}$. It is obvious for $d = 1$. Thus let $d \geq 2$ and suppose the lemma is true for $d - 1$. Also we observe that there is a constant $c > 0$ such that for all $k \geq 0$ and $0 \leq n \leq k$ we have

$$c \log^{d-2}(k + 2) \leq \log^{d-2}(n + 2) + \log^{d-2}(k - n + 2) \leq 2 \log^{d-2}(k + 2).$$

Then for $k \geq 0$

$$a_d(k) = \sum_{n_1 + \ldots + n_d = k} \frac{1}{(n_1 + 1) \ldots (n_d + 1)}$$

$$= \sum_{n=0}^{k} \frac{1}{n + 1} \sum_{n_2 + \ldots + n_d = k-n} \frac{1}{(n_2 + 1) \ldots (n_d + 1)}$$

$$\approx \sum_{n=0}^{k} \frac{1}{n + 1} \log^{d-2}(k - n + 2) \frac{1}{k - n + 1} \text{ by the inductive assumption}$$

$$= \frac{1}{2} \sum_{n=0}^{k} \frac{\log^{d-2}(n + 2) + \log^{d-2}(k - n + 2)}{(n + 1)(k - n + 1)}$$

$$\approx \log^{d-2}(k + 2) \sum_{n=0}^{k} \frac{1}{(n + 1)(k - n + 1)} \text{ by the earlier observation}$$

$$= \frac{\log^{d-2}(k + 2)}{k + 1} \sum_{n=0}^{k} \frac{1}{n + 1} + \frac{1}{k - n + 1}$$

$$\approx \frac{\log^{d-1}(k + 2)}{k + 1}.$$

Next, we prove the equivalence $[\varphi]_{HS_d} \approx \|\varphi\|_{HS_d}$ which appears in (5).

**Lemma 1.3.** Let $d \in \mathbb{N}$. Then

$$\int_{0}^{1} t^k \left( \frac{1}{t} \log \frac{1}{1-t} \right)^{d-1} dt \approx \frac{\log^{d-1}(k + 2)}{k + 1}, \quad k \geq d.$$

Given the Lemma, we expand

$$\|\varphi\|_{HS_d}^2 = |\hat{\varphi}(0)|^2 + \int_{C} \left( \sum_{k=1}^{\infty} \hat{\varphi}(k) k z^{k-1} \right)^2 \log^{d-1} \frac{1}{1-|z|^2} \frac{dA(z)}{\pi}$$

$$= |\hat{\varphi}(0)|^2 + \sum_{k=1}^{\infty} k^2 |\hat{\varphi}(k)|^2 \int_{0}^{1} \log^{d-1} \frac{1}{1-t} t^{k-1} dt.$$
\[ \approx |\hat{\psi}(0)|^2 + \sum_{k=1}^{\infty} k^2 \left| \hat{\psi}(k) \right|^2 \frac{\log^{d-1}(k+2)}{k+1} \]

obtaining the desired conclusion.

**Proof of Lemma 1.3.** The case \( d = 1 \) is obvious, leaving us to consider \( d \geq 2 \). We will also assume that \( k \geq 2 \).

Then by Lemma 1.2 we have

\[ \int_0^1 t^k \left( \frac{1}{t} \log \left( \frac{1}{1-t} \right) \right)^{d-1} dt \approx \int_0^1 t^k \sum_{n=0}^{\infty} \frac{\log^{d-2}(n+2)}{n+1} t^n dt = \sum_{n=0}^{\infty} \frac{\log^{d-2}(n+2)}{(n+1)(n+k+1)} = S_1 + S_2, \]

where

\[ S_1 = \sum_{n=0}^{k-1} \frac{\log^{d-2}(n+2)}{(n+1)(n+k+1)} \approx \frac{1}{k+1} \sum_{n=0}^{k-1} \frac{\log^{d-2}(n+2)}{n+1} \approx \frac{1}{k+1} \int_1^{k+2} \frac{\log^{d-2}(t)}{t} dt \]

and

\[ S_2 = \sum_{n=k}^{\infty} \frac{\log^{d-2}(n+2)}{(n+1)(n+k+1)} \leq \sum_{n=k}^{\infty} \frac{\log^{d-2}(n+1)}{n^2} \leq \sum_{j=1}^{\infty} \sum_{n=k^j}^{k^{j+1}-1} \frac{\log^{d-2}(n+1)}{n^2} \leq \sum_{j=1}^{\infty} (j+1)^{d-2} k^{j+1} \frac{1}{n^2} \leq \log^{d-2}(k+2) k \sum_{j=1}^{\infty} (j+1)^{d-2} \frac{k+1}{k+1} \sum_{j=1}^{\infty} (j+1)^{d-2} \frac{k+1}{(k-1)k^{j-1}} \]

\[ \leq \frac{\log^{d-2}(k+2)}{k+1} \sum_{j=1}^{\infty} (j+1)^{d-2} \frac{3}{2^{j-1}} = o \left( \frac{\log^{d-1}(k+2)}{k+1} \right). \]

Now, the duality between \( D_d \) and \( HS_d \) under the duality pairing given by the inner product of \( D \) is easily seen by considering \([\cdot]_d \) and \([\cdot]_{HS_d} \). They are weighted \( L^2 \) norms and duality is established by means of the Cauchy-Schwarz inequality.

Next we will prove that \([\varphi]_d \approx [\varphi]_d \). This is equivalent to proving that the dual space of \( HS_d \), with respect to the Dirichlet inner product \((\cdot, \cdot)_D\), is the Hilbert space with the norm \([\cdot]_d \).

Let \( d \in \mathbb{N} \) and set, for \( 0 < t < 1 \), \( w_d(t) = \left( \frac{1}{t} \log \frac{1}{1-t} \right)^d \) and, for \( 0 < |z| < 1 \), \( W_d(z) = w_d(|z|^2) \) and \( W_d(0) = 1 \).

**Lemma 1.4.** Let \( d \in \mathbb{N} \). Then

\[ \int_{1-\varepsilon}^1 w_d(t) dt \cdot \int_{1-\varepsilon}^1 \frac{1}{w_d(t)} dt \approx \varepsilon^2 \text{ as } \varepsilon \to 0. \]

**Proof.** Write \( \tilde{w}(t) = (\log \frac{1}{1-t})^d \), and note that it suffices to establish the lemma for \( \tilde{w} \) in place of \( w_d \). Let \( \varepsilon > 0 \).

Then \( \tilde{w} \) is increasing in \((0, 1)\) and \( \tilde{w}(1-\varepsilon^{k+1}) = (k+1)^d (\log \frac{1}{\varepsilon})^d \), hence

\[ \int_{1-\varepsilon}^1 \tilde{w}(t) dt = \sum_{k=1}^{\infty} \int_{1-\varepsilon^k}^{1-\varepsilon^{k+1}} \tilde{w}(t) dt \leq \sum_{k=1}^{\infty} \tilde{w}(1-\varepsilon^{k+1}) (\varepsilon^k - \varepsilon^{k+1}) \]

\[ \approx \sum_{k=1}^{\infty} (k+1)^d (\log \frac{1}{\varepsilon})^d (\varepsilon^k - \varepsilon^{k+1}) \]

\[ \approx \varepsilon^2 \text{ as } \varepsilon \to 0. \]
\[= \sum_{k=1}^{\infty} (k + 1)^d (\log \frac{1}{\varepsilon})^d \varepsilon^k (1 - \varepsilon) \approx \varepsilon (\log \frac{1}{\varepsilon})^d \frac{1}{(1 - \varepsilon)^d}\]

For \(1/\varepsilon\) we just notice that it is decreasing and hence
\[
\int_{1-\varepsilon}^{1} \frac{1}{\tilde{w}(t)} dt \leq \frac{1}{\tilde{w}(1-\varepsilon)} \varepsilon = \frac{\varepsilon}{(\log \frac{1}{\varepsilon})^d}
\]

Thus as \(\varepsilon \to 0\) we have
\[
\varepsilon^2 \leq \int_{1-\varepsilon}^{1} \tilde{w}(t) dt \int_{1-\varepsilon}^{1} \frac{1}{\tilde{w}(t)} dt = O(\varepsilon^2).
\]

For \(0 < h < 1\) and \(s \in [-\pi, \pi]\) let \(S_h(e^{is})\) be the Carleson square at \(e^{is}\), i.e.
\[S_h(e^{is}) = \{r e^{it} : 1 - h < r < 1, |t - s| < h\}.
\]

A positive function \(W\) on the unit disc is said to satisfy the Bekollé-Bonami condition (B2) if there exists \(c > 0\) such that
\[
\int_{S_h(e^{is})} W dA \cdot \int_{S_h(e^{is})} \frac{1}{W} dA \leq c h^4
\]
for every Carleson square \(S_h(e^{is})\). If \(d \in \mathbb{N}\) and if \(W_d(z)\) is defined as above, then
\[
\int_{S_h(e^{is})} W_d dA \cdot \int_{S_h(e^{is})} \frac{1}{W_d} dA = h^2 \int_{1-h}^{1} w_d(t) dt \cdot \int_{1-h}^{1} \frac{1}{w_d(t)} dt \approx h^4
\]
by Lemma 1.4, at least if \(0 < h < 1/2\). Observe that both \(W_d\) and \(1/W_d\) are positive and integrable in the unit disc, hence it follows that the estimate holds for all \(0 < h \leq 1\).

Thus \(W_d\) satisfies the condition (B2). Furthermore, note that if \(f(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k\) is analytic in the open unit disc, then
\[
\int_{|z|<1} |f(z)|^2 w_d(|z|^2) \frac{dA(z)}{\pi} = \sum_{k=0}^{\infty} w_k |\hat{f}(k)|^2.
\]
where \(w_k = \int_{0}^{1} t^k w_d(t) dt \approx \frac{\log^d(k+2)}{k+2}3.
\]

A special case of Theorem 2.1 of Luecking’s paper [7] says that if \(W\) satisfies the condition (B2) by Bekollé and Bonami [5], then one has a duality between the spaces \(L^2_0(WdA)\) and \(L^2_0(\frac{1}{W}dA)\) with respect to the pairing given by \(\int_{|z|<1} f \overline{g} dA\). Thus, we have
\[
\int_{|z|<1} |g(z)|^2 \frac{1}{W_d(z)} dA \approx \sup_{f \neq 0} \left| \frac{\int_{|z|<1} g(z) f(z) \overline{f(z)} \frac{dA(z)}{\pi}}{\int_{|z|<1} |f(z)|^2 W_d(z) dA} \right|^2 = \sup_{f \neq 0} \left| \frac{\sum_{k=0}^{\infty} \frac{\hat{g}(k)}{(k+1) \sqrt{w_k}} \sqrt{w_k} \hat{f}(k)}{\sum_{k=0}^{\infty} w_k |\hat{f}(k)|^2} \right|^2
\]
\[
= \sum_{k=0}^{\infty} \frac{1}{(k+1)^2 w_k} |\hat{g}(k)|^2
\]

This finishes the proof of (5). It remains to demonstrate (6). We defer its proof to the next section.

By Theorem 1.1 we have the following chain of inclusions:

\[\ldots \hookrightarrow HS_{d+1} \hookrightarrow HS_d \hookrightarrow \ldots \hookrightarrow HS_2 \hookrightarrow HS_1 = D = D_1 \hookrightarrow D_2 \hookrightarrow \ldots \hookrightarrow D_d \hookrightarrow D_{d+1} \hookrightarrow \ldots\]

with duality w.r.t. \(D\) linking spaces with the same index. It might be interesting to compare this sequence with the sequence of Banach spaces related to the Dirichlet spaces studied in [3]. Note that for \(d \geq 3\) the reproducing kernel of \(HS_d\) is continuous up to the boundary. Hence functions in \(HS_d\) extend continuously to the closure of the unit disc, for \(d \geq 3\).
Hilbert-Schmidt norms of Hankel-type operators

Let \( \{e_n\} \) be the canonical orthonormal basis of \( D, e_n(z) = \frac{z^n}{\sqrt{n+1}} \). Equation (6) follows from the computation

\[
\sum_{k=0}^{\infty} \sum_{|n|=k} |(e_{n_1} \cdots e_{n_d}, \psi)|^2 = \sum_{k=0}^{\infty} \sum_{|n|=k} \frac{1}{(n_1 + 1) \cdots (n_d + 1)} |(z^{n_1} \cdots z^{n_d}, \psi)|^2
\]

\[
= \sum_{k=0}^{\infty} \sum_{|n|=k} \frac{1}{(n_1 + 1) \cdots (n_d + 1)} |(k^k, \psi)|^2 = \sum_{k=0}^{\infty} \sum_{|n|=k} \frac{(k+1)^2}{(n_1 + 1) \cdots (n_d + 1)} |\hat{\psi}(k)|^2
\]

\[
= \sum_{k=0}^{\infty} (k+1) \alpha_k |\hat{\psi}(k)|^2 \approx \sum_{k=0}^{\infty} \log^{d-1}(k+2) |\hat{\psi}(k)|^2.
\]

Polarizing this expression for \( \| \cdot \|_{HS_d} \), the inner product of \( HS_d \) can be written

\[
\langle \psi_1, \psi_2 \rangle_{HS_d} = \sum_{(n_1, \ldots, n_d)} \langle \psi_1, e_{n_1} \cdots e_{n_d} \rangle_D \langle e_{n_1} \cdots e_{n_d}, \psi_2 \rangle_D.
\]

Hence, for any \( \lambda, \xi \in \mathbb{D} \),

\[
\langle k_\lambda, k_\xi \rangle_{HS_d} = \sum_{n \in \mathbb{N}^d} \langle k_\lambda, e_{n_1} \cdots e_{n_d} \rangle_D \langle e_{n_1} \cdots e_{n_d}, k_\xi \rangle_D = \sum_{n \in \mathbb{N}^d} e_n(\lambda) \cdots e_n(\lambda) e_n(\xi) \cdots e_n(\xi)
\]

\[
= \left( \sum_{i=0}^{\infty} \frac{e_i(\lambda)}{i!} e_i(\xi) \right)^d = k_\lambda(\xi)^d = \langle k_\lambda, k_\xi \rangle_{D_d}.
\]

That is,

**Proposition 1.5.** The map \( U : k_\lambda \mapsto k_\lambda^d \) extends to a unitary map \( HS_d \to D_d \).

When \( d = 2 \), \( HS_2 \) contains those functions \( b \) for which the Hankel operator \( H_b : D \to \overline{D} \), defined by \( \langle H_b e_j, \overline{e_k} \rangle = \langle e_j, e_k \rangle_D \), belongs to the Hilbert-Schmidt class.

Analogous interpretations can be given for \( d \geq 3 \), but then function spaces on polydiscs are involved. We consider the case \( d = 3 \), which is representative. Consider first the operator \( T_b : D \to \overline{D} \otimes \overline{D} \) defined by

\[
\left( T_b f, \overline{g} \otimes \overline{h} \right)_{\overline{D} \otimes \overline{D}} = \langle fgh, b \rangle_D.
\]

The formula uniquely defines an operator, whose action is

\[
T_b f(z, w) = \left( T_b f, \overline{z} \overline{w} \right)_{\overline{D} \otimes \overline{D}}
\]

\[
= \langle f z k_w, b \rangle_D
\]

\[
= \sum_{n,m,j} \hat{f}(j) \frac{\overline{w}^n}{n+1} \frac{z^{m-j}}{m+1} \langle \zeta^{n+m+j}, b \rangle_D
\]

\[
= \sum_{n,m,j} \hat{f}(j) \overline{b}(n+m+j) \frac{\zeta^{n+m+j}}{(n+1)(m+1)} \frac{\overline{w}^n}{z^{m-j}}
\]

Then, the Hilbert-Schmidt norm of \( T_b \) is:

\[
\sum_{l,m,n} |\langle T_b e_l, e_m e_n \rangle_{\overline{D} \otimes \overline{D}}|^2 = \sum_{l,m,n} |\langle e_l e_m e_n, b \rangle_D|^2 = \|b\|_{HS_3}^2.
\]

Similarly, we can consider \( U_b : D \otimes D \to \overline{D} \) defined by

\[
\left( U_b f \otimes g, \overline{h} \right)_{\overline{D}} = \langle fgh, b \rangle_D.
\]
The action of this operator is given by

\[ U_b(f \otimes g)(z) = \sum_{l,m,n=0}^{\infty} f(l)\overline{g}(m) \frac{(l+m+n+1)\overline{b}(l+m+n)}{n+1} z^n. \]

The Hilbert-Schmidt norm of \( U_b \) is still \( \| b \|_{HS} \).

### Carleson measures for the spaces \( D_d \) and \( HS_d \)

The (B2) condition allows us to characterize the Carleson measures for the spaces \( D_d \) and \( HS_d \). Recall that a nonnegative Borel measure \( \mu \) on the open unit disc is Carleson for the Hilbert function space \( H \) if the inequality

\[ \int_{|z|<1} |f|^2 d\mu \leq C(\mu) \| f \|_H^2 \]

holds with a constant \( C(\mu) \) which is independent of \( f \). The characterization [2] shows that, since the (B2) condition holds, then

**Theorem 1.6.** For \( d \in \mathbb{N} \), a measure \( \mu \geq 0 \) on \( \{ |z| < 1 \} \) is Carleson for \( D_d \) if and only if for \( |a| < 1 \) we have:

\[ \int_{S(a)} \log^{d-1} \left( \frac{1}{1-|z|^2} \right) \frac{(1-|z|^2)\mu(S(z) \cap S(a))^2}{(1-|z|^2)^2} \frac{dx dy}{2} \leq C_1(\mu)\mu(S(a)), \]

where \( S(a) = \{ z : 0 < 1 - |z| < 1 - |a|, \ |\arg(z\overline{a})| < 1 - |a| \} \) is the Carleson box with vertex \( a \) and \( \tilde{S}(a) = \{ z : 0 < 1 - |z| < 2(1 - |a|), \ |\arg(z\overline{a})| < 2(1 - |a|) \} \) is its “dilation”.

The characterization extends to \( HS_2 \), with the weight \( \log^{-1} \left( \frac{1}{1-|z|^2} \right) \). Since functions in \( HS_d \) are continuous for \( d \geq 3 \), all finite measures are Carleson measures for these spaces. Once we know the Carleson measures, we can characterize the multipliers for \( D_d \) in a standard way.

### The complete Nevanlinna-Pick property for \( D_d \)

Next, we prove that the spaces \( D_d \) have the Complete Nevanlinna-Pick (CNP) Property. Much research has been done on kernels with the CNP property in the past twenty years, following seminal work of Sarason and Agler. See the monograph [1] for a comprehensive and very readable introduction to this topic. We give here a definition which is simple to state, although perhaps not the most conceptual. An irreducible kernel \( k : X \times X \rightarrow \mathbb{C} \) has the CNP property if there is a positive definite function \( F : X \rightarrow \mathbb{D} \) and a nowhere vanishing function \( \delta : X \rightarrow \mathbb{C} \) such that:

\[ k(x, y) = \frac{\overline{F(x)} \delta(y)}{1 - F(x,y)}, \]

whenever \( x, y \) lie in \( X \). The CNP property is a property of the kernel, not of the Hilbert space itself.

**Theorem 1.7.** There are norms on \( D_d \) such that the CNP property holds.

**Proof.** A kernel \( k : \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{C} \) of the form \( k(w, z) = \sum_{k=0}^{\infty} a_k (\overline{w})^k \) has the CNP property if \( a_0 = 1 \) and the sequence \( \{a_n\}_{n=0}^{\infty} \) is positive and log-convex:

\[ \frac{a_{n-1}}{a_n} \leq \frac{a_n}{a_{n+1}}. \]
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Consider $\eta(x) = \alpha \log \log(x) - \log(x)$, with real $\alpha$. Then, $\eta''(x) = \frac{\log^2(x) - \alpha \log(x) - \alpha}{x^2 \log^2(x)}$, which is positive for $x \geq M_\alpha$, depending on $\alpha$. Let now

$$a_n = \frac{\log^{d-1}(M_d(n+1))}{\log(M_d) \cdot (n+1)} \approx \frac{1}{n+1} + \frac{\log^{d-1}(n+1)}{n+1} \quad (7)$$

Then, the sequence $\{a_n\}_{n=0}^\infty$ provides the coefficients for a kernel with the CNP property for the space $D_d$.

The CNP property has a number of consequences. For instance, we have that the space $D_d$ and its multiplier algebra $M(D_d)$ have the same interpolating sequences. Recall that a sequence $Z = \{z_n\}_{n=0}^\infty$ is interpolating for a RKHS $H$ with reproducing kernel $k^H$ if the weighted restriction map $R : \varphi \mapsto \{k^H(z_n, \cdot)^{\gamma/2}\}_{n=0}^\infty$ maps $H$ boundedly onto $\ell^2$; while $Z$ is interpolating for the multiplier algebra $M(H)$ if $Q : \psi \mapsto \{\psi(z_n)^{\gamma/2}\}_{n=0}^\infty$ maps $M(H)$ boundedly onto $\ell^\infty$. The reader is referred to [1] and to the second chapter of [8] for more on this topic.

It is a reasonable guess that the universal interpolating sequences for $D_d$ and for its multiplier space $M(D_d)$ are characterized by a Carleson condition and a separation condition, as described in [8] (see the Conjecture at p. 33). See also [6], which contains the best known result on interpolation in general RKHS spaces with the CNP property. Unfortunately we do not have an answer for the spaces $D_d$.

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