Status of the Kazakov–Migdal Model

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ABSTRACT

In this talk I discuss both the present status and some recent work on the Kazakov–Migdal Model which was originally proposed as a soluble, large $N$ realization of QCD. After a brief description of the model and a discussion of its solubility in the large $N$ limit I discuss several of the serious problems with the model which lead to the conclusion that it does not induce QCD. The model is nonetheless a very interesting example of a Gauge Theory and it is related to some very interesting Matrix Models. I then outline a technique \[1\] which uses “Loop Equations” for solving such models. A Penner–like model is then discussed with two logarithmic singularities. This model is distinguished by the fact that it is exactly and explicitly soluble in spite of the fact that it is not Gaussian. It is shown how to analyze this model using both a technical approach and from a more physical point of view.

1. INTRODUCTION

The Kazakov–Migdal Model is an SU(N) Lattice Gauge model which was proposed several years ago\[2\] in the hope that it would be both soluble in the large $N$ limit and that it would give the correct long distance behavior of QCD in the continuum limit. This model was thus also known as “Induced QCD”. Although it is now widely believed that this model does not induce QCD much has been learned from the study of this model, both about ordinary and about gauged Matrix Models. Work is still in progress on a class of “soluble”, non–Gaussian Kazakov–Migdal Models.

This talk will begin with a brief review of the Kazakov–Migdal Model (the KM Model) and why it was originally though that it might induce QCD. This will

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be followed by a discussion of the solubility (i.e. tractability) of the model at large \( N \). Some of the main problems with the model will then be described including a discussion of the extra local \( Z_N \) symmetry and the absence of critical behaviour in the Gaussian Model. This will then explain why it is extremely unlikely that this model could induce QCD. This will be followed by an outline of a powerful method for solving KM Models. I shall conclude this talk with a discussion of a soluble, non–Gaussian, (“Penner”) model which is explicitly soluble in the large \( N \) limit and which is the focus of continued research.

2. The Model

The KM Model is a \( d \)-dimensional lattice Gauge Theory containing an SU(\( N \)) Gauge Field \( U_{xy} \) defined on the links of the lattice and a Hermitian (adjoint) scalar “Higgs” Field \( \Phi_x \) defined on the sites of the lattice. The model is defined in terms of the Euclidean Partition Function

\[
Z = \int \prod_{\text{links}} \mathcal{D}U_{xy} \prod_{\text{sites}} \mathcal{D}\Phi_x \exp [-S]
\]

(1)

with

\[
S = N \sum_{\text{sites},x} \text{tr} (V(\Phi_x)) - N \sum_{\text{links},xy} \text{tr} (\Phi_x U_{xy} \Phi_y U_{yx}^{-1})
\]

(2)

where \( V(\Phi) \) is, at this stage, an unspecified potential for the field \( \Phi \) and where the second term is the standard gauge invariant kinetic term for the adjoint field \( \Phi \). Note that this action is simply the lattice action for an adjoint scalar field except that the Wilson term which is the kinetic term for the Gauge Field is missing. This Wilson term

\[
\sum_{\text{plaquettes},1234} \text{Tr} (U_1 U_2 U_3 U_4) + h.c.
\]

(3)

is missing in order to assure the solubility of the model at large \( N \).

The original hope was that the integration over \( \Phi \) would induce a kinetic term for the Gauge Field which would lead to a theory which, in the continuum limit, would be QCD (or, more precisely, Quantum Gluodynamics, since quarks have not been introduced at this stage). The arguments supporting this claim can be found in the original papers \(^3\).
3. LARGE $N$ SOLUBILITY

The missing Wilson term (3) in Eq. (2) allows for exact integration over all the Gauge Fields $U_{xy}$ on the lattice. To see this note that each term in the action contains one and only one $U_{xy}$ and each such Gauge Field appears in only one term in the action. The integral over $U$ can be done by using the formula [4]

$$
\int D U \exp \left( \sum_{i,j} \phi_i \psi_j |U_{ij}|^2 \right) = \text{const} \frac{\det_{ij} e^{\phi_i \psi_j}}{\Delta(\phi) \Delta(\psi)}
$$

where

$$
\Delta(\phi) = \prod_{i<j} (\phi_i - \phi_j)
$$

is the Vandermonde determinant for $\phi$. This formula can be applied to the integral in Eq. (1) by diagonalizing the Hermitian Matrices $\Phi_x$ and $\Phi_y$ which appear in Eq. (2).

The main point is that the result of the integral depends only on the eigenvalues of the $\Phi$’s. The integral over each matrix $\Phi$ (which is an integral with the Hermitian measure) can now be written as an integral over the eigenvalues $\phi_i$ of $\Phi$ since

$$
\int D \Phi \cdots = \int \prod d \phi_i \Delta^2(\phi) \cdots
$$

when the integrand depends only on the eigenvalues of $\Phi$. It follows that the Partition Function, Eq. (1), is of the form

$$
Z = \int \prod d \phi_{x,i} \Delta^2(\phi_x) \exp \left[ -S_{\text{eff}} \right]
$$

where $S_{\text{eff}}$ depends only on the eigenvalues of the $\Phi$’s. In fact it contains one term which involves the $\phi_{x,i}$ at neighbouring sites and a potential term involving only one site at a time.

In the limit of large $N$ the above integral can be done by finding the Classical Minimum of the action $S_{\text{eff}}$. This is assumed to occur when the eigenvalues of $\Phi_x$ are independent of $x$. The basic problem is then to minimize the effective action

$$
-\log \left\{ \left( \frac{\det_{ij} e^{N \phi_i \phi_j}}{\Delta^2(\phi)} \right)^d \Delta^2(\phi) e^{-N \text{Tr} V(\phi)} \right\}
$$

with respect to all the $\phi_i$. The result will be a specific set of eigenvalues $\phi_1...\phi_N$ which, in the limit of large $N$, is described by a density of eigenvalues $\rho(\phi)$ (proportional to
the number of eigenvalues in the vicinity of \( \phi \) which is normalized so that \( \int d\lambda \rho(\lambda) = 1 \). This minimization can be carried out in practice for only a very limited number of special choices for the potential \( V \). We shall discuss this further later in this talk.

4. Problems with “Induced QCD”

Despite the solubility of the KM Model at large \( N \), several serious problems were recognized immediately with the hypothesis that the KM Model induces QCD. The first of these problems is the presence of an additional local \( Z_N \) symmetry in the model\(^5\). The absence of the Wilson term allows us to multiply the Gauge Field \( U_{xy} \) on any link \((xy)\) by an element of the center of the Gauge Group without affecting the action. More formally, the transformation

\[
U_{xy} \to \exp \left( \frac{2\pi ik}{N} \right) U_{xy}
\]

which is also an element of SU\((N)\) if \( k \) is an integer leaves the action Eq. (2) invariant even if a different \( k \) is chosen for each link \((xy)\). (A Wilson term would ruin this invariance.)

The main problem with this symmetry is that it implies the vanishing of all Wilson Loops since

\[
< \prod_{\text{loop}} U > = \exp \left( \frac{2\pi ik}{N} \right) < \prod_{\text{loop}} U >
\]

if we multiply any one of the links in the loop by \( \exp \left( \frac{2\pi ik}{N} \right) \). This forces all Wilson Loops to vanish. Furthermore, since the symmetry is local, it cannot be broken\(^6\). This is a disaster if the model is to induce QCD since, in QCD, we expect an area law for Wilson Loops.

Several proposals have been made to solve this problem, none of which have been particularly successful. Attempts at solving this problem have included the introduction of Fundamental Representation Fermions\(^7\) or a subleading (in \( 1/N \)) Wilson term\(^8\) both of which break the local \( Z_N \) symmetry at the expense of the exact solubility of the model. It has also been suggested that “Filled Wilson Loops” which are invariant under the local \( Z_N \) replace the ordinary Wilson Loops in this model.

A second very serious problem with the idea of inducing QCD from the KM Model was discovered when Gross\(^9\) solved the Gaussian KM Model, i.e., the model with \( V(\phi) = m^2 \phi^2 / 2 \), explicitly. The argument is as follows. It turns out
that the perturbative mass of the “Higgs” scalar is \( m^2 - 2d \) (\( d \) is the dimension of the spacetime.) The most convincing arguments that the KM Model induced QCD were made for precisely this Gaussian potential for which it was expected that the model would have critical behaviour and a continuum limit when \( m^2 \to 2d \). The exact solution to the model proved that no such critical behaviour was present and thus the Gaussian model certainly did not induce QCD.

This, by itself, does not rule out that some non-Gaussian model may induce QCD but, in light of the fact that the main argument that the model induced QCD did work for the Gaussian case and in light of the \( Z_N \) problem it seems very unlikely that any non-Gaussian model would induce QCD. There are, in fact, several other problems with the idea of Induced QCD which are discussed in the literature.

Despite the failure of the model to induce QCD, the KM Model is still a very interesting Gauge Theory and it is very interesting to find non-Gaussian models which can be solved exactly. Fortunately there is a Penner-like model with a logarithmic potential which can be solved explicitly (Ref. 1) and which will be discussed below.

5. A Technique for Solving KM Models

In this section I shall review briefly the method of Dobroliubov, Makeenko and Semenoff (Ref. 1) for finding the extremum of the action (8). The basic idea is to begin with the original action (2) and to consider the following two quantities:

\[
E(\lambda) = \left\langle \frac{1}{N} \text{Tr} \left( \frac{1}{\lambda - \Phi} \right) \right\rangle = \int d\phi \frac{\rho(\phi)}{\lambda - \phi}
\]

and

\[
G(\lambda, \nu) = \left\langle \frac{1}{N} \text{Tr} \left( \frac{1}{\lambda - \Phi} U_{xy} \frac{1}{\nu - \Phi} U^{-1}_{xy} \right) \right\rangle
\]

where the average is with respect to the original Path Integral and where the expression of \( E(\lambda) \) in terms of the density of eigenvalues \( \rho \) is valid in the large \( N \) limit.

The main idea is to write a set of equations, similar to Schwinger–Dyson Equations and often called “Loop Equations” for these quantities. These equations are of the form

\[
\int \mathcal{D}\Phi \frac{d}{d\Phi_{x,ij}} \left( \left[ \frac{1}{\lambda - \Phi} \right]_{ij} e^{-S} \right) = 0
\]

and

\[
\int \mathcal{D}\Phi \frac{d}{d\Phi_{x,ij}} \left( \left[ \frac{1}{\lambda - \Phi} U_{xy} \frac{1}{\nu - \Phi} U^{-1}_{xy} \right]_{ij} e^{-S} \right) = 0
\]
The first equation Eq. (13) will yield an equation for \(E(\lambda)\) and \(G(\lambda, \nu)\) whereas the second equation will, in general, involve quantities with more factors of \(1/(\lambda - \phi)\).

In solving these equations one needs to make heavy use of the large \(N\) limit in which the integral is dominated entirely by a single matrix \(\Phi\) which minimizes the effective action Eq. (8). Even averages over the Gauge fields \(U\) which appear in Eq. (14) are performed in this background \(\Phi\) field. In fact it is useful to treat

\[
\int \mathcal{D}U U_{(xy)}^{ij} U_{(xy)}^{-1} e^{-S} \tag{15}
\]

as an unknown whose only nonvanishing components are defined as

\[
C_{ij} = \frac{1}{Z} \int \mathcal{D}U |U_{ij}|^2 e^N \text{Tr}\Phi U \Phi^{-1} \tag{16}
\]

where \(\Phi\) is the minimum of the effective action which has yet to be determined. Obviously \(C_{ij}\) depends on \(\Phi\). It is also useful to define a quantity

\[
\Lambda_i = C_{ij} \phi_j \tag{17}
\]

where \(\phi_j\) are the eigenvalues of \(\Phi\). Now the unknown but fixed values of \(\phi_i\) allow us to identify an integer with every eigenvalue of \(\Phi\). We can thus replace the quantity \(\Lambda_i\) with a quantity \(\Lambda(\phi)\) which is defined via the correspondence of \(\phi\) with \(i\).

The quantities defined above are very useful in solving the “loop equations.” It turns out that there are several cases in which these loop equations close and one can solve explicitly for \(E(\lambda)\) and \(G(\lambda, \nu)\). In these cases the density of eigenvalues can be extracted from \(E(\lambda)\) using Eq. (11) which implies that \(E(\lambda)\) has a branch cut along the support of \(\rho\) whose discontinuity is proportional to \(\rho\). \(C_{ij}\) can also be determined as the double discontinuity across the cut of \(G(\lambda, \nu)\) in \(\lambda\) and in \(\nu\).

The simplest case in which these equations close is the Gaussian case in which \(V(\phi) = m^2 \phi^2/2\). This case is discussed in detail in Ref. 1. What is even more interesting is that this system can be solved for a non-Gaussian, though singular potential of the Penner\(^{10}\) (logarithmic) type. An outline of the solution follows.

Begin with the “Loop Equations” Eqs. (13) and (14). The first equation Eq. (13) leads to the equation

\[
E(\lambda)^2 - \left< \frac{V'(\phi) - 2d\Lambda(\phi)}{\lambda - \phi} \right> = 0 \tag{18}
\]

Here \(V' = dV/d\phi\) The second equation Eq. (14) can be simplified if we assume a simple form for the quantity \(V' - 2(d - 1)\Lambda\). The soluble “Penner” case consists of
requiring this quantity to have a simple pole at some point $\xi$ so that

$$V'(\phi) - 2(d - 1)\Lambda(\phi) = \frac{q}{\phi - \xi} + B \quad (19)$$

This “ansatz” allows a significant simplification of the equations. Using the asymptotic conditions

$$E(\lambda) \sim \frac{1}{\lambda} + \frac{\bar{\phi}}{\lambda^2} + \cdots \quad (20)$$

$$G(\lambda, \nu) = \frac{E(\lambda)}{\nu} + \cdots \quad (21)$$

one eventually derives a quadratic equation for $E(\lambda)$ in terms of a single, unknown quantity $\bar{\phi}$ which must be determined self consistently. This quadratic equation for $E(\lambda)$ is simply

$$E(\lambda)^2 \left[ (\lambda - B)(\lambda - \xi) \right] + E(\lambda) \left[ (\lambda - B)(\lambda - \xi)(\xi - B) + q(B - \xi) - (\lambda - \xi) \right]$$

$$+ \left[ (\lambda + \bar{\phi})(B - \xi) + (\xi^2 - B^2) \right] = 0 \quad (22)$$

Besides allowing us to solve for $E(\lambda)$ and thus for the density of eigenvalues $\rho(\lambda)$ we can now find the potential which leads to the ansatz (19). It is not difficult to show that if we were to substitute the expression

$$V'(\lambda) - 2d\Lambda(\lambda) = \frac{q}{\lambda - \xi} + \frac{(1 - q)}{\lambda - B} + (\xi - B) \quad (23)$$

into equation (18) we would obtain precisely the same quadratic equation (Eq. (22)) for $E(\lambda)$ as Equation (22). It thus follows that

$$\Lambda(\lambda) = \frac{(q - 1)}{\lambda - B} + \xi \quad (24)$$

and thus the potential $V$ is given by

$$V'(\lambda) = \frac{q}{\lambda - \xi} + \frac{(2d - 1)(q - 1)}{\lambda - B} + B + (2d - 1)\xi \quad (25)$$

We are thus able to solve the KM Model for a potential with two logarithmic singularities plus a linear term when the various coefficients are related as

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1 There is actually a subtle assumption being made here. It is not guaranteed that $V$ will simply be the integral of $V'$ unless $V$ is of the special form $\text{Tr} f(\phi)$ where $f$ is some ordinary function of $\phi$. The point is that since we have made an ansatz for $V' - (2d - 1)\Lambda$ rather than for $V$ we are not guaranteed that $V$ will be of this simple form.
6. Relationship with an Ordinary Matrix Model

We are now ready to see that the solution to the above KM Model is closely related to the solution of an ordinary Matrix model with the potential $V' - 2d\Lambda$ given by Eq. (23). To see this consider an ordinary matrix model defined as an integral over a single $N \times N$ Hermetian matrix $\Phi$

$$Z = \int \mathcal{D}\Phi \exp \left[ -N\text{Tr}W(\Phi) \right]$$  \hspace{1cm} (26)

This model can be solved by defining $E(\lambda)$ in analogy with Eq. (11) as

$$E(\lambda) = \left\langle \frac{1}{\lambda - \Phi} \right\rangle$$  \hspace{1cm} (27)

and writing a “loop equation” in analogy with Eq. (13) which leads to an equation identical to Eq. (18)

$$E(\lambda)\lambda^2 - \left\langle \frac{W'(\phi)}{\lambda - \phi} \right\rangle = 0$$  \hspace{1cm} (28)

It follows that if, for $W'$, we choose $V' - 2d\Lambda$ given by Eq. (23) we arrive at precisely the same equation for $E(\lambda)$ and thus the same distribution of eigenvalues $\rho(\lambda)$.

We are thus led to consider an ordinary Matrix Model with two logarithmic singularities so that $W$ is given by

$$W'(\lambda) = \frac{r_1}{\lambda - \eta_1} + \frac{r_2}{\lambda - \eta_2} + C$$  \hspace{1cm} (29)

This model is related to the KM Model above provided (see Eq. (23))

$$r_1 + r_2 = 1; \quad C = \eta_2 - \eta_1$$  \hspace{1cm} (30)

It is most interesting to consider the general two pole Penner Model above and to discuss the KM case as the special case when Eq. (30) is satisfied. In the
general case Eq. (28) can be written as:

\[ E(\lambda)^2 - W'(\lambda)E(\lambda) + \sum_{i=1}^{2} \frac{r_i E(\eta_i)}{\lambda - \eta_i} = 0 \] (31)

Using the asymptotic expansion Eq. (20) one can trade the two unknowns \( E(\eta_1) \) and \( E(\eta_2) \) for \( \bar{\phi} \) via the relations

\[ \sum_{i=1}^{2} r_i E(\eta_i) = C \]

\[ \sum_{i=1}^{2} r_i E(\eta_i)\eta_i = r_1 + r_2 - 1 + C\bar{\phi} \] (32)

The solution for \( E(\lambda) \) is given by

\[ E(\lambda) = \frac{1}{2} \left( W'(\lambda) \pm \sqrt{W'(\lambda)^2 - 4 \sum_{i=1}^{2} \frac{r_i E(\eta_i)}{\lambda - \eta_i}} \right) \] (33)

Now recall that \( E(\lambda) = \int d\phi \rho(\phi)/(\lambda - \phi) \) has a branch cut precisely on the support of \( \rho \). Furthermore on such a branch cut of \( E(\lambda) \)

\[ E(\lambda + i\epsilon) - E(\lambda - i\epsilon) = 2\pi i \rho(\lambda) \] (34)

Thus the first step in finding the density of eigenvalues is to find the location of the branch points of \( E(\lambda) \) which can be done by finding the places where the term inside the square root vanishes. This is a quartic equation and thus, in general, there will be two branch cuts. So the general solution will be a two cut solution. Of course there will be cases when there are no real solutions to the quartic equation and there will be cases when we have a one cut solution.

The technical details of the various solutions for various values of the parameters will be discussed in more detailed papers on this subject. What I would like to discuss in this talk is a general physical picture of what kind of solutions one should expect. Before doing so, however, there is one further important technical point. When any of the \( r_i \) are positive the potential has an infinitely attractive logarithmic well. In such cases it is known from the single pole Penner Model that the branch cuts may have to circle the singularity. Mathematically this means that there is no real solution for \( \rho \) though physically it may be possible to interpret this as a “condensation of eigenvalues” at the singularity.
7. Physical Picture of Solutions

It is instructive and not difficult to get a good physical picture of the location of the eigenvalues of $\Phi$ for the solution of both the simple Matrix Model and the KM Model. In particular it is not difficult to decide whether solutions exist, whether they are unique, where the eigenvalues are relative to the singularities and whether there are one–cut or two–cut solutions.

We begin with the simple Matrix Model given by Equation (26) which can be written, using Eq. (6) as

$$\int \prod \phi d\phi \Delta^2(\phi) \exp \left[ -N \sum \Phi_i \right]$$ (35)

In large $N$, we must minimize the effective Action

$$- \sum_{(i \neq j)} \log (\phi_i - \phi_j)^2 + N \sum \Phi_i$$ (36)

Notice that the minimization of this Action corresponds to an analogue classical mechanical problem in which a large number $N$ of particles are located at locations $\phi_1...\phi_N$ along a line. They are each subjected to a central potential $W(\phi)$ and to a two–body repulsive potential $\log(\phi_i - \phi_j)^2$. Thus, for example, if the potential has a single minimum at some point $\phi_0$, one expects the eigenvalues to collect near this minimum with some distribution which is controlled by the logarithmic repulsion. If the potential has several minima then, presumably, there are a whole class of classical solutions in which a varying proportion of particles are located in each of the wells. In the Penner–like model in which there are at most two such minima, these various cases can be distinguished by the value of $\bar{\phi}$.

Let us now apply these ideas to the potential (29). The simplest case occurs when both $r_1$ and $r_2$ are negative. Recall that Eq. (29) gives the derivative of the potential so that in this case the potential has two repulsive singularities at $\eta_1$ and $\eta_2$ and a linear term with either a positive or a negative coefficient. In either case the potential has precisely two minima. If, without loss of generality, we consider the case in which $C>0$ then one minimum is between $\eta_1$ and $\eta_2$ and the other is at a point greater than $\eta_2$ (assuming $\eta_2 > \eta_1$). The potential is unbounded below for $\phi < \eta_1$. The fact that the potential is unbounded does not affect the existence of solutions. In fact classically the analogue “particles” can exist in the two minima without being significantly affected by what occurs for $\phi < \eta_1$. Thus we expect a whole class of solutions which can be parameterized by $\bar{\phi}$ which are, in general, two–cut solutions, in which some of the eigenvalues live in a compact region between $\eta_1$ and $\eta_2$ and the remainder live in another compact region near the minimum beyond $\eta_2$. 
The next case to consider is that in which both \( r_1 \) and \( r_2 \) are positive. In this case the potential has two **attractive** singularities at \( \eta_1 \) and at \( \eta_2 \). It has two maxima but no minima. In this case there are clearly no nonsingular solutions to the analogue problem though, as discussed previously, there may be some solutions in which the cuts of \( E(\lambda) \) circle the singularities and which can be interpreted as a condensation of eigenvalues. These turn out to be cases in which the quartic form under the square root in Eq. (33) has real solutions but, in order to get both the asymptotic (large \( \lambda \)) behaviour of \( E(\lambda) \) and its behaviour near the singularities correctly the cuts must circle one or more singularities.

The final case, in which the \( r_i \) have opposite sign, is the most interesting case for the KM Model. If, without loss of generality, we assume that \( \eta_1 < \eta_2 \) and \( C > 0 \) then we identify two cases. If \( r_1 < 0 \) and \( r_2 > 0 \) then the potential is repulsive at \( r_1 \) and attractive at \( r_2 \). In this case we expect no real solutions, though “condensation of eigenvalues” may occur at \( r_2 \). In the case \( r_1 > 0 \) and \( r_2 < 0 \) the potential is attractive at \( r_1 \) and repulsive at \( r_2 \). Thus there is a minimum for \( \phi > \eta_2 \) and we should expect at least one real one–cut solution to this problem for some specific value of \( \phi \).

To see that this last case is the one relevant to the KM Model recall Equation (30) which states that for the KM case \( r_1 + r_2 = 1 \). Now in order to get a real solution we need at least one of the \( r_i \) to be negative. In this case we see that the other one must be positive so that the sum equals 1. We see from the preceding paragraph (and from the fact that \( C = \eta_2 - \eta_1 > 0 \) that if \( r_2 > 0 \) we expect no real solutions whereas if \( r_2 < 0 \) we expect a single cut solution, where the cut is near the minimum at \( \phi > \eta_2 \).

Let us now compare this with Eq. (23). We should identify \( \eta_2 \) as \( \eta_1 \) as \( B \) and \( r_2 \) as \( q \). Thus we expect a real single cut solution only if \( q < 0 \). The form of the KM potential can then be deduced from Eq. (25). Note that the qualitative form of \( V \) is not uniquely determined. Even though \( B - \xi < 0 \), the sign of \( B + (2d - 2)\xi \) depends on the precise values of \( B \) and \( \xi \). What is however uniquely determined is the sign of the singularities (provided \( d \geq 1 \)). If \( q < 0 \) then \((2d - 1)(q - 1) < 0\) and both singularities in \( V \) as opposed to \( W \) are repulsive. The integral over the Gauge Fields force a renormalization of the analogue potential from \( V \) to the effective Matrix Model form \( W \).

In the previous paragraphs we have shown how to estimate the approximate location of the eigenvalues for an ordinary Matrix Model or for a KM Model given the relationship between a KM Model and an ordinary Matrix Model. It is, however possible to get some information from the KM Potential directly. To see this suppose we have a KM Model with an arbitrary potential \( V \). Our job is to find the extrema of the Effective Potential given by Eq. (8). This can be rewritten in the following
form”

\[ S_{\text{eff}} = N \sum_i \left( V(\phi_i) - d \phi^2 \right) + (d - 1) \sum_{i \neq j} \log (\phi_i - \phi_j)^2 - d \log \left( \det_{i,j} e^{N(\phi_i - \phi_j)^2/2} \right) \]

Using the fact that the Integral (4) is finite and nonzero when \( \phi = \psi \) and when any two eigenvalues are equal, it is easy to see that minimization of Eq. (37) corresponds to the following analogue mechanical problem. We imagine again \( N \) particles living on a line with a central potential which is \( V - d \phi^2 \) (not simply \( V \)), with an interparticle potential which is no longer 2–body but which has the following property: Whenever any two eigenvalues approach each other there is a repulsion (the Action diverges) and whenever any two eigenvalues are separated from each other by a large distance, there is an attraction which tries to bring them closer together. This is quite different from the physical picture which emerges from the simple Matrix Model.†

8. Summary

At this point there is no evidence that the Kazakov–Migdal Model provides a soluble, large \( N \) realization of QCD and such a hypothesis has many serious difficulties. We have, however, learned much from the study of the KM Model both about Matrix Models and about this new, interesting class of Gauge Theories.

† Unfortunately the compatibility between this picture and the one which emerges from writing an effective one Matrix Model and using the Physical Picture in that case, is not at all clear. In fact, as mentioned briefly earlier in this article, there is some possibility that the solution which corresponds to the one Matrix Model is not the KM Model with the simple potential \( \text{Tr}V \) given as the integral of \( V' \) but one with a more complicated matrix structure. These issues will be discussed in a more detailed publication.
REFERENCES

1. DMS paper M.I. Dobroliubov, Yu. Makeenko, G.W. Semenoff, Mod. Phys. Lett. A8 P.2387 1993; Yu Makeenko, Mod. Phys. Lett A8 (209) 1993; Yu Makeenko, Mod. Phys. Lett. B422 (237) 1994

2. V.A. Kazakov and A.A. Migdal, Nucl. Phys. B397 (214) 1993

3. V.A. Kazakov and A.A. Migdal, Ref V.A. Kazakov and A.A. Migdal, Nucl. Phys. B397 (214) 1993; A.A. Migdal, Mod. Phys. Lett. A8 (359) 1993; A.A. Migdal, Mod Phys. Lett. A8 (153) 1993

4. Harish-Chandra, Amer. J. Math. 79 (87) 1957; C. Itzykson and J.B. Zuber, J. Math. Phys. 21 (411) 1980; M.L. Mehta, Comm. Math. Phys. 79 (327) 1981

5. I. Kogan, G. Semenoff, N. Weiss, Phys. Rev. Lett 69 (3435) 1992

6. S. Elitzur, Phys. Rev. D12, 3978.

7. A.A. Migdal, Mod. Phys. Lett. A8 (359) 1993

8. Phys.Lett. B302 (283) 1993.

9. D.Gross, Phys.Lett. B293 (181) 1992

10. Yu. Makeenko, “Critical Scaling and Continuum Limits in the Dgl Kazakov-Migdal Model” hep-th 9408029; See also Ref. 11

11. L.D. Paniak “Singular Matrix Models and the Kazakov–Migdal Model.” M.Sc. Thesis, University of British Columbia, (in preparation). L.D. Paniak, N. Weiss, “Kazakov–Migdal Model with Logarithmic Potential and the Double Penner Matrix Model” hep-th 9501034

12. S. Chauduri, H. Dykstra, J. Lykken, Mod. Phys. Lett A V6 1665 1991