Abstract. Let \((X^{m+1}, g)\) be a globally hyperbolic spacetime with Cauchy surface diffeomorphic to an open subset of \(\mathbb{R}^m\). The Legendrian Low conjecture formulated by Natário and Tod says that two events \(x, y \in X\) are causally related if and only if the Legendrian link of spheres \(S_x, S_y\) whose points are light geodesics passing through \(x\) and \(y\) is non-trivial in the contact manifold of all light geodesics in \(X\). The Low conjecture says that for \(m = 2\) the events \(x, y\) are causally related if and only if \(S_x, S_y\) is non-trivial as a topological link. We prove the Low and the Legendrian Low conjectures. We also show that similar statements hold for any globally hyperbolic \((X^{m+1}, g)\) such that a cover of its Cauchy surface is diffeomorphic to an open domain in \(\mathbb{R}^m\).

1. Introduction. The space \(\mathfrak{N}\) of non-parameterised future pointing null geodesics in a globally hyperbolic spacetime \((X^{m+1}, g), m \geq 2\), has a natural structure of a contact \((2m - 1)\)-manifold obtained by identifying \(\mathfrak{N}\) with the spherical cotangent bundle \(ST^*M\) of a smooth spacelike Cauchy surface \(M^m \subset X\). Null geodesics passing through a point \(x \in X\) form a Legendrian \((m - 1)\)-sphere \(S_x \subset \mathfrak{N}\) called the sky of \(x\). (Details and definitions may be found in §§3-4 below.)

All skies in \(\mathfrak{N}\) are Legendrian isotopic. The situation is more interesting for links formed by pairs of disjoint skies. It was observed by Low \[23\] that the isotopy class of the link \(S_x \sqcup S_y\) may depend on whether the points \(x \) and \(y\) are causally related, that is, connected by a future pointing non-spacelike curve in \(X\).

It is not hard to show that all links \(S_x \sqcup S_y\) formed by skies of causally unrelated points belong to the same Legendrian isotopy class in \(\mathfrak{N}\) represented by a pair of fibres of \(ST^*M\). It is therefore natural to call \(S_x \sqcup S_y\) topologically unlinked (respectively, Legendrian unlinked) if they are disjoint and the link \(S_x \sqcup S_y\) is smoothly (respectively, Legendrian) isotopic to a link in that ‘trivial’ isotopy class. It is also natural to ask whether the skies of causally related points are in some sense linked. This question was raised in different forms by Low \[23, 24, 25, 26, 27\] and Natário and Tod \[29\]. It appeared on Arnold’s problem lists as a problem communicated by Penrose \[3, Problem 8\], \[4, Problem 1998-21\]. The following result was conjectured in \[29, Conjecture 6.4\] for the case when the Cauchy surface is diffeomorphic to an open subset of \(\mathbb{R}^3\).

Theorem A (Legendrian Low Conjecture). Assume that a smooth spacelike Cauchy surface of a globally hyperbolic spacetime \((X, g)\) has a cover diffeomorphic to an open subset of \(\mathbb{R}^m, m \geq 2\). Then the skies of causally related points in \(X\) are Legendrian linked.

Explicit examples (see \[29, §6\]) show that causally related points in a globally hyperbolic spacetime with Cauchy surface diffeomorphic to \(\mathbb{R}^3\) can have topologically unlinked skies. That is, Legendrian linking is a strictly weaker condition for \(m \geq 3\). On the other hand, combining Theorem A with a recent result of Ding and Geiges \[17\] on the classification of Legendrian links in \(ST^*\mathbb{R}^2\), we obtain the following result for \((2 + 1)\)-dimensional space-times. This result was conjectured by Low \[23\] for the case when the Cauchy surface is diffeomorphic to an open subset of \(\mathbb{R}^2\).

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Theorem B (Low’s Conjecture). Assume that the universal cover of a smooth spacelike Cauchy surface of a globally hyperbolic (2+1)-dimensional spacetime \((X, g)\) is diffeomorphic to \(\mathbb{R}^2\). Then the skies of causally related points in \(X\) are topologically linked.

The proof of Theorem A is based on the methods of the theory of generating functions developed in the context of contact topology by Traynor [33] and Bhupal [12] following the seminal work of Viterbo [34]. (After our paper was submitted for publication, we discovered that results very similar to those contained in §§5–6 below were obtained earlier by Colin, Ferrand, and Pushkar’ [16].) One more application of this approach shows that Legendrian linking, unlike topological linking, can distinguish between past and future. For the standard Minkowski spacetime, this result is an interpretation of the main result of [33] in terms of skies, see [29, Theorem 6.2].

Theorem C. Assume that a smooth spacelike Cauchy surface of a globally hyperbolic spacetime \((X, g)\) has a cover diffeomorphic to an open subset of \(\mathbb{R}^m\), \(m \geq 2\). Let \(x, y \in X\) be causally related points with disjoint skies. Then the links \(\mathcal{S}_x \sqcup \mathcal{S}_y\) and \(\mathcal{S}_y \sqcup \mathcal{S}_x\) are not Legendrian isotopic.

The universal cover of a 2-dimensional manifold \(M \neq S^2, \mathbb{R}P^2\) with \(\partial M = \emptyset\) is diffeomorphic to \(\mathbb{R}^2\). In particular, we see that Theorems A, B, C hold for \((2+1)\)-dimensional globally hyperbolic spacetimes with Cauchy surfaces other than \(S^2\) or \(\mathbb{R}P^2\).

According to Agol’s talk [1], Thurston’s geometrisation conjecture proved recently by Perelman [31], [32] implies that the universal cover of a closed 3-manifold \(M\) is diffeomorphic either to \(S^3\) or to an open subset of \(\mathbb{R}^3\). If \(M\) is universally covered by \(S^3\), then the geometrisation conjecture tells us that \(M\) is diffeomorphic to a quotient of \(S^3\) by a finite group of isometries of the standard round metric. If \(M\) is the interior of a compact 3-manifold \(M\) with boundary, then the double of \(M\) is a closed 3-manifold and hence \(M\) is covered by an open subset of \(\mathbb{R}^3\). Since any open 3-manifold can be exhausted by compact 3-manifolds with boundary, it follows (see Remarks 5.2 and 6.3) that in the physically interesting case of \((3+1)\)-dimensional globally hyperbolic spacetimes, Theorems A and C hold assuming that the Cauchy surface is not diffeomorphic to a metric quotient of the standard sphere \(S^3\).

If \(M\) is a metric quotient of the standard round sphere \(S^m\) by a finite group of isometries, then it is easy to construct a globally hyperbolic spacetime with Cauchy surface \(M\) for which Theorems A, B, C are false. One can take the product Lorentz manifold \((M \times \mathbb{R}, g \oplus -dt^2)\), where \(g\) is the quotient Riemann metric on \(M\), see [15, Example 3].

It is worth pointing out that all known examples of this sort are refocussing, see [28] and [15, Definition 22]. (This notion seems to be related to Riemann \(Y^x\)-manifolds studied by Béard-Bergery [6] and Besse [11], see [15, Remark 7].) On the other hand, it was proved by Rudyak and the first author [15, Corollary 1] that if a globally hyperbolic spacetime is non-refocussing, then skies of causally related points cannot be unlinked by a Legendrian isotopy consisting of skies of points. So it is conceivable that our results remain valid for any globally hyperbolic spacetime that is not diffeomorphic to a refocussing spacetime.

Contents of the paper. The key notion of non-negative Legendrian isotopy is introduced in §2. Necessary facts and definitions from Lorentz geometry are recalled in §3. Contact geometry of the space of null geodesics and its relation with causality are discussed in §4. Generating functions are used to study Legendrian isotopies in 1-jet bundles in §5. The hodograph transformation is applied in §6. The proofs of the results stated in the introduction occupy §§7–9. The last §10 contains a tentative application to physics.

Conventions. All manifolds, maps etc. are assumed to be smooth unless the opposite is explicitly stated, and the word smooth means \(C^\infty\). The connected components of a
disconnected manifold (such as a link) are assumed to be ordered; maps between disconnected manifolds are assumed to preserve the order of components. Contactomorphisms of co-oriented contact structures are assumed to be co-orientation preserving.

2. Non-negative Legendrian isotopies. Let \( Y \) be a contact manifold with a co-oriented contact structure defined by a contact form \( \alpha \). A submanifold \( \Lambda \subset Y \) is called Legendrian if it is tangent to the contact distribution, i.e., if \( \alpha|_\Lambda \equiv 0 \). A \textit{Legendrian isotopy} in \( Y \) is a smooth family \( \{ \Lambda_t \}_{t \in [0,1]} \) of Legendrian submanifolds. Two Legendrian submanifolds are called Legendrian isotopic if they can be connected by a Legendrian isotopy.

A basic fact about Legendrian isotopies is the \textit{Legendrian isotopy extension theorem} (see, e.g., [20, Theorem 2.6.2]). It asserts that for any Legendrian isotopy \( \{ \Lambda_t \}_{t \in [0,1]} \) of compact submanifolds, there exists a smooth family of compactly supported contactomorphisms \( \Psi_{t \in [0,1]} : Y \to Y \) such that \( \Psi_0 = \text{id}_Y \) and \( \Psi_t(\Lambda_0) = \Lambda_t \) for all \( t \in [0,1] \). In particular, isotopic compact Legendrian submanifolds are isotopic by an ambient contact isotopy.

\textbf{Definition 2.1.} A Legendrian isotopy \( \{ \Lambda_t \}_{t \in [0,1]} \) in a contact manifold \((Y, \ker \alpha)\) is called \textit{non-negative} if it has a parameterisation \( F : \Lambda_0 \times [0,1] \to Y \) such that \( (F^*\alpha) \left( \frac{\partial}{\partial t} \right) \geq 0 \).

Clearly, this definition does not depend on the choice of the parameterisation \( F \) of the Legendrian isotopy and on the choice of the contact form defining the co-oriented contact structure. It is also obvious that if \( \Psi : Y \to Y' \) is a contactomorphism, then the image of a non-negative Legendrian isotopy in \( Y \) is a non-negative Legendrian isotopy in \( Y' \).

\textbf{Example 2.2} (Non-negative isotopy in \( ST^*M \)). Let \( \pi : ST^*M \to M \) be the spherical cotangent bundle of an \( m \)-dimensional manifold \( M \). A point \( p \in ST^*M \) may be regarded as a linear form \( \tilde{p} \) on \( T_{\pi(p)}M \) defined up to a multiplication by a positive scalar. Thus, \( p \in ST^*M \) is determined by the co-oriented hyperplane \( \ell_p = \ker \tilde{p} \subset T_{\pi(p)}M \). (The co-orientation is given by the half-space in \( T_{\pi(p)}M \) on which \( \tilde{p} \) is positive.) The standard contact structure on \( ST^*M \) is the co-oriented hyperplane distribution
\[ \chi = \{(d\pi)^{-1}(\ell_p) \subset T_p(ST^*M) \mid p \in ST^*M \}. \]

Let \( \{ \Lambda_t \}_{t \in [0,1]} \) be a Legendrian isotopy parameterised by \( F : \Lambda_0 \times [0,1] \to ST^*M \). It follows from the definitions that this isotopy is non-negative if and only if the value of a linear form corresponding to \( F(s, \tau) \in ST^*_{\pi(F(s,\tau))}M \) on the vector \( d\pi \circ dF|_{(s,\tau)}(\frac{\partial}{\partial t}) \) is non-negative for all \( (s, \tau) \in \Lambda_0 \times [0,1] \).

Suppose now that the projection \( \pi|_{\Lambda_t} : \Lambda_t \to M \) is an embedding for all \( t \in [0,1] \). For any point \( p \in \Lambda_t \), the tangent hyperplane \( T_{\pi(p)}\pi(\Lambda_t) \) coincides with \( \ker \tilde{p} \) and therefore has a canonical co-orientation. The co-oriented hypersurface \( \pi(\Lambda_t) \subset M \) is called the wave front of the Legendrian submanifold \( \Lambda_t \subset ST^*M \). In this situation, the isotopy \( \{ \Lambda_t \}_{t \in [0,1]} \) is non-negative if and only if the wave fronts \( \pi(\Lambda_t) \) move in the direction of their co-orientation. \( \Box \)

\textbf{Remark 2.3.} A closely related notion of non-negative contact isotopy was studied by Eliashberg and Polterovich [19], Eliashberg, Kim, and Polterovich [18], and Bhupal [12].

3. Lorentz geometry: definitions and terminology. Let \((X, g)\) be a Lorentz manifold of dimension \( m + 1 \) and signature \((+\ldots,+,-)\). A non-zero vector \( v \in T_pX \) is called \textit{timelike}, \textit{non-spacelike}, \textit{null} (\textit{lightlike}), or \textit{spacelike} if \( g(v, v) \) is respectively negative, non-positive, zero, or positive. A piecewise smooth curve is timelike if all of its velocity vectors are timelike. Non-spacelike and null (lightlike) curves are defined similarly. Since \((X, g)\) has a unique Levi-Civita connection, see for example [5, p. 22], we can talk about spacelike, timelike, and null (light) geodesics. A submanifold \( M \subset X \) is \textit{spacelike} if \( g \) restricted to \( TM \) is a Riemann metric.
All non-spacelike vectors in $T_pX$ form a cone consisting of two hemicones, and a continuous with respect to $p \in X$ choice of one of the two hemicones is called a time orientation of $(X, g)$. The vectors from the chosen hemicones are called future pointing. A time oriented connected Lorentz manifold is called a spacetime and its points are called events.

For an event $x$ in a spacetime $(X, g)$, its causal future $J^+(x) \subset X$ (respectively, chronological future $I^+(x)$) is the set of all $y \in X$ that can be reached by a future pointing non-spacelike (respectively, timelike) curve from $x$. The causal past $J^-(x)$ and the chronological past $I^-(x)$ of the event $x \in X$ are defined similarly.

Two events $x, y$ are said to be causally related if $x \in J^+(y)$ or $y \in J^+(x)$; and they are said to be chronologically related if and only if $x \in I^+(y)$ or $y \in I^+(x)$.

An open subset $U \subset X$ is itself a Lorentz manifold and for $x \in U$ we denote by $J^\pm(x, U)$, $I^\pm(x, U)$ the causal and the chronological past and future of $x$ with respect to $U$.

**Definition 3.1.** A spacetime is causal if it does not contain closed future pointing curves. A spacetime $X$ is said to be globally hyperbolic if it is causal and $J^+(x) \cap J^-(y)$ is compact for all $x, y \in X$.

This definition of global hyperbolicity is equivalent to the more classical one used in [5] and [22] by a recent result of Bernal and Sanchez [10] Theorem 3.2.

A Cauchy surface in $(X, g)$ is a subset such that every inextensible future pointing non-spacelike curve $\gamma(t)$ intersects it at exactly one value of $t$. It is a classical result that $(X, g)$ is globally hyperbolic if and only if it contains a Cauchy surface, see [22, pp.211–212]. Geroch [21] proved that every globally hyperbolic $(X, g)$ is homeomorphic to the product of $\mathbb{R}$ and a Cauchy surface. Bernal and Sanchez [7, Theorem 1], [8, Theorem 1.1], [9, Theorem 1.2] proved that every globally hyperbolic $(X^{m+1}, g)$ has a smooth spacelike Cauchy surface $M^m$, any two smooth spacelike Cauchy surfaces of $(X^{m+1}, g)$ are diffeomorphic, and that moreover for every smooth spacelike Cauchy surface $M$ there is a diffeomorphism $h_M : M \times \mathbb{R} \to X$ such that

a) $h_M(M \times t)$ is a smooth spacelike Cauchy surface for all $t$;

b) $h_M(x \times \mathbb{R})$ is a future pointing timelike curve for all $x \in M$;

c) $h_M(M \times 0) = M$ with $h_M|_{M \times 0} : M \to M$ being the identity map.

This deep result has the following useful corollary (cf. [15] Proof of Theorem 8)).

**Lemma 3.2.** Let $x_1, x_2$ be causally unrelated points in $X$. Then they can be smoothly moved into $M$ so that they remain causally unrelated in the process.

**Proof.** Let $(\bar{x}_i, t_i) = h_M^{-1}(x_i)$ and assume that the points are ordered so that $t_1 \leq t_2$. Note that moving $x_1$ into the future along the segment $h_M(\bar{x}_1, [t_1, t_2])$ will not create causal relations between $x_1$ and $x_2$. Indeed, if $h_M(\bar{x}_1, t)$ were causally related to $x_2$ for some $t \in [t_1, t_2]$, then $x_2$ would have to lie in $J^+(h_M(\bar{x}_1, t))$. The union of a future pointing curve connecting $h_M(\bar{x}_1, t)$ to $x_2$ with the future pointing timelike curve $h_M(\bar{x}_1, [t_1, t])$ would be a future pointing non-spacelike curve connecting $x_1$ to $x_2$.

Thus, we can move $x_1$ into the future till $t = t_2$ so that both points lie on the Cauchy surface $h_M(M \times t_2)$. Since points lying on the same Cauchy surface are causally unrelated, it remains to use the obvious isotopy of Cauchy surfaces $M_t := h_M(M \times t t_2)$ connecting $h_M(M \times t_2)$ and $M = h_M(M \times 0)$. \[\Box\]

4. **Contact geometry of the space of null geodesics.** Let $\mathfrak{N}$ be the space of all (inextensible) future pointing null geodesics in a globally hyperbolic spacetime $(X, g)$ considered up to affine orientation preserving reparameterisation.

**Definition 4.1.** The sky $\mathfrak{S}_x$ is the set of all null geodesics in $\mathfrak{N}$ passing through $x \in X$. 

Let \( M \) be a spacelike Cauchy surface in \( X \). A null geodesic \( \gamma = \gamma(t) \) intersects \( M \) at a time \( t \). Since \( M \) is spacelike and \( \gamma \) is a null geodesic, the linear form on \( T_{\gamma(t)}M \) given by \( \mathbf{v} \mapsto g(\gamma'(t), \mathbf{v}) \) is non-zero and therefore defines a point in the spherical cotangent bundle \( ST^*M \) of \( M \). Thus, we have an identification

\[
\rho_M : \mathfrak{N} \xrightarrow{\sim} ST^*M.
\]

Note that if \( \mathfrak{S}_x \) is the sky of a point \( x \in M \), then \( \rho_M(\mathfrak{S}_x) = ST^*_x M \) is the fibre of \( ST^*M \) over \( x \).

Consider the contact structure on \( \mathfrak{N} \) induced by \( \rho_M \) from the standard contact structure on \( ST^*M \). Low \cite{24} showed that if \( M' \) is another spacelike Cauchy surface, then the map \( \rho_M \circ \rho_M^{-1} : ST^*M' \to ST^*M \) is a contactomorphism (see also \cite{29} pp. 252–253). Thus, this contact structure on \( \mathfrak{N} \) does not depend on the choice of the Cauchy surface \( M \).

It follows, in particular, that any sky \( \mathfrak{S}_x \) is a Legendrian sphere in \( \mathfrak{N} \) because it corresponds to the fibre \( ST^*_x M' \) for a Cauchy surface \( M' \) passing through \( x \). Note also that if two points \( x_1, x_2 \in X \) are connected by a curve \( x = x(t) \), then \( \mathfrak{S}_{x(t)} \) is a Legendrian isotopy connecting the skies \( \mathfrak{S}_{x_1} \) and \( \mathfrak{S}_{x_2} \), and hence all skies are Legendrian isotopic.

**Proposition 4.2.** Let \( x, y \in X \) be causally related points not lying on the same null geodesic and such that \( y \in J^+(x) \). Then \( \mathfrak{S}_y \) is connected to \( \mathfrak{S}_x \) by a non-negative Legendrian isotopy.

**Warning.** Note that the non-negative isotopy goes to the past. This is caused by the use of future pointing geodesics in the definition of \( \rho_M \).

**Proof.** Since \( x, y \) are not on the same null geodesic, we have \( y \in I^+(x) \) by \cite{5} Corollary 4.14. (It was Corollary 3.14 in the first edition of \cite{5}.) Choose a past directed smooth timelike curve \( \gamma : [0, 1] \to X \) from \( y \) to \( x \). We claim that the Legendrian isotopy \( \{ \mathfrak{S}_{\gamma(t)} \}_{t \in [0, 1]} \) connecting \( \mathfrak{S}_y \) to \( \mathfrak{S}_x \) is non-negative.

Fix \( \tau \in [0, 1] \) and put \( q = \gamma(\tau) \). Choose a spacelike Cauchy surface \( M' \) of the type \( h_M(M \times s) \) passing through \( q \). We will show that there exists \( \varepsilon > 0 \) such that the Legendrian isotopy \( \{ \rho_{M'}(\mathfrak{S}_{\gamma(t)}) \}_{t \in [\tau, \tau + \varepsilon]} \) in \( ST^*M' \) is non-negative for all \( t \in (\tau, \tau + \varepsilon) \). The claim will follow because \( \rho_{M'} : \mathfrak{N} \to ST^*M' \) is a contactomorphism and \( \tau \) can be chosen arbitrarily.

![Figure 1. Foliation of a neighbourhood of q in M' by embedded spheres](attachment:image.png)

**Figure 1.** Foliation of a neighbourhood of \( q \) in \( M' \) by embedded spheres

Using \cite{30} Chapter 5, Proposition 7], choose a convex neighbourhood \( U \) of \( q \) in \( X \). For every \( p \in U \), the intersection of the exponent of the future null cone in \( T_pX \) with \( M' \) is either an embedded hypersurface or a single point in \( M' \). The intersection of \( M' \) with the exponent
of the null cone at \( q \) is just one point \( q \). Moving \( p \) from \( q \) slightly into the past along \( \gamma \), we see that the intersection of \( M' \) with the exponent of the future null cone at \( p \) is an embedded sphere \( \Sigma_p \subset M' \). Two such spheres corresponding to different points \( q_1, q_2 \in \epsilon \cap U \) do not intersect. Indeed, assume they do intersect at \( z \in M' \). Without loss of generality assume that \( q_2 \in I^+(q_1) \) and hence that \( q_2 \in I^+(q_1, U) \). Then \( z \) is on an arc of a null geodesic passing through \( q_1 \). The arc is contained in \( U \) and thus \( z \) is in \( J^+(q_1, U) \setminus I^+(q_1, U) \) by \[30\], Chapter 14, Lemma 2]. Similarly, \( z \) is in \( J^+(q_2, U) \setminus I^+(q_2, U) \) and, since \( q_2 \in I^+(q_1, U) \), we have \( z \in I^+(q_1, U) \) by \[30\], Chapter 14, Corollary 1]. Contradiction. Thus, a small deleted neighbourhood of \( q \) in \( M' \) is foliated by the spheres \( \Sigma_{\gamma(t)} \), \( t \in (\tau, \tau + \varepsilon) \), that are expanding as the point \( \gamma(t) \) moves into the past along \( \gamma \), see Figure \[1\].

It follows from the definition of \( \rho_{M'} \) that for any \( t \in (\tau, \tau + \varepsilon) \) the sphere \( \Sigma_{\gamma(t)} \subset M' \) with the outward co-orientation coincides with the wave front of the Legendrian submanifold \( \rho_{M'}(\Sigma_{\gamma(t)}) \subset ST^*M' \). Since the family of spheres is expanding, the wave fronts move in the direction of their co-orientation and hence the Legendrian isotopy \( \{ \rho_{M'}(\Sigma_{\gamma(t)}) \}_{t \in (\tau, \tau + \varepsilon)} \) is non-negative, see Example \[2,2\].

The notion of (un)linking in \( \mathcal{N} \) is based on the following observation going back to Low \[23\] (see also \[29\] and \[15\]).

**Lemma 4.3.** The Legendrian isotopy class of the link \( \mathfrak{S}_x \sqcup \mathfrak{S}_y \subset \mathcal{N} \) formed by the skies of two causally unrelated points \( x, y \in X \) does not depend on the points \( x \) and \( y \).

**Proof.** Pick a spacelike Cauchy surface \( M \subset X \). By Lemma \[3,2\] we can move \( x \) and \( y \) into \( M \) keeping them causally unrelated. Since the skies of causally unrelated points are disjoint, we obtain a Legendrian isotopy connecting \( \mathfrak{S}_x \sqcup \mathfrak{S}_y \) to a link of the form \( \rho^{-1}_M(\mathcal{ST}_{x'}^*M \sqcup \mathcal{ST}_{y'}^*M) \) for \( x' \neq y' \in M \). Once \( M \) is fixed, any two such links are obviously Legendrian isotopic.

**Definition 4.4.** Let \( \mathcal{U} \) be the Legendrian isotopy class of links in \( \mathcal{N} \) containing \( \mathfrak{S}_x \sqcup \mathfrak{S}_y \) for all causally unrelated points \( x, y \in X \). Two skies \( \mathfrak{S}_x \) and \( \mathfrak{S}_y \) are called Legendrian unlinked if they are disjoint and \( \mathfrak{S}_x \sqcup \mathfrak{S}_y \) belongs to \( \mathcal{U} \). Two skies \( \mathfrak{S}_x \) and \( \mathfrak{S}_y \) are called topologically unlinked if they are disjoint and \( \mathfrak{S}_x \sqcup \mathfrak{S}_y \) is smoothly isotopic to a link in \( \mathcal{U} \). Skies are called topologically (respectively, Legendrian) linked if they are not topologically (respectively, Legendrian) unlinked.

Lemma \[4,3\] shows that the Legendrian isotopy class \( \mathcal{U} \) is well-defined. A natural way to represent it in \( \mathcal{N} \) is to identify \( \mathcal{N} \) with \( \mathcal{ST}^*M \) for a spacelike Cauchy surface \( M \subset X \) and consider the link in \( \mathcal{ST}^*M \) formed by any two different fibres \( \mathcal{ST}_x^*M \) and \( \mathcal{ST}_y^*M \). In other words, we take the skies of two different points \( x \neq y \) lying on the same Cauchy surface.

5. **Generating functions and Legendrian isotopies.** Let \( \mathcal{J}^1(L) \) denote the 1-jet bundle of a compact connected manifold \( L \) equipped with the standard contact form \( du - pdq \), where \( u \) is the fibre coordinate in \( \mathcal{J}^0(L) \) and \( pdq \) denotes the Liouville form on \( T^*L \). Let \( \Lambda \subset \mathcal{J}^1(L) \) be a Legendrian submanifold. A function \( S = S(q, \xi) : L \times \mathbb{R}^N \to \mathbb{R} \) is a generating function for \( \Lambda \) if zero is a regular value of the partial differential \( d_\xi S \) and the map

\[
\{ d_\xi S(q, \xi) = 0 \} \ni (q, \xi) \mapsto (q, d_\eta S(q, \xi), S(q, \xi)) \in \mathcal{J}^1(L) \quad (5.1)
\]

is a diffeomorphism onto \( \Lambda \). A generating function is said to be quadratic at infinity if \( S(q, \xi) = Q(q, \xi) + \sigma(q, \xi) \), where \( \sigma \) has compact support and \( Q(q, \cdot) \) is a non-degenerate quadratic form in the variable \( \xi \).
Given a quadratic at infinity function $S : L \times \mathbb{R}^N \to \mathbb{R}$, there is a topological procedure for selecting one of its critical values $c_-(S)$ (see [34, §2]). Consider the sublevel sets

$$S^c := \{(q, \xi) \in L \times \mathbb{R}^N \mid S(q, \xi) \leq c\}$$

and denote by $S^{-\infty}$ the set $S^c$ for a sufficiently negative $c \ll 0$. Let $\mathcal{N}(S)$ denote the negative index of the quadratic form $Q(q, \cdot)$. Pick a point $q_0 \in L$ and a negative linear subspace $V \subset \{q_0\} \times \mathbb{R}^N$ for $Q(q_0, \cdot)$ of maximal possible dimension $\mathcal{N}(S)$. The relative homology class $[V] \in H_\mathcal{N}(L \times \mathbb{R}^N, S^{-\infty})$ does not depend on the choices made. Define

$$c_-(S) := \inf\{c \in \mathbb{R} \mid [V] \in \iota_* H_\mathcal{N}(S^c, S^{-\infty})\},$$

where $\iota_* : H_\mathcal{N}(S^c, S^{-\infty}) \to H_\mathcal{N}(L \times \mathbb{R}^N, S^{-\infty})$ is the homomorphism of relative homology groups induced by the inclusion $\iota : S^c \to L \times \mathbb{R}^N$.

**Example 5.1.** Let $\Lambda^f := \{(q, df(q), f(q)) \mid q \in L\} \subset J^1(L)$ be the graph of the 1-jet of a smooth function $f : L \to \mathbb{R}$. It follows from the definitions and Morse theory that

$$c_-(S) = \min_{\{d\_\, S = 0\}} S = \min_L f$$

for any quadratic at infinity generating function $S : L \times \mathbb{R}^N \to \mathbb{R}$ of the Legendrian submanifold $\Lambda^f \subset J^1(L)$. In particular, $c_-(S) = 0$ for any quadratic at infinity generating function of the zero section $\Lambda^0 \subset J^1(L)$. \[\square\]

Suppose now that $\{\Lambda_t\}_{t \in [0,1]}$ is a Legendrian isotopy such that $\Lambda_0$ is the zero section of $J^1(L)$. By Chekanov’s theorem [14] (see also [13] and [33]) there exists a smooth family of quadratic at infinity generating functions $S_t : L \times \mathbb{R}^N \to \mathbb{R}$ for $\Lambda_t$.

**Lemma 5.2.** If the Legendrian isotopy $\{\Lambda_t\}_{t \in [0,1]}$ is non-negative, then $t \mapsto c_-(S_t)$ is a non-decreasing function on $[0,1]$.

**Proof.** It follows from the definition of a non-negative isotopy and formula (5.1) that $\frac{\partial c_-(S_t)}{\partial t}(q, \xi) \geq 0$ when $d\xi S_t(q, \xi) = 0$ and, in particular, when $d(q, \xi) S_t(q, \xi) = 0$. Hence, $c_-(S_t)$ is a non-decreasing (continuous) function of $t$ by [34, Lemma 4.7]. \[\square\]

Combining Lemma 5.2 with Example 5.1 we obtain the principal result of this section.

**Proposition 5.3.** Assume that there exists a non-negative Legendrian isotopy connecting the zero section of $J^1(L)$ with the graph $\Lambda^f$ of the 1-jet of a smooth function $f$. Then $f \geq 0$ everywhere on $L$. \[\square\]

Here are a couple of immediate corollaries of Proposition 5.3.

**Corollary 5.4.** If there exists a non-negative Legendrian isotopy connecting $\Lambda^h$ to $\Lambda^f$, then $h \leq f$ everywhere on $L$.

**Proof.** The ‘downshift’ map $T^h(q, p, u) := (q, p - \frac{\partial h}{\partial q}, u - h)$ is a contactomorphism of $J^1(L)$. Clearly, $T^h(\Lambda^h) = \Lambda^0$ and $T^h(\Lambda^f) = \Lambda^{f-h}$. Thus, $T^h$ will take a non-negative Legendrian isotopy between $\Lambda^h$ and $\Lambda^f$ to a non-negative Legendrian isotopy between $\Lambda^0$ and $\Lambda^{f-h}$. Hence, $f - h \geq 0$ by Proposition 5.3. \[\square\]

**Corollary 5.5.** A non-negative Legendrian isotopy connecting the zero section with itself is constant.

**Proof.** For any non-constant Legendrian isotopy $\{\Lambda_t\}_{t \in [0,1]}$ of the zero section, there exists $t' > 0$ such that $\Lambda_{t'} = \Lambda^h$ for a function $h$ not identically equal to zero. (Note that $\Lambda_t$ is the graph of the 1-jet of a function for all sufficiently small $t \geq 0$.) Assume that the isotopy is non-negative. Proposition 5.3 shows that $h \geq 0$. On the other hand, applying Corollary 5.4 to the isotopy $\{\Lambda_t\}_{t \in [0,1]}$, we see that $h \leq 0$. So $h \equiv 0$, a contradiction. \[\square\]
6. Application of the hodograph transformation. Let \( \langle \cdot, \cdot \rangle \) denote the standard scalar product on \( \mathbb{R}^m \) and let \( S^{m-1} \subset \mathbb{R}^m \) be the unit sphere. The map
\[
\mathbb{R}^m \times S^{m-1} \ni (x, q) \longmapsto \langle q, \cdot \rangle \in ST^*_x \mathbb{R}^m
\]
provides a trivialisation of \( ST^* \mathbb{R}^m \). The hodograph transformation is then defined by the formula
\[
ST^* \mathbb{R}^m \ni (x, q) \longmapsto \langle q, \langle x, \cdot \rangle \rangle \in J^1(S^{m-1}).
\]
(6.1)

It is easy to see that this map is a contactomorphism of the standard contact structures on \( ST^* \mathbb{R}^m \) and \( J^1(S^{m-1}) \) (see [2, pp. 48–49]).

In the case \( m = 2 \), we can trivialise \( J^1(S^1) \) using the angle coordinate \( \varphi \) on \( S^1 \) and the corresponding momentum coordinate on the fibre of \( T^* S^1 \). Formula (6.1) then becomes
\[
(x_1, x_2, \varphi) \longmapsto (\varphi, -x_1 \sin \varphi + x_2 \cos \varphi, x_1 \cos \varphi + x_2 \sin \varphi).
\]
(6.2)

It should be clear from formula (6.1) that the hodograph image of the fibre of \( ST^* \mathbb{R}^m \) over a point \( x \in \mathbb{R}^m \) is the graph of the 1-jet of the function \( q \mapsto \langle x, q \rangle \) on \( S^{m-1} \). In particular, the fibre over the origin is mapped to the zero section of \( J^1(S^{m-1}) \).

**Corollary 6.1.** If a non-negative Legendrian isotopy connects two fibres of \( ST^* \mathbb{R}^m \), then the fibres coincide and the isotopy is constant.

**Proof.** Applying a parallel shift in \( \mathbb{R}^m \), we may assume that the first fibre is the fibre over the origin. Then the image of our isotopy under the hodograph transformation is a non-negative Legendrian isotopy in \( J^1(S^{m-1}) \) connecting the zero section \( \Lambda^0 \) to the graph \( \Lambda^f \) of the 1-jet of the function \( f(q) = \langle x, q \rangle \) for some \( x \in \mathbb{R}^m \). Proposition 5.3 shows that \( f \geq 0 \), which is only possible if \( x = 0 \) because otherwise \( f(-x/|x|) = -|x| < 0 \). Thus, \( \Lambda^f = \Lambda^0 \) and the isotopy is constant by Corollary 5.5.

**Corollary 6.2.** Let \( M \) be a manifold smoothly covered by an open subset \( \tilde{M} \subset \mathbb{R}^m \). If a non-negative Legendrian isotopy connects two fibres of \( ST^* M \), then the fibres coincide and the isotopy is constant.

**Proof.** Let \( ST^* \tilde{M} \to ST^* M \) be the fibrewise covering associated to the covering \( \tilde{M} \to M \). A non-negative Legendrian isotopy connecting two fibres of \( ST^* M \) lifts to a non-negative Legendrian isotopy connecting two fibres of \( ST^* \tilde{M} \subset ST^* \mathbb{R}^m \). Thus, the result follows from Corollary 6.1.

**Remark 6.3.** Somewhat more generally, one can assume that every compact subset of \( M \) is contained in an open set smoothly covered by an open subset of \( \mathbb{R}^m \).

**Example 6.4.** There exist manifolds \( M \) for which the assertion of Corollary 6.2 is false. The simplest example is provided by the sphere \( S^m \). Indeed, moving a fibre of \( ST^* S^m \) along the (co-)geodesic flow of the standard round metric \( \tilde{g} \) defines a positive Legendrian isotopy connecting it to the fibre over the antipodal point and then to itself. Note that a very similar argument shows that the Legendrian Low conjecture is false for the product Lorentz manifold \( (S^m \times \mathbb{R}, \tilde{g} \oplus -dt^2) \) with Cauchy surface \( S^m \), see [15, Example 3].

7. Proof of the Legendrian Low conjecture. Let \( x \) and \( y \) be causally related points in a globally hyperbolic spacetime with a Cauchy surface \( M \) covered by an open subset of \( \mathbb{R}^m \). Identify the space of future pointing null geodesics \( \mathfrak{N} \) with \( ST^* M \). Since intersecting skies are linked by definition, we may assume that \( x \) and \( y \) do not lie on the same null geodesic. We may also assume that \( y \) lies in the causal past of \( x \). By Proposition 4.2 there is a non-negative Legendrian isotopy \( \{ \Lambda_t \}_{t \in [0,1]} \) in \( ST^* M \) such that \( \Lambda_0 = \mathfrak{S}_x \) and \( \Lambda_1 = \mathfrak{S}_y \).
Suppose that \( \mathcal{G}_x \) and \( \mathcal{G}_y \) are Legendrian unlinked, i.e., that the link \( \mathcal{G}_x \cup \mathcal{G}_y \) is Legendrian isotopic to the link \( F \sqcup F' \) formed by two different fibres of \( ST^*M \). By the Legendrian isotopy extension theorem, there exists a contactomorphism \( \Psi : ST^*M \to ST^*M \) such that \( \Psi(\mathcal{G}_x \cup \mathcal{G}_y) = F \sqcup F' \). Hence, \( \{ \Psi(\Lambda_t) \}_{t \in [0,1]} \) is a non-negative Legendrian isotopy connecting two different fibres of \( ST^*M \), which contradicts Corollary 6.2. Thus, \( \mathcal{G}_x \) and \( \mathcal{G}_y \) are Legendrian linked.

8. Proof of Theorem C. Consider again a globally hyperbolic spacetime with a Cauchy surface \( M \) covered by an open subset of \( \mathbb{R}^m \). Identify \( \mathcal{N} \) with \( ST^*M \).

Let \( x \) and \( y \) be causally related points with disjoint skies. Suppose that the links \( \mathcal{G}_x \cup \mathcal{G}_y \) and \( \mathcal{G}_y \cup \mathcal{G}_x \) are Legendrian isotopic. By the Legendrian isotopy extension theorem, there exists a contactomorphism \( \Psi \) such that \( \Psi(\mathcal{G}_x \cup \mathcal{G}_y) = \mathcal{G}_y \cup \mathcal{G}_x \). Assume that \( y \) lies in the causal past of \( x \) (otherwise rename the points). Let \( \{ \Lambda_t \}_{t \in [0,1]} \) be a non-negative Legendrian isotopy in \( ST^*M \) connecting \( \mathcal{G}_x \) to \( \mathcal{G}_y \) provided by Proposition 4.2. Then \( \{ \Psi(\Lambda_t) \}_{t \in [0,1]} \) is a non-negative Legendrian isotopy connecting \( \mathcal{G}_y \) to \( \mathcal{G}_x \). Composing these two isotopies, we obtain a non-constant non-negative Legendrian isotopy connecting \( \mathcal{G}_x \) to itself.

Recall now that any sky \( \mathcal{G}_x \) is Legendrian isotopic to a fibre of \( ST^*M \). By the Legendrian isotopy extension theorem, there exists a contactomorphism \( \Phi \) taking \( \mathcal{G}_x \) to that fibre. The Legendrian isotopy connecting \( \mathcal{G}_x \) to itself constructed above is taken by \( \Phi \) to a non-constant non-negative Legendrian isotopy connecting the fibre to itself, which is impossible by Corollary 6.2. Hence, the links \( \mathcal{G}_x \cup \mathcal{G}_y \) and \( \mathcal{G}_y \cup \mathcal{G}_x \) cannot be Legendrian isotopic.

Remark 8.1. The proof shows that the links \( \mathcal{G}_x \cup \mathcal{G}_y \) and \( \mathcal{G}_y \cup \mathcal{G}_x \) are not even contactomorphic in \( \mathcal{N} \). Similarly, the proof of Theorem A in §7 shows that the link formed by the skies of causally related points \( x, y \in X \) is not contactomorphic to any link formed by the skies of a pair of causally unrelated points.

Remark 8.2. The proofs of Theorems A and C in §§7–8 work for any globally hyperbolic spacetime such that the assertion of Corollary 6.2 is true for its Cauchy surface \( M \).

9. Proof of the Low conjecture for \((2+1)\)-dimensional spacetimes. Consider the (nonoriented) link in \( ST^*\mathbb{R}^2 \) formed by two distinct fibres. In the terminology of [17], the image of this link under the hodograph transformation (6.1) is the \((1,1)\)-cable link in \( J^1(S^1) \). (Indeed, it is clear from formula (6.2) that if the image of the first fibre is the zero section, then the image of the second one goes along it once and makes a single turn around it in the \((p,u)\)-plane.) According to the main result of [17], Legendrian links smoothly isotopic to the \((1,1)\)-cable link are classified up to Legendrian isotopy by the classical invariants of their components.

Now let \( X \) be a globally hyperbolic \((2+1)\)-dimensional spacetime with Cauchy surface \( M \) covered by \( \mathbb{R}^2 \). Let \( \mathcal{G}_x \cup \mathcal{G}_y \) be a link in \( \mathcal{N} \cong ST^*M \) formed by the skies of two causally related points. We may assume that \( y \in J^-(x) \). Let \( \{ \Lambda_t \}_{t \in [0,1]} \) be a non-negative Legendrian isotopy in \( ST^*M \) connecting \( \mathcal{G}_x \) to \( \mathcal{G}_y \) provided by Proposition 4.2. Consider the covering \( ST^*\mathbb{R}^2 \to ST^*M \) associated to the covering of \( M \) by \( \mathbb{R}^2 \). Lift the isotopy \( \{ \Lambda_t \}_{t \in [0,1]} \) to a non-negative Legendrian isotopy \( \{ \tilde{\Lambda}_t \}_{t \in [0,1]} \) in \( ST^*\mathbb{R}^2 \) and set \( \tilde{\mathcal{G}}_x = \tilde{\Lambda}_0 \) and \( \tilde{\mathcal{G}}_y = \tilde{\Lambda}_1 \).

Now we argue by contradiction. Suppose that the link \( \mathcal{G}_x \cup \mathcal{G}_y \) is smoothly isotopic to a pair of fibres in \( ST^*M \). Since this isotopy lifts to \( ST^*\mathbb{R}^2 \), it follows that \( \tilde{\mathcal{G}}_x \cup \tilde{\mathcal{G}}_y \) is smoothly isotopic to a pair of fibres in \( ST^*\mathbb{R}^2 \). Each component of \( \tilde{\mathcal{G}}_x \cup \tilde{\mathcal{G}}_y \) is Legendrian isotopic to the fibre of \( ST^*\mathbb{R}^2 \) and therefore has the same classical invariants (in \( J^1(S^1) \)). Hence, \( \tilde{\mathcal{G}}_x \cup \tilde{\mathcal{G}}_y \) is Legendrian isotopic to a pair of fibres by the aforementioned result from [17]. Applying the Legendrian isotopy extension theorem in the same way as in §7 we see that this contradicts Corollary 6.1.
Remark 9.1. Let \( \tilde{X} \to X \) be the covering corresponding to the covering \( \mathbb{R}^2 \to M \). Then \((\tilde{X}, \tilde{g})\) with the induced Lorentz metric is a globally hyperbolic spacetime with a spacelike Cauchy surface \( \tilde{M} \cong \mathbb{R}^2 \), see [13] Proof of Theorem 14. For a pair of causally related points \( x, y \in X \) with \( y \in J^+(x) \), we can pick a pair of lifts \( \tilde{x}, \tilde{y} \in \tilde{X} \) connected by the lift of a past pointing curve connecting \( x \) to \( y \). In this way, Theorem [B] can be reduced to the case when the Cauchy surface is itself diffeomorphic to \( \mathbb{R}^2 \). A similar argument may be used to reduce Theorem [A] to the case when the Cauchy surface is diffeomorphic to an open subset of \( \mathbb{R}^m \). Note also that the Legendrian circles \( \tilde{S}_x, \tilde{S}_y \subset ST^*\mathbb{R}^2 \) appearing in the proof of Theorem [B] correspond to the skies \( S_x, S_y \).

10. Application to relativity. Suppose that \( M \) is a manifold smoothly covered by an open subset of \( \mathbb{R}^m \). It follows from Theorems [A] and [C] that it is impossible to find two globally hyperbolic Lorentz metrics \( g_1 \) and \( g_2 \) on \( X = M \times \mathbb{R} \) and two pairs of events \( x_1, y_1 \in (X, g_1) \) and \( x_2, y_2 \in (X, g_2) \) such that

\begin{enumerate}
  \item \( M_0 = M \times 0 \) is a spacelike Cauchy surface for both \((X, g_1)\) and \((X, g_2)\);
  \item the Legendrian links in \( ST^*M_0 \) corresponding to \((x_1, y_1)\) and \((x_2, y_2)\) are the same (in any natural sense, see Remark [8.1]);
  \item \( x_1, y_1 \) are causally related and \( x_2, y_2 \) are causally unrelated (or \( y_1 \in J^+(x_1) \) and \( y_2 \notin J^+(x_2) \)).
\end{enumerate}

The Legendrian links in (b) are completely determined by the intersections of the exponentials of the null cones at \( x_1, y_1 \) and \( x_2, y_2 \) with an arbitrarily thin (in fact, infinitesimal) neighbourhood of the Cauchy surface \( M_0 \). Hence, the result admits the following physical or, perhaps, ‘science fictional’ interpretation. Even if you are given absolute freedom to change the past, you will not be able to destroy causal relations between events so that the change cannot be observed on spacelike Cauchy surfaces in the immediate present.

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