Scalar $QCD_4$ on the null-plane

R. Casana$^*$, B.M. Pimentel$^†$ and G. E. R. Zambrano$^‡§$

1 Departamento de Física, Universidade Federal do Maranhão (UFMA),
Campus Universitário do Bacanga, CEP 65080-040, São Luís - MA, Brasil.
2 Instituto de Física Teórica (IFT/UNESP), UNESP - São Paulo State University
Rua Pamplona 145, CEP 01405-900, São Paulo, SP, Brazil

Abstract

We have studied the null-plane hamiltonian structure of the free Yang-Mills fields and the scalar chromodynamics ($SQCD_4$). Following the Dirac's procedure for constrained systems we have performed a detailed analysis of the constraint structure of both models and we give the generalized Dirac brackets for the physical variables. In the free Yang-Mills case, using the correspondence principle in the Dirac's brackets we obtain the same commutators present in the literature.

1 Introduction

To quantize the theory on the null-plane, initial conditions on the hyperplane $x^+ = cte$ and equal $x^+$-commutation relations must be given and the hamiltonian must describe the time evolution from an initial value surface to other parallel surface that intersects the $x^+$-axis at some later time. Although the prescription has a lot of similarities with the conventional approach there are significant differences when we perform the quantization of the theory. Inside the null-plane framework, the lagrangian which describes a given field theory is singular and at least second class constraints appear, these can be eliminated by constructing Dirac’s brackets (DB) and the theory can be quantized, via correspondence principle, in terms of a reduced number of independent fields, the physical ones. Thus, the Dirac’s method [1] allows built the null-plane hamiltonian and the canonical commutation relations in terms of the independent fields of the theory.

The quantization of relativistic field theory at the null plane time, proposed by Dirac [2], has found important applications [3] in both gauge theories and string theory [4]. It is interesting to observe that the null-plane quantization of a non-abelian gauge theory using the null-plane gauge condition, $A_1 = 0$, identified the transverse components of the gauge field as the degrees of freedom of the theory and, therefore, the ghost fields can be eliminated of the quantum action [5].

In [6], Tomboulis has quantized the massless Yang-Mills field in the null-plane gauge $A^a_2 = 0$ and derived the Feynman rules. However, in [7], McKeon has shown that the null-plane quantization of this theory leads a set of second-class constraints in addition to the usual first-class constraints, characteristics of the usual instant form quantization, what implies in the introduction of additional constraints.
ghost fields in the effective lagrangian. Moreover, in [8], Morara, Soldati and McCartor have quantized the theory in the framework of the standard perturbation approach and they have explained that the difficulties appearing in the null-plane gauge are overcome using the gauge \( A^a_{\mu} = 0 \), such gauge provides a generating functional for the renormalized Green’s functions that takes to the Mandelstam-Leibbrandt’s prescription for the free gluon propagator.

On the other hand, in [9], Neville and Rohlich have studied the scalar electrodynamics and have obtained the commutation relations between free fields from the commutations relations of the free field operators at unequal times but the commutation relation representing the interaction was not computed but they affirmed to be derived solving a quantum constraint. This last commutation relation was determined in [10], Casana, Pimentel and Zambrano have calculated all the commutation relations following a careful analysis of the constraint structure of the theory and the results obtained are consistent with the specified in the literature [9].

In this paper we will discuss firstly the null-plane structure of the pure Yang-Mills fields and after its interaction with a scalar complex field (SQCD\(_4\)) following the Dirac’s formalism for constrained systems. The Hamiltonian analysis follows the spirit outlined in [10]. The work is organized as follow: In the section 2, we study the free Yang-Mills field, being its constrained structure analyzed in detail, thus, we classify the constraints and the appropriated equations of motion of the dynamical variables are determined by using the extended hamiltonian. The null-plane gauge is imposed to transform the set of first class constraints in second class one and, the Diracs’s brackets (DB) among the independent fields are obtained by choosing appropriate boundary conditions on the fields. In the section 3, the constraint structure of the scalar chromodynamics (SQCD\(_4\)) is analyzed, the set of constraints is classified and the correct equations of motion are checked by using extended hamiltonian as the generator of temporal evolution. Next, we invert the second class matrix by imposing appropriated boundary conditions on the fields and we calculate the DB among the fundamental dynamical variables. Finally, we give our conclusions and remarks.

## 2 Free Yang-Mills field

For any semi-simple Lie group with structure constant \( f_{bc}^a \) the Yang-Mill lagrangian density is

\[
\mathcal{L} = -\frac{1}{4} F^{\mu\nu}_a F^a_{\mu\nu},
\]

with \( F^{\mu\nu}_a = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{bc}^a A^b_\mu A^c_\nu \), the gauge index \( a, b, c \) runs from 1 to \( n \). Such lagrangian is invariant under the following infinitesimal gauge transformations

\[
\delta A^a_\mu (x) = f_{bc}^a A^b_\mu (x) A^c_\mu (x) + \frac{1}{g} \partial^\mu \Lambda_a (x) ,
\]

with \( \Lambda_a (x) \) a infinitesimal function.

In the present work, we specialize for convenience to the \( SU(2) \) gauge group that only has three generators and \( f_{bc}^a = \varepsilon_{abc} \), where \( \varepsilon_{abc} \) is the Levi-Civita totally antisymmetric tensor in three dimensions, thus, we can define everything in such way that we can forget about raising and lowering group indexes. From (1) we find the Euler-Lagrange equations

\[
(D_\nu)^{ab} F^{\nu\mu}_b = 0,
\]

where we have defined the covariant derivative \( (D_\nu)^{ab} \equiv \delta^a_b \partial_\nu - g \varepsilon_{abc} A^c_\nu \).
2.1 Structure Constraints and Classification

In the null-plane dynamics, the canonical conjugate momenta are

$$\pi^\mu_a \equiv \frac{\partial L}{\partial (\partial_+ A^\mu_a)} = -F^{+\mu}_a, \quad (4)$$

this equation gives the following set of primary constraints

$$\phi_a \equiv \pi^+_a \approx 0 \quad \Rightarrow \quad \phi^k_a \equiv \pi^+_a - \partial_+ A^a_k + \partial_k A^a_+ - g\varepsilon_{abc}A^b_+A^c_+ \approx 0. \quad (5)$$

and the dynamical relation for $A^a_-$

$$\pi^-_a = \partial_+ A^-_a - \partial_- A^a_+ - g\varepsilon_{abc}A^b_+A^c_-, \quad (6)$$

At once, the canonical hamiltonian is given by

$$H_C = \int d^3 y H_C = \int d^3 y \left\{ \frac{1}{2} (\pi^-_a)^2 + \pi^-_a (D^a_\tau)_{ab} A^b_+ + \pi^+_a (D^a_\tau)_{ab} A^b_+ + \frac{1}{4} (F^a_{ij})^2 \right\}. \quad (7)$$

Following the Dirac procedure, we define the primary hamiltonian adding to the canonical hamiltonian the primary constraints

$$H_P = \int d^3 y \left\{ \frac{1}{2} (\pi^-_a)^2 + \pi^-_a (D^a_\tau)_{ab} A^b_+ + \pi^+_a (D^a_\tau)_{ab} A^b_+ + \frac{1}{4} (F^a_{ij})^2 + u^b \phi_b + \lambda_b^b \phi_b \right\}, \quad (8)$$

where $u^b$ and $\lambda_b^b$ are their respective Lagrange multipliers.

The fundamental Poisson brackets (PB) among fields are

$$\{ A^a_\mu(x), \pi^\nu_b(y) \} = \delta^\nu_\mu \delta_0^3(x-y). \quad (9)$$

Requiring that $H_P$ is the generator of temporal evolutions, the consistency condition of the primary constraints, i.e. $\{ \phi, H_P \} = 0$, give us for $\phi_a$

$$\{ \phi_a(x), H_P \} = (D^a_\tau)_{ab} \pi^-_b + (D^a_\tau)_{ab} \pi^+_b \equiv G_a(x) \approx 0, \quad (10)$$

a genuine secondary constraint, which is the Gauss’s law. Also, for $\phi^k_a$ we obtain

$$\{ \phi^k_a(x), H_P \} = (D^a_\tau)_{ab} F^b_{+-} + (D^a_\tau)_{ab} F^b_{ik} - 2 (D^a_\tau)_{ab} \lambda^b_k \approx 0, \quad (11)$$

a differential equation which allows to compute $\lambda^b_k$ after imposition of appropriated boundary conditions. The consistency condition of the secondary constraint yields

$$\{ G_a(x), H_P \} = g\varepsilon_{abc}A^c_+(x) G_b(x) \approx 0, \quad (12)$$

the Gauss’s law is automatically conserved. Then, there are not more constraints and the equations and give the full set of constraints.

The set of first class constraints is $\{ \pi^+_a, G_a \}$ and the set of second class constraints is $\{ \phi^k_a \}$ whose PB’s are

$$\{ \phi^k_a(x), \phi^l_b(y) \} = -2 \delta^l_k (D^a_\tau)_{ab} \delta^3(x-y) \quad (13)$$
2.2 Equations of motion

Now we check the equations of motion. The time evolution of the fields is determined by computing their PB’s with the so called extended hamiltonian $H_E$, which is obtained by adding to the primary hamiltonian all the first class constraints of the theory:

$$H_E = \int d^3y \left\{ \frac{1}{2} (\pi_b)^2 + \pi_b^{-} (D_i^y)^{bc} A_i^c + \pi_b^{-} (D_i^y)^{bc} A_i^c + \frac{1}{4} (F_{ij}^b)^2 + \lambda_i^b \phi_i^b + u^b \phi_b + v^b G_b \right\}, \quad (14)$$

thus, we have the time evolution of the dynamical variables, i.e., $\dot{\phi} = \{ \phi, H_E \}$, gives

$$\dot{A}_i^a = u^a \quad (15)$$
$$\dot{\pi}_a^{-} = \pi_a^{-} + (D_i^x)^{ac} A_i^c - (D_i^x)^{ab} v^b \quad (16)$$
$$\dot{A}_k^x = (D_k^x)^{ac} A_i^c + \lambda_k^a - (D_k^x)^{ab} v^b \quad (17)$$
$$\dot{\pi}_a^k = G_a \quad (18)$$
$$\pi_a^{-} = -g \varepsilon_{abc} \pi_b^{-} A_i^c + (D_i^x)^{ab} \lambda_j^b - g \varepsilon_{cba} v^b \pi_c^{-} \quad (19)$$
$$\pi_a^k = -g \varepsilon_{cba} \pi_b^k A_i^c + (D_i^x)^{ab} F_{kj}^b - (D_i^x)^{ab} \lambda_j^b - g \varepsilon_{abc} v^b \pi_c^k, \quad (20)$$

if we demand consistence with the Euler-Lagrange equation of motion (23), we must to choose $v^b = 0$, however, the multiplier $u^a$ remains indeterminate.

The Dirac’s algorithm requires as many gauge conditions as first class constraints there are, nevertheless these conditions should be compatible with the Euler-Lagrange equations and together with the first class set they should form a second class set, in such way that the Lagrange multipliers, corresponding to the first class set, are determined. Under such considerations, we choose as the first gauge condition

$$A_a^a \approx 0, \quad (21)$$

whose consistency condition $\dot{A}_a^a = \{ A_a^a, H_E \} \approx 0$ must be compatible with the dynamical equation (20) thus we see that if we choose $v^b = 0$ in (15) then the Eq.(21) will hold for all times only if

$$\pi_a^{-} + \partial_x^a A_i^c \approx 0, \quad (22)$$

therefore, the equations (21) and (22) constitute our gauge conditions on the null-plane and they are known as the null-plane gauge.

2.3 Dirac Brackets

The gauge fixing conditions transform the first class set into second class one, the following stage in the Dirac’s procedure is to transform the second class constraints in strong identities. This demands an alteration of the canonical brackets (PB) to form a new brackets set, the Dirac’s brackets (DB), with which the second class constraints are automatically satisfied. Thus, the prescription for determine the DB implies in calculating the inverse of the second class matrix, for this purpose, we rename the second class constraints as

$$\Theta_1 \equiv \pi_a^+ \quad (23)$$
$$\Theta_2 \equiv (D_i^x)^{ab} \pi_b^{-} + (D_i^x)^{ab} \pi_b^+$$
$$\Theta_3 \equiv A_a^a \quad (24)$$
$$\Theta_4 \equiv \pi_a^{-} + \partial_x^a A_i^c$$
$$\Theta_5 \equiv \pi_a^+ - \partial_x^a A_i^c + \partial_x^a A_i^c - g \varepsilon_{abc} A_i^b A_i^c,$$
and we define the elements of the second class matrix as $F_{ab}(x,y) \equiv \{\Theta_a(x),\Theta_b(y)\}$. With these considerations, the Dirac’s bracket of two dynamical variables, $A_a(x)$ and $B_b(y)$, is then defined as
\[
\{A_a(x),B_b(y)\}_D = \{A_a(x),B_b(y)\} - \int \delta^3u \delta^3v \left\{A_a(x),\Theta_c(u)\right\}(F^{-1})^{cd}(u,v)\left\{\Theta_d(v),B_b(y)\right\},
\]
where $F^{-1}$ is the inverse of the constraint matrix.

The explicit evaluation of $F^{-1}$ involve the determination of an arbitrary function of the variables $x^+$ and $x^-$ which can be fixed by considering appropriate boundary conditions on the fields $A_{\mu}^a$ eliminating the ambiguity in the definition of the inverse of the operator $\partial_-$ related to their zero modes that give origin to hidden subset of first class constraints which generate improper gauge transformations what is characteristic of the null-plane constraint structure. Thus, from (24) we obtain the DB among the independent variables of the theory
\[
\{A_k^a(x),A_l^b(y)\}_D = -\frac{1}{4}\delta^a_b\delta_{kL}(x-y)\delta^2(x^\top - y^\top)
\]
\[
\{A_k^a(x),A_+^b(y)\}_D = \frac{1}{4}|x-y|(D_k^a)_{ab}\delta^2(x^\top - y^\top).
\]
At once, via the correspondence principle we obtain the commutators among the fields
\[
[A_k^a(x),A_l^b(y)] = -\frac{i}{4}\delta^a_b\delta_{kL}(x-y)\delta^2(x^\top - y^\top),
\]
\[
[A_k^a(x),A_+^b(y)] = \frac{i}{4}|x-y|(D_k^a)_{ab}\delta^2(x^\top - y^\top).
\]
The first relationship is exactly that obtained by Tomboulis and starting from it is possible to calculate the other two expressions determined by him, meanwhile the equation is our contribution to the commutation relations.

3 Scalar chromodynamics $SQCD_4$

The model describing the interaction of Yang-Mills and complex scalar field is given the following lagrangian density
\[
\mathcal{L} = y^{\mu\nu}(D_\mu)^{ab}\Phi_b^\dagger(D_\nu)^{ac}\Phi_c - m^2\Phi_a^\dagger\Phi_a - \frac{1}{4}F^{\mu\nu}F_{\mu\nu},
\]
where the field strength $F_{\mu\nu}^{ab}$ and the covariant derivative $(D_\mu)^{ab}$ are defined in the $SU(2)$ adjoint representation by
\[
F_{\mu\nu}^{ab} = \partial_\mu A_\nu^{ab} - \partial_\nu A_\mu^{ab} + g\varepsilon_{abc}A_\mu^{bc}A_\nu^{ab}, \quad (D_\mu)^{ab} \equiv \delta^b_a\partial_\mu - g\varepsilon_{abc}A_\mu^{bc},
\]
respectively. $\Phi_c$ is the complex scalar field which has three components in an internal space and the gauge transformation are rotations in this space what gives a conserved vector quantity named isospin. The field equations are given for
\[
(D_\nu)^{ab}F_b^{\nu\mu} = J_\mu^a
\]
\[
(D_\mu)^{ab}(D_\nu)^{bc}\Phi_c + m^2\Phi_a = 0
\]
\[
(D_\mu)^{ab}(D_\nu)^{bc}\Phi_\xi + m^2\Phi_\xi = 0,
\]
where $J_\mu^a$ is the current density defined by
\[
J_\mu^a \equiv g\varepsilon_{abc}\left\{\left[(D_\mu)^{bd}\Phi_d^\dagger\right]\Phi_c + \Phi_\xi^\dagger\left[(D_\mu)^{bd}\Phi_d\right]\right\}.
\]
3.1 Structure Constraints and Classification

The canonical conjugate momenta of the gauge field is

$$\pi^\mu_a \equiv \frac{\partial L}{\partial (\partial_+ A^a_\mu)} = -F^+_a \mu,$$

and for the fields $\Phi_a$, $\Phi^\dagger_a$ are

$$\Pi^a \equiv \frac{\partial L}{\partial (\partial_+ \Phi_a)} = (D_-)^{ab} \Phi^\dagger_b,$$

respectively.

From (32) we get one dynamical relation for $A^a_-$

$$\pi^a_{-} = \partial_+ A^a_- - \partial_- A^a_+ - g\partial_{abc}A^b_+ A^c_-,$$

and the following set of primary constraints for the gauge sector

$$\phi_a \equiv \pi^a_+ \approx 0 \quad , \quad \phi^k_a \equiv \pi^a_k - \partial_- A^a_k - g\partial_{abc}A^b_- A^c_k \approx 0,$$

and from (33) we obtain a set of primary constraints of the scalar sector

$$\Theta_a \equiv \Pi_a - (D_-)^{ab} \Phi_b \approx 0 \quad , \quad \Theta^\dagger_a \equiv \Pi^\dagger_a - (D_-)^{ab} \Phi^\dagger_b \approx 0.$$  

The canonical hamiltonian is

$$H_C = \int d^3y \left\{ \frac{1}{2} (\pi^a_+)^2 + \pi_{-} (D_-)^{bc} A^c_+ + \pi^k_{b} (D_k)^{bc} A^c_+ + J^+_a A^a_+ \right\}$$

$$+ \int d^3y \left\{ (D^+_k)^{ab} \Phi^\dagger_b \left[ (D^+_k)^{cd} \Phi_d \right] + m^2 \Phi_b \Phi_b + \frac{1}{4} (F^b_{jk})^2 \right\}$$

and the primary hamiltonian is

$$H_P = H_C + \int d^3y \left\{ u^b \phi_b + \lambda^b_k \phi^k_b + U^\dagger_b \Theta_b + \Theta^\dagger_b U_b \right\},$$

where $u^b$ and $\lambda^b_k$ are the Lagrange multipliers associated to the vector constraints and $U^\dagger_b$ and $U_b$ are the multipliers associated with the scalar ones.

The fundamental Poisson brackets are

$$\{ A^a_\mu (x), \pi^\nu_b (y) \} = \delta^\nu_a \delta^\mu_b \delta^3 (x - y)$$

$$\{ \Phi_a (x), \Pi_b (y) \} = \delta^b_a \delta^3 (x - y) \quad , \quad \{ \Phi^\dagger_a (x), \Pi_b (y) \} = \delta^b_a \delta^3 (x - y),$$

and the non null PB’s among the primary constraints

$$\{ \phi_a^k (x), \phi^k_b (y) \} = -2\delta^k_a \delta^3 (x - y) \quad ,$$

$$\{ \Theta_a (x), \Theta^\dagger_b (y) \} = -2 \delta^3 (x - y).$$

Following the Dirac’s procedure, we compute the consistence condition of every primary constraint. Thus, the consistence condition of the scalar constraints yields:

$$\dot{\Theta}_a = -g\partial_{abc} \pi^a_b \Phi_c - 2g\partial_{bcd} (D_-)^{ab} \left[ \Phi_d A^c_+ \right] + (D_k)^{ab} (D_k)^{bc} \Phi_c - m^2 \Phi_a - 2 (D_-)^{ab} U_b,$$

$$\dot{\Theta}^\dagger_a = -g\partial_{abc} \pi^a_b \Phi^\dagger_c - 2g\partial_{bcd} (D_-)^{ab} \left[ \Phi^\dagger_d A^c_+ \right] + (D_k)^{ab} (D_k)^{bc} \Phi^\dagger_c - m^2 \Phi^\dagger_a - 2 (D_-)^{ab} U^\dagger_b.$$
these relations allow to determine the multipliers $U_b$ and $U_b^\dagger$, respectively. In this way, there are not more constraints associated with the scalar sector.

In the gauge sector, the consistency condition of $\phi_a^k$ provides

$$\dot{\phi}_a^k = (D_k)^{ab} \pi_b^- + (D_j)^{ab} F_{jk}^b - J_a^k - 2 (D_-)^{ab} \lambda_k^b \approx 0$$

(43)

that is an equation which determines the multiplier $\lambda_k^b$. Finally, the consistence condition of $\pi_a^+$ contributes with a secondary constraint

$$\dot{\phi}_a = (D_-)^{ab} \pi_b^- + (D_\|)^{ab} \pi_b^+ - J_a^b \equiv G_a \approx 0,$$

(44)

which is the Gauss’s law for the scalar chromodynamics. After a laborious work, it is possible to verify that no more further constraints are generated from the consistence condition of the Gauss’ law because it is automatically conserved

$$\dot{G}_a = g\varepsilon_{abc} \left[ \Phi_c \dot{\Theta}_b + \Phi_b \dot{\Theta}_c^\dagger \right] \approx 0.$$

(45)

Therefore, the equation (35), (36) and (44) constitute the full set of constraints of the theory.

The non null PB’s among the constraints of the theory are

$$\begin{align*}
\{ \phi_a^k (x), \phi_b^j (y) \} & = -2 \Theta_k (D_-)^{ab} \delta^3 (x - y), \\
\{ \Theta_a^k (x), \Theta_b^j (y) \} & = -2 (D_-)^{ab} \delta^3 (x - y), \\
\{ G_a (x), \Theta_b^j (y) \} & = -2 g\varepsilon_{acf} \Phi_f^j (x) (D_-)^{cb} \delta^3 (x - y), \\
\{ G_a (x), \Theta_b (y) \} & = -2 g\varepsilon_{acf} \Phi_f (x) (D_-)^{cb} \delta^3 (x - y),
\end{align*}$$

(46)

thus, it is easy to note that $\pi_a^+$ is vanishing PB with all the other constraints, therefore, it is a first class constraint. The remaining set, $\{ \phi_a^k, \Theta_a^k, \Theta_b^j, G_a \}$, is apparently a second class set, however, it is possible to show that their constraint matrix is singular and its zero mode eigenvector provides a linear combination of constraints which is the first class constraint [10]. Such second first class constraint is

$$\Sigma_a \equiv G_a - g\varepsilon_{abc} \left[ \Phi_c \Theta_b + \Phi_b \Theta_c^\dagger \right].$$

(47)

Then, the first class constraints set is

$$\phi_a = \pi_a^+ \quad , \quad \Sigma_a = G_a - g\varepsilon_{abc} \left[ \Phi_c \Theta_b + \Phi_b \Theta_c^\dagger \right].$$

(48)

it is the maximal number of first class constraints and, the second class set is

$$\begin{align*}
\phi_a^k & = \pi_a^k - \partial_- A_a^k + \partial_+ A_a^k - g\varepsilon_{abc} A_b^a A_c^k, \\
\Theta_a & = \Pi_a - (D_-)^{ab} \Phi_b, \\
\Theta_a^j & = \Pi_a^j - (D_-)^{ab} \Phi_b^j.
\end{align*}$$

(49)

3.2 Equations of Motion

At this point we need to check that we have the correct equation of motion. The time evolution of the fields is determined by computing their PB with the extended hamiltonian which is defined as

$$H_E = H_C + \int d^3 y \left\{ u^b \phi_b + \lambda_b^j \phi_b^j + U_b^\dagger \Theta_b + \Theta_b^\dagger U_b + w_b \Sigma_b \right\}$$

(50)
Thus, the time evolution of the gauge field yields
\[
\begin{align*}
\dot{A}_+^a &= u^a, \\
\dot{A}_-^a &= \pi_+^a + (D_-)^{ac} A_+^c - (D_-)^{ab} w_b, \\
\dot{A}_k^a &= (D_k)^{ab} A_+^b + \lambda_k^a - (D_k)^{ab} w_b, \\
\dot{\pi}_a^+ &= G_a, \\
\dot{\pi}_a^- &= g\varepsilon_{abc} \pi_b^c A_+^a + 2g^2 A_+^a \Phi_+^b \Phi_b - g^2 A_+^a \Phi_+^b \Phi_b - g^2 A_+^a \Phi_+^b \Phi_b \\
&+ (D_k)^{ab} \lambda_k^b - g\varepsilon_{abc} U_+^b \Phi_c - g\varepsilon_{abc} \Phi_+^b U_b + g\varepsilon_{bca} w_b \pi_c^- \\
\dot{\pi}_a^k &= g\varepsilon_{abc} \pi_b^c A_+^a - J_a^k + (D_j)^{ab} F_{jk}^b - (D_-)^{ab} \chi_k^b + g\varepsilon_{abc} w_b \pi_c^k
\end{align*}
\]
and for the scalar fields the dynamics is given for
\[
\begin{align*}
\dot{\Phi}_a &= U_a + g\varepsilon_{abc} w_b \Phi_c, \\
\dot{\Phi}_a^\dagger &= U_a^\dagger + g\varepsilon_{abc} w_b \Phi_c^\dagger, \\
\dot{P}_a &= g\varepsilon_{cde} (D_-)^{ad} \left[ A_+^c \Phi_e \right] - g\varepsilon_{abc} A_+^b \left[ (D_-)^{cd} \Phi_d \right] + (D_k)^{ab} \left[ (D_k)^{bc} \Phi_c \right] \\
&- m^2 \Phi_a - (D_-)^{ab} U_b - g\varepsilon_{abc} w_b \Phi_c^\dagger, \\
\dot{P}_a^\dagger &= -g\varepsilon_{abc} A_+^b \left[ (D_-)^{cd} \Phi_d^\dagger \right] + g\varepsilon_{bcd} (D_-)^{ac} \left[ A_+^b \Phi_d \right] + (D_k)^{ab} \left[ (D_k)^{bc} \Phi_c^\dagger \right] \\
&- m^2 \Phi_a^\dagger - (D_-)^{ab} U_b^\dagger + g\varepsilon_{abc} w_b \Pi_d^\dagger
\end{align*}
\]
We can note that the set of equation \((51)\) and \((52)\) only will be compatible with the Euler-Lagrange equations \((30)\) if we set \(w_b = 0\) however the multiplier \(u^a\) still remains undetermined in this way the Dirac's formalism tell us to impose one set of gauge conditions, one for every first class constraint. The gauge conditions are chosen in such a way that they are compatible with the Euler-Lagrange equations of motion, thus one such set is the null-plane gauge conditions is given by relations \((21)\) and \((22)\).

### 3.3 Dirac Brackets

We have the following set of second class constraints:
\[
\begin{align*}
\Psi_1 &= \pi_+^a, \quad \Psi_2 \equiv G_a - g\varepsilon_{abc} \left( \Phi_+^b \Theta_b + \Phi_c \Theta_c^\dagger \right) \\
\Psi_3 &= A_-^a, \quad \Psi_4 \equiv \pi_+^a - \partial_- A_-^a \\
\Psi_5 &= \pi_+^k - \partial_- A_-^a + \partial_k A_-^a - g\varepsilon_{abc} A_-^a A_-^b \\
\Psi_6 &= \Pi_a - (D_-)^{ab} \Phi_b, \quad \Psi_7 \equiv \Pi_+^a - (D_-)^{ab} \Phi_+^b
\end{align*}
\]
with these, we define the following constraint matrix \(M_{ab} (x, y) \equiv \{ \Psi_a (x), \Psi_b (y) \} \), from where the DB for the dynamical variables are determined via evaluation of the inverse of this matrix. Then, by considering appropriate boundary conditions on the fields, \([10] [11]\), a unique inverse of the constraint matrix is obtained and after a laborious work we obtain the DB for the independent dynamical
4 Remarks and conclusions

In this work we have studied the null plane Hamiltonian structure of the free Yang-Mills field and its interaction with a complex scalar that we named as scalar chromodynamics (SQCD).

Performing a careful analysis of the constraint structure of Yang-Mills field, we have determined in addition of the usual first class constraints set a second class ones set, which is a characteristic of the null-plane dynamics [10]. The imposition of appropriated boundary conditions on the fields fixes the hidden subset of first class constraints [13] and eliminates the ambiguity on the operator $\partial_-$, that allows to get a unique inverse for the second class constraint matrix [10]. The Dirac’s brackets of the theory are quantized via correspondence principle; the commutators obtained are the same derived by Tomboulis [9].

The scalar chromodynamics SQCD Hamiltonian analysis has shown further of the free Yang-Mills structure, a contribution of the scalar sector with an additional constraints set. However, as a consequence of the constraint associated with the scalar part, one of the first class constraints is a linear combination of the Gauss’ law with the scalar constraints, in a similar way to the scalar electrodynamics case [10], such first class constraint is given by the zero mode eigenvector of the constraint matrix. Finally, choosing the null-plane gauge condition, which transforms first class constraints in second class ones and imposing appropriated boundary conditions on the fields to get a unique inverse of the second class constraints matrix and following the Dirac’s procedure we obtain the Dirac’s brackets of the canonical variables of the theory. Our results are consistent with those reported in the literature [9,10] when the abelian case is considered.

As the null-plane hamiltonian structure is well-defined, the null-plane quantization, of the models reported here and [10], via the path-integral formalism are now in advanced and whose result will be reported elsewhere.

Acknowledgements

RC thanks to CNPq for partial support, BMP thanks CNPq for partial support and GERZ thanks CNPq (grant 142695/2005-0) for full support.

References

[1] P. A. M. Dirac, Lectures in Quantum Mechanics, Benjamin, New York, 1964.

A. Hanson, T. Regge and C. Teitelboim, Constrained Hamiltonian Systems, Acc. Naz. dei Lincei, Roma, 1976.
[2] P. A. M. Dirac, Rev. Mod. Phys. 21, 392 (1949).

[3] Stanley J. Brodsky, Hans-Christian Pauli and Stephen S. Pinsky, Phys. Rep. 301, 299 (1988).
   P. P. Srivastava, Nuovo Cimento A107, 549 (1994).

[4] D. Biatti and L. Susskind, Phys. Lett. B425, 351 (1998).

[5] G. I. Lapage and S. J. Brodsky, Phys. Rev. D22, 2157 (1980).
   A. Bassetto, M. Dalbosco and R. Soldati, Phys. Rev. D36, 3138 (1987).
   A. Bassetto, G. Heinrich, Z. Kunszt and W. Volgelsang, Phys. Rev. D58, 94020 (1998).

[6] E. Tomboulis, Phys. Rev. D8, 2736 (1971).

[7] G. McKeon, Can. J. Phys. 64, 549 (1986).

[8] M. Marara, R. Soldati and G. McCartor, IAP Conf. Proc. 494, 284 (199).

[9] R. A. Neville and F. Rohrlich, Phys. Rev. D3, 1692(1971).

[10] R. Casana, B. M. Pimentel and G. E. R. Zambrano, SQED\textsubscript{4} and QED\textsubscript{4} on the null-plane, arXiv:0803.2677 [hep-th].

[11] F. Rohrlich, Acta Phys. Austriaca, Suppl. VIII, 277(1971).
   R. A. Neville and F. Rohrlich, Nuovo Cimento A1, 625 (1971).

[12] R. Benguria, P. Cordero and C. Teitelboim, Nucl. Phys. B122, 61 (1976).

[13] P. J. Steinhardt, Ann. Phys 128, 425 (1980).