Variance Breakdown of Huber \((M)\)-estimators: \(n/p \to m \in (1, \infty)\)

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Abstract

Huber’s gross-errors contamination model considers the class \(\mathcal{F}_\varepsilon\) of all noise distributions \(F = (1 - \varepsilon)\Phi + \varepsilon H\), with \(\Phi\) standard normal, \(\varepsilon \in (0, 1)\) the contamination fraction, and \(H\) the contaminating distribution. A half century ago, Huber evaluated the minimax asymptotic variance in scalar location estimation,

\[
\min_\psi \max_{F \in \mathcal{F}_\varepsilon} V(\psi, F) = \frac{1}{I(F^*_\varepsilon)}
\]

where \(V(\psi, F)\) denotes the asymptotic variance of the \((M)\)-estimator for location with score function \(\psi\), and \(I(F^*_\varepsilon)\) is the minimal Fisher information \(\min_{F \in \mathcal{F}_\varepsilon} I(F)\).

We consider the linear regression model \(Y = X\theta_0 + W\), \(W_i \sim\text{i.i.d.} F\), and iid Normal predictors \(X_{i,j}\), working in the high-dimensional-limit asymptotic where the number \(n\) of observations and \(p\) of variables both grow large, while \(n/p \to m \in (1, \infty)\); hence \(m\) plays the role of ‘asymptotic number of observations per parameter estimated’. Let \(V_m(\psi, F)\) denote the per-coordinate asymptotic variance of the \((M)\)-estimator of regression in the \(n/p \to m\) regime \([EKBBL13, DM13, Kar13]\). Then \(V_m \neq V\); however \(V_m \to V\) as \(m \to \infty\).

In this paper we evaluate the minimax asymptotic variance of the Huber \((M)\)-estimate. The statistician minimizes over the family \((\psi_\lambda)_{\lambda > 0}\) of all tunings of Huber \((M)\)-estimates of regression, and Nature maximizes over gross-error contaminations \(F \in \mathcal{F}_\varepsilon\). Suppose that \(I(F^*_\varepsilon) \cdot m > 1\). Then

\[
\min_\lambda \max_{F \in \mathcal{F}_\varepsilon} V_m(\psi_\lambda, F) = \frac{1}{I(F^*_\varepsilon) - 1/m}.
\]

Of course, the RHS of (2) is strictly bigger than the RHS of (1). Strikingly, if \(I(F^*_\varepsilon) \cdot m \leq 1\), then

\[
\min_\lambda \max_{F \in \mathcal{F}_\varepsilon} V_m(\psi_\lambda, F) = \infty.
\]

In short, the asymptotic variance of the Huber estimator breaks down at a critical ratio of observations per parameter. Classically, for the minimax \((M)\)-estimator of location, no such breakdown occurs \([DH83]\). However, under this paper’s \(n/p \to m\) asymptotic, the breakdown point is where the Fisher information per parameter equals unity:

\[
\varepsilon^* \equiv \varepsilon_m^* (\text{Minimax Huber-}(M) \text{ Estimate}) = \inf\{\varepsilon : m \cdot I(F^*_\varepsilon) \geq 1\}.
\]

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Dedication. Based on a lecture delivered at a special colloquium honoring the 50th anniversary of the Seminar für Statistik (SfS) at ETH Zürich, November 25, 2014. The year 2014 was simultaneously: the 80th birthday year of Peter Huber, the 50th anniversary of his great 1964 paper on Robust Estimation, and the 50th anniversary of SfS. All of these events are causes for celebration, and we thank especially Peter Bühlmann, Sara van de Geer, Hansruedi Künsch, Marloes Maathuis, Nicolai Meinshausen, and indeed everyone at SfS for creating a wonderful commemoration event. Special congratulations to Peter J. Bickel on receiving his Doctor Honoris Causa from ETH as part of this celebration!

1 Introduction

Fifty years ago, Peter Huber published the masterwork [Hub64] in the Annals of Mathematical Statistics. His paper, ‘Robust Estimation of a Location Parameter’ revealed robust statistics to be amenable to mathematical analysis, producing a new optimal robust estimator – now called the Huber (M)-estimator – that has proven practical, elegant and lasting. Richard Olshen once called Peter’s paper ‘an out-of-the-park, grand-slam home run’.

Only 8 years after this initial paper in statistics, Peter delivered the Wald Lectures [Hub73], recognition from the profession of the exceptional importance of his Œuvre. While Huber’s 1964 paper considered the estimation of a scalar location parameter, his Wald Lectures summarized work showing that much of the framework of the 1964 paper generalized immediately to regression estimation.

1.1 (M)-estimates of Regression

Consider the traditional linear regression model

\[ Y = X \theta_0 + W, \]  

(3)

with \( Y = (Y_1, \ldots, Y_n)^T \in \mathbb{R}^n \) a vector of responses, \( X \in \mathbb{R}^{n \times p} \) a known design matrix, \( \theta_0 \in \mathbb{R}^p \) a vector of parameters, and \( W \in \mathbb{R}^n \) a random noise vector with i.i.d. components having marginal distribution \( F = F_W \).

To estimate \( \theta_0 \) from observed data \((Y, X)\) we use an (M)-estimator. Picking a non-negative even convex function \( \rho : \mathbb{R} \to \mathbb{R}_{\geq 0} \), we solve the optimization problem\(^3\)

\[
\hat{\theta}(Y; X) \equiv \arg \min_{\theta \in \mathbb{R}^p} \sum_{i=1}^{n} \rho(Y_i - X_i \cdot \theta),
\]  

(4)

\(^1\)Terminology from American baseball. The highest-impact scoring outcome that can ever be delivered by a batsman, and not at all frequent. Wikipedia states that over 112 annual World Series, comprising more than 500 games, and ten thousand at-bats, this has happened only eighteen times.

\(^2\)With a slight abuse of notation, we also use \( W \) to denote a scalar random variable with the same marginal distribution \( F_W \).

\(^3\) \( X_1, \ldots, X_n \) denote the rows of \( X \); while \( \theta \) denotes a column vector. \( \hat{\theta} \) is chosen arbitrarily if there are multiple minimizers.
Of course the prescription is broad enough to encompass traditional least squares \(- \rho_{LS}(t) = t^2\) - however, this would not be robust to outliers\(^4\). Better choices might include least absolute deviations \(- \rho_{LAD}(t) = |t|\) - and of course the Huber \(\rho = \rho_H(t; \lambda) = \min(t^2/2, \lambda|t| - \lambda^2/s)\).\(^5\)

1.2 Fixed \(p\), large \(n\) Minimax Robustness

Consider the random design case where \(X_i \sim iid \ N(0, I_p)\), and let \(\psi = \rho'\) denote the score function associated to the (M) estimator of interest. Let \(n \to \infty\) with \(p\) fixed, and consider the per-coordinate asymptotic variance

\[
V_\infty(\psi, F) = a.s. \lim_{n \to \infty} \frac{n}{p} \cdot \text{Tr}(\text{Var}_F(\hat{\theta})).
\]

Huber proposed to consider \(V_\infty(\psi, F)\) as the payoff function in a game between the statistician and nature. The two arguments of \(V_\infty\) represent the two choices being made here: the statistician is choosing the estimator, by specifying \(\psi\), and ‘nature’ is choosing the error distribution, by specifying \(F = F_W\). The statistician pays out the amount \(V_\infty(\psi, F)\) and, planning for all eventualities, wants to minimize the worst-case payout. The statistician envisions that \(F\) might contain a fraction \(\varepsilon\) of ‘bad data’, and so assumes that the action space of Nature is the class \(F_\varepsilon\) of all contaminated normal distributions \(F = (1 - \varepsilon)\Phi + \varepsilon H\). Here \(\Phi\) notes the standard normal, \(\varepsilon \in (0, 1)\) the contamination fraction, and \(H\) the contaminating distribution.

For a given choice \(\psi\), the maximal payout that can arise is \(\max_{F \in F_\varepsilon} V_\infty(\psi, F)\). Huber proposed that the statistician should minimize this quantity across \(\psi\), thus obtaining the minimax asymptotic variance and the associated minimax score. He found the least-informative distribution, \(F_\varepsilon^*\) - the cdf \(F\) solving \(\min_{F \in F_\varepsilon} I(F)\) with \(I\) the Fisher information for location, and Huber obtained the formula

\[
\min_{\psi} \max_{F \in F_\varepsilon} V_\infty(\psi, F) = \frac{1}{I(F_\varepsilon^*)}. \tag{5}
\]

He also discovered the minimax-optimal score function, now called the Huber score; it has the form

\[
\psi_\lambda(x) = \min(\lambda, \max(-\lambda, x)),
\]

for a specific \(\kappa = \kappa^*(\varepsilon)\), achieving the minimax. Numerous textbooks cover this material, including of course [HR09]; see also Section 2.1 below.

1.3 High-Dimensional Asymptotics

In his Wald lectures [Hub73] Page 802] Peter Huber called attention to the fertile regime beyond the fixed \(p\), large \(n\) asymptotic,

We intend to build an asymptotic theory for \(n \to \infty\); but there are several possibilities for the concomitant behavior of \(p\). In particular, with decreasing restrictiveness:

(a) \(\lim sup p < \infty\)

\(^4\)As can be documented by Frank Hampel’s notions of Influence Curve [Ham74], which shows that least squares has unbounded influence, and Breakdown Point, which documents that a single bad observation can cause the least squares solution to misbehave arbitrarily.

\(^5\)Other seemingly good choices, like \(\rho(t) = -\log(1 + t^2)\) are ruled out by lack of convexity.
(b) $\lim p^3/n = 0$
(c) $\lim p^2/n = 0$
(d) $\lim p/n = 0$
(e) $\limsup p/n < 1$
(f) $\limsup n - p = \infty$.

P.J. Huber, Annals of Statistics, 1, 802.

Huber also initiated the attack on this hierarchy of new asymptotic settings, addressing cases (b)-(d).

Though this was 40 years ago, it has taken the profession a while to catch up. In recent years, the focus of mathematical statistics research has finally gone beyond the fixed $p$, large $n$ asymptotic, to consider regimes (d)-(e), where $n$ and $p$ are both large.

In this paper, we consider a precise version of case (e), which we call the Proportional-Limit asymptotic PL($m$); in this regime $n,p \rightarrow \infty$ and $n/p \rightarrow m \in (1,\infty)$. Thus $m$ measures the number of observations per parameter to be estimated. This parameter seems to recur frequently in practitioner thinking: Huber specifically mentions in his 1972 Wald lectures the advice from crystallographers to keep $n/p > 5$.

In this paper the assumption PL($m$) will further entail a random Gaussian design, normalized so for each $n$, $X_i \sim \text{iid } N(0, \frac{1}{n}I_{p\times p})$; and the regression parameter $\theta_0 = \theta_{0,n} \in \mathbb{R}^p$ will be normalized so that the per-coordinate size $p^{-1}\|\theta_{0,n}\|_2 \rightarrow_{a.s.} \tau_0^2$. In this model $\mathbb{E}\{X^T X\} = I_{p\times p}$, and so under standard Gaussian errors $F = \Phi$, the per-coordinate Fisher Information is 1 for every $n$. Because of the finiteness of the total Fisher Information per coordinate, we are not entitled to expect highly precise estimation; hence it should be no surprise to find that the MSE $p^{-1}\|\hat{\theta}_n - \theta_0\|_2^2 \rightarrow_{a.s.} \text{AMSE}(\hat{\theta}, \theta_0) \neq 0$.

(Here and below AMSE stands for asymptotic mean square error.) Consider as performance measure the per-coordinate asymptotic variance:

$$V_m(\psi, F) = \lim_{n \rightarrow \infty} \frac{1}{p} \cdot \text{Tr}(\text{Var}_F(\hat{\theta}_n)).$$

The notation $V_m(\psi, F)$ emphasizes both the dependence of the asymptotic variance on $\psi$ and $F$ as in the classical case, but also the dependence on $m \in (1,\infty)$. Recent work on (M)-estimates in PL($m$) by [EKBBL13, DM13] shows that $V_m(\psi, F) > V_\infty(\psi, F)$, while $V_m(\psi, F) \rightarrow V_\infty(\psi, F)$ as $m \rightarrow \infty$.

Here we will carry out the Huber program of evaluating the minimax asymptotic variance of the Huber estimate – this time for $V_m(\psi, F)$ for $m \in (1,\infty)$, rather than the classical case $V_\infty$. The

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6Peter Bloomfield entered this area already in 1974 [Blo74], and Stephen Portnoy in 1984 [Por84]. Soviet-era mathematicians also began studying the high-dimensional asymptotic in the late 1960’s just when Huber was also thinking about it; and so Serdobolskii [Ser10] speaks of the Kolmogorov asymptotic, crediting Andrei Kolmogorov with calculations in the proportional-limit asymptotic already in 1967. Nevertheless, Huber’s 1972 Wald Lectures were certainly the earliest high-profile venue marking out this asymptotic for future research.

7A few references here may suffice: [CT07, BRT09, BvdG11, EKBBL13].

8Huber’s wife Effi Huber-Buser was trained as a crystallographer and in the experience of DLD is an insightful scientist, even knowing quite a lot even about the field of statistics and the statistical profession.

9In DLD’s first linear models statistics course, based on the classic Daniel and Wood [DW99], the instructor specifically mentioned $n/p > 10$ as a desirable ratio. It will be clear from the main results of this paper that the prescription to keep $n/p > 5$ was very good advice indeed.
statistician minimizes $V_m$ over the family $(\psi_\lambda)_{\lambda > 0}$ of all tunings of Huber (M)-estimates of regression, and Nature maximizes over gross-error contaminations $F \in \mathcal{F}_\varepsilon$.

The classical solution $m = \infty$ plays an important role even in the $PL(m)$ case. Suppose that Huber’s least-informative distribution $F^*_\varepsilon$ obeys $I(F^*_\varepsilon) \cdot m > 1$. In dimensional analysis $I(F^*_\varepsilon)$ is the Fisher information per observation, while $m$ is the number of observations per parameter. Hence this product is the Fisher information per parameter. Suppose that this exceeds 1. Then our main result (Corollary 5.6) shows that

$$\min_{\lambda} \max_{F \in \mathcal{F}_\varepsilon} V_m(\psi_\lambda, F) = \frac{1}{I(F^*_\varepsilon) - 1/m}. \quad (6)$$

Of course, the RHS of (6) is strictly bigger than the RHS of the classical ($m = \infty$) case [5]. As compared to the classical $m = \infty$ case, when $1 < m < \infty$ the worst-case asymptotic variance is no longer given by the reciprocal of the worst-case Fisher Information. However, the discrepancy grows small as $m \to \infty$. Hence, new phenomena emerge in the high-dimensional situation.

### 1.4 Variance Breakdown

Suppose now that the minimal Fisher information per parameter does not exceed 1 - i.e. that $I(F^*_\varepsilon) \cdot m \leq 1$. Then our main result additionally states that

$$\min_{\lambda} \max_{F \in \mathcal{F}_\varepsilon} V_m(\psi_\lambda, F) = \infty.$$

In short, the asymptotic variance of the Huber estimator breaks down at a critical ratio $m = m^*(\varepsilon)$ of observations per parameter. Hampel (1968) defined the breakdown point - the minimal fraction of gross errors that can drive the estimator beyond all bounds. Later, in connection with non-convex (M) estimators - such as Hampel’s redescending (M)-estimator - the phenomenon of breakdown of asymptotic variances arose; see [DH83, Section 5.2]. For Huber’s minimax (M)-estimator of classical location, no such breakdown occurs: for each $\varepsilon \in (0, 1)$,

$$\min_{\lambda} \max_{F \in \mathcal{F}_\varepsilon} V_\infty(\psi_\lambda, F) < \infty.$$

Huber, in personal communication, at one time considered this non-breakdown of the asymptotic variance to be a notable advantage of the Huber estimator in comparison to some other procedures, such as the Hampel ‘redescending’ score function.

Under this paper’s $PL(m)$ asymptotic, variance breakdown of Huber (M)-estimates indeed occurs. For a fixed ratio $m$ of observations per parameter, the variance breakdown point is exactly the critical fraction of contamination $\varepsilon$ where the minimal Fisher Information per parameter drops to 1 or smaller:

$$\varepsilon^* \equiv \varepsilon^*_m(\text{Minimax Huber-(M) Estimate}) = \inf\{\varepsilon : m \cdot I(F^*_\varepsilon) = 1\}.$$

### 1.5 Illustration

As a first deliverable of this paper, consider Figure 1, which displays the minimax asymptotic variance as a function of the contamination fraction $\varepsilon$ and the degrees of freedom per parameter estimated $m$. Below the critical curve $-1/m = I(F^*_\varepsilon)$ – we present contours of the minimax asymptotic variance;
Contours of $V^*_{m}(\epsilon)$ and $I(F^*_{\epsilon})$ ($\cdot$)

Figure 1: Minimax asymptotic variance $V^*_{m}(\epsilon)$. Each pair $(\epsilon, m)$ is represented by the $(x, y)$-point with $x = \epsilon$ and $y = 1/m$. The resulting parameter space $0 \leq \epsilon, 1/m \leq 1$ is divided into two phases - below and above the critical curve indicated by the dashdot line. Contours of the asymptotic variance $V^*_{m}(\epsilon)$ are depicted in the lower phase; they are undefined in the upper phase, where the asymptotic variance cannot be bounded: $V^*_{m}(\epsilon) = +\infty$. The boundary separating the two phases is indicated by the dashdot curve, at $1/m = I(F^*_{\epsilon})$.

in the lower left corner, the asymptotic variance is nearly 1, as it would be in the classical $m = \infty$ $\epsilon = 0$ case, The minimax asymptotic variance blows up as we approach the dashdot curve.

A second deliverable is provided by Figure 2 which presents contours of the minimax tuning parameter $\lambda^*(\epsilon, m)$; this selects the Huber $\rho_{\lambda}$ that achieves the minimax asymptotic ($V_{m^*}$) variance. Figure 2 shows that how $\lambda^*$ decays towards zero as $(\epsilon, m)$ approaches the critical curve.

Table 1 gives some specific numerical values of the minimax asymptotic variance $V^*_{m}(\epsilon)$. When $m = 2$, it turns out that the minimax asymptotic variance breaks down at exactly $\epsilon^* = 0.1924...$, this is the value of $\epsilon$ where $I(F_{\epsilon^*}) = 1/2$; the dramatic increase in variance as $\epsilon \uparrow \epsilon^*$ is plain from the table.

We conducted a small Monte-Carlo experiment to illustrate these concepts. With $n = 500$ and $p = 250$, so $m = 2$, we considered the linear model with iid Normal predictors $X_{i,j}$, and contaminated normal errors $W_i$, where $F_W = G_{\epsilon, \mu} = (1 - \epsilon)\Phi + \epsilon H_{\mu}$, and $H_{\mu}$ denotes the symmetric Heaviside CDF, with mass spread equiprobably at $\pm \mu$.

The reader can see in Table 2 that, for small $\epsilon = 1/20$, even as we make the contamination increasingly large, by setting $\mu = 100$, the empirical standard error stays bounded, independently of contamination amplitude $\mu$. However, as $\epsilon$ approaches the breakdown point $\epsilon_2^* = 0.1924...$, the variance grows considerably as $\mu$ grows large.
| $\varepsilon$ | $0.05$ | $0.10$ | $0.15$ | $0.175$ | $0.1875$ | $0.20$ | $0.25$ |
|---------------|--------|--------|--------|--------|--------|--------|--------|
| $\mathcal{V}_2^*(\varepsilon)$ | 3.38   | 5.84   | 13.9   | 35.0   | 136.4  | $\infty$ | $\infty$ |

Table 1: Worst-case asymptotic variance of minimax-tuned Huber (M)-estimator, at various levels of contamination; degrees of freedom per parameter $m = 2$.

| $\varepsilon$ | $\mu$ | $\text{Var}(\hat{\theta}_n^\lambda)^{1/2}$ |
|---------------|-------|---------------------------------|
| $0.05$        | 2     | $1.5883$                        |
| $0.05$        | 5     | $1.8662$                        |
| $0.05$        | 10    | $1.8801$                        |
| $0.05$        | 20    | $1.8594$                        |
| $0.05$        | 100   | $1.8436$                        |
| $0.1875$      | 2     | $1.9900$                        |
| $0.1875$      | 5     | $3.5099$                        |
| $0.1875$      | 10    | $5.5643$                        |
| $0.1875$      | 20    | $8.7302$                        |
| $0.1875$      | 100   | $37.8817$                       |

Table 2: Empirical Standard Error of minimax-tuned Huber (M)-estimator, at various amplitudes $\mu$ of contamination; degrees of freedom per parameter $m = 2$. Here the amplitude of the contamination is $\mu$. These empirical data reflect this paper’s theoretical; conclusion that for $\varepsilon$ small, variability stays controlled as $\mu \to \infty$, but as $\varepsilon$ approaches the breakdown point (here $0.1924...$), variability grows very large as $\mu$ increases, even though it will still ultimately stay bounded below the breakdown point.
Figure 2: Minimax \( \lambda^*(\epsilon; m) \). Each pair \((\epsilon, m)\) is represented by the point \( x = \epsilon \) and \( y = 1/m \). Contours of the minimax \( \lambda \) parameter \( \lambda^*(\epsilon; m) \) are depicted in the region below the dashdot curve at \( 1/m = I(F_\epsilon^*) \).

2 Reminders

2.1 Classical \((M)\) Estimation and minimax asymptotic variance

Huber (1964) supposed we have real scalar observations \( Y_i = \theta_0 + W_i \) where \( W_i \) are iid and symmetrically distributed, so that \( P(W > x) = P(W < -x) \). Hence \( \theta_0 \in \mathbb{R} \) is the center of symmetry of the distribution of \( Y_i \), and so also the mean, median, etc. He introduced the \((M)\)-estimator as a solution \( \hat{\theta}_n \) of

\[
(M) \quad \min_\theta \sum_{i=1}^{n} \rho(X_i - \theta),
\]

where \( \rho \) is an even convex function, \( \rho(x) = \rho(-x) \), so the score function \( \psi = \rho' \) was monotone nondecreasing. Under additional regularity conditions, he showed that any solution \( \hat{\theta}_n \) obeys

\[
\sqrt{n}(\hat{\theta}_n - \theta_0) \Rightarrow_{D} N(0, V(\psi, F)), \quad n \to \infty,
\]

where the asymptotic variance is given by

\[
V(\psi, F) = \frac{\int \psi^2 dF}{(\int \psi dF)^2}. \tag{7}
\]

For further discussion of regularity conditions, see [HR09].
Huber considered the situation where the random variable $W_i$ was distributed roughly as $N(0, 1)$, but is subject to gross-errors contamination. He evaluated

$$v^*(\varepsilon) \equiv \min_{\psi} \max_{F \in \mathcal{F}_\varepsilon} V(\psi, F),$$

and found the following insightful form. Let $I(F) = \int (f'(x))^2/f(x)dx$ denote the Fisher information for location; the least informative distribution $F^*_\varepsilon$ minimizes this quantity:

$$i^*(\varepsilon) \equiv \min_{F \in \mathcal{F}_\varepsilon} I(F);$$

Huber characterized the minimax asymptotic variance as the reciprocal of the minimal information:

$$v^*(\varepsilon) = \frac{1}{i^*(\varepsilon)},$$

and using this was able to write closed formulas for the optimal shape of $\psi$ – now called the Huber score function. In the original paper this was denoted

$$\psi_\kappa(x) = \min(\kappa, \max(x, -\kappa))$$

with so-called capping parameter $\kappa$, such that errors larger in absolute value than $\kappa$ get capped. Huber obtained closed form expressions\textsuperscript{10} for the minimax capping parameter $\kappa = \kappa^*(\varepsilon)$, the least favorable $F = F^*_\varepsilon$, and the minimax asymptotic variance $v^*(\varepsilon) = V(\psi_{\kappa^*(\varepsilon)}, F^*_\varepsilon)$. Figure 3 displays the behavior of $v^*(\varepsilon)$ and $\kappa^*(\varepsilon)$, as well as $i^*(\varepsilon)$.

### 2.2 Regularized Score Functions

Huber’s $(M)$ estimator of regression uses, for some fixed $\lambda > 0$,

$$\rho_\lambda(z) = \begin{cases} z^2/2 & \text{if } |z| \leq \lambda, \\ \lambda |z| - \lambda^2/2 & \text{otherwise}. \end{cases} \quad (8)$$

Huber’s $\rho$ is quadratic in the middle, has linear tails, and is continuous with a continuous derivative. This is straight out of Huber’s theory for the location problem, so no-one should be confused by the switch from $\kappa$ to $\lambda$ to denote the threshold for transition from quadratic to linear; it simply is convenient below to use $\lambda$ rather than $\kappa$ in the regression case.

For the AMP algorithm discussed below, we need the family of regularized $\rho$-functions, where for each regularization parameter $r > 0$,

$$\rho(z; r) \equiv \min_{x \in \mathbb{R}} \left\{ r\rho_\lambda(x) + \frac{1}{2}(x - z)^2 \right\}. \quad (9)$$

Associated to this is a regularized score function $\Psi(z) = \Psi(z; r)$. [DM13] writes it in terms of Huber’s original score $\psi_\lambda$:

$$\Psi_\lambda(z; r) = r \cdot \psi_\lambda\left( \frac{z}{1+r} \right). \quad (10)$$

\textsuperscript{10} For example, $i^*(\varepsilon) = j(\kappa^*(\varepsilon), \varepsilon)$, where $j(\kappa, \varepsilon) = (1 - \varepsilon) \int_{-\kappa}^{\kappa} x^2 \phi(x)dx + \kappa^2 \cdot (\varepsilon + (1 - \varepsilon) \cdot 2 \cdot \Phi(-\kappa))$ and $\kappa^*(\varepsilon) = \arg\min_{\kappa} j(\kappa, \varepsilon)$. 

9
In particular the shape of each $\Psi$ is similar to $\psi$, but the slope of the central part is now $\|\Psi'(\cdot; r)\|_\infty = \frac{r}{1+r} < 1$.

As explained in [DM13], although one uses the Huber $\psi$ as the basis of a high-dimensional regression estimation, the effective score function of that ($M$)-estimator belongs to the family $\Psi(\cdot; r)$, for a particular choice of $r$, defined below.

2.3 AMP algorithm

The approximate message passing (AMP) algorithm we proposed in [DM13] for the optimization problem (4) is iterative, starting at iteration 0 with an initial estimate $\hat{\theta}^0 \in \mathbb{R}^p$. At iteration $t = 0, 1, 2, \ldots$ it applies a simple procedure to update its estimate $\hat{\theta}^t \in \mathbb{R}^p$, producing $\hat{\theta}^{t+1}$. The procedure involves three steps at each iteration.

Adjusted residuals. Using the current estimate $\hat{\theta}^t$, we compute the vector of adjusted residuals $R^t \in \mathbb{R}^n$,

\begin{equation}
R^t = Y - \hat{X} \hat{\theta}^t + \Psi(R^{t-1}; r_{t-1}); \tag{11}
\end{equation}

where to the ordinary residuals $Y - \hat{X} \hat{\theta}^t$ we here add the extra term\footnote{Here and below, given $f : \mathbb{R} \to \mathbb{R}$ and $v = (v_1, \ldots, v_m)^T \in \mathbb{R}^m$, we define $f(v) \in \mathbb{R}^m$ by applying $f$ coordinate-wise to $v$, i.e. $f(v) \equiv (f(v_1), \ldots, f(v_m))^T$.} $\Psi(R^{t-1}; r_{t-1})$.\footnotetext{Here and below, given $f : \mathbb{R} \to \mathbb{R}$ and $v = (v_1, \ldots, v_m)^T \in \mathbb{R}^m$, we define $f(v) \in \mathbb{R}^m$ by applying $f$ coordinate-wise to $v$, i.e. $f(v) \equiv (f(v_1), \ldots, f(v_m))^T$.}
Effective Score. We choose a scalar \( r_t > 0 \), so that the effective score \( \Psi(\cdot; r_t) \) has empirical average slope \( p/n \in (0, 1) \). Setting \( m = m(n) = n/p > 1 \), we take any solution\(^{12}\) (for instance the smallest solution) to

\[
\frac{1}{m} = \frac{1}{n} \sum_{i=1}^{n} \Psi'(R^t_i; r). \tag{12}
\]

Scoring. We apply the effective score function \( \Psi(R^t_i; r_t) \):

\[
\hat{\theta}^{t+1} = \hat{\theta}^t + mX^T \Psi(R^t_i; r_t). \tag{13}
\]

We emphasize that the above procedure, although presented as an algorithm, will in fact be used simply a tool in proving results about \((M)\)-estimates.

2.4 State evolution description of AMP

State Evolution (SE) is a formal procedure for computing the operating characteristics of the AMP iterates \( \hat{\theta}^t \) and \( R^t \) for arbitrary fixed \( t \), under the \( PL(m) \) asymptotic \( n,p \to \infty \), \( n/p \to m \). The ideas have been described at length in [DM13]. Namely, for the \( t \)-th iteration of AMP, consider the quantity

\[
\tau^2_t \equiv \lim_{n \to \infty} \frac{1}{pm} \|\hat{\theta}^t - \theta_0\|^2_2 = \frac{1}{m} \text{AMSE}(\hat{\theta}^t; \theta_0).
\]

SE offers a way to calculate \( \tau_t \) using \( \tau_{t-1} \), and by extension calculating the limiting AMSE \( m \lim_{t \to \infty} \tau^2_t \).

At the heart of State Evolution are the effective noise level \( \sigma_t = \sqrt{1 + \tau^2_t} \), which changes iteration by iteration as the statistical properties of the AMP iterates evolve; it reflects the combined impact on the estimation of a parameter of observational noise \( W \) with standard deviation 1 (on the uncontaminated data) together with estimation noise \( \tau \) that ‘leaks’ from the other estimated parameters.

Also there is the notion of the effective slope: the well-defined value \( r = \mathcal{R}(\tau; m, \lambda, F_W) \) giving the smallest solution \( r \geq 0 \) to

\[
\frac{1}{m} = \mathbb{E}\left\{ \Psi'_\lambda(W + \tau Z; r) \right\},
\]

where \( W \sim F_W \), and, independently, \( Z \sim N(0, 1) \). Informally, \( \mathcal{R} \) measures the value of the regularization parameter \( r \) that satisfies the population analog of the AMP empirical average slope condition\(^{12}\).

Similarly, define the variance map

\[
\mathcal{A}(\tau^2, r; F_W) = \mathbb{E}\left\{ \Psi^2_\lambda(W + \tau Z; r) \right\},
\]

\( \mathcal{A} \) measures the variance of the resulting effective score. Evidently, for \( r > 0 \), \( 0 \leq \mathcal{A}(\tau^2, r) \leq (\text{Var}(W) + \tau^2) \).

In the last two displays, the reader can see that extra Gaussian noise of variance \( \tau^2 \) is being added to the underlying noise \( W \).

\(^{12}\)This equation always admits at least one solution; cf [DM13] Proposition A.1]
**Definition 2.1.** State Evolution is an iterative process for computing the sequence of scalars \( \{\tau_t^2\}_{t \geq 0} \), starting from an initial condition \( \tau_0^2 \in \mathbb{R}_{\geq 0} \) following the recursion

\[
\tau_{t+1}^2 = m \cdot A(\tau_t^2, \mathcal{R}(\tau_t)) = m \cdot A(\tau_t^2, \mathcal{R}(\tau_t; m, \lambda, F_W; \kappa, F_W)).
\]

(14)

Defining \( T(\tau^2) = m \cdot A(\tau^2, \mathcal{R}(\tau)) \), we see that the evolution of \( \tau_t^2 \) follows the iterations of the map \( T \). In particular, we make these observations:

- \( T(0) > 0 \),
- \( T(\tau^2) \) is a continuous, nondecreasing function of \( \tau \).
- \( T(\tau^2) < c \cdot \tau^2 \) for some \( c \in (0, 1) \) and all sufficiently large \( \tau \).

As a consequence of Theorem 2.2 below, \( T \) has a unique fixed point \( \tau^2_\infty \), i.e.

\[
T(\tau^2_\infty) = \tau^2_\infty.
\]

If follows from the above properties that this fixed point is stable and attracts \( (\tau_t^2) \) from any starting value. Explicitly, for each initial value \( \tau_0 \in (0, \infty) \), the sequence defined for \( t = 1, 2, \ldots \) by \( \tau_t^2 = T(\tau_{t-1}^2) \) converges to the above fixed point:

\[
\tau_t^2 \to \tau^2_\infty, \quad \text{as } t \to \infty.
\]

**2.5 Correctness of State Evolution**

The paper [DM13] considers \((M)\) estimates with strongly convex \( \psi \)-functions – this excludes the Huber estimator for technical reasons. In that paper, [DM13, Theorem 3.1] shows that State Evolution correctly computes the operating characteristics of the AMP algorithm. In particular, the AMP algorithm has \( m \cdot \tau^2_\infty \) for its \( t \to \infty \) limiting AMSE in estimating \( \theta_0 \).

Within the strongly convex setting, [DM13, Theorem 4.1] shows that the AMP algorithm converges in mean square to the \((M)\)-estimator, which is therefore also described by the fixed point of State Evolution.

Define the asymptotic variance of the \((M)\)-estimator \( \hat{\theta} \) by

\[
\text{AVar}(\hat{\theta}) = \lim_{n,p \to \infty} \text{Ave}_{i \in [p]} \text{Var}(\hat{\theta}_i),
\]

where \( \text{Ave}_{i \in [p]} \) denotes the average across indices \( i \). [DM13, Corollary 4.2] shows that the asymptotic variance of \( \hat{\theta} \) obeys

\[
\text{AVar}(\hat{\theta}_i) = m \tau^2_\infty.
\]

(15)

It follows that State Evolution describes not only the operating characteristics of the large \( t \)-limit of the AMP algorithm, but any algorithm for obtaining the \((M)\)-estimate in the PL\((m)\) asymptotic. So the fixed point of the one-dimensional dynamical system \( \tau^2 \mapsto T(\tau^2) \) is fundamental.

All these results extend to the Huber estimator itself. The companion paper [DM15] proves the following extension of the results in [DM13].

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Theorem 2.2. [DM15] Suppose that \((\tau_\infty, r_\infty)\) solve the two equations

\[
\frac{1}{m} = \mathbb{E}\left\{ \Psi'_\lambda(W + \tau Z; r) \right\},
\]

\[
\tau^2 = m \mathbb{E}\left\{ \Psi^2_\lambda(W + \tau Z; r) \right\}.
\]

Then under the PL(m)-limit, the Huber \((M)\)-estimator \(\hat{\theta} = (\hat{\theta}_i)_i\) obeys:

\[
\text{AVar}(\hat{\theta}) = m \cdot \tau^2_\infty.
\]

In particular, this implies that such fixed point is unique.

We note that, at the fixed point \((\tau_\infty, r_\infty)\), we have

\[
\text{AVar}(\hat{\theta}) = \mathbb{E}\left\{ \Psi^2_\lambda(W + \tau Z; r) \right\} \mathbb{E}\left\{ \Psi'_\lambda(W + \tau Z; r) \right\}^2.
\]

the expression on the RHS can be written in terms of Huber’s asymptotic variance formula \(V(\Psi_\lambda; \cdot, r)\) evaluated at the effective score function \(\Psi_\lambda(\cdot; r)\) with respect to the effective error distribution \(F_W \ast N(0, \tau^2_\infty)\); see [EKBBL13] for another approach to this formula.

3 Least-Favorable State Evolution

In this section we develop an upper bound on the behavior of State Evolution. We first introduce a variant of SE, in which \(\lambda\) evolves rather than staying fixed. This variant can be conveniently analyzed. In a later section, we tie the results obtained for this evolution to the original state evolution.

3.1 Floating-Threshold State Evolution

Recall the notion of effective noise level \(\sigma_t = \sqrt{1 + \tau^2_t}\) in state evolution, and consider a variant of SE where the threshold parameter \(\lambda_t\) ‘floats’ proportionally to the noise level \(\sigma_t\), as follows \(\lambda_t = \kappa \cdot \sigma_t\). Here \(\kappa\) may be viewed as the capping parameter for data which are presumed to be standardized, and so the floating \(\lambda_t\) is actually invariant across iteration – when expressed in multiples \(\kappa\) of the effective noise level.

In an abuse of notation, define \(A\) with a \(\kappa\) (rather than \(\lambda\)) as argument to be the variance map, based on floating \(\lambda\):

\[
A(\tau^2, r; \kappa, F_W) = \mathbb{E}\left\{ \Psi^2_{\kappa, \sigma}(W + \tau Z; r) \right\}.
\]

Compounding the abuse, define \(r = R(\tau; m, \kappa, F_W)\) analogously, so that

\[
\frac{1}{m} = \mathbb{E}\left\{ \Psi'_{\kappa, \sigma}(W + \tau Z; r) \right\}.
\]

Similarly, we define \(T(\tau^2; m, \kappa, F_W) = m \cdot A(\tau^2, R(\tau; m, \kappa, F_W))\), without any warning to the reader that the same symbols are being used as in the earlier state evolution with fixed \(\lambda\) while here and
above the appearance of \( \kappa \) in the argument always refers to the floating \( \lambda \) evolution. For example, we might write \( \tau^2_\infty(m, \kappa, F) \) for the fixed point of a floating-\( \lambda \) evolution and \( \tau^2_\infty(m, \lambda, F) \) for the (in general different) fixed point of a fixed-\( \lambda \) evolution. As a first justification for this, note that the fixed points of the two different dynamical systems (fixed- and floating- \( \lambda \) dynamical systems) are in one-one correspondence, via

\[
\lambda = \kappa \cdot \sqrt{1 + \tau^2_\infty},
\]

i.e.

- The fixed-\( \lambda \) fixed point \( \tau^2_\infty(\lambda) \) is identical to the floating-\( \lambda \) fixed point \( \tau^2_\infty(\kappa_\infty(\lambda)) \), under the floating-\( \lambda \) parameter \( \kappa_\infty(\lambda) = \lambda / \sqrt{1 + \tau^2_\infty(\lambda)} \); while

- The floating-\( \lambda \) fixed-point \( \tau^2_\infty(\kappa) \) is identical to the fixed-\( \lambda \) fixed point \( \tau^2_\infty(\lambda_\infty(\kappa)) \) at parameter \( \lambda_\infty = \kappa \cdot \sqrt{1 + \tau^2_\infty(\kappa)} \).

Setting \( \lambda = \lambda_\infty(m, \kappa, F_\varepsilon) \) and \( \kappa = \kappa_\infty(m, \lambda, F_\varepsilon) \) establishes the correspondence. Hence characterizing the fixed points of the floating \( \lambda \) scheme will also characterize those of the fixed lambda scheme; see also Definition 5.1 et seq. below.

### 3.2 Least-Favorable SE

Let \( H_\infty \) denote the improper distribution with its probability mass placed evenly on \( \{\pm \infty\} \); with this notation, set \( F_\varepsilon = (1 - \varepsilon) \Phi + \varepsilon H_\infty \). We now describe an extremal form of floating-threshold state evolution.

**Definition 3.1.** Least Favorable State Evolution (LFSE) is an iterative process for computing a sequence of scalars \( \{\tau^2_t\}_{t \geq 0} \), starting from an initial condition \( \tau^2_0 \in \mathbb{R}_{\geq 0} \). An instance of LFSE is determined by \( \tau^2_0 \) together with fixed positive scalars \( m, \kappa \) and \( \varepsilon \).

At the \( t \)-th iteration, one needs the \( (t-1) \)'th result \( \tau_{t-1} \) and sets

\[
\bar{\tau}_t = \mathcal{R}(\tau_{t-1}; m, \kappa, F_\varepsilon), \\
\bar{\tau}^2_t = m \cdot \mathcal{A}(\tau_{t-1}, \bar{\tau}_t; \kappa, F_\varepsilon)
\]

The procedure is then repeated at the next iteration \( t + 1 \), and so on.

Letting \( \Phi_\sigma \) denote the CDF for \( N(0, \sigma^2) \), let \( F_{\varepsilon, \sigma} = (1 - \varepsilon) \Phi_\sigma + \varepsilon H_\infty \), and define an improper random variable \( X_{\varepsilon, \sigma} \sim F_{\varepsilon, \sigma} \), taking infinite values with positive probability. Setting \( \sigma = (1 + \tau^2)^{1/2} \), we have \( F_{\varepsilon, \sigma} = F_\varepsilon \ast \Phi_\sigma \). Definition 3.1, written in terms of the improper random variable \( X_{\varepsilon, \sigma_t-1} \), and the floating threshold \( \lambda_t = \kappa \cdot \bar{\sigma}_{t-1} \), gives:

\[
\frac{1}{m} = \mathbb{E} \Psi_{\lambda_t}(X_{\varepsilon, \sigma_{t-1}}, \bar{\tau}_t),
\]

and

\[
\bar{\tau}^2_t = m \cdot \mathbb{E} \Psi^2_{\lambda_t}(X_{\varepsilon, \sigma_{t-1}}, \bar{\tau}_t).
\]

Although \( X_{\varepsilon, \sigma_{t-1}} \) is an improper random variable, these expectations are well defined\(^{13}\). We refer to instances where state evolution is applied to proper distributions in \( F_\varepsilon \) as proper state evolutions.

\(^{13}\)given the boundedness and differentiability of the underlying Huber \( \psi \)
Lemma 3.2. (LFSE Dominates.) Consider a given instance \((m, \tau_0, \kappa, F)\) of floating-threshold state evolution where \(F \in F_\varepsilon\). The LFSE instance \((m, \tau_0, \kappa, \varepsilon)\) dominates this proper state evolution, namely: with \(\bar{\tau}_t\) the sequence of LFSE regularizing parameters and \(\tau_t\) the sequence of proper SE regularizing parameters,

\[ \bar{\tau}_t \geq \tau_t, \quad t = 1, 2, \ldots, \]

while for \(\bar{\tau}_t^2\) the MSE under LFSE and \(\tau_t^2\) under proper SE, respectively, we have:

\[ \bar{\tau}_t^2 \geq \tau_t^2, \quad t = 0, 1, 2, \ldots \]

Figure 4 illustrates the dominance of LFSE; it shows that the corresponding dynamical maps obey \(\bar{T} \geq T\).

The proof - given in the appendix - will depend on the following sequence of observations:

Lemma 3.3. Monotonicity in \(x\), \(r\), and \(\lambda\). Let \(\Psi(x, r)\) denote the regularized score function based on Huber’s \(\psi_\lambda\). (With \(\lambda\) fixed unless stated otherwise.)

1. For each fixed \(r \in \mathbb{R}_+\), \(\Psi_\lambda(x, r)\) is a monotone increasing function of \(|x|\);
2. For each fixed \(r \in \mathbb{R}_+\), \(\Psi_\lambda'(x, r)\) is a monotone nonincreasing function of \(|x|\);
3. For each fixed \(x \in \mathbb{R}\); \(r \mapsto |\Psi_\lambda(x, r)|\) is monotone nondecreasing in \(r\); and
4. For each fixed \(x \in \mathbb{R}\), \(\lambda \mapsto |\Psi_\lambda(x, r)|\) is monotone nondecreasing in \(\lambda\).

It will also need the following invariances, which are very special to the extremal improper RV’s \(\bar{X}_{\varepsilon, \sigma}\) and \(\bar{X}_{\varepsilon, \sigma}\) together with the fact that the proper SE and LFSE use exactly the same \(\kappa\) in forming their respective floating \(\lambda\)’s.

Lemma 3.4. For \(r > 0\), and \(t \geq 1\), let \(0 < \sigma_{t-1} < \sigma_{t-1}\) and \(\lambda_t = \kappa \sigma_{t-1}\), and \(\bar{\lambda}_t = \kappa \cdot \bar{\sigma}_{t-1}\).

\[
\mathbb{E} \Psi_\lambda'(\bar{X}_{\varepsilon, \sigma_{t-1}}, r) = \mathbb{E} \Psi_\lambda'(\bar{X}_{\varepsilon, \bar{\sigma}_{t-1}}, r);
\]

\[
\mathbb{E} \Psi_\lambda^2(X_{\varepsilon, \sigma_{t-1}}, r) = \left(\frac{\sigma_{t-1}}{\sigma_{t-1}}\right)^2 \cdot \mathbb{E} \Psi_\lambda^2(X_{\varepsilon, \bar{\sigma}_{t-1}}, r).
\]

3.3 The envelope functionals \(\bar{A}\) and \(\bar{B}\)

To make LFSE more transparent, we introduce some helpful notation. In this subsection, we are again in Huber’s original location setting. The evaluation of \(v^*(\varepsilon)\) is made significantly easier by helpful notation. Suppose that \(F\) is a sub distribution, i.e. a CDF on the extended reals, and put

\[
A(\psi_\kappa, F) = \int_{-\infty}^{\infty} \psi_\kappa^2(w)dF(w) = \mathbb{E}_F \psi_\kappa^2(W)
\]

\[
B(\psi_\kappa, F) = \int_{-\infty}^{\infty} \psi_\kappa'(w)dF(w) = \mathbb{E}_F \psi_\kappa'(W)
\]

where \(W \sim F\). Calculating explicitly for the Huber score function, we can equally well write

\[
A(\psi_\kappa, F) = \int_{-\kappa}^{\kappa} w^2dF(w) + \kappa^2 \cdot \mathbb{P}_F\{|W| \geq \kappa\}.
\]
\[
B(\psi, F) = \mathbb{P}_F\{|W| \leq \kappa\}.
\]

Now define the envelope functions \(\tilde{A}\) and \(\tilde{B}\), so that
\[
\tilde{A}(\kappa, \varepsilon) = \sup\{A(\psi, F) : F \in \mathcal{F}_\varepsilon\}, \quad (20)
\]
\[
\tilde{B}(\kappa, \varepsilon) = \inf\{B(\psi, F) : F \in \mathcal{F}_\varepsilon\}. \quad (21)
\]
More explicitly, with \(\Phi\) denoting the standard normal CDF, and \(Z \sim N(0, 1)\),
\[
\tilde{A}(\kappa, \varepsilon) = (1 - \varepsilon)A(\psi, \Phi) + \varepsilon\kappa^2, \quad (22)
\]
\[
\tilde{B}(\kappa, \varepsilon) = \mathbb{P}_F\{|Z| \leq \kappa\} = 2\Phi(-|\kappa|) - 1. \quad (23)
\]
Defining \(V(\psi, F) = A(\psi, F)/B^2(\psi, F)\), and correspondingly,
\[
\tilde{V}(\kappa, \varepsilon) = \sup\{V(\psi, F) : F \in \mathcal{F}_\varepsilon\}.
\]
It follows from Huber(1964) that
\[
\tilde{V}(\kappa, \varepsilon) = \frac{\tilde{A}(\kappa, \varepsilon)}{\tilde{B}^2(\kappa, \varepsilon)},
\]
(the inequality LHS \(\leq\) RHS is obvious) and also that
\[
v^*(\varepsilon) = \inf_{\kappa} \tilde{V}(\kappa, \varepsilon);
\]
(the inequality LHS \(\leq\) RHS again being immediate).

### 3.4 Explicit Solution of Least Favorable State Evolution

We now put Huber’s notation from the previous subsection to work, giving explicit formulas for LFSE.

**Lemma 3.5.** For a given tuple \((m, \varepsilon, \kappa)\) obeying \((1 - \varepsilon) > 1/m\), there is a unique positive solution \(\tilde{r}(m, \varepsilon, \kappa)\) to
\[
\left(\frac{\tilde{r}}{1 + \tilde{r}}\right) \cdot \tilde{B}(\kappa \cdot (1 + \tilde{r}), \varepsilon) = \frac{1}{m}. \quad (24)
\]

Using this notation, we give an explicit characterization of LFSE. Let \(\tilde{\kappa} = \tilde{\kappa}(m, \varepsilon, \kappa) = \kappa \cdot (1 + \tilde{r})\) as in the first argument of \(\tilde{B}\) in [24].

**Lemma 3.6.** LFSE with parameters \((m, \varepsilon, \kappa)\) satisfies, with \(\tilde{\kappa} = \tilde{\kappa}(m, \varepsilon, \kappa)\)
\[
\bar{T}(\tau^2; m, \varepsilon, \kappa) = (1 + \tau^2) \cdot \tilde{V}(\tilde{\kappa}, \varepsilon)/m,
\]
and, if \(\tilde{V}(\tilde{\kappa}, \varepsilon) < m\), LFSE has the unique stable fixed point
\[
\tilde{\tau}_\infty^2(m, \varepsilon, \kappa) = \frac{\tilde{V}(\tilde{\kappa}, \varepsilon)/m}{1 - \tilde{V}(\tilde{\kappa}, \varepsilon)/m}.
\]
To prove this, consider a seemingly different evolution, which we call double-bar evolution: with \( \bar{r} \) as introduced above, define

\[
\bar{A}(\tau^2; m, \varepsilon, \kappa) = (1 + \tau^2) \cdot \bar{V}(\bar{\kappa}, \varepsilon)/m^2
\]

and

\[
\bar{T}(\tau^2) = m \cdot \bar{A}(\tau^2).
\]

With \( m, \varepsilon, \kappa \) and thus \( \bar{r} \) and \( \bar{\kappa} \) fixed, define a sequence \( \bar{\tau}_t^2 \) for \( t = 0, 1, 2, \ldots \). At iteration \( t = 0 \), we pick a starting value \( \bar{\tau}_0 \geq 0 \), we then proceed inductively, setting all later iterates by:

\[
\bar{\tau}_t^2 = \bar{T}(\bar{\tau}_{t-1}^2), \quad t = 1, 2, \ldots
\]

Now (25) sets up the dynamical system \( \bar{\tau}^2 \mapsto \bar{T}(\bar{\tau}^2) \) as an affine dynamical system (in the variable \( \bar{\tau}^2 \)). Its fixed point (if it exists at all) must obey

\[
\bar{\tau}_\infty^2 = (1 + \bar{\tau}_\infty^2) \cdot \bar{V}(\bar{\kappa}, \varepsilon)/m.
\]

So double-bar evolution has the following explicit solution:

**Lemma 3.7.** Consider the double-bar evolution introduced in this section, with parameters \((m, \varepsilon, \kappa)\). If \( \bar{V}(\bar{\kappa}, \varepsilon) < m \), it has the unique stable fixed point

\[
\bar{\tau}_\infty^2(m, \varepsilon, \kappa) = \frac{\bar{V}(\bar{\kappa}, \varepsilon)/m}{1 - \bar{V}(\bar{\kappa}, \varepsilon)/m}.
\]

Otherwise there is no fixed point, and successive iterates run off to infinity.

In fact, double-bar evolution is really just LFSE, in disguise. Results of the next subsection will prove:

**Lemma 3.8.** With \((\bar{r}_t)\) and \((\bar{\tau}_t)\) defined by the procedure of Section 3.2, and \((\bar{\tau}_t)\) defined by the procedure of this section, each initialized identically \( \bar{\tau}_0 = \bar{\tau}_0 \) – we have

\[
\bar{r}_t = \bar{r}, \quad t = 0, 1, 2, \ldots,
\]

\[
\bar{\tau}_t = \bar{\tau}_t, \quad t = 0, 1, 2, \ldots,
\]

and

\[
\bar{T}(\cdot) = \bar{T}(\cdot).
\]

**Lemma 3.6** then follows from the last two lemmas. In turn, **Lemma 3.8** follows immediately from the following:

**Lemma 3.9.**

\[
\sup_{F \in \mathcal{F}} R(\tau; m, \kappa, F) = \bar{r}(m, \varepsilon, \kappa),
\]

\[
\sup_{F \in \mathcal{F}} A(\tau^2, R(\tau; m, \kappa, F); m, \kappa, F) = \bar{A}(\tau^2; m, \varepsilon, \kappa).
\]

This shows that the affine evolution (25) indeed implements LFSE, and proves Lemma 3.8.

The proof of Lemma 3.9 is given in the Appendix; it depends on terminology and results of the next subsection.
Figure 4: MSE maps of proper state evolutions and of LFSE. Here $\varepsilon = 0.05$, $m = 5$, and $\mu = 2, 5, 7.5, 10$. The variance map of LFSE is the green straight line, which lies above the variance maps of all the proper SE’s as depicted by red curves. Correspondingly, its fixed point is also higher.

3.5 Bounds for $\mathcal{A}$

The quantity $\mathcal{A}$ occurring in LFSE is defined using moments of $E\Psi^2$; however, Section 3.4 defines $\bar{\mathcal{A}}$ in terms of $\tilde{\mathcal{A}}$, which uses moments of $\psi^2$. To explain the connection – and prove Lemma 3.8 – we need to relate the two kinds of moments.

Indeed, (10) says that

$$\Psi(z; r) = \frac{r}{1+r} \psi_{\lambda(1+r)}(z),$$

and we also have, for any random variable $X$,

$$E\psi^2_{\lambda}(cX) = c^2 E\psi^2_{\lambda/c}(X),$$

while

$$E\psi'_{\lambda}(cX) = E\psi'_{\lambda/c}(X).$$

Furthermore, supposing that $W$ has distribution $(1 - \varepsilon)\Phi + \varepsilon H$ and that $Z \sim \Phi$ while $U \sim H$, then

$$E|\Psi(W + \tau Z; r)|^2 = (1 - \varepsilon) E|\Psi(\sqrt{1 + \tau^2} \cdot Z; r)|^2 + \varepsilon \cdot E|\Psi(U + \tau Z; r)|^2.$$  

Now introducing $a \equiv \sqrt{1 + \tau^2}/(1 + r)$ where $r$ is some fixed positive scalar kept the same in all the coming displays,

$$E|\Psi(\sqrt{1 + \tau^2} \cdot Z; r)|^2 = (ar)^2 \cdot E\psi^2_{\lambda/a}(Z) = (ar)^2 A(\lambda/a, \Phi).$$
We have the upper bound
\[ \mathbb{E}\Psi(U + \tau \cdot Z; r)^2 = \left( \frac{r}{1 + r} \right)^2 \mathbb{E}\psi_{\lambda(1+r)}^2(U + \tau Z) = \left( \frac{r}{1 + r} \right)^2 A(\lambda \cdot (1 + r), H \ast \Phi_r). \]

Similarly,
\[ \mathbb{E}\Psi'(W + \tau Z; r) = (1 - \varepsilon)\mathbb{E}\Psi'(!1 + \tau^2 \cdot Z; r) + \varepsilon \cdot \mathbb{E}\Psi'(U + \tau Z; r); \]
and
\[ \mathbb{E}\Psi'(!1 + \tau^2 \cdot Z; r) = \left( \frac{r}{1 + r} \right) \cdot \mathbb{E}\psi_{\lambda(1+r)}'(!1 + \tau^2 \cdot Z). \]

But \( \mathbb{E}\psi_{\lambda/a}'(Z) = B(\lambda/a, \Phi) \); so
\[ \mathbb{E}\Psi'(\sqrt{!1 + \tau^2} \cdot Z; r) = \left( \frac{r}{1 + r} \right) \cdot B(\lambda/a, \Phi). \]

We have the upper bound \( A(\lambda \cdot (1 + r), H \ast \Phi_r) \leq (\lambda \cdot (1 + r))^2 \) because \( \|\psi_\kappa\|_\infty = \kappa \), and the lower bound \( B(\lambda \cdot (1 + r), H \ast \Phi_r) \geq 0 \) because \( \psi_\kappa' \geq 0 \). Moreover, both bounds are tight, as can be seen by choosing the point mass with \( H = \delta_\mu \) as \( \mu \to \infty \). Combining all the above, we obtain the following.

**Lemma 3.10.** With \( r > 0 \), \( a \equiv \sqrt{!1 + \tau^2}/(1 + r) \), and \( F = (1 - \varepsilon)\Phi + \varepsilon H \),
\[
A(\tau^2, r; m, \kappa, F) = (1 - \varepsilon)(ar)^2 A(\kappa \cdot (1 + r), \Phi) + \varepsilon \cdot \left( \frac{r}{1 + r} \right)^2 A(\lambda \cdot (1 + r), H \ast \Phi_r),
\]
\[
A(\tau^2, r; m, \kappa, F) \leq (ar)^2 \cdot \bar{A}(\kappa \cdot (1 + r), \varepsilon),
\]
\[
A(\tau^2, r; m, \kappa, F) \to (ar)^2 \cdot \bar{A}(\kappa \cdot (1 + r), \varepsilon), \quad H = \delta_\mu, \quad \mu \to \infty.
\]

**Lemma 3.11.** With \( B = \mathbb{E}\Psi'(W + \tau Z; r) \) and \( W \sim F = (1 - \varepsilon)\Phi + \varepsilon H \),
\[
B(\tau^2, r; m, \kappa, F) = \left( \frac{r}{1 + r} \right) \cdot \left( (1 - \varepsilon)B(\kappa \cdot (1 + r), \Phi) + \varepsilon \cdot \mathbb{E}\psi_{\lambda(1+r)}'(U + \tau Z) \right),
\]
\[
B(\tau^2, r; m, \kappa, F) \geq \left( \frac{r}{1 + r} \right) \cdot \bar{B}(\kappa \cdot (1 + r), \varepsilon),
\]
\[
B(\tau^2, r; m, \kappa, F) \to \left( \frac{r}{1 + r} \right) \cdot \bar{B}(\kappa \cdot (1 + r), \varepsilon), \quad H = \delta_\mu, \quad \mu \to \infty.
\]

The proof of Lemma 3.9 in the Appendix, combines the last two lemmas to obtain the equivalence of LFSE and double-bar evolution.

### 4 Minimax Asymptotic Variance of Floating Threshold SE

#### 4.1 Minimax Formal Variance

**Definition 4.1.** Define the formal variance
\[ \mathcal{V}_m(\kappa, F) = m \cdot \tau_\infty^2(m, \kappa, F). \]

where \( \tau_\infty^2(m, \kappa, F) \) denotes the fixed point of the associated floating-threshold State Evolution.

Define the minimax formal variance to be
\[ \mathcal{V}_m(\varepsilon) = \inf_{\kappa} \sup_{F \in F_\varepsilon} \mathcal{V}_m(\kappa, F). \]
The minimax problem identifies a distinguished choice of the capping parameter, offering the best guarantee applicable across all $F \in \mathcal{F}_\varepsilon$. Here is the solution:

**Lemma 4.2.** The mapping $\kappa \mapsto \bar{r}(\kappa; m, \varepsilon)$ is continuous and strictly monotone decreasing. For each $\bar{\kappa} > 0$, the equation

$$\bar{\kappa} = \kappa \cdot (1 + \bar{r}(\kappa))$$

has an unique solution $\kappa = \kappa(\bar{\kappa})$.

**Theorem 4.3.** Let $\kappa^*(\varepsilon)$ denote Huber’s minimax capping parameter in the scalar estimation problem [Hub64]. Let $\kappa(\cdot)$ denote the re-calibrated function defined by Lemma 4.2. Define the re-calibrated parameter

$$\kappa^*(\varepsilon) = \kappa(\kappa^*(\varepsilon)).$$

Suppose that $m \cdot I(F^*_\varepsilon) > 1$; then every instance of floating-threshold state evolution having parameters $(m, \tau_0, \kappa^*(\varepsilon), F)$ with proper $F \in \mathcal{F}_\varepsilon$ has a fixed point at $\tau^2_\infty \equiv \tau^2_\infty(m, \kappa^*(\varepsilon), F)$ obeying

$$\tau^2_\infty \leq \tau^2_\infty(m, \varepsilon, \kappa^*(\varepsilon)) \equiv \frac{v^*(\varepsilon)/m}{1 - v^*(\varepsilon)/m}.$$

More specifically, we have the saddlepoint relation:

$$\inf_{\kappa} \sup_{F \in \mathcal{F}_\varepsilon} \mathcal{V}_m(\kappa, F) = \mathcal{V}_m(\kappa^*(\varepsilon), \bar{F}_\varepsilon) = \sup_{F \in \mathcal{F}_\varepsilon} \inf_{\kappa} \mathcal{V}_m(\kappa, F),$$

with saddle point at $(\kappa^*(\varepsilon), \bar{F}_\varepsilon)$, and where the minimax value $\mathcal{V}^*_m(\varepsilon) = \mathcal{V}_m(\kappa^*(\varepsilon), \bar{F}_\varepsilon)$ obeys:

$$\mathcal{V}^*_m(\varepsilon) = \frac{v^*(\varepsilon)}{1 - v^*(\varepsilon)/m} \equiv \frac{1}{I(F^*_\varepsilon) - 1/m}.$$

Figure 4 presents a diagram showing contours of $\mathcal{V}^*_m(\varepsilon)$. The diagram employs the unit square $\{(\varepsilon, 1/m) : 0 \leq \varepsilon, 1/m \leq 1\}$ where the $x$-axis shows the contamination fraction $\varepsilon$, and the $y$ axis shows $1/m$ for plotting purposes. Only the part of the diagram where $1/m < I(F^*_\varepsilon)$ is populated with contours. The reader can see how the asymptotic variance ‘blows up’ as $1/m$ approaches $I(F^*_\varepsilon)$.

Figure 5 shows contours of the minimax capping parameter $\kappa^*(\varepsilon; m)$. The reader can see how the capping parameter shrinks to zero as $1/m$ approaches $I(F^*_\varepsilon)$.

4.2 State Evolution in the Unbounded Phase

Figure 4 has a ‘bounded’ phase, where the formal variance is bounded across all contaminating distributions, and a complementary so-far undescribed phase. It seems that the formal variance must be unbounded in this phase, since the phase consist of cases with smaller $m$ than the bounded ones, and so therefore of ‘harder’ cases. Validating this intuition, we have:

**Corollary 4.4.** Suppose that $m \cdot I(F^*_\varepsilon) \leq 1$; then for each $\tau < \infty$, and each $\kappa > 0$, some instance of proper state evolution with parameters $(m, \tau_0, \kappa, F)$ and proper $F \in \mathcal{F}_\varepsilon$ has a unique fixed point at $\tau^2_\infty \equiv \tau^2_\infty(m, \kappa, F)$ obeying

$$\tau^2_\infty \geq \tau^2.$$
Goings-on in the unbounded phase are documented in Figure 6. In the unbounded phase, every LFSE map \( \hat{T} \) has no fixed point, whatever be the parameter \( \kappa \). Proper state evolutions still have unique stable fixed points, but there is no upper bound on their size. Hence the worst-case fixed point \( \bar{\tau}_2^\infty \) is infinite.

This is an instance of what Donoho and Huber [DH83] called breakdown of asymptotic variance. Breakdown occurs, in the \((\varepsilon, 1/m)\) phase diagram, where-ever \( mI(F^*_\varepsilon) \leq 1 \), and the breakdown point is \( mI(F^*_\varepsilon) = 1 \), the dashdot curve in our figures.

Note that as \( m \to \infty \), we converge to the classical case, where the asymptotic variance of \((M)\)-estimates does not break down. In the high-dimensional case \( n/p \to m \in (1, \infty) \), the asymptotic variance does break down.

5 Minimax Variance of the Huber \((M)\)-estimates

We now develop our main result about \((M)\)-estimates. The analysis in the last section concerns floating-\(\lambda\) state evolution; while Theorem 2.2 shows that fixed-\(\lambda\) state evolution describes the asymptotic variance of the Huber \((M)\)-estimate. We show how to bridge this difference.

5.1 Minimax Formal Variance

Definition 5.1. Calibration Relation. Suppose the proper floating threshold state evolution with parameters \((m, \tau_0, \kappa, F)\) has a unique fixed point \( \bar{\tau}_2^\infty \). We formally associate this to a Huber \((M)\)-
Figure 6: SE in the unbounded phase. Examples of proper state evolutions with \( \mu = 2, 5, 7.5, 10, \varepsilon = 0.05, m = 5 \). The LFSE dynamical system has no fixed point. The proper SE’s have fixed points, but the location of the fixed point is unbounded above.

Estimate in the linear model under asymptotic regime \( PL(m) \) with parameter \( \lambda \) satisfying

\[
\lambda = \kappa \cdot \sqrt{1 + \tau_2^2(m, \kappa, F)}.
\]

We denote this correspondence by \( \lambda = \lambda_\infty(m, \kappa, F) \) and the inverse correspondence with \( \kappa = \kappa_\infty(m, \lambda, F) \).

**Definition 5.2.** The formal asymptotic variance of the Huber \((M)\)-estimator under the \( PL(m) \) asymptotic framework is

\[
V_m^\circ(\lambda, F) \equiv m \cdot \tau_\infty^2(m, \kappa, F),
\]

where \( \tau_\infty^2(m, \kappa, F) \) denotes the fixed point of the floating threshold state evolution with parameter \( \kappa \) and where \( \lambda = \lambda_\infty(m, \kappa, F) \).

Theorem 2.2 shows that this formula is rigorously correct – the Huber estimator with the specified parameter \( \lambda \) indeed has almost surely an asymptotic variance and it is equal to the formal asymptotic variance.

**Lemma 5.3.** Let \( \overline{V}_m(\kappa, \varepsilon) = \sup_{F \in \mathcal{F}_\varepsilon} V_m^\circ(\kappa, F) \) denote the worst case formal variance, across the full \( \varepsilon \)-neighborhood, of the floating-threshold state evolution fixed point under capping parameter \( \kappa \). Set

\[
\kappa^+(m, \varepsilon) = \sup \{ \kappa : \overline{V}_m(\kappa, \varepsilon) < \infty \};
\]
there is \( m_0(\varepsilon) \in (1, \infty) \) so that, for \( m > m_0(\varepsilon) \), we have \( \overline{V}_m(\kappa, \varepsilon) < \infty \) throughout \((0, \kappa^+(m, \varepsilon))\). Define
\[
\bar{\lambda}(\kappa; m, \varepsilon) = \sup_{F \in F_\varepsilon} \lambda_{\infty}(m, \kappa, F).
\]
For \( m > m_0(\varepsilon) \), the mapping
\[
\kappa \mapsto \bar{\lambda}(\kappa; m, \varepsilon)
\]
is strictly increasing for \( 0 < \kappa < \kappa^+(m, \varepsilon) \).

Figure 7 displays \( \bar{\lambda}(\kappa) \) for a variety of choices of \( \varepsilon, m \); the monotonicity is evident. Numerics show that we may take \( m_0 \equiv \pi/2 \); however our proof only attempts to show that some \( m_0 \) sufficiently large will work.

![Figure 7](image-url)

Figure 7: Monotonicity of \( \kappa \mapsto \bar{\lambda}(\kappa) \). Each subplot depicts \( \bar{\lambda}(\kappa; m, \varepsilon) \) as a function of \( \kappa \), for \( \varepsilon \in \{0.01, 0.02, 0.05, 0.10\} \), at one particular \( m \). Evidently, as \( m \to \infty \), \( \bar{\lambda} \to \kappa \).

The monotonicity condition on \( \bar{\lambda} \) ensures that the least-favorable contamination for the Huber (M)-estimator is achieved by the improper distribution \( \tilde{F}_\varepsilon \).

**Theorem 5.4. Evaluation of Minimax Asymptotic Variance of Huber (M)-estimator.** If the mapping \( \kappa \mapsto \bar{\lambda}(\kappa; m, \varepsilon) \) is strictly increasing for \( 0 < \kappa < \kappa^+ \) we have
\[
\inf_{\lambda} \sup_{F \in F_\varepsilon} \mathcal{V}_m^\circ(\lambda, F) = \inf_{\kappa} \sup_{F \in F_\varepsilon} \mathcal{V}_m^\circ(\kappa, F),
\]
where the minimax on the left concerns the formal variance of Huber (M)-estimates parametrized by \( \lambda \), and that on the right concerns the formal variance of floating-threshold state evolutions parametrized by \( \kappa \). The minimax tuning of the Huber (M)-estimator is achieved by the tuning parameter
\[
\lambda^*(\varepsilon) = \bar{\lambda}(m, \tilde{\kappa}^*(\varepsilon), \varepsilon).
\]
It follows of course that we have the formula
\[
\inf_{\lambda} \sup_{F \in F_{\varepsilon}} V_m^*(\lambda, F) = \frac{1}{I(F_{\varepsilon}^*) - 1/m},
\]
which agrees in the limit \( m \to \infty \) with Huber’s classical formula for the scalar location problem:
\[
\inf_{\lambda} \sup_{F \in F_{\varepsilon}} V(\psi, F) = \frac{1}{I(F_{\varepsilon}^*)}.
\]

Figure 2 shows contours of the minimax thresholding parameter \( \lambda^*(\varepsilon; m) \). The reader can see how this parameter shrinks to zero as \( 1/m \) approaches \( I(F_{\varepsilon}^*) \). While the story is much the same as for the \( \kappa \) parameter in Figure 5, the \( \lambda \)-parameter is the one relevant to practice, because the \( \kappa \) parameter is a theoretical construct while the corresponding \( \lambda \) parameter can actually be used to specify the desired Huber estimator in statistical software packages.

Since the formal variance \( V_m^*(\lambda, F) \) has the saddlepoint property, Theorem 2.2 shows that the rigorous asymptotic variance \( \text{AVar} = \text{AVar}(\hat{\theta}_n^\lambda, F) \) (say) has it as well.

**Definition 5.5.** Let \( F_{2\varepsilon}^2 \) denote the subset of distributions in \( F_{\varepsilon} \) with finite variance: \( \mu^2(F) = \int w^2 dF(w) < \infty \).

**Corollary 5.6.** Fix \( \varepsilon > 0 \), and \( m > m_0(\varepsilon) \). We are in the asymptotic regime \( PL(m) \).

- Suppose that \( V_m^*(\varepsilon) \) is finite. Consider the formally minimax parameter \( \lambda = \lambda^*(\varepsilon) \); let \( \hat{\theta}_n^* \) denote a corresponding solution of the Huber (M)-equation with that \( \lambda \). For every error distribution \( F \in F_{2\varepsilon}^2 \), we have
  \[
  \text{AVar}(\hat{\theta}_n^*, F) \leq V_m^*(\varepsilon).
  \]
  For every \( \lambda \neq \lambda^*(\varepsilon) \) there is a proper \( \varepsilon \)-contaminated normal error distribution \( F \in F_{2\varepsilon}^2 \) so that the Huber estimator \( \hat{\theta}_n^\lambda \) obeys
  \[
  \text{AVar}(\hat{\theta}_n^\lambda, F) > V_m^*(\varepsilon).
  \]
  Consequently,
  \[
  \inf_{\lambda} \sup_{F \in F_{2\varepsilon}^0} \text{AVar}(\hat{\theta}_n^\lambda, F) = \text{AVar}(\hat{\theta}_n^*, F_{\varepsilon}) = V_m^*(\varepsilon).
  \]
- Suppose that \( V_m^*(\varepsilon) \) is infinite. For every \( \lambda > 0 \) and each \( V > 0 \), there is a proper \( \varepsilon \)-contaminated normal error distribution \( F \in F_{2\varepsilon}^2 \) with
  \[
  \text{AVar}(\hat{\theta}_n^\lambda, F) > V.
  \]
  Consequently,
  \[
  \inf_{\lambda} \sup_{F \in F_{2\varepsilon}^2} \text{AVar}(\hat{\theta}_n^\lambda, F) = +\infty = V_m^*(\varepsilon).
  \]
6 Discussion

Under the high-dimensional $PL(m)$ asymptotic - as shown in [BBEKY13] - the maximum likelihood estimator is no longer an efficient estimator. It follows that the Huber estimator is no longer asymptotically minimax among all (M)-estimators. Hence the asymptotic minimax in (6) should better be called the asymptotic minimax among Huber estimates. The degree of sub optimality can be controlled explicitly. By [DM13, Corollary 3.7], the asymptotic variance under $PL(m)$ obeys the following inequality, which is strictly stronger than the Cramér-Rao bound when $1 < m < \infty$:

$$V_m(\psi, F) \geq \frac{1}{1 - 1/m} \cdot \frac{1}{I(F)},$$

and so the minimax asymptotic variance obeys:

$$\min_\lambda \max_{F \in \mathcal{F}} V_m(\psi, F) \geq \frac{1}{1 - 1/m} \cdot \frac{1}{I(F^*_\epsilon)}.$$  \hspace{2cm} (28)

It follows that provided $mI(F^*_\epsilon) > 1$,

$$\text{minimax Huber asymptotic variance} \leq K \cdot \text{minimax asymptotic variance},$$

where

$$K = K(m, \varepsilon) = \frac{1 - 1/m}{1 - I(F^*_\epsilon)/m}.$$  

One sees directly that the sub-optimality of the Huber estimator is well controlled provided that $I(F^*_\epsilon)$ is close to one; i.e., in the regime where $\varepsilon$ is small enough (though this is $m$-dependent). Of course in the regime $I(F^*_\epsilon)m \leq 1$, some other estimators could be dramatically more robust.

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Appendix: Proofs

Proof of Lemma 3.2

The desired relations are true for iteration $t = 0$ by assumption (note that no assertion about the sequence $(r_t)_{t \geq 1}$ is made at stage $t = 0$, only about $r_0$).

Suppose that we have proved the desired relations up to iteration $t - 1$ and we now must show that they hold for iteration $t$.

We observe that $\bar{F}_{\varepsilon, \sigma}$ is stochastically more spread than any proper distribution in $\mathcal{F}_{\varepsilon, \sigma}$ - that is, every distribution with all its mass on the reals $\mathbb{R}$ rather than the extended reals $\mathbb{R} \cup \{\pm \infty\}$. Hence for every function $\xi(x)$ monotone increasing in $|x|$,

$$\sup \{ \mathbb{E}_\xi(X) : X \sim F \in \mathcal{F}_{\varepsilon, \sigma} \} = \mathbb{E}_\xi(\bar{X}_{\varepsilon, \sigma}).$$
In this sense $X_{\bar{\varepsilon}, \bar{\sigma}}$ is extremal among contaminated normals. Moreover, we note that for $\bar{\sigma} > \sigma$, $\bar{F}_{\varepsilon, \bar{\sigma}}$ is more spread than $\bar{F}_{\varepsilon, \sigma}$. Hence, again for $\xi$ monotone increasing in $|x|$,

$$\mathbb{E}\xi(\bar{X}_{\varepsilon, \sigma}) \leq \mathbb{E}\xi(\bar{X}_{\varepsilon, \bar{\sigma}}).$$

Applying Lemma 3.3 Claim 2,

$$\inf\left\{ \mathbb{E}F'_{\lambda}(X, r) : X \sim F \in \mathcal{F}_{\varepsilon, \sigma} \right\} \equiv \mathbb{E}F'_{\lambda}(\bar{X}_{\varepsilon, \sigma}, r), \quad \forall b, \lambda > 0. \quad (29)$$

So in particular, for $X \sim F \in \mathcal{F}_{\varepsilon, \sigma_{t-1}}$ we have $\mathbb{E}F'_{\lambda}(\bar{X}_{\varepsilon, \sigma_{t-1}}, r_t) \leq \mathbb{E}F'(X, r_t)$. Hence we must have

$$\frac{1}{m} = \mathbb{E}F'_{\lambda_t}(X, r_t) \geq \mathbb{E}F'_{\lambda_t}(\bar{X}_{\varepsilon, \sigma_{t-1}}, r_t) = \mathbb{E}F'_{\lambda_t}(\bar{X}_{\varepsilon, \sigma_{t-1}}, \bar{r}_t).$$

The first step is just the definition of $r_t$, the second step used (29), and the third step used (16). Now since $r \mapsto \frac{r}{1+r}$ is increasing in $r$, while $r \mapsto \mathbb{P}\{|\bar{X}_{\varepsilon, \sigma_{t-1}}| > \kappa \cdot \sigma_{t-1} \cdot (1 + r)\}$ is monotone increasing in $r$. Hence the product $- r \mapsto \mathbb{E}F'_{\lambda_t}(\bar{X}_{\varepsilon, \sigma_{t-1}}, r)$ is monotone increasing in $r$; so in order to satisfy the definition of $\bar{r}_t$ -

$$\frac{1}{m} = \mathbb{E}F'_{\lambda_t}(\bar{X}_{\varepsilon, \sigma_{t-1}}, \bar{r}_t)$$

- we must have

$$\bar{r}_t \geq r_t.$$

Now turn to the dominance relation concerning $\tau^2_t$.

$$\sup\{\mathbb{E}F'_{\lambda_t}(X, r_t) : X \sim F \in \mathcal{F}_{\varepsilon, \sigma_{t-1}}\} \equiv \mathbb{E}F'_{\lambda_t}(\bar{X}_{\varepsilon, \sigma_{t-1}}, r_t)$$

$$= \left(\frac{\sigma_{t-1}}{\bar{\sigma}_{t-1}}\right)^2 \cdot \mathbb{E}F'_{\lambda_t}(\bar{X}_{\varepsilon, \sigma_{t-1}}, r_t)$$

$$\leq \mathbb{E}F'_{\lambda_t}(\bar{X}_{\varepsilon, \sigma_{t-1}}, \bar{r}_t).$$

where in the first inequality we substituted (17) and in the second inequality we substituted $r_t \mapsto \bar{r}_t$ by Lemma 3.3 Claim 3. We conclude that

$$\tau_t \leq \bar{\tau}_t,$$

which completes iteration $t$ of the claimed result and sets up the assumptions for the next iteration.

\textbf{Proof of Lemma 3.3}

To prove the $i$-th claim, for $i = 1, \ldots, 4$, combine formula $\Psi(x; r) = r \psi_\lambda(x/(1 + r))$ with the corresponding numbered observation:

1. From $|\psi_\lambda(x)| = \min(|x|, \lambda)$, $|x| \mapsto |\psi_\lambda(x)|$ is nondecreasing.

2. From $\psi_\lambda(x) = 1_{\{|x| \leq \lambda\}}$, $|x| \mapsto \psi_\lambda'(x)$ is nonincreasing.

3. $r \mapsto r/(1 + r)$ is monotone increasing.

4. $\lambda \mapsto \min(|x|, \lambda)$ is nondecreasing.
Proof of Lemma 3.4

These are simple scaling invariances, combined with $\Psi(x; r) = \frac{r}{1+r} \psi_\lambda(x)$.

The first, (16), says simply that for any $0 < \sigma < \bar{\sigma}$,

$$E_{\kappa \sigma} \psi'(\tilde{X}_{\epsilon, \sigma}) = \mathbb{P}\{|\tilde{X}_{\epsilon, \sigma}| < \kappa \cdot \sigma\} = \Phi(-\kappa, \kappa) = \mathbb{P}\{|\tilde{X}_{\epsilon, \bar{\sigma}}| < \kappa \cdot \bar{\sigma}\} = \mathbb{E} \psi'(\tilde{X}_{\epsilon, \bar{\sigma}}).$$

The second, (17), combines two invariances. If $Z \sim \mathcal{N}(0,1)$, then

$$E_{\kappa \sigma} \psi'^2(\kappa \sigma (\sigma Z)) = \sigma^2 \cdot E_{\kappa} \psi'^2(Z);$$

while, if $U \sim H_\infty$ is the degenerate improper random variable supported at $\infty$,

$$E_{\kappa \sigma} \psi'^2(U) = \kappa^2 \cdot \sigma^2.$$

Hence

$$E_{\kappa \sigma} \psi'^2(X_{\epsilon, \sigma}) = \sigma^2 \cdot ((1 - \epsilon)E_{\kappa} \psi'^2(Z) + \epsilon \kappa^2),$$

is thus proportional to $\sigma^2$. Applying this both to $\sigma = \sigma_{l-1}$ and $\sigma = \bar{\sigma}_{l-1}$ gives (17).

Proof of Lemma 3.9.

Note that $r \mapsto \frac{r}{1+r}$ and $r \mapsto B(\kappa \cdot (1+r), F)$ are each strictly monotone increasing in $r > 0$. Moreover,

$$\inf_{F \in \mathcal{F}_\epsilon} B(\kappa \cdot (1+r), F) = B(\kappa \cdot (1+r), \epsilon).$$

Set $R(b) = 1/(mb - 1)$; this is monotone decreasing in $b > 1/m$. Then $\mathcal{R} = R(\mathcal{B})$ and this relationship is monotone decreasing in $\mathcal{B} > 1/m$. Hence

$$\sup_{F \in \mathcal{F}_\epsilon} R(\mathcal{B}(\tau^2, r; m, \kappa, F)) = R(B(\kappa \cdot (1+r), \epsilon))$$

Also note that $\bar{r} = R(B(\kappa \cdot (1 + \bar{r}), \epsilon)$. It follows that for each $\eta > 0$, for some $r > \bar{r} - \eta$ we can find $F \in \mathcal{F}_\epsilon$ satisfying

$$r = R(B(\tau^2, r; m, \kappa, F)).$$

Hence,

$$\sup_{F \in \mathcal{F}_\epsilon} \mathcal{R}^{\tau^2; m, \kappa, F} \geq \bar{r}.$$
and hence, for such $r$ $r > \sup_{F\in\mathcal{F}_e} R(\mathcal{B}(\tau^2, r; m, \kappa, F))$, implying that $r > \sup_{F\in\mathcal{F}_e} \mathcal{R}(\tau^2; m, \kappa, F)$, and so also $\bar{r} \geq \sup_{F\in\mathcal{F}_e} \mathcal{R}(\tau^2; m, \kappa, F)$, which proves (26).

We turn to Eq. (27). Set $\bar{a} \equiv \sqrt{1 + \tau^2/(1 + \bar{r})}$. We have

$$\sup_{F\in\mathcal{F}_e} A(\tau^2; \mathcal{R}(\tau; m, \kappa, F); m, \kappa, F) = \sup_{F\in\mathcal{F}_e} A(\tau^2, \bar{r}; m, \kappa, F)$$

$$= (\bar{a}r)^2 \cdot \bar{A}(\kappa \cdot (1 + \bar{r}), \varepsilon)$$

$$= (1 + \tau^2) \left(1 + \bar{r}\right)^2 \bar{A}(\kappa, \varepsilon)$$

$$= (1 + \tau^2) \frac{\bar{A}(\kappa, \varepsilon)}{m^2 B(\kappa, \varepsilon)}$$

$$= (1 + \tau^2) \cdot \bar{V}(\kappa, \varepsilon)/m^2$$

$$\equiv \bar{A}(\tau^2; m, \varepsilon, \kappa).$$

In the first step we used monotonicity of $r \mapsto A(\tau^2, r; m, \kappa, F)$, and in the second step, we used Lemma 3.10. In each step inequality is clear, while equality is demonstrated by choosing a sequence of contamination cdfs $H = \delta_\mu$, $\mu \to \infty$. 

Proof of Lemma 4.2

For fixed $(\varepsilon, m)$, consider the relationship between $(\bar{r}, \bar{\kappa})$ implied by

$$\frac{\bar{r}}{1 + \bar{r}} \cdot \bar{B}(\varepsilon, \bar{\kappa}) = \frac{1}{m}.$$ 

Note that $\bar{B}(\varepsilon, \bar{\kappa}) = (1 - \varepsilon)(2\Phi(\bar{\kappa}) - 1)$ where $\Phi(x)$ is the standard normal CDF, which is a bijection between $(-\infty, \infty)$ and $(0, 1)$. One can check that, for fixed $(\varepsilon, m)$, $(\bar{r}, \bar{\kappa})$ are in one-one correspondence by the functions

$$\bar{r}(\bar{\kappa}) = \frac{(1 - \varepsilon)(2\Phi(\bar{\kappa}) - 1)}{1/m - (1 - \varepsilon)(2\Phi(\bar{\kappa}) - 1)}$$

and

$$\bar{\kappa}(\bar{r}) = \Phi^{-1}\left(1 + \frac{1 + 1/\bar{r}}{m(1 - \varepsilon)}/2\right),$$

acting as bijections $\bar{r} \leftrightarrow \bar{\kappa}$ between domains $(0, \infty)$ and $(0, \bar{\kappa}^*)$, where $\bar{\kappa}^*(\varepsilon, m) = \Phi^{-1}((1 + \frac{1}{m(1 - \varepsilon)})/2)$. Defining

$$\kappa(\bar{\kappa}) = \bar{\kappa}/(1 + \bar{r}(\bar{\kappa})),$$
the pair \((\kappa(\tilde{\kappa}), \tilde{r}(\tilde{\kappa}))\) will obey the relation

\[
\frac{\tilde{r}}{1 + \tilde{r}} B_2(\varepsilon, \kappa(1 + \tilde{r})) = \frac{1}{m}.
\]

We obtain the explicit expression

\[\kappa(\tilde{\kappa}) = \frac{\tilde{\kappa}}{1 + \tilde{r}(\tilde{\kappa})},\]

showing directly that \(\kappa\) is uniquely defined in terms of \(\tilde{\kappa}\), for given \((\varepsilon, m)\).

**Proof of Lemma 4.3**

By Lemma 3.9, the variance map \(\bar{T}\) is the pointwise supremum of all variance maps of proper floating-threshold state evolutions with \(F \in \mathcal{F}_\varepsilon\). Hence, no proper FTSE can have a larger fixed point; i.e.

\[
\tau_\infty^2(m, \kappa, F) \leq \tau_\infty^2(m, \kappa, \varepsilon), \quad \forall F \in \mathcal{F}_\varepsilon.
\]

From \(V_m(\kappa, F) = m \cdot \tau_\infty^2(m, \kappa, F)\), we have

\[
\sup_{F \in \mathcal{F}_\varepsilon} V_m(\kappa, F) = m \cdot \tau_\infty^2(m, \kappa, \varepsilon),
\]

and so, if \(\tau_\infty^2(m, \kappa, \varepsilon) < \infty\) - implying \(\bar{V}(\bar{\kappa}(\kappa), \varepsilon) < m\) -

\[
\sup_{F \in \mathcal{F}_\varepsilon} V_m(\kappa, F) = \frac{\bar{V}(\bar{\kappa}(\kappa), \varepsilon)}{1 - \bar{V}(\bar{\kappa}(\kappa), \varepsilon)/m}.
\]

Setting \(K_0 = \{\kappa : \bar{V}(\bar{\kappa}(\kappa), \varepsilon) < m\}\),

\[
\inf_{\kappa} \sup_{F \in \mathcal{F}_\varepsilon} V_m(\kappa, F) = \inf_{\kappa \in K_0} \frac{\bar{V}(\bar{\kappa}(\kappa), \varepsilon)}{1 - \bar{V}(\bar{\kappa}(\kappa), \varepsilon)/m}.
\]

Now by construction,

\[
\bar{V}(\bar{\kappa}(\kappa^*(\varepsilon)), \varepsilon) = \bar{V}(\kappa^*(\varepsilon), \varepsilon) = v^*(\varepsilon);
\]

and moreover for \(\kappa \neq \kappa^*(\varepsilon), \bar{\kappa}(\kappa) \neq \kappa^*(\varepsilon)\); so

\[
\bar{V}(\bar{\kappa}(\kappa), \varepsilon) = \bar{V}(\bar{\kappa}(\kappa), \varepsilon) > \bar{V}(\kappa^*(\varepsilon), \varepsilon) = v^*(\varepsilon).
\]

Now \(v \mapsto v/(1 - v/m)\) is monotone increasing on \(\{v : v \leq m\}\). Consequently, if \(v^*(\varepsilon) < m\)

\[
\inf_{\kappa} \sup_{F \in \mathcal{F}_\varepsilon} V_m(\kappa, F) = \frac{v^*(\varepsilon)}{1 - v^*(\varepsilon)/m}.
\]

By hypothesis \(v^*(\varepsilon)/m \equiv 1/(mI(F^*_e)) < 1\), and so this formula indeed holds.

Now note that automatically

\[
\inf_{\kappa} \sup_{F \in \mathcal{F}_\varepsilon} V_m(\kappa, F) \geq \sup_{F \in \mathcal{F}_\varepsilon} \inf_{\kappa} V_m(\kappa, F);
\]
hence the argument will be completed by showing that
\[
\sup_{F \in \mathcal{F}_\varepsilon} \inf_{\kappa} \mathcal{V}_m(\kappa, F) = \frac{v^*(\varepsilon)}{1 - v^*(\varepsilon)/m}.
\]

But we have already shown by (30) that
\[
\inf_{\kappa} \mathcal{V}_m(\kappa, \tilde{F}_\varepsilon) = \frac{v^*(\varepsilon)}{1 - v^*(\varepsilon)/m}.
\]

For all but purists, this completes the proof of the saddlepoint relation
\[
\sup_{F \in \mathcal{F}_\varepsilon} \inf_{\kappa} \mathcal{V}_m(\kappa, F) = \inf_{\kappa} \mathcal{V}_m(\kappa, \tilde{F}_\varepsilon).
\]

Purists who want everything stated using proper RV’s will want the following spelled out. Let
\[G_{\varepsilon, \mu} = (1 - \varepsilon)\Phi + \varepsilon H_\mu.\]
For \(\eta > 0\), there is \(\mu \in \mathbb{R}\) with
\[
\inf_{\kappa} \mathcal{V}_m(\kappa, G_{\varepsilon, \mu}) > \inf_{\kappa} \mathcal{V}_m(\kappa, \tilde{F}_\varepsilon) - \eta.
\]

Now note also that, for \(\mu > \kappa\), the Huber \(\psi_\kappa\) obeys
\[
V(\kappa, G_{\varepsilon, \mu}) = V(\kappa, G_{\varepsilon, \infty}) \equiv V(\kappa, \tilde{F}_\varepsilon) = \tilde{V}(\kappa, \varepsilon) > v^*(\varepsilon).
\]
with similar statements also being true for \(A\) and \(B\). This observation can be elaborated into a full proof, exploiting
\[
V(\kappa, (1 - \varepsilon)\Phi + \varepsilon N(\mu, \gamma)) \rightarrow V(\kappa, \tilde{F}_\varepsilon)
\]
as \(\mu \rightarrow \infty\). We omit the details. \Box

**Proof of Lemma 4.4**

If \(m \cdot I(F^*_\varepsilon) \leq 1\), then \(\tilde{V}(\kappa, \varepsilon)/m \geq v^*(\varepsilon)/m = 1/(m I(F^*_\varepsilon)) \geq 1\) and so \(\tilde{V}(\tilde{\kappa}(\kappa), \varepsilon)/m \geq 1\) for each \(\kappa > 0\).

The variance map of the LFSE with parameters \((m, \kappa, \varepsilon)\) is affine:
\[
\bar{T}(\tau^2) = \frac{\tilde{V}(\tilde{\kappa}(\kappa), \varepsilon)}{m} (1 + \tau^2);
\]
so both the slope and intercept equal \(\tilde{V}(\psi_{\tilde{\kappa}}, \varepsilon)/m \geq 1\). Hence there is no fixed point, and in fact there is a strict vertical gap between the identity line and the graph of \(\bar{T}\) - a gap of size \(\tilde{V}(\psi_{\tilde{\kappa}}, \varepsilon)/m \geq 1\).

Now \(\bar{T}\) is the pointwise supremum of all the variance maps of proper state evolutions. Hence for any \(\tau\) we choose, there is a variance map \(T\) of some proper state evolution lying above the diagonal line at \(\tau^2\):
\[
T(\tau^2) > \tau^2,
\]
which implies that the corresponding highest fixed point \(T(\tau^2_\infty) = \tau^2_\infty\) obeys \(\tau^2_\infty > \tau^2\). For all but purists, this completes the proof.

Purists will want to know that among the highest such fixed points are in fact unique fixed points, which then represent variances that are in fact achieved. We will show this for contaminated distributions of the form \(G_{\varepsilon, \mu}\), for large \(\mu\).
For such \(G_{\varepsilon, \mu}, \mu\) sufficiently large, we will show that the variance map \(T\) is star shaped; namely, defining \(T = T(\tau^2; G_{\varepsilon, \mu})\) by
\[
T(\tau^2) = (1 + \tau^2) \cdot T(\tau^2),
\]
then we will show that for \(\mu\) large, \(\tau^2 \mapsto T(\tau^2)\) is a monotone nonincreasing function of \(\tau^2\).

Any such star-shaped map has a unique fixed point; if \(\tau_1^2 < \tau_2^2\) are two distinct purported fixed points then because the line \(\tau^2 \mapsto (1 + \tau^2)T(\tau_1^2)\) has a unique fixed point at \(\tau^2 = \tau_1^2\), then
\[
\tau^2 > (1 + \tau^2)T(\tau_1^2), \quad \tau^2 > \tau_1^2.
\]
(31)

Hence
\[
(1 + \tau_2^2)T(\tau_2^2) \leq (1 + \tau_2^2)T(\tau_1^2) = (1 + \tau^2)T(\tau_1^2)|_{\tau^2=\tau_2^2} < \tau_2^2.
\]

In the last step we use (31), evaluated at \(\tau^2 = \tau_2^2\). The last display contradicts the supposed fixed-point nature of \(\tau_2^2\) and proves that the second fixed point \(\tau_2^2\) cannot exist.

To explain the star-shapedness, we need to develop some rescaling relationships. Let \(S^\sigma F\) denote the rescaling operator on CDF’s, producing \((S^\sigma F)(x) = F(x/\sigma)\). For a given \(F \in \mathcal{F}_\varepsilon\) and a given \(\tau^2\) and associated \(\sigma^2 = 1 + \tau^2\), let \(\tilde{F}^\sigma \equiv S^\sigma(F \ast \Phi_\tau)\). We then have
\[
\tilde{F}^\sigma = (1 - \varepsilon)\Phi + \varepsilon \tilde{H}^\sigma,
\]
where the contamination CDF \(\tilde{H}^\sigma = S^\sigma(H \ast \Phi_\tau)\). Because of the scale invariance \(\lambda = \kappa \sigma\),
\[
\int \psi^2_{\kappa \sigma}(x)d(F \ast \Phi_\tau)(x) = \sigma^2 \int \psi_{\kappa}(u)d\tilde{F}^\sigma(u).
\]
(32)

Similarly,
\[
\int \psi'_{\kappa \sigma}(x)d(F \ast \Phi_\tau)(x) = \int \psi'_{\kappa}(u)d\tilde{F}^\sigma(u).
\]

It follows that
\[
T(\tau^2) \equiv \frac{T(\tau^2)}{1 + \tau^2} \equiv \frac{T(\tau^2)}{\sigma^2} = \left(\frac{r}{1 + r}\right)^2 \cdot A(\psi_{\tilde{\kappa}}, \tilde{F}^\sigma),
\]
where \(\tilde{\kappa} = \kappa(1 + r)\) solves
\[
\frac{r}{1 + r} \cdot B(\psi_{\tilde{\kappa}}, \tilde{F}^\sigma) = \frac{1}{m}.
\]

The reader should check that the following claims, if established, would combine to prove the desired monotonicity of \(T\).

- \(A(\psi_{\kappa_0}, \tilde{F}^\sigma)\) is monotone decreasing in \(\sigma\), for fixed \(\kappa_0\).
- \(B(\psi_{\kappa_0}, \tilde{F}^\sigma)\) is monotone decreasing in \(\sigma\), for fixed \(\kappa_0\).
- \(\kappa \mapsto A(\psi_{\kappa}, F)\) is increasing in \(\kappa\).
• $\kappa \mapsto B(\psi_\kappa, F)$ is increasing in $\kappa$.

• $\sigma \mapsto r$ is monotone decreasing in $\sigma$.

• $\sigma \mapsto \tilde{\kappa}$ is decreasing in $\sigma$.

Some of these are obvious - for example, monotonicity of $\kappa \mapsto A(\psi_\kappa, F)$ and $\kappa \mapsto B(\psi_\kappa, F)$. Others follow from earlier items - monotonicity of $\sigma \mapsto \tilde{\kappa}$ follows from that of $\sigma \mapsto r$, while monotonicity of $\sigma \mapsto r$ follows from the two earlier claims about $B$. Finally, the first two claims will be shown for $F = G_{\varepsilon, \mu}$ for all sufficiently large $\mu$.

In the coming two paragraphs, let $\kappa$ be fixed independent of $\sigma$. Now of course

$$A(\psi_\kappa, \tilde{F}^\sigma) = \int \psi_\kappa^2(u)d\tilde{F}^\sigma(u)$$

$$= (1 - \varepsilon) \int \psi_\kappa^2(u)d\Phi + \varepsilon \int \psi_\kappa^2(u)d\tilde{H}^\sigma(u);$$

$$= I + II.$$  \hspace{1cm} (33)

the term $I$ being independent of $\sigma$, we focus on the second one, $II$. Similarly,

$$B(\psi_\kappa, \tilde{F}^\sigma) = \int \psi_\kappa'(u)d\tilde{F}^\sigma(u)$$

$$= (1 - \varepsilon) \int \psi_\kappa'(u)d\Phi + \varepsilon \int \psi_\kappa'(u)d\tilde{H}^\sigma(u);$$

$$= III + IV.$$  \hspace{1cm} (34)

We again focus on the $\sigma$-varying term; this time $IV$. Letting $H^\sigma = S^\sigma H$ we have $\tilde{H}^\sigma = \Phi_{\tau/\sigma} * H^\sigma$. By associativity of convolution,

$$\int \psi_\kappa^2(u)d\tilde{H}^\sigma(u) = \int (\psi_\kappa^2 * \Phi_{\tau/\sigma})(u)dH^\sigma(u).$$

Similarly,

$$\int \psi_\kappa'(u)d\tilde{H}^\sigma(u) = \int (\psi_\kappa' * \Phi_{\tau/\sigma})(u)dH^\sigma(u).$$

Now note that, for all sufficiently large $u$, $u \mapsto (\psi_\kappa^2 * \Phi_{\tau/\sigma})(u)$ is strictly monotone increasing. At the same time, again for all sufficiently large $u$, $\sigma \mapsto (\psi_\kappa^2 * \Phi_{\tau/\sigma})(u)$ is strictly monotone decreasing in $\sigma$. Also, let $H_\mu$ denote the CDF of a point mass at $\mu$, then $H_{\mu/\sigma} = S^\sigma H_\mu$. Consequently, $\sigma \mapsto S^\sigma H_\mu$ is increasingly concentrated (rather than spread) as $\sigma$ increases. It follows that, for large enough $\mu > 0$,

$$\sigma \mapsto \int (\psi_\kappa^2 * \Phi_{\tau/\sigma})(u)dH_\mu^\sigma(u) = (\psi_\kappa^2 * \Phi_{\tau/\sigma})(\mu/\sigma)$$

is monotone decreasing in $\sigma$. Similarly, for large enough $\mu > 0$,

$$\sigma \mapsto \int (\psi_\kappa' * \Phi_{\tau/\sigma})(u)dH_\mu^\sigma(u) = (\psi_\kappa' * \Phi_{\tau/\sigma})(\mu/\sigma)$$

is monotone increasing in $\sigma$. 

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Because

\[ A(\psi_{\kappa}, \tilde{H}^\sigma) = (\psi_\kappa^2 * \Phi_{\tau/\sigma})(\mu/\sigma) \]

and

\[ B(\psi_{\kappa}, \tilde{H}^\sigma) = (\psi_\kappa' * \Phi_{\tau/\sigma})(\mu/\sigma), \]

and the decompositions \( I + II \) and \( III + IV \), our claims about the behavior of the RHS’s in these displays, for large \( \mu \), imply the needed monotonocities of \( \sigma \mapsto A(\psi_{\kappa}, \tilde{F}^\sigma) \) and \( \sigma \mapsto B(\psi_{\kappa}, \tilde{F}^\sigma) \).

**Proof of Lemma 5.3**

Putting \( \tilde{\sigma}(m, \kappa, \varepsilon) \equiv \sqrt{1 + \tau^2_{m, \kappa, \varepsilon}} \), we note that

\[ \tilde{\lambda}(m, \kappa, \varepsilon) = \kappa \cdot \tilde{\sigma}(m, \kappa, \varepsilon). \]

Now

\[ \tilde{\sigma}(m, \kappa, \varepsilon)^2 = (1 + \frac{\tilde{V}(\tilde{\kappa}, \varepsilon)/m}{1 - \tilde{V}(\tilde{\kappa}, \varepsilon)/m}) = \frac{1}{1 - \tilde{V}(\tilde{\kappa}, \varepsilon)/m}. \]

By direct evaluation, the function \( \kappa \mapsto \tilde{V}(\kappa, \varepsilon) \) is at first strictly decreasing on \((0, \infty)\) to a minimum at the Huber minimax parameter \( \kappa^*(\varepsilon) \), after which it is strictly increasing, tending to infinity as \( \kappa \to \infty \).

Consequently, on the interval \( \kappa \in (\kappa^*(\varepsilon), \infty) \), the function \( \kappa \mapsto \tilde{V}(\kappa, \varepsilon) \) is strictly increasing. On the interval \( \mathcal{K}_+ = (\kappa^*(\varepsilon), \kappa^+(\varepsilon; m)) \) the function \( \kappa \mapsto \tilde{V}(\tilde{\kappa}(\kappa), \varepsilon) \) is likewise strictly increasing. Hence on \( \mathcal{K}_+ \) \( \kappa \mapsto \tilde{\sigma} \) is strictly increasing, and so also is \( \kappa \cdot \tilde{\sigma} \).

Fix \( m_0 > \tilde{V}(0, \varepsilon) = \frac{\pi}{2(1 - \varepsilon)} \). For each \( m > m_0 \), \( \tilde{\sigma}(m, 0, \varepsilon) < \infty \), and this is the largest that \( \tilde{\sigma}(m, \kappa, \varepsilon) \) ever gets on \( \kappa \in (0, \kappa^*(\varepsilon)) \). On the interval \( \mathcal{K}_- = (0, \kappa^*(\varepsilon)) \) the function \( \kappa \mapsto \tilde{V}(\tilde{\kappa}(\kappa), \varepsilon) \) is bounded and has bounded derivative. It follows that, as \( m \to \infty \)

\[ \sup_{\kappa \in \mathcal{K}_-} |\tilde{\sigma}(m, \kappa, \varepsilon) - 1| \to 0, \quad m \to \infty; \]

and also

\[ \sup_{\kappa \in \mathcal{K}_-} \left| \frac{\partial}{\partial \kappa} \tilde{\sigma}(m, \kappa, \varepsilon) - 0 \right| \to 0, \quad m \to \infty, \]

together implying

\[ \sup_{\kappa \in \mathcal{K}_-} \left| \frac{\partial}{\partial \kappa} \tilde{\lambda}(m, \kappa, \varepsilon) - 1 \right| \to 0, \quad m \to \infty, \]

yielding \( \frac{\partial}{\partial \kappa} \tilde{\lambda}(m, \kappa, \varepsilon) > 0 \) throughout \( \mathcal{K}_- \).

We have shown that \( \tilde{\lambda} \) is strictly increasing, as a function of \( \kappa \), throughout the whole domain \((0, \kappa^+(m, \varepsilon)) = \mathcal{K}_- \cup \mathcal{K}_+ \).
Proof of Theorem 5.4

We first remark that for any specific \( \lambda > 0 \),

\[
V_m^o(\lambda, \bar{F}_\varepsilon) = V_m(\kappa(\lambda, \bar{F}_\varepsilon), \bar{F}_\varepsilon),
\]

\[
= m \cdot \tau^2_\infty(m, \kappa(\lambda, \bar{F}_\varepsilon), \bar{F}_\varepsilon),
\]

\[
\geq m \cdot \min_{\kappa} \tau^2_\infty(m, \kappa, \bar{F}_\varepsilon),
\]

\[
= m \cdot \tau^2_\infty(m, \kappa^*(\varepsilon), \varepsilon)
\]

\[
= V_m(\kappa^*, \bar{F}_\varepsilon) = V^*_m(\varepsilon).
\]

and so

\[
\inf_{\lambda} \sup_{F \in \bar{F}_\varepsilon} V_m^o(\lambda, \bar{F}_\varepsilon) \geq V^*_m(\varepsilon).
\]

We complete the argument by showing that

\[
\sup_{F \in \bar{F}_\varepsilon} V_m^o(\lambda, \bar{F}_\varepsilon) \leq V^*_m(\varepsilon),
\]

or in other words:

\[
V_m^o(\lambda^*, F) \leq V^*_m(\varepsilon), \quad \forall F \in \bar{F}_\varepsilon.
\]

To show this, we need merely to show that for each \((\kappa, F)\) yielding an instance where \(\lambda(m, \kappa, F) = \lambda^*(m, \varepsilon, \kappa^*(\varepsilon))\), we have

\[
\tau^2_\infty(m, \kappa, F) \leq \tau^2_\infty(m, \kappa^*(\varepsilon), \varepsilon),
\]

since then

\[
V_m^o(\lambda^*, F) = V_m(\kappa, F)
\]

\[
= m \cdot \tau^2_\infty(m, \kappa, F),
\]

\[
\leq m \cdot \tau^2_\infty(m, \kappa^*(\varepsilon), \varepsilon)
\]

\[
= V_m(\kappa^*, \bar{F}_\varepsilon) = V^*_m(\varepsilon).
\]

Suppose that \( \kappa \geq \kappa(\varepsilon) \), then from

\[
\kappa \cdot \sqrt{1 + \tau^2_\infty(m, \kappa, F)} = \lambda(m, \kappa, F)
\]

\[
= \lambda^*
\]

\[
= \kappa^*(\varepsilon) \cdot \sqrt{1 + \tau^2_\infty(m, \kappa^*(\varepsilon), \varepsilon)}
\]

we conclude that

\[
\sqrt{1 + \tau^2_\infty(m, \kappa, F)} \cdot \frac{\kappa}{\kappa^*(\varepsilon)} = \sqrt{1 + \tau^2_\infty(m, \kappa^*(\varepsilon), \varepsilon)}
\]

and since \( \frac{\kappa}{\kappa^*(\varepsilon)} \geq 1 \), we indeed obtain (37).

To finish, we argue that \( \kappa < \kappa^*(\varepsilon) \) can never arise in a pair \((\kappa, F)\) obeying \(\lambda(m, \kappa, F) = \lambda^*\). By the monotonicity property of Lemma 5.3, if we have \( \kappa < \kappa^*(\varepsilon) \),

\[
\sup_{F \in \bar{F}_\varepsilon} \lambda(m, \kappa, F) = \bar{\lambda}(m, \kappa, \varepsilon)
\]

\[
< \tilde{\lambda}(m, \kappa^*(\varepsilon), \varepsilon),
\]

proving that it can never happen that \(\kappa(m, \lambda^*, F) < \kappa^*(\varepsilon)\), for any \(F \in \bar{F}_\varepsilon\).
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