Upper Bound of Fully Entangled Fraction

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Abstract

We study the fully entangled fraction of quantum states. An upper bound is obtained for arbitrary dimensional bipartite systems. This bound is shown to be exact for the case of two-qubit systems. An inequality related the fully entangled fraction of two qubits in a three-qubit mixed state has been also presented.

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The fully entangled fraction (FEF) is tightly related to many quantum information processing such as dense coding [1], teleportation [2], entanglement swapping [3], and quantum cryptography (Bell inequalities) [4]. As the optimal fidelity of teleportation is given by FEF [5], experimentally measurement of FEF can be also used to determine the entanglement of the non-local source used in teleportation. Thus an analytic formula for FEF is of great importance. In [6] an elegant formula for two-qubit system is derived analytically by using the method of Lagrange multipliers. Concerning the estimation of entanglement of formation and concurrence, exact results have been obtained not only for two-qubit case, but also for some high dimensional states, isotropic and Werner states. And analytical lower bounds have been obtained for general cases [7]. While the analytical computation of FEF remains formidable and less result has been known for high dimensional quantum states.

In this paper, we study the fully entangled fraction of arbitrary dimensional quantum bipartite states: the upper bound of FEF, its relations to the filtering operations in the generalized distillation protocol of entanglement, the relations between FEF of two qubits in a three-qubit mixed state and the related concurrence.

Let $\mathcal{H}$ be a $d$-dimensional complex vector space with computational basis $|i\rangle$, $i = 1, ..., d$. The fully entangled fraction of a density matrix $\rho \in \mathcal{H} \otimes \mathcal{H}$ is defined by

$$\mathcal{F}(\rho) = \max_U \langle \psi_+ | (I \otimes U^\dagger) \rho (I \otimes U) | \psi_+ \rangle$$

under all unitary transformations $U$, where $|\psi_+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle$ is the maximally entangled states and $I$ is the corresponding identity matrix.

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Let $\lambda_i, i = 1, ..., d^2 - 1$, be the generators of the $SU(d)$ algebra with $Tr\{\lambda_i\lambda_j\} = 2\delta_{ij}$. A bipartite state $\rho \in \mathcal{H} \otimes \mathcal{H}$ can be expressed as
\[
\rho = \frac{1}{d^2}I \otimes I + \frac{1}{d} \sum_{i=1}^{d^2-1} r_i(\rho)\lambda_i \otimes I + \frac{1}{d} \sum_{j=1}^{d^2-1} s_j(\rho)I \otimes \lambda_j + \sum_{i,j=1}^{d^2-1} m_{ij}(\rho)\lambda_i \otimes \lambda_j, \tag{2}
\]
where $r_i(\rho) = \frac{1}{2} Tr\{\rho \lambda_i(1) \otimes I\}$, $s_j(\rho) = \frac{1}{2} Tr\{\rho I \otimes \lambda_j(2)\}$ and $m_{ij}(\rho) = \frac{1}{4} Tr\{\rho \lambda_i(1) \otimes \lambda_j(2)\}$. Let $M(\rho)$ denote the correlation matrix with entries $m_{ij}(\rho)$.

**Theorem 1:** For any $\rho \in \mathcal{H} \otimes \mathcal{H}$, the fully entangled fraction $\mathcal{F}(\rho)$ satisfies
\[
\mathcal{F}(\rho) \leq \frac{1}{d^2} + 4||M^T(\rho)M(P_+)||_{KF}, \tag{3}
\]
where $M^T$ stands for the transpose of $M$ and $||M||_{KF} = Tr\sqrt{MM^\dagger}$ is the Ky Fan norm of $M$.

**Proof:** First, we note that
\[
P_+ = \frac{1}{d^2}I \otimes I + \sum_{i,j=1}^{d^2-1} m_{ij}(P_+)\lambda_i \otimes \lambda_j,
\]
where $m_{ij}(P_+) = \frac{1}{4} Tr\{P_+\lambda_i \otimes \lambda_j\}$.

By definition (1), one obtains
\[
\mathcal{F}(\rho) = \max_U \langle \psi_+|(I \otimes U^\dagger)\rho(I \otimes U)|\psi_+\rangle = \max_U Tr\{\rho(I \otimes U)P_+(I \otimes U^\dagger)\} = \max_U \left[\frac{1}{d^2} Tr\{\rho\} + \sum_{i,j=1}^{d^2-1} m_{ij}(P_+)Tr\{\rho \lambda_i \otimes U\lambda_j U^\dagger\}\right].
\]

Since $U\lambda_i U^\dagger$ is a traceless Hermitian operator, it can be expanded according to the $SU(d)$ generators,
\[
U\lambda_i U^\dagger = \sum_{j=1}^{d^2-1} \frac{1}{2} Tr\{U\lambda_i U^\dagger \lambda_j\}\lambda_j \equiv \sum_{j=1}^{d^2-1} O_{ij}\lambda_j. \tag{4}
\]

Entries $O_{ij}$ defines a real $(d^2 - 1) \times (d^2 - 1)$ matrix $O$. From the completeness relation of $SU(d)$ generators
\[
\sum_{j=1}^{d^2-1} (\lambda_j)_{ki}(\lambda_j)_{mn} = 2\delta_{im}\delta_{kn} - \frac{2}{d}\delta_{ki}\delta_{mn}, \tag{5}
\]


one can show that $O$ is an orthonormal matrix. Using (4) we have

$$\mathcal{F}(\rho) \leq \frac{1}{d^2} + \max_O \sum_{i,j,k} m_{ij}(P_+)O_{jk} \{\rho \lambda_i \otimes \lambda_k\}$$

$$= \frac{1}{d^2} + 4 \max_O \sum_{i,j,k} m_{ij}(P_+)O_{jk} m_{ik}(\rho)$$

$$= \frac{1}{d^2} + 4 \max_O Tr\{M(\rho)^TM(P_+)O\}$$

$$= \frac{1}{d^2} + 4\|M(\rho)^TM(P_+)\|_{KF}.$$  

□

For the case $d = 2$, we can get an exact result from (3):

**Corollary:** For two qubits system, we have

$$\mathcal{F}(\rho) = \frac{1}{4} + 4\|M(\rho)^TM(P_+)\|_{KF},$$

i.e. the upper bound derived in Theorem 1 is exactly the $FEF$. 

**Proof:** We have shown in (4) that given an arbitrary unitary $U$, one can always obtain an orthonormal matrix $O$. Now we show that in two-qubit case, for any $3 \times 3$ orthonormal matrix $O$ there always exits $2 \times 2$ unitary matrix $U$ such that (4) holds.

For any vector $t = \{t_1, t_2, t_3\}$ with unit norm, define an operator $X \equiv \sum_{i=1}^{3} t_i \sigma_i$, where $\sigma_i$s are Pauli matrices. Given an orthonormal matrix $O$ one obtains a new operator $X' \equiv \sum_{i=1}^{3} O_{ij} t_j \sigma_i$.

$X$ and $X'$ are both hermitian traceless matrices. Their eigenvalues are given by the norms of the vectors $t$ and $t' = \{t'_1, t'_2, t'_3\}$ respectively. As the norms are invariant under orthonormal transformations $O$, they have the same eigenvalues: $\pm \sqrt{t'_1^2 + t'_2^2 + t'_3^2}$. Thus there must be a unitary matrix $U$ such that $X' = UXU^\dagger$. Hence the inequality in the proof of Theorem 1 becomes an equality. The upper bound (3) then becomes exact at this situation, which is in accord with the result in (6). □

**Remark** The upper bound of $FEF$ (3) and the $FEF$ (6) for a state $\rho$ depend on the correlation matrices $M(\rho)$ and $M(P_+)$. They can be calculated directly according to a given set of $SU(d)$ generators $\lambda_i$, $i = 1, \ldots, d^2 - 1$. Nevertheless the $FEF$ and its upper bound do not depend on the choice of the $SU(d)$ generators.

The upper bound can give rise to not only an estimation of the fidelity in quantum information processing such as teleportation, but also an interesting application in entanglement distillation of quantum states. In [8, 9], a separability criterion called reduction criterion has been proposed. It says that if a bipartite quantum state $\rho$ is separable, then $(\rho_1 \otimes I) - \rho \geq 0$ and $(I \otimes \rho_2) - \rho \geq 0$, where $\rho_1 = Tr_2(\rho)$ (resp. $\rho_2 = Tr_1(\rho)$) is the reduced density matrix.
FIG. 1: Upper bound of $\mathcal{F}(\rho) - \frac{1}{3}$ from (3) (solid line) and fidelity $F(\rho) - \frac{1}{3}$ (dashed line).

obtained by tracing over the second (resp. first) subsystem. Here a matrix $X \succeq 0$ means that all the eigenvalues of $X$ are greater than or equal to 0. In [8] a generalized distillation protocol has been presented. It is shown that a quantum state $\rho$ violating the reduction criterion can always be distilled. For such states if their single fraction of entanglement $F(\rho) = \langle \psi_+ | \rho | \psi_+ \rangle$ is greater than $\frac{1}{d}$, then one can distill these states directly by using the generalized distillation protocol. However if even the $\text{FEF}$ (the largest value of single fraction of entanglement under local unitary transformations) is less than or equal to $\frac{1}{d}$, then a proper filtering operation has to be used at first to transform $\rho$ to another state $\rho'$ so that $F(\rho') > \frac{1}{d}$. For $d = 2$, one can compute $\text{FEF}$ analytically according to the corollary. For $d \geq 3$ our upper bound (3) can supply a necessary condition in the distillation:

**Theorem 2:** For an entangled state $\rho \in \mathcal{H} \otimes \mathcal{H}$ violating the reduction criterion, if the upper bound (3) is less than or equal to $\frac{1}{d}$, then the filtering operation has to be applied before using the generalized distillation protocol.

As an example we consider a $3 \times 3$ state

$$\rho = \frac{8}{9} \sigma + \frac{1}{9} |\psi_+ \rangle \langle \psi_+|,$$

where $\sigma = (x|0\rangle\langle 0| + (1 - x)|1\rangle\langle 1|) \otimes (x|0\rangle\langle 0| + (1 - x)|1\rangle\langle 1|)$. It is direct to verify that $\rho$ violates the reduction criterion for $0 \leq x \leq 1$, as $(\rho_1 \otimes I) - \rho$ has a negative eigenvalue $-\frac{27}{27}$. Therefore the state is distillable. From Fig. 1 we see that for $0 \leq x < 0.0722$ and $0.9278 < x \leq 1$, the fidelity is already greater than $\frac{1}{3}$, thus the generalized distillation protocol can be applied without the filtering operation. However for $0.1188 \leq x \leq 0.8811$, even the upper bound of the fully entangled fraction is less than or equal to $\frac{1}{3}$, hence the filtering operation has to be applied first, before using the generalized distillation protocol.
The upper bound of $FEF$ has also interesting relations to the entanglement measure concurrence. Let us consider tripartite case. Let $\rho_{ABC}$ be a state of three-qubit systems denoted by $A$, $B$ and $C$. We study the upper bound of the $FEF$, $F(\rho_{AB})$, between qubits $A$ and $B$, and its relations to the concurrence under bipartite partition $AB$ and $C$. For convenience we normalize $F(\rho_{AB})$ to be

$$F_N(\rho_{AB}) = \max\{2F(\rho_{AB}) - 1, 0\}.$$  
(8)

For a bipartite pure state $|\psi\rangle \in \mathcal{H} \otimes \mathcal{H}$, the concurrence \[12\] is defined by $C(|\psi\rangle) = \sqrt{2(1 - Tr(\rho_2^2))}$. The concurrence is extended to mixed states $\rho$ by the convex roof, $C(\rho) \equiv \min \{p_iC(|\psi_i\rangle)\} \text{ for all possible ensemble realizations } \rho = \sum_i p_i|\psi_i\rangle \langle \psi_i|, p_i \geq 0, \sum_i p_i = 1$. Let $C(\rho_{AB|C})$ denote the concurrence between subsystems $AB$ and $C$.

**Theorem 3:** For any triqubit state $\rho_{ABC}$, $F_N(\rho_{AB})$ satisfies

$$F_N(\rho_{AB}) \leq \sqrt{1 - C^2(\rho_{AB|C})}.$$  
(9)

**Proof:** We first consider the case that $\rho_{ABC}$ is pure, $\rho_{ABC} = |\psi\rangle_{ABC}\langle \psi|$. By using the Schmidt decomposition between qubits $A$, $B$ and $C$, $|\psi\rangle_{ABC}$ can be written as:

$$|\psi\rangle_{AB|C} = \sum_{i=1}^{2} \eta_i |i_{AB}\rangle |i_C\rangle, \eta_1^2 + \eta_2^2 = 1, \eta_1 \geq \eta_2$$  
(10)

for some othonormalized bases $|i_{AB}\rangle$, $|i_C\rangle$ of subsystems $AB$, $C$ respectively. The reduced density matrix $\rho_{AB}$ has the form

$$\rho_{AB} = Tr_C\{\rho_{ABC}\} = \sum_{i=1}^{2} \eta_i^2 |i_{AB}\rangle \langle i_{AB}| = U^T \Lambda U^*,$$

where $\Lambda$ is a $4 \times 4$ diagonal matrix with diagonal elements $\{\eta_1^2, \eta_2^2, 0, 0\}$, $U$ is a unitary matrix and $U^*$ denotes the conjugation of $U$.

The $FEF$ of the two-qubit state $\rho_{AB}$ can be calculated by using formula \[6\] or the one in \[6\]. Let

$$M = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & i \\ 0 & i & -1 & 0 \\ 0 & i & 1 & 0 \\ 1 & 0 & 0 & -i \end{pmatrix}$$

be the $4 \times 4$ matrix constituted by the four Bell bases. The $FEF$ of $\rho_{AB}$ can be written as

$$F(\rho_{AB}) = \eta_{max}(Re\{M^\dagger \rho_{AB} M\}) = \frac{1}{2} \eta_{max}(M^\dagger \rho_{AB} M + M^T \rho_{AB}^* M^*)$$

$$\leq \frac{1}{2} [\eta_{max}(M^\dagger U^T \Lambda U^* M) + \eta_{max}(M^T U^* \Lambda U^* M^*)] = \eta_1^2$$  
(11)
FIG. 2: $\mathcal{F}_N(\rho_{AB})$ (dashed line) and Upper bound $\sqrt{1 - C^2(|W_{AB}^\prime\rangle_{AB(C)})}$ (solid line) of state $|W_{AB}^\prime\rangle_{AB(C)}$ at $|\alpha| = |\beta|$.

where $\eta_{\text{max}}(X)$ stands for the maximal eigenvalues of the matrix $X$.

For pure state $(\text{11})$ in bipartite partition $AB$ and $C$, we have

$$C(|\psi\rangle_{AB|C}) = \sqrt{2(1 - \text{Tr}\{\rho_{AB}^2\})} = 2\eta_1\eta_2.$$  
(12)

From (8), (11) and (12) we get

$$\mathcal{F}_N(\rho_{AB}) \leq \sqrt{1 - C^2(|\psi\rangle_{AB|C})}.$$  
(13)

We now prove that the above inequality (13) also holds for mixed state $\rho_{ABC}$. Let $\rho_{ABC} = \sum_i p_i |\psi_i\rangle_{ABC}\langle\psi_i|$ be the optimal decomposition of $\rho_{ABC}$ such that $C(\rho_{AB|C}) = \sum_i p_i C(|\psi_i\rangle_{AB|C})$. We have

$$\mathcal{F}_N(\rho_{AB}) \leq \sum_i p_i \mathcal{F}_N(\rho_{AB}^i) \leq \sum_i p_i \sqrt{1 - C^2(\rho_{AB|C}^i)}$$

$$\leq \sqrt{1 - \sum_i p_i C^2(\rho_{AB|C}^i)} \leq \sqrt{1 - C^2(\rho_{AB|C})},$$

where $\rho_{AB|C}^i = |\psi_i\rangle_{ABC}\langle\psi_i|$ and $\rho_{AB|C}^i = Tr_C\{\rho_{AB|C}^i\}$.  
□

From Theorem 2 we see that the FEF of qubits $A$ and $B$ are bounded by the concurrence between qubits $A$, $B$ and qubit $C$. The upper bound of FEF for $\rho_{AB}$ decreases when the entanglement between qubits $A$, $B$ and $C$ increases. As an example, we consider the generalized W state defined by $|W^\prime\rangle = \alpha|100\rangle + \beta|010\rangle + \gamma|001\rangle$, $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$. The
reduced density matrix is given by

\[ \rho_{AB}' = \begin{pmatrix}
|\gamma|^2 & 0 & 0 & 0 \\
0 & |\beta|^2 & \alpha^* \beta & 0 \\
0 & \alpha \beta^* & |\alpha|^2 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}. \]

The FEF of \( \rho_{AB}' \) is given by

\[ \mathcal{F}_N(\rho_{AB}') = -\frac{1}{2} + 2|\alpha||\beta| + \frac{1}{2}||\alpha|^2 + |\beta|^2 - |\gamma|^2|. \]

While the concurrence of \( |W'\rangle \) has the from \( C_{AB|C}(|W'\rangle) = 2|\gamma|\sqrt{|\alpha|^2 + |\beta|^2} \). We see that (9) always holds. In particular for \( |\alpha| = |\beta| \) and \( |\gamma| \leq \frac{\sqrt{2}}{2} \), the inequality (9) is saturated (see Fig. 2).

We have studied the fully entangled fraction of arbitrary dimensional quantum bipartite states. We obtained an analytic upper bound of FEF, which is exact the FEF for two-qubit systems. This upper bound of FEF gives a necessary condition for which the filtering step has to be performed in the generalized distillation protocol of entanglement. An inequality related the fully entangled fraction of two qubits in a three-qubit mixed state has been also presented. As the fully entangled fraction is directly related to dense coding, teleportation, entanglement swapping and quantum cryptography, the results could shed new lights on the study of relevant quantum information processing both theoretically and experimentally.

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