Streaming, Memory Limited Matrix Completion with Noise

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Abstract

In this paper, we consider the streaming memory-limited matrix completion problem when the observed entries are noisy versions of a small random fraction of the original entries. We are interested in scenarios where the matrix size is very large so the matrix is very hard to store and manipulate. Here, columns of the observed matrix are presented sequentially and the goal is to complete the missing entries after one pass on the data with limited memory space and limited computational complexity. We propose a streaming algorithm which produces an estimate of the original matrix with a vanishing mean square error, uses memory space scaling linearly with the ambient dimension of the matrix, i.e. the memory required to store the output alone, and spends computations as much as the number of non-zero entries of the input matrix.

Keywords: matrix completion, streaming input, limited memory, computational complexity

1. Introduction

Reconstructing a structured (e.g. low rank) matrix from noisy observations of a subset of its entries constitutes a fundamental problem in collaborative filtering [Rennie and Srebro (2005)], and has recently attracted much interest, see e.g. [Candès and Recht (2009), Candès and Tao (2010), Keshavan et al. (2010), Recht (2011)]. The recent development of matrix completion algorithms has been largely motivated by the design of efficient recommendation systems. These systems (amazon, netflix, google) aim at proposing items or products from large catalogues to targeted users based on the ratings provided by users of a small subset of items. This goal naturally translates to a matrix completion problem where the rows (resp. the columns) of the matrix correspond to items (resp. to users). And often, the (item, user) rating matrix is believed to exhibit a low rank structure due to the inherent similarities among users and among items.

In this paper, we address the problem of matrix completion in scenarios where the matrix can be extremely large, so that (i) it might become difficult to manipulate or even store, and (ii) the complexity of the proposed algorithms should not rapidly increase with the matrix dimensions. In other words, we aim at designing matrix completion algorithms under memory and computational constraints. Memory-limited algorithms are particularly relevant in the streaming data model, where observations (e.g. ratings in recommendation systems) are collected sequentially. We assume here that the columns of the matrix are revealed one by one to the algorithm. More specifically, a subset of noisy entries of an arriving column is observed, and may be stored, but the algorithm cannot request these entries later if they were not stored. The streaming model seems particularly appropriate to model recommendation systems, where users actually seek for recommendations sequentially. Recently, motivated by the need to understand high-dimensional data, several machine learning techniques, such as PCA [Mitliagkas et al. (2013)] or low-rank matrix approximation.
Clarkson and Woodruff (2009), have been revisited considering memory and computational constraints. To our knowledge, this paper provides the first analysis of the matrix completion problem under these constraints (refer to the related work section for a detailed description of the connection of our problem to existing work).

Throughout the paper, we use the following notations. For any \( m \times n \) matrix \( A \), we denote by \( A^\top \) its transpose. We also denote by \( s_1(A) \geq \cdots \geq s_{\min(m,n)}(A) \geq 0 \), the singular values of \( A \). The SVD of matrix \( A \) is \( A = U \Sigma V^\top \) where \( U \) and \( V \) are unitary matrices and \( \Sigma = \text{diag}(s_1(A), \ldots , s_{\min(m,n)}(A)) \). \( A^{-1} \) denotes the pseudo-inverse matrix of \( A \), i.e. \( A^{-1} = V \Sigma^{-1} U^\top \). Finally, for any vector \( v \), \( \| v \| \) denotes its Euclidean norm, whereas for any matrix \( A \), \( \| A \|_F \) denotes its Frobenius norm, \( \| A \|_2 \) its operator norm, and \( \| A \|_\infty \) its \( \ell_\infty \)-norm, i.e., \( \| A \|_\infty = \max_{i,j} | A_{ij} | \).

**Contributions.** Let \( M \in [0, 1]^{m \times n} \) denote the \( m \times n \) ground-truth matrix we wish to recover from noisy observations of some of its entries. \( M \) is assumed to exhibit a sparse structure (refer to Assumption 1 (ii) for a formal definition). \( m \) and \( n \) are typically very large, and can be thought as tending to \( \infty \). We assume that each entry of \( M \) is observed (but corrupted by noise) with probability \( \delta \) (independently over entries). The random set of observed entries is denoted by \( \Omega \), and we introduce the following operator from \( \mathbb{R}^{m \times n} \) to itself: for all \( Y \in \mathbb{R}^{m \times n} \),

\[
[\mathcal{P}_\Omega(Y)]_{ij} = \begin{cases} Y_{ij}, & \text{if } (i, j) \in \Omega \\ 0, & \text{otherwise.} \end{cases}
\]

Then, we wish to reconstruct \( M \) from the observed matrix \( A = \mathcal{P}_\Omega(M + X) \), where \( X \) is a noise matrix with independent and zero-mean entries, and such that \( M_{ij} + X_{ij} \in [0, 1] \). Note that \( \delta \) typically depends of \( n \) and \( m \), and tends to zero as \( n \) and \( m \) tend to infinity. Finally, we analyze the matrix completion problem under the streaming model: we assume that in each round, a column of \( A \) is observed. This column is uniformly distributed among the set of columns that have not been observed so far.

We present SMC (Streaming Matrix Completion), a memory-limited and low-complexity algorithm which, based on the observed matrix \( A \), constructs an estimator \( \hat{M} \) of \( M \). We prove, under mild assumptions on \( M \) and the proportion \( \delta \) of observed entries, that \( \hat{M} \) is asymptotically accurate, in the sense that its average mean-square error converges to 0 as both \( n \) and \( m \) grows large, i.e., \( \| \hat{M} - M \|_F^2 / mn = o(1) \). More precisely, we make the following assumption.

**Assumption 1.** (i) \( \| M \|_F^2 = \Theta(mn) \).

(ii) (Structural sparsity of \( M \)) there exists \( i \leq \min(n, m) \) such that \( s_i(M) = \omega(1) \) and \( \sum_{j=i+1}^{m \wedge n} s^2_j(M) = O(mn) \). We denote by \( k \) the smallest \( i \) satisfying this condition.

(iii) \( \delta = \omega(k \max(\frac{k}{n}, \frac{\log^2 m}{m}, \frac{\log m}{m})) \), and \( \delta = o(\frac{1}{\log^4 m}) \).

The main result of this paper is a direct consequence of Theorems 5, 6, and 7. It states that under Assumption 1, with high probability, the SMC algorithm provides an asymptotically accurate estimate \( \hat{M} \) of \( M \) using one pass on the observed matrix \( A \), and requires \( O(km + kn) \) memory space and \( O(\delta mnk) \) operations.

Note that Assumption 1 (ii) is satisfied as soon as \( M \) has low rank. More precisely, when \( \text{rank}(M) = K \), then (ii) is satisfied when \( k = K \). In such a case, there is a non-empty set of sampling rates \( \delta \) for which SMC yields an asymptotically accurate estimate of \( M \) as soon as \( K = o(\frac{1}{\log(m)^2}) \) (if for example \( m \) and \( n \) grows at the same pace to infinity).
Note also that \( O(km + kn) \) is the dimension of the ambient space for \( M \), i.e. \( M \) can be well-approximated by a rank\((k)\) matrix and hence \((km + kn)\) is the minimum memory size required to output a good estimate of \( M \). Our algorithm SMC is optimal in the sense that it only requires the amount of memory required to store the output.

The SMC algorithm consists in three main steps.

- **Step 1.** We first treat the \( \ell = \frac{1}{\delta \log m} \) first arriving columns. These columns do not contain enough information to learn the right singular vectors of \( M \) since there are many rows with no observed entries. Instead, we can extract the top \( k \) right singular vectors for the submatrix of \( M \) corresponding to the \( \ell \) arriving columns. Let \( A^{(B)} \) be the \( \ell \) arriving columns and \( Q \) be the top \( k \) right singular vectors extracted from the \( A^{(B)} \). After finding \( Q \), we compute and keep \( W = A^{(B)} Q \) for the next step. \( W \) will be used to recover the top \( k \) right singular vectors of \( M \).

- **Step 2.** We extract the top \( k \) right singular vectors of \( M \) using \( W \). We show that the linear span of the columns of \( \hat{V} = A^\dagger \cdot W \) is similar to the linear span of the top \( k \) right singular vectors of \( M \) (Theorem 4). Although \( \hat{V} \) is noisy, the matrix product amplifies the linear span of the top \( k \) right singular vectors of \( M \).

- **Step 3.** Once we know \( \hat{V}^\dagger \), it is easy to find column vectors \( \hat{U} \) such that \( \| \hat{U} \hat{V} - M \|_F^2 = o(1) \). First, using the Gram-Schmidt process, we find \( \hat{R} \) such that \( \hat{V} \hat{R} \) is an orthonormal matrix and compute \( \hat{U} = \frac{1}{\delta} A^\dagger \hat{V} \hat{R} \hat{R}^\dagger \). Then, \( \hat{U} \hat{V}^\dagger = \frac{1}{\delta} A^\dagger \hat{V} \hat{R} \hat{R}^\dagger \) where \( \hat{V} \hat{R} \hat{R}^\dagger \) is the projection matrix onto the linear span of the top \( k \) right singular vectors of \( M \). Therefore, \( \hat{U} \hat{V}^\dagger \) becomes very close to the best rank \( k \) approximation.

We show that these three steps can be realized in a memory-efficient manner, and using low complexity algorithms.

**Additional Notations.** When matrices \( A \) and \( B \) have the same number of rows, \([A, B]\) to denote the matrix whose first columns are those of \( A \) followed by those of \( B \). For any matrix \( A \), \( A_L \) denotes an orthonormal basis of the subspace perpendicular to the linear span of the columns of \( A \). \( A_i \), \( A^j \), and \( A_{ij} \) denote the \( i \)-th column of \( A \), the \( j \)-th row of \( A \), and the \( (i, j) \) entry of \( A \), respectively. For \( b \geq a \), \( A_{a:b} \) and \( A_{a:b} \) are submatrices of \( A \) respectively defined as \( A_{a:b} = (A^j)_{j=a,\ldots,b} \) and \( A_{a:b} = (A_i)_{i=a,\ldots,b} \). Also, we will abbreviate \( A_{1:k}^i \) to \( A_{[k]}^i \). Finally, we define the following thresholding operator for matrices. The operator is defined by two real positive numbers \( a \) and \( b \), with \( b \geq a \), and if applied to \( A \), it returns the matrix \( |A|^{b}_{a} \) such that

\[
|A|^{b}_{a} = \begin{cases} 
  b & \text{if } A_{ij} \geq b, \\
  A_{ij} & \text{if } a < A_{ij} < b, \\
  a & \text{if } A_{ij} \leq a.
\end{cases}
\]

**2. Related Work**

This section surveys existing work on the design of matrix completion algorithms. We also provide a description of recent work on rank-\(k\) approximation and PCA algorithms, as these algorithms could be seen as building blocks of matrix completion methods. The section is organised as follows. We
first review algorithms for matrix completion. We then focus on streaming algorithms for rank-$k$
approximation, and PCA. Finally we discuss algorithms designed to be computationally efficient.

**Matrix completion algorithms.** Candès and Recht (2009) first showed that in absence of noise
(i.e., $X = 0$), the matrix $M$, with low rank $k$, can be recovered exactly using convex relaxation under
some conditions on the sampling rate $\delta$ and the singular vectors. These conditions were improved
in Candès and Tao (2010) and Recht (2011), and the approach was also extended to the case of
noisy observed entries Candès and Plan (2010). The proposed algorithms involves solving a convex
program, which can be computationally expensive. If the rank $k$ of the matrix is known, $M$ can be
recovered using simpler spectral methods. For example, in Keshavan et al. (2010), the authors show
that in absence of noise, $M$ can be reconstructed asymptotically accurately using $O(\delta kmn \log n)$
operations under the conditions that the rank $k$ does not depend on $n$ and $m$, $\delta m = \omega(1)$ and
$\delta n = \omega(1)$. Again these results can be adapted to the presence of noise Keshavan et al. (2009).
In this paper, we improve the spectral method used in Keshavan et al. (2010) and Keshavan et al.
(2009), so that it becomes memory-efficient, and so that it has performance guarantees even if the
rank $k$ of $M$ scales with $m$ and $n$.

**Streaming algorithms.** Clarkson and Woodruff (2009) proposes an algorithm to provide a rank-$k$
approximation of a fully observed matrix $A$, using 1-pass on the columns of $A$. The algorithm
uses a random $m \times \ell$ Rademacher matrix $S$, with an appropriate choice of $\ell$, and outputs a rank-$k$
matrix $\widehat{A}^{(k)}$ constructed from $A^\dagger S$ and $AA^\dagger S$. When setting $\ell = O(k\epsilon^{-1} \log(1/\eta))$ which requires
$O(k \epsilon^{-1} (m + n) \log(1/\eta))$ memory space, it is shown that with probability at least $1 - \eta$,

$$\|A - \widehat{A}^{(k)}\|_F \leq (1 + \varepsilon)\|A - \bar{A}^{(k)}\|_F,$$

where $\bar{A}^{(k)}$ is the optimal rank-$k$ approximation of $A$. We could think of applying this algorithm to
our problem. If the observed matrix $A$ is $A = P_\Omega(M + X)$, it would make sense to estimate $M$
by $\frac{1}{\delta} \widehat{A}^{(k)}$ where $\widehat{A}^{(k)}$ is the output of the algorithm in Clarkson and Woodruff (2009) applied to $A$. Indeed, it is easy to check that $\|M - \frac{1}{\delta} \widehat{A}^{(k)}\|_F^2 = o(mn)$ (i.e., the optimal rank-$k$ approximation
of $\frac{1}{\delta} A$ estimates $M$ asymptotically accurately). However, in general, $\frac{1}{\delta} \widehat{A}^{(k)}$ is not asymptotically
accurate:

$$\|M - \frac{1}{\delta} \widehat{A}^{(k)}\|_F^2 \geq \left(\|A - \bar{A}^{(k)}\|_F - \|A - \bar{A}^{(k)}\|_F - \|\bar{A}^{(k)} - \delta M\|_F\right)^2$$

$$\frac{\delta^2 mn}{\delta^2 mn}$$

Now, one can also easily check that $\|A - \bar{A}^{(k)}\|_F = \Theta(\sqrt{\delta mn})$ and $\|\delta M - \bar{A}^{(k)}\|_F = o(\delta \sqrt{mn})$, so
that if we choose $\epsilon = \sqrt{\delta}$, we get $\|M - \frac{1}{\delta} \bar{A}^{(k)}\|_F^2 = \Omega(1)$. As a consequence, using the algorithm
in Clarkson and Woodruff (2009), we cannot reconstruct $M$ asymptotically accurately using
$O(k \sqrt{\delta (m + n) \log(1/\eta)})$ memory space. Recall that our algorithm reconstructs $M$ accurately
with $O(k (m + n))$ memory space.

We could also think of using sketching and streaming PCA algorithms to reconstruct $M$. When
the columns arrive sequentially, these algorithms identify the left singular vectors in 1-pass on the
matrix. We would then need a second pass on the data to estimate the right singular vectors, and
complete the matrix. For example, Liberty (2013) proposes a sketching algorithm that updates the
$\ell$ most frequent directions when a new column of $A$ is (fully) observed. This algorithm outputs a
Algorithm 1 Spectral PCA (SPCA)

**Input:** $A \in [0, 1]^{m \times \ell}$, $k$

$\hat{\delta} \leftarrow \frac{1}{m} \sum_{(i,j)} 1([A[k]]_{ij} > 0)$

(Trimming) $\tilde{A} \leftarrow$ erase rows of $A$ with more than $\max \{10, 10\hat{\delta}\ell\}$ non-zero entries

$\Phi \leftarrow \tilde{A}^\dagger \tilde{A} - \text{diag}(\tilde{A}^\dagger \tilde{A})$

$\hat{V}_{1:k} \leftarrow \text{QR}(\Phi, k)$

**Output:** $\hat{V}_{1:k}$

Algorithm 2 QR Algorithm

**Input:** $\Phi$ (of size $\ell \times \ell$), $k$

**Initialization:** $Q(0) \leftarrow$ Randomly choose $k$ orthonormal vectors

for $\tau = 1$ to $\lceil 10 \log(\ell) \rceil$ do

$Q^{(\tau)} R^{(\tau)} \leftarrow \text{QR decomposition of } \Phi Q^{(\tau-1)}$

end for

**Output:** $Q^{(\tau)}$

Sketch $\hat{A}$ of $A$ and has the following performance guarantee: $\|AA^\dagger - \hat{A} \hat{A}^\dagger\|_2 \leq \frac{2\|A\|_F^2}{\|A\|_F}$. It also uses $O(m\ell)$ memory space. Again if we apply the algorithm to our matrix completion problem, i.e., to the observed matrix $A = \mathcal{P}_\Omega(M + X)$, where $M$ is of rank $k$, then $\|A\|_F^2 = \Theta(\delta mn)$ and $\sigma_k(AA^\dagger) = \Theta(\delta^2 \sigma_k^2(M)) = \Theta(\frac{\delta mn}{k})$. Hence to efficiently extract the top $k$ left singular vectors, we would need that $\frac{2\|A\|_F^2}{\ell} = o(\sigma_k(AA^\dagger))$, which implies $\ell = \omega(k/\delta)$. Therefore, the required memory space would be $O(\frac{km}{\delta} + kn)$. Our algorithm is more efficient, and uses only 1-pass on the matrix. Note that the streaming PCA algorithm proposed in Mitliagkas et al. (2013) does not apply to our problem (in Mitliagkas et al. (2013), the authors consider the spiked covariance model where a column is randomly generated in an i.i.d. every time).

Low complexity algorithms. There have been recently an intense research effort to propose low-complexity algorithms for various linear algebra problems. Randomization has appeared as an efficient way to reduce the complexity of algorithms, see Halko et al. (2011) for a survey. For example, Sarlos (2006) and Clarkson and Woodruff (2009) devise algorithms for rank-$k$ approximation with guarantees (1) and that use $O(\delta mn(k/\varepsilon + k \log k) + n \text{poly}(k/\varepsilon))$ operations. When the input matrix is sparse, Clarkson and Woodruff (2013) leverages sparse embedding techniques, and reduces the required complexity to $O(\delta mn) + O((nk^2\varepsilon^{-4} + k^3\varepsilon^{-5}) \cdot \text{polylog}(m + n))$ operations. But once again, as explained above, these results do not apply to our framework ((1) is not enough to guarantee an asymptotically accurate matrix completion).

3. Extracting Right-Singular Vectors

As mentioned in the introduction, the SMC algorithm deals with batches of arriving columns. Information from each batch will be extracted and aggregated as more columns arrive. In this section, we present an algorithm that will be used as a building block for extracting information from a batch of columns. For concreteness, let assume that the size of a batch is $\ell$. In the SMC algorithm, $\ell$ will be chosen much smaller than $m$, so as to guarantee that the algorithm does not require large memory space.
The algorithm presented in this section addresses the following problem. Let \( M \in [0, 1]^{m \times \ell} \) with singular value decomposition \( M = U \Sigma V^\dagger \). Given \( 0 < k \leq \ell \) and \( A = \mathcal{P}_\Omega (M + X) \), we wish to estimate the \( k \) dominant right-singular vectors of \( M \), \( V_{1:k} \). At first, this might appear as a standard PCA task, but we are only interested in cases where \( A \) is very sparse. Indeed \( A \) only has a vanishing proportion \( \delta \) of non-zero entries. Note that on average, we have \( \delta \ell \) observed entries per row of \( M + X \). Moreover, as this will become clear in the design of the SMC algorithm, we need to consider the case where \( \delta \ell = o(1) \). In particular, there are many rows of \( A \) with no observed entry. As a consequence, we do not get any information about the corresponding rows of \( U \) in the singular value decomposition of \( M \). Hence, we are here only interested in providing an estimate of the right-singular vectors \( V \).

The algorithm to extract the dominant right-singular vectors, referred to as SPCA (Spectral Principal Component Analysis), is simple and its design relies on the following observation. If we had access to the matrix \( M \), then estimating the right-singular vectors of \( M \) would be obvious. Indeed \( M^\dagger M = V \Sigma^2 V^\dagger \), so that a standard QR algorithm would output \( V \). Now \( A \) constitutes a subsampled noisy version of \( M \) and we could try to apply this algorithm directly to \( A \). From basic random matrix theory, we expect that the eigenvalues associated to the signal (i.e., the subsampled version of \( M^\dagger M \)) to be of the order of \( \delta^2 s_k^2(M) \). On the other hand, the eigenvalues associated with the noise (i.e., the subsampled version of \( X^\dagger X \)) should be of the order \( \delta \sqrt{m \ell} \). Thus, one could believe that the eigenvectors obtained by applying the QR algorithm to \( A \) provide a good estimate of \( V_{1:k} \) as soon as the ratio \( \frac{\delta^2 s_k^2(M)}{\sqrt{m \ell}} \) is large enough. However, this is not quite true, because of the sparsity of the matrix \( A \). To overcome this issue, we need to regularize the matrix \( A \) before applying the QR algorithm. This is done in two steps:

(a) **Trimming:** The rows of the subsampled matrix \( A \) with too many non-zero entries are first removed. This trimming step is standard and avoids rows with too many entries to perturb the spectral decomposition.

(b) **Removing diagonal entries:** Let \( \tilde{A} \) denote the trimmed matrix. The diagonal entries of the covariance matrix \( \tilde{A}^\dagger \tilde{A} \) are then removed: \( \Phi = \tilde{A}^\dagger \tilde{A} - \text{diag}(\tilde{A}^\dagger \tilde{A}) \). This step is needed because the diagonal entries of \( \tilde{A}^\dagger \tilde{A} \) scale as \( \delta \), whereas its off-diagonal entries scale as \( \delta^2 \).

Hence, when \( \delta \to 0 \), if the diagonal entries are not removed, they would be clearly dominant in the spectral decomposition.

In summary, the SPCA algorithm consists in applying the QR algorithm to the regularized version of \( A \), i.e., to \( \Phi \). Its pseudo-code is presented in Algorithm 1. The following theorem provides a performance analysis of SPCA, and is of independent interest.

**Theorem 1** Let \( \ell < m, \ell = o(1/\delta) \), and \( M \in [0, 1]^{m \times \ell} \) with singular value decomposition \( M = U \Sigma V^\dagger \), where \( \Sigma = \text{diag}(s_1(M), \ldots, s_\ell(M)) \) with \( s_1(M) \geq \cdots \geq s_\ell(M) \geq 0 \). Let \( A = \mathcal{P}_\Omega (M + X) \). Assume that there exists \( k \leq \ell \) such that \( s_k(M) = \omega(\sqrt{m}) \), \( \frac{s_k(M)}{s_{k+1}(M)} = \omega(1) \), and \( \frac{\delta^2 s_k^2(M)}{\sqrt{m \ell \log \ell}} = \omega(1) \). Let \( \hat{V}_{1:k} \) be the output of SPCA with input \( A \) and \( k > 0 \). Then we have \( \| (V_{1:k})^\dagger \cdot (\hat{V}_{1:k})_\perp \|_2 = o(1) \) with high probability.

Note that the condition \( \frac{\delta^2 s_k^2(M)}{\sqrt{m \ell \log \ell}} = \omega(1) \) in Theorem 1 is similar to that suggested by the random matrix theory argument presented above. However we loose a log factor here because we use, in the
Algorithm 3 Streaming Matrix Completion (SMC)

**Input:** \{A_1, \ldots, A_n\}, k, \ell
1. \(A^{(B)} \leftarrow [A_1, \ldots, A_k]\)
2. \(\hat{\delta} \leftarrow \frac{1}{m\ell} \sum (i,j) 1(|A^{(B)}|_{ij} > 0)\)
3. \(A^{(B_1)}, A^{(B_2)}, A^{(B_3)}, A^{(B_4)} \leftarrow \text{Split}(A^{(B)}, 4, 4, \hat{\delta})\)
4. (PCA for the first block) \(Q \leftarrow \text{SPCA}(A^{(B_1)}, k)\)
5. (Trimming rows and columns)
   - \(A^{(B_2)} \leftarrow \) make the rows having more than two observed entries to zero rows
   - \(A^{(B_2)} \leftarrow \) make the columns having more than 10\(m\hat{\delta}\) non-zero entries to zero columns
6. (Reference Columns) \(W \leftarrow A^{(B_2)}Q\)
7. (Principle row vectors) \(V^{1:\ell} \leftarrow (A^{(B_3)})^\dagger W\)
8. (Principle column vectors) \(\hat{I} \leftarrow A^{(B_4)}V^{1:\ell}\)

**Remove** \(A^{(B)}, A^{(B_1)}, A^{(B_2)}, A^{(B_3)}, A^{(B_4)}, Q\) from the memory space

for \(t = \ell + 1\) to \(n\) do
9. \(A^{(1)}_t, A^{(2)}_t \leftarrow \text{Split}(A_t, 2, 4, \hat{\delta})\)
10. (Principle row vectors) \(\hat{V}^{t} \leftarrow (A^{(1)}_t)^\dagger W\)
11. (Principle column vectors) \(\hat{I} \leftarrow \hat{I} + A^{(2)}_t\hat{V}^{t}\)

**Remove** \(A_t\) and \(A_t^\dagger\) from the memory space
end for
12. \(\hat{R} \leftarrow \text{find} \hat{R}\) using the Gram-Schmidt process such that \(\hat{V}\hat{R}\) is an orthonormal matrix.
13. \(\hat{U} \leftarrow \frac{1}{\delta} \hat{I} \hat{R}\)

**Matrix completion:** \(|\hat{U}\hat{V}^\dagger|_0^\dagger\)

proof, the Matrix Bernstein inequality (Theorem 6.1 of Tropp (2012)). The condition \(\frac{s_k(M)}{s_{k+1}(M)} \to \infty\) ensures a good separation in the spectrum of \(M\) and is needed to ensure that the space spanned by \(V_{k+1}^{1:\ell}\) is nearly orthogonal to the space spanned by \(V_{1:k}\) by Davis-Kahan \(\sin \Theta\) Theorem (Theorem VII.3.2 in Bhatia (1997)). We conclude this section by analyzing the memory required by the SPCA algorithm, and its computational complexity.

**Required memory.** SPCA needs to store \(A, \Phi = \tilde{A}^\dagger \tilde{A} = \text{diag}(\tilde{A}^\dagger \tilde{A})\), and \(\hat{V}\). The number of non-zero entries of \(A\) is \(O(\delta m\ell)\), and for each entry we need to store its id and its value. Hence for \(A, O(\delta m\ell \log(m))\) memory is required. Similarly, the required memory for \(\Phi\) is \(O(\delta^2 m\ell^2 \log(\ell))\). Finally, storing \(V_{1:k}\) requires \(O(\ell k)\) memory. Overall the required memory is \(O(\delta m\ell \log(m) + \ell k)\).

**Computational complexity.** To run SPCA, we have to compute \(\Phi\) and apply the QR algorithm to \(\Phi\). The computation of \(\Phi\) requires to perform \(\ell(\ell-1)/2\) inner products of columns of \(\tilde{A}\). Each inner product requires \(O(\delta^2 m)\) floating-point operations, and thus the computational complexity to compute \(\Phi\) is \(O(\delta^2 m\ell^2)\). Now in the QR algorithm, we compute \(\Phi Q_T\) and run the QR decomposition \(\log(\ell)\) times. The matrix product \(\Phi Q_T\) requires \(O(\delta^2 m\ell^2 k)\) floating-point operations, while the QR decomposition requires \(O(\ell k^2)\) operations. Hence, the QR algorithm needs \(O(\ell k(\delta^2 m\ell + k) \log(\ell))\) operations. Overall, the computational complexity of SPCA is \(O(\ell k(\delta^2 m\ell + k) \log(\ell))\).

4. **Matrix completion with Streaming Input**
In this section, we present our main algorithm, SMC, that reconstructs a matrix $M \in [0,1]^{m \times n}$ from a few noisy observations on its entries, i.e., from $A = P_{\Omega} (M + X)$. The pseudo-code of SMC is presented in Algorithm 3. SMC consists in three main steps: Step 1) Generate reference columns denoted by $W$, Step 2) Find principle row vectors $\hat{V}$ using $W$, and Step 3) Find $\hat{U}$ such that $\hat{U} \cdot \hat{V}^\dagger \approx M$. In what follows, we explain each of these steps in details and show for each step which conditions of Assumption 1 are needed. All proofs are presented in Appendix. The singular value decomposition of $M$ is $M = U \Sigma V^\dagger$.

4.1. Step 1: Finding reference columns $W$

We now explain the first step of the algorithm leading to a $m \times k$ matrix $W$ containing reference columns. This step corresponds to lines 1 to 6 in the pseudo-code.

Let $A^{(B)} = A_{1: \ell}$ be the batch of the $\ell$ first arriving columns of $A$. Note in particular that we have:

$$A^{(B)} = \mathcal{P}_\Omega(M^{(B)} + X_{1: \ell}) \quad \text{with}, \quad M^{(B)} = M_{1: \ell} = U \Sigma \left(V_{1: \ell}^\dagger\right)^\dagger.$$

In line 2, we compute $\hat{\delta}$, an estimate of the sampling rate $\delta$. In line 3, we construct 4 undersampled copies of $A^{(B)}$. For $i \in \{1, 2, 3, 4\}$, the different $A^{(B)_i}$’s are independent given $M + X$ and have the same distribution as $A^{(B)}$, except that the parameter $\delta$ is now replaced by $\delta/4$.

The first non-trivial operation is presented in line 4 where we apply the algorithm SPCA described in previous section to the matrix $A^{(B)_1}$. In order to apply our Theorem 1, we need to have:

$$\frac{s_k(M^{(B)})}{s_{k+1}(M^{(B)})} = \omega(1) \quad \text{and} \quad \frac{\delta s_k^2(M^{(B)})}{\sqrt{m\ell \log \ell}} = \omega(1). \quad (2)$$

Note that there is a slight abuse of notation as the distribution of $A^{(B)_1}$ is the same as the one of $A^{(B)}$ if we change $\delta$ to $\delta/4$ but a constant factor 4 is clearly irrelevant here. Our first task is to translate the conditions (2) on the original matrix $M$. To this aim, we state the following lemma:
Lemma 2 Let $M = U \Sigma V^\dagger$ be a $m \times n$ matrix and $\ell \leq n$. Denote by $M^{(B)} = M_{1:\ell}$. If $s_k^2(M) = \frac{\omega(mn \log m)}{\ell}$ and $s_{k+1}(M) = \omega(1)$, then with high probability,

$$s_k(U_{1:k}U_{1:k}^\dagger M^{(B)}) \geq \sqrt{\frac{\ell}{2n}} s_k(M) \quad \text{and} \quad \frac{s_k(U_{1:k}U_{1:k}^\dagger M^{(B)})}{s_1((I - U_{1:k}U_{1:k}^\dagger)M^{(B)})} = \omega(1).$$

Its proof is given in Appendix A.2 and follows from the matrix Chernoff bound (Theorem 2.2 of Tropp (2011)).

Note that $U_{1:k}U_{1:k}^\dagger$ is the orthogonal projection on the span of $U_{1:k}$. As a result, we have $s_k(M^{(B)}) \geq s_k(U_{1:k}U_{1:k}^\dagger M^{(B)})$ by a simple application of the Courant-Fischer variational formulas for singular values. In particular, as soon as $s_{k}^2(M) \rightarrow \infty$, we see that the second condition in (2) is satisfied. To get the first condition in (2), we write:

$$M^{(B)} = U_{1:k}U_{1:k}^\dagger M^{(B)} + (I - U_{1:k}U_{1:k}^\dagger)M^{(B)},$$

note that the first matrix is of rank $k$ and we can use Lidskii’s inequality $s_{k+1}(A + B) \leq s_k(A) + s_1(B)$ to get:

$$s_{k+1}(M^{(B)}) \leq s_1((I - U_{1:k}U_{1:k}^\dagger)M^{(B)}).$$

Hence we have

$$\frac{s_k(M^{(B)})}{s_{k+1}(M^{(B)})} \geq \frac{s_k(U_{1:k}U_{1:k}^\dagger M^{(B)})}{s_1((I - U_{1:k}U_{1:k}^\dagger)M^{(B)})},$$

and the first condition in (2) follows from the second statement in Lemma 2 as soon as its conditions are satisfied. Combined with Lemma 2, Theorem 1 allows us to get the properties of $Q$ computed in line 4 of the Algorithm SMC:

Corollary 3 Assume that there exists $k$ and $\ell$ such that $s_k(M) = \omega(1)$, $\frac{\ell^2 s_k(M)}{mn^2 \log \ell} = \omega(1)$, and $s_k^2(M) = \omega\left(\frac{mn \log m}{\ell}\right)$. Let $V^{1: \ell}$ be an orthonormal basis of the linear span of $V_{1:k}^{1: \ell}$. Then we have $\|V^{1: \ell\dagger}Q \| = o(1)$ with high probability, where $Q$ is the $\ell \times k$ matrix obtained in line 4 of the Algorithm SMC.

Once we have $Q$, we compute what we call the reference columns as follows:

$$W = A^{(B_2)} \cdot Q.$$ 

Note that $W$ will be kept in memory during the whole algorithm. It is relatively easy to see that the linear span of the columns of $W$ is a noisy version of the linear span of $U_{1:k}$. Indeed, note that $\mathbb{E}[A^{(B_2)}] = \frac{\delta}{4} M^{(B)}$, moreover we have $M^{(B)} = U \Sigma (V^{1: \ell\dagger} \approx U_{1:k} \Sigma [k] (V_{1:k}^{1: \ell\dagger}$ thanks to Lemma 2. Hence we have

$$W = A^{(B_2)} Q \approx \frac{\delta}{4} U_{1:k} \Sigma [k] (V_{1:k}^{1: \ell\dagger}) Q.$$ 

By Corollary 3, the span of the columns of $Q$ is approximately the span of the column of $V_{1:k}^{1: \ell\dagger}$ so that the singular values associated to the linear span of $U_{1:k}$ are $\Omega(\delta s_k(M^{(B)})) = \Omega(\delta \sqrt{\ell/n} s_k(M))$ by Lemma 2. This value has to be compared to the noise level. For the same reason as in Section 3, we first trim the matrix $A^{(B_2)}$ (note that the first trimming phase in line 5 is made to ease the technical
proof). After the trimming process, the singular values of $(A^{(B_2)} - E[A^{(B_2)}]) \cdot Q$ are bounded by $O(\sqrt{\delta m \ell})$. Unfortunately, in our setting this can be much larger than $\delta \sqrt{\ell / ns_k(M)}$. However, the hidden signal in $W$ is in the span of the columns of $U_{1:k}$ and all the columns that arrive belong (approximately) to this span. In the sequel, we use this fact in order to amplify the signal in $W$ when estimating $V$ and then $U$.

4.2. Step 2: Finding principle row vectors $\hat{V}$

In this section, we explain how we recover $V_{1:k}$ or at least $k$ vectors having the same linear span as $V_{1:k}$.

Let $A^{(1)} = [A^{(B_3)}, A^{(1)}_{\ell+1}, \ldots, A^{(1)}]$.

Note that thanks to the splitting procedure in line 9, the columns of $A^{(1)}$ are i.i.d. with sampling rate $\delta/4$. In the SMC algorithm, we simply get an estimate of $V$ as follows: $\hat{V} = (A^{(1)})^\dagger W$. The linear span of the columns of $\hat{V}$ becomes very close to the linear span of the columns of $V_{1:k}$ when

$$\frac{s_k(V_{1:k}V_{1:k}^\dagger \hat{V})}{s_1((I - V_{1:k}V_{1:k}^\dagger)\hat{V})} = o(1).$$

This can be seen as in Section 4.1 since $V_{1:k}V_{1:k}^\dagger$ is simply the orthogonal projection on $V_{1:k}$.

The above condition holds for the following reasons:

- The signal is amplified (Lemma 12 in Appendix). Since $E[A^{(1)}] = \frac{\delta}{4} M$, we see that

$$\hat{V} = (A^{(1)})^\dagger W \approx \frac{\delta^2}{16} V \Sigma U_1 U_{1:k} \Sigma \{1:k\}^\dagger Q$$

$$\approx \frac{\delta^2}{16} V \Sigma \{1:k\}^\dagger Q.$$

Roughly, the signal which was $\Omega(\sqrt{\ell / ns_k(M)})$ is now multiplied by $\delta s_k(M)$ and we get:

$$s_k(V_{1:k}V_{1:k}^\dagger \hat{V}) = \Omega(V_{1:k}V_{1:k}^\dagger E[A^{(1)}]^\dagger E[A^{(B_2)}]Q) = \Omega(\delta^2 s_k^2(M) \sqrt{\ell / n}).$$

- The noise is cancelled (Lemma 13 in Appendix). Since the two noise matrices $A^{(B_2)} - E[A^{(B_2)}]$ and $A^{(1)} - E[A^{(1)}]$ are independent, the noise directions are not amplified as much as the signals. We can bound the noise as follows:

$$s_1((I - V_{1:k}V_{1:k}^\dagger)\hat{V}) = o(\delta^2 s_k^2(M) \sqrt{\ell / n}).$$

Putting things togetehr, we obtain the following result:

**Theorem 4** Assume that there exists $k$ and $\ell$ such that $s_k^2(M) = \omega(\frac{mn \log m}{\ell})$, $s_k(M), s_k(M)/s_{k+1}(M) = \omega(1)$, and $\frac{\delta^2 \ell s_k^4(M)}{mn^2(k + \log \ell)} = o(1)$. Then we have $\|V_{1:k}^\dagger(\hat{V}_{1:k})\| = o(1)$ with high probability.
4.3. Step 3: Finding principle column vectors $\hat{U}$

In the previous step, we identified a $n \times k$ matrix $\hat{V}$ estimating the principle row vectors of $M$. From this estimate, we now extract the matrix $\hat{U}$ such that $\|\hat{U}\hat{V}^\dagger - M\|_F = o(mn)$.

Let $A^{(2)} = [A^{(B_1)}, A^{(B_2)}, \ldots, A^{(B_s)}]$. For simplicity, suppose that the linear span of the rows of $\hat{V}^\dagger$ is exactly the same as the linear span of the rows of $M$. From $\hat{V}$, we can generate a $k \times k$ matrix $\hat{R}$ using the Gram-Schmidt process so that $\hat{V} \hat{R}$ becomes an orthogonal matrix. Since $\hat{V} \hat{R}$ is an orthonormal basis of the linear span of the rows of $M$, we have

$$M = \frac{4}{\delta} E[A^{(2)}] \hat{V} \hat{R} (\hat{V} \hat{R})^\dagger = \left( \frac{4}{\delta} E[A^{(2)}] \hat{V} \hat{R} \right) \cdot \hat{V}^\dagger = \tilde{U} \hat{V}^\dagger,$$

where $\tilde{U} = \frac{4}{\delta} E[A^{(2)}] \hat{V} \hat{R} \hat{R}^\dagger$.

From the above observation, we propose to compute $\hat{U}$ as follows:

$$\hat{U} = \frac{4}{\delta} \hat{I} \hat{R} \hat{R}^\dagger = \frac{4}{\delta} A^{(2)} \hat{V} \hat{R} \hat{R}^\dagger = \frac{4}{\delta} E[A^{(2)}] \hat{V} \hat{R} \hat{R}^\dagger = \left( 1 + o(1) \right) \hat{U}.$$

This is true only if $n$ is large enough, indeed $n = \omega(k/\delta)$ (see Appendix).

We are now ready to analyze the performance of the SMC algorithm. We first need to check that Assumption 1 implies the technical conditions required in our previous results. When $M$ has $k$ dominant singular values such that $\frac{s_k(M)}{s_{k+1}(M)} = \omega(1)$ and $\sum_{i > k} s_i^2(M) = o(mn)$, then, $s_k^2(M) = \Omega\left( \frac{mn}{k} \right)$. To see this, assume this is not the case so that there exists $k' < k$ such that $s_{k'} = \omega\left( \frac{mn}{k} \right)$ and $s_{k'+1}^2(M) = o(mn)$. But then $\frac{s_{k'}(M)}{s_{k'+1}(M)} = \omega(1)$ and $\sum_{i > k'} s_i^2 = o(mn)$ which contradicts the minimality of $k$. Therefore, the conditions $s_k^2(M) = \omega\left( \frac{mn \log m}{k^2} \right)$ and $\frac{s_k^2(M)}{mn^2 \log (k+\log \ell)} = \omega(1)$ become $\ell = \omega(k \log m)$ and $\frac{s_k^2(m)}{k^2(\log m)} = \omega(1)$, which are satisfied by Assumption 1 when $\ell = O(m)$ and $\ell = \Omega\left( \frac{k}{\delta \log m} \right)$. Hence we obtain the following result:

**Theorem 5** Assume that Assumption 1 is satisfied with $\ell = \Omega\left( \frac{k}{\delta \log m} \right)$ and $\ell = O(m)$. Then with high probability, the SMC algorithm provides an asymptotically accurate estimate of $M$:

$$\frac{\|M - [\hat{U}\hat{V}^\dagger]_0\|_F}{mn} = o(1).$$

4.4. Required Memory

Next we analyze the memory required by the SMC algorithm. From line 1 to 8 in the pseudo-code. We need to store $A^{(B)}$, $A^{(B_1)}$, $A^{(B_2)}$, $A^{(B_3)}$, and $A^{(B_4)}$. Since these matrices are sparse with sampling rate $\delta$ or $\delta/4$, we need to store only $O(\delta \ell \ell)$ of their elements and $O(\delta m \ell \log m)$ bits to store the ID of the non-zero entries. From the previous section, we know that the SPCA algorithm requires $O(\delta m \ell \log m + k \ell)$ memory to find $Q$. Finally we need
to store $\hat{V}$ and $\hat{I}$. Thus, when $\ell = \frac{k}{\delta \log m}$, this first part of the algorithm requires $O(km + kn)$.

From line 9 to 11. Here we treat the remaining columns. Note that before doing that, $A(B), A(B_1), A(B_2), A(B_3)$, and $Q$ are removed from the memory. Using this memory, for the $t$-th arriving column, we can store it, compute $\hat{V}^t$ and $\hat{I}$, and remove the column to save memory. Therefore, we do not need additional memory to treat the remaining columns.

Lines 12 and 13. From $\hat{I}$ and $\hat{V}$, we compute $\hat{U}$. To this aim, the memory required is $O(km + kn)$.

In summary, we have:

**Theorem 6** When $\ell = \frac{k}{\delta \log(m)}$, the memory required to run the SMC algorithm is $O(km + kn)$.

### 4.5. Computational Complexity

The computational complexity of the SMC (Algorithm 3) depends on the number of non-zero elements of $A$ and $\ell$. More precisely:

From line 1 to 8. From the previous section, the SPCA algorithms requires $O(\ell k (\delta^2 m \ell + k) \log(\ell))$ floating-point operations to compute $Q$. The computations of $W$, $V$, and $I$ are just inner products, and require $O(\ell (\delta^2 m \ell + k) \log(\ell))$ operations.

From line 9 to 11. To compute $\hat{V}^t$ and $\hat{I}$ when the $t$-th column arrives, we need $O(km \delta)$ operations. Since there are $n - \ell$ remaining columns, the total number of operations is $O(kmn \delta)$.

Lines 12 and 13 $\hat{R}$ is computed from $\hat{V}$ using the Gram-Schmidt process which requires $O(k^2 m)$ operations. We then compute $\hat{R}^\dagger$ using $O(k^2 m)$ operations.

When $\ell = \frac{k}{\delta \log(m)}$ and $k^2 = O(\delta n)$, the number of operations to treat the first $\ell$ columns is

$$O(\ell k (\delta^2 m \ell + k) \log(\ell)) = O(k \delta^2 m \ell^2 \log(\ell)) + O(\ell k^2 \log(\ell))$$

$$= O(k^3 m \frac{\log \ell}{\log^2 m}) + O(\delta mn) = O(kmn \delta).$$

Since the remaining part of the algorithm requires $O(\delta kmn)$ operations as well, we conclude: Theorem 7.

**Theorem 7** Assume that Assumption 1 is satisfied with $\ell = \frac{k}{\delta \log(m)}$. Then, the computational complexity of the SMC algorithm is $O(\delta kmn)$.

### 5. Conclusion

This paper investigated the streaming memory-limited matrix completion problem when the observed entries are noisy versions of a small random fraction of the original entries. We proposed a streaming algorithm which produces an estimate of the original matrix with a vanishing mean square error, uses memory space scaling linearly with the ambient dimension of the matrix, i.e. the memory required to store the output alone, and spends computations as much as the number of non-zero entries of the input matrix. Our algorithm is relatively simple, and in particular, it does exploit elaborated techniques (such as sparse embedding techniques) recently developed to reduce the memory requirement and complexity of algorithms addressing various problems in linear algebra.
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Appendix A. Appendix

A.1. Proof of Theorem 1

We can split \( \Phi \) as follows:

\[
\Phi = \delta^2 V_{1:k} V_{1:k}^\dagger M + \Phi - \delta^2 V_{1:k} V_{1:k}^\dagger M. 
\]

The power method can find \( \hat{V} \) such that \( \| \hat{V}^\dagger (V_{1:k})_\perp \|_2 = o(1) \) when \( \frac{\delta^2 s_k(M^\dagger M)}{\| \Phi - \delta^2 V_{1:k} V_{1:k}^\dagger M \|_2} = \omega(1) \) which is shown in Lemma 11 of Yun et al. (2014). Since

\[
\| \Phi - \delta^2 V_{1:k} V_{1:k}^\dagger M \|_2 \leq \| \mathbb{E}[\Phi] - \delta^2 V_{1:k} V_{1:k}^\dagger M \|_2 + \| \Phi - \mathbb{E}[\Phi] \|_2 \\
\leq \| \delta^2 \text{diag}(M^\dagger M) \|_2 + \| \delta^2 (I - V_{1:k} V_{1:k}^\dagger) M^\dagger M \|_2 + \| \Phi - \mathbb{E}[\Phi] \|_2 \\
\leq \delta^2 m + \delta^2 s_{k+1}(M) + \| \Phi - \mathbb{E}[\Phi] \|_2, 
\]

in the remaining part, we transform \( \Phi - \mathbb{E}[\Phi] \) as a sum of random matrices, and then using Matrix Bernstein inequality we get an upper bound for \( \| \Phi - \mathbb{E}[\Phi] \|_2 \) to conclude this proof.

Recall that \( A^i \) is the \( i \)-th low of \( A \) and

\[
\Phi - \mathbb{E}[\Phi] = \sum_{i=1}^m \left( (A^i)^\dagger A^i - \text{diag}((A^i)^\dagger A^i) - \mathbb{E}[(A^i)^\dagger A^i] - \text{diag}((A^i)^\dagger A^i) \right).
\]

Let \( X^{(i)} = (A^i)^\dagger A^i - \text{diag}((A^i)^\dagger A^i) - \mathbb{E}[(A^i)^\dagger A^i] - \text{diag}((A^i)^\dagger A^i) \). Then \( X^{(i)} \) is a self-adjoint \( \ell \times \ell \) matrix and \( \mathbb{E}[X^{(i)}] = 0. \)

The Matrix Bernstein inequality (Theorem 6.1 Tropp (2012)) is a matrix concentration inequality for the sum of zero mean random matrices.

**Proposition 8 (Matrix Bernstein)** Consider a finite independent random matrix set \( \{X^{(i)}\}_{1 \leq i \leq m} \), where every \( X^{(i)} \) is self-adjoint with dimension \( n \), \( \mathbb{E}[X^{(i)}] = 0 \), and \( \|X^{(i)}\|_2 \leq R \) almost surely. Let \( \rho^2 = \| \sum_{i=1}^m \mathbb{E}[X^{(i)} X^{(i)}]\|_2 \). Then,

\[
\mathbb{P}\{ \| \sum_{i=1}^m X^{(i)} \|_2 \geq x \} \leq n \exp \left( \frac{-x^2/2}{\rho^2 + Rx/3} \right).
\]

In order to use the Matrix Bernstein inequality, we have to find upper bounds for \( \|X^{(i)}\|_2 \) and \( \rho^2 \). Since \( A^i \) are independently sampled with probability \( \delta \), \( [X^{(i)}]_{uv} \) has a some constant value if both \( u \) and \( v \) are sampled in \( A_i \) and \( O(\delta^2) \) otherwise. Using these, the following lemmas bound \( \|X^{(i)}\|_2 \) and \( \rho^2 \).

**Lemma 9** When \( n = \omega(1) \), for \( 1 \leq i \leq m \), there exists a constant \( C_1 \) such that

\[
\|X^{(i)}\|_2 \leq C_1 \max\{1, \delta \ell\}.
\]

**Proof:** Since the number of non-zero entries of \( A^i \) is bounded by \( \max\{10, 10\delta \ell\} \), we can easily compute \( r_u = \sum_{i \neq u} |X^{(i)}|_{uv} | \leq \max\{10, 10\delta \ell\} + \delta \ell \) for all \( 1 \leq i \leq m \) and \( 1 \leq u \leq \ell \). By the Gershgorin circle theorem, therefore, for all \( i \)

\[
\|X^{(i)}\|_2 \leq \max\{10, 10\delta \ell\} + \delta \ell.
\]

\( \blacksquare \)
Lemma 10 There exists a constant $C_2$ such that
\[
\|\sum_{i=1}^{m} E[X^{(i)} X^{(i)}]\|_2 \leq C_2 m \max\{\delta^2 \ell, \delta^3 \ell^2\}.
\]

**Proof:** Since the number of non-zero entries of $A^i$ is bounded by $\max\{10, 10\delta \ell\}$, every $|E[X^{(i)} X^{(i)}]|_{uv} = O(\delta^2 (1 + \delta \ell))$ when $u \neq v$ and every $|E[X^{(i)} X^{(i)}]|_{uu} = O(\delta^2 \ell (1 + \delta \ell))$. By the Gershgorin circle theorem, therefore
\[
\|\sum_{i=1}^{m} E[X^{(i)} X^{(i)}]\|_2 = O(\delta^2 m \ell (1 + \delta \ell)).
\]

Let $C = 16 \max\{C_1, C_2\}$. From Lemma 9 and 10 and Proposition 8,
\[
P\left\{ \|\Phi - E[\Phi]\|_2 \geq \sqrt{C \log(n)} \max\{1, \delta^2 m \ell, \delta^3 m \ell^2\} \right\} \leq \frac{1}{\ell^2}. \tag{4}
\]

**Proof of Theorem 1:** This proof starts with
\[
\Phi = \delta^2 V_{1:k} V_{1:k}^\dagger M^M + \Phi - \delta^2 V_{1:k} V_{1:k}^\dagger M^M = \delta^2 V_{1:k} V_{1:k}^\dagger M^M + Y,
\]
where $Y = \Phi - \delta^2 V_{1:k} V_{1:k}^\dagger M^M$. From (3) and (4)
\[
\|Y\|_2 \leq \delta^2 m + \delta^2 s_{k+1}^2(M) + \sqrt{C \log(\ell)} \max\{1, \delta^2 m \ell, \delta^3 m \ell^2\}
\]
\[
= o(\delta^2 s_k^2(M)) + \sqrt{C \log(\ell)} \max\{1, \delta^2 m \ell\},
\]
where the last equality stems from $s_k^2(M) = \omega(m)$ and $s_{k+1}^2(M) = \omega(1)$ the conditions of this theorem. Since the condition $\frac{\delta^2 s_k^2(M)}{m \ell \log \epsilon} = \omega(1)$ implies $\delta^2 m \ell = \omega(k^2 \log \ell)$ and $\frac{\delta^2 s_k^2(M)}{C \log(\ell) \max\{1, \delta^2 m \ell\}} = \omega(1)$, we can deduce $s_k^2(V_{1:k} V_{1:k}^\dagger M^M) = o(1)$ from Lemma 11 of Yun et al. (2014).

A.2. Proof of Lemma 2

Let $F = U_{1:k}^\dagger M^{(B)}$ and $G = (I - U_{1:k} U_{1:k}^\dagger) M^{(B)}$. We find a lower bound for $s_k(F)$ and an upper bound $s_1(G)$ using the matrix Chernoff bound (Theorem 2.2 in Tropp (2011)).

**Proposition 11 (Matrix Chernoff)** Let $\mathcal{X}$ be a finite set of positive-semidefinite matrices with dimension $d$ and satisfy $\max_{X \in \mathcal{X}} s_1(X) \leq \alpha$. Let
\[
\beta_{\min} = \frac{\ell}{|\mathcal{X}|} s_d(\sum_{X \in \mathcal{X}} X) \quad \text{and} \quad \beta_{\max} = \frac{\ell}{|\mathcal{X}|} s_1(\sum_{X \in \mathcal{X}} X).
\]

When $\{X^{(1)}, \ldots, X^{(\ell)}\}$ are sampled uniformly at random from $\mathcal{X}$ without replacement,
\[
P\left\{ s_1(\sum_{i=1}^{\ell} X^{(i)}) \geq (1 + \varepsilon) \beta_{\max} \right\} \leq d \left( \frac{e^\varepsilon}{(1 + \varepsilon)^{1+\varepsilon}} \right)^{\beta_{\max}/\alpha} \quad \text{for } \varepsilon \geq 0 \quad \text{and}
\]
\[
P\left\{ s_d(\sum_{i=1}^{\ell} X^{(i)}) \leq (1 + \varepsilon) \beta_{\min} \right\} \leq d \left( \frac{e^{-\varepsilon}}{(1 - \varepsilon)^{1-\varepsilon}} \right)^{\beta_{\min}/\alpha} \quad \text{for } \varepsilon \in [0, 1).
\]

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i) $s_k(F)$: $FF^\dagger$ is the sum of $\ell$ matrices which are sampled uniformly at random from $\mathcal{X} = \{U_{1:k}^\dagger M_1 (U_{1:k}^\dagger M_1)^\dagger, \ldots, U_{1:k}^\dagger M_n (U_{1:k}^\dagger M_n)^\dagger\}$ without replacement where the matrix dimension is $k$. We can obtain the other parameters to compute the matrix Chernoff as follows: $\alpha = m$ since $\|M_i\|^2 \leq m$ for all $1 \leq i \leq n$ and $\beta_{\min} = \frac{\ell}{n}s_k^2(M)$. From Proposition 11,
\[
\mathbb{P}\left\{s_k(FF^\dagger) \leq (1 - \varepsilon)\frac{\ell}{n}s_k^2(M) \right\} \leq k \left(\frac{e^{-\varepsilon}}{(1 - \varepsilon)^{1-\varepsilon}}\right) \frac{\ell}{mn}s_k^2(M) \quad \text{for } \varepsilon \in [0, 1).
\]
Therefore, when $s_k^2(M) = \omega\left(\frac{mn\log m}{\ell}\right)$,
\[
\mathbb{P}\left\{s_k^2(F) \leq \frac{\ell}{2n}s_k^2(M) \right\} \leq \frac{1}{m}.
\]
ii) $s_1(G)$: $GG^\dagger$ is the sum of matrices sampled uniformly at random without replacement from $\mathcal{X} = \{(I - U_{1:k}U_{1:k}^\dagger)M_1((I - U_{1:k}U_{1:k}^\dagger)M_1)^\dagger, \ldots, (I - U_{1:k}U_{1:k}^\dagger)M_n((I - U_{1:k}U_{1:k}^\dagger)M_n)^\dagger\}$. Here, the dimension is $m$, $\alpha = m$ and $\beta_{\max} = \frac{\ell}{n}s_{k+1}^2(M)$. From Proposition 11,
\[
\mathbb{P}\left\{s_1(GG^\dagger) \geq (1 + \varepsilon)\frac{\ell}{n}s_{k+1}^2(M) \right\} \leq m \left(\frac{e^\varepsilon}{(1 + \varepsilon)^{1+\varepsilon}}\right) \frac{\ell}{mn}s_{k+1}^2(M) \quad \text{for } \varepsilon \geq 0.
\]
When we set $\varepsilon^* = \max\{2, \frac{2mn\log m}{\ell s_{k+1}^2(M)}\}$, $\mathbb{P}\left\{s_1(GG^\dagger) \geq (1 + \varepsilon^*)\frac{\ell}{n}s_{k+1}^2(M) \right\} \leq \frac{1}{m}$ and $(1 + \varepsilon^*)s_{k+1}^2(M) \leq \max\{3s_{k+1}^2(M), \frac{3mn\log m}{\ell}\}$. Therefore,
\[
\frac{s_k(U_{1:k}U_{1:k}^\dagger M^{(B)})}{s_1((I - U_{1:k}U_{1:k}^\dagger)M^{(B)})} = \omega(1),
\]
since $s_k^2(M) = \omega\left(\frac{mn\log m}{\ell}\right)$ and $\frac{s_k(M)}{s_{k+1}(M)} = \omega(1)$.

### A.3. Proof of Theorem 4

We can rewrite $(A^{(1)})^\dagger W$ as follows:
\[
(A^{(1)})^\dagger W = \mathbb{E}[(A^{(1)})^\dagger W + ((A^{(1)})^\dagger - \mathbb{E}[(A^{(1)})^\dagger])W] = V_{1:k}V_{1:k}^\dagger \mathbb{E}[(A^{(1)})^\dagger W] + (I - V_{1:k}V_{1:k}^\dagger)\mathbb{E}[(A^{(1)})^\dagger W] + ((A^{(1)})^\dagger - \mathbb{E}[(A^{(1)})^\dagger])W.
\]
In the above equation, the columns of $(V_{1:k}V_{1:k}^\dagger \mathbb{E}[(A^{(1)})^\dagger W])$ have the same space what we want to recover and the remaining part is noise. Thus, we can easily recover $\hat{V}$ satisfying $\|V_{1:k}^\dagger \hat{V}\perp\| = o(1)$ when
\[
\frac{s_k(V_{1:k}V_{1:k}^\dagger \mathbb{E}[(A^{(1)})^\dagger W])}{\|(I - V_{1:k}V_{1:k}^\dagger)\mathbb{E}[(A^{(1)})^\dagger W]\|_2 + \|((A^{(1)})^\dagger - \mathbb{E}[(A^{(1)})^\dagger])W\|_2} = \omega(1). \tag{5}
\]
Before giving the proof of (5) to conclude the proof of Theorem 4, we introduce key lemmas. Lemma 12 finds a lower bound for $s_k(V_{1:k}V_{1:k}^\dagger \mathbb{E}[(A^{(1)})^\dagger W])$ and an upper bound for $\|(I - V_{1:k}V_{1:k}^\dagger)\mathbb{E}[(A^{(1)})^\dagger W]\|_2$ and Lemma 13 induces an upper bound for $\|((A^{(1)})^\dagger - \mathbb{E}[(A^{(1)})^\dagger])W\|_2$. 

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Lemma 12 When \( s_k^2(M) = \omega\left(\frac{mn \log m}{\ell}\right) \), \( s_k(M) = \omega(1) \), and \( \frac{\delta^2 s_k^2(M)}{mn^2(k + \log \ell)} = \omega(1) \), with high probability,

\[
s_k(V_{1:k}V_{1:k}^\dagger)\mathbb{E}[(A^{(1)})^\dagger W] = \Omega\left(\delta^2 s_k^2(M)\sqrt{\frac{\ell}{n}}\right) \quad \text{and} \quad \| (I - V_{1:k}V_{1:k}^\dagger)\mathbb{E}[(A^{(1)})^\dagger W] \|_2 = o\left(\delta^2 s_k^2(M)\sqrt{\frac{\ell}{n}}\right).
\]

Proof: The proof is given in Section A.4.

Lemma 13 For given \( Q \) and \( A^{(B_2)} \), \( \mathbb{E}[\|((A^{(1)})^\dagger - \mathbb{E}[(A^{(1)})^\dagger])W\|_F^2] = O(\delta^2 kmn) \).

Proof: Since every entry of \( A^{(1)} \) is randomly sampled with probability \( \delta/4 \) and \( W \) and \( A^{(1)} \) are independent, for all \( 1 \leq i \leq n \) and \( 1 \leq j \leq k \),

\[
\mathbb{E}\left[\|((A^{(1)}) - \mathbb{E}[A^{(1)}])^\dagger W\|_{ij}^2\right] = \mathbb{E}\left[\left(\sum_{u=1}^{n}[A^{(1)} - \mathbb{E}[A^{(1)}]]_{ui}[W]_{uj}\right)^2\right] \\
\leq \frac{\delta}{4}\|W\|^2 = O(\delta^2 m),
\]

where the last equality stems from the trimming process on \( A^{(B_2)} \). Thus,

\[
\mathbb{E}[\|((A^{(1)}) - \mathbb{E}[(A^{(1)})^\dagger])W\|_F^2] = O(\delta^2 kmn).
\]

Proof of Theorem 4: When \( \frac{\delta^2 s_k^2(M)}{kmn^2} = \omega(1) \), from Lemma 12, Lemma 13, and the Markov inequality, \( \|((I - V_{1:k}V_{1:k}^\dagger)^\dagger V')\|_2 = \omega(1) \), with high probability. Let \( \hat{V} = V\Sigma' (U')^\dagger \) be the singular value decomposition of \( \hat{V} \). Since

\[
\| (I - V_{1:k}V_{1:k}^\dagger)\hat{V}\|_2 \geq \| (I - V_{1:k}V_{1:k}^\dagger) V'\|_2 s_k(\hat{V}) = \| (V_{1:k})_{\perp} V'\|_2 s_k(\hat{V})
\]

and \( s_k(\hat{V}) = \Omega(s_k(V_{1:k}V_{1:k}^\dagger \hat{V})) \) from the Lidskii inequality \( s_{k+1}(A + B) \geq s_k(A) - s_{k+1}(A) \),

\[
\frac{s_k(V_{1:k}V_{1:k}^\dagger \hat{V})}{\| (I - V_{1:k}V_{1:k}^\dagger)^\dagger V'\|_2} = \omega(1) \text{ implies } \| (V_{1:k})_{\perp} V'\|_2 = o(1).
\]

Therefore, with high probability,

\[
\| V_{1:k}^\perp (\hat{V})_{\perp} \|_2 = \sqrt{1 - s_k^2(V_{1:k}^\dagger V')} = \| (V_{1:k})_{\perp} V'\|_2 = o(1).
\]

A.4. Proof of Lemma 12

Since \( W = A^{(B_2)}Q = \mathbb{E}[A^{(B_2)}]Q + (A^{(B_2)} - \mathbb{E}[A^{(B_2)}])Q \), we find a lower bound for \( s_k(V_{1:k}V_{1:k}^\dagger \mathbb{E}[(A^{(1)})^\dagger W]) \) and an upper bound for \( \| (I - V_{1:k}V_{1:k}^\dagger)^\dagger \mathbb{E}[(A^{(1)})^\dagger W] \|_2 \) from

\[
s_k(V_{1:k}V_{1:k}^\dagger \mathbb{E}[(A^{(1)})^\dagger W]) \geq s_k(V_{1:k}V_{1:k}^\dagger \mathbb{E}[(A^{(1)})^\dagger W] \mathbb{E}[A^{(B_2)}]Q - \| (\mathbb{E}[A^{(1)})^\dagger (A^{(B_2)} - \mathbb{E}[A^{(B_2)}])Q) \|_2 \quad \text{and}
\]

\[
\| (I - V_{1:k}V_{1:k}^\dagger)^\dagger \mathbb{E}[(A^{(1)})^\dagger W] \|_2 \leq \| (\mathbb{E}[A^{(1)})^\dagger (A^{(B_2)} - \mathbb{E}[A^{(B_2)}])Q) \|_2.
\]
\[ \| (I - V_{1:k} V_{1:k}^\dagger) E[(A^{(1)})\dagger] W \|_2 \leq \| (I - V_{1:k} V_{1:k}^\dagger) E[(A^{(1)})\dagger] E[A^{(B_2)}] Q \|_2 + \| (E[(A^{(1)})\dagger] E[A^{(B_2)}]) Q \|_2. \]  

(6)

**Key lemmas:** The following lemmas bound each element of the above inequalities. To show the lemmas, we use Corollary 3: \[ \| (V_{1:k})^\dagger Q_\perp \| = o(1) \] with high probability when \( \sigma_k^2(M) = \omega(\frac{mn \log m}{\epsilon^2}) \), \( \frac{s_k(M)}{s_{k+1}(M)} = \omega(1) \), and \( \frac{\delta^2 \epsilon_k^4(M)}{mn^2 \log \epsilon} = \omega(1) \).

**Lemma 14** When \( s_k^2(M) = \omega(\frac{mn \log m}{\epsilon}) \), \( \frac{s_k(M)}{s_{k+1}(M)} = \omega(1) \), and \( \frac{\delta^2 \epsilon_k^4(M)}{mn^2 \log \epsilon} = \omega(1) \), with high probability,

\[
s_k(V_{1:k} V_{1:k}^\dagger) E[(A^{(1)})\dagger] E[A^{(B_2)}] Q) = \Omega \left( \delta^2 s_k^2(M) \sqrt{\frac{\epsilon}{n}} \right) .
\]

**Proof:** Since every entry of \( A^{(B_2)} \) and \( A^{(1)} \) is randomly sampled with probability \( \delta/4 \), we know that \( E[(A^{(1)})\dagger] = \frac{\delta}{4} V \Sigma U^\dagger \) and \( E[A^{(B_2)}] = \frac{\delta}{4} U \Sigma (V^{1:\ell})^\dagger \). Under the conditions of this lemma, from Corollary 3 \( \| (V^{1:\ell})^\dagger Q_\perp \| = o(1) \) and from Lemma 2 \( s_k(U_{1:k}^\dagger M(B)) \geq \sqrt{\frac{\epsilon}{2n}} s_k(M) \) with high probability. Let \( \hat{R}^B \) be the \( k \times k \) matrix satisfying \( V_{1:k}^\dagger = \hat{V}^{1:\ell} \hat{R}^B \). Then,

\[
s_k(V_{1:k} V_{1:k}^\dagger) E[(A^{(1)})\dagger] E[A^{(B_2)}] Q) = \frac{\delta^2}{16} s_k(V_{1:k} V_{1:k}^\dagger M^{(B)}(Q))
\]

\[
= \frac{\delta^2}{16} s_k(V_{1:k} \Sigma_{1:k} \Sigma_{1:k}^\dagger (V^{1:\ell})^\dagger Q))
\]

\[
\geq \frac{\delta^2}{16} s_k(M) s_k(\Sigma_{1:k}^\dagger (V^{1:\ell})^\dagger Q))
\]

\[
= \frac{\delta^2}{16} s_k(M) s_k(\Sigma_{1:k}^\dagger (\hat{R}^B)^\dagger (V^{1:\ell})^\dagger Q))
\]

\[
\geq \frac{\delta^2}{16} s_k(M) s_k(\Sigma_{1:k}^\dagger (\hat{R}^B)^\dagger) s_k((V^{1:\ell})^\dagger Q))
\]

\[
= \Omega \left( \delta^2 s_k^2(M) \sqrt{\frac{\epsilon}{n}} \right) ,
\]

where the last equality stems from the fact that \( s_k(\Sigma_{1:k}^\dagger (\hat{R}^B)^\dagger) = s_k(M(B)) \) and \( s_k((V^{1:\ell})^\dagger Q)) = 1 - o(1) \).

**Lemma 15** When \( s_k^2(M) = \omega(\frac{mn \log m}{\epsilon}) \), \( \frac{s_k(M)}{s_{k+1}(M)} = \omega(1) \), and \( \frac{\delta^2 \epsilon_k^4(M)}{mn^2 \log \epsilon} = \omega(1) \), with high probability,

\[ \| (I - V_{1:k} V_{1:k}^\dagger) E[(A^{(1)})\dagger] E[A^{(B_2)}] Q \|_2 = o \left( \delta^2 s_k^2(M) \sqrt{\frac{\epsilon}{n}} \right) . \]

**Proof:** Since \( E[(A^{(1)})\dagger] = \frac{\delta}{4} V \Sigma U^\dagger \) and \( E[A^{(B_2)}] = \frac{\delta}{4} U \Sigma (V^{1:\ell})^\dagger \),

\[
(I - V_{1:k} V_{1:k}^\dagger) E[(A^{(1)})\dagger] E[A^{(B_2)}] Q) = \frac{\delta^2}{16} V_{k+1:n}^\dagger M_{k+1:n}^B Q
\]

\[
= \frac{\delta^2}{16} V_{k+1:n}^\dagger \Sigma_{k+1:n} M_{k+1:n}^B Q.
\]
Under the conditions of this lemma, \( s_1(U_{k+1:n \wedge m}^\dagger M(B)) = o\left(\sqrt{\frac{\ell}{n}}\sigma_k(M)\right) \) with high probability from Lemma 2. Therefore,

\[
\begin{align*}
\hspace{10pt} s_1((I - V_{1:k}V_{1:k}^\dagger)\mathbb{E}[(A^{(1)})^\dagger]\mathbb{E}[A^{(B_2)}]Q) &= \frac{\delta^2}{16} s_1(V_{k+1:n \wedge m}^\dagger \sum_{k+1:n \wedge m} \sum_{k+1:n \wedge m} V_{k+1:n \wedge m}^\dagger Q) \\
&\leq \frac{\delta^2}{16} s_{k+1}(M) s_1(\sum_{k+1:n \wedge m} V_{k+1:n \wedge m}^\dagger Q) \\
&\leq \frac{\delta^2}{16} s_{k+1}(M) s_1(\sum_{k+1:n \wedge m} V_{k+1:n \wedge m}^\dagger) \\
&= o\left(\frac{\delta^2 s_k^2(M)}{\sqrt{n}}\right),
\end{align*}
\]

where the last equality stems from the fact that \( s_k(M) = \omega(1) \) and \( s_1(\sum_{k+1:n \wedge m} V_{k+1:n \wedge m}^\dagger) = s_1(U_{k+1:n \wedge m}^\dagger M(B)) = o(s_k(M)\sqrt{\ell/n}) \).

**Lemma 16** With probability \( 1 - 1/\delta, \|\mathbb{E}[A^{(1)}]^\dagger ((A^{(B_2)} - \mathbb{E}[A^{(B_2)}])Q_{1:k})\|_2 = O(\sqrt{\delta^2 k m n}) \).

**Proof:** Since entries of \( A^{(B_2)} \) are randomly sampled with probability \( \delta/4 \) and independent with \( Q \), for all \( 1 \leq i \leq n \) and \( 1 \leq j \leq k \),

\[
\mathbb{E} \left[ \left(\mathbb{E}[A^{(1)}]^\dagger ((A^{(B_2)} - \mathbb{E}[A^{(B_2)}])Q)_{ij} \right)^2 \right] \\
= \mathbb{E} \left[ \left(\frac{\delta}{4} \sum_{u=1}^{m} \sum_{v=1}^{\ell} [M]_{ui} [A^{(B_2)}]_{uv} [Q]_{vj} \right)^2 \right] \\
= \frac{\delta^2}{16} \sum_{u=1}^{m} \sum_{v=1}^{\ell} [M]_{ui}^2 [Q]_{vj}^2 \mathbb{E} \left[ ([A^{(B_2)}] - \mathbb{E}[A^{(B_2)}])_{uv}^2 \right] \\
\leq \frac{\delta^2}{16} \sum_{u=1}^{m} \sum_{v=1}^{\ell} [M]_{ui}^2 [Q]_{vj}^2 \frac{\delta}{4} \leq \left(\frac{\delta}{4}\right)^3 m.
\]

From the above inequality, \( \mathbb{E}[(\mathbb{E}[A^{(1)}]^\dagger ((A^{(B_2)} - \mathbb{E}[A^{(B_2)}])Q))]_2^2 = \left(\frac{\delta}{4}\right)^3 k m n. \) Therefore, by the Markov inequality, we conclude this proof. \( \square \)

**Proof of Lemma 12:** When \( \frac{\delta^2 s_k^2(M)}{\sqrt{mn}^{2}(k+log \ell)} = \omega(1) \), \( s_k(V_{1:k}V_{1:k}^\dagger \mathbb{E}[(A^{(1)})^\dagger]W) = \omega(1) \). Inserting Lemma 14, Lemma 15, and Lemma 16 into (6), therefore, we conclude this proof:

\[
\begin{align*}
s_k(V_{1:k}V_{1:k}^\dagger \mathbb{E}[(A^{(1)})^\dagger]W) &= \Omega\left(\frac{\delta^2 s_k(M^\dagger M) \sqrt{\frac{\ell}{n}}}{\sqrt{n}}\right) \quad \text{and} \\
\|(I - V_{1:k}V_{1:k}^\dagger)\mathbb{E}[(A^{(1)})^\dagger]W\|_2 &= o\left(\frac{\delta^2 s_k(M^\dagger M) \sqrt{\frac{\ell}{n}}}{\sqrt{n}}\right).
\end{align*}
\]
A.5. Proof of Theorem 5

Let \( P_\hat{V} = \hat{V} \hat{R} \hat{R}^\dagger \hat{V}^\dagger \) which is an orthogonal projection matrix onto the linear span of \( \hat{V} \). Then, \( \hat{U} \hat{V} = \frac{1}{\delta} A^{(2)} P_\hat{V} \). We can bound \( \| \hat{U} \hat{V} \|_F^2 - M \|_F^2 \) using the projection \( P_\hat{V} \) as follows:

\[
\frac{4}{\delta} A^{(2)} P_\hat{V} \|_F^2 - M \|_F^2 = \| (M + \frac{4}{\delta} (A^{(2)} - \frac{\delta}{4} M)) P_\hat{V} \|_F^2 - M \|_F^2 \leq 2 \| M P_\hat{V} - M \|_F^2 + 2 \| \frac{4}{\delta} (A^{(2)} - \frac{\delta}{4} M) P_\hat{V} \|_F^2 \leq 2 \| M P_\hat{V} - M \|_F^2 + o(mn).
\]

where (a) stems from Lemma 17, (b) uses the fact that \( \| (I - U_{1:k} U_{1:k}^\dagger) M \|_F^2 = o(mn) \), and (c) holds since \( \| V^\dagger V \|_F^2 = o(1) \) from Theorem 4.

**Lemma 17** When \( n = \omega(K/\delta) \), with high probability, \( \frac{4}{\delta} (A^{(2)} - \frac{\delta}{4} M) P_\hat{V} \|_F^2 = o(mn) \).

**Proof:** Since entries of \( A^{(2)} \) are randomly sampled with probability \( \delta/4 \) and independent with \( \hat{V} \), for all \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \),

\[
\mathbb{E} \left[ \left( \left( \frac{A^{(2)} - \frac{\delta}{4} M}{P_\hat{V}} \right)_{ij} \right)^2 \right] = \mathbb{E} \left[ \left( \sum_{v=1}^{\infty} [A^{(2)} - \frac{\delta}{4} M]_{iv} [P_\hat{V}]_{vj} \right)^2 \right] = \sum_{v=1}^{\infty} [P_\hat{V}]_{ij}^2 \mathbb{E} \left[ \left( [A^{(2)} - \frac{\delta}{4} M]_{iv} \right)^2 \right] \leq \frac{\delta}{4} \sum_{v=1}^{\infty} [P_\hat{V}]_{ij}^2.
\]

Since \( \sum_{w=1}^{\infty} \sum_{v=1}^{\infty} [P_\hat{V}]_{vw}^2 = k \), from the above inequality,

\[
\mathbb{E} \left[ \left( \frac{4}{\delta} (A^{(2)} - \frac{\delta}{4} M) P_\hat{V} \|_F^2 \right) \right] = \left( \frac{4}{\delta} \right)^2 \sum_{i=1}^{m} \sum_{j=1}^{n} \mathbb{E} \left[ \left( [A^{(2)} - \frac{\delta}{4} M]_{ij} \right)^2 \right] \leq \frac{4km}{\delta}.
\]

Therefore, by the Markov inequality, we conclude this proof. \( \blacksquare \)