A global existence result for two-dimensional semilinear strongly damped wave equation with mixed nonlinearity in an exterior domain

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Abstract. We study two-dimensional semilinear strongly damped wave equation with mixed nonlinearity $|u|^p + |u_t|^q$ in an exterior domain, where $p, q > 1$. Assuming the smallness of initial data in exponentially weighted spaces and some conditions on powers of nonlinearity, we prove global (in time) existence of small data energy solution with suitable higher regularity by using a weighted energy method.

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1. Introduction

In this paper, we consider the following initial boundary value problem for two-dimensional semilinear strongly damped wave equation with mixed nonlinearity:

$$\begin{cases}
    u_{tt} - \Delta u - \Delta u_t = |u|^p + |u_t|^q, & x \in \Omega, \ t > 0, \\
    u(0, x) = u_0(x), \ u_t(0, x) = u_1(x), & x \in \Omega, \\
    u = 0, & x \in \partial \Omega, \ t > 0,
\end{cases} \tag{1.1}$$

with $p, q > 1$, where $\Omega \subset \mathbb{R}^2$ is an exterior domain with a compact smooth boundary $\partial \Omega$. Without loss of generality, we may assume that $0 \notin \overline{\Omega}$.

In the whole space $\mathbb{R}^n$, the Cauchy problem for the linear strongly damped wave equation can be modeled by

$$\begin{cases}
    u_{tt} - \Delta u - \Delta u_t = 0, & x \in \mathbb{R}^n, \ t > 0, \\
    u(0, x) = u_0(x), \ u_t(0, x) = u_1(x), & x \in \mathbb{R}^n,
\end{cases} \tag{1.2}$$

where $n \geq 1$. The authors of [23] and [24] investigated some $L^p - L^q$ estimates away of the conjugate line for the Cauchy problem (1.2). Recently, asymptotic profiles of solutions in $L^2$-norm to the Cauchy problem (1.2) under a framework of weighted $L^1$ data were derived in [11]. Concerning asymptotic profiles of solutions to the corresponding abstract form of strongly damped waves were derived in [17]. The authors of [6] obtained ($L^2 \cap L^1$) – $L^2$ estimate for (1.2) by using the partial Fourier transform and WKB analysis. By employing some energy estimates with suitable regularities, additionally, they proved global (in time) existence of small data solution to the semilinear Cauchy problem. More precisely, [6] considered the semilinear Cauchy problem with power nonlinearity $|u|^p$, namely,

$$\begin{cases}
    u_{tt} - \Delta u - \Delta u_t = |u|^p, & x \in \mathbb{R}^n, \ t > 0, \\
    u(0, x) = u_0(x), \ u_t(0, x) = u_1(x), & x \in \mathbb{R}^n,
\end{cases} \tag{1.3}$$
where \( p > 1 \) and \( n \geq 2 \). The authors of [6] proved global existence results for \( n \geq 2 \) if \( p > 1 + 3/(n-1) \) and \( p \in [2, n/(n-4)] \). Besides, applying the test function method, the result for nonexistence of global (in time) solutions has been proved providing that \( 1 < p \leq 1 + 2/(n-1) \). Here, we refer to Theorem 4.2 in [7].

Let us turn to initial boundary value problem with an exterior domain. To the best of the authors’ knowledge, there exist few results on the exterior problem for strongly damped wave equation. We refer to [14] for some estimates of solutions to the linearized exterior problem by using suitable energy method. Later, [15] proved global existence result for the semilinear strongly damped wave equation with \(|u|^p\) in 2D when \( p > 6 \). To be specific, there is a uniquely determined energy solution

\[
 u \in \mathcal{E} \left( [0, \infty), H^1_0(\Omega) \right) \cap \mathcal{C}^1 \left( [0, \infty), L^2(\Omega) \right)
\]
to the exterior problem

\[
\begin{cases}
 u_{tt} - \Delta u - \Delta u_t = |u|^p, & x \in \Omega, \ t > 0, \\
 u(0, x) = u_0(x), \ u_t(0, x) = u_1(x), & x \in \Omega, \\
 u = 0, & x \in \partial \Omega, \ t > 0,
\end{cases}
\]  

(1.4)

where \( \Omega \subset \mathbb{R}^2 \) is an exterior domain with a compact smooth boundary, for small initial data taken from weighted energy space and \( p > 6 \). On the other hand, the blow-up result for the exterior problem (1.4) for all dimensions has been recently derived by [8]. Considering the semilinear exterior problem for strongly damped wave with nonlinearity of derivative-type \(|u|^q\) or nonlinearity of mixed-type \(|u|^p + |u_t|^q\), the authors of [2] proved local (in time) existence of mild solution by energy estimates associated with the Banach fixed-point theorem and blow-up of solutions by the test function method in all dimension \( n \geq 1 \). For the other studied of damped wave equations in an exterior domain, we refer to [13, 20, 12, 10, 21, 18, 9, 25, 5] and references therein. So far the global (in time) existence of energy solution with higher regularity for semilinear strongly damped wave equations in an exterior domain is still unknown. Hence, in the present paper, we will give the answer to it by constructing some exponentially weighted energy estimates of higher-order.

Our main approach in proving global existence result is a exponentially weighted energy method, which has been introduced firstly in the pioneering work [27] and later in [16] with certain exponentially weighted spaces. In some sense, this method strongly relies on the choice of the weighted function. In this paper, motivated by [15], we introduce the weighted function by the following way:

\[
 \psi(t, x) = \frac{1}{\rho(1+t)^p} + \frac{|x|^2}{2(1+t)^{2+p}}
\]  

(1.5)

with a constant \( \rho > \rho_0 \), where

\[
 \rho_0 = \frac{-3 + \sqrt{13}}{4} \approx 1.386
\]

is the positive root of the quadratic equation

\[
 2\rho^2 + 3\rho_0 - 8 = 0.
\]

In order to prove existence of higher-order energy solution, we have to construct the exponentially weighted estimates for higher-order energy to (1.1), that is

\[
 \left\| e^{\psi(t, \cdot)} (\partial_t, \nabla) u(t, \cdot) \right\|_{L^2}^2 \text{ and } \left\| e^{\psi(t, \cdot)} \nabla (\partial_t, \nabla) u(t, \cdot) \right\|_{L^2}^2.
\]

We would like to remark that the estimate of the second energy mentioned in the above is more complex than those of the first energy, which will be shown in Propositions 3.1 and 3.2 in Section 3.

Let us point out that the study of the exterior problem with \(|u|^p + |u_t|^q\) is not simply a generalization of what happens for the exterior problem with \(|u|^p\) shown in [15]. On one hand, the application of the Gagliardo-Niremberg inequality with a weighted function (e.g. Lemma 2.3 in [15]) allows us to estimate the nonlinear term \(|u_t|^q\) in a weighted \( L^q \) space by its gradient in \( L^2 \) space and it in weighted \( L^2 \) space. Thus, we need to control a new energy of nonlinear exterior problems. On the other hand,
the interplay between the power nonlinearity $|u|^p$ and nonlinearity of derivative-type $|u|^q$ should be considered, which does not happen for the nonlinear problem carrying $|u|^p$ only.

Before stating our global existence result, we show some notations will be used in this paper.

By direct computations, the next properties for the function $\psi(t, x)$ are fulfilled:

$$\psi(t, x) < 0, \quad \Delta \psi(t, x) = \frac{2}{(1 + t)^{2+\rho}} \quad \text{and} \quad -\psi(t, x) \leq \frac{C_\rho}{1 + t} \psi(t, x),$$

where the positive constant $C_\rho$ independent of $x$ and $t$. Furthermore, it holds that

$$|\nabla \psi(t, x)|^2 - \psi(t, x)|\nabla \psi(t, x)|^2 - |\psi(t, x)|^2 \leq 0. \quad (1.6)$$

We assume $\varepsilon > 0$ be an auxiliary constant satisfying

$$\frac{4\rho + 14}{(2 + \rho)(2\rho + 3)} \leq \varepsilon < 1. \quad (1.7)$$

Next, we define the space-dependent function (see Theorem 1.1 in [14])

$$d(x) = |x| \log(B|x|)$$

with a positive constant $B$ such that $\inf_{x \in \Omega} |x| \geq 2/B > 0$.

Finally, we introduce a norm for initial data such that

$$\mathcal{J}[u_0, u_1] = \sum_{j=0, 1} \left( \|u_j\|_{L^2}^2 + \|\nabla u_j\|_{L^2}^2 + \|\Delta u_0\|_{L^2}^2 + \|\Delta u_1\|_{L^2}^2 + \|d(\cdot) \Delta u_0\|_{L^2}^2 + \|d(\cdot) u_1\|_{L^2}^2 + I_{\exp}[u_0, u_1]\right),$$

where the exponentially weighted norm for initial data is defined by

$$I_{\exp}[u_0, u_1] = \int_{\Omega} e^{2\psi(0, x)} \left( |\nabla u_1(x)|^2 + |\Delta u_0(x)|^2 + |u_1(x)|^2 + |\nabla u_0(x)|^2 \right) \, dx. \quad (1.8)$$

Let us state our main result.

**Theorem 1.1.** Let us assume

$$p > 6 + 2\rho_0 \quad \text{and} \quad q > 6 + 2\rho_0. \quad (1.9)$$

Then, there exists a constant $\varepsilon_0 > 0$ such that for any

$$(u_0, u_1) \in \left( H^2(\Omega) \cap H^1_0(\Omega) \right) \times H^1(\Omega)$$

with $\mathcal{J}[u_0, u_1] \leq \varepsilon_0$, there is a uniquely determined energy solution of higher-order

$$u \in \mathcal{C} \left( [0, \infty), H^2(\Omega) \cap H^1_0(\Omega) \right) \cap \mathcal{C}^1 \left( [0, \infty), H^1(\Omega) \right)$$

to (1.1). Furthermore, the solution satisfies the following estimates:

$$\|u(t, \cdot)\|_{L^2}^2 \leq C \mathcal{J}[u_0, u_1],$$

$$\|u(t, \cdot)\|_{L^2}^2 + \|\nabla u(t, \cdot)\|_{L^2}^2 + \|\Delta u(t, \cdot)\|_{L^2}^2 \leq C(1 + t)^{-1} \mathcal{J}[u_0, u_1],$$

$$\left\| e^{\psi(t, \cdot)} u(t, \cdot) \right\|_{L^2}^2 + \left\| e^{\psi(t, \cdot)} \nabla u(t, \cdot) \right\|_{L^2}^2 + \left\| e^{\psi(t, \cdot)} \Delta u(t, \cdot) \right\|_{L^2}^2 \leq C \mathcal{J}[u_0, u_1],$$

for any $t \geq 0$.

**Remark 1.1.** Here, we emphasize that one cannot not compare the hereinbefore proposed result with those result of the previous research [15]. First of all, the nonlinear term of what we treat is different from [15]. What’s more, as mentioned before, the authors of [15] proved global (in time) existence for classical energy solution

$$u \in \mathcal{C} \left( [0, \infty), H^1_0(\Omega) \right) \cap \mathcal{C}^1 \left( [0, \infty), L^2(\Omega) \right)$$

for semilinear strongly damped wave equation with power nonlinearity $|u|^p$, where initial data are taken from $(H^2(\Omega) \cap H^1_0(\Omega)) \times L^2(\Omega)$. What we do in this paper is to derive the global (in time) existence of energy solution of higher-order. It seems reasonable to have a stronger condition on $p$. 
Remark 1.2. Considering the weighted function $\psi(t, x)$ defined in (1.5), roughly speaking, the reason for us to consider $\rho > \rho_0$ is in the derivation of weighted estimates for higher-order energy. We will see later in Proposition 3.1.

Remark 1.3. The critical curve $\Upsilon(n)$ of the $n$-dimensional semilinear strongly damped wave equation in an exterior domain carrying mixed nonlinear term $|u|^p + |u_t|^q$ is still open. Here, the critical curve means that if a pair of exponents $p$ and $q$ are above the curve $\Upsilon(n)$, there exists global (in time) small data solution; on the contrary, if the exponents are on or below the curve $\Upsilon(n)$, every local (in time) solutions blows up in finite time even with small data. However, under the assumption of initial data taken from energy space with suitable higher regularity, the blow-up of solutions with suitable condition on the exponent is still unknown. For this reason, so far we cannot tell whether or not the restriction (1.9) is the critical curve in 2D.

The remaining part of the present paper is organized as follows. In Section 2, we derive energy estimates for the corresponding linear homogeneous problem to (1.1). In Section 3, some exponentially weighted $L^2$ estimates for nonlinear strongly damped wave equation are obtained. In Section 4, we prove Theorem 1.1. Finally, final remark in Section 5 completes the paper.

2. Energy estimates for linear homogeneous strongly damped wave equation

In the section, we are concerned with energy estimates for the corresponding linearized equation to (1.1), namely,

$$
\begin{cases}
  u_{tt} - \Delta u - \Delta u_t = 0, & x \in \Omega, \ t > 0, \\
  u(0, x) = u_0(x), \ u_t(0, x) = u_1(x), & x \in \Omega, \\
  u = 0, & x \in \partial \Omega, \ t > 0.
\end{cases}
$$

(2.1)

To do this, let us define an energy containing higher-order derivative of solutions for (2.1) firstly

$$
E[u](t) = \frac{1}{2} \left( \|u_t(t, \cdot)\|_{L^2}^2 + \|\nabla u(t, \cdot)\|_{L^2}^2 + \|\nabla u_t(t, \cdot)\|_{L^2}^2 + \|\Delta u(t, \cdot)\|_{L^2}^2 \right).
$$

We found that this energy is in a higher-order sense, which is different from total energy defined in [14]. Therefore, we need to derive a new estimate of the energy $E[u](t)$ in the next lemma.

Lemma 2.1. Let us assume

$$(u_0, u_1) \in (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1(\Omega)$$

satisfying $\|d(\cdot)(u_1 - \Delta u_0)\|_{L^2} < \infty$. Then, the following energy estimate for (2.1) holds:

$$E[u](t) \leq (1 + t)^{-1} I_2[u_0, u_1],$$

(2.2)

where the constant $I_2[u_0, u_1]$ with respect to initial data will be defined in (2.11) later.

Proof. First of all, multiplying the equation in (2.1) by $u_t$ and integrating the resulting identity over $\Omega$, one gets

$$
\frac{1}{2} \frac{d}{dt} \int_\Omega (|u_t(t, x)|^2 + |\nabla u(t, x)|^2) \, dx + \int_\Omega |\nabla u_t(t, x)|^2 \, dx = 0.
$$

To construct the energy, let us integrate the above equation over $[0, t]$ such that

$$
\frac{1}{2} \left( \|u_t(t, \cdot)\|_{L^2}^2 + \|\nabla u(t, \cdot)\|_{L^2}^2 \right) + \int_0^t \|\nabla u_t(s, \cdot)\|_{L^2}^2 \, ds = \frac{1}{2} \left( \|u_1\|_{L^2}^2 + \|\nabla u_0\|_{L^2}^2 \right).
$$

(2.3)

With the aim of deriving higher-order energy, we multiply the equation in (2.1) by $\Delta u_t$ and integrate over $\Omega$ to have

$$
\frac{1}{2} \frac{d}{dt} \int_\Omega (|\nabla u(t, x)|^2 + |\Delta u(t, x)|^2) \, dx + \int_\Omega |\Delta u_t(t, x)|^2 \, dx = 0.
$$

(2.4)
Integrating the resulting equation over \([0, t]\) leads to
\[
\frac{1}{2} \left( \| \nabla u(t, \cdot) \|^2_{L^2} + \| \Delta u(t, \cdot) \|^2_{L^2} \right) + \int_0^t \| \Delta u(t, s, \cdot) \|^2_{L^2} \, ds = \frac{1}{2} \left( \| \nabla u_1 \|^2_{L^2} + \| \Delta u_0 \|^2_{L^2} \right). \tag{2.5}
\]

Similarly as the previous two steps, we multiply the equation in (2.1) by \(\Delta u\) and integrate over \(\Omega\) as well as the interval \([0, t]\). It implies
\[
-\frac{d}{dt} \int_\Omega \nabla u_t(t, x) \cdot \nabla u(t, x) \, dx + \int_\Omega |\nabla u_t(t, x)|^2 \, dx - \int_\Omega |\Delta u(t, x)|^2 \, dx - \frac{1}{2} \frac{d}{dt} \int_\Omega |\Delta u(t, x)|^2 \, dx = 0
\]
and then,
\[
\int_0^t \int_\Omega |\Delta u(s, x)|^2 \, dx \, ds + \frac{1}{2} \int_\Omega |\Delta u(t, x)|^2 \, dx = \int_0^t \int_\Omega |\nabla u_t(s, x)|^2 \, ds \, dx - \int_\Omega \nabla u_t(t, x) \cdot \nabla u(t, x) \, dx + \frac{1}{2} \int_\Omega |\Delta u_0(x)|^2 \, dx + \int_\Omega \nabla u_1(x) \cdot \nabla u_0(x) \, dx.
\]
By employing Young’s inequality, we may conclude that
\[
\int_0^t \| \Delta u(s, \cdot) \|^2_{L^2} \, ds + \frac{1}{2} \| \Delta u(t, \cdot) \|^2_{L^2} \leq \int_0^t \| \nabla u_t(s, \cdot) \|^2_{L^2} \, ds + \frac{1}{2} \| \nabla u_t(t, \cdot) \|^2_{L^2} + \frac{1}{2} \| \nabla u(t, \cdot) \|^2_{L^2}
\]
\[
+ \frac{1}{2} \| \Delta u_0 \|^2_{L^2} + \frac{1}{2} \| \nabla u_1 \|^2_{L^2} + \frac{1}{2} \| \nabla u_0 \|^2_{L^2}
\]
\[
\leq \| \Delta u_0 \|^2_{L^2} + \| \nabla u_1 \|^2_{L^2} + \| \nabla u_0 \|^2_{L^2} + \frac{1}{2} \| u_1 \|^2_{L^2}, \tag{2.6}
\]
where we have used the equations (2.3) and (2.5).

According to Theorem 1.1 in [14], we know under our assumption of initial data in Lemma 2.1, the integration with respect to time variable of classical energy is bounded such that
\[
\int_0^t \left( \| u_t(s, \cdot) \|^2_{L^2} + \| \nabla u(s, \cdot) \|^2_{L^2} \right) \, ds \leq I_0[u_0, u_1], \tag{2.7}
\]
where the constant \(I_0[u_0, u_1]\) is denoted by
\[
I_0[u_0, u_1] \doteq 2 \| u_0 \|^2_{L^2} + \| u_1 \|^2_{L^2} + \frac{1}{2} \| \nabla u_0 \|^2_{L^2} + 3C_0 \| d(\cdot)(u_1 - \Delta u_0) \|^2_{L^2},
\]
with Hardy’s constant \(C_0 > 0\). On the other hand, from (2.3) and (2.6) one observes
\[
\int_0^t \left( \| \nabla u_t(s, \cdot) \|^2_{L^2} + \| \Delta u(s, \cdot) \|^2_{L^2} \right) \, ds \leq I_1[u_0, u_1], \tag{2.8}
\]
where the constant \(I_1[u_0, u_1]\) is defined by
\[
I_1[u_0, u_1] \doteq \| \Delta u_0 \|^2_{L^2} + \| \nabla u_1 \|^2_{L^2} + \frac{3}{2} \| \nabla u_0 \|^2_{L^2} + \| u_1 \|^2_{L^2}.
\]
Furthermore, we notice from (2.4) that
\[
\frac{d}{dt} \left( (1 + t) \left( \| \nabla u_t(t, \cdot) \|^2_{L^2} + \| \Delta u(t, \cdot) \|^2_{L^2} \right) \right)
\]
\[
= \| \nabla u_t(t, \cdot) \|^2_{L^2} + \| \Delta u(t, \cdot) \|^2_{L^2} + (1 + t) \frac{d}{dt} \left( \| \nabla u_t(t, \cdot) \|^2_{L^2} + \| \Delta u(t, \cdot) \|^2_{L^2} \right)
\]
\[
\leq \| \nabla u_t(t, \cdot) \|^2_{L^2} + \| \Delta u(t, \cdot) \|^2_{L^2}.
\]
The next decay estimate for higher-order energy yields immediately by integrating the above derived inequality over \([0, t]\):
\[
\| \nabla u_t(t, \cdot) \|^2_{L^2} + \| \Delta u(t, \cdot) \|^2_{L^2} \leq (1 + t)^{-1} \left( I_1[u_0, u_1] + \| \nabla u_1 \|^2_{L^2} + \| \Delta u_0 \|^2_{L^2} \right). \tag{2.9}
\]
Here, we applied (2.8). The estimate (2.9) will be used later.
From the derived estimates (2.3) as well as (2.5), we see
\[
\frac{d}{dt} E[u](t) = - (\|\nabla u_t(t, \cdot)\|_{L^2}^2 + \|\Delta u_t(t, \cdot)\|_{L^2}^2) \leq 0,
\]
which implies
\[
\frac{d}{dt} ((1 + t) E[u](t)) = E[u](t) + (1 + t) \frac{d}{dt} E[u](t) \leq E[u](t).
\]
We integrate the above inequality over \([0, t]\) to have
\[
(1 + t) E[u](t) \leq \int_0^t E[u](s) \, ds + E[u](0).
\]
Finally, taking the next definition of the constant with respect to initial data:
\[
I_2[u_0, u_1] = \frac{1}{2} \left( I_0[u_0, u_1] + I_1[u_0, u_1] + \|u_1\|_{L^2}^2 + \|\nabla u_0\|_{L^2}^2 + \|\nabla u_1\|_{L^2}^2 + \|\Delta u_0\|_{L^2}^2 \right)
\]
and applying (2.7) and (2.8), it follows from (2.10) that
\[
(1 + t) E[u](t) \leq I_2[u_0, u_1].
\]
Thus, the proof is complete.

To end this section, we have to mention that the estimate from [14] holds
\[
\|u(t, \cdot)\|_{L^2}^2 \leq CI_2[u_0, u_1]
\]
if the assumptions mentioned Lemma 2.1 are satisfied.

3. Weighted estimates for nonlinear strongly damped wave equation

Throughout this section, our motivation is to derive some exponentially weighted estimates for the next nonlinear problem:
\[
\begin{aligned}
&u_{tt} - \Delta u - \Delta u_t = F(u), & x \in \Omega, \ t > 0, \\
u(0, x) = u_0(x), & u_t(0, x) = u_1(x), & x \in \Omega, \\
u = 0, & u \in \partial \Omega, \ t > 0.
\end{aligned}
\]
We assume that \(e^{\psi(t, \cdot)} F(u)(t, \cdot) \in L^2(\Omega)\), where the weight function \(\psi\) was introduced (1.5).

In order to derive exponentially weighted estimates in the \(L^2\) norm, we introduce some lemmas, which play an important role in the future.

Lemma 3.1. The weighted function \(\psi\) fulfills two inequalities as follows:
\[
\frac{\varepsilon(2 + \rho) + 6}{\varepsilon(2 + \rho) - 2} |\nabla \psi(t, x)|^2 - |\psi_t(t, x)|^2 \leq 0
\]
for any \(t > 0\) and \(\rho > \rho_0\), and
\[
\frac{|\nabla \psi(t, x)|^2}{-\psi_t(t, x)} \leq \frac{2}{2 + \rho}
\]
for any \(t > 0\) and \(\rho > 0\).
Proof. We now begin to prove (3.2). By simple calculations, we get
\[
\frac{\varepsilon(2 + \rho) + 6}{\varepsilon(2 + \rho) - 2} |\nabla \psi(t, x)|^2 - |\psi_t(t, x)|^2
= \left( \frac{\varepsilon(2 + \rho) + 6}{\varepsilon(2 + \rho) - 2} \frac{|x|}{(1 + t)^2 + \rho} \right)^2 - \left( \frac{2 + \rho}{2} \frac{|x|^2}{(1 + t)^2 + \rho} + \frac{1}{(1 + t)^2 + \rho} \right)^2
= \frac{1}{(1 + t)^1 + \rho} \left( \frac{\varepsilon(2 + \rho) + 6}{\varepsilon(2 + \rho) - 2} X - \frac{2 + \rho}{2} X^2 - 1 \right) \left( \frac{\varepsilon(2 + \rho) + 6}{\varepsilon(2 + \rho) - 2} X + \frac{2 + \rho}{2} X^2 + 1 \right) \leq 0,
\]
where \( X = |x|/(1 + t) \), because the discriminant of the first polynomial is
\[
\Delta = \frac{\varepsilon(2 + \rho) + 6}{\varepsilon(2 + \rho) - 2} - 2(2 + \rho) \leq 0.
\]
In the above, we have used the condition (1.7) that
\[
\frac{2}{2 + \rho} \leq \left( 2 + \frac{8}{2\rho + 3} \right) \frac{1}{2 + \rho} = \frac{4\rho + 14}{(2 + \rho)(2\rho + 3)} \leq \varepsilon.
\]
Next, the desired estimate (3.3) can be proved by the following way:
\[
-\psi_t(t, x) = \frac{2 + \rho}{2} \frac{|x|^2}{(1 + t)^2 + \rho} + \frac{1}{(1 + t)^2 + \rho} \geq \frac{2 + \rho}{2} \frac{|x|^2}{(1 + t)^2 + \rho} = \frac{2 + \rho}{2} |\nabla \psi(t, x)|^2.
\]
So, the proof is completed \( \square \)

Lemma 3.2. Let \( u \) be a regular solution of (3.1). Then, under the condition (1.7), the following estimate holds:
\[
\frac{|\nabla \psi(t, x)|^2}{-\psi_t(t, x)} e^{2\psi(t,x)} |u_{tt}(t, x)|^2 \leq -\psi_t(t, x) e^{2\psi(t,x)} |\Delta u(t, x)|^2 + \varepsilon e^{2\psi(t,x)} |\nabla u_t(t, x)|^2 + C_{\varepsilon, \rho} e^{2\psi(t,x)} |F(u)(t, x)|^2,
\]
(3.4)
for all \( t > 0 \) and \( \rho > \rho_0 \). Here, the positive constant \( C_{\varepsilon, \rho} \) will be determined in (3.6).

Proof. Taking the consideration on (3.1), we may expand the quadratic term by
\[
|u_{tt}(t, x)|^2 = |\Delta u(t, x) + \Delta u_t(t, x) + F(u)(t, x)|^2
= |\Delta u(t, x) + \Delta u_t(t, x)|^2 + |F(u)(t, x)|^2 + 2\Delta u(t, x) F(u)(t, x) + 2 \Delta u_t(t, x) F(u)(t, x).
\]
Later, we will employ Young’s inequality such that
\[
ab \leq \frac{1}{4\varepsilon_1} a^2 + \varepsilon_1 b^2
\]
(3.5)
with \( \varepsilon_1 = (\varepsilon(2 + \rho) - 2)/8 > 0 \) (here we used the assumption (1.7)).
Plugging \( a = |\Delta u(t, x)| \) and \( b = |\Delta u_t(t, x)| \) into the above Young’s inequality, one immediately has
\[
|\Delta u(t, x) + \Delta u_t(t, x)|^2 = |\Delta u(t, x)|^2 + |\Delta u_t(t, x)|^2 + 2\Delta u(t, x) \Delta u_t(t, x)
\leq \left( 1 + \frac{1}{2\varepsilon_1} \right) |\Delta u(t, x)|^2 + (1 + 2\varepsilon_1) |\Delta u_t(t, x)|^2,
\]
which implies that
\[
|u_{tt}(t, x)|^2 \leq \left( 1 + \frac{1}{2\varepsilon_1} \right) |\Delta u(t, x)|^2 + (1 + 2\varepsilon_1) |\Delta u_t(t, x)|^2 + |F(u)(t, x)|^2 + 2\Delta u(t, x) F(u)(t, x) + 2 \Delta u_t(t, x) F(u)(t, x).
\]
On the other hand, by employing Young’s inequality (3.5) again, we can compute
\[ 2\Delta u(t, x)F(u)(t, x) + 2\Delta u(t, x)F(u)(t, x) \]
\[ \leq \frac{1}{2\varepsilon_1} |\Delta u(t, x)|^2 + 2\varepsilon_1 |\Delta u(t, x)|^2 + \left(2\varepsilon_1 + \frac{1}{2\varepsilon_1}\right) |F(u)(t, x)|^2. \]
Hence,
\[ |u_{tt}(t, x)|^2 \leq \left(1 + \frac{1}{\varepsilon_1}\right) |\Delta u(t, x)|^2 + (1 + 4\varepsilon_1) |\Delta u(t, x)|^2 + \left(1 + 2\varepsilon_1 + \frac{1}{2\varepsilon_1}\right) |F(u)(t, x)|^2. \]

Finally, we may deduce that
\[
\frac{|\nabla \psi(t, x)|^2}{-\psi_1(t, x)} e^{2\psi(t, x)} |u_{tt}(t, x)|^2 \leq \left(\varepsilon(2 + \rho) + \frac{6}{\varepsilon(2 + \rho) - 2}\right) \frac{|\nabla \psi(t, x)|^2}{-\psi_1(t, x)} e^{2\psi(t, x)} |\Delta u(t, x)|^2 \\
+ \varepsilon \left(1 + \frac{\rho}{2}\right) |\nabla \psi(t, x)|^2 e^{2\psi(t, x)} |\Delta u(t, x)|^2 + C_{\varepsilon, \rho} e^{2\psi(t, x)} |F(u)(t, x)|^2 \\
\leq -\psi_1(t, x) e^{2\psi(t, x)} |\Delta u(t, x)|^2 + \varepsilon e^{2\psi(t, x)} |\Delta u(t, x)|^2 \\
+ C_{\varepsilon, \rho} e^{2\psi(t, x)} |F(u)(t, x)|^2,
\]
where we have used (3.2) as well as (3.3), and the fact that \(1 + 4\varepsilon_1 = \varepsilon(1 + \rho/2).\) In the above inequality, we denote the positive constant
\[ C_{\varepsilon, \rho} = \frac{2}{2 + \rho} \left(1 + \frac{\varepsilon(2 + \rho) - 2}{4} + \frac{4}{\varepsilon(2 + \rho) - 2}\right) > 0. \quad (3.6)\]
Thus, we complete the proof of this lemma. \( \square \)

By using our derived lemmas, we may prove the next propositions for weighted estimates for higher-order energy to (3.1). At this time, we restrict ourselves \(\rho > \rho_0\) to control the higher-order term in the energy estimate.

**Proposition 3.1.** Let \(u\) be a regular solution of (3.1). Then, we have the estimate
\[
\left\| e^{\psi(t, \cdot)} \nabla u(t, \cdot) \right\|_{L^2}^2 + \left\| e^{\psi(t, \cdot)} \Delta u(t, \cdot) \right\|_{L^2}^2 + (1 - \varepsilon) \int_0^t \left\| e^{\psi(s, \cdot)} \Delta u(t, \cdot) \right\|_{L^2}^2 ds \\
\leq \left\| e^{\psi(0, \cdot)} \nabla u_0 \right\|_{L^2}^2 + \left\| e^{\psi(0, \cdot)} \Delta u_0 \right\|_{L^2}^2 + \tilde{C}_{\varepsilon, \rho} \int_0^t \left\| e^{\psi(s, \cdot)} F(u)(s, \cdot) \right\|_{L^2}^2 ds \quad (3.7)
\]
for all \(t > 0\) and \(\rho > \rho_0.\) Here, the positive constant \(\tilde{C}_{\varepsilon, \rho}\) will be shown in (3.11).

**Proof.** Firstly, we multiply (3.1) by \(e^{2\psi(t, x)} \Delta u_t\) to have
\[
e^{2\psi(t, x)} u_{tt}(t, x) \Delta u_t(t, x) - e^{2\psi(t, x)} \Delta u(t, x) \Delta u_t(t, x) = e^{2\psi(t, x)} |\Delta u(t, x)|^2 \\
= e^{2\psi(t, x)} F(u)(t, x) \Delta u_t(t, x). \quad (3.8)
\]
Due to the computations that
\[
e^{2\psi(t, x)} u_{tt}(t, x) \Delta u(t, x) = e^{2\psi(t, \cdot)} \text{div}(u(t, x) \nabla u(t, x)) - e^{2\psi(t, \cdot)} (\nabla u(t, x))_t \cdot \nabla u(t, x) \\
= \text{div} \left(e^{2\psi(t, x)} u_{tt}(t, x) \nabla u(t, x) - 2e^{2\psi(t, x)} u_{tt}(t, x) \nabla \psi(t, x) \cdot \nabla u(t, x) \\
- \frac{1}{2} \frac{d}{dt} \left(e^{2\psi(t, x)} |\nabla u(t, x)|^2 \right) + e^{2\psi(t, x)} \psi(t, x) |\nabla u(t, x)|^2, \right.
\]
and
\[
e^{2\psi(t, x)} \Delta u(t, x) \Delta u_t(t, x) = \frac{1}{2} \frac{d}{dt} \left(e^{2\psi(t, x)} |\Delta u(t, x)|^2 \right) - e^{2\psi(t, x)} \psi(t, x) |\Delta u(t, x)|^2,
\]

\[ \quad \frac{1}{2} \frac{d}{dt} \left(e^{2\psi(t, x)} |\nabla u(t, x)|^2 \right) - e^{2\psi(t, x)} \psi(t, x) |\nabla u(t, x)|^2, \]
it follows from (3.8) that
\[
\begin{align*}
\frac{d}{dt} & \left( \frac{e^{2\psi(t,x)}}{2} \right) (|\nabla u(t,x)|^2 + |\Delta u(t,x)|^2) - \text{div} \left( e^{2\psi(t,x)} u_{tt}(t,x) \nabla u(t,x) \right) \\
& + 2e^{2\psi(t,x)} u_{tt}(t,x) \nabla \psi(t,x) \cdot \nabla u_{tt}(t,x) - e^{2\psi(t,x)} \psi_{t}(t,x) |\nabla u(t,x)|^2 \\
& + e^{2\psi(t,x)} (-\psi_{t}(t,x))|\Delta u(t,x)|^2 + e^{2\psi(t,x)} |\Delta u(t,x)|^2 \\
& = -e^{2\psi(t,x)} F(u)(t,x) \Delta u_{t}(t,x).
\end{align*}
\]

Clearly,
\[
\begin{align*}
2e^{2\psi(t,x)} u_{tt}(t,x) \nabla \psi(t,x) \cdot \nabla u_{t}(t,x) - e^{2\psi(t,x)} \psi_{t}(t,x) |\nabla u(t,x)|^2 \\
= \frac{e^{2\psi(t,x)}}{-\psi_{t}(t,x)} |\psi_{t}(t,x) \nabla u_{t}(t,x) - \nabla \psi(t,x) u_{tt}(t,x)|^2 + \frac{|\nabla \psi(t,x)|^2}{\psi_{t}(t,x)} e^{2\psi(t,x)} |u_{tt}(t,x)|^2.
\end{align*}
\]

It leads that (3.9) can be written by
\[
\begin{align*}
\frac{d}{dt} & \left( \frac{e^{2\psi(t,x)}}{2} \right) (|\nabla u(t,x)|^2 + |\Delta u(t,x)|^2) - \text{div} \left( e^{2\psi(t,x)} u_{tt}(t,x) \nabla u_{t}(t,x) \right) \\
& + \frac{e^{2\psi(t,x)}}{-\psi_{t}(t,x)} |\psi_{t}(t,x) \nabla u_{t}(t,x) - \nabla \psi(t,x) u_{tt}(t,x)|^2 \\
& + e^{2\psi(t,x)} (-\psi_{t}(t,x))|\Delta u(t,x)|^2 + e^{2\psi(t,x)} |\Delta u(t,x)|^2 \\
& \leq \frac{|\nabla \psi(t,x)|^2}{-\psi_{t}(t,x)} e^{2\psi(t,x)} |u_{tt}(t,x)|^2 + e^{2\psi(t,x)} |F(u)(t,x)||\Delta u_{t}(t,x)| \\
& \leq e^{2\psi(t,x)} (-\psi_{t}(t,x))|\Delta u(t,x)|^2 + e^{2\psi(t,x)} |\Delta u_{t}(t,x)|^2 \\
& + C_{\epsilon,\rho} e^{2\psi(t,x)} |F(u)(t,x)|^2 + e^{2\psi(t,x)} |F(u)(t,x)||\Delta u_{t}(t,x)|.
\end{align*}
\]

where we have used (3.4) in the last step of the above estimate. Then, the above inequality (3.10) can be simplified as follows:
\[
\begin{align*}
\frac{d}{dt} & \left( \frac{e^{2\psi(t,x)}}{2} \right) (|\nabla u_{t}(t,x)|^2 + |\Delta u_{t}(t,x)|^2) - \text{div} \left( e^{2\psi(t,x)} u_{tt}(t,x) \nabla u_{t}(t,x) \right) \\
& + \frac{e^{2\psi(t,x)}}{-\psi_{t}(t,x)} |\psi_{t}(t,x) \nabla u_{t}(t,x) - \nabla \psi(t,x) u_{tt}(t,x)|^2 + (1 - \epsilon)e^{2\psi(t,x)} |\Delta u_{t}(t,x)|^2 \\
& \leq C_{\epsilon,\rho} e^{2\psi(t,x)} |F(u)(t,x)|^2 + e^{2\psi(t,x)} |F(u)(t,x)||\Delta u_{t}(t,x)|.
\end{align*}
\]

The application of Young’s inequality yields
\[
e^{2\psi(t,x)} |F(u)(t,x)||\Delta u_{t}(t,x)| \leq \frac{1 - \epsilon}{2} e^{2\psi(t,x)} |\Delta u_{t}(t,x)|^2 + \frac{1}{2(1 - \epsilon)} e^{2\psi(t,x)} |F(u)(t,x)|^2,
\]
since our setting of the constant that \(\epsilon < 1\). This inequality shows that
\[
\begin{align*}
\frac{d}{dt} & \left( \frac{e^{2\psi(t,x)}}{2} \right) (|\nabla u_{t}(t,x)|^2 + |\Delta u_{t}(t,x)|^2) - \text{div} \left( e^{2\psi(t,x)} u_{tt}(t,x) \nabla u_{t}(t,x) \right) \\
& + \frac{e^{2\psi(t,x)}}{-\psi_{t}(t,x)} |\psi_{t}(t,x) \nabla u_{t}(t,x) - \nabla \psi(t,x) u_{tt}(t,x)|^2 + (1 - \epsilon)e^{2\psi(t,x)} |\Delta u_{t}(t,x)|^2 \\
& \leq \left( C_{\epsilon,\rho} + \frac{1}{2(1 - \epsilon)} \right) e^{2\psi(t,x)} |F(u)(t,x)|^2 = C_{\epsilon,\rho} e^{2\psi(t,x)} |F(u)(t,x)|^2.
\end{align*}
\]

Consequently, integrating the above inequality over \(\Omega \times [0, t]\) and using the boundary condition and the fact that \(\psi_{t} < 0\) we may derive our desired result.

Next, the lower-order weighted energy can be estimated by the next lemma. We should emphasize that the next lemma is hold for all \(\rho > 0\).
Proposition 3.2. Let \( u \) be a regular solution of (3.1). Then, we have the estimate

\[
\left\| e^{\psi(t, \cdot)} u(t, \cdot) \right\|_{L^2}^2 + \left\| e^{\psi(t, \cdot)} \nabla u(t, \cdot) \right\|_{L^2}^2 \lesssim \left\| e^{\psi(0, \cdot)} u_1 \right\|_{L^2}^2 + \left\| e^{\psi(0, \cdot)} \nabla u_0 \right\|_{L^2}^2 + 2 \int_0^t \left\| e^{\psi(s, \cdot)} F(u(s, \cdot)) \right\|_{L^2} \left\| e^{\psi(s, \cdot)} u_t(s, \cdot) \right\|_{L^2} \, ds
\]

for all \( t > 0 \) and \( \rho > 0 \).

Proof. Similarly to the proof of Lemma 2.1 in [15], we multiply (3.1) by \( e^{2\psi(t, x)} u_t \) and get

\[
e^{2\psi(t, x)} u_{tt}(t, x) u_t(t, x) - e^{2\psi(t, x)} \Delta u(t, x) u_t(t, x) - e^{2\psi(t, x)} \Delta u_t(t, x) u_t(t, x) = e^{2\psi(t, x)} F(u(t, x)) u_t(t, x).
\]

Hence, the equality can be deduced as follows:

\[
\frac{d}{dt} \left( \frac{e^{2\psi(t, x)}}{2} \left( |u_t(t, x)|^2 + |\nabla u(t, x)|^2 \right) \right) - \text{div} \left( e^{2\psi(t, x)} u_t(t, x) \nabla (u(t, x) + u_t(t, x)) \right)
\]

\[
+ \frac{e^{2\psi(t, x)}}{\psi_t(t, x)} |u_t(t, x)|^2 \left( |\nabla \psi(t, x)|^2 - \psi_t(t, x) |\nabla \psi(t, x)|^2 - |\psi_t(t, x)|^2 \right)
\]

\[
+ \frac{e^{2\psi(t, x)}}{-\psi_t(t, x)} |\psi_t(t, x) \nabla u(t, x) - u_t(t, x) \nabla \psi(t, x)|^2 + e^{2\psi(t, x)} |\nabla u_t(t, x) + u_t(t, x) \nabla \psi(t, x)|^2
\]

\[
e^{2\psi(t, x)} F(u(t, x)) u_t(t, x).
\]

Using (1.6) and the property \( \psi_t < 0 \), we claim that

\[
\frac{d}{dt} \left( \frac{e^{2\psi(t, x)}}{2} \left( |u_t(t, x)|^2 + |\nabla u(t, x)|^2 \right) \right) - \text{div} \left( e^{2\psi(t, x)} u_t(t, x) \nabla (u(t, x) + u_t(t, x)) \right)
\]

\[
\lesssim e^{2\psi(t, x)} F(u(t, x)) u_t(t, x).
\]

Let us integrate (3.13) over \( \Omega \times [0, t] \) to obtain

\[
\frac{1}{2} \left\| e^{\psi(t, \cdot)} u(t, \cdot) \right\|_{L^2}^2 + \frac{1}{2} \left\| e^{\psi(t, \cdot)} \nabla u(t, \cdot) \right\|_{L^2}^2 \lesssim \frac{1}{2} \left\| e^{\psi(0, \cdot)} u_1 \right\|_{L^2}^2 + \frac{1}{2} \left\| e^{\psi(0, \cdot)} \nabla u_0 \right\|_{L^2}^2 + \int_0^t \left\| e^{2\psi(s, \cdot)} F(u(s, \cdot)) u_t(s, \cdot) \right\|_{L^2} \, ds.
\]

The proof can be completed after using Hölder’s inequality in the above inequality. \( \square \)

4. Proof of Theorem 1.1

Before proving our main theorem, let us denote by \( E_0(t, x) \) and \( E_1(t, x) \) the fundamental solutions to the linear problem (2.1) with initial data \(( u_0, u_1 ) = ( \delta_0, 0 )\) and \(( u_0, u_1 ) = ( 0, \delta_0 )\), respectively. Here, \( \delta_0 \) is the Dirac distribution in \( x = 0 \) with respect to spatial variables. Therefore, the solution \( u^{\text{lin}} = u^{\text{lin}}(t, x) \) to the exterior problem (2.1) is given by

\[
u^{\text{lin}}(t, x) = E_0(t, x) * (x) u_0(x) + E_1(t, x) * (x) u_1(x).
\]

Let us define an evolution space

\[
X(T) = C \left( [0, T], H^2(\Omega) \cap H_0^1(\Omega) \right) \cap C^1 \left( [0, T], H^1(\Omega) \right)
\]

carrying the corresponding norm

\[
M[u](T) = \sup_{t \in [0, T]} W[u](t) = \sup_{t \in [0, T]} \left( \left\| e^{\psi(t, \cdot)} \nabla u(t, \cdot) \right\|_{L^2}^2 + (1 + t) \left\| \nabla u(t, \cdot) \right\|_{L^2}^2 + \left\| u(t, \cdot) \right\|_{L^2}^2 \right),
\]

where the space-time differential operator is denoted by \( \mathcal{D} \triangleq (\partial_t, \nabla, \nabla \partial_t, \Delta) \).
According to Duhamel’s principle, we introduce the operator
\[ N : u \in X(T) \rightarrow Nu = u^{\text{lin}}(t, x) + u^{\text{non}}(t, x) \]
\[ = u^{\text{lin}}(t, x) + \int_0^t E_1(t - s, x) * (F(u)(s, x)) \, ds, \]
where we choose \( F(u)(t, x) \equiv |u(t, x)|^p + |u_t(t, x)|^q \) in this section. Furthermore, let us define
\[ J_0[u_0, u_1] = I_2[u_0, u_1] + I_{\exp}[u_0, u_1], \]
where \( I_{\exp}[u_0, u_1] \) has been defined in (1.8). We should remark that if \( J_0[u_0, u_1] \leq \varepsilon_0 \), then it is trivial that \( J_0[u_0, u_1] \leq C\varepsilon_0 \) for some constant \( C > 0 \).

We will prove as the global in time solution to (1.1) the fixed points of operator \( N \). In other words, our first aim is to derive
\[ M[Nu](T) \leq \tilde{C}_0 J_0[u_0, u_1] + \tilde{C}_1 \left( \sum_{r=p,q(\rho+1)/2} M[u](T)^r \right) \]
with positive constants \( \tilde{C}_0 \) and \( \tilde{C}_1 \). Here, we denote
\[ \tilde{M}[u](T; p, q) = I_{\exp}[u_0, u_1]^{\frac{q-1}{q}} + M[u](t)^{\frac{p(q-1)}{q}} + M[u](t)^{q-1}. \]
Moreover, to guarantee uniqueness of global (in time) small data solution, our second aim is to prove the next estimates:
\[ M[Nu - Nv](T) \leq \tilde{C}_2 M[u - v], \]
for any \( u, v \in X(T) \), with a positive constant \( \tilde{C}_2 \).

From Lemma 2.1 and (2.12), it is sufficient for us to show
\[ \|u^{\text{lin}}(t, \cdot)\|_{L^2}^2 + (1 + t) \|D u^{\text{lin}}(t, \cdot)\|_{L^2}^2 \leq CI_2[u_0, u_1]. \]
Furthermore, the association of Lemmas 3.1 and 3.2 shows that
\[ \left\| e^{\psi(t, \cdot)} D u^{\text{lin}}(t, \cdot) \right\|_{L^2}^2 \leq CI_{\exp}[u_0, u_1]. \]
Together with them, we claim that \( u^{\text{lin}} \in X(T) \).

Therefore, the next part of this section is to prove \( u^{\text{non}} \in X(T) \). To estimate the nonlinear term in the weighted space, we introduce some lemmas, which will be used later.

**Lemma 4.1.** Let \( p, q > 4 + \rho \) for all \( \rho > 0 \). Then, the next estimate holds:
\[ \int_0^t \left\| e^{\psi(s, \cdot)} F(u)(s, \cdot) \right\|_{L^2}^2 \, ds \leq C \left( M[u](t)^p + M[u](t)^q \right) \]
for all \( t > 0 \).

**Proof.** To begin with the proof, let us split the estimate into two parts
\[ \int_0^t \left\| e^{\psi(s, \cdot)} F(u)(s, \cdot) \right\|_{L^2}^2 \, ds \leq 2 \int_0^t \left\| e^{\psi(s, \cdot)} |u_t(s, \cdot)|^p \right\|_{L^2}^2 \, ds + 2 \int_0^t \left\| e^{\psi(s, \cdot)} |u_t(s, \cdot)|^q \right\|_{L^2}^2 \, ds \]
\[ = 2 \left( A_1(t) + A_2(t) \right). \]
Now, we will estimate each part step by step. By using Lemma A.2, i.e. the Gagliardo-Nirenberg type inequality associated with weighted function $\psi$, we get

$$A_1(t) \leq C \int_0^t \left( (1 + s)^{(2 + \rho)(1 - \theta(2p))} \| \nabla u(s, \cdot) \|_{L^2}^{1 - \frac{1}{q}} \left\| e^{\psi(s, \cdot)} \nabla u(s, \cdot) \right\|_{L^2}^{\frac{1}{q}} \right)^{2p} ds$$

$$\leq C \int_0^t (1 + s)^{-(1 + \eta)} \left( (1 + s)^{(2 + \rho)(1 - \theta(2p))} + \frac{1 + \eta}{\rho} - (1 - \frac{1}{p}) \right) ds$$

$$\leq C \int_0^t (1 + s)^{-(1 + \eta)} ds \left( \sup_{s \in [0, t]} (1 + s) \beta_1 W[u](s) \right)^p$$

$$\leq C \left( \sup_{s \in [0, t]} (1 + s) \beta_1 W[u](s) \right)^p,$$

where

$$\beta_1 = (2 + \rho)(1 - \theta(2p)) + \frac{1 + \eta}{\rho} - \left(1 - \frac{1}{p}\right)$$

and $\theta(2p) = 2\left(\frac{1}{\rho} - \frac{1}{2p}\right)$ for all $\eta > 0$. Choosing a sufficiently small constant $\eta > 0$, we found that

$$\beta_1 = (2 + \rho) \left( 1 - 2 \left( 1 - \frac{1}{2p} \right) \right) + \frac{2}{p} \frac{\eta}{p} - 1 = \frac{\eta}{p} - \frac{p - 4 - \rho}{p} < 0,$$

since our assumption $p > 4 + \rho$. Therefore, the estimate for $A_1(t)$ is

$$A_1(t) \leq CM[u](t)^p.$$

Similarly as the above, one has

$$A_2(t) \leq C \int_0^t \left( (1 + s)^{(2 + \rho)(1 - \theta(2q))} \| \nabla u_t(s, \cdot) \|_{L^2}^{1 - \frac{1}{q}} \left\| e^{\psi(s, \cdot)} \nabla u_t(s, \cdot) \right\|_{L^2}^{\frac{1}{q}} \right)^{2q} ds$$

$$\leq C \int_0^t (1 + s)^{-(1 + \eta)} \left( (1 + s)^{(2 + \rho)(1 - \theta(2q))} + \frac{1 + \eta}{q} - (1 - \frac{1}{q}) \right) ds$$

$$\leq C \int_0^t (1 + s)^{-(1 + \eta)} ds \left( \sup_{s \in [0, t]} (1 + s) \beta_2 W[u](s) \right)^q$$

$$\leq C \left( \sup_{s \in [0, t]} (1 + s) \beta_2 W[u](s) \right)^q,$$

where

$$\beta_2 = (2 + \rho)(1 - \theta(2q)) + \frac{1 + \eta}{q} - \left(1 - \frac{1}{q}\right)$$

and $\theta(2q) = 2\left(\frac{1}{\rho} - \frac{1}{2q}\right)$ for all $\eta > 0$. By choosing a sufficiently small constant $\eta > 0$. It is clear that $\beta_2 < 0$, when $q > 4 + \rho$. Thus, the second time-dependent function can be estimated by

$$A_2(t) \leq CM[u](t)^q.$$

Summarizing the above estimates, we may complete the proof. \qed

Additionally, by the similar method as the proof of Lemma 4.1, we will prove the following result.

**Lemma 4.2.** Let $p, q > 5 + \rho$ for all $\rho > 0$. Then, the next estimate holds:

$$\int_0^t \left\| e^{\psi(s, \cdot)} F(u)(s, \cdot) \right\|_{L^2} \left\| e^{\psi(s, \cdot)} u_t(s, \cdot) \right\|_{L^2} ds \leq C \left( M[u](t)^{\frac{n+1}{n}} + M[u](t)^{\frac{n+1}{n}} \right) \quad (4.4)$$

for all $t > 0$. 
We apply Lemma A.2 again to derive
\[ \int_0^t \left\| e^{\psi(s, \cdot)} F(u)(s, \cdot) \right\|_{L^2} \left\| e^{\psi(s, \cdot)} u_t(u, \cdot) \right\|_{L^2} \, ds \]
\[ \leq 2 \int_0^t \left\| e^{\psi(s, \cdot)} u(s, \cdot) \right\|^p \left\| e^{\psi(s, \cdot)} u_t(u, \cdot) \right\|_{L^2} \, ds + 2 \int_0^t \left\| e^{\psi(s, \cdot)} u_t(u, \cdot) \right\|^q \left\| e^{\psi(s, \cdot)} u_t(u, \cdot) \right\|_{L^2} \, ds \]
\[ \leq 2 (B_1(t) + B_2(t)). \]

We apply Lemma A.2 again to derive
\[ B_1(t) \leq C \int_0^t \left( (1 + s)^{(2+\rho)(1-\theta(2p))} \left\| \nabla u(s, \cdot) \right\|^{1-\frac{n}{2}} \left\| e^{\psi(s, \cdot)} \nabla u(s, \cdot) \right\|^{\frac{1}{2}} \right)^p W[u](s)^{\frac{p}{2}} \, ds \]
\[ \leq C \int_0^t (1 + s)^{-1+\eta} \left( (1 + s)^{(2+\rho)(1-\theta(2p))} + 1 + \frac{n}{p} - \frac{1}{2} \right) W[u](s)^{\frac{p}{2}} \, ds \]
\[ \leq C \int_0^t (1 + s)^{-1+\eta} \, ds \left( \sup_{s \in [0, t]} (1 + s)^{\beta_3} W[u](s)^{\frac{p}{2}} \right)^p W[u](t)^{\frac{p}{2}} \]
\[ \leq C \left( \sup_{s \in [0, t]} (1 + s)^{\beta_3} W[u](s)^{\frac{p}{2}} \right)^p M[u](t)^{\frac{p}{2}} \]
for all \( \eta > 0 \). Here, choosing sufficiently small constant \( \eta > 0 \), we observe that
\[ \beta_3 = \frac{(2 + \rho)(1 - \theta(2p))}{2} + 1 + \frac{n}{p} - \frac{1}{2} \left( 1 - \frac{1}{p} \right) = \frac{\eta}{p} - \frac{p - 5 - \rho}{2p} < 0, \]
where we used our assumption \( p > 5 + \rho \). So, the estimate holds
\[ B_1(t) \leq CM[u](t)^{\frac{\eta}{p}}. \]

Analogously, we may compute
\[ B_2(t) \leq C \int_0^t \left( (1 + s)^{(2+\rho)(1-\theta(2q))} \left\| \nabla u(s, \cdot) \right\|^{1-\frac{n}{2}} \left\| e^{\psi(s, \cdot)} \nabla u_t(s, \cdot) \right\|^{\frac{1}{2}} \right)^q W[u](s)^{\frac{q}{2}} \, ds \]
\[ \leq C \int_0^t (1 + s)^{-1+\eta} \, ds \left( \sup_{s \in [0, t]} (1 + s)^{\beta_4} W[u](s)^{\frac{q}{2}} \right)^q W[u](t)^{\frac{q}{2}} \]
\[ \leq C \left( \sup_{s \in [0, t]} (1 + s)^{\beta_4} W[u](s)^{\frac{q}{2}} \right)^q M[u](t)^{\frac{q}{2}}, \]
for all \( \eta > 0 \). The parameter \( \beta_4 \) satisfies
\[ \beta_4 = \frac{(2 + \rho)(1 - \theta(2q))}{2} + 1 + \frac{n}{q} - \frac{1}{2} \left( 1 - \frac{1}{q} \right) = \frac{\eta}{q} - \frac{q - 5 - \rho}{2q} < 0, \]
by using our assumption \( q > 5 + \rho \) and the choice of sufficiently small constant \( \eta > 0 \). We conclude
\[ B_2(t) \leq CM[u](t)^{\frac{\eta}{q}}. \]
This implies the desired estimate. \( \square \)

The next proposition plays an important role in proving global (in time) existence of small data solution by using Lemmas 4.1 and 4.2 directly.

**Proposition 4.1.** Let \( p, q > 5 + \rho \) for all \( \rho > \rho_0 \). The next estimate holds:
\[
\left\| e^{\psi(t, \cdot)} \mathcal{D} u_{\text{non}}(t, \cdot) \right\|_{L^2}^2 \leq C \left( M[u](t)^p + M[u](t)^q + M[u](t)^{\frac{p+q}{2}} + M[u](t)^{\frac{p+q}{2}} \right) \quad (4.5)
\]
for all \( t > 0 \).
In conclusion, we have completed the estimate of weighted energy even for higher-order. So, we will estimate
\[ \| \mathcal{D} u^\text{non}(t, \cdot) \|_{L^2}^2 \quad \text{and} \quad \| u^\text{non}(t, \cdot) \|_{L^2}^2 \]
in the following part.

Let us define integral operators
\[ u_{N_1}(t, x) = \int_0^t E_1(t - s, x) *_{(x)} |u(s, x)|^p \, ds \quad \text{and} \quad u_{N_2}(t, x) = \int_0^t E_1(t - s, x) *_{(x)} |u_t(s, x)|^q \, ds, \]
which implies
\[ u^\text{non}(t, x) = u_{N_1}(t, x) + u_{N_2}(t, x). \]
Moreover, we define the differential operators
\[ \mathcal{D}_1 = (\partial_t, \nabla) \quad \text{and} \quad \mathcal{D}_2 = (\partial_t \nabla, \Delta). \]
Hence, we notice that \( \mathcal{D} = (\mathcal{D}_1, \mathcal{D}_2). \)

From the proof of Theorem 1.1 in [15], under our assumption \( p > 6 + 2\rho_0 \) it is sufficient to derive
\[ \| u_{N_1}(t, \cdot) \|_{L^2}^2 + (1 + t) \| \mathcal{D}_1 u_{N_1}(t, \cdot) \|_{L^2}^2 \leq CM[u](t)^p. \] (4.6)

Next, we will estimate \( \| \mathcal{D}_2 u_{N_1}(t, \cdot) \|_{L^2}. \) The application of (2.9) indicates that
\[ \| \mathcal{D}_2 E_1(t - s, \cdot) *_{(\cdot)} |u(s, \cdot)|^p \|_{L^2} \]
\[ \leq C(1 + t - s)^{-\frac{1}{2}} \left( \| u(s, \cdot) \|_{L^2}^{1-\theta_1(2p)} \| \nabla u(s, \cdot) \|_{L^2}^{\theta_1(2p)} + \| u(s, \cdot) \|_{L^2}^{p-1} \| \nabla u(s, \cdot) \|_{L^2} \right) \]
\[ \leq C(1 + t - s)^{-\frac{1}{2}} \left( \| u(s, \cdot) \|_{L^2}^{p-1} \| \nabla u(s, \cdot) \|_{L^2} \right) \]
\[ = C(1 + t - s)^{-\frac{1}{2}} \left( \| u(s, \cdot) \|_{L^2}^{p-1} \| \nabla u(s, \cdot) \|_{L^2} \right), \]
where we have used H"older's inequality.

Moreover, due to \( u(t, \cdot) \in H^2(\Omega) \) for \( t \in [0, T], \) the Gagliardo-Nirenberg inequality shows
\[ \| u(s, \cdot) \|_{L^2} \leq C \| u(s, \cdot) \|_{L^2}^{1-\theta_1(2p)} \| \nabla u(s, \cdot) \|_{L^2}^{\theta_1(2p)}, \]
\[ \| \nabla u(s, \cdot) \|_{L^2} \leq C \| \nabla u(s, \cdot) \|_{L^2}^{1-\theta_1(2p)} \| \Delta u(s, \cdot) \|_{L^2}^{\theta_1(2p)}, \]
where \( \theta_1(2p) := \theta(2p) = 2(\frac{1}{2} - \frac{1}{2p}) = 1 - \frac{1}{p}. \)

By applying the estimates from the definition of solution space \( X(T) \) such that
\[ \| u(s, \cdot) \|_{L^2} \leq C(1 + s)^{-\frac{\theta_1(2p)}{2}} W[u](s)^{\frac{1}{2}}, \] (4.7)
\[ \| \nabla u(s, \cdot) \|_{L^2} \leq C(1 + s)^{-\frac{1}{2}} W[u](s)^{\frac{1}{2}}, \] (4.8)
we derive
\[ \| \mathcal{D}_2 E_1(t - s, \cdot) *_{(\cdot)} |u(s, \cdot)|^p \|_{L^2} \leq C(1 + t - s)^{-\frac{1}{2}} (1 + s)^{-\frac{\theta_1(2p)}{2}} W[u](s)^{\frac{1}{2}}. \]

From the definition of \( u_{N_1}, \) we immediately obtain
\[ \| \mathcal{D}_2 u_{N_1}(t, \cdot) \|_{L^2} \leq \int_0^t \| \mathcal{D}_2 E_1(t - s, \cdot) *_{(\cdot)} |u(s, \cdot)|^p \|_{L^2} \, ds \]
\[ \leq C \int_0^t (1 + t - s)^{-\frac{1}{2}} (1 + s)^{-\frac{\theta_1(2p)}{2}} W[u](s)^{\frac{1}{2}} \, ds \]
\[ \leq CM[u](t)^{\frac{1}{2}} \int_0^t (1 + t - s)^{-\frac{1}{2}} (1 + s)^{-\frac{\theta_1(2p)}{2}} \, ds \]
\[ \leq C(1 + t)^{-\frac{1}{2}} M[u](t)^{\frac{1}{2}}, \] (4.9)
where we used Lemma 4.1 in [1] and the condition that \( \theta(2p)/2 > 1 \) under the assumption \( p > 3. \)
Now, we begin with the estimate of \(\|D_2 u_{N_2}(t, \cdot)\|_{L^2}\). Taking the consideration of (2.9) again with Hölder’s inequality, we have

\[
\|D_2 E_1(t-s, \cdot) |u_t(s, \cdot)|^q\|_{L^2} \\
\leq C(1 + t - s)^{-\frac{q}{2}} (\|u_t(s, \cdot)|^q \|_{L^2} + \|\nabla u_t(s, \cdot)|^q \|_{L^2}) \\
\leq C(1 + t - s)^{-\frac{q}{2}} (\|u_t(s, \cdot)|^q \|_{L^2} + \|u_t(s, \cdot)|^{q-1} \|\nabla u_t(s, \cdot)\|_{L^2}) \\
\leq C(1 + t - s)^{-\frac{q}{2}} (\|u_t(s, \cdot)|^q \|_{L^2} + \|u_t(s, \cdot)|^{q-1} \|\nabla u_t(s, \cdot)\|_{L^2}) \\
= C(1 + t - s)^{-\frac{q}{2}} (\|u_t(s, \cdot)|^q \|_{L^2} + \|u_t(s, \cdot)|^{q-1} \|\nabla u_t(s, \cdot)\|_{L^2}) \\
\leq C(K_1(t, s) + K_2(t, s)),
\]

where

\[
K_1(t, s) = (1 + t - s)^{-\frac{q}{2}} \|u_t(s, \cdot)|^q \|_{L^2}, \\
K_2(t, s) = (1 + t - s)^{-\frac{q}{2}} \|u_t(s, \cdot)|^{q-1} \|\nabla u_t(s, \cdot)\|_{L^2}.
\]

Again, by \(u(t, \cdot) \in H^2(\Omega)\) for \(t \in [0, T]\), the Gagliardo-Nirenberg inequality can be applied to get

\[
\|u_t(s, \cdot)|^q \|_{L^2} \leq C \|u_t(s, \cdot)|^{1-\theta_2(2q)} \|\nabla u_t(s, \cdot)\|_{L^2}^{\theta_2(2q)}, \\
\|\nabla u_t(s, \cdot)\|_{L^2} \leq C \|\nabla u_t(s, \cdot)|^{1-\theta_2(2q)} \|u_t(s, \cdot)\|_{L^2}^{\theta_2(2q)},
\]

where \(\theta_2(2q) := \theta(2q) = 2(\frac{1}{2} - \frac{1}{q'}) = 1 - \frac{1}{q}\).

Due to the fact that \(\psi(s, x) > 0\) for all \(x \in \Omega\) and \(s \in [0, t]\), the next two estimates hold:

\[
\|u_t(s, \cdot)|^q \|_{L^2} \leq C(1 + s)^{-\frac{q}{2}} W[u](s)^{\frac{q}{2}},
\]

\[
\|\nabla u_t(s, \cdot)\|_{L^2} \leq C(1 + s)^{\frac{1}{2} - \theta_2(2q)} W[u](s)^{\frac{1}{2}} \|e^{\psi(s, \cdot)} \Delta u_t(s, \cdot)\|_{L^2}^{\theta_2(2q)}.
\]

Therefore, the combination of (4.10) and (4.11) yields

\[
\int_0^t K_1(t, s) ds \leq C \int_0^t (1 + t - s)^{-\frac{q}{2}} (1 + s)^{-\frac{q}{2}} W[u](s)^{\frac{q}{2}} ds \\
\leq CM[u](t)^{\frac{q}{2}} \int_0^t (1 + t - s)^{-\frac{q}{2}} (1 + s)^{-\frac{q}{2}} ds \\
\leq C(1 + t)^{-\frac{q}{2}} M[u](t)^{\frac{q}{2}},
\]

where Lemma 4.1 in [1] has been applied again with \(q > 2\).

To estimate another term with respect to \(K_2(t, s)\), using (4.11) and (4.12) one may obtain

\[
\int_0^t K_2(t, s) ds \leq C \int_0^t (1 + t - s)^{-\frac{q}{2}} (1 + s)^{-\frac{q}{2}} W[u](s)^{\frac{q}{2}} \|e^{\psi(s, \cdot)} \Delta u_t(s, \cdot)\|_{L^2}^{\theta_2(2q)} ds \\
\leq C \left( \int_0^t (1 + t - s)^{-\frac{q}{2}} (1 + s)^{-\frac{q}{2}} W[u](s)^{\frac{q}{2}} \|e^{\psi(s, \cdot)} \Delta u_t(s, \cdot)\|_{L^2}^{\theta_2(2q)} ds \right)^{2(1-\theta_2(2q))/2} \\
\times \left( \int_0^t \|e^{\psi(s, \cdot)} \Delta u_t(s, \cdot)\|_{L^2}^{\theta_2(2q)} ds \right)^{\theta_2(2q)/2} \\
\leq CM[u](t)^{\frac{1}{2}-\theta_2(2q)} \left( \int_0^t (1 + t - s)^{-\frac{q}{2}} (1 + s)^{-\frac{q}{2}} W[u](s)^{\frac{q}{2}} \|e^{\psi(s, \cdot)} \Delta u_t(s, \cdot)\|_{L^2}^{\theta_2(2q)} ds \right)^{(2\theta_2(2q))/2} \\
\times \left( \int_0^t \|e^{\psi(s, \cdot)} \Delta u_t(s, \cdot)\|_{L^2}^{\theta_2(2q)} ds \right)^{\theta_2(2q)/2} \\
\leq C(1 + t)^{-\frac{q}{2}} M[u](t)^{\frac{1}{2}-\theta_2(2q)} \left( \int_0^t \|e^{\psi(s, \cdot)} \Delta u_t(s, \cdot)\|_{L^2}^{2} ds \right)^{\theta_2(2q)/2},
\]

(4.14)
where Hölder’s inequality is employed since \( \theta_2(2q) < 1 \) for \( q > 2 \). Note that here we have also used Lemma 4.1 in [1], because

\[
\frac{1}{2 - \theta_2(2q)} < 1 \quad \text{and} \quad \frac{q - \theta_2(2q)}{2 - \theta_2(2q)} > 1.
\]

Moreover, Proposition 3.1 and (4.3) tell us the integration of higher-order weighted energy can be controlled by the following way:

\[
\int_0^t \left\| e^{\psi(t-s)} \Delta u(t,s) \right\|_{L^2}^2 \, ds \leq \int_0^t \left\| e^{\psi(0-s)} \nabla u_0 \right\|_{L^2}^2 + \int_0^t \left\| e^{\psi(s)} \Delta u_0 \right\|_{L^2}^2 + C \int_0^t \left\| e^{\psi(s)} F(u)(s) \right\|_{L^2}^2 \, ds 
\]

\[
\leq I_{\exp}[u_0,u_1] + C (M[u](t)^p + M[u](t)^q) .
\]

All in all, we may conclude

\[
\int_0^t K_2(t,s) \, ds \leq C (1 + t)^{-\frac{\theta_2(2q)}{2}} M[u](t)^{\frac{\theta_2(2q)}{2}} 
\times \left( I_{\exp}[u_0,u_1] + M[u](t)^{\frac{\theta_2(2q)}{2}} + M[u](t)^{\frac{\theta_2(2q)}{2}} \right) .
\]

Using the derived estimates (4.13) and (4.15), we claim that

\[
(1 + t)^{-\frac{\theta_2(2q)}{2}} \left\| \mathcal{D}_t u_{N_2}(t,\cdot) \right\|_{L^2} \leq C (1 + t)^{-\frac{\theta_2(2q)}{2}} M[u](t)^{\frac{\theta_2(2q)}{2}} 
\times \left( I_{\exp}[u_0,u_1] + M[u](t)^{\frac{\theta_2(2q)}{2}} + M[u](t)^{\frac{\theta_2(2q)}{2}} \right) .
\]

It remains to estimate \( \left\| \mathcal{D}_t^2 u_{N_2}(t,\cdot) \right\|_{L^2} \) for \( j = 0, 1 \). Let us apply Theorem 2.1 in [15] to have

\[
\left\| \mathcal{D}_t^2 E_1(t-s,\cdot) |u_t(s,\cdot)|^q \right\|_{L^2} \leq C (1 + t-s)^{-\frac{\theta_2(2q)}{2}} \left( \left\| |u_t(s,\cdot)|^q \right\|_{L^2} + \left\| d(\cdot) |u_t(s,\cdot)|^q \right\|_{L^2} \right) .
\]

Using Lemma 2.5 in [15] and replacing \( u \) by \( u_t \), one derives

\[
\left\| |u_t(s,\cdot)|^q \right\|_{L^2} \leq C (1 + s)^{\frac{2(\rho+\delta)(1+\varepsilon)}{2}} \left\| e^{\delta \psi(s,\cdot)} |u_t(s,\cdot)| \right\|_{L^{2q}}^q
\]

for any \( \varepsilon_1 > 0, \rho > 0 \) and \( \delta > 0 \). It is obvious that

\[
\left\| |u_t(s,\cdot)|^q \right\|_{L^{2q}} \leq C (1 + s)^{\frac{2(\rho+\delta)(1+\varepsilon)}{2}} \left\| e^{\delta \psi(s,\cdot)} |u_t(s,\cdot)| \right\|_{L^{2q}}^q .
\]

Eventually, the following estimate holds:

\[
\left\| \mathcal{D}_t^2 u_{N_2}(t,\cdot) \right\|_{L^2} \leq C \int_0^t \left\| \mathcal{D}_t^2 E_1(t-s,\cdot) |u_t(s,\cdot)|^q \right\|_{L^2} \, ds
\]

\[
\leq C \int_0^t (1 + t-s)^{-\frac{\theta_2(2q)}{2}} \left( 1 + s \right)^{\frac{2(\rho+\delta)(1+\varepsilon)}{2}} \left\| e^{\delta \psi(s,\cdot)} |u_t(s,\cdot)| \right\|_{L^{2q}}^q \, ds
\]

\[
\leq C \left( 1 + t \right)^{-\frac{\theta_2(2q)}{2}} \int_0^t \left( \sup_{s \in [0,t]} \left\| e^{\delta \psi(s,\cdot)} |u_t(s,\cdot)| \right\|_{L^{2q}}^q \right) \, ds.
\]
where the constant is denoted by
\[ \beta_5 \doteq \frac{(2 + \rho)(1 + \varepsilon_1)}{2q} + \frac{1 + \eta}{q}. \]

By Lemma 2.3 from [15], we get for \( j = 0, 1 \)
\[
\left\| D^j u_{N_2}(t, \cdot) \right\|_{L^2} \leq C(1 + t)^{-\frac{j}{2}} \left( \sup_{s \in [0, t]} (1 + s)^{\beta_2} (1 + s)^{\frac{2 + \rho(1 - \theta_2)(2q)}{2}} \left\| \nabla u_t(s, \cdot) \right\|_{L^2}^{1 - \delta} \left\| e^{\psi(s, \cdot) \nabla u_t(s, \cdot)} \right\|_{L^2}^{\delta} \right)^q
\]
\[
= C(1 + t)^{-\frac{j}{2}} \left( \sup_{s \in [0, t]} (1 + s)^{\beta_6} \left( 1 + s \right)^{\frac{q}{2}} \left\| \nabla u_t(s, \cdot) \right\|_{L^2}^{1 - \delta} \left\| e^{\psi(s, \cdot) \nabla u_t(s, \cdot)} \right\|_{L^2}^{\delta} \right)^q
\]
\[
\leq C(1 + t)^{-\frac{j}{2}} \left( \sup_{s \in [0, t]} (1 + s)^{\beta_6} W[u](s)^{\frac{q}{2}} \right)^q
\]
where the constant is defined by
\[ \beta_6 \doteq \beta_5 + \frac{(2 + \rho)(1 - \theta_2(2q))}{2} - \frac{1 - \delta}{2} = \varepsilon_1(2 + \rho) + 2\eta + \delta - \left( \frac{1}{2} - \frac{6 + 2\rho}{2q} \right) < 0 \]
if \( q > 6 + 2\rho \) by taking sufficiently small constants \( \varepsilon_1, \eta \) and \( \delta \).

In conclusion, we derive
\[
\left\| D^j u_{N_2}(t, \cdot) \right\|_{L^2} \leq C(1 + t)^{-\frac{j}{2}} M[u](t)^{\frac{q}{2}}. \tag{4.17}
\]

Finally, summarizing the derive estimates (4.6), (4.9), (4.16) and (4.17) we conclude
\[
\left\| u^{\text{non}}(t, \cdot) \right\|_{L^2}^2 + (1 + t) \left\| D u^{\text{non}}(t, \cdot) \right\|_{L^2}^2 \leq C \left( M[u](t)^p + M[u](t)^q + M[u](t)^{p+\frac{q}{2}} \right) \left( M[u](t; p, q) \right),
\]
where we would like to show again that
\[
\tilde{M}[u](t; p, q) = I_{\exp[u_0, u_1]}^{\frac{q}{2} - 1} + M[u](t)^{\frac{p(q-1)}{2} + M[u](t)^{q-1}}.
\]
Then, we derive our desired estimate (4.1). Namely, we can claim that \( u \in X(T) \) and \( N \) maps \( X(T) \) into itself.

To prove the Lipschitz condition (4.2), we may compute
\[
M[Nu - Nv](T) = M \left[ \int_0^T E_1(t - s, x) *(x) (F(u)(s, x) - F(v)(s, x)) \, ds \right](T)
\]
\[
= M[Nw](T),
\]
where
\[
w(t, x) = u(t, x) - v(t, x).
\]

In other words, we need to estimate the next four norms:
\[
\left\| e^{\psi(t, \cdot) D N w(t, \cdot)} \right\|_{L^2}^2, \quad \left\| D_1 N w(t, \cdot) \right\|_{L^2}^2, \quad \left\| D_2 N w(t, \cdot) \right\|_{L^2}^2 \quad \text{and} \quad \left\| N w(t, \cdot) \right\|_{L^2}^2.
\]

First of all, by applying
\[
\left\| u(s, x)^p - v(s, x)^p \right\| \leq C \left| w(s, x) \right| \left( \left| u(s, x) \right|^{p-1} + \left| v(s, x) \right|^{p-1} \right), \tag{4.18}
\]
and Hölder’s inequality, one has the estimate for solution in the weighted \( L^2 \)-norm
\[
\left\| e^{\psi(s, \cdot) \left\| u(s, \cdot) \right\|^{p}} \left\| u(s, \cdot) \right\|^{p} \right\|_{L^2} \]
\[
\leq C \left\| e^{\psi(s, \cdot)} w(s, \cdot) \right\| \left( \left| u(s, \cdot) \right|^{p-1} + \left| v(s, \cdot) \right|^{p-1} \right) \left\| L^2 \right\|
\]
\[
\leq C \left\| e^{\frac{1}{2} \psi(s, \cdot)} w(s, \cdot) \right\|_{L^2(p)} \left( \left\| e^{\frac{1}{2} \psi(s, \cdot) u(s, \cdot)} \right\|_{L^2(p)}^{p-1} + \left\| e^{\frac{1}{2} \psi(s, \cdot) v(s, \cdot)} \right\|_{L^2(p)}^{p-1} \right),
\]
and the estimate for solution in the $L^2$-norm
\[
\| |u(s, \cdot)|^p - |v(s, \cdot)|^p \|_{L^2} \leq C \| w(s, \cdot) \|_{L^{2p}} \left( \| u(s, \cdot) \|_{L^{2p}}^{p-1} + \| v(s, \cdot) \|_{L^{2p}}^{p-1} \right).
\]
Next, repeating the same procedure as the proof of (4.1), we may derive
\[
\left\| e^{\psi(t, \cdot) \phi N w(t, \cdot)} \right\|_{L^2}^2 \leq CM[w](t) \sum_{r=p-1, q-1, (p-1)/2, (q-1)/2} (M[u](t)^r + M[v](t)^r), \tag{4.19}
\]
\[
\| N w(t, \cdot) \|_{L^2}^2 + (1 + t) \| \phi N w(t, \cdot) \|_{L^2}^2 \leq CM[w](t) \sum_{r=p-1, q-1} (M[u](t)^r + M[v](t)^r), \tag{4.20}
\]
providing that our assumptions $p, q > 6 + 2p_0$. Then, to conclude the remaining part of the Lipschitz condition, we just need the estimate of
\[
L(t) \doteq \| \nabla (F(u)(s, \cdot) - F(v)(s, \cdot)) \|_{L^2}.
\]
We divide the proof by two parts
\[
\begin{align*}
L_1(t) & \doteq \| \nabla (|u(s, \cdot)|^p - |v(s, \cdot)|^p) \|_{L^2} , \\
L_2(t) & \doteq \| \nabla (|u(s, \cdot)|^q - |v(s, \cdot)|^q) \|_{L^2} .
\end{align*}
\]
Obviously, it follows $L(t) \leq CL_1(t) + CL_2(t)$.

Let us sketch the proof due to the fact the proof is standard (see, for example, [22, 19, 3]). Setting $g(f) = f|f|^{p-2}$, we may rewrite the different of nonlinearity by
\[
|u(s, x)|^p - |v(s, x)|^p = p \int_0^1 w(s, x) g (\nu u(s, x) + (1 - \nu) v(s, x)) \, dv.
\]
Consequently, some applications of Minkowski’s inequality and the Leibniz rule show that
\[
\| \nabla (|u(s, \cdot)|^p - |v(s, \cdot)|^p) \|_{L^2} \leq C \int_0^1 \| \nabla w(s, \cdot) \|_{L^{r_1}} \| g (\nu u(s, \cdot) + (1 - \nu) v(s, \cdot)) \|_{L^{r_2}} \, dv
\]
\[
+ C \int_0^1 \| w(s, \cdot) \|_{L^{r_3}} \| \nabla g (\nu u(s, \cdot) + (1 - \nu) v(s, \cdot)) \|_{L^{r_4}} \, dv,
\]
where $1/r_1 + 1/r_2 = 1/r_3 + 1/r_4 = 1/2$ and these parameters will be determined later.

We notice from the Gagliardo-Nirenberg inequality that
\[
\int_0^1 \| g (\nu u(s, \cdot) + (1 - \nu) v(s, \cdot)) \|_{L^{r_2}} \, dv \leq C \left( \| u(s, \cdot) \|_{L^{2(p-1)}}^{p-1} + \| v(s, \cdot) \|_{L^{2(p-1)}}^{p-1} \right),
\]
\[
\| u(s, \cdot) \|_{L^{2(p-1)}} \leq C(1 + s) \frac{\theta(r_2(p-1))}{\theta(r_2)} W[u](s)^{\frac{1}{2}},
\]
\[
\| v(s, \cdot) \|_{L^{2(p-1)}} \leq C(1 + s) \frac{\theta(r_2(p-1))}{\theta(r_2)} W[v](s)^{\frac{1}{2}},
\]
\[
\| \nabla w(s, \cdot) \|_{L^{r_1}} \leq C(1 + s)^{\frac{1}{2}} W[w](s)^{\frac{1}{2}},
\]
\[
\| w(s, \cdot) \|_{L^{r_3}} \leq C(1 + s)^{\frac{1}{2}} W[w](s)^{\frac{1}{2}},
\]
where $\theta(r_2(p-1)) = 1 - 1/(r_2(p-1))$ and $\theta(r_3) = 1 - 2/r_3$.

Furthermore, the applications of the chain rule and the Gagliardo-Nirenberg inequality imply
\[
\| \nabla g (\nu u(s, \cdot) + (1 - \nu) v(s, \cdot)) \|_{L^{r_4}} \leq C \| \nu u(s, \cdot) + (1 - \nu) v(s, \cdot) \|_{L^{r_5}} \| \nabla (\nu u(s, \cdot) + (1 - \nu) v(s, \cdot)) \|_{L^{r_6}},
\]
\[
\leq C \left( \| u(s, \cdot) \|_{L^{r_5}} + \| v(s, \cdot) \|_{L^{r_5}} \right)^{p-2} \| \nabla u(s, \cdot) \|_{L^{r_6}} + \| \nabla v(s, \cdot) \|_{L^{r_6}}
\]
\[
\leq C(1 + s)^{\theta(r_5)(p-1)/2} \left( W[u](s)^{\frac{1}{2(p-1)}} + W[v](s)^{\frac{1}{2(p-1)}} \right).
\]
with $1/r_4 = (p - 2)/r_5 + 1/r_6$ and $\theta(r_5) = 1 - 2/r_5$.
Combining with the above derived estimates, we obtain

$$L_1(t) \leq C (1 + s)^{d_1} + (1 + s)^{d_2} \left[ W[u](s)^{\frac{p-1}{2}} + W[v](s)^{\frac{p-1}{2}} \right],$$

where the parameters in the estimate are defined by

$$d_1 = \frac{p}{2} + \frac{1}{r_2} \quad \text{and} \quad d_2 = \frac{1}{r_3} + \frac{p-2}{r_5} - \frac{p}{2}.$$ 

By choosing

$$r_1 = r_4 = \frac{1}{\varepsilon_2}, \quad r_2 = r_3 = \frac{2}{1-2\varepsilon_2}, \quad r_5 = r_6 = \frac{2(p-2)}{\varepsilon_2}$$

with sufficiently small constant $\varepsilon_2 \to 0^+$, we can show when $p > 3$, the constants satisfy

$$d_1 < -1 \quad \text{and} \quad d_2 < -1.$$ 

By using the same approach of the above, we get

$$L_2(t) \leq C(1 + s)^{-\frac{2p-1}{2}} W[w](s)^{\frac{1}{2}} \left( W[u](s)^{\frac{1}{2}} + W[v](s)^{\frac{1}{2}} \right) \left\| e^{\psi(s,\cdot)} \Delta w_t(s,\cdot) \right\|_{L^2}^{\frac{1}{2}}$$

$$+ C(1 + s)^{-\frac{2p-1}{2}} W[w](s)^{\frac{1}{2}} \left( W[u](s)^{\frac{1}{2}} + W[v](s)^{\frac{1}{2}} \right) \left\| e^{\psi(s,\cdot)} \Delta u_t(s,\cdot) \right\|_{L^2}^{\frac{1}{2}} + \left\| e^{\psi(s,\cdot)} \Delta v_t(s,\cdot) \right\|_{L^2}^{\frac{1}{2}},$$

where

$$\tilde{W}[u,v](t;q) \doteq \left( W[u](t)^{\frac{1}{2}} + W[v](t)^{\frac{1}{2}} \right)^{q-2} \left( W[u](t)^{\frac{1}{2}} + W[v](t)^{\frac{1}{2}} \right).$$

Finally, we conclude

$$\| \mathcal{D}_2 N w(t,\cdot) \|_{L^2}^2 \leq C M[w](t) \left( M[u](t)^{p-1} + M[v](t)^{p-1} \right)$$

$$+ C M[w](t) \left( M[w](t)^{p-1} + M[w](t)^{q-1} \right)^{\frac{1}{2}} \left( M[u](t) + M[v](t) \right)$$

$$+ C M[w](t) M[u,v](t) \left( M[u](t)^p + M[v](t)^q \right)^{\frac{1}{2}}. \quad (4.21)$$

Summarizing the derived estimates (4.19), (4.20) and (4.21), we claim (4.2) holds.

Applying the Banach fixed-point theorem, our proof is complete.

5. Final remark

In this paper, we prove global (in time) existence of small data solution

$$u \in \mathcal{C} ([0, \infty), H^2(\Omega) \cap H^1_0(\Omega)) \cap \mathcal{C}^1 ([0, \infty), H^1(\Omega))$$

to the exterior problem (1.1) with $p, q > 6 + 2\rho_0$. Let us give some explanations for the conditions of $p$ and $q$. Actually, we may observe from the proof that the condition for the exponent $p$ is influenced by the value of $\rho$. Because we apply the weighted function $e^{\psi(t,x)}$ with parameter $\rho$ on the energy estimates in this paper, the interplay between the power nonlinearity $|u|^p$ and $|u|^q$ comes. Moreover, to control the higher-order energy, we choose a suitable parameter $\rho$ in the weighted function such that $\rho > \rho_0$.

To end the paper, we give some remarks on the semilinear strongly damped wave equations with an exterior domain for higher-dimensional case, namely,

$$\begin{aligned}
& u_{tt} - \Delta u - \Delta u_t = f(u, u_t; p, q), \quad x \in \Omega, \ t > 0, \\
& u(0, x) = u_0(x), \ u_t(0, x) = u_1(x), \quad x \in \Omega, \\
& u = 0, \quad x \in \partial\Omega, \ t > 0, \\
\end{aligned} \quad (5.1)$$
with $p, q > 1$, where $\Omega \subset \mathbb{R}^n$ for $n \geq 3$ is an exterior domain with a compact smooth boundary $\partial \Omega$. Without loss of generality, we assume again that $0 \notin \overline{\Omega}$. In (5.1), the nonlinearity can be represented by

$$f(u, u_t; p, q) = a|u|^p + b|u_t|^q$$

with $a, b \geq 0$ but $a + b \neq 0$, and $p, q > 1$. From [15], we understand the difficulties to study global (in time) existence of small data solutions in higher-dimension ($n \geq 3$) is that we must restrict the power $p$ to the range $1 < p, q \leq n/(n-2)$, which is restricted by the application of the Gagliardo-Nirenberg inequality or Sobolev inequality. Nevertheless, by applying the weighted energy method, we always proposed the a strong condition such as $p > 6$ for $f(u, u_t; p, q) = |u|^p$ in [15], and $p, q > 6 + 2\rho_0$ for $f(u, u_t; p, q) = |u|^p + |u_t|^q$ in this paper for 2D. It will immediately leads to the empty range of $p$ and $q$. To solve this difficulty, motivated by the main approach of this paper, we observe that there exists a possibility to consider higher-order energy solutions with even large regular data. At this time, we may apply the embedding $H^s(\Omega) \hookrightarrow L^\infty(\Omega)$ for $s > n/2$ rather than apply the Gagliardo-Nirenberg inequality for estimating the solutions in the $L^{2p}$ and $L^{2q}$ norms. Furthermore, the benefit of this approach is to weaken the upper bound restrictions of $p, q$ from $n/(n-2)$ to $\infty$ (One may see this effect in Section 6.2 of [22]). Thus, we just need to derive higher-order energy estimates with large regular data. But, we should emphasize that higher-order energy estimates with exponentially weighted function are still open.

### Appendix A. Gagliardo-Nirenberg type inequalities

Let us introduce a well-known interpolation inequality, i.e., the Gagliardo-Nirenberg inequality in exterior domains in 2D. This result has been proved by [4].

**Lemma A.1.** Let $1 \leq r \leq q \leq +\infty$. If $v \in H^1(\Omega)$ with a exterior domain $\Omega \subset \mathbb{R}^2$ with compact boundary, having the cone property, then the inequality holds

$$\|v\|_{L^r} \leq M \|v\|_{L^2}^{1-\theta(q)} \|\nabla v\|_{L^2}^{\theta(q)},$$

where $M > 0$ is a constant independent of $v$ and $\theta(q) = 2(1/2 - 1/q) \in (0, 1]$.

Next, we give a weighted version of Gagliardo-Nirenberg inequality in exterior domain. For $\sigma > 0$, $\rho > 0$ and $t \geq 0$, we may define a family of weighted function space by

$$H^1_{\psi(t, \cdot)}(\Omega) = \left\{ f \in H^1(\Omega) : \left\| e^{\sigma \psi(t, \cdot)} f \right\|_{L^2}^2 + \left\| e^{\sigma \psi(t, \cdot)} \nabla f \right\|_{L^2}^2 < \infty \right\}.$$  

According to our choice of weighted function $\psi(t, x)$, the following Gagliardo-Nirenberg type inequality holds (cf. Lemma 2.3 in [15])

**Lemma A.2.** Let $\theta(q) = 2(1/2 - 1/q)$ and $0 \leq \theta(q) < 1$ and let $0 < \sigma \leq 1$, $\rho > 0$. If $v \in H^1(\psi(t, \cdot))$ with $t \geq 0$, then the inequality holds

$$\left\| e^{\sigma \psi(t, \cdot)} v \right\|_{L^r} \leq C(1 + t)^{(2+\rho)(1-\theta(q))/2} \left\| \nabla v \right\|_{L^2}^{1-\sigma} \left\| e^{\psi(t, \cdot)} \nabla v \right\|_{L^2}^{\sigma}$$

for each $t \geq 0$, where $C = C_\sigma > 0$ is a positive constant.

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