Quantum billiards in multidimensional models with fields of forms

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Abstract

Cosmological Bianchi-I type model in the \((n + 1)\)-dimensional gravitational theory with several forms is considered. When electric non-composite brane ansatz is adopted the Wheeler-DeWitt (WDW) equation for the model, written in the conformally-covariant form, is analyzed. Under certain restrictions asymptotic solutions to WDW equation near the singularity are found which reduce the problem to the so-called quantum billiard on the \((n - 1)\)-dimensional Lobachevsky space \(H^{n-1}\). Two examples of quantum billiards are considered: 2-dimensional quantum billiard for 4-dimensional model with three 2-forms and 9-dimensional quantum billiard for \(D = 11\) model with 120 4-forms which mimic \(M2\)-brane solutions of \(D = 11\) supergravity. For certain asymptotic solutions the vanishing of the wave function at the singularity is proved.

1 Introduction

In this paper we deal with the quantum billiard approach for multidimensional cosmological-type models defined on the manifold \((u_-, u_+) \times \mathbb{R}^n\), where \(n \geq 3\).

In classical case the billiard approach was suggested by Chitre [1] for explanation the BLK-oscillations [2] in the Bianchi-IX model [3, 4] by using a simple triangle billiard in the Lobachevsky space \(H^2\).

The billiard approach in dimension \(D = 4\) in classical and quantum case was also considered in papers of A.A. Kirillov [5].

In multidimensional case the billiard representation for cosmological model with multicomponent “perfect” fluid was introduced in [6, 7, 8]. In [8] the finiteness of the billiard volume was formulated in terms of the so-called illumination problem. The inequalities on Kasner parameters were written and the quantum billiard was also considered.

The billiard approach for multidimensional models with scalar fields and fields of forms was suggested in [9], where inequalities on Kasner parameters were found. These inequalities played an important role in the proof of “chaotic” behavior in superstring-inspired (e.g. supergravitational) models [10, 11].

Recently the quantum billiard approach for a multidimensional gravitational model with several forms was considered in [12, 13]. The asymptotic solutions to WDW equation presented in these papers are equivalent to those obtained earlier in [8]. In refs. [9, 12, 13] a semi-quantum approach was used, when gravity was quantum but the matter (e.g. fluids,
forms) was considered as the classical one. It should be noted that a semi-quantum form of WDW equation for the model with fields of forms and scalar field was suggested earlier in [14].

Here we use another form of the WDW equation with enlarged minisuperspace which include the form potentials $\Phi_s$ [15]. We suggest another version of the quantum billiard approach by deducing asymptotic solutions to WDW equation which obey the master equation with anomaly term $A < 0$. We consider two examples of quantum billiards: triangle 2-dimensional Chitre’s billiard for $D = 4$ model with three 2-forms and 9-dimensional billiard for $D = 11$ model with 120 4-forms which mimics $M2$-brane sector of $D = 11$ supergravity. For certain asymptotic solutions the vanishing of the wave function is proved.

2 The model

Here we consider the multidimensional gravitational model governed by the action

$$S_{\text{act}} = \frac{1}{2\kappa^2} \int_M d^Dz \sqrt{|g|} \{ R[g] - \sum_{s \in S} \frac{\theta_s}{n_s^s} (F_s^s)^2 \} + S_{\text{YGH}},$$

where $g = g_{MN} dz^M \otimes dz^N$ is the metric on the manifold $M$, $\dim M = D$, $\theta_s \neq 0$, $F_s^s = dA_s^s = \frac{1}{n_s^s} F_{M_1...M_{n_s^s}} dz^{M_1} \wedge \ldots \wedge dz^{M_{n_s^s}}$ is a $n_s$-form ($n_s \geq 2$) on a $D$-dimensional manifold $M$, $s \in S$. In (2.1) we denote $|g| = |\det (g_{MN})|$, $(F_s^s)^2 = F_{M_1...M_{n_s^s}} F_{N_1...N_{n_s^s}} g^{M_1N_1} \ldots g^{M_{n_s^s}N_{n_s^s}}$, $s \in S$, where $S$ is some finite set of indices and $S_{\text{YGH}}$ is the standard York-Gibbons-Hawking boundary term [16,17].

In the models with one time and usual form fields all $\theta_s > 0$ when the signature of the metric is $(-1, +1, \ldots, +1)$. For this choice of the signature $\theta_s < 0$ corresponds to a phantom form field $F_s^s$.

2.1 Ansatz for non-composite brane configurations

Let us consider the manifold

$$M = \mathbb{R}_+ \times \mathbb{R}^n,$$

with the metric

$$g = we^{2\gamma(u)} du \otimes du + \sum_{i=1}^n e^{2\phi_i(u)} \varepsilon(i) dx^i \otimes dx^i,$$

where $\mathbb{R}_+ = (u_-, u_+)$, $w = \pm 1$ and $\varepsilon(i) = \pm 1$, $i = 1, \ldots, n$. The dimension of $M$ is $D = 1 + n$. For $w = -1$ and $\varepsilon(i) = 1$, $i = 1, \ldots, n$, we deal with cosmological solutions while for $w = 1$, and $\varepsilon(1) = -1$, $\varepsilon(j) = 1$, $j = 2, \ldots, n$, we get static solutions (e.g. wormholes etc).

Let $\Omega = \Omega(n)$ be a set of all non-empty subsets of $\{1, \ldots, n\}$. For any $I = \{i_1, \ldots, i_k\} \in \Omega$, $i_1 < \ldots < i_k$, we denote $\tau(I) \equiv dx^{i_1} \wedge \ldots \wedge dx^{i_k}$, $\varepsilon(I) \equiv \varepsilon(i_1) \ldots \varepsilon(i_k)$, $d(I) = |I| \equiv k$.

For the fields of forms we consider the following non-composite electric ansatz

$$A_s^s = \Phi^s \tau(I_s), \quad F_s^s = d\Phi^s \wedge \tau(I_s),$$

where $\Phi^s = \Phi^s(u)$ is smooth function on $\mathbb{R}_+$ and $I_s \in \Omega$, $s \in S$. Due to (2.4) we have $d(I_s) = n_s - 1$, $s \in S$. 

2
2.2 Sigma-model action

It was proved in [18] that the equations of motion for the model (2.1) with the fields from (2.3) and (2.4) are equivalent to equations of motion for the \( \sigma \)-model governed by the action

\[
S_{\sigma 0} = \frac{\mu}{2} \int du N \left\{ G_{ij} \dot{\phi}^i \dot{\phi}^j + \sum_{s \in S} \varepsilon_s \exp \left( -2U_s^i \phi^i \right) (\dot{\Phi}_s)^2 \right\},
\]

where \( \dot{X} \equiv dX/du \), \( \mu \neq 0 \), \( \gamma_0(\phi) \equiv \sum_{i=1}^n \phi^i \) and \( N = \exp(\gamma_0 - \gamma) > 0 \) is modified lapse function,

\[
G = G_{ij} d\phi^i \otimes d\phi^j, \quad G_{ij} = \delta_{ij} - 1,
\]

is truncated target space metric and co-vectors \( U^s \) read

\[
U^s(\phi) = U_i^s \phi^i = \sum_{i \in I_s} \phi^i, \quad U^s = (U^s_i) = \delta_{iI_s},
\]

\( s \in S \).

Here

\[
\delta_{iI} = \sum_{j \in I} \delta_{ij}
\]

is an indicator of \( i \) belonging to \( I \): \( \delta_{iI} = 1 \) for \( i \in I \) and \( \delta_{iI} = 0 \) otherwise; and

\[
\varepsilon_s = \varepsilon(I_s)\theta_s,
\]

\( s \in S \).

In what follows we will use the scalar product

\[
(U, U') = G^{ij} U_i U'_j,
\]

for \( U = (U_i), U' = (U'_i) \in \mathbb{R}^n \), where \( (G^{ij}) \) is the matrix inverse to the matrix \( (G_{ij}) \)

\[
G^{ij} = \delta^{ij} + \frac{1}{2 - D},
\]

\( i, j = 1, \ldots, n. \)

The scalar products of \( U \)-vectors (2.7) read [18]

\[
(U^s, U^{s'}) = d(I_s \cap I_{s'}) + \frac{d(I_s) d(I_{s'})}{2 - D},
\]

\( s, s' \in S. \)

Action (2.5) may be also written in the form

\[
S_{\sigma} = \frac{\mu}{2} \int du N \left\{ \mathcal{G}_{AB}(X) \dot{X}^A \dot{X}^B \right\},
\]

where \( X = (X^A) = (\phi^i, \Phi^s) \in \mathbb{R}^N \), \( N = n + m \), \( m = |S| \) is the number of branes and minisupermetric \( \mathcal{G} = \mathcal{G}_{AB}(X) dX^A \otimes dX^B \) on minisuperspace \( \mathcal{M} = \mathbb{R}^N \) is defined by the relation
\[
G = G + \sum_{s \in S} \varepsilon_s e^{-2U^s(\phi)} d\Phi^s \otimes d\Phi^s,
\] (2.14)

where \( G \) is defined in (2.6) and \( U^s(\phi) = U_i^s \phi^i \) is defined in (2.7).

In what follows we use the notation

\[
U^\Lambda (\phi) = U^\Lambda_i \phi^i = \gamma_0(\phi), \quad U^\Lambda = 1.
\] (2.15)

The vector \( U^\Lambda = (U^\Lambda_i) \) is time-like, since

\[
(U^\Lambda, U^\Lambda) = -\frac{D-1}{D-2} < 0.
\] (2.16)

3 Quantum billiard approach

In this section we develop a quantum analogue of the billiard approach which is understood as finding of certain asymptotic solutions to Wheeler–DeWitt (WDW) equation.

3.1 Restrictions.

First we outline two restrictions which will be used in derivation of the quantum billiard

\[
\begin{align*}
(i) \quad &d(I_s) < D - 2, \\
(ii) \quad &\varepsilon_s > 0,
\end{align*}
\] (3.1)

for all \( s \). The first restriction on the dimensions of the brane worldvolumes excludes domain walls. The second one is a necessary condition for the formation of infinite “wall” potential in certain limit (see below).

Due to the first restriction we get

\[
(U^s, U^s) = d(I_s) \left( 1 + \frac{d(I_s)}{2 - D} \right) > 0, \quad s \in S.
\] (3.3)

3.2 Wheeler-DeWitt equation

Let us fix the temporal gauge as follows

\[
\gamma_0 - \gamma = 2f(X), \quad \mathcal{N} = e^{2f},
\] (3.4)

where \( f: \mathcal{M} \rightarrow \mathbb{R} \) is a smooth function. Then we obtain the Lagrange system with the Lagrangian

\[
L_f = \frac{\mu}{2} e^{2f} G_{AB}(X) \dot{X}^A \dot{X}^B
\] (3.5)

and the energy constraint

\[
E_f = \frac{\mu}{2} e^{2f} G_{AB}(X) \dot{X}^A \dot{X}^B = 0.
\] (3.6)
Using the standard prescriptions of (covariant and conformally covariant) quantization of the energy constraint, see \[19\, 20\, 21\, 15\] and refs. therein, we are led to the Wheeler-DeWitt (WDW) equation

\[\hat{H}^f \Psi^f \equiv \left( -\frac{1}{2\mu} \Delta \left[ e^{2f} G \right] + \frac{a}{\mu} R \left[ e^{2f} G \right] \right) \Psi^f = 0, \quad (3.7)\]

where

\[a = a_N = \frac{(N - 2)}{8(N - 1)}, \quad (3.8)\]

\[N = n + m.\]

Here \(\Psi^f = \Psi^f(X)\) is the wave function corresponding to the \(f\)-gauge \(3.4\) and satisfying the relation

\[\Psi^f = e^{bf} \Psi^f=0, \quad b = (2 - N) / 2. \quad (3.9)\]

In (3.7) we denote by \(\Delta[G^f]\) and \(R[G^f]\) the Laplace-Beltrami operator and the scalar curvature corresponding to the metric

\[G^f = e^{2f} G, \quad (3.10)\]

respectively.

Let us put \(f = f(\phi)\). Then we get

\[\Delta[G^f] = e^{2f} |G|^{-1/2} \frac{\partial}{\partial \phi^i} \left( \tilde{G}^{ij} e^{-U} |\tilde{G}|^{1/2} \frac{\partial}{\partial \phi^j} \right) + \sum_{s \in S} e^{2U^s(\phi)} \left( \frac{\partial}{\partial \Phi^s} \right)^2, \quad (3.11)\]

where

\[\tilde{U} = \sum_{s \in S} \tilde{U}^s, \quad \tilde{U}^s = U^s(\phi) - f \quad (3.12)\]

and

\[G_{ij} = e^{2f} G_{ij}, \quad G^{ij} = e^{-2f} G^{ij}, \quad (3.13)\]

\[|G| = | \det (G_{ij}) |.\]

Here we are interested in a special class of asymptotical solutions to WDW-equation. The metrics \(G, \mathcal{G}\) have pseudo-Euclidean signatures \((-+, +, ..., +\) (the last one - due to \(3.2\)). We put

\[e^{2f} = -(G_{ij} \phi^i \phi^j)^{-1}, \quad (3.14)\]

where \(G_{ij} \phi^i \phi^j < 0.\)

In what follows we will use a diagonalization of \(\phi\)-variables

\[\phi^i = S^i_a z^a, \quad (3.15)\]

\(a = 0, ..., n - 1,\) obeying \(G_{ij} \phi^i \phi^j = \eta_{ab} z^a z^b,\) where \((\eta_{ab}) = \text{diag}(-1, +1, ..., +1).\)
We restrict the WDW equation to the lower light cone $V_\pm = \{ z = (z^0, \vec{z}) | z^0 < 0, \eta_{ab} z^a z^b < 0 \}$ and introduce Misner-Chitre-like coordinates

\begin{align*}
z^0 &= -e^{-y^0} \frac{1 + \vec{y}^2}{1 - \vec{y}^2}, \quad (3.16) \\
\vec{z} &= -2e^{-y^0} \frac{\vec{y}}{1 - \vec{y}^2}, \quad (3.17)
\end{align*}

where $y^0 < 0$ and $\vec{y}^2 < 1$. We note that in these variables $f = y^0$.

Using the relation $f, i = \bar{G}^{ij} \phi^j$, following from (3.14), we obtain

$$\Delta [\bar{G}] f = 0, \quad \bar{G}^{ij} f, i f, j = -1. \quad (3.18)$$

These relations may readily deduced from the following formula

$$G = -dy^0 \otimes dy^0 + h_L, \quad (3.19)$$

where

$$h_L = 4\delta_{rs} dy^r \otimes dy^s \frac{(1 - \vec{y}^2)^2}{(1 - \vec{y}^2)^2}, \quad (3.20)$$

(with summation over $r, s = 1, ..., n - 1$ assumed). Here the metric $h_L$ is defined on the unit ball $D^{n-1} = \{ \vec{y} \in \mathbb{R}^{n-1} | \vec{y}^2 < 1 \}$. The pair $(D^{n-1}, h_L)$ is one of the realization of $(n - 1)$-dimensional analogue of the Lobachevsky space.

For the wave function we consider the following ansatz

$$\Psi^f = e^{C(\phi)} \Psi_s, \quad (3.21)$$

where the pre-factor $e^{C(\phi)}$ is chosen in order to cancel the terms linear in derivatives $(\Psi_{s,i})$ arising in calculation of $\Delta [G^f] \Psi^f$. This takes place if we put

$$C(\phi) = \frac{1}{2} \bar{U} = \frac{1}{2} \sum_{s \in S} U^s \phi^s - mf. \quad (3.22)$$

From (3.21) and (3.22) we get

$$e^{C(\phi)} \left(\frac{1}{2} \Delta [e^{2f} \bar{G}] + aR [e^{2f} \bar{G}] \right) (e^{C(\phi)} \Psi_s) =$$

$$= e^{C(\phi)} \left( -\frac{1}{2} \Delta [\bar{G}] - \frac{1}{2} \sum_{s \in S} e^{2U^s} \left( \frac{\partial}{\partial \Phi^s} \right)^2 + \delta V \right) \Psi_s, \quad (3.23)$$

where

$$\delta V = Ae^{-2f} - \frac{1}{8}(n - 2)^2. \quad (3.24)$$

Here we denote

$$A = \frac{1}{8(N - 1)} \sum_{s, s' \in S} (e^{U^s}, e^{U^{s'}}) - (N - 2) \sum_{s \in S} (U^s, U^s). \quad (3.25)$$
Now we put
$$\Psi^f = e^{C(\phi)} e^{iQ_s\Phi^s} \Psi_{0,L},$$
(3.26)
where parameters \(Q_s \neq 0\) correspond to charge densities of branes and \(e^{iQ_s\Phi^s} = \exp(i \sum_{s \in S} Q_s \Phi^s)\).

Using relation (3.23) we get
$$\hat{H}^f \Psi^f = \mu -1 e^{C(\phi)} \left(-\frac{1}{2} \Delta [\bar{G}]+ \frac{1}{2} \sum_{s \in S} Q_s^2 e^{-2f+2U^s(\phi)} + \delta V\right) \Psi_{0,L} = 0.$$  
(3.27)

### 3.3 Asymptotic behavior of the solutions for \(y^0 \to -\infty\)

Here we deal with asymptotic solutions to WDW equation in the limit \(y^0 \to -\infty\). Due to relations (3.26) and (3.27) this equation reads
$$\left(-\frac{1}{2} \Delta [\bar{G}]+ \frac{1}{2} \sum_{s \in S} Q_s^2 e^{-2f+2U^s(\phi)} + \delta V\right) \Psi_{0,L} = 0.$$  
(3.28)

It was shown in our paper on the classical billiard approach [9] that
$$\frac{1}{2} \sum_{s \in S} Q_s^2 e^{-2f+2U^s(\phi)} \to V_\infty,$$  
(3.29)
as \(y^0 = f \to -\infty\).

In this relation \(V_\infty\) is the potential of infinite walls which are produced by branes:
$$V_\infty = \sum_{s \in S} \theta_\infty (\vec{v}_s^2 - 1 - (\vec{y} - \vec{v}_s)^2)$$  
(3.30)

Here we use the notation \(\theta_\infty(x) = +\infty\) for \(x \geq 0\) and \(\theta_\infty(x) = 0\) for \(x < 0\). The vectors \(\vec{v}_s, s \in S\), belonging to \(\mathbb{R}^{n-1}\) are defined by the formulae
$$\vec{v}_s = -\vec{u}_s/u_{s0},$$  
(3.31)
where \(n\)-dimensional vectors \(u_s = (u_{s0}, \vec{u}_s) = (u_{sa})\) are obtained from \(U^s\)-vectors using a diagonalization matrix \((S^s_i)\) from (3.15)
$$u_{sa} = S^s_i U^s_i.$$  
(3.32)

Due to condition (3.1)
$$(U^s, U^s) = -(u_{s0})^2 + (\vec{u}_s)^2 > 0$$  
(3.33)
for all \(s\). Here we use a diagonalization (3.15) from [9] obeying
$$u_{s0} > 0$$  
(3.34)
for all $s \in S$. The inverse matrix $(S_t^a) = (S_a^i)^{-1}$ defines the map inverse to (3.15)

$$z^a = S_t^a \phi_i,$$

(3.35)

$a = 0, \ldots, n - 1$.

The inequalities (3.33) imply $|\vec{v}_s| > 1$ for all $s$. The potential $V_\infty$ corresponds to the billiard $B$ in the multidimensional Lobachevsky space $(D^{n-1}, h_L)$. This billiard is an open domain in $D^{n-1}$ which is defined by a set of inequalities:

$$|\vec{y} - \vec{v}_s| < \sqrt{\vec{v}_s^2 - 1} = r_s,$$

(3.36)

$s \in S$. The boundary $\partial B$ is formed by parts of hyper-spheres with centers in $\vec{v}_s$ and radii $r_s$.

The condition (3.34) is also obeyed for the diagonalization (3.35) with

$$z^0 = U_i \phi^i / \sqrt{|(U, U)|},$$

(3.37)

where $U$-vector is time-like

$$(U, U) < 0$$

(3.38)

and

$$(U, U_s) < 0$$

(3.39)

for all $s \in S$.

The inequalities (3.38) and (3.39) are satisfied identically if $U = kU^A$, $k > 0$, see (2.15).

Conditions (3.39) and hence (3.34) may be relaxed. In this case we obtain a more general prescription for the drawing of the billiard walls (e.g. for $u_{s0} < 0$ and $u_{s0} = 0$) described in [22].

Thus, we are led to an asymptotic relation for the function $\Psi_{0,L}(y^0, \vec{y})$

$$\left( -\frac{1}{2} \Delta[G] + \delta V \right) \Psi_{0,L} = 0$$

(3.40)

with $\vec{y} \in B$ and the zero boundary condition $\Psi_{0,L}|_{\partial B} = 0$ imposed. Due to (3.19) we get

$$\Delta[G] = - (\partial_0)^2 + \Delta[h_L],$$

where $\Delta[h_L] = \Delta_L$ is the Laplace-Beltrami operator corresponding to the $(n - 1)$-dimensional Lobachevsky metric $h_L$.

By splitting the variables

$$\Psi_{0,L} = \Psi_0(y^0) \Psi_L(\vec{y})$$

(3.41)

we are led to the asymptotic relation (for $y^0 \to -\infty$)

$$\left( \left( \frac{\partial}{\partial y^0} \right)^2 - \Delta_L + 2Ae^{-2y^0} + E - \frac{1}{4} (n - 2)^2 \right) \Psi_0 = 0$$

(3.42)

equipped with the relations

$$\Delta_L \Psi_L = -E, \quad \Psi_L|_{\partial B} = 0.$$  (3.43)

Here we assume that the operator $(-\Delta_L)$ with the zero boundary condition imposed has a spectrum obeying

$$E \geq \frac{1}{4} (n - 2)^2.$$  (3.44)
This is valid at least when the billiard $B$ is (sub)compact and small enough. The examples of non-(sub)compact billiards obeying (3.44) are considered in the next section.

Here we put

$$A < 0.$$  \hspace{1cm} (3.45)

Solving equation (3.42) we get for $A < 0$ the following basis of solutions

$$\Psi_0 = B_{i\omega} \left( \sqrt{2|A|} e^{-y^0} \right),$$  \hspace{1cm} (3.46)

where $B_{i\omega}(z) = I_{i\omega}(z), K_{i\omega}(z)$ are modified Bessel functions and

$$\omega = \sqrt{E - \frac{1}{4}(n - 2)^2} \geq 0.$$  \hspace{1cm} (3.47)

In semi-quantum case (with quantum gravity and classical matter source) [8, 12, 13] the anomaly term is absent, i.e. $A = 0$.

Using asymptotical relations

$$I_{\nu} \sim \frac{e^z}{\sqrt{2\pi z}}, \quad K_{\nu} \sim \frac{e^{-z}}{\sqrt{2\pi z}},$$  \hspace{1cm} (3.48)

for $z \to +\infty$, we get

$$\Psi_0 \sim C_{\pm} \exp \left( \pm \sqrt{2|A|} e^{-y^0} + \frac{1}{2} y^0 \right),$$  \hspace{1cm} (3.49)

for $y^0 \to -\infty$. Here $C_{\pm}$ are nonzero constants and “+” corresponds to $B = I$ and “-” - to $B = K$. Now we evaluate the pre-factor $e^{C(\phi)}$ in (3.26), where

$$C(\phi) = \frac{1}{2}(U(\phi) - mf)$$  \hspace{1cm} (3.50)

and

$$U(\phi) = U_i\phi^i = \sum_{s \in S} U^s_i \phi^i, \quad U_i = \sum_{s \in S} U^s_i.$$  \hspace{1cm} (3.51)

Now we use $U = (U_i)$ as a time-like vector which defines $z^0$ in (3.37). Thus, we need to impose the restriction $(U, U) < 0$. Then, using (3.37), (3.16) and $f = y^0$ we obtain

$$C(\phi) = \frac{1}{2} \left( -q e^{-y^0} \frac{(1 + \bar{y}^2)}{1 - \bar{y}^2} - my^0 \right),$$  \hspace{1cm} (3.52)

where

$$q = \sqrt{-(U, U)} > 0.$$  \hspace{1cm} (3.53)

Combining relations (3.26), (3.41), (3.49) and (3.52) we find

$$\Psi^f \sim C_{\pm} \exp \left( \theta^\pm(|\bar{y}|) e^{-y^0} - \frac{1}{2}(m - 1)y^0 \right) e^{iQ \cdot \Phi^0} \Psi_L(\bar{y}),$$  \hspace{1cm} (3.54)

as $y^0 \to -\infty$ for fixed $\bar{y} \in B$. Here $C_{\pm} \neq 0$ and

$$\theta^\pm(|\bar{y}|) = -\frac{q}{2} \frac{(1 + \bar{y}^2)}{1 - \bar{y}^2} \pm \sqrt{-2A},$$  \hspace{1cm} (3.55)
where “+” corresponds to the Bessel function $B = I$ and “-” - to $B = K$.

Relation (3.25) may be rewritten as

$$A = \frac{1}{8(N-1)}[(U, U) - (N - 2) \sum_{s \in S} (U^s, U^s)].$$  \hspace{1cm} (3.56)$$

where we have used identity

$$(U, U) = \sum_{s, s' \in S} (U^s, U^{s'}),$$  \hspace{1cm} (3.57)$$

following from the definition of $U$ in (3.51). It should be noted that restrictions $(U, U) < 0$ and $(U^s, U^s) > 0$, $s \in S$, imply $A < 0$.

Now we study the asymptotical behaviour of the wave function (3.26)

$$\Psi^f = e^{C(\phi)} e^{iQ_s \Phi^s} B_{i\omega} \left( \sqrt{2|A|} e^{-y^0} \right) \Psi_L(y),$$  \hspace{1cm} (3.58)$$

with $C(\phi)$ from (3.52) and $(U, U) < 0$, $A < 0$.

Let i) $B = K$. Then

$$\Psi^f \to 0$$  \hspace{1cm} (3.59)$$
as $y^0 \to -\infty$ for fixed $y \in B$ and $\Phi^s \in \mathbb{R}$, $s \in S$. This follows just from (3.54).

Now we consider the case ii) $B = I$. First we put

$$\frac{1}{2} q > \sqrt{2|A|},$$  \hspace{1cm} (3.60)$$

We get

$$\Psi^f \to 0$$

as $y^0 \to -\infty$ for fixed $y \in B$ and $\Phi^s \in \mathbb{R}$, $s \in S$. This also follows from (3.54).

Let us consider the second case

$$\frac{1}{2} q = \sqrt{2|A|},$$  \hspace{1cm} (3.61)$$

We obtain

$$\Psi^f \to 0$$

as $y^0 \to -\infty$ for fixed $y \in B \setminus \{0\}$ and $\Phi^s \in \mathbb{R}$, $s \in S$. This also follows from (3.54). Moreover, $|\Psi^f| \to +\infty$ as $y^0 \to -\infty$, when $y = 0$ and $\Psi_L(0) \neq 0$.

Now we consider the third case

$$\frac{1}{2} q < \sqrt{2|A|},$$  \hspace{1cm} (3.62)$$

If the point $\{0\}$ belongs to the billiard $B$ and $\Psi_L(0) \neq 0$ then, it may be readily verified that there exists $\varepsilon > 0$ such that for all $y$ obeying $|y| < \varepsilon$ (and all $\Phi^s \in \mathbb{R}$, $s \in S$)

$$|\Psi^f| \to +\infty$$  \hspace{1cm} (3.63)$$
as $y^0 \to -\infty$. 

10
4 Examples

Here we consider two examples of quantum billiards in dimensions $D = 11$ and $D = 4$. In this section we deal with $(n + 1)$-dimensional cosmological metrics (2.3) with $w = -1$.

4.1 9-dimensional billiard in $D = 11$ model

Let us consider 11-dimensional gravitational model with several 4-forms, which gives non-composite analogous of $M$-brane solutions in $D = 11$ supergravity [23]. The action reads as follows

$$ S_{11} = \frac{1}{2\kappa^2_{11}} \int_M d^{11}z \sqrt{|g|} \{ R[g] + \mathcal{L} \} + S_{YGH}, \quad (4.1) $$

where

$$ \mathcal{L} = -\frac{1}{4!} \sum_{I \in S} (F^I_4)^2. \quad (4.2) $$

Here $F^I_4$ is 4-form with the index $I \in S$, where here $S$ is the set of all subsets with three elements: $I = \{i_1, i_2, i_3\}$, $1 \leq i_1 < i_2 < i_3 \leq 10$. The number of elements in $S$ is 120.

The action (4.1) with $\mathcal{L}$ from (4.2) mimics non-composite $SM2$-brane solutions which are given by the metric (2.3) with $w = -1$, $n = 10$ and

$$ F^I_4 = d\Phi^I \wedge \tau(I), \quad (4.3) $$

$I \in S$.

We consider the non-trivial case when all charge densities of branes $Q_I$, $I \in S$, are non-zero. In the classical case we get a 9-dimensional billiard $B \in H^9$ with 120 “electric” walls. This billiard was found in [10]. It has a finite volume.

The minus Laplace-Beltrami operator $(-\Delta_L)$ on $B$ with zero boundary conditions has a spectrum obeying restriction (3.44) with $n = 10$ [13].

Let us us calculate $(U, U)$, where $U = \sum_{s \in S} U^s$. We get $U_i = \sum_{I \in S} \delta_{iI}$ where $\delta_{iI}$ is defined in (2.8). Thus, $U_i$ is the number of sets $I \in S$ which contain the point $i$. It is obvious that $U_i = G^2_9 = 36$. Thus, $U = 36U^A$ and hence we may use the $z$-variables from [8, 9].

We get (see (2.11))

$$ (U, U) = G^{ij}U_iU_j = \sum_{i,j=1}^{10} (\delta^{ij} - \frac{1}{9})(36)^2 = -1440 < 0 \quad (4.4) $$

in agreement with our restriction (3.38). Since $N = 130$ ($m = 120$) and all $(U^s, U^s) = 2$ we obtain from (3.56) the following value for the anomaly number

$$ A = -\frac{1340}{43}. \quad (4.5) $$

In this case the inequality (3.60) is satisfied and hence we get from the previous analysis that the wave function $\Psi^f \to 0$ is asymptotically vanishing as $y^0 \to -\infty$. 

11
4.2 2-dimensional billiard in $D = 4$ model

Let us consider the 4-dimensional gravitational model with three 2-forms which gives the two-dimensional Chitre’s billiard.

The action reads
\[
S_4 = \frac{1}{2\kappa^2} \int_M d^4z \sqrt{|g|} \left\{ R[g] + \mathcal{L} \right\} + S_{YGH},
\]
(4.6)

where
\[
\mathcal{L}_e = -\frac{1}{2} \sum_{i=1,2,3} (F_2^i)^2.
\]
(4.7)

Here $F_2^i$ is a 2-form, $i = 1, 2, 3$.

By using the ansatz
\[
F_2^i = d\Phi^i(u) \wedge dx^i,
\]
(4.8)
i = 1, 2, 3, with non-zero charges $Q_i$ and the metric (2.3) with $w = -1$, $n = 3$, we are led to the Chitre’s triangle billiard (see [9]) which has a finite volume. The energy restriction (3.14) is also obeyed in this case [13].

The calculations give us $(U^s, U^s) = \frac{1}{2}$, $(U, U) = -\frac{3}{2}$ and
\[
A = -\frac{3}{16}.
\]
(4.9)

According to the analysis which was performed above we get the asymptotic vanishing of the wave function: $\Psi^f \to 0$ as $y^0 \to -\infty$, when either i) $B = K$, or ii) $B = I$ and $\vec{y} \neq \vec{0}$ (see (3.61)).

5 Conclusions

Here we have considered the quantum billiard approach for the cosmological-type model with $n$ one-dimensional factor-spaces in the theory with several forms. When electric non-composite brane ansatz was adopted and certain restrictions on parameters of the model were imposed the Wheeler-DeWitt (WDW) equation for the model, written in conformally-covariant form, was analyzed.

We have imposed certain restrictions on parameters of the model and have obtained asymptotic solutions to WDW equation. These solutions are of quantum billiard form since they are governed by the spectrum of the Lapalace-Beltrami operator on the billiard with the zero boundary condition imposed. The billiard belongs to the $(n-1)$-dimensional Lobachevsky space $H^{n-1}$.

We have presented two examples of quantum billiards: (a) the quantum $d = 2$ billiard in $D = 4$ gravitational model with three 2-forms and (b) the quantum $d = 9$ billiard for $D = 11$ gravitational model with 120 4-forms which mimics the quantum billiard with $M2$-branes in $D = 11$ supergravity. We have shown the asymptotic vanishing of the wave function: $\Psi^f \to 0$, in the case (b) for all basis solutions and in the case (a) for the Bessel function $B = K$ and for $B = I$ when $\vec{y} \neq \vec{0}$. 

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References

[1] D.M. Chitre, Ph. D. Thesis (University of Maryland) 1972.

[2] V.A. Belinskii, E.M. Lifshitz and I.M. Khalatnikov, Usp. Fiz. Nauk 102, 463 (1970) [in Russian]; Adv. Phys. 31, 639 (1982).

[3] C.W. Misner, Quantum cosmology, Phys. Rev. 186, 1319 (1969).

[4] C.W. Misner, The Mixmaster cosmological metrics, preprint UMCP PP94-162; gr-qc/9405068.

[5] A.A. Kirillov, Sov. Phys. JETP 76, 355 (1993) [ZhETF 76, 705 (1993), in Russian]; Int. Jour. Mod. Phys. D3, 431 (1994).

[6] V.D. Ivashchuk, A.A. Kirillov and V.N. Melnikov, On Stochastic Properties of Multidimensional Cosmological Models near the Singular Point, Izv. Vuzov (Fizika) 11, 107 (1994) (in Russian) [Russian Physics Journal 37, 1102 (1994)].

[7] V.D. Ivashchuk, A.A. Kirillov and V.N. Melnikov, On Stochastic Behaviour of Multidimensional Cosmological Models near the Singularity, Pis’sma ZhETF 60, No 4, 225 (1994) (in Russian) [JETP Lett. 60, 235 (1994)].

[8] V.D. Ivashchuk and V.N. Melnikov, Billiard Representation for Multidimensional Cosmology with Multicomponent Perfect Fluid near the Singularity, Class. Quantum Grav. 12, No 3, 809-826 (1995); [gr-qc/9407028].

[9] V.D. Ivashchuk and V.N. Melnikov, Billiard Representation for Multidimensional Cosmology with Intersecting p-branes near the Singularity, J. Math. Phys. 41, No 9, 6341-6363 (2000); [hep-th/9904077].

[10] T. Damour and M. Henneaux, Chaos in Superstring Cosmology, Phys. Rev. Lett. 85, 920-923 (2000); [hep-th/0003139].

[11] T. Damour, M. Henneaux and H. Nicolai, Cosmological billiards, topical review, Class. Quantum Grav. 20, R145-R200 (2003); [hep-th/0212256].

[12] A. Kleinschmidt, M. Koehn and H. Nicolai, Supersymmetric quantum cosmological billiards, Phys. Rev. D 80: 061701 (2009); arxiv: 0907.3048.

[13] A. Kleinschmidt and H. Nicolai, Cosmological quantum billiards, arxiv: 0912.0854.

[14] H. Liu, J. Maharana, S. Mukherji and C.N. Pope, Cosmological Solutions, p-branes and the Wheeler De Witt Equation, Phys. Rev. D 57, 2219-2229 (1997); [hep-th/9707182].

[15] V.D. Ivashchuk and V.N. Melnikov, Multidimensional classical and quantum cosmology with intersecting p-branes, J. Math. Phys. 39, 2866-2889 (1998); [hep-th/9708157].

[16] J.W. York, Role of conformal three-geometry in the dynamics of gravitation, Phys. Rev. Lett. 28 (16), 1082 (1972).

[17] G.W. Gibbons and S.W. Hawking, Action integrals and partition functions in quantum gravity, Phys. Rev. D 15, 2752 (1977).
[18] V.D. Ivashchuk and V.N. Melnikov, Sigma-model for the Generalized Composite p-branes, *Class. Quantum Grav.* **14**, 3001-3029 (1997); Corrigendum *ibid.* **15** (1998) 3941-3942; [hep-th/9705036](https://arxiv.org/abs/hep-th/9705036)

[19] C.W. Misner, In: ”Magic without Magic: John Archibald Wheeler”, ed. J.R. Klauder, Freeman, San Francisco, 1972.

[20] J.J. Halliwell, Derivation of the Wheeler-De Witt Equation from a Path Integral for Minisuperspace Models, *Phys. Rev. D* **38**, 2468 (1988).

[21] V.D. Ivashchuk, V.N. Melnikov and A.I. Zhuk, On Wheeler-DeWitt Equation in Multidimensional Cosmology, *Nuovo Cimento, B* **104**, No 5, 575-581 (1989).

[22] V.D. Ivashchuk and V.N. Melnikov, On billiard approach in multidimensional cosmological models, *Grav. Cosmol.**15**, No. 1, 49-58 (2009); arXiv: 0811.2786.

[23] E. Cremmer, B. Julia and J. Scherk, Supergravity Theory in Eleven-Dimensions, *Phys. Lett. B* **76**, 409-412 (1978).