GROTHENDIECK DUALITY AND TRANSITIVITY II: TRACES
AND RESIDUES VIA VERDIER’S ISOMORPHISM

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Abstract. For a smooth map between noetherian schemes, Verdier relates
the top relative differentials of the map with the twisted inverse image func-
tor “upper shriek” [V]. We show that the associated traces for smooth proper
maps can be rendered concrete by showing that the resulting theory of residues
satisfy the residue formulas (R1)–(R10) in Hartshorne’s Residues and Dual-
ity [RD]. We show that the resulting abstract transitivity map relating the
twisted image functors for the composite of two smooth maps satisfies an ex-
plcit formula involving differential forms. We also give explicit formulas for
traces of differential forms for finite flat maps (arising from Verdier’s isomor-
phism) between schemes which are smooth over a common base, and use this to
relate Verdier’s isomorphism to Kunz and Waldi’s regular differentials. These
results also give concrete realisations of traces and residues for Lipman’s fun-
damental class map via the results of Lipman and Neeman [LN2] relating the
fundamental class to Verdier’s isomorphism.

All schemes formal or ordinary are assumed to be noetherian. The category
of \( \mathcal{O}_X \)-modules for a formal scheme \( \mathcal{X} \) is denoted \( \mathcal{A}(\mathcal{X}) \) and its derived category
\( D(\mathcal{X}) \) as in [NS1]. In general we use the notations of ibid. Thus the torsion functor
\( \Gamma'_\mathcal{X} \) on \( \mathcal{O}_\mathcal{X} \)-modules is defined by the formula
\[
\Gamma'_\mathcal{X} := \varinjlim_n \mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{O}_{\mathcal{X}}/\mathcal{I}^n, -)
\]
where \( \mathcal{I} \) is any ideal of definition of the formal scheme \( \mathcal{X} \). A torsion module
\( \mathcal{F} \) is an object in \( \mathcal{A}(\mathcal{X}) \) such that \( \Gamma'_\mathcal{X} \mathcal{F} = \mathcal{F} \). The reader is advised to look
at [NS1 §2] for further definitions and notations, especially the definitions of
\( \mathcal{A}_c(\mathcal{X}), \mathcal{A}_{qc}(\mathcal{X}), \mathcal{A}_{qct}(\mathcal{X}), \) and the triangulated full subcategories of
\( D(X), D_c(\mathcal{X}), D_{qc}(\mathcal{X}), D_{qct}(\mathcal{X}), \) and their various bounded versions
(e.g., \( D^+_c(\mathcal{X}) \ldots \)).

1. Introduction

The principal aim of this paper is to describe explicitly the residues—and the
trace, when the map in question is proper—associated with Verdier’s isomorphism
\[
f^! \cong f^*(-) \otimes \Omega^\bullet_{X/Y}[n]
\]
(see [V, p. 397, Thm. 3]) for a smooth map \( f: X \to Y \) of relative dimension \( n \). The
foundations of Grothendieck duality (GD) that we use are the ones initiated by
Deligne in [D1].

For this introduction, unless otherwise stated, schemes are ordinary noetherian
schemes. In the main body of the paper, we use formal schemes as way around
compactifications of separated finite type maps, so that complications involving
compatibilities between different compactifications do not need to be addressed.
The principal input, when we use formal schemes, is [NS1] which should be regarded
as a companion paper, written mainly with this manuscript in mind. There are two quite different ways that GD is constructed. The foundations for GD used in [RD] and [C1] are based on residual complexes. In this approach, the functor \( f^! \) (for a suitable finite type map \( f \)), as well as its attendant trace \( \text{Tr} f \): \( Rf_* f^! \rightarrow 1 \) (when \( f \) is proper) have a certain concreteness built into their construction. One then has to work out a large array of compatibilities between the various concrete representations of \( f^! \) and \( \text{Tr} f \) and there often are different concrete representations of these for the same map, e.g., a finite map which also factors as a closed immersion followed by a smooth map. In a different direction, in his appendix to [RD], Deligne initiated an approach to GD which is conceptually attractive [D1]. From this point of view, \( f^! \) for a proper map \( f \) is the right adjoint to \( Rf_* \), and exists for very general category theoretic reasons. These foundations have been worked on, extended, and new techniques introduced over the years by Lipman, Neeman, and their collaborators. Residual complexes and dualizing complexes are not needed to build GD in this approach. We mention [D1], [D2], [D2'], [V] for literature on this approach before the 1980s, and recent work found in [Ne1], [Ne3], which do much to extend (via a conceptually different approach to finding right adjoints) the work initiated by Deligne and Verdier to more general situations, often bypassing the old annoying hypotheses on boundedness for the existence of \( f^! \) or for its base change. The stable version of these can be found in Lipman’s elegant and carefully written book [L4]. We also mention Neeman’s recent manuscript [Ne4] which gives a coherent account of the difficulties and the recent simplifications of many matters. Since we rely on formal schemes as a way to our results on maps between ordinary schemes, we have been influenced enormously by the work of Lipman, Alonso Tarrío, and Jeremías López, in [AJL1], [AJL2], and [AJL3]. We rely on our results on abstract transitivity on formal schemes and related matter in [NS1].

Given the highly abstract methods of construction, and definitions based on universal properties, the question arises:

To what extent can we render concrete realisations of the various constructions occurring in this version of GD?

The issue of concrete representations of \( f^! \) (for our preferred version of GD) was addressed partially, soon after [D1] appeared, by Verdier when \( f \) is smooth [V]. The answer is \( f^! \cong f^*(-) \otimes_{\mathcal{O}_X} \Omega^0_{X/Y} \otimes [n] \) where \( n \) is the relative dimension of \( f : X \rightarrow Y \). This isomorphism in turn depends on the concrete representation \( i^! \cong i^*(-) \otimes \wedge^d \mathcal{N} [-d] \) (via the fundamental local isomorphism) for a regular immersion \( i : U \hookrightarrow V \) of codimension \( d \), with \( \mathcal{N} \) the normal bundle of \( U \) in \( V \). Verdier’s answer for smooth maps is only a partial answer because the associated trace map (when \( f \) is proper)

\[
\text{tr}_f : R^n f_*(\Omega^n_{X/Y}) \rightarrow \mathcal{O}_Y,
\]

denoted \( f \) in [V], is seemingly intractable via this approach. In fact \( \text{tr}_f \) has not been worked out in the literature even when \( A \) is the spectrum of a field \( k \) (e.g., \( k = \mathbb{C} \)), \( X = \mathbb{P}^n_k \), and \( f \) the structure map. However, from the abstract properties of \( f^! \) and the fact that \( f^! \mathcal{O}_Y \) is concentrated in degree \(-n \) (for

\[\text{tr}_f \text{ is H}^0(-) \text{ applied to the composite } \cdots \text{ where the first arrow is Verdier’s isomorphism.}\]
example by Verdier’s isomorphism), the pair \((\Omega^n_{X/Y}, \text{tr}_f)\) is easily seen to represent the functor \(\text{Hom}_X(R^n f_*(-), \mathcal{O}_Y)\) on quasi-coherent sheaves on \(X\) when \(f\) is proper (the only situation where \(\text{tr}_f\) is defined), and in fact \(\text{tr}_f\) and the composite \(R f_*\Omega^n_{X/Y}[n] \xrightarrow{\sim} R f_* f^! \mathcal{O}_Y \xrightarrow{\text{tr}_f(\mathcal{O}_Y)} \mathcal{O}_Y\) determine each other.

When \(Z\) is a closed subscheme of \(X\), finite over \(Y\), defined locally by an \(\mathcal{O}_X\)-sequence, and \(\mathcal{E}xt^i_f(\mathcal{O}_Z, -)\) the \(i\)th right derived functor of \(f_* \mathcal{H}om_X(\mathcal{O}_Z, -)\), Verdier asserts (see top of p. 400 of [V]) that the composite

\[
\mathcal{E}xt^n_f(\mathcal{O}_Z, \Omega^n_{X/Y}) \rightarrow R^n f_*(\Omega^n_{X/Y}) \xrightarrow{\text{tr}_f} \mathcal{O}_Y
\]

is governed by the residue symbol of [RD] Chap. III, §9. It is certainly true that if this is so, following (essentially) the argument given in [V] bottom of p. 399, the trace map \(\text{tr}_f\) can be realised in an explicit way. However, the proof that (I) (denoted \(\text{Res}_Z\) in [V]) is governed by the residue symbol is not there in the literature. In the over 50 years that have passed since Verdier’s assertion, it has been recognised by experts that this is in fact a non-trivial problem (see our quote of Conrad below). One difficulty is the assertion (R4) in [V, p. 400], namely that (I) commutes with arbitrary base change. This needs, at the very least, for one to show that the isomorphism \(f^! \mathcal{O}_Y \cong \Omega^n_{X/Y}[n]\) of Verdier commutes with arbitrary base change, in a sense we will make more precise in a moment. This compatibility with base change was only established in 2004 by the second author [S2]. In slightly greater detail, here is what the compatibility entails. Suppose \(u: Y' \rightarrow Y\) is a map and \(g: X \times_Y Y' \rightarrow Y'\) and \(v: X \times_Y Y' \rightarrow X\) are the two projections. To show the compatibility of Verdier’s isomorphism \(f^! \mathcal{O}_Y \cong \Omega^n_{X/Y}[n]\) with arbitrary base change, first one needs to show that there is an isomorphism

\[
\theta^u_f: v^* f^! \mathcal{O}_Y \xrightarrow{\sim} g^! \mathcal{O}_Y,
\]

for our smooth \(f\) (even when it is not proper). This is a delicate point, especially if one demands that in the proper case \(\text{Tr}_f(\mathcal{O}_Y)\) should be compatible with this base change isomorphism (remember, \(\text{Tr}_f\) in Deligne’s approach, is defined as a co-adjoint unit and is not explicit), and that the isomorphism is also compatible with open immersions into \(X\). After this is established, one has to check that this base change isomorphism \(\theta^u_f\) when grafted on to Verdier’s isomorphisms for \(f\) and for \(g\) give the canonical isomorphism of differential forms. It is easier to carry out the first part in the slightly more general situation of \(f\) being Cohen-Macaulay, and this is one of the main results of [S2]. In [C1], the base change isomorphism \(\theta^u_f\) is proven using the foundations of GD in [RD]. However, the isomorphism between \(f^! \mathcal{O}_Y\) and \(\Omega^n_{X/Y}[n]\) in [RD] and [C1] is by fiat, and it is not clear that it is the same as Verdier’s isomorphism. In other words, it is not clear that the trace \(R^n f_*(\Omega^n_{X/Y}) \rightarrow \mathcal{O}_Y\) built using the foundations of GD in [RD] is the same as the one that arises when using the foundations initiated in [D1]. In fact, we are back to the frustrating detail that we do not know the \(\text{tr}_f\) explicitly when we work with the foundations initiated in [D1]. Even with the compatibility of Verdier’s isomorphism with arbitrary base change in hand, showing that (I) is governed by the residue symbol of [RD] Chap. III, §9 is not trivial. In fact it takes all this paper. We
can do no better than quote Conrad from his introduction to his book [C1] (using however our labelling of the citations given there):

“...The methods in [V] take place in derived categories with "bounded below" conditions. This leads to technical problems for a base change such as \( p : \text{Spec}(A/\mathfrak{m}) \to \text{Spec}(A) \) with \((A, \mathfrak{m})\) a non-regular local ring, in which case the right exact \( p^* \) does not have finite homological dimension (so \( Lp^* \) does not make sense as a functor between "bounded below" derived categories). Moreover, Deligne’s construction of the trace map in [RD Appendix], upon which [V] is based, is so abstract that it is a non-trivial task to relate Deligne’s construction to the sheaf \( R^n f_*(\Omega^n_{X/Y}) \). However, a direct relation between the duality theorem and differential forms is essential for many important calculations (e.g., [Maz, §6, §14(p.121)]).”

In other words the task of finding a concrete expression for \( \text{tr}_f \) is not simple. In this paper we take up this task, and believe we give a satisfactory answer to the problem. Briefly, any theory of traces comes with an associated theory of residues, and we show that residues associated with \( \text{tr}_f \) satisfy the formulas [RD III, §9], which are stated without proof in loc.cit.

The prime object of study in this paper is Verdier’s isomorphism [V] p.397, Thm.3]

\[ \Omega^n_{X/Y}[n] \xrightarrow{\sim} f^! \mathcal{O}_Y \]

for a smooth separated morphism \( f : X \to Y \) of ordinary schemes of relative dimension \( n \). Strictly speaking, the isomorphism in loc.cit. is from \( f^! \mathcal{O}_Y \) to \( \Omega^n_{X/Y}[n] \), and thus, we are talking about the inverse of the map in loc.cit. In view of recent results of Lipman and Neeman, this is the fundamental class map \( c_f \) associated with \( f \) [LN2, p.152, (4.4.1)], but we use the description given in [V] and hence call it the Verdier isomorphism. In [LN3], Lipman outlines a programme for a global residue theorem via the fundamental class map (see [ibid., §5.5 and §5.6]). This paper is intimately related to that programme via the just mentioned results of Lipman and Neeman. However, we do not use the results on the fundamental class map of [ibid.]. Since the isomorphism we use (between \( \Omega^n_{X/Y}[n] \) and \( f^! \mathcal{O}_Y \)) is that described by Verdier, we call it the Verdier isomorphism rather than the fundamental class.

We also recommend Beauville’s expository paper [Be] for an overview (without proofs) of residues, especially for the concrete expressions for them. Our attention was drawn to it recently by Joe Lipman.

We now give a more more detailed description of the contents of the paper. We are concerned, mainly with three (intertwined) aspects:

1. Understanding the abstract traces

\[ \text{Tr}_f(\mathcal{O}_Y) : Rf_*f^! \mathcal{O}_Y \to \mathcal{O}_Y \]

and

\[ \text{Tr}_{f,Z}(\mathcal{O}_Y) : Rf_*f^! \mathcal{O}_Y \to \mathcal{O}_Y \]

in concrete terms (using differential forms via Verdier’s isomorphism) when \( f : X \to Y \) is smooth and separated. The first map is meaningful when \( f \)

\(^3\)though all the formulas stated in [RD] (labelled (R1)-(R10) there) have been proved with great care by Conrad in [C1] Appendix A]
1.1. The twisted image pseudofunctor $-^1$. GD is concerned with constructing a variance theory, i.e., a pseudofunctor, “upper shriek”, which we denote $-^1$, on a suitable subcategory of schemes and finite type maps. For a fixed scheme $Z$, $Z^1$ is a suitable full subcategory of $\mathbf{D}(Z)$ containing $\mathbf{D}_c(Z)$. We will say more about these subcategories later. For now we wish to paint with broad strokes. Whichever way one approaches the foundations of GD, the resulting pseudofunctor $-^1$ should be local (more on that in a moment), stable under, at least, flat base change, and

is proper, as the co-adjoint unit for the adjoint pair $(\mathbf{R} f_\ast, f^!)$ [NSI (1.1.2)]. The second is meaningful when $Z$ is a closed subscheme of $X$ proper over $Y$ [NSI (3.3.1)]. If $Z = X$, $\text{Tr}_{Z, X} = \text{Tr}_f$. In fact, we will concentrate on the case when $Z$ is finite over $Y$, in which case we are talking about abstract residues. The aim to is realise these abstractions concretely when we substitute $\Omega^n_{X/Y}[n]$ for $f^!\mathcal{O}_Y$ via Verdier’s isomorphism ($n$ being the relative dimension of $f$). Understanding $\text{Tr}_{Z, X}$ for such $Z$ is tantamount to understanding $\text{Tr}_f$ for $f$ proper via the so-called Residue Theorem.

2. Making concrete the abstract transitivity map

$$\chi_{[a, f]} : L f^* g! \mathcal{O}_X \otimes_{\mathcal{O}_X} f^! \mathcal{O}_Y \to (gf)^! \mathcal{O}_Z$$

of [L4 §4.9] and [NSI (7.2.16)] concrete in terms of differential forms (again using Verdier’s isomorphism) when $f : X \to Y$ and $g : Y \to Z$ are separated finite-type maps in certain situations. Our main interest is in the following two situations:

(i) The maps $f$ and $g$ are smooth, say of relative dimensions $m$ and $n$ respectively, and we use Verdier’s isomorphisms to identify $g! \mathcal{O}_Z$, $f^! \mathcal{O}_Y$, and $(gf)^! \mathcal{O}_Z$ with $\Omega^m_{Y/Z}[n]$, $\Omega^n_{X/Y}[m]$, and $\Omega^{m+n}_{X/Z}[m + n]$ respectively.

This is closely related to the results in [LS].

(ii) The map $f$ is a closed immersion say of codimension $d$, and the maps $g$ and $gf$ are smooth, say of relative dimensions $n + d$ and $n$ respectively.

In fact these two cases are essentially enough to develop a theory of residues which gives the formulas (R1) to (R10) in [RD, Chap. III, §9].

3. Finding a concrete expression for the abstract trace map

$$h_* f^! \mathcal{O}_Z \cong h_* h^! g! \mathcal{O}_Z \xrightarrow{\text{Tr}_h} g! \mathcal{O}_Z$$

where $f : X \to Z$ and $g : Y \to Z$ are smooth separated maps and $h : X \to Y$ is a finite flat map. This concrete expression is in terms of differential forms (via our now familiar way of identifying $f^! \mathcal{O}_Z$ and $g! \mathcal{O}_Z$ with differential forms). In fact we show that it is the trace of Lipman and Kunz, defined in [Ku §16]. One consequence is that if $f : X \to Y$ is an equidimensional map of relative dimension $n$ such that $X$ and $Y$ are excellent with no embedded points and the smooth locus of $f$ is dense in $X$, then $H^{-n}(f^! \mathcal{O}_Y)$ can be identified, via the Verdier isomorphism on the smooth locus of $f$, with a coherent subsheaf of the sheaf of meromorphic differentials $\wedge^n_{k(X)} \Omega^1_{k(X)/k(Y)}$, namely the sheaf of regular differentials of Kunz and Waldi [KW §3, §4]. One therefore recovers the main results in [HK1], [HK2], [HS], and [LS] via our approach.

We elaborate on these points in the rest of this introduction.
such that when $f$ is proper, $f^!$ is right adjoint to $Rf_*$. By local, this is what we mean: If $U \to Y$ is an open $Y$-subscheme of $g: V \to Y$ as well as of $h: W \to Y$ ($g$, $h$ of finite type), then $g^!|_U$ and $h^!|_U$ are canonically isomorphic — canonical enough that if we have a third finite type $Y$-scheme $f: X \to Y$ which contains $U$ as an open $Y$-subscheme, then the isomorphisms between $f^!|_U$, $g^!|_U$, and $h^!|_U$ are compatible. All of this (and much more) can be found in [L4] for the theory of $-^!$ initiated in [DI]. For schemes with finite Krull dimension, the local nature of upper-shruck was proved by Deligne in [DI], and using his flat base change result, by Verdier in [V].

Additionally, one wants a theory which specializes to the familiar Serre duality for smooth complete varieties, with the top differential forms playing a critical dualizing role. For a slightly more general situation, this means that from the theory of upper shriek one should recover the duality isomorphisms

$$E\text{xt}^i_f(Y, \Omega^d_{X/Y}) \cong \mathcal{H}\text{om}_Y(R^{d-i}f_*, \Omega^d_Y) \quad (0 \leq i \leq d)$$

when $f: X \to Y$ is smooth and proper of relative dimension $d$, $\mathcal{V}$ is a finite rank vector bundle on $X$. This amounts to showing that $f^!\Omega^d_Y \cong \Omega^d_{X/Y}[d]$ for such a smooth map $f$.

1.2. Traces and residues. Given a theory of upper shriek there is an associated theory of traces and residues related to it in a very close manner. Briefly, for $f: X \to Y$ proper, since $f^!$ is a right adjoint to $Rf_*$, there is a co-adjoint unit

$$\text{Tr}_f: Rf_*f^! \to 1$$

called the trace map, namely the image of the the identity transformation $f^! \to f^!$ under $\text{Hom}_{X^!}(f^!, f^!) \to \text{Hom}_{Y^!}(Rf_*f^!, 1)$ where $1$ is the identity functor on $Y^!$. More generally, if $f$ is separated of finite type and $Z$ is closed subscheme of $X$ which is proper over $Y$, then one has a map (the trace along $Z$)

$$\text{Tr}_{f,Z}: Rf_*f^! \to 1$$

which for this introduction can be defined as follows: If $F: P \to Y$ is a compactification of $f$, and $i: X \to P$ the open $Y$-immersion associated to this compactification, then using the isomorphism $f^! \sim i^!F^!$, we have the composite

$$Rf_*f^! \sim Rf_*i^!(F^!) \sim R_{i(Z)}F_*F^! \to R F_* F^! \overset{\text{Tr}_f}{\to} 1.$$  

The above composite is independent of the compactification data $(i, F)$, and we define $\text{Tr}_{f,Z}$ to be this composite. We give an equivalent but more useful definition in [NS1 §3.3]. The map $\text{Tr}_{f,Z}$ depends only on the formal completion of $X$ along $Z$ and not on the exact subscheme structure of $Z$. Note that when $Z = X$ (this implies $f$ is proper), then $\text{Tr}_{f,Z} = \text{Tr}_f$.

If $f$ is smooth of relative dimension $n$, $Z$ as above is cut out by $t = (t_1, \ldots, t_n) \in \Gamma(X, \mathcal{O}_X)$ with $Z \to Y$ finite, and $Z_m$ is the thickening of $Z$ defined by $t^n = (t_1^n, \ldots, t_n^n)$, and say $Y = \text{Spec} A$, then [I] gives us maps (one for each $m$)

$$\text{Ext}_X^n(\mathcal{O}_{Z_m}, \Omega^d_{X/Y}) \to A.$$  

As Verdier argued in [VI] bottom of p.399, by passing to the completion of a localisation of $A$ (via flat base change), and making étale base changes, to know the

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5i.e., $F$ is proper and contains $X$ as an open $Y$-subscheme—such an $F$ can always be arranged.

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[NS1 §3.3]
above map (for any m) is to know $\text{Tr}_f$ when f is proper. If one passes to the direct limit as $m \to \infty$, then we get a map

$$H^0_Z(X, \Omega^n_{X/Y}) \to A.$$ 

The above map is easily seen to be $H^0(-)$ applied to the composite

$$(1.2.1) \quad \mathcal{R}\Gamma_Z(X, \Omega^n_{X/Y}[n]) \cong \mathcal{R}\Gamma_Z(X, f^!\mathcal{O}_Y) \xrightarrow{\text{Tr}_f,Z} A.$$ 

This map, which we denote $\text{res}_Z$, also determines $\text{Tr}_f$ if f is proper. We prefer to work with cohomology with supports (rather than with $\text{Ext}^n_X(\mathcal{O}_Z, \Omega^n_{X/Y})$), following the general philosophy underlying Lipman’s body of work, especially [L2]. Berthelot in [Ber] also makes the connection between the map on $\text{Ext}^n_X(\mathcal{O}_Z, \Omega^n_{X/Y})$ and the map on cohomology with supports. However Berthelot uses the foundations of GD based on residual complexes.

The relationship between upper shriek and the associated traces is intimate. To assert that one has a concrete understanding of upper shriek in a particular situation is tantamount to asserting that one understands $\text{Tr}_{f,Z}(\mathcal{O}_Y)$ for Z which are finite and flat over Y is to “know” duality for f.

Returning to the case we are discussing (f smooth of relative dimension n), suppose $Z \hookrightarrow X$ is a closed immersion cut out by a sequence of global sections $t = (t_1, \ldots, t_n)$ of $\mathcal{O}_X$, and $Y = \text{Spec} A$. Assume $Z \to Y$ is an isomorphism and contained in an affine open subscheme $U = \text{Spec} B$ of X, something that can be achieved by shrinking Y, since $Z \to Y$ is an isomorphism. Let

$$\text{res}_Z : H^0_Z(X, \Omega^n_{X/Y}) = H^0_Z(U, \Omega^n_{U/Y}) \longrightarrow A$$

be $H^0((1.2.1))$. It is well known that elements of $H^0_B(\Omega^n_{B/A})$ are finite $A$-linear combinations of elements of the form $\left[ \frac{dt_1 \wedge \cdots \wedge t_n}{t_1^{\beta_1} \cdots t_n^{\beta_n}} \right]$ with $\beta_i$ positive integers. Ideally one would like

$$(1.2.2) \quad \text{res}_Z \left[ \frac{dt_1 \wedge \cdots \wedge t_n}{t_1^{\beta_1} \cdots t_n^{\beta_n}} \right] = \begin{cases} 1 & \text{when } \beta_i = 1 \text{ for all } i = 1, \ldots, n \\ 0 & \text{otherwise.} \end{cases}$$

The exact answer depends on the isomorphism $f^!\mathcal{O}_Y \cong \Omega^n_{X/Y}[n]$ chosen. This is at the heart of this paper, since our choice is the isomorphism Verdier gives in [V] p. 397, Thm. 3. In fact we show that Verdier’s isomorphism does give the above formula in the case being considered, i.e., when Z is a section of f. This is the critical case, and we deduce other residue formulas from this one by either making étale base changes, or base changing f by itself and using the diagonal section $X \twoheadrightarrow X \times_Y X$ of the first projection (which is to be thought of as the base change of f).

We could obtain the above explicit description of $\text{res}_Z$ when Z is a section of f because of the results in [S2]. The main results there state that if $f : X \to Y$ is Cohen-Macaulay of relative dimension d, then for any base change $u : Y' \to Y$, there is a natural isomorphism $\mathcal{O}_Z^p \cong \mathcal{O}_Z^{p'}$, where $v : X \times_Y Y' \to X$ and $g : X \times_Y Y' \to Y'$ are the respective projections. When f is proper, this isomorphism is compatible with traces. If f is smooth (proper or not), then this
isomorphism when transferred to \( \nu^* \Omega^n_{X/Y} \) and \( \Omega^n_{X \times Y'/Y'} \) is the identity map under the standard identification of differential forms. These are very similar to the main results in [C1]. The difference is that in [S2] the foundations of GD are based on the one initiated by Deligne in [D1], whereas in [C1] it is the based on residual complexes. In [C1] the identification of differential forms is built into the definition of the base change isomorphism between \( \nu^* \omega_f^p \) and \( \omega_f^p \), since the strategy is to embed \( X \) into schemes smooth over \( Y \). The challenge in [C1] is to show that the result is compatible with traces when \( f \) is proper.

In our approach to finding explicit formulas for \( \text{res}_Z \), the role played by \( \theta_u^f \), when \( u \) is non-flat, is crucial. Roughly speaking, Verdier’s isomorphism can be regarded as the residue formula \( \text{res}_\Delta \left[ \frac{\partial x_i}{\partial s_1} \wedge \ldots \wedge \frac{\partial x_n}{\partial s_n} \right] = 1 \) for the diagonal section \( \Delta \) in \( X \times_Y X \) where the diagonal is cut out by \( s \) in \( X \times_Y X \). If \( Z \hookrightarrow X \) is a section of \( f \), cut out by \( t_1, \ldots, t_n \in \Gamma(X, \mathcal{O}_X) \) then pulling back the diagonal via the base change \( Z \to X \), we get

\[
(1.2.3) \quad \text{res}_Z \left[ dt_1 \wedge \ldots \wedge dt_n \right] = 1.
\]

We can do this because Verdier’s isomorphism is compatible with arbitrary base change – the result in [S2, p.740, Thm. 2.3.5 (b)] that we alluded to above. The formula (1.2.3) says that (1.2.2) is true when all the \( \beta_i \) are 1. If \( X = \mathbb{P}^n \), \( f \) the standard projection \( \mathbb{P}^n \to Y \), and \( Z = \bigcap_{i=1}^n \{ T_i \neq 0 \} \), where \( T_i, i = 0, \ldots, n \) are homogeneous co-ordinates on \( \mathbb{P}^n \) (and \( t_i = T_i/T_0 \), for \( i = 1, \ldots, n \), then one can show easily that (1.2.3) implies (1.2.2). The crucial ingredient needed is the simple and elegant computation of Lipman in [L2, pp.79–80, Lemma (8.6)]. The proof is essentially carried out in the proof of Proposition 5.2.3(ii). Since \( \text{res}_Z \) depends only on the formal completion of \( X \) along \( Z \), therefore if \( Z \) and \( t \) satisfy the hypotheses given when stating (1.2.2), then formula (1.2.2) holds. This is the first, and a very important step in our proofs in Section 10 of the residue formulas (R1)–(R10) of [R13, Chap. III, §9].

If the closed subscheme \( Z \) of \( X \) cut out by \( t = (t_1, \ldots, t_n) \) is finite over \( Y \) (and hence necessarily flat over \( Y \)) of constant rank (not necessarily an isomorphism), and of constant rank then it turns out that the right side of (1.2.3) needs to be replaced by \( \text{rank}(Z/Y) \).

**Remark 1.2.4.** One way to think about formula (1.2.3) is to regard

\[
\varphi \mapsto \text{res}_Z \left[ \varphi \, dt_1 \wedge \ldots \wedge dt_n \right] \quad (t_1, \ldots, t_n),
\]

(for \( \varphi \) a section of \( \mathcal{O}_X \) in an open, or even formal, neighbourhood of \( Z \)) as the Dirac distribution along \( Z \). Indeed, (with \( Y = \text{Spec}(A) \)), since \( Z \) is a section of \( f \), the completion of \( X \) along \( Z \) is the power series ring \( A[[t_1, \ldots, t_n]] \), and according to (1.2.3), the right side is \( \varphi(0, \ldots, 0) \), after developing \( \varphi \) as a power-series in \( t \).

If \( A = \mathbb{C} \), and field of complex numbers (so that \( Z = \{ p \} \), a point), this can be interpreted as the fact that the Dolbeault representative of the Cauchy kernel \( dt_1 \wedge \ldots \wedge dt_n/t_1 \ldots t_n \) at the point \( p \) is the Dirac distribution at \( p \) (see [ST]).

1.3. **Transitivity.** Finding concrete expressions (when we have two finite type separable maps \( f : X \to Y \) and \( g : Y \to Z \)) for the abstract transitivity map

\[
\chi_{[g,f]} : Lf^* \gamma Z \otimes \mathcal{O}_X \to Lf^* \mathcal{O}_Y \to (gf)^* \mathcal{O}_Z
\]
of [L4] § 4.9] and [NSI] (7.2.16) is perhaps the most important technical task undertaken in this paper. To establish this, we rely heavily on the abstract transitivity results on formal schemes in [NSI]. As we pointed out earlier (see item (2) on p.5) there are two key situations where concrete manifestations of $\chi_{[-, -]}$ are important. The first situation of importance is when we have two smooth separated maps, $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, say of relative dimensions $m$ and $n$ respectively. If we use Verdier’s isomorphisms to identify $g^* \mathcal{O}_Z$, $f^* \mathcal{O}_Y$, and $(gf)^* \mathcal{O}_Z$ with $\Omega^m_{Y/Z}[n]$, $\Omega^m_{X/Y}[m]$, and $\Omega^m_{X/Z}[m + n]$ respectively, then $\chi_{[f,g]}$ transforms to the map $f^* \mu \otimes \nu \mapsto \nu \wedge f^* \mu$ (see Theorem 7.2.2). In fact we show this at the level of formal schemes, and formal schemes enter in an essential way in our proof (via transitivity for residues [NSI] (8.3.2) and Theorem 8.1.1) even when $X$ and $Y$ are ordinary schemes. The proof is carried out in Section 4.

The second situation of importance occurs when the smooth map $f : X \rightarrow Y$ factors as $f = \pi \circ i$, where $i : X \hookrightarrow P$ is a closed immersion say of codimension $d$, and $\pi : P \rightarrow Y$ is smooth of relative dimension $n + d$. The concrete expression for $\chi_{i,\pi}$ then is governed by

$$i^*(\eta \wedge dt_1 \wedge \cdots \wedge dt_d) \otimes 1/t \mapsto i^* \eta$$

where $\eta$ is a section of $\Omega^{n+d}_{P/Y}$, $t_i \in \Gamma(P, \mathcal{O}_P)$, $i = 1,\ldots, d$, are sections which cut out $X$ and $1/t$ is a well-defined generating section, depending upon $t = (t_1,\ldots, t_d)$, of the top exterior product $\wedge^d \mathcal{N}$ of the normal bundle $\mathcal{N}$ of $X$ in $P$, which exterior product, by the fundamental local isomorphism is identified with $f^* \mathcal{O}_P[d]$. The proof of this concrete representation of $\chi_{i,\pi}$ is carried out in Subsection 8.2.

In both situations, we need the residue formula (1.2.2) for residues along sections of smooth maps. We turn this around later, and use the concrete expressions for $\chi_{[-, -]}$ to arrive at formulas for $\text{res}_Z$ for smooth maps $f : X \rightarrow Y$ when $Z \rightarrow Y$ is not an isomorphism (but is finite).

There is one interesting way in which (1.2.3) brings in concrete answers. Let $A$ be a ring, and $C = A[T_1,\ldots, T_n]/(f_1,\ldots, f_n)$ be a finite flat algebra over $A$. Let $Z = \text{Spec} C$, $X = \mathbb{A}^n_A$, and $Y = \text{Spec} A$. Let $I = f_A[T]$, so that $I$ is the ideal of $Z$ in the polynomial ring $A[T]$. By the general calculus of generalised fractions, if $p(T) \in A[T]$ then the element

$$[p(T) dT_1 \wedge \cdots \wedge dT_n]_{f_1,\ldots, f_n} \in H^0_i(\Omega^n_{A[T]/A})$$

depends only on the image of $p(T)$ in $C$. We show that the map

$$c \mapsto \text{res}_Z \left[ p(T) dT_1 \wedge \cdots \wedge dT_n \right]_{f_1,\ldots, f_n}$$

with $p(T)$ a pre-image of $c$, is the Tate trace described in [MR, Appendix]. We prove this in Theorem 8.1.1 and (1.2.3) plays an important role. The point is, knowing the residue in a very special situation allows us to deduce formulas for residues in many other situations.

Perhaps the most important way that that (1.2.2) comes into play is that it characterises the Verdier isomorphism (or more accurately the fundamental class). Continuing with the situation where $f : X \rightarrow Y$ is smooth of relative dimension $n$, suppose we have some isomorphism $\psi : \Omega^n_{X/Y}[n] \xrightarrow{\sim} f^* \mathcal{O}_Y$. When is $\psi$ Verdier’s isomorphism? The answer is, if and only if, for every étale base change $u^* Y' \rightarrow Y$ and every section $Z$ of the base change map $f' : X \times_Y Y' \rightarrow Y'$, the composite

$$R^n_Z f'_* \Omega^n_{X \times_Y Y'/Y'} \xrightarrow{\text{via } \psi} H^0(\mathcal{R}_Z f'_* f'^* \mathcal{O}_{Y'}) \xrightarrow{\text{Tr}_{f'/Y'}} \mathcal{O}_{Y'}$$
is given by \((1.2.2)\). Note that the first isomorphism involves flat base change for \(f^!\). The precise statement is given in Theorem 5.4.8. This characterisation of Verdier’s isomorphism allows us to relate the fundamental class with the regular differentials of Kunz. We work this out in Section 6. We give a different proof later of the relationship between the fundamental class and regular differentials.

1.4. Trace for finite flat maps. Suppose the smooth map \(f: X \to Z\) of relative dimension \(n\) can be factored as \(f = g \circ h\), where \(h: X \to Y\) is finite and \(g: Y \to Z\) is smooth of relative dimension \(n\) (so that \(h\) is in fact flat). Then the composite \(h^!* f^! \simeq h^* h^! g^! \simeq g^!\), gives, via Verdier’s isomorphism for \(f\) and \(g\), a map \(\text{tr}_h : h^! \Omega^n_{X/Z} \to \Omega^n_{Y/Z}\).

In [Ku], Kunz, based on a suggestion by Lipman (who in turn was influenced by residue formulas stated without proof in [RD, Chap. III, §9]) defined an explicit trace \(\sigma_h : h^! \Omega^n_{X/Z} \to \Omega^n_{Y/Z}\). We show that \(\text{tr}_h = \sigma_h\). In fact, we use the two concrete versions of transitivity that we mention above. Assuming \(h\) factors as a closed immersion \(i: X \to P\) followed by a smooth map \(\pi: P \to Y\), where \(P\) is an open subscheme of \(\mathbb{A}^{n+d}_Y\) and \(\pi\) the structure map, (a situation we can achieve, retaining finiteness of \(h\), if we pass to completions of local rings of points on \(Y\)) then the assertion \(\text{tr}_h = \sigma_h\) amounts to the compatibilities between the abstract transitivity maps \(\chi_{(h,g)}\), \(\chi_{(i,\pi)}\), \(\chi_{(\pi,g)}\), and \(\chi_{(i,\pi,g)}\) given in [NST, Prop.-Def. 7.2.4 (ii)] or in [L4, p. 238]. The map \(\text{tr}_h\) occurs in formula (R10) for residues, and it is satisfying that there is a more explicit description of it in terms of the Kunz-Lipman trace \(\sigma_h\).

1.5. Regular Differential Forms. The regular differentials of Kunz and Waldi developed in [KW] is a vast generalisation of Rosenlicht’s differentials for singular curves [R]. We have already alluded to the connection between the regular differential forms and Verdier’s isomorphism. Regular differential forms are defined when \(f: X \to Y\) is a generically smooth equidimensional map between excellent schemes having no embedded points. In such a case, if \(X^{\text{sm}}\) is the smooth locus of \(f\), and \(f^{\text{sm}} : X^{\text{sm}} \to Y\) the restriction of \(f\), there is an isomorphism \(\Omega^n_{X^{\text{sm}}/Y}[n] \to (f^{\text{sm}})^!\).

The isomorphism is based on the construction of regular differentials in [KW] and the principal results of [HS]. What we show in this paper is that this isomorphism is Verdier’s isomorphism. We give two proofs. The first using the characterisation of Verdier’s isomorphism via \((1.2.2)\) that we alluded to before. The other, more satisfying, proof relies on the equality of traces \(\text{tr}_h = \sigma_h\) for finite flat maps \(h\) between schemes smooth over a base that we discussed above. (See Subsection 9.3.)

2. Preliminaries

2.1. Flat base change. Suppose we have a cartesian square \(s\) of noetherian formal schemes

\[
\begin{array}{ccc}
\mathcal{V} & \xrightarrow{u} & \mathcal{X} \\
\downarrow g & & \downarrow f \\
\mathcal{W} & \xrightarrow{v} & \mathcal{Y}
\end{array}
\]

with \(f\) in \(\mathcal{G}\) and \(u\) flat. The flat-base-change theorem for \((-)^*\) (see [NST, §§3.2]) states that is if \(\mathcal{F} \in D_c^+ (\mathcal{W})\), or if \(u\) is open or if \(\mathcal{V}\) is an ordinary scheme, we have
an isomorphism (see [NS1] (3.2.2)):
\[(2.1.1)\quad v^* f^# F \sim g^# u^* F.\]

If \(f\) is pseudo-proper, (2.1.1) is proved in [AJL2, Theorem 8.1, Corollary 8.3.3].

There is a somewhat more explicit version (2.1.1) when \(f\) is a regular immersion i.e., \(\mathcal{X}\) is a closed subscheme of \(\mathcal{Y}\) given by the vanishing of a regular sequence. In such a case let \(\mathcal{I}\) be the ideal sheaf of \(\mathcal{X}\) in \(\mathcal{Y}\), and suppose the co-dimension of the immersion is \(r\), i.e., \(\mathcal{I}\) is locally generated by regular sequences of length \(r\). As in [NS1, (C.2.10)], for \(F \in D^+_c(\mathcal{Y})\), we write
\[(2.1.2)\quad f^!(\mathcal{I}) := \mathcal{L}f^! \mathcal{Y} \otimes_{\mathcal{O}_{\mathcal{X}}} (\wedge^r_{\mathcal{O}_{\mathcal{X}}} (\mathcal{I}/\mathcal{I}^2)[r]).\]

According [NS1] (C.1.13), for \(\mathcal{F} \in D^+_c(\mathcal{Y})\) we have a functorial isomorphism
\[(2.1.3)\quad \eta_f^*(\mathcal{F}) : f^!(\mathcal{I}) \sim f^*(\mathcal{F}).\]

Now suppose we have a cartesian diagram \(\mathbb{G}\) of formal schemes
\[(2.1.4)\quad \begin{array}{ccc} \mathbb{G}' & \xrightarrow{j} & \mathbb{G} \\ \downarrow{\kappa} & \quad & \downarrow{\kappa_0} \\ \mathcal{X}' & \xrightarrow{i} & \mathcal{X} \end{array}\]

such that \(i\) is a regular immersion and \(\kappa_0\) is the completion of \(\mathcal{X}'\) with respect to a closed subscheme given by a coherent ideal. Then according to [NS1] (C.4.2) we have an isomorphism
\[(2.1.5)\quad \kappa^* i^! \sim j^! \kappa_0^*.\]

The compatibility of (2.1.5) with (2.1.1) (with \(f = i, g = j, u = \kappa_0, \) and \(v = \kappa\)) given the isomorphisms \(\eta_f^!\) and \(\eta^!\) (see (2.1.3)), is proven in [NS1, Prop. C.4.3].

Further properties of the base-change map are explored [NS1, §§ A.1].

2.2. If \(f : \mathcal{X} \to \mathcal{Y}\) is a map in \(\mathbb{G}\) which is formally étale and \(\mathcal{F} \in D^+_c(\mathcal{Y})\), then we have an isomorphism
\[(2.2.1)\quad f^* \mathcal{F} \sim f^! \mathcal{F}\]

which is pseudofunctorial over the category of formally étale maps [NS1, (3.1.3)].

2.3. Completing direct image with support. Let \(Z\) be a closed subscheme of an ordinary scheme \(X\) and \(\kappa : \mathcal{X} \to X\) the completion of \(X\) along \(Z\). The isomorphism \(\kappa_* R^j_{\mathcal{X}} \kappa^* \sim R^j_{\mathcal{X}}\) gives rise to isomorphisms (one for every \(j\))
\[(2.3.1)\quad R^j_\mathcal{F} f_* \mathcal{F} \sim H^j_\mathcal{F} (R \mathcal{F} \kappa_0^* \mathcal{F}) = R^j_\mathcal{F} \mathcal{F} \kappa^* \mathcal{F}\]

which are functorial in \(\mathcal{F}\) varying over quasi-coherent \(\mathcal{O}_\mathcal{X}\)-modules. In affine terms, if \(X = \text{Spec} R, M\) an \(R\)-module, and \(Z\) is given by the ideal \(I\), then writing \(\hat{R}\) for the \(I\)-adic completion of \(R\), and \(J = I \hat{R}\), the above isomorphism is the well-known one
\[H^j_\mathcal{F} (M) \sim H^j_\mathcal{F} (M \otimes_R \hat{R}).\]

In the above situation, suppose \(f : X \to Y\) is a separated finite-type map of ordinary schemes such that the induced map \(Z \to Y\) is proper. Let \(\hat{f} = f \circ \kappa\). Then according to [NS1, (A.3.1)] we have a functorial isomorphism
\[(2.3.2)\quad R\hat{f}_* R^j_{\mathcal{X}} f^* \sim R\hat{f}_* R^j_\mathcal{F} \hat{f}^*.\]
2.4. Abstract traces and residues. Suppose \( f: X \to Y \) is a pseudo-proper map. As in [NS1 (3.1.2)] we define the trace map associated to \( f \),

\[
\text{Tr}_f: Rf_*Rf^! f^* \to 1.
\]

(2.4.1)

to be the co-adjoint unit associated to the right adjoint \( f^* \) of \( Rf_*Rf^! \).

In [NS1, Def. 4.1.2] we defined a map of formal schemes \( f: X \to Y \) to be Cohen-Macaulay (CM) of relative dimension \( r \) if it is flat, locally in \( G \) with \( H^i(f^* O_Y) = 0 \) for \( i \neq -r \), and \( \omega^f = H^{-r}(f^* O_Y) \) is flat over \( Y \). The coherent \( O_X \)-module \( \omega^f \) is called the relative dualizing sheaf for the CM map \( f \).

For a CM map \( f: X \to Y \) of relative dimension \( r \) which is pseudo-proper,

\[
\text{tr}_f^*: R^r Z f_* O_Y \to O_Y
\]

(2.4.2)

will denote the abstract trace map on \( R^r Z f_* O_Y \) and is defined by the formula

\[
\text{tr}_f^* = H^0(\text{Tr}_f(O_Y))
\]
as in [NS1 (5.1.2)].

Next, let \( f: X \to Y \) be a separated map of finite-type between ordinary schemes, and \( Z \) a closed subscheme of \( X \) which is proper over \( Y \). We recall the notion of the trace of \( f \) along \( Z \) from [NS1, §§3.3]. Now, the completion map \( \kappa: X \to X \) of \( X \) along \( Z \), is formally étale and affine and the composition \( \hat{f} := f \kappa \) is pseudo-proper.

We define the trace map for \( f \) along \( Z \),

\[
\text{Tr}_{f,Z}: R_{Z f_*} f^* \to 1
\]

(2.4.3)

to be the composite

\[
R_{Z f_*} f^* \xrightarrow{\sim} Rf_* Rf^! f^* \xrightarrow{\text{Tr}_f} 1.
\]

Suppose \( f \) above is Cohen-Macaulay of relative dimension \( r \). As in [NS1 (5.2.2)], the abstract residue along \( Z \)

\[
\text{res}^r_Z: R^r Z f_* O_Y \to O_Y
\]

(2.4.4)

is defined as the composite

\[
R^r Z f_* O_Y \xrightarrow{\text{H}^0(\text{Tr}_f)} Rf_* Rf^! f^* \xrightarrow{\text{tr}_f^*} O_Y.
\]

It is clear from the definitions that

\[
\text{res}^r_Z = H^0(\text{Tr}_{f,Z}(O_Y)).
\]

3. Verdier’s isomorphism

3.1. Let \( f: X \to Y \) be a smooth map of relative dimension \( r \) between (formal) schemes. Assume \( f \) is a composite of compactifiable maps. Set \( X'' := X \times_Y X \) and let \( \Delta: X \to X'' \) the diagonal immersion, which is closed by our hypotheses. Denote by \( p_1 \) and \( p_2 \) the two projections from \( X'' \) on to \( X \), and by \( \mathcal{N}_\Delta \) the locally free \( O_X \)-module corresponding to the “normal bundle” of the regular immersion \( \Delta \). In other words, if \( \mathcal{I}_\Delta \) is the ideal sheaf of \( \Delta \) in \( X'' \), then \( \mathcal{N}_\Delta = (\mathcal{I}_\Delta/\mathcal{I}_\Delta^2)^* \), the dual of \( \mathcal{I}_\Delta/\mathcal{I}_\Delta^2 \). As in [NS1 (C.2.8)] and (2.1.2), set

\[
\mathcal{N}_\Delta^r = \mathcal{N}_\Delta^r \cap \mathcal{N}_\Delta.
\]
and
\[ \Delta^* = L \Delta^*(-) \otimes_{\mathcal{O}_Y} \mathcal{N}_Y[-r] \]

We then have an isomorphism
\[ f^* \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{N}_Y^X[-r] \xrightarrow{\sim} \mathcal{O}_X \]
defined by the commutativity of the following diagram:
\[ \begin{array}{c}
\text{(3.1.2)} \quad f^* \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{N}_Y^X[-r] \xrightarrow{\sim} \Delta^* \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{N}_Y^X[-r] \\
\mathcal{O}_X \xrightarrow{\eta}_{\Delta} \Delta^* \mathcal{O}_X
\end{array} \]

The map \( \eta^\prime \) is as in (2.1.3). The unlabelled arrow on the top row is the one arising from \( L \Delta^* p^*_Y \xrightarrow{\sim} 1 \) and the one on the bottom row from the functorial isomorphism \( \Delta^* p^*_Y \xrightarrow{\sim} 1_F^* \).

Writing \( \mathcal{L}^* \) for the dual of an invertible \( \mathcal{O}_X \)-module \( \mathcal{L} \) we see that \( \mathcal{N}_Y^X = \omega^*_f \).

Using this and (3.1.1) one deduces, as Verdier did in \([V, p. 397, \text{Theorem 3}]\), that \( f^* \mathcal{O}_Y \) and \( \omega_f[r] \) are isomorphic. However, there is some ambiguity about the exact isomorphism (see the discussion around (7.2) in p. 758 of \([S2]\)). But at the very least we note that \( f \) is Cohen-Macaulay. We give the isomorphism we will work with in Definition 3.1.5 after some necessary preliminaries.

As usual, let \( \omega^*_f = H^{-r}(f^* \mathcal{O}_Y) \) and make the identification
\[ f^* \mathcal{O}_Y = \omega^*_f[r]. \]

Applying \( H^0 \) to (3.1.1) we get an isomorphism
\[ \omega^*_f \otimes_{\mathcal{O}_X} \omega^*_f \xrightarrow{\sim} \mathcal{O}_X \]

Let
\[ \overline{v}_f (= \overline{v}) : \omega_f \xrightarrow{\sim} \omega^*_f \]
be the canonical isomorphism induced by (3.1.3).

**Definition 3.1.5.** The **Verdier isomorphism** for the smooth map \( f \) is the isomorphism
\[ \overline{v}_f (= \overline{v}) : \omega_f[r] \xrightarrow{\sim} \omega^*_f[r] = f^* \mathcal{O}_Y \]
given by \( \overline{v}_f = \overline{v}_f[r] \).

We will often refer to \( \overline{v}_f \) also as the Verdier isomorphism. Indeed \( v_f \) and \( \overline{v}_f \) determine each other.

**Remark 3.1.6.** The isomorphism \( p^*_Y f^* \mathcal{O}_Y \xrightarrow{\sim} p^*_Y \mathcal{O}_X \) of (2.1.1) induces (on applying \( H^0 \)) an isomorphism
\[ \theta : p^*_Y \omega^*_f \xrightarrow{\sim} \omega^*_{p^*_Y}. \]

Note that the original isomorphism \( p^*_Y f^* \mathcal{O}_Y \xrightarrow{\sim} p^*_Y \mathcal{O}_X \) is \( \theta[r] \) under the identifications we have agreed to make throughout, namely, \( f^* \mathcal{O}_Y = \omega^*_f[r] \) and \( p^*_Y \mathcal{O}_X = \omega^*_{p^*_Y}. \)
Applying the functor $H^0$ to the commutative diagram (3.1.2) we get the following commutative diagram, showing the relationship between the pairing (3.1.3) and maps of the form $\tau_i$ defined in [NSI (5.3.2)] (below, $h$ is the identity map).

\begin{equation}
(3.1.6.1)
\begin{array}{ccc}
\omega_i^\# \otimes_{\mathcal{O}_X} \omega_j^\# & \xrightarrow{\Delta^*(p_i^*\omega_i^\# \otimes_{\mathcal{O}_{X'}} \Delta_!\omega_j^\#)} & \\
\mathcal{O}_Y & \xrightarrow{\tau_{i,p_1,\Delta}} & \Delta^*(\omega_{p_1}^\# \otimes_{\mathcal{O}_{X'}} \Delta_!\omega_j^\#) \end{array}
\end{equation}

Here the map on the bottom row is as in [NSI (5.3.2)], with $h = 1_X$, $i = \Delta$, and $f = p_i$. It is an isomorphism because $h = 1_X$ is an isomorphism.

**Definition 3.1.7.** Suppose $f : \mathcal{X} \to \mathcal{Y}$ is pseudo-proper and smooth of relative dimension $r$. The Verdier integral (or simply the integral)

\begin{equation}
(3.1.8)
\tau_f : R^r f_* \omega_f \to \mathcal{O}_Y
\end{equation}

is the composite $R^r f_* \omega_f \xrightarrow{\tau_f} R^r f_* \omega_f \to \mathcal{O}_Y$. If in the above situation, $\mathcal{X} = \text{Spf}(R, J)$ and $\mathcal{Y} = \text{Spf}(A, I)$, then we write

\begin{equation}
(3.1.9)
\tau_{R/A} : H^r_f(\omega_{R/A}) \to A
\end{equation}

for the global sections of $\tau_f$. Here $\omega_{R/A}$ is the $r$-th exterior power of the universally finite module of differentials for the $\mathcal{A}$-algebra $R$. If we wish to emphasise the adic structure on $R$ and $A$, we will use the inconvenient notation $\tau_{(R,J)/(A,I)}$ for $\tau_{R/A}$.

**Remark 3.1.10.** While we have defined $\tau_f$ in general, our interest is really in the case where $R^j_f(\mathcal{X}) = 0$ for every $j > r$ and every $\mathcal{X} \in \mathcal{A}_k(\mathcal{X})$, for then $(\omega_f, \tau_f)$ represents the functor $\text{Hom}_Y(R^r_f(\mathcal{X}), \mathcal{O}_Y)$ on coherent $\mathcal{O}_X$-modules (see the result in [NSI Cor. (5.1.4)]). Even here the notion is most useful in this paper when $\mathcal{Y}$ is an ordinary scheme and either $\mathcal{X}$ is also ordinary (and hence proper over $\mathcal{Y}$) or else $\mathcal{Y} = \text{Spec} A$ and $\mathcal{X} = \text{Spf} R$ where $R$ is an adic ring, one of whose defining ideals $I$ is generated by a quasi-regular sequence of length $r$ and such that $R/I$ is finite and flat over $A$.

### 3.2. Local description of Verdier’s isomorphism

In the above situation suppose $\mathcal{X} = \text{Spf} R$, $\mathcal{Y} = \text{Spf} A$, so that $\mathcal{Y}'' = \text{Spf} R''$ where $R'' = R \otimes_A R$ is the complete tensor product of $R$ with itself over $A$. The diagonal map $\Delta : \mathcal{X}'' \to \mathcal{X}'''$ corresponds to the surjective map $R'' \to R$ given by $t_1 \otimes t_2 \to t_1 t_2$. Let us assume that the kernel of this map, i.e., the ideal $I$ of the diagonal immersion, is generated by $r$-elements $\{s_1, \ldots, s_r\}$. Since $R$ is smooth over $A$ of relative dimension $r$, the sequence $\mathbf{s} = (s_1, \ldots, s_r)$ is necessarily a $R''$-sequence. This condition on the diagonal is locally (in $\mathcal{X}$ and $\mathcal{X}'''$) always achievable.

Let $R_1$ and $R_2$ be the two $R$-algebra structures on $R''$ corresponding to the projections $p_i : \mathcal{X}'' \to X$, $i = 1, 2$. For specificity, if $a \in R$, then the $R$-algebra structure on $R_1$ is given by $a(b \otimes c) = (ab) \otimes c$ whilst on $R_2$ it is given by $a(b \otimes c) = b \otimes (ac)$. Let $\omega_{R/A}^\#, \omega_{R_1/A}^\#, \omega_{R_2/A}^\#$ be the global sections of $\omega_f^\#, \omega_{p_1}^\#, \omega_{p_2}^\#$ respectively, where $i \in \{1, 2\}$. Similarly, Verdier’s isomorphism in this context is the isomorphism

\[ \tilde{\nu}_{R/A} : \omega_{R/A} \cong \omega_{R/A}^\# \]

obtained by taking global sections of $\tilde{\nu}_i : \omega_i \cong \omega_i^\#$. 

The isomorphism (3.1.3) is equivalent to the isomorphism of finitely generated $R$-modules obtained by taking global sections:

\[(3.2.1) \quad \omega_{R/A} \otimes_R \omega_{R/A}^* \sim R.\]

Here is the promised local description of Verdier’s isomorphism. The module of differentials $\omega_{R/A} = \wedge^r R/I/I^2$ is a free rank one $R$-module with generator $d_{s} := (s_1 + I^2) \wedge \cdots \wedge (s_r + I^2)$.

Let $1/s$ be the element of $(\wedge^r R/I/I^2)^* = \omega_{R/A} = \text{Hom}_R(\omega_{R/A}, R)$ which sends $d_{s}$ to 1, i.e., it is the generator of the rank one free module $(\wedge^r R/I/I^2)^*$ which is dual to $d_{s}$.

**Proposition 3.2.2.** In the above situation we have the following:

(a) Let $v_0(s) \in \omega_{R/A}^*$ be the unique element such that $v_0(s) \otimes 1/s \mapsto 1$ under (3.2.1). Verdier’s isomorphism $\bar{\nu}_{R/A} : \omega_{R/A} \sim \omega_{R/A}^*$ is given by the formula

$$\bar{\nu}_{R/A}(r d_{s}) = r v_0(s) \quad (r \in R).$$

(b) Suppose further that the adic rings $A$ and $R$ have discrete topology so that $\text{Spf } A = \text{Spec } A$, $\text{Spf } R = \text{Spec } R$ and $A \to R$ is of finite type. The following formula holds:

$$\text{res}_{\Delta, p_1} R \left[ \bar{\nu}_{R_1/R}(ds_1 \wedge \cdots \wedge ds_r) \right] = 1.$$  

Remarks: Here $ds_1 \wedge \cdots \wedge ds_r \in \omega_{R_1/R}$ and the notation $\text{res}_{\Delta, p_1}$ is to indicate that the residue is to be taken for the map $p_1$ and not for $p_2$. The hypotheses in part (b) regarding the adic topologies on $A$ and $R$ is there because we need the result that Verdier’s isomorphism is compatible with base change. This is one of the main results of [S2] (see [S2, p.740, Theorem 2.3.5 (b)]). Unfortunately the results in [S2] are for maps between ordinary schemes. Since certain special compactifications are locally used, and these are unavailable for arbitrary formal schemes, we decided it is best not pursue these issues in this paper, except in the following special case. Suppose the base change is flat. Then the proof in [S2] works mutatis mutandis, and we see that Verdier’s isomorphism is compatible with flat base change whether we are working with ordinary schemes or formal schemes. See Theorem 3.4.1.

**Proof.** Part (a) is an immediate consequence of the definition of $\bar{\nu}_{R/A}$ in (3.1.4). It remains to prove part (b).

Let us save on notation and write

$$\theta : \omega_{R/A}^* \otimes_R R_2 \sim \omega_{R_1/R}^*$$

for the $R''$-isomorphism corresponding to $\theta : p_1^* \omega_{f} \sim \omega_{p_1}^*$ of Remark 3.1.6. Then the affine version of the commutative diagram (3.1.6.1) is the commutative diagram

\[(3.2.3) \quad \begin{array}{ccc}
\omega_{R/A}^* \otimes_R \omega_{R/A}^* & \sim & (\omega_{R/A}^* \otimes_R R_2) \otimes_{R''} \omega_{R/A}^* \\
\downarrow \text{(3.2.2)} & & \downarrow \text{via } \theta \\
R & \sim & \omega_{R_1/R}^* \otimes_{R''} \omega_{R/A}^* 
\end{array}\]
Now if $s \in U$ is an element of the $\omega$-module $\Omega^2_{I/I^2}$, whence an isomorphism $\varphi \otimes_R R_1 \otimes_{R_1} \cdots \otimes_{R_1} \varphi \otimes_R R_1 \otimes_{R_1} \cdots \otimes_{R_1}$.

It is immediate that

$$\text{res}_{\Delta, \mathbb{P}^1}^a \left[ \left( \varphi \otimes_{R_{\mathbb{P}^1}} \left( \varphi \otimes_{R_{\mathbb{P}^1}} \cdots \right) \right) \right] = 1.$$

Next, if $M$ and $N$ are finitely generated modules over $R$ and $\varphi: M \to N$ a map of $R$-modules, then we denote the map $\varphi \otimes_R R_1$ by $p^*_1(\varphi)$. One checks easily that

$$(p^*_1(\varphi))(m \otimes 1) = (\varphi(m)) \otimes 1.$$

This means in particular that in the $\omega$-module $\Omega^2_{I/I^2}$, we have the equality $s + I^2 = \sum a_i db_i$.

We will be done if we can show that $\theta \circ (p^*_1(\varphi)) = \varphi_{R_{\mathbb{P}^1}}$. This statement about the compatibility of Verdier’s isomorphism with base change follows from [S2, p.740, Theorem 2.3.5] (see also [ibid., pp.739–740, Remark 2.3.4]). Incidentally, this is where we need our hypothesis that our formal schemes are ordinary schemes and our map is of finite type.

**Remarks 3.2.4.** Two observations are worth making.

1. $\check{\omega} = H^{-r}(\nu)$.

2. If $\mathcal{O}$ is an open subscheme of $\mathcal{X}$, and $f_\mathcal{O}: \mathcal{O} \to \mathcal{Y}$ is the structural morphism on $\mathcal{O}$, then we have a natural isomorphism $f^*_\mathcal{O}|_{\mathcal{O}} \simeq f^*_\mathcal{O}$ from the main results of [Nay], whence an isomorphism $\omega^*_\mathcal{O}|_{\mathcal{O}} \simeq \omega^*_\mathcal{O}$. From the definitions of $\Omega$ and $\nu$, it is easy to see that the composition of isomorphisms $\omega_{f_{\mathcal{O}}} = \omega_{f_{\mathcal{O}}}|_{\mathcal{O}} \simeq \omega_{f_{\mathcal{O}}}|_{\mathcal{O}} \simeq \omega_{f_{\mathcal{O}}}$ is $\nu_{f_{\mathcal{O}}}$, where the first arrow is $\nu_{f_{\mathcal{O}}}|_{\mathcal{O}}$ and the second the just mentioned isomorphism.

### 3.3. Compatibility of Verdier’s isomorphism with completions.

We now wish to show the compatibility of Verdier’s isomorphism with completion. More precisely if $f: \mathcal{X} \to \mathcal{Y}$ is a smooth map and $\tilde{f}: \mathcal{O} \to \mathcal{Y}$ its “completion” along a closed subscheme of $\mathcal{X}$, then Verdier’s isomorphism (i.e., (3.1.5)) for $\tilde{f}$ is the “completion” of the Verdier isomorphism for $f$. The formal statement is given in Theorem 3.3.2.

Let $f: \mathcal{X} \to \mathcal{Y}$ be a smooth map of relative dimension $r$ between formal schemes. Suppose $\mathcal{I}$ is a defining ideal of $\mathcal{X}$ and $\mathcal{O} \subset \mathcal{O}_\mathcal{X}$ a coherent ideal containing $\mathcal{I}$ (so that $\mathcal{I}$ is the ideal of an ordinary scheme $Z$ which is a closed subscheme of $\mathcal{X}$). Let $\mathcal{O}$ be the completion of $\mathcal{X}$ along $\mathcal{I}$ (i.e., along $Z$). Let $\kappa: \mathcal{O} \to \mathcal{X}$ be the completion map and $\tilde{f}: \mathcal{O} \to \mathcal{Y}$ the composite $\tilde{f} = f \circ \kappa$. We wish to show that the Verdier isomorphism for $\tilde{f}$ “is” $\kappa^*$ of the Verdier isomorphism for $f$. As before let $\mathcal{X}'' = \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}'$, and $\Delta: \mathcal{X} \to \mathcal{X}''$ the diagonal immersion.

Let $\kappa: \mathcal{O} \to \mathcal{X}''$ be the map $\kappa_\mathcal{O} = \kappa \times \kappa$. As usual, we have projections $p_i: \mathcal{X}'' \to \mathcal{X}$ and $\pi_i: \mathcal{O}_{\mathcal{X}''} \to \mathcal{O}$ for $i = 1, 2$. The following commutative diagrams may help the reader map the relative positions of the schemes and maps involved:
In what follows, let

\[(3.3.1)\]

\[\kappa^* f^\# \sim \widehat{f}^\#\]

be the composite \[L\kappa^* f^\# \sim \kappa^* f^\# \sim \widehat{f}^\#\], and let \(\kappa^* \omega_f[r] \sim \omega_f[r]\) be the one induced by the canonical isomorphism \(\kappa^* \omega_f \sim \omega_f\).

**Theorem 3.3.2.** The following diagram commutes

\[
\begin{array}{ccc}
\text{Lk}^* \omega_f[r] & \sim & \kappa^* \omega_f[r] \\
\downarrow \phi & & \downarrow \phi \\
\text{Lr}^*(f^* O_y) & \sim & \widehat{f}^* O_y
\end{array}
\]

**Proof.** Since \(\nu_f\) and \(\nu_f\) are isomorphisms, we assume \(f^* O_y\) and \(\widehat{f}^* O_y\) are complexes which are zero in all degrees except at the \((-r)\)-th spot, where each is locally free (in fact invertible). This means we write \(h^*(f^* O_y) = Lh^*(\widehat{f}^* O_y)\) (resp. \(h^*(f^* O_y) = Lh^*(\widehat{f}^* O_y)\)) for any map of schemes to \(X\) (resp. \(Y\)). Similarly \(Lh^* \omega_f[d] = h^* \omega_f[d]\) etc. Let

\[\phi: \kappa^*(N^\tau_D) \sim N^\tau_G\]

be the canonical isomorphism. We have to show that the diagram \(\bullet\) below commutes.

\[
\begin{array}{ccc}
\widehat{f}^* O_y \otimes N^\tau_G[-r] & \sim & \kappa^* f^* O_y \otimes N^\tau_D[-r] \\
\downarrow \phi & & \downarrow \phi \\
O_y & \sim & \kappa^* O_y
\end{array}
\]
We expand $\bullet$ as follows (with the label (C.4.2), referring to the label in \[NS1\]):

\[
\tilde{f}^* F \otimes N^\Delta_{-r} \xrightarrow{\phi} \kappa^* f^* F \otimes N^\Delta_{-r} \xrightarrow{\text{via } \phi} \kappa^* (f^* F \otimes N^\Delta_{-r})
\]

The maps $\eta_1$ and $\eta_2$ are the maps defined in (2.1.1). The maps $\alpha_i^{-1}$ are induced by the isomorphism $\pi_i^* \kappa^* \mathcal{O}_Y \xrightarrow{\sim} \kappa^* \pi_i^* \mathcal{O}_X$ resulting from the following composite of natural maps

\[
\pi_i^* \kappa^* \xrightarrow{\sim} \kappa^* \pi_i^* \kappa^* \xrightarrow{\sim} \kappa^* \pi_i^* \kappa^* \xrightarrow{\sim} \kappa^* \pi_i^* \kappa^* \xrightarrow{\sim} \kappa^* \pi_i^* \kappa^* .
\]

In the above expansion of $\bullet$, the unlabelled sub-rectangles clearly commute. Sub-rectangle $\square_2$ commutes by definition of the isomorphism \[NS1\] (C.4.2). Prop. C.4.3 of \[NS1\] gives the commutativity of $\square_3$. For $\square_4$ we apply the outer border of the following diagram on $\mathcal{O}_Y$:

\[
\delta^* \pi_i^* \kappa^* \xrightarrow{\sim} \delta^* \pi_i^* \kappa^* \pi_i^* \kappa^* \xrightarrow{\sim} \delta^* \kappa^* \pi_i^* \pi_i^* \kappa^* \xrightarrow{\sim} \delta^* \kappa^* \pi_i^* \pi_i^* \kappa^* \xrightarrow{\sim} \delta^* \kappa^* \pi_i^* \pi_i^* \kappa^* \xrightarrow{\sim} \delta^* \kappa^* \pi_i^* \pi_i^* \kappa^* .
\]

The unlabelled arrows are the obvious ones. The rectangle $\square_4$ commutes by \[NS1\] Lemma A.1.4 while the remaining commute for obvious reasons.

It remains to prove that $\square_1$ commutes. To that end, it suffices to prove that the outer border of the following diagram commutes where the unlabelled arrows are the obvious ones coming from pseudofunctoriality of $(-)^*$ or $(-)^\#$, the ones labelled b-ch are induced by suitable base-change isomorphisms as given in (2.1.1), the ones labelled $\# = ^*$ are induced by (2.2.1) and the ones labelled $\gamma_i$ are induced by the
composite $\kappa_2^* \kappa_1^* \xrightarrow{\sim} \overline{\kappa}^* \xrightarrow{\sim} \overline{\kappa}^*$. 

Now $\dagger$ commutes by transitivity of the base-change isomorphism, see [NS1, Prop. A.1.1]. For the diagrams labelled $\triangle$, we refer to [NS1, Lemma A.1.4], while $\triangledown$ commutes because of the pseudofunctorial nature of the isomorphism $(-)^\# \cong (-)^*$ of (2.2.1) over the category of formally étale maps. The unlabelled diagrams commute for trivial reasons. □

3.3.3. There is a related result. Suppose $f : \mathcal{X} \to \mathcal{Y}$ is a smooth map of relative dimension $d$ in $\mathcal{G}$ and suppose $f$ factors as

$$
\xymatrix{ \hat{\mathcal{X}} \ar[r]^{\hat{f}} \ar[d]_{\hat{\kappa}} & \hat{\mathcal{Y}} \ar[d]^{\kappa} & \mathcal{Y} \\
\mathcal{X} \ar[r]^{f} & \mathcal{Y} \ar@{.>}[ur]_{\kappa}}
$$

where $\kappa$ is the completion of $\mathcal{Y}$ along a coherent $\mathcal{O}_\mathcal{Y}$-ideal $I$, and $\hat{f}$ is smooth (necessarily of relative dimension $d$). Note that

$$
\overline{\mathcal{Y}} \cong \mathcal{Y}.
$$

Consider the commutative diagram of cartesian squares:

$$
\xymatrix{ \mathcal{X} \ar[r]^{\hat{f}} \ar[d]_{\hat{\kappa}} & \hat{\mathcal{Y}} \ar[d]_{\kappa} & \mathcal{Y} \\
\mathcal{Y} \ar[r]^{f} & \mathcal{Y} \ar@{.>}[ur]_{\kappa}}
$$

From (2.1.1) we conclude that we have an isomorphism

$$(*) \quad f^\# \sim 1^\#_f f^\# \sim \hat{f}^\# \kappa^*.$$
Now clearly, \( \omega_f = \omega_{\hat{f}} \). Call the common \( \mathcal{O}_X \)-module \( \omega \). We have two related isomorphisms, namely, \( \check{\nu}_f : \omega[d] \sim \hat{f}^* \mathcal{O}_{\hat{X}} \) and \( \check{\nu}_f : \omega[d] \sim f^* \mathcal{O}_Y \). With these notations, we have the following Proposition, related to Theorem 3.3.2:

**Proposition 3.3.4.** With notations as above, the following diagram commutes:

\[
\begin{array}{ccc}
\omega[d] & \xrightarrow{\check{\nu}_f} & \hat{f}^* \mathcal{O}_{\hat{X}} \\
\downarrow \check{\nu}_f & & \\
\hat{f}^* \mathcal{O}_Y & \sim & \hat{f}^* \mathcal{O}_Y.
\end{array}
\]

**Proof.** We have the following commutative diagram with all squares cartesian:

\[
\begin{array}{ccc}
\mathcal{X}'' & \xrightarrow{p_2} & \mathcal{Y} \\
\downarrow p_1 & & \downarrow f \\
\mathcal{X} & \xrightarrow{\hat{f}} & \mathcal{Y} \\
\downarrow \check{f} & & \downarrow \kappa \\
\mathcal{X} & \xrightarrow{\check{f}} & \mathcal{Y}.
\end{array}
\]

We claim that diagram (**) below commutes:

\[
\begin{array}{ccc}
p_2^* f^* & \sim & p_1^* f^* \\
\downarrow (\ast) & & \downarrow (\ast) \\
p_2^* \hat{f}^* \kappa^* & \sim & p_1^* \hat{f}^* \kappa^*
\end{array}
\]

Indeed, this follows immediately from the horizontal transitivity of the base-change isomorphism (see [NS1, Prop. A.1.1 (i)]) corresponding to the “composite” of base-change diagrams:

\[
\begin{array}{ccc}
\mathcal{X}'' & \xrightarrow{p_2} & \mathcal{Y} \\
\downarrow p_1 & & \downarrow f \\
\mathcal{X} & \xrightarrow{\hat{f}} & \mathcal{Y} \\
\downarrow \check{f} & & \downarrow \kappa \\
\mathcal{X} & \xrightarrow{\check{f}} & \mathcal{Y}
\end{array}
\]

We therefore have the following commutative diagram, where the square on the left in induced by (**)..

\[
\begin{array}{ccc}
\Delta^*(p_2^* f^* \mathcal{O}_Y \otimes \omega^{-1}[-d]) & \sim & \Delta^*(p_1^* f^* \mathcal{O}_Y) \\
\downarrow (\ast) & & \downarrow (\ast) \\
\Delta^*(p_2^* \hat{f}^* \mathcal{O}_{\hat{X}} \otimes \omega^{-1}[-d]) & \sim & \Delta^*(p_1^* \hat{f}^* \mathcal{O}_{\hat{X}})
\end{array}
\]
In other words
\[
\begin{align*}
\xymatrix{
\hat{f}^* \mathcal{O}_Y \otimes \omega^{-1}[-d] & \mathcal{O}_X \\
\circ & \\
\circ & \mathcal{O}_X
}\end{align*}
\]
commutes. This is equivalent to the statement of the Proposition. □

3.4. Base change and Verdier’s isomorphism. As mentioned earlier, according to [S2, p.740, Theorem 2.3.5 (a)], for any Cohen-Macaulay map between ordinary schemes \( f: X \to Y \), and any base change \( u: Y' \to Y \), with \( X' = X \times_Y Y' \), \( f': X' \to Y' \) and \( v: X' \to X \) the base change maps, there is a natural isomorphism \( \theta^u_{f'}: v^* \omega^d_{f'} \to \omega^d_f \). In the event \( f \) is smooth, then using Verdier’s isomorphisms for \( f \) and \( f' \) to identify \( \omega^d_{f} \) with \( \omega_{f} \) and \( \omega^d_{f'} \) with \( \omega_{f'} \), the map \( \theta^u_{f} \) corresponds to the obvious canonical map (see [Ibid., p.740, Theorem 2.3.5 (b)]). The difficulty in transferring this statement to formal schemes is that defining \( \theta^u_{f} \) required certain special local compactifications of \( f \) which may or may not be available for general formal scheme maps. However these difficulties disappear if the map \( u \) is flat, and the proof in loc.cit. works mutatis mutandis. The precise statement is:

Theorem 3.4.1. Suppose

\[
\xymatrix{
\mathcal{Y}' & \mathcal{Y} \\
\mathcal{Y}' \ar[u]_v \\
}\xymatrix{
\mathcal{X}' & \mathcal{X} \\
\mathcal{X}' \ar[u]_v \\
\mathcal{Y} \ar[l]_u}
\]

is a cartesian square with \( f \) smooth, in \( \mathcal{G} \), of relative dimension \( d \), and \( u \) flat.

Let \( \theta: v^* f^* \mathcal{O}_Y \to (f')^* u^* \mathcal{O}_Y \) be the resulting base change isomorphism (see \[2.1.1]\)). Then the isomorphism \( v_{f'}^{-1} \circ \theta \circ v^*(v_{f'}) : v^* \omega_{f'}[d] \to \omega_{f}[d] \) is the obvious canonical map.

4. Residues

4.1. Verdier residue. Let \( f: X \to Y \) be smooth of relative dimension \( r \) and let \( Z \to X \) be a closed subscheme proper over \( Y \). Analogous to to the abstract residue \( \text{res}^*_Z \) in [NS1, (5.2.2)] one has the Verdier residue along \( Z \)

\[
\text{res}_Z : R^r f_* \omega_f \to \mathcal{O}_Y
\]

(4.1.1)

defined as the composite

\[
\xymatrix{
R^r_Z f_* \omega_f \\
\ar[r]_-{\text{2.3.3}} & \\
R^r_Y f_* \kappa^* \omega_f \\
\ar[r]_-{\text{1r.f.}} & \\
R^r_Y f_* \omega_f \\
\ar[r]_-{\text{tr.f.}} & \\
\mathcal{O}_Y
}
\]

where the middle isomorphism is induced by the canonical one \( \kappa^* \omega_f \to \omega_f \). By compatibility of the Verdier isomorphism with completions (Theorem 3.3.2), the following diagram commutes (where, as before, \( \text{res}^*_Z \) is the abstract residue map defined in [NS1, (5.2.2)]):

\[
\xymatrix{
R^r_Z f_* \omega_f \\
\ar[r]_-{\text{res}_Z} & \\
\ar[r]_-{\text{res}_Z} & \mathcal{O}_Y
}
\]

(4.1.2)
Remark 4.1.3. While we have defined residues in general, our interest is really in the case where $R'_X(F) = 0$ for every $j > r$ and every $F \in \mathcal{A}(\mathcal{X})$, for then $(\omega_f, \text{tr}_f)$ represents the functor $\text{Hom}_Y(R'_X(\_), O_Y)$ on coherent $O_X$-modules (cf. [NS1, Cor. 5.1.4]). Even here the most useful situation is when $Y = \text{Spec} A$ and $\mathcal{X} = \text{Spf} R$ where $R$ is an adic ring, with a defining ideal $I$ generated by $r$ elements, with $R/I$ finite and flat over $A$.

The various relationships between the abstract residue, Verdier residue, the trace and the Verdier integral are captured in the following commutative diagram:

$$\begin{align*}
R'_X f_* \omega_f & \sim R'_Y \hat{f}_* \omega_f \\
\text{res}_Z & \downarrow v & \downarrow v \\
R'_Z f_* \omega_f & \sim R'_Y \hat{f}_* \omega_f \\
\text{tr}_f & \downarrow \text{tr}_f & \downarrow \text{tr}_f \\
O_Y & \rightarrow O_Y
\end{align*}$$

(4.1.4)

In the event $f: X \to Y$ is proper we have the following commutative diagram

$$\begin{align*}
R'_X f_* \omega_f & \rightarrow R' f_* \omega_f \\
\text{res}_Z & \downarrow v & \downarrow v \\
R'_Z f_* \omega_f & \rightarrow R' f_* \omega_f \\
\text{tr}_f & \downarrow \text{tr}_f & \downarrow \text{tr}_f \\
O_Y & \rightarrow O_Y
\end{align*}$$

(4.1.5)

4.2. Some residue formulas. Suppose $A \to R$ is a finite type map of rings which is smooth. Set $R'' = R \otimes_A R$. As before, the two $R$-algebra structures on $R''$ will be denoted $R_1$ and $R_2$, with $R_k$ denoting the algebra corresponding to the projection $p_k: X'' := X \times_Y X \to X$ for $k \in \{1, 2\}$. The diagonal map $\Delta: X' \to X''$ corresponds to the surjective map $R'' \to R$ given by $t_1 \otimes t_2 \mapsto t_1 t_2$. Suppose the kernel of this map, i.e., the ideal of the diagonal immersion, is generated by $r$-elements $\{s_1, \ldots, s_r\}$. Since $R$ is smooth over $A$ of relative dimension $r$, the sequence $s = (s_1, \ldots, s_r)$ is necessarily a $R'$-sequence. By part (b) of Proposition 3.2.2 we get the following formula, which is at the heart of much of what we do in this paper.

$$\text{res}_{\Delta, p_1} \left[ \frac{ds_1 \wedge \cdots \wedge ds_r}{s_1, \ldots, s_r} \right] = 1.$$

(4.2.1)

**Proposition 4.2.2.** Let $X = \text{Spec} A$ and $Y = \text{Spec} R$ be affine schemes, and suppose $f: X \to Y$ is a smooth map of relative dimension $r$. Suppose further that we have a closed subscheme $Z$ of $X$ such that $Z \to Y$ is an isomorphism and the ideal $J$ of $R$ giving the closed subscheme $Z$ of $X$ is generated by $r$-elements.
\{t_1, \ldots, t_r\} of R. Then

\begin{equation}
\tag{4.2.3}
\text{res}_Z \left[ \frac{dt_1 \wedge \cdots \wedge dt_r}{t_1, \ldots, t_r} \right] = 1.
\end{equation}

Proof. First note that since \( f \) is smooth, \( t = (t_1, \ldots, t_r) \) is an \( R \)-regular sequence. Next note that the question is local on \( Y \) and so we may assume, without loss of generality, that the diagonal immersion \( \Delta: X \to X'' \) is cut out by \( r \)-elements \( \{s_1, \ldots, s_r\} \) in \( R'' = R \otimes_A R \). As in Subsection 3.2, we write \( I \) for the ideal of the diagonal and use the notations of that subsection. Let \( Z = \text{Spec} B \). Let \( \sigma: Y \to X \) be the section defined by \( Z \), and \( i: Z \hookrightarrow X \) the natural closed immersion. We have a commutative diagram with all sub-rectangles cartesian.

\[
\begin{array}{ccc}
Z & \xrightarrow{\sigma} & X \\
\downarrow i & & \downarrow \Delta \\
X & \xrightarrow{\sigma} & X'' \\
\downarrow f & & \downarrow p_1 \\
Y & \xrightarrow{\sigma} & X
\end{array}
\]

We now need some results from \([S2]\) regarding non-flat base-change. Since \( \sigma \) is a closed immersion, the usual flat-base-change results do not apply. Nevertheless, we do have the following. First, there is a base change isomorphism \( \theta = \theta_f: \sigma_*\omega^\#_f \cong \omega^\#_{p_1} \) as in [Ibid., p.740, Theorem 2.3.5 (a)]. Next, by [Ibid., Prop. 6.2.2, pp.755–756], under this isomorphism, residues are compatible. In other words, the diagram

\[
\begin{array}{ccc}
\sigma^* R^p_{\Delta} (\omega^\#_{p_1}) & \xrightarrow{\theta} & R^p f_* (\sigma_* \omega^\#_{p_1}) \\
\sigma^* \text{res}_\Delta & & \downarrow \theta \\
\text{res}_{\sigma_*} & & \\
\sigma^* \omega^\#_{p_1} & \xrightarrow{\theta} & R^p f_* \omega^\#_f
\end{array}
\]

commutes. Finally on replacing \( \omega^\#_f \) by \( \omega_f \) and \( \omega^\#_{p_1} \) by \( \omega_{p_1} \) via \( v_f \) and \( v_{p_1} \), according to [Ibid., p.740, Theorem 2.3.5 (b)], the map \( \theta \) reduces to the standard identity \( \sigma_* \omega_{p_1} = \omega_f \). Thus it follows that if \( u_i, i = 1, \ldots, r \), are the images of \( s_i \) in \( R \) under the map \( R'' \to R \) corresponding to \( \sigma_*: X \to X'' \), (so that \( J \) is generated by the set \( \{u_1, \ldots, u_r\} \)) we have (via (4.2.1))

\[
\text{res}_Z \left[ \frac{du_1 \wedge \cdots \wedge du_r}{u_1, \ldots, u_r} \right] = 1.
\]

Since \( \left[ \frac{du_1 \wedge \cdots \wedge du_r}{u_1, \ldots, u_r} \right] = \left[ \frac{dt_1 \wedge \cdots \wedge dt_r}{t_1, \ldots, t_r} \right] \), hence the result. \( \square \)

5. Residues along sections

Let \( f: X \to Y \) be a smooth map of relative dimension \( r \), which is separated. We begin with some notations and conventions. In general, if we are working over affine schemes (ordinary or formal) we will use the same notations for maps between modules as the corresponding sheaves. For example if \( A \to R \) is smooth map of rings of relative dimension \( r \), and \( I \) an \( R \) ideal generated by a regular sequence \( \{t_1, \ldots, t_r\} \) such that \( A \to B := R/I \) is finite, then with \( X = \text{Spec} R \),
Y = Spec A and Z = Spec B, and \( f: X \to Y \) the map given by \( A \to R \), we will write \( \text{res}_Z: H^r_Z(X, \omega_f) \to A \) instead of \( \Gamma(Y, \text{res}_Z) \). As another illustration of this principle, in the above situation, if \( \omega_{R/A} \) is the A-module given by \( \omega_{R/A} = \Gamma(X, \omega_f) \), then we will make no distinction between \( H^r_Z(X, \omega_f) \) and \( H^r_f(\omega_{R/A}) \).

5.1. Suppose \( Y = \text{Spec} A \) and \( Z \hookrightarrow X \) is a closed subscheme such that \( Z \to Y \) is an isomorphism and \( Z \) is given in an open affine subscheme \( U = \text{Spec} R \) of \( X \) such that \( Z \) is described as follows. It should be pointed out that if \( Z \) is described as \( (5.1.1) \), but as we will see later, it is independent of it. It should be pointed out that if \( Z \) is also defined (in \( U \)) by the vanishing of \( s_1, \ldots, s_r \), then by [NST Thm. C.7.2 (iii)]

\[
\begin{align*}
\text{res}_t & \colon H^r_Z(X, \omega_f) \to A \\
& \begin{cases}
1 & \text{when } \alpha_1 = \cdots = \alpha_r = 1 \\
0 & \text{otherwise}.
\end{cases}
\end{align*}
\]

This map depends \textit{a priori} on the choice of \( t = (t_1, \ldots, t_r) \), but as we will see later, it is independent of it. Moreover, there is an \( A \)-module direct sum decomposition

\[
H^r_f(\omega_f) = \bigoplus_{\alpha} A[\frac{dt_1 \wedge \cdots \wedge dt_r}{t_1^{\alpha_1} \cdots t_r^{\alpha_r}}]
\]

with \( \alpha = (\alpha_1, \ldots, \alpha_r) \) running over \( r \)-tuples of positive integers. The summands are free \( A \)-modules. While this decomposition depends on \( t \), the summand generated by \( \frac{dt_1 \wedge \cdots \wedge dt_r}{t_1^{\alpha_1} \cdots t_r^{\alpha_r}} \) is independent of \( t \) by (5.1.2). In what follows, let

\[
\theta_x = \left[ \frac{dt_1 \wedge \cdots \wedge dt_r}{t_1^{\alpha_1} \cdots t_r^{\alpha_r}} \right].
\]

5.2. Relative projective space. Let \( \mathbb{P} = \mathbb{P}^r_Y \), the relative projective space of relative dimension \( r \) over an ordinary scheme \( Y \). We regard \( \mathbb{P} = \text{Proj}(O_Y[T_0, \ldots, T_r]) \). Let \( \pi: \mathbb{P} \to Y \) be the structure map and

\[
\int_{\mathbb{P}/Y} : R^r \pi_* \omega_\pi \xrightarrow{\sim} O_Y
\]

be the standard trace map (known to be an isomorphism) defined, for example in [EG] III.1.2 or [RD] p.152, Theorem 3.4. The generating section \( \mu = \mu_\pi \) of \( R^r \pi_* \omega_\pi \) corresponding to the standard section 1 of \( O_Y \) is described as follows. Let \( \mathcal{U} = \{ U_i \mid i = 0, \ldots, r \} \) be the open cover of \( \mathbb{P} \) given by \( U_i = \{ T_i \neq 0 \} \). On \( U_0 \cap \cdots \cap U_r \) we have inhomogeneous coordinates \( t_i = T_i/T_0, \ i = 1, \ldots, r \) whence a section

\[
\bar{\mu}_T := \frac{dt_1 \wedge \cdots \wedge dt_r}{t_1 \cdots t_r} \in \Gamma(U_0 \cap \cdots \cap U_r, \omega_\pi).
\]

We have an isomorphism

\[
H^r(\pi_* \mathcal{C}^*(\mathcal{U}, \omega_\pi)) \xrightarrow{\sim} R^r \pi_* \omega_\pi
\]

and \( \bar{\mu}_T \) has a natural image in the left side as a Čech cohomology class. Let \( \mu \) be the corresponding element on the right side. The section \( \mu \) does not depend on
the choice of homogeneous coordinates $T_0, \ldots, T_r$ of $\mathbb{P}$ (cf. [C1 p.34, Lemma 2.3.1]) and is the sought after section.

Let $Z_0$ be the closed subscheme of $\mathbb{P}$ defined by $\{ T_i = 0 \mid i = 1, \ldots, r \}$, i.e., the intersection of the relative hyperplanes $H_i = \{ T_i = 0 \}$, $i = 1, \ldots, r$. Then $Z_0 \to Y$ is an isomorphism. The section $\sigma_0: Y \to \mathbb{P}$ defined by $Z_0$ is the $Y$-valued point of the $Y$-scheme $\mathbb{P}$ given by the "homogeneous co-ordinates" $(1, 0, 0, \ldots, 0)$.

Now suppose $Y = \text{Spec} \ A$. It is well known (see for example [L2 p.74, Prop. (8.4)], the proof of which generalizes to our situation) that the following diagram commutes.

\[
\begin{array}{ccc}
\text{H}^r_{Z_0}(\mathbb{P}, \omega_\pi) & \xrightarrow{\text{res}} & \text{H}^r(\mathbb{P}, \omega_\pi) \\
\downarrow{f_{\mathbb{P}/Y}} & & \downarrow{\text{A}} \\
0 & & \text{A}
\end{array}
\]

We now indicate how the commutativity of (5.2.1) is proved in [L2]. For an $n$-tuple of positive integers $\alpha = (\alpha_1, \ldots, \alpha_r)$ one can regard fractions of the form $dt_1 \wedge \ldots \wedge dt_r / t_1^{\alpha_1} \ldots t_r^{\alpha_r}$ as $r$-cocycles in the Čech complex $C^*(\mathcal{U}, \omega_\pi) = \Gamma(\mathbb{P}, C^*(\mathcal{U}, \omega_\pi))$. Let us write $\nu(\alpha)$ for the image of this fraction in $\text{H}^r(\mathbb{P}, \omega_\pi)$. (Note that $\nu(1, \ldots, 1) = \mu$.) According to [L2 pp.79–80, Lemma (8.6)] the natural map

\[
\text{H}^r_{Z_0}(\mathbb{P}, \omega_\pi) \to \text{H}^r(\mathbb{P}, \omega_\pi)
\]

is described by

\[
(5.2.2) \quad \begin{bmatrix} dt_1 \wedge \ldots \wedge dt_r \\ t_1^{\alpha_1} \ldots t_r^{\alpha_r} \end{bmatrix} \mapsto \nu(\alpha)
\]

In particular $\theta_{Z_0} \mapsto \mu = \nu(-1, \ldots, -1)$. It is well known that if $\alpha \neq (-1, \ldots, -1)$ the Čech $r$-cocycle $dt_1 \wedge \ldots \wedge dt_r / t_1^{\alpha_1} \ldots t_r^{\alpha_r}$ for the complex $C^*(\mathcal{U}, \omega_\pi)$ is a coboundary, whence in this case $\nu(\alpha) = 0$. This establishes the commutativity of (5.2.1).

If $K_{Z_0}$ is the kernel of $\text{res}_t$, we have a split short exact sequence of $A$-modules, with $\mu \mapsto \theta_{Z_0}$ giving the splitting:

\[
0 \to K_{Z_0} \longrightarrow \text{H}^r_{Z_0}(\mathbb{P}, \omega_\pi) \xrightarrow{\text{canonical}} \text{H}^r(\mathbb{P}, \omega_\pi) \to 0.
\]

**Proposition 5.2.3.** With the above notations we have:

(i) The Verdier integral for $\pi$ equals the standard trace for the relative projective space $\mathbb{P}_Y$, i.e.,

\[
\text{tr}_\pi = \int_{\mathbb{P}/Y}.
\]

(ii) Let $A$ be a ring, $t = (t_1, \ldots, t_r)$ analytically independent variables over $A$, and $J \subset A[[t]]$ the ideal of $A[[t]]$ generated by $t$. Then the Verdier integral $\text{tr}_{A[[t]]/A}: \text{H}^r_J(\omega_{A[[t]]}/A) \to A$ defined in (3.1.9) is given by

\[
\begin{bmatrix} dt_1 \wedge \ldots \wedge dt_r \\ t_1^{\alpha_1} \ldots t_r^{\alpha_r} \end{bmatrix} \mapsto \begin{cases}
1 & \text{when } \alpha_1 = \cdots = \alpha_r = 1 \\
0 & \text{otherwise}.
\end{cases}
\]

**Proof.** For part (i) without loss of generality we may assume $Y = \text{Spec} \ A$. Let $\mu$ be the canonical generator of the free rank one $A$-module $\text{H}^r(\mathbb{P}, \omega_\pi)$ corresponding to $1 \in A$ under the isomorphism $\int_{\mathbb{P}/Y}: \text{H}^r(\mathbb{P}, \omega_\pi) \xrightarrow{\sim} A$. It is enough to show that
\[ \text{tr}_r(\mu) = \int_{\mathcal{P}/Y} (\mu), \text{i.e., it is enough to show that } \text{tr}_r(\mu) = 1. \] According to (5.2.2), the image of \( \theta_{Z_0} \in H^r_{Z_0}(\mathcal{P}, \omega_{\mathcal{P}}) \) in \( H^r(\mathcal{P}, \omega_{\mathcal{P}}) \) is \( \mu \). We have

\[
\text{tr}_r(\mu) = \text{res}_{Z_0}(\theta_{Z_0}) \quad \text{(by (11.1.3))}
\]

\[ = 1 \quad \text{(via Proposition 4.2.2)} \]

and hence we are done for part (i).

For part (ii), let us agree to write \( Y = \text{Spec } A \). Let us write \( \mathcal{P} = \mathcal{P}_A^r \) for \( \text{Spf } A[[t]] \), and \( \pi: \mathcal{P} \to Y \) for the structure map. With \( \mathcal{P}, \pi, Z_0 \) as above, we can identify \( \mathcal{P} \) with the completion of \( \mathcal{P} \) along \( Z_0 \). We thus have a completion map \( \kappa: \mathcal{P} \to \mathcal{P} \), which factors through the open subscheme \( U_0 \) of \( \mathcal{P} \) where \( T_0 \neq 0 \) as \( \mathcal{P} \to U_0 \subset \mathcal{P} \). Moreover, if \( U_0 \) is identified in the usual way with Spec \( A[t_1, \ldots, t_r] \) (via \( t_i = T_i/T_0 \)), then the first map in the factorization arises from the inclusion of the polynomial ring \( A[t] \) into the power series ring \( A[[t]] \). Now, by part (a), (11.1.5), and (5.2.1), we have \( \text{res}_i = \text{res}_{Z_0} \). Since the composite

\[
\text{res}_{Z_0} \pi_\ast \omega_Y \xrightarrow{\sim} \text{res}_{\mathcal{P}} \pi_\ast \omega_{\mathcal{P}} \xrightarrow{\text{tr}_r} \mathcal{O}_Y
\]

is \( \text{res}_{Z_0} \) by (11.1.2), and \( \text{res}_{Z_0} = \text{res}_i \), by taking global sections we are done. \( \square \)

5.3. Let us return to our smooth map \( f: X \to Y \) of relative dimension \( r \), and suppose \( Y = \text{Spec } A \) and \( Z \to X \) as before a closed subscheme such that \( Z \to Y \) is an isomorphism, \( Z \) lies in affine open set \( U = \text{Spec } R \) of \( X \), and \( Z \) is cut out in \( U \) by the vanishing or \( r \) elements \( t_1, \ldots, t_r \) in \( R \).

**Proposition 5.3.1.** In the above situation \( \text{res}_i = \text{res}_Z \). In particular, if \( s = (s_1, \ldots, s_r) \) is another sequence in \( R \) generating the ideal defining \( Z \), then \( \text{res}_i = \text{res}_s \).

**Proof.** Let \( I \) be the ideal generated by \( t \). Suppose \( I \) is also generated by \( s = (s_1, \ldots, s_r) \). The completion of \( R \) in the \( I \)-adic topology is \( A[[t]] = A[[s]] \) and both are the completion of \( \hat{R} \) of \( R \) in the \( I \)-adic topology. It follows that \( \text{tr}_A[[t]]/A = \text{tr}_A[[s]]/A \). Part (ii) of Proposition 5.2.3 and the relationship between \( \text{res}_Z \) and \( \text{tr}_A[[t]]/A \) then proves out assertion. \( \square \)

Consider again the \( A \)-module decomposition

\[
H^r_X(X, \omega_f) = H^r_I(\omega_{R/A}) = \bigoplus_{\alpha} A[dt_1, \ldots, dt_r]^{d_1, \ldots, d_r}
\]

with \( \alpha = (\alpha_1, \ldots, \alpha_r) \) running over \( r \)-tuples of positive integers. Each summand is a free \( A \)-module. While this decomposition depends on \( t = (t_1, \ldots, t_r) \), we have seen that summand generated by \( \theta_Z = [dt_1, \ldots, dt_r]^{d_1, \ldots, d_r} \) is independent of \( t \) by (6.1.2). Moreover, since the the sum of the remaining summands in the direct sum is the kernel \( K_Z \) of \( \text{res}_Z \), it too is independent of \( t \). Thus, we have a canonical decomposition of \( A \)-modules

\[
(5.3.2) \quad H^r_X(X, \omega_f) = K_Z \oplus A \cdot \theta_Z
\]

which is independent of \( t \) with \( K_Z = \ker(\text{res}_Z) \).

**Remark 5.3.3.** Let \( A \) and \( t = (t_1, \ldots, t_r) \) be as in Proposition 5.2.3(ii). Then a little thought shows that for \( f \in A[[t]] \), with \( \mu(t_1, \ldots, t_r) \) the coefficient of \( t_1^{i_1} \ldots t_r^{i_r} \)
in the power series expansion of \( f \), one has the formula:
\[
\text{tr}_{A[[t]]/A} \left[ f \cdot dt_1 \wedge \cdots \wedge dt_r \right] = \mu(\alpha_1 - 1, \ldots, \alpha_r - 1).
\]
In particular, we have
\[
\text{tr}_{A[[t]]/A} \left[ f \cdot dt_1 \wedge \cdots \wedge dt_r \right] = f(0, \ldots, 0).
\]
Similarly, if \( A, t, Z, R \) are as in Proposition 5.3.1 then
\[
\text{res}_Z \left[ f \cdot dt_1 \wedge \cdots \wedge dt_r \right] = \bar{f}
\]
where \( \bar{f} \in A \) is the image of \( f \) in \( A \) under the natural surjection \( R \to R/I \cong A \).

More generally, given positive integers \( \alpha_1, \ldots, \alpha_r \) one can write
\[
f = \sum_{i_1, \ldots, i_r} \mu(i_1, \ldots, i_r) t_1^{i_1} \cdots t_r^{i_r} + g,
\]
where \( i_k \) are non-negative integers such that \( \sum_j i_j < \alpha_1 + \cdots + \alpha_r \), \( \mu(i_1, \ldots, i_r) \in A \), and \( g \in I^{\alpha_1 + \cdots + \alpha_r} \). In this case we have
\[
\text{res}_Z \left[ f \cdot dt_1 \wedge \cdots \wedge dt_r \right] = \mu(\alpha_1 - 1, \ldots, \alpha_r - 1).
\]

5.4. The Verdier residue for sections of smooth maps. Now suppose \( Y \) is not necessarily affine, and as above we have a closed subscheme \( Z \to X \) is such that \( Z \to Y \) is an isomorphism. Let \( z \in Z \) be a point. Pick affine open subschemes \( U' \) in \( X \) and \( V' \) in \( Y \) such that \( z \in U' \) and \( f(U') \subset V' \) and such that \( U' \cap Z \) is given in \( U' \) by the vanishing of \( r \)-elements \( t_1, \ldots, t_r \in \Gamma(U', \mathcal{O}_X) \). Let \( V = f(U' \cap Z) \).

Then \( V \) is an affine open subscheme of \( V' \) (since it is isomorphic to \( U' \cap Z \) which, being a closed subscheme of \( U' \) is affine). Moreover \( U' \to V' \) is affine, whence \( U := f^{-1}(V) \cap U' \) is affine. Note that \( U' \cap Z = U \cap Z, f(U) = V, Z \cap U \) is given by the vanishing of \( t_1, \ldots, t_r \) and \( Z \cap U \to V = f(Z \cap U) \) is an isomorphism. Thus locally we can reduce to the situation in Subsection 5.1.

If \( Z_U = U \cap Z \), then from (5.1.2), it is clear that \( \theta_z \) glue to give a section \( \theta_z \) of \( R_Z \cdot f \cdot \omega_f \):
\[
\theta_z \in \Gamma(X, R_Z \cdot f \cdot \omega_f).
\]

Moreover, the \( A \)-module \( K_Z \) in (5.3.2) being independent of \( t \) means that its construction globalizes to give a quasi-coherent submodule \( \mathcal{K}_Z \) of \( R_Z \cdot f \cdot \omega_f \). Finally, since the decomposition (5.3.2) is canonical, it globalizes to give a decomposition:
\[
(5.4.1) \quad R_Z \cdot f \cdot \omega_f = \mathcal{K}_Z \oplus (\mathcal{O}_Y \cdot \theta_z).
\]

Theorem 5.4.2. Let \( Z \) be a closed subscheme of \( X \) such that \( Z \to Y \) is an isomorphism. Then \( \text{res}_Z \) is the composite
\[
R_Z \cdot f \cdot \omega_f = \mathcal{K}_Z \oplus (\mathcal{O}_Y \cdot \theta_z) \xrightarrow{\text{projection}} \mathcal{O}_Y \cdot \theta_z \xrightarrow{\sim} \mathcal{O}_Y
\]
where the direct sum decomposition is (5.4.1) and the last isomorphism is \( \theta_z \mapsto 1 \).

Proof. Without loss of generality we may assume \( X = \text{Spec} R, Z = \text{Spec} R/I \) where \( I \) is an ideal of \( R \) generated by \( r \) elements \( \{t_1, \ldots, t_r\} \) and \( Y = \text{Spec} A \). The result then follows from Proposition 5.3.1 and the explicit description of \( \text{res}_Z \).

Before stating the next theorem we need some notation. If \( \psi: \omega_f[r] \to f^* \mathcal{O}_Y \) is a map of \( \mathcal{O}_X \)-modules, then \( \bar{\psi}: \omega_f \to \omega_f^* \) will denote the map \( \bar{\psi} = H^{-r}(\psi) \). We
Now suppose we have an isomorphism $\psi$. In other words, $Z$ is a closed subscheme such that

\[ \psi : \omega_f[r] \xrightarrow{\sim} f^!\mathcal{O}_Y. \]

We alert the reader to one notational issue. In this subsection, for good bookkeeping purposes we will write $\bar{\psi} : \omega_f \xrightarrow{\sim} \omega_f^\#$ for $H^{-1}(\psi)$. For most of the paper we do not put the “bar” over $\psi$ for this map, as that abuse of notation is usually harmless. (Cf. also Remark 5.2.3)

**Lemma 5.4.3.** Let $Z$ be a closed subscheme of $X$ such that $Z \rightarrow Y$ is finite and flat. Suppose we have an isomorphism $\psi : \omega_f[r] \xrightarrow{\sim} f^!\mathcal{O}_Y$ such that the composite

\[ R_Z^r f_* \omega_f \xrightarrow{\text{via } \psi} R_Z^r f_* \omega_f^\# \xrightarrow{\text{res}_Z^*} \mathcal{O}_Y \]

is the residue map $\text{res}_Z$. Then there is an open neighbourhood $U$ of $Z$ in $X$ such that $\psi|_U = v|_U$.

**Proof.** It is enough to prove that there is an open neighbourhood $U$ of $Z$ such that $\bar{\psi}|_U = \bar{v}|_U$. Let $\varphi : \omega_f^\# \xrightarrow{\sim} \omega_f^\#$ be the automorphism given by $\varphi = \bar{v} \circ \bar{\psi}^{-1}$. Let $\kappa : \mathcal{A} = X/Z \rightarrow X$ be the completion of $X$ along $Z$ and $\hat{f} = f \circ \kappa$. By the hypothesis we have $\text{res}_Z^* \circ R_Z^r f_* (\varphi) = \text{res}_Z^*$, and hence by definition of $\text{tr}^* \hat{f}$ we get $\text{tr}^* \hat{f} \circ R_Z^r f_* (\kappa^*(\varphi)) = \text{tr}^* \hat{f}$. Thus by local duality [NS1, Cor. 5.1.4], we see that $\kappa^*(\varphi)$ is the identity map, whence there is an open neighbourhood $U$ of $Z$ such that $\varphi|_U$ is the identity map. \(\square\)

We need a little more notation in order to state the next Lemma. Consider a cartesian diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{v} & X \\
\downarrow{f'} & & \downarrow{f} \\
Y' & \xrightarrow{n} & Y
\end{array}
\]

where $f$ is smooth (and hence Cohen-Macaulay) of relative dimension $r$. We will use the notation of [S2] and denote by

\[ \theta'_u : v^*\omega_f^\# \xrightarrow{\sim} \omega_f^\#. \]

the corresponding base change isomorphism (see [S2, p.740, Theorem 2.3.5 (a)]). Now suppose we have an isomorphism $\psi : \omega_f[r] \xrightarrow{\sim} f^!\mathcal{O}_Y$ and suppose $Z \hookrightarrow X'$ is a closed subscheme such that $Z \rightarrow Y'$ is an isomorphism. We write

\[ \text{res}_{\psi, Z}^* : R_Z^r f'_* \omega_{f'} \rightarrow \mathcal{O}_{Y'}. \]

for the composite:

\[ R_Z^r f'_* \omega_{f'} = R_Z^r f'_* v^*\omega_f \xrightarrow{\text{via } \bar{\psi}} R_Z^r f'_* v^*\omega_f^\# \xrightarrow{\text{via } \theta'_u} R_Z^r f'_* \omega_f^\# \xrightarrow{\text{res}_Z^*} \mathcal{O}_{Y'}. \]

In other words $\text{res}_{\psi, Z} = \text{res}_Z^* \circ R_Z^r f_* (\theta'_u \circ v^*(\bar{\psi}))$.

**Lemma 5.4.6.** Let $u : Y' \rightarrow Y$ be an étale map, and let $X'$, $f'$, $v$, $\theta'_u$ be as above. Suppose we have an isomorphism $\psi : \omega_f[r] \xrightarrow{\sim} f^!\mathcal{O}_Y$, and a closed subscheme $Z$ of $X'$ such that $Z \rightarrow Y'$ is finite and flat and such that $\text{res}_{\psi, Z} = \text{res}_Z$. Then there is an open neighbourhood $U$ of the locally closed set $v(Z)$ such that $\psi|_U = v|_U$. 

Proof. By the hypothesis on $\psi$ and by \[5.4.3\] we can find an open neighbourhood $V$ of $Z$ in $X'$ such that $(\theta'_u \circ v^*(\bar{\psi}))|_V = \bar{\psi}_j|_V$. On the other hand, by [S2] p. 740, Theorem 2.3.5 (b)), we have $\theta'_u \circ v^* \bar{\psi}_j = \bar{v}_{j'}$. It follows that $v^*(\bar{\psi}))|_V = v^* \bar{\psi}_j|_V$. Set $U = v(V)$. Since $v$ is étale, $U$ is open, and $V \to U$ is faithfully flat, whence $\bar{\psi}|_U = \bar{\psi}_j|_U$. □

**Remark 5.4.7.** Since if $f$ is smooth, if $x$ is an associated point of $X$, then $y = f(x)$ is an associated point of $Y$, and $x$ is a generic point of the fibre $f^{-1}(y)$. This means that if an open subscheme $V$ of $X$ is such that $V \cap f^{-1}(y)$ is dense in $f^{-1}(y)$ for every associated point $s$ of $Y$, then $V$ is scheme theoretically dense in $X$, since it contains every associated point of $X$. We use this fact in what follows.

**Theorem 5.4.8.** Let $\psi : \omega_f[r] \rightarrow f^! \mathcal{O}_Y$. A necessary and sufficient condition that $\psi$ is the Verdier isomorphism $v$, is the following:

For every étale map $u : Y' \to Y$ and every closed subscheme $Z$ of $X'$ such that $Z \to Y'$ is an isomorphism, we have $\text{res}_{\psi, Z} = \text{res}_{Z}$. Here $X'$, $f'$, $v$ are as in diagram \[5.4.3\].

Proof. For $u : Y' \to Y$, $f' : X' \to Y'$, $v : X' \to X$ as above, according to [S2] p. 740, Theorem 2.3.5 (b) we have $\theta'_u \circ v^* \bar{\psi}_j = \bar{v}_{j'}$. The necessity part of the theorem then follows from Theorem \[5.4.2\].

Conversely, suppose we have an isomorphism $\psi : \omega_f[r] \rightarrow f^! \mathcal{O}_Y$ satisfying the condition stated in the theorem. We have to show that $\psi = \bar{\psi}_j$. Fix $y \in Y$. Since $f$ is smooth, the set $W$ of points $x \in f^{-1}(y)$ such that $k(x)$ is finite and separable over $k(y)$, is dense in $f^{-1}(y)$ by [BLR] p. 42, §2.2, Cor. 13. Let $x$ be such a point. We can find an étale map $u : Y' \to Y$ such that (with the usual notations) there is a section of $f'$ passing through a point $x'$ satisfying $v(x') = x$. [BLR] p. 43, §2.2, Prop. 14). Let $Z$ be the image of this section. Then $Z$ is closed, and $Z \to Y'$ is an isomorphism, whence by our hypotheses on $f$ and by Lemma \[5.4.6\] there is an open neighbourhood $U$ of $v(Z)$ on which $\psi = \bar{\psi}_j$. Since $x \in v(Z)$, this equality holds in an open neighbourhood of $x$. Varying $x$ over $W_y$, and varying $y$ over $Y$, by Remark \[5.4.7\] the equality holds in a scheme theoretically dense open subset of $X$ and hence everywhere, for $\omega_f$ and $\omega^f$ are invertible $\mathcal{O}_X$-modules.

Recall that given a point $x \in X$, closed in its fibre, with $k(x)$ separable over $k(f(x))$, since $f$ is smooth we can find an étale neighbourhood $Y' \to Y$ of $f(x)$ and a section of $f'$ (with the usual notations for base change that we have been following) passing through one of the points of $v^{-1}(x)$. It is immediate that one can find an open cover $\{U_\alpha\}$ of $Y$, étale surjective maps $u_\alpha : Y_\alpha \to U_\alpha$, such that (with $X_\alpha := X \times_Y Y_\alpha$, and $f_\alpha : X_\alpha \to Y_\alpha$, $v_\alpha : X_\alpha \to X$ the projections) there is a closed subscheme $Z_\alpha$ of $X_\alpha$ which maps isomorphically on to $Y_\alpha$. Let $Y' = \coprod_\alpha Y_\alpha$, $X' = \coprod_\alpha X_\alpha$, $f' = \coprod_\alpha f_\alpha$, $u = \coprod_\alpha u_\alpha$. Then we have a closed subscheme $Z$ of $X'$ such that $Z \to Y'$ is an isomorphism (take $Z = \coprod_\alpha Z_\alpha$). Note that $u : Y' \to Y$ is étale and surjective, whence it is faithfully flat.

**Proposition 5.4.9.** Let $\psi : \omega_f[r] \rightarrow f^! \mathcal{O}_Y$ be an isomorphism.

(a) If the fibres of $f$ are connected, and $Z$ is a closed subscheme of $X$ such that $Z \to Y$ is an isomorphism and $\text{res}_{\psi, Z}^f \circ R^f_\alpha(\psi) = \text{res}_{Z}$, then $\psi = \bar{v}_{j'}$.

(b) Suppose the fibres of $f$ are geometrically connected. Then $\psi = \bar{v}_{j'}$ if and only if there is an étale surjective map $u : Y' \to Y$ and (with the usual notation) a closed subscheme $Z$ of $X' = X \times_Y Y'$ with $Z \to Y'$ an isomorphism such that $\text{res}_{\psi, Z} = \text{res}_{Z}$. 

Proof. For part (a), we note that if \( \kappa: X \to X \) is the completion of \( Z \) along \( X \), then \( \kappa^*\psi = \kappa^*\nu_j \). We therefore have an open subscheme \( V \) containing \( Z \) such that \( \psi|_V = \nu_j|_V \). Since \( f \) is smooth, it has a (locally) a factorization \( f = \pi \circ h \), where \( h \) is étale and \( \pi \) is the structural map \( A^Y \to Y \). Since the fibres of \( f \) are connected, and \( f^{-1}(y) \cap V \supset f^{-1}(y) \cap Z \neq \emptyset \), it follows that \( V \cap f^{-1}(y) \) is dense in \( f^{-1}(y) \). Thus \( V \) is scheme-theoretically dense in \( X \) by Remark 5.4.7. Now \( \psi^{-1} \circ \nu_j \) is the identity automorphism on \( \omega_f \) on \( V \), which is scheme theoretically dense on \( X \), and \( \omega_f \) is invertible on \( X \). It follows that it \( \psi^{-1} \circ \nu_j \) is the identity automorphism on all of \( X \).

For part (b), first suppose \( \psi = \nu_j \). By the remarks made above the statement of the theorem, there is an étale surjective map \( u: Y' \to Y \), and (with the usual meaning attached to \( X', f' \) and \( v \)) a closed subscheme \( Z' \) of \( X' \) such that \( Z \to Y' \) is an isomorphism. Now \( \text{res}_{\psi,Z} = \text{res}_{\nu_j,Z}^* \circ R^f_{Y',Y} (\theta'_\psi \circ \nu^*(\psi)) = \text{res}_{\nu_j,Z}^* \circ R^f_{Y',Y} (\theta'_\nu \circ \nu^*(\nu_j)) \). On the other hand, by [S2, p.740, Theorem 2.3.5 (b)], \( v \) behaves well with respect to base change, i.e., \( \theta'_\psi \circ \nu^*(\nu_j) = \nu_{j'} \). Thus \( \text{res}_{\psi,Z} = \text{res}_{\nu_j,Z}^* \circ R^f_{Y',Y} (\nu_{j'}) = \text{res}_{\nu_j,Z}^* \).

Conversely, suppose we have an étale surjective map \( u: Y' \to Y \) and a closed subscheme \( Z' = X \times_Y Y' \), with \( Z \to Y \) an isomorphism satisfying \( \text{res}_{\psi,Z} = \text{res}_{\nu_j,Z}^* \). Let \( f': X' \to Y' \) and \( v: X' \to X \) be the projections. Since the fibres of \( f' \) are connected, by part (a) we have \( \theta'_\psi \circ \nu^*(\psi) = \nu_{j'} \). Now \( \nu_{j'} = \theta'_\nu \circ \nu^*(\nu_j) \) (by [S2, p.740, Theorem 2.3.5 (b)] again) from which it is immediate that \( \nu^*(\psi) = \nu^*(\nu_j) \). The map \( v: X' \to X \) is étale surjective, and hence faithfully flat, giving the result.

6. Regular Differential Forms

The results in this section do not affect the results in the rest of the paper, and so may be skipped on first reading. These results are here to give a non-trivial application of the characterisation of Verdier’s map in the previous section. The main results of this section, connecting the Kunz-Waldi regular differentials with Verdier’s isomorphism, are proved again in Subsection 9.3 without making use of the results in [KW] or the results in this section. There is a fleeting reference to a definition of the map (6.4.1) of this section in Subsection 9.3 (see in §§§9.3.3).

There is a fleeting reference to the definition of the map (6.4.1) of this section in Subsection 9.3. All schemes in this section, unless otherwise stated, are ordinary schemes. The aim is to relate the concrete form of Grothendieck duality via Kunz’s regular differential forms to Verdier’s isomorphism. In somewhat greater detail, regular differential forms defined for certain types of maps \( f: X \to Y \) are concrete representations of many aspects of Grothendieck duality. A well-known special case is that of Rosenlicht’s differentials on singular curves [R]. Kunz defined generalization of these to more general situations (higher dimensions) in a series of papers, and in [KW], Kunz and Waldi defined the sheaf of (relative) regular differentials for dominant finite type equidimensional maps \( f: X \to Y \) between excellent schemes which do not have embedded components. When such an \( f \) is generically smooth, this was related to duality theory by Kunz, Lipman, Hübli, Sastry (see [L2], [HK1], [HK2], [HS]). All the papers just mentioned work within the framework of a simpler version of Grothendieck duality (one eschewing derived categories) due to Kleiman [K]. We now review this, taking a slightly revisionist view, in that we interpret the principal objects (\( r \)-dualizing pairs) in terms of the full blown duality theory of Grothendieck.
6.1. Overview of Kleiman’s functor. Regular differentials are a vast generalisation of the differentials Rosenlicht used for describing describing duality for singular curves \([R]\). To put the theory in context one we give a quick account of Kleiman’s theory of \(r\)-dualizing pairs given in \([K]\). Let \(f : X \to Y\) be a proper map such that \(\dim(X \otimes k(y)) \leq r\) for every \(y \in Y\). For any scheme \(Z\), let \(Z_{qc}\) denote the category of quasi-coherent \(\mathcal{O}_Z\)-modules. According to \([\text{loc. cit.}, \text{pp. 41–42, Definition (1)}]\), an \(r\)-dualizing pair \((f^K, t_f)\) consists of a covariant functor \(f^K : Y_{qc} \to X_{qc}\) and a natural transformation \(t_f : \mathcal{R}^f f^K \to \mathfrak{I}_{Y_{qc}}\) inducing a bifunctorial isomorphism of quasi-coherent sheaves,

\[
f_* \mathcal{H}om_X(\mathcal{F}, f^K \mathcal{G}) \xrightarrow{\sim} \mathcal{H}om_Y(\mathcal{R}^f f_* \mathcal{F}, \mathcal{G})
\]

for each \(\mathcal{F} \in X_{qc}\) and each \(\mathcal{G} \in Y_{qc}\). Kleiman explicitly eschewed derived categories in his paper, and shows the existence of an \(r\)-dualizing pair (for \(f\) of the kind we are considering) using the special adjoint functor theorem. From our point of view \(f^K\) can identified with \(H^{-r}(f(\_))\). Our hypotheses on \(f\) ensure that \(H^j(f(\_)) = 0\) for \(j < -r\), whence we get a map (of functors from \(Y_{qc}\) to \(\text{D}_{qc}(X)\)) \(f^K(\_)[r] \to f^!\). The map \(t_f\) is then the composite

\[
\mathcal{R}^f f_* f^K = H^0(\mathcal{R}^f f_* f^K(-)[r]) \to H^0(\mathcal{R}^f f_* f^\delta \mathcal{O}_Y) \xrightarrow{\text{Tr}_f} H^0(\mathcal{O}_Y) = \mathcal{O}_Y.
\]

When \(f\) is not proper, \(f^K\) still makes sense (even if \(f\) is not compactifiable, i.e., even if \(f\) is not separated), since \(H^n(f^\delta)\) makes sense even for every integer \(n\), even if \(f^\delta\) is not defined (see comment above \([NS1, \text{Def. 4.1.2]}\)), and hence one can set \(f^K = H^{-r}(f^\delta)\). If \(f\) is Cohen-Macaulay of relative dimension \(r\), \(f^K \mathcal{O}_Y = \omega_f^\delta\), and if further \(f\) is smooth we have, via Verdier’s isomorphism \(\omega_f \xrightarrow{\sim} f^K \mathcal{O}_Y\). In the proper, Cohen-Macaulay case we have \((\omega_f^\delta, \text{tr}_f) = (f^K \mathcal{O}_Y, t_f(\mathcal{O}_Y))\). If \(f\) is in addition smooth, we have a unique isomorphism of pairs \((\omega_f, \text{tr}_f) \xrightarrow{\sim} (f^K \mathcal{O}_Y, t_f(\mathcal{O}_Y))\).

6.2. Regular Differentials. Let \(f : X \to Y\) be a finite type map. Following Kunz in \([\text{Ku}, \text{B.17}]\) say it is equidimensional of dimension \(r\) if

- the generic point of \(X\) are mapped to the generic points of \(Y\), and
- the non-empty fibres of \(f\) are such that irreducible component of these fibres are all of dimension \(r\).

Now suppose the map \(f : X \to Y\) satisfies the following conditions

- \(X\) and \(Y\) are excellent schemes, and neither have embedded points amongst their associated points;
- \(f\) is equidimensional of dimension \(r\), and
- the smooth locus of \(f\) is scheme-theoretically dense in \(X\) (which, given our hypotheses, means that the smooth locus of \(f\) contains all the generic points of \(X\)).

Next let \(X_0\) be the artinian scheme

\[
X_0 = \coprod_s \text{Spec } \mathcal{O}_{X,s}
\]

where \(s\) runs through the set of associated (= maximal in this case) points and \(i_X : X_0 \to X\) the natural affine map. Similarly, we have the artinian scheme \(Y_0\) constructed out of the generic points of \(Y\), and an affine map \(i_Y : Y_0 \to Y\). We write

\[
k(X) = i_{X,*} \mathcal{O}_{X_0}
\]
where as before $s$ runs over generic points of $X$. The sheaf of relative meromorphic $r$-forms $\Omega^r_{k(X)/k(Y)}$ on $X$ is then the quasi-coherent $\mathcal{O}_X$-module given by the formula

$$\Omega^r_{k(X)/k(Y)} = i^* \Omega^r_{X_0/Y_0} = \varphi_f \otimes_{\mathcal{O}_X} k(X).$$

Under our hypotheses on $f$ the $\mathcal{O}_X$-module of regular differentials $\omega^r_f$ (denoted $\omega^r_{X/Y}$ in [HK1], [HK2], and [HS]) is defined in [KW, §3, §4]. It is coherent and is an $\mathcal{O}_X$ submodule of the module of meromorphic $r$-differentials $\Omega^r_{k(X)/k(Y)}$, and hence is torsion-free. On the smooth locus $X^s$ of $f$, and writing $f^s: X^s \to Y$ for the smooth map obtained by restricting $f$, we have $\omega^r_f|_{X^s} = \omega^r_{f^s} = \omega^r_{f^s}$.

When $f$ is proper we have a trace map (denoted $\int_{X/Y}$ in [HK1], [HK2], [HS])

$$\int_{f}^{\text{reg}}: R^r f_* \omega^r_f \to \mathcal{O}_Y.$$ 

This map is defined when $f$ is projective in [HK1], and is generalized to proper $f$ in [HS]. One of the main results of [HS] is that the resulting map $\omega^r_f \to f^K \mathcal{O}_Y$ is an isomorphism (a fact proved in [HK2] for projective maps $f$). There is also a notion of a residue map $R^r_Z f_* \omega^r_f \to \mathcal{O}_Y$ (denoted $\int_{X/Y,Z}$ in [HK1], [HK2], and [HS]) for certain special closed subschemes $Z$ of $X$ which are finite over $Y$ (see [HK1, pp.77–78, Assumption 4.3 and Theorem 4.4]).

To avoid notational confusion we denote this

$$\text{res}^r_{Z f}: R^r_Z f_* \omega^r_f \to \mathcal{O}_Y.$$ 

6.3. **Summary of the main result of [HS].** The complete statement concerning $\omega^r_f (= \omega^r_{X/Y}), \int_{f}^{\text{reg}} (= \int_{X/Y})$ and $\text{res}^r_{Z f} (= \int_{X/Y,Z})$ can be found in [HS, pp.750–752, Theorem]. In brief, here are the main points of this result:

(i) One has a canonical isomorphism $\varphi = \varphi_f: \omega^r_f \xrightarrow{\sim} f^K \mathcal{O}_Y$ such that when $f$ is proper $\varphi$ is the unique isomorphism for which the diagram

$$\begin{array}{ccc}
R^r f_* \omega^r_f & \xrightarrow{\text{via } \varphi} & R^r f_* f^K \mathcal{O}_Y \\
\downarrow_{\int_{f}^{\text{reg}}} & & \downarrow_{f^K (\mathcal{O}_Y)} \\
\mathcal{O}_Y & \xrightarrow{\varphi_f} & (f^K) \mathcal{O}_Y
\end{array}$$

commutes [HS pp.750–751, (i) (The Duality Theorem)].

(ii) The isomorphism $\varphi_f$ is compatible with open immersions into $X$. In greater detail, if $j: U \to X$ is an open immersion, as submodules of the $\mathcal{O}_X$ module $\Omega_{k(U)/k(Y)}$, $i^* \omega^r_f = \omega^r_{f^i}$ and the diagram

$$\begin{array}{ccc}
i^* \omega^r_f & \xrightarrow{\sim} & i^* f^K \mathcal{O}_Y \\
\downarrow_{\varphi_{f^i}} & & \downarrow_{f^i (\mathcal{O}_Y)} \\
i^* \omega^r_{f^i} & \xrightarrow{\sim} & (f^i)^K \mathcal{O}_Y
\end{array}$$

commutes [Ibid, pp.750–751, (i) and (ii)].
(iii) If $Z$ is a closed subscheme of $X$ satisfying Assumption 4.3 of [HK1, p.77] then the diagram

\[
\begin{array}{ccc}
R^pf_*\omega^\text{reg}_{Z/X} & \xrightarrow{\text{canonical}} & R^pf_*\omega^\text{reg}_f \\
| & | & | \\
\text{res}^u_Z & \downarrow & \text{res}^u_f \\
\mathcal{O}_Y & \rightarrow & \mathcal{O}_Y
\end{array}
\]

commutes [HS, p.752, (iii) (The Residue Theorem)].

(iv) If $Z$ is a closed subscheme of $X$ such that $Z$ lies in the smooth locus of $f$ and $Z \rightarrow Y$ is an isomorphism, then $\text{res}^\text{reg}_Z = \text{res}_Z$ (see [HK1, p.62, Cor. 1.13] and [HK1] p.78, 4.4] as well as the formulae in Remark 5.3.3.

(v) The map $\varphi$ is compatible with flat base change to excellent schemes without embedded associated points [KW, 3.13] and [HS, pp.751–752, (ii) and (iv)]. In particular $\varphi$ is compatible with étale base change.

6.4. Regular Differentials and Verdier. Now suppose $f$ is smooth. Then $f^K\mathcal{O}_Y = \omega^f_Y$ and $\omega^\text{reg}_f = \omega_f$. Let $\psi = \varphi[r]$. Identifying $f^r\mathcal{O}_Y$ with $\omega^f_Y[r]$ we have an isomorphism

\[
\psi: \omega_f[r] \xrightarrow{\sim} f^r\mathcal{O}_Y.
\]

Then using the notations of Subsection 5.4, we have $\varphi = \psi$. In light of the properties listed above for $\varphi$, and $\omega^\text{reg}_f$ we see that if $u: Y' \rightarrow Y$ is an étale map and $Z$ is a closed subscheme of $X' = X \times_Y Y'$ such that $Z \rightarrow Y$ is an isomorphism, then $\text{res}^\text{reg}_Z = \text{res}_Z$. However, the left side is the map $\text{res}_{\psi, Z}$ of (5.3.5), whence we conclude from Theorem 5.4.8 that $\psi = \nu_f$, where the right side is the Verdier isomorphism of (3.1.4).

Our next observation is one that was made by J. Lipman in pp. 33–34 of [L2] for varieties over fields in his discussion leading to Lemma (2.2) of [L2]. Suppose $f$ is as in the previous subsection, and $U$ is the smooth locus of $f$. Let $j: U \rightarrow X$ be the open immersion and $g = f \circ j: U \rightarrow Y$ the resulting smooth map. By our hypotheses, $U$ contains all the associated points of $X$, whence it is scheme theoretically dense. Without getting into the notions of canonical structures and dualizing structures, we have a composition

\[
\begin{align*}
\stackrel{(6.4.1)}{f^K\mathcal{O}_Y} \hookrightarrow j_*g^K\mathcal{O}_Y \xrightarrow{\text{can}} j_*\omega_g \hookrightarrow \Omega^k_{k(X)/k(Y)}
\end{align*}
\]

with every arrow an inclusion since $f^K\mathcal{O}_Y$, $g^K\mathcal{O}_Y$ and $\omega_g$ are torsion free and $j^*f^K\mathcal{O}_Y \hookrightarrow g^K\mathcal{O}_Y$. The image of $f^K\mathcal{O}_Y$ in $\Omega^k_{k(X)/k(Y)}$ must be $\omega^\text{reg}_f$ since $\nu_g$ is $\varphi_g$ of item (1) of §5.3 In greater detail, if $\bar{\omega}$ is the image of $f^K\mathcal{O}_Y$ in $\Omega^k_{k(X)/k(Y)}$ under (6.4.1), and $\alpha: f^K\mathcal{O}_Y \hookrightarrow \bar{\omega}$ the resulting isomorphism, then we have an isomorphism $\beta: \omega^\text{reg}_f \xrightarrow{\sim} \bar{\omega}$ such that $\alpha = \beta \circ \varphi_f$. Now $j^*\beta = 1_{\omega_g}$ since $\varphi_g = \bar{\nu}_g$. Since $U$ is scheme theoretically dense in $X$ and the sheaves involved are torsion free, the assertion follows. In other words Verdier’s isomorphism gives us the regular differential forms of Kunz and Waldi, as well as the dualizing structure on them.

Here is the formal statement of the result(s) we just proved.

**Theorem 6.4.2.** Let $f: X \rightarrow Y$ be a finite type map between excellent schemes such that $X$ and $Y$ have no embedded points, $f$ is equidimensional of dimension $r$, and the smooth locus of $f$ contains all the associated points of $X$ (i.e., the smooth locus of $X$ is scheme-theoretically dense in $X$).
(a) If \( f \) is smooth then the map \( \varphi_f \) of item (1) in Subsection 6.3 is the Verdier isomorphism \( \bar{\varphi}_f \) defined in (3.1.1).

(b) If \( j: U \to X \) is the open immersion from the smooth locus of \( f \) to \( X \), and \( g: U \to Y \) is the composite \( g = f \circ i \), then the module of regular differential \( r \)-forms \( \omega_f^{\text{reg}} \) of Kunz and Waldi [KW §3, §4] is the image of \( f^K \mathcal{O}_Y \) under injective composite (6.4.1). Moreover the resulting isomorphism \( f^K \mathcal{O}_Y \xrightarrow{\sim} \omega_f^{\text{reg}} \) is inverse of the map \( \varphi_f \).

7. Transitivity for smooth maps

7.1. The map \( \zeta_{g,f} \) between differential forms. Suppose \( f: \mathcal{X} \to \mathcal{Y} \) and \( g: \mathcal{Y} \to \mathcal{Z} \) are maps in \( G \), with \( f \) a smooth map of relative dimension \( e \), and \( g \) a smooth map of relative dimension \( d \). We have a map of differential forms

\[
(7.1.1) \quad \zeta_{g,f} : f^* \omega_g[d] \otimes_{\mathcal{O}_X} \omega_f[e] \to \omega_{gf}[d + e]
\]

defined by the commutativity of the following diagram

\[
\begin{array}{ccc}
f^* \omega_g[d] & \otimes_{\mathcal{O}_X} & \omega_f[e] \\
\downarrow^{\zeta_{g,f}} & \downarrow \quad \downarrow^{\varphi_f} & \\
f^* g^* \mathcal{O}_\mathcal{X} \otimes_{\mathcal{O}_\mathcal{Y}} f^* \mathcal{O}_\mathcal{Y} & \xrightarrow{\chi_{[g,f]}} & (gf)^* \mathcal{O}_\mathcal{X}
\end{array}
\]

where \( \chi_{[g,f]} : f^* g^* \mathcal{O}_\mathcal{X} \otimes_{\mathcal{O}_\mathcal{Y}} f^* \mathcal{O}_\mathcal{Y} \to (gf)^* \mathcal{O}_\mathcal{X} \) is the map defined in [NS1, Def. 7.2.16].

**Proposition 7.1.2.** The following hold:

(a) (Flat Base Change) Suppose

\[
\begin{array}{ccc}
\mathcal{U} & \xrightarrow{u} & \mathcal{X} \\
p & \mathcal{Y} & \xrightarrow{f} \\
\varphi & \mathcal{V} & \xrightarrow{g} \mathcal{Z}
\end{array}
\]

is a cartesian square with \( u \) flat, \( f \) and \( g \) smooth and in \( G \). Then

\[
u^* \zeta_{g,f} = \zeta_{g,f}.
\]

(b) Suppose \( f: \mathcal{X} \to \mathcal{Y} \) and \( g: \mathcal{Y} \to \mathcal{Z} \) are smooth maps and in \( G \). Let \( \kappa: \mathcal{X}^\ast \to \mathcal{X} \) be the completion of \( \mathcal{X} \) with respect to an open coherent ideal. Then

\[
\zeta_{g,f} = \kappa^* \zeta_{g,f}.
\]

(c) Suppose \( \mathcal{X} \xrightarrow{\iota} \mathcal{Y}_1 \xrightarrow{\iota} \mathcal{Y}_2 \to \mathcal{Z} \) is a sequence of maps in \( G \) with \( f \) and \( g \) smooth, and \( \kappa \) a completion map with respect to an open coherent ideal. Then

\[
\zeta_{g,f} = \zeta_{g,f}.
\]
(d) Suppose

\[
\begin{array}{c}
\mathcal{X} \\
\downarrow f \\
\mathcal{Y} \\
\downarrow g \\
\mathcal{Z}
\end{array}
\xrightarrow{\kappa_1} \xrightarrow{\kappa_2} \begin{array}{c}
\mathcal{X} \\
\downarrow f \\
\mathcal{Y} \\
\downarrow g \\
\mathcal{Z}
\end{array}
\]

is a commutative diagram in $\mathbb{G}$ with $f$ and $g$ smooth, and $\kappa_1$ and $\kappa_2$ completions with respect to open coherent ideals of $\mathcal{O}_Y$ and $\mathcal{O}_X$ respectively. Then

$$\kappa_2^* \zeta_{g,f} = \hat{\zeta}_{g,f}.$$  

Proof. Follows from the properties for $\chi_{g,f}$ listed in [NS1, §§7.2], Theorem 3.3.2, Theorem 3.3.4, and the fact that the Verdier isomorphism is compatible with flat base change. \hfill \Box

7.2. The map $\varphi_{g,f}$ between differential forms. For a smooth map between ordinary schemes $f: X \to Y$ of relative dimension $d$, $\omega_f := \land^d \Omega^1_{X/Y}$. Let $X$, $Y$, and $Z$ be ordinary schemes. Suppose $f: X \to Y$ is a smooth map of schemes of relative dimension $d$ and $g: Y \to Z$ is smooth of relative dimension $e$. Let

$$\varphi_{g,f}: f^* \omega_g \otimes \omega_f \xrightarrow{\sim} \omega_{gf}$$

be the map which is locally given by

$$f^*(dt_1 \wedge \cdots \wedge dt_e) \otimes ds_1 \wedge \cdots \wedge ds_d \mapsto dt_1 \wedge \cdots \wedge dt_1.$$ 

Here $t = (t_1, \ldots, t_e)$ and $s = (s_1, \ldots, s_d)$ are local relative “co-ordinates”, i.e., $t$ gives an étale map $U \to \mathbb{A}^d_Y$ on an open subscheme $U$ of $Y$, and on an open subscheme $V$ of $f^{-1}(U)$, $s$ gives an étale map $V \to \mathbb{A}^d_Y$. The local map given above (i.e., $f^*(dt) \otimes ds \mapsto ds \wedge dt$) is independent of these local relative co-ordinates and hence globalises to give $\varphi_{g,f}$.

Using the recipe that gives us $\psi$ in [NS1 (8.1.2)] from $\varphi_{g,f}$ we get a well defined isomorphism in $D_c(X)$

$$\varphi_{g,f}: f^* \omega_g \otimes \omega_f \xrightarrow{\sim} \omega_{gf}[d + e].$$

Note that

$$H^{-(d+e)}(\varphi_{g,f}) = \varphi_{g,f}$$

and hence one can go back and forth between $\varphi_{g,f}$ and $\varphi_{g,f}$.

Here is the main theorem:

**Theorem 7.2.4.** Let $f: \mathcal{X} \to \mathcal{Y}$ and $g: \mathcal{Y} \to \mathcal{Z}$ be maps in $\mathbb{G}$ which are smooth. Then

$$\zeta_{g,f} = \varphi_{g,f}.$$  

Proof. We divide the proof into cases.

**Case 1.** Let $A$ be a noetherian ring, $u_1, \ldots, u_d$, $v_1, \ldots, v_e$ analytically independent variables over $A$, and consider the $A$-algebras $R$, $S$, and $T$ given by $R = A[[u_1, \ldots, u_d]]$, $S = R[[v_1, \ldots, v_e]] = A[[u_1, \ldots, u_d, v_1, \ldots, v_e]]$. Let $I = uR$ be the $R$-ideal generated by $u = (u_1, \ldots, u_d)$, $J = vS$ the $S$-ideal generated by
\( v = (v_1, \ldots, v_e) \), and \( L = IS + J \). In other words, \( J \) is the \( S \)-ideal generated by \( (u, v) \).

Suppose \( \mathscr{X} = \text{Spf}(S, L) \), \( \mathscr{Y} = \text{Spf}(R, I) \), \( \mathscr{Z} = Z = \text{Spec} A \), and that our smooth maps \( f: \mathscr{X} \to \mathscr{Y} \), \( g: \mathscr{Y} \to \mathscr{Z} \) are the natural maps corresponding to the maps of adic rings \( (R, I) \to (S, L) \) and \( (A, 0) \to (R, I) \).

We have additional schemes, namely \( Y = \text{Spec} R \), and \( Y' = \text{Spf}(S, J) \). The natural maps between the adic rings involved give us a commutative diagram with the square on top being a cartesian square:

\[
\begin{array}{ccc}
\mathscr{X} & \xrightarrow{\kappa'} & \mathscr{Y} \\
\downarrow f & & \downarrow p \\
\mathscr{Y} & \xrightarrow{\kappa} & \mathscr{Z} \\
\downarrow g & & \downarrow q \\
& & Y^* \\
& & Z
\end{array}
\]

The maps \( \kappa \) and \( \kappa' \) are completion maps and \( f, g, p, q \) are the obvious maps. Note that \( p, f, g, \) and \( q \) are smooth and pseudo-proper (however this is not true for the map \( g \), which is not of pseudo-finite-type unless \( d = 0 \)).

The rank one free \( \mathcal{O}_{\mathscr{Y}} \)-modules \( \omega_f \) and \( \omega_{gf} \) correspond to the universal finite \( S \)-module of differentials \( \omega_{S/R} := \tilde{\Omega}_{S/R}^e \) and \( \omega_{S/A} := \tilde{\Omega}_{S/A}^{d+e} \). The rank one free \( \mathcal{O}_{\mathscr{Y}} \)-module \( \omega_g \) corresponds to the universal finite \( R \)-module of degree \( d \) differentials \( \omega_{R/A} := \tilde{\Omega}_{R/A}^d \). Thus

\[
\begin{align*}
\omega_{S/R} &= Sdv_1 \land \cdots \land dv_e, \\
\omega_{S/A} &= Sdu_1 \land \cdots \land du_d \land dv_1 \land \cdots \land dv_e, \\
\omega_{R/A} &= Rdu_1 \land \cdots \land du_d.
\end{align*}
\]

The \( S \)-module \( \omega_{S/R} \) gives us a rank one free \( \mathcal{O}_{\mathscr{Y}} \)-module. A little thought shows us that this module is in fact \( \omega_p \). Define \( \omega_q \) as the rank one free \( \mathcal{O}_{\mathscr{Y}} \)-module corresponding to \( \omega_{R/A} \). The equations \( \Gamma(\mathscr{X}, \omega_f) = \omega_{S/R} = \Gamma(\mathscr{Y}, \omega_p) \) and \( \Gamma(\mathscr{Y}, \omega_g) = \omega_{R/A} = \Gamma(Y, \omega_q) \) can be re-written as

\[
\omega_f = (\kappa')^*\omega_p \quad \text{and} \quad \omega_g = \kappa^*\omega_q.
\]

Write \( \varphi \) and \( \varphi' \) for (the global sections of) the maps \( \varphi_{g,f} \) and \( \varphi_{g,f} \). Then the \( S \)-module isomorphism

\[
\varphi: \omega_{R/A} \otimes_R \omega_{S/R} \xrightarrow{\sim} \omega_{S/A}
\]

is given by \( \varphi(du \otimes dv) = dv \land du \). We have the following formula, where \( \text{tr}_{A[[u,v]]/A} \) and \( \text{tr}_{R[[v]]/R} \) are as in \((3.1.9)\).

\[
(*) \quad \text{tr}_{A[[u]]/A} \left[ \text{tr}_{R[[v]]/R} \left[ \begin{array}{c} dv \\
\frac{u_1^{\alpha_1} \cdots u_e^{\alpha_e}}{v_1^{\beta_1} \cdots v_e^{\beta_e}} \end{array} \right] du \right] = \text{tr}_{A[[u,v]]/A} \left[ \begin{array}{c} v_1^{\beta_1} \cdots v_e^{\beta_e} u_1^{\alpha_1} \cdots u_e^{\alpha_e} \\
v_1^{\beta_1} \cdots v_e^{\beta_e} \end{array} \right].
\]

Indeed, if any of the \( \alpha_i \)'s or \( \beta_k \)'s is not equal to 1, then both sides equal zero. If \( \alpha_1 = \beta_k = 1 \) for \( l = 1, \ldots, d, k = 1, \ldots, e \), both sides equal 1. This means that the
following diagram commutes.

\[
\begin{array}{ccc}
H_L^{d+e}((\omega_R/A \otimes_R \omega_S/R)) & \xrightarrow{\phi} & H_L^{d+e}(\omega_S/A) \\
\text{NSI (8.2.4)} & & \\
H_I^f((\omega_R/A \otimes_R H_J^f(\omega_S/R))) & \xrightarrow{\text{tr}_{A[u,v]}/A} & A \\
\text{via tr}_{R[[v]]/R} & & \\
H_I^f(\omega_R/A) & \xrightarrow{\text{tr}_{A[u]}/A} & A
\end{array}
\]

If, in the above diagram, we replace \(\phi\) by \(\zeta_{g,f}\), then by \text{NSI} Prop. 8.3.1(b), the resulting diagram commutes. (See also \text{NSI} Remark 8.3.4.) By the universal property of the pair \((\omega_S^A, \text{tr}_{S/A})\) we see that \(\bar{\phi} = \zeta_{g,f}\), i.e., \(\varphi_{g,f} = \zeta_{g,f}\).

**Case 2.** Suppose we have a section \(\sigma: \mathcal{F} \to \mathcal{X}\) and \(\tau := f \circ \sigma\), and \(\mathcal{X}\) and \(\mathcal{Y}\) are the completions of \(\mathcal{X}\) and \(\mathcal{Y}\) along the closed subschemes given by the closed immersions \(\sigma: \mathcal{F} \to \mathcal{X}\) and \(\tau: \mathcal{F} \to \mathcal{Y}\) respectively. More precisely, if \(\mathcal{I}_1 \subset \mathcal{O}_{\mathcal{X}}\) and \(\mathcal{I}_2 \subset \mathcal{O}_{\mathcal{Y}}\) are the coherent ideals giving the embeddings of \(\mathcal{F}\) into \(\mathcal{X}\) and \(\mathcal{Y}\) (via \(\sigma\) and \(\tau\)), and \(\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}\) is an ideal of definition of \(\mathcal{X}\), then \(\mathcal{I} + \mathcal{I}_1\) and \(\mathcal{I} + \mathcal{I}_2\) are ideals of definition of \(\mathcal{X}\) and \(\mathcal{Y}\) respectively. Since the source and target of \(\zeta_{g,f}\) and \(\varphi_{g,f}\) are concentrated in one degree, the question of their equality is a local question on \(\mathcal{X}\) hence, without loss of generality, we may assume that the schemes involved are affine, say \(\mathcal{X} = \text{Spf}(S,L)\), \(\mathcal{Y} = \text{Spf}(R,I)\) and \(\mathcal{F} = \text{Spf}(A,I_0)\) respectively. In fact we may assume that \(\tau\) and \(\sigma\) are given by regular sequences \((u_1, \ldots, u_d)\) and \((u_1, \ldots, y_d, v_1, \ldots, v_e)\) respectively, and \(u\) is analytically independent over \(A\), and \(v\) is analytically independent over \(R\). We then have a cartesian diagram (where the power series rings \(A[[u_1, \ldots, u_d]] = A[[u]]\) and \(R[[v_1, \ldots, v_e]] = R[[v]]\) are given the adic topologies from the ideals \((u_1, \ldots, u_d)\) and \((v_1, \ldots, v_e)\) respectively)

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{v} & \text{Spf } R[[v]] \\
\downarrow f & & \downarrow p \\
\mathcal{Y} & \xrightarrow{u} & \text{Spf } A[[u]] \\
\downarrow s & & \downarrow q \\
\mathcal{F} & \xrightarrow{w} & \text{Spec } A
\end{array}
\]

with the horizontal arrows being the natural ones. Note that \(w\) is flat being a completion map. Therefore flat base change applies (see Proposition 7.1.2(a)) and we have \(\zeta_{g,f} = v^*\zeta_{q,p}\). Clearly \(\varphi_{g,f} = v^*\varphi_{q,p}\) from the explicit description of \(\varphi_{q,p}\) and \(\varphi_{g,f}\). By Case 1, we have \(\zeta_{q,p} = \varphi_{q,p}\). Applying \(v^*\) to both sides, we get the result for this case.

**Case 3 (The General Case).** In the general case, let \(\mathcal{Y} \times \mathcal{F} = \mathcal{F}\), \(\mathcal{X} \times \mathcal{F} = \mathcal{X}\), and let \(p: \mathcal{F} \to \mathcal{F}\), \(q: \mathcal{F} \to \mathcal{X}\) be the base changes of \(f\) and \(g\), and let \(\pi_1: \mathcal{P} \to \mathcal{X}\) and \(\pi_2\) be the projections \(\mathcal{X} \times \mathcal{F} \to \mathcal{X}\), with \(\pi_1 = q \circ p\). It \(\Delta: \mathcal{X} \to \mathcal{P}\) is the diagonal immersion, then let \(\kappa: \mathcal{Y} \to \mathcal{X}\) be the completion of \(\mathcal{X}\) with respect to \(\Delta(\mathcal{X})\), and let \(\kappa': \mathcal{Y} \to \mathcal{Y}\) be the completion of \(\mathcal{Y}\) along \((p \circ \Delta)(\mathcal{X})\). We have a natural map \(\hat{p}: \mathcal{Y} \to \mathcal{Y}\) such that \(\kappa' \circ \hat{p} = p \circ \kappa\). Let
\( q = q \circ \kappa' \) and let \( \delta : \mathcal{X} \to \mathcal{W} \) be the natural closed immersion. We then have a commutative diagram with the two rectangles on the right being cartesian:

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\delta} & \mathcal{W} \\
\downarrow & & \downarrow \kappa & \cong \\
\mathcal{Y} & \xrightarrow{\nu} & \mathcal{P} & \xrightarrow{\pi_2} & \mathcal{X} \\
\downarrow & & \downarrow \kappa' & \cong & \downarrow \pi_1 & \cong \\
\mathcal{Y} & \xrightarrow{q} & \mathcal{P} & \xrightarrow{f} & \mathcal{X} \\
\end{array}
\]

Now, using the explicit formula for \( \phi_{g,f}, \phi_{q,p}, \) and \( \phi_{\hat{q},\hat{p}} \), we see that \( \pi_2^* \phi_{g,f} = \phi_{q,p} \) and \( \kappa^* \phi_{q,p} = \phi_{\hat{q},\hat{p}} \). Thus \( \kappa^* \pi_2^* \phi_{g,f} = \phi_{\hat{q},\hat{p}} \).

A similar relationship holds for the \( \zeta_{\bullet} \) maps. Indeed, by Proposition 7.1.2(a) we have \( \pi_2^* \zeta_{g,f} = \zeta_{q,p} \) and by Proposition 7.1.2(d) we have \( \kappa^* \zeta_{q,p} = \zeta_{\hat{q},\hat{p}} \), giving \( \kappa^* \pi_2^* \zeta_{g,f} = \zeta_{\hat{q},\hat{p}} \). On the other hand, by Case 2 considered above, we have \( \zeta_{\hat{q},\hat{p}} = \phi_{\hat{q},\hat{p}} \). Thus

\[ \kappa^* \pi_2^* \zeta_{g,f} = \kappa^* \pi_2^* \phi_{g,f}. \]

Applying \( \delta^* \) to both sides of this equation, and noting that \( \pi_2 \circ \kappa \circ \delta = 1_{\mathcal{X}} \), we get the result.

8. Applications of Transitivity

8.1. Iterated residues. Suppose \( f : X \to Y \) is smooth of relative dimension \( e \), \( g : Y \to Z \) smooth of relative dimension \( d \), \( W_1 \hookrightarrow X \) a closed subscheme, finite and flat over \( Y \), \( W_2 \hookrightarrow Y \) a closed subscheme which is finite and flat over \( Z \). Let \( W = W_1 \cap f^{-1}(W_2) \to X \). Suppose further that \( W_1 \) is cut out by a quasi-regular sequence \( v = (v_1, \ldots, v_e) \) in \( S \) and \( W_2 \) is cut out by a quasi-regular sequence \( u = (u_1, \ldots, u_d) \) in \( R \).

**Theorem 8.1.1.** In the above situation, for \( \nu \in \Gamma(\mathcal{O}_Y, \omega_g) \), \( \mu \in \Gamma(\mathcal{O}_X, \omega_f) \), we have

\[
\text{res}_{W_2} \left[ \text{res}_{W_1} \left[ \begin{array}{c} \mu \\
\begin{array}{c}
u \\
u_{v_1}^{\beta_1}, \ldots, v_{v_e}^{\beta_e} \\
u_{u_1}, \ldots, u_d^{\alpha_d} \end{array} \end{array} \right] \right] = \text{res}_{W} \left[ \begin{array}{c} \mu \land f^* \nu \\
\begin{array}{c}v_{1}^{\beta_1}, \ldots, v_{e}^{\beta_e} \\
u_{1}, \ldots, u_d^{\alpha_d} \end{array} \end{array} \right] 
\]

where, for notational simplicity, we denote the image of \( u_i \) in \( S \) also by \( u_i \).

**Proof.** Recall from the definition of \( \zeta_{g,f} \) in (7.1.1) that \( \zeta_{g,f} \) is the transform of \( \chi_{[g,f]} \) after applying Verdier’s isomorphism to \( f^! \mathcal{O}_Y, g^! \mathcal{O}_Z \) and \( (gf)^! \mathcal{O}_Z \). From Theorem 7.2.4 and [NS1] (8.3.2) we get

\[
\text{res}_{W_2} \left[ \text{res}_{W_1} \left[ \begin{array}{c} \mu \\
\begin{array}{c}
u \\
u_{v_1}^{\beta_1}, \ldots, v_{v_e}^{\beta_e} \\
u_{u_1}, \ldots, u_d^{\alpha_d} \end{array} \end{array} \right] \right] = \text{res}_{W} \left[ \begin{array}{c} \phi_{g,f} (\nu \otimes \mu) \\
\begin{array}{c}v_{1}^{\beta_1}, \ldots, v_{e}^{\beta_e} \\
u_{1}, \ldots, u_d^{\alpha_d} \end{array} \end{array} \right].
\]

The result then follows from the definition of \( \phi_{g,f} \) in (7.2.1). \( \square \)
8.2. The Restriction Formula. An important application of our transitivity result is the so-called Restriction Formula, namely the formula in Corollary 8.2.7 below. The formula is related to the following problem. Suppose

\[
\begin{array}{c}
X \\ \downarrow f \\
Y \\ \downarrow \pi
\end{array}
\]

(8.2.1)

is a commutative diagram of ordinary schemes, with \( \pi \) and \( f \) smooth and separated and \( i \) a closed immersion. Let the relative dimension of \( \pi \) be \( n = d + e \) and the relative dimension of \( f \) be \( e \). As usual, let \( \mathcal{N}^d_i \) be the \( d \)-th exterior power of the normal bundle \( \mathcal{N}_i \) of \( X \) in \( P \). We have, via Verdier’s isomorphism and the isomorphism \( \eta'_i \) of (2.1.3), an isomorphism

\[
a_{X/P} : i^*\omega_{\pi}[n] \otimes_{\mathcal{O}_X} \mathcal{N}^d_i[-d] \xrightarrow{\sim} \omega_f[e],
\]

(8.2.2)

defined as the composite

\[
i^*\omega_{\pi}[n] \otimes_{\mathcal{O}_X} \mathcal{N}^d_i[-d] \xrightarrow{\eta'_i} \omega_{\pi}[n] \xrightarrow{\nu_i} \omega_f[e] = f^!\mathcal{O}_Y \xrightarrow{\nu_i^{-1}} \omega_f[e].
\]

(8.2.2)

The question then is, what is the concrete form of \( a_{X/P} \) in terms of local relative coordinates? We answer the question in Theorem 8.2.6 below.

We leave it to the reader to check that \( a_{X/P} \) is compatible with open immersions into \( Y \) and \( P \). Indeed every map in the composition defining \( a_{X/P} \) is well behaved with respect to open immersions into \( Y \) and \( P \). Thus we may assume that \( P \) and \( Y \) are affine and that \( X \) is defined by an ideal generated by a quasi-regular sequence, which is part of a relative system of co-ordinates for \( \pi : P \to Y \).

We now assume that \( Y = \text{Spec} \, A, P = \text{Spec} \, S, \) and \( X = \text{Spec} \, R \) where \( R = S/I \), and \( I \) is generated by a quasi-regular sequence \( t = (t_1, \ldots, t_d) \) in \( S \), and there is an étale map \( A[T_1, \ldots, T_d, V_1, \ldots, V_k] \to S, \) (where \( T_1, l = 1, \ldots, d, \) and \( V_k, k = 1, \ldots, e \) are algebraically independent variables), and \( t_i \) is the image of \( T_i \) for \( i = 1, \ldots, d \). This can always be achieved by shrinking \( Y \) and \( P \) (see [BLR pp. 39–40, Prop. 7(c)]). Now every \( \mu \in \omega_{S/A} \) can be written uniquely as

\[
\mu = dt_1 \wedge \cdots \wedge dt_d \wedge \nu
\]

(8.2.3)

with \( \nu \in \wedge^e S\Omega^1_{S/A} \). Define

\[
b_{X/P} : i^*\omega_{\pi} \otimes_{\mathcal{O}_X} \mathcal{N}^d_i \xrightarrow{\sim} \omega_f
\]

(8.2.4)

by the formula

\[
\mu \otimes 1/t \mapsto i^*\nu
\]

where \( \mu \) and \( \nu \) are related by (8.2.3). We should clarify that \( i^*\nu \in \omega_{R/A} \) is the pull-back of \( \nu \) as a differential form. In other words, \( i^*\nu \) is the image of \( \nu \) under the composite of maps \( \wedge^e S\Omega^1_{S/A} \to R \otimes S \wedge^e S\Omega^1_{S/A} \to \wedge^e R\Omega^1_{R/A} = \omega_{R/A} \).

In what follows, let

\[
\bar{a}_{X/P} : i^*\omega_{\pi} \otimes_{\mathcal{O}_X} \mathcal{N}^d_i \to \omega_f
\]

(8.2.5)
be the map
\[ \tilde{a}_{X/P} = H^0(a_{X/P}). \]

The notation follows the conventions we have been using throughout, and as observed earlier, \( a_{X/P} \) can be recovered from \( \tilde{a}_{X/P} \) (see [NS1 §§ 8.1]).

**Theorem 8.2.6.** Under the above assumptions on \( i, \pi, f, A, S, \) and \( R \), we have
\[ \tilde{a}_{X/P} = b_{X/P}. \]

**Proof.** Since we will be using results from [NS1] which have been stated in terms of the abstract dualizing sheaves of the form \( \omega_f^* \), rather than in terms of \( \omega_f \), it is convenient for us to have an analogue of \( a_{X/P} \) taking values in \( \omega_f^*[e] \). To that end, suppose \( k: W \to P \) is a regular immersion of codimension \( m \leq n \), and that \( g = \pi \circ k \) is flat over \( Y \), so that \( g: W \to Y \) is Cohen-Macaulay of relative dimension \( n - m \). Define
\[ a_w^*: k^* \omega[n] \otimes_{\mathcal{O}_W} \mathcal{M}_k^m[-m] \to \omega_g[n - m] \]
as the composite:
\[ k^* \omega[n] \otimes_{\mathcal{O}_W} \mathcal{M}_k^m[-m] \xrightarrow{\eta} k^* \omega[n] \xrightarrow{\nu} k^* \omega[n] = k^! \omega^*[n] \]
\[ = g^! \omega = \omega_g[n - m] \]

(Similarly we have an isomorphism \( a_{w/Y}^* \) for a regular immersion \( W \to X \) of \( Y \)-schemes such that \( W \to Y \) is flat.) If \( W \to Y \) is finite (in addition to being flat), so that \( m = n \), and \( W \) is given by the vanishing of \( v = (v_1, \ldots, v_n) \), then by the definition of the map \( \tau_{g^* \pi^*} \) in [NS1 (5.3.2)], we have:

\[ \text{tr}^g \circ h_\pi(a_{w/X}^*) = \tau_{g^* \pi^*}. \]

By definition of \( a_{X/P}^* \), it is clear that \( \nu_0 \circ a_{X/P}^* = a_w^* \).

We first prove that the map \( \tilde{a}_{X/P} \) is compatible with base change. In greater detail, suppose \( w: Y' \to Y \) is a map, \( Y' := X \times Y \), \( P' := P \times Y' \), and let \( f': X' \to Y' \), \( \pi': P' \to Y' \), \( t': X' \to P' \), \( w': X' \to X \) be the resulting maps obtained from base change. We then clearly have \( \nu(i^* \omega \otimes \mathcal{M}_X^m) = i^* \omega \otimes \mathcal{M}_X^m \) and \( v^* \omega_f = \omega_f'. \) We claim that \( v^* \tilde{a}_{X/P} = \tilde{a}_{X'/P} \). In what follows, \( Y' = \text{Spec} \, A' \), \( R' = R \otimes_A A' \), \( S' = S \otimes_A A' \).

Consider the base change isomorphism \( \theta = \theta_{u'}: v^* \omega_f' \to v^* \omega_f \), of part (a) of [S2, p.740, Theorem 2.3.5 (a)]. According to loc.cit. (b) we have a commutative diagram

\[ \begin{array}{ccc}
  v^* \omega_f & \xrightarrow{\sim} & v^* \omega_f' \\
  \downarrow \theta & & \downarrow \theta' \\
  \omega_f & \xrightarrow{\sim} & \omega_f'
\end{array} \]

Since, \( \nu_0 \circ a_{X/P}^* = a_w^* \), from the above diagram we see that it is enough to show that \( \theta \circ v^* a_{X/P}^* = a_w^* \), in order to show that \( v^* a_{X/P} = a_{X'/P} \).

Let \( \eta_1 \) and \( \eta_1' \) be the maps defined in (C.2.11) and (C.2.13) of [NS1]. For the next few lines, all labels of the form (B.x.y) or (C.x.y) refer to the labels in [NS1].
Appendix. By definition, \( \eta'_i = (B.1.2) \circ \eta_i \). It follows that \( \bar{a}^i_{X/p} \) is the composite of isomorphisms:

\[
i^* \omega_\pi \otimes_{\mathcal{O}_X} \mathcal{N}^d_i \xrightarrow{(C.2.7)} \mathcal{E}xt^d_{\mathcal{O}_p}(\mathcal{O}_X, \omega_\pi) \xrightarrow{(B.1.2)} H^0(i^* \omega_\pi^n[n]) \xrightarrow{\sim} \omega_f^i.
\]

Let the composite of the last two maps in the above composition be denoted \( c_{X/p} : \mathcal{E}xt^d_{\mathcal{O}_p}(\mathcal{O}_X, \omega_\pi) \xrightarrow{\sim} \omega_f^i \). Consider the diagram

\[
\begin{array}{ccc}
\vspace{3cm}
\end{array}
\]

where the isomorphism in the middle is the natural one, which we now describe. Let \( Q^\bullet \to R \) be a projective resolution of the \( S \)-module \( R \). Then \( Q^\bullet \otimes_R R' = Q^\bullet \otimes_A A' \to R \otimes_A A' = R' \) is an \( S' \)-projective resolution of the \( S' \)-module \( R' \). Now

\[
\text{Hom}_S^\bullet(Q^\bullet, \omega_{S/[d]}(d) \otimes_A A') = \text{Hom}_S^\bullet(Q^\bullet \otimes_A A', \omega_{S/[d]}(d) \otimes_A A') = \text{Hom}_S^\bullet(Q^\bullet \otimes_A A', \omega_{S/[d]}(d)).
\]

Since \( t \) is a quasi-regular sequence in \( S \), we can (and will) pick \( Q^\bullet \) to be the version of the Koszul homology complex on \( t \) such that \( \text{Hom}_S^\bullet(Q^\bullet, S) = K^\bullet(t) \), and the equality \( \text{Hom}_S^\bullet(Q^\bullet, \omega_{S/[d]}(d) \otimes_A A') = \text{Hom}_S^\bullet(Q^\bullet \otimes_A A', \omega_{S/[d]}(d)) \) reduces to the well-known equality \( \omega_{S/[d]}(d) \otimes_S K^\bullet(t) \otimes_S S' = \omega_{S/[d]}(d) \otimes_S K^\bullet(t') \), where \( t' = t \otimes 1 \).

By right-exactness of tensor products, we get:

\[
H^0(\omega_{S/[d]}(d) \otimes_S K^\bullet(t)) \otimes_S S' = H^0(\omega_{S/[d]}(d) \otimes_S K^\bullet(t) \otimes_S S')
\]

The isomorphism \( v^* \mathcal{E}xt^d_{\mathcal{O}_p}(\mathcal{O}_X, \omega_\pi) \xrightarrow{\sim} \mathcal{E}xt^d_{\mathcal{O}_p}(\mathcal{O}_X', \omega_\pi') \) then follows from the isomorphism in [NS1] (C.2.3)]. (See also the proof of Lemma 1 of [L1] pp.39–40 as well as [S2] p.762, (8.9)) for the case when \( X \to P \) is not necessarily a regular immersion, but \( R \) is relatively Cohen-Macaulay over \( A \). The description of the isomorphism we have given also shows that the rectangle on the left in diagram \((1)\) above commutes. The rectangle on the right commutes by [S2] p.741, Theorem 2.3.6]. Thus diagram \((1)\) commutes and \( \delta^q_{u} \circ v^* \tilde{a}^q_{X/p} = \tilde{a}^q_{X'/p'}, \) i.e., \( v^* \delta_{X/p} = \delta_{X'/p'} \).

Next suppose we have a closed subscheme \( j : Z \hookrightarrow X \) such that (a) \( h = f \circ j : Z \to Y \) is an isomorphism, and (b), if \( L \subset R \) is the ideal of \( R \) which defines \( Z \), then \( L \) is generated by a quasi-regular sequence \( \bar{u} = (\bar{u}_1, \ldots, \bar{u}_e) \). Let \( u_i \in S \) be lifts of \( \bar{u}_i \in R \) for \( i = 1, \ldots, e \). Let \( L \) be the ideal generated by \( (t, u) \). Then \( L \) is the ideal defining the closed immersion \( i : Z \to P \). Let \( B = \Gamma(Z, \mathcal{O}_Z) \). For a sequence of positive integers \( m = (m_1, \ldots, m_e) \), let \( u^m = (u_1^{m_1}, \ldots, u_e^{m_e}) \), \( \bar{u}^m = (\bar{u}_1^{m_1}, \ldots, \bar{u}_e^{m_e}) \), \( L^m \) the \( R \)-ideal generated by \( \bar{u}^m \), \( B_m = R/L^m \), \( Z_m = \text{Spec} B_m \) and \( j_m : Z_m \to X \) the natural closed immersion. Then \( h_m : Z_m \to Y \) be the finite flat map \( h_m = f \circ j_m \).

Finally let \( k : \mathcal{A} \to X \) be the completion of \( X \) along \( Z \).

For \( \mu \in \omega_{S/A} \), and positive integers \( m_i \), \( i = 1, \ldots, e \), it is easy to see that

\[
(\star) \quad \text{res}_{Z,P} \left[ t_1, \ldots, t_d, u_1^{m_1}, \ldots, u_e^{m_e} \right] = \text{res}_{Z,F} \left[ \frac{h_m}{t} \right].
\]
Indeed, we can write $\mu$ in a unique manner as $\mu = f dt_1 \wedge \cdots \wedge dt_d \wedge du_1 \wedge \cdots \wedge du_e$, with $f \in S$. Then $b_{X/P}(\mu \otimes 1/t) = \bar{f} \wedge d\bar{u}_1 \wedge \cdots \wedge d\bar{u}_e$, where $\bar{f}$ is the image of $f$ in $R$. Both sides of $(\star)$ are then realised as the coefficient of $u_1^{m_1-1}u_2^{m_2-1} \cdots u_e^{m_e-1}$ in the power series expansion of $\bar{f}$, whence $(\star)$ holds. On the other hand, according to [NSI Prop. C.6.6],

$$a^g_{z/m/P}(\mu \otimes 1/t, u^m) = \bar{a}^g_{z/m/X}(\bar{a}_{X/P}(\mu \otimes 1/t)) \otimes 1/\bar{u}^m).$$

Apply $\text{tr}^\#_{h_m} \circ h_{m*}$ to both sides. By $(\dagger)$ and [NSI Prop. 5.4.4], this yields,

$$(**) \quad \text{res}_{Z, \pi}\left[ \frac{\mu}{t_1, \ldots, t_d, u_1^{m_1}, \ldots, u_e^{m_e}} \right] = \text{res}_{Z, f}\left[ \frac{\bar{a}_{X/P}(\mu \otimes 1/t)}{\bar{u}_1^{m_1}, \ldots, \bar{u}_e^{m_e}} \right].$$

From $(\star)$ and $(**)$ we conclude that $\text{res}_{Z}\left[ \frac{\bar{a}_{X/P}(\mu \otimes 1/t)}{\bar{a}^m_{\bar{u}}/\bar{u}} \right] = \text{res}_{Z}\left[ \frac{b_{X/P}(\mu \otimes 1/t)}{a^m_{\bar{u}}/\bar{u}} \right]$. Now apply local duality, i.e. [NSI Cor. 5.1.4], to conclude that $\kappa^* \bar{a}_{X/P} = \kappa^* b_{X/P}$. This means that on a Zariski open neighbourhood of $Z$, $a_{X/P} = b_{X/P}$. We point out that the hypothesis that $Z$ be defined globally by the vanishing of a quasi-regular sequence is not necessary to reach this conclusion, since $j$ is a regular immersion and locally, one can arrange this. In other words, if we have a section of $f$, then in an open neighbourhood $U$ of the image of the section, $a_{X/P}|U = b_{X/P}|U$.

In the general case, let $X'' = X \times_Y X, P'' = P \times_Y X$, and consider the cartesian square

$$\begin{array}{ccc}
\begin{array}{c}
X'' \xrightarrow{p_2} X \\
\downarrow \quad \quad \downarrow f \\
X \xrightarrow{f} Y
\end{array}
\end{array}$$

We know that $p_2^* a_{X/P} = a_{X''/P''}$. It is clear from the description of $b_{X/P}$ that it is compatible with arbitrary base change and hence $p_2^* b_{X/P} = b_{X''/P''}$. Then by what we have proven, there is a Zariski open subscheme $V$ of $X''$ containing the diagonal such that $p_2^* a_{X/P}|V = p_2^* b_{X/P}|V$.

Let $\Delta : X \hookrightarrow V$ be the map induced by the diagonal immersion $X \hookrightarrow X''$. Applying $\Delta^*$ to both sides of the displayed equation above, we see that $a_{X/P} = b_{X/P}$. □

**Corollary 8.2.7.** Let $\nu = (v_1, \ldots, v_e) \in \Gamma(P, \mathcal{O}_P) = S, J$ the ideal in $S$ generated by $(t, \nu)$, $Z = \text{Spec} S/J$, and $\nu'|\nu$ the restriction of $v_i$ to $Z$ for $i = 1, \ldots, e$. If $Z \to Y$ is finite and flat, then

$$\text{res}_{Z, \pi}\left[ \frac{dt_1 \wedge \cdots \wedge dt_d \wedge \nu}{t_1, \ldots, t_d, v_1, \ldots, v_e} \right] = \text{res}_{Z, f}\left[ \frac{i^* \nu|\nu'}{v_1', \ldots, v_e'} \right].$$

for $\nu \in \wedge^e \Omega^1_{S/A}$.

**Proof.** Theorem 8.2.6 together with [NSI Prop. C.6.6] yields

$$\bar{a}^g_{z/P}(dt_1 \wedge \cdots \wedge dt_d \wedge \nu \otimes 1/(t, \nu)) = \bar{a}^g_{z/X}(i^* \nu \otimes 1/\nu')$$

where $\bar{a}^g_{z/P}$ and $\bar{a}^g_{z/X}$ are as in the proof of Theorem 8.2.6 and $\nu'$ is $(v_1', \ldots, v_e')$. Let $h : Z \to Y$ be the composite $Z \to X \xrightarrow{f} Y$. Applying $\text{tr}^\#_{h \circ h_*}$ to both sides, we get the result. (See [NSI Prop. 5.4.4].) □
8.2.8. Quasi-finite maps. Suppose the map $\pi: P \to Y$ in [8.2.1] factors as $P \xrightarrow{\pi} W \xrightarrow{g} Y$, with $p$ smooth of relative dimension $d$, and $g$ smooth of relative dimension $e$, and assume $h = p \circ \pi$ is quasi-finite. In other words we have a commutative diagram of ordinary schemes

$$
\begin{array}{ccc}
X & \xrightarrow{i} & P \\
\downarrow \hspace{0.5cm} h & & \downarrow \pi \\
W & \xrightarrow{p} & Y
\end{array}
$$

with $h$ quasi-finite and $f = g \circ h = p \circ i$, and with $p$, $\pi$, $g$ and $f$ smooth of relative dimensions $d$, $d + e$, $e$ and $e$, respectively. To lighten notation, we write

$${\mathcal N} = N^d_i = (\wedge^d \mathcal{O}_X / \mathcal{I}^2)$$

where $\mathcal{I}$ is the quasi-coherent ideal sheaf in $\mathcal{O}_P$ defining $i: X \hookrightarrow P$.

Since $h$ is quasi-finite and flat over $W$ (the latter because $p$ is smooth, and $i$ is a local complete intersection map), for quasi-coherent $\mathcal{O}_W$-module $\mathcal{F}$, $h^! \mathcal{F}$ can be identified with $H^0(h^! \mathcal{F})$ in the standard way, and we will do so in what follows. With this convention, we have three isomorphisms which we now describe. First, we clearly have

(8.2.8.1) $\quad h^! \omega_g \xrightarrow{\sim} \omega_f$ via the isomorphism $h^! g^! \xrightarrow{\sim} f^!$, and Verdier’s isomorphisms for $f$ and $g$.

Next, for a quasi-coherent $\mathcal{O}_W$-module $\mathcal{F}$, we have the transitivity isomorphism (8.2.8.2) $\chi^h(\mathcal{F}, \mathcal{O}_W): h^* \mathcal{F} \otimes_{\mathcal{O}_X} h^! \mathcal{O}_W \xrightarrow{\sim} h^! \mathcal{F}$ of [NS1 (7.2.1)]. Since we are dealing with ordinary schemes, taking account of our choice of order of tensor product, this is the same as the map $\chi^h_{\mathcal{F}, \mathcal{O}_W}$ of [L4 p. 231, (4.9.1.1)]. The map $\chi^h(\mathcal{F}, \mathcal{O}_W)$ is an isomorphism since $h$ is flat and hence perfect [L4 pp. 234–235, Thm. 4.9.4].

Finally, we have an isomorphism (8.2.8.3) $\quad i^* \omega_p \otimes \mathcal{N} \xrightarrow{\sim} h^! \mathcal{O}_W$ given by $\eta_i(\omega_p[d]): i^*(\omega_p[d]) \xrightarrow{\sim} i^! \omega_p$ of [NS1 (C.2.13)], the isomorphism $i^! p^! \xrightarrow{\sim} h^!$, and Verdier’s isomorphism $\mathbf{v}: \omega_p[d] \xrightarrow{\sim} p^! \mathcal{O}_W$.

These three isomorphisms are related in the following way.

**Proposition 8.2.8.4.** The following diagram of isomorphisms commutes

$$
\begin{array}{ccc}
h^* \omega_g \otimes_{\mathcal{O}_X} (i^* \omega_p \otimes_{\mathcal{O}_X} \mathcal{N}) & \xrightarrow{\varphi_{g,p}} & i^* \omega_g \otimes_{\mathcal{O}_X} \mathcal{N} \\
\downarrow \hspace{0.5cm} \mathbf{v} & & \downarrow \mathbf{v} \\
\mathcal{H}^* \omega_g \otimes_{\mathcal{O}_X} \mathcal{H}^0 \mathcal{O}_W (i^* \mathcal{N}) & \xrightarrow{\sim} & \mathcal{H}^0 \mathcal{O}_W \mathcal{H}^1 \omega_g
\end{array}
$$

where $\varphi_{g,p}$ is the explicit map described in [7.2.1] and $\mathbf{v}_{X/P}$ is the map [8.2.5] described locally, via Theorem 8.2.6, by the explicit map $b_{X/P}$ in [8.2.4].
Proof. The essential point is that other than (8.2.8.1), all other maps in the diagram are various avatars of transitivity maps. The map η′ which is used in the definition of (8.2.8.3) is
\[ \chi^i(-, \mathcal{O}_P): L \mu^* (-) \otimes_{\mathcal{O}_X} \mathcal{O}_P \to \mathcal{O}_P \]
with \( \mathcal{N}[-d] \) substituted for \( i! \mathcal{O}_P \), via the canonical isomorphism
\[ \eta'(\mathcal{O}_P): \mathcal{N}[-d] \to i! \mathcal{O}_P \]
(see also [NS1, C.2.14.1] for another way of looking at this).

Next, according to Theorem 7.2.4, and the definition in (7.1.1), the map \( \tilde{\varphi}_{g,p} \) is \( H^{d+e} \) of the composite (after substituting \( g^! \mathcal{O}_Y, \pi^! \mathcal{O}_Y, p^! \mathcal{O}_W \) with \( \omega_g[e], \omega_\pi[d+e] \), and \( \omega_p[d] \) respectively, via Verdier’s isomorphisms):
\[ p^* g^! \mathcal{O}_Y \otimes_{\mathcal{O}_P} p^! \mathcal{O}_W \to p^! g^! \mathcal{O}_Y \to \pi^! \mathcal{O}_Y, \]
where the first arrow is the transitivity map \( \chi^p(g^! \mathcal{O}_Y, \mathcal{O}_W) \), which is an isomorphism since \( p \) is flat and hence perfect.

The map \( \bar{a}_{X/P} \) is, according to (8.2.2) and (8.2.2) (after the usual Verdier substitutions and the substitution \( \mathcal{N}[-d] \to i! \mathcal{O}_P \), \( H^{d+e} \) applied to the composite
\[ L i^* \pi^! \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_P \]
where the first arrow is the transitivity map \( \chi^i(\pi^! \mathcal{O}_Y, \mathcal{O}_P) \). This is an isomorphism since a regular immersion is a perfect map.

Finally, (8.2.8.1) is by definition \( \chi^h(\omega_\pi, \mathcal{O}_W) \).

Consider the diagram below, in which the arrows are either natural ones arising from the pseudofunctorial nature of \( -^! \) or from abstract transitivity maps, and in which:
\[ i^* = L i^* \text{ and } \otimes = \otimes. \]

The diagram commutes by [NS1, Prop.-Def. 7.2.4 (ii)] and [L4] p. 238]. The Proposition follows. □

9. Traces of differential forms for finite maps

9.1. Tate traces. Let \( A \) be a ring, and \( C \) an \( A \)-algebra which is finite and free as an \( A \)-module. We have the canonical trace
\[ \text{Tr}_{C/A}: C \to A \]
given by the composite
\[ C \rightarrow \text{End}_A(C, C) \rightarrow A \]
where the first arrow is the map \( c \mapsto (x \mapsto cx) \) and the second the standard trace of an endomorphism of a finite free \( A \)-module.

If the \( C \)-module \( \text{Hom}_A(C, A) \) is a free \( C \)-module of rank one (this happens if and only if, in addition to \( C \) being a finite free \( A \)-module, its fibres are Gorenstein) then, following Kunz in [Ku], we regard any free generator of \( \text{Hom}_A(C, A) \) as a “trace” for the \( A \)-algebra \( C \) (cf. [Ku, F8 (b), pp. 362–363]). If there is one, then clearly we have exactly as many as the units of \( C \). We point out that the canonical trace, \( \text{Tr}_{C/A} \), need not be a trace in this sense on \( C \). Indeed if \( A \) and \( C \) are fields and \( C \) is a purely inseparable extension of \( A \), then \( \text{Tr}_{C/A} = 0 \) and hence cannot be a free generator of \( \text{Hom}_A(C, A) \).

Tate studies the existence and characterisation of traces in an important situation which includes the case of \( C \) being a complete intersection algebra over \( A \).

In the rest of this sub-section we make the following assumptions and use the following notations. The \( A \)-algebra \( C \) (which is free of finite rank as an \( A \)-module) is such that the canonical map \( A \rightarrow C \) factors as
\[ A \rightarrow B \xrightarrow{\pi} C \]
with \( \pi \) a surjective map, the kernel \( I \) of \( \pi \) generated by a regular \( B \)-sequence \( f = (f_1, \ldots, f_n) \), and the kernel \( J \) of the canonical map
\[ s: B \otimes_A C \rightarrow C \]
is generated by a \( B \otimes_A C \)-sequence \( g = (g_1, \ldots, g_n) \). In somewhat greater detail, if \( m: C \otimes_A C \rightarrow C \) is the \( A \)-algebra map \( c \otimes c' \mapsto cc' \), then \( s \) is the composition
\[ B \otimes_A C \xrightarrow{\pi \otimes 1_C} C \otimes_A C \xrightarrow{m} C. \]
Note that \( f_i \otimes 1 \in J \) and hence we have \( h_{ij} \in B \otimes_A C \) such that \( f_i \otimes 1 = \sum_{j=1}^n h_{ij} g_j \) for \( i = 1, \ldots, n \). Let
\[ (9.1.2) \quad \Delta = \det (h_{ij}), \]
and
\[ (9.1.3) \quad \overline{\Delta} = (\pi \otimes 1_C)(\Delta). \]
Set \( \overline{J} = \ker m = (\pi \otimes 1_C)(J) \). We have the following commutative diagram with \( \overline{s}: C \otimes_A B \rightarrow C \)
being the composite \( m \circ (1_C \otimes \pi) \).

\[ \begin{array}{ccccccccc}
C & \xrightarrow{\pi} & C \otimes_A B & \xrightarrow{1_C \otimes \pi} & C \otimes C & \xrightarrow{s} & C \\
\pi & & \pi \otimes 1_B & & \pi \otimes 1_C & & \\
B & \xrightarrow{\pi \otimes 1_B} & B \otimes_A B & \xrightarrow{1_B \otimes \pi} & B \otimes_A C & & \\
A & \xrightarrow{\pi} & B & \xrightarrow{\pi \otimes 1_C} & C & & \\
\end{array} \]
In the above situation it is shown in [MR, Appendix] that traces exist (i.e., \( \text{Hom}_A(C, A) \) is a rank one free \( C \)-module) and there is a canonical free generator (i.e., a trace) \( \lambda = \lambda(f, g) \) of \( \text{Hom}_A(C, A) \). We summarise the results of Tate as given in [MR, Appendix] in the following two theorems in which we make the standard identifications \( B \otimes_A \text{Hom}_A(C, A) = \text{Hom}_B(B \otimes_A C, B) \) and \( C \otimes_A \text{Hom}_A(C, A) = \text{Hom}_C(C \otimes_A C, C) \). Under these identifications it is clear that

\[
\pi \circ (1_B \otimes \phi) = (1_C \otimes \phi) \circ (\pi \otimes 1_C) \quad (\phi \in \text{Hom}_A(C, A)).
\]

**Theorem 9.1.6.** (Tate) [MR, p.231, Lemma (A.10)] The map

\[ t : \text{Hom}_A(C, A) \longrightarrow C \]

given by

\[ \phi \mapsto \pi((1_B \otimes \phi)(\Delta)) = (1_C \otimes \phi)(\overline{\Delta}) \]

is an isomorphism of \( C \)-modules.

In loc.cit. the description of \( t \) is \( \phi \mapsto \pi((1_B \otimes \phi)(\Delta)) \). Using (9.1.5) it is clear that \( t \) can also be described as \( \phi \mapsto (1_C \otimes \phi)(\Delta) \).

The results in [MR, Appendix] are perhaps more useful when stated in the following way.

**Theorem 9.1.7.** (Tate) Let \( \lambda = \lambda(f, g) \) be the free \( C \)-module generator of \( \text{Hom}_A(C, A) \) given by

\[ \lambda = t^{-1}(1) \]

where \( t : \text{Hom}_A(C, A) \rightarrow C \) is the isomorphism in Theorem 9.1.6.

(a) If \( \phi \in \text{Hom}_A(C, A) \), then the constant of proportionality \( c \in C \) such that \( \phi = c\lambda \) is given by

\[ c = \pi((1_B \otimes \phi)(\Delta)) = (1_C \otimes \phi)(\overline{\Delta}). \]

(b) If \( \psi \in \text{Hom}_B(B \otimes_A C, B) \) and \( \phi \in \text{Hom}_A(C, A) \) are such that \( 1_B \otimes \phi - \psi \in J \text{Hom}_B(B \otimes_A C, B) \), then

\[ \pi((1_B \otimes \phi)(\Delta)) = \pi\psi(\Delta). \]

(c) If \( \psi \in \text{Hom}_C(C \otimes_A C, C) \) and \( \phi \in \text{Hom}_A(C, A) \) are such that \( 1_C \otimes \phi - \psi \in J \text{Hom}_C(C \otimes_A C, C) \), then

\[ (1_C \otimes \phi)(\overline{\Delta}) = \psi(\overline{\Delta}). \]

(d) If \( \text{Tr}_{C/A} : C \rightarrow A \) is the canonical trace given in (9.1.1), then

\[ \text{Tr}_{C/A} = m(\overline{\Delta})\lambda. \]

**Proof.** These are all results in [MR, Appendix], stated in perhaps a different way. Part (a) is [ibid, pp. 229–230, 3. of Theorem (A.3)] (together with (9.1.5)). Part (b) is an immediate consequence of [ibid, p. 230, Lemma (A.9)] and (c) is the same, together with (9.1.5). Part (d) is [ibid, pp. 229–230, 4. of Theorem (A.3)].

The first application of Tate’s result we give is the following (this is (R6) of [RD, p.198] but for our version of residues).

**Theorem 9.1.8.** In the above situation, suppose \( B \) is smooth of relative dimension \( n \) over \( A \), \( f : X \rightarrow Y \) the corresponding smooth map from \( X = \text{Spec} B \) to \( Y = \text{Spec} A \), and \( Z = \text{Spec} C \). Then

\[ \text{res}_{Z,f} \left[ b df_1 \wedge \cdots \wedge df_n \right] = \text{Tr}_{C/A}(b|_Z). \]
Proof. It is important to keep diagram \([9.1.4]\) in mind when following this proof. There is an annoying issue that \(\Delta\) is defined in terms of \(f_i \otimes 1_C\) and \(g_i\), but in dealing with the base change \(1_C \otimes \phi\), for \(\phi \in \text{Hom}_A(C, A)\), the natural elements that show up are \(1_C \otimes f_i \in C \otimes_A B\). One has to do somewhat careful book-keeping to avoid confusion. Since \(C \otimes_A B\) and \(B \otimes_A C\) play different roles, let us agree to write \(x\) for the element of \(C \otimes_A B\) corresponding to \(x \in B \otimes_A C\) under the standard isomorphism between \(B \otimes_A C\) and \(C \otimes_A B\).

In what follows, the \(C\)-algebra structures on \(C \otimes_A B\) and \(C \otimes_A C\) are \(c \mapsto c \otimes 1_B\) and \(c \mapsto c \otimes 1_C\) respectively. Let \(N = (I/I^2)^*\) and \(N_C = C \otimes_A N\). Let \(h : Z \to Y\) be the natural finite flat map corresponding to \(A \to C\) and \(i : Z \hookrightarrow X\) the natural closed immersion, with normal bundle \(\mathcal{N}\). If \(\tau_{C/A} : \Omega^n_{B/A} \otimes^C N \to A\) is the map arising from the composite (all isomorphisms being the obvious ones, e.g., the fundamental local isomorphism, Verdier’s isomorphism, \ldots)

\[
h_* (i^*(\Omega^n_{X/Y}) \otimes_Z \mathcal{N}) \stackrel{\sim}{\to} H^0(h_* (i^! f_1 \mathcal{O}_Y)) \stackrel{\sim}{\to} H^0(h_* h^! \mathcal{O}_Y) \xrightarrow{H^0(\tau_{B/A})} \mathcal{O}_Y
\]

then \((\Omega^n_{B/A} \otimes B \wedge^n N, \tau_{C/A})\) represents the functor \(M \mapsto \text{Hom}_A(M, A)\) from finite \(C\)-modules to finite \(A\)-modules, whence we have an isomorphism of \(C\)-modules

\[
\Phi : \Omega^n_{B/A} \otimes^C \wedge^n N \stackrel{\sim}{\to} \text{Hom}_A(C, A)
\]

with \(\tau_{C/A}\) corresponding to “evaluation at 1” under this isomorphism. According to [NS1, Prop. 5.4.4], we have

\[
\tau_{C/A}(\mu \otimes 1/f) = \text{res}_Z \left[ \frac{\mu}{f_1, \ldots, f_n} \right] \quad (\mu \in \Omega^n_{B/A}).
\]

Thus

\[
\Phi(\mu \otimes 1/f)(c) = \text{res}_Z \left[ \frac{b \cdot \mu}{f_1, \ldots, f_n} \right] \quad (c \in C)
\]

where \(b \in B\) is any pre-image of \(c\). If \(b \in I\), then \(b \partial f_1 \wedge \cdots \wedge \partial f_n \otimes 1/f = 0\) in \(\Omega^n_{B/A} \otimes B \wedge^n N\) and hence the right side of the above displayed formula is well-defined as a function of \(c \in C\).

Similarly we have an isomorphism of \(C \otimes_A C\) modules

\[
\Phi' : \Omega^n_{(C \otimes_A B)/C} \otimes \wedge^n N_C \stackrel{\sim}{\to} \text{Hom}_C(C \otimes_A C, C)
\]

given by

\[
\Phi'(\nu \otimes 1/(1_C \otimes f))(x) = \text{res}_{Z \times_Y Z_p} \left[ \frac{\bar{x} \cdot \mu}{(1_C \otimes f_1), \ldots, (1_C \otimes f_n)} \right] \quad (x \in C \otimes_A C)
\]

where \(\bar{x} \in C \otimes_A B\) is any pre-image of \(x\) and \(p : Z \times_Y X \to Z\) is the natural projection.

Let \(s' : C \otimes_A B \to C\) be as in \([9.1.4]\), i.e., \(s' = m \circ (1_C \otimes \pi)\). Then \(J' := \ker s'\) is generated by \(g_1^\circ, \ldots, g_n^\circ\).

Let \(\phi \in \text{Hom}_A(C, A)\) and \(\psi \in \text{Hom}_C(C \otimes_A C, C)\) be the maps defined by

\[
\phi = \Phi(\partial f_1 \wedge \cdots \wedge \partial f_n \otimes 1/f),
\]

and

\[
\psi = \Phi'(\Delta' \cdot (\partial g_1^\circ \wedge \cdots \wedge \partial g_n^\circ) \otimes 1/(1_C \otimes f)).
\]

We have to show that \(\phi = \text{Tr}_{C/A}\). By Theorem \([9.1.4](d)\), this is equivalent to showing that \((1_C \otimes \phi)(\Delta) = m(\Delta)\). It is easier to show that \(\psi(\Delta) = m(\Delta)\), and we can reduce to this via Theorem \([9.1.4](c)\). The details are as follows. First, we claim
that \(1 \otimes \phi - \psi \in \mathcal{J}\text{Hom}_C(C \otimes_A C, C)\) so that Theorem \[9.1.7\](c) applies. Before we prove the claim, we point out that

\[
1 \otimes \phi = \Phi'(\mu((d(1 \otimes f_1) \wedge \cdots \wedge d1_C \otimes f_n)) \otimes 1/(1 \otimes f)).
\]

Since \(1 \otimes f_i = \sum_j h_{ij} g^\nu_j\) we have

\[
d(1 \otimes f_1) \wedge \cdots \wedge d(1 \otimes f_n) = \mu + \Delta^\nu g^\nu_1 \wedge \cdots \wedge g^\nu_n
\]

where \(\mu \in J^\nu \Omega^n_{(C \otimes_A B)/C}\) (for \(h_{ij} \in J^\nu\)). It follows that

\[
1 \otimes \phi - \psi = \Phi'(\mu(1/(1 \otimes f))) \in \mathcal{J}\text{Hom}_C(C \otimes_A C, C)
\]
as claimed.

We then have, with \(\delta \in C \otimes_A B\) a lift of \(\Delta \in C \otimes_A C\),

\[
(1 \otimes \phi)(\Delta) = \psi(\Delta) = \text{res}_{Z \times_Y Z, p} \left[ \delta \Delta^\nu g^\nu_1 \wedge \cdots \wedge g^\nu_n \right]
\]

\[
= \text{res}_{Z, p} \left[ \delta g^\nu_1 \wedge \cdots \wedge g^\nu_n \right]
\]

\[
= s^\nu(\delta)
\]

\[
= m(\Delta).
\]

In the above sequence, the first equality is from Theorem \[9.1.7\](c), the one in the second line from [NS1] Thm. 5.4.5, the third from the fact that the composite \(Z \xrightarrow{\psi^\nu} Z \xrightarrow{p} Z\) is an isomorphism, which means the formulae in Remark \[5.3.3\] apply. The last equality is from the definition of \(s^\nu\) as \(m \circ (1 \otimes \pi)\). From (\*) and Theorem \[9.1.7\](d) we get that \(\phi = \text{Trc}_{C/A}\), and from this the Theorem follows.

**Remarks 9.1.9.** 1) The above proof would be easier if one could show that \(\Delta^\nu\) is a pre-image of \(\Delta\) under \(C \otimes_A B \xrightarrow{1 \otimes \pi} C \otimes_A C\). But there is no guarantee it is so. However, in the special case where \(B\) is a polynomial ring over \(A\), something like this be arranged as the proof Proposition \[9.1.10\] below shows.

2) If \(B\) is flat over \(A\), then \(\lambda\) is stable under any base change of \(A\). In somewhat greater detail, if \(A \to A'\) is a map of rings, \(B', C', f', g'\) the obvious base changes of \(B, C, f, g\), then, under the identification \(\text{Hom}_A(C, A') = A' \otimes_A \text{Hom}_A(C, A)\), we have \(\lambda(f', g') = 1 \otimes \lambda(f, g)\). This is because, if \(B\) is flat over \(A\), then \(f'\) and \(g'\) are regular sequences.

**Proposition 9.1.10.** Let \(q \in A[T_1, \ldots, T_n] = A[T]\). Suppose \(B\) is the \(A\)-algebra \(B = A[T]/q\). For \(i = 1, \ldots, n\) let \(\gamma_i = \pi(T_i)\) and

\[
g_i = T_i \otimes 1_C - 1_B \otimes \gamma_i.
\]

Let \(Z = \text{Spec } C\). Then \(\lambda = \lambda(f, g) \in \text{Hom}_A(C, A)\) is given by

\[
\lambda(c) = \text{res}_Z \left[ \frac{b d T_1 \wedge \cdots \wedge d T_n}{f_1, \ldots, f_n} \right] \quad (c \in C)
\]

where \(b \in B = A[T_1, \ldots, T_n]\) is any pre-image of \(c\).

**Proof.** It is straightforward to see that the \(g_i\), as defined in the Proposition, generate \(J = \ker s\), and form a regular \(B \otimes_A C\)-sequence. As before, let \(X = \text{Spec } B\), \(Y = \text{Spec } A, Z = \text{Spec } C\), and let \(p: Z \times_Y X \to Z\) be the projection map. As we
did earlier, we need to distinguish between \( B \otimes_A C \) and \( C \otimes_A B \), and so between \( X \times_Y Z \) and \( Z \times_Y X \), and \( p \) corresponds to the map \( C \to C \otimes_A B \) given by \( c \mapsto c \otimes 1 \).

For the proof of the theorem, it is simpler to regard the two copies of \( B \) in Diagram (9.1.4), the one in the middle of the bottom row, and the one in the middle of the left column, as two different copies of \( A[T]_q \), say \( A[X_1, \ldots, X_n]_{q_1(q_X)} = A[X]_{q_1(q_Y)} \) and \( A[Y_1, \ldots, Y_n]_{q_2(q_Y)} = A[Y]_{q_2(q_Y)} \), respectively. Then \( B \otimes_A B \) can be regarded as \( A[X, Y]_{q_1(q_Y)} \). Moreover, \( B \otimes_A C \) is then identified with \( C[Y]_{q_1(q_Y)} \) and \( C \otimes_A B \) with \( C[X]_{q_1(q_X)} \). Diagram (9.1.4) translates to

\[
\begin{array}{cccc}
A & \rightarrow & A[X]_{q_1(q_X)} & \rightarrow & A[Y]_{q_1(q_Y)} & \rightarrow & C[Y]_{q_1(q_Y)} \\
\downarrow \pi_1 & & \downarrow \pi'_i & & \downarrow \pi_{i'} & & \downarrow \pi''_i & \downarrow m & \downarrow \pi''_i & \downarrow \pi_2 & \rightarrow & C
\end{array}
\]

Here \( \pi_1 \) is the map \( X_i \mapsto \gamma_i \), and \( \pi_2 \) is \( Y_i \mapsto \gamma_i \). The maps \( \pi'_i \) and \( \pi''_i \) are the base changes of \( \pi_1 \), and \( \pi'_2, \pi''_2 \) the base changes of \( \pi_2 \). We point out that

\[
\pi''_1 \left( \sum_i c_i X_i \right) = \sum_i \gamma_i \otimes c_i
\]

and

\[
\pi''_2 \left( \sum_i c_i Y_i \right) = \sum_i \gamma_i \otimes c_i.
\]

For any \( h \in B = A[T]_q \), the element \( h \otimes 1_B \) (resp. \( h \otimes 1_C \)) is identified with the element \( h(Y) \) of \( A[X, Y]_{q_1(q_Y)} \) (resp. the element \( h(X) \) of \( C[Y]_{q_1(q_Y)} = B \otimes_A C \)), whereas \( 1_B \otimes h \) and \( 1_C \otimes h \) are identified with \( h(X) \) (regarded as elements of \( A[X, Y]_{q_1(q_X)} \), and of \( C[X]_{q_1(q_X)} \) respectively). Finally \( s'(\sum \gamma_i Y_2) = \sum c_i \gamma_i \) and \( s'(\sum \gamma_i X_2) = \sum c_i \gamma_i \). It follows that

\[
g_i = Y_i - \gamma_i, \quad g_i' = X_i - \gamma_i \quad (i = 1, \ldots, n).
\]

Now there exist \( h_{ij}(X, Y) \in A[X, Y] \) such that

\[
f_i(X) - f_i(Y) = \sum_j h_{ij}(X, Y)(X_j - Y_j).
\]

Then \( f_i(Y) = \sum_j h_{ij}(\gamma, Y)(Y_j - \gamma_i) \) and \( f_i(X) = \sum_j h_{ij}(X, \gamma)(X_j - \gamma_j) \). Let \( \delta(X, Y) = \det(h_{ij}(X, Y)) \).

If \( \Delta \) is defined as in (9.1.2), then

\[
\Delta = \delta(\gamma, Y).
\]

Note that

\[
(*) \quad \Delta = \pi''_2(\Delta) = \pi''_2(\delta(X, \gamma)).
\]
On the other hand, since $f_i(X) = \sum_{i,j} h_{ij}(X,\gamma)(X_i - \gamma_i)$, according to [NSI, Thm. 5.4.5] we have

$$f_i(X, \gamma) = \sum_{i,j} g_{ij}(\gamma_i - \gamma_i),$$

where $\mu \in \Omega^n_{C[X]/C}$. 

Let $\phi: C \to A$ be defined by

$$\phi(c) = \text{res}_{Z \times Y} \left[ b dT_1 \wedge \cdots \wedge dT_n \right],$$

where $b \in B = A[T]$ is any element in $\pi^{-1}(c)$. Since $dX_i = d(X_i - \gamma_i)$, therefore for $x \in C \otimes A$ and $\tilde{x} \in C[X]$ such that $\pi''(\tilde{x}) = x$, we have

$$\text{(1)} \otimes \phi(x) = \text{res}_{Z \times Y} \left[ d\tilde{x} \wedge \cdots \wedge d(X_n - \gamma_n) \right].$$

By (*) and (**) we get

$$\text{(1)} \otimes \phi(\Delta) = \text{res}_{Z \times Y} \left[ d\tilde{g}_{1} \wedge \cdots \wedge d\tilde{g}_{n} \right] = 1.$$

Theorem 9.1.7 (a) then gives $\phi = \lambda$. 

9.2. Traces of differential forms. Suppose we have a commutative diagram of ordinary schemes

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{h} & & \downarrow{g} \\
Z & & \\
\end{array}
\]

with $f$ and $g$ smooth of relative dimension $n$ and $h$ a finite map (necessarily flat). The composite $h_* f^* \mathcal{O}_Z \to h_! h^! \mathcal{O}_Z \xrightarrow{\text{Tr}_h} g^* \mathcal{O}_Z$ gives a $\mathcal{O}_Y$-map map (after applying Verdier’s isomorphism to $f^* \mathcal{O}_Z$ and $g^* \mathcal{O}_Z$ and applying $H^{-n}(-)$)

$$\text{tr}_h: h_* \omega_f \to \omega_g.$$

A note of caution. We have used the symbol tr$_p$ earlier for the trace map $R^m p_* \omega_f \to \mathcal{O}_W$ for a smooth proper map $p: V \to W$ of relative dimension $m$. The context will make the meaning of the symbol clear.

**Proposition 9.2.3.** Let $W$ be a closed subscheme of $Y$, proper over $Z$, and let $W' = h^{-1}(W)$. Assume $g$ (and hence $f$) is separated. Then the following diagram commutes:

\[
\begin{array}{ccc}
R^n_{W,f_!} & \xrightarrow{\sim} & R^n_{W,g^*h_*}\omega_f \\
\text{res}_{W'} & & \text{via tr}_h \\
\mathcal{O}_Z & \xrightarrow{\text{res}} & R^n_{W,g^*}\omega_g
\end{array}
\]

**Proof.** By Nagata’s compactification [N] we have an open immersion $u: Y \to \overline{Y}$ together with a proper map $\overline{g}: \overline{Y} \to Z$ such that $\overline{g} \circ u = g$. By Zariski’s Main Theorem the quasi-finite map $u \circ h: X \to \overline{Y}$ can be completed to a finite map, i.e., we can find an open immersion $v: X \to \overline{X}$ and a finite map $\overline{h}: \overline{X} \to \overline{Y}$ such that
\[ u \circ h = \bar{h} \circ v. \] Moreover, we may assume \( X \) is scheme-theoretically dense in \( \bar{X} \) so that \( \bar{h}^{-1}(u(Y)) = v(X) \). Let \( \bar{f} = g \circ \bar{h} \). We have a composite

\[(\dag) \quad \bar{h}_* \bar{f}^! \xrightarrow{\sim} \bar{h}_* \bar{h}^! \bar{g}^! \xrightarrow{Tr_h} \bar{g}^!.\]

Consider the commutative diagram

\[ \begin{array}{ccc}
Rg_*R\Gamma_W^1\omega_\mathfrak{g}[n] & \xrightarrow{\sim} & R\bar{g}_*R\Gamma_{u(W)}\bar{g}^! \mathcal{O}_Z \\
\downarrow{\text{tr}_h} & & \downarrow{(i)} \\
Rg_*R\Gamma_W^1h_*\omega_f[n] & \xrightarrow{\sim} & R\bar{g}_*R\Gamma_{v(W')}h_*^! \mathcal{O}_Z \\
\downarrow{\text{tr}_f} & & \downarrow{(i)} \\
Rg_*R\Gamma_W^1\omega_f[n] & \xrightarrow{\sim} & R\bar{f}_*R\Gamma_{u(W')}^! \bar{h}_*^! \mathcal{O}_Z \\
\end{array} \]

The rectangle on the right commutes by definition of \((\dag)\) (especially of the isomorphism \( \bar{h}^! \bar{g}^! \xrightarrow{\sim} \bar{f}^! \)) which drives \((\dag)\).

Applying \( H^0(-) \) to the above diagram we get the asserted result. \(\square\)

**Proposition 9.2.4.** Let \( f, g, h \) be as above, and suppose \( u: Z' \to Z \) is a map of ordinary schemes. Let

\[ \begin{array}{ccc}
X' & \xrightarrow{w} & X \\
\downarrow{h'} & & \downarrow{h} \\
Y' & \xrightarrow{v} & Y \\
\downarrow{g'} & & \downarrow{g} \\
Z' & \xleftarrow{u} & Z
\end{array} \]

be the corresponding base change diagram. Then \( v^* \text{tr}_h = \text{tr}_{h'} \).

**Proof.** By [EGA IV, (13.3.2)], \( Y \) can be covered by open subschemes \( U \) such that \( U \to Y \) is the composite of a quasi-finite map \( U \to \mathbb{P}^n_Z \) followed by the structural map \( \mathbb{P}^n_Z \to Z \). Since the question is local on \( Y \), we replace \( Y \) by \( U \) if necessary, and assume we have a quasi-finite map \( Y \to \mathbb{P}^n_Z \). Using Zariski’s Main Theorem we can find a finite map \( \bar{Y} \to \mathbb{P}^n_Z \) such that \( Y \) is an open \( \mathbb{P}^n_Z \)-scheme of \( \bar{Y} \).

Since \( h \) is finite, the composite \( X \to Y \to \bar{Y} \) is quasi-finite, and another application of Zariski’s Main Theorem tells us that \( X \to Y \) factors as an open immersion \( X \to \bar{X} \) followed by a finite map \( \bar{h}: \bar{X} \to \bar{Y} \). Replacing \( \bar{X} \) by the scheme theoretic closure of its open subscheme \( X \) if necessary, we may assume that \( X \) is scheme theoretically dense in \( \bar{X} \). This forces \( X = \bar{h}^{-1}(Y) \). We thus have a cartesian diagram,
with horizontal arrows being open immersions

\[
\begin{array}{ccc}
X & \xrightarrow{h} & X \\
\downarrow & & \downarrow \circ h \\
Y & \xrightarrow{\bar{h}} & Y
\end{array}
\]

We write \( \bar{g} : \bar{Y} \to Z \) for the composite \( \bar{Y} \to \mathbb{P}^n_Z \to Z \), and set \( \bar{f} = \bar{g} \circ \bar{h} \). The important point is that \( f : X \to Z \) and \( \bar{g} : \bar{Y} \to Z \) are proper over \( Z \) and the fibres of \( f \) and \( \bar{g} \) have dimension \( \leq n \). This means \( H^j(\bar{f}^*\mathcal{O}_Z) = H^j(\bar{g}^*\mathcal{O}_Z) = 0 \) for \( j < -n \). It follows that if \( \omega^\#_f := H^{-n}(\bar{f}^*\mathcal{O}_Z) \), and if \( \text{tr}^\#_f : R^n\bar{g}_*\omega^\#_f \to \mathcal{O}_Z \) is the map induced by \( \text{Tr}_f(\mathcal{O}_Z) : R^n\bar{f}_*\mathcal{O}_Z \to \mathcal{O}_Z \) then \( (\omega^\#_f, \text{tr}^\#_f) \) represents the functor \( \mathcal{F} \to \text{Hom}(R^n\bar{f}_*, \mathcal{F}, \mathcal{O}_Z) \) of quasi-coherent sheaves \( \mathcal{F} \) on \( X \) (see [NS1] (5.1.5) for this argument). Along these lines, if \( \omega^\#_g := H^{-n}(\bar{g}^*\mathcal{O}_Z) \), and \( \text{tr}^\#_g : R^n\bar{g}_*\omega^\#_g \to \mathcal{O}_Z \), the map induced by \( \text{Tr}_g(\mathcal{O}_Z) \), then one can make a similar statement about \( (\omega^\#_g, \text{tr}^\#_g) \).

Let \( \bar{X} = X \times_Z Z' \), \( \bar{Y} = Y \times_Z Z' \), and let \( \bar{f}', \bar{g}', \bar{h}', \bar{u}, \bar{v} \) be the obvious base changes of \( f, g, h, u, \) and \( v \), respectively. Let \( \text{tr}^\#_h : h_*\omega^\#_f \to \omega^\#_g \) be the obvious analogue of \( \text{tr}_h \), namely

\[(9.2.4.1) \quad \text{tr}^\#_h = H^{-n}(h_*\bar{f}'^*\mathcal{O}_Z \iff h_*h^!\bar{g}'^*\mathcal{O}_Z \xrightarrow{\text{tr}_h} \bar{g}'^*\mathcal{O}_Z).
\]

Similarly define \( \text{tr}^\#_{h'}, \text{tr}^\#_{\bar{h}} \), and \( \text{tr}^\#_{\bar{h}'} \). Since \( \bar{g} \) and \( \bar{f} \) are proper, \( \text{tr}^\#_h : h_*\omega^\#_f \to \omega^\#_g \) has the following alternative description: It is the adjoint to the element of \( \text{Hom}_Y(R^n\bar{g}_*, h_*\omega^\#_f, \mathcal{O}_Z) \) given by the composite

\[
R^n\bar{g}_*h_*\omega^\#_f \xleftarrow{\bar{h}_*\bar{v}^*\omega^\#_f} R^n\bar{f}_*\omega^\#_f \xrightarrow{\text{tr}^\#_f} \mathcal{O}_Z.
\]

Let \( \theta^\#_{\bar{f}} : \bar{w}^*\omega^\#_f \to \omega^\#_f \), and \( \theta^\#_{\bar{g}} : \bar{v}^*\omega^\#_g \to \omega^\#_g \) be the base change isomorphisms defined in [S2] pp. 738–739, Rmk. 2.3.2, especially (2.5)]. We claim that the following diagram commutes:

\[
\begin{array}{ccc}
\bar{v}^*\bar{h}_*\omega^\#_f & \xrightarrow{\bar{h}_*\theta^\#_{\bar{f}}} & \bar{h}_*\omega^\#_f \\
\downarrow & & \downarrow \text{tr}^\#_h \\
\bar{v}^*\omega^\#_g & \xrightarrow{\theta^\#_{\bar{g}}} & \omega^\#_g
\end{array}
\]

Suppose \((*)\) commutes. Restricting \((*)\) to \( Y \), and using the Verdier isomorphisms for \( f, \bar{f}, g, \) and \( \bar{g} \) and [S2] p. 739, Theorem 2.3.3, especially (c]) (which states that via these isomorphisms \( \theta^\#_{\bar{f}} \) and \( \theta^\#_{\bar{g}} \) are the identity maps) we get \( v^*\text{tr}_h = \text{tr}_{h'} \) as we wish. The commutativity of \((*)\) is equivalent to

\[(\dagger) \quad \text{tr}^\#_{\bar{g}}(R^n\bar{g}_*(\text{tr}^\#_{h'} \circ \bar{h}_*\theta^\#_{\bar{f}})) = \text{tr}^\#_{\bar{g}}(R^n\bar{g}_*(\theta^\#_{\bar{g}} \circ \bar{v}^*\text{tr}^\#_h))
\]
The proof of (†) rests on the fact that the following diagram of functors commutes

\[
\begin{array}{ccc}
\mathbf{R}^n f_*\omega_f^\#$ & \longrightarrow & \mathbf{R}^n \bar{g}_*\bar{h}_* \\
\mathbf{R}^n f'_*\omega_f^\# & \longrightarrow & \mathbf{R}^n \bar{g}'_*\bar{h}'_* \bar{h}_* \\
\end{array}
\]

In greater detail, consider the following diagram:

\[
\begin{array}{ccc}
\mathbf{R}^n f_*\omega_f^\# & \longrightarrow & \mathbf{R}^n \bar{g}_*\bar{h}_* \omega_f^\#
\end{array}
\]

The outer border commutes because of our alternate description of \(\text{tr}^*_h\). The rectangle on the left commutes because of the definition of \(\theta^f_u\). The rectangle on the lower right commutes because of the definition of \(\theta^g_u\). The remaining rectangle bordering the bottom edge commutes because of the alternate description of \(\text{tr}^*_h\). The rectangle on the top right is simply (†) and so commutes. All other rectangles, save \(\blacksquare\), commute for functorial reasons. Consider \(\blacksquare\). We have two possible routes from its northeast vertex to \(\mathcal{O}_{Z'}\) lying directly below its southwest vertex, namely, south followed by west followed by south, and west followed by south all the way. We have to show that the two routes give the same map. This follows from the fact that all the subrectangles (except possibly \(\blacksquare\)) and the outer border commute. This establishes (†) and hence the theorem.

We wish to understand (9.2.2) more explicitly. For that we need to work more locally, with affine schemes, and often in a “punctual way”, i.e., by working with completions of local rings at points. With this in mind, let us assume that we are in the situation of diagram (9.2.1), with a small change in hypothesis, namely we assume \(h\) is separated and quasi-finite, rather than finite. The maps \(f\) and \(g\) remain smooth of relative dimension \(n\).
We are interested in duality for \( h \) in terms of \( \omega_g \) and \( \omega_f \) “at a point \( x \in X \).” To that end we make the following further assumptions.

- \( Z = \text{Spec } A \)
- \( Y = \text{Spec } R \) and \( X = \text{Spec } S \).

Let \( y \in Y \) assume \( h^{-1}(y) \) consists of exactly one point \( x \).

Let \( R' = \hat{O}_y \) be the completion of the local ring \( \mathcal{O}_{Y,y} \), \( S' = \hat{O}_x \) the completion of \( \mathcal{O}_{X,x} \), and set \( Y' = \text{Spec } R' \), \( X' = \text{Spec } S' \). Since \( h^{-1}(y) = \{x\} \), we have a cartesian square

\[
\begin{array}{ccc}
X' & \xrightarrow{h'} & X \\
\downarrow h' & & \downarrow h \\
Y' & \xrightarrow{u} & Y
\end{array}
\]

with \( h' \) finite, even though \( h \) need not be finite.

To lighten notation, write \( \omega_R = \omega_{R/A} \), and \( \omega_S = \omega_{S/A} \). Set \( \omega_{R'} = \omega_R \otimes_R R' \), and \( \omega_{S'} = \omega_S \otimes_S S' = \omega_S \otimes_R R' \).

Since \( h \) is flat and Gorenstein of relative dimension 0, for any quasi-coherent \( \mathcal{O}_Y \)-module \( F \) we have \( H^k(h^! F) = 0 \) for \( k \neq 0 \), and so we identify \( h^! F \) with \( H^0(h^! F) \).

Similarly, we identify \( h'^! G \) with \( H^0(h'^! G) \) for every quasi-coherent \( \mathcal{O}_{Y'} \)-module \( G \).

For an \( R \)-module \( M \), \( h^! M \) is defined to be \( \Gamma(X, h^! \tilde{M}) \). Similarly, for an \( R' \)-module \( N \), \( (h')^! N \) will denote \( \Gamma(X', (h')^! \tilde{N}) \).

Let

\[
(9.2.5) \quad \zeta: h^! \omega_R \xrightarrow{\sim} \omega_S
\]

denote the isomorphism obtained from \( h^! g^! \mathcal{O}_Y \xrightarrow{\sim} f^! \mathcal{O}_Y \) and the Verdier isomorphisms \( \nu_g \) and \( \nu_f \). By (flat) base change, we have

\[
(9.2.6) \quad \zeta': (h')^! \omega_{R'} \xrightarrow{\sim} \omega_{S'}.
\]

In particular we have a trace map (for \( h' \) is finite)

\[
(9.2.7) \quad \text{tr}_{S'}: \omega_{S'} \rightarrow \omega_{R'}
\]

corresponding to

\[
h'_* \tilde{\omega}_{S'} \xleftarrow{(\ast)_{R'}} h'_*(h')^! \tilde{\omega}_{R'} \xrightarrow{\text{Tr}_{h'}} \tilde{\omega}_S
\]

Our interest is in making \( \text{tr}_{S'} \) explicit. We point out that to define it, it was not necessary to assume that \( x \) is the only point of \( X \) lying over \( y \). However, by shrinking \( X \) around \( x \), we can be in the situation we are in.

Now suppose \( h: X \rightarrow Y \) factors as in the following commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{i} & P \\
\downarrow h & & \downarrow p \\
Y & &
\end{array}
\]
with \( P = \text{Spec} \, E, \, p: P \to Y \) smooth of relative dimension \( d \), and \( i \) a closed immersion. We have a commutative diagram with each square cartesian

\[
\begin{array}{ccc}
X' & \xrightarrow{v} & X \\
\downarrow j & & \downarrow i \\
Y' & \xrightarrow{p'} & Y \\
\end{array}
\]

(9.2.8)

with \( h = p \circ i \) and \( h' = p' \circ j \).

Let \( E' = E \otimes_R R' \), \( P' = \text{Spec} \, E' \), \( \pi = g \circ p' \), and consistent with out notations above, let \( \omega_E = \omega_{E/A} \), and \( \omega_{E'} = \omega_{E \otimes_R R'} \).

We remark that \( \omega_{R'} \) and \( \omega_{S'} \) are the \( e \)-th graded pieces of the differential graded algebras \( \wedge^*_R(\Omega^1_{E/A} \otimes_R R') \) and \( \wedge^*_S(\Omega^1_{S/A} \otimes_R R') \) respectively. Similarly, \( \omega_{E'} \) is the \((n+e)\)-th graded piece of \( \wedge^{*}_{E'}(\Omega^1_{E/A} \otimes E E') = \wedge^{*}_{E'}(\Omega^1_{E/A} \otimes R R') \).

Let

\[
\phi: \omega_R \otimes_R \omega_{E/R} \sim \omega_E
\]

be the isomorphism \( \phi = \Gamma(Y, \tilde{\varphi}_{g,p}) \), where \( \tilde{\varphi}_{g,p} \) is the map defined in (7.2.1). In other words \( \phi(v \otimes \mu) = \mu \wedge p^*v \). Let

\[
\phi': \omega_{R'} \otimes_{R'} \omega_{E'/R'} \sim \omega_{E'}
\]

be the base change of \( \phi \). In greater detail, we have \( \omega_{E'/R'} = \omega_{E/R} \otimes_R R' \), and therefore \( \omega_{R'} \otimes_{R'} \omega_{E'/R'} = (\omega_R \otimes_R \omega_{E/R}) \otimes_R R' \). Set \( \phi' = \phi \otimes 1 \).

Next, let \( I = \ker \, E \to S \), \( J = \ker \, E' \to S' \). Write \( N = (\wedge^0 I/J^2)^* \) and \( N' = (\wedge^0 J/J^2)^* \). Let

\[
b: \omega_E \otimes_E N \sim \omega_S
\]

be the map given by (8.2.3). By base change, as in the definition of \( \phi' \), we have a map \( b':= b \otimes 1 \):

\[
b': \omega_{E'} \otimes_{E'} N' \sim \omega_{S'}.
\]

Let \( q: \omega_R \otimes_R \omega_{E/R} \otimes_E N \to \omega_S \) and \( q': \omega_{R'} \otimes_{R'} \omega_{E'/R'} \otimes_{E'} N' \to \omega_{S'} \) be the maps

(9.2.9) \[ q = b \circ (\phi \otimes 1_N) \quad \text{and} \quad q' = b' \circ (\phi' \otimes 1_{N'}) = q \otimes 1_{N'} \]

Finally, let

(9.2.10) \[ \psi: \omega_{E/R} \otimes_E N \sim \omega_S \quad \text{and} \quad \psi': \omega_{E'/R'} \otimes_{E'} N' \sim \omega_{S'} \]

be the maps defined as in (8.2.3).

**Proposition 9.2.11.** In the above situation, assume \( I = \ker \, (E \to S) \) is generated by \( u = (u_1, \ldots, u_d) \) and set \( f_k = u_k \otimes 1 \in E' \), so that \( f = (f_1, \ldots, f_d) \) generates \( J = \ker \, (E' \to S') \). Set \( \mathcal{N} = \mathcal{A}_i^{d}(= \tilde{N}) \) and \( \mathcal{N}' = \mathcal{A}_j^{d}(= \tilde{N}') \). Let \( \omega \in \omega_{S'} \).
(i) The following diagram commutes:

\[
\begin{array}{c}
\omega_R^* \otimes_{R'} \omega_{E^f/R'} \otimes_{S'} N' \\
\downarrow \downarrow \\
\omega_R^* \otimes_{R'} (\omega_{E^f/R'} \otimes_{S'} N') \\
\downarrow 1 \otimes \nu' \\
\omega_R^* \otimes_{R'} h^f(R') \sim h^f(\omega_R)
\end{array}
\]

where \( \chi^{\nu'} \) is the transitivity map defined in \( \text{[2.8.2]} \).

(ii) If \( \omega = s \cdot (h')^*(\nu) \), where \( \nu \in \omega_{R'} \) and \( s \in S' \), then

\[ \text{tr}_{S'}(\omega) = \text{Tr}_{S'/R'}(s) \cdot \nu. \]

(iii) Let \( \eta \in \Omega^n_{E/A} \otimes_{R'} R' \) be any element such that \( j^*\eta = \omega \). Then

\[ \text{tr}_{S'}(\omega) = \nu \cdot \text{res}_{X', R'} \left[ x \cdot \mu \right] f_1, \ldots, f_d \]

where \( x \in E' \), \( \mu \in \omega_{E' / R'} \), and \( \nu \in \omega_{R'} \) are related via the formula

\[ df_1 \wedge \cdots \wedge df_d \wedge \eta = x \cdot \mu \wedge p^* \nu. \]

(iv) Let \( \mu, \nu, \eta \) and \( x \) be as in (ii). Suppose \( E = R[T_1, \ldots, T_d]_q(T) \), where \( q(T) \in R[T] \). Let \( g_i \in E' \otimes_{R'} S' \) be the elements \( g_i = T_i \otimes 1 - 1 \otimes \gamma_i \), where the \( \gamma_i \in S' \) is the images of \( T_i \), \( i = 1, \ldots, d \), and let

\[ \lambda: S' \to R' \]

be the map \( \lambda = \lambda(f, g) \) of Theorem \( \text{[9.1.7]} \). Then

\[ \text{tr}_{S'}(\omega) = \lambda(x|_{X'}) \nu. \]

Proof. We point out that \( u \) and \( f \) are necessarily quasi-regular. We first prove (i).

Consider the following diagram.

The rectangle \( \blacklozenge \) on the top commutes by definition of \( \rho' \). The sub-diagram on the right, the one labelled \( \square \), squeezed between the curved arrow and the vertical column, commutes by definition of \( \zeta' \). The rectangle labelled \( \diam \) at the bottom commutes by [NS1] Prop.7.2.10].
We now show that the sub-diagram on the left, labelled ■, squeezed between the curved arrow and the vertical column on the left, commutes. First, the composite of isomorphisms, with the middle arrow the base change isomorphism

\[ v^\ast \mathcal{N}[-d] \xrightarrow{v^\ast \eta_f} v^\ast \mathcal{O}_P \xrightarrow{\sim} j^! \mathcal{O}_P, \quad \eta_f \mapsto \eta_f^{-1} \]

is the identity map on \( \mathcal{N}'[-d] \) [NS1, Remark 6.2.5]. Next, the composite of isomorphisms

\[ w^\ast \omega_p[d] \xrightarrow{w^\ast v_p} w^\ast p^! \mathcal{O}_Y \xrightarrow{\sim} p^! \mathcal{O}_Y, \quad \omega_p \mapsto \omega_p' \]

is the identity map on \( \omega_p'[d] \) [S2, p. 740, Prop. 2.3.5 (b)]. Finally, the transitivity property of base change [NS1, Prop. A.1.1 (ii)] (see also [L4, p. 183, Prop. 4.6.8]) tells us that the base change of the composite \( p \circ i \) with respect to \( u \) is compatible with the base change for \( p \) and \( i \) with respect to \( u \) and \( w : P' \to P \) respectively. Putting these together, we see that ■ also commutes.

The outer border commutes by Proposition 8.2.8.4 after using Theorem 8.2.6 to realise \( b \) as a concrete representation of the map \( a_{X/P} \).

It follows that the rectangle in the middle also commutes. This proves (i).

Next note that the following diagram commutes, by definition of the various isomorphisms involved.

\[
\begin{array}{ccc}
\omega_{R'} \otimes_R h^!(R') & \xrightarrow{\sim} & h^!(R') \\
\downarrow \text{id} \otimes \text{Tr}_{S'/R'} & & \downarrow \text{id} \\
\omega_{R'} & \xrightarrow{\sim} & \omega_{S'}
\end{array}
\]

Let \( \tau_h = \tau_{h^!} : h^!(j^* \omega_p \otimes \mathcal{N}') \xrightarrow{\sim} \mathcal{O}_{Y'} \) be the map in [NS1 (5.3.2)]. Define

\[ \tau_{h'} : h^!(j^* \omega_p' \otimes \mathcal{N}') \xrightarrow{\sim} \mathcal{O}_{Y'} \]

in the obvious way, namely by substituting \( \omega_p' \) in the definition of \( \tau_{h'} \) by \( \omega_p' \) via the Verdier isomorphism \( v_p' \). Write \( \tau_{S'/R'} \) for the global sections of \( \tau_{h'} \). From part (i) and the above commutative diagram, we see that

\[ \text{tr}_{S'} \circ \rho' = 1 \otimes \tau_{S'/R'} \quad (\ast) \]

Now suppose \( \mu \in \omega_{E'/R'} \) and \( \nu \in \omega_{R'} \). By [NS1, Prop. 5.4.4] and (\ast) we get

\[ \text{tr}_{S'}(\rho'(\nu \otimes \mu \otimes 1/f)) = \mathbf{res}_{X', p'} \left[ \begin{array}{c} \mu \\ f_1, \ldots, f_d \end{array} \right] \cdot \nu \quad (\dagger) \]

Now if \( \omega = s \cdot h^!(\nu) \), then by definition of \( \rho' \), if \( x \in E' \) is a lift of \( s \), we have

\[ \rho'(x \cdot (\nu \otimes df_1 \wedge \cdots \wedge df_d \otimes 1/f)) = \omega, \]

whence by (\dagger)

\[ \text{tr}_{S'}(\omega) = \mathbf{res}_{X', p'} \left[ \begin{array}{c} x \cdot df_1 \wedge \cdots \wedge df_d \\ f_1, \ldots, f_d \end{array} \right] \cdot \nu. \]

The right side is equal to \( \text{Trc}_{S'/R'}(s) \cdot \nu \) by Theorem 9.1.8. This proves (ii). Part (iii) is a re-statement of (\dagger). Indeed

\[ \text{tr}_{S'}(\omega) = \text{tr}_{S'}(j^* \eta) = \text{tr}_{S'}(b'(df_1 \wedge \cdots \wedge df_d \otimes \eta \otimes 1/f)) \]

\[ = \text{tr}_{S'}(\rho(x \cdot (\nu \otimes \mu \otimes 1/f))) \]

\[ = \nu \cdot \mathbf{res}_{X', p'} \left[ \begin{array}{c} x \cdot \mu \\ f_1, \ldots, f_d \end{array} \right] \]
Part (iv) follows from (iii) and Proposition 9.1.10.

Remark 9.2.12. We have already observed that it was not necessary to assume $h^{-1}(y)$ consisted of exactly one point, namely $x$, in order to define $\text{tr}_{S'}$. In fact more can be said. Suppose $u: X \to \overline{X}$ is an open immersion of (ordinary) $Y$-schemes such that the structure map $\overline{h}: \overline{X} \to Y$ is quasi-finite, and $h^{-1}(y) = \{x_1, \ldots, x_m\}$, with $x_1 = x$. In this case, the fibre dimension of $f = g \circ \overline{h}$ is $n$. As before, set $\omega^f = H^{-n}(f^\ast \mathcal{O}_Z)$. Now, $\overline{X} = \overline{X} \times_Y Y'$ is finite over $Y$, since $Y'$ is the spectrum of a complete local ring. Let $\overline{h}': \overline{X} \to Y'$ be the base change of $\overline{h}$. Now $\overline{X} = \text{Spec} \prod_{i=1}^m S'_i$, where $S'_i$ is the completion of the local ring $S_i = \mathcal{O}_{\overline{X}, x_i}$. Let $\overline{S} = \prod_i S'_i$, and let $X'_i = \text{Spec} S'_i$, so that $X'_i$ is open and closed in $\overline{X}$, and $\overline{X} = \prod_i X'_i$. Let $h'_1: X'_1 \to Y'$ be the restriction of $\overline{h}'$ to $X'_1$. Note $X'_1 = X'$, $S'_1 = S'$, and $h'_1 = h'$. If $\omega^g_{S_i} = \omega_{f, x_i} \otimes S_i S'_i$, and $\omega^g_S = \oplus \omega^g_{S'_i}$ (the direct sum thought of as an $\overline{S}$-module, then, as in the argument used in 9.2.4.1), we have, analogous to $\zeta^i$, isomorphisms $\text{H}^0(h'_1' \omega_{Y'}) \xrightarrow{\sim} \omega^g_{S'_i}$ and $\text{H}^0((h') \omega_{Y'}) \xrightarrow{\sim} \omega^g_S$\footnote{Regarding $(h') \omega_{Y'}$ as a complex of $\overline{S}$-modules associated to $(h') \omega_{Y'}$, etc.} whence abstract trace maps

$$\text{tr}^g_{S'_i}: \omega^g_{S'_i} \to \omega_{Y'} \quad (i = 1, \ldots, m)$$

and

$$\text{tr}^g_{S}: \omega^g_S \to \omega_{Y'}.$$

Clearly $\text{tr}^g_{S'} = \sum_i \text{tr}^g_{S'_i}$. The Verdier isomorphism $\nu_f: \omega_f \xrightarrow{\sim} \omega^g_{S'}$ base changes to $\nu_{S'}: \omega_{S'} \xrightarrow{\sim} \omega^g_{S'}$, and it is clear that $\text{tr}_{S'} = \text{tr}^g_{S'} \circ \nu_{S'}$. Finally, if $h: \overline{X} \to Y$ is finite, say $\overline{X} = \text{Spec} \overline{S}$, then we have a map $\text{tr}^h: \text{tr}^g_{\overline{S}} \to \omega^g_{\overline{S}}$ defined in (9.2.4.1). Consistent with the above notations, set $\omega^g_{\overline{S}} = \Gamma(\overline{X}, \omega^g_{\overline{S}})$. Let $\text{tr}^g_{\overline{S}}: \omega^g_{\overline{S}} \to \omega_{Y}$ be the map $\Gamma(Y, \nu_f^{-1} \circ \text{tr}^f_{\overline{S}})$. Then clearly

$$\text{tr}^g_{\overline{S}} \otimes_R R' = \text{tr}^g_{\overline{S}} = \sum_k \text{tr}^g_{S'_k},$$

where $\text{tr}^g_{S}$ is the global sections of $\text{tr}^g_{S}$ defined in (9.2.4.1). In particular, if $f: \overline{X} \to Z$ is smooth, then with $\text{tr}^g_{\overline{S}} = \Gamma(\overline{X}, \omega^g_{\overline{S}})$ we have

(9.2.12.1) \[
\text{tr}^g_{\overline{S}} \otimes_R R' = \sum_k \text{tr}^g_{S'_k}.
\]

9.2.13. The Kunz-Lipman trace. Suppose, as we have for most of this section, $X = \text{Spec} S$, $Y = \text{Spec} R$, and $Z = \text{Spec} A$, and as before suppose $f: X \to Z$ and $g: Y \to Z$ are smooth of relative dimension $n$, $f = g \circ h$, and now assume $h$ is finite, and not merely separated and quasi-finite. In this case (and in more general situations) we have a trace map

$$\sigma_{S/R}: \omega_S \to \omega_R$$

or, in sheaf-theoretic terms,

$$\sigma_h: h_* \omega_f \to \omega_g$$

due to Lipman and Kunz, defined in Kunz’s book \cite[p. 254, 16.4]{Ku}. The idea is attributed by Kunz to Lipman (see footnote in loc.cit.)

The Kunz-Lipman trace $\sigma_{S/R}$ can be understood punctually. In greater detail, the Tate trace $\lambda(f, g)$ of Theorem 9.1.7 is denoted $\tau^f_g$ in \cite[p. 370, (F.20)]{Ku} (and...
studied in some detail in F.18–F.28 of ibid). Now suppose \( y \) is a point in \( Y \). Fix \( x \in h^{-1}(y) \) and pick an affine open subscheme \( U = \text{Spec} \, S_U \) of \( X \) such that \( h^{-1}(y) \cap U = \{x\} \), and a presentation

\[
R[T_1, \ldots, T_d]_{q(T)}/(u_1, \ldots, u_d) = S_U.
\]

Such a \( U \) and presentation always exists. Let \( R' \) be as before, the completion of the local ring \( \mathcal{O}_{Y,y} \), and let \( S' \) be the completion of the local ring \( \mathcal{O}_{X,x} \). Let \( E = R[T]_{q(T)} \), and \( E' = E \otimes_R R' \). Let \( f_1, \ldots, f_d \) be the images of \( u_1, \ldots, u_d \) in \( E' \). We continue to denote the image of the variables \( T_k \) in \( E' \) as \( T_k \). Let \( \omega_R, \omega_{S_U}, \omega_{S'}, \text{tr}_{S'} \) etc., be as before. Let \( \gamma_k \in S' \) be the image of \( f_k \), and set \( g_k = X_k \otimes 1 - 1 \otimes \gamma_k \in E' \otimes_R S' \). Finally let

\[
\lambda: S' \to R'
\]

be the Tate trace \( \lambda(f,g) \) of Theorem 9.1.17. Since \( \omega_{S'} \) is a direct summand of \( \omega_S \otimes_R R' \), the map \( \sigma_{S/R} \) restricts to a map

\[
\sigma_{S'}: \omega_{S'} \to R'.
\]

For \( \omega \in \omega_{S'} \) and \( \eta \in \omega_{E'} \) a pre-image of \( \omega \) under the natural surjective map \( \omega_{E'} \to \omega_{S'} \), suppose \( x \in E' \), \( \nu \in \omega_{E'} \) are such that

\[
df_1 \wedge \cdots \wedge df_d \wedge \eta = x \cdot dT_1 \wedge \cdots \wedge dT_d \wedge \nu.
\]

Using properties Tr 3) and Tr 4) of [Ku, pp. 245-246, §16], proved in [ibid, p. 254, Thm. 16.1], the definition of the Kunz-Kipman trace in [ibid, p. 254, 16.4] gives

\[
\sigma_{S'} = \lambda(\bar{x}) \cdot \nu,
\]

where \( \bar{x} \in S' \) is the image of \( x \in E' \). This means, by the formula in Proposition 9.2.11(iv),

\[
\sigma_{S'} = \text{tr}_{S'}.
\]

Once again by the above mentioned properties Tr 3) and Tr 4) of \( \sigma_{S/R} \), and by (9.2.12.1), this gives \( \sigma_{h,y} = \text{tr}_{h,y} \), where \( (\cdot) \) denote completion of an \( \mathcal{O}_{Y,y} \)-module with respect to the maximal ideal. Since \( y \) is arbitrary in \( Y \), we have

\[
(9.2.13.1) \quad \sigma_{h} = \text{tr}_{h}.
\]

Clearly, we don’t need \( X, Y, \) and \( Z \) to be affine for the argument to go through.

**Theorem 9.2.14.** Let \( f: X \to Z \) and \( g: Y \to Z \) be smooth separated maps of ordinary schemes of relative dimension \( n \), and suppose \( f = g \circ h \), where \( h: X \to Y \) is a finite map.

(i) Suppose \( Z = \text{Spec} \, A, Y = \text{Spec} \, R \) and \( X = \text{Spec} \, S \). Let \( \omega = s(h^*(\nu)) \) where \( s \in S \) and \( \nu \in \Omega^n_{R/A} \). Then

\[
\text{tr}_{h}(\omega) = \text{Trc}_{S/R}(s) \nu.
\]

(ii) If \( \sigma: h_*\omega_f \to \omega_g \) is the Kunz-Lipman trace then

\[
\sigma_{h} = \text{tr}_{h}.
\]

**Proof.** Part (ii) of Proposition 9.2.11 gives (i). Part (ii) is simply (9.2.13.1). \( \square \)
9.3. **Regular Differentials again.** The fact that $\text{tr}_h$ agrees with the Lipman-Kunz trace $\sigma_h$ allows us to prove Theorem 6.4.2 in a different way. Suppose $f, g, h$ are as above, with the caveat that we no longer assume that $f$ is smooth, but assume $f$ is of finite type, the smooth locus of $f$, $X^{sm}$ contains all the associated points of $X$. The map $g$ remains smooth, and $h$ finite. Assume further that:

(i) $X$, $Y$ and $Z$ are excellent have no embedded points;
(ii) $X = \text{Spec} \, S$, $Y = \text{Spec} \, R$, and $Z = \text{Spec} \, A$;
(iii) $R \to S$ is injective.

We use the notations of Section 6. Thus $f^K = H^{-n}(f^!)$ is as in Subsection 6.1 and $\omega_{X/Z}^\text{reg}$ is the sheaf of regular differential $n$-forms discussed in Subsection 6.4. Since we are in the affine situation, we work with modules and algebras over $A$, $R$, and $S$, and choose appropriate notations. To that end, let $k(R)$ and $k(S)$ be the total ring of fractions of $R$ and $S$ respectively. Set $\omega_R = \Gamma(X, \omega_g)$, $\omega_{k(S)} = \Gamma(X, \Omega^n_{k(X)/k(Z)})$, $\Omega_{k(R)} = \Gamma(X, \Omega^n_{k(Y)/k(Z)})$.

Standard arguments show that there is an scheme theoretically dense open sub-scheme $U$ of $Y$, such that $h^{-1}(U)$ is in $X^{sm}$ and is scheme theoretically dense in $X$ (e.g., $U = Y \setminus h(X \setminus X^{sm})$). We have the trace map $\text{tr}_{h_U}: (h_U)_\ast \omega_{h^{-1}U} \to \omega_{h(U)}$, where $h_U: h^{-1}(U) \to U$ is the restriction of $h$. By taking stalks at generic points (we have no embedded points!) we get a map

$\text{tr}_{k(S)}: \omega_{k(S)} \to \omega_{k(R)}$.

We point out that $\omega_R \subset \omega_{k(S)}$. The content of the next result is that $\overline{\omega}_S$ is a “complementary module” in the sense of Kunz and Waldi [KW § 4]. It is equivalent to Theorem [6.4.2] via Theorem 9.2.14, but we give a direct proof along the lines of the proof given of a related statement in [L2].

**Theorem 9.3.1.** Let $\overline{\omega}_S \subset \omega_{k(S)}$ be the image of the injective map $\omega^g_S \to \omega_{k(S)}$ defined in 6.4.1. Then

$$\omega^g_S = \{ \nu \in \omega_{k(S)} \mid \text{tr}_{k(S)}(s\nu) \in \omega_R, \forall s \in S \}.$$

**Proof.** The proof is mutatis mutandis the proof given in [L2] p. 34, Lemma (2.2)]. We give it here, with the necessary changes, for completeness. We have a natural isomorphism $\text{Hom}_R(S, \omega_R) \cong \omega_S^g$ obtained by applying $H^{-n}$ to $h^! \omega_g[n] \cong f^! \Omega_Z$, whence an isomorphism $\overline{\omega}_S \cong \text{Hom}_R(S, \omega_R)$.

One checks (by using the open set $U = Y \setminus h(X \setminus X^{sm})$ as an intermediary if necessary) that the following diagram commutes

$$\begin{align*}
\text{Hom}_R(S, \omega_R) \ar[r] \ar[d] & \text{Hom}_{k(R)}(k(S), \omega_{k(R)}) \ar[d] \\
\overline{\omega}_S \ar[r] & \omega_{k(S)}
\end{align*}$$

where the isomorphism on the right is $\nu \mapsto (x \mapsto \text{tr}_{k(S)}(x\nu))$, for $\nu \in \omega_{k(S)}$ and $x \in k(S)$. The result follows since the image of $\text{Hom}_R(S, \omega_R)$ in $\text{Hom}_{k(R)}(k(S), \omega_{k(R)})$ consists of $k(R)$-linear maps $\psi: k(S) \to \omega_{k(R)}$ such that $\psi(s) \in \omega_R$ for every $s \in S$. In other words, such $\psi$ are characterised by the property that $e(s\psi) \in \omega_R$ for every $s \in S$, where

$$e: \text{Hom}_{k(R)}(k(S), \omega_{k(R)}) \to \omega_{k(R)}$$
is “evaluation at 1”. Since $e$ corresponds to $\text{tr}_{k(S)}$ under the upward arrow on the right in the above diagram, we are done. □

The next statement is a re-statement of Theorem 6.4.2, but the point is that it is also a consequence of Theorem 9.3.1.

**Corollary 9.3.2.** Let $\omega^\text{reg}_S$ be the $S$-module whose associated quasi-coherent sheaf is $\omega^\text{reg}_{X/Y}$. Then $\omega^\text{reg}_S = \omega_S$.

**Proof.** Let $U = Y \setminus h(X \setminus X^\text{sm})$ and $h_U : h^{-1}(U) \to U$ the restriction of $h$. From Theorem 9.2.14(ii), $\text{tr}_{h_U} = \sigma_{h_U}$. The result follows from the characterisation of $\omega_S$ as a complementary module in Theorem 9.3.1 and the definition of regular differentials in [HK1, p.58]. □

9.3.3. We would like draw out the differences between the approach in Section 6 and that of this subsection. In the former, we treat the theory of regular differential forms as a settled theory, and freely use the results in [HK1], [HK2], and [HS] to arrive at a proof of Theorem 6.4.2 using our characterisation of the Verdier isomorphism in terms of standard residues along sections. In the “settled theory” mentioned above, $\omega^\text{reg}_{X/Z}$ is defined via local quasi-normalisations, i.e. via quasi-finite maps from open subschemes of $X$ to $\mathbb{A}^n_Z$, their compactifications by Zariski’s Main Theorem and complementary modules [KW]. The theory of residues and traces used there make no reference to Verdier’s isomorphism, and are developed ab initio for the purpose at hand. In Subsection 6, we mapped our theory on to all of that.

In contrast, from the results in this subsection, if $\omega^\text{reg}_{X/Z}$ is defined as the image of the injective map $f^*\mathcal{O}_Y \to \Omega^n_{\mathcal{O}(X/Z)}$ as in (6.4.1), then we show that every time one has a finite dominant $Z$-map $h : X \to Y$ of schemes, such that $Y \to Z$ is smooth, then $\omega^\text{reg}_{X/Z}$ is necessarily the complementary module on the right side of Theorem 9.3.1 (which can clearly be defined even when $Y$ is not affine). Using [EGA] IV.3, (13.3.2)] and Zariski’s main theorem, as in the first two paragraphs proof of Proposition 9.2.4 we see that locally we can always arrange matters so that $X$ is covered by affine open subschemes, each of which is finite over an affine smooth $Z$-scheme (in fact an affine open subscheme of $\mathbb{P}^n_Z$), and hence $\omega^\text{reg}_{X/Z}$ has a local description via complementary modules. This gives a different proof, than that given in [KW], that these complementary modules glue, and do not depend on the choice of the various finite maps of the sort just discussed. Finally the theory of residues and traces presented here in this paper means that all the important results in [HK1], [HK2], and [HS] can be recovered.

Our approach (in this subsection) is closer in spirit to the approach to these matters in [L2], though even here it is necessarily different, since we use, consistently, Verdier’s isomorphism, and we work over an arbitrary (noetherian) base rather than over a perfect field. It should be said that in [KW] and [Ku], the theory is for general differential algebras, and that in [KW], generic complete intersection algebras $A \to S$ are considered.

10. The Residue Symbol

10.1. **Definition.** Let $f : X \to Y$ be a separated smooth map of relative dimension $r, t_1, \ldots, t_r \in \Gamma(X, \mathcal{O}_X)$ such that if $\mathcal{I}$ is the quasi-coherent ideal sheaf generated by $t = (t_1, \ldots, t_r)$, then $Z := \text{Spec}(\mathcal{O}_X/\mathcal{I})$ is finite over $Y$. Let $i : Z \to X$ be the closed immersion and $h : Z \to Y$ the finite map. In this case it is well-known that
$h$ is flat and $t$ is a regular $\mathcal{O}_{X,z}$-sequence for every $z \in Z$ (EGA IV$_3$, Théorème (11.3.8) or [M, p. 177, Corollary to Thm. 22.5]). In particular $h_*\mathcal{O}_Z$ locally free over $\mathcal{O}_Y$.

In this situation, according to [NS1, (5.3.2)], we have a map

$$\tau_h^a = \tau_{h,f,i}^a : h^*\omega_f^a \otimes_{\mathcal{O}_Z} \mathcal{M}_i^r \longrightarrow \mathcal{O}_Y,$$

allowing us to define

\begin{equation}
(10.1.1) \tau_h = \tau_{h,f,i} : h^*\omega_f \otimes_{\mathcal{O}_Z} \mathcal{N}_i^r \longrightarrow \mathcal{O}_Y
\end{equation}

as the composite

$$h^*\omega_f \otimes_{\mathcal{O}_Z} \mathcal{N}_i^r \longrightarrow \mathcal{O}_Y.$$

If $\bar{t}_i \in \Gamma(Z, \mathcal{I}/\mathcal{I}^2)$ is the section generated by the image of $t_i$, then $\bar{t}_1 \wedge \cdots \wedge \bar{t}_i$ is a generator of the free rank one $\mathcal{O}_Z$-module $\wedge^r \mathcal{O}_Z/\mathcal{I}/\mathcal{I}^2$. As before, let $1/t \in \Gamma(Z, \mathcal{N}_i^r) = \text{Hom}_Z(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Z)$ be the dual generator. For $\omega \in \Gamma(X, \omega_f)$ let $\omega/t \in \Gamma(Z, i^*\omega_f \otimes_{\mathcal{O}_Z} \mathcal{N}_i^r)$ be the image of $\omega \otimes 1/t \in \Gamma(Z, i^*\omega_f) \otimes \Gamma(Z, \mathcal{N}_i^r)$. With these notations, we follow [RD] and [C1] and define the residue symbol as

\begin{equation}
(10.1.2) \text{Res}_{X/Y} \left[ \omega_{t_1, \ldots, t_r} \right] := \Gamma(Z, \tau_h)(\omega/t) \in \Gamma(Z, \mathcal{O}_Z).
\end{equation}

In [RD III, §9] a list of statements about the residue symbol are made without proof. The statements (with minor corrections to the statements in [RD]) have been proved by Conrad in [C1, A.2, Appendix A]. Since our approach to residues and the residue symbol follows a different route (via Verdier’s isomorphism) we provide independent proofs of these statements in §10.2 below. Here are the statements (R1)–(R10), as in [C1], with modifications to take care of our conventions. In the statements, $\omega, f : X \to Y, Z, t_1, \ldots, t_r$ are as above, except in (R4).

**(R1).** Let $s_i = \sum j u_{ij} t_j$ where $u_{ij} \in \Gamma(X, \mathcal{O}_X), 1 \leq i, j \leq r$, and suppose the closed subscheme of $X$ cut out by the $s_i$'s is finite over $Y$. Then

$$\text{Res}_{X/Y} \left[ \omega_{t_1, \ldots, t_r} \right] = \text{Res}_{X/Y} \left[ \det (u_{ij}) \omega_{s_1, \ldots, s_r} \right].$$

***(R2).** (Localisation) We give the version in [C1] p. 239. Suppose $g : X' \to X$ is separated and étale, $Z' = g^{-1}(Z)$, and the map $g' : Z' \to Z$ is finite, where $g'$ is induced from $g$. We have a commutative diagram of schemes

\[
\begin{array}{ccc}
Z' & \xrightarrow{i'} & X' \\
\downarrow g' & & \downarrow g \\
Z & \xrightarrow{i} & X \\
\downarrow h & & \downarrow f \\
Y & \xrightarrow{f} & Y
\end{array}
\]

where, as indicated in the diagram, the square on the top is cartesian. Assume that the function on $Z$ given by $z \mapsto \text{rank}_{\mathcal{O}_{Z,z}} g'(\mathcal{O}_{Z'})_z$ extends to a locally constant
function \( \text{rk}_{Z'/Z} \) in a Zariski open neighbourhood \( V \) of \( Z \) in \( X \). Then, for every \( \omega \in \Gamma(X, \omega_f) \), we have,

\[
\text{Res}_{V/Y}[\omega \cdot \text{rk}_{Z'/Z} t_1, \ldots, t_r] = \text{Res}_{X/Y}[\omega' t'_1, \ldots, t'_r],
\]

where \( t'_i = g^*(t_i) \in \Gamma(X', \mathcal{O}_{X'}) \) and \( \omega' = g^*(\omega) \in \Gamma(X', \omega_{fY}) \).

**(R3). (Restriction)** Suppose we have a commutative diagram of schemes

\[
\begin{array}{ccc}
X & \xrightarrow{i} & P \\
\downarrow f & & \downarrow \pi \\
Y & \xrightarrow{j} & Z
\end{array}
\]

with \( f \) smooth and separated of relative dimension \( r \), \( \pi \) smooth and separated of relative dimension \( n = d + r \), \( i \) a closed immersion, with \( X \) cut out by \( s_1, \ldots, s_d \in \Gamma(Y, \mathcal{O}_Y) \), and suppose \( t'_1, \ldots, t'_r \in \Gamma(Y, \mathcal{O}_Y) \) are such that \( s_1, \ldots, s_d, t'_1, \ldots, t'_r \) cut out a scheme \( Z \) which is finite over \( Y \), and finally suppose \( t_j \) is the restriction of \( t'_j \) to \( X \) for \( j = 1, \ldots, r \). Then for every \( \nu \in \Gamma(P, \Omega^r_{P/Y}) \),

\[
\text{Res}_{P/Y}^{[d s_1 \wedge \cdots \wedge d s_d \wedge \nu]} = \text{Res}_{X/Y}^{[i^* \nu t_1, \ldots, t_r]}.
\]

**(R4). (Transitivity)** Suppose we have a pair of separated maps \( X \xrightarrow{f} Y \xrightarrow{g} Z \) are a pair of separated smooth maps, \( f \) of relative dimension \( e \) and \( g \) of relative dimension \( d \). Suppose \( s_1, \ldots, s_d \in \Gamma(Y, \mathcal{O}_Y) \) cuts out a scheme \( W' \) in \( Y \) which is finite over \( Z \), and with \( s'_j = f^*(s_j) \), suppose we have \( t_1, \ldots, t_e \in \Gamma(X, \mathcal{O}_X) \) such that \( s'_1, \ldots, s'_d, t_1, \ldots, t_e \) cut out a scheme \( W \) in \( X \) which is finite over \( Z \). For \( \mu \in \Gamma(\mathcal{O}_Y, \omega_f) \) and \( \nu \in \Gamma(\mathcal{O}_X, \omega_g) \) we have:

\[
\text{Res}_{Y/Z}^{[\text{Res}_{X/Y}^{[\mu t_1, \ldots, t_e]} \nu]} = \text{Res}_{X/Z}^{[\mu \wedge f^* \nu t_1, \ldots, t_e, s'_1, \ldots, s'_d]}.
\]

**(R5). (Base Change)** Formation of the residue symbol commutes with base change.

**(R6). (Trace Formula)** For any \( \varphi \in \Gamma(X, \mathcal{O}_X) \)

\[
\text{Res}_{X/Y}^{[\varphi \cdot dt_1 \wedge \cdots \wedge dt_r]} = \text{Tr}_{Z/Y}(\varphi|_Z).
\]

**(R7). (Intersection Formula)** For any collection of positive integers \( k_1, \ldots, k_r \) not all equal to 1,

\[
\text{Res}_{X/Y}^{[dt_1 \wedge \cdots \wedge dt_r]} = 0.
\]
(R8). (Duality) (See [C1] p. 240, (R8).) If \( \omega|_Z = 0 \), then
\[
\text{Res}_{X/Y} \left[ \frac{\omega}{t_1, \ldots, t_r} \right] = 0.
\]
Conversely, let \( \{Y_j\} \) be an étale covering of \( Y \) such that \( Y_j \) is affine, \( Z_j = Z \times_Y Y_j \) decomposes into a finite disjoint union of \( Z_{jk} \)'s with each \( Z_{jk} \) contained in an open subscheme \( X_{jk} \) of \( X := X \times_Y Y_j \), with \( X_{jk} \cap Z_{jm} = \emptyset \) for \( m \neq k \). Also assume that \( \Gamma(X_{jk}, \mathcal{O}_{X_{jk}}) \to \Gamma(Z_{jk}, \mathcal{O}_{Z_{jk}}) \) is surjective. If
\[
\text{Res}_{X_{jk}/Y_j} \left[ \frac{f \omega}{t_1, \ldots, t_r} \right] = 0
\]
for all \( f \in \Gamma(X_{jk}, \mathcal{O}_{X_{jk}}) \), then \( \omega|_Z = 0 \).

(R9). (Exterior Differentiation) For \( \nu \in \Gamma(X, \Omega_{X/Y}^{r-1}) \) and positive integers \( k_1, \ldots, k_r \),
\[
\text{Res}_{X/Y} \left[ \frac{d^k}{k_1, \ldots, k_n} \right] = \sum_{i=1}^r k_i \cdot \text{Res}_{X/Y} \left[ \frac{dt_i \wedge \nu}{t_1, \ldots, t_r} \right].
\]

(R10). (Residue Formula) Let \( h: X' \to X \) be a finite map, with \( X' \) smooth over \( Y \) of relative dimension \( r \). Let \( t'_j = h^*(t_j) \in \Gamma(X', \mathcal{O}_{X'}) \). Then
\[
\text{Res}_{X'/Y} \left[ \frac{\nu}{t'_1, \ldots, t'_r} \right] = \text{Res}_{X/Y} \left[ \frac{\text{tr}_h(\nu)}{t_1, \ldots, t_r} \right],
\]
for every \( \nu \in \Gamma(X', \omega_{X'}) \), where \( \text{tr}_h: h_*\omega_{X'} \to \omega_Y \) is the map in (10.2.2).

10.2. Proofs. For a quasi-coherent \( \mathcal{O}_X \)-module \( \mathcal{F} \), let
\[
\psi = \psi(\mathcal{F}): h_*(i^*\mathcal{F} \otimes_{\mathcal{O}_Z} (\wedge_{\mathcal{O}_Z} \mathcal{I}/\mathcal{I}^2)^*) \longrightarrow R^r_Z f_* \mathcal{F}
\]
be defined by applying \( H^r \) to the composite
\[
h_i^* \mathcal{F}[r] \xrightarrow{\eta_i} Rf_* i^* \mathcal{F}[r] \xrightarrow{\tau^*_h} Rf_* \mathcal{I} \mathcal{F}[r] \longrightarrow Rf_* R\mathcal{I} Z \mathcal{F}[r].
\]
where \( \eta_i : i^* \xrightarrow{\sim} i^* \) is the isomorphism in [NS1] (C.2.11). (See [NS1] (5.3.3) and (5.3.4).) According to [NS1] Thm.5.3.8, the following diagram commutes
\[
\begin{array}{ccc}
h_*(i^*\omega_f \otimes_{\mathcal{O}_Z} (\wedge_{\mathcal{O}_Z} \mathcal{I}/\mathcal{I}^2)^*) & \xrightarrow{\tau^*_h} & \mathcal{O}_Z \\
\psi(\omega_f) & \downarrow & \\
R^r_Z f_* \omega_f & \xrightarrow{\text{res}_Z} & \mathcal{O}_Z
\end{array}
\]
A few things are worth pointing out. First, \( \text{Res}_{X/Y} \left[ \frac{\omega}{t_1, \ldots, t_r} \right] \) is linear in \( \omega \), and since \( \tau^*_h \) is a map of sheaves, the residue symbol is local over \( Y \). Moreover, according to [NS1] Remark 5.3.9, if \( U \) is an open subscheme of \( X \) containing \( Z \), then
\[
\text{Res}_{X/Y} \left[ \frac{\omega}{t_1, \ldots, t_r} \right] = \text{Res}_{U/Y} \left[ \frac{\omega}{t_1, \ldots, t_r} \right].
\]
\footnote{Such \( Y_j \)'s, \( Z_{jk} \)'s, and \( X_{jk} \)'s always exist, using direct limit arguments.}
\footnote{We point out that \( \text{tr}_h \) has an explicit description (in terms of the Kunz-Lipman trace) given in Theorem [2.2.1(i)].}
From (10.2.3) we see easily that if $Z$ is a disjoint union of $Z_1, \ldots, Z_m$ and $X_i$ is open in $X$ with $X_i \cap Z = Z_i$, then as in [Cl] p. 239, (A.1.5)], we have,

$$\text{Res}_{X/Y} \left[ \frac{\omega}{t_1, \ldots, t_r} \right] = \sum_{i=1}^{m} \text{Res}_{X_i/Y} \left[ \frac{\omega}{t_1, \ldots, t_r} \right].$$  

(10.2.5)

We also note that by [NS1 Thm. 6.3.2], and [S2 p. 740, Thm. 2.3.5 (b)], the residue symbol (10.1.2) is stable under arbitrary (noetherian) base change. This proves (R5).

If $Y = \text{Spec } A$ and there is an open affine subscheme $U = \text{Spec } R$ of $X$ containing $Z$, then by [NS1 Prop. 5.4.4] and (10.2.4), we see that

$$\text{Res}_{X/Y} \left[ \frac{\omega}{t_1, \ldots, t_r} \right] = \text{res}_{Z} \left[ \frac{\omega}{t_1, \ldots, t_r} \right].$$  

(10.2.6)

Since the formation of the residue symbol is compatible with arbitrary noetherian base change, i.e., since (R5) is true, we can prove a number of things by assuming $Y$ is the spectrum of an artin local ring, or of a complete local ring. In greater detail, many of the formulas we have to prove are of the form $\alpha = \beta$ where $\alpha, \beta \in \Gamma(Y, O_Y)$.

It is clearly enough to prove that the germs $\alpha_y$ and $\beta_y$ are equal at every $y \in Y$. So suppose $y \in Y$ and $A = O_y$, the local ring at $y$, and $m$ is the maximal ideal of $A$. To show $\alpha_y = \beta_y$, it is clearly enough to show $\alpha_y \otimes_A A/m^n = \beta_y \otimes_A A/m^n$ are equal for every $n \in \mathbb{N}$, and by (R5), to prove this for a given positive integer $n$, it is enough to assume $Y = \text{Spec } A/m^n$. Once we are in this situation, using (10.2.4), (10.2.5), we are in a situation where (10.2.6) applies. Note that we are in the situation where (10.2.6) applies even when pass to the completion of $A$ with respect to $m$. Occasionally, by a further faithful flat base change on $Y$, we may assume $Y$ is a strictly henselian local ring, or even a strictly henselian artin local ring.

With this in mind, (R1) follows from [NS1 Thm. 5.4.5], (R3) from Corollary (R4) from Theorem 5.4.1 (R6) from Theorem 9.1.8. For (R10), first note that $\text{tr}_h$ is compatible with arbitrary noetherian base change by Proposition 9.2.4. So once again, the problem is stable under base change, and we may assume we are in a situation where (10.2.6) applies. This gives us (R10) via Proposition 0.2.3. We have already seen that (R5) is true.

It remains to prove (R2), (R7), (R8) and (R9).

For (R2), we may assume, as in the proof of (R2) in [Cl], that $Y$ is the spectrum of a strictly henselian artin local ring. We are immediately reduced, via (10.2.5), to the case where $Z$ and $Z'$ consist of a single component each, $Z' = Z$, and $q$ is the identity map. In this case the completion of $X'$ along $Z'$ is the same as the completion of $X$ along $Z$, whence, since $\text{res}_Z$ and $\text{res}_{Z'}$ are really only dependent on the formal schemes, we are done.

To prove (R7) we assume without loss of generality that $Y = \text{Spec } A$, where $A$ is an artin local ring, that $Z$ is supported at one point, say $z_0$. By shrinking $X$ around $Z$ (via (10.2.3)) if necessary, we may assume that the map $\pi: X \to A'T_1, \ldots, T_r$ defined by $t$ is a quasi-finite and that $\pi^{-1}(W) = Z$, where $W$ is the closed subscheme of $A'T_1, \ldots, T_r$. By Zariski’s Main Theorem, we have a finite map $\bar{\pi}: \bar{X} \to A_\pi$, which is a compactification of $\pi$, in the sense that there exists an open immersion $u: X \to \bar{X}$ such that $\bar{\pi} \circ u = \pi$. Let $P = A_\pi \setminus \bar{\pi}(\bar{X} \setminus X)$. Then $P$ is open in $A_\pi$, $W \subset P$, and $\pi^{-1}(P) \subset X$. Replacing
X by \( \pi^{-1}(P) \) if necessary, we may assume \( \pi: X \to P \) is finite. Shrinking \( P \) around \( \mathbb{Z} \), we may assume \( P \) and \( \mathbb{Z} \) are affine, say \( P = \text{Spec} \mathcal{D} \) and \( \mathbb{Z} = \text{Spec} \mathcal{E} \). The map \( \pi \) is flat by [M, p. 174, Thm. 22.3 (3')] since \( \mathbb{Z} \) is flat over \( \mathbb{Y} \), and \( \text{Tor}_{r}^{1}(A, E) = 0 \) (the latter by noting that \( K^{\bullet}(\mathcal{T}) \otimes_{D} E = K^{\bullet}(\mathcal{Z}) \)). By (R10) and Theorem [9.2.13(i)] we have

\[
\text{Res}_{X/Y} \left[ \frac{dt_{1} \wedge \cdots \wedge dt_{r}}{k_{1}, \ldots, t_{r}} \right] = \text{Res}_{P/Y} \left[ \frac{\text{Tr}_{Z/W}(1) \cdot dt_{1} \wedge \cdots \wedge dt_{r}}{T_{1}^{k_{1}}, \ldots, T_{r}^{k_{r}}} \right]
= \text{rk}_{B/A} \cdot \text{Res}_{P/Y} \left[ \frac{dt_{1} \wedge \cdots \wedge dt_{r}}{T_{1}^{k_{1}}, \ldots, T_{r}^{k_{r}}} \right]
\]

where \( B = \mathcal{O}_{\mathbb{Z}, z_{0}} \). The last expression is zero if \( k_{1}, \ldots, k_{r} \) are not all equal to 1, since \( W \to Y \) is an isomorphism. This proves (R7)

For (R8), one direction is obvious, namely if \( \omega|_{\mathbb{Z}} = 0 \) then \( \text{Res}_{X/Y} \left[ \frac{\omega}{t_{1}, \ldots, t_{r}} \right] = 0 \), for in this case \( i^{*} \omega \otimes 1/t = 0 \). For the “converse”, by faithful flat descent we may assume \( j = 1 \), \( Y = \text{Spec} \mathcal{A} \), i.e., we may assume \( Y = \text{Spec} A \). Moreover, via (10.2.4) and (10.2.5), we may replace \( X \) by \( X_{ij} \) if necessary, and assume that \( \Gamma(X, \mathcal{O}_{X}) \to \Gamma(Z, \mathcal{O}_{Z}) \) is surjective. Since \( h: Z \to Y \) is finite, \( Z \) is affine, say \( Z = \text{Spec} \mathcal{B} \). Write \( \omega_{B/A} \) for \( \Gamma(Z, i^{*} \omega_{f} \otimes \mathcal{N}^{\nu}) \). The map \( \tau_{h} \) induces a natural isomorphism \( \omega_{B/A} \xrightarrow{\sim} \text{Hom}_{A}(B, A) \), which for any \( \nu \in \Gamma(X, \omega_{f}) \), sends \( \nu/t \in \omega_{B/A} \) to \( \varphi_{\nu} \in \text{Hom}_{A}(B, A) \) where

\[ \varphi_{\nu}(g) = \text{Res}_{X/Y} \left[ \frac{g \cdot \nu}{t_{1}, \ldots, t_{r}} \right] \quad (g \in B) \]

where \( g \in \Gamma(X, \mathcal{O}_{X}) \) is any pre-image of \( g \) under the surjective map \( \Gamma(X, \mathcal{O}_{X}) \to \Gamma(Z, \mathcal{O}_{Z}) \). It is clear that under our hypotheses, \( \varphi_{\omega} = 0 \), whence the section \( \omega/t = 0 \). This means \( \omega|_{\mathbb{Z}} = 0 \).

It remains to prove (R9)

Proof of (R9). As before, we reduce to the case where \( Y = \text{Spec} A \), \( A \) an artin local ring, \( X = \text{Spec} \mathcal{D} \) and \( Z_{\text{red}} = \{ z_{0} \} \), where \( z_{0} \) is a closed point of \( X \) lying over the closed point of \( Y \). We may assume \( A \) has an algebraically closed residue field \([\text{EGA} \ 0\text{III}, \ 10.3.1]\). Recall from [NS1] §8 C.5, especially \([\text{ibid.}, \ (C.5.1)]\), that for an \( R \)-module \( M \), we have the notion of a stable Koszul complex \( K^{\bullet}_{\omega}(t, M) \). We need this notion for arbitrary sheaves \( \mathcal{F} \) of abelian groups on \( X \) (which need not even be \( \mathcal{O}_{X} \)-modules). To that end, let \( U_{i} = \{ t_{i} \neq 0 \} = \text{Spec} R_{t_{i}}, \ i = 1, \ldots, r, \ \mathcal{U} = \{ U_{i} \} \), and for a sheaf of abelian groups \( \mathcal{F} \) on \( X \), and \( C^{\bullet}(\mathcal{U}, \mathcal{F}) \) the ordered Čech complex associated with \( \mathcal{U} \). Since \( H^{0}(\mathcal{U}, \mathcal{F}) = \mathcal{F}(U) \), the natural restriction map \( \mathcal{F}(X) \to \mathcal{F}(U) \) gives us a complex \( K^{\bullet}_{\omega}(t, \mathcal{F}) \) defined as

\[
0 \to \mathcal{F}(X) \to C^{0}(\mathcal{U}, \mathcal{F}) \to C^{1}(\mathcal{U}, \mathcal{F}) \to \cdots \to C^{r-1}(\mathcal{U}, \mathcal{F}) \to 0
\]

with \( K^{0}_{\omega}(t, \mathcal{F}) = \mathcal{F}(X), \ K^{i+1}_{\omega}(t, \mathcal{F}) = C^{i}(\mathcal{U}, \mathcal{F}) \) for \( i \geq 0 \), the first map being the composite

\[ \mathcal{F}(X) \to \mathcal{F}(U) \to C^{0}(\mathcal{U}, \mathcal{F}) \]

and the remaining maps the usual coboundary maps on Čech cohomology. If \( M \) is an \( R \)-module, clearly \( K^{\bullet}_{\omega}(t, M) = K^{\bullet}_{\omega}(t, M) \). Note that \( K^{\bullet}_{\omega}(t, \mathcal{F}) \) is functorial in \( \mathcal{F} \), as \( \mathcal{F} \) varies over sheaves of abelian groups on \( X \). In what follows, following standard conventions, we write \( U_{i_{1}, \ldots, i_{p}} := U_{i_{1}} \cap \cdots \cap U_{i_{p}} \) for \( 1 \leq i_{1} < \cdots < i_{p} \leq r \).
Since $H^0(K^\bullet_\infty(t, \mathcal{F})) = \Gamma_X(X, \mathcal{F})$, we have a functorial map of complexes

$$\Gamma_Z(X, \mathcal{F})[0] \rightarrow K^\bullet_\infty(t, \mathcal{F})$$

which is one readily checks is a quasi-isomorphism when $\mathcal{F}$ is flasque. If $\mathcal{F}^\bullet$ is a complex of flasque sheaves of abelian groups on $X$, and $\mathcal{D}(|X|)$ denotes the derived category of sheaves of abelian groups on $X$, then (10.2.7) gives us a pair of isomorphisms in $\mathcal{D}(|X|)$

$$\mathcal{R} \Gamma_Z(X, \mathcal{F}^\bullet) \rightarrow \Gamma_Z(X, \mathcal{F}^\bullet) \rightarrow \operatorname{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)).$$

The first isomorphism is from general principles (since flasque sheaves have no higher cohomologies with support), and the second is from the fact that (10.2.7) is a quasi-isomorphism on flasque sheaves.

Now suppose $\mathcal{F}$ is a sheaf of abelian groups on $X$ and $\mathcal{F} \rightarrow \mathcal{F}^\bullet$ a flasque resolution of $\mathcal{F}$. Since $\mathcal{R} \Gamma_Z(X, \mathcal{F}) \rightarrow \Gamma_Z(X, \mathcal{F}^\bullet)$, (10.2.8) gives us an isomorphism

$$\mathcal{R} \Gamma_Z(X, \mathcal{F}) \rightarrow \operatorname{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)),$$

where the right side is the total complex of the double complex $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)$.

By examining the “columns” of the double complex $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)$ one obtains a map of complexes

$$K^\bullet_\infty(t, \mathcal{F}) \rightarrow \operatorname{Tot}(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)).$$

We therefore have a map in $\mathcal{D}(|X|)$, which is functorial in $\mathcal{F}$ varying over sheaves of abelian groups,

$$K^\bullet_\infty(t, \mathcal{F}) \rightarrow \mathcal{R} \Gamma_Z(X, \mathcal{F})$$

given by $\mathcal{R} \mathcal{O}_X \tau^{-1} \circ (10.2.10)$.

If $\mathcal{F}$ is quasi-coherent then (10.2.10) is a quasi-isomorphism, since $U_{i_1} \cdots i_p$ and $X$ are affine, whence $\mathcal{F}(U_{i_1} \cdots i_p)[0] \rightarrow \mathcal{F}^\bullet(U_{i_1} \cdots i_p)$ and $\mathcal{F}(X)[0] \rightarrow \mathcal{F}^\bullet(X)$ are quasi-isomorphisms. This means, (10.2.11) is an isomorphism when $\mathcal{F}$ is quasi-coherent. In fact, in this case, by definition it agrees with $[\text{NS}1]$ (C.5.2).

Let $d_{X/Y}^{-1}: \mathcal{O}_{X/Y}^{-1} \rightarrow \mathcal{O}_{X/Y}$ be the standard exterior derivative map. Note that $d_{X/Y}^{-1}$ is not $\mathcal{O}_X$-linear. Nevertheless our discussion above gives us a commutative diagram:

$$\begin{array}{ccc}
\mathcal{K}^\bullet_\infty(t, \mathcal{O}_{X/Y}^{-1}) & \xrightarrow{d_{X/Y}^{-1}} & \mathcal{K}^\bullet_\infty(t, \mathcal{O}_{X/Y}^r) \\
\mathcal{R} \Gamma_Z(X, \mathcal{O}_{X/Y}^{-1}) & \xrightarrow{d_{X/Y}^{-1}} & \mathcal{R} \Gamma_Z(X, \mathcal{O}_{X/Y}^r)
\end{array}$$

Using the generalised fraction notation in $[\text{NS}1]$ (C.5.3) and the fact that (10.2.11) is described for quasi-coherent sheaves by $[\text{NS}1]$ (C.5.2), the commutativity of (10.2.12) gives:

$$H^r_Z(d_{X/Y}^{-1}) \left[ \begin{array}{c} \eta \\ t^{k_1}_1, \ldots, t^{k_r}_r \end{array} \right] = \left[ \begin{array}{c} dt_j \\ t^{k_1}_j, \ldots, t^{k_r}_j \end{array} \right] - \sum_{j=1}^r k_j \left[ \begin{array}{c} t^{k_1}_j, \ldots, t^{k_j+1}_j, \ldots, t^{k_r}_r \end{array} \right].$$

(10.2.13)
Thus (R9) is equivalent to:

\[(10.2.14) \quad \text{res}_Z \circ H^r_Z(\Omega^{r-1}_{X/Y}) = 0.\]

If \( I \) is the ideal of \( R \) generated by \( t \), \( R^* \) the completion of \( I \) in the \( I \)-adic topology, then (10.2.14) is equivalent to

\[(10.2.15) \quad \text{tr}_{X/Y} \circ H^r_I(\Omega^{r-1}_{X/Y}) = 0,\]

where \( \Omega^{r-1}_{X/Y} : \Omega^r_{X/Y} \to 
\Omega^r_{X/Y} \) is the exterior differentiation on the exterior algebra of universally finite differential forms on \( X/Y \).

Since the residue field of \( A \) is algebraically closed, the formal \( Y \)-scheme \( \mathcal{X} \) is isomorphic as a \( Y \)-scheme to \( \text{Spf} A[[T_1, \ldots, T_r]] \), where \( A[[T_1, \ldots, T_r]] \) is given the \( T \)-adic topology. And using the equivalence of (10.2.14) and (10.2.15) the other way, we are done if we prove (10.2.14) for \( R = A[T] \) and \( Z \) the scheme cut out by \( T \). In this case, \( \eta \) is a finite sum of \((r-1)\)-forms of the kind

\[ \eta_{j,a_1,\ldots,a_r} = T_1^{a_1} \cdots T_r^{a_r} \cdot \text{d}T_1 \wedge \cdots \wedge \text{d}T_j \wedge \cdots \wedge \text{d}T_r, \]

where \( a_i \) are non-negative integers. Since \( Z \to Y \) is an isomorphism in this case, \( \text{res}_Z \) is the standard residue which we know explicitly, the right side of (10.2.13) \( \eta = \eta_{j,a_1,\ldots,a_r} \) and \( t_i = T_i \), is trivially seen to vanish.

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