DYNAMICAL BOUNDARY CONDITIONS IN A NON-CYLINDRICAL DOMAIN FOR THE LAPLACE EQUATION

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Abstract. In this paper, we study existence, uniqueness and asymptotic behavior of the Laplace equation with dynamical boundary conditions on regular non-cylindrical domains. We write the problem as a non-autonomous Dirichlet-to-Neumann operator and use form methods in a more general framework to accomplish our goal. A class of non-autonomous elliptic problems with dynamical boundary conditions on Lipschitz domains is also considered in this same context.

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1. INTRODUCTION

In this paper, we consider the following problem:

\[
\begin{aligned}
(\lambda + \Delta) u (t, x) &= 0, (t, x) \in D, t > t_0, \\
\frac{\partial u}{\partial t} (t, x) &= -\frac{\partial u}{\partial n} (t, x) + f (t, x), (t, x) \in S, t > t_0, \\
u (t_0, x) &= u_0 (x), x \in \partial \Omega_{t_0},
\end{aligned}
\]

where $\lambda < 0$ is a constant, $t_0 \geq 0$, $D \subset \mathbb{R}^{n+1}$ is an appropriate open set of real variables $(t, x) = (t, x_1, ..., x_n)$ bounded by a bounded domain $\Omega_0 \subset \mathbb{R}^n$ at $t = 0$ and a $n$-dimensional surface $S$ on the half space $t > 0$. We denote by $\Omega_\tau$, and by $\partial \Omega_\tau$, $\tau > 0$, the intersections $D \cap \{ (t, x) \in \mathbb{R}^{n+1}; t = \tau \}$ and $S \cap \{ (t, x) \in \mathbb{R}^{n+1}; t = \tau \}$, respectively. Therefore $D = \bigcup_{t>0} \Omega_t$ and $S = \bigcup_{t>0} \partial \Omega_t$. The outward normal derivative at the point $x \in \partial \Omega_t$ is denoted by $\frac{\partial u}{\partial n} (t, x)$, and $n$ is the unit outward normal vector to the boundary $\partial \Omega_t$.
Our main goal is to show existence and uniqueness of the solutions of this Laplace equation with dynamical boundary conditions. Notice that this task is not trivial since problem (1.1) is posed in a non-cylindrical domain $D$. Furthermore, we study the asymptotic behavior of the solutions at infinite time, when $\Omega_t$ converges to a domain $\Omega$, and $f(t,\cdot)$ converges to a function $f_\infty(\cdot)$. In this situation, we obtain that the solutions converge to the stationary problem
\[
\begin{aligned}
(\lambda + \Delta) u_\infty (x) &= 0, \quad x \in \Omega \\
\frac{\partial u_\infty}{\partial n} (x) &= f_\infty (x), \quad x \in \partial \Omega.
\end{aligned}
\]

We set the non-cylindrical domain $D$ by smooth perturbations of a fixed open set $\Omega$ which are defined by diffeomorphisms according, for instance, to D. Henry in [17]. Performing a change of variable, we transform (1.1) in a non-autonomous Dirichlet-to-Neumann operator posed in the cylindrical set $[0, \infty] \times \Omega$, which leads us to consider non-autonomous elliptic equations with dynamical boundary conditions. We introduce these non-autonomous equations in the following way:

Let $P(t,x,D)$ be a second order elliptic operator acting on the bounded domain $\Omega \subset \mathbb{R}^n$ given by
\[
P(t,x,D) := - \sum_{i,j=1}^{n} \partial_{x_i} (a_{ij}(t,x) \partial_{x_j} u(x)) + \sum_{j=1}^{n} b_j(t,x) \partial_{x_j} u(x) \]
\[
- \sum_{j=1}^{n} \partial_{x_j} (c_j(t,x) u(x)) + d(t,x) u(x).
\]

Suppose that for every suitable function $g$ defined on the boundary $\partial \Omega$, there exists a unique solution to the Dirichlet problem: $P(t,x,D) u = 0$ and $u|_{\partial \Omega} = g$. In this case, the Dirichlet-to-Neumann operator, denoted here by $A(t)$, is the operator that maps $g$ to the conormal derivative of $u$ on $\partial \Omega$:
\[
A(t)(g) = C(t,x,D)(u) := \sum_{i,j=1}^{n} a_{ij}(t,x) \nu_j(u(x)) \partial_{x_j} u(x) + \sum_{j=1}^{n} c_j(t,x) u(x) \nu_j(x),
\]
where $\nu(x) = (\nu_1(x),...,\nu_n(x))$ denotes the outer unit normal at $x \in \partial \Omega$.

Thereby, we introduce the non-autonomous problem defined on the boundary $\partial \Omega$:
\[
\begin{aligned}
\frac{du}{dt} (t) + A(t) u(t) &= f(t), \quad t \geq t_0 \\
\quad u(t_0) &= u_0.
\end{aligned}
\]

Here, we give simple conditions that guarantee that the above problem is well-posed. In order to do that on $H^{-\frac{1}{2}}(\partial \Omega)$, we use form methods and assume Hölder continuity with exponent $\alpha \in [0,1]$ in time of the forms that define the operators $\{A(t)\}_{t \geq t_0}$. We are then able to show that the operators satisfy the Tanabe-Sobolevskii conditions. On $L^2(\partial \Omega)$, we assume Hölder continuity of the forms with an exponent $\alpha \in [\frac{1}{2},1]$ and prove that the operators satisfy the Yagi conditions. Our assumptions also allow us to study the asymptotic behavior at infinity of the Problem (1.4). Consequently, the results concerned with (1.4) are directly obtained as an application of the results obtained for (1.4).

Our approach follows W. Arendt and A. Elst [14], see also [3, 10], where form methods were used to deal with the Dirichlet-to-Neumann problem defined by the Laplace operator. Their technique has been also used for general second order problems by J. Abreu and E. Capelato [11], and E. M. Ouhabaz [10]. The novelty here is to study non-autonomous problems in bounded domains combining form methods for the Dirichlet-to-Neumann problem with the Tanabe-Sobolevskii and the Yagi conditions [20, 24, 25].
Notice that dynamical boundary conditions have been studied by many authors. Among them, we mention: J. Lions \[19\], J. Escher \[11\]12, J. Escher and J. Seiler \[13\], T. Hintermann \[18\], A. Friedman and Shinbrot \[15\] and L. Vazquez and E. Vitillaro \[27\]. Although there exists a big literature on the study of parabolic equations on non-cylindrical domains, among which we can mention the pioneering work of A. Friedman \[14\], as well as Bonaccorsi and G. Guatteri \[8\], the recent papers by Ma To Fu et al. \[22\] and J. Calvo et al. \[9\], we could not find any result for the Laplace equation with dynamical boundary conditions on non-cylindrical domains. Thus, we are fulfilling this gap with this work, besides our study of non-autonomous equations with dynamical boundary conditions. It is interesting to note that convergence of the Dirichlet-to-Neumann operators associated with the Laplacian on varying domains was considered by A.F.M. ter Elst and E.M. Ouhabaz \[10\], but they did not studied it as a change in time and their results are quite different from ours.

The organization of the paper is as follows: In the Section 2, we give a rigorous definition of the Dirichlet-to-Neumann operators on Lipschitz domains and we characterize them as bounded operators from \(H^\frac{1}{2}(\partial\Omega)\) to \(H^{-\frac{1}{2}}(\partial\Omega)\) using forms. Next, we recall the Tanabe-Sobolevski\'s conditions discussing how to recover them using form methods, to finally study existence, uniqueness and asymptotic behavior to Problem (1.4) on \(H^{-\frac{1}{2}}(\partial\Omega)\). Using the Yagi conditions, we are then able to extend the study to \(L^2(\partial\Omega)\) functions. In Section 3, we apply the results of Section 2 to a non-autonomous elliptic equation with dynamic boundary conditions in bounded Lipschitz domains, and then, to the Laplace equation with dynamic boundary conditions on non-cylindrical domains.

2. Non-autonomous Dirichlet-to-Neumann problem

In this section, \(\Omega \subset \mathbb{R}^n\) is a Lipschitz bounded domain. We set some notations and recall some facts, whose proofs can be found, for instance, in P. Grisvard \[16\]. For \(m \in \mathbb{Z}\), the Sobolev spaces are defined by:

\[
H^m(\Omega) := \left\{ u \in L^2(\Omega) ; \sum_{|\alpha| \leq m} \|\partial^\alpha u\|_{L^2(\Omega)}^2 < \infty, \text{ if } m \geq 0 \right\}
\]

\[
H^{m+1}(\Omega) := \left\{ u \in \mathcal{D}'(\Omega) ; \exists u_\alpha \in L^2(\Omega), |\alpha| \leq m, \text{ such that } u = \sum_{|\alpha| \leq m} \partial^\alpha u_\alpha, \text{ if } m < 0 \right\}
\]

The norms are denoted by \(u \in H^m(\Omega) \mapsto \|u\|_{H^m(\Omega)}\). The Sobolev space \(H^s(\partial\Omega)\), \(0 < s < 1\), is defined as the space of all measurable functions \(u : \partial\Omega \to \mathbb{C}\) such that

\[
\|u\|_{H^s(\partial\Omega)}^2 := \int_{\partial\Omega} \|u(x)\|^2 d\sigma_x + \int_{\partial\Omega \times \partial\Omega} \frac{|u(x) - u(y)|^2}{\|x-y\|^{n+1+2s}} d\sigma_x d\sigma_y < \infty,
\]

where \(\sigma_x\) is the surface measure of \(\partial\Omega\). We denote by \(C^m(\overline{\Omega})\) the set of all functions \(u\) such that the derivatives \(\partial^\alpha u\) exist for all \(|\alpha| \leq m\) and are continuous functions up to the boundary. The set of functions of class \(C^m\) whose support is contained in \(\Omega\) will be denoted by \(C^m_c(\Omega)\). The trace operator \(\gamma_0 : H^1(\Omega) \to H^\frac{1}{2}(\partial\Omega)\) is the unique continuous extension of the function \(C^1(\overline{\Omega}) \ni u \mapsto u|_{\partial\Omega} \in C(\partial\Omega)\). As usual we denote \(H_0^1(\Omega) := \ker(\gamma_0)\). Note that there exists a continuous extension operator \(E : H^\frac{1}{2}(\partial\Omega) \to H^1(\Omega)\) such that \(\gamma_0 \circ E\) is the identity. Denoting by \((\cdot, \cdot)_{L^2(\Omega)}\) and by \((\cdot, \cdot)_{L^2(\partial\Omega)}\) the usual scalar products of \(L^2(\Omega)\) and \(L^2(\partial\Omega)\), respectively, it is well known that \((\cdot, \cdot)_{L^2(\Omega)} \times C^1_c(\Omega) \to \mathbb{C}\) extends uniquely to a sesquilinear form \((\cdot, \cdot)_{H^{-1}(\Omega) \times H_0^1(\Omega)} : H^{-1}(\Omega) \times H_0^1(\Omega) \to \mathbb{C}\) that allows the identification of \(H^{-1}(\Omega)\) with the set of all continuous anti-linear functionals of \(H_0^1(\Omega)\). The anti-dual space of \(H^1(\Omega)\) is denoted by \(H^1(\Omega)^*\) and the dual product of \(u \in H^1(\Omega)^*\) and \(v \in H^1(\Omega)\) is denoted by \(\langle u, v \rangle_{H^1(\Omega)^*, H^1(\Omega)}\).
Similarly we define $H^{-\frac{1}{2}}(\partial\Omega)$ as the anti-dual space of $H^{\frac{1}{2}}(\partial\Omega)$ and denote the dual product of $u \in H^{-\frac{1}{2}}(\partial\Omega)$ and $v \in H^{\frac{1}{2}}(\partial\Omega)$ as $\langle u, v \rangle_{H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)}$. As the inclusion $H^{\frac{1}{2}}(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$ is injective with dense range, we can define an injective map with dense range $L^2(\partial\Omega) \hookrightarrow H^{-\frac{1}{2}}(\partial\Omega)$ as $\langle u, v \rangle_{H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)} := \langle u, v \rangle_{L^2(\partial\Omega)}$, when $u \in L^2(\partial\Omega)$ and $v \in H^{\frac{1}{2}}(\partial\Omega)$.

If $E$ and $F$ are Banach spaces, $B(E, F)$ is the set of all continuous maps from $E$ to $F$ and $B(E) := B(E, E)$. The space $C_0^\alpha([t_0, \infty[, E)$ is the space of all uniformly $\alpha$-Hölder continuous functions from $[t_0, \infty[ \cap E$, that is, if $f \in C_0^\alpha([t_0, \infty[, E)$, then there is $C > 0$ such that

$$
\|f(t) - f(s)\|_E \leq C|t-s|^\alpha, \forall t, s \geq t_0.
$$

Similarly $C_0^1([t_0, \infty[ , E)$ is the set of all $C^1$ functions $f$ such that $f$ and $\frac{df}{dt}$ belong to $C_0^\alpha([t_0, \infty[, E)$.

In order to study the Dirichlet-to-Neumann problem, we consider the following forms $a_t : H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$, $t \in [t_0, \infty[ $,

$$
(2.1) \quad a_t(u, v) = \int_\Omega \left( \sum_{i,j=1}^n a_{ij}(t,x) \partial_{x_i} u(x) \overline{\partial_{x_j} v(x)} + \sum_{j=1}^n b_j(t,x) \partial_{x_j} u(x) \overline{v(x)} 
+ \sum_{j=1}^n c_j(t,x) u(x) \partial_{x_j} v(x) + d(t,x) u(x) v(x) \right) dx.
$$

These are the forms associated with the differential operators $P(t, x, D)$ defined in (1.2). Below are the basic assumptions we shall use in this paper.

**Assumption 1.** The coefficients of the forms $\{a_t\}_{t \in [t_0, \infty[}$ satisfy the assumptions for $H^{-\frac{1}{2}}(\partial\Omega)$ if:

1) There is an $\alpha \in [0, 1]$ such that $a_{ij}, b_j, c_j$, and $d \in C_0^\alpha([t_0, \infty[, L^\infty(\Omega))$, for all $i, j$.
2) $a_{ij}([t_0, \infty[, b_j([t_0, \infty[, c_j([t_0, \infty[)$ and $d([t_0, \infty[) \in L^\infty(\Omega)$. 
3) There is a constant $C > 0$ such that

$$
(2.2) \quad Re(a_t(u, u)) \geq C \|u\|_{H^1(\Omega)}^2, \forall t \in [t_0, \infty[ , \forall u \in H^1(\Omega).
$$

4) $\lim_{t \to \infty} a_{ij}(t, .) = a_{ij}(\infty, .), \lim_{t \to \infty} b_j(t, .) = b_j(\infty, .), \lim_{t \to \infty} c_j(t, .) = c_j(\infty, .)$ and $\lim_{t \to \infty} d(t, .) = d(\infty, .)$ in $L^\infty(\Omega)$, for all $i, j$.

**Assumption 2.** The coefficients of the forms $\{a_t\}_{t \in [t_0, \infty[}$ satisfy the assumptions for $L^2(\partial\Omega)$ if they satisfy all conditions of Assumption 1 for $\alpha \in [\frac{1}{2}, 1]$. 

We will always assume that, at least, the Assumption 1 holds. The stronger Assumption 2 will only be necessary when we deal with the problem on $L^2(\partial\Omega)$, as it will be the case in Sections 2.1.2 and 3.2.

**Remark 1.** Conditions 1 and 4 of the Assumption 1 imply that the coefficients are bounded. Hence there is a constant $M > 0$ such that

$$
\|a_t(u, v)\| \leq M \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}, \forall t \in [t_0, \infty[ , u, v \in H^1(\Omega).
$$

The above conditions together with the Lax-Milgram Theorem can be used to define two important operators: $B_{t,D} : H^1_0(\Omega) \to H^{-1}(\Omega)$ and $B_{t,N} : H^1(\Omega) \to H^1(\Omega)^*$. They are the unique isomorphisms that satisfy

$$
\langle a_t(u, v) \rangle_{H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)}, \quad \text{for all } u, v \in H^1_0(\Omega),
$$

and

$$
\langle a_t(u, v) \rangle_{H^{1}(\Omega)^* \times H^{1}(\Omega)}, \quad \text{for all } u, v \in H^1(\Omega).
$$
It is easy to see that $B_{t,N}$ is equal to the operator $P(t,x,D)$ acting on $H^1_0(\Omega)$ in the sense of distributions and that, for all $t \in [t_0, \infty]$: 

$$
\|B_{t,N}\|_{B(H^1_0(\Omega),H^{-1}(\Omega))} \leq M, \quad \|B_{t,N}\|_{B(H^1(\Omega),H^1(\Omega)^*)} \leq M
$$

$$
\|B_{t,D}^{-1}\|_{B(H^{-1}(\Omega),H^1_0(\Omega))} \leq \frac{1}{C} \quad \text{and} \quad \|B_{t,N}^{-1}\|_{B(H^1(\Omega)^*,H^{-1}(\Omega))} \leq \frac{1}{C}.
$$

**Definition 2.** Let $u \in H^1(\Omega)$. We say that $C(t,x,D)u$ exists in the $H^{-\frac{1}{2}}(\partial\Omega)$-weak sense and it is equal to $y \in H^{-\frac{1}{2}}(\partial\Omega)$ if

$$
\langle a_t(u,v) \rangle_{H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)} = \langle y, \gamma_0(v) \rangle_{H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)}, \forall v \in H^1(\Omega).
$$

If $y \in L^2(\partial\Omega)$, then we say that $C(t,x,D)u$ exists in the $L^2(\partial\Omega)$-weak sense.

By the divergence theorem, the above definition coincides with the usual conormal derivative if $P(t,x,D)u = 0$ and if we impose sufficiently regularity to $u$ and to the coefficients. Our definition of weak conormal derivative can be found in a similar way in [1],[4],[10].

If $u \in H^1(\Omega)$ is such that $C(t,x,D)u$ exists in the $H^{-\frac{1}{2}}(\partial\Omega)$-weak sense, then considering $v \in C_0^\infty(\Omega)$ in the Definition 2, we conclude that $P(t,x,D)u = 0$. On the other hand, we have:

**Proposition 3.** Let $u \in H^1(\Omega)$ be such that $P(t,x,D)u = 0$. Then $C(t,x,D)u$ exists in the $H^{-\frac{1}{2}}(\partial\Omega)$-weak sense. Moreover, there exists a constant $C > 0$, which does not depend on $t \in [t_0, \infty]$, such that

$$
\|C(t,x,D)u\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq C\|u\|_{H^1(\Omega)},
$$

whenever $t \in [t_0, \infty]$ and $u \in H^1(\Omega)$ is such that $P(t,x,D)u = 0$.

**Proof.** Let $u \in H^1(\Omega)$ be such that $P(t,x,D)u = 0$. We define $y \in H^{-\frac{1}{2}}(\partial\Omega)$ as

$$
\langle y, z \rangle_{H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)} = a_t(u, \mathcal{E}(z)), z \in H^{\frac{1}{2}}(\partial\Omega).
$$

First we show that the above definition is independent of the extension of $z$ we choose: if $\tilde{z} \in H^1(\Omega)$ is such that $\gamma_0(\tilde{z}) = z$, then

$$
\langle y, z \rangle_{H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)} = a_t(u, \tilde{z}).
$$

Indeed, by the construction of $y$, the expression (2.3) holds if $\tilde{z} = \mathcal{E}(z)$. Let us now suppose that $\tilde{z} \in H^1(\Omega)$ is any other function such that $\gamma_0(\tilde{z}) = z$. As $C_0^\infty(\Omega)$ is dense in $H^1_0(\Omega)$, we know that

$$
a_t(u, \mathcal{E}(\tilde{z}) - \tilde{z}) = (P(t,x,D)u, \mathcal{E}(\tilde{z}) - \tilde{z})_{H^{-1}(\Omega) \times H^1_0(\Omega)}, \forall v \in H^1(\Omega) \quad \text{and} \quad u \in H^1_0(\Omega).
$$

Since $\mathcal{E}(\tilde{z}) - \tilde{z} \in H^1_0(\Omega)$ and $P(t,x,D)u = 0$, we have

$$
a_t(u, \mathcal{E}(\tilde{z}) - \tilde{z}) = (P(t,x,D)u, \mathcal{E}(\tilde{z}) - \tilde{z})_{H^{-1}(\Omega) \times H^1_0(\Omega)} = 0.
$$

This implies that

$$
\langle y, z \rangle_{H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)} = a_t(u, \mathcal{E}(z)) = a_t(u, \tilde{z}).
$$

In particular,

$$
a_t(u,v) = \langle y, \gamma_0(v) \rangle_{H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)}, \forall v \in H^1(\Omega).
$$

Therefore $C(t,x,D)u$ exists in the $H^{-\frac{1}{2}}(\partial\Omega)$-sense and it is equal to $y$.

Finally note that (2.3) implies that

$$
\|C(t,x,D)u\|_{H^{-\frac{1}{2}}(\partial\Omega)} = \|y\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq M \|E\|_{B(H^{\frac{1}{2}}(\partial\Omega),H^{-1}(\Omega))} \|u\|_{H^1(\Omega)},
$$

where $M$ is the constant of Remark 1.

\[\square\]
The definition of the Dirichlet-to-Neumann operator and the study of its asymptotic behavior require the next simple proposition.

**Proposition 4.** 1) (Dirichlet Problem) Let \( y \in H^{1/2}_*(\partial \Omega) \). Then there is a unique \( u_t \in H^1(\Omega) \) such that \( P(t, x, D) u_t = 0 \) and \( \gamma_0(u_t) = y \). It is given by \( E(y) - \mathcal{B}_{t,N}^{-1} P(t, x, D) E(y) \).

2) (Neumann Problem) Let \( y \in H^{-1/2}_*(\partial \Omega) \). Then there is a unique \( u_t \in H^1(\Omega) \) such that \( P(t, x, D) u_t = 0 \) and \( C(t, x, D) u_t = y \). It is given by \( \mathcal{B}_{t,N}^{-1} \circ k(y) \), where \( k : H^{-1/2}_*(\partial \Omega) \to H^1(\Omega)^* \) is defined as

\[
\langle k(w), v \rangle_{H^1(\Omega)^* \times H^1(\Omega)} = \langle w, \gamma_0(v) \rangle_{H^{-1/2}_*(\partial \Omega) \times H^{1/2}_*(\partial \Omega)}, v \in H^1(\Omega).
\]

**Proof.** 1) \( u_t = E(y) - \mathcal{B}_{t,N}^{-1} P(t, x, D) E(y) \) is clearly a solution of the Dirichlet problem. If \( v_t \in H^1(\Omega) \) is another solution, then \( \gamma_0(u_t - v_t) = 0 \). Hence \( u_t - v_t \in H^1_0(\Omega) \) and, due to item 3 of Assumption[1],

\[
C \| u_t - v_t \|^2_{H^1(\Omega)} \leq \text{Re} \left( a_t(u_t - v_t, u_t - v_t) \right) = \text{Re} \left( \langle P(t, x, D)(u_t - v_t), (u_t - v_t) \rangle_{H^{-1}(\Omega) \times H^1_0(\Omega)} \right) = 0.
\]

2) Let \( u_t = \mathcal{B}_{t,N}^{-1} \circ k(y) \). Then, for \( v \in H^1(\Omega) \),

\[
\langle a_t \left( \mathcal{B}_{t,N}^{-1} \circ k(y), v \right), v \rangle_{H^1(\Omega)^* \times H^1(\Omega)} = \langle y, \gamma_0(v) \rangle_{H^{-1/2}_*(\partial \Omega) \times H^{1/2}_*(\partial \Omega)}.
\]

We conclude that \( y \) is the \( H^{-1/2}_*(\partial \Omega) \)-weak conormal derivative of \( u_t \) and, therefore, the solution of the Neumann problem. If \( v_t \in H^1(\Omega) \) is another solution, then

\[
a_t(u_t - v_t, u_t - v_t) = \langle C(t, x, D)(u_t - v_t), \gamma_0(u_t - v_t) \rangle_{H^{-1/2}_*(\partial \Omega) \times H^{1/2}_*(\partial \Omega)} = 0.
\]

Hence \( u_t = v_t \). \( \square \)

Finally we give a precise definition of the Dirichlet-to-Neumann operator.

**Definition 5.** For each \( t \in [t_0, \infty] \), we define a bounded operator \( A(t) : H^{1/2}_*(\partial \Omega) \to H^{-1/2}_*(\partial \Omega) \), called Dirichlet-to-Neumann operator, as

\[
A(t)y = C(t, x, D) \left( E(y) - \mathcal{B}_{t,D}^{-1}(P(t, x, D) E(y)) \right).
\]

**Proposition 6.** The operators \( A(t) \) are invertible for all \( t \in [t_0, \infty] \). Moreover the families

\[
\left\{ A(t) \in \mathcal{B}(H^{1/2}_*(\partial \Omega), H^{-1/2}_*(\partial \Omega)) \right\}_{t \in [t_0, \infty]} \quad \text{and} \quad \left\{ A(t)^{-1} \in \mathcal{B}(H^{-1/2}_*(\partial \Omega), H^{1/2}_*(\partial \Omega)) \right\}_{t \in [t_0, \infty]}
\]

are uniformly bounded.

**Proof.** In order to conclude that the family \( \{ A(t) \}_{t \in [t_0, \infty]} \) is uniformly bounded, it is enough to note that \( \mathcal{B}_{t,D}^{-1} : H^{-1}_0(\Omega) \to H^1_0(\Omega) \), \( P(t, x, D) : H^1(\Omega) \to H^{-1}(\Omega) \) and \( C(t, x, D) : \ker(P(t, x, D)) \subset H^1(\Omega) \to H^{-1/2}_*(\partial \Omega) \) are uniformly bounded.

For the family \( \{ A(t)^{-1} \}_{t \in [t_0, \infty]} \), we first need a good representation of the inverse. We have seen that \( A(t)y = C(t, x, D) u_t \), where \( u_t = E(y) - \mathcal{B}_{t,D}^{-1}(P(t, x, D) E(y)) \). Hence \( u_t \) solves the Neumann problem \( P(t, x, D) u_t = 0 \) and \( C(t, x, D) u_t = A(t)y \). Proposition[4] (2) implies that \( u_t = \mathcal{B}_{t,N}^{-1} \circ k(A(t)y) \). Therefore \( y = \gamma_0(u_t) = \gamma_0 \circ \mathcal{B}_{t,N}^{-1} \circ k(A(t)y) \).
On the other hand, if \( z \in H^{-\frac{1}{2}}(\partial \Omega) \), then, due to (2.24), \( v_t = B_{\perp_N}^{-1} \circ k(z) \) is such that

\[
a_t(v_t, v) = \langle z, \gamma_0(v) \rangle_{H^{-\frac{1}{2}}(\partial \Omega) \times H^{\frac{1}{2}}(\partial \Omega)}, \quad \forall v \in H^1(\Omega).
\]

Thus we conclude that \( A(t) \left( \gamma_0 \circ B_{\perp_N}^{-1} \circ k(z) \right) = A(t) (\gamma_0(v_t)) = z \).

The above discussion implies that \( A(t)^{-1} = \gamma_0 \circ B_{\perp_N}^{-1} \circ k \). Therefore \( \{ A(t)^{-1} \}_{t \in [t_0, \infty)} \) is uniformly bounded, since \( B_{\perp_N}^{-1} : H^1(\Omega)^* \rightarrow H^1(\Omega) \) is uniformly bounded. \( \square \)

We end this subsection giving a characterization of the Dirichlet-to-Neumann operators from \( H^\frac{1}{2}(\partial \Omega) \) to \( H^{-\frac{1}{2}}(\partial \Omega) \) using form methods.

**Theorem 7.** For every \( y \in H^\frac{1}{2}(\partial \Omega) \), there is a \( u_t \in H^1(\Omega) \) such that

\[
\gamma_0(u_t) = y \quad \text{and} \quad a_t(u_t, v) = \langle A(t)y, \gamma_0(v) \rangle_{H^{-\frac{1}{2}}(\partial \Omega) \times H^{\frac{1}{2}}(\partial \Omega)},
\]

for all \( v \in H^1(\Omega) \). The Dirichlet-to-Neumann operators \( A(t) \in B \left( H^\frac{1}{2}(\partial \Omega), H^{-\frac{1}{2}}(\partial \Omega) \right) \) are the only operators with this property.

**Proof.** By definition, \( A(t)y = C(t, x, D) u_t \), where \( u_t = E(y) - B_{\perp_D}^{-1} P(t, x, D) E(y) \). Using the definition of \( C(t, x, D) u_t \), we see that

\[
a_t(u_t, v) = \langle C(t, x, D) u_t, \gamma_0(v) \rangle_{H^{-\frac{1}{2}}(\partial \Omega) \times H^{\frac{1}{2}}(\partial \Omega)}, \quad v \in H^1(\Omega),
\]

which is equivalent to (2.6).

Let us now prove uniqueness. Suppose that \( A(t) : H^\frac{1}{2}(\partial \Omega) \rightarrow H^{-\frac{1}{2}}(\partial \Omega) \) and \( \tilde{A}(t) : H^\frac{1}{2}(\partial \Omega) \rightarrow H^{-\frac{1}{2}}(\partial \Omega) \) are operators that satisfy the properties stated in the theorem. Then, for every \( y \in H^\frac{1}{2}(\partial \Omega) \), there exist \( u_t \in H^1(\Omega) \) and \( \tilde{u}_t \in H^1(\Omega) \) such that \( \gamma_0(u_t) = \gamma_0(\tilde{u}_t) = y \), and, for all \( v \in H^1(\Omega) \),

\[
a_t(u_t, v) = \langle A(t)y, \gamma_0(v) \rangle_{H^{-\frac{1}{2}}(\partial \Omega) \times H^{\frac{1}{2}}(\partial \Omega)} \quad \text{and} \quad a_t(\tilde{u}_t, v) = \langle \tilde{A}(t)y, \gamma_0(v) \rangle_{H^{-\frac{1}{2}}(\partial \Omega) \times H^{\frac{1}{2}}(\partial \Omega)}.
\]

In this case,

\[
a_t(u_t - \tilde{u}_t, v) = \langle A(t)y - \tilde{A}(t)y, \gamma_0(v) \rangle_{H^{-\frac{1}{2}}(\partial \Omega) \times H^{\frac{1}{2}}(\partial \Omega)}, \quad \forall v \in H^1(\Omega).
\]

Choosing \( v = u_t - \tilde{u}_t \), we have \( a_t(u_t - \tilde{u}_t, u_t - \tilde{u}_t) = 0 \). Hence \( u_t = \tilde{u}_t \) and

\[
\langle A(t)y - \tilde{A}(t)y, \gamma_0(v) \rangle_{H^{-\frac{1}{2}}(\partial \Omega) \times H^{\frac{1}{2}}(\partial \Omega)} = 0, \quad \forall v \in H^1(\Omega),
\]

which implies that \( A(t) = \tilde{A}(t) \). \( \square \)

**Remark 8.** The proof of Theorem 7 implies that the function \( u_t \) associated to \( y \) is unique and it is given by \( E(y) - B_{\perp_D}^{-1} P(t, x, D) E(y) \). As \( B_{\perp_D}^{-1} : H^{-1}(\Omega) \rightarrow H^1(\Omega) \) and \( P(t, x, D) : H^1(\Omega) \rightarrow H^{-1}(\Omega) \) are uniformly bounded, there is a constant \( C > 0 \), which does not depend on \( t \in [t_0, \infty] \), such that

\[
\|u_t\|_{H^1(\Omega)} \leq C \|y\|_{H^\frac{1}{2}(\partial \Omega)} = C \|\gamma_0(u_t)\|_{H^\frac{1}{2}(\partial \Omega)}, \quad \forall y \in H^\frac{1}{2}(\partial \Omega).
\]

2.1. Well-posedness and asymptotic behavior.
2.1.1. The Tanabe-Sobolevskii conditions. Let $H$ and $D$ be Hilbert spaces such that $D \subset H$ is a dense subset and the injection $D \hookrightarrow H$ is continuous. We consider a family of bounded operators \( \{S(t)\}_{t \in \mathbb{R}} \) in $H$.

**Definition 9.** The family of operators \( \{S(t)\}_{t \in [t_0, \infty]} \) satisfies the Tanabe-Sobolevskii conditions if

1) The set $\{\lambda \in \mathbb{C}, \text{Re} \lambda \leq 0 \}$ is contained in the resolvent set of the linear operator $S(t): H \to H$, $t \in [t_0, \infty]$, and there is a constant $C > 0$ such that

$$
\left\| (\lambda - S(t))^{-1} \right\|_{\mathcal{B}(H)} \leq \frac{C}{1 + |\lambda|}, \quad \text{Re} \lambda \leq 0, \quad t \in [t_0, \infty].
$$

2) The function $[t_0, \infty] \ni t \to S(t) \in \mathcal{B}(D, H)$ belongs to $C^\alpha_u ([t_0, \infty), \mathcal{B}(D, H))$, for some $\alpha \in (0, 1]$.

3) $\lim_{t \to \infty} \|S(t) - S(\infty)\|_{\mathcal{B}(D, H)} = 0$.

4) The families $\{S(t) \in \mathcal{B}(D, H)\}_{t \in [t_0, \infty]}$ and $\left\{S(t)^{-1} \in \mathcal{B}(H, D)\right\}_{t \in [t_0, \infty]}$ are uniformly bounded.

**Theorem 10.** (Tanabe-Sobolevskii) Let $f \in C^\alpha_u ([t_0, \infty), H)$. Then, for every $u_0 \in H$, there is a unique function $u \in C([t_0, \infty), H) \cap C^1([t_0, \infty), H) \cap C([t_0, \infty), D)$ such that

$$
\frac{du}{dt} (t) + S(t) u(t) = f(t), \quad t > t_0.
$$

The operator $S(\infty): D \to H$ is invertible and if $\lim_{t \to \infty} f(t) = f_\infty \in H$, then $u_\infty = S(\infty)^{-1} f_\infty \in D$ is such that $\lim_{t \to \infty} \|u(t) - u_\infty\|_{D} = 0$. In other words, $u(t)$ converges to the stationary solution $S(\infty)u_\infty = f_\infty$.

**Proof.** The existence and uniqueness of $u$ follows from Theorem 6.8 in [23, Chapter 5.6]. One can even show that the solution is Hölder continuous [23, Theorem 1.2.1].

From Tanabe [26, Theorem 5.6.1] (see also A. Pazy [23]), we know that $S(\infty): D \to H$ is a bijective operator and, for $u_\infty = S(\infty)^{-1} f_\infty \in D$, we have

$$
\lim_{t \to \infty} \|u(t) - u_\infty\|_{H} = 0 \quad \text{and} \quad \lim_{t \to \infty} \left\| \frac{du}{dt} (t) \right\|_{H} = 0.
$$

As $\frac{du}{dt} (t) + S(t) u(t) = f(t)$, we conclude that $\lim_{t \to \infty} \|u(t) - u_\infty\|_{D} = 0$. In fact, we have that

$$
\|u(t) - u_\infty\|_{D} = \left\| S(t)^{-1} f(t) - S(t)^{-1} \frac{du}{dt} (t) - S(\infty)^{-1} f_\infty \right\|_{D}
\leq \left\| S(t)^{-1} \right\|_{\mathcal{B}(H,D)} \|f(t) - f_\infty\|_{H} + \left\| S(t)^{-1} - S(\infty)^{-1} \right\|_{\mathcal{B}(H,D)} \|f_\infty\|_{H}
\rightarrow 0.
$$

Note that the first and last terms on the right hand side of the above inequality go to zero due to (2.8), to the convergence of the functions $f(t)$ and to the uniform boundedness of the set $\{S(t)^{-1}\}_{t \in [t_0, \infty]}$. Also the second one goes to zero, due to the third and fourth items of Definition 9 and the inequality below

$$
\left\| S(t)^{-1} - S(\infty)^{-1} \right\|_{\mathcal{B}(H,D)} \leq \left\| S(t)^{-1} \right\|_{\mathcal{B}(H,D)} \left\| S(t) - S(\infty) \right\|_{\mathcal{B}(D,H)} \left\| S(\infty)^{-1} \right\|_{\mathcal{B}(H,D)}.
$$

\(\square\)
2.1.2. The Dirichlet-to-Neumann operator in \( H^{-\frac{1}{2}}(\partial \Omega) \). The scalar product of \( L^2(\partial \Omega) \) allows the definition of the map \( y \in H^\frac{1}{2}(\partial \Omega) \mapsto \left( x \in H^\frac{1}{2}(\partial \Omega) \mapsto (y, x)_{L^2(\partial \Omega)} \in \mathbb{C} \right) \in H^{-\frac{1}{2}}(\partial \Omega) \). Using this map, we can identify \( H^\frac{1}{2}(\partial \Omega) \) as a dense subspace of \( H^{-\frac{1}{2}}(\partial \Omega) \). As always, we assume that Assumption 1 holds.

**Theorem 11.** The family \( \left\{ A(t) \in \mathcal{B} \left( H^\frac{1}{2}(\partial \Omega), H^{-\frac{1}{2}}(\partial \Omega) \right) \right\} \) of Dirichlet-to-Neumann operators defined by Definition 5 satisfies the Tanabe-Sobolevskii conditions.

**Proof.** Let us check all conditions of Definition 5

1. We define the form \( a_{t,\lambda} : H^1(\Omega) \times H^1(\Omega) \to \mathbb{C} \) by
   \[
a_{t,\lambda}(u, v) = a_t(u, v) - \lambda \langle \gamma_0(u), \gamma_0(v) \rangle_{H^{-\frac{1}{2}}(\partial \Omega) \times H^\frac{1}{2}(\partial \Omega)}.
   \]

The form \( a_{t,\lambda} \) is continuous and, as \( \text{Re}(\lambda) \leq 0 \), it satisfies \( \text{Re}(a_{t,\lambda}(u, u)) \geq C \|u\|_{H^1(\Omega)}^2 \). Hence, by the Lax-Milgram Theorem, there is an isometry \( B_{t,\lambda,N} : H^1(\Omega) \to H^1(\Omega)^* \) such that
   \[
a_{t,\lambda}(u, v) = \langle B_{t,\lambda,N}(u), v \rangle_{H^1(\Omega) \times H^1(\Omega)}, \quad u, v \in H^1(\Omega).
   \]

Using this form, we conclude that \( A(t) - \lambda \) is invertible and that \( (A(t) - \lambda)^{-1} = \gamma_0 \circ B_{t,\lambda,N}^{-1} \circ k \), where \( k \) is the map defined in Proposition 3. In fact, using the characterization provided by Theorem 7, we have
   \[
y \in H^\frac{1}{2}(\partial \Omega), \quad (A(t) - \lambda)y = f \iff \exists u_t \in H^1(\Omega) \text{ s.t. } \gamma_0(u_t) = y \text{ and } a_{t,\lambda}(u_t, v) = \langle f, \gamma_0(v) \rangle_{H^{-\frac{1}{2}}(\partial \Omega) \times H^\frac{1}{2}(\partial \Omega)}, \forall v \in H^1(\Omega).
   \]

Now suppose that \( (A(t) - \lambda)y = f \). Then there is a unique \( u_t \in H^1(\Omega) \) such that \( \gamma_0(u_t) = y \) and
   \[
a_{t,\lambda}(u_t, v) = \langle f, \gamma_0(v) \rangle_{H^{-\frac{1}{2}}(\partial \Omega) \times H^\frac{1}{2}(\partial \Omega)}, \forall v \in H^1(\Omega).
   \]

Setting \( v = u_t \) in Equation (2.9) and recalling that \( \text{Re}(\lambda) \leq 0 \), we obtain that
   \[
C \|u_t\|_{H^1(\Omega)} \leq \text{Re} a_t(u_t, u_t) \leq \text{Re} \langle f, \gamma_0(u_t) \rangle_{H^{-\frac{1}{2}}(\partial \Omega) \times H^\frac{1}{2}(\partial \Omega)} \leq \|f\|_{H^{-\frac{1}{2}}(\partial \Omega)} \|\gamma_0(u_t)\|_{H^\frac{1}{2}(\partial \Omega)}.
   \]

Equation (2.10) and the boundedness of \( \gamma_0 : H^1(\Omega) \to H^\frac{1}{2}(\partial \Omega) \) imply that
   \[
\|\gamma_0(u_t)\|_{H^\frac{1}{2}(\partial \Omega)} \leq C \|\gamma_0\|_{B(H^1(\Omega), H^\frac{1}{2}(\partial \Omega))} \|f\|_{H^{-\frac{1}{2}}(\partial \Omega)}.
   \]

The Equations (2.9) and (2.10) show us that, for all \( z \in H^\frac{1}{2}(\partial \Omega) \), we have
   \[
(1 + |\lambda|) \left\| \gamma_0(u_t), z \right\|_{H^{-\frac{1}{2}}(\partial \Omega) \times H^\frac{1}{2}(\partial \Omega)} \leq |a_t(u_t, E(z))| + \left\| f, z \right\|_{H^{-\frac{1}{2}}(\partial \Omega) \times H^\frac{1}{2}(\partial \Omega)} + \left\langle \gamma_0(u_t), z \right\|_{H^{-\frac{1}{2}}(\partial \Omega) \times H^\frac{1}{2}(\partial \Omega)} \leq M \|u_t\|_{H^1(\Omega)} \|E(z)\|_{H^1(\Omega)} + \left\| f \right\|_{H^{-\frac{1}{2}}(\partial \Omega)} + \|\gamma_0(u_t)\|_{H^\frac{1}{2}(\partial \Omega)} \right\|_{H^\frac{1}{2}(\partial \Omega)}.
   \]

Using the Remark 8 the boundedness of \( E \) and Equation (2.11), we conclude that
   \[
\left\| \gamma_0(u_t), z \right\|_{H^{-\frac{1}{2}}(\partial \Omega) \times H^\frac{1}{2}(\partial \Omega)} \leq \frac{C}{1 + |\lambda|} \left\| f \right\|_{H^{-\frac{1}{2}}(\partial \Omega)} \|z\|_{H^\frac{1}{2}(\partial \Omega)}, \forall z \in H^\frac{1}{2}(\partial \Omega).
   \]
The above expression implies that
\[ \left\| (A(t) - \lambda I)^{-1} f \right\|_{H^{-\frac{1}{2}}(\partial \Omega)} \leq \frac{C}{1 + |\lambda|} \left\| f \right\|_{H^{-\frac{1}{2}}(\partial \Omega)}. \]

2. First, we prove that \( \| B_{s, N} - B_{t, N} \|_{B(H^1(\Omega), H^1(\Omega)^*)} \leq C |t - s|^\alpha. \) Indeed, due to Assumptions 1 we have
\[ \left| (B_{s, N}(u) - B_{t, N}(u) , v)_{H^1(\Omega)^*} \right| = |a_s(u, v) - a_t(u, v)| \leq C |t - s|^\alpha \| u \|_{H^1(\Omega)} \| v \|_{H^1(\Omega)}. \]

Second, we show that \( \| B_{s, N}^{-1} - B_{t, N}^{-1} \|_{B(H^1(\Omega)^*, H^1(\Omega))} \leq C |t - s|^\alpha. \) This follows from:
\[ \| B_{s, N}^{-1} - B_{t, N}^{-1} \|_{B(H^1(\Omega)^*, H^1(\Omega))} = \| B_{s, N}(B_{s, N} - B_{t, N}) B_{t, N}^{-1} \|_{B(H^1(\Omega)^*, H^1(\Omega))} \leq \| B_{s, N}^{-1} \|_{B(H^1(\Omega)^*, H^1(\Omega))} \| B_{s, N} - B_{t, N} \|_{B(H^1(\Omega), H^1(\Omega)^*)} \| B_{t, N}^{-1} \|_{B(H^1(\Omega)^*, H^1(\Omega))}. \]

Finally, the uniform boundedness of the family of operators \( \{ A(t) \in B \left( H^\frac{1}{2}(\partial \Omega), H^{-\frac{1}{2}}(\partial \Omega) \right) \} \) and the fact that \( A(t)^{-1} = \gamma_0 \circ B_{t, N}^{-1} \circ k \) imply the second condition of Definition 2 due to
\[ \| A(t) - A(s) \|_{B \left( H^\frac{1}{2}(\partial \Omega), H^{-\frac{1}{2}}(\partial \Omega) \right)} = \left\| A(s) \left( A(s)^{-1} - A(t)^{-1} \right) A(t) \right\|_{B \left( H^\frac{1}{2}(\partial \Omega), H^{-\frac{1}{2}}(\partial \Omega) \right)}. \]

3. The Assumption 1 implies that \( \lim_{t \to \infty} \left( \sup_{\| u \|_{H^1(\partial \Omega)} = 1} \| a_t(u, v) \|_{H^1(\partial \Omega)} \right) = 0. \) The proof then follows the same arguments of the second item.

4. The forth condition follows from Proposition 3.

**Corollary 12.** Let \( f \in C^{\alpha}_a \left( [t_0, \infty[, H^{-\frac{1}{2}}(\partial \Omega) \right) \) and \( A(t) : H^\frac{1}{2}(\partial \Omega) \subset H^{-\frac{1}{2}}(\partial \Omega) \to H^{-\frac{1}{2}}(\partial \Omega) \) be the Dirichlet-to-Neumann operators associated with the forms \( \{ a_t : H^1(\Omega) \times H^1(\Omega) \to \mathbb{C} \}_{t \in [t_0, \infty]} \) defined by (2.4) and satisfying Assumption 3.

Then, for every \( u_0 \in H^{-\frac{1}{2}}(\partial \Omega) \), there is a unique \( u \in C \left( [t_0, \infty[, H^{-\frac{1}{2}}(\partial \Omega) \right) \cap C^1 \left( [t_0, \infty[, H^\frac{1}{2}(\partial \Omega) \right) \cap C \left( [t_0, \infty[, H^\frac{1}{2}(\partial \Omega) \right) \) that solves Problem (1.3).

Moreover, \( A(\infty) : H^\frac{1}{2}(\partial \Omega) \to H^{-\frac{1}{2}}(\partial \Omega) \) is invertible and if \( \lim_{t \to \infty} f(t) = f_\infty \in H^{-\frac{1}{2}}(\partial \Omega) \), then \( \lim_{t \to \infty} \| u(t) - u_\infty \|_{H^\frac{1}{2}(\partial \Omega)} = 0 \), where \( u_\infty \) is the unique solution of \( A(\infty)u_\infty = f_\infty \).

**Proof.** It is a consequence of Theorem 10 and 11.

2.1.3. The Yagi conditions. It is natural to consider the operator Dirichlet-to-Neumann acting on functions instead of distribution spaces. In order to study the Dirichlet-to-Neumann problem in \( L^2(\partial \Omega) \), we will apply the results of A. Yagi [25, Chapter 3]. Let us recall them in this section. We fix a complex Hilbert space \( H \).

**Definition 13.** We say that a family \( \{ S(t) : D(S(t)) \subset H \to H \}_{t \in [t_0, \infty]} \) of closed and densely defined operators satisfies the Yagi conditions if there exist constants \( M \geq 1, 0 < \nu \leq 1, 0 < \alpha < 1 \) with \( \alpha + \nu > 1 \), such that
Proposition 16. For all Re(\(\lambda\)) \leq 0 and t \in [t_0, \infty], the operators 
\[ A(t)|_{L^2(\partial\Omega)} - \lambda \quad \text{and} \quad A(t)|_{H^\frac{1}{2}(\partial\Omega)} - \lambda \]
are invertible. Moreover, there is a constant $C > 0$ such that
\[ \left\| (A(t)|_{L^2(\partial \Omega)} - \lambda)^{-1} \right\|_{\mathcal{B}(L^2(\partial \Omega))} \leq \frac{C}{1 + |\lambda|} \]

and
\[ \left\| (A(t)|_{H^{1/2}(\partial \Omega)} - \lambda)^{-1} \right\|_{\mathcal{B}(H^{1/2}(\partial \Omega))} \leq \frac{C}{1 + |\lambda|} \]

for all $\text{Re}(\lambda) \leq 0$ and $t \in [t_0, \infty]$.

**Proof.** It is clear $A(t)|_{L^2(\partial \Omega)} - \lambda$ and $A(t)|_{H^{1/2}(\partial \Omega)} - \lambda$ are bijections, since $A(t) - \lambda I : H^{1/2}(\partial \Omega) \to H^{-1/2}(\partial \Omega)$ is also one.

Let us prove the inequality for $A(t)|_{L^2(\partial \Omega)}$. We consider $\text{Re}(\lambda) \leq 0$, $y \in H^{1/2}(\partial \Omega)$ and $f \in L^2(\partial \Omega)$ such that $(A(t) - \lambda)y = f$. Then there exists a $u_t \in H^1(\Omega)$ that satisfies Equation (2.9) and is such that $\gamma_0(u_t) = y$. Consequently, we obtain that
\[ \begin{aligned}
(1 + |\lambda|) \| \gamma_0(u_t) \|^2_{L^2(\partial \Omega)} & \leq |a_t(u_t, u_t)| + \left| \langle f, \gamma_0(u_t) \rangle \right|_{H^{-1/2}(\partial \Omega) \times H^{1/2}(\partial \Omega)} + \left| \langle \gamma_0(u_t), \gamma_0(u_t) \rangle \right|_{H^{-1/2}(\partial \Omega) \times H^{1/2}(\partial \Omega)} \\
& \leq C \left( \| u_t \|^2_{H^1(\Omega)} + \| f \|_{L^2(\partial \Omega)} \| \gamma_0(u_t) \|_{L^2(\partial \Omega)} + \| \gamma_0(u_t) \|^2_{L^2(\partial \Omega)} \right).
\end{aligned} \]

Arguing as in Equation (2.10), we obtain that, if $f \in L^2(\partial \Omega)$, then
\[ \langle f, \gamma_0(u_t) \rangle_{H^{-1/2}(\partial \Omega) \times H^{1/2}(\partial \Omega)} = \langle f, \gamma_0(u_t) \rangle_{L^2(\partial \Omega)}, \]

we get that
\[ \| u_t \|^2_{H^1(\Omega)} \leq C \| f \|_{L^2(\partial \Omega)} \| \gamma_0(u_t) \|_{L^2(\partial \Omega)}. \]

Now, using the continuity of the trace and Equation (2.14), we have that
\[ \begin{aligned}
\| \gamma_0(u_t) \|^2_{L^2(\partial \Omega)} & \leq \| \gamma_0(u_t) \|^2_{H^{1/2}(\partial \Omega)} \leq \| \gamma_0 \|^2_{\mathcal{B}(H^{1/2}(\Omega), H^{1/2}(\partial \Omega))} \| u_t \|^2_{H^1(\Omega)} \\
& \leq C \| f \|_{L^2(\partial \Omega)} \| \gamma_0(u_t) \|_{L^2(\partial \Omega)}.
\end{aligned} \]

Applying Equations (2.14) and (2.15) to the Equation (2.13), we conclude that
\[ \| \gamma_0(u_t) \|_{L^2(\partial \Omega)} \leq \frac{C}{1 + |\lambda|} \| f \|_{L^2(\partial \Omega)}. \]

Finally, let us prove the inequality for $A(t)|_{H^{1/2}(\partial \Omega)}$. We take $\text{Re}(\lambda) \leq 0$ and $f \in H^{1/2}(\partial \Omega)$. As
\[ \lambda (\lambda - A(t))^{-1} f = (\lambda - A(t))^{-1} A(t) f + f, \]

we conclude that
\[ \begin{aligned}
|\lambda| \left\| (\lambda - A(t))^{-1} f \right\|_{H^{1/2}(\partial \Omega)} & = \left( \left\| (\lambda - A(t))^{-1} \right\|_{\mathcal{B}(H^{-1/2}(\partial \Omega), H^{1/2}(\partial \Omega))} \| A(t) \|_{\mathcal{B}(H^{1/2}(\partial \Omega), H^{-1/2}(\partial \Omega))} + 1 \right) \| f \|_{H^{1/2}(\partial \Omega)} \\
& \leq C \| f \|_{H^{1/2}(\partial \Omega)}. \end{aligned} \]
The last inequality was obtained using Equation (2.11) and the uniform boundedness of \( A(t) \). The inequality from the statement of the Theorem follows then from Equation (2.10) and the fact that \( \lambda - A(t) \big|_{H^{\frac{1}{2}} (\partial \Omega)} \) is invertible in a neighborhood of the origin. \( \square \)

The Dirichlet-to-Neumann problem on \( L^2 (\partial \Omega) \) can now be defined by:

\[
\begin{align*}
\frac{du}{dt} (t) + A(t) \big|_{L^2(\partial \Omega)} u (t) &= f (t), \ t > t_0 \\
u (t_0) &= u_0,
\end{align*}
\]

where \( u_0 \in L^2 (\partial \Omega) \) and \( f \in C_u \left( [t_0, \infty[ , L^2 (\partial \Omega) \right) \).

The well-posedness of the above problem is the main Theorem of this section:

**Theorem 17.** Suppose that the forms \( 2.1 \) satisfy the Assumption 2, that is, \( \alpha \in \left] \frac{1}{2}, 1 \right] \), and let \( f \in C_u \left( [t_0, \infty[ , L^2 (\partial \Omega) \right) \). Then there is a unique \( u \in C \left( [t_0, \infty[ , L^2 (\partial \Omega) \right) \cap C_1 \left( [t_0, \infty[, L^2 (\partial \Omega) \right) \cap C \left( [t_0, \infty[, H^{\frac{1}{2}} (\partial \Omega) \right) \) such that \( u (t) \in D \left( A(t) |_{L^2(\partial \Omega)} \right) \), for all \( t > t_0 \), and such that \( u \) is a solution of the Problem 2.17.

An elegant way to prove Theorem 17 is due to A. Yagi. In [28, 29], well-posedness of non-autonomous problems were obtained to operators defined by forms in the traditional way, when the domain of the operator is a subset of the domain of the form. In the case we are considering here, the domain of the form is \( H^1 (\Omega) \), and the domain of the operator is a set contained in \( H^{\frac{1}{2}} (\partial \Omega) \). One set is not even included in the other. It is clear that such changes are necessary.

In order to provide a full proof, we argue as Yagi. Lemma 20 below is essentially contained in [28, Theorem 2.32, page 110]. The idea of proof of Theorem 15 comes from [28, page 149], although here it requires some results obtained in the previous sections of this paper. A different proof could be given using the methods of Tanabe [20, Section 5.4], although it would also require some modifications and the same hypothesis \( \alpha \in \left] \frac{1}{2}, 1 \right] \).

Theorem 17 is obtained as a consequence of Corollary 12 and the following result.

**Theorem 18.** If the Assumption 2 is fulfilled, then for all \( \nu \in \left] 1 - \alpha, \frac{1}{2} \right[ \), the family of operators

\[
\left\{ A (t) |_{L^2(\partial \Omega)} : D \left( A(t) |_{L^2(\partial \Omega)} \right) \subset L^2 (\partial \Omega) \rightarrow L^2 (\partial \Omega) \right\}_{t \in [t_0, \infty[}
\]

satisfies the Yagi conditions.

First, we fix some notation. Using the duality of \( H^\frac{1}{2} (\partial \Omega) \) and \( H^-\frac{1}{2} (\partial \Omega) \), for each \( A : H^\frac{1}{2} (\partial \Omega) \rightarrow H^-\frac{1}{2} (\partial \Omega) \), we define \( A^* : H^-\frac{1}{2} (\partial \Omega) \rightarrow H^\frac{1}{2} (\partial \Omega) \) as the operator such that

\[
\langle Au, v \rangle_{H^-\frac{1}{2} (\partial \Omega) \times H^\frac{1}{2} (\partial \Omega)} = \langle A^* v, u \rangle_{H^-\frac{1}{2} (\partial \Omega) \times H^\frac{1}{2} (\partial \Omega)} \quad \forall u, v \in H^\frac{1}{2} (\partial \Omega).
\]

If \( B : H^-\frac{1}{2} (\partial \Omega) \rightarrow H^\frac{1}{2} (\partial \Omega) \), we define \( B^* : H^-\frac{1}{2} (\partial \Omega) \rightarrow H^\frac{1}{2} (\partial \Omega) \) as the operator such that

\[
\langle u, Bu \rangle_{H^-\frac{1}{2} (\partial \Omega) \times H^\frac{1}{2} (\partial \Omega)} = \langle v, B^* u \rangle_{H^-\frac{1}{2} (\partial \Omega) \times H^\frac{1}{2} (\partial \Omega)} \quad \forall u, v \in H^-\frac{1}{2} (\partial \Omega).
\]

By the above definitions, it is clear that \( A^* \) and \( B^* \) are uniquely defined and \( (A^*)^{-1} = (A^{-1})^* \). Moreover, if \( A \) is the Dirichlet-to-Neumann operator associated with the form \( a : H^1 (\Omega) \times H^1 (\Omega) \rightarrow \mathbb{C} \), then \( A^* \) is operator associated with \( a^* : H^1 (\Omega) \times H^1 (\Omega) \rightarrow \mathbb{C} \) defined by \( a^* (u, v) = a (v, u) \).
Lemma 19. Let $A : H^{\frac{1}{2}}(\partial \Omega) \subset H^{-\frac{1}{2}}(\partial \Omega) \to H^{-\frac{1}{2}}(\partial \Omega)$ be an operator such that $\text{Re} (\lambda) \leq 0$ is contained in the resolvent set and $\| (\lambda - A)^{-1} \| \leq \frac{C}{|\lambda|^{1/2}}$. Then, if $u \in H^{-\frac{1}{2}}(\partial \Omega)$ and $v \in H^{\frac{1}{2}}(\partial \Omega)$, we have $\langle A^{-\theta} u, v \rangle_{H^{-\frac{1}{2}}(\partial \Omega) \times H^{\frac{1}{2}}(\partial \Omega)} = \left\langle u, (A^*)^{-\theta} v \right\rangle_{H^{-\frac{1}{2}}(\partial \Omega) \times H^{\frac{1}{2}}(\partial \Omega)}$ for all $\theta \in [0, 1]$.

Proof. It follows from the definitions given. In fact,

$$
\pi \sin (\theta \pi) \langle A^{-\theta} u, v \rangle_{H^{-\frac{1}{2}}(\partial \Omega) \times H^{\frac{1}{2}}(\partial \Omega)} = \int_0^\infty \rho^{-\theta} \left\langle (\rho + A)^{-1} u, v \right\rangle_{H^{-\frac{1}{2}}(\partial \Omega) \times H^{\frac{1}{2}}(\partial \Omega)} \, d\rho
$$

$$
= \int_0^\infty \rho^{-\theta} \langle v, (\rho + A)^{-1} u \rangle_{H^{-\frac{1}{2}}(\partial \Omega) \times H^{\frac{1}{2}}(\partial \Omega)} \, d\rho = \int_0^\infty \rho^{-\theta} \left\langle u, (\rho + A)^{-1} v \right\rangle_{H^{-\frac{1}{2}}(\partial \Omega) \times H^{\frac{1}{2}}(\partial \Omega)} \, d\rho
$$

As $A(t)$, $A(t)|_{L^2(\partial \Omega)}$, and $A(t)|_{H^{\frac{1}{2}}(\partial \Omega)}$ are sectorial, we can define $A(t)^{-\theta} \in B \left( H^{-\frac{1}{2}}(\partial \Omega) \right)$, 

$$
\left( A(t)|_{L^2(\partial \Omega)} \right)^{-\theta} \in B \left( L^2(\partial \Omega) \right) \text{ and } \left( A(t)|_{H^{\frac{1}{2}}(\partial \Omega)} \right)^{-\theta} \in B \left( H^{\frac{1}{2}}(\partial \Omega) \right) \text{ for all } \theta \in [0, 1].
$$

By the definition of the fractional powers of an operator, it is clear that $A(t)^{-\theta}$ coincides with $\left( A(t)|_{L^2(\partial \Omega)} \right)^{-\theta}$ and $\left( A(t)|_{H^{\frac{1}{2}}(\partial \Omega)} \right)^{-\theta}$ in $L^2(\partial \Omega)$ and $H^{\frac{1}{2}}(\partial \Omega)$, respectively. In particular, $A(t)^{-\theta}$ takes elements from $L^2(\partial \Omega)$ into $L^2(\partial \Omega)$ and elements from $H^{\frac{1}{2}}(\partial \Omega)$ into $H^{\frac{1}{2}}(\partial \Omega)$. Actually we can say a little more about their mapping properties.

Lemma 20. Let $\theta \in \left[ \frac{1}{2}, 1 \right]$. If $y \in L^2(\partial \Omega)$, then $A(t)^{-\theta} y \in H^{\frac{1}{2}}(\partial \Omega)$ and $A(t)^{-\theta} : L^2(\partial \Omega) \to H^{\frac{1}{2}}(\partial \Omega)$ is a continuous operator. In particular, $\left( A(t)^* \right)^{-\theta} : L^2(\partial \Omega) \to H^{\frac{1}{2}}(\partial \Omega)$ is continuous.

Proof. First, note that $\left( H^{-\frac{1}{2}}(\partial \Omega), D(A(t)) \right)^{\frac{1}{\theta}, \infty} \subset D(A(t)^\theta)$, [21] Proposition 1.1.4 and 4.1.7, for all $0 < \sigma < \frac{1}{\theta}$, where the real interpolation space $\left( H^{-\frac{1}{2}}(\partial \Omega), D(A(t)) \right)^{\frac{1}{\theta}, \infty}$ is equal to

$$
\left\{ y \in H^{-\frac{1}{2}}(\partial \Omega) \mid \sup_{t \in [0, \infty]} \left\| \rho^\frac{1}{\theta} A(t) (\rho + A(t))^{-1} y \right\|_{H^{-\frac{1}{2}}(\partial \Omega)} < \infty \right\}
$$

by [20] Propositions 2.2.2 and 2.2.6. We divide the proof into steps.

First step: If $(A(t) - \lambda) y = f$, for $y \in H^{\frac{1}{2}}(\partial \Omega)$, $f \in L^2(\partial \Omega)$ and $\text{Re} (\lambda) < 0$, then

$$
\| y \|_{H^{\frac{1}{2}}(\partial \Omega)} \leq C |\lambda|^{-\frac{1}{2}} \| f \|_{L^2(\partial \Omega)}.
$$

In order to prove it, let $u_t = \mathcal{E}(y) - B_{L^2}^{-1}(P(t, x, D)f(y))$. Since $y = \gamma_0(u_t)$, we have

$$
\| y \|_{H^{\frac{1}{2}}(\partial \Omega)}^2 \leq \| u_t \|^2_{H^{1/2}(\Omega)} \leq C \| f \|_{L^2(\partial \Omega)} \| y \|_{L^2(\partial \Omega)} \leq C |\lambda|^{-1} \| f \|_{L^2(\partial \Omega)}^2.
$$

We have used the continuity of the trace in (1), Equation (2.2) in (2), Equation (2.14) in (3) and Proposition [10] in (4). The constants $C > 0$ can change from one inequality to another.

Second step: If $0 < \sigma < \frac{1}{\theta}$, then $L^2(\partial \Omega) \subset D(A(t)^\sigma)$ and the inclusion is continuous.
Let \( y \in L^2(\partial \Omega) \). Then
\[
\rho^{\frac{1}{2}} \left\| A(t) (\rho + A(t))^{-1} y \right\|_{H^{-\frac{1}{2}}(\partial \Omega)} \leq \left\| A(t) \right\|_{B(\mathcal{H}^\pi(\partial \Omega), H^{-\frac{1}{2}}(\partial \Omega))} \rho^{\frac{1}{2}} \left\| (\rho + A(t))^{-1} y \right\|_{H^{\frac{1}{2}}(\partial \Omega)} \leq C \left\| A(t) \right\|_{B(\mathcal{H}^\pi(\partial \Omega), H^{-\frac{1}{2}}(\partial \Omega))} \left\| y \right\|_{L^2(\partial \Omega)}.
\]

This implies that \( y \in \left( H^{-\frac{1}{2}}(\partial \Omega), D(A(t)) \right)_{\frac{1}{2}, \infty} \subset D(A(t)^\sigma) \) and that the inclusions are continuous. Hence \( \left\| A(t)^\sigma y \right\|_{H^{-\frac{1}{2}}(\partial \Omega)} \leq C \left\| y \right\|_{L^2(\partial \Omega)} \), where the constant \( C \) does not depend on \( t \), due to the uniform boundedness of \( A(t) \) and its resolvent \( (\rho + A(t))^{-1} \), for \( \rho > 0 \).

Third step: If \( \theta \in [\frac{1}{2}, 1] \), then \( A(t)^{-\theta} : L^2(\partial \Omega) \to H^\pi(\partial \Omega) \) continuously.

Let \( y \in H^{-\frac{1}{2}}(\partial \Omega) \) and \( x \in H^\frac{1}{2}(\partial \Omega) \). Then
\[
\left\| A(t)^{-\theta} x \right\|_{H^\frac{1}{2}(\partial \Omega)} \leq C \left\| x \right\|_{L^2(\partial \Omega)}, \text{ where } C \text{ is again a constant that does not depend on } t.
\]

As \( H^\pi(\partial \Omega) \) is dense in \( L^2(\partial \Omega) \), we obtain the result.

Finally, we see that \( (A(t)^*)^{-\theta} : L^2(\partial \Omega) \to H^\pi(\partial \Omega) \) is also a continuous operator, as \( A(t)^* \) is the operator associated to the sesquilinear form \( a^*_t : H^1(\Omega) \times H^1(\Omega) \to \mathbb{C} \) defined as \( a^*_t(u, v) = a_t(v, u) \), which has the same properties of \( a_t \).

We now proceed to prove Theorem 18.

**Proof.** (of the Theorem 18)

We have to check all conditions of Definition 18. The Item 1) follows from Proposition 10. It remains to prove Item 2).

For all \( x, y \in H^\pi(\partial \Omega) \), we have
\[
\left\| A(t)^{-\theta} x \right\|_{H^\pi(\partial \Omega)} \leq C \left\| x \right\|_{L^2(\partial \Omega)}, \text{ where } C \text{ is again a constant that does not depend on } t.
\]

As \( H^\pi(\partial \Omega) \) is dense in \( L^2(\partial \Omega) \), we obtain the result.

Finally, we see that \( (A(t)^*)^{-\theta} : L^2(\partial \Omega) \to H^\pi(\partial \Omega) \) is also a continuous operator, as \( A(t)^* \) is the operator associated to the sesquilinear form \( a^*_t : H^1(\Omega) \times H^1(\Omega) \to \mathbb{C} \) defined as \( a^*_t(u, v) = a_t(v, u) \), which has the same properties of \( a_t \).
We have used Lemma 19 in (1), Proposition 4 in (2). In (3), we use that \( A(t)^{-1} = \gamma_0 \circ B_{t,\Lambda}^{-1} \circ k \) as proved in Proposition 4 and that \( B_{t,\Lambda}^{-1} \) is Hölder continuous as proved in Theorem 11. Finally, in (4), we have used Lemma 20.

\[ \square \]

3. Applications

3.1. Non-autonomous elliptic equations with dynamic boundary conditions. In this section, we first consider the following problem:

\[
\begin{aligned}
P(t, x, D) u(t, x) &= 0, \ (t, x) \in [t_0, \infty[ \times \Omega \\
\frac{\partial u}{\partial t}(t, x) &= -C(t, x, D) u(t, x) + f(t, x), \ (t, x) \in [t_0, \infty[ \times \partial \Omega \\
u(t_0, x) &= u_0(x), \ x \in \partial \Omega,
\end{aligned}
\]

where \( \Omega \subset \mathbb{R}^n \) is a bounded Lipschitz domain, \( P(t, x, D) \) and \( C(t, x, D) \) are the operators defined in (1.2) and (1.3) with coefficients that satisfy the conditions of Assumption 1 for the \( H^{-\frac{1}{2}}(\Omega) \) problem and Assumption 2 for the \( L^2(\Omega) \) problem.

**Theorem 21.** Suppose that the Assumption 1 holds, that is, \( \alpha \in [0, 1] \), and let \( u_0 \in H^{-\frac{1}{2}}(\partial \Omega) \) and \( f \in C^\alpha_u \left( [t_0, \infty[, H^{1/2}(\partial \Omega) \right) \). Then, there is a unique function \( u \in C \left( [t_0, \infty[, H^{1/2}(\partial \Omega) \right) \cap C \left( [t_0, \infty[, H^{-1/2}(\partial \Omega) \right) \) such that:

1. \( \gamma_0(u) \in C^1 \left( [t_0, \infty[, H^{-1/2}(\partial \Omega) \right) \cap C \left( [t_0, \infty[, H^{-1/2}(\partial \Omega) \right) \) and \( (3.2) \) holds.

\[
\begin{aligned}
P(t, x, D) u(t) &= 0, \forall t \in [t_0, \infty[ \\
\frac{d\gamma_0(u)}{dt}(t) &= -C(t, x, D) u(t) + f(t), \ t \in [t_0, \infty[.
\end{aligned}
\]

Moreover, if \( \lim_{t \to \infty} f(t) = f_{\infty} \) in \( H^{-1/2}(\partial \Omega) \), then \( \lim_{t \to \infty} u(t) = u_{\infty} \) in \( H^{1/2}(\Omega) \), where \( u_{\infty} \) is the unique solution of

\[
\begin{aligned}
P(\infty, x, D) u_{\infty} &= 0 \\
C(\infty, x, D) u_{\infty} &= f_{\infty}.
\end{aligned}
\]

If the Assumption 2 holds, that is, \( \alpha \in [1/2, 1] \), and if \( u_0 \in L^2(\partial \Omega) \) and \( f \in C^\alpha_u \left( [t_0, \infty[, L^2(\partial \Omega) \right) \), then there is a unique \( u \in C \left( [t_0, \infty[, H^{1/2}(\Omega) \right) \) such that its trace \( \gamma_0(u) \in C^1 \left( [t_0, \infty[, L^2(\partial \Omega) \right) \cap C \left( [t_0, \infty[, H^{1/2}(\partial \Omega) \right) \) and (3.2) holds. In particular, \( C(t, x, D) u(t) \) exists in the \( L^2(\partial \Omega) \)-weak sense.

Moreover, if \( \lim_{t \to \infty} f(t) = f_{\infty} \) in \( L^2(\partial \Omega) \), then \( \lim_{t \to \infty} u(t) = u_{\infty} \) in \( H^{1/2}(\Omega) \), where \( u_{\infty} \) is the unique solution of (3.2).

**Proof.** Suppose that \( u \in C \left( [t_0, \infty[, H^{1/2}(\Omega) \right) \) satisfies the conditions of item 1 and 2 of the theorem. As \( P(t, x, D) u(t) = 0 \), the expression \( C(t, x, D) u(t) \) is equivalent to \( A(t) (\gamma_0(u(t))) \), where \( A(t) \) is the Dirichlet-to-Neumann operator. Hence \( \gamma_0(u) \) must be the solution of the Equation 14 and it is uniquely determined by \( u_0 \) and \( f \). By Proposition 4, we conclude that

\[
\begin{aligned}
u(t) &= E(\gamma_0(u(t))) - B_{t,\Lambda}^{-1}(P(t, x, D) E(\gamma_0(u(t)))).
\end{aligned}
\]

Thus \( u \) is also uniquely determined by \( u_0 \) and \( f \). On the other hand, if we use (3.2) as the definition of \( u(t) \), where \( \gamma_0(u) \) is the solution of Equation 14, then \( u \) satisfies properties 1 and 2 stated in the theorem. This proves existence.
If \( \lim_{t \to \infty} f(t) = f_{\infty} \) in \( H^{-\frac{1}{2}}(\partial \Omega) \), then, by Corollary 12 \( \lim_{t \to \infty} \gamma_{0}(u)(t) = \gamma_{0}(u)(\infty) \) in \( H^{\frac{1}{2}}(\partial \Omega) \), where \( A(\infty)(\gamma_{0}(u)(\infty)) = f_{\infty} \). Hence \( \lim_{t \to \infty} u(t) = u_{\infty} \) in \( H^{1}(\Omega) \), where

\[
   u_{\infty} = E(\gamma_{0}(u)(\infty)) - B_{1,1}^{-1}(P(t, x, D)E(\gamma_{0}(u)(\infty)))
\]

and, therefore, it is the unique solution of Equation 3. The results for \( L^{2}(\partial \Omega) \) case follow from similar arguments and Theorem 17.

### 3.2. Dynamical boundary conditions on non-cylindrical domains

Finally, we consider the Laplace equation with dynamic boundary conditions on a non-cylindrical domain as described in the Introduction:

\[
   \begin{cases}
   (\lambda + \Delta) u(t, x) = 0, \ (t, x) \in D, \ t > t_0 \\
   \frac{\partial u}{\partial t}(t, x) = -\frac{\partial u}{\partial n}(t, x) + f(t, x), \ (t, x) \in S, \ t > t_0 \\
   u(t_0, x) = u_0(x), \ x \in \partial \Omega_{t_0},
   \end{cases}
\]

where \( \lambda < 0 \) and \( t_0 \geq 0 \).

In order to define the set \( D \), we consider a bounded domain \( \Omega \subset \mathbb{R}^{n} \) with \( C^{2} \)-regular boundary \( \partial \Omega \) and a map

\[
   h \in C_{u}^{\alpha}([0, \infty[, Diff^{2}(\Omega)) \cap C_{u}^{1, \alpha}([0, \infty[, C(\overline{\Omega}, \mathbb{R}^{n}))
\]

where \( Diff^{2}(\Omega) \) is an open set of \( C^{2}(\overline{\Omega}, \mathbb{R}^{n}) \), defined as

\[
   Diff^{2}(\Omega) := \left\{ g \in C^{2}(\overline{\Omega}, \mathbb{R}^{n}) : g \text{ is injective and } \det \left( \frac{\partial g}{\partial x_{j}}(y) \right) \neq 0, \ \forall y \in \overline{\Omega} \right\}.
\]

The set \( D \subset \mathbb{R}^{n+1} \) is defined as the image of the function \( H : [0, \infty[ \times \Omega \to [0, \infty[ \times \mathbb{R}^{n} \), given by \( H(t, y) = (t, h(t, y)) \). According to (3.6), \( D \) is an open set of \( \mathbb{R}^{n+1} \) since \( H \) is a diffeomorphism onto its image. We notice that \( h \) defines a family of diffeomorphisms \( \{h_{t} : \Omega \to \Omega_{t}\}_{t \geq 0} \) given as \( h_{t}(y) = h(t, y) \) and that

\[
   \Omega_{t} = \{(t, h_{t}(y)) : y \in \Omega\}, \quad \partial \Omega_{t} = \{(t, h_{t}(y)) : y \in \partial \Omega\}, \quad \text{for } t \geq 0,
\]

and \( S = h([0, \infty[ \times \partial \Omega) \).

#### Assumption 3

The function \( h \) satisfies:

1. \( \lim_{t \to \infty} h(t, \cdot) = I \) in \( C^{2}(\overline{\Omega}, \mathbb{R}^{n}) \) where \( I : \Omega \to \Omega \) is the identity.

2. There is a function \( c \in C_{0}^{\alpha}([0, \infty[), \alpha \in \left[ \frac{1}{2}, 1 \right] \) such that \( \frac{\partial h}{\partial t}(t, y) = c(t) n(t, h(t, y)) \), for all \( y \in \partial \Omega \), where \( n(t, h(t, y)) \) is the outward normal vector to \( \Omega_{t} \) at the point \( h(t, y) \in \partial \Omega_{t} \).

The item i) gives a precise meaning to the convergence of the sets \( \Omega_{t} \) to \( \Omega \) as \( t \to \infty \). Intuitively it also says that \( \Omega_{t} \) are temporal perturbations of the set \( \Omega \). The Hölder continuity of \( h \) and \( \frac{\partial h}{\partial t} \) assumed in (3.6) implies that the perturbations and its rate of variation do not change rapidly. The second item of Assumption 3 says that we allow only small perturbations of the domain along the normal vector.

#### Example 22

Let \( \Omega \) be a bounded domain with \( C^{3} \)-regular boundary. Let \( \nu : \partial \Omega \to \mathbb{R}^{n} \) be the outward normal vector field and \( B_{r}^{\mathbb{N}}(y) := \{y + t\nu(y), \ |t| \leq r\} \), for \( y \in \partial \Omega \). By the collar neighborhood theorem, there is an \( r > 0 \) such that \( U := \cup_{y \in \partial \Omega} B_{r}^{\mathbb{N}}(y) \) is an open set and \( B_{r}^{\mathbb{N}}(w) \cap B_{r}^{\mathbb{N}}(y) = \emptyset \), if \( w \neq y, \ w, y \in \partial \Omega \). Moreover, there is a unique function \( \pi : U \to \partial \Omega \) of class \( C^{2} \) such
that $\pi(z) = y$, when $z \in B^N \{y\}$. Let $\chi \in C^\infty (U)$ be equal to 1 in a neighborhood of $\partial \Omega$. Now take $f \in C^2 ([0, \infty])$ and define $h : [0, \infty] \times \Omega \to \mathbb{R}^n$ as

$$h(t, y) = y + f(t) \chi(y) \nu(y).$$

If $\|f\|_{L^\infty([0, \infty])}$ is small, it is clear that the above formula defines a $C^2$ diffeomorphism, for all $t \in [0, \infty[$. Moreover, take $y \in \partial \Omega$ and let $\phi : B_1(0) = \{w \in \mathbb{R}^{n-1}; \|w\| < 1\} \subset \mathbb{R}^{n-1} \to \partial \Omega$ be an embedding of class $C^2$ such that $\phi(0) = y$. Hence $x \in B_1(0) \mapsto h(t, \phi(x)) \in \partial \Omega_t$ is an embedding of class $C^2$. The tangent space $T_{h(t,y)} \partial \Omega_t$ consists of the linear span of the vectors

$$\left. \frac{\partial}{\partial x_j} h(t, \phi(x)) \right|_{x=0} = \left. \frac{\partial}{\partial x_j} \phi(x) \right|_{x=0}, \quad j = 1, \ldots, n - 1.$$

Denoting by $\langle a, b \rangle_{\mathbb{R}^n}$ the usual scalar product of vectors $a$ and $b$ in $\mathbb{R}^n$, we have

$$\langle \frac{\partial}{\partial x_j} (h(t, \phi(x))), \nu(\phi(x)) \rangle_{\mathbb{R}^n} = \langle \frac{\partial \phi}{\partial x_j}(x), \nu(\phi(x)) \rangle_{\mathbb{R}^n} + f(t) \langle \frac{\partial \nu}{\partial x_j}(\phi(x)), \nu(\phi(x)) \rangle_{\mathbb{R}^n} = \langle \frac{\partial \phi}{\partial x_j}(x), \nu(\phi(x)) \rangle_{\mathbb{R}^n} + f(t) \frac{1}{2} \frac{\partial}{\partial x_j} (\nu(\phi(x)), \nu(\phi(x)) \rangle_{\mathbb{R}^n} = 0.$$}

Hence $\nu(y) = \nu(\phi(0))$ is the normal vector at $h(t, y)$, that is, $\nu(t, h(t, y)) = \nu(y)$. In particular, $\frac{\partial}{\partial t} (t, y) = \frac{\partial}{\partial t} (t) n(t, h(t, y))$, when $y \in \partial \Omega$. Hence, all items of Assumption 3 hold if $\|f\|_{L^\infty}$ is sufficiently small, if the first and second derivatives of $f$ are bounded and if $\lim_{t \to \infty} f(t) = 0$. In particular, we can take $f$ behaving as $t^{-\beta} \sin(t^\alpha)$, for $t$ large enough, and positive constants $\alpha$ and $\beta$ satisfying $2(\alpha - 1) < \beta$. Consequently, our assumptions also allow a kind of oscillatory behavior to the boundary $S$ of the non-cylindrical domain $D$ at infinite time.

**Remark 23.** Assumption 3 implies the following convergences, for all $i, j, k \in \{1, \ldots, n\}$:

$$\lim_{t \to \infty} \left\| \frac{\partial h_i}{\partial y_j} (t, \cdot) - \delta_{ij} \right\|_{L^\infty(\Omega)} = \lim_{t \to \infty} \left\| \frac{\partial h_i}{\partial y_j} \partial y_k (t, \cdot) \right\|_{L^\infty(\Omega)} = \lim_{t \to \infty} \left\| \frac{\partial h_i}{\partial t} (t, \cdot) \right\|_{L^\infty(\Omega)^n} = 0.$$

The first two limits follow directly from to item i) of Assumption 3. For the third one, we consider $(t, y)$ such that $\frac{\partial h_i}{\partial t} (t, y) > 0$, for some $i \in \{1, \ldots, n\}$, and $k : = \left( \frac{1}{2C} \frac{\partial h_i}{\partial t} (t, y) \right)^{\frac{1}{2}}$, where $C > 0$ is the constant of Hölder continuity expressed in (3.6). Then

$$\left\| \frac{\partial h_i}{\partial t} (t + \theta k, \cdot) - \frac{\partial h_i}{\partial t} (t, \cdot) \right\|_{L^\infty(\Omega)} \leq C (\theta k)^\alpha \leq \frac{1}{2} \frac{\partial h_i}{\partial t} (t, y).$$

Therefore, for all $\theta \in [0, 1]$, we have $\frac{\partial h_i}{\partial t} (t + \theta k, y) \geq \frac{1}{2} \frac{\partial h_i}{\partial t} (t, y)$. We then conclude that

$$\frac{1}{2} \left( \frac{1}{2C} \right)^{\frac{1}{2}} \right| \frac{\partial h_i}{\partial t} (t, y) \right|^{\frac{1}{\alpha}} \leq k \int_0^1 \frac{\partial h_i}{\partial t} (t + \theta k, y) \, d\theta \leq |h_i (t + k, \cdot) - h_i (t, \cdot)|.$$}

The above estimate holds also if $\frac{\partial h_i}{\partial t} (t, y) < 0$ by same arguments. As $\lim_{t \to \infty} \|h(t, \cdot) - L\|_{L^\infty(\Omega)^n} = 0$, we conclude that $\lim_{t \to \infty} \|h_i (t + k, \cdot) - h_i (t, \cdot)\|_{L^\infty(\Omega)} = 0$ and that $\|\frac{\partial h_i}{\partial t} (t, \cdot)\|_{L^\infty(\Omega)^n}$ converges to zero. In particular, $\lim_{t \to \infty} c(t) = 0$.

In order to understand the Problem (3.5), let us consider a function $u : \{(t, x) \in D; t > t_0\} \to \mathbb{C}$ that can be extended to a continuous function in $\{(t, x) \in \overline{D}; t \geq t_0\}$ and such that $\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x_j}$ and $\frac{\partial^2 u}{\partial x_i \partial x_j}$ exist and are continuous up to $\{(t, x) \in \overline{D}; t > t_0\}$, for all $i, j$. Suppose that $u$ is a
classical solution of the Problem (3.5). It is worth to mention that, in a point \((t, x) \in [t_0, \infty[ \times S\), \(\frac{\partial u}{\partial t} (t, x)\) must be interpreted as the continuous extension of \(\frac{\partial u}{\partial t}\) to this point. In fact, the limit \(\lim_{h \to 0} u(t+h,x)-u(t,x)\) does not always make sense, as it is not even clear that \((t+h, x) \in \overline{D}\) for some \(h \neq 0\), when \((t, x) \in S\).

For such a function, we can make a change of variables as in [17 Chapter 2]. We define \(v(t, y) := u(t, h(t, y))\), and consider the matrix \((\frac{\partial h}{\partial y})\), whose entries are \(\frac{\partial h}{\partial y}(t, y)\), and by \(\frac{\partial h}{\partial y}\) is the inverse of this matrix. Moreover we denote by \(\nu(y)\) the normal vector at \(y \in \partial \Omega\) and by \(n(t, x)\) the normal vector at \(x \in \partial \Omega_t\). We then have

\[
\begin{align*}
\text{i)} \frac{\partial u}{\partial x}(t, h(t, y)) &= \sum_{k=1}^{n} \left( \frac{\partial h}{\partial y} \right)^{-1}_{kj} (t, y) \frac{\partial u}{\partial y}_{ky}(t, y), \\
\text{ii)} \frac{\partial u}{\partial t}(t, h(t, y)) &= \frac{\partial u}{\partial t}(t, y) - \sum_{l=1}^{n} \left( \sum_{k=1}^{n} \left( \frac{\partial h}{\partial y} \right)^{-1}_{lk} (t, y) \frac{\partial h}{\partial t}(t, y) \right) \frac{\partial u}{\partial y}_{ky}(t, y). \\
\text{iii)} \frac{\partial u}{\partial t}(t, h(t, y)) &= \sum_{k=1}^{n} \left( \sum_{j=1}^{n} \left( \frac{\partial h}{\partial y} \right)^{-1}_{kj} (t, y) n_j(t, h(t, y)) \right) \frac{\partial u}{\partial y}_{ky}(t, y), \\
\text{iv)} \nu_j(y) &= \sqrt{\sum_{k=1}^{n} \left( \sum_{j=1}^{n} \left( \frac{\partial h}{\partial y} \right)^{-1}_{kj} (t, y) n_k(t, h(t, y)) \right)^2}.
\end{align*}
\]

Using item ii) of Assumption [3] we conclude that the Equation (3.5) is formally equivalent to the following non-autonomous elliptic equation with dynamic boundary conditions:

\[
0 = \lambda v + \sum_{j=1}^{n} \sum_{k=1}^{n} \left( \frac{\partial h}{\partial y} \right)^{-1}_{lj} (t, y) \frac{\partial h}{\partial y}_{kj} (t, y) \frac{\partial u}{\partial y}_{ky}(t, y), \quad [t_0, \infty[ \times \Omega \\
\frac{\partial u}{\partial t}(t, y) = \sum_{l=1}^{n} \left( \sum_{k=1}^{n} \left( \frac{\partial h}{\partial y} \right)^{-1}_{lk} (t, y) (c(t) - 1) n_k(t, h(t, y)) \right) \frac{\partial u}{\partial y}_{ky}(t, y), \\
v(t_0, y) = u_0(h(t_0, y)), \quad y \in \partial \Omega.
\]

The above equation can be studied using suitable forms. To define them, we fix a \(C^1\) extension of \(\nu: \partial \Omega \to \mathbb{R}^n\) to \(\overline{\Omega}\) and call it, with a slight abuse of notation, by the same letter \(\nu: \overline{\Omega} \to \mathbb{R}^n\). The normal vector \(n\) can also be extended by the expression below:

\[
n_j(t, h(t, y)) = \sqrt{\sum_{k=1}^{n} \left( \sum_{j=1}^{n} \left( \frac{\partial h}{\partial y} \right)^{-1}_{kj} (t, y) n_k(t, h(t, y)) \right)^2}.
\]

Again we have used the same letter to denote its extension, \(n = (n_1, ..., n_n): \overline{D} \to \mathbb{R}^n\).

Let \(U \subset \mathbb{R}^n\) be an open set that contains \(\partial \Omega\) and such that \(\nu(y) \neq 0\) for all \(y \in \overline{\Omega} \cap U\). We fix a function \(\chi \in C^\infty_c(U)\) that satisfies \(\chi|_{\partial \Omega} \equiv 1\) and define \(N: [0, \infty[ \times \overline{\Omega} \to \mathbb{R}\) as

\[
N(t, y) = \chi(y) \left( \sum_{j=1}^{n} \left( \sum_{k=1}^{n} \left( \frac{\partial h}{\partial y} \right)^{-1}_{kj} (t, y) n_k(t, h(t, y)) \right)^2 \right)^{-\frac{1}{2}} + 1 - \chi(y).
\]

This function has the following easily verified properties:

- \(N(t, y) > 0\), for all \((t, y) \in [0, \infty[ \times \overline{\Omega}\).
• $t \in [0, \infty[ \rightarrow (x \in \Omega \rightarrow \mathcal{N}(t, x))$ is a function that belongs to $C^\infty_\alpha ([0, \infty[ , C^1 (\overline{\Omega}))$.

• $\lim_{t \to \infty} \| \mathcal{N}(t, .) - 1 \|_{C^1 (\overline{\Omega})} = 0$.

We finally set the forms $\{ a_t : H^1 (\Omega) \times H^1 (\Omega) \to \mathbb{C} \}_{t \in [t_0, \infty[}$. The definition is obtained by the multiplication of Equation (3.8) by operators that take zero and $(t, \nu)$. The above discussion together with Theorem 21 implies: Using (3.10) and (3.7), iv), we see that the conormal derivative is equal to

\[ a_t \begin{pmatrix} v \\ w \end{pmatrix} = \int_{\Omega} \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^n \left( \left( \frac{\partial h}{\partial y} \right)_{kj}^{-1} (t, y) \frac{\partial v}{\partial y_k} (y) \right) \times \frac{\partial}{\partial y_l} \left( \left( \frac{\partial h}{\partial y} \right)_{lj}^{-1} (t, y) \right) \mathcal{N}(t, y) w(y) dy - \lambda v(y) \mathcal{N}(t, y) w(y) \] dy.

We conclude that the Dirichlet-to-Neumann operator associated with the forms (3.11) are the conormal derivative associated to the form $a_k = \sum_{j=1}^n \left( \left( \frac{\partial h}{\partial y} \right)_{kj}^{-1} (t, y) \frac{\partial v}{\partial y_k} (y) \right) \left( \left( \frac{\partial h}{\partial y} \right)_{lj}^{-1} (t, y) (1 - c(t)) \mathcal{N}(t, y) \right) = \mathcal{N}(t, y)$.

Comparing the forms (3.11) and (2.4), we see that, in this case, the coefficients $c_j$ are equal to zero and

\[ a_{kl} = \sum_{j=1}^n \left( \left( \frac{\partial h}{\partial y} \right)_{kj}^{-1} (t, y) \right) \left( \left( \frac{\partial h}{\partial y} \right)_{lj}^{-1} (t, y) (1 - c(t)) \mathcal{N}(t, y) \right). \]

Using (3.10) and (3.7), iv), we see that the conormal derivative is equal to (3.12)

\[ \mathcal{A}(t, y, D)v = \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^n \left( \left( \frac{\partial h}{\partial y} \right)_{kj}^{-1} (t, y) \frac{\partial v}{\partial y_k} (y) \right) \left( \left( \frac{\partial h}{\partial y} \right)_{lj}^{-1} (t, y) \mathcal{N}(t, y) (1 - c(t)) \nu_l (y) \right) = \mathcal{N}(t, y) \mathcal{A}(t, y, D)v.
\]

We conclude that the Dirichlet-to-Neumann operator associated with the forms (3.11) are the operators that take $g \in H^2 (\partial \Omega)$ to the $H^2 (\partial \Omega)$-weak conormal derivative $\mathcal{A}(t, x, D) u_t$, defined in (3.12), where $u_t \in H^1 (\Omega)$ is the unique solution of the Dirichlet problem:

(3.13)

\[ \mathcal{B}(t, y, D)u = \lambda u_t (y) + \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^n \left( \left( \frac{\partial h}{\partial y} \right)_{kj}^{-1} (t, y) \frac{\partial v}{\partial y_k} (y) \right) \left( \left( \frac{\partial h}{\partial y} \right)_{lj}^{-1} (t, y) \mathcal{N}(t, y) \frac{\partial u_t}{\partial y_l} (y) \right) = 0, \quad \gamma_0 (u_t) = g. \]

The above discussion together with Theorem 21 implies:

**Theorem 24.** Let $u_0 \in L^2 (\partial \Omega, \nu_0)$ and $f : \mathcal{S} \to \mathbb{C}$ be such that the map defined by $t \in [t_0, \infty[ \rightarrow (y \in \partial \Omega \rightarrow f (t, h(t, y)))$ belongs to $C^\infty_\alpha ([t_0, \infty[, L^2 (\partial \Omega))$, where, as always in this section, $\alpha \in ]\frac{3}{2}, 1]$. Then, there exists a unique function $u : D \to \mathbb{C}$ such that the function $v$ defined by $v(t, y) = u(t, h(t, y))$ belongs to $C ([t_0, \infty[, H^1 (\Omega))$ and satisfies:
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1) \( \gamma_0 (v) \in C^1 \left( \left[ t_0, \infty \right], L^2 (\partial \Omega) \right) \cap C \left( \left[ t_0, \infty \right], L^2 (\partial \Omega) \right) \cap C \left( \left[ t_0, \infty \right], H^{\frac{1}{2}} (\partial \Omega) \right) \),

2) \[
\begin{align*}
\bar{P} (t, y, D) v (t) &= 0 \\
\bar{\gamma}_0 (v) (t_0) &= v_0
\end{align*}
\]

with \( \bar{f} (t, y) = f (t, h (t, y)) \) and \( v_0 (y) = u_0 (h (t_0, y)) \).

Moreover, if \( \lim_{t \to \infty} \bar{f} (t) = f^\infty \) in \( L^2 (\partial \Omega) \), then \( \lim_{t \to \infty} v (t) = u_\infty \) in \( H^1 (\Omega) \), where \( u_\infty \) is the unique solution of

\[
\begin{cases}
(\lambda + \Delta) u_\infty (x) = 0, & x \in \Omega \\
\frac{\partial u_\infty}{\partial n} (x) = f^\infty (x), & x \in \partial \Omega.
\end{cases}
\]

It is clear, by our discussion, that \( \bar{P} (t, y, D) v (t) = 0 \) is equivalent to \( (\lambda + \Delta) u (t) = 0 \), and

\[
\frac{d}{dt} \gamma_0 (v) (t) = -\bar{C} (t, y, D) v (t) + \bar{f} (t)
\]
is formally equivalent to the dynamic boundary conditions of [Kab], after change of variables.

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