LAUDAL’S LEMMA IN POSITIVE CHARACTERISTIC

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Abstract. Laudal’s Lemma states that if $C$ is a curve of degree $d > s^2 + 1$ in $\mathbb{P}^3$ over an algebraically closed field of characteristic 0 such that its plane section is contained in an irreducible curve of degree $s$, then $C$ lies on a surface of degree $s$. We show that the same result does not hold in positive characteristic and we find different bounds $d > f(s)$ which ensure that $C$ is contained in a surface of degree $s$.

1. Introduction

Let $C$ be a curve in $\mathbb{P}^3_k$, being $k$ an algebraically closed field. Let $\Gamma$ be the generic plane section of $C$. In this paper we study the problem of finding bounds on the degree of $C$ in such a way that, if $\Gamma$ is contained in a plane curve of degree $s$, then $C$ is contained in a surface of the same degree. In the case that $\text{char } k = 0$ the following result has been proved:

Theorem 1.1 (Laudal’s Lemma, [11, Corollary, p.147], [6]). If $\Gamma$ is contained in an integral plane curve of degree $s$ and $\deg C > s^2 + 1$, then $C$ is contained in a surface of degree $s$.

The bound on the degree of the curve found in this result is sharp. Indeed, there are examples of curve of degree $s^2 + 1$ whose the generic plane section is contained in an irreducible plane curve of degree $s$ and that are not contained in any surface of degree $s$ (see [7], [6] and [13, Proposition 1]).

In this paper, following the proof of Gruson and Peskine of Laudal’s Lemma in [6], we prove an analogous result in the case that the field $k$ has positive characteristic:

Theorem 1.2. Let $C \subset \mathbb{P}^3$ be a non degenerate reduced curve of degree $d$ in characteristic $p > 0$. Suppose that $\Gamma$ is contained in an integral plane curve of degree $s$. Then $C$ is contained in a surface of degree $s$, if one of the following conditions is satisfied:

(1) $C$ is connected, $p \geq s$ and $d > s^2 + 1$;
(2) $C$ is connected, $p < s$ and $d > s^2 + p^{2n}$, with $p^n < s \leq p^{n+1}$; in particular this holds if $d > 2s^2 - 2s + 1$;
(3) $p > s$ and $d > s^2 + 1$;
(4) $p \leq s$ and $d > s^2 + p^{2n}$, with $p^n \leq s < p^{n+1}$. In particular this holds if $d > 2s^2$.

Let us make a note about terminology. Given the incidence variety $T = \{(H, P) \in \mathbb{P}^3 \times \mathbb{P}^3 \mid P \in C \cap H\}$ associated to $C$, the fibre of the projection $T \rightarrow \mathbb{P}^3$ over the generic point $\eta \in \mathbb{P}^3$ is the generic plane section $\Gamma$. In particular we consider the open subset $U \subset \mathbb{P}^3$ such that any $[H] \in U$ corresponds to a plane $H \subset \mathbb{P}^3$ that $C$ meets transversally and such planes are generic for the curve $C$. 

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Let us give now a sketch of the proof of Theorem 1.2, given in Section 4. We follow the idea of the proof of Theorem 1.1 given by Gruson and Peskin in [6]. So we take $S \subset \mathbb{P}^3 \times \mathbb{P}^3$ containing $T$ such that the fibre over $\eta$ is an integral plane curve of degree $s$ containing $J$. Then we suppose that $h^0(\mathcal{I}_C(s)) = 0$ and, using Theorem 3.3 that is the main result of Section 3, we factor the projection $S \rightarrow \mathbb{P}^3$ through a generically smooth morphism $S_r \rightarrow \mathbb{P}^3$, with $S_r = S \times_{\mathbb{P}^3, F^s} \mathbb{P}^3$ and $F^r$ some $r$-th power of the absolute Frobenius of $\mathbb{P}^3$. Then, proceeding as in [6], we arrive to the inequality $d \leq s^2 + p^{2r}$. Remarking that it must be $h^1(\mathcal{I}_C(s - p^r)) \neq 0$ we find the desired inequalities.

Looking at the proof we see that the assumption that $C$ is reduced is required to find a suitable bound to the power $p^r$. Moreover, in the case that $C$ is connected this bound is sharp, as we will see in Example 5.3. Indeed, generalizing the example given in [6] and [13, Proposition 1] to prove that the bound in Theorem 1.1 is sharp, we consider the sheaf $E^\prime$ given in [6] and [13, Proposition 1] to prove that the bound in Theorem 1.1 is sharp, as we will see in Example 5.3. Indeed, generalizing the example given in [6] and [13, Proposition 1] to prove that the bound in Theorem 1.1 is sharp, we consider the sheaf $E^\prime = \mathcal{F}^\ast(\mathcal{E}_0)$, with $\mathcal{E}_0$ null-correlation bundle and $F$ absolute Frobenius on $\mathbb{P}^3$. Then, the zero locus of a generic global section of $E^\prime(s)$, for $s > p^2$, is an integral curve of degree $s^2 + p^{2n}$ not lying on any surface of degree $s$ such that its generic plane section is contained in an integral plane curve of degree $s$.

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2. The Frobenius morphism

First let us recall the definition of the relative Frobenius morphism (we follow Ein’s notation in [3]):

**Definition 2.1.** The absolute Frobenius morphism of a scheme $X$ of characteristic $p > 0$ is $F_X : X \rightarrow X$, where $F_X$ is the identity as a map of topological spaces and on each $U$ open set $F_X^\# : O_X(U) \rightarrow O_X(U)$ is given by $f \mapsto f^p$ for each $f \in O_X(U)$. Given $X \rightarrow S$ for some scheme $S$ and $X^{p/S} = X \times_{S, F^p} S$, the absolute Frobenius morphisms on $X$ and $S$ induce a morphism $F_{X/S} : X \rightarrow X^{p/S}$, called the Frobenius morphism of $X$ relative to $S$.

Let now $r \in \mathbb{N}$ and $n \in \mathbb{Z}$ be integers. Let $F : \mathbb{P}^3 \rightarrow \mathbb{P}^3$ be the absolute Frobenius and let us consider the sheaf $\mathcal{F} = (F^\ast)^\ast(\Omega_{p^3})$. The following result will be needed later:

**Lemma 2.2.**

(i) $h^0(\mathcal{F}(2p^r)) = 6$,
(ii) $h^0(\mathcal{F}(n)) \neq 0$ if and only if $n \geq 2p^r$,
(iii) $h^2(\mathcal{F}(n)) = 0$ for every $n \in \mathbb{Z}$.

**Proof.** First let us make some remarks. The sheaf $\Omega_{p^3}$ is determined by the Euler sequence $0 \rightarrow \Omega_{p^3} \rightarrow O_{\mathbb{P}^3}^\oplus(-1) \rightarrow O_{p^3} \rightarrow 0$, which is part of the Koszul complex $0 \rightarrow O_{p^3}(-4) \rightarrow O_{\mathbb{P}^3}^\oplus(-3) \rightarrow O_{p^3}^\oplus(-2) \rightarrow O_{p^3}^\oplus(-1) \rightarrow O_{p^3} \rightarrow 0$. So $\mathcal{F}$, by the flatness of the absolute Frobenius, is determined by the exact sequence:

\[
0 \rightarrow \mathcal{F} \rightarrow O_{\mathbb{P}^3}^\oplus(-p^r) \rightarrow O_{p^3} \rightarrow 0,
\]
which is part of the following long exact sequence:

\[(2) \quad 0 \to \mathcal{O}_{\mathbb{P}^3}(-4p^r) \to \mathcal{O}_{\mathbb{P}^3}^{\oplus 4}(-3p^r) \to \mathcal{O}_{\mathbb{P}^3}^{\oplus 6}(-2p^r) \to \mathcal{O}_{\mathbb{P}^3}^{\oplus 4}(-p^r) \to \mathcal{O}_{\mathbb{P}^3} \to 0.\]

(iii) follows immediately from (ii). Now we prove (i) and (ii). Considered the cokernel $\mathcal{G}$ of the first nonzero map in (2):

\[(3) \quad 0 \to \mathcal{O}_{\mathbb{P}^3}(-4p^r) \to \mathcal{O}_{\mathbb{P}^3}^{\oplus 4}(-3p^r) \to \mathcal{G} \to 0\]

$\mathcal{F}$ and $\mathcal{G}$ are related by the exact sequence:

\[(4) \quad 0 \to \mathcal{G} \to \mathcal{O}_{\mathbb{P}^3}^{\oplus 6}(-2p^r) \to \mathcal{F} \to 0.\]

Since $\mathcal{F}$ and $\mathcal{G}$ are vector bundles and $\mathcal{F}^\vee \cong \mathcal{G}(4p^r)$, then they are reflexive and $\mathcal{G}^\vee \cong \mathcal{F}(4p^r)$. So by (3) and (4):

\[h^0(\mathcal{F}(n)) = 6h^0(\mathcal{O}_{\mathbb{P}^3}(n-2p^r)) - 4h^0(\mathcal{O}_{\mathbb{P}^3}(n-3p^r)) + h^0(\mathcal{O}_{\mathbb{P}^3}(n-4p^r)).\]

From this we get (i) and (ii). \[\blacksquare\]

In the notation of Lemma 2.2 let us consider the sheaf $\mathcal{X} = \mathcal{F}(p^r)|_H$, restriction of $\mathcal{F}(p^r)$ to a plane $H$ in $\mathbb{P}^3$.

**Lemma 2.3.** For every $m \in \mathbb{Z}$:

\[h^0(\mathcal{X}(m)) = h^0(\mathcal{O}_H(m)) + 3h^0(\mathcal{O}_H(m-p^r)) - h^0(\mathcal{O}_H(m-2p^r)).\]

**Proof.** Let us make the position $\mathcal{F}_H = (\mathcal{F}^r)^\ast(\Omega_H)$. We can construct a surjective morphism of sheaves $\varphi : \mathcal{O}_H^{\oplus 4} \to \mathcal{O}_H^{\oplus 3}$ in such a way that we get the following commutative diagram:

\[
\begin{array}{ccccccccc}
0 & \to & \mathcal{X} & \to & \mathcal{O}_H^{\oplus 4} & \to & \mathcal{O}_H(p^r) & \to & 0 \\
0 & \to & \mathcal{F}_H(p^r) & \to & \mathcal{O}_H^{\oplus 3} & \to & \mathcal{O}_H(p^r) & \to & 0.
\end{array}
\]

Indeed, if $k[x_0, x_1, x_2, x_3]$ is the coordinate ring associated to $\mathbb{P}^3$ and $H$ has equation $x_3 = \sum_{i=0}^2 a_i x_i$, then we can define $\varphi$ as given by $\mathcal{O}_H^{\oplus 4} \ni (\sigma_0, \sigma_1, \sigma_2, \sigma_3) \mapsto (\sigma_0 + a_0 \sigma_3, \sigma_1 + a_1 \sigma_3, \sigma_2 + a_2 \sigma_3) \in \mathcal{O}_H^{\oplus 3}$. So by the snake lemma and by the fact that $\text{Ker} \varphi \cong \mathcal{O}_H$ we find the exact sequence:

\[(5) \quad 0 \to \mathcal{O}_H \to \mathcal{X} \to \mathcal{F}_H(p^r) \to 0.\]

Proceeding as in Lemma 2.2, we see that $\mathcal{F}_H(p^r)$ comes from the Koszul complex $0 \to \mathcal{O}_H(-2p^r) \to \mathcal{O}_H^{\oplus 3}(-p^r) \to \mathcal{O}_H^{\oplus 3} \to \mathcal{O}_H(p^r) \to 0$, which implies that $\mathcal{F}_H = (\mathcal{F}_H^{\vee})^\ast(-3p^r)$. Now:

\[(6) \quad h^0(\mathcal{F}_H(p^r + m)) = h^0((\mathcal{F}_H^{\vee}(m-2p^r)) = h^0(\mathcal{F}_H(2p^r - m)) = h^2(\mathcal{F}_H(2p^r - m) \otimes \mathcal{O}_H(-3)).\]

From $0 \to \mathcal{F}_H(2p^r - m - 3) \to \mathcal{O}_H^{\oplus 3}(p^r - m - 3) \to \mathcal{O}_H(2p^r - m - 3) \to 0$ we see that $h^2(\mathcal{F}_H(2p^r - m - 3)) = -h^0(\mathcal{O}_H(m-2p^r)) + 3h^0(\mathcal{O}_H(m-p^r))$, that, together with (5) and (6), leads us to the conclusion. \[\blacksquare\]
3. Incidence varieties in characteristic $p$

Let us consider the bi-projective space $\tilde{\mathbb{P}}^3 \times \mathbb{P}^3$ and let $r \in \mathbb{N}$ be a non negative integer. Let $k[t]$ and $k[x]$ be the coordinate rings for $\tilde{\mathbb{P}}^3$ and $\mathbb{P}^3$, respectively. Let $M_r$ be the hypersurface of equation:

$$h_r := \sum_{i=0}^{3} t_i x_i^p = 0.$$ 

First we need the following result:

**Lemma 3.1.** Let $q \in k[t, x]$ be a homogeneous polynomial of bi-degree $(\alpha, s)$ such that:

$$x_i^p \frac{\partial q}{\partial t_j} - x_j^p \frac{\partial q}{\partial t_i} \in (h_r) \quad \forall \ i, j$$

and $q \notin (h_r)$. Then there exists $q' = q + h_r m$ bi-homogeneous of bi-degree $(\alpha, s)$ such that:

$$\frac{\partial q'}{\partial t_i} = 0 \quad \forall \ i.$$

Since the proof of this lemma requires some computations, we leave it to the end of this section. Let us now remark that in the case $r = 0$ $M_r$ is usual incidence variety $M$ of equation $\sum t_i x_i = 0$. If $r \geq 1$, $M_r$ is determined by the following fibred product:

$$M_r \leftarrow \tilde{\mathbb{P}}^3 \times \mathbb{P}^3 \quad \pi \rightarrow M$$

where $F : \mathbb{P}^3 \rightarrow \mathbb{P}^3$ is the absolute Frobenius. Moreover $M = \mathbb{P}(\Theta_{\mathbb{P}^3}(-1))$ and so by [3 Lemma 1.5] we get $M_r = \mathbb{P}(F^* (\Theta_{\mathbb{P}^3}(-1)))$. By [3] Ch.II, ex. 7.8] this implies:

$$\text{Pic}(M_r) = \mathbb{Z} \times \mathbb{Z}$$

for any $r \geq 0$.

Let us consider an integral hypersurface $V \subset M_r$ and let us suppose that the projection $\pi : V \rightarrow \mathbb{P}^3$ is dominant. Using the previous lemma we prove the following result:

**Proposition 3.2.** If $\pi$ is not generically smooth, then there exists $s \geq 1$, such that $V \subset \tilde{\mathbb{P}}^3 \times \mathbb{P}^3$ is the complete intersection determined by $g = h_r$, for some $g \in k[t^p, x].$

**Proof.** Since $M_r \subset \tilde{\mathbb{P}}^3 \times \mathbb{P}^3$ is a hypersurface of bi-degree $(1, p^r)$, the structure sheaf $\mathcal{O}_{M_r}$ is given by $0 \rightarrow \mathcal{O}_{\tilde{\mathbb{P}}^3 \times \mathbb{P}^3}(-1, -p^r) \rightarrow \mathcal{O}_{\tilde{\mathbb{P}}^3 \times \mathbb{P}^3} \rightarrow \mathcal{O}_{M_r} \rightarrow 0$. By the Künneth formula ([12] Ch. VI, Corollary 8.13) $H^1(\mathcal{O}_{\tilde{\mathbb{P}}^3 \times \mathbb{P}^3}(m, n)) = 0$ for every $m, n \in \mathbb{Z}$, so that the morphism $H^0(\mathcal{O}_{\tilde{\mathbb{P}}^3 \times \mathbb{P}^3}(m, n)) \rightarrow H^0(\mathcal{O}_{M_r}(m, n))$ is surjective for every $m$,
Together with (11), this implies that \( V \subset \mathbb{P}^3 \times \mathbb{P}^3 \) is a complete intersection given by \( g = h_r = 0 \) for some \( g \in k[t, x] \) bi-homogeneous of bi-degree \((m, n)\) for some \( m, n \in \mathbb{N} \).

Let \( P_0 = (a, b) \in V \) be a regular point. By hypothesis the map on the projective tangent spaces \( T_{V,P_0} \) and \( T_{P,\pi(P_0)} \) is not surjective. The projective tangent space \( T_{V,P_0} \) at \( P_0 \in V \) is given by the equations:

\[
3 \sum_{i=0}^{3} \frac{\partial g}{\partial x_i}(P_0)x_i + 3 \sum_{i=0}^{3} \frac{\partial g}{\partial t_i}(P_0)t_i = 3 \sum_{i=0}^{3} (a_i x_i + b_i t_i) = 0
\]

if \( r = 0 \) and by the equations:

\[
3 \sum_{i=0}^{3} \frac{\partial g}{\partial x_i}(P_0)x_i + 3 \sum_{i=0}^{3} \frac{\partial g}{\partial t_i}(P_0)t_i = 3 \sum_{i=0}^{3} b_i t_i = 0
\]

if \( r \geq 1 \). In both cases the projection on \( T_{P,\pi(P_0)} \) is not surjective if and only if there exists \( \lambda \in k \) such that:

\[
\frac{\partial g}{\partial t_i}(P_0) = \lambda b_i t_i \quad \forall i = 0, \ldots, 3.
\]

So in such a situation:

\[
b_i t_i \frac{\partial g}{\partial t_j}(P_0) - b_j t_i \frac{\partial g}{\partial t_j}(P_0) = 0 \quad \forall i, j.
\]

This means that for every \( i, j \) the hypersurface \( V_{ij} \subset \mathbb{P}^3 \times \mathbb{P}^3 \) given by \( x_i t_j \frac{\partial g}{\partial t_j} - x_j t_i \frac{\partial g}{\partial t_i} = 0 \) contains \( \text{Reg}(V) \), the open subset of the regular points of \( V \). So \( V_{ij} \supset V \) for all \( i, j \), which means that:

\[
x_i t_j \frac{\partial g}{\partial t_j} - x_j t_i \frac{\partial g}{\partial t_i} \in (g, h_r) \quad \forall i, j.
\]

If \( x_i t_j \frac{\partial g}{\partial t_j} - x_j t_i \frac{\partial g}{\partial t_i} \) is a nonzero polynomial, then it is a bi-homogeneous polynomial of bi-degree \((m - 1, n + p)\). Since \( g \) is bi-homogeneous of bi-degree \((m, n)\), then

\[
x_i t_j \frac{\partial g}{\partial t_j} - x_j t_i \frac{\partial g}{\partial t_i} \in (h_r) \quad \forall i, j.
\]

Applying Lemma 3.1, we see that there exists \( m \in k[t, x] \) such that, given \( g' = g + mh_r \), we have \( \frac{\partial g'}{\partial t_i} = 0 \) for every \( i \). So by replacing \( g \) by \( g' \) we can suppose that \( g \in k[t^{p'}, x] \), for some \( s \geq 1 \).

Now we can prove the main result of this section:

**Theorem 3.3.** Let \( V \subset \mathbb{P}^3 \times \mathbb{P}^3 \) be an integral hypersurface in \( M \) such that the projection \( \pi: V \to \mathbb{P}^3 \) is dominant and not generically smooth. Then there exist \( r \geq 1 \), and \( V_r \subset M_r \) integral hypersurface such that \( \pi \) can be factored in the following way:

\[
\begin{array}{ccc}
V & \xrightarrow{\pi} & \mathbb{P}^3 \\
V_r & \xleftarrow{\pi_r} & M_r
\end{array}
\]
where the projection \( \pi_r \) is dominant and generically smooth and \( F_r \) is induced by the commutative diagram:

\[
\begin{array}{ccc}
V & \xrightarrow{F_r} & V_r \\
\downarrow j & & \downarrow i \\
M & \xrightarrow{F_{M_r}} & M_r.
\end{array}
\]

Proof. First note that by hypothesis and by Proposition 3.2 it follows that \( V \subset \mathbb{P}^3 \times \mathbb{P}^3 \) is the complete intersection determined by \( h = 0 \) and \( q = 0 \), for some \( q \in k[t^p, x] \) and \( r \geq 1 \). We can suppose that \( q \in k[t^p, x] \) and \( q' \notin k[t^p+1, x] \) for any \( q' \equiv q \mod (h) \). So we can say that \( q(t, x) = f(t^p, x) \) for some bi-homogeneous \( f \in k[t, x] \).

Let us now return to Lemma 3.1.

Proof of Lemma 3.1. From (7) we have:

\[
\left( \sum_{i=0}^{3} t_i x_i t^p \right) \frac{\partial q}{\partial t_j} - x_j t^p \left( \sum_{i=0}^{3} t_i \frac{\partial q}{\partial t_i} \right) \in (h_r, \forall j).
\]

Using that \( \sum t_i x_i t^p = h_r \) and that \( h_r \) is irreducible, we deduce \( \sum t_i \partial q / \partial t_i \in (h_r) \). However \( \sum_{i=0}^{3} t_i \partial q / \partial t_i = aq \), where \( a \) is the remainder of the division of \( \alpha \) by \( p \), because \( q \) is homogeneous of degree \( \alpha \) in the \( \{t_i\} \). So \( aq \in (h_r) \) and by hypothesis the only possibility is that \( a = 0 \), which means that \( p \mid \alpha \).

By (7) for every \( i, j = 0, 1, 2, 3 \) there exists \( l_{ij} \) bi-homogeneous in \( k[t, x] \) such that:

\[
x_j t^p \frac{\partial q}{\partial t_j} - x_j t^p \frac{\partial q}{\partial t_i} = l_{ij} h_r.
\]
The identity:
\[ x^{k^p_r} \left( x^i \frac{\partial q}{\partial t_j} - x^j \frac{\partial q}{\partial t_i} \right) - x^i \frac{\partial q}{\partial t_i} \left( x^k \frac{\partial q}{\partial t_j} - x^j \frac{\partial q}{\partial t_k} \right) + x^j \frac{\partial q}{\partial t_i} \left( x^k \frac{\partial q}{\partial t_j} - x^j \frac{\partial q}{\partial t_k} \right) = 0 \]
for every \( i, j, k \) determines the equality \( x^{k^p_r} l_{ij} - x^{i^p_r} l_{kj} + x^{j^p_r} l_{ki} = 0 \). So on \( D_+(x, x, x, x) \) we have the equality:
\[ \frac{l_{ij}}{(x, x, x)^{p^p_r}} - \frac{l_{ki}}{(x, x, x)^{p^p_r}} + \frac{l_{ki}}{(x, x, x)^{p^p_r}} = 0. \]
Considered now the open covering \( \mathcal{U} = \{ D_+(x, i) \mid i = 0, \ldots, 3 \} \) and \( n = \deg l_{ij} \), we get a Čech cocycle in \( H^1(\mathcal{U}, \mathcal{O}_X((n - 2p^r))) \cong H^1(\mathcal{O}_X((n - 2p^r))) = 0 \). So the cocycle is a coboundary and for every \( i, j \) there exist \( m_i, m_j \in k[\mathcal{L}, x] \) such that \( l_{ij} = m_i x^{i^p_r} - m_j x^{j^p_r} \). By replacing \( q \) by \( q - mh_r \), we may assume that \( \partial q / \partial t_i = m_i h_r \) for every \( i \). We want to prove that \( \partial q / \partial t_i = U_i h_r^p \) for some \( U_i \) and so let us suppose that:
\[ \frac{\partial q}{\partial t_i} = v_i h_r^n \forall i \]
for some \( n < p - 1 \). Then:
\[ \frac{\partial^2 q}{\partial t_i \partial t_j} = \frac{\partial v_i}{\partial t_j} h_r^n + n v_i x^i p^r h_r^{n-1} \forall i, j. \]
But we have also:
\[ \frac{\partial^2 q}{\partial t_i \partial t_j} = \frac{\partial v_i}{\partial t_i} h_r^n + n v_j x^j p^r h_r^{n-1} \forall i, j. \]
So:
\[ \frac{\partial^n}{\partial t_j} h_r^n + n v_i x^i p^r h_r^{n-1} = \frac{\partial v_i}{\partial t_i} h_r^n + n v_j x^j p^r h_r^{n-1} \]
\[ \Rightarrow v_i x^i p^r - v_j x^j p^r = \frac{h_r}{n} \left( \frac{\partial v_i}{\partial t_i} - \frac{\partial v_j}{\partial t_j} \right) \forall i, j. \]
This implies that \( v_i = u x^i p^r + h_r u_i \) for every \( i \). By replacing \( q \) by \( q - \frac{1}{n+1} v h_r^{n+1} \), we may assume that:
\[ \frac{\partial q}{\partial t_i} = V_i h_r^{p-1} \forall i. \]
We know that:
\[ \frac{\partial p q}{\partial t_i^p} = 0. \]
This means that:
\[ \frac{\partial^{p-1}(V_i h_r^{p-1})}{\partial t_i^{p-1}} = 0 \]
\[ \Rightarrow \sum_{n=0}^{p-1} \binom{p-1}{n} \frac{\partial^n V_i}{\partial t_i^n} \frac{\partial^{p-1-n}(h_r^{p-1})}{\partial t_i^{p-1-n}} = 0 \]
⇒ \( h_r \mid (p - 1)!x_i^{p+1-p}V_i \Rightarrow h_r \mid V_i \forall i \).

So we can suppose that:
\[
\frac{\partial q}{\partial t_i} = U_i h_r^p \forall i.
\]

Now (12) leads us to the conclusion that:
\[
\frac{\partial^{p-1} U_i}{\partial t_i^{p-1}} = 0
\]

which means that in \( U_i \), for each \( i \), there are no terms of type \( t_i^{kp-1} \) for any \( k \geq 1 \).

So in particular we can say that:
\[
U_0 = \frac{\partial M_0}{\partial t_0}
\]

for some \( M_0 \) bi-homogeneous. Now \( q' = q - M_0 h_r^p \) is such that:
\[
\frac{\partial q'}{\partial t_0} = 0 \quad \text{and} \quad \frac{\partial q'}{\partial t_i} = U_i' h_r^p, \quad i = 1, 2, 3
\]

⇒ \[
\frac{\partial U''_i}{\partial t_0} = 0, \quad i = 1, 2, 3.
\]

So we can find \( U''_1 \) such that:
\[
\frac{\partial U''_1}{\partial t_0} = 0 \quad \text{and} \quad \frac{\partial U''_1}{\partial t_1} = U_1'.
\]

If we consider \( q'' = q' - U''_1 h_r^p \) we see that:
\[
\frac{\partial q''}{\partial t_i} = 0, \quad i = 0, 1 \quad \text{and} \quad \frac{\partial q''}{\partial t_i} = U''_i h_r^p, \quad i = 2, 3.
\]

Proceeding in this way we get \( \partial q / \partial t_i = 0 \) for every \( i \).

\[\blacksquare\]

4. Proof of the main theorem

Let us consider now a curve \( C \subset \mathbb{P}^3 \) and, following the notation of Theorem 3.3 the projections \( p_M : M_r \to \mathbb{P}^3 \) and \( g_M : M_r \to \mathbb{P}^3 \). Let \( T_r = p_{M_r}^{-1}(C) \) and:
\[
\mathcal{I}_r(m, n) = g_{M_r}^* (\mathcal{O}_{\mathbb{P}^3}(m)) \otimes g_{M_r}^* (\mathcal{O}_{\mathcal{C}}(n))
\]

for every \( m, n \in \mathbb{Z} \).

**Proposition 4.1.** If \( \mathcal{I}_r = \mathcal{I}_r(0, 0) \) and \( \mathcal{I}_{T_r} \) is the ideal sheaf of \( T_r \) in \( M_r \), then \( \mathcal{I}_r = \mathcal{I}_{T_r} \).

**Proof.** First note that \( \mathcal{I}_r(m, n) = \mathcal{O}_{M_r}(m, n) \otimes \mathcal{O}_{M_r} p_{M_r}^* (\mathcal{I}_{\mathcal{C}}) \) for any \( m, n \in \mathbb{Z} \). Moreover \( p_M \) is smooth, in particular flat. So, by base change (see \[\mathbb{S}\]), \( p_M \) is flat too and we can apply \[\mathbb{S}\] Ch. III, Proposition 9.3] to the following commutative diagram:

\[
\begin{array}{ccc}
T_r & \xrightarrow{\pi} & C \\
\downarrow j & & \downarrow i \\
M_r & \xrightarrow{p_{M_r}} & \mathbb{P}^3
\end{array}
\]

to get that \( p_{M_r}^* \mathcal{I}_{\mathcal{C}} \cong j_* \pi^* \mathcal{I}_{\mathcal{C}} \cong j_* \mathcal{I}_{T_r} \). This fact together with the exact sequence \( 0 \to p_{M_r}^* \mathcal{I}_{\mathcal{C}} \to p_{M_r}^* \mathcal{O}_{\mathbb{P}^3} \to p_{M_r}^* i_* \mathcal{O}_{\mathcal{C}} \to 0 \), consequence of the flatness of \( p_{M_r} \), leads us to the desired conclusion. \[\blacksquare\]
Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. We divide the proof in different steps.

Step 1. There exists \( S \subset \mathbb{P}^3 \times \mathbb{P}^3 \) integral such that the generic fibre of the projection \( S \to \mathbb{P}^3 \) is an integral plane curve of degree \( s \) containing \( \Gamma \).

Proof of Step 1. Let \( I_C \) be the ideal sheaf of \( C \) in \( \mathbb{P}^3 \) and let \( M \subset \mathbb{P}^3 \times \mathbb{P}^3 \) be the incidence variety. Let us consider the two projections:

\[
\begin{array}{c}
P^3 \downarrow \quad \downarrow g \\
p \quad S \\
\end{array}
\]

and the \( \mathcal{O}_M \)-module \( \mathcal{I}(m, n) = \omega_{\mathbb{P}^3}(m) \otimes \mathcal{O}_M p^*(\mathcal{I}_C(n)) \). As we have seen in Proposition 4.1 in the case \( r = 0 \), \( \mathcal{I} \) is the ideal sheaf of \( T = p^{-1}(C) \) in \( \mathcal{O}_M \). Moreover there exists \( \alpha \) such that \( h^0(\mathcal{I}(\alpha, s)) \neq 0 \). Indeed, if \( \eta \in \mathbb{P}^3 \) denotes the generic point and \( \Gamma \) is the generic plane section of \( C \), then \( h^0(p^*(\mathcal{I}(s)|_{g^{-1}(\eta)}) = H^0(\mathcal{I}_C(s)) \neq 0 \) and so this global section determines an effective divisor in \( M_{k(\eta)} = M \times_{\mathbb{P}^3} \text{Spec } k(\eta) \cong \mathbb{P}^2_{k(\eta)} \). Then there exists \( U \subset \mathbb{P}^3 \) such that this divisor extends to an effective divisor \( D \subset M_U = M \times_{\mathbb{P}^3} U \) containing \( T \times_{\mathbb{P}^3} U \). The closure \( D \subset M \) is an effective divisor containing \( T \) and, since \( \text{Pic}(M) = \mathbb{Z} \times \mathbb{Z} \), it is a divisor determined by a global section of \( \mathcal{I}(\alpha, s) \), for some \( \alpha \geq 0 \).

Taken the least \( \alpha \) such that \( h^0(\mathcal{I}(\alpha, s)) \neq 0 \), there exists \( q \in H^0(\mathcal{I}(\alpha, s)) \) that determines a hypersurface \( S \) in \( M \) such that \( S \cap g^{-1}(\eta) \) is an integral curve of degree \( s \) containing \( \Gamma \). Moreover, as we saw in Proposition 5.2, \( S \) is a complete intersection of codimension 2 in \( \mathbb{P}^3 \times \mathbb{P}^3 \). This implies that \( S \) is irreducible. \( \square \)

To prove Theorem 1.2, we now assume that the curve \( C \) is not contained in any surface of degree \( s \), in other words, \( h^0(\mathcal{I}_C(s)) = 0 \). Then \( p_S : S \to \mathbb{P}^3 \) is dominant and, since \( \alpha \geq 0 \), in such a situation it must be \( \alpha > 0 \).

Step 2. We can factor \( p_S \) through a generically smooth morphism \( S_r \to \mathbb{P}^3 \), with \( S_r \) scheme of zeroes of a global section of \( \mathcal{I}_r(\beta, s) \), being \( \mathcal{I}_r = \pi_{M_r*}\mathcal{I}_C \), and \( \alpha = \beta p^r \), for some \( r \geq 0 \).

Proof of Step 2. If \( p_S \) is not generically smooth, then by Theorem 3.3 it follows that there exist \( r \geq 1 \), and \( S_r \subset M_r \) integral hypersurface such that \( p_S \) can be factored in the following way:
where the projection \( p_S \) is dominant and generically smooth and \( F^r \) is induced by the commutative diagram:

\[
\begin{array}{c}
S \\
\downarrow j \\
M \\
\end{array}
\quad \begin{array}{c}
\quad \begin{array}{c}
F^r \\
\downarrow i \\
S_r \\
\end{array}
\quad \begin{array}{c}
\quad \begin{array}{c}
M_r \\
\end{array}
\end{array}
\end{array}
\]

Moreover, we also get that \( \alpha = p^r \beta \), for some \( \beta \in \mathbb{N} \), \( \beta > 0 \).

Considered the sheaf \( \mathcal{F}_r = \pi_{M_r}^* \mathcal{F} \) and the scheme \( T_r = \pi_{M_r}^{-1}(C) \), by Proposition \[\[ \] \] we see that \( \mathcal{F}_r \) is the ideal sheaf of \( T_r \) in \( M_r \). Given \( T = \pi^{-1}(C) \), since \( S \supset T \) and \( F_M(T) = T_r \), then \( S_r \supset T_r \). So \( S_r \subset M_r \) is the scheme of zeros of a global section in \( H^0(\mathcal{F}_r(\beta, s)) \).

Hence in both cases we find \( S_r \) integral hypersurface in \( M_r \), with \( r \geq 0 \), such that the projection \( p_{S_r} : S_r \to \mathbb{P}^3 \) is generically smooth and \( S_r \subset M_r \) is the scheme of zeros of a global section in \( H^0(\mathcal{F}_r(\beta, s)) \), for some \( \beta > 0 \).

\[\square\]

Let us now follow the proof of Gruson and Peskine given in \[\[ \] \].

**Step 3.** There exists a 3-dimensional scheme \( Y \), with \( T_r \subseteq Y \subset S_r \), such that we have:

\[
0 \to \Omega^1_{S_r/\mathbb{P}^3} \to \Omega^1_{M_r/\mathbb{P}^3} \otimes \mathcal{O}_{M_r/\mathbb{P}^3} \mathcal{O}_{S_r} \to \mathcal{F}_Y(\beta, s) \to 0
\]

with \( \mathcal{F}_Y \subset \mathcal{O}_{S_r} \) ideal sheaf of \( Y \).

**Proof of Step 3.** Since \( S_r \) is generically smooth over \( \mathbb{P}^3 \), we have the exact sequence

\[
0 \to \mathcal{O}_{S_r}(-\beta, -s) \to \Omega^1_{M_r/\mathbb{P}^3} \otimes \mathcal{O}_{M_r/\mathbb{P}^3} \mathcal{O}_{S_r} \to \Omega^1_{S_r/\mathbb{P}^3} \to 0.
\]

Dualizing with respect to \( \mathcal{O}_{S_r/\mathbb{P}^3} \), we get:

\[
0 \to \Omega^1_{S_r/\mathbb{P}^3}^* \to \Omega^1_{M_r/\mathbb{P}^3}^* \otimes \mathcal{O}_{M_r/\mathbb{P}^3} \mathcal{O}_{S_r} \to \mathcal{O}_{S_r} \mathcal{O}_{S_r}(\beta, s).
\]

Since all the fibres of the projection \( T_r \to C \) have dimension 2 and \( \dim S_r = 4 \), \( p_{S_r} \) is not regular in any of the points of \( T_r \). It means that the last map in \([14]\) has image inside \( \mathcal{F}_T_r(\beta, s) \), the ideal sheaf of \( T_r \) in \( S_r \). So this image is an ideal sheaf of type \( \mathcal{F}_Y(\beta, s) \), where \( Y \subset S_r \) is a scheme containing \( T_r \), and \( 3 = \dim T_r \leq \dim Y \leq \dim S_r = 4 \). Since \( S_r \) is reduced and irreducible, if \( \dim Y = 4 \), then \( p_{S_r} \) would be non regular almost everywhere in \( S_r \). This contradicts the fact that \( p_{S_r} \) is generically smooth. So \( \dim Y = \dim T_r = 3 \) and \( T_r \subseteq Y \).

\[\square\]

Let us consider the projection \( g_M : M_r \to \mathbb{P}^3 \) and take any \( (\mathbf{b}) = (d^p) \in \mathbb{P}^3 \). Then \( g_M^{-1}(\mathbf{b}) = \{ (x_i, d^p) \mid (\sum d_i x_i)^p^r = 0 \} \). If \( H = g_M^{-1}(\mathbf{b})_{\text{red}} \) and \( D = p(g^{-1}(\mathbf{b})_{\text{red}}) \), then there exists \( U \subset \mathbb{P}^3 \) open such that, taken \( (\mathbf{b}) \in U \), \( D \) is an irreducible curve of degree \( s \) containing the plane section of \( C \) with \( H \). Let \( \Gamma \) denote such a section and let \( \mathcal{F}_T \subset \mathcal{O}_D \) be the its ideal sheaf.

**Step 4.** If \( \mathcal{M} = (\Omega_{M_r/\mathbb{P}^3})_{|H} \), there exist a rank two vector bundle \( \mathcal{N} \) and a zero-dimensional scheme \( \Delta \), with \( \Gamma \subseteq \Delta \subset D \), such that we have:

\[
0 \to \mathcal{N} \to \mathcal{M} \to \mathcal{I}_\Delta(\beta, s) \to 0
\]

being \( \mathcal{I}_\Delta \subset \mathcal{O}_D \) the ideal sheaf of \( \Delta \).
Proof of Step 4. Since \( M = \mathbb{P}(\Theta_{\mathfrak{p}_3}(-1)) \), by [8] and by [3] Lemma 1.5 we see that \( M_r = \mathbb{P}(F^*(\Theta_{\mathfrak{p}_3}(-1))) \), where we denoted by \( F \), as in [3], the absolute Frobenius on \( \mathbb{P}^3 \). The sheaf \( \mathcal{E} = F^*(\Theta_{\mathfrak{p}_3}(-1)) \) is determined by the exact sequence
\[ 0 \to \mathcal{O}_{\mathbb{P}^3}(-p^r) \to \mathcal{O}_{\mathbb{P}^3}(\mathfrak{p}^4) \to \mathcal{E} \to 0 \]
and by [8] Ch. III, Ex. 8.4(b) we have also
\[ 0 \to \Omega_{\mathbb{P}^3(\mathfrak{p}^4)} (p_{M_r}^* \mathcal{E}(-1)) \to \Omega_{M_r} \to 0. \]
When we restrict to \( H \), by the fact that the sequences locally split it follows that the following sequences are exact:
\[ 0 \to \mathcal{O}_H(-p^s) \to \mathcal{O}_H(\mathfrak{p}^4) \to \mathcal{E} \to 0 \]
and
\[ 0 \to \mathcal{M} \to (p_{M_r}^* \mathcal{E})|_H(-1) \to \mathcal{O}_H \to 0 \]
where \( \mathcal{M} = (\Omega_{\mathbb{P}^3} \mathfrak{p}^4)|_H. \) Since \( (p_{M_r}^* \mathcal{E})|_H(-1) = (p_{M_r}^* \mathcal{E}(-1,0))|_H = \mathcal{E}_H \), we have:
\[ 0 \to \mathcal{M} \to \mathcal{E}_H \to \mathcal{O}_H \to 0. \]
Restricting \( \mathcal{E}_H \) to \( H \), we get a surjective map \( \mathcal{M} \cap \mathcal{E}_H \mathcal{O}_D \to \mathcal{I}_\Delta(s) \), with \( \mathcal{I}_\Delta \subset \mathcal{O}_D \) ideal sheaf of a zero-dimensional scheme \( \Delta \) containing \( \Gamma \). The kernel of this map is a locally free sheaf of rank 2 that determines the exact sequence \( (15) \).

Step 5. \( d \leq s^2 + 2p^r. \)

Proof of Step 5. Note that \( c_1(\mathcal{I}_\Delta(s)) = s \) and \( c_2(\mathcal{I}_\Delta(s)) = \deg \Delta = \delta \geq d \). Now we compute the Chern classes of the other sheaves. From \( (13) \) we have \( c_1(\mathcal{E}_H) = p^r \)
and \( c_2(\mathcal{E}_H) = p^{2r}. \) So by \( (17) \) \( c_1(\mathcal{M}) = p^r \) and \( c_2(\mathcal{M}) = p^{2r} \), from which it follows that \( c_1(\mathcal{M}^\vee) = -p^r \) and \( c_2(\mathcal{M}^\vee) = p^{2r}. \) By \( (15) \) we see that:
\[ c_1(\mathcal{M}) = -p^r - s \quad \text{and} \quad c_2(\mathcal{M}) = p^{2r} - \delta + s^2 + p^r s. \]

Let \( m \in \mathbb{Z} \) be the smallest number such that \( H^0(\mathcal{M}^\vee(m-1)) = 0 \) and \( H^0(\mathcal{M}^\vee(m)) > 0. \) Dualizing \( (17) \), since \( \mathcal{E}_H \) is a locally free sheaf, we get \( 0 \to \mathcal{E}_H \to \mathcal{E}_H^\vee \to \mathcal{M}^\vee \to 0 \), from which it follows that:
\[ h^0(\mathcal{M}^\vee(m)) = h^0(\mathcal{E}_H^\vee(m)) - h^0(\mathcal{E}_H(m)) \quad \forall m \in \mathbb{Z}. \]
From the exact sequence \( (16) \) we see that, in the notation of Lemma 2.3 \( \mathcal{E}_H^\vee = \mathcal{M} \), so that, by Lemma 2.3 we see that:
\[ h^0(\mathcal{E}_H^\vee(m)) = h^0(\mathcal{E}_H(m)) + 3h^0(\mathcal{E}_H(m-p^s)) - h^0(\mathcal{E}_H(m-2p^r)) \]
for every \( m \in \mathbb{Z}. \) So:
\[ h^0(\mathcal{M}^\vee(m)) = 3h^0(\mathcal{E}_H(m-p^r)) - h^0(\mathcal{E}_H(m-2p^r)) \quad \forall m \in \mathbb{Z}. \]
This implies that \( h^0(\mathcal{M}^\vee(p^r-1)) = 0 \) and \( h^0(\mathcal{M}^\vee(p^r)) > 0. \) So \( h^0(\mathcal{M}(p^r-1)) = 0 \)
and \( p^{2r} + p^r(-s - p^r) + c_2(\mathcal{M}) = c_2(\mathcal{M}(p^r)) \geq 0. \) So we get that \( p^{2r} + s^2 \geq \delta \)
and, since \( \delta \geq d \):
\[ p^{2r} + s^2 \geq d. \]

Step 6. If \( C \) is connected, \( p^r < s; \) if \( C \) is merely reduced, \( p^r \leq s. \)
Proof of Step 6. Let us now consider a generic plane $H = V(l)$, with $l$ linear form in the $\{x_i\}$, and the non reduced surface $H_r$ in $\mathbb{P}^3$ given by $l^{r'} = 0$. Let $\Gamma_r \subset H_r$ be the section of $C$ with $H_r$. Then there is the following exact sequence:

$$0 \rightarrow I_C(-p^r) \xrightarrow{\varphi_H} I_C \rightarrow i_*I_{\Gamma_r} \rightarrow 0$$

where $i:\Gamma_r \hookrightarrow \mathbb{P}^3$ and $\varphi_H$ is the multiplication by $l^{r'}$. The long cohomology exact sequence associated to the previous exact sequence shifted by $s$ determines the following one:

$$(19) \quad H^0(I_C(s)) \rightarrow H^0(I_{\Gamma_r}(s)) \rightarrow H^1(I_C(s - p^r)) \xrightarrow{\varphi_H} H^1(I_C(s)).$$

Let $[H] \in \mathbb{P}^3$ be a point such that the fibre of the projection $M_r \rightarrow \mathbb{P}^3$ at $[H]$ is isomorphic to $H_r$. Then, taking $[H]$ in a suitable open $U \subset \mathbb{P}^3$, $g_{s,-1}([H])$ is the complete intersection of $H_r$ and of a surface of degree $s$ containing $C \cap H_r$, because $T_r \subset S_r$. It means that $H^0(I_{\Gamma_r}(s)) \neq 0$ and so by $(19)$ and by hypothesis it must be $h^1(I_C(s - p^r)) \neq 0$.

Let us suppose that $C$ is connected. Then $h^1(I_C(n)) = 0$ for $n \leq 0$. So $s - p^r \geq 1$, because otherwise $h^0(I_C(s)) \neq 0$, which contradicts the hypothesis made at the beginning.

If $C$ is merely reduced, we still have $h^1(I_C(n)) = 0$ for $n < 0$. So, as before, it must be $s - p^r \geq 0$. □

Let us suppose that $C$ is connected. By Step 6, if $p \geq s$, then the only possibility is that $r = 0$, which implies $d \leq s^2 + 1$. If $p < s$, then $p^r \leq s - 1$ and, in particular, $p^r \leq p^n$, being $p^n < s \leq p^{n+1}$. So by $(18)$ $d \leq s^2 + p^{2n}$ and in particular we see that $d \leq 2s^2 - 2s + 1$.

Let us suppose now that $C$ is merely reduced. If $p > s$, then it must be $r = 0$, so that by $(18)$ we have $d \leq s^2 + 1$. If $p \leq s$, then $p^r \leq s$ and, in particular, $p^r \leq p^n$, being $p^n \leq s < p^{n+1}$. Now by $(18)$ we see that $d \leq s^2 + p^{2n}$. In particular, $d \leq 2s^2$.

5. Example

In this section we show that for any $p$ there exist smooth integral curves of degree $d = s^2 + p^{2n}$, being $s > p$ and $n$ such that $p^n < s \leq p^{n+1}$, that are not contained in any surface of degree $s$ and that have the generic plane section contained in an integral plane curve of degree $s$.

First, let us recall the following definition.

Definition 5.1. A rank 2 vector bundle $E_0$ on $\mathbb{P}^3$ is said to be a null-correlation bundle if there exists an exact sequence:

$$(20) \quad 0 \rightarrow O_{\mathbb{P}^3} \xrightarrow{\tau} \Omega_{\mathbb{P}^3}(2) \rightarrow E_0(1) \rightarrow 0$$

where $\tau$ is a nowhere vanishing section of $\Omega_{\mathbb{P}^3}(2)$.

Remark 5.2. It is possible to prove (see [1], [13] and [9] Example 8.4.1)) that $E$ is a stable rank 2 vector bundle on $\mathbb{P}^3$ with $c_1(E) = 0$ and $c_2(E) = 1$ if and only if $E$ is isomorphic to a null-correlation bundle.

Example 5.3. Let $E_0$ be a null-correlation bundle. Let $n, s \in \mathbb{N}$ be positive integers and let $F: \mathbb{P}^3 \rightarrow \mathbb{P}^3$ be the absolute Frobenius on $\mathbb{P}^3$. Let us consider the sheaf $E(s) = F^{n*}(E_0) \otimes O_{\mathbb{P}^3}(s)$. Since $c_1(F^{n*}(E_0)) = 0$ and $c_2(F^{n*}(E_0)) = p^{2n}$, we see that $c_1(E(s)) = 2s$ and $c_2(E(s)) = p^{2n} + s^2$. 

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Let $\sigma \in H^0(\mathcal{E}(s))$ be a global section such that the zero locus of $\sigma$ is a curve $C$. Then we get the exact sequence:

$$(21) \quad 0 \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{E}(s) \to \mathcal{I}_C(2s) \to 0$$

so that $h^0(\mathcal{I}_C(2s)) = h^0(\mathcal{E}(s))$ and $\deg C = c_2(\mathcal{E}(s)) = p^2 + s^2$. Let $H$ be a plane transversal to $C$ and $\Gamma = C \cap H$. Restricting to $H$ the exact sequence (21) we have:

$$(22) \quad 0 \to \mathcal{O}_H \to \mathcal{E}(s)|_H \to \mathcal{I}_T(2s) \to 0$$

so that:

$$(23) \quad h^0(\mathcal{I}_T(s)) = h^0(\mathcal{E}|_H).$$

By [10, Theorem 3.2] $\mathcal{E}$ is stable and we can choose $H$ sufficiently general in such a way that $\mathcal{E}|_H$ is semi-stable, but not stable. Since $\mathcal{E}$ is stable and $c_1(\mathcal{E}) = 0$, then by Lemma 3.1 $h^0(\mathcal{E}) = 0$, which implies that $h^0(\mathcal{I}_C(s)) = 0$. So $C$ is not contained in any surface of degree $s$. Since $\mathcal{E}|_H$ is semi-stable, but not stable and $c_1(\mathcal{E}|_H) = 0$, it must be $h^0(\mathcal{E}|_H) \neq 0$, so that $h^0(\mathcal{I}_T(s)) \neq 0$. Moreover by (22) $h^0(\mathcal{I}_T(s - 1)) = h^0(\mathcal{E}|_H((s - 1)))$. Now note that by (20) the sheaf $\mathcal{E}$ is determined by the exact sequence:

$$(24) \quad 0 \to \mathcal{O}_{\mathbb{P}^3}(-p^n) \to (F^n)\ast (\Omega_{\mathbb{P}^3}(p^n)) \to \mathcal{E} \to 0.$$

so that, considered the sheaf $\mathcal{F} = (F^n)\ast (\Omega_{\mathbb{P}^3})$, we have the exact sequence $0 \to \mathcal{O}_H(-p^n - 1) \to \mathcal{F}|_H(p^n - 1) \to \mathcal{E}|_H(-1) \to 0$, which implies that $h^0(\mathcal{E}|_H(-1)) = h^0(\mathcal{F}|_H(p^n - 1)) = 0$ by Lemma 2.3. So the plane curves of degree $s$ containing the generic plane section of $C$ are the minimal ones. Moreover by the previous exact sequence and by Lemma 2.3 we have the equality $h^0(\mathcal{E}|_H) = h^0(\mathcal{F}|_H(p^n)) = 1$, which implies by (23) that $h^0(\mathcal{I}_T(s)) = 1$. So there is a unique plane curve of degree $s$ containing $\Gamma$, which means that this plane curve of degree $s$ is the minimal plane curve containing $\Gamma$.

Now we want to know when $h^0(\mathcal{E}(s)) \neq 0$. By (24) we get for each $s \in \mathbb{N}$:

$$(25) \quad 0 \to \mathcal{O}_{\mathbb{P}^3}(-p^n + s) \to \mathcal{F}(p^n + s) \to \mathcal{E}(s) \to 0.$$

By Lemma 2.2 $h^0(\mathcal{F}(2p^n)) = 6$ and $h^0(\mathcal{F}(p^n + s)) \neq 0$ if and only if $s \geq p^n$. So from (25):

$$(26) \quad h^0(\mathcal{E}(p^n)) = 5$$

and $h^0(\mathcal{E}(s)) \neq 0$ if and only if $s \geq p^n$.

So we have global sections only for $s \geq p^n$. We want to prove there exist global sections of $\mathcal{E}(s)$, for every $s \geq p^n$, whose zero locus is a curve. First we must prove that $\mathcal{E}$ is not split. If this was the case, then, being $\mathcal{E}$ a locally free sheaf of rank 2, we would have $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^3}(a) \oplus \mathcal{O}_{\mathbb{P}^3}(b)$, for some $a, b \in \mathbb{Z}$. Since $h^0(\mathcal{E}(p^n - 1)) = 0$, it must be $a + p^n - 1 < 0$ and $b + p^n - 1 < 0$, so that $h^0(\mathcal{E}(p^n)) \leq 2$, but this contradicts (20). So $\mathcal{E}$ is not split. Moreover, since $h^0(\mathcal{E}(p^n)) = 5 > h^0(\mathcal{E}_{\mathbb{P}^3}) = 1$, by [5, Theorem 0.1] we get that every general nonzero global section of $\mathcal{E}(s)$, for $s \geq p^n$, has as zero locus a curve in $\mathbb{P}^3$.

Now we want to know when there are connected curves. By [3, Proposition 1.4] if $h^1(\mathcal{E}(\mathcal{V})(s)) = 0$, then a generic global section of $\mathcal{E}(s)$, for $s \geq p^r$, determines a connected curve. Note that $h^1(\mathcal{E}(\mathcal{V})(s)) = h^2(\mathcal{E}(s - 4))$. By (25) and by Lemma 2.2 $h^2(\mathcal{E}(s - 4)) \leq h^3(\mathcal{O}_{\mathbb{P}^3}(s - p^n - 4)) = 0$ for $s > p^n$. So for $s > p^n$ the generic global section of $\mathcal{E}(s)$ is connected.
Now we want to know when we have nonsingular curves. By [9, Proposition 1.4], if $\mathcal{E}(s-1)$ is generated by its global sections, then a sufficiently generic global section in $H^0(\mathcal{E}(s))$ will determine a nonsingular zero locus (not necessarily connected).

Note now that in the proof of Lemma 2.2 we have seen that there is a surjective morphism of sheaves $\mathcal{O}^{\oplus 6}_{\mathbb{P}^3} \twoheadrightarrow \mathcal{F}(2p^n)$ (see (4)). From (25) we see that we have also the surjective morphism $\mathcal{F}(2p^n) \twoheadrightarrow \mathcal{E}(p^n)$, which means that $\mathcal{E}(p^n)$ is generated by its global sections and so $\mathcal{E}(s)$ is generated by its global sections for $s \geq p^n$.

In this way we construct, for any $p$, $n$, $s$, with $s \geq p^n$, examples of curves $C \subset \mathbb{P}^3$ of degree $p^{2n} + s^2$ not contained in any surface of degree $s$ such that the minimal curve containing its generic plane section has degree $s$ and such that:

1. $C$ is nonsingular, in particular reduced;
2. $C$ is nonsingular and connected, which means nonsingular and irreducible, in the case $s > p^n$. In this situation the minimal curve of degree $s$ containing the generic plane section of $C$ is integral by [2, Theorem 4.1].

In particular, we see that the bound in Theorem 1.2 for connected curves is sharp. Moreover, taking $s = p^n + 1$, we see that there exist connected and reduced curves (in particular nonsingular) of degree $d = 2s^2 - 2s + 1$, not lying on any surface of degree $s$, whose generic plane section is contained in an integral plane curve of degree $s$.

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