Oscillating wandering domains for $p$-adic transcendental entire maps

Adrián Esparza-Amador$^1$ | Jan Kiwi$^2$

$^1$Instituto de Matemáticas, Pontificia Universidad Católica de Valparaíso, Cerro Barón, Valparaíso, Chile
$^2$Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Macul, Santiago, Chile

Correspondence
Adrián Esparza-Amador, Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Vicuña Mackenna 4860, Macul, Santiago 7820436, Chile. Email: adrian.esparza@pucv.cl

Abstract
We give examples of transcendental entire maps over $\mathbb{C}_p$ having an oscillating wandering Fatou component.

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1 | INTRODUCTION

The general aim of this paper is to advance on the study of iterations of transcendental entire maps over the $p$-adic complex numbers $\mathbb{C}_p$. The emphasis is on the Fatou–Julia theory, which was initially considered by Bézivin in [6] and further developed by Fan and Wang in [7].

As it is well established in one-dimensional non-Archimedean dynamics, the action occurs on the corresponding Berkovich analytic space. Consequently, a transcendental entire map $f$ will be regarded as a dynamical system acting on the Berkovich affine line $\mathbb{A}^1_{an}$ over $\mathbb{C}_p$. The Fatou set of $f$, denoted $\mathcal{F}(f)$, consists of the elements of $\mathbb{A}^1_{an}$ having a neighborhood where the iterates of $f : \mathbb{A}^1_{an} \to \mathbb{A}^1_{an}$ form a normal family, in the sense of [8]. A connected component $U$ of $\mathcal{F}(f)$ is necessarily bounded (see [8, Theorem 6.5]), and $f(U)$ is again a component of $\mathcal{F}(f)$. Such a component is called eventually periodic if $f^{on}(U) = f^{om}(U)$ for some $n > m \geq 0$, and otherwise is called wandering. According to Fan and Wang [7, Section 4], eventually periodic components must be Berkovich disks whose behavior is analogous to eventually periodic components of polynomial maps (see, for example, [4]). The focus of this article is on wandering Fatou components, sometimes simply called wandering domains.
An a priori classification for wandering domains of entire maps, disregarding existence issues, is as follows. We say that a wandering Fatou component $U$ of an entire map $f$ is **escaping** if $f^{\circ n}(x) \to \infty$, for all $x \in U$, and **dynamically bounded** if $\{f^{\circ n}(x) : n \in \mathbb{N}\}$ is precompact in $\mathbb{A}_1^n$, for all $x \in U$. When there exists a point $x$ in $U$ whose orbit is unbounded but not converging to $\infty$, then every point in $U$ has this property, and we say that $U$ is an **oscillating wandering domain** (see Proposition 2.1).

Dynamically bounded examples are known or follow from the literature. In fact, Benedetto [2, 3] provided the first examples of polynomials in $\mathbb{C}_p$ with a wandering Fatou component (necessarily a dynamically bounded disk). According to Fernández [10], a large class of families obtained as perturbations of those considered by Benedetto still possess members with wandering domains. The existence of transcendental entire maps with dynamically bounded wandering domains follows.

Examples of escaping wandering domains are also known and not hard to construct. According to Fan and Wang [7, Theorem 1.2], any Fatou component which is not a Berkovich open disk is an escaping domain. A transcendental entire map with escaping annuli Fatou components was provided in [8, Example 6.7]. Examples with escaping disk components are easy to produce. On the contrary, we do not know if every entire transcendental map has an escaping disk component.

The aim of this article is to provide the first examples of transcendental entire maps over $\mathbb{C}_p$ with oscillating wandering domains. Thus, we answer in the positive a question by Favre, Trucco and the second author (see [8, Question 6.12]). We build on the beautiful ideas introduced by Benedetto [2, 3] to produce (dynamically bounded) wandering domains in the context of polynomial dynamics.

Given a strictly increasing sequence $r = (R_j)_{j \geq 1}$ in $|\mathbb{C}_p|^\times$ converging to infinity such that $R_1 > p$, we will consider a family of transcendental entire maps parametrized by

$$\mathcal{C}(r) = \{c = (c_j)_{j \in \mathbb{N}} : c_j \in \mathbb{C}_p \text{ and } |c_j| = R_j\}.$$ \hspace{1cm} (1)

Specifically, given $c = (c_j) \in \mathcal{C}(r)$, let

$$f_c(z) := \frac{z^p}{p} \prod_{j \geq 1} \left(1 - \frac{z}{c_j}\right).$$

Under an extra mild condition on $r = (R_j)$, we will say that $r$ is a **generic sequence of radii** (see Definition 3.1), and our main result proves that for any $c \in \mathcal{C}(r)$ the map $f_c$ can be perturbed to one possessing an oscillating wandering domain:

**Theorem A.** Let $r$ be a generic sequence of radii and let $(\varepsilon_j)$ be a sequence of positive real numbers. Then, for all $c = (c_j) \in \mathcal{C}(r)$, there exists $c' = (c'_j) \in \mathcal{C}(r)$ such that $f_{c'}$ has an oscillating wandering domain and $|c_j - c'_j| < \varepsilon_j$, for all $j \geq 1$.

The local degree of $f_c$ at $z = 0$ is $p$, for all $c$. That is, all the maps $f_c$ possess a **wild critical point** at $z = 0$ whose dynamical influence plays a crucial role in the proof of our main result. Following Trucco [12], we say that an entire transcendental map $f : \mathbb{A}_1^{an} \to \mathbb{A}_1^{an}$ is **tame** if the local degree

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1 When working over more general fields, one has to further distinguish dynamically bounded wandering domains according to whether $(f^{\circ n}(x))$ converges to a type II periodic orbit or not.
of \( f \) at any \( y \in \mathbb{A}^1_{an} \) is not divisible by \( p \). It is natural to ask if there are tame transcendental entire maps with oscillating wandering Fatou components.

Let us now briefly outline the organization of the paper.

In Section 2, we review some fundamentals about the Berkovich affine line and analytic maps with the primary purpose of introducing notation. We refer the reader to \([1, 4, 5, 9]\) for a detailed exposition. We also recall the definitions and main properties of the Julia and Fatou sets following \([8]\) and \([7]\).

In Section 3, we start discussing the main dynamical features of the maps \( f_\epsilon \) and introduce symbolic dynamics according to a partition of \( \mathbb{A}^1_{an} \) specially adapted for our purpose. Then, we give a detailed outline of the proof of Theorem A. More precisely, we discuss intermediate results and state, without proof, Lemmas 3.6–3.8. Then, assuming these lemmas, we proceed to establish Theorem 3.10 which is a quantified version of Theorem A.

Sections 4–6 are devoted to proving Lemmas 3.6–3.8, respectively.

\section{Preliminaries}

\subsection{Berkovich space}

For a detailed exposition on the Berkovich affine line, specially adapted to non-Archimedean dynamics, we refer the reader to \([4, \text{Part 3}]\). See also \([7–9]\). Our primary purpose here is to agree on notation.

For any \( z \in \mathbb{A}^1 \equiv \mathbb{C}_p \) and \( r > 0 \),

\begin{align*}
D(z, r) &:= \{ w \in \mathbb{C}_p : |w - z| < r \}, \\
\overline{D}(z, r) &:= \{ w \in \mathbb{C}_p : |w - z| \leq r \},
\end{align*}

are open and closed disks in \( \mathbb{A}^1 \), respectively. The corresponding disks in Berkovich space \( \mathbb{A}^1_{an} \) are denoted by \( D(z, r) \) and \( \overline{D}(z, r) \).

Recall that the Berkovich affine line \( \mathbb{A}^1_{an} \) is the space of multiplicative semi-norms in the ring \( \mathbb{C}_p[z] \) that extend the \( p \)-adic absolute value in \( \mathbb{C}_p \) endowed with the Gel’fand topology. With this topology \( \mathbb{A}^1_{an} \) is a locally compact space.

The sup-norm on \( D(z, r) \), denoted \( \xi_{z,r} \), is often referred to as the point associated to \( D(z, r) \). Also, to each element \( z \) of \( \mathbb{A}^1 \) corresponds the semi-norm given by \( f \mapsto |f(z)| \). Via this correspondence, the affine line \( \mathbb{A}^1 \) is identified with a (dense) subset of \( \mathbb{A}^1_{an} \).

In general, for \( x \in \mathbb{A}^1_{an} \) and \( f \in \mathbb{C}_p[z] \), it is convenient to denote by \( |f(x)| \) the semi-norm \( x \) evaluated at \( f \). In particular, \( |x| \) denotes the semi-norm \( x \) evaluated at \( f(z) = z \). Given \( x \in \mathbb{A}^1_{an} \), let

\[ \overline{D}_x := \{ y \in \mathbb{A}^1_{an} : |f(y)| \leq |f(x)| \text{ for all } f \in \mathbb{C}_p[z] \}. \]

According to Berkovich, \( \overline{D}_x \cap \mathbb{A}^1 \) is either a singleton, a closed disk with radius in \( |\mathbb{C}_p^X| \), a disk with radius not in \( |\mathbb{C}_p^X| \) or the empty set. The point \( x \) is, respectively, called of type I, II, III or IV. Type I points coincide with the elements of \( \mathbb{A}^1 \) and sometimes are called ‘classical points’.
Between any two points \( x, y \in \mathbb{A}^1_{an} \), there is a unique arc connecting them, which we denote by \([x, y]\). It is convenient to consider the one point compactification \( \mathbb{A}^1_{an} \cup \{\infty\} \). We denote by \([x, \infty]\) the unique arc in \( \mathbb{A}^1_{an} \) with one endpoint at \( x \) that converges to \( \infty \) at the other end.

Given \( y \in \mathbb{A}^1_{an} \), two points \( z, z' \), both distinct from \( y \), are said to be in the same direction at \( y \) if they lie in the same connected component of \( \mathbb{A}^1_{an} \setminus \{y\} \). Equivalently, \([z, y] \cap [z', y] \neq \emptyset \). The set of directions at \( y \), denoted by \( T_y \mathbb{A}^1_{an} \), is called the (projectivized) tangent space or space of directions at \( y \). Given a direction \( \mathbf{v} \) at \( y \), we denote by \( \mathbb{R}_y(\mathbf{v}) \) the set of points in that direction. There is a unique unbounded direction \( \mathbb{R}_y(\mathbf{v}) \), which we call the direction of \( \infty \). All other directions are maximal open disks contained in \( \mathbb{R}_y(\mathbf{v}) \). In particular, the tangent space of type I and IV points are trivial (that is, a singleton). Type III points have two tangent directions. The tangent space of type II points is naturally endowed with the structure of a projective line over the residue field \( \overline{\mathbb{C}}_p \) with a distinguished point \( \infty \) corresponding to the direction of \( \infty \).

The action of an entire map \( f : \mathbb{A}^1_{an} \to \mathbb{A}^1_{an} \) induces an action \( T_x f : T_x \mathbb{A}^1_{an} \to T_{f(x)} \mathbb{A}^1_{an} \) between tangent spaces for all \( x \in \mathbb{A}^1_{an} \). More precisely, given \( \mathbf{v} \in T_x \mathbb{A}^1_{an} \), the direction \( \mathbf{w} = T_x f(\mathbf{v}) \) is defined by the property that \( f(D_x(\mathbf{v}) \cap U) \subset D_{f(x)}(\mathbf{w}) \) for all sufficiently small neighborhoods \( U \) of \( x \). It follows that \( T_x f \) maps a bounded direction (respectively, the direction of infinity) onto a bounded direction (respectively, the direction of infinity). Thus, this map is rather trivial for type I, III, and IV points. For a type II point \( x \), the tangent map \( T_x f \) is a non-constant polynomial map between the corresponding tangent spaces endowed with their \( \mathbb{P}^1(\overline{\mathbb{C}}_p) \)-structure and the distinguished point \( \infty \). Moreover, for any bounded direction \( \mathbf{v} \), we have \( f(D_x(\mathbf{v})) = D_{f(x)}(\mathbf{w}) \) if and only if \( \mathbf{w} = T_x f(\mathbf{v}) \).

### 2.2 Fatou–Julia sets

Recall from [8] that given an open subset \( U \) of \( \mathbb{A}^1_{an} \), we say that a family \( G \) of analytic functions \( g : U \to \mathbb{A}^1_{an} \) is normal if for any sequence \((g_n) \subset G\) and any \( x \in U \) there exists a neighborhood \( V \) of \( x \), and a subsequence \((g_{n_k}) \) such that \( g_{n_k} : U \to \mathbb{P}^1_{an} \) converges pointwise to a continuous function. The main result in [8] implies that families \( G \) for which there exists an open disk \( D \) such that \( g(U) \cap D = \emptyset \), for all \( g \in G \), are normal.

Given an entire function \( f : \mathbb{A}^1_{an} \to \mathbb{A}^1_{an} \), we say that a point \( x \in \mathbb{A}^1_{an} \) lies in the Fatou set \( F(f) \) of \( f \) if there exists a neighborhood of \( x \) where \( \{f^n : n \in \mathbb{N}\} \) is a normal family. The complement is the Julia set \( J(f) \) of \( f \). Both of these sets are completely invariant under \( f \) and the Julia set is always non-empty. It is an open question posed by Bézivin [6] to determine if the Fatou set can be empty. See [7] for a detailed discussion of the Fatou and Julia sets of entire transcendental maps. As in rational dynamics, periodic points of \( f \) in \( \mathbb{A}^1_{an} \) are classified into attracting, indifferent, and repelling. According to Fan and Wang [7], repelling periodic points are dense in the Julia set. Also, \( \{x, \infty \} \cap J(f) \neq \emptyset \) for all \( x \in \mathbb{A}^1_{an} \) [8, Theorem 6.5].

As claimed in the introduction, the following result shows that once a point \( z \) in a Fatou component \( U \) has an unbounded non-escaping orbit, the orbit of all the points in \( U \) have the same behavior:
**Proposition 2.1.** Let $U$ be a Fatou component of a transcendental entire map $f : A^1_{an} \to A^1_{an}$. Then, the following hold.

1. If there exists $z \in U$ with bounded orbit, then there exists $R > 0$ such that $f^{on}(U) \subset D(0, R)$ for all $n \geq 0$.
2. If there exists $z \in U$ such that $f^{on}(z) \to \infty$, then for all $R > 0$, there exists $N$ such that $f^{on}(U) \subset A^1_{an} \setminus D(0, R)$ for all $n \geq N$.

**Proof.**

(1) If $z \in U$ has bounded orbit, then $f^{on}(z) \in D(0, r)$ for some $r$ and all $n$. Hence, there exists $R > r$ such that $\xi_{0,R} \in \mathcal{J}(f)$ where $\xi_{0,R}$ denotes the boundary point of $D(0, R)$. Therefore, $f^{on}(U) \subset D(0, R)$ for all $n$, since $D(0, R)$ is a connected component of $A^1_{an} \setminus \{\xi_{0,R}\}$.

(2) Let $R > 0$ and assume that $f^{on}(z) \to \infty$. There exists $R' > R$ such that $\xi_{0,R'} \in \mathcal{J}(f)$ and therefore, $f^{on}(U) \subset A^1_{an} \setminus D(0, R')$ for sufficiently large $n$. □

### 2.3 Non-Archimedean mean value theorem

The usual Mean Value Theorem does not apply verbatim over $C_p$. For the sake of completeness we state its non-Archimedean version below, since it is intensively employed throughout this work. The reader may find a detailed discussion in [11, Section 6.2.4].

The so called exponential radius

$$\varphi := p^{-1/(p-1)}$$

naturally arises in this context:

**Theorem 2.2** (Non-Archimedean Mean Value Theorem). Let $f : D(a, r) \to C_p$ be an analytic map. Then, for all $z, w \in D(a, r)$ with $|z - w| < \varphi r$, there exists $u \in D(a, r)$ such that

$$|f(z) - f(w)| = |f'(u)| \cdot |z - w|.$$

### 3 OSCILLATING ITINERARIES

This section is devoted to discussing the dynamics of the family of entire transcendental maps $f_c$ defined in the introduction. We restrict to maps $f_c$ where the sequence $(R_j)$ is generic, in the sense discussed in Section 3.1. Symbolic dynamics according to a suitable partition of $A^1_{an}$ is introduced in Section 3.2. Specifically, we define the cylinder set associated to sequences $m$ and $\ell$ of positive integers. Oscillating wandering domains will arise as interior components of these cylinders after carefully choosing $m$ and $\ell$, and proving that the interior is non-empty. In Section 3.3, we give a detailed outline of the proof of Theorem A by reducing it to three main lemmas.
3.1 | **Generic sequence of radii**

Let \( \mathbf{r} = (R_j) \) be an (strictly) increasing sequence in \( |\mathbb{C}_p^\times| \) converging to infinity, and for convenience, bounded below by \( p \) (that is, \( R_1 > p \)). Recall from the introduction that \( \mathcal{C}(\mathbf{r}) \) denotes the set of sequences \( (c_j) \) such that \( |c_j| = R_j \) for all \( j \), and for each \( \mathbf{c} \in \mathcal{C}(\mathbf{r}) \), we consider the map \( f_\mathbf{c} \) introduced in (1).

The arc \( ]0, \infty[ \subset \mathbb{A}^1 \) is invariant under \( f_\mathbf{c} \) and the action \( f_\mathbf{c} : ]0, \infty[ \to ]0, \infty[ \) is described by

\[
\varphi(r) \coloneqq \sup \{|f_\mathbf{c}(z)| : |z| \leq r\},
\]

where \( r > 0 \). In fact,

\[
f_\mathbf{c}(\xi_{0,r}) = \xi_{0,\varphi(r)},
\]

where \( \xi_{0,r} \) denotes the element of \( \mathbb{A}^1 \) given by the sup-norm on the disk \( D(0, s) \). Often in the literature, \( \varphi \) is denoted by \( |f_\mathbf{c}|_r \). The function \( \varphi \) is increasing, piecewise monomial, and only depends on \( \mathbf{r} \). Indeed, a rather straightforward calculation shows that for all \( \mathbf{c} \in \mathcal{C}(\mathbf{r}) \),

\[
\varphi(r) = \begin{cases} 
pr^p & 0 < r \leq R_1, \\
p(R_1 \cdots R_{j-1})^{-1} r^{p+j-1} & R_{j-1} < r \leq R_j.
\end{cases}
\]

It will also be convenient to declare \( \varphi(0) \coloneqq 0 \). A sequence of radii will be called generic if the radii involved are dynamically independent. More precisely:

**Definition 3.1.** We say that a sequence \( \mathbf{r} = (R_j) \) in \( |\mathbb{C}_p^\times| \) is a generic sequence of radii if it is increasing, converges to \( \infty \), \( R_1 > p \), and for all integers \( j > i \geq 1 \), we have:

\[
\varphi^\circ n(R_i) \neq R_j,
\]

for all \( n \geq 1 \).

Generic sequences exist and are, in a certain sense, ‘generic’:

**Lemma 3.2.** Given an increasing sequence \( \mathbf{r} = (R_j) \) in \( |\mathbb{C}_p^\times| \) converging to \( \infty \) such that \( R_1 > p \), and a sequence \( (\delta_j) \) with \( \delta_j > 0 \) for all \( j \), there exists a generic sequence of radii \( \mathbf{r}' = (R'_j) \) such that \( |R_j - R'_j| < \delta_j \) for all \( j \).

**Proof.** The \( \varphi \)-orbit of \( R_i > \varphi = p^{-1/(p-1)} \) is strictly increasing for all \( i \geq 1 \). Moreover, \( \varphi \) restricted to \( [0, R_j] \) is independent of \( R_j \). Thus, given any \( j \geq 2 \), if necessary, we may find \( R'_j \in |\mathbb{C}_p^\times| \) arbitrarily close to \( R_j \) so that \( R'_j \) is not in the forward orbit of \( R_i \) for all \( i < j \). The lemma follows after recursively adjusting \( R_j \) for all \( j \).

\( \square \)

In the sequel, we let \( \mathbf{r} = (R_j) \) be a generic sequence of radii.
3.2  Symbolic dynamics

3.2.1  Dynamical partition

Given $c \in C(r)$, to study the dynamics of $f_c$, we partition $\mathbb{A}_1$ into the sets:

\[
B_0 := D(0, 1),
\]
\[
B_j := D(c_j, R_j), \text{ for } j \geq 1, \text{ and}
\]
\[
A := \mathbb{A}_1 \setminus \left( \bigcup_{j \geq 0} B_j \right).
\]

We omit the dependence of $B_j$ and $A$ on $c$. In fact, unless otherwise stated, given a parameter $c$, we will only consider parameters $c'$ such that $|c_j - c'_j| < R_j$. For all such $c'$, we have $D(c_j, R_j) = D(c'_j, R_j)$.

Note that $\varphi : ]0, \infty[ \to ]0, \infty[$ has as unique fixed point the exponential radius

\[
\varphi = p^{-1/(p-1)},
\]

which is repelling. It corresponds to the type II point $\xi_{0, \varphi} \in B_0$, fixed under the action of $f_c$, for all $c \in C(r)$. Moreover, as we show in the proposition below, Julia fixed points of $f_c$ are in natural correspondence with the disks $B_j$ for $j \geq 0$:

**Proposition 3.3.** For all $c \in C(r)$, the following statements hold.

1. Every fixed point of $f_c$ is contained in $B_j$, for some $j \geq 0$.
2. $B_0$ contains a unique repelling fixed point $w_0 = \xi_{0, \varphi}$. In appropriate coordinates, the tangent map $T_{w_0} f_c$ at $w_0$ is the Frobenius morphism.
3. $B_j$ contains a unique fixed point $w_j(c)$, which is a repelling fixed point of type I with multiplier $\lambda_j$ such that $|\lambda_j| = \frac{\varphi(R_j)}{R_j}$.

We will consistently use the notation of the above proposition. That is, $w_j(c)$ will denote the fixed point of $f_c$ in $B_j$ with multiplier $\lambda_j$, for all $j \geq 1$. We will ignore the dependence of $\lambda_j$ on $c$, since only its absolute value $|\lambda_j|$, which solely depends on $r$, will be relevant to us. The proposition will be a consequence of the following:

**Lemma 3.4.** For all $c \in C(r)$, if $z \in A \cup B_0$, then

\[
|f_c(z)| = \varphi(|z|).
\]

Moreover, if $j \geq 1$, then $f_c : B_j \to D(0, \varphi(R_j))$ is an analytic isomorphism and, for all type I points $z, w \in B_j$,\n
\[
|f_c(z) - f_c(w)| = \frac{\varphi(R_j)}{R_j} \cdot |z - w|.
\]
Proof. We set $R_0 := 0$ for convenience. Consider the auxiliary monomials $\mu_j$ defined by $\mu_1(z) := z^p/p$, and for $j \geq 2$, by

$$\mu_j(z) := (-1)^{j-1} (p \cdot c_1 \cdots c_{j-1})^{-1} z^{p+j-1}.$$ 

Let $z \in \mathbb{A}^1$. For all $j \geq 1$, if $R_{j-1} < |z| < R_j$, then $|\mu_j(z)| = \varphi(|z|)$ and

$$|f_c(z) - \mu_j(z)| < |f_c(z)|.$$ 

Therefore, $|f_c(z)| = \varphi(|z|)$ for all $z \in A \cup B_0$. Hence, the disk $B_j \cap \mathbb{A}^1 = \{ z : |z - c_j| < R_j \}$, is mapped onto $\{ z : |z| < \varphi(R_j) \}$ with degree 1, since $c_j$ is the unique (simple) zero of $f_c$ in $B_j$. □

Proof of Proposition 3.3. For all $z \in A$, observe that $\varphi(|z|) > |z|$ and apply Lemma 3.4 to conclude that $A$ is fixed point free. Thus (1) holds.

To prove (2), note that if $z \in B_0$ and $|z| > \varphi$, then $|f_c(z)| = \varphi(|z|) > |z|$. Moreover, if $|z| \leq \varphi$, then $|f_c(z)| \leq \varphi$. Hence, every point in $\overline{D}(0, \varphi)$ distinct from $\xi_{0,\varphi}$ lies in the Fatou set. A computation shows that, after moving $\xi_{0,\varphi}$ to the Gauss point via $z \mapsto p^{-1/(p-1)}z$, the map $f_c$ reduces to $\tilde{z} \mapsto \tilde{z}^p$. Therefore, $\xi_{0,\varphi}$ is a repelling fixed point having as tangent map the Frobenius morphism.

Assertion (3) follows at once from the Lemma since the inverse of $f_c : B_j \cap \mathbb{A}^1 \to D(0, \varphi(R_j))$ is a contraction. □

3.2.2 Symbolic dynamics

The dynamical partition of $\mathbb{A}^1_{an}$ furnishes a symbolic coding for the dynamics of $f_c$ via an itinerary function. Indeed, let

$$\Sigma := \{A, B_0, B_1, \ldots \}^{\mathbb{N}_0},$$

and

$$\text{itin}_c : \mathbb{A}^1_{an} \to \Sigma,$$

$$x \mapsto (\alpha_n),$$

if $f_c^{\alpha_n}(x) \in \alpha_n$.

We employ the usual multiplicative notation for concatenation of symbols. For example, the unique fixed point of $f_c$ in $B_j$, denoted $\omega_j(c)$, has itinerary $B_j^\infty$. Also, since $f_c(\xi_{0,r}) = \xi_{0,\varphi(r)}$, the itinerary of $\xi_{0,r}$ is $B_0^\infty$ if $r \leq \varphi$ and $B_0^m A^\infty$ if $r > \varphi$, for some $m \geq 0$.

Given a finite or infinite word $\alpha = \alpha_0 \alpha_1 \cdots$ in the alphabet $\{A, B_0, B_1, \ldots \}$, we denote by $|\alpha|$ its length (maybe $\infty$) under the agreement that $|\alpha_0 \alpha_1 \cdots \alpha_k| = k + 1$. The associated cylinder set of points having itinerary prescribed by $\alpha$ is

$$C_c(\alpha) := \{ x \in \mathbb{A}^1_{an} : f_c^{\alpha_n}(x) \in \alpha_n \text{ for all } 0 \leq n < |\alpha| + 1 \},$$

with the understanding that $\infty + 1 = \infty$. 

There are some cylinder sets which are empty. In fact, given \( j \geq 1 \), set

\[
n_j := \min\{n \geq 1 : f_c^{-(n+1)}(B_j) \cap B_0 \neq \emptyset\}.
\]

That is, \( n_j + 1 \) is the minimal number of iterations required for a point in \( B_0 \) to reach \( B_j \). If the itinerary of a point has a sub-word of the form \( B_0 A^n B_j \) for some \( n \geq 1 \), then \( n \geq n_j \). Since \( n_j \) is also the minimal \( n \) such that \( \varphi^{-(n+1)}(R_j) < 1 \), it is not difficult to see that \( n_j \to \infty \) as \( j \to \infty \). Recall that we require \( R_1 > p \). This requirement is convenient since it forces \( n_1 \geq 1 \), and therefore \( n_j \geq 1 \) for all \( j \geq 1 \).

### 3.2.3 Cylinder sets

We adapt the remarkable strategy introduced by Benedetto to produce (dynamically bounded) wandering domains in a family of polynomial maps. We will obtain an oscillating wandering domain consisting of points with itinerary prescribed by two sequences \( m = (m_j)_{j \geq 1} \) and \( \ell = (\ell_j)_{j \geq 2} \) of positive integers. The **itinerary associated to \( m \) and \( \ell \)** is

\[
\alpha = \alpha(m, \ell) := B_{m_1}^0 A_{n_1} B_{\ell_2}^1 A_{n_2} B_{m_3}^2 A_{n_3} B_{\ell_4}^3 A_{n_4} B_{m_5}^4 A_{n_5} \cdots \in \Sigma.
\]

A point with itinerary \( \alpha \) ‘oscillates’. Indeed, the orbit starts at \( B_0 \) and then visits \( B_1 \) to come back to \( B_0 \) and then visit \( B_2 \), and so on.

**Proposition 3.5.** Consider the itinerary \( \alpha \) associated to sequences \( m = (m_j)_{j \geq 1} \) and \( \ell = (\ell_j)_{j \geq 2} \) of positive integers. Let \( c \in \mathcal{C}(r) \). Then \( C_c(\alpha) \) is a non-empty closed set. Moreover, given a connected component \( X \) of \( C_c(\alpha) \) one of the following statements hold.

- \( X = \{y\} \) for some type I point \( y \) in \( J(f_c) \).
- \( X \) is a closed disk whose boundary point \( y \) is a type II or III point in \( J(f_c) \). Every connected component of the interior of \( X \) is an oscillating wandering domain.
- \( X = \{y\} \) for some type IV point \( y \) in \( J(f_c) \).

**Proof.** For all \( j \geq 1 \), consider the word \( \eta_j = B_{m_j}^0 A_{n_j} B_{\ell_j+1}^j \). Given \( k \geq 1 \), let \( X_k = C_c(\eta_1 \cdots \eta_k B_0) \) and observe that \( \cap X_k = C_c(\alpha) \).

We claim that \( \cap X_k = \cap \overline{X_k} \). Indeed, if \( x \in \partial X_k \) for some \( k \), then there exists \( n \geq 0 \) such that \( f_c^{on}(x) \) is the boundary point of \( B_i \) for some \( i \leq k \). Thus, \( f_c^{on+m}(x) \in A \) for all \( m \geq 1 \). Therefore, \( x \notin C_c(\alpha) = \cap X_k \) and the claim follows. In particular, \( C_c(\alpha) \) is a closed set.

To show that \( \overline{X_k} \) is non-empty, we proceed recursively. Assume that \( Y = C_c(\eta_{j+1} \cdots \eta_k B_0) \) is not the empty set. Now we pull-back \( Y \) according to \( \eta_j \) in order to prove that the cylinder \( C_c(\eta_j \eta_{j+1} \cdots \eta_k B_0) \) is also non-empty. More precisely, denote by \( h_j \) the inverse of \( f_c : B_j \to D(0, \varphi(R_j)) \). Let \( Z = h_j^{\sigma \ell+1}(Y) \). Consider \( r_j = \varphi^{-(m_j+n_j)}(R_j) \). Since \( n_j \) is the minimal \( n \) such that \( \varphi^{-(n+1)}(R_j) < 1 \), it follows that \( f_c^{on}(\xi_0 r_j) \) is \( B_0 \) for all \( 0 \leq m < m_j \). For all \( n < m_j + n_j \), we have that \( \varphi^{on}(r_j) \) is not an element of the generic sequence \( \sigma = (R_j) \). Thus from Lemma 3.4, we have that \( f_c \) maps \( \{z : |z| = \varphi^{on}(r_j)\} \) onto \( \{z : |z| = \varphi^{on+1}(r_j)\} \) and, all the elements of \( \{z : |z| = r_j\} \) have itinerary \( B_0^{m_j} A^{n_j} \). Moreover, at least one direction \( D \) at \( \xi_0 r_j \), contained in \( \{z : |z| = r_j\} \), maps onto...
B_j under f_c^{o m_{j}+n_{j}} : D \to B_j is onto. Then the pre-image Y' \subset D of Z under this map is such that \( \emptyset \neq Y' \subset C_{c}(\eta_{j} B_{0}) \) and \( f^{o [\eta_{j}]}(Y') = Y \). Hence, Y' is contained in \( C_{c}(\eta_{j} ... \eta_{k} B_{0}) \).

Thus, \( \overline{X}_{k} \neq \emptyset \), for all \( k \geq 1 \).

Let \( X \) be a connected component of \( C_{c}(\alpha) \). The itineraries of points in \( ]0, \infty[ \) are either \( B_{\infty}^{0} \) or \( B_{m}^{0} A_{\infty} \) for some \( m \geq 0 \). In particular, \( ]0, \infty[ \) is disjoint from \( C_{c}(\alpha) \). Thus, given \( x \in X \), if \( y = f_{c}^{o n}(x) \) lies in \( A \) or in some \( B_{j} \), it follows that the associated disk \( \overline{D}_{y} \) is contained in \( A \) or in \( B_{j} \), respectively. Therefore, if \( x \in X \), then \( \overline{D}_{x} \subset X \) and, after choosing a reference point \( x_{0} \in X \), we have that \( X \cap [x_{0}, \infty[ = [x_{0}, y] \) for some \( y \in X \), by compactness. Since \( X \) is connected and \( \overline{D}_{y} \) is a connected component of \( \mathbb{A}_{1} an \setminus ]y, \infty[ \), it follows that \( X = \overline{D}_{y} \).

To finish, we show that the singleton \( \partial X \) lies in the Julia set. Suppose otherwise. Let \( U \) be the Fatou component containing \( \partial X \). Note that \( 0 \not\in f^{o n}(U) \), since the itinerary of points in the Fatou component of \( z = 0 \) is \( B_{\infty}^{0} \). In particular, if \( U \) is a disk, then \( f_{c}^{o n}(U) \cap [0, \infty[ = \emptyset \); therefore \( U \subset C_{q}(\alpha) \). If \( U \) is not a disk, then \( U \) is an escaping Fatou component, according to [7, Theorem 1.2]. Both alternatives lead to a contradiction so \( \partial X \subset J(f_{c}) \). □

### 3.3 Outline of the proof

The strategy to prove Theorem A is to make appropriate choices of \( m \) and \( l \) so that, after perturbation of a given parameter \( c \), a connected component of the cylinder set with itinerary \( \alpha \) is neither a type I nor a type IV singleton. In this section, we introduce the definitions and notations required to state three lemmas and explain how to deduce from them our main result. The rest of the paper is devoted to prove these lemmas.

Consider two sequences \( m = (m_{j})_{j \geq 1} \) and \( l = (l_{j})_{j \geq 2} \) of positive integers, and let \( \alpha \) be the associated itinerary. Our first lemma will show that, under certain conditions, once the cylinder set \( C_{c}(\alpha) \) contains a type I point, it automatically contains a disk and therefore an oscillating wandering domain. To explain this phenomenon, suppose that \( c \) is a parameter such that \( C_{c}(\alpha) \) contains a type I point \( x \). One should think of \( m_{j} \) as the duration of the \( j \)th excursion to the zone under the influence of the ramified type II fixed point \( w_{0} \). The fixed point \( w_{0} \) is closely related to the wild critical point \( z = 0 \). We think of these parts of the orbit as the wild excursions. During the wild excursions the dynamics in \( \mathbb{A}_{1} \) is contracting. The numbers \( l_{j} \) should be thought as the duration of the excursion near the \( (j - 1) \)-th repelling fixed point. During these excursions the dynamics in \( \mathbb{A}_{1} \) is expanding. Thus, these are the expanding excursions. The numbers \( n_{j} \) are transition times between the wild and expanding excursions. During the \( j \)th wild excursion, we will show that the derivative along with the orbit contracts by a factor of \( p^{-m_{j}} \) (modulo constants). Along with the expanding excursion that follows, around the fixed point \( w_{j}(c) \), the derivative expands by a factor of \( |\lambda_{j}|^{l_{j}+1} \) (modulo constants). If each wild excursion is long enough compared to the expanding one that follows, then the corresponding cylinder cannot be a type I singleton.

Indeed, in Section 4, we prove the following:

**Lemma 3.6** (Uniform disk). Consider positive integer sequences \( l = (l_{j}) \), \( m = (m_{j}) \) with associated itinerary \( \alpha \) such that, for all \( k \geq 1 \),

\[
\varphi^{-1}R_{k} p^{-m_{k}}|\lambda_{k}|^{l_{k}+1} < 1. \tag{3}
\]
If there exists a type I point $x$ and a parameter $c \in \mathcal{C}(r)$ such that

$$x \in C_c(\alpha),$$

then

$$D(x, \varphi^2) \subset C_c(\alpha).$$

In view of Proposition 3.5, given $m$ and $\varphi$ such that (3) holds for all $k$, to obtain a wandering domain, the idea is to perturb an initial parameter $c$ to a nearby parameter $c'$ such that $C_{c'}(\alpha)$ contains a type I point.

### 3.3.1 Notation

Before we continue with the outline, for future reference, let us summarize and introduce some of the notation that we will freely employ throughout this work.

Recall that $r = (R_j)_{j \geq 1}$ is a generic sequence of radii as in Definition 3.1. Associated to such a sequence, we have the parameter space

$$\mathcal{C}(r) = \{c = (c_j)_{j \in \mathbb{N}} : c_j \in \mathbb{C}_p \text{ and } |c_j| = R_j\}.$$ 

Given $c \in \mathcal{C}(r)$, the corresponding entire map $f_c$ is given by (1). The function

$$\varphi : ]0, \infty[ \rightarrow ]0, \infty[,$$

which is defined by

$$\varphi(r) := \sup\{|f_c(z)| : |z| \leq r\},$$

is independent of $c \in \mathcal{C}(r)$. An explicit formula for $\varphi$ can be found in Section 3.1. It will be convenient to set, for all $j \geq 1$,

$$S_j := \varphi^{-1}(R_j).$$

For any $z \in \mathbb{A}^1$ and $s > 0$, the sup-norm on a disk $D(z, s) \subset \mathbb{A}^1$ regarded as an element of $\mathbb{A}^1_{an}$ is denoted by $\xi_{z,s}$. So $f_c(\xi_{0,r}) = \xi_{0,\varphi(r)}$. The exponential radius is

$$\varphi = p^{-1/(p-1)}.$$

According to Proposition 3.3, the unique Julia fixed point of $f_c$ in $B_0$ is $w_0 = \xi_{0,\varphi}$, and, for $j \geq 1$, the unique fixed point in $B_j$ is $w_j(c)$. In order to lighten notation, the multiplier of $w_j(c)$ is simply denoted by $\lambda_j$, although it depends on $c$. However, we are mainly concerned with

$$|\lambda_j| = \frac{\varphi(R_j)}{R_j}$$

which is independent of $c \in \mathcal{C}(r)$. 
Consider sequences \( m = (m_j)_{j \geq 1} \), \( \ell = (\ell_j)_{j \geq 2} \) of positive integers with associated itinerary \( \alpha \) as defined in Section 3.2.3. Given \( k \geq 1 \), the \( k \)th truncation of \( \alpha \) is the word:

\[
\alpha^{(k)} = \alpha^{(k)}(m, \ell) := B_0^{m_1} A^{n_1} B_1^{\ell_2} B_0^{m_2} A^{n_2} B_2^{\ell_3} ... B_k^{\ell_k} B_0^{m_k} A^{n_k} B_k,
\]

where \( n_j \) is the minimal number of iterations required for a point in \( B_0 \) to reach \( B_j \) as defined by (2).

Let \( L_1 = 0, M_1 = m_1 \), and for all \( k \geq 1 \), we set:

\[
N_k := |\alpha^{(k)}| - 1, \\
L_{k+1} := N_k + \ell_{k+1}, \\
M_{k+1} := L_{k+1} + m_{k+1}.
\]

Thus, \( N_{k+1} = M_{k+1} + n_{k+1} \). Given \( j \geq k \), note that the \( L_k \)-th iterate of an element in \( C_\ell(\alpha^{(j)}) \) is the first one of the \( k \)-th wild excursion. The \( M_k \)-th iterate is the beginning of the transition to the \( k \)-th expanding excursion that starts in the \( N_k \)-th iterate.

Given a parameter \( c \in C(r) \) and a sequence \( \varepsilon = (\varepsilon_j)_{j \geq 1} \) of positive real numbers, we consider perturbations of \( c \) in two types of sets. That is, for \( k \geq 2 \), let

\[
U_k(c, \varepsilon_k) := \{ c' = (c'_j) : |c'_j - c_j| < \varepsilon_k, c'_j = c_j \text{ if } j \neq k \}, \\
\Delta_k(c, \varepsilon) := \{ c' = (c'_j) : c_j = c'_j \text{ if } j < k \text{ and } |c_j - c'_j| < \varepsilon_j \text{ if } j \geq k \}.
\]

In the sequel, \( m = (m_j)_{j \geq 1} \), \( \ell = (\ell_j)_{j \geq 2} \) will be sequences of positive integers and \( \varepsilon = (\varepsilon_j)_{j \geq 1} \) will be a sequence of positive real numbers such that \( \varepsilon_j < R_j \) for all \( j \). When clear from context, \( \alpha \) will denote the itinerary associated to the sequences \( m \) and \( \ell \) with truncations \( \alpha^{(k)} \). Also, \( L_k, M_k \), and \( N_k \) will denote the corresponding numbers introduced above.

When the parameter \( c \in C(r) \) is clear from context and \( y \in \mathbb{A}_1^{1n} \), we freely employ \( y_n \) to denote \( f^{\circ n}_c(y) \).

### Perturbation lemmas

Given an initial parameter \( c \), the idea is to start with a type I point \( x \) such that

\[
\text{itin}_c(x) = B_0^{m_1} A^{n_1} B_1^{\infty}.
\]

That is, \( x \) eventually maps onto the fixed point \( w_1(c) \), under iteration of \( f_c \). It is easy to show that such a point \( x \) always exists. To prove our main result, we will successively perturb \( c = c^{(1)} \) to obtain a sequence of parameters \( (c^{(k)}) \) where \( c^{(k+1)} \) is obtained as a perturbation of \( c^{(k)} \) for all \( k \geq 1 \). These parameters will be such that the exact same point \( x \) eventually maps, under \( f_{c^{(k)}} \), onto the fixed point \( w_k(c^{(k)}) \) in \( B_k \) according to the itinerary

\[
\text{itin}_{c^{(k)}}(x) = \alpha^{(k)} B_k^{\infty}.
\]
In Section 5, we give an explicit upper bound on the size of the perturbations to guarantee that the cylinder sets $C_c(\alpha^{(k)})$ remain stable. This will provide us a well-defined range of perturbations around $c^{(k)}$ to adjust the parameter without changing the initial segment of the itinerary of $x$. Recall that, for all $j \geq 1$, $S_j = \varphi^{-1}(R_j)$.

**Lemma 3.7** (Cylinder stability). Consider $k_0 \geq 1$. Let $\varepsilon, \ell, m$ be such that the following hold:

\[
\varepsilon_k < \varepsilon^k R_k \quad \text{and},
\]

\[
\varepsilon_k < \varepsilon^{k-j} \cdot \frac{R_k^2}{R_{j-1}S_{j-1}} |\lambda_{j-1}|^{-\ell_j} \quad \text{for all } 2 \leq j < k,
\]

for all $k > k_0$ and,

\[
\varepsilon^{-2} R_{j-1} S_{j-1} |\lambda_{j-1}|^{\ell_j} p^{-m_j} < 1 \quad \text{for all } 2 \leq j \leq k_0.
\]

If $c \in \mathcal{C}(r)$, then for all $c' \in \Delta_{k_0+1}(c, \varepsilon)$,

\[
C_{c'}(\alpha^{(k_0)}) = C_c(\alpha^{(k_0)}).
\]

Given $k \geq 2$, it will be convenient to simply say that (5) holds for $k$ if (5) holds after replacing $k_0$ by $k$.

The crucial observation here is that the upper bound in (4) on $\varepsilon_k$ does not depend on the lengths $m_j$ of the wild excursions.

For $\ell_{k+1}$ sufficiently large and any parameter $c \in \mathcal{C}(r)$, there exist points $y(c) \in B_k$ near $w_k(c)$ with itinerary $B_k^{\ell_{k+1}} B_0^{m_{k+1}} A_{n_{k+1}} B_{k+1}^\infty$. To obtain $c^{(k+1)}$ from $c^{(k)}$, the idea is to ‘connect’ the $N_k$-th iterate of $x$ with $y(c)$ for some $c$ close to $c^{(k)}$. More precisely, in Section 6, we prove the following:

**Lemma 3.8** (Connecting). Consider sequences $\varepsilon, \ell, m$ and let $k \geq 1$. Assume that (4) holds for $k+1$, (5) holds for $k$ and

\[
p \cdot \varepsilon^{-2} R_{k+1}^2 S_k |\lambda_{k+1}|^{-\ell_{k+1}} < \varepsilon_{k+1}.
\]

If $c = (c_j) \in \mathcal{C}(r)$ is a parameter and $x$ is a type I point such that

\[
\text{itit}_c(x) = \alpha^{(k)} B_k^\infty,
\]

then there exists $c' = (c'_j) \in U_{k+1}(c, \varepsilon_{k+1})$ such that

\[
\text{itit}_{c'}(x) = \alpha^{(k+1)} B_{k+1}^\infty.
\]

As noted above, the upper bound on the size of the perturbation required in (4) is independent of $m$. This grants the possibility of constructing sequences that satisfy the hypothesis of Lemmas 3.6–3.8:
Lemma 3.9. Let \((\xi_j)\) be a sequence of positive real numbers. Then there exist \(\varepsilon = (\varepsilon_j)\) such that \(0 < \varepsilon_j < \xi_j\) for all \(j\) and, \(m, \ell\) such that, for all \(k \geq 1\), (3), (4), (5), and (6) hold.

Proof. We start by recursively choosing \(\varepsilon_k\) and \(\ell_k\) so that (4) and (6) hold. That is, let \(\varepsilon_i < \min\{\xi_i, \xi' R_i\}\) for \(i = 1, 2\). Choose \(\ell_2\) sufficiently large such that (6) holds for \(k = 1\). Now assume that, for some \(k \geq 2\), we have already defined \(\varepsilon_j\) and \(\ell_j\), for all \(j \leq k\). Pick \(0 < \varepsilon_{k+1} < \xi_{k+1}\) so that (4) holds for \(k + 1\). Now choose \(\ell_{k+1}\) large enough so that (6) holds.

Once we have chosen \(\varepsilon = (\varepsilon_j)\) and \(\ell = (\ell_j)\), we finish by selecting, for all \(j \geq 1\), integers \(m_j\) sufficiently large so that both (3) and (5) hold for all \(k\).

Now we deduce from Lemmas 3.6–3.8 a quantified version of our main result:

Theorem 3.10. Consider a generic sequence of radii \(r\) and a parameter \(c \in C(r)\). Assume that \(\varepsilon, m, \ell\) are sequences such that (3)–(6) hold for all \(k \geq 1\). Denote by \(\alpha\) the itinerary associated to \(m\) and \(\ell\).

If \(x\) is a type I point such that \(\text{itin}_c(x) = B_{m_1}^{B_{n_1}} A_{1}^{B_{\infty}}\), then there exists \(c' \in \Delta_2(c, \varepsilon)\) such that

\[
\text{itin}_{c'}(x) = \alpha.
\]

Moreover, \(x\) lies in an oscillating wandering Fatou component of \(f_{c'}\).

The proof relies on the recursive construction of a sequence of parameters \((c^{(k)})_{k \geq 1}\) in \(C(r)\). That is, each \(c^{(k)}\) is itself an infinite sequence:

\[
c^{(k)} = \left(c_1^{(k)}, c_2^{(k)}, \ldots\right) \in C(r).
\]

Then, we will consider the 'diagonal' parameter

\[
c' = \left(c_1^{(1)}, c_2^{(2)}, \ldots\right) \in C(r).
\]

Proof (Assuming Lemmas 3.6, 3.7, and 3.8). Let \(c^{(1)} = c\). For all \(k \geq 1\), given a parameter \(c^{(k)} \in \Delta_2(c, \varepsilon)\) such that \(\text{itin}_{c^{(k)}}(x) = \alpha^{(k)}B_{\infty}^{k}\), apply Lemma 3.8 to obtain a parameter \(c^{(k+1)} \in U_{k+1}(c^{(k)}, \varepsilon_{k+1})\) such that \(\text{itin}_{c^{(k+1)}}(x) = \alpha^{(k+1)}B_{\infty}^{k+1}\). Let \(c' = (c_j^{(j)})\) and observe that

\[
c' \in \Delta_{k+1}(c^{(k)}, \varepsilon)
\]

for all \(k \geq 1\). Thus, applying Lemma 3.7:

\[
x \in C_{c'}(\alpha^{(k)}),
\]

for all \(k\). Therefore

\[
\text{itin}_{c'}(x) = \alpha.
\]
Hence, by Lemma 3.6,

\[ D(x, \varphi^2) \subset C_c(\alpha). \]

Finally, in view of Proposition 3.5, the map \( f_{c'} \) has an oscillating wandering domain that contains \( x \).

## 4 | UNIFORM DISK

The purpose of this section is to prove Lemma 3.6. The proof employs the following estimates on the derivative \( f'_c \):

**Lemma 4.1.** Let \( z \in \mathbb{A} \). For all \( c \in C(\mathbb{R}) \),

\[
|f'_c(z)| = \begin{cases} 
|f_c(z)|/p|z| & \text{if } z \in B_0, \\
|\lambda_j| & \text{if } z \in B_j \text{ for some } j \geq 2,
\end{cases}
\]

and if \( z \in A \), then

\[
|f'_c(z)| \leq \frac{|f_c(z)|}{|z|}.
\]

**Proof.** By Lemma 3.4, we only need to consider \( z \in A \cup B_0 \). For such \( z \), the lemma is a consequence of the ultrametric inequality applied to the following formula:

\[
f'_c(z) = f_c(z) \left[ \sum_{j \geq 1} \left( \frac{1}{z-c_j} \right) - \frac{p}{z} \right].
\]

**Lemma 4.2.** Let \( c \in C(\mathbb{R}) \) and \( m, k \geq 1 \). If \( z \in C_c(B_0^m A^n B_k) \) is a type I point, then

\[
D(z, \varphi) \subset C_c(B_0^m A^n B_k),
\]

and

\[
|(f_{c^m+n_k}^c)'(z)| < \varphi^{-1}R_k p^{-m}.
\]

**Proof.** Recall that open disks map onto open disks under \( f_c \). Moreover, given \( z \in \mathbb{A} \), if \( z \in A \cup B_0 \), then \( f_c(D(z, |z|)) = D(f_c(z), |f_c(z)|) \).

Let \( z \in C_c(B_0^m A^n B_k) \) be a type I point. Then, \( f_{c^m}^n \) maps \( D(z, |z|) \), for all \( n \leq m + n_k \), onto the open disk \( D(f_{c^m}^n(z), |f_{c^m}^n(z)|) \). It follows that \( D(z, \varphi) \subset C_c(B_0^m A^n B_k) \). Recall that \( z_n = f_{c^m}^n(z) \). Now, we apply Lemma 4.1 to obtain

\[
|(f_{c^m}^n)'(z)| = \prod_{j=0}^{m-1} |f'_c(z_j)| = \prod_{j=0}^{m-1} \frac{|z_{j+1}|}{p|z_j|} = \frac{|z_m|}{p^n|z|} < \frac{|z_m|}{p^n \varphi}.
\]
since \( z_j \in B_0 \), for \( 0 \leq j < m \), and \(|z| > \varphi\). Taking into account that \( f_c^{\text{on}}(z_m) \in A \) for \( 0 \leq n \leq n_k - 1 \), Lemma 4.1 yields

\[
|(f_c^{\text{on}_k})'(z_m)| \leq \frac{|z_m + n_k|}{|z_m|} = \frac{R_k}{|z_m|}.
\]

The inequality claimed in the statement of the lemma now follows from the chain rule. \( \square \)

**Proof of Lemma 3.6 (Uniform disks).** Assume that \( x \) is a type I point in \( C_c(\alpha) \). It is sufficient to prove, for all \( k \geq 1 \), the following inclusions:

\[
D(x, \varphi^2) \subset C_c(\alpha^{(k)}),
\]

\[
f_c^{\ell k}(D(x, \varphi^2)) \subset D(x_{\ell k}, \varphi^2).
\]

Indeed, if the first inclusion above holds for all \( k \geq 1 \), then the lemma follows.

We proceed by induction. The inclusions are easily verified for \( k = 1 \) with the aid of Lemma 4.2 and recalling that \( L_1 = 0 \). Suppose that the inclusions hold for \( k \). Note that \( D(x, \varphi^2) \subset C_c(\alpha^{(k+1)}) \) if and only if

\[
f_c^{\text{on}_k}(D(x, \varphi^2)) \subset C_c(B_{\ell k+1}^0 A_{n_k+1} B_{\ell k+1}).
\]

Since \( x \in C_c(\alpha) \), we have that \( x_{\ell k} \in C_c(B_0^m A_n B_k) \). By Lemma 4.2, \( D(x_{\ell k}, \varphi) \) is also contained in this cylinder set and, for all \( z \in D(x_{\ell k}, \varphi) \), we have

\[
|(f_c^{\text{on}_k})'(z)| < \varphi^{-1} R_k p^{-m_k} < |\lambda_k|^{-\ell_{k+1}},
\]

where the last inequality is granted by (3). From Theorem 2.2,

\[
f_c^{\text{on}_k}(D(x_{\ell k}, \varphi^2)) \subset D(x_{\ell k}, |\lambda_k|^{-\ell_{k+1}} \varphi^2).
\]

By Lemma 3.4, it follows that

\[
f_c^{\text{on}_k_{\ell k+1}}(D(x, \varphi^2)) \subset f_c^{\text{on}_k}(D(x_{\ell k}, |\lambda_k|^{-\ell_{k+1}} \varphi^2)) = D(x_{\ell k+1}, |\lambda_k|^{-\ell_{k+1}} \varphi^2),
\]

for all \( 0 \leq \ell \leq \ell_{k+1} \). Hence, \( f_c^{\text{on}_k}(D(x, \varphi^2)) \subset C_c(B_{\ell k+1}^0) \) and

\[
f_c^{\ell k+1}(D(x, \varphi^2)) \subset D(x_{\ell k+1}, \varphi^2) \subset C_c(B_{\ell k+1}^0 A_{n_k} B_{k+1}),
\]

so \( D(x, \varphi^2) \subset C_c(\alpha^{(k+1)}) \), and the assertion holds for \( k + 1 \). \( \square \)

5 | CYLINDER SET STABILITY

In this section, we prove Lemma 3.7. The proof relies on first establishing stability results for certain cylinder sets as well as studying the dependence on parameters of points in an appropriate
orbit. This study involves estimating the partial derivatives of \( c \mapsto f_c^{on}(z) \). The estimates will be useful when combined with the non-Archimedean Mean Value Theorem (see Theorem 2.2).

We start by showing that tangent maps, acting on a portion of the ‘axis’ \([0, \infty[\), remain constant under perturbations. The stability of cylinders of the form \( C_c(B_0^m A^n B_k) \) will then follow.

**Lemma 5.1.** Let \( k \geq 1 \). Consider two parameters \( c, c' \in \mathcal{C}(r) \) such that \( |c_j - c'_j| < R_j \), for all \( 1 \leq j < k \). Then,

\[
T_y f_c = T_y f_{c'}
\]

for all \( y = \xi_{0,r} \in ]0, R_k[ \).

**Proof.** For all \( j \geq 1 \), let

\[
\eta_j(z) = \frac{1 - z/c'_j}{1 - z/c_j} - 1.
\]

Consider \( z \in \mathbb{A}^1 \) such that \( |z| < R_k \) and \( z \in A \cup B_0 \). A calculation shows that if \( |z| \geq R_j \), then

\[
|n_j(z)| = |1/c_j - 1/c'_j| \cdot R_j < 1,
\]

since \( |c'_j - c_j| < R_j \). Moreover, if \( |z| < R_j \), then

\[
|n_j(z)| = |1/c_j - 1/c'_j| \cdot |z| < 1.
\]

Therefore,

\[
\left| 1 - \frac{f_{c'}(z)}{f_c(z)} \right| = \left| 1 - \prod_{j \geq 1} (1 + \eta_j(z)) \right| < 1.
\]

Thus, given \( 0 < r < R_k \), we have \(|f_{c'}(z) - f_c(z)| < \varphi(r)\) for all \( z \) in the set \((A \cup B_0) \cap \overline{D}(0, r)\) which omits at most two directions at \( y = \xi_{0,r} \). It follows that \( T_y f_c = T_y f_{c'} \).

**Corollary 5.2.** Consider \( k \geq 1 \). If \( c, c' \in \mathcal{C}(r) \) are such that \( |c_j - c'_j| < R_j \), for all \( 1 \leq j < k \), then

\[
C_c(B_0^m A^n B_k) = C_c'(B_0^m A^n B_k),
\]

for all \( m \geq 1 \).

**Proof.** Let \( r = \varphi^{-(m+n_k)}(R_k) \). Note that \( z \in C_c(B_0^m A^n B_k) \) if and only if \( z \) lies in a direction at \( \xi_{0,r} \) which maps under \( f_c^{om+n_k} \) onto \( B_k \). By Lemma 5.1, these directions are independent of \( c' \), under our assumptions on \( c' \).

Recall from Section 3.3.1 that \( U(c, r) \) is the subset of \( \mathcal{C}(r) \) formed by parameters \( c' = (c'_j) \) so that \( c'_i = c_i \) for all \( i \neq j \) and \( |c_j - c'_j| < r \). Also \( |\lambda_k| = \varphi(R_k)/R_k \) where \( \lambda_k \) is the multiplier of the fixed point in \( B_k \).
The next result deals with the dependence of \( f_c(z) \) on \( c_j \):

**Lemma 5.3.** Suppose that \( c' \in U_j(c, R_j) \) for some \( j > 1 \). Then, for all \( z \in \mathbb{A}^1 \) such that \( |z| < R_j \),

\[
|f_{c'}(z) - f_c(z)| = \frac{|z| \cdot |f_c(z)|}{R_j^2} |c_j - c'|.
\]

Moreover, if \( k < j \) and \( |z| \leq R_k \), then

\[
|f_{c'}(z) - f_c(z)| \leq |\lambda_k| \cdot \frac{R_k^2}{R_j^2} |c_j - c'|.
\]

**Proof.** If \( |z| < R_j \), then

\[
|f_{c'}(z) - f_c(z)| = \left| \frac{z^p}{p} \prod_{i \neq j} \left| 1 - \frac{z}{c_i} \right| \right| \cdot |z| \cdot \left| \frac{z}{c'_j} - \frac{z}{c_j} \right|
\]

\[
= \left| f_c(z) \right| \cdot \left| \frac{z}{c'_j} - \frac{z}{c_j} \right|
\]

\[
= \frac{|z| \cdot |f_c(z)|}{R_j^2} |c_j - c'|,
\]

where the second line follows from the first since \( 1 = |1 - (z/c_j)| \) when \( |z| < R_j \). If moreover \( k < j \) and \( |z| \leq R_k \), then \( |f_c(z)| \leq \varphi(R_k) = |\lambda_k| R_k \) and the lemma follows. \( \square \)

In the sequel, we denote the partial derivative with respect to \( c_j \) by \( \partial_j \). In particular, \( \partial_j f_c^o(z) \) denotes the partial derivative of \( c \mapsto f_c^o(z) \), for certain \( z \in \mathbb{A}^1 \) and \( n \geq 1 \), with respect to \( c_j \) for some \( j \).

Recall that we aim to show that under sufficiently small perturbations, the cylinder \( C_c(\alpha) \) is constant. Via partial derivatives, the following three results will allow us to control the dependence on parameters of certain orbit elements.

**Proposition 5.4.** For all \( z \in \mathbb{A}^1 \) such that \( |z| < R_j \),

\[
|\partial_j f_c(z)| = \frac{|z| \cdot |f_c(z)|}{R_j^2}.
\]

**Proof.** By formula (1) of \( f_c \), we have

\[
\partial_j f_c(z) = \frac{zf_c(z)}{c_j^2(1-c_j^{-1}z)} = \frac{zf_c(z)}{c_j(c_j - z)}.
\]

Since \( |z| < R_j \) implies that \( |c_j - z| = |c_j| = R_j \), the proposition follows. \( \square \)
Corollary 5.5. Let \( j \geq 1 \) and \( z \in \mathbb{A}^1 \) be such that \( |z| > \varphi \). Assume that \( f_c^{o_n}(z) \in D(0, R_j) \setminus (B_1 \cup \cdots \cup B_{j-1}) \) for all \( 0 \leq n < N \). Then,

\[
|\partial_j f_c^{o_N}(z)| = \frac{\varphi^{o_N}(|z|) \cdot \varphi^{o_N-1}(|z|)}{R_j^2}.
\]

**Proof.** We proceed by induction. Note that Proposition 5.4 establishes the case \( N = 1 \). For the inductive step, first apply the chain rule:

\[
\partial_j f_c^{o_{1+N}}(z) = f'_c(f_c^{o_N}(z)) \cdot \partial_j f_c^{o_N}(z) + \partial_j f_c(f_c^{o_N}(z)).
\]

Then, observe that

\[
|f'_c(f_c^{o_N}(z))| \cdot |\partial_j f_c^{o_N}(z)| \leq \frac{\varphi^{o_{N+1}}(|z|) \cdot \varphi^{o_N}(|z|)}{\varphi^{o_N}(|z|)} \cdot \frac{\varphi^{o_N-1}(|z|)}{R_j^2} < \frac{\varphi^{o_N+1}(|z|) \cdot \varphi^{o_N}(|z|)}{R_j^2} = |\partial_j f_c(f_c^{o_N}(z))| = |\partial_j f_c^{o_{N+1}}(z)|,
\]

where the last line follows from the ultrametric triangle inequality.\( \square \)

For all \( k \geq 1 \), recall that \( S_k = \varphi^{-1}(R_k) > 1 \).

Lemma 5.6. Let \( j > k \geq 1 \). Assume that \( z \) is a type I point such that

\[
z \in C_z(\alpha^{(k)} B_k^{\ell'}),
\]

for some \( \ell' \geq 0 \). If (5) holds for \( k \), then

\[
|\partial_j f_c^{o_{N_k+\ell'}}(z)| = \frac{R_k S_k}{R_j^2} |\lambda_k|^{\ell'}
\]

(7)

for all \( 0 \leq \ell' \leq \ell \). If moreover \( |f_c^{o_{N_k+\ell'+1}}(z)| \leq R_k \), then (7) also holds for \( \ell'' = \ell' + 1 \).

Before proving the lemma, let us establish a basic inequality. From the formula of \( \varphi \) given in Section 3.1, it is not difficult to conclude that \( \varphi(r) \geq pr^b \) for all \( r > 0 \). In particular, \( \varphi(r) > r^2 \) and since \( S_k > 1 \), we have

\[
|\lambda_k| = \frac{\varphi(R_k)}{R_k} > \frac{R_k}{S_k} > 1.
\]

(8)
Proof. Consider a type I point \( z \in C_c(\alpha^k B^\ell_k) \). Recall that, to simplify notation, we write \( z_n = f_c^n(z) \). It follows that \( z_{N_k}, \ldots, z_{N_k+\ell'} \in B_k \). So automatically \(|z_{N_k+\ell'}| = R_k\) for all \( 0 \leq \ell' \leq \ell \). Thus, in order to prove the lemma it suffices to establish the following:

Claim. Given \( j > k \geq 1 \) and \( \ell' \geq 0 \), assume that \( z \in C_c(\alpha^k B^\ell_k) \) is such that \(|f_c\circ N_k+\ell'(z)| \leq R_k\), for all \( 0 \leq \ell' \leq \ell + 1 \). If (5) holds for \( k \), then (7) holds for all \( 0 \leq \ell' \leq \ell + 1 \).

Let us first suppose that, under the hypothesis of the claim, (7) holds for \( \ell' = 0 \) and prove that it holds for \( 0 \leq \ell' < \ell + 1 \). We proceed by induction on \( \ell' \). So suppose (7) holds for some \( 0 \leq \ell' < \ell + 1 \). Then,

\[
|\partial_j f_c^{N_k+\ell'+1}(z)| = |f_c'\circ N_k+\ell' \partial_j f_c^{N_k+\ell'}(z) + \partial_j f_c(z_{N_k+\ell'})|.
\]

Observe that

\[
|f_c'\circ N_k+\ell' \partial_j f_c^{N_k+\ell'}(z)| = |\lambda_k| \cdot |\lambda_k|^{\ell'} \frac{R_k S_k}{R_j^2} \geq |\lambda_k| \cdot \frac{R_k^2}{R_j^2} \geq |\partial_j f_c(z_{N_k+\ell'})|,
\]

where the second line follows from (8) and the third is a consequence of Proposition 5.4 since \(|z_{N_k+\ell'}| = R_k \geq |f_c(z_{N_k+\ell'})|\). Therefore,

\[
|\partial_j f_c^{N_k+\ell'+1}(z)| = |\lambda_k|^{\ell'+1} \frac{R_k S_k}{R_j^2}.
\]

We proceed by induction on \( k \) to prove the claim. If \( z \in C_c(\alpha^1 B^\ell_1) \), then, by Corollary 5.5,

\[
|\partial_j f_c^{N_1}(z)| = \frac{R_1 S_1}{R_j^2}.
\]

By the previous discussion, the claim holds for \( k = 1 \).

Suppose the claim is true for \( k \) and \( j > k + 1 \). Assume that (5) holds for \( k + 1 \),

\[
z \in C_c(\alpha^{k+1} B^\ell_{k+1})
\]

for some \( \ell' \geq 0 \), and \(|z_{N_k+\ell'}| \leq R_k\) for all \( 0 \leq \ell' \leq \ell + 1 \). Then

\[
z \in C_c(\alpha^k B^\ell_{k+1-1})
\]

and \( f_c^{N_{k+1}}(z) \in B_0 \). By the inductive hypothesis,

\[
|\partial_j f_c^{N_{k+1}}(z)| = \frac{R_k S_k}{R_j^2} |\lambda_k|^{\ell_{k+1}}.
\]
Recall that $M_{k+1} = m_{k+1} + L_{k+1}$ and apply the chain rule:

$$|\partial_j f_c^{oM_{k+1}}(z)| = |(f_c^{oM_{k+1}})'(z_{L_{k+1}}) \cdot \partial_j f_c^{oL_{k+1}}(z) + \partial_j f_c^{oM_{k+1}}(z_{L_{k+1}})|.$$

From Lemma 4.1,

$$|(f_c^{oM_{k+1}})'(z_{L_{k+1}})| \leq p^{-m_{k+1}} \frac{|z_{M_{k+1}}|}{|z_{L_{k+1}}|}.$$

Recall that $|z_{L_{k+1}}| > \varphi$. By the inductive hypothesis and taking $k + 1$ in the place of $j$ in (5):

$$|(f_c^{oM_{k+1}})'(z_{L_{k+1}}) \cdot \partial_j f_c^{oL_{k+1}}(z)| \leq p^{-m_{k+1}} \frac{|z_{M_{k+1}}| R_k S_k}{|z_{L_{k+1}}|} |\lambda_k'|^k < \frac{|z_{M_{k+1}}| \cdot \varphi}{R_j^k}.$$

Corollary 5.5 applied to $z_{L_{k+1}} \in C_c(B_{0}^{m_{k-1}})$ gives

$$|\partial_j f_c^{oM_{k+1}}(z_{L_{k+1}})| = \frac{|z_{M_{k+1}}| \cdot |z_{M_{k+1}}-1|}{R^2_j} > \frac{|z_{M_{k+1}}| \cdot \varphi}{R^2_j}.$$

Hence,

$$|\partial_j f_c^{oM_{k+1}}(z)| = |\partial_j f_c^{oM_{k+1}}(z_{L_{k+1}})| = \frac{|z_{M_{k+1}}| \cdot |z_{M_{k+1}}-1|}{R^2_j}. \quad (9)$$

Now we apply the chain rule taking into account that $N_{k+1} = M_{k+1} + n_{k+1}$:

$$|\partial_j f_c^{oN_{k+1}}(z)| = |\partial_j f_c^{on_{k+1}}(z_{M_{k+1}}) + (f_c^{on_{k+1}})'(z_{M_{k+1}}) \cdot \partial_j f_c^{oM_{k+1}}(z)|$$

and, by Corollary 5.5, since $z_{M_{k+1}} \in C_c(A^{n_{k+1}} B_{k+1})$, we have

$$|\partial_j f_c^{on_{k+1}}(z_{M_{k+1}})| = \frac{R_{k+1} S_{k+1}}{R^2_j} \geq \frac{R_{k+1} |z_{M_{k+1}}| \cdot |z_{M_{k+1}}-1|}{|z_{M_{k+1}}| R^2_j} \geq |(f_c^{on_{k+1}})'(z_{M_{k+1}}) \cdot \partial_j f_c^{oM_{k+1}}(z)|.$$

The second line follows at once from $|z_{M_{k+1}}-1| < 1 < S_{k+1}$. For the third line, use the upper bound for $|(f_c^{on_{k+1}})'(z_{M_{k+1}})|$ given by repeated applications of Lemma 4.1 after using the chain rule and the formula for $|\partial_j f_c^{oM_{k+1}}(z)|$ furnished by (9).

Thus,

$$|\partial_j f_c^{on_{k+1}}(z)| = \frac{R_{k+1} S_{k+1}}{R^2_j}.$$
It follows that (7) holds for $k+1$ and $\ell' = 0$. From the discussion at the beginning of the proof, (7) holds for $k+1$ and for all $0 \leq \ell' \leq \ell + 1$, which finishes the proof of the claim and the lemma follows.

We are now in position to apply the non-Archimedean Mean Value Theorem 2.2 to prove the stability of the truncated cylinder $C_c(\alpha^{(k)})$ under perturbations in the set $U_{k+1}(c, \varepsilon_{k+1})$, introduced in Section 3.3.1.

**Lemma 5.7.** Let $k \geq 1$. Assume that (4) holds for $k+1$ and (5) holds for $k$. If $c' \in U_{k+1}(c, \varepsilon_{k+1})$, then

$$C_{c'}(\alpha^{(k)}) = C_c(\alpha^{(k)}).$$

Before proving the lemma, let us observe that, by the Maximum Principle, each connected component of the pre-image of a Berkovich disk, under an entire map, is again a Berkovich disk. In the case of our maps $f_c$, if $D$ is an open disk disjoint from $[0, \infty[$, then each component $D'$ of $f_c^{-1}(D)$, is again a disk disjoint from $[0, \infty[$. Thus, $D'$ is contained in $A$ or $B_j$ for some $j \geq 0$. By repeatedly applying this observation, given $c$ and $k \geq 1$, we conclude that each connected component of $C_c(\alpha^{(k)})$ is an open disk. It follows that the cylinder $C_c(\alpha^{(k)})$ is uniquely determined by $C_c(\alpha^{(k)}) \cap \mathbb{A}^1$.

**Proof.** The lemma will follow once we have proven that, for all $1 \leq j \leq k$, if $c' \in V_j := U_{k+1}(c, \varphi^{j-k}\varepsilon_{k+1})$ and $z \in C_c(\alpha^{(j)}) \cap \mathbb{A}^1$, then $z \in C_{c'}(\alpha^{(j)})$. We identify $V_j$ with the open disk $D(c_{k+1}, \varphi^{j-k}\varepsilon_{k+1})$. We will prove this assertion by induction on $j$.

For $j = 1$, the assertion holds by Corollary 5.2 since by (4) we have $\varphi^{1-k}\varepsilon_{k+1} < \varphi^{2}R_{k+1} < R_{k+1}$. Suppose that it holds for some $j < k$. Let $z \in C_c(\alpha^{(j+1)})$ be a type I point and $c' \in V_{j+1}$. Then, $z \in C_{c'}(\alpha^{(j)})$ and $|f_{c'}^{oN_j+1}(z)| \leq R_j$, for all $c' \in V_{j+1}$. In Lemma 5.6, with $\ell' = 0$ and using $j$ in the role of $k$ and $k+1$ in the role of $\ell$, we obtain

$$|\partial_{k+1} f_{c'}^{oN_j+1}(z)| = \frac{R_j S_j}{R_{k+1}^2} |\lambda_j|. $$

Thus we may apply the non-Archimedean Mean Value Theorem 2.2 to

$$g : V_j \equiv D(c_{k+1}, \varphi^{j-k}\varepsilon_{k+1}) \to \mathbb{C}_p,$$

given by $g(c'_{k+1}) = f_{c'}^{oN_j+1}(z)$ and conclude that for all $c' \in V_{j+1} \equiv D(c_{k+1}, \varphi^{j-k+1}\varepsilon_{k+1})$,

$$|f_{c}^{oN_j+1}(z) - f_{c'}^{oN_j+1}(z)| = \frac{R_j S_j}{R_{k+1}^2} |\lambda_j| \cdot |c_{k+1} - c'_{k+1}| \leq \frac{R_j S_j}{R_{k+1}^2} |\lambda_j| \cdot \varphi^{j-k+1}\varepsilon_{k+1}. $$
Now replacing the upper bound for $\varepsilon_{k+1}$ given by (4) with $j + 1$ in the role of $j$, 

$$|f^{oNj+1}_c(z) - f^{oNj+1}_{c'}(z)| < \varphi |\lambda_j|^{-\ell_{j+1}+1}$$

$$\leq \varphi < R_j,$$

since $\ell_{j+1} \geq 1$.

In particular, $z \in C_c(\alpha^{(j)}B_j)$. For $1 \leq \ell \leq \ell_{j+1}$, we proceed recursively on $\ell$ to show that

$$|f^{oNj+\ell}_c(z) - f^{oNj+\ell}_{c'}(z)| < \varphi |\lambda_j|^{-\ell_{j+1}+\ell}. \quad (10)$$

This recursion will show that $z \in C_c(\alpha^{(j+1)}B_{j+1})$, for all $1 \leq \ell \leq \ell_{j+1}$. Assume that (10) holds for some $\ell < \ell_{j+1}$. Observe that if $z \in B_j$, $|f_c(z)| \leq R_j$ and $c' \in V_{j+1}$, from Lemma 5.3, we have

$$|f_c(z) - f_{c'}(z)| \leq \frac{R_j}{R_{k+1}} |c_{k+1} - c'_{k+1}|$$

$$\leq \frac{R_j}{R_{k+1}} \varphi^{-k+j+1} \varepsilon_{k+1}$$

$$< \frac{R_j}{R_{k+1}} \varphi^{-k+1} |\lambda_j|^{-\ell_{j+1}+\ell+1} < \varphi |\lambda_j|^{-\ell_{j+1}+\ell+1},$$

where the second line is obtained from the definition of $V_{j+1}$, and the third from the bound on $\varepsilon_{k+1}$ given by (4) and then using that $R_j/S_j < |\lambda_j|$ furnished by (8).

Write $|f^{oNj+\ell+1}_c(z) - f^{oNj+\ell+1}_{c'}(z)|$ as

$$|f_c(f^{oNj+\ell}_c(z)) - f_{c'}(f^{oNj+\ell}_{c'}(z)) + f_{c'}(f^{oNj+\ell}_c(z)) - f_{c'}(f^{oNj+\ell}_{c'}(z))|.$$

After replacing $z$ by $f^{oNj+\ell}_c(z)$ in the previous observation yields

$$|f_c(f^{oNj+\ell}_c(z)) - f_{c'}(f^{oNj+\ell}_{c'}(z))| < \varphi |\lambda_j|^{-\ell_{j+1}+\ell+1}.$$

Applying Lemma 3.4 and the inductive hypothesis, we obtain

$$|f_{c'}(f^{oNj+\ell}_c(z)) - f_{c'}(f^{oNj+\ell}_{c'}(z))| < \varphi |\lambda_j|^{-\ell_{j+1}+\ell+1}.$$

Thus, (10) holds for $\ell + 1$ and, therefore, for $\ell = \ell_{j+1}$.

Hence, the distance between $f^{oNj+\ell_{j+1}}_c(z)$ and $f^{oNj+\ell_{j+1}}_{c'}(z)$ is bounded above by $\varphi$. By Lemma 4.2 and Corollary 5.2,

$$f^{oNj+\ell_{j+1}}_{c'}(z) \in C_c(B_{j+1}^mA^{N_{j+1}}B_{j+1}) = C_{c'}(B_{j+1}^mA^{N_{j+1}}B_{j+1}).$$

It follows that $z \in C_{c'}(\alpha^{(j+1)})$, which ends the proof of the lemma. In fact, since $c \in U_{k+1}(c', \varepsilon_{k+1})$, the same argument proves that $C_{c'}(\alpha^{(k)}) \subset C_c(\alpha^{(k)})$. □
Our results so far are only concerned with perturbations in $U_{k+1}(c, \varepsilon_{k+1})$, rather than in the larger set $\Delta_{k+1}(c, \varepsilon)$, considered in the statement of the Cylinder Stability Lemma 3.7. In order to remedy this situation, we show that maps in $\Delta_{k+1}(c, \varepsilon)$ converge to $f_c$, as $k \to \infty$, uniformly on any bounded set. To be precise, given $R > 0$ and an entire map $g$, let

$$|g|_R := \sup\{|g(z)| : z \in D(0, R)\}.$$  

**Lemma 5.8.** Assume that $\varepsilon = (\varepsilon_j)_{j \geq 2}$ converges to 0, as $j \to \infty$. Then, given any $R > 0$,

$$\lim_{k \to \infty} \sup \{|f_c - f_{c'}|_R : c' \in \Delta_k(c, \varepsilon)\} = 0.$$

**Proof.** Without loss of generality, we assume that $\varepsilon_j < 1$ for all $j$, and $c' \in \Delta_2(c, \varepsilon)$. Set

$$a_j(z) := \frac{z}{c'_j \cdot \overline{c_j} - c'_j},$$

and observe that for all $|z| < R$, and all $j$ such that $R_j \geq R$, we have

$$|a_j(z)| < \frac{R}{R_j} \cdot \varepsilon_j < \varepsilon_j.$$

Now, if $c' \in \Delta_k(c, \varepsilon)$ for some $k$, then

$$\left| \frac{f_{c'}(z)}{f_c(z)} - 1 \right| = \prod_{j \geq k} (1 + a_j(z)) - 1 \leq \max_{j \geq k} |a_j(z)| < \max_{j \geq k} \varepsilon_j,$$

for all $|z| < R$. Since $|f_c(z)| < \varphi(R)$ for all $|z| < R$, the lemma follows. \hfill \Box

**Proof of Lemma 3.7.** Consider $k_0 \geq 1$, and $c' \in \Delta_{k_0+1}(c, \varepsilon)$. For $k \geq k_0$, let $c^{(k)}_j$ be the sequence defined by $c^{(k)}_j := c'_j$ for all $j < k$, and $c^{(k)}_j := c_j$ for all $j \geq k$. Note that $c^{(k_0)} = c$.

Note that $c^{(k+1)}_j \in U_{k+1}(c^{(k)}_j, \varepsilon_{k+1})$, for all $k \geq k_0$. Hence, for all $k > k_0$, by Lemma 5.7, $C_{c^{(k)}}(c^{(k_0)}) = C_c(c^{(k_0)})$. Also, $c^{(k+1)}_j \in \Delta_{k+1}(c', \varepsilon)$, so $f^{(n)}_{c^{(k)}}(z) \to f^{(n)}_{c'}(z)$, as $k \to \infty$, for all $z \in C_c(c^{(k_0)}) \cap \mathbb{A}_1$, and all $n \leq N_{k_0}$, by Lemma 5.8. We conclude that $C_{c'}(c^{(k_0)}) \subset C_c(c^{(k_0)})$. Equality of the cylinder sets follows, since $c \in \Delta_{k_0+1}(c', \varepsilon)$. \hfill \Box

### 6 | CONNECTING LEMMA

The Connecting Lemma 3.8 will be the outcome of two perturbations. First, we consider a parameter $c$ and a point $x$ with itinerary $\alpha^{(k)}_B \infty$, that is, $f^{\circ N_k}_c(x)$ is the repelling fixed point $u_k(c)$ in $B_k$. Near $f^{\circ N_k}_c(x)$, there are points with itinerary $B^{\ell_{k+1}}_k B_0^\infty$. The idea is to obtain a parameter $c''$ perturbing $c$, to connect $f^{\circ N_k}_c(x)$ with a point with itinerary $B^{\ell_{k+1}}_k B_0^\infty$. Then, the itinerary of $x$ under $f_{c''}$ becomes $\alpha^{(k)}_B B^{\ell_{k+1}}_k B_0^\infty$. To achieve this, a priori estimates are provided in Lemma 6.1 and $c''$
is obtained in Lemma 6.2. Then, in the proof of the Connecting Lemma 3.8, we produce a second perturbation that yields a parameter \( c' \) near \( c'' \). This second perturbation connects \( f_{c''}^{\circ L_{k+1}}(x) \) with a point whose itinerary is \( B_0^{m_{k+1}} A^{n_{k+1}} B_{k+1}^{\infty} \).

**Lemma 6.1.** Given \( k \geq 1 \), let

\[
h_c(z) := \left( f_c|_{B_k} \right)^{-1}.
\]

Then, for all \( \ell \geq 1 \),

\[
|h_c^{\circ \ell}(0) - w_k(c)| = R_k |\lambda_k|^{-\ell}.
\]

Moreover, \( \partial_{k+1} h_c(0) = 0 \) and for all \( \ell \geq 2 \):

\[
|\partial_{k+1} h_c^{\circ \ell}(0)| = \frac{R_k^2}{R_{k+1}^2 |\lambda_k|}.
\]

**Proof.** Recall that \( c_k \in B_k \). Thus, \( h_c(0) = c_k \). In view of Lemma 3.4, the map \( f_c \) expands distances by a factor of \( |\lambda_k| \) in \( B_k \cap \mathbb{A}^1 \). In particular, \( |\lambda_k| \cdot |c_k - w_k(c)| = |0 - w_k(c)| = R_k \) and, by induction,

\[
|h_c^{\circ \ell}(0) - w_k(c)| = R_k |\lambda_k|^{-\ell},
\]

for all \( \ell \geq 1 \).

We also proceed by induction to establish the second part of the lemma. Indeed, since \( h_c(0) = c_k \) we have \( \partial_{k+1} h_c(0) = 0 \). Now let \( \ell \geq 1 \) and assume that the lemma holds for \( \ell' \). Since \( h_c^{\circ \ell'}(0) = f_c(h_c^{\circ \ell+1}(0)) \), from the chain rule,

\[
\partial_{k+1} h_c^{\circ \ell}(0) = \partial_{k+1}(f_c \circ h_c^{\circ \ell+1}(0))
\]

\[
= \partial_{k+1} f_c(h_c^{\circ \ell+1}(0)) + f'_c(h_c^{\circ \ell+1}(0)) \cdot \partial_{k+1} h_c^{\circ \ell+1}(0).
\]

Therefore,

\[
|\lambda_k| \cdot |\partial_{k+1} h_c^{\circ \ell+1}(0)| = |\partial_{k+1} h_c^{\circ \ell}(0) - \partial_{k+1} f_c(h_c^{\circ \ell+1}(0))|.
\]

Taking into account that \( R_k = |h_c^{\circ \ell}(0)| = |h_c^{\circ \ell+1}(0)| \), we may apply Lemma 5.3 to obtain

\[
|\partial_{k+1} f_c(h_c^{\circ \ell+1}(0))| = \frac{R_k^2}{R_{k+1}^2} \geq \frac{R_k^2}{R_{k+1}^2 |\lambda_k|} \geq |\partial_{k+1} h_c^{\circ \ell}(0)|.
\]

The formula for \( |\partial_{k+1} h_c^{\circ \ell+1}(0)| \) follows. □

**Lemma 6.2.** Let \( k \geq 1 \). Consider \( c \in \mathcal{C}(r) \) and \( x \in C_c(\alpha^{(k)}) \) such that

\[
f_c^{\circ N_k}(x) = w_k(c).
\]
Assume that (4) holds for \( k + 1 \) and, (5), (6) hold for \( k \). Then, there exists \( \mathbf{c''} \in U_{k+1}(\mathbf{c}, \varepsilon_{k+1}) \) such that

\[
f_{\mathbf{c''}} \circ N_k(x) = h_{\mathbf{c''}}^{\sigma \mathbf{c''} k+1}(0).
\]

**Proof.** Let \( H : U_{k+1}(\mathbf{c}, \varepsilon_{k+1}) \to \mathbb{A}^1 \) be defined by

\[
H(\mathbf{a}) := f_{\mathbf{a}}^{\circ N_k}(x) - h_{\mathbf{a}}^{\sigma \mathbf{c''} k+1}(0).
\]

As before, we identify \( U_{k+1}(\mathbf{c}, \varepsilon_{k+1}) \) with the disk \( D(c_{k+1}, \varepsilon_{k+1}) \subset \mathbb{C}_p \). After this identification \( H : U_{k+1}(\mathbf{c}, \varepsilon_{k+1}) \to \mathbb{A}^1 \) is analytic and, therefore, maps disks onto disks. For all \( \mathbf{a} \in U_{k+1}(\mathbf{c}, \varepsilon_{k+1}) \), in view of Lemma 5.7, we have that \( x \in C_{\mathbf{a}}(\alpha^{(k)}) \); thus we may apply Lemma 5.6, (8) and Lemma 6.1 to conclude that:

\[
|\partial_{k+1} f_{\mathbf{a}}^{\circ N_k}(x)| = \frac{R_k S_k}{R_{k+1}^2} > \frac{R_k^2}{R_{k+1}^2} \cdot |\lambda_k| \geq |\partial_{k+1} h_{\mathbf{a}}^{\sigma \mathbf{c''} k+1}(0)|.
\]

Thus,

\[
|\partial_{k+1} H(\mathbf{a})| = \frac{R_k S_k}{R_{k+1}^2},
\]

for all \( \mathbf{a} \in U_{k+1}(\mathbf{c}, \varepsilon_{k+1}) \). By Theorem 2.2,

\[
H(U_{k+1}(\mathbf{c}, \varphi \varepsilon_{k+1})) = D(H(\mathbf{c}), \varphi \varepsilon_{k+1}, \frac{R_k S_k}{R_{k+1}^2}).
\]

From Lemma 6.1 and (6),

\[
|H(\mathbf{c})| = R_k |\lambda_{k+1}|^{-\varphi \varepsilon_{k+1}} < \varphi \varepsilon_{k+1},
\]

Hence, there exists \( \mathbf{c''} \in U_{k+1}(\mathbf{c}, \varphi \varepsilon_{k+1}) \) such that \( H(\mathbf{c''}) = 0 \) and the lemma follows. \( \square \)

**Proof of Lemma 3.8.** Let \( r \) be such that \( \varphi^{n_{k+1} + m_{k+1}}(r) = R_{k+1} \). From the definition of transition times (see (2)), we have \( \varphi^{-(n_{k+1} + 1)}(R_{k+1}) < 1 \). Therefore, \( \varphi < r < 1 \). Consider

\[
r' := r \frac{R_{k+1}^2}{R_k S_k} |\lambda_k|^{-\varphi \varepsilon_{k+1}}.
\]

Note that, from (6), \( r' < \varphi \varepsilon_{k+1} \). Consider \( \mathbf{c''} = (c_j'') \) as in Lemma 6.2 and let

\[
\overline{U} := \{ \mathbf{a} = (a_j) \in U_{k+1}(\mathbf{c}'', \varepsilon_{k+1}) : |a_{k+1} - c''_{k+1}| \leq r' \},
\]

\[
U := \{ \mathbf{a} = (a_j) \in U_{k+1}(\mathbf{c}'', \varepsilon_{k+1}) : |a_{k+1} - c''_{k+1}| < r' \}.
\]
Naturally, we may identify $\overline{U}$ and $U$ with $\overline{D}(0, r') \subset \mathbb{C}_p$ and $D(0, r') \subset \mathbb{C}_p$, respectively. We will find a parameter $c' \in \overline{U} \setminus U$, as in the statement of the lemma.

Consider

$$F : \overline{U} \to \mathbb{C}_p$$
$$a \mapsto f_a^{\circ L_{k+1}}(x).$$

\[\square\]

**Claim.** The map $F : \overline{U} \to \overline{D}(0, r)$ is an analytic isomorphism.

**Proof of the Claim.** Let

$$F_0 : U_{k+1}(c'', \varepsilon_{k+1}) \to \mathbb{C}_p$$
$$a \mapsto f_a^{\circ \ell_{k+1}}(x).$$

By Lemma 5.6, the derivative $F'_0(a)$ has constant absolute value $\frac{R_k S_k}{R_{k+1}^2}$. By Theorem 2.2, for all $a \in \overline{U}$,

$$|F_0(c'') - F_0(a)| = \frac{R_k S_k}{R_{k+1}^2} |c''_{k+1} - a_{k+1}|.$$

From the definition of $r'$, it follows that $F_0$ maps $\overline{U}$ isomorphically onto a disk of radius $r|\lambda_k|^{-\ell_{k+1}} < r < R_k$, which is contained in $B_k$. Now, for $1 \leq \ell \leq \ell_{k+1}$, consider

$$F_\ell : \overline{U} \to \mathbb{C}_p$$
$$a \mapsto f_a^{\circ N_k + \ell}(x).$$

The claim will follow once we prove the following assertion: for all $0 \leq \ell \leq \ell_{k+1}$,

$$|F_\ell(c'') - F_\ell(a)| = |\lambda_k|^{-\ell_{k+1}} \frac{R_k S_k}{R_{k+1}^2} |c''_{k+1} - a_{k+1}|,$$

and $F_\ell$ is an analytic isomorphism whose image is a disk of radius $r|\lambda_k|^{-\ell_{k+1} + \ell}$. Indeed, recursively, suppose the assertion true for $\ell < \ell_{k+1}$. Then, $F_\ell(c'') = h_\ell c''^{\circ \ell_{k+1} - \ell}(0) \in B_k$. Note that $F_\ell(\overline{U}) \subset B_k$ since $F_\ell(\overline{U})$ is a disk of radius $r|\lambda_k|^{-\ell_{k+1} + \ell} < r < R_k$ containing $F_\ell(c'') \in B_k$. Moreover, $F_{\ell+1}(c'') \in B_k$ if $\ell + 1 < \ell_{k+1}$ and $F_{\ell+1}(c'') = 0$ if $\ell + 1 = \ell_{k+1}$. Hence, in both cases, $|f_\ell(c''(F_\ell(c''))| \leq R_k = |F_\ell(c'')|$. Thus, we may apply Lemma 5.3 to conclude that, for all $a \in \overline{U}$,

$$|f_\ell(c''(F_\ell(c'')) - f_a(F_\ell(c''))| \leq \frac{R_k^2}{R_{k+1}^2} |c''_{k+1} - a_{k+1}|$$

$$< |\lambda_k|^{-\ell_{k+1} + \ell} \frac{R_k S_k}{R_{k+1}^2} |c''_{k+1} - a_{k+1}|$$

$$= |f_a(F_\ell(c'')) - f_a(F_\ell(a))|,$$
where the second line follows from $R_k/S_k < |\lambda_k|$, and the third line is a consequence of Lemma 3.4 since $F_\epsilon(U) \subset B_k$. Then,

$$|F_{\epsilon+1}(c') - F_{\epsilon+1}(a)| = |f_\epsilon(F_\epsilon(c')) - f_\epsilon(F_\epsilon(c'')) + f_\epsilon(F_\epsilon(c'')) - f_\epsilon(F_\epsilon(a))|$$

$$= |\lambda_k^{\epsilon+1} \frac{R_k S_k}{R_{k+1}^2} c' - a_{k+1}|,$$

and the claim follows.

Note that $F(c'') = 0$ and, for all $a \in \overline{U} \setminus U$, we have $|f_\epsilon^{o_k+1}(x)| = |F(a)| = r$. Thus, the lemma reduces to proving the existence of a solution in $\overline{U} \setminus U$ to the equation:

$$f_\epsilon^{o_k+1}(x) - w_{k+1} = 0. \tag{11}$$

To show the existence of such a solution, given $a \in \overline{U}$, let

$$g_a := f_\epsilon^{n_k+1+m_{k+1}} : D(0, r) \to D(0, R_{k+1}).$$

Note that, for all $a \in \overline{U}$, the map $g_a$ is onto and its Weierstrass degrees on $D(0, r)$ and $D(0, r)$ are both $\delta$, for some positive integer $\delta$ independent of $a$. Also $F : U \to \overline{D}(0, r)$ has Weierstrass degrees $1$ both on $U$ and on $\overline{U}$. As we will show after finishing the proof of the lemma, it is not difficult to check from here that the Weierstrass degrees of $a \mapsto g_a(F(a))$ on $\overline{U}$ and on $U$ are both $\delta$. Observe that, for all $a \in \overline{U}$, the fixed point $w_{k+1}(a)$ lies in $B_{k+1}$. In particular $|w_{k+1}(a)| = R_{k+1}$. Therefore, the Weierstrass degree of

$$a \mapsto g_a(F(a)) - w_{k+1}(a) = f_\epsilon^{o_k+1}(x) - w_{k+1}(a)$$

on $\overline{U}$ is $\delta$ and on $U$ is $0$. Hence, there exists a parameter $c' \in \overline{U} \setminus U$ which solves (11) and the lemma follows.

It remains to show that the Weierstrass degrees of $a \mapsto g_a(F(a))$ on $\overline{U}$ and on $U$ are both $\delta$. Indeed, without loss of generality, after changing coordinates, we may assume that $U = D(0, 1)$, $\overline{U} = \overline{D}(0, 1)$ and $R_k = r = 1$. Let $g : U \times U \to \overline{U}$ be an analytic map of the form $g(a, z) = g_a(z)$ such that, for all $a \in \overline{U}$, the Weierstrass degrees of $g_a$ on $\overline{U}$ and on $U$ are both $\delta$. Then, for all $a \in \overline{U}$, the map $g_a : U \to \overline{U}$ has reduction $\overline{g}_a(z) = \alpha(\hat{a}) z^\delta$ with $\alpha(\hat{a}) \neq 0$, where $\alpha(\hat{a})$ is the reduction of the coefficient of $z^\delta$ in the expansion of $g_a(z)$. Hence, $\alpha(\hat{a})$ is constant. Now consider an analytic isomorphism $F : \overline{U} \to \overline{U}$ such that $F(U) = U$. Then the reduction of $F$ is $F(\hat{z}) = \beta \hat{z}$ for some $0 \neq \beta \in \overline{C_p}$. Therefore, $a \mapsto g_a(F(a))$ reduces to a degree $\delta$ monomial and the assertion follows. □

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