Extremal function for Moser-Trudinger type Inequality with Logarithmic weight

February 16, 2016

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Abstract
On the space of weighted radial Sobolev space, the following generalization of Moser-Trudinger type inequality was established by Calanchi and Ruf in dimension 2: If $\beta \in [0, 1)$ and $w_0(x) = |\log |x||^\beta$ then

$$\sup_{f_B |\nabla u|^2 w_0 \leq 1, u \in H^1_0, rad(w_0, B)} \int_B e^{\alpha u^2} dx < \infty,$$

if and only if $\alpha \leq \alpha_\beta = 2 [2\pi (1 - \beta)]^{\frac{1}{1 - \beta}}$. We prove the existence of an extremal function for the above inequality for the critical case when $\alpha = \alpha_\beta$ thereby generalizing the result of Carleson-Chang who proved the case when $\beta = 0$.

1 Introduction
Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^n$. For $p < n$ it is well known from Sobolev embedding that the space

$$W^{1,p}_0(\Omega) \hookrightarrow L^p(\Omega)$$

continuously, if $1 \leq p \leq p^* = \frac{np}{n-p}$. The Moser-Trudinger inequality concerns about the borderline case, that is when $p = n$. In this case

$$W^{1,n}_0(\Omega) \hookrightarrow L^p(\Omega)$$

for all $p \in [1, \infty)$. The embedding is not true for case $p = \infty$. For $n = 2$ one can take the function

$$f(x) = \log (1 - \log |x|)$$

in unit ball $B$ centered at origin, then it is easy to check that $f \in W^{1,2}_0(B)$ but clearly it is not in $L^\infty(B)$. Hence one may look for maximal growth function $g$ such that

$$\int_{\Omega} g(u) < \infty, \forall u \in W^{1,n}_0(\Omega).$$
It was shown by Trudinger in [20], that such a \( g \) should be of the form
\[
g(t) = e^{t^\alpha_n}.\]

There was a further improvement by Moser who proved the following theorem:

**Theorem 1** [Moser] Let \( u \in W^{1,n}_0(\Omega) \) with
\[
\int_\Omega |\nabla u|^n \leq 1.
\]
Then there exist a constant \( C > 0 \) depending only on \( n \), such that
\[
\int_\Omega e^{\alpha u} \leq C|\Omega|, \quad \text{if } \alpha \leq \alpha_n := n\omega^{\frac{1}{n-1}}
\]
where \( \omega_{n-1} \) is the \((n-1)\) dimensional surface measure of the unit sphere and the above result is false if \( \alpha > \alpha_n \). Here \(|\Omega|\) denotes the \(n\) dimensional Lebesgue measure of the set \( \Omega \).

Then the next natural question was if there exist an extremal function for the inequality in (1). Carleson-Chang [6] showed that the answer to the above question is positive if the domain is a ball. In [18] Struwe proved the case when the domain is close to a ball. Flucher in [9] provided a positive answer for the case of any general domain \( \Omega \) in 2 dimensions. The higher dimension case for general domains was done by Lin in [11]. Moser-Trudinger type inequality had been an interesting topic of research for several authors. We list a few of them [1, 2, 3, 7, 8, 10, 12, 14, 15, 18] and the references there in for the available literature in this direction.

Let \( n = 2 \) and \( w_0 = |\log |x||^\beta \)
be defined on the unit ball \( B \) and \( H^1_0(w_0, B) \) denotes the usual weighted Sobolev space defined as the completion of \( C_c^\infty(B) \) (the space of smooth functions with compact support) functions with respect to the norm
\[
||u||_{w_0} := \left( \int_B |\nabla u|^2 w_0 dx \right)^{\frac{1}{2}}.
\]
The subspace of radial functions in \( H^1_0(w_0, B) \) is denoted by \( H^1_{0, rad}(w_0, B) \).

The following theorem by Calanchi-Ruf in [5] generalizes the Moser-Trudinger inequality for balls. The work of this paper is based on this generalization.

**Theorem 2** Let \( \beta \in (0,1) \) and \( n = 2 \), then
\[
\sup_{||u||_{w_0} \leq 1, u \in H^1_{0, rad}(w_0, B)} \int_B e^{\alpha u} dx < \infty \tag{2}
\]
if and only if \( \alpha \leq \alpha_\beta := 2\left[\frac{2\pi(1-\beta)^{\frac{1}{n}}}{}\right] \) and the inequality above is false if \( \alpha > \alpha_\beta \).
They have also obtained the optimal Moser-Trudinger type inequality for the case \( \beta = 1 \) which we do not mention here. For \( \beta = 0 \) the above theorem is precisely Moser-Trudinger inequality for balls in 2 dimensions. This is obvious after using a symmetrization argument. In this work we are concerned with the existence of extremal function for the inequality in \( \beta \) for the critical case i.e. \( \alpha = \alpha_\beta \). In the sub critical case \( (\alpha < \alpha_\beta) \) the issue of existence of an extremizer is not very difficult, as one can use Vitali’s convergence theorem to pass through the limit. This issue is addressed in [3] . Also very recently in [4] Calanchi-Ruf had also established such logarithmic Moser-Trudinger type inequalities in higher dimensions.

For each \( \beta \in [0,1) \), let \( J_\beta : H_{0,rad}^1(w_0, B) \to \mathbb{R} \) defined by

\[
J_\beta(u) := \frac{1}{|B|} \int_B e^{\alpha_\beta u} \, dx
\]

denotes Logarithmic Moser-Trudinger functional. The following is our main result:

**Theorem 3 [Main Result]** There exist \( u_\beta \in H_{0,rad}^1(w_0, B) \) such that

\[
J_\beta(u_\beta) = \sup_{||u||_{w_0} \leq 1, u \in H_{0,rad}^1(w_0, B)} J_\beta(u),
\]

for all \( \beta \in [0,1) \).

The main difficulty in proving the existence of extremizer lies in the fact that the functional \( J_\beta \) is not continuous with respect to the weak convergence of the space \( H_{0,rad}^1(w_0, B) \). The following sequence \( w_k \in H_{0,rad}^1(w_0, B) \) (power of Moser sequence), defined as

\[
w_k(x) = \alpha_\beta^{-1} \begin{cases} \frac{k^{-\frac{\alpha}{2}}}{2 \log \frac{|x|}{\sqrt{k}}} \left( \frac{2 \log |x|}{\sqrt{k}} \right)^{1-\beta} & \text{in } 0 \leq |x| \leq e^{-\frac{k}{2}}, \\ 0 & \text{on } e^{-\frac{k}{2}} \leq |x| \leq 1, \end{cases}
\]

has the property that \( w_k \rightharpoonup 0 \) in \( H_{0,rad}^1(w_0, B) \) but \( J_\beta(w_k) \nrightarrow J_\beta(0) \). We refer to Section [2] for the proof of the last statement.

**Definition (Concentration):** A sequence of functions \( u_k \in H_{0,rad}^1(w_0, B) \) is said to concentrate at \( x = 0 \), denoted by

\[
|\nabla u_k|^2 w_0 \rightharpoonup \delta_0,
\]

if \( ||u_k||_{w_0} \leq 1 \) and for any given \( 1 > \delta > 0 \),

\[
\int_{B \setminus B_\delta} |\nabla u_k|^2 w_0 \to 0,
\]

where \( B_\delta \) denotes the ball of radius \( \delta \) at origin.

**Definition (Concentration Level \( J_\delta^\beta(0) \)):**

\[
J_\delta^\beta(0) := \sup_{\{w_m\} \in H_{0,rad}^1(w_0, B)} \left\{ \limsup_{m \to \infty} J(w_m) \mid |\nabla w_m|^2 w_0 \rightharpoonup \delta_0 \right\}.
\]
The method of the proof of our main result follows similar idea as it is done in [6]. First in Lemma 5 we show that a maximizing sequence can lose compactness only if it concentrates at 0. Then in Lemma 9 the concentration level is explicitly calculated. It is shown that for all $\beta \in [0, 1)$

\[ J_\beta'(0) \leq 1 + e. \]  

Note that the right hand side of the above inequality is independent of $\beta$.

Finally we finish the proof by showing that for all $\beta \in [0, 1)$ one can find $v_\beta \in H^1_{0, \text{rad}}(w_0, B)$ such that

\[ J_\beta(v_\beta) > 1 + e. \]

As a direct application of the above theorem one obtains existence of radial solution of the following nonlinear elliptic Dirichlet (mean field type) problem for $\beta \in [0, 1)$:

\[
\begin{cases}
-\text{div} (w_0 \nabla u) = \frac{\beta + 1}{w_0^\frac{\beta + 1}{2}} e^{\alpha \beta u^1_{1-\beta}} & \text{in } B, \\
u = 0 & \text{on } \partial B, \\
u > 0 & \text{on } B.
\end{cases}
\]

2 Some supporting results for the proof of main theorem

At first we will deduce an equivalent formulation of the problem with which we will work in this paper. Let $\gamma = \frac{1}{1-\beta}$. For $u \in H^1_{0, \text{rad}}(w_0, B)$ first change the variable as

\[ |x| = e^{-t} \]

and set

\[ \psi(t) = \frac{1}{\beta} u(x). \]

Then the functional changes as

\[ I_\beta(\psi) := \int_0^\infty e^{\psi^2(t)} - t dt = \frac{1}{|B|} \int_B e^{\alpha \beta u^2} dx = J_\beta(u). \]

and the weighted gradient norm changes as

\[ \Gamma(\psi) := \int_0^\infty \frac{\psi^2(t) t^\beta}{1-\beta} dt = \int_B |\nabla u|^2 |\log |x||^\beta dx. \]

For $\delta \in (0, 1]$, define

\[ \tilde{\Lambda}_\delta := \{ \phi \in C^1(0, \infty) \mid \phi(0) = 0, \; \Gamma(\phi) \leq \delta \} \]

and now since

\[
\sup_{||u||_{w_0} \leq 1, u \in H^1_{0, \text{rad}}(w_0, B)} \int_B e^{\alpha \beta u^1_{1-\beta}} dx = \sup_{||u||_{w_0} \leq 1, u \in H^1_{0, \text{rad}}(w_0, B), \text{smooth}} \int_B e^{\alpha \beta u^1_{1-\beta}} dx.
\]
the problem reduces in finding \( \psi_0 \in \tilde{\Lambda}_1 \) such that
\[
M_\beta := I_\beta(\psi_0) = \sup_{\psi \in \tilde{\Lambda}_1} I_\beta(\psi). \tag{7}
\]

**Lemma 4** Let \( w_k \) be as in (3), then for all \( \beta \in [0, 1) \),
\[
\liminf_{k \to \infty} J_\beta(w_k) > 1 + \frac{1}{e} > J_\beta(0) = 1.
\]

**Proof** It is clear that \( w_k \to 0 \) in \( H_{0, \text{rad}}^1(w_0, B) \). Let \( \psi_k(t) = \alpha_{1/\beta} w_k(x) \) and \( |x| = e^{-t} \). Then in view of (6), it is enough to show that
\[
\liminf_{k \to \infty} I_\beta(\psi_k) > 1.
\]

Then
\[
I_\beta(\psi_k) = \int_0^k e^{t^2 - t} dt + \int_k^\infty e^{k - t} dt = k \int_0^1 e^{k(t^2 - t)} dt + 1.
\]

Note that the function \( e^{t^2 - t} \) is monotone decreasing on \( [0, 1/2] \), monotone increasing on \( [1/2, 1] \) and strictly positive. Therefore one has
\[
k \int_0^1 e^{k(t^2 - t)} > ke^{k(1/2 - 1/2)} = k e^{-1} = e^{-1}.
\]

This implies that
\[
\liminf_{k \to \infty} I_\beta(\psi_k) > 1 + \frac{1}{e}.
\]

This completes the proof of the lemma. \( \blacksquare \)

Let \( \tilde{g}_m \) be a maximizing sequence i.e. \( J_\beta(\tilde{g}_m) \to M_\beta \). Since
\[
\int_B |\nabla \tilde{g}_m|^2 \log |x|^\beta \, dx \leq 1,
\]
one can find up to a subsequence (which we again denote by \( \tilde{g}_m \)) and for some function \( \tilde{g}_0 \in H_{0, \text{rad}}^1(w_0, B) \)
\[
\tilde{g}_m \rightharpoonup \tilde{g}_0 \quad \text{in} \ H_{0, \text{rad}}^1(w_0, B),
\]
\[
\tilde{g}_m \to \tilde{g}_0 \quad \text{pointwise.} \tag{8}
\]

The next lemma is equivalent to concentration-compactness alternative, for Moser-Trudinger case, by P. L. Lions in [12]. As a consequence of the next lemma it would be enough to prove that the sequence \( \tilde{g}_m \) does not concentrates at 0, in order to pass thought the limit in the functional.

**Lemma 5** [Concentration-Compactness alternative] For any sequence \( \tilde{w}_m, \tilde{w} \in H_{0, \text{rad}}^1(w_0, B) \) such that \( \tilde{w}_m \rightharpoonup \tilde{w} \) in \( H_{0, \text{rad}}^1(w_0, B) \), then for a subsequence, either
\[
(1) \ I_\beta(\tilde{w}_m) \to I_\beta(\tilde{w}),
\]
or
\[
(2) \ \tilde{w}_m \text{ concentrates at } x = 0.
\]
Proof Let us assume that (1) does not hold. Then it is enough to show that for each $A > 0$, as $m \to \infty$, it implies that

$$\int_{B \setminus B_r} |\nabla \tilde{w}_m| |\log |x|| |x| dx = \int_0^A \frac{|\tilde{w}_m(t)|^2 t^\beta}{1 - \beta} dt \to 0$$

where

$$\alpha^{\frac{1}{2}} \tilde{w}_m(x) = w_m(t), \; \dot{w}(x) = w(t) \text{ and } |x| = e^{-\frac{t}{2}}.$$

We argue by contradiction. Then there exists some $A > 0$ and $\delta > 0$ with

$$\int_0^A \frac{|w_m'|^2 t^\beta}{1 - \beta} dt \geq \delta,$$

for all $m \geq m_0$, for some $m_0$. Using Fundamental theorem of calculus and Hölder’s inequality, we obtain for $t \geq A$,

$$w_m(t) - w_m(A) = \int_A^t w'_m(s) = \sqrt{1 - \beta} \int_A^t \frac{w'_m(s) s^{\frac{\gamma}{2}}}{\sqrt{1 - \beta} s^{\frac{\gamma}{2}}} \leq \sqrt{1 - \beta} \left( \int_A^t \frac{|w'_m|^2 s^{\beta}}{1 - \beta} \right)^{\frac{1}{2}} \left( \int_A^t s^{-\beta} \right)^{\frac{1}{2}} \leq \left( \int_A^t \frac{|w'_m|^2 s^{\beta}}{1 - \beta} \right)^{\frac{1}{2}} \left( 1 - \beta \right)^{\frac{1}{2}} \left( 1 - \frac{\gamma}{2} \right) t^{\frac{1 - \beta}{2}} \leq (1 - \delta) \left( 1 - \frac{\gamma}{2} \right) t^{\frac{1 - \beta}{2}}.$$

Now using the inequality $w_m(A) \leq A^{\frac{1 - \beta}{2}}$ for all $m$, we have for $t \geq N$, (for sufficiently large $N$)

$$w_m(t) \leq \left( A^{\frac{1 - \beta}{2}} + (1 - \delta) \frac{1}{2} t^{\frac{1 - \beta}{2}} \right) \leq A + \left( 1 - \frac{\delta}{2} \right) t^{\frac{1 - \beta}{2}}. \quad (9)$$

In the last step we have used the following inequality: If $\mu > \gamma > 0$, $p > 1$ then for sufficiently large $y \in \mathbb{R}$, one has

$$\left( 1 + \gamma y \right)^p \leq 1 + \mu y^p.$$

Therefore from (9), we have

$$I_m^1(w_m) := \int_0^N e^{w_m(t)} t^{\frac{2 - \beta}{2} - 1} dt \leq e^A \int_0^N e^{\left( 1 - \frac{\beta}{2} \right) t^{\frac{2 - \beta}{2}} - 1} t,$$

which can be made less than any arbitrary positive number $\epsilon$, after choosing $N$ large enough. Since we know that $\tilde{w}_m$ converges pointwise to $\tilde{w}_m$, this implies that $w_m$ also converges pointwise to $w$.

Now we split $I_\beta(w_m) = I_1(w_m) + I_2(w_m)$ where

$$I_1^m(w_m) := \int_0^N e^{w_m(t)} t^{\frac{2 - \beta}{2} - 1} dt.$$

Using the bound $w(t) \leq t^{\frac{1 - \beta}{2}}$ and dominated convergence theorem, one obtains

$$I_1(w_m) \to I_1(w).$$
and for $I^n_t(w_m)$ we already know that it can made arbitrarily small. Therefore $I_\beta(w_m) \to I_\beta(w)$ which is a contradiction. ■

An Inequality: For any $w \in C^1(0, \infty)$ and $t \geq A \geq 0$, then we get after using Hölder’s inequality, that
\[
w(t) - w(A) \leq \left( \int_A^t |w'|^{2s\beta} \right)^\frac{1}{s} \left( t^{1-\beta} - A^{1-\beta} \right)^{\frac{1}{s}}. \tag{10}
\]
We will recall this inequality several times.

The following lemma is proved in [6], Lemma 1. Here we will use it without giving the proof.

Let for $\delta > 0$,
\[\Lambda_\delta := \left\{ \phi \in C^1(0, \infty) \mid \phi(0) = 0, \int_0^\infty |\phi'|^2 dt \leq \delta \right\}.
\]

**Lemma 6** [Carleson-Chang] For each $c > 0$, we have
\[
\sup_{\phi \in \Lambda_\delta} \int_a^\infty e^{c\phi(t)-t} dt \leq e^{\frac{c^2}{4}+1}.
\]

The next lemma is a technical result that will be useful in proving Lemma 9.

**Lemma 7** For $a > 0$, if $1 - \gamma\delta > 0$ then
\[
\sup_{\phi \in \tilde{\Lambda}_\delta} \int_a^\infty e^{\phi^2(t)-t} dt \leq \frac{e^{1-a}}{(1-\gamma\delta)} e^{\phi^2(a)+\frac{2\phi^2(a)\delta}{1-\gamma\delta}}
\]
where $\gamma = \frac{1}{1-\beta}$.

**Proof** Put $w = \sqrt{1-\beta}\phi^\gamma$ for $\phi \in \tilde{\Lambda}_\delta$. Then easy computation gives
\[
\int_0^\infty |w'|^2 dt = \int_0^\infty \frac{|\phi'|^2 \phi^\frac{2\delta}{1-\beta}}{1-\beta} dt \leq \int_0^\infty \frac{|\phi'|^2 t^\beta}{1-\beta} dt \leq \delta
\]
and
\[I := \int_a^\infty e^{\phi^2(t)-t} dt = \int_a^\infty e^{\gamma w^2(t)-t} dt.
\]
In one of the inequality above we have used (10) with $A = 0$. Now changing the variable as $x = t - a$ and $w(t) = \psi(x) + w(a)$ for all $x \geq 0$. Then it is easy to see that
\[
\int_0^\infty |\psi'|^2 dx \leq \delta.
\]
From (10) with $\beta = 0$, $A = 0$ we get $\psi^2(x) \leq \delta x$. Using this the functional changes as
\[
I = \int_a^\infty e^{\gamma w^2(t)-t} dt = e^{\gamma w^2(a)-a} \int_0^\infty e^{\gamma(2\psi(x)w(a)+\psi^2(x))} - x dx \\
\leq e^{\gamma w^2(a)-a} \int_0^\infty e^{2\gamma w(a)\psi(x)-(1-\gamma\delta)x} dx.
\]
Further changing the variable to $\chi(y) = \psi(x)$ and $y = (1 - \gamma\delta)x$, we get
\[
\int_0^\infty |\chi'(y)|^2 dy = \frac{1}{1 - \gamma\delta} \int_0^\infty |\psi'|^2 dx \leq \frac{\delta}{1 - \gamma\delta}.
\]
Therefore $\chi \in \Lambda_{\frac{\delta}{1 - \gamma\delta}}$ whenever $1 - \gamma\delta > 0$. The functional changes as
\[
I = e^{\gamma w^2(a)} \int_0^\infty e^{2\gamma w(a)y} \chi(y) - y dy = \frac{1}{1 - \gamma\delta} \int_0^\infty e^{2\gamma w(a)\chi(y) - y} dy.
\]
Finally applying Lemma 4, we obtain if $1 - \gamma\delta > 0$,
\[
I \leq e^{\phi^2(\gamma w^2(a))} \int_0^\infty e^{2\gamma w(a)\chi(y) - y} dy.
\]
This finishes the proof of the lemma. ■

3 Proof on the main theorem

Let $\tilde{f}_m \in H_{0,rad}^1(w_0, B)$ such that $|\nabla \tilde{f}_m|^2 w_0 \rightarrow \delta_0$. Then we know that $\tilde{f}_m \rightarrow 0$ in $H_{0,rad}^1(w_0, B)$. We further consider $f_m$ such that $J_\beta(\tilde{f}_m) \rightarrow J_\beta(0) = 1$. Define $f_m$, from $\tilde{f}_m$, using the same transformation introduced in (4) and (5).

Lemma 8 Let $f_m$ be as above. There exists $a_m$, the first point in $[1, \infty)$, with
\[
f_m^{\frac{2}{\beta}}(a_m) - a_m = -2\log(a_m) \tag{11}
\]
and also this $a_m \rightarrow \infty$ as $m \rightarrow \infty$.

Proof STEP 1: Existence of $a_m$

Since $f_m(t) \leq t^{\frac{1}{2\beta}}$, this implies that $f_m(t)^{\frac{2}{\beta}} - t \leq 0$ if $t \in [0, 1)$, while $-2\log(t) > 0$ if $t \in [0, 1)$. Therefore $f_m^{\frac{2}{\beta}}(t) - t < -2\log(t)$ which implies non existence of such $a_m$ satisfying (11) on the interval $[0, 1)$.

Now let us assume the non existence of such $a_m$’s in the interval $[1, \infty)$. This implies that $f_m^{\frac{2}{\beta}}(t) - t < -2\log(t)$ on $[1, \infty)$. Or in other words we have
\[
e^{f_m(t)^{\frac{2}{\beta}} - t} \leq \frac{1}{t^2}, \text{ if } t \in [1, \infty).
\]
One can use dominated convergence theorem, with the dominating function
\[
g(t) = \begin{cases} 
1 & \text{ in } (0, 1), \\
\frac{1}{t^2} & \text{ on } [1, \infty), 
\end{cases}
\]
to show that $I_\beta(f_m) \rightarrow 1$. This is a contradiction to our assumption.

STEP 2: $a_m \rightarrow \infty$
Given $K$ arbitrary large number. It suffices to show that for all $m \geq m_0$, one has $a_m \geq K$. First choose $\eta$ small, such that

$$\eta t < t - 2 \log(t), \quad \text{for all } t \in [0, K).$$

(12)

Now using the last lemma we get for $t \in [0, K)$ and $\forall \ m \geq m_0$,

$$f_m(t)^{\frac{2}{1-\beta}} \leq \left( \int_0^K \frac{|w'_m|^{2\beta}}{1-\beta} dt \right)^{\frac{1}{1-\beta}} \ t < \eta t \leq t - 2 \log(t).$$

This says that $a_m > K$ forall $m \geq m_0$. ■

**Lemma 9** [Estimate for Concentration level] For $\beta \in [0, 1)$ it implies that

$$J_{\beta}^2(0) \leq 1 + e.$$  

(13)

**Proof** First note that it is enough to consider concentrating sequences $\tilde{f}_m$ such that $J_{\beta}(\tilde{f}_m) \nRightarrow J_{\beta}(0) = 1$, because in this case the required inequality (13) is already satisfied.

**Step 1 :**

$$\lim_{m \to \infty} \int_0^a m e^{f_m(t)^{\frac{2}{1-\beta}} - t} dt = 1,$$

where $f_m$ and $a_m$ is as in the previous lemma.

Using Lemma 5 and (10) we notice that $f_m \to 0$ uniformly on compact subsets of $R^+$. Therefore for each $A, \epsilon > 0$, we have $f_m(t)^{\frac{2}{1-\beta}} \leq \epsilon$ for all $t \leq A$ and sufficiently large $m$. Using the property of $a_m$, that for all $t \leq a_m$ one has $f_m(t)^{\frac{2}{1-\beta}} \leq t - 2 \log(t)$, we get

$$\int_0^a m e^{f_m(t)^{\frac{2}{1-\beta}} - t} dt = \int_0^A e^{f_m(t)^{\frac{2}{1-\beta}} - t} dt + \int_A^a m e^{f_m(t)^{\frac{2}{1-\beta}} - t} dt$$

$$\leq e^{\epsilon} \int_0^A e^{-t} dt + \int_A^a m e^{2 \log(t)} dt$$

$$= e^{\epsilon} (1 - e^{-A}) + \left( \frac{1}{A} - \frac{1}{a_m} \right) \leq 1,$$

as $\epsilon \to 0$ and for large $A$. For the other way round

$$\int_0^a m e^{f_m(t)^{\frac{2}{1-\beta}} - t} dt \geq \int_0^a m e^{-t} dt = 1 - e^{-a_m} \to 1.$$

**Step 2 :** We claim that

$$\lim_{m \to \infty} \int_{a_m}^\infty e^{f_m(t)^{\frac{2}{1-\beta}} - t} dt \leq \epsilon.$$

Set

$$\delta_m = \int_{a_m}^\infty \frac{|f'_m|^{2\beta}}{1-\beta} dt.$$
Then using the relation (which is obtained from (10))

\[ f_{m}^{2\gamma}(t) \leq \left( \int_{0}^{t} \frac{|f'_{m}|^2 t^\beta}{1-\beta} \, dt \right)^\gamma t \]

one obtains

\[ \delta_{m} := 1 - \int_{0}^{a_{m}} \frac{|f'_{m}|^2 t^\beta}{1-\beta} \, dt \leq 1 - \left( \frac{f_{m}^{2\gamma}(a_{m})}{a_{m}} \right)^{\frac{1}{\gamma}} = 1 - \left( 1 - \frac{2 \log(a_{m})}{a_{m}} \right)^{\frac{1}{\gamma}}. \]  

(14)

In the last inequality we have used the property of the points \( a_{m} \).

Set \( \delta = \delta_{m} \), \( a = a_{m} \) and \( \phi = f_{m} \) in Lemma 7, then clearly

\[ \int_{a_{m}}^{\infty} e^{f_{m}(t)} \frac{2}{\gamma} - t \, dt \leq \frac{e^{K_{m}+1}}{1-\gamma\delta_{m}} \]

where

\[ K_{m} = \left( f_{m}^{2\gamma}(a_{m}) - a_{m} \right) + \frac{\delta_{m} \gamma f_{m}^{2\gamma}(a_{m})}{1-\gamma\delta_{m}}. \]

From the expression in (14), \( \delta_{m} \rightarrow 0 \) as \( m \rightarrow \infty \) and therefore \( 1 - \gamma\delta_{m} > 0 \) which is one of the requirement in Lemma 7. Clearly the lemma will be proved if we show

\[ \lim_{m \rightarrow \infty} K_{m} = 0. \]

First of all notice that \( K_{m} > 0 \). Now using Lemma 8 and \( f_{m}^{2\gamma}(t) \leq t \), we get

\[ K_{m} \leq -2 \log(a_{m}) + \frac{\delta_{m} a_{m} \gamma}{1-\gamma \delta_{m}}. \]

Using (14), it implies

\[ K_{m} \leq \frac{1}{\gamma_{m}} \left[ -2 \log(a_{m}) \gamma_{m} + \gamma a_{m} \left\{ 1 - \left( 1 - \frac{2 \log(a_{m})}{a_{m}} \right)^{\frac{1}{\gamma}} \right\} \right], \]

where \( \gamma_{m} = 1 - \gamma \delta_{m} = 1 - \gamma \left\{ 1 - \left( 1 - \frac{2 \log(a_{m})}{a_{m}} \right)^{\frac{1}{\gamma}} \right\} \). Note that \( \gamma_{m} \rightarrow 1 \) as \( m \rightarrow \infty \). The lemma will be proved if we show that the function, as \( x \rightarrow \infty \),

\[ -2 \log(x) + \gamma x \left\{ 1 - \left( 1 - \frac{2 \log(x)}{x} \right)^{\frac{1}{\gamma}} \right\} + 2 \gamma \log(x) \left\{ 1 - \left( 1 - \frac{2 \log(x)}{x} \right)^{\frac{1}{\gamma}} \right\} \rightarrow 0. \]

First let us consider the third part

\[ \lim_{x \rightarrow \infty} \log(x) \left\{ 1 - \left( 1 - \frac{2 \log(x)}{x} \right)^{\frac{1}{\gamma}} \right\} = \lim_{x \rightarrow \infty} \frac{1 - \left( 1 - \frac{2 \log(x)}{x} \right)^{\frac{1}{\gamma}}}{(\log(x))^{-1}}. \]
Using L'Hospital’s rule, we obtain
\[
\lim_{x \to \infty} 1 - \left(1 - \frac{2 \log(x)}{x}\right)^{\frac{1}{x}} = \frac{2}{\gamma} \lim_{x \to \infty} \left(1 - \frac{2 \log(x)}{x}\right)^{\frac{1}{x}} \log^2(x)(\log(x) - 1) \to 0.
\]

We consider the first and the second term together now,
\[
\lim_{x \to \infty} -2 \log(x) + \gamma x \left\{1 - \left(1 - \frac{2 \log(x)}{x}\right)^{\frac{1}{x}}\right\} = \lim_{x \to \infty} \left\{-2 \frac{\log(x)}{x} + \gamma \left\{1 - \left(1 - \frac{2 \log(x)}{x}\right)^{\frac{1}{x}}\right\}\right\}.
\]

Using L’Hospital rule again
\[
\lim_{x \to \infty} \frac{-2 \frac{\log(x)}{x} + \gamma \left\{1 - \left(1 - \frac{2 \log(x)}{x}\right)^{\frac{1}{x}}\right\}}{x^{-1}} = 2 \lim_{x \to \infty} (\log(x) - 1) \left\{1 - \left(1 - \frac{2 \log(x)}{x}\right)^{\frac{1}{x}}\right\} = 0.
\]

To see the last equality one has to to use the following inequality, to show that the limiting value is less than or equal to 0: If $\mu > 0$, then for small $x > 0$
\[
(1 - x)^{\mu} \geq 1 - (\mu + 1)x.
\]

The other side of the follows since for large $x$, it implies that
\[
(\log(x) - 1) \left\{1 - \left(1 - \frac{2 \log(x)}{x}\right)^{\frac{1}{x}}\right\} > 0.
\]

Summing up the inequalities in Step 1 and Step 2, we get
\[
\lim_{m \to \infty} J_\beta(\tilde{g}_m) = \lim_{m \to \infty} I_\beta(f_m) \leq 1 + e.
\]

Since $f_m$ is any arbitrary sequence this finishes the proof of the lemma. 

Now we present the proof of the main theorem.

**Proof of Theorem**

If possible let $J_\beta(\tilde{g}_m)$ does not converges to $J_\beta(\tilde{g}_0)$, where $\tilde{g}_m$, $\tilde{g}_0$ as in [3]. Then from previous lemma’s we know that
\[
M_\beta = \lim_{m \to \infty} J_\beta(\tilde{g}_m) \leq 1 + e.
\]

If we show that, there exist some $\phi \in \tilde{A}_1$ such that $J_\beta(\phi) > 1 + e$, then clearly $M_\beta > 1 + e$ and this would be a contradiction. Consider the function $f \in A_1$, defined as
\[
f(t) = \begin{cases} 
\frac{t}{2} & \text{in } 0 \leq t \leq 2, \\
(t - 1)^{\frac{1}{2}} & \text{on } 2 \leq t \leq e^2 + 1, \\
e & \text{on } t \geq e^2 + 1.
\end{cases}
\]
Set $\phi = f^{1-\beta}$. It has been verified in [6] that $f \in \tilde{\Lambda}_1$ and
\[
I_\beta(f) = \int_0^\infty e^{f^2(t) - t} dt = 1 + e + \varsigma^* > 1 + e
\]
for some $\varsigma^* > 0$. We are left to verify that $\phi \in \tilde{\Lambda}_1$. Since $\phi' = 0$ for $t \geq e^2 + 1$, we have
\[
\int_0^\infty \frac{|\phi'|^2 t^\beta}{1-\beta} dt = \int_0^2 \frac{|\phi'|^2 t^\beta}{1-\beta} dt + \int_2^{e^2+1} \frac{|\phi'|^2 t^\beta}{1-\beta} dt := I_1 + I_2.
\]
Now after simple calculation one obtains
\[
I_1 = (1-\beta) \int_0^2 f^{-2\beta} |f'|^2 t^\beta dt = 2^{\beta-1}
\]
and
\[
I_2 = \frac{(1-\beta)}{4} \int_2^{e^2+1} (t-1)^{-\beta-1} t^\beta dt = \frac{(1-\beta)}{4} \int_1^{e^2} \frac{(m+1)^{\beta}}{m^{\beta+1}} dm \leq \frac{(1-\beta)}{2}.
\]
In the last step above we have used that for $\beta > 0$, it implies that $\psi(\beta) \leq \psi(0) = 2$ where
\[
\psi(\beta) = \int_1^{e^2} \frac{(m+1)^{\beta}}{m^{\beta+1}} dm.
\]
Therefore
\[
\int_0^\infty \frac{|\phi'|^2 t^\beta}{1-\beta} dt = 2^{\beta-1} + \frac{1-\beta}{2} \leq 1, \forall \beta \in [0,1).
\]
This proves that $f \in \tilde{\Lambda}_1$.

Acknowledgments The research work of the author is supported by “Innovation in Science Pursuit for Inspired Research (INSPIRE)” under the IVR Number: 2014000099.

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