Fully coupled numerical simulation techniques for 3D hydraulic fracturing

V Borisov, A Ivanov, B Kritskiy, I Menshov, M Ramazanov and E Savenkov
Keldysh Institute of Applied Mathematics RAS, Moscow, Russian Federation
E-mail: e.savenkov@gmail.com

Abstract. In this paper a self-consistent mathematical model for large scale hydraulic fracture development in poroelastic medium is proposed together with appropriate simulation techniques. The proposed model is fully three-dimensional self-consistent and includes Biot poroelasticity equations, fracture fluid flow model, mechanical conditions of fracture development and interface conditions on lateral fracture surfaces. In the description of fracture development physically-consistent breakdown criteria are used taking into account the complex loading state in the neighborhood of the fracture front. The proposed set of algorithms is based on X-FEM approach for poroelastic medium and utilizes closest-point projection technique for both fracture geometrical representation and fracture flow solver. Numerical results which illustrate application of the developed numerical techniques are presented.

1. Introduction
In this work we present a mathematical model and simulation technique for the analysis of hydraulic fracture propagation (HF) in fully 3D setting and accounting for all major flow and geomechanical effects. The approach can be used for hydraulic fracture design as well as for analysis of such processes as production/injection stress re-orientation, spontaneous fracture growth during reservoir development etc.

The reservoir is described by 3D Biot poroelastic model which couples flow and mechanical processes in medium [1]. The fracture is described as an arbitrary sufficiently smooth surface with boundary. The flow in fracture is described by 2D Reynolds lubrication equation. Only simple Newtonian flow model will be considered but generalizations to more realistic rheological models are straightforward. These (sub)models (for the fracture and for the reservoir) are coupled by appropriate interface conditions which ensure continuity of mass and momentum fluxes at the fracture/reservoir interface. The complete model is fully coupled and self consistent. The numerical approach is based on finite elements method. To account for discontinuous displacement and pressure fields an eXtended Finite Element (X-FEM) method is used [2,3]. In contrast to most X-FEM implementations we do not use level set approach for fracture representation and evolution. Instead the so called closest point projection method [4] is utilized for both solution of the fracture flow Reynolds lubrication equations and fracture geometry representation and evolution in the X-FEM context. The vector version of J-integral for poroelastic medium was used for describing pointwise direction field of fracture evolution [5]. The unified way of HF representation provides flexibility and efficiency of the complete coupled algorithm. The complete system of equations is solved in a partitioned way which provides flexibility for applications.

The applicability of developed techniques is demonstrated by a number of simulation results.
2. Model of the media
Mechanical equilibrium equations in matrix medium are given by:

$$\nabla \cdot \mathbf{T} + \rho \mathbf{g} = 0,$$

with $\rho = \rho(\mathbf{x})$ being a density of fluid-saturated medium, $\mathbf{g}$ being gravity acceleration, $\mathbf{T}$ – Cauchy stress tensor. In the general anisotropic case and in linear approximation the constitutive relation is given by [1]:

$$\Delta \mathbf{T} = \mathbf{C} : \mathbf{E} - \mathbf{B} \Delta \rho, \Delta \mathbf{T} = \mathbf{B} : \mathbf{E} + \frac{1}{N} \Delta \rho$$

Here $p$ is fluid pressure, $\mathbf{E} = \left[ \nabla \otimes \mathbf{u} + (\nabla \otimes \mathbf{u})^T \right] / 2$ – deformation tensor, $\mathbf{u}$ – displacements field, $\mathbf{C} = \left[ C_{ijkl} \right]$ – symmetric tensor of elastic coefficients, $\mathbf{B} = \left[ B_{ij} \right]$ – symmetric Biot tensor, $\phi$ – porosity, $N$ – Biot coefficient that describes dependence of porosity on fluid pressure, $\Delta f = f - f_0$ for any variable $f$, $\rho = (1 - \phi) \rho_s + \phi \rho_f$, where $\rho_s, f$ is matrix and fluid density respectively.

Fluid mass conservation equation is given by:

$$\frac{\partial m_f}{\partial t} + \nabla \cdot (w_m) = q, \quad \mathbf{w} = \frac{\mathbf{w}_m}{\rho_f} = \mathbf{K} \cdot (-\nabla p + \rho_f \mathbf{g}),$$

where $m_f$ is fluid mass in a unit volume of a saturated medium, $\mathbf{w}_m = \rho_f \mathbf{v}$ – mass flux density vector, $\mathbf{v}$ – the filtration velocity determined by Darcy’s law, $\mathbf{K}$ – symmetric tensor of permeability coefficients, $q$ – mass density of sources.

Further we assume that

$$\frac{\Delta \rho_f}{\rho_f^0} = \frac{\Delta p}{K_f}, \quad \Delta m_f = \rho_f^0 v_f, \quad v_f = \mathbf{B} : \Delta \mathbf{E} + \frac{1}{M} \Delta \rho,$$

where $p$ is fluid pressure, $\rho_f$ – fluid density, $K_f$ – bulk fluid compression module, $v_f$ is relative change of fluid volume in an elementary volume of a saturated medium.

3. Fracture geometrical model
Geometrically fracture is considered as of zero thickness with mid-surface being a smooth and simply-connected manifold

$$\mathcal{F} : D_f \rightarrow \mathbb{R}^3, \quad D_f \subset \mathbb{R}^2,$$

with boundary (fracture front) $\partial \mathcal{F}$. At the time moment $t$ fracture mid-surface is denoted by $\mathcal{F}_t$. The condition $\mathcal{F}_{t_1} \subset \mathcal{F}_{t_2}$ is hold $t_1 \geq t_2$. At each time moment and for point on $\gamma(t) = \partial \mathcal{F}_t$ the fracture front velocity field is given by $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ so the fracture extension is described by the equation

$$\frac{d \mathbf{x}}{dt} = \mathbf{v}(\mathbf{x}, t), \quad \mathbf{x}_{b,0} = \mathbf{x}_0 \in \gamma(0),$$

where velocity field $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ is smooth enough function of point $\mathbf{x} \in \gamma(t)$ at each fixed time moment and a smooth function of time for each fixed (Lagrangian) point of the boundary.
4. Flow model in fracture
We assume that the fluid propagates along the fracture with a given geometry, trying to fill the whole one. Part of the middle surface $F$ of fracture corresponds only to a part of the fracture occupied by the liquid. Crack opening is a function of the point of surface $F$, $w = w(x)$, $x \in F$. Lateral fracture surfaces $F^\pm = x \pm n(x)w(x)/2$ are at a distance $w/2$ from $F$, where $n(x)$ – unit normal to $F$ at point $x$ directed to the fracture face $F^-$ (thus, $n^+ = n$, $n^- = -n$). Function $w$ is smooth, nonnegative and equals zero on $\partial F$.

The equation of flow of a compressible fluid in a fracture has the form:

$$\frac{\partial}{\partial t}(\rho_w v) = Q_f, \quad v = -D(\nabla_x p_f - \rho_f g_x),$$

where $v_x$ is the velocity of fluid flow in the fracture, averaged over the opening of the crack, coefficient $D$ is an effective permeability, $g_x$ is the projection of the vector of gravity acceleration on the mid-surface of fracture, $g_x = (1 - n \otimes n)g$.

In the case of a Newtonian fluid, $D = \mu_f / (12 \mu_f)$, where $\mu_f$ is dynamic viscosity of fluid.

Source $Q_f$ can be represented as $Q_f = Q^w_f + Q^\omega_f$. Here the first term describes the flow rate from the well to the fracture, the second one – the leakage of fluid into the poroelastic reservoir (see Section 5).

Equation (6) is considered in a bounded domain $\Omega_f = \Omega_j \subset F$ with boundary $\partial \Omega_f$. At $\partial \Omega_f$ the appropriate boundary conditions are to be set, see Section 6.

Equation (6) can be supplemented by an equation describing the proppant transport in the fracture [6]. This equation practically does not change the structure of the considered model and is not considered here.

5. Interface conditions
$F$ is jump discontinuity for the variables that describe reservoir state and can be determined by the boundary conditions. $x \in F$ denotes point on fracture middle surface, $x^\pm$ are the corresponding points on the fracture faces $F^\pm$.

The first group of interface conditions are kinematic conditions given by:

$$w(x) = (u^+ - u^-) \cdot n = [u] \cdot n = [u_n], \quad u_n^+ = u^+ \cdot n, \quad [u_n] = u_n^+ - u_n^-, \quad u^\pm = u(x^\pm).$$

Dynamic interface conditions are given by:

$$T^+ \cdot n^+ = -p^+ n^+ , \quad T^- \cdot n^- = -p^- n^-.$$

Consider now interface conditions for the hydrodynamics part of problem. They can be given in different forms, depending on the assumptions imposed on the fracture flow.

In the first case fracture and medium are considered to be saturated with the same fluid that occupies entire volume of the fracture ($\Omega_f = F$), and the hydrodynamic contact between fracture and the medium is ideal. In that case at each point $x \in F$ of the mid-surface the pressure on fracture faces are given by $p^+ = p^- = p$, with $p^\pm = p(x^\pm)$. The continuity of the material flux conditions with the corresponding expressions for the flow in the medium (2) and the form of equation (6), expressing the material balance in the fracture, have the form $Q_f = Q^w_f + Q^\omega_f$, $Q^\pm = w \cdot n^\pm |_{x=x^\pm}$ where $p^*$ is defined as $p^*(x) = p_f(x)$.
In the second case it is assumed that fluid in the fracture does not flow into the medium, i.e. the fracture volume is isolated. In this case it is natural to assume that \( \Omega_f \subset F \cup \partial \Omega_f = \emptyset \) (see figure 1). Interface conditions for the mass flux in domain \( \Omega_f \) are given by \( Q_f = 0, \ Q^\pm = 0, \ x \in \Omega_f \).

![Figure 1. Relative position of fracture mid-surface and fluid domain.](image)

In other words, the interface conditions degenerate into the Neumann boundary conditions for the flow equation in the medium. In this case conditions for pressure in the medium cannot be formulated, and the fracture pressure is given by (6) with non-zero right-hand side \( Q_f \) with \( p^* \) defined as \( p^*(x) = p_f(x) \).

Interface conditions in the domain \( F \setminus \Omega_f \) (“lag” domain) can be set in different ways depending on whether the fracture faces are permeable or not for the fluid in media. In the first case at points \( x \in F \setminus \Omega_f \) the relations \( Q^\pm = 0 \) and \( p^*(x) = 0 \) are satisfied. In the second case it is natural to consider that fluid in porous media fills fracture (sub)domain \( F \setminus \Omega_f \) without discontinuity in passing through the faces of the fracture, i.e. \( [p] = 0 \) \([Q] = 0\).

Finally, consider practically important case in which fluid does not penetrate medium, but contains component, that can be penetrate medium. Then it is usually assumed that the flows on the faces of the fracture are defined as functions of the fluid properties, the filtration-capacitive properties of the medium and the pressure fields in the fracture and its area. In the framework of the model under consideration, this leads to conditions \( Q_f = Q^+ + Q^- , \ Q^\pm = Q^\pm(p_f, p^\pm) = w^\pm_m \cdot n^\pm \). The last dependence can be given, for example, in accordance with the Carter leakage model or its generalizations [7].

6. Boundary conditions at the fracture front
During fluid injection in the fracture it starts to evolve. From mathematical point of view this corresponds to fracture mid-surface evolution. In what follows we assume that this evolution is quasi equilibrium, i.e., at each time moment reservoir medium and fracture are in mechanical equilibrium. The latter conditions define not only stress state of the reservoir but also fracture configuration as a part of the solution of the complete coupled problem.
In this work we assume that fracture evolution can be described by linear elastic fracture mechanics using energy fracture growth criterion. In this approach certain amount of energy can be assigned to unit element of the fracture surface which is interpreted as an amount of energy which is needed to form it. Correspondently, fracture mid-surface evolution produce an energy flux related to the fracture front. This considerations lead to the concept of Rice and Cherepanov \( J \)-integral of Cherepanov and Rice [8]. In the presented model we use vector valued \( J \)-integral [8], generalized to the poroelastic case [5].

Let us assume that at each \( x \in \partial F \) a vector valued function \( J = J(x) \), can be defined such that energy released when fracture propagates in direction given by unit vector \( \mathbf{l} \) (\( \| \mathbf{l} \| = 1 \)) is equal to \( J = J \cdot \mathbf{l} \). In accordance to energy growth criterion fracture evolves in the direction which maximize value of \( J \), 
\[
J = J_{\text{max}} = J(l_{\text{max}}).
\]
In the case when \( J_{\text{max}} \leq J_* \), where \( J_* \) is some critical value fracture growth does not occur. When \( J_{\text{max}} = J_* \) the fracture is in equilibrium. In the case when \( J_{\text{max}} > J_* \) fracture develops in an unstable way. Consequently, if the fracture growth occur, the growth direction is given \( \mathbf{J} \), i.e. one have \( \mathbf{v} = \alpha \mathbf{J} \), \( \alpha \geq 0 \) in (5).

At the same moment, linear elastic fracture mechanics under quasi-equilibrium conditions does not provide a way to determine fracture growth velocity in the case \( \alpha > 0 \). Hence, (5) is a kind of “kinematic” equation and has to be supplemented by equation which determines fracture front profile under the given conditions: \( J_{\text{max}} \leq J_* \). This equation is nonlocal (both in space and time) and introduce dependency of the future fracture configuration on the history of the process, its current configuration, problem (fracture and reservoir) solution at the given time in the whole domain of the interest, boundary conditions, etc.

Let us now consider the case of fluid filled fracture. Let at each moment of time \( t \) one have \( \Omega_f(t) \subset \mathcal{F} \) and \( \partial \Omega_f(t) \cap \partial \mathcal{F} = \emptyset \). Hence, at each point of the fluid front it holds \( w(x) \geq w_0 > 0 \), \( x \in \Gamma(t) \). Let \( \mathbf{n} \) be a unit normal to \( \partial \Omega_f \) tangent to the fracture mid-surface, \( \mathbf{V} \) be velocity of \( \Gamma(t) = \partial \Omega_f(t) \), \( \mathbf{v}_f \) averaged over fracture opening fluid particle velocity, \( \rho_f \) – fracture fluid density, \( Q_f \) – external volumetric inflow rate, \( q_f = Q_f / \rho_f \) – external mass flow rate. From (6) one have:
\[
\int_{\Omega_f(t)} \frac{\partial \rho_f}{\partial t} w ds + \int_{\Gamma(t)} \rho_f w v_n d \gamma = \int_{\Omega_f(t)} Q_f ds, \quad \mathbf{v}_n = \mathbf{v}_f \cdot \mathbf{n} \big|_{\Gamma(t)}.
\]
In this case fluid velocity normal component \( V_n \) is equal to the normal component of the fracture front velocity \( v_n \), i.e.
\[
\lim_{x \to x_f} v_n = V_n = \frac{dx_f}{dt} \cdot \mathbf{n} = \lim_{x \to x_f} \frac{q_f \cdot \mathbf{n}}{\rho_f w} = \mathbf{q}_f \cdot \mathbf{v}_f, \quad x_f \in \Gamma(t).
\]
Eq. (8) (called “speed equation”, see [9]) is quite obvious but nevertheless is fundamental one and relates fluid front velocity to fluid particles velocity. The tangential component of fluid velocity usually can be neglected which leads to \( \mathbf{V} = \mathbf{v} \). Note that eq. (8) fulfills even in the case when fracture opening is vanishing at the fluid front (that is exactly why we use \( x \to x_f \) and not \( x = x_f \)).

Now let us discuss boundary conditions for equation (6) keeping (8) in mind.

The first option is to define Dirichlet boundary condition for fracture fluid pressure at the fluid front, \( p_f(x) = p_f^*(x), \quad x \in \Gamma(t) \), with \( p_f^* \) being given function. In that case the fluid velocity at the front can be determined using (8). Temporal evolution of the fluid domain id described by equation
similar to (5) with fluid front velocity defined by (8). The value of \( p_f^* \) at the point of the fluid front can be set to zero when reservoir fluid does not enter fracture (“empty” lag zone). Otherwise it is natural to set to be equal to the reservoir pressure at the given point.

The second option assumes that at each point of the fluid front its average particle is defined (or, which the same, the mass flux at the fluid front, see (8)). Temporal evolution of the fluid front is described in the same way as above. The fluid front velocity in that case can be defined in an arbitrary way the velocity value has to be in agreement with mass flux at the boundary at fracture opening at the fluid front in accordance with (8).

Let us note that the first option is more natural in the considered case. Also note that there are two fronts are present in the model: the fracture and the fluid one.

To sum up let us consider the case when fluid occupies complete fracture volume, i.e., the situation when, when two fronts fracture and fluid ones coincide. From mathematical point of view such a case is the most complex for analysis. The reason is that, since fluid and fracture fronts coincides, opening is vanishing at the fluid front. Nevertheless the mass flux in (6) is finite since fluid front propagates with finite velocity. This leads to the singularity in the pressure field. The correct statement of the boundary conditions for fluid equations in fracture is nontrivial in this case. The “natural” form of boundary conditions is fluid front velocity which, in turn, is defined by fracture front velocity and fracture propagation criterion. Simultaneously, equation (8) can be considered not as definitive relation for fluid front velocity but rather as equation which relates fracture and fluid front velocities.

Finally let us note that (i) for practically important cases it is more important to account for lag itself but not for it precise value and (ii) consideration of lag is of the importance only in the relatively small neighborhood of the fluid front and at the initial stage of fracture development. This consideration usually allows obtaining lag values using some additional considerations or keeps being small but fixed in time and space quantity (see, e.g., [9]).

7. Numerical approach and results
The core of developed computational algorithm is eXtended Finite Elements Method (X-FEM) [2,3]. The method is based on the extension of a standard finite-element basis by special additional basis functions that allow one to correctly represent the discontinuous fields of the displacement, stresses and deformations on the middle surface of fracture. One important part of X-FEM method is an algorithm of fracture surface representing in a finite-dimensional problem, which should provide convenience of computing the all necessary quantities for the method implementation (for example, the stiffness matrix of the problem). In the modern version of X-FEM, the most commonly used is the level set method, see [10]. For the evolution of the level set functions equations of Hamilton-Jacobi type are used, see [11, 12].

In this paper we used an approach based on the method of the nearest point projection to describe the geometry of a surface. Algorithms for describing the operator evolution of the closest point projection are considered in [4]. In the context of fracture evolutions problems it is as general as the level set method, but it has a number of advantages when the boundary conditions on the lateral surfaces of a crack are determined by the solution of some equations (in our case, it is the equations of the lubricating layer). In addition, the method of closest point projection is currently a convenient tool for developing computational algorithms for solving equations on surfaces of arbitrary shape, see [13].

Thus, in this paper the CP projection method is used both to represent the fracture surface in X-FEM, and to solve the equation of the lubricating layer in the fracture. A description of the X-FEM / CP approach is presented in [14].

We present below some simulations results that illustrate the applicability of the presented algorithms.
On figure 2 an example of fracture mid-surface evolution is shown. On the left hand side panel initial fracture geometry is shown. The result of its evolution under prescribed velocity field is shown on center of figure. On right hand side panel we present stress components in the reservoir computed for
certain fracture configuration using algorithms discussed above. On the figure 3 (left) CP projection representation is shown where each FE mesh node is connected with its projection onto the surface and its boundary. On the right fracture pressure and opening filed evolution is shown obtained using finite element CP projection solver for surface PDEs.

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Figure 2. Fracture mid-surface evolution using CP projection method and solution for fractured reservoir.

Figure 3. CP projection fracture representation (left) and fracture flow parameters obtained using CP solver (right).
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