ON RANDOM SURFACE AREA

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Abstract. Consider a random smooth Gaussian field $G(x) : F \to \mathbb{R}$, where $F$ is a compact in $\mathbb{R}^d$. We derive a formula for average area of a surface generated by the equation $G(x) = 0$ and give some applications. As an auxiliary result we obtain an integral expression for area of a surface induced by zeros of a non-random smooth field.

Keywords: random Gaussian field, surface area, Favard measure, coarea formula, Rice formula.

1. Results

Consider a compact set $F \subset \mathbb{R}^d$. By $\partial F$ denote the boundary of $F$. We assume that the area of $\partial F$ is finite (the notion of area is defined below). Let $G(x) : F \to \mathbb{R}$ be a random Gaussian field. Put $m(x) = \mathbb{E}G(x)$ and $\sigma^2(x) = \text{Var}G(x)$. Here and below we assume that $\sigma(x) > 0$ for all $x \in F$ and $G \in C^1(F)$ a.s. It is known that the supremum of a continues Gaussian field defined on a compact is summable (see [10]). Therefore, by Kolmogorov’s Theorem on differentiation of mathematical expectations with respect to a parameter (see [4]), we have $m, \sigma \in C^1(F)$. Let $G'_i, \sigma'_i$ denote partial derivatives of $G, \sigma$ with respect to $i$th variable. By $\nabla$ denote a gradient of a function (a vector field whose components are partial derivatives).

Consider a zero set of the field $G$

$G^{-1}(0) = \{ x \in F | G(x) = 0 \}$.

With probability one $G^{-1}(0)$ is a compact smooth $(d - 1)$-dimensional submanifold in $\mathbb{R}^d$, i.e., a compact smooth surface.

The problem we are interested in is a calculation of average area of the surface $G^{-1}(0)$. Substituting $G/\sigma$ for $G$ does not change $G^{-1}(0)$. Therefore we may assume that $\sigma \equiv 1$. We prove that

$$\mathbb{E} \lambda_{d-1}[G^{-1}(0)] = \frac{1}{\sqrt{2\pi}} \int_F e^{-m^2(x)/2} \mathbb{E} \left\| \nabla G(x) \right\| \, dx. \quad (1)$$

For this purpose we derive an auxiliary formula for area of a surface generated by zeros of a non-random smooth field $g(x) : F \to \mathbb{R}$:

$$\lambda_{d-1}[g^{-1}(0)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} du \int_F \cos[ug(x)] \left\| \nabla g(x) \right\| \, dx. \quad (2)$$

Before we proceed with the exact results formulation, we need to define the notion of area. There exist several well-known definitions of area of a $(d - 1)$-dimensional submanifold in $\mathbb{R}^d$: a surface Lebesgue measure, a Hausdorff measure, a Favard measure. In general they are not equivalent. However in case of compact $C^1$-smooth manifolds all three definitions coincide. Therefore we may choose any one. To prove (2) the best choice for $\lambda_{d-1}$ is a Favard measure (for exact definition see Sect. 3). If $d = 1$, then by $\lambda_0(A)$ we denote the cardinality of a set $A$ (may be infinite).

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Recall that $F$ is supposed to be compact and $\lambda_{d-1}[\partial F] < \infty$.

**Theorem 1.** Suppose $g \in C^1(F)$ and

(a) $\lambda_{d-1}[(\nabla g)^{-1}(0)] < \infty$;
(b) $\lambda_{d-1}[g^{-1}(0) \cap \partial F] = 0$.

Then (2) holds.

**Remark.** The proof of Theorem 1 shows that it is possible to get rid of condition (b). Then (2) becomes

$$\lambda_{d-1}[g^{-1}(0)] - \frac{1}{2} \lambda_{d-1}[g^{-1}(0) \cap \partial F] = \frac{1}{2\pi} \int_{-\infty}^{\infty} du \int_F \cos[ug(x)] \|\nabla g(x)\| \, dx .$$

We shall not exploit this generalization at a later stage.

**Theorem 2.** Suppose a random field $G \in C^1(F)$ a.s. and

(a') $E \lambda_{d-1}[(\nabla G)^{-1}(0)] < \infty$;
(b') $\sigma(x) > 0$ for all $x \in F$.

Then

$$E \lambda_{d-1}[G^{-1}(0)] = \frac{1}{\sqrt{2\pi}} \int_F \exp \left\{ - \frac{\sigma^2(x)}{2\sigma^2(x)} \right\} E \left\| \nabla G(x) / \sigma(x) \right\| \, dx .$$

The proofs of the theorems are in Sect. 4. The auxiliary lemmas are in Sect. 3. The applications of Theorem 2 are in Sect. 2.

2. Applications of Theorem 2

2.1. Coarea formula.

**Example 1.** Suppose a function $g$ satisfies the conditions of Theorem 2. Then

$$\int_{-\infty}^{\infty} \lambda_{d-1}[g^{-1}(u)] \, du = \int_F \|\nabla g(x)\| \, dx .$$

**Proof.** Consider $G(x) = g(x) - \xi$, where $\xi$ is a Gaussian r.v. with $E\xi = 0$ and $D\xi = \sigma^2$. Then (3) becomes

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \lambda_{d-1}[g^{-1}(u)] e^{-\frac{u^2}{2\sigma^2}} \, du = \frac{1}{\sqrt{2\pi}} \int_F e^{-g(x)/(2\sigma^2)} \frac{\|\nabla g(x)\|}{\sigma} \, dx .$$

To obtain (4) it remains to multiply both sides by $\sqrt{2\pi\sigma^2}$ and apply the Monotone convergence theorem (as $\sigma \to \infty$). □

Relation (4) is called “the coarea formula”. It was obtained by H. Federer in 7.

2.2. Centered Gaussian field. By $S^{d-1}$ denote a $(d - 1)$-dimensional unit sphere with a Lebesgue measure $\mu_{d-1}(ds)$.

**Example 2.** If $G(x)$ satisfies the conditions of Theorem 2 and $m(x) \equiv 0$, then

$$E \lambda_{d-1}[G^{-1}(0)] = \frac{\Gamma(d+1)}{2\pi^{(d+1)/2}} \int_F \int_{S^{d-1}} \sqrt{s^T \Sigma(x)s} \mu_{d-1}(ds) ,$$

where $\Sigma(x)$ is a covariation matrix of $\nabla G(x) / \sigma(x)$.

**Proof.** The proof is by Lemma 7 (see Sect. 3) which we apply to (5). □

**Remark.** Relation (5) is easily extended to the case of $m(x) \equiv u$, $\sigma(x) \equiv 1$:

$$E \lambda_{d-1}[G^{-1}(0)] = \frac{\Gamma(d+1)}{2\pi^{(d+1)/2}} \int_F e^{-u^2/2} \, dx \int_{S^{d-1}} \sqrt{s^T \Sigma(x)s} \mu_{d-1}(ds) .$$
Remark. We have $\Sigma = \left(\frac{E G' G' - \sigma_i \sigma_j}{\sigma^2}\right)_{i,j=1}^d$.

2.3. Linear Gaussian field.

Example 3. Suppose $G(x) = \langle h(x), \xi \rangle$, where $h = (h^1, \ldots, h^n)^T : F \to \mathbb{R}^n$ is a vector function from the class $C^1(F)$ and $\xi$ is a $n$-dimensional centered Gaussian vector with the identity covariance matrix. Then

$$E \lambda_{d-1}[G^{-1}(0)] = \frac{\Gamma\left(\frac{d+1}{2}\right)}{2\pi^{d+1/2}} \int_F dx \int_{S^{d-1}} \|J_h(x)s\| \mu_{d-1}(ds),$$

where $J_h$ is the Jacobian $n$-by-$d$ matrix of $h/\|h\|$.

Proof. We have $\Sigma = J_h^T J_h$ in (6). □

Remark. If we consider a centered Gaussian vector with an arbitrary covariance matrix $\Lambda$, then $\Sigma = J_h^T \Lambda J_h$ and (6) becomes

$$E \lambda_{d-1}[G^{-1}(0)] = \frac{\Gamma\left(\frac{d+1}{2}\right)}{2\pi^{d+1/2}} \int_F dx \int_{S^{d-1}} \sqrt{\langle J_h(x)s \rangle \Lambda J_h(x)s} \mu_{d-1}(ds).$$

For $d = 1$ this formula was obtained by A. Edelman and E. Kostlan in [6, Theorem 3.1].

Corollary. Suppose under the conditions of Example 3 the rank of $J_h$ equals $k$. By $\sigma_1, \ldots, \sigma_k$ denote the nonzero singular values of the matrix $J_h$, i.e., the nonnegative square roots of the eigenvalues of $J_h J_h^T$. Then

$$E \lambda_{d-1}[G^{-1}(0)] = \frac{\Gamma\left(\frac{d+1}{2}\right)}{2\pi^{d+1/2}} \int_F dx \int_{S^{d-1}} \left(\sum_{j=1}^k \sigma_j(x)s_j^2\right)^{1/2} \mu_{d-1}(ds).$$

Proof. It is known from linear algebra (see, e.g., [5]) that the matrix $J_h$ may be written in the singular form $J_h = V Q W$, where $V, W$ are $n$-by-$n$ and $d$-by-$d$ unitary matrices. The $n$-by-$d$ matrix $Q$ is diagonal. The diagonal elements are the singular values of the matrix $J_h$. We have $\|J_h s\| = \|V Q W s\| = \|Q W s\|$.

To conclude the proof, it remains to apply this to (5) and make a change of variables $s' = W s$. □

Now we derive another form of $E \lambda_{d-1}[G^{-1}(0)]$ which will be useful for us later.

Example 4. Under the conditions of Example 3

$$E \lambda_{d-1}[G^{-1}(0)] = \frac{\Gamma\left(\frac{d+1}{2}\right)}{2\pi^{d+1/2}} \int_F dx \int_{S^{d-1}} \left(\sum_{i,j=1}^d \frac{|h|^2 \langle h_i^*, h_j^* \rangle - \langle h, h_i^* \rangle \langle h, h_j^* \rangle}{\|h\|^4} s_i s_j\right)^{1/2} \mu_{d-1}(ds),$$

where $h_i^*$ and $h_j^*$ are the $i$-th and $j$-th components of $h$. □
where
\[ h'_i = \left( \frac{\partial h_1}{\partial x_i}, \ldots, \frac{\partial h_n}{\partial x_i} \right)^\top. \]

Proof. We have
\[ \sigma = \|h\|, \quad \mathbb{E} G'_i G'_j = \langle h'_i, h'_j \rangle, \quad \sigma'_i = \|h\|^{-1}\langle h, h'_i \rangle. \]
It remains to apply (7). \(\square\)

2.4. Zeros of random polynomial.

Example 5. Consider \(G(t) = \xi_n t^n + \cdots + \xi_1 t + \xi_0, t \in F \subset \mathbb{R}\), where \(\{\xi_i\}\) are independent standard Gaussian random variables. Then
\[ \mathbb{E} \lambda_0[G^{-1}(0)] = \frac{1}{\pi} \int_F \frac{[A_n(t)C_n(t) - B_n^2(t)]^{1/2}}{A_n(t)} \, dt, \]
where
\[ A_n(t) = \sum_{j=0}^n t^{2j}, \quad B_n(t) = \sum_{j=0}^n j t^{2j-1}, \quad C_n(t) = \sum_{j=0}^n j^2 t^{2j-2}. \]

Proof. The proof follows from (9). \(\square\)

This formula was obtained by M. Kac in [8]. He also derived the asymptotic relation
\[ \mathbb{E} \lambda_0[G^{-1}(0)] \approx \frac{2}{\pi} \log n \cdot (1 + o(1)), \quad n \to \infty, \]
for \(F = [-\infty, \infty]\).

2.5. Random algebraic surface.

Example 6. Consider \(G(x) = \sum \xi_\alpha x^\alpha\), where \(\alpha = (\alpha_1, \ldots, \alpha_d)\) is a multi-index, the summation is taken over all such \(\alpha\) as such that \(0 \leq \alpha_j \leq n\), and \(\xi_\alpha\) are independent standard Gaussian random variables. Then
\[ \mathbb{E} \lambda_{d-1}[G^{-1}(0)] \]
\[ = \Gamma(\frac{d+1}{2}) \frac{2^{d-1}}{2\pi^{(d+1)/2}} \int_F \int_{S^{d-1}} \left( \sum_{i=1}^d \frac{A_n(x_i)C_n(x_i) - B_n^2(x_i)}{A_n^2(x_i)} \right)^{1/2} \mu_{d-1}(ds). \]

Proof. Using the notations of Subsection 2.3, we get
\[ \|h(x)\|^2 = \sum_{\alpha} x_\alpha^2 = \prod_{k=1}^d A_n(x_k), \]
\[ \langle h(x), h'_i(x) \rangle = \frac{1}{2} \frac{\partial}{\partial x_i} \|h(x)\|^2 = B_n(x_i) \prod_{k \neq i} A_n(x_k) \]
and
\[ \langle h'_i(x), h'_j(x) \rangle = \sum_{\alpha} \alpha_i x_\alpha^{n-\epsilon_i} x_\alpha^{n-\epsilon_j} = \begin{cases} B_n(x_i)B_n(x_j) \prod_{k \neq i,j} A_n(x_k) & \text{for } i \neq j, \\ C_n(x_i) \prod_{k \neq i} A_n(x_k) & \text{for } i = j, \end{cases} \]
where \(\epsilon_i\) denotes the multi-index in which the \(i\)-th position is occupied by one and all the other positions are occupied by zeros. These relations imply that for \(i \neq j\)
\[ \|h\|^2 \langle h'_i, h'_j \rangle - \langle h, h'_i \rangle \langle h, h'_j \rangle = 0 \]
and for \(i = j\)
\[ \|h\|^2 \langle h'_i, h'_i \rangle - \langle h, h'_i \rangle \langle h, h'_i \rangle = \|h\|^4 \frac{A_n(x_i)C_n(x_i) - B_n^2(x_i)}{A_n^2(x_i)}. \]
It remains to apply (9). \(\square\)
Formula (10) was obtained by I.A. Ibragimov and S.S. Podkorytov in [2]. They also derived the asymptotic relation

\[ \mathbb{E} \lambda_{d-1}[G^{-1}(0)] = \frac{\log d}{\pi} \lambda_{d-1}[F \cap \Gamma] \cdot (1 + o(1)), \quad n \to \infty, \]

where

\[ \Gamma = \bigcup_{j=1}^{d} \{ x \mid |x_j| = 1 \}, \]

provided that \( \lambda_{d-1}[\partial F \cap \Gamma] = 0. \)

2.6. Random surface of Kostlan-Shub-Smale.

**Example 7.** Consider \( G(x) = \sum_{\alpha} \xi_{\alpha} x^{\alpha} \), where the summation is taken over all nonnegative \( \alpha \) such that \( \alpha_1 + \cdots + \alpha_d \leq n \) and \( \xi_{\alpha} \) are independent Gaussian random variables with \( \mathbb{E} \xi_{\alpha} = 0 \) and \( \mathbb{D} \xi_{\alpha} = C_n^\alpha \), where

\[ C_n^\alpha = \frac{n!}{\alpha_1! \cdots \alpha_d! (n - \alpha_1 - \cdots - \alpha)!}. \]

Then

\[ \mathbb{E} \lambda_{d-1}[G^{-1}(0)] = \sqrt{n} \frac{\Gamma(\frac{d+1}{2})}{2\pi^{(d+1)/2}} \int_F \frac{dx}{1 + \|x\|^2} \int_{S_{d-1}} \sqrt{1 + \|x\|^2 - \langle x, s \rangle^2} \mu_{d-1}(ds). \]

**Proof.** Using the notations of Subsection 2.3, we get

\[ \|h(x)\|^2 = \sum_{\alpha} C_n^\alpha x^{2\alpha} = \left( 1 + \sum_{k=1}^{d} x_k^2 \right)^n, \]

\[ \langle h(x), h'(x) \rangle = \frac{1}{2} \frac{\partial}{\partial x_i} \|h(x)\|^2 = nx_i \left( 1 + \sum_{k=1}^{d} x_k^2 \right)^{n-1}. \]

For \( i \neq j \)

\[ \langle h'_i(x), h'_j(x) \rangle = \sum_{\alpha} C_n^\alpha \alpha_i x^{\alpha - \epsilon_i} \alpha_j x^{\alpha - \epsilon_j} = n(n-1)x_i x_j \sum_{\alpha} C_{n-2}^{\alpha - \epsilon_i - \epsilon_j} x^{2\alpha - 2\epsilon_i - 2\epsilon_j} = n(n-1)x_i x_j \left( 1 + \sum_{k=1}^{d} x_k^2 \right)^{n-2} \]

and for \( i = j \)

\[ \langle h'_i(x), h'_j(x) \rangle = \sum_{\alpha} C_n^\alpha \alpha_i x^{\alpha - \epsilon_i} \alpha_i x^{\alpha - \epsilon_i} = \sum_{\alpha} C_n^\alpha \alpha_i x^{2\alpha - 2\epsilon_i} + \sum_{\alpha} C_n^\alpha (\alpha_i - 1)x^{2\alpha - 2\epsilon_i} = n \sum_{\alpha} C_{n-1}^{\alpha - \epsilon_i} x^{2\alpha - 2\epsilon_i} + n(n-1)x_i^2 \sum_{\alpha} C_{n-2}^{\alpha - 2\epsilon_i} x^{2\alpha - 4\epsilon_i} = n \left( 1 + \sum_{k=1}^{d} x_k^2 \right)^{n-1} + n(n-1)x_i^2 \left( 1 + \sum_{k=1}^{d} x_k^2 \right)^{n-2}. \]

These relations imply that for \( i \neq j \)

\[ \|h\|^2 \langle h'_i, h'_j \rangle - \langle h, h'_i \rangle \langle h, h'_j \rangle = -n \left( 1 + \sum_{k=1}^{d} x_k^2 \right)^{2n-2} x_i x_j. \]
and for \( i = j \)

\[
\|h\|^2 \langle h'_i, h'_j \rangle - \langle h, h'_i \rangle \langle h, h'_j \rangle = n \left( 1 + \sum_{k=1}^{d} x_k^2 \right)^{2n-2} \left( 1 + \sum_{k \neq i}^{d} x_k^2 \right).
\]

Therefore, using (9) we get

\[
\mathbb{E}\lambda_{d-1}[G^{-1}(0)] = \frac{\Gamma\left(\frac{d+1}{2}\right)}{2\pi^{(d+1)/2}} \sqrt{n} \int_{F} \left( 1 + \sum_{k=1}^{d} x_k^2 \right)^{-1} dx
\]

\[
\times \int_{\mathbb{R}^{d-1}} \left( - \sum_{i \neq j}^{d} x_i x_j s_i s_j + \sum_{i=1}^{d} \left( 1 + \sum_{k \neq i}^{d} x_k^2 \right) s_i^2 \right)^{1/2} \mu_{d-1}(ds)
\]

\[
= \frac{\Gamma\left(\frac{d+1}{2}\right)}{2\pi^{(d+1)/2}} \sqrt{n} \int_{F} (1 + \|x\|^2)^{-1/2} dx \int_{\mathbb{R}^{d-1}} \sqrt{1 + \|x\|^2} dx \mu_{d-1}(ds).
\]

\[\square\]

Remark. Thus,

\[
\mathbb{E}\lambda_{d-1}[G^{-1}(0)] = C_F \sqrt{n},
\]

where \( C_F \) depends only on \( F \) and \( d \). M. Shub and S. Smale obtained a similar result for the number of zeros of a system of \( d \) polynomials in \([15]\).

Corollary. For \( d = 1 \) we get

\[
\mathbb{E}\lambda_{0}[G^{-1}(0)] = \sqrt{n} \int_{F} \frac{dx}{\pi(1 + x^2)}.
\]

This relation was obtained by E. Kostlan in \([9]\).

2.7. Random trigonometric surface. By \(|F|\) denote a volume of \( F \) (i.e., a Lebesgue measure in \( \mathbb{R}^d \)).

Example 8. Consider

\[
G(x) = \sum_{\alpha} [\xi_\alpha \cos(\alpha, x) + \eta_\alpha \sin(\alpha, x)],
\]

where the summation is taken over all \( \alpha \) such that \( 0 \leq \alpha_j \leq n \) and \( \xi_\alpha, \eta_\alpha \) are independent standard Gaussian random variables. Then

\[
\mathbb{E}\lambda_{d-1}[G^{-1}(0)] = n \frac{\Gamma\left(\frac{d+1}{2}\right)}{4\pi^{(d+1)/2}} |F| \int_{\mathbb{R}^{d-1}} \left( s_1 + \cdots + s_d \right)^{1/2} \mu_{d-1}(ds).
\]

Proof. Using the notations of Subsection 2.2 we get

\[
\|h(x)\|^2 = (n + 1)^d, \quad \langle h(x), h'_i(x) \rangle = \frac{1}{2} \frac{\partial}{\partial x_i} \|h(x)\|^2 = 0
\]

and

\[
\langle h'_i(x), h'_j(x) \rangle = \sum_{\alpha} \alpha_i \alpha_j = \begin{cases} (n + 1)^{d-2} \left( \frac{n(n+1)}{2} \right)^2 & \text{for } i \neq j, \\ (n + 1)^{d-1} \frac{n(n+1)(2n+1)}{6} & \text{for } i = j. \end{cases}
\]

It remains to apply (9). \[\square\]

Corollary (1).

\[
\mathbb{E}\lambda_{d-1}[G^{-1}(0)] = c_d |F| n \cdot (1 + o(1)), \quad n \to \infty,
\]

where \( c_d \) depends only on the dimension \( d \).
Corollary (2). For $d = 1$ we get
\[ E \lambda_0[G^{-1}(0)] = \frac{1}{\pi} |F| \sqrt{\frac{n(2n+1)}{6}}. \]

This formula was obtained by C. Qualls in [11].

2.8. Level sets of homogeneous Gaussian field.

Example 9. Let $G(x)$ be a homogeneous Gaussian field with a spectral measure $\nu$. Suppose $\nu$ satisfies the conditions of Theorem 7. For the sake of simplicity, we assume that $m(x) \equiv 0$ and $\sigma(x) \equiv 1$. Then
\[ E \lambda_{d-1}[G^{-1}(u)] = \frac{\Gamma(d+1)}{2\pi^{(d+1)/2}} |F| e^{-u^2/2} \int_{S_{d-1}} \left( \int_{R^d} \langle s, z \rangle^2 \nu(dz) \right)^{1/2} \mu_{d-1}(ds). \]

Proof. By the spectral representation theorem,
\[ E G(x)G(y) = \int_{R^d} e^{i(x-y,z)} \nu(dz). \]
Differentiating this twice and putting $x = y = 0$, we get
\[ E G_i'(0)G_j'(0) = \int_{R^d} z_i z_j \nu(dz). \]
Applying (6) to $G(x) - u$, we obtain
\[ E \lambda_{d-1}[G^{-1}(u)] = \frac{\Gamma(d+1)}{2\pi^{(d+1)/2}} |F| e^{-u^2/2} \int_{S_{d-1}} \left( \int_{R^d} \langle s, z \rangle^2 \nu(dz) \right)^{1/2} \mu_{d-1}(ds). \]

\[ \Box \]

Corollary (1). We have
\[ \frac{1}{\pi} \gamma_1 e^{-u^2/2} |F| \leq E \lambda_{d-1}[G^{-1}(0)] \leq \frac{\Gamma(d+1)}{\sqrt{\pi} \Gamma(\frac{d}{2})} \gamma_2 e^{-u^2/2} |F|, \]
where
\[ \gamma_k = \left( \int_{R^d} \| z \|^k \nu(dz) \right)^{1/k}. \]

Proof. By Jensen’s inequality, Fubini’s theorem and Lemma 2 (see Sect. 3), we get
\[ \int_{S_{d-1}} \left( \int_{R^d} \langle s, z \rangle^2 \nu(dz) \right)^{1/2} \mu_{d-1}(ds) \leq \int_{S_{d-1}} \mu_{d-1}(ds) \int_{R^d} \| \langle s, z \rangle \| \nu(dz) = \int_{R^d} \nu(dz) \int_{R^d} \| \langle s, z \rangle \| \mu_{d-1}(ds) = \frac{2\pi^{(d-1)/2}}{\Gamma(d/2)} \| z \| \nu(dz) = \frac{2\pi^{(d-1)/2}}{\Gamma(d/2)} \gamma_1. \]
On the other hand, it follows from the Cauchy—Schwarz inequality that $\| \langle s, z \rangle \| \leq \| s \| \| z \| = \| z \|$. Therefore,
\[ \int_{S_{d-1}} \left( \int_{R^d} \langle s, z \rangle^2 \nu(dz) \right)^{1/2} \mu_{d-1}(ds) \leq \omega_{d-1} \left( \int_{R^d} \| z \|^2 \nu(dz) \right)^{1/2} \leq \frac{2\pi^{d/2}}{\Gamma(d/2)} \gamma_2. \]

\[ \Box \]
Corollary (2). For \( d = 1 \) we get
\[
E \lambda_0[G^{-1}(u)] = \frac{72 \pi^2}{\pi} u^2/2 |F|.
\]

This formula was obtained by S. O. Rice in [12].

3. Auxiliary lemmas

Let us recall that to define a \((d - 1)\)-dimensional Favard measure of a set \( A \), project it onto a \((d - 1)\)-dimensional linear hyperplane, take the Lebesgue measure (counting multiplicities), average over all such projections, and normalize properly:

\[
\lambda_{d-1}[A] = \frac{ \Gamma\left(\frac{d+1}{2}\right) }{2^{d-1} \pi^{\frac{d-1}{2}}} \int_{S^{d-1}} \mu_{d-1}(ds) \int_{s^\perp} \lambda_0\left(s^\perp \cap A\right) dy,
\]

where \( s^\perp \) is the linear hyperplane orthogonal to the unit vector \( s \in S^{d-1} \) and \( \{s^\perp\}^\perp \) is the line through \( y \in s^\perp \) orthogonal to \( s^\perp \).

Let us introduce the notations which we shall use in this section. Put
\[
M = \sup_{R > 0} \left| \int_{-R}^{R} \sin t \int_{-R}^{R} \frac{1}{u} du \right|.
\]

It follows from Lemma 1 (see below) that \( M < \infty \). By \( \omega_k \) denote area of a \( k \)-dimensional sphere:
\[
\omega_k = \frac{2 \pi^{(k+1)/2}}{\Gamma\left(\frac{k+1}{2}\right)}.
\]

Throughout this section we assume that a function \( g \) satisfies the conditions of Theorem 1. By \( g'_s \) denote a partial derivative of \( g \) with respect to the direction \( s \in S^{d-1} \).

Lemma 1. For all \( t \in \mathbb{R} \)
\[
\frac{1}{\pi} \int_{-\infty}^{\infty} \sin tu \frac{1}{u} du = \text{sign } t.
\]

Proof. See, i.e., [1]. \( \square \)

Lemma 2. For all \( x \in \mathbb{R}^d \)
\[
\int_{S^{d-1}} |(x, s)| \mu_{d-1}(ds) = \frac{2 \pi^{(d-1)/2}}{\Gamma\left(\frac{d+1}{2}\right)} |x|.
\]

Proof. Omit the trivial case when \( x = 0 \). Consider a Borel set \( A \) such that \( A \subset x^\perp \) and \( \lambda_{d-1}[A] = |x| \). Let us apply (11). It is clear that the integrand \( \int_{s^\perp} \lambda_0\left\{s^\perp \cap A\right\} dy \) is equal to area of the projection of \( A \) onto the linear hyperplane \( s^\perp \). On the other hand, if we project a set from one hyperplane to another, then area of the set multiplies by the cosine of the angle between the hyperplanes. Therefore,
\[
\int_{s^\perp} \lambda_0\left\{s^\perp \cap A\right\} dy = \lambda_{d-1}[A] \cdot \left| \frac{x}{\|x\|}, s \right| = |(x, s)|.
\]

Applying this to (11) and replacing \( \lambda_{d-1}[A] \) by \( |x| \), we obtain (13). \( \square \)

The next lemma is due to M. Kac (see, e.g., [3]).

Lemma 3. If \( f(t) \) continuous for \( a \leq t \leq b \) and continuously differentiable for \( a < t < b \) has a finite number of turning points (i.e., only a finite number of points at which \( f'(t) \) vanishes in \((a, b)\)) then the number of zeros of \( f(t) \) in \((a, b)\) is given by the formula
\[
\lambda_0[f^{-1}(0)] = \frac{1}{2 \pi} \int_{-\infty}^{\infty} du \int_{a}^{b} \cos[u f(t)] |f'(t)| dt.
\]

Multiple zeros are counted once and if either $a$ or $b$ is a zero it counted as 1/2.

**Remark.** This statement can be easily extended to the case of the union of a finite number of intervals. We shall use this form in the sequel.

**Proof.** For the reader’s convenience we present the proof from [3]. Let $\alpha_1, \ldots, \alpha_k$ be the abscissas of the turning points:

$$a = \alpha_0 < \alpha_1 < \cdots < \alpha_k \leq \alpha_{k+1} = b.$$  

We have

$$\int_a^b \cos[u f(t)] \, |f'(t)| \, dt = \sum_{j=0}^k \int_{\alpha_j}^{\alpha_{j+1}} \cos[u f(t)] \, |f'(t)| \, dt$$

$$= \sum_{j=0}^k \left\{ \pm \int_{\alpha_j}^{\alpha_{j+1}} \cos[u f(t)] \, f'(t) \, dt \right\} = \sum_{j=0}^k \left\{ \pm \sin[u f(\alpha_{j+1})] - \sin[u f(\alpha_j)] \right\},$$

where the sign $+$ is attached if $f(t)$ is increasing between $\alpha_j$ and $\alpha_{j+1}$ and the sign $-$ if it is decreasing. Thus using (12) we have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} du \int_a^b \cos[u f(t)] \, |f'(t)| \, dt$$

$$= \sum_{j=0}^k \left\{ \pm \frac{1}{2\pi} \int_{-\infty}^{\infty} \sin[u f(\alpha_{j+1})] - \sin[u f(\alpha_j)] \, du \right\}$$

$$= \sum_{j=0}^k \left\{ \pm \frac{\text{sign}[f(\alpha_{j+1})] - \text{sign}[f(\alpha_j)]}{2} \right\} = \lambda_0[f^{-1}(0)].$$

□

**Lemma 4.** If $f(t)$ continuous for $a \leq t \leq b$ and continuously differentiable for $a < t < b$ has $k$ turning points, then uniformly for $R > 0$

$$\left| \int_{-R}^{+R} du \int_a^b \cos[u f(t)] \, |f'(t)| \, dt \right| \leq 2M(k + 1).$$

**Proof.** In the same way as in Lemma 3 we have

$$\left| \int_{-R}^{+R} du \int_a^b \cos[u f(t)] \, |f'(t)| \, dt \right|$$

$$= \left| \sum_{j=0}^k \left\{ \pm \int_{-R}^{+R} \frac{\sin[u f(\alpha_{j+1})] - \sin[u f(\alpha_j)]}{u} \, du \right\} \right|$$

$$= \left| \sum_{j=0}^k \pm \left\{ \int_{-R f(\alpha_{j+1})}^{+R f(\alpha_j)} \frac{\sin u}{u} \, du - \int_{-R f(\alpha_j)}^{-R f(\alpha_{j+1})} \frac{\sin u}{u} \, du \right\} \right|$$

$$\leq 2(k + 1) \sup_{t \in \mathbb{R}} \left| \int_{-t}^{+t} \frac{\sin u}{u} \, du \right| = 2M(k + 1).$$

□

**Corollary.** If we replace $[a, b]$ by a set $H$ consisting of the union of $l$ intervals, then uniformly for $R > 0$

$$\left| \int_{-R}^{+R} du \int_H \cos[u f(t)] \, |f'(t)| \, dt \right| \leq 2M(k + l).$$

(15)
Lemma 6. The following inequality holds:

$$
\int_{S^{d-1}} \lambda_{d-1}[g'_s^{-1}(0)] \mu_{d-1}(ds) \leq \omega_{d-1}\lambda_{d-1}[(\nabla g)^{-1}(0)] + \omega_{d-2}|F|.
$$

Proof. We have

$$
\int_{S^{d-1}} \lambda_{d-1}[g'_s^{-1}(0)] \mu_{d-1}(ds) = \int_{S^{d-1}} \mu_{d-1}(ds) \int_F 1\{g'_s(y) = 0\} \lambda_{d-1}(dy)
$$

$$
\leq \omega_{d-1}\lambda_{d-1}[(\nabla g)^{-1}(0)] + \int_{S^{d-1}} \mu_{d-1}(ds) \int_{F \setminus (\nabla g)^{-1}(0)} 1\{g'_s(y) = 0\} \lambda_{d-1}(dy).
$$

It remains to estimate the second summands. If $\nabla g(y) \neq 0$, then the set $S(y) = \{s \in S^{d-1} | g'_s(y) = 0\}$ is contained in a unit hypersphere of the sphere $S^{d-1}$ orthogonal to $\nabla g(y)$. Consequently $\lambda_{d-2}[S(y)] \leq \omega_{d-2}$ and by Fubini's theorem,

$$
\int_{S^{d-1}} \mu_{d-1}(ds) \int_{F \setminus (\nabla g)^{-1}(0)} 1\{g'_s(y) = 0\} \lambda_{d-1}(dy)
$$

$$
= \int_{F \setminus (\nabla g)^{-1}(0)} dx \int_{S^{d-1}} 1\{f'_s(x) = 0\} \mu_{d-2}(ds)
$$

$$
= \int_{F \setminus (\nabla g)^{-1}(0)} \lambda_{d-2}[S(y)] dx \leq \int_{F \setminus (\nabla g)^{-1}(0)} \omega_{d-2} dx = \omega_{d-2}|F|.
$$

□

Lemma 6. For all $R > 0$

$$
\int_{S^{d-1}} \mu_{d-1}(ds) \int_{\{y, s = 0\}} dy \int_{\{y + ts \in F\}} du \cos[u g(y + ts)] |g'_s(y + ts)| dt
$$

$$
\leq 2M \left( \omega_{d-1}\lambda_{d-1}[(\nabla g)^{-1}(0)] + \omega_{d-2}|F| + \frac{\pi^{(d-1)/2}}{\Gamma(\frac{d+1}{2})} \lambda_{d-1}[\partial F] \right)
$$

and

$$
\left| \int_{\{y + ts \in F\}} \frac{du}{\int_{F} \cos[u g(x)] \|\nabla g(x)\| dx} \right|
$$

$$
\leq \frac{\Gamma(\frac{d+1}{2})}{\pi^{(d+1)/2}} M \left( \omega_{d-1}\lambda_{d-1}[(\nabla f)^{-1}(0)] + \omega_{d-2}|F| + \frac{\pi^{(d-1)/2}}{\Gamma(\frac{d+1}{2})} \lambda_{d-1}[\partial F] \right).
$$

Proof. By $k(s, y)$ denote the number of zeros of $g'_s(y + ts)$ (may be infinite) in the set $\{t | y + ts \in F\}$ and by $l(s, y)$ denote the number of intervals of this set (if the set is not the union of a finite number of intervals, then we put $l(s, y) = \infty$). It follows from (15) that

$$
\left| \int_{\{y + ts \in F\}} \frac{du}{\int_{F} \cos[u g(x)] \|\nabla g(x)\| dx} \right|
$$

$$
\leq \frac{\Gamma(\frac{d+1}{2})}{\pi^{(d+1)/2}} M \left( \omega_{d-1}\lambda_{d-1}[(\nabla f)^{-1}(0)] + \omega_{d-2}|F| + \frac{\pi^{(d-1)/2}}{\Gamma(\frac{d+1}{2})} \lambda_{d-1}[\partial F] \right).
$$

If we project the set $g'_s^{-1}(0)$ onto the hyperplane $\{y | y, s = 0\}$, then $k(s, y)$ is equal to the multiplicity of the projection at the point $y$. A measure does not increase under the action of projection, therefore

$$
\int_{\{y, s = 0\}} k(s, y) dy \leq \lambda_{d-1}[g'_s^{-1}(0)],
$$

which together with Lemma 5 implies

$$
\int_{S^{d-1}} \mu_{d-1}(ds) \int_{\{y, s = 0\}} k(s, y) dy \leq \omega_{d-1}\lambda_{d-1}[(\nabla g)^{-1}(0)] + \omega_{d-2}|F|.
$$
Further, applying the definition of a Favard measure to the boundary of $F$, we get

\begin{equation}
\int_{S^{d-1}} \mu_{d-1}(ds) \int_{\{y,s\}=0} 2I(s,y) \, dy = \frac{2\pi^{(d-1)/2}}{\Gamma\left(\frac{d+1}{2}\right)} \lambda_{d-1} \partial F.
\end{equation}

Combining (13), (19) and (20) we obtain (16).

Let us prove (17). It follows from (13) that

$$\|\nabla g(x)\| = \frac{\Gamma\left(\frac{d+1}{2}\right)}{2\pi^{(d-1)/2}} \int_{S^{d-1}} \|\nabla g(x), s\| \mu_{d-1}(ds).$$

Consequently, using Fubini’s Theorem we get

$$\left| \int_{-R}^{R} du \int_{F} \cos[ug(x)] \|\nabla g(x)\| \, dx \right| = \frac{\Gamma\left(\frac{d+1}{2}\right)}{2\pi^{(d-1)/2}} \times \int_{-R}^{R} du \int_{S^{d-1}} \cos[ug(x)] \|\nabla g(x), s\| \mu_{d-1}(ds) \right| = \frac{\Gamma\left(\frac{d+1}{2}\right)}{2\pi^{(d-1)/2}} \times \int_{S^{d-1}} \mu_{d-1}(ds) \int_{\{y,s\}=0} dy \int_{-R}^{R} du \int_{\{y+ts\in F\}} \cos[ug(y+ts)] |g_t'(y+ts)| \, dt \right|.
$$

To complete the proof it remains to apply (16).

\begin{lemma}
Consider an $n$-dimensional centered Gaussian vector $\xi$ with a covariance matrix $\Sigma$. Then

$$E \|\xi\| = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{2\pi}^{d+1}} \int_{S^{d-1}} \sqrt{s_\Sigma} \mu_{d-1}(ds).$$

\end{lemma}

\begin{proof}
It follows from (13) and Fubini’s theorem that

$$E \|\xi\| = \frac{\Gamma\left(\frac{d+1}{2}\right)}{2\pi^{(d-1)/2}} \int_{S^{d-1}} E \|\xi, s\| \mu_{d-1}(ds).$$

Moreover,

$$E \|\xi, s\| = E \|N(0,1)\| \sqrt{D\|\xi, s\|} = \left(\frac{2}{\pi}\right)^{1/2} \sqrt{s_\Sigma} \tau,$$

which completes the proof.
\end{proof}

\section{4. Proofs of theorems}

\begin{proof}[Proof of Theorem 7]
Using (11) and Lemma 3 we get

$$\lambda_{d-1}[g^{-1}(0)] = \frac{\Gamma\left(\frac{d+1}{2}\right)}{4\pi^{(d+1)/2}} \int_{S^{d-1}} \mu_{d-1}(ds) \int_{\{y,s\}=0} dy \times \int_{-\infty}^{\infty} du \int_{\{y+ts\in F\}} \cos[ug(y+ts)] |g_t'(y+ts)| \, dt \right| = \frac{\Gamma\left(\frac{d+1}{2}\right)}{4\pi^{(d+1)/2}} \int_{S^{d-1}} \mu_{d-1}(ds) \int_{\{y,s\}=0} dy \times \lim_{R\to\infty} \int_{-R}^{R} du \int_{\{y+ts\in F\}} \cos[ug(y+ts)] |g_t'(y+ts)| \, dt \right|.$$

\end{proof}
It follows from the choice of $F$, condition (b), and (15) that we may apply Lebesgue’s theorem:

$$\lambda_{d-1}(g^{-1}(0)) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{4\pi^{(d+1)/2}} \lim_{R \to \infty} \int_{\partial F} \mu_{d-1}(ds) \int_{\{x,s\}=0} dy$$

$$\times \int_{-R}^{R} du \int_{\{x+ts \in F\}} \cos[ug(x+ts)] |g'(x+ts)| dt.$$ 

All the domains of integration are of finite measure and the integrands are bounded. Therefore we may apply Fubini’s Theorem:

$$\lambda_{d-1}(g^{-1}(0)) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{4\pi^{(d+1)/2}} \lim_{R \to \infty} \int_{-R}^{R} du \int_{\partial F} \mu_{d-1}(ds) \int_{\{x,s\}=0} dy$$

$$\times \int_{\{x+ts \in F\}} \cos[ug(x+ts)] |g'(x+ts)| dt$$

$$= \frac{\Gamma\left(\frac{d+1}{2}\right)}{4\pi^{(d+1)/2}} \lim_{R \to \infty} \int_{-R}^{R} du \int_{\partial F} \mu_{d-1}(ds) \int_{F} \cos[ug(x)] |\nabla g(x), s| dx$$

$$= \frac{\Gamma\left(\frac{d+1}{2}\right)}{4\pi^{(d+1)/2}} \int_{-\infty}^{\infty} du \int_{\partial F} \mu_{d-1}(ds) \int_{F} \cos[ug(x)] |\nabla g(x), s| dx.$$ 

To complete the proof it remains to apply Lemma [2].

Let us proceed to the proof of the second theorem.

*Proof of Theorem 3.* To apply Theorem [1] we have to show that $G$ satisfies conditions (a), (b) almost surely. It easily follows from (a’) that (a) holds almost surely. Further, using (b’), Fubini’s theorem, and $\lambda_{d-1}[^{\partial F}] < \infty$, we obtain

$$E \lambda_{d-1}[G^{-1}(0) \cap \partial F] = E \int_{\partial F} 1 \{G(y) = 0\} d\lambda_{d-1}(y) = \int_{\partial F} P \{G(y) = 0\} d\lambda_{d-1}(y) = 0,$$

which implies that (b) holds a.s.

First let us prove the theorem for the case when $\sigma \equiv 1$. From [2] we get

$$E \lambda_{d-1}(G^{-1}(0)) = E \frac{1}{2\pi} \int_{-\infty}^{\infty} du \int_{F} \cos[uG(x)] \|\nabla G(x)\| dx$$

$$= \frac{1}{2\pi} E \lim_{R \to \infty} \int_{-R}^{R} du \int_{F} \cos[uG(x)] \|\nabla G(x)\| dx.$$ 

It follows from the choice of $F$, condition (a’), and (17) that we may apply Lebesgue’s theorem:

$$E \lambda_{d-1}(G^{-1}(0)) = \frac{1}{2\pi} \lim_{R \to \infty} E \int_{-R}^{R} du \int_{F} \cos[uG(x)] \|\nabla G(x)\| dx$$

$$= \frac{1}{2\pi} \lim_{R \to \infty} \int_{F} dx \int_{-R}^{R} E \left\{ \cos[uG(x)] \|\nabla G(x)\| \right\} du.$$ 

We may use Fubini’s Theorem in the last equality on account of

$$|\cos[uG(x)]| \|\nabla G(x)\| \leq \|\nabla G(x)\| \leq \sum_{j=1}^{d} \lambda_{j}'(x)$$

and

$$E \int_{-R}^{R} du \int_{F} \sum_{j=1}^{d} \lambda_{j}'(x) dx \leq 2R|F| \sum_{j=1}^{d} E \sup_{x \in F} \lambda_{j}'(x) < \infty.$$
The right-hand side is finite because the supremum of a continues Gaussian field defined on a compact is summable (see [10]).

Differentiating $\sigma^2 \equiv 1$, we get

$$\frac{\partial (E G^2)}{\partial x_i} - 2E G \frac{\partial (E G)}{\partial x_i} = 0.$$ 

Therefore, by Kolmogorov’s Theorem on differentiation of mathematical expectations with respect to a parameter (see [4]), we have

$$E G G_i = \frac{1}{2} E G \frac{\partial (E G^2)}{\partial x_i} - E \frac{\partial (E G)}{\partial x_i} = E \frac{\partial (E G)}{\partial x_i}.$$

In other words, $G$ does not correlate with the components of the vector $\nabla G$ which is equivalent to the independence in the Gaussian case. Thus,

$$E \left\{ \cos[uG(x)] \| \nabla G(x) \| \right\} = E \cos[uG(x)] E \| \nabla G(x) \| = \text{Re} \left\{ e^{iuG(x)} E \| \nabla G(x) \| \right\},$$

which implies

$$E \lambda_{d-1}(G^{-1}(0)) = \frac{1}{2\pi} \lim_{R \to \infty} \int_F E \| \nabla G(x) \| dx \int_{-R}^R \cos[um(x)] e^{-u^2/2} du.$$ 

Using Lebesgue’s Theorem and the formula

$$\int_{-\infty}^{\infty} \cos[um(x)] e^{-u^2/2} du = \sqrt{2\pi} \text{Re} \left\{ e^{im(x)\mathcal{N}(0,1)} \right\} = \sqrt{2\pi} e^{-m^2(x)/2},$$

we obtain

$$(21) \quad E \lambda_{d-1}(G^{-1}(0)) = \frac{1}{2\pi} \int_F E \| \nabla G(x) \| dx \lim_{R \to \infty} \int_{-R}^R \cos[um(x)] e^{-u^2/2} du$$

$$= \frac{1}{\sqrt{2\pi}} \int_F e^{-m^2(x)/2} E \| \nabla G(x) \| dx.$$ 

We have proved the theorem for the case when $\sigma \equiv 1$. To treat the general one consider the field $G/\sigma$. It has unit variance and its zero set coincides with the zero set of $G$. Thus to complete the proof it remains to apply (21) to $G/\sigma$. \qed

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