Global existence and decay estimates for a viscoelastic plate equation with nonlinear damping and logarithmic nonlinearity

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Abstract

In this article, we consider a viscoelastic plate equation with a logarithmic nonlinearity in the presence of nonlinear frictional damping term. Using the the Faedo-Galerkin method we establish the global existence of the solution of the problem and we also prove few general decay rate results.

Keywords: Viscoelasticity, Global existence, Decay estimates, Convexity, Logarithmic nonlinearity.

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1 Introduction

This work deals with the existence and decay of solutions to the following plate problem:

\[
\begin{align*}
&|u_t|^\rho u_t + \Delta^2 u + \Delta^2 u_t + u - \int_0^t b(t-s)\Delta^2 u(s)ds + h(u_t) = ku \ln |u|, & (x, t) &\in \Omega \times (0, \infty), \\
&u(x, t) = \frac{\partial u}{\partial \nu}(x, t) = 0, & (x, t) &\in \partial \Omega \times (0, \infty), \\
&u(x, 0) = u_0(x), & u_t(x, 0) = u_1(x) &\text{in } \Omega
\end{align*}
\]

where \(\Omega \subset \mathbb{R}^2\) is a bounded domain, \(\nu\) is the unit outer normal to \(\partial \Omega\), \(k\) and \(\rho\) are positive constants and \(0 < \rho \leq \frac{2}{n-2}\) if \(n \geq 3\). We use the Lebesgue space \(L^2(\Omega)\) and \(H_0^2(\Omega)\) with their usual scalar product and norms. Through out this paper, we consider the following hypotheses:

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(H1) Let $b : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a $C^1$-nonincreasing function satisfying
\[ b(0) > 0, \quad 1 - \int_0^t b(\tau) d\tau = l > 0. \] (2)

(H2) Assume that there exist a nonincreasing positive differentiable function $\xi$ such that $\xi(0) > 0$ and $\xi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$. Further assume that there exists a $C^1$ function $B : (0, \infty) \rightarrow (0, \infty)$ which is linear or strictly convex $C^2$ function and strictly increasing on $(0, r_1]$, $r_1 \leq b(0)$, $B(0) = B'(0) = 0$ and $B$ satisfies
\[ b'(t) \leq -\xi(t)B(b(t)), \quad \forall t \geq 0. \] (3)

(H3) Let $h : \mathbb{R} \rightarrow \mathbb{R}$ is a nondecreasing continuous function such that there exist a $h_1 \in C^1(\mathbb{R}^+)$ with $h_1(0) = 0$ which is strictly increasing function and $h_1$ satisfies
\[ h_1(|s|) \leq |h(s)| \leq h_1^{-1}(|s|), \quad \forall |s| \leq \epsilon, \]
\[ c_1 |s| \leq |h(s)| \leq c_2 |s|, \quad \forall |s| \geq \epsilon, \]
where $c_1, c_2, \epsilon$ are positive constants. Moreover, define $H$ to be a strictly convex $C^2$ function in $(0, r_2]$ for some $r_2 \geq 0$ such that $H(s) = \sqrt{s}h_1(\sqrt{s})$ when $h_1$ is nonlinear.

(H4) The constant $k$ in (1) is such that
\[ 0 < k < k_0 = \frac{2\pi le^3}{c_p}, \]
where $c_p$ is the smallest positive number satisfying
\[ \|\nabla u\|_2^2 \leq c_p \|\Delta u\|_2^2, \quad u \in H_0^2(\Omega), \] (5)
where $\|\cdot\|_2 = \|\cdot\|_{L^2(\Omega)}$.

Throughout this article, we use $c$ to denote a generic positive constant.

Remark 1.1. Hypothesis (H3) implies that $\tau h(\tau) > 0$, $\forall \tau > 0$.

Remark 1.2. If $B$ is a strictly convex $C^2$ function and strictly increasing on $(0, r]$ for some $r > 0$, then we can extend $B$ to $\bar{B}$. Moreover, $\bar{B}$ is also a strictly convex $C^2$ function and strictly increasing on $(0, \infty)$ (see [16]). Similarly, we denote the extension of $H$ to be $\bar{H}$.
Plate problems have been broadly explored by mathematicians and other scientists. This type of problems have a lot of applications in different areas of science and engineering such as material engineering, mechanical engineering, nuclear physics and optics.

Let us discuss some work related to the plate problems. In [1], the authors treated the following problem
\[
\begin{aligned}
& u_{tt} - \Delta u + \alpha(t) h(u_t) = 0, \\
& u = 0, \\
& (x, t) \in \Omega \times (0, \infty),
\end{aligned}
\]
where \( h \) is a function having a polynomial growth near the origin, and they have established few energy decay results. Decay results for arbitrary growth of the damping term have been considered and studied for the first time in the work of Lasiecka and Tataru (see [9]). They have found that the energy decays as fast as the solution of an associated differential equation whose coefficients depend on the damping term. In [27], Liu considered the following problem
\[
\begin{aligned}
& |u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} - \int_0^t g(t - s) \Delta u(s) ds + \alpha(t) h(u_t) = b|u|^{\rho - 2} u, \\
& u(x, t) = 0, \\
& u(x, 0) = u_0(x), \\
& u_t(x, 0) = u_1(x) \\
& \text{in } \Omega \times (0, \infty),
\end{aligned}
\]
and they have proved a general decay result that depends on the behavior of \( g, \alpha \) and \( h \) without imposing any restrictive growth assumption on the damping term at origin. For more results in the direction of the plate problems, see [5, 8, 11, 10, 12, 23, 21, 26] and the references there in.

Now, let us review some work with a logarithmic term that are related to the problem (1). Cazenave and Haraux [22] studied the following problem:
\[
\begin{aligned}
& u_{tt} - \Delta u = u \ln |u|^k, \\
& \text{in } \mathbb{R}^3
\end{aligned}
\]
and established the existence and uniqueness of the solution for the Cauchy problem. Gorka [19] obtained the global existence of weak solutions in the one-dimensional case by using compactness arguments, for all \((u_0, u_1) \in H_0^1([a, b]) \times L^2([a, b])\), to the initial-boundary value problem (6). The authors in [13] considered the one dimensional Cauchy problem for equation (6) and they have proved the existence of classical solutions and also they have investigated the weak solutions. Birula and Mycielski [6, 7] considered
\[
\begin{aligned}
& u_{tt} - u_{xx} + u - \epsilon u \ln |u|^2 = 0, \\
& u(a, t) = u(b, t) = 0, \\
& u(x, 0) = u_0(x), \\
& u_t(x, 0) = u_1(x)
\end{aligned}
\]
which is a relativistic version of logarithmic quantum mechanics. Moreover, it can also be obtained for the p-adic string equation by taking the limit as $p \to 1$ (see [20, 23]). Mohammad M. Al-Gharabli [17] considered equation (1) without damping term and they have established the existence of solution and proved the decay rates and stability result. Mohammad M. Al-Gharabli et.al. (in [3]) have considered the viscoelastic problem with variable exponent and logarithmic nonlinearities:

$$u_{tt} - \Delta u + u + \int_0^t b(t-s)\Delta u(s)ds + |u_t|^\gamma \Delta u = u \ln |u|^\alpha$$

and they have established a global existence result using the well-depth method and then they have also established explicit and general decay results under a wide class of relaxation functions. Gongwei Liu ([15]) considered the differential equation

$$u_{tt} + \Delta^2 u + |u_t|^{m-2}u_t = u |u|^{p-2} \log |u|^k \quad (x,t) \in \Omega \times R^+$$

with the boundary conditions given in (1). They have established the local existence result by the fixed point techniques. The global existence and decay estimate of the solution at sub-critical initial energy is obtained, and they additionally prove that the solution with negative initial energy blows up in finite time under some suitable conditions. Moreover, they find out the blow-up in finite time of solution at the arbitrarily high initial energy for linear damping (i.e., $m = 2$). In [4], Adel M. Al-Mahdi considered viscoelastic plate equation with infinite memory and logarithmic nonlinearity:

$$|u_t|^\rho u_{tt} + \Delta^2 u + \Delta^2 u_{tt} + u - \int_0^\infty b(s)\Delta^2 u(t-s)ds = \alpha u \ln |u|, \quad \text{in } \Omega \times (0, \infty),$$

$$u(x,t) = \frac{\partial u}{\partial \nu}(x,t) = 0, \quad \text{in } \partial \Omega \times (0, \infty),$$

$$u(x,-t) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in } \Omega.$$

By imposing minimal conditions on the relaxation function the authors in [4] established an explicit and general decay rate results. See [2, 14] (and the references there in) for more results in this direction.

In this article, we are engaged with the global existence and stability of the plate problem (1) with kernels $b$ having an arbitrary growth at infinity. This article organized as follows. In Section 2, we establish the local existence of the solutions to the problem (1). The global existence is proved in Section 3. Finally, in the last section we derive few stability results.

## 2 Local existence

In this section, we state and prove the local existence result for the problem (1). The energy associated with problem (1) is
Lemma 2.1. (Logarithmic Sobolev inequality) Let $H \in H^1_0(\Omega)$ be any number. Then

\[
E(t) = \frac{1}{p+2} \|u_t\|^{p+2}_{L^2} + \frac{1}{2} \int_0^t \int_\Omega \left( 1 - \int_0^t b(s) ds \right) \|\Delta u\|^2 dx + \int_\Omega \|\Delta u_t\|^2 dx - k \int_\Omega u_t^2 \ln |u| dx + \|u\|^2_2 + b \circ \Delta u \right] + \frac{k}{2} \|u\|^2_2.
\]

(7)

where the product $\circ$ is defined by

\[(b \circ \Delta u)(t) = \int_0^t b(t-s) \|\Delta u(s) - \Delta u(t)\|^2_2 ds.
\]

Direct differentiation of (7) with respect to $t$ and using (11) we observe that

\[
E'(t) = \frac{1}{2} (b' \circ \Delta u)(t) - \frac{1}{2} b(t) \|\Delta u\|^2_2 - \int_\Omega u_t h(u_t)
\leq \frac{1}{2} (b' \circ \Delta u)(t) - \int_\Omega u_t h(u_t)
\leq 0.
\]

Lemma 2.1. Let $u \in H^1_0(\Omega)$ and $a > 0$ be any number. Then

\[
\int_\Omega u^2 \ln |u| dx \leq \frac{1}{2} \|u\|^2_2 \ln \|u\|^2_2 + \frac{a^2}{2\pi} \|\nabla u\|^2_2 - (1 + \ln a) \|u\|^2_2
\]

(9)

Corollary 2.1. Let $u \in H^2_0(\Omega)$ and $a > 0$ be any number. Then

\[
\int_\Omega u^2 \ln |u| dx \leq \frac{1}{2} \|u\|^2_2 \ln \|u\|^2_2 + \frac{ca^2}{2\pi} \|\Delta u\|^2_2 - (1 + \ln a) \|u\|^2_2
\]

(10)

Lemma 2.2. Let $\epsilon_0 \in (0,1)$, then there exists $d_{\epsilon_0} > 0$ such that

\[
s \ln s \leq s^2 + d_{\epsilon_0} s^{1-\epsilon_0} \forall s > 0
\]

(11)

Definition 2.1. A function

\[u \in C^1([0,T], H^2_0(\Omega))\]

is called a weak solution of (7) on $[0,T]$ if, for any $t \in [0,T]$ and $\forall w \in H^2_0(\Omega)$, $u$ satisfies

\[
\left\{
\begin{array}{l}
\int_\Omega |u_t|^p u_t \Delta u dx + \int_\Omega \Delta u(x,t) \Delta w(x) dx + \int_\Omega \Delta u_t(x,t) \Delta w(x) dx \\
+ \int_\Omega u(x,t) w(x) dx - \int_\Omega \Delta w(x) \int_0^t b(t-s) \Delta u(s) ds \\
+ \int_\Omega h(u_t(x,t)) w(x) dx = k \int_\Omega u(x,t) w(x) \ln(|u(x,t)|) dx,
\end{array}
\right\}
\]

(12)

\[u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x).
\]
**Theorem** Assume that the hypothesis \((H1) - (H4)\) hold. Let \((u_0, u_1) \in H_0^2(\Omega) \times H_0^2(\Omega)\). Then the problem (1) has weak solution on \([0, T]\).

**Proof.** To prove the existence of a solution to the problem (1), we use the Faedo-Galerkin approximations. Let \((w_j)_{j=1}^\infty\) be an orthogonal basis of the separable space \(H_0^2(\Omega)\). Let \(V_m = \text{span}(w_1, w_2, \ldots, w_m)\) and let the projections of the initial data on the finite dimensional subspace \(V_m\) be given by

\[ u_0^m(x) = \sum_{j=1}^m \alpha_j w_j(x), \quad u_1^m(x) = \sum_{j=1}^m \beta_j w_j(x). \quad (13) \]

We search for an approximation solution

\[ u_0^m(x) = \sum_{j=1}^m g_j^m(t)w_j(x), \quad (14) \]

of the approximate problem in \(V_m\):

\[
\begin{align*}
\int_\Omega \left[ |u_t^m|^\rho u_{tt}^m w + \Delta u^m \Delta w + \Delta u_t^m \Delta w + u^m w + h(u_t^m) w \right]
&\quad \int_0^t b(t-s) \Delta u^m(s) \Delta w ds \right] dx = k \int_\Omega w u^m \ln |u^m| dx, \quad \forall w \in V_m, \\
u^m(0) := u_0^m &= \sum_{j=1}^m (u_0, w_j)w_j, \\
u_t^m(0) := u_1^m &= \sum_{j=1}^m (u_1, w_j)w_j. \\
&\quad (15)
\end{align*}
\]

This gives a system of ordinary differential equation (ODE’s) for the unknown functions \(g_j^m(t)\). Using the standard existence theory for ODE’s, one can obtain functions

\[ g_j : [0, t_m) \to \mathbb{R}, \quad j = 1, 2, \ldots, m, \]

which satisfy (15) in a maximal interval \([0, t_m), t_m \in (0, T]\). Later, we show that \(t_m = T\) and the local solution is uniformly bounded which is independent of \(m\) and \(t\). To do this, substitute \(w = u_t^m\) in (15) and using integration by parts to obtain

\[ \frac{d}{dt} E^m(t) \leq \frac{1}{2} (b' \circ \Delta u^m) - \int_\Omega u_t^m h(u_t^m) dx \leq 0, \quad (16) \]

where

\[
E^m(t) = \frac{1}{\rho + 2} \|u_t^m\|_{\rho+2}^2 + \frac{1}{2} \left(1 - \int_0^t b(s) ds \right) \|\Delta u^m\|^2_2 + \|\Delta u_t^m\|^2_2 \\
- k \int_\Omega |u|^2 \ln |u^m| dx + \frac{k+2}{4} \|u^m\|^2_2 + \frac{1}{2} (b \circ \Delta u^m), \quad (17)
\]

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from (16), we have
\[ E^m(t) \leq E^m(0), \forall t \geq 0. \]
the logarithmic Sobolev inequality together with last inequality, we observe that
\[
\|u^m_t\|_{\rho+2}^2 + \|\Delta u^m_t\|_2^2 + \left( t - \frac{ka^2c_p}{2}\pi \right) \|\Delta u^m\|_2^2 \\
+ \left[ \frac{k+2}{2} + k(1 + \ln a) \right] \|u^m\|_2^2 + b \circ \Delta u^m \leq 2E^m(0) + \|u^m\|_2^2 \ln \|u^m\|_2^2,
\]
Choose \( a \) such that
\[ e^{-3/2} < a < \sqrt{\frac{2\pi l}{kc_p}}, \]
then \( a \) satisfies
\[ l - \frac{ka^2c_p}{2\pi} > 0, \]
and
\[ \frac{k+2}{2} + k(1 + \ln a) > 0. \]
So, we obtain
\[
\|u^m_t\|_{\rho+2}^2 + \|\Delta u^m_t\|_2^2 + \|\Delta u^m\|_2^2 + \|u^m\|_2^2 \\
+ b \circ \Delta u^m \leq c \left( 1 + \|u^m\|_2^2 \ln \|u^m\|_2^2 \right),
\]
And we know that
\[ u^m(., t) = u^m(., 0) + \int_0^t \frac{\partial u^m}{\partial s}(., s)ds. \]
Using Cauchy Schwarz inequality, observe that
\[
\|u^m(t)\|_2^2 \leq 2\|u^m(0)\|_2^2 + 2\left\| \int_0^t \frac{\partial u^m}{\partial s}(s)ds \right\|_2^2 \\
\leq 2\|u^m(0)\|_2^2 + 2T \int_0^t \|u^m_t(s)\|_2^2 ds,
\]
therefore inequality (22) gives
\[
\|u^m\|_2^2 \leq 2\|u^m(0)\|_2^2 + 2Tc \left( 1 + \int_0^t \|u^m\|_2^2 \ln \|u^m\|_2^2 ds \right),
\]
if we substitute $c_1 = \max\{2Tc, 2\|u(0)\|_2^2\}$, then (24) leads to
\[
\|u^m\|_2^2 \leq 2c_1 \left(1 + \int_0^t \|u^m\|_2^2 \ln(\|u^m\|_2^2)ds\right),
\]
since $c_1 \geq 0$, we get
\[
\|u^m\|_2^2 \leq 2c_1 \left(1 + \int_0^t (c_1 + \|u^m\|_2^2) \ln(c_1 + \|u^m\|_2^2)ds\right).
\]
When Logarithmic Gronwall inequality applied to (25), we get the following estimate:
\[
\|u^m\|_2^2 \leq 2c_1 e^{2c_1 T} = c_2.
\]
Hence, from inequality (22) it follows that
\[
(b \circ \Delta u^m)(t) + \|u^m\|_{p+2}^p + \|\Delta u^m\|_2^2 + \|\Delta u^m\|_2^2 + \|u^m\|_2^2 \leq c(1 + c_2 \ln c_2) \leq c_3.
\]
This implies
\[
\sup_{t \in (0, t_m)} [(go \Delta u^m)(t) + \|u^m\|_{p+2}^p + \|\Delta u^m\|_2^2 + \|\Delta u^m\|_2^2 + \|u^m\|_2^2] \leq c_3.
\]
So, we have
\[
\begin{cases}
  u^m \text{ is uniformly bounded in } L^\infty(0, T; H^2_0(\Omega)), \\
  u^m_t \text{ is uniformly bounded in } L^\infty(0, T; L^{p+2}(\Omega)) \cap L^\infty(0, T; H^2_0(\Omega)),
\end{cases}
\]
therefore, these satisfies a subsequence of $(u_m)$, such that
\[
\begin{cases}
  u^m \rightharpoonup u \text{ in } L^\infty(0, T; H^2_0(\Omega)), \\
  u^m_t \rightarrow u_t \text{ in } L^\infty(0, T; L^{p+2}(\Omega)) \cap L^\infty(0, T; H^2_0(\Omega)), \\
  u^m \rightarrow u \text{ in } L^2(0, T; H^2_0(\Omega)), \\
  u^m_t \rightarrow u_t \text{ in } L^2(0, T; L^{p+2}(\Omega)) \cap L^2(0, T; H^2_0(\Omega)),
\end{cases}
\]
where $\rightharpoonup$ represent the weak * convergence and $\rightarrow$ represent weak convergence. Therefore, the approximate solution is uniformly bounded and it is independent of $m$ and $t$. Therefore we can extend $t_m$ to $T$.

Next we prove that $u^m_{tt}$ is bounded in $L^2(0, T; H^2_0(\Omega))$. To do this, we substitute $w = u^m_{tt}$ in (12). Using (11), we see that
\[
\begin{align*}
\int_\Omega |u^m_t|^p |u^m_{tt}|^2 dx + \|\Delta u^m_{tt}\|_2^2 &= -\int_\Omega (\Delta u^m \Delta u^m_{tt} + u^m u^m_{tt}) \\
+ \int_0^t \int_\Omega b(t-s)\Delta u^m(s)\Delta u^m_{tt}(t)dsdx - \int_\Omega h(u^m_t)u^m_t dx + k \int_\Omega u^m_t u^m \ln |u^m| dx.
\end{align*}
\]
By using the Cauchy-Schwarz’ inequality, Young’s inequality, and the embedding inequality we obtain,

\[
\int_{\Omega} |u^m_t|^p |u^m_{tt}|^2 \, dx + \| \Delta u^m_{tt} \|_2^2 \leq \delta \| \Delta u^m_t \|_2^2 + \frac{1}{4\delta} \| \Delta u^m(t) \|_2^2 + \delta \| u^m_t \|_2^2 + \frac{1}{4\delta} \| u^m(t) \|_2^2 + \delta \| \Delta u^m_t \|_2^2
\]

\[
+ \frac{1}{4\delta} \left( \int_0^t b(t-s) \| \Delta u^m(s) \| \, ds \right)^2 + \frac{1}{4\delta} \int_{\Omega} h^2(u^m) \, dx + \delta \| u^m_t \|_2^2
\]

\[
+ k \int_{\Omega} u^m_t u^m \ln |u^m| \, dx
\]

\[
\leq 2\delta \| \Delta u^m_t \|_2^2 + 2\delta \| u^m_t \|_2^2 + k \int_{\Omega} u^m_t u^m \ln |u^m| \, dx
\]

\[
+ \frac{1}{4\delta} \left[ \| \Delta u^m(t) \|_2^2 + \left( \int_0^t b(t-s) \| \Delta u^m(s) \| \, ds \right)^2 + \int_{\Omega} h(u^m) \, dx + \| u^m \|_2^2 \right]
\]

\[
\leq c\delta \| \Delta u^m_t \|_2^2 + k \int_{\Omega} u^m_t u^m \ln |u^m| \, dx
\]

\[
+ \frac{1}{4\delta} \left[ \| \Delta u^m(t) \|_2^2 + \left( \int_0^t b(t-s) \| \Delta u^m(s) \| \, ds \right)^2 + \int_{\Omega} h(u^m) \, dx + \| u^m \|_2^2 \right].
\]  

(30)

Using Lemma 2.2 with \( \epsilon_0 = \frac{1}{2} \), the second term in the right hand side of (30) is estimated as follows:

\[
k \int_{\Omega} u^m_t u^m \ln |u^m| \, dx \leq c \int_{\Omega} u^m_t \left( |u^m|^2 + d_1 \sqrt{u^m} \right) \, dx
\]

\[
\leq c \left( \delta \int |u^m_t|^2 \, dx + \frac{1}{4\delta} \int |u^m|^2 + d_1 \sqrt{u^m} \right)^2 \, dx
\]

\[
\leq c\delta \| u^m_t \|_2^2 + \frac{c}{4\delta} \left( \int_{\Omega} |u^m|^4 \, dx + \int_{\Omega} |u^m| \, dx \right)
\]

\[
\leq c\delta \| \Delta u^m_t \|_2^2 + \frac{c}{4\delta} \left( \| \Delta u^m \|_2^4 + \| u^m \|_2^4 \right).
\]  

(31)

from (30) and (31) we have

\[
\int_{\Omega} |u^m_t|^p |u^m_{tt}|^2 \, dx + (1 - c\delta) \| \Delta u^m_t \|_2^2 \leq \frac{c}{4\delta} \left( \| \Delta u^m \|_2^4 + \| u^m \|_2^4 \right)
\]

\[
+ \frac{1}{4\delta} \left[ \| \Delta u^m(t) \|_2^2 + \left( \int_0^t b(t-s) \| \Delta u^m(s) \| \, ds \right)^2 + \int_{\Omega} h(u^m) \, dx + \| u^m \|_2^2 \right]
\]

(32)

Now we prove that \( h(u^m) \) is bounded in \( L^2(0,T;L^2(\Omega)) \). For this purpose, we consider two cases:

Case 1. When \( h \) is linear, we can directly prove that

\[
\int_0^t \int_{\Omega} h^2(u^m_t) \, dx \, dt \leq C
\]

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Case 2. When $h_1$ is nonlinear,
Consider $|s| \leq \epsilon$, then

$$H(h^2(s)) = |h(s)|h_1(|h(s)|) \leq sh(s),$$

this implies that

$$H^{-1}(sh(s)) \geq h^2(s), \quad \forall |s| \leq \delta.$$ 

then using the similar lines from Theorem 3.2 in [18], we get $\int_0^T \int_\Omega h^2(u_m^m) dx dt \leq C_T$. We conclude that $h(u_m^m)$ is bounded in $L^2(0, T; L^2(\Omega))$.

Integrating (32) from (0, T), using the hypothesis (H1) and (26), we obtain

$$\int_0^T \int_\Omega |u_m^m|^p |u_m^m|^2 dx dt + (1 - 2c\delta) \int_0^T \Delta u_m^m \|_{L^2}^2 dt$$ 

$$\leq c \delta \int_0^T \left( (b \circ \Delta u_m^m)(t) + \|\Delta u_m^m\|_2^2 + \|\Delta u_m^m\|_2^4 + \|u_m^m\|_2^2 \right) dx$$ 

$$- c \delta \int_0^T \int_\Omega h^2(u_m^m) dx dt. \quad (33)$$

From (33) and using the fact that $h(u_m^m)$ is bounded in $L^2(0, T; L^2(\Omega))$, it is easy to observe that for $\delta$ small enough,

$$u_m^m$$

is bounded in $L^2(0, T; H^2_0(\Omega))$. \quad (34)

Taking $m \to \infty$ to (12) and from (28) and (34) (thanks to Aubin-Lions’ theorem), for all $w \in H^2_0(\Omega)$ and a.e. $t \in (0, T)$, we see that

$$\int_\Omega |u| \rho u_{tt} w dx + \int_\Omega \Delta u \Delta w dx + \int_\Omega \Delta u_t \Delta w dx + \int_\Omega uw dx + \int_\Omega h(u_t) w dx$$ 

$$- \int_\Omega \left( \int_0^t g(t-s) \Delta u(s) ds \right) \Delta w dx = k \int_\Omega w u \ln |u| dx ds.$$ 

This complete the proof of this theorem. \qed

3 Global existence

In this section, under smallness condition on the initial data, we state and prove global existence result. For the sake of simplicity, we introduce the following functionals:

$$J(t) := J(u(t))$$ 

$$= \frac{1}{2} \left[ \|\Delta u_t\|_2^2 + (1 - \int_0^t b(s) ds) \|\Delta u\|_2^2 + \|u\|_2^4 + (b \circ \Delta u) - \int_\Omega u^2 \ln |u| \right]$$ 

$$+ \frac{k}{4} \|u\|_2^2, \quad (35)$$
and
\[ I(t) := I(u(t)) = \| \Delta u_t \|^2 + (1 - \int_0^t b(s) ds) \| \Delta u \|^2 + \| u \|^2 + (b \circ \Delta u) - \int_\Omega u^2 \ln |u|^k, \]

(36)

Note: From (7), (35) and (36), it is clear that
\[ J(t) = \frac{1}{2} I(t) + \frac{k}{4} \| u \|^2 \]

(37)
\[ E(t) = \frac{1}{\rho + 2} \| u_t \|^2 + J(t), \]

(38)

Notation: Define \( \tilde{\rho} = e^{\frac{2Q_0 - k}{2}} \), \( d = \frac{1}{2} Q_0 \tilde{\rho}^2 - \frac{k}{4} \tilde{\rho}^2 \ln \tilde{\rho}^2 \) and
\[ Q_0 = \frac{k+2}{2} + k(1 + \ln a), \]

where \( 0 < a < \sqrt{\frac{2\pi e}{k}} \).

Lemma 3.1. Let \( (u_0, u_1) \in H^2_0(\Omega) \times H^2_0(\Omega) \) and assume that the hypothesis (H1) holds. Further assume that \( \| u_0 \| < \overline{\rho} \) and \( 0 < E(0) < d \). Then \( I(u) > 0 \) \( \forall t \in [0, T] \).

Proof. We divide the proof for this lemma into two steps. In step (1), we prove that \( \| u \| < \overline{\rho} \) \( \forall t \in [0, T] \) and in step (2), we prove \( I(t) > 0 \) \( \forall t \in [0, T] \).

Step 1. From (7) and (35), and using Logarithmic Sobolev inequality it is easy to see that
\[ E(t) \geq J(t) \]
\[ \geq \frac{1}{2} \left( 1 - \frac{c_p k a^2}{2\pi} \right) \| \Delta u \|^2 + \frac{1}{2} \left( \frac{k + 2}{2} + k(1 + \ln a) - \frac{k}{2} \ln \| u \|^2 \right) \| u \|^2. \]

Using (20), we obtain
\[ E(t) \geq Q(\tilde{\rho}) = \frac{1}{2} Q_0 \tilde{\rho}^2 - \frac{k}{4} \tilde{\rho}^2 \ln \tilde{\rho}^2 \]

(39)

where \( \tilde{\rho} = \| u \|_2 \). Observe that (39) implies \( Q \) is increasing on \((0, \overline{\rho})\) and decreasing on \((\overline{\rho}, \infty)\). Also, note that \( Q(\tilde{\rho}) \to -\infty \) as \( \tilde{\rho} \to +\infty \). Let
\[ \max_{0 < \rho < +\infty} Q(\tilde{\rho}) = \frac{1}{2} Q_0 \tilde{\rho}^2 - \frac{k}{4} \tilde{\rho}^2 \ln \tilde{\rho}^2 = Q(\overline{\rho}) = d. \]

Suppose that \( \| u \| < \overline{\rho} \) does not hold in \([0, T]\), then there exist \( t_0 \in (0, T) \) and \( \| u(x, t_0) \| = \overline{\rho} \). Using (39), we get \( E(t_0) \geq Q(\| u(x, t_0) \|_2) = Q(\overline{\rho}) = d \), which is contradiction to the fact \( E(t) \leq E(0) < d \) \( \forall t > 0 \). Hence \( \| u \|_2 < \overline{\rho} \) \( \forall t \in [0, T] \). Hence, \( \| u \| < \overline{\rho} \) for all \( t \in [0, T] \).

Step 2. Using the definition of \( I(t) \) and (20), we notice that for \( t \in [0, T] \),
\[ I(t) \geq \left( 1 - \frac{c_p k a^2}{2\pi} \right) \| \Delta u_t \|^2 + \left( 1 + k(1 + \ln a) - \frac{k}{2} \ln \| u \|^2 \right) \| u \|^2 \]
\[ \geq \left( 1 - \frac{c_p k a^2}{2\pi} \right) \| \Delta u \|^2 + \| u \|^2 \]
\[ \geq 0. \]
This complete the proof of this lemma.

**Remark 3.1.** Under the assumptions of Lemma (3.1), for \( t \in [0, T) \), we have

\[
\|u_t\|_{p+2}^{p+2} \leq (\rho + 2)E(t) \leq (\rho + 2)E(0),
\]

and

\[
\|\Delta u_t\|^2 \leq 2E(t) \leq 2E(0).
\]

This shows that the solution is global and bounded in time (in the above mentioned norm).

4 Stability

In this section, we state and prove the decay of the solutions of the problem (1). At first we establish some lemmas which are useful to prove our main results.

**Lemma 4.1.** Assume that \( b \) satisfies the hypothesis (H1). Then, for \( u \in H^2_0(\Omega) \), we have

\[
\int_\Omega \left( \int_0^t b(t-s)(u(t) - u(s))ds \right)^2 dx \leq c(b \circ \Delta u)(t),
\]

and

\[
\int_\Omega \left( \int_0^t b'(t-s)(u(t) - u(s))ds \right)^2 dx \leq -c(b' \circ \Delta u)(t),
\]

for some \( c > 0 \).

For the proof of Lemma 4.1 refer to [17].

**Lemma 4.2.** Assume that \( h \) satisfies the hypothesis (H3). Then, the solution of (1) satisfies the following estimates:

\[
\int_\Omega h^2(u_t)dx \leq c \int_\Omega u_t h(u_t)dx \leq -cE'(t), \quad \text{if } h_1 \text{ is linear} \quad (40)
\]

\[
\int_\Omega h^2(u_t)dx \leq cH^{-1}(G(t)) - cE'(t), \quad \text{if } h_1 \text{ is nonlinear} \quad (41)
\]

where \( G(t) := \frac{1}{|\Omega_1|} \int_{\Omega_1} u_t h(u_t)dx \leq -cE'(t) \) and \( \Omega_1 = \{x \in \Omega : |u_t| \leq \epsilon\} \) for some \( c, \epsilon > 0 \).

**Lemma 4.3.** Assume that \( b \) satisfies the hypothesis (H1) and (H2). Then the solution of (1) satisfies the following estimate:

\[
\int_{t_1}^t b(s) \int_\Omega |\Delta u(t) - \Delta u(t-s)|^2 dxds \leq \frac{t - t_1}{\delta} B^{-1}\left(\frac{\delta M(t)}{(t-t_1)\xi(t)}\right), \quad (42)
\]
where $\delta \in (0, 1)$, $\bar{B}$ is an extension of $B$ and

$$M(t) := - \int_1^t b'(s) \int_\Omega |\Delta u(t) - \Delta u(t - s)|^2 dxds \leq -cE'(t). \quad (43)$$

The proofs of Lemma’s 4.2 and 4.3 follows from the similar lines as in [16]. So, we skip the proof’s.

**Lemma 4.4.** Under the hypothesis $(H1) - (H4)$, the functional $\Psi_1(t)$

$$\Psi_1(t) := \frac{1}{\rho + 1} \int_\Omega |u|^{\rho + 2} u dx + \int_\Omega \Delta u \Delta u_t dx,$$

satisfies the estimate:

$$\Psi_1'(t) \leq \frac{1}{\rho + 1} \|u_t\|^{\rho + 2} + \frac{1}{4} \|\Delta u\|^2 - \frac{1}{2} \|u\|^2 + c \int_\Omega h^2(u_t) dx$$

$$+ c(b \circ \Delta u)(t) + k \int_\Omega u^2 \ln |u| dx. \quad (44)$$

**Proof.** Differentiating $\Psi_1$ with respect to $t$ and using (1), we get

$$\Psi_1'(t) = \frac{1}{\rho + 1} \|u_t\|^{\rho + 2} + k \int_\Omega u^2 \ln |u| dx - \|\Delta u\|^2 - \|u\|^2 + \|\Delta u_t\|^2$$

$$+ \int_\Omega \Delta u(t) \int_0^t b(t - s) \Delta u(s) ds dx - \int_\Omega uh(u_t) dx. \quad (45)$$

At first, we estimate the sixth term in the right hand side of (45). Using the hypothesis $(H1)$ and Lemma 4.1, we have

$$\int_\Omega \Delta u(t) \int_0^t b(t - s) \Delta u(s) ds dx$$

$$= \int_\Omega \Delta u(t) \int_0^t b(t - s)(\Delta u(s) - \Delta u(t)) ds dx + \int_\Omega (\Delta u(t))^2 \int_0^t b(t - s) ds$$

$$\leq \frac{1}{2} \int_\Omega |\Delta u|^2 + \frac{1}{2} \int_\Omega \left( \int_0^t b(t - s)|\Delta u(s) - \Delta u(t)| ds \right)^2 dx + (1 - l) \int_\Omega |\Delta u|^2$$

$$= (1 - l + \frac{1}{2}) \int_\Omega |\Delta u|^2 + \frac{l}{\eta} (b \circ \Delta u)(t)$$

$$= (1 - l - \frac{1}{2}) \int_\Omega |\Delta u|^2 + \frac{c}{\eta} (b \circ \Delta u)(t).$$

The last line in the above inequality is obtained by choosing $\eta = l$. Now, we estimate the last term in (45).

$$\int_\Omega uh(u_t) \leq \delta c \|
abla u\|^2 + \frac{c}{\eta} \int_\Omega h^2(u_t) dx$$

$$\leq \delta c \|\Delta u\|^2 + \frac{c}{\eta} \int_\Omega h^2(u_t) dx.$$

Now choose $\delta = \frac{l}{2c}$ to get (44). Hence proved.
**Lemma 4.5.** Under the hypothesis \((H1) - (H4)\), the functional \(\Psi_2(t)\)

\[
\Psi_2(t) := \int_{\Omega} \left( \Delta^2 u_t + \frac{1}{\rho + 1} |u_t|^\rho u_t \right) \int_0^t b(t - s)(u(t) - u(s)) ds dx
\]
satisfies the estimate:

\[
\Psi_2'(t) \leq \frac{1}{\rho + 1} \left( \int_0^t b(s) ds \right) \|u_t\|^{\rho + 2}_{\rho + 2} + \left( \frac{\delta}{2} + \delta_1 + 2\delta_1(1 - \ell)^2 \right) \|\Delta u\|_{L^2}^2 + \left( \delta + c\delta_2(E(0))^{\rho} - \int_0^t b(s) ds \right) \|u_t\|_{L^2}^2 - \frac{c}{\delta} (b' \circ \Delta u)(t) + (c + \frac{c}{\delta} + c\delta_1 + \frac{c}{\delta_1})(b \circ \Delta u)(t) + c(b \circ \Delta u)^{1-\alpha}(t) + c \int_\Omega h^2(u_t) dx,
\]
for some \(\epsilon_0 \in (0, 1)\).

**Proof.** Differentiating \(\Psi_2\) with respect to \(t\) and using \((1)\), we get

\[
\Psi_2'(t) = -\int_\Omega u \ln |u|^k \left( \int_0^t b(t - s)(u(t) - u(s)) ds \right) dx \\
+ \int_\Omega \Delta u(t) \left( \int_0^t b(t - s)(\Delta u(t) - \Delta u(s)) ds \right) dx \\
+ \int_\Omega u \left( \int_0^t b(t - s)(u(t) - u(s)) ds \right) dx \\
- \int_\Omega \left( \int_0^t b(t - s) \Delta u(s) ds \right) \left( \int_0^t b(t - s)(\Delta u(t) - \Delta u(s)) ds \right) dx \\
+ \int_\Omega h(u_t) \left( \int_0^t b(t - s)(u(t) - u(s)) ds \right) dx \\
- \frac{1}{\rho + 1} \|u_t\|^\rho \left( \int_0^t b'(t - s)(u(t) - u(s)) ds \right) dx \\
- \int_\Omega \frac{1}{\rho + 1} |u_t|^{\rho} u_t \left( \int_0^t b(s) ds \right) dx \\
- \int_\Omega \Delta u_t \left( \int_0^t b'(t - s)(\Delta u(t) - \Delta u(s)) ds \right) dx \\
- \left( \int_0^t b(s) ds \right) \left( \int_\Omega |u_t|^2 dx \right).
\]

In order to estimate the all the nine terms in the right hand side of \((47)\), we will use Young’s inequality, Cauchy-Schwarz’ inequality and Lemma 4.1. All the terms except for the third and fifth term, are estimated in the similar lines as in \([18]\). For the sake of completeness, we will write the final estimates.
\((\text{I})\) \quad - \int_{\Omega} u \ln |u| k \left( \int_0^t b(t-s)(u(t)-u(s))ds \right) dx \\
\quad \leq \frac{\delta}{4} \|\Delta u\|^2_2 + \frac{c}{\delta}(b \circ \Delta u)(t) + c(b \circ \Delta u)^{1 + \epsilon_0}(t),

\((\text{II})\) \quad \int_{\Omega} \Delta u(t) \left( \int_0^t b(t-s)(\Delta u(t) - \Delta u(s))ds \right) dx \\
\quad \leq \delta_1 \|\Delta u\|^2_2 + \frac{c}{\delta_1}(b \circ \Delta u)(t),

\((\text{III})\) \quad \int_{\Omega} u \left( \int_0^t b(t-s)(u(t)-u(s))ds \right) dx \leq \frac{\delta}{4} \|\Delta u\|^2_2 + \frac{c}{\delta}(b \circ \Delta u)(t),

\((\text{IV})\) \quad - \int_{\Omega} \left( \int_0^t b(t-s)\Delta u(s)ds \right) \left( \int_0^t b(t-s)(\Delta u(t) - \Delta u(s))ds \right) dx \\
\quad \leq (c\delta_1 + \frac{c}{\delta_1})(b \circ \Delta u)(t) + 2\delta_1 (1-l)^2 \|\Delta u\|^2_2,

\((\text{V})\) \quad \int_{\Omega} h(u_t) \left( \int_0^t b(t-s)(u(t)-u(s))ds \right) dx \\
\quad \leq c \int_{\Omega} h^2(u_t) dx + (1-l)c^2(b \circ \Delta u)(t),

\((\text{VI})\) \quad - \int_{\Omega} \frac{1}{\rho + 1} |u_t|^\rho u_t \left( \int_0^t b'(t-s)(u(t)-u(s))ds \right) dx \\
\quad \leq c\delta_2 (E(0))^{\rho} \|\Delta u_t\|^2_2 - \frac{c}{\delta_2}(b' \circ \Delta u)(t),

\((\text{VII})\) \quad - \int_{\Omega} \frac{1}{\rho + 1} |u_t|^\rho u_t^2 \left( \int_0^t b(s)ds \right) dx = - \frac{1}{\rho + 1} \left( \int_0^t b(s)ds \right) \|u_t\|^\rho+2_{\rho+2},

\((\text{VIII})\) \quad \int_{\Omega} \Delta u_t \left( \int_0^t b'(t-s)(\Delta u(t) - \Delta u(s))ds \right) dx \\
\quad \leq \delta \|\Delta u_t\|^2_2 - \frac{c}{\delta}(b' \circ \Delta u)(t),

and

\((\text{IX})\) \quad - \left( \int_0^t b(s)ds \right) \left( \int_{\Omega} |\Delta u_t|^2 dx \right) = - \left( \int_0^t b(s)ds \right) \|\Delta u_t\|^2_2.

Combining all the estimates we will arrive at \((46)\). \qed
Lemma 4.6. Assume that the hypothesis of Lemma 3.1 and the hypothesis of (H1) – (H4) holds. Then there exists \( N, \epsilon > 0 \) such that the functional
\[
L(t) = NE(t) + c\Psi_1(t) + \Psi_2(t)
\]
satisfies,
\[
L \sim E,
\]
and for all \( t \geq 0 \), there exists \( m > 0 \) and \( \epsilon_0 \in (0, 1) \) such that
\[
L'(t) \leq -mE(t) + c(b \circ \Delta u)(t) + c(b \circ \Delta u)^{\frac{1}{1+\alpha}}(t) + c \int_\Omega h^2(u_t)dx,
\]
(48)

Proof. In order to prove (48), we use the Sobolev embedding \( H_0^1(\Omega) \rightarrow L^{p+2}(\Omega) \), \( \int_\Omega |u_t|^{2(p+1)}dx \leq c\|\Delta u_t\|^2 \) and Remark 3.1. From (49), we observe that
\[
|L(t) - NE(t)| \leq c|\Psi_1| + |\Psi_2|
\]
\[
\leq \frac{1}{p+1} \int_\Omega |u_t|^\frac{p+1}{p}|u|dx + |\epsilon \int_\Omega \Delta u \Delta u_t dx|
\]
\[
+ \frac{1}{p+1} |u_t|^\frac{p}{p+1} \int_0^t b(t - s)(u(t) - u(s))dsdx|
\]
\[
+ \frac{1}{p+1} (\Delta u_t)^2 \int_0^t b(t - s)(u(t) - u(s))dsdx
\]
\[
\leq \frac{1}{p+2} \|u_t\|_{p+2}^\frac{p+2}{p+1} + \frac{c_p(b \circ \Delta u)}{2(p+1)} + \frac{\|\Delta u\|^2}{2} + \frac{\|\Delta u_t\|^2}{2} + \frac{\|\Delta u_t\|^2}{2} + \frac{1}{2} (b \circ \Delta u)(t)
\]
\[
\leq c(1 + \epsilon) E(t).
\]
Therefore, by choosing \( N \) large enough we obtain (48). For the inequality (49), we follow similar lines given in [17]. \( \square \)

Remark 4.1. Since,
\[
E(t) \geq J(t) \geq \frac{1}{2} (b \circ \Delta u)(t)
\]
that imply
\[
(b \circ \Delta u)(t) \leq 2E(t) \leq 2E(0).
\]

Hence,
\[
(b \circ \Delta u)(t) \leq (b \circ \Delta u)^{\frac{\alpha}{1+\alpha}}(t) \leq c(b \circ \Delta u)^{\frac{1}{1+\alpha}}(t)
\]
(50)

Remark 4.2. If \( b \) is linear then, we have
\[
\xi(t)(b \circ \Delta u)^{\frac{1}{1+\alpha}}(t) = [\xi^\alpha(t)(b \circ \Delta u)(t)]^{\frac{1}{1+\alpha}}
\]
\[
\leq [\xi^\alpha(0)(b \circ \Delta u)(t)]^{\frac{1}{1+\alpha}}
\]
\[
\leq c(\xi(t)(b \circ \Delta u)(t))^{\frac{1}{1+\alpha}} (51)
\]
\[
\leq c(-E'(t))^{\frac{1}{1+\alpha}}
\]

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Theorem 4.1. Let \((u_0, u_1) \in H^2_0(\Omega) \times H^2_0(\Omega)\). Assume that the hypothesis 
\(H1 - H4\) holds and \(h_1\) is linear. Then for all \(t \geq t_0\), we have

if B is linear then \(E(t) \leq c \left(1 + \int_{t_0}^t \xi^{1+\epsilon_0}(s)ds\right)^{-\frac{1}{\epsilon_0}}, \forall t \geq t_0\), \((52)\)

and

if B is nonlinear then \(E(t) \leq c(t-t_0)^{\frac{1}{1+\epsilon_0}}\mathcal{K}^{-1}\left(\frac{c}{(t-t_0)^{\frac{1}{1+\epsilon_0}}} \int_0^t \xi(s)ds\right), \forall t \geq t_1\), \((53)\)

where \(\mathcal{K}(t) = t\mathcal{K}'(\epsilon_1t)\) and \(\mathcal{K}'(t) = \left((\bar{B}^{-1})^{\frac{1}{1+\epsilon_0}}\right)^{-1}(t)\).

Proof. We will divide the proof of this theorem into two cases.

Case 1: \(G\) is linear. Using \((49)\),

\[L'(t) \leq -mE(t) + c(b \circ \Delta u)(t) + c(b \circ \Delta u)^{\frac{1}{1+\epsilon_0}}(t) + c \int_\Omega h^2(u)dx\]

\[\leq -mE(t) + c(b \circ \Delta u)(t) + c(b \circ \Delta u)^{\frac{1}{1+\epsilon_0}}(t) + c(-E'(t)).\]

Denote \(L_1(t) = L(t) + cE(t)\), then the above inequality becomes

\[L'_1(t) \leq -mE(t) + c(b \circ \Delta u)(t) + c(b \circ \Delta u)^{\frac{1}{1+\epsilon_0}}(t)\]

(54)

Multiplying \(\xi(t)\) to \((54)\) and using Remarks \(4.1\) and \(4.2\) and for all \(t \geq t_0\) we get

\[\xi(t)L'_1(t) \leq -m\xi(t)E(t) + c(-E'(t))^{\frac{1}{1+\epsilon_0}}.\]

Multiplying the above inequality by \(\xi^{\epsilon_0}(t)E^{\epsilon_0}(t)\) and using the Young’s inequality and for any \(\epsilon_1 > 0\) and \(t \geq t_0\), we obtain

\[\xi^{1+\epsilon_0}(t)E^{\epsilon_0}(t)L'_1(t) \leq -m\xi^{1+\epsilon_0}(t)E^{1+\epsilon_0}(t) + c(\xi E)^{\epsilon_0}(t)(-E'(t))^{\frac{1}{1+\epsilon_0}}\]

\[\leq -m\xi^{1+\epsilon_0}(t)E^{1+\epsilon_0}(t) + c\left(\epsilon_1 \xi^{1+\epsilon_0}(t)E^{1+\epsilon_0}(t) - \epsilon_1 E'(t)\right)\]

\[\leq -(m - \epsilon_1 c)\xi^{1+\epsilon_0}(t)E^{1+\epsilon_0}(t) - cE'(t).\]

Denote \(L_2(t) = \xi^{1+\epsilon_0}(t)E^{\epsilon_0}(t)L_1(t) + cE(t)\) and choosing \(\epsilon_1 < \frac{m}{c}\) and using the properties of \(\xi\) and \(E\) to get

\[L_2'(t) \leq -c\xi^{1+\epsilon_0}(t)E^{1+\epsilon_0}(t), \forall t \geq t_0,\]

(55)

where \(c_1 = m - \epsilon_1 c\). Since, \(L_2 \sim E\) and from \((55)\), it is easy to see that

\[E'(t) \leq -c\xi^{1+\epsilon_0}(t)E^{1+\epsilon_0}(t), \forall t \geq t_0,\]

Integrating the above inequality from \((t_0, t)\), we obtain \((42)\).
Case 2: $G$ is nonlinear. Using (49),

$$L'(t) \leq -mE(t) + c(b \circ \Delta u)(t) + c(b \circ \Delta u)\frac{1}{t-\xi}(t) + c \int_{\Omega} h^2(u_t)dx$$

$$\leq -mE(t) + c(b \circ \Delta u)(t) + c(b \circ \Delta u)\frac{1}{t-\xi}(t) + (-E'(t)).$$

Denote $L_1(t) = L(t) + cE(t)$, then using Lemma 4.3 and Remark 4.1 the above inequality becomes

$$L_1'(t) \leq -mE(t) + c(b \circ \Delta u)(t) + c(b \circ \Delta u)\frac{1}{t-\xi}(t)$$

$$\leq -mE(t) + c(t-t_0)\frac{1}{t-\xi} \tilde{B}^{-1}\left(\frac{\delta M(t)}{(t-t_0)\xi(t)}\right)^{1+\alpha},$$

the last line in the above inequality follows from the fact that, there exists $t_1 > t_0$ such that $\frac{1}{t-t_0} < 1$ when ever $t > t_1$. Hence we have

$$\tilde{B}^{-1}\left(\frac{\delta M(t)}{(t-t_0)\xi(t)}\right) \leq \tilde{B}^{-1}\left(\frac{\delta M(t)}{(t-t_0)^{1+\alpha}\xi(t)}\right) \forall t > t_1.$$ 

Denote

$$\mathcal{K}(t) = \left[\left(\tilde{B}^{-1}\right)^{1+\alpha}\right]^{-1}(t), \quad \alpha(t) = \frac{\delta M(t)}{(t-t_0)^{1+\alpha}\xi(t)}. \quad (56)$$

From the definition of $\mathcal{K}$, it is clear that $\mathcal{K}'(t) > 0$ and $\mathcal{K}''(t) > 0$ on $(0, r]$ where $r = \min\{r_1, r_2\}$. Using these notations, we obtain that for all $t \geq t_1$,

$$L_1'(t) \leq -mE(t) + c(t-t_0)^{1+\alpha} \mathcal{K}^{-1}(\alpha(t)). \quad (57)$$

Now, we define the functional $L_2$ as follows

$$L_2(t) := \mathcal{K}'\left(\frac{\epsilon_1}{(t-t_0)^{1+\alpha}} \frac{E(t)}{E(0)}\right) L_1(t),$$

for some $\epsilon_1 \in (0, r)$. Using the fact that $\mathcal{K}'(t) > 0$, $\mathcal{K}''(t) > 0$ and $E'(t) \leq 0$ on $(0, r]$, and using (57) we obtain

$$L_2'(t) \leq -mE(t)\mathcal{K}'\left(\frac{\epsilon_1}{(t-t_0)^{1+\alpha}} \frac{E(t)}{E(0)}\right) + c(t-t_0)^{1+\alpha}\mathcal{K}'\left(\frac{\epsilon_1}{(t-t_0)^{1+\alpha}} \frac{E(t)}{E(0)}\right) \mathcal{K}^{-1}(\alpha(t)), \forall t \geq t_1, \quad (58)$$

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and

\[ L_2 \sim E. \]  

(59)

Let \( \mathcal{H}^* \) denote the convex conjugate of \( \mathcal{H} \) in the sense of Young (see [24]), then we have

\[ \mathcal{H}^*(\tau) = \tau(\mathcal{H}')^{-1}(\tau) - \mathcal{H}(\lfloor \mathcal{H}' \rfloor^{-1}(\tau)), \quad \text{if} \quad \tau \in (0, \mathcal{H}'(r)), \]

here \( \mathcal{H}^* \) satisfies the generalized Young Inequality:

\[ ab \leq \mathcal{H}^*(a) + \mathcal{H}(b), \quad \text{if} \quad a \in (0, \mathcal{H}'(r)), \ b \in (0, r]. \]

So, by assuming \( a = \mathcal{H}' \left( \frac{\epsilon_1}{(t-t_0)^{1+\sigma_0}} \frac{E(t)}{E(0)} \right) \) and \( b = \mathcal{H}^{-1}(\alpha(t)) \) and using \( (58) \) we obtain

\[
L'_2(t) \leq -mE(t)\mathcal{H}' \left( \frac{\epsilon_1}{(t-t_0)^{1+\sigma_0}} \frac{E(t)}{E(0)} \right) \\
+ c(t-t_0)^{\frac{1}{1+\sigma_0}} \mathcal{H}^* \left[ \mathcal{H}' \left( \frac{\epsilon_1}{(t-t_0)^{1+\sigma_0}} \frac{E(t)}{E(0)} \right) \right] + c(t-t_0)^{\frac{1}{1+\sigma_0}} \alpha(t) \\
\leq -mE(t)\mathcal{H}' \left( \frac{\epsilon_1}{(t-t_0)^{1+\sigma_0}} \frac{E(t)}{E(0)} \right) + c\epsilon_1 \frac{E(t)}{E(0)} \mathcal{H}' \left( \frac{\epsilon_1}{(t-t_0)^{1+\sigma_0}} \frac{E(t)}{E(0)} \right) \\
+ c(t-t_0)^{\frac{1}{1+\sigma_0}} \alpha(t)
\]

Multiplying the above inequality by \( \xi(t) \) and using \( (43) \) and \( (56) \) for all \( t \geq t_1 \) we get

\[
\xi(t)L'_2(t) \leq -m\xi(t)E(t)\mathcal{H}' \left( \frac{\epsilon_1}{(t-t_0)^{1+\sigma_0}} \frac{E(t)}{E(0)} \right) \\
+ c\epsilon_1 \xi(t) \frac{E(t)}{E(0)} \mathcal{H}' \left( \frac{\epsilon_1}{(t-t_0)^{1+\sigma_0}} \frac{E(t)}{E(0)} \right) - cE'(t).
\]

By setting \( L_3 := \xi L_2 + cE \) (notice that \( L_3 \sim E \)), we get for all \( t \geq t_1 \)

\[
L'_3(t) \leq - (mE(0) - c\epsilon_1) \xi(t) \mathcal{H}' \left( \frac{\epsilon_1}{(t-t_0)^{1+\sigma_0}} \frac{E(t)}{E(0)} \right)
\]

by choosing \( \epsilon_1 \) such that \( mE(0) - c\epsilon_1 > 0 \), we obtain

\[
L'_3(t) \leq -c\xi(t) \frac{E(t)}{E(0)} \mathcal{H}' \left( \frac{\epsilon_1}{(t-t_0)^{1+\sigma_0}} \frac{E(t)}{E(0)} \right) \\
\leq -c\xi(t) \frac{E(t)}{E(0)} \mathcal{H}' \left( \frac{\epsilon_1}{(t-t_0)^{1+\sigma_0}} \frac{E(t)}{E(0)} \right) \\
\leq -c\xi(t) \frac{E(t)}{E(0)} \mathcal{H}' \left( \frac{\epsilon_1}{(t-t_0)^{1+\sigma_0}} \frac{E(t)}{E(0)} \right) \\
(60)
\]
Lemma 4.7. Assume that there exists some constant \( c > 0 \) such that 
\[
\frac{d}{dt} H(t) < \frac{c}{t^{1/\theta}} H(t) \quad \text{for all } t > 0.
\]
Using the strict convexity and strictly increasing properties of \( \bar{H} \), \( \bar{H} > 0 \), the above inequality reduces to 
\[
\frac{d}{dt} H(t) < \frac{c}{t^{1/\theta}} H(t) \quad \text{for all } t > 0.
\]
Hence, 
\[
\frac{c}{t^{1/\theta}} H(t) < H(t) \quad \text{for all } t > 0.
\]
Setting \( \bar{H}_1(\tau) = \tau \bar{H}'(\epsilon_1 \tau) \), the above inequality reduces to 
\[
c\bar{H}_1 \left( \frac{\epsilon_1}{(t-t_0)^{1+\theta}} E(0) \right) \int_{t_0}^t \xi(\tau) d\tau \leq \frac{c_1}{(t-t_0)^{1+\theta}}.
\]
After rearranging the terms in (61), we conclude that 
\[
E(t) \leq c(t-t_0)^{1/\theta} \bar{H}_1^{-1} \left( \frac{c_1}{(t-t_0)^{1+\theta}} \int_{t_0}^t \xi(\tau) d\tau \right), \quad \forall t \geq t_1.
\]
Hence, the theorem is proved.

\[\square\]

Remark 4.3. If \( H(0) = 0 \) and \( H \) is strictly convex on \((0, r] \), then 
\[
H(\theta s) \leq \theta H(s), \quad \theta \in [0, 1], \quad \text{and } s \in (0, r].
\]

(62)

Lemma 4.7. Assume that there exists \( C \) such that \( \tau h(\tau) \leq C \). Then for some constant \( c > 0 \), we have the following estimate:

\[
\bar{H}^{-1}(G(t)) \leq c(t-t_1)^{1/\theta} \bar{H}^{-1} \left( \frac{M(t)}{(t-t_1)^{1+\theta}} \right). \quad (63)
\]

Proof. Since, \( \lim_{t \to \infty} \frac{1}{t-t_1} = 0 \), \( \exists t_2 \) such that \( \frac{1}{t-t_1} < 1 \) whenever \( t > t_2 \). Using the strict convexity and strictly increasing properties of \( \bar{H} \), with \( \theta = \frac{1}{(t-t_1)^{1+\theta}} < 1 \) and using (62), we get 
\[
\bar{H}^{-1}(G(t)) \leq (t-t_1)^{1/\theta} \bar{H}^{-1} \left( \frac{M(t)}{(t-t_1)^{1+\theta}} \right). \quad (64)
\]
Since, \( \tau h(\tau) \leq C \) it is easy to see that \( M(t) \leq C \) and also \( \frac{M(t)}{(t-t_1)^{\frac{1}{\theta}}} \leq C \). Therefore

\[
\bar{H}^{-1}\left( \frac{M(t)}{(t-t_1)^{\frac{1}{\theta}}} \right) = \bar{H}^{-1}\left( \frac{M(t)}{(t-t_1)^{\frac{1}{\theta}}} \right)^{\frac{1}{1+\theta}} \bar{H}^{-1}\left( \frac{M(t)}{(t-t_1)^{\frac{1}{\theta}}} \right)^{\frac{1}{1+\theta}} \leq c \bar{H}^{-1}\left( \frac{M(t)}{(t-t_1)^{\frac{1}{\theta}}} \right)^{\frac{1}{1+\theta}}
\]  

(65)

Therefore (63) follows from (64) and (64). Hence, the lemma is proved. \( \Box \)

**Theorem 4.2.** Let \((u_0, u_1) \in H^2_0(\Omega) \times H^2_0(\Omega)\) and \(h_1, B\) are nonlinear. Assume that the hypothesis of Lemma 4.7 and the hypothesis \((H1) - (H4)\) holds. Then for all \(t \geq t_1\), we have

\[
E(t) \leq c(t-t_0)^{\frac{1}{1+\theta}} W_2^{-1}\left( \frac{c}{(t-t_0)^{\frac{1}{1+\theta}} \int_{t_0}^t \xi(s)ds} \right),
\]  

(66)

where \(W_2(t) = tW'(e_1 t)\) and \(W(t) = \left( (\bar{B}^{-1})^{\frac{1}{1+\theta}} + (\bar{H}^{-1})^{\frac{1}{1+\theta}} \right)^{-1}(t)\).

**Proof.** Using (49) and Lemma 4.2, observe that

\[
L'(t) \leq -mE(t) + c(b \circ \Delta u)(t) + c(b \circ \Delta u)^{\frac{1}{\theta_0}}(t) + c \int_{\Omega} h^2(u_t)dx
\]

\[
\leq -mE(t) + c(b \circ \Delta u)(t) + c(b \circ \Delta u)^{\frac{1}{\theta_0}}(t) + cH^{-1}(M(t)) + c(-E'(t)).
\]

Denote \(L_1(t) = L(t) + cE(t)\), then using Remark 4.1 Lemma’s 4.3 and 4.7 and for all \(t \geq t_1\) the above inequality becomes

\[
L_1'(t) \leq -mE(t) + c(b \circ \Delta u)(t) + c(b \circ \Delta u)^{\frac{1}{\theta_0}}(t) + cH^{-1}(M(t))
\]

\[
\leq -mE(t) + c(b \circ \Delta u)^{\frac{1}{\theta_0}}(t) + cH^{-1}(M(t))
\]

\[
\leq -mE(t) + c\int_{t_0}^t \frac{1}{\theta_0} B^{-1}\left( \frac{\delta M(t)}{(t-t_0)^{\frac{1}{\theta}}} \right)^{\frac{1}{1+\theta}} + c(t-t_0)^{\frac{1}{1+\theta}} H^{-1}\left( \frac{M(t)}{(t-t_0)^{\frac{1}{\theta}}} \right)^{\frac{1}{1+\theta}}
\]

\[
\leq -mE(t) + c\int_{t_0}^t \frac{1}{\theta_0} B^{-1}\left( \frac{\delta M(t)}{(t-t_0)^{\frac{1}{\theta}}} \right)^{\frac{1}{1+\theta}} + c(t-t_0)^{\frac{1}{1+\theta}} H^{-1}\left( \frac{M(t)}{(t-t_0)^{\frac{1}{\theta}}} \right)^{\frac{1}{1+\theta}}
\]

\[
= -mE(t) + c\int_{t_0}^t \frac{1}{\theta_0} B^{-1}\left( \frac{\delta M(t)}{(t-t_0)^{\frac{1}{\theta}}} \right)^{\frac{1}{1+\theta}} + c(t-t_0)^{\frac{1}{1+\theta}} H^{-1}\left( \frac{M(t)}{(t-t_0)^{\frac{1}{\theta}}} \right)^{\frac{1}{1+\theta}}
\]

\[
\leq -mE(t) + c\int_{t_0}^t \frac{1}{\theta_0} B^{-1}\left( \frac{\delta M(t)}{(t-t_0)^{\frac{1}{\theta}}} \right)^{\frac{1}{1+\theta}} + c(t-t_0)^{\frac{1}{1+\theta}} H^{-1}\left( \frac{M(t)}{(t-t_0)^{\frac{1}{\theta}}} \right)^{\frac{1}{1+\theta}}
\]
Denote
\[ W(t) = \left( (\bar{B}^{-1})^{1+\frac{1}{2\gamma}} + (\bar{H}^{-1})^{1+\frac{1}{2\gamma}} \right)^{-1}(t), \] (67)

and
\[ \beta(t) = \max \left\{ \frac{\delta M(t)}{(t-t_0)^{1+\alpha} \xi(t)}, \frac{M(t)}{(t-t_1)^{1+\alpha}} \right\}. \] (68)

From the definition of \( W \), it is clear that \( W'(t) > 0 \) and \( W''(t) > 0 \) on \((0, r] \), where \( r = \min\{r_1, r_2\} \). Using these notations, we obtain for all \( t \geq t_1 \),
\[ L'_1(t) \leq -mE(t) + c(t-t_0)^{1+\alpha} W^{-1}(\beta(t)). \] (69)

To conclude (66), we follow the similar lines as shown in the proof of Theorem 4.1. This completes the proof of this theorem.

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