Cloning and Broadcasting in Generic Probabilistic Models

Howard Barnum\textsuperscript{1} Jonathan Barrett\textsuperscript{2} Matthew Leifer\textsuperscript{2,3} Alexander Wilce\textsuperscript{4}

\textsuperscript{1}CCS-3: Modeling, Algorithms, and Informatics, Mail Stop B256, Los Alamos National Laboratory, Los Alamos, NM 87545 USA.

\textsuperscript{2}Perimeter Institute for Theoretical Physics, 31 Caroline Street North, Waterloo, Ontario, Canada, N2L 2Y5

\textsuperscript{3}Centre for Quantum Computation, Department of Applied Maths and Theoretical Physics, University of Cambridge, Wilberforce Road, Cambridge, CB3 0WA, UK

\textsuperscript{4}Department of Mathematical Sciences, Susquehanna University, Selinsgrove, PA 17870 USA.

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Abstract

We prove generic versions of the no-cloning and no-broadcasting theorems, applicable to essentially any non-classical finite-dimensional probabilistic model that satisfies a no-signaling criterion. This includes quantum theory as well as models supporting “super-quantum” correlations that violate the Bell inequalities to a larger extent than quantum theory. The proof of our no-broadcasting theorem is significantly more natural and more self-contained than others we have seen: we show that a set of states is broadcastable if, and only if, it is contained in a simplex whose vertices are cloneable, and therefore distinguishable by a single measurement. This necessary and sufficient condition generalizes the quantum requirement that a broadcastable set of states commute.

1 Introduction

The growth of quantum information science has led many to wonder which aspects of quantum mechanics are responsible for its enhanced information processing powers. Some have compared quantum and classical theories in frameworks broad enough to encompass both of them and more \cite{27, 28, 2, 3, 4, 9, 19}, and others have constructed toy theories that capture qualitative features of quantum information protocols \cite{26, 51, 50}. Beyond simply understanding the conceptual sources of the power of quantum theory, researchers have become interested in information-processing as a source of axioms that could characterize probabilistic physical theories \cite{23, 24, 17, 2, 3, 51, 9}, shedding light on the conceptual essence of quantum mechanics and potentially giving new stimulus to the longstanding program \cite{42, 37, 38, 39, 40, 1} of axiomatic characterization of quantum theory. It has even been suggested that this approach might ease the integration of quantum theory with general relativity and gravitation \cite{29}.
As part of this development several authors, notably Barrett [9] and Spekkens [51], have recently taken up the question of how far the information-theoretic novelties presented by quantum mechanics are in fact generic in other types of probabilistic theory. Spekkens constructs an ingenious “toy model” in which a limitation on the amount of knowledge available to observers is sufficient to yield, among many other things, a no-cloning property. Working in a framework in which essentially any finite-dimensional compact convex set counts as a state space, Barrett shows that universal probabilistic cloning is impossible in any non-classical finite-dimensional probabilistic theory.

A major motivation for Barrett’s work was to come up with a reasonable physical framework in which arbitrary nonsignaling correlations may be obtained from measurements on a bipartite system. Such correlations can be more non-local than quantum theory allows, and include the super-quantum correlations that have come to be known as Popescu-Rohrlich (PR) boxes, or Non-Local Machines [33, 45, 10, 58, 47, 15, 14, 11, 32, 53, 54, 13]. His framework is based on that of Hardy, and in developing it Barrett and Hardy have essentially reinvented the finite-dimensional version of a much older framework for generalized probabilistic models, based on convex sets [42, 37, 38, 39, 40, 41, 21, 12, 25, 30], which grew out of attempts to axiomatize quantum theory within the quantum logic tradition, and which we adopt here.

Popescu and Rohrlich [45] originally raised the question of why nature does not allow super-quantum correlations, given that they would not violate relativistic causality. In this regard, it is important to distinguish the unique features of quantum theory from those that would still hold in theories permitting more general correlations. Placing PR boxes within a framework that also includes quantum theory and classical probability theory as special cases, helps to understand the common features that these have been found to exhibit (see [9] for a discussion of many of these).

In this paper, we completely characterize the sets of states that can be cloned or broadcast in any finite-dimensional probabilistic model within the convex sets framework, obtaining along the way a simple, natural, and self-contained proof of the quantum no-broadcasting theorem that is substantially simpler than the original proof of Barnum, Caves, Fuchs, Jozsa, and Schumacher [5], and substantially more intuitive and self-contained than that based on Lindblad’s Theorem [36] (which, however, provided some suggestive ideas).

In section 2, we sketch the standard framework for generalized probability theory, in which arbitrary compact convex sets are construed as state-spaces. We restrict our attention, in the main, to finite-dimensional state spaces. In this context, a state space is classical iff it is a simplex. In section 3, we discuss the maximal, or injective, tensor product of convex sets, pointing out along the way some familiar aspects of entanglement (e.g., entanglement monogamy) that hold generically for all non-classical models. In section 4, we prove our generic no-cloning theorem. We show that the set of states cloned by an affine mapping must be distinguishable from one another, with certainty, by a single observable. It follows that only when the state-space is a simplex is it possible to clone all pure states.

In section 5, we show that the set of states broadcast by an affine mapping is contained in a possibly larger set of states, the extreme points of which are cloned by an affine map. It follows that the extreme points of this larger set are distinguishable. In fact we show that a set of states is broadcastable if, and only if, it is contained in a simplex whose vertices are jointly distinguishable. In the quantum-mechanical setting, convex combinations of distinguishable states commute, so we obtain the quantum no-broadcasting theorem as a corollary. Finally, we extend this result to show that for any affine map, the set of states it broadcasts is a (possibly empty) simplex whose vertices are distinguishable states. To prove this, we use an extension of the classical Perron-Frobenius theory for (possibly reducible) non-negative real square matrices. The necessary technical apparatus is collected in an appendix.
2 The Framework

To survey all possible probabilistic theories requires some altitude. That is, one needs to work in a mathematical framework that imposes only the most minimal constraints on the structure of probabilistic models. Such a framework was constructed, for just this purpose, by Mackey [42] in the late 1950s; refinements and stylistic variants of this can be found in the work of many other authors, including Ludwig [37, 38, 39, 40, 41], Foulis and Randall [21], Beltrametti and Bugajski [12], Gudder et al. [25], and Holevo [30]. The framework developed by Hardy [27, 28] (see also [43]) for an axiomatic derivation of quantum mechanics is essentially a finite-dimensional version. What follows is simply a sketch of this common, more or less canonical, framework.

States and Effects

We assume that a physical system is characterized by its state-space $\Omega$, which we take to be convex. We write $\mathcal{A}(\Omega)$ for the space of all affine linear functionals $f : \Omega \to \mathbb{R}$, and $\mathcal{A}(\Omega)_+$ for the space of all nonnegative linear functionals $f : \Omega \to \mathbb{R}_+$. Note that $\mathcal{A}(\Omega)$ is an ordered linear space, with $f \leq g$ iff $f(\omega) \leq g(\omega)$ for all $\omega \in \Omega$. The order unit of $\mathcal{A}(\Omega)$ is the unit functional $u$ given by $u(\omega) = 1$ for all $\omega \in \Omega$; the unit interval in $\mathcal{A}(\Omega)$ is the set $[0, u]$ consisting of all functionals $a \in \mathcal{A}(\Omega)$ satisfying $0 \leq a \leq u$ (in the pointwise ordering on $\Omega$).

We interpret each $a \in [0, u]$ as representing an “effect” – that is, some possible event or occurrence associated with the system – and $a(\omega)$, as the probability of this occurrence when the system is in state $\omega$. There is a natural embedding of $\Omega$ in $\mathcal{A}(\Omega)^*$, given by $\omega \mapsto \hat{\omega}$, where $\hat{\omega}(a) = a(\omega)$ for all $a \in \mathcal{A}(\Omega)$. Henceforth, we identify $\omega$ with $\hat{\omega}$, writing $\omega(a)$ in place of $a(\omega)$, as this is in better keeping with the the idea of states assigning probabilities to effects (rather than effects assigning expected values to states).

We write $V(\Omega)$ for the span of $\Omega$ in $\mathcal{A}(\Omega)^*$. The space $V$ is ordered by the cone $V_+$ consisting of all $\mu \in V$ with $\mu(a) \geq 0$ for every $a \in \mathcal{A}(\Omega)_+$. Equivalently, $\mu \in V_+$ iff $\mu$ is a non-negative multiple of a state $\omega \in \Omega$. Accordingly, we call elements of $V(\Omega)$ weights. We say that $\Omega$ is finite-dimensional iff $V(\Omega)$ is finite-dimensional, and compact iff $\Omega$ is compact in the weakest topology making evaluation at each $a \in [0, u]_0$ continuous. For the remainder of this paper, we make the standing assumption that all state spaces are finite-dimensional and compact (equivalently, closed) as subsets of $\mathcal{A}(\Omega)^*$. This guarantees that $\Omega$ is the closed convex hull of its extreme points, which are referred to as pure states.

Examples

In constructing examples, one often begins with a test space (or manual) [21, 34, 35]: that is, a collection $\mathcal{A}$ of (not necessarily disjoint) sets $E, F, \ldots$, called tests, interpreted as the outcome-sets of various measurements. Let $X = \bigcup \mathcal{A}$ be the set of all outcomes of all tests $E \in \mathcal{A}$. A state on $\mathcal{A}$ is defined to be a mapping $\omega : X \to [0, 1]$ summing independently to 1 over each $E \in \mathcal{A}$. The collection $\Omega(\mathcal{A})$ of all such states is obviously convex. If each $E \in \mathcal{A}$ is finite, then it is also compact in the topology of pointwise convergence on $X$ [55]. A state is deterministic (dispersion-free) iff its value on each outcome $x \in E$ is either 0 or 1.

(a) If $\mathcal{A}$ consists of a single test $E$, with a finite number of outcomes then $\Omega(\mathcal{A})$ is the set of all classical probability distributions over $E$. This is a simplex, which we denote by $\Delta(E)$.

(b) If $\mathcal{A}$ consists of two two-outcome tests $E_0 = \{a_{00}, a_{01}\}$ and $E_1 = \{a_{10}, a_{11}\}$, then $\Omega(\mathcal{A})$ is a square. The index $i$ in $a_{ij}$ can be thought of as the “input”, corresponding to the choice of measurement to be performed on the system, and the index $j$ can be thought of as a binary “output”. Then, the states $\omega \in \Omega(\mathcal{A})$ can be thought of as conditional probability distributions (or equivalently 2 × 2 stochastic matrices) where $p(\text{output} = j | \text{input} = i) = \omega(a_{ij})$, and any conditional probability distribution likewise defines a valid state. The four vertices of the state space are the four deterministic states corresponding to the choice of a definite output for each possible input. Clearly, this construction can be repeated for a test space with any number of nonoverlapping tests, the resulting state space being an appropriate
set of conditional probability distributions. Such test spaces are often called “semi-classical” test spaces in the quantum logic literature \[56, 20\].

(c) If $\mathfrak{A}$ is the collection of all maximal orthonormal subsets (i.e., orthonormal bases) of a Hilbert space $\mathbf{H}$ of dimension at least 3, then $\Omega(\mathfrak{A})$ is canonically isomorphic to the convex set of density operators on $\mathbf{H}$, by Gleason’s Theorem.

(d) An interesting model, well known in the quantum logic literature \[56, 20\], consists of three, three-outcome tests with a self-adjoint operator.

(e) For another example, let $\mathfrak{A}$ consist of the rows and columns of a $3 \times 3$ array: then $\Omega(\mathfrak{A})$ is the convex set of doubly-stochastic $3 \times 3$ matrices, which is not a simplex in spite of the fact that the pure states, corresponding to permutation matrices, are deterministic.

**Observables**

By a (discrete) observable on a system with state-space $\Omega$, we mean a function $F : x \mapsto F_x$ from a finite set $E$ into $A(\Omega)$, satisfying (i) $F_x \geq 0$ for all $x \in E$, and $\sum_{x \in E} F_x = u$. Any state $\omega \in \Omega$ pulls back along $F$ to a probability weight $p \in \Delta(E) \equiv p(x) = F_x(\omega)$. This provides a dual map $F^* : \Omega \to \Delta(E)$ defined as $F^*(\omega) = p$. Note that this definition of an observable generalizes the notion of a Positive Operator Valued Measure (POVM) in quantum theory, rather than the more specialized notion of an observable associated with a self-adjoint operator.

A special case of an observable is a list $(a_1, \ldots, a_k)$ of positive elements of $A(\Omega)$ that sums to $u$ (in this case, the mapping $F : \{1, \ldots, k\} \to [0, u]$ taking $i$ to $a_i$ is implicit.) Most of the observables considered below will be of this type.

An observable $F$ is said to be **informationally complete**, or IC, if and only if the set of functionals $\{F_x|x \in E\}$ separates states, i.e., if $F_x(\omega) = F_x(\mu)$ for all $x \in E$ implies $\omega = \mu$ for all states $\omega, \mu \in \Omega$. (This is equivalent to saying that the dual mapping $F^* : \Omega \to \Delta(E)$ is an affine injection.) Note that $F$ is IC if and only if $\{F_x|x \in E\}$ spans $A(\Omega)$. If this set is a basis for $A(\Omega)$, we shall say the observable is minimally IC. The following result is not new (see \[49\] for an infinite-dimensional version), but we include a proof for completeness.

**Lemma 1** Any finite-dimensional state space supports a minimal informationally complete observable.

Proof: It suffices to produce a sequence $(a_1, \ldots, a_n)$ of vectors $a_i \in [0, u]$, with $n = \dim(A(\Omega))$ distinct entries, summing to the order unit $u$. Let $B = \{b_1, \ldots, b_n\}$ be any basis for $A(\Omega)$. Without loss of generality, suppose that $\sum_i b_i = ku$, a multiple of the order unit. (If not, apply a suitable invertible linear transformation). Let $c$ be the minimum of $\inf\{b_i(\omega)|\omega \in \Omega\}$. Then $b_i - cu$ is positive. Now, $\sum_i (b_i - cu) = (k - nc)u$, with $k - nc \geq 0$. Hence, if $a_i = (b_i - cu)/(k - nc)$, we have $a_i \geq 0$ and $\sum_i a_i = u$. Obviously, $(a_i | i = 1, \ldots, n)$ spans $A(\Omega)$, so $(a_1, \ldots, a_n)$ is a minimal IC observable. $\square$

**Operations**

Any physically performable operation on a system should respect probabilistic mixtures of states, and hence, should be representable by an affine mapping $\phi : \Omega \to \Omega'$, where $\Omega$ is the state space of the system prior to the operation being performed, and $\Omega'$ is the post-operation state space. Generally, the set of allowed operations in a given model could be a strict subset of the set of all affine maps. This should be familiar from the quantum case given in example (c), since in that case the affine maps are the set of all positive, trace-preserving, linear maps on operators, whereas quantum operations are usually taken to be completely positive. Since we are concerned with proving restrictions on the set of operations available in any model, we assume that all affine maps represent possible operations, but the restrictions obviously still apply to any subset of these maps.

**Lemma 2** Let $E = (a_1, \ldots, a_n)$ be any observable on $\Omega$, and let $\delta_1, \ldots, \delta_n \in \Omega'$ be any states in $\Omega'$. Then...
The mapping \( \phi : \Omega \to \Omega' \) given by
\[
\phi : \omega \mapsto \sum_i \omega(a_i) \delta_i
\]
for all \( \omega \in \Omega \) is affine, i.e., an operation.

The proof is routine. Physically, such a process could be implemented by measuring \( E \) and then preparing the indicated state.

Notice that any operation \( \kappa : \Omega \to \Omega' \) determines a dual linear transformation \( \kappa^* : A(\Omega') \to A(\Omega) \), given by \( \kappa^*(f)(\omega) = f(\kappa(\omega)) \) for all effects \( f \in A(\Omega') \) and all states \( \omega \in \Omega \). This mapping preserves positivity and the order unit, and hence, allows us to pull observables on \( \Omega' \) back to observables on \( \Omega \). (In this connection, notice also that if \( \kappa \) is injective, \( \kappa^* \) will pull informationally complete observables on \( \Omega' \) back to informationally complete observables on \( \Omega \).)

### 3 Tensor Products

Given two systems with state spaces \( \Omega \) and \( \Omega' \), we’d like to construct a state space to represent a coupled system with these as components. There is in general no unique way to do this, but rather there is a spectrum of candidates, bounded by a maximal and a minimal tensor product.

**Definition:** The maximal tensor product of two state spaces \( \Omega \) and \( \Omega' \), which we’ll denote by \( \Omega \otimes \Omega' \), is the set of all bilinear functionals \( \mu : A(\Omega) \times A(\Omega') \to \mathbb{R} \) that are (i) positive on pairs \( (a, b) \) with \( a, b \geq 0 \), and (ii) normalized by \( \mu(u, u') = 1 \) (where \( u \) and \( u' \) are the order-units of \( A(\Omega) \) and \( A(\Omega') \), respectively).

One can show that the maximal tensor product corresponds to the largest set of joint probability assignments to measurements on the two component systems, subject to a “no-signaling” condition [9, 21, 35].

Given states \( \alpha \in \Omega \) and \( \beta \in \Omega' \), one has a product state \( \alpha \otimes \beta \in \Omega \otimes \Omega' \) given by \( (\alpha \otimes \beta)(a, b) = \alpha(a)\beta(b) \) for all \( (a, b) \in A(\Omega) \times A(\Omega') \).

**Definition:** The minimal tensor product of \( \Omega \) and \( \Omega' \) is the the convex hull of the set of product states in \( \Omega \otimes \Omega' \). We term such a convex combination a separable state, and accordingly denote the minimal tensor product by \( \Omega \otimes_{sep} \Omega' \). A non-separable state in \( \Omega \otimes \Omega' \) will be termed entangled.

In the present finite-dimensional setting, \( V(\Omega \otimes \Omega') = V(\Omega) \otimes V(\Omega') \) and \( A(\Omega \otimes \Omega') = A(\Omega) \otimes A(\Omega') \) [31, 35, 40, 41, 55]. It follows that \( \Omega \otimes \Omega \) and \( \Omega \otimes_{sep} \Omega \) have the same affine dimension. Hence, every state in \( \Omega \otimes \Omega \) can be expressed as an affine combination \( \sum t_i \alpha_i \otimes \beta_i \), where \( \sum t_i = 1 \), but the \( t_i \) need not be positive.

**Examples**

(a) If \( \Omega \) and \( \Omega' \) are both classical state spaces, say \( \Omega = \Delta(E) \) and \( \Omega' = \Delta(E') \), then \( \Omega \otimes \Omega' = \Omega \otimes_{sep} \Omega' \), both being isomorphic to \( \Delta(E \times E') \).

(b) If \( \Omega \) and \( \Omega' \) are the state spaces associated with the semiclassical binary-input, binary-output test space discussed in example (b) in section 2, then \( \Omega \otimes \Omega' \) supports all bipartite nonsignaling correlations obtainable with two binary inputs and two binary outputs. The extreme points are the local deterministic states specifying a definite output for each input, and states supporting nonlocal PR-box type correlations. On the other hand \( \Omega \otimes_{sep} \Omega' \) only contains local states from which no Bell-inequality violations can be obtained. More generally, for any pair of semiclassical test spaces \( \Omega \otimes \Omega' \) supports all bipartite nonsignaling correlations with the appropriate cardinality of inputs and outputs, whereas \( \Omega \otimes_{sep} \Omega' \) contains only local states from which no Bell-inequality violations can be obtained.

(c) If \( \Omega \) and \( \Omega' \) are the usual state spaces associated with complex Hilbert spaces \( H \) and \( H' \), then \( \Omega \otimes \Omega' \) is properly larger than the usual quantum state space associated with \( H \otimes H' \) [21, 35, 34, 6]; the minimal tensor product, consisting of separable states, is properly smaller.

Henceforth, by a tensor product for state spaces \( \Omega \) and \( \Omega' \), we’ll simply mean some convex set containing \( \Omega \otimes_{sep} \Omega' \) and contained in \( \Omega \otimes \Omega \).
Remark: Given affine mappings \( \phi : \Omega \rightarrow \Gamma \) and \( \phi' : \Omega' \rightarrow \Gamma' \), there is a unique affine mapping
\[
\phi \otimes \phi' : \Omega \otimes \Omega' \rightarrow \Gamma \otimes \Gamma'
\]
satisfying \((\phi \otimes \phi')(\alpha \otimes \beta) = \phi(\alpha) \otimes \phi'(\beta)\) for all \( \alpha \in \Omega \) and all \( \alpha' \in \Omega' \). In particular, there is no notion of “complete positivity” for either the minimal or maximal tensor product. That is, the tensor product of “complete positivity” for either the minimal or conditional states.

Marginal and Conditional States

A state \( \omega \in \Omega \otimes \Omega' \) has well-defined marginal states \( \omega_1 \in \Omega \) and \( \omega_2 \in \Omega' \) given, respectively, by
\[
a(\omega_1) = (a \otimes u')(\omega) \quad \text{and} \quad b(\omega_2) = (u \otimes b)(\omega)
\]
for all effects \( a \in [0, u] \), \( b \in [0, u'] \). This fact allows us to define conditional states \( \omega_{2,a} \) and \( \omega_{1,b} \) by
\[
\omega_{2,a}(b) := \frac{\omega(a, b)}{\omega_1(a)} \quad \text{and} \quad \omega_{1,b}(a) := \frac{\omega(a, b)}{\omega_2(b)}.
\]
We have the expected identities
\[
\omega(a, b) = \omega_1(a)\omega_{2,a}(b) = \omega_{1,b}(a)\omega_2(b).
\]

The following observation is familiar in the setting of both classical and quantum probability theory:

Lemma 3 If either marginal, \( \omega_1 \) or \( \omega_2 \), of a bipartite state \( \omega \) in \( \Omega \otimes \Omega' \) is pure (i.e. extremal), then \( \omega = \omega_1 \otimes \omega_2 \).

Proof: Suppose \( \omega_2 \) is pure. We wish to show that \( \omega(a, b) = \omega_1(a)\omega_2(b) \) for all effects \( a, b \in [0, u] \). Let \( E \subseteq [0, u] \) be any observable. Then we have,
\[
\omega_2 = \sum_{a \in E} \omega_1(a)\omega_{2,a}.
\]
This gives us \( \omega_2 \) as a convex combination of the states \( \omega_{2,a} \) with coefficients \( \omega_1(a) \). As \( \omega_2 \) is pure, we have for each \( a \in E \) either \( \omega_1(a) = 0 \) or \( \omega_{2,a} = \omega_2 \); in either case, \( \omega(a, b) = \omega_1(a)\omega_2(b) \) for all \( b \in [0, u] \). Since \( E \) was chosen arbitrarily, this holds also for all \( a \in [0, u] \). \( \square \).

The tensor product construction can be iterated – we can form
\[
\Omega^n := \underbrace{\Omega \otimes \cdots \otimes \Omega}_n \quad \text{n times}
\]
Applying Lemma 3 to this setting, we see that the “monogamy of entanglement” [52] is an entirely generic phenomenon. Thus, for instance, if \( \omega \) is a tripartite state in \( \Omega_1 \otimes \Omega_2 \otimes \Omega_3 \), then we can form various marginals, e.g., \( \omega_{12} \in \Omega_1 \otimes \Omega_2 \), etc. If \( \omega_{12} \) is a pure entangled state, then \( \omega = \omega_{12} \otimes \omega_3 \) – whence, \( \omega_{23} = \omega_2 \otimes \omega_3 \) and \( \omega_{13} = \omega_1 \otimes \omega_3 \).

Remarks:

(1) In the context of abstract convex sets, the maximal tensor product (more usually called the injective tensor product) seems first to have been discussed by Namioka and Phelps [44]; see also Wittstock [57] for a survey. As a model for coupled physical systems, it was discussed (implicitly) by Foulis and Randall [21], Kläy, Randall and Foulis [35], and Kläy [34]. (See also [6] and [55]).

(2) The definition of an entangled state as a state not contained in \( \Omega \otimes_{\text{sep}} \Omega' \) naturally generalizes the quantum definition. A pure state is entangled iff it has a mixed marginal, and a mixed state is entangled if it cannot be written as a convex combination of pure product states. (See [7, 8] for an even more broadly applicable generalization of this definition of entanglement to convex operational settings.) With this definition, it is easy to see from Lemma 3 that any tensor product properly larger than the minimal one contains entangled states.

4 Cloning

A deterministic cloning procedure for a state \( \alpha \in \Omega \) involves preparing the system in state \( \alpha \), preparing a second copy of the system in a particular state \( \beta \), and performing an operation on the combined system \( \Omega \otimes \Omega \) that takes the initial state \( \alpha \otimes \beta \) the final state...
\( \alpha \otimes \alpha \). Since the initial ancillary state \( \beta \) is supposed to be fixed, we can equally well regard such a procedure as an affine mapping \( \kappa : \Omega \to \Omega \otimes \Omega \) such that \( \kappa(\alpha) = \alpha \otimes \alpha \). One can also consider probabilistic cloning, in which there is a non-zero probability that the cloning procedure will simply fail (but we will know if it does). Barrett has shown in [9] that universal probabilistic cloning is generically impossible in (finite-dimensional) non-classical theories. Here, we consider only deterministic cloning, and accordingly drop the adjective.

Our aim is to show that a set of states simultaneously cloneable, must also be sharply distinguishable from one another by a single observable and vice versa. Our proof of this is essentially just crystallized folklore: cloning allows us to produce large ensembles of independent copies of each cloneable state; performing the same measurement on each of these defines an observable on the original system, which distinguishes among the cloned states to arbitrary accuracy, by the law of large numbers. Conversely, if a set of states is sharply distinguishable then they may be cloned by measuring the distinguishing observable and then preparing another copy of the corresponding state.

This observation has already been made in the quantum case (see [16] for example) and it has also been noted that the argument does not seem to depend on the details of quantum mechanics, which is confirmed by the present result. However, the argument need not be true in all conceivable frameworks for physical theories, as it depends on the idea that any state can be reliably prepared and that distinct states are separated by some measurement. This is true in the present framework, but theories in which the notion of state includes “hidden variables” provide counterexamples to this. As a rather extreme example, consider a theory just like the ones described here, except that the state of each system is supplemented by a hidden bit that can have value 0 or 1, but which has absolutely no effect on measurement outcomes. Suppose further that any operation from a single system to a bipartite composite system copies the value of the hidden bit to both output systems. In such a world, we can clone states just as well as in the present framework, but nevertheless we cannot distinguish between two states that have differing values of the hidden bit.

In the present framework, the existence of a cloning procedure will depend not only on the structure of the convex set of states, but also on what kinds of affine mappings one admits as “physical” operations. Indeed, the constant mapping that takes every state in \( \Omega \) to the state \( \alpha \otimes \alpha \) is affine; thus, on a liberal understanding of physical operations, in which any affine mapping between state spaces is physically realizable, every state – mixed as well as pure – is (deterministically) cloneable if we do not demand that the same map clone more than this one state.

**Definitions:** Call a finite collection \( \alpha_1, \ldots, \alpha_n \) of states

(a) **co-cloneable** iff there exists a single cloning map \( \kappa : \Omega \to \Omega^2 \) that clones them all, i.e., \( \kappa(\alpha_i) = \alpha_i \otimes \alpha_i \) for every \( i = 1, \ldots, n \), and

(b) **jointly distinguishable** iff there exists an observable \( E = (a_0, \ldots, a_n) \) with \( \alpha_i(a_j) = \delta_{ij} \). In this case, we say that the \( \alpha_i \) are **distinguishable** by \( E \), or that \( E \) is **distinguishing** for \( \alpha_1, \ldots, \alpha_n \).

In discrete classical probability theory, any finite collection of pure states is jointly distinguishable. It is important to note that, in general, a pairwise-distinguishable set of states will not be jointly distinguishable. Indeed, in the case of a binary input, binary output, semiclassical test space (see Example (b) of section 2), any two extreme states are distinguishable by one of the two tests, but no observable will sharply distinguish between any three pure states. (See also the remark following Corollary 1 below.)

In finite-dimensional quantum probability theory, the pure states corresponding to two vectors \( v \) and \( w \) are distinguishable in the foregoing sense iff the vectors \( v \) and \( w \) are orthogonal. More generally, we have the following

**Lemma 4** Quantum states \( \rho \) and \( \rho' \) are distinguishable iff the corresponding density operators satisfy \( \rho \rho' = \rho' \rho = 0 \).
Proof: \( \rho \) and \( \rho' \) are distinguishable iff there exists a self-adjoint operator \( 0 \leq A \leq 1 \) with \( \text{Tr}(A\rho) = 1 \) and \( \text{Tr}(A\rho') = 0 \). Let \( \rho = \sum_i t_i P_i \) where the \( P_i \) are rank-one projections associated with unit vectors \( v_i \), and where the convex coefficients \( t_i \) are all non-zero. If \( \text{Tr}(A\rho) = 1 \), then \( \sum_i t_i (A v_i, v_i) = 1 \). Since \( 0 \leq A \leq 1 \), \( 0 \leq (A v_i, v_i) \leq 1 \), so we must have \( (A v_i, v_i) = 1 \) for each \( i \). In other words, each \( v_i \) belongs to the eigenspace of \( A \) corresponding to eigenvalue 1. By the same argument, if \( \rho' = \sum_j r_j Q_j \) is a convex combination of rank-one projections \( Q_j \) (with \( r_j > 0 \) for all \( j \)), the vectors in the range of \( Q_j \) must belong to the 0-eigenspace of \( A \). Accordingly, \( P_i \perp Q_j \) for every \( i \) and every \( j \), so that \( \rho' = \rho' \rho = 0 \). \( \Box \).

An easy extension of this argument shows that a set of quantum states is jointly distinguishable iff all pairs \( \rho, \rho' \) (with \( \rho \neq \rho' \)) of corresponding density operators satisfy \( \rho \rho' = 0 \). That is, a pairwise distinguishable set of quantum states is jointly distinguishable. As noted above, this is not generally the case. This is one of many respects in which quantum probabilistic models are relatively well-behaved.

**Theorem 1** In any finite-dimensional probabilistic theory, using any tensor product, distinct states are co-cloneable iff they are jointly distinguishable.

In outline, the proof is simply the observation that, to distinguish among the states to any given accuracy, it suffices to produce, by iterated cloning, a sufficiently large ensemble of independent copies of each cloneable state, and then to apply to each copy any observable on which these states have distinct distributions. The details are as follows:

Proof: Suppose first that \( \alpha_1, ..., \alpha_n \) are distinguishable by \( E = \{ a_1, ..., a_n \} \). Define \( \kappa : \Omega \to \Omega^2 \) by
\[
\kappa(\omega) = \sum_{i=0}^{n} a_i(\omega) \alpha_i \otimes \alpha_i
\]
where \( \alpha_0 \) is chosen arbitrarily. As observed in Lemma 2, this mapping is affine; obviously, \( \kappa(\alpha_i) = \alpha_i \otimes \alpha_i \) for \( i = 1, ..., n \).

For the converse, we use the fact that—regardless of what tensor product we use!—cloning maps can be iterated. Let \( E \subseteq [0, u] \) be an informationally complete observable (as afforded by Lemma 1), and consider the \( N \)-fold iterated cloning map \( \kappa_N : \Omega \to \Omega^{2N} \), where \( N \) is a large positive integer. The set \( E_N := E^{2N} \) is a partition of unity in \( A(\Omega^{2N}) \). Every sequence \( x = (x_1, ..., x_{2N}) \) in \( E_N \) determines an empirical distribution \( p_x \) on \( E \), given by
\[
p_x(x) = \frac{|\{j | x_j = x\}|}{2^N},
\]
for each \( i = 1, ..., n \), let
\[
A_{i,N,\epsilon} = \{ x \in E_N \mid \| p_x - \alpha_i \| < \epsilon \},
\]
where \( \| f \|_E \) denotes the maximum absolute value of a function \( f \) over \( E \). By the weak law of large numbers, if \( \alpha_{i,N} := (\alpha_i)^{2N} = \kappa_N(\alpha_i) \), then \( \alpha_{i,N}(A_{i,N,\epsilon}) > 1 - \epsilon \) for sufficiently large \( N \).

Let \( a_{i,N,\epsilon} \) be the unique functional in \([0, u]\) defined by \( a_{i,N,\epsilon}(\omega) = \kappa_N(\omega)(A_{i,N,\epsilon}) \) for all \( \omega \in \Omega \), and pull-back of the set \( A_{i,N,\epsilon} \) along \( \kappa_N \). We then have \( a_{i,N}(\alpha_i, N, \epsilon) > 1 - \epsilon \) for sufficiently large \( N \). Note that, since only finitely many \( \alpha_i \) are involved, we can choose \( N \) large enough to make this hold simultaneously for all \( i = 1, ..., N \). We claim that, for sufficiently large \( N \) and sufficiently small \( \epsilon \), \( \{ a_{i,N,\epsilon} \} \) is summable in \( E \), hence, extends to a partition of unity. It is sufficient to show that \( A_{i,N,\epsilon} \cap A_{k,N,\epsilon} = \emptyset \) for \( i \neq k \). To this end, note that since \( E \) is informationally complete, the distinct states \( \alpha_i \) induce distinct probability distributions on \( E \). In particular, there is some \( \delta > 0 \) such that \( \| \alpha_i - \alpha_k \|_E > \delta \) for all \( i \neq k \). Let \( \epsilon < \delta/2 \). If \( x \in A_{i,N,\epsilon} \cap A_{k,N,\epsilon} \), then
\[
\| \alpha_i - p_x \|_E < \epsilon \text{ and } |p_x - \alpha_k|_E < \epsilon,
\]
so \( \| \alpha_i - \alpha_k \|_E < 2\epsilon < \delta \) a contradiction. Thus, \( A_{i,N,\epsilon} \cap A_{k,N,\epsilon} = \emptyset \), as claimed.

Now let \( a_0 = \kappa^*(E_N \setminus \bigcup_i A_{i,N,\epsilon}) \). We now have an observable \( E_{N,\epsilon} = (a_{i,N,\epsilon} | i = 0, 1, ..., N) \), such that \( \alpha_i(E_{N,\epsilon}) > 1 - \epsilon \) for each \( i \). Since \([0, e]^N \) is compact, we can choose from among the \( E_{N,\epsilon} \) a convergent sequence of observables \( E_m = (a_{i,m}, ..., a_{N,m}) \) with \( a_{i,m}(\alpha_i) > 1 - 1/m \) for all \( i \). Thus, for each \( i \), the
sequence \((a_{i,m})\) converges in \([0, u]\) to an effect \(a_i\) with \(a_i(\alpha_i) = 1\). We also have

\[
\sum_{i=0}^{N} a_i = \lim_{m} \sum_{i=1}^{N} a_{i,m} = \lim_{m} u = u.
\]

Thus, \((a_0, ..., a_n)\) is a distinguishing observable for \(\alpha_1, ..., \alpha_n\), as advertised. \(\square\)

The familiar quantum no-cloning result follows, in view of the remarks about orthogonality preceding the proof. The following result shows that only classical systems – i.e., those the state spaces of which are simplices – allow universal deterministic cloning.

**Corollary 1** Suppose that \(\alpha_1, ..., \alpha_n\) are co-cloneable. Then the convex hull of \(\alpha_1, ..., \alpha_n\) in \(\Omega\) is a simplex. Hence, if every finite set of pure (extremal) states in \(\Omega\) is co-cloneable then \(\Omega\) is a simplex.

**Proof:** A simplex is the only finite dimensional convex set for which each element has a unique decomposition into extremal states. Hence, let \(\alpha_1, ..., \alpha_n\) be jointly distinguishable states, and let \(\sum s_i \alpha_i = \sum t_i \alpha_i = \omega \in \Omega\), where \(s_1, ..., s_n\) and \(t_1, ..., t_n\) are convex coefficients. Let \(E = (a_0, a_1, ..., a_n)\) be a discriminating observable for \(\alpha_1, ..., \alpha_n\). Then \(s_i = a_i(\omega) = t_i\). \(\square\)

**Remark:** One can certainly construct non-classical theories in which any pair of extremal states is distinguishable, and hence cloneable. For example, consider a semi-classical test space, that is, a pairwise disjoint collection of outcome-sets. A pure state on such a test space amounts to a selection of one outcome per test, and any two such states are distinguished by any test on which they differ. (Single systems in both of the theories GNST and GLT considered in [9] are of this form.)

## 5 Broadcasting

We say that a state \(\rho \in \Omega\) is broadcast by an affine mapping \(B : \Omega \to \Omega \otimes \Omega\) iff the bipartite state \(B(\rho)\) has marginals equal to \(\rho\). The quantum no-broadcasting result of Barnum et al. [5] tells us that two quantum states are jointly broadcastable iff, regarded as density operators, they commute. Our aim in this section is to obtain a characterization of joint broadcastability for arbitrary systems.

Let \(B : \Omega \to \Omega \otimes \Omega\) be an affine mapping. We define the marginal mappings \(B_1, B_2 : \Omega \to \Omega\) by \(B_1(\rho)(a) = B(\rho)(a \otimes u)\) and \(B_2(\rho)(b) = B(\rho)(u \otimes b)\).

**Definition:** We say that \(\rho \in \Omega\) is broadcast by \(B\) iff \(B_1(\rho) = B_2(\rho) = \rho\) – that is, iff \(\rho\) is simultaneously a fixed point of both \(B_1\) and \(B_2\). Let \(\Gamma\) be the set of all states \(\rho \in \Omega\) broadcast by \(B\). Note that \(\Gamma\) is a convex subset of \(\Omega\). Indeed, it is \(\Omega\)-affine, meaning it is the intersection of \(\Omega\) with an affine subspace.

Cloning is a special case of broadcasting. Indeed, for pure states of \(\Omega\), broadcasting reduces to cloning: if \(\alpha\) is extreme and \(B(\alpha)\) has marginals equal to \(\alpha\), then by Lemma 3, \(B(\alpha) = \alpha \otimes \alpha\). Thus, no-cloning theorem, there can be no universally broadcasting map on a non-simplicial state space. On the other hand, all states in the convex hull of a distinguishable set of states can be broadcast, simply by cloning the extreme points. To be explicit, let \(\rho = \sum t_i \alpha_i\) be a convex combination of co-cloneable states \(\alpha_1, ..., \alpha_n\), and let \(E = (a_0, ..., a_n)\) be a distinguishing observable for \(\alpha_1, ..., \alpha_n\). Then the very map \(\kappa\) used to clone the \(\alpha_i\) in the proof of Theorem 1, namely,

\[
\kappa : \omega \mapsto \sum_i \omega(a_i) \alpha_i \otimes \alpha_i.
\]

applied to \(\rho\), yields

\[
\kappa(\rho) = \sum_i t_i \alpha_i = \sum_i t_i \alpha_i \otimes \alpha_i.
\]

Taking the first marginal of this, we have

\[
a(\kappa(\rho)_1) = \sum_i t_i a(\alpha_i) = a(\sum_i t_i \alpha_i) = a(\rho);
\]

similarly, the second marginal is also \(\rho\). Thus, \(\kappa\) is broadcasting on \(\Delta(\{\alpha_1, \alpha_2, ..., \alpha_n\})\).

In fact, the convexity of the set of states broadcast by any map \(B\) shows that any map that broadcasts \(\Gamma\)’s extreme points broadcasts \(\Gamma\). If \(\Gamma\)’s extreme
points are extremal in $\Omega$ then, as mentioned above, a broadcasting map for $\Gamma$ must clone them, but this is not so in general. Any map of the form

$$B : \omega \mapsto \sum_{i} \omega(a_i)\rho_i,$$

(2)

where $\rho_i$'s marginals are both equal to $\alpha_i$ and $[a_i]$ as usual distinguish the $\alpha_i$, broadcasts $\omega \in \Delta(\{\alpha_i\})$, even though $\rho_i$ may not be $\alpha_i \otimes \alpha_i$.

If $\Gamma$ is a convex subset of a convex set $\Omega$, then every affine functional $a \in A(\Omega)$ defines, by restriction, an affine functional $a_\Gamma$ on $\Gamma$. This gives us a natural positive linear mapping $a \mapsto a_\Gamma$ from $A(\Omega)$ to $A(\Gamma)$, taking the order unit $u \in A(\Omega)$ to the order unit $u_\Gamma$ in $A(\Gamma)$. By a compression of a convex set $\Omega$ onto $\Gamma$, we mean an idempotent affine mapping $P : \Omega \to \Omega$ having range $\Gamma$. The existence of a compression implies that the natural mapping $A(\Omega) \to A(\Gamma)$ is surjective.

**Lemma 5** Let $A : \Omega \to \Omega$ be any affine mapping taking $\Omega$ into itself. Then there exists a compression of $\Omega$ onto the set of fixed points of $A$.

**Proof:** For each $n \in \mathbb{N}$, let

$$P_n = \frac{1}{n} \sum_{k=1}^{n} A^k : \Omega \to \Omega.$$

Since $\Omega$ is compact, we may assume (passing to a subsequence if necessary) that $(P_n)$, converges to a limiting affine map $P : \Omega \to \Omega$. If $A(\rho) = \rho$, then clearly $P(\rho) = \rho$; conversely, if $\rho = P(\mu)$ for some $\mu \in \Omega$, then we have

$$A(\rho) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} A^k(\mu) = \lim_{n \to \infty} \frac{1}{n} A(\mu) = \frac{1}{n} \sum_{k=1}^{n} A^k(\mu) - \lim_{n \to \infty} \frac{1}{n} A(\mu) = \frac{1}{n} A^{n+1}(\mu)$$

$$= P(\mu) = \rho.$$

Thus, the range of $P$ is exactly the fixed-point set of $A$, as advertised. Note also that, as $P(\mu)$ is a fixed point of $A$, we have $P(P(\mu)) = P(\mu)$ for any $\mu$, i.e., $P$ is idempotent. $\square$

**Lemma 6** Let $P : \Omega \to \Omega$ be a compression of a convex set $\Omega$ onto a convex subset $\Gamma \subseteq \Omega$. Then (i) $\Gamma \otimes \Gamma$ can be regarded as a convex subset of $\Omega \otimes \Omega$, and (ii) the mapping $P \otimes P : \Omega \otimes \Omega \to \Omega \otimes \Omega$ has range contained in $\Gamma \otimes \Gamma$.

**Proof:** We can regard $P$ as a surjective mapping from $\Omega$ to $\Gamma$. If $a$ is a positive affine functional on $\Gamma$, then $P^*(a) := a \circ P$ is an extension of $a$ to a positive affine functional on $\Omega$. Now for every $\omega$ belonging to $\Gamma \otimes \Gamma$, define a bilinear form $\rho : A(\Omega) \times A(\Omega) \to \mathbb{R}$ by $\rho(a, b) = \omega(a \circ b)$; this is obviously positive and normalized, so $\rho \in \Omega \otimes \Omega$. The mapping $\omega \mapsto \rho$ is clearly affine; it is also injective, by the aforementioned extension property. Identifying $\omega$ with $\rho$, we can (and shall) regard $\Gamma \otimes \Gamma$ as a convex subset of $\Omega \otimes \Omega$. It now follows (see the remark at the bottom of page 5) that $P \otimes P : \Omega \otimes \Omega \to \Gamma \otimes \Gamma$ is a well-defined affine mapping; composing this with the injection $\omega \mapsto \rho$, we have that $P \otimes P$ takes $\Omega \otimes \Omega$ into itself, with range contained in $\Gamma \otimes \Gamma$. $\square$

**Theorem 2** Let $\Gamma$ be the set of states broadcast by an affine mapping $B : \Omega \to \Omega \otimes \Omega$. Then $\Gamma$ is contained in the simplex generated by a set of distinguishable states in $\Omega$.

**Proof:** Let $\sigma : \Omega \otimes \Omega \to \Omega \otimes \Omega$ be the affine isomorphism that interchanges the two factors. Given the broadcasting map $B : \Omega \to \Omega \otimes \Omega$, define another affine mapping $B' : \Omega \to \Omega \otimes \Omega$ by $B' = (B + \sigma \circ B) / 2$. Note that $B'$ broadcasts every state $\rho \in \Gamma$. Call a state $\rho \in \Omega$ symmetrically broadcastable iff it is broadcast by $B'$, and denote by $\Gamma'$ the set of all such states. As just observed, $\Gamma \subseteq \Gamma'$.

Observe that $\rho \in \Gamma'$ iff $\rho$ is a fixed point of the mapping $B'_1$ sending $\rho$ to the marginal $B'_1(\rho)$ of $B'$. By Lemma 5, we have a compression $P$ onto $\Gamma'$. Notice that $P^* : A(\Gamma') \to A(\Omega)$ is a positive linear injection, with $P^*(u_{\Gamma'}) = u$ (since $P^*(u_{\Gamma'})(\omega) = u_{\Gamma'}(P(\omega)) = \omega \in \Gamma'$).
The claim is that this is universally broadcasting on $\Gamma'$. For if $\rho \in \Gamma'$, we have, for all $a \in [0, u_{\Gamma'}]$, 
\[
Q_1(\rho)(a) = Q(\rho)(a \otimes u_{\Gamma'}) = B(\rho)(P^*a \otimes P^*u_{\Gamma'}) = B_1(\rho)(P^*a) = \rho(P^*a) = P(\rho)(a) = \rho(a)
\]
(usually, in the last step, the fact that $P(\rho) = \rho$), since $\rho \in \Gamma'$. It follows that $Q_1(\rho) = \rho$: in the same way, one has that $Q_2(\rho) = \rho$. Since $Q$ is universally broadcasting on $\Gamma'$, it must in particular broadcast every extreme state $\alpha \in \Gamma'$. But then Lemma 3 implies that $Q(\alpha)$, being a state in $\Gamma' \otimes \Gamma'$ with extreme marginals, must be a product state, namely, $\alpha \otimes \alpha$. Thus, $Q$ is (jointly) cloning for all of $\Gamma'$'s extreme points. It follows now from Theorem 1 that these extreme points are distinguishable in $\Gamma'$ – hence, also in $\Omega$ (since any observable on $\Gamma'$ lifts to one on $\Omega$). □

We now have a quantum no-broadcasting theorem as an easy

**Corollary 2** Let $\Gamma$ be a set of density operators on a Hilbert space $H$. Suppose that there exists a positive map $\phi : B(H) \to B(H)$ broadcasting each $\rho \in \Gamma$. Then the operators in $\Gamma$ are pairwise commuting.

**Proof:** By Theorem 2, $\Gamma$ is contained in a simplex generated by distinguishable – hence, by Lemma 4, commuting – density operators. It follows that the operators in $\Gamma$ also commute. □

**Remarks:**

1. The standard quantum no-broadcasting theorem applies to a completely positive broadcasting map. Our result gives, in the form of the above Corollary, a stronger formulation: that no positive map between matrix algebras can broadcast two non-commuting states.

2. As stated, Theorem 2 tells us little about the convex structure of the set $\Gamma$ of states broadcast by a map $B$ (since any convex set can be embedded in a simplex). Combining it with the simple observation made above near Definition 5 that $\Gamma$ is $\Omega$-affine, we can say more: that $\Gamma$ is an affine section of a simplex generated by distinguishable states. Our next result is that $\Gamma$ in fact is a simplex generated by distinguishable states.

**Theorem 3** Let $\Gamma$ be the set of states broadcast by an affine mapping $B : \Omega \to \Omega \otimes \Omega$. Then $\Gamma$ is a simplex generated by jointly distinguishable states in $\Omega$.

**Proof:** We maintain the definitions used in the proof of Theorem 2. Any state $\omega \in \Gamma'$ has a unique representation $\omega = \sum i \omega_i \alpha_i$ as a convex combination of the extremal points $\alpha_i$ of the simplex $\Gamma'$. Let $[a_i]_{i=0}^n$ be a measurement that distinguishes the vertices of $\Gamma'$. The $a_0$ outcome has probability 0 on all states in $\Gamma'$, so we may set $a_1 = a_0 + a_1'$ and $a_i = a_i'$ for $2 \leq i \leq n$ to obtain an observable $[a_i]_{i=1}^n$ that still satisfies $\alpha_i(a_j) = \delta_{ij}$. This observable can be used to define a restriction map $r : \Omega \to \Gamma$ via

\[
r(\omega) = \sum_{i=1}^n \omega(a_i) \alpha_i,
\]
which is affine and surjective. For any $\omega \in \Omega$, this induces a unique “reduced state” $\omega^r \in \Gamma'$ defined as $\omega^r = r(\omega)$. All these “reduced states” $\omega^r$ are determined uniquely by an $n$-vector $v^\omega$ of probabilities, with components $v^\omega_i = \omega(a_i)$.

Any state $\omega \in \Gamma$ satisfies $(B_\omega(\omega))^r = B_\omega(\omega) = \omega$ for $m = 1, 2$. Therefore $B_\omega(\omega) = (B_m(\omega))^r = (\sum_i \omega_i B_m(\alpha_i))^r = \sum_i \omega_i (B_m(\alpha_i))^r$. Since $(B_\omega(\alpha_i))^r \in \Gamma'$, the restriction to $\Gamma'$ of the map $\omega \mapsto (B_m(\omega))^r$ is a classical stochastic map on the simplex $\Gamma'$. This map can be represented as a column stochastic matrix $M_m$ that acts on the vector $v^\omega$. The $i$th column of $M_m$ is just the vector representative of the image of the vertex $\alpha_i$ under the map $B_m$, i.e. $v^{B_m(\alpha_i)}$. Thus a state $\omega \in \Gamma'$ is broadcastable if and only if $M_m v^\omega = v^\omega$ for $m = 1, 2$, that is, if $v^\omega$ is in the intersection of
the fixed-point subspaces of both stochastic matrices $M_m$. We can understand these fixed point spaces using the extension of the Perron-Frobenius theory of eigenvectors and eigenvalues of irreducible nonnegative square matrices to the case of general (i.e. possibly reducible) nonnegative square matrices. Appendix A summarizes this theory and proves two Lemmas we use. Lemma 7, following easily from the extended Perron-Frobenius theory, gives a basis for the space of fixed points of a stochastic map consisting of disjointly supported nonnegative vectors, which correspond to distinguishable states when normalized. The main technical work of the present proof is in deriving from this Lemma 8, stating that the intersection of the fixed-point spaces of two such stochastic matrices also has (when it is not $\{0\}$) a basis of disjointly supported nonnegative vectors $v^\alpha$, so that the set of normalized states that are fixed points of both maps is the simplex $\Delta(\{v^\alpha\})$ generated by these distinguishable states. Since we established above that $\Gamma$ is the set of states fixed by two stochastic maps, it is a simplex generated by distinguishable states. (If the intersection is $\{0\}$ (as it will be for a generic map $B$), the $\Gamma = \Delta(0) = \emptyset$, which we view as a degenerate case of a simplex generated by a set of distinguishable states.) □

Remark: Although for a given $B$ both $\Gamma'$ and $\Gamma$ are simplices generated by distinguishable states, it is easily shown by example that $\Gamma$ may be a proper subset of $\Gamma'$. For instance, let $\Omega = \Delta(\{(\alpha_1, \alpha_2)\})$ and let $B : \alpha_1 \mapsto \alpha_1 \otimes \alpha_2, \alpha_2 \mapsto \alpha_2 \otimes \alpha_1$. Then $\Gamma = \emptyset$ while $\Gamma' = \{(\alpha_1 + \alpha_2)/2\}$.

6 Conclusions

In order to understand the nature of information processing in quantum mechanics, it is important to be able to delineate clearly those probabilistic and information-theoretic phenomena that are indeed essentially quantum, from those that are more generically non-classical. We have established here that several specific features of quantum information are generic: entanglement monogamy, and, in finite-dimensional theories, the connection between cloning and state-discrimination and the no-broadcasting theorem.

One might wonder at this point whether every qualitative result of quantum information will turn out to be similarly generic, either in non-classical theories or in all theories. This is not the case, however. For example, not every finite-dimensional probabilistic theory allows for teleportation (this is shown in [9] and also follows from the results of [48] on entanglement swapping.)

Finally, it is worth commenting on the program of deriving quantum theory from information theoretic axioms [23, 24, 17] in the light of the present work. Any such attempt must begin with a framework that delineates the set of theories under consideration. The framework must be narrow enough to allow the axioms to be succinctly expressed mathematically, but broad enough that the main substantive assumptions are contained in the axioms rather than in the framework itself. The generalized probability models discussed in this paper would appear to be a natural choice for this task.

In [17], Clifton, Bub and Halvorson attempt an information theoretic axiomatization within a $C^*$-algebraic framework, which is narrower than the framework adopted here. In fact, the $C^*$ framework is already very close to quantum theory, in the sense that all theories in the framework have Hilbert space representations. In the finite dimensional case, quantum theory, classical probability and quantum theory with superselection rules are the only options available. The information theoretic axioms used in [17] are: no-signaling, no-broadcasting and no-bit-commitment. From these it is shown that there must be noncommuting observables in the theory and there must be some entangled states. Given the restricted nature of the $C^*$ framework, this already yields a theory that looks quite close to quantum theory.

In contrast, the generalized probabilistic framework adopted here automatically satisfies no-signaling, and we have shown that no-broadcasting is generically true of any nonclassical model. Such generic models can look very different from quantum theory. For example, they include models that support super-quantum correlations. An open question is whether no-bit-commitment is also generic in the
present framework, and it is possible that it does place nontrivial constraints on the choice of tensor product. Nevertheless, it seems unlikely that these three axioms alone would get one particularly close to quantum theory. In the light of this, it seems that the best hope for future progress in axiomatization would be to supplement or replace these axioms with things that do not appear to be generic, such as the existence of a teleportation protocol.

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References

[1] H. Araki. On a characterization of the state space of quantum mechanics. Commun. Math. Phys., 75:1–24, 1980.

[2] H. Barnum. Quantum information processing and quantum logic: toward mutual illumination. arXiv.org e-print quant-ph/0205129, 2002.

[3] H. Barnum. Quantum information processing, operational quantum logic, convexity, and the foundations of physics. Studies in the History and Philosophy of Modern Physics, 34:343–379, 2003. See also arXiv.org e-print quant-ph/0304159.

[4] H. Barnum. The view from everywhere: Convex operational theories, quantum information, quantum foundations, and the coordination of quantum agents’ perspectives. In Yu. A. Khrennikov, editor, Proceedings of International Conference: “Quantum Theory: Reconsideration of Foundations–2”, pages 553–637, Växjö, Sweden, 2004. Växjö University Press. arXiv.org e-print version is quant-ph/0611110.

[5] H. Barnum, C. Caves, C. Fuchs, and B. Schumacher. Noncommuting mixed states cannot be broadcast. Phys. Rev. Lett., 76:2818–2821, 1996.

[6] H. Barnum, C. Fuchs, J. Renes, and A. Wilce. Influence-free states on compound quantum systems. arXiv.org e-print quant-ph/0507108, 2005.

[7] H. Barnum, E. Knill, G. Ortiz, and L. Viola. Generalizations of entanglement based on coherent states and convex sets. Phys. Rev. A, 68:032308, 2003.

[8] H. Barnum, G. Ortiz, R. Somma, and L. Viola. A generalization of entanglement to convex operational theories: Entanglement relative to a subspace of observables. Int. J. Theor. Phys., 44:2127–2145, 2005. (Special issue, Proceedings of “Quantum Structures 2004: Biennial Meeting of the International Quantum Structures Association”).

[9] J. Barrett. Information processing in generalized probabilistic theories. arXiv.org e-print quant-ph/0508211, 2005.

[10] J. Barrett, N. Linden, S. Massar, S. Pironio, S. Popescu, and D. Roberts. Nonlocal correlations as an information-theoretic resource. Phys. Rev. A, 71:022101, 2005.

[11] J. Barrett and S. Pironio. Popescu-Rohrlich correlations as a unit of nonlocality. Phys. Rev. Lett., 95:140401, 2005.

[12] E. G. Beltrametti and S. Bugajski. Effect algebras and statistical physical theories. J. Math. Phys., 38:3020–3030, 1997.

[13] G. Brassard, H. Buhrman, N. Linden, A. A. Méthot, A. Tapp, and F. Unger. A limit on nonlocality in any world in which communication complexity is not trivial. Phys. Rev. Lett., 96:250401, 2006. arXiv.org e-print quant-ph/0508042.

[14] A. Broadbent and A. A. Méthot. On the power of non-local boxes. Theor. Comput.
[15] H. Buhrman, M. Christandl, F. Unger, S. Wehner, and A. Winter. Implications of superstrong nonlocality for cryptography. Proc. Royal Soc. A. 462(2071):1919–1932, 2006. arXiv.org e-print quant-ph/0504133.

[16] A. Chefles and S. M. Barnett. Quantum state separation, unambiguous discrimination and exact cloning. J. Phys. A, 31:10097–10103, 1998. arXiv.org e-print quant-ph/9808018.

[17] R. Clifton, J. Bub, and H. Halvorson. Characterizing quantum theory in terms of information-theoretic constraints. Found. Phys., 33:1561–1591, 2003. arXiv.org e-print quant-ph/0211089.

[18] C. D. H. Cooper. On the maximum eigenvalue of a reducible non-negative real matrix. Math. Z., 13:213–217, 1973.

[19] G. M. d’Ariano. How to derive the hilbert-space formulation of quantum mechanics from purely operational axioms. arXiv.org e-print quant-ph/0603011, 2006.

[20] D. J. Foulis. Mathematical metascience. J Natural Geometry, 13:1–50, 1998.

[21] D. J. Foulis and C. H. Randall. Empirical logic and tensor products. In H. Neumann, editor, Interpretations and Foundations of Quantum Mechanics. Bibliographisches Institut, Wissenschaftsverlag, Manheim, 1981.

[22] G. Frobenius. Über Matrizen aus nicht negativen Elementen. Sitzungsberichte Preussische Akademie der Wissenschaft, Berlin, pages 456–477, 1912.

[23] C. A. Fuchs. Quantum mechanics as quantum information (and only a little more). arXiv.org e-print quant-ph/0205039, 2002.

[24] C. A. Fuchs. Quantum mechanics as quantum information, mostly. J. Mod. Opt., 50:987, 2003.

[25] S. Gudder, S. Pulmannová, S. Bugajski, and E. Beltrametti. Convex and linear effect algebras. Rep. Math. Phys, 44:359–379, 1999.

[26] L. Hardy. Disentangling nonlocality and teleportation. arXiv.org e-print quant-ph/9906123, 1999.

[27] L. Hardy. Quantum theory from five reasonable axioms. arXiv.org e-print quant-ph/0101012, 2001.

[28] L. Hardy. Why quantum theory? arXiv.org e-print quant-ph/0111068. Contribution to NATO Advanced Research Workshop "Modality, Probability, and Bell’s Theorem, Cracow, Poland 19–23.8.01, 2001.

[29] L. Hardy. Probability theories with dynamic causal structure: A new framework for quantum gravity. arXiv.org e-print gr-qc/0509120, 2005.

[30] A. S. Holevo. Probabilistic and Statistical Aspects of Quantum Mechanics. North-Holland, 1983.

[31] M. Horodecki, P. Horodecki, R. Horodecki, and M. Piani. Quantumness of ensemble from no-broadcasting principle. arXiv.org e-print quant-ph/0506174, 2005.

[32] N. S. Jones and L. Masanes. Interconversion of nonlocal correlations. Phys. Rev. A, 72:052312, 2005.

[33] L. A. Khalfi and B. S. Tsirelson. In P. Lahti and P. Mittelstaedt, editors, Symposium on the Foundations of Modern Physics, pages 441–460. World Scientific, Singapore, 1985.

[34] M. Kläy. Einstein-Podolski-Rosen experiments: the structure of the sample space i, ii. Found. Phys. Lett., 1:205–244, 1988.

[35] M. Kläy, C. H. Randall, and D. J. Foulis. Tensor products and probability weights. Int. J. Theor. Phys., 26:199–219, 1987.

[36] G. Lindblad. A general no-cloning theorem. Lett. Math. Phys., 47:189–196, 1999.
[37] G. Ludwig. Versuch einer axiomatischen Grundlegung der Quanten Mechanik und allgemeinerer physikalischer Theorien. *Z. Phys.*, 181:233, 1964.

[38] G. Ludwig. Attempt of an axiomatic foundation of quantum mechanics and more general theories: II. *Commun. Math. Phys.*, 4:331, 1967.

[39] G. Ludwig. An axiomatic basis of quantum mechanics. In H. Neumann, editor, Interpretations and foundations of quantum mechanics: proceedings of a conference held in Marburg 28-30 May 1979, Zürich, 1981. Bibliographisches Institut.

[40] G. Ludwig. *Foundations of Quantum Mechanics I*. Springer, 1983.

[41] G Ludwig. *Foundations of Quantum Mechanics II*. Springer, 1985.

[42] G. Mackey. *Mathematical Foundations of Quantum Mechanics*. Addison-Wesley, 1963.

[43] P. G. L. Mana. Probability tables. In Yu. A. Khrennikov, editor, *Proceedings of International Conference: Quantum Theory: Reconsideration of Foundations–2”,* pages 387–402, Växjö, Sweden, 2004. Växjö University Press. arXiv.org e-print quant-ph/0403084.

[44] I. Namioka and R. Phelps. Tensor products of compact convex sets. *Pacific J. Math.*, 9:469–480, 1969.

[45] S. Popescu and D. Rohrlich. Quantum nonlocality as an axiom. *Found. Phys.*, 24:379–385, 1994.

[46] H. Schneider. The influence of the marked reduced graph of a nonnegative matrix on the Jordan form and on related properties: A survey. *Lin. Alg. Appl.*, 84:161–189, 1986.

[47] A. Short, N. Gisin, and S. Popescu. The physics of no-bit-commitment: Generalized quantum non-locality versus oblivious transfer. *Quantum Information Processing*, 5(2):131–138, 2006. arXiv.org e-print quant-ph/0504134.

[48] A. Short, S. Popescu, and N. Gisin. Entanglement swapping for generalized non-local correlations. *Phys. Rev. A*, 73:012101, 2006. arXiv.org e-print quant-ph/0508120.

[49] Singer and W. Stulpe. Phase space representations of general statistical physical theories. *Journal of Mathematical Physics*, 33:131–142, 1992.

[50] J. Smolin. Can quantum cryptography imply quantum mechanics? arXiv.org e-print quant-ph/0310067, 2001.

[51] R. W. Spekkens. In defense of the epistemic view of quantum states: a toy theory. arXiv.org e-print quant-ph/0401052; to appear in *Physical Review A*, 2004.

[52] B. Terhal. Is entanglement monogamous? *IBM Journal of Research and Development*, 48:71–78, 2004.

[53] W. van Dam. *Nonlocality and communication complexity*. PhD thesis, University of Oxford, 2000.

[54] W. van Dam. Implausible consequences of superstrong nonlocality. arXiv.org e-print quant-ph/0501159, 2005.

[55] A. Wilce. Tensor products in generalized measure theory. *Int. J. Theor. Phys.*, 31:1915–1928, 1992.

[56] A. Wilce. Test spaces and orthoalgebras. In B. Coecke, D. Moore, and A. Wilce, editors, *Current research in operational quantum logic*, Dordrecht, 2000. Kluwer.

[57] G. Wittstock. Ordered normed tensor products. In H. Neumann, editor, *Foundations of Quantum Mechanics and Ordered Linear Spaces*, Springer Lecture Notes In Physics. Springer, 1974.

[58] S. Wolf and J. Wullschleger. Oblivious transfer and quantum non-locality. In *Proceedings of International Symposium on Information Theory (ISIT)*, pages 1745–1748, 2005.
A Perron-Frobenius Theory, Fixed Points of Classical Stochastic Maps, and Lemmas Used in Proving Theorem 3

By a nonnegative matrix (or row or column vector) we mean one with real nonnegative entries. By a semipositive matrix or vector, we mean one with nonnegative entries at least one of which is positive, and by a positive matrix or vector, we mean one for which every entry is strictly positive. A nonnegative matrix is called reducible if there exists a permutation matrix \( P \) such that \( PMP^t \) has the form:

\[
\begin{pmatrix}
M^{11} & 0 & 0 & \cdots & 0 \\
M^{12} & M^{22} & 0 & \cdots & 0 \\
M^{31} & M^{32} & M^{33} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
M^{K1} & M^{K2} & M^{K3} & \cdots & M^{KK}
\end{pmatrix}
\]

 irreducible if there does not. Some such permutation \( P \) will put a general nonnegative square matrix \( M \) in Frobenius normal form

\[
\begin{pmatrix}
M^{11} & 0 & 0 & \cdots & 0 \\
M^{12} & M^{12} & 0 & \cdots & 0 \\
M^{31} & M^{32} & M^{33} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
M^{K1} & M^{K2} & M^{K3} & \cdots & M^{KK}
\end{pmatrix}
\]

where each diagonal block \( M^{IJ}, I \in \{1,\ldots,K\} \) is irreducible.

The standard Perron-Frobenius theory applies to irreducible nonnegative square matrices \( M \), guaranteeing a strictly positive eigenvector with a real positive eigenvalue \( \rho(M) \) greater than or equal to the modulus of any other eigenvalue, real or complex (thus \( \rho(M) \) is the spectral radius of \( M \)).

A result explicitly stated and proved in [18], and also stated in [46] (where its proof is said to be essentially present in Frobenius [22]) partially characterizes the real nonnegative eigenvectors of general (possibly reducible) nonnegative square matrices that correspond to positive eigenvalues. The eigenvalues of such nonnegative eigenvectors are \( \rho_I := \rho(M^{IJ}) \), and for each diagonal block \( M^{IJ} \) in the Frobenius normal form of \( M \) having a given \( \rho_I \), there is an eigenvector \( v^I \) whose components with indices (after the permutation that gives Frobenius normal form) in block \( J \) and above are nonnegative, and whose lower-indexed components are zero. It is also possible to characterize the eigenvectors \( v^I \) in a way which is independent of Frobenius normal form by introducing the following terminology. An index \( i \) has access to an index \( j \) if there is some finite power \( p \) such that \( (MP)^p_{ij} > 0 \). In the context of column-stochastic matrices interpreted as transition matrices, this means that probability can eventually leak from state \( j \) to state \( i \) (note the directionality, which is not obvious from the term “has access to”). Equivalently, \( i \) has access to \( j \) if in the directed “transition graph” having edges \((i, j)\) (thought of as directed “from \( i \) to \( j \)”) where, and only where, \( M_{ij} \neq 0 \) (note again the nonintuitive directionality opposite the flow of probability), there is a (directed) path from \( i \) to \( j \). The indices in a given subset \( I \), on which an eigenvector \( v^I \) has positive components, can be characterized as mutually having access to each other (a condition which identifies those subsets without the need to mention Frobenius normal form as we did above). Finally, the eigenvectors with a given real positive eigenvalue \( \lambda \) are precisely the real semipositive linear combinations of the eigenvectors, among those whose existence is asserted above, having eigenvalue \( \lambda \).

The next result concerns the fixed point states, that is to say the real nonnegative normalized eigenvectors \( v \) (\( \sum_i v_i = 1 \)) with eigenvalue-1 of the column-stochastic matrix \( M \).

Lemma 7 A column-stochastic matrix \( M \) may be put into Frobenius normal form in such a way each of its fixed point states is supported precisely on one of the \( L \leq K \) blocks numbered \( K - L + 1, \ldots, K \). The restriction of \( M \) to these blocks will then be block-diagonal.

Proof: Without loss of generality suppose \( M \) is in Frobenius normal form, with blocks \( M^{IJ}, (I, J \in \{1,\ldots,K\} \).

The real positive eigenvalues of a column-stochastic matrix must be equal to 1 (because it preserves normalization). Those of an irreducible prop-
erly substochastic matrix (i.e. one for which all column sums are less than 1, and at least one strictly so) must be strictly less than 1.

$M_{LK}$ is column-stochastic, so it follows easily from the irreducible Perron-Frobenius theory that $\rho_K = 1$ and there is an eigenvector whose support is $K$ with eigenvalue 1. For any other diagonal block $M_{LL}$ to have an eigenvalue-1 nonnegative eigenvector, it must be the case that all blocks $M_{LM}$ below it ($M < L$) are zero matrices, for if one of them is not, then $M_{LL}$ is properly column-substochastic. Any such diagonal blocks $M_{LL}$ with $\rho(M_{LL}) = 1$ can be put at the end of the ordering of blocks (indeed, in arbitrary order at the end) by an index permutation preserving Frobenius normal form. Assume this has been done, and let them be blocks $K - L + 1$ through $K$. Thus by the Cooper/Frobenius result discussed above Lemma 7 $M$ has $L$ disjointly supported fixed-point eigenvectors, one supported on each of the subsets $K - L + 1, \ldots, K$. The indices belonging to $1, \ldots, K - L$ thus correspond to vertices on which the fixed points of $M$ have zero support. □

**Lemma 8** Let $M_1$, $M_2$ be two column-stochastic matrices. The intersection of their fixed-point subspaces is spanned by a set of distinguishable states, so the set of normalized states that are fixed points of both maps is a simplex generated by distinguishable states.

**Proof:**

We cannot necessarily put both $M_1$ and $M_2$ in Frobenius normal form simultaneously. However, the block indices in the Frobenius normal form of $M_m$ correspond (for each fixed $m$) to a partition of the set of vertex indices into subsets.

Thus each map’s fixed-point space is defined by a partition $\Lambda_m$ of the vertices into a set $Z_m$ on which it has no support, and sets, for which we use variables $I, I', \ldots$ for $m = 1$, and $J, J', \ldots$ for $m = 2$, of vertices each of which supports a strictly positive fixed-point vector $v^I$ (resp. $w^I$), with components $v^I_k$ (resp. $w^I_k$). Thus e.g. $v^I_k = 0$ whenever $k \not\in I$. We will also define vectors $v = \sum_I v^I$ with components $v_k$, $w = \sum_J w^J$ with components $w_k$). We will use the notation $I(l)$ to mean the subset of the pertinent partition to which the vertex-index $l$ belongs.

If $\omega$ is in the intersection of the fixed-point spaces of $M_{1,2}$ then there exist nonnegative $\lambda_I, \mu_J$ such that

$$\omega = \sum_I \lambda_I v^I = \sum_J \mu_J w^J.$$ (6)

The first way of expressing $\omega$ enforces that it is a fixed point of $M_1$, the second, that it is a fixed point of $M_2$. We now give a procedure for expressing the condition $\omega = \sum_I \mu_J w^I$ as further constraints on the $\lambda_I$’s taking the form that for some of the $I$, $\lambda_I$ must be zero, while some of the ratios $\lambda_I/\mu_J$ are fixed by the data $\mu_J, w^I$ when $I, I'$ are both incident on the same $J$.

To do this, it will be useful to define some relations $R, S$ on $\Lambda := \Lambda_1 \cup \Lambda_2$. We say $G \subseteq H$ iff $G \subseteq H$. $R$ is reflexive and symmetric. Let $S$ be its transitive closure (i.e. $G S P$ iff there is a finite chain $H_1, \ldots, H_n$ such that $G R H_1 \ R H_2 \ R \cdots \ R H_n \ R H$). $S$ is an equivalence relation, so its equivalence classes $[I]_S, [J]_S$ partition $\Lambda$. Moreover, it is easy to see that its restrictions $S_1, S_2$ to $\Lambda_1, \Lambda_2$ are also equivalence relations, and for any given equivalence class $[I]_S$ or $[J]_S$ of $S$, the equivalence classes $[I]_S$, or $[J]_S$, satisfy $\cup [I]_S = \cup [I]_S$, (or $\cup [J]_S$), i.e. the sets in them contain the same vertices.

A fact that will be useful below is that if $\lambda^I = 0$ (or $\mu^I = 0$), then $\lambda^{I'} = \mu^{I'} = 0$ for all $I', J' \in [I]_S$ (or $[J]_S$). The reason is that $\lambda^I = 0$ implies $\omega_k = 0$ for all $k \in I$, so for such $k$, $\mu_J w_k^J$ = 0, implying (since $w^J_k > 0$) $\mu^J = 0$. In other words, $\lambda^I = 0$ and $I R J$ imply $\mu_J = 0$; the same argument shows that $\mu_J = 0$ and $J R I'$ implies $\lambda_J = 0$; thus the same statements hold with $S$ in place of $R$ and we see that zero coefficients for $I$ (or $J$) propagate throughout $[I]_S$ (or $[J]_S$).

Note that if $I \cap Z_2 \neq \emptyset$ then $\lambda^I = 0$, $\mu^I = 0$ for $I, J \in [I]_S$. This is because the vectors $v^I$ have positive components $v_k^I$ for $k \in I$, but for $k \in Z_2$ we have $\omega_k = 0$, which therefore requires $\lambda^I = 0$; the above observation then applies.

Let $Z'$ be $Z_1$ plus the set of all the vertices that this argument shows to have $\omega_k = 0$, and $\Lambda_m$ the partitions of the remainder of the vertices agreeing with $\Lambda_m$. 

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Recall from (6) that the components of $\omega$ satisfy:

$$\mu_{J(k)} w_k = \lambda_{I(k)} v_k ,$$

for all $k$. Thus if $I \cap J \neq \emptyset$ then either $\mu_J, \lambda_I = 0$ or $v_k/w_k$ for $k \in I \cap J$ is some constant $\beta_{I,J} := \lambda_J/\lambda_I$ independent of $k$. So $\lambda_I$ must be zero if there is any $J$ with $J \cap I \neq \emptyset$ for which $v$ is not proportional to $w$ on $J \cap I$. As before, the upshot is that if an equivalence class $X$ of $S$ contains sets $J, I$ such that $v$ is not proportional to $w$ on $I \cap J$, the coefficients of all sets in $X$ must be zero. This constraint removes more indices from the subset on which joint fixed-points can be supported (implying the fixed-points lie in the subsimplex of $\Gamma'$ with those vertices deleted).

We therefore define $Z''', \Lambda'''_m$ similarly to $Z', \Lambda'_m$. Now let $J(k) = J(l) = J$ but $I(k) = I \neq I(l) = I'$ and consider

$$\frac{v_k/w_k}{v_l/w_l} = \frac{\mu_{J(k)} \lambda_{I(k)}}{\mu_{J(l)} \lambda_{I(l)}} .$$

(8)

If $\lambda_I \neq 0$ then $\mu_J, \lambda_{I'} \neq 0$ and we get the requirement:

$$\frac{\lambda_{I(l)}}{\lambda_{I(k)}} = \frac{v_k}{w_k} \frac{w_l}{v_l} ,$$

(9)

i.e.

$$\frac{\lambda_{I'}}{\lambda_I} = \beta_{I,I'/J} .$$

(10)

As promised, some of the constraints coming from $M_2$ have fixed the ratio of $\lambda_I$ and $\lambda_{I'}$. Any $I' \neq J$ incident on both $I$ and $I'$ must give rise to the same ratio $\lambda_{I'}/\lambda_I$; that is,

$$\beta_{I,J}/\beta_{I',J} = \beta_{I',I}/\beta_{I',I'} .$$

(11)

Should this not be the case, our assumption that $\lambda_I \neq 0$ must be false, so all $\lambda_{I''} = 0$ for $I'' \in [I]_S$.

Thus, the ratios $\lambda_{I'}/\lambda_I$ are fixed to $\gamma_{I,I'} := \beta_{I,J}/\beta_{I',J}$ within those $S$-equivalence class for which the RHS is independent of $J$, while all $\lambda_J = 0$ in the other $S$-equivalence classes. No constraints on the $\lambda_J$ arise across $S$-equivalence classes. We add the zeroed-out vertices to $Z'''$ to obtain $Z''''$, and similarly obtain $\Lambda''''_m$ as the remaining $S_1$-equivalence classes.

Some obvious consistency conditions must be satisfied by the ratios $\gamma_{I,I'} = \lambda_{I'}/\lambda_I$ thus obtained, namely the transitivity conditions:

$$\gamma_{I'I''} \gamma_{I''I'} = \gamma_{II'} .$$

(12)

It may be the case that one side of this is defined while the other side is not, because, for example, although some $J$ is incident on both $I$ and $I'$, no $J$ is incident on both $I$ and $I''$, in which case no further constraint arises; but when all are defined, we have (recalling the definition of $\beta_{I,J}$) that:

$$\frac{v_k/w_k}{v_{k'}/w_{k'}} = \frac{v_{l}/w_{l}}{v_{l'}/w_{l'}} .$$

(13)

Canceling, we obtain an identity so no further constraints arise.

We have just expressed all the constraints arising from $\omega = \sum_{I} \mu_J w^I$ as constraints on the $\lambda_I$. The other constraint $\omega = \sum_{I} \lambda_I v^I$ gives $\omega$ as a convex combination of distinguishable states $v^I$, i.e. $\omega \in \Delta(\{v^I\})$. It is evident that fixing $\omega_k = 0$ for $k \in Z''''$ just says the states are in a subsimplex of $\Delta(\{v^I\})$, while fixing the ratios of vertices $v^I$ within the elements of a partition just says that the states are convex combinations of a particular set of disjointly supported, and therefore still distinguishable, states in this subsimplex.

To be rigorous we give an explicit expression for $\omega$ as a convex combination of distinguishable states. Without loss of generality suppose that $\sum_k v_k = \sum_I \lambda_I = 1$, so that $\omega, v^I, w^I$ are normalized states. Picking representatives $I \in [\hat{I}]$ from each element $[\hat{I}]$ of the partition $\Lambda'''_m$ we begin with $\omega = \sum_{I \cup \Lambda'''_m} \lambda_I v^I$ and impose the constraints, getting:

$$\omega = \sum_{[\hat{I}]} \lambda_I \sum_{I' \in [\hat{I}]} (\lambda_{I'}/\lambda_I) v^{I'} \equiv \sum_{[\hat{I}]} \lambda_I \sum_{I' \in [\hat{I}]} \gamma_{I'I'} v^{I'} .$$

(14)

Define normalized vectors

$$v^I_{[\hat{I}]} := \left( \sum_{I' \in [\hat{I}]} \gamma_{I'I'} \right) / \left( \sum_{I' \in [\hat{I}]} \gamma_{I'I'} \right),$$

(15)

and scalars

$$\lambda_{[\hat{I}]} := \lambda_I \left( \sum_{I' \in [\hat{I}]} \gamma_{I'I'} \right) .$$

(16)
To see that these definitions are independent of the choice of representative $\hat{I}$ of $[\hat{I}]$, recall (cf. (9)) that

$$\gamma_{\hat{I}I} := \frac{\lambda_I}{\hat{\lambda}_I} = \frac{v_p w_l}{w_p v_l}, \quad (17)$$

for any $p \in \hat{I}$, $l \in I$ (independently of our choice of such $p, l$). Now from (15),

$$v_{k[I]} = \sum_{I' \in [\hat{I}]} \sum_{k \in I'} \gamma_{I'I} v_{k[I']} \equiv \sum_{I' \in [\hat{I}]} \gamma_{I'I} v_{k[I']}, \quad (18)$$

and we see that the $\hat{I}$ dependence, which is only through the factor $\gamma_{\hat{I}I}$ on top and $\gamma_{I'I}$ on the bottom, takes the form of factors $v_p/w_p$ for some $p \in \hat{I}$ on the top and bottom, which cancel establishing the claimed independence from the choice of $I \in [\hat{I}]$. Also,

$$\lambda'_{[\hat{I}]} := \lambda_I \sum_{I \in [\hat{I}]} \gamma_{I[I]}$$

$$= \lambda_{[\hat{I}]} \sum_{I \in [\hat{I}]} (\lambda_I/\hat{\lambda}_I) = \sum_{I \in [\hat{I}]} \lambda_I,$$  \quad (19)

showing that this too depends only on $[\hat{I}]$.

With these definitions, (14) becomes:

$$\omega = \sum_{[\hat{I}]} \lambda'_{[\hat{I}]} v^{[\hat{I}]}.$$  \quad (20)

Since the sets of vertices $\cup \cup [\hat{I}]$ supporting each $v^{[\hat{I}]}$ are disjoint, the $v^{[\hat{I}]}$ are distinguishable, and since in addition the nonnegative coefficients $\lambda_{[\hat{I}]}$ are free except for overall normalization, $\Gamma$ is the simplex $\Delta(\{v^{[\hat{I}]}\}_{[\hat{I}]})$ with distinguishable vertices $v^{[\hat{I}]}$. □