New Examples of Duality in Chiral and Non-Chiral Supersymmetric Gauge Theories

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We present evidence for renormalization group fixed points with dual magnetic descriptions in fourteen new classes of four-dimensional N = 1 supersymmetric models. Nine of these classes are chiral and many involve two or three gauge groups. These theories are generalizations of models presented earlier by Seiberg, by Kutasov and Schwimmer, and by the present authors. The different classes are interrelated; one can flow from one class to another using confinement or symmetry breaking.
1. Introduction

Recent work has shown that $N = 1$ supersymmetry is a fruitful arena for studying strongly coupled dynamics of gauge theories. See [1,2] for recent reviews and references therein for earlier work. One of the most intriguing of the recent results is the discovery by Seiberg that $N = 1$ theories can have interacting superconformal fixed points, in a non-Abelian Coulomb phase, which have dual descriptions in terms of different, “magnetic”, gauge theories [3]. This duality is a generalization [3,4,5] of the Montonen-Olive electric-magnetic duality [6] of $N = 4$ theories [7] and finite $N = 2$ theories [8,9,10] to $N = 1$ supersymmetric theories.

The original examples of $N = 1$ duality include $SU(N_c)$ with matter in the fundamental representation [3], $SO(N_c)$ with matter in the vector representation [3,4], and $Sp(N_c)$ with matter in the fundamental representation [3,11]. New examples of non-trivial infrared fixed points with dual descriptions have recently been discovered by Kutasov [12], Kutasov and Schwimmer [13], and the present authors [14,15]; related models appear in [16,17]. The new theories are all based on the original theories discussed in [3] along with an additional field $X$ and a superpotential $W(X)$. The superpotential plays an important role, controlling the infrared fixed point. The need in these models for the superpotential is a disappointment — it would be preferable to have the dual for the theory without the superpotential, since perturbing it with a superpotential and flowing to the infrared would give the dualities for all of the theories with superpotentials. Unfortunately, the dual of the theory without any superpotential for the field $X$ is not yet known. In addition, there are limits to our understanding of the physics even in the theories with the added superpotential [12,13]; many dynamical questions remain. Given the gaps in our present understanding of duality, it is important to have as many examples as possible. In this paper we will present fourteen new classes of theories which are generalizations of the models of [12-15].

In [3,4] it was suggested that the duality of $N = 1$ models is closely related to that of finite $N = 2$ theories. The explicit relation between the $N = 1$ theories studied in [3] and certain finite $N = 2$ theories was presented in [1]. Finite $N = 2$ models have a marginal coupling constant (the gauge coupling) with an associated continuous space of fixed points. The duality transformation involves a reflection on this space, along with a transformation of the flavor representations. An analogous situation occurs for the $N = 1$ models of [3] when the number of flavors is such that the gauge group is self dual. In
all classes studied in [12-15] and in this paper, it has been found that the models with self dual gauge group always have exactly marginal composite meson operators. The full significance of this observation is still under investigation. As in [4], it appears to play an important role in the duality transformation [13,15].

Because a theory in one class often flows to a theory in another class under a relevant perturbation, new examples lead to non-trivial consistency checks of duality. Many of our examples have flat directions in which they flow to a previously known model. Additional terms in the superpotential can have a similar effect. Perhaps the most interesting phenomenon that we observe is associated with confinement; in cases with more than one gauge group we find that the duality is consistent even when one gauge coupling is much larger than the other. This is essential for the renormalization group flow to be sensible at all values of the gauge couplings. A detailed illustration of how this occurs is presented in one class of models (sect. 6.3).

Nine of our new classes of models are chiral. Previous studies of duality in chiral models have been carried out by Berkooz [17] and by Seiberg [18]. In principle, chiral models can be very different from non-chiral ones; for example, they can dynamically break supersymmetry. In our examples, however, the duality in the chiral theories appears to be completely analogous to that in the non-chiral theories. We do not know if this is an indication of a general similarity — it could be that some unsolved chiral models are completely unlike those that we have discovered. One feature of our chiral models is that the $R$-charges, and therefore the dimensions at the fixed point, of some chiral operators are not uniquely specified; only dimensions of operators which are uncharged under all other $U(1)$ symmetries are determined.

Because we will present so many models, we will discuss each example only briefly. In each case we will provide a few pieces of evidence for the duality, mention interesting features, and show how it is connected to other models when subjected to a relevant perturbation. We hope that our extensive collection of interrelated theories will serve as a compelling demonstration of the validity and generality of the phenomenon of $N = 1$ duality.

The structure of this paper is as follows. In the next section we provide an overview, reviewing the models of [12-15] and summarizing the duality in our new examples. Readers interested only in our main results should read this section. In sects. 3–16, each of our new classes of models is discussed in more detail.
2. Overview of the old and new theories

In this section we list the models of [12-15], and then present a short summary of each of our new models. For every new model, a later section provides additional details.

Each summary follows roughly the same pattern. First, we will introduce the electric theory. Each model consists of one or two 2-index tensors and some fields in the defining representation (fundamental representation for $SU(N_c)$ and $Sp(N_c)$, vector representation for $SO(N_c)$) and has a superpotential for the two-index tensor(s). (When the superpotential is a mass term, the duality always reduces to that of [3,5,11].) Next, we will present its magnetic dual, which is of the same type but with a different number of colors and with singlet fields which enter in the superpotential. We then will discuss the effect of a perturbation of the theory by a certain superpotential which causes both the electric and magnetic theories to flow to a product of simpler theories; in all cases the low energy theories are dual to one another. Lastly, any additional special features of the model, including interesting flat directions and behavior under confinement, will be outlined.

Before beginning we discuss a few notational issues. We refer to the symplectic group whose fundamental representation is $2N_c$-dimensional as $Sp(N_c)$. A “flavor” of $Sp(N_c)$ represents two fields in the fundamental representation; we will use $N_f$ when referring to the number of flavors and $n_f$ when referring to the number of fields. Similarly a “flavor” of $SU(N_c)$ represents a field in the fundamental representation and another in the antifundamental representation; we will use $N_f$ when referring to the number of flavors and $(\tilde{m}_f)m_f$ when referring to the number of (anti)fundamental representations. A “defining model” is a theory which only contains charged fields in the defining representation of the gauge group, as in [3,4,11].

2.1. $SU(N_c)$ with an adjoint field and $N_f$ fundamental flavors

These models were presented by Kutasov and Schwimmer [13]. The field $X$ is in the adjoint representation of $SU(N_c)$ and the $N_f$ fields $Q^f$ ($\tilde{Q}^\dot{g}$) are in the (anti)fundamental representation. The superpotential $W = \text{Tr} \, X^{k+1}$ truncates the chiral ring; the chiral mesons are $(M_j)^{f\dot{g}} \equiv Q^f X^j \tilde{Q}^{\dot{g}}, j = 0 \ldots k - 1$.

The dual theory has gauge group $SU(\tilde{N}_c)$, with $\tilde{N}_c = kN_f - N_c$. It has an adjoint field $Y$, $N_f$ flavors $q_f, \tilde{q}^\dot{g}$, and singlet fields $(M_j)^{f\dot{g}}$. The superpotential is

$$W = \text{Tr} \, Y^{k+1} + \sum_{j=0}^{k-1} M_{k-j-1} q Y^j \tilde{q}. \quad (2.1)$$
Under perturbation by a mass term, \( W = \text{Tr} \, X^{k+1} + m \text{Tr} \, X^2 \), the gauge group breaks to a product of decoupled defining models

\[
SU(n_1) \times U(n_2) \cdots \times U(n_k)
\]  

(2.2)

where \( \sum n_j = N_c \); each group has \( N_f \) flavors in the fundamental representation. The mass term is mapped under duality to a mass term for \( Y \); the magnetic theory flows to a similar product of defining models, which are dual to those above [3].

If \( N_c \) is divisible by \( k \), then the model also has a flat direction under which it breaks to \( [SU(N_c/k)]^k \), with each factor a defining model with \( N_f \) flavors. The dual theory breaks to \( [SU(\tilde{N}_c/k)]^k = [SU(N_f - N_c/k)]^k \), which satisfies the duality of [3].

2.2. \( SO(N_c) \) with a traceless symmetric tensor and \( N_f \) vectors

These models were presented in [14]. The field \( X \) is in the traceless symmetric tensor representation of \( SO(N_c) \) and the \( N_f \) fields \( Q^f \) are in the vector representation. The superpotential \( W = \text{Tr} \, X^{k+1} \) truncates the chiral ring; the chiral mesons are \( (M_j)^{fg} \equiv Q^f X^j Q^g \), \( j = 0 \ldots k - 1 \), which are symmetric tensors of the flavor group.

The dual theory has gauge group \( SO(\tilde{N}_c) \), with \( \tilde{N}_c = k(N_f + 4) - N_c \). It has a traceless symmetric tensor field \( Y \), \( N_f \) vectors \( q_f \), and singlet fields \( (M_j)^{fg} \). The superpotential is

\[
W = \text{Tr} \, Y^{k+1} + \sum_{j=0}^{k-1} M_{k-j-1} q Y^j q .
\]

(2.3)

Under perturbation by a mass term, \( W = \text{Tr} \, X^{k+1} + m \text{Tr} \, X^2 \), the gauge group breaks to a product of decoupled defining models

\[
SO(n_1) \times SO(n_2) \cdots \times SO(n_k)
\]

(2.4)

where \( \sum n_j = N_c \); each group has \( N_f \) fields in the vector representation. The magnetic theory flows to the dual of this product.

If \( N_c \) is divisible by \( k \), then the model also has a flat direction under which it breaks to \( [SO(N_c/k)]^k \), with each factor a defining model with \( N_f \) fields in the vector representation. The dual theory breaks to \( [SU(\tilde{N}_c/k)]^k = [SU(N_f + 4 - N_c/k)]^k \), which satisfies the duality of [3].
2.3. $Sp(N_c)$ with a traceless antisymmetric tensor and $N_f$ flavors

These models were also presented in [14]. The field $X$ is in the traceless antisymmetric tensor representation of $Sp(N_c)$ and the $N_f$ flavors ($2N_f$ fields) $Q^j$ are in the $2N_c$ dimensional fundamental representation. The superpotential $W = \text{Tr} \; X^{k+1}$ truncates the chiral ring; the chiral mesons are $(M_j)^{fg} \equiv Q^f X^j Q^g$, $j = 0 \ldots k - 1$, which are antisymmetric tensors of the flavor group.

The dual theory has gauge group $Sp(\tilde{N}_c)$, with $\tilde{N}_c = k(N_f - 2) - N_c$. It has a traceless antisymmetric tensor field $Y$, $N_f$ flavors ($2N_f$ fields) $q_f$, and singlet fields $(M_j)^{fg}$. The superpotential is

$$W = \text{Tr} \; Y^{k+1} + \sum_{j=0}^{k-1} M_{k-j-1} q Y^j q . \tag{2.5}$$

Under perturbation by a mass term, $W = \text{Tr} \; X^{k+1} + m \text{Tr} \; X^2$, the gauge group breaks to a product of decoupled defining models

$$Sp(n_1) \times Sp(n_2) \cdots \times Sp(n_k) \tag{2.6}$$

where $\sum n_j = N_c$; each group has $N_f$ flavors in the fundamental representation. The magnetic theory flows to the dual of this product.

If $N_c$ is divisible by $k$, then the model also has a flat direction under which it breaks to $[Sp(N_c/k)]^k$, with each factor a defining model with $N_f$ flavors. The dual theory breaks to $[Sp(\tilde{N}_c/k)]^k = [Sp(N_f - 2 - N_c/k)]^k$, which satisfies the duality of [3].

2.4. $Sp(N_c)$ with an adjoint field and $N_f$ flavors

These models were presented in [15]. The field $X$ is in the adjoint representation of $Sp(N_c)$ and the $N_f$ flavors $Q^j$ are in the fundamental representation. The superpotential $W = \text{Tr} \; X^{2(k+1)}$ truncates the chiral ring; the chiral mesons are $(M_j)^{fg} \equiv Q^f c X^j Q^g$, $j = 0 \ldots 2k$, which are (anti)symmetric tensors of the flavor group for odd (even) $j$.

The dual theory has gauge group $Sp(\tilde{N}_c)$, with $\tilde{N}_c = (2k + 1)N_f - N_c - 2$. It has a field $Y$ in the adjoint representation, $N_f$ flavors $q_f$, and singlet fields $(M_j)^{fg}$. The superpotential is

$$W = \text{Tr} \; Y^{2(k+1)} + \sum_{j=0}^{2k} M_{2k-j} q Y^j q . \tag{2.7}$$
Under perturbation by a mass term, \( W = \text{Tr} \ X^{2(k+1)} + m \text{Tr} \ X^2 \), the gauge group breaks to a product of decoupled defining models

\[
Sp(n_0) \times U(n_1) \cdots \times U(n_k)
\]  

(2.8)

where \( \sum_0^k n_j = N_c \); the symplectic group factor has \( N_f \) flavors (2\( N_f \) fields) while each unitary group has 2\( N_f \) flavors in the fundamental representation. The magnetic theory flows to the dual of this product.

2.5. \( SO(N_c) \) with an adjoint field and \( N_f \) vectors

These models were also presented in [15]. The field \( X \) is in the adjoint representation of \( SO(N_c) \) and the \( N_f \) flavors \( Q^f \) are in the vector representation. The superpotential \( W = \text{Tr} \ X^{2(k+1)} \) truncates the chiral ring; the chiral mesons are \( (M_j)^{fg} \equiv Q^f X^j Q^g \), \( j = 0 \ldots 2k \), which are (anti)symmetric tensors of the flavor group for even (odd) \( j \).

The dual theory has gauge group \( SO(\tilde{N}_c) \), with \( \tilde{N}_c = (2k + 1)N_f - N_c + 4 \). It has a field \( Y \) in the adjoint representation, \( N_f \) vectors \( q_f \), and singlet fields \( (M_j)^{fg} \). The superpotential is

\[
W = \text{Tr} \ Y^{2(k+1)} + \sum_{j=0}^{2k} M_{2k-j} q Y^j q .
\]  

(2.9)

Under perturbation by a mass term, \( W = \text{Tr} \ X^{2(k+1)} + m \text{Tr} \ X^2 \), the gauge group breaks to a product of decoupled defining models

\[
SO(n_0) \times U(n_1) \cdots \times U(n_k)
\]  

(2.10)

where \( n_0 + 2 \sum_1^k n_j = N_c \); the orthogonal group factor has \( N_f \) vectors while each unitary group has \( N_f \) flavors in the fundamental representation. The magnetic theory flows to the dual of this product.

2.6. \( SU(N_c) \) with an antisymmetric flavor and \( N_f \) fundamental flavors

The fields \( X \) and \( \tilde{X} \) are a flavor of antisymmetric tensor representations of \( SU(N_c) \) and the \( N_f \) fields \( Q^f \) (\( \tilde{Q}^\dot{g} \)) are in the (anti)fundamental representation. The superpotential

\[
W = \text{Tr} \ (X \tilde{X})^{k+1}
\]  

(2.11)
truncates the chiral ring; the chiral mesons are \((M_j)^{fg} \equiv Q^f(\tilde{X}X)^j\tilde{Q}^g\), \(j = 0 \ldots k\), 
\((P_r)^{fg} \equiv Q^f(\tilde{X}X)^r\tilde{X}Q^g\), and \((\tilde{P}_r)^{fg} \equiv \tilde{Q}^fX(\tilde{X}X)^r\tilde{Q}^g\), \(r = 0 \ldots k - 1\). The \(P\) and \(\tilde{P}\) operators are antisymmetric in their flavor indices.

The dual theory has gauge group \(SU(\tilde{N}_c)\), with \(\tilde{N}_c = (2k+1)N_f - 4k - N_c\). It has fields \(Y\) and \(\tilde{Y}\) which form a flavor of antisymmetric tensor representations, \(N_f\) fundamental flavors \(q_f, \tilde{q}_g\), and singlet fields \((M_j)^{fg}, (P_r)^{fg}, (\tilde{P}_r)^{fg}\). The superpotential is

\[
W = \operatorname{Tr} (Y\tilde{Y})^{k+1} + \sum_{j=0}^{k} M_{k-j}q(\tilde{Y}Y)^j\tilde{q} + \sum_{r=0}^{k-1} P_{k-r-1}q(\tilde{Y}Y)^r\tilde{q} + \tilde{P}_{k-r-1}\tilde{q}Y(\tilde{Y}Y)^r\tilde{q} \]

\[(2.12)\]

The duality exhibits a feature not previously observed. There are two vector-like \(U(1)\) symmetries, one counting \(X\)-number and one counting \(Q\)-number. Under duality these are mixed; \(Y\)-number and \(q\)-number are linear combinations of \(X\)-number and \(Q\)-number.

Under perturbation by a mass term, \(W = \operatorname{Tr} (X\tilde{X})^{k+1} + m\operatorname{Tr} (X\tilde{X})\), the gauge group breaks to a product of decoupled defining models

\[
SU(n_0) \times Sp(n_1) \times \cdots \times Sp(n_k)
\]

\[(2.13)\]

where \(n_0 + 2\sum n_j = N_c\); each group has \(N_f\) flavors of fundamental representations. The magnetic theory flows to the dual of this product.

The model has a set of flat directions in which an operator \(B_n \equiv X^nQ^{N_c-2n}\), with gauge indices contracted using an epsilon tensor, gets an expectation value. The gauge group is broken to \(Sp(n)\) with an antisymmetric tensor \(X\) and \(N_f\) flavors; the superpotential is \(W = \operatorname{Tr} \tilde{X}^{k+1}\). This is the theory considered in [14] and discussed in sect. 2.3. Under duality the operator \(B_n\) is mapped to \(Y^{k(N_f-2)-n}q^{N_f+2n-N_c}\). Its expectation value causes the magnetic theory to flow to the dual expected from sect. 2.3 [14].

Additional discussion of this model is presented in sect. 3.

### 2.7. \(SU(N_c)\) with a symmetric flavor and \(N_f\) fundamental flavors

The fields \(X\) and \(\tilde{X}\) are a flavor of symmetric tensor representations of \(SU(N_c)\) and the \(N_f\) fields \(Q^f(\tilde{Q}^g)\) are in the (anti)fundamental representation. The superpotential

\[
W = \operatorname{Tr} (X\tilde{X})^{k+1}
\]

\[(2.14)\]

truncates the chiral ring; the chiral mesons are \((M_j)^{fg} \equiv Q^f(\tilde{X}X)^j\tilde{Q}^g\), \(j = 0 \ldots k\), 
\((P_r)^{fg} \equiv Q^f(\tilde{X}X)^r\tilde{X}Q^g\), and \((\tilde{P}_r)^{fg} \equiv \tilde{Q}^fX(\tilde{X}X)^r\tilde{Q}^g\), \(r = 0 \ldots k - 1\). The \(P\) and \(\tilde{P}\) operators are symmetric in their flavor indices.
The dual theory has gauge group $SU(\tilde{N}_c)$, with $\tilde{N}_c = (2k + 1)N_f + 4k - N_c$. It has fields $Y$ and $\tilde{Y}$ which form a flavor of symmetric tensor representations, $N_f$ fundamental flavors $q_f, \tilde{q}_g$, and singlet fields $(M_j)^{fg}, (P_r)^{fg},$ and $(\tilde{P}_r)^{fg}$. The superpotential is

$$W = \text{Tr} (Y\tilde{Y})^{k+1} + \sum_{j=0}^{k} M_{k-j}q(\tilde{Y}Y)^j\tilde{q} + \sum_{r=0}^{k-1} \left[ P_{k-r-1}q(\tilde{Y}Y)^r\tilde{q} + \tilde{P}_{k-r-1}\tilde{q}Y(\tilde{Y}Y)^r\tilde{q} \right].$$

(2.15)

As in the previous case there are two vector-like $U(1)$ symmetries, one counting $X$-number and one counting $Q$-number, which are mixed under duality.

Under perturbation by a mass term, $W = \text{Tr} (X\tilde{X})^{k+1} + m\text{Tr} (X\tilde{X})$, the gauge group breaks to a product of decoupled defining models

$$SU(n_0) \times SO(n_1) \times \cdots \times SO(n_k)$$

(2.16)

where $n_0 + \sum n_j = N_c$; the unitary factor has $N_f$ flavors of fundamental representations while each orthogonal group has $2N_f$ fields in the vector representation. The magnetic theory flows to the dual of this product.

The model has a set of flat directions in which an operator $B_n \equiv X^n Q_{N_c-n} Q_{N_c-n}^{N_c-n}$, with gauge indices contracted using two epsilon tensors, gets an expectation value. The gauge group is broken to $SO(n)$ with a symmetric tensor $\tilde{X}$ and $2N_f$ vectors; the superpotential is $W = \text{Tr} \tilde{X}^{k+1}$. This is the theory considered in [14] and discussed in sect. 2.2. Under duality the operator $B_n$ is mapped to $Y^{k(2N_f+4)-n} q^{N_f-n-N_c} q^{N_f+n-N_c}$. Its expectation value causes the magnetic theory to flow to the dual expected from sect. 2.2 [14].

Additional discussion of this model is presented in sect. 4.

2.8. $SU(N_c)$ with an antisymmetric tensor and a symmetric tensor; a chiral theory

The gauge group is $SU(N_c)$; the field $X$ is in the $\frac{1}{2}N_c(N_c - 1)$ representation, the field $\tilde{X}$ is in the $\frac{1}{2}N_c(N_c + 1)$ representation, and the $m_f(\tilde{m}_f)$ fields $Q^f (\tilde{Q}^\tilde{f})$ are in the (anti)fundamental representation of $SU(N_c)$. This model is chiral; anomaly cancellation requires that $m_f = \tilde{m}_f + 8$. The superpotential

$$W = \text{Tr} (X\tilde{X})^{2(k+1)}$$

(2.17)

truncates the chiral ring; the chiral mesons are $(M_j)^{fg} \equiv Q^f (\tilde{X}X)^j\tilde{Q}^g, j = 0 \ldots 2k + 1$, $(P_r)^{fg} \equiv Q^f (\tilde{X}X)^r\tilde{X}Q^g$ and $(\tilde{P}_r)^{fg} \equiv \tilde{Q}^j X(\tilde{X}X)^r\tilde{Q}^\tilde{g}, r = 0 \ldots 2k$. The $P_r$ ($\tilde{P}_r$) operators
are symmetric in their flavor indices for even (odd) \( r \) and antisymmetric in their flavor indices for odd (even) \( r \).

The dual theory has gauge group \( SU(\tilde{N}_c) \), with \( \tilde{N}_c = \frac{1}{2}(4k+3)(m_f + \tilde{m}_f) - N_c \). It has a field \( Y \) in the \( \frac{1}{2}N_c(\tilde{N}_c - 1) \) representation, a field \( \tilde{Y} \) in the \( \frac{1}{2}N_c(\tilde{N}_c + 1) \) representation, \( m_f (\tilde{m}_f) \) fields \( q_f (\tilde{q}_f) \) in the (anti)fundamental representation of \( SU(\tilde{N}_c) \), and singlet fields \( (M_j)^f g \), \( (P_r)^f g \) and \( (\tilde{P}_r)^f g \). The gauge anomaly vanishes since we still have \( m_f = \tilde{m}_f + 8 \). The superpotential is

\[
W = \text{Tr} (Y \tilde{Y})^{2(k+1)} + \sum_{j=0}^{2k+1} M_{2k-j+1} q_f (\tilde{Y} Y)^j \tilde{q}_f + \sum_{r=0}^{2k} \left[ P_{2k-r} q_f (\tilde{Y} Y)^r \tilde{Y} q_f + \tilde{P}_{2k-r} \tilde{q}_f (\tilde{Y} Y)^r \tilde{Y} \right].
\]

As in earlier cases there are two \( U(1) \) global symmetries, in addition to the \( R \) symmetry, which are rotated under duality.

Under the perturbation by a term \( W = \text{Tr} (X \tilde{X})^{2(k+1)} + \lambda \text{Tr} (X \tilde{X})^2 \) the gauge group breaks to a product of decoupled models

\[
SU(n_0) \times U(n_1) \times \cdots \times U(n_k)
\]

where \( n_0 + 2 \sum n_j = N_c \). The first factor is a model of the type discussed in this section, with \( k = 0 \); it has an antisymmetric tensor \( \tilde{X} \), a symmetric tensor \( \tilde{X} \), a superpotential \( \tilde{W} = \text{Tr} (\tilde{X} \tilde{X})^2 \), and \( m_f (\tilde{m}_f) \) fields in the (anti)fundamental representation. The other factors are defining models with \( m_f + \tilde{m}_f \) flavors. The magnetic theory flows to the dual of this product.

The model has a set of flat directions in which an operator \( B_n \equiv X^n Q^{N_c-2n} \), with gauge indices contracted using an epsilon tensor, gets an expectation value. The gauge group is broken to \( Sp(n) \) with a symmetric (adjoint) tensor \( \tilde{X} \) and \( m_f + \tilde{m}_f \) fundamental fields; the superpotential is \( W = \text{Tr} (\tilde{X} \tilde{X})^{2(k+1)} \). This is the theory considered in [13] and discussed in sect. 2.4. Under duality the operator \( B_n \) is mapped to \( Y^{(k+1)(m_f + \tilde{m}_f) - n - 2} q_{m_f + 2n - N_c} \). Its expectation value causes the magnetic theory to flow to the dual expected from sect. [2,3,15].

The model also has a set of flat directions in which an operator \( \tilde{B}_n \equiv \tilde{X}^n \tilde{Q}^{N_c-n} \tilde{Q}^{N_c-n} \), with gauge indices contracted using two epsilon tensors, gets an expectation value. The gauge group is broken to \( SO(n) \) with an antisymmetric (adjoint) tensor \( X \) and \( m_f + \tilde{m}_f \) vectors; the superpotential is \( W = \text{Tr} (X^{2(k+1)}) \). This is the theory considered in [13] and discussed in sect. 2.3. Under duality the operator \( \tilde{B}_n \) is mapped to \( \tilde{Y}^{(k+1)(m_f + \tilde{m}_f) - n + 4} \tilde{q}_{m_f + n - N_c} \tilde{q}_{m_f + n - N_c} \). Its expectation value causes the magnetic theory to flow to the dual expected from sect. [2,3,15].

Additional discussion of this model is presented in sect. 5.
2.9. $Sp(N_c) \times Sp(N'_c)$

The gauge group is $Sp(N_c) \times Sp(N'_c)$; the field $X$ is in the $(2N_c, 2N'_c)$, and the $N_f$ $(N'_f)$ flavors $Q^f$ $(Q'^{g'})$ are in the fundamental representation of $Sp(N_c)$ $(Sp(N'_c))$. The superpotential

$$W = \text{Tr} \ X^{2(k+1)}$$

(2.20)

truncates the chiral ring; the chiral mesons are $(P_r)^{f g'} = Q^f X^{(2r+1)} Q^{g'}, \ r = 0 \ldots k - 1$ ($M_j)^{f g} = Q^j X^{2j} Q^g$, and $(M'_j)^{f' g'} = Q^{f'} X^{2j} Q'^{g'}, \ j = 0 \ldots k$. The $M$ and $M'$ operators are antisymmetric in their flavor indices.

The dual theory has gauge group $Sp(\tilde{N}_c) \times Sp(\tilde{N}'_c)$, with $\tilde{N}_c = (k+1)(N_f + N'_f - 2) - N_f - N'_c$ and $\tilde{N}'_c = (k+1)(N_f + N'_f - 2) - N'_f - N_c$. It has a field $Y$ in the $(\tilde{N}_c, \tilde{N}'_c)$, $N_f$ $(N'_f)$ flavors $q^{f'}$ $(q'_f)$ in the fundamental representation of $Sp(\tilde{N}_c)$ $(Sp(\tilde{N}'_c))$, and singlet fields $(P_r)^{f g'}$, $(M_j)^{f g}$, and $(M'_j)^{f' g'}$. The superpotential is

$$W = \text{Tr} \ Y^{2(k+1)} + \sum_{r=0}^{k-1} P_{k-r-1} q^{2r+1} q' + \sum_{j=0}^{k} [M_{k-j} q'^{2j} q' + M_{k-j}' q'^{2j} q] .$$

(2.21)

Note that the mesons of the $Sp(N_c)$ group couple to the dual quarks of $Sp(\tilde{N}_c)$; this sort of transformation on the flavor indices will be seen repeatedly.

Under perturbation by a mass term, $W = \text{Tr} \ X^{2(k+1)} + m \text{Tr} \ X^2$, the gauge group breaks to a product of decoupled defining models

$$Sp(N_c - N'_c + p_0) \times Sp(p_0) \times Sp(p_1) \times \ldots \times Sp(p_k)$$

(2.22)

(for $N_c \geq N'_c$) with $\sum_{\ell=0}^{k} p_\ell = N'_c$. The first (second) factor has $N_f$ ($N'_f$) fundamental flavors, while the others have $N_f + N'_f$ flavors. The magnetic theory flows to the dual of this product.

When the gauge coupling of one $Sp$ factor is much larger than that of the other, the former can be dualized as in [3,11] with the latter carried along as a weakly coupled spectator. For example, for $N'_f + N_c = N'_c + 2$ the $Sp(N'_c)$ factor confines [11], and the $Sp(N_c)$ theory below the confinement scale is a model of [14] (see sect. 2.3) with an antisymmetric tensor, $N_f + N'_f$ flavors and $W = X^{k+1}$. In the magnetic theory $Sp(\tilde{N}'_c)$ confines similarly (since $N'_f + \tilde{N}_c = \tilde{N}'_c + 2$) leaving a theory with $Sp(\tilde{N}_c) = Sp[k(N_f + N'_f - 2) - N_c]$, which is the dual [14] of the confined electric model.* With $N'_f + N_c > N'_c + 2$

* In addition, both the electric and magnetic theory have extra singlets and a non-perturbative term in the superpotential which are preserved by the duality.
both theories again flow to models studied in \[14\], with part of the flavor group weakly gauged and with a superpotential. The preservation of duality in this way is essential for the consistency of the renormalization group flow given arbitrary gauge couplings. The details of the general case are given in sect. 6.3.

Additional discussion of this model is presented in sect. 6.

2.10. \(SO(N_c) \times SO(N'_c)\)

The gauge group is \(SO(N_c) \times SO(N'_c)\); the field \(X\) is in the \((N_c, N'_c)\), and the \(N_f \ (N'_f)\) fields \(Q^f \ (Q'^{g'})\) are in the vector representation of \(SO(N_c) \ (SO(N'_c))\). The superpotential truncates the chiral ring; the chiral mesons are \((M_j)^fg = Q^f X^{(2r+1)} Q'^g\), \(r = 0 \ldots k - 1\) \((M'_j)^f g' = Q'^f X^{2j} Q'^g\), \(j = 0 \ldots k\). The \(M\) and \(M'\) operators are symmetric in their flavor indices.

The dual theory has gauge group \(SO(\tilde{N}_c) \times SO(\tilde{N}'_c)\), with \(\tilde{N}_c = (k+1)(N_f + N'_f + 4) - N_f - N'_c\) and \(\tilde{N}'_c = (k+1)(N_f + N'_f + 4) - N'_f - N_c\). It has a field \(Y\) in the \((\tilde{N}_c, \tilde{N}'_c)\), \(N'_f \ (N_f)\) fields \(q'_g \ (q'_f)\) in the vector representation of \(SO(\tilde{N}_c) \ (SO(\tilde{N}'_c))\), and singlet fields \((P_r)^fg'\), \((M_j)^fg\), and \((M'_j)^f g'\). The superpotential is

\[
W = \text{Tr} \ X^{2(k+1)}
\]

Note that the mesons of the \(SO(N_c)\) group couple to the dual quarks of \(SO(\tilde{N}'_c)\).

Under perturbation by a mass term, \(W = \text{Tr} \ X^{2(k+1)} + m \text{Tr} \ X^2\), the gauge group breaks to a product of decoupled defining models

\[
SO(N_c - N'_c + p_0) \times SO(p_0) \times SO(p_1) \times \ldots \times SO(p_k)
\]

(\(N_c \geq N'_c\)) with \(\sum_{\ell=0}^k p_\ell = N'_c\). The first (second) factor has \(N_f \ (N'_f)\) fields in the vector representation, while the others have \(N_f + N'_f\) vectors. The magnetic theory flows to the dual of this product.

As in sect. 2.9, if \(N'_f + N_c = N'_c - 4\) the \(SO(N'_c)\) factor can confine \((M_j)^fg\), leaving a theory of \([4]\) (see sect. 2.2) with \(SO(N_c)\), a symmetric tensor, \(N_f + N'_f\) flavors and \(W = X^{k+1}\). In the magnetic theory \(SO(\tilde{N}'_c)\) similarly confines (since \(N'_f + \tilde{N}_c = \tilde{N}'_c - 4\)) leaving a theory with \(SO[k(N_f + N'_f + 4) - N_c]\), which is the dual \([4]\) of the confined electric model. More general cases are discussed in sect. 7.3.

Additional discussion of this model is presented in sect. 7.
2.11. $SU(N_c) \times SU(N'_c)$

The gauge group is $SU(N_c) \times SU(N'_c)$; the field $X$ is in the $(N_c,N'_c)$, the field $\tilde{X}$ is in the conjugate representation, and the $N_f$ $(N'_f)$ flavors $Q^f, \tilde{Q}^\dot{g}$ $(Q'^{f'}, \tilde{Q}'^{\dot{g}'})$ are in the fundamental representation of $SU(N_c)$ ($SU(N'_c)$). The superpotential

$$W = Tr (X \tilde{X})^{k+1}$$

(2.26)

truncates the chiral ring; the chiral mesons are $(M_j)^{f\dot{g}} = Q^f (\tilde{X}X)^j \tilde{Q}^{\dot{g}}$, $(M'_j)^{f'\dot{g}'} = Q'^{f'} (\tilde{X}X)^j \tilde{Q}'^{\dot{g}'}$, $(P_r)^{f\dot{g}} = \tilde{Q}(\tilde{X}X)^{-1}Q'\dot{g}'$ and $(\tilde{P}_r)^{f'\dot{g}'} = \tilde{Q}'(\tilde{X}X)^{-1}Q\dot{g}$. where $j = 0 \ldots k$ and $r = 0 \ldots k - 1$.

The dual theory has gauge group $SU(\tilde{N}_c) \times SU(\tilde{N}'_c)$, with $\tilde{N}_c = (k+1)(N_f + N'_f) - N_f - N'_c$ and $\tilde{N}'_c = (k+1)(N_f + N'_f) - N'_f - N_c$. It has fields $Y$ and $\tilde{Y}$ in the $(\tilde{N}_c,\tilde{N}'_c)$ and its conjugate, $(N'_f)$ $(N_f)$ flavors $q^{f'}, \tilde{q}^{\dot{g}'}$ ($q^{f}, \tilde{q}^{\dot{g}}$) in the fundamental representation of $SU(\tilde{N}_c)$ ($SU(\tilde{N}'_c)$), and singlet fields $(M_j)^{f\dot{g}}$, $(M'_j)^{f'\dot{g}'}$, $(P_r)^{f\dot{g}}$, and $(\tilde{P}_r)^{f'\dot{g}'}$. The superpotential is

$$W = Tr (Y \tilde{Y})^{k+1} + \sum_{j=0}^{k} \left[ M_{k-j} q^{f}(\tilde{Y}Y)^j \tilde{q} + M_{k-j} \tilde{q}(Y \tilde{Y})^j q \right]$$

(2.27)

$$+ \sum_{r=0}^{k-1} \left[ P_{k-r-1} q \tilde{Y}(Y \tilde{Y})^r q' + \tilde{P}_{k-r-1} q' (Y \tilde{Y})^r Y \tilde{q} \right].$$

Note that the mesons of the $SU(N_c)$ group couple to the dual quarks of $SU(\tilde{N}'_c)$.

As in sects. 2.6 and 2.7, the three vector-like $U(1)$ symmetries, counting $X$-number, $Q$-number and $Q'$-number, are mixed under duality.

Under perturbation by a mass term, $W = Tr (X \tilde{X})^{k+1} + m Tr X \tilde{X}$, the gauge group breaks to a product of decoupled defining models

$$SU(N_c - N'_c + p_0) \times SU(p_0) \times U(p_1) \times \ldots \times U(p_k)$$

(2.28)

(for $N_c \geq N'_c$) with $\sum_{\ell=0}^{k} p_{\ell} = N'_c$. The first (second) factor has $N_f$ ($N'_f$) fundamental flavors, while the others have $N_f + N'_f$ flavors. The magnetic theory flows to the dual of this product.

As in sect. 2.3, if $N'_f + N_c = N'_c + 1$ the $SU(N'_c)$ factor can confine 3 leaving a theory of 13 (see sect. 2.1) with $SU(N_c)$, an adjoint tensor $\hat{X} \sim X \tilde{X}$, $N_f + N'_f + 1$ flavors and $W = Tr \hat{X}^{k+1} + B X \tilde{B}$, where $B$, a baryon of $SU(N'_c)$, is in the fundamental representation.
of $SU(N_c)$. In the magnetic theory $SU(N'_c)$ confines similarly (since $N_f + N'_c = N'_c + 1$) leaving a theory with $SU(N_c) = SU[k(N_f + N'_f) - N_c + 1]$ and a similar superpotential; this is dual \[12,16,13\] to the confined electric model. More general cases are discussed in section 8.4.

The model also has a set of flat directions in which an operator $B_n \equiv X^n Q^{N_c-n} Q'^{N'_c-n}$, with gauge indices contracted using two epsilon tensors, gets an expectation value. The gauge group is broken to $SU(n)$ with an adjoint field $\tilde{X}$ and $N_f + N'_f$ flavors, along with some additional singlets; the infrared superpotential is $W = \text{Tr} \, \tilde{X}^{k+1}$. This is the theory considered in \[12\] and \[13\] and discussed in sect. 2.4. Under duality the operator $B_n$ is mapped to $Y^{k(N_f+N'_f)-n} q^{N'_f+n-N'_c} q'^{N_f+n-N_c}$. Its expectation value causes the magnetic theory to flow to the dual expected from \[13\].

Additional discussion of this model is presented in sect. 8.

2.12. $SO(N_c) \times Sp(N'_c)$

The gauge group is $SO(N_c) \times Sp(N'_c)$; the field $X$ is in the $(N_c, 2N'_c)$, the $N_f$ fields $Q^f$ are in the vector representation of $SO(N_c)$, and the $n'_f$ fields $Q'^{g'}$ are in the fundamental representation of $Sp(N'_c)$. The $Sp(N'_c)$ global anomaly of \[19\] requires $n'_f + N_c$ to be even. This model is chiral in that mass terms cannot be written for all the matter fields. The superpotential

$$W = \text{Tr} \, X^{4(k+1)}$$

(2.29) truncates the chiral ring; the chiral mesons are $(P_r)^{fg'} = Q^f X^{(2r+1)} Q'^{g'}$, $r = 0 \ldots 2k$ $(M_j)^{fg} = Q^f X^{2j} Q^g$, and $(M'_j)^{fg'} = Q'^{f'} X^{2j} Q'^{g'}$, $j = 0 \ldots 2k + 1$. The $M_j$ operators are (anti-)symmetric in their flavor indices for (odd) even $j$; the reverse is true for the $M'_j$ operators.

The dual theory has gauge group $SO(N'_c) \times Sp(N'_c)$, with $N'_c = 2(k+1)(N_f + n'_f) - N_f - 2N'_c$ and $2\tilde{N}'_c = 2(k+1)(N_f + n'_f) - n'_f - N_c$. (Note that the anomaly cancellation in the electric $Sp(N'_c)$ theory ensures that $\tilde{N}'_c$ is an integer.) It has a field $Y$ in the $(\tilde{N}_c, 2\tilde{N}'_c)$, $n'_f$ ($N_f$) fields $q^{g'}$ ($q^f$) in the vector (fundamental) representation of $SO(\tilde{N}_c)$ ($Sp(\tilde{N}'_c)$), and singlet fields $(P_r)^{fg'}$, $(M_j)^{fg}$, and $(M'_j)^{fg'}$. (Note that $\tilde{N}_c + N_f$ is even, ensuring that $Sp(\tilde{N}'_c)$ is not anomalous.) The superpotential is

$$W = \text{Tr} \, Y^{4(k+1)} + \sum_{r=0}^{2k} P_{2k-r} \, q Y^{2r+1} q'$$

(2.30)

$$+ \sum_{j=0}^{2k+1} \left[ M_{2k-j+1} \, q' Y^{2j} q' + M'_{2k-j+1} \, q Y^{2j} q \right].$$
Notice the mesons of the $SO(N_c)$ group couple to the dual quarks of the $Sp(\tilde{N}_c')$ group.

When the theory is perturbed by taking $W = \text{Tr} \ X^{4(k+1)} + \lambda \text{Tr} \ X^4$ (no mass term for $X$ can be written), the gauge group breaks to a product of models

$$SO(N_c - 2N'_c + 2p_0) \times Sp(p_0) \times U(p_1) \times \ldots \times U(p_k)$$

(2.31)

with $2p_0 + \sum_{\ell=1}^k p_\ell = 2N'_c$. (Here we consider $N_c \geq 2N'_c$; the other case is similar.)

The first and second factor form a model of the same type as the original, but with $k = 0$; the unitary factors are defining models with $N_f + n'_f$ flavors. The magnetic theory flows to the dual of this product.

As in sect. 2.13, if $N_f + 2N'_c = N_c - 4$ the $SO(N_c)$ factor confines [5] leaving a theory of $[15]$ (see sect. 2.4) with $Sp(N'_c)$, an adjoint tensor $\tilde{X} \sim X^2$, $N_f + n'_f$ fundamentals and $W = \text{Tr} \ \tilde{X}^{2(k+1)}$. In the magnetic theory $SO(\tilde{N}_c')$ confines similarly (since $n'_f + N_c = \tilde{N}_c' - 4$) leaving a theory with $Sp[\frac{1}{2}(2k + 1)(N_f + n'_f) - N'_c - 2]$ which is dual [15] to the confined electric model. Similarly, if $N_c + n'_f = 2(N'_c + 2)$, the $Sp(N'_c)$ and $Sp(\tilde{N}_c')$ factors confine; the confined theories are those of sect. 2.13 and satisfy the duality of [15].

Additional discussion of this model is presented in sect. 9.

2.13. $SU(M) \times SO(N)$ with a symmetric tensor of $SU(M)$

The gauge group is $SU(M) \times SO(N)$; the field $X$ is in the $(M, N)$, the field $\tilde{X}$ is a symmetric tensor in the $(\frac{1}{2}M(M+1), 1)$, the $m_f(\tilde{m}_f)$ fields $Q^f(\tilde{Q}^\tilde{f})$ are in the (anti)fundamental representation of $SU(M)$, and the $n_f$ fields $S^i$ are in the vector representation of $SO(N)$. For $SU(M)$ to be non-anomalous requires that $m_f + N = \tilde{m}_f + M + 4$. The superpotential

$$W = \text{Tr} \ (X \tilde{X} X)^{(k+1)}$$

(2.32)

truncates the chiral ring; the chiral mesons are $(M_j)^{f\tilde{g}} \equiv Q^f(\tilde{X} X^2)^j \tilde{Q}^{\tilde{g}}$, $(P_j)^{f\tilde{g}} \equiv Q^f(\tilde{X} X^2)^j \tilde{X} Q^{\tilde{g}}$, $(N_j)^{tu} \equiv S^t(X \tilde{X} X)^j S^{u}$, $(\tilde{R}_j)^{\tilde{g}t} \equiv \tilde{Q}^{\tilde{g}}(X^2 \tilde{X})^j X S^t$, $j = 0 \ldots k$, $(\tilde{P}_r)^{\tilde{f}t} \equiv \tilde{Q}^{\tilde{f}} X^2(\tilde{X} X^2)^r \tilde{X} X S^t$, $r = 0 \ldots k - 1$. The $N$, $P$ and $\tilde{P}$ operators are symmetric in their flavor indices.

The dual theory has gauge group $SU(\tilde{M}) \times SO(\tilde{N})$, with $\tilde{M} = (k+1)(m_f + \tilde{m}_f + n_f + 4) - m_f - N$ and $\tilde{N} = (k+1)(m_f + \tilde{m}_f + n_f + 4) - n_f - M$. It has a field $Y$ in the $(\tilde{M}, \tilde{N})$, a field $\tilde{Y}$ in the $(\frac{1}{2}M(M+1), 1)$, $n_f(\tilde{m}_f)$ fields $q_t(\tilde{q}_\tilde{g})$ in the (anti)fundamental representation of $SU(\tilde{M})$, $m_f$ fields $s_f$ in the vector representation of $SO(\tilde{N})$, and singlet
fields $(M_j)^{f\bar{g}}, (P_j)^{f\bar{g}}, (\bar{P}_r)^{f\bar{g}}, (N_j)^{tu}, (R_r)^{ft}$ and $(\bar{R}_j)^{\bar{g}t}$. All gauge anomalies automatically vanish. The superpotential is

$$W = \text{Tr} (Y\tilde{Y}Y)^{(k+1)} + \text{singlet meson couplings}. \quad (2.33)$$

We omit the meson couplings for brevity; these terms are analogous to those of all other models and are determined by the flavor symmetries (see sect. 10). We note that the mesons of the $SO(N)$ group couple to the dual quarks of the $SU(\tilde{M})$ group while the antiquark mesons of $SU(M)$ couple to the antiquarks of $SU(\tilde{M})$.

If the $SU(M)$ factor confines in the electric theory, then the $SU(\tilde{M})$ factor confines in the magnetic theory; in the infrared the models are electric and magnetic duals under the duality of [14] (see sect. 2.2). If the $SO(N)$ factor confines, then $SO(\tilde{N})$ confines also, and the infrared duality is that of sect. 2.7.

Additional discussion of this model is presented in sect. 10.

2.14. $SU(M) \times SO(N) \times SO(N')$

The gauge group is $SU(M) \times SO(N) \times SO(N')$; the field $X$ is in the $(\mathbf{M}, \mathbf{N}, \mathbf{1})$, the field $\tilde{X}$ is in the $(\mathbf{\tilde{M}}, \mathbf{1}, \mathbf{N'})$, the $m_f(\bar{m}_f)$ fields $Q^f \ (\bar{Q}^g)$ are in the (anti)fundamental representation of $SU(M)$, and the $n_f(n'_f)$ fields $S^t(S'^t)$ are in the vector representation of $SO(N)$ ($SO(N')$). Anomaly cancellation requires that $m_f + N = \bar{m}_f + N'$. The superpotential which truncates the chiral ring is

$$W = \text{Tr} (X\tilde{X})^{2(k+1)}. \quad (2.34)$$

The chiral mesons are $M_j \equiv Q(\tilde{X}X)^{2j}Q$, $P_j \equiv Q(\tilde{X}X)^{2j}\bar{X}^2Q$, $\bar{P}_j \equiv \bar{Q}X^2(\tilde{X}X)^{2j}\bar{Q}$, $N_j \equiv S(\tilde{X}X)^{2j}S$, $N'_j \equiv S'(\tilde{X}X)^{2j}S'$, $L_j \equiv SX(\tilde{X}X)^{2j}S', \bar{R}_j \equiv \bar{Q}(X\tilde{X})^{2j}XS$, $R'_j \equiv Q(\tilde{X}X)^{2j}\bar{X}S'$, $R_r \equiv Q(\tilde{X}X)^{2r}\bar{X}^2XS$ and $\bar{R}_r' \equiv \bar{Q}(X\tilde{X})^{2r}X^2\bar{X}S'$, with $j = 0 \ldots k$ and $r = 0 \ldots k - 1$. The $N$, $N'$, $P$ and $\bar{P}$ operators are symmetric in their flavor indices.

The dual theory has gauge group $SU(\tilde{M}) \times SO(\tilde{N}) \times SO(\tilde{N}')$, with $\tilde{M} = (k+1)(m_f + \bar{m}_f + n_f + n'_f + 4) - m_f - N$, $\tilde{N} = (k+1)(m_f + \bar{m}_f + n_f + n'_f + 4) - n_f - M$ and $\tilde{N}' = (k+1)(m_f + \bar{m}_f + n_f + 4) - n'_f - M$. It has a field $Y$ in the $(\mathbf{\tilde{M}}, \mathbf{\tilde{N}}, \mathbf{1})$, a field $\tilde{Y}$ in the $(\mathbf{\tilde{M}}, \mathbf{1}, \mathbf{\tilde{N}}')$, $n_f$ $(n'_f)$ fields $q_t \ (\bar{q}_{\bar{t}})$ in the (anti)fundamental representation of $SU(\tilde{M})$, $m_f$ $(\bar{m}_f)$ fields $s_f \ (s'_f)$ in the vector representation of $SO(\tilde{N})(SO(\tilde{N}'))$, and singlet fields $M_j, P_j, \bar{P}_j, N_j, N'_j, L_j, R_r, \bar{R}_j, R'_j$ and $\bar{R}_r'$. All gauge anomalies automatically vanish. The superpotential is

$$W = \text{Tr} (Y\tilde{Y})^{2(k+1)} + \text{singlet meson couplings}. \quad (2.35)$$
We omit the meson couplings for brevity; these terms are analogous to those of all other models and are determined by the flavor symmetries (see sect. 11). We note that the mesons of the $SU(M)$ group couple to the dual quarks of the $SO(\tilde{N}) \times SO(\tilde{N}')$ group.

If the $SU(M)$ factor confines in the electric theory, then the $SU(\tilde{M})$ factor confines in the magnetic theory; in the infrared the models are electric and magnetic duals under the duality of sect. 2.10. If the $SO(N')$ factor confines, then $SO(\tilde{N}')$ confines also, and the infrared duality is that of sect. 2.13.

Additional discussion of this model is presented in sect. 11.

2.15. $SU(M) \times Sp(N)$ with an antisymmetric tensor of $SU(M)$

The gauge group is $SU(M) \times Sp(N)$; the field $X$ is in the $(M, 2N)$, the field $\tilde{X}$ is an antisymmetric tensor in the $(\frac{1}{2}M(M-1), 1)$, the $m_f(\tilde{m}_f)$ fields $Q^f(\tilde{Q}^{\tilde{g}})$ are in the (anti)fundamental representation of $SU(M)$, and the $n_f$ fields $S^t$ are in the fundamental representation of $Sp(N)$. Anomaly cancellation requires that $m_f + 2N = \tilde{m}_f + M - 4$ and that $n_f + M$ be even. The superpotential

$$W = \text{Tr} \left( X\tilde{X}X \right)^{(k+1)}$$

(2.36)

trunicates the chiral ring; the chiral mesons are $(M_j)^{f\tilde{g}} \equiv Q^f(XX^2)^j\tilde{Q}^{\tilde{g}}$, $(P_j)^{f\tilde{g}} \equiv Q^f(XX^2)^j\tilde{X}Q^g$, $(N_j)^{tu} \equiv S^t(XXX)^jS^u$, $(\tilde{P}_j)^{\tilde{g}t} \equiv \tilde{Q}^{\tilde{g}}(XX\tilde{X})^jXS^t$, $j = 0 \ldots k$, $(\tilde{P}_r)^{\tilde{g}t} \equiv \tilde{Q}^{\tilde{g}}X^2(XX)\tilde{X}XS^t$ and $(R_r)^{ft} \equiv Q^f(XX^2)^r\tilde{X}XS^t$, $r = 0 \ldots k - 1$. The $N$, $P$ and $\tilde{P}$ operators are antisymmetric in their flavor indices.

The dual theory has gauge group $SU(\tilde{M}) \times Sp(\tilde{N})$, with $\tilde{M} = (k+1)(m_f + \tilde{m}_f + n_f - 4) - m_f - 2N$ and $2\tilde{N} = (k+1)(m_f + \tilde{m}_f + n_f - 4) - n_f - M$. (Anomaly cancellation in the electric theory ensures that $m_f + \tilde{m}_f + n_f$ is even.) It has a field $Y$ in the $(\frac{1}{2}M(M-1), 1)$, a field $\tilde{Y}$ in the $(\frac{1}{2}M(M-1), 1)$, $m_f$ $(\tilde{m}_f)$ fields $q_t$ $(\tilde{q}_{\tilde{g}})$ in the (anti)fundamental representation of $SU(\tilde{M})$, $m_f$ fields $s_f$ in the fundamental representation of $Sp(\tilde{N})$, and singlet fields $(M_j)^{f\tilde{g}}, (P_j)^{f\tilde{g}}, (\tilde{P}_j)^{\tilde{g}t}, (N_j)^{tu}, (R_r)^{ft}$ and $(\tilde{R}_j)^{\tilde{g}t}$. All gauge anomalies automatically vanish. The superpotential is

$$W = \text{Tr} \left( Y\tilde{Y}Y \right)^{(k+1)} + \text{singlet meson couplings}.$$ 

(2.37)

We omit the meson couplings for brevity; these terms are analogous to those of all other models and are determined by the flavor symmetries (see sect. 12). We note that the
mesons of the $Sp(N)$ group couple to the dual quarks of the $SU(\tilde{M})$ group while the
antiquark mesons of $SU(M)$ couple to the antiquarks of $SU(\tilde{M})$.

If the $SU(M)$ factor confines in the electric theory, then the $SU(\tilde{M})$ factor confines
in the magnetic theory; in the infrared the models are electric and magnetic duals under
the duality of $[14]$ (see sect. 2.3.) If the $Sp(N)$ factor confines, then the $Sp(\tilde{N})$ confines
also, and the infrared duality is that of sect. 2.4.

Additional discussion of this model is presented in sect. 12.

2.16. $SU(M) \times Sp(N) \times Sp(N')$

The gauge group is $SU(M) \times Sp(N) \times Sp(N')$; the field $X$ is in the $(\mathbf{M}, \mathbf{2N}, \mathbf{1})$, the
field $\tilde{X}$ is in the $(\mathbf{\bar{M}}, \mathbf{1}, \mathbf{2N'})$, the $m_f(\tilde{m}_f)$ fields $Q^f(\tilde{Q}^g)$ are in the (anti)fundamental
representation of $SU(M)$, and the $n_f(n'_f)$ fields $S^f(S'^{n_f})$ are in the fundamental representation
of $Sp(N)(Sp(N'))$. Anomaly cancellation requires that $m_f + 2N = \tilde{m}_f + 2N'$ and
that $n_f + M$ and $n'_f + M$ be even.

The superpotential

$$W = \text{Tr} \ (X \tilde{X})^{2(k+1)} \quad (2.38)$$

truncates the chiral ring; the chiral mesons are $M_j \equiv Q(\tilde{X}X)^{2j}Q$, $P_j \equiv Q(\tilde{X}X)^{2j}\tilde{X}Q$,
$\tilde{P}_j \equiv \tilde{Q}X^2(\tilde{X}X)^{2j}\tilde{Q}$, $N_j \equiv S(\tilde{X}X)^{2j}S$, $N'_j \equiv S'(\tilde{X}X)^{2j}S'$, $L_j \equiv SX\tilde{X}(\tilde{X}X)^{2j}S'$,
$\tilde{R}_j \equiv \tilde{Q}(X\tilde{X})^{2j}XS$, $R'_j \equiv Q(\tilde{X}X)^{2j}\tilde{X}S'$, $j = 0 \ldots k$, $R_r \equiv Q(\tilde{X}X)^{2r}\tilde{X}XS$ and
$\tilde{R}'_r \equiv \tilde{Q}(X\tilde{X})^{2r}X^2\tilde{X}S'$, $r = 0 \ldots k - 1$. The $N$, $N'$, $P$ and $\tilde{P}$ operators are antisym-metric in their flavor indices.

The dual theory has gauge group $SU(\tilde{M}) \times Sp(\tilde{N}) \times Sp(\tilde{N}')$, with $\tilde{M} = (k+1)(m_f + \tilde{m}_f + n_f + n'_f - 4) - m_f - 2N$, $2\tilde{N} = (k+1)(m_f + \tilde{m}_f + n_f + n'_f - 4) - n_f - M$ and
$2\tilde{N}' = (k+1)(m_f + \tilde{m}_f + n_f + n'_f - 4) - n'_f - M$. (Anomaly cancellation in the electric theory
ensures that $m_f + \tilde{m}_f + n_f + n'_f$ is even.) It has a field $Y$ in the $(\mathbf{\bar{M}}, \mathbf{2N}, \mathbf{1})$, a field $\tilde{Y}$ in
the $(\mathbf{\bar{M}}, \mathbf{1}, \mathbf{2N'})$, $n_f (n'_f)$ fields $q_t(\tilde{q}_t)$ in the (anti)fundamental representation of $SU(\tilde{M})$,
$m_f (\tilde{m}_f)$ fields $s_f (s'_f)$ in the fundamental representation of $Sp(\tilde{N})(Sp(\tilde{N}'))$, and singlet
fields $M_j, P_j, \tilde{P}_j, N_j, N'_j, L_j, R_r, \tilde{R}_j, R'_j$ and $\tilde{R}'_r$. All gauge anomalies automatically vanish.

The superpotential is

$$W = \text{Tr} \ (Y \tilde{Y})^{2(k+1)} + \text{singlet meson couplings} \quad (2.39)$$

We omit the meson couplings for brevity; these terms are analogous to those of all other
models and are determined by the flavor symmetries (see sect. 13). We note that the
mesons of the $SU(M)$ group couple to the dual quarks of the $Sp(\tilde{N}) \times Sp(\tilde{N}')$ group.
If the $SU(M)$ factor confines in the electric theory, then the $SU(\tilde{M})$ factor confines in the magnetic theory; in the infrared the models are electric and magnetic duals under the duality of sect. 2.9. If the $Sp(N')$ factor confines, then the $Sp(\tilde{N}')$ confines also, and the infrared duality is that of sect. 2.13.

Additional discussion of this model is presented in sect. 13.

2.17. $SU(M) \times Sp(N)$ with a symmetric tensor of $SU(M)$

The gauge group is $SU(M) \times Sp(N)$; the field $X$ is in the $(M, 2N)$, the field $\tilde{X}$ is a symmetric tensor in the $(\frac{1}{2}M(M+1), 1)$, the $m_f(\tilde{m}_f)$ fields $Q^f (\tilde{Q}^{\tilde{g}})$ are in the (anti)fundamental representation of $SU(M)$, and the $n_f$ fields $S^t$ are in the fundamental representation of $Sp(N)$. Anomaly cancellation requires that $m_f + 2N = \tilde{m}_f + M + 4$ and that $n_f + M$ be even. The superpotential

$$W = \text{Tr} \left( X \tilde{X} X \right)^{2(k+1)}$$

(2.40)

truncates the chiral ring; the chiral mesons are $(M_f)^{f \tilde{g}} \equiv Q_f^f (\tilde{X} X^2)^j \tilde{Q}^{\tilde{g}}$, $(P_j)^{f \tilde{g}} \equiv Q_f^f (\tilde{X} X^2)^j \tilde{X} Q^g$, $(N_j)^{t u} \equiv S^t (X \tilde{X} X)^j S^u$, $(\tilde{R}_j)^{\tilde{g} \tilde{t}} \equiv \tilde{Q}^g (X^2 \tilde{X})^j \tilde{X} S^t, \quad j = 0 \ldots 2k + 1,$

$(\tilde{P}_r)^{f \tilde{g}} \equiv \tilde{Q}^f X^2 (\tilde{X} X^2)^r \tilde{Q}^{\tilde{g}}$ and $(R_r)^{f t} \equiv Q_f^f (\tilde{X} X^2)^r \tilde{X} S^t, \quad r = 0 \ldots 2k$. The $P_j$ ($N_j$ and $\tilde{P}_j$) operators are symmetric in their flavor indices for even (odd) $j$ and antisymmetric in their flavor indices for odd (even) $j$.

The dual theory has gauge group $SU(\tilde{M}) \times Sp(\tilde{N})$, with $\tilde{M} = 2(k+1)(m_f + \tilde{m}_f + n_f) - m_f - 2N$ and $2\tilde{N} = 2(k+1)(m_f + \tilde{m}_f + n_f) - n_f - M$. (Anomaly cancellation in the electric theory ensures that $m_f + \tilde{m}_f + n_f$ is even.) It has a field $Y$ in the $(\tilde{M}, 2\tilde{N})$, a field $\tilde{Y}$ in the $(\frac{1}{2}M(M+1), 1), n_f (\tilde{m}_f)$ fields $q_t (\tilde{q}_{\tilde{g}})$ in the (anti)fundamental representation of $SU(\tilde{M})$, $m_f$ fields $s_f$ in the fundamental representation of $Sp(\tilde{N})$, and singlet fields $(M_f)^{f \tilde{g}}, (P_j)^{f \tilde{g}}, (\tilde{P}_r)^{f \tilde{g}}, (N_j)^{t u}, (R_r)^{f t}$ and $(\tilde{R}_j)^{\tilde{g} \tilde{t}}$. All gauge anomalies automatically vanish. The superpotential is

$$W = \text{Tr} \left( Y \tilde{Y} Y \right)^{2(k+1)} + \text{singlet meson couplings.}$$

(2.41)

We omit the meson couplings for brevity; these terms are analogous to those of all other models and are determined by the flavor symmetries (see sect. 14). We note that the mesons of the $Sp(N)$ group couple to the dual quarks of the $SU(\tilde{M})$ group while the antiquark mesons of $SU(M)$ couple to the antiquarks of $SU(\tilde{M})$.

If the $SU(M)$ factor confines in the electric theory, then the $SU(\tilde{M})$ factor confines in the magnetic theory; in the infrared the models are electric and magnetic duals under the duality of [13] (see sect. 2.4). If the $Sp(N)$ factor confines, then the $Sp(\tilde{N})$ confines also, and the infrared duality is that of sect. 2.8.

Additional discussion of this model is presented in sect. 14.
2.18. $SU(M) \times SO(N)$ with an antisymmetric tensor of $SU(M)$

The gauge group is $SU(M) \times SO(N)$; the field $X$ is in the $(M,N)$, the field $\tilde{X}$ is an antisymmetric tensor in the $(\frac{1}{2}M(M-1), 1)$, the $m_f(\tilde{m}_f)$ fields $Q^f(\tilde{Q}^g)$ are in the (anti)fundamental representation of $SU(M)$, and the $n_f$ fields $S^t$ are in the vector representation of $SO(N)$. Anomaly cancellation requires that $m_f + N = \tilde{m}_f + M - 4$. The superpotential

$$ W = \text{Tr} (X\tilde{X}X)^{2(k+1)} $$

(2.42)

truncates the chiral ring; the chiral mesons are $(M_j)^{f\tilde{g}} \equiv Q^f(\tilde{X}X^2)^j\tilde{Q}^g$, $(P_j)^{f\tilde{g}} \equiv Q^f(\tilde{X}X^2)^j\tilde{X}Q^g$, $(N_j)^tu \equiv S^t(X\tilde{X}X)^jS^u$, $(\tilde{R}_j)^{\tilde{g}t} \equiv \tilde{Q}^\tilde{g}(X^2\tilde{X})^jXS^t$, $j = 0 \ldots 2k + 1$, $(\tilde{P}_j)^{f\tilde{g}} \equiv \tilde{Q}^fX^2(\tilde{X}X^2)^r\tilde{Q}^\tilde{g}$ and $(R_r)^{ft} \equiv Q^f(X^2\tilde{X})^rXXS^t$, $r = 0 \ldots 2k$. The $P_j$ ($N_j$ and $\tilde{P}_j$) operators are antisymmetric in their flavor indices for even (odd) $j$ and symmetric in their flavor indices for odd (even) $j$.

The dual theory has gauge group $SU(\tilde{M}) \times SO(\tilde{N})$, with $\tilde{M} = 2(k+1)(m_f + \tilde{m}_f + n_f + n_f') - m_f - N$, and $\tilde{N} = 2(k+1)(m_f + \tilde{m}_f + n_f) - n_f - M$. It has a field $Y$ in the $(\tilde{M}, \tilde{N})$, a field $\tilde{Y}$ in the $(\frac{1}{2}M(M-1), 1)$, $n_f$ ($\tilde{n}_f$) fields $q_t$ ($\tilde{q}_\tilde{g}$) in the (anti)fundamental representation of $SU(\tilde{M})$, $m_f$ fields $s_f$ in the vector representation of $SO(\tilde{N})$, and singlet fields $(M_j)^{f\tilde{g}}, (P_j)^{f\tilde{g}}, (\tilde{P}_r)^{f\tilde{g}}, (N_j)^tu,(R_r)^{ft}$ and $(\tilde{R}_j)^{\tilde{g}t}$. All gauge anomalies automatically vanish. The superpotential is

$$ W = \text{Tr} (Y\tilde{Y}Y)^{2(k+1)} + \text{singlet meson couplings}. $$

(2.43)

We omit the meson couplings for brevity; these terms are analogous to those of all other models and are determined by the flavor symmetries (see sect. 15). We note that the mesons of the $SO(N)$ group couple to the dual quarks of the $SU(\tilde{M})$ group while the antiquark mesons of $SU(M)$ couple to the antiquarks of $SU(M)$.

If the $SU(M)$ factor confines in the electric theory, then the $SU(\tilde{M})$ factor confines in the magnetic theory; in the infrared the models are electric and magnetic duals under the duality of [15] (see sect. 2.3.) If the $SO(N)$ factor confines, then $SO(\tilde{N})$ confines also, and the infrared duality is that of sect. 2.8.

Additional discussion of this model is presented in sect. 15.
2.19. $SU(M) \times Sp(N) \times SO(N')$

The gauge group is $SU(M) \times Sp(N) \times SO(N')$; the field $X$ is in the $(M, 2N, 1)$, the field $\tilde{X}$ is in the $(\overline{M}, 1, N')$, the $m_f(\tilde{m}_f)$ fields $Q^f (\tilde{Q}^\theta)$ are in the (anti)fundamental representation of $SU(M)$, and the $n_f(n'_f)$ fields $S^t(S'^t)$ are in the fundamental (vector) representation of $Sp(N)(SO(N'))$. Anomaly cancellation requires that $m_f + 2N = \tilde{m}_f + N'$ and that $n_f + M$ be even. The superpotential

$$W = \text{Tr} (X\tilde{X})^{4(k+1)}$$

(2.44)

truncates the chiral ring; the chiral mesons are

$$M_j \equiv Q(\tilde{X}X)^{2j}\tilde{Q}, \quad P_j \equiv Q(\tilde{X}X)^{2j}\tilde{X}^2Q,$$

$$\tilde{P}_j \equiv \tilde{Q}X^2(\tilde{X}X)^{2j}\tilde{Q}, \quad N_j \equiv S(\tilde{X}X)^{2j}S, \quad N'_j \equiv S'(\tilde{X}X)^{2j}S', \quad L_j \equiv SX\tilde{X}(\tilde{X}X)^{2j}S', \quad \tilde{R}_j \equiv \tilde{Q}(X\tilde{X})^{2j}XS,$$

$$R'_j \equiv Q(\tilde{X}X)^{2j}\tilde{X}S', \quad j = 0 \ldots 2k + 1, \quad R_r \equiv Q(\tilde{X}X)^{2r}\tilde{X}^2XS$$

and

$$\tilde{R}_r \equiv \tilde{Q}(X\tilde{X})^{2r}X^2\tilde{X}S', \quad r = 0 \ldots 2k.$$  

The $N'$ and $P$ ($\tilde{N}$ and $\tilde{P}$) operators are symmetric in their flavor indices for even (odd) $j$ and antisymmetric in their flavor indices for odd (even) $j$.

The dual theory has gauge group $SU(\widetilde{M}) \times Sp(\widetilde{N}) \times SO(\widetilde{N'})$, with $\widetilde{M} = 2(k + 1)(m_f + \tilde{m}_f + n_f + n'_f) - m_f - 2N$, $2\widetilde{N} = 2(k + 1)(m_f + \tilde{m}_f + n_f + n'_f) - n_f - M$, and $2\widetilde{N}' = 2(k + 1)(m_f + \tilde{m}_f + n_f + n'_f) - n'_f - M$. It has a field $Y$ in the $(\overline{M}, 2\overline{N}, 1)$, a field $\tilde{Y}$ in the $(\overline{M}, 1, \overline{N}')$, $n_f$ ($n'_f$) fields $q_t$ ($\tilde{q}_{\nu'}$) in the (anti)fundamental representation of $SU(\widetilde{M})$, $m_f$ ($\tilde{m}_f$) fields $s_f$ ($s'_\nu$) in the fundamental (vector) representation of $Sp(\widetilde{N})(SO(\widetilde{N}')$, and singlet fields $M_j, P_j, \tilde{P}_j, N_j, N'_j, L_j, R_r, \tilde{R}_j, R'_j$ and $\tilde{R}_r$. All gauge anomalies automatically vanish: $2\widetilde{N} + n_f = \widetilde{N}' + n'_f$ and $\widetilde{M} + m_f$ is even. The superpotential is

$$W = \text{Tr} (Y\tilde{Y})^{4(k+1)} + \text{singlet meson couplings}.$$  

(2.45)

We omit the meson couplings for brevity; these terms are analogous to those of all other models and are determined by the flavor symmetries (see sect. 16). We note that the mesons of the $SU(M)$ group couple to the dual quarks of the $Sp(\widetilde{N}) \times SO(\widetilde{N}')$ group.

If the $SU(M)$ factor confines in the electric theory, then the $SU(\widetilde{M})$ factor confines in the magnetic theory; in the infrared the models are electric and magnetic duals under the duality of sect. 2.12. If the $SO(N')$ factor confines, then the $SO(\widetilde{N}')$ confines also, and the infrared duality is that of sect. 2.17. Confinement of the $Sp$ groups leads to the duality of sect. 2.18.

Additional discussion of this model is presented in sect. 16.
3. SU($N_c$) with an antisymmetric flavor and $N_f$ fundamental flavors

In this and all later sections we provide details on the models which were summarized in sect. 2.

We consider the model of sect. 2.6. This theory has an anomaly free SU($N_f$)$_L \times SU(N_f)_R \times U(1)_X \times U(1)_B \times U(1)_R$ global symmetry with matter transforming as $SU(N_c) \times SU(N_f)_L \times SU(N_f)_R \times U(1)_X \times U(1)_B \times U(1)_R$ global symmetry with matter transforming as

|    | SU($N_c$) | SU($N_f$)$_L$ | SU($N_f$)$_R$ | U(1)$_X$ | U(1)$_B$ | U(1)$_R$ |
|----|-----------|---------------|---------------|----------|----------|----------|
| $Q$ | $N_c$     | $N_f$         | 1             | 0        | $\frac{1}{N_c}$ | $1 - \frac{N_c+2k}{(k+1)N_f}$ |
| $\tilde{Q}$ | $N_c$     | 1             | $N_f$         | 0        | $-\frac{1}{N_c}$ | $1 - \frac{N_c+2k}{(k+1)N_f}$ |
| $X$ | asym      | 1             | 1             | 1        | $\frac{2}{N_c}$  | $\frac{1}{k+1}$   |
| $\tilde{X}$ | asym      | 1             | 1             | $-1$     | $-\frac{2}{N_c}$ | $\frac{1}{k+1}$   |

The theory (2.11) also has a discrete $\mathbb{Z}_{2N_f(k+1)}$ symmetry generated by

$$X, \tilde{X} \rightarrow \alpha^{N_f} X, \tilde{X}$$

$$Q, \tilde{Q} \rightarrow \alpha^{-(N_c-2)} Q, \tilde{Q},$$

with $\alpha = e^{2\pi i / 2N_f(k+1)}$.

3.1. Duality

The dual theory is $SU(\tilde{N}_c)$, where $\tilde{N}_c = (2k+1)N_f - 4k - N_c$, as described in sect. 2.6. Taking the singlets $M_j$, $P_r$, and $\tilde{P}_r$ to transform as the mesons of the electric theory, the dual theory (2.12) has the global $SU(N_f)_L \times SU(N_f)_R \times U(1)_X \times U(1)_B \times U(1)_R$ symmetry with the matter transforming as:

|    | SU($\tilde{N}_c$) | SU($N_f$)$_L$ | SU($N_f$)$_R$ | U(1)$_X$ | U(1)$_B$ | U(1)$_R$ |
|----|------------------|---------------|---------------|----------|----------|----------|
| $q$ | $\tilde{N}_c$    | $\overline{N_f}$ | 1             | $\frac{k(N_f-2)}{N_c}$ | $\frac{1}{N_c}$ | $1 - \frac{\tilde{N}_c+2k}{(k+1)N_f}$ |
| $\tilde{q}$ | $\tilde{N}_c$    | 1             | $\overline{N_f}$ | $-\frac{k(N_f-2)}{N_c}$ | $-\frac{1}{N_c}$ | $1 - \frac{\tilde{N}_c+2k}{(k+1)N_f}$ |
| $Y$ | asym             | 1             | 1             | $\frac{N_c-N_f}{N_c}$ | $\frac{2}{N_c}$ | $\frac{1}{k+1}$   |
| $\tilde{Y}$ | asym             | 1             | 1             | $-\frac{N_c-N_f}{N_c}$ | $-\frac{2}{N_c}$ | $\frac{1}{k+1}$   |
| $M_j$ | 1                | $N_f$         | $N_f$         | 0        | 0        | $\frac{N_c-N_c+(2j+1)N_f}{N_f(k+1)}$ |
| $P_r$ | 1                | asym          | 1             | $-1$     | 0        | $\frac{N_c-N_c+2N_f(r+1)}{N_f(k+1)}$ |
| $\tilde{P}_r$ | 1                | 1             | asym          | 1        | 0        | $\frac{N_c-N_c+2N_f(r+1)}{N_f(k+1)}$ |
These are anomaly free in the dual gauge theory (which is our first check on the duality).

Taking the gauge singlets of the dual theory to transform as the mesons of the electric theory under the discrete symmetry (3.1), the dual superpotential respects the symmetry provided that the other fields transform as

\[ Y, \tilde{Y} \rightarrow C \alpha^{N_f} Y, \tilde{Y} \]

\[ q, \tilde{q} \rightarrow C N_f \alpha^{-2} (\tilde{N}_c - 2) q, \tilde{q}, \]

where \( C \) is charge conjugation. The transformation (3.2) is indeed non-anomalous in the dual gauge theory: it is not violated by instantons in the dual gauge group. Note the mixing of the discrete symmetry with charge conjugation and with the \( \mathbb{Z}_{N_f} \) centers of the \( SU(N_f)_L \times SU(N_f)_R \) flavor symmetries.

At \( \langle X \rangle = \langle \tilde{X} \rangle = \langle Q \rangle = \langle \tilde{Q} \rangle = 0 \) the full flavor symmetry is unbroken and the ’t Hooft anomalies computed with the fermions of the electric theory must match those computed with the fermions of the magnetic theory. These conditions are indeed satisfied; in both the electric and magnetic theories, we find:

\[
\begin{align*}
U(1)_{R} & \quad - \frac{N_c(N_c + 3k)}{k + 1} - 1 \\
U(1)_{R}^3 & \quad N_c^2 - 1 - \frac{2N_c(N_c + 2k)^3}{N_f^2(k + 1)^3} - N_c(N_c - 1) \frac{k^3}{(k + 1)^3} \\
SU(N_f)^3 & \quad N_c d_3(N_f) \\
SU(N_f)^2 U(1)_{R} & \quad - \frac{N_c(N_c + 2k)}{(k + 1) N_f} d_2(N_f) \\
SU(N_f)^2 U(1)_{X} & \quad 0 \\
SU(N_f)^2 U(1)_{B} & \quad d_2(N_f) \\
U(1)_{X} U(1)_{R} & \quad - N_c(N_c - 1) \frac{k}{k + 1} \\
U(1)_{B} U(1)_{R} & \quad - 2 \frac{(2k + 1)}{k + 1} \\
U(1)_{X} U(1)_{B} U(1)_{R} & \quad - 2(N_c - 1) \frac{k}{k + 1},
\end{align*}
\]

where \( d_2(N_f) \) and \( d_3(N_f) \) are the quadratic and cubic \( SU(N_f) \) Casimirs of the fundamental representation.

The electric theory has a variety of baryon-like operators, with gauge indices contracted with an \( \epsilon \)-tensor. Under the duality transformation, these operators are mapped...
in a non-trivial manner to baryon-like operators of the magnetic gauge theory. An example of this mapping is

\[ X^n Q^{N_c-2n} \rightarrow Y^{k(N_f-2)-n} q^{N_f+2n-N_c}, \]  

(3.4)

where we have suppressed the flavor indices. Note that this map is consistent with all of the global symmetries, including the discrete symmetries mentioned above (the charge conjugation \(C\) in (3.2) is needed for the map to respect the discrete symmetry). We will see in the next section that deforming by such an operator in the electric theory leads to the correct physics in the dual theory.

### 3.2. Deformations: superpotential

Consider deforming (2.11) to include lower order terms:

\[ W_{pert} = \sum_{\ell=0}^{k} \lambda_{\ell} \text{Tr} \left( X \tilde{X} \right)^{\ell+1}, \]  

(3.5)

The theory has multiple vacua with \(\langle Q \rangle = 0\) and \(\langle X \tilde{X} \rangle \neq 0\) satisfying \(\partial W / \partial X = \partial W / \partial \tilde{X} = 0\). For \(N_c = 2n_c + 1\) the D-flat directions with \(\langle Q \rangle = 0\) are

\[ \langle X \rangle = \langle \tilde{X} \rangle = \begin{pmatrix} x_1 \sigma_2 \\ x_n \sigma_2 \end{pmatrix}, \]  

(3.6)

generically breaking \(SU(2n_c + 1)\) to \(Sp(1)^{n_c}\). For \(N_c = 2n_c\) the D-flat directions with \(\langle Q \rangle = 0\) are

\[ \langle X \rangle = \begin{pmatrix} x_1 \sigma_2 \\ x_n \sigma_2 \end{pmatrix}, \quad \langle \tilde{X} \rangle = \begin{pmatrix} \tilde{x}_1 \sigma_2 \\ \tilde{x}_n \sigma_2 \end{pmatrix} \quad \text{with} \quad |x_i|^2 - |\tilde{x}_i|^2 = \text{const.}, \]  

(3.7)

again generically breaking \(SU(2n_c)\) to \(Sp(1)^{n_c}\). The theory with superpotential (3.5) has multiple vacua with expectation values of the form (3.6) or (3.7) with \(x_i\) and \(\tilde{x}_i\) satisfying \(W'(x_i \tilde{x}_i) x_i = W'(x_i \tilde{x}_i) \tilde{x}_i = 0\). The equation \(W'(z) = 0\) has \(k\) solutions \(z_\ell\) which are generically distinct. In addition, there is the solution \(x_i = \tilde{x}_i = 0\). Let \(p_0\) be the number of eigenvalues with \(x_i = \tilde{x}_i = 0\) and \(p_\ell\) be the number with \(x_i \tilde{x}_i\) equal to \(z_\ell\), with \(\sum_{\ell=0}^{k} p_\ell = n_c\) for \(N_c = 2n_c + \eta\) with \(\eta \equiv 0 \ (1)\) for \(N_c\) even (odd). In such a vacuum the gauge group is broken by \(\langle X \tilde{X} \rangle\) as:

\[ SU(N_c) \rightarrow SU(2p_0 + \eta) \times Sp(p_1) \times Sp(p_2) \times \ldots \times Sp(p_k). \]  

(3.8)
In the generic situation $X$ and $\tilde{X}$ are massive in each vacuum and can be integrated out. The $SU(2p_0 + \eta)$ gauge group has $N_f$ flavors in the fundamental and each $Sp(p_1)$ factor has $2N_f$ matter fields in the $2p_1$-dimensional fundamental representation. There is a stable vacuum (in this theory as well as that with superpotential (2.14)) provided $N_f \geq 2p_0 + \eta$ \[\text{(2.14)}\] and $N_f > p_\ell$, $\ell > 0$ \[\text{[14]}\].

An interesting example of the deformation (3.5) is a mass term $mX\tilde{X}$. In any one vacuum, the unbroken symmetry will be of the form (3.8). In the magnetic theory the analysis is much the same and the gauge group is broken to

$$SU(\tilde{N}_c) \to SU(2\tilde{p}_0 + \tilde{\eta}) \times Sp(\tilde{p}_1) \times \ldots \times Sp(\tilde{p}_k).$$

The fields $Y$ and $\tilde{Y}$ are massive in each factor and can be integrated out. The duality mapping is $2\tilde{p}_0 + \tilde{\eta} = N_f - (2p_0 + \eta)$, $\tilde{p}_\ell = N_f - 2 - p_\ell$ for $\ell > 0$; in the $SU$ factor the duality is that of [3] and in each $Sp$ factor it is that of [3, 14].

Next we consider deforming the theory by giving a mass to one flavor of the electric quarks $Q$. The low energy theory is an $Sp(N_c)$ theory with matter $X$ and superpotential (2.11) with one fewer flavor, $\tilde{N}_f = N_f - 1$; having fewer flavors, it is more strongly coupled in the infrared. In the dual theory this perturbation corresponds to adding a term $m(M_0)^{N_f,N_f}$ to (2.12). The $M_j$ equations of motion imply that the vacua of this theory satisfy

$$q_{N_f}\tilde{Y}Y^{\ell-1}q_{N_f} = -m\delta_{\ell,k+1}; \quad \ell = 1, \ldots, k + 1$$

which, along with some additional conditions, give expectation values proportional to:

$$\langle q_{N_f} \rangle_c = \delta_{c,1};$$

$$\langle \tilde{q}_{N_f} \rangle_c = \delta_{c,2k+1};$$

$$Y_{c,d} = \tilde{Y}^{c+1,d+1} = \begin{cases} 
\delta_{c+1,d} & c = 2r - 1; \quad r = 1, \ldots, k \\
-d_{d+1,c} & d = 2r - 1; \quad r = 1, \ldots, k \\
0 & \text{otherwise.}
\end{cases}$$

(3.11)

These expectation values break the magnetic $SU((2k + 1)N_f - 4k - N_c)$ gauge group to $SU((2k + 1)(N_f - 1) - 4k - N_c)$ with $N_f - 1$ remaining light flavors. The low energy magnetic theory is at weaker coupling and is the dual of the low energy electric theory.

We can also consider perturbing the electric theory by the other meson operators. The corresponding perturbation in the dual theory yields a low energy theory which is dual to the perturbed electric theory.
3.3. Deformations: flat directions

The electric theory can also be deformed by giving fields expectation values along flat directions. Such a perturbation generally breaks the gauge group and takes the theory to weaker coupling in the infrared. The corresponding perturbation in the dual theory should lead to an infrared theory which is dual to the infrared electric theory. We will only mention an especially interesting flat direction: that along which the baryon operator $B_n = X^n Q^{N_c-2n}$ gets an expectation value. In terms of the elementary fields, this flat direction is

$$\langle Q_{f,c} \rangle = \sqrt{2} a \delta_{f,c}$$

for $c \leq N_c - 2n$ and zero otherwise, $\langle X^{cd} \rangle = a(\delta_{c,d+1} - \delta_{d,c+1})$ for $c, d > N_c - 2n$ and zero otherwise, with all other expectation values zero. Along this flat direction, which is not lifted by the superpotential (2.11), the gauge group is broken to $Sp(\tilde{n})$. In the low energy theory the matter field $\tilde{X}$ yields an antisymmetric $Sp(\tilde{n})$ tensor $\tilde{\chi}$ and $N_c - 2n$ fields in the $2n$-dimensional fundamental representation of $Sp(\tilde{n})$. The $\tilde{Q}$ and the $N_f - N_c + 2n$ flavors of $Q$ not entering in $B_n$ yield additional matter fields in the fundamental representation of $Sp(n)$, bringing the total number to $2N_f$. In addition, there are a variety of singlets. The low energy theory has a superpotential inherited from (2.11), $W = \text{Tr} \tilde{\chi}^{k+1}$, where indices are contracted using $\langle X \rangle$. This low energy theory is that considered in [14], along with some additional singlets.

In the dual theory the operator $B_n$ labeling the above flat direction is mapped as $b_n = Y^{k(N_f-2)-n} q^{N_f+2n-N_c}$, which gets an expectation value along the flat direction $\langle q_{f,c} \rangle = \sqrt{2} a \delta_{f,c}$ for $c > N_c - 2n$ and zero otherwise, $\langle Y_{cd} \rangle = a(\delta_{c,d+1} - \delta_{d,c+1})$ for $c, d \leq k(N_f - 2) - n$ and zero otherwise. The dual $SU(\tilde{N}_c)$ gauge group is broken to $Sp(\tilde{n})$, with $\tilde{n} = k(N_f - 2) - n$. The low energy $Sp(\tilde{n})$ theory has $2N_f$ matter fields in the fundamental representation, with $N_f - N_c + 2n$ coming from $\tilde{Y}$ and the remainder coming from $\tilde{q}$ and the $N_c - 2n$ fields $q$ not entering in $b_n$. In addition, there is an antisymmetric tensor coming from $\tilde{Y}$, along with some singlets, some of which are eliminated by (2.12). The low energy magnetic theory gets a superpotential from (2.12), such that it is precisely that shown in [14] to be the dual of the above low energy electric theory. The $Sp(n)$ duality discussed in [14] is thus inherited from the $SU(N_c)$ model discussed here.

4. $SU(N_c)$ with a symmetric flavor and $N_f$ fundamental flavors

We now consider the theory described in sect. 2.7. It is very similar to the model of the previous section. The theory has an anomaly-free $SU(N_f)_L \times SU(N_f)_R \times U(1)_X \times U(1)_B \times U(1)_R$ global symmetry with matter in the representations
In addition the theory (2.14) has a discrete $\mathbb{Z}_2$ symmetry generated by

\[
\begin{align*}
X, & \quad \bar{X} \to \alpha X, \quad \bar{X} \\
Q, & \quad \bar{Q} \to \alpha^{-N_c+2} Q, \quad \bar{Q},
\end{align*}
\]

where \( \alpha = e^{2\pi i/2N_f(k+1)} \).

4.1. Duality

The dual theory is $SU(\tilde{N}_c)$, where \( \tilde{N}_c = (2k+1)N_f + 4k - N_c \), as described in sect. 2.7. Taking the singlets $M_j$, $P_r$, and $\tilde{P}_r$ to transform as in the electric theory, the dual theory (2.27) has the $SU(N_f)_L \times SU(N_f)_R \times U(1)_X \times U(1)_B \times U(1)_R$ global flavor symmetry with the matter transforming as:

|        | $SU(\tilde{N}_c)$ | $SU(N_f)_L$ | $SU(N_f)_R$ | $U(1)_X$ | $U(1)_B$ | $U(1)_R$ |
|--------|-------------------|-------------|-------------|----------|----------|----------|
| $q$    | $\tilde{N}_c$     | $\overline{N}_f$ | 1           | $\frac{k(N_f+2)}{N_c}$ | $\frac{1}{N_c}$ | $1 - \frac{\tilde{N}_c-2k}{(k+1)N_f}$ |
| $\tilde{q}$ | $\overline{N}_c$ | 1           | $\overline{N}_f$ | $-\frac{k(N_f+2)}{N_c}$ | $-\frac{1}{N_c}$ | $1 - \frac{\tilde{N}_c-2k}{(k+1)N_f}$ |
| $Y$    | sym               | 1           | 1           | $\frac{N_c-N_f}{N_c}$ | $\frac{2}{N_c}$ | $\frac{1}{k+1}$ |
| $\bar{Y}$ | sym               | 1           | 1           | $-\frac{N_c-N_f}{N_c}$ | $-\frac{2}{N_c}$ | $\frac{1}{k+1}$ |
| $M_j$  | 1                 | $N_f$       | $N_f$       | 0         | 0         | $\frac{N_c-N_e+(2j+1)N_f}{N_f(k+1)}$ |
| $P_r$  | 1                 | sym         | 1           | $-1$      | 0         | $\frac{N_c-N_e+2N_f(r+1)}{N_f(k+1)}$ |
| $\tilde{P}_r$ | 1             | 1           | sym         | 1         | 0         | $\frac{N_c-N_e+2N_f(r+1)}{N_f(k+1)}$ |

It is a check on the duality that these are anomaly free in the dual gauge theory. Similarly, the discrete symmetry (4.1) is a symmetry of the dual theory with the singlets transforming as the mesons of the electric theory provided the other fields transform as

\[
\begin{align*}
Y, & \quad \bar{Y} \to \alpha^N Y, \quad \bar{Y} \\
q, & \quad \bar{q} \to e^{2\pi i(2/N_f)} \alpha^{-(\tilde{N}_e+2)} q, \quad \bar{q},
\end{align*}
\]
which is indeed a non-anomalous discrete symmetry of the dual gauge theory. (As in (3.2) and [3,5], a factor of charge conjugation may also be needed in (4.2) in order for the baryons to transform properly under duality.)

We have verified that the ’t Hooft anomaly matching conditions are satisfied, providing a highly non-trivial check on the duality.

The electric theory has a variety of baryon-like operators, with gauge indices contracted with $\epsilon$-tensors. Under the duality transformation, these operators are mapped in a non-trivial manner to baryon-like operators of the magnetic gauge theory. An example of this mapping is

$$X^n Q^{N_c-n} Q^{N_c-n} \rightarrow Y^k (2N_f+4) - n q^{N_f+n-N_c} q^{N_f+n-N_c}, \quad (4.3)$$

contracted with two $\epsilon$-tensors, where we have suppressed all indices. Note that this map is consistent with all of the global symmetries, including the discrete symmetries mentioned above. (Because it involves two epsilon tensors it is insensitive to the possible $C$ in (4.2)). We will see in the next section that deforming by such an operator in the electric theory leads to the correct physics in the dual theory.

### 4.2. Deformations: superpotential

Consider deforming (2.14) to include lower order terms:

$$W_{pert} = \sum_{\ell=0}^{k} \lambda_\ell \text{Tr} (X X)^{\ell+1}, \quad (4.4)$$

The theory has multiple vacua with $\langle Q \rangle = 0$ and $\langle X X \rangle \neq 0$ satisfying $\partial W/\partial X = \partial W/\partial X = 0$. The D-flat directions with $\langle Q \rangle = 0$ are

$$\langle X \rangle = \begin{pmatrix} x_1 \\ x_n \end{pmatrix}, \quad \langle \tilde{X} \rangle = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_n \end{pmatrix} \quad \text{with} \quad |x_i|^2 - |\tilde{x}_i|^2 = \text{const.}, \quad (4.5)$$

generically completely breaking $SU(N_c)$. The theory with superpotential (4.4) has multiple vacua with expectation values of the form (4.3) with $x_i$ and $\tilde{x}_i$ satisfying $W'(x_i, \tilde{x}_i) x_i = W'(x_i, \tilde{x}_i) \tilde{x}_i = 0$. The equation $W'(z) = 0$ has $k$ solutions $z_\ell$ which are generically distinct. There is also the solution $x_i = \tilde{x}_i = 0$. Let $p_0$ be the number of eigenvalues with $x_i = \tilde{x}_i = 0$.
and let \( p_\ell \) be the number of \( x_i \tilde{x}_i \) equal to \( z_\ell \), with \( \sum_{\ell=0}^{k} p_\ell = N_c \). In such a vacuum the gauge group is broken by \( \langle X \, \tilde{X} \rangle \) as:

\[
SU(N_c) \rightarrow SU(p_0) \times SO(p_1) \times SO(p_2) \times \cdots \times SO(p_k). \tag{4.6}
\]

In the generic situation \( X \) and \( \tilde{X} \) are massive in each vacuum and can be integrated out. The \( SU(p_0) \) factor has \( N_f \) fundamental flavors and each \( SO(p_\ell) \) factor has \( 2N_f \) matter fields in the \( p_\ell \)-dimensional vector representation. The vacuum is stable provided \( N_f \geq p_0 \) [20] and \( 2N_f \geq p_\ell - 4 \) for every \( \ell = 1, \ldots, k \) [5]..

An interesting example of the deformation (4.4) is a mass term \( mX \tilde{X} \). In any one vacuum, the unbroken symmetry will be of the form (4.6). In the magnetic theory the analysis is much the same and the gauge group is broken to

\[
SU(\tilde{N}_c) \rightarrow SU(\tilde{p}_0) \times SO(\tilde{p}_1) \times SO(\tilde{p}_2) \times \cdots \times SO(\tilde{p}_k). \tag{4.7}
\]

The fields \( Y \) and \( \tilde{Y} \) are massive in each factor and can be integrated out. The duality mapping is \( \tilde{p}_0 = N_f - p_0, \tilde{p}_\ell = 2N_f + 4 - p_\ell \) for \( \ell > 0 \); in the \( SU \) factor the duality is that of [3] and in each \( SO \) factor it is that of [3,5]..

Next consider deforming the electric theory by adding a mass for the \( N_f \)-th quark flavor. In the magnetic theory the term \( m(M_0)^{N_f, N_f} \) is added to (2.15). The vacuum has

\[
q_{N_f}(\tilde{Y}Y)^{j-1}q_{N_f} = -m\delta_{j,k+1}; \quad j = 1, \ldots, k + 1
\]

\[
q_{N_f}(\tilde{Y}Y)^{r-1}\tilde{Y}q_{N_f} = 0
\]

\[
\tilde{q}_{N_f}Y(\tilde{Y}Y)^{r-1}\tilde{q}_{N_f} = 0,
\]

which give expectation values \( \langle q_{N_f} \rangle, \langle \tilde{q}_{N_f} \rangle, \langle Y \rangle \) and \( \langle \tilde{Y} \rangle \), breaking \( SU((2k+1)N_f+4k-N_c) \) to \( SU((2k+1)(N_f-1) + 4k - N_c) \) with \( N_f - 1 \) flavors. The low energy magnetic theory is the dual of the low energy electric theory.

4.3. Deformations: flat directions

Consider the flat direction along which the operator \( B_n = X^nQ^{N_c-n}Q^{N_c-n} \) gets an expectation value. In terms of the elementary fields this flat direction is \( \langle Q^{f,c} \rangle = \sqrt{2}a\delta^{f,c} \) for \( c \leq N_c - n \) and zero otherwise, \( \langle X^{c,d} \rangle = a\delta^{c,d} \) for \( c > N_c - n \) and zero otherwise, with all other expectation values zero. Along this flat direction, which is not lifted by the superpotential (2.14), the gauge group is broken to \( SO(n) \). In the low energy theory
there are $2N_f$ matter fields in the $n$ dimensional vector representation of $SO(n)$, with some coming from $\tilde{X}$ and others from the $Q$ and $\tilde{Q}$ not entering in $B_n$. There is also a symmetric $SO(n)$ tensor $\hat{X}$ from $\tilde{X}$, along with a variety of singlets. The low energy theory has a superpotential inherited from (2.14), $W = \text{Tr} \hat{X}^{k+1}$, where indices are contracted using $\langle X \rangle$. Up to some additional singlets, this low energy theory is that considered in [14].

In the dual theory the operator $B_n$ labeling the above flat direction is mapped as in (4.3) to $b_n = Y^{k(2N_f+4)-n}q^{N_f+n-N_c}q^{N_f+n-N_c}$, which gets an expectation value along the flat direction $\langle q_{f,c} \rangle = \sqrt{2}a\delta_{f,c}$ for $c > N_c - n$ and zero otherwise, $\langle Y_{cd} \rangle = a\delta_{c,d}$ for $c \leq k(2N_f+4) - n$ and zero otherwise. When this operator gets an expectation value the dual $SU(\tilde{N}_c)$ gauge theory is therefore broken to $SO(\tilde{n})$, with $\tilde{n} = k(2N_f+4) - n$. This low energy $SO(\tilde{n})$ theory has $2N_f$ matter fields in the $\tilde{n}$-dimensional vector representation, with some coming from $\tilde{Y}$ and others from $\tilde{q}$ and the $q$ not entering in $b_n$. There is also a symmetric tensor coming from $\tilde{Y}$, and some singlets, some of which are eliminated by (2.15). The low energy magnetic theory gets a superpotential from (2.15), such that it is precisely that shown in [14] to be the dual of the above low energy $SO(n)$ electric theory. The $SO(n)$ duality considered in [14] is thus inherited from the $SU(N_c)$ model discussed here.

5. $SU(N_c)$ with an antisymmetric tensor and a symmetric tensor

We now consider the theory described in sect. 2.8. This is a chiral theory; anomaly cancellation requires $m_f - \tilde{m}_f = 8$. The theory has an anomaly-free $SU(m_f) \times SU(\tilde{m}_f) \times U(1)_X \times U(1)_B \times U(1)_R$ global symmetry with matter in the representations

|          | $SU(N_c)$ | $SU(m_f)$ | $SU(\tilde{m}_f)$ | $U(1)_X$   | $U(1)_B$   | $U(1)_R$   |
|----------|-----------|-----------|-------------------|------------|------------|------------|
| $Q$      | $N_c$     | $m_f$     | 1                 | $-(2k+1)$  | $\frac{1}{N_c}$| $1 - \frac{N_c+2(4k+3)}{2(k+1)m_f}$|
| $\tilde{Q}$ | $\overline{N}_c$ | 1          | $\tilde{m}_f$    | $2k+1 + \frac{2(4k+3)}{m_f}$ | $-\frac{1}{N_c}$ | $1 - \frac{N_c-2(4k+3)}{2(k+1)m_f}$ |
| $X$      | asym      | 1         | 1                 | $2$        | $\frac{1}{N_c}$| $\frac{1}{2(k+1)}$|
| $\tilde{X}$ | sym      | 1         | 1                 | $-1$       | $-\frac{2}{N_c}$| $\frac{1}{2(k+1)}$|

The dimensions of chiral operators at an interacting fixed point are determined by the $R$ charge appearing in the same multiplet with the energy momentum tensor; however, in this case $U(1)_R$ may differ from $U(1)_R$ in the above table by a linear combination of $U(1)_X$ and $U(1)_B$. The charge $U(1)_R$ thus determines dimensions only of operators neutral under


$U(1)_X$ and $U(1)_B$ (for example, terms in the superpotential.) These statements will be true in all our chiral models.

In addition the theory (2.17) has a discrete $\mathbb{Z}_{2(m_f + \tilde{m}_f)(k+1)}$ symmetry generated by

$$X, \tilde{X} \rightarrow \alpha^{(m_f + \tilde{m}_f)/2} X, \tilde{X}$$

$$Q, \tilde{Q} \rightarrow \alpha^{-N_c} Q, \tilde{Q},$$

where $\alpha = e^{2\pi i/2(m_f + \tilde{m}_f)(k+1)}$.

5.1. Duality

The dual theory is $SU(\tilde{N}_c)$, where $\tilde{N}_c = \frac{1}{2}(4k + 3)(m_f + \tilde{m}_f) - N_c$, as described in sect. 2.8. Taking the singlets $M_j, P_r, \tilde{P}_r$ to transform as in the electric theory, the dual theory (2.18) has the $SU(m_f) \times SU(\tilde{m}_f) \times U(1)_X \times U(1)_B \times U(1)_R$ global flavor symmetry with the other matter transforming as:

| Field | $SU(\tilde{N}_c)$ | $SU(m_f)$ | $SU(\tilde{m}_f)$ | $U(1)_X$ | $U(1)_B$ | $U(1)_R$ |
|-------|------------------|------------|------------------|----------|----------|----------|
| $q$   | $\tilde{N}_c$    | $m_f$      | 1                | $2k + 1 - \frac{2(4k+3)}{m_f}$ | $\frac{1}{N_c}$ | $1 - \frac{N_c+2(4k+3)}{2(k+1)m_f}$ |
| $\tilde{q}$ | $\tilde{N}_c$ | 1          | $\tilde{m_f}$    | $-(2k + 1) - \frac{2(4k+3)}{m_f}$ | $-\frac{1}{N_c}$ | $1 - \frac{N_c-2(4k+3)}{2(k+1)m_f}$ |
| $Y$   | asym             | 1          | 1                | $-1$     | $\frac{2}{N_c}$ | $\frac{1}{2(k+1)}$ |
| $\tilde{Y}$ | sym          | 1          | 1                | 1        | $-\frac{2}{N_c}$ | $\frac{1}{2(k+1)}$ |

It is a check on the duality that these are anomaly free in the dual gauge theory.

Similarly, the discrete symmetry (5.1) is a symmetry of the dual theory, with the singlets transforming as the mesons of the electric theory, provided the other fields transform as

$$Y \rightarrow e^{2\pi i B_Y p} \alpha^{(m_f + \tilde{m}_f)/2} Y,$$

$$\tilde{Y} \rightarrow e^{-2\pi i B_Y p} \alpha^{(m_f + \tilde{m}_f)/2} \tilde{Y},$$

$$q \rightarrow e^{2\pi i B_q p} \alpha^{-N_c} q,$$

$$\tilde{q} \rightarrow e^{-2\pi i B_q p} \alpha^{-\tilde{N}_c} \tilde{q}.$$

(5.2)

Here, $p = \frac{1}{4} \left[ m_f + \tilde{m}_f + 2 \frac{m_f + \tilde{m}_f + 4N_c}{(k+1)(m_f + \tilde{m}_f)} \right]$, and $B_Y, B_q$ are the $U(1)_B$ charges of $Y$ and $q$. This is indeed a non-anomalous discrete symmetry of the dual gauge theory.

We have verified that the ’t Hooft anomaly matching conditions are satisfied, providing a highly non-trivial check on the duality.
The electric theory has a variety of baryon-like operators, with gauge indices contracted with $\epsilon$-tensors. Under the duality transformation, these operators are mapped in a non-trivial manner to baryon-like operators of the magnetic gauge theory. Two examples of this mapping are

$$X^n Q_{Nc-2n} \to Y^{(k+1/2)(m_f+\tilde{m}_f)-n-2} q_{m_f+2n-Nc}^{m_f+2n-Nc},$$

$$\tilde{X}^n Q_{Nc-n} \tilde{Q}_{Nc-n} \to \tilde{Y}^{(2k+1)(m_f+\tilde{m}_f)-n+4} \tilde{q}_{m_f+n-Nc}^{m_f+n-Nc},$$

contracted with $\epsilon$-tensors, where we have suppressed all indices. Note that this map is consistent with all of the global symmetries, including the discrete symmetries mentioned above. We will see in the next section that deforming by such an operator in the electric theory leads to the correct physics in the dual theory.

5.2. Deformations: superpotential

Consider deforming (2.17) to include lower order terms:

$$W_{pert} = \sum_{\ell=0}^{k} \lambda_{\ell} \text{Tr} (X \tilde{X})^{2(\ell+1)},$$

The theory has multiple vacua with $\langle Q \rangle = 0$ and $\langle X \tilde{X} \rangle \neq 0$ satisfying $\partial W / \partial X = \partial W / \partial \tilde{X} = 0$. The D-flat directions with $\langle Q \rangle = 0$ require both $X$ and $\tilde{X}$ to be $2 \times 2$ block diagonal

$$\langle X \rangle = \begin{pmatrix} x_1 \sigma_2 \\ x_n \sigma_2 \end{pmatrix}, \quad \langle \tilde{X} \rangle = \begin{pmatrix} \tilde{x}_1 1 \\ \tilde{x}_n 1 \end{pmatrix} \quad \text{with} \quad |x_i|^2 - |\tilde{x}_i|^2 = \text{const.},$$

generically breaking $SU(N_c)$ to $U(1)^{N_c}$.

If $N_c$ is odd then there is a zero in the lower right corner of $X$ and $\tilde{X}$. The theory with superpotential (5.4) has multiple vacua with expectation values of the form (5.5) with $x_i$ and $\tilde{x}_i$ satisfying $W'(x_i \tilde{x}_i) x_i = W'(x_i \tilde{x}_i) \tilde{x}_i = 0$. The equation $W'(z) = 0$ has $k$ solutions $z_\ell$ which are generically distinct. There is also the solution $x_i = \tilde{x}_i = 0$. Let $p_0$ be the number of eigenvalues $x_i = \tilde{x}_i = 0$ and let $p_\ell$ be the number of $x_i \tilde{x}_i$ equal to $z_\ell$; then $2(p_0 + \sum_{\ell=0}^{k} p_\ell) + \eta = N_c$, where $\eta = 0$ for even $N_c$ and $\eta = 1$ for odd $N_c$. In such a vacuum the gauge group is broken by $\langle X \tilde{X} \rangle$ to

$$SU(N_c) \to SU(2p_0 + \eta) \times U(p_1) \times U(p_2) \times \cdots \times U(p_k).$$

(5.6)
The first group factor is a model of the type studied in this section, with \( k = 0 \); it has both a symmetric and an antisymmetric tensor and \( m_f(\tilde{m}_f) \) fields in the (anti)fundamental representation. In the remaining factors \( X \) and \( \tilde{X} \) are generically massive and can be integrated out. Each \( U(p) \) factor \((\ell > 0)\) is a defining model with \( m_f + \tilde{m}_f \) flavors of fields in the fundamental and antifundamental representation.

In the magnetic theory the analysis is much the same and the gauge group is broken to

\[
SU(\tilde{N}_c) \to SU(2\tilde{p}_0 + \eta) \times U(\tilde{p}_1) \times U(\tilde{p}_2) \times \ldots \times U(\tilde{p}_k). \tag{5.7}
\]

The duality mapping is \( 2\tilde{p}_0 + \eta = \frac{3}{2}(m_f + \tilde{m}_f) - (2p_0 + \eta), \tilde{p}_\ell = m_f + \tilde{m}_f - p_\ell \) for \( \ell > 0 \); in the first factor the duality is that of this section and in the other factors it is that of [3].

Next consider deforming the electric theory by adding a mass term \( W_{tree} = mQ^{m_f, \tilde{m}_f} \). In the magnetic theory the term \( m(M_0)^{m_f, \tilde{m}_f} \) is added to (2.13). The vacuum has

\[
q_{m_f}(\tilde{Y}Y)^{\ell}\tilde{q}_{\tilde{m}_f} = -m_\delta_{j,2k+1}
q_{m_f}(\tilde{Y}Y)^{\ell}\tilde{q}_{\tilde{m}_f} = 0
\tilde{q}_{\tilde{m}_f}Y(\tilde{Y}Y)^{\ell}\tilde{q}_{\tilde{m}_f} = 0,
\tag{5.8}
\]

which give expectation values \( \langle q_{m_f} \rangle, \langle \tilde{q}_{\tilde{m}_f} \rangle, \langle Y \rangle \) and \( \langle \tilde{Y} \rangle \), breaking \( SU(\frac{1}{2}(4k + 3)(m_f + \tilde{m}_f) - N_c) \) to \( SU(\frac{1}{2}(4k + 3)(m_f - 1 + \tilde{m}_f - 1) - N_c) \) with \( m_f - 1 \) and \( \tilde{m}_f - 1 \) fields. The low energy magnetic theory is the dual of the low energy electric theory.

5.3. Deformations: flat directions

The flat directions in this model are very similar to those of the previous two models. We therefore limit our discussion to a general sketch.

Consider the flat direction along which the baryon operator \( B_n = X^nQ^{N_c - 2n} \) gets an expectation value. Along this flat direction the gauge group is broken to \( Sp(n) \). The low energy theory has a symmetric (adjoint) \( Sp(n) \) tensor \( \tilde{X} \), \( 2N_f \) fields in the \( 2n \)-dimensional fundamental representation of \( Sp(n) \), some additional singlets, and the superpotential \( W = Tr \tilde{X}^{2(k+1)} \). In the dual theory the operator \( B_n \) is mapped as in (3.4) to \( Y^{(k+1/2)(m_f + \tilde{m}_f) - n - 2}q^{m_f + 2n - N_c} \), whose expectation value breaks the dual \( SU(\tilde{N}_c) \) gauge group to \( Sp(\tilde{n}) \), with \( \tilde{n} = (2k + 1)N_f - n - 2 \). The low energy \( Sp(\tilde{n}) \) theory has \( 2N_f \) matter fields in the fundamental representation, a symmetric tensor coming from \( Y \), and some singlets. This theory gets a superpotential from (2.18), such that it is precisely that shown in [15] to be the dual of the above low energy electric theory (see section 2.4).
There is another flat direction along which the baryon operator \( \tilde{B}_n = \tilde{X}^n \tilde{Q}^{N_c-n} \tilde{Q}^{N_c-n} \)
gets an expectation value. Along this flat direction the gauge group is broken to \( SO(n) \). The low energy theory has an antisymmetric (adjoint) \( SO(n) \) tensor \( \tilde{X} \), \( N_f \) fields in the \( n \)-dimensional vector representation of \( SO(n) \), some additional singlet fields, and the superpotential \( W = \text{Tr} \tilde{X}^{2(k+1)} \). In the dual theory the operator \( \tilde{B}_n \) is mapped as in (3.4) to \( \tilde{Y}^{(2k+1)(m_f+n_f)-n+4} \tilde{q}^{m_f+n-c} \tilde{q}^{m_f+n-c} \), whose expectation value breaks the dual \( SU(\tilde{N}_c) \) gauge group to \( SO(\tilde{n}) \), with \( \tilde{n} = (2k+1)N_f-n+4 \). The low energy \( SO(\tilde{n}) \) theory has \( N_f \) matter fields in the vector representation, an antisymmetric tensor coming from \( Y \), and some singlets. This theory gets a superpotential from (2.18), such that it is precisely that shown in [15] to be the dual of the above low energy electric theory (see section 2.5.). Thus, the \( Sp(n) \) and \( SO(n) \) duality considered in [15] is inherited from the \( SU(N_c) \) model discussed here.

6. \( Sp(N_c) \times Sp(N'_c) \)

We consider the theory discussed in sect. 2.9. The electric theory (2.20) has an \( SU(2N_f) \times SU(2N'_f) \times U(1)_R \) flavor symmetry with matter transforming as:

|       | \( Sp(N_c) \) | \( Sp(N'_c) \) | \( SU(2N_f) \) | \( SU(2N'_f) \) | \( U(1)_R \) |
|-------|---------------|---------------|----------------|----------------|----------------|
| \( Q \) | \( 2N_c \)    | 1             | \( 2N_f \)    | 1              | \( R_Q = 1 - \frac{(N_c+1)(k+1)-kN'_f}{N_f(k+1)} \) |
| \( Q' \) | 1             | \( 2N'_c \)   | 1              | \( 2N'_f \)    | \( R'_Q = 1 - \frac{(N'_c+1)(k+1)-kN_f}{N'_f(k+1)} \) |
| \( X \) | \( 2N_c \)    | \( 2N'_c \)   | 1              | 1              | \( R_X = \frac{1}{k+1} \) |

In addition, the theory has a discrete \( \mathbb{Z}_{2N_fN'_f(k+1)} \) symmetry generated by

\[
X \rightarrow \alpha^{N_fN'_f} X, \quad Q \rightarrow \alpha^{-N_cN'_f} Q, \quad Q' \rightarrow \alpha^{-N_cN_f} Q',
\]

with \( \alpha = e^{2\pi i/2N_fN'_f(k+1)} \).

6.1. The dual model

The dual theory is an \( Sp(\tilde{N}_c) \times Sp(\tilde{N}'_c) \) gauge theory with \( \tilde{N}_c = (k + 1)(N_f + N'_f - 2) - N_f - N'_c \) and \( \tilde{N}'_c = (k + 1)(N_f + N'_f - 2) - N'_f - N_c \) as described in sect. 2.9. Taking the singlets \( P_r, M_j \) and \( M'_j \) to transform as in the electric theory, the magnetic theory (2.21) has a \( SU(2N_f) \times SU(2N'_f) \times U(1)_R \) flavor symmetry with matter fields transforming as:

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These are indeed anomaly free in the dual theory. Similarly, the discrete $\mathbb{Z}_{2N_fN'_f(k+1)}$ symmetry (6.1) of the electric theory is a symmetry of the dual theory with the singlets transforming as the associated mesons of the electric theory provided that the other fields transform as

$$Y \to \alpha N_f N'_f Y, \quad q \to e^{2\pi i(N_f-2)/2N'_f} \alpha^{-\tilde{N}_c N_f} q, \quad q' \to e^{2\pi i(N'_f-2)/2N_f} \alpha^{-\tilde{N}_c N'_f} q', \quad (6.2)$$

which is indeed non-anomalous in the dual gauge theory. There is a freedom to modify (6.2) to give an additional minus sign to both $q$ and $q'$; this additional transformation is a symmetry of the dual theory which is free to mix with the $\mathbb{Z}_{2N_fN'_f(k+1)}$ discrete symmetry.

As another check on the duality we verify that that 't Hooft anomaly matching conditions are satisfied. With both the electric and the magnetic fermions we find

\[
\begin{align*}
U(1)_R & \quad N_c(2N_c + 1) + N'_c(2N'_c + 1) - 4N_cN'_c \frac{k}{k+1} \\
& \quad - 4N_c \frac{(k+1)(N_c + 1) - kN'_c}{(k+1)} - 4N'_c \frac{(k+1)(N'_c + 1) - kN_c}{(k+1)} \\
U(1)_R^3 & \quad N_c(2N_c + 1) + N'_c(2N'_c + 1) - 4N_cN'_c \left[ \frac{k}{k+1} \right]^3 \\
& \quad - 4N_c \frac{(k+1)(N_c + 1) - kN'_c}{(k+1)}^3 - 4N'_c \frac{(k+1)(N'_c + 1) - kN_c}{(k+1)}^3 \\
SU(2N_f)^3 & \quad 2N_c d_3(2N_f) \\
SU(2N_f)^2U(1)_R & \quad - 2N_c \frac{(k+1)(N_c + 1) - kN'_c}{N_f(k+1)} d_2(2N_f). \quad (6.3)
\end{align*}
\]
6.2. Deformations: superpotential

Consider perturbing the superpotential as:

\[ W_{\text{pert}} = \sum_{n=0}^{k} \lambda_n \text{Tr} \ X^{2(n+1)}. \]  

(6.4)

In the D-flat vacua with \( \langle Q \rangle = \langle Q' \rangle = 0 \), we may write:

\[ \langle X \rangle = \begin{pmatrix} x_1 \sigma_2 \\ x_2 \sigma_2 \\
\vdots \\
x_{N'_c} \sigma_2 \end{pmatrix}, \]

(6.5)

where we have assumed \( N_c \geq N'_c \). The vanishing of the F-terms then give \( x_i W'(x_i^2) = 0 \) which in general has \( k \) distinct solutions for \( x \) plus the solution \( x = 0 \). Each eigenvalue \( x_i \) of \( \langle X \rangle \) may take any of these values. Thus the general form of the unbroken symmetry in a given vacuum labeled by integers \( \{p_0, p_1, \ldots, p_k\} \) is:

\[ \text{Sp}(N_c) \times \text{Sp}(N'_c) \rightarrow \text{Sp}(N_c - N'_c + p_0) \times \text{Sp}(p_0) \times \text{Sp}(p_1) \times \ldots \times \text{Sp}(p_k) \]

(6.6)

with \( \sum_{\ell=0}^{k} p_\ell = N'_c \). In the generic situation, \( X \) is massive in each vacuum and can be integrated out. The \( \text{Sp}(N_c - N'_c + p_0) \) factor has \( N_f \) fundamental flavors (2\( N_f \) fields); the \( \text{Sp}(p_0) \) factor has \( N'_f \) fundamental flavors; and each \( \text{Sp}(p_\ell) \) factor has \( N_f + N'_f \) flavors in the fundamental representation. The vacuum is stable provided \( N_f > N_c - N'_c + p_0 \), \( N'_f > p_0 \), and \( N_f + N'_f > p_\ell \) for every \( \ell = 1, \ldots, k \).

In particular, consider the special case where we add just a mass term for \( X \) to the superpotential (2.20). The electric gauge group is broken as described in eq. (6.4). In the dual theory, \( Y \) is also massive. For \( \tilde{N}_c \leq \tilde{N}_c' \), the magnetic gauge group, in a vacuum labeled by integers \( \{q_0, q_1, \ldots, q_k\} \), is broken to

\[ \text{Sp}(\tilde{N}_c) \times \text{Sp}(\tilde{N}_c') \rightarrow \text{Sp}(\tilde{N}_c - \tilde{N}_c' + q_0) \times \text{Sp}(q_0) \times \text{Sp}(q_1) \times \ldots \times \text{Sp}(q_k) \]

(6.7)

The first factor has \( N'_f \) flavors, the second \( N_f \) flavors and each of the remaining factors has \( N_f + N'_f \) flavors. The duality between (6.3) and (6.7) is \( q_0 = N_f - (N_c - N'_c + p_0) - 2 \), \( q_{\ell \geq 1} = N_f + N'_f - p_\ell - 2 \), the duality of [3][1]. The situation is similar for \( \tilde{N}_c > \tilde{N}_c' \).
Next, we consider perturbing the theory (2.20) by a mass term $W = m Q^{2N_f - 1} Q^{2N_f}$ for one flavor of the $Q$ fields. In the infrared limit of the electric theory the number of flavors $N_f$ will be reduced by one. In the magnetic theory, adding $(M_0)^{2N_f - 1,2N_f}$ to the superpotential, the vacua satisfy

$$q_{2N_f - 1} Y^{2j} q_{2N_f} = -m \delta_{j,k}$$

(6.8)

which, along with some other conditions, give expectation values proportional to

$$q_{2N_f - 1,c} = \delta_{c,1};$$

$$q_{2N_f,c} = \delta_{c,2(k+1)};$$

$$Y_{c,d'} = \begin{cases} \delta_{c,d'+1} & d' = 1 \ldots 2k; \\ 0 & \text{otherwise.} \end{cases}$$

(6.9)

These expectation values break $Sp(\tilde{N}_c) \times Sp(\tilde{N}_c')$ to $Sp(\tilde{N}_c - (k+1)) \times Sp(\tilde{N}_c - k)$, exactly corresponding to reducing $N_f$ by one. The low energy magnetic theory is the dual of the low energy electric theory.

6.3. “Confining” or “dualizing” one gauge group

Consider the limit where the gauge coupling for $Sp(N_c)$ on the electric side is much weaker than that of $Sp(N_c')$. Turning off $Sp(N_c)$ entirely leaves $Sp(N_c')$ with $N_f^{eff}(N_c') = N_c + N_f$ flavors. With $N_c'$ chosen so that $N_f^{eff} = N_c' + 2$, this theory confines to give a theory with mesons in the antisymmetric representation of the $SU(2N_f^{eff})$ flavor symmetry with a pfaffian superpotential [11]. Upon weakly gauging a $Sp(N_c)$ subgroup of the $SU(2N_f^{eff})$ flavor symmetry, the theory is $Sp(N_c)$ with an antisymmetric tensor $\hat{X} \sim X^2$, $2N_f'$ fundamentals $T \sim XQ'$, $2N_f$ fundamentals $Q$ as in the original theory, and singlets $U \sim Q'Q'$. In addition to the superpotential $W \sim Tr \hat{X}^{k+1}$ inherited from (2.20), there is a pfaffian superpotential [11] involving $\hat{X}$, $T$, and $U$. This is a model of the type described in sect. 2.3.

The corresponding limit in the dual theory is when $Sp(\tilde{N}_c)$ is weakly gauged. Turning off $Sp(\tilde{N}_c)$ leaves the $Sp(\tilde{N}_c')$ theory with $N_f^{eff} = \tilde{N}_c + N_f = \tilde{N}_c' + 2$ flavors and thus, as in the electric side, it confines. With $Sp(\tilde{N}_c)$ weakly gauged, the dual theory is $Sp(\tilde{N}_c)$, where $\tilde{N}_c = k(N_f + N_f' - 2) - N_c$, with an antisymmetric tensor $\hat{Y} \sim YY$, $2N_f$ fundamentals $t \sim Yq'$, the $2N_f'$ original fundamentals $q$, and singlets $u \sim q'q'$. In addition to the superpotential inherited from (2.21), there is a pfaffian superpotential involving $\hat{Y}$, $t$, and $u$. Up to the singlets and the pfaffian superpotential, the duality between the above $Sp(N_c)$
theory and the dual $Sp(\tilde{N}_c)$ theory is the duality of $[14]$ discussed in sect. 2.3. It is possible to show that the additional singlets and pfaffian superpotential preserve the duality.

More generally, we can consider the electric theory with $Sp(N_c)$ weakly gauged for arbitrary $N'_f$. The $Sp(N'_c)$ theory can be given a dual description as in $[3, 11]$ in terms of an $Sp(\tilde{N}'_c)$ gauge theory with $\tilde{N}'_c = N_c + N'_f - 2 - N'_c$. Weakly gauging $Sp(N_c)$, the theory becomes an $Sp(N_c) \times Sp(\tilde{N}'_c)$ gauge theory with matter in the representations

$$\hat{X} \quad (N_c, 1; 1, 1)$$
$$T \quad (N_c; 1, 1, 2N'_f)$$
$$U \quad (1; 1, 1, N'_f(2N'_f - 1))$$
$$V \quad (1; \tilde{N}'_c; 1, 2N'_f)$$
$$Z \quad (N_c, \tilde{N}'_c; 1, 1)$$
$$Q \quad (N_c, 1; 2N_f, 1)$$

of the $Sp(N_c) \times Sp(\tilde{N}'_c) \times SU(2N_f) \times SU(2N'_f)$ gauge and flavor group with a superpotential

$$W = \text{Tr} \hat{X}^{k+1} + UVV + \hat{X}ZZ + TVZ.$$  \hfill (6.10)

As above, $\hat{X} \sim X^2$, $T \sim XQ'$, $U \sim Q'Q'$. The mesons $M_j, M'_j,$ and $P_r$ of the original theory reduce in part to mesons $\tilde{M}_j = \hat{Q}\hat{X}^j\hat{Q}$, where $\hat{Q}$ is either $Q$ or $T$ in the theory (6.10), along with the meson $U$ and an extra meson $M_k = Q\hat{X}^kQ$, which is a redundant operator in the theory (6.11).

Likewise, consider the dual theory (2.21) in the limit where $Sp(\tilde{N}_c)$ is turned off. In this limit the $Sp(\tilde{N}'_c)$ theory has the dual $Sp(\tilde{N}'_c)$, with $\tilde{N}'_c = N_c + N'_f - 2 - N'_c$ as above, and upon weakly gauging $Sp(\tilde{N}_c)$, the theory can be described by a $Sp(\tilde{N}_c) \times Sp(\tilde{N}'_c)$ gauge theory, with matter in the representations

$$\hat{Y} \quad (\tilde{N}_c, 1, 1, 1)$$
$$t \quad (\tilde{N}_c, 1, 2N'_f, 1)$$
$$u \quad (1, 1, \tilde{N}_f(2N'_f - 1), 1)$$
$$v \quad (1, \tilde{N}'_c, 2N_f, 1)$$
$$z \quad (\tilde{N}_c, \tilde{N}'_c; 1, 1)$$
$$q \quad (\tilde{N}_c, 1, 1, 2N'_f)$$

(6.12)
of the $Sp(\tilde{N}_c) \times Sp(\tilde{N}_c') \times SU(2N_f) \times SU(2N'_f)$ gauge and flavor group and a superpotential

$$W = \text{Tr} \  \hat{Y}^{k+1} + uuv + \hat{Y} zz + tvz + M_k u + U q Y^k q + \sum_{j=1}^k \hat{M}_j \hat{q} \hat{Y}^{k-j} \hat{q}, \quad (6.13)$$

with the additional terms those in (2.21) and $\hat{q}$ referring to $t$ or $q$. Note that the $u$ equations of motion eliminate the redundant $M_k$ and that the $U$ equation of motion sets the redundant $qY^k q$ to zero. Similarly, the $M_k$ equation of motion sets $u$ to zero.

It follows from the $Sp(N_c) \times Sp(N'_c) \leftrightarrow Sp(\tilde{N}_c) \times Sp(\tilde{N}_c')$ duality of this section that the above $Sp(N_c) \times Sp(\tilde{N}_c')$ theory with matter (6.10) and superpotential (6.11) is dual to the above $Sp(\tilde{N}_c) \times Sp(\tilde{N}_c')$ theory with matter (6.12) and superpotential (6.13). This duality again is a consequence of that considered in [14]. Consider starting from $Sp(N_c)$ with an antisymmetric traceless tensor, a superpotential, and $2(N_f + N'_f + \tilde{N}_c')$ fields in the fundamental as in sect. 2.3. From [14] the dual theory is $Sp(k(N_f + N'_f + \tilde{N}_c' - 2) - N_c)$. Adding the third term in the electric superpotential (6.11) causes the magnetic theory to be broken to $Sp(k(N_f + N'_f - 2) + \tilde{N}_c' - N_c) = Sp(\tilde{N}_c)$, and its superpotential picks up the third term in (6.13). The duality found above then follows upon adding $Sp(N_c)$ singlets $U$ and $V$ to the electric theory and $Sp(\tilde{N}_c)$ singlets $u$ and $v$ to the magnetic theory, along with the remaining terms in (6.11) and (6.13), and upon gauging a particular $Sp(\tilde{N}_c')$ subgroup of the flavor symmetry in both the electric and magnetic theories.

7. $SO(N_c) \times SO(N'_c)$

We consider the theory described in sect. 2.10. The flavor group of these models is $SU(N_f) \times SU(N'_f) \times U(1)_R$; the matter fields transform as:

|       | $SO(N_c)$ | $SO(N'_c)$ | $SU(N_f)$ | $SU(N'_f)$ | $U(1)_R$ |
|-------|-----------|------------|-----------|------------|-----------|
| $Q$   | $N_c$     | 1          | 1         | 1          | $R_Q = 1 - \frac{(N_c-2)(k+1)-kN'_c}{N_f(k+1)}$ |
| $Q'$  | 1         | $N'_c$    | 1         | $N'_f$    | $R_{Q'} = 1 - \frac{(N'-2)(k+1)-kN_c}{N'_f(k+1)}$ |
| $X$   | $N_c$     | $N'_c$    | 1         | 1         | $R_X = \frac{1}{k+1}$ |

The theory has a discrete $\mathbb{Z}_{2N_f N'_f(k+1)}$ symmetry generated by

$$X \to \alpha^{N_f N'_f} X, \quad Q \to \alpha^{-N'_c N_f} Q, \quad Q' \to \alpha^{-N_c N'_f} Q', \quad (7.1)$$

with $\alpha = e^{2\pi i/2N_f N'_f(k+1)}$. (There are also $\mathbb{Z}_{2N_f}$ and $\mathbb{Z}_{2N'_f}$ symmetries acting on $Q$ and $Q'$ respectively.)
7.1. The dual model

The dual magnetic theory is an $SO(\tilde{N}_c) \times SO(\tilde{N}_c')$ gauge theory with $\tilde{N}_c = (k+1)(N_f + N'_f + 4) - N_f - N'_c$ and $\tilde{N}_c' = (k+1)(N_f + N'_f + 4) - N'_f - N_c$, as described in sect. 2.10. Taking $M_j$, $M'_j$, and $P_r$ to transform as in the electric theory, the theory (2.24) has an $SU(N_f) \times SU(N'_f) \times U(1)_R$ flavor symmetry with matter fields transforming as:

| Field | $SO(\tilde{N}_c)$ | $SO(\tilde{N}_c')$ | $SU(N_f)$ | $SU(N'_f)$ | $U(1)_R$ |
|-------|-------------------|-------------------|-----------|-----------|-----------|
| $q$   | $\tilde{N}_c$     | 1                 | $\overline{N}_f'$ | 1         | $1 - \frac{(N_c-2)(k+1)-kN'_f}{N'_f(k+1)}$ |
| $q'$  | 1                 | $\tilde{N}_c'$   | $\overline{N}_f$   | 1         | $1 - \frac{(N'_c-2)(k+1)-kN_c}{N_f(k+1)}$ |
| $Y$   | $\tilde{N}_c$     | $\tilde{N}_c'$   | 1           | 1         | $\frac{1}{k+1}$ |
| $P_r$ | 1                 | 1                 | $N_f$       | $N'_f$    | $R_Q + R_{Q'} + (2r + 1)R_X$ |
| $M_j$ | 1                 | 1                 | sym         | 1         | $2R_Q + 2jR_X$ |
| $M'_j$| 1                 | 1                 | 1           | sym       | $2R_{Q'} + 2jR_X$ |

These are indeed anomaly free in the dual theory. Similarly, the discrete $\mathbb{Z}_{2N_fN'_f(k+1)}$ symmetry (7.1) of the electric theory is a symmetry of the dual theory with the singlets transforming as the associated mesons of the electric theory provided that the other fields transform as

$$Y \rightarrow \alpha^{N_fN'_f}Y, \quad q \rightarrow e^{2\pi i(N_f+4)/N'_f}\alpha^{-\tilde{N}_cN_f}q, \quad q' \rightarrow e^{2\pi i(N'_f+4)/2N_f}\alpha^{-\tilde{N}_cN'_f}q', \quad (7.2)$$

which is indeed non-anomalous in the dual gauge theory. There is a freedom to modify (7.2) to give an extra minus sign to both $q$ and $q'$; this operation is a symmetry of the dual theory which can mix with the $\mathbb{Z}_{2N_fN'_f(k+1)}$ symmetry.

We have verified that the 't Hooft anomaly matching conditions are indeed satisfied.

7.2. Deformations

We consider the case $N_c \geq N'_c$. Consider deforming (2.23) to include lower order terms:

$$W_{\text{pert}} = \sum_{n=0}^{k} \lambda_n \text{Tr} \ X^{2(n+1)}. \quad (7.3)$$
The theory has vacua with $\langle Q \rangle = \langle Q' \rangle = 0$ for which the D-terms give

$$
\langle X \rangle = \begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_{N'_\ell}
\end{pmatrix}
$$

(7.4)

generically breaking the gauge group to $SO(N_c - N'_c)$. The theory with superpotential (7.3) has multiple vacua with expectation values of the form (7.4) with $x_i$ satisfying $x_i W'(x_i^2) = 0$ which generically has $k$ distinct non-zero solutions $z_\ell$; zero is also a solution. Let $p_0$ be the number of eigenvalues with $x_i = 0$ and let $p_\ell$ be the number of $x_i = z_\ell$, with $\sum_{\ell=0}^k p_\ell = N'_c$. In such a vacuum the gauge group is broken by $\langle X \rangle$ to

$$
SO(N_c) \times SO(N'_c) \to SO(N_c - N'_c + p_0) \times SO(p_0) \times SO(p_1) \times \ldots \times SO(p_k).
$$

(7.5)

In the generic situation $X$ is massive in each vacuum and can be integrated out. The $SO(N_c - N'_c + p_0)$ factor has $N_f$ vector representations; the $SO(p_0)$ factor has $N'_f$ vector representations; and each $SO(p_\ell)$ factor has $N_f + N'_f$ matter fields in the vector representation. The vacuum is stable provided $N_f \geq N_c - N'_c + p_0 - 4$, $N'_f \geq p_0 - 4$, and $N_f + N'_f \geq p_\ell - 4$ for every $\ell = 1, \ldots, k$.

A similar analysis in the dual theory leads to a dual version of (7.5) with the duality in each factor that of (7.4).

We now consider adding a mass term $m Q^{N_f} \cdot Q^{N_f}$, giving a mass to a quark flavor in the electric theory. In the magnetic theory the added term to the superpotential gives expectation values

$$
q_{N_f} Y^{2j} q_{N_f} = -m \delta_{j,k}
$$

(7.6)

which, along with the other $F$ terms and the $D$ terms, give expectation values $\langle q_{N_f} \rangle$ and $\langle Y \rangle$ which break $SO(\tilde{N}_c) \times SO(\tilde{N}'_c)$ to $SO(\tilde{N}_c - (k + 1)) \times SO(\tilde{N}'_c - k)$. For example, for $k = 1$ the expectation values are $(q_{N_f})_c = \delta_{c,1} + i \delta_{c,2}$ and $Y_{c,d'} = (\delta_{c,1} - i \delta_{c,2}) \delta_{d',1}$, which reduces $N_c$ by two and $N'_c$ by one. The low energy magnetic theory is the dual of the low energy electric theory.
7.3. Dualizing one gauge group

Consider the limit where the gauge coupling for $SO(N_c)$ on the electric side is much weaker than that of $SO(N'_c)$. Turning off $SO(N_c)$ entirely leaves $SO(N'_c)$ with $N_f^{eff}(N'_c) = N_c + N'_f$ fields in the vector representation. If $N'_f$ is chosen so that $N_f^{eff} = N'_c - 4$, this theory confines to give a theory with mesons in the symmetric representation of the $SU(N_f^{eff})$ flavor symmetry [5]. When the $SO(N_c)$ subgroup of $SU(N_f^{eff})$ is weakly gauged, the theory has a symmetric tensor $\hat{X} \sim X^2$ of $SO(N_c)$, $N_f$ vectors $T \sim XQ'$, $N_f$ vectors $Q$ as in the original theory, and singlets $U \sim Q'Q'$. This is a theory of [14] (sect. 2.2).

The corresponding limit in the dual theory is when $SO(\tilde{N}_c)$ is weakly gauged. Turning off $SO(\tilde{N}_c)$ leaves the $SO(\tilde{N}'_c)$ theory with $N_f^{eff} = \tilde{N}_c + N_f = \tilde{N}'_c - 4$ vectors, and thus, as on the electric side, it confines. When $SO(\tilde{N}_c)$ is weakly gauged, where $\tilde{N}_c = k(N_f + N'_f + 4) - N_c$, the theory has a symmetric tensor $\hat{Y} \sim YY$ of $SO(\tilde{N}_c)$, $N_f$ vectors $t \sim Yq'$, the $N'_f$ original vectors $q$, and singlets $u \sim q'q'$. Up to the singlets, the duality between the above $SO(N_c)$ theory and the dual $SO(\tilde{N}_c)$ theory is the duality of [14] discussed in sect. 2.2. It is possible to show that the additional singlets preserve the duality.

More generally, we can consider the electric theory with $SO(N_c)$ weakly gauged for arbitrary $N'_f$. The analysis is very similar to the case studied in sect. 3.3, and we refer the reader to that section for details. The main point is that the electric and magnetic theories flow to $SO(N_c) \times SO(\tilde{N}'_c)$ and $SO(\tilde{N}_c) \times SO(\tilde{N}'_c)$ respectively. The matter content is such that the electric model is an example studied in [14] (sect. 2.2) with extra singlets, a superpotential, and an $SO(\tilde{N}'_c)$ subgroup of its flavor group gauged. The magnetic theory is the dual theory under the duality of sect. 2.2, along with extra singlets, the dual superpotential, and the same $SO(\tilde{N}'_c)$ subgroup of its flavor group gauged.

8. $SU(N_c) \times SU(N'_c)$

This theory was outlined in sect. 2.11. The theory (2.26) has an $[SU(N_f) \times SU(N'_f)]^2 \times U(1)^3 \times U(1)_R$ global flavor symmetry with the fields transforming as
We have chosen $U(1)_B$ and $U(1)_{B'}$ such that all mesons are invariant.

There is also a $\mathbb{Z}_{2N_fN'_f(k+1)}$ discrete symmetry generated by

$$X, \bar{X} \rightarrow \alpha^{N_fN'_f}X, \bar{X} \quad Q, \bar{Q} \rightarrow \alpha^{-N_cN'_c}Q, \bar{Q} \quad Q', \bar{Q}' \rightarrow \alpha^{-N_cN'_c}Q', \bar{Q'},$$

(8.1)

with $\alpha = e^{2\pi i/2N_fN'_f(k+1)}$.

8.1. The dual theory

The dual is an $SU(\tilde{N}_c) \times SU(\tilde{N}'_c)$ gauge theory with $\tilde{N}_c = (k+1)(N'_f + N_f) - N_f - N'_c$ and $\tilde{N}'_c = (k+1)(N'_f + N_f) - N'_f - N_c$, as described in sect. 2.11. Taking the singlets to transform as the mesons of the electric theory, the dual theory has the same flavor group with the various fields transforming as:

|          | $Q; \tilde{Q}$ | $Q'; \tilde{Q}'$ | $X; \tilde{X}$ |
|----------|----------------|----------------|---------------|
| $SU(N_c)$ | $N_c; \overline{N}_c$ | $1; 1$ | $N_c; \overline{N}_c$ |
| $SU(N'_c)$ | $1; 1$ | $N'_c; \overline{N}'_c$ | $N'_c; \overline{N}'_c$ |
| $SU(N_f)_L$ | $N_f; 1$ | $1; 1$ | $1; 1$ |
| $SU(N'_f)_L$ | $1; 1$ | $N'_f; 1$ | $1; 1$ |
| $SU(N_f)_R$ | $1; N_f$ | $1; 1$ | $1; 1$ |
| $SU(N'_f)_R$ | $1; 1$ | $1; N'_f$ | $1; 1$ |
| $U(1)_B$ | $\pm \frac{1}{N_c}$ | $\pm \frac{1}{N'_c}$ | $\pm \left(\frac{1}{N_c} + \frac{1}{N'_c}\right)$ |
| $U(1)_{B'}$ | $\pm \frac{1}{N_c}$ | $\pm \frac{1}{N'_c}$ | $\pm \left(\frac{1}{N_c} - \frac{1}{N'_c}\right)$ |
| $U(1)_X$ | $0$ | $0$ | $\pm 1$ |
| $U(1)_R$ | $1 + \frac{(kN'_f - (k+1)N_c)}{N_f(k+1)}$ | $1 + \frac{(kN_c - (k+1)N'_f)}{N'_f(k+1)}$ | $\frac{1}{k+1}$ |
Here \( B \) is indeed anomaly free in the dual theory. Similarly, the discrete symmetry \((\Sigma.1)\) is a symmetry of the dual theory with the singlets transforming as the corresponding mesons of the electric theory provided it acts on the other fields as

\[
Y \rightarrow e^{2\pi i B_Y N_f'/2} e^{2\pi i B_Y' N_f/2} \alpha^{N_f N_f'} Y \\
\tilde{Y} \rightarrow e^{-2\pi i B_Y N_f'/2} e^{-2\pi i B_Y' N_f/2} \alpha^{-N_f N_f'} \tilde{Y} \\
q \rightarrow e^{2\pi i B_q N_f'/2} e^{2\pi i B_q' N_f/2} \alpha^{-N_f \tilde{N}_c} e^{2\pi i N_f/2 N_f'} q \\
\tilde{q} \rightarrow e^{-2\pi i B_q N_f'/2} e^{-2\pi i B_q' N_f/2} \alpha^{-N_f \tilde{N}_c} e^{2\pi i N_f/2 N_f'} \tilde{q} \\
q' \rightarrow e^{2\pi i B_{q'} N_f'/2} e^{2\pi i B_{q'}' N_f/2} \alpha^{-N_f' \tilde{N}_c} e^{2\pi i N_f'/2 N_f} q' \\
\tilde{q}' \rightarrow e^{-2\pi i B_{q'} N_f'/2} e^{-2\pi i B_{q'}' N_f/2} \alpha^{-N_f' \tilde{N}_c} e^{2\pi i N_f'/2 N_f} \tilde{q}'
\]

which are indeed anomaly free in the dual theory.

**Table:**

| Group                  | Representation | Charge | Charge | \( Y; \tilde{Y} \) |
|------------------------|----------------|--------|--------|---------------------|
| \( SU(\tilde{N}_c) \) | \( \tilde{N}_c; \tilde{N}_c \) | \( \tilde{N}_c; \tilde{N}_c \) | \( \tilde{N}_c; \tilde{N}_c \) |
| \( SU(N_f') \)        | \( 1; 1 \)     | \( \tilde{N}_c; \tilde{N}_c \) | \( \tilde{N}_c; \tilde{N}_c \) |
| \( SU(N_f)_L \)       | \( 1; 1 \)     | \( \tilde{N}_c; \tilde{N}_c \) | \( \tilde{N}_c; \tilde{N}_c \) |
| \( SU(N_f)_R \)       | \( 1; 1 \)     | \( 1; \tilde{N}_f \)      | \( 1; 1 \)      |
| \( SU(N_f')_L \)      | \( \tilde{N}_f; 1 \) | \( 1; 1 \)     | \( 1; 1 \)     |
| \( SU(N_f')_R \)      | \( 1; \tilde{N}_f \) | \( 1; 1 \)     | \( 1; 1 \)     |
| \( U(1)_B \)          | \( \pm \frac{1}{N_c} \) | \( \pm \frac{1}{N_c} \) | \( \pm \left( \frac{1}{N_c} + \frac{1}{N_c'} \right) \) |
| \( U(1)_{B'} \)       | \( \pm \frac{1}{N_c} \) | \( \mp \frac{1}{N_c} \) | \( \pm \left( \frac{1}{N_c} - \frac{1}{N_c'} \right) \) |
| \( U(1)_X \)          | \( \pm \frac{k N_f}{N_c} \) | \( \pm \frac{k N_f'}{N_c} \) | \( \pm \left( 1 - \frac{k N_f}{N_c} - \frac{k N_f'}{N_c} \right) \) |
| \( U(1)_R \)          | \( 1 + \frac{(k N_f - (k+1) N_f')}{N_f(k+1)} \) | \( 1 + \frac{(k N_f - (k+1) N_f')}{N_f(k+1)} \) | \( \frac{1}{k+1} \) |

Here \( B_Y \) and \( B_{Y'} \) are the \( U(1)_B \) and \( U(1)_{B'} \) charges of \( Y \), and similarly for the other fields. This is indeed a non-anomalous discrete symmetry of the dual gauge theory.

We have verified that all of the 't Hooft anomaly matching conditions are indeed satisfied.

Under the duality the baryons of the electric theory,

\[
B\{n_j, m_r\} = \prod_{j=0}^{k} Q^{m_j}_{(j)} \prod_{r=0}^{k-1} S^{m_r}_{(r)}, \tag{8.3}
\]
with gauge indices contracted with an \( \epsilon \)-tensor of \( SU(N_c) \), where \( Q(j) = (X \tilde{X})^j Q \) and \( S_{(r)} = (X \tilde{X})^r X \tilde{Q}' \), are mapped to the analogous baryons

\[
 b\{\tilde{n}_j, \tilde{m}_r\} = \prod_{j=0}^{k} q_{(j)}^{\tilde{m}_j} \prod_{r=0}^{k-1} s_{(r)}^{\tilde{m}_r}
\]

of the dual theory, where \( q_{(j)}' = (Y \tilde{Y})^j q' \) and \( s_{(r)}' = (Y \tilde{Y})^r Y \tilde{q} \). The mapping is \( B\{n_j, m_r\} \rightarrow b\{\tilde{n}_j, \tilde{m}_r\} \), where \( \tilde{n}_j = N_f - n_{k-j} \) and \( \tilde{m}_r = N'_f - m_{k-1-r} \). This transformation is consistent with all of the flavor symmetries. There is a similar mapping for all other baryons and antibaryons in the theory.

### 8.2. Deformations: superpotential

Consider deforming the superpotential as:

\[
 W_{\text{pert}} = \sum_{n=0}^{k} \lambda_n \text{Tr} (X \tilde{X})^{n+1}.
\]

Let us assume that \( N'_c \geq N_c \); the general D-flat vacua are of the form

\[
 \langle X \rangle = \begin{pmatrix} x_1 & x_2 & \cdots & x_{N_c} \\ x_1 & x_2 & \cdots & x_{N_c} \\ \vdots & \ddots & \ddots & \vdots \\ x_1 & x_2 & \cdots & x_{N_c} \end{pmatrix} \quad ; \quad \langle \tilde{X} \rangle = \begin{pmatrix} \tilde{x}_1 & \tilde{x}_2 & \cdots & \tilde{x}_{N_c} \\ \tilde{x}_1 & \tilde{x}_2 & \cdots & \tilde{x}_{N_c} \\ \vdots & \ddots & \ddots & \vdots \\ \tilde{x}_1 & \tilde{x}_2 & \cdots & \tilde{x}_{N_c} \end{pmatrix}
\]

where \( |x_i| = |\tilde{x}_i| \). The equations of motion for \( X \) and \( \tilde{X} \) give \( x_i W'(x_i \tilde{x}_i) = \tilde{x}_i W'(x_i \tilde{x}_i) = 0 \). There are generically \( k \) distinct solutions to these equations \( z_{\ell} = x_{\ell} \tilde{x}_{\ell} \), plus the solution \( x = \tilde{x} = 0 \). A vacuum state is thus labeled by integers \( \{p_0, p_1, \ldots, p_k\} \), with \( \sum_{j=0}^{k} p_j = N_c \), where \( p_0 \) is the number of eigenvalues which are zero. The unbroken symmetry in such a vacuum is

\[
 SU(N_c) \times SU(N'_c) \rightarrow SU(N'_c - N_c + p_0) \times SU(p_0) \times SU(p_1) \cdots \times SU(p_k).
\]

In the generic situation \( X \) and \( \tilde{X} \) are massive in each vacuum and can be integrated out. In the first two factors there are \( N_f \) and \( N'_f \) flavors of quarks respectively and in each of the other factors there are \( N_f + N'_f \) flavors of quarks. In particular, this is the case when adding a mass term \( W = mX \tilde{X} \) to (2.17).

In the dual theory there is a breaking similar to (8.7) and the duality between the two theories is the duality of [3] in each factor.
8.3. Deformations: flat directions

Consider the flat direction along which the operator \( B_n = X^n Q^{N_c-n} Q^{N'_c-n} \), with gauge indices contracted with two \( \epsilon \)-tensors, gets an expectation value. In terms of the elementary fields this flat direction is \( \langle Q^{f,c} \rangle = \sqrt{2} a \delta^{f,c} \) for \( c \leq N_c - n \) and zero otherwise, \( \langle X^{cd} \rangle = a \delta^{c,d} \) for \( c > N_c - n \) and zero otherwise, with all other expectation values zero. Along this flat direction, which is not lifted by the superpotential \( (2.26) \), the gauge group is broken to \( SU(n) \). In the low energy theory there are \( N_f + N'_f \) matter fields in the fundamental representation of \( SU(n) \), with some coming from \( \tilde{X} \) and some from the quarks not entering in \( B_n \). There is also an adjoint \( \tilde{X} \) from \( \tilde{X} \). The low energy theory has a superpotential inherited from \( (2.26) \), \( W = \text{Tr} \tilde{X}^{k+1} \), where indices are contracted using \( \langle X \rangle \). Up to some additional singlets, this low energy theory is that considered in \([12,13]\).

In the dual theory the operator \( B_n \) labeling the above flat direction is mapped to \( b_n = Y^{k(N_f+N'_f)-n} q^f q^{N_f-N_c+n} q^{N'_f-N'_c+n} \). When this operator gets an expectation value the dual \( SU(\tilde{N}_c) \times SU(\tilde{N'}_c) \) gauge theory is broken to \( SU(\tilde{n}) \), with \( \tilde{n} = k(N_f+N'_f) - n \). This low energy \( SU(\tilde{n}) \) theory has \( N_f + N'_f \) matter fields in the fundamental representation, with some coming from \( \tilde{Y} \) and some from the dual quarks not entering in \( b_n \). There is also an adjoint coming from \( \tilde{Y} \), and some singlets, some of which are eliminated by \( (2.27) \). The low energy magnetic theory gets a superpotential from \( (2.27) \), such that it is precisely that shown in \([12,13]\) to be the dual of the above low energy electric theory. The \( SU(n) \) duality discussed in \([12,13]\) is thus inherited from the model discussed here.

8.4. Dualizing one gauge group

Consider the limit where the gauge coupling for \( SU(N_c) \) on the electric side is much weaker than that of \( SU(N'_c) \). Turning off \( SU(N_c) \) entirely leaves \( SU(N'_c) \) with \( N_{f,\text{eff}}(N'_{c}) = N_c + N'_f \) flavors in the fundamental representation. If \( N'_f \) is chosen so that \( N_{f,\text{eff}} = N'_c + 1 \), this theory confines to give a theory with mesons in the \( (N_{f,\text{eff}},N_{f,\text{eff}}) \) representation of the \( SU(N_{f,\text{eff}})^{\times} SU(N_{f,\text{eff}}) \) flavor symmetry \([1]\). When the diagonal \( SU(N_c) \) subgroup of \( SU(N_{f,\text{eff}})^{\times} SU(N_{f,\text{eff}}) \) is weakly gauged, the theory has an adjoint tensor \( \tilde{X} \sim X^2 \) of \( SU(N_c) \), \( N'_f \) flavors \( T \sim X \tilde{Q}' \), \( \tilde{T} \sim \tilde{X} Q' \), \( N_f \) flavors \( Q, \tilde{Q} \) as in the original theory, a flavor \( B, \tilde{B} \) from the baryons of \( SU(N'_c) \), and some singlets. The superpotential is \( W = \text{Tr} \tilde{X}^{k+1} + B\tilde{X}\tilde{B} \) plus other terms. This is a theory of \([12,13,16]\) (sect. 2.1).

The corresponding limit in the dual theory is when \( SU(\tilde{N}_c) \) is weakly gauged. Turning off \( SU(\tilde{N}_c) \) leaves the \( SU(\tilde{N}'_c) \) theory with \( N_{f,\text{eff}} = \tilde{N}_c + N_f = \tilde{N}'_c + 1 \) flavors, and thus,
as on the electric side, it confines. When $SU(\tilde{N}_c)$ is weakly gauged, where $\tilde{N}_c = k(N_f + N'_f) + 1 - N_c$, the theory has an adjoint tensor $\tilde{Y} \sim YY$ of $SU(\tilde{N}_c)$, $N_f$ flavors $t \sim Yq'$, $\tilde{t} \sim Yq'$, the $N'_f$ original flavors $q, \tilde{q}$, a flavor $b, \tilde{b}$ from the baryons of $SU(\tilde{N}_c)$, and some singlets. The superpotential is $W = \text{Tr} \tilde{Y}^{k+1} + b\tilde{Y}b$ plus other terms. Up to the singlets and the terms in the superpotential which we have not written, the duality between the above $SU(N_c)$ theory and the dual $SU(\tilde{N}_c)$ theory is the duality of [12,13,16] discussed in sect. 2.1. It is possible to show that the additional singlets and superpotential preserve the duality.

More generally, we can consider the electric theory with $SU(N_c)$ weakly gauged for arbitrary $N'_f$. The analysis is very similar to the case studied in sect. 2.1, and we refer the reader to that section for details. The main point is that the electric and magnetic theories flow to $SU(N_c) \times SU(\tilde{N}_c)$ and $SU(\tilde{N}_c) \times SU(\tilde{N}_c)$ respectively. The matter content is such that the electric model is an example studied in [13] (sect. 2.1) with extra singlets, a superpotential, and an $SU(\tilde{N}_c)$ subgroup of its flavor group gauged. The magnetic theory is the dual theory under the duality of sect. 2.1, along with extra singlets, the dual superpotential, and the same $SU(\tilde{N}_c)$ subgroup of its flavor group gauged.

9. $SO(N_c) \times Sp(N'_f)$

Consider the $SO(N_c) \times Sp(N'_f)$ theories described in sect. 2.12. These are chiral theories in that the field $X$ can not be given a mass term, as there is no gauge invariant $\text{Tr} X^2$; the basic gauge invariants involving $X$ are $\text{Tr} X^{4(j+1)}$. Also, to cancel the global anomaly of the $Sp$ factor we must take $N_c + n'_f$ even. This theory has an anomaly-free $SU(N_f) \times SU(n'_f) \times U(1)_R$ flavor symmetry with matter fields in the representations

\[
\begin{array}{|c|c|c|c|c|}
\hline
& SO(N_c) & Sp(N'_f) & SU(N_f) & SU(n'_f) & U(1)_R \\
\hline
Q & N_c & 1 & N_f & 1 & R_Q = 1 - \frac{1}{N_f} \left( N_c - 2 - \frac{N'_f(2k+1)}{2(k+1)} \right) \\
Q' & 1 & 2N'_f & 1 & n'_f & R_{Q'} = 1 - \frac{1}{n'_f} \left( 2N'_f + 2 - \frac{N_c(2k+1)}{2(k+1)} \right) \\
X & N_c & 2N'_f & 1 & 1 & R_X = \frac{1}{2(k+1)} \\
\hline
\end{array}
\]

In addition, the theory (2.29) has a $\mathbb{Z}_{4N_f n'_f (k+1)}$ discrete symmetry generated by

$X \to \alpha^{N_f n'_f} X$, \quad $Q \to \alpha^{-2N'_f n'_f} Q$, \quad $Q' \to \alpha^{-N_c N_f} Q'$, \quad (9.1)

with $\alpha = e^{2\pi i / 4N_f n'_f (k+1)}$, and a $\mathbb{Z}_{2N_f}$ symmetry generated by

$Q \to \beta Q$, \quad (9.2)

where $\beta = e^{2\pi i / 2N_f}$. 

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9.1. Dual theory

The dual theory is $SO(\tilde{N}_c) \times Sp(\tilde{N}_c')$, where $\tilde{N}_c = 2(k+1)(n_f' + N_f) - N_f - 2N_c'$ and $2\tilde{N}_c' = 2(k+1)(n_f' + N_f) - n_f' - N_c$, as described in sect. 2.14. Taking the singlets $P_r, M_j$, and $M'_j$ to transform as the mesons of the electric theory, the theory (2.30) has a $SU(N_f) \times SU(n_f') \times U(1)_R$ global symmetry with matter in the representations

| | $SO(\tilde{N}_c)$ | $Sp(\tilde{N}_c')$ | $SU(N_f)$ | $SU(n_f')$ | $U(1)_R$ |
|---|---|---|---|---|---|
| $q$ | $\tilde{N}_c$ | 1 | 1 | $n_f'$ | $1 - \frac{1}{N_f} \left( \tilde{N}_c - 2 - \frac{N_f'(2k+1)}{k+1} \right)$ |
| $q'$ | 1 | $2\tilde{N}_c'$ | $\tilde{N}_f$ | 1 | $1 - \frac{1}{n_f'} \left( 2\tilde{N}_c' + 2 - \frac{\tilde{N}_c(2k+1)}{2(k+1)} \right)$ |
| $Y$ | $\tilde{N}_c$ | $2\tilde{N}_c'$ | 1 | 1 | $\frac{1}{(k+1)}$ |
| $P_r$ | 1 | 1 | $n_f'$ | $R_Q + R'_Q + (2r + 1)R_X$ |
| $M_j$ | 1 | 1 | sym, $j$ even | asym, $j$ odd | $2R_Q + 2jR_X$ |
| $M'_j$ | 1 | 1 | sym, $j$ even | asym, $j$ odd | $2R_Q' + 2jR_X$ |

which are indeed anomaly free in the dual gauge theory. Taking the singlets to transform as the mesons of the electric theory under (9.1), the superpotential (2.30) is invariant under the $\mathbb{Z}_{4N_fn_f'(k+1)}$ symmetry if the other fields transform as

$$Y \to \alpha^{N_fn_f'}Y, \quad q \to e^{2\pi in_f'/2n_f'}\alpha^{-2\tilde{N}_cN_f}q, \quad q' \to e^{2\pi in_f'/2N_f}\alpha^{-n_f'}\tilde{N}_c'q'. \quad (9.3)$$

Similarly, (2.30) is invariant under the $\mathbb{Z}_{2N_f}$ transformation on the singlets corresponding to (9.2) if the other fields transform as

$$q' \to \beta^{-1}q'. \quad (9.4)$$

However, both (9.3) and (9.4) are not quite symmetries of the dual theory: under both of them the $Sp(\tilde{N}_c')$ instanton 't Hooft vertex picks up a minus sign. (For odd $N_f$ this can be corrected by combining the transformations (9.3) and (9.4) with a transformation which gives a minus sign to both $q$ and $q'$.) Perhaps this suggests that the symmetries (9.1) and (9.2) of the electric theory take the magnetic dual to a dyonic dual in which the $Sp(\tilde{N}_c')$ theta angle is shifted by $\pi$ relative to that of the magnetic dual. Dyonic duals with theta angles shifted by $\pi$ appeared in [3].

We have verified that all of the 't Hooft anomaly matching conditions are satisfied.
9.2. Deformations

Consider deforming (2.29) to include lower order terms:

\[ W_{\text{pert}} = \sum_{n=0}^{k} \lambda_n \text{Tr} \ X^{4(n+1)}. \]  

(9.5)

We consider the case \( N_c \geq 2N'_c \); the other case is very similar. The theory has vacua with \( \langle Q \rangle = \langle Q' \rangle = 0 \) for which the D-terms give

\[ \langle X \rangle = \begin{pmatrix}
    x_1 1_2 \\
    x_2 1_2 \\
    \vdots \\
    x_{N'_c} 1_2 
\end{pmatrix}. \]  

(9.6)

A vacuum is labeled by integers \( \{p_0, p_1, \ldots, p_k\} \), with \( \sum_{\ell=0}^{k} p_\ell = N'_c \), with the gauge group broken by \( \langle X \rangle \) to

\[ SO(N_c) \times Sp(N'_c) \rightarrow SO(N_c - 2N'_c + 2p_0) \times Sp(p_0) \times U(p_1) \times U(p_2) \times \ldots \times U(p_k). \]  

(9.7)

Because \( X \) cannot be given a mass term, it will not be massive in the \( SO(2p_0 + \eta) \times Sp(N'_c - n_c + p_0) \) factor; this factor will have a field \( \hat{X} \) transforming in the defining representation of both the \( SO \) and \( Sp \) groups. For example, consider the effect of the superpotential \( W = \text{Tr} \ X^{4(k+1)} + \lambda \text{Tr} \ X^4 \). The \( SO(N_c - 2N'_c + 2p_0) \times Sp(p_0) \) factor in (9.7) is then a theory of the type considered in this section, with a field \( \hat{X} \) in the defining representation of both gauge groups and a superpotential \( W \sim \hat{X}^4 \) (i.e. \( k = 0 \)), along with \( N_f \) fields in the vector representation of the \( SO \) factor and \( n'_f \) fields in the fundamental representation of the \( Sp \) factor. In the \( U(p_\ell) \) factors in (9.7) the field \( X \) is massive and can be integrated out, leaving a defining model; each has \( N_f + n'_f \) flavors in the fundamental representation. The vacuum is stable provided \( N_f \geq 2p_0 + \eta - 4, N'_f > N'_c - n_c + p_0 \), and \( N_f + N'_f \geq p_\ell \) for every \( \ell = 1, \ldots, k \).

A similar analysis in the dual theory yields vacua labeled by \( q_\ell \), with \( \sum_{\ell=0}^{k} q_\ell = 2\tilde{N}'_c \), with the gauge group broken as

\[ SO(\tilde{N}_c - 2\tilde{N}'_c + 2q_0) \times Sp(q_0) \times U(q_1) \times U(q_2) \times \ldots \times U(q_k) \]  

(9.8)

with a matter field \( \hat{Y} \) and a superpotential \( W \sim \text{Tr} \ \hat{Y}^4 \) in the \( SO \times Sp \) factor, and with the matter field \( Y \) massive and decoupled in all of the \( U(q_\ell) \) factors. The duality mapping between (9.7) and (9.8) is the \( k = 0 \) case of the duality of this section for the \( SO \times Sp \) factor and the duality of \([3]\) for the \( U \) factors: \( 2q_0 = 2N_f + n'_f - (N_c - 2N'_c + 2p_0) \) and, for \( \ell > 0, q_\ell = N_f + n'_f - p_\ell \).
9.3. Dualizing one gauge group

Consider the limit where the gauge coupling for $SO(N_c)$ on the electric side is much weaker than that of $Sp(N'_c)$. Turning off $SO(N_c)$ entirely leaves $Sp(N'_c)$ with $n^{eff}_f(N'_c) = N_c + n'_f$ fields in the fundamental representation. If $n'_f$ is chosen so that $n^{eff}_f = 2(N'_c + 2)$, this theory confines to give a theory with mesons in the antisymmetric representation of the $SU(n^{eff}_f)$ flavor symmetry [11]. When the $SO(N_c)$ subgroup of $SU(n^{eff}_f)$ is weakly gauged, the theory has an antisymmetric (adjoint) tensor $\hat{X} \sim X^2$ of $SO(N_c)$, $n'_f$ vectors $T \sim XQ'$, $N_f$ vectors $Q$ as in the original theory, and singlets $U \sim Q'Q'$. This is a theory of [15] (sect. 2.5).

The corresponding limit in the dual theory is when $SO(\tilde{N}_c)$ is weakly gauged. Turning off $SO(\tilde{N}_c)$ leaves the $Sp(\tilde{N}'_c)$ theory with $n^{eff}_f = \tilde{N}_c + N_f = 2(\tilde{N}'_c + 2)$ fields in the fundamental representation, and thus, as on the electric side, it confines. When $SO(\tilde{N}_c)$ is weakly gauged, where $\tilde{N}_c = k(N_f + N'_f + 4) - N_c$, the theory has an antisymmetric (adjoint) tensor $\hat{Y} \sim YY$ of $SO(\tilde{N}_c)$, $N_f$ vectors $t \sim Yq'$, the $n'_f$ original vectors $q$, and singlets $u \sim q'q'$. Up to the singlets, the duality between the above $SO(N_c)$ theory and the dual $SO(\tilde{N}_c)$ theory is the duality of [15] discussed in sect. 2.5. It is possible to show that the additional singlets preserve the duality.

More generally, we can consider the electric theory with $SO(N_c)$ weakly gauged for arbitrary $n'_f$. The analysis is very similar to the case studied in sect. 6.3, and we refer the reader to that section for details. The main point is that the electric and magnetic theories flow to $SO(N_c) \times Sp(\tilde{N}'_c)$ and $SO(\tilde{N}_c) \times Sp(\tilde{N}'_c)$ respectively. The matter content is such that the electric model is an example studied in [15] (sect. 2.5) with extra singlets, a superpotential, and an $Sp(\tilde{N}'_c)$ subgroup of its flavor group gauged. The magnetic theory is the dual theory under the duality of sect. 2.5, along with extra singlets, the dual superpotential, and the same $Sp(\tilde{N}'_c)$ subgroup of its flavor group gauged.

If the gauge coupling for $Sp(N'_c)$ is much weaker than that of $SO(N_c)$, the analysis is very similar to the above with appropriate replacements of $N_f$ with $n'_f$, “antisymmetric” with “symmetric”, etc. The low energy electric and magnetic theories flow to $Sp(N'_c) \times SO(\tilde{N}_c)$ and $Sp(\tilde{N}_c) \times SO(\tilde{N}_c)$ respectively; the matter content is such that the models are those of sect. 2.4 with an $SO(\tilde{N}_c)$ subgroup of the flavor symmetry gauged. They are dual under the duality of [15].
10. $SU(M) \times SO(N)$ with a symmetric tensor of $SU(M)$

Consider the $SU(M) \times SO(N)$ theories described in sect. 2.13. These models have an anomaly-free $SU(m_f) \times SU(\tilde{m}_f) \times SU(n_f) \times U(1)_R$ flavor symmetry with matter fields in the representations

|        | $SU(M)$ | $SO(N)$ | $SU(m_f)$ | $SU(\tilde{m}_f)$ | $SU(n_f)$ | $U(1)_R$ |
|--------|---------|---------|-----------|-------------------|-----------|----------|
| $Q$    | M       | 1       | $m_f$     | 1                 | 1         | $R_Q$    |
| $Q'$   | $\overline{M}$ | 1   | 1         | $\tilde{m}_f$     | 1         | $R_{Q'}$ |
| $S$    | 1       | N       | 1         | 1                 | $n_f$     | $R_S$    |
| $X$    | M       | N       | 1         | 1                 | 1         | $R_X = \frac{1}{2(k+1)}$ |
| $\tilde{X}$ | sym | 1       | 1         | 1                 | 1         | $R_{\tilde{X}} = \frac{1}{2(k+1)}$ |

(There are two additional $U(1)$ symmetries which we have omitted from this table.) This is a chiral theory; cancellation of gauge anomalies requires $m_f + 2N = \tilde{m}_f + M + 4$. The $R$ charges can be taken to be

$$R_Q = 1 - \frac{(k+1)(2[M-N] + m_f + n_f - \tilde{m}_f) + N}{2(k+1)m_f},$$

$$R_{\tilde{Q}} = 1 - \frac{(k+1)(-m_f - n_f + \tilde{m}_f) + 2(M-2k)}{2(k+1)\tilde{m}_f},$$

$$R_S = 1 - \frac{2(k+1)(N-M+2) + M}{2(k+1)n_f}. \quad (10.1)$$

10.1. The dual theory

The dual theory is $SU(\tilde{M}) \times SO(\tilde{N})$ where

$$\tilde{M} = (k+1)(n_f + m_f + \tilde{m}_f + 4) - m_f - N,$$

$$\tilde{N} = (k+1)(n_f + m_f + \tilde{m}_f + 4) - n_f - M.$$

Taking the singlets $M_j, \ldots, R_r$ to transform as the mesons of the electric theory, the magnetic theory (2.33) has charged matter in the representations
\[
\begin{array}{cccccccc}
| & SU(\tilde{M}) & SO(\tilde{N}) & SU(m_f) & SU(\tilde{m}_f) & SU(n_f) & U(1)_R \\
\hline
q & \tilde{M} & 1 & 1 & 1 & \bar{n}_f & R_q \\
q' & \tilde{M} & 1 & 1 & \tilde{m}_f & 1 & R_{q'} \\
s & 1 & \tilde{N} & \bar{m}_f & 1 & 1 & R_s \\
Y & \tilde{M} & \tilde{N} & 1 & 1 & 1 & R_Y = \frac{1}{2(k+1)} \\
\tilde{Y} & \text{sym} & 1 & 1 & 1 & 1 & R_{\tilde{Y}} = \frac{1}{2(k+1)} \\
\end{array}
\]

The formulas for the \( R \) charges are the same as in the electric theory with \( M, N \) replaced by \( \tilde{M}, \tilde{N} \) and with \( m_f \) exchanged with \( n_f \).

11. \( SU(M) \times SO(N) \times SO(N') \)

Consider the \( SU(M) \times SO(N) \times SO(N') \) theories described in sect. 2.14. These models have an anomaly-free \( SU(m_f) \times SU(\tilde{m}_f) \times SU(n_f) \times SU(n'_f) \times U(1)_R \) flavor symmetry with matter fields in the representations

\[
\begin{array}{cccccccc}
| & SU(M) & SO(N) & SO(N') & SU(m_f) & SU(\tilde{m}_f) & SU(n_f) & SU(n'_f) & U(1)_R \\
\hline
Q & M & 1 & 1 & m_f & 1 & 1 & 1 & R_Q \\
Q' & \tilde{M} & 1 & 1 & 1 & \tilde{m}_f & 1 & 1 & R_{Q'} \\
S & 1 & N & 1 & 1 & n_f & 1 & & R_S \\
S' & 1 & 1 & N' & 1 & 1 & n'_f & & R_{S'} \\
X & M & N & 1 & 1 & 1 & 1 & 1 & R_X = \frac{1}{2(k+1)} \\
\tilde{X} & \tilde{M} & \tilde{N} & 1 & 1 & 1 & 1 & 1 & R_{\tilde{X}} = \frac{1}{2(k+1)} \\
\end{array}
\]

(There are two additional \( U(1) \) symmetries which we have omitted from this table.) This is a chiral theory; cancellation of gauge anomalies requires \( m_f + N = \tilde{m}_f + N' \). The \( R \) charges can be taken to be

\[
\begin{align*}
R_Q &= 1 - \frac{(k + 1)(2[M - N] + m_f + n_f - \tilde{m}_f - n'_f) + N}{2(k + 1)m_f}, \\
R_{\tilde{Q}} &= 1 - \frac{(k + 1)(2[M - N'] - m_f - n_f + \tilde{m}_f + n'_f) + N'}{2(k + 1)\tilde{m}_f}, \\
R_S &= 1 - \frac{2(k + 1)(N - M - 2) + M}{2(k + 1)n_f}, \\
R_{S'} &= 1 - \frac{2(k + 1)(N' - M - 2) + M}{2(k + 1)n'_f}. \\
\end{align*}
\] (11.1)
11.1. The dual theory

The dual theory is \( SU(\tilde{M}) \times SO(\tilde{N}) \times SO(\tilde{N}') \) where

\[
\begin{align*}
\tilde{M} &= (k+1)(n_f + n'_f + m_f + \bar{m}_f + 4) - m_f - N, \\
\tilde{N} &= (k+1)(n_f + n'_f + m_f + \bar{m}_f + 4) - n_f - M, \\
\tilde{N}' &= (k+1)(n_f + n'_f + m_f + \bar{m}_f + 4) - n'_f - M.
\end{align*}
\]

Taking the singlets \( M_j, \ldots, \tilde{R}_{i} \) to transform as the mesons of the electric theory, the magnetic theory (2.35) has charged matter in the representations

| \( SU(\tilde{M}) \) | \( SO(\tilde{N}) \) | \( SO(\tilde{N}') \) | \( SU(m_f) \) | \( SU(\tilde{m}_f) \) | \( SU(n_f) \) | \( SU(n'_f) \) | \( U(1)_R \) |
|---------------------|-------------------|-------------------|-------------|--------------|-------------|--------------|-------------|
| \( q \)             | \( \tilde{M} \)   | 1                 | 1           | 1           | \( \bar{n}_f \) | 1            | \( R_q \)   |
| \( q' \)            | \( \tilde{M} \)   | 1                 | 1           | 1           | 1           | \( \bar{n}'_f \) | \( R_{q'} \) |
| \( s \)             | 1                 | \( \tilde{N} \)   | 1           | \( \bar{m}_f \) | 1           | 1            | \( R_s \)   |
| \( s' \)            | 1                 | 1                 | \( \tilde{N}' \) | 1           | \( \bar{m}_f \) | 1            | \( R_{s'} \) |
| \( Y \)             | \( \tilde{M} \)   | \( \tilde{N} \)   | 1           | 1           | 1           | 1            | \( R_Y = \frac{1}{2(k+1)} \) |
| \( \tilde{Y} \)      | \( \tilde{M} \)   | 1                 | \( \tilde{N}' \) | 1           | 1           | 1            | \( R_{\tilde{Y}} = \frac{1}{2(k+1)} \) |

The formulas for the \( R \) charges are the same as in the electric theory with \( M, N, N' \) replaced by \( \tilde{M}, \tilde{N}, \tilde{N}' \) and with \( m_f, \bar{m}_f \) exchanged with \( n_f, n'_f \).

12. \( SU(M) \times Sp(N) \) with an antisymmetric tensor of \( SU(M) \)

Consider the \( SU(M) \times Sp(N) \) theories described in sect. 2.15. These models have an anomaly-free \( SU(m_f) \times SU(\tilde{m}_f) \times SU(n_f) \times U(1)_R \) flavor symmetry with matter fields in the representations

| \( SU(M) \) | \( Sp(N) \) | \( SU(m_f) \) | \( SU(\tilde{m}_f) \) | \( SU(n_f) \) | \( U(1)_R \) |
|-------------|-------------|-------------|---------------------|-------------|-------------|
| \( Q \)     | \( M \)     | 1           | \( m_f \)           | 1           | 1           | \( R_Q \)   |
| \( Q' \)    | \( \tilde{M} \) | 1           | 1                   | \( \tilde{m}_f \) | 1           | \( R_{Q'} \) |
| \( S \)     | 1           | \( 2N \)    | 1                   | 1           | \( n_f \)   | \( R_S \)   |
| \( X \)     | \( M \)     | \( 2N \)    | 1                   | 1           | 1           | \( R_X = \frac{1}{2(k+1)} \) |
| \( \tilde{X} \) | \( \text{asym} \) | 1           | 1                   | 1           | 1           | \( R_{\tilde{X}} = \frac{1}{2(k+1)} \) |
(There are two additional $U(1)$ symmetries which we have omitted from this table.) This is a chiral theory; cancellation of gauge anomalies requires $m_f + 2N = \tilde{m}_f + M - 4$, and $M + n_f$ even. The $R$ charges can be taken to be

$$R_Q = 1 - \frac{(k+1)(2[M - 2N] + m_f + n_f - \tilde{m}_f) + 2N}{2(k+1)m_f},$$

$$R_{\tilde{Q}} = 1 - \frac{(k+1)(-m_f - n_f + \tilde{m}_f) + 2(M + 2k)}{2(k+1)\tilde{m}_f},$$

$$R_S = 1 - \frac{2(k+1)(2N - M + 2) + M}{2(k+1)n_f}.$$

### 12.1. The dual theory

The dual theory is $SU(\tilde{M}) \times Sp(\tilde{N})$ where

$$\tilde{M} = (k+1)(n_f + m_f + \tilde{m}_f - 4) - m_f - 2N,$$

$$2\tilde{N} = (k+1)(n_f + m_f + \tilde{m}_f - 4) - n_f - M.$$

Taking the singlets $M_j, \ldots, \tilde{R}_r$ to transform as the mesons of the electric theory, the magnetic theory (2.37) has charged matter in the representations

|        | $SU(\tilde{M})$ | $Sp(\tilde{N})$ | $SU(m_f)$ | $SU(\tilde{m}_f)$ | $SU(n_f)$ | $U(1)_R$ |
|--------|-----------------|-----------------|-----------|-------------------|-----------|-----------|
| $q$    | $\tilde{M}$     | 1               | 1         | 1                 | $m_f$     | $R_q$     |
| $q'$   | $\tilde{M}$     | 1               | 1         | $\tilde{m}_f$     | 1         | $R_{q'}$  |
| $s$    | 1               | $2\tilde{N}$    | $\tilde{m}_f$ | 1                 | 1         | $R_s$     |
| $Y$    | $\tilde{M}$     | 1               | 1         | 1                 | 1         | $R_Y = \frac{1}{2(k+1)}$ |
| $\tilde{Y}$ | asym | 1               | 1         | 1                 | 1         | $R_{\tilde{Y}} = \frac{1}{2(k+1)}$ |

The formulas for the $R$ charges are the same as in the electric theory with $M, N$ replaced by $\tilde{M}, \tilde{N}$ and with $m_f$ exchanged with $n_f$.

### 13. $SU(M) \times Sp(N) \times Sp(N')$

Consider the $SU(M) \times Sp(N) \times Sp(N')$ theories described in sect. 2.16. These models have an anomaly-free $SU(m_f) \times SU(\tilde{m}_f) \times SU(n_f) \times SU(n'_f) \times U(1)_R$ flavor symmetry with matter fields in the representations
(There are two additional $U(1)$ symmetries which we have omitted from this table.) This is a chiral theory; cancellation of gauge anomalies requires $m_f + 2N = \tilde{m}_f + 2N'$, and $M + n_f, M + n'_f$ both even. The $R$ charges can be taken to be

\[
R_Q = 1 - \frac{(k+1)(2[M - 2N] + m_f + n_f - \tilde{m}_f - n'_f) + 2N}{2(k+1)m_f},
R_{\tilde{Q}} = 1 - \frac{(k+1)(2[M - 2N'] - m_f - n_f + \tilde{m}_f + n'_f) + 2N'}{2(k+1)\tilde{m}_f},
R_S = 1 - \frac{2(k+1)(2N - M + 2) + M}{2(k+1)n_f},
R_{\tilde{S}'} = 1 - \frac{2(k+1)(2N' - M + 2) + M}{2(k+1)n'_f}.
\]

13.1. The dual theory

The dual theory is $SU(\tilde{M}) \times Sp(\tilde{N}) \times Sp(\tilde{N}')$ where

\[
\tilde{M} = (k+1)(n_f + n'_f + m_f + \tilde{m}_f - 4) - m_f - 2N,
2\tilde{N} = (k+1)(n_f + n'_f + m_f + \tilde{m}_f - 4) - n_f - M,
2\tilde{N}' = (k+1)(n_f + n'_f + m_f + \tilde{m}_f - 4) - n'_f - M.
\]

Taking the singlets $M_j, \ldots, \tilde{R}'_r$ to transform as the mesons of the electric theory, the magnetic theory (2.39) has charged matter in the representations
The formulas for the $R$ charges are the same as in the electric theory with $M, N, N'$ replaced by $\tilde{M}, \tilde{N}, \tilde{N}'$ and with $m_f, \tilde{m}_f$ exchanged with $n_f, n'_f$.

**14. SU($M$) × Sp($N$) with a symmetric tensor of SU($M$)**

Consider the $SU(M) \times Sp(N)$ theories described in sect. 2.17. These models have an anomaly-free $SU(m_f) \times SU(\tilde{m}_f) \times SU(n_f) \times U(1)_R$ flavor symmetry with matter fields in the representations

|       | SU($\tilde{M}$) | Sp($\tilde{N}$) | SU($\tilde{N}'$) | SU($m_f$) | SU($\tilde{m}_f$) | SU($n_f$) | SU($n'_f$) | U(1)$_R$ |
|-------|-----------------|-----------------|-----------------|-----------|-----------------|-----------|-----------|-----------|
| $q$   | $\tilde{M}$     | 1               | 1               | 1         | 1               | $\tilde{n}_f$ | 1         | $R_q$     |
| $q'$  | $\tilde{M}$     | 1               | 1               | 1         | 1               | $\tilde{n}_f'$ | 1         | $R_{q'}$  |
| $s$   | 1               | 2$\tilde{N}$    | 1               | $\tilde{m}_f$ | 1         | 1         | 1         | $R_s$     |
| $s'$  | 1               | 1               | 2$\tilde{N}'$   | 1         | $\tilde{m}_f$ | 1         | 1         | $R_{s'}$  |
| $Y$   | $\tilde{M}$     | 2$\tilde{N}$    | 1               | 1         | 1               | 1         | 1         | $R_Y = \frac{1}{2(k+1)}$ |
| $\tilde{Y}$ | $\tilde{M}$     | 1               | 2$\tilde{N}'$   | 1         | 1               | 1         | 1         | $R_{\tilde{Y}} = \frac{1}{2(k+1)}$ |

(There are two additional $U(1)$ symmetries which we have omitted from this table.) This is a chiral theory; cancellation of gauge anomalies requires $m_f + 2N = \tilde{m}_f + M + 4$, and $M + n_f$ even. The $R$ charges can be taken to be

\[
R_Q = 1 - \frac{2(k+1)(2[M - 2N] + m_f + n_f - \tilde{m}_f) + 2N}{4(k+1)m_f},
\]

\[
R_{\tilde{Q}} = 1 - \frac{2(k+1)(-m_f - n_f + \tilde{m}_f) + 2(M + 2k)}{4(k+1)\tilde{m}_f},
\]

\[
R_S = 1 - \frac{4(k+1)(2N - M + 2) + M}{4(k+1)n_f}.
\]

(14.1)
14.1. The dual theory

The dual theory is $SU(\tilde{M}) \times Sp(\tilde{N})$ where

\[
\tilde{M} = 2(k+1)(n_f + m_f + \tilde{m}_f) - m_f - 2N,
\]
\[
2\tilde{N} = 2(k+1)(n_f + m_f + \tilde{m}_f) - n_f - M.
\]

Taking the singlets $M_j, \ldots, \tilde{R}_r$ to transform as the mesons of the electric theory, the magnetic theory (2.41) has charged matter in the representations

|       | $SU(\tilde{M})$ | $Sp(\tilde{N})$ | $SU(m_f)$ | $SU(\tilde{m}_f)$ | $SU(n_f)$ | $U(1)_R$ |
|-------|-----------------|-----------------|-----------|-------------------|------------|-----------|
| $q$   | $\tilde{M}$     | 1               | 1         | 1                 | $\tilde{n}_f$ | $R_q$     |
| $q'$  | $\tilde{M}$     | 1               | 1         | $\tilde{m}_f$     | 1          | $R_{q'}$  |
| $s$   | 1               | $2\tilde{N}$    | $\tilde{m}_f$ | 1                 | 1          | $R_s$     |
| $Y$   | $\tilde{M}$     | $2\tilde{N}$    | 1         | 1                 | 1          | $R_Y = \frac{1}{2(k+1)}$ |
| $\tilde{Y}$ | sym            | 1               | 1         | 1                 | 1          | $R_{\tilde{Y}} = \frac{1}{2(k+1)}$ |

The formulas for the $R$ charges are the same as in the electric theory with $M, N$ replaced by $\tilde{M}, \tilde{N}$ and with $m_f$ exchanged with $n_f$.

15. $SU(M) \times SO(N)$ with an antisymmetric tensor of $SU(M)$

Consider the $SU(M) \times SO(N)$ theories described in sect. 2.18. These models have an anomaly-free $SU(m_f) \times SU(\tilde{m}_f) \times SU(n_f) \times U(1)_R$ flavor symmetry with matter fields in the representations

|       | $SU(M)$ | $SO(N)$ | $SU(m_f)$ | $SU(\tilde{m}_f)$ | $SU(n_f)$ | $U(1)_R$ |
|-------|---------|---------|-----------|-------------------|------------|-----------|
| $Q$   | $M$     | 1       | $m_f$     | 1                 | 1          | $R_Q$     |
| $Q'$  | $M$     | 1       | 1         | $\tilde{m}_f$     | 1          | $R_{Q'}$  |
| $S$   | 1       | $N$     | 1         | 1                 | $n_f$      | $R_S$     |
| $X$   | $M$     | $N$     | 1         | 1                 | 1          | $R_X = \frac{1}{2(k+1)}$ |
| $\tilde{X}$ | asym | 1       | 1         | 1                 | 1          | $R_{\tilde{X}} = \frac{1}{2(k+1)}$ |
(There are two additional $U(1)$ symmetries which we have omitted from this table.) This is a chiral theory; cancellation of gauge anomalies requires $m_f + N = \tilde{m}_f + M - 4$. The $R$ charges can be taken to be

\[
R_Q = 1 - \frac{2(k + 1)(2[M - N] + m_f + n_f - \tilde{m}_f) + N}{4(k + 1)m_f},
\]
\[
R_{\tilde{Q}} = 1 - \frac{2(k + 1)(-m_f - n_f + \tilde{m}_f) + 2(M - 2)}{4(k + 1)\tilde{m}_f},
\]
\[
R_S = 1 - \frac{4(k + 1)(N - M - 2) + M}{4(k + 1)n_f}.
\]

(15.1)

15.1. The dual theory

The dual theory is $SU(\tilde{M}) \times SO(\tilde{N})$ where

\[
\tilde{M} = 2(k + 1)(n_f + m_f + \tilde{m}_f) - m_f - N,
\]
\[
\tilde{N} = 2(k + 1)(n_f + m_f + \tilde{m}_f) - n_f - M.
\]

Taking the singlets $M_j, \ldots, \tilde{R}_r$ to transform as the mesons of the electric theory, the magnetic theory (2.43) has charged matter in the representations

|          | $SU(\tilde{M})$ | $SO(\tilde{N})$ | $SU(m_f)$ | $SU(\tilde{m}_f)$ | $SU(n_f)$ | $U(1)_R$ |
|----------|----------------|-----------------|------------|-------------------|------------|-----------|
| $q$      | $\tilde{M}$    | 1               | 1          | 1                 | $\tilde{m}_f$ | $R_q$     |
| $q'$     | $\tilde{M}$    | 1               | 1          | $\tilde{m}_f$    | 1          | $R_{q'}$  |
| $s$      | 1              | $\tilde{N}$    | $\tilde{m}_f$ | 1                 | 1          | $R_s$     |
| $Y$      | $\tilde{M}$    | $\tilde{N}$    | 1          | 1                 | 1          | $R_Y = \frac{1}{2(k+1)}$ |
| $Y'$     | $\text{asym}$ | 1               | 1          | 1                 | 1          | $R_{\tilde{Y}} = \frac{1}{2(k+1)}$ |

The formulas for the $R$ charges are the same as in the electric theory with $M, N$ replaced by $\tilde{M}, \tilde{N}$ and with $m_f$ exchanged with $n_f$.

16. $SU(M) \times Sp(N) \times SO(N')$

Consider the $SU(M) \times Sp(N) \times SO(N')$ theories described in sect. 2.19. These models have an anomaly-free $SU(m_f) \times SU(\tilde{m}_f) \times SU(n_f) \times SU(n_f') \times U(1)_R$ flavor symmetry with matter fields in the representations
(There are two additional $U(1)$ symmetries which we have omitted from this table.) This is a chiral theory; cancellation of gauge anomalies requires $m_f + 2N = \tilde{m}_f + N'$ and $M + n_f$ be even. The $R$ charges can be taken to be

$$
R_Q = 1 - \frac{2(k+1)(2[M - 2N] + m_f + n_f - \tilde{m}_f - n'_f - 4) + 2N}{4(k+1)m_f},
$$

$$
R_{\tilde{Q}} = 1 - \frac{2(k+1)(2[M - N'] - m_f - n_f + \tilde{m}_f + n'_f + 4) + N'}{4(k+1)\tilde{m}_f},
$$

$$
R_S = 1 - \frac{4(k+1)(2N - M + 2) + M}{4(k+1)n_f},
$$

$$
R_{S'} = 1 - \frac{4(k+1)(N' - M - 2) + M}{4(k+1)n'_f}.
$$

16.1. The dual theory

The dual theory is $SU(\tilde{M}) \times Sp(\tilde{N}) \times SO(\tilde{N}')$ where

$$
\tilde{M} = 2(k+1)(n_f + n'_f + m_f + \tilde{m}_f) - \tilde{m}_f - N',
$$

$$
2\tilde{N} = 2(k+1)(n_f + n'_f + m_f + \tilde{m}_f) - n_f - M,
$$

$$
\tilde{N}' = 2(k+1)(n_f + n'_f + m_f + \tilde{m}_f) - n'_f - M.
$$

Taking the singlets $M_j, \ldots, \tilde{R}'_r$ to transform as the mesons of the electric theory, the magnetic theory (2.45) has charged matter in the representations

| $Q$ | $M$ | 1 | 1 | $m_f$ | 1 | 1 | 1 | $R_Q$ |
| $Q'$ | $\overline{M}$ | 1 | 1 | 1 | $\tilde{m}_f$ | 1 | 1 | $R_{Q'}$ |
| $S$ | 1 | 2N | 1 | 1 | 1 | $n_f$ | 1 | $R_S$ |
| $S'$ | 1 | 1 | $N'$ | 1 | 1 | 1 | $n'_f$ | $R_{S'}$ |
| $X$ | $M$ | 2N | 1 | 1 | 1 | 1 | 1 | $R_X = \frac{1}{4(k+1)}$ |
| $\tilde{X}$ | $\overline{M}$ | 1 | $N'$ | 1 | 1 | 1 | 1 | $R_X = \frac{1}{4(k+1)}$ |
The formulas for the $R$ charges are the same as in the electric theory with $M, N, N'$ replaced by $\tilde{M}, \tilde{N}, \tilde{N}'$ and with $m_f, \tilde{m}_f$ exchanged with $n_f, n'_f$.

17. Conclusion

We have presented many generalizations of previously known examples of duality in $N=1$ supersymmetry [3,12-15]. In particular we have discussed chiral models which are very similar in appearance to vector-like ones. The models are all interconnected under superpotential perturbations, symmetry breaking and confinement. We have by no means presented a complete list of models of this type; many more remain to be explored. Furthermore, much work needs to be done to clarify certain aspects of the physics of these theories, and to discover the duality in similar models without a superpotential. Still, we hope that these new examples will help guide us to an understanding of duality in supersymmetric field theory.

Acknowledgments

We would like to thank N. Seiberg for discussions. This work was supported in part by DOE grant #DE-FG05-90ER40559. R.G.L. thanks the CERN Theory Division, and R.G.L. and M.J.S. thank the Aspen Center for Physics, where part of this work was done.
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