Hidden Kerr/CFT at finite frequencies

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Abstract

Massless fields propagating in a generic Kerr black hole background enjoy a hidden $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ symmetry. We determine how the exact mode functions decompose into representations of this symmetry group. This extends earlier results on the low frequency limit of the massless scalar case to finite frequencies and general spin. As an application, we numerically determine the parameters of the representations that appear in quasinormal modes. These results represent a first step to formulating a more precise mapping to a holographic dual conformal field theory for generic black holes.
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References
I. INTRODUCTION

It is a curious result that the entropy formula for a generic Kerr black hole can be simply explained using a two-dimensional conformal field theory at finite temperature as a holographic dual [1]. This has motivated attempts to determine the properties of such a conformal field theory. In the near extremal-limit there has been some success in this direction [2–6]. In addition, results have been claimed in the low frequency limit of generic Kerr. A review of these various approaches is [7]. Clearly this is a very important problem, as a complete description of the holographic theory could lead to an exact quantum description of black holes beyond the semiclassical limit commonly studied, or the exact descriptions found in special limits in string theory.

In the present work we study in more detail the symmetry structure of the equations of motion for massless fields of general spin in a generic Kerr background without taking any other special limits. We find a hidden $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ symmetry structure which matches that of the global conformal group of a two-dimensional conformal field theory.

In order to construct the parameters that characterize the representations that appear, it is necessary to solve an eigenvalue problem that may be expressed as a continued fraction equation. The solutions to this problem are exhibited as a low frequency expansion, and also computed numerically for quasinormal modes. The highly damped quasinormal modes are thought to have a close connection to the exact description of black hole entropy, and hence the holographic conformal field theory. We develop the numerical solution of the eigenvalue problem in this limit to determine the associated representations.

The Kerr mode functions lead to non-unitary representations of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ for the mode functions which reflect the non-invariance of the coordinate patch under this group. However the representations that appear match exactly what one expects of the BTZ black hole, for which the correspondence between gravity and a CFT at finite left/right temperatures is well-understood [8]. These results provide useful clues and constraints on the structure of the holographic dual to a generic Kerr black hole.
II. \( SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \) AND MASSLESS FIELDS IN KERR

The equations of motion of a massless field in the Kerr geometry can be organized using the symmetry group \( SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \). To see this, we use Teukolsky’s separation of the equation of motion [9] in Boyer-Lindquist coordinates:

\[
    ds^2 = \frac{\Delta}{\rho^2} \left( dt - a \sin^2 \theta d\phi \right)^2 - \frac{\sin^2 \theta}{\rho^2} \left( (r^2 + a^2) d\phi - adt \right)^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2,
\]

defining \( \Delta = r^2 - 2Mr + a^2 \), \( \rho^2 = r^2 + a^2 \cos^2 \theta \). The solution takes the form, for angular quantum numbers \( \ell \) and \( m \)

\[
    \psi = e^{-i\omega t} e^{im\phi} S^m_{\ell}(\theta) R_{\omega \ell m}(r),
\]

with \( S^m_{\ell} \) being a spin weighted spheroidal harmonic, dependent on the spin weight \( s \). This satisfies the angular equation

\[
    \left( \frac{d}{dy} (1 - y^2) \frac{d}{dy} + \frac{1}{4} \epsilon^2 y^2 - sq\epsilon y - \frac{m^2 + s^2 + 2msy}{1 - y^2} + E \right) S^m_{\ell}(y) = 0, \tag{1}
\]

with angular eigenvalue \( E \). The radial equation takes the form

\[
    \left( \Delta^{-s} \frac{d}{dr} \Delta^{s+1} \frac{d}{dr} + [V_0(r) + V_{\omega \ell m}(r)] \right) R_{\omega \ell m}(r) = 0, \tag{2}
\]

where the potentials \( V_0 \) and \( V_{\omega \ell m} \) are defined as

\[
    V_0(r) = \frac{r_+ - r_-}{r - r_+} \omega_+ (\omega_+ - is) - \frac{r_+ - r_-}{r - r_-} \omega_- (\omega_- + is)
\]
\[
    V_{\omega \ell m}(r) = s(s+1) - E + \epsilon(\epsilon - is) + r^2 \omega^2 + r\omega(\epsilon + 2is),
\]

and we have introduced

\[
    \omega_\pm = \frac{2Mr_\pm \omega - am}{r_+ - r_-}, \quad \epsilon = 2M\omega, \quad q = \frac{a}{M}, \quad y = \cos \theta.
\]

Around a low frequency limit the radial equation solutions may be expanded in terms of hypergeometric functions [10] [11]

\[
    R_{\omega \ell m}(x) = e^{i\epsilon x} (-x)^{-s - \frac{1}{2}(\epsilon + \tau)} (1 - x)^{\frac{1}{2}(\epsilon - \tau)} \sum_{n=-\infty}^{\infty} a_{\nu}^{\ell} \, _2F_1 \left( n + \nu + 1 - i\tau, -n - \nu - i\tau; 1 - s - i\epsilon - i\tau; x \right), \tag{3}
\]
where the solution is chosen to satisfy ingoing boundary conditions at the horizon. Here we find it convenient to use $x = \omega (r_+ - r) / \epsilon \kappa$, $\kappa = \sqrt{1 - q^2}$, $\tau = (\epsilon - mq) / \kappa$. The parameter $\nu$ will ultimately determine the $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ representations that appear in the mode function, and will be determined momentarily. In a low frequency expansion, $\nu = \ell + \mathcal{O}(\epsilon^2)$. For sufficiently small $\epsilon$ usually a single term in the expansion survives at leading order. Remarkably, it was found the series converges for all $r < \infty$ beyond the small $\epsilon$ limit.

If we define

$$B_{\omega \ell m}(x) = (-x)^s/2 (1 - x)^s/2 e^{-i\kappa x} P^\nu_{\omega \ell m} (x), \quad (4)$$

we transform the radial equation into

$$\left( \frac{d}{dx} \Lambda \frac{d}{dx} + \left[ \dot{\omega}^2 + \frac{\dot{\omega}^2}{x} + 1 - x - \nu (\nu + 1) \right] \right) B_{\omega \ell m} =$$

$$= - \left[ 2i\kappa \epsilon \Lambda \frac{d}{dx} - i\kappa (s - 1) \frac{d\Lambda}{dx} - \epsilon^2 \kappa \frac{d\Lambda}{dx} - E + \nu (\nu + 1) + \frac{\epsilon^2}{4} (7 + k^2) \right] B_{\omega \ell m} \quad (5)$$

where

$$\Lambda = -x(1 - x), \quad \dot{\omega}_\pm = \frac{s}{2} \pm i\omega.$$

Now the first line of (5) is (up to a trivial modification) what is known as the $q$-form of the hypergeometric equation. This form is useful for exhibiting the $SL(2, \mathbb{R})$ structure of the solution. There are two ways to rewrite this first line as a quadratic Casimir of $SL(2, \mathbb{R})$, so all together we find a $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ structure. The generators are

$$J^\pm = e^{\pm 2\pi T_R \phi} \left[ -\sqrt{\Lambda} \partial_x \pm \frac{M}{T_R} \frac{T_L \Lambda' - T_R}{\sqrt{\Lambda}} \partial_t \pm \frac{1}{4\pi T_R} \frac{\Lambda'}{\sqrt{\Lambda}} \partial_\phi \pm \frac{s}{2\sqrt{\Lambda}} \right] \quad (6)$$

$$J^3 = 2M \frac{T_L}{T_R} \partial_t + \frac{1}{2\pi T_R} \partial_\phi$$

and

$$\bar{J}^\pm = e^{\pm (2\pi T_L \phi - t/2M)} \left[ -\sqrt{\Lambda} \partial_x \pm \frac{M}{T_R} \frac{T_L - T_R \Lambda'}{\sqrt{\Lambda}} \partial_t \pm \frac{1}{4\pi T_R} \frac{1}{\sqrt{\Lambda}} \partial_\phi \pm \frac{s\Lambda'}{2\sqrt{\Lambda}} \right] \quad (7)$$

$$\bar{J}^3 = -2M \partial_t + s$$

where we have introduced the parameters

$$T_R = \frac{r_+ - r_-}{4\pi a}, \quad T_L = \frac{r_+ + r_-}{4\pi a}. \quad (8)$$

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These generators satisfy the $SL(2, \mathbb{R})$ algebra in the form

\[ [J^+, J^-] = 2J^3 \]
\[ [J^\pm, J^3] = \mp J^\pm, \]

and likewise for barred generators. The barred and unbarred generators commute. This generalizes the proposed algebra of [1] to general spin weight $s$, up to algebra isomorphism. The Casimir operators

\[ C = J^+ J^- + J^3 J^3 - J^3 \]
\[ \bar{C} = \bar{J}^+ \bar{J}^- + \bar{J}^3 \bar{J}^3 - \bar{J}^3 \]

each reproduce the term in (5)

\[ C = \bar{C} = \frac{d}{dx} \Lambda \frac{d}{dx} + \frac{\hat{\omega}_+^2}{x} + \frac{\hat{\omega}_-^2}{1-x}. \]

It is worth pointing out that differential generators (6) and (7) appear in the literature on the relation between representations of $SL(2, \mathbb{R})$ and hypergeometric functions [12]. A related $SU(2)$ algebra may also be defined for the angular equation, and was used in [13] to give a compact derivation of the Press-Teukolsky identities [14]. All together, the expansion for a mode function in the Kerr geometry exhibits a hidden $SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \times SU(2) \times SU(2)$ symmetry, which curiously matches the near-horizon isometry symmetry of the string theory black holes for which the microscopic entropy counting is well-understood.

From (3) we observe $R_{\omega \ell m}(x)$ is invariant under the exchange $\nu \to -\nu - 1, n \to -n$. In order to build representations of $SL(2, \mathbb{R})$ under the action of (6) and (7), we decompose $B_{\omega \ell m}(x)$ into two independent modes:

\[ B_{\omega \ell m}(x) = \sum_{n=-\infty}^{\infty} \left[ \tilde{a}_n B^{n+\nu}_{\omega \ell m}(x) + \tilde{a}_{n-1} B^{n-\nu-1}_{\omega \ell m}(x) \right], \]

where

\[ B^{n+\nu}_{\omega \ell m}(x) = (-x)^{n+\nu-\frac{1}{2} - \frac{1}{2}(\epsilon-\tau)}(1-x)^{-\frac{1}{2} + \frac{1}{2}(\epsilon-\tau)} \]
\[ 2F_1(-n-\nu - i\tau, -n-\nu + s + i\epsilon; -2n - 2\nu; 1/x) \]

(9)

and
\[
\ddot{a}_n' = a_n' \frac{\Gamma (1 - s - i \epsilon - i \tau) \Gamma (2n + 2 \nu + 1)}{\Gamma (n + \nu + 1 - i \tau) \Gamma (n + \nu + 1 - s - i \epsilon)}.
\]

The invariance of \(B_{\omega \ell m}(x)\) under \(\{\nu \to -\nu - 1, n \to -n\}\) is enforced via the assignment \(a_0' = a_0^{-\nu-1}\).

One can define the irreducible \(SL(2, \mathbb{R})\) representations we will be interested in as realized on a set of basis functions \(f_{m_0}^u\) via

\[
\begin{align*}
J^3 f_{m_0}^u &= m_0 f_{m_0}^u, \\
J^\pm f_{m_0}^u &= (-u \pm m_0) f_{m_0 \pm 1}^u.
\end{align*}
\]

With the help of the hypergeometric identity

\[
\frac{d}{dz} [z^a F(a, b, c; z)] = az^{a-1} F(a + 1, b, c; z),
\]

where \(z = 1/x\), we conclude that \(e^{-i\omega t + im\phi} B_{\omega \ell m}^{n+\nu}(x) = f_{-i\tau}^{n+\nu}\) under the action of (6). For barred generators \(\{7\}\), we swap the first two arguments in the hypergeometric function and use the same identity to show \(e^{-i\omega t + im\phi} B_{\omega \ell m}^{n+\nu}(x) = f_{s+i\epsilon}^{n+\nu}\).

In the notation \(D(u, m_0)\) of \([12]\), with respect to the product algebra \(SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R\) the representation associated with the \(n\)-th term of the expansion (4) is a direct product

\[
D(n + \nu, s + i2M\omega)_L \times D\left(n + \nu, -i\frac{2M^2\omega - am}{\sqrt{M^2 - a^2}}\right)_R,
\]

which is a non-unitary representation of the algebra, with Casimir \((|n| + \nu)(|n| + \nu + 1)\). Later we will see that \(\nu\) will be real provided \(\omega\) is real, but will be complex for quasinormal modes. With respect to the generators defined above, the weights of representations are \(ik_L - 2M\omega, ik_R - \frac{2M^2\omega - am}{\sqrt{M^2 - a^2}}\), where \(k_{L,R}\) is an integer. The condition that the representations collapse to a highest weight or lowest weight representation is that \(\nu + i2M\omega\) or \(\nu + i\frac{2M^2\omega - am}{\sqrt{M^2 - a^2}}\) is an integer. This is generally not satisfied for real non-vanishing frequencies or momenta.

The original motivation for the expansion (3) was as a low frequency expansion. As we have seen, this is equivalent to organizing the expansion according to the \(SL(2, \mathbb{R}) \times SL(2, \mathbb{R})\) symmetry, which moreover leads to a convergent expansion for general frequencies\(^1\).

\(^1\) In \([15]\) a more general one-parameter family of related \(SL(2, \mathbb{R}) \times SL(2, \mathbb{R})\) symmetries was found in the
This expansion straightforwardly produces low frequency scattering amplitudes. In particular, the results of Page/Starobinsky \cite{16,17} can easily be recovered by retaining only the $n = 0$ term in (3), as shown in \cite{10}. We discuss this point further in section III.C.

We will find the parameter $\nu$ becomes a function of frequency, determined by picking out a convergent solution to (3). We solve for this parameter in various different limits. But first we will investigate in more detail the connection to conformal field theory.

### III. CFT/GRAVITY MAPPING

In the above we have shown how a general mode function decomposes into irreducible representations of the $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ algebra. To make the meaning of these representations more clear it is helpful to compare to the analogous computation for the three-dimensional black hole in asymptotically anti-de Sitter spacetime \cite{18}, where the same symmetry structure appears, and the holographic dictionary is well-known.

#### A. BTZ example

Quasinormal modes have been studied in this context in \cite{8} so we find it helpful to follow their notation. The BTZ metric can be written in the form

$$ds^2 = \frac{dz^2}{4z(1-z)^2} + \frac{1}{1-z} \left( -r_- dt + r_+ d\phi \right)^2 - \frac{z}{1-z} \left( r_+ dt - r_- d\phi \right)^2,$$

where infinity is $z = 1$ and the outer horizon sits at $z = 0$. Here $r_+$ and $r_-$ are the radii of the outer and inner horizons. These are related to left and right temperatures in the CFT via (8) by replacing $a \to 1/2$:

$$T_R = \frac{r_+ - r_-}{2\pi}, \quad T_L = \frac{r_+ + r_-}{2\pi}.$$

For a scalar field of mass $\tilde{m}$, or a vector field of mass $\tilde{m}$ (with spin parameter $s = \pm 1$), an analog of the Teukolsky solution is

$$\Phi = e^{-ik_+ x^+ - ik_- x^-} B(z)$$

low frequency limit. In the present work, we are able to treat the finite frequency regime, which seems to single out the particular $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ considered here.
where
\[ x^+ = r_+ t - r_- \phi, \quad x^- = r_+ \phi - r_- t \]
and
\[ k_+ + k_- = \frac{\omega - k}{2\pi T_R}, \quad k_+ - k_- = \frac{\omega + k}{2\pi T_L}. \]

The radial equation takes the hypergeometric form
\[
z(1 - z) \frac{d^2 B}{dz^2} + (1 - z) \frac{dB}{dz} + \left[ \frac{k_+^2}{4z} - \frac{k_-^2}{4z} - \frac{\tilde{m}^2 + 2s\tilde{m}}{4(1 - z)} \right] B = 0 \tag{13}
\]
with solution
\[
B(z) = z^\alpha (1 - z)^\beta _2 F_1(a, b; c; z) \tag{14}
\]
with
\[
\alpha = -\frac{ik_+}{2}, \quad \beta = \frac{1}{2} \left( 1 - \sqrt{1 + \tilde{m}^2 + 2s\tilde{m}} \right),
\]
\[
a = \frac{k_+ - k_-}{2i} + \beta, \quad b = \frac{k_+ + k_-}{2i} + \beta, \quad c = 1 + 2\alpha.
\]

The radial equation (13) is of the same form as the first line in (5) with the replacement
\[ B(z) \to (1 - z)^{1/2} \tilde{B}(z). \]

However one subtlety we should address is that a given hypergeometric equation has 24 different equivalent ways of writing the solution in terms of hypergeometric functions. These different solutions are related by Kummer transformations. From the viewpoint of the black hole, these presumably correspond to mode functions in different coordinate patches. To compare with the form of the solution (9), where \( x \) runs over the range \((-\infty, 0)\) we perform a \( z \to y = 1 - 1/z \) Kummer transformation which maps the solution (14) to
\[
B(x) = (-y)^{c-a-b+\beta} (1 - y)^{b-a-\beta} _2 F_1(c - a, 1 - a; c + 1 - a - b; y).
\]

From here we may read off the parameters of the \( SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \) representation. In the notation \( D(\nu, m_0) \) of [12] we find
\[
D_L \left( \beta - 1, i \frac{k_+ - k_-}{2} \right) \times D_R \left( \beta - 1, -i \frac{k_+ + k_-}{2} \right) \tag{15}
\]
for which the quadratic Casimir is
\[
C = \bar{C} = \frac{\tilde{m}^2 + 2s\tilde{m}}{4}.
\]
These are non-unitary representations of $SL(2, \mathbb{R})$'s although the Casimir agrees with what
we expect for a discrete highest weight irreducible representation of $SL(2, R)$ with conformal
weight $h = \beta$. The non-unitarity may be attributed to the fact that the external coordinate
patch of the BTZ black hole (11) is not invariant under global AdS isometries. When
one takes a pure AdS limit, the coordinates (11) only cover one of an infinite number of
patches needed to cover AdS (or more precisely the covering space of AdS). Thus, while
modes on the covering space of AdS transform as a unitary highest weight representation
of $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ they are related to a nontrivial composition of non-unitary repre-
sentations of modes on different patches. At the level of the mode functions, the mapping
between the global mode functions and the BTZ patch mode functions will (and conse-
quently the group representations will) follow analogous results in [20] for Rindler versus
Minkowski spacetime. We conclude that the non-unitary representations (15) for a sin-
gle BTZ patch can be viewed as descending from a unitary highest weight representation
of $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ corresponding to a primary operator in the CFT with conformal
dimension $\beta$.

B. Conjecture for Kerr/CFT

We conclude that the representations of $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ for BTZ modes (15) match
exactly those of Kerr (10). This leads us to conjecture that an exact mode of Kerr may
be reconstructed from some more fundamental primary conformal field of weight $\nu$ and its
descendants.

Let us note that the expansion for a Kerr mode (3) may be expressed as a sum from
$n = 0$ to $\infty$ simply by using the symmetry of the hypergeometric function under its first
two arguments. This swaps $\nu \to -\nu - 1$ and $n \to -n$. As is well-known in the AdS/CFT
correspondence, both terms may be viewed as originating from correlators of a conformal
primary of weight $\nu$ in the presence of a source term [21].

A new issue that arises in Kerr/CFT is that of the proper normalization of the modes
with physical boundary conditions at spatial infinity. Certainly one may use the expansion
(3) to compute scattering correlators on some surface of large fixed $r$, but to properly impose

\[2\] This is treated explicitly for $AdS_4$ in [19], and the result for $AdS_3$ follows by decomposing the highest
weight representations of $SO(3, 2)$ into $SO(2, 2)$. 

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incoming/outgoing boundary conditions at spatial infinity, another expansion must be used that is convergent at \( r = \infty \). Such an expansion is crucial for obtaining quasinormal mode boundary conditions, or computing genuine scattering amplitude computations from past infinity to future infinity. To achieve this at the level of mode functions, one must instead expand in terms of a different set of basis functions. In [10, 11, 22] this is chosen to be a set of Coulomb functions, though other choices are possible [23].

From the CFT point of view, one may view the evolution from \( r = \infty \) to finite \( r \) as a renormalization group flow. However we know little about the holographic dual of asymptotically flat spacetime from which the theory is flowing to the Kerr/CFT in the infrared. Without such a description we cannot formulate the proper boundary conditions in the holographic dual that would allow us, for example, to reproduce quasinormal mode frequencies. Some related works that study the problem of holographic duals to asymptotically flat spacetime include [24–26].

### C. Absorption probability

By matching the solutions obtained via an expansion in Coulomb functions convergent at \( r = \infty \) to the expansion (3) valid at all finite \( r \), and with the help of Teukolsky-Starobinsky identities and a few other simplifying relations, [10] write down the exact Kerr absorption rate for all finite frequencies:

\[
\sigma_{\text{abs}} = \frac{(2\epsilon \kappa)^{2\nu + 1} e^{\pi \epsilon}}{\pi} \sinh \pi (\epsilon + \tau) \frac{D_{\nu}}{|N_{\nu}|^2}
\]

where

\[
N_{\nu} = 1 + \frac{i}{\pi} (2\epsilon \kappa)^{2\nu + 1} (-1)^{2\nu} e^{i\pi \nu} \sin \pi (\nu - i\tau) \left( \frac{\sin \pi (\nu - s - i\epsilon)}{\sin 2\pi \nu} \right)^2 D_{\nu}
\]

\[
D_{\nu} = \left| \frac{\Gamma (\nu + 1 - i\tau) \Gamma (\nu + 1 - s + i\epsilon) \Gamma (\nu + 1 + s + i\epsilon)}{\Gamma (2\nu + 1) \Gamma (2\nu + 2)} \right|^2 d_{\nu}
\]

\[
d_{\nu} = \sum_{n \leq 0} \frac{(-1)^{n}}{(2\nu + 2)^n} \left( \frac{(\nu + 1 + s - i\epsilon)}{n!} \frac{d_{\nu}}{d_{\nu}} \right)^2
\]

\[
\times \sum_{n \geq 0} \frac{(2\nu + 1)^{n}}{n!} \left( \frac{(\nu + 1 + s - i\epsilon)}{n!} \frac{d_{\nu}}{d_{\nu}} \right)^{-2}
\]
In the small frequency approximation, $\epsilon \ll 1$, the absorption rate properly reproduces the Page formula $[16, 17]$. By keeping $O(\epsilon)$ terms in (16), we can set $E = \ell(\ell + 1) + O(\epsilon^2)$ in (5). Choosing the integer shift in $\nu$ so that $\nu = \ell + O(\epsilon)$ makes the $n = 0$ term the leading term in the series expansion.

To leading order in $\epsilon$, the formula (16) reduces to:

$$\sigma_{abs}^\epsilon = (2\epsilon\kappa)^{2\ell+1} \frac{e^{\pi\epsilon}}{\pi} \frac{\omega_R}{2T_R} \prod_{k=1}^{\ell} \left[k^2 + \left(\frac{\omega_R}{2\pi T_R}\right)^2\right] \times (1 + O(\epsilon))$$

where we introduced the frequencies

$$\omega_L = \frac{2M^2\omega}{a}, \quad \omega_R = \frac{2M^2\omega}{a} - m.$$

In the $\epsilon \to 0$ limit the dependence on the left-moving frequency $\omega_L \propto \epsilon$ is rather trivial, however there is a highly non-trivial dependence on the right-moving frequency which remains finite $\omega_R = -m$. This is a familiar behavior of the effective string absorption cross section for massless bosons in $N = 4$ supergravity $[27]$. With the representations surviving this limit we associate conformal weight $h_R = \ell + 1$, compatible with results of $[28]$ and conformal weight identification of $[27]$.

**Connection with extremal Kerr/CFT**

Encouraged with our findings, here we speculate on a possible connection with results obtained in the extremal limit $a \to M$. More precisely, we search for regime in which we find representations corresponding to near horizon extreme Kerr (NHEK) modes near the superradiant bound. In $[3]$ it was found that the absorption probability of the modes saturating the superradiant bound in the near-extremal Kerr background corresponds to a thermal CFT 2-point function:

$$\sigma \propto T_H^{2\beta} e^{\pi m} \sinh \pi \left(m + \frac{\omega - m/2M}{2\pi T_H}\right) \left|\Gamma\left(\frac{1}{2} + \beta + im\right)\right|^4 \left|\Gamma\left(\frac{1}{2} + \beta + i\frac{\omega - m/2M}{2\pi T_H}\right)\right|^2$$

where $\beta^2 = \frac{1}{4} - 2m^2 + \bar{A}_{lm}$, and $\bar{A}_{lm}$ is the angular eigenvalue evaluated for $a\omega = m/2$.

We do not observe this truncation of (16) for quasinormal mode excitations (QNM). In terms of the transmission and reflection coefficients, the QNMs correspond to frequencies at
which both $T$ and $R$ develop poles, in such way that $|T| \approx |R|$. There is also another set of modes one can consider, compatible with the purely outgoing boundary condition at infinity, called total reflection modes (TRM). These correspond to frequencies at which transmission coefficient vanishes, making them standing waves at $r = \infty$. We observe that total reflection modes with exact frequencies given by \[ \omega_{TRM} = m\Omega - 2\pi iT_H(n - s), \quad n \in \mathbb{N} \]
correctly reproduce the near-superradiant frequencies in the extremal limit. Here $\Omega = \frac{a}{r_+^2 + a^2}$ is the angular velocity at the outer horizon and $T_H = \frac{r_+ - r_-}{4\pi(r_+^2 + a^2)}$ is the Hawking temperature.

Taking the extremal limit on the parameters of our $SL(2, \mathbb{R})$ representations,

$$\lim_{a \to M} \frac{2M^2 \omega_{TRM} - ma}{\sqrt{M^2 - \alpha^2}} = n - s$$

we arrive at an interesting analogy: just as the BTZ scattering amplitudes for quasinormal mode frequencies reproduce the pole structure of a CFT 2-point function [8], so do the hidden Kerr/CFT representations in the case of extremal total reflection modes. As $\kappa \to 0$, the denominator in (16) is exactly equal to 1, and the Kerr absorption cross section reproduces (17), provided we identify $\nu + 1$ with $\beta + 1/2$.

The absorption cross section accounts for poles of a chiral 2-dimensional CFT 2-point function; we suspect the present understanding of NHEK asymptotic boundary conditions, providing an enhancement to one Virasoro only, can then be recast in terms of the monodromy analysis in the highly damped regime.

The fact that the total reflection modes have no bulk degrees of freedom corroborates with the findings of [30], which makes the extremal Kerr/CFT a topological theory. As we know how to count the microscopic degrees of freedom for $AdS_3$ quotients at any value of the angular momentum [31], it is reasonable to assume same can be achieved in Kerr/CFT. We expect that the two hidden $SL(2, \mathbb{R})$’s enhance to a full $\text{vir}_L \times \text{vir}_R$, with central charges given by $c_L = c_R = 12J$. As originally hinted in [1], the Cardy formula with this value of central charges, together with temperatures (8), exactly reproduces the classical Bekenstein-Hawking entropy of the Kerr black hole.
IV. EIGENVALUE EQUATION

The expansion of the mode functions (3) converges only for special values of the parameter $\nu$. To find this parameter a continued fraction is set up. Similar methods are used to construct the angular eigenvalue, and the frequency of quasinormal modes. To proceed, one expresses the radial equation (5) as a three-term recurrence relation

$$\alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} = 0$$  \hspace{1cm} (18)

where

$$\alpha_n = \frac{i\epsilon\kappa(n + \nu + 1 + s + i\epsilon)(n + \nu + 1 + s - i\epsilon)(n + \nu + 1 + i\tau)}{(n + \nu + 1)(2n + 2\nu + 3)}$$

$$\beta_n = -\lambda - s(s + 1) + (n + \nu)(n + \nu + 1) + \epsilon^2 + \epsilon(\epsilon - mq) + \frac{\epsilon(\epsilon - mq)(s^2 + \epsilon^2)}{(n + \nu)(n + \nu + 1)}$$

$$\gamma_n = \frac{-i\epsilon\kappa(n + \nu - s + i\epsilon)(n + \nu - s - i\epsilon)(n + \nu - i\tau)}{(n + \nu)(2n + 2\nu - 1)}$$

and we have defined

$$\lambda = E - s(s + 1) - 2ma\omega + a^2\omega^2.$$  

The eigenvalue equation is expressed using the continued fractions

$$R_n = \frac{a_n}{a_{n-1}} = -\frac{\gamma_n}{\beta_n + \alpha_n R_{n+1}}, \quad L_n = \frac{a_n}{a_{n+1}} = -\frac{\alpha_n}{\beta_n + \gamma_n L_{n-1}}.$$  \hspace{1cm} (19)

For general values of $\nu$ the solution to the recurrence relation (18) will diverge as $|n| \to \infty$. To avoid this and find the so-called minimal solution, one must demand that $\nu$ solve an additional eigenvalue equation

$$R_n L_{n-1} = 1.$$  \hspace{1cm} (20)

There is an equivalence of solutions under $\nu \to \nu + k$ where $k$ is an integer (apparently not noticed in [10, 11]). One convention, useful for real frequencies, is to choose the integer shift in $\nu$ such that $E - \nu(\nu + 1)$ term on the right-hand side of (5) is minimized, in order that the $n = 0$ term tends to be the leading order term in the expansion. Another convention, that will be useful when discussing quasinormal modes, will be to simply shift the real part of $\nu$ into the range $[0, 1/2)$ using the combined symmetries of the expansion under $\nu \to -\nu - 1$ and $\nu \to \nu + k$. 

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When \( \nu \) satisfies (20), one finds

\[
\lim_{n \to -\infty} nR_n = \lim_{n \to -\infty} nL_n = \frac{i \epsilon \kappa}{2}.
\]

This leads to convergence of the series (3) for all finite radii \( r \). [11]

A. Low frequency expansion

In a low frequency expansion, the solution of (20) is

\[
\nu = \ell - \epsilon^2 \frac{\ell(\ell + 1)(-11 + 15(1 + \ell)) + 6 (-1 + \ell + \ell^2) s^2 + 3 s^4}{2\ell(1 + 2\ell)(1 + \ell)(-1 + 2\ell)(3 + 2\ell)} + \epsilon^3 \frac{mq}{\ell(1 + \ell)(-1 + 2\ell)(1 + 2\ell)(3 + 2\ell)} \left[ 5\ell(1 + \ell) - 3 + \frac{s^2 (3\ell(\ell + 1)(\ell^2 + \ell - 3) + 11)}{(-1 + \ell)(1 + \ell)(2 + \ell)} \right] + \cdots
\]

which is obtained by substituting the low frequency expansion for \( E \) of [13] into (20). Note for real frequencies \( \nu \) is also real. As we will see in the following, \( \nu \) becomes complex for quasinormal modes, as does the angular eigenvalue. The first two terms match the expression in [11] and the next term is new. As we have seen in section II, with \( \nu \) and \( \omega \) given, the irreducible representations that appear in the mode function are fully determined.

B. Numerical solutions

The solution of closely related continued fraction eigenvalue equations has been considered for spheroidal harmonics in [13, 32]. Quasinormal modes have been studied using similar techniques in [22, 23]. In this case one imposes quasinormal mode boundary conditions on solutions of the radial equation. Imposing the purely outgoing boundary condition at infinity requires a different expansion of the radial solution near infinity. This is then matched to the boundary condition that the mode be purely infalling on the future horizon, leading to a radial eigenvalue problem that determines the quasinormal mode frequencies.

One method that is helpful in improving the convergence of such algorithms has been developed in [33]. When one numerically computes a continued fraction, such as (19), some cutoff on \( n \) is needed. By choosing an initial value for \( R_{\text{cutoff}} \) judiciously, it is possible to improve the convergence of the continued fraction, as well as its domain of convergence.
In our numerical code, we apply this method at high order in both the determination of the quasinormal mode frequencies, as well as the determination of the $\nu$ eigenvalues. The numerical determination of quasinormal modes has been studied more recently in [34–38] and our results for the frequencies match well with those found there.

1. Low order quasinormal modes

In the case of low order quasinormal modes the spheroidal eigenvalue equation and the frequency equation must be solved simultaneously. Towards that end we adopt the technique of Leaver [23], with the Nollert improvement of continued fractions [33], for both the radial and spheroidal eigenvalue equation. This is implemented using a high precision iterative method in Mathematica, as described in the Appendix.

Once the quasinormal mode frequency is known to high precision, we determine $\nu$ by using similar numerical methods to solve the equation (20) for $n = 1$. In this case both rising and lowering infinite continued fractions $R_n$ and $L_n$ need to be truncated to some $n_{\text{cutoff}}$, where we use Nollert approximation described in Appendix to estimate the remainder of the continued fraction.

Some representative results are shown in Tables I and II, for $s = -2$, $\ell = 2$ and $m = 0, 1$ quasinormal modes. The quasinormal mode frequencies precisely agree with results presented in [23]. Furthermore, some sample computations of $\nu$ appear in [39], and we have checked our numerics correctly reproduces those results. We conventionally normalize $2M = 1$. For clarity of presentation we utilize the invariance under $\nu \rightarrow -\nu - 1$ and $\nu \rightarrow \nu + k$, $k \in \mathbb{Z}$ to map our $\nu$ numeric values into the range $0 < \text{Re}(\nu) < 1/2$.

2. Highly damped quasinormal modes

In the highly damped regime the simultaneous solution of the spheroidal eigenvalue equation and the frequency equation becomes numerically unstable. To make progress, we follow the method of [35] and use a conjectured asymptotic expansion for the high order eigenvalues of spheroidal equation

$$E = (2L + 1)i\epsilon/2 + \mathcal{O}(\epsilon^0)$$
Table I: Numerical results for $s = -2$, $\ell = 2$ and $m = 0$ quasinormal modes.

| $a/M$ | $\omega$          | $A_{lm}$          | $\nu$     |
|-------|-------------------|-------------------|-----------|
| 0.0   | $0.747343 - 0.177925i$ | $4.000000 + 0.000000i$ | $0.271625 - 0.233965i$ |
| 0.1   | $0.748064 - 0.177796i$ | $3.999309 + 0.000348i$ | $0.272325 - 0.234104i$ |
| 0.2   | $0.750248 - 0.177401i$ | $3.997216 + 0.001395i$ | $0.274456 - 0.234519i$ |
| 0.3   | $0.753970 - 0.176707i$ | $3.993667 + 0.003142i$ | $0.278128 - 0.235204i$ |
| 0.4   | $0.759363 - 0.175653i$ | $3.988560 + 0.005596i$ | $0.283541 - 0.236149i$ |
| 0.5   | $0.766637 - 0.174138i$ | $3.981738 + 0.008757i$ | $0.291024 - 0.237327i$ |
| 0.6   | $0.776108 - 0.171989i$ | $3.972969 + 0.012620i$ | $0.301108 - 0.238673i$ |
| 0.7   | $0.788259 - 0.168905i$ | $3.961901 + 0.017153i$ | $0.314673 - 0.240044i$ |
| 0.8   | $0.803835 - 0.164313i$ | $3.947997 + 0.022256i$ | $0.333269 - 0.241085i$ |
| 0.9   | $0.824009 - 0.156965i$ | $3.930384 + 0.027633i$ | $0.359906 - 0.240780i$ |

Table II: Numerical results for $s = -2$, $\ell = 2$ and $m = 1$ quasinormal modes.

| $a/M$ | $\omega$          | $A_{lm}$          | $\nu$     |
|-------|-------------------|-------------------|-----------|
| 0.0   | $0.747343 - 0.177925i$ | $4.000000 + 0.000000i$ | $0.271625 - 0.233965i$ |
| 0.1   | $0.760865 - 0.177597i$ | $3.948480 + 0.012231i$ | $0.281465 - 0.236237i$ |
| 0.2   | $0.776496 - 0.176977i$ | $3.893150 + 0.025197i$ | $0.293393 - 0.238774i$ |
| 0.3   | $0.794661 - 0.176000i$ | $3.833210 + 0.038881i$ | $0.307963 - 0.241559i$ |
| 0.4   | $0.815958 - 0.174514i$ | $3.767570 + 0.053241i$ | $0.326020 - 0.244553i$ |
| 0.5   | $0.841265 - 0.172346i$ | $3.694740 + 0.068178i$ | $0.348933 - 0.247684i$ |
| 0.6   | $0.871937 - 0.169128i$ | $3.612470 + 0.083470i$ | $0.379077 - 0.250847i$ |
| 0.7   | $0.910243 - 0.164170i$ | $3.517150 + 0.098594i$ | $0.420976 - 0.254022i$ |
| 0.8   | $0.960461 - 0.155910i$ | $3.402280 + 0.112173i$ | $0.484170 - 0.258022i$ |
| 0.9   | $1.032583 - 0.139609i$ | $3.253450 + 0.119510i$ | $0.493335 + 0.266455i$ |

where $L = \min(\ell - |m|, \ell - |s|)$. We obtain the quasinormal mode frequencies by numerically searching for a solution of an inverted continued fraction equation, as proposed by Leaver [23]. At high damping, the numeric solutions can be shown to follow the analytic result $\omega \simeq \omega_0 + 4\pi iT_0 (N + 1/2)$, where $\omega_0$, $T_0$ can also be computed within the WKB
We numerically solve for $\nu$ by finding solutions of the equation quadratic in the continued fractions, (20), where we find better convergence if we shift index $n$ by the imaginary part of the quasinormal mode frequency. As a rule of thumb, we use the number of terms in the continued fraction of the order of the overtone number and implement the Nollert improvement for the remainder of the fraction. With each overtone we increase computational precision until the result converges. The numerics appear most stable when a starting value $\nu \sim \mathcal{O}(N/2)$ is used. As the solutions to (20) are determined up to integer shift, $\nu + k$, and reflection $\nu \rightarrow -\nu - 1$, we can always find a solution such that $\text{Re}(\nu) < 1/2$. We use this symmetry when we plot our data.

In figure 1 we show quasinormal mode frequencies at high damping for different values of $s$, and $m$ with $\ell = 2$. For $m$ fixed, we observe the real part of the frequency converges to the same value at high overtones, irrespective of spin weight. The results are compatible with known numerical computations of Berti et al. [35].

In figure 2 we display our numerical solutions for $s = -2$, $\ell = 2$, and $m = 2$ modes. We plot quasinormal mode data as a function of $a/M$ and for fixed overtone number. We find the frequencies approach the expected asymptotic behavior with increasing overtones. Our data for the 400th overtone is comparable to the extrapolated asymptotic curve computed in [35], with the plot in figure 2 suggesting the overtones above 240 already give a qualitatively good estimate of the asymptotic regime. The improvement in convergence for $\nu$ equation happens roughly around this value of frequency. The plot of $\nu$ values displays two strands of solutions corresponding to even and odd valued overtones, which persist at all values of quasinormal frequency. This arises from the approximate half-integer spacing between frequencies at high damping, analytically computed in [29], and also numerically confirmed in [35]. In figure 3 we compare $\nu$ values corresponding to a single strand, for fixed $\ell = 2$ and independently varied $s = 0, -2$ and $m = 1, 2$, where we map out the solutions of (20) up to 1000th overtone.

V. CONCLUSIONS

For a general massless excitation in a Kerr black hole background it is possible to compute a universal function $\nu(\omega, l, m, s)$ which determines a sequence of irreducible representations
Figure 1: Quasinormal mode frequencies for $\ell = 2$ modes and $a/M = 0.2$ with $s = 0$ and $m$ varied (left) and $m = 2$, $s$ varied (right).

Figure 2: Numerical values for $s = -2$, $\ell = 2$, $m = 2$ modes: real part of quasinormal mode frequencies for 100th, 240th and 400th overtone as a function of $a/M$ (left) and real and imaginary $\nu$ values at high overtones for $a/M = 0.2$ (right).

of the group $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ which gives an exact expression for the mode function. These representations for general black hole rotation parameter $a$ do not correspond to highest/lowest weight representations as originally conjectured in the Kerr/CFT literature, but we have pointed out this feature also arises for the BTZ black hole. By analogy we may view this as evidence for an underlying CFT formulation where these modes are related to unitary primary operators.

From the dual holographic field theory, correlators in the bulk are reproduced by placing
the dual CFT at finite left/right moving temperatures. However to resolve detailed scattering amplitudes from past null infinity to future null infinity in the Kerr background, it is also necessary to understand the renormalization group flow from a purported holographic dual to asymptotically flat spacetime to the Kerr/CFT structure that takes over at finite values of the radius.

**Appendix - Numerical evaluation of quasinormal frequencies**

Here we present a brief numerical recipe. The solution to the angular or radial eigenvalue equation can be expressed as an expansion in polynomials, with coefficients satisfying a recurrence equation \[23\] of the form:

\[
\begin{align*}
\alpha_{\theta,r}^0 a_{\theta,r}^0 + \beta_{\theta,r}^0 a_{\theta,r}^0 &= 0 \\
\alpha_{\theta,r}^n a_{\theta,r}^{n-1} + \beta_{\theta,r}^n a_{\theta,r}^{n} + \gamma_{\theta,r} a_{\theta,r}^{n-1} &= 0
\end{align*}
\]

where superscript \(\theta, r\) is to denote one or the other equation. The angular separation constant \(E\) and quasinormal mode frequency \(\omega\) can be determined as roots of the corresponding continued fraction equations:

\[
\beta_{\theta,r}^0 = \frac{\alpha_{\theta,r}^0 \gamma_{\theta,r}^{\theta,r} \alpha_{\theta,r}^1 \gamma_{\theta,r}^{\theta,r}}{\beta_{\theta,r}^1 - \beta_{\theta,r}^2 - \ldots - \beta_{\theta,r}^{n} - \ldots}
\]

Figure 3: Comparison of real and imaginary \(\nu\) values at high overtones for \(m = 1\) and \(m = 2\) and fixed \(s = 2\), \(\ell = 2\) at \(a/M = 0.2\) (left) and for \(s = 0\), \(s = -2\) with fixed \(\ell = 2\), \(m = 2\) (right).
Numerical evaluation of this problem requires truncation to a finite number of terms. To improve convergence, we implement the Nollert algorithm [33] to evaluate the remainder of the continued fraction,

$$R^θ, r_N ≃ \frac{γ^θ, r_{N+1}}{β^θ, r_{N+1}} \frac{α^θ, r_{N+1}}{β^θ, r_{N+2}} \ldots$$

for some large $N$. For $N \gg 1$ the remainder $R^θ, r_N$ will be well approximated with a large $N$ expansion,

$$R^θ, r_N(ω, A_{lm}) ≃ C^θ, r_0 + C^θ, r_1 N^{-1/2} + C^θ, r_2 N^{-1} + \ldots + C^θ, r_k N^{-k} + \ldots$$

By rewriting [22] in an implicit form, $R^θ, r_N - γ^θ, r_{N+1}/(β^θ, r_{N+1} - α^θ, r_{N+1} R^θ, r_{N+1})$ and expanding for large $N$, we read off the coefficients $C_k^θ, r(ω, E)$. For the radial continued fraction we impose $Re(C_1) > 0$ following [33]; for the angular continued fraction all half-integer powers vanish.

At low overtones, we assume an initial value for $ω_0, E_0$ around which we look for solutions, and evaluate the remainder $R_N(ω_0, E_0)$ for a set number of terms where we apply cutoff, $N = n_{\text{cutoff}}$. We are then in a position to solve [21]. We utilize an iterative procedure, separately computing radial and angular continued fractions [21], and feeding solutions back into the next iteration. We increase $n_{\text{cutoff}}$ with each iteration, until results converge to the desired precision.

At high overtones a much more reliable method is to replace [21] with its inversion [23],

$$β^r_0 - \frac{α^r_{n-1} γ^r_n}{β^r_{n-1}} - \frac{α^r_{n-2} γ^r_{n-1}}{β^r_{n-2}} - \ldots = \frac{α^r_0 γ^r_1}{β^r_0} = \frac{α^r_n γ^r_{n+1}}{β^r_{n+1}} - \frac{α^r_{n+1} γ^r_{n+2}}{β^r_{n+2}} - \ldots$$

for any positive integer $n$. We find good convergence for the number of terms in the inverted continued fraction of the order of the overtone number.

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