Mixed XFEM formulation for the simulation of heterogeneities including elasto-plastic material behaviour

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Metal matrix composites often show large isochoric plastic deformations within the matrix material. For dominant plastic deformations this may lead to volumetric locking effects similar to locking in incompressible materials. Lower order mixed finite element formulations can alleviate this difficulty. Complex microstructures often require extensive effort to generate a mesh suitable for finite element simulations. The eXtended Finite Element Method may improve this situation by allowing for discontinuities in the displacement and/or strain field within any finite element. In this contribution a mixed XFEM formulation is presented combining the advantages of the XFEM and mixed FEM. The formulation is applied to small deformation elasto-plasticity with linear isotropic and linear kinematic hardening. An example demonstrates its superior convergence behaviour compared to a standard XFEM formulation.

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1 Introduction

It is well known that lower order continuum finite elements show locking effects for thin structures under bending dominated deformations, for nearly incompressible materials and for metal elasto-plasticity with significant isochoric plastic deformations. Classical remedies are mixed finite elements like the Q1P0 formulation or enhanced assumed strain techniques. To be able to simulate arbitrary microstructures with heterogeneities without the need for sophisticated meshing tools required to generate high quality meshes, the eXtended Finite Element Method (XFEM) can be employed in combination with non-conforming and usually rather regular meshes. Using the XFEM enrichment published in [1] it is possible to accurately reflect the discontinuity in the strain field within elements intersected by the material boundary. Here a mixed hexahedral XFEM element based on the Q1P0 formulation is developed and applied to small strain von Mises elasto-plasticity with linear isotropic and linear kinematic hardening.

2 The mixed XFEM formulation for heterogeneities and elasto-plastic material

Using the kinematic assumption of an additive split of the linearised total strain into an elastic and a plastic part \( \varepsilon = \varepsilon_e + \varepsilon_i \) and an isotropic Hooke material model for the elastic stresses the deviatoric part and the volumetric part of the stresses become

\[
\sigma^D = \mu \varepsilon^D = \mu \left( \varepsilon_e - \frac{1}{3} \text{tr}(\varepsilon_e) \mathbf{1} \right) \quad \sigma^V = p \mathbf{1} = K \text{tr}(\varepsilon) \mathbf{1} \quad \sigma = \sigma^D + \sigma^V
\]  

(1)

For von Mises elasto-plasticity with linear isotropic and linear kinematic hardening the yield function \( f \) and the isotropic hardening \( q \) yield

\[
f(\sigma^D, \chi, q) = ||\sigma^D - \chi||_2 - \sqrt{\frac{2}{3}}(\sigma_y - q) \quad q = -H^{\text{iso}} \xi
\]  

(2)

and using an associated flow rule the evolution equations for the plastic strain \( \varepsilon_i \), the backstress \( \chi \) and the accumulated plastic strain \( \xi \) are

\[
\varepsilon_i = \dot{\lambda} \frac{\partial f}{\partial (\sigma - \chi)} \quad \dot{\chi} = \frac{\dot{\lambda}}{2} H^{\text{kin}} \frac{\partial f}{\partial (\sigma - \chi)} \quad \dot{\xi} = \dot{\lambda} \frac{\partial f}{\partial q}
\]  

(3)

with \( \dot{\lambda} \) being the plastic slip rate and \( H^{\text{iso}} \) and \( H^{\text{kin}} \) are the isotropic and kinematic hardening modulus, respectively. Using a volumetric-deviatoric split and as primary field variables the displacements \( \mathbf{u} \) and the volumetric stress \( p \) the mixed form of the weak form of equilibrium and the additional mixed equation are

\[
\int_{\Omega} (e^D(\delta \mathbf{u}) : \sigma^D(\mathbf{u}) + p \text{div}(\delta \mathbf{u})) \, d\Omega = \int_{\Omega} \delta \mathbf{u} \cdot \mathbf{f} \, d\Omega + \int_{\partial \Omega} \delta \mathbf{u} \cdot \mathbf{n} \, d\Gamma
\]  

(4)
The chosen mixed XFEM ansatz for all elements intersected by the material interface is

\[ u^h = \sum_{I \in N} N_I(X) u_I + \sum_{J \in N^*} N_J(X) \psi(X) a_J \quad \text{with} \quad \psi = |\phi(X)| - \sum_K N_K(X) |\phi(X_K)| \]

where \( \phi(X) \) is a level set function describing the position of the interface and \( H \) is the modified Heaviside enrichment function. The shape functions \( N_I \) connected to the nodal degrees of freedom \( u_I \) and \( a_I \) are chosen to be the standard tri-linear shape functions of a first order hexahedral element, and \( p^e \) as well as \( \tilde{p}^e \) are element-wise constant degrees of freedom that can be eliminated on element level. All elements that are not intersected by a material interface are not enriched. Because this element formulation is based on the well-known Q1P0 element [2] it is called XQ1XP0 element.

### 3 Numerical example

To demonstrate the improved convergence behaviour of this formulation for significant plastic strains a cube of the size \( 2\,\text{mm} \times 2\,\text{mm} \times 2\,\text{mm} \) containing a spherical inclusion with radius \( 0.5\,\text{mm} \) in its center is subjected to displacement controlled uniaxial tension. The applied displacement in normal direction on one of the surfaces is \( 0.1\,\text{mm} \). The inclusion material is chosen to be a rather stiff purely elastic material with shear modulus \( \mu = 416.0\,\text{GPa} \) and bulk modulus \( K = 554.3\,\text{GPa} \). The matrix material behaves elasto-plastic and has the material parameters \( \mu = 38.0\,\text{GPa}, \ K = 302.3\,\text{GPa}, \sigma_y = 3.0\,\text{GPa}, \ H^{\text{iso}} = 5.0\,\text{GPa} \) and \( H^{\text{kin}} = 5\,\text{GPa} \). Figure 1 shows the geometry and boundary conditions of the example as well as the comparison of the total reaction force as a function of the total number of degrees of freedom between a standard first order XFEM hexahedral element (XQ1) and the presented mixed XFEM element (XQ1XP0). It can be observed that the mixed XFEM element converges much faster than the standard XFEM element.

### 4 Conclusions and outlook

In this contribution a mixed XFEM element is presented for elasto-plastic material behaviour within heterogeneous microstructures showing superior convergence properties compared to a standard XFEM element requiring the same global degrees of freedom. The formulation is presented for the small deformation case. It can be extended to finite deformations, different material models and other XFEM applications such as fracture mechanics problems.

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