Kontsevich - Miwa Transform of the Virasoro Constraints as Null-Vector Decoupling Equations *

A.M. Semikhatov

Theory Division, P.N. Lebedev Physics Institute
53 Leninsky prosp., Moscow SU 117924 USSR

(November 25, 1991)

Abstract

We use the Kontsevich–Miwa transform to relate the Virasoro constraints on integrable hierarchies with the David-Distler-Kawai formalism of gravity-coupled conformal theories. The derivation relies on evaluating the energy-momentum tensor on the hierarchy at special values of the spectral parameter. We thus obtain in the Kontsevich parametrization the ‘master equations’ which implement the Virasoro constraints and at the same time coincide with null-vector decoupling equations in an ‘auxiliary’ conformal field theory on the complex plane of the spectral parameter. This gives the operators their gravitational scaling dimensions (for one out of four possibilities to choose signs), with the $\alpha_+$ being equal to the background charge $Q$ of an abstract $bc$ system underlying the structure of the Virasoro constraints. The formalism also generalizes to the $N$-KdV hierarchies.

*This is a revised version of the author’s virtual paper A Kontsevich - Miwa Transform of the Virasoro Constraints on KP and Generalized KdV Hierarchies (Oct. 1991) which consisted to a considerable degree of arithmetical errors and is herewith nullified.
1 Introduction

The interest in Matrix Models \cite{3, 13, 21} has been stimulated, besides their applications to matter+gravity systems \cite{6, 20, 7, 18}, also by intriguing relations they have with completely integrable equations and the intersection theory on the moduli space of curves \cite{12, 39, 31, 25, 40}. On the other hand, as to the foundation of the matrix model approach by itself, a challenging problem is the direct proof of its equivalence to the conformal field theory formalism for quantum gravity \cite{23, 10, 6}. Assuming that this equivalence exists, as is suggested by a circumstantial evidence, one then has to believe that certain ingredients of conformal field theory satisfy integrable equations. These, however, seem to be a long way from the equations which are known to hold for conformal field theory correlators \cite{11, 24}.

As we will show, the role of an ‘intermediary’ on the way from non-linear KdV-like equations to conformal field theory is played by the Virasoro constraints on integrable hierarchies (which in another guise are recursion relations in topological theories) \cite{17, 8, 22, 29}. The Virasoro constraints are the heart of the matrix models’ applications to both gravity-coupled theories and the intersection theory. The case studied in most detail is the Virasoro-constrained KdV hierarchy whose relation to the intersection theory on moduli space of Riemann surfaces has been discussed in \cite{39}.

To reveal this role of the Virasoro constraints, we will adopt the approach which has proved fruitful in establishing the relation between matrix models and the universal moduli space: That is, we borrow from the matrix model due to Kontsevich the choice of independent variables. The Kontsevich matrix model provides a combinatorial model of the universal moduli space \cite{25} and, as such, serves as an important step in demonstrating the KdV hierarchy in the intersection theory on the moduli space. Note, however, that the Kontsevich model is not of the form of the matrix models considered previously, which raises the question of its equivalence to one of the “standard” models. The crucial point in studying this equivalence is, again, the proof of the Virasoro constraints satisfied by the Kontsevich matrix integral \cite{10, 27}. Once the constraints are established, one is left with “only” the proof that they specify the model uniquely.

The Kontsevich model by itself, as well as the existing derivations of the Virasoro constraints, appear to be tied up to the KdV hierarchy and thus to the \( l = 2 \) minimal models. There exist, at the same time, matrix models of other minimal conformal theories coupled to gravity, which correspond to higher generalized ‘KdV’ hierarchies \cite{14}. Although neither the interpretation of Virasoro-constrained \( N \)-KdV hierarchies in terms of moduli spaces, nor the corresponding Kontsevich-type matrix integrals are known, it would not be natural if the \( N = 2 \) case were exceptional. Thus another problem is whether a unified approach exists which allows one to recast Virasoro constraints on \( N \)-KdV (and hopefully other) hierarchies into Ward identities of a would-be Kontsevich-type matrix integral.

In this paper we take as a starting point the Virasoro constraints in the usual parametrization and then investigate whether they can be recast into the Kontsevich variables. There are obvious similarities between Kontsevich’ parametrization and the Miwa transform used in

\footnote{In this paper we will restrict ourselves to the series associated to \( \mathfrak{sl}(N) \) Kac-Moody algebras (and therefore the \( A \)-series minimal models \cite{3}, and we will call these the \( N \)-KdV hierarchies. Virasoro constraints on the \( N \)-KdV hierarchies admit a unified treatment, which is in turn a specialization of a general construction applicable to hierarchies of the \( r \)-matrix type \cite{18}.}
We thus attempt to proceed with a general Miwa transformation. As we will see, the Virasoro constraints are \textit{not} reformulated nicely unless one restricts the Miwa transformation to the Kontsevich one, yet we find it instructive to see at which step the Miwa parametrization fails to work.

Pulling back the Virasoro generators to the Kontsevich parametrization seems only possible for the combination $\sum_{n \geq -1} L_n z^{-n-2}$ of the Virasoro generators, and only at special values of the spectral parameter $z$. The resulting relations are candidates for the Ward identities corresponding to a Kontsevich-type matrix integral. For the $N$-KdV hierarchies these relations are the analogues of the “master equation” of ref.\cite{26}. Further, we show in the KP case that our version of the master equation (which, though very similar to, is not quite the same as the one from \cite{26}) happens to be nothing but an equation on correlation functions in an ‘auxiliary’ conformal field theory, stating the decoupling of a certain null vector. This is in the classical spirit of \cite{2}, yet the applicability to Virasoro-constrained hierarchies seems to be new. Recall in particular that according to the basic matrix models ideology, the resulting equations that hold on an abstract (spectral-parameter) $\mathbb{C}P^1$ are in fact non-perturbative. It is thus very reassuring to recover in our approach the results of refs.\cite{10,6}!

## 2 Virasoro action on the KP hierarchy

2.1. The KP hierarchy is described in terms of $\psi$Diff operators \cite{3} as an infinite set of mutually commuting evolution equations

\[
\frac{\partial K}{\partial t_r} = -(KD'K^{-1})_r K, \quad r \geq 1
\]

on the coefficients $w_n(x, t_1, t_2, t_3, \ldots)$ of a $\psi$Diff operator (more precisely, a $\psi$Diff symbol) $K$ of the form (here and in the sequel, $D = \partial/\partial x$)

\[
K = 1 + \sum_{n \geq 1} w_n D^{-n}
\]

Introduce a ‘matrix model potential’

\[
\xi(t, z) = \sum_{r \geq 1} t_r z^r
\]

The wave function and the adjoint wave function are then defined by

\[
\psi(t, z) = Ke^{\xi(t,z)}, \quad \psi^*(t, z) = K^{*-1}e^{-\xi(t,z)}
\]

where $K^*$ is the formal adjoint to $K$. The wave function $\psi$ is an eigenfunction of the Lax operator:

\[
Q\psi(t, z) = z\psi(t, z), \quad Q \equiv KDK^{-1}
\]

functions. The basic property of the wave functions is their relation to the tau function:

\[
\psi(t, z) = e^{\xi(t,z)} \frac{\tau(t-[z^{-1}])}{\tau(t)}, \quad \psi^*(t, z) = e^{-\xi(t,z)} \frac{\tau(t+[z^{-1}])}{\tau(t)}
\]
where,
\[
t \pm [z^{-1}] = (t_1 \pm z^{-1}, t_2 \pm \frac{1}{2} z^{-2}, t_3 \pm \frac{1}{3} z^{-3}, \ldots)
\]  

(2.8)

2.2. Now we introduce a Virasoro action on the tau function: The Virasoro generators read,
\[
L_{p>0} = \frac{1}{2} \sum_{k=1}^{p-1} \frac{\partial^2}{\partial t_{p-k} \partial t_k} + \sum_{k \geq 1} k t_k \frac{\partial}{\partial t_{p+k}} + (a_0 + (J - \frac{1}{2} p) \frac{1}{24}) \frac{\partial}{\partial t_p}
\]
\[
L_0 = \sum_{k \geq 1} k t_k \frac{\partial}{\partial t_k} + \frac{1}{2} a_0^2 - \frac{1}{24}
\]
\[
L_{p<0} = \sum_{k \geq 1} (k - p) t_{k-p} \frac{\partial}{\partial t_k} + \frac{1}{2} \sum_{k=1}^{p-1} k(-p-k) t_{k-p} + (a_0 + (J - \frac{1}{2} p)(-p)t_{-p}
\]

(2.9)

They satisfy the algebra
\[
[L_p, L_q] = (p - q)L_{p+q} + \delta_{p+q,0}(-p^3)(J^2 - J + \frac{1}{6})
\]

(2.10)

which shows, in particular, the role played by the parameter $J$. (Shifting $L_0$ as $L_0 \mapsto L_0 - \frac{1}{24}(J^2 - J + \frac{1}{6})$ we recover in (2.10) the standard central term $-\delta_{p+q,0}(p^3 - p)(J^2 - J + \frac{1}{6})$.) It will be quite useful to introduce an “energy-momentum tensor”
\[
T(u) = \sum_{p \in \mathbb{Z}} u^{-p-2} L_p
\]

(2.11)

Using this to deform the tau function as
\[
\tau(t) \mapsto \tau(t) + \delta \tau(t) = \tau(t) + T(u) \tau(t),
\]

(2.12)

we translate this action into the space of dressing operators $K$. The result is \[33\] that $K$ gets deformed by means of a left multiplication,
\[
\delta K = -\mathcal{T}(u) K,
\]

where $\mathcal{T}(u)$ is the energy-momentum tensor in another guise, now a pseudodifferential operator
\[
\mathcal{T}(u) = (1 - J) \frac{\partial \psi(t,u)}{\partial u} \circ D^{-1} \circ \psi^*(t,u) - J \psi(t,u) \circ D^{-1} \circ \frac{\partial \psi^*(t,u)}{\partial u}
\]

(2.13)

Thus, $\mathcal{T}(u)$ reproduces the structure of the energy-momentum tensor of a spin–$J$ bc theory \[13\]. In its own turn, $\mathcal{T}(u)$ can be expanded in powers of the variable $u$, which was introduced in (2.11) and has now acquired the role of a spectral parameter, as
\[
\mathcal{T}(u) = \sum_{p \in \mathbb{Z}} u^{-p-2} L_p
\]

(2.14)

\[2\] We have chosen the irrelevant parameter $a_0 = \frac{1}{2}$, see \[33\] where $a_0 = N + \frac{1}{2}$. 4
This gives the individual Virasoro generators (which are a particular case of the general construction applicable to integrable hierarchies of the $r$–matrix type [38])

$$\mathfrak{L}_r \equiv K(J(r+1)D^r + PD^{r+1})K^{-1}, \quad P \equiv x + \sum_{r \geq 1} rt_r D^{r-1}$$ (2.15)

2.3. The KP hierarchy can be reduced to generalized N-KdV hierarchies [3] by imposing the constraint

$$Q^N \equiv L \in \text{Diff} \quad (\Rightarrow Q^{Nk} \in \text{Diff}, k \geq 1)$$ (2.16)

requiring that the $N^{th}$ power of the Lax operator be purely differential. Then, in a standard manner, the evolutions along the times $t_{Nk}$, $k \geq 1$, drop out and these times may be set to zero. The rest of the $t_n$ are conveniently relabelled as $t_{a,i} = t_{N_i+a}$, $i \geq 0$, $a = 1, \ldots, N-1$.

As to the Virasoro generators, only $\mathfrak{L}_{Nj}$ are compatible with the reduction in the sense that they remain symmetries of the reduced hierarchy without imposing any further constraints [35]. The value of $J$ can be set to zero [33, 39]. Then, after a rescaling, the generators

$$\mathfrak{L}^{[N]}_j = \frac{1}{N}(K \sum_{a,i} (Ni + a)t_{a,i}D^{N(i+j)+a}K^{-1})$$ (2.17)

span a Virasoro algebra of their own.

Again, we find it very useful to construct the energy-momentum tensor corresponding to these generators. Recall that the spectral parameter of the $N$-KdV hierarchy is $\zeta = z^N$. Then

$$\mathcal{T}^{[N]}(\zeta)(d\zeta)^2 \equiv \sum_{j \in \mathbb{Z}} \zeta^{-j-2} \mathfrak{L}^{[N]}_j(d\zeta)^2$$

$$= N \left( K \sum_{b,j} (Nj + b)t_{b,j}D^{Nj+b} \frac{1}{2z^2} \delta(D^N, z^N)K^{-1} \right) (dz)^2$$ (2.18)

where

$$\delta(u, v) = \sum_{n \in \mathbb{Z}} \left( \frac{u}{v} \right)^n$$ (2.19)

denotes the formal delta function. The essential point is that $\delta(z, D)$ is a projector onto an eigenspace of $D$ with the eigenvalue $z$. Then it is obvious that

$$\delta(D^N, z^N) = \frac{1}{N} \sum_{c=0}^{N-1} \delta(z^{(c)}, D), \quad z^{(c)} = \omega^c z, \quad \omega = \exp \left( \frac{2\pi \sqrt{-1}}{N} \right)$$ (2.20)

Using this we bring the above energy-momentum tensor to the form

$$\mathcal{T}^{[N]}(E) = \sum_{c=0}^{N-1} \omega^c \frac{\partial \psi(t, z^{(c)})}{\partial z} \circ D^{-1} \circ \psi^*(t, z^{(c)}) = \frac{1}{N} \sum_{c=0}^{N-1} \omega^{2c} \mathcal{T}(z^{(c)})$$ (2.21)

where the following notations have been used: recall that the spectral parameter of an $N$-KdV hierarchy lies on a complex curve defined in $\mathbb{C}^2 \ni (z, E)$ by an equation $z^N = P(E)$. Then, $\psi$ and $\psi^*$ are defined on this curve, and after the projection onto $\mathbb{CP}^1$ yield $N$ wave functions $\psi^{(a)}(t, E)$, distinct away from the branch points. That is, we have defined

$$\psi^{(a)}(t, E) = Ke^{\xi(t, z^{(a)})} \equiv w(t, z^{(a)})e^{\xi(t, z^{(a)})}, \quad \xi(t, z^{(a)}) = \sum_{j,b} t_{b,j}(z^{(a)})^{Nj+b}$$ (2.22)
Note a striking similarity between (2.21) and the energy-momentum tensor of conformal theories on $Z_N$-curves $\mathbb{I}$.

2.4. To conclude this review, we outline the basic steps of how the Virasoro action on the $N$–KdV tau function, by the generators of the type of (2.9), can be recovered from the “energy-momentum tensor” (2.21). The usual way to derive objects pertaining to the tau function is through the use of the equation

$$\text{res} K = -\partial \log \tau,$$

whence

$$\delta \partial \log \tau = -\text{res} \delta K = \text{res} \mathfrak{T}^{[N]}(z) K = \text{res} \mathfrak{T}^{[N]}(z)$$  \hspace{1cm} (2.23)

The residue of $\mathfrak{T}^{[N]}(z)$ is immediately read off from (2.21). To the combination of wave functions thus appearing we apply the formula

$$\partial \left( \frac{1}{u-z} \frac{\xi(t,u)-\xi(t,z)}{\tau(t)} \right) = \psi(t,u) \psi^*(t,z)$$ \hspace{1cm} (2.24)

(it follows directly by applying the vertex operator $\exp \left( -\sum_{r \geq 1} \frac{1}{r} \frac{z^{-r} - u^{-r}}{z-u} \frac{\partial}{\partial t} \right)$ to the bilinear identity of the KP hierarchy $\mathbb{II}$ and then evaluating the integral as a sum over residues). It follows from (2.24) by expanding it at $z \to u$ that

$$\frac{\partial \psi(t,z)}{\partial z} \psi^*(t,z) = \partial \left\{ \frac{1}{2 \tau(t)} \frac{1}{z} \nabla(t,z) \nabla(t,z) \tau(t) + \frac{1}{\tau(t)} \frac{\partial \xi(t,z)}{\partial z} \frac{1}{z} \nabla(t,z) \tau(t) \right\}$$

$$+ \frac{1}{2} \frac{1}{\tau(t)} \frac{1}{z} \nabla(t,z) \tau(t) + \frac{1}{2} \left( \frac{\partial \xi(t,z)}{\partial z} \right)^2$$ \hspace{1cm} (2.25)

This expression by itself would lead us back to the KP Virasoro generators (2.9) (with $J = 0$). Now the $N$–KdV generators follow according to the formula (2.21), by substituting $z \to z^{(c)}$ and summing over $c$. The sum over $c$ plays the role of a projector onto the identity of the group of $N^{th}$ roots of unity. Therefore,

$$\sum_{c=0}^{N-1} \omega^c \frac{\partial \psi(t,z^{(c)})}{\partial z^{(c)}} \psi^*(t,z^{(c)}) = N \partial \left\{ \frac{1}{2} \sum_{a=1}^{N-1} \sum_{i,j \geq 0} (N_j + a)(N(j + 1) - a) t_{a,i} t_{N-a,j} z^{N(i+j+1)-2} \right\}$$

$$+ \frac{1}{2} \frac{1}{\tau(t)} \sum_{a=1}^{N-1} \sum_{i,j \geq 0} z^{-N(i+j+1)-2} \frac{\partial^2 \tau(t)}{\partial t_{a,i} \partial t_{N-a,j}} + \frac{1}{\tau(t)} \sum_{a=1}^{N-1} \sum_{i,j \geq 0} (N_i + a) t_{a,i} z^{N(i-j)-2} \frac{\partial \tau(t)}{\partial t_{a,j}}$$ \hspace{1cm} (2.26)
from which the Virasoro generators can be read off as

\[ L_n^{[N]} = \frac{1}{N} \sum_{a=1}^{N-1} \sum_{i=0}^{n-1} \frac{\partial^2}{\partial t_{a,i} \partial t_{N-a,n-i-1}} + \frac{1}{N} \sum_{a=1}^{N-1} (Ni + a) t_{a,i} \frac{\partial}{\partial t_{a,i+n}}, \]

\[ L_0^{[N]} = \frac{1}{N} \sum_{a=1}^{N-1} \sum_{i=0}^{n-1} (Ni + a) t_{a,i} \frac{\partial}{\partial t_{a,i}}, \]

\[ n < 0 : \quad L_n^{[N]} = \frac{1}{N} \sum_{a=1}^{N-1} \sum_{i=0}^{n-1} (Ni + a) (-N(i + n) - a) t_{a,i} t_{N-a,-i-n-1} \]

These generators act on the tau functions \( \tau(t) \) of the \( N \)-KdV hierarchy.

3 Miwa–Kontsevich transform

Now we are going to use the same strategy as was used to derive (2.27), but this time in the Miwa–Kontsevich parametrization of the times of the hierarchy. As in the above, we start with the simplest case, the KP hierarchy.

The Miwa reparametrization of the KP times is accomplished by the substitution

\[ t_r = \frac{1}{r} \sum_j n_j z_j^{-r} \]  

(3.1)

where \( \{z_j\} \) is a set of points on the complex plane and \( n_j \) are integers. This parametrization puts, in a sense, the times and the spectral parameter on equal ground. It may in some cases be conceptually advantageous to write (3.1) as

\[ t_r = \frac{1}{r} \sum_{z \in \mathbb{C} \mathbb{P}^1} n_z z^{-r} \]  

(3.2)

where \( n_z \) is nonvanishing for only a finite (countable) set of points. Then the tau function becomes a functional \( \tau[n] \) of a function \( n \) on \( \mathbb{C} \mathbb{P}^1 \) which must be from the class of functions in some sense close to linear combinations (with integer coefficients) of delta-functions. On the other hand, the way Kontsevich has used a parametrization of this type implied setting all the \( n_j \) equal to unity. We will in fact see why a restriction of this kind is necessary, but this will require working with the general \( n_j \) for as long as possible.

The Miwa substitution turns out very inconvenient with regard to the use of the standard machinery of the KP hierarchy (e.g., proceeding along the usual chain (tau function) \( \rightarrow \) (wave function) \( \rightarrow \) (dressing operator), etc.); instead, it serves to construct a quite different, “discrete” formalism for the KP and related hierarchies [32]. Now, the above expressions (2.15), (2.17) and (2.21) for the Virasoro generators involve all the standard ingredients such as the wave functions, the spectral parameter etc., which complicates re-expressing them in the Miwa parametrization. That is, taking (3.2) seriously, and even viewing it as

\[ t_r = \frac{1}{r} \int_{\mathbb{C} \mathbb{P}^1} d\mu(z) z^{-r} n(z) \]  

(3.3)
one can formally define the wave functions as

\[ \psi[n](z) = \prod_j \left(1 - \frac{z}{z_j}\right)^{n_j} \frac{1}{\tau[n]} e^{\frac{i}{\tau[n]} \tau[n]} \]  

(3.4)

Short-‘distance’ expansion as in (2.24) – (2.25) would then require making sense out of this formula and similarly out of expressions such as \( \frac{\partial}{\partial z} \frac{\delta}{\delta \tau[z]} \). Even this, however, would not be quite satisfactory, as one would still have had to express the result in terms of the derivatives with respect to \( z \), which are the parameters of the Kontsevich model: for us, the tau function must be a function \( \tau\{z_j\} \) of points scattered over \( \mathbb{CP}^1 \).

There are two circumstances, however, that save the day. First, we will be interested not in all the Virasoro generators, but rather in those with non-negative (and, in addition, \(-1\)) mode numbers \( \mathfrak{L}_{n \geq -1} \). Picking these out amounts to retaining in \( \mathfrak{T}(z) \) only terms with \( z \) to negative powers, \( i.e., \) the terms vanishing at \( z \to \infty \). This part of \( \mathfrak{T}(z) \) is singled out as

\[ \mathfrak{T}(\infty)(v) = \sum_{n \geq 0} v^{-n-1} \frac{1}{2\pi i} \oint dz z^n \mathfrak{T}(z) = \frac{1}{2\pi i} \oint dz \frac{1}{v - z} \mathfrak{T}(z) \]  

(3.5)

where \( v \) is from a neighbourhood of the infinity and the integration contour encompasses this neighbourhood.

Second, a crucial simplification will be achieved by evaluating \( \mathfrak{T}(\infty)(v) \) only at the points from the above set \( \{z_j\} \) (one has to take care that they lie in the chosen neighbourhood). We use the formulas (2.23), (2.21), (2.13) and (2.24) to find the variation of the tau function \( \tau\{z_j\} \), which amounts to evaluating, for a fixed index \( i \),

\[ \frac{1}{2\pi i} \oint dz \frac{1}{z_i - z} \left\{ (1 - J) \frac{\partial \psi\{z_j\}(z)}{\partial z} \psi^*\{z_j\}(z) - J \psi\{z_j\}(z) \frac{\partial \psi^*\{z_j\}(z)}{\partial z} \right\} \equiv \partial \left( \frac{1}{\tau} T(z_i) \tau \right) \]  

(3.6)

or, after the use of (2.24),

\[ \frac{1}{\tau} T(z_i) \tau = \frac{1}{2\pi i} \oint dz \frac{1}{z_i - z} \left\{ (1 - J) \frac{\partial \psi\{z_j\}(z)}{\partial u} - J \frac{\partial \psi\{z_j\}(z)}{\partial z} \right\} \frac{1}{u - z} e^{\xi(t,u) - \xi(t,z)} \frac{\tau(t + [z^{-1}] - [u^{-1}])}{\tau(t)} \]  

(3.7)

where reg implies subtracting the pole \( \frac{1}{(u-z)^r} \), and everything has to be reexpressed through the \( \{z_j\} \) variables. This latter task, however, will be achieved not until the final stage of the derivation. Now we perform an expansion using \( \textit{time} \) derivatives acting on the tau function and thus find:

\[ \mathcal{T}(z_i) = \frac{1}{2\pi i} \oint dz \frac{1}{z_i - z} \left\{ (J - \frac{1}{2}) \frac{z - \sum_{r \geq 1} z^{-r-1} \frac{\partial}{\partial t_r} + \frac{1}{2} \sum_{r,s} z^{-r-s-2} \frac{\partial^2}{\partial t_r \partial t_s}}{z - \sum_{r \geq 1} z^{-r-1} \frac{\partial}{\partial t_r} + \frac{1}{2} \sum_{r,s} z^{-r-s-2} \frac{\partial^2}{\partial t_r \partial t_s}} \right\} + \sum_j n_j \frac{1}{z_j - z} \sum_{r \geq 1} z^{-r-1} \frac{\partial}{\partial t_r} + \frac{1}{2} \sum_j \frac{n_j + n_j^2}{(z_j - z)^2} + \frac{1}{2} \sum_{j \neq k} \frac{n_j n_k}{(z_j - z)(z_k - z)} \]  

\[ - \frac{1}{2} \sum_{j \geq 1} z^{-r-2} \frac{\partial}{\partial t_r} \]  

(3.8)

\(^3\) It is these Virasoro generators that are used to define \textit{Virasoro-constrained} hierarchies, simply as \( \mathfrak{L}_n = 0 \), \( n \geq -1 \).
Evaluating the residue is the crucial step which allows one to bring (3.8) to a tractable form in terms of the $z_j$. As the integration contour encompasses all the points $\{z_j\}$, the residues at both $z = z_i$ and $z = z_j, j \neq i$, contribute to (3.8). The residue at $z_i$ consists of the following parts: first, the terms with the first-order pole contribute

$$
\left( J - \frac{1}{2} - \frac{1}{2n_i} \right) \frac{1}{n_i} \frac{1}{z_i} \frac{\partial}{\partial z_i} - \frac{1}{2n_i^2} \frac{\partial^2}{\partial z_i^2} + \frac{1}{n_i} \sum_{j \neq i} \frac{n_j}{z_j - z_i} \frac{\partial}{\partial z_i}
$$

$$
- \left( J - \frac{1}{2} - \frac{1}{2n_i} \right) \sum_{r \geq 1} r z_i^{-r-2} \frac{\partial}{\partial t_r}
- \frac{1}{2} \sum_{j \neq i} \frac{n_j + n_j^2 - 2Jn_j}{(z_j - z_i)^2} - \frac{1}{2} \sum_{j \neq i, k \neq j} \frac{n_j n_k}{(z_j - z_i)(z_k - z_i)}
$$

(3.9)

where we have substituted

$$
\sum_{r,s} z_i^{-r-s-2} \frac{\partial^2}{\partial t_r \partial t_s} = \frac{1}{n_i^2} \frac{\partial^2}{\partial z_i^2} + \frac{1}{n_i} \frac{\partial}{\partial z_i} - \frac{1}{n_i} \sum_{r \geq 1} z_i^{-r-2} r \frac{\partial}{\partial t_r},
$$

(3.10)

Next, second-order poles occur in the double sum over $j, k$ in (3.8):

$$
\frac{1}{2\pi i} \oint dz \frac{1}{z_i - z} \sum_{j \neq i} \frac{n_j n_i}{(z_j - z)(z_i - z)} = \sum_{j \neq i} \frac{n_j n_j}{(z_i - z_j)^2}
$$

Now, to get rid of the $\partial/\partial t_r$-terms in (3.9) which cannot be expressed through $\partial/\partial z_j$, we set the coefficient in front of these equal to zero:

$$
n_i = \frac{1}{2J - 1} = \frac{1}{Q}
$$

(3.11)

Then the contribution of the residue at $z = z_i$ equals

$$
\mathcal{T}^{(i)}(z_i) = - \frac{1}{z_j - z_i} \frac{\partial}{\partial z_j} + \frac{1}{z_j - z_i} \sum_{k \neq j} \frac{n_j n_k}{z_k - z_j} \frac{\partial}{\partial z_j}
$$

$$
- \frac{1}{2} \sum_{j \neq i, k \neq j} \frac{n_j n_k}{(z_j - z_i)(z_k - z_i)} - \frac{1}{2} \sum_{j \neq i} \frac{n_j + n_j^2 - 2Jn_j - 2n_i n_j}{(z_j - z_i)^2}
$$

(3.12)

Similarly, each of the residues at $z_j, j \neq i$, contributes

$$
\mathcal{T}_{(j)}(z_i) = - \frac{1}{z_j - z_i} \frac{\partial}{\partial z_j} + \frac{1}{z_j - z_i} \sum_{k \neq j} \frac{n_j n_k}{z_k - z_j} + \frac{1}{2} \frac{n_j + n_j^2 - 2Jn_j}{(z_i - z_j)^2}
$$

(3.13)

and thus,

$$
\mathcal{T}(z_i) = \mathcal{T}^{(i)}(z_i) + \sum_{j \neq i} \mathcal{T}_{(j)}(z_i)
$$

$$
= - \frac{1}{2n_i^2} \frac{\partial^2}{\partial z_i^2} + \frac{1}{n_i} \sum_{j \neq i} \frac{n_j}{z_j - z_i} \left( \frac{\partial}{\partial z_i} - n_i \frac{\partial}{\partial z_j} \right)
$$

(3.14)
(We have used the identity

\[ \sum_{j \neq i} \sum_{k \neq j} \frac{1}{(z_j - z_i)(z_k - z_j)} = \frac{1}{2} \sum_{j \neq i} \sum_{k \neq j} \frac{1}{(z_j - z_i)(z_k - z_i)}. \]

Now, the above treatment can be applied equally well to each of the \( T(z_j) \), and thus (3.11) must hold for all the \( n_j \). Finally,

\[ T(z_i) = -\frac{Q^2}{2} \frac{\partial^2}{\partial z_i^2} - \sum_{j \neq i} \frac{1}{z_j - z_i} \left( \frac{\partial}{\partial z_j} - \frac{\partial}{\partial z_i} \right) \]

(3.15)

We thus see that, indeed, in order that the \( \Lambda \geq -1 \) Virasoro generators translate into the \( \{ z_j \} \) variables, one has to restrict the general Miwa transform (3.1) to a Kontsevich form with all the \( n_j \) equal to each other.4

From the above derivation of (3.15) we see that \( z_i \) is nothing but a value taken by the spectral parameter and thus the trick, described in 2.3, with building up invariants with respect to \( Z_N \) applies here as well. That is, to perform the reduction to an \( N \)-KdV hierarchy, it suffices to substitute

\[ z_i \mapsto \omega^c z_i \]

and then sum over \( Z_N \) as in (2.21).5 We thus arrive at

\[ T_i^{[N]} = \frac{1}{N} \sum_{c=0}^{N-1} \omega^{2c} T(z_i^{(c)}) \]

(3.16)

Note that \( z^N \equiv \zeta \) can be viewed as a spectral parameter of the \( N \)-KdV hierarchy, as the \( N \)-KdV Lax operator \( L \) (see (2.16)) satisfies \( L \psi(t, z) = z^N \psi(t, z) \). In terms of these variables, the operator (3.16) becomes, up to an overall factor,

\[ -\frac{Q^2 N}{2} \zeta_i \frac{\partial^2}{\partial \zeta_i^2} - \frac{Q^2(N - 1)}{2} \frac{\partial}{\partial \zeta_i} + \sum_{j \neq i} \frac{1}{\zeta_j - \zeta_i} \left( \zeta_j \frac{\partial}{\partial \zeta_j} - \zeta_i \frac{\partial}{\partial \zeta_i} \right) \]

(3.17)

4Restricting to only integer \( n_j \) would fix two (equivalent) values \( J = 0, 1 \) of conformal spin of the underlying abstract bc system. For our purposes in Sect.4, however, we will need more general \( n_j \) and \( J \).

5 Clearly, having defined the reduced \( T \)-operator as (see (2.21))

\[ T_i^{[N]} = \frac{1}{2\pi i} \oint dz \sum_{c=0}^{N-1} \omega^{2c} \Xi(\omega^c z) \]

one continues this as

\[ = \frac{1}{2\pi i} \sum_{c=0}^{N-1} \frac{d\omega^{-c}}{z_i - \omega^{-c} z} \Xi(z) = \frac{1}{2\pi i} \sum_{c=0}^{N-1} \frac{d\omega^{2c}}{\omega^c z_i - \omega^{-c} z} \Xi(z) = \frac{1}{N} \sum_{c=0}^{N-1} \omega^{2c} T(\omega^c z_i). \]
When imposing Virasoro constraints on the $N$–reduced hierarchy, it is these $\zeta_i$ that are candidates for eigenvalues of the “source” matrix in a Kontsevich-type matrix integral, at least for $Q^2 = 1$. We consider the reformulation of the Virasoro constraints in more detail in the next section.

### 4 A la recherche de Liouville perdu

Obviously now, if one starts with the Virasoro-constrained KP hierarchy, i.e.,

$$\mathfrak{T}^{(\infty)}(z) = 0,$$

one ends up in the Kontsevich parametrization with the KP Virasoro master equation (cf. ref. [26])

$$\mathcal{T}(z_i), \tau\{z_j\} = 0$$

(4.2)

The above derivation of (4.2), (3.15), with the $z_j$ (which in the alternative approach are the eigenvalues of the ‘source’ matrix in a matrix integral) viewed as coordinates on the spectral parameter complex plane, suggests an interpretation of the master equation in terms of a conformal field theory living on this complex plane. First, it is natural to assume that (with a possible ‘background’ insertion at infinity)

$$\tau\{z_j\} = \lim_{n \to \infty} \langle \Psi(z_1) \ldots \Psi(z_n) \Phi(\infty) \rangle$$

(4.3)

with the pre-limit correlators satisfying,

$$\left\{ -\frac{Q^2}{2} \frac{\partial^2}{\partial z_i^2} + \sum_{j \neq i}^{n} \frac{1}{z_i - z_j} \left( \frac{\partial}{\partial z_j} - \frac{\partial}{\partial z_i} \right) \right\} \langle \Psi(z_1) \ldots \Psi(z_n) \Phi(\infty) \rangle = 0$$

(4.4)

Further, one can imagine a conformal theory of a $U(1)$ current $j(z)$ and an energy-momentum tensor $T(z)$:

$$j(z) = \sum_{n \in \mathbb{Z}} j_n z^{-n-1}, \quad T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$$

(4.5)

$$[j_m, j_n] = km \delta_{m+n,0}$$

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{d+1}{12} (m^3 - m) \delta_{m+n,0}$$

(4.6)

$$[L_m, j_n] = -nj_{m+n}$$

(We have parametrized the central charge as $d+1$). Then, in the standard conformal field theory setting [2], let us look for a null vector at level 2:

$$|\Upsilon\rangle = (\alpha L_{-1}^2 + L_{-2} + \beta j_{-2} + \gamma j_{-1}^2 + \epsilon j_{-1} L_{-1}) |\Psi\rangle$$

(4.7)

where $\Psi$ is a primary field with conformal dimension $\Delta$ and $U(1)$ charge $q$. We will in fact need the specific case $\gamma = 0$. Then (4.7) is a null vector when

$$\alpha = \frac{k}{2q^2}, \quad \beta = -\frac{q}{k} - \frac{1}{2q}, \quad \epsilon = -\frac{1}{q}, \quad \Delta = -\frac{q^2}{k} - \frac{1}{2}$$

(4.8)
with \( q \) given by,

\[
\frac{q^2}{k} = \frac{d - 13 \pm \sqrt{(d - 25)(d - 1)}}{24}
\]  

(4.9)

(so that,

\[
\Delta = \frac{1 - d \mp \sqrt{(d - 25)(d - 1)}}{24}.
\]  

(4.10)

Factoring the state (4.7) out from the Hilbert space leads in the usual manner to the equation

\[
\begin{aligned}
\left\{ \frac{k}{2q^2} \frac{\partial^2}{\partial z^2} - \frac{1}{q} \sum_j & \frac{1}{z_j - z} \left( q \frac{\partial}{\partial z_j} - q_j \frac{\partial}{\partial z} \right) + \frac{1}{q} \sum_j q \Delta_j - q_j \Delta \right\} \langle \Psi(z) \Psi_1(z_1) \ldots \Psi_n(z_n) \rangle = 0
\end{aligned}
\]  

\[
\text{(4.11)}
\]

where \( \Psi_j \) are primaries of dimension \( \Delta_j \) and \( U(1) \) charge \( q_j \). In particular,

\[
\begin{aligned}
\left\{ \frac{k}{2q^2} \frac{\partial^2}{\partial z_i^2} + \sum_{j \neq i} & \frac{1}{z_i - z_j} \left( \frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j} \right) \right\} \langle \Psi(z_1) \ldots \Psi(z_n) \rangle = 0
\end{aligned}
\]  

\[
\text{(4.12)}
\]

This is to be compared with (4.4) (with the insertion at the infinity disregarded)\footnote{Note that the energy-momentum tensor \( T \) introduced in (4.5) appears to have a priori nothing to do with the energy-momentum tensor on the hierarchy we have started with. In terms of the latter tensor, eq.(4.4) comprises the contribution of all the positive-moded Virasoro generators, while out of \( T(z) \) only \( L_{-1} \) and \( L_{-2} \) enter in the equivalent equation (4.7).}. We thus arrive at the identification

\[
Q^2 = -\frac{k}{q^2} = \frac{13 - d \pm \sqrt{(d - 25)(d - 1)}}{6}
\]  

(4.13)

and therefore find ourselves in the friendly realm of minimal models [2, 11, 16], tensored with the \( U(1) \) current. Moreover, the theory on the \( z \)-plane also incorporates the gravitational dressing of the matter theory. To see this let us first examine closer the constraint \( \langle \Upsilon \rangle = 0 \). Writing the Hilbert space as \( (\text{matter}) \otimes (\text{current}) \equiv \mathcal{M} \otimes \mathcal{C} \), \( |\Psi\rangle = |\psi\rangle \otimes |\tilde{\Psi}\rangle \), we introduce the matter Virasoro generators \( l_n \) by,

\[
L_n = l_n + \bar{L}_n \equiv l_n + \frac{1}{2k} \sum_{m \in \mathbb{Z}} : j_{n-m} j_m : 
\]  

(4.14)

They then have central charge \( d \). Singling out the \( j \)-independent terms, we write

\[
|\Upsilon\rangle = \left( \frac{k}{2q^2} l_{-1}^2 + l_{-2} \right) |\Psi\rangle + \ldots
\]  

(4.15)

By virtue of (4.13) the term written out explicitly is by itself a null vector, and can therefore be set to zero in \( \mathcal{M} \). As to the other terms on the RHS of (4.15), we substitute

\[
\begin{aligned}
\bar{L}_{-1}|\bar{\Psi}\rangle &= \frac{q}{k} j_{-1}|\bar{\Psi}\rangle, \\
\bar{L}_{-2}|\bar{\Psi}\rangle &= \left( \frac{q}{k} j_{-2} + \frac{1}{2k} j_{-1}^2 \right) |\bar{\Psi}\rangle, \\
\bar{L}_{-1}^2|\bar{\Psi}\rangle &= \left( \frac{q}{k} j_{-2} + \frac{q^2}{k^2} j_{-1}^2 \right) |\bar{\Psi}\rangle
\end{aligned}
\]
Then, with the coefficients chosen as in (4.8), all the other terms cancel out, and thus the ellipsis in (4.15) vanishes.

We are thus left with the null vector

\[
\left( \frac{k}{2q^2} l_1^2 + l_2 \right) |\psi\rangle
\]

in the matter Hilbert space \( \mathcal{M} \). The dimension of \( |\psi\rangle \) in the matter sector is found from

\[
L_0 |\Psi\rangle = \left( l_0 + \frac{1}{2k} j_0^2 \right) |\Psi\rangle
\]

and equals

\[
\delta = \Delta - \frac{1}{2k} q^2 = \frac{5 - d \mp \sqrt{(1 - d)(25 - d)}}{16}
\]

Viewing this as the ‘bare’ dimension we see that tensoring with the current \( j \) is equivalent to the gravitational dressing: evaluating the gravitational scaling dimension according to \([6, 10]\),

\[
\hat{\delta}_\pm = \pm \sqrt{1 - d + 24\delta} - \sqrt{1 - d}
\]

we find

\[
\delta_+ = \frac{3}{8} \pm \frac{d - 4 - \sqrt{(1 - d)(25 - d)}}{24}
\]

The sign on the RHS corresponds to that in (4.17) and the previous formulae. In particular, choosing the lower signs throughout, we have

\[
\hat{\delta}_+ = \Delta + \frac{1}{2}
\]

Therefore, up to the shift by 1/2 (which seems somewhat mysterious), the dimension \( \Delta \) with respect to the full Virasoro algebra acting in the tensor product space \( \mathcal{M} \otimes \mathcal{C} \), is equal to the scaling dimension of the gravity-dressed operators. Thus, the role of the gravitational dressing is effectively played by tensoring with the theory defined by

\[
[j_m, j_n] = km\delta_{m+n,0},
\]

\[
j_{n>0}|0\rangle = 0, \quad j_0|0\rangle = q|0\rangle
\]

with central charge 1 and negative \( q^2/k \). The reason for a current to appear at all is that it serves to represent the hierarchy flows and thus signifies the “hierarchical” origin of the theory. This also provides a new insight into the theory of completely integrable evolutions: for Virasoro-constrained hierarchies these amount to the Liouville dynamics in the conformal gauge.

To return to the relation with the formalism of \([3, 14]\), recall that the Coulomb-gas realization of the matter theory requires introducing a scalar field \( \varphi \) with the energy-momentum tensor

\[
T_m = -\frac{1}{2} \partial \varphi \partial \varphi + i \frac{Q_m}{2} \partial^2 \varphi
\]
Then the matter central charge is equal to $1 - 3Q_m^2$, and equating this with $d$ we invert (4.13) as

$$d = 1 - 3\frac{(Q^2 - 2)^2}{Q^2},$$

and thus

$$Q_m^2 = \left(Q - \frac{2}{Q}\right)^2$$

(4.23)

On the other hand, the parameter $Q$ was introduced initially in the Virasoro constraints (2.9) (where $J = \frac{Q+1}{2}$) as the background charge of an abstract $bc$ system (cf. eq.(2.13)). Now, it has to be tuned as

$$Q^2 = \left(\frac{Q_m \pm \sqrt{Q_m^2 + 8}}{2}\right)^2 = \left(\frac{Q_m \pm Q_L}{2}\right)^2 = \left(-\frac{Q_L \mp Q_m}{2}\right)^2$$

(4.24)

where

$$Q_L = \sqrt{Q_m^2 + 8}$$

(4.25)

is the background charge of the ‘Liouville’ scalar field [10, 6] with the energy-momentum tensor

$$T_L = -\frac{1}{2} \partial \phi \partial \phi - \frac{Q_L}{2} \partial^2 \phi$$

(4.26)

Equivalently, one sees that (for the respective signs in (4.13)),

$$Q^2 = \alpha^2_\pm$$

(4.27)

where $e^{\alpha + \phi}$ is the gravitational dressing of the identity operator. This establishes the physical meaning of $Q$ (note that $Q$ enters explicitly in the Kontsevich transform through (3.11)). - It looks like the $bc$ system underlying eqs.(2.13) and (2.15) describes (upon imposing the Virasoro constraints) a ‘mixture’ of the matter and Liouville theories.

5 Concluding remarks

1. It remains an open problem to represent the $N$–reduced master equation as a Ward identity of a matrix integral.

2. Our approach was based on a general construction of Virasoro generators on the phase space of integrable hierarchies [36, 38], and, as the $N$-KdV hierarchies do not seem so much formally distinguished in any way, it must apply also to other Virasoro-constrained hierarchies, including the “discrete” ones, e.g. Toda [19, 28, 36]. This may be especially interesting in view of the lack of a “discretized” version of the Kontsevich model (which does by itself seem to be ‘discrete’), while, on the other hand, Virasoro constraints on the Toda hierarchy [19, 29] have been shown [34] to undergo a continuum limit into Virasoro constraints on a KP hierarchy obtained from Toda also as a result of the scaling. It would be interesting to investigate what kind of a Kontsevich-type matrix integral the corresponding master equation may be derived from.
3. Various aspects of the conversion of Virasoro constraints into decoupling equations would be interesting, in particular, from the ‘Liouville’ point of view. The Kontsevich-type matrix integral whose Ward identities coincide with our master equation may thus provide a candidate for a discretized definition of the Liouville theory.

It was implicitly understood in Sect.4 that the matter central charge $d$ should be fixed to the minimal-models series; then factoring out the null-vector leads to a bona fide minimal model (and our $\psi$ thus becomes the ‘21’ operator). Now, thinking in terms of the minimal models, how can the higher null-vectors be arrived at starting from the Virasoro-constrained hierarchies? If these vectors correspond to higher symmetries of Virasoro-constrained hierarchies, then the whole Kac table must have a relation to the $W_\infty$ algebra.

Acknowledgements. I am grateful to O.Andreev, A.Zabrodin and A.Mironov for useful remarks and to A.Subbotin and R.Metsaev for valuable suggestions on the manuscript.
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