Hecke relations among 2d fermionic RCFTs

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ABSTRACT: Recently, Harvey and Wu proposed a suitable Hecke operator for vector-valued $SL(2, \mathbb{Z})$ modular forms to connect the characters of different 2d rational conformal field theories (RCFTs). We generalize such an operator to the 2d fermionic RCFTs and call it fermionic Hecke operator. The new Hecke operator naturally maps the Neveu-Schwarz (NS) characters of a fermionic theory to the NS characters of another fermionic theory. Mathematically, it is the natural Hecke operator on vector-valued $\Gamma_0$ modular forms of weight zero. We find it can also be extended to $\tilde{\text{NS}}$ and Ramond (R) sectors by combining the characters of the two sectors together. We systematically study the fermionic Hecke relations among 2d fermionic RCFTs with up to five NS characters and find that almost all known supersymmetric RCFTs can be realized as fermionic Hecke images of some simple theories such as supersymmetric minimal models. We also study the coset relations between fermionic Hecke images with respect to $c = 12k$ holomorphic SCFTs.

KEYWORDS: Conformal and W Symmetry, Scale and Conformal Symmetries

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1 Introduction

In recent years there has been a resurgence of interest in 2d fermionic and supersymmetric conformal field theories. Much progress has been made from various approaches including fermionic modular bootstrap [1–3], super modular category (SMC) [4–9], super vertex operator algebra (SVOA) [10–13], topological modular forms [14], fermionization by non-anomalous $\mathbb{Z}_2$ symmetry [15–21] and so on. In particular, many new fascinating examples of fermionic and supersymmetric CFTs were found which largely broaden our scope beyond the classical constructions in the 1980s such as the $\mathcal{N} = 1, 2$ supersymmetric minimal models. Not only many WZW models now are shown to have emergent supersymmetry [20], many RCFTs associated to sporadic groups are also known to be supersymmetric. For example, a series of renowned examples is the Suzuki chain [12].

In this work, we introduce a different approach called the fermionic Hecke operator which emphasizes more on the modularity of the fermionic characters of 2d fermionic rational CFTs. The Hecke operator for 2d bosonic RCFTs was introduced by Harvey and Wu [22] and later developed in [23–25]. Such an operator exploits the modularity of characters of 2d RCFTs and elegantly connects the characters of 2d RCFTs with different central charges. It also reveals new interesting number-theoretic properties of characters as well as nontrivial relations on the modular representations and fusion algebras in the space of RCFTs. The Hecke operator along with the coset with respect to $c = 8k$ holomorphic theories also provides a new paradigm to classify 2d bosonic RCFTs [25]. It is natural to consider how to generalize such an operator to the fermionic cases and utilize it to study various aspects of 2d fermionic RCFTs. This question was raised at the end of [25].

As the first step, we find that the most natural setting to define fermionic Hecke operator is in the Neveu-Schwarz (NS) sector of fermionic RCFTs. The main reason is that the NS characters transform to themselves under the $S$ modular action. More precisely, they form a weight-zero vector-valued modular form of a level-two congruence subgroup $\Gamma_\theta$ of $\text{SL}(2,\mathbb{Z})$, which is generated by the $S$ and $T^2$ actions. This may be not surprising since the super modular category is also defined by the NS data. We define the fermionic Hecke operator $T^F_p$ on the space of weight-zero $\Gamma_\theta$ vector valued modular forms such that it sometimes can naturally map the NS characters of a fermionic RCFT of central charge $c$ to the NS characters of another fermionic RCFT of central charge $pc$. This resembles the basic property of the bosonic Hecke operator. On the other hand, as we will show later, the $\bar{\text{NS}}$ and R characters combined together also form a $\Gamma_\theta$ vector-valued modular form, thus the fermionic Hecke operator can be applied on the $(\bar{\text{NS}}, \text{R})$ sector as well. The meaning of $(\bar{\text{NS}}, \text{R})$ sector will be explained in section 8.

To make the fermionic Hecke operator $T^F_p$ consistent with the bosonic Hecke operator $T_p$ defined in [22], a prerequisite is that it should commute with fermionization $\mathcal{F}$. This means that for a bosonic theory $B$ allowing fermionization, $T^F_p(\mathcal{F}B)$ should be the same as $\mathcal{F}(T_pB)$. We find that our definition of $T^F_p$ indeed satisfies this nontrivial condition. Note that such commutativity does not diminish the value of the fermionic Hecke operator for two reasons. One main reason is that when a (potential) fermionic theory is bootstrapped, e.g. from the fermionic modular linear differential equations (MLDEs) for NS characters,
or a (potential) object of super modular category is found, it is sometimes not easy to
determine precisely what the underlying full bosonic theory is or what the full bosonic
characters are.\textsuperscript{1} This urges an independent approach to establish maps just by the NS
data. Our definition of fermionic Hecke operator successfully realizes this goal, where a
key ingredient interestingly is a function very recently proposed in [9] in the study of the
super modular category. Besides, from the computational aspects, it is also much easier to
directly proceed with just some fermionic characters, rather than to first recover the full
bosonic characters, perform the bosonic Hecke operator and take fermionization in the end,
which sometimes is even impossible.

We summarize some nice examples of fermionic Hecke images we find in the following
table 1, which are also the main results of the current paper. Here we use $SM_{eff}(p',p)$
to denote the effective theory of supersymmetric minimal model $SM(p',p)$. We use $F$
for a single chiral fermion and $S$ for supersymmetrization. The $S^2$ stands for $\mathcal{N} = 2$
supersymmetry. The details will be given later.

The bosonic Hecke operator is closely related to the cosets with respect to holomorphic
CFTs [22–24]. These special CFTs are well-known to carry central charge $8k, k \in \mathbb{Z}$ and a
single bosonic character related to the Klein $J$ function. The coset relations with respect
to holomorphic CFTs along with the implication in MLDEs were nicely discussed in [27].
In [22], it was realized that many Hecke images can pair together to form a (potential)
c = 8k holomorphic theory, which suggests that the characters of a pair of Hecke images
satisfy a simple bilinear relation. For example, the bilinear combination of the characters of
a $c = 24$ pair of Hecke images usually gives $J(\tau) + N$, where $N$ is the sum of the number
of spin-1 currents for the two Hecke images. In the cases where both Hecke images are
physical, i.e. the characters of genuine 2d RCFTs, this gives the character $J(\tau) + N$ of
a $c = 24$ holomorphic CFTs in the list of Schellekens [28]. We find these nice properties
can be generalized to the fermionic cases, where we are interested in the pairs of fermionic
Hecke images forming $c = 12k$ holomorphic SCFTs which have one single NS character. The
bilinear combination of such a pair of fermionic Hecke images produces a degree $k$ polynomial
of a function $K(\tau) = (\theta_3/\eta)^{12} - 24$. The simplest case, i.e. the $c = 12$ holomorphic SCFTs
have been classified by Creutzig, Duncan and Riedler in 2017 [11] which contain only three
cases: supersymmetric $E_{8,1}$ theory $SE_8$, Conway SCFT [10] and the theory of 24 free chiral
fermions $F_{24}$. We will show that indeed the characters of a $c = 12$ pair of physical fermionic
Hecke images always have bilinear combination equal to $K(\tau), K(\tau) + 8$ or $K(\tau) + 24$, which
are the single NS character of Conway SCFT, $SE_8$ or $F_{24}$ respectively.

Hecke operators are also known to have a deep connection with the modular tensor
categories (MTCs) [23]. The data of an MTC is defined by central charge modulo 8 and
conformal weights modulo 1. It was found in [23] that Hecke operators can induce Galois
conjugation in an MTC. The classification results with a small number of characters in [25]
also suggest that Hecke operators along with $c = 8k$ cosets should be able to produce

\textsuperscript{1}The main reason is that the $\tilde{R}$ sector of a fermionic RCFT can not be obtained from the modular
transformation of the NS sector, and only when all four sectors NS, $\tilde{N}$S, $R$ and $\tilde{R}$ are known, one can recover
the underlying bosonic RCFT.
| Theory          | $c$ | $h_{NS}$ | $T_{F_p}^F$ | Theory          | $\hat{c}$ | $\hat{h}_{NS}$ |
|-----------------|-----|----------|-------------|-----------------|------------|----------------|
| $F$             | $\frac{1}{2}$ | $-$ | $T_{51}^F$  | $\mathcal{F}(E_8)_{2}$ | $\frac{31}{2}$ | $-$ |
|                 |     |         | $T_{35}^F$  | $\mathcal{F}(C_{10})_{1}$ | $\frac{35}{2}$ | $-$ |
|                 |     |         | $T_{37}^F$  | $\mathbb{B}$ | $\frac{47}{2}$ | $-$ |
| $\mathcal{F}(A_1)_{3}$ | $2$ | $-$ | $T_{7}^F$   | $\mathcal{F}(E_7)_{1}$ | $14$ | $-$ |
| $\mathcal{F}(A_3)_{1}$ | $3$ | $-$ | $T_{5}^F$   | $\mathcal{F}(A_{15})_{1}$ | $15$ | $-$ |
| $SM_{eff}(8, 2)$ | $\frac{3}{4}$ | $\frac{1}{4}$ | $T_{3}^F$   | $\mathcal{F}(A_1)_{6}$ | $\frac{9}{4}$ | $\frac{1}{4}$ |
|                 |     |         | $T_{13}^F$  | $\mathcal{F}(C_6)_{1}$ | $\frac{39}{4}$ | $\frac{3}{4}$ |
| $SM_{eff}(8, 2)_{2}$ | $\frac{3}{2}$ | $\frac{1}{2}$ | $T_{5}^F$   | $\mathcal{F}(E_7)_{2}$ | $\frac{133}{10}$ | $\frac{9}{10}$ |
| $E_8$ inv of $SM_{eff}(60, 2)$ | $\frac{7}{6}$ | $\frac{1}{6}$ | $T_{19}^F$  | $\mathcal{F}(B_7)_{3}$ | $\frac{91}{6}$ | $\frac{11}{10}$ |
| $SA_1$          | $\frac{3}{2}$ | $\frac{1}{2}$ | $T_{1}^F$   | $SE_7$ | $\frac{21}{2}$ | $\frac{3}{4}$ |
| $S^2A_1$        | $1$ | $\left(\frac{1}{2}\right)_2$ | $T_{11}^F$  | $\mathcal{F}(A_{11})_{1}$ | $11$ | $\left(\frac{6}{5}\right)_2$ |
| $S^2A_1 \otimes F$ | $\frac{3}{2}$ | $\left(\frac{1}{2}\right)_2$ | $T_{5}^F$   | $\mathcal{F}(E_6)_{4}$ | $\frac{39}{2}$ | $\left(\frac{1}{2}\right)_2$ |
| $SM_{eff}(12, 2)$ | $1$ | $\frac{1}{6}, \frac{1}{2}$ | $T_{5}^F$   | $\mathcal{F}(B_3)_{3}$ | $5$ | $\frac{1}{3}, \frac{1}{2}$ |
|                 |     |         | $T_{7}^F$   | $\mathcal{F}(C_3)_{2}$ | $7$ | $\frac{1}{2}, \frac{2}{3}$ |
| $SM_{eff}(8, 2)_{2}$ | $\frac{3}{2}$ | $\left(\frac{1}{2}\right)_2, \frac{1}{2}$ | $T_{5}^F$   | $\mathcal{F}(A_3)_{4}$ | $\frac{15}{2}$ | $\left(\frac{1}{2}\right)_2, 1$ |
| $D_6$ inv of $SM_{eff}(20, 2)$ | $\frac{6}{5}$ | $\frac{1}{10}, \left(\frac{3}{10}\right)_2$ | $T_{3}^F$   | $\mathcal{F}(A_1)_{3}$ | $\frac{18}{5}$ | $\frac{3}{10}, \left(\frac{2}{5}\right)_2$ |
| $SM_{eff}(16, 2)$ | $\frac{9}{4}$ | $\frac{1}{8}, \frac{3}{8}, \frac{3}{4}^2$ | $T_{7}^F$   | $\mathcal{F}(B_3)_{3}$ | $\frac{63}{8}$ | $\frac{1}{3}, \frac{3}{8}, \frac{3}{4}$ |
| $SM_{eff}(16, 2)$ | $1$ | $\frac{1}{10}, \frac{1}{5}, \frac{1}{2}$ | $T_{11}^F$  | $\mathcal{F}(D_6)_{2}$ | $11$ | $\frac{11}{10}, \frac{1}{5}, \frac{1}{2}$ |
| $SM_{eff}(6, 4)$ | $\frac{5}{4}$ | $\frac{1}{12}, \frac{1}{4}, \frac{1}{2}^2_2$ | $T_{11}^F$  | $\mathcal{F}(A_5)_{2}$ | $\frac{55}{4}$ | $\frac{7}{12}, \frac{3}{4}, \frac{1}{2}^2_2$ |
| $SM_{eff}(8, 6)$ | $\frac{5}{4}$ | $\frac{1}{12}, \frac{1}{4}, \frac{1}{2}^2_2$ | $T_{11}^F$  | $\mathcal{F}(A_5)_{2}$ | $\frac{55}{4}$ | $\frac{7}{12}, \frac{3}{4}, \frac{1}{2}^2_2$ |
| $D_8$ inv of $SM_{eff}(28, 2)$ | $\frac{7}{2}$ | $\frac{1}{14}, \frac{1}{7}, \frac{1}{2}^2_2$ | $T_{5}^F$   | $\mathcal{F}(D_3)_{3}$ | $\frac{47}{7}$ | $\frac{5}{14}, \frac{9}{14}, \frac{1}{2}^2_2$ |
| $D_{10}$ inv of $SM_{eff}(36, 2)$ | $\frac{4}{3}$ | $\frac{1}{15}, \frac{1}{5}, \frac{1}{3}, \frac{1}{2}^2_2$ | $T_{7}^F$   | $\mathcal{F}(D_4)_{3}$ | $\frac{28}{3}$ | $\frac{7}{15}, \frac{5}{15}, \frac{2}{3}$ |

Table 1. Examples of fermionic Hecke relations. The inv is short for non-diagonal $\Gamma$ modular invariant. The $h_{NS}$ are the weights of non-vacuum NS primaries with degeneracy marked. The $\mathbb{B}$ stands for Baby Monster SVOA [26].

all objects in an MTC. We notice that this paradigm can be generalized to fermionic Hecke operator and super modular category in which the SMC data is defined by central charge and NS weights both modulo 1/2. In fact, we conjecture that the fermionic Hecke operator and some proper cosets can uniformly generate all objects in an SMC. We leave the discussion on the relationship between fermionic Hecke operator and SMCs to future work and focus on characters of RCFTs in the current one.

This paper is organized as follows. In section 2, we give an overview on the 2d fermionic and supersymmetric RCFTs. We will review many known examples and the concept of cosets.
with respect to $c = 12$ holomorphic SCFTs. In section 3, we introduce our new approach — fermionic Hecke operator and discuss its main properties. In particular, we will discuss the simplest yet still nontrivial example of free chiral fermions. In sections 4, 5, 6, 7, we will discuss the fermionic Hecke operation on some small fermionic theories with 2, 3, 4, 5 NS characters (modulo degeneracy) respectively. We will show that almost all known supersymmetric RCFTs can be realized as some fermionic Hecke images. In section 8, we briefly discuss the fermionic Hecke operation for $\tilde{\text{NS}}$ and R sectors. In section 9, we conclude and raise some questions for future work. Following our previous paper [25], we use $T^F_p, \star$ to represent fermionic Hecke images with NS quasi-characters which means there are negative Fourier coefficients. These images usually have coefficient $-1$ for the NS vacuum state, thus are unphysical theories. Besides, we adopt the convention of affine Lie algebras as in SageMath [29].

2 Basics of 2d fermionic RCFTs

2.1 Basics of fermionic RCFTs

2d RCFTs have rational central charge, rational conformal weights and a finite number of conformal primaries. For bosonic RCFTs, the torus partition function can be written as a sesquilinear combination of characters

$$Z(\tau, \bar{\tau}) = \sum_{i,j=0}^{d-1} M_{ij} \chi_i(\tau) \chi_j(\bar{\tau}), \quad M_{ij} \in \mathbb{N}. \quad (2.1)$$

The characters $\chi_i(\tau)$ together form a $d$-dimensional vector valued modular form of $\text{SL}(2, \mathbb{Z})$ such that the torus partition function is modular invariant. More precisely, there exists a $d$-dimensional representation $\rho : \text{SL}(2, \mathbb{Z}) \to \text{GL}(d, \mathbb{C})$ such that for any $\gamma \in \text{SL}(2, \mathbb{Z})$,

$$\chi_i(\gamma \tau) = \sum_j \rho(\gamma)_{ij} \chi_j(\tau). \quad (2.2)$$

For the two generators of $\text{SL}(2, \mathbb{Z})$,

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (2.3)$$

the matrices $\rho(T)$ and $\rho(S)$ are usually called the $T$ and $S$ matrices of the bosonic RCFT. The $T$ and $S$ matrices satisfy some constraints. For example, $C = \rho(S)^2 = (\rho(T)\rho(S))^3$ gives the charge conjugation matrix of the theory.

In the presence of fermions, it is necessary to specify the boundary conditions along the two cycles of the torus. They are either periodic, i.e., Ramond (R) condition or antiperiodic, i.e., Neveu-Schwarz (NS) condition for each cycle. Clearly, there exist four possible combinations of boundary conditions along the two cycles: (NS,NS), (R,NS), (NS,R), (R,R). It is traditional to call them the NS, $\tilde{\text{NS}}$, R, $\tilde{\text{R}}$ sectors respectively. These are also called the spin structures. In each sector, there is torus partition function defined by the trace over the Hamiltonians as

$$Z_{\text{NS}}(\tau, \bar{\tau}) = \text{Tr}_{\text{NS}} \left( q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} \right), \quad Z_{\text{NS}}(\tau, \bar{\tau}) = \text{Tr}_{\text{NS}} \left( q^{L_0 - \frac{1}{2}} \bar{q}^{\bar{L}_0 - \frac{c}{24} (-1)^F} \right),$$

$$Z_{\text{R}}(\tau, \bar{\tau}) = \text{Tr}_{\text{R}} \left( q^{L_0 - \frac{1}{2}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} \right), \quad Z_{\text{R}}(\tau, \bar{\tau}) = \text{Tr}_{\text{R}} \left( q^{L_0 - \frac{1}{2}} \bar{q}^{\bar{L}_0 - \frac{c}{24} (-1)^F} \right). \quad (2.4)$$
Each of these partition functions can be written as a sesquilinear combination of the NS, \(\tilde{\text{NS}},\) R, \(\tilde{\text{R}}\) characters respectively. For example,

\[
Z_{\text{NS}}(\tau, \bar{\tau}) = \sum_{i,j=0}^{n-1} M_{ij}^{\text{NS}} \chi_{i}^{\text{NS}}(\tau) \chi_{j}^{\text{NS}}(\bar{\tau}), \quad M_{ij}^{\text{NS}} \in \mathbb{N}_{\geq 0}.
\]

Unlike the bosonic case, the four sectors of the fermionic theory do not necessarily transform to the same sector under \(\text{SL}(2, \mathbb{Z})\). By careful analysis on the boundary conditions, it can be found that the NS characters transform to themselves under \(S\), and to \(\tilde{\text{NS}}\) characters under \(T\). We summarize the transformation among different sectors by the following formula:

\[
T : \begin{pmatrix} \chi_{\text{NS}} \\ \chi_{\tilde{\text{NS}}} \\ \chi_{\text{R}} \\ \chi_{\tilde{\text{R}}} \end{pmatrix} (\tau + 1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi_{\text{NS}} \\ \chi_{\tilde{\text{NS}}} \\ \chi_{\text{R}} \\ \chi_{\tilde{\text{R}}} \end{pmatrix} (\tau). \tag{2.6}
\]

\[
S : \begin{pmatrix} \chi_{\text{NS}} \\ \chi_{\tilde{\text{NS}}} \\ \chi_{\text{R}} \\ \chi_{\tilde{\text{R}}} \end{pmatrix} (-\frac{1}{\tau}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi_{\text{NS}} \\ \chi_{\tilde{\text{NS}}} \\ \chi_{\text{R}} \\ \chi_{\tilde{\text{R}}} \end{pmatrix} (\tau). \tag{2.7}
\]

With the above \(T\) and \(S\) transformations, one can check that the following combination of the fermionic partition functions

\[
Z = \frac{1}{2} \left( Z_{\text{NS}} + Z_{\tilde{\text{NS}}} + Z_{\text{R}} + Z_{\tilde{\text{R}}} \right)
\]

is still modular invariant under \(\text{SL}(2, \mathbb{Z})\). Besides, the above transformations imply that the number of NS, \(\tilde{\text{NS}}\) and R characters are also the same, while the number of \(\tilde{\text{R}}\) characters is independent.

Although NS, \(\tilde{\text{NS}}\) and R characters are not modular under \(\text{SL}(2, \mathbb{Z})\), they are vector-valued modular forms for the following level-two congruence subgroups respectively (see e.g. [1])

\[
\Gamma_0 = \left\{ \gamma \in \text{SL}(2, \mathbb{Z}) | \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ mod 2} \right\},
\]

\[
\Gamma^0(2) = \left\{ \gamma \in \text{SL}(2, \mathbb{Z}) | \gamma \equiv \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \text{ mod 2} \right\}, \tag{2.9}
\]

\[
\Gamma_0(2) = \left\{ \gamma \in \text{SL}(2, \mathbb{Z}) | \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \text{ mod 2} \right\}.
\]

For example, suppose there are \(n\) number of NS characters, then there exists a \(n\)-dimensional representation \(\rho^F : \Gamma_0 \to \text{GL}(n, \mathbb{C})\) such that for any \(\gamma \in \Gamma_0\),

\[
\chi_{i}^{\text{NS}}(\gamma \tau) = \sum_{j} \rho_{ij}^{F}(\gamma) \chi_{j}^{\text{NS}}(\gamma \tau). \tag{2.10}
\]

It is easy to see that the \(\Gamma_0\) congruence subgroup is generated by \(S\) and \(T^2\) of \(\text{SL}(2, \mathbb{Z})\). Therefore, it is convenient to call \(\rho^F(S)\) and \(\rho^F(T^2)\) the \(S\) and \(T^2\) matrices of (the NS
characters of) a fermionic RCFT. We sometimes omit the $\rho^F$ when there is no room for confusion. From now on, we focus on the $S$ and $T^2$ matrices of the NS sector. Such matrices are known to satisfy the following constraints

$$\rho^F(S)^4 = \text{Id}, \quad \rho^F(S)^2 \rho^F(T^2) = \rho^F(T^2) \rho^F(S)^2. \quad (2.11)$$

Using $\rho^F(S)$ one can define the fusion rules of NS primaries inside the NS sector by the usual Verlinde formula [30].

The NS, $\tilde{\text{NS}}$ and R characters in general have the following kinds of Fourier expansion

$$\chi_{\text{NS}} = q^{-\frac{c}{2} + h_{\text{NS}}} \left( m_0 + m_1 q^{1/2} + m_1 q + m_3/2 q^{3/2} + \ldots \right),$$

$$\chi_{\tilde{\text{NS}}} = q^{-\frac{c}{2} + h_{\tilde{\text{NS}}}} \left( m_0 - m_1 q^{1/2} + m_1 q - m_3/2 q^{3/2} + \ldots \right), \quad (2.12)$$

$$\chi_{\text{R}} = q^{-\frac{c}{2} + h_{\text{R}}} \left( s_0 + s_1 q + s_2 q^2 + \ldots \right).$$

All coefficients $m_i$ and $s_i$ should be non-negative integers and $m_0$ for the NS vacuum should be 1. The $m_{1/2}$ measures the number of free fermions. The $k$ number of NS characters $\chi_{i\text{NS}}$ in general satisfy an order $k$ fermionic MLDE of $\Gamma_0$, while the $\chi_{i\tilde{\text{NS}}}$ and $\chi_{i\text{R}}$ satisfy fermionic MLDEs of $\Gamma_0(2)$ and $\Gamma_0(2)$ respectively [1]. The three different fermionic MLDEs are equivalent by modular transformations. In the current paper, we will be mainly interested in the NS characters. As a final remark, the $\tilde{\text{R}}$ characters are isolated, and do not have modular connection with NS, $\tilde{\text{NS}}$ and R characters. It is known for supersymmetric RCFTs, there is just one single $\tilde{\text{R}}$ character and it is a constant.

### 2.2 Examples of fermionic RCFTs

A large class of $\mathcal{N} = 1$ supersymmetric RCFTs are the *supersymmetric minimal models* $SM(p', p)$ for $2 \leq p \leq p' - 2$, $p' - p \in 2\mathbb{Z}$ and $\text{gcd}(\frac{p' - p}{2}, p) = 1$ [31–33]. They have only super Virasoro symmetry and are also called the minimal SCFTs. The central charge of $SM(p', p)$ is given by

$$c = \frac{3}{2} \left( 1 - \frac{2(p' - p)^2}{pp'} \right) \quad (2.13)$$

and the fermionic conformal weights are

$$h_{r,s} = \frac{(sp - p' r)^2 - (p' - p)^2}{8pp'} + \frac{2\epsilon_{r-s} - 1}{16}, \quad \epsilon_a = \begin{cases} \frac{1}{2} & a \in 2\mathbb{Z}, \\ 1 & a \in 2\mathbb{Z} + 1. \end{cases} \quad (2.14)$$

Here $r = 1, 2, \ldots, p - 1$ and $s = 1, 2, \ldots, p' - 1$. The $h_{r,s}$ with $r - s \in 2\mathbb{Z}$ belong to the NS sector, while those with $r - s \in 2\mathbb{Z} + 1$ belong to the R sector. Owing to the symmetry $(r, s) \leftrightarrow (p - r, p' - s)$, we only need to consider the $sp \leq p'r$ part. Denote $(q)_{\infty} = \prod_{n=1}^{\infty} (1 - q^n)$. The fermionic characters of $SM(p', p)$ are given by [34]

$$\chi_{r,s}(q) = \chi_{p-r,p'-s}(q) = \frac{(-q^{r-s})}{(q)_{\infty}} \sum_{\ell \in \mathbb{Z}} q^{\ell(\ell p' + r p' - s p')/2} - q^{\ell(p r + r p' + s p')/2}. \quad (2.15)$$

It is well-known that only for $p' = p + 2$, the $SM(p', p)$ is unitary [31–33]. The non-unitary supersymmetric minimal models and their non-diagonal modular invariants to
our knowledge are less studied. Most of them have negative central charges and negative fermionic conformal weights. It is useful to introduce the effective theory of non-unitary minimal models where one change the labels of vacuum and non-vacuum primaries to get a positive effective central charge. For the effective theory of bosonic non-unitary RCFTs, we refer to a good discussion in [23, section 3]. For example, the non-unitary Lee-Yang minimal model \( M(5,2) \) has central charge \( c = -22/5 \) and non-vacuum conformal weight \( h = -1/5 \). One can exchange the notion of vacuum and non-vacuum primaries by shifting \( c \) and \( h \) simultaneously while keeping \( -c^2/24 + h \) invariant. It is easy to see after such shifting, the central charge becomes \( c_{\text{eff}} = 2/5 \), while non-vacuum weight becomes \( h_{\text{eff}} = 1/5 \). This is called the effective Lee-Yang theory. The effective minimal model \( M_{\text{eff}}(p', p) \) has effective central charge

\[
\frac{3}{2} \left( 1 - \frac{8}{pp'} \right),
\]

and non-negative effective weights

\[
h_{r,s}^{\text{eff}} = \frac{(sp - p'r)^2 - 4}{8pp'} + 2\epsilon_{r-s} - 1/16.
\]

Mimicking the level \( k \) Lee-Yang model defined in our previous paper [25], we introduce the notion of level \( k \) supersymmetric Lee-Yang models defined by

\[
(SLY)_k := SM_{\text{eff}}(2, 4k + 4).
\]

The series of non-unitary \( SM(2, 4k + 4) \) models was extensively studied in the 1990s, see for example in [36–39]. They have effective central charge

\[
c = \frac{3}{2} \left( 1 - \frac{1}{1 + k} \right) = \frac{3k}{2(1 + k)}
\]

and \( k + 1 \) NS characters with effective weights

\[
h_{r,s}^{\text{NS}} = \frac{i(i + 1)}{4(k + 1)}, \quad i = 0, 1, \ldots, k.
\]

The NS characters have the following infinite product expression [37] up to \( q^{-c/24 + h_i} \):

\[
\chi_i(q) = \prod_{n \in \mathcal{I}_{k,i}} (1 - q^{n/2})^{-1},
\]

where \( \mathcal{I}_{k,i} = \{ n \in \mathbb{N} | n \neq 2 \mod 4, \text{ and } n \neq 0, \pm(2k + 1 - 2i) \mod (4k + 4) \} \). The \( S \)-matrix for the NS characters \( \chi_i \) can be determined as

\[
S_{ij} = \sqrt{\frac{2}{k + 1}} \cos \left( \frac{\pi(2i + 1)(2j + 1)}{4k + 4} \right), \quad i, j = 0, 1, \ldots, k.
\]
Similar to the bosonic minimal models, the $\mathcal{N} = 1$ minimal models also have interesting and rich modular invariants. The modular invariants of unitary $SM(p + 2, p)$ models have been classified by [40–42]. The modular invariants of non-unitary $\mathcal{N} = 1$ minimal models to our knowledge are less studied. In this paper, we find several non-diagonal $\Gamma_g$ modular invariants of non-unitary $\mathcal{N} = 1$ minimal models which serve as good input to study fermionic Hecke operations. Besides, some special $\mathcal{N} = 1$ minimal models can be realized as fermionization of bosonic minimal models including the famous examples $SM(5, 3) \equiv \mathcal{FM}(5, 4)$ and $SM(8, 2) \equiv \mathcal{FM}(8, 3)$. A less known example found in [37] is that $SM(7, 3)$ can be realized as the fermionization of the $E_6$ modular invariant of $M(12, 7)$. We will encounter all these examples in what follows.

There also exist a series of unitary $\mathcal{N} = 2$ minimal models for $k = 1, 2, 3, \ldots$, which comes from the coset construction of $\mathcal{N} = 2$ superconformal algebras by SU(2) affine Lie algebras [43, 44]. We denote these theories as $S^2A_k$. They have central charge $c = 3k/(k + 2)$ and NS weights $h^\text{NS}_{ab} = (ab - 1/4)/(k + 2)$ with $a, b \in \mathbb{Z} + 1/2$ and $0 < a, b, (a + b) < k + 1$. In the current paper, we will encounter the $A_1$ case with $c = 1$, which in fact can be realized as a non-diagonal modular invariant of $SM(6, 4)$ theory. For the NS character formulas of $S^2A_k$, we refer to e.g. [1, equation (5.41)].

Another large class of $\mathcal{N} = 1$ RCFTs are the supersymmetric ADE WZW models of level 1. For $G$ of ADE type, we can couple the bosonic WZW model $(G)_1$ with rank$(G)$ number of free chiral fermions to produce obviously supersymmetric theories. We denote these models obtained from the supersymmetrization of WZW $(G)_1$ as $SG$ theories. Apparently the central charge of $SG$ is just $3/2$ times the central charge of WZW $(G)_1$, and the NS weights are the same as the bosonic conformal weights of WZW $(G)_1$. The $S$ matrix of $SG$ is also identical to the one of WZW $(G)_1$. In the analysis of second order $\Gamma_g$ MLDEs [1], the $SG$ theories for $G = A_1, A_2, D_4, E_6, E_7$ already appeared as solutions. We notice that the non-vacuum NS weights of these theories $h^\text{NS} = \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}$ are just the entries not equal to 0 or 1 in the Farey sequence of order 4. This resembles the observation in [22] that the non-vacuum conformal weights of the solutions of bosonic second order MLDEs [45] are the entries not equal to 0 or 1 in the Farey sequence of order 5.

In the above, we have been reviewing the examples of supersymmetric RCFTs. There are of course many fermionic RCFTs that are not supersymmetric. For example, a series of renowned fermionic RCFTs comes from the fermionization of WZW $(A_1)_{4k + 2}$ models, see e.g. [5]. The $\mathcal{F}(A_1)_{4k + 2}$ theory has $k + 1$ NS, $\bar{\text{NS}}$ and R characters for each sector and $k$ $\bar{\text{R}}$ characters. It is known that only for $\mathcal{F}(A_1)_6$, the fermionic theory becomes supersymmetric. The fermionic characters and the affine $(A_1)_6$ characters have the following well-known relations:

\begin{equation}
\begin{aligned}
\lambda_0^\text{NS} &= \lambda_{(A_1)_6}^{(A_1)_6} + \lambda_{3/2}^{(A_1)_6}, & \lambda_0^{\text{NS}} &= \lambda_1^{(A_1)_6} + \lambda_3^{(A_1)_6}, \\
\tilde{\lambda}_0^\text{NS} &= \lambda_{(A_1)_6}^{(A_1)_6} - \lambda_{3/2}^{(A_1)_6}, & \tilde{\lambda}_0^\text{NS} &= \lambda_1^{(A_1)_6} - \lambda_3^{(A_1)_6}, \\
\lambda_{3/32}^R &= \lambda_{3/32}^{(A_1)_6} + \lambda_{35/32}^{(A_1)_6}, & \lambda_{3/32}^R &= \sqrt{2}\lambda_{15/32}^{(A_1)_6}, \\
\lambda_{3/32}^{\bar{R}} &= \lambda_{3/32}^{(A_1)_6} - \lambda_{35/32}^{(A_1)_6} = 2, & \lambda_{3/32}^{\bar{R}} &= \sqrt{2}\lambda_{15/32}^{(A_1)_6}.
\end{aligned}
\end{equation}
where the last constant equality is the result of a Macdonald identity. There exist many more supersymmetric RCFTs that come from the fermionization of WZW models. See good summaries in [12], [20, table 1] and [1, 2]. From a modern viewpoint, these come from the orbifold of a non-anomalous $Z_2$ symmetry and the generalized Jordan-Wigner transformation [16, 18]. We will show later that almost all known WZW models with emergent supersymmetry can be realized as fermionic Hecke images.

### 2.3 Cosets with respect to $c = 12k$ holomorphic SCFTs

Holomorphic 2d SCFTs or the so-called self-dual SVOAs with central charge $c = 12$ are the natural supersymmetric analogies of holomorphic CFTs with $c = 24$. Similar to the renowned Schellekens’ list [28] of the 71 holomorphic CFTs with $c = 24$, there is also a complete classification of $c = 12$ holomorphic SCFTs. It has been proved in [11] that there are only three cases of such holomorphic SCFTs: supersymmetric WZW $E_{8,1}$ theory $SE_8$, Conway SCFT [10] and the theory of 24 chiral fermions $F_{24}$. Furthermore, the $F_{24}$ allows eight possible affine Lie algebra structures also called $\mathcal{N} = 1$ structures [13]:

$$(A_1)_2^8, (A_2)_3^3, (A_4)_5, (A_3)_4(A_1)_2^2, (B_2)_3(G_2)_4, (B_2)_3(A_2)_3(A_1)_2^2, (B_3)_5(A_1)_2, (C_3)_4(A_1)_2.$$  

Interestingly, these eight affine Lie algebra structures also appeared earlier in [46] in the study of Borcherds products and theta blocks. It is worthwhile to point out that all these $c = 12$ holomorphic SCFTs have further hyperbolic structures, i.e., Borcherds-Kac-Moody superalgebras. For example, the $SE_8$ has hyperbolization called the fake Monster superalgebra, describing the physical states of 10-dimensinal superstring moving on torus [47]. The BKM superalgebra with Conway symmetry was constructed in [48]. The BKM superalgebras associated to $F_{24}$ were constructed in [13]. The eight affine Lie algebras in $F_{24}$ and the 69 ones in the Schellekens’ list [28] have an uniform description as the hyperbolization of affine Lie algebras in [49].

As we mentioned earlier, a holomorphic SCFT with $c = 12$ has one single NS character $K(\tau) + n$ where $K(\tau)$ is defined by\footnote{The Dedekind $\eta$ function is defined as $\eta = q^{1/24} \prod_{j=1}^{\infty} (1 - q^j)$. The Jacobi theta functions are defined as $\theta_2 = \sum_{j=-\infty}^{\infty} q^{(j+1/2)^2/2}, \theta_3 = \sum_{j=-\infty}^{\infty} q^{j^2/2}$ and $\theta_4 = \sum_{j=-\infty}^{\infty} (-1)^j q^{j^2/2}$.}

$$K(\tau) = (\theta_3/n)^{1/2} - 24 = q^{-1/2} + 276q^{1/2} + 2048q + 11202q^{3/2} + 49152q^2 + \ldots \quad (2.24)$$

The $n = 0$ case corresponds to Conway SCFT, while $n = 8$ and $n = 24$ correspond to $SE_8$ and $F_{24}$. More generally, a holomorphic SCFT with $c = 12k$ has one single NS character that can be written as a degree $k$ polynomial $P_k(K(\tau))$. The $\tilde{NS}$ and $R$ characters can easily be obtained from the modular transformation of $K(\tau)$, while the $\bar{R}$ character is also some constant.

We are interested in the cosets with respect to $c = 12k$ holomorphic SCFTs. They are completely parallel to the bosonic generalized cosets discussed in [27], thus we only state the results. Consider a formal coset $\mathcal{C} = \mathcal{G}/\mathcal{H}$ where $\mathcal{G}$ is a $c = 12k$ holomorphic SCFT, and $\mathcal{H}$ is a fermionic sub-theory. Then $\mathcal{C}$ can be defined by the chiral algebra generators which
have trivial OPE with all those in $\mathcal{H}$ and it is automatically a fermionic CFT. The central charge obviously satisfies $c_C = c_H - c_H$. More importantly, the weights of non-vacuum NS primaries should satisfy

$$h_i^H + h_i^C = n_i, \quad \text{with } 2n_i \in \mathbb{N}. \quad (2.25)$$

We call $C$ and $H$ as a $c = 12$ pair of two fermionic CFTs. The main feature concerning us is the bilinear relation of the NS characters of a $c = 12$ pair of two fermionic CFTs. Given the classification in [11], any such pair of physical theories should have

$$\chi_{NS}^{(c)} \chi_{NS}^{(12-c)} = K(\tau) + n, \quad n = 0, 8, 24. \quad (2.26)$$

This equation is actually a very strong constraint. We will show later that some $c = 12$ pairs of fermionic Hecke images produce bilinear relation $K(\tau) + n$ with $n$ different from $0, 8, 24$. In those cases, the $c = 12$ pair does not correspond to a consistent holomorphic SCFT. On the other hand, we also find many good pairs of fermionic Hecke images indeed producing the admissible $K(\tau) + n$. For example, $SA_1$ and its $T_{17}$ image $SE_7$ can pair together to produce $SE_8$. More generally, one can consider holomorphic SCFTs with central charge $c = 12k$. The bilinear relation of a $c = 12k$ pair of fermionic theories can be written as

$$\chi_{NS}^{(c)} \chi_{NS}^{(12k-c)} = P_k(K(\tau)). \quad (2.27)$$

For $k > 1$, to our knowledge there is no classification of $c = 12k$ holomorphic SCFTs yet. An interesting constraint was recently proposed in [14] from the viewpoint of topological modular forms. We expect pairs of fermionic Hecke images can produce many potential holomorphic SCFTs.

3 Hecke relations

3.1 Bosonic Hecke relations

To introduce fermionic Hecke operator, we first briefly review the definition of bosonic Hecke operator given by Harvey and Wu [22]. Consider an arbitrary bosonic RCFT with central charge $c$, conformal weights $h_i$ and characters $\chi_i$. Denote the least common denominator of $h_i - c/24$ as $N$, which is called the conductor of the theory. Apparently, an equivalent definition is the smallest number $N$ such that $\rho(T)^N = \text{Id}$. It was proved by Bantay [50] that each character $\chi_i$ itself is invariant under $\tau \to \gamma \tau$ for any $\gamma \in \Gamma(N)$ defined as

$$\Gamma(N) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}(2, \mathbb{Z}) | a \equiv d \equiv 1 \pmod{N}, \ b \equiv c \equiv 0 \pmod{N} \right\}. \quad (3.1)$$

Clearly, the congruence subgroup $\Gamma(N)$ is the kernel of the canonical mod $N$ map $\mu_N : \text{SL}(2, \mathbb{Z}) \to \text{SL}(2, \mathbb{Z}_N)$.

For a $\text{SL}(2, \mathbb{Z})$ modular form $f(\tau)$ of weight 0, the Hecke operator $T_p$ for prime number $p$ is defined by

$$(T_p f)(\tau) := p^{-1} \sum_{a,d > 0, ad = p b(\text{mod } d)} f \left( \frac{a \tau + b}{d} \right). \quad (3.2)$$
For modular forms of $\Gamma(N)$, one has to take into consideration the nature of vector-valued modular forms as well. The proper generalization of the Hecke operator to such circumstances was found by Harvey and Wu. Denote $\bar{p}$ as the multiplicative inverse of $p$ modulo $N$ and $\sigma_p$ as $\mu_N^{-1}\text{diag}(\bar{p},p)$. Then Hecke operator $T_p$ acts on $f_i(\tau)$ as

$$
(T_p f)_i(\tau) := \sum_j \rho_{ij}(\sigma_p)f_j(\tau) + \sum_{b=0}^{p-1} f_i\left(\frac{\tau + bN}{p}\right).
$$

This definition does not preserve $\rho(S)$ and $\rho(T)$, thus is not technically an automorphism. However, this is also an advantage as it releases more possibilities which are encoded in the matrix $\rho(\sigma_p)$. We will call each $\rho(\sigma_p)$ as a Hecke class. All Hecke classes form a finite abelian group related to the quadratic residue modulo $N$. In practice, the transfer matrix $\rho(\sigma_p)$ can be computed by the $S$ and $T$ matrices as

$$
\rho(\sigma_p) = \rho(T^\bar{p}S^{-1}T^pST^\bar{p}S). \tag{3.4}
$$

The Hecke operator can also equivalently be defined by the map of Fourier coefficients. Suppose $f_i(\tau) = \sum_n a_i(n)q^n$ and $(T_p f_i)(\tau) = \sum_n a_i^{(p)}(n)q^n$. Then the Hecke operator (3.3) gives the map

$$
a_i^{(p)}(n) = \begin{cases} 
pa_i(pm), & p \nmid n, 
\rho_{ij}(\sigma_p)a_j\left(\frac{n}{p}\right), & p \mid n.
\end{cases} \tag{3.5}
$$

The $p$ divisibility of this kind of series $a_i^{(p)}$ is called the mod $p$ property. The Hecke operator (3.3) can be further generalized to non-prime $p$ with $\text{gcd}(p,N) = 1$ by

$$
\begin{cases}
T_{rs} = T_r \circ T_s, & \text{gcd}(r,s) = 1, 
T_p^{n+1} = T_p^n - p\sigma_p \circ T_p^{n-1}, & p \text{ prime}.
\end{cases} \tag{3.6}
$$

The $S$ and $T$ matrices of the Hecke image $T_p$ are related to those of the input theory by

$$
\rho^{(p)}(T) = \rho(T^\bar{p}), \quad \rho^{(p)}(S) = \rho(\sigma_p S). \tag{3.7}
$$

These nice properties make it often possible to pair two Hecke images together to form a holomorphic CFT with $c = 8k$, while the bilinear relation of the characters of the two Hecke images goes to simple functions related to Klein $J$ function.

### 3.2 Fermionic Hecke relations

In this section, we define the fermionic Hecke operator for the NS characters of fermionic RCFTs. As we mentioned earlier, the reason why the NS sector is the most natural setting is that it transforms to itself under $S$ and $T^2$ of $\text{SL}(2,\mathbb{Z})$, i.e., the NS characters form a $\Gamma_\theta$ vector-valued modular form of weight 0. The conductor $N$ of a fermionic RCFT is always the same as the conductor of its bosonic theory. We can also define the conductor just by the NS data as the minimal positive integer $N$ such that $\rho^F(T^2)^{N/2} = \text{Id}$. Thus the conductor of any fermionic RCFT is always even.
To generalize from bosonic Hecke operator to the fermionic one, the key point is to define a fermionic transfer matrix $\rho^F_I(\sigma_p)$ as a sub-matrix of the bosonic one:

$$\rho^F_I(\sigma_p) \subset \rho_{ij}(\sigma_p).$$

(3.8)

Here $I, J$ are the indices of NS characters. This is required because fermionization and Hecke operation should be commutative. Most non-trivially, we find that the matrix $\rho^F_I(\sigma_p)$ can be independently defined just by the NS data, i.e., $\rho^F(S)$ and $\rho^F(T^2)$, as

$$\rho^F(\sigma_p) = \rho^F\left(S^2(T^2)^{\frac{p^2-\rho}{2}}S(T^2)\left(-\frac{p_1}{p}\right)S(T^2)S^{p-1}\right).$$

(3.9)

This combination is a slight modification of a function proposed very recently in [9, appendix B] in the study of SMC.® It is well-defined owing to the fact that as long as $p$ is odd, $\frac{p^2-\rho}{2}$ and $\frac{p_1}{p}$ are integers. A direct computation shows that

$$\left(S^2(T^2)^{\frac{p^2-\rho}{2}}S(T^2)\left(-\frac{p_1}{p}\right)S(T^2)S^{p-1}\right) = \begin{pmatrix} \bar{p}(p\bar{p} - 1)^2 + \bar{p} & -\bar{p} - (\bar{p} - 1)^2(p\bar{p} - 1) \\ p\bar{p} - 1 & 1 - p\bar{p} + p \end{pmatrix}$$

\equiv \begin{pmatrix} \bar{p} & 0 \\ 0 & p \end{pmatrix} \mod N.

(3.10)

Therefore this combination produces the correct $\sigma_p$ as the preimage $\mu_N^{-1}\text{diag}(\bar{p}, p)$. With these notions, we can introduce the following fermionic Hecke operator $T^F_p$ for the NS characters $f^\text{NS}_I$:

$$(T^F_p f^\text{NS}_I)(\tau) := \sum_J \rho^F_J(\sigma_p)f^\text{NS}_J(p\tau) + \sum_{b=0}^{p-1} f^\text{NS}_I\left(\tau + bN/p\right).$$

(3.11)

Apparently, this operator gives a similar map of the Fourier coefficients as (3.5). Suppose $f_I(\tau) = \sum_n b_I(n) q^{\frac{n}{N}}$ and $(T^F_p f_I)(\tau) = \sum_n b_I^{(p)}(n) q^{\frac{n}{N}}$. Then we have

$$b_I^{(p)}(n) = \begin{cases} p b_I(pn), & p \nmid n, \\ p b_I(pn) + \sum_J \rho^F_{IJ}(\sigma_p)b_J(n), & p \mid n. \end{cases}$$

(3.12)

This operator can also be further generalized to non-prime $p$ with $\gcd(p, N) = 1$. Analogous to the bosonic case, we find

$$\begin{cases} T^F_{rs} = T^F_r \circ T^F_s, & \gcd(r, s) = 1, \\ T^F_{p^{r+1}} = T^F_p \circ T^F_{p^r} - p \sigma_p \circ T^F_{p^{r-1}}, & p \text{ prime}. \end{cases}$$

(3.13)

These formulas hold owing to the important property found in [9, appendix B] that for arbitrary $p, q$ coprime to $N$,

$$\rho^F(\sigma_p)\rho^F(\sigma_q) = \rho^F(\sigma_{pq}).$$

(3.14)

We call each $\rho^F(\sigma_p)$ as a fermionic Hecke class. Similar to the bosonic case, all fermionic Hecke classes form a finite abelian group. We will compute this finite abelian group for many examples later.

3In [9], the authors defined a function $H(p) := S^2(T^2)^{\frac{p^2+\rho}{2}}S(T^2)\left(-\frac{p_1}{p}\right)S(T^2)S^{p-1}$. Three good properties were found there: $H(-1) = S^2$, $H(a)H(b) = H(ab)$ and $SH(a) = H(a)S$, for $a, b \in Z_N^\times$. This function is related to our $\sigma_p$ by $H(p) = \sigma_p^\rho = \sigma_p^{-1}$.}

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We find that the fermionic Hecke operator has some basic properties resembling the bosonic Hecke operator:

- **The multiple \( p \) requirement.** The central charge \( c^{(p)} \) and NS weights \( h^{(p)}_{\text{NS}} \) of \( T^F_p \) image of a theory with \((c, h_{\text{NS}})\) satisfy the following property:
  \[
  c^{(p)} = p c, \quad \text{and} \quad h^{(p)}_{\text{NS}} \equiv p h_{\text{NS}} \mod 1/2 
  \]  
  (3.15)
  
  By contrast, in the bosonic case, the conformal weights of the Hecke image satisfy multiple \( p \) property mod 1.

- **The homogeneous property.** We find that when the Hecke image \( T^F_p \) has central charge less or equal than 12, i.e. \( c^{T^F_p} \leq 12 \), any NS character of \( T^F_p \) can be written as a degree \( p \) homogeneous polynomial of the NS characters of the starting theory. When \( c^{T^F_p} > 12 \), this may not be always possible, but it is still always possible to write as the combination of a degree \( p \) homogeneous polynomial and some lower degree homogeneous polynomials. This is slightly different from the bosonic cases, where the homogeneous property holds for arbitrary \( c \).

- **The \( S \) and \( T^2 \) matrices of the fermionic Hecke image \( T^F_p \) are related to those of the input theory by**
  \[
  \rho^{T^F_p}_{(p)}(T^2) = \rho^F(T^{2p}), \quad \rho^{T^F_p}_{(p)}(S) = \rho^F(\sigma_p S). 
  \]  
  (3.16)
  
  These properties make it sometimes possible to pair two fermionic Hecke images together to form a holomorphic SCFT with \( c = 12k \), while the bilinear relation of the characters of the two fermionic Hecke images gives a simple functions related to \( K(\tau) \) defined in (2.24).

To establish a Hecke relation, i.e., to identify a Hecke image with a known RCFT, it is always sufficient to check the Fourier coefficients of the characters up to a finite \( q \) order. This is owing to the finite generation property of modular forms [51]. In the fermionic cases, the same applies. For all fermionic Hecke relations we find in this paper, we have checked the Fourier coefficients of the NS characters to rather high \( q \) orders, typically up to \( q^{10} \) for each component, against the characters computed by other methods such as Sagemath software for WZW models.

In our previous paper [25, section 2.5], we introduced the concept of **generalized Hecke relations** \( T_p \) between bosonic theories for some \( p \) not coprime to the conductor \( N \). We find this concept is also useful in fermionic RCFTs. Analogously, we define the **generalized fermionic Hecke relations** \( T^F_p \) by the following three conditions when \( p \) is not coprime to the conductor \( N \):

1. The central charge and NS conformal weights satisfy the multiple \( p \) requirement as in (3.15).

2. The degeneracy for each non-vacuum NS primary is inherited. This means if a NS primary with weight \( h_{\text{NS}} \) has degeneracy \( M \), then the NS primary of the \( T^F_p \) with weight \( h^{(p)}_{\text{NS}} \equiv p h_{\text{NS}} \mod 1/2 \) should also have degeneracy \( M \).

3. The homogeneous property holds.
The examples of generalized fermionic Hecke relations are not as rich as the bosonic cases. Nevertheless, we will encounter many such relations for free chiral fermions which will be discussed in the next subsection 3.3 and for SO($m$)$^3$ supersymmetric theories in section 5.6. We observe from examples that generalized fermionic Hecke relations have similar properties as ordinary $T_F^p$, e.g., the Fourier coefficients of $T_F^p$ still satisfy the mod $p$ properties. However, the conductor $N$ will become $N/p$. These properties also exist for bosonic generalized Hecke relations [25].

### 3.3 Example of free chiral fermions

As a simple yet still nontrivial example, let us consider the fermionic Hecke images of a free Majorana fermion $F$. A free Majorana fermion is well-known to be the fermionization of 2d critical Ising model. The NS index $I$ only takes the vacuum 0 and the single NS character is just $\psi_{NS} = \sqrt{\theta_3(\tau)/\eta(\tau)}$. The conductor $N = 48$ is of course the same with the Ising model. The bosonic Hecke operation on the Ising model has been discussed in [22], see also [25, table 9]. Recall there are two Hecke classes forming $\mathbb{Z}_2$ group for the bosonic Hecke operation of Ising model [22]. However, for NS character, the fermionic Hecke operation only has one single class, i.e., $\rho_F^p(\sigma_p) = 1$ for arbitrary admissible $p$. Besides, the homogeneous property implies the following simple relations

$$T_F^p = \begin{cases} (T_F^1)^p, & p < 24, \\ (T_F^1)^p - p(T_F^1)^{p-24}, & 24 < p < 48. \end{cases} \tag{3.17}$$

This is just consistent with the fact that the fermionic Hecke image $T_F^p$ with $p < 24$ describes the theory of $p$ free chiral fermions. Besides, the $T_F^p$ and $T_{24-p}^F$ fermionic Hecke images naturally form the $F_{24}$ SCFT of $c = 12$. The bilinear relations of the NS characters of such pairs give the well-known identity $(\psi_{NS})^{24} = K(\tau) + 24$. We summarize the $c = 12$ pairs in table 2. For $24 < p < 48$, the fermionic Hecke image $T_F^p$ no longer describes $p$ free fermions, but some more nontrivial interacting fermionic theories. For example, from the bosonic Hecke operation [22], we can see that the $T_{31}^F$ and $T_{35}^F$ images should describe the fermionization of the WZW ($E_8$)$_2$ and ($C_{10}$)$_1$ models. Moreover, the $T_{47}^F$ image should describe the SCFT associated with the Baby Monster group [26]. We have checked these are indeed correct.

In the above discussion, $p$ can only be coprime to 48. It turns out we are not limited by such a condition. This is actually an excellent playground for generalized fermionic Hecke relations. Let us consider the theory of two free fermions $2F$. This can be regarded as the fermionization of WZW SO(2)$_1$ model. The central charge doubles to $c = 1$ and the conductor halves to $N = 24$. The single NS character is just $(T_F^1)^2$. Thus we can regard $2F$ as a generalized fermionic Hecke $T_2^F$ image of $F$. Consider the fermionic Hecke images

| $c$ | $h_{NS}$ | $m_{1/2}$ | remark | $\hat{c}$ | $\hat{h}_{NS}$ | $\hat{m}_{1/2}$ | remark | $K(\tau) + n$ |
|-----|----------|-----------|--------|----------|----------------|----------------|--------|-------------|
| $\frac{p}{2}$ | $\frac{p}{16}$ | $p$ | $T_F^p$ | $\frac{24-p}{2}$ | $\frac{24-p}{16}$ | $24 - p$ | $T_{24-p}^F$ | 24 |

Table 2. (Generalized) fermionic Hecke images of one free chiral fermion.
of $2F$. From [25, section 4.2], we know there is only one bosonic Hecke class for $\text{SO}(2)_1$. Thus after fermionization there can only be one single class of $\rho^F(\sigma_p)$, still as 1. Therefore, all relations in (3.17) and table 2 still hold as long as for $p = 2k, k \in \mathbb{Z}$, we regard $T^F_p$ as generalized fermionic Hecke image $T_k$ of $2F$.

As a step forward, let us consider the theory of three free fermions $3F$. The central charge is $\frac{3}{2}$. This theory can be regarded as the fermionization of WZW $\text{SO}(3)_1$ or equivalently the $\text{SU}(2)_2$ model. The bosonic $\text{SU}(2)_2$ theory has conformal weights 0, $\frac{3}{16}, \frac{1}{2}$. The relation between the NS character of $3F$ and $(A_1)_2$ affine characters is well-known to be

$$\chi^3_F = (T^F_1)^3 = \chi^{(A_1)_2}_0 + \chi^{(A_1)_2}_{1/2}. \quad (3.18)$$

The conductor becomes $N = 16$. The above relation also shows $F(A_1)_2$ can be regarded as a generalized fermionic Hecke $T^F_3$ image of $F$. We find equation (3.17) and table 2 still hold if for $p = 3k, k \in \mathbb{Z}$, we regard $T^F_p$ as generalized fermionic Hecke image $T_k$ of $3F$. For example, the $T^F_{21} F$ image, i.e., $T^F_{21}$ of $F(A_1)_2$ describes a theory $F(A_1)^{21}_2$ of central $\frac{21}{2}$. The single NS character of $F(A_1)^{21}_2$ can be written as

$$\frac{F(A_1)^{21}_2}{\chi_0} = T^F_1 (F(A_1)_2) = (T^F_1)^{21} = (\chi^{(A_1)_2}_0 + \chi^{(A_1)_2}_{1/2})^7. \quad (3.19)$$

Together, $F(A_1)_2$ and $F(A_1)^{21}_2$ form the $c = 12$ holomorphic SCFT $F(A_1)^{8}_2$ mentioned in section 2.3.

The theory of four free fermions $4F$ describes the fermionization of the double product $(A_1)_4^2$ owing to the appearance of weight $\frac{1}{2}$ field. It is easy to check that

$$\chi^{4F}_0 = (T^F_1)^4 = (\chi^{(A_1)_4}_0)^2 + (\chi^{(A_1)_4}_{1/4})^2. \quad (3.20)$$

Thus, $F(A_1)_2^4$ can be regarded as a generalized $T^F_3$ image of $2F$ or a generalized $T_4$ image of $F$. Moreover, the theories of $6F$, $8F$, $12F$ can be regarded as generalized $T^F_3$, $T^F_8$ and $T^F_{12}$ images of $F$ which describe $F(A_3)_1$, $F(D_4)_1$ and $F(D_6)_1$ respectively. In summary, (3.17) holds for all positive integers $p < 48$, and table 2 holds for all positive integers $p < 24$.

As a final remark, we can also consider the $c = 24$ pairs between $T^F_p$ and $T^F_{48-p}$. Suppose $1 \leq p \leq 24$. The bilinear relation of $T^F_p$ and $T^F_{48-p}$ leads to a potential holomorphic SCFT of $c = 24$ with single NS character

$$T^F_p, T^F_{48-p} = (K(\tau) + 24)^2 - (48 - p)(K(\tau) + 24) = q^{-1} + p q^{-1/2} + 24(p - 1) + \ldots. \quad (3.21)$$

4 Two NS characters

4.1 Type $(SLY)_1$

Supersymmetric minimal model $SM(8, 2)$ has $c = -\frac{21}{4}$ and $h = -\frac{1}{4}$, while $SM_{\text{eff}}(8, 2)$ has $c_{\text{eff}} = \frac{3}{2}$ and $h_{\text{eff}} = \frac{1}{2}$. It is well-known that $SM(8, 2)$ can be realized as the fermionization of the bosonic minimal model $M(8, 3)$. As $SM(8, 2)$ is the simplest non-unitary supersymmetric minimal model, it was dubbed with the name “supersymmetric Lee-Yang model”. The
effective theory by our notation is denoted as $(SLY)_1$. It has the following two NS characters

$$\chi_0 = q^{-\frac{1}{16}} \prod_{n=0}^{\infty} \frac{1}{(1 - q^{\frac{2}{16}(8n+1)})(1 - q^{\frac{2}{16}(8n+4)})(1 - q^{\frac{2}{16}(8n+7)})},$$

$$\chi_\frac{1}{4} = q^{\frac{3}{16}} \prod_{n=0}^{\infty} \frac{1}{(1 - q^{\frac{2}{16}(8n+3)})(1 - q^{\frac{2}{16}(8n+4)})(1 - q^{\frac{2}{16}(8n+5)})}.$$  

Clearly the conductor is $N = 32$. The $S$-matrix and $T^2$-matrix are known to be

$$S = \begin{pmatrix} \cos(\frac{\pi}{8}) & \sin(\frac{\pi}{8}) \\ \sin(\frac{\pi}{8}) & -\cos(\frac{\pi}{8}) \end{pmatrix}, \quad T^2 = \begin{pmatrix} e^{-\frac{\pi i}{4}} & 0 \\ 0 & e^{\frac{3\pi i}{4}} \end{pmatrix}. $$

This $S$ matrix leads to the NS fusion rule $\phi_1 \times \phi_1 = \phi_0 - 2\phi_1$. We notice the two NS characters satisfy the following degree 8 homogeneous polynomial identity

$$\chi_0 \chi_\frac{1}{4} \left( \chi_0^6 - 7\chi_0^4\chi_\frac{1}{4}^2 + 7\chi_0^2\chi_\frac{1}{4}^4 - \chi_\frac{1}{4}^6 \right) = 1.$$  

This resembles the famous Ramanujan identity of the two Lee-Yang characters, see e.g. [25, equation (2.48)]. The two $R$ characters can be found in e.g. [1]. The $\tilde{R}$ character is just a constant 1.

Let us consider the fermionic Hecke operation of $(SLY)_1$. We find there are in total four classes for $\rho^F(\sigma_p)$ for $p$ modulo 16 which are represented by

$$\rho^F(\sigma_{1,15}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho^F(\sigma_{3,13}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \rho^F(\sigma_{5,11}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho^F(\sigma_{7,9}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

Clearly, the four fermionic Hecke classes form a $\mathbb{Z}_4$ group, which is just the $\mathbb{Z}_4$ group of $p$ with the same quadratic residue modulo 32. More precisely the quotient in the group of all $p$ coprime to 32 is defined by the equivalence relation $p_1^2 \equiv p_2^2 \mod 32$. With the above $\rho^F(\sigma_p)$, we compute all fermionic Hecke images $T^F \rho^F_p$ up to $p < 16$ and organize them in pairs w.r.t. $c = 12$ in table 3. These fermionic images are consistent with the computation on bosonic Hecke images of $M(8,3)$ in [25, table 38].

For example, we find that the fermionic Hecke image $T^F_3$ of $(SLY)_1$ describes exactly the fermionization of $(A_1)_6$. This is a renowned RCFT with emergent supersymmetry with true supersymmetric vacua. This means that a bosonic CFT can be mapped to a SCFT via

| $c$ | $h_{NS}$ | $m_{1/2}$ | remark | $\tilde{c}$ | $\tilde{h}_{NS}$ | $\tilde{m}_{1/2}$ | remark | $K(\tau) + n$ |
|-----|----------|-----------|--------|----------|----------------|----------------|--------|-------------|
| $\frac{3}{4}$ | $\frac{1}{4}$ | 1 | $T^F_1$ | $(SLY)_1$ | $\frac{15}{4}$ | 3 | 15 | $T^F_{15}$ | 16 |
| $\frac{9}{4}$ | $\frac{1}{4}$ | 0 | $T^F_3, F(A_1)_6$ | $\frac{39}{4}$ | $\frac{3}{4}$ | 0 | $T^F_{13}, F(C_6)_1$ | 0 |
| $\frac{15}{4}$ | $\frac{1}{4}$ | 5 | $T^F_5, \star$ | $\frac{33}{4}$ | $\frac{3}{4}$ | 11 | $T^F_{11}, \star$ | $-16$ |
| $\frac{21}{4}$ | $\frac{1}{4}$ | 14 | $T^F_7, \star$ | $\frac{27}{4}$ | $\frac{3}{4}$ | 18 | $T^F_9, \star$ | $-32$ |

Table 3. Fermionic Hecke images of supersymmetric Lee-Yang model $(SLY)_1$.
the generalized Jordan-Wigner transformation, see a detailed recent discussion in [20]. The two NS characters are related to the affine characters and the \((SLY)_1\) characters by the following combination:

\[
\begin{align*}
\mathcal{F}^0_{1}(A_1) &= \chi^0_{A_1} + \chi^3_{A_1} = \chi^0_{T} - 3\chi^0_{T} = 0, \\
\mathcal{F}^3_{1}(A_1) &= \chi^3_{A_1} + \chi^0_{A_1} = 3\chi^2_{T} - \chi^3_{T}. 
\end{align*}
\] (4.4)

Besides, we find the fermionic Hecke image \(T^F_{13}\) describes the fermionization of \((C_6)_1\). To be precise, we find the following relations

\[
\begin{align*}
\mathcal{F}^0_{1}(C_6) &= \chi^0_{C_6} + \chi^3_{C_6} = \chi^0_{T} \left( 10 - 13\chi^0_{T} + 130\chi^2_{T} - 338\chi^4_{T} + 221\chi^6_{T} - 65\chi^8_{T} \right), \\
\mathcal{F}^3_{1}(C_6) &= \chi^3_{C_6} + \chi^0_{C_6} = \chi^3_{T} \left( 65\chi^0_{T} - 221\chi^2_{T} + 338\chi^4_{T} - 130\chi^6_{T} + 13\chi^8_{T} - \chi^1_{T} \right).
\end{align*}
\]

The \(F(C_6)_1\) is also a famous example with emergent supersymmetry with supersymmetric vacua, see e.g. [12] and the fermionic characters in [20, table 14]. As a \(c=12\) pair, these two fermionic Hecke images form the Conway SCFT. Their NS characters give the following bilinear identity

\[
K(\tau) = T^F_{3} \cdot T^F_{13} = \chi^4_{C_1} - 16\chi^0_{C_1} + 364\chi^6_{C_1} - 1456\chi^8_{C_1} + 2470\chi^8_{C_1} + 364\chi^4_{C_1} - 16\chi^2_{C_1} + \chi^2_{C_1}.
\] (4.5)

The \(T^F_{15}\) image is not unitary, but it may serve as a certain analogy of the famous WZW \((E_7)_{1}\) theory, which was found in [45] by MLDEs and realized as a \(T_{19}\) Hecke image of Lee-Yang model in [22]. We show the \(T^F_{15}\) characters in the following:

\[
\begin{align*}
\mathcal{F}^0_{0}(T_{15}) &= \chi^3_{0} \left( 10 + 105\chi^0_{T} + 280\chi^6_{T} + 435\chi^4_{T} + 35\chi^6_{T} - 168\chi^0_{T} + 35\chi^6_{T} \right), \\
&= q^{-\frac{10}{2}} \left( 1 + 15q^{1/2} + 225q + 1555q^{3/2} + 7920q^{2} + 32580q^{5/2} + \ldots \right), \\
\mathcal{F}^3_{0}(T_{15}) &= \chi^3_{3} \left( 35\chi^0_{T} - 168\chi^0_{T} + 280\chi^6_{T} + 435\chi^4_{T} + 35\chi^0_{T} + 105\chi^6_{T} + \chi^0_{T} \right), \\
&= q^{-\frac{3}{2}} \left( 35 + 252q^{1/2} + 1485q + 6805q^{3/2} + 25845q^{2} + 86220q^{5/2} + \ldots \right).
\end{align*}
\] (4.6)

The \(T^F_{15}\) and \((SLY)_1\) do not pair as a consistent holomorphic SCFT of \(c=12\). However, from \(K(\tau) + 16 = T^F_{3} \cdot T^F_{15}\) we can obtain a different expression for \(K(\tau)\). The difference from (4.5) is exactly 16 times the square of identity (4.3).

**4.2 Type SM(5, 3)**

\(SM(5, 3)\) is the simplest unitary supersymmetrical minimal model. It can be realized as the fermionization of the bosonic minimal model \(M(5, 4)\) [31–33]. It has central charge \(c = \frac{7}{10}\) and two NS primaries with weights 0 and \(\frac{1}{10}\). The conductor is \(N = 240\). The \(S\)-matrix for the two NS characters is

\[
S = \frac{1}{\sqrt{30}} \begin{pmatrix} \alpha_- & \alpha_+ \\ \alpha_+ & -\alpha_- \end{pmatrix}, \quad \alpha_{\pm} = \sqrt{15 \pm 3\sqrt{5}}.
\] (4.7)

This leads to the simple NS fusion rule \(\phi_1 \times \phi_1 = \phi_0 + \phi_1\).
We now consider a theory rather similar to $SM(5,3)$, but non-unitary. The supersymmetric minimal model $SM(60,2)$ has $c = -\frac{413}{5}$ and $c_{\text{eff}} = \frac{7}{5}$. The effective theory, i.e., $(SLY)^{14}$ has 15 NS characters $\chi_i, i = 0, 1, 2, \ldots, 14$ with weights

$$0, \frac{1}{30}, \frac{1}{10}, \frac{1}{5}, \frac{1}{3}, \frac{2}{2}, \frac{7}{10}, \frac{14}{15}, \frac{6}{5}, \frac{3}{2}, \frac{11}{10}, \frac{11}{5}, \frac{13}{5}, \frac{91}{6}, \frac{7}{2}$$

The four fermionic Hecke classes form a $Z_4$ group. We remark that this $Z_4$ is actually different from the $Z_4$ group of $p$ with the same quadratic residue modulo 240. For example, it is easy to check that all $p$ satisfying $p^2 \equiv 1 \mod 240$ are $p = 1, 31, 41, 49, 71, 79, 89, 109 \mod 120$. We compute all fermionic Hecke images $T^F_p$ for $p < 24$ and organize them in table 4. These are all consistent with the computation on the bosonic Hecke images of $M(5,4)$ in [25].

Table 4. Fermionic Hecke images of $SM(5,3)$.

| $c$ | $h_{\text{NS}}$ | $m_{1/2}$ | remark |
|-----|----------------|----------|--------|
| $\frac{7}{10}$ | $\frac{1}{10}$ | 0 | $T^F_1, SM(5,3)$ |
| $\frac{9}{10}$ | $\frac{1}{5}$ | 7 | $T^F_3, \ast$ |
| $\frac{77}{10}$ | $\frac{3}{5}$ | 11 | $T^F_{11}, \ast$ |
| $\frac{91}{10}$ | $\frac{4}{5}$ | 13 | $T^F_{13}, \ast$ |
| $\frac{119}{10}$ | $\frac{7}{10}$ | 17 | $T^F_{17}, \ast$ |
| $\frac{133}{10}$ | $\frac{9}{10}$ | 0 | $T^F_{19}, F(E_7)_2$ |
| $\frac{161}{10}$ | $\frac{4}{5}$ | 0 | $T^F_{23}$ |

The four fermionic Hecke classes form a $Z_4$ group. We remark that this $Z_4$ is actually different from the $Z_4$ group of $p$ with the same quadratic residue modulo 240. For example, it is easy to check that all $p$ satisfying $p^2 \equiv 1 \mod 240$ are $p = 1, 31, 41, 49, 71, 79, 89, 109 \mod 120$. We compute all fermionic Hecke images $T^F_p$ for $p < 24$ and organize them in table 4. These are all consistent with the computation on the bosonic Hecke images of $M(5,4)$ in [25].

Notably, the $T^F_{19}$ image describes the fermionization of WZW $(E_7)_2$ theory. The $F(E_7)_2$ is a well-known example with emergent supersymmetry, yet having a non-supersymmetric Ramond ground state. The relation between the NS characters of $F(E_7)_2$ and affine $(E_7)_2$ characters can be found in e.g. [1, equation (5.94)].

It is straightforward to further compute the fermionic Hecke images of double product $SM(5,3)^2$, which produce lots of three NS-character theories with degeneracy $(1,2,1)$. The conductor halves to $N = 120$. As a remark, we notice that $SM(5,3)^2$ describes exactly the fermionization of $Z_8$ parafermion CFT discussed in [20, equation (A.3)].

4.3 Type $SM_{\text{sub}}(60,2)$

We now consider a theory rather similar to $SM(5,3)$, but non-unitary. The supersymmetric minimal model $SM(60,2)$ has $c = -\frac{413}{5}$ and $c_{\text{eff}} = \frac{7}{5}$. The effective theory, i.e., $(SLY)^{14}$ describes exactly the fermionization of $Z_8$ parafermion CFT discussed in [20, equation (A.3)].
We construct a non-diagonal $\Gamma_\theta$ modular invariant of $E_8$ type out of $(SLY)_{14}$ by
\[
\begin{align*}
\chi^S_{\text{sub}(60,2)} &= \chi_0 + \chi_5 + \chi_9 + \chi_{14} = q^{-\frac{7}{30}}\left(1 + 2\sqrt{q} + 2q + 4q^{3/2} + 6q^2 + 8q^{5/2} + \ldots\right), \\
\chi^S_{\text{sub}(60,2)}/5 &= \chi_3 + \chi_6 + \chi_8 + \chi_{11} = q^{\frac{17}{30}}\left(1 + 2\sqrt{q} + q + 2q^{3/2} + 5q^2 + 6q^{5/2} + \ldots\right).
\end{align*}
\]

Note for all $\chi_i$ appearing here, $2i + 1$ give exactly the eight exponents $E_8$ Lie algebra, and $2i + 2$ give the eight degrees of fundamental invariants which are 2, 8, 12, 14, 18, 20, 24, 30. It is easy to check
\[
Z^S_{\text{NS}} = |\chi^S_{\text{sub}(60,2)}|^2 + |\chi^S_{\text{sub}(60,2)/5}|^2
\]
is $\Gamma_\theta$ modular invariant. The $S$-matrix can be easily deduced from the one of $(SLY)_{14}$ as
\[
S = \frac{1}{\sqrt{30}}\begin{pmatrix} \alpha_+ & \alpha_- \\ \alpha_- & -\alpha_+ \end{pmatrix}, \quad \alpha_\pm = \sqrt{15 \pm 3\sqrt{5}}.
\]

Note $S^2 = \text{Id}$. We denote this sub-theory of $(SLY)_{14}$ as $SM_{\text{sub}}(60,2)$. Clearly the conductor is $N = 120$. Notice that the $S$-matrix here is just different from the one (4.7) of $SM(5,3)$ by exchanging the plus and minus in the subscripts of $\alpha$. This leads to the non-unitary NS fusion rule $\phi_1 \times \phi_1 = \phi_0 - \phi_1$.

Consider the fermionic Hecke images of this $E_8$ type sub-theory. We find there are in total four classes for the fermionic Hecke operation of the NS characters: for $p \equiv 1, 19, 41, 59 \mod 60$, $\rho^F(\sigma_p) = \text{Id}$, for $p \equiv 11, 29, 31, 49 \mod 60$, $\rho^F(\sigma_p) = -\text{Id},$
\[
\begin{align*}
&\text{for } p \equiv 17, 23, 37, 43 \mod 60, \quad \rho^F(\sigma_p) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\
&\text{for } p \equiv 7, 13, 47, 53 \mod 60, \quad \rho^F(\sigma_p) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\end{align*}
\]
The four fermionic Hecke classes still form a $\mathbb{Z}_4$ group. We remark that the first two classes of $p$ have quadratic residue 1 modulo 120, while the last two classes have quadratic residue 49 modulo 120. We compute all fermionic Hecke images $T^F_p$ up to $p < 20$ and organize them in table 5. Interestingly, we find the $T^F_{13}$ image describes exactly the fermionization of WZW $(D_7)_{3}$. The relation between the NS characters of $F(D_7)_{3}$ and the affine characters can be found in e.g. [1, equation (5.99)]. We remark that $F(D_7)_{3}$ and $SM(5,3)$ belong to the same object in the super-modular category of rank 4, see e.g. [9].

### 4.4 Type $SA_1$

The $\mathcal{N} = 1$ supersymmetric $A_1$ is the supersymmetrization of the lattice $A_1$ CFT by coupling it with one free chiral fermion. Clearly it has central charge $c = \frac{3}{2}$ and NS weights 0, $\frac{1}{2}$. The fermionic Hecke operation for this case is almost trivial as it is parallel to the bosonic Hecke operation for WZW $(A_1)_{1}$ theory. However, it is interesting to observe how much the supersymmetrization changes the theory. For example, the conductor $N$ is changed from 24 to 16. Nevertheless, the $S$-matrix remains the same. Consider the fermionic Hecke
Table 5. Fermionic Hecke images of $SM_{sub}(60, 2)$.

| $c$  | $h_{NS}$ | $m_{1/2}$ | remark         |
|------|----------|-----------|----------------|
| $\frac{7}{5}$ | $\frac{1}{5}$ | 2        | $T_{1}^{F}, SM_{sub}(60, 2)$ |
| $\frac{44}{5}$ | $\frac{2}{5}$ | 14       | $T_{7}^{F}$    |
| $\frac{77}{5}$ | $\frac{7}{10}$ | 0        | $T_{11}^{F}, \ast$ |
| $\frac{91}{5}$ | $\frac{11}{10}$ | 0        | $T_{13}^{F}, F(D_{7})_{3}$ |
| $\frac{149}{5}$ | $\frac{7}{5}$ | 0        | $T_{17}^{F}, \ast$ |
| $\frac{133}{5}$ | $\frac{13}{10}$ | 0        | $T_{19}^{F}$   |

Table 6. Fermionic Hecke images of supersymmetric $A_{1}$ theory $SA_{1}$.

| $c$  | $h_{NS}$ | $m_{1/2}$ | remark | $\tilde{c}$ | $\tilde{h}_{NS}$ | $\tilde{m}_{1/2}$ | $K(\tau + n)$ |
|------|----------|-----------|--------|------------|-----------------|-----------------|---------------|
| $\frac{3}{2}$ | $\frac{1}{4}$ | 1        | $T_{1}^{F}, SA_{1}$ | $\frac{21}{4}$ | $\frac{3}{4}$ | 7 | $T_{7}^{F}, SE_{7}$ | 8 |
| $\frac{9}{2}$ | $\frac{1}{4}$ | 9        | $T_{3}^{F}, \ast$ | $\frac{15}{2}$ | $\frac{3}{4}$ | 15 | $T_{5}^{F}, \ast$ | $-24$ |

operation $T_{p}^{F}$ for $SA_{1}$. We find there exist in total two classes $\rho^{F}(\sigma_{p})$ for $p \equiv 1, 7 \mod 8$, $\rho^{F}(\sigma_{p}) = Id$, while for $p \equiv 3, 5 \mod 8$, $\rho^{F}(\sigma_{p}) = -Id$. We compute all fermionic Hecke images $T_{p}^{F}$ up to $p < 8$ and organize them in pairs w.r.t. $c = 12$ in table 6. Just like the bosonic Hecke operation $T_{7}(A_{1})_{1} = (E_{7})_{1}$ [22], the fermionic Hecke operation produces $T_{7}SA_{1} = SE_{7}$. Together $SA_{1}$ and $SE_{7}$ form a holomorphic SCFT of $c = 12$ that is $SE_{8}$. On the other hand, $T_{3}^{F}$ and $T_{5}^{F}$ images have negative NS vacuum, thus are unphysical.

4.5 Type $S^{2}A_{1}$

The $N = 2$ supersymmetric $A_{1}$ theory $S^{2}A_{1}$ is the first one of a unitary $N = 2$ series. It has central charge $c = 1$ and NS weights $0, (\frac{1}{2})_{2}$. Note the non-vacuum NS primary has degeneracy two. This theory can be realized as a non-diagonal modular invariant of $SM(6, 4)$ minimal model, see e.g. [1, equation (5.29)] for the NS characters. It can also be regarded as the fermionization of bosonic $U(1)_{6}$ theory. The conductor is $N = 24$. The full $S$-matrix for the three NS primaries is well-known to be the same with the $S$-matrix of WZW $(A_{2})_{1}$ as

$$S = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega_{1} & \omega_{1}^{2} \\ 1 & \omega_{1}^{2} & \omega_{1} \end{pmatrix}, \quad \omega_{1} = e^{\frac{2\pi i}{3}}. \quad (4.14)$$

It is also equivalent to consider a reduced $S$-matrix for just two NS characters:

$$S_{\text{reduced}} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}. \quad (4.15)$$

Let us study the fermionic Hecke operation for the reduced $S$-matrix as it is computationally easier. We find there exist two classes of $\rho^{F}(\sigma_{p})$. For $p \equiv 1, 11 \mod 12$, $\rho^{F}(\sigma_{p}) = Id$, while for $p \equiv 5, 7 \mod 12$, $\rho^{F}(\sigma_{p}) = -Id$. We compute all fermionic Hecke images $T_{p}^{F}$
for \( p < 12 \) and organize them in pairs w.r.t. \( c = 12 \) in Table 7. These are consistent with the computation on bosonic Hecke images of \( U(1)_6 \) theory in our previous work [25, table 37]. Notably, the \( T^F_{11} \) image describes the fermionization of WZW \((A_{11})_1\) model. The relation between the NS characters of \( \mathcal{F}(A_{11})_1 \) and affine characters can be found in e.g. [20, equation (3.12)]. The \( \mathcal{F}(A_{11})_1 \) theory has unbroken supersymmetry, and its automorphism group is known to be related to \( Suz : 2 \) sporadic group [12]. The homogeneous property of fermionic Hecke relation implies the following nice relations between the NS characters of \( S^2A_1 \) and \( \mathcal{F}(A_{11})_1 \):

\[
\chi_0^{\mathcal{F}(A_{11})_1} = \chi_0^{11} + 132\chi_0^{5}\chi_1^{6} + 110\chi_0^{2}\chi_1^{9}, \\
\chi_{5/6}^{\mathcal{F}(A_{11})_1} = 3\left(4\chi_0^{11} + 22\chi_0^{6}\chi_1^{5} + 55\chi_0^{3}\chi_1^{8}\right).
\]

(4.16)

Here we use \( \chi_0 \) and \( \chi_1 \) to denote the two NS characters of \( S^2A_1 \). Together they form the bilinear identity

\[
K(\tau) = T^F_1 \cdot T^F_{11} = \chi_0^{12} + 2\chi_1^{12} + 264\chi_0^{6}\chi_1^{6} + 440\chi_0^{3}\chi_1^{9} + 24\chi_1^{12}.
\]

(4.17)

Different \( c = 12 \) pairs lead to a simple identity

\[
\chi_{1/6}(\chi_0^{3} - \chi_1^{3}) = 1.
\]

(4.18)

It is easy to prove this identity from the \( S \)-matrix (4.15).

Consider a product type theory composed of a \( S^2A_1 \) and a free chiral fermion. The central charge is \( c = \frac{3}{2} \). Adding a free fermion does not change the \( S \)-matrix of \( S^2A_1 \), but changes its \( T^2 \)-matrix and conductor. The two NS characters have the following Fourier expansion

\[
\chi_0 = q^{-\frac{1}{12}}\left(1 + \sqrt{q} + q + 4q^{3/2} + 5q^2 + 6q^{5/2} + 9q^3 + \ldots\right), \\
\chi_{1/6} = q^{\frac{5}{24}}\left(1 + 2\sqrt{q} + 2q + 3q^{3/2} + 5q^2 + 8q^{5/2} + 10q^3 + \ldots\right).
\]

(4.19)

The second character has degeneracy two. The conductor becomes \( N = 48 \).

Let us study the fermionic Hecke operation for this new theory. We find there exist two classes \( \rho^F(\sigma_p) \) for \( p \equiv 1, 11, 13, 23 \mod 24 \), \( \rho^F(\sigma_p) = Id \), while for \( p \equiv 5, 7, 17, 19 \mod 24 \), \( \rho^F(\sigma_p) = -Id \). We compute all fermionic Hecke images \( T^F_p \) up to \( p < 14 \) and summarize them in Table 8. Interestingly, we notice the \( T^F_{13} \) image describes exactly the fermionization of WZW \((E_6)_{14}\) model. The relation between the NS characters of \( \mathcal{F}(E_6)_{14} \) and the affine characters can be found in for example [1, equation (5.104)].

| \( c \) | \( h_{NS} \) | \( m_{1/2} \) | remark | \( \tilde{c} \) | \( \tilde{h}_{NS} \) | \( \tilde{m}_{1/2} \) | remark | \( K(\tau) + n \) |
|-------|--------|--------|-------|------|--------|--------|-------|--------|
| 1     | \( \frac{1}{6} \) | 0      | \( T^F_1, S^2A_1 \) | 11   | \( \frac{5}{6} \) | 0      | \( T^F_{11}, \mathcal{F}(A_{11})_1 \) | 0     |
| 5     | \( \frac{1}{3} \) | 10     | \( T^F_5, \star \) | 7    | \( \frac{2}{3} \) | 14     | \( T^F_5, \star \) | -24   |

Table 7. Fermionic Hecke images of \( S^2A_1 \). The non-vacuum NS character has degeneracy 2.
Table 8. Fermionic Hecke images of $S^2 A_1 \otimes F$. The non-vacuum NS character has degeneracy 2.

5 Three NS characters

5.1 Type $(SLY)_2$

Supersymmetric minimal model $SM(12, 2)$ has $c = -11$ and $h = -\frac{1}{3}, -\frac{1}{2}$, while the effective theory $SM_{\text{eff}}(12, 2)$, i.e., $(SLY)_2$ has $c_{\text{eff}} = 1$ and $h_{\text{eff}} = \frac{1}{6}, \frac{1}{2}$. The three NS characters of $(SLY)_2$ can be easily computed from (2.15) or (2.21). The conductor is $N = 24$. We find the $S$-matrix of the three NS characters to be

$$S = \frac{1}{2\sqrt{3}}\begin{pmatrix} \sqrt{3} + 1 & 2 & \sqrt{3} - 1 \\ 2 & -2 & -2 \\ \sqrt{3} - 1 & -2 & \sqrt{3} + 1 \end{pmatrix}. \quad (5.1)$$

It is easy to check $S^2 = \text{Id}$. This $S$ matrix gives the NS fusion rules $\phi_1 \times \phi_1 = \phi_0 = \phi_1 + \phi_2$, $\phi_1 \times \phi_2 = \phi_1 - 2\phi_2$ and $\phi_2 \times \phi_2 = \phi_0 - 2\phi_1 + 2\phi_2$.

Consider the fermionic Hecke operation on $(SLY)_2$. We find there exist two classes $\rho^F(\sigma_p)$ for $p$ mod 12,

$$\rho^F(\sigma_{11}) = \text{Id}, \quad \text{and} \quad \rho^F(\sigma_{5,7}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (5.2)$$

Obviously, they form a $\mathbb{Z}_2$ group. We compute all fermionic Hecke images $T^F_p$ for $p < 24$ and summarize those with $p < 12$ in $c = 12$ pairs in table 9. Notably, we find the $T^F_5$ image describes exactly the fermionization of WZW $(B_2)_3$ model, while the $T^F_7$ image describes exactly the fermionization of WZW $(C_3)_2$ model. Together they form the character of holomorphic Conway SCFT of $c = 12$. For the explicit relations between the NS characters of $F(B_2)_3$ and affine characters, we refer to [2, equation (3.71)], while for $F(C_3)_2$ we refer to [2, equation (3.73)]. From the homogeneous property of fermionic Hecke images, we find that the NS character of $F(B_2)_3$ can be written as

$$\begin{align*}
\chi^F_{(B_2)_3} &= \chi_0 \left( \chi_0^4 - 5\chi_1/2\chi_0^3 + 15\chi_1/2\chi_0^2 - 5\chi_1/2\chi_0 + 10\chi_{1/2}^4 \right), \\
\chi^F_{(B_2)_3} &= \chi_{1/6}^2 \left( 5\chi_0^3 - 15\chi_1/2\chi_0^2 + 15\chi_1/2\chi_0^3 + 4\chi_{1/6}^3 - 5\chi_{1/2}^3 \right), \\
\chi^F_{(B_2)_3} &= \chi_{1/2} \left( 10\chi_0^4 - 5\chi_1/2\chi_0^3 + 15\chi_1/2\chi_0^2 - 5\chi_1/2\chi_0 + \chi_{1/2}^4 \right). \quad (5.3)
\end{align*}$$
As we reviewed earlier, the critical Ising model can be fermionized to a free chiral fermion. This kind of vanishing identities of characters also begin to appear in the bosonic minimal model. The full S-matrix can be easily deduced from the one of the Ising model, see e.g. \[9\]. The fermionic Hecke operation of the fermionic RCFT with \(c = 1\) is unique, here the expression is no longer unique due to vanishing identities like

\[
0 = \chi_{1/2}^3 + \chi_{1/2}^3\chi_0 - \chi_0\chi_{1/6}^3 + \chi_{1/6}\chi_{1/2}.
\]

This kind of vanishing identities of characters also begin to appear in the bosonic minimal model \((LY)\)\(_2\), i.e. \(M_{\text{eff}}(7,2)\).

### 5.2 Type \((SLY)\)\(_2\)

Consider the double product of \(SM_{\text{eff}}(8,2)\) theory which we denote as \((SLY)\)\(_2\). Clearly, the central charge is \(c = \frac{3}{2}\) and the NS weights are \(0, \frac{1}{2}, \frac{1}{2}\). The conductor becomes \(N = 16\). The full \(S\)-matrix can be easily deduced from the one of \((SLY)\)\(_1\) in (4.2). Consider the fermionic Hecke operation on \((SLY)\)\(_1\). We find there exist two classes \(\rho(\sigma_p)^F\) for \(p\) mod 8,

\[
\rho^F(\sigma_{1,7}) = \text{Id}, \quad \rho^F(\sigma_{3,5}) = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}.
\]

Obviously, they form a \(\mathbb{Z}_2\) group. We compute all fermionic Hecke images \(T^F_p\) up to \(p < 16\) and summarize those with \(p < 8\) in \(c = 12\) pairs in table 10. Interestingly, we find the \(T^F_3\) image describes exactly the SU(4)\(_4\)/\(\mathbb{Z}_2\) theory studied in \[20\]. This orbifold theory has unbroken supersymmetry. See \[20\], equation (3.19)\] for the relation between fermionic characters and affine characters. Besides, the \(T^F_3\) image describes \(\mathcal{F}(A_1)\)\(_3\) as inherited from the fermionic Hecke operation of \((SLY)\)\(_1\) discussed in section 4.1.

### 5.3 Type Ising \(\otimes F\)

As we reviewed earlier, the critical Ising model can be fermionized to a free chiral fermion \(F\). Let us consider the double product of the Ising model, but only fermionize one of them. Obviously, we obtain a \(c = 1\) fermionic RCFT with \(h_{\text{NS}} = 0, \frac{1}{2}, \frac{1}{16}\). This is a unitary theory appearing in the rank-6 SMC, see e.g. \[9\]. The \(S\)-matrix of Ising model can be found in

| \(c\) | \(h_{\text{NS}}\) | \(m_{1/2}\) | \(\tilde{c}\) | \(\tilde{h}_{\text{NS}}\) | \(\tilde{m}_{1/2}\) | \(K(\tau) + n\) | \(\text{remark}\) |
|---|---|---|---|---|---|---|---|
| 1 | \(\frac{1}{5}, \frac{1}{2}\) | 1 | \(T^F_1\), \((SLY)\)_2 | 11 | \(\frac{1}{5}, \frac{5}{6}\) | 11 | \(T^F_{11}\) | 12 |
| 5 | \(\frac{1}{3}, \frac{1}{2}\) | 0 | \(T^F_5\), \(\mathcal{F}(B_2)\)\(_3\) | 7 | \(\frac{1}{2}, \frac{7}{3}\) | 0 | \(T^F_7\), \(\mathcal{F}(C_3)\)\(_2\) | 0 |

Table 9. Fermionic Hecke images of \((SLY)\)\(_2\).

| \(c\) | \(h_{\text{NS}}\) | \(m_{1/2}\) | \(\tilde{c}\) | \(\tilde{h}_{\text{NS}}\) | \(\tilde{m}_{1/2}\) | \(K(\tau) + n\) | \(\text{remark}\) |
|---|---|---|---|---|---|---|---|
| \(\frac{3}{2}\) | \((\frac{1}{4})_2, \frac{1}{2}\) | 2 | \(T^F_1\), \((SLY)\)\(_1\)\(_2\) | \(\frac{21}{2}\) | \(\frac{1}{2}, (\frac{3}{4})_2\) | 14 | \(T^F_7\) | 16 |
| \(\frac{9}{2}\) | \((\frac{1}{4})_2, \frac{1}{2}\) | 0 | \(T^F_5\), \(\mathcal{F}(A_1)\)\(_6\) | \(\frac{15}{2}\) | \(\frac{1}{2}, (\frac{3}{4})_2\) | 0 | \(T^F_5\), \(\mathcal{F}(A_3)\)\(_4\) | 0 |

Table 10. Fermionic Hecke images of \((SLY)\)\(_1\).
One can easily recognize some irreducible representations e.g. \([25, \text{equation (4.2)}]\). Coupling with a free fermion does not change the \(S\)-matrix. The conductor also remains the same as \(N = 48\).

The bosonic Hecke images of Ising model were studied in \([22]\), see also \([25, \text{section 4.1}]\), while the bosonic Hecke images of Ising\(^2\) model were studied in \([25, \text{section 7.7}]\). The current study on the fermionic Hecke images of Ising \(\otimes F\) is somewhat between them. We find there exist in total two classes of \(\rho^F(\sigma_p)\) for \(p \mod 24\): for \(p \equiv 1, 7, 17, 23 \mod 24\), i.e., \(p^2 \equiv 1 \mod 24\), \(\rho^F(\sigma_p) = 1d\), for \(p \equiv 5, 11, 13, 19 \mod 24\), i.e., \(p^2 \equiv 25 \mod 24\),

\[
\rho^F(\sigma_p) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\] (5.6)

Not surprisingly, these are just the same with the two bosonic Hecke classes for Ising model given in \([25, \text{section 4.1}]\). We find the fermionic Hecke images \(T_p^F(\text{Ising} \otimes F)\) for \(p < 12\) is just the bosonic Hecke images \(T_p(\text{Ising})\) coupling with \(p\) number of fermions. One can also introduce the generalized fermionic Hecke relations for \(p = 3k, k \in \mathbb{Z}\), where \(T_p^F\) is defined by \((A_1)_2 \otimes F^3\). We compute all fermionic Hecke images \(T_p^F\) for \(p < 24\) and summarize them in \(c = 24\) pairs in table 11. Note in this case, it is not possible to pair two \(T_p^F\) images as a \(c = 12\) holomorphic theory. Nevertheless, for \(p_1 + p_2 = 8\), two images \(T_{p_1}^F\) and \(T_{p_2}^F\) can form a holomorphic SCFT of 16 free chiral fermions.

In the end, we remark that the \(T_{23}^F\) image describes a SCFT associated to the second largest Conway group \(\text{Co}_2\), or more precisely a multi-covering \(2 \cdot 2^{1+22}\text{Co}_2\). We compute the NS characters of \(T_{23}^F\) image as

\[
\begin{align*}
\chi_0^{T_{23}^F} &= q^{-24} \left( 1 + 2300q^{3/2} + 46851q^2 + 529828q^{5/2} + 4310154q^3 + \ldots \right), \\
\chi_1^{T_{23}^F} &= q^{24} \left( 23 + 2300\sqrt{q} + 46598q + 529828q^{3/2} + 4311948q^2 + \ldots \right), \\
\chi_{\frac{3}{16}}^{T_{23}^F} &= q^{12} \left( 2048 + 47104\sqrt{q} + 565248q + 4757504q^{3/2} + 31700992q^2 + \ldots \right).
\end{align*}
\] (5.7)

One can easily recognize some irreducible representations \(23, 2048, 2300, 47104, \ldots\) of \(2 \cdot 2^{1+22}\text{Co}_2\) from the Fourier coefficients. See a collection of the dimensions of the irreducible representations of \(2 \cdot 2^{1+22}\text{Co}_2\) in e.g. \([52]\). This is not surprising as the bosonic Hecke image \(T_{23}\) of Ising\(^2\) has been associated to \(2 \cdot 2^{1+22}\text{Co}_2\) in \([53, \text{section 3.2.8}]\).
5.4 Type $SM_{\text{eff}}(7, 3)$

Supersymmetric minimal model $SM(7, 3)$ has $c = -\frac{11}{12}$ and $c_{\text{eff}} = \frac{13}{12}$. The NS conformal weights are $h^{\text{NS}} = 0, -\frac{1}{14}, \frac{2}{7}$, while $h^{\text{NS}}_{\text{eff}} = 0, \frac{1}{14}, \frac{5}{14}$. The conductor is $N = 336$. It was noticed in [37] that $SM(7, 3)$ can be realized as the fermionization of the $E_6$ invariant of bosonic minimal model $M(12, 7)$. See [37, equation (4.5)] for the character relations. We find the $S$-matrix of the NS characters of $SM_{\text{eff}}(7, 3)$ to be

$$S = \frac{2}{\sqrt{7}} \begin{pmatrix}
\cos\left(\frac{\pi}{14}\right) & \sin\left(\frac{\pi}{14}\right) & \cos\left(\frac{3\pi}{14}\right) \\
\sin\left(\frac{\pi}{14}\right) & \cos\left(\frac{3\pi}{14}\right) - \cos\left(\frac{\pi}{14}\right) \\
\cos\left(\frac{3\pi}{14}\right) - \cos\left(\frac{\pi}{14}\right) & -\sin\left(\frac{\pi}{14}\right)
\end{pmatrix}.$$ \hspace{1cm} (5.8)

It is easy to check $S^2 = \text{Id}$. We remark that this is the same $S$-matrix as the bosonic minimal model $M_{\text{eff}}(7, 2)$ in the order of weights $0, \frac{3}{7}, \frac{4}{7}$, see e.g. [25, equation (4.13)]. We find there exist in total three classes for the fermionic Hecke operation of $SM_{\text{eff}}(7, 3)$: for $p \equiv 1, 13, 29, 41, 43, 55, 71, 83, 85, 97, 113, 125, 127, 139, 155, 167$ mod $168$, i.e., $p^2 \equiv 1$ mod $168$, $\rho^F(\sigma_p) = \text{Id}$, for $p \equiv 5, 19, 23, 37, 47, 61, 65, 79, 89, 103, 107, 121, 131, 145, 149, 163$ mod $168$, i.e., $p^2 \equiv 25$ mod $168$,

$$\rho^F(\sigma_p) = \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix},$$ \hspace{1cm} (5.9)

finally, for $p \equiv 11, 17, 25, 31, 53, 59, 67, 73, 95, 101, 109, 115, 137, 143, 151, 157$ mod $168$, i.e., $p^2 \equiv 121$ mod $168$,

$$\rho^F(\sigma_p) = \begin{pmatrix}
0 & -1 & 0 \\
0 & 0 & -1 \\
1 & 0 & 0
\end{pmatrix}. $$ \hspace{1cm} (5.10)

Clearly, the three fermionic Hecke classes form a $\mathbb{Z}_3$ group. We compute all fermionic Hecke images $T^F_p$ up to $p < 14$ and summarize them in table 12. Although we do not recognize any interesting fermionic Hecke images, it still interesting to see that all six objects in rank-6 SMC related to divisor 7 (see e.g. [9, table 2]) can be generated by $T^F_9$ on $SM_{\text{eff}}(7, 3)$. One can check that for each SMC $6^S_5$, the central charge in [9, table 2]) is equal to the $c$ of fermionic Hecke image modulo $1/2$, while the topological spins are equal to the $h_{\text{NS}}$ (including the vacumm NS weight 0) modulo $1/2$.

5.5 Type $SM_{\text{sub}}(20, 2)$

Supersymmetric minimal model $SM(20, 2)$ has $c = -\frac{14}{5}$ and $c_{\text{eff}} = \frac{6}{5}$. The NS weights are $h^{\text{NS}} = 0, -\frac{2}{5}, -\frac{7}{10}, -\frac{9}{10}, -1$, while $h^{\text{NS}}_{\text{eff}} = 0, \frac{1}{5}, \frac{3}{10}, \frac{3}{5}, 1$. This suggests $SM(20, 2)$ is a degenerate theory. Let us consider a sub-theory of $SM_{\text{eff}}(20, 2)$, i.e., a D-type non-diagonal modular invariant composed of

$$\chi_0 = \chi^{SM_{\text{eff}}(20, 2)}_0 + \chi^{SM_{\text{eff}}(20, 2)}_1 = q^{-\frac{1}{20}} \left(1 + \sqrt{q} + 2q + 2q^{3/2} + 3q^2 + \ldots\right),$$

$$\chi_{1/10} = \chi^{SM_{\text{eff}}(20, 2)}_{1/10} - \chi^{SM_{\text{eff}}(20, 2)}_{3/5} = q^{\frac{1}{20}} \left(1 + q^{3/2} + 2q^2 + 2q^{5/2} + 2q^3 + \ldots\right),$$

$$\chi_{3/10} = \chi^{SM_{\text{eff}}(20, 2)}_{3/10} = q^{\frac{1}{5}} \left(1 + \sqrt{q} + q + 2q^{3/2} + 3q^2 + 3q^{5/2} + 4q^3 + \ldots\right).$$ \hspace{1cm} (5.11)
It is easy to check
\[ Z_{\text{NS}} = |\chi_0|^2 + |\chi_{1/10}|^2 + 2|\chi_{3/10}|^2 \] (5.12)
is \( \Gamma_0 \) modular invariant. Therefore the weight-\( \frac{3}{10} \) NS primary has degeneracy 2. The extended\( S \)-matrix can be deduced from the full \( S \)-matrix of \( SM_{\text{eff}}(20, 2) \) as
\[
S = \sqrt{\frac{2}{5}} \begin{pmatrix}
\sin\left(\frac{\pi}{20}\right) + \cos\left(\frac{\pi}{20}\right) & \cos\left(\frac{3\pi}{20}\right) - \sin\left(\frac{3\pi}{20}\right) & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\cos\left(\frac{3\pi}{20}\right) - \sin\left(\frac{3\pi}{20}\right) & \sin\left(\frac{\pi}{20}\right) + \cos\left(\frac{\pi}{20}\right) & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \alpha_+ & -\frac{\alpha_+}{\sqrt{2}} \\
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{\alpha_+}{\sqrt{2}} & \alpha_+ \end{pmatrix}.
(5.13)
Here \( \alpha_{\pm} = \sqrt{3 \pm \sqrt{5}} \). We denote this sub-theory of \( SM_{\text{eff}}(20, 2) \) as \( SM_{\text{sub}}(20, 2) \). Clearly the conductor is \( N = 20 \).

Consider the fermionic Hecke operation on \( SM_{\text{sub}}(20, 2) \). From the above \( S \)-matrix, we find there exist two classes of \( \rho^F(\sigma_p) \) for \( p \mod 10 \):
\[
\rho^F(\sigma_{1,9}) = \text{Id}, \quad \rho^F(\sigma_{3,7}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.
(5.14)
Obviously, they form a \( \mathbb{Z}_2 \) group. We compute all fermionic Hecke images \( \mathsf{T}^F_p \) for \( p < 20 \) and summarize them as \( c = 12 \) pairs in table 13.
same with those of \( SM_{\text{sub}}(20, 2) \) itself. Therefore the \( c = \frac{66}{5} \) theory if exists is non-unitary and cannot be realized as the fermionization of a WZW model.

As a side remark, we also studied the fermionic Hecke operation on the full \( SM_{\text{eff}}(20, 2) \), i.e., \( (SLY)_4 \), as a theory with five NS primaries. Unfortunately, we did not find any interesting fermionic Hecke images.

### 5.6 Type \( \mathcal{F}(\text{Ising}^3) \)

A large class of supersymmetric RCFTs is known as the fermionization of WZW \( SO(m)_1^3 \) theories [12]. These theories have central charge \( c = \frac{3m}{2} \) and NS weights with degeneracy \( h_{\text{NS}} = 0, \left( \frac{1}{2} \right)_3, \left( \frac{m}{2} \right)_3 \). Recall WZW \( SO(m)_1 \) model has conformal weights \( 0, \frac{m}{16}, \frac{1}{2} \). The NS characters of \( \mathcal{F}(SO(m)_1^3) \) is defined by

\[
\begin{align*}
\chi_0^{\mathcal{F}(SO(m)_1^3)} &= (\chi_0^{SO(m)_1})^3 + (\chi_{1/2}^{SO(m)_1})^3, \\
\chi_{1/2}^{\mathcal{F}(SO(m)_1^3)} &= \chi_0^{SO(m)_1} \chi_{1/2}^{SO(m)_1} (\chi_0^{SO(m)_1} + \chi_{1/2}^{SO(m)_1}), \\
\chi_{m/8}^{\mathcal{F}(SO(m)_1^3)} &= (\chi_{1/2}^{SO(m)_1})^2 (\chi_0^{SO(m)_1} + \chi_{1/2}^{SO(m)_1}).
\end{align*}
\] (5.15)

When \( m = 1 \), as \( SO(1)_1 \) is just the Ising model, we denote the supersymmetric theory as \( \mathcal{F}(\text{Ising}^3) \). Note it is different from \( (\mathcal{F}\text{Ising}^3) = 3F \), or Ising \( \otimes 2F \), or Ising\(^2 \) \( \otimes F \). The \( S \)-matrix of the three NS characters in (5.15) can be easily determined from the one of \( SO(m)_1 \) as

\[
S = \frac{1}{4} \begin{pmatrix} 1 & 3 & 6 \\ 1 & 3 & -2 \\ 2 & -2 & 0 \end{pmatrix}.
\] (5.16)

Note it is independent from \( m \). Considering the degeneracy, the \( 7 \times 7 \) full \( S \)-matrix can be found in e.g. [2, equation (3.47)]. Both \( S \)-matrices can be used to compute fermionic Hecke images, here for simplicity we use the reduced one (5.16).

Consider the fermionic Hecke operation on \( \mathcal{F}(\text{Ising}^3) \). The conductor is \( N = 16 \). From (5.16), we find for arbitrary odd \( p \), \( \rho_p^\mathcal{F}(\sigma_p) = \text{Id} \), i.e., there is only one fermionic Hecke class. By computing \( T_p^\mathcal{F} \) for all \( p < 16 \), we find that for \( p < 8 \) the \( T_p^\mathcal{F} \) images exactly describes \( \mathcal{F} SO(p)_1^3 \). We summarize those with \( p < 8 \) in \( c = 12 \) pairs in table 14. The bilinear relation of the NS characters of each pair equals to \( K(\tau) \). For \( 8 < p < 16 \), \( T_p^\mathcal{F} \) is different from \( \mathcal{F} SO(p)_1^3 \). Nevertheless, similar with the one free fermion case, we find the NS characters of \( \mathcal{F} SO(p)_1^3 \) can be written as the linear combinations of \( T_p^\mathcal{F} \) and \( T_{p-8}^\mathcal{F} \) images.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|}
\hline
\( c \) & \( h_{\text{NS}} \) & \( m_{1/2} \) & remark & \( \hat{c} \) & \( \hat{h}_{\text{NS}} \) & \( \hat{m}_{1/2} \) & remark & \( K(\tau) + n \) \\
\hline
\( \frac{6}{5} \) & \( \frac{1}{10}, (\frac{3}{10})_2 \) & 1 & \( T_{\mathcal{F}}^5, SM_{\text{sub}}(20, 2) \) & \( \frac{54}{5} \) & \( (\frac{7}{10})_2, \frac{9}{10} \) & 9 & \( T_{9}^\mathcal{F} \) & 10 \\
\( \frac{18}{5} \) & \( \frac{3}{10}, (\frac{7}{10})_2 \) & 0 & \( T_{\mathcal{F}}^5, \mathcal{F}(D_2)_3 \) & \( \frac{42}{5} \) & \( (\frac{5}{2})_2, \frac{7}{10} \) & 0 & \( T_{7}^\mathcal{F}, \mathcal{F}(C_3)_1^2 \) & 0 \\
\hline
\end{tabular}
\caption{Fermionic Hecke images of \( SM_{\text{sub}}(20, 2) \).}
\end{table}
The non-unitary supersymmetric minimal model

This procedure can be further extended to even

This shows that

It is easy to check

| $c$ | $h_{NS}$ | $m_{1/2}$ | remark | $\hat{c}$ | $\hat{h}_{NS}$ | $\hat{m}_{1/2}$ | remark | $K(\tau) + n$ |
|-----|----------|-----------|--------|------|------------|-----------|--------|-------------|
| $\frac{3p}{2}$ | $(\frac{1}{2})_3, (\frac{p}{8})_3$ | 0 | $T^F_p$ | $\frac{3(8-p)}{2}$ | $(\frac{1}{2})_3, (\frac{8-p}{8})_3$ | 0 | $T^F_{16-p}$ | 0 |

Table 14. (Generalized) fermionic Hecke images of $\mathcal{F}$(Ising$^3$).

For example, for $p = 11$, we find the three NS characters of weights $0, \frac{1}{2}, \frac{11}{8}$ can be written as

$$
\chi^{FSO(11)^3} = T^F_{11} + 11M \cdot T^F_3, \quad M = \begin{pmatrix}
0 & 3 & 0 \\
1 & 2 & 0 \\
0 & 0 & -1
\end{pmatrix}.
$$

(5.17)

Now we would like to show that the above results can be safely extended to even $p$ by including generalized fermionic Hecke images $T^F_2$, $T^F_4$ and $T^F_8$ which describe $\mathcal{F}SO(2)^3_1$, $\mathcal{F}SO(4)^3_1$ and $\mathcal{F}SO(8)^3_1$ respectively. This is very similar with the situation of one free chiral fermion in section 3.3. Here we just show for generalized $T^F_p$. Denote the three NS characters of $\mathcal{F}$(Ising$^3$) as $\chi_0, \chi_{1/2}, \chi_{1/8}$. We find the three NS characters of $\mathcal{F}SO(2)^3_1$ can be written as degree 2 polynomials of $\chi_0, \chi_{1/2}, \chi_{1/8}$ as

$$
\begin{align*}
\chi^{FSO(2)^3_1}_0 &= \chi_0^2 + 3\chi_{1/2}^2, \\
\chi^{FSO(2)^3_1}_{1/2} &= 2\chi_0\chi_{1/2} + 2\chi_{1/2}^2, \\
\chi^{FSO(2)^3_1}_{1/4} &= \chi_{1/8}^2.
\end{align*}
$$

(5.18)

This shows that $\mathcal{F}SO(2)^3_1$ can be regarded as a generalized $T^F_2$ image of $\mathcal{F}$(Ising$^3$). Then we can use the $T^F_p$ images of $\mathcal{F}SO(2)^3_1$ to define the generalized $T^F_{2k}$ images of $\mathcal{F}$(Ising$^3$), with $k$ coprime to the new conductor $8$. There is still only one fermionic Hecke class $\rho^F(\sigma_p) = \text{Id}$. This procedure can be further extended to $\mathcal{F}SO(4)^3_1$ and $\mathcal{F}SO(8)^3_1$. In summary, we find all information in table 14 still holds for all generalized $T^F_p$ images.

6 Four NS characters

6.1 Type (SLY) characters

The non-unitary supersymmetric minimal model $SM(16, 2)$ has $c = -\frac{135}{8}$ and $h = 0, -\frac{3}{8}, -\frac{5}{8}, -\frac{3}{4}$, while $SM_{\text{eff}}(16, 2)$, i.e., $(SLY)_3$ has $c_{\text{eff}} = \frac{9}{8}$ and $h_{\text{eff}} = 0, \frac{1}{8}, \frac{3}{8}, \frac{3}{4}$. The conductor is $N = 64$. We find the $S$-matrix for $(SLY)_3$ is

$$
S = \frac{1}{\sqrt{2}} \begin{pmatrix}
\cos(\frac{\pi}{16}) & \cos(\frac{3\pi}{16}) & \sin(\frac{3\pi}{16}) & \sin(\frac{\pi}{16}) \\
\cos(\frac{3\pi}{16}) & -\sin(\frac{\pi}{16}) & -\cos(\frac{\pi}{16}) & -\sin(\frac{3\pi}{16}) \\
\sin(\frac{3\pi}{16}) & -\cos(\frac{\pi}{16}) & \sin(\frac{\pi}{16}) & \cos(\frac{3\pi}{16}) \\
\sin(\frac{\pi}{16}) & -\sin(\frac{3\pi}{16}) & \cos(\frac{3\pi}{16}) & -\cos(\frac{\pi}{16})
\end{pmatrix}.
$$

(6.1)

It is easy to check $S^2 = \text{Id}$. 

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Consider the fermionic Hecke operation of \((Sly)_3\). We find there exist in total 8 classes of \(\rho^F(\sigma_p)\) for \(p \mod 32\):

\[
\begin{align*}
\rho^F(\sigma_{1,31}) &= -\rho^F(\sigma_{15,17}) = \text{Id}, \\
\rho^F(\sigma_{3,29}) &= -\rho^F(\sigma_{13,19}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\
\rho^F(\sigma_{9,27}) &= -\rho^F(\sigma_{11,21}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \\
\rho^F(\sigma_{7,25}) &= -\rho^F(\sigma_{9,23}) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.
\end{align*}
\]

We find the 8 fermionic Hecke classes form a \(\mathbb{Z}_8\) group. We compute all fermionic Hecke images \(T^F_p\) up to \(p < 14\) and summarize the relevant information in table 15. Notably, we find the \(T^F_7\) image describes exactly the fermionization of WZW \(SO(7)_3\) model. The relation between the NS characters of and the affine characters can be found in e.g. [25, equation (4.45)].

### 6.2 Type \(SM(6, 4)\)

The supersymmetric minimal model \(SM(6, 4)\) is a degenerate unitary theory with \(c = 1\). There are four NS primaries with weights \(0, \frac{1}{16}, \frac{1}{6}, 1\). It is easy to find the \(S\)-matrix is

\[
S = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & \sqrt{6} & 2 & 1 \\ \sqrt{6} & 0 & 0 & -\sqrt{6} \\ 2 & 0 & -2 & 2 \\ 1 & -\sqrt{6} & 2 & 1 \end{pmatrix}.
\]

In section 4.5, we have discussed a sub-theory of \(SM(6, 4)\) that is \(S^2A_1\). Note here the conductor \(N = 48\) is bigger than its sub-theory.
Consider the fermionic Hecke operation of $SM(6, 4)$. We find there exist four classes of $ho^F(\sigma_p)$ for $p$ mod 24:

$$\rho^F(\sigma_{1,23}) = -\rho^F(\sigma_{5,19}) = \text{Id}, \quad \rho^F(\sigma_{7,17}) = -\rho^F(\sigma_{11,13}) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. $$

We compute all fermionic Hecke images $T^F_p$ up to $p < 24$ and summarize them in $c = 24$ pairs in table 16. It is worthwhile to point out that the $T^F_{11}$ image of $SM(6, 4)$ describes the fermionization of $(D_6)_2$, while the $T^F_{11}$ image of the sub-theory of $SM(6, 4)$ describes the fermionization of $SU(12)_1$. For the precise relation between the fermionic characters of $(D_6)_2$ and the affine characters, we refer to [20, equation (3.27)]. It is easy to check the characters satisfy the bilinear relations $T^F_{11} \cdot T^F_{13} = K(\tau)^2 - 408 = q^{-1} + 144 + \ldots$. Similarly, we find $T^F_{1} \cdot T^F_{23} = K(\tau)^2 - 552$.

### 6.3 Type $SM_{\text{sub}}(28, 2)$

Supersymmetric minimal model $SM(28, 2)$ has central charge $c = -\frac{252}{7}$ and NS weights $h = 0, -\frac{3}{7}, -\frac{11}{14}, -\frac{15}{14}, -\frac{9}{7}, -\frac{10}{7}, -\frac{3}{2}$, while $SM_{\text{eff}}(28, 2)$ has $c_{\text{eff}} = \frac{1}{2}$ and $h_{\text{eff}} = 0, \frac{1}{14}, \frac{3}{14}, \frac{3}{7}, \frac{5}{7}, \frac{15}{14}, \frac{3}{2}$. The conductor is $N = 56$. Let us consider a sub-theory of $SM_{\text{eff}}(28, 2)$, viz. a D-type non-diagonal modular invariant by

$$
\begin{align*}
\chi_0 &= \chi_0^{SM_{\text{eff}}(28, 2)} - \chi_{3/2}^{SM_{\text{eff}}(28, 2)} = q^{-\frac{3}{2}} \left( 1 + \sqrt{q} + q + q^{3/2} + 3q^2 + 4q^{5/2} + \ldots \right), \\
\chi_{1/14} &= \chi_{1/14}^{SM_{\text{eff}}(28, 2)} + \chi_{15/14}^{SM_{\text{eff}}(28, 2)} = q^{\frac{1}{2}} \left( 1 + \sqrt{q} + 2q + 3q^{3/2} + 4q^2 + 5q^{5/2} + \ldots \right), \\
\chi_{3/14} &= \chi_{3/14}^{SM_{\text{eff}}(28, 2)} - \chi_{5/14}^{SM_{\text{eff}}(28, 2)} = q^{\frac{3}{2}} \left( 1 + q^{3/2} + q^2 + q^{5/2} + 2q^3 + 3q^{7/2} + \ldots \right), \\
\chi_{3/7} &= \chi_{3/7}^{SM_{\text{eff}}(28, 2)} = q^3 \left( 1 + \sqrt{q} + q + 2q^{3/2} + 3q^2 + 4q^{5/2} + 5q^3 + 6q^{7/2} + \ldots \right).
\end{align*}
$$

It is easy to check

$$
Z_{\text{NS}} = |\chi_0|^2 + |\chi_{1/14}|^2 + |\chi_{3/14}|^2 + 2|\chi_{3/7}|^2
$$
Table 17. Fermionic Hecke images of $SM_{\text{sub}}(28, 2)$.

is $\Gamma_0$ modular invariant. Clearly the weight-$\frac{3}{7}$ primary has degeneracy 2. The extended $S$-matrix can be deduced from the full $S$-matrix of $SM_{\text{eff}}(28, 2)$ as

$$S = \sqrt{\frac{2}{7}} \begin{pmatrix}
\cos\left(\frac{\pi}{28}\right) - \sin\left(\frac{\pi}{28}\right) & \sin\left(\frac{\pi}{28}\right) + \cos\left(\frac{3\pi}{28}\right) & \cos\left(\frac{5\pi}{28}\right) - \sin\left(\frac{3\pi}{28}\right) & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\sin\left(\frac{\pi}{28}\right) + \cos\left(\frac{3\pi}{28}\right) & \cos\left(\frac{5\pi}{28}\right) - \sin\left(\frac{3\pi}{28}\right) & \cos\left(\frac{9\pi}{28}\right) - \sin\left(\frac{5\pi}{28}\right) & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
\cos\left(\frac{\pi}{28}\right) - \sin\left(\frac{\pi}{28}\right) & \cos\left(\frac{5\pi}{28}\right) - \sin\left(\frac{\pi}{28}\right) & \cos\left(\frac{9\pi}{28}\right) - \sin\left(\frac{5\pi}{28}\right) & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{\alpha_+}{2} & -\frac{\alpha_-}{2} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{\alpha_-}{2} & \frac{\alpha_+}{2}
\end{pmatrix}.$$  

Here $\alpha_{\pm} = \sqrt{4 \pm \sqrt{7}}$. One can check $S^2 = \text{Id}$. We denote this sub-theory as $SM_{\text{sub}}(28, 2)$.

Consider the fermionic Hecke operation of $SM_{\text{sub}}(28, 2)$. We find there exist in total 6 classes of $\rho^F(\sigma_p)$ for $p \mod 28$: $\rho(\sigma_{1,27}) = -\rho(\sigma_{13,15}) = \text{Id},$

$$\rho^F(\sigma_{3,25}) = -\rho^F(\sigma_{11,17}) = \begin{pmatrix}
0 & 0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad \rho^F(\sigma_{5,23}) = -\rho^F(\sigma_{9,19}) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & 0
\end{pmatrix}.$$  

They form a $\mathbb{Z}_6$ abelian group, where the group elements 0, 1, 2, 3, 4, 5 are represented by $p = 1, 5, 3, 13, 9, 11$ respectively. We compute all $T_p^F$ for $p < 18$ and summarize the relevant data in table 17. Notably, we find the $T_5^F$ image describes exactly the fermionization of WZW $SO(6)_3$ model. The relation between the NS characters of and the affine characters can be found in for example [25, equation (4.33)].

6.4 Type $SM_{\text{sub}}(8, 6)$

Unitary supersymmetric minimal model $SM(8, 6)$ has $c = \frac{5}{4}$ and nine NS primaries with weights $h_{\text{NS}} = 0, \frac{1}{32}, \frac{1}{17}, \frac{5}{32}, \frac{1}{4}, \frac{5}{6}, \frac{33}{32}, \frac{5}{7}, 3$. To produce interesting fermionic Hecke images, let
us consider the following non-diagonal modular invariant of $SM(8, 6)$ composed of

$$\chi_0 = \chi_0^{SM(8, 6)} + \chi_3^{SM(8, 6)} = q^{\frac{1}{72}} \big(1 + q^{3/2} + q^2 + q^{5/2} + 2q^3 + 3q^{7/2} + \ldots\big),$$

$$\chi_{1/12} = \chi_{1/12}^{SM(8, 6)} = q^{\frac{1}{12}} \big(1 + \sqrt{q} + q + 2q^{3/2} + q^2 + 4q^{5/2} + 5q^3 + 7q^{7/2} + \ldots\big),$$

$$\chi_{1/4} = \chi_{1/4}^{SM(8, 6)} + \chi_{5/4}^{SM(8, 6)} = q^{\frac{10}{12}} \big(1 + \sqrt{q} + 2q + 2q^{3/2} + 3q^2 + 5q^{5/2} + \ldots\big),$$

$$\chi_{5/6} = \chi_{5/6}^{SM(8, 6)} = q^{\frac{22}{12}} \big(1 + \sqrt{q} + q^{5/2} + 2q^3 + 3q^{5/2} + 3q^3 + 4q^{7/2} + \ldots\big).$$

(6.5)

It is easy to check

$$Z_{NS} = |\chi_0|^2 + 2|\chi_{1/12}|^2 + |\chi_{1/4}|^2 + 2|\chi_{5/6}|^2$$

(6.6)

is $\Gamma_\theta$ modular invariant. The weight-$\frac{1}{2}$ and $\frac{5}{6}$ NS primaries both have degeneracy 2. This $\Gamma_\theta$ modular invariant has been studied in [40–42]. Let us denote this sub-theory as $SM_{sub}(8, 6)$. Obviously the conductor is $N = 96$. In fact, it can be also realized as a supersymmetric coset

$$SM_{sub}(8, 6) = \frac{F(A_1)_6}{SM_{sub}(6, 4)}.$$

(6.7)

The character relations of this coset can be found in e.g. [1, equation (5.51)]. The $S$-matrix of the four characters in (6.5) can be determined as

$$S = \frac{1}{\sqrt{3}}\begin{pmatrix}
\sin\left(\frac{\pi}{6}\right) & 2\cos\left(\frac{\pi}{6}\right) & \cos\left(\frac{\pi}{6}\right) & 2\sin\left(\frac{\pi}{6}\right) \\
\cos\left(\frac{\pi}{6}\right) & \sin\left(\frac{\pi}{6}\right) & -\sin\left(\frac{\pi}{6}\right) & -\cos\left(\frac{\pi}{6}\right) \\
\cos\left(\frac{\pi}{6}\right) & -2\sin\left(\frac{\pi}{6}\right) & -\sin\left(\frac{\pi}{6}\right) & 2\cos\left(\frac{\pi}{6}\right) \\
\sin\left(\frac{\pi}{6}\right) & -\cos\left(\frac{\pi}{6}\right) & \cos\left(\frac{\pi}{6}\right) & -\sin\left(\frac{\pi}{6}\right)
\end{pmatrix}.$$  

(6.8)

To lighten the computation, we use this non-symmetric reduced $S$-matrix instead of the $6 \times 6$ full $S$-matrix.

Let us consider the fermionic Hecke operation on $SM_{sub}(8, 6)$. We find there exist in total 8 classes of $\rho^F(\sigma_p)$ for $p$ mod 48: $\rho^F(\sigma_{1, 7, 41, 47}) = -\rho^F(\sigma_{17, 23, 25, 31}) = \text{Id}$, while

$$\rho^F(\sigma_{5, 13, 35, 43}) = -\rho^F(\sigma_{11, 19, 29, 37}) = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}.$$  

(6.9)

The four Hecke classes form a $\mathbb{Z}_4$ abelian group. We compute all fermionic Hecke images $T^F_p$ for $p < 20$ and list the results in table 18. Notably, we find the $T^F_{13}$ image describes the renowned fermionization of WZW SU(6)$_2$ theory. This theory has emergent supersymmetry and true supersymmetric vacua. The relation between the NS characters of $F(A_5)_2$ and the affine characters can be found in e.g. [20, table 9].

7 Five NS characters

7.1 Type $SM_{sub}(36, 2)$

Supersymmetric minimal model $SM(36, 2)$ has central charge $c = -\frac{140}{3}$ and NS weights $h = 0, -\frac{4}{9}, -\frac{5}{6}, -\frac{7}{6}, -\frac{13}{9}, -\frac{5}{3}, -\frac{11}{6}, -\frac{35}{18}, -2$, while the effective theory $SM_{eff}(36, 2)$ has $c_{eff} = \ldots$
and $h_{\text{eff}} = 0, 1/18, 1/6, 5/6, 5/8, 7/8, 7/9, 2$. Let us consider a sub-theory of $SM_{\text{eff}}(36, 2)$, that is a D-type non-diagonal modular invariant composed by

$$
\begin{align*}
\chi_0 &= \chi_0^{SM_{\text{eff}}(36, 2)} + \chi_2^{SM_{\text{eff}}(36, 2)} = q^{-\frac{1}{18}} \left(1 + \sqrt{q} + 3q^{3/2} + 2q^2 + 4q^{5/2} + \ldots\right), \\
\chi_{1/18} &= \chi_{1/18}^{SM_{\text{eff}}(36, 2)} - \chi_{14/9}^{SM_{\text{eff}}(36, 2)} = 1 + \sqrt{q} + q^{3/2} + 2q^2 + 3q^{5/2} + 4q^3 + \ldots, \\
\chi_{1/6} &= \chi_{3/14}^{SM_{\text{eff}}(36, 2)} + \chi_{7/6}^{SM_{\text{eff}}(36, 2)} = q^{\frac{3}{2}} \left(1 + \sqrt{q} + 2q + 3q^{3/2} + 4q^2 + 6q^{5/2} + \ldots\right), \\
\chi_{1/3} &= \chi_{1/3}^{SM_{\text{eff}}(36, 2)} - \chi_{5/6}^{SM_{\text{eff}}(36, 2)} = q^{\frac{5}{3}} \left(1 + q^{3/2} + q^2 + q^{5/2} + q^3 + 2q^7/2 + \ldots\right), \\
\chi_{5/9} &= \chi_{5/9}^{SM_{\text{eff}}(36, 2)} = q^{\frac{5}{3}} \left(1 + \sqrt{q} + q + 2q^{3/2} + 3q^2 + 4q^{5/2} + 5q^3 + 7q^{7/2} + \ldots\right).
\end{align*}
$$

It is easy to check

$$
Z_{\text{NS}} = |\chi_0|^2 + |\chi_{1/18}|^2 + |\chi_{1/6}|^2 + |\chi_{1/3}|^2 + 2|\chi_{5/9}|^2
$$

is $\Gamma_0$ modular invariant. Clearly, the weight $\frac{5}{6}$ NS character has degeneracy 2. The extended $S$-matrix can be deduced from the full $S$-matrix of $SM_{\text{eff}}(36, 2)$ as

$$
\begin{pmatrix}
1 & \sqrt{2} (\sin(\frac{7\pi}{36}) + \cos(\frac{7\pi}{36})) & \sqrt{2} (\cos(\frac{7\pi}{36}) - \sin(\frac{7\pi}{36})) & 1 & 1 \\
1 & 2 & -1 & 1 & -1 & -1 \\
1 & -\sqrt{2} (\sin(\frac{7\pi}{36}) + \cos(\frac{7\pi}{36})) & -\sqrt{2} (\sin(\frac{7\pi}{36}) + \cos(\frac{7\pi}{36})) & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & 2 & -1 \\
1 & -1 & -1 & 1 & -1 & 2
\end{pmatrix}
$$

We denote this sub-theory as $SM_{\text{sub}}(36, 2)$. Clearly the conductor is $N = 18$. Note this is a degenerate theory.
They form a \( T \) under the \( S \) action, which can still generalize it to the \( T \) sector. The relations between affine characters and fermionic characters including fermionization of WZW results in exchanges the affine node and vector node of affine \( \Gamma \). Clearly, this \( \Gamma \) is natural since there are the symmetry between the spinor and conjugate spinor nodes. We compute all \( T_p^F \) images for \( p < 18 \) and summarize the results in \( c = 24 \) pairs in Table 19. Notably, we find the \( T_7^F \) image describes exactly the fermionization of WZW \( \text{SO}(8)_3 \) model:

\[
\begin{align*}
\chi_0^{T_F} &= \chi_{0,0}^{\text{SO}(8)_3} + \chi_{2,3w_1}^{\text{SO}(8)_3} = q^{-7/8} \left( 1 + 28q + 112q^{3/2} + 434q^2 + 1568q^{5/2} + \ldots \right), \\
\chi_7^{T_F} &= \chi_7^{\text{SO}(8)_3} + \chi_7^{\text{SO}(8)_3} = 8 + 35\sqrt{q} + 224q + 980q^{3/2} + 3472q^2 + \ldots, \\
\chi_2^{T_F} &= \chi_2^{\text{SO}(8)_3} + \chi_4^{\text{SO}(8)_3} = q^{5/8} \left( 28 + 160\sqrt{q} + 784q + 3080q^{3/2} + \ldots \right), \\
\chi_5^{T_F} &= \chi_5^{\text{SO}(8)_3} + \chi_5^{\text{SO}(8)_3} = q^{5/8} \left( 56 + 350\sqrt{q} + 1568q + 5704q^{3/2} + \ldots \right), \\
\chi_7^{T_F} &= \chi_7^{\text{SO}(8)_3} + \chi_7^{\text{SO}(8)_3} = 35\sqrt{q} + 224q + 980q^{3/2} + 3472q^2 + \ldots.
\end{align*}
\]

Clearly, this \( \Gamma \) modular invariant of WZW \( \text{SO}(8)_3 \) is induced by the simple current \( v \) that exchanges the affine node and vector node of affine \( \text{SO}(8) \). The degeneracy two of the weight-9/8 fields is natural since there are the symmetry between the spinor and conjugate spinor nodes. The relations between affine characters and fermionic characters including other sectors can also be found in [20, equation (3.31)]. The \( T_7^F \) and \( T_{17}^F \) images should form a holomorphic SCFT of \( c = 24 \). The NS characters satisfy the bilinear relation \( T_7^F \cdot T_{11}^F = K(\tau)^2 - 384 \). Similarly, we find \( T_1^F \cdot T_{17}^F = K(\tau)^2 + K(\tau) - 516 \).

### 8 Comments on ∼NS and R characters

Although the most natural setting for fermionic Hecke operator is in the NS sector, one can still generalize it to the ∼NS and R sectors by some trick. As we reviewed in section 2.1, under \( S \) action, the ∼NS and R sectors transform into each other. Therefore, it is necessary
to consider them together, which we denote as the \((\tilde{N}S, R)\) sector. The \(T^2\) and \(S\) actions are now closed in this combined sector. Denote the combined character as \(\chi_{(\tilde{N}S,R)} = \{\chi_{\tilde{N}S}, \chi_R\}\). Suppose there are \(n\) number of NS characters, then there exists a \(2n\)-dimensional representation \(\rho^F : \Gamma_\theta \rightarrow \text{GL}(2n, \mathbb{C})\) such that for any \(\gamma \in \Gamma_\theta,

\[
\chi^F_{i(\tilde{N}S,R)}(\gamma\tau) = \sum_j \rho^F(\gamma)_{ij}\chi^F_{j(\tilde{N}S,R)}(\tau), \quad i = 1, 2, 3, \ldots, 2n.
\]  

(8.1)

To define the fermionic Hecke operator for \((\tilde{N}S, R)\) sector, we still need the transfer matrix \(\rho^F(\sigma_p)\). We can find it can be defined by \(\rho^F(S)\) and \(\rho^F(T^2)\) just analogous to (3.9) as

\[
\rho^F(\sigma_p) = \rho^F \left( S^2(T^2)\frac{e^{2\pi i}}{\eta^2} S(T^2) - \frac{e^{-\pi i}}{\tau^2} S(T^2 S)^{\sigma - 1} \right).
\]

(8.2)

Then the fermionic Hecke operator \(T^F_p\) is defined in the same way as (3.11) and all properties of Hecke images still hold. One prominent feature here is \(T^F_p\) always maps the \((\tilde{N}S, R)\) characters of one theory to the \((\pm\tilde{N}S, R)\) characters of another theory. The reason is that there are always even numbers of \(S\) actions in the right hand side of (8.2), which protects the order. We will show three examples here: one chiral fermion, \((SLY)_1\) and \(SM(5,3)\). Other types are just similar.

For one single chiral fermion, it is well-known that the \(\tilde{N}S\) and R characters are \(\tilde{\psi}_{NS} = \sqrt{\theta_1/\eta}\) and \(\psi_R = \sqrt{\theta_2/\eta}\). Under \(S\) action, the \(\tilde{N}S\) and R characters are exchanged. Thus

\[
\rho^F(S) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho^F(T^2) = \begin{pmatrix} e^{-\frac{17\pi i}{12}} & 0 \\ 0 & e^{\frac{17\pi i}{12}} \end{pmatrix}.
\]

(8.3)

By (8.2) we find there exist two classes of \(\rho^F(\sigma_p)\): for \(p = 1, 7, 17, 23\) mod 24, i.e., \(p^2 \equiv 1\) mod 48, \(\rho^F(\sigma_p) = \text{Id}\), while for \(p = 5, 11, 13, 19\) mod 24, i.e., \(p^2 \equiv 25\) mod 48, \(\rho^F(\sigma_p) = -\text{Id}\). For fermionic Hecke images with \(p < 24\), we find that for \(p = 1, 7, 17, 23\), \(T^F_p\) describes the \(\{\chi_{\tilde{N}S}, \chi_R\}\) of fermionization \(\mathcal{F}(SO(p))\), while for \(p = 5, 11, 13, 19\), it describes the \(\{-\chi_{\tilde{N}S}, \chi_R\}\) of fermionization \(\mathcal{F}(SO(p))\). Here we adopt the usual convention that the leading Fourier coefficient of vacuum \(\chi_{\tilde{N}S}\) is 1. It is easy to check the bilinear relation \(T^F_p \cdot T^F_{24-p} = K(\tau) + 24\), which resembles the bilinear relation in the NS sector.

For supersymmetric Lee-Yang model \((SLY)_1\), the R weights are \(h_R = \frac{1}{32}, \frac{5}{32}\). The \(\tilde{N}S\) and R characters can be defined from the bosonic minimal model \(M_{\text{eff}}(13,2)\) by \(\{\chi_{\tilde{N}S}, \chi_R\} = \{\chi_0 - \chi_{1/2}, \chi_{1/4} - \chi_{7/4}, \chi_{1/32} + \chi_{33/32}, \sqrt{2}\chi_{5/32}\}\). See the expression of \(M_{\text{eff}}(13,2)\) characters in e.g. [25, equation (8.3)]. It is worthy to point out that the \(\tilde{R}\) character here is just a constant \(\chi_{1/32} - \chi_{33/32} = 1\). We find the \(S\) and \(T^2\) matrices for the \(\{\chi_{\tilde{N}S}, \chi_R\}\) are

\[
\rho^F(S) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}, \quad \rho^F(T^2) = \begin{pmatrix} e^{-\frac{17\pi i}{24}} & 0 & 0 & 0 \\ 0 & e^{\frac{7\pi i}{24}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{\frac{17\pi i}{24}} \end{pmatrix}.
\]

(8.4)
Consider the fermionic Hecke operation on \( \{ \chi_{\text{NS}}, \chi_{\text{R}} \} \). We find there are in total two fermionic Hecke classes: for \( p \equiv 1, 7 \mod 8 \), \( \rho^{F}(\sigma_{p}) = \text{Id} \), for \( p \equiv 3, 5 \mod 8 \),

\[
\rho^{F}(\sigma_{p}) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\] (8.5)

We explicitly compute all \( T^{F}_{p} \) for \( p < 16 \) and find that the \( T^{F}_{5} \) image exactly describes the \( \{-\chi_{\text{NS}}, \chi_{\text{R}}\} \) of \( F(A_{1})_{6} \) theory given in \( (2.23) \), while the \( T^{F}_{13} \) image exactly describes the \( \{-\chi_{\text{NS}}, \chi_{\text{R}}\} \) of \( F(C_{6})_{1} \) theory. It is easy to check the bilinear relation \( T^{F}_{5} \cdot T^{F}_{13} = K(\tau) + 24 \).

For unitary supersymmetric minimal model \( SM(5,3) \), the R weights are \( h_{R} = \frac{3}{307} \). The NS and R characters can be defined from the bosonic tricritical Ising model \( M(5,4) \) by \( \{ \chi_{\text{NS}}, \chi_{\text{R}} \} = \{ \chi_{0} - \chi_{3/2}, \chi_{1/10} - \chi_{3/5}, \sqrt{2}\chi_{3/80}, \sqrt{2}\chi_{7/16} \} \). The S matrix here is related to the \( \rho^{F}(S) \) matrix of the NS characters of \( SM_{\text{sub}}(60,2) \) in \( (4.12) \) by a simple relation

\[
\rho^{F}(S) = \begin{pmatrix} 0 & \rho^{F}(S) \\ \rho^{F}(S) & 0 \end{pmatrix}.
\] (8.6)

Consider the fermionic Hecke operation on \( \{ \chi_{\text{NS}}, \chi_{\text{R}} \} \). We find there exist four Hecke classes: for \( p \equiv 1, 11, 29, 41, 79, 91, 109, 119 \mod 120 \), \( \rho^{F}(\sigma_{p}) = \text{Id} \), for \( p \equiv 19, 31, 49, 59, 61, 71, 89, 101 \mod 120 \), \( \rho^{F}(\sigma_{p}) = -\text{Id} \),

\[
\text{for } p \equiv 7, 37, 43, 47, 73, 77, 83, 113 \mod 120 , \quad \rho^{F}(\sigma_{p}) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
\] (8.7)

\[
\text{for } p \equiv 13, 17, 23, 53, 67, 97, 103, 107 \mod 120 , \quad \rho^{F}(\sigma_{p}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.
\]

These four fermionic Hecke classes form a \( \mathbb{Z}_{4} \) group. However, we remark that these are different from the four Hecke classes of the NS sector discussed in section 4.2. By explicitly computing the fermionic Hecke images, we find that the \( T^{F}_{10} \) image describes the \( \{-\chi_{\text{NS}}, \chi_{\text{R}}\} \) characters of fermionization \( F(E_{7})_{2} \). We refer to e.g. [1] for the relations between \( \{ \chi_{\text{NS}}, \chi_{\text{R}}\} \) and affine \( (E_{7})_{2} \) characters.

9 Summary and outlook

This work gives an affirmative answer to the question raised at the end of [25] whether there exists a natural fermionic Hecke operator for 2d fermionic RCFTs. We find the natural setting is the NS characters of fermionic RCFTs, as they transform to themselves by \( S \) and \( T^{2} \) actions. In a mathematical sense, we find the natural definition of Hecke operator for vector-valued \( \Gamma_{0} \) modular forms of weight zero. We also discuss how to apply fermionic
Hecke operator to the $\tilde{\text{NS}}$ and $\text{R}$ sectors, that is to combine them together such that the characters $\{\chi_{\tilde{\text{NS}}}, \chi_{\text{R}}\}$ still form a vector-valued $\Gamma_g$ modular form.

We discover many fermionic Hecke relations among the 2d fermionic and supersymmetric RCFTs, most of which are unknown from the bosonic side. In particular, for all supersymmetric theories appearing in [1, 2, 12, 20], the only one we couldn’t properly realize as a fermionic Hecke image is the fermionization of $\text{SU}(8)_2/\mathbb{Z}_2$ theory [20, equation (4.5)], with central charge $c = \frac{63}{5}$ and NS weights $h_{\text{NS}} = \left(\frac{4}{5}, \frac{9}{10}\right)$. The weights and degeneracy reminds us of $SM(5,3)^2$ with $c = \frac{7}{5}$ and NS weights $h_{\text{NS}} = (\frac{1}{10}, \frac{1}{5})$, which was briefly mentioned in section 4.2. One can easily check a $T^F_9$ (if it exists) on $SM(5,3)^2$ would produce the correct degeneracy and multiple 9 condition for the central charge and NS weights. Unfortunately, 9 is not coprime to the conductor 120 of $SM(5,3)^2$. It would be interesting to determine whether there exists a middle fermionic theory serving as the generalized $T^F_3$ image with $c = \frac{21}{5}$ and conductor $N = 40$. Such theory should be non-unitary according to the classification of rank-8 SMC, see e.g. [9].

The fermionic Hecke operator also helps us to rule out some previously undetermined theory. For example, by studying the third order fermionic MLDEs, [2] bootstrapped the three NS characters of a potential $c = \frac{66}{5}$ theory. As we discussed in section 5.5, we notice this theory can be realized a $T^F_{11}$ image of $SM_{\text{sub}}(20,2)$ and share the same $S$-matrix and fusion rules as $SM_{\text{sub}}(20,2)$. Therefore, it is non-unitary and can not be the fermionization of any WZW models.

The relation between fermionic Hecke operator and super modular category should be further clarified. It should be possible to generalize the results on Galois symmetry in [23] to the fermionic cases. It is also intriguing to consider whether fermionic Hecke operation can produce the fermionic characters of the new putative objects in the rank-10 SMC recently proposed in [9]. We leave these for future work.

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