CHARACTERIZING ROBUST WEAK SHARP SOLUTION SETS OF CONVEX OPTIMIZATION PROBLEMS WITH UNCERTAINTY

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Abstract. We introduce robust weak sharp and robust sharp solution to a convex programming with the objective and constraint functions involved uncertainty. The characterizations of the sets of all the robust weak sharp solutions are obtained by means of subdifferentials of convex functions, DC functions, Fermat rule and the robust-type subdifferential constraint qualification, which was introduced in X.K. Sun, Z.Y. Peng and X. Le Guo, Some characterizations of robust optimal solutions for uncertain convex optimization problems, Optim Lett. 10. (2016), 1463-1478. In addition, some applications to the multi-objective case are presented.

1. Introduction. The notion of a weak sharp minimizer in general mathematical programming problems was first introduced in [15]. It is an extension of a sharp minimizer (or equivalently, strongly unique minimizer) in [22] to include the possibility of non-unique solution set. It has been acknowledged that the weak sharp minimizer plays important roles in stability/sensitivity analysis and convergence analysis of a wide range of numerical algorithms in mathematical programming (see [6, 7, 19, 21, 8, 9] and references therein).

In the context of optimization, much attention has been paid to concerning sufficient and/or necessary conditions for weak sharp minimizers/solutions and characterizing weak sharp solution sets (of such weak sharp minimizers) in various types of problems. Particularly, the study of characterizations of the weak sharp solution sets covers both single-objective and multi-objective optimization problems (see [10, 11, 12, 30] and references therein) and, recently, is extended to mathematical programs with inequality constraints and semi-infinite programs (see, e.g.,

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As it might be seen, the study of characterizations of the weak sharp solution sets has been popular in many optimization problems. How about the issue of this study, particularly, in a robust optimization?

Robust (convex) optimization has been known as an important class of convex optimization deals with uncertainty in the data of the problems [2, 3]. The goal of robust optimization is to immunize an optimization problem against uncertain parameters in the problem. In the last two decades, it has been through a rapid development owing to the practical requirement and its effective implementation in real-world applications of optimization (see, e.g., [17, 23, 18, 20] and the references therein). A successful treatment of the robust optimization approaches to convex optimization problems under data uncertainty was given in ([2, 3, 4, 5, 24]).

While the characterizations of optimal solution sets have been in the limelight presently, there has been no research concerning the characterizations robust weak sharp solution sets for such problems. Indeed, a robust weak sharp solution of an uncertain optimization problem is the weak sharp minimizer of the robust counterpart of such problem. Our main goal in this paper is to establish characterizations of the robust weak sharp solution set of the convex optimization problem under the data uncertainty.

This paper is organized as follows. In section 2, we recall the basic definitions. In Section 3, we establish necessary conditions for a robust weak sharp solution, constancy of Lagrangian-type function on the robust weak sharp solution set, and some characterizations of robust weak sharp solution sets are established respectively. Some properties of subdifferentials of convex functions and the (RSCQ), which was introduced in [24], are employed in the section. Finally, in section 4, we consider the characterizations of the robust weak sharp weakly efficient solutions for the multi-objective optimization problem under data uncertainty.

2. Preliminary. Throughout the paper, let \( \mathbb{R}^n \), \( n \in \mathbb{N} \), be the \( n \)-dimensional Euclidean space, and the inner product and the norm of \( \mathbb{R}^n \) are denoted respectively by \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \). The symbol \( B(x, r) \) stands for the open ball centered at \( x \in \mathbb{R}^n \) with the radius \( r > 0 \) while the \( \mathcal{B}_{\mathbb{R}^n} \) stands for the closed unit ball in \( \mathbb{R}^n \). For a nonempty subset \( A \subseteq \mathbb{R}^n \), we denote the notations of the closure, boundary and convex hull of \( A \) by \( \text{cl} A \), \( \text{bd} A \), and \( \text{co} A \), respectively. In particular, when \( \lambda x \in E \subseteq \mathbb{R}^n \) for every \( \lambda \geq 0 \) and every \( x \in E \), the set \( E \) in \( \mathbb{R}^n \) is said to be a cone. A dual cone \( E^* \) of the cone \( E \) is given as \( E^* := \{ x \in \mathbb{R}^n : \langle x, y \rangle \geq 0 \text{ for all } y \in E \} \). Observe that the dual cone \( E^* \) is always closed and convex (regardless of \( E \)).

In general, for a given nonempty set \( A \subseteq \mathbb{R}^n \), the indicator function \( \delta_A : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) of \( A \) and the support function \( \sigma_A : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) of \( A \) are, respectively, defined by

\[
\delta_A(x) = \begin{cases} 
0, & \text{if } x \in A; \\
+\infty, & \text{otherwise},
\end{cases}
\]

and

\[
\sigma_A(x^*) := \sup_{x \in A} \langle x^*, x \rangle.
\]

The distance function \( d_A : \mathbb{R}^n \to \mathbb{R}_+: [0, +\infty) \) is defined by

\[
d_A(x) := \inf_{y \in A} \| x - y \|.
\]
A normal cone of the set \( A \) at the point \( x \) is the following set:

\[
N_A(x) = \begin{cases} 
\{ y \in \mathbb{R}^n : \langle y, a-x \rangle \leq 0 \text{ for all } a \in A \}, & \text{if } x \in A; \\
\emptyset, & \text{otherwise}.
\end{cases}
\]

The normal cone \( N_A(x) \) is always closed and convex for any set \( A \).

For any extended real-valued function \( h : \mathbb{R}^n \to \overline{\mathbb{R}} := [-\infty, +\infty] \) the following notations stand, respectively, for its effective domain and epigraph:

\[
\text{dom} h := \{ x \in \mathbb{R}^n : h(x) < +\infty \},
\]

and

\[
\text{epi} h := \{ (x, r) \in \mathbb{R}^n \times \mathbb{R} : h(x) \leq r \}.
\]

The function \( h \) is said to be a proper function if and only if \( h(x) > -\infty \) for every \( x \in \mathbb{R}^n \) and \( \text{dom} h \) is nonempty. Further, it is said to be a convex function if for any \( x, y \in \mathbb{R}^n \) and \( \lambda \in [0, 1] \),

\[
h(\lambda x + (1-\lambda)y) \leq \lambda h(x) + (1-\lambda)h(y),
\]

or equivalently, \( \text{epi} h \) is convex. On the other hand, the function \( h \) is said to be a concave function if and only if \(-h \) is a convex function. In the case of vector valued function, let \( h : \mathbb{R}^n \to \mathbb{R}^p \) be a given function and \( D \subseteq \mathbb{R}^p \) is a convex set. The function \( \tilde{h} \) is said to be \( D \)-convex if and only if for any \( x, y \in \mathbb{R}^n \) and \( \lambda \in [0, 1] \),

\[
\tilde{h}(\lambda x + (1-\lambda)y) - \lambda \tilde{h}(x) - (1-\lambda)\tilde{h}(y) \in -D.
\]

Simultaneously, the function \( h \) is called a lower semicontinuous at \( x \in \mathbb{R}^n \) if for every sequence \( \{x_k\} \subseteq \mathbb{R}^n \) converging to \( x \),

\[
h(x) \leq \liminf_{k \to \infty} h(x_k).
\]

Equivalently,

\[
h(x) \leq \liminf_{y \to x} h(y),
\]

where the term on the right-hand side of the inequality denotes the lower limit of the function \( h \) defined as

\[
\liminf_{y \to x} h(y) = \lim_{r \downarrow 0} \sup_{y \in B(x, r)} \inf h(y).
\]

For any proper and convex function \( h : \mathbb{R}^n \to \mathbb{R} \), the subdifferential of \( h \) at \( \hat{x} \in \text{dom} h \), is defined by

\[
\partial h(\hat{x}) := \{ \xi \in \mathbb{R}^n : \langle \xi, x-\hat{x} \rangle \leq h(x) - h(\hat{x}), \forall x \in \mathbb{R}^n \}.
\]

More specifically, for each \( \varepsilon \geq 0 \), the \( \varepsilon \)-subdifferential of \( h \) at \( \hat{x} \in \text{dom} h \), is defined by

\[
\partial_{\varepsilon} h(\hat{x}) := \{ \xi \in \mathbb{R}^n : \langle \xi, x-\hat{x} \rangle \leq h(x) - h(\hat{x}) + \varepsilon, \forall x \in \mathbb{R}^n \}.
\]

It is obvious that for \( \varepsilon \geq \varepsilon' \), we have \( \partial_{\varepsilon} h(\hat{x}) \subseteq \partial_{\varepsilon'} h(\hat{x}) \). In particular, if \( h \) is a proper lower semicontinuous convex function, then for every \( \hat{x} \in \text{dom} h \), the \( \varepsilon \)-subdifferential \( \partial_{\varepsilon} h(\hat{x}) \) is a nonempty closed convex set and

\[
\partial h(\hat{x}) = \bigcap_{\varepsilon > 0} \partial_{\varepsilon} h(\hat{x}).
\]

If \( x \notin \text{dom} h \), then we set \( \partial h(x) = \emptyset \). Simultaneously, for the nonempty subset \( A \) of \( \mathbb{R}^n \), we get \( \partial \delta_A(x) = N_A(x) \) and \( \partial \delta_A(x) = \mathcal{B}_{\mathbb{R}^n} \cap N_A(x) \).
The conjugate function $h^* : \mathbb{R}^n \rightarrow \mathbb{R}$ of any $h : \mathbb{R}^n \rightarrow \mathbb{R}$ is defined by

$$
h^*(x^*) := \sup \{ \langle x^*, x \rangle - h(x) \}
$$

for all $x \in \mathbb{R}^n$. The function $h^*$ is lower semicontinuous convex irrespective of the nature of $h$ but for $h^*$ to be proper, we need $h$ to be a proper convex function.

Next, let us recall some basic concepts dealing a DC problem/programming. A DC function is the difference of two convex functions. The minimization (or maximization) problem of a DC function is called a DC problem, i.e., the DC problem concerned about finding

$$
\inf_{x \in \mathbb{R}^n} h(x) := f(x) - \phi(x)
$$

where $f, \phi : \mathbb{R}^n \rightarrow \mathbb{R}$ are convex. Note that the function $h$ is DC and it is not expected to be convex.

It shall be found later that some DC problems are considered and their properties, in particular the following lemma, are employed.

**Lemma 2.1.** [16] Let $h_1, h_2 : \mathbb{R}^n \rightarrow \mathbb{R}$ be two proper lower semicontinuous convex functions. Then

(i) A point $\hat{x} \in \text{dom } h_1 \cap \text{dom } h_2$ is a (global) minimizer of the DC problem:

$$
\inf_{x \in \mathbb{R}^n} \{h_1(x) - h_2(x)\}
$$

if and only if for any $\varepsilon \geq 0$, $\partial \varepsilon h_2(\hat{x}) \subseteq \partial \varepsilon h_1(\hat{x})$.

(ii) If $\hat{x} \in \text{dom } h_1 \cap \text{dom } h_2$ is a local minimizer of the DC problem:

$$
\inf_{x \in \mathbb{R}^n} \{h_1(x) - h_2(x)\}
$$

then $\partial h_2(\hat{x}) \subseteq \partial h_1(\hat{x})$.

**Lemma 2.2.** [18] Let $\mathcal{U} \subseteq \mathbb{R}^p$ be a convex compact set, and $f : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}$ be a function such that, $f(\cdot, u)$ is a convex function for any $u \in \mathcal{U}$, and $f(x, \cdot)$ is a concave function for any $x \in \mathbb{R}^n$. Then,

$$
\partial \left( \max_{u \in \mathcal{U}} f(\cdot, u) \right)(\hat{x}) = \bigcup_{u \in \mathcal{U}(\hat{x})} \partial f(\cdot, u)(\hat{x}),
$$

where

$$
\mathcal{U}(\hat{x}) := \left\{ \hat{u} \in \mathcal{U} : f(\hat{x}, \hat{u}) = \max_{u \in \mathcal{U}} f(\hat{x}, u) \right\}.
$$

Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set. Let $D \subseteq \mathbb{R}^p$ be a nonempty closed convex cone. Consider the following convex optimization problem:

$$
\min f(x) \text{ s.t. } x \in C, \ g(x) \in -D \quad \text{(P)}
$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a convex function and $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a $D$-convex function. The feasible set of (P) is defined by

$$
K_0 := \{ x \in C : g(x) \in -D \}.
$$

The problem (P) in the face of data uncertainty both in the objective and constraints can be captured by the following uncertain optimization problem:

$$
\min \{ f(x, u) : x \in C, \ g(x, v) \in -D \} \quad \text{(UP)}
$$
where \( U \subseteq \mathbb{R}^p \) and \( V \subseteq \mathbb{R}^q \) are convex and compact uncertainty sets, \( f : \mathbb{R}^n \times U \to \mathbb{R} \) is a given real-valued function such that, for any uncertain parameter \( u \in U \), \( f(\cdot, u) \) is convex as well as \( f(x, \cdot) \) is concave for any \( x \in \mathbb{R}^n \); \( g : \mathbb{R}^n \times V \to \mathbb{R}^m \) is a vector-valued function such that, for any uncertain parameter \( v \in V \), \( g(\cdot, v) \) is \( D \)-convex as well as \( g(x, \cdot) \) is \( D \)-concave for any \( x \in \mathbb{R}^n \). The uncertain sets can be apprehended in the sense that the parameter vectors \( u \) and \( v \) are not known exactly at the time of the decision.

For examining the uncertain optimization problem \((UP)\), one usually associates with its robust (worst-case) counterpart, which is the following problem:

\[
\min \left\{ \max_{u \in U} f(x, u) : x \in C, g(x, v) \in -D, \forall v \in V \right\}. 
\]

(RUP)

It is worth observing here that the robust counterpart, which is termed as the robust optimization problem, finds a worst-case possible solution that can be immunized against data uncertainty.

The problem \((RUP)\) is said to be feasible if the robust feasible set \( K \) is nonempty where it is denoted by

\[
K := \{ x \in C : g(x, v) \in -D, \forall v \in V \}. 
\]

(1)

Now, we recall the following concept of solutions, which was introduced in [1].

**Definition 2.3.** [1] A point \( \hat{x} \in K \) is said to be a robust optimal solution for \((UP)\) if it is an optimal solution for \((RUP)\), i.e., for all \( x \in K \),

\[
\max_{u \in U} f(x, u) - \max_{u \in U} f(\hat{x}, u) \geq 0. 
\]

The robust optimal solution set of \((UP)\) is the set which consists of all robust optimal solutions of \((UP)\) and is given by

\[
S := \left\{ x \in K : \max_{u \in U} f(x, u) = \max_{u \in U} f(\hat{x}, u) \right\}. 
\]

In this paper, using the idea of weak sharp minimizer, and the robust optimal solution, we introduce a new concept of solutions for \((UP)\), which related to the sharpness, namely the robust weak sharp solution.

**Definition 2.4.** A point \( \hat{x} \in K \) is said to be a (or an optimal) weak sharp solution for \((RUP)\) if there exist a real number \( \eta > 0 \) such that for all \( x \in K \),

\[
\max_{u \in U} f(x, u) - \max_{u \in U} f(\hat{x}, u) \geq \eta d_{\tilde{K}}(x) 
\]

where \( \tilde{K} := \left\{ x \in K : \max_{u \in U} f(x, u) = \max_{u \in U} f(\hat{x}, u) \right\} \).

**Definition 2.5.** A point \( \hat{x} \in K \) is said to be a (or an optimal) robust weak sharp solution for \((UP)\) if it is a weak sharp solution for \((RUP)\). The robust weak sharp solution set of \((UP)\) is given by

\[
\tilde{S} := \left\{ \hat{x} \in K : \exists \eta > 0 \text{ s.t. } \max_{u \in U} f(y, u) - \max_{u \in U} f(\hat{x}, u) \geq \eta d_{\tilde{K}}(y), \forall y \in K \right\}. 
\]

Throughout the paper, we assume that \( \tilde{S} \) is nonempty.

**Remark 1.** It is worthwhile to be noted that every robust weak sharp solution for \((UP)\) is a robust optimal solution. In general, the reverse implication need not to be valid.
3. Characterizations of robust weak sharp solutions. In this section, we establish some optimality conditions for the robust weak sharp solution in convex uncertain optimization problems as well as obtain characterizations of the robust weak sharp solution sets for the considered problems. For any \( \hat{x} \in \mathbb{R}^n \), we use the following notations:

\[
U(\hat{x}) := \left\{ \hat{u} \in U : f(\hat{x}, \hat{u}) = \max_{u \in U} f(\hat{x}, u) \right\},
\]

and

\[
V(\hat{x}) := \left\{ \hat{v} \in V : g(\hat{x}, \hat{v}) = \max_{v \in V} g(\hat{x}, v) \right\}.
\]

The following definition, which was introduced in [24], plays a vital role in determining characterizations of robust optimal weak sharp solution sets.

**Definition 3.1.** [24] The robust type subdifferential constraint qualification (RSCQ) is said to be satisfied at \( \hat{x} \in K \) if

\[
\partial \delta_K(\hat{x}) \subseteq \partial \delta_C(\hat{x}) + \bigcup_{\mu \in D^*, v \in V} \partial (\mu g)(\cdot, v)(\hat{x}).
\]

**Remark 2.** In an excellent work, [24], Sun et. al. introduced the (RSCQ) and then obtained some characterizations of the the robust optimal solution set, for an uncertain convex optimization problem.

Although it has been used as a guideline for dealing with the (UP), our attention is paid to characterizing the sets containing the robust weak sharp solutions of such problem. Furthermore, the presence of the term \( d_{\tilde{K}}(\hat{x}) \) in this paper has led us to deal with some different tools and methods from those in work of Sun et.al.

The following theorem presents that the robust type subdifferential constraint qualification (RSCQ) defined in Definition 3.1 is fulfilled if and only if optimality conditions for a robust weak sharp solution of (UP) are satisfied.

**Theorem 3.2.** Let \( f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R} \) and \( g : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^m \) satisfy the following properties:

(i) for any \( u \in U \) and \( v \in V \), \( f(\cdot, u) \) is convex and continuous as well as \( g(\cdot, v) \) is \( D \)-convex on \( \mathbb{R}^n \);

(ii) for any \( x \in \mathbb{R}^n \), \( f(x, \cdot) \) is concave on \( U \) and \( g(x, \cdot) \) is \( D \)-concave on \( V \).

Then, the following statements are equivalent:

(a) The (RSCQ) is fulfilled at \( \hat{x} \in K \);

(b) \( \hat{x} \in \mathbb{R}^n \) is a robust weak sharp solution of (UP) if and only if there exists a positive constant \( \eta \) such that

\[
N_{\tilde{K}}(\hat{x}) \cap \eta B_{\mathbb{R}^n} \subseteq \bigcup_{u \in U(\hat{x})} \partial f(\cdot, u)(\hat{x}) + \partial \delta_C(\hat{x}) + \bigcup_{\mu \in D^*, v \in V} \partial (\mu g)(\cdot, v)(\hat{x}).
\]

**Proof.** (a) ⇒ (b) Assume that the (RSCQ) is satisfied at \( \hat{x} \in K \). Let \( \hat{x} \) be a robust weak sharp solution of (UP). Consequently, there exists \( \eta > 0 \) such that

\[
\max_{u \in U} f(x, u) - \max_{u \in U} f(\hat{x}, u) \geq \eta d_{\tilde{K}}(x).
\]
By (3), we obtain that for all $x \in K$,
\[
\max_{u \in U} f(x, u) + \delta_K(x) - \eta d_{\tilde{K}}(x) \geq \max_{u \in U} f(\hat{x}, u) = \max_{u \in U} f(\hat{x}, u) + \delta_K(\hat{x}) - \eta d_{\tilde{K}}(\hat{x}),
\]
thereby implying that, for all $\xi_d \in \partial \eta d_{\tilde{K}}(x)$,
\[
\left( \max_{u \in U} f(\cdot, u) + \delta_K \right)(x) - \left( \max_{u \in U} f(\cdot, u) + \delta_K \right)(\hat{x}) \geq \eta d_{\tilde{K}}(x) - \eta d_{\tilde{K}}(\hat{x}) \geq \langle \xi_d, x - \hat{x} \rangle.
\]
Thus, $\xi_d \in \partial (\max_{u \in U} f(\cdot, u) + \delta_K)(\hat{x})$. Hence,
\[
\partial (\eta d_{\tilde{K}})(\hat{x}) \subseteq \partial \left( \max_{u \in U} f(\cdot, u) + \delta_K \right)(\hat{x}).
\]
As $\max f(\cdot, u)$ is continuous on $\mathbb{R}^n$ and $\delta_K$ is proper lower semicontinuous convex on $\mathbb{R}^n$, we have
\[
\partial (\eta d_{\tilde{K}})(\hat{x}) \subseteq \partial (\max_{u \in U} f(\cdot, u))(\hat{x}) + \partial \delta_K(\hat{x}).
\]
It can be noted that $\partial d_{\tilde{K}}(x) = N_{\tilde{K}}(x) \cap \mathbb{B}_{\mathbb{R}^n}$. Since (RSCQ) is satisfied at $\hat{x}$, we have the following:
\[
\mathbb{N}_{\tilde{K}}(x) \cap \mathbb{B}_{\mathbb{R}^n} = \partial (\eta d_{\tilde{K}})(\hat{x}) = \bigcup_{\mu \in \mathcal{D}^*, \nu \in \mathcal{V}} \partial f(\cdot, u)(\hat{x}) + \partial \delta_C(\hat{x}) + \bigcup_{\mu \in \mathcal{D}^*, \nu \in \mathcal{V}} \partial (\mu g)(\cdot, v)(\hat{x}),
\]
which implies that (2) holds.

Conversely, assume that there is a positive number $\eta$ such that (2) holds. Since $\mathbb{N}_{\tilde{K}}(\hat{x}) \cap \eta \mathbb{B}_{\mathbb{R}^n}$ always contains 0, it is a nonempty set and so is $\bigcup_{\varepsilon > 0} \partial_\varepsilon (\eta d_{\tilde{K}})(\hat{x})$. Thus, for any $\varepsilon \geq 0$, $\partial_\varepsilon (\eta d_{\tilde{K}})(\hat{x}) \neq \emptyset$. Let $\varepsilon > 0$ be arbitrary and let $\xi \in \partial_\varepsilon (\eta d_{\tilde{K}})(\hat{x})$. Then for any $x \in K$,
\[
\eta d_{\tilde{K}}(x) - \eta d_{\tilde{K}}(\hat{x}) \geq \langle \xi, x - \hat{x} \rangle - \varepsilon. \tag{4}
\]
Note that $0 \in \partial_\varepsilon (\eta d_{\tilde{K}})(\hat{x})$. It follows that
\[
\eta d_{\tilde{K}}(\hat{x}) \leq \inf_{x \in \mathbb{R}^n} \eta d_{\tilde{K}}(x) + \varepsilon \leq \inf_{x \in K} \eta d_{\tilde{K}}(x) + \varepsilon.
\]
Above inequality and (4) imply that
\[
0 \geq \langle \xi, x - \hat{x} \rangle - \varepsilon. \tag{5}
\]
Simultaneously, there exist $\hat{u} \in U(\hat{x})$, $\hat{\mu} \in \mathbb{D}^*$, $\hat{\nu} \in V(\hat{x})$ \(\xi_f \in \mathcal{D}^*(\hat{u}) \cap V(\hat{x})\), $\xi_\delta \in \partial \delta_C(\hat{x})$, and $\hat{\xi}_{\mu g} \in \partial ((\hat{\mu} g)(\cdot, \hat{\nu})) (\hat{x})$ such that
\[
\xi_f + \xi_\delta + \hat{\xi}_{\mu g} = 0, \tag{6}
\]
and for any $x \in \mathbb{R}^n$, we have
\[
f(x, \hat{u}) - f(\hat{x}, \hat{u}) \geq \langle \xi_f, x - \hat{x} \rangle, \quad \delta_C(x) - \delta_C(\hat{x}) \geq \langle \xi_\delta, x - \hat{x} \rangle, \quad \text{and} \quad \hat{\mu g}(x, \hat{\nu}) - \hat{\mu g}(\hat{x}, \hat{\nu}) \geq \langle \hat{\xi}_{\mu g}, x - \hat{x} \rangle.
\]
Adding these above inequalities implies that for each \( x \in K \)
\[
f(x, \hat{u}) - f(\hat{x}, \hat{u}) + (\hat{\mu}g)(x, \hat{v}) - (\hat{\mu}g)(\hat{x}, \hat{v}) \geq (0, x - \hat{x}) = 0.
\]

Since \( \hat{u} \) belongs to \( \mathcal{U}(\hat{x}) \), for each \( x \in K \), above inequality becomes
\[
\max_{u \in \mathcal{U}} f(x, u) - \max_{u \in \mathcal{U}} f(\hat{x}, u) + (\hat{\mu}g)(x, \hat{v}) - (\hat{\mu}g)(\hat{x}, \hat{v}) \geq 0.
\]

This along with \( (\hat{\mu}g)(x, \hat{v}) \leq 0, (\hat{\mu}g)(\hat{x}, \hat{v}) = 0 \), and (6) imply
\[
\max_{u \in \mathcal{U}} f(x, u) - \max_{u \in \mathcal{U}} f(\hat{x}, u) \geq 0,
\]
for all \( x \in K \). Observe that, combining inequalities (5) and (7) leads to
\[
\max_{u \in \mathcal{U}} f(x, u) - \max_{u \in \mathcal{U}} f(\hat{x}, u) \geq \langle \xi, x - \hat{x} \rangle - \varepsilon, \forall x \in K.
\]

This means \( \xi \in \partial_\varepsilon (\max_{u \in \mathcal{U}} f(\cdot, u))(\hat{x}) \), and so \( \partial_\varepsilon (\eta d_{\tilde{R}})(\hat{x}) \subseteq \partial_\varepsilon (\max_{u \in \mathcal{U}} f(\cdot, u))(\hat{x}) \). Since the inclusion holds for arbitrary \( \varepsilon \geq 0 \), it follows from the Lemma 2.1 that \( \hat{x} \) is a minimizer of the DC problem: \( \inf_{x \in \mathbb{R}^n} \{ \max_{u \in \mathcal{U}} f(x, u) - \eta d_{\tilde{R}}(x) \} \) and hence for any \( x \in K \)
\[
\max_{u \in \mathcal{U}} f(x, u) - \max_{u \in \mathcal{U}} f(\hat{x}, u) - (\eta d_{\tilde{R}}(x) - \eta d_{\tilde{R}}(\hat{x})) \geq 0.
\]

Therefore, for any \( x \in K \),
\[
\max_{u \in \mathcal{U}} f(x, u) - \max_{u \in \mathcal{U}} f(\hat{x}, u) \geq \eta d_{\tilde{R}}(x).
\]

This means \( \hat{x} \) is a robust weak sharp solution of (UP).

(b) \(\Rightarrow\) (a) Let \( \xi_\delta \in \partial \delta_K(\hat{x}) \) be given. Then, we have
\[
0 = \delta_K(x) - \delta_K(\hat{x}) \geq \langle \xi_\delta, x - \hat{x} \rangle
\]
holds for all \( x \in K \). Let \( \tilde{\eta} > 0 \) be given, and then, set \( f(x, u) := -\langle \xi_\delta, x \rangle + \tilde{\eta} d_{\tilde{R}}(x) \). Thus, for any \( x \in K \),
\[
\max_{u \in \mathcal{U}} f(x, u) - \tilde{\eta} d_{\tilde{R}}(x) = -\langle \xi_\delta, x \rangle
\]
\[
\geq -\langle \xi_\delta, \hat{x} \rangle + \tilde{\eta} d_{\tilde{R}}(\hat{x})
\]
\[
= \max_{u \in \mathcal{U}} f(\hat{x}, u).
\]

Thus, \( \hat{x} \) is a robust weak sharp solution of (UP). By hypothesis, there is \( \eta := \tilde{\eta} \) such that (2) is fulfilled. Since for any \( u \in \mathcal{U}, \partial f(\cdot, u)(\hat{x}) \subseteq \{-\xi_\delta\} + \partial (\eta d_{\tilde{R}})(\hat{x}) \), we obtain that for any \( x^* \in N_{\tilde{R}}(\hat{x}) \cap \eta B_{\mathbb{R}^n} \), there exist \( \hat{u} \in \mathcal{U}(\hat{x}), \hat{v} \in \mathbb{V} \) and \( \hat{\mu} \in D^* \) such that
\[
x^* \in \{-\xi_\delta\} + \partial (\eta d_{\tilde{R}})(\hat{x}) + \partial \delta_C(\hat{x}) + \partial ((\hat{\mu}g)(\cdot, \hat{v}))(\hat{x}) \text{ and } (\hat{\mu}g)(\hat{x}, \hat{v}) = 0.
\]
As \( 0 \in N_{\tilde{R}}(\hat{x}) \cap \eta B_{\mathbb{R}^n} \), we obtain
\[
\xi_\delta \in \partial \delta_C(\hat{x}) + \partial ((\hat{\mu}g)(\cdot, \hat{v}))(\hat{x}) \text{ and } (\hat{\mu}g)(\hat{x}, \hat{v}) = 0.
\]

It follows that
\[
\xi_\delta \in \partial \delta_C(\hat{x}) + \bigcup_{\mu \in D^*, v \in \mathbb{V}} \partial ((\mu g)(\cdot, v))(\hat{x}),
\]
and so we get the desired inclusion. Therefore, the proof is complete. \( \square \)
Remark 3. In [26], the necessary conditions for weak sharp minima in cone constrained optimization problems, which can be captured by weak sharp minima in cone constrained robust optimization problems, were established by means of upper Studniarski or Dini directional derivatives. With the result in Theorem 3.2, the mentioned necessary conditions are established by an alternative method different from the referred work.

The following result is established easily by means of the basic concepts of variational analysis.

**Corollary 1.** Let \( f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R} \) and \( g : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^p \) satisfying the following properties:
1. for any \( u \in \mathcal{U} \), and \( v \in \mathcal{V} \), \( f(\cdot, u) \) is convex and continuous as well as \( g(\cdot, v) \) is \( D \)-convex on \( \mathbb{R}^n \);
2. for any \( x \in \mathbb{R}^n \), \( f(x, \cdot) \) is concave on \( \mathcal{U} \) and \( g(x, \cdot) \) is \( D \)-concave on \( \mathcal{V} \), respectively.

The following two below statements are equivalent:
(a) The (RSCQ) is fulfilled at \( \hat{x} \in K \);
(b) \( \hat{x} \in \mathbb{R}^n \) is a robust weak sharp solution of \( (UP) \) if and only if there exists a real number \( \eta > 0 \) such that for any \( \hat{x} \) such that for any \( x^* \in N_-(\hat{x}) \cap \eta \mathcal{B}_{\mathbb{R}^n} \), there exist \( \hat{u} \in \mathcal{U}(\hat{x}), \hat{v} \in \mathcal{V} \) and \( \hat{\mu} \in D^\ast \) yield
\[
x^* \in \partial f(\cdot, \hat{u}) + \partial \delta_C(\hat{x}) + \partial ((\hat{\mu}g)(\cdot, \hat{v})) (\hat{x}), \text{ and } (\hat{\mu}g)(\hat{x}, \hat{v}) = 0.
\]
(8)

The result, which deals with a special case that \( \mathcal{U} \) and \( \mathcal{V} \) are singleton sets, can be obtained easily and be presented as follows:

**Corollary 2.** Let \( f : \mathbb{R}^n \to \mathbb{R} \) is convex and continuous and \( g : \mathbb{R}^n \to \mathbb{R}^m \) is \( D \)-convex. The following statements are equivalent:
1. The (SCQ) is fulfilled at \( \hat{x} \in K \);
2. \( \hat{x} \in \mathbb{R}^n \) is a weak sharp solution of \( (P) \) if and only if there exists a real number \( \eta > 0 \) such that for any \( x^* \in N_-(\hat{x}) \cap \eta \mathcal{B}_{\mathbb{R}^n} \), there exist \( \hat{\mu} \in D^\ast \) such that
\[
x^* \in \partial f(\hat{x}) + \partial \delta_C(\hat{x}) + \partial (\hat{\mu}g)(\hat{x}) \text{ and } (\hat{\mu}g)(\hat{x}) = 0.
\]
(9)

Next, a characterization of robust weak sharp solution sets in terms of a given robust weak sharp solution point of our considered problem is also illustrated in this section. In order to present the mentioned characterization, we first prove that the Lagrangian-type function associated with fixed Lagrange multiplier and uncertainty parameters corresponding to a robust weak sharp solution is constant on the robust weak sharp solution set under suitable conditions. In what follows, let \( u \in \mathcal{U}, v \in \mathcal{V} \) and \( \mu \in D^\ast \). The Lagrangian-type function \( \mathcal{L}(\cdot, \mu, u, v) \) is given by
\[
\mathcal{L}(x, \mu, u, v) = f(x, u) + (\mu g)(x, v), \forall x \in \mathbb{R}^n.
\]
Now, we denote by
\[
\bar{S} := \left\{ x \in K : \exists \eta > 0 \text{ s.t. } \max_{u \in \mathcal{U}} f(y, u) \geq \max_{u \in \mathcal{U}} f(x, u) + \eta d_K(y), \forall y \in K \right\}.
\]
the robust weak sharp solution set of \( (UP) \), and then we prove that the Lagrangian-type function associated with a Lagrange multiplier corresponding to a robust weak sharp solution is constant on the robust weak sharp solution set.
Theorem 3.3. Let \( \hat{x} \in \tilde{S} \) be given. Suppose that the (RSCQ) is satisfied at \( \hat{x} \). Then, there exist uncertainty parameters \( \hat{u} \in \mathcal{U}, \hat{v} \in \mathcal{V} \), and Lagrange multiplier \( \hat{\mu} \in D^* \), such that for any \( x \in \tilde{S} \),

\[
(\hat{\mu}g)(x, \hat{v}) = 0, \quad \hat{u} \in \mathcal{U}(x), \quad \text{and} \quad \mathcal{L}(x, \hat{\mu}, \hat{u}, \hat{v}) \text{ is a constant on } \tilde{S}.
\]

Proof. Since \( \hat{x} \in \tilde{S} \) with the real number \( \eta_1 > 0 \) and the (RSCQ) is satisfied at this point \( \hat{x} \), by Theorem 3.2 we have that (2) holds for \( \eta := \eta_1 \). Clearly \( N_{\tilde{K}}(\hat{x}) \cap \eta B_{\mathbb{R}^n} \) contains 0, then it is nonempty and so is any \( \partial_x(\eta d_{\tilde{K}})(\hat{x}) \) where \( \varepsilon > 0 \). Let \( \varepsilon > 0 \) and \( x^* \in \partial_x(\eta d_{\tilde{K}})(\hat{x}) \) be arbitrary. Again, we obtain that there exist \( \hat{u} \in \mathcal{U}, \hat{v} \in \mathcal{V} \) and \( \hat{\mu} \in D^* \) such that (2) is fulfilled. Let \( x \in \tilde{S} \) be arbitrary, then we have

\[
f(x, \hat{u}) - f(\hat{x}, \hat{u}) + (\hat{\mu}g)(x, \hat{v}) - (\hat{\mu}g)(\hat{x}, \hat{v}) \geq (x^*, x - \hat{x}),
\]

and so

\[
f(x, \hat{u}) - f(\hat{x}, \hat{u}) + (\hat{\mu}g)(x, \hat{v}) - (\hat{\mu}g)(\hat{x}, \hat{v}) \geq (x^*, x - \hat{x}) - \varepsilon. \tag{10}
\]

Since \( f(\cdot, u) \) and \( g(\cdot, v) \) are convex, for all \( u \in \mathcal{U} \) and \( v \in \mathcal{V} \) respectively,

\[
x^* \in \partial_x(f(\cdot, u) + \lambda g(\cdot, v))(\hat{x}).
\]

Therefore, we obtain \( \partial_x(\eta d_{\tilde{K}})(\hat{x}) \subseteq \partial_x(f(\cdot, u) + \lambda g(\cdot, v))(\hat{x}) \), and so

\[
f(x, \hat{u}) + (\hat{\mu}g)(x, \hat{v}) - \eta d_{\tilde{K}}(x) \geq f(\hat{x}, \hat{u}) = \max_{u \in \mathcal{U}} f(\hat{x}, u). \tag{11}
\]

Note that, as \( x \in \tilde{S} \), there exists \( \eta_2 > 0 \) such that

\[
\max_{u \in \mathcal{U}} f(y, u) \geq \max_{u \in \mathcal{U}} f(x, u) + \eta_2 d_{\tilde{K}}(y), \quad \forall y \in \tilde{S},
\]

and so

\[
\max_{u \in \mathcal{U}} f(\hat{x}, u) \geq \max_{u \in \mathcal{U}} f(x, u) + \eta_2 d_{\tilde{K}}(\hat{x}) = \max_{u \in \mathcal{U}} f(x, u). \tag{12}
\]

From \( \hat{\mu} \in D^*, g(x, \hat{v}) \in -D \), and (11), it is not hard to see that

\[
(\hat{\mu}g)(x, \hat{v}) = 0. \tag{13}
\]

Then, by (11) and the positivity of \( \eta d_{\tilde{K}}(x) \), we see that

\[
\max_{u \in \mathcal{U}} f(x, u) \geq f(x, \hat{u}) \geq \max_{u \in \mathcal{U}} f(\hat{x}, u) + \eta d_{\tilde{K}}(x) \geq \max_{u \in \mathcal{U}} f(\hat{x}, u), \tag{14}
\]

which together with (12) leads to

\[
\max_{u \in \mathcal{U}} f(x, u) = f(x, \hat{u}). \tag{15}
\]

It follows that \( \mathcal{L}(x, \hat{\mu}, \hat{u}, \hat{v}) = f(\hat{x}, \hat{u}) \), which is constant. Since \( x \in \tilde{S} \) was arbitrary, we finish the proof. \( \square \)

Theorem 3.4. For the problem (UP), let \( \tilde{S} \) be the robust weak sharp solutions set of (UP) and \( \hat{x} \) belongs to it. Suppose that the (RSCQ) is satisfied at \( \hat{x} \in \tilde{S} \). Then, there exist uncertain parameters \( \hat{u} \in \mathcal{U}, \hat{v} \in \mathcal{V} \) and Lagrange multiplier \( \hat{\mu} \in D^* \) such that

\[
\tilde{S} = \left\{ x \in K : \exists \eta > 0, \exists \xi \in \partial_x f(\cdot, \hat{u}))(\hat{x}) \cap \partial_x f(\cdot, \hat{u}))(\hat{x}), \exists \xi > \eta d_{\tilde{K}}(x), \langle \xi, \hat{x} - x \rangle = \eta d_{\tilde{K}}(x), (\xi g)(x, \hat{v}) = 0, \max_{u \in \mathcal{U}} f(x, u) = f(x, \hat{u}) \right\}. \tag{16}
\]
Proof. \((\subseteq)\) Let \(x \in \tilde{S}\) be given. Then there exists \(\eta > 0\) such that (2) holds. Hence, there exist \(\xi_f \in \partial f(\cdot, \hat{u})(x)\), \(\xi_{\hat{u}} \in \partial (\hat{u})\) and \(\xi_{\hat{u}g} \in \partial ((\hat{u}g)(\cdot, \hat{v}))(\hat{x})\) such that
\[
0 = \xi_f + \xi_{\hat{u}} + \xi_{\hat{u}g} \text{ since } 0 \in N_{R^*}(\hat{x}) \cap \eta B_{R^*},
\]
and
\[
(\hat{u}g)(\hat{x}, \hat{v}) = 0.
\]
Since \(\xi_{\hat{u}} \in \partial (\hat{u})\) and \(\xi_{\hat{u}g} \in \partial ((\hat{u}g)(\cdot, \hat{v}))(\hat{x})\),
\[
\delta(\hat{x}) - \delta(\hat{x}) + (\hat{u}g)(\hat{x}, \hat{v}) - (\hat{u}g)(\hat{x}, \hat{v}) \geq (\xi_f + \xi_{\hat{u}g}, x - \hat{x}).
\]
By the same fashion in the proof of Theorem 3.2, we have
\[
(\hat{u}g)(\hat{x}, \hat{v}) = (\hat{u}g)(\hat{x}, \hat{v}) = 0,
\]
and
\[
\max_{u \in U} f(x, u) = f(x, \hat{u}).
\]
Therefore, it follows from (19) that
\[
0 \geq (\xi_f + \xi_{\hat{u}g}, x - \hat{x}),
\]
and so by (17), we obtain
\[
\eta d_{\hat{K}}(x) \geq (\xi_f, \hat{x} - x).
\]
Simultaneously, since \(\xi_f \in \partial f(\cdot, \hat{u})(\hat{x})\), we have
\[
\langle \xi_f, \hat{x} - x \rangle \geq f(\hat{x}, \hat{u}) - f(x, \hat{u}).
\]
By (15) in the proof of Theorem 3.2, we obtain
\[
\langle \xi_f, \hat{x} - x \rangle \geq \max_{u \in U} f(x, u) = f(x, \hat{u}) \geq 0 = \eta d_{\hat{K}}(x).
\]
Hence, we have that \(\langle \xi_f, \hat{x} - x \rangle = \eta d_{\hat{K}}(x)\). Now, we prove that for \(\xi_f \in \partial f(\cdot, \hat{u})(x)\), there is an \(\varepsilon > \eta d_{\hat{K}}(x) \geq 0\). In fact, we can show that for any \(y \in \mathbb{R}^n\),
\[
\langle \xi_f, y - x \rangle = \langle \xi_f, y - \hat{x} \rangle + \langle \xi_f, \hat{x} - x \rangle \leq \langle \xi_f, y - \hat{x} \rangle
\]
as \(\langle \xi_f, \hat{x} - x \rangle \leq 0\). Since \(\xi_f \in \partial f(\cdot, \hat{u})(\hat{x})\) and \(f(x, \hat{u}) = f(\hat{x}, \hat{u})\) by (14) and (12),
\[
\langle \xi_f, y - x \rangle \leq f(y, \hat{u}) - f(\hat{x}, \hat{u}) = f(y, \hat{u}) - f(x, \hat{u}),
\]
which means \(\xi_f \in \partial f(\cdot, \hat{u})(x)\).
\((\supseteq)\) Let
\[
x \in \left\{ x \in K : \exists \eta > 0, \exists \xi_f \in \partial f(\cdot, \hat{u})(\hat{x}) \cap \partial f(\cdot, \hat{u})(x), \exists \varepsilon > \eta d_{\hat{K}}(x), \langle \xi_f, x - \hat{x} \rangle = \eta d_{\hat{K}}(x), (\hat{u}g)(x, \hat{v}) = 0, \max_{u \in U} f(x, u) = f(x, \hat{u}) \right\}.
\]
Since \(\hat{x} \in \tilde{S}\), it is clear that \(\eta d_{\hat{K}}(\hat{x}) = 0\). By assumption and \(\xi_f \in \partial f(\cdot, \hat{u})(x)\) for some \(\varepsilon > 0\), we get
\[
-\eta d_{\hat{K}}(\hat{x}) = 0
\]
\[
= \langle \xi_f, \hat{x} - x \rangle - \eta d_{\hat{K}}(x)
\]
\[
\leq f(\hat{x}, \hat{u}) - f(x, \hat{u}) + \varepsilon - \eta d_{\hat{K}}(x)
\]
\[
= f(\hat{x}, \hat{u}) - f(x, \hat{u}) - \eta d_{\hat{K}}(x) + \eta d_{\hat{K}}(x)
\]
\[
= f(\hat{x}, \hat{u}) - f(x, \hat{u}).
\]
Therefore, we obtain
\[
\max_{u \in U} f(x, u) \leq \max_{u \in U} f(x, u) + \eta d_{\hat{K}}(\hat{x}).
\]
Since $\hat{x} \in \tilde{S}$ and $x \in K$, the conclusion that $x \in \tilde{S}$ is satisfied.

In the case that $D := \mathbb{R}_+$, which is a closed convex (and pointed) cone in $\mathbb{R}$, the problem is reduced to be an inequality constrain problem. Suppose that $f : \mathbb{R}^n \times U \to \mathbb{R}$ is a function such that $f(\cdot, u)$ is convex for any $u \in U$ and $f(x, \cdot)$ is concave for any $x \in \mathbb{R}^n$ as well as $g : \mathbb{R}^n \times V \to \mathbb{R}$ is a function such that $g(\cdot, v)$ is convex for any $v \in V$ and $g(x, \cdot)$ is concave for any $x \in \mathbb{R}^n$. Here, the problem (UP) is represented as

$$\min \{ f(x, u) : g(x, v) \leq 0, \forall v \in V \},$$

and its robust counter part is

$$\min \left\{ \max_{u \in U} f(x, u) : g(x, v) \leq 0, \forall v \in V \right\}.$$

In this case, we can see that robust feasible set $K$ is denoted by

$$K := \{ x \in \mathbb{R}^n : g(x, v) \leq 0, \forall v \in V \}.$$

**Corollary 3.** Let $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ and $g : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$ satisfying the following properties:

1. for any $u \in U$, and $v \in V$, $f(\cdot, u)$ is convex and continuous as well as $g(\cdot, v)$ is convex on $\mathbb{R}^n$;
2. for any $x \in \mathbb{R}^n$, $f(x, \cdot)$ and $g(x, \cdot)$ are concave on $U$ and $V$, respectively.

The following statements are equivalent:

(a) The (RSCQ) is fulfilled at $\hat{x} \in K$;

(b) $\hat{x} \in \mathbb{R}^n$ is a robust weak sharp solution of (UP) if and only if there exists a real number $\eta > 0$ such that for any $x^* \in N_{\tilde{K}}(\hat{x}) \cap \eta \mathbb{B}_{\mathbb{R}^n}$, there exist $\hat{u} \in U(\hat{x}), \hat{v} \in V$ and $\hat{\mu} \geq 0$ yield

$$x^* \in \partial f(\cdot, \hat{u})(\hat{x}) + \partial \delta_C(\hat{x}) + \partial(\hat{\mu}g)(\cdot, \hat{v})(\hat{x}), \text{ and } (\hat{\mu}g)(\hat{x}, \hat{v}) = 0.$$

**Corollary 4.** Let $\hat{x} \in \tilde{S}$ be given. Suppose that the (RSCQ) is satisfied at $\hat{x}$. Then, there exist uncertain parameters $\hat{u} \in U, \hat{v} \in V$, and Lagrange multiplier $\hat{\mu} \geq 0$ such that for any $x \in \tilde{S},$

$$(\hat{\mu}g)(x, \hat{v}) = 0, \hat{u} \in U(x), \text{ and } L(x, \hat{\mu}, \hat{u}, \hat{v}) \text{ is constant on } \tilde{S}.$$

**Corollary 5.** For the problem (UP), let $\tilde{S}$ be the robust weak sharp solutions set of (UP) and $\hat{x}$ belongs to it. Suppose that the (RSCQ) is satisfied at $\hat{x} \in \tilde{S}$. Then, there exist uncertain parameters $\hat{u} \in U, \hat{v} \in V$ and Lagrange multiplier $\hat{\mu} \geq 0$ such that

$$\tilde{S} = \left\{ x \in K : \exists \eta > 0, \exists a \in \partial \epsilon f(\cdot, \hat{u})(\hat{x}) \cap \partial \epsilon f(\cdot, \hat{u})(\hat{x}), \exists \epsilon > \eta d_{\tilde{K}}(x),
\langle a, \hat{x} - x \rangle = \eta d_{\tilde{K}}(x), (\hat{\mu}g)(x, \hat{v}) = 0, \max_{u \in U} f(x, u) = f(x, \hat{u}) \right\}.$$ (22)

4. **Applications to multi-objective optimization.** In this section, in order to apply our general results of the previous section, we investigate the class multi-objective optimization problem

$$\min_{x \in \mathbb{R}^n} \{ (f_1(x), f_2(x), \ldots, f_l(x)) : x \in C, g(x) \in -D \},$$ (MP)
where where $C \subseteq \mathbb{R}^n$ is a nonempty convex set, $D \subseteq \mathbb{R}^m$, $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is a convex function for any $i \in I$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a $D$-convex function. The feasible set of (MP) is defined by

$$K_0 := \{ x \in C : g(x) \in -D \}.$$  

The problem (MP) in the face of data uncertainty both in the objective and constraint can be captured by the following multi-objective optimization problem

$$\min_{x \in \mathbb{R}^n} \{ (f_1(x, u_1), f_2(x, u_2), \ldots, f_l(x, u_l)) : x \in C, g(x, v) \in -D \},$$  

(UMP)

where $f_i : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}$, $i = 1, \ldots, l$, and $g : \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}^m$. $u_i, i = 1, \ldots, l$, and $v$ are uncertain parameters, and they belong to the corresponding convex and compact uncertainty sets $U \subseteq \mathbb{R}^p$, and $V \subseteq \mathbb{R}^q$. Suppose that for any $u_i \in U_i, i \in I$, the function $f_i(\cdot, u_i)$ is convex on $\mathbb{R}^n$ and for any $x \in \mathbb{R}^n$, $f_i(x, \cdot)$ is concave on $U_i, i \in I$. Besides, suppose that for any $v \in V$, the function $g(\cdot, v)$ is $D$-convex on $\mathbb{R}^n$ and for any $x \in \mathbb{R}^n$, $g(x, \cdot)$ is $D$-concave on $V$.

Similarly, we obtain some characterizations of the robust weak sharp weakly efficient solutions of (UMP) by using investigation of its robust (worst case) counterpart:

$$\min_{x \in \mathbb{R}^n} \left\{ \max_{u_i \in U_i} f_i(x, u_1), \ldots, \max_{u_i \in U_i} f_l(x, u_l) : x \in C, g(x, v) \in -D \right\},$$  

(RUMP)

where the robust feasible set of (UMP) is also defined by

$$K := \{ x \in C : g(x, v) \in -D \}.$$  

Now, we recall the following concepts of robust weak sharp weakly efficient solutions in multi-objective optimization, which can be found in the literature; see e.g.,[20] and [30].

**Definition 4.1.** [20] A point $\hat{x} \in K$ is said to be a weakly robust efficient solution of (UMP) if it is a weakly efficient solution for (RUMP) i.e., there does not exist $x \in K$ such that

$$\max_{u_i \in U_i} f_i(x, u_i) < \max_{u_i \in U_i} f_i(\hat{x}, u_i), \text{ for all } i \in I.$$  

**Definition 4.2.** [30] A point feasible element $\hat{x}$ is said to be a weakly robust efficient solution for (MP) if there exists a real number $\eta > 0$ such that for any $x \in K$

$$\max_{1 \leq k \leq l} \{ f_k(x) - f_k(\hat{x}) \} \geq \eta d_K(x)$$

where $\hat{K} := \{ x \in K : f(x) = f(\hat{x}) \}$.

Now, we introduce a new concept of solution, which related to the sharpness, namely the robust weak sharp weakly efficient solutions.

**Definition 4.3.** A point $\hat{x} \in K$ is said to be a weak sharp weakly efficient solution for (RUMP) if and only if there exist a real number $\eta > 0$ such that there does not exist $y \in K \setminus \{ \hat{x} \}$ satisfying

$$\max_{u_i \in U_i} f_i(y, u_i) - \max_{u_i \in U_i} f_i(\hat{x}, u_i) < \eta d_K(y), \text{ for all } i \in I,$$

or equivalently, for all $x \in K$

$$\max_{i \in I} \left\{ \max_{u_i \in U_i} f_i(x, u_i) - \max_{u_i \in U_i} f_i(\hat{x}, u_i) \right\} \geq \eta d_K(x).$$
where $\bar{K} := \left\{ x \in K : \max_{u \in U_i} f_i(x, u) = \max_{u \in U_i} f_i(\hat{x}, u), i \in I \right\}$.

**Definition 4.4.** A point $\hat{x} \in K$ is said to be a robust weak sharp weakly efficient solution for (UMP) if it is a weakly weak sharp weakly efficient solution for (RUMP).

The following lemma is useful for establishing our results in this section.

**Lemma 4.5.** [25] Let $U_1, \ldots, U_i$ be nonempty convex and compact sets of $\mathbb{R}^p$ and for any $u_i \in U_i, i \in I$, the function $f_i(\cdot, u_i) : \mathbb{R}^n \to \mathbb{R}$ be convex as well as for any $x \in \mathbb{R}^n$, $f_i(x, \cdot) : U_i \to \mathbb{R}$ be concave where $i \in I$. Then, for any $\lambda_i \geq 0, i \in I$,

$$
\partial \left( \max_{u \in \prod_{i \in I} U_i(\hat{x})} \sum_{i \in I} \lambda_i f_i(\cdot, u_i) \right)(\hat{x}) = \bigcup_{u \in \prod_{i \in I} U_i(\hat{x})} \sum_{i \in I} \lambda_i (f_i(\cdot, u_i))(\hat{x}),
$$

where

$$
\prod_{i \in I} U_i(\hat{x}) := \left\{ (\hat{u}_1, \ldots, \hat{u}_i) \in \prod_{i \in I} U_i : \sum_{i \in I} \lambda_i f_i(\hat{x}, \hat{u}_i) = \max_{u \in \prod_{i \in I} U_i(\hat{x})} \sum_{i \in I} \lambda_i f_i(\hat{x}, u_i) \right\}
$$

Now, by using the similar methods of Section 3, we can characterize the corresponding robust weak sharp weakly efficient solutionss of (UMP).

**Theorem 4.6.** Let $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^l$ and $g : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^m$ satisfying the following properties:

1. for any $u_i \in U_i, i \in I$ and $v_j \in V_j, j \in J, f_i(\cdot, u_i)$ is convex and continuous as well as $g(\cdot, v)$ is $D$-convex on $\mathbb{R}^n$;
2. for any $x \in \mathbb{R}^n$, $f_i(x, \cdot)$ is concave on $U_i, i \in I$ and $g(x, \cdot)$ is $D$-concave on $V$.

Then, the following statements are equivalent:

(a) The (RSCQ) is fulfilled at $\hat{x} \in K$;

(b) $\hat{x} \in \mathbb{R}^n$ is a robust weak sharp weakly efficient solutions of (UMP) if and only if there exists $\eta > 0$ such that for any $x^* \in N_K(\hat{x}) \cap \eta \mathbb{B}_{\mathbb{R}^n}$, there exist $\hat{u}_i \in U_i(\hat{x}), \sigma_i \geq 0, i \in I$, not all zero, $\hat{v} \in V$, and $\hat{\mu} \geq 0$ such that

$$
0 \in \{-x^*\} + \sum_{i \in I} \sigma_i (\partial f_i(\cdot, \hat{u}_i)(\hat{x})) + \partial \delta_C(\hat{x}) + \partial ((\hat{\mu}g)(\cdot, \hat{v}))(\hat{x}) \quad (23)
$$

and

$$
(\hat{\mu}_j g_j)(\hat{x}, \hat{v}_j) = 0, \quad (24)
$$

and

$$
\sigma_i f_i(\hat{x}, \hat{u}_i) = \sigma_i \max_{u_i \in U_i} f_i(\hat{x}, u_i), i \in I. \quad (25)
$$

**Proof.** (a)⇒(b) Assume that the (RSCQ) is satisfied at $\hat{x} \in \mathbb{R}^n$. Let $\hat{x}$ be a robust weak sharp weakly efficient solutions of (UMP) i.e., there exists $\eta > 0$ such that there does not exist $y \in K \setminus \{\hat{x}\}$ satisfying

$$
\max_{u_i \in U_i} f_i(y, u_i) - \max_{u_i \in U_i} f_i(\hat{x}, u_i) < \eta d_K(y), \text{ for all } i \in I,
$$
or equivalently, for any $x \in K$,

$$
\max_{i \in I} \left\{ \max_{u_i \in U_i} f_i(x, u_i) - \max_{u_i \in U_i} f_i(\hat{x}, u_i) \right\} \geq \eta d_K(x). \quad (26)
$$
By (26), there is $s \in I$ such that for all $x \in K$,

$$\max_{u_s \in U_s} f_s(x, u_s) + \delta_K(x) - \eta_d(x) \geq \max_{u_s \in U_s} f(x, u_s)$$

$$= \max_{u_s \in U_s} f_s(\hat{x}, u_s) + \delta_K(\hat{x}) - \eta_d(\hat{x}). \quad (27)$$

Besides, according to (27), we follow the techniques used in Theorem 3.2 and obtain that for any $\xi \in \partial \eta_d(\hat{x})$,

$$\langle \xi, x - \hat{x} \rangle \leq \max_{u_s \in U_s} f_s(x, u_s) + \delta_K(x) - \max_{u_s \in U_s} f_s(\hat{x}, u_s) - \delta_K(\hat{x}). \quad (28)$$

Therefore,

$$\partial(\eta_d)(\hat{x}) \subseteq \partial \left( \max_{u_s \in U_s} f_s(\cdot, u) + \delta_K(\hat{x}) \right) \quad (29)$$

Note that the right hand side term of above inclusion is in the subdifferential of the max function:

$$\phi(x) = \max_{i \in I} \phi_i(x) := \max_{i \in I} \left( \max_{\bar{u}_l \in U_l} f_i(\cdot, \bar{u}_l) + \delta_K(\hat{x}) \right) (x).$$

Due to the well-known fact, subdifferential of maximum of functions at $x$ is the convex hull of the union of subdifferentials of the active functions at $x$, the inclusion (29) becomes

$$\partial(\eta_d)(\hat{x}) \subseteq \text{co} (\cup \{ \partial \phi_i(\hat{x}) : \phi_i(\hat{x}) = \phi(x) \}),$$

thereby

$$\partial(\eta_d)(\hat{x}) \subseteq \sum_{i \in I(\hat{x})} \sigma_i \partial \phi_i(\hat{x}),$$

where $\sigma_i \geq 0, i \in I(\hat{x})$ with $\sum_{i \in I(\hat{x})} \sigma_i = 1$ and $I(\hat{x}) := \{ k \in I : \phi_k(\hat{x}) = \phi(\hat{x}) \}$. Further, setting $\sigma_i = \bar{\sigma}_i, i \in I(\hat{x})$, and otherwise equals to 0 leads to

$$\partial(\eta_d)(\hat{x}) \subseteq \sum_{i \in I} \bar{\sigma}_i \partial \phi_i(\hat{x}).$$

By the definition of $\phi_i$, $i \in I$, the continuity of $\max_{u_s \in U_s} f_i(\cdot, u_i), i \in I$ and the lower semicontinuity and convexity of $\delta_K$, we have

$$\partial(\eta_d)(\hat{x}) \subseteq \sum_{i \in I} \bar{\sigma}_i \partial \left( \max_{u_i \in U_i} f(\cdot, u_i) \right) (\hat{x}) + \sum_{i \in I} \bar{\sigma}_i \partial \delta_K(\hat{x}).$$

It follows from Lemma 4.5 and the hypothesis such (RSCQ) is satisfied at $\hat{x} \in K$ that

$$\partial(\eta_d)(\hat{x}) \subseteq \bigcup_{u \in \prod_{i \in I} U_i(\hat{x}) \in I} \sum_{i \in I} \bar{\sigma}_i \partial f_i(\cdot, u_i)(\hat{x}) + \sum_{i \in I} \bar{\sigma}_i \partial \delta_K(\hat{x})$$

$$+ \bigcup_{\mu \in D^*, \nu \in V} \partial ((\mu g)(\cdot, v))(\hat{x}).$$
Because $\hat{\sigma}_i \geq 0$, $i = 1, 2, \ldots, l$, all nonzero, thereby
\[
\partial (\eta d_{\hat{K}})(\hat{x}) \subseteq \bigcup_{u = (u_i)_{i=1}^l \in I} \sum_{i \in I} \hat{\sigma}_i (\partial f_i(\cdot, u_i)(\hat{x})) + \partial \delta_C(\hat{x}) + \partial ((\hat{\mu} g)(\cdot, \hat{v}))(\hat{x}) + \partial ((\hat{\mu} g)(\cdot, \hat{v}))(\hat{x}).
\]
As $\partial d_{\hat{K}}(x) = N_{\hat{K}}(x) \cap B_{\mathbb{R}^n}$, we obtain (23) as desired.

Conversely, assume that there is $\eta > 0$ such that (23)-(25) hold. Then, for any $x^* \in N_{\hat{K}}(\hat{x}) \cap \eta B_{\mathbb{R}^n}$, there exist $\hat{u} := (\hat{u}_1, \ldots, \hat{u}_l) \in \prod_{i \in I} U_i(\hat{x}), \hat{v} \in V$ and $\hat{\mu} \in D^*$ such that
\[
x^* \in \sum_{i \in I} \hat{\sigma}_i (\partial f_i(\cdot, \hat{u}_i)(\hat{x})) + \partial \delta_C(\hat{x}) + \partial ((\hat{\mu} g)(\cdot, \hat{v}))(\hat{x}), \text{ and}
\]
\[
(\hat{\mu} g)(\hat{x}, \hat{v}) = 0. \tag{30}
\]
Since $0 \in N_{\hat{K}}(\hat{x}) \cap \eta B_{\mathbb{R}^n} = \bigcap_{\epsilon > 0} \partial_{\epsilon} (\eta d_{\hat{K}})(\hat{x})$, for each positive $\epsilon$, $\partial_{\epsilon} (\eta d_{\hat{K}})(\hat{x})$ is nonempty. Let $\epsilon > 0$ and $\xi \in \partial_{\epsilon} (\eta d_{\hat{K}})(\hat{x})$ be arbitrary, then for any $x \in K$
\[
\eta d_{\hat{K}}(x) - \eta d_{\hat{K}}(\hat{x}) \geq \langle \xi, x - \hat{x} \rangle - \epsilon. \tag{31}
\]
Therefore, we obtain
\[
\eta d_{\hat{K}}(\hat{x}) \leq \inf_{x \in \mathbb{R}^n} \eta d_{\hat{K}}(x) + \epsilon \leq \inf_{x \in K} \eta d_{\hat{K}}(x) + \epsilon.
\]
Above inequality and (31) imply that
\[
0 \geq \langle \xi, x - \hat{x} \rangle - \epsilon. \tag{32}
\]
Further, since $0 \in N_{\hat{K}}(\hat{x}) \cap \eta B_{\mathbb{R}^n}$, we have that there exist $\xi_f \in \sum_{i \in I} \hat{\sigma}_i (\partial f_i(\cdot, \hat{u}_i)(\hat{x})), \xi_\delta \in \partial \delta_C(\hat{x})$, and $\xi_{\hat{\mu} g} \in \partial ((\hat{\mu} g)(\cdot, \hat{v}))$ such that
\[
\xi_f + \xi_\delta + \xi_{\hat{\mu} g} = 0. \tag{33}
\]
As $\xi_f \in \sum_{i \in I} \hat{\sigma}_i (\partial f_i(\cdot, \hat{u}_i)(\hat{x})) = \partial \left( \sum_{i \in I} \hat{\sigma}_i f_i(\cdot, \hat{u}_i) \right)(\hat{x}), \xi_\delta \in \partial \delta_C(\hat{x})$ and $\xi_{\hat{\mu} g} \in \partial ((\hat{\mu} g)(\cdot, \hat{v}))(\hat{x})$, we have
\[
\sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i) - \sum_{i \in I} \hat{\sigma}_i f_i(\hat{x}, \hat{u}_i) \geq \langle \xi_f, x - \hat{x} \rangle,
\]
\[
\delta_C(x) - \delta_C(\hat{x}) \geq \langle \xi_\delta, x - \hat{x} \rangle, \text{ and}
\]
\[
(\hat{\mu} g)(x, \hat{v}) - (\hat{\mu} g)(\hat{x}, \hat{v}) \geq \langle \xi_{\hat{\mu} g}, x - \hat{x} \rangle.
\]
Then, adding these inequalities yields
\[
\langle 0, x - \hat{x} \rangle \leq \sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i) - \sum_{i \in I} \hat{\sigma}_i f_i(\hat{x}, \hat{u}_i)
\]
\[
+ (\hat{\mu} g)(x, \hat{v}) - (\hat{\mu} g)(\hat{x}, \hat{v}).
\]
Since $\hat{u}_i$ belongs to $\mathcal{U}_i(\hat{x})$, above inequality becomes the following one:

$$0 \leq \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \sum_{i \in I} \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) + (\hat{\mu}g)(x, \hat{v}) - (\hat{\mu}g)(\hat{x}, \hat{v}).$$

This together with $(\hat{\mu}g)(x, \hat{v}) \leq 0, (\hat{\mu}g)(\hat{x}, \hat{v}) = 0$, and (33), for any $x \in K$,

$$\sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) \geq \langle 0. \quad (34) \rangle$$

By summing (34) with (31), for any $x \in K$, we obtain

$$\sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) \geq \langle \xi, x - \hat{x} \rangle - \varepsilon,$n

which means $\xi \in \partial_\varepsilon\left( \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\cdot, u_i) \right)(\hat{x})$, and so $\partial_\varepsilon(\eta d_K)(\hat{x}) \subseteq \partial_\varepsilon\left( \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\cdot, u_i) \right)(\hat{x})$. As $\varepsilon > 0$ was arbitrary, for each $x \in K$,

$$0 \leq \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i)$$

$$- (\eta d_K(x) - \eta d_K(\hat{x})), $$

which is equivalent to the following inequality: for all $x \in K$

$$\sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \eta d_K(x) \geq \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) - \eta d_K(\hat{x}).$$

It follows that

$$\sum_{i \in I} \hat{\sigma}_i \left( \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \eta d_K(x) \right) \geq \sum_{i \in I} \hat{\sigma}_i \left( \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) - \eta d_K(\hat{x}) \right),$$

for any $x \in K$, which yields for any $i \in I$,

$$\max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \eta d_K(x) \geq \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) - \eta d_K(\hat{x}), \forall x \in K.$$ 

Therefore, for any $x \in K$

$$\max_{i \in I} \left\{ \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) \right\} \geq \eta d_K(x).$$

This means $\hat{x}$ is a robust weak sharp weakly efficient solutions of (UMP).

(b) $\Rightarrow$ (a) Let $\bar{\eta} > 0$ be given. Consider $f_i(x, u_i) := - \langle \xi_i, x \rangle + \bar{\eta} d_K(x), i \in I$. Thus, for any $x \in K$,

$$\max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \bar{\eta} d_K(x) = - \langle \xi_i, x \rangle$$

$$\geq - \langle \xi_i, \hat{x} \rangle + \bar{\eta} d_K(\hat{x})$$

$$= \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i).$$
Thus, \( \hat{x} \) is a robust weak sharp weakly efficient solution of \((UMP)\). By hypothesis, there is \( \eta := \bar{\eta} \) such that (23) is fulfilled. Since for any \( u_i \in U_i, \partial f_i(\cdot, u_i)(\hat{x}) \subseteq \{-\xi_\delta\} + \partial(\eta d_{\bar{K}})(\hat{x}) \), one has
\[
\sum_{i \in I} \hat{\sigma}_i (\partial f_i(\cdot, u_i)(\hat{x})) \subseteq \{-\xi_\delta\} + \partial(\eta d_{\bar{K}})(\hat{x}),
\]
where \( \hat{\sigma}_i \geq 0, i \in I \) and all nonzero. Thus, we obtain that for any \( x^* \in N_{\bar{K}}(\hat{x}) \cap \eta B_{\mathbb{R}^n} \), there exist \( \hat{u}_i \in U_i(\hat{x}), \hat{v} \in V \) and \( \hat{\mu} \in D^* \) such that
\[
x^* \in \{-\xi_\delta\} + \partial(\eta d_{\bar{K}})(\hat{x}) + \partial \delta_C(\hat{x}) + \partial ((\hat{\mu} g)(\cdot, \hat{v})) (\hat{x}) \text{ and } (\hat{\mu} g)(\hat{x}, \hat{v}) = 0.
\]
As \( 0 \in N_{\bar{K}}(\hat{x}) \cap \eta B_{\mathbb{R}^n} \), we obtain
\[
\xi_\delta \in \partial \delta_C(\hat{x}) + \partial ((\mu g)(\cdot, \hat{v})) (\hat{x}) \text{ and } (\mu g)(\hat{x}, \hat{v}) = 0.
\]
It follows that
\[
\xi_\delta \in \partial \delta_C(\hat{x}) + \bigcup_{\mu \in D^*, v \in V} (\partial ((\hat{\mu} g)(\cdot, v)) (\hat{x}), \forall \mu \geq 0.
\]
and so we get the desired inclusion. Therefore, the proof is complete.

Remark 4. (i) In [13] and [14], the authors presented the necessary condition for the local sharp efficiency for the semi-infinite vector optimization problem by using the different method with Theorem 4.6. In fact, they employed the exact sum rule for Fréchet subdifferentials to obtained their results. This means Theorem 4.6 use the different method from the mentioned work.

(ii) In [29], the exact sum rule for Mordukhovich subdifferentials was used as a vital tool under some regularity and differentiability assumptions for establishing their results. This means Theorem 4.6 use the different method from the mentioned work.

Next, by using the similar methods of section 3, a characterization of robust weak sharp weakly efficient solution sets in terms of a given robust weak sharp weakly efficient solution point of the problem is also illustrated in this section. In order to present the mentioned characterization, we start by deriving constant Lagrangian-type property for robust weak sharp weakly efficient solution sets of \((MP)\). In what follows, let \( u = (u_1, \ldots, u_l) \in U_1 \times \ldots \times U_l, \sigma = (\sigma_1, \ldots, \sigma_l) \in \mathbb{R}^l, v \in V \) and \( \mu \geq 0 \). The Lagrangian-Type function \( L(\cdot, \sigma, \mu, u, v) \) is given by
\[
L(x, \sigma, \mu, u, v) = \sum_{i \in I} \sigma_i f_i(x, u_i) + (\mu g)(x, v), \forall x \in \mathbb{R}^n.
\]

Theorem 4.7. Let \( x \in \bar{S} \) be given. Suppose that the (RSCQ) is fulfilled at \( \hat{x} \). Then, there exist a positive valued vector \( \hat{\sigma} := (\hat{\sigma}_1, \ldots, \hat{\sigma}_l) \in \mathbb{R}^l, \hat{\sigma}_i, i \in I \) all nonzero, uncertain parameters \( \hat{u} := (u_1, \ldots, u_l) \in U = U_1 \times \ldots \times U_l, \hat{v} \in V \), and Lagrange multiplier \( \hat{\mu} \geq 0 \) such that for any \( x \in \bar{S} \),
\[
(\hat{\mu} g)(x, \hat{v}) = 0, \quad \hat{u} \in U(x), \quad \text{and } L(x, \hat{\sigma}, \hat{\mu}, \hat{u}, \hat{v}) \text{ is a constant on } \bar{S}.
\]

Proof. Since \( \hat{x} \in \bar{S} \) with the real number \( \eta_1 > 0 \) and the (RSCQ) is satisfied at this point \( \hat{x} \), by Theorem 4.6, (23) holds for \( \eta := \eta_1 \). Since \( N_{\bar{K}}(\hat{x}) \cap \eta B_{\mathbb{R}^n} \) is nonempty we can let \( \varepsilon > 0 \) be arbitrary and \( x^* \in \partial_{\varepsilon}(\eta d_{\bar{K}})(\hat{x}) \) be given. Besides, there exist \( \hat{\sigma} \in \mathbb{R}^l, \) all nonzero, \( \hat{u} \in U, \hat{v} \in V \) and \( \hat{\mu} \in D^* \) such that (23) is fulfilled. Let \( x \in \bar{S} \)
be arbitrary. By the same fashion using in the proof of Theorem 3.2 we have
\[ \langle x^*, x - \hat{x} \rangle \leq \sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i) - \sum_{i \in I} \hat{\sigma}_i f_i(\hat{x}, \hat{u}_i) + (\hat{\mu}_g)(x, \hat{v}) - (\hat{\mu}_g)(\hat{x}, \hat{v}), \]
and so
\[ \langle x^*, x - \hat{x} \rangle - \varepsilon \leq \sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i) - \sum_{i \in I} \hat{\sigma}_i f_i(\hat{x}, \hat{u}_i) + (\hat{\mu}_g)(x, \hat{v}) - (\hat{\mu}_g)(\hat{x}, \hat{v}), \]
(35)
As \( f_i(\cdot, u_i), i \in I \) and \( g(\cdot, v) \) are convex, for any \( u_i \in \mathcal{U}_i \) and \( v \in \mathcal{V} \), we have
\[ x^* \in \partial_{\varepsilon} \left( \sum_{i \in I} \hat{\sigma}_i (f_i(\cdot, u_i) + \lambda g(\cdot, v)) \right)(\hat{x}). \]
Hence, one has
\[ \partial_{\varepsilon} (\eta d_{\hat{K}})(\hat{x}) \subseteq \partial_{\varepsilon} \left( \sum_{i \in I} \hat{\sigma}_i (f_i(\cdot, u_i) + \lambda g(\cdot, v)) \right)(\hat{x}), \]
thereby
\[ \sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i) + (\hat{\mu}_g)(x, \hat{v}) - \eta d_{\hat{K}}(x) \geq \sum_{i \in I} \hat{\sigma}_i f_i(\hat{x}, \hat{u}_i) = \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i). \]
(36)
Note that, as \( x \in \mathcal{S} \), then there exists \( \eta_2 > 0 \) such that for all \( y \in K \),
\[ \max_{u_i \in \mathcal{U}_i} f_i(y, u_i) \geq \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) + \eta_2 d_{\hat{K}}(y), \]
which implies
\[ \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(y, u_i) \geq \sum_{i \in I} \hat{\sigma}_i \left( \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) + \eta_2 d_{\hat{K}}(y) \right) = \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) + \eta_2 d_{\hat{K}}(y) = \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i), \]
for all \( y \in \mathcal{S} \). Since \( \hat{x} \in \mathcal{S} \),
\[ \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) \geq \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i). \]
(37)
From \( \hat{\mu} \geq 0, g(\cdot, \hat{v}) \leq 0 \), and (36), it is not hard to see that
\[ (\hat{\mu}_g)(x, \hat{v}) = 0. \]
(38)
Moreover, by (36) and the positivity of \( \eta d_{\hat{K}}(x) \), we see that
\[ \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) \geq \sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i) \geq \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i) + \eta d_{\hat{K}}(x) \geq \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in \mathcal{U}_i} f_i(\hat{x}, u_i). \]
(39)
This together with (38) leads to
\[ \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in U_i} f_i(x, u_i) = \sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i). \] (40)

Thus, \( \mathcal{L}(. \hat{\sigma}, \hat{\mu}, \hat{u}, \hat{v}) \) is constant on \( \tilde{S} \) as follows:
\[
\mathcal{L}(x, \hat{\sigma}, \hat{\mu}, \hat{u}, \hat{v}) = \sum_{i \in I} \hat{\sigma}_i f_i(x, u_i) + (\hat{\mu} g)(x, \hat{v})
= \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in U_i} f_i(x, u_i) + (\hat{\mu} g)(x, \hat{v})
= \sum_{i \in I} \hat{\sigma}_i f_i(x, u_i) + (\hat{\mu} g)(x, \hat{v})
= \sum_{i \in I} \hat{\sigma}_i f_i(x, u_i).
\]

This completes the proof. \( \square \)

**Theorem 4.8.** For the problem (UMP), let \( \tilde{S} \) be the robust weak sharp weakly efficient solution set of (UMP) and \( \hat{x} \in \tilde{S} \). Suppose that the (RSCQ) is fulfilled at \( \hat{x} \in \tilde{S} \). Then, there exist \( \hat{\sigma}_i > 0, i \in I \), all non zero, \( \hat{u} := (\hat{u}_1, \ldots, \hat{u}_I) \in U = U_1 \times \ldots, \times U_I, \hat{v} \in V \) and \( \hat{\mu} \geq 0 \) such that
\[
\tilde{S} = \left\{ x \in K : \exists \eta > 0, \exists a \in \bigcap_{y \in \{x, \hat{x}\}} \partial \left( \sum_{i \in I} \hat{\sigma}_i f_i(\cdot, \hat{u}_i) \right)(\hat{y}), \exists \varepsilon > \eta d_{\tilde{K}}(x), (a, \hat{x} - x) = \eta d_{\tilde{K}}(x), (\hat{\mu} g)(x, \hat{v}) = 0, \right. \nonumber
\]
\[ \left. \max_{u_i \in U_i} f_i(x, u_i) = f_i(x, \hat{u}_i), i \in I \right\}. \]

**Proof.** (\( \subseteq \)) Let \( x \in \tilde{S} \) be given. Then there exists \( \eta > 0 \) such that (23) holds. Thus, there exist \( \hat{u} \in U, \hat{v} \in V \) and \( \hat{\mu} \geq 0 \) such that (23) is fulfilled. Hence, we have that there exist \( \xi_f \in \sum_{i \in I} \hat{\sigma}_i (\partial f_i(\cdot, \hat{u}_i)(\cdot)), \xi_\delta \in \partial \delta_C(\hat{x}) \) and \( \xi_{\hat{\mu} g} \in \partial (\hat{\mu} g)(\cdot, \hat{v})(\cdot) \) such that
\[ 0 = \xi_f + \xi_\delta + \xi_{\hat{\mu} g}, \text{ since } 0 \in N_{\tilde{K}}(\hat{x}) \cap \eta B_{\mathbb{R}^n}, \] (41)
and
\[ (\hat{\mu} g)(\hat{x}, \hat{v}) = 0. \] (42)

Since \( \xi_\delta \in \partial \delta_C(\hat{x}) \) and \( \xi_{\hat{\mu} g} \in \partial (\hat{\mu} g)(\cdot, \hat{v})(\cdot), \)
\[ \delta_C(x) - \delta_C(\hat{x}) + (\hat{\mu} g)(x, \hat{v}) - (\hat{\mu} g)(\hat{x}, \hat{v}) \geq (\xi_\delta + \xi_{\hat{\mu} g}, x - \hat{x}). \] (43)

By the same fashion in the proof of Theorem 4.6, we have
\[ (\hat{\mu} g)(\hat{x}, \hat{v}) = (\hat{\mu} g)(\hat{x}, \hat{v}) = 0, \]
and
\[ \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in U_i} f_i(x, u_i) = \sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i). \]

Therefore, it follows from (43) that
\[ \eta d_{\tilde{K}}(x) = 0 \geq \langle b + c, x - \hat{x} \rangle, \]
and so by (41), we obtain
\[ \eta d_{\tilde{K}}(x) \geq \langle \xi_f, \hat{x} - x \rangle. \]
Simultaneously, since \( \xi_f \in \sum_{i \in I} \hat{\sigma}_i (\partial f_i(\cdot, \hat{u}_i)(\hat{x})) = \partial \left( \sum_{i \in I} \hat{\sigma}_i f_i(\cdot, \hat{u}_i) \right)(\hat{x}) \), we have
\[
\langle \xi_f, \hat{x} - x \rangle \geq \sum_{i \in I} \hat{\sigma}_i f_i(\hat{x}, \hat{u}_i) - \sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i).
\]

By (34) in the proof of Theorem 4.6, we obtain
\[
\langle \xi_f, \hat{x} - x \rangle \geq \sum_{i \in I} \hat{\sigma}_i f_i(\hat{x}, \hat{u}_i) - \sum_{i \in I} \hat{\sigma}_i f_i(x, u_i) = 0 = \eta d_{\tilde{K}}(x).
\]

Hence, we have that \( \langle \xi_f, \hat{x} - x \rangle = \eta d_{\tilde{K}}(x) \). Next, we shall prove that there is \( \varepsilon > \eta d_{\tilde{K}}(x) \geq 0 \) such that
\[
\xi_f \in \partial \left( \sum_{i \in I} \hat{\sigma}_i f_i(\cdot, \hat{u}_i) \right)(x).
\]

In fact, we can show that \( \xi_f \in \partial \left( \sum_{i \in I} \hat{\sigma}_i f_i(\cdot, \hat{u}_i) \right)(x) \). For any \( y \in \mathbb{R}^n \),
\[
\langle \xi_f, y - x \rangle = \langle \xi_f, y - \hat{x} \rangle + \langle \xi_f, \hat{x} - x \rangle \leq \langle \xi_f, y - \hat{x} \rangle
\]
as \( \langle \xi_f, \hat{x} - x \rangle \leq 0 \). Since \( a \in \partial \left( \sum_{i \in I} \hat{\sigma}_i f_i(\cdot, \hat{u}_i) \right)(\hat{x}) \) and \( f_i(x, \hat{u}_i) = f_i(\hat{x}, \hat{u}_i), i \in I \),
\[
\langle \xi_f, y - x \rangle \leq \sum_{i \in I} \hat{\sigma}_i f_i(y, \hat{u}_i) - \sum_{i \in I} \hat{\sigma}_i f_i(\hat{x}, \hat{u}_i)
\]
\[
= \sum_{i \in I} \hat{\sigma}_i f_i(y, \hat{u}_i) - \sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i),
\]
which means \( \xi_f \in \partial \left( \sum_{i \in I} \hat{\sigma}_i f_i(\cdot, \hat{u}_i) \right)(x) \).

(\( \geq \)) Let
\[
x \in \left\{ x \in K : \exists \eta > 0, \forall \xi_f \in \bigcap_{y \in \{x, \hat{x}\}} \partial \left( \sum_{i \in I} \hat{\sigma}_i f_i(\cdot, \hat{u}_i) \right)(\hat{x}), \exists \varepsilon > \eta d_{\tilde{K}}(x), \langle \xi_f, x - \hat{x} \rangle = \eta d_{\tilde{K}}(x), (\mu g)(x, \hat{v}) = 0, \max_{u_i \in U_i} f_i(x, u_i) = f_i(x, \hat{u}_i) \right\}.
\]

Since \( \hat{x} \in \bar{S}, \eta d_{\tilde{K}}(\hat{x}) = 0 \) and so the assumption dealing with \( \xi_f \) lead to
\[
-\eta d_{\tilde{K}}(\hat{x}) = 0
\]
\[
= \langle \xi_f, \hat{x} - x \rangle - \eta d_{\tilde{K}}(x)
\]
\[
\leq \sum_{i \in I} \hat{\sigma}_i f_i(\hat{x}, \hat{u}_i) - \sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i) - \eta d_{\tilde{K}}(x) + \varepsilon
\]
\[
= \sum_{i \in I} \hat{\sigma}_i f_i(\hat{x}, \hat{u}_i) - \sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i) - \eta d_{\tilde{K}}(x) + \eta d_{\tilde{K}}(x)
\]
\[
= \sum_{i \in I} \hat{\sigma}_i f_i(\hat{x}, \hat{u}_i) - \sum_{i \in I} \hat{\sigma}_i f_i(x, \hat{u}_i) + \eta d_{\tilde{K}}(x),
\]
(45)
for any $\hat{\sigma}_i \geq 0, i \in I$, all nonzero. Therefore, we obtain
\[
\sum_{i \in I} \hat{\sigma}_i \max_{u_i \in U_i} f_i(x, u_i) \leq \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in U_i} f_i(\hat{x}, u_i) + \eta d_{\hat{K}}(\hat{x}) = \sum_{i \in I} \hat{\sigma}_i \max_{u_i \in U_i} f_i(\hat{x}, u_i).
\]
Since $\hat{x} \in \bar{S}$ and $x \in K$, the conclusion that $x \in \bar{S}$ is satisfied. \hfill \Box

**Conclusion.** In this paper, we examined convex optimization problems with uncertain constraints and have defined a robust weak sharp solution by studying weak sharp solution of robust convex optimization problems where the uncertain constraints are enforced for all possible uncertainties within prescribed uncertainty sets. By employing tools of convex analysis and the valuable of constraint qualifications, we have established the necessary and sufficient conditions of robust weak sharp solutions, and characterizations of robust weak sharp solution set. As an application, we provided the characterization of the robust weak sharp weakly efficient solution sets for multi-objective convex optimization problems with uncertain constraints.

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