Green’s functions for non-classical transport with general anisotropic scattering

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Abstract

In non-classical linear transport the chord length distribution between collisions is non-exponential and attenuation does not respect Beer’s law. Generalized radiative transfer (GRT) extends the classical theory to account for such two-point correlation between collisions and neglects all higher order correlations. For this form of transport, we derive the exact time-independent Green’s functions for the isotropic point source in infinite 3D homogeneous media with general anisotropic scattering. Green’s functions for both collision rate density, which characterizes absorption and reaction rates in the system, and radiance/flux, which characterizes displacement of radiation/particles, are solved in Fourier space. We validate the derivations using gamma random flights to produce the first anisotropic scattering benchmark solutions for the generalized linear Boltzmann equation. For gamma-4 flights with linearly anisotropic scattering and gamma-6 flights with Rayleigh scattering the collision rate density is found explicitly in real space as a sum of diffusion modes.

Keywords: generalized radiative transfer, non-classical Boltzmann, Green’s function, point source, random flight, diffusion

1 Introduction

Linear transport of monoenergetic neutral particles in stochastic geometries can significantly deviate from the predictions of classical transport theory [Chandrasekhar 1960; Davison 1957] when there are significant spatial correlations in the geometries. This has motivated the use of non-exponential random flights to form generalized radiative transfer (GRT) theories that exactly exhibit some desired non-exponential distribution of chord lengths between collisions [Burrows 1960; Doub 1961; Randall 1964; Rybicki 1965; Alt 1980; Sahni 1989; Audic and Frisch 1993; Kostinski 2001; Davis and Marshak 2004; Davis 2006; Moon et al. 2007; Taine et al. 2010; Frank et al. 2010; Larsen and Vasques 2011; Zarrouati et al. 2013; Vasques and Larsen 2014; Davis and Xu 2014; Frank et al. 2015; Xu et al. 2016; Rukolaine 2016; Liemert and Kienle 2017; Binzoni et al. 2018; Frank and Sun 2018; d’Eon 2018; Jarabo et al. 2018; Bitterli et al. 2018]. We derive new green’s functions for these theories in the case of anisotropic scattering.

Point source Green’s functions for monoenergetic transport in infinite homogeneous medium with a completely general symmetric phase function are known exactly in classical exponential media via Fourier transform [Davison 2000; Wallace 1948; Grosjean 1951; Vanmassenhove and Grosjean 1967; Williams 1977; Paasschens 1997; Ganapol 2003; Zoia et al. 2011] and have been studied using other approaches, such as path integrals [Tessendorf 2011]. These functions have a number of direct applications [Narasimhan and Nayar 2003], they provide important benchmarks for checking correctness of more advanced codes [Ganapol 2008] and they play a role in bounded media via Placzek’s lemma [Case et al. 1953]. The goal of this paper is to derive such Green’s functions for GRT.

In the case of isotropic scattering, the Fourier transform approach has been extended for GRT and the Green’s functions are known [d’Eon 2013; d’Eon 2019]. In this paper, we show how to use the derivations of Grosjean [1951] to derive the exact forms for general anisotropic scattering. To the best of our knowledge, Grosjean’s 1951 derivation has never been validated numerically in the non-exponential case with anisotropic scattering. We perform Monte Carlo simulation of gamma random flights in 3D and find good agreement. We omit the lengthy details of Grosjean’s full derivation and refer the interested reader to his monograph. Our present goal is to briefly highlight only those details that are required to form benchmark solutions in GRT and discuss their role in these theories.
Figure 1: (a) We consider the random flight where a particle leaves an isotropic point source in 3D (black circle) and travels at constant speed along straight paths between collision events (white circles). The free-path lengths between collisions are random variates drawn from the distribution \( p_c(s) \). The initial free-path length \( p_1(s) \) may be different from \( p_c(s) \). At each collision the particle scatters according to a given phase function \( P(\Omega_i \rightarrow \Omega_o) \) with probability \( c \) and is absorbed otherwise, terminating the flight. (b) We seek the rate density \( C(r, \mu) \) that particles enter collisions at some distance \( r \) from the point source restricted to a cone of directions indexed by cosine \( \mu \) to the position vector.

2 General Theory

2.1 The random flights of GRT

Let us consider a unit isotropic point source at the origin of a homogeneous infinite 3D medium emitting particles that travel at constant speed \( v \) along piecewise straight paths (Figure 1). Collisions between the particle and the medium result either in absorption, with probability \( 1 - c \), or scattering into a new direction using a given phase function with probability \( 0 < c \leq 1 \). We consider a general symmetric phase function with Legendre expansion

\[
P(\Omega_i \rightarrow \Omega_o) = \frac{1}{4\pi} \sum_{l=0}^{\infty} (2l + 1) j_l P_l(\Omega : \Omega_o) = \frac{1}{4\pi} \sum_{l=0}^{\infty} A_l P_l(\Omega : \Omega_o)
\]

(1)

where \( \Omega_i \) is the direction before collision and \( \Omega_o \) is the scattered direction. The phase function is normalized over the unit sphere, \( A_0 = 1 \).

Free path-lengths between collisions are drawn from a given normalized distribution \( p_c(s) \), which can be estimated by Monte Carlo tracking in specific realizations of the stochastic geometry [Audic and Frisch 1993; Moon et al. 2007; Larsen and Vasques 2011] or derived from a known attenuation law in the system [Torquato and Lu 1993; d’Eon 2018]. We denote the initial free-path length distribution from the point source \( p_1(s) \), which is either \( p_c(s) \) or a related distribution, \( p_a(s) \). If emission is spatially correlated to the first scattering event in the same way that spatial correlation arises between collisions then \( p_1(s) = p_c(s) \) and the full random flight is homogeneous. However, emission from a deterministic location in the stochastic geometry must consider statistics to the next collision using all realizations of the geometry and not only those that have a particle at the origin from which to begin the flight. For this reason, emission from a so called uncorrelated origin has free-path length statistics \( p_a(s) \) that are distinct from the intercollision statistics \( p_c(s) \) [Audic and Frisch 1993; d’Eon 2018]. An equivalent distinction has been recognized for continuous-time random walks [Feller 1971; Tunaley 1974; Tunaley 1976; Weiss and Rubin 1983] where it was proposed that

\[
p_c(s) = -p_a(s) / \langle s_c \rangle, \quad \langle s_c \rangle \equiv \int_0^\infty p_c(s)sds
\]

(2)

1 Specifically, this occurs in the case of tightly-coupled walks where the wait time is proportional to the displacement, which corresponds to the constant speed model.
where \( \langle s \rangle \) is the mean free path length between collisions. Eq.(2) was also found to be a necessary condition for achieving Helmholtz reciprocity in GRT [d’Eon 2018], which is desired given that reciprocity is observed in any realization of the stochastic geometry and therefore expected of the mean transport.

2.1.1 Collision-rate density

Now consider a single particle leaving the isotropic point source. Our primary quantity of interest in GRT is the collision rate density, since most reaction rates in a given system will be proportional to this quantity. The angular collision-rate density \( C(r, \mu) \) is defined such that \( 4\pi \delta C(r, \mu) \, d\mu \) is the mean number of collisions (either scattering or absorbing) experienced by the particle within a shell of radii \([r, r + dr]\) about the point source who directions before collision have cosines in \([\mu, \mu + d\mu] \). Direction cosines \( \mu \) are measured with respect to the normalized position vector and index cones of directions with \( \mu = 1 \) pointing away from the point source (Figure 1). We denote the scalar collision rate density by

\[
C(r) = \int_0^1 C(r, \mu) \, d\mu .
\]

To distinguish between correlated and uncorrelated emission we also write \( C_c(r) \) and \( C_u(r) \), respectively for the scalar collision rate densities. With non-stochastic single-scattering albedo \( c \) at every collision, the mean number of collisions is \( 1/(1 - c) \) and therefore the 0th moment of \( C \) is [Ivanov 1994]

\[
\int_0^\infty 4\pi r^2 C(r) \, dr = \frac{1}{1 - c}.
\]

(4)

for all phase functions and free-path length distributions. In a system with spherical symmetry, which will always be the case in this paper, the angular collision rate density \( C(x, \Omega) \) at position \( x \) and direction \( \Omega \) is related to \( C(r, \mu) \) by \( C(x, \Omega) = C(r, \mu)/(2\pi) \).

The phase function and single-scattering albedo immediately give the distribution of particles in the system as they leave collisions. The in-scattering rate density \( B(x, \Omega) \) is related to the collision rate density by [Larsen and Vasques 2011]

\[
B(x, \Omega) = c \int_{4\pi} P(\Omega' \cdot \Omega) C(x, \Omega') \, d\Omega'
\]

(5)

and satisfies the generalized Peierls integral equation,

\[
B(x, \Omega) = c \iiint P \left( \frac{x - x'}{|x - x'|} \right) \left\{ \left[ B \left( \frac{x' \cdot x}{|x - x'|^2} \right) + \frac{Q_c(|x - x'|)}{4\pi} \right] p_c(|x - x'|) + \frac{Q_a(|x' - x|)}{4\pi} \right\} \, dV'.
\]

(6)

This is an integral equation of non-homogeneous non-exponential random flights [Grosjean 1951; Weiss and Rubin 1983] and is the basis of our derivation for Green’s functions in GRT.

2.1.2 Radiance and fluence

We are also interested in finding Green’s functions for the radiance and fluence in the system. In GRT, the semi-Markov nature of the transport breaks the classical proportionality between collision rate and radiance [d’Eon 2019], and the two are no longer related by the inverse mean free path, requiring separate Green’s function derivations and separate Monte Carlo estimators. The radiance in GRT satisfies a more complicated equation, the generalized linear Boltzmann equation (GLBE). However, the GLBE has been proven equivalent to Eq.(7) [Larsen and Vasques 2011], and the radiance \( I(x, \Omega) \) (specific intensity/angular flux) is simply related to \( B \) by [Larsen and Vasques 2011]

\[
I(x, \Omega) = \int_0^\infty \left[ B(x - s\Omega, \Omega) + \frac{Q_c(x - s\Omega)}{4\pi} \right] X_c(s) + \frac{Q_a(x - s\Omega)}{4\pi} \, X_a(s) \, ds.
\]

(7)

where

\[
X_c(s) = \int_s^\infty p_c(s') \, ds'
\]

(8)

is the attenuation law leaving a collision [Larsen and Vasques 2011] and

\[
X_a(s) = \int_s^\infty p_a(s') \, ds'
\]

(9)

is the uncorrelated-origin attenuation law [d’Eon 2018]. Finally, the fluence (scalar flux) is

\[
\phi(x) = \int_{4\pi} I(x, \Omega) \, d\Omega .
\]

(10)
For a single point source, where either one of \( Q_x \) or \( Q_y \) is a unit dirac delta at the origin and the other is zero, it suffices to solve Eq.(6) for \( B(r, \mu) \), and then the radiance and fluence are determined after by Eq.(7). We solve for these Green’s functions in the next section using Fourier transforms.

### 2.1.3 Relationship to previous GRT models

This formulation of random flights is an extension of the generalized models of Larsen [2011] and Davis [2006] to support reciprocal transport in bounded domains by including \( p_n(s) \), following [Audic and Frisch 1993], who proposed such an extension for Markovian binary mixtures. Boundary conditions for this extension were given in an earlier paper [d’Eon 2019]. The extended integral equations Eq.(6) and (7) are also analogous to Sahni’s [1989], where he considered a different class of Markovian binary mixtures than Audic and Frisch in what could be considered the first proposal of a non-exponential two-point transport theory. Our extended formulation also supports several proposed forms of non-exponential random flights where all free paths use \( p_n(s) \) [Davis 2006; Taine et al. 2010; Davis and Xu 2014; Wrenninge et al. 2017; Liemert and Kienle 2017; Binzoni et al. 2018]. However, application of these models and related Green’s functions to bounded domains are known to result in non-reciprocal transport.

### 2.2 Grosjean’s solution

In his thesis, Grosjean [1951] solved a fully general random flight in an infinite 3D medium with an isotropic point source. He solved for \( C(r, \mu) \) and \( B(r, \mu) \), given in spherical harmonic expansions whose coefficient functions are given by Fourier inversion (requiring numerical inversion in most cases). In the most general form of his work, Grosjean permitted free-path-length distributions between collisions \( p_n(s) \), single-scattering albedos \( \tau_n(s) \), and scattering kernels (phase functions) \( P_n(\Omega_i \rightarrow \Omega_o) \) that are chosen independently for each collision order \( n > 0 \), to build fully heterogeneous random flights. He also presented simplifications for the case of a completely homogeneous random flight where the intercollision free-path length distribution \( p_0(s) \) and phase function \( P(\Omega_i \rightarrow \Omega_o) \) are identical for all collision orders \( n \) and the single-scattering albedo \( 0 < c \leq 1 \) at every collision is the same constant.

For our homogeneous random flight \( (p_1(s) = \tau_1(s)) \), Grosjean showed that the angular collision rate density has a spherical harmonic expansion given by

\[
C(r, \mu) = \sum_{\ell=0}^\infty C_\ell(r)(2\ell + 1)P_\ell(\mu)
\]  

where

\[
C_\ell(r) = \frac{1}{8\pi c} \int_0^\infty h^{(\ell)}(u) j_\ell(r u) du
\]

in terms of expansion functions \( h^{(\ell)}(u) \) that are the solution of the linear system ([Grosjean 1951], p. 77)

\[
h^{(\ell)}(u) = \frac{2}{\pi} u F^{(\ell,0)}(u) + \sum_{m=0}^\infty A_m h^{(m)}(u) F^{(\ell,m)}(u).
\]

The functions \( F \) are given by ([Grosjean 1951], p. 70)

\[
F^{(\ell,m)}(u) = c \frac{u^{m-l}}{2} \int_0^u \int_{-1}^1 \tau_p(z) P_l(\mu) P_m(\mu) e^{i\mu y} d\mu dz
\]

\[
= c \int_0^u \tau_p(y) dy \left[ \int_0^\pi P_m \left( \frac{d}{dz} \right) (j_l(z)) z \right] \equiv y_{\text{w}}
\]

where \( j_\ell(z) \) is the spherical Bessel function

\[
j_\ell(z) = \frac{\sqrt{\pi} I_{\ell+1/2}(z)}{\sqrt{2}}.
\]

The notation

\[
P_m \left( \frac{d}{dz} \right) (j_l(z))
\]

is understood to mean the differential operator formed from replacing \( z^n \) in the expansion of Legendre polynomial \( P_m(z) \) with

\[
\frac{\partial^n}{\tau^n} z^n
\]
applied to $j_i(z)$. The functions $F$ obey a symmetry $F^{(l,m)}(u) = (-1)^{(l+m)}F^{(m,l)}(u)$.

The Fourier integrals for $C_i(r)$ may only be convergent in the sense of Cesaro summability, prompting the separation of the density of first collisions to express the total collision-rate density as ([Grosjean 1951], p.75)

$$C(r, \mu) = \frac{p_\nu(r)}{4\pi r^2} \delta(1 - \mu) + \sum_{l=0}^\infty C_l^+ (r)(2l + 1)P_l(\mu)$$

with

$$C_l^+ (r) = \frac{1}{8\pi \nu} \int_0^\infty \left( \frac{h^{(l)}(u)}{u} - \frac{2u}{\pi} F^{(l,0)}(u) \right) u j_l(r u) du.$$  \hspace{1cm} (20)

From these derivations we immediately have the scalar collision-rate density $C(r) = 2C_0(r)$ for correlated emission.

For the case of uncorrelated emission, $p_1(s) = p_\nu(s)$, and we refer to Grosjean’s more general derivation. For this almost homogeneous random flight, we find a modified system of equations for the expansion functions $h$ in Eq.(23)

$$h^{(l)}(u) = \frac{2}{\pi} u F_1^{(l,0)}(u) + \sum_{m=0}^\infty A_m h^{(m)}(u) F^{(l,m)}(u)$$

where the functions $F_1$ arise from appropriately modifying Eq.(14) to include the free-path length distribution $p_1(s)$ instead of always $p_\nu(s)$, giving

$$F_1^{(l,m)}(u) = c \int_0^\infty p_1(y)dy \left[ \mu^m P_m \left( \frac{d}{dz} \right) (j_i(z)) \right]_{z=yu}.$$  \hspace{1cm} (22)

For the radiance and fluence, we use Grosjean’s solutions for the Neumann series of a heterogeneous flight ([Grosjean 1951], Eqs.(259,259’)). The rate density for the particle to enter its nth collision at $r$ with cosine $\mu$ is

$$C(r, \mu|n) = \sum_{l=0}^\infty C_l(r|n)(2l + 1)P_l(\mu)$$

where

$$C_l(r|n) = \frac{1}{8\pi \nu} \int_0^\infty h^{(l)}(u) u j_l(r u) du$$

and the Neumann series $h$ functions are

$$h^{(l)}(u) = \sum_{m=0}^\infty A_m h_{n-1} F_n^{(l,m)}, \hspace{1cm} (n = 2, 3, 4...)
$$

$$h^{(l)}_1(0) = 2u F_1^{(l,0)}$$

where $F_n^{(l,m)}$ are defined using the free-path length distribution for the nth free path $p_\nu(s)$. The homogeneous system of equations in Eq.(13) follows from Eqs.(25) using the definition $h^{(l)}(u) = \sum_{n=1}^\infty h^{(l)}_n(0)$ and that $F_n^{(l,m)} = F^{(l,m)}$ for the homogeneous flight. We now add an additional segment to each term in this Neumann series using a free-path distribution that is proportional to the attenuation law for leaving a collision. This creates a fictitious collision density that essentially applies Eq.(7) to the in-scattering rate density, which follows our previous approach for the case of isotropic scattering [d’Eon 2019]. From Eq.(25) it is clear how a given $h_n$ relates to the previous order $h_{n-1}$, and we find

$$h^{(l)}_n(0) = \sum_{m=0}^\infty A_m h^{(m)} F^{(l,m)}_X$$

where the $F$ functions use the attenuation law for leaving a collision (Eq.(8)) instead of a free path distribution,

$$F_X^{(l,m)}(u) = c \int_0^\infty X_0(y)dy \left[ \mu^m P_m \left( \frac{d}{dz} \right) (j_i(z)) \right]_{z=yu}.$$  \hspace{1cm} (28)

This accounts for all flux that arises from collisions in the system. Adding the uncollided flux from the source we find the total fluence

$$\phi(r) = X_0(s) + \frac{1}{4\pi c} \int_0^\infty \frac{h^{(0)}(u)}{u} j_k(r u) du$$

where

$$X_0(s) = \int_x p_1(s') ds'.$$  \hspace{1cm} (30)
The full radiance integrated around a given cone with cosine $\mu$ is

$$I(r, \mu) = \frac{X_0(s)}{4\pi r^2} \delta(1-\mu) + \sum_{l=0}^{\infty} I_l(r)(2l+1)P_l(\mu)$$

(31)

with

$$I_l(r) = \frac{1}{8\pi c} \int_0^\infty h_0^{(l)}(u) u j_l(r u) du.$$  

(32)

Radiance at some position $x$ and direction $\Omega$ in the system is given by $I(x, \Omega) = I(r, \mu)/(2\pi)$.

2.2.1 The case of classical exponential random flights

For clarity, we briefly examine classical radiative transfer under the present formalism.

In classical linear transport with no spatial correlation between collisions in homogeneous media, the above derivation is equivalent to alternatives involving Kuscer/Chandrasekhar polynomials that satisfy a two-term recurrence [Davison 2000; Kuščer 1955; Grosjean 1963; Ganapol 2003]. With $p_c(s) = e^{-s}$, the $F$ functions reduce to

$$F^{(l,m)}(x) = c \frac{e^{-al}}{2} \int_{-1}^1 \frac{P_l(\mu) P_m(\mu)}{1-i\mu} d\mu$$

(33)

with known general solutions in terms of Legendre Q functions [Grosjean 1963; Vanmassenhove and Grosjean 1967], the first few low order terms being

$$F^{(0,0)}(u) = c \frac{\tan^{-1}(u)}{u}, \quad F^{(0,1)}(u) = c \frac{\tan^{-1}(u) - u}{u^2}, \quad F^{(1,1)}(u) = c \frac{u - \tan^{-1}(u)}{u^3}. \quad (34)

3 Gamma random flights in 3D

To test Grosjean’s derivations for the case of non-exponential random flights we chose gamma random flights [Beghin and Orsingher 2010; Le Caër 2011; Pogorui and Rodríguez-Dagnino 2011; d’Eon 2013], which admit explicit solutions in some cases and have a number of interesting properties with respect to diffusion theory. Intercollision free-path lengths are distributed according the normalized gamma distribution

$$p_c(s) = \frac{e^{-\alpha s}}{\Gamma(a)}, \quad a > 0,$$

(35)

which includes classical exponential transport when $a = 1$. For Monte Carlo validation, random free-path lengths are easily sampled from

$$s = - \log \xi_1 \xi_2 \ldots \xi_a,$$

(36)

where $\xi_n \in [0, 1]$ are $a$ independent random uniform variates. We used the generalized collision estimator for collision-rate density and radiance [d’Eon 2019] to compute the Monte Carlo reference solutions below.

Combining Eq.(35) with (14) we require the integrals

$$F^{(l,m)}(u) = c \frac{e^{-al}}{2} \int_{-1}^1 \frac{e^{-a\mu}}{\Gamma(a)} P_l(\mu) P_m(\mu) e^{a\mu u} d\mu d\zeta = c \frac{e^{-al}}{2} \int_{-1}^1 \frac{P_l(\mu) P_m(\mu)}{(1-i\mu a)^a} d\mu$$

(37)

in the general case $a > 0$. For the case $m = 0$, we found a general solution

$$F^{(l,0)}(u) = \frac{\sqrt{\pi a} c^{1/2-l-1} u^{1-l} \Gamma(a+l)}{\Gamma(a)} 2F_1 \left( \frac{a+l+1}{2}; \frac{1}{2}; \frac{3}{2} - a^2 \right)$$

(38)

using the regularized hypergeometric function $2F_1$. We suspect a completely general solution is possible using a two-term recurrence, similar to the exponential case, but we did not find it.

Isotropic Scattering  For the case of isotropic scattering, the expansion coefficients in Eq.(1) are

$$A_0 = 1, \quad A_{l>0} = 0.$$  

(39)

The general solution of the linear system for correlated emission (13) with expansion coefficients (39) yields

$$h^{(l)} = \frac{2u F^{(l,0)}}{\pi - \pi F^{(0,0)}}$$

(40)
and, for the uncorrelated point source, solution of Eq. (21) is
\[
h^{(l)} = \frac{2u}{\pi} \left( \frac{F^{(l,0)}_1 F^{(0,0)}_1}{1 - F^{(0,0)}_1} + F^{(l,0)}_1 \right).
\] (41)

**Linearly-Anisotropic Scattering** With linearly-anistropic scattering with parameter \(-1 < b < 1\), we find
\[
A_0 = 1, \quad A_1 = b, \quad A_{l>1} = 0,
\] (42)
yielding expansion functions for correlated emission
\[
h^{(l)} = -\frac{2u((bF^{(1,1)}(u) - 1)F^{(l,0)}_1(u) + bF^{(0,1)}_1 F^{(l,1)}_1(u))}{\pi (b ((F^{(0,1)}_1(u))^2 - F^{(1,1)}_1(u)) + F^{(0,0)}_1 (bF^{(1,1)}_1(u) - 1) + 1)}.
\] (43)
The bulky expressions for the uncorrelated case are omitted.

**Rayleigh Scattering** We also consider a simple three-term phase function due to Rayleigh that has application in light scattering [Chandrasekhar 1960], with
\[
A_0 = 1, \quad A_1 = 0, \quad A_2 = \frac{1}{2}, \quad A_{l>2} = 0
\] (44)
yielding correlated expansion functions
\[
h^{(l)} = \frac{2u((2 - F^{(2,2)}_1(u)))F^{(l,0)}_1(u) + F^{(0,2)}_1(u) F^{(l,2)}_1(u))}{\pi ((F^{(0,2)}_1(u))^2 + F^{(0,0)}_1(u) F^{(2,2)}_1(u) - 2 - F^{(2,2)}_1(u) + 2)}.
\] (45)
The bulky expressions for the uncorrelated case are omitted.

### 3.1 Gamma-2 random flight in 3D

With \(a = 2\) and an intercollision FPD \(p_c(s) = e^{-s}\), we find, using Eq. (14),
\[
F^{(0,0)}_1(u) = \frac{c}{1 + u^2}, \quad F^{(0,1)}_1(u) = c \left( \frac{1}{u^2 + u} - \frac{\tan^{-1}(u)}{u^2} \right), \quad F^{(1,1)}_1(u) = c \frac{2 \tan^{-1}(u) - \frac{u(u^2 + 2)}{u^2 + 1}}{u^3}
\]
\[
F^{(0,2)}_1(u) = c \left( \frac{u}{u^2 + 1} + 2 - 3 \tan^{-1}(u) \right), \quad F^{(1,2)}_1(u) = c \left( \frac{u^2 + 1}{u^2 + 9} \right) \tan^{-1}(u) - u \left( \frac{7u^2 + 9}{2(u^6 + u^4)} \right)
\] (46)
\[
F^{(2,2)}_1(u) = c \left( \frac{u^2 + 1}{u^2 + 6} + 8 - 3 \left( u^2 + 3 \right) \tan^{-1}(u) \right).
\]

#### 3.1.1 Linearly-Anisotropic scattering

Combining Eq. (46) with Eq. (43) we find
\[
h^{(0)} = \frac{2cu (bcu^2 - bc(u^2 + 1) \tan^{-1}(u)^2 + u^4)}{\pi (bc^2(u^2 + 1) \tan^{-1}(u)^2 + bcu^2(-c + u^2 + 2) - 2bc(u^2 + 1)u \tan^{-1}(u) + u^4(-c + u^2 + 1))}
\] (47)
yielding scalar collision-rate density
\[
C_c(r) = \frac{1}{2\pi^2 r} \int_0^\infty \frac{u (bcu^2 - bc(u^2 + 1) \tan^{-1}(u)^2 + u^4)}{(u^4((b - 1)c + 1) - b(c - 2)cu^2 + bc(u^2 + 1)\tan^{-1}(u)(c \tan^{-1}(u) - 2u) + u^6) \sin(ru) du}.
\] (48)

A comparison of this result to Monte Carlo reference is provided in Figure 2. While diffusion is an exact result for collision-rate density in 3D with gamma-2 flights and isotropic scattering [d’Eon 2013], we see that this does not extend to more general phase functions.

For the fluence, we find
\[
F^{(0,0)}_X = \frac{1}{u^2 + 1} + \frac{\tan^{-1}(u)}{u}, \quad F^{(0,1)}_X = -\frac{u}{u^2 + 1}
\] (49)
Figure 2: Scalar collision-rate density $C_c(r)$ about an isotropic point source in 3D with linearly-anisotropic scattering and intercollision free-path lengths drawn from $e^{-s}$. Validation of Eq.(48) (continuous) with respect to Monte Carlo (dots).

Figure 3: Scalar flux/fluence $\phi_c(r)$ about a correlated isotropic point source in 3D with linearly-anisotropic scattering and intercollision free-path lengths drawn from $e^{-s}$. Validation of Eq.(50) (continuous) with respect to Monte Carlo (dots) and comparison of the diffusion approximation, Eq.(51).

and

$$h_0^{(0)} = -\frac{2c^2}{\pi} \left( - (u^2 + 1) u^2 \tan^{-1}(u) + bc (u^2 + 1) u \tan^{-1}(u)^2 + bc (u^2 + 1)^2 u \tan^{-1}(u)^3 - (b (c + u^2) + u^2) \right)$$

The fluence then follows from Eq.(29). Taking a $(0, 2)$ order Pade approximant of the Fourier-transformed density $\pi h_0^{(0)}/(2cu)$ we find the diffusion approximation for the fluence [d’Eon 2013]

$$\phi(r) \approx \frac{e^{-r}(r + 1)}{4\pi r^2} - \frac{3c(bc - 3) \exp \left(-\sqrt{r/v_0}\right)}{2\pi r (bc(2c^2 + 4c + 3) - 6c + 15)}$$

A comparison of these results to Monte Carlo reference is provided in Figure 3.

In Figure 4 we compare the angular distributions $C(r, \mu)$ and $I(r, \mu)/\langle s_c \rangle$ using 4 term Legendre expansions about a correlated point source at a radius $r = 11.4437$. At this distance from the point source this low order expansion seems
Figure 4: Comparison of the angular collision rate density $C(r, \mu)$ (continuous) and the classically scaled radiance $I(r, \mu)/\langle s_\mu \rangle$ (dashed) for gamma-2 flights and linearly-anisotropic scattering showing agreement with Monte Carlo (dots) and how the two densities are not proportional in GRT.

Figure 5: Scalar collision-rate density $C_\mu(r)$ about an uncorrelated-emission isotropic point source in 3D with linearly-anisotropic scattering and intercollision free-path lengths drawn from $e^{-s}$’s. Validation of Eq.(54) (continuous) with respect to Monte Carlo (dots).

reasonably accurate with respect to Monte Carlo and illustrates how collision rate and flux are not proportional in GRT.

For the uncorrelated source, we find (22)

$$F_1^{(0,0)}(u) = \frac{c}{2u^2 + 2} + \frac{c \tan^{-1}(u)}{2u}, \quad F_1^{(0,1)}(u) = -\frac{cu}{2(u^2 + 1)}$$

(52)
giving (21)

$$h^{(0)} = \frac{cu \left(2bcu^2 - 2bc \left(u^2 + 1\right) \tan^{-1}(u)^2 + u^4 + \left(u^5 + u^3\right) \tan^{-1}(u)\right)}{\pi \left(bc^2 \left(u^2 + 1\right) \tan^{-1}(u)^2 + bcu^2 (-c + u^2 + 2) - 2bc \left(u^2 + 1\right) u \tan^{-1}(u) + u^4 (-c + u^2 + 1)\right)}$$

(53)

for the scalar collision rate density

$$C_\mu(r) = \frac{e^{-r}(r + 1)}{8\pi r^2} + \int_0^\infty \frac{c \sin(ru) \left(- \left(u^2 + 1\right) \tan^{-1}(u) \left(-bcu^2 + bc \tan^{-1}(u) \left((u^2 + 1) \tan^{-1}(u) + u\right) + (b - 1)u^4\right) + bu^3 \left(c + u^2\right) + u^5\right)}{4\pi^2 r \left(u^2 + 1\right) \left(u^4((b - 1)c + 1) - b(c - 2)cu^2 + bc \left(u^2 + 1\right) \tan^{-1}(u) \left(c \tan^{-1}(u) - 2u\right) + u^6\right)} du.$$  

(54)

A comparison of this result to Monte Carlo reference is provided in Figure 5.
Infinite 3D, isotropic point source, linearly-anisotropic scattering, Gamma-3 random flight – correlated emission
Collision-rate density $C_c(r)$, $c = 0.8, b = -0.9$

Figure 6: Scalar collision-rate density $C_c(r)$ about an isotropic point source in 3D with linearly-anisotropic scattering and intercollision free-path lengths drawn from $e^{-s^2/2}$. Validation of Eq.(57) (continuous) with respect to Monte Carlo (dots).

3.2 Gamma-3 random flight in 3D

With intercollision FPD $p_c(s) = \frac{1}{2} e^{-s^2} (a = 3)$ we find

$$F^{(0,0)}(u) = \frac{c}{(u^2 + 1)^2}, \quad F^{(0,1)}(u) = -\frac{cu}{(u^2 + 1)^2}, \quad F^{(1,1)}(u) = \frac{c}{u^3}\left(\frac{2u^3 + u}{(u^2 + 1)^2} - \tan^{-1}(u)\right)$$

$$F^{(0,2)}(u) = \frac{1}{2}c \left(3 \tan^{-1}(u) - \frac{-5u^2 - 3}{u^3 + u}\right), \quad F^{(1,2)}(u) = \frac{3c}{2u^4}\left(u\left(\frac{1}{u^2 + 1} + 2\right) - 3\tan^{-1}(u)\right) - \frac{cu}{u^3 + 1})^2$$

$$F^{(2,2)}(u) = \frac{c}{2u^3}\left(3(u^2 + 9) \tan^{-1}(u) - \frac{u(19u^6 + 48u^2 + 27)}{(u^2 + 1)^2}\right).$$

3.2.1 Linearly-anisotropic scattering

Combining Eq.(55) with Eq.(43) we find

$$h^{(0)} = \frac{2cu(-bcu + bc \tan^{-1}(u) + u^3)}{\pi bcu(c - 2u^2 - 1) + bc\left(u^2 + 1\right)^2 - c} \tan^{-1}(u) + u^3\left(\left(u^2 + 1\right)^2 - c\right)$$

$$\tan^{-1}(u) + u^3\left(\left(u^2 + 1\right)^2 - c\right)$$

yielding scalar collision-rate density

$$C_c(r) = \frac{1}{2\pi^2r} \int_0^{\infty} \frac{u(-bcu + bc \tan^{-1}(u) + u^3)}{bcu(c - 2u^2 - 1) + bc\left(u^2 + 1\right)^2 - c} \tan^{-1}(u) + u^3\left(\left(u^2 + 1\right)^2 - c\right) \sin(ru) \, du.$$
3.3 Gamma-4 random flight in 3D

With \( a = 4 \) we find that the exact fluence about the point source with linearly-anisotropic scattering can be expressed explicitly as a sum of diffusion modes. With the intercollision FPD \( p_c(s) = \frac{1}{8} e^{-s^3} \) we find

\[
F^{(0,0)}(u) = -\frac{c \left( u^2 - 3 \right)}{3 (u^2 + 1)^3}, \quad F^{(0,1)}(u) = -\frac{4cu}{3 (u^2 + 1)^3}, \quad F^{(1,1)}(u) = \frac{c - 3cu^2}{3 (u^2 + 1)^3}, \quad F^{(0,2)}(u) = \frac{4cu^2}{3 (u^2 + 1)^3}
\]

\[
F^{(1,2)}(u) = \frac{c \left( 9 \tan^{-1}(u) - \frac{u (2u^2 + 4uv + 27)}{u^2 + 1} \right)}{6u^2}, \quad F^{(2,2)}(u) = \frac{c \left( u (11u^2 + 60u^4 + 72u^6 + 27) - 27 \tan^{-1}(u) \right)}{3u^2}.
\] (58)

Combining Eq.(58) with Eq.(43) we find

\[
h^{(0)} = -\frac{2cu (bc + u^4 - 2u^2 - 3)}{\pi \left( bc^2 + c \left( u^2 + 1 \right) \left( b (3u^2 - 1) + u^2 - 3 \right) \right)}
\] (59)

yielding scalar collision-rate density

\[
C_c(r) = \frac{1}{2\pi^2 r} \int_0^\infty \frac{u \left( bc + u^4 - 2u^2 - 3 \right)}{\left( bc^2 + c \left( u^2 + 1 \right) \left( b (3u^2 - 1) + u^2 - 3 \right) \right) \sin(ru) du}.
\] (60)

A comparison of this result to Monte Carlo reference is provided in Figure 7. The complete scalar collision rate density can be solved by standard contour integration ([Grosjean 1963], pp.73–75), yielding

\[
C_c(r) = \sum_{v \in v^+} \frac{e^{-c \int_0^r u \sin(ru) du}}{4\pi^2 2c (b (2c + 3v^4 - 3) + v^4 + 4v^2 - 5)}
\] (61)

where \( v^+ \) is the set of roots with positive real part of the dispersion equation

\[
c^2 \left( v^4 + 10v^2 + 1 \right) - c \left( 11v^4 + 26v^2 + 11 \right) \left( 1 - v^2 \right) + 10 \left( 1 - v^2 \right)^8 = 0
\] (62)

for which we found two real and two complex roots in \( v^+ \). Figure 7 also includes comparisons of this exact result to two forms of moment-preserving diffusion approximation found using the methods in [d’Eon 2019]. For the Classical diffusion approximation, we find

\[
C_c(r) \approx \frac{3 - bc}{8\pi r (bc + 5)} e^{-\sqrt{\frac{c^2}{2b} - \frac{bc}{(bc + 5)^2}}}.
\] (63)

and removing the first-collided portion, we find the Grosjean-form diffusion approximation

\[
C_c(r) \approx \frac{e^{-c \int_0^r u \sin(ru) du}}{24\pi^2} + \frac{c \left( e^{-c \int_0^r u \sin(ru) du} \right)}{1 - c 4\pi^2 r^2}, \quad v = \frac{2b(5(c - 2) c + 8) - 30(c - 2)}{3(c - 1)(bc - 3)}.
\] (64)

3.4 Gamma-6 flights in 3D

With intercollision FPD \( p_c(s) = \frac{1}{120} e^{-s^5} \) we find

\[
F^{(0,0)}(u) = \frac{c \left( u^4 - 10u^2 + 5 \right)}{5 (u^2 + 1)^3}, \quad F^{(0,1)}(u) = \frac{2cu \left( 3u^2 - 5 \right)}{5 (u^2 + 1)^3}, \quad F^{(1,1)}(u) = \frac{c \left( 5u^4 - 38u^2 + 5 \right)}{15 (u^2 + 1)^3}
\]

\[
F^{(0,2)}(u) = -\frac{2cu^2 \left( u^2 - 7 \right)}{5 (u^2 + 1)^3}, \quad F^{(1,2)}(u) = \frac{4cu \left( 3u^2 - 1 \right)}{5 (u^2 + 1)^3}, \quad F^{(2,2)}(u) = \frac{c \left( 1 - 3u^2 \right)^2}{5 (u^2 + 1)^5}.
\] (65)

3.4.1 Rayleigh scattering

Similar to the gamma-4 case with linearly-anisotropic scattering, with Rayleigh scattering and gamma-6 flights we find a solvable scalar collision rate density. Combining Eq.(65) with Eq.(45) we find

\[
h^{(0)} = -\frac{2cu \left( c \left( u^4 - 10u^2 + 1 \right) - 2 \left( u^2 + 1 \right)^3 \left( u^4 - 10u^2 + 5 \right) \right)}{\pi \left( c^2 \left( u^4 - 10u^2 + 1 \right) - c \left( 11u^4 - 26u^2 + 11 \right) (u^2 + 1)^3 + 10 (u^2 + 1)^8 \right)}
\] (66)
Infinite 3D, isotropic point source, linearly-anisotropic scattering, Gamma-4 random flight - correlated emission

Collision-rate density \( C_c[r] \), \( c = 0.5, b = 0.7 \)

**Figure 7:** Scalar collision-rate density \( C_c[r] \) about an isotropic point source in 3D with linearly-anisotropic scattering and intercollision free-path lengths drawn from \( \frac{1}{5} e^{-s^2} \). Validation of Eq.(60) (continuous) with respect to Monte Carlo (dots). Comparisons to a classical diffusion approximation (Eq.(63), dot-dashed) and modified-diffusion approximation (Eq.(64), dashed) are also shown.

Infinite 3D, isotropic point source, Rayleigh scattering, Gamma-6 random flight - correlated emission

Collision-rate density \( C_0[r] \), \( c = 0.999 \)

**Figure 8:** Scalar collision-rate density \( C_0[r] \) about an isotropic point source in 3D with Rayleigh scattering and intercollision free-path lengths drawn from \( \frac{1}{15} e^{-s^2} \). Validation of Eq.(67) (continuous) with respect to Monte Carlo (dots).

Yielding scalar collision-rate density

\[
C_c(r) = \frac{1}{2\pi^2 r} \int_0^\infty - \frac{u^2 \left(c (u^4 - 10u^2 + 1) - 2 (u^2 + 1)^3 (u^4 - 10u^2 + 5)\right)}{u \left(c^2 (u^4 - 10u^2 + 1) - c (11u^4 - 26u^2 + 11) (u^2 + 1)^3 + 10 (u^2 + 1)^8\right)} \sin(ru) \, du. \tag{67}
\]

A comparison of this result to Monte Carlo reference is provided in Figure 8.

The complete scalar collision rate density is then solved by standard contour integration ([Grosjean 1963], pp.73–75)

\[
C_c(r) = \sum_{\nu \in \nu^+} \frac{(1 - \nu^2) \left(c (\nu^4 + 10\nu^2 + 1) + 2 (\nu^4 + 10\nu^2 + 5) (\nu^2 - 1)^3\right)}{3c \left(2c (\nu^4 + 12\nu^2 + 3) + (\nu^2 + 3) (11\nu^4 + 9) (\nu^2 - 1)^3\right)} e^{-\nu r} \tag{68}
\]

where \( \nu^+ \) is the set of roots with positive real part of the dispersion equation

\[
c^2 (\nu^4 + 10\nu^2 + 1) - c (11\nu^4 + 26\nu^2 + 11) (1 - \nu^2)^3 + 10 (1 - \nu^2)^8 = 0. \tag{69}
\]

In this case we always noted two real roots and six complex roots for a variety of absorption levels \( c \).
4 Conclusion

We have derived the point source Green’s functions for infinite media with anisotropic scattering in non-classical linear transport where the free-path distributions between collisions and attenuation laws are non-exponential. The general solutions are expressed as Fourier inversions, which were validated numerically using gamma random flights in 3D. Distinct solutions for both collision rate, and fluence and their angular counterparts were derived and tested using Monte Carlo. For low integer-order gamma flights and low Legendre orders we found the solutions to be numerically straightforward to manage. Higher order angular expansions and more challenging free-path distribution such as power-law flights [Davis 2006] will require more numerical care when dealing with the oscillatory Fourier inversions. Nevertheless, the provided gamma flight solutions provide important benchmarks for GRT and for validating more efficient approximations, such as SPN [Palmer and Vasques 2020].

5 Acknowledgements

We thank M.M.R. Williams for helpful feedback on the manuscript and Forrest Brown for bringing several older works [Randall 1964; Doub 1961] to our attention.

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