Perturbation theory for the modified nonlinear Schrödinger solitons

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Abstract

The perturbation theory based on the Riemann-Hilbert problem is developed for the modified nonlinear Schrödinger equation which describes the propagation of femtosecond optical pulses in nonlinear single-mode optical fibers. A detailed analysis of the adiabatic approximation to perturbation-induced evolution of the soliton parameters is given. The linear perturbation and the Raman gain are considered as examples.

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1. Introduction

The study of the dynamical processes associated with the propagation of high-power optical pulses in single-mode nonlinear fibers is based as a rule on the integrable nonlinear Schrödinger equation (NLSE) \cite{1}. Various realistic effects accompanying the soliton propagation and destroying the integrability of the NLSE are usually treated as perturbations. There are different approaches to describe analytically perturbation-induced dynamics of the NLS solitons \cite{2-8}, for review see Ref. 9. It is evident that the "quality" of taking into account for the above effects depends crucially on smallness of a parameter responsible for a definite perturbation. Just this situation takes place with the Kerr nonlinearity dispersion effect. Being sufficiently small in the picosecond pulse duration region, it becomes essential for femtosecond solitons, having a parameter of the order $10^{-2} - 10^{-1}$. Hence, strictly speaking, this effect cannot be treated as a perturbation for the femtosecond region of soliton pulse duration.
The natural approach to treat analytically the dynamics of femtosecond solitons is to consider the so-called perturbed modified nonlinear Schrödinger equation (MNLSE) [10]

\begin{equation}
\begin{aligned}
    iq_z + \frac{1}{2}q_{\tau\tau} + i\alpha (|q|^2q)_\tau + |q|^2q = r,
\end{aligned}
\end{equation}

where the term with the real parameter \( \alpha \) governs the effect of the Kerr nonlinearity dispersion (self-steepening) and \( r \) accounts for small effects which we will consider as perturbation. Here \( q(\tau, z) \) is the normalized slowly varying amplitude of the complex field envelope, \( z \) is the normalized propagation distance along the fiber, \( \tau \) is the normalized time measured in a frame of reference moving with the pulse at the group velocity (the retarded time). It is remarkable that MNLSE (1) with zero r.h.s. is still integrable by the inverse scattering transform (IST) method [11], though the linear spectral problem associated with the MNLSE is different from that for the NLSE.

Our primary goal is to develop a simple formalism to treat analytically the femtosecond soliton dynamics governed by Eq. (1). Three points should be stressed which differ our approach from the previously known ones. First, we account for the Kerr nonlinearity dispersion effect exactly. In other words, we do not make any hypothesis about smallness of \( \alpha \) in Eq. (1). Moreover, we consider as a background solution not the sech-like pulse of the NLS type but precisely the MNLS soliton. Finally, the third point is relevant to the formalism, namely, we employ the Riemann-Hilbert (RH) problem, which was proved to be effective for treating perturbations to nonlinear evolution equations integrable by means of the Zakharov-Shabat spectral problem [12-15]. Recently, we developed the RH problem-based approach [16-18] for solving nonlinear equations integrable by the Wadati-Konno-Ichikawa spectral problem [11]. This approach includes the MNLSE and some its generalizations. This development serves as a base for taking into account small perturbations.

The paper is arranged as follows. In Sec. 2 the basic results on the RH problem-based approach to the MNLSE is summarized. In Sec. 3 the general one-soliton solution of the MNLSE is derived in a form which, we believe, is as simple as possible. Here we also discuss the limiting transition to the NLS soliton. In Sec. 4 we obtain the perturbation-induced evolution equations for the RH problem data related to the soliton parameters and discuss peculiarities of the perturbation theory for gauge equivalent equations. Sec. 5 is devoted to the adiabatic approximation. Here we consider as an example the linear perturbation (excess gain or fiber loss) and the Raman self-frequency shift [19-21]. Concluding remarks are contained in the last section.

2. Riemann-Hilbert problem for MNLSE

In this section we summarize the basic results concerning the approach to the MNLSE based on the RH problem. Let us write the MNLSE in the general form

\begin{equation}
\begin{aligned}
    iq_z + \frac{1}{2}q_{\tau\tau} + i\alpha (|q|^2q)_\tau + \beta|q|^2q = 0,
\end{aligned}
\end{equation}

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where $\alpha$ and $\beta$ are real parameters. Eq. (2) is integrable via the IST method and can be represented as the compatibility condition $U_z - V_\tau + [U, V] = 0$ for the following system of two linear matrix equations

$$
\Phi_\tau = \Lambda(k)[\sigma_3, \Phi] + 2ikQ\Phi \equiv U\Phi - \Lambda(k)\Phi\sigma_3,
$$

$$
\Phi_z = \Omega(k)[\sigma_3, \Phi] + \left( \frac{4i}{\alpha} k^3 Q + 2i k^2 Q^2 \sigma_3 - \frac{i \beta}{\alpha} kQ + kQ_\tau \sigma_3 - 2i\alpha kQ^3 \sigma_3 \right) \Phi
\equiv V\Phi - \Omega(k)\Phi\sigma_3.
$$

Here the Hermitian matrix $Q = \begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix}$ represents the potential of the linear spectral problem (3a), $\Lambda(k) = -(2i/\alpha) \left( k^2 - \beta/4 \right)$, $\Omega(k) = -(4i/\alpha^2) \left( k^2 - \beta/4 \right)^2$, the bar stands for complex conjugation and $k$ is a spectral parameter.

As seen from Eq. (3a), the MNLSE (2) belongs to the class of equations integrable by means of the Wadati-Koike-Ichikawa spectral problem [11]. However, Eq. (2) is not canonical among the equations of this class [16]. The canonical equation

$$
i q'z + \frac{1}{2} q''_\tau - i\alpha q'^2 q'_\tau + \beta |q'|^2 q'_\tau + \alpha^2 |q'|^4 q' = 0
$$

(4)
does not admit as obvious a physical interpretation as the MNLSE, but possesses the Lax representation too,

$$
\Phi'_\tau = \Lambda(k)[\sigma_3, \Phi'] + \left( 2i k Q' + i\alpha Q'^2 \sigma_3 \right) \Phi' \equiv U'\Phi' - \Lambda(k)\Phi'\sigma_3,
$$

$$
\Phi'_z = \Omega(k)[\sigma_3, \Phi'] + \left( \frac{4i}{\alpha} k^3 Q' + 2i k^2 Q'^2 \sigma_3 - \frac{i \beta}{\alpha} kQ' + kQ'_\tau \sigma_3 + \frac{\alpha}{2}[Q', Q'_\tau] \right) \Phi' \equiv V'\Phi' - \Omega(k)\Phi'\sigma_3,
$$

(5b)

where $Q' = \begin{pmatrix} 0 & q' \\ \bar{q}' & 0 \end{pmatrix}$. The spectral problem (5a) associated with Eq. (4), as distinct from the spectral problem (3a), is compatible with the canonical normalization condition $\Phi'(k = \infty) = I$, where $I$ is the $2 \times 2$ identity matrix. Eqs. (2) and (4) are gauge equivalent equations interrelated by the following gauge transformation:

$$
Q = g^{-1} Q' g,
$$

(6)

where $g(\tau, z) = \Phi'(k = 0)$. Thereby, solutions of the MNLSE can be obtained from those of Eq. (4) by means of simple algebraic transformation (6). The RH problem formalism can be developed equivalently for either
the MNLSE (2) or Eq. (4), the RH problem data being invariant under the gauge transformation. Because
the formulation of the RH problem with the canonical normalization condition has a number of technical
advantages in calculation of soliton solutions, we will develop the RH formalism for Eq. (4) and give the
transition relations to the MNLSE.

To construct the RH problem associated with Eq. (4), consider the matrix Jost-type solutions \( J'_\pm \) of
Eq. (5a) which satisfy the asymptotic conditions \( J'_\pm \to I \) at \( \tau \to \pm \infty \). By the standard analysis of the
Volterra-type integral equations for \( J'_\pm \) which follow from Eq. (5a) and the above asymptotic properties, we
conclude that the following matrix function ((\( J'_\pm \)) \( l \) means the \( l \)-th column of \( J'_\pm \))

\[
\Phi'_+ (k) = \begin{pmatrix} (J'_+)_{1}(k), (J'_+)_{2}(k) \end{pmatrix},
\]

being a solution of Eq. (5a), is holomorphic in the two quadrants of the complex \( k \)-plane which are defined
by the condition \( \alpha \text{Re}(k) \text{Im}(k) \leq 0 \). The scattering matrix \( S'(k) \) is defined in the usual way:

\[
J'_E = J'_+ ES', \quad E \equiv \exp(\Lambda(k)\sigma_3 \tau).
\]

Note that \( \text{det}S' = 1 \) due to \( \text{det}J'_\pm = 1 \). The Zakharov-Shabat factorization [22] of the scattering matrix,

\[
S'_+ = S'S'_-, \quad S'_+ = \begin{pmatrix} 1 & S'_{12} \\ 0 & S'_{22} \end{pmatrix}, \quad S'_- = \begin{pmatrix} (S'^{-1})_{11} & 0 \\ (S'^{-1})_{21} & 1 \end{pmatrix},
\]

allows us to represent \( \Phi'_+(k) \) in two equivalent forms:

\[
\Phi'_+ = J'_+ ES'_+ E^{-1} = J'_- E S'_- E^{-1}.
\]

Since the potential \( Q' \) is Hermitian, we have the following identities:

\[
J'^{-1}_\pm (k) = J'^{-1}_\pm (\overline{k}), \quad S'^{-1}_\pm (k) = S'^{-1}_\pm (\overline{k}), \quad \text{for } k \in \{ k : \text{Re}(k) \text{Im}(k) = 0 \}.
\]

Hermiticity of the potential also enables to define the matrix function conjugated to \( \Phi'_+(k) \) and holomorphic
in the rest two quadrants of the complex \( k \)-plane, i.e., which are given by the condition \( \alpha \text{Re}(k) \text{Im}(k) \geq 0 \):

\[
\Phi'_-^{-1}(k) = \Phi'^+(\overline{k}) = \begin{pmatrix} (J'^{-1}_-)_{1}(k), (J'^{-1}_-)_{2}(k) \end{pmatrix}^t,
\]

where \( (J'^{-1}_\pm)_{lt} \) denotes the \( l \)-th row of the matrix \( J'^{-1}_\pm \) and superscript \( t \) means transposition. The linear
spectral problem (5a) possesses the parity symmetry [16]. It can be summarized to the following important
identities:

\[
\mathcal{P} J'_\pm (k) = J'_\pm (k), \quad \mathcal{P} S'(k) = S'(k), \quad \mathcal{P} S'_\pm (k) = S'_\pm (k),
\]

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where $\mathcal{P}$ is the parity operator defined by $\mathcal{P} F(k) = F^{(d)}(-k) - F^{(off)}(-k) \equiv \sigma_3 F(-k) \sigma_3$, $F^{(d)}$ and $F^{(off)}$ are diagonal and off-diagonal parts of a matrix $F$. These identities give

$$\mathcal{P} \Phi'(k) = \Phi'(k), \quad \Phi'(k) = \begin{pmatrix} \Phi'_+(k), & k \in \{ k : \alpha \Re(k) \Im(k) \leq 0 \}; \\ \Phi'_-(k), & k \in \{ k : \alpha \Re(k) \Im(k) \geq 0 \}. \end{pmatrix} \quad (12)$$

Here $\Phi'(k)$ is a matrix function piecewise meromorphic in the complex $k$-plane and discontinuous through the curve $k \in \{ k : \Re(k) \Im(k) = 0 \}$.

Let us return to the gauge equivalence between Eqs. (2) and (4). From Eq. (12) taken at $k = 0$ it follows that the gauge transformation matrix $g$ is diagonal, while Eqs. (5a) and (10) lead to the following expression for $g$:

$$g \equiv \Phi'(k = 0) = \begin{pmatrix} \exp \left(-i \alpha \int_{\tau}^\infty \mathrm{d} \tau' |q'|^2 \right) & 0 \\ 0 & \exp \left(-i \alpha \int_{-\infty}^\tau \mathrm{d} \tau' |q'|^2 \right) \end{pmatrix}. \quad (13)$$

The matrix function $\Phi(k)$ which results from $\Phi'(k)$ by means of the transformation

$$\Phi(k) = g^{-1} \Phi'(k), \quad (14)$$

is a solution of the linear problem (3a), possesses the same conjugation and meromorphic properties as $\Phi'(k)$ does, and is an eigenfunction of the parity operator $\mathcal{P}$ too. For the linear problem (3a), the Jost-type solutions $J_\pm(k)$, the scattering matrix $S(k)$, and its factorization $S_\pm(k)$ are constructed similarly to the problem (5a). The matrix function $\Phi(k)$ is given through the Jost-type solutions and the factorization by the the same formulas as $\Phi'(k)$ does (see Eq.(10)). The following relations are valid

$$J_\pm = g^{-1} J'_\pm g_\pm, \quad S = g_+^{-1} S' g_-, \quad S_\pm = g_\pm^{-1} S'_\pm, \quad (15)$$

where $g_\pm = \lim_{\tau \to \pm \infty} g$.  

Now we can formulate the RH problem associated with Eq. (4). Indeed, using Eqs. (7) and (11) as well as the relation $E(k) = E(k)$ for $k \in \{ k : \Re(k) \Im(k) = 0 \}$, we write ($S'_1(k) \equiv S'_1(k)$)

$$\Phi'^{-1}(k) \Phi'_+(k) = E(k) S'_+ \left( \mathcal{T} \right) S'_+(k) E^{-1}(k) = E(k) \begin{pmatrix} 1 & S'_1(k) \\ S'_1(k) & 1 \end{pmatrix} E^{-1}(k) \equiv G(k), \quad (16a)$$

$$\Phi'(k) \to I, \quad k \to \infty, \quad (16b)$$
where \( k \in \{ k : \text{Re}(k) \text{Im}(k) = 0 \} \). It is a problem of analytic factorization of the nondegenerate matrix \( G(k) \) given on the contour defined by the following disconnected oriented set:

\[
C_\alpha = \text{sgn}(\alpha) \left( \{(\infty, 0), (0, 0)\} \cup \{(0, 0), (-\infty, 0)\} \right)
\]

\[
\cup \{(0, 0), (0, i\infty)\} \}
\]

of the \( k \)-plane axes. Here \( \text{sgn}(\alpha) \) means bypassing the contour in the reverse direction for \( \alpha < 0 \). The functions \( \Phi'_{\pm}(k) \) are just a solution of the RH problem (16). The uniqueness of this solution is provided by the canonical normalization condition (16b).

Substituting the asymptotic decomposition of \( \Phi'(k) \) in the inverse power series of \( k \),

\[
\Phi'(k) = 1 + k^{-1}\Phi'^{[1]} + \ldots,
\]

into the spectral equation (5a) and taking advantage of Eq. (12), we reconstruct the potential \( Q' \):

\[
Q' = \frac{1}{\alpha}\sigma_3, \Phi'^{[1]} = 1 \frac{1}{\alpha} \lim_{k \to \infty} k\sigma_3, \Phi'(k). \tag{17}
\]

Hence, to solve the MNLSE (2), we should at first solve the RH problem (16), then obtain the potential \( Q' \) (17) and transform it by Eq. (6). It should be noted that the RH problem (16) remains unchanged under the gauge transformation (14), except for the normalization of the matrix \( \Phi(k) \) at infinity, namely, \( \Phi(k) \to g^{-1} \) at \( k \to \infty \).

In general, the functions \( \det\Phi'_+(k) \) and \( \det\Phi'_-^{-1}(k) \) have zeros in their regions of analyticity, the RH problem being said to be nonregular (or with zeros). It follows from Eq. (11) that zeros of the above determinants are complex conjugate, while the parity symmetry (Eq. (12)) tells us that zeros appear by pairs, i.e., \( \det\Phi'_{\pm}(\pm k_j) = 0 \) for the \( j \)-th zero \( k_j \). The solution of the RH problem with zeros can be factorized [23]

\[
\Phi'_\pm(k) = \Phi'_{o\pm}(k)\Gamma(k), \quad \det\Phi'_{o\pm}(k) \neq 0 \tag{18}
\]

by means of the solution \( \Phi'_{o\pm} \) of the regular RH problem:

\[
(\Phi'_{o-})^{-1}(k)\Phi'_{o+}(k) = \Gamma(k)G(k)\Gamma^{-1}(k), \tag{19a}
\]

\[
\Phi'_{o}(k) \to I, \quad k \to \infty, \tag{19b}
\]

which is posed on the same contour \( C_\alpha \). The matrix function \( \Gamma(k) \) represents the contribution of zeros. In the case of the single pair of zeros \( \pm k_1 \equiv k_{\pm 1} \) this function is given by (see also Ref. 18)

\[
\Gamma = I - \sum_{j,l=\pm 1} \frac{|p_j|D^{-1}_{jl}(p_l)}{k - \bar{k}_l}, \quad \Gamma^{-1} = I + \sum_{j,l=\pm 1} \frac{|p_j|D^{-1}_{jl}(p_l)}{k - k_j}, \tag{20}
\]

6
where \( D_{ln} = (k_n - \bar{k}_l)^{-1}(p_l \mid p_n) \), \( D^{-1} \equiv (D^{-1})_i \), \( \langle p_l \mid p_n \rangle = (\overline{p_l})_1(p_n)_1 + (\overline{p_l})_2(p_n)_2 \), vector-columns \( \langle p_{\pm 1} \rangle \) are related to vector-columns \( \langle p_{\pm 1} \rangle \) by conjugation, i.e., \( \langle p_{\pm 1} \rangle = \overline{\langle p_{\pm 1} \rangle} \), and the latter are given by

\[
\Phi_\pm'(k_{\pm 1}) \mid p_{\pm 1} \rangle = 0. \tag{21}
\]

The parity symmetry immediately gives a relation between \( \mid p_{-1} \rangle \) and \( \mid p_{+1} \rangle \). Indeed, the identity \( \Phi_\pm'(k_1) \mid p_{+1} \rangle = \sigma_3 \Phi'(k_{-1}) \sigma_3 \mid p_{+1} \rangle = 0 \) leads, due to uniqueness of \( \mid p_{-1} \rangle \), to the relation \( \mid p_{-1} \rangle = \sigma_3 \mid p_{+1} \rangle \). From Eq. (11) it follows that \( \langle p_{\pm 1} \mid \Phi_{\pm}^{-1}(k_{\pm 1}) = 0 \). This identity and Eq. (21) ensure that the matrix functions \( \Phi_0'(k) \) and \( (\Phi_0')^{-1}(k) \) are holomorphic in the respective quadrants of the complex \( k \)-plane.

In the general case of the RH problem with \( N \) pairs of zeros of \( \det \Phi'(k) \) the matrix function \( \Gamma(k) \) is given by Eq. (20) with \( j, l \in \{-N, -N + 1, \ldots, -1, 1, \ldots N - 1, N\} \) and \( k_{-s} = -k_s \).

3. Soliton solution of MNLSE

The RH problem data are divided into two parts: discrete data \( \{k_j, \mid p_{j+k}\}, j = 1, \ldots, N \) (\( 2N \) is the whole number of zeros of \( \det \Phi'(k) \)) and continuous datum \( G(k) \). Soliton solutions correspond to the RH problem with zeros provided \( G(k) = 1 \), i.e., \( \Phi'(k) = I \). In other words, only the matrix \( \Gamma(k) \) is responsible for solitons. We will consider the simplest case of a one-soliton solution of the MNLSE (2). We have from Eqs. (6) and (17):

\[
Q = \frac{1}{\alpha} \Gamma^{-1}(k = 0) \lim_{k \to \infty} k[\sigma_3, \Gamma(k)]\Gamma(k = 0), \tag{22}
\]

where the matrices \( \Gamma(k) \) and \( \Gamma^{-1}(k) \) are given by Eq. (18) and it is taken into account that \( g = \Gamma(k = 0) \). As regards the coordinate dependence of the vector-column \( \mid p_{+1} \rangle \), it is determined by differentiation of Eq. (21) with respect to \( \tau \) and \( z \) and taking advantage of Eqs. (5). It should be noted that Eq. (21) determines \( \mid p_{+1} \rangle \) only up to an arbitrary norm. We obtain in a particular case

\[
\mid p_{+1} \rangle = \Lambda(k_1) \sigma_3 \mid p_{+1} \rangle, \quad \mid p_{+1} \rangle = \Omega(k_1) \sigma_3 \mid p_{+1} \rangle. \tag{23}
\]

Integration of the above equations gives

\[
\mid p_{+1} \rangle = e^{f \sigma_3} \mid p^0 \rangle, \quad \mid p^0 \rangle = \begin{pmatrix} p^0_1 \\ p^0_2 \end{pmatrix},
\]

where \( f = \Lambda(k_1) \tau + \Omega(k_1) z \) and \( \mid p^0 \rangle \) is an integration constant determined up to an arbitrary norm. Let us define \( p^0_1/p^0_2 = \exp(a + i\varphi) \), where \( a \) and \( \varphi \) are real constants. Ultimately, we have

\[
\mid p_{+1} \rangle = \begin{pmatrix} \exp(a + i\varphi + f) \\ e^{-f} \end{pmatrix}, \quad f = -\frac{2i}{\alpha} \left( k^2 - \beta \right) \tau - \frac{4i}{\alpha^2} \left( k^2 - \beta \right)^2 z. \tag{24}
\]
Substituting Eq. (24) into Eq. (20), we obtain for $\Gamma(k)$ the following expression

$$
\Gamma(k) = 1 - \frac{2e^a}{k_1^2 - k^2} \left( k_1 e^{i\psi} \left( D^{-1}_{++} + D^{-1}_{-+} \right) - k e^{-i\psi} \left( D^{-1}_{++} - D^{-1}_{-+} \right) \right). 
$$

(25)

Here

$$x = a + 2\text{Re} f = a - \frac{8\xi \eta}{\alpha} \left( \tau + \frac{4}{\alpha} \left[ \xi^2 - \eta^2 - \frac{\beta}{4} \right] z \right),$$

(26)

and $k_1 = \xi - i\eta$ (due to the condition $\alpha \text{Re}(k_1) \text{Im}(k_1) < 0$ we will have $\xi > 0$ and $\eta > 0$ for $\alpha > 0$). From the definition $D_{ln} = (k_n - k_{ln})^{-1} \langle p_l | p_n \rangle$ we derive the following properties of the matrix $D$: $D_{++} = -D_{--}$, $D_{-+} = -D_{+-}$. Hence,

$$D^{-1}_{++} \pm D^{-1}_{-+} = - (D_{++} \mp D_{-+}) (\det D)^{-1} = (D_{++} \mp D_{-+})^{-1}$$

$$= e^{-a} \left( \frac{i}{\eta} \chi x \mp \frac{1}{\xi} \chi x \right)^{-1}. 
$$

(27)

Now, we have all to calculate one-soliton solution $q_s$ of the MNLSE (2). Substituting Eqs. (26) and (27) into Eq. (25), then, in its turn, Eq. (25) into Eq. (22) we obtain

$$q_s = Q_{12} = 8i \frac{\xi \eta}{\alpha} \frac{k_1 e^{-x} + k_1 e^x}{(k_1 e^x + k_1 e^{-x})^2} e^{i\psi}.$$ 

(28)

This is a general form of the one-soliton solutions to the MNLSE (2) which depends on four real parameters $\xi$, $\eta$, $a$, and $\varphi$. The solution (28) is written in terms of the coordinates $x$ and $\psi$ (26) comprising linear combinations of the normalized retarded time $\tau$ and distance along the fiber $z$. It should be stressed that $\alpha$ enters the denominator of the soliton solution (28). In other words, we account nonperturbatively for the pulse self-steepening effect. Similarly to the NLSE, the parameters $a$ and $\psi$ play the role of the initial position and phase, respectively, while the other two parameters $\xi$ and $\eta$ do not admit so obvious interpretation. In any case, we see from Eq. (28) that the normalized half-width $w$ and velocity $v$ of the soliton are represented by

$$w = \alpha \frac{2}{2\xi \eta}, \quad v = \frac{\beta}{\alpha} - \frac{4}{\alpha} (\xi^2 - \eta^2).$$

(29a)
As regards the soliton amplitude $A$, it is natural to admit it in the following form:

$$A = \frac{4\xi \eta}{\alpha (\xi^2 + \eta^2)^{\frac{3}{2}}}.$$  \hfill (29b)

The MNLS soliton (28) has a number of peculiarities which distinguish it from the NLS soliton. First, the MNLS soliton has nonzero phase difference at its limits. Indeed,

$$\frac{k_1 e^{-x} + k_1 e^{x}}{(k_1 e^{x} + k_1 e^{-x})^2} \rightarrow \begin{cases} \frac{k_1^2}{k_1^2}, & x \to \infty, \\ \frac{k_1}{k_1^2}, & x \to -\infty \end{cases}$$

and the said phase difference reads

$$\Delta \psi = \arg(q_s(z \to \infty)) - \arg(q_s(z \to -\infty)) = 6\arg(k_1).$$

Further, the integral of the soliton amplitude

$$\int_{-\infty}^{\infty} d\tau |q_s|^2 = \frac{\pi}{2(\xi^2 + \eta^2)^{\frac{3}{2}}} \sum_{n=0}^{\infty} \left( \frac{\alpha}{\pi} \right)^n \frac{2\xi \eta}{(\xi^2 + \eta^2)^{\frac{3}{2}}},$$

where $\left( \frac{1}{4} \right)_n \equiv \left( \frac{1}{4} + 1 \right) \left( \frac{1}{4} + 2 \right) \cdots \left( \frac{1}{4} + n - 1 \right)$, depends on the parameters $\xi$ and $\eta$, in contrast to the same integral of the NLS soliton which does not depend on the soliton parameters. Moreover, the important invariant of Eq. (2), namely, the number of particles or the optical energy of the soliton,

$$E = \int_{-\infty}^{\infty} d\tau |q_s|^2 = \frac{4}{\alpha} \arg(k_1), \quad 0 < |\arg(k_1)| < \frac{\pi}{2},$$

has the upper limit $2\pi/|\alpha|$. The phase difference and optical energy of the MNLS soliton are related:

$$\Delta \psi = -3E\alpha/2.$$  

The above properties of the MNLS soliton (28) resemble those of the dark NLS soliton, which also has nonzero phase difference and a relation between its energy and the phase difference [24]. Nevertheless, the soliton (28) with $\beta > 0$ reduces to a bright NLS soliton at $\alpha \to 0$. To carry out this limit one should take into account that the Lax pair for the NLSE should be produced at $\alpha \to 0$ from the Lax pair (3) for the MNLSE. This condition implies that the spectral parameter $k$ depends on $\alpha$ and gives the following prescription:

$$2(k_{MNLS}^2 - \beta/4)/\alpha \to -k_{NLS} \text{ at } \alpha \to 0$$

or

$$\frac{2 \left( \xi^2 - \frac{\beta}{4} \right)}{\alpha} \to -\xi_0, \quad \frac{4\xi \eta}{\alpha} \to \eta_0, \quad \alpha \to 0,$$
where \( k_{\text{MNLS}} = \xi - i\eta \) and \( k_{\text{NLS}} = \xi_0 + i\eta_0 \). In other words, we have the decomposition

\[
\xi = \frac{\sqrt{\beta}}{2} - \frac{\alpha}{2\sqrt{\beta}} \xi_0 + O(\alpha^2), \quad \eta = \frac{\alpha}{2\sqrt{\beta}} \eta_0 + O(\alpha^2), \quad \alpha \to 0,
\]

which transforms the MNLS soliton (28) to the NLS soliton \( q_s = (2i\eta_0/\sqrt{\beta}) e^{i\psi_0} \text{sech} x_0 \), where \( \psi_0 = \varphi + 2\xi_0 \tau - 2(\xi_0^2 - \eta_0^2) z \), \( x_0 = a - 2\eta_0 (\tau - 2\xi_0 z) \).

4. Perturbation-induced evolution of RH problem data

Following our strategy, to find correction to the soliton solution of the MNLSE caused by a perturbation, we should at first derive the perturbation-induced evolution equations for the RH problem data. Decomposition of these equations in the asymptotic power series with respect to the perturbation will produce the consequent corrections to the soliton solution.

In Sec. II we have seen that the gauge equivalent Eqs. (2) and (4) have resembling IST schemes with simple mutual relations (15). As the RH problem data are invariant under the gauge transformation (6), we can choose between the two IST formulations the most convenient one for calculation of corrections to soliton solutions. Consider eq. (2) with a small perturbation at the r.h.s. of the equation, i.e., Eq. (1). The perturbation causes a variation \( \delta U \) of the potential, what, in its turn, leads to a variation \( \delta J^\pm \) of the Jost-type solutions. From the spectral problem (3a) we obtain the following equation for \( \delta J^\pm \)

\[
(\delta J^\pm)_\tau = -\Lambda(k)\delta J^\pm \sigma_3 + \delta U J^\pm + U \delta J^\pm, \quad \lim_{\tau \to \pm \infty} (J^\pm_1 \delta J^\pm) = 0.
\]

Therefore,

\[
\delta J^\pm = J^\pm E \left( \int_{\pm \infty}^\tau d\tau E^{-1} J^\pm_1 \delta U J^\pm E \right) E^{-1},
\]

where \( \delta U = (\delta U/\delta z) \delta z \). It should be stressed that the same representation of the variation \( \delta J^\pm_1 \) of the Jost-type solutions to the spectral problem (5a) follows from Eq. (31) by the simple substitutions \( U \to U' \) and \( J^\pm \to J'^\pm \) and such a procedure can be carried out at any step below.

Now we introduce a useful matrix function

\[
\gamma(\pm \infty, \tau) = \int_{\pm \infty}^\tau d\tau E^{-1} \Phi_1 \frac{\delta U}{\delta z} \Phi_1 E.
\]

Then the variation derivative \( \delta J^\pm/\delta z \) takes the form

\[
\frac{\delta J^\pm}{\delta z} = J^\pm ES^\pm \gamma(\pm \infty, \tau) S^-_1 E^{-1}.
\]
From the evident relation $S = \lim_{\tau \to \infty} E^{-1} J_\tau E$ we obtain the variation of the scattering matrix:

$$\frac{\delta S}{\delta z} = S_+ \gamma(-\infty, \infty) S_\tau^{-1},$$

(34)

as well as of its factorization (9):

$$\frac{\delta S_+}{\delta z} = S_+ \gamma(-\infty, \infty) M_{22}, \quad M_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

(35)

What regards the variation $\delta \Phi_+(k)$ for $k \in \{\text{Re}(k)\text{Im}(k) = 0\}$, we obtain from Eqs. (10), (33), and (35):

$$\delta \Phi_+(k) = \frac{\delta \Phi_+(k)}{\delta z} = \Phi_+(k) E \left( \Phi_+(k) - \frac{1}{E} \delta \Phi_+(k)/\delta z \right) E^{-1},$$

(36)

where $M_{11} = \text{diag}(1, 0)$. Hence, in the case of perturbation the evolution equation for $\Phi_+$ with respect to $z$ (3a) gains an additional term responsible for the perturbation:

$$\Phi_+(k) = \Phi_+(k) E \left( \Phi_+(k) - \frac{1}{E} \delta \Phi_+(k)/\delta z \right) E^{-1},$$

(37)

Eq. (36) was derived for $k$ belonging to the curve $\{\text{Re}(k)\text{Im}(k) = 0\}$, involved in the formulation of the RH problem, but it can be analytically continued into the domain $\{k : \alpha\text{Re}(k)\text{Im}(k) \leq 0\}$, where $\Phi_+(k)$ is holomorphic. Indeed, from the linear problem (3a) we get

$$(\Phi_+^{-1} \delta \Phi_+)_\tau = \Phi_+^{-1} \delta U \Phi_+ + \Lambda(k)[\sigma_3, \Phi_+^{-1} \delta \Phi_+].$$

Hence, the integrand of Eq. (32) is represented as $(E^{-1} \Phi_+^{-1} (\delta \Phi_+ / \delta z) E)_\tau$ and the matrix function $\gamma(\pm\infty, \tau)$ reads

$$\gamma(\pm\infty, \tau) = E^{-1} \Phi_+^{-1} \frac{\delta \Phi_+}{\delta z} E - S_\tau^{-1} \frac{\delta S_\tau}{\delta z},$$

(38)

where we have used Eq. (10) to compute the limits $\tau \to \pm\infty$. Now, in virtue of the explicit structure of the matrices $S_\pm$ (9) and analyticity of their diagonal elements in the domain $\{k : \alpha\text{Re}(k)\text{Im}(k) \leq 0\}$, we conclude
that the entries of the matrix Π(±k) (37) are functions meromorphic in this domain and having simple poles at the zeros of the RH problem, i.e., at the zeros of detΦ+(k). Analiticity of Φ+(k) in the above domain gives immediately an important identity

\[ \Phi_+(±k_j)E(±k_j)\text{Res}\{Π(k),±k_j\} = 0, \]  

(39)

where \text{Res}\{·,±k_j\} stands for the residue at \( k = ±k_j \), \( \det\Phi_+(±k_j) = 0 \).

In order to derive the perturbation-induced evolution of the RH problem data, let us consider for simplicity the case of a single pair of zeros \( ±k_1 \). The full set of the independent data involves in this case \( k_1, |p+1|, G(k) \). By the definition of \( k_1 \), we have

\[
0 = \frac{d}{dz}\det\Phi_+(k_1) = \frac{∂}{∂z}\det\Phi_+(k_1) + \frac{dk_1}{dz}\left(\frac{∂}{∂k}\det\Phi_+(k)ight)_{k=k_1} \\
= \left(\text{trΠ}(k)\det\Phi_+(k)\right)_{k=k_1} + \frac{dk_1}{dz}\left(\frac{∂}{∂k}\det\Phi_+(k)\right)_{k=k_1} .
\]

Here we took advantage of Eq. (36) and of the identity \( \text{trV} = 0 \). Decomposition of \( Π(k) \) into the regular and singular parts at \( k = k_1 \),

\[ Π(k) = Π_r(k) + \frac{\text{Res}\{Π(k),k_1\}}{k - k_1}, \]  

(40)

and use of the formula \( \det\Phi_+ = \det\Phi_0 + \detΓ = \det\Phi_0 + (k^2 - k_1^2)(k^2 - k_2^2)^{-1} \) which follows from Eqs. (14), (18), and (20) with \( \Phi_o ≡ g^{-1}\Phi_o' \) help us to compute the limit \( k → k_1 \):

\[
\frac{dk_1}{dz} = -\left(\text{trΠ}(k) + (k - k_1)^{-1}\text{Res}\{Π(k),k_1\}\right) \left(\frac{\partial}{\partial k}\right) \left(\frac{1}{(k^2 - k_1^2)(k^2 - k_2^2)^{-1}\det\Phi_0 + (k)}\right)_{k=k_1} \]

\( = -\text{Res}\{\text{trΠ}(k),k_1\} \).  

(41)

The perturbation-induced evolution equation for the vector \( |p+1| \) can be obtained by differentiation of Eq. (21):

\[
\left(\frac{d}{dz}\Phi_+(k)\right)_{k=k_1} |p+1| + \Phi_+(k_1)\frac{d}{dz}\left|p+1\right| = 0.
\]  

(42)

Here the full derivative of \( \Phi_+(k) \) with respect to \( z \) is given, due to Eq. (36), as follows:

\[
\left(\frac{d}{dz}\Phi_+(k)\right)_{k=k_1} |p+1| = \left(\frac{∂}{∂z}\Phi_+(k)\right)_{k=k_1} |p+1| + \frac{dk_1}{dz}\left(\frac{∂}{∂k}\Phi_+(k)\right)_{k=k_1} |p+1|
\]
where we have used the identity (21) to exclude an evidently vanishing term: \( V(k_1)\Phi_+(k_1) \mid p_{+1} \rangle = 0. \)

Turning again to Eqs. (36) and (40), we obtain the following relation:

\[
[(k - k_1)\Phi_+(k)\zeta(k)]_{k=k_1} \mid p_{+1} \rangle = E(k_1)\text{Res}\{\Pi(k), k_1\}E^{-1}(k) \mid p_{+1} \rangle.
\]

On the other hand, the l.h.s. of this relation can be transformed in that way:

\[
[(k - k_1)\Phi_+(k)\zeta(k)]_{k=k_1} \mid p_{+1} \rangle = -\frac{dk_1}{dz} \mid p_{+1} \rangle
\]

where we have once again used Eq. (21). Hence, we have derived an important identity

\[
E(k_1)\text{Res}\{\Pi(k), k_1\}E^{-1}(k) \mid p_{+1} \rangle = -\frac{dk_1}{dz} \mid p_{+1} \rangle.
\]

Using now Eq. (39) we can write

\[
\Phi_+(k)E(k)\frac{\text{Res}\{\Pi(k), k_1\}}{k - k_1}E^{-1}(k) \mid p_{+1} \rangle,
\]

\[
= \left( \frac{\partial}{\partial k} \Phi_+(k) \right)_{k=k_1} E(k_1)\text{Res}\{\Pi(k), k_1\}E^{-1}(k) \mid p_{+1} \rangle.
\]

Collecting Eqs. (42)-(45), we obtain the perturbation-induced evolution of the vector \( \mid p_{+1} \rangle \),

\[
\frac{d}{dz} \mid p_{+1} \rangle = \Omega(k_1)\sigma_3 \mid p_{+1} \rangle - E(k_1)\Pi_r(k_1)E^{-1}(k_1) \mid p_{+1} \rangle,
\]

or, for \( \tau \)-independent vector \( \mid p^{(o)} \rangle \equiv F^{-1}(k_1)E^{-1}(k_1) \mid p_{+1} \rangle \), where \( F(k) \equiv \exp \left( \int dz\Omega(k)\sigma_3 \right) \),

\[
\frac{d}{dz} \mid p^{(o)} \rangle = -F^{-1}(k_1)\Pi_r(k_1)F(k_1) \mid p^{(o)} \rangle.
\]
Finally, we should derive an evolution equation for the continuous datum \( G(k) \). Eqs. (36), (11), (16) (written for the quantities without prime), and (36) give

\[
G_z(k) = \Omega(k) [\sigma_3, G(k)] + G(k) E(k) \Pi(k) E^{-1}(k) + E(k) \Pi^\dagger(k) E^{-1}(k) G(k).
\]

This equation is simplified by introducing the \( \tau \)-independent matrix \( G_0 \equiv F^{-1} E^{-1} G F \):

\[
G_0^\tau(k) = G_0(k) F^{-1}(k) \Pi(k) F(k) + F^{-1}(k) \Pi^\dagger(k) F(k) G_0(k).
\] (47)

It is evident that \( \tau \)-independence of the l.h.s. of Eqs. (41), (46), and (47) enables us to simplify further these equations by taking one of the limits \( \tau \to \pm \infty \), at which the fundamental matrix \( \Pi(k) \) (37) has only one nonzero column.

As was pointed above, we can use the gauge equivalent IST schemes for the derivation of perturbation-induced evolution equations for the RH problem data, due to the gauge invariance of the latter. Indeed, from Eqs. (38), (14), and (15), using the parity symmetry, one can easily find the following relation between the gauge equivalent fundamental matrices:

\[
\Pi'(k) = \Pi(k) - E^{-1}(k) \Phi_+^{-1}(k) \Pi^{(d)}(k \to \infty) \Phi_+(k) E(k)
\]

\[
= \Pi(k) + E^{-1}(k) \Phi_+^{-1}(k) \left( \begin{array}{cc} \gamma_{11}(\tau, \infty) & 0 \\ 0 & -\gamma_{22}(\infty, \tau) \end{array} \right)_{k \to \infty} \Phi_+(k) E(k),
\]

where \( \Pi'(k) \) is associated with the canonical equation (4). It is easy to see that the second term in the r.h.s. of this expression for \( \Pi'(k) \) gives no contribution to the evolution equations (41), (46), and (47). For instance, \( \text{Res} \{ \text{tr} \Pi'(k), k_1 \} = \text{Res} \{ \text{tr} \Pi(k), k_1 \} \). This once again ensures that one can use either of the gauge equivalent fundamental matrices \( \Pi \) or \( \Pi' \) expressed through \( \Phi_+, \delta U/\delta z \) or \( \Phi'_+, \delta U'/\delta z \), respectively, for calculation of corrections. It will be more convenient to use the former fundamental matrix \( \Pi \), because precisely \( \delta U_{12}/\delta z \) is proportional to the r.h.s. of Eq. (1), treated as the perturbation to the MNLSE.

5. Adiabatic approximation

It should be stressed that the evolution equations (41), (46), and (47) of the preceding section are exact, but just this circumstance prevents their direct use for concrete calculations. They include unknown function \( \Phi_+(k) \), a solution of the perturbed equation (36). The way to determine \( \Phi_+(k) \) explicitly defines the sort of the approximation used. Here we develop the simplest variant to account for perturbation, the so called adiabatic approximation. In the adiabatic approximation, soliton shape is considered to be unchanged instantaneously under the action of a small perturbation, while the soliton parameters, being constant in the integrable case, acquire slow \( z \)-dependence. In other words, we impose the condition \( G(k) = I \) and,
consequently, $\Phi'_o+(k) = I$ (19a). Then, Eqs. (14), (13) and (18) give $\Phi_{+}(k) = g^{-1}\Phi'_o(k) = \Gamma^{-1}(k = 0)\Gamma(k)$. Let us define the perturbation matrix $R$ as

$$\frac{\delta U}{\delta z} = 2ik\frac{\delta Q}{\delta z} = 2kR. \quad (48)$$

Then the r.h.s. of Eq. (1) is nothing but $R_{12}$. As was mentioned above, taking the limit $\tau \to \pm \infty$ considerably simplifies the evolution equations (41) and (46) for soliton parameters. From Eq. (32) we obtain

$$\gamma(k) \equiv \gamma(-\infty, \infty) = 2k \int_{-\infty}^{\infty} d\tau E^{-1}(k)\Gamma^{-1}(k = 0)\Gamma(k = 0)\Gamma(k)E(k). \quad (49)$$

At the limit $\tau \to \infty$ the fundamental matrix $\Pi(k)$ defined by Eq. (37) takes the form

$$\Pi_{\tau \to \infty}(k) = \begin{pmatrix} 0 & \gamma_{12}(k) \\ 0 & \gamma_{22}(k) \end{pmatrix}_{\tau \to \infty}. \quad (50)$$

With this representation for $\Pi(k)$, the evolution equation (41) is written as $dk_1/d\tau = -\text{Res}\{\gamma_{22}(k), k_1\}$. Taking into account Eqs. (25) and (27), from Eqs. (41) and (49) we obtain

$$\frac{dk_1}{dz} = i\alpha k_1^2 \int_{-\infty}^{\infty} dx e^{x} \left( r_{o}(x) + r_{o}(-x) \right), \quad (51)$$

where $r_{o}(x, z) \equiv e^{-i\psi(x, z)}r(x, z)$ and the variables $x$ and $\psi$ are given by Eqs. (26) with account of possible $z$-dependence of the soliton parameters $\xi$ and $\eta$:

$$x = a - \frac{8\xi \eta}{\alpha} \tau - \frac{32}{\alpha^2} \int dz \xi \eta \left( \xi^2 - \eta^2 - \frac{\beta}{4} \right),$$

$$\psi = \varphi - \frac{4}{\alpha} \left\{ \left( \xi^2 - \eta^2 - \frac{\beta}{4} \right) \tau + \frac{2}{\alpha} \int dz \left[ \left( \xi^2 - \eta^2 - \frac{\beta}{4} \right)^2 - 4\xi^2\eta^2 \right] \right\}. \quad (52)$$

For the soliton parameters $\xi$ and $\eta$, $k_1 = \xi - i\eta$, we get from Eq. (51):

$$\frac{d\xi}{dz} = i\alpha \frac{1}{2} \int_{-\infty}^{\infty} dx \frac{k_1^2 e^{x} - k_1 e^{-x}}{(k_1 e^{-x} + k_1 e^{x})^2} (r_{o}(x) + r_{o}(-x)), \quad (52)$$

where

$$r_{o}(x) + r_{o}(-x) = \left( \begin{array}{c} r_{o}(x) \\ r_{o}(-x) \end{array} \right) \left( \begin{array}{c} k_1 e^{-x} + k_1 e^{x} \\ k_1 e^{-x} - k_1 e^{x} \end{array} \right).$$

\[
\frac{d\eta}{dz} = -\frac{\alpha}{2} \int_{-\infty}^{\infty} dx \frac{k_1^2 e^x + \overline{k}_1^2 e^{-x}}{\left(k_1 e^{-x} + \overline{k}_1 e^x\right)^2} \left(r_o(x) + \overline{r}_o(-x)\right). \tag{53}
\]

It is noteworthy that only the symmetric part of the perturbation, i.e., such that \(\overline{r}_o(-x, z) = r_o(x, z)\), changes the soliton parameters \(\xi\) and \(\eta\).

In order to obtain the adiabatic evolution for the rest two soliton parameters \(a\) and \(\varphi\), we turn to Eq. (46). Rewriting it by entries with the matrix \(\Pi\) given by Eq. (50) and taking into account Eq. (40), we obtain

\[
\frac{dp^0_1}{dz} = \exp\left(-2 \int dz \Omega(k_1)\right) \gamma_{r12}(k_1) p^2_1, \quad \frac{dp^0_2}{dz} = -\gamma_{r22}(k_1) p^0_2
\]

and for \(p^0_1/p^0_2 \equiv e^{a+iv}:

\[
\frac{d}{dz} e^{a+iv} = e^{a+iv} \gamma_{r22}(k_1) - \exp\left(-2 \int dz \Omega(k_1)\right) \gamma_{r12}(k_1). \tag{54}
\]

Now from Eq. (54) we obtain

\[
\frac{da}{dz} = -\frac{\alpha}{2\xi \eta} \int_{-\infty}^{\infty} dx \frac{d}{dx} \left(k_1^2 e^x + \overline{k}_1^2 e^{-x}\right) \left(C \left(k_1^3 e^x + \overline{k}_1^3 e^{-x}\right) \left(r_o(x) + \overline{r}_o(-x)\right)\right)
\]

Now we should extract the regular part of \(\gamma(k)\) according to the decomposition \(\gamma_2(k) = \gamma_{r2}(k) + \text{Res} \{\gamma_2(k), k_1\} (k - k_1)^{-1}\) (40). After long but not difficult calculations with account of the explicit expression for \(\Gamma(k)\) (25) we get

\[
\gamma_{r12}(k_1) = \frac{\alpha k_1}{2\xi \eta} \exp\left(2 \int dz \Omega(k_1)\right) e^{a+iv} \int_{-\infty}^{\infty} dx \frac{d}{dx} \left(k_1 e^{-x} + \overline{k}_1 e^x\right)\]

\[
\times \left[\left(2k_1^2 C - 3i\xi \eta\right) (r_o(x) + \overline{r}_o(-x)) + (\xi^2 - \eta^2 - 2k_1^2) (r_o(x) - \overline{r}_o(-x)) \right] e^x
\]

\[
+ 2|k_1|^2 r_o(x) e^{-x}\right] = \frac{\alpha k_1}{2\xi \eta} \int_{-\infty}^{\infty} dx \frac{r_o(x) + \overline{r}_o(-x)}{\left(k_1 e^{-x} + \overline{k}_1 e^x\right)^2} \left(|k_1|^2 e^{-x} - 3i\xi \eta e^x\right).
\]

Now from Eq. (54) we obtain

\[
\frac{da}{dz} = -\frac{\alpha}{2\xi \eta} \int_{-\infty}^{\infty} dx \frac{d}{dx} \frac{\left(k_1^3 e^x + \overline{k}_1^3 e^{-x}\right) (r_o(x) + \overline{r}_o(-x))}{\left(k_1 e^{-x} + \overline{k}_1 e^x\right)^2} C\left(k_1^3 e^x + \overline{k}_1^3 e^{-x}\right) (r_o(x) + \overline{r}_o(-x))
\]

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\[-\left[ i\xi \eta \left( k_1 e^x + \overline{k}_1 e^{-x} \right) + x \left( k_1^2 e^x + \overline{k}_1^2 e^{-x} \right) \right] \left( r_o(x) - \tau_o(-x) \right), \quad (55) \]

\[
\frac{d\varphi}{dz} = \frac{i\alpha}{2\xi \eta} \int_{-\infty}^{\infty} \frac{dx}{(k_1 e^{-x} + \overline{k}_1 e^x)^2} \left\{ C \left( k_1^3 e^x - \overline{k}_1^3 e^{-x} \right) \left( r_o(x) + \tau_o(-x) \right) \right. \\
\left. \left[ k_1 e^x \left( \overline{k}_1^2 - i\xi \eta \right) + \overline{k}_1 e^{-x} \left( k_1^2 + i\xi \eta \right) - x \left( k_1^3 e^x - \overline{k}_1^3 e^{-x} \right) \right] \left( r_o(x) - \tau_o(-x) \right) \right\}. \quad (56) \]

For the case of the symmetric perturbation, i.e., when \( r_o(-x, z) = r_o(x, z) \) holds, the evolution of \( a \) and \( \varphi \) is determined by the evolution of \( k_1 \):

\[
\frac{da}{dz} = -\frac{C}{\xi \eta} \text{Im} \left\{ k_1 \frac{dk_1}{dz} \right\}, \quad \frac{d\varphi}{dz} = \frac{C}{\xi \eta} \text{Re} \left\{ k_1 \frac{dk_1}{dz} \right\}. \quad (57) \]

Hence, in the symmetric case the adiabatic evolution of the soliton parameters is determined by a single integral involved in the r.h.s. of Eq. (51). It is remarkable that the symmetry condition \( r_o(-x, z) = r_o(x, z) \) is rather natural for the MNLSE, because the soliton (28) itself satisfies this condition: \( q_s(-x, z)e^{-i\psi} = q_s(x, z)e^{-i\psi} \).

**Example**

As an example of using the above formalism, we analyse here in the adiabatic approximation the effect of two different perturbations to the soliton propagation. We will consider a perturbation comprising two terms, namely, the linear perturbation which describes the fiber loss or excess linear gain and the Raman driving which is responsible for the soliton self-frequency shift effect [19-21]. Such a choice of perturbation types does not in any way exhaust the set of all perturbations relevant for the femtosecond soliton propagation in optical fibers and serves mainly as an illustration of the perturbation theory developed above. The results will be presented in a form suitable for numerical calculations.

Let us write the perturbation in the following form:

\[
r = i\varepsilon_1 q + \varepsilon_2 |q|^2 q, \quad (58)\]

where \( \varepsilon_1 \) and \( \varepsilon_2 \) are small parameters characterizing the linear gain (\( \varepsilon_1 > 0 \)) or the fiber losses (\( \varepsilon_1 < 0 \)) and the Raman driving, respectively. Substituting the soliton solution (28) into Eq. (58), we obtain the characteristic perturbation \( r_o \) in the form

\[
r_o = -4\varepsilon_1 \frac{\xi \eta}{\alpha} \frac{k_1 e^{-x} + \overline{k}_1 e^x}{(k_1 e^x + \overline{k}_1 e^{-x})^2}.
\]

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\[ +i\varepsilon_2 \left( \frac{8\xi\eta}{\alpha} \right)^4 \frac{e^{2x} - e^{-2x}}{(k_1 e^x + \bar{k}_1 e^{-x})^4 (k_1 e^x + \bar{k}_1 e^{-x})}. \]  

(59)

Note that this perturbation is symmetric in the sense of the preceding section, i.e., \( r_o(-x, z) = r_o(x, z) \). Hence, evolution of the soliton parameters \( a \) and \( \varphi \) is determined by evolution of \( k_1 \) in accordance with Eq. (57). With the perturbation (59) Eq. (51) takes surprisingly simple form:

\[ \frac{dk_1}{dz} = -\varepsilon_1 k_1 \left( 1 - 2\theta \cot(2\theta) - 2i\theta \right) \]

\[ -16\varepsilon_2 \frac{|k_1|^4}{\alpha^3} \left( 1 - 2\theta \cot(2\theta) - \frac{1}{3}\sin^2(2\theta) \right), \]  

(60)

with

\[ \frac{d\theta}{dz} = 2\varepsilon_1 \theta. \]  

(61)

Here \( \theta \equiv \arctan(\eta/\xi) \), \( 0 < \theta < \pi/2 \), \( \cot(2\theta) = (\xi^2 - \eta^2)/(2\xi\eta) \), and \( \sin(2\theta) = 2\xi\eta/ (\xi^2 + \eta^2) \).

It should be noted that the parameter \( \alpha \) enters Eq. (60) in a denominator representing thereby a non-perturbative account of selfsteepening of the soliton. In the limit (30) Eq. (60) reduces to the well-known equations [5] for the NLS soliton parameters. From Eq. (60) one can derive evolution equations for the soliton parameters (29):

\[ \frac{dw^{-1}}{dz} = -\left[ \varepsilon_1 \left( 1 - 2\theta \cot(2\theta) \right) + 32\varepsilon_2 \frac{|k_1|^4}{\alpha^3} \left( 1 - 2\theta \cot(2\theta) - \frac{1}{3}\sin^2(2\theta) \right) \right] w^{-1} \]

\[ + \frac{\varepsilon_1 \theta}{2} \left( v - \frac{\beta}{\alpha} \right), \]

\[ \frac{dv}{dz} = -\varepsilon_1 \left( 1 - 2\theta \cot(2\theta) \right) + 32\varepsilon_2 \frac{|k_1|^4}{\alpha^3} \left( 1 - 2\theta \cot(2\theta) - \frac{1}{3}\sin^2(2\theta) \right) \left( v - \frac{\beta}{\alpha} \right) \]

\[ -8\varepsilon_1 \theta w^{-1}, \]

\[ \frac{dA}{dz} = \left[ \varepsilon_1 \left( 6\theta \cot(2\theta) - 1 \right) - 16\varepsilon_2 \frac{|k_1|^4}{\alpha^3} \left( 1 - 2\theta \cot(2\theta) - \frac{1}{3}\sin^2(2\theta) \right) \right] A. \]

It follows from Eqs. (61) that the soliton energy \( E = \int_{-\infty}^{\infty} d\tau |q|^2 = 4\theta/|\alpha| \) does not depend on the Raman driving:

\[ \frac{dE}{dz} = 2\varepsilon_1 E, \]
just as in the case of the NLSE [5].

6. Conclusion

We have developed a perturbation theory for the MNLS soliton. It has been shown recently [25] that the MNLSE utilizing the notion of slowly varying envelope is still valid up to 3 - 5 periods of field oscillations within the envelope. Our formalism is based on the RH problem associated with the Wadati-Konno-Ichikawa linear spectral problem. We have demonstrated previously the efficiency of the similar approach to the perturbed Manakov system [15], as well as to nonlinear evolution equations integrable by the $N \times N$ Zakharov-Shabat spectral problem [14]. It is not mere a chance that the formalisms for both linear and quadratic spectral bundles resemble each other. Indeed, the main idea behind the RH-based approach consists in the analysis of analytical behavior of the Jost-type solutions to the spectral problem. Besides, we have stressed the gauge invariance of the RH problem data. Just this property is responsible for the definite unification of the objects used in both cases. In the recent paper [26] the RH problem has been used for studying asymptotics of the MNLSE solution associated with the continuous spectral data.

As regards the examples of perturbations considered above, we confine ourselves to the adiabatic approximation. This restriction is by no means concerned with the present approach. Moreover, we have derived the perturbation-induced evolution equation (47) for the continuous RH problem datum. It is precisely this equation which is necessary for the description of the soliton shape distortion and emission of linear waves. This problem will be solved in a forthcoming paper.

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