REAL STRUCTURES AND THE Pin$^-(2)$-MONOPOLE EQUATIONS

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Abstract. We investigate the Pin$^-(2)$-monopole invariants of symplectic 4-manifolds and Kähler surfaces with real structures. We prove the nonvanishing theorem for real symplectic 4-manifolds which is an analogue of Taubes’ nonvanishing theorem of the Seiberg-Witten invariants for symplectic 4-manifolds. Furthermore, the Kobayashi-Hitchin type correspondence for real Kähler surfaces is given.

1. Introduction

In the study of the Seiberg-Witten invariants, the computations of the invariants of Kähler surfaces are fundamental [3,6,7,16,27]. These are based on a certain type of Kobayashi-Hitchin correspondence. On the other hand, Taubes’ works on the Seiberg-Witten theory on symplectic 4-manifolds begin with the non-triviality theorem [23] for the canonical Spin$^c$ structure.

The purpose of this paper is to prove the theorems for Pin$^-(2)$-monopole invariants [18] parallel to the above results. The Pin$^-(2)$-monopole equations [17,18] are a variant of the Seiberg-Witten equations twisted along a local system or a double cover. In general, the Pin$^-(2)$-monopole theory is related with the Seiberg-Witten theory on the double cover. In fact, there exists an anti-linear involution $I$ on the Spin$^c$ structure on the double cover, and the Pin$^-(2)$-monopole theory can be considered as the $I$-invariant part of the Seiberg-Witten theory on the double cover. Our results are on Kähler surfaces and symplectic 4-manifolds with real structure. We start from the observation that the aforementioned $I$-action can be understood through the real structure.

Let us state our results more precisely. Let $(X,\omega,\iota)$ be a closed real symplectic 4-manifold, which is a triple consisting of a closed smooth 4-manifold $X$, a symplectic form $\omega$ and an involution $\iota$ on $X$ such that $\iota^*\omega = -\omega$. Let $J$ be a compatible almost complex structure such that $\iota^* \circ J = -J \circ \iota$, and $K$ the canonical complex line bundle associated with $J$. We assume $(X,\omega,\iota)$ has empty real part, that is, the involution $\iota$ is free. Let $\tilde{X}$ be the quotient manifold $X/\iota$ and $\pi: X \to \tilde{X}$ the projection. Since $\iota$ induces an anti-linear involution on $K$, the quotient bundle $\tilde{K} = K/\iota$ is a nonorientable $\mathbb{R}^2$ bundle. Let $\ell = X \times_{\{\pm 1\}} \mathbb{Z}$ be the local system ($\mathbb{Z}$-bundle) associated to the double cover $X \to \tilde{X}$ and set $\ell_R = \ell \otimes \mathbb{R} = \det \tilde{K}$.

In general, Pin$^-(2)$-monopole equations are defined on a Spin$^c$- structure, which is a Pin$^-(2)$-analogue of Spin$^c$ structure. In the situation above, we

2010 Mathematics Subject Classification. 57R57, 53C55, 53D05.
define the canonical Spin$^c$ structure $\hat{s}_0$ on $X \to \hat{X}$ (1.2). The following theorem is an analogue of Taubes’ nonvanishing theorem \[23\].

**Theorem 1.1.** Suppose

1. $w_2(\hat{X}) + w_2(\hat{K}) + w_1(\ell_{\mathbb{R}})^2 = 0$,
2. $\pi^*: H^1(\hat{X}; \mathbb{Z}_2) \to H^1(X; \mathbb{Z}_2)$ is surjective.

Then there exists a unique canonical Spin$^c$ structure $\hat{s}_0$ on $X \to \hat{X}$. Furthermore, suppose $b_1^+ = \dim H^+(\hat{X}; \ell_{\mathbb{R}}) \geq 2$. Then the Pin$^-(2)$-monopole invariant $SW^{\text{Pin}}(\hat{X}, \hat{s}_0)$ is $\pm 1$.

**Remark 1.2.** We refer the readers to \[17\-18\] for the generality of the Pin$^-(2)$-monopole theory. In general, Pin$^-(2)$-monopole invariants are defined as $\mathbb{Z}_2$-valued invariants. However $\mathbb{Z}$-valued invariants can be defined in some special situations, e.g., in the case when the moduli space is 0-dimensional and orientable. Theorem \[1.1\] is true for both cases, that is, the $\mathbb{Z}_2$-valued invariant for $s_0$ is 1(\(\neq 0\)) in $\mathbb{Z}_2$, and, if defined, the $\mathbb{Z}$-valued invariant is $\pm 1$ in $\mathbb{Z}$.

As in the ordinary Seiberg-Witten case, there is a symmetry of conjugation in the Pin$^-(2)$-monopole theory (1.3). On the other hand, the anti-canonical Spin$^c$ structure $\hat{s}_0 \otimes \hat{K}$ is defined as a Spin$^c$ structure obtained by twisting $\hat{s}_0$ by $\hat{K}$. Theorem \[1.1\] with Corollary \[2.14\] immediately implies the following.

**Corollary 1.3.** $SW^{\text{Pin}}(\hat{X}, \hat{s}_0 \otimes \hat{K}) = \pm 1$.

For a Spin$^c$ structure, it is associated an O(2) bundle $\hat{L}$ called characteristic bundle with characteristic class $c_1(\hat{L}) \in H^2(\hat{X}; \ell)$. Since $\iota^*\omega = -\omega$, there is a $\ell_{\mathbb{R}}$-valued self-dual closed 2-form $\hat{\omega} \in \Omega^2(\hat{X}; \ell_{\mathbb{R}})$ such that $\pi^*\hat{\omega} = \omega$. The following is an analogue of [24, Theorem 2].

**Theorem 1.4.** Under the assumptions of Theorem \[1.1\], if the Pin$^-(2)$-monopole invariant for a Spin$^c$ structure $\hat{s}$ on $X \to \hat{X}$ is nonzero, then its characteristic bundle $\hat{L}$ satisfies

\[
\left| c_1(\hat{L}) \cdot [\hat{\omega}] \right| \leq c_1(\hat{K}) \cdot [\hat{\omega}],
\]

and the virtual dimension $d(\hat{s})$ of the moduli space is 0.

Suppose further that $(X, \omega)$ is a compact Kähler surface and $\iota$ is an anti-holomorphic free involution. In such a case, a certain kind of Kobayashi-Hitchin correspondence is proved \[14\]. In fact, the Pin$^-(2)$-monopole moduli space for $\hat{X}$ can be identified with the $I$-invariant part of the spaces of simple holomorphic pairs consisting of holomorphic structures on a line bundle with nonzero holomorphic sections, or, effective divisors on $X$. By using such descriptions, we can compute the Pin$^-(2)$-monopole invariants for the quotient manifolds of several kind of Kähler surfaces. The following is an analogue of \[16\, Theorem 7.4.1\].

**Theorem 1.6.** Let $X$ be a minimal Kähler surface of general type. Suppose $\iota: X \to X$ is an anti-holomorphic involution without fixed points satisfying
the assumptions in Theorem [1.1]. Then

\[
SW^{\text{Pin}}(\hat{X}, \hat{s}) = \begin{cases} 
\pm 1 & \hat{s} = \hat{s}_0 \text{ or } \hat{s}_0 \otimes \hat{K} \\
0 & \text{otherwise}
\end{cases}
\]

Remark 1.7. Theorem [1.6] is true for \(\mathbb{Z}_2\) and \(\mathbb{Z}\)-valued invariants.

A series of concrete examples for Theorem 1.6 is given by hypersurfaces in \(\mathbb{C}P^3\) with complex conjugation (§5.1). We also give some computations for elliptic surfaces in §5.2.

Acknowledgements. The author is supported in part by Grant-in-Aid for Scientific Research (C) 25400096.

2. Spin\(^c\) structures induced from the real structure

2.1. Reduction of the frame bundle. Recall the isomorphism \(U(2) \cong (U(1) \times SU(2))/\{\pm 1\}\). Define the group \(\hat{U}(2)\) by

\[
\hat{U}(2) = (\text{Pin}^- (2) \times SU(2))/\{\pm 1\}.
\]

Then \(\hat{U}(2)/\text{Pin}^- (2) = SO(3), \hat{U}(2)/SU(2) = O(2)\), the identity component of \(\hat{U}(2)\) is \(U(2)\), and \(\hat{U}(2)/U(2) = \{\pm 1\}\). We have an exact sequence

\[
1 \rightarrow \{\pm 1\} \rightarrow \hat{U}(2) \rightarrow O(2) \times SO(3) \rightarrow 1.
\]

Note that \(\hat{U}(2)\) is embedded in \(SO(4)\) as

\[
\hat{U}(2) = \frac{\text{Pin}^- (2) \times SU(2)}{\{\pm 1\}} \subset \frac{SU(2) \times SU(2)}{\{\pm 1\}} = SO(4).
\]

Suppose we have a manifold \(\hat{Y}\) with a double cover \(Y \rightarrow \hat{Y}\) and a principal \(\hat{U}(2)\)-bundle \(P\) over \(\hat{Y}\) such that \(P/U(2) \cong Y\). Then an \(O(2)\)-bundle \(P_O\) such that \(P_O/\text{SO}(2) \cong Y\) and an \(SO(3)\)-bundle \(P_S\) are associated via the homomorphism \(\sigma\) in (2.1). Conversely, the following holds

Proposition 2.3. For a double covering \(Y \rightarrow \hat{Y}\), let \(\ell_\mathbb{R} = Y \times \{\pm 1\} \mathbb{R}\) and suppose an \(O(2)\)-bundle \(P_O\) such that \(P_O/\text{SO}(2) \cong Y\) and an \(SO(3)\)-bundle \(P_S\) are given. If \(w_2(P_O) + w_1(\ell_\mathbb{R})^2 = w_2(P_S)\), then there exists a \(U(2)\)-bundle \(P\) such that

\[
P/\text{Pin}^- (2) \cong P_S, \quad P/SU(2) \cong P_O, \quad P/U(2) \cong Y.
\]

Proof. (Cf. [17], Proposition 11.) Note that the image of \(\text{Pin}^- (2) \subset \text{Sp}(1) = \text{Spin}(3)\) by the canonical homomorphism \(\text{Spin}(3) \rightarrow \text{SO}(3)\) is a copy of \(O(2)\) embedded in \(\text{SO}(3)\). The embedding \(O(3) \subset \text{SO}(3)\) is given by \(A \mapsto A \oplus \det A\). Embed \(O(2) \times \text{SO}(3)\) into \(\text{SO}(6)\) by using this embedding. Then we have a commutative diagram

\[
\begin{array}{cccccc}
1 & \rightarrow & \{\pm 1\} & \rightarrow & \hat{U}(2) & \rightarrow & O(2) \times SO(3) & \rightarrow & 1 \\
& & \| & & \downarrow & & \downarrow & & \\
1 & \rightarrow & \{\pm 1\} & \rightarrow & \text{Spin}(6) & \rightarrow & \text{SO}(6) & \rightarrow & 1.
\end{array}
\]
The diagram leads to a commutative diagram of fibrations
\[ K(\mathbb{Z}_2,1) \longrightarrow B\hat{U}(2) \longrightarrow BO(2) \times BSO(3) \longrightarrow K(\mathbb{Z}_2,2) \]
\[ K(\mathbb{Z}_2,1) \longrightarrow BSpin(6) \longrightarrow BSO(6) \xrightarrow{w_2} K(\mathbb{Z}_2,2). \]
From these, we see that
\[ w_2(P_S \oplus P_O \oplus \det P_O) = w_2(P_S) + w_2(P_O) + w_1(\ell_R)^2 = 0 \]
is the required condition. \qed

Remark 2.4. The choice of \( P \) is not unique. The possibility of \( P \) is parametrized by \( H^1(\hat{Y};\mathbb{Z}_2) \).

Recall the embedding
\[ U(2) = (U(1) \times SU(2))/\{\pm 1\} \subset (SU(2) \times SU(2))/\{\pm 1\} = SO(4) \]
and a commutative diagram
\[ 
\begin{array}{ccc}
1 & \longrightarrow & \{\pm 1\} \\
\| & & \| \\
1 & \longrightarrow & SO(4) \\
\| & & \| \\
1 & \longrightarrow & SO(3) \times SO(3) & \xrightarrow{1} & 1.
\end{array}
\]
Let \((X,\omega,J)\) be a symplectic 4-manifold with compatible almost complex structure \( J \). Fixing a Hermitian metric on \( TX \), we obtain a \( U(2) \) reduction \( P_F \) of the \( SO(4) \)-frame bundle. Then a \( U(1) \)-bundle \( P_K \) and an \( SO(3) \)-bundle \( P_S \) are associated via the homomorphism \( \sigma' \). Let \( \Lambda = \Lambda^{2,0}(X) \) and \( K^{-1} = \Lambda^{0,2}(X) \) be respectively the canonical and anti-canonical line bundles associated with the almost complex structure \( J \). Note that \( \Lambda^+(X) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}\omega \oplus K \oplus K^{-1} \). Then we can identify
\[ P_K \times_{U(1)} \mathbb{C} \cong K \cong K^{-1} \]
as real vector bundles. We assume \( P_K \times_{U(1)} \mathbb{C} = K \). On the other hand,
\[ \Lambda^- (X) \cong P_S \times_{SO(3)} \mathbb{R}^3. \]
Let \((X,\omega,\iota)\) be a closed real symplectic 4-manifold without real part. Then \( X \) admits an almost complex structure \( J \) compatible to \( \omega \) such that \( \iota\circ J = -J \circ \iota \). Fixing such a \( J \), we have a \( U(2) \) reduction \( P_F \) of the \( SO(4) \)-frame bundle. Let \( P_K \) and \( P_S \) be the induced \( U(1) \) and \( SO(3) \) bundles. Let \( \hat{X} \) be the quotient manifold \( \hat{X} = X/\iota \) and \( \ell_R = X \times \{\pm 1\} \mathbb{R} \). The involution \( \iota \) induces a bundle automorphism \( \iota \) of \( P_S \) such that \( \iota^2 = 1 \), and its quotient bundle \( \hat{P}_S = P_S/\iota \) over \( \hat{X} \) has the property that
\[ \hat{P}_S \times_{SO(3)} \mathbb{R}^3 = \Lambda^-(\hat{X}). \]
On the other hand, \( \iota \) does not induce a bundle automorphism on \( P_K \) since \( \iota \) is not complex linear. However \( \iota \) induces an anti-linear involution on the canonical bundle \( K = P_K \times_{U(1)} \mathbb{C} \). Then the quotient bundle \( \hat{K} = K/\iota \) is a nonorientable \( \mathbb{R}^2 \) bundle over \( \hat{X} \) such that \( \det \hat{K} = \ell_R \). Let \( \hat{P}_K \) be the \( O(2) \)-bundle over \( \hat{X} \) of orthogonal frames on \( \hat{K} \). By Proposition 2.3 we have a \( U(2) \)-bundle \( \hat{P} \) which induces \( \hat{P}_S \) and \( \hat{P}_K \) if \( w_2(\hat{X}) = w_2(\hat{K}) + w_1(\ell_R)^2 \).
Note that the $\{\pm 1\}$-bundle $\hat{P}/U(2) \to \hat{X}$ is isomorphic to $\pi: X \to \hat{X}$. Fix an isomorphism between them. Then $\hat{P} \to P/\text{U}(2)$ can be considered as a $\text{U}(2)$-bundle over $X$. This $\text{U}(2)$-bundle $\hat{P} \to P/\text{U}(2) = X$ is denoted by $P'$.

**Proposition 2.5.** Suppose

1. $w_2(\hat{X}) + w_2(\hat{K}) + w_1(\mathbb{R})^2 = 0$,
2. $\pi^*: H^1(\hat{X};\mathbb{Z}_2) \to H^1(X;\mathbb{Z}_2)$ is surjective.

Then we can take a $\text{U}(2)$-bundle $\hat{P} \to \hat{X}$ such that

\[ \hat{P}/\text{Pin}^-(2) \cong \hat{P}_S, \quad \hat{P}/\text{SU}(2) \cong \hat{P}_K, \quad \hat{P}/\text{U}(2) \cong X, \quad P' \cong P_F. \]

Furthermore $\hat{T} = \hat{P} \times_{\text{U}(2)} \mathbb{R}^4$ is isomorphic to $T\hat{X}$, where $\hat{T}$ is defined via the embedding $\text{(2.2)}$.

**Proof.** By the proof of Proposition 2.3 we see that the set of isomorphism classes of $\text{U}(2)$-bundle $\hat{P}$ which induces the fixed $\hat{P}_K$ and $\hat{P}_S$ is parametrized by $H^1(\hat{X};\mathbb{Z}_2)$. If a choice of $\hat{P}$ is given, then every other choice is obtained by tensoring a real line bundle. Similarly, the set of isomorphism classes of $\text{U}(2)$-bundle $P_F$ which induces the fixed $P_K$ and $P_S$ is parametrized by $H^1(X;\mathbb{Z}_2)$. Now suppose a choice of $\hat{P}$ is given. Then it follows from the construction that the $\text{U}(2)$-bundle $P'$ induces $P_K$ and $P_S$. Thus the difference between $P_F$ and $P'$ is given by an element of $H^1(X;\mathbb{Z}_2)$. Under the assumption, the difference can be annihilated by tensoring an appropriate real line bundle over $X$ with given $\hat{P}$.

Since $\pi^*\hat{P} \times_{\text{U}(2)} \mathbb{R}^4 = P' \times_{\text{U}(2)} \mathbb{R}^4 = P_F \times_{\text{U}(2)} \mathbb{R}^4 = TX$, we have $\pi^*\hat{T} \cong TX \cong \pi^*T\hat{X}$. From this, it follows that $e(\hat{T}) = e(T\hat{X})$ and $p_1(\hat{T}) = p_1(T\hat{X})$. Consider the homomorphisms $\hat{U}(2) \xrightarrow{\sigma} O(2) \times SO(3) \xrightarrow{\pi} SO(3)$ where $p$ is the projection to the second factor. Then the composite map $p \circ \sigma: \hat{U}(2) \to SO(3)$ factors through $\hat{U}(2) \hookrightarrow SO(4) \to SO(3)$. Then we have a commutative diagram

\[
\begin{array}{ccc}
B\hat{U}(2) & \xrightarrow{\tau} & BSO(3) \\
\downarrow & & \downarrow \\
BSO(4) & & 
\end{array}
\]

From this, it follows that $w_2(\hat{T}) = w_2(\hat{P}_S) = w_2(\hat{X})$. Therefore $\hat{T} \cong T\hat{X}$. \qed

**Remark 2.7.** The choice of $\hat{P}$ is not unique. The possibility of $\hat{P}$ is parametrized by $\ker(\pi^*: H^1(\hat{X};\mathbb{Z}_2) \to H^1(X;\mathbb{Z}_2))$.

2.2. **Canonical Spin$^c$-structure.** Recall that the canonical Spin$^c$ structure $\mathfrak{g}_0$ over $X$ with respect to the almost complex structure $J$ is defined from the $U(2)$-reduction $P_F$, and it has the positive spinor bundle $W_0^+$ of the form $W_0^+ = \mathbb{C} \oplus K^{-1}$. In this subsection, we define the canonical Spin$^c$-structure over $X \to \hat{X}$ induced from the real structure on $X$.

Recall that

\[
\text{Spin}^c(4) = \frac{\text{SU}(2) \times \text{SU}(2) \times \text{Pin}^-(2)}{\{\pm 1\}} = \frac{\text{Sp}(1) \times \text{Sp}(1) \times \text{Pin}^-(2)}{\{\pm 1\}}.
\]
A Spin$^c$-structure $\mathfrak{s}$ on $X \to \hat{X}$ consists of a Spin$^c$-(4)-bundle $Q$ over $\hat{X}$, an isomorphism of $\mathbb{Z}/2$-bundles $Q/\text{Spin}^c(4) \cong X$, and an isomorphism between the SO(4)-frame bundle and $Q/\text{Pin}^-(2)$. The O(2)-bundle $L = Q/\text{Spin}(4)$ is called the characteristic bundle of $\mathfrak{s}$. It has a $\ell$-coefficient orientation and its Euler class is denoted by $\tilde{c}_1(L) \in H^2(\hat{X}; \ell)$. We often make no distinction between $L$ and its associated $\mathbb{R}^2$-bundle. Let $\mathbb{H}_\pm$ be the Spin$^c$-(4) modules which are copies of $\mathbb{H}$ as vector spaces such that the action of $[q_+, q_-, u] \in \text{Spin}^c(4) = (\text{Sp}(1) \times \text{Sp}(1) \times \text{Pin}^-(2))/\{\pm 1\}$ on $\phi \in \mathbb{H}_\pm$ is given by $q_\pm \phi u^{-1}$. Then the associated bundles $W_\pm = Q \times_{\text{Spin}^c(4)} \mathbb{H}_\pm$ are the spinor bundles of $\mathfrak{s}$.

Note that the embedding $\hat{U}(2) \hookrightarrow \text{SO}(4)$ factors through another embedding $\varepsilon : \hat{U}(2) \hookrightarrow \text{Spin}^c(4)$ which is defined by

$$\varepsilon : \hat{U}(2) = \frac{\text{Pin}^-(2) \times SU(2)}{\{\pm 1\}} \to \frac{SU(2) \times SU(2) \times \text{Pin}^-(2)}{\{\pm 1\}} = \text{Spin}^c(4),$$

$$(u, q) \mapsto (u, q, u).$$

For a $\hat{U}(2)$-bundle $\hat{P}$ as in Proposition 2.5 a Spin$^c$-structure $\mathfrak{s}$ over $X$ whose characteristic $\text{O}(2)$-bundle is $\hat{P}_K$ is defined via the embedding $\varepsilon$. That is, the Spin$^c$-(4)-bundle $Q$ of $\mathfrak{s}$ is given by

$$Q = \hat{P} \times_{\hat{U}(2)} \text{Spin}^c(4),$$

and the positive spinor bundle $\hat{W}^+$ is defined by the adjoint action of $\text{Pin}^-(2)$ on the space of quaternions $\mathbb{H} = \mathbb{C} \oplus j\mathbb{C}$:

$$\hat{W}^+ = Q \times_{\text{Spin}^c(4)} \mathbb{H}_+ = \hat{P} \times_{\text{Pin}^-(2)} \mathbb{H}.$$

For $u \in U(1)$ and $z \in \mathbb{C}$, the adjoint action is given by

$$\text{ad}_u(z) = uz u^{-1} = z,$$

$$\text{ad}_{ju}(z) = jzu^{-1}j^{-1} = \bar{z},$$

$$\text{ad}_u(jz) = u^2j\bar{z} = u^2\bar{z}j,$$

$$\text{ad}_{ju}(jz) = u^{-2}j\bar{z} = u^{-2}\bar{z}j.$$

This action preserves the components $\mathbb{C}$ and $j\mathbb{C}$. It follows from (2.8) that $\hat{W}^+$ is decomposed into the direct sum of two $\mathbb{R}^2$ bundles as $\hat{W}^+ = \hat{E}_1 \oplus \hat{E}_2$ such that $\det \hat{E}_1 = \det \hat{E}_2 = \ell_\mathbb{R}$. Define the $\mathbb{R}^2$-bundle $\mathbb{C}$ by $\mathbb{C} = X \times_{\{\pm 1\}} \mathbb{C}$, where $\{\pm 1\}$ acts on $\mathbb{C}$ by complex conjugation. Note that $\mathbb{C} = \mathbb{R} \oplus \ell_\mathbb{R}$. Since $\pi^*\hat{W}^+ = W_0^+ = \mathbb{C} \oplus K^{-1}$, we see that $\hat{W}^+$ has a form of

$$\hat{W}^+ = (\mathbb{C} \oplus \hat{K}^{-1}) \otimes \lambda',$$

where $\hat{K}^{-1} = (K^{-1})/\ell$ (which is the characteristic bundle of the Spin$^c$-structure) and $\lambda'$ is a real line bundle over $\hat{X}$ with $\pi^*\lambda'$ trivial. Note that tensoring $\lambda'$ to $\hat{P}$ changes $\hat{W}^+$ into $\hat{W}_0^+ = \mathbb{C} \oplus \hat{K}^{-1}$. Now we define the canonical Spin$^c$-structure

**Definition 2.9.** A Spin$^c$-structure $\mathfrak{s}_0$ on $X \to \hat{X}$ is **canonical** if it is defined from a $\hat{U}(2)$-bundle $\hat{P}$ satisfying (2.6) and its positive spinor bundle $\hat{W}_0^+$ has a form of
\[ \hat{W}_0^+ = \hat{\mathcal{C}} \oplus \hat{K}^{-1}. \]

The discussion above immediately implies the following.

**Corollary 2.10.** Suppose (1) and (2) in Theorem 1.1. Then there exists a unique canonical Spin\(^c\) structure on \( X \rightarrow \hat{X} \).

Recall \( \mathbb{R}^2 \)-bundles \( \hat{E} \) such that \( \det \hat{E} = \ell_\mathbb{R} \) with \( \ell_\mathbb{R} \)-coefficient orientation are classified by \( \tilde{c}_1(\hat{E}) \in H^2(\hat{X}; \ell) \). We call an \( \mathbb{R}^2 \)-bundle \( \hat{E} \) such that \( \det \hat{E} = \ell_\mathbb{R} \) an \( \mathbb{R}^2 \)-bundle twisted along \( \ell_\mathbb{R} \). When \( \mathbb{R}^2 \)-bundles \( \hat{E}_1 \) and \( \hat{E}_2 \) twisted along \( \ell_\mathbb{R} \) are given, there exists another \( \mathbb{R}^2 \)-bundle \( \hat{E} \) twisted along \( \ell_\mathbb{R} \) such that \( \tilde{c}_1(\hat{E}) = \tilde{c}_1(\hat{E}_1) + \tilde{c}_1(\hat{E}_2) \), which can be considered as a “twisted tensor product” of \( \hat{E}_1 \) and \( \hat{E}_2 \). We write \( \hat{E} = \hat{E}_1 \hat{\otimes} \hat{E}_2 \).

On the other hand, if \( X \rightarrow \hat{X} \) admits a Spin\(^c\) structure, then the set of equivalence classes of Spin\(^c\) structures is also parametrized by \( H^2(X; \ell) \). When a Spin\(^c\) structure is given, the other Spin\(^c\) structures are given by “tensoring” an \( \mathbb{R}^2 \)-bundle \( \hat{E} \) twisted along \( \ell_\mathbb{R} \). In fact, when a canonical Spin\(^c\) structure \( \hat{s}_0 \) is given, the Spin\(^c\) structure made from \( \hat{s}_0 \) and \( \hat{E} \) has the positive spinor bundle

\[ \hat{W} = \hat{E} \oplus (\hat{E} \hat{\otimes} \hat{K}^{-1}). \]

Such a Spin\(^c\) structure is denoted by \( \hat{s}_0 \hat{\otimes} \hat{E} \).

**Definition 2.11.** The Spin\(^c\) structure \( \hat{s}_0 \hat{\otimes} \hat{K} \) is called the anti-canonical Spin\(^c\) structure. This has the spinor bundle of the form

\[ \hat{W} = \hat{K} \oplus \hat{\mathcal{K}}. \]

**Remark 2.12.** If we pull back a Spin\(^c\) structure \( \hat{s} \) over \( \pi: X \rightarrow \hat{X} \) to \( X \), the pulled-back Spin\(^c\) structure \( \pi^* \hat{s} \) has two Spin\(^c\) reductions, and one of them is the canonical reduction \cite[§2.4]{18}. Then it can be seen that the canonical reduction of the pull-back \( \pi^* \hat{s}_0 \) of the canonical Spin\(^c\) structure \( \hat{s}_0 \) is the canonical Spin\(^c\) structure \( s_0 \) on \( X \), and the canonical reduction of \( \pi^* (\hat{s}_0 \hat{\otimes} \hat{K}) \) is the anti-canonical Spin\(^c\) structure \( s_0 \hat{\otimes} K \).

### 2.3. A symmetry in the Pin\(^-(2)\)-monopole theory

It is well-known that there is a symmetry of complex conjugation in the Seiberg-Witten theory \cite[§6.8]{16}. This subsection explains a similar symmetry in the Pin\(^-(2)\)-monopole theory. The conjugation of a quaternion \( z \in \mathbb{H} \) is given by

\[ z = a + ib + jc + kd \mapsto \bar{z} = a - ib - jc - kd. \]

Define the conjugation \( \alpha: \text{Spin}^c(-4) \rightarrow \text{Spin}^c(-4) \) by

\[ \alpha([q, z]) = [q, \bar{z}] \text{ for } [q, z] \in \text{Spin}^c(-4) = \text{Spin}(4) \times \{ \pm 1 \} \text{ Pin}^-(2). \]

For a Spin\(^c\)(-4)-bundle \( P \), let \( P^c \) be the Spin\(^c\)(-4)-bundle such that the total space is same with \( P \), but the action of Spin\(^c\)(-4) is given by \( p \cdot \alpha(q) \) for \( p \in P = P^c \) and \( q \in \text{Spin}^c(-4) \).

For a Spin\(^c\) structure \( \hat{s} \) with Spin\(^c\)(-4)-bundle \( P \), we obtain another Spin\(^c\) structure \( \hat{s}^c \) by replacing \( P \) in \( \hat{s} \) with \( P^c \). We call \( \hat{s}^c \) the conjugate of \( \hat{s} \).
Recall that $\ell_{\mathbb{R}}$-oriented $\mathbb{R}^2$-bundles $\hat{E}$ twisted along $\ell_{\mathbb{R}}$ are classified by $\hat{c}_1(\hat{E})$. For such an $\hat{E}$, let $\hat{E}^c$ be an $\mathbb{R}^2$-bundle such that $\hat{c}_1(\hat{E}^c) = -\hat{c}_1(\hat{E})$. We collect several facts on conjugate which can be easily seen.

**Proposition 2.13.** For a Spin$^c$-structure $\hat{s}$ and its conjugate $\hat{s}^c$, we have the following:

1. If $\hat{L}$ is the characteristic bundle for $\hat{s}$, then $\hat{L}^c$ can be identified with the characteristic bundle of $\hat{s}^c$. In particular, $\hat{c}_1(\hat{L}^c) = -\hat{c}_1(\hat{L})$.
2. For an $\mathbb{R}^2$-bundle $E$ twisted along $\ell_{\mathbb{R}}$, $(\hat{s} \hat{\otimes} \hat{E})^c = \hat{s}^c \hat{\otimes} \hat{E}^c$.
3. The conjugate of the canonical Spin$^c$-structure is the anti-canonical Spin$^c$-structure, i.e., $\hat{s}_0^c = \hat{s}_0 \hat{\otimes} \hat{K}$.
4. If $s$ is the canonical reduction of $\pi^* \hat{s}$, then the canonical reduction of $\pi^* \hat{s}^c$ is the complex conjugate $\overline{s}$ of $s$.

Let $\hat{s}$ be a Spin$^c$-structure on $\pi: \hat{X} \to \hat{\hat{X}}$ and $s$ the canonical reduction of $\pi^* \hat{s}$. By [17 §4.5] (see also [18 §2.5]), there is an involution $I$ on the Seiberg-Witten theory on $(X, s)$, and a bijective correspondence between the Pin$^-(2)$-monopole solutions on $(\hat{X}, \hat{s})$ and the $I$-invariant Seiberg-Witten solutions on $(X, s)$. Let us recall the relation between the downstairs and upstairs more precisely. Note that $I^* s$ is isomorphic to the complex conjugation $\overline{s}$ of $s$.

The gauge transformation group of the Pin$^-(2)$-monopole theory is given by

$$\hat{\mathcal{G}} = \Gamma(X \times \{ \pm 1 \} U(1)),$$

where $\{ \pm 1 \}$ acts on $U(1)$ by $u \mapsto u^{-1}$. Then $\hat{\mathcal{G}}$ can be identified with the $I$-invariant gauge transformation group on the upstairs $X$. That is, the $I$-action on $\hat{\mathcal{G}} = C^\infty(X, U(1))$ is given by $f \mapsto I^* f$, and we have a natural identification $\hat{\mathcal{G}} = \mathcal{G}^f$.

The Pin$^-(2)$-monopole moduli space is

$$\hat{\mathcal{M}}(\hat{X}, \hat{s}) = \{ \text{Pin}^-(2)\text{-monopole solutions on } \hat{s} \} / \hat{\mathcal{G}}.$$

and this is identified with the $I$-invariant moduli space,

$$\mathcal{M}(X, s)^I = \{ \text{Seiberg-Witten solutions on } s \}^I / \mathcal{G}^I.$$

By Proposition 2.13 we have the identifications,

$$\hat{\mathcal{M}}(\hat{X}, \hat{s}) \cong \mathcal{M}(X, s)^I \cong \mathcal{M}(X, \overline{s})^I \cong \hat{\mathcal{M}}(\hat{X}, \hat{s}^c).$$

The second identification is the isomorphism of complex conjugation in the ordinary Seiberg-Witten theory.

**Corollary 2.14.** $\text{SW}^{\text{Pin}}(\hat{X}, \hat{s}^c) = \pm \text{SW}^{\text{Pin}}(\hat{X}, \hat{s})$. 
3. Real symplectic 4-manifolds

In this section, we prove Theorem 1.1 and Theorem 1.4. Suppose a closed real symplectic 4-manifold \((X, \omega, i)\) satisfies the assumption of Theorem 1.1.

First we discuss about the Pin\(^-\)(2)-monopole equations on the canonical Spin\(^c\) structure. Let \(\tilde{s}_0\) be the canonical Spin\(^c\) structure on \((X, \omega)\) and \(\tilde{s}_0\) the canonical Spin\(^c\) structure on \(X \to \tilde{X}\). Recall \(\tilde{\omega}\) is a \(\ell_\mathbb{R}\)-valued self-dual 2-form such that \(\omega = \pi^* \tilde{\omega}\). Normalize the metric on \(X\) so that \(|\omega| = \sqrt{2}\) and pull it back to \(\tilde{X}\) so that \(|\omega| = \sqrt{2}\). Recall the splitting

\[\Lambda^+(X) \otimes_\mathbb{R} \mathbb{C} = \mathbb{C} \cdot \omega \oplus K \oplus K^{-1}.\]

The Clifford multiplication by \(\omega\) induces the splitting \(W_0^+ = \mathbb{C} \oplus K^{-1}\). In fact, \((\omega/i)\) acts on \(W^+\) as an involution, and \(\mathbb{C}\) and \(K^{-1}\) are (+2) and \((-2)\)-eigenspaces, respectively.

On the Spin\(^c\) structure \(\tilde{s}_0\), we have a twisted Clifford multiplication \(\rho: \Lambda^1(\tilde{X}) \otimes i\ell_\mathbb{R} \to \text{Hom}(W_0^+, W_0^-)\) [17], and this extends to

\[\rho: \Lambda^+(\tilde{X}) \otimes i\ell_\mathbb{R} \to \text{End}(W_0^+).\]

Then \((\tilde{\omega}/i)\) induces the splitting \(\tilde{W}_0^+ = \mathbb{C} \oplus \tilde{K}^{-1}\). Since the real part of \(\mathbb{C}\) is trivial, there is a constant section \(\tilde{u}_0\) such that \(|\tilde{u}_0| = 1\). Mimicking the argument of Taubes [23], we obtain the following.

**Proposition 3.1.** There is a unique O(2)-connection \(\tilde{A}_0\) (up to gauge) on \(\tilde{P}_K\) whose induced covariant derivative \(\nabla_{\tilde{A}_0}\) on \(\tilde{W}^+\) has the property that

\[\left(1 + \frac{1}{2} \rho(\tilde{\omega}/i)\right) \nabla_{\tilde{A}_0} \tilde{u}_0 = 0.\]

Furthermore, \(D_{\tilde{A}_0} \tilde{u}_0 = 0\) if and only if \(d\tilde{\omega} = 0\), where \(D_{\tilde{A}_0}\) is the Dirac operator associated with \(\tilde{A}_0\).

Let us consider the Pin\(^-\)(2)-monopole equations rescaled and perturbed as follows:

\[(3.2) \quad D_{\tilde{A}} \tilde{\phi} = 0, \quad F_{\tilde{A}}^+ = rq(\tilde{\phi}) - \frac{r}{4} i\tilde{\omega} + F_{\tilde{A}_0}^+,\]

where \(\tilde{A}\) is an O(2)-connection on \(\tilde{P}_K\), \(\tilde{\phi} \in \Gamma(\tilde{W}_0^+)\), \(q\) is the quadratic form defined in [17] and \(r\) is a positive real constant. (This is an analogue of Taubes’ perturbation [24].) Then we can see that \((\tilde{A}_0, \tilde{u}_0)\) is a solution to (3.2) for every \(r\).

To proceed further, it is convenient to move to the upstairs and consider the \(I\)-invariant part. Let \((A_0, u_0)\) be the configuration corresponding to \((\tilde{A}_0, \tilde{u}_0)\), i.e., \(u_0 = \pi^* \tilde{u}_0\) and \(A_0\) is the canonical U(1)-reduction of the induced O(2)-connection \(\pi^* \tilde{A}_0\). Then a spinor \(\phi \in \Gamma(W_0^+)\) can be written as \(\phi = \alpha u_0 + \beta\), where \(\alpha\) is a complex-valued function on \(X\) and \(\beta \in \Gamma(K^{-1})\).

Then a solution to the equation (3.2) corresponds to an \(I\)-invariant solution to the perturbed equation due to Taubes [25]:

\[D_A \phi = 0,\]

\[(3.3) \quad F_A^+ - F_{A_0}^+ = -\frac{ir}{8} (1 - |\alpha|^2 + |\beta|^2) \omega + \frac{ir}{4} (\alpha \bar{\beta} + \bar{\alpha} \beta).\]
**Proof of Theorem 1.1.** Certainly, \((A_0, u_0)\) is an \(I\)-invariant solution to \((3.3)\) for every \(r\). Taubes [23, 25](see also Kotschick [14]) proved that there is no solution to \((3.3)\) except \((A_0, u_0)\) for large \(r\). It follows from this that \((\hat{A}_0, \hat{u}_0)\) is a unique solution to \((3.2)\) for large \(r\). These implies Theorem 1.1. □

**Proof of Theorem 1.4.** Suppose the Pin\(^{-}\)(2)-monopole invariant on a Spin\(^{c}\)-structure \(\hat{s}\) is nonzero. Then the equations \((3.2)\) considered on \(\hat{s}\) has a solution for every \(r\). Correspondingly, there is a Spin\(^{c}\) structure \(s\) on \(X\) and an \(I\)-invariant solution to \((3.3)\) for every \(r\). Then, by Kotschick [14](Cf. Taubes [24]), the existence of solutions for large \(r\) implies that

\[
|c_1(L) \cdot [\omega]| \leq c_1(K) \cdot [\omega],
\]

where \(L\) is the determinant line bundle of \((X, s)\). Let \(\hat{L}\) be the characteristic bundle for \((\hat{X}, \hat{s})\). Then \(L = \pi^* \hat{L}\). The inequality \((1.5)\) follows from \((3.4)\).

By [25], we can find an embedded symplectic curve \(C\) in \(X\) such that \(\epsilon = P.D.\{C\}\) satisfies \(\epsilon^2 = c_1(K \otimes L)\). If \(X\) contains embedded 2-spheres with self-intersection number \(-1\), then blowing down them makes a minimal symplectic manifold \(X'\) with another embedded symplectic curve \(C'\) (see, e.g., [20]). Then the proof of Theorem 0.2(6) of [25] implies that the virtual dimension \(d(s)\) of the moduli space for \((X, s)\) is 0. Therefore \(d(\hat{s}) = \frac{1}{2}d(s) = 0\). □

### 4. Real Kähler surfaces

The purpose of this section is to prove that the Pin\(^{-}\)(2)-monopole moduli space on a real Kähler surface can be identified with the \(I\)-invariant moduli space of holomorphic simple pairs, or the space of \(I\)-invariant effective divisors. The moduli space of vortices is also introduced for the intermediate one. The goal of this section is Corollary 4.22 and Corollary 4.26.

Let \((X, \omega, \iota)\) be a compact Kähler surface with anti-holomorphic free involution \(\iota\) such that \(\iota^* \omega = -\omega\). Note that the pull-back of a \((p, q)\)-form by the anti-linear map \(\iota\) is a \((q, p)\)-form, and the complex conjugation of a \((q, p)\)-form is a \((p, q)\)-form. Then the involution \(\iota\) and complex conjugation induce an involution \(I\) on the space of \((p, q)\)-forms \(\Omega^{p,q}(X)\) defined by

\[
I(\alpha) = \overline{\iota^* \alpha}, \quad \alpha \in \Omega^{p,q}(X).
\]

Note that \(K = \Lambda^{2,0}(X)\), \(K^{-1} = \Lambda^{0,2}(X)\), \(\iota^* K = K^{-1}\).

Suppose there is a canonical Spin\(^{c}\)-structure \(\hat{s}_0\) on \(X \to \hat{X} = X/\iota\). As explained in [22], every Spin\(^{c}\)-structure on \(X \to \hat{X}\) is made from \(\hat{s}_0\) and an \(\mathbb{R}^2\)-bundle \(\hat{E}\) twisted along \(\iota_\mathbb{R}\) as \(\hat{s}_0 \otimes \hat{E}\).

For a Spin\(^{c}\)-structure \(\hat{s} = \hat{s}_0 \otimes \hat{E}\) on \(X \to \hat{X}\), there exists a Spin\(^{c}\) structure \(s = s_0 \otimes E\) on \(X\) which is the canonical Spin\(^{c}\) reduction of \(\pi^* \hat{s}\), whose positive spinor bundle is \(W^+ = E \oplus (E \otimes K^{-1})\) such that \(E \cong \pi^* \hat{E}\) as \(\mathbb{R}^2\)-bundles. Note that \(\iota^* s = \hat{s}\). Then \(\iota^* E = \hat{E}\) and \(E\) naturally admits a Hermitian metric \(h\) such that \(\iota^* h = \hat{h}\).

Let \(C\) be a Hermitian connection on \(K^{-1}\) induced by the Chern connection on \(TX\) associated with the Kähler structure.
Recall that the Dirac operator $D$ on the canonical Spin$^c$ structure $s_0$ is identified with

$$D = \sqrt{2}(\partial + \bar{\partial}^*) : \Omega^{0,0}(X) \oplus \Omega^{0,2}(X) \to \Omega^{1,0}(X).$$

Since $\iota$ is anti-holomorphic, the pull-back of $D$ by $\iota$ is

$$\iota^*D = \sqrt{2}(\partial + \bar{\partial}^*) : \Omega^{0,0}(X) \oplus \Omega^{2,0}(X) \to \Omega^{1,0}(X).$$

Then we see that the Dirac operator $D$ is $I$-equivariant.

Next we consider Dirac operators on a Spin$^c$ structure $s = s_0 \otimes E$. For a Hermitian connection $A$ on $\text{det}(W^+) = E^2 \otimes K^{-1}$, there is a unique Hermitian connection $B$ on $E$ such that $A = C \otimes B \otimes \2$. Then the Dirac operator $D_A$ associated with $A$ is identified with

$$D_A = \sqrt{2}(\bar{\partial}_B + \bar{\partial}_B^*) : \Omega^{0,0}(E) \oplus \Omega^{0,2}(E) \to \Omega^{1,0}(E).$$

The pull-back $B' = \iota^*B$ is a Hermitian connection on $\tilde{E} = \iota^*E$, and the pull-back $\iota^*D_A$ can be written as

$$\iota^*D_A = \sqrt{2}(\bar{\partial}_{B'} + \bar{\partial}_{B'}^*) : \Omega^{0,0}(E) \oplus \Omega^{2,0}(E) \to \Omega^{1,0}(E).$$

For an $O(2)$-connection $\hat{B}$ on $\tilde{E}$, we have a Hermitian connection $B$ on $E$ which is the $U(1)$-reduction of $\pi^*B$. Then $B$ is $I$-invariant, i.e., $B = \iota^*B$. For such a connection $B$, the Dirac operator $D_A = \sqrt{2}(\bar{\partial}_B + \bar{\partial}_B^*)$ is also $I$-equivariant.

Recall the identifications:

$$H^2(X; \mathbb{C}) = H^{1,1} \oplus H^{2,0} \oplus H^{0,2}, \quad H_+(X; i\mathbb{R}) = i\mathbb{R}\omega \oplus H^{0,2}.$$ 

Note that the $I = \iota^*(\cdot)$-action preserves $H_+(X; i\mathbb{R})$ and

$$H_+(\hat{X}; \ell_{\mathbb{R}}) \cong H_+(X; i\mathbb{R})^I = i\mathbb{R}\omega \oplus (H^{0,2})^I.$$ 

In particular, we have

**Proposition 4.1.** If $b_2^+ = \text{rank } H_+(\hat{X}; \ell_{\mathbb{R}}) \geq 2$, then $(H^{2,0})^I \cong (H^{0,2})^I \neq 0$.

4.1. **Seiberg-Witten equations.** The Seiberg-Witten equations on Kähler surfaces can be written as follows([10],[27]):

$$\bar{\partial}_B\alpha + \bar{\partial}_B\beta = 0$$

$$2F_B^{0,2} + 2\pi\eta^{0,2} - \frac{1}{2}\beta\bar{\alpha} = 0$$

$$2F_B^{2,0} + 2\pi\eta^{2,0} + \frac{1}{2}\alpha\beta = 0$$

$$\{\Lambda_g(F_B + \pi\eta) - \frac{i}{2}s_g + \frac{i}{8}(|\beta|^2 - |\alpha|^2)\} \omega = 0$$

These are equations for Hermitien connections $B$ on $E$ and sections $(\alpha, \beta) \in (\Omega^{0,0} \oplus \Omega^{0,2})(E)$. The perturbation term is given by $\eta \in \Omega^2(X)$, $\Lambda_g$ denotes the adjoint of the multiplication operator $\omega \wedge \cdot : i\Omega^{0,0} \to i\Omega^{1,1}$, and $s_g$ is the scalar curvature. (Here we use the fact that $i\Lambda_gC = s_g$ for the Chern connection $C$.) If we take an $I$-invariant $\eta$, then (4.2) is $I$-equivariant.

The discussion below is largely indebted to Teleman’s excellent exposition [27]. The general principle is to “consider in the upstairs and take the $I$-invariant part”. The next two theorems are obtained by restricting everything to the $I$-invariant part in the corresponding theorems of [27].
Theorem 4.3 (27, Théorème 8.1.7). Suppose $\eta$ is an $I$-invariant closed $(1,1)$-form, and

$$\Theta := \frac{1}{2}((\eta) - 2c_1(E) + c_1(K)) \cup [\omega], [X]) \neq 0$$

Then an $I$-invariant triple $(B, \alpha, \beta)$ is a solution to (4.2) if and only if:

I. $\Theta > 0$ and

$$\beta = 0, \quad \bar{\partial}_B \alpha = 0, \quad F^0_{B} = 0, \quad i\Lambda_g F_B + \frac{1}{8}|\alpha|^2 = \pi \Lambda_g \eta - \frac{s_g}{2}$$

II. $\Theta < 0$ and

$$\alpha = 0, \quad \bar{\partial}_B \beta = 0, \quad F^0_{B} = 0, \quad i\Lambda_g F_B + \frac{1}{8}|\beta|^2 = \pi \Lambda_g \eta - \frac{s_g}{2}$$

Let $\mathcal{C}^\nu$ be the Hermitian connection on $K$ induced from the Chern connection $C$. For a Hermitian connection on $E$, let $B'$ be the Hermitian connection on $K \otimes \bar{E}$ such that $B \otimes B' = \mathcal{C}^\nu$. For $\beta \in \Omega^{0,2}(E) = \Gamma(E \otimes K^{-1})$, let $\varphi = \bar{\nu} \in \Gamma(E \otimes K)$. Then the condition $\bar{\partial}_B \beta = 0$ is equivalent to $\bar{\partial}_B \varphi = 0$ by the Serre duality, and (4.5) can be rewritten as

$$\alpha = 0, \quad \bar{\partial}_B \varphi = 0, \quad F^0_{B'} = 0, \quad i\Lambda_g F_B + \frac{1}{8} |\varphi|^2 = -\pi \Lambda_g \eta + \frac{s_g}{2}$$

If $\eta$ is not $(1,1)$, then we have the following.

Theorem 4.6 (27, Théorème 9.3.1). Suppose an $I$-invariant 2-form $\eta$ has a form of $\eta = \eta^{2,0} \oplus \eta^{1,1} \oplus \overline{\eta^{2,0}}$ where $\eta^{2,0}$ is an $I$-invariant non-zero holomorphic 2-form. Then an $I$-invariant triple $(B, \alpha, \beta)$ is a solution to (4.2) if and only if:

$$\alpha \beta = -8 \pi i \eta^{2,0}, \quad \bar{\partial}_B \alpha = \bar{\partial}_B \beta = 0, \quad F^0_{B} = 0,$$

$$i\Lambda_g F_B + \frac{1}{8}(|\beta|^2 - |\alpha|^2) = \pi \Lambda_g \eta^{1,1} - \frac{s_g}{2}$$

Let $\mathcal{C}^\nu$, $B'$ and $\varphi$ be as above. Then (4.7) can be rewritten as

$$\alpha \varphi = -8 \pi i \eta^{2,0}, \quad B \otimes B' = \mathcal{C}^\nu,$$

$$\bar{\partial}_B \alpha = \bar{\partial}_B \varphi = 0, \quad F^0_{B} = F^0_{B'} = 0,$$

$$\frac{i}{2} \Lambda_g (F_B - F_B') + \frac{1}{8}(|\varphi|^2 - |\alpha|^2) = \pi \Lambda_g \eta$$

4.2. Vortex equations. Let $(X, \omega, \iota)$ be a compact Kähler surface with anti-holomorphic free involution $\iota$. Suppose we have a $C^\infty$ Hermitian line bundle $(E, h)$ over $X$ with an isomorphism $\iota^*(E, h) \cong (E, h)$. This isomorphism defines the bundle map $I = \iota^*(\cdot)$ covering $\iota$ which is the composite map of

$$E \xleftarrow{\iota^*} \iota^* E \cong \bar{E} \xrightarrow{(\cdot)} E.$$

We suppose $I$ generates an order-2 action (involution) on $E$. We define the $I$-action on $\Omega^0(E)$ also by $I = \iota^*(\cdot)$. Let $\mathcal{A}(E, h)$ be the space of Hermitian connections on $E$. Then the involution $I$ naturally induces an involution on $\mathcal{A}(E, h)$, also denoted by $I$. The gauge transformation group $\mathcal{G} = C^\infty(X; S^1)$ acts on $\mathcal{A}(E, h) \times \Omega^0(E)$ by

$$(B, \phi) \cdot f = (B + f^{-1}df, f^{-1}\phi)$$

for $(B, \phi) \in \mathcal{A}(E, h) \times \Omega^0(E), f \in \mathcal{G}$. 

$$E \xleftarrow{\iota^*} \iota^* E \cong \bar{E} \xrightarrow{(\cdot)} E.$$
A configuration \((B, \phi)\) with \(\phi \neq 0\) is called an irreducible. The \(G\)-action on the space of irreducibles is free. We define the involution \(I\) on \(G\) by \(I(f) = \overline{f^*}\). Then the \(G\)-action on \(A(E, h) \times \Omega^0(E)\) is \(I\)-equivariant.

**Definition 4.9.** Let \(t : X \to \mathbb{R}\) be a \(C^\infty\)-function. A \(t\)-vortex is a solution \((B, \phi) \in A(E, h) \times \Omega^0(E)\) to the system of the equations

\[
\begin{align*}
\bar{\partial}_B \phi &= 0 \\
F^0_B &= 0 \\
i\Lambda_g F_B + \frac{1}{2} |\phi|^2 - t &= 0
\end{align*}
\]

(4.10)

If \((B, \phi)\) is a solution to (4.10), then

\[
\Xi := \frac{1}{2\pi} \int_X t \text{vol}_g - (c_1(E) \cup [\omega], [X]) = \frac{1}{2\pi} \int_X (t - i\Lambda_g F_B) \text{vol}_g = \frac{1}{4\pi} ||\phi||^2_{L^2} \geq 0.
\]

If \(t\) is \(\iota\)-invariant, that is, \(\iota^* t = t\), then the system (4.10) is \(I\)-equivariant. Define \(I\)-invariant moduli spaces as follows:

\[
\begin{align*}
\mathcal{V}_I(E)^I &= \{ \text{I-invariant } t\text{-vortices } \}/G^I, \\
\mathcal{V}^*_I(E)^I &= \{ \text{I-invariant irreducible } t\text{-vortices } \}/G^I.
\end{align*}
\]

If \(\Xi > 0\), then \(\mathcal{V}^*_I(E)^I = \mathcal{V}_I(E)^I\).

As usual, we take \(L^2_k\)-completion of \(C^*_k(E) := A(E, h) \times (\Omega^0(E) \setminus \{0\})\) and \(L^2_{k+1}\)-completion of \(G\) for sufficiently large \(k\). We use the notation \((\cdot)_k\) for the completed spaces. For a generic choice of \(I\)-invariant \(t\) with positive \(\Xi\), \(\mathcal{V}^*_I(E)^I = \mathcal{V}_I(E)^I\) is a submanifold of the Hilbert manifold

\[
(B^*_k)^I := (C^*_k(E))^I/G^I_{k+1}.
\]

For an orbit \([v] = [(B, \phi)] \in (B^*_k)^I\), the tangent space of \((B^*_k)^I\) at \([v]\) is given by

\[
T_{[v]}(B^*_k)^I = \{ (\dot{B}, \dot{\phi}) | d^* \dot{B} = -i \text{ Im}(\dot{\phi} \overline{\phi}) = 0 \}^I.
\]

The following is a direct consequence of Theorem 4.3.

**Corollary 4.12.** Suppose \(\eta\) is an \(I\)-invariant closed \((1,1)\)-form. Let \(t = \pi \Lambda_g \eta - s_g/2\). Then we have the following identifications:

1. \(\mathcal{M}(X, s_0 \otimes E)^I \cong \mathcal{V}^*_I(E)^I\), if \(\Theta > 0\).
2. \(\mathcal{M}(X, s_0 \otimes E)^I \cong \mathcal{V}^*_I(K \otimes E^{-1})^I\), if \(\Theta < 0\).

### 4.3 Holomorphic simple pairs.

Let \((X, \omega, \iota)\) be a compact Kähler surface with anti-holomorphic involution \(\iota\), and \(E\) a \(C^\infty\) complex line bundle such that \(\iota^* E \cong \bar{E}\). As before, we suppose \(I = \iota^* I\) generates an involution on \(E\). We define the \(I\)-action on \(\Omega^0(E)\) also by \(I = \iota^* I\). Let \(A^{0.1}(E)\) be the space of semiconnections on \(E\). Note that a semiconnection \(\delta \in A^{0.1}(E)\) can be written as \(\delta = \partial_B\) for some complex linear connection \(B\) on \(E\). The involution \(I\) naturally induces an involution on \(A^{0.1}(E)\), also denoted by \(I\). The complex gauge transformation group \(G^C = C^\infty(X, \mathbb{C}^*)\) acts on \(\mathcal{P}(E) = A^{0.1}(E) \times \Omega^0(E)\) by

\[
(\delta, \phi) \cdot f = (\delta \cdot f, f^{-1} \phi) \text{ for } (\delta, \phi) \in \mathcal{P}(E), \quad f \in G^C,
\]
where \( \delta \cdot f = f^{-1} \circ \delta \circ f = \delta + f^{-1} \partial f \). A pair \((\delta, \phi)\) with nonzero \(\phi\) is called simple. Let \(\mathcal{P}^s(E)\) be the space of simple pairs. Then \(\mathcal{G}_C\) acts on \(\mathcal{P}^s(E)\) freely. We define the involution \(I\) on \(\mathcal{G}_C\) by \(I(f) = \overline{f}^+\). Then the \(\mathcal{G}_C\)-action on \(\mathcal{P}(E)\) is \(I\)-equivariant.

Let \(\mathcal{H}(E)\) be the space of holomorphic pairs:

\[
\mathcal{H}(E) = \{ (\delta, \phi) \in \mathcal{A}^{0,1}(E) \times \Omega^0(E) \mid \delta \circ \delta = 0, \delta \phi = 0 \}.
\]

A pair \((\delta, \phi)\) in \(\mathcal{H}(E)\) with non-zero \(\phi\) is called a holomorphic simple pair. Let \(\mathcal{H}^s(E)\) be the space of holomorphic simple pairs.

We consider the \(I\)-invariant moduli space of holomorphic simple pairs:

\[
\mathcal{M}^s(E)^I = \mathcal{H}^s(E)^I/(\mathcal{G}_C)^I.
\]

Deformation complex for an \(I\)-invariant holomorphic simple pair \(p = (\delta, \phi)\):

\[
(\mathcal{C}_p)^I = (\mathcal{C}_0)^I \xrightarrow{\partial_0^I} (\mathcal{C}_1)^I \xrightarrow{\partial_1^I} (\mathcal{C}_2)^I \xrightarrow{\partial_2^I} (\mathcal{C}_3)^I,
\]

where

\[
(\partial_0^I, \partial_1^I, \partial_2^I) = (\delta, \partial, -\delta \sigma - \alpha \phi).
\]

The moduli space \(\mathcal{M}^s(E)^I\) has a Kuranishi model as follows.

**Proposition 4.15** (Cf. [27], Proposition 8.2.10). Let \(H^i((\mathcal{C}_p)^I), \mathbb{H}^i((\mathcal{C}_p)^I)\) be the cohomology group and harmonic space of the elliptic complex \((\mathcal{C}_p)^I\). There exists a neighborhood \(U_p\) of \(0 \in \mathbb{H}^1((\mathcal{C}_p)^I)\) and a smooth map

\[
t_p : U_p \to \mathbb{H}^2((\mathcal{C}_p)^I)
\]

such that a neighborhood of \(p \in \mathcal{M}^s(E)^I\) is homeomorphic to \(t_p^{-1}(0)\). Furthermore, if \(H^2((\mathcal{C}_p)^I) = 0\), then \(\mathcal{M}^s(E)^I\) is a smooth manifold of dimension \(\dim H^1((\mathcal{C}_p)^I)\) near \(|p|\), and the tangent space of \(\mathcal{M}^s(E)^I\) at \(|p|\) is identified with \(H^1((\mathcal{C}_p)^I)\).

The proof is standard.

### 4.4. \(I\)-invariant divisors.

(A reference of this subsection is [21], I.4.) A Weil divisor is a formal linear combination \(\sum_i n_i D_i\) of irreducible analytic hypersurfaces. Define the \(I\)-action on divisors by \(I \cdot D = \sum_i n_i \iota(D_i)\). We call a divisor \(D\) \(I\)-invariant if \(D = I \cdot D\). We will mainly consider effective divisors, i.e., \(D = \sum_i n_i D_i\) with \(n_i \geq 0\).

When \(D\) is considered as a Cartier divisor, the \(I\)-action can be written as follows. For an open subset \(U \subset X\) and a holomorphic function \(f \in \mathcal{O}_X(U)\), define \(I \cdot f \in \mathcal{O}_X(\iota(U))\) by \((I \cdot f)(\iota(x)) = \overline{f}(\iota(x))\). Let \(\mathcal{S}\) be the set of pairs \((U_\lambda, \lambda)\) where \(U_\lambda\) is an open set and \(\lambda \in \mathcal{O}_X(U_\lambda)\). Then define the \(I\)-action on \(\mathcal{S}\) by

\[
I \cdot (U_\lambda, \lambda) = (\iota(U_\lambda), I \cdot \lambda)
\]

An effective Cartier divisor is given as a subset \(\mathcal{F} \subset \mathcal{S}\) whose elements \((U_\lambda, \lambda)\) satisfy the following:

1. \(\lambda\) is not identically zero.
2. \(\bigcup_{\lambda \in \mathcal{F}} U_\lambda = X\).
(3) For every \( \lambda, \mu \in \mathcal{F} \), there exists \( g_{\lambda,\mu} \in \mathcal{O}_{\mathcal{X}}^\ast (U_\lambda \cap U_\mu) \) such that \( \lambda = g_{\lambda,\mu} \mu \).

We will take a maximal one of such systems for \( \mathcal{F} \). The effective Weil divisor corresponding to an effective Cartier divisor is obtained by considering \( \lambda \) as local defining equations. Let

\[
I \cdot \mathcal{F} = \{ I \cdot (U_\lambda, \lambda) \mid (U_\lambda, \lambda) \in \mathcal{F} \}.
\]

Then \( I \cdot D \) corresponds to \( I \cdot \mathcal{F} \). Note that \( g_{I\lambda,I\mu} = Ig_{\lambda,\mu} \). If \( D \) is \( I \)-invariant, then we can take \( \mathcal{F} \) corresponding to \( D \) such that \( \mathcal{F} = I \cdot \mathcal{F} \).

The system of cocycles \( \{ g_{\lambda,\mu} \} \) and local functions \( \{ \lambda \} \) define a holomorphic line bundle \( \mathcal{L} \) with a holomorphic section \( \phi \).

\[
\mathcal{L} = \left( \bigcup_{\lambda \in \mathcal{F}} \{ \lambda \} \times U_\lambda \times \mathbb{C} \right) / \sim,
\]

\[
\{ \mu \} \times (U_\lambda \cap U_\mu) \times \mathbb{C} \ni (\mu, u, \zeta) \sim (\lambda, u, g_{\lambda,\mu} \zeta) \in \{ \lambda \} \times (U_\lambda \cap U_\mu) \times \mathbb{C},
\]

\[
\phi(u) = [(\lambda, u, \lambda(u))] \mod \sim (u \in U_\lambda).
\]

Then the corresponding divisor \( D \) is \( D = Z(\phi) \). When \( (\mathcal{L}_D, \phi_D) \) is associated with \( D \) (or \( \mathcal{F} \)), note that the line bundle with section associated with \( I \cdot D \) (or \( I \cdot \mathcal{F} \)) is

\[
(\mathcal{L}_{I \cdot D}, \phi_{I \cdot D}) = (\iota^* \mathcal{L}_D, \iota^* \phi_D).
\]

If \( D \) is \( I \)-invariant, then an antilinear involution \( I \) on \( \mathcal{L}_D \) covering \( \iota \) is naturally defined by

\[
I \cdot [(\lambda, u, \zeta)] = [(I\lambda, \iota(u), \zeta)].
\]

### 4.5. \( I \)-equivariant sheaves.

The \( I \)-action makes the structure sheaf \( \mathcal{O}_\mathcal{X} \) an \( I \)-equivariant sheaf in the sense of [10][22], i.e., the sheaf projection \( \mathcal{O}_\mathcal{X} \to \mathcal{X} \) is \( I \)-equivariant. If \( D \) is \( I \)-invariant, then \( \mathcal{O}_{\mathcal{X}}(D) \) and \( \mathcal{O}_D(D) = \mathcal{O}_{\mathcal{X}}(D)/\mathcal{O}_X \) are also \( I \)-equivariant. For an \( I \)-equivariant sheaf \( \mathcal{E} \), the equivariant sheaf cohomology \( H^p(\mathcal{X}; I, \mathcal{E}) \) is defined: For an \( I \)-invariant open set \( U \subset \mathcal{X} \), let \( \Gamma^I(U; \mathcal{E}) \) be the module of \( I \)-invariant sections. Take an injective resolution \( \mathcal{J}^\ast(\mathcal{E}) \) of \( \mathcal{E} \) in the category of \( I \)-equivariant sheaves. Then \( H^p(\mathcal{X}; I, \mathcal{E}) \) is defined by

\[
H^p(\mathcal{X}; I, \mathcal{E}) = H^p(\Gamma^I(\mathcal{X}; \mathcal{J}^\ast(\mathcal{E}))).
\]

The equivariant direct image \( \pi^I \mathcal{E} \) of \( \mathcal{E} \) is the sheaf on \( \hat{\mathcal{X}} = \mathcal{X}/\iota \) which is generated by the presheaf,

\[
\hat{U} \mapsto \Gamma^I(\pi^{-1}(\hat{U}); \mathcal{E}), \quad \hat{U} \subset \hat{\mathcal{X}} = \mathcal{X}/\iota \quad \text{open}.
\]

In general, \( \pi^G \) is a left exact functor for \( G \)-sheaves. However our case is much simple. Since \( I \) covers the free involution \( \iota \) on \( \mathcal{X} \), \( \pi^I \) is an exact functor. That is, for an exact sequence of \( I \)-sheaves on \( \mathcal{X} \),

\[
0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{H} \to 0,
\]

we have an exact sequence of sheaves on \( \hat{\mathcal{X}} \),

\[
0 \to \pi^I \mathcal{E} \to \pi^I \mathcal{F} \to \pi^I \mathcal{H} \to 0.
\]

In particular,

\[
\pi^I \mathcal{F}/\pi^I \mathcal{E} = \pi^I(\mathcal{F}/\mathcal{E}).
\]
The fact that \( I \) covers the free involution \( \iota \) on \( X \) also implies that \[
H^i(X; I, \mathcal{E}) = H^i(X/\iota; \pi^i(\mathcal{E})).
\]
([10], p.204, Corollaire; [22], Collorary 5.6.)

There is an \( I \)-equivariant exact sequence.

\[
0 \longrightarrow \mathbb{Z} \xrightarrow{i} \mathcal{O}_X \xrightarrow{\exp 2\pi i} \mathcal{O}_X^* \longrightarrow 0,
\]
where \( \tilde{\mathbb{Z}} \) is the constant sheaf on which \( I \) acts via multiplication of \(-1\). This induces the sequence

\[
0 \rightarrow H^1(X; I, \tilde{\mathbb{Z}}) \rightarrow H^1(X; I, \mathcal{O}_X) \rightarrow H^1(X; I, \mathcal{O}_X^*) \xrightarrow{\tilde{\pi}} H^2(X; I, \tilde{\mathbb{Z}}) \rightarrow \cdots.
\]

Note that \( H^i(X; I, \tilde{\mathbb{Z}}) \cong H^i(\tilde{X}; \ell) \). Let \( \text{NS}^I(X) = \text{Im} \tilde{c}_1 \). For \( e \in \text{NS}^I(X) \), let \( \mathcal{D}(e) \) be the set of effective divisors representing \( e \).

**Proposition 4.16** ([27], Proposition 8.2.13). Let \( e = \tilde{c}_1(E) \).

1. The map \((\delta, \phi) \mapsto Z(\phi)\) induces a bijection \( \mathcal{M}^s(E)^I \xrightarrow{\sim} \mathcal{D}(e)^I \).
2. \[
H^0((\mathcal{E}_p)^I) = 0, \quad H^i((\mathcal{E}_p)^I) \cong H^{i-1}(X; I, \mathcal{O}_D(D)) = H^{i-1}(\tilde{X}; \pi^i \mathcal{O}_D(D)),
\]
   for each positive integer \( i \).

**Proof.** With ([14]) understood, (1) is easy. The proof of (2) is parallel to that of [27] Proposition 8.2.13. Let \( \mathcal{A}^{p,q}(X) \) and \( \mathcal{A}^{p,q}(E) \) be the sheaves of \( C^\infty \)-sections of \( \mathcal{A}^{p,q} \) and \( \mathcal{A}^{p,q}(E) \). Let

\[
\mathcal{C}^0 := \mathcal{A}^0(X), \quad \mathcal{C}^i := \mathcal{A}^0(X) \oplus \mathcal{A}^{0,i-1}(E), (i = 1, 2), \quad \mathcal{C}^3 := \mathcal{A}^{0,2}(E).
\]

Then the \( I \)-action makes \( \mathcal{C}^i \) \( I \)-equivariant sheaves. The formula of \( \mathcal{D}_p^I \) in ([14]) defines the sequence of \( I \)-equivariant sheaves:

\[
0 \rightarrow \mathcal{C}^0 \xrightarrow{\delta^0} \mathcal{C}^1 \xrightarrow{\delta^1} \mathcal{C}^2 \xrightarrow{\delta^2} \mathcal{C}^3 \rightarrow 0.
\]

This induces the sequence of sheaves over \( \tilde{X} = X/\iota \):

\[
(4.17) \quad 0 \rightarrow \pi^i \mathcal{C}^0 \xrightarrow{\delta^0} \pi^i \mathcal{C}^1 \xrightarrow{\delta^1} \pi^i \mathcal{C}^2 \xrightarrow{\delta^2} \pi^i \mathcal{C}^3 \rightarrow 0.
\]

Since \( \iota \) is free, it can be seen from the \( \partial \)-Poincaré lemma that the sequence (4.17) is exact unless \( i \neq 1 \). Furthermore, the following map is an isomorphism:

\[
\pi^i \mathcal{O}_X(\mathcal{E}_\delta)/\phi \pi^i \mathcal{O}_X \rightarrow \ker \delta^1_{\pi^i} / \im \delta^0_{\pi^i}, \quad [\lambda] \mapsto (0, \lambda).
\]

For \( i > 0 \),

\[
H^i(\pi^i \mathcal{C}^1 / \im \delta^0_{\pi^i}) = 0.
\]

Then we obtain a resolution of \( \pi^i \mathcal{O}_X(\mathcal{E}_\delta)/\phi \pi^i \mathcal{O}_X = \pi^i \mathcal{O}_D(D) \) as follows:

\[
0 \rightarrow \pi^i \mathcal{O}_X(\mathcal{E}_\delta)/\phi \pi^i \mathcal{O}_X = \ker \delta^0_{\pi^i} / \im \delta^0_{\pi^i} \xrightarrow{\delta^1_{\pi^i}} \pi^i \mathcal{C}^1 / \im \delta^0_{\pi^i} \xrightarrow{\delta^2_{\pi^i}} \pi^i \mathcal{C}^2 / \pi^i \mathcal{C}^3 \rightarrow 0.
\]

Since \( H^1(\im \delta^0_{\pi^i}) \cong H^1(\pi^i \mathcal{C}^0) \), we have

\[
H^0(\pi^i \mathcal{C}^1 / \mathcal{D}_p^I(\pi^i \mathcal{C}^0)) \cong H^0(\pi^i \mathcal{C}^1 / \im \delta^0_{\pi^i}).
\]
Then we obtain
\[ H^0(\pi^\dagger \mathcal{O}_D(D)) \cong \ker \left( H^0(\pi^\dagger \mathcal{O}_D(D)^{\dagger}) / \mathfrak{h}_p \right) \cong H^1(\mathcal{C}^1) = (\mathcal{C}^2), \]
\[ H^1(\pi^\dagger \mathcal{O}_D(D)) \cong \ker \mathfrak{D}_p^{1+1} / \mathfrak{D}_p = H^{1+1}((\mathcal{C}_p)^{\dagger}). \]

**Corollary 4.18.** For \( D \in \mathcal{D}(e)^I \), if \( H^1(X; I, \mathcal{O}_D(D)) = H^1(\tilde{X}; \pi^\dagger \mathcal{O}_D(D)) = 0 \), then \( \mathcal{D}(e)^I \) is smooth at \( D \), and the tangent space of \( \mathcal{D}(e)^I \) at \( D \) is identified with \( H^0(X; I, \mathcal{O}_D(D)) = H^0(\tilde{X}; \pi^\dagger \mathcal{O}_D(D)) \).

We call the following the Zariski tangent space of \( \mathcal{D}(e)^I \) at \( D \):
\[ T_D(\mathcal{D}(e)^I) := H^0(X; I, \mathcal{O}_D(D)) = H^0(\tilde{X}; \pi^\dagger \mathcal{O}_D(D)). \]

### 4.6. Correspondence

Define the map \( \tilde{J} : \mathcal{C}^*_k \rightarrow \mathcal{P}^s_k \) by
\[ \tilde{J}(B, \phi) = (\tilde{\delta}B, \phi). \]

Then the restriction of \( \tilde{J} \) to the \( I \)-invariant part \( \mathcal{C}^*_k \) induces a submersion
\[ \mathcal{J}' : (\mathcal{B}^{\dagger}_k)^I = \mathcal{C}^*_k \, |\, \mathcal{G}^{I+1}_k \rightarrow (\mathcal{B}^{\dagger}_k)^I := \mathcal{P}^s_k \, |\, \mathcal{G}^{I+1}_k. \]

The goal of this subsection is the next proposition.

**Theorem 4.19.** If \( \Xi > 0 \), then the map \( \mathcal{J}' \) induces a homeomorphism
\[ \mathcal{J} : \mathcal{V}^*_k(E)^I \xrightarrow{\sim} \mathcal{M}^s_k(E)^I. \]

For the proof, we need some preparation. Define \( \tilde{\mu}_t : \mathcal{C}^*_k \rightarrow \Omega^0(E)_{k-1} \) by the left hand side of the third equation of (4.14) as
\[ \tilde{\mu}_t(B, \phi) = i\Lambda_g F_B + \frac{1}{2} |\phi|^2 - t. \]

Restrict \( \tilde{\mu}_t \) to the \( I \)-invariant part \( \mathcal{C}^*_k \). Then the restriction \( \tilde{\mu}_t \, |\, \mathcal{C}^*_k \) descends to the map \( \mu^I_t : (\mathcal{B}^{\dagger}_k)^I \rightarrow \Omega^0(E)_{k-1} \). Let \( Z(\mu^I_t) = (\mu^I_t)^{-1}(0) \). For \( v = (B, \phi) \in (\mathcal{C}^*_k)^I \), by using the Kähler identities \( \tilde{\delta}^* = i[\Lambda, \tilde{\delta}], \tilde{\delta}^* = -i[\Lambda, \tilde{\delta}] \), we have
\[ T[v]Z(\mu^I_t) = \mathcal{T}_v[(\mathcal{B}_k^{\dagger})^I] \cap \ker d\tilde{\mu}_t = \{(B, \phi) | d' \dot{B} - i \text{Im} (\dot{\phi} \bar{\phi}) = 0, -iA_g d\tilde{B} - \text{Re} (\bar{\phi} \phi) = 0\}^I \]
\[ = \{(B, \phi) | 2\tilde{\delta}^* \tilde{\delta} B^{0,1} - \phi \bar{\phi} = 0\}^I \]

Note that the last space can be identified with the \( L^2 \)-orthogonal complement of the tangent space of the orbit \( p \cdot (\mathcal{G}^{I+1}_k)^I \) in \( T_p(\mathcal{P}^s_k)^I \) where \( p = (B, \phi) \in \mathcal{P}^s_k \). Then we can see the following:

**Proposition 4.20.** The map \( \mathcal{J}' \, |\, Z(\mu^I_t) : (\mathcal{B}^{\dagger}_k)^I \rightarrow (\mathcal{B}^{\dagger}_k)^I \) is a local homeomorphism.

**Proof of Theorem 4.19.** (Cf. [27], Proposition 8.2.20.) By Proposition 4.20, it suffices to see that \( \mathcal{J} \) is bijective. First, we prove \( \mathcal{J} \) is surjective. Suppose \((\delta, \phi) \in \mathcal{M}^s_k(E)^I \). We have an Hermitian connection \( B = A_{\bar{t}, \delta} \) associated with the holomorphic structure \( \delta \). We want to find an \( I \)-invariant function...
\[ \psi \in C^\infty(X; \mathbb{R})^I \text{ such that } (A_{h,\delta,f}, \phi) \text{ is a solution to } (4.10) \text{ for } f = e^{-\psi}. \]
Note that
\[ A_{h,\delta,f} = A_{h,\delta} - \partial \psi + \partial \psi, \]
and \( (A_{h,\delta,f}, \phi) \) is a \( t \)-vortex if and only if
\[ i\Lambda_g \partial \psi + \frac{1}{2}e^{2\psi} |\phi|^2 = t - i\Lambda_g F_B. \]
Since \( \phi \) and \( \theta := t - i\Lambda_g F_B \) are \( I \)-invariant section and function, they descend on \( \hat{X} \), i.e., we find \( \hat{\phi} \) and \( \hat{\theta} \) such that \( \phi = \pi^* \hat{\phi} \) and \( \theta = \pi^* \hat{\theta} \). Consider the following equation for \( \hat{\psi} \in C^\infty(\hat{X}; \mathbb{R}) \):
\[ \Delta \hat{\psi} + \frac{1}{2}e^{2\hat{\psi}} |\hat{\phi}|^2 = \hat{\theta}. \]
This is a Kazdan-Warner type equation \[4.21\], and has a unique solution \( \hat{\psi} \) since \( \int_{\hat{X}} \theta d\text{vol}_g = \frac{1}{2} \int_X \theta d\text{vol}_g = \frac{1}{2}\Theta > 0 \). Then \( \psi = \pi^* \hat{\psi} \) is an \( I \)-invariant solution to \((4.21)\).
We prove \( J \) is injective. Suppose \((B_1, \phi_1), (B_2, \phi_2)\) are \( I \)-invariant solutions to \((4.10)\) such that \( (\partial B_1, \phi_1) = (\partial B_2, \phi_2) \cdot f \) for some \( f \in (\mathcal{G}^C)^I \).
By replacing \((B_2, \phi_2)\) with \( \mathcal{G}^I \)-equivalent one, if necessary, we may assume \( f = e^{-\psi} \) for some \( I \)-invariant function \( \psi \). Since \((B_2, \phi_2)\) is an \( I \)-invariant solution, \( \psi \) satisfies \((4.21)\). Since \((B_1, \phi_1)\) is an \( I \)-invariant solution, \( \psi = 0 \) is a solution to \((4.21)\). Moving to the downstairs, we see that the uniqueness of the solution to Kazdan-Warner’s equation implies that \( \psi = 0 \).

For a Spin\(^{-}\) structure \( \hat{s}_0 \circledS \hat{E} \) on \( X \to \hat{X}, \hat{s}_0 \circledS E \) is the canonical reduction of \( \hat{s}_0 \circledS \hat{E} \) where \( E \) is the canonical \( U(1) \)-reduction of \( \hat{E} \). We choose an \( I \)-invariant closed \((1,1)\)-form \( \eta \) for the perturbation term of the \( I \)-invariant Seiberg-Witten equation \[4.12\].

**Corollary 4.22.** Let \( t = \pi \Lambda \eta - s_g/2, e = \hat{c}_1(\hat{E}) \) and \( k = \hat{c}_1(\hat{K}) \).

1. If \( \Theta > 0 \), then
   \[ \mathcal{M}(\hat{X}, \hat{s}_0 \circledS \hat{E}) \cong \mathcal{M}(X, s_0 \circledS E)^I \cong \mathcal{V}^*_e(E)^I \cong \mathcal{M}^e(E)^I \cong \mathcal{D}(e)^I. \]
2. If \( \Theta < 0 \), then
   \[ \mathcal{M}(\hat{X}, \hat{s}_0 \circledS \hat{E}) \cong \mathcal{M}(X, s_0 \circledS E)^I \cong \mathcal{V}^*_e(E^{-1} \circledS K)^I \cong \mathcal{M}^e(E^{-1} \circledS K)^I \cong \mathcal{D}(k-e)^I. \]

4.7. **Witten’s perturbation.** In the previous subsection, we consider the perturbation by an \( I \)-invariant \((1,1)\)-form \( \eta \), and the \( \text{Pin}^{-}(2) \)-monopole moduli space is identified with the \( I \)-invariant moduli space of vortices and holomorphic simple pairs. In this subsection, we consider the perturbation as in Theorem \[4.10\].

Let \( (X, \omega, \iota) \) be a compact Kähler surface with anti-holomorphic involution \( \iota \), and \( E \) and \( E' \) two \( C^\infty \) complex line bundles such that \( \iota^* E \cong \hat{E} \) and \( \iota^* E' \cong \hat{E}' \). We suppose \( I = \iota^* (\cdot) \) defines involutions on \( E \) and \( E' \). Consider \( \mathcal{P}^* (E) \times \mathcal{P}^* (E') \). Let \( \mathcal{G}^C = C^\infty(X, \mathbb{C}^*) \) act on \( \mathcal{P}^* (E) \times \mathcal{P}^* (E') \) by
\[ (p, p') \cdot f := (p \cdot f, p' \cdot f^{-1}) \text{ for } (p, p') \in \mathcal{P}^* (E) \times \mathcal{P}^* (E'), f \in \mathcal{G}^C. \]

Fix a holomorphic structure \( N \) on \( N = E \circledS E' \), and let \( \delta_N \) be the corresponding integrable semiconnection. (Later we assume \( N \), and therefore \( \delta_N \),
are $I$-invariant.) Put
\[ \mathcal{H}^s(E) \times_N \mathcal{H}^s(E') := \{(\delta, \phi), (\delta', \phi')) \in \mathcal{H}^s(E) \times \mathcal{H}^s(E') \mid \delta \otimes \delta' = \delta_N\}. \]
A natural map $\mathcal{I}: \mathcal{P}^s(E) \times \mathcal{P}^s(E') \to \mathcal{P}^s(N)$ given by $((\delta, \phi), (\delta', \phi')) \mapsto (\delta \otimes \delta', \phi \otimes \phi')$ is $\mathcal{G}_C$-invariant. We have a $\mathcal{G}_C$-equivariant commutative diagram
\[ \begin{array}{ccc}
\mathcal{H}^s(E) \times \mathcal{H}^s(E') & \xrightarrow{\mathcal{I}} & \mathcal{H}^s(N) \\
\uparrow & & \uparrow \\
\mathcal{H}^s(E) \times_N \mathcal{H}^s(E') & \xrightarrow{\mathcal{I}_N} & \{\delta_N\} \times (H^0(N) \setminus \{0\})
\end{array} \]
Now suppose $N$ is $I$-invariant and $(H^0(N) \setminus \{0\})^f \neq \emptyset$, and choose an $I$-invariant holomorphic section $\xi \in (H^0(N) \setminus \{0\})^f$. Let
\[ \mathcal{M}^s(E, E', N, \xi)^{I} = (\mathcal{I}^{-1}_N(\xi))^f / \mathcal{G}_C. \]
For $e = c_1(E)$ and $e' = c_1(E')$, consider the map defined by sum of divisors
\[ \theta: \mathcal{D}(e) \times \mathcal{D}(e') \to \mathcal{D}(e + e'). \]
For $\Delta \in \mathcal{D}(e + e')$, let
\[ \mathcal{D}_b(\Delta) := \theta^{-1}(\Delta). \]
Then, for $\Delta = Z(\xi)$, we have a natural identification
\[ \mathcal{M}^s(E, E', N, \xi)^{I} \cong \mathcal{D}_b(\Delta)^{I}. \]
The Zariski tangent space $T_{(D, D')} \left( \mathcal{D}_b(\Delta)^{I} \right)$ of $\mathcal{D}_b(\Delta)^{I}$ at $(D, D')$ is given by
\[ T_{(D, D')} \left( \mathcal{D}_b(\Delta)^{I} \right) = \ker \left( \theta_*: T_D \left( \mathcal{D}(e)^{I} \right) \oplus T_D' \left( \mathcal{D}(e')^{I} \right) \to T_{\Delta} \left( \mathcal{D}(e + e')^{I} \right) \right). \]
For $I$-invariant $D$, $D'$, $\Delta = D + D'$, the inclusions $\emptyset \subset C \subset \Delta$, $\emptyset \subset D' \subset \Delta$ induces the inclusions
\[ \pi^I_0 \mathcal{O}_X \subset \pi^I_0 \mathcal{O}_X(D) \subset \pi^I_0 \mathcal{O}_X(\Delta), \quad \pi^I_0 \mathcal{O}_X \subset \pi^I_0 \mathcal{O}_X(D') \subset \pi^I_0 \mathcal{O}_X(\Delta), \]
\[ \pi^I_0 \mathcal{O}_D(D) = \pi^I_0 \left( \mathcal{O}_X(D)/\mathcal{O}_X \right) \subset \pi^I_0 \left( \mathcal{O}_X(\Delta)/\mathcal{O}_X \right) = \pi^I_0 \mathcal{O}_{\Delta}(\Delta), \]
\[ \pi^I_0 \mathcal{O}_{D'}(D') = \pi^I_0 \left( \mathcal{O}_X(D')/\mathcal{O}_X \right) \subset \pi^I_0 \left( \mathcal{O}_X(\Delta)/\mathcal{O}_X \right) = \pi^I_0 \mathcal{O}_{\Delta}(\Delta), \]
and therefore the injective maps
\[ T_D \left( \mathcal{D}(e)^{I} \right) = H^0(\pi^I_0 \mathcal{O}_D(D)) \xrightarrow{i} H^0(\pi^I_0 \mathcal{O}_{\Delta}(\Delta)) = T_{\Delta} \left( \mathcal{D}(e + e')^{I} \right), \]
\[ T_D \left( \mathcal{D}(e')^{I} \right) = H^0(\pi^I_0 \mathcal{O}_{D'}(D')) \xrightarrow{i'} H^0(\pi^I_0 \mathcal{O}_{\Delta}(\Delta)) = T_{\Delta} \left( \mathcal{D}(e + e')^{I} \right). \]
Then it can be seen that
\[ \theta_*(a, a') = \iota(a) + \iota'(a'). \]

**Proposition 4.23.** Let $D_0$ be the maximal effective divisor such that $D_0 \leq D$ and $D_0 \leq D'$. (Note that $D_0$ is also $I$-invariant.) Then there exists an isomorphism $T_{(D, D')} \left( \mathcal{D}_b(\Delta)^{I} \right) \cong H^0(\pi^I_0 \mathcal{O}_{D_0}(D_0))$. 

Proof. This is proved by considering the $I$-invariant part or applying $\pi^I$ to everything in the proof of [27, Lemma 9.3.3]. The commutative diagram

\[
\begin{array}{ccc}
\pi^I\mathcal{O}(D) & \to & \pi^I\mathcal{O}(D_0) \\
\pi^I\mathcal{O}(D) & \to & \pi^I\mathcal{O}(D') \\
\pi^I\mathcal{O}(D) & \to & \pi^I\mathcal{O}(\Delta)
\end{array}
\]

 induces another commutative diagram

\[
\begin{array}{ccc}
H^0(\pi^I\mathcal{O}(D)) & \to & H^0(\pi^I\mathcal{O}\Delta(\Delta)) \\
H^0(\pi^I\mathcal{O}_D(D_0)) & \to & H^0(\pi^I\mathcal{O}_D(D)) \\
H^0(\pi^I\mathcal{O}_D(D')) & \to & H^0(\pi^I\mathcal{O}_D(\Delta))
\end{array}
\]

where all of maps are linear monomorphisms. The image of the monomorphism

\[(-u) \oplus u': H^0(\pi^I\mathcal{O}_D(D_0)) \to H^0(\pi^I\mathcal{O}_D(D)) \oplus H^0(\pi^I\mathcal{O}_D(D'))\]

is contained in $\ker(\theta_s)$. Therefore

\[H^0(\pi^I\mathcal{O}_D(D_0)) \subset \ker(\theta_s) = T_{[D,D']}\left(D_b(\Delta)^I\right)\].

Conversely, let $(a,a')$ be an element of $\ker(\theta_s)$. Then $i(a) + i'(a') = 0$. We have the exact sequences

\[
0 \to \pi^I\mathcal{O}_D(-D) \to \pi^I\mathcal{O}_D \xrightarrow{\rho} \pi^I\mathcal{O}_D \to 0,
\]

\[
0 \to \pi^I\mathcal{O}_D(-D') \to \pi^I\mathcal{O}_{D'} \xrightarrow{\rho'} \pi^I\mathcal{O}_D \to 0,
\]

\[
0 \to \pi^I\mathcal{O}_{D'}(D') \xrightarrow{i'} \pi^I\mathcal{O}_D(\Delta) \xrightarrow{r} \pi^I\mathcal{O}_D(\Delta) \to 0,
\]

\[
0 \to \pi^I\mathcal{O}_D(D) \xrightarrow{i} \pi^I\mathcal{O}_D(\Delta) \xrightarrow{r'} \pi^I\mathcal{O}_{D'}(\Delta) \to 0.
\]

Then $\rho$, $\rho'$, $r$, $r'$ are restriction maps, since the corresponding maps in the exact sequences without $\pi^I$ called the decomposition sequences [11, p.62] are restriction maps. Since $r' \circ i = r \circ i' = 0$ and $i(a) = -i'(a')$, we have

\[
r'(i(a)) = 0, \quad r(i(a)) = -r(i'(a')) = 0.
\]

Hence the restrictions of $i(a) \in H^0(\pi^I\mathcal{O}_D(\Delta))$ to $D$ and $D'$ are 0. Let $\tilde{D} \leq \Delta$ be the smallest effective divisor such that $D \leq \tilde{D}$, $D' \leq \tilde{D}$. Then the restriction of $i(a)$ is also 0. By using the decomposition $\Delta = D_0 + \tilde{D}$, we obtain the exact sequence

\[
0 \to \pi^I\mathcal{O}_{D_0}(D_0) \xrightarrow{u} \pi^I\mathcal{O}_D(\Delta) \xrightarrow{i} \pi^I\mathcal{O}_{D}(\Delta) \to 0.
\]
Since $\hat{r}(i(a)) = 0$, we have an element $b \in H^0(\pi^*\Omega_{D_0}(D_0))$ such that $v(b) = i(a)$. Since $v = i \circ u$ and $i$ is injective, $a = u(b)$. Then $i'(a' + u'(b)) = -i(a) + (i' \circ u')(b) = -i(a) + v(b) = 0$, and hence $a' = -u'(b)$. \hfill \square

Fix Hermitian metrics $h$, $h'$ on $E$ and $E'$, a function $t \in C^\infty(X, \mathbb{R})$, an integrable connection $\Sigma$ on $N$ and a nonzero $\partial_\Sigma$-holomorphic section $\xi \in \Omega^0(N) \setminus \{0\}$. (Later we assume that all of them are $I$-invariant.) Let $N$ be the holomorphic structure on $N$ induced from $\Sigma$. Let $\mathcal{G} = C^\infty(X, S^1)$ act on $(\mathcal{A}(E, h) \times \Omega^0(E)) \times (\mathcal{A}(E', h') \times \Omega^0(E'))$ by

$$((B, \phi), (B', \phi')) \cdot f = ((B, \phi) \cdot f, (B', \phi') \cdot f^{-1}).$$

Consider the following system of equations:

$$
\begin{align*}
\bar{\partial}_B \phi &= \bar{\partial}_{B'} \phi' = 0 \\
F^0_{B'} &= F^0_{B'} = 0 \\
i\Lambda_g(F_B - F_{B'}) + \frac{1}{2}(|\phi|^2 - |\phi'|^2) &= t \\
B \otimes B' &= \Sigma \\
\phi \otimes \phi' &= \xi
\end{align*}
$$

(4.24)

Suppose that all of $h$, $h'$, $t$, $\Sigma$ and $\eta$ are $I$-invariant. Let

$$\mathcal{V}_I(E, E', \Sigma, \xi) = \{ I\text{-invariant solutions to } (4.24) \} / \mathcal{G}I.$$

**Theorem 4.25.** The map

$$((B, \phi), (B', \phi')) \mapsto ((\bar{\partial}_B, \phi), (\bar{\partial}_{B'}, \phi'))$$

induces a homeomorphism

$$\mathcal{V}_I(E, E', \Sigma, \xi) \cong \mathcal{M}(E, E', \Sigma, \xi).$$

**Proof.** The proof is similar to that of Theorem 4.19. In this case, we need to find an $I$-invariant function $\psi: X \to \mathbb{R}$ so that

$$i\Lambda_g \bar{\partial} \partial \psi + \frac{1}{2}e^{2\psi}|\phi|^2 - \frac{1}{2}e^{-2\psi}|\phi'|^2 = t - i\Lambda_g(F_B - F_{B'}).$$

As before, this equation descends to $\hat{X}$, and it has a unique smooth solution. (See [2] or [19, §3.2]). The rest of the proof is similar. \hfill \square

**Corollary 4.26.** For $\eta$ as in Theorem 4.6, let $t = \pi\Lambda_g \eta^{1,1} - s_g/2$.

$$\mathcal{M}(\hat{X}, \hat{\mathbf{s}}_0 \otimes \hat{E}) \cong \mathcal{M}(X; \mathbf{s}_0 \otimes E)^I$$

$$\cong \mathcal{V}_I(E, E^{-1} \otimes K, C^\vee, \eta^{2,0}) \cong \mathcal{M}(E, E^{-1} \otimes K, K, \eta^{2,0}) \cong \mathcal{D}_b(\Delta)^I$$

5. Calculation and Examples

The purpose of this section is to compute $\text{Pin}^{-}(2)$-monopole invariants of several concrete examples.
5.1. **Surfaces of general type.** In this subsection, we prove Theorem 1.6 on the surfaces of general type and give a series of examples.

**Proof of Theorem 1.6.** (Cf. [10], Theorem 7.4.1.) The results on the canonical and anticanonical Spin$^c$-structures follow from Theorem 1.1 and Corollary 1.9.

Since $X$ is minimal and of general type, $K^2_X > 0$ and $K_X$ is numerically effective. The latter condition implies $K_X \cdot \omega \geq 0$. But if $K_X \cdot \omega = 0$, then the Hodge index theorem implies $K^2_X \leq 0$. Therefore $K_X \cdot \omega > 0$.

Suppose a Spin$^c$-structure $\hat{s}$ has a nonvanishing Pin$^-(2)$-monopole invariant $SW^{\text{Pin}}_X(\hat{s})$. Let $s$ be the Spin$^c$ structure which is the canonical reduction of $\pi^*\hat{s}$. Let $L$ be the determinant line bundle of $s$. Then there exists a complex line bundle $E$ such that $s = s_0 \otimes E$. Note that $L = 2E - K_X$.

Since $SW^{\text{Pin}}_X(\hat{s}) \neq 0$, $d(\hat{s}) = 2d(\hat{s}) \geq 0$, and therefore $L^2 \geq K^2_X > 0$. Then $c_1(L)^+$ is not a torsion class, and this implies that there is no reducible solution and $L \cdot \omega \neq 0$.

Suppose $L \cdot \omega > 0$. Since $SW^{\text{Pin}}_X(\hat{s}) \neq 0$, there is an $I$-invariant holomorphic structure on $E$ and an $I$-invariant non-zero holomorphic section. Hence $K_X \cdot E \geq 0$ because $K_X$ is numerically effective. Since $E$ can be written as $E = (K_X + L)/2$, $K_X \cdot E \geq 0$ implies $K^2_X \geq -K_X \cdot L$. Since $K_X \cdot \omega > 0$ and $L \cdot \omega > 0$, there is $t \geq 0$ such that

$$\omega \cdot (K_X + tL) = 0.$$ 

By the Hodge index theorem, we have

$$0 \geq (K_X + tL)^2 = K^2_X + 2tK_X \cdot L + t^2L^2 =: f(t).$$

The quadratic function $f(t)$ attains its minimum at $t = -(K_X \cdot L)/L^2$ and the minimum is

$$K^2_X - \frac{(K_X \cdot L)^2}{L^2}.$$ 

Since $L^2 \geq K^2_X \geq -K_X \cdot L$, this quantity is non-negative, and therefore equal to 0. Then we have $L^2 = K^2_X = -K_X \cdot L$, and we see that $f(t) \leq 0$ only when $t = 1$. Hence $(K_X + L)^2 = 0$ and $(K_X + L) \cdot \omega = 0$. By the Hodge index theorem, we have $K_X + L$ is a torsion class, and therefore $E$ is also a torsion class. Since $E$ has an $I$-invariant non-zero holomorphic section, $E$ is an $I$-equivariant trivial bundle. This means $\hat{s}$ is the canonical Spin$^c$-structure.

On the other hand, in the case when $L \cdot \omega > 0$, $SW^{\text{Pin}}_X(\hat{s}) \neq 0$ implies the existence of an $I$-invariant holomorphic structure on $E - K_X$, and an $I$-invariant holomorphic section on it. Arguing similarly, we can prove that $K_X - L$ is a torsion class, and $\hat{s}$ is the anti-canonical Spin$^c$-structure. \( \square \)

As a series of examples for minimal Kähler surfaces of general type with free antiholomorphic involutions, we have hypersurfaces $M_{4k}$ in $\mathbb{CP}^3$ defined by real polynomials of degree $4k$, e.g., $\sum_{j=0}^{3} w_j^4$, with involution $\iota$ given by

$$[x_0, x_1, x_2, x_3] \mapsto [\overline{x}_1, -\overline{x}_0, \overline{x}_3, -\overline{x}_2].$$

Let $\hat{M}_{4k} = M_{4k}/\iota$. We check the assumptions.

**Lemma 5.1.** There is a lift of $w_2(\hat{M}_{4k})$ in the torsion part of $H^2(\hat{M}_{4k}; \mathbb{Z})$.
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\begin{proof}
See \cite{11}. The proof in \cite{11} is on \(\tilde{M}_4\), but it works well for \(\tilde{M}_{4k}\). \qed
\end{proof}

\textbf{Proposition 5.2.} \(w_2(\tilde{K}) = 0\) and \(w_2(M_{4k}) = w_1(\ell_\mathbb{R})^2\). \(\pi^* : H^1(\tilde{M}_{4k}; \mathbb{Z}_2) \to H^1(M_{4k}; \mathbb{Z}_2)\) is surjective.

\begin{proof}
The fact \(w_2(\tilde{K}) = 0\) follows from that the canonical bundle \(K\) of \(M_{4k}\) is given by
\[ K = (4k - 4)H, \]
where \(H\) is the hyperplane section. By Lemma 5.1 and \cite{17} §1, Remark 3(2)], there exists a class \(\alpha \in H^1(M_{4k}; \ell)\) such that \(w_2(M_{4k}) = \alpha \cup \alpha\). Since \(\pi_1(M_{4k}) = 1, \pi_1(M_{4k}) = \mathbb{Z}/2\), \(\alpha\) must be \(w_1(\ell_\mathbb{R})\) and \(\pi^*\) is surjective. \qed
\end{proof}

\textbf{Corollary 5.3.} There exists a canonical Spin\(^c\)-structure \(s_0\) on \(M_{4k} \to \tilde{M}_{4k}\).

\textbf{Proposition 5.4.} The moduli space for \((\tilde{M}_{4k}, s_0)\) is orientable and its virtual dimension is 0. Therefore the Pin\(^{-}(2)\)-monopole invariant of \((\tilde{M}_{4k}, s_0)\) can be defined as a \(\mathbb{Z}\)-valued invariant.

\begin{proof}
Note \(b_+^\ell(\tilde{M}_{4k}) = 0\). In order to prove the orientability of the moduli space, it suffices to prove the Dirac index, \(\text{ind} \, D\), of \(\tilde{s}_0\) is even by \cite{13} Proposition 2.15. Let \(d(\tilde{s}_0)\) be the virtual dimension of the moduli space. Since \(\ell\) is free, we have \(d(\tilde{s}_0) = \frac{1}{2}d(s_0) = 0\). Then
\[ 0 = d(\tilde{s}_0) = \text{ind} \, D - (b_0^\ell - b_1^\ell + b_+^\ell). \]
Since \(\ell\) is nontrivial, \(b_0^\ell\) is 0. Therefore
\[ \text{ind} \, D = b_+^\ell = \frac{1}{2}(1 + b_+(M_{4k})) = \frac{1}{2} \left( \frac{4k}{3}(16k^2 - 24k + 11) \right). \]
(For the calculation of \(b_+(M_{4k})\), see e.g. \cite{13} Example 4.27.) Therefore \(\text{ind} \, D\) is even. \qed
\end{proof}

\textbf{Remark 5.5.} Note that \(\tilde{M}_4\) is diffeomorphic to an Enriques surface. On the other hand, all of the ordinary Seiberg-Witten invariants of \(\tilde{M}_{4k}\) for \(k > 1\) are zero by a theorem due to S. Wang \cite{30}.

\textbf{5.2. Elliptic surfaces.} In this subsection, the Pin\(^{-}(2)\)-monopole invariants of the quotient manifolds of some elliptic surfaces are computed. First, we construct anti-holomorphic involutions on certain elliptic surfaces over \(\mathbb{C}\mathbb{P}^1\).

A method to construct elliptic fibrations by using hyperelliptic involutions is given in Gompf-Stipsicz's book \cite{9}, §3.2. Let \(\Sigma_k\) be a Riemannian surface of genus \(k\), and \(h_k : \Sigma_k \to \Sigma_k\) be a hyperelliptic involution. Take the diagonal \(\mathbb{Z}_2\)-action \(h_k \times h_1\) on \(\Sigma_k \times \Sigma_1\). Dividing by the \(\mathbb{Z}_2\)-action, we obtain the quotient \((\Sigma_k \times \Sigma_1)/\mathbb{Z}_2\) with \(4(2k+2)\) singular points. Resolving the singular points makes a complex manifold \(X(k+1)\). Dividing the projection \(pr_1 : \Sigma_k \times \Sigma_1 \to \Sigma_k\) and extending it to the resolution, we obtain the elliptic fibration \(\varpi : X(k+1) \to \mathbb{C}\mathbb{P}^1\). It is well-known that \(X(n)\) is diffeomorphic to \(E(n)\), the fiber sum of \(E(1) = \mathbb{C}\mathbb{P}^2 \# 9\mathbb{C}\mathbb{P}^2\).

We construct an anti-holomorphic free involution on \(X(2n)\). Take the antipodal map \(a_0\) on \(\mathbb{C}\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}\) defined by \(z \mapsto z^* := -1/z\). Choose \(k\) distinct points \(a_1, \ldots, a_k\) on \(\mathbb{C}\mathbb{P}^1\) satisfying \(0 < |a_i| < 1\). Let \(\Sigma_k\) be the hyperelliptic curve defined by the equation
\[ w^2 = z(z - a_1)(z - a_1^*) \cdots (z - a_k)(z - a_k^*), \]
and $\Sigma_k \to \mathbb{C}P^1$ the associated double covering branched at $a_1, a_1^*, \ldots, a_k, a_k^*, 0, \infty$. Then the antipodal map $\iota_0$ on the base $\mathbb{C}P^1$ can be lifted to an anti-holomorphic map $\sigma_k$ on $\Sigma_k$ with order 2 if $k$ is odd, and with order 4 if $k$ is even.

Suppose $k = 2n - 1$ for a positive integer $n$. Take the diagonal action $\sigma_{2n-1} \times \sigma_1: \Sigma_{2n-1} \times \Sigma_1 \to \Sigma_{2n-1} \times \Sigma_1$. Then $\sigma_{2n-1} \times \sigma_1$ descends to a free involution on the quotient $(\Sigma_{2n-1} \times \Sigma_1)/\mathbb{Z}_2$. Furthermore we can easily extend it to an anti-holomorphic free involution $\iota$ on $X(2n)$ which covers the antipodal map $\iota_0$ on the base $\mathbb{C}P^1$.

**Proposition 5.6.** The surface $X(2n)$ admits a Kähler form $\omega$ such that $\iota^* \omega = -\omega$.

**Proof.** We can easily construct a Kähler form $\omega_0$ on $\Sigma_k \times \Sigma_1$ such that $(h_k \times h_1)^* \omega_0 = \omega_0$ and $(\sigma_k \times \sigma_1)^* \omega_0 = -\omega_0$. Then $\omega_0$ induces a singular Kähler form $\hat{\omega}_0$ on $(\Sigma_k \times \Sigma_1)/\mathbb{Z}_2$. By the results due to Fujiki [3], we can obtain a Kähler form $\omega$ on $X(2n)$. Moreover we can choose $\omega$ such that $\iota^* \omega = -\omega$. \hfill $\square$

Let $\hat{X}(2n) = X(2n)/\iota$. By construction, $\varpi: X(2n) \to \mathbb{C}P^1$ descends to $\hat{\varpi}: \hat{X}(2n) \to \mathbb{R}P^2$. The general fiber of $\hat{\varpi}$ is a torus. Note that $X(2) = E(2)$ is a $K3$ surface.

**Proposition 5.7.** The quotient manifold $\hat{X}(2) = X(2)/\iota$ is diffeomorphic to an Enriques surface.

**Proof.** ([5] and [4], §15.1.) Take an $I$-invariant holomorphic form $\phi$ on $X(2).$ By the Calabi-Yau theorem, there exists a unique Kähler-Einstein metric. Then $\phi$ and $\omega$ induce a hyper-Kähler structure on $X(2)$. There exists a complex structure for which $\iota$ is a holomorphic free involution. Thus $\hat{X}(2)$ is an Enriques surface. \hfill $\square$

Since $X(2n)$ is diffeomorphic to the fiber sum of $E(2) = K3$ with $E(2n - 2)$, $\hat{X}(2n)$ is diffeomorphic to the fiber sum of the fibration $\hat{\varpi}: \hat{X}(2) \to \mathbb{R}P^2$ with $E(n - 1)$.

**Proposition 5.8.** If $k \equiv 2$ modulo 4, then there exists a canonical Spin$^c$-structure $\hat{\xi}_0$ on $X(k)$.

**Proof.** Since $\hat{X}(4m + 2)$ is the fiber sum of $\hat{\varpi}: \hat{X}(2) \to \mathbb{R}P^2$ with $E(2m)$, it is easy to see that $\hat{X}(4m + 2)$ is a non-spin manifold with $\pi_1(\hat{X}(4m + 2)) = \mathbb{Z}/2$ whose intersection form is isomorphic to

\[(2m + 1)(-E_8) \oplus (4m + 1)H,\]

where $H$ is a hyperbolic form. Then it follows from a result of Hambleton-Kreck [12] (Cf. [29]) that $\hat{X}(4m + 2)$ is homeomorphic to the connected sum

\[\Sigma \# (2m + 1)|E_8| \# (4m + 1)(S^2 \times S^2),\]

where $\Sigma$ is a rational homology 4-sphere such that $\pi_1(\Sigma) = \mathbb{Z}/2$ and $w_2(\Sigma) \neq 0$, and $|E_8|$ is the $E_8$-manifold, i.e., the simply-connected topological manifold whose intersection form is isomorphic to $-E_8$. Since $|E_8|$ and $S^2 \times S^2$ are spin and $w_2(\Sigma)$ is a torsion class, $w_2(\hat{X}(4m + 2))$ is a torsion class.
Proposition 5.11. Every \( D \) divisor of the proof is similar to that of Proposition 5.2. 

Take an \( I \)-invariant divisor \( D_k \) of \( X(n) \) of the form

\[
D_k = \sum_{i=1}^{k} (F_i + IF_i),
\]

where \( F_i \) are general fibers. Let \( E_k \) be the line bundle associated to \( D_k \). Then \( E_k \) can be written as the pull-back \( E_k = \varpi^* L \) where \( \varpi \colon X(n) \to \mathbb{C}P^1 \) is the elliptic fibration and \( L \) is a line bundle over \( \mathbb{C}P^1 \) of degree \( 2k \). Let \( \hat{E}_k = E_k/I \).

The next is an analogue of [7, Proposition 4.2] or [3, Proposition 42].

Theorem 5.9. Let \( \hat{X} = \hat{X}(4m + 2) \) and \( \hat{s}_k = \hat{s}_0 \otimes \hat{E}_k \). The moduli space \( \mathcal{M}(\hat{X}, \hat{s}_k) \) is orientable and the corresponding invariant is

\[
\text{SW}^{\text{Pin}}(\hat{X}, \hat{s}_k) = \pm \left( \frac{2m}{k} \right).
\]

The rest of this section is devoted to the proof of Theorem 5.9. For \( \varinjlim \), let \( q^* = t_0(q) \) where \( t_0 \) is the antipodal map. Choose \( 2m \) distinct points \( q_1, \ldots, q_{2m} \) on \( \mathbb{C}P^1 \) such that all of \( q_1, \ldots, q_{2m} \) and \( q_1, \ldots, q_{2m} \) are distinct. Let \( F_1 = \varpi^{-1}(q_i) \). Then \( IF_i = \varpi^{-1}(q_i) \), and we obtain an \( I \)-invariant canonical divisor \( D_m = \sum_{i=1}^{2m} (F_i + IF_i) \) of \( \hat{X} = X(4m + 2) \). Let \( \eta \) be the corresponding \( I \)-invariant holomorphic section on the canonical bundle \( \mathcal{K} \). By Corollary 4.20, the Pin\(^{-}(2)\)-monopole moduli space \( \mathcal{M}(\hat{X}, \hat{s}_k) \) can be identified with

\[
\mathcal{V}_I(E, E', \Sigma, \eta)^I \cong M^\ast(E, E', \mathcal{K}, \Sigma, \eta)^I \cong \mathcal{D}_b(\Delta)^I,
\]

where \( E' = K \otimes E^{-1} \) and \( \Delta = Z(\eta) \).

Lemma 5.10. The moduli space \( \mathcal{M}(\hat{X}, \hat{s}_k) \) is 0-dimensional and orientable.

Proof. Let \( s_k \) be the Spin\(^c\) structure on \( X \) of the canonical reduction of \( \pi^* s_k \). Note that \( c_1(L)^2 = 0 \), \( \tau(X) = -16(2m + 1) \) and \( e(X) + 3\tau(X) = 0 \), where \( L = K^{-1} \otimes E_k^2 \) is the determinant line bundle, and \( e(X) \) and \( \tau(X) \) are the Euler characteristic and signature of \( X \). Then the virtual dimension \( d(s_k) \) of the Seiberg-Witten moduli space of \( (X, s_k) \) is

\[
d(s_k) = \frac{1}{4}(c_1(L)^2 - e(X) - 3\tau(X)) = 0.
\]

Then we have \( d(\hat{s}_k) = d(s_k)/2 = 0 \).

Since the index of the Dirac operator \( D_{\hat{A}} \) on \( \hat{s}_k \) is a half of that on \( s \), we have

\[
\text{ind} D_{\hat{A}} = \frac{1}{2} \left( \frac{1}{4}(c_1(L)^2 - \tau(X)) \right) = 4m + 2.
\]

Especially, \( \text{ind} D_{\hat{A}} \) is even. Then the moduli space is orientable by [18, Proposition 2.15].

Proposition 5.11. Every Pin\(^{-}(2)\)-monopole solution corresponding to a divisor \( D \in \mathcal{D}_b(\Delta)^I \) is non-degenerate.
Proof. Since $D_m - D$ and $D$ have no intersection for $D \in \mathcal{D}_b(\Delta)^I$, Proposition \[1.23\] implies that the first cohomology $H^1$ of the deformation complex of the solution corresponding to $D$ is 0. Therefore the second cohomology $H^2$ is also 0 since $d(\mathbf{r}_k) = 0$ and $H^1 = 0$. \[\square\]

For a subset $\{i_1, \ldots, i_k\} \subset \{1, \ldots, 2m\}$, we can take an $I$-invariant divisor $\sum_{j=1}^k (F_{i_j} + IF_{i_j}) \in \mathcal{D}_b(\Delta)^I$. Since this correspondence is bijective, the number of elements in $\mathcal{D}_b(\Delta)^I$ is $\binom{2m}{k}$. The only left task to prove Theorem \[5.9\] is that all of the divisors in $\mathcal{D}_b(\Delta)^I$ have same orientation.

**Proposition 5.12.** For a set of distinct points $b_1, \ldots, b_k \in \mathbb{C}P^1$, let $B = b_1 + \cdots + b_k$ be its divisor. Let $F_j = \mathcal{O}^*(b_j)$ and $D = F_1 + \cdots + F_j$. Then $\mathcal{O}^* : H^0(\mathcal{O}_{\mathbb{C}P^1}(B)) \to H^0(\mathcal{O}_X(D))$ is an isomorphism, and

$$H^0(\mathcal{O}_X(D)) = \mathbb{C}^{k+1}, \quad H^1(\mathcal{O}_X(D)) = 0, \quad H^2(\mathcal{O}_X(D)) = \mathbb{C}^{4m-k}$$

**Proof.** Consider the short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{O}_D(D) \to 0$$

and its associated long exact sequence. Note that $\mathcal{O}_{F_j}(D)$ is the holomorphic normal bundle of $F_j \subset X$. Since this is holomorphically trivial and the genus of $F_j$ is 1, we have

$$H^0(\mathcal{O}_D(D)) = H^1(\mathcal{O}_D(D)) = \mathbb{C}^k.$$ Since $X$ is simply-connected, $H^1(\mathcal{O}_X) = 0$ and therefore

$$H^1(\mathcal{O}_X(D)) = H^0(\mathcal{O}_X) \oplus H^1(\mathcal{O}_D(D)) = \mathbb{C}^{k+1}.$$ By the Serre duality,

$$H^2(\mathcal{O}_X(D)) = H^0(\mathcal{O}_X(K - D)) = \mathbb{C}^{4m-k}.$$ Also we have $H^1(\mathcal{O}_X(D)) = 0$. \[\square\]

Let $t = (B, \phi, \psi) \in \mathcal{A}(E, h) \times \Omega^{0,0}(E) \times \Omega^{2,0}(E^{-1})$ be an $I$-invariant solution to \[1.24\], i.e., $((B, \phi), (B', \phi'))$ where $B' = \Sigma \otimes B^{-1}$ and $\phi' = \psi$ which is $I$-invariant and satisfies \[1.24\]. The deformation complex at $t$ is given by

$$D^0_t(f) = \left( \begin{array}{c} df \\ f\phi \\ -f\psi \end{array} \right), \quad D^1_t(\hat{B}, \hat{\phi}, \hat{\psi}) = \left( \begin{array}{c} \hat{B}^{0,1} \phi + w(\hat{B}^{0,1})^* \bar{\psi} + \bar{\partial}_B \bar{\phi} + \bar{\partial}_B \bar{\psi} \\ \bar{\partial}_B \hat{B}^{0,1} - \frac{i}{2} \bar{\psi} \bar{\phi} - \frac{1}{2} \bar{\psi} \bar{\phi} \\ \Lambda \psi + i[\text{Re}(\bar{\phi} \bar{\psi}) - \text{Re}(\hat{\bar{\psi}} \bar{\phi})] \end{array} \right),$$

where $w(\hat{B}^{0,1})^*$ is the adjoint of the multiplication operator $\hat{B}^{0,1} \wedge$. The adjoint of $D^0_t$ is

$$(D^0_t)^* (\hat{B}, \hat{\phi}, \hat{\psi}) = d^* \hat{B} - i \text{Im}(\bar{\psi} \hat{\phi}) + i \text{Im}(\hat{\bar{\psi}} \bar{\phi}).$$
Set \( \hat{b} = \sqrt{2} \hat{B}, \) \( U = \frac{1}{\sqrt{2}} \hat{\phi}, \) \( V = \frac{1}{\sqrt{2}} \hat{\psi}. \) Replace \( \hat{b} \) by \( \hat{b}^{0,1} \) via the identification \( i\Omega_X \cong \Omega^{0,1}. \) We introduce the operators \( Q_t \) and \( Q_t^0 \) by

\[
Q_t = \begin{pmatrix}
-\text{id} & 0 & 0 & 0 \\
0 & \sqrt{2}\text{id} & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}}\omega & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{2}}
\end{pmatrix}
(D_t^{1})^{*},
\]

\[
Q_t \begin{pmatrix} \hat{b} \\ \hat{\phi} \\ \psi \end{pmatrix} = \begin{pmatrix}
-\hat{b}^{0,1}U - w(\hat{b}^{0,1})^* V - \partial_B \hat{\phi} - \partial_B \hat{\psi} \\
\partial \hat{b}^{0,1} - V \hat{\phi} - \hat{\psi} U \\
\partial^* \hat{b}^{0,1} - \hat{\phi} \hat{U} + \hat{\psi} \hat{V}
\end{pmatrix},
\]

\[
Q_t^0 \begin{pmatrix} \hat{b} \\ \hat{\phi} \\ \psi \end{pmatrix} = \begin{pmatrix}
-\partial_B \hat{\phi} - \partial_B \hat{\psi} \\
\partial \hat{b}^{0,1} \\
\partial^* \hat{b}^{0,1}
\end{pmatrix}.
\]

Let \( K \) and \( C \) be the kernel and cokernel of \( Q_t^0, \) respectively,

\[ K := \ker Q_t^0 = \ker(\partial_B + \partial_B^*)^I \oplus \mathbb{H}^1(\mathcal{O}_X)^I, \]

\[ C := \text{coker} Q_t^0 = \ker(\partial_B^* + \partial_B)^I \oplus \mathbb{H}^2(\mathcal{O}_X)^I \oplus \mathbb{H}^0(\mathcal{O})^I. \]

Note that \( \mathbb{H}^1(\mathcal{O}_X) = 0 \) since \( b_1(X) = 0. \) By Proposition 5.12 \( \ker(\partial_B^* + \partial_B) = H^1(\mathcal{O}_X(D)) = 0. \)

To see the orientation of the solution \( t, \) we consider

\[ \text{pr}_C \circ (Q_t^0|_K) : K = \ker(\partial_B + \partial_B^*)^I \rightarrow C = H^2(\mathcal{O}_X)^I \oplus H^0(\mathcal{O}_X)^I. \]

This can be identified with

\[ R_t : H^0(D)^I \oplus H^0(K - D)^I \rightarrow H^0(K)^I \oplus H^0(\mathcal{O}_X)^I, \]

\[ R_t = \begin{pmatrix} \hat{\phi} \\ \hat{\psi} \end{pmatrix} = \begin{pmatrix}
-V \hat{\phi} - U \hat{\psi} \\
f(-\langle \hat{\phi}, U \rangle + \langle \hat{\psi}, V \rangle) d\text{vol}_g
\end{pmatrix}. \]

Note that \( R_t \) is a linear isomorphism. If we fix orientations of the domain and target of \( R_t, \) the orientation of the solution \( t \) is determined by the sign of the determinant of \( R_t. \) (See e.g. [27], [19].) We want to represent \( R_t \) by a matrix with some explicit bases of \( H^0(D)^I \oplus H^0(K - D)^I \) and \( H^0(K)^I \oplus H^0(\mathcal{O}_X)^I. \)

Consider a complex manifold \( Z \) with a divisor \( D. \) Let \( \mathcal{L}_D \) be the holomorphic line bundle associated with \( D. \) Then \( H^0(\mathcal{L}_D) \) can be identified with the space of meromorphic functions \( f \) such that \( \lambda f \) are holomorphic for every local defining function \( \lambda \) of \( D. \) This space is denoted by \( \mathfrak{M}(D). \)

Note that \( 1 \in \mathfrak{M}(D) \) corresponds to the holomorphic section \( \phi_D \) defined by the divisor \( D. \)

Suppose \( \iota \) is an anti-holomorphic free involution on \( Z \) and the divisor \( D \) is \( I- \)invariant. The \( I- \)action on \( H^0(\mathcal{L}_D) = \mathfrak{M}(D) \) is given by \( f \mapsto \iota^* f. \)

Consider \( (Z, \iota) = (\mathbb{C}^1, \iota_0) \) where \( \iota_0 \) is the antipodal map. Choose \( p \in \mathbb{C} \subset \mathbb{C}^1 \) such that \( |p| = 1. \) Let \( p^* = \iota_0(p) = -1/\bar{p} = -p. \) Take the \( \iota_0- \)invariant divisor \( B = p + p^*. \) In terms of meromorphic functions, we can take the following for a complex basis of \( \mathfrak{M}(B) = H^0(\mathcal{L}_B). \)

\[ 1, \quad \frac{z - p^*}{z - p}, \quad \frac{z - p}{z - p^*}. \]
We want to have a (real) basis for $\mathfrak{M}(B)^I = H^0(\mathcal{L}_B)^I$. Via the projection to the $I$-invariant part, $f \mapsto \frac{1}{2}(f + if)$, we can see that the following is a real basis for $\mathfrak{M}(B)^I = H^0(\mathcal{L}_B)^I$:

$$1, \quad \frac{z^2 + p^2}{z^2 - p^2}, \quad \frac{2pz}{z^2 - p^2}.$$ 

Let us consider the case of $\varpi : X = X(4m + 2) \to \mathbb{C}P^1$ with the antiholomorphic involution $\iota$. Choose $2m$ distinct points $p_1, \ldots, p_k, q_1, \ldots, q_{2m-k}$ on $\mathbb{C}P^1$ such that $|p_j| = 1$, $|q_l| = 1$, and all of $p_j$, $p_j^*$, $q_l$, $q_l^*$ are distinct. Take the following divisors on $\mathbb{C}P^1$:

$$B_D = \sum_{j=1}^k (p_j + p_j^*), \quad B_{K-D} = \sum_{l=1}^{2m-k} (q_l + q_l^*), \quad B_K = B_D + B_{K-D}.$$ 

Then $K = \varpi^* B_K$ is a canonical divisor of $X$. Let $D = \varpi^* B_D$. Then $K - D = \varpi^* B_{K-D}$. Let

$$P_j^1 = \frac{z^2 + p_j^2}{z^2 - p_j^2}, \quad P_j^2 = \frac{2p_jz}{z^2 - p_j^2}, \quad Q_l^1 = \frac{z^2 + q_l^2}{z^2 - q_l^2}, \quad Q_l^2 = \frac{2q_lz}{z^2 - q_l^2}.$$ 

Then $\{1, P_j^1, P_j^2\} (j = 1, \ldots, k)$ gives a basis for $\mathfrak{M}(B_D)^I = H^0(B_D)^I \cong H^0(D)^I$. Similarly, $\{1, Q_l^1, Q_l^2\} (l = 1, \ldots, 2m - k)$ and $\{1, P_j^1, P_j^2, Q_l^1, Q_l^2\} (j = 1, \ldots, k, l = 1, \ldots, 2m - k)$ give bases for $\mathfrak{M}(B_{K-D})^I = H^0(B_{K-D})^I \cong H^0(K-D)^I$ and $\mathfrak{M}(B_K) = H^0(B_K)^I \cong H^0(K)^I$, respectively.

Now the divisor $D$ corresponds to an $I$-invariant solution $t = (B, \phi, \psi)$ such that $\varphi^{-1}(0) = D$ and $\psi^{-1}(0) = K - D$. We may assume the following correspondence,

$$1 \in \mathfrak{M}(B_D)^I \longleftrightarrow U = \frac{1}{\sqrt{2}} \phi \in H^0(D)^I,$$

$$1 \in \mathfrak{M}(B_{K-D})^I \longleftrightarrow V = \frac{1}{\sqrt{2}} \psi \in H^0(K-D)^I,$$

$$1 \in \mathfrak{M}(B_K)^I \longleftrightarrow W := U \otimes V \in H^0(K)^I.$$ 

and, for $f \in \mathfrak{M}(D)$, let $fU$ denote the holomorphic section in $H^0(D)$ corresponding to $f$.

Let $\{e\}$ be a real basis of $H^0(\mathcal{O}_X)^I \cong \mathbb{R}$. We choose the basis for $H^0(D)^I \oplus H^0(K-D)^I$ as follows,

$$\{U, P_j^1 U, P_j^2 U\}_{j=1,\ldots,k} \cup \{V, Q_l^1 V, Q_l^2 V\}_{l=1,\ldots,2m-k},$$

and for $H^0(K)^I \oplus H^0(\mathcal{O}_X)^I$,

$$\{W, P_j^1 W, P_j^2 W, Q_l^1 W, Q_l^2 W\}_{j=1,\ldots,k} \cup \{e\},$$

and for $H^0(K)^I \oplus H^0(\mathcal{O}_X)^I$,
With respect to the bases above, the isomorphism $R_t$ is represented by the matrix

$$
\begin{pmatrix}
U & P^1_t U & P^2_t U & \cdots & V & Q^1_t V & Q^2_t V & \cdots \\
-1 & 0 & 0 & \cdots & -1 & 0 & 0 & \cdots \\
0 & -1 & 0 & \cdots & 0 & 0 & 0 & \cdots \\
0 & 0 & -1 & \cdots & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
0 & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots \\
0 & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix}
$$

It is easy to see that the determinant of the matrix above is $-\|U\|^2 - \|V\|^2$.

To prove any other solution $t'$ has the same orientation with the solution $t$ for $D$ above, we need to prove that the determinant of the matrix $R_{t'}$ associated with $t'$ has the same sign with det $R_t$.

For simplicity, let us consider the case when $m = 2, k = 1$. The general case will be obvious. Take the solution $t$ corresponding to the divisors $D = \varphi^*(p_1 + p_1^* + p_2 + p_2^*)$ and $K - D = \psi^*(q_1 + q_1^* + q_2 + q_2^*)$. Exchanging $p_1$ and $q_1$, we obtain another $I$-invariant solution $t'$ corresponding to $D' = \varphi^*(p_1 + p_1^* + p_2 + q_2^*)$ and $K - D' = \psi^*(p_1 + p_1^* + q_2 + q_2^*)$. Let $U, V$ be the holomorphic sections for $t$, and $U', V'$ for $t'$. Without loss of generality, we may assume $U', V'$ are related with $U, V$ by

$$
U' = \frac{p_1}{q_1} z^2 - \frac{q_1}{p_1} U, \quad V' = \frac{q_1}{p_1} z^2 - \frac{p_1}{q_1} V.
$$

The meaning of this is as follows: When the holomorphic section $U'$ associated with the solution $t'$ is considered as an element of $\mathcal{H}^0(D)^I$ (not of $\mathcal{M}(D')^I$), it is represented by the meromorphic function $\frac{p_1}{q_1} z^2 - \frac{q_1}{p_1}$.

We want to represent the map $R_{t'}$ by a matrix with respect to the bases (5.13), (5.14).

Before that we note several useful relations. For $p_j$ and $q_l$ with $|p_j| = |q_l| = 1$, let $a, b$ be the real numbers such that $\frac{p_j}{q_l} = a + ib$. Then we have the following relations.

$$
P^1_j Q^1_l = -1 + \frac{a}{b} P^1_j - \frac{a}{b} Q^1_l
$$

$$
P^1_j Q^2_l = \frac{1}{b} P^2_j - \frac{a}{b} Q^2_l
$$

$$
P^2_j Q^1_l = -\frac{1}{b} Q^1_l^2 + \frac{a}{b} P^2_j
$$

$$
P^2_j Q^2_l = -\frac{1}{b} P^2_j^2 + \frac{1}{b} Q^1_l^2
$$

Let $a, b, c, d, e, f$ be the real numbers such that

$$
\frac{p_1}{q_1} = a + ib, \quad \frac{p_2}{q_1} = c + id, \quad \frac{p_1}{q_2} = e + if.
$$
Then we have $U' = (a + bP_1^i)U$, $V' = (a - bQ_1^i)V$, and the map $R_\nu$ is represented by the matrix

$$
\begin{pmatrix}
W & P_1^i U & P_2^i U & P_1^i U & P_2^i U & V & Q_1^i V & Q_2^i V & Q_3^i V \\
P_1^i W & -a & -b & -b & -a & b & -\frac{b}{\pi} & \frac{b}{\pi} & b \\
P_2^i W & -\frac{ad-bc}{d} & -1 & -\frac{ad-bc}{d} & -1 & -b \\
Q_1^i W & b & -a & -\frac{bc}{\pi} & -\frac{bc}{\pi} & -\frac{ad-bc}{d} \\
Q_2^i W & -1 & -\frac{ad-bc}{d} & -\frac{ad-bc}{d} & -1 & -\frac{b}{\pi} \\
Q_3^i W & -\frac{ad-bc}{d} & -1 & -\frac{ad-bc}{d} & -\frac{ad-bc}{d} & -\frac{b}{\pi} \\
Q_4^i W & -\frac{ad-bc}{d} & -1 & -\frac{ad-bc}{d} & -\frac{ad-bc}{d} & -\frac{b}{\pi} \\
e & -u_0 & -u_1 & -u_2 & -v_0 & v_1 & v_2 & v_3 & v_4
\end{pmatrix}
$$

where $u_0 = \langle U, U' \rangle$, $u_1^2 = \langle P_1^i U, U' \rangle$, $v_0 = \langle V, V' \rangle$, $v_1^2 = \langle Q_1^i V, V' \rangle$. Furthermore, we take one more orientation-preserving basis change given by

$$
(U, P_1^i U) \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = (U', (-b + aP_1^i)U),
$$

$$(V, Q_1^i V) \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = (V', (b + aQ_1^i)V).$$

Then the matrix above is transformed into

$$
\begin{pmatrix}
-1 & -b & -1 & -1 & -\frac{bc}{\pi} & -\frac{b}{\pi} & -\frac{ad-bc}{d} & -\frac{ad-bc}{d} \\
-\frac{ad-bc}{d} & -1 & -\frac{ad-bc}{d} & -1 & -\frac{bc}{\pi} & -\frac{b}{\pi} \\
-1 & -\frac{bc}{\pi} & -\frac{bc}{\pi} & -1 & -\frac{ad-bc}{d} & -\frac{ad-bc}{d} \\
-\frac{ad-bc}{d} & -1 & -\frac{ad-bc}{d} & -1 & -\frac{bc}{\pi} & -\frac{b}{\pi} \\
-\|U'\|^2 & * & * & * & \|V'\|^2 & * & * & * & * \\
\end{pmatrix}
$$

where $*$ are some numbers of inner products of sections. It is easy to see that the determinant of the above matrix is

$$
- \left(\frac{ad-bc}{d}\right)^2 \left(\frac{af-be}{f}\right)^2 (\|U'\|^2 + \|V'\|^2).
$$

Since $\det R_t$ and $\det R_\nu$ have the same sign, the orientation of the solution $t'$ is same with that of $t$.

The general cases when $k \neq 1$ or $m \neq 2$ are similar. In fact, by an exchange of some $p_j$ and $q_l$, the orientation does not change. Thus Theorem 5.9 is proved.

6. Concluding remarks

6.1. More examples with nontrivial $Pin^{-}(2)$-monopole invariants.

By using the gluing formulae in [15], we obtain more examples with nontrivial $Pin^{-}(2)$-monopole invariants. Let $Z \to \hat{Z}$ be a nontrivial double covering which satisfies the following:
(1) $b'_\ell(\hat{Z}) = 0$ for $\ell' = Z \times \{\pm 1\} \subset Z$.
(2) There is a Spin$^c$-structure $\sigma'$ on $Z \to \hat{Z}$ whose characteristic bundle $\hat{L}'$ satisfies $c_1(\hat{L}')^2 = \text{sign}(Z)$.

(For instance, a connected sum of several $S^2 \times \Sigma_g$ and $S^1 \times W$ has a double cover satisfying the above conditions [18, §1.2], where $\Sigma_g$ is a Riemann surface with genus $g \geq 1$ and $W$ is a closed 3-manifold.) Then [18, Theorem 3.11] implies that $\hat{M}_{4k}#\hat{Z}$ and $\hat{X}(4m+2)#\hat{Z}$ has nontrivial Pin$^-(2)$-monopole invariants.

On the other hand, [18, Theorem 3.13] implies that any connected sum $\hat{Y}_1#\cdots#\hat{Y}_N$ such that each $\hat{Y}_i$ is $\hat{M}_{4k}$ or $\hat{X}(4m+2)$ for any $k$ or $m$ has nontrivial Pin$^-(2)$-monopole invariants.

As an application of the nontriviality of the Pin$^-(2)$-monopole invariants, we have the adjunction inequality for local-coefficient classes [18, Theorem 1.15].

6.2. Problems. We suggest several problems for future researches.

- Generalize the results to the case of the real structures with real parts.
  If we can drop the condition that $\iota$ is free in our story, then we might expect some applications to, say, real algebraic geometry. For this purpose, we need to generalize the notion of the Spin$^c$-structure.
- Analogy of SW=Gr [26]. Can SW$^\text{Pin}$ be identified with some kind of real Gromov-Witten invariant? Cf. Tian-Wang [28].
- What is the counter part of Pin$^-(2)$-monopole equations in Donaldson theory? Is there a version of Witten’s conjecture between Pin$^-(2)$-monopole theory and Donaldson theory? More concretely, is SW$^\text{Pin}$ equivalent to some kind of Donaldson invariants?

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