Genericity of Bumpy Metrics, Bifurcation and Stability in Free Boundary CMC Hypersurfaces

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Abstract

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In this thesis we prove the genericity of the set of metrics on a manifold with boundary $M^{n+1}$, such that all free boundary constant mean curvature (CMC) embeddings $\varphi: \Sigma^n \to M^{n+1}$, being $\Sigma$ a manifold with boundary, are non-degenerate (Bumpy Metrics), (Theorem 2.4.1). We also give sufficient conditions to obtain a free boundary CMC deformation of a CMC immersion (Theorems 3.2.1 and 3.2.2), and a stability criterion for this type of immersions (Theorem 3.3.3 and Corollary 3.3.5). In addition, given a one-parametric family, $\{\varphi_t: \Sigma \to M\}_{t \in I}$, of free boundary CMC immersions for the family $\{\varphi_t\}$, via the implicit function theorem when the kernel of the Jacobi operator $J$ is non-trivial, (Theorems 4.2.3 and 4.3.2), and we study stability and instability problems for hypersurfaces in this bifurcated branches (Theorems 5.3.1 and 5.3.3).

**Keywords:** Constant Mean Curvature, Free Boundary, Bumpy Metrics, Jacobi Operator, Stability, Bifurcation.
Resumo

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Nesta tese, provamos a genericidade do conjunto de métricas em uma variedade com fronteira
$M^{n+1}$, de modo que todos os mergulhos de curvatura média constante (CMC) e fronteira livre
$\varphi : \Sigma^n \to M^{n+1}$, sendo $\Sigma$ uma variedade com fronteira, sejam não-degenerados (Métricas Bumpy),
(Teorema 2.4.1). Nós também fornecemos condições suficientes para obter uma deformação CMC
e fronteira livre de uma imersão CMC (Teoremas 3.2.1 and 3.2.2), e um critério de estabilidade
para este tipo de imersões (Teorema 3.3.3 and Corolario 3.3.5). Além disso, dada uma família
1-paramétrica, $\{\varphi_t : \Sigma \to M\}_{t \in I}$, de imersões de CMC e fronteira livre, damos os critérios para a
existência de ramos de bifurcação suaves de imersões CMC e fronteira livre para a família $\{\varphi_t\}$, por
meio de o teorema da função implícita quando o kernel do operador Jacobi $J$ é não-trivial, (Teore-
mas 4.2.3 and 4.3.2), e estudamos o problema da estabilidade e instabilidade para hipersuperfícies
em naqueles ramos de bifurcação (Teoremas 5.3.1 and 5.3.3).

Palavras-chave: Curvatura Média Constante, Fronteira Livre, Métricas Bumpy, Operador de
Jacobi, Estabilidade, Bifurcação.
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Introduction

In variational calculus there exists a class of problems called *isoperimetric*; one of them is to find the minimum area among all hypersurfaces of a Riemannian manifold enclosing the same volume. We know that solutions to this problem are hypersurfaces with constant mean curvature (in short CMC). More precisely, if \( \varphi : \Sigma \to M \) is an immersion of an orientable \( n \)-dimensional compact manifold \( \Sigma \) into the \( (n + 1) \)-dimensional Riemannian manifold \( M \), the condition that \( \varphi \) has nonzero constant mean curvature \( H_0 \) is equivalent to the fact that \( \varphi \) is a critical point of such isoperimetric problem (see [6]).

If \( \varphi_t \) is a smooth variation of \( \varphi \), \( t \in (-\epsilon, \epsilon) \), \( \varphi_0 = \varphi \), such that \( V_t = V_0 \), for all \( t \in (-\epsilon, \epsilon) \), where \( V_t \) is the volume of \( \varphi_t \), a standard approach to find the solution of such a isoperimetric problem is by looking at the critical points of functional \( f(t) = A_t + \lambda V_t \), where \( A_t \) is the area of \( \varphi_t \) and \( \lambda \) is constant, which is analogous to Lagrange multipliers method. When \( \lambda = nH_0 \), we have the equivalence aforementioned.

In the case where \( M \) is a manifold with boundary, the isoperimetric problem can be described as follows. One wants to minimize the area between all compact hypersurfaces in \( M \) with boundary contained in the boundary of \( M \) and whose interior lies in the interior of \( M \) (\( \varphi(\Sigma) \cap \partial M = \varphi(\partial \Sigma) \)), which divides \( M \) in two regions such that the closure of one of them is compact and with prescribed volume. Solutions of this problem are the orthogonal constant mean curvature hypersurfaces, which will be called *free boundary CMC hypersurfaces*. Let \( H_0 \) denote the value of the (constant) mean curvature of one such hypersurface. If \( H_0 = 0 \) then we say that \( \varphi(\Sigma) \) is a *orthogonal free boundary minimal hypersurface*. A. Ros and E. Vergasta obtain results on the stability of solutions of this isoperimetric problem in the case where \( M \) is compact and convex, see [24]. There are well-known results obtained by Abraham [1] and Anosov [5] which are related to properties of geodesic flows for generic Riemannian metrics on a closed smooth manifold.

In this dissertation we study four problems related to free boundary CMC hypersurfaces: i) The genericity of the metrics that makes these hypersurfaces are non-degenerate critical points of the functional area (Chapter 2); ii) The existence and stability of variations of this type of hypersurfaces (Chapter 3); iii) The existence of branches of bifurcation in families of immersions with these characteristics (Chapters 4); and iv) The stability in those branches of bifurcation (Chapters 5). We now turn to explain the content of each chapter and the most important results obtained.

In Chapter 1 we define the objects and notation that we will use. We are going to give some basic definitions; some results about the function spaces, operators and applications that are necessary for the work. In particular, given a orthogonal CMC immersion \( \varphi_0 : \Sigma^n \to M^{n+1} \), we define
the Banach space of the Hölder functions in $\Sigma$, with regularity $C^{i,\alpha}$, and satisfying $g(\nabla f, \bar{n}_{\partial M}) + \Pi^{\partial M}(\bar{\eta}_{\Sigma_0}, \bar{\eta}_{\Sigma_0}) f = 0$, where $g$ is a background riemannian metric in $M$, $\nabla$ is the gradient in $\Sigma$ defined by the pullback metric of $g$ generated by $\varphi_0$, $\bar{n}_{\partial M}$ and $\bar{\eta}_{\Sigma_0}$ are the orthonormal vectors of $\partial M$ and $\Sigma_0$ respectively, and $\Pi^{\partial M}$ the second fundamental form of the boundary of $M$, we denote as $C_{\partial}^{i,\alpha}(\Sigma)$. There is an identification of $C_{\partial}^{i,\alpha}(\Sigma)$ with the set of orthogonal immersions in $M$, $f \mapsto \varphi_f$, defined by $\varphi_f(p) := \exp_{\varphi_0(p)}(f(p)\bar{n}_{\Sigma_0}(p))$, where $\exp$ is the exponential map in $M$ defined by the metric $g$ (see Proposition 2.3.1). We will define the Jacobi Operator of $\varphi_0$ as $J_{\varphi_0}(f) = \Delta_{\Sigma_0} f - (||\Pi^{\Sigma_0}||_{HS}^2 + \text{Ric}(\eta_{\Sigma_0}, \eta_{\Sigma_0})) f$, where $\varphi_0(\Sigma) = \Sigma_0$, which, restricted to space $C_{\partial}^{i,\alpha}(\Sigma)$, is a Fredholm operator of zero index, extremely important condition in the development of all this work. Also, we give a characterization of free boundary CMC hypersurfaces non-degenerate in terms of the Jacobi Operator kernel.

In Chapter 2, we prove the genericity of Riemannian metrics of $M$ for which every free boundary orthogonal minimal immersion $\varphi : \Sigma^n \to M^{n+1}$ is non-degenerate, which we will call $(M, \Sigma)$-bumpy metrics. The group of diffeomorphisms of $\Sigma$ acts freely on the set of embeddings of $\Sigma$ into $M$. We denote by $[\varphi]$ its equivalence class with respect to this action and as $E_{\partial, \gamma}^+(\Sigma, M)$ the set of $[\varphi]$ such that $\varphi(\Sigma) \cap \partial M = \varphi(\partial \Sigma)$ and $\varphi$ is an $\gamma$-orthogonal embedding with CMC. So, our main theorem is as follows:

**Theorem 2.4.1** Let $M^{n+1}$ be a differential manifold with smooth boundary $\partial M \neq \emptyset$, and $\Sigma^n$ a compact differential manifold with smooth boundary $\partial \Sigma \neq \emptyset$. $\text{Met}^k(M)$ the set of all $C^k$ metric tensors in $M$, $k \geq 2$, $\Gamma \subset \text{Met}^k(M)$ be an open subset with a structure of separable Banach space. We define the following set

$$\mathcal{M} = \{ (\gamma, [\varphi]) \in \Gamma \times E_{\partial}(\Sigma, M) : [\varphi] \in E_{\partial, \gamma}^+(\Sigma, M), \varphi \text{ is } \gamma\text{-minimal} \}.$$

Then,

1. $\mathcal{M}$ is a separable Banach manifold modelled on $\Gamma$.
2. $\Pi : \mathcal{M} \to \Gamma$, defined by $\Pi(\gamma, [\varphi]) = \gamma$, is a Fredholm map with index 0.
3. $\gamma_0$ is critical value of $\Pi$ if and only if there is a $\gamma_0$-minimal embedding $\varphi_0 : \Sigma \to M$ which is degenerate.

In (2) we see that $\Pi$ meets the conditions of the Sard-Smale theorem (see [26]), which we use in (3) to test the genericity of regular points of $\Pi$. We note that, when $M$ is noncompact, $\text{Met}^k(M)$ has no Banach space structure; so, we have to choose a suitable subset $\Gamma \subset \text{Met}^k(M)$ where a Banach structure can be found. Typically, $\Gamma$ consist of metrics satisfying some growth control at infinity (see definition 2.2.1). This theorem is the version for minimal hypersurfaces with free boundary analogous to results of Brian White [28], and Biliotti-Javaloyes-Piccione [9]. This result is valid also for the case where the mean curvature is a non-zero constant (see section 2.4.3).

In Chapter 3 we find results concerning to existence and uniqueness of CMC deformation of free boundary CMC immersions, $\{\varphi_t\}_{t \in I}$, in addition to some criteria of stability for this type of deformations. Within the Theory of Optimization and Stability in different areas of science and technology is important criteria that serves as tools for study of phenomena such as energy conservation, minimization of materials, optimization of resources, etc. The term stability refers
to the fact that surfaces with constant mean curvature must minimize the area of enclosed volume among all nearby surfaces that enclose the same volume. A hypersurface with constant mean curvature is stable if the second variation of the functional area is nonnegative. This is a version for free boundary CMC hypersurfaces in $n$-dimension obtained by Miyuki Koiso (see [16]), who study the case for surfaces in $\mathbb{R}^3$ with fixed boundary.

To obtain these results we have based on the properties of the eigenvalues and eigenfunctions of the Jacobi Operator $J_{\varphi_0}$, which is interpreted as the linearization of the mean curvature $H_{\varphi_0}$ at $\varphi_0$ (see Section 1.3). So, the first perturbation existence theorem, when the eigenvalues of $J_{\varphi_0}$ are non-zero, is the following:

**Theorem 3.2.1.** Let $\varphi_0 \in C^{j+1,\alpha}(\Sigma, M)$ a free boundary CMC immersion, with mean curvature $H_0$. Suppose that $\dim(\ker(J_{\varphi_0})) = 0$, this is, the eigenvalues of problem $J_{\varphi_0}(f) = \lambda f$, $f \in C^{j,\alpha}_0(\Sigma_0)$, are nonzero. Then, there is a neighborhood $\hat{I}$ of $H_0 \in \mathbb{R}$ and a unique injective $C^1$ mapping, $\zeta : \hat{I} \to C^{j,\alpha}_0(\Sigma_0)$, such that $\zeta(H_0) = 0$ and $\varphi_{\zeta(H)}$ is a free boundary CMC immersion with mean curvature $H$.

Moreover, if $\psi : \Sigma \to M$ is an free boundary CMC immersion sufficiently close to $\varphi_0$, in the topology of $C^{j,\alpha}$, then $\psi$ must be equal (up to diffeomorphisms) to some $\varphi_{\zeta(H)}$.

Now, if $J_{\varphi_0}$ has some eigenvalue equal to zero, then we have the following theorem:

**Theorem 3.2.2** Let $\varphi_0 \in C^{j+1,\alpha}(\Sigma, M)$ a free boundary CMC immersion, with mean curvature $H_0$. Suppose that:

- $\dim(\ker(J_{\varphi_0})) = 1$. This is, $\lambda = 0$ is an eigenvalue of multiplicity 1 for $J_{\varphi_0}$, and
- $\int_{\Sigma_0} f_0 \, \text{vol}_{\varphi_0(g)} \neq 0$ for some $f_0 \in \ker(J_{\varphi_0}) - \{0\}$.

Then there exist a neighborhood $W \subset \ker(J_{\varphi_0})$ of 0 and a unique injective map $C^1$

$$((\xi, \eta) : W \to (C^{j,\alpha}_0(\Sigma_0) \cap \ker(J_{\varphi_0}))^\perp \times \mathbb{R},$$

such that $(\xi, \eta)(0) = (0, H_0)$ and such that $\varphi_{f+\xi(f)} : \Sigma \to M$, with $f \in W$, is an free boundary CMC immersion, with mean curvature $\eta(f)$.

Moreover, if $\psi : \Sigma \to M$ is an free boundary CMC immersion sufficiently close to $\varphi_0$, in the topology of $C^{j,\alpha}$, then $Y$ must be equal (up to diffeomorphisms) to some $\varphi_{f+\xi(f)}$.

In both cases, we also obtain uniqueness in this perturbation, up to parameterizations.

Now, regarding the stability of the free boundary CMC hypersurfaces, we obtained the following result, where $\lambda_1 < \lambda_2 \leq \ldots$ are the eigenvalues of $J_{\varphi_0}$ (see remark 3.3.1):

**Theorem 3.3.3.** Let $\varphi_0 : \Sigma \to M$ a free boundary CMC immersion. Let $\lambda_i$, $i \geq 1$ be the eigenvalues $J_{\varphi_0}$.

1. If $\lambda_1 \geq 0$, then $\varphi_0$ is stable.

2. If $\lambda_1 < 0 < \lambda_2$, then there is a unique function $\kappa \in C^{j,\alpha}(\Sigma)$ such that $J(\kappa) = 1$ and we have that:

   - (2-a) If $\int_{\Sigma} \kappa \, \text{vol}_{\varphi_0(g)} \leq 0$, then $\varphi_0$ is stable.
   - (2-b) If $\int_{\Sigma} \kappa \, \text{vol}_{\varphi_0(g)} > 0$, then $\varphi_0$ is unstable.
(3) If $\lambda_1 < 0 = \lambda_2$, then we have:

(3-a) If there exist a $\lambda_2$-eigenfunction $f_2$ such that $\int_\Sigma f_2 \text{vol}_{\phi_0(\Sigma)} \neq 0$, then $\phi_0$ is unstable.

(3-b) If $\int_\Sigma h_2 \text{vol}_{\phi_0(\Sigma)} = 0$ for all $\lambda_2$-eigenfunction $h_2$, then there exist a unique function $\tilde{h}_2 \in (\ker(J_{\phi_0}))^\perp$ such that $J(\tilde{h}_2) = 1$ and

(3-b-i) If $\int_\Sigma \tilde{h}_2 \text{vol}_{\phi_0(\Sigma)} < 0$, then $\phi_0$ is stable.

(3-b-ii) If $\int_\Sigma \tilde{h}_2 \text{vol}_{\phi_0(\Sigma)} > 0$, then $\phi_0$ is unstable

(4) If $\lambda_2 < 0$, then $\phi_0$ is unstable.

For cases (2) and (3) the calculation of eigenvalues and integrals of the eigenfunctions can be difficult. In this sense, the following criterion gives us a geometric interpretation of stability based on the existence of a deformation family.

**Corollary 3.3.5.** Let $\phi : \Sigma \to M$ a free boundary CMC immersion of class $C^{j+1,\alpha}$. We assume that $\lambda_1 < 0 \leq \lambda_2$. If there exist a deformation $\phi_t$ of $\phi$, $-\epsilon < t < \epsilon$, with $\phi_0 = \phi$, such that $\phi_t$ is a free boundary CMC immersion of class $C^{j,\alpha}$ for all $t \in (-\epsilon,\epsilon)$, and such that $\frac{dH_t}{dt}|_{t=0} = H_0^t = \text{constant} \neq 0$, where $H_t$ is the constant mean curvature of $\phi_t$ and $V_t$ is the volume of $\phi_t$, we have that:

(1) If $H_0^tV_0^t \leq 0$, then $\phi$ is stable.

(2) if $H_0^tV_0^t > 0$, then $\phi$ is unstable.

If there is no such deformation, then $\phi$ is unstable

In Chapter 4, we study the existence of bifurcation points in a 1-parametric family $\{\phi_t\}_{-\epsilon < t < \epsilon}$ of free boundary CMC hypersurfaces. A bifurcation occurs when a small smooth change made to the parameter values of a system causes a sudden qualitative or topological change in its behaviour. Natural phenomena such as the buckling of the Euler rod, the appearance of Taylor vortex and the start of oscillations in an electric circuit, have a common cause: a specific physical parameter that exceeds a threshold, which forces the system to organize a new state that is considerably different from the one initially observed (see [15]). The Theory of Bifurcation explains there different phenomena. Examples of bifurcation appear experimentally long ago; Plateau’s research on Delaunay surfaces shows how surfaces belonging to a certain family were spontaneously transformed into surfaces of an adjacent family when a certain stability limit was reached (see [23]). We prove the existence of smooth bifurcating families of free boundary CMC hypersurfaces from the family $\{\phi_t\}_{-\epsilon < t < \epsilon}$. For the proof of our results we follow the the approach of Koiso-Palmer-Piccione [18], who made the version for surfaces in $\mathbb{R}^3$ with fixed boundary. We use the theory of stability for smooth variations of free boundary CMC hypersurfaces developed in Chapter 3. These results are based on the Abstract Bifurcation Theory of Crandall and Rabinowitz ([10] and [11]), which can be applied, under appropriate conditions, to produce a smooth family of free boundary CMC hypersurfaces via an Implicit Function Theorem when $\ker(J_{\phi}) \neq \{0\}$. Under the conditions of the main problem of Bifurcation Theory: Let $F$ be a mapping of a subset of a Banach space $W$ in a Banach space $X$ and let $\alpha(t)$ be a curve in $W$ such that $F(\alpha(t)) = 0$. Crandall and Rabinowitz imposed simple conditions in order to find a neighborhood $U_p$ of $p$ such that $F^{-1}(0) \cap U_p$, is topologically (or diffeomorphaly) equivalent to $(-1,1) \times \{0\} \cup \{0\} \times (-1,1)$. Also they study the behavior of
$F$ in $F^{-1}(0) \cap U_p$ and the topological structure of the same set. This theory is developed in the framework of the so-called “Theory of Bifurcation from a simple eigenvalue” and offer a wide spectrum of examples, including some generalizations of known results related to problems of non-linear eigenvalues for partial and ordinary differential equations.

We present two results on the existence of bifurcations. The first result provides a smooth bifurcating family of free boundary CMC hypersurfaces, whose mean curvatures coincide with the mean curvatures of the original family, that is the bifurcation parameter is the mean curvature,

**Theorem 4.2.3.** Let $\{\varphi_t\}_{-\epsilon < t < \epsilon}$ a family of free boundary CMC hypersurfaces. $H_t$ the constant mean curvature of $\varphi_t$. $H_0 \neq 0$. dim$(\ker(J_{\varphi_0}|_{C_0^{j,\alpha}(\Sigma_0)})) = 1$. Then,

1. $\int_{\Sigma_0} e \text{vol}_{\varphi_0}(g) = 0$, $e \neq 0$, $e \in \ker(J_{\varphi_0}|_{C_0^{j,\alpha}(\Sigma_0)})$, and there exists a differentiable map $\lambda : (-\epsilon_0, \epsilon_0) \to \mathbb{R}$, $0 < \epsilon_0 \leq \epsilon$, such that $\lambda(0) = \lambda_0 = 0$, $\lambda(t) = \lambda_t$ is a simple eigenvalue of $J_t$, and there is no other eigenvalue of $J_t$ near 0.

2. Assume further that $\lambda'(0) \neq 0$ holds. Then, $\varphi_0$ is a bifurcation point with respect to $\{\varphi_t\}_{-\epsilon < t < \epsilon}$, where the bifurcation branch is an analytic family of free boundary CMC immersions. More precisely, there exist an open interval $\hat{I} \subset \mathbb{R}$, $0 \in \hat{I}$, and $C^1$ functions $\zeta : \hat{I} \to E_+$ and $\psi : \hat{I} \to \mathbb{R}$, such that $\psi(t(0)) = 0$, $\zeta(0) = 0$, and

$$\psi_s = \exp_{\varphi_t(s)} (s \zeta(s)), \quad (\hat{I},\varphi_t(s)),$$

is a free boundary CMC immersion with mean curvature $\hat{H}_s = H_{s}(s)$.

3. Every free boundary CMC immersion sufficiently close, in the topology of $C^{j,\alpha}$, to $\varphi_0$, is equal, up to parameterization, to some element of families $\{\varphi_t\}_{t \in \hat{I}}$, $0 \in \hat{I} \subset (-\epsilon_0, \epsilon_0)$, or $\{\psi_s\}_{s \in \hat{I}}$. Furthermore, the surfaces $\{\varphi_t\}_{t \in \hat{I}}$ and $\{\psi_s\}_{s \in \hat{I}}$ are pairwise distinct except for $\varphi_0 = \psi_0$.

The second result provides a smooth bifurcating branch of free boundary CMC hypersurfaces, whose volumes coincide with the volumes of surfaces in the original family, that is the bifurcation parameter is the volume,

**Theorem 4.3.2.** Let $\{\varphi_t\}_{-\epsilon < t < \epsilon}$ a family of free boundary CMC hypersurfaces. $H_t$ and $V(t)$ the constant mean curvature and the volume of $\varphi_t$ respectively. $H'_0 \neq 0$ and $dV(0) \neq 0$. dim$(\ker(J_{\varphi_0}|_{C_0^{j,\alpha}(\Sigma_0)})) = 1$. Then,

1. $\int_{\Sigma_0} e \text{vol}_{\varphi_0}(g) = 0$, $e \neq 0$, $e \in \ker(J_{\varphi_0}|_{C_0^{j,\alpha}(\Sigma_0)})$, and there exists a differentiable map $\tilde{\lambda} : (-\epsilon_0, \epsilon_0) \to \mathbb{R}$, $0 < \epsilon_0 \leq \epsilon$, such that $\tilde{\lambda}(0) = \tilde{\lambda}_0 = 0$, $\tilde{\lambda}(t) = \tilde{\lambda}_t$ is a simple eigenvalue of $\tilde{J}_t = \frac{\partial}{\partial t} \tilde{F}(t,0)$, and there is no other eigenvalue of $\tilde{J}_t$ near 0.

2. Assume further that $\frac{\partial \tilde{\lambda}}{\partial t}|_{t=0} = \tilde{\lambda}_0' \neq 0$ holds. Then, $\varphi_0$ is a bifurcation point with respect to $\{\varphi_t\}_{-\epsilon < t < \epsilon}$, where the bifurcation branch is an smooth family of free boundary CMC immersions. More precisely, there exist an open interval $\hat{I} \subset \mathbb{R}$, $0 \in \hat{I}$, and $C^1$ functions $\eta : \hat{I} \to B_0$ and $\tau : \hat{I} \to \mathbb{R}$, such that $\tau(0) = 0$, $\eta(0) = 0$, and

$$\psi_s = \exp_{\varphi_0}(\tilde{F}(\tau(s),s\eta(s)) \tilde{\varphi}_0),$$

is a free boundary CMC immersion with volume $\tilde{V}(s) = V(\tau(s))$. 
Every free boundary CMC immersion sufficiently close, in the topology of $C^{1,\alpha}$, to $\varphi_0$, is equal, up to parameterization, to some element of families $\{\varphi_t\}_{t \in I}$, $0 \in I \subset (-\epsilon_0, \epsilon_0)$, or $\{\psi_s\}_{s \in I}$. Furthermore, the surfaces $\{\varphi_t\}_{t \in I}$ and $\{\psi_s\}_{s \in I}$ are pairwise distinct except for $\varphi_0 = \psi_0$.

In Chapter 5, we study stability of free boundary CMC hypersurfaces that we presented in the previous chapter. Bifurcation for a family $\{\varphi_t\}$ only occurs in the case where $\lambda_k = 0$, for some $k \geq 2$, where $\{\lambda_k\}$ is the countable set of eigenvalues of $J$ (see remark 3.3.1). Therefore, in view of results of Chapter 3, we only need to study the case where $\lambda_2 = 0$. If $\varphi_0$ is a stable bifurcation point in the family $\{\varphi_t\}$, then only three types of bifurcation occurs: Supercritical pitchfork bifurcation (see Figure 5.1); Subcritical pitchfork bifurcation (see Figure 5.2) and Transcritical bifurcation (see Figure 5.3). In the proof of these stability criteria we use again the theory of Crandall and Rabinowitz (see [11]). Crandall-Rabinowitz estimated the eigenvalue of the minimum module in the spectrum of the Frechet derivative of the nonlinear operator $F$ along of a bifurcated curve $\alpha(t)$, where $F(\alpha(t)) = 0$. Being $X$ and $Y$ Banach spaces and $F : \mathbb{R} \times X \to Y$ twice continuously differentiable, they show that the zero eigenvalue of $F_x(t_0,0)$ corresponds to small real values $\lambda(t)$ of $F_x(t,0)$ and $\mu(s)$ of $F_x(t(s),x(s))$, where $\lambda(t_0) = 0 = \mu(0)$. The main interest is studying the relationship between $t(s)$, $\lambda(t)$ and $\mu(s)$. They show that if $F_x(t_0,0)$ satisfies the condition of simplicity, then $\mu(s)$ and $-s\ell'(s)\lambda'(t_0)$ have the same zeros and, in the case where $\mu(s) \neq 0$, the same sign. We make a generalization of these Crandall-Rabinowitz results for the case when $F$ is in terms of the mean curvatures of $\{\varphi_t\}$, $F_x(t_0,0)$ is the Jacobi Operator, $X = C^{j,\alpha}_0(\Sigma)$ and $Y = C^{j-2,\alpha}_0(\Sigma)$ (see Lemmas 5.2.4 and 5.2.3).

When the bifurcation parameter is the volume, we provide two stability criteria:

The first is given in terms of the sign of the derivative of the volume function of hypersurfaces in the bifurcation branch, in addition to the sign of the derivative of the function of the first eigenvalue of $\tilde{J}_t$, where $\tilde{J}_t$ is the restriction of Jacobi operator $J_t$ to scalar fields with integral zero,

**Theorem 5.3.1** Let $\{\psi_s\}_{s \in I}$ be the bifurcation branch formed by free boundary CMC hypersurfaces obtained in Theorem 4.3.2. $H'_0 \neq 0$. $\ker(J_{f_0}|_{C^{j,\alpha}_0(\Sigma)}) = \text{Span}\{e \neq 0\}$. $\lambda_2 = 0$ (Equivalently $\tilde{\lambda}_1 = 0$).

$\tilde{\lambda}_1(0) \neq 0$ ($\tilde{\lambda}_1(t)$ the first eigenvalue of $\tilde{J}_t = J_t - \frac{1}{\tilde{\lambda}_1(t)} \int_{\Sigma} J_t \text{vol}_{\varphi_0}(\varphi_0)$). If moreover $V'(0) > 0$ (if necessary, change the parameter $t$ to $-t$). Then,

(1) In the case $H'_0 < 0$ (in this case $\varphi_0$ is stable by proposition 5.1.2 and $\tilde{\lambda}_1 = 0$)

(1-a) If $\tilde{V}'(s) = 0$ for $s$ near 0 (that is, if $\tilde{V}$ is locally constant), then, $\psi_s$ is stable for $s$ near 0.

(1-b) If $\tilde{V}'(s) \neq 0$, $s \neq 0$, $s$ near 0, then, for a sufficiently small $s_0 > 0$, in each interval $[-s_0,0]$ and $(0,s_0]$ we have:

(1-b-i) If $\tilde{\lambda}_1'(0)s\tilde{V}'(s) < 0$, then, $\psi_s$ is stable.

(1-b-ii) If $\tilde{\lambda}_1'(0)s\tilde{V}'(s) > 0$, then, $\psi_s$ is unstable.

In particular, supercritical and subcritical pitchfork bifurcations are presented where $s\tilde{V}'(s)$ does not change sign at $s = 0$, and transcritical bifurcation occurs when $s\tilde{V}'(s)$ changes sign at $s = 0$.

(2) In the case $H'_0 > 0$ (in this case $\varphi_0$ is unstable by proposition 5.1.2 and $\tilde{\lambda}_2 = 0$), we have that $\psi_s$ is unstable for small $|s|$.
The Second is given in terms of the sign of the first and second derivative of the volume function of the hypersurfaces in the bifurcation branch, by fixing the sign positively for the derivative of the first eigenvalue function of $\tilde{J}_t$.

**Corollary 5.3.2** Let $\{\psi_s\}_{s \in I}$ be the bifurcation branch formed by free boundary CMC hypersurfaces obtained in Theorem 4.3.2. $H_0' < 0$, $V_0' > 0$, $\tilde{\lambda}_1'(0) > 0$. Then, there exist positive constants $t_0 \in (0, \epsilon)$ and $s_0 \in \hat{I}$ such that:

1. $\varphi_t$ is stable for all $t \in [0, t_0]$ and unstable for all $t \in [-t_0, 0)$.

2. If $\hat{V}'(0) \neq 0$, then we have transcritical bifurcation for $\{\psi_s\}_{s \in [s_0, s_0]}$. This is,

   (2-a) If $\hat{V}'(0) > 0$, then $\psi_s$ is stable for $s \in [-s_0, 0]$ and unstable for $s \in (0, s_0]$.

   (2-b) If $\hat{V}'(0) < 0$, then $\psi_s$ is stable for $s \in [0, s_0]$ and unstable for $s \in [-s_0, 0)$.

3. If $\hat{V}'(0) = 0$ and there is the second derivative of $\hat{V}$ in $s = 0$, we have

   (3-a) If $\hat{V}''(0) < 0$, then $\psi_s$ is stable for all $s \in [-s_0, s_0]$. Here is a supercritical pitchfork bifurcation.

   (3-b) If $\hat{V}''(0) > 0$, then $\psi_s$ is unstable for all $s \in [-s_0, 0) \cup (0, s_0]$. Here is a subcritical pitchfork bifurcation.

If $H_0' < 0$ and $V_0' > 0$, inverting the parameterization of $\varphi_t$, $t \mapsto -t$, similar conclusions are obtained.

Now, when the bifurcation parameter is mean curvature, we also provide two stability criteria similar to the previous case and whose proofs are similar (see Theorema 5.3.3 and Corollary 5.3.4). The first is given in terms of the sign of derivative of mean curvature function of the hypersurfaces in the bifurcation branch, in addition to the sign of derivative of the function of second eigenvalue of $J_t$. The second is given in terms of the sign of the first and second derivative of the mean curvature function of the hypersurfaces in the bifurcation branch, by fixing the sign positively for the derivative of the second eigenvalue function of $J_t$.

In Chapter 6 we write the conclusions, obtained results and some statements of possible examples of the impact that our results could have on the study and research that occur in other areas of knowledge, such as Physics, Engineering, Chemistry, Economics, Biology, Medicine, etc. We also raised some problems to be solved in this theory. Among other things, the implementation of an example of a family of free boundary CMC hypersurfaces that has a bifurcation point and the description of the type of hypersurfaces in the bifurcation branch.

Although in the case of fixed boundary CMC surfaces there are examples of families with and without bifurcation points (see [17] and [18]), for our case (free boundary) to present an example it is more complex. The difficulty is the following: the bifurcation point must be a degenerated hypersurface, that is, the class $C^{j,\alpha}$ function associated this hypersurface must be a solution of a differential equation that may contain variable coefficients, such as the norm of the Second Fundamental Form and the Ricci curvature. In addition, this same function must comply with the linearized free boundary condition. Additionally, we also intend studying, in the near future, the same results but in the case of semi-Riemannian metrics.
Now we present a simple example of a family of compact surfaces with constant mean curvature and fixed boundary without bifurcation.

Fixed a circle in a plane of $\mathbb{R}^3$, we consider the set of the truncated spheres contained in one of the hemispheres and whose edge is the fixed circle (see Figure 1)

![Family of truncated spheres with fixed border.](image1)

Clearly, each surface has constant mean curvature equals $-\frac{1}{r}$, where $r$ is the radius of the sphere. Since the mean curvature function defined in this family has a minimum in the Semi-Sphere with center in the center of the fixed circle (see Figure 2), then, clearly, this surface is stable.

![Semi-sphere, point of minimum of the mean curvature function.](image2)

The geometrical reason why there is no bifurcation points in this family of truncated spheres is the following:

In 1841 Delaunay proved that the only surfaces of revolution with constant mean curvature are the surfaces obtained by rotating the roulette of the conics (see [12]). These are the plane, cylinder, sphere, the catenoid, the unduloid and nodoid (see Figure 3).

By definition, surfaces in the bifurcation branch are not congruent with any of the original branch, except at the bifurcation point. That is, if $\{\varphi_t : \Sigma \to \mathbb{R}^3\}_{-c_0<t<c_0}$ is the original family and
Figure 3: Delaunay surfaces: cylinder, unduloid, nodoid, catenoid.

$\{\psi_s : \Sigma \to \mathbb{R}^3\}_{-\epsilon_1 < s < \epsilon_1}$ is the bifurcation branch, we have $\varphi_0 = \psi_0$, but there is no diffeomorphism $\phi : \Sigma \to \Sigma$ such that $\psi_s = \varphi_t \circ \phi$ for $t, s \neq 0$. However, all the elements of the family of truncated spheres considered are homotopically equivalent to a sphere without a point, while compact portions of cylinders, catenoids, unduloids and nodoids are homotopically equivalent to a sphere without two points. Therefore, none of the truncated spheres can be congruent to any of these other Delaunay surfaces, that is, there is no bifurcation point.
Chapter 1

Preliminaries

Throughout this work we will consider $M$ as a $(n + 1)$-dimensional differential manifold with smooth boundary $\partial M \neq \emptyset$ and $\Sigma$ as $n$-dimensional differential manifold with smooth boundary $\partial \Sigma \neq \emptyset$. In this chapter we introduce the concepts of admissibility and orthogonality of hypersurfaces with boundary in a manifold with smooth boundary, we will give the definition of mean curvature and free boundary CMC (Constant Mean Curvature) hypersurface. Also, we give the meaning of nondegeneracy of hypersurfaces with CMC, under the Hölder condition of regularity $C^{j,\alpha}$, through Jacobi operator restricted to the Banach space of the Hölder functions defined on a manifold with boundary that fulfill the linearized free boundary condition.

1.1 Orthogonal sub-manifolds and mean curvature

**Definition 1.1.1.** Let $\varphi : \Sigma \to M$ be an embedding. We identify $\varphi$ with its image $\varphi(\Sigma) \subset M$. $\vec{n}_{\partial M}$ is the outer unit normal field along the boundary of $M$. We call $\varphi$ admissible if it satisfies (a) and (b), and normal (orthogonal) if it also satisfies (c):

(a) $\varphi(\Sigma) \cap \partial M = \varphi(\partial \Sigma)$,

(b) the normal bundle $T(\varphi(\Sigma))^\perp$ is orientable,

(c) and for each point $p \in \varphi(\partial \Sigma)$, $\vec{n}_{\partial M}(p) \in T_p \varphi(\Sigma)$,

The admissible hypersurface $\varphi(\Sigma)$ is said to bound a finite volume if

(d) $M \setminus \varphi(\Sigma) = \Omega_1 \cup \Omega_2$, with $\overline{\Omega_1}$ compact and $\Omega_1 \cap \Omega_2 = \emptyset$.

If $\varphi : \Sigma \to M$ is an orthogonal admissible embedding, then $\varphi(\Sigma)$ it is compact and $\varphi(\Sigma)$ and $\partial M$ are transverse sub-manifolds (see Definition A.0.1). We say that $\varphi(\Sigma)$ is a orthogonal sub-manifold of $M$ (see Figure 1.1).

Let $g$ be a Riemannian metric on $M$ and $\varphi_0 : \Sigma \to M$ an orthogonal immersion. And write $\Sigma_0 := \varphi_0(\Sigma)$. We define the second fundamental form on $\Sigma_0$ as

$$I_{\Sigma_0}(X,Y) := g(\nabla_X Y, \vec{n}_{\Sigma_0}), \quad (1.1.1)$$

where $\vec{n}_{\Sigma_0}$ is the unit normal vector field to $\Sigma_0$ in the orientable normal bundle, $\nabla$ is the Levi-Civita connection in $M$, and $X,Y$ are vector fields in $T\Sigma_0$. 

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The mean curvature function $H_{\Sigma_0} : \Sigma_0 \to \mathbb{R}$ is defined as trace of the second fundamental form $\Pi^{\Sigma_0}$. The mean curvature vector of $\Sigma_0$ is defined as $\vec{H}_{\Sigma_0} = H_{\Sigma_0} \vec{\eta}_{\Sigma_0}$. If $H_{\Sigma_0}$ is constant, $\Sigma_0$ is called a constant mean curvature hypersurface (or CMC hypersurface), and if $H_{\Sigma_0} = 0$, $\Sigma_0$ is a minimal hypersurface.

1.2 Variational problem

In the theory of variational problems is known that the hypersurfaces with CMC of $M$ minimize the area among all hypersurfaces enclosing a fixed volume. In the case where $\partial \Sigma$ is allowed to move freely along $\partial M$ the variational problem is called free boundary CMC problem. The solutions of this problem are orthogonal hypersurfaces with CMC which are called free boundary CMC hypersurfaces.

We introduce the following notation:

- $\text{Emb}_0(\Sigma, M)$ be the space of admissible embeddings of $\Sigma$ in $M$,
- $\text{Emb}_0(\Sigma, M) \subset \text{Emb}(\Sigma, M)$ the sub-space of normal admissible embeddings and bounding a finite volume.

We have (see for example the article by Barbosa-do Carmo [6]) that $\varphi_0 \in \text{Emb}_0(\Sigma, M)$ have CMC $H$ if only if is a critical point of funtional $f_H : \text{Emb}_0(\Sigma, M) \to \mathbb{R}$, defined by

$$f_H(\varphi) = \int_\Sigma \text{vol}(\varphi^*(g)) - H \int_{\Omega_1} \text{vol}_g,$$  \hspace{1cm} (1.2.1)

Note that if $H = 0$ then $\varphi_0(\Sigma)$ has the minimal volume over all hypersurfaces $\varphi(\Sigma), \varphi \in \text{Emb}_0(\Sigma, M)$. In this case, $\varphi_0$ is said to be a free boundary minimal hypersurface.

We say that $\varphi_0 : \Sigma \to M$ is non-degenerate if $\varphi_0$ is a non-degenerate critical point of $f_H$.

**Definition 1.2.1.** A metric $g$ on $M$ is called "$(\Sigma, M)$-Bumpy", if all $\varphi \in \text{Emb}_0(\Sigma, M)$, with CMC in the metric $g$, is non-degenerate.

We discuss below some other characterizations of non-degenerate hypersurfaces.
1.3 Jacobi operator

One of the most important elements for the study of our theory is the Jacobi Operator. The properties of this operator will allow us to prove different results concerning the genericity, bifurcation and stability of the free boundary CMC hypersurfaces. Let \( \Sigma_0 = \varphi_0(\Sigma) \) be an orthogonal CMC hypersurface, \( \varphi_0 \in \text{Emb}_{\partial \perp}(\Sigma, M) \). \( C^j(\Sigma_0) \) is the set of functions \( f : \Sigma_0 \to \mathbb{R} \) with continuous derivatives to \( j \) order, \( j \) could be infinite. The second-order linear differential operator \( J_{\varphi_0} : C^j(\Sigma_0) \to C^{j-2}(\Sigma_0), j \geq 2 \), defined by

\[
J_{\varphi_0}(f) := \Delta_{\Sigma_0} f - (||\Pi^{\Sigma_0}||^2_{HS} + \text{Ric}_g(\eta_{\Sigma_0}, \eta_{\Sigma_0})) f,
\]

is called Jacobi operator, where \( \Delta_{\Sigma_0} \) is the (nonnegative) laplacian of \( (\Sigma_0, \gamma) \) and \( ||\Pi^{\Sigma_0}||^2_{HS} \) is the square of Hilbert-Schmidt norm (see Definition A.0.3) of the second fundamental form of \( \varphi_0 \). A Jacobi scalar field along of \( \varphi_0 \) is a smooth function \( f \in C^j(\Sigma_0) \) such that \( J_{\varphi_0}(f) = 0 \).

We consider a smooth variation of \( \varphi_0 \) as follows:

\[
\Phi : \Sigma \times (-\epsilon, \epsilon) \to M, \quad \epsilon > 0,
\]

such that \( \Phi(\Sigma, s) = \varphi_s(\Sigma) = \Sigma_s \subset M, \varphi_s \in \text{Emb}_{\partial \perp}(\Sigma, M) \) with CMC \( H_s \). Let \( V = \frac{\partial}{\partial s}|_{s=0} \Phi \) be the corresponding variational vector field. Then \( \xi_0 = g(V, \tilde{\eta}_{\Sigma_0}) \) satisfies

\[
\frac{d}{ds}|_{s=0} H_s = \Delta_{\Sigma_0} \xi_0 - (||\Pi^{\Sigma_0}||^2_{HS} + \text{Ric}_g(\eta_{\Sigma_0}, \eta_{\Sigma_0})) \xi_0 = J_{\varphi_0}(\xi_0),
\]

(see [29], A.2). Then \( J_{\varphi_0} \) represents the second variation \( d^2f_H(\varphi_0) \) of \( f_H \) at the critical point \( \varphi_0 \), with respect to \( L^2 \) inner product.

**Remark 1.3.1.** Note that \( \xi_0 \) is a Jacobi field exactly when \( H_s \equiv H_0 \).

**Lemma 1.3.2.** If each \( \varphi_s \) is normal, that is \( \varphi_s \in \text{Emb}_{\partial \perp}(\Sigma, M) \), with CMC, then \( \xi_0 \) satisfies the so-called linearized free boundary condition

\[
g(\nabla \xi_0, \tilde{\eta}_{\partial M}) + \Pi^{BM}(\tilde{\eta}_{\Sigma_0}, \tilde{\eta}_{\Sigma_0}) \xi_0 = 0,
\]

where \( \nabla \xi_0 \) is the g-gradient of \( \xi_0 \) in \( \Sigma_0 \).

**Proof.** We can decompose \( V \) in its tangent and normal components

\[
V = V^T + \xi_0 \tilde{\eta}_{\Sigma_0},
\]

and

\[
\nabla_V \tilde{\eta}_{\Sigma_0} = \nabla_{V^T} \tilde{\eta}_{\Sigma_0} - \nabla \xi_0,
\]
(see Proposition 15 of Ambrozio, [3]). So, if $V^T = 0$ then
\[
V g(\vec{\eta}_{\Sigma_0}, \vec{\eta}_{\partial M}) = g(\nabla V \vec{\eta}_{\Sigma_0}, \vec{\eta}_{\partial M}) + g(\vec{\eta}_{\Sigma_0}, \nabla V \vec{\eta}_{\partial M}) \\
= g(\nabla V \vec{\eta}_{\Sigma_0} - \nabla \xi_0, \vec{\eta}_{\partial M}) + g(\vec{\eta}_{\Sigma_0}, \nabla V \vec{\eta}_{\partial M}) \\
= -g(\nabla \xi_0, \vec{\eta}_{\partial M}) + g(\vec{\eta}_{\Sigma_0}, \nabla \vec{\eta}_{\Sigma_0} \vec{\eta}_{\partial M}) \\
= -\frac{\partial \xi_0}{\partial \vec{\eta}_{\partial M}} + g(\vec{\eta}_{\Sigma_0}, \nabla \vec{\eta}_{\Sigma_0} \vec{\eta}_{\partial M}) \xi_0.
\]
Thus
\[
\frac{\partial \xi_0}{\partial \vec{\eta}_{\partial M}} = g(\vec{\eta}_{\Sigma_0}, \nabla \vec{\eta}_{\Sigma_0} \vec{\eta}_{\partial M}) \xi_0 - V g(\vec{\eta}_{\Sigma_0}, \vec{\eta}_{\partial M}).
\]
Therefore, if each $\varphi_s$ is a free boundary CMC hypersurface
\[
\frac{\partial \xi_0}{\partial \vec{\eta}_{\partial M}} = g(\vec{\eta}_{\Sigma_0}, \nabla \vec{\eta}_{\Sigma_0} \vec{\eta}_{\partial M}) \xi_0.
\]
Then,
\[
g(\nabla \xi_0, \vec{\eta}_{\partial M}) + II_{\partial M}(\vec{\eta}_{\Sigma_0}, \vec{\eta}_{\Sigma_0}) \xi_0 \\
= \frac{\partial \xi_0}{\partial \vec{\eta}_{\partial M}} + g(\vec{\eta}_{\Sigma_0}, \nabla \vec{\eta}_{\Sigma_0} \vec{\eta}_{\partial M}) \xi_0 \\
= \frac{\partial \xi_0}{\partial \vec{\eta}_{\partial M}} - g(\vec{\eta}_{\Sigma_0}, \nabla \vec{\eta}_{\Sigma_0} \vec{\eta}_{\partial M}) \xi_0 \\
= \frac{\partial \xi_0}{\partial \vec{\eta}_{\partial M}} - \frac{\partial \xi_0}{\partial \vec{\eta}_{\partial M}} \\
= 0.
\]
\] 1.4 Regularity
Sard’s theorem or Sard-Smale theorem in the case of infinite dimensions (see [26]) is the main tool when it comes to solving problems of genericity of regular points for a certain map between Banach manifolds. Said map needs the condition of being Fredholm with a certain index (For the Fredholm operator definition see A.0.5). To obtain this condition it is necessary to establish a regularity condition type Hölder, $C^{j, \alpha}$ (see definition A.0.4), for our embeddings. We can endow space of functions defined from $\Sigma$ to $\mathbb{R}$, $C^{j, \alpha}(\Sigma)$, with regularity $C^{j, \alpha}$, with the following norm:
\[
\|f\|_{C^{j, \alpha}} = \|f\|_{C^j} + \max_{|\beta| = j} \left| D^\beta f \right|_{C^{0, \alpha}}, \tag{1.4.1}
\]
where $\beta$ ranges over multi-indices and
\[
\|f\|_{C^j} = \max_{|\beta| \leq j, x \in \Sigma} \left| D^\beta f(x) \right|, \quad |Df|_{C^{0, \alpha}} = \sup_{x \neq y \in \Sigma} \frac{|Df(x) - Df(y)|}{|x - y|^\alpha}.
\]
It is well-known that $C^{j, \alpha}(\Sigma)$ endowed with this norm is a (non separable) Banach space.

Remark 1.4.1. When the operator $f_H$ defined in (1.2.1) is considered on the space of $C^{j, \alpha}$-
embeddings, the Jacobi operator acts on the corresponding tangent space at $\varphi_0$, which can be identified with $C^{j,\alpha}(\Sigma_0)$ (see Proposition 2.3.1).

We define the following space,

$$C^{j,\alpha}_{\partial}(\Sigma_0) := \{ f \in C^{j,\alpha}(\Sigma_0) : g(\nabla f, \eta_{\partial M}) + \Pi^{\partial M}(\vec{n}_{\Sigma_0}, \vec{n}_{\Sigma_0}) f = 0 \}.$$  \hfill (1.4.2)

The restriction of $J_{\varphi_0} : C^{j,\alpha}_{\partial}(\Sigma) \to C^{j-2}(\Sigma)$ is a Fredholm operator of index zero (see \cite[section 2]{21}).

Since $J_{\varphi_0}$ is the representation of the second variation of the area functional, the following proposition is immediately.

**Proposition 1.4.2.** The embedding $\varphi_0 \in \text{Emb}_{\partial \perp}(\Sigma, M)$ with $g$-CMC is called non-degenerate if $J_{\varphi_0} \big|_{C^{j,\alpha}_{\partial}(\Sigma)}$ is an isomorphism of Banach spaces, i.e. $\ker J_{\varphi_0} \cap C^{j,\alpha}_{\partial}(\Sigma) = \emptyset$.

The degenerate embeddings will be very important objects in the following chapters: for example, they are a fundamental part of the demonstration of the main theorem of Chapter 2 on the genericity of the bumpy metrics. We will also see that from degenerate free boundary CMC embeddings we can decide about the existence of pertubations and also about the generation of bifurcations branch and changes in the stability of families o free boundary CMC hipersurfaces, as will be seen in Chapters 3, 4 and 5.
Chapter 2

Genericity Of Nondegenerate Free Boundary CMC Embeddings

In this chapter we prove two important theorems aimed at demonstrating the genericity of the Bumpy metrics (defined in 1.2.1). The first, Theorem 2.1.1, is a result given in general terms over Banach spaces $\Gamma, X$ and $Y$, where it is proved that the kernel of a certain application $H : \Gamma \times X \rightarrow Y$, defined from a functional $A : \Gamma \times X \rightarrow \mathbb{R}$, is a Banach manifold and the first projection $\Pi$, defined on that manifold is a Fredholm operator of zero index, whose critical points $(\gamma_0, u_0)$ are elements such that $u_0$ is a degenerate critical point of $A(\gamma_0, \cdot)$. The second, Theorem 2.4.1, where we prove that for the spaces $\Gamma = W \cap Met^k(M)$, ($W$ is a Banach subspace of type $C^k$-Whitney of the symmetric tensor fields on $M$, defined in the section 2.2), $X = C^{j, \alpha}(\Sigma)$ and $Y = C^{j-2, \alpha}(\Sigma)$, the functional area $A : \Gamma \times X \rightarrow \mathbb{R}$ and the mean curvature of the operator $H : \Gamma \times X \rightarrow Y$, the conditions of the Theorem 2.1.1 are fulfilled. Thus, as an immediate consequence of the Theorem 2.4.1 and the Sard-Smale Theorem, we obtained the genericity of non-degenerate free boundary CMC embeddings, Corollary 2.4.6.

Definition 2.0.1. A subset of metrical space is said to be generic if it is the countable intersection of dense open subsets. By Baire’s theorem, a generic set is dense.

2.1 Important Theorem

The Theorem 2.1.1 is of great importance for us since it shows the genericness of the regular points of a defined mapping between Banach spaces in general terms and will be used in the proof of the main theorem of this Chapter. It will be supposed that $A : \Gamma \times X \rightarrow \mathbb{R}$ is a map of class $C^j$, with $j \geq 2$, where $\Gamma$ and $X$ are Banach spaces; we also assume that $X$ has an inner product. We require a condition of transversality (see Definition A.0.2) between the function $\frac{\partial}{\partial x} A : \Gamma \times X \rightarrow TX^*$ and the zero section of $TX^*$, this is equivalent to saying that $\forall (\gamma_0, x_0), \frac{\partial A}{\partial x} (\gamma_0, x_0)$ and $w \neq 0, w \in \ker(\frac{\partial^2 A}{\partial x^2})$, $\exists v \in T_{\gamma_0} \Gamma$ such that $\frac{\partial^2 A}{\partial x^2}(\gamma_0, x_0)(w, v) \neq 0$ (see [9, Proposition 3.1]).

It is necessary to give the following definition of locally fibered sub-manifold that is imposed as a condition in the Theorem 2.1.1

Definition 2.1.1. Let $\Gamma$ and $X$ be Banach spaces and $\Pi : \Gamma \times X \rightarrow \Gamma$ the first projection. Let $\mathcal{X} \subset \Gamma \times X$ be a sub-manifold. We say that $\mathcal{X}$ is a locally fibered if it meets the following condition: For each $(u_0, v_0) \in \mathcal{X}$, there is an open $U \subset \Gamma$ and a closed subspace $W \subset X$, with $u_0 \in U$
Theorem 2.1.1. Let $\Gamma$, $X$ and $Y$ be Banach spaces, $\mathcal{H}$ a Hilbert space with inner product $\langle \cdot , \cdot \rangle$, and $X \subset Y \subset \mathcal{H}$. Let
\begin{equation}
A : \Gamma \times X \to \mathbb{R} \tag{2.1.1}
\end{equation}
a $C^j$ function, $j \geq 2$. Suppose that there exists a map $H : \Gamma \times X \to Y$ such that
\begin{equation}
\frac{d}{dt} \bigg|_{t=0} A(\gamma, u + tv) = \langle H(\gamma, u), v \rangle, \tag{2.1.2}
\end{equation}
for all $\gamma \in \Gamma$ and $u, v \in X$.

Let $\mathfrak{X} \subset \Gamma \times X$ be a locally fibred sub-manifold such that for all $(\gamma_0, u_0) \in \mathfrak{X}$ the operator
\begin{equation}
\frac{\partial H}{\partial u}(\gamma_0, u_0)\bigg|_{T_{u_0}X} : T_{u_0}X \to Y \tag{2.1.3}
\end{equation}
is a Fredholm with index zero ($T_{u_0}X \cong X$).

Furthermore, suppose that for all $k \in \ker \frac{\partial H}{\partial u}(\gamma_0, u_0) \cap T_{u_0}X$, $k \neq 0$, there exists a family $(\gamma(s), u(t)) \in \mathfrak{X}$, such that $\gamma(0) = \gamma_0$, $u(0) = u_0$, $u'(0) = k$ and
\begin{equation}
\frac{\partial^2}{\partial s \partial t} \bigg|_{s=t=0} A(\gamma(s), u(t)) \neq 0. \tag{2.1.4}
\end{equation}

Then

(1) $H|_{\mathfrak{X}} : \mathfrak{X} \to Y$ is a submersion near $(\gamma_0, u_0)$, so there exists a neighborhood $W \subset \mathfrak{X}$, $(\gamma_0, u_0) \in W$, such that
\[
\mathcal{M} = \{ (\gamma, u) \in W : H(\gamma, u) = 0 \}
\]
is a sub-manifold of $\mathfrak{X}$, and
\[
T_{(\gamma,u)}\mathcal{M} = \ker \left( \frac{dH}{du}(\gamma,u) \bigg|_{T_{(\gamma,u)}\mathfrak{X}} \right).
\]

(2) The projection
\[
\Pi : \mathcal{M} \to \Gamma, \quad \Pi(\gamma, u) = \gamma
\]
is an Fredholm operator with index zero.

(3) The critical points of $\Pi|_{\mathcal{M}}$ are elements $(\gamma_0, u_0) \in \mathcal{M}$ such that $u_0$ is a degenerate critical point of the functional $A(\gamma_0, \cdot)$.

Proof. (1) In order to simplify the notation we write $J = \frac{\partial H}{\partial u}(\gamma_0, u_0)|_{T_{u_0}X,\gamma_0}$. We show that $J$ is symmetric with respect to the product in $\mathcal{H}$. Indeed,
\[
\frac{\partial^2}{\partial s \partial t} \bigg|_{s=t=0} A(\gamma_0, u_0 + sv + tw) = \frac{d}{ds} \bigg|_{s=0} \left( \frac{d}{dt} \bigg|_{t=0} A(\gamma_0, u_0 + sv + tw) \right) \\
= \frac{d}{ds} \bigg|_{s=0} \langle H(\gamma_0, u_0 + sv), w \rangle \\
= \langle Jv, w \rangle.
\]

On the other hand
\[
\frac{\partial^2}{\partial t \partial s} \bigg|_{s=t=0} A(\gamma_0, u_0 + sv + tw) = \langle Jw, v \rangle.
\]

Now, for all \( u \in \ker J \) and \( v \in X_{\gamma_0} \), we have \( \langle J u, v \rangle = \langle u, J v \rangle = 0 \). Thus, \( \text{im}(J) \subset (\ker J)^\perp \).

Since \( J \) is Fredholm with index 0, we obtain
\[
\text{im}(J) = (\ker J)^\perp.
\]

In order to prove that \( H : \mathcal{X} \to Y \) is a submersion we have to prove that \( dH(\gamma_0, u_0) \big|_{T(\gamma_0, u_0)X} \) is surjective and its kernel is complemented.

First we prove surjectivity. Let \((\gamma(s), u(t))\) be a family compatible with \( \mathcal{X} \), such that \( \gamma(0) = \gamma_0, u(0) = u_0 \), \( u'(0) = k \neq 0, k \in \ker J \) (for example, take \( u(t) = u_0 + tk \)). Note that \( \gamma'_0 = \frac{d}{ds} \bigg|_{s=0} \gamma(s) \), \( \mathbb{R} \cdot \gamma'_0 \subset T_{\gamma_0} \Gamma \) and \( (\mathbb{R} \cdot \gamma'_0) \times \{0\} \subset T(\gamma_0, u_0)\mathcal{X} \). So,

\[
0 \neq \frac{\partial^2}{\partial s \partial t} A(\gamma(s), u(t)) \\
= \frac{d}{ds} \bigg|_{s=0} \frac{d}{dt} \bigg|_{t=0} A(\gamma(s), u(t)) \\
= \frac{d}{ds} \bigg|_{s=0} \langle H(\gamma(s), u_0), k \rangle \\
= \langle \frac{\partial H}{\partial \gamma}(\gamma_0, u_0)\gamma'_0, k \rangle.
\]

Since \( \frac{\partial H}{\partial \gamma}(\gamma_0, u_0)\gamma'_0 \in \text{Im}(dH(\gamma_0, u_0) \big|_{T(\gamma_0, u_0)\mathcal{X}}) \), we use Lemma B.0.5, with \( V = Y \), \( W = \ker J \) and \( Z = \text{Im}(dH(\gamma_0, u_0) \big|_{T(\gamma_0, u_0)\mathcal{X}}) \), which shows that \( dH(\gamma_0, u_0) \) restricted to \( T(\gamma_0, u_0)\mathcal{X} \) is surjective.

Now, we have
\[
\ker(dH(\gamma_0, u_0) \big|_{T(\gamma_0, u_0)\mathcal{X}}) \subset \ker J,
\]
and from the fact that \( J \) is Fredholm we infer that
\[
\dim(\ker(dH(\gamma_0, u_0) \big|_{T(\gamma_0, u_0)\mathcal{X}})) \leq \dim(\ker J) < \infty.
\]

Therefore \( \ker(dH(\gamma_0, u_0) \big|_{T(\gamma_0, u_0)\mathcal{X}}) \) is complemented in \( T(\gamma_0, u_0)\mathcal{X} \).

Thus \( H : \mathcal{X} \to Y \) is a submersion. Whence there is a neighborhood \( U \) of \( (\gamma_0, u_0) \) such that
\[
\mathcal{M} = H^{-1}(0) \cap U
\]
is a sub-manifold and
\[ T_{(\gamma_0, u_0)} \mathcal{M} = \ker(dH(\gamma_0, u_0)|_{T_{(\gamma_0, u_0)} X}) \]

(2) Let

\[ \Pi : \Gamma \times X \longrightarrow \Gamma \]
\[ (\gamma, u) \longmapsto \gamma \]

be the projection on the first factor. Take the restriction of \( \Pi \) to \( \mathcal{M} \), \( \Pi|_{\mathcal{M}} \). We have to prove that \( \text{Dim}(\ker(d\Pi|_{T_{(\gamma_0, u_0)} \mathcal{M}})) < \infty \) and \( \text{im}(d\Pi|_{T_{(\gamma_0, u_0)} \mathcal{M}}) \) is closed and has finite co-dimension.

We have that
\[ \ker(d\Pi|_{T_{(\gamma_0, u_0)} \mathcal{M}}) = \ker \Pi \cap T_{(\gamma_0, u_0)} \mathcal{M} \]
\[ = (\{0\} \times X) \cap \ker(dH(\gamma_0, u_0)|_{T_{(\gamma_0, u_0)} X}) \]
\[ = \{0\} \times \ker J, \]

which is of finite dimension.

Now, by hypothesis about \( \mathcal{X} \), there is a neighborhood \( U \subset \Gamma \), \( \gamma \in U \), and a diffeomorphism \( \varphi : U \times W \to \Pi^{-1}(U) \), \( W \subset X \) a closed sub-space, such that the follow diagram is commutative

\[ \text{Diagram:} \]
\[ \Pi^{-1}(U) \xleftarrow{\varphi} U \times W \]
\[ \quad \Pi \quad \xrightarrow{\text{proj}_1} \quad U \]

So, locally \( \Pi \) is as a projection from a product space.

In particular for \( \gamma_0 \) fixed, we can take \( \varphi \) such that \( \varphi(\gamma_0, u) = (\gamma_0, u) \), for all \( u \in W \). Set
\[ \overline{H} := H \circ \varphi : U \times W \to Y. \]

Then
\[ \frac{\partial \overline{H}}{\partial u}(\gamma_0, u_0) = \left. \frac{d}{dt} \right|_{t=0} \overline{H}(\gamma_0, u_0 + tv) \]
\[ = \left. \frac{d}{dt} \right|_{t=0} H(\varphi(\gamma_0, u_0 + tv)) \]
\[ = \left. \frac{d}{dt} \right|_{t=0} H(\gamma_0, u_0 + tv) \]
\[ = \frac{\partial H}{\partial u}(\gamma_0, u_0). \]

And we have
\[ \ker(d\overline{H}(\gamma_0, u_0)) = \{(\xi, \omega) : \frac{\partial \overline{H}}{\partial \gamma}(\gamma_0, u_0)\xi + \frac{\partial \overline{H}}{\partial u}(\gamma_0, u_0)\omega = 0\} \]

and \( \xi \in \left[ \frac{\partial \overline{H}}{\partial \gamma}(\gamma_0, u_0) \right]^{-1}(\text{im} J) \).
We prove that \( \text{im}(d\Pi|_{T(\gamma_0, u_0), \mathcal{M}}) \) has finite co-dimension.

\[
\text{im}(d\Pi|_{T(\gamma_0, u_0), \mathcal{M}}) = \Pi(T(\gamma_0, u_0), \mathcal{M}) \\
= \Pi(\ker(dH(\gamma_0, u_0))) \\
= \left[ \frac{\partial H}{\partial \gamma}(\gamma_0, u_0) \right]^{-1}(\text{im}(\frac{\partial H}{\partial u}(\gamma_0, u_0))) \\
= \left[ \frac{\partial H}{\partial \gamma}(\gamma_0, u_0) \right]^{-1}(\text{im}J).
\]

The Fredholmness of \( J \) implies that its image is closed and finite co-dimensional. Thus, we use Lemma B.0.6 with \( U = T_{\gamma_0}\Gamma, V = Y, S = \text{im}\frac{\partial H}{\partial u}(\gamma_0, u_0) \) and \( L = \frac{\partial H}{\partial \gamma}(\gamma_0, u_0) \). Now, we have

\[
\text{im}\left( \frac{\partial H}{\partial \gamma}(\gamma_0, u_0) + \frac{\partial H}{\partial u}(\gamma_0, u_0) \right) = \text{im}(dH(\gamma_0, u_0) |_{T(\gamma_0, u_0), \mathcal{M}}),
\]

but \( dH(\gamma_0, u_0) |_{T(\gamma_0, u_0), \mathcal{M}} \) is surjective, so

\[
\text{Codim}_Y(\text{im}\left( \frac{\partial H}{\partial \gamma}(\gamma_0, u_0) + \frac{\partial H}{\partial u}(\gamma_0, u_0) \right) = 0
\]

and

\[
\text{Codim}_Y(\text{im}\left( \frac{\partial H}{\partial u}(\gamma_0, u_0) \right) = \text{Codim}_{T_{\gamma_0}\Gamma}(\left[ \frac{\partial H}{\partial \gamma}(\gamma_0, u_0) \right]^{-1}(\text{im}J)).
\]

On the other hand,

\[
\text{Codim}_Y(\text{im}\left( \frac{\partial H}{\partial u}(\gamma_0, u_0) \right) = \text{Dim}(\text{im}J)^\perp = \ker(J)
\]

Thus, \( \Pi \) is Fredholm with index 0.

(3) Recall that \( (\gamma_0, u_0) \) is a regular point of \( \Pi|_{\mathcal{M}} \) if

\[
d\Pi(\gamma_0, u_0)|_{T(\gamma_0, u_0), \mathcal{M}}
\]

is surjective, and also remember that

\[
\text{im}(d\Pi|_{T(\gamma_0, u_0), \mathcal{M}}) = \left[ \frac{\partial H}{\partial \gamma}(\gamma_0, u_0) \right]^{-1}\left( \text{im}\left( \frac{\partial H}{\partial u}(\gamma_0, u_0) \right) \right),
\]

as we see in 2.1.6. Then, \( (\gamma_0, u_0) \) is a regular point of \( \Pi|_{\mathcal{M}} \) if and only if

\[
\text{im}(\frac{\partial H}{\partial \gamma}(\gamma_0, u_0)) \subset \text{im}(\frac{\partial H}{\partial u}(\gamma_0, u_0)),
\]
but
\[ \text{im}(dH(\gamma_0, u_0)) = \text{im}(\frac{\partial H}{\partial \gamma}(\gamma_0, u_0)) + \text{im}(\frac{\partial H}{\partial u}(\gamma_0, u_0)) = \text{im}(\frac{\partial H}{\partial u}(\gamma_0, u_0)) = \text{im}J. \]

Now,
\[ \text{im}(dH(\gamma_0, u_0)) = \text{im}(dH(\gamma_0, u_0))|_{T(\gamma_0,u_0)} \]
and \(dH(\gamma_0, u_0)|_{T(\gamma_0,u_0)}\) is surjective. Hence \(\ker J = \{0\}\), since \(J\) is Fredholm of index 0.

Therefore, \((\gamma_0, u_0) \in \mathcal{M}\) is a regular point to \(\Pi|_{\mathcal{M}}\) if and only if \(\ker J = \{0\}\). Whence \(\gamma_0\) is a critical value of \(\Pi|_{\mathcal{M}}\) if and only if there exists \(u_0\) which is a degenerate critical point of \(A(\gamma_0, \cdot)\). \(\square\)

2.2 \(C^k\)-Whitney type Banach space of tensor fields

The definitions of this section are taken from [7] and [9]. We denote by \(\mathcal{G}^k_{\text{sym}}(TM^* \otimes TM^*)\) the vector space of all sections \(\sigma\) of class \(C^k\), \(k \geq 2\), of the vector bundle \(TM^* \otimes TM^*\) such that \(\sigma_p : T_pM \times T_pM \to \mathbb{R}\) is symmetric for all \(p\). Let \(\text{M}et^k(M) \subset \mathcal{G}^k_{\text{sym}}(TM^* \otimes TM^*)\) be the set of all metric tensors \(g\) on \(M\) of class \(C^k\). The set \(\mathcal{G}^k_{\text{sym}}(TM^* \otimes TM^*)\) does not have necessarily a canonical Banach space structure, for example if \(M\) is noncompact. In order to give a structure of Banach space to this space of tensors, we introduce the following definition (see [9, section 4.1]).

**Definition 2.2.1.** A vector subspace \(\mathcal{W} \subset \mathcal{G}^k_{\text{sym}}(TM^* \otimes TM^*)\) will be called \(C^k\)-Whitney type Banach space of tensor fields over \(M\) if complies the following conditions:

1. \(\mathcal{W}\) contains all tensor fields in \(\mathcal{G}^k_{\text{sym}}(TM^* \otimes TM^*)\) having compact support;

2. \(\mathcal{W}\) has a Banach space norm \(|| \cdot ||_{\mathcal{W}}\) with the property that \(|| \cdot ||_{\mathcal{W}}\)-convergence of a sequence implies convergence in the weak Whitney \(C^k\)-topology\(^1\).

The second condition means that given any sequence \((b_n)_{n \in \mathbb{N}}\) and \(b_\infty \in \mathcal{W}\) such that \(\lim_{n \to \infty} ||b_n - b_\infty||_{\mathcal{W}} = 0\), then for each compact set \(K \subset M\), the restriction \(b_n|_K\) converges to \(b_\infty|_K\) in the \(C^k\)-topology as \(n \to \infty\).

We can construct a \(C^k\)-Whitney type Banach space of tensors on \(M\) by using an auxiliary Riemannian metric \(g_R\) on \(M\) as follows (see [9, Example 1]). The Levi-Civita connection \(\nabla^R\) of \(g_R\) induces a connection on all vector bundles over \(M\) obtained with functorial constructions from the tangent bundle \(TM\). Also for each \(r, s \in \mathbb{N}\), \(g_R\) induces canonical Hilbert space norms on each tensor bundle \(TM^{(r)} \otimes TM^{(s)}\), which will be denoted \(|| \cdot ||_R\). Now, we define \(\mathcal{G}^k_{\text{sym}}(TM^* \otimes TM^*; g_R)\) as the subset of \(\mathcal{G}^k_{\text{sym}}(TM^* \otimes TM^*)\) consisting of all sections \(\sigma\) such that

\[ ||\sigma||_k = \max_{i=0,...,k} \left[ \sup_{x \in M} ||(\nabla^R)^i \sigma(x)||_R \right] < +\infty. \]  \hspace{1cm} (2.2.1)

The norm \(|| \cdot ||_k\) in (2.2.1) turns \(\mathcal{G}^k_{\text{sym}}(TM^* \otimes TM^*; g_R)\) into a separable normed space (see [25]), which is complete if the Riemannian metric \(g_R\) is complete. Thus, we have that \(\mathcal{G}^k_{\text{sym}}(TM^* \otimes TM^*; g_R)\) is a \(C^k\)-Whitney type Banach space of tensors.

\(^1\)For definition and properties of Whitney \(C^k\)-topology see [25]
When \( M \) is compact, \( \mathcal{G}_k^g(TM^* \otimes TM^*; g_R) = \mathcal{G}_k^g(TM^* \otimes TM^*) \), and \( \text{Met}^k(M) \) is an open subset.

### 2.3 Smooth structure of non-degenerate hypersurface

Since the group of diffeomorphisms of \( \Sigma \) acts freely on the set of embeddings of \( \Sigma \) into \( M \), the appropriate setup for studying the set of submanifolds of a given diffeomorphism type is obtained by considering the notion of \textit{unparameterized embeddings}.

**Definition 2.3.1.** Two embeddings \( \varphi_1 \) and \( \varphi_2 \) from \( \Sigma \) into \( M \) will be \textit{equivalent} if there exists a diffeomorphism \( \phi : \Sigma \to \Sigma \) such that \( \varphi_2 = \varphi_1 \circ \phi \), i.e., if they are different parametrizations of the same submanifold of \( M \) diffeomorphic to \( \Sigma \). For \( \varphi \in \text{Emb}(\Sigma, M) \), we denote by \( [\varphi] \) the class of all embedding that are equivalent to \( \varphi \). We say that \( [\varphi] \) is a \textit{unparametrized embedding} of \( \Sigma \) in \( M \).

**Definition 2.3.2.** We define the following sets:

- \( \mathcal{E}(\Sigma, M) := \{ [\varphi] : \varphi \text{ is an embedding of order } C^{j,\alpha} \} \),
- \( \mathcal{E}_\partial(\Sigma, M) := \{ [\varphi] \in \mathcal{E}(\Sigma, M) : \varphi(\Sigma) \cap \partial M = \varphi(\partial\Sigma) \} \),
- Let \( \gamma \in \text{Met}(M) \),
  \[ \mathcal{E}_{\partial,\gamma}(\Sigma, M) := \{ [\varphi] \in \mathcal{E}_\partial(\Sigma, M) : \varphi \text{ is } \gamma - \text{orthogonal} \} . \]

There is a smooth Banach manifold structure, of infinite dimension, for a sufficiently small neighborhood of \( [\varphi_0] \in \mathcal{E}_{\partial,\gamma}(\Sigma, M) \) in some suitable topology.

**Proposition 2.3.1.** [8, Proposition 4.1] Let \( \Sigma \) be a compact manifold with boundary and \( \varphi_0 \in \text{Emb}_\partial(\Sigma, M) \). Let \( \mathcal{U} \subset \mathcal{E}_{\partial,\gamma}(\Sigma, M) \) be a sufficiently small neighborhood of \( [\varphi_0] \), then \( \mathcal{U} \) can be identified with an infinite-dimensional smooth submanifold \( \mathcal{N} \) of Banach space \( C^{j,\alpha}(\Sigma) \), with \( 0 \in \mathcal{N} \) corresponding to \( [\varphi_0] \), such that \( T_0\mathcal{N} = C^{j,\alpha}_{\partial}(\Sigma) \) (see 1.4.2)

### 2.4 Genericity of Bumpy metrics

In this section we prove that the set of all Riemannian metrics \( \gamma \) in \( M \) for which every minimal free boundary embedding is non-degenerate (Bumpy metrics) is generic in the space of Riemannian metrics of \( M \).

**Theorem 2.4.1.** Let \( M \) be a \((n+1)\)-dimensional differential manifold with smooth boundary \( \partial M \neq \emptyset \), and \( \Sigma \) a \( n \)-dimensional compact differential manifold with smooth boundary \( \partial\Sigma \neq \emptyset \). Let \( \mathcal{W} \subset \mathcal{G}_k^g(TM^* \otimes TM^*) \) be a \( C^k \)-Whitney type Banach sub-space of the symmetrical tensor fields over \( M \), with \( k > j \geq 2 \), let \( \Gamma \subset \mathcal{W} \cap \text{Met}^k(M) \) be an open subset of \( \mathcal{W} \). Let \( \mathcal{M} \) be the set defined as

\[ \mathcal{M} = \{ (\gamma, [\varphi]) \in \Gamma \times \mathcal{E}_\partial(\Sigma, M) : [\varphi] \in \mathcal{E}_{\partial,\gamma}^+(\Sigma, M), \varphi \text{ is } \gamma - \text{minimal} \} . \]

Then,

(1) \( \mathcal{M} \) is a separable Banach manifold modelled on \( \Gamma \).
(2) $\Pi : \mathcal{M} \rightarrow \Gamma$, defined by $\Pi(\gamma, [\varphi]) = \gamma$, is a Fredholm map with index 0.

(3) $\gamma_0$ is critical value of $\Pi$ if and only if there is a $\gamma_0$-minimal embedding $\varphi_0 : \Sigma \rightarrow M$ which is degenerate.

2.4.1 Analytic preliminaries

We begin by proving some lemmas to clarify the ideas in the proof of Theorem 2.4.1.

**Lemma 2.4.2.** Let $\Sigma_0 = \varphi_0(\Sigma)$ be a free boundary minimal surface and $f : \Sigma_0 \rightarrow \mathbb{R}$ satisfying the linearized free boundary condition

$$\gamma_0(\nabla f, \vec{n}_\partial M) + \Pi^{\partial M}(\vec{n}_{\Sigma_0}, \vec{n}_{\Sigma_0}) f = 0.$$ 

Then, there is a map $o : (-\epsilon, \epsilon) \rightarrow C^{j,\alpha}(\Sigma_0)$, with $o(t) \rightarrow 0$ if $t \rightarrow 0$, such that $\varphi_t : \Sigma_0 \rightarrow M$ defined by

$$\varphi_t(p) := \exp_{\varphi_0(p)}(t \vec{f}(p) + o(t)(p)\vec{n}_{\Sigma_0}(p))$$

is orthogonal.

**Proof.** By Proposition 2.3.1 there is a bijective correspondence between a neighborhood $U \subset E_\partial M (\Sigma, M)$ of $[\varphi_0]$ and a infinite-dimensional smooth sub-manifold $N$ of the Banach space $C^{j,\alpha}(\Sigma)$, with $0 \in N$ corresponding to $[\varphi_0]$ and $T_0N = C^{j,\alpha}(\Sigma)$. So, there is a diffeomorphism, given by the Inverse Mapping Theorem (see A.0.1), between $U$ and a neighborhood $V \subset T_0N$ of 0, such that $\varphi_t \mapsto tf$. On the other hand, $Exp_{\varphi_0}$ also generates a diffeomorphism between $U$ and some neighborhood $V' \subset T_0N$, such that $\varphi_t \mapsto t\bar{g}_t$, with $g_0 = 0$ and $g'_0 = \bar{f}$. Since $V$ and $V'$ are diffeomorphic we have $g_t = tf + o(t)$, where $o(t)$ is differentiable and $\frac{o(t)}{t} \rightarrow 0$ if $t \rightarrow 0$.

**Lemma 2.4.3.** Let $\Sigma$ be a $n$-dimensional compact sub-manifold embedding in $M$, $f : \Sigma \rightarrow \mathbb{R}$ function, $f \in C^j$, $f \neq 0$ and $n_\Sigma$ the unit normal vector field to $\Sigma$. Then, there is a map $\psi : M \rightarrow \mathbb{R}$ such that

i) $\psi(p) = 0$, for all $p \in \Sigma$,

ii) $\int_\Sigma d\psi(n_\Sigma) : f(p)d\Sigma \neq 0$

**Proof.** Let $p_0 \in \Sigma$ be such that $f(p_0) > 0$. There is a local coordinate chart around of $p_0$, $(U, x = (x_1, ..., x_{n+1}))$, such that

1) $x(U) = B_1(0) \subset \mathbb{R}^{n+1}$,

2) $x|_{U \cap \Sigma} = (x_1, ..., x_n, 0)$ and

3) $f(p) > 0$ for all $p \in U \cap \Sigma$.

We have

$$dx_p(\vec{n}_p) = \vec{v}_x(p),$$
where \( v_{x(p)} = (v_1, ..., v_{n+1}) \) with \( v_{n+1} \neq 0 \). We may assume that \( v_{n+1} > 0 \).

Set

\[
\begin{align*}
  h: \quad & B_1(0) \quad \rightarrow \quad \mathbb{R} \\
  (x_1, ..., x_{n+1}) \quad & \mapsto \quad l(x_{n+1}),
\end{align*}
\]

where \( l: \mathbb{R} \rightarrow \mathbb{R} \) is a smooth function such that

(i) \( \text{Supp}(l) \subset [-1, 1] \),

(ii) \( l(0) = 0 \) and

(iii) \( l'(0) > 0 \).

For example \( l \) can be taken as follows (see Figure 2.1):

\[
l(t) = \begin{cases} 
-2(2(t - 1))^3 + 3(2(t - 1))^2, & \text{if } -1 < t \leq -1/2 \\
\sin(\pi t), & \text{if } -1/2 < t < 1/2 \\
2(2(t - 1))^3 + 3(2(t - 1))^2, & \text{if } 1/2 \leq t < -1/2 \\
0 & \text{otherwise.}
\end{cases}
\]

![Figure 2.1: Graph of \( l(t) \).](image)

Under these conditions we have

\[
h(B_1(0) \cap \mathbb{R}^n) = 0 \\
\nabla h_p = (0, ..., 0, l'(0)) \quad \text{for all } \quad p \in B_1(0) \cap \mathbb{R}^n
\]

So, we define \( \psi: M \rightarrow \mathbb{R} \) as

\[
\psi(p) = \begin{cases} 
h \circ x(p) & \text{if } p \in U \\
0 & \text{if } p \in M \setminus U.
\end{cases}
\]

Therefore, \( \psi \) fulfills the affirmation i).
Now, for all \( p \in U \cap \Sigma \) we have
\[
d\psi_p(\vec{n}_p) = \nabla h(x(p)) \cdot dx_p(\vec{n}_p) = \nabla h(x(p)) \cdot \vec{v}_x(p) = l'(0) \cdot v_{n+1} > 0
\]
and
\[
d\psi_p(\vec{n}_p) = 0 \quad \text{for} \quad p \in \Sigma \setminus U.
\]
Then
\[
\int_{\Sigma} d\psi_p(\vec{n}_p) \cdot f(p) d\Sigma = \int_{\Sigma \cap U} d\psi_p(\vec{n}_p) \cdot f(p) d\Sigma > 0
\]
\[\square\]

**Lemma 2.4.4.** Let \((X, d)\) be a non-separable metric space and let \((Y, \tau_s)\) be a separable topological space, with \(\emptyset \neq Y \subset X\). Assume that \(\tau_d|Y \subset \tau_s\), where \(\tau_d\) is the topology generated by \(d\). Then, the closure of \(Y\) in \(\tau_d\) is separable.

**Proof.** Let \(\overline{Y}\) be the closure of \(Y\) in \(\tau_d\), and \(y \in \overline{Y}\). Let \(B_r(y)\) be a \(d\)-ball with center \(y\) and radius \(r\), thus \(B_r(y) \cap Y \in \tau_s\). If \(D\) is a dense subset of \(Y\) then there is \(x \in D \cap [B_r(y) \cap Y]\), so \(x \in B_r(y)\). Therefore, \(\overline{Y}\) is separable in \(\tau_d\).

\[\square\]

### 2.4.2 Proof of the Genericity Theorem of Metrics Bumpy

In this section will be dedicated to the proof of Theorem 2.4.1, the main result of this chapter. Under suitable constraints on the set of metrics in \(M\) and the space of embeddings of \(\Sigma\) in \(M\), the its proof follows from theorem 2.1.1.

Parts (2) and (3) are immediate consequence of Theorem 2.1.1.

To prove (1), let \(g\) be a metric in \(M\), \(g\) of class \(C^\infty\), such that \(\partial M\) is \(g\)-totally geodesic. Take \((\gamma_0, [\varphi_0]) \in M\). Let \(\vec{n}_0\) be the unit normal vector field along of \(\varphi_0(\Sigma)\). For each \(f: \Sigma \to \mathbb{R}\) of class \(C^{k,\alpha}\) sufficiently small, we associate the embedding \(\varphi_f: \Sigma \to M\) defined by
\[
\varphi_f(p) := \exp_{\varphi_0(p)}(f(p)\vec{n}_0(p)),
\]
where \(\exp\) is the exponential map in \(M\) defined by \(g\). We set \(\vec{n}_0(p) := \vec{n}_{\varphi_0(p)}\). Since \(\varphi_0\) is orthogonal then \(\vec{n}_0(p) \in T_{\varphi_0(p)}(\partial M)\), for all \(p \in \varphi_0^{-1}(\varphi_0(\Sigma) \cap \partial M)\). Hence \(\varphi_f(\partial \Sigma) \subset \partial M\) since \(\partial M\) is totally geodesic. Note that if \(f \equiv 0\) then \(\varphi_f = \varphi_0\).

Now, let \(U \subset C^{j,\alpha}(\Sigma)\) be a sufficiently small neighborhood of 0 such that \(\exp\) is a diffeomorphism in a neighborhood \(V\) generated by \(U\),
\[
\exp_{\varphi_0(p)}: C^{j,\alpha}(\Sigma) \to M \\
f \mapsto \exp(f, \vec{n}_0) =: \varphi_f(p).
\]

Note that the map \(f \mapsto [\varphi_f]\) is a diffeomorphism between \(U\) and a neighborhood \(\tilde{U}\) of \([\varphi_0] \in \mathcal{E}_0(\Sigma, M)\).
Defined the following spaces:

\[ X = C^{j,\alpha}(\Sigma) \]
\[ Y = C^{j-2,\alpha}(\Sigma) \]
\[ \mathcal{H} = L^2(\Sigma), \]

and the set

\[ \mathcal{X} = \{ (\gamma, f) : f \in U, \varphi_f \text{ is } \gamma \text{-orthogonal to } \partial M \}. \]

Let’s see that \( \mathcal{X} \subset \Gamma \times X \) is a locally fiber sub-bundle.

**Proposition 2.4.5.** \( \mathcal{X} \) is a locally fibered sub-manifold over \( \Gamma \)

**Proof.** Let’s denote \( X = C^{j,\alpha}(\Sigma) \) to simplify writing. We define the following map, clearly differentiable,

\[ \phi : \Gamma \times X \longrightarrow C^{j-1,\alpha}(\partial \Sigma) \]
\[ (\gamma, f) \longmapsto \gamma(\vec{n}_{\varphi_f}, \vec{n}_{\partial M}). \]

where \( \vec{n}_{\varphi_f} \) is a unitary normal field to \( \varphi_f(\Sigma) \) and \( \vec{n}_{\partial M} \) is a unitary normal field to \( \partial M \). So \( \mathcal{X} = \phi^{-1}(0) \). Therefore, if \( \phi \) is a submersion for all \( (\gamma, f) \in \mathcal{X} \), then \( \mathcal{X} \) is a sub-manifold.

Let \( (\gamma_0, f_0) \in \mathcal{X} \), we have to \( T_{\gamma_0} \Gamma \times \{0\} \subset T_{(\gamma_0, f_0)} \mathcal{X} \subset T_{\gamma_0} \Gamma \times T_{f_0} X \).

Now,

\[ d\phi(\gamma, f) : T_\gamma \Gamma \times T_f X \longrightarrow T_{\phi(\gamma, f)} C^{j-1,\alpha}(\partial \Sigma), \]

then, we must prove that \( d\phi(\gamma, f) \) is surjective and \( \text{Ker}(d\phi(\gamma, f)) \) is complemented.

We have that

\[ d\phi(\gamma, f)(\tilde{\gamma}, \tilde{f}) = \frac{\partial \phi}{\partial \gamma} \tilde{\gamma} + \frac{\partial \phi}{\partial f} \tilde{f} \]
\[ = \tilde{\gamma}(\vec{n}_{\varphi_f}, \vec{n}_{\partial M}) + \frac{\partial \tilde{f}}{\partial \vec{n}_{\partial M}} \nabla_{\vec{n}_{\varphi_f}} \vec{n}_{\varphi_f} + \gamma(\vec{n}_{\varphi_f}, \vec{n}_{\partial M}) \tilde{f} \]
\[ = \tilde{\gamma}(\vec{n}_{\varphi_f}, \vec{n}_{\partial M}) + \frac{\partial \tilde{f}}{\partial \vec{n}_{\partial M}} + \Pi^M(\vec{n}_{\varphi_f}, \vec{n}_{\varphi_f}) \tilde{f}. \]

To prove that \( d\phi(\gamma, f) \) is surjective we observe that \( \tilde{\gamma} \in T_\gamma \Gamma \) is a symmetric bilinear form, then for any \( h \in T_{\phi(\gamma, f)} C^{j-1,\alpha}(\partial \Sigma) \), we choose \( \tilde{\gamma} \) such that

\[ \tilde{\gamma}(\vec{n}_{\varphi_f}, \vec{n}_{\partial M}) = h - \left( \frac{\partial \tilde{f}}{\partial \vec{n}_{\partial M}} + \Pi^M(\vec{n}_{\varphi_f}, \vec{n}_{\varphi_f}) \tilde{f} \right). \]

Now, to see that \( d\phi(\gamma, f) \) is complemented, notice that the projection in the second factor,

\[ P_2 : \text{Ker}(d\phi(\gamma, f)) \longrightarrow T_f X, \]

is surjective. Indeed, if \( \tilde{f} \in T_f X \), we can take \( \tilde{\gamma} \) such that \( \tilde{\gamma}(\vec{n}_{\varphi_f}, \vec{n}_{\partial M}) = -\left( \frac{\partial \tilde{f}}{\partial \vec{n}_{\partial M}} + \Pi^M(\vec{n}_{\varphi_f}, \vec{n}_{\varphi_f}) \tilde{f} \right) \).

For fixed \( \tilde{f} \in T_f X \), we define the following space.

\[ \{ \tilde{\gamma} : (\tilde{\gamma}, \tilde{f}) \in \text{Ker}(d\phi(\gamma, f)) \} = \tilde{\gamma} + \{ \tilde{\gamma}(\vec{n}_{\varphi_f}, \vec{n}_{\partial M}) = 0 \}. \]
where \( \tilde{\gamma} = - \left( \frac{\partial f}{\partial \varphi} + \Pi^{\partial M}(\tilde{n}_{\varphi f}, \tilde{n}_{\varphi f}) \dot{f} \right) \).

Let’s see that 

\[
\Gamma_0 = \{ \tilde{\gamma}(\tilde{n}_{\varphi f}, \tilde{n}_{\varphi f}) = 0 \}
\]

is complemented. In general, if \( N_1 \) and \( N_2 \) are differentiable fields in \( M \), such that \( N_1|_{\partial M} = \tilde{n}_{\partial M}, \)

\( N_2|_{\varphi f} = \tilde{n}_{\varphi f} \)

and let \( \eta_0 \in \Gamma^{0,2}_{\text{Sym}}(M) \) be such that, in a neighborhood of \( \partial M \cup \Sigma_f \),

\[
\eta_0(N_1, N_2) = 1.
\]

Then, for all \( \eta \in \Gamma^{0,2}_{\text{Sym}}(M) \)

\[
\eta = \eta_1 + \eta_2,
\]

where \( \eta_1 = -\eta(N_1, N_2) \cdot \eta_0 \) and \( \eta_2 = \eta - \eta(N_1, N_2) \cdot \eta_0 \). So, \( \eta_1 \) is in the generated by \( \eta_0, \eta_2(N_1, N_2) = 0 \), then \( \eta_2 \in \Gamma_0 \). Therefore, \( \Gamma_0 \) is complemented.

Thus, the complement of \( \text{Ker}(d\phi(\gamma, f)) \) is the set

\[
\{ (\tilde{f}, \Gamma_0 - \tilde{\gamma} f) : \tilde{f} \in T_f X \}.
\]

Now, by the Local Form of the Submersions (see statement in A.0.3), we get that \( X \) is locally fibred.

Now, we define the function

\[
A = \Gamma \times X \longrightarrow \mathbb{R}
\]

\[
(\gamma, f) \mapsto A(\gamma, f) := \gamma\text{-area of } \varphi_f(\Sigma),
\]

and the operator

\[
H = \mathcal{X} \longrightarrow \mathcal{Y}
\]

\[
(\gamma, f) \mapsto H(\gamma, f) := \gamma\text{-mean curvature of } \varphi_f(\Sigma).
\]

We have therefore that

\[
\frac{\partial H}{\partial f}(\gamma_0, 0) = J_{(\gamma_0, \varphi_0)},
\]

where \( J_{(\gamma_0, \varphi_0)} \) is the Jacobi operator \( J_{\varphi_0} \) defined in the metric \( \gamma_0 \) (see section 1.3), which restricted to \( C^j_\partial(\Sigma) \) is Fredholm with index 0.

Also, we have that

\[
\frac{d}{dt}\bigg|_{t=0} A(\gamma, f + tl) = \frac{d}{dt}\bigg|_{t=0} \int_{\Sigma} \text{Vol}_{\varphi_f(\Sigma)} \gamma
\]

\[
= \int_{\Sigma} H(\gamma, f) \cdot l
\]

\[
= (H(\gamma, f), l)_H
\]

It remains to show that the condition of transversality defined by equation 2.1.4 is fulfilled, in order to verify the hypotheses of Theorem 2.1.1.

Let \( \tilde{f} \in \ker(\frac{\partial H}{\partial f}(\gamma_0, 0)) = \ker(J_{\varphi_0}) \) be non-zero. As \( J_{\varphi_0} \) is restricted to \( C^j_\partial(\Sigma) \), then \( \tilde{f} \) satisfied
the linearized free boundary condition. Take a smooth variation of \( \varphi_0, \varphi_t := \varphi_{ft}, -\epsilon < t < \epsilon, \) such that
\[
\varphi_t(p) = \text{Exp}_{\varphi_0(p)}([tf(p) + o(t)]\vec{n}_0(p)),
\]
where \( o(t) : (-\epsilon, \epsilon) \to C^{k,\alpha}(\Sigma_0) \) is a differential application with \( \frac{o(t)}{t} \to 0 \) if \( t \to 0 \). By Lemma 2.4.2, \( \varphi_t \) is orthogonal to \( \partial M \).

Now, take a variation \( \gamma_s, -\delta < s < \delta, \) of \( \gamma_0 \) by conformal metrics as follows:
\[
\gamma_s(q) = (1 + s\psi(q))\gamma_0(q),
\]
where \( \psi : M \to \mathbb{R} \) is a smooth function such that \( \psi(q) = 0 \) for all \( q \in \varphi_0(\Sigma) \), this is, if \( q = \varphi_0(p), \psi(\text{Exp}_q(0)) = \psi(q) = 0. \)

Let \( \{\Omega, (x_1, \ldots, x_n)\} \) be a local coordinate chart of \( \Sigma, p \in \Omega \). The volume form associated to the metric \( \gamma_s \) is given by
\[
\text{vol}_{\varphi_s^*(\gamma_s)}|_p = \sqrt{\text{Det}[\varphi_s^*(\gamma_s)]} \, dx^1 \wedge \ldots \wedge dx^n|_p
\]
\[
= \sqrt{\text{Det}\left[\gamma_0(d\varphi_f(\frac{\partial}{\partial x_i}), d\varphi_f(\frac{\partial}{\partial x_j}))\right]} \, dx^1 \wedge \ldots \wedge dx^n|_p
\]
\[
= \sqrt{\text{Det}\left[\left(1 + s\psi(\text{Exp}_{\varphi_0(p)}(f(p)\vec{n}_0(p)))\right)\gamma_0(d\varphi_f(\frac{\partial}{\partial x_i}), d\varphi_f(\frac{\partial}{\partial x_j}))\right]} \, dx^1 \wedge \ldots \wedge dx^n
\]
\[
= \left(1 + s\psi(\text{Exp}_{\varphi_0(p)}(f(p)\vec{n}_0(p)))\right)^{n/2} \sqrt{\text{Det}[\gamma_0(d\varphi_f(\frac{\partial}{\partial x_i}), d\varphi_f(\frac{\partial}{\partial x_j}))]} \, dx^1 \wedge \ldots \wedge dx^n
\]
\[
= \left(1 + s\psi(\text{Exp}_{\varphi_0(p)}(f(p)\vec{n}_0(p)))\right)^{n/2} \text{vol}_{\varphi_s^*(\gamma_0)}
\]

To simplify the notation we write
\[
f_t = tf + o(t)
\]
and
\[
v_t = ft\vec{n}_0.
\]
Note that \( f_0 \equiv 0, \) so \( v_0 = 0. \) Also take \( q = \varphi_0(p) \) and \( \varphi_t = \varphi_{ft}. \) Hence, the area function over the variations of \( \varphi_0 \) and \( \gamma_0 \) given by equations 2.4.1 and 2.4.2 have the following form:
\[
A(\gamma_s, f_t) = \int_\Sigma \text{vol}_{\varphi_s^*(\gamma_s)}
\]
\[
= \int_\Sigma \left(1 + s\psi(\text{Exp}_q(v_t(p)))\right)^{n/2} \text{vol}_{\varphi_s^*(\gamma_0)}.
\]
On the other hand,

\[
\frac{\partial^2}{\partial t \partial s} A(\gamma_s, f_t) \bigg|_{s=t=0} = \frac{d}{dt} \bigg|_{t=0} \frac{d}{ds} \bigg|_{s=0} \int_{\Sigma} \left( 1 + s\psi(Exp_q(v_i)) \right)^{n/2} vol_{\varphi^*_t(\gamma_0)} \\
= \frac{d}{dt} \bigg|_{t=0} \int_{\Sigma} \frac{n}{2} \psi(Exp_q(v_i)) \left( 1 + s\psi(Exp_q(v_i)) \right)^{n/2-1} \bigg|_{s=0} \psi(Exp_q(v_i)) vol_{\varphi^*_t(\gamma_0)} \\
= \frac{d}{dt} \bigg|_{t=0} \int_{\Sigma} \frac{n}{2} \psi(Exp_q(v_i)) vol_{\varphi^*_t(\gamma_0)} \\
= \frac{d}{dt} \bigg|_{t=0} \int_{\Sigma} \frac{n}{2} \left( \frac{d}{dt} \psi(Exp_q(v_i)) \right) vol_{\varphi^*_t(\gamma_0)} + \psi(Exp_q(v_i)) \frac{d}{dt} vol_{\varphi_t(\gamma_0)} \bigg|_{t=0} \\
= \frac{d}{dt} \bigg|_{t=0} \int_{\Sigma} \frac{n}{2} \left( \frac{d}{dt} (Exp_q(v_i)) \cdot dExp_q(v_i) \left( \frac{d}{dt} v_i \right) vol_{\varphi^*_t(\gamma_0)} \right)_{t=0} + 0,
\]

but \( \frac{d}{dt} (v_i) = \frac{d}{dt} ((\vec{f} + O(t))\vec{n}_0) = (\vec{f} + O'(t))\vec{n}_0, \) then

\[
= \int_{\Sigma} \frac{n}{2} \left[ d\psi(Exp_q(0)) \cdot dExp_q(0) (\vec{f}(p)\vec{n}_0(p)) \right] vol_{\varphi^*_0(\gamma_0)} \\
= \int_{\Sigma} \frac{n}{2} \left[ d\psi(\varphi_0(p)) \cdot \vec{f}(p)\vec{n}_0(p) \right] vol_{\varphi^*_0(\gamma_0)} \\
= \int_{\Sigma} \frac{n}{2} \vec{f}(p) \left[ d\psi(\varphi_0(p)) (\vec{n}_0(p)) \right] vol_{\varphi^*_0(\gamma_0)}
\]

so, by lemma 2.4.3 we can to choose \( \psi \) such that the last integral be non zero.

Therefore, by Theorem 2.1.1, there is a neighborhood \( W \) of \( (\gamma_0, 0) \) such that

\[ \tilde{\mathcal{M}} = \{ (\gamma, f) \in W : H(\gamma, f) = 0 \} \]

is a Banach sub-manifold of \( \Gamma \times X \). Now, since \( h(\gamma, f) = (\gamma, [\varphi_f]) \) is a diffeomorphism between \( \tilde{\mathcal{M}} \) and an open subset of \( \mathcal{M} \), we conclude that \( \mathcal{M} \) is a Banach sub-manifold of \( \Gamma \times \mathcal{E}_T(\Sigma, M) \).

It remains to show the separability of \( \mathcal{M} \). For a coordinate system \( (x_1, \ldots, x_n) \) in \( \varphi_f(\Sigma) = \Sigma_f \), the \( \gamma \)-mean curvature of \( \Sigma_f \) is

\[ H(\gamma, f) = \gamma^{ij}\gamma^j(\nabla_{\partial x_i} \frac{\partial \varphi_f}{\partial x_j}, \tilde{\varphi}_f), \]

where \( \gamma_{ij} = \gamma(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) \) and \( \gamma^{ij}\gamma_{ij} = \delta_{ij} \). If \( H(\gamma_0, 0) = 0 \), then \( \varphi_0 \) is solution of a linear elliptic partial differential equation. So by the Schauder theory for elliptic equations we have that \( \varphi_0 \) is \( C^k \) (see [2, Sec. 10]). Let \( S \) be the set of \( \gamma \)-orthogonal minimal embeddings of \( \Sigma \) in \( M \). \( S \subset C^k(\Sigma, M) \). Since \( C^k(\Sigma, M) \) is a separable space and \( C^k(\Sigma, M) \subset C^{j,\alpha}(\Sigma, M) \) the Lemma 2.4.4 implies that the closure of \( S \) in the topology \( C^{j,\alpha} \) is also separable.

Now we state Smale’s theorem which is a generalization of Sard’s theorem for infinite-dimensional spaces. For a proof, see [26]. We will say “almost all” instead of “except for a set of first category”.

**Sard-Smale Theorem.** Let \( V \) and \( W \) be two differentiable Banach manifolds, connected and second-countable. Let \( \phi : V \to W \) be a \( C^k \) Fredholm map with \( k > \max\{ \text{index } \phi, 0 \} \). Then, the set of regular values of \( \phi \) is generic in \( W \).

Whence, as a consequence of Theorem 2.4.1 and the Sard-Smale theorem, we infer the following
Corollary 2.4.6. Under hypotheses of Theorem 2.4.1, the set of $(M, \Sigma)$-Bumpy metrics is generic in $\Gamma$.

2.4.3 Nonzero constant mean curvature

Let’s see now that the main result is valid also when the mean curvature is a constant different from zero.

Let $\gamma \in \text{Met}(M)$, recall the following notation (see 2.3.2):

$$\text{Emb}^{\gamma}\_\partial (\Sigma, M) := \{ \varphi : \varphi : \Sigma \to M \text{ is } \gamma \text{-orthogonal admissible embedding and bounding a finite volume} \}$$

$\varphi \in \text{Emb}^{\gamma}\_\partial (\Sigma, M)$ has CMC $\hbar \neq 0$ if and only if it is a critical point for the functional

$$f_{\hbar}(\varphi) = \int_{\Sigma} \text{vol}_{\varphi^*}^{\gamma} - \hbar \int_{\Omega_{\varphi}} \text{vol}_{\gamma},$$

where $\Omega_{\varphi}$ is the finite volume that bounded $\varphi$.

If in the statement of the main theorem we modify the condition of zero mean curvature by the condition of constant mean curvature $h \neq 0$ in the definition of $M$, that is,

$$\mathcal{M} = \{ (\gamma, [\varphi]) \in \Gamma \times \mathcal{E}_0(\Sigma, M) : [\varphi] \in \mathcal{E}_0^{1,\gamma}(\Sigma, M), \varphi \text{ with } \gamma - \text{CMC } h \neq 0 \},$$

the proof of the theorem is not modified, except for transversality. More explicitly, we must verify that, given $(\gamma_0, [\varphi_0]) \in M$, for all $\bar{f} \in \text{Ker}(\frac{\partial H}{\partial f}(\gamma_0, 0))$, $\bar{f} \neq 0$, there exists a family $(\gamma_s, f_t) \in \mathcal{X}$, $s, t \in (-\epsilon, \epsilon)$,

$$\mathcal{X} = \{ (\gamma, f) \in \Gamma \times C^3 : f \in U, \varphi_f \in \text{Emb}^{\gamma}_{\partial, \perp}(\Sigma, M) \},$$

with $f'(0) = \bar{f}$, such that

$$\frac{\partial^2}{\partial s \partial t} \bigg|_{s=t=0} A(\gamma_s, f_t) \neq 0$$

where

$$A(\gamma_s, f_t) = \int_{\Sigma} \text{vol}_{\varphi_f^*}^{\gamma_s} - \hbar \int_{\Omega_{\varphi_f}} \text{vol}_{\gamma_s}.$$
Now, 

\[
\frac{\partial}{\partial t} \bigg|_{t=0} \int_{\Omega_{\varphi_t}} \text{vol}_{\gamma_s} = \int_{\Sigma_0} \gamma_s (\frac{\partial \varphi_t}{\partial t} \bigg|_{t=0}, \vec{n}_{\Sigma_0}) \text{vol}_{\varphi_0^*(\gamma_s)} \\
= \int_{\Sigma_0} \gamma_0 (\frac{\partial \varphi_t}{\partial t} \bigg|_{t=0}, \vec{n}_{\Sigma_0}) \text{vol}_{\varphi_0^*(\gamma_0)}.
\]

The last integral does not depend on \( s \), so we get the conclusion.

**Corollary 2.4.7.** If in Theorem 2.4.1 we replace the hypothesis that \( \varphi \) is \( \gamma \)-minimal by \( \varphi \) has nonzero CMC, the set of \( (M, \Sigma) \)-Bumpy metrics is generic in \( \Gamma \).
Chapter 3

Deformation of Free Boundary CMC Immersions and Stability Criteria

In this Chapter, given a free boundary CMC immersion \( \varphi \in C^{j+1,\alpha}(\Sigma, M) \), we prove, using properties of eigenvalues and eigenfunctions of the Jacobi operator \( J_\varphi \), that if \( \text{Dim}(\text{Ker}(J_\varphi)) = 0 \), or if \( \text{Dim}(\text{Ker}(J_\varphi)) = 1 \) and, for \( f \in \text{Ker}(J_\varphi) \), \( f \neq 0 \), \( \int_\Sigma f \, \text{vol}_{\varphi^*(g)} \neq 0 \), \( \varphi \) has a free boundary CMC deformation and this is unique up to diffeomorphism (see Theorems 3.2.1 and 3.2.2). In addition, we give stability and unstability criteria for free boundary CMC immersions (see Theorem 3.3.3 and Corollary 3.3.5). We have based on the article by Miyuki Koiso (see [16]), who studies the case for CMC surfaces in \( \mathbb{R}^3 \) with fixed boundary. Using the properties of the eigenvalues and the eigenfunctions of a problem of eigenvalues associated with the second variation of the functional area, Koiso gives sufficient conditions under which it has a CMC deformation that fixes the boundary and also a criterion of stability for the CMC immersions. For a special case, combine these results and obtain a geometric way to study stability.

3.1 Analytical preliminaries

Following with the same notation, \( \Sigma^n \) and \( M^{n+1} \) be smooth manifolds, with smooth boundaries \( \partial \Sigma \) and \( \partial M \) respectively. Let \( g \) be a \( C^\infty \) riemannian metric in \( M \), and \( \varphi_0 : \Sigma \to M \) an free boundary CMC immersion, \( H_0 \) the mean curvature of \( \varphi_0 \). Let \( \vec{n}_{\Sigma_0} \) be the unit normal vector field to \( \Sigma_0 = \varphi_0(\Sigma) \) in the orientable normal bundle and \( \vec{n}_{\partial M} \) the unit normal vector field to \( \partial M \). \( C^{j,\alpha}_0(\Sigma_0) \) the space of functions \( C^{j,\alpha} \) that complies with the linearized free boundary condition \( g(\nabla f, \vec{n}_{\partial M}) + \Pi^M(\vec{n}_{\Sigma_0}, \vec{n}_{\Sigma_0})f = 0 \), defined in (1.4.2). We have that \( C^{j,\alpha}_0(\Sigma_0) \subset L^2(\Sigma_0) \).

Remember that by the Proposition 2.3.1 we have the bijection \( f \mapsto \varphi_f \) from a suitable neighborhood \( U \) of \( 0 \in C^{j,\alpha}(\Sigma_0) \) to a neighborhood \( V \) of \( \varphi_0 \in C^{j,\alpha}(\Sigma, M) \) defined by

\[
\varphi_f(p) := \exp_{\varphi_0(p)}[[f(p) + o(f)(p)]\vec{n}_{\Sigma_0}(p)],
\]

where \( \exp \) is the exponential function of \( g \), \( o : U \to C^{j,\alpha}(\Sigma_0) \) is such that \( \frac{o(f)}{f} \to 0 \) if \( f \to 0 \) and such that \( \varphi_f \) is orthogonal to \( M \).

Remember too that restriction of Jacobi operator, define as \( J_{\varphi_0}(f) := \Delta_{\Sigma_0} f - (||\Pi_{\Sigma_0}||_{\tilde{H}S}^2 + \text{Ric}_g(\vec{n}_{\Sigma_0}, \vec{n}_{\Sigma_0}))f \), to the closed subspace \( C^{j,\alpha}_0(\Sigma_0) \) is a Fredholm operator of index zero that takes values in \( C^{j-2,\alpha}(\Sigma_0) \).
To simplify the notation we set

\[ J = J_{\varphi_0} : C^{j,\alpha}_\sigma(\Sigma_0) \to C^{j-2,\alpha}_\sigma(\Sigma_0). \]

Let \( E = \text{Ker}(J) \) and \( E^\perp \) the orthogonal of \( E \) in \( L^2(\Sigma) \).

By definition \( \Delta_{\Sigma_0} f = \text{Div}(-\nabla f) \) and if \( V \) is a vector field, then \( \text{Div}(V) = \nabla \cdot V \). Now, if \( h \) is a scalar field we have to \( \text{Div}(hV) = \nabla h \cdot \vec{V} + h\text{Div}(V) \). On the other hand by the Gauss theorem (divergence theorem)

\[
\int_{\Sigma} \text{Div}(hV) \, d\Sigma = \int_{\partial \Sigma} hV \cdot \vec{n}_{\Sigma} \, d\theta_{\Sigma}.
\]

**Proposition 3.1.1.** \( J \) is symmetric with the \( L^2(\Sigma_0) \) product inner.

**Proof.** From the definition of \( J_{\varphi_0} \) we see that it is sufficient to prove that \( \Delta_{\Sigma_0} \) is symmetric. Let \( f, h \in C^{j,\alpha}_\sigma(\Sigma_0) \). Then, by the divergence theorem

\[
\int_{\Sigma_0} \Delta_{\Sigma_0}(f)h \, d\Sigma_0 = \int_{\Sigma_0} \text{Div}(-\nabla f)h \, d\Sigma_0 = -\int_{\Sigma_0} \nabla f \cdot \nabla h \, d\Sigma_0 - \int_{\partial \Sigma_0} h \left( \frac{\partial f}{\partial \vec{n}_M} \right) \, d\theta_{\Sigma_0}
\]

\[
= -\int_{\Sigma_0} \nabla f \cdot \nabla h \, d\Sigma_0 - \int_{\partial \Sigma_0} h \left( g(\nabla f, \vec{n}_M) \right) \, d\theta_{\Sigma_0}
\]

\[
= -\int_{\Sigma_0} \nabla f \cdot \nabla h \, d\Sigma_0 + \int_{\partial \Sigma_0} h \left( f\vec{M}(\vec{n}_{\Sigma_0}, \vec{n}_{\Sigma_0}) \right) \, d\theta_{\Sigma_0}
\]

\[
= \int_{\Sigma_0} \Delta_{\Sigma_0}(h)f \, d\Sigma_0.
\]

\( \square \)

**Lemma 3.1.2.** Let \( \lambda \) be a real number. Let

\[
J(f) - \lambda f = h \tag{3.1.1}
\]

a equation, where \( h \in C^{j-2,\alpha}(\Sigma_0) \). \( (3.1.1) \) has a solution in the following cases:

(a) If \( \lambda \) is not an eigenvalue of \( J \), then \( (3.1.1) \) has a unique solution in \( C^{j,\alpha}(\Sigma_0) \) for all \( h \in C^{j-2,\alpha}(\Sigma_0) \)

(b) If \( \lambda \) is an eigenvalue of \( J \), \( E_{\lambda} \subset C^{j,\alpha}_\sigma(\Sigma_0) \) be the eigenspace associated to \( \lambda \) and \( E_{\lambda}^\perp \subset L^2(\Sigma_0) \cap C^{j,\alpha}_\sigma(\Sigma_0) \) the orthogonal to \( E_{\lambda} \), then \( (3.1.1) \) have solution in \( C^{j,\alpha}_\sigma(\Sigma_0) \) if and only if \( h \in E_{\lambda}^\perp \), this is

\[
\int_{\Sigma_0} h\sigma \, \text{vol}_{\varphi_0(g)} = 0,
\]

for all \( \sigma \in E_{\lambda} \).

**Proof.** For (a), we have that \( J \) is a Fredholm operator of index zero. Then so is the operator \( (J - \lambda) : C^{j,\alpha}_\sigma(\Sigma_0) \to C^{j-2,\alpha}(\Sigma_0) \) (as the sum of Fredholm and compact operators). If \( \lambda \) is not an eigenvalue then \( (J - \lambda) \) is injective and hence also surjective \( \text{Codim}(J - \lambda) = 0 \).
For (b), let \( h \in C^{j-2,\alpha}(\Sigma_0) \) be such that \( h = J(f) - \lambda(f) \) for some \( f \in C^{j,\alpha}_\partial(\Sigma_0) \) and let \( \sigma \in E_\lambda \). Then,

\[
\int_{\Sigma_0} h\sigma \operatorname{vol}_{\varphi_0}(g) = \int_{\Sigma_0} (J(f) - \lambda f)\sigma \operatorname{vol}_{\varphi_0}(g) = \int_{\Sigma_0} J(f)\sigma - J(\sigma)f \operatorname{vol}_{\varphi_0}(g) = \int_{\Sigma_0} J(f)\sigma - J(\sigma)f \operatorname{vol}_{\varphi_0}(g) = 0.
\]

The equality holds because \( J \) is symmetric (by proposition 3.1.1). Therefore \( \text{Im}(J - \lambda) \subset E_\lambda^\perp \).

Again, as \((J - \lambda)\) is Fredholm of index zero, \( \text{Im}(J - \lambda) \) is closed and

\[
\text{Codim}(\text{Im}(J - \lambda)) = \text{Dim}(\text{Ker}(J - \lambda)) = \text{Dim}(E_\lambda),
\]

and since \( \text{Im}(J - \lambda) \subset E_\lambda^\perp \), then \( \text{Im}(J - \lambda) = E_\lambda^\perp \). \( \square \)

### 3.2 Existence and uniqueness of CMC deformation

Next, we enunciate our first perturbation theorem. Here we assume that the kernel of the Jacobi operator is trivial. The perturbation obtained is unique, up to parameterizations.

**Theorem 3.2.1.** Let \( \varphi_0 \in C^{j+1,\alpha}(\Sigma, M) \) a free boundary CMC immersion, with mean curvature \( H_0 \). Suppose that \( \text{Dim}(E) = 0 \), this is, the eigenvalues of problem

\[
\begin{cases}
J(f) = \lambda f \\
f \in C^{j,\alpha}_\partial(\Sigma_0)
\end{cases}
\]  

(3.2.1)

are nonzero. Then, there is a neighborhood \( \hat{I} \) of \( H_0 \in \mathbb{R} \) and a unique injective \( C^1 \) mapping, \( \zeta : \hat{I} \to C^{j,\alpha}_\partial(\Sigma_0) \), such that \( \zeta(H_0) = 0 \) and \( \varphi_{\zeta(H)} \) is a free boundary CMC immersion with mean curvature \( H \).

Moreover, if \( \psi : \Sigma \to M \) is an free boundary CMC immersion sufficiently close to \( \varphi_0 \), in the topology of \( C^{j,\alpha} \), then \( \psi \) must be equal (up to diffeomorphisms) to some \( \varphi_{\zeta(H)} \).

**Proof.** Let \( U \) a suitably small neighborhood of \( 0 \in C^{j,\alpha}_\partial(\Sigma_0) \) such that \( \varphi_f \) is orthogonal to \( M \) for each \( f \in U \).

We define the following map

\[
F : U \times \mathbb{R} \longrightarrow C^{j-2,\alpha}(\Sigma_0)
\]

\[
(f, H) \mapsto F(f, H) = H - H_f,
\]

where \( H_f \) is the mean curvature of \( \varphi_f \). Then we have that

i) \( F(0, H_0) = 0 \) and

ii) \( \varphi_f \) has CMC if and only if \( F(f, H) = 0 \) for some \( H \in \mathbb{R} \).
Now,
\[ \frac{\partial F}{\partial f}(0, H_0) : C^j_\partial(\Sigma_0) \rightarrow C^{j-2,\alpha}(\Sigma_0) \]
is given by
\[ \frac{\partial F}{\partial f}(0, H_0)(h) = J(h). \]

Let us see that if 0 is not an eigenvalue of \( J(f) = \lambda f \) then \( \frac{\partial F}{\partial f}(0, H_0) \) is bijective. The injectivity is immediate because \( \text{Dim}(\text{Ker} J) = \text{Dim}(E) = 0 \). Now, applying part (a) of Lemma 3.1.2 with \( \lambda = 0 \) the surjectivity is obtained.

Thus, the conditions for applying the Implicit Mapping Theorem (see the statement in A.0.2) are satisfied for \( F \). Then there is a neighborhood \( \tilde{I} \) of \( H_0 \) and a map \( \zeta : \tilde{I} \rightarrow U \) such that \( \zeta(H_0) = 0 \) and \( F(\zeta(H), H) = 0 \). So \( \varphi_{\zeta(H)} \) is a free boundary CMC immersion with mean curvature \( H \).

A second result on existence of perturbation is given below. Here we assume that the Jacobi operator kernel is generated by a single non-zero function in \( C^{j+1,\alpha}(\Sigma) \). Again, we get uniqueness up to parameterizations.

**Theorem 3.2.2.** Let \( \varphi_0 \in C^{j+1,\alpha}(\Sigma, M) \) a free boundary CMC immersion, with mean curvature \( H_0 \). Suppose that:

1. \( \text{Dim}(E) = 1 \). This is, \( \lambda = 0 \) is an eigenvalue of multiplicity 1 for the problem (3.2.1), and
2. \( \int_{\Sigma_0} f_0 \text{vol}_{\varphi_0} \neq 0 \) for some \( f_0 \in E - \{0\} \).

Then there exist a neighborhood \( W \subset E \) of 0 and a unique injective map \( C^1 \)
\[ (\xi, \eta) : W \rightarrow (C^j_\partial(\Sigma_0) \cap E^\perp) \times \mathbb{R}, \]
such that \( (\xi, \eta)(0) = (0, H_0) \) and such that \( \varphi_{f+\xi(f)} : \Sigma \rightarrow M \), with \( f \in W \), is an free boundary CMC immersion, with mean curvature \( \eta(f) \).

Moreover, if \( \psi : \Sigma \rightarrow M \) is an free boundary CMC immersion sufficiently close to \( \varphi_0 \), in the topology of \( C^{j,\alpha} \), then \( Y \) must be equal (up to diffeomorphisms) to some \( \varphi_{f+\xi(f)} \).

**Proof.** We take \( F : U \times \mathbb{R} \rightarrow C^{j-2,\alpha}(\Sigma_0) \), defined as \( F(f, H) = H - H_f \), as in the proof of Theorem 3.2.1. We have to

i) \( F(0, H_0) = 0 \) and

ii) \( \varphi_f \) have CMC if and only if \( F(f, H) = 0 \) for some \( H \in \mathbb{R} \).

Now we defined the following map:
\[ \tilde{F} : (U \cap E) \times (U \cap E^\perp) \times \mathbb{R} \rightarrow C^{j-2,\alpha}(\Sigma_0) \]
\[ (f_1, f_2, H) \mapsto \tilde{F}(f_1, f_2, H) = F(f_1 + f_2, H). \]

Thus,
\[ \tilde{F}(0, 0, H_0) = 0. \]
Let’s see that
\[ \frac{\partial \bar{F}}{\partial (f_2, H)}(0, 0, H) : (C^j_\alpha(\Sigma_0) \cap E^\perp) \times \mathbb{R} \rightarrow C^{j-2,\alpha}(\Sigma_0) \]
is bijective.

We have that
\[ \frac{\partial \bar{F}}{\partial (f_2, H)}(0, 0, H)(f, H) = H - J(f) \]
and
\[ \text{Ker}(\frac{\partial \bar{F}}{\partial (f_2, H)}(0, 0, H)) = \{(f, H) : H - J(f) = 0\} \]
i.e. if \((f_2, H) \in \text{Ker}(\frac{\partial \bar{F}}{\partial (f_2, H)}(0, 0, H))\), then \(J(f_2) = H\). Let \(f_0 \in E\). Then, by (b) in Lemma 3.1.2 we have
\[ H \int_{\Sigma_0} f_0 \, \text{vol}_{\varphi_0^\alpha(g)} = 0, \]
which implies that \(H = 0\), i.e., \(f_2 \in E\). But \(f_2 \in E^\perp\), so \(f_2 = 0\). Therefore, \(\text{Ker}(\frac{\partial \bar{F}}{\partial (f_2, H)}(0, 0, H)) = \{(0, 0)\}\) and we have the injectivity.

To prove the surjectivity, we take \(h \in C^{j-2,\alpha}(\Sigma_0)\) and set
\[ H = \frac{\int_{\Sigma_0} h f_0 \, \text{vol}_{\varphi_0^\alpha(g)}}{\int_{\Sigma_0} f_0 \, \text{vol}_{\varphi_0^\alpha(g)}}. \]
Again by (b) in Lemma 3.1.2 there exist \(f \in C^j_\alpha(\Sigma_0)\) such that
\[ J(f) = h - H. \]
Now, we decompose \(f = f_1 + f_2\) such that \(f_1 \in E\), \(f_2 \in E^\perp\). So,
\[ J(f_2) = h - H. \]
Then,
\[ h = H - J(f_2) = \frac{\partial \bar{F}}{\partial (f_2, H)}(0, 0, H)(f_2, H). \]

Now we can used The Implicit Mapping Theorem in \(\bar{F} : (U \cap E) \times (U \cap E^\perp) \times \mathbb{R} \rightarrow C^{j-2,\alpha}(\Sigma_0)\).
Then, there exist a neighborhood \(W \subset E \cap U\) of 0 and a map
\[ (\xi, \eta) : W \rightarrow (U \cap E^\perp) \times \mathbb{R}, \]
such that
\[ (\xi, \eta)(0) = (0, H_0) \]
and such that, for all \(f \in W\),
\[ 0 = \psi(f, (\xi, \eta)(f)) = \psi(f, \xi(f), \eta(f)) = \phi(f + \xi(f), \eta(f)). \]
Thus, \(\varphi_{f + \xi(f)}\) have CMC \(\eta(f)\). If \(W\) is suitable small then \((\xi, \eta)\) is unique. \(\square\)
3.3 Stability Criteria

Let $\varphi_t : \Sigma \rightarrow M$ be a variation of $\varphi_0$, $-\epsilon < t < \epsilon$, such that $\varphi_t$ is a free boundary CMC immersion. Let $A(t)$ and $V(t)$ be the area and volume of $\varphi_t$ respectively. We will say that $\varphi_t$ is volume-preserving if $V(t) = V(0)$ for all $t \in (-\epsilon, \epsilon)$. Remember that $\varphi_t$ have CMC $H_t$ if only if is a critical point of functional

$$f(t) = A(t) - H_t V(t) = \int_{\Sigma} \text{vol} \varphi_t^* (g) - H_t \int_{\Omega_t} \text{vol} g,$$

where $\Omega_t$ is the volume enclosed by $\varphi_t$. Let $\xi_t = g(\frac{\partial \varphi_t}{\partial t}, \vec{n}_\Sigma)$, then we have (see [3, Appendix])

$$A'(t) = \int_{\Sigma} H_t \xi_t \text{vol} \varphi_t^* (g).$$

Then,

$$A''(0) = \int_{\Sigma} J(\xi_0) \xi_0 \text{vol} \varphi_0^* (g).$$

Also,

$$V'(t) = \int_{\Sigma} \xi_t \text{vol} \varphi_t^* (g).$$

**Definition 3.3.1.** A free boundary CMC immersion $\varphi : \Sigma \rightarrow M$ is said to be stable if only if $A''(0)(f) \geq 0$, for all $f \in C^{j,\alpha}_0(\partial \Sigma)$ such that $\int_{\Sigma} f \text{vol} \varphi_0^* (g) = 0$. When $\varphi$ is not stable, it is said to be unstable.

To simplify the notation we write

$$I(f) = \int_{\Sigma} J(f) f \text{vol} \varphi_0^* (g).$$

Since the stability criteria, presented below, depend of eigenvalues of Jacobi operator, it is important to highlight the following in relation to them:

**Remark 3.3.1.** In the problem (3.2.1), the eigenvalues of $J(f) = \lambda f$ are a countable set (Smale, [27, lemma 1]), such that $\lambda_1 < \lambda_2 \leq ..., \lambda_i \rightarrow \infty$ if $i \rightarrow \infty$. Each eigenfunction is in $C^{j+1,\alpha}_0(\Sigma)$ (Gilbarg-Trudinger, [13, Theorem 8.13]), and Ladyzhenskaya-Ural’tseva [19, Chap. 3, Theorem 12.1]). And we can choose an orthonormal basis $B = \{f_i\}$ for $L^2(\Sigma)$ where each $f_i$ is associated to $\lambda_i$. The first eigenvalue $\lambda_1$, which has a special role in the spectral theory of $J$, is always simple, i.e., of multiplicity 1, and the $\lambda_1$-eigenfunction are positives. Let $f_k$ be a (non zero) eigenfunction corresponding to the eigenvalue $\lambda_k$, for some $k \geq 1$. The connected components of the set $\Sigma \setminus f_k^{-1}(0)$ are called the nodal domains of $f_k$. Then, the number of nodal domains of $f_k$ is less than or equal to $k$; this is known as Courant’s nodal domain theorem (see [4]).

For the demonstration of our stability criteria we need the following Smale’s Lemma.

**Lemma 3.3.2.** [27, lemma 4][Smale Lemma]

a) Let $\mathcal{H}$ be the prehilbert space $\mathcal{H}_0^0(E)$ with the inner product $\langle , \rangle_L$ and $B_L$ quadratic form on $\mathcal{H}$, $B_L(h) = B_L(h, h)$. Then $B_L$ has a minimum $\lambda_1$ at $f_1$ on the unit sphere $S$ of $\mathcal{H}$ and on $\{f_1, ..., f_{q-1}\}^\perp \cap S$ has its minimum value $\lambda_q$ and $f_q$. 

b) Let \( d(v) \) be the minimum of \( B_L \) on \( V^\perp \cap S \), where \( V \) is a finite dimensional subspace of \( H \). Then

\[
\lambda_n = \max_{\dim V < n} d(V)
\]

Thus, we can write our first stability criterion in the following way:

**Theorem 3.3.3.** Let \( \varphi_0 : \Sigma \to M \) a free boundary CMC immersion. Let \( \lambda_i, i \geq 1 \) be the eigenvalues \( J\varphi_0 \).

1. If \( \lambda_1 \geq 0 \), then \( \varphi_0 \) is stable.
2. If \( \lambda_1 < 0 < \lambda_2 \), then there is a unique function \( \kappa \in C^{j,\alpha}(\Sigma) \) such that \( J(\kappa) = 1 \) and we have that:
   - (2-a) If \( \int_{\Sigma} \kappa \, \text{vol}_{\varphi_0}(g) \leq 0 \), then \( \varphi_0 \) is stable.
   - (2-b) If \( \int_{\Sigma} \kappa \, \text{vol}_{\varphi_0}(g) > 0 \), then \( \varphi_0 \) is unstable.
3. If \( \lambda_1 < 0 = \lambda_2 \), then we have:
   - (3-a) If there exist a \( \lambda_2 \)-eigenfunction \( f_2 \) such that \( \int_{\Sigma} f_2 \, \text{vol}_{\varphi_0}(g) \neq 0 \), then \( \varphi_0 \) is unstable.
   - (3-b) If \( \int_{\Sigma} h \, \text{vol}_{\varphi_0}(g) = 0 \) for all \( \lambda_2 \)-eigenfunction \( h_2 \), then there exist a unique function \( \tilde{h}_2 \in (\ker(J\varphi_0))^\perp \) such that \( J(\tilde{h}_2) = 1 \) and
     - (3-b-i) If \( \int_{\Sigma} \tilde{h}_2 \, \text{vol}_{\varphi_0}(g) \leq 0 \), then \( \varphi_0 \) is stable.
     - (3-b-ii) If \( \int_{\Sigma} \tilde{h}_2 \, \text{vol}_{\varphi_0}(g) > 0 \), then \( \varphi_0 \) is unstable.
4. If \( \lambda_2 < 0 \), then \( \varphi_0 \) is unstable.

**Proof.** By Lemma 3.3.2 we have that:

\[
\lambda_1 = I(f_1) = \int_{\Sigma} J(f_1) f_1 \, \text{vol}_{\varphi_0}(g) = \min\left\{ I(f) : f \in C_j^{\alpha}(\Sigma) \text{ and } \int_{\Sigma} f^2 \, \text{vol}_{\varphi_0}(g) = 1 \right\}
\]

\[
\lambda_i = I(f_i) = \int_{\Sigma} J(f_i) f_i \, \text{vol}_{\varphi_0}(g) = \min\left\{ I(f) : f \in C_j^{\alpha}(\Sigma), \int_{\Sigma} f^2 \, \text{vol}_{\varphi_0}(g) = 1, \text{ and } \int_{\Sigma} f f_k \, \text{vol}_{\varphi_0}(g) = 0, \text{ for } k \in \{1, ..., i - 1\} \right\},
\]

where \( i = 2, 3, ... \).

So, if \( \lambda_1 \geq 0 \) we have (1).

Now we assume that \( \lambda_1 < 0 \). We know that \( f_1 \) does not change sign (see Remark 3.3.1), then

\[
\int_{\Sigma} f_1 \, \text{vol}_{\varphi_0}(g) \neq 0.
\]

For \( \kappa \in C_j^{j+1,\alpha}(\Sigma) \), let

\[
a = \frac{\int_{\Sigma} \kappa \, \text{vol}_{\varphi_0}(g)}{\int_{\Sigma} f_1 \, \text{vol}_{\varphi_0}(g)}.
\]
and

$$\xi = af_1 + \kappa.$$ 

Then,

$$\int_{\Sigma} \xi \text{vol}_{\varphi_0^*(g)} = \int_{\Sigma} af_1 + \kappa \text{vol}_{\varphi_0^*(g)}$$

$$= \int_{\Sigma} \frac{\int_{\Sigma} \kappa \text{vol}_{\varphi_0^*(g)} f_1 + \kappa \int_{\Sigma} f_1 \text{vol}_{\varphi_0^*(g)}}{\int_{\Sigma} f_1 \text{vol}_{\varphi_0^*(g)}}$$

$$= \frac{1}{\int_{\Sigma} f_1 \text{vol}_{\varphi_0^*(g)}} \left[ - \int_{\Sigma} \kappa \text{vol}_{\varphi_0^*(g)} f_1 \text{vol}_{\varphi_0^*(g)} + \int_{\Sigma} \kappa \text{vol}_{\varphi_0^*(g)} f_1 \text{vol}_{\varphi_0^*(g)} \right]$$

$$= 0$$

By Lemma 3.1.2-(a) (with \(\lambda = 0\) and \(h = 1\)) there exists \(\kappa \in C_0^j+1,\alpha(\Sigma)\) such that \(J(\kappa) = 1\).

Using the symmetry of \(J\) we have:

$$I(\xi) = \int_{\Sigma} \xi J(\xi) \text{vol}_{\varphi_0^*(g)} = \int_{\Sigma} (af_1 + \kappa)J(af_1 + \kappa) \text{vol}_{\varphi_0^*(g)}$$

$$= \int_{\Sigma} \left[ a^2 f_1 J(f_1) + af_1 J(\kappa) + af_1 J(\kappa) + \kappa J(f_1) \right] \text{vol}_{\varphi_0^*(g)}$$

$$= a^2 \lambda_1 \int_{\Sigma} f_1^2 \text{vol}_{\varphi_0^*(g)} + 2a \int_{\Sigma} f_1 J(\kappa) \text{vol}_{\varphi_0^*(g)}$$

$$+ a \int_{\Sigma} \left[ \kappa J(f_1) - f_1 J(\kappa) \right] \text{vol}_{\varphi_0^*(g)} + \int_{\Sigma} \kappa \text{vol}_{\varphi_0^*(g)}$$

$$= a^2 \lambda_1 + 2a \int_{\Sigma} f_1 J(\kappa) \text{vol}_{\varphi_0^*(g)} + \int_{\Sigma} \kappa \text{vol}_{\varphi_0^*(g)}$$

$$= a^2 \lambda_1 - \int_{\Sigma} \kappa \text{vol}_{\varphi_0^*(g)}.$$ 

Since \(\lambda_1 < 0\), then \(I(\xi) < 0\) if \(\int_{\Sigma} \kappa \text{vol}_{\varphi_0^*(g)} > 0\). Thus, (2-b) is satisfied.

Now, if \(\kappa = f_2, f_2 \in B\), then

$$I(\xi) = a^2 \lambda_1 \int_{\Sigma} f_2^2 \text{vol}_{\varphi_0^*(g)} + 2a \int_{\Sigma} f_1 J(f_2) \text{vol}_{\varphi_0^*(g)}$$

$$+ a \int_{\Sigma} \left[ f_2 J(f_1) - f_1 J(f_2) \right] \text{vol}_{\varphi_0^*(g)} + \int_{\Sigma} f_2 J(f_2) \text{vol}_{\varphi_0^*(g)}$$

$$= a^2 \lambda_1 \int_{\Sigma} f_2^2 \text{vol}_{\varphi_0^*(g)} + 2a \lambda_2 \int_{\Sigma} f_1 f_2 \text{vol}_{\varphi_0^*(g)} + \lambda_2 \int_{\Sigma} f_2^2 \text{vol}_{\varphi_0^*(g)}$$

$$= \lambda_1 \frac{\left( \int_{\Sigma} f_2 \text{vol}_{\varphi_0^*(g)} \right)^2}{\left( \int_{\Sigma} f_1 \text{vol}_{\varphi_0^*(g)} \right)^2} + \lambda_2,$$

therefore if \(\int_{\Sigma} f_2 \text{vol}_{\varphi_0^*(g)} \neq 0\) and \(\lambda_2 = 0\), then \(I(\xi) < 0\), so we have (3-a), e.i., \(\varphi_0\) is unstable. If \(\lambda_2 < 0\), (4) is fulfilled.

To proof (2-a), we define:

$$E_1 = \{af_1 : a \in \mathbb{R}\}, \quad E_1^+ = \left\{ f \in C_0^{j+1,\alpha}(\Sigma) : \int_{\Sigma} f \text{vol}_{\varphi_0^*(g)} = 0 \right\}.$$
Again by Lemma 3.1.2-(a), there is a function $\kappa \in C^{j+1,\alpha}_\partial(\Sigma)$ such that $J(\kappa) = 1$. If $\int_{\Sigma} \kappa \, \text{vol}_{\varphi_0}(g) \leq 0$, then

$$I(\kappa) = \int_{\Sigma} \kappa J(\kappa) \, \text{vol}_{\varphi_0}(g) = \int_{\Sigma} \kappa \, \text{vol}_{\varphi_0}(g) \leq 0$$

As

$$\lambda_2 = I(f_2) = \min \left\{ I(f) : f \in C^{j,\alpha}_\partial(\Sigma) \cap E_1^+, \int_{\Sigma} f^2 \, \text{vol}_{\varphi_0}(g) = 1 \right\}$$

and $\lambda_2 > 0$, we have that $\kappa \notin E_0^\perp$. Therefore, for any $\xi \in C^{j,\alpha}_\partial(\Sigma)$, where $\int_{\Sigma} \xi \, \text{vol}_{\varphi_0}(g) = 0$, we can write $\xi = b\kappa + w$, $b \in \mathbb{R}$, $w \in E_1^+$. So,

$$I(\xi) = \int_{\Sigma} \xi J(\xi) \, \text{vol}_{\varphi_0}(g) = \int_{\Sigma} (b\kappa + w)[bJ(\kappa) + J(w)] \, \text{vol}_{\varphi_0}(g)$$

$$= \int_{\Sigma} \left[b^2 \kappa J(\kappa) + bwJ(\kappa) + b\kappa J(w) + wJ(w) \right] \, \text{vol}_{\varphi_0}(g)$$

$$= b^2 \int_{\Sigma} \kappa \, \text{vol}_{\varphi_0}(g) + 2b \int_{\Sigma} wJ(\kappa) \, \text{vol}_{\varphi_0}(g)$$

$$+ b \int_{\Sigma} \kappa J(w) - wJ(\kappa) \, \text{vol}_{\varphi_0}(g) + I(w)$$

$$= b^2 \int_{\Sigma} \kappa \, \text{vol}_{\varphi_0}(g) + 2b \int_{\Sigma} w \, \text{vol}_{\varphi_0}(g) + I(w)$$

$$= -b^2 \int_{\Sigma} \kappa J(\kappa) \, \text{vol}_{\varphi_0}(g) + 2b \int_{\Sigma} (b\kappa + w) \, \text{vol}_{\varphi_0}(g) + I(w)$$

$$= -b^2 I(\kappa) + I(w) \geq 0.$$

Thus, $\varphi_0$ is stable.

Now, under the hypotheses of (3-b), $\lambda_2 = 0$ and $\int_{\Sigma} h_2 \, \text{vol}_{\varphi_0}(g) = 0$ for all $\lambda_2$-eigenfunctions $h_2$. Let $E_0$ be the eigenspace associated with $\lambda_2 = 0$,then $E = Ker(J) = E_0$. By Lemma 3.1.2-(b) (with $h = 1$ and $\lambda = 0$) there exist a unique function $\bar{h}_2 \in E^\perp \cap C^{j+1,\alpha}_\partial(\Sigma)$ such that $J(\bar{h}_2) = 1$. So,

(i) If $\int_{\Sigma} h_2 \, \text{vol}_{\varphi_0}(g) = 0$,

$$I(\bar{h}_2) = \int_{\Sigma} \bar{h}_2 J(\bar{h}_2) \, \text{vol}_{\varphi_0}(g) = \int_{\Sigma} \bar{h}_2 \, \text{vol}_{\varphi_0}(g) = 0 = \lambda_2.$$

assuming that $\bar{h}_2 \in E_1^+$ and since

$$\lambda_2 = I(f_2) = \min \left\{ I(f) : f \in C^{j,\alpha}_\partial(\Sigma) \cap E_1^+, \int_{\Sigma} f^2 \, \text{vol}_{\varphi_0}(g) = 1 \right\},$$

then $\bar{h}_2$ is a $\lambda_2$-function, e.i., $J(\bar{h}_2) = 0$. This is a contradiction. Therefore, $\bar{h}_2 \notin E_1^\perp$. So, stability is proved as in (2-a). Similary, if $\int_{\Sigma} h_2 \, \text{vol}_{\varphi_0}(g) < 0$, stability is also proved as in (2-a).

(ii) If $\int_{\Sigma} h_2 \, \text{vol}_{\varphi_0}(g) > 0$, then the proof is the same as for (2-b). Thus, $\varphi_0$ is unstable.

\[ \square \]
The following proposition shows that when the operator $J_{\varphi_0}$ has an eigenvalue equal to zero, this is equivalent to the fact that the function of mean curvatures $H_t$, for a free boundary CMC immersions variation, has a critical point at $t = 0$.

**Proposition 3.3.4.** If $\{\varphi_t\}_{-\epsilon < t < \epsilon}$, is a perturbation of $\varphi_0$, such that $\varphi_t : M \to \Sigma$ is a $C^{j,\alpha}$ free boundary CMC immersion, and $\frac{dH_t}{dt}|_{t=0} = H'_0 = 0$, where $H_t$ is the constant mean curvature of $\varphi_t$, then some eigenvalue $\lambda_i$ of $J_{\varphi_0}$ must vanish.

Conversely, if some eigenvalue $\lambda_i$ of $J_{\varphi_0}$ vanishes, and some $\lambda_j$-eigenfunction $f_i$ satisfies $\int_\Sigma f_i \text{vol}_{\varphi_0}(g) \neq 0$, then for every free boundary CMC perturbation $\varphi_t$ of $\varphi_0$ we have $H'_0 = 0$.

**Proof.** Let $\xi_0 = g(\frac{\partial \varphi_0}{\partial t}|_{t=0}, \vec{n}_\Sigma_0)$. We have that $J(\xi_0) := J_{\varphi_0}(\xi_0) = H'_0$. If $H'_0 = 0$, then $\xi_0 \in \text{Ker}(J)$, $\xi_0 \neq 0$. Thus, there is some eigenvalue $\lambda_i = 0$.

Conversely, if $\lambda_i = 0$ and $f_i$ is a $\lambda_i$-eigenfunction such that $\int_\Sigma f_i \text{vol}_{\varphi_0}(g) = \langle 1, f_i \rangle_{L^2(\Sigma)} \neq 0$, then $1 \notin (\text{Ker}(J))^\perp$. Therefore, the only constant function in the image of $J$ must be 0. Given any free boundary CMC perturbation $\varphi_t$ of $\varphi_0$ and $\xi_0 = g(\frac{\partial \varphi_0}{\partial t}|_{t=0}, \vec{n}_\Sigma_0)$, as $J(\xi_0) = H'_0$ is a constant function in $\text{Im}(J)$ so $H'_0 = 0$. \hfill \Box

The second stability criteria is given in the following corollary. It’s possible check the positivity of the first eigenvalue or the negativity of the second eigenvalue in Theorem 3.3.3. However, for cases II and III the calculation of the eigenvalues can be quite difficult. In this sense, the following criterion facilitates the understanding of the geometric meaning for these cases and is based on the existence of families of 1-parameter deformations.

**Corollary 3.3.5.** Let $\varphi : \Sigma \to M$ a free boundary CMC immersion of class $C^{j+1,\alpha}$. We assume that $\lambda_1 < 0 \leq \lambda_2$. If there exists a deformation $\varphi_t$ of $\varphi$, $-\epsilon < t < \epsilon$, with $\varphi_0 = \varphi$, such that $\varphi_t$ is a free boundary CMC immersion of class $C^{j,\alpha}$ for all $t \in (-\epsilon, \epsilon)$, and such that $\frac{dH_t}{dt}|_{t=0} = H'_0 = \text{constant} \neq 0$, where $H_t$ is the constant mean curvature of $\varphi_t$ and $V_t$ is the volume of $\varphi_t$, we have that:

1. If $H'_0V'_0 \leq 0$, then $\varphi$ is stable;

2. if $H'_0V'_0 > 0$, then $\varphi$ is unstable.

If there is no such deformation, then $\varphi$ is unstable.

**Proof.** In the case of $\lambda_1 < 0 < \lambda_2$, by Theorem 3.2.1 there exists a strictly monotonous deformation $\{\varphi_t\}_{-\epsilon < t < \epsilon}$, with $H_t = t$. So, $H'_t = 1$, in particular $H'_0 = 1$. We have that $H'_0 = J(\xi_0)$, where $\xi_0 = g(\frac{\partial \varphi_0}{\partial t}|_{t=0}, \vec{n}_\Sigma)$ and $V'_0 = \int_\Sigma \xi_0 \text{vol}_{\varphi_0}(g)$. That way, the conditions of the Theorem 3.3.3-(2) fulfilled and then we have (1) and (2).

Now, if $\lambda_1 < 0 = \lambda_2$, one of the implications of proposition 3.3.4 says that: If $\lambda_i = 0$, for some $i = 1, 2, ...$, and, for some $f_i \in E_{\lambda_i}$, $\int_\Sigma f_i \text{vol}_{\varphi_0}(g) \neq 0$, then, for every perturbation $\{\varphi_t\}_{t \in I}$ of $\varphi_0$ we have that $H'_0 = 0$. This is equivalent to: If $\lambda_i = 0$, for some $i = 1, 2, ...$, and there is a perturbation $\{\varphi_t\}$ of $\varphi_0$ such that $H'_0 \neq 0$, then, for all $f_i \in E_{\lambda_i}$, $\int_\Sigma f_i \text{vol}_{\varphi_0}(g) = 0$. Therefore, since $H'_0 = c \neq 0$, $c$ constant, then the conditions for Theorem 3.3.3-(3-b) are met. So, let’s take $h = \frac{1}{c} \xi_0$, we have

$$J(h) = \frac{1}{c} J(\xi_0) = \frac{1}{c} H'_0 = 1.$$
Now, $J(h) = 1$ if only if $h = u + h_0$, where $u \in E^\perp \cap C_{\lambda^2}^{j+1}(\Sigma)$ is the only one such that $J(u) = 1$, (by Lemma 3.1.2-(b)), and $h_0 \in E = E_{\lambda^2}$. So, if we take $\bar{h}_2 = h - h_0$, and as $\int_\Sigma h_0 \ vol_{\bar{\varphi}_0}(g) = 0$, we have

$$H_0'V_0' = H_0' \int_\Sigma \xi_0 \ vol_{\bar{\varphi}_0}(g) = c \int_\Sigma \xi_0 \ vol_{\varphi_0}(g)$$

$$= c^2 \int_\Sigma \epsilon_0 \ vol_{\varphi_0}(g) - c^2 \int_\Sigma h_0 \ vol_{\varphi_0}(g)$$

$$= c^2 \int_\Sigma (\frac{1}{c} \xi_0 - h_0) \ vol_{\varphi_0}(g)$$

$$= c^2 \int_\Sigma \bar{h}_2 \ vol_{\varphi_0}(g)$$

Thus, (1) and (2) are equivalent to (i) and (ii) in the part III.b) of Theorem 3.3.3.

\[\square\]

We will use these criteria in Chapter 5 to study the stability in bifurcation branches that will be presented in Chapter 4.
Chapter 4

Bifurcation of Free Boundary CMC Immersions

In this Chapter we present two results on the existence of bifurcation points for a 1-parametric family \( \{ \varphi_t \}_{t \in I} \) of free boundary CMC hypersurfaces. The first result, Theorem 4.2.3, furnishes a smooth bifurcating family where the parameter is the mean curvature, i.e., a family where mean curvatures coincide with the mean curvatures of the original family. The second result, Theorem 4.3.2, provides a smooth bifurcating branch of free boundary CMC hypersurfaces, whose volumes coincide with the volumes of surfaces in the original family, that is the bifurcation parameter is the volume. In both cases, the conditions are imposed that the kernel of the Jacobi operator is one-dimensional and that the mean curvature function defined on the family \( \{ \varphi_t \}_{t \in I} \) is strictly monotonous. In the second case, additionally, it is imposed that the function volume defined on \( \{ \varphi_t \}_{t \in I} \) is also strictly monotonous. For prove of our criteria, we will base ourselves on the Koiso-Palmer-Piccione article [18] and use the Crandall and Rabinowitz bifurcation theory presented in [10] and [11].

4.1 Definition of bifurcation

In this Chapter we will continue using the notation established in the previous Chapters, to remember:

- \( \Sigma^n \) and \( M^{n+1} \) smooth manifolds, with smooth boundaries \( \partial \Sigma \) and \( \partial M \) respectively;
- \( g \) a \( C^\infty \) riemannian metric in \( M \);
- \( \varphi : \Sigma \to M \) an free boundary CMC immersion;
- \( \vec{n}_\Sigma \) the unit normal vector field to \( \varphi(\Sigma) \) in the orientable normal bundle;
- \( \{ f \in C^{j,\alpha}(\Sigma) : g(\nabla f, \vec{n}_\partial M) + \| \mathbb{I} M(\vec{n}_\Sigma, \vec{n}_\Sigma) \| f = 0 \} \);
- \( J = J_{\varphi}|_{C^{j,\alpha}(\Sigma)} : C^{j,\alpha}(\Sigma) \to C^{j-2,\alpha}(\Sigma) \);
- the Jacobi operator of \( \varphi \) defined as

\[
J_{\varphi}(f) := \Delta_\Sigma f - (\| \mathbb{I} M \|_{HS} + \text{Ric}_g(\vec{n}_\Sigma, \vec{n}_\Sigma)) f;
\]

\( E = \text{Ker}(J) \) and \( E^\perp = (\text{Ker}(J))^\perp \cap C^{j,\alpha}(\Sigma) \), this orthogonal with respect to the internal product of \( L^2(\Sigma) \); \( \{ \varphi_t : \Sigma \to M \}_{-\epsilon<t<\epsilon} \) be a variation of \( \varphi \), such that \( \varphi_0 = \varphi \) and \( \varphi_t \) is a free boundary
CMC immersion, with mean curvature $H_{\ell}$, and defined as

$$\varphi_{\ell}(p) = \text{Exp}_{\varphi_{0}(p)}(\phi_{\ell}(p)\mathbf{n}_{\Sigma_{0}}(p)), $$

where $\phi_{\ell} = t\xi_{0} + o(t)$, $\xi_{0} = g(\frac{\partial \varphi_{0}}{\partial t}|_{t=0}, \mathbf{n}_{\Sigma_{0}}) \in C^{j,\alpha}(\Sigma)$, $o : (-\epsilon, \epsilon) \to C^{j,\alpha}(\Sigma)$, $\frac{o(t)}{t} \to 0$ if $t \to 0$; $J_{\ell}$ the Jacobi operator of $\varphi_{\ell}$.

Next we give the general definition of bifurcation point for an application $F$ defined between general Banach spaces.

**Definition 4.1.1.** Let $W$ and $Y$ be Banach spaces, $\Omega \subset W$ open and $F : \Omega \to Y$ a continuous map. Suppose there is a simple arc in $\Omega$ parameterized by $\omega : I \to \Omega$, where $I$ is an interval in $\mathbb{R}$, such that $F(\omega(t)) = 0$ for all $t \in I$. If there exists a number $t_{0} \in I$ such that every neighborhood of $w(t_{0})$ contains zeros of $F$ that are not in $\text{Im}(\omega)$, then $w(t_{0})$ is called a bifurcation point for the equation $F = 0$ with respect to curve $\omega$.

What is sought when studying Bifurcation Theory is to find the bifurcation points for the equation $F = 0$ with respect to curve $\omega$ and to understand the structure of $F^{-1}\{0\}$ around such points. In our case $W$ is going to be of the form $I \times X$, where $X$ is a real Banach space and the curve will be of the form $\text{Im}(\omega) = \{(t, 0) : t \in I, 0 \in X\}$.

In our context, $\varphi_{0}$ is a bifurcation point with respect to $\{\varphi_{\ell}\}_{t \in (-\epsilon, \epsilon)}$ if and only if, for all $s > 0$, given a $s$-neighborhood of $\varphi_{0}$, $V_{s} \subset C^{j,\alpha}(\Sigma, M)$, there exists $\psi_{s} \in V_{s}$, such that, $\psi_{s}$ is a free boundary CMC immersion and $\psi_{s}(\Sigma)$ is not congruent with any $\varphi_{\ell}(\Sigma)$.

### 4.2 Existence of Bifurcation with parameter the mean curvature

Next, we will give the definition of $K$-simple own value of $T$, where $K$ and $T$ are bounded linear operators. Also, a lemma and a theorem that are part of the Crandall-Rabinowitz Theory that are important in the development of the demonstration of our criteria.

**Definition 4.2.1.** Let $X$ and $Y$ be Banach Spaces. Let $B(X, Y)$ denote the set of bounded linear maps of $X$ into $Y$. Let $T, K \in B(X, Y)$. Then $\mu \in \mathbb{R}$ is a $K$-simple eigenvalue of $T$ if

1. $\dim (\ker(T - \mu K)) = \text{codim (Im}(T - \mu K)) = 1$,

   and, if $\ker(T - \mu K) = \text{Span}\{x_{0}\}$,

2. $K(x_{0}) \notin \text{Im}(T - \mu K)$.

The next lemma is about of existence of $K$-simple eigenvalues for operators $T$ nearby to the fixed operator $T_{0}$.

**Lemma 4.2.1.** ([11, Lemma 1.3], Crandall-Rabinowitz) Let $T_{0}, K \in B(X, Y)$ and assume that $r_{0}$ is a $K$-simple eigenvalue of $T_{0}$. Then there exists a value $\delta > 0$ such that whenever $T \in B(X, Y)$ and $\|T - T_{0}\| < \delta$, there is a unique $r(T) \in \mathbb{R}$ satisfying $\|r(T) - r_{0}\| < \delta$ for which $T - r(T)K$ is singular. The map $T \mapsto r(T)$ is analytic and $r(T)$ is a $K$-simple eigenvalue of $T$. Finally, if $\ker(T_{0} - r_{0}K) = \text{Span}\{x_{0}\}$ and $Z$ is a complement of $\text{Span}\{x_{0}\}$ in $X$, there is a unique null vector $x(T)$ of $T - r(T)K$ satisfying $x(T) - x_{0} \in Z$. The map $T \mapsto x(T)$ is also analytic.
The following theorem guarantees the existence of bifurcation branches, under some conditions, for a function $F : I \times V \to Y$, where $I \subset \mathbb{R}, V \subset X, X Y$ Banach spaces in general.

**Theorem 4.2.2.** ([10, Teorema 1.7] Crandall-Rabinowitz) Let $X, Y$ be Banach spaces, $V$ a neighborhood of $0$ in $X, I = (-\epsilon, \epsilon)$ be a non-empty open interval, and let

$$F : I \times V \to Y$$

with the following properties

1. $F(t, 0) = 0$ for all $t \in I$,
2. The partial derivatives $F_t, F_x$ and $F_{tx}$ exist and are continuous,
3. Dim(Ker($F_x(0, 0)$)) = Codim(Im($F_x(0, 0)$)) = 1,
4. If Ker($F_x(0, 0)$) = Span{$x_0$}, $F_{tx}(0, 0)(x_0) \notin \text{Im}(F_x(0, 0))$.

If $Z$ is any complement Ker($F_x(0, 0)$) in $X$, then there is a neighborhood $U$ of $(0, 0)$ in $\mathbb{R} \times X$, an interval $(-a, a)$, and continuous functions $\zeta : (-a, a) \to Z$ and $\tau : (-a, a) \to \mathbb{R}$, such that $\tau(0) = 0$, $\zeta(0) = 0$, and

$$F^{-1}(0) \cap U = \{(\tau(s), sx_0 + s\zeta(s)) : -a < s < a\} \cup \{(t, 0) : (t, 0) \in U\}.$$ 

If $F_{xx}$ is also continuous, the functions $\tau$ and $\zeta$ are once continuously differentiable.

Now, we give our first criterion of existence of bifurcation for a family of free boundary CMC hypersurfaces, so that the mean curvatures of the hypersurfaces in the bifurcation branch are equal to those of the original family.

**Theorem 4.2.3.** Let $\{\varphi_t : \Sigma \to M\}_{-\epsilon < t < \epsilon}$ be a smooth variation of $\varphi$, such that $\varphi_0 = \varphi$ and $\varphi_t$ is a free boundary CMC immersion, with mean curvature $H_t$ and:

1. $\frac{dH}{dt}|_{t=0} = H'_0 \neq 0$
2. Dim($E$) = 1. That is, $E = \{be : b \in \mathbb{R}\}$, for some $e \in C^{0,\alpha}_0(\Sigma_0), e \neq 0$.

Then,

1. $\int_{\Sigma} e \text{vol}_{\varphi_0^*g} = 0$, and there exists a differentiable map $\lambda : (-\epsilon_0, \epsilon_0)$ $\to \mathbb{R}, 0 < \epsilon_0 \leq \epsilon$, such that $\lambda(0) = \lambda_0 = 0$, $\lambda(t) = \lambda_t$ is a simple eigenvalue of $J_t$, and there is no other eigenvalue of $J_t$ near 0.

2. Assume further that $\lambda'(0) \neq 0$ holds. Then, $\varphi_0$ is a bifurcation point with respect to $\{\varphi_t\}_{-\epsilon < t < \epsilon}$, where the bifurcation branch is an analytic family of free boundary CMC immersions. More precisely, there exist an open interval $I \subset \mathbb{R}, 0 \in I$, and $C^1$ functions $\zeta : I \to E$ $\perp$ and $t : I \to \mathbb{R}$, such that $t(0) = 0$, $\zeta(0) = 0$, and

$$\psi_s = \text{Exp}_{\varphi_t(s)}([\phi_t(s) + se + s\zeta(s)]n_{\varphi_t(s)}),$$
is a free boundary CMC immersion with mean curvature $\hat{H}_s = H_{t(s)}$.

(3) Every free boundary CMC immersion sufficiently close, in the topology of $C^{j,\alpha}$, to $\varphi_0$, is equal, up to parameterization, to some element of families $\{\varphi_t\}_{t \in I}$, $0 \in I \subset (-\epsilon_0,\epsilon_0)$, or $\{\psi_s\}_{s \in I}$.

Furthermore, the surfaces $\{\varphi_t\}_{t \in I}$ and $\{\psi_s\}_{s \in I}$ are pairwise distinct except for $\varphi_0 = \psi_0$.

Proof. Let $v_0 = \frac{\partial \varphi_t}{\partial t} |_{t=0}$ be the variational field of $\varphi_t$ in $\varphi_0$, and $\xi_0 = g(v_0, \vec{n}_{\Sigma_0})$. We have that

$J(\xi_0) = \frac{d}{dt} \bigg|_{t=0} H_t = H_0' \neq 0$.

So, as $e \in E = \text{Ker}(J)$

$$H'(0) \int_{\Sigma} e \ vol_{\varphi_0^*(g)} = \int_{\Sigma} eJ(\xi_0) \ vol_{\varphi_0^*(g)} = \int_{\Sigma} \xi_0 J(e) \ vol_{\varphi_0^*(g)} = 0,$$

then

$$\int_{\Sigma} e \ vol_{\varphi_0^*(g)} = 0$$

Now, we must verify that the conditions to apply the Lemma 4.2.1 are met with $X = C^{j,\alpha}_0(\Sigma)$, $Y = C^{j,\alpha}(\Sigma)$, $Z = E^\perp$ and let $\iota : X \hookrightarrow Y$ the inclusion map, and thus complete the proof of (1).

Now, by the Fredholm Alternative Theorem

$$y \in \text{Im}(J) \iff y = J(x) \text{ for some } x \in X \iff \int_{\Sigma} ye \ vol_{\varphi_0^*(g)} = 0 \iff y \in E^\perp.$$ 

So,

$$\text{Im}(J) = E^\perp.$$ 

Therefore $\text{Codim}(\text{Im}(J)) = 1$ in $C^{j,\alpha}_0(\Sigma)$ and $\iota(e) = e \notin \text{Im}(J)$. Thus, 0 is a $\iota$-simple eigenvalue of $J$. Then, by Lemma 4.2.1, there exist a differentiable map $\lambda : (-\epsilon_0,\epsilon_0) \rightarrow \mathbb{R}$, such that $\lambda_0 = 0$, $\lambda_t$ is a simple eigenvalue of $J_t$ and there is no another eigenvalue of $J_t$ near to 0.

For the proof of (2), recall from of Proposition 2.3.1 that we have the bijection $f \mapsto \varphi_f$ from a suitable neighborhood $U$ of $0 \in C^{j,\alpha}(\Sigma_0)$ to a neighborhood $W$ of $\varphi_0 \in C^{j,\alpha}(\Sigma, M)$ defined by

$$\varphi_f(p) := \text{Exp}_{\varphi_0(p)}([f(p) + o(f)(p)]\vec{n}_{\Sigma_0}(p)),$$

where $\text{Exp}$ is the exponential function generated by $g$, $o : U \rightarrow C^{j,\alpha}(\Sigma_0)$ is such that $\frac{o(f)}{f} \rightarrow 0$ if $f \rightarrow 0$ and such that $\varphi_f$ is orthogonal to $M$. Let $H_f$ be the mean curvature of $\varphi_f$. Now, we define

$$F : I \times U \rightarrow C^{j-2,\alpha}(\Sigma_0)$$

$$(t,f) \mapsto F(t,f) = H_{\phi_t+f} - H_{\phi_t}$$

where $I = (-\epsilon,\epsilon)$, $\phi_t = t\xi_0 + o(t)$, $\xi_0 \in C^{j,\alpha}_0(\Sigma_0)$ fixed, $o : I \rightarrow C^{j,\alpha}(\Sigma_0)$, $\frac{o(t)}{t} \rightarrow 0$ if $t \rightarrow 0$. We want to prove that $F$ fulfills the conditions of Theorem 4.2.2.

It is clear that $F$ is a twice continuously Fréchet differentiable mapping by the smoothness of the variation.

It is also evident that $F(t,0) = 0$ for all $t \in I$.

Now, if $F(t,f) = 0$ then $H_{\phi_t+f} = H_{\phi_t} = \text{const.}$
Conversely, if for some \( h \in C^{j,\alpha}_0(\Sigma) \), \( H_h = c \), \( c \) a constant, and \( c \) is near enough to \( H_0 \), then there is some \( t_c \in I \) such that \( H_{\phi_{t_c}} = c \). The latter because \( H'_0 \neq 0 \). If we do \( f = h - \phi_{t_c} \) we have to \( H_{\phi_{t_c} + f} = c \), thus \( F(t_c, f) = 0 \). Therefore, in a neighborhood of \( \varphi_0 \) any free boundary CMC immersion is obtained as a solution of \( F(t, f) = 0 \).

Let \( T_h U \) be the tangent space to \( U \) in \( h \in U \). We have that \( T_h U \cong C^{j,\alpha}_0(\Sigma) \). So,

\[
\frac{\partial}{\partial f} F(t, 0) : C^{j,\alpha}_0(\Sigma) \to C^{j-2,\alpha}_0(\Sigma),
\]

and

\[
\frac{\partial}{\partial f} F(t, 0)(u) = J_t(u),
\]

where \( J_t \) is the Jacobi operator of \( \varphi_t \) and \( J_0 = J \).

Thus, \( \text{Ker}\left(\frac{\partial}{\partial f} F(0, 0)\right) = \text{Ker}(J) = E \) and therefore one-dimensional. Also, since \( J \) is Fredholm with zero index, then \( \text{Codim}(\text{Im}(J)) = \text{Codim}(\text{Im}(\frac{\partial}{\partial f} F(0, 0))) = 1 \).

It remains to prove that

\[
\frac{\partial^2}{\partial t \partial f} F(0, 0)(e) \notin \text{Im}\left(\frac{\partial}{\partial f} F(0, 0)\right).
\]

From part (1) we have

\[
\frac{\partial}{\partial f} F(t, 0)(e_t) = J_t(e_t) = \lambda_t e_t.
\]

\( e_t \) the unit eigenvector associated with \( \lambda_t \). If we differentiate the second equation with respect to \( t \) and evaluate at \( t = 0 \) then we have

\[ J'_0(e_0) + J_0(e'_0) = \lambda_0 e_0. \]

Remember that \( \lambda_0 = 0 \), \( e_0 = e \) and \( J_0 = J \). Further, \( \lambda'_0 \neq 0 \). So \( \lambda'_0 e \in E - \{0\} \), that is \( \lambda'_0 e \notin E^\perp = \text{Im}(J) = \text{Im}\left(\frac{\partial}{\partial f} F(0, 0)\right) \). Thus,

\[
\frac{\partial^2}{\partial t \partial f} F(0, 0)(e) = J'_0(e) = \lambda'_0 e - J(e'_0) \notin \text{Im}\left(\frac{\partial}{\partial f} F(0, 0)\right).
\]

Then, by Theorem 4.2.2, there exist a neighborhood \( \tilde{U} \) of \( (0, 0) \) in \( \mathbb{R} \times C^{j,\alpha}_0(\Sigma) \), an interval \( \tilde{I} = (-a, a) \subset \mathbb{R} \) and continuous functions \( \zeta : \tilde{I} \to E^\perp \) \( t : \tilde{I} \to \mathbb{R} \), such that \( \zeta(0) = 0 \), \( t(0) = 0 \) and

\[
F^{-1}\{0\} \cap \tilde{U} = \{(t, s, e) \in \tilde{I} \times E^\perp : t : \tilde{I} \to \mathbb{R} \} = \{(t, 0) : t \in \tilde{I}, 0 \in C^{j,\alpha}_0(\Sigma)\}.
\]

Therefore, we have the existence of unique free boundary CMC immersions, with mean curvature \( H_{\varphi(t)} \), of the form

\[
\psi_s = \exp_{\varphi(s)}([\varphi_{s(t)} + se + s\zeta(s)]\tilde{n}_{\varphi_{t(s)}(\Sigma)}).
\]

Now, for (3) the variational vector field of \( \varphi_t \) at \( \varphi_0 \) is \( v_0 \tilde{n}_{\Sigma_0} \), while the variational vector field of \( \psi_s \) at \( \varphi_0 \) is \( (t'(0)v_0 + e)\tilde{n}_{\Sigma_0} \). Since \( J(v_0) = \text{const} \neq 0 \) and \( J(e) = 0 \), then \( v_0 \) and \( e \) are linearly independent. So, the families \( \{\varphi_t : t \in I\} \) and \( \{\psi_s : s \in \tilde{I}\} \) have non equivalent elements, except for \( \varphi_0 = \psi_0 \).
4.3 Existence of Bifurcation with parameter the volume

In this section we will prove a second smooth bifurcation result of the hypersurfaces with CMC and free boundary, whose volumes coincide with the volumes of the surfaces in the original family. For this we need to define some objects and develop some preliminaries.

For simplicity we denote:

\[ B = C_{\alpha}^{l}(\Sigma), \]

\[ B_{0} = \left\{ h \in B : \int_{\Sigma} h \text{ vol}_{\varphi_{0}^{*}(g)} = 0 \right\} \]

and

\[ V : B \longrightarrow \mathbb{R} \]

\[ f \mapsto V(f), \]

where \( V(f) \) is the volume of \( \varphi_{f} : \Sigma \rightarrow M \). Also, let

\[ l := dV(0) : B \longrightarrow \mathbb{R} \]

\[ w \mapsto l(w) = dV(0)(w) = \int_{\Sigma} w \text{ vol}_{\varphi_{0}^{*}(g)}. \]

Let \( w_{0} \in B \) be such that \( l(w_{0}) = 1 \) \( (w_{0} = \frac{1}{dV(0)(\xi_{0})}\xi_{0}) \).

Now, we define

\[ P : B \longrightarrow \mathbb{R} \times B_{0} \]

\[ f \mapsto (V(f), f - dV(0)(f)w_{0}). \]

We have that \( P(0) = (V(0), 0) \). Let us see that \( dP(0) \) is an isomorphism:

\[ dP(0) : B \rightarrow \mathbb{R} \times B_{0}, \]

where

\[ dP(0)(w) = (dV(0)(w), w - dV(0)(w)w_{0}) \]

\[ = (l(w), w - l(w)w_{0}) \]

Therefore, \( \text{Ker}(dP(0)) = \{0\} \). So, \( dP(0) \) is injective.

Now, to prove that is surjective, let \( a \in \mathbb{R} \) and \( h \in B_{0} \). There exist \( f \in B \) such that \( l(f) = \int_{\Sigma} f \text{ vol}_{\varphi_{0}^{*}(g)} = a \) and \( h = f - aw_{0} \). In effect, if we take \( f = h + aw_{0} \), then

\[ \int_{\Sigma} h \text{ vol}_{\varphi_{0}^{*}(g)} = \int_{\Sigma} f \text{ vol}_{\varphi_{0}^{*}(g)} - a \int_{\Sigma} w_{0} \text{ vol}_{\varphi_{0}^{*}(g)} = a - a = 0. \]

So,

\[ dP(0)f = (l(f), f - l(f)w_{0}) = (a, h + aw_{0} - aw_{0}) = (a, h). \]

Then, by the Inverse Mapping Theorem (see Theorem A.0.1), there exists a neighborhood of
where $P$ is invertible. Thus, for a neighborhood $U_0$ of 0 in $B_0$ and an interval $I \subset \mathbb{R}$ such that $V(0) \in I$, we can define

$$Q = P^{-1} : I \times U_0 \rightarrow B.$$  \hspace{1cm} (4.3.3)

For $(t, h) \in I \times U$, with suitable neighborhoods $I$ of 0 in $\mathbb{R}$ and $U$ of 0 in $B_0$, we write

$$f_{(t,h)} = Q(V(\phi_t), \gamma(t) + h),$$  \hspace{1cm} (4.3.4)

where

$$\gamma(t) = \phi_t - l(\phi_t)w_0,$$  \hspace{1cm} (4.3.5)

$$\phi_t = t\xi_0 + o(t)$$

as defined before. As $\phi'_t = \xi_0$ and $\gamma'(t) = \phi'_t - \frac{\xi_0}{\int_\Sigma \text{vol}_{\phi_0^*}(g)} l(\phi'_b)$, then

$$\gamma(0) = 0 \quad \text{and} \quad \gamma'(0) = 0.$$  \hspace{1cm} (4.3.6)

We can observe that, fixing $t$ and varying $h$, $\varphi_{f_{(t,h)}}$ are immersions that have the same volume as $\varphi_t = \varphi_{\phi_t}$. In particular $f_{(t,0)} = \phi_t$.

Let

$$C^{j-2,\alpha}_\ast(\Sigma) := \left\{ f \in C^{j-2,\alpha}(\Sigma) : \int_\Sigma f \text{vol}_{\phi_0^*(g)} = 0 \right\},$$

and define

$$F : I \times U \rightarrow C^{j-2,\alpha}_\ast(\Sigma)$$

$$(t, h) \mapsto F(t, h) = H_{f_{(t,h)}} - H_{\phi_t},$$

where $I \times U$ a suitable neighborhood of $(0,0)$ in $\mathbb{R} \times B_0$, and

$$\kappa_0 : C^{j-2,\alpha}(\Sigma) \rightarrow C^{j-2,\alpha}_\ast(\Sigma)$$

$$f \mapsto f - \frac{1}{|\Sigma|} \int_\Sigma f \text{vol}_{\phi_0^*(g)},$$

where $|\Sigma| := \int_\Sigma \text{vol}_{\phi_0^*(g)}$ the area of $\Sigma$. We can see that

$$\text{Ker}(\kappa_0) = \{ \text{The constant functions in } C^{j-2,\alpha}(\Sigma) \}.$$

Also, clearly $\kappa_0$ is an surjective linear operator.

Then, since $F(t,0) = 0$ and $\kappa_0(0) = 0$, the composition

$$\bar{F} := \kappa_0 \circ F : I \times U \rightarrow C^{j-2,\alpha}_\ast(\Sigma)$$  \hspace{1cm} (4.3.9)

satisfies

$$\bar{F}(t,0) = 0.$$  \hspace{1cm} (4.3.10)

The following lemma characterizes the free boundary CMC immersions that have fixed volume $\nu$, as solutions of the equation $\bar{F}(t, h) = 0$ and that are represented in terms of the functions $f_{(t,h)}$ defined in (4.3.4).
**Lemma 4.3.1.** Let $\Omega$ be a neighborhood of 0 in $B$ and $f \in \Omega$. Suppose that $dV(0) \neq 0$. Then:

1) If there exist $(t, h) \in I \times U$ such that $f = f_{(t,h)}$, $V(\phi_t) = \nu$ and $F(t, h) = 0$, then $H_{f_{(t,h)}}$ is constant and $V(f_{(t,h)}) = \nu$. Here $H_{f_{(t,h)}}$ is the mean curvature of $\varphi_{f_{(t,h)}} : \Sigma \to M$.

2) Conversely, if $H_f$ is constant and $V(f) = \nu$, then there exist $(t, h) \in I \times U$ such that $f = f_{(t,h)}$, $V(\phi_t) = \nu$ and $F(t, h) = 0$.

**Proof.**  
1) If 
$$0 = F(t, h) = \kappa_0(F) = \kappa_0(H_{f_{(t,h)}} - H_{\phi_t}),$$
then $H_{f_{(t,h)}} - H_{\phi_t}$ is constant, because Ker($\kappa_0$) consists of constant functions. As $H_{\phi_t}$ is constant we have that $H_{f_{(t,h)}}$ is constant. Now, if $t$ is fixed, the functions $f_{(t,h)} = Q(V(\phi_t), \gamma(t) + h)$ have the same volume $V(\phi_t)$. Therefore, $V(f_{(t,h)}) = V(\phi_t) = \nu$.

2) Being $\Omega$ a neighborhood of 0, small enough, and $f \in \Omega$, we have that $P(f) = (\nu, \eta)$ is near $(V(0), 0)$ ($P$ was defined in 4.3.2). As $dV(0) \neq 0$ there is a unique $t$ close to 0 such that $V(\phi_t) = \nu$. Now, $f = P^{-1}(V(\phi_t), \eta)$, $\eta \in B_0$. It is enough to see that $\eta$ is of the form $\eta = \gamma(t) + h$ ($h \in B_0$, $\gamma(t)$ was defined in 4.3.5). Indeed, it is enough to take $h = \eta - \gamma(t)$. Therefore, $f = P^{-1}(V(\phi_t), \gamma(t) + h) = Q(V(\phi_t), \gamma(t) + h) = f_{(t,h)}$. Also, by hypothesis, $H_{f_{(t,h)}} = H_f$ is constant. So, $H_{f_{(t,h)}} - H_{\phi_t}$ is constant. Then 
$$F(t, h) = \kappa_0(H_{f_{(t,h)}} - H_{\phi_t}) = 0$$

Now, we present our second criterion of existence of bifurcation, the case in which the parameter is the volume. Again, the idea of the proof is to see that the conditions of Crandall-Rabinowitz Theorema are met (see Theorema 4.2.2).

**Theorem 4.3.2.** Let $\{\varphi_t : \Sigma \to M\}_{-\epsilon < t < \epsilon}$ be a smooth variation of $\varphi$, such that $\varphi_0 = \varphi$ and $\varphi_t$ is a free boundary CMC immersion, with mean curvature $H_t$ and:

1. $\frac{dH_t}{dt}|_{t=0} = H'_0 \neq 0$
2. $\frac{dV(\phi_t)}{dt}|_{t=0} = dV(0) \neq 0$
3. $\dim(E) = 1$. That is, $E = \{b \in \mathbb{R} : \text{for some } e \in C^1_0(\Sigma_0), e \neq 0\}$.

Then,

1. $\int_{\Sigma} e \, \text{vol}_{\varphi_0} = 0$, there exists a differentiable map $\tilde{\lambda} : (-\epsilon_0, \epsilon_0) \to \mathbb{R}$, $0 < \epsilon_0 \leq \epsilon$, such that $\lambda(0) = \lambda_0 = 0$, $\tilde{\lambda}(t) = \lambda_t$ is a simple eigenvalue of $\tilde{J}_t$ = $\frac{\partial}{\partial t} F(t, 0)$, and there is no other eigenvalue of $\tilde{J}_t$ near 0.

2. Assume further that $\frac{dH_t}{dt}|_{t=0} = \lambda'_0 \neq 0$ holds. Then, $\varphi_0$ is a bifurcation point with respect to $\{\varphi_t\}_{-\epsilon < t < \epsilon}$, where the bifurcation branch is a smooth family of free boundary CMC immersions. More precisely, there exist an open interval $\hat{I} \subset \mathbb{R}$, $0 \in \hat{I}$, and $C^1$ functions $\eta : \hat{I} \to B_0$ and $\tau : \hat{I} \to \mathbb{R}$, such that $\tau(0) = 0$, $\eta(0) = 0$, and 
$$\psi_s = \exp_{\varphi_0}(f(\tau(s), s \eta(s)) \tilde{\eta}) \varphi_0,$$
is a free boundary CMC immersion with volume $\tilde{V}(s) = V(\tau(s))$.

(3) Every free boundary CMC immersion sufficiently $C^{1,\alpha}$-close, to $\varphi_0$, is equal, up to reparametrization, to some element of families $\{\varphi_t\}_{t \in I}$, $I \subset (-\epsilon_0, \epsilon_0)$, or $\{\psi_s\}_{s \in I}$. Furthermore, the hypersurfaces in the families $\{\varphi_t\}_{t \in I}$ and $\{\psi_s\}_{s \in I}$ are different, except for $\varphi_0 = \psi_0$.

Proof. (1) The proof of $\int_{\Sigma} e \text{vol}_{\tau_0^*(g)} = 0$ is equal to the proof that was made in the first part of Theorem 4.2.3. Now,

$$\mathbf{J}_t = \frac{\partial}{\partial h} F(t,0) : B_0 \rightarrow C^{2,\alpha}_{*}(\Sigma)$$

$$\mathbf{J}_t = \frac{\partial}{\partial h} F(t,0) = \frac{\partial}{\partial h} \kappa_0(F(t,0)) = \kappa_0\left( \frac{\partial}{\partial h} F(t,0) \right)$$

$$= \kappa_0 \left( \frac{\partial}{\partial h} (H_{f(t,h)} - H_{\phi_t})\big|_{h=0} \right)$$

So,

$$\mathbf{J}_t(u) = \kappa_0\left( \mathbf{J}_t\left( \frac{\partial f(t,0)}{\partial h} u \right) \right).$$

Since $PQ$ is the identity in $\hat{I} \times U_0$, we have

$$P(f(t,h)) = PQ(V(\phi(t)), \gamma(t) + h) = (V(\phi(t)), \gamma(t) + h). \quad (4.3.11)$$

Differentiating (4.3.11) with respect to $h$ and evaluating at $u \in B_0$, we have:

$$dP(f(t,h))\left( \frac{\partial f(t,h)}{\partial h} \right)(u) = (0, u).$$

On the other hand, from the definition of $P$ (see 4.3.2) we have

$$dP(f(t,h))\left( \frac{\partial f(t,h)}{\partial h} \right)(u) = \left( \int_{\Sigma} \frac{\partial f(t,h)}{\partial h}(u) \xi_t \text{vol}_{\tau_{f(t,h)}^*(g)} , \right)$$

$$= \left( \int_{\Sigma} \frac{\partial f(t,h)}{\partial h}(u) \right) - \left( \int_{\Sigma} \frac{\partial f(t,h)}{\partial h}(u) \text{vol}_{\tau_{f(t,h)}^*(g)} w_0 \right)$$

where $\xi_t = g(\frac{\partial \varphi_{f(t,h)}}{\partial h}, \overline{n}_{f(t,h)})$, $\overline{n}_{f(t,h)}$ the orthonormal field to $\varphi_{f(t,h)}$. So, for any $u \in B_0$

$$\int_{\Sigma} \frac{\partial f(t,h)}{\partial h}(u) \xi_t \text{vol}_{\tau_{f(t,h)}^*(g)} = 0,$$

and

$$\left( \frac{\partial f(t,h)}{\partial h} \right)(u) - \left( \int_{\Sigma} \frac{\partial f(t,h)}{\partial h}(u) \text{vol}_{\tau_{f(t,h)}^*(g)} w_0 \right) = u.$$

Therefore, in the particular case $(t, h) = (0, 0)$, we have, for any $u \in B_0$,

$$\int_{\Sigma} \frac{\partial f(0,0)}{\partial h}(u) \text{vol}_{\tau_{0}^*(g)} = 0 \quad \text{and} \quad \frac{\partial f(0,0)}{\partial h}(u) = u.$$

Thus,

$$\mathbf{J}_0(u) = \frac{\partial F}{\partial h}(0,0)(u) = \kappa_0(J_0(u)) = J_0(u) - \frac{1}{|\Sigma|} \int_{\Sigma} J_0(u) \text{vol}_{\tau_0^*(g)},$$

here $J_0 = J$. Then, $\mathbf{J}_0(u) = 0$ if and only if $J(u)$ is a constant.
If \( J(u) = c, c \) a constant, \( \xi_0 = g\left( \frac{\partial g}{\partial t} \right)_{t=0}, n_0 \) and as \( H'_0 \neq 0 \) we have to

\[
J\left( u - \frac{c}{H'_0} \xi_0 \right) = J(u) - \frac{J(u)}{H'_0} J(\xi_0) = 0,
\]

since \( J(\xi_0) = H'_0 \). Therefore, \( u - \frac{c}{H'_0} \xi_0 = ae, a \in \mathbb{R} \). So, \( u = ae + \frac{c}{H'_0} \xi_0 \). Moreover, if \( u \in B_0 \), that is

\[
0 = \int_{\Sigma} u \ vol_{u'_0}(g) = \int_{\Sigma} ae + \frac{c}{H'_0} \xi_0 \ vol_{u'_0}(g)
= a \int_{\Sigma} e \ vol_{u'_0}(g) + \frac{c}{H'_0} \int_{\Sigma} \xi_0 \ vol_{u'_0}(g)
= \frac{c}{H'_0} dV(0),
\]

since \( \int_{\Sigma} e \ vol_{u'_0}(g) = 0 \). Now, as \( dV(0) \neq 0 \) by hypothesis, then \( c = J(u) = 0 \). Thus,

\[
\text{Ker}(\tilde{J}_0) = \text{Ker}\left( \frac{\partial F}{\partial h} (0,0) \right) = \text{Ker}(J) \quad (4.3.12)
\]

So,

\[
\dim(\text{Ker}(\tilde{J}_0)) = 1 \quad (4.3.13)
\]

We have \( J : B \to C^{j-2,\alpha}(\Sigma) \) is a linear Fredholm operator with index \( i_F(J) = 0 \). This is \( \text{Codim}(\text{Im}(J)) = 1 \). Now, as \( \int_{\Sigma} e \ vol_{u'_0}(g) = 0 \), so,

\[
\text{Ker}(J) \subset B_0.
\]

Since \( \text{Codim}_B(B_0) = 1 \), then \( \text{Codim}(J(B_0)) = 2 \). Let \( \iota : B_0 \hookrightarrow B \) be the inclusion map. \( \iota \) is Fredholm with index \( i_F(\iota) = -1 \). On the other hand \( \text{Ker}(\kappa_0) \) consist of the constant functions. So, \( \dim(\text{Ker}(\kappa_0)) = 1 \). As \( \kappa_0 \) is surjective, then \( \text{Codim}(\text{Im}(\kappa_0)) = 0 \). Thus, the Fredholm index of \( \kappa_0 \) is \( i_F(\kappa_0) = 1 \). Also, \( J \circ \iota : B_0 \to C^{j-2,\alpha}(\Sigma) \) have \( \text{Ker}(J \circ \iota) = \text{Ker}(J) \). So \( \dim(\text{Ker}(J \circ \iota)) = 1 \). Therefore, \( \text{Codim}(J \circ \iota) = -1 \).

Thus

\[
\tilde{J}_0 = \frac{\partial F}{\partial h} (0,0) = \kappa_0 \circ J \circ \iota
\]

is Fredholm with index \( i_f(\tilde{J}_0) = i_F(\kappa_0 \circ J \circ \iota) = i_F(\kappa_0) + i_F(J) + i_f(\iota) = 0 \).

So,

\[
\text{Codim}(\text{Ker}(\tilde{J}_0)) = 1 \quad (4.3.14)
\]

Now, let \( \iota_0 : B_0 \hookrightarrow C^{j-2,\alpha}_{\iota}(\Sigma) \) be the inclusion map. We must see that \( \iota_0(e) \notin \text{Im}(\tilde{J}_0) \). We know that \( e \notin J(B) = E_{-1} \), then \( e \notin J(B_0) \). \( \kappa_0(e) = e - \frac{1}{H'_0} \int_{\Sigma} e \ vol_{u'_0}(g) = e \). Now, suppose there is \( u \in B_0 \) such that \( \kappa_0(J(u)) = e \). This is, \( 0 = \kappa_0(J(u)) - e = \kappa_0(J(u)) - \kappa_0(e) = \kappa_0(u - \frac{c}{H'_0} \xi_0) = H'_0 \kappa_0(u - \frac{c}{H'_0} \xi_0) = 0 \).

So,

\[
\text{Codim}(\text{Ker}(\tilde{J}_0)) = 1 \quad (4.3.14)
\]
existence of bifurcation with parameter the volume

\( \kappa_0(J(u) - c) \). So, \( J(u) - c = c \), where \( c \) is a constant. \( J(u) = c + e \).

\[
0 < \int_{\Sigma} e^2 \text{vol}_{\varphi_0^*(g)} = c \int_{\Sigma} e \text{vol}_{\varphi_0^*(g)} + \int_{\Sigma} e^2 \text{vol}_{\varphi_0^*(g)} \\
= \int_{\Sigma} e(c + e) \text{vol}_{\varphi_0^*(g)} \\
= \int_{\Sigma} eJ(u) \text{vol}_{\varphi_0^*(g)} \\
= \int_{\Sigma} uJ(e) \text{vol}_{\varphi_0^*(g)} = 0,
\]

which is a contradiction. Therefore, \( \omega_0(e) = e \notin \text{Im}(T_0) \). Thus, 0 is a \( \omega_0 \)-simple eigenvalue of \( T_0 \).

Then, by Lemma 4.2.1 there exists \( \omega_0 \)-simple eigenvalues \( \tilde{\lambda}_t \) of \( \tilde{J}_t = \frac{\partial F}{\partial h}(t, 0) \), where \( \tilde{\lambda}_0 = 0 \) and \( \tilde{\lambda}_t \) is the only one eigenvalue of \( \tilde{J}_1 \) near to 0.

(2) To prove the existence of a bifurcation we are going to apply again the Crandall-Rabinowitz Theorem (see Theorem 4.2.2) in the spaces \( X = B_0, Y = C^{j-2,\alpha}_0(\Sigma), Z = E^\perp \cap B_0 \) and to the operator \( \bar{F} : I \times U \to C^{j-2,\alpha}_0(\Sigma) \). It remains to prove that

\[
\frac{\partial^2 F}{\partial t \partial h}(0, 0)(e) \notin \text{Im} \left( \frac{\partial F}{\partial h}(0, 0) \right).
\]

Let \( \tilde{e}_t \) be the unitary eigenvector of \( T_t \) associated to eigenvalue \( \tilde{\lambda}_t \). So,

\[
\tilde{J}_t(\tilde{e}_t) = \tilde{\lambda}_t \tilde{e}_t. \tag{4.3.15}
\]

Deriving 4.3.15 with respect to \( t \) at \( t = 0 \), we have

\[
\left. \frac{d}{dt} \right|_{t=0} \tilde{J}_t(\tilde{e}_t) = \frac{\partial^2 F}{\partial t \partial h}(0, 0)(\tilde{e}_0) + \tilde{J}_0(\tilde{e}_0') \\
= \tilde{\lambda}_0 \tilde{e}_0 + \tilde{\lambda}_0 \tilde{e}_0'
\]

where \( \tilde{e}_0 = e \) and \( \tilde{\lambda}_0 = 0 \). Thus,

\[
\frac{\partial^2 F}{\partial t \partial h}(0, 0)(e) = \tilde{\lambda}_0 e - \tilde{J}_0(e_0').
\]

If we assume that \( \tilde{\lambda}_0' \neq 0, \tilde{\lambda}_0 e \neq 0 \) and \( \tilde{\lambda}_0' e \notin \text{Im}(\tilde{J}_0) \). So,

\[
\frac{\partial^2 F}{\partial t \partial h}(0, 0)(e) \notin \text{Im} \left( \frac{\partial F}{\partial h}(0, 0) \right).
\]

Then, by Theorem 4.2.2 there exist a neighborhood \( \tilde{U} \) of \( (0, 0) \) in \( \mathbb{R} \times B_0 \), an interval \( \tilde{I} = (-a, a) \subset \mathbb{R} \) and continuous functions \( \eta : \tilde{I} \to E^\perp \cap B_0, \tau : \tilde{I} \to \mathbb{R} \), such that, \( \eta(0) = 0, \tau(0) = 0 \) and

\[
\tilde{F}^{-1}(0) \cap \tilde{U} = \{(\tau(s), se + s\eta(s)) : s \in \tilde{I}\} \cup \{(t, 0) : (t, 0) \in \tilde{U}\}.
\]
Therefore, we have the existence of unique free boundary CMC immersions with volume $\hat{V}(s) = V(\phi_{\tau(s)})$, which are of the form

$$\psi_s = \exp_{\varphi_0}(f(t),s + \eta(t))(\tilde{n}_{\varphi_0}).$$

(3) As we had seen before, the variational vector field of $\varphi_t$ in $\varphi_0$ is $v_0 \tilde{n}_{\varphi_0}$, $v_0 = \frac{\partial \varphi_0}{\partial t}|_{t=0}$. Now let us see how is the variational vector field of $\psi_s$ in $\varphi_0$.

Differentiating (4.3.11) with respect to $t$ and calculating in $r \in \mathbb{R}$ we have:

$$dP(f(t,h))(\frac{\partial f(t,h)}{\partial t})(r) = \left( \frac{dV(\phi(t))}{dt} \right)(r),$$

On the other hand

$$dP(f(t,h))(\frac{\partial f(t,h)}{\partial t})(r) = \left( \int_{\Sigma} \frac{\partial f(t,h)}{\partial t}(r) \xi_t \operatorname{vol}_{f(t,h)}(g) \right)\left(\frac{\partial f(t,h)}{\partial t}(r) - \left( \int_{\Sigma} \frac{\partial f(t,h)}{\partial t}(r) \operatorname{vol}_{f(t,h)}(g) \right)w_0 \right).$$

That is, for all $r \in \mathbb{R}$,

$$\int_{\Sigma} \frac{\partial f(t,h)}{\partial t}(r) \xi_t \operatorname{vol}_{f(t,h)}(g) = \frac{dV(\phi(t))}{dt}(r),$$

and

$$\left(\frac{\partial f(t,h)}{\partial t}(r) - \left( \int_{\Sigma} \frac{\partial f(t,h)}{\partial t}(r) \operatorname{vol}_{f(t,h)}(g) \right)w_0 \right) = \frac{d\gamma(t)}{dt}(r).$$

In the particular case $(t,h) = (0,0)$, as $\gamma'(0) = 0$ (see 4.3.6), then

$$\int_{\Sigma} \frac{\partial f(0,0)}{\partial t}(r) \operatorname{vol}_{f(0,0)}(g) = dV(0)(\xi_0)(r),$$

and

$$\left(\frac{\partial f(0,0)}{\partial t}(r) - dV(0)(\xi_0)(r)w_0 \right) = 0.$$

But, as $w_0 = \frac{1}{dV(0)(\xi_0)} \xi_0$ (see 4.3.1), then

$$\left(\frac{\partial f(0,0)}{\partial t}(r) \right) = r \xi_0$$

So, using ??, we have

$$\frac{d}{ds}\bigg|_{s=0} f(\tau(s),s + \eta(s)) = \frac{d}{ds}\bigg|_{s=0} \left( Q(\phi_{\tau(s)},\gamma(\tau(s)) + se + \eta(s)) \right) = \frac{\partial Q}{\partial \tau}(V_0,0)\tau'(0) + \frac{\partial Q}{\partial h}(V_0,0)(\gamma'(0)\tau'(0) + e) = \frac{\partial f(0,0)}{\partial \tau}(\tau'(0)) + \frac{\partial f(0,0)}{\partial h}(\tau'(0))(e) = \tau'(0)\xi_0 + e$$

So, the variational field of $\psi_s$ in $\varphi_0$ is $(\tau'(0)\xi_0 + e)\tilde{n}_{\varphi_0}$. Since $J(\xi_0) = H_0^\prime \neq 0$ and $J(e) = 0$, \ldots
then $\xi_0$ and $\varepsilon$ are linearly independent. Thus, the families \{\varphi_t : t \in I\} and \{\psi_s : s \in \hat{I}\} are different except for $\varphi_0 = \psi_0$. 

\qed
Chapter 5

Study of Stability in the Bifurcation Branches

Given the results on the existence of bifurcation in the Theorems 4.2.3 and 4.3.2 in the previous chapter, we are now going to study the stability of the immersions in these branches of bifurcation. One in terms of the sign of derivative of the mean curvature function (or volume function, according to the case) of the hypersurfaces in the bifurcation branch, in addition to the sign of the derivative of the function first eigenvalue of $\tilde{J}_t$; and other in terms of the sign of first and second derivative of the mean curvature function (or volume function, according to the case) of the hypersurfaces in the bifurcation branch. Again, in this study we based on Koiso-Palmer-Piccione [14] and Crandall-Rabinowitz [9] and [10].

5.1 A first result in stability

Recall from Remark 3.3.1, let $\lambda_1 < \lambda_2 \leq \ldots, \lambda_n \to \infty$ ($n \to \infty$) the eigenvalues of $J$. When Theorem 3.2.2, of existence and uniqueness of deformation, is applied in particular to the case $\lambda_1 = 0$ we have that $E_{\lambda_1} = E = \text{Ker}(J)$. In this case the multiplicity of $\lambda_1$ is 1 and of functional analysis we know that the $\lambda_1$-eigenfunctions do not change of sign. In particular, we have that $\int_{\Sigma} e\ vol_{\varphi_0(g)} \neq 0$, for $e \in E$. Therefore, the bifurcation for a family $\{\varphi_t\}$ only occurs in the case where $\lambda_k = 0$, for some $k \geq 2$.

Now, remember that in Theorem 4.3.2 we had defined the operator $\tilde{J}_0 : B_0 \to C^{1,-2,\alpha}_\ast(\Sigma)$ ($\tilde{J}_0 = \tilde{J}$ for simplicity) as

$$J(u) = \frac{\partial F}{\partial h}(0,0)(u) = J(u) - \frac{1}{|\Sigma|} \int_{\Sigma} J(u) \ vol_{\varphi_0(g)}.$$  

We can see that $\tilde{J}$ is the representation operator of the restriction of $A''(0)$ to $B_0$. As $u \mapsto \frac{1}{|\Sigma|} \int_{\Sigma} J(u) \ vol_{\varphi_0(g)}$ is an operator of rank 1, then $\tilde{J}$ is a symmetric compact finite-rank perturbation of $J$. Thus, $\tilde{J}$ admits a sequence of eigenvalues, $\tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \tilde{\lambda}_3 \ldots, \tilde{\lambda}_n \to \infty$ ($n \to \infty$); where each eigenspace $E_{\tilde{\lambda}_i}$ is finite-dimensional, and there is a sequence of eigenfunctions, $\{\tilde{e}_i : \tilde{J}(\tilde{e}_i) = \tilde{\lambda}_i \tilde{e}_i, i = 1, 2, \ldots\}$, that form an orthonormal basis for $B_0$. In that way, a CMC immersion is stable if and only if $A''(0)$ is positive semi-definite on $B_0$ (see [6]).

Thus, the following Lemma provides us a characterization of the stability of a CMC immersion from the first eigenvalue of $\tilde{J}$.
Lemma 5.1.1. \( \varphi_0 \) is stable if and only if \( \tilde{\lambda}_1 \geq 0 \)

Proof. \[
A''(0) \big|_{B_0} (\tilde{e}_1) = \int_{\Sigma} \tilde{J}(\tilde{e}_1) \tilde{e}_1 \ vol_{\varphi_0^*(g)} = \int_{\Sigma} \tilde{\lambda}_1 \tilde{e}_1^2 \ vol_{\varphi_0^*(g)}.
\]
Therefore, \( \varphi_0 \) is stable if only if \( \tilde{\lambda}_1 \geq 0 \). \( \square \)

An immediate consequence of the second criterion of stability, Corollary 3.3.5, is the following proposition, a first result in the stability of \( \varphi_0 \) when \( \lambda_2 = 0 \) holds (equivalently, \( \tilde{\lambda}_1 = 0 \) holds)

and under the conditions of Theorem 4.3.2 we obtain in the following proposition

**Proposition 5.1.2.** Under the conditions of Theorem 4.3.2 and assuming that:

(i) \( \frac{dH_0}{dt} \big|_{t=0} = H_0' \neq 0 \).

(ii) \( E = \{be : b \in \mathbb{R}\} \), for some \( e \in C^j_0(\Sigma_0) \), \( e \neq 0 \).

(iii) \( \lambda_2 = 0 \).

Then,

(1) If \( H_0' V_0' \leq 0 \), then \( \varphi \) is stable and \( \tilde{\lambda}_1 = 0 \).

(2) If \( H_0' V_0' > 0 \), then \( \varphi \) is unstable and \( \tilde{\lambda}_2 \leq 0 \)

\( \square \)

Proof. From (1) and (2) we have that there exist \( j \geq 2 \) and \( k \geq 1 \) such that \( \lambda_j = \tilde{\lambda}_k = 0 \). Taking into account the Remark 3.3.1 and the variational characterization of the eigenvalues of Jacobi operator \( J \), presented in (3.3.1) and (3.3.2), we can write

\[
\lambda_k = \max_{\text{Dim}(V) = k} \left[ \min_{u \in V \setminus \{0\}} \int_{\Sigma} \frac{\tilde{J}(u) u \ vol_{\varphi_0^*(g)}}{\int_{\Sigma} |u|^2 \ vol_{\varphi_0^*(g)}} \right],
\]

and

\[
\tilde{\lambda}_k = \max_{\text{Dim}(W) = k} \left[ \min_{u \in W \setminus \{0\}} \int_{\Sigma} \frac{\tilde{J}(u) u \ vol_{\varphi_0^*(g)}}{\int_{\Sigma} |u|^2 \ vol_{\varphi_0^*(g)}} \right].
\]

Now, as

\[
\int_{\Sigma} \tilde{J}(u) u \ vol_{\varphi_0^*(g)} = \int_{\Sigma} \left[ J(u) - \frac{1}{|\Sigma|} \int_{\Sigma} J(u) \ vol_{\varphi_0^*(g)} \right] u \ vol_{\varphi_0^*(g)}
\]

\[
= \int_{\Sigma} J(u) u \ vol_{\varphi_0^*(g)} - c \int_{\Sigma} u \ vol_{\varphi_0^*(g)}
\]

\[
= \int_{\Sigma} J(u) u \ vol_{\varphi_0^*(g)}
\]

Therefore, \( \tilde{\lambda}_k \leq \lambda_k \). \( \square \)
5.2 Preliminaries for stability on Bifurcation

For the study of stability at the branches of bifurcation of free boundary CMC hypersurfaces, it is necessary to enunciate some theorems and lemmas that are based on the Crandall-Rabinowitz theory.

Corollary 5.2.1. (\cite{11}, Corollary 1.13) Crandall-Rabinowitz) Let $X$ and $Y$ be Banach Spaces. $B(X,Y)$ denote the set of bounded linear maps of $X$ into $Y$. Let $K \in B(X,Y)$ and 0 a $K$-simple eigenvalue of $F_x(t_0,0)$ (F as in the Theorem 4.2.2). $\text{Ker}(F_x(t_0,0)) = \text{Span}\{x_0\}$. Then, there exist open intervals $\tilde{L}, \tilde{I} \in \mathbb{R}$ and continuously differential functions $\lambda : \tilde{L} \to \mathbb{R}$, $\mu : \tilde{I} \to \mathbb{R}$, $u : \tilde{L} \to X$, $w : \tilde{I} \to X$, such that

(i) $F_x(t,0)u(t) = \lambda(t)K(u(t))$, for $t \in \tilde{L}$

(ii) $F_x(\tau(s),x(s))w(s) = \mu(s)K(w(s))$, for $s \in \tilde{I}$. Where $x(s) = sx_0 + s\zeta(s)$, $\zeta$ and $\tau$ as in Theorem 4.2.2.

Moreover

$$\lambda(t_0) = \mu(0) = 0, \quad u(t_0) = x_0 = w(0),$$

and

$$u(t) - x_0 \in Z, \quad w(s) - x_0 \in Z.$$

The main result relating $\tau$, $\mu$ and $\lambda$ is

Theorem 5.2.2. (\cite{11}, Theorem 1.16) Crandall-Rabinowitz) We assume the same hypotheses and the notation of corollary 5.2.1. Then, $\lambda'(t_0) \neq 0$ and near to $s = 0$ the functions $\mu(s)$ and $-s\tau'(s)\lambda(s)$ have the same zeroes, and, whenever $\mu(s) \neq 0$, the same sign. More precisely,

$$\lim_{s \to 0} \frac{-s\tau'(s)\lambda(t_0)}{\mu(s)} = 1.$$

Moreover, there is a constant $c$ such that

$$||x'(s) - w(s)|| \leq c\text{Min}\{|s\tau'(s)|, |\mu(s)|\}$$

near $s = 0$

The following lemmas provide us with the existence of eigenvalues for the Jacobi operator associated with the hypersurfaces that belong to the bifurcation branches.

Lemma 5.2.3. assuming the hypotheses of the Theorem 4.2.3. Let $H^e_0 \neq 0$, $E = \text{Span}\{e\}$, $e \in C^1_0(\Sigma)$, $e \neq 0$. Then, there exist an interval $\tilde{I} \subset \tilde{I}$, $0 \in \tilde{I}$ and continuously differentiable functions $\mu : \tilde{I} \to \mathbb{R}$ and $w : \tilde{I} \to C^1_0(\Sigma)$, whit $\mu_0 = 0$ and $w_s \neq 0$ for all $s \in \tilde{I}$, such that:

(1) $J_{\psi_s}(w_s) = \mu_sw_s$, for all $s \in \tilde{I}$. ($J_{\psi_s}$ is the Jacobi operator of $\psi_s$, $\psi_s$ the free boundary CMC immersion provided by Theorem 4.2.3.)

(2) $\lim_{s \to 0} \frac{-s\tau'(s)\lambda(t_0)}{\mu_s} = 1$. ($t(s)$ and $\lambda(s)$ as in the Theorem 4.2.3.$)
Thus, we have the part (1).

We assume the hypotheses of Theorem 4.3.2. Let \( H'_s \neq 0 \), \( E = \text{Span}(e) \), \( e \in C^1_0(\Sigma) \), \( e \neq 0 \), \( J_{\psi_s} \), the Jacobi operator of \( \psi_s \) the free boundary CMC immersion provided by Theorem 4.3.2. Then, there exist an interval \( \bar{I} \subset I \), \( 0 \in \bar{I} \) and continuously differentiable functions \( \mu : \bar{I} \to \mathbb{R} \), \( w : \bar{I} \to X \) such that

\[
J_{\psi_s}(w_s) = \mu_s w_s, \quad \text{for all } s \in \bar{I}.
\]

Thus, we have the part (1).

Now, \( \tilde{H}'_s = H'_s t'(s) \). So \( t'(s) = \frac{H'_s}{\tilde{H}'_s} \). Thus, parts (2) and (3) follow from Theorem 5.2.2 (Crandall-Rabinowitz).

**Lemma 5.2.4.** We assume the hypotheses of Theorem 4.3.2. Let \( H'_0 \neq 0 \), \( E = \text{Span}(e) \), \( e \in C^1_0(\Sigma) \), \( e \neq 0 \), \( J_{\psi_s} \), the Jacobi operator of \( \psi_s \) the free boundary CMC immersion provided by Theorem 4.3.2. Then, there exist an interval \( \bar{I} \subset I \), \( 0 \in \bar{I} \) and continuously differentiable functions \( \mu : \bar{I} \to \mathbb{R} \), \( w : \bar{I} \to C^1_0(\Sigma) \), with \( \mu_0 = 0 \) and \( w_s \neq 0 \) for all \( s \in \bar{I} \), such that:

1. \( J_{\psi_s}(w_s) = \mu_s w_s + c_s \), for all \( s \in \bar{I} \), \( c_s \) a constant that depends on \( s \).
2. \( \lim_{s \to 0} \frac{-s \tau'(s) \lambda'(0)}{\mu_s} = 1 \). (\( \tau(s) \) and \( \lambda(s) \) as in the Theorem 4.3.2.)
3. \( ||x'(s) - w_s|| \leq c \text{Min} \{|s \tau'(s)|, |\mu_s|\}, \) for all \( s \in \bar{I} \). Where \( x(s) = s e + s q(s) \) and \( c \) is a positive constant. (\( q(s) \) as in the Theorem 4.3.2.) We can observe, from the previous limit, that \( \mu_s \)
Now, and 0 is a following. In part (1) of Theorem 5.3.1 we will prove this. Recall that parts (2) and (3) follow from Theorem 5.2.2. Thus, we have the part (1).

\[ s \tau'(s) \dot{\lambda}(0) = \frac{-s \dot{\psi}^{\prime}}{\tau(s)} \lambda(0) \]

have the same zeroes and, where \( \mu_s \neq 0 \), the same sign. (\( \dot{\psi} \) is the volume of \( \psi_s \).)

**Proof.** In the proof of Theorem 4.3.2 we observed that for the spaces \( X = B_0, Y = C^{j-2,\alpha}_x(\Sigma) \) and \( Z = E^\perp \cap B_0 \), together with the inclusion map \( \iota_0 : B_0 \hookrightarrow Y \) and the operator \( \hat{F} : I \times U \rightarrow Y \), 0 \( \in I \subset \mathbb{R} \), the conditions of Theorem 4.2.2 (Crandall-Rabinowitz) are satisfied, where \( U \subset B_0 \), \( \hat{F}(t, h) = \kappa_0(F(t, h)) = F(t, h) - \frac{1}{|\Sigma|} \int_{\Sigma} F(t, h) \text{vol}_{\varphi_0^*(g)} \) and \( F(t, h) = H_{f(t, h)} - H_{\psi_t} \). We have that

\[ \frac{\partial}{\partial h} \hat{F}(t, 0) = \kappa_0 \left( \frac{\partial}{\partial h} F(t, 0) \right). \]

So,

\[ \frac{\partial}{\partial h} \hat{F}(0, 0) u = J(u) - \frac{1}{|\Sigma|} \int_{\Sigma} J(u) \text{vol}_{\varphi_0^*(g)}, \]

and 0 is a \( \iota_0 \)-simplies eigenvalue of \( \frac{\partial}{\partial h} \hat{F}(0, 0) \). Then, again by corollary 5.2.1 (ii), whit \( K = \iota_0 \), there exist an interval \( \tilde{I} \subset I \), 0 \( \in \tilde{I} \) and functions \( \mu : \tilde{I} \rightarrow \mathbb{R} \), \( w : \tilde{I} \rightarrow X \) such that

\[ \frac{\partial}{\partial h} \hat{F}(\tau(s), x(s))(w_s) = \mu_s w_s, \quad \text{for all } s \in \tilde{I}. \]

Now,

\[ \frac{\partial}{\partial h} \hat{F}(\tau(s), x(s))(u) = J_{\psi_s}(u) - \frac{1}{|\Sigma|} \int_{\Sigma} J_{\psi_s}(u) \text{vol}_{\varphi_0^*(g)}. \]

So,

\[ J_{\psi_s}(w_s) = \mu_s w_s + \frac{1}{|\Sigma|} \int_{\Sigma} J_{\psi_s}(w(s)) \text{vol}_{\varphi_0^*(g)}. \]

Thus, we have the part (1).

We have that \( \hat{V}(s) = V(\tau(s)) \). Then \( \hat{V}'(s) = V'(\tau(s)) \tau'(s) \). So, \( \tau'(s) = \frac{\hat{V}'(s)}{V'(\tau(s))} \). Thus, the parts (2) and (3) follow from Theorem 5.2.2.

**5.3 Stability in the Bifurcation Branch**

Now we are going to present the stability criteria in the bifurcation branches. If a free boundary immersion \( \varphi_0 \) is stable, then only three types of bifurcation can occur, which we will enunciate following. In part (1) of Theorem 5.3.1 we will prove this. Recall that \( \varphi_0 \) is stable if \( H_0' V(0)' < 0 \). The same three types of bifurcation occur when \( \hat{H}'(s) \neq 0 \), as we can see in part 2 of Theorem 5.3.3.

a) Supercritical pitchfork bifurcation; if \( \hat{V}'(0) = 0 \) and \( \hat{V}''(0) < 0 \) then \( \psi_s \) is stable for \( s \in [-s_0, 0] \) (Figure 5.1)

b) Subcritical pitchfork bifurcation; if \( \hat{V}'(0) = 0 \) and \( \hat{V}''(0) > 0 \) then \( \psi_s \) is unstable for \( s \in [-s_0, 0] \cup (0, s_0] \) (Figure 5.2)

c) Transcritical bifurcation: If \( \hat{V}'(0) > 0 \) then \( \psi_s \) is stable for \( s \in [-s_0, 0] \) and unstable for \( (0, s_0] \) (Figure 5.3)
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Figure 5.1: Supercritical pitchfork bifurcation.

Figure 5.2: Subcritical pitchfork bifurcation.

Similar conclusions hold in the case when \( H'_0 < 0 \) and \( V(0) < 0 \), by reversing the parameterization of \( \varphi_t, t \mapsto -t \).

We start with the case when the bifurcation parameter is the volume. That is, the bifurcation branch obtained in Theorem 4.3.2.

**Theorem 5.3.1.** Let \( \{\psi_s\}_{s \in I} \) be the bifurcation branch formed by free boundary CMC hypersurfaces obtained in Theorem 4.3.2. \( H'_0 \neq 0 \). \( \ker(J_{\psi_0}(\Sigma)) = \text{Span}\{e \neq 0\} \). \( \lambda_2 = 0 \) (Equivalently \( \tilde{\lambda}_1 = 0 \)). \( \tilde{\lambda}_1'(0) \neq 0 \) (\( \tilde{\lambda}_1(t) \) the first eigenvalue of \( \tilde{J}_t = J_t - \frac{1}{|\Sigma|} \int_{\Sigma} J_t \text{vol}_{\varphi_0(g)} \)). If moreover \( V'(0) > 0 \) (if necessary, change the parameter \( t \) to \( -t \)). Then,

(1) In the case \( H'_0 < 0 \) (in this case \( \varphi_0 \) is stable by proposition 5.1.2 and \( \tilde{\lambda}_1 = 0 \))

(1-a) If \( \tilde{V}'(s) = 0 \) for \( s \) near 0 (that is, if \( \tilde{V} \) is locally constant), then, \( \psi_s \) is stable for \( s \) near 0.

(1-b) If \( \tilde{V}'(s) \neq 0, s \neq 0, s \) near 0, then, for a sufficiently small \( s_0 > 0 \), in each interval \([−s_0, 0)\) and \((0, s_0]\) we have:

(1-b-i) If \( \tilde{\lambda}_1'(0)s\tilde{V}'(s) < 0 \), then, \( \psi_s \) is stable.

(1-b-ii) If \( \tilde{\lambda}_1'(0)s\tilde{V}'(s) > 0 \), then, \( \psi_s \) is unstable.

In particular, supercritical and subcritical pitchfork bifurcations are presented where \( s\tilde{V}'(s) \) does not change sign at \( s = 0 \), and transcritical bifurcation occurs when \( s\tilde{V}'(s) \) changes sign at \( s = 0 \).

(2) In the case \( H'_0 > 0 \) (in this case \( \varphi_0 \) is unstable by proposition 5.1.2 and \( \tilde{\lambda}_2 = 0 \)), we have that \( \psi_s \) is unstable for small \( |s| \).

**Proof.** (1) If \( H'_0V'(0) < 0 \), then \( \tilde{\lambda}_1 = 0 \) by proposition 5.1.2 (\( \tilde{\lambda}_1 \) eigenvalue of operator \( J - \frac{1}{|\Sigma|} \int_{\Sigma} J \text{vol}_{\varphi_0(g)} \)). Let \( \mu_s \) be the eigenvalue of \( \tilde{J}_{\psi_s} = J_{\psi_s} - \frac{1}{|\Sigma|} \int_{\Sigma} J_{\psi_s} \text{vol}_{\varphi_0(g)} \) obtained in Lemma 5.2.4. So, for \( s \) sufficiently small, \( \mu_s \) is the eigenvalue closest to 0 of \( \tilde{J}_{\psi_s} \). This is because the condition \( \tilde{\lambda}_2 > 0 \) is open in the space of Jacobi operators \( (J_{\varphi}(f) = \Delta_{\Sigma}f - \text{vol}_{\varphi_0(g)}) \).
we put stronger conditions for derivative of Corollary 5.3.2.

H obtained in Theorem 4.3.2. Under the assumptions of the theorem 5.3.1, we also assume that $H$ obtained.

(2) If $\hat{\phi}_{(1)}(2)$

As a consequence of the previous theorem we obtain the following stability criterion, where

If $\hat{\phi}_{(1)}(2)$

Let

$\hat{\phi}_{(1)}(2)$

have the same zeros and, where $\mu_s \neq 0$, have the same sign. As we also assume that $\hat{\lambda}(0) \neq 0$, so

(1-a) If $\hat{V}'(s) = 0$, then $\mu_s = 0$ and $\psi_s$ is stable.

(1-b) If $\hat{V}'(s) \neq 0$, let $s_0 > 0$ sufficiently small, then, in each interval $[-s_0, 0)$ and $(0, s_0]$, we have the following options:

1-b-i If $s\hat{V}'(s)\hat{\lambda}_1'(0) < 0$, then $\mu_s > 0$. Therefore, $\psi_s$ is stable.

1-b-ii If $s\hat{V}'(s)\hat{\lambda}_1'(0) > 0$, then $\mu_s < 0$. Therefore, $\psi_s$ is unstable.

(2) If $H'_0 V'(0) > 0$, then $\hat{\lambda}_2 = 0$ by proposition 5.1.2. So, $\hat{\lambda}_1(s) < 0$ for $s$ enough small (the condition $\hat{\lambda}_1 < 0$ is open in the set of Jacobi operators). Therefore, $\psi_s$ is unstable.

\[\square\]

As a consequence of the previous theorem we obtain the following stability criterion, where we put stronger conditions for $H'_0, V'_0$ and $\hat{\lambda}_1'(0)$ and the criterion is given in terms of the second derivative of $\hat{V}$. We obtain conditions for pitchfork or transcritical bifurcation.

Corollary 5.3.2. Let $\{\psi_s\}_{s \in \hat{I}}$ be the bifurcation branch formed by free boundary CMC hypersurfaces obtained in Theorem 4.3.2. Under the assumptions of the theorem 5.3.1, we also assume that $H'_0 < 0, V'_0 > 0$ and $\hat{\lambda}_1'(0) > 0$. Then, there exist positive constants $t_0 \in (0, \epsilon)$ and $s_0 \in \hat{I}$ such that:

(1) $\varphi_t$ is stable for all $t \in [0, t_0]$ and unstable for all $t \in [-t_0, 0)$.

(2) If $\hat{V}'(0) \neq 0$, then we have transcritical bifurcation for $\{\psi_s\}_{s \in [-s_0, s_0]}$ (see Figure 5.3). This is,

(2-a) If $\hat{V}'(0) > 0$, then $\psi_s$ is stable for $s \in [-s_0, 0]$ and unstable for $s \in (0, s_0]$.

(2-b) If $\hat{V}'(0) < 0$, then $\psi_s$ is stable for $s \in [0, s_0]$ and unstable for $s \in [-s_0, 0)$.

(3) If $\hat{V}'(0) = 0$ and there is the second derivative of $\hat{V}$ in $s = 0$, we have

(3-a) If $\hat{V}''(0) < 0$, then $\psi_s$ is stable for all $s \in [-s_0, s_0]$. Here is a supercritical pitchfork bifurcation (see Figure 5.1).

(3-b) If $\hat{V}''(0) > 0$, then $\psi_s$ is unstable for all $s \in [-s_0, 0) \cup (0, s_0]$. Here is a subcritical pitchfork bifurcation (see Figure 5.2).

If $H'_0 > 0$ and $V'_0 < 0$, inverting the parameterization of $\varphi_t$, $t \mapsto -t$, similar conclusions are obtained.
Proof. (1) Since \( \varphi_0 \) is stable if and only if \( \hat{\lambda}_1 \geq 0 \), this is by lemma 5.1.1, and additionally, by part (1) of proposition 5.1.2, \( \hat{\lambda}_1 = 0 \), and by the hypothesis that \( \hat{\lambda}'_1(0) > 0 \), we have the first statement.

(2) Part (2) follows immediately from I.2. of theorem 5.3.1.

(3) For part (3) we must see the sign of \( s \hat{V}'(s) \) and use again I.2. of Theorem 5.3.1.

(3-a) If \( \hat{V}''(0) < 0 \), then \( \hat{V}'(s) \) is decreasing in \([−s_0, s_0]\). So, for \( s ∈ [−s_0, 0) \), \( s \hat{V}'(0) < 0 \), thus \( ψ_s \) is stable. For \( s ∈ (0, s_0] \), \( s \hat{V}'(0) < 0 \), thus \( ψ_s \) is stable too. Therefore, \( ψ_s \) is stable for \( s ∈ [−s_0, s_0] \)

(3-b) If \( \hat{V}''(0) > 0 \), then \( \hat{V}'(s) \) is increasing in \([−s_0, s_0]\). So, for \( s ∈ [−s_0, 0) \), \( s \hat{V}'(0) > 0 \), thus \( ψ_s \) is unstable. For \( s ∈ (0, s_0] \), \( s \hat{V}'(0) > 0 \), thus \( ψ_s \) is unstable too.

Now, we going to study the case when then bifurcation parameter is the mean curvature. That is, stability in the bifurcation branch obtained in the theorem 4.2.3. Here also we obtain conditions on the eigenvalues of the Jacobi operator for pitchfork or transcritical bifurcation.

**Theorem 5.3.3.** Let \( \{ψ_s\}_{s ∈ I} \) be the bifurcation branch formed by free boundary CMC hypersurfaces obtained in Theorem 4.2.3. Under the same conditions and the same notation:

(i) \( H'_0 \neq 0 \).

(ii) \( E = \text{Span}\{e\} \), for some \( e ∈ \mathcal{C}^{j,α}_0(Σ_0) \), \( e ≠ 0 \).

(iii) \( λ_2 = 0 \).

(iv) \( λ_2'(0) ≠ 0 \). \( (λ_2(t) \) the second eigenvalue of \( J_t \).)

And additionally assuming that \( H'_0 < 0 \) holds, (if necessary, change the parameter \( t \) to \( −t \)). Let \( μ_s \) be the eigenvalue of \( J_{ψ_s} \) obtained in Lemma 5.2.3. Then,

(1) If \( \hat{H}'_s = 0 \) for \( s \) near to 0 (that is, if \( \hat{H} \) is locally constant, \( \hat{H}_s \) the constant mean curvature of \( ψ_s \), then \( μ_s = 0 \) for \( s \) near 0.

(2) If \( \hat{H}'_s ≠ 0 \) for \( s \) near to 0, \( s ≠ 0 \), then, for \( s_0 > 0 \), sufficiently small, in each interval \([−s_0, 0) \) and \((0, s_0] \)

(2-a) \( μ_s > 0 \) if \( s \hat{H}'_s λ_2'(0) > 0 \)

(2-b) \( μ_s < 0 \) if \( s \hat{H}'_s λ_2'(0) < 0 \)

In particular, supercritical and subcritical pitchfork bifurcations correspond to the cases where \( s \hat{H}'_s \) does not change sign at \( s = 0 \), and transcritical bifurcation occurs when \( s \hat{H}'_s \) changes sign at \( s = 0 \).

Proof. By Lemma 5.2.3, for \( s \) enough close to 0, \( μ_s \) and \( \frac{s \hat{H}'_s λ_2'(0)}{H'_{r(s)}} \) have the same zeros and, where \( μ_s ≠ 0 \), have the same sign. Therefore, having to \( H'_{r(s)} < 0 \)

(1) If \( \hat{H}'_s = 0 \), then \( μ_s = 0 \).
If \( \dot{H}'_s \neq 0 \), let \( s_0 > 0 \), sufficiently small. So, in each interval \([-s_0, 0)\) and \((0, s_0]\), we have the following options:

1. If \( s\dot{H}'_s \lambda'_2(0) > 0 \), then \( \mu_s > 0 \)
2. If \( s\dot{H}'_s \lambda'_2(0) < 0 \), then \( \mu_s < 0 \)

Again we have that \( \psi_s \) is stable if and only if \( \mu_s \geq 0 \). Therefore, we have the following corollary with results analogous to Corollary 5.3.2

**Corollary 5.3.4.** Let \( \{\psi_s\}_{s \in \hat{I}} \) be the bifurcation branch formed by free boundary CMC hypersurfaces obtained in Theorem 4.2.3. Under the conditions of the theorem 5.3.3, we assuming that \( \lambda'_2(0) > 0 \). Then, there exist positives constants \( t_0 \in (0, \epsilon) \) and \( s_0 \in \hat{I} \) such that:

1. \( \varphi_t \) is stable for all \( t \in [0, t_0] \) and unstable for all \( t \in [-t_0, 0) \).
2. If \( \dot{H}'_s(0) \neq 0 \), then we have transcritical bifurcation for \( \{\psi_s\}_{s \in [-s_0, s_0]} \) (see Figure 5.3). This is,
   - If \( \dot{H}'_s(0) < 0 \), then \( \psi_s \) is stable for \( s \in [-s_0, 0] \) and unstable for \( s \in (0, s_0] \).
   - If \( \dot{H}'_s(0) > 0 \), then \( \psi_s \) is stable for \( s \in [0, s_0] \) and unstable for \( s \in [-s_0, 0) \).
3. If \( \dot{H}'_s(0) = 0 \) and there is the second derivative of \( \dot{H} \) in \( s = 0 \), we have
   - If \( \dot{H}''_s(0) > 0 \), then \( \psi_s \) is stable for all \( s \in [-s_0, s_0] \). Here is a supercritical pitchfork bifurcation (see Figure 5.1).
   - If \( \dot{H}''_s(0) < 0 \), then \( \psi_s \) is unstable for all \( s \in [-s_0, 0) \cup (0, s_0] \). Here is a subcritical pitchfork bifurcation (see Figure 5.2).

If \( H'_0 > 0 \), inverting the parameterization of \( \varphi_t, t \mapsto -t \), similar conclusions are obtained.

**Proof.** The proof is completely analogous to of Corollary 5.3.2. \qed
Chapter 6

Conclusions

6.1 Obtained results and impacts

Variational problems are commonly known in areas such as Physics, Physics-Chemistry, Engineering, Economics, Biology, Medicine, etc. Some very specific examples in these areas range from Determining the advance of invasive biological species; The healing of a tissue of the corneal epithelium after the occurrence of a wound; The front of propagation of natural phenomena such as tsunamis, forest fires and avalanches; Speculation on the rise and fall of prices in financial events; Even theoretical-physical phenomena such as the boundary that separates the plasma and vacuum zone inside certain types of experimental reactors of nuclear fusion by magnetic confinement (see [22]).

In this type of problems, involved solutions must satisfy certain conditions in the limits of a known domain. They are called Boundary Problems. In many cases, the domain boundary in which is defined the solution functions is unknown. Therefore, these solutions must also satisfy additional conditions at the boundary. Two types of solutions can be imposed in this case, solutions with fixed boundary and solutions with free boundary. The latter are precisely those concerning this dissertation.

Genericity, existence of perturbation, stability and bifurcation are important characteristics to study when natural phenomena of this nature occurs. Our study in variations of free boundary CMC hypersurfaces generates tools for the study of this type of variational problems. We mainly use the ideas from White (see [28]), Biliotti-Javaloyes-Piccione (see [9]), Bettiol-Piccione-Santoro (see [8]), Koiso (see [16]), Koiso-Palmer-Piccione (see [18]), Crandall-Rabinowitz (see [10] and [11]), to obtain generalizations and results in the aspects mentioned above in families of free boundary CMC hypersurfaces.

We establish results in genericity of Bumpy metrics (Chapter 2). In fact, we prove that the set of Riemannian metrics, which makes all orthogonal embedding with CMC non-degenerate, is generic; in other words, almost all Riemannian metrics meet this condition.

We establish criteria for the existence of perturbation or deformation with free boundary hypersurfaces with CMC from a given free boundary hypersurface $\varphi_0 : \Sigma \rightarrow M$, by setting conditions at the Jacobi Operator’s kernel in $\varphi_0$ and at the derivation of the volume function in $\varphi_0$ (Chapter 3). Given a deformation of this type, we establish criteria to study the stability of $\varphi_0$, that is, by testing if $\varphi_0$ corresponds to a critical point of the area functional where there is a minimum. These criteria are established from the sign of the eigenvalues of the Jacobi Operator in $\varphi_0$ and of the
derivatives of the volume and mean curvature functions in $\varphi_0$.

We also establish criteria for the existence of bifurcation points in a free boundary CMC hypersurfaces family, for the following cases: firstly, the elements of the branch of bifurcation have the same values in the constant mean curvatures as the elements of the family original; secondly, the volumes of elements of the branch of bifurcations coincide with those of the original branch (Chapter 4).

Finally, given the existence of bifurcation branches of free boundary CMC hypersurfaces, we study the stability of the elements of these branches (Chapter 5), by using the results of Chapter 3.

6.2 Open problems

• In the theory developed by Koiso-Palmer-Piccione there are specific examples of bifurcation and stability in families of fixed boundary CMC surfaces in $\mathbb{R}^3$ (see [18] and [17]).

For our case (free boundary), examples are difficult to construct. It requires the solution of systems of differential equations that involve variable coefficients, such as the norm of the second fundamental form and and the Ricci curvature of the ambient manifold,

$$
\Delta_{\Sigma_0} f - \left( ||\Pi_{\Sigma_0}||_{HS}^2 + \text{Ric}_g(\eta_{\Sigma_0}, \eta_{\Sigma_0}) \right) f = 0,
$$

and that also meet with the linearized free boundary condition

$$
g(\nabla f, \eta_{\partial M}) + \Pi_{\partial M}(\vec{n}_{\Sigma_0}, \vec{n}_{\Sigma_0}) f = 0.
$$

Therefore, it remains finding a specific example of a free boundary CMC hypersurfaces family having a degenerate point where there is a bifurcation and we can study the stability of the members of this family.

• Additionally, we still need to study in this theory if the same results are obtained in genericity, bifurcation and stability in the case of Semi-Riemannian metrics.
Appendix A

In this appendix we will give some definitions and important results of Functional Analysis, which were necessary during the development of the whole thesis.

Definition A.0.1. If \( f : N \to M \) is a smooth map and \( S \subset M \) is an embedded sub-manifold, we say that \( f \) is transverse to \( S \) if, for every \( p \in f^{-1}(S) \), \( T_{f(p)}M = T_{f(p)}S + df_p(T_pN) \).

Definition A.0.2. The definition of transversality between a map \( F : X \to Y \) and \( Z \subset Y \) a smooth submanifold, where \( X \) and \( Y \) are Banach manifolds, is that presented in the Definition A.0.1 but with the additional assumption that \( dF^{-1}(T_{f(x)}Z) \) is a complemented subspace of \( T_{f(x)}X \), i.e., there is a subspace \( V \subset T_{f(x)}X \) such that \( T_{f(x)}X = dF^{-1}(T_{f(x)}Z) \oplus V \).

Definition A.0.3. If \( A \) is a bounded operator in a Hilbert space \( H \) and \( \{ e_i : i \in I \} \) is an orthonormal bases for \( H \), is defined the Hilbert-Schmidt norm as \( ||A||_{HS}^2 = Tr(A^*A) = \sum_{i \in I} ||Ae_i||_H^2 \), where \( || \cdot ||_H \) is the norm of \( H \).

Definition A.0.4. The Hölder space \( C^{j,\alpha}(\Omega) \), where \( \Omega \) is an open subset of some Euclidean space and \( j \geq 0 \) an integer, consists of those functions on \( \Omega \) having continuous derivatives up to order \( j \) and such that the \( j \)-th partial derivatives are Hölder continuous with exponent \( \alpha \), where \( 0 < \alpha \leq 1 \). A real valued function \( f \) on \( n \)-dimensional Euclidean space is Hölder continuous, when there are nonnegative real constants \( c \), such that

\[
|f(x) - f(y)| \leq c||x - y||^\alpha
\]

Definition A.0.5. A linear continuous operator \( T : E \to F \) between normed spaces is Fredholm if \( \text{Ker} \, T \) is finite dimensional and \( \text{Im} \, T \) is close and finite codimensional, the index of \( T \) is \( \text{ind} \, T = \dim \text{Ker} \, T - \dim \text{coker} \, T \). A Fredholm map is a \( C^1 \) map \( f : M \to N \), \( M \) and \( N \) being differentiable Banach manifolds, such that for each \( x \in M \), the derivative \( df_x : T_x(M) \to T_{f(x)}(N) \) is a Fredholm operator. The index of \( f \) is defined to be the index of \( df_x \) for some \( x \). The definition doesn’t depend on \( x \), see [14].

Theorem A.0.1. [20, The Inverse Mapping Theorem, 5.2] Let \( E, F \) Banach spaces, \( U \) an open sub-set of \( E \), and Let \( f : U \to F \) a \( C^p \)-morphism with \( p \geq 1 \). Assume that for some point \( x_0 \in U \) The derivative \( f'(x_0) : E \to F \) is a toplinear isomorphism. Them \( f \) is a local \( C^p \)-isomorphism at \( x_0 \).
Theorem A.0.2. [20, The Implicit Mapping Theorem, 5.9] Let $U, V$ be open sets in Banach Spaces $E, F$ respectively, and set $f : U \times V \to G$ be a $C^p$ mapping. Let $(a, b) \in U \times V$, and assume that $D_2 f(a, b) : F \to G$ is an isomorphism. Let $f(a, b) = 0$. Then there exist a continuous map $g : U_0 \to V$ defined on an open neighborhood $U_0$ of $a$ such that $g(a) = 0$ and such that $f(x, g(x)) = 0$ for all $x \in U_0$. If $U_0$ is taken to be a sufficiently small ball, then $g$ is uniquely determined, and is also of class $C^p$.

Theorem A.0.3 (Local Form of the Submersions). Let $X$ and $Y$ be Banach spaces and let $f : X \to Y$ be a submersion in $x_0$, e.i, $df(x_0) : T_{x_0} X \to T_{f(x_0)} Y$ is surjective and $\text{Ker}(df(x_0))$ is complemented. Then, there are open sets $U \subset X$ and $V \subset \text{Ker}(df(x_0))$, with $x_0 \in U$ and $0 \in V$, and a diffeomorphism $\varphi : V \times W \to U$, $W \subset Y$ closed sub-space, such that $f \circ \varphi(x, w) = w$, for all $(x, w) \in V \times W$. 
Appendix B

Appendix B

In this appendix we prove some lemmas of linear algebra in spaces of infinite dimension that are necessary in the proof of the genericity of the Bumpy Metrics in Chapter 2.

Lemma B.0.4. Let $V$ be a infinite dimensional vector space with inner product $\langle \cdot, \cdot \rangle$ and $W \subset V$ an finite dimensional sub-space. Let’s suppose that $V = W \oplus W^\perp$. Let $Z \subset V$ be a sub-space, $Z \supset W^\perp$. Then

1) $(W^\perp)^\perp = W$.

2) $V = Z \oplus Z^\perp$

Proof. 1) Clearly $W \subset (W^\perp)^\perp$. Now, if $w \in (W^\perp)^\perp$, we can write $w = w_1 + w_2$, with $w_1 \in W$ and $w_2 \in W^\perp$. From the fact $W \subset (W^\perp)^\perp$, we have $w_1 \in (W^\perp)^\perp$. Thus $w_2 = w + w_1 \in (W^\perp)^\perp$ (recall that $(W^\perp)^\perp$ is a closed subspace of $V$). Whence $w_2 \in W^\perp \cap (W^\perp)^\perp = \{0\}$, that is, $w = w_1 \in W$. So, we conclude that $(W^\perp)^\perp \subset W$.

2) Note that the quotient spaces $Z/W^\perp$ and $V/W^\perp$ are both finite dimensional and $Z/W^\perp \subset V/W^\perp$. Given $v \in V$, there are unique $v_1 \in W$ and $v_2 \in W^\perp$ such that $v = v_1 + v_2$. Therefore the map $\pi_1 : V/W^\perp \to W$, $\pi_1(v + W^\perp) = v_1$, is an isomorphism. Let $i : Z/W^\perp \hookrightarrow V/W^\perp$ be the inclusion map. Consider $L = \pi_1 \circ i : Z/W^\perp \to W$ and let $\tilde{Z} = \text{Im} L \subset W$. We claim that $x_1, \cdots, x_k \in Z^\perp$. Let $z + W^\perp \in Z/W^\perp$, $z = z_1 + z_2$, $z_1 \in W$, $z_2 \in W^\perp \subset Z$ and $W = \tilde{Z} \oplus \tilde{Z}$. Let $\{x_1, \cdots, x_k\} \subset W$ be a basis for $\tilde{Z}$. We claim that $x_1, \cdots, x_k \in Z^\perp$. Let $z + W^\perp \in Z/W^\perp$, $z = z_1 + z_2 \in Z$, $z_1 \in W$ and $z_2 \in W^\perp$. We have

$$
\langle z, x_i \rangle = \langle z, x_i \rangle + \langle W^\perp, x_i \rangle
= \langle z + W^\perp, x_i \rangle
= \langle z_1 + z_2 + W^\perp, x_i \rangle
= \langle z_1 + W^\perp, x_i \rangle
= \langle z_1, x_i \rangle + \langle W^\perp, x_i \rangle = 0,
$$

since $z_1 \in \tilde{Z}$, $x_i \in \tilde{Z}^\perp \subset W$, $i = 1, \cdots, k$.

Now we prove that $\text{Span}\{Z, x_1, \cdots, x_k\} = V$. Let $v = v_1 + v_2 \in V$ be given, with $v_1 \in W$ and $v_2 \in W^\perp \subset Z$. There holds that $v_1 = \tilde{z}_1 + \sum_{i=1}^{k} a_i x_i$, $\tilde{z}_1 \in \tilde{Z}$, where $a_i = 1, \cdots, k$, are scalars. Whence $v = (\tilde{z} + v_2) + \sum_{i=1}^{k} a_i x_i$.

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It remains to show that \( \text{Span}\{x_1, \cdots, x_k\} = Z^\perp \). In fact, it suffices to show that \( \text{Span}\{x_1, \cdots, x_k\} \supseteq Z^\perp \). Let \( i : Z^\perp \hookrightarrow V = \mathbb{Z} \oplus \text{Span}\{x_1, \cdots, x_k\} \)

be the inclusion map, and

\[
\pi_2 : \mathbb{Z} \oplus \text{Span}\{x_1, \cdots, x_k\} \rightarrow \text{Span}\{x_1, \cdots, x_k\}
\]

the natural projection. We have that \( \text{Ker}(\pi_2 \circ i) = \{0\} \), so \( \pi_2 \circ i \) is injective, thus \( \dim(Z^\perp) \leq k \). □

**Lemma B.0.5.** Let \( V \) be a infinite-dimensional vector space with inner product \( \langle , \rangle \) and \( W \subset V \) an finite dimensional subspace. Suppose that \( V = W \oplus W^\perp \). Let \( Z \subset V \) be a subspace, \( Z \supseteq W^\perp \), such that for all \( w \in W \setminus \{0\} \), there exist \( z \in Z \) with \( \langle z, w \rangle \neq 0 \). Then \( Z = V \)

**Proof.** We show that

\[
\forall \ w \in W \setminus \{0\} \exists \ z \in Z \langle z, w \rangle \neq 0 \iff W \cap Z^\perp = \{0\} \quad (B.0.1)
\]

First suppose that \( \forall w \in W \setminus \{0\} \exists z \in Z \langle z, w \rangle \neq 0 \). Let \( w \in W \cap Z^\perp \) be given, then \( \forall z \in Z, \langle z, w \rangle = 0 \), so \( w = 0 \), that is, \( W \cap Z^\perp = \{0\} \).

Now suppose that \( W \cap Z^\perp = \{0\} \) and let \( w \in W \setminus \{0\} \), then \( w \notin Z^\perp \), so there is \( z \in Z \) such that \( \langle z, w \rangle \neq 0 \).

Since \( Z \supseteq W^\perp \), by 1) of Lemma B.0.4 we have \( Z^\perp \subset (W^\perp)^\perp = W \). From the fact \( W \cap Z^\perp = \{0\} \), we deduce that \( Z^\perp = \{0\} \). Also by 2) of Lemma B.0.4 we have \( V = Z \oplus Z^\perp \). So, \( V = Z \). □

The next lemma and its proof were taken from [9, Lemma 2.2].

**Lemma B.0.6.** Let \( L : U \rightarrow V \) be a linear map between vector spaces, and let \( S \subset V \) be a subspace of finite codimension. Then, \( L^{-1}(S) \) is finite co-dimensional in \( U \), and

\[
\text{Codim}_U(L^{-1}(S)) = \text{Codim}_V(S) - \text{Codim}_V(\text{Im}(L) + S)
\]

**Proof.** If \( \pi : V \rightarrow V/S \) is the canonical projection, the linear map \( \pi \circ L : U \rightarrow V/S \) has kernel \( L^{-1}(S) \). Hence, \( \pi \circ L \) defines an injective linear map from \( U/L^{-1}(S) \) to \( V/S \), and we have

\[
\text{Codim}_V(S) = \dim(V/S)
= \dim(U/L^{-1}(S)) + \text{Codim}_{V/S}(\text{Im}(\pi \circ L))
= \text{Codim}_U(L^{-1}(S)) + \text{Codim}_V(\text{Im}(L) + S). \quad \square
\]
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