DESCENT-INVERSION STATISTICS IN RIFFLE SHUFFLES

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Abstract. This paper studies statistics of riffle shuffles by relating them to random word statistics with the use of inverse shuffles. Asymptotic normality of the number of descents and inversions in riffle shuffles with convergence rates of order \(1/\sqrt{n}\) in the Kolmogorov distance are proven. Results are also given about the lengths of the longest alternating subsequences of random permutations resulting from riffle shuffles. A sketch of how the theory of multisets can be useful for statistics of a variation of top \(m\) to random shuffles is presented.

1. Introduction

For a sequence \(x = (x_1, ..., x_n)\) of real numbers, the number of descents and inversions are defined as
\[
des(x) = \sum_{i=1}^{n-1} \mathbb{1}(x_i > x_{i+1})
\]
and
\[
inv(x) = \sum_{i<j} \mathbb{1}(x_i > x_j),
\]
respectively. For a permutation \(\pi\) in the symmetric group \(S_n\), we write \(des(\pi)\) for the number of descents in the sequence \((\pi(1), \pi(2), ..., \pi(n))\). Similar notation will be used for inversions and other permutation statistics. In this paper, we will analyze \(des(\rho)\) and \(inv(\rho)\) when \(\rho\) is a random permutation with riffle shuffle distribution (which is defined in the next section precisely), and will discuss some other related problems.

Our interest in descent-inversion statistics in riffle shuffles started with the following elementary observation for uniformly random permutations: Let \(\pi\) be a uniformly random permutation in \(S_n\) and \(X = (X_1, ..., X_n)\) be a random vector where \(X_i\)'s are independent and identically distributed (i.i.d.) \(U(0,1)\) random variables. For \(i = 1, ..., n\), let \(R_i\) and \(R'_i\) be the ranks of \(\pi(i)\) and \(X_i\) in \((\pi(1), ..., \pi(n))\) and \((X_1, ..., X_n)\) respectively. Then \((R_1, ..., R_n) =_d (R'_1, ..., R'_n)\) where \(=_d\) denotes equality in distribution.

This simple result, which can be proven by a simple induction (or, by a measure theoretic argument as in [11]), makes it easier to study problems regarding uniform permutation statistics by transforming them into independent \(U(0,1)\) random variable statistics. As an example, we have
\[
inv(\pi) =_d \sum_{i<j} \mathbb{1}(X_i > X_j)
\]
giving an alternative representation of \(inv(\pi)\) that can be quite useful for asymptotic problems.

A natural question at this point is: What would \(\sum_{i<j} \mathbb{1}(X_i > X_j)\) represent if \(X_i\)'s were instead i.i.d. over \([a] := \{1, ..., a\}\) with distribution \(p = (p_1, ..., p_a)\) where \(a \geq 2\)? Recently, Bliem and Kousidis [3] and Janson [12] considered this problem in terms of the generalized Galois numbers and provided several different probabilistic explanations.

In this paper, we give a different interpretation of this using random permutations which is analogous to the discussion given above for uniformly random permutations. This time, the equivalent distribution turns out to be a biased riffle
shuffle with \(a\) hands. Using this transformation, we are able to obtain asymptotic normality of the number of inversions in riffle shuffles (which was questioned in [8], pg 10) with convergence rates, and also understand some other related statistics.

The organization of this paper is as follows. Section 2 provides background in riffle shuffles and makes the connection to random words using inverse shuffles. It also discusses how similar results can be obtained for a variation of top \(m\) to random shuffles. Section 3 treats the asymptotic distribution of the number of descents and inversions in riffle shuffles. Section 4 provides asymptotic results for the lengths of longest alternating subsequences in uniformly random permutations and riffle shuffles.

2. Riffle shuffles and connection to random words

The method most often used to shuffle a deck of cards is the following: first, cut the deck into two piles and then riffle the piles together, that is, drop the cards from the bottom of each pile to form a new pile. The first mathematical models for riffle shuffles were introduced [9] and [16]. These were further developed in [2] and [8]. Now following [8], we will give two equivalent descriptions of riffle shuffles in the most general sense. For other alternative descriptions (which will not be used in this paper), see [1] and [8].

**Description 1:** Cut the \(n\) card deck into \(a\) piles by picking pile sizes according to the \(\text{mult}(a; p)\) distribution, where \(p = (p_1, \ldots, p_a)\). That is, choose \(b_1, \ldots, b_a\) with probability

\[
\binom{n}{b_1, \ldots, b_a} \prod_{i=1}^{a} p_i^{b_i}.
\]

Then choose uniformly one of the \(\binom{n}{b_1, \ldots, b_a}\) ways of interleaving the packets, leaving the cards in each pile in their original order.

**Definition 2.1.** The probability distribution on \(S_n\) resulting from Description 1 will be called as the **riffle shuffle distribution** and will be denoted by \(P_{n,a,p}\). When \(p = (1/a, 1/a, \ldots, 1/a)\), the shuffle is said to be **unbiased** and the resulting probability measure is denoted by \(P_{n,a}\). Otherwise, shuffle is said to be **biased**.

Note that the usual way of shuffling \(n\) cards corresponds to \(P_{n,2}\) (assuming that the shuffler is not cheating). Before moving on to Description 2, let’s give an example using unbiased 2-shuffles. The permutation

\[
\rho_{n,2} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 5 & 3 & 6 & 7 & 4 \end{pmatrix}
\]

is a possible outcome of the \(P_{n,2}\) distribution. Here the first four cards form the first pile, the last three form the second one and these two piles are riffl ed together.

The following alternative description will be important in the following discussion.

**Description 2:** (Inverse \(a\)-shuffles) The inverse of a biased \(a\)-shuffle has the following description. Assign independent random digits from \(\{1, \ldots, a\}\) to each card with distribution \(p = (p_1, \ldots, p_a)\). Then sort according to digit, preserving relative order for cards with the same digit.
Corollary 2.3. Consider the setting in Lemma 2.2 and let \( X \) only if \( (1 \leq \sigma^{-1} \sim P_{n,a,p}. A \) proof of the equivalence of these two descriptions (with two other formulations) for unbiased shuffles can be found in [2]. Extension to biased case is straightforward. Now let’s give an example of generating a random permutation with distribution \( P_{n,2} \) using inverse shuffles.

Consider a deck of 7 cards. We wish to shuffle this deck with the unbiased 2-shuffle distribution using inverse shuffles. Let \( X = (X_1, \ldots, X_7) = (1, 1, 2, 1, 2, 2, 1) \) be a sample from \( U(\{1, 2\}^7) \). Then, sorting according to digits preserving relative order for cards with the same digit gives the new configuration of cards as \( (1, 2, 4, 7, 3, 5, 6) \). In the usual permutation notation, the resulting permutation after the inverse shuffle is

\[
\sigma = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 2 & 4 & 7 & 3 & 5 & 6
\end{pmatrix},
\]

and the resulting sample from \( P_{n,2} \) is

\[
\rho_{n,2} := \sigma^{-1} = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 2 & 5 & 3 & 6 & 7 & 4
\end{pmatrix}.
\]

In the following, we will sometimes call the random vector \( X = (X_1, \ldots, X_n) \) where \( X_i \)'s are independent with distribution \( p = (p_1, \ldots, p_n) \) as a random word.

Next we formalize the relation between riffle shuffles and random words. Let \( \rho_{n,a,p} \) be a random permutation with distribution \( P_{n,a,p} \) that is generated using inverse shuffles with the random word \( X = (X_1, \ldots, X_n) \) and observe that

\[
\rho_{n,a,p}(i) = \#\{j : X_j < X_i\} + \#\{j < i : X_j = X_i\}.
\]

Thus for \( i, k \in [n] \), we have \( \rho_{n,a,p}(i) > \rho_{n,a,p}(k) \) if and only if \( \#\{j : X_j < X_i\} + \#\{j : j < i, X_j = X_i\} > \#\{j : X_j < X_k\} + \#\{j : j < k, X_j = X_k\} \).

Using this, for the case \( i < k \), we immediately arrive at the following important lemma.

Lemma 2.2. Let \( X = (X_1, \ldots, X_n) \) where \( X_i \)'s are independent with distribution \( p = (p_1, \ldots, p_n) \). Also let \( \rho_{n,a,p} \) be the corresponding permutation as described above so that \( \rho_{n,a,p} \) has distribution \( P_{n,a,p} \). Then for \( i < k \), \( \rho_{n,a,p}(i) > \rho_{n,a,p}(k) \) if and only if \( X_i > X_k \).

This has the following corollary:

Corollary 2.3. Consider the setting in Lemma 2.2 and let \( S \subset \{(i, j) \in [n] \times [n] : i < j\} \). Then

\[
\sum_{(i, j) \in S} \mathbb{I}(\rho_{n,a,p}(i) > \rho_{n,a,p}(j)) = \sum_{(i, j) \in S} \mathbb{I}(X_i > X_j).
\]

In the following two sections, we will make use of this connection to study various statistics of riffle shuffles. Before that, we demonstrate the use of random words approach with two other examples. The first one will be relating riffle shuffles to uniformly random permutations and the second one will give a different interpretation of a variation of top to random shuffles. As a general remark, we note that the results in this paper will be mostly given for unbiased shuffles to keep the notations simple. However, all the results in this paper are extendible to the biased case in a straightforward way.
We start with a total variation result relating riffle shuffle statistics and uniform permutation statistics. Although the result is given for des and inv, it is much more general as can be seen from the proof easily.

**Theorem 2.4.** Let \( \rho_{n,a} \) and \( \pi \) be random permutations in \( S_n \) with unbiased \( a \)-shuffle distribution and uniform distribution, respectively. If \( f = \text{des} \) or \( f = \text{inv} \), then for any \( a \geq n \),

\[
d_{TV}(f(\rho_{n,a}), f(\pi)) \leq 1 - \frac{a!}{(a-n)!} \frac{1}{a^n}.
\]

In particular, \( d_{TV}(f(\rho_{n,a}), f(\pi)) \to 0 \) as \( a \to \infty \).

**Proof.** Let \( \pi \) be a uniformly random permutation and \( \rho_{n,a} \) be a random permutation that is generated using inverse shuffling with the random vector \( \mathbf{X} = (X_1, ..., X_n) \). Also let \( T \) be the number of different digits in the vector \( \mathbf{X} \). Then we have

\[
\mathbb{P}(f(\rho_{n,a}) \in A) = \mathbb{P}(f(\rho_{n,a}) \in A, T = n) + \mathbb{P}(f(\rho_{n,a}) \in A, T < n)
\]

\[
= \mathbb{P}(f(\rho_{n,a}) \in A | T = n) \mathbb{P}(T = n) + \mathbb{P}(f(\rho_{n,a}) \in A, T < n)
\]

\[
\leq \mathbb{P}(f(\pi) \in A) + \mathbb{P}(f(\rho_{n,a}) \in A, T < n) \tag{2.1}
\]

where \( 2.1 \) follows by observing \( \mathbb{P}(f(\rho_{n,a}) \in A | T = n) = \mathbb{P}(\pi \in A) \) since \( \rho_{n,a} \) has uniform distribution conditional on \( T = n \). This yields

\[
\mathbb{P}(f(\rho_{n,a}) \in A) - \mathbb{P}(f(\pi) \in A) \leq \mathbb{P}(f(\rho_{n,a}) \in A, T < n) \leq \mathbb{P}(T < n). \tag{2.2}
\]

Similarly, we have

\[
\mathbb{P}(f(\pi) \in A) = \mathbb{P}(f(\pi) \in A) \mathbb{P}(T = n) + \mathbb{P}(f(\pi) \in A) \mathbb{P}(T < n)
\]

\[
\leq \mathbb{P}(\rho_{n,a} \in A) + \mathbb{P}(T < n)
\]

implying

\[
\mathbb{P}(f(\pi) \in A) - \mathbb{P}(\rho_{n,a} \in A) \leq \mathbb{P}(T > n). \tag{2.3}
\]

Hence combining (2.2) and (2.3), for \( a \geq n \), we get

\[
d_{TV}(f(\rho_{n,a}), f(\pi)) \leq \mathbb{P}(T < n) = \mathbb{P}\left( \bigcup_{i \neq j} \{X_i = X_j\} \right) = 1 - \mathbb{P}\left( \bigcap_{i \neq j} \{X_i \neq X_j\} \right)
\]

\[
= 1 - \frac{\binom{n}{a} n!}{a^n} \frac{1}{a^n}
\]

proving the first claim. The second assertion is immediate from the bound we obtained. \( \square \)

**Remark 2.5.** As can be seen easily from the proof, the result is actually true for a large class of functions \( f \).

**Remark 2.6.** Note that Theorem 2.4 is also informative for understanding multiple 2-shuffles by a nice convolution property of riffle shuffles given by Fulman [8]. Letting \( \mathbf{p} = (p_1, ..., p_a) \), \( \mathbf{p}' = (p'_1, ..., p'_b) \) be two probability measures and defining the product \( \otimes \) by \( \mathbf{p} \otimes \mathbf{p}' = (p_1 p'_1, ..., p_1 p'_b, ..., p_a p'_1, ..., p_a p'_b) \), Fulman’s result gives that the convolution of \( P_{n,a,p} \) and \( P_{n,a,p'} \) is \( P_{n,ab,p\otimes p'} \). In particular, when \( a = b = 2 \) and \( p_1 = p_2 = 1/2 \), the case of multiple 2-shuffles is handled.
Since convergence in total variation implies convergence in distribution, we also have

**Corollary 2.7.** If the shuffle is unbiased, then \( f(\rho_{n,a}) \longrightarrow_d f(\pi) \) as \( a \to \infty \).

We close this section by describing how one can use above ideas to study a variation of top \( m \) to random shuffles which was first introduced in [7]. Consider a deck of \( n \) cards and let \( 0 \leq m \leq n \) be fixed. Now cut off the top \( m \) cards and insert them randomly among the remaining \( n - m \) cards, keeping both packets in the same relative order. We will call this shuffling method as *ordered top \( m \) to random shuffles*.

An ordered top \( m \) to random shuffle is actually equivalent to a 2-shuffle in which exactly \( m \) cards are cut off (whereas for the 2-shuffles case, \( m \) is a binomial random variable). It is not hard to see that the following result gives an inverse description of ordered top \( m \) to random shuffles.

**Theorem 2.8.** The inverse of an ordered top \( m \) to random shuffle has the following description. Assign card \( i \in [n] \) a random bit \( X_i \) where the random vector \( X = (X_1, \ldots, X_n) \) is uniformly distributed over \( \{0, 1\}^n \) with the restriction that \( \sum_{i=1}^n X_i = n - m \). Then sort according to digit, preserving relative order for cards with the same digit.

Now letting \( \tau \) be a random permutation in \( S_n \) with ordered top \( m \) to random shuffle distribution, Theorem 2.8 allows us to rewrite \( \text{des}(\tau) \) or \( \text{inv}(\tau) \) in a useful way exactly as we did in Corollary 2.3. Namely, we have

\[
\text{des}(\tau) =_d \sum_{i=1}^{n-1} \mathbb{1}(X_i > X_{i+1}) \quad \text{and} \quad \text{inv}(\tau) =_d \sum_{i<j} \mathbb{1}(X_i > X_j)
\]

where \( X = (X_1, \ldots, X_n) \) is uniformly distributed over \( \{0, 1\}^n \) with the restriction that \( \sum_{i=1}^n X_i = n - m \). Hence the problem is transformed into a problem of uniform permutations of a fixed multiset which is well studied in the literature. See, for example, [6]. We will revisit this at the end of Section 3.

### 3. Convergence rates for the number of descents and inversions

In this section we will discuss the asymptotic normality of the number of descents and inversions in riffle shuffles and will provide convergence rates of order \( 1/\sqrt{n} \) in the Kolmogorov distance. Recall that the Kolmogorov distance between two probability measures \( \mu \) and \( \nu \) on \( \mathbb{R} \) is defined to be

\[
d_K(\mu, \nu) = \sup_{z \in \mathbb{R}} |\mu((-\infty, z]) - \nu((-\infty, z])|.
\]

We start with the asymptotic normality of the number of inversions after an \( a \) shuffle which was conjectured by Fulman in [5] for unbiased 2-shuffles. Our strategy will be using Corollary 2.3 to transform the problem into random words language, use Janson’s U-statistic construction [12] for the random words case and finally use Chen and Shao’s results on asymptotics of U-statistics [5]. Before moving on to the main result, we provide some pointers to the literature and give the necessary background on U-statistics.

First we note that the asymptotic normality of the number of inversions in random words is recently proven by Bliem and Kousidis [3] without convergence rates in a more general framework. In [12], Janson gave equivalent descriptions of the
random words problem and analyzed the asymptotic behavior using U-statistics
theory. Naturally, the convergence rate result given here will also apply to Janson’s
case.

Now recall that for a real valued symmetric function \( h : \mathbb{R}^m \to \mathbb{R} \) and for a
random sample \( X_1, \ldots, X_n \) with \( n \geq m \), a U-statistic with kernel \( h \) is defined as

\[
U_n = U_n(h) = \frac{1}{\binom{n}{m}} \sum_{i \in [m]} h(X_{i_1}, \ldots, X_{i_m})
\]

where the summation is over the set \( C_{m,n} \) of all \( \binom{n}{m} \) combinations of \( m \) integers,
\( i_1 < i_2 < \ldots < i_m \) chosen from \( \{1, \ldots, n\} \). The next result of Chen and Shao will be
useful for obtaining convergence rates in the Kolmogorov distance. We note that
throughout this paper, \( Z \) will denote a standard normal random variable. Also in
the following statement \( h_1(X_1) \) := \( \mathbb{E}[h(X_1, \ldots, X_m)|X_1] \).

**Theorem 3.1.** Let \( X_1, \ldots, X_n \) be i.i.d. random variables, \( U_n \) be a U-statistic with symmetric kernel \( h \), \( \mathbb{E}[h(X_1, \ldots, X_m)] = 0 \), \( \sigma^2 = \text{Var}(h(X_1, \ldots, X_m)) < \infty \) and \( \sigma_1^2 = \text{Var}(h_1(X_1)) > 0 \). If in addition \( \mathbb{E}[h_1(X_1)]^3 < \infty \), then

\[
d_K \left( \frac{\sqrt{n}}{m \sigma_1} U_n, Z \right) \leq \frac{6.1 \mathbb{E}[h_1(X_1)]^3}{\sqrt{\pi \sigma_1^2}} + \frac{(1 + \sqrt{2})(m-1)\sigma_1}{(m(n-m+1))^{1/2} \sigma_1}.
\]

Now we are ready to state and prove our main result on the number of inversions
in riffle shuffles.

**Theorem 3.2.** Let \( \rho_{n,a} \) be a random permutation with distribution \( P_{n,a} \) with \( a \geq 2 \). Then

\[
d_K \left( \frac{\text{inv}(\rho_{n,a}) - (n(n-1)a^{-1})}{\sqrt{n(n-1)\sigma_1^2}}, Z \right) \leq \frac{C}{\sqrt{n}}
\]

where \( C \) is a constant independent of \( n \).

**Proof.** Let \( a \geq 2 \) and \( \rho_{n,a} \) have distribution \( P_{n,a} \). Using Corollary 2.3, we have

\[
\text{inv}(\rho_{n,a}) := \sum_{i<j} \mathbb{1}(\rho_{n,a}(i) > \rho_{n,a}(j)) = \sum_{i<j} \mathbb{1}(X(i) > X(j))
\]

where \( X_i \)'s are independent and uniformly distributed over \( [a] \). This immediately
yields

\[
\mathbb{E}[\text{inv}(\rho_{n,a})] = \mathbb{E} \left[ \sum_{i<j} \mathbb{1}(X_i > X_j) \right] = \left( \begin{array}{c} n \ 2 \end{array} \right) \mathbb{P}(X_1 > X_2) = \frac{n(n-1)a^{-1}}{4}.
\]

Using similar elementary computations one gets

\[
\sigma_1^2 = \text{Var}(\text{inv}(\rho_{n,a})) = \frac{n(n-1)(2n+5)a^2 - 1}{72}.
\]

See [3] or [12] for details. Now following [12], we will find a U-statistic representation
of \( \text{inv}(\rho_{n,a}) \). All details are included for the sake of completeness.

Let \( U_1, \ldots, U_n \) be independent random variables uniformly distributed over \( (0, 1) \).
Order \( U_i \)'s as \( U_{\sigma(1)} < U_{\sigma(2)} < \ldots < U_{\sigma(n)} \) where \( \sigma \in S_n \) is properly chosen. Since \( \sigma \)
has uniform distribution over \( S_n \), we have

\[
(X_1, \ldots, X_n) = d(X_{\sigma(1)}, \ldots, X_{\sigma(n)}).
\]
Now we get
\[ \text{inv}(\rho_{n,a}) = d \sum_{i<j} \mathbb{1}(X_i > X_j) = d \sum_{i<j} \mathbb{1}(X_{\sigma(i)} > X_{\sigma(j)}) \]
\[ = \sum_{i,j=1}^{n} \mathbb{1}(X_{\sigma(i)} > X_{\sigma(j)}, i < j). \]

where the second equality follows from (3.1). Observing \( i < j \) if and only if \( U_{\sigma(i)} < U_{\sigma(j)} \), we obtain
\[ \text{inv}(\rho_{n,a}) = d \sum_{i,j=1}^{n} \mathbb{1}(X_{\sigma(i)} > X_{\sigma(j)}, U_{\sigma(i)} < U_{\sigma(j)}) = \sum_{i,j=1}^{n} \mathbb{1}(X_i > X_j) \mathbb{1}(U_i < U_j). \tag{3.2} \]

Next let \( Z_i = (X_i, U_i) \) for \( i = 1, \ldots, n \) and observe that \( Z_i \)'s are i.i.d. random variables. Define the functions \( f \) and \( g \) by
\[ f((x_i, u_i), (x_j, u_j)) := \binom{n}{2} \mathbb{1}(x_i > x_j) \mathbb{1}(u_i < u_j) \]
and
\[ g((x_i, u_i), (x_j, u_j)) = f((x_i, u_i), (x_j, u_j)) + f((x_j, u_j), (x_i, u_i)). \]

Then clearly \( g \) is a real valued symmetric function and
\[ \sum_{k,l=1}^{n} \mathbb{1}(X_k > X_l) \mathbb{1}(U_k < U_l) = \frac{1}{\binom{n}{2}} \sum_{k<l} g(Z_k, Z_l). \tag{3.3} \]

Thus, by (3.2) and (3.3) we conclude that \( \text{inv}(\rho_{n,a}) \) is a U-statistic with
\[ \text{inv}(\rho_{n,a}) = d \binom{n}{2}^{-1} \sum_{i<j} \binom{n}{2} (\mathbb{1}(X_i > X_j) \mathbb{1}(U_i < U_j) + \mathbb{1}(X_i < X_j) \mathbb{1}(U_i > U_j)). \]

So in terms of Theorem 3.1 we have \( h((x_1, u_1), (x_2, u_2)) = \binom{n}{2} k((x_1, u_1), (x_2, u_2)) \) where
\[ k((x_1, u_1), (x_2, u_2)) = \mathbb{1}(x_1 > x_2) \mathbb{1}(u_1 < u_2) + \mathbb{1}(x_1 < x_2) \mathbb{1}(u_1 > u_2) - \frac{a - 1}{2a}. \]

Defining
\[ k_1(x_1, u_1) = \mathbb{E}[k(X_1, U_1), (X_2, U_2)|X_1 = x_1, U_1 = u_1], \]
we have \( h_1(x_1, u_1) = \binom{n}{2} k_1(x_1, u_1) \). Also
\[ k_1(X_1, U_1) = \mathbb{E}[\mathbb{1}(X_1 > X_2) \mathbb{1}(U_1 < U_2) + \mathbb{1}(X_1 < X_2) \mathbb{1}(U_1 > U_2)|X_1, U_1] - \frac{a - 1}{2a} \]
\[ = \frac{X_1 - 1}{a} (1 - U_1) + \frac{a - X_1}{a} U_1 - \frac{a - 1}{2a} \]
\[ = \frac{1}{a} (X_1 - 2X_1 U_1 + (a + 1)U_1 - 1) - \frac{a - 1}{2a}. \]

Now doing some elementary computations, we obtain
\[ \sigma_1^2 = \text{Var}(h_1(X_1, U_1)) = \binom{n}{2}^2 \text{Var}(k_1(X_1, U_1)) = \binom{n}{2}^2 \frac{a^2 - 1}{36a^2} \]
and also
\[ \mathbb{E}[h_1(X_1, U_1)]^3 \leq 9 \binom{n}{2}^3. \]
Hence using Theorem 3.1 we arrive at
\[
\begin{align*}
d_K \left( \frac{\text{inv}(\rho_{n,a}) - \frac{n(n-1)a-1}{2}}{\sqrt{n}(n-1)\sqrt{\frac{a^2-1}{36a^2}}} \right) & \leq \frac{2(1 + \sqrt{2})\sqrt{n(n-1)(2n+5)}}{\sqrt{\pi^3} \left( \frac{a^2-1}{36a^2} \right)^{3/2}} + \frac{\sqrt{2} \sqrt{\pi - \frac{1}{2}}} {\sqrt{n}\left(\frac{a^2-1}{36a^2} \right)^{3/2}}
\end{align*}
\]
which in particular implies the existence of a constant $C$ independent of $n$ as in the statement of the theorem. This completes the proof. □

Remark 3.3. U-statistics construction given above will still work when the shuffle is biased. So under certain conditions on the distribution vector $\mathbf{p} = (p_1, ..., p_a)$ (namely, by excluding the case $p_j = 1$ for some $j \in [a]$), one can extend Theorem 3.2 to the case of biased riffle shuffles (or random words).

Remark 3.4. By the nice convolution property discussed in Remark 2.6, Theorem 3.2 also gives convergence rates for multiple unbiased 2-shuffles (with explicit constants, as can be seen easily from the proof).

Next we move on to the number of descents in riffle shuffles which is much easier due to the underlying local dependence. Recall that, if we define the distance between two subsets of $A$ and $B$ of $\mathbb{N}$ by
\[
\rho(A, B) := \inf \{|i - j| : i \in A, j \in B\},
\]
the sequence of random variables $Y_1, Y_2, ...$ is said to be $m$-dependent if $\{Y_i, i \in A\}$ and $\{Y_j, j \in B\}$ are independent whenever $\rho(A, B) > m$ with $A, B \subset \mathbb{N}$. Now we recall the following result from \[4\] about $m$-dependent random variables.

**Theorem 3.5.** If $\{Y_i\}_{i \geq 1}$ is a sequence of zero mean $m$-dependent random variables, $W = \sum_{i=1}^{n} Y_i$ and $E[W^2] = 1$, then for all $p \in (2, 3]$,
\[
d_K (W, Z) \leq 75(10m + 1)^{p-1} \sum_{i=1}^{n} E|Y_i|^p.
\]

Now, letting $\rho_{n,a}$ be a sample from $P_{n,a}$, we know from Corollary 2.34 that
\[
des(\rho_{n,a}) = \sum_{i=1}^{n-1} I(\rho_{n,a}(i) > \rho_{n,a}(i + 1)) = \sum_{i=1}^{n-1} I(X_i > X_{i+1})
\]
where $X_i$'s are independent and uniform over $[a]$. Setting $V = \sum_{i=1}^{n-1} Y_i$ with $Y_i = I(X_i > X_{i+1})$, we have $E[V] = (n-1)\frac{a-1}{2a}$. Also since $\text{Var}(Y_i) = \frac{a^2-1}{4a^2}$ and $\text{Cov}(Y_i, Y_{i+1}) = -\left(\frac{a^2-1}{12a^2}\right)$ for $i = 1, ..., n-1$, we have
\[
\text{Var}(V) = \sum_{i=1}^{n-1} \text{Var}(Y_i) + 2 \sum_{i<j} \text{Cov}(Y_i, Y_j) = (n-1)\frac{a^2-1}{4a^2} - 2(n-1) \left(\frac{a^2-1}{12a^2}\right) = \frac{(a^2-1)(n-1)}{12a^2}.
\]
Now noting that $Y_i$’s are 1-dependent and using Theorem 3.5 with $p = 3$, we arrive at
Theorem 3.6. Let $\rho_{n,a}$ be distributed according to $P_{n,a}$. Then

$$d_K \left( \frac{\text{des}(\rho_{n,a}) - \frac{(a-1)(n-1)}{2a}}{\sqrt{\frac{(a^2-1)(n-1)}{12n^2}}}, Z \right) \leq \frac{C}{\sqrt{n}}$$

where $C$ is a constant independent of $n$.

Remark 3.7. The discussion from Section 2 and a simple coupling argument gives the following stochastic dominance result, say, for the number of inversions:

$$\text{Inv}(\rho_{n,2}) \leq_s \text{Inv}(\rho_{n,a}) \leq_s \text{Inv}(\pi)$$

where $a \geq 2$, $\pi$ is a uniformly random permutation and $\leq_s$ denotes stochastic ordering. Since the means and variances of these three statistics are of the same order, it wouldn’t be surprising to obtain the asymptotic normality of $\text{Inv}(\rho_{n,a})$ by the corresponding results for $\text{Inv}(\rho_{n,2})$ and $\text{Inv}(\pi)$. We will pursue this idea in a future work.

We conclude this section with a discussion of the asymptotic normality of the number of inversions after ordered top $m$ to random shuffles which were defined at the end of Section 2. We start by recalling a special case of a result of Congar and Viswanath on multisets. Let $\beta \in [1/2, 1)$. Then there exists a constant $C > 0$ depending only on $\beta$ so that whenever $\tau$ is a uniform permutation of the multiset $\{0^n_1, 1^n_1\}$ with $n_0, n_1 \in \mathbb{N}$, $n_0 + n_1 = n$, $\max\{n_0, n_1\} \leq \beta n$,

$$d_K \left( \frac{\text{des}(\tau) - \mu}{\sigma}, Z \right) \leq \frac{C}{\sqrt{n}}$$

is satisfied where $\mu = \mathbb{E}[\text{des}(\tau)]$ and $\sigma^2 = \text{Var}(\text{des}(\tau))$ (For details, see [6]). It is easily seen from this result and Theorem 3.6 that, one can analyze the asymptotic behavior of the number of inversions in ordered top $m$ to random shuffles under the assumption that $\max\{m, n - m\} \leq \beta n$. Note that this also suggests a natural generalization of riffle shuffles. To see this, consider the case where the number of cards in the hands are $(n_0, n_1)$ where $(n_0, n_1)$ is uniform over the set $\{(n_0, n_1) \in [n] \times [n]: n_0 + n_1 = n, \min\{n_0, n_1\} \geq \alpha n\}$ for some $1 > \alpha \geq 0$. When $\alpha = 0$, we get $P_{n,2}$. Using $\alpha > 0$, we get a different model which can be meaningful since when one shuffles a deck, there will be at least a few cards in each hand.

4. Another Related Statistic: Longest Alternating Subsequences

In this section we will study the asymptotic behavior of lengths of longest alternating subsequences (which are closely related to descents) in uniform permutations and riffle shuffles. Letting $x := (x_i)_{i=1}^n$ be a sequence of real numbers, a subsequence $x_{i_k}$, where $1 \leq i_1 < ... < i_k \leq n$, is called an alternating subsequence if $x_{i_1} > x_{i_2} < x_{i_3} > ... x_{i_k}$. The length of the longest alternating subsequence of $x$ is defined as

$$LA_n(x) := \max\{k: x \text{ has an alternating subsequence of length } k\}.$$ 

For an example, let $x = (3, 1, 7, 4, 2, 6, 5)$. Then $(3, 1, 7, 2, 6, 5)$ is an alternating subsequence and it is easy to see that $LA_7(x) = 6$. For an excellent survey on longest alternating subsequence problem, see [18]. The following lemma, whose proof can be found in [11] and [17], is very useful to understand $LA_n(x)$ when $x$ is a sequence of random variables.
Lemma 4.1. [17] Let \( x := (x_i)_{i=1}^n \) be a sequence of distinct real numbers. Then

\[
LA_n(x) = 1 + \mathbb{I}(x_1 > x_2) + \# \text{ local extremum of } x
\]

\[
= 1 + \mathbb{I}(x_1 > x_2) + \sum_{k=2}^{n-1} \mathbb{I}(x_{k-1} > x_k < x_{k+1}) + \sum_{k=2}^{n-1} \mathbb{I}(x_{k-1} < x_k > x_{k+1}).
\]

Example 4.2. Let \( x = (3, 1, 7, 4, 2, 6, 5) \). Then the local maximums are \( \{x_3, x_6\} = \{7, 6\} \) and the local minimums are \( \{x_2, x_5\} = \{1, 2\} \). Noting that \( x_1 > x_2 \) and using Lemma 4.1, we get \( LA_n(x) = 1 + 1 + 2 + 2 = 6 \). Indeed, the subsequence \( (3, 1, 7, 2, 6, 5) \) has length 6 and \( x \) does not have a longer alternating subsequence.

Now we move on to discussing longest alternating subsequence of a uniformly random permutation \( \pi \). In this direction, [11] and [17] find the expectation and variance as

\[
\mathbb{E}[LA_n(\pi)] = \frac{2n}{3} + \frac{1}{6} \quad \text{and} \quad \text{Var}(LA_n(\pi)) = \frac{8n}{45} - \frac{13}{180}.
\]

They also prove asymptotic normality of \( LA_n(\pi) \) by using an alternative representation and the underlying local dependence. We contribute to their result by obtaining convergence rates in the Kolmogorov distance.

Theorem 4.3. Let \( \pi \) be a uniformly random permutation in \( S_n \). Then for every \( n \geq 1 \),

\[
d_K \left( \frac{LA_n(\pi) - \left( \frac{2n}{3} + \frac{1}{6} \right)}{\sqrt{\frac{8n}{45} - \frac{13}{180}}}, Z \right) \leq C \sqrt{n}
\]

where \( C \) is a constant independent of \( n \).

Proof. Let \( \pi \) be a uniformly random permutation and \( X_1, \ldots, X_n \) be independent uniform random variables over \( (0, 1) \). Letting

\[
E_k = \{X_{k-1} > X_k < X_{k+1}\} \cup \{X_{k-1} < X_k > X_{k+1}\} \quad \text{for } k = 2, \ldots, n-1, \tag{4.1}
\]

we have

\[
LA_n(\pi) = 1 + \mathbb{I}(\pi(1) > \pi(2)) + \sum_{k=2}^{n-1} \mathbb{I}(\pi(k-1) > \pi(k) < \pi(k+1))
\]

\[
+ \sum_{k=2}^{n-1} \mathbb{I}(\pi(k-1) < \pi(k) > \pi(k+1))
\]

\[
= d \quad 1 + \mathbb{I}(X_1 > X_2) + \sum_{k=2}^{n-1} \mathbb{I}(X_{k-1} > X_k < X_{k+1}) + \sum_{k=2}^{n-1} \mathbb{I}(X_{k-1} < X_k > X_{k+1})
\]

\[
= 1 + \mathbb{I}(X_1 > X_2) + \sum_{k=2}^{n-1} \mathbb{I}(E_k) \quad \text{for } k = 2, \ldots, n-1, \tag{4.2}
\]

where in the second equality, we used the discussion from the Introduction (or see [17] for a precise statement). Now, clearly \( LA_n(\pi) \) is a sum of 4-dependent random variables and result follows from Theorem 3.5. \( \square \)
Remark 4.4. Using the representation of $L A_n(\pi)$ given in [12], one can easily obtain a concentration inequality for $L A_n(\pi)$ by using, for example, McDiarmid’s well known bounded differences inequality [14]. By [12], we have

$$L A_n(\pi) = d f(X_1, ..., X_n) := 1 + 1(X_1 > X_2) + \sum_{k=2}^{n-1} 1(E_k)$$

where $X_i$’s are independent random variables and $E_k$’s are defined in terms of $X_i$’s as in (4.1). Now by a case analysis, it is easy to see that bounded differences property holds with $c_k = 3$ for $k = 1, ..., n$ and one immediately arrives at

$$P(|L A_n(\pi) - \mu| \geq t) \leq 2e^{-2t^2/9n}.$$ 

Next we will work on the same problem for riffle shuffles. First note that, with its close connection to the number of extremum points and number of runs, longest alternating subsequences can be quite useful in non-parametric tests. Indeed, our motivation here comes from the practical discussions of this issue in [15] on cheating in card games.

We start by recalling the development of longest alternating subsequences in random words given in [11]. This time we need to be careful about defining maxima and minima properly as we may have repeated values in the sequence. We say that a sequence $x = (x_1, ..., x_n) \in [a]^n$, has a local minimum at $k$, if (i) $x_k < x_{k+1}$ or $k = n$, and if (ii) for some $j < k$, $x_j > x_{j+1} = ... = x_{k-1} = x_k$. Similarly, $x$ has a local maximum at $k$, if $x_k > x_{k+1}$ or $k = n$, and if (ii) for some $j < k$, $x_j < x_{j+1} = ... = x_{k-1} = x_k$, or for all $j < k$, $x_j = x_k$. With these definitions, a useful representation of $L A_n(x)$ is found by Houdre and Restrepo [11] as

$$L A_n(x) = \text{# of local maxima of } x + \text{# of local minima of } x.$$ 

Letting $X = (X_1, ..., X_n)$ be a random word where $X_i$’s are independent and uniform over $[a]$, they also show that

$$\frac{L A_n(X) - n(2/3 - 1/3a)}{\sqrt{n}\gamma} \rightarrow_d Z$$

as $n \rightarrow \infty$ where

$$\gamma = \frac{8}{45} \left( \frac{(1 + 1/a)(1 - 3/4a)(1 - 1/2a)}{1 - 2/(a + 1)} \right).$$

(Note there is a typo in [11] for the expression of $\gamma^2$. This can be checked from [13] by taking limits in the corresponding variance formula). Now, Lemma 2.2, the discussion just before it with Houdre and Restrepo’s result immediately gives

**Theorem 4.5.** Let $\rho_{n,a}$ be a random permutation with distribution $P_{n,a}$. Then

$$\frac{L A_n(\rho_{n,a}) - n(2/3 - 1/3a)}{\sqrt{n}\gamma} \rightarrow_d N(0, 1)$$

as $n \rightarrow \infty$ where $\gamma$ is as defined in (4.3).

This result can be generalized to biased shuffles as in previous problems in a straightforward way. Asymptotic mean and variance of this case are described in detail in [11]. Also note that, due to the lack of local dependence, obtaining convergence rates is not as easy as the case of uniform random permutations for a
shuffles and it will be studied in a subsequent work. However, when one focuses on \( \rho_{n,2} \), one still has local dependence as we describe in the rest of this section.

The ease of the 2-shuffle case comes from the following proposition which gives a characterization of extremum points of 2-shuffles in terms of the descents. Note that this result also gives the asymptotic behavior of the number of local maxima or minima with the use of Theorem 3.6.

**Proposition 4.6.** Let \( \rho_{n,2,p} \) be a random permutation with distribution \( P_{n,2,p} \) generated by inverse shuffling with the random vector \( \pi = (X_1, \ldots, X_n) \) where \( X_i \)'s are independent with distribution \( p = (p_1, p_2) \) with \( 0 < p_1 < 1 \). Then for \( k = 2, \ldots, n-1 \),

1. \( \rho_{n,2,p} \) has a local maximum at \( k \) if and only if \( \rho_{n,2,p} \) has a descent at \( k \).
2. \( \rho_{n,2,p} \) has a local minimum at \( k \) if and only if \( \rho_{n,2,p} \) has a descent at \( k - 1 \).

**Proof.**

1. (\( \Rightarrow \)) Obvious. (\( \Leftarrow \)) Assume \( \pi(k) > \pi(k+1) \). We should show \( \pi(k-1) < \pi(k) \). Since \( \pi(k) > \pi(k+1) \), we see that \( k^{th} \) card comes from the second pile and \( k + 1^{st} \) from the first pile. Now whether card \( k - 1 \) comes from the first pile or the second pile, we have \( \pi(k-1) < \pi(k) \) since the relative orders of the piles are preserved.

2. Proof is similar to the maximum case and we skip it.

Now we are ready to give a useful representation of \( LA_n(\rho_{n,2,p}) \). First recall that

\[
LA_n(\pi) = 1 + \mathbb{1}(\rho_{n,2,p}(1) > \rho_{n,2,p}(2)) + \sum_{k=2}^{n-1} \mathbb{1}(\rho_{n,2,p}(k-1) > \rho_{n,2,p}(k) < \rho_{n,2,p}(k+1)) + \sum_{k=2}^{n-1} \mathbb{1}(\rho_{n,2,p}(k-1) < \rho_{n,2,p}(k) > \rho_{n,2,p}(k+1)).
\]

Using Proposition 4.6 we obtain

\[
LA_n(\rho_{n,2,p}) = d 1 + \mathbb{1}(X_1 > X_2) + \sum_{i=2}^{n-1} \mathbb{1}(X_i > X_{i+1}) + \sum_{i=1}^{n-2} \mathbb{1}(X_i > X_{i+1})
\]

where \( X_i \)'s are independent with distribution \( p = (p_1, p_2) \). This immediately gives

\[
LA_n(\rho_{n,2,p}) = d 2 \left( \sum_{i=1}^{n-1} \mathbb{1}(X_i > X_{i+1}) \right) + \mathbb{1}(X_{n-1} < X_n).
\]

By the representation in (4.4), it is clear that we still have local dependence for \( LA_n(\rho_{n,2,p}) \) and thus, we can still use Theorem 3.6 with \( p = 3 \) to obtain a convergence rate of order \( 1/\sqrt{n} \) for \( LA_n(\rho_{n,2,p}) \).

5. Concluding Remarks

In this note, after relating riffle shuffle statistics to random word statistics, we were able to obtain asymptotic normality results with convergence rates for the number of descents and inversions after an arbitrary number of \( a \)-shuffles. Throughout the way, we also discussed how similar ideas can be used for a variant of top \( m \) to random shuffles and provided small contributions to Houdre and Restrepo’s work on longest alternating subsequences.

In subsequent work, we will provide convergence rates for the length of longest alternating subsequences in \( a \)-shuffles for \( a \geq 2 \). We also hope to find out a general...
framework for establishing the asymptotic normality of a large class of shuffle statistics. One possible direction for this can be using the stochastic dominance idea introduced in Remark 3.7 as in many cases it can be easier to prove the results for 2-shuffles and uniformly random permutations.

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