THE CUSPED HYPERBOLIC 3-ORBIFOLD OF MINIMUM VOLUME

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An orbifold is a space locally modelled on $\mathbb{R}^n$ modulo a finite group action. We will restrict our attention to complete orientable hyperbolic 3-orbifolds $Q$; thus, we can think of $Q$ as $H^3/\Gamma$, where $\Gamma$ is a discrete subgroup of $\text{Isom}_+(H^3)$, the orientation-preserving isometries of hyperbolic 3-space. An orientable hyperbolic 3-manifold corresponds to a discrete, torsion-free subgroup of $\text{Isom}_+(H^3)$. We will work in the upper-half-space model $H^3$ of hyperbolic 3-space, in which case $\text{PGL}(2, \mathbb{C})$ acts as isometries on $H^3$ by extending the action of $\text{PGL}(2, \mathbb{C})$ on the Riemann sphere (boundary of $H^3$) to $H^3$. If the discrete group $\Gamma$ corresponding to $Q$ has parabolic elements, then $Q$ is said to be cusped. (For more details on this paragraph see [T, Chapter 13].)

Unless otherwise stated, we will assume all manifolds and orbifolds are orientable. Mostow's theorem implies that a complete, hyperbolic structure of finite volume on a 3-orbifold is unique. Consequently, hyperbolic volume is a topological invariant for orbifolds admitting such structures. Jørgensen and Thurston proved (see [T, §6.6]) that the set of volumes of complete hyperbolic 3-manifolds is well-ordered and of order type $\omega^\omega$. In particular, there is a complete hyperbolic 3-manifold of minimum volume $V_1$ among all complete hyperbolic 3-manifolds and a cusped hyperbolic 3-manifold of minimum volume $V_\omega$. Further, all volumes of closed manifolds are isolated, while volumes of cusped manifolds are limits from below (thus the notation $V_\omega$).

Modifying the proofs in the Jørgensen-Thurston theory yields similar results for complete hyperbolic 3-orbifolds (but see the remark at the end of this paper). In particular, there is a hyperbolic 3-orbifold of minimum volume, and a cusped hyperbolic 3-orbifold of minimum volume. We prove

**Theorem.** Let $Q_1 = H^3/\Gamma_1$ where $\Gamma_1 = \text{PGL}(2, O_3)$ and $O_3 = \text{ring of integers in } \mathbb{Q}(\sqrt{-3})$. The orbifold $Q_1$ has minimum volume among all orientable cusped hyperbolic 3-orbifolds.

*Note.* $Q_1$ is the orientable double-cover of the (nonorientable) tetrahedral orbifold with Coxeter diagram $\ldots \ldots$ (see [T, Theorem 13.5.4] and [H, §1]). This tetrahedral orbifold has fundamental domain $1/24$ of the ideal regular hyperbolic tetrahedron (use the symmetries). In particular, $Q_1$ has a cusp and its volume is $1/12$ the volume of the ideal regular tetrahedron $T$, i.e. $\text{vol}(Q_1) = V/12 \approx 0.0846$, where $V = \text{vol}(T)$.
**Proof (of Theorem).** In Parts I and II of the proof we will get a lower bound for the volume of $H^3/\Gamma$ for arbitrary cusped discrete $\Gamma$.

**PART I: Volume Contributions of Cusped Neighborhoods in $H^3/\Gamma$.**

**Manifold Case** (i.e., $\Gamma$ such that $H^3/\Gamma$ is a manifold with a cusp): We can assume (using a suitable conjugation) that the cusp corresponds to the point at $\infty$ in $H^3$, and that the parabolic transformation $z \mapsto z + 1$ is the “shortest” element in $\Gamma_\infty$, the stabilizer of $\infty$ in $\Gamma$ ($\Gamma_\infty$ has no hyperbolic elements; see [Be, Theorem 5.1.2]). Construct the horoball $C_\infty$, centered at $\infty$, for which $\text{Too}$ has minimum translation length one (in the Euclidean metric) on the horosphere boundary of $C_\infty$. Our set-up has been rigged so that $C_\infty = \{(x, y, z): t \geq 1\}$. Construct such “length one” cusped neighborhoods at all parabolic fixed points (for some element of $\Gamma$). It is a standard fact (see [Be, Theorem 5.4.4]) that all such cusped neighborhoods are disjoint. Thus $C_\infty/\text{Too}$ is an embedded “cusp neighborhood” in $M = H^3/\Gamma$.

What is the volume of $C_\infty/\Gamma_\infty$? If $z \mapsto z + 1$ is the “shortest element” in $\Gamma_\infty$, then any other element $z \mapsto z + w$ in $\Gamma_\infty$ must have $|w| \geq 1$ and $|\text{Im}(w)| \geq \sqrt{3}/2$. Thus, we can compute $\text{vol}(C_\infty/\Gamma_\infty) \geq \sqrt{3}/4$ (see [M1, §5]).

**Orbifold Case.** The only additional complication from the manifold case is that $\text{Too}$ may include elliptic elements. If so, then the elliptic and parabolic elements comprising $\Gamma_\infty$ act as rigid motions on the (Euclidean) horosphere at height $1$ in $H^3$. Thus, we need only study the oriented wall-paper groups to understand the effect of the elliptic elements on the volume estimate for $C_\infty/\Gamma_\infty$. There are 5 such wall-paper groups, and the worst case reduces volume by a factor of 6.

The cusp neighborhoods contribute at least $\sqrt{3}/24$ to the volume of a complete orientable cusped hyperbolic 3-orbifold.

**PART II: Volume Contributions Outside the Cusp Neighborhoods.** By Part I, we have some control over the size of a cusped neighborhood. However, this cusp neighborhood is only a portion of the fundamental domain for $\Gamma$. Can we gain some control over the size of the fundamental domain outside of the cusp neighborhood? Yes, by sphere-packing. First, we fix a particular fundamental domain $D$ for $\Gamma$: Let $D_\infty = \{p \in H^3: p$ is closer to $C_\infty$ than to any conjugate (under $\Gamma$) of $C_\infty\}$. Then we take $D$ to be a fundamental domain for the action of $\Gamma_\infty$ on $D_\infty$.

Next, consider 4 horospheres in $H^3$, each touching all the others. (Their centers (points of tangency with $\partial H^3$) will determine an ideal regular tetrahedron $T$. Let $B$ be the union of the 4 horoballs bounded by the 4 horospheres. Böröczky’s theorem (see [B, Theorem 4]) says that this is, in some sense, the densest packing of horospheres in hyperbolic 3-space. In terms of $C_\infty$ and $D$, Böröczky’s theorem implies that $\text{vol}(C_\infty \cap D)/\text{vol}(D) \leq \text{vol}(B \cap T)/\text{vol}(T) = 4(\sqrt{3}/8)/V = \sqrt{3}/2V$ (for more details, see [M2]).

Thus, $\text{vol}(H^3/\Gamma) = \text{vol}(D) \geq \text{vol}(C_\infty \cap D)/(\sqrt{3}/2V) \geq (\sqrt{3}/24)(2V/\sqrt{3}) = V/12.$
PART III: SUMMARY. As mentioned above, the orbifold $Q_1$ has a cusp and has volume $V/12$. Parts I and II tell us that all cusped orbifolds have volume at least $V/12$. Thus $Q_1$ realizes the minimum volume and it is $V/12 \approx 0.0846$. □

REMARK. There are cusped orbifolds on which Dehn surgery cannot be performed. Consequently, unlike the manifold case, there are cusped hyperbolic 3-orbifolds whose volumes are isolated—$Q_1$ is such an orbifold. The question of finding "the least limiting orbifold" remains open.

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BIBLIOGRAPHY

[Be] A. Beardon, The geometry of discrete groups, Springer-Verlag, New York, 1983.
[B] K. Böröczky, Packing of spheres in spaces of constant curvature, Acta Math. Acad. Sci. Hungar. 32 (1978), 243–261.
[H] A. Hatcher, Hyperbolic structures of arithmetic type on some link complements, J. London Math. Soc (2) 27 (1983), 345–355.
[M1] R. Meyerhoff, A lower bound for the volume of hyperbolic 3-manifolds, preprint.
[M2] ———, Sphere-packing and volume in hyperbolic 3-space, preprint.
[T] W. Thurston, The geometry and topology of 3-manifolds, Princeton Univ. preprint 1978.

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