Discrete-time quantum walk approach to high-dimensional quantum state transfer and quantum routing

Heng-Ji Li,1,2 Jian Li,2,* and Xiu-Bo Chen1,3

1Information Security Center, State Key Laboratory Networking and Switching Technology, Beijing University of Posts Telecommunications, Beijing 100876, China
2School of Computer Science, Beijing University of Posts Telecommunications, Beijing 100876, China
3Guizhou University, Guizhou Provincial Key Laboratory of Public Big Data, Guizhou Guiyang, 550025, China

(Dated: August 12, 2021)

High-dimensional quantum systems can offer extended possibilities and multiple advantages while developing advanced quantum technologies. In this paper, we propose a class of quantum-walk architecture networks that admit the efficient routing of high-dimensional quantum states. Perfect state transfer of an arbitrary unknown qudit state can be achieved between two arbitrary nodes via a one-dimensional lackadaisical discrete-time quantum walk. In addition, this method can be generalized to the high-dimensional lattices, where it allows distillable entanglement to be shared between arbitrary input and output ports. Implementation of our scheme is more feasible through exploiting the coin degrees of freedom and the settings of the coin flipping operators are simple. These results provide a direct application in a high-dimensional computational architecture to process much more information.

PACS numbers: Valid PACS appear here

I. INTRODUCTION

High-dimensional quantum systems (qudits) has emerging as an alternative to two-dimensional quantum systems (qubits), because they can offer extended possibilities and multiple advantages while developing advanced quantum technologies. For computation, implementing qudits brings about efficient distillation of resource states [1] and simplified gates [2]. For quantum communication, qudits can lead to both higher information capacity [3] and increased noise resilience [4]. The extensions of various protocols from qubits to qudits have been proposed, such as universal quantum computation [5], quantum cryptography [6] and even implemented in experiment [7]. Additionally, the application of qudits will enhance and deepen our understanding of quantum computation and communication.

Quantum state transfer [8-12] between two selected nodes is a crucial task for quantum science technologies. The ability to perfectly transfer an arbitrary quantum state between different parts in the interior of quantum computers [13] is essential and it is interesting from the perspective of distributed quantum computation [14] while combining local quantum processing with quantum state transfer. In addition, quantum state transfer can be used to realize the distribution of arbitrary unknown multi-particle entanglement state in quantum network, which opens the possibility for a variety of novel applications ranging from teleportation [15], error correction [16], and purification [17].

In this paper we consider two related fields of research: lackadaisical quantum walks (LQWs) on one-dimensional or higher-dimensional lattices and quantum state transfer. In particular we are interested in the discrete-time quantum walk realization of a high-dimensional quantum state transfer in a quantum-walk architecture. Lackadaisical quantum walks (LQWs) was proposed by Wong [18], as the generalization of the original quantum walks (QWs), in which each vertex in a standard quantum walks is attached to $l$ self-loops. QWs, as the quantum mechanical analogs of classical random walks, were first introduced by Aharonov et al. in 1993 [19]. Due to the quantum interference effects, they are computationally more efficient than their classical counterparts and offer an alternative approach to implementing better quantum algorithms, such as database search [20, 21], element distinctness [22], graph isomorphism [23] and so on. Later it was shown that they are capable of universal quantum computation [24-27].

Some work [28-30] on quantum state transfer has been investigated showing the promising application of QWs. One approach to the problem is that one performs the local coin operators at each individual node with the full control of the walk-coin system. It is essentially the discrete-time variant of the engineered coupling protocol [9] in spin chains. They have shown that an arbitrary qubit can be transferred with unit fidelity over arbitrary distance on the line [29], cycle [28] and square lattice [29]. In our work, we focus on the high-dimensional state transfer.

In this manuscript, we propose a class of quantum-walk architecture networks that admit an efficient quantum routing, where an arbitrary unknown qudit can be perfectly transferred between arbitrary input and output ports. Consequently, sharing entangled qudits to multiple arbitrary nodes can be achieved by extending the method. For achieving it, we need to perform inhomogeneous coin flipping operators at every step, which are...
high-dimensional quantum gate: the identity operator, the generalized Pauli gate and swap gate. It is shown that the time scaling between arbitrary sites is linear to the distance to be covered. By introducing the coin state, transferring the qudit is more feasible and easier to extend to multiqubit entanglement transfer.

The paper is structured as follows. In Sec. II the preliminaries are provided about the knowledge of one-dimensional LQW on the line and three high-dimensional coin operators that are crucial for realizing our scheme. In Sec. III, the scheme of transferring an arbitrary qutrit is presented and then it can be generalized to the transfer of an arbitrary qudit. In Sec. IV, We extend the scheme from one-dimensional quantum routing to the multi-dimensional case, routing entangled qudits to m arbitrary positions. Finally, Sec. V contains our conclusions.

II. PRELIMINARIES

A. One-dimensional LQW on the line

Lackadaisical quantum walks (LQWs) on the line was first introduced in 2015 [13], as quantum analog of lazy random walks where each vertex is attached to λ self-loops. It is defined as a quantum system with two Hilbert spaces, the coin space $H_c$, spanned by $d(d = \lambda + 2)$ basis states $\{|0\rangle, |1\rangle, \ldots, |d-1\rangle\}$, and the position Hilbert space $H_p$ spanned by $\{|x\rangle | x \in \mathbb{Z}\}$. The whole system is in the Hilbert space $H = H_p \otimes H_c$.

One-step time evolution of QWs is controlled by the unitary operator $U = S(I \otimes \mathcal{C})$, where $\mathcal{C} \in SU(d)$ is the coin flipping operator and $S$ is the conditional shift operator described the following unitary operator

$$S = \sum_{x} |x - 1, 0\rangle \langle x, 0| + \left( \sum_{i=1}^{\lambda} |x, i\rangle \langle x, i| \right) + |x + 1, \lambda + 1\rangle \langle x, \lambda + 1|,$$  \hfill (1)

where the index $x$ runs over $\mathbb{Z}$. One finds clearly that the coin state $|0\rangle$ and $|d-1\rangle$ correspond to the left and right and that the coin state $|i\rangle$ corresponds to the neutral state for the motion and denote the direction of the walk as $\{l, s, r\}$. An illustrative example (taking $\lambda = 1$) is given in Figure I. If the particle and coin start in state $|\psi_0\rangle$, the state of the system after $t$ steps of the walk is $|\psi_t\rangle = U^t |\psi_0\rangle$.

B. High-dimensional quantum gate

Before we show our scheme, we introduce three kinds of high-dimensional quantum gate performed on the coin state, the identity operator $I$, the generalized Pauli gate $\mathcal{X}$ which is also called increment gate with $\mathcal{X} = (k + 1 \text{mod} d) |k\rangle$ and swap gate $\mathcal{X}_{i \leftrightarrow j}$ [22], which act on the two-dimensional subspace $H_{i,j}$ of $d$-dimensional Hilbert space with $\mathcal{X}_{i \leftrightarrow j} = |i\rangle \langle j| + |j\rangle \langle i| + \sum_{k \neq i,j} |k\rangle \langle k|$.

FIG. 1. The one-dimensional LQWs with one self-loop on the line. The quantum state $|0\rangle$, $|1\rangle$ and $|2\rangle$ determines that the walker move left, stay put and move right, respectively.

We call the operators $\mathcal{X}$ and $\mathcal{X}_{i \leftrightarrow j}$ as the special coin operators. For the identity coin operator $I$, the coin states remain unchanged and therefore the direction of the walker remains the same as that for the previous step. For the special coin operators, it will change the coin state with the following forms

$${\|0\rangle \rightarrow |i\rangle (t \rightarrow s), |0\rangle \rightarrow |d - 1\rangle (s \rightarrow r),}$$

{\|i\rangle \rightarrow |0\rangle (s \rightarrow l), |j\rangle \rightarrow |j\rangle (s \rightarrow l), |i\rangle \rightarrow |d - 1\rangle (s \rightarrow r),}$$

{\|d - 1\rangle \rightarrow |0\rangle (r \rightarrow s), |d - 1\rangle \rightarrow |0\rangle (r \rightarrow l).}$$

III. TRANSFERRING AN ARBITRARY QUDIT VIA ONE-DIMENSIONAL LQW

A. Perfect state transfer of an unknown qutrit

To begin with, we introduce a basic scheme to transfer the three-dimensional coin state also called qutrit $|\Psi\rangle_0 = \alpha |0\rangle + \beta |1\rangle + \gamma |2\rangle$, where $\alpha$, $\beta$, and $\gamma$ are complex numbers that fulfill $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$. For accomplishing it, take $\lambda = 1$ in the shift operator $\{1\}$.

Our goal is to transfer the coin state $|\Phi\rangle_0$ to a certain position $p$ from the original position 0 after $n$-step QWs

$$|\Psi\rangle_0 = |0\rangle |\Phi\rangle_0 \xrightarrow{n \text{ steps}} |\Psi\rangle_n = |p\rangle |\Phi\rangle_0,$$  \hfill (3)

where $|\Psi\rangle_i$ denotes quantum state after the $i$ steps. Consequently, it can be derived that

$$|\Psi\rangle_{n-1} = \sum_{j=-1,0,1} a_{n-1,p+j} |p + j\rangle |\phi\rangle_{n-1,p+j},$$  \hfill (4)

where $|a\rangle_{n,p}$ and $|\phi\rangle_{n,p}$, respectively, are the complex amplitude and the coin state corresponding to the walker in position $p$ after the $n$-th step. The position state in $|\Psi\rangle_{n-1} = p - 1$, $p - 1$ or $p$, due to the fact that the states in other positions cannot walk to the position in $p$ in one step.

Next, by using the iteration relation

$$|\Psi\rangle_{n} = S \sum_{x} (I \otimes C_n,x) |\Psi\rangle_{n-1},$$  \hfill (5)
where $C_{n,x}$ stands for the coin flipping operator performed on the position $x$ of the $n$-th step, and substituting (1) and (3) into (4), it can be concluded that
\[
\begin{align*}
    a_{n-1,p+1} &= \alpha, C_{n,p+1} |\phi\rangle_{n-1,p+1} = |0\rangle, \\
    a_{n-1,p} &= \beta, C_{n,p} |\phi\rangle_{n-1,p} = |1\rangle, \\
    a_{n-1,p-1} &= \gamma, C_{n,p-1} |\phi\rangle_{n-1,p-1} = |2\rangle.
\end{align*}
\]
By analysis, we will take $C_{1,0} = \lambda_{0+2}$ to swap the information flow $\alpha$ and $\gamma$, and therefore after the first step the whole walk-coin system will be
\[
|\Psi\rangle_1 = \gamma |{-1}\rangle|0\rangle + \alpha|0\rangle|1\rangle + \alpha|1\rangle|2\rangle
\]
Thus, we need to determine the coin unitary operators from second to $(n-1)$-th step
\[
|\Psi\rangle_1 \xrightarrow{C_{x}} |\Psi\rangle_2 \xrightarrow{C_{n-1,x}} |\Psi\rangle_{n-1}
\]
which can achieve the goal, propagating to position $x-1$, $x$, or $x+1$ from position $-1$, 0, or 1 after the $(n-1)$-th step and correspondingly, we need to realize the transmission of the information flow of $\alpha$, $\beta$ and $\gamma$ as follows
\[
\begin{align*}
    \alpha |1\rangle|2\rangle &\rightarrow \alpha |p+1\rangle|\phi\rangle_{n-1,p+1} , \\
    \beta |0\rangle|1\rangle &\rightarrow \beta |p\rangle|\phi\rangle_{n-1,p} , \\
    \gamma |{-1}\rangle|0\rangle &\rightarrow \gamma |p-1\rangle|\phi\rangle_{n-1,p-1}.
\end{align*}
\]
For solving it, the set of the coin unitary operator used will be $\{I, X, X^2\}$ and by using the model of the classical lazy random walks, it can be derived
\[
\begin{align*}
    n_t + n_s + n_r &= n - 2, \\
    n_r - n_t &= p,
\end{align*}
\]
where $n_t, n_s$ and $n_r$, respectively, are the number of moving left, staying put, and moving right which will give
\[
n - 2 \geq |p|.
\]
which restricts the range of the position that the coin state can be perfectly transferred. Furthermore, it will yield
\[
\begin{align*}
    n_t &= \frac{n - 2 - n_s - p}{2}, \\
    n_r &= \frac{n - 2 - n_s + p}{2}, \\
    n_s &= \frac{n - 2 - |p| - 2n_\Delta}{2}.
\end{align*}
\]
where $n_\Delta = \min\{n_r, n_t\}$. Hence the number of such walks that satisfy the equation (10) will be
\[
N(n, p) = \sum_{n_s=0}^{n-2-|p|} \binom{n-2}{n_s} \binom{n-2-n_s+p}{(n-2-n_s+p)/2},
\]
which is depending on the step $n$ and the target position $p$. Correspondingly there will be $N(n, p)$ kinds of coin unitary operators and denote the solution space as $\Omega = \{\tau_1, \tau_2, \ldots, \tau_{n(p)}\}$.
Denote $C(\tau_i)$ as the sum of the numbers of the special coin operators $\{X, X^2\}$ for the case $\tau_i$. The target is finding out $\tau^*_i$ that satisfies the condition
\[
C(\tau^*_i) = \min C(\tau_i), \forall \tau_i \in \Omega.
\]
It can be easily obtained that the solution should have the following form
\[
d_1d_2 \cdots d_1 d_2 d_2 \cdots d_2 d_3d_3 \cdots d_3
\]
where $d_1, d_2, d_3 \in \{l, s, r\}$ with $d_1 \neq d_2 \neq d_3$.
(i)The walking of the information flow $\alpha$ from 1 to $p+1$, it can be derived that
\[
\begin{align*}
rr \cdots r &ss \cdots ss ll \cdots l
\end{align*}
\]
Due to the constraint $n + p = 2(n_s + 1 + \frac{n_\Delta}{2})$, while $n_s$ is even or odd, the parity of $n$ and $p$ will be the same or different. In order to transfer the state to position $p$ after an arbitrary $n$-step QW, both two cases where $n_s$ is even or odd need to be considered.
**Case 1:** While $n_s$ is even, the solution is that $X$ needs to be performed to make $|2\rangle \rightarrow |0\rangle$ at the location $\frac{p+2}{2}$ in the $\frac{n+p+2}{2}$ step with $n_s = 0$. It is because that if $n_s \neq 0$, $X^2$ will be performed twice for making $|2\rangle \rightarrow |1\rangle$ and $|1\rangle \rightarrow |0\rangle$.
**Case 2:** While $n_s$ is odd, the solution is $X^2$ needs to be performed respectively for achieving $|2\rangle \rightarrow |1\rangle$ and $|1\rangle \rightarrow |0\rangle$ at the location $\frac{n+p-n_s}{2}$ in the $\frac{n+p+2-n_s}{2}$ and $\frac{n+p+2+n_s}{2}$ steps. Without loss of generality, take $n_s = n-2-|p|$ to achieve the transfer.
(ii)For the case that the walking of the information flow $\gamma$ from $-1$ to $p-1$, similarly it can be deduced that
\[
\begin{align*}
ll \cdots ll ss \cdots ss rr \cdots r
\end{align*}
\]
and then we will show the result directly.
**Case 1:** While $n_s$ is even, the solution is that $X^2$ is needed to render $|0\rangle \rightarrow |2\rangle$ at the location $\frac{p-n}{2}$ in the $\frac{n-p+2}{2}$ step with $n_s = 0$.
**Case 2:** While $n_s$ is odd, the solution is that $X$ is used to achieve $|0\rangle \rightarrow |1\rangle$ and $|1\rangle \rightarrow |2\rangle$ at the location $\frac{p+n-n_s}{2}$ and $\frac{n+p+2-n_s}{2}$ step. And without loss of generality, take $n_s = n-2-|p|$ to achieve the transfer.
(iii)For the case that the walking of the information flow $\beta$ from 0 to $p$, the solution will be
\[
\begin{align*}
ss \cdots ss uu \cdots uu ss \cdots ss
\end{align*}
\]
where $n_s = n - 2 - |p|$, $0 \leq j \leq n_s$ and $w = \{l, p < 0\}$. Therefore, at position 0 and $p$ in the $j+2$ and $j+2 + |p|$
step, (i) for $p > 0$, $X$ and $X^2$ are respectively performed to make $|1\rangle \rightarrow |2\rangle$ and $|2\rangle \rightarrow |1\rangle$; (ii) for $p < 0$, $X^2$ and $X$ are respectively performed to make $|1\rangle \rightarrow |0\rangle$ and $|0\rangle \rightarrow |1\rangle$. Without loss of generality, take $j = 0$ to achieve the transmission task.

To sum up, the transmission of the information flow $\alpha$, $\beta$ and $\gamma$ have been achieved. The setting of the special coin flipping operators depends on the target position and the step numbers and they are as follows with leaving the others equal to $I$. There are 5 or 7 special coin operators while the parity of $n$ and $p$ are the same or different. Specially, the common coin flipping operators for achieving the transfer of the information flow $\beta$ are shown below.

$$X_{0+2} : (1, 0), X : (|a_2| + 2, a_2), X^2 : (|a_1| + 2, a_1);$$

As for achieving the transfer of the information flow $\alpha$ and $\gamma$, there are two different sets of the coin operators.

(i) While the parity of $n$ and $p$ are the same:

$$X : (b^+ + 1, b^+), X^2 : (b^- + 1, -b^-);$$

(ii) While the parity of $n$ and $x$ are different:

$$\begin{cases} X^2 : (|a_1| + 2, a_1 + 1), (n - |a_2|, a_1 + 1); \\ X : (|a_2| + 2, a_2 - 1), (n - |a_1|, a_2 - 1). \end{cases}$$

where $(1,)$ is the two-dimensional array of the step and position, $a_i = p\delta_1(p)(i = 1, 2), b^\pm = \frac{n + p}{2} \delta_1(p) = \begin{cases} 1, p > 0; \\ 0, p < 0 \end{cases}$ and $\delta_2(p) = \begin{cases} 0, p > 0; \\ 1, p < 0. \end{cases}$ Two examples of transferring three-dimensional quantum state from position 0 to position 2 via a four and five-step discrete-time QW are shown in Figure 2 and 3.

By taking $\lambda = 0$, it can be deduced that the two-dimensional state (qubit) $|\Phi\rangle_0 = \alpha |0\rangle + \gamma |1\rangle$ can be transferred to target position $p$ after $n$-step walks with the coin unitary operator $\sigma_x$ at $(1, 0), (b^- + 1, -b^-), (b^+ + 1, b^+)$. It should be noted that for even (odd) step numbers the coin state can be transferred only to the even (odd) positions because of the lacking of “staying put”.

B. Perfect state transfer of an unknown qudit

We now consider the transfer of arbitrary $d$-dimensional quantum state

$$|\Phi\rangle_0 = \alpha |0\rangle + \sum_{i=1}^{d-2} \beta_i |i\rangle + \gamma |d - 1\rangle,$$

where $\alpha$, $\beta_i$ and $\gamma$ are complex numbers, and $|\alpha|^2 + \sum_i |\beta_i|^2 + |\gamma|^2 = 1$. We will take $C_{1,0} = X_{0+d-1}$, and it will yield

$$|\Psi\rangle_n = \gamma |1\rangle |0\rangle + \sum_{i=1}^{d-2} \beta_i |0\rangle |i\rangle + \alpha |1\rangle |d - 1\rangle,$$

$$|\Psi\rangle_{n-1} = \alpha |p + 1\rangle |\phi\rangle_{n-1,p+1} + \gamma |p - 1\rangle |\phi\rangle_{n-1,p-1} + \sum_{i=1}^{d-2} \beta_i |p\rangle |\phi^{(i)}\rangle_{n-1,p}.$$  

Then take the following coin unitary operators in the $n$-th step

$$\begin{align*}
C_{n,p+1} |\phi\rangle_{n-1,p+1} & = |0\rangle, \\
C_{n,p} \sum_{i=1}^{d-2} \beta_i |\phi^{(i)}\rangle_{n-1,p} & = \sum_{i=1}^{d-2} \beta_i |i\rangle, \\
C_{n,p-1} |\phi\rangle_{n-1,p-1} & = |d - 1\rangle.
\end{align*}$$

and thereby it will deduce that

$$|\Psi\rangle_{n-1} \xrightarrow{C_{n,p}} |\Psi\rangle_n = |p\rangle |\phi\rangle_0.$$  

Therefore, our goal is to realize the transmission of the information flow of $\alpha$, $\beta_i$ and $\gamma$ as follows

$$\begin{align*}
\langle \alpha | & d - 1\rangle \rightarrow \alpha |p + 1\rangle |\phi\rangle_{n-1,p+1}, \\
\langle \beta_i | & 0\rangle |i\rangle \rightarrow \beta_i |p\rangle |\phi^{(i)}\rangle_{n-1,p}, 1 \leq i \leq d - 2, \\
\langle \gamma & | -1\rangle |0\rangle \rightarrow \gamma |p - 1\rangle |\phi\rangle_{n-1,p-1}.
\end{align*}$$

Considering the walking of the information flow $\alpha$ and $\gamma$, the corresponding solution is the same as the transfer of qutrit. Consequently, our work is gain access to achieving the transmission of the information $\beta_i$ from 0 to $x$. The main idea is to perform the proper coin unitary operator at the original position to break the coherence. The detailed process is shown below.
operators are needed depending on the parity of $n$ is shown in Figure 4. position 0 to position 2 via a five-step discrete-time QW of transferring the four-dimensional quantum state from linearly as the dimensionality increases. An examples of transferring the information flow $\alpha$, $\beta_1$, $\beta_2$ and $\gamma$, respectively.

(i) Keep performing the coin unitary operator $\mathcal{X}^{k(p)}$ at position 0 from second to $(d-1)$-th step and in the $j$-th step $(2 \leq j \leq d - 1)$ it will make $\beta f_{j(p)} |f(j,p)) \rangle$ become $\beta f_{j(p)} |d(p)\rangle$ with $k(p) = \delta_1 (p) + (d - 1) \delta_2 (p), f(j,p) = (d - j) \delta_1 (p) + (j - 1) \delta_2 (p)$ and $d(p) = (d - 1) \delta_1 (p)$.

(ii) While $\beta f_{j(p)} |d(p)\rangle$ arriving at position $p$ one by one, $\mathcal{X}_{d(p)+f_{j(p)}}$ is performed one by one and it will make $\beta f_{j(p)} |d(p)\rangle$ become $\beta f_{j(p)} |f(j,p)\rangle$. As a result, we achieve the transfer of to position $p$ of all the information flow of $\beta_1$, and it needs to meet the condition $n \geq |p| + d - 1$.

Next, we will show the coin flipping operators. The common special coin operators which can achieve the transfer of the information flow $\beta_3$ are shown below.

$$
\mathcal{X}_{0 \rightarrow d-1} : (1,0); \mathcal{X}^{k(p)} : (j,0); \mathcal{X}_{d(p)+f_{j(p)}} : (|p| + j,p);
$$

And two cases are considered below, in order to achieve the transfer of the information flow $\alpha$ and $\gamma$.

(i) While the parity of $n$ and $p$ are the same:

$$
\mathcal{X} : (b^+ + 1, b^+), \mathcal{X}^{d-1} : (b^- + 1, b^-);
$$

(ii) While the parity of $n$ and $x$ are different:

$$
\begin{align*}
\mathcal{X}^2 : (&|a_1| + 2, a_1 + 1), \mathcal{X}^{d-1} : (n - |a_2|, a_1 + 1); \\
\mathcal{X} : (&|a_2| + 2, a_2 - 1), \mathcal{X}^{d-2} : (n - |a_1|, a_2 - 1).
\end{align*}
$$

In summary, in our scheme $2d-1$ or $2d+1$ special coin operators are needed depending on the parity of $n$ and $p$, meaning that the number of the special operators grows linearly as the dimensionality increases. An examples of transferring the four-dimensional quantum state from position 0 to position 2 via a five-step discrete-time QW is shown in Figure 4

IV. ROUTING MULTIDIMENSIONAL ENTANGLED STATE VIA HIGH-DIMENSIONAL QWS

For realizing the transferring of an arbitrary unknown $m$-qudit entangled state, we now consider the $m$-dimensional discrete-time QWs, in which $m$ coins control the walkers to walk in different directions. Our goal is to achieve the transfer from the initial position $(0, \cdots, 0)$ to target position $p = (p_1, \cdots, p_N)$ after $n$-step walks. It also means the unknown multi-particle high-dimensional quantum state can be successfully delivered to the users corresponding to the positions. While the unknown quantum state is entangled, a quantum information process based on entanglement distribution can be carried out successfully after efficient routing. For the sake of simplicity, here we explain it using the example of a 2-dimensional system.

Define the unitary operator of the $i$-th walk-coin system for achieving the transfer to position $p_i$ after $n$-step as $U_i(n, p_i)$ abbreviated as $U_i$ and thus we can obtain

$$
U_i(0) |\Phi_1\rangle = |p_1\rangle |\Phi_1\rangle
$$

$$
U_i(0) |\Phi_2\rangle = |p_2\rangle |\Phi_2\rangle
$$

which will yield

$$
(U_i \otimes U_2)(|0,0\rangle |\Phi_1\rangle |\Phi_2\rangle) = |p_1, p_2\rangle |\Phi_1\rangle |\Phi_2\rangle
$$

where $|\Phi_1\rangle |\Phi_2\rangle$ is a separate state with the form $\sum_{i} |a_i| |\sum_{j} a_j\rangle$. Because $U_i$ is independent on the amplitude of state transferred, we can rewrite as

$$
(U_i \otimes U_2)(|0,0\rangle |\Phi_s\rangle) = |p_1, p_2\rangle |\Phi_s\rangle
$$

where $|\Phi_s\rangle = \sum_{i} |a_i\rangle |\sum_{j} a_j\rangle$ which is an arbitrary two-particle quantum state. It means that an arbitrary quantum state $|\Phi_s\rangle$ can be transferred to the position $(p_1, p_2)$, by performing the unitary operator $U_i \otimes U_2$. The coin flipping operator in position $(x_1, x_2)$ at the $k$-th step will be $C_{k,x_1}^{(n-1)} \otimes C_{k,x_2}^{(2)}$ defined by the formulas (23), (24) and (25), which is a local operation and does not break the entanglement between the two walk-coin systems. Especially, while taking $a_{ij} = 0 (i \neq j)$, the state will turn to be $\sum_{i} |a_i\rangle |\tilde{i}\rangle$, which is more interesting and significant for many quantum information processes.

For example, take $n = 6$, $(p_1, p_2) = (3, -3)$ and the coin operators in the first step will be $C_{1,0}^{(1)} \otimes C_{1,0}^{(2)} = \mathcal{X}_{0 \rightarrow d-1} \otimes \mathcal{X}_{0 \rightarrow d-1}$. An examples of transferring the four-dimensional quantum entanglement state $|\Phi_s\rangle = \sum_{i=0}^{3} |\Phi_s\rangle$ from position $(0,0)$ to position $(3, -3)$ via a six-step discrete-time QW is shown in Figure 4.

Furthermore, for multiple coins in the architecture of a $N$-dimensional discrete-time QW, the efficient routing scheme can be developed based on the way we develop one dimensional system to two-dimensional system. Then it can be easily generalized to the $m$-dimensional case, that is

$$
\otimes_{i=1}^{m} U_i(0, |\Phi_m\rangle) = \otimes_{i=1}^{m} |x_i\rangle |\Phi_{m}^{i}\rangle
$$

where $|\Phi_{m}^{i}\rangle = \sum_{i_1}^{d} \cdots \sum_{i_m}^{d} a_{i_1 \cdots i_m} |i_1 \cdots i_m\rangle$ and it means that an arbitrary $m$-particle quantum state $|\Phi_{m}^{i}\rangle$ can be transferred to the arbitrary position $p$, by performing the unitary operator $\otimes_{i=1}^{m} U_i$.

For every walk-coin system, two kinds of special coin flipping operators are utilized that depends on the target position and the step numbers. The time of routing multidimensional quantum states between arbitrary sites is linear to the distance to be covered. Efficiently routing $m$ coins needs $m(2d + 1)$ special settings, which grows linearly with then number of the coins.
FIG. 5. Perfect state transfer of the coin state $\alpha|00\rangle + \beta_1|11\rangle + \beta_2|22\rangle + \gamma|33\rangle$ from position $(0,0)$ to $(3,-3)$ after six-step walks in the two-dimensional case. The first walker walks along the $x_1$ axis and the second walker walks along the $x_2$ axis. The red, green, purple and blue arrows indicate the directions of the information flow of $\alpha$, $\beta_1$, $\beta_2$ and $\gamma$, respectively.

V. CONCLUSION

In this paper, we have demonstrated that an arbitrary unknown qudit can be transferred with unit fidelity over arbitrary distances. This is a direct application to communicate between two remote registers in a computational architecture using high-dimensional systems, which can construct a bigger Hilbert space to process much more information than the two-dimensional ones. One can perfectly transfer the unknown high-dimensional coin state in a one-dimensional quantum-walk architecture from the initial position to the target position. $2d-1$ or $2d+1$ special coin operators are needed depending on the parity of $n$ and $p$, while leaving the others equal to $I$. Consequently, routing multiqubit entanglement can be realized based on the state transfer on the regular network. Efficiently routing $m$ coins needs $m(2d \pm 1)$ special settings, which grows linearly with then number of the coins. The settings of the scheme are simple and independent of the number of target positions, which makes our protocol feasible with the current experimental technology.

[1] E. T. Campbell, H. Anwar, and D. E. Browne, Physical Review X 2, 041021 (2012).
[2] B. P. Lanyon, M. Barbieri, M. P. Almeida, T. Jennewein, T. C. Ralph, K. J. Resch, G. J. Pryde, J. L. O’Brien, A. Gilchrist, and A. G. White, Nature Physics 5, 134 (2009).
[3] M. Erhard, R. Fickler, M. Krenn, and A. Zeilinger, Light: Science & Applications 7, 17146 (2018).
[4] H. Bechmann-Pasquinucci and W. Tittel, Physical Review A 61, 062308 (2000).
[5] Y. Wang, Z. Hu, B. C. Sanders, and S. Kais, Frontiers in Physics 8, 479 (2020).
[6] F.-X. Wang, W. Chen, Z.-Q. Yin, S. Wang, G.-C. Guo, and Z.-F. Han, Physical Review Applied 11, 024070 (2019).
[7] X.-M. Hu, C. Zhang, B.-H. Liu, Y. Cai, X.-J. Ye, Y. Guo, W.-B. Xing, C.-X. Huang, Y.-F. Huang, C.-F. Li, et al., Physical Review Letters 125, 230501 (2020).
[8] S. Bose, Physical review letters 91, 207901 (2003).
[9] M. Christandl, N. Datta, A. Ekert, and A. J. Landahl, Physical review letters 92, 187902 (2004).
[10] M. Christandl, N. Datta, T. C. Dorlas, A. Ekert, A. Kay, and A. J. Landahl, Physical Review A 71, 032312 (2005).
[11] A. Wojciech, T. Luczak, P. Kurzyński, A. Grudka, T. Gdala, and M. Bednarska, Physical Review A 75, 022330 (2007).
[12] C. Chudzicki and F. W. Strauch, Physical review letters 105, 260501 (2010).
[13] A. Erhard, J. J. Wallman, L. Postler, M. Meth, R. Stricker, E. A. Martinez, P. Schindler, T. Monz, J. Emerson, and R. Blatt, Nature communications 10, 1 (2019).
[14] I. Cohen and K. Mølmer, Physical Review A 98, 030302 (2018).
[15] Q.-C. Sun, Y.-L. Mao, S.-J. Chen, W. Zhang, Y.-F. Jiang, Y.-B. Zhang, W.-J. Zhang, S. Miki, T. Yamashita, H. Terai, et al., Nature Photonics 10, 671 (2016).
[16] P. Mazurek, M. Farkas, A. Grudka, M. Horodecki, and M. Studziński, Physical Review A 101, 042305 (2020).
[17] X.-M. Hu, C.-X. Huang, Y.-B. Sheng, L. Zhou, B.-H. Liu, Y. Guo, C. Zhang, W.-B. Xing, Y.-F. Huang, C.-F. Li, et al., Physical Review Letters 126, 010503 (2021).
[18] K. Wang, N. Wu, P. Xu, and F. Song, Journal of Physics A: Mathematical and Theoretical 50, 505303 (2017).
[19] Y. Aharonov, L. Davidovich, and N. Zagury, Physical Review A 48, 1687 (1993).
[20] N. Shenvi, J. Kempe, and K. B. Whaley, Physical Review A 67, 052307 (2003).
[21] A. Ambainis, J. Kempe, and A. Rivosh, in Proceedings of the sixteenth annual ACM-SIAM symposium on Discrete algorithms (Society for Industrial and Applied Mathematics, 2005) pp. 1099–1108.
[22] A. Tuls, Physical Review A 78, 012310 (2008).
[23] A. Ambainis, SIAM Journal on Computing 37, 210 (2007).
[24] J. K. Gamble, M. Friesen, D. Zhou, R. Joynt, and S. Coppersmith, Physical Review A 81, 052313 (2010).
[25] A. M. Childs, Physical review letters 102, 180501 (2009).
[26] N. B. Lovett, S. Cooper, M. Everitt, M. Trevers, and V. Kendon, Physical Review A 81, 042330 (2010).
[27] A. M. Childs, D. Gosset, and Z. Webb, Science 339, 791 (2013).
[28] P. Kurzyński and A. Wojcik, Physical Review A 83, 062315 (2011).
[29] X. Zhan, H. Qin, Z.-h. Bian, J. Li, and P. Xue, Physical Review A 90, 012331 (2014).
[30] İ. Yalçınkaya and Z. Gedik, Journal of Physics A: Mathematical and Theoretical 48, 225302 (2015).
[31] W. Qin, C. Wang, and G. L. Long, Physical Review A 87, 012339 (2013).
[32] Y.-M. Di and H.-R. Wei, Physical Review A 87, 012325 (2013).