Topological Theory of Classical and Quantum Phase Transition

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We presented the topological current of Ehrenfest definition of phase transition. It is shown that different topology of the configuration space corresponds to different phase transition, it is marked by the Euler number of the interaction potential. The two phases separated by the coexistence curve is assigned with different winding numbers of opposite sign. We also found an universal equation of coexistence curve, from which one can arrive the phase diagram of any order classical and quantum phase transition. The topological quantum phase transition theory is established, and is applied to the Bose-Hubbard model, the phase diagram of the first order quantum PT is in agreement with recent progress.

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A. Introduction

A phase transition (PT) brings about a sudden change of the macroscopic properties of a system while smoothly varying a physical parameter. While in topology, this sudden change is always marked by Euler number. For example, when the radii $R$ of the mouth of a rubber bag smoothly shrinks to nothing, the Euler number jumps from 1 (Euler number of a rubber bag) to 2 (Euler number of a rubber sphere) at $R = 0$. One can not help asking: does PT bear a topological origin?

On the origin of PT, Caiani et al proposed the topological hypothesis \([8,9]\): the PT is determined by a change in the topology of the configuration space, and the loss of analyticity in the thermodynamic observables is nothing but a consequence of such topological change. This hypothesis has received direct evident support from numerical and analytical computation of the Euler characteristic in \(\phi^4\) lattice model \([3]\), mean-field XY model \([4]\) and k-trigonometric model \([5]\). It is also supported by the Bishop-Peyard model of DNA thermal denaturation \([6]\). Recently, it is proposed that the occurrence of a PT is connected to certain properties of the potential energy \(V\). Franzosi, Pettini and Spinelli provided a rigorous theoretical support to this topological hypothesis \([8,9]\).

In this letter, we shall start from a more practical point of view to study this topological hypothesis of PT. It will be shown that the topological number of the critical points of the microscopic interaction potential naturally appeared in the topological current of Euler number. Furthermore, based on Ehrenfest definition of PT, we found an universal equation of coexistence curve, from which one can arrive the phase diagram of any order PT. This equation holds both for classical and quantum PT. From our definition of quantum PT, we derived the phase diagram of superfluid-Mott-insulating quantum PT in the Bose-Hubbard model, it is in perfect agreement with the well known phase diagram.

B. Topological theory of classical phase transition

For a standard Hamiltonian system, which is described by \(H = \sum_{i=1}^{N} \frac{1}{2} \mathbf{p}_i^2 + V(q_1, ..., q_N)\), the Helmholtz free energy reads \(F(\beta) = -(2\beta)^{-1} \text{Log}(\pi/\beta) - f(\beta)/\beta\), where \(\beta = 1/T\) is the inverse of temperature. The first part of \(F(\beta)\) comes from the kinetic energy and the second term \(f(\beta) = \frac{1}{\beta} \text{Log} \int d^N q \exp[-\beta V(q)]\) comes from the microscopic interaction potential \(V(q)\).

As we know, a PT happens when we change the temperature or pressure of the system. In a more practical approach, we introduce two extra microscopic practical parameter \(\{\gamma_1, \gamma_2\}\), which represent pressure, external magnetic field, electric field, and so on. The interaction potential \(\{\gamma_1, \gamma_2\}\) is generalized to \(V(\vec{q}, \gamma_1, \gamma_2)\). In the frame work of Ehrenfest definition, the free energy \(F(\gamma_1, \gamma_2)\) is utterly depend on the two extra parameters.

According to Ehrenfest definition \([10]\) of a PT, when the system transforms from phase A to phase B, the first order phase transition requires that the free energy \(F(\gamma_1, \gamma_2)\) is continuous at the transition, i.e., \(\delta F = F^B(\gamma_1, \gamma_2) - F^A(\gamma_1, \gamma_2) = 0\), but

\[
\partial_{\gamma_1} \delta F \neq 0, \quad \partial_{\gamma_2} \delta F \neq 0, \quad (1)
\]

where \(\partial_{\gamma_1} = \frac{\partial}{\partial \gamma_1}\), \(\partial_{\gamma_2} = \frac{\partial}{\partial \gamma_2}\). For the second order
transition, it requires
\[ \delta F = 0, \ \partial_{\gamma_1} \delta F = 0, \ \partial_{\gamma_2} \delta F = 0, \]
\[ \partial_{\gamma_1}^2 \delta F \neq 0, \ \partial_{\gamma_1} \partial_{\gamma_2} \delta F \neq 0, \ \partial_{\gamma_2}^2 \delta F \neq 0. \] (2)

The pth-order PT is characterized by a discontinuity in the pth derivative of the free energy. Ehrenfest definition suggests that the change of the free energy between the two phases \( \delta F = F^B(\gamma_1, \gamma_2) - F^A(\gamma_1, \gamma_2) \) plays an essential role in defining the order of PT. So we choose the \( \delta F \) as the fundamental order parameter field.

The order parameter of the pth-order PT, \( \vec{\phi} \), can be chosen as the \((p - 1)\)th derivative of \( \delta F \), i.e., \( \phi^1 = \partial_{\gamma_1}^p \delta F, \phi^2 = \partial_{\gamma_2}^p \delta F \). In order to investigate the topological properties of a PT, we map the order parameter into a two dimensional sphere by introducing an unit vector field \( \vec{n} \) with \( n^a = \phi^a / ||\phi|| \), \( n^a n^a = 1 \), where \( ||\phi|| = \sqrt{\phi^a \phi^a} \) (\( a = 1, 2 \)). It can be proved that the Gaussian curvature on the two dimensional extra-space can be expressed in terms of \( \vec{n} \):
\[ J = \sum_{i,j,a,b} \epsilon^{ij} \epsilon_{ab} \partial_a n^a \partial_b n^b / ||\phi||^4 \] (3)

In light of Duan’s \( \vec{\phi} \)-mapping topological current theory [11], using \( \partial_{\phi} \phi^a / ||\phi|| = \phi^a / ||\phi|| \) and the Green function relation in \( \phi \)-space: \( \partial_{\phi} \partial_{\phi} \ln ||\phi|| = 2 \pi \delta^2(\phi), (\partial_a = \partial / \partial \phi^a) \), we can prove that
\[ J = \delta^2(\phi) D(\phi / \gamma) = \delta^2(\phi) \{ \phi^1, \phi^2 \}, \] (4)
where \( D(\phi / \gamma) = \frac{1}{2} \epsilon^{ijk} \epsilon_{ab} \partial_j \phi^a \partial_k \phi^b \) is the Jacobian vector, in the extra-two dimensional space, it is actually the Poisson bracket of \( \phi^1 \) and \( \phi^2 \), \( \{ \phi^1, \phi^2 \} = \sum_i (\partial_i \phi^1 \partial_i \phi^2 - \partial_i \phi^2 \partial_i \phi^1) / ||\phi||^4 \).

In Ehrenfest definition, the \((p - 1)\)th order PT requires that \( \phi^a \neq 0, (a = 1, 2) \), the pth order PT requires \( \phi^a = 0, (a = 1, 2) \). Eq. (4) suggests that the topological current of Gaussian curvature: \( J = 0 \), iff \( \vec{\phi} \neq 0 \); \( J \neq 0 \), iff \( \vec{\phi} = 0 \). So this topological current vanishes for the \((p - 1)\)th PT, and only exist for the pth order PT. In other words, the Euler number of the \((p - 1)\)th PT is zero, and it is not zero for the pth order PT.

Therefore the topological information of the system focused on the zero points of the vector field \( \vec{\phi} \). In fact, zero points of the vector field \( \vec{\phi} \) just corresponds to the extremal point of the variation of free energy, since the vector field is defined by the gradient of \( \delta F \).

For a classical system, when the microscopic interaction potential \( V(\vec{q}) \) depends on the two microscopic parameters \( \{ \gamma_1, \gamma_2 \} \), the Helmoltz free energy reads \( F(\gamma_1, \gamma_2) = - (2 \beta)^{-1} \log (\pi / 2) - f(\beta, \gamma_1, \gamma_2) / \beta \), with \( f(\beta, \gamma_1, \gamma_2) = \frac{1}{\beta} \log \int d^n q \exp[- \beta V(\vec{q}, \gamma_1, \gamma_2)] \). Then the gradient field of free energy is
\[ \frac{\partial F}{\partial \gamma_i} = \frac{1}{N} \int d^n q \exp[- \beta V(\vec{q})] \frac{\partial V}{\partial \gamma_i} = \frac{1}{N} (\partial V / \partial \gamma_i). \] (5)

Now we see the topological current relies on the critical points of \( V \) in configuration space. Therefore it is the extremal point of the interaction potential that decides a PT. Eq. (5) also suggests that the higher order is decided by powers of a few basic ingredient, such as \( \partial V / \partial \gamma_1, \partial^2 V / \partial \gamma_1 \partial \gamma_j, \partial^3 V / \partial \gamma_1 \partial \gamma_j \partial \gamma_k \), and so on.

The implicit function theory shows, under the regular condition \( D(\phi / \gamma) \neq 0 \), i.e., \( \{ \phi^1, \phi^2 \} \neq 0 \), we can solve the equations \( \phi = 0 \) and derive n isolated solutions: \( \vec{z}_k = (\gamma_k, \vec{q}_k), k = 1, 2, \ldots, n \). According to the \( \delta \)-function theory [13], one can expand \( \delta(\vec{\phi}) \) at these solutions, the topological current of Gaussian curvature is rewritten as
\[ J = \sum_{k=1}^{n} \beta_k \eta_k \partial(\gamma - \gamma_k) \delta(\gamma - \gamma_k), \] (6)
where \( \beta_k = |W_k| \) is the Hopf index, \( W_k \) is the winding number around \( z_k \), \( \eta_k = \text{sign} D(\phi / \gamma) \) is the winding index in Morse theory. Its absolute value \( \beta_k \) measures the strength of the phase. \( \eta_k = \text{sign} \{ \phi^1, \phi^2 \} = \pm 1 \) represents different phases. For example, in the phase diagram of ultracold atoms in an optical lattice (FIG. 1), in the area circulated by the curve, \( \eta_k = \text{sign} \{ \phi^1, \phi^2 \} = 1 \), represents the Mott-Insulator phase, while outside the curve, \( \eta_k = \text{sign} \{ \phi^1, \phi^2 \} = -1 \), it represents the superfluid phase. On the curve, \( \{ \phi^1, \phi^2 \} = 0 \), that is where a quantum PT takes place. Loosely speaking, the PT which happened at the critical point with higher winding number is obviously different from the process happened at the point with lower winding number.

For a more sophisticated case, when there are many phases in the phase diagram, the total winding number of different phases, \( \chi = \int J d^2 q = \sum_{k=1}^{n} W_k \), is the Euler number. There would be a substantial change of topology of the configuration space, when the phase diagram jumps to a higher order PT or a lower order PT. Such
as, the manifold can be jumps from a sphere to a torus, and to a torus with many holes, the Euler number would jumps from $\chi = 2$ to $\chi = 0$, then to $\chi = 2(1 - h)$ with $h$ as the number of holes of the torus. During these process, the phase transition jumps from first order to the second order, and to the $p$th order.

C. The universal equation of coexistence curve

The most important thing we concerns about is the phase diagram of a PT. If we have found the coexistence curve equation, the phase diagram can be derived by solving this equation. Our topological theory provide us an universal coexistence curve equation:

$$\{\phi^1, \phi^2\} = 0. \quad (7)$$

It is an unification of the special coexistence equations of different order PT. As all know, in quantum mechanics, if two operator $\hat{A}$ and $\hat{B}$ commutate with each other, i.e., $[\hat{A}, \hat{B}] = 0$, they share the same eigenfunction. So Eq. (7) reflects the same properties of different phase operators $\phi^1$ and $\phi^2$.

In the bifurcation theory of topological current\[14\], the topological current will split into several branches, when $D(\phi/q) = 0$, i.e., $\{\phi^1, \phi^2\} = 0$. It is rather surprising that the coexistence curve in the phase diagram happened to be the bifurcation curve of topological current.

We first take the second order PT for example to demonstrate this. The order parameter of the second order PT is $\phi^1 = \partial_T \delta F$ and $\phi^2 = \partial_P \delta F$, substituting them into the Jacobian vector

$$\{\phi^1, \phi^2\} = \frac{\partial \phi^1}{\partial T} \frac{\partial \phi^2}{\partial P} - \frac{\partial \phi^1}{\partial P} \frac{\partial \phi^2}{\partial T} = 0, \quad (8)$$

and using the relations

$$\partial_T \partial_T \delta F = \frac{C_T^B - C_T^A}{T}, \quad \partial_P \partial_P \delta F = V(\kappa_T^A - \kappa_T^B),$$

$$\partial_P \partial_T \delta F = V(\alpha^B - \alpha^A), \quad (9)$$

we arrive

$$D(\phi/q) = \frac{V}{T}(C_T^B - C_T^A)(\kappa_T^B - \kappa_T^A) - (V\alpha^B - V\alpha^A). \quad (10)$$

Recalling the Ehrenfest equations

$$\frac{dP}{dT} = \frac{\alpha^B - \alpha^A}{k_B^T - k_A^T}, \quad \frac{dT}{dP} = \frac{C_T^B - C_T^A}{TV(\alpha^B - \alpha^A)}. \quad (11)$$

it is easy to verify that the equation above is in consistent with the bifurcation condition Eq. (10). So the bifurcation equation $D(\phi/q) = 0$ is an equivalent expression of the coexistence curve equation.

For the first order PT, we chose the order parameter as $\phi = \partial^0 \delta F$, here ‘0’ means no derivative of the free energy. The generalized Jacobian vector of the first order PT with $\phi = \partial^0 \delta F$ is given by

$$D(\phi/q) = (\frac{\partial F^B}{\partial T} - \frac{\partial F^A}{\partial T}) - (\frac{\partial F^B}{\partial P} - \frac{\partial F^A}{\partial P}) = 0, \quad (12)$$

and using the relations

$\frac{\partial F}{\partial T} = -S$ and $\frac{\partial F}{\partial P} = V$, and considering $D(\phi/q) = 0$, we have

$$\frac{dP}{dT} = \frac{(S^B - S^A)}{(V^B - V^A)}. \quad (13)$$

This is just the famous Clapeyron equation.

The coexistence equation $\{\phi^1, \phi^2\} = 0$ also holds for a higher-order PT. We consider a system whose free energy is a function of temperature $T$ and magnetic field $B$, then the Clausius-Clapeyron equation becomes $dB/dT = -\Delta S/\Delta M$. If the entropy and the magnetization are continuous across the phase boundary, the transition is of higher order. For the $p$th order PT, the vector field is chosen as the $(p - 1)$th derivative of $\delta F$, $\phi^1 = \partial_T^{p-1} \delta F, \phi^2 = \partial_B^{p-1} \delta F$. Substituting $(\phi^1, \phi^2)$ into Eq. (10), we arrive

$$D(\phi/q) = \frac{\partial p \delta F \partial T \delta F}{\partial T^p - \partial B^p} - \frac{\partial p^{p-1} \delta F}{\partial T \partial B^{-p-1}} = 0. \quad (14)$$

Considering the heat capacity $\frac{\partial^2 F}{\partial T^2} = -\frac{C_p}{T}$ and the susceptibility $\frac{\partial^2 F}{\partial B^2} = \chi$, the bifurcation condition $D(\phi/q) = 0$ is rewritten as

$$\left[\frac{dP}{dT}\right]^p = (-1)^p \frac{\Delta \partial^{p-2}C/\partial T^{p-2}}{T^2 \Delta \partial^{p-2} \chi/\partial B^{p-2}}. \quad (15)$$

This is in excellent agreement with Kumar’s results[15].

So different phases coexists on the the bifurcation curve, when PT occurs, the system bifurcate into several phases, each of them is assigned with a winding number, but the total topological number is conserved, it is a constant which relies on the topology of configuration space.

In fact, the Jacobian vector also provide us a new spectacles to the scaling law[16]. As all known, at the critical temperature, the correlation length $\xi$ becomes infinite, from which the divergence of the physical parameters arises.\[17\] The topological current $\xi$ is invariant under renormalization group transformation $q' = R(q)$. The new topological current $J'$ expressed by the new parameter $q' = (q^{1'}, q^{2'})$ share the same fixed point with $J, J' = J = \infty$. This is an open question which need further study.
D. Topological quantum phase transition theory

So far there is still no such a general topological theory of quantum PT, in the frame work of this paper, it is easy to establish an unified topological quantum PT theory.

The usual thermodynamic phase transition at finite temperature is driven by thermal fluctuations, as temperature is lowered, the thermal fluctuations are suppressed. But the quantum fluctuations still exist and play a vital role in driving the transition from one phase to another. At the absolute zero of temperature, quantum PT can be accomplished by changing not the temperature, but some parameter in the Hamiltonian of the system. This parameter might be the charging energy in Josephson-junction arrays (which controls their superconductor-insulator transition), the magnetic field of the variation of the ground state energy. In fact, they are equivalent. we considering a hermitian Hamiltonian $H(t) = H(\gamma(t))$ which is time-dependent through a set of parameters $\gamma_i(t), i = 1, ..., n$. Suppose, that for any fixed $t$ on the time interval $[0, T]$, the spectrum of $H$ is non-degenerate: $H(\gamma)|n(\gamma)\rangle = E_n(\gamma)|n(\gamma)\rangle$, with $n(\gamma)$ as an eigenstate of Hamiltonian.

As seen above, the gradient field of the free energy plays the keys role in a classical PT, here we also choose the order parameter of quantum PT as

$$E_g = \frac{i}{T} \ln Z, \quad Z = \int D\phi e^{i \int dx dt L(\phi)}. \quad (16)$$

It plays a similar role as free energy

$$F = -\frac{1}{\beta} \ln Z, \quad Z = Tr \ e^{-\beta H(\gamma(t))}. \quad (17)$$

In fact, they are equivalent. we considering a hermitian Hamiltonian $H(t) = H(\gamma(t))$ which is time-dependent through a set of parameters $\gamma_i(t), i = 1, ..., n$. Suppose, that for any fixed $t$ on the time interval $[0, T]$, the spectrum of $H$ is non-degenerate: $H(\gamma)|n(\gamma)\rangle = E_n(\gamma)|n(\gamma)\rangle$, with $n(\gamma)$ as an eigenstate of Hamiltonian.

As seen above, the gradient field of the free energy plays the keys role in a classical PT, here we also choose the order parameter of quantum PT as

$$\frac{\partial F}{\partial \gamma_i} = -\frac{1}{\beta} \left( \frac{Tr e^{-\beta H(\gamma(t))} \partial H}{\partial \gamma_i} \right) = \left( \frac{\partial H}{\partial \gamma_i} \right). \quad (18)$$

In mind of Feynman-Hellman theorem: $\langle n(\gamma) | \frac{\partial H(\gamma)}{\partial \gamma_i} | n(\gamma) \rangle = \frac{\delta E_g(\gamma)}{\delta \gamma_i}$, one can sees that the order parameter of quantum PT is actually the gradient field of the variation of the ground state energy $\delta E_g$.

In analogy with Ehrenfest definition, the first order quantum PT, in which the system transforms from phase A to phase B, requires that the ground state energy $E_g(\gamma_1, \gamma_2)$ is continuous at the transition, $\delta E_g = E_g^{B}(\gamma_1, \gamma_2) - E_g^{A}(\gamma_1, \gamma_2) = 0$, and

$$\partial_{\gamma_1} \delta E_g \neq 0, \quad \partial_{\gamma_2} \delta E_g \neq 0, \quad (19)$$

where $\partial_{\gamma_i} = \frac{\partial}{\partial \gamma_i}$, $\partial_{\gamma_2} = \frac{\partial}{\partial \gamma_2}$. For the second order quantum PT, it requires

$$\delta E_g = 0, \quad \partial_{\gamma_1} \delta E_g = 0, \quad \partial_{\gamma_2} \delta E_g = 0,$$

$$\partial_{\gamma_1}^2 \delta E_g \neq 0, \quad \partial_{\gamma_1} \partial_{\gamma_2} \delta E_g \neq 0, \quad \partial_{\gamma_2}^2 \delta E_g \neq 0. \quad (20)$$

The $p$th-order quantum PT is characterized by a discontinuity in the $p$th derivative of difference of ground state energy $\delta E_g$:

$$\partial_{\gamma_1}^{p-1} \delta E_g = 0, \quad \partial_{\gamma_1}^p \partial_{\gamma_2}^{p-m} \delta E_g = 0, (m = 1, 2, ..., p - 1),$$

$$\partial_{\gamma_2}^{p} \delta E_g = 0.$$

In the topological quantum PT theory, we can chose the vector order parameter $\phi$ as the $(p - 1)$th derivative of $\delta E_g$:

$$\phi^1 = \partial_{\gamma_1}^{-1} \delta E_g, \quad \phi^2 = \partial_{\gamma_2}^{-1} \delta E_g. \quad (22)$$

Then the phase diagram can be obtained from the coexistence curve equation,

$$\{\phi^1, \phi^2\} = \frac{\partial \phi^1}{\partial \gamma_1} \frac{\partial \phi^2}{\partial \gamma_2} - \frac{\partial \phi^1}{\partial \gamma_2} \frac{\partial \phi^2}{\partial \gamma_1} = 0. \quad (23)$$

Here $\gamma_i$ is the physical parameters of a quantum system.

A gas of ultra cold atoms in an optical lattice has provided us a very good experimental observation of superfluid-Mott-insulator phase transition. In gapped quantum spin systems, an explicit divergence of the entanglement length appears in the ground states at a topological phase transition. Using Matrix Products States, it was found that the localizable entanglement possesses a discontinuity at its first derivative at a Kosterlitz-Thouless phase transition. In the following, we shall establish a complete theory of topological quantum PT.

This system is described by the Bose-Hubbard model

$$H = -J \sum_{\langle i, j \rangle} (a_i^\dagger a_j + a_j^\dagger a_i) + \sum_i \varepsilon_i n_i + \frac{U}{2} \sum_i n_i(n_i - 1), \quad (24)$$

the last term $U$ corresponding to the on site repulsion between atoms, while the first term $J$ describes the tunnelling of atoms between neighboring sites.

Using mean-field approaches, the ground state energy of Bose-Hubbard model can be generally expressed as the functional of $\varepsilon_i$, $J$ and $U$, i.e., $E_g = E_g(\varepsilon_i, J, U)$. 
Following perturbation theory up to the second order, the variation of ground state energy, \( \delta E_g \) is
\[
\delta E_g = \left\{ \frac{g}{U(g-1)-J} + \frac{g+1}{J-Ug} + 1 \right\}, \tag{25}
\]
From our definition of the first order phase transition, the boundary between the superfluid and the Mott insulator phases should be decided by the equation \( \delta E_g = 0 \). We have plotted the phase diagram (FIG. 1). This phase diagram is in perfect agreement with the well-known phase diagram of superfluid-Mott-insulator phase transition. It is also the same as the phase boundary obtained from the minimizing the free energy, when treat the hopping term as perturbation. So most of the recent phase diagram of the quantum phase transition are focused on the first order.

As for the \( p \)th-order quantum PT, it is characterized by a discontinuity in the \( p \)th derivative of the variation of ground state energy. The order parameter field of the \( p \)th order Quantum PT can be chosen as the \((p-1)\)th derivative of \( \delta E_g \) with respect to \( U \) and \( J \), i.e., \( \phi^1 = \frac{\partial}{\partial J}^{-1} \delta E_g \), \( \phi^2 = \frac{\partial}{\partial U}^{-1} \delta E_g \).

The equation of coexistence curve is given by \( \{ \phi^1, \phi^2 \} = D(g) = 0 \),
\[
\{ \phi^1, \phi^2 \} = \frac{\partial^p \delta E_g}{\partial J^p} \frac{\partial^p \delta E_g}{\partial U^p} - \frac{\partial^p \delta E_g}{\partial U \partial J^p} \frac{\partial^p \delta E_g}{\partial J \partial U^p} = 0 \tag{26}
\]
here we have taken \( \gamma_1 = U \), \( \gamma_2 = J \). From this equation, one can arrive the phase diagram of the \( p \)th order quantum PT from the equation above, and find some new quantum phases.

E. conclusion

In summary, when a PT jumps from lower order to higher order, the Euler number of the extra configuration space would jump from one integer to another. In other word, different order of PT corresponds to different manifold with different Euler number.

We have found an universal equation of coexistence curve equation, \( \{ \phi^1, \phi^2 \} = 0 \), from which one can derive the phase diagram of any order. In the phase diagram, different phase are assigned with different winding numbers, the sum of these winding numbers is Euler number which bifurcated on the coexistence curve.

In analogy with Ehrenfest definition of classical PT, we proposed the higher order quantum PT, in which the ground state energy correction \( \delta E_g \) plays the key role. The universal equation of coexistence curve equation, \( \{ \phi^1, \phi^2 \} = 0 \), also holds for quantum PT.

Now we proposed the general process to a obtain the phase diagram of \( p \)th order quantum PT. First, we calculate the ground state energy correction using perturbation theory. Then, substitute \( \delta E_g \) into \( \{ \phi^1, \phi^2 \} = 0 \), and solve this equation. This method is an universal theory, it can be generalized to a great variety of classical and quantum many body system, such as t-J model, Hubbard model, and so on. It can help us to predict some new quantum PT.

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