A Tight \((1.5 + \epsilon)\)-Approximation for
Unsplittable Capacitated Vehicle Routing on Trees

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Abstract

In the unsplittable capacitated vehicle routing problem (UCVRP) on trees, we are given a rooted tree with edge weights and a subset of vertices of the tree called terminals. Each terminal is associated with a positive demand between 0 and 1. The goal is to find a minimum length collection of tours starting and ending at the root of the tree such that the demand of each terminal is covered by a single tour (i.e., the demand cannot be split), and the total demand of the terminals in each tour does not exceed the capacity of 1.

For the special case when all terminals have equal demands, a long line of research culminated in a quasi-polynomial time approximation scheme [Jayaprakash and Salavatipour, SODA 2022] and a polynomial time approximation scheme [Mathieu and Zhou, ICALP 2022].

In this work, we study the general case when the terminals have arbitrary demands. Our main contribution is a polynomial time \((1.5 + \epsilon)\)-approximation algorithm for the UCVRP on trees. This is the first improvement upon the 2-approximation algorithm more than 30 years ago [Labbé, Laporte, and Mercure, Operations Research, 1991]. Our approximation ratio is essentially best possible, since it is \(\text{NP}\)-hard to approximate the UCVRP on trees to better than a 1.5 factor.

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1 Introduction

In the unsplittable capacitated vehicle routing problem (UCVRP) on trees, we are given a rooted tree with edge weights and a subset of vertices of the tree called terminals. Each terminal is associated with a positive demand between 0 and 1. The root of the tree is called the depot. The goal is to find a minimum length collection of tours starting and ending at the depot such that the demand of each terminal is covered by a single tour (i.e., the demand cannot be split), and the total demand of the terminals in each tour does not exceed the capacity of 1.

The UCVRP on trees has been well studied in the special setting when all terminals have equal demands: Hamaguchi and Katoh [HK98] gave a polynomial time 1.5-approximation; the approximation ratio was improved to 1.35078 by Asano, Katoh, and Kawashima [AKK01] and was further reduced to 4/3 by Becker [Bec18]; Becker and Paul [BP19] gave a bicriteria polynomial time approximation scheme; and very recently, Jayaprakash and Salavatipour [JS22] gave a quasi-polynomial time approximation scheme, based on which Mathieu and Zhou [MZ22] designed a polynomial time approximation scheme.

In this work, we study the UCVRP on trees in the general setting when the terminals have arbitrary demands. Our main contribution is a polynomial time $(1 + \epsilon)$-approximation algorithm (Theorem 1). This is the first improvement upon the 2-approximation algorithm of Labbé, Laporte, and Mercure [LLM91] more than 30 years ago. Our approximation ratio is essentially best possible, since it is NP-hard to approximate the UCVRP on trees to better than a 1.5 factor [GW81].

Theorem 1. For any $\epsilon > 0$, there is a polynomial time $(1 + \epsilon)$-approximation algorithm for the unsplittable capacitated vehicle routing problem on trees.

The UCVRP on trees generalizes the UCVRP on paths. The latter problem has been studied extensively due to its applications in scheduling, see Section 1.1. Previously, the best approximation ratio for the UCVRP on paths was 1.6 due to Wu and Lu [WL20]. As an immediate corollary of Theorem 1, we obtain a polynomial time $(1 + \epsilon)$-approximation algorithm for the UCVRP on paths. This ratio is essentially best possible, since it is NP-hard to approximate the UCVRP on paths to better than a 1.5 factor (Appendix A.1).

1.1 Related Work

UCVRP on paths. The UCVRP on paths is equivalent to the scheduling problem of minimizing the makespan on a single batch processing machine with non-identical job sizes [Uzs94]. Many heuristics have been proposed and evaluated empirically, e.g., [Uzs94, DDF02, MDC04, DMS06, KKJ06, PKK10, CDH11, AS15, Mut20].

The UCVRP on paths has also been studied in special cases. For example, in the special case when the optimal value is at least $\Omega(1/\epsilon^6)$ times the maximum distance between any terminal and the depot, asymptotic polynomial time approximation schemes are known [DMM10, Rot12, CGHH13]. In contrast, the algorithm in Theorem 1 applies to any path instance (and more generally any tree instance).

UCVRP on general metrics. The first constant-factor approximation algorithm for the UCVRP on general metrics is due to Altinkemer and Gavish [AG87]. The approximation ratio was only recently improved in work by Blauth, Traub, and Vygen [BTV21], and then further by Friggstad,

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1Up to scaling, the equal demand setting is equivalent to the unit demand version of the capacitated vehicle routing problem in which each terminal has unit demand, and the capacity of each tour is a positive integer $k$.

2The UCVRP on paths is called the train delivery problem in [DMM10, Rot12, CGHH13].
Mousavi, Rahgoshay, and Salavatipour [FMRS22], so that the best-to-date approximation ratio stands at roughly 3.194 [FMRS22].

2 Overview of Techniques

At a high level, our approach is to modify the problem and add enough structural constraints so that the structured problem contains a \((1.5 + O(\epsilon))\)-approximate solution and can be solved in polynomial time by dynamic programming.

2.1 Preprocessing

We start by some preprocessing as in [MZ22]. We assume without loss of generality that every vertex in the tree has two children, and the terminals are the leaf vertices of the tree [MZ22]. Furthermore, we assume that the tree has bounded distances (Section 3.1). Next, we decompose the tree into components (Fig. 1 and Section 3.2).

![Figure 1: Decomposition of the tree into components](image)

Figure 1: Decomposition of the tree into components. Figure extracted from [MZ22]. Each brown triangle represents a component. Each component has a root vertex and at most one exit vertex.

2.2 Solutions Within Each Component

A significant difficulty is to compute solutions within each component. It would be natural to attempt to extend the approach in the setting when all terminals have equal demands [MZ22]. In that setting, the demands of the subtours in each component are among a polynomial number of values; since the component is visited by a constant number of tours in a near-optimal solution, that solution inside the component can be computed exactly in polynomial time using a simple dynamic program. However, when the terminals have arbitrary demands, the demands of the subtours in each component might be among an exponential number of values.\(^4\) Indeed, unless \(P = NP\), we cannot compute in polynomial time a better-than-1.5 approximate solution inside a component, since that problem generalizes the bin packing problem (Appendix A.2).

To compute in polynomial time good approximate solutions within each component, at a high level, we simplify the solution structure in each component, so that the demands of the subtours in that component are among a constant \(O(\epsilon)\) number of values, while increasing the cost of the solution by at most a multiplicative factor \(1.5 + O(\epsilon)\).

Where does the 1.5 factor come from? Intuitively, our construction creates an additional subtour to cover a selected subset of terminals, charging each edge on that subtour to two existing subtours using that edge, thus adding a 0.5 factor to the cost.

\(^3\)The demand of a subtour is the total demand of the terminals visited by that subtour.

\(^4\)For example, consider a component that is a star graph with \(\Theta(n)\) leaves, where the \(i^{th}\) leaf has demand \(1/2^i\).
(a) Decomposition of a component into blocks. The orange nodes represent the big terminals in the component. The black nodes represent the root and the exit vertices of the component (defined in Lemma 6). The gray nodes are the branching vertices in the subtree spanning the orange and the black nodes. Splitting the component at the orange, the black, and the gray nodes results in a set of blocks, represented by green triangles. Each block has a root vertex and at most one exit vertex. See Section 4.1.

(b) Decomposition of a block into clusters. The green triangle represents a block. Each blue triangle represents a cluster. Each cluster has a root vertex and at most one exit vertex. A cluster is passing if it has an exit vertex, and is ending otherwise. Each passing cluster has a spine (dashed). See Section 4.2.

(c) Decomposition of a passing cluster into cells. The blue triangle represents a passing cluster. Removing the thick edges from the cluster results in a set of at most $1/\epsilon$ cells. Each red triangle represents a cell. Each of those cells has a root vertex, an exit vertex, and a spine (dashed). See Section 4.3.

Figure 2: Three-level decomposition of a component.
In the rest of this section, we explain our approach in more details.

### 2.2.1 Multi-Level Decomposition (Section 4)

First, we distinguish big and small terminals depending on their demands. The number of big terminals in a component is $O_\epsilon(1)$. Next, we partition the small terminals of a component into $O_\epsilon(1)$ parts using a multi-level decomposition as follows. In the first level, a component is decomposed into $O_\epsilon(1)$ blocks so that all terminals strictly inside a block are small; see Fig. 2a and Section 4.1. In the second level, each block is decomposed into $O_\epsilon(1)$ clusters so that the overall demand of each cluster is roughly an $\epsilon$ fraction of the demand of a component; see Fig. 2b and Section 4.2. Intuitively, the clusters are such that, if we assign the small terminals of each cluster to a single subtour, the subtour capacities would be violated only slightly. We define the spine of a cluster to be the path traversing that cluster. In the third level, each cluster is decomposed into $O_\epsilon(1)$ cells so that the spine of each cell is roughly an $\epsilon$ fraction of the spine of a cluster; see Fig. 2c and Section 4.3.

![Figure 3: Relation of the multiple levels in the decomposition.](image)

### 2.2.2 Simplifying the Local Solution (Section 5)

The main technical contribution in this paper is the Local Theorem (Theorem 12), which simplifies a local solution inside a component so that, in each cell, a single subtour visits all small terminals, while increasing the cost of the local solution by at most a multiplicative factor $1.5 + O(\epsilon)$. The Local Theorem builds upon techniques from [MZ22] together with substantial new ideas.

A first attempt is to combine all subtours of a cluster into a single subtour. However, there are two obstacles. First, the resulting subtours in the component are no longer connected; reconnecting those subtours would require including the spines of the clusters, which would be too expensive. Secondly, the resulting subtours in the component may exceed their capacities, so an extra cost is needed to reduce the demands of those subtours; in the equal demand setting [MZ22], that extra cost is at most an $\epsilon$ fraction of the solution cost, but this is no longer the case in the arbitrary demand setting.

We overcome those obstacles thanks to the decomposition of a cluster into cells. In the analysis, we introduce the technical concept of threshold cells (Fig. 4a), and we ensure that each cluster contains at most one threshold cell. A crucial step is to include the spine subtour of the threshold cell into the solution (Fig. 4b). This enables us to reassign all small terminals of each cell to a single subtour, without losing connectivity, while only slightly violating the subtour capacities.

To reduce the demand of each subtour exceeding capacity, we select some cells from that subtour, and we remove all pieces in that subtour belonging to those cells. We show that each removed piece is connected to the root through at least two subtours in the solution (Lemma 18, see Fig. 5). That property is a main technical novelty in this paper. It enables us to reconnect all removed pieces
with an extra cost of at most half of the solution cost (Lemma 19), hence an approximation ratio of $1.5 + O(\epsilon)$.

### 2.3 Postprocessing

As in [MZ22], we modify the tree so that it has only $O_\epsilon(1)$ levels of components; see Section 6. Consider a near-optimal solution in the resulting tree (Theorem 24). After applying the Local Theorem (Theorem 12) to simplify the local solutions in all components, we obtain a global solution (Theorem 27); see Section 7. We observe that the possible subtour demands in that global solution are within a polynomial number of values (Fact 29). Furthermore, we leverage the adaptive rounding technique due to Jayaprakash and Salavatipour [JS22] and we show that, in each subtree, the subtour demands are among a constant $O_\epsilon(1)$ number of values (Theorem 31); see Section 8.

Finally, we design a polynomial time dynamic program to compute the best solution that satisfies the structural constraints established previously. The computed solution is a $(1.5 + O(\epsilon))$-approximation. See Section 9.

**Remark 2.** For more general metrics, such as graphs of bounded treewidth and the Euclidean space, it is challenging to simplify the solution structure due to the lack of the unique spine in a subproblem. It is an open question to design optimal approximation algorithms in those metrics.

### 3 Preliminaries

Let $T$ be a rooted tree $(V,E)$ with edge weights $w(u,v) \geq 0$ for all $(u,v) \in E$. Let $n$ denote the number of vertices in $V$. The cost of a tour (resp. a subtour) $t$, denoted by $\text{cost}(t)$, is the overall weight of the edges on $t$. For a set $S$ of tours (resp. subtours), the cost of $S$, denoted by $\text{cost}(S)$, is $\sum_{t \in S} \text{cost}(t)$.

**Definition 3 (UCVRP on trees).** An instance of the unsplittable capacitated vehicle routing problem (UCVRP) on trees consists of

- an edge weighted tree $T = (V,E)$ with root $r \in V$ representing the depot,
- a set $V' \subseteq V$ of terminals,
- for each terminal $v \in V'$, a demand of $v$, denoted by $\text{demand}(v)$, which belongs to $(0,1]$.

A feasible solution is a set of tours such that

- each tour starts and ends at $r$,
- the demand of each terminal is covered by a single tour,
- the total demand of the terminals covered by each tour does not exceed the capacity of 1.

The goal is to find a feasible solution of minimum cost.

For any two vertices $u,v \in V$, let $\text{dist}(u,v)$ denote the distance between $u$ and $v$ in the tree $T$.

We say that a tour (resp. a subtour) visits a terminal if it covers the demand of that terminal. For technical reasons, we allow dummy terminals of appropriate demands to be included artificially. The demand of a tour (resp. a subtour) $t$, denoted by $\text{demand}(t)$, is defined to be the total demand of all terminals (including dummy terminals) visited by $t$. 

3.1 Reduction to Bounded Distance Instances

**Definition 4** (Definition 3 in [MZ22]). Let $D_{\text{min}}$ (resp. $D_{\text{max}}$) denote the minimum (resp. maximum) distance between the depot and any terminal in the tree $T$. We say that $T$ has bounded distances if $D_{\text{max}} < (1/\epsilon)^{(1/\epsilon)-1} \cdot D_{\text{min}}$.

The next theorem (Theorem 5) enables us to assume without loss of generality that the tree $T$ has bounded distances.

**Theorem 5** (Theorem 5 in [MZ22]). For any $\rho \geq 1$, if there is a polynomial time $\rho$-approximation algorithm for the UCVRP on trees with bounded distances, then there is a polynomial time $(1+5\epsilon)\rho$-approximation algorithm for the UCVRP on trees with general distances.

3.2 Decomposition Into Components (Fig. 1)

We decompose the tree $T$ into components as in [MZ22].

**Lemma 6** (Lemma 9 in [MZ22]). Let $\Gamma = 12/\epsilon$. There is a polynomial time algorithm to compute a partition of the edges of the tree $T$ into a set $C$ of components, such that all of the following properties are satisfied:

1. Every component $c \in C$ is a connected subgraph of $T$; the root vertex of the component $c$, denoted by $r_c$, is the vertex in $c$ that is closest to the depot.
2. A component $c$ shares vertices with other components at vertex $r_c$ and possibly at one other vertex, called the exit vertex of the component $c$ and denoted by $e_c$. We say that $c$ is an internal component if $c$ has an exit vertex, and is a leaf component otherwise.
3. The total demand of the terminals in each component $c \in C$ is at most $2\Gamma$.
4. The number of components in $C$ is at most $\max\{1, 3 \cdot \text{demand}(T)/\Gamma\}$, where $\text{demand}(T)$ denotes the total demand of the terminals in the tree $T$.

**Definition 7** ([MZ22]). Let $c \in C$ be any component. A subtour in component $c$ is a path $t$ that starts and ends at the root $r_c$ of component $c$, and such that every vertex on $t$ is in component $c$. We say that $t$ is a passing subtour if $c$ has an exit vertex and that vertex belongs to $t$, and is an ending subtour otherwise.

4 Multi-Level Decomposition in a Component

Let $c \in C$ be any component. We distinguish big and small terminals in $c$ depending on their demands.

**Definition 8** (big and small terminals). Let $\alpha = \epsilon(1/\epsilon)+1$. Let $\Gamma' = \epsilon \cdot \alpha/\Gamma$, where $\Gamma$ is defined in Lemma 6. We say that a terminal $v$ is big if $\text{demand}(v) > \Gamma'$ and small otherwise.

We partition the small terminals in $c$ using a multi-level decomposition: first, the component $c$ is decomposed into blocks (Section 4.1); next, each block is decomposed into clusters (Section 4.2); and finally, each cluster is decomposed into cells (Section 4.3).

We introduce some common notations for blocks, clusters, and cells. Each block (resp. cluster or cell) has a root vertex and at most one exit vertex. We say that a terminal $v$ is strictly inside a block (resp. cluster or cell) if $v$ belongs to the block (resp. cluster or cell) and is different from the root vertex and the exit vertex of the block (resp. cluster or cell). Note that any terminal strictly inside a block (resp. cluster or cell) is small. The demand of a block (resp. cluster
or cell) is defined as the total demand of all terminals strictly inside that block (resp. cluster or cell). We say that a block (resp. cluster or cell) is passing if it has an exit vertex and is ending otherwise. The spine of a passing block (resp. passing cluster or passing cell) is the path between the root vertex and the exit vertex of that block (resp. cluster or cell).

4.1 Decomposition of a Component Into Blocks (Fig. 2a)

Let \( c \) be any component. Let \( U \subseteq V \) denote the set of vertices consisting of the big terminals in \( c \), the root vertex of \( c \), and possibly the exit vertex of \( c \) if \( c \) is an internal component (see Lemma 6 for definitions). Let \( T_U \) denote the subtree of \( c \) spanning the vertices in \( U \). We say that a vertex in \( T_U \) is a key vertex if either it belongs to \( U \) or it has two children in \( T_U \). We define a block to be a maximally connected subgraph of component \( c \) in which any key vertex has degree 1; in other words, blocks are obtained by splitting the component at the key vertices. The blocks form a partition of the edges of component \( c \).

4.2 Decomposition of a Block Into Clusters (Fig. 2b)

As an adaptation from Lemma 6, we decompose a block into clusters in Lemma 9.

Lemma 9. Let \( b \) be any block. There is a polynomial time algorithm to compute a partition of the edges of the block \( b \) into a set of clusters, such that all of the following properties are satisfied:

1. Every cluster \( x \) is a connected subgraph of \( b \); the root vertex of the cluster \( x \), denoted by \( r_x \), is the vertex in \( x \) that is closest to the depot.
2. A cluster \( x \) shares vertices with other clusters at vertex \( r_x \) and possibly at one other vertex, called the exit vertex of the cluster \( x \) and denoted by \( e_x \). If block \( b \) has an exit vertex \( e_b \), then there is a cluster \( x \) in \( b \) such that \( e_x = e_b \).
3. The demand of each cluster in \( b \) is at most \( 2\Gamma' \).
4. The number of clusters in \( b \) is at most \( 3 \cdot (\text{demand}(b)/\Gamma' + 1) \).

The proof of Lemma 9 is in Appendix B.

4.3 Decomposition of a Cluster Into Cells (Fig. 2c)

Let \( x \) be any cluster. If \( x \) is an ending cluster, then the decomposition of \( x \) consists of a single cell, which is the entire cluster \( x \). If \( x \) is a passing cluster, then we decompose \( x \) into cells as follows. Let \( \ell \) denote the cost of the spine of cluster \( x \). When \( \ell = 0 \), the decomposition of \( x \) consists of a single cell, which is the entire cluster \( x \). Next, we assume that \( \ell > 0 \). For each integer \( i \in [1, (1/\epsilon) - 1] \), there exists a unique edge \((u, v)\) on the spine of cluster \( x \) satisfying \( \min(\text{dist}(r_x, u), \text{dist}(r_x, v)) \leq i \cdot \epsilon \cdot \ell < \max(\text{dist}(r_x, u), \text{dist}(r_x, v)) \); let \( e_i \) denote that edge. Removing the edges \( e_1, e_2, \ldots, e_{(1/\epsilon)-1} \) from cluster \( x \) results in at most \( 1/\epsilon \) connected subgraphs; each subgraph is called a cell. Observe that those cells form a partition of the vertices of cluster \( x \).

From the construction, the (unique) cell inside an ending cluster is an ending cell, and the cells inside a passing cluster are passing cells. The cost of the spine of any passing cell in a passing cluster is at most an \( \epsilon \) fraction of the cost of the spine of that cluster.

Fact 10. In any component \( c \), the number of cells and the number of big terminals are both \( O_\epsilon(1) \).

Proof. By Lemma 6, the total demand of the terminals in component \( c \) is at most \( 2\Gamma' \). Since the demand of a big terminal is at least \( \Gamma' \), there are at most \( 2\Gamma'/\Gamma' = O_\epsilon(1) \) big terminals in \( c \).
From the construction in Section 4.1, the set \( U \) consists of at most \( 2 + 2\Gamma'/\Gamma' \) vertices. Since each vertex in \( c \) has at most two children, the number of blocks in \( c \) is at most \( 2|U| \leq 4 + 4\Gamma'/\Gamma' \). From the construction in Section 4.2, each block \( b \) is partitioned into at most \( 3 \cdot (\text{demand}(b)/\Gamma' + 1) \) clusters, where \( \text{demand}(b) \) is at most the total demand of the terminals in component \( c \), which is at most \( 2\Gamma' \). From the construction in Section 4.3, each cluster is partitioned into at most \( 1/\epsilon \) cells. So the number of cells in \( c \) is at most \( (4 + 4\Gamma'/\Gamma') \cdot (3 \cdot (2\Gamma'/\Gamma' + 1)) \cdot (1/\epsilon) = O_c(1) \). \( \square \)

**Definition 11** (Adaptation from Definition 7). A subtour in a cluster (resp. cell) is a path \( t \) that starts and ends at the root of that cluster (resp. cell), and such that every vertex on \( t \) is in that cluster (resp. cell). We say that \( t \) is a passing subtour if that cluster (resp. cell) has an exit vertex and that vertex belongs to \( t \), and is an ending subtour otherwise. The spine subtour in a passing cluster (resp. passing cell) consists of the spine of that cluster (resp. cell) in both directions.

### 5 Simplifying the Local Solution

In this section, we prove the Local Theorem (Theorem 12).

**Theorem 12** (Local Theorem). Let \( c \) be any component. Let \( S_c \) denote a set of at most \( (2\Gamma/\alpha) + 1 \) subtours in component \( c \) visiting all terminals in \( c \). Then there exists a set \( S_c^* \) of subtours in component \( c \) visiting all terminals in \( c \), such that all of the following properties hold:

1. For each cell in \( c \), a single subtour in \( S_c^* \) visits all small terminals in that cell;

2. \( S_c^* \) contains one particular subtour \( \bar{t} \) of demand at most 1, and the subtours in \( S_c^* \setminus \{\bar{t}\} \) are in one-to-one correspondence with the subtours in \( S_c \), such that for every subtour \( t \) in \( S_c \) and its corresponding subtour \( t^* \) in \( S_c^* \setminus \{\bar{t}\} \), the demand of \( t^* \) is at most the demand of \( t \), and in addition, if \( t \) is a passing subtour in \( c \), then \( t^* \) is also a passing subtour in \( c \);

3. The cost of \( S_c^* \) is at most \( 1.5 + 2\epsilon \) times the cost of \( S_c \).

#### 5.1 Construction of \( S_c^* \)

The construction of \( S_c^* \) starts from \( S_c \) and proceeds in 5 steps. Step 2 uses a new concept of threshold cells and is the main novelty in the construction. Step 1 and Step 3 are based on the following lemma due to Becker and Paul [BP19].

**Lemma 13** (Assignment Lemma, Lemma 1 in [BP19]). Let \( G = (V[G], E[G]) \) be an edge-weighted bipartite graph with vertex set \( V[G] = A \uplus B \) and edge set \( E[G] \subseteq A \times B \), such that each edge \( (a, b) \in E[G] \) has a weight \( w(a, b) \geq 0 \). For each vertex \( b \in B \), let \( N(b) \) denote the set of vertices \( a \in A \) such that \( (a, b) \in E[G] \). We assume that \( N(b) \neq \emptyset \) and the weight \( w(b) \) of the vertex \( b \) satisfies \( 0 \leq w(b) \leq \sum_{a \in N(b)} w(a, b) \). Then there exists a function \( f : B \rightarrow A \) such that each vertex \( b \in B \) is assigned to a vertex \( a \in N(b) \) and, for each vertex \( a \in A \), we have

\[
\sum_{b \in B | f(b) = a} w(b) - \sum_{b \in B | (a, b) \in E[G]} w(a, b) \leq \max_{b \in B} \{ w(b) \}.
\]

**Step 1: Combining ending subtours within each cluster.** Let \( A_0 \) denote \( S_c \). We define a weighted bipartite graph \( G \) in which the vertices in one part represent the subtours in \( A_0 \) and the vertices in the other part represent the clusters in \( c \). There is an edge in \( G \) between a subtour

\[^5\text{With a slight abuse, we identify a vertex in } G \text{ with either a subtour in } A_0 \text{ or a cluster in } c.\]
Figure 4: The threshold cell and the extension of an ending subtour. The outermost triangle in blue represents a cluster $x$. In Fig. 4(a), the black segments represent the ending subtour $t_e$ in $x$. The threshold cell of cluster $x$ is the deepest cell visited by $t_e$ and is represented by the yellow triangle. In Fig. 4(b) subtour $t_e$ is extended within the threshold cell: the green segment represents the part of the spine subtour of the threshold cell that is added to $t_e$, resulting in a subtour $\tilde{t}_e$.

$a \in A_0$ and a cluster $x$ in $c$ if and only if $a$ contains an ending subtour $t$ in $x$; the weight of the edge is defined to be $\text{demand}(t)$. For each cluster $x$ in $c$, we define the weight of $x$ in $G$ to be the sum of the weights of its incident edges in $G$. We apply the Assignment Lemma (Lemma 13) to the graph $G$ (deprived of the vertices of degree 0) and obtain a function $f$ that maps each cluster $x$ in $c$ to some subtour $a \in A_0$ such that $(a, x)$ is an edge in $G$.

We construct a set of subtours $A_1$ as follows: for every cluster $x$ in $c$ and for every subtour $a \in A_0$ containing an ending subtour $t$ in $x$, the subtour $t$ is removed from $a$ and added to the subtour $f(x)$. Observe that each resulting subtour in $A_1$ is connected. From the construction, for each cluster $x$, at most one subtour in $A_1$ has an ending subtour in $x$. In particular, for any ending cell, which is equivalent to an ending cluster, a single subtour in $A_1$ visits all small terminals in that cell.

**Step 2: Extending ending subtours within threshold cells.** Let $x$ be any passing cluster in $c$ such that there is a subtour in $A_1$ containing an ending subtour in $x$. From Step 1 of the construction, such a subtour in $A_1$ is unique; let $t_e$ denote the corresponding ending subtour in $x$. Define the threshold cell of cluster $x$ to be the deepest cell in $x$ containing vertices of $t_e$. See Fig. 4(a). We add to $t_e$ the part of the spine subtour in the threshold cell of $x$ that does not belong to $t_e$, resulting in a subtour $\tilde{t}_e$; see Fig. 4(b).

Let $A_2$ denote the resulting set of subtours in $c$ after the extension within all threshold cells. From the construction, for each passing cell $s$, all subtours in $s$ that are contained in $A_2$ are passing subtours in $s$.

**Step 3: Combining passing subtours within each passing cell.** We define a weighted bipartite graph $G'$ in which the vertices in one part represent the subtours in $A_2$ and the vertices in the other part represent the passing cells in $c$. There is an edge in $G'$ between a subtour $a \in A_2$ and a passing cell $s$ in $c$ if and only if $a$ contains a non-spine passing subtour $t$ in $s$; the weight of the edge is defined to be the total demand of the small terminals on $t$. For each passing cell $s$ in $c$, we define the weight of $s$ in $G'$ to be the sum of the weights of its incident edges in $G'$. We apply the Assignment Lemma (Lemma 13) to the graph $G'$ (deprived of the vertices of degree 0) and obtain a function $f'$ that maps each passing cell $s$ in $c$ to some subtour $a \in A_2$ such that $(a, s)$

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6With a slight abuse, we identify a vertex in $G'$ with either a subtour in $A_2$ or a passing cell in $c$. 9
is an edge in $G'$.

We construct a set of subtours $A_3$ as follows: for every passing cell $s$ in $c$ and for every subtour $a \in A_2$ containing a non-spine passing subtour $t$ in $s$, the subtour $t$ is removed from $a$ except for the spine subtour of $s$; the removed part is added to the subtour $f'(s)$. Observe that each resulting subtour in $A_3$ is connected. From the construction, for each passing cell $s$, a single subtour in $A_3$ visits all small terminals in $s$.

**Step 4: Correcting subtour capacities.** For each subtour $t_3$ in $A_3$, let $t_0$ denote the corresponding subtour in $A_0$. As soon as the demand of $t_3$ is greater than the demand of $t_0$, we repeatedly modify $t_3$ as follows: find a terminal $v$ that is visited by $t_3$ but not visited by $t_0$; let $s$ denote the cell containing $v$ and let $t_s$ denote the subtour of $t_3$ in cell $s$; if $s$ is an ending cell, then remove $t_s$ from $t_3$; and if $s$ is a passing cell, then remove $t_s$ from $t_3$ except for the spine subtour of $s$.

Let $A_4$ denote the resulting set of modified subtours. Observe that each subtour in $A_4$ is connected. From the construction, the demand of each subtour in $A_4$ is at most the demand of the corresponding subtour in $A_0$.

Let $R$ denote the set of the removed pieces. We claim that the total demand of the pieces in $R$ is at most 1 (Lemma 20).

**Step 5: Creating an additional subtour.** We connect all pieces in $R$ by a single subtour in component $c$; let $\bar{t}$ be that subtour.

Finally, let $S_c^*$ denote $A_4 \cup \{\bar{t}\}$.

### 5.2 Analysis on the Cost of $S_c^*$

From the construction of $S_c^*$, we observe that the cost of $S_c^*$ equals the cost of $S_c$ plus the extra costs in Step 2 and in Step 5 of the construction, denoted by $W_2$ and $W_5$, respectively.

To analyze the extra costs, first, in a preliminary lemma (Lemma 14), we bound the overall cost of the spines of the threshold cells. Lemma 14 will be used to analyze both $W_2$ (Corollary 15) and $W_5$ (Lemma 19).

**Lemma 14.** The overall cost of the spines of all threshold cells in the component $c$ is at most $(\epsilon/2) \cdot \text{cost}(S_c)$.

**Proof.** Consider any threshold cell $s$. Let $x$ be the passing cluster that contains $s$. As observed in Section 4.3, the cost of the spine of cell $s$ is at most an $\epsilon$ fraction of the cost of the spine of the cluster $x$. Since $x$ is a passing cluster, at least one subtour in $S_c$ contains a passing subtour in $x$; let $t_x$ denote that passing subtour in $x$. Observe that $t_x$ contains each edge of the spine of cluster $x$ in both directions (Definition 11), so the cost of the spine of $x$ is at most $\text{cost}(t_x)/2$. Thus the cost of the spine of $s$ is at most $(\epsilon/2) \cdot \text{cost}(t_x)$. We charge the cost of the spine of $s$ to $t_x$.

From the construction, each cluster contains at most one threshold cell. Thus the costs of the spines of all threshold cells are charged to disjoint parts of $S_c$. The claim follows.

Observe that the extra cost in Step 2 of the construction is at most the overall cost of the spine subtours in all threshold cells in the component $c$, which equals twice the overall cost of the spines of those cells by Definition 11.

---

7 Any big terminal cannot be removed, since it is the exit vertex of some cell, thus belongs to the spine of that cell.
Corollary 15. The extra cost $W_2$ in Step 2 of the construction is at most $\epsilon \cdot \text{cost}(S_c)$.

Next, we bound the extra cost in Step 5 of the construction.

Fact 16. Let $t$ denote any subtour in $S_c$. Let $x$ denote any cluster in $c$. Let $r_c$ and $r_x$ denote the root vertices of component $c$ and of cluster $x$, respectively; let $e_x$ denote the exit vertex of cluster $x$. If the $r_c$-to-$r_x$ path (resp. the $r_c$-to-$e_x$ path) belongs to $t$, then that path belongs to the corresponding subtour of $t$ throughout the construction in Section 5.7.

Definition 17 (nice edges). We say that an edge $e$ in component $c$ is nice if $e$ belongs to at least two subtours in $A_2$.

The next Lemma is the main novelty in the analysis.

Lemma 18. Any piece in $R$ is connected to the root $r_c$ of component $c$ through nice edges in $c$.

Proof. Consider any piece $q \in R$. Let $s$ be the cell containing $q$. Let $x$ be the cluster containing $q$. See Fig. 5. Let $r_s$ and $r_x$ denote the root vertices of cell $s$ and of cluster $x$, respectively. Observe that the terminals in $x$ are visited by at least two subtours in $S_c$. This is because, if all terminals in cluster $x$ are visited by a single subtour in $S_c$, then those terminals belong to the corresponding subtour throughout the construction, thus none of those terminals belongs to a piece in $R$, contradiction. Thus the $r_c$-to-$r_x$ path belongs to at least two subtours in $S_c$. By Fact 16 the $r_c$-to-$r_x$ path belongs to at least two subtours in $A_2$, thus every edge on the $r_c$-to-$r_x$ path is nice. It suffices to show the following Claim:

Piece $q$ is connected to vertex $r_x$ through nice edges in $c$. (*)

There are two cases:

Case 1: $x$ is an ending cluster. See Fig. 5a. From the decomposition in Section 4.3 $s$ is an ending cell and $s$ equals $x$. Piece $q$ is an ending subtour in $x$ and in particular contains $r_x$. Claim (*) follows trivially.

Case 2: $x$ is a passing cluster. Let $e_s$ and $e_x$ denote the exit vertices of cell $s$ and of cluster $x$, respectively. Observe that at least one subtour in $S_c$ contains a passing subtour in $x$. There are two subcases.

Subcase 2(i): At least two subtours in $S_c$ contain passing subtours in $x$. See Fig. 5b. Then the $r_c$-to-$e_x$ path belongs to at least two subtours in $S_c$. By Fact 16 the $r_c$-to-$e_x$ path belongs to at least two subtours in $A_2$, thus each edge on the spine of $x$ is nice. Since piece $q$ contains a vertex on the spine of $x$, Claim (*) follows.

Subcase 2(ii): Exactly one subtour in $S_c$ contains a passing subtour in $x$. See Figs. 5c and 5d. Let $t_p$ denote that passing subtour in $x$. As observed previously, at least two subtours in $S_c$ visit terminals in $x$, so there must be at least one subtour in $S_c$ that contains an ending subtour in $x$. Let $t^1, \ldots, t^m$ (for some $m \geq 1$) denote the ending subtours in $x$ contained in the subtours in $S_c$. In Step 1 of the construction, the $m$ ending subtours are combined into a single ending subtour, denoted by $t_e$ (recall that the threshold cell of $x$ is defined with respect to $t_e$); and in Step 2 of the construction, subtour $t_e$ is extended to a subtour $t_e$ (Fig. 4). Note that the passing subtour $t_p$ remains unchanged in Steps 1 and 2 of the construction. We observe that cell $s$ is either above or equal to the threshold cell of $x$. This is because, if cell $s$ is below the threshold cell of $x$, then all terminals in $s$ are visited by a single subtour in $S_c$, i.e., the subtour $t_p$, so those terminals belong to the corresponding subtour of $t_p$ throughout the construction, thus none of those terminals belongs to a piece in $R$, contradiction. Hence the following two subsubcases.
Figure 5: Illustrations for the different cases in the proof of Lemma 18. A removed piece $q \in R$ is in brown. The cell $s$ containing that piece is represented by the triangle in red; the cluster $x$ containing that piece is represented by the outermost triangle in blue. The black node $r_c$ is the root of component $c$. In Fig. 5a $x$ is an ending cluster. In Fig. 5b $x$ is a passing cluster, and the solution $S_c$ contains two passing subtours in $x$. In Figs. 5c and 5d $x$ is a passing cluster, and the solution $S_c$ contains a unique passing subtour in $x$: the yellow triangle represents the threshold cell of $x$. In the case when $q$ belongs to the threshold cell (Fig. 5d), $q$ is connected to $r_c$ through at least two subtours, thanks to the extension of the ending subtour within the threshold cell.

Subcase 2(ii)(α): $s$ is above the threshold cell of $x$, see Fig. 5c. Each edge on the $r_x$-to-$e_s$ path belongs to both subtours $t_p$ and $t_e$, hence is nice. Since $q$ contains some vertex on the spine of $s$, Claim (*) follows.

Subcase 2(ii)(β): $s$ equals the threshold cell of $x$, see Fig. 5d. Observe that each edge on the $r_x$-to-$e_s$ path belongs to $t_e$ due to the extension of the ending subtour $t_e$ within the threshold cell (Step 2 of the construction). Thus each edge on the $r_x$-to-$e_s$ path belongs to both subtours $t_p$ and $t_e$, hence is nice. Since $q$ contains some vertex on the spine of $s$, Claim (*) follows. □

Lemma 19. The extra cost $W_5$ in Step 5 of the construction is at most $(0.5 + \epsilon) \cdot \text{cost}(S_c)$.

Proof. First, we argue that the extra cost $W_5$ in Step 5 of the construction is at most twice the overall cost of the nice edges in $c$. Let $H$ be the multi-subgraph in $c$ that consists of the pieces in $R$ and two copies of each nice edge in $c$ (one copy for each direction). Since any piece in $R$ is connected to the root $r_c$ of component $c$ through nice edges (Lemma 18), $H$ induces a connected subtour in $c$.

Second, we bound the overall cost of the nice edges in $c$. From the construction, any nice edge $e$ in $c$ is of at least one of the two classes:

1. edge $e$ belongs to at least two subtours in $S_c$;
2. edge $e$ belongs to the spine of a threshold cell in component $c$.

Each nice edge $e$ of the first class has at least 4 copies in $S_c$, since each subtour to which $e$ belongs contains 2 copies of $e$ (one for each direction). Thus the overall cost of the nice edges of the first class is at most $0.25 \cdot \text{cost}(S_c)$. By Lemma 14, the overall cost of the nice edges of the second

Subsubcase 2(ii)(α): $s$ is above the threshold cell of $x$, see Fig. 5c. Each edge on the $r_x$-to-$e_s$ path belongs to both subtours $t_p$ and $t_e$, hence is nice. Since $q$ contains some vertex on the spine of $s$, Claim (*) follows.

Subsubcase 2(ii)(β): $s$ equals the threshold cell of $x$, see Fig. 5d. Observe that each edge on the $r_x$-to-$e_s$ path belongs to $t_e$ due to the extension of the ending subtour $t_e$ within the threshold cell (Step 2 of the construction). Thus each edge on the $r_x$-to-$e_s$ path belongs to both subtours $t_p$ and $t_e$, hence is nice. Since $q$ contains some vertex on the spine of $s$, Claim (*) follows. □

Lemma 19. The extra cost $W_5$ in Step 5 of the construction is at most $(0.5 + \epsilon) \cdot \text{cost}(S_c)$.

Proof. First, we argue that the extra cost $W_5$ in Step 5 of the construction is at most twice the overall cost of the nice edges in $c$. Let $H$ be the multi-subgraph in $c$ that consists of the pieces in $R$ and two copies of each nice edge in $c$ (one copy for each direction). Since any piece in $R$ is connected to the root $r_c$ of component $c$ through nice edges (Lemma 18), $H$ induces a connected subtour in $c$.

Second, we bound the overall cost of the nice edges in $c$. From the construction, any nice edge $e$ in $c$ is of at least one of the two classes:

1. edge $e$ belongs to at least two subtours in $S_c$;
2. edge $e$ belongs to the spine of a threshold cell in component $c$.

Each nice edge $e$ of the first class has at least 4 copies in $S_c$, since each subtour to which $e$ belongs contains 2 copies of $e$ (one for each direction). Thus the overall cost of the nice edges of the first class is at most $0.25 \cdot \text{cost}(S_c)$. By Lemma 14, the overall cost of the nice edges of the second
class is at most \((\epsilon/2) \cdot \text{cost}(S_c)\). Hence the overall cost of the nice edges of both classes is at most \((0.25 + \epsilon/2) \cdot \text{cost}(S_c)\).

Since the extra cost \(W_5\) is at most twice the overall cost of the nice edges in \(c\), we have \(W_5 \leq (0.5 + \epsilon) \cdot \text{cost}(S_c)\).

From Corollary 15 and Lemma 19 we conclude that
\[
\text{cost}(S^*_c) = \text{cost}(S_c) + W_2 + W_5 \leq (1.5 + 2\epsilon) \cdot \text{cost}(S_c).
\]
Hence the third property of the claim in the Local Theorem (Theorem 12).

5.3 Feasibility

From the construction, \(S^*_c\) is a set of subtours in \(c\) visiting all terminals in \(c\). The first property of the claim in the Local Theorem (Theorem 12) follows from the construction. The second property of the claim follows from the construction, Fact 16, and the following Lemma 20.

**Lemma 20.** The total demand of the pieces in \(R\) is at most 1.

**Proof.** Observe that the pieces in \(R\) are removed from subtours in \(A_3\). Let \(t_3\) denote any subtour in \(A_3\). Let \(t_0, t_1, t_2, t_4\) denote the corresponding subtours of \(t_3\) in \(A_0, A_1, A_2, A_4\), respectively. Let \(\Delta\) denote the overall demand of the pieces that are removed from \(t_3\) in Step 4 of the construction. Observe that \(\Delta = \text{demand}(t_3) - \text{demand}(t_4)\). To bound \(\Delta\), first, by Step 1 of the construction and the Assignment Lemma (Lemma 13), \(\text{demand}(t_1) - \text{demand}(t_0)\) is at most the maximum demand of a cluster, which is at most \(2\Gamma\) by the definition of clusters (Section 4.2). By Step 2 of the construction, \(\text{demand}(t_2) = \text{demand}(t_1)\). By Step 3 of the construction and the Assignment Lemma (Lemma 13), \(\text{demand}(t_3) - \text{demand}(t_2)\) is at most the maximum demand of a cell, which is at most \(2\Gamma\) by the definition of cells (Section 4.3). By Step 4 of the construction, \(\text{demand}(t_0) - \text{demand}(t_4)\) is at most the maximum demand of a cell, which is at most \(2\Gamma\). Combining, we have \(\Delta = \text{demand}(t_3) - \text{demand}(t_4) \leq 6\Gamma\).

The number of subtours in \(A_3\) equals the number of subtours in \(S_c\), which is at most \((2\Gamma/\alpha) + 1\) by assumption. Thus total demand of the pieces in \(R\) is at most \(6\Gamma' \cdot ((2\Gamma/\alpha) + 1) < 13\epsilon < 1\), assuming \(\epsilon < 1/13\).

This completes the proof of the Local Theorem (Theorem 12).

6 Height Reduction

In this section, we transform the tree \(T\) into a tree \(\hat{T}\) so that \(\hat{T}\) has \(O(1)\) levels of components.

All results in this section are already given in [MZ22] for the equal demand setting. The arguments for the arbitrary demand setting are identical, except for the proof of Theorem 24, which is a minor adaptation of the proof in [MZ22], see Remark 25.

**Lemma 21** (Lemma 21 in [MZ22]). Let \(\hat{D} = \alpha \cdot \epsilon \cdot D_{\min}\), where \(\alpha\) is defined in Definition 8 and \(D_{\min}\) is defined in Definition 2. Let \(H_\epsilon = (1/\epsilon)^{(2/\epsilon)} + 1\). For each \(i \in [1, H_\epsilon]\), let \(C_{c_i} \subseteq C\) denote the set of components \(c \in C\) such that \(\text{dist}(r, r_c) \in [(i - 1) \cdot \hat{D}, i \cdot \hat{D})\). Then any component \(c \in C\) belongs to a set \(C_{c_i}\) for some \(i \in [1, H_\epsilon]\).

**Definition 22** (Definition 22 in [MZ22]). We say that a set of components \(\tilde{C} \subseteq C_i\) is maximally connected if the components in \(\tilde{C}\) are connected to each other and \(\tilde{C}\) is maximal within \(C_i\). For a maximally connected set of components \(\tilde{C} \subseteq C_i\), we define the critical vertex of \(\tilde{C}\) to be the root vertex of the component \(c \in \tilde{C}\) that is closest to the depot.
Algorithm 1 Construction of the tree \( \hat{T} \) ([MZ22]).

1: for each \( i \in [1, H] \) do
2: for each maximally connected set of components \( \tilde{C} \subseteq C_i \) do
3: \( z \leftarrow \) critical vertex of \( \tilde{C} \)
4: for each component \( c \in \tilde{C} \) do
5: \( \delta \leftarrow r_c \)-to-\( z \) distance in \( T \)
6: Split the tree \( T \) at the root vertex \( r_c \) of the component \( c \)
7: Add an edge between the root of the component \( c \) and \( z \) with weight \( \delta \)
8: \( \hat{T} \leftarrow \) the resulting tree

Fact 23 (Fact 6 in [MZ22]). Algorithm 1 constructs in polynomial time a tree \( \hat{T} \) such that:
- The components in \( \hat{T} \) are the same as those in the tree \( T \);
- Any solution to the UCVRP on the tree \( \hat{T} \) can be transformed in polynomial time into a solution to the UCVRP on the tree \( T \) without increasing the cost.

Theorem 24 (Adaptation of Theorem 23 in [MZ22]). Consider the UCVRP on the tree \( \hat{T} \). There exist dummy terminals of appropriate demands and a solution \( \text{OPT}_2 \) visiting all of the real and the dummy terminals, such that all of the following properties hold:
1. For each component \( c \), there are at most \((2\Gamma/\alpha) + 1\) tours visiting terminals in \( c \);
2. For each component \( c \) and each tour \( t \) visiting terminals in \( c \), the total demand of the terminals in \( c \) visited by \( t \) is at least \( \alpha \);
3. The cost of \( \text{OPT}_2 \) is at most \( 1 + 3\epsilon \) times the optimal cost for the UCVRP on the tree \( T \).

Remark 25. In the proof of Theorem 24, everything in [MZ22] carries over to the arbitrary demand setting, except that the Iterated Tour Partitioning (ITP) algorithm, which is used to reduce the demand of the tours exceeding capacity, is adapted as follows. Let \( t_{TSP} \) be a traveling salesman tour visiting a selected subset of terminals (Section 3.1 in [MZ22]). We partition \( t_{TSP} \) into segments, such that all segments, except possibly the last segment, have demands between \( 1 - \alpha \) and \( 1 \), instead of being exactly \( 1 \). This is achievable since every terminal on \( t_{TSP} \) has demand at most \( \alpha \). The analysis is identical to [MZ22] except for a suitable adaptation of the analysis of the ITP algorithm.

7 Combining Local Solutions

Definition 26. For any component \( c \in C \), let \( Q_c \) denote the partition of the terminals of component \( c \), such that each part of the partition consists of either all small terminals in a cell in \( c \), or a single big terminal in \( c \). We define the set

\[
Y_c = \{\alpha\} \cup \{\text{demand}(\tilde{Q}_c) : \tilde{Q}_c \subseteq Q_c \cap (\alpha, 1]\}.
\]

Theorem 27. Consider the UCVRP on the tree \( \hat{T} \). There exist dummy terminals of appropriate demands and a solution \( S^* \) visiting all of the real and the dummy terminals, such that all of the following properties hold:
1. For each cell, a single tour visits all small terminals in that cell;
2. For each component $c$ and each tour $t$ visiting terminals in $c$, the total demand of the terminals in $c$ visited by $t$ belongs to $Y_c$.

3. The cost of $S^*$ is at most $1.5 + 7\epsilon$ times the optimal cost for the UCVRP on the tree $T$.

Proof. Let $S$ denote the solution to the UCVRP on the tree $\hat{T}$ that is identical to $\text{OPT}_2$ defined in Theorem 24 except for ignoring dummy terminals in $\text{OPT}_2$. To construct $S^*$, we modify the solution $S$ component by component as follows. Consider any component $c$. Let $S_c$ denote the set of subtours in $c$ obtained by restricting the tours of $S$ visiting terminals in $c$. By Theorem 24, $|S_c| \leq (2\Gamma/\alpha) + 1$. We apply the Local Theorem (Theorem 12) on $S_c$ to obtain a set $S^*_c$ of subtours in $c$, such that $S^*_c$ contains one particular subtour $t$ of demand at most 1, and the subtours in $S^*_c \setminus \{\tilde{t}\}$ are in one-to-one correspondence with the subtours in $S_c$. For the subtour $t$, we create a new tour from the depot by adding the $r$-to-$r_c$ connection in both directions. Next, consider any subtour $t^{(1)}$ in $S_c$; let $t^{(2)}$ be the corresponding subtour in $S^*_c$. We replace $t^{(1)}$ in the solution $S$ by a subtour $t$ defined as follows: $t$ is identical to $t^{(2)}$, except that if $t^{(2)}$ visits terminals in $c$ and the demand of $t^{(2)}$ is less than $\alpha$, then we add to $t$ a dummy terminal at $r_c$ of demand $\alpha - \text{demand}(t^{(2)})$.

Let $S^*$ denote the resulting solution.

By the second property of the Local Theorem (Theorem 12), in any component $c$, if a subtour in $S_c$ is a passing subtour, then the corresponding subtour in $S^*_c$ is a passing subtour, thus each tour in $S^*$ is a connected tour starting and ending at the depot. Again by the Local Theorem, in any component $c$, the subtours in $S^*_c$ together visit all terminals in $c$, thus the tours in $S^*$ together visit all of the real and the dummy terminals in $\hat{T}$.

To see that the tours in $S^*$ are within the capacity, first, by the second property of the Local Theorem (Theorem 12), any additional tour created from a subtour $\tilde{t}$ in some set $S^*_c$ is within the capacity. Next, consider any tour $a$ among the remaining tours in $S^*$. Consider any component $c$ that contains terminals visited by $a$; let $t$ be the subtour in $c$ from tour $a$. Let $t^{(0)}$, $t^{(1)}$, and $t^{(2)}$ denote the corresponding subtours of $t$ in $\text{OPT}_2$, $S_c$, and $S^*_c$, respectively. By Theorem 24, $\text{demand}(t^{(0)}) \geq \alpha$. By definition of $S$, $\text{demand}(t^{(1)}) \leq \text{demand}(t^{(0)})$. Now we use the second property of the Local Theorem (Theorem 12), which ensures that $\text{demand}(t^{(2)}) \leq \text{demand}(t^{(1)})$. Thus $\text{demand}(t) = \max\{\text{demand}(t^{(2)}), \alpha\} \leq \text{demand}(t^{(0)})$. Let $a^{(0)}$ denote the tour in $\text{OPT}_2$ corresponding to $a$. Then $\text{demand}(a) \leq \text{demand}(a^{(0)})$. Since $\text{OPT}_2$ is a solution to the UCVRP (Theorem 24), the total demand of the (real and the dummy) terminals in $a^{(0)}$ is at most 1. Thus tour $a$ is within the capacity.

The first property of the claim follows from the first property of the Local Theorem (Theorem 12). The second property of the claim follows from the first property of the claim and the construction of $S^*$. It remains to analyze the cost of $S^*$.

From the construction of $S^*$, we have

$$\text{cost}(S^*) = \sum_c \text{cost}(S^*_c) + \sum_c 2 \cdot \text{dist}(r, r_c),$$

(1)

where we use the fact that, for each component $c$, the distance in the tree $\hat{T}$ between the depot $r$ and the root $r_c$ of component $c$ equals that distance in the tree $T$. By Lemma 17 in \cite{MZ22},

$$\sum_c \text{dist}(r, r_c) \leq \epsilon \cdot \text{opt}.$$  

(2)

By the third property of the Local Theorem (Theorem 12), we have

$$\sum_c \text{cost}(S^*_c) \leq \sum_c (1.5 + 2\epsilon) \cdot \text{cost}(S_c) = (1.5 + 2\epsilon) \cdot \text{cost}(S).$$

(3)
By the definition of \( S \) and Theorem 24, we have
\[
\text{cost}(S) = \text{cost}(\text{OPT}_2) \leq (1 + 3\epsilon) \cdot \text{opt},
\]
where \( \text{opt} \) denotes the optimal cost for the UCVRP on the tree \( T \).

The last property of the claim follows from Eqs. (1) to (4).

8 Structure Theorem

Definition 28. Let \( Y \subseteq [\alpha, 1] \) denote the set of values \( y \in [\alpha, 1] \) such that \( y \) equals the sum of the elements in a multi-subset of \( \bigcup_c Y_c \).

Fact 29. \( \{Q_c\}_c, \{Y_c\}_c, \) and \( Y \) satisfy the following properties:

1. For any component \( c \), the set \( Q_c \) consists of \( O_\epsilon(1) \) parts and the set \( Y_c \) consists of \( O_\epsilon(1) \) values;
2. For any component \( c \), we have \( Y_c \subseteq Y \);
3. For any values \( y \in Y \) and \( y' \in Y \) such that \( y + y' \leq 1 \), we have \( y + y' \in Y \);
4. The set \( Y \) consists of \( n^{O_\epsilon(1)} \) values.

Proof. The first property of the claim follows from Fact 10. The second and the third properties of the claim follow from Definition 28. For any component \( c \), each value in \( Y_c \) is at least \( \alpha \), so each value \( y \in Y \) is the sum of at most \( 1/\alpha \) values in \( \bigcup_c Y_c \). Since the number of components in the tree \( \hat{T} \) is at most \( n \), the fourth property of the claim follows.

Definition 30 (subtours at a vertex). For any vertex \( v \in V \), we say that a path is a subtour at \( v \) if that path starts and ends at \( v \) and only visits vertices in the subtree of \( \hat{T} \) rooted at \( v \).

We build on Theorem 27 to obtain the following Structure Theorem.

Theorem 31 (Structure Theorem). Let \( \beta = \frac{1}{4} \cdot 4^{(4/\epsilon)+1} \). Consider the UCVRP on the tree \( \hat{T} \). There exist dummy terminals of appropriate demands and a solution \( \hat{S} \) visiting all of the real and the dummy terminals, such that all of the following properties hold:

1. For each cell, a single tour visits all small terminals in that cell;
2. For each component \( c \) and each tour \( t \) visiting terminals in \( c \), the total demand of the terminals in \( c \) visited by \( t \) belongs to \( Y_c \);
3. For the root of each component and for each critical vertex (Definition 22), the demand of each subtour at that vertex belongs to \( Y \);
4. For each critical vertex, there exist \( \frac{1}{\beta} \) values in \( Y \) such that demand of each subtour at a child of that vertex is among those values;
5. The cost of \( \hat{S} \) is at most \( 1.5 + 8\epsilon \) times the optimal cost for the UCVRP on the tree \( T \).

In the rest of the section, we prove the Structure Theorem (Theorem 31).
8.1 Construction of \( \hat{S} \)

We construct the solution \( \hat{S} \) by modifying the solution \( S^* \) defined in Theorem 27. The construction is an adaptation of Section 5.1 in [MZ22].

Definition 32. Let \( I \subseteq V \) denote the set of vertices \( v \in V \) such that \( v \) is either the root of a component or a critical vertex.

We consider the vertices in \( I \) in the bottom up order.

Let \( v \) be any vertex in \( I \). Let \( S^*(v) \) denote the set of subtours at \( v \) in \( S^* \). We construct a set \( A(v) \) of subtours at \( v \) satisfying the following invariants:

- the subtours in \( A(v) \) have a one-to-one correspondence with the subtours in \( S^*(v) \); and
- for each subtour in \( A(v) \), its demand belongs to \( Y \) and is at most the demand of the corresponding subtour in \( S^*(v) \).

The construction of \( A(v) \) is according to one of the following three cases on \( v \).

Case 1: \( v \) is the root vertex \( r_c \) of a leaf component \( c \) in \( \hat{T} \). Let \( A(v) = S^*(v) \). For each subtour in \( A(v) \), its demand belongs to \( Y_c \) by Theorem 27, thus belongs to \( Y \) by Fact 29.

Case 2: \( v \) is the root vertex \( r_c \) of an internal component \( c \) in \( \hat{T} \). For each subtour \( a \in S^*(v) \), if \( a \) contains a subtour at the exit vertex \( e_c \) of component \( c \), letting \( t \) denote this subtour and \( t' \) denote the subtour in \( A(e_c) \) corresponding to \( t \), we replace the subtour \( t \) in \( a \) by the subtour \( t' \). Let \( a' \) denote the resulting subtour. By induction, the demand of \( t' \) is at most that of \( t \), thus the demand of \( a' \) is at most the demand of \( a \). Again by induction, the demand of \( t' \) belongs to \( Y \).

Using the same argument as before, the demand of the subtour in \( c \) that is contained in \( a \) belongs to \( Y \). Since the set \( Y \) is closed under addition (Fact 29), the demand of \( a' \) is in \( Y \).

Let \( A(v) \) be the resulting set of subtours at \( v \).

Case 3: \( v \) is a critical vertex in \( \hat{T} \). Let \( r_1, \ldots, r_m \) be the children of \( v \) in \( \hat{T} \). For each subtour \( a \in S^*(v) \) and for each \( i \in [1, m] \), if \( a \) contains a subtour at \( r_i \), letting \( t \) denote this subtour and \( t' \) denote the subtour in \( A(r_i) \) corresponding to \( t \), we replace \( t \) in \( a \) by \( t' \). Let \( A_1(v) \) denote the resulting set of subtours at \( v \).

Let \( W_v \) denote the set of the subtours at the children of \( v \) in \( A_1(v) \), i.e., \( W_v = A(r_1) \cup \cdots \cup A(r_m) \). By induction, the demand of each subtour in \( W_v \) belongs to \( Y \). If \( |W_v| \leq \frac{1}{2} \), let \( A(v) = A_1(v) \).

Next, consider the non-trivial case when \( |W_v| > \frac{1}{2} \). We sort the subtours in \( W_v \) in non-decreasing order of their demands, and partition these subtours into \( \frac{1}{2} \) groups of equal cardinality. We round the demands of the subtours in each group to the maximum demand in that group. For each \( i \in [1, m] \) and each subtour at \( r_i \), the demand of that subtour is increased to the rounded value by adding a dummy terminal of appropriate demand at vertex \( r_i \). We rearrange the subtours in \( W_v \) as follows.

- Each subtour \( t \in W_v \) in the last group is discarded, i.e., detached from the subtour in \( A_1(v) \) to which it belongs.
- Each subtour \( t \in W_v \) in other groups is associated in a one-to-one manner to a subtour \( t' \in W_v \) in the next group. Letting \( a \) (resp. \( a' \)) denote the subtour in \( A_1(v) \) to which \( t \) (resp. \( t' \)) belongs, we detach \( t \) from \( a \) and reattach \( t \) to \( a' \).

\(^8\)We add empty subtours to the first groups if needed in order to achieve equal cardinality among all groups.
Let \( A(v) \) be the set of the resulting subtours at \( v \) after the rearrangement for all \( t \in W_v \). From the construction, the demand of each subtour in \( A(v) \) is at most the demand of the corresponding subtour in \( S^*(v) \). Since the demand of each subtour in \( A(v) \) is the sum of values in \( Y \) and the set \( Y \) is closed under addition (Property 3 of Fact 29), the demand of that subtour belongs to \( Y \).

To construct a solution \( \hat{S} \) visiting all terminals, it remains to cover those subtours that are discarded in the construction. For each discarded subtour \( t \), we complete \( t \) into a separate tour by adding the connection to the depot in both directions. Observe that the demand of \( t \) belongs to \( Y \). Let \( B \) denote the set of those newly created tours.

Let \( \hat{S} = A(r) \cup B \), where \( r \) denotes the root of the tree \( \hat{T} \).

### 8.2 Analysis of \( \hat{S} \)

It is easy to see that \( \hat{S} \) is a feasible solution to the UCVRP, i.e., each tour in \( \hat{S} \) starts and ends at the depot and has total demand at most 1, and each terminal is covered by some tour in \( \hat{S} \). The solution \( \hat{S} \) satisfies the first four properties of the claim. Following the analysis in Section 5.2 in [MZ22], we obtain that \( \text{cost}(\hat{S}) \leq \frac{2}{2-\epsilon} \cdot \text{cost}(S^*) \), where we use the bounded distance property of \( T \) and the properties of \( \hat{T} \). Combined with the bound on \( \text{cost}(S^*) \) in Theorem 27, the last property of the claim follows.

This completes the proof of the Structure Theorem (Theorem 31).

### 9 Dynamic Program

In this section, we prove Theorem 33.

**Theorem 33.** There is a polynomial time dynamic program that computes a solution for the UCVRP on the tree \( \hat{T} \) with cost at most \( 1.5 + 8\epsilon \) times the optimal cost on the tree \( T \).

Theorem 1 follows immediately from Fact 23 and Theorem 33.

To design the dynamic program in Theorem 33, we compute the best solution on the tree \( \hat{T} \) that satisfies the properties in the Structure Theorem (Theorem 31). The algorithm consists of two phases: the first phase computes local solutions inside components (Section 9.1) and the second phase computes solutions in subtrees in the bottom up order (Section 9.2). Properties 1 and 2 of the Structure Theorem are used in Section 9.1. Properties 3 and 4 of the Structure Theorem are used in Section 9.2. The analysis on the cost of the output solution is given in Section 9.3.

### 9.1 Computing Solutions Inside Components

**Definition 34.** A local configuration \((c, A)\) is defined by a component \( c \) and a list \( A \) consisting of \( \ell(A) \) pairs \((y_1, b_1), (y_2, b_2), \ldots, (y_{\ell(A)}, b_{\ell(A)})\) such that

- \( \ell(A) \leq |Q_c| = O_\epsilon(1) \);
- for each \( i \in [1, \ell(A)] \), \( y_i \in Y_c \) and \( b_i \in \{\text{passing, ending}\} \).

The value of the local configuration \((c, A)\), denoted by \( f(c, A) \), is the minimum cost of a collection of \( \ell(A) \) subtours \( t_1, \ldots, t_{\ell(A)} \) in \( c \) such that:

- \( t_1, \ldots, t_{\ell(A)} \) together cover all terminals in \( c \);
• For each $i \in [1, \ell(A)]$, the demand of $t_i$ is at most $y_i$ and the type of $t_i$ is $b_i$.

• For each cell in $c$, some $t_i$ visits all small terminals in that cell.

When the local configuration $(c, A)$ corresponds to the solution $\hat{S}$ defined in the Structure Theorem (Theorem 31), that solution satisfies the above constraints using the first two properties in that Theorem. Thus the value of that local configuration is at most the cost of $\hat{S}$ in $c$.

The computation for the value of a local configuration is done by exhaustive search and detailed in Algorithm 2.

Algorithm 2 Computation for local configurations inside a component $c$

1: for each list $A = ( (y_1, b_1), (y_2, b_2), \ldots, (y_{\ell(A)}, b_{\ell(A)}) )$ do
2: $f(c, A) \leftarrow \infty$
3: for each partition of $Q_c$ into $\ell$ parts $Q_c^{(1)}, \ldots, Q_c^{(\ell)}$ s.t. $\forall i$, demand($Q_c^{(i)}$) $\leq y_i$ do
4: for each $i \in [1, \ell]$ do
5: $U \leftarrow \{ \text{terminals in } Q_c^{(i)} \}$
6: if $b_i = \text{passing}$ then $U \leftarrow U \cup \{ \text{the exit vertex of } c \}$
7: $t_i \leftarrow \text{the subtour in } c \text{ spanning } U$
8: $f(c, A) \leftarrow \min(f(c, A), \sum_i \text{cost}(t_i))$
9: return $f(c, \cdot)$

Running time. Since $|Q_c| = O_\epsilon(1)$ (Fact 29), the number of partitions of $Q_c$ is $O_\epsilon(1)$, and for each partition, the subtour costs can be computed in polynomial time, hence polynomial time to compute the value of a local configuration. Observe that the number of local configurations in $c$ is $O_\epsilon(1)$. Thus the overall running time to compute the values of all local configurations in $c$ is polynomial.

9.2 Computing Solutions in Subtrees

The algorithm to compute solutions in a subtree is identical to the dynamic program in Section 6.2 in [MZ22] in the equal demand setting, except for a suitable adaptation of the values of the subtour demands. In the arbitrary demand setting, the subtour demands are values in the set $Y$ by Property 3 in the Structure Theorem (Theorem 31). Since the set $Y$ is closed under addition (Fact 29), the dynamic program in [MZ22] carries over to the arbitrary demand setting (this is the place where we use Property 4 in the Structure Theorem). Since $Y$ consists of a polynomial number of values (Fact 29), the polynomial running time of the algorithm in [MZ22] carries over to the arbitrary demand setting.

For completeness, the algorithm and the running time analysis are given in Appendix C.

9.3 Cost Analysis

The cost of the output solution is at most the cost of the solution $\hat{S}$ in the Structure Theorem (Theorem 31), which is at most $1.5 + 8\epsilon$ times the optimal cost for the UCVRP on the tree $T$.

This completes the proof of Theorem 33.

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9When the demand of $t_i$ is strictly less than $y_i$, a dummy terminal will be added to $t_i$ so that the total demand of the real and the dummy terminals on $t_i$ equals $y_i$. 
A Hardness on the Approximation

A.1 UCVRP on Paths

We reduce the bin packing problem to the UCVRP on paths. Consider a bin packing instance with \( n \) items of sizes \( a_1, \ldots, a_n \), where \( 0 < a_i \leq 1 \) for each \( i \in [1, n] \). We construct an instance of the UCVRP on paths as follows. The path consists of \( n + 1 \) vertices \( r, v_1, \ldots, v_n \), such that \( r \) is the depot and \( v_1, \ldots, v_n \) are the terminals. The weight of the edge \( rv_1 \) equals 1; the weight of the edge \( v_i v_{i+1} \) equals 0 for each \( i \in [1, n - 1] \). The demand of \( v_i \) equals \( a_i \) for each \( i \in [1, n] \). Observe that a solution to this instance of the UCVRP is equivalent to a solution to the bin packing instance. Since it is NP-hard to approximate the bin packing problem to better than a 1.5 factor \[WS11\], it is NP-hard to approximate the UCVRP on paths to better than a 1.5 factor.

A.2 UCVRP Inside a Component

We reduce the bin packing problem to the UCVRP inside a component. Consider a bin packing instance with \( n \) items of sizes \( a_1, \ldots, a_n \), where \( 0 < a_i \leq 1 \) for each \( i \in [1, n] \). We construct a component in the UCVRP as follows. The component consists of \( n + 2 \) vertices \( r, v_0, v_1, \ldots, v_n \), which form a star at \( v_0 \), such that \( r \) is the depot and \( v_1, \ldots, v_n \) are the terminals. The weight of the edge \( rv_0 \) equals 1; the weight of the remaining edges \( v_0v_i \) equals 0 for all \( i \in [1, n] \). The demand of \( v_i \) equals \( a_i \) for each \( i \in [1, n] \). Observe that a solution to this instance of the UCVRP is equivalent to a solution to the bin packing instance. Since it is NP-hard to approximate the bin packing problem to better than a 1.5 factor \[WS11\], it is NP-hard to approximate the UCVRP in a component to better than a 1.5 factor.

B Proof of Lemma 9

The proof of Lemma 9 is a minor adaptation from the proof of Lemma 6 in \[MZ22\].

For any subgraph \( H \) of \( b \), the demand of \( H \), denoted by \( \text{demand}(H) \), is defined as the total demand of all terminals in \( H \) that are strictly inside \( b \).

First, we construct leaf clusters as follows. For any vertex \( v \in b \) such that the subtree of \( b \) rooted at \( v \) has demand at least \( \Gamma' \) and each of the subtrees rooted at the children of \( v \) has demand strictly less than \( \Gamma' \), we create a leaf cluster that equals the subtree of \( b \) rooted at \( v \). If the block \( b \) has an exit vertex \( e_b \), we need to ensure that \( e_b \) is the exit vertex of some cluster. To that end, we distinguish two cases: if \( e_b \) belongs to some existing leaf cluster, then we set \( e_b \) to be the exit vertex of that cluster; otherwise, we create a (trivial) leaf cluster consisting of the singleton \( \{e_b\} \) and set \( e_b \) to be the root vertex and the exit vertex of that cluster. See Algorithm 3 for a formal description of the construction of leaf clusters. Observe that the leaf clusters are disjoint subtrees of \( b \).

Definition 35 (key vertices). The backbone of \( b \) is the subgraph of \( b \) spanning the root of \( b \) and the roots of the leaf clusters. We say that a vertex \( v \in b \) is a key vertex if it belongs to one of the three cases: (1) \( v \) is the root of the block \( b \); (2) \( v \) is the root of a leaf cluster; (3) \( v \) has two children in the backbone.

We say that two key vertices \( v_1 \) and \( v_2 \) are consecutive if the \( v_1 \)-to-\( v_2 \) path in the tree does not contain any other key vertex. For each pair of consecutive key vertices \( (v_1, v_2) \), we consider the subgraph between \( v_1 \) and \( v_2 \), and decompose that subgraph into internal clusters, each of demand
Algorithm 3 Construction of leaf clusters in block b.

1: for each vertex \( v \in b \) do
2: \( T(v) \leftarrow \) subtree of \( b \) rooted at \( v \)
3: for each non-leaf vertex \( v \in b \) do
4: Let \( v_1 \) and \( v_2 \) denote the two children of \( v \) in \( b \)
5: if demand\((T(v))\) \( \geq \Gamma' \) and demand\((T(v_1))\) \( < \Gamma' \) and demand\((T(v_2))\) \( < \Gamma' \) then
6: Create a leaf cluster that equals \( T(v) \) and has root vertex \( v \)
7: if \( b \) has an exit vertex \( e_b \) then
8: if \( e_b \) belongs to some existing leaf cluster \( x \) then
9: \( e_x \leftarrow e_b \)
10: else
11: Create a leaf cluster \( x^* \) that equals a singleton \( \{ e_b \} \)
12: \( e_{x^*} \leftarrow e_b \)
13: \( r_{x^*} \leftarrow e_b \)

Algorithm 4 Construction of internal clusters in block b.

1: for each pair of consecutive key vertices \( v_1, v_2 \) such that \( v_1 \) is an ancestor of \( v_2 \) do
2: \( v'_1 \leftarrow \) child of \( v_1 \) on the \( v_1 \)-to-\( v_2 \) path
3: for each vertex \( v \) on the \( v'_1 \)-to-\( v_2 \) path do \( H(v) \leftarrow T(v) \setminus T(v_2) \)
4: \( x \leftarrow v_2 \)
5: while \( H(v'_1) \) has demand at least \( \Gamma' \) do
6: \( v \leftarrow \) the deepest vertex on the \( v'_1 \)-to-\( x \) path such that \( H(v) \) has demand at least \( \Gamma' \)
7: Create an internal cluster that equals \( H(v) \) with root vertex \( v \) and exit vertex \( x \)
8: \( x \leftarrow v \)
9: for each vertex \( v' \) on the \( v'_1 \)-to-\( x \) path do \( H(v') \leftarrow T(v') \setminus T(x) \)
10: Create an internal cluster that equals \( \{(v_1, v'_1)\} \cup H(v_1) \) with root vertex \( v_1 \) and exit vertex \( x \)

at most \( 2 \Gamma \), such that all of these cluster are big (i.e., of demand at least \( \Gamma \)) except possibly for the upmost cluster. See Algorithm 4 for the detailed construction of internal clusters.

The first three properties in the claim follow from the construction. It remains to show the last property in the claim. The following lemma is crucial in the analysis.

**Lemma 36.** We say that a cluster in \( b \) is good if it is a leaf cluster or a big cluster, and is bad otherwise. There is a map from all clusters in \( b \) to good clusters in \( b \) such that each good cluster in \( b \) has at most three pre-images.

**Proof.** First, we define a map \( \phi_{\text{good}} \) that maps each good cluster to itself. Next, we consider the bad clusters. Observe that the root vertex \( r_x \) of any bad cluster \( x \) is a key vertex. We say that a bad cluster \( x \) is a left bad cluster (resp. right bad cluster) if \( x \) contains the left child (resp. right child) of \( r_x \). We define a map \( \phi_{\text{left}} \) from left bad clusters to leaf clusters, such that the image of a left bad cluster is the leaf cluster that is rightmost among its descendants. We show that \( \phi_{\text{left}} \) is injective. Let \( c_1 \) and \( c_2 \) be any left bad clusters. Observe that \( r_{c_1} \) and \( r_{c_2} \) are distinct key vertices. If \( r_{c_1} \) is ancestor of \( r_{c_2} \) (the case when \( r_{c_2} \) is ancestor of \( r_{c_1} \) is similar), then \( \phi_{\text{left}}(c_2) \) is in the left subtree of \( r_{c_2} \) whereas \( \phi_{\text{left}}(c_1) \) is outside the left subtree of \( r_{c_2} \), so \( \phi_{\text{left}}(c_1) \neq \phi_{\text{left}}(c_2) \). In the remaining case, the subtrees rooted at \( r_{c_1} \) and at \( r_{c_2} \) are disjoint, so \( \phi_{\text{left}}(c_1) \neq \phi_{\text{left}}(c_2) \). Thus \( \phi_{\text{left}} \) is injective. Note that every leaf cluster is good, so \( \phi_{\text{left}} \) is an injective map from left bad clusters to
good clusters. Similarly, we obtain an injective map $\phi_{\text{right}}$ from right bad clusters to good clusters. The claim follows by combining the three injective maps $\phi_{\text{good}}$, $\phi_{\text{left}}$, and $\phi_{\text{right}}$. \qedsymbol

Observe that all good clusters are big except possibly one cluster, i.e., the cluster $x^*$ containing the exit vertex $e_b$ of the block. Thus the number of good clusters is at most $\text{demand}(b)/\Gamma' + 1$.

By Lemma 36, the number of clusters in $b$ is at most three times the number of good clusters in $b$, hence the last property of the claim. This completes the proof of Lemma 9.

C Computing Solutions in Subtrees

This section is a slight adaptation from [MZ22]. Everything is identical to [MZ22] except that we use the properties on the set $Y$ for the subtour demands (Fact 29).

Definition 37. A subtree configuration $(v, A)$ is defined by a vertex $v$ and a list $A$ consisting of $\ell(A)$ pairs $(\tilde{y}_1, n_1), (\tilde{y}_2, n_2), \ldots, (\tilde{y}_\ell, n_{\ell(A)})$ such that

- $v$ belongs to the set $I$ (Definition 32);

- $\ell(A) = O_\epsilon(1)$; in particular, when $v$ is a critical vertex, $\ell(A) \leq \left(\frac{1}{\beta}\right)^{1/\beta}$;

- for each $i \in [1, \ell(A)]$, $\tilde{y}_i$ belongs to the set $Y$ (Definition 28) and $n_i$ is an integer in $[0, n]$.

The value of the subtree configuration $(v, A)$, denoted by $g(v, A)$, is the minimum cost of a collection of $\ell(A)$ subtours in the subtree of $\hat{T}$ rooted at $v$, each subtour starting and ending at $v$, that together visit all of the real terminals of the subtree rooted at $v$, such that, for each $i \in [1, \ell(A)]$, there are $n_i$ subtours each with total demand of the real and the dummy terminals that equals $\tilde{y}_i$.

When $v$ is the root $r$, any subtree configuration $(r, A)$ corresponds to a solution to the UCVRP on the tree $\hat{T}$. The output of the algorithm is the minimum value of $g(r, A)$ among all lists $A$.

To compute the values of subtree configurations, we consider the vertices $v \in I$ in the bottom up order. For each vertex $v \in I$ that is the root of a component, we compute the values $g(v, \cdot)$ using Algorithm 5 in Appendix C.1 and for each vertex $v \in I$ that is a critical vertex, we compute the values $g(v, \cdot)$ using Algorithm 6 in Appendix C.2. See Fig. 6.
C.1 Subtree Configurations at the Root of a Component

In this subsection, we compute the values of the subtree configurations at the root $r_c$ of a component $c$.

From Section 5, we have already computed the values of the local configurations in the component $c$. If $c$ is a leaf component, the local configurations in $c$ induce the subtree configurations at $r_c$, where $\tilde{y}_i$ is the demand of the $i$-th subtour in the local configuration and $n_i = 1$ for each $i$. In the following, we consider the case when $c$ is an internal component. From Definition 22, we observe that the exit vertex $e_c$ of the component $c$ is a critical vertex. Thus the values of subtree configurations at $e_c$ have already been computed using Algorithm 6 in Appendix C.2 according to the bottom up order of the computation. To compute the value of a subtree configuration at $r_c$, we combine a subtree configuration at $e_c$ and a local configuration in $c$, in the following way.

Consider a subtree configuration $(e_c, A_e)$ and a local configuration $(c, A_c)$, where

$$A_e = ((\tilde{y}_1, n_1), (\tilde{y}_2, n_2), \ldots, (\tilde{y}_{\ell_e}, n_{\ell_e})), \quad A_c = ((y_1, b_1), (y_2, b_2), \ldots, (y_{\ell_c}, b_{\ell_c})).$$

To each $i \in [1, \ell_c]$ such that $b_i$ is “passing”, we associate $y_i$ with $\tilde{y}_j$ for some $j \in [1, \ell_e]$ with the constraints that $y_i + \tilde{y}_j \leq 1$ and for each $j \in [1, \ell_e]$ at most $n_j$ elements are associated to $\tilde{y}_j$ (because in the subtree rooted at $e_c$ we only have $n_j$ subtours of demand $\tilde{y}_j$ at our disposal). Observe that, for each association $(y_i, \tilde{y}_j)$, we have $y_i \in Y_c \subseteq Y$ (Fact 29), $\tilde{y}_j \in Y$, and $y_i + \tilde{y}_j \leq 1$. Thus $y_i + \tilde{y}_j \in Y$ by Fact 29. Consequently, we obtain the list $A$ of a subtree configuration $(r_c, A)$ as follows:

- For each association $(y_i, \tilde{y}_j)$, we put in $A$ the pair $(y_i + \tilde{y}_j, 1)$.
- For each pair $(\tilde{y}_j, n_j) \in A_e$, we put in $A$ the pair $(\tilde{y}_j, n_j - (\text{number of } y_i \text{'s associated to } \tilde{y}_j))$.
- For each pair $(y_i, \text{“ending”}) \in A_c$, we put in $A$ the pair $(y_i, 1)$.

From the construction, $\ell(A) \leq \ell(A_e) + \ell(A_c)$. Since $e_c$ is a critical vertex, $\ell(A_e) \leq \left( \frac{1}{\alpha} \right)^{\frac{1}{\alpha}}$ by Definition 37. From Definition 34, $\ell(A_c) = O_c(1)$. Thus $\ell(A) \leq \left( \frac{1}{\alpha} \right)^{\frac{1}{\alpha}} + O_c(1) = O_c(1)$.

Next, we compute the cost of the combination of the subtree configuration $(e_c, A_e)$ and the local configuration $(c, A_c)$; let $x$ denote this cost. For any subtour $t$ at $e_c$ that is not associated to any non-spine passing subtour in the component $c$, we pay an extra cost to include the spine subtour of the component $c$, which is combined with the subtour $t$. The number of times that we include the spine subtour of $c$ is the number of subtours at $e_c$ minus the number of passing subtours in $A_e$, which is $\sum_{j \leq \ell_e} n_j - \sum_{i \leq \ell_c} 1 \{ b_i \text{ is “passing”} \}$. Thus we have

$$x = f(c, A_c) + g(e_c, A_e) + \text{cost(spine}_c) \cdot \left( \left( \sum_{j \leq \ell_e} n_j \right) - \left( \sum_{i \leq \ell_c} 1 \{ b_i \text{ is “passing”} \} \right) \right). \quad (5)$$

The algorithm is described in Algorithm 5.

**Running time** Since $|Y| = n^{O_c(1)}$ (Fact 29), the number of subtree configurations $(e_c, A_e)$ and the number of local configurations $(c, A_c)$ are both $n^{O_c(1)}$. For fixed $(e_c, A_e)$ and $(c, A_c)$, the number of ways to combine them is $O_c(1)$. Thus the running time of Algorithm 5 is $n^{O_c(1)}$. 

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Definition 38 (sum list). A sum list \( A \) consists of \( \ell(A) \) pairs \((y_1,n_1), (y_2,n_2), \ldots, (y_{\ell(A)},n_{\ell(A)})\) such that

1. \( \ell(A) \leq (\frac{1}{2})^{\frac{1}{n}} \);
2. For each \( i \in [1,\ell(A)] \), \( y_i \in Y \) is the sum of a multiset of values in \( X \) and \( n_i \) is an integers in \([0,n]\).

From the Structure Theorem, we only need to consider subtree configurations \((z,A)\) such that the list \( A \) is a sum list.

Let \( r_1,r_2,\ldots,r_m \) be the children of \( z \). For each \( i \in [1,m] \), let

\[
A_i = ((y^{(i)}_1,n^{(i)}_1),(y^{(i)}_2,n^{(i)}_2),\ldots,(y^{(i)}_{\ell_i},n^{(i)}_{\ell_i}))
\]

denote the list in a subtree configuration \((r_i,A_i)\). We round the list \( A_i \) to a list

\[
\overline{A}_i = ((\overline{y}^{(i)}_1,\overline{n}^{(i)}_1),(\overline{y}^{(i)}_2,\overline{n}^{(i)}_2),\ldots,(\overline{y}^{(i)}_{\ell_i},\overline{n}^{(i)}_{\ell_i}))
\]

where \( \overline{x} \) denotes the smallest value in \( X \) that is greater than or equal to \( x \), for any value \( x \). The rounding is represented by adding a dummy terminal of demand \( \overline{x} - x \) at vertex \( r_i \) to each subtour of initial demand \( x \).

Let \( S \subseteq Y \) denote a multiset such that for each \( i \in [1,m] \) and for each \( j \in [1,\ell_i] \), the multiset \( S \) contains \( n^{(i)}_j \) copies of \( \overline{y}^{(i)}_j \).
Algorithm 6 Computation for subtree configurations at a critical vertex $z$.

1: for each list $A$ do
2: \quad $g(z, A) \leftarrow +\infty$
3: for each set $X \subseteq Y$ of \( \frac{1}{\beta} \) values do
4: \quad for each $i \in [0, m]$ and each list $A$ do
5: \quad \quad $\mathsf{DP}_i(A) \leftarrow +\infty$
6: \quad \quad $\mathsf{DP}_0(\emptyset) \leftarrow 0$
7: for each $i \in [1, m]$ do
8: \quad for each subtree configuration $(r_i, A_i)$ do
9: \quad \quad $\mathcal{A}_i \leftarrow \text{round}(A_i)$
10: \quad \quad for each sum list $A_{\leq i-1}$ do
11: \quad \quad \quad for each way to combine $A_{\leq i-1}$ and $\mathcal{A}_i$ do
12: \quad \quad \quad \quad $A_{\leq i} \leftarrow$ the resulting sum list
13: \quad \quad \quad \quad $x \leftarrow \mathsf{DP}_{i-1}(A_{\leq i-1}) + g(r_i, A_i) + 2 \cdot n(A_i) \cdot w(r_i, z)$
14: \quad \quad \quad \quad $\mathsf{DP}_i(A_{\leq i}) \leftarrow \min(\mathsf{DP}_i(A_{\leq i}), x)$
15: \quad \quad for each list $A$ do
16: \quad \quad \quad $g(z, A) \leftarrow \min(g(z, A), \mathsf{DP}_m(A))$
17: return $g(z, \cdot)$

Definition 39 (compatibility). A multiset $\mathcal{S} \subseteq Y$ and a sum list

$$A = ((y_1, n_1), (y_2, n_2), \ldots, (y_{\ell(A)}, n_{\ell(A)}))$$

are compatible if there is a partition of $\mathcal{S}$ into $\sum_{i=1}^{\ell(A)} n_i$ parts and a correspondence between the parts of the partition and the values $y_i$’s, such that for each $y_i$, there are $n_i$ associated parts, and for each of those parts, the elements in that part sum up to $y_i$.

For a sum list $A = ((y_1, n_1), (y_2, n_2), \ldots, (y_{\ell(A)}, n_{\ell(A)}))$, the value $g(z, A)$ of the subtree configuration $(z, A)$ equals the minimum, over all sets $X$ and all subtree configurations $\{(r_i, A_i)\}_{1 \leq i \leq m}$ such that $\mathcal{S}$ and $A$ are compatible, of

$$\sum_{i=1}^{m} g(r_i, A_i) + 2 \cdot n(A_i) \cdot w(r_i, z), \tag{6}$$

where $n(A_i)$ denotes $\sum_{j=1}^{\ell_i} n_j^{(i)}$. We note that $n_1 y_1 + n_2 y_2 + \cdots + n_{\ell(A)} y_{\ell(A)}$ is equal to the total demand of the (real and dummy) terminals in the subtree rooted at $z$.

Fix any set $X \subseteq Y$ of $\frac{1}{\beta}$ values. We show how to compute the minimum cost of Eq. (6) over all subtree configurations $\{(r_i, A_i)\}_{1 \leq i \leq m}$ such that $\mathcal{S}$ and $A$ are compatible. For each $i \in [1, m]$ and for each subtree configuration $(r_i, A_i)$, the value $g(r_i, A_i)$ has already been computed using Algorithm 5 in Appendix C.1 according to the bottom up order of the computation. We use a dynamic program that scans $r_1, \ldots, r_m$ one by one: those are all siblings, so here the reasoning is not bottom-up but left-right. Fix any $i \in [1, m]$. Let $\mathcal{S}_i \subseteq Y$ denote a multiset such that for each $i' \in [1, i]$ and for each $j \in [1, \ell_{i'}]$, the multiset $\mathcal{S}_i$ contains $n_j^{(i')} \cdot y_{j}^{(i')}$. We define a dynamic program table $\mathsf{DP}_i$. The value $\mathsf{DP}_i(A_{\leq i})$ at a sum list $A_{\leq i}$ equals the minimum, over all subtree configurations $\{(r_{i'}, A_{i'})\}_{1 \leq i' \leq i}$ such that $\mathcal{S}_i$ and $A_{\leq i}$ are compatible, of

$$\sum_{i'=1}^{i} g(r_{i'}, A_{i'}) + 2 \cdot n(A_{i'}) \cdot w(r_{i'}, z).$$
When \( i = m \), the values \( DP_m(\cdot) \) are those that we are looking for. It suffices to fill in the tables \( DP_1, \ldots, DP_m \).

To compute the value \( DP_i \) at a sum list \( A_{\leq i} \), we use the value \( DP_{i-1} \) at a sum list \( A_{\leq i-1} \) and the value \( g(r_i, A_i) \) of a subtree configuration \((r_i, A_i)\). Let \( A_{\leq i-1} = ((\hat{y}_1, \hat{n}_1), (\hat{y}_2, \hat{n}_2), \ldots, (\hat{y}_\ell, \hat{n}_\ell)) \).

We combine \( A_{\leq i-1} \) and \( \overline{A}_i \) as follows. For each \( p \in [1, \ell] \) and each \( j \in [1, \ell_i] \) such that \( \hat{y}_p + y_j^{(i)} \leq 1 \), we observe that \( \hat{y}_p + y_j^{(i)} \) is the sum of a multiset of values in \( X \), thus belongs to \( Y \) by Fact 29.

We create \( n_{p,j} \) copies of the association \( (\hat{y}_p, y_j^{(i)}) \), where \( n_{p,j} \in [0, n] \) is an integer variable that we enumerate in the algorithm. We require that for each \( p \in [1, \ell] \), \( \sum_{j=1}^{\ell_i} n_{p,j} \leq \hat{n}_p \); and for each \( j \in [1, \ell_i] \), \( \sum_{p=1}^{\ell} n_{p,j} \leq n_j^{(i)} \). The resulting sum list \( A_{\leq i} \) is obtained as follows.

- For each association \((\hat{y}_p, y_j^{(i)})\), we put in \( A_{\leq i} \) the pair \((\hat{y}_p + y_j^{(i)}, n_{p,j})\).
- For each pair \((\hat{y}_p, \hat{n}_p) \in A_{\leq i-1} \), we put in \( A_{\leq i} \) the pair \((\hat{y}_p, \hat{n}_p - \sum_{j=1}^{\ell_i} n_{p,j})\).
- For each pair \((y_j^{(i)}, n_j^{(i)}) \in \overline{A}_i \), we put in \( A_{\leq i} \) the pair \((y_j^{(i)} - \sum_{p=1}^{\ell} n_{p,j}, n_j^{(i)} - \sum_{p=1}^{\ell} n_{p,j})\).

The algorithm is described in Algorithm 6.

**Running time** Since \(|Y| = n^{O(1)}\), the numbers of subtree configurations and of sum lists are \( n^{O(1)} \). Observe that the number of ways to combine them and the number of sets \( X \) are \( n^{O(1)} \). Thus the overall running time of Algorithm 6 is \( n^{O(1)} \).

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