Solving non-abelian loop Toda equations

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Abstract

We construct soliton solutions for non-abelian loop Toda equations associated with general linear groups. Here we consider the untwisted case only and use the rational dressing method based upon appropriate block-matrix representation suggested by the initial \(\mathbb{Z}\)-gradation.

1 Introduction

The Toda systems [1,2,3] associated with loop groups possess features attractive from both mathematical and physical perspectives. The fact that they have the so-called soliton solutions is certainly among such interesting properties. Here, by an \(N\)-soliton solution we simply mean a solution depending on \(N\) linear combinations of independent variables. In particular, the investigation of soliton solutions imply developing methods of solving nonlinear partial differential equations and besides, also modeling various nonlinear phenomena in particle physics and field theory, see, for example, the paper [4] and references therein.

There are various approaches to constructing soliton solutions for loop Toda systems. The best known and elaborated among them are, probably, the rational dressing formalism [5,6], that is a version of the inverse scattering method, and the Hirota’s approach [7,8,9,10,11,12] based on an appropriate change of the field variables. Also certain combinations of these two methods prove to be quite efficient in the purpose of finding soliton solutions of Toda equations [4,13]. Besides, it is worth while mentioning generalizations of the Leznov–Saveliev [14,15,16] and the Bäcklund–Darboux [17,18,19,20,21] methods that were employed at Toda systems.

In a recent paper [22], we have carried out a comparative analysis of the Hirota’s and rational dressing methods in application to abelian Toda systems associated with the untwisted loop groups of complex general linear groups and, in particular, explicitly reproduced the corresponding multi-soliton solutions. Further, in the subsequent paper [23], we have constructed soliton solutions for abelian twisted loop Toda systems. And now, we are going to investigate the non-abelian loop Toda equations being a direct generalization of the systems considered in [22]. Here we work within the

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rational dressing formalism based upon appropriate block-matrix representation. The latter is naturally suggested by the $\mathbb{Z}$-gradation under consideration and turns out to be most suitable to the non-abelian Toda systems.

Inasmuch as the abelian soliton solutions allow for such a physical interpretation as of interacting extended particle-like objects, so their non-abelian generalizations should be very interesting as such objects having additionally certain internal structures. And since this physical interpretation promises a good basis for a consistent modeling of various nonlinear phenomena, the mathematical part consisting in developing the corresponding integration methods and constructing explicit soliton solutions becomes crucial.

Note finally that since the pioneering paper [6] where simplest non-abelian loop Toda equations were presented, certain efforts have been made to solve them by means of various methods. Thus, in [24] the notion of quasi-determinants was exploited for the purpose, see also [25]; in [26] a simplest matrix generalization of the sine-Gordon equation was treated by the rational dressing method. An approach based on the dressing (gauge) transformation method was developed in a series of papers [27, 28, 29, 30] for a simplest case of non-abelian affine Toda systems where a specific gradation leads to a minimal extension of the abelian counterpart, and then the vertex operator method was also used there in order to construct some soliton solutions.

2 Formulation of loop Toda equations

The formulation of Toda systems, in a way most appropriate to our purposes, is based on their simple differential-geometry and group-algebraic background, and here we generally follow the monographs [1, 2] and the papers [31, 32, 33].

Let the trivial fiber bundle $\mathbb{R}^2 \times G \rightarrow \mathbb{R}^2$, with the structure Lie group $G$ and its Lie algebra $\mathfrak{g}$, be given. We identify a connection in this fiber bundle with a $G$-valued 1-form $\mathcal{O}$ on $\mathbb{R}^2$ and decompose it over basis 1-forms,

$$\mathcal{O} = \mathcal{O}_-dz^- + \mathcal{O}_+dz^+,$$

where $z^-$, $z^+$ are the standard coordinates on the base manifold $\mathbb{R}^2$, and the components $\mathcal{O}_-, \mathcal{O}_+$ are $\mathfrak{g}$-valued functions on it. We assume that the connection $\mathcal{O}$ is flat, and it means that its curvature is zero. Then, in terms of the components, we have

$$\partial_-\mathcal{O}_+ - \partial_+\mathcal{O}_- + [\mathcal{O}_-, \mathcal{O}_+] = 0, \quad (2.1)$$

where we use the notation $\partial_- = \partial/\partial z^-$ and $\partial_+ = \partial/\partial z^+$. One can consider this relation as a system of partial differential equations. The general solution of this system is well known,

$$\mathcal{O}_- = \Phi^{-1}\partial_-\Phi, \quad \mathcal{O}_+ = \Phi^{-1}\partial_+\Phi,$$

where $\Phi$ is an arbitrary mapping of $\mathbb{R}^2$ to $G$. Actually, the zero-curvature condition as a system of partial differential equations is trivial due to the gauge invariance. Indeed, if a connection $\mathcal{O}$ satisfies (2.1), then for an arbitrary mapping $\Psi$ of $\mathbb{R}^2$ to $G$ the gauge-transformed connection

$$\mathcal{O}^\Psi = \Psi^{-1}\mathcal{O}\Psi + \Psi^{-1}d\Psi, \quad (2.2)$$

\[\text{However, it is not quite clear how the soliton solutions of [26] were obtained without averaging over the action of the corresponding automorphism group, which is one of the principal ingredients of the rational dressing procedure.}\]
satisfies (2.1) as well.

To obtain nontrivial integrable systems out of the zero-curvature condition we impose on the connection $\mathcal{O}$ some restrictions which destroy the gauge invariance. To come specifically to Toda systems, we should use certain grading and gauge-fixing conditions.

Suppose that $\mathfrak{G}$ is endowed with a $\mathbb{Z}$-gradation,

$$
\mathfrak{G} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{G}_k, \quad [\mathfrak{G}_k, \mathfrak{G}_l] \subset \mathfrak{G}_{k+l},
$$

and $L$ is such a positive integer that the grading subspaces $\mathfrak{G}_k$ and $\mathfrak{G}_{-k}$, where $0 < k < L$, are trivial. The grading condition states that the components of $\mathcal{O}$ have the form

$$
\mathcal{O}_- = \mathcal{O}_0 + \mathcal{O}_{-L}, \quad \mathcal{O}_+ = \mathcal{O}_0 + \mathcal{O}_{+L}, \quad (2.3)
$$

where $\mathcal{O}_0$ and $\mathcal{O}_0$ take values in $\mathfrak{G}_0$, while $\mathcal{O}_{-L}$ and $\mathcal{O}_{+L}$ take values in $\mathfrak{G}_{-L}$ and $\mathfrak{G}_{+L}$ respectively. There is a residual gauge invariance. Indeed, the gauge transformation (2.2) with $\Psi$ taking values in the connected Lie subgroup $\mathcal{G}_0$ corresponding to the subalgebra $\mathfrak{G}_0$ does not violate the grading condition (2.3). Therefore, we additionally impose a gauge-fixing condition of the form

$$
\mathcal{O}_{+0} = 0.
$$

Now the components of the connection $\mathcal{O}$ can be represented as

$$
\mathcal{O}_- = \Xi^{-1} \partial_- \Xi + \mathcal{F}_-, \quad \mathcal{O}_+ = \Xi^{-1} \mathcal{F}_+ \Xi, \quad (2.4)
$$

where $\Xi$ is a mapping of $\mathbb{R}^2$ to $\mathcal{G}_0$, $\mathcal{F}_-$ and $\mathcal{F}_+$ are some mappings of $\mathbb{R}^2$ to $\mathfrak{G}_{-L}$ and $\mathfrak{G}_{+L}$. One can easily see that the zero-curvature condition is equivalent to the equality

$$
\partial_+ (\Xi^{-1} \partial_- \Xi) = [\mathcal{F}_-, \Xi^{-1} \mathcal{F}_+ \Xi] \quad (2.5)
$$

and the relations

$$
\partial_+ \mathcal{F}_- = 0, \quad \partial_- \mathcal{F}_+ = 0. \quad (2.6)
$$

We suppose that the mappings $\mathcal{F}_-$ and $\mathcal{F}_+$ are fixed and consider (2.5) as an equation for $\Xi$ called the Toda equation. When the group $\mathcal{G}_0$ is abelian the corresponding Toda equations are called abelian. In other cases we have non-abelian Toda systems.

Thus, a Toda equation associated with a Lie group $\mathcal{G}$ is specified by a choice of a $\mathbb{Z}$-gradation of the Lie algebra $\mathfrak{G}$ of $\mathcal{G}$ and mappings $\mathcal{F}_-$, $\mathcal{F}_+$ satisfying the conditions (2.6). To classify the Toda equations associated with a Lie group $\mathcal{G}$ one should classify the $\mathbb{Z}$-gradations of the Lie algebra $\mathfrak{G}$ of $\mathcal{G}$.

We consider the case where $\mathcal{G}$ is a loop group of a finite-dimensional Lie group, $\mathcal{L}_{a,M}(G)$, where $a$ is an automorphism of $G$ of order $M$. The corresponding Lie algebra $\mathfrak{G}$ is thus the loop Lie algebra $\mathcal{L}_{A,M}(g)$, where $g$ is the Lie algebra of the Lie group $G$, with $A$ being the respective automorphism of $g$ of order $M$. This is a subalgebra of the loop Lie algebra $\mathcal{L}(g)$ formed by elements $\xi$ satisfying the equality

$$
\xi(\epsilon_{Ms}) = A(\xi(s)),
$$

We assume for simplicity that $\mathcal{G}$ is a subgroup of the group formed by invertible elements of some unital associative algebra $A$. In this case $\mathfrak{G}$ can be considered as a subalgebra of the Lie algebra associated with $A$. Our consideration can be generalized to the case of an arbitrary Lie group $\mathcal{G}$. 

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where \( \epsilon_M = e^{2\pi i/M} \) is the \( M \)th principal root of unity and \( s \in S^1 \). Similarly, the loop group \( \mathcal{L}_{a,M}(G) \) is defined as the subgroup of the loop group \( \mathcal{L}(G) \) formed by the elements \( \chi \) satisfying the equality

\[
\chi(\epsilon_M s) = a(\chi(s)).
\]

For a consistent description of these objects given in a way most suitable for the matter of loop Toda equations we refer to [31, 32, 33]. To have a wider list of publications on such systems see also [34, 35, 36, 37] and references therein.

To specify the Toda equations associated with the loop group \( \mathcal{L}_{a,M}(G) \) we first note that the group \( \mathcal{L}_{a,M}(G) \) and its Lie algebra \( \mathcal{L}_{A,M}(\mathfrak{g}) \) are infinite-dimensional manifolds. However, using the so-called exponential law [38, 39], that can generally be expressed by the canonical identification

\[
C^\infty(\mathcal{M}, C^\infty(\mathcal{N}, \mathcal{P})) = C^\infty(\mathcal{M} \times \mathcal{N}, \mathcal{P}),
\]

where \( \mathcal{M}, \mathcal{N}, \mathcal{P} \) are finite-dimensional manifolds and \( \mathcal{N} \) is besides compact, we reformulate the zero-curvature representation of the Toda equations associated with \( \mathcal{L}_{a,M}(G) \) in terms of finite-dimensional manifolds.

In the case under consideration the connection components \( O_- \) and \( O_+ \) entering the equality (2.1) are mappings of \( \mathbb{R}^2 \) to the loop Lie algebra \( \mathcal{L}_{A,M}(\mathfrak{g}) \). We denote the corresponding mappings of \( \mathbb{R}^2 \times S^1 \) to \( \mathfrak{g} \) by \( \omega_- \) and \( \omega_+ \), and call them also the connection components. The mapping \( \Phi \) generating the connection is a mapping of \( \mathbb{R}^2 \) to \( \mathcal{L}_{a,M}(G) \). Denoting the corresponding mapping of \( \mathbb{R}^2 \times S^1 \) by \( \varphi \) we write

\[
\varphi^{-1} \partial_- \varphi = \omega_-, \quad \varphi^{-1} \partial_+ \varphi = \omega_+.
\]

Seeing that the mapping \( \varphi \) uniquely determines the mapping \( \Phi \), we say that the mapping \( \varphi \) also generates the connection under consideration.

We follow the classification of loop Toda systems performed in [31, 32, 33]. Important for our purposes here is that the initial Toda equation associated with \( \mathcal{L}_{a,M}(G) \) is equivalent to a Toda equation associated with \( \mathcal{L}_{a',M'}(G) \) arising when \( \mathcal{L}_{A',M'}(\mathfrak{g}) \) is supplied with the standard \( \mathbb{Z} \)-gradation.

The grading subspaces for the standard \( \mathbb{Z} \)-gradation of a loop Lie algebra \( \mathcal{L}_{A,M}(\mathfrak{g}) \) are

\[
\mathcal{L}_{A,M}(\mathfrak{g})_k = \{ \xi \in \mathcal{L}_{A,M}(\mathfrak{g}) \mid \xi = \lambda^k x, A(x) = e^{kx}_M \},
\]

where by \( \lambda \) we denote the restriction of the standard coordinate on \( \mathbb{C} \) to \( S^1 \). It is clear that every automorphism \( A \) of the Lie algebra \( \mathfrak{g} \) satisfying the relation \( A^M = \text{id}_\mathfrak{g} \) induces a \( \mathbb{Z}_M \)-gradation of \( \mathfrak{g} \) with the grading subspaces

\[
\mathfrak{g}_{[k]_M} = \{ x \in \mathfrak{g} \mid A(x) = e^{kx}_M \}, \quad k = 0, \ldots, M - 1,
\]

where by \([k]_M \) we denote the element of the ring \( \mathbb{Z}_M \) corresponding to the integer \( k \). Vice versa, any \( \mathbb{Z}_M \)-gradation of \( \mathfrak{g} \) obviously defines an automorphism \( A \) of \( \mathfrak{g} \) satisfying the relation \( A^M = \text{id}_\mathfrak{g} \). In terms of the corresponding \( \mathbb{Z}_M \)-gradation the grading subspaces for the standard \( \mathbb{Z} \)-gradation of a loop Lie algebra \( \mathcal{L}_{A,M}(\mathfrak{g}) \) are

\[
\mathcal{L}_{A,M}(\mathfrak{g})_k = \{ \xi \in \mathcal{L}_{A,M}(\mathfrak{g}) \mid \xi = \lambda^k x, x \in \mathfrak{g}_{[k]_M} \}.
\]

It is evident that for the standard \( \mathbb{Z} \)-gradation the subalgebra \( \mathcal{L}_{A,M}(\mathfrak{g})_0 \) is isomorphic to the subalgebra \( \mathfrak{g}_{[0]_M} \) of \( \mathfrak{g} \) and the Lie group \( \mathcal{L}_{a,M}(G)_0 \) is isomorphic to the
connected Lie subgroup $G_0$ of $G$ corresponding to the Lie algebra $\mathfrak{g}[0]_M$. Hence, the relations (2.4) are equivalent to the relations

$$\omega_- = \gamma^{-1} \partial_- \gamma + \lambda^{-L} c_-,$$  
$$\omega_+ = \lambda^L \gamma^{-1} c_+ \gamma,$$  

(2.8)

where $\gamma$, taken as a smooth mapping of $\mathbb{R}^2 \times S^1$ to $G$ corresponding to the mapping $\Xi$ in accordance with the exponential law, is actually a mapping of $\mathbb{R}^2$ to $G_0$, and respecting the mappings $\mathcal{F}_-$ and $\mathcal{F}_+$, the mappings $c_-$ and $c_+$ above are mappings of $\mathbb{R}^2$ to $\mathfrak{g}_-[-L]_M$ and $\mathfrak{g}_+[L]_M$ respectively. The Toda equation can subsequently be written as

$$\partial_+ (\gamma^{-1} \partial_- \gamma) = [c_-, \gamma^{-1} c_+ \gamma].$$  

(2.9)

The conditions (2.6) imply that

$$\partial_+ c_- = 0, \quad \partial_- c_+ = 0.$$  

(2.10)

We call an equation of the form (2.9) also a Toda equation.

Let us consider the transformations

$$\gamma' = \eta_+^{-1} \gamma \eta_-,$$  
$$c'_- = \eta_+^{-1} c_- \eta_-, \quad c'_+ = \eta_+^{-1} c_+ \eta_+,$$  

(2.11)

(2.12)

where $\eta_-$ and $\eta_+$ are some mappings of $\mathbb{R}^2 \times S^1$ to $G_0$ that satisfy the conditions

$$\partial_+ \eta_- = 0, \quad \partial_- \eta_+ = 0.$$  

If a mapping $\gamma$ satisfies the Toda equation (2.9), then the mapping $\gamma'$ satisfies the Toda equation (2.9) where the mappings $c_-, c_+$ are replaced by the mappings $c'_-$ and $c'_+$. If the mappings $\eta_-$ and $\eta_+$ are such that

$$\eta_+^{-1} c_- \eta_- = c_-,$$  
$$\eta_+^{-1} c_+ \eta_+ = c_+,$$

then the transformation (2.11) is a symmetry transformation for the Toda equation under consideration.

3 Untwisted loop Toda equations

The complete classification of Toda equations associated with twisted loop groups of complex classical Lie groups, where the corresponding twisted loop Lie algebras are endowed with integrable $\mathbb{Z}$-gradations with finite-dimensional grading subspaces, is given in the series of papers [31, 32, 33]. We will use these results related to the particular case of untwisted loop groups of the complex general linear groups. The $\mathbb{Z}$-gradations of the corresponding loop Lie algebras are thus generated by an inner automorphism of the initial finite-dimensional complex Lie algebra $\mathfrak{gl}_n(\mathbb{C})$,

$$A(x) = h x h^{-1},$$

where $x$ is an arbitrary element of $\mathfrak{gl}_n(\mathbb{C})$, and $h$ is a diagonal matrix of the form

$$h = \begin{pmatrix} 
\epsilon_{M}^m I_{n_1} \\
\epsilon_{M}^m I_{n_2} \\
\vdots \\
\epsilon_{M}^m I_{n_p}
\end{pmatrix},$$

where $\epsilon_{M}$ is a diagonal matrix.
where $I_{n_{\alpha}}$ denotes the $n_{\alpha} \times n_{\alpha}$ unit matrix and $M \geq m_1 > m_2 > \ldots > m_p > 0$. Here $n_{\alpha}, \alpha = 1, \ldots, p$, are positive integers, such that $\sum_{\alpha=1}^{p} n_{\alpha} = n$. According to the block-matrix structure of $h$, it is convenient to represent the element $x$ as a $p \times p$ block matrix $(x_{\alpha\beta})$, where $x_{\alpha\beta}$ is an $n_{\alpha} \times n_{\beta}$ matrix,

$$
x = \begin{pmatrix}
x_{11} & x_{12} & \cdots & x_{1p} \\
x_{21} & x_{22} & \cdots & x_{2p} \\
\vdots & \vdots & \ddots & \vdots \\
x_{p1} & x_{p2} & \cdots & x_{pp}
\end{pmatrix}.
$$

(3.1)

Here the inner automorphism $a$ acts on an arbitrary element $g$ of $GL_n(\mathbb{C})$ as $a(g) = hgh^{-1}$, with the same diagonal matrix $h$ given above.

The mapping $\gamma$ has the block-diagonal form

$$
\gamma = \begin{pmatrix}
\Gamma_1 \\
\Gamma_2 \\
\vdots \\
\Gamma_p
\end{pmatrix}.
$$

For each $\alpha = 1, \ldots, p$ the mapping $\Gamma_\alpha$ is a mapping of $\mathbb{R}^2$ to the Lie group $GL_{n_{\alpha}}(\mathbb{C})$.

The mapping $c_+$ has the following block-matrix structure:

$$
c_+ = \begin{pmatrix}
0 & C_{+1} & \cdots & 0 \\
0 & \cdots & \cdots & 0 \\
C_{+0} & 0 & \cdots & 0 \\
& & & 
\end{pmatrix},
$$

(3.2)

where for each $\alpha = 1, \ldots, p - 1$ the mapping $C_{+\alpha}$ is a mapping of $\mathbb{R}^2$ to the space of $n_{\alpha} \times n_{\alpha+1}$ complex matrices, and $C_{+0}$ is a mapping of $\mathbb{R}^2$ to the space of $n_p \times n_1$ complex matrices. The mapping $c_-$ has a similar block-matrix structure:

$$
c_- = \begin{pmatrix}
0 & C_{-0} \\
C_{-1} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
C_{-(p-1)} & 0 & \cdots & 0 \\
& & & 
\end{pmatrix},
$$

(3.3)

where for each $\alpha = 1, \ldots, p - 1$ the mapping $C_{-\alpha}$ is a mapping of $\mathbb{R}^2$ to the space of $n_{\alpha+1} \times n_{\alpha}$ complex matrices, and $C_{-0}$ is a mapping of $\mathbb{R}^2$ to the space of $n_1 \times n_p$ complex matrices. The conditions (2.10) imply

$$
d_+C_{-\alpha} = 0, \quad d_-C_{+\alpha} = 0, \quad \alpha = 0, 1, \ldots, p - 1.
$$

It is not difficult to show that the Toda equation (2.9) is equivalent to the following
system of equations for the mappings $\Gamma_{\alpha}$:

\[
\begin{align*}
\partial_{+} \left( \Gamma_{1}^{-1} \partial_{-} \Gamma_{1} \right) &= -\Gamma_{1}^{-1} C_{+1} \Gamma_{2} C_{-1} + C_{-0} \Gamma_{1}^{-1} C_{+0} \Gamma_{1}, \\
\partial_{+} \left( \Gamma_{2}^{-1} \partial_{-} \Gamma_{2} \right) &= -\Gamma_{2}^{-1} C_{+2} \Gamma_{3} C_{-2} + C_{-1} \Gamma_{1}^{-1} C_{+1} \Gamma_{2}, \\
&\vdots \\
\partial_{+} \left( \Gamma_{p-1}^{-1} \partial_{-} \Gamma_{p-1} \right) &= -\Gamma_{p-1}^{-1} C_{+p} \Gamma_{p} C_{-p} + C_{-p+2} \Gamma_{p-2}^{-1} C_{+p-2} \Gamma_{p-1}, \\
\partial_{+} \left( \Gamma_{p}^{-1} \partial_{-} \Gamma_{p} \right) &= -\Gamma_{p}^{-1} C_{+0} \Gamma_{1} C_{-0} + C_{-p+1} \Gamma_{p-1}^{-1} C_{+p-1} \Gamma_{p}.
\end{align*}
\]

(3.4)

As is shown in [32, 33], if for some $\alpha$ we have $C_{-\alpha} = 0$ or $C_{+\alpha} = 0$, then the system of equations (3.4) is equivalent to a system of equations associated with a respective finite-dimensional Lie group, or to a set of two such systems. Hence, to deal actually with Toda equations associated with a loop group, we assume that all mappings $C_{-\alpha}$ and $C_{+\alpha}$ are nontrivial. This is possible only if $m_{\alpha} = (p - \alpha + 1)L$ and $M = pL$. Moreover, it appears that in the case under consideration we can assume, without any loss of generality, that the positive integer $L$ is equal to 1.

The Toda equations (3.4) can also be written as

\[
\partial_{+} (\Gamma_{\alpha}^{-1} \partial_{-} \Gamma_{\alpha}) + \Gamma_{\alpha}^{-1} C_{+\alpha} \Gamma_{\alpha+1} C_{-\alpha} - C_{-(\alpha-1)} \Gamma_{\alpha-1}^{-1} C_{+\alpha-1} \Gamma_{\alpha} = 0,
\]

(3.5)

with $\Gamma_{\alpha}$ subject to the periodicity condition $\Gamma_{\alpha+p} = \Gamma_{\alpha}$. If transformed according to (2.11), (2.12), the submatrices entering the Toda equations would look here as follows:

\[
\Gamma'_{\alpha} = \eta_{+\alpha}^{-1} \Gamma_{\alpha} \eta_{-\alpha}, \quad C'_{-\alpha} = \eta_{-(\alpha+1)}^{-1} C_{-\alpha} \eta_{-\alpha}, \quad C_{+\alpha} = \eta_{+\alpha}^{-1} C_{+\alpha} \eta_{+(\alpha+1)},
\]

(3.6)

with the block-diagonal matrices $\eta_{\pm}$ defined by $(\eta_{\pm})_{\alpha\beta} = \eta_{\pm\alpha} \delta_{\alpha\beta}$.

Similarly to the abelian case [22], it can be shown that the determinant of the mapping $\gamma$ can be represented in a factorized form as

\[
\det \gamma = \prod_{\alpha=1}^{p} \det \Gamma_{\alpha} = \Gamma_{+} \Gamma_{-}^{-1},
\]

where

\[
\partial_{+} \Gamma_{-} = 0, \quad \partial_{-} \Gamma_{+} = 0.
\]

Then, setting

\[
\eta_{-\alpha} = \Gamma_{-}^{1/n} I_{n_{\alpha}}, \quad \eta_{+\alpha} = \Gamma_{+}^{1/n} I_{n_{\alpha}}
\]

in (3.6), we can see that it is possible to make the determinant of the transformed mapping $\gamma'$ be equal to 1,

\[
\det \gamma' = \prod_{\alpha=1}^{p} \det \Gamma'_{\alpha} = 1.
\]

Therefore, the reduction to the non-abelian Toda systems associated with the loop groups of the special linear groups is possible, just as well as it was in the abelian case [22].
We require that for any $m \in \mathbb{R}^2$ the matrices $c_-(m)$ and $c_+(m)$ commute, that is equivalent to the relations
\[
C_{-(\alpha-1)}C_{+(\alpha-1)} - C_+ C_- = 0. \tag{4.1}
\]
Then it is obvious that
\[
\gamma = I_n, \tag{4.2}
\]
where $I_n$ is the $n \times n$ unit matrix, is a solution to the Toda equation (2.9). Denote a mapping of $\mathbb{R}^2 \times S^1$ to $\text{GL}_n(\mathbb{C})$ which generates the corresponding connection by $\varphi$. Using the equalities (2.7) and (2.8) and remembering that in our case $L = 1$, we write
\[
\varphi^{-1} \partial_- \varphi = \lambda^{-1} c_-, \quad \varphi^{-1} \partial_+ \varphi = \lambda c_+,
\]
where the matrices $c_+$ and $c_-$ are defined by the relations (3.2), (3.3).

To construct more interesting solutions to the Toda equations we will look for a mapping $\psi$, such that the mapping
\[
\varphi' = \varphi \psi \tag{4.3}
\]
would generate a connection satisfying the grading condition and the gauge-fixing constraint $\omega_{+,0} = 0$.

For any $m \in \mathbb{R}^2$ the mapping $\tilde{\psi}_m$ defined by the equality $\tilde{\psi}_m(s) = \psi(m,s)$, $s \in S^1$, is a smooth mapping of $S^1$ to $\text{GL}_n(\mathbb{C})$. We treat the unit circle $S^1$ as a subset of the complex plane which, in turn, is a subset of the Riemann sphere. Assume that it is possible to extend analytically each mapping $\tilde{\psi}_m$ to all of the Riemann sphere. As the result we get a mapping of the direct product of $\mathbb{R}^2$ and the Riemann sphere to $\text{GL}_n(\mathbb{C})$ which we also denote by $\psi$. Suppose that for any $m \in \mathbb{R}^2$ the analytic extension of $\tilde{\psi}_m$ results in a rational mapping regular at the points 0 and $\infty$, hence the name rational dressing. Below, for each point $s$ of the Riemann sphere we denote by $\psi_s$ the mapping of $\mathbb{R}^2$ to $\text{GL}_n(\mathbb{C})$ defined by the equality $\psi_s(m) = \psi(m,s)$.

We work with the Toda equations described in section 3. It means that the mapping $\psi$ is generated by a mapping of the Euclidean plane to the loop group $L_{a,p}(\text{GL}_n(\mathbb{C}))$ with the corresponding inner automorphism of order $p$. Hence, for any $m \in \mathbb{R}^2$ and $s \in S^1$ we should have
\[
\psi(m, \epsilon_p s) = h \psi(m, s) h^{-1}, \tag{4.4}
\]
where $h$ is a block-diagonal matrix explicitly given by the expression
\[
h_{\alpha,\beta} = \epsilon_p^{-\alpha+1} I_{n_\alpha} \delta_{\alpha\beta}, \quad \alpha, \beta = 1, 2, \ldots, p. \tag{4.5}
\]
The equality (4.4) means that two rational mappings coincide on $S^1$, therefore, they must coincide on the entire Riemann sphere.

A mapping, satisfying the equality (4.4), can be constructed by the following procedure. Let $\chi$ be an arbitrary mapping of the direct product of $\mathbb{R}^2$ and the Riemann sphere to the algebra $\text{Mat}_n(\mathbb{C})$ of $n \times n$ complex matrices. Let $\hat{a}$ be a linear operator acting on $\chi$ as
\[
\hat{a} \chi(m, s) = h \chi(m, \epsilon_p^{-1} s) h^{-1}.
\]
It is easy to get convinced that the mapping

$$\psi = \sum_{k=1}^{p} \hat{a}^k \chi$$

satisfies the relation $\hat{a} \psi = \psi$ which is equivalent to the equality (4.4). Here we have $\hat{a}^p \chi = \chi$. Note that $\chi$ is in fact a mapping to the Lie group $\text{GL}_n(\mathbb{C})$, but, to justify the above averaging relation, we should consider $\text{GL}_n(\mathbb{C})$ as a subset of $\text{Mat}_n(\mathbb{C})$.

To construct a rational mapping satisfying (4.4) we start with a rational mapping regular at the points $0$ and $\infty$ and having poles at $r$ different nonzero points $\mu_i$, $i = 1, \ldots, r$. Concretely speaking, we consider a mapping $\chi$ of the form

$$\chi = \left( I_n + p \sum_{i=1}^{r} \frac{\lambda_{i}}{\lambda - \mu_i} P_i \right) \chi_0,$$

where $P_i$ are some smooth mappings of $\mathbb{R}^2$ to the algebra $\text{Mat}_n(\mathbb{C})$ and $\chi_0$ is a mapping of $\mathbb{R}^2$ to the Lie subgroup of $\text{GL}_n(\mathbb{C})$ formed by the elements $g \in \text{GL}_n(\mathbb{C})$ satisfying the equality

$$hgh^{-1} = g,$$  \hspace{1cm} (4.6)

where $h$ is given by the expression (4.5). Actually this subgroup coincides with the subgroup $G_0$. The averaging procedure leads to the mapping

$$\psi = \left( I_n + p \sum_{i=1}^{r} \sum_{k=1}^{p} \frac{\lambda_{i}}{\lambda - e_{i}^{p} \mu_i} h^k P_i h^{-k} \right) \psi_0,$$  \hspace{1cm} (4.7)

where $\psi_0 = p\chi_0$. It is convenient to assume that $\mu_i^p \neq \mu_j^p$ for all $i \neq j$.

Denote by $\psi^{-1}$ the mapping of $\mathbb{R}^2 \times S^1$ to $\text{GL}_n(\mathbb{C})$ defined by the relation

$$\psi^{-1}(m, s) = (\psi(m, s))^{-1}.$$

Suppose that for any fixed $m \in \mathbb{R}^2$ the mapping $\tilde{\psi}^{-1}_m$ of $S^1$ to $\text{GL}_n(\mathbb{C})$ can be extended analytically to a mapping of the Riemann sphere to $\text{GL}_n(\mathbb{C})$ and as the result we obtain a rational mapping of the same structure as the mapping $\psi$,

$$\psi^{-1} = \psi_0^{-1} \left( I_n + \sum_{i=1}^{r} \sum_{k=1}^{p} \frac{\lambda_{i}}{\lambda - e_{i}^{p} \nu_i} h^k Q_i h^{-k} \right),$$  \hspace{1cm} (4.8)

with the pole positions satisfying the conditions $\nu_i \neq 0$, $\nu_i^p \neq \nu_j^p$ for all $i \neq j$, and additionally $\nu_i^p \neq \mu_j^p$ for any $i$ and $j$. We will denote the mapping of the direct product of $\mathbb{R}^2$ and the Riemann sphere to $\text{GL}_n(\mathbb{C})$ again by $\psi^{-1}$.

By definition, the equality

$$\psi^{-1} \psi = I_n$$

is valid at all points of the direct product of $\mathbb{R}^2$ and $S^1$. Since $\psi^{-1} \psi$ is a rational mapping, the above equality is valid at all points of the direct product of $\mathbb{R}^2$ and the Riemann sphere. Hence, the residues of $\psi^{-1} \psi$ at the points $\nu_i$ and $\mu_i$ should be equal to
zero. Explicitly we have

\[
Q_i \left( I_n + \sum_{j=1}^r \sum_{k=1}^p \frac{v_i}{v_i - \epsilon_i^k \mu_j} h^k P_j h^{-k} \right) = 0,
\]

(4.9)

\[
\left( I_n + \sum_{j=1}^r \sum_{k=1}^p \frac{\mu_i}{\mu_i - \epsilon_i^k v_j} h^k Q_j h^{-k} \right) P_i = 0.
\]

(4.10)

We will discuss later how to satisfy these relations, and now let us consider what connection is generated by the mapping \( \varphi' \) defined by (4.3) with the mapping \( \psi \) possessing the prescribed properties.

Using the representation (4.3), we obtain for the component \( s \) of the connection generated by \( \varphi' \) the expressions

\[
\omega_- = \psi^{-1} \partial_- \psi + \lambda^{-1} \psi^{-1} c_- \psi,
\]

(4.11)

\[
\omega_+ = \psi^{-1} \partial_+ \psi + \lambda \psi^{-1} c_+ \psi.
\]

(4.12)

We see that the component \( \omega_- \) is a rational mapping which has simple poles at the points \( \mu_i, v_i \) and zero. Similarly, the component \( \omega_+ \) is a rational mapping which has simple poles at the points \( \mu_i, v_i \) and infinity. We are looking for a connection which satisfies the grading and gauge-fixing conditions. The grading condition in our case is the requirement that for each point of \( \mathbb{R}^2 \) the component \( \omega_- \) is rational and has the only simple pole at zero, while the component \( \omega_+ \) is rational and has the only simple pole at infinity. Hence, we demand that the residues of \( \omega_- \) and \( \omega_+ \) at the points \( \mu_i \) and \( v_i \) should vanish.

The residues of \( \omega_- \) and \( \omega_+ \) at the points \( v_i \) are equal to zero if and only if

\[
(\partial_- Q_i - v_i^{-1} Q_i c_-) \left( I_n + \sum_{j=1}^r \sum_{k=1}^p \frac{v_i}{v_i - \epsilon_i^k \mu_j} h^k P_j h^{-k} \right) = 0,
\]

(4.13)

\[
(\partial_+ Q_i - v_i Q_i c_+) \left( I_n + \sum_{j=1}^r \sum_{k=1}^p \frac{v_i}{v_i - \epsilon_i^k \mu_j} h^k P_j h^{-k} \right) = 0,
\]

(4.14)

respectively. Similarly, the requirement of vanishing of the residues at the points \( \mu_i \) gives the relations

\[
\left( I_n + \sum_{j=1}^r \sum_{k=1}^p \frac{\mu_i}{\mu_i - \epsilon_i^k v_j} h^k Q_j h^{-k} \right) (\partial_- P_i + \mu_i^{-1} c_- P_i) = 0,
\]

(4.15)

\[
\left( I_n + \sum_{j=1}^r \sum_{k=1}^p \frac{\mu_i}{\mu_i - \epsilon_i^k v_j} h^k Q_j h^{-k} \right) (\partial_+ P_i + \mu_i c_+ P_i) = 0.
\]

(4.16)

To obtain the relations (4.13)–(4.16) we made use of the equalities (4.9), (4.10).

Suppose that we have succeeded in satisfying the relations (4.9), (4.10) and (4.13)–(4.16). In such a case from the equalities (4.11) and (4.12) it follows that the connection under consideration satisfies the grading condition.

---

3Here and below discussing the holomorphic properties of mappings and functions we assume that the point of the space \( \mathbb{R}^2 \) is arbitrary but fixed.
It is easy to see from (4.12) that
\[ \omega_+(m,0) = \psi_0^{-1}(m)\partial_+\psi_0(m). \]

Taking into account that \( \omega_{+0}(m) = \omega_+(m,0) \), we conclude that the gauge-fixing constraint \( \omega_{+0} = 0 \) is equivalent to the relation
\[ \partial_+\psi_0 = 0. \quad (4.17) \]

Assuming that this relation is satisfied, we come to a connection satisfying both the grading condition and the gauge-fixing condition.

Recall that if a flat connection \( \omega \) satisfies the grading and gauge-fixing conditions, then there exist a mapping \( \gamma \) from \( \mathbb{R}^2 \) to \( G \) and mappings \( c_- \) and \( c_+ \) of \( \mathbb{R}^2 \) to \( g_- \) and \( g_+ \), respectively, such that the representation (2.8) for the components \( \omega_- \) and \( \omega_+ \) is valid. In general, the mappings \( c_- \) and \( c_+ \) parameterizing the connection components may be different from the mappings \( c_- \) and \( c_+ \) which determine the mapping \( \varphi \). Let us denote the mappings corresponding to the connection under consideration by \( \gamma' \), \( c'_- \) and \( c'_+ \). Thus, we have
\[ \psi^{-1}\partial_-\psi + \lambda^{-1}\psi^{-1}c_-\psi = \gamma'^{-1}\partial_-\gamma' + \lambda^{-1}c'_-, \quad (4.18) \]
\[ \psi^{-1}\partial_+\psi + \lambda\psi^{-1}c_+\psi = \lambda\gamma'^{-1}c'_+. \quad (4.19) \]

Note that \( \psi_\infty \) is a mapping of \( \mathbb{R}^2 \) to the Lie subgroup of \( \text{GL}_n(\mathbb{C}) \) defined by the relation (4.6). Recall that this subgroup coincides with \( G_0 \), and denote \( \psi_\infty \) by \( \gamma \). From the relation (4.18) we obtain the equality
\[ \gamma'^{-1}\partial_-\gamma' = \gamma^{-1}\partial_-\gamma. \]

The same relation (4.18) gives
\[ \psi_0^{-1}c_-\psi_0 = c'_-. \]

Impose the condition \( \psi_0 = I_n \), which is consistent with (4.17). Here we have
\[ c'_- = c_- . \]

Finally, from (4.19) we obtain
\[ \gamma'^{-1}c'_+\gamma' = \gamma^{-1}c_+\gamma. \]

We see that if we impose the condition \( \psi_0 = I_n \), then the components of the connection under consideration have the form given by (2.8) where \( \gamma = \psi_\infty \).

Thus, to find solutions to Toda equations under consideration, we can use the following procedure. Fix 2r complex numbers \( \mu_i \) and \( \nu_i \). Find matrix-valued functions \( P_i \) and \( Q_i \) satisfying the relations (4.9), (4.10) and (4.13)–(4.16). With the help of (4.7), (4.8), assuming that
\[ \psi_0 = I_n , \]
construct the mappings \( \psi \) and \( \psi^{-1} \). Then, the mapping
\[ \gamma = \psi_\infty \quad (4.20) \]
satisfies the Toda equation (2.9).
Let us return to the relations (4.9), (4.10). It can be shown that, if we suppose that the matrices \( P_i \) and \( Q_i \) are of maximum rank, then we get the trivial solution of the Toda equation given by (4.2). Hence, we will assume that \( P_i \) and \( Q_i \) are not of maximum rank. The simplest case here is given by matrices of rank one which can be represented as

\[
P_i = u_i^t w_i, \quad Q_i = x_i^t y_i, \tag{4.21}\]

where \( u, w, x \) and \( y \) are \( n \)-dimensional column vectors.

The \( \mathbb{Z} \)-gradation suggests that it is convenient to consider the \( n \times n \) matrix-valued functions \( P_i \) and \( Q_i \) in the corresponding block-matrix form. According to the representation (3.1), we can write

\[
P_i = \begin{pmatrix} (P_{1i})_{11} & (P_{1i})_{12} & \cdots & (P_{1i})_{1p} \\ (P_{2i})_{21} & (P_{2i})_{22} & \cdots & (P_{2i})_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ (P_{pi})_{p1} & (P_{pi})_{p2} & \cdots & (P_{pi})_{pp} \end{pmatrix},
\]

and make similar block-matrix partition for \( Q_i \), where the submatrices \( (P_i)_{\alpha \beta} \) and \( (Q_i)_{\alpha \beta} \) are complex \( n_\alpha \times n_\beta \) matrices. Then, in terms of such block submatrices, the relations (4.21) take the forms

\[
(P_i)_{\alpha \beta} = u_{i,\alpha}^t w_{i,\beta}, \quad (Q_i)_{\alpha \beta} = x_{i,\alpha}^t y_{i,\beta},
\]

where the standard matrix multiplication of the \( n_\alpha \times 1 \) submatrices \( u_{i,\alpha}, x_{i,\alpha} \) by the \( 1 \times n_\beta \) submatrices \( w_{i,\beta}, y_{i,\beta} \) is implied as respective. We see that, from the point of view of the \( \mathbb{Z} \)-gradation, also the \( n \times 1 \) matrices \( u_i, w_i, x_i \) and \( y_i \) receive a natural representation in a block-matrix form,

\[
t u_i = (u_{i,1}^t u_{i,2} \cdots u_{i,\alpha} \cdots u_{i,p}), \quad t y_i = (y_{i,1}^t y_{i,2} \cdots y_{i,\alpha} \cdots y_{i,p}),
\]

where \( u_{i,\alpha} \) and \( y_{i,\alpha}, \alpha = 1, \ldots, p, \) are complex \( n_\alpha \times 1 \) matrices. We have similar expressions also for \( w_i \) and \( x_i \). This representation, together with the block-matrix form (4.5) of \( h \), allows us to write the relations (4.9) and (4.10) as follows:

\[
t y_{i,\alpha} + \sum_{j=1}^{r} \sum_{\delta=1}^{p} \frac{v_i e_p^{-\beta(\delta-\alpha)}}{v_i - e_p^{-\beta} u_{j,\delta}} (y_{j,\delta} u_{i,\delta})_i^t w_{j,\alpha} = 0, \tag{4.22}
\]

\[
u u_{i,\alpha} + \sum_{j=1}^{r} \sum_{\delta=1}^{p} \frac{\mu_i e_p^{-\beta(\alpha-\delta)}}{\mu_i - e_p^{-\beta} v_{j,\delta}} x_{j,\alpha} (y_{j,\delta} u_{i,\delta})_i^t = 0. \tag{4.23}
\]

Using the identity

\[
\sum_{a=0}^{p-1} \frac{z e_p^{-\beta a}}{z - e_p^{-\beta}} = \begin{cases} p & z = e_p^{-|\beta|_p} \\ 0 & z \neq e_p^{-|\beta|_p} \end{cases}, \tag{4.24}
\]

where \( |\beta|_p \) is the residue of division of \( \beta \) by \( p \), we can rewrite (4.22) in terms of the block submatrices,

\[
t y_{i,\alpha} + p \sum_{j=1}^{r} (R_{\alpha})_{ij}^t w_{j,\alpha} = 0. \tag{4.25}
\]
Here the $r \times r$ matrices $R_\alpha$ are defined as
\[
(R_\alpha)_{ij} = \frac{1}{v_i - \mu_j} \sum_{\beta=1}^{p} v_i^{p-|\beta-\alpha|p} \mu_j^{p-|\alpha-\beta|p} y_{i,\beta} u_{j,\beta}.
\]
The identity (4.24) allows us to write also the submatrix form of (4.23) as
\[
u_i, \alpha + p \sum_{j=1}^{r} x_{j,\alpha} (S_\alpha)_{ji} = 0, \tag{4.26}
\]
where
\[
(S_\alpha)_{ji} = -\frac{1}{v_j - \mu_i} \sum_{\beta=1}^{p} v_j^{p-|\alpha-\beta|p} \mu_i^{p-|\alpha-\beta|p} y_{j,\beta} u_{i,\beta}.
\]
With the help of the equality
\[
p - 1 - |\alpha - 1|_p = | - \alpha|_p
\]
it is straightforward to demonstrate that
\[
(S_\alpha)_{ji} = -\frac{\mu_i}{v_j} (R_{\alpha+1})_{ji},
\]
and so, (4.26) can be written as
\[
u_i, \alpha - p \mu_i \sum_{j=1}^{r} x_{j,\alpha} \frac{1}{v_j} (R_{\alpha+1})_{ji} = 0. \tag{4.27}
\]
We use the equations (4.25) and (4.27) to express the vectors $w_i$ and $x_i$ via the vectors $u_i$ and $y_i$,
\[
t^tw_{i,\alpha} = -\frac{1}{p} \sum_{j=1}^{r} (R^{-1}_\alpha)_{ij} t^j y_{j,\alpha}, \quad x_{i,\alpha} = \frac{1}{p} \sum_{j=1}^{r} u_{j,\alpha} \frac{1}{\mu_j} (R^{-1}_{\alpha+1})_{ji} v_i.
\]
Apart from the summation over the pole indices $j$, there are the corresponding matrix multiplications of the submatrices entering the last two relations. As a result, we come to the following solution of the relations (4.9) and (4.10):
\[
(P_i)_{\alpha\beta} = -\frac{1}{p} u_{i,\alpha} \sum_{j=1}^{r} (R^{-1}_\beta)_{ij} t^j y_{j,\beta}, \quad (Q_i)_{\alpha\beta} = \frac{1}{p} \sum_{j=1}^{r} u_{j,\alpha} \frac{1}{\mu_j} (R^{-1}_{\alpha+1})_{ji} v_i t^j y_{i,\beta}.
\]
Using (4.7) and (4.20), we get
\[
\gamma = \psi_\infty = I_n + \sum_{i=1}^{r} \sum_{\alpha=1}^{p} h^\alpha P_i h^{-\alpha}.
\]
For the submatrices of $\gamma$ this gives the expression
\[
\gamma_{\alpha\beta} = \delta_{\alpha\beta} \left( I_{n_{\alpha}} + p \sum_{i=1}^{r} (P_{i})_{\alpha\alpha} \right) = \delta_{\alpha\beta} \left( I_{n_{\alpha}} - \sum_{i,j=1}^{r} u_{i,\alpha} (R^{-1}_{\alpha})_{ij} t^j y_{j,\alpha} \right).
\]
Hence, in view of the block-diagonal structure of $\gamma$, we have
\[
\Gamma_\alpha = 1 - \sum_{i,j=1}^r u_{i,\alpha} (R_\alpha^{-1})_{ij} t_{y_{j,\alpha}}.
\]

According to our general convention, we assume that the $n_\alpha \times 1$ matrix-valued functions $u_{i,\alpha}$ and $y_{i,\alpha}$ are defined for arbitrary integer values of $\alpha$ and
\[
u_{i,\alpha} + p = \nu_{i,\alpha}, \quad \nu_{i,\alpha} = \nu_{i,\alpha}.
\]
The periodicity of $(R_\alpha)_{ij}$ actually follows from its definition,
\[(R_{\alpha+p})_{ij} = (R_{\alpha})_{ij}.
\]

It appears that it is more convenient to use quasi-periodic quantities $\tilde{u}_{i,\alpha}, \tilde{y}_{i,\alpha}$ and $(\tilde{R}_\alpha)_{ij}$ defined by
\[
\tilde{u}_{i,\alpha} = u_{i,\alpha} \nu_{i,\alpha}, \quad \tilde{y}_{i,\alpha} = y_{i,\alpha} v_{i,\alpha}^{-1},
\]
\[(\tilde{R}_\alpha)_{ij} = v_{i,\alpha}^{-1}(R_{\alpha})_{ij} \nu_{i,\alpha}.
\]

For these quantities we have
\[
\tilde{u}_{i,\alpha+p} = \tilde{u}_{i,\alpha} \nu_{i,\alpha}, \quad \tilde{y}_{i,\alpha+p} = \tilde{y}_{i,\alpha} v_{i,\alpha}^{-1},
\]
\[(\tilde{R}_{\alpha+p})_{ij} = v_{i,\alpha}^{-1}(\tilde{R}_{\alpha})_{ij} \nu_{i,\alpha}.
\]

The expression of the matrix elements of the matrices $\tilde{R}_\alpha$ through the functions $\tilde{y}_{i,\alpha}$ and $\tilde{u}_{i,\alpha}$ has a nicely simplified form
\[
(\tilde{R}_\alpha)_{ij} = \frac{1}{v_{i,\alpha}^{-1} - \nu_{i,\alpha}} \left( \mu_{j,\alpha}^{p-1} \sum_{\beta=1}^{p} \tilde{y}_{i,\alpha} \tilde{u}_{j,\alpha} + v_{i,\alpha}^{p} \sum_{\beta=1}^{p} \tilde{y}_{i,\alpha} \tilde{u}_{j,\alpha} \right). \tag{4.28}
\]

In terms of the quasi-periodic quantities, for the matrix-valued functions $\Gamma_\alpha$ we have
\[
\Gamma_\alpha = I_{n_\alpha} - \sum_{i,j=1}^r \tilde{u}_{i,\alpha} (\tilde{R}_\alpha^{-1})_{ij} t_{\tilde{y}_{j,\alpha}}. \tag{4.29}
\]

It is useful to have also the explicit expression of the inverse mapping $\gamma^{-1}$. Using the relation
\[
\gamma^{-1} = \psi^{-1}_\infty = I_n + \sum_{i=1}^r \sum_{\alpha=1}^p h_{i,\alpha} Q_i h^{-\alpha}
\]
we derive
\[
\Gamma_\alpha^{-1} = I_{n_\alpha} + \sum_{i,j=1}^r \tilde{u}_{i,\alpha} (\tilde{R}_{\alpha+1}^{-1})_{ij} t_{\tilde{y}_{j,\alpha}}. \tag{4.30}
\]

Using the definition of $\tilde{R}_\alpha$, we come to the equality
\[
(\tilde{R}_{\alpha+1})_{ij} = (\tilde{R}_{\alpha})_{ij} - t_{\tilde{y}_{i,\alpha}} \tilde{u}_{i,\alpha}.
\]
It is clear that in the case under consideration we do not have any determinant representation specific to the abelian case [6, 22], and the last two relations are just helpful for verifying the equations of motion by the obtained solutions.

Further, it follows from (4.22) and (4.23) that, to fulfill also (4.13)–(4.16), it is sufficient to satisfy the equations

\[
\begin{align*}
\partial^- y_i &= v_i^{-1} c_- y_i, \\
\partial^+ y_i &= v_i^t c_+ y_i, \\
\partial^- u_i &= -\mu_i^{-1} c_- u_i, \\
\partial^+ u_i &= -c_+ u_i.
\end{align*}
\]

(4.31) (4.32)

The general solution to these equations in the case when \( c_- \) and \( c_+ \) are constant is formally

\[
\begin{align*}
y_i(z^-, z^+) &= \exp(v_i^{-1} c_- z^- + v_i^t c_+ z^+) y_i^0, \\
u_i(z^-, z^+) &= \exp(-\mu_i^{-1} c_- z^- - c_+ z^+) u_i^0,
\end{align*}
\]

where \( y_i^0 = y_i(0, 0) \) and \( u_i^0 = u_i(0, 0) \).

Thus we have shown that it is possible to satisfy (4.9), (4.10) and (4.13)–(4.16) and construct in this way a wide class of solutions to the non-abelian loop Toda equations (3.4). In what follows we will suppose that \( c_- \) and \( c_+ \) represent constant mappings and shall make the above formal solution to the equations (4.31), (4.32) explicit.

5 Deriving soliton solutions

5.1 The eigenvalue problems

Seeing the formal expressions for \( u_i \) and \( y_i \), we understand that we need to somehow handle the exponentials of the matrices \( c_- \) and \( c_+ \). To this end, it is customary to treat them as matrices of linear operators. Assume that the submatrices entering the mappings \( c_- \) and \( c_+ \) are of maximum ranks, that is

\[
\begin{align*}
\text{rank } C_{-\alpha} &= \min (n_{\alpha+1}, n_{\alpha}), \\
\text{rank } C_{+\alpha} &= \min (n_{\alpha}, n_{\alpha+1}),
\end{align*}
\]

and they respect the commutativity of \( c_- \) and \( c_+ \) according to (4.1). Here we consider the case where these matrices are such that the corresponding \( n_{\alpha+1} \times n_{\alpha} \) and \( n_{\alpha} \times n_{\alpha+1} \) submatrices \( C_{-\alpha} \) and \( C_{+\alpha} \) can be brought to the forms

\[
\begin{align*}
(I_{n_{\alpha}} & 0), \\
0 & (I_{n_{\alpha}})
\end{align*}
\]

if \( n_{\alpha} \leq n_{\alpha+1} \), and

\[
\begin{align*}
(I_{n_{\alpha+1}} & 0), \\
0 & (I_{n_{\alpha}})
\end{align*}
\]

if \( n_{\alpha+1} \leq n_{\alpha} \), respectively, by implementing the transformations (2.12), or the same in the submatrix form (3.6), accompanied by an appropriate change of independent variables.

Denote by \( n_s \) the minimum value of the positive integers \( \{n_{\alpha}\} \). Consider the eigenvalue problems for the linear operators \( c_- \) and \( c_+ \). The corresponding characteristic polynomial is \((-1)^n t^{n-pn_s} (t^p - 1)^{n_s}\) giving rise to the characteristic equation

\[t^{n-pn_s} \prod_{\alpha=1}^p (t - e_{\alpha}^p)^{n_s} = 0.\]
Therefore, the spectrum consists of the zero eigenvalue having the algebraic multiplicity \( n - pn_* \) and nonzero eigenvalues being powers of the \( p \)th root of unity having the algebraic multiplicity \( n_* \) each. We also take into account that the spectra of similar matrices coincide.

The eigenvalue problem relations
\[
c_- \Psi_\beta = \epsilon_p^{-\beta} \Psi_\beta, \quad c_+ \Psi_\beta = \epsilon_p^{\beta} \Psi_\beta
\]
are satisfied by the eigenvectors\(^4\)
\[
\Psi_\beta = (\psi_{\beta,1}, \ldots, \psi_{\beta,a}, \ldots, \psi_{\beta,p})
\]
with
\[
\Psi_{\beta,a} = \epsilon_p^{a\beta} \theta_a,
\]
where constant \( n_\alpha \times n_* \) submatrices \( \theta_a \) are subject to the conditions
\[
C_{-a} \theta_a = \theta_{a+1}, \quad C_{+a} \theta_{a+1} = \theta_a. \tag{5.1}
\]

Denote by \( k \) the rank of the matrix \( c_- \). Then we have \( n - k = \text{dim ker } c_- \). It is clear that for the case under consideration \( k = \text{rank } c_- = \sum_{a=1}^p \min \left( n_\alpha, n_{a+1} \right) \geq pn_* \). In general, the algebraic multiplicity of an eigenvalue does not coincide with its geometric multiplicity, the former is just non less than the latter. To be precise, here we have that the algebraic and geometric multiplicities of the nonzero eigenvalues \( t = \epsilon_p^{\pm a} \) are one and the same and equal to \( n_* \) for all \( a = 1, 2, \ldots, p \). Indeed, it can be shown that there are no generalized eigenvectors of \( c_- \) corresponding to its nonzero eigenvalues, that is, no nontrivial eigenvectors of the form \( \Psi_\beta' = (c_- - \epsilon_p^{-\beta} I_n) \Psi_\beta \) for any \( \beta = 1, 2, \ldots, p \).

In contrast, there are nontrivial generalized eigenvectors of \( c_- \) corresponding to the zero eigenvalue. As a consequence, the algebraic multiplicity of the zero eigenvalue, equal to \( n - pn_* \) as is seen from the characteristic equation, does not coincide with its geometric multiplicity equal to \( n - k \).

Hence, treating \( c_- \) as a matrix of a linear operator acting on an \( n \)-dimensional vector space \( V \), we see that the latter can be decomposed into a direct sum as
\[
V = V_0 \oplus V_1,
\]
where \( V_1 \) is a \( pn_* \)-dimensional subspace spanned by the \( \Psi \)-eigenvectors of \( c_- \) with nonzero eigenvalues, actually, \( V_1 = \text{im } c_- \), and \( V_0 \) is simply defined to be its orthogonal complement spanned by the null vectors and generalized null vectors of \( c_- \). In this, the null-subspace spanned by the generalized null vectors has the dimension \( k - pn_* \).

Similar consideration can be given for the matrix \( c_+ \) as well. Note also that the above decomposition induces the corresponding dual decomposition.

Now, seeing the structure of the general solution for \( y_i \) and \( u_i \), we can proceed as follows. Expand the initial values \( y_i^0 \) and \( u_i^0 \) over the basis vectors of \( V \), taking into account its decomposition:
\[
y_i^0 = \Psi_0 d_{i,0} + \sum_{a=1}^p \Psi_a d_{i,a}, \quad u_i^0 = \Psi_0 c_{i,0} + \sum_{a=1}^p \Psi_a c_{i,a},
\]

\(^4\)Here \( \Psi_\beta \) is in fact an \( n \times n_* \) matrix being thus a collection of different \( n_* \) eigenvectors of \( c_- \) corresponding to one and the same eigenvalue \( \epsilon_p^{-\beta} \).
where $\Psi_0$ is an $n \times (n - pn_s)$ matrix whose columns are an appropriate collection of basis vectors of $V_0$, the $n \times n_s$ matrices $\Psi_{\alpha}$ are giving basis vectors of $V_1$ as introduced earlier, $c_{i,0}$ and $d_{i,0}$ are $(n - pn_s) \times 1$ matrices, while $c_{i,\alpha}^{n_s \times 1}$ and $d_{i,\alpha}$ are matrices, altogether encoding the initial value data for $u_i$ and $y_i$. Then, in view of the above consideration of the properties of the matrices $c_{\pm}$, the general solutions to (4.31), (4.32) take the forms

$$y_i(z^-, z^+) = \Psi_0 q_i_-(z^-)q_i_+(z^+)d_{i,0} + \sum_{a=1}^{p} \Psi_{\alpha} \exp \left( v_i^{-1} e_p^a z^- + v_i e_p^a z^+ \right) d_{i,\alpha},$$

$$u_i(z^-, z^+) = \Psi_0 p_i_-(z^-)p_i_+(z^+)c_{i,0} + \sum_{a=1}^{p} \Psi_{\alpha} \exp \left( -\mu_i^{-1} e_p^a z^- - \mu_i e_p^a z^+ \right) c_{i,\alpha},$$

where $p_i^{\pm}(z^\pm)$ and $q_i^{\pm}(z^\pm)$ are $(n - pn_s) \times (n - pn_s)$ matrices with matrix elements $(p_i^{\pm}(z^\pm))_{ab}$ and $(q_i^{\pm}(z^\pm))_{ab}$, $a, b = 1, 2, \ldots, n - pn_s$, being polynomials in $z^\pm$ of degrees not greater than $k - pn_s$. More specifically, these polynomials should be such that the equations (4.31) and (4.32) should be satisfied.

Further, for the submatrices introduced earlier according to the $\mathbb{Z}$-grading structure we obtain

$$t y_{i,\delta} u_{i,\delta} = t d_{i,0} t q_i_+ t q_i_- (\Psi_{0,\delta} \Psi_{0,\delta}) p_j - p_j + c_{j,0}$$

$$+ \sum_{a,\beta=1}^{p} \exp \left( Z_{\alpha}(v_i) - Z_{\beta}(\mu_j) \right) t d_{i,\alpha} (\Psi_{\alpha,\delta} \Psi_{\beta,\delta}) c_{j,\beta},$$

where for convenience we have introduced the notation

$$Z_{\alpha}(\mu_i) = \mu_i^{-1} e_p^{-a} z^- + \mu_i e_p^{a} z^+. \tag{5.2}$$

Recalling that $\Psi_{\beta,\delta} = \delta_{\beta} e_p \theta_{\delta}$ and passing to quasi-periodic quantities, we find

$$t y_{i,\delta} u_{i,\delta} = t d_{i,0} t q_i_+ t q_i_- (\Psi_{0,\delta} \Psi_{0,\delta}) \mu_j^\delta p_j - p_j + c_{j,0}$$

$$+ \sum_{a,\beta=1}^{p} v_i^{-\delta} \mu_j^\delta e_p^{\delta(\alpha + \beta)} \exp \left( Z_{\alpha}(v_i) - Z_{\beta}(\mu_j) \right) t d_{i,\alpha} (\theta_{\delta} \theta_{\delta}) c_{j,\beta}. \tag{5.3}$$

Suppose that the $n_\delta \times n_s$ submatrices $\theta_{\delta}$ are such that

$$t \theta_{\delta} \theta_{\delta} = \Theta,$$ \hspace{1cm} \tag{5.4}

where $\Theta$ is one and the same non-degenerate $n_s \times n_s$ matrix for all $\delta = 1, 2, \ldots, p$. Consequently, we have from (4.28)

$$(\tilde{R}_{\delta})_{ij} = t d_{i,0} t q_i_+ t q_i_- (\Psi^0_{\alpha})_{ij} p_j - p_j + c_{j,0}$$

$$+ \sum_{\beta,\delta=1}^{p} e^{Z_{-\beta}(v_i) - Z_{\delta}(\mu_j)} \frac{e_p^{\alpha(\beta + \delta)}}{1 - \mu_j v_i^{-1} e_p^{\beta + \delta}} t d_{i,\beta} \Theta c_{j,\delta} (\mu_j^\alpha v_i^{-\alpha}), \tag{5.4}$$

where, for sake of brevity, we have used the notation

$$(\Psi^0_{\alpha})_{ij} = \frac{1}{v_i^{-p} - H_j^p} \left( \mu_j^{p-1} v_i^{\epsilon} \Psi_{0,\epsilon}^0 \Psi_{0,\epsilon} \mu_j^\epsilon + v_i^{\epsilon} \sum_{\epsilon=1}^{p} v_i^{-\epsilon} \Psi_{0,\epsilon}^0 \Psi_{0,\epsilon} \mu_j^\epsilon \right).$$

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for this constant \((n - pk_+) \times (n - pk_+)\) matrix. The relations (5.2), (5.3) and (5.4) allow us to construct general solutions \(\Gamma_{\alpha}\) by (4.29) (it should be instructive to compare our general solution with the corresponding construction of [24] where the notion of quasi-determinants was exploited for this purpose).

### 5.2 One-soliton solution

To construct simplest one-soliton solutions \((r = 1)\) to the Toda equations, we assume that the initial data of the system are such that the coefficients \(c_{\alpha}\) are nonzero only for one value of the index \(\alpha\), which we denote by \(I\), and the coefficients \(d_{\alpha}\) are nonzero only for two values of the index \(\alpha\), which we denote by \(J\) and \(K\). Besides, let \(c_0\) and \(d_0\) be zero.\[5\] Thus, for \(n_{\alpha} \times 1\) submatrices of the matrix-valued functions \(\tilde{u}\) and \(\tilde{y}\) we have

\[
\tilde{u}_{\alpha} = \mu^{\alpha} \epsilon_p e^{-Z_l(\mu)} \theta_{\alpha} c_I,
\]

\[
\tilde{y}_{\alpha} = \nu^{\alpha} \epsilon_p e^{Z_l(y)} t_d l^{\alpha} \theta_{\alpha} + \nu^{\alpha} \epsilon_p e^{Z_{-1}(v)} t_d l^{\alpha} \theta_{\alpha}.
\]

The matrix \(\widetilde{R}_\alpha\) is simply a function for this case and is defined by the expression

\[
\widetilde{R}_\alpha = \mu^{\alpha} \nu^{-\alpha} e^{I_{\alpha}} e^{-Z_l(\mu)} \left( e^{Z_l(v)} \frac{\epsilon_p^{I_{\alpha}}}{1 - \mu \nu^{-1} e^{I_{\alpha}}} \left( t_d l c_I \right) + e^{Z_{-1}(v)} \frac{\epsilon_p^{K_{\alpha}}}{1 - \mu \nu^{-1} e^{K_{\alpha}}} \left( t_d l c_I \right) \right).
\]

And for the \(n_{\alpha} \times n_{\alpha}\) matrix-valued functions \(\Gamma_{\alpha}\) this gives

\[
\Gamma_{\alpha} = I_{n_{\alpha}} - (1 - \mu \nu^{-1} e^{I_{\alpha} + J}) Y_{\alpha}(J) + \tilde{d} \epsilon_p^{a(K-J)} e^{Z_{-1}(v) - Z_{-1}(v)} \left( I_{n_{\alpha}} - (1 - \mu \nu^{-1} e^{I_{\alpha} + K}) Y_{\alpha}(K) \right),
\]

where we have introduced constant idempotent \(n_{\alpha} \times n_{\alpha}\) matrices

\[
Y_{\alpha}(A) = \frac{\theta_{\alpha} c_I t_d A \theta_{\alpha}}{t_d A \theta_{\alpha} c_I}, \quad (Y_{\alpha}(A))^2 = Y_{\alpha}(A), \quad A = J, K,
\]

satisfying the relations

\[
Y_{\alpha}(J) Y_{\alpha}(K) = Y_{\alpha}(K), \quad Y_{\alpha}(K) Y_{\alpha}(J) = Y_{\alpha}(J),
\]

and also the notation

\[
\tilde{d} = \frac{t_d l c_I 1 - \mu \nu^{-1} e^{I_{\alpha} + J}}{t_d l c_I 1 - \mu \nu^{-1} e^{I_{\alpha} + K}}.
\]

The expression for \(\Gamma_{\alpha}\) can be rewritten as

\[
\Gamma_{\alpha} = \left[ I_{n_{\alpha}} - (1 - \mu \nu^{-1} e^{I_{\alpha} + J}) Y_{\alpha}(J) \right] \frac{I_{n_{\alpha}} + \epsilon_p^{a \alpha} e^{Z_{\alpha}(\mu) + \tilde{\delta} X_{\alpha}}}{1 + \epsilon_p^{a \alpha} e^{Z_{\alpha}(\mu) + \tilde{\delta} X_{\alpha}}},
\]

\[5\] Note that, essentially unlike the twisted abelian case [23], keeping these null-eigenspace coefficients nonzero here we do not obtain any soliton-like solutions.
where 
\[
\tilde{X}_\alpha = \left( I_{n_\alpha} - (1 - \mu \nu^{-1} \epsilon_p^{l+1}) Y_\alpha^{(l)} \right)^{-1} \left( I_{n_\alpha} - (1 - \mu \nu^{-1} \epsilon_p^{l+K}) Y_\alpha^{(K)} \right)
\]
and we use the notations
\[
\rho = K - l, \quad \kappa_\rho = 2 \sin \frac{\pi \rho}{p}, \quad \zeta = -i \nu \epsilon_p^{-(K+1)} / 2, \quad \tilde{\delta} = \exp \delta,
\]
and the function \(Z\) in the exponent takes the most familiar form
\[
Z(\zeta) = \kappa_\rho (\zeta^{-1} z^- + \zeta z^+).
\]

With the help of the above properties of the matrices \(Y_\alpha^{(l)}\), it is not difficult to make sure that
\[
h_{a,j} C_{+a} h_{a+1,j}^{-1} = C_{+a}
\]
for
\[
h_{a,j} = I_{n_\alpha} - (1 - \mu \nu^{-1} \epsilon_p^{l+1}) Y_\alpha^{(l)}, \quad h_{a,j}^{-1} = I_{n_\alpha} - (1 - \mu^{-1} \nu \epsilon_p^{-(l+1)}) Y_\alpha^{(l)}.
\]
Therefore the transformation
\[
h_{a,j} \Gamma_\alpha \to \Gamma_\alpha \quad (5.5)
\]
is a symmetry transformation of the Toda equations \((3.5)\) realizing a particular case of the simplest WZNW-type symmetry transformation described by the relations \((3.6)\) with \(\eta_{+a} = h_{a,j}, \eta_{-a} = I_{n_\alpha}\). Similarly, one can use the relations
\[
h_{a+1,j}^{-1} C_{-a} h_{a,j} = C_{-a}
\]
to show that also the transformation
\[
\Gamma_\alpha h_{a,j} \to \Gamma_\alpha
\]
is a symmetry transformation of the Toda equations \((3.5)\) corresponding to the transformations \((3.6)\) with \(\eta_{-a} = h_{a,j}^{-1}, \eta_{+a} = I_{n_\alpha}\).

Now, performing the symmetry transformation \((5.5)\), we write the one-soliton solution to the equations \((3.5)\) as follows:
\[
\Gamma_\alpha = \frac{I_{n_\alpha} + \epsilon_p^{a+1} e^{Z(\zeta) + \tilde{\delta}} \tilde{X}_\alpha'}{1 + \epsilon_p^{a+1} e^{Z(\zeta) + \tilde{\delta}}},
\]
with all entries defined above. Using the properties of the idempotent matrices \(Y_\alpha^{(A)}\), we can also rewrite the expressions for the matrices \(\tilde{X}_\alpha\) as
\[
\tilde{X}_\alpha = h_{a,j}^{-1} h_{a,K} = I_{n_\alpha} + \mu^{-1} \nu \epsilon_p^{-(l+1)} \left( (1 - \mu \nu^{-1} \epsilon_p^{l+1}) Y_\alpha^{(l)} - (1 - \mu \nu^{-1} \epsilon_p^{l+K}) Y_\alpha^{(K)} \right).
\]
Using \((4.30)\) we can also show that
\[
\tilde{\Gamma}_\alpha^{-1} = \frac{I_{n_\alpha} + \epsilon_p^{a+1} e^{Z(\zeta) + \tilde{\delta}} \tilde{X}_\alpha'}{1 + \epsilon_p^{a+1} e^{Z(\zeta) + \tilde{\delta}}},
\]
where
\[
\tilde{X}_\alpha' = h_{a,K}^{-1} h_{a,j}.
\]
Here it is obvious that \(\tilde{X}_\alpha' = \tilde{X}_\alpha^{-1}\), and it is not difficult to show that \(\tilde{\Gamma}_\alpha^{-1} \Gamma_\alpha = I_{n_\alpha}\). It is worthwhile noting that when \(n_\alpha = 1\), we get \(Y_\alpha^{(l)} = Y_\alpha^{(K)} = 1\), so that \(\tilde{X}_\alpha = \epsilon_p^{a}\), while \(\tilde{\Gamma}_\alpha = \epsilon_p^{a+1}\), and also \(\tilde{d} \to d\), and so we recover precisely the abelian case \([22]\).
5.3 Multi-soliton solutions

Now, to obtain solutions depending on \( r \) linear combinations of independent variables we assume that for each value of the index \( i = 1, \ldots, r \) the matrix-valued coefficients \( c_{i,\alpha} \) are different from zero for only one value of \( \alpha \), which we denote by \( l_i \), and that the matrix-valued coefficients \( d_{i,\alpha} \) are different from zero for only two values of \( \alpha \), which we denote by \( J_i \) and \( K_i \). And we also use the following slightly simplified notation for such nonvanishing initial-data \( n_a \times 1 \) matrix-valued coefficients:

\[
d_{l_i} = d_{i,l_i}, \quad d_{K_i} = d_{i,K_i} \quad c_{l_i} = c_{i,l_i},
\]

Then we have from the equality (5.4) that

\[
(\tilde{R}_a)_{ij} = v_i^{-\alpha} \epsilon_p^{a_{j}l_i} e^{Z_{l_i}(v_i)} \left( \frac{t^d_{l_i} \Theta c_{l_i}}{1 - \mu_j v_i^{-1} \epsilon_p^{l_i+j}} + \epsilon_p^{(K_i-l_i)\alpha} e^{Z_{K_i}(v_i)-Z_{l_i}(v_i)} \frac{t^d_{K_i} \Theta c_{K_i}}{1 - \mu_j v_i^{-1} \epsilon_p^{l_i+K_i}} \right) \mu_j^a \epsilon_p^{a_{l_i}} e^{-Z_{l_i}(\mu_j)}.
\]

With account of explicit forms of \( u_{i,\alpha} \) and \( t_y_{j,\alpha} \), the expression for \( \Gamma_\alpha \) can be written in the form

\[
\Gamma_\alpha = I_{n_a} - \sum_{i,j=1}^r (\tilde{R}_a^{-1})_{ij} \left( \tilde{Y}_{a,ji}^{(l_i)\alpha} + \epsilon_p^{a_{ji}l_i} e^{Z_{l_i}(v_i)} \tilde{Y}_{a,ji}^{(K_i)\alpha} \right),
\]

where

\[
(\tilde{R}_a^{-1})_{ij} = \tilde{D}_{ij}(v \epsilon_p^{-A}, \mu \epsilon_p^{l_i}) + \epsilon_p^{a_{ji}l_i} e^{Z_{l_i}(v_i)} \tilde{D}_{ij}(v \epsilon_p^{-K}, \mu \epsilon_p^{l_i})
\]

and (for \( A = J, K \))

\[
\tilde{D}_{ij}(v \epsilon_p^{-A}, \mu \epsilon_p^{l_i}) = (t^d_{A_i} \Theta c_{l_i}) D_{ij}(v \epsilon_p^{-A}, \mu \epsilon_p^{l_i}) = (t^d_{A_i} \Theta c_{l_i}) \frac{v_i \epsilon_p^{-A_i}}{v_i \epsilon_p^{-A_i} - \mu_j \epsilon_p^{l_i}}
\]

(cp. with the notation used for the abelian case [22]) and besides,

\[
\tilde{Y}_{a,ij}^{(A)\alpha} = (t^d_{A_i} \Theta c_{l_i}) \tilde{Y}_{a,ij}^{(A)\alpha} = \theta_{a} c_{l_i} t^d_{A_i} \theta_{a}.
\]

The idempotent \( n_a \times n_a \) matrices \( \tilde{Y}_{a,ij}^{(A)\alpha} \) satisfy the following remarkable properties:

\[
\tilde{Y}_{a,ij}^{(A)\alpha} \tilde{Y}_{a,k\ell}^{(B)\alpha} = \left( \frac{t^d_{A_i} \Theta c_{l_i}}{t^d_{A_i} \Theta c_{l_j}} \right) \left( \frac{t^d_{B_k} \Theta c_{l_j}}{t^d_{B_k} \Theta c_{l_{\ell}}^{k'}} \right) \tilde{Y}_{a,k\ell}^{(B)\alpha}\quad A, B = J, K, \tag{5.6}
\]

while their tilded counterparts are subject to the relations of simpler forms,

\[
\tilde{Y}_{a,ij}^{(A)\alpha} \tilde{Y}_{a,k\ell}^{(B)\alpha} = (t^d_{A_i} \Theta c_{l_i}) \tilde{Y}_{a,k\ell}^{(B)\alpha}.
\]

Further, we can write

\[
\Gamma_\alpha = \frac{I_{n_a} \det \tilde{R}_a' - \sum_{i,j=1}^r (\tilde{R}_a')_{ij} \left( \tilde{Y}_{a,ij}^{(l_i)\alpha} + \epsilon_p^{a_{ji}l_i} e^{Z_{l_i}(v_i)} \tilde{Y}_{a,ij}^{(K_i)\alpha} \right)}{\det \tilde{R}_a'},
\]

where

\[
\tilde{R}_a' = \left( \frac{t^d_{l_i} \Theta c_{l_i}}{1 - \mu_j v_i^{-1} \epsilon_p^{l_i+j}} \right) \mu_j^a \epsilon_p^{a_{l_i}} e^{-Z_{l_i}(\mu_j)}.
\]
meaning that, according to Leibniz,

\[ \det \tilde{R}_\alpha' = \sum_{\sigma \in S_r} \text{sgn}(\sigma) \prod_{\ell=1}^{r} \left( \tilde{D}_{\ell,\sigma(\ell)}(v_{e_p}^{-1}, \mu e_p^I) + e_p^{\alpha_p} e^{Z(\xi_\ell)} \tilde{D}_{\ell,\sigma(\ell)}(v_{e_p}^{-K}, \mu e_p^I) \right) \]

and

\[ (\tilde{R}_\alpha')_{ij} = \sum_{\sigma \in S_r} \text{sgn}(\sigma) \prod_{\ell \neq i, \sigma(\ell) \neq j}^{r} \left( \tilde{D}_{\ell,\sigma(\ell)}(v_{e_p}^{-1}, \mu e_p^I) + e_p^{\alpha_p} e^{Z(\xi_\ell)} \tilde{D}_{\ell,\sigma(\ell)}(v_{e_p}^{-K}, \mu e_p^I) \right). \]

Here \( S_r \) is the symmetric group on the set of integers \( \{1, 2, \ldots, r\} \), and \( \text{sgn}(\sigma) \) denotes the signature of the permutation \( \sigma \).

For sake of brevity, it is also convenient to denote \( \tilde{D}_{ij}(A) = \tilde{D}_{ij}(v_{e_p}^{-A}, \mu e_p^I) \). Seeing that

\[ \det \tilde{R}_\alpha' = \det \tilde{D}(v_{e_p}^{-1}, \mu e_p^I) \cdot \det \tilde{R}_\alpha'' \]

where

\[ (\tilde{R}_\alpha'')_{ij} = \delta_{ij} + e_p^{\alpha_p} e^{Z(\xi_\ell)} \sum_{k=1}^{r} \tilde{D}_{ik}(v_{e_p}^{-K}, \mu e_p^I) \tilde{D}_{kj}^{-1}(v_{e_p}^{-1}, \mu e_p^I), \]

we can also write

\[ I_n a \det \tilde{R}_\alpha'' - \sum_{i, j, k=1}^{r} \tilde{D}_{ik}(K) (\tilde{R}_\alpha'')_{kj} \left( \tilde{Y}_{a,ji}^{(j)} + e_p^{\alpha_p} e^{Z(\xi_\ell)} \tilde{Y}_{a,ji}^{(K)} \right) \]

\[ \Gamma_\alpha = \frac{I_n a \det \tilde{R}_\alpha''}{\det \tilde{R}_\alpha''}, \]

where

\[ (\tilde{R}_\alpha'')_{kj} = \sum_{\sigma \in S_r} \text{sgn}(\sigma) \prod_{\ell \neq i, \sigma(\ell) \neq j}^{r} \left( \delta_{\ell,\sigma(\ell)} + e_p^{\alpha_p} e^{Z(\xi_\ell)} \sum_{m=1}^{r} \tilde{D}_{\ell m}(K) \tilde{D}_{m,\sigma(\ell)}^{-1}(J) \right). \]

Let us introduce a new notation for sake of certain brevity:

\[ \tilde{H}_{ij}(K, J) = \sum_{k=1}^{r} \tilde{D}_{ik}(K) \tilde{D}_{kj}^{-1}(J). \]

Then we find that

\[ \Gamma_\alpha = h_{a,j} T_\alpha^{-1} T_\alpha^X, \]

where

\[ h_{a,j} = I_n a - \sum_{i, j=1}^{r} \tilde{D}_{ij}^{-1}(J) \tilde{Y}_{a,ji}^{(j)}, \]

and the quantities \( \tilde{T}_\alpha \) and \( n_\alpha \times n_\alpha \) matrices \( \tilde{T}_\alpha^X \) together represent a non-abelian analogue of the Hirota’s \( \tau \)-functions,

\[ \tilde{T}_\alpha = 1 + \sum_{i=1}^{r} E_{a,i} + \sum_{\ell=2}^{r} \sum_{i_1 < i_2 < \ldots < i_\ell}^{r} \tilde{\eta}_{i_1 i_2 \ldots i_\ell} E_{a,i_1} E_{a,i_2} \ldots E_{a,i_\ell}, \]

\[ \tilde{T}_\alpha^X = I_{n_\alpha} + \sum_{i=1}^{r} E_{a,i} \tilde{X}_{a,i} + \sum_{\ell=2}^{r} \sum_{i_1 < i_2 < \ldots < i_\ell}^{r} \tilde{\eta}_{i_1 i_2 \ldots i_\ell} E_{a,i_1} E_{a,i_2} \ldots E_{a,i_\ell} \tilde{X}_{a,i_1 i_2 \ldots i_\ell}. \]
Here we also use our standard notation, coming yet from the abelian case \cite{[22]},

\[ E_{\alpha,i} = e^{\alpha_p} e^{Z(\xi_i)} + \bar{\delta}_i, \]

with the quantities \( \bar{\delta}_i \) defined by

\[ e^{\bar{\delta}_i} \equiv \bar{H}_{ii} = \sum_{k=1}^{r} \bar{D}_{ik}(K) \bar{D}_{ki}^{-1}(J), \]

and the ‘soliton interaction coefficients’ given by

\[ \tilde{\eta}_{i_1i_2...i_\ell} = \frac{\sum_{\sigma \in S_\ell} \text{sgn}(\sigma) \prod_{m=1}^{\ell} \tilde{H}_{im}\alpha_{\nu(m)}}{\prod_{k=1}^{\ell} \tilde{H}_{ik}k}. \]

Note that in the abelian case \cite{[22]} these quantities factorize into pairwise interaction coefficients. To be noted also here the relationship with our former notation from \cite{[22]}, \( \bar{d}_i \equiv \bar{H}_{ii} \). Finally, the matrices \( \bar{X}_\alpha... \) are defined as follows:

\[ \bar{X}_{\alpha,i_1i_2...i_\ell} = h^{-1}_{\alpha,J} X_{\alpha,i_1i_2...i_\ell}, \]

where for \( \ell = 1 \)

\[ X_{\alpha,i} = I_{n_\alpha} - \frac{1}{\bar{H}_{ii}} \sum_{j,k=1}^{r} \left( \bar{D}_{kj}^{-1}(J) \bar{H}_{ji} - \bar{D}_{ki}^{-1}(J) \bar{H}_{ij} \right) \bar{Y}_{\alpha,jk}^{(J)} + \frac{1}{\bar{H}_{ii}} \sum_{k=1}^{r} \bar{D}_{ki}^{-1}(J) \bar{Y}_{\alpha,jk}^{(K)} \]

and the higher order quantities are

\[ X_{\alpha,i_1i_2...i_\ell} = I_{n_\alpha} - \sum_{\sigma \in S_\ell} \text{sgn}(\sigma) \prod_{m=1}^{\ell} \tilde{H}_{im}\alpha_{\nu(m)} \times \left( \sum_{j,k=1}^{r} \sum_{\sigma \in S_\ell+1} \text{sgn}(\sigma) \bar{D}_{kj}^{-1}(J) \bar{H}_{ji}\alpha_{\nu(1)} \bar{H}_{i2}\alpha_{\nu(2)} ... \bar{H}_{i\ell}\alpha_{\nu(\ell)} \bar{Y}_{\alpha,jk}^{(J)} \right) + \sum_{k=1}^{r} \sum_{\sigma \in S_\ell} \text{sgn}(\sigma) \bar{D}_{ki}^{-1}(J) \bar{H}_{i2}\alpha_{\nu(2)} \bar{H}_{i3}\alpha_{\nu(3)} ... \bar{H}_{i\ell}\alpha_{\nu(\ell)} \bar{Y}_{\alpha,jk}^{(K)} \right), \quad \ell \geq 2, \]

with \( S_\ell \) being the symmetric group on the set \{1, 2, ..., \ell\}, \( S_{\ell+1}' \) the symmetric group on the set \{j, i_1, i_2, ..., i_\ell\}, everywhere admitting that \( \sigma(i_m) = i_{\sigma(m)} \).

In particular, for \( \ell = r \), we obtain a remarkably simple equality

\[ X_{\alpha,12...r} = h_{\alpha,K} = I_{n_\alpha} - \sum_{j,k=1}^{r} \bar{D}_{kj}^{-1}(K) \bar{Y}_{\alpha,jk}^{(K)} \]

and

\[ \tilde{\eta}_{12...r} = \frac{\det \bar{H}}{\prod_{k=1}^{r} \bar{H}_{ik}k} = \exp \left( -\sum_{k=1}^{r} \delta_k \right) \det \bar{H}. \]

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We notice that \( h_{a,j}^{-1} \) is a linear combination of \( \tilde{Y}_{a,ij}' \),

\[
h_{a,j}^{-1} = I_{n_a} + \sum_{i,j=1}^{r} \tilde{B}_{ji}^{-1}(J) \tilde{Y}_{a,ij}'\]

\[
\tilde{B}_{ij}(J) = (t_{ij} \Theta c_{ij}) \frac{\mu_j e_p}{v_i e_p} \]

and so, as well as the matrices \( X_{a,\ldots} \), also the \( \tilde{X}_{a,\ldots} \) can easily be written with the help of (5.7) just in form of \( I_{n_a} \) minus certain linear combinations of \( \tilde{Y}_{a,ij}' \) and \( \tilde{Y}_{a,jk}' \).

With the help of the relations (5.1) and (5.6), (5.7) it can be seen that

\[
C_{a} = h_{a+1,1} C_{a} h_{a,j} = C_{a}, \quad C_{a}' = h_{a,j}^{-1} C_{a} h_{a+1,j} = C_{a}.
\]

Hence, again, as it was the case for the one-soliton solutions, the transformation

\[
h_{a,j} \Gamma_a \rightarrow \Gamma_a
\]

is a symmetry transformation of the Toda equations (3.5). Consequently, we can write the multi-soliton solution to the nonlinear matrix differential Toda equations (3.5) as the ‘ratio’

\[
\Gamma_a = \tilde{T}_a^{-1} \tilde{T}_a^X,
\]

where the ‘numerator’ and ‘denominator’ are given explicitly by (5.8). Observe also that for \( p = n \) one has \( n_a = 1 \) and then \( \tilde{X}_{a,ij12\ldots i_\ell} \) turns into \( e_p^{(\rho_1+\rho_2+\ldots+\rho_p)} \), and so, one obtains \( \tilde{X}_a^X \rightarrow \tilde{T}_a+1, \) with \( \tilde{T}_a \) reproducing the Hirota’s \( \tau \)-functions for abelian Toda systems [22].

It is also useful to have an explicit expression for the inverse mapping, that is

\[
\Gamma_a^{-1} = \tilde{T}_a^{-1} \tilde{T}_a^{X}. \]

Here, the entries of this expression are defined according to the relations (5.8), with \( \tilde{X}_{a,\ldots} \) replaced by

\[
\tilde{X}_{a,ij12\ldots i_\ell}^{-1} = X_{a,ij12\ldots i_\ell}^{-1} h_{a,j},
\]

where for \( \ell = 1 \) we have

\[
X_{a,i}^{-1} = I_{n_a} + \frac{1}{\tilde{F}_{ii}} \sum_{j,k=1}^{r} \left( \tilde{B}_{kj}^{-1}(J) \tilde{F}_{ji} - \tilde{B}_{ki}^{-1}(J) \tilde{F}_{ij} \right) \tilde{Y}_{a,ijk}^{(j)} + \frac{1}{\tilde{F}_{ii}} \sum_{k=1}^{r} \tilde{B}_{ki}^{-1}(J) \tilde{Y}_{a,ijk}^{(K)}
\]

with the quantities \( \tilde{F} \) defined similarly to \( \tilde{H} \), only that by means of \( \tilde{B} \),

\[
\tilde{F}_{ij}(K, J) = \sum_{k=1}^{r} \tilde{B}_{ik}(K) \tilde{B}_{kj}^{-1}(J),
\]

and for \( \ell \geq 2 \) the other \( n_a \times n_a \) inverse matrices \( X_{a,\ldots}^{-1} \) are

\[
X_{a,ij12\ldots i_\ell}^{-1} = I_{n_a} + \frac{1}{\tilde{F}_{ii}} \sum_{\sigma \in S_\ell} \text{sgn}(\sigma) \prod_{m=1}^{\ell} \tilde{F}_{iml_{\sigma(m)}}^{-1} \times
\]

\[
\times \left( \sum_{j,k=1}^{r} \text{sgn}(\sigma) \tilde{B}_{kj}^{-1}(j) \tilde{F}_{i1l_{\sigma(1)}} \tilde{F}_{i2l_{\sigma(2)}} \cdots \tilde{F}_{i\ell l_{\sigma(\ell)}} \tilde{Y}_{a,jk}^{(j)} + \sum_{k=1}^{r} \text{sgn}(\sigma) \tilde{B}_{ki}^{-1}(j) \tilde{F}_{i1l_{\sigma(1)}} \tilde{F}_{i2l_{\sigma(2)}} \cdots \tilde{F}_{i\ell l_{\sigma(\ell)}} \tilde{Y}_{a,ik}^{(K)} \right).
\]
Now, to make our construction a little bit more transparent, we add the following observation. Let us consider an $r \times r$ matrix $\tilde{\Delta}_i$ being explicitly of the form

$$\tilde{\Delta}_i = \begin{pmatrix}
\tilde{D}_{11}(J) & \tilde{D}_{12}(J) & \cdots & \tilde{D}_{1r}(J) \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{D}_{i,1}(K) & \tilde{D}_{i,2}(K) & \cdots & \tilde{D}_{i,r}(K) \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{D}_{r1}(J) & \tilde{D}_{r2}(J) & \cdots & \tilde{D}_{rr}(J)
\end{pmatrix}.$$ 

That is, one takes the matrix $\tilde{D}(J)$ and simply changes its $i$th row by the corresponding row of the matrix $\tilde{D}(K)$. Then we can write

$$X_{a,i} = I_{n_a} - \sum_{j,k=1}^{r} (\tilde{\Delta}_i^{-1})_{kj} \tilde{Y}_{a,jk} - \sum_{k=1}^{r} (\tilde{\Delta}_i^{-1})_{ki} \tilde{Y}^{(K)}_{a,ik}. $$

And, besides, one also has

$$e^{\tilde{\Delta}_i} = \tilde{H}_{ii} = \frac{\det \tilde{\Delta}_i}{\det \tilde{D}(J)}.$$ 

In general, for $\ell \geq 2$, introduce the following $r \times r$ matrix:

$$\tilde{\Delta}_{1i_2 \cdots i_\ell} = \begin{pmatrix}
\tilde{D}_{11}(J) & \tilde{D}_{12}(J) & \cdots & \tilde{D}_{1r}(J) \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{D}_{i_1,1}(K) & \tilde{D}_{i_1,2}(K) & \cdots & \tilde{D}_{i_1,r}(K) \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{D}_{i_\ell,1}(K) & \tilde{D}_{i_\ell,2}(K) & \cdots & \tilde{D}_{i_\ell,r}(K) \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{D}_{r1}(J) & \tilde{D}_{r2}(J) & \cdots & \tilde{D}_{rr}(J)
\end{pmatrix}.$$ 

Here, similarly to the preceding, we have taken the matrix $\tilde{D}(J)$ and replaced its rows $\tilde{D}_{ij_1}, \tilde{D}_{ij_2}, \ldots, \tilde{D}_{ij_r}$, for $j$ running from 1 to $r$, by the corresponding matrix elements of $\tilde{D}(K)$. And with the help of the introduced matrices we immediately find out that

$$X_{a,i_1i_2 \cdots i_\ell} = I_{n_a} - \sum_{j,k=1}^{r} (\tilde{\Delta}_{1i_2 \cdots i_\ell}^{-1})_{kj} \tilde{Y}_{a,jk} - \sum_{j=1}^{r} (\tilde{\Delta}_{i_1i_2 \cdots i_\ell}^{-1})_{kj} \tilde{Y}^{(K)}_{a,jk}.$$ 

Also the corresponding inverse matrices $X_{a, \ldots}^{-1}$ can easily be found by replacing $\tilde{D}$ by $\tilde{B}$ in the above construction and putting there the sign $-$ instead of $-$. 

Note finally that if $n_s = 1$ some relations simplify, and we recover for them certain expressions specific to the abelian case [22].
6 Conclusion

In this paper we have considered the non-abelian Toda systems associated with the untwisted loop groups of the complex general linear groups. Developing the rational dressing method, we have constructed multi-soliton solutions for these equations. Here, the block-matrix representation of the group and algebra elements, as suggested by the $\mathbb{Z}$-gradation, turned out to be most appropriate to the problem under consideration. The solutions are presented in a form of a direct matrix generalization of the expressions obtained earlier for the abelian case \[22\]. In particular, the fact of non-abelian generalization shows up explicitly through the special matrices $\tilde{X}_a,...$ and non-factorability of the ‘soliton interaction coefficients’ $\tilde{\eta}_{i_1i_2}...$.

We have also observed that the reduction to the non-abelian loop Toda systems associated with the complex special linear groups can be performed.

Now, it would be interesting to generalize our consideration to other non-abelian loop Toda systems described in the classification of \[32,33\], that is to Toda systems associated with twisted loop groups of general linear groups and twisted and untwisted loop groups of the complex orthogonal and symplectic Lie groups.

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