Quantum acoustic bremsstrahlung of impurity atoms in a Bose–Einstein condensate

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We study the process of scattering of two impurity atoms accompanied by generation of an elementary excitation in a surrounding Bose–Einstein condensate. This process, unlike the phonon generation by a single impurity atom, has no velocity threshold and can be regarded as a quantum acoustic analog of a bremsstrahlung in quantum electrodynamics.

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Sixty years ago Landau proposed a phenomenologic microscopic description of the effect of superfluidity [1]. He considered a massive body moving in a medium with the elementary excitation energy spectrum \( \epsilon(q) \), where \( q \) is the kinetic momentum of an elementary excitation.

The energy conservation law avoids excitation generation and, therefore, dissipation of the kinetic energy of the moving body, if the velocity of the body is less than the critical one, \( v_{cr} = \min_{q} [\epsilon(q)/q] \). If the velocity \( v_{cr} \) of the body is less than the critical velocity, the energy conservation law avoids excitation generation and, therefore, dissipation of the kinetic energy of the moving body. The energy conservation law avoids excitation generation and, therefore, dissipation of the kinetic energy of the moving body, if the velocity of the body is less than the critical one, \( v_{cr} = \min_{q} [\epsilon(q)/q] \). In the case of the Bogoliubov tipe spectrum of elementary excitations in a dilute Bose–condensed gas [2], \( v_{cr} \) coincides with the speed of sound in such a degenerate quantum gas. It has been known, however, that the Landau criterion applies in its strict form only to microscopic objects moving through a Bose–Einstein condensate (BEC) [3]. If the size of an object exceeds the healing length of the BEC, the object must be regarded as a macroscopic one, and its motion can result in generation of vortex pairs in the BEC. Reduction, due to effects of the moving object size, of the critical velocity with respect to the value given by the Landau’s theory has been demonstrated in the MIT group experiments [4], where a laser beam has been used as a macroscopic object stirring the BEC. From the other hand, the MIT group also performed an experiment [5] on probing BEC superfluidity by microscopic objects, namely, by atoms transferred from the BEC to the untrapped hyperfine state by a Raman laser pulse. A good agreement of the results with the Landau criterion was demonstrated.

The probability of an elementary excitation generation by an impurity atom in a dilute BEC has been calculated by Timmermans and Côté [6].

The processes described above are not related to collisions between impurity particles in the presence of a BEC. In the present paper we try to fill this gap in the theory of ultracold gases. We consider a homogeneous BEC of neutral atoms of the first kind at zero temperature with a small admixture of atoms of the second kind (impurity atoms) traveling through the BEC at different velocities. The mass of an atom of the \( j \)th kind is denoted by \( m_j \). The number density of atoms in the BEC is \( n_1 \).

Let us consider two colliding impurity atoms. Before the collision their momenta are \( \frac{1}{2}P \pm p \), respectively, where \( P \) is the center-of-mass momentum and \( p \) is the momentum of their relative motion in the center-of-mass frame of reference. If the collision is accompanied by a generation of an elementary excitation with the momentum \( q \), in the final state the center-of-mass momentum is \( P' = P - q \), and the momentum of relative motion \( p' \) is less by absolute magnitude than \( p \), unlike a case of elastic collision. The Bogoliubov spectrum for elementary excitations gives \( \epsilon(q) = \sqrt{q^2/(2m_1)} \sqrt{q^2/(2m_1) + 2m_2 c_s^2} \), where \( c_s \) is the speed of sound in the BEC. The number density of impurity atoms is assumed to be small enough to neglect interaction with them in the expression for the energy spectrum of elementary excitations in the BEC. The energy conservation law

\[
\frac{P^2}{4m_2} + \frac{p^2}{2m_2} = \frac{(P-q)^2}{4m_2} + \frac{p'^2}{2m_2} + \epsilon(q)
\]

always admits solutions with \( q \neq 0 \) and the initial velocities of the colliding impurity atoms less (and even much less) than the critical velocity \( v_{cr} = c_s \).

For the sake of simplicity, we shall consider hereafter the case of \( P = 0 \). In this case Eq.(1) takes the form

\[
\epsilon(q) + \frac{q^2}{4m_2} + \frac{p'^2 - p^2}{2m_2} = 0.
\]

It is obvious that even if \( p \ll m_2 c_s \), the generation of a phonon with \( q \leq p^2/(2m_2 c_s) \) is possible.

In Fig. 1 we plot the five diagrams describing the process mentioned above. There are three different kinds of vertices corresponding to the different multipliers appearing in the transition matrix element \( M(p, p') \). The first one is the only kind of vertex appearing in the diagrams in Fig. 1 (a, b), where it is denoted by small filled circles. It corresponds to the multiplier \( g_{12} \sqrt{n_1(u_K - v_K)/\sqrt{V}}. \) Here \( V \) is the quantization volume, \( u_K = \sqrt{K^2/(2m_1) + m_1 c_s^2}/2\pi(K) + \frac{1}{2} \) and \( v_K = \sqrt{K^2/(2m_1) + m_1 c_s^2}/2\pi(K) - \frac{1}{2} \) are the Bogoliubov transformation coefficients [3]. \( K \) is the kinetic momentum of an elementary excitation, \( K = Q \) for a virtual phonon in the intermediate state and \( K = q \) for the actual phonon in the final state. The coupling constant \( g_{12} \) describes interaction of atoms of the 1st kind with atoms of the 2nd kind and is introduced according to the common definition \( g_{ij} = 2\pi \hbar^2 (m_i + m_j) a_{ij}/(m_i m_j) \), \( i, j = 1, 2 \), where \( a_{ij} \) is the scattering
length for $s$-wave scattering of an atom of the $i$th kind on an atom of the $j$th kind. Then in Fig. 1(c, d) we see vertices denoted by large filled circles and proportional to $g_{22}/V$. In Fig. 1(e), there is a vertex denoted by an open circle and proportional to $g_{11}\sqrt{m_1}\frac{1}{u_q(u_Qu_{Q'}-u_Qv_{Q'}-v_Qv_Q)-v_q(v_Qv_{Q'}-u_Qv_{Q'}-u_Qv_Q)}/\sqrt{V}$.

FIG. 1. The five diagrams describing the process of scattering of two impurity atoms (solid lines) accompanied by emission of a phonon (dashed line).

It is worth to note that the diagrams of Fig. 1 (a, b) are in certain sense analogous of the diagrams describing bremsstrahlung of charged particles in quantum electrodynamics [9]; impurity atoms and BEC elementary excitations in our case play role of charged massive particles and photons, respectively. However, in our case, the attraction potential between two impurity atoms emerging due to virtual phonon exchange is a short-range potential. For a pair impurity atoms moving with velocities below $c_s$ this potential is just of the Yukawa type, and its Fourier transform is

$$\mathcal{U}_{eff}(Q) = \frac{g_{22}^2}{g_{11}1 + [Q/(2m_1c_s)]^2}. \quad (3)$$

Thus, the diagrams of Fig. 1 (a, b) are similar to those of Fig. 2 (c, d) where the direct interaction between impurity atoms is short-range, too. To make the analogy with quantum electrodynamics closer, we refer to the bremsstrahlung process in a collision between a charged particle and a neutral atom (see, e.g., the recent paper by Korol and co-workers [10] and numerous references therein related to ordinary as well as to polarizational bremsstrahlung). The last diagram shown in Fig. 1 (e) is distinct from the previous ones. It has no analog among third-order processes in quantum electrodynamics (only in fifth order a similar process appears, if annihilation of two virtual photon and creation of one photon in a final state require creation of an electron-positron pair in an intermediate state). Therefore, it is reasonable to represent the transition matrix element as the sum

$$\mathcal{M}(\mathbf{p}, \mathbf{p}') = \mathcal{M}^{(1)}(\mathbf{p}, \mathbf{p}') + \mathcal{M}^{(2)}(\mathbf{p}, \mathbf{p}'), \quad (4)$$

where the first term in the right hand side corresponds to the four diagrams of Fig. 1 (a — d) together, and the second term corresponds to the diagram shown in Fig. 1 (e).

We consider here the case of impurity atom velocities less than the speed of sound in the BEC. Under this condition elementary excitations both in the final and intermediate state belong to the phonon range of the Bogoliubov spectrum, and their magnitudes differ significantly: $Q \approx |\mathbf{p} - \mathbf{p}'| \gg q$. For the diagram of Fig. 1 (e) the momenta of the two virtual phonons are practically opposite, since their sum $Q + Q' = q$ is relatively small. Therefore the effects of Bose statistics of phonons are not pronounced for the diagrams with two elementary excitations in the intermediate state [Fig. 1 (b, e)], because they occupy essentially different modes.

If the relative velocity of the colliding impurity atoms is small compared to the sound velocity then we have also $Q \ll m_2c_s$ and, hence, can the processes contributing to $\mathcal{M}^{(1)}(\mathbf{p}, \mathbf{p}')$ within the simplified model of effective contact interaction. I.e., we describe them by the diagrams similar to those of Fig. 1 (c, d) where the vertex corresponding to the contact interaction of two impurity atoms is substituted by $g_{eff}/V$, where $g_{eff} = g_{22} - g_1\sqrt{g_1}$. It is also convenient to define the effective scattering length $a_{eff} = a_{22} - (1 + m_1/m_2)^2 m_2a_{12}^2/(4m_1a_{11})$, so that $g_{eff} = 4\pi\hbar^2a_{eff}/m_2$.

Using the conditions and approximations discussed above, we calculate the matrix elements

$$\mathcal{M}^{(1)}(\mathbf{p}, \mathbf{p}') = -\frac{4g_{eff}g_{11}\sqrt{m_1}}{V^{3/2}c_s\sqrt{2m_1c_sq}|\mathbf{p} - \mathbf{p}'|}, \quad (5)$$

and

$$\mathcal{M}^{(2)}(\mathbf{p}, \mathbf{p}') = \frac{g_{22}^2g_{11}a_{12}^{3/2}}{2V^{3/2}c_s\sqrt{2m_1c_sq}|\mathbf{p} - \mathbf{p}'|}. \quad (6)$$

For evaluation of an energetic denominator $(p^2 - p'^2)/m_2 + q^2/(4m_2)$ appearing in derivation of Eq. (5), the energy conservation law [Eq. (2)] was used. For the further calculations involving $\mathcal{M}^{(2)}(\mathbf{p}, \mathbf{p}')$, it is useful to recall the well-known expansions [11] $|\mathbf{p} \mp \mathbf{p}'|^{-1} = p^{-1}\sum_{\ell=0}^{\infty} (\pm p'/p)^\ell P_\ell (\cos \vartheta)$, where $P_\ell$’s are Legendre polynomials, $\vartheta$ is the angle between $\mathbf{p}$ and $\mathbf{p}'$, and $p < p'$. One can note that the matrix element given by Eq. (6) contributes only to s-wave scattering while the matrix element given by Eq. (5) contributes also to higher angular momentum scattering channels.

The total cross-section of the process of two impurity atoms collision accompanied by a phonon emission can be written, taking into account the effects of impurity atoms quantum statistics, as [11]

$$\sigma_{tot} = \frac{12\pi}{\hbar} \int \frac{V d^3\mathbf{p'}}{(2\pi\hbar)^2} \int \frac{V d^3q}{(2\pi\hbar)^2} \frac{\sqrt{2}}{2} |\mathcal{M}(\mathbf{p}, \mathbf{p}')| \pm |\mathcal{M}(\mathbf{p}, -\mathbf{p}')|^2 \delta \left( \epsilon(q) + \sqrt{\frac{q^2}{4m_2} + \frac{p^2 - p'^2}{m_2}} \right). \quad (7)$$
where the main term of the ratio of smallness of this cross-section is the smallness of the BEC diluteness parameter, and its reverse process. The first term in the right hand side corresponds to phonon scattering on impurity atoms and the second term corresponds to quantum acoustic bremsstrahlung and its reverse process. The first term can be written explicitly as follows:

\[
N_{ph}^{(col)} = -8\pi a_{12}^2 n_2 m_{12}^{-1} q N_{ph}. \tag{13}
\]

This expression is exact if \( m_2 \gg m_1 \). However, even if \( m_1 \) and \( m_2 \) are comparable, Eq. (13) provides a reasonable approximation, provided that \( p \ll m_{2c_s} \).

Calculation of the rate of change of phonon number in the sound wave due to the processes involving a pair of impurity atoms results in the following expression:

\[
N_{ph}^{(ph)} = \frac{N_{ph} n_2 V m_{2c_s}^2}{2\pi \hbar^2} \left[ \frac{p'_f M(p, p')}{2} \right]^{1/2} \tag{12}
\]
\[ N_{ph}^{(q)b} = \frac{\pi}{8} N_{ph} \frac{a_{12}^2}{a_{11}} \frac{m_2 c_s}{p} \frac{m_1 c_s}{q} \left[ \frac{p'}{p} \xi(p) \right] \times \Theta \left( \frac{p^2 - q^2}{2m_2} - \epsilon(q) - \frac{q^2}{4m_2} \right) \]  

where \( \xi(p) = \left( \frac{2a_{11} - 16a_{eff}}{a_{11}m_2 c_s} \right)^2 \). The net effect of the two processes contributing to Eq.(13) is always sound wave attenuation.

As an example, consider the case when \( m_2 \approx m_1 \equiv m \), all the scattering lengths \( a_{ij} \approx a_{12} \) (hence, \( a_{eff} \ll a_{12} \)), \( p/(m c_s) \approx 0.3 \), and \( q/(m c_s) \approx 0.07 \). If, e.g., we set \( p = 0.3 m c_s \) and \( q = 0.07 m c_s \), Eq.(12) gives

\[ N_{ph} = -8 \pi a_{12}^2 c_s N_{ph} \left( 0.07 + 0.21 n_2/n_1 \right) . \]

Thus increase of sound wave dissipation rate by few dozens percents seems experimentally feasible.

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