Liouville Type Theorem about $p$-harmonic 1 form, $p$-harmonic map and harmonic $q$ form

Xiangzhi Cao ‡

Contents

1 Introduction 2
2 Preliminary 6
3 Liouville type theorem in terms of integral curvature condition 8
4 Liouville theorem in terms of BiRic curvature 15
5 Liouville type theorem for harmonic $q$ form 21

Abstract

In this paper, we will use the normalized integral Ricci curvature to investigate Liouville type property of $p$ harmonic function on Riemannian manifold. secondly, we will use the BiRic curvature to obtain Liouville theorem for $p$ harmonic function or $p$ harmonic 1 form. Lastly,

*School of information engineering, Nanjing Xiaozhuang University, Nanjing 211171, China
‡aaa7756kijlp@163.com
we we will use the BiRic curvature to obtian Liouville theorem for harmonic \( q(q \geq 2) \) form.

1 Introduction

Let \((M, g)\) and \((N, h)\) be two Riemannian manifold, \(u: (M, g) \rightarrow (N, h)\) The energy functional of \(p\) harmonic map is defined by

\[
E_p(u) = \int_M \frac{|\nabla u|^p}{2} dv_g,
\]
whose Euler-lagrange equation is as follows:

\[
\tau_p(u) = \text{div}(|du|^{p-2} du), \tag{1.1}
\]

In particular, when \(N = \mathbb{R}\), \(p\) harmonic map reduces to \(p\) harmonic function. It is well known that the regularity of \(p\) harmonic function is not better than \(C^{1,\alpha}\) (c.f. [39][35][25]).

There are a lot of works on Liouville theorem for \(p\) harmonic map and \(p\) harmonic function. Generally speaking, the common strategy is to use Bochner formula and Kato inequality by imposing the curvature of the domain manifold and target manifold. In [38], Wang studied \(p\) harmonic map and \(p\) harmonic function from submanifold in partially negative manifold to Cartan Hardmard manifold. In [5], we extended the results of [38] to \(p\) harmonic function and \(p\) harmonic map. One can also refer to [1, 20, 28, 33, 42, 2, 22, 12, 29] and reference therein for the works on Liouville theorem of \(p\) harmonic maps.

In previous literatures, Liouville type theorem for \(p\)-harmonic map was obtained under certain conditions, for example, lower bound on Ricci curvature of \(M\), the Sobolev inequality or weighted Poincare inequality holds on \(M\). If \(M\) is a submanifold in Riemannian manifold, then the second fundementan form shall satisfy some conditons, meanwhile \(M\) is stable submanifold. However, little is known for Liouville type theorem of \(p\) harmonic map if the domain manifold has small normalized integral Ricci curvature (c.f. (2.2)).
Integral Ricci curvature is an important notion which can be used to study comparison theom([40][27]) and gradient estimates. In [9], Dai used the integral Ricci curvature to study the heat kernel bound. In [36], Wang and Wei used the notion of integral Bakry-Émery Ricci curvature to obtain the local sobolev inequality.

In this paper, we will prove Liouville type theorem for $p$ harmonic function on manifold $M$ whose normalized integral Ricci curvature is sufficiently small.

Let $(M, g, e^{-f}dv_g)$ be metric measure space, let $\pi : E \to M$ be a vector bundle of rank $m$ over $M$, we call $\omega$ is $E$-valued weighted $p$ harmonic form if it satisfies

$$d\omega = 0, \delta_f(|\omega|^{p-2}\omega) = 0.$$ 

where $\delta_f = \delta + i\nabla_f$. One can refer to [30, section 3] for the definition of the two operators $d$ and $\delta$. In fact, they are just a slightly generalizations of exterior differential operator and codifferential operator on Riemannian manifold.

When $f = 0$, we call $\omega$ $E$ valued $p$ harmonic form if it satisfies

$$d\omega = 0, \delta(|\omega|^{p-2}\omega) = 0.$$ 

Moreover, let $(M, g) \to (N, h)$, take $E = u^{-1}TN$, then it is easy to see that $du$ is $u^{-1}TN$ valued $p$ harmonic form on $M$.

It is interesting to obtain Liouville theorem for $p$ harmonic form on Riemannian manifold. The research of $p$ harmonic $l$ form mainly focus on vanishing property and finiteness of the vector space $p$ harmonic $l$ form. On the one hand, there are a lot researches for vanishing theorem of $p$ harmonic $l$ form on submanifold in Riemannian manifold with special curvature property, such as [18][15][16]. On the other hand, the present research mainly focus on the vanishing theorem of $p$ harmonic $l$ form on manifold with special structure such as conformally flat or special curvature property, such as [19][43][10][6]. One can refer to [18] for the studies on finiteness property of vector space $p$ harmonic $l$ form on Riemannian manifold.
Seo and Yun [30] proved that if manifold has nonnegative Bakry-Émery Ricci curvature, then weighted $p$ harmonic 1 form must vanish. As an corollary, they got the Liouville theorem of weighted $p$ harmonic map on manifold with nonnegative Bakry-Émery Ricci curvature.

In [31] [32], Shen and Ye introduced the concept of BiRic curvature. Bi-Ricci curvature is an important tool to obtain vanishing theorem for differential forms and to estimate Ricci curvature of submanifold when the Bi-Ricci curvature is bounded below by a suitable constant. Bi-Ricci curvature can also be used to study stable submanifolds immersed in a Riemannian manifold. Moreover, nonnegative BiRic curvature is weaker than nonnegative sectional curvature.

Tanano [34] obtained Liouville theorem of $L^2$ harmonic 1-forms on $M$ which complete noncompact orientable stable minimal hypersurface in a Riemannian manifold with non-negative BiRic curvature. Wang [37] proved the Liouville theorem for harmonic map with finite energy in terms of BiRic curvature. Cheng [7] used the condition that $BiRic > \frac{n-5}{4}H^2$ to improve the results in [21], and obtained the vanishing theorem. Li and Zou [23] further generalized the results in [7] by relaxing the lower bound of BiRic. Cheng [8] studied the topologic property of complete finite index hypersurface immersed in a manifold $N$ with constant mean curvature, if $BiRic > \frac{n-5}{4}H^2$. We refer the readers to the just mentioned references and references therein for further discussions.

One can refer to [11] for the tensor.

$$BiRic^\delta_f(X, n) := Ric_f(X, X) + \delta Ric_f(n) - K(X, n).$$

where $X, n$ are two unit vector field. When $f = 0, \delta = 1$, it is just the BiRicci tensor.

Let $M$ be an $n(\geq 3)$-dimensional complete $f$-stable noncompact $f$-minimal submanifold isometrically immersed in an $(n+k)$-dimensional metric measure space $(\bar{M}, \bar{g}, e^{-f}d\bar{v}_g)$. In [11], they proved that if $\bar{M}$ has $\overline{BiRic}_f \geq 0$, any weighted harmonic 1-form with finite $L^2$-energy on $M$ vanishes.
However, up to now, there is rare study about $p$ harmonic map ($p \neq 2$) in terms of BiRic curvature or weighted BiRic curvature. In this paper, we will prove that if $\bar{M}$ has $\overline{\text{BiRic}}^\delta \geq 0$, any $p$-harmonic 1-form with finite $L^{2p-2}$-energy on $M$ vanishes.

Vanishing property of harmonic form is important topic in differential geometry, one can refer to these works ([?, ?, ?, ?]) and reference therein. However, the conditions which were used to obtain vanishing theorem of harmonic form on submanifold or Riemmanian manifold are, for example, requiring the lower bound of Weitzenböck curvature operator, or requiring Sobolev inequality holds on $M$ or weighted poincaré inequality holds on $M$, or imposing some lower bound on the first eigenvalue of the operator $\Delta_g$. If harmonic forms on Riemqannian submanifold are considered, one may require the submanifold is stale or impose some conditions on the second fundamental form or mean curvature of submanifold. Of course, there are many extra conditions to obtain vanishing property of harmonic form on manifold. Generally speaking, the studies on harmonic $q$ ($q \geq 2$) form are more difficulty than that of $L^2$ harmonic 1 form. For harmonic 1 form in terms of BiRic curvature, one can refer to [7] [21] and [11]. However, at present, there is rare study about harmonic $q$ ($q \geq 2$) form in terms of BiRic curvature or weighted BiRic curvature. In this paper, we will prove that if $\bar{M}$ has $\overline{\text{BiRic}}^\delta \geq 0$, any $L^2$ harmonic $q$-form on $M$ vanishes.

The organization of this paper is as follows: in section 2, we give some Lemmas, formula and definitions in the proof of our Theorems. in section 3, we will derive Liouville theorem of $p$ harmonic function on Riemannian manifold in terms of integral curvature; in section 4, we will obtain Liouville theorem of $p$ harmonic function or $p$ harmonic 1 form in terms of BiRic curvature; in section 5, we will study vanishing property for harmonic $q$ form in terms of BiRic curvature.
2 Preliminary

In this section, we give some lemmas which will be used in this paper. We firstly recall Kato inequality for \( p \)-harmonic function (cf. lemma 2.4 in [17]):

\[
|\nabla (|\! du|^{p-2} du)|^2 \geq \frac{n}{n-1} |\nabla |du||^{p-1}|^2. \tag{2.1}
\]

However, for \( p \)-harmonic map, we know that ([3])

\[
|\nabla (|\! du|^{p-2} du)|^2 \geq |\nabla |du||^{p-1}|^2.
\]

A map \( u \) is called \( L^q \)-finite energy if

\[
\int_M |\nabla u|^q d\nu < \infty.
\]

In the literature (c.f. [40][36]), we can find the quantity

\[
\text{Ric}_f^H = \max \{0, (n-1)H - \rho_f(x)\},
\]

where \( \rho_f(x) \) is the smallest eigenvalue of \( \text{Ric}_f \) tensor of \( M \).

**Remark 1.** \( \text{Ric}_f^H = 0 \) if and only if \( \text{Ric}_f \geq (n-1)H \).

**Definition 1** (c.f. [40]).

\[
\|\phi\|_{p}(r) := \sup_{x \in M} \left( \int_{B(x,r)} |\phi|^p \cdot A_f e^{-at} dt d\theta_{n-1} \right)^{\frac{1}{p}},
\]

where \( \partial_r f \geq -a \) for some constant \( a \geq 0 \), along a minimal geodesic segment from \( x \in M \). Here \( A_f(t, \theta) \) is the volume element of weighted form \( e^{-f}dv_g = A_f(t, \theta)dt \wedge d\theta_{n-1} \) in polar coordinate, and \( d\theta_{n-1} \) is the volume element on unit sphere \( S^{n-1} \). Sometimes it is convenient to work with the normalized curvature quantity (c.f. [40][36])

\[
\bar{k}(p, f, H, a, r) := \sup_{x \in M} \left( \frac{1}{V_f(x, r)} \cdot \int_{B(x,r)} \left( \text{Ric}_f^H \right)^p A_f e^{-at} dt d\theta_{n-1} \right)^{\frac{1}{p}}, \tag{2.2}
\]

where \( V_f(x, r) := \int_{B(x,r)} e^{-f}dv \). The reader should care about the difference of the same notation \( k(p, f, H, a, r) \) in [40][36]. Obviously, \( \|\text{Ric}_f^H\|_p (r) = 0 \) (or \( \bar{k}(p, f, H, a, r) = 0 \)) iff \( \text{Ric}_f \geq (n-1)H \). When \( f = 0 \) (and \( a = 0 \)), all above notations recover the usual integral curvature on manifolds.
One can refer to these works [9] [27] for integral Ricci curvature. A map $u$ is called $f$-weighted $L^q$-finite energy if $\int_M |\nabla u|^q e^{-f} dv_g < \infty$.

**Lemma 1** (c.f. (2.5) in [24]). For any differential $q$ form, assume that $M$ is submanifold of $N$, $M$ has flat normal bundle and $N$ has pure curvature tensor.

$$\sum (R_{ij} + f_{ij}) \omega^{ii_2...i_q}_{ii_2...i_q} - \frac{q-1}{2} \sum R_{kjih} \omega^{k}_{i_3...i_q} \omega^{ih_{i_3...i_q}} = F_1(\omega) + F_2(\omega),$$

where

$$F_1(\omega) = \sum_{k=1}^{n} \bar{R}_{ikik} \omega^{i_2...i_q}_{i_2...i_q}$$
$$- \frac{q-1}{2} \sum \left( \bar{R}_{iijj} \omega^{ij_{i_3...i_q}}_{i_3...i_q} \omega^{ij}_{i_3...i_q} + \bar{R}_{ijji} \omega^{ij_{i_3...i_q}}_{i_3...i_q} \omega^{ji}_{i_3...i_q} \right)$$
$$= \sum_{k=1}^{n} K_{ik} \omega^{i_2...i_q}_{i_2...i_q} - (q-1) \sum K_{ij} \omega^{ij_{i_3...i_q}}_{i_3...i_q} \omega^{ji}_{i_3...i_q}$$
$$= \sum_{k=1}^{n} K_{1i} \omega^{i_2...i_q}_{1i_2...i_q}$$

(2.3)

$$F_2(\omega) \geq \frac{1}{q} \left( \inf_{i_1,...i_n} \sum_{t=1}^{q} \sum_{h=q+1}^{n} K_{iti_h} \omega^{i_2...i_q}_{i_2...i_q} \omega_{i_2...i_q} \right) |\omega|^2.$$

and

$$F_2 \geq \frac{1}{2} (n^2 |H|^2 - C_{n,q} |A|^2) |\omega|^2,$$

where $C_{n,q} = \max\{q, n-q\}$.
Lemma 2 (c.f. Lemma 2.4 in [15] or Lemma 2.5 in [13]). For any closed q-form ω and φ ∈ C∞(M), we have

\[ |d(φω)| = |dφ ∧ ω| ≤ |dφ| · |ω|. \]

Let

\[ A_{p,n,q} = \begin{cases} 
1 + \frac{1}{\max\{q,n-q\}} & \text{if } p = 2 \\
1 + \frac{1}{(p-1)^2} \min\{1, \frac{(p-1)^2}{n-1}\} & \text{if } p > 2 \text{ and } q = 1 \\
1 & \text{if } p > 2 \text{ and } 1 < q ≤ n-1
\end{cases} \]

Lemma 3 (Kato’s inequality, c.f. Lemma 2.2 in [13]). For p ≥ 2, q ≥ 1, let ω be an p-harmonic q-form on a complete Riemannian manifold M^n. The following inequality holds

\[ |∇ (|ω|^{p-2}ω)|^2 ≥ A_{p,n,q}|∇|ω|^{p-1}|^2. \]

Moreover, when p = 2, q > 1 then the equality holds if and only if there exists a 1-form α such that

\[ ∇ω = α ⊗ ω - \frac{1}{\sqrt{q+1}}θ_1(α ∧ ω) + \frac{1}{\sqrt{n+1-2}}θ_2(i_αω). \]

3 Liouville type theorem in terms of integral curvature condition

Lemma 4 (c.f. Propostion 1.3 in [36]). Let (M, g, e^{-f}dv_g) be a complete smooth metric measure space. Assume that ∂_r f ≥ -a along all minimal geodesic segments for some constant a ≥ 0. For p > \frac{n}{2}, there exists \varepsilon = \varepsilon(n,p,a) > 0 such that if \bar{r}^2\bar{κ}(p,f,a,r) ≤ \varepsilon, Then the Sobolev inequality holds

\[ \int_{B_x(R)} |∇h|e^{-f}dv ≥ 10^{-2n}e^{-2a}R^{-1} \left( \int_{B_x(R)} h^{\frac{n}{n-1}}e^{-f}dv \right)^{\frac{n-1}{n}}, \]

holds for all h ∈ C^∞_0(B_x(R)).
Theorem 3.1. Let \((M, g)\) be an \(n \geq 3\)-dimensional complete noncompact simply connected complete metric measure space. If there exists a positive constant \(\epsilon\), such that \(\tilde{k}(n, 0, H, 0, R) \leq \epsilon\) and \(\frac{1}{n-1} \geq \left(\frac{p-2}{p-1}\right)^2, p \geq 2, 0 < H\).

We also assume that there is a point \(q_1 \in M\) such that \(\text{Ric}(X, X)|_{q_1} \neq 0\) for all \(0 \neq X \in T_{q_1}M\). If one of the following conditions holds

(1) \(H < \frac{1}{n-1} \left[ \frac{1}{n-1} - \left(\frac{p-2}{p-1}\right)^2 \right] \frac{\epsilon_1 \epsilon_2 + 1}{\epsilon_1^2}, \) where \(\epsilon_1, \epsilon_2\) are arbitrary small positive constants.

(2) \(\text{Vol}(B(R)) \leq CR^{n-\epsilon},\) where \(C\) is a constant independent of \(R, \epsilon\) is arbitrary small positive constant.

then any \(p\)-harmonic function with finite \(L^{2p-2}\)-energy on \(M\) is a constant.

Remark 2. \(R^2 \tilde{k}(n, 0, H, 0, R)\) is scaling invariant with regard to \(R\). \(R^2 \tilde{k}(n, 0, H, 0, R) \leq \epsilon\) doesn’t imply that \(M\) is compact. However, the condition \(\tilde{k}(n, 0, H, 0, R) \leq \epsilon\) may imply \(M\) is compact.

Remark 3. The proof depends on the good bound in Kato’s inequality, thus the theorem can not be applied to \(p\) harmonic map or \(p\) harmonic \(k\) form \((k \geq 2)\). However, our methods can be applied to study \(p\) harmonic 1 form on manifold. The strategy is almost the same, we omit it.

Proof. Here we modify the proof in [5]. First we recall Bochner formula for \(p\)-harmonic function(cf.lemma 2.1 in [17]): for any smooth function \(u\), we have

\[
\frac{1}{2} \Delta |du|^{2p-2} = |\nabla |du|^{p-2} du|^2 - \langle |du|^{2p-2} du, \Delta (|du|^{p-2} du) \rangle + |du|^{2p-4} \langle du(\text{Ric}^M(e_k), du(e_k)) \rangle, \tag{3.1}
\]

and the following Kato inequality for \(p\)-harmonic function (c.f. [17, lemma 2.4] or [5]):

\[
|\nabla u|^2 \geq \frac{n}{n-1} |\nabla u|^2. \tag{3.2}
\]
If we set \( \lambda = |du|^{p-1} \), we have (c.f. [5])

\[
\frac{1}{2} \Delta \lambda^2 \geq \frac{n}{n-1} |\nabla \lambda|^2 - \langle |du|^{2p-2}du, \delta d \left( |du|^{p-2}du \right) \rangle + |du|^{2p-4} \langle du \left( \text{Ric} M(e_k), du(e_k) \right) \rangle \\
\geq \frac{n}{n-1} |\nabla \lambda|^2 - \langle |du|^{2p-2}du, \delta d \left( |du|^{p-2}du \right) \rangle + ((n-1)H - \text{Ric}_H) \lambda^2
\]

(3.3)

Now we choose a cut-off function on \( M \), which satisfies that (c.f. c.f. [5] or [38]):

\[
\begin{align*}
0 \leq \phi(x) &\leq 1, \quad x \in M \\
\phi(x) &\equiv 1, \quad x \in B_R(x_0) \\
\phi(x) &\equiv 0, \quad x \in M - B_3R(x_0) \\
|\nabla \phi(x)| &\leq \frac{C}{R}, \quad x \in M.
\end{align*}
\]

(3.4)

Integrating on \( M \) after multiplying both sides of (2.3) by \( \phi^2 \), we get

\[
\int_M \frac{1}{2} \phi^2 \Delta \lambda^2 dv_g + \int_M \text{Ric}_H \lambda^2 \phi^2 dv_g \\
\geq \frac{n}{n-1} \int_M \phi^2 |\nabla \lambda|^2 dv_g - \int_M \phi^2 \langle |du|^{2p-2}du, \delta d \left( |du|^{p-2}du \right) \rangle dv_g \\
+ (n-1)H \int_M \lambda^2 \phi^2 dv_g.
\]

It is not hard to get that

\[
\frac{1}{\text{Vol}(B_{3R}(x))} \int_M \text{Ric}_H \phi^2 \lambda^2 dv_g \\
\leq \frac{1}{\text{Vol}(B_{3R}(x)) \frac{1}{n} \| \text{Ric}_H \|_{L^n(B_{3R}(x))}} \frac{1}{\text{Vol}(B_{3R}(x)) \frac{n-1}{n} \| \phi^2 \lambda^2 \|_{L^n(B_{3R}(x))}}
\]

Thus, in Lemma 4, we take \( f = 0 \) and \( a = 0 \), the condition that \( R^2 k(n, 0, H, 0, R) \leq \epsilon \) implies that

\[
\int_M \text{Ric}_f^H \phi^2 \lambda^2 dv_g \leq \epsilon \left\{ 10^2 n R^{-1} \text{vol}(B_{3R}) \frac{1}{n} \right\} \int_M |\nabla (\phi \lambda)|^2 dv_g \\
\leq \epsilon \left\{ 10^2 n R^{-1} \text{vol}(B_{3R}) \frac{1}{n} \right\} \left[ \int_M \frac{1}{\epsilon} |\nabla \phi|^2 \lambda^2 + \epsilon \phi^2 \lambda^2 + \epsilon_1 |\nabla \lambda|^2 \phi^2 + \frac{1}{\epsilon_1} \phi^2 \lambda^2 dv_g \right],
\]

10
and we have (c.f. [5])
\[
\int_M \phi^2 (|du|^{2p-2} du, \delta d (|du|^{2p-2} du)) dv_g 
\leq 2\frac{p-2}{p-1} \int_M \phi |\nabla f| |f| dv_g + \left(\frac{p-2}{p-1}\right)^2 \int_M \phi^2 |\nabla f|^2 dv_g. 
\tag{3.5}
\]

Thus combining the above estimates yields that
\[
\int_M \frac{1}{2} \phi^2 \Delta \lambda^2 dv_g 
+ \epsilon \left\{10^{2n} \epsilon^2 a R \text{vol}(B_{3R})^{-\frac{1}{2}} \right\} \int_M \left[ \frac{1}{\epsilon^2} |\nabla \phi|^2 \lambda^2 dv_g + \frac{\epsilon_1}{\epsilon} |\nabla \phi|^2 dv_g + \left(\frac{\epsilon_2}{\epsilon_1}\right) \phi^2 \lambda^2 dv_g \right] 
\geq -2\frac{p-2}{p-1} \int_M \phi |\nabla \phi| |\nabla \lambda| |\lambda| dv_g - \left(\frac{p-2}{p-1}\right)^2 \int_M |\nabla \phi|^2 |\nabla \lambda|^2 dv_g + \frac{\epsilon_1}{\epsilon} |\phi|^2 |\nabla \lambda|^2 dv_g 
+ (n-1) H \int_M \lambda^2 \phi^2 dv_g 
\tag{3.6}
\]

where the constant $p \geq 2$. Integrating by parts, we have
\[
\int_M \frac{1}{2} \phi^2 \Delta \lambda^2 dv_g = -2 \int_M \phi |\lambda| (\nabla \phi, \nabla \lambda) dv_g \leq \delta \int_M |\phi|^2 |\nabla \lambda|^2 dv_g + C_\delta \int_M |\nabla \phi|^2 |\lambda|^2 dv_g. 
\tag{3.7}
\]

Combining (3.7) and (3.6), we have
\[
\left(\frac{n}{n-1} - \delta - 2\epsilon_3 - \left(\frac{p-2}{p-1}\right)^2 - \epsilon_1 \left\{10^{2n} R^{-1} \text{vol}(B_{3R})^{-\frac{1}{n}}\right\}\right) \int_M \phi^2 |\nabla \lambda|^2 dv_g 
\leq \left(C_\delta + \frac{1}{\epsilon_2} \left\{10^{2n} R \text{vol}(B_{3R})^{-\frac{1}{n}}\right\} + \frac{1}{2\epsilon_3} \left(\frac{p-2}{p-1}\right)^2 \right) \int_M \lambda^2 |\nabla \phi|^2 dv_g 
+ \left(\epsilon \left\{10^{2n} R^{-1} \text{vol}(B_{3R})^{-\frac{1}{n}}\right\} \left(\epsilon_2 + \frac{1}{\epsilon_1}\right) - (n-1) H \right) \int_M \lambda^2 \phi^2 dv_g. 
\tag{3.8}
\]
Firstly, we choose sufficiently small $\epsilon_1, \epsilon_2$, such that
\[
\epsilon \left\{ 10^n R^{-1} \text{vol}(B_{3R})^{-\frac{1}{n}} \right\} \left( \epsilon + \frac{1}{\epsilon_1} \right) - (n - 1) H \leq 0.
\]

Secondly, when $R$ is sufficiently large, we choose sufficiently small $\epsilon_3, \delta$, such that
\[
\frac{n}{n - 1} - \delta - 2\epsilon_3 - \left( \frac{p - 2}{p - 1} \right)^2 - \epsilon\epsilon_1 \left\{ 10^n R^{-1} \text{vol}(B_{3R})^{-\frac{1}{n}} \right\} > 0,
\]

then we can easily deduce that $|du|$ is a constant. By (3.3), we see that $\text{Ric}(\omega^*, \omega^*) = 0$. We conclude that $u$ is a constant. \qed

Motivated by [37], we have

**Theorem 3.2.** Let $N^n$ be a compact Riemannian manifold with nonpositive sectional curvature. Let $(M, g)$ be an $m(\geq 3)$-dimensional complete non-compact metric measure space with $\lambda_1(\Delta) > 0$ and

\[
\text{Ric} \geq -a\lambda_1(\Delta) + \delta,
\]

for some $a \geq 0, \delta > 0$. If $u$ is $p$-harmonic map with finite $L^{2p-2}$-energy from $M$ to $N$, then $u$ is a constant.

**Remark 4.** This Theorem generalizes Theorem 1.1 in [37].

**Proof** First we recall Bochner formula for $p$-harmonic map: for any smooth function $u$, we have (c.f. [5, (3.13)])
\[
\frac{1}{2} \Delta |du|^{2p-2} = |\nabla |du|^{p-2} du|^2 - |du|^{2p-2} du, \Delta (|du|^{p-2} du) = |du|^{2p-4} \langle du(\text{Ric}^M(e_k), du(e_k)) \rangle,
\]

and the following Kato inequality for $p$-harmonic map (cf. [5]):
\[
|\nabla (|du|^{p-2} du)|^2 \geq |\nabla |du|^{p-1}|^2.
\]
If we set $f = |du|^{p-1}$, we have
\[
\frac{1}{2} \Delta f^2 + \left[ - (1 + \frac{1}{2ns}) \lambda_1(M) + \delta \right] f^2 \geq |\nabla f|^2 - \langle |du|^{2p-2} du, \delta d(|du|^{2p-2} du) \rangle.
\]
(3.11)

Now we choose a cut-off function in (3.4) on $M$, which satisfies that:
\[
\begin{cases}
0 \leq \phi(x) \leq 1, & x \in M \\
\phi(x) = 1, & x \in B_r(x_0) \\
\phi(x) = 0, & x \in M - B_{3r}(x_0) \\
|\nabla \phi(x)| \leq \frac{C}{r}, & x \in M.
\end{cases}
\]

Integrating on $M$ after multiplying both sides of (3.11) by $\phi^2$, we get
\[
- \int_M \frac{1}{2} \phi^2 \Delta f^2 dv_g \\
\leq \left[ - a \lambda_1(M) + \delta \right] \int_M f^2 \phi^2 dv_g - \int_M \phi^2 |\nabla f|^2 dv_g + \int_M \phi^2 \langle |du|^{2p-2} du, \delta d(|du|^{2p-2} du) \rangle dv_g.
\]
(3.12)

Thus we can cite the following inequality (c.f. [5, (3.8)]),
\[
\int_M \phi^2 \langle |du|^{2p-2} du, \delta d(|du|^{2p-2} du) \rangle dv_g
\]
(3.13)
\[
\leq 2p-2 \int_M \phi |\nabla \phi| |\nabla f| |f| dv_g + \left( \frac{p-2}{p-1} \right)^2 \int_M \phi^2 |\nabla f|^2 dv_g
\]
(3.14)
\[
\leq 2\epsilon \int_M \phi^2 |\nabla f|^2 dv_g + \frac{1}{2\epsilon} \left( \frac{p-2}{p-1} \right)^2 \int_M f^2 |\nabla \phi|^2 dv_g + \left( \frac{p-2}{p-1} \right)^2 \int_M \phi^2 |\nabla f|^2 dv_g,
\]
(3.15)

Hence, we get
\[
- \int_M \frac{1}{2} \phi^2 \Delta f^2 dv_g \\
\leq \left[ - a \lambda_1(M) + \delta \right] \int_M f^2 \phi^2 dv_g + \frac{1}{2\epsilon} \left( \frac{p-2}{p-1} \right)^2 \int_M f^2 |\nabla \phi|^2 dv_g + \left( \frac{p-2}{p-1} \right)^2 \int_M \phi^2 |\nabla f|^2 dv_g,
\]
(3.16)
Notice also that

\[-\int_M \frac{1}{2} \phi^2 \Delta f^2 = 2 \int_M \phi f \nabla f \nabla \phi \leq \frac{1}{l} \int_M |\nabla f|^2 \phi^2 + l \int_M f |\nabla \phi|^2. \quad (3.17)\]

From the proof of Theorem 1.1 in [14], we know that:

\[\lambda_1(M) \int_M f^2 \phi^2 dv_g \leq \int_M f^2 |\nabla \phi|^2 dv_g + \int_M \phi^2 |\nabla f|^2 dv_g - \int_M \frac{1}{2} \phi^2 \Delta f^2 dv_g. \quad (3.18)\]

Combining (3.18) and (3.16), we get

\[2 \int_{M^n} f \phi \nabla f \nabla \phi dv_g = (b_1 + b_2) \int_{M^n} f \phi \nabla f \nabla \phi dv_g \]

\[\leq 2b_1 \left\{ \left( (2 \epsilon + \frac{p-2}{p-1})^2 - 1 \right) \int_{M^n} \phi^2 |\nabla f|^2 dv_g + (a \lambda_1 - \delta) \int_{M^n} f^2 \phi^2 dv_g \right\} + \frac{1}{2 \epsilon} \left( \frac{p-2}{p-1} \right)^2 \int_M f^2 |\nabla \phi|^2 dv_g \]

\[+ 2b_2 \left\{ \frac{1}{l} \int_{M^n} \phi^2 |\nabla f|^2 dv_g + l \int_{M^n} f^2 |\nabla \phi|^2 dv_g \right\} \]

\[= 2b_1 (a \lambda_1(M) - \delta) \int_{M^n} f^2 \phi^2 dv_g \]

\[+ 2 \left\{ b_1 \left( (2 \epsilon + \frac{p-2}{p-1})^2 - 1 \right) + \frac{b_2}{l} \right\} \int_{M^n} \phi^2 |\nabla f|^2 dv_g \]

\[+ 2b_2 \left\{ \frac{1}{l} \int_{M^n} f^2 |\nabla \phi|^2 + \frac{1}{2 \epsilon} \left( \frac{p-2}{p-1} \right)^2 \int_M f^2 |\nabla \phi|^2 dv_g, \right\} \]

where \(b_1 + b_2 = 1\).
Plugging the above inequality into (3.18), we get

\[
\lambda_1(M) \int_M f^2 \phi^2 \, dv_g \leq \int_M f^2 \vert \nabla \phi \vert^2 \, dv_g + \int_M \phi^2 \vert \nabla f \vert^2 \, dv_g \\
+ 2b_1 \left( a\lambda_1(M) - \delta \right) \int_M f^2 \phi^2 \, dv_g \\
+ 2 \left[ b_1 \left( (2\epsilon + \frac{p-2}{p-1})^2 - 1 \right) + \frac{b_2}{l} \right] \int_{M^n} \phi^2 \vert \nabla f \vert^2 \, dv_g \\
+ 2b_2 l \int_{M^n} f^2 \vert \nabla \phi \vert^2 \, dv_g + \frac{1}{2\epsilon} \left( \frac{p-2}{p-1} \right)^2 \int_M f^2 \vert \nabla \phi \vert^2 \, dv_g,
\]

which is written as

\[
(\lambda_1(M) - 2ab_1\lambda_1(M) + 2b_1\delta) \int_M f^2 \phi^2 \, dv_g \\
\leq \int_M f^2 \vert \nabla \phi \vert^2 \, dv_g \\
+ 2 \left[ b_1 \left( (2\epsilon + \frac{p-2}{p-1})^2 - 1 \right) + \frac{b_2}{l} + 1 \right] \int_{M^n} \phi^2 \vert \nabla f \vert^2 \, dv_g \\
+ 2b_2 l \int_{M^n} f^2 \vert \nabla \phi \vert^2 \, dv_g + \frac{1}{2\epsilon} \left( \frac{p-2}{p-1} \right)^2 \int_M f^2 \vert \nabla \phi \vert^2 \, dv_g,
\]

(3.19)

we can choose \( b_1 \) such that

\[
\frac{-(1 + \frac{1}{7})}{2\epsilon + \left( \frac{p-2}{p-1} \right)^2 - 1 - \frac{1}{7}} < b_1 < \min\{1, \frac{1}{2a}\}
\]

Thus, we have \( f = 0 \), thus, \( u \) is constant.

4 Liouville theorem in terms of BiRic curvature

**Theorem 4.1.** Let \( M \) be an \( n(\geq 3) \)-dimensional complete stable non-compact \( f \)-minimal hypersurface isometrically immersed in an \( (n + 1) \)-dimensional
metric measure space $(\bar{M}, g)$. If $\bar{M}$ has $\overline{\text{BiRic}}^\delta \geq 0$ with $\delta \geq 1$ and there exists at least a point $x_0 \in M$ such that $\overline{\text{BiRic}}^\delta(x_0) > 0$, moreover, assume that $\frac{1}{n-1} > \left( \frac{p-2}{p-1} \right)^2$ then any $p$-harmonic function with finite $L^{2p-2}$-energy on $M$ vanishes. If $\min\{1, \frac{(p-1)^2}{(n-1)^2}\} > \left( \frac{p-2}{p-1} \right)^2$, then any $f$ weighted $p$-harmonic 1 forms with finite $L^{2p-2}$ energy on $M$ vanishes.

**Remark 5.** This theorem generalize Theorem in [11] and Theorem in [21].

**Proof.** Since $M$ is $\delta$-stable, $$I(h) = \int_M \left( |\nabla \phi|^2 - \delta(|A|^2 + \overline{\text{Ric}}_f(n))\phi^2 \right) dv_g \geq 0.$$ Take $\phi = h|\omega|^{p-1}$, we derive

$$I(h) = -\int_M |h|^2 \left( |\omega|^{p-1}\Delta_f|\omega|^{p-1} + \delta(|A|^2 + \overline{\text{Ric}}_f(n))|\omega|^{2p-2} \right)dv_g + \int_M |\nabla h|^2 |\omega|^{2p-2}dv_g. \tag{4.1}$$

However,

$$|\omega|^{p-1}\Delta_f|\omega|^{p-1} = |\nabla(|\omega|^{p-2}\omega)|^2 - |\nabla|\omega|^{p-1}|^2 + |\omega|^{2p-4} \sum_{i=1}^n \langle \omega(\overline{\text{Ric}}_f^M(e_i), \omega(e_i) \rangle \tag{4.2}$$

where weighted Hodge laplacian

$$\Delta_f = \delta_f d + d\delta_f.$$

Plugging (4.2) into (4.1) to infer that

$$\int_M |\nabla h|^2 |\omega|^{2p-2}dv_g \geq \int_M |h|^2 \left( |\omega|^{p-1}\Delta_f|\omega|^{p-1} + (|A|^2 + \overline{\text{Ric}}_f(n))|\omega|^{2p-2} \right)dv_g \geq \int_M |h|^2 \left( F(\omega) - \langle |\omega|^{p-2}\omega, \Delta_f(|\omega|^{p-2}\omega) \rangle + \Phi_2 + (|A|^2 + \overline{\text{Ric}}_f(n))|\omega|^{2p-2} \right)dv_g. \tag{4.3}$$
Since $Ric(X, X) \geq -\delta (Ric(n, n) + |A|^2)$

$$
\Phi_2 + (|A|^2 + Ric(n))|\omega|^{2p-2} = |\omega|^{2p-4} \sum R_{ij} \omega^i \omega_j + \delta (|A|^2 + Ric(n))|\omega|^{2p-2}
$$

(4.4)

$$
\geq 0.
$$

Combining (4.3) and (4.4), we get

$$
\int_M h^2 |\nabla (|\omega|^{p-2}\omega)|^2 - |\nabla |\omega|^{p-1}|^2 + \langle |\omega|^{p-2}\omega, \Delta_f (|\omega|^{p-2}\omega) \rangle \, dv_g
\leq \int_M |\nabla h|^2 |\omega|^{2p-2} \, dv_g.
$$

(4.5)

Thus, by Kato’s inequality for $p$ harmonic function,

$$
|\nabla (|du|^{p-2} du)|^2 \geq \frac{n}{n-1} |\nabla (|du|^{p-1})|^2.
$$

we have

$$
\int_M \frac{1}{n-1} |\nabla |\omega|^{p-1}|^2 \, dv_g \leq \int_M |\nabla h|^2 |\omega|^{2p-2} \, dv_g - \langle |\omega|^{p-2}\omega, \Delta (|\omega|^{p-2}\omega) \rangle h^2 \, dv_g
\leq \int_M |\nabla h|^2 |\omega|^{2p-2} \, dv_g + \frac{p-2}{p-1} \int_M h |\nabla h| |\nabla |du|^{p-1}| |du|^{p-1} \, dv_g
+ \left( \frac{p-2}{p-1} \right)^2 \int_M h^2 |\nabla |du|^{p-1}|^2 \, dv_g.
$$

(4.6)

Hence, we deduce that

$$
\left( \frac{1}{n-1} - \left( \frac{p-2}{p-1} \right)^2 - \epsilon \right) \int_M |\nabla |\omega|^{p-1}|^2 \, dv_g
\leq \left[ 1 + \frac{1}{4\epsilon^4} \left( \frac{p-2}{p-1} \right)^2 \right] \int_M |\nabla h|^2 |\omega|^{2p-2} \, dv_g
$$

(4.7)

By (4.2), we infer that $|\omega|$ is constant. We may assume $|\omega| \neq 0$, otherwise the theorem has been proved.
By (4.3), we have

$$Ric(\omega^*, \omega^*) = 0$$

By (4.3) and (4.4), we have

$$Ric(\omega^*, \omega^*) + \delta(|A|^2 + Ric(n))\omega|^{2p-2} = 0.$$  

However, by [11], we know that

$$Ric(X, X) \geq BiRic^\delta(X, N) - \delta(|A|^2 + Ric_f(n))$$

Thus we have $BiRic = 0$, this is a contradiction to our assumption in the theorem. The last statement follows from the Kato inequality for $p$ harmonic 1 form. \hfill \Box

Follow the proof of Cheng [21] and Li and Zou [23], for constant mean curvature submanifold, we can get

**Theorem 4.2.** Let $M$ be an $n$-dimensional complete and non compact orientable hypersurface with constant mean curvature $H$ in a Riemannian manifold with biRicci curvature satisfying along $M$

$$BiRic^\delta \geq \frac{(n-5)n^2}{4}H^2.$$

If $M$ is strongly $\delta$-stable and $p \geq 2, \frac{1}{n-1} > \left(\frac{p-2}{p-1}\right)^2$, then there are no non-trivial $L^{2p-2}$ $p$-harmonic function on $M$. Moreover, If $2 \leq p < 2 + \frac{1}{\sqrt{n-1}}$, then there are no nontrivial $L^p$ $p$-harmonic 1-forms on $M$. Generally, if $\min\{1, \frac{(p-1)^2}{(n-1)p}\} > (p-2)^2$, there are no nontrivial $L^p$ $p$-harmonic 1-forms on $M$

**Remark 6.** When $p=2$, it is just Theorem 1 in Cheng [21].

18
Remark 7. The proof depends on the good bound in Kato’s inequality, thus the theorem can not be applied to $p$ harmonic map or $p$ harmonic $k$ form ($k \geq 2$).

Proof. We slightly modify the proof of Theorem 4.1. Let $\omega = du$,

$$|\omega|^{p-1}\Delta |\omega|^{p-1} = F(\omega) + \Phi_2 - \langle |\omega|^{p-2}\omega, \Delta(|\omega|^{p-2}\omega) \rangle.$$  \hspace{1cm} (4.8)

where $\Delta = d\delta + \delta d$, $\Phi_2 = |\omega|^{2p-4} \sum R_{ij} \omega^i \omega_j$.

Since $M$ is $\delta$ stable,

$$\int_M |\nabla h|^2 |\omega|^{2p-2} dv_g$$

$$\geq \int_M |h|^2 \left( |\omega|^{p-1}\Delta|\omega|^{p-1} + \delta(|A|^2 + \overline{Ric}(n))|\omega|^{2p-2} \right) dv_g$$

$$\geq \int_M |h|^2 \left( F(\omega) + \langle |\omega|^{p-2}\omega, \Delta(|\omega|^{p-2}\omega) \rangle + \Phi_2 + \delta(|A|^2 + \overline{Ric}(n))|\omega|^{2p-2} \right) dv_g.$$  \hspace{1cm} (4.9)

where $F(\omega) = |\nabla(|\omega|^{p-2}\omega)|^2 - |\nabla|\omega|^{p-1}|^2$.

Notice also that

$$\Phi_2 + \delta(|A|^2 + \overline{Ric}(n))|\omega|^{2p-2}$$

$$= |\omega|^{2p-4} \sum R_{ij} \omega^i \omega_j + \delta(|A|^2 + \overline{Ric}(n))|\omega|^{2p-2}$$

$$= |\omega|^{2p-4} \sum \left( \overline{Ric}_{ij} - \overline{R}(n, e_j, n, e_i) + nHA_{ij} - \sum_k A_{ik}A_{jk} \right) \omega^i \omega_j$$

$$+ \delta(|A|^2 + \overline{Ric}(n))|\omega|^{2p-2}$$

By [21, Lemma 2.1], we have

$$\left( nHA_{ij} - \sum_k A_{ik}A_{jk} \right) \omega^i \omega_j + |A|^2 |\omega|^2$$

$$= nHA(\omega^*, \omega^* - \langle A\omega^*, A\omega^* \rangle + |A|^2 |\omega|^2 \geq -\frac{(n-5)n^2H^2}{4} |\omega|^2.$$  \hspace{1cm} (4.10)

Since $BiRic \delta \geq \frac{(n-5)n^2H^2}{4}$, we have

$$Ric(X, X) + \delta Ric(n)|X|^2 - K(X, n)|X|^2 \geq \frac{(n-5)n^2H^2}{4} |X|^2.$$  \hspace{1cm} (4.11)
where $X = \omega^*$.

\[
\int_M |\nabla h|^2 |\omega|^{2p-2} dv_g \\
\geq \int_M |h|^2 \left( |\omega|^{p-1} \Delta |\omega|^{p-1} + \delta(|A|^2 + \overline{Ric}(n))|\omega|^{2p-2} \right) dv_g \\
\geq \int_M |h|^2 \left( F(\omega) - \langle |\omega|^{p-2}\omega, \Delta(|\omega|^{p-2}\omega) \rangle + BiRic(X, n) - \frac{(n-5)n^2H^2}{4} |\omega|^2 \right) dv_g.
\]

(4.10)

By Kato inequality for $p$ harmonic function, we have

\[
\frac{1}{n-1} \int_M h^2 |\nabla|\omega|^{p-1}|^2 dv_g \\
\leq \int_M |\nabla h|^2 |\omega|^{2p-2} dv_g - \langle |\omega|^{p-2}\omega, \Delta(|\omega|^{p-2}\omega) \rangle h^2 dv_g \\
\leq \int_M |\nabla h|^2 |\omega|^{2p-2} dv_g + 2 \frac{p-2}{p-1} \int_M h |\nabla h||\nabla|du|^{p-1|du|^{p-1} dv_g \\
+ \left( \frac{p-2}{p-1} \right)^2 \int_M h^2 |\nabla|du|^{p-1|2} dv_g.
\]

(4.11)

Thus, we have

\[
\left( \frac{1}{n-1} - \left( \frac{p-2}{p-1} \right)^2 - \epsilon \right) \int_M |\nabla|\omega|^{p-1}|^2 dv_g \leq C(p) \int_M |\nabla h|^2 |\omega|^{2p-2} dv_g.
\]

(4.12)

By Kato’s inequity for $p$ harmonic 1 form ($p \geq 2$), from (4.10), we see that

\[
\left( \frac{1}{(n-1)(p-1)^2} - \left( \frac{p-2}{p-1} \right)^2 - \epsilon \right) \int_M |\nabla|\omega|^{p-1}|^2 dv_g \leq C(p) \int_M |\nabla h|^2 |\omega|^{2p-2} dv_g.
\]

(4.13)

By (4.8), we have

\[
Ric(\omega^*, \omega^*) = 0
\]
By (4.10), we have
\[ \text{Ric}(\omega^*, \omega^*) + \delta(|A|^2 + \text{Ric}(n))||\omega||^{2p-2} = 0. \]

However, by Gauss equation and the definition of \( \text{BiRic}^\delta \), we know that (c.f. [11] and [21, Theorem 1])
\[
\text{Ric}(X, X) = \text{Ric}(X, X) + \delta \text{Ric}(n, n)|X|^2 - K(X, n)|X|^2 \\
- \delta \left( \text{Ric}(n, n) + |A|^2 \right) |X|^2 + nH\langle AX, X \rangle - |AX|^2 + a|A|^2|X|^2 \\
\geq \text{BiRic}^\delta \left( \frac{X}{|X|}, N \right)|X|^2 - \frac{(n-5)n^2H^2}{4}|X|^2 - \delta (|A|^2 + \text{Ric}(n))|X|^2
\]

As in Cheng [21, Theorem 1], we can conclude that
\[ \text{Ric}_M \geq 0. \]

It is well known that \( M \) has infinite volume (c.f. Yau[41]). Since \( |\omega| \) is a positive constant, this is a contradiction to the fact that \( \omega \) has finite \( L^{2p-2} \) energy.

The last statement follows from the Kato inequality in Lemma 3 and
\[
\left( \min\{1, \frac{(p-1)^2}{(n-1)^2} \} - \left( \frac{p-2}{p-1} \right)^2 - \epsilon \right) \int_M |\nabla |\omega||^{p-1}d_{g} \leq C(p) \int_M |\nabla \eta|^2 |\omega|^{2p-2}d_{g}.
\]

(4.14)

5 Liouville type theorem for harmonic q form

Inspired by the method in [21], we get

**Theorem 5.1.** Let \( M \) be an \( n(2 < n \leq 4) \)-dimensional complete 2-stable noncompact minimal hypersurface isometrically immersed in an \( (n + 1) \)-dimensional a complete noncompact metric measure space (\( M, g \)), which satisfies and \( \text{BiRic} \geq 0 \). Moreover, assume that \( R_{kjih}T^{kj}T^{ih} \leq 0 \) for any antisymmetric tensor \( T \), then any harmonic 2-form with finite \( L^2 \)-energy on \( M \) vanishes.
Remark 8. The theorem can be applied to the case where $\bar{M} = \mathbb{R}^{n+1}$ or $\mathbb{R}^{n+1}$, thus we generalize Theorem 6.1 in [21].

Proof. For any differential $q$ form $\omega$, it is well known that

$$
\Delta \omega = \Delta \omega_{i_1 \ldots i_q} = \nabla^r \nabla_r \omega_{i_1 \ldots i_q} - \sum_{s=1}^{q} R^r_{i_s} \omega_{i_1 \ldots r \ldots i_q} + \frac{q}{2} \sum_{t<s} R^r_{i_t i_s} \omega_{i_1 \ldots t \ldots u \ldots i_q} = 0.
$$

Putting $\|\omega\|^2 = \sum \omega_{i_1 \ldots i_q} \omega^{i_1 \ldots i_q}$ and $\|\nabla \omega\|^2 = \sum \nabla_r \omega_{i_1 \ldots i_q} \nabla^r \omega^{i_1 \ldots i_q}$, we obtain (c.f. [21])

$$
\frac{1}{2} \Delta \|\omega\|^2 = \|\nabla \omega\|^2 + \sum \omega_{i_1 \ldots i_q} \nabla^r \nabla_r \omega^{i_1 \ldots i_q} = \|\nabla \omega\|^2 + \sum R^r_{i_s} \omega_{i_1 \ldots r \ldots i_q} - \sum_{t<s} R^r_{i_t i_s} \omega_{i_1 \ldots t \ldots u \ldots i_q} \omega^{i_1 \ldots i_q}
$$

$$
= \|\nabla \omega\|^2 + q \sum R^r_{i_1} \omega_{i_2 \ldots i_q} \omega^{i_1 \ldots i_q} - \sum_{t<s} R^r_{i_t i_s} \omega_{i_1 \ldots t \ldots u \ldots i_q} \omega^{i_1 \ldots i_q}
$$

$$
= \|\nabla \omega\|^2 + q \sum R^r_{i_s} \omega_{i_1 \ldots i_q} \omega^{i_1 \ldots i_q} - \sum_{t<s} R^r_{i_t i_s} \omega_{i_1 \ldots t \ldots u \ldots i_q} \omega^{i_1 \ldots i_q}
$$

$$
= \|\nabla \omega\|^2 + q \sum R^r_{i_1} \omega_{i_2 \ldots i_q} \omega^{i_1 \ldots i_q} - \sum_{t<s} R^r_{i_t i_s} \omega_{i_1 \ldots t \ldots u \ldots i_q} \omega^{i_1 \ldots i_q} + \Delta \omega.
$$

(5.1)

On the other hand, we have

$$
\frac{1}{2} \Delta \|\omega\|^2 = \|\omega\| \|\Delta \omega\| + \|\nabla \|\omega\||^2 = \|\omega\| \|\Delta \omega\| + \|\nabla \omega\|^2 - F(\omega),
$$

where

$$
F(\omega) = \|\nabla \omega\|^2 - \|\nabla \|\omega\||^2
$$

We take cut off function $h(x)$ as in (3.4),

$$
\begin{align*}
0 & \leq h(x) \leq 1, \quad x \in M \\
h(x) & = 1, \quad x \in B_r(x_0) \\
h(x) & = 0, \quad x \in M - B_{2r}(x_0) \\
|\nabla h| & \leq \frac{C}{r}, \quad x \in M.
\end{align*}
$$
Since $M$ is 2-stable, we have
\[
\int_M |\nabla h|^2 |\omega|^2 dv_g \\
\geq \int_M |h|^2 \left( |\omega| |\Delta \omega| + 2(|A|^2 + \overline{Ric}(n)) |\omega|^2 \right) dv_g \\
\geq \int_M |h|^2 \left( F(\omega) + \Phi_2 + 2(|A|^2 + \overline{Ric}(n)) |\omega|^2 \right) dv_g. \tag{5.3}
\]
where
\[
\Phi_2 = 2 \sum R_{ij} \omega^i_{j2} \omega^{j2}_{i2} - \sum R_{kjih} \omega^{kj}_{i} \omega^{ih}_{j} \\
= 2 \sum R_{ij} \omega^i_{j2} \omega^{j2}_{i2} - \sum R_{kjih} \omega^{kj}_{i} \omega^{ih}_{j}. \tag{5.4}
\]
Notice that
\[
\Phi_2 + 2(|A|^2 + \overline{Ric}(n)) |\omega|^2 \\
= 2 \sum R_{ij} \omega^i_{j2} \omega^{j2}_{i2} - \sum R_{kjih} \omega^{kj}_{i} \omega^{ih}_{j} \\
+ 2(|A|^2 + \overline{Ric}(n)) |\omega|^{2p-2} \\
= 2|\omega|^{2p-4} \sum \left( (\overline{Ric})_{ij} - \overline{R}(n, e_j, n, e_i) + n HA_{ij} - \sum_k A_{ik} A_{jk} \right) \omega^i_{j2} \omega^{j2}_{i2} \\
- \sum \left( R_{kjih} + A_{ki} A_{jh} - A_{kh} A_{ji} \right) \omega^{kj}_{i} \omega^{ih}_{j} + q(|A|^2 + \overline{Ric}(n)) |\omega|^2. \tag{5.5}
\]
Since BiRic$_f \geq 0$, we have
\[
Ric_f(X, X) + Ric_f(n) - K(X, n) \geq 0,
\]
where $X = \frac{\omega^*}{|\omega^*|}$. By [21, Lemma 6.1], when $2 \leq n \leq 4$, we have
\[
2 \left( n HA_{ij} - \sum_k A_{ik} A_{jk} \right) \omega^i_{j2} \omega^{j2}_{i2} \\
- (A_{ki} A_{jh} - A_{kh} A_{ji}) \omega^{kj}_{i} \omega^{ih}_{j} + |A|^2 |\omega|^2 \geq 0. \tag{5.6}
\]
Moreover by our assumption, we have

\[-R_{kjih}k^j\omega^i h \geq 0.\]

Plugging all the above estimates into (5.5), we have

\[\Phi_2 + (|A|^2 + \overline{\text{Ric}}(n))|\omega|^{2n-2} \geq 0.\]

By Kato inequality for harmonic 2 form (c.f.[4]), we have

\[\frac{1}{n-2} \int_M h^2 |\nabla |\omega||^2 dv_g \leq \int_M |\nabla h|^2 |\omega|^2 dv_g \quad (5.7)\]

Thus, we have

\[\int_M h^2 |\nabla |\omega||^2 dv_g \leq (n-2) \int_M |\nabla h|^2 |\omega|^2 dv_g \quad (5.8)\]

It is standard to infer that $\omega$ is parallel. One can refer to the argument in in [21, page 9].

**Theorem 5.2.** Let $M$ be an $n(\geq 3)$-dimensional complete $q$-stable non-compact submanifold isometrically immersed in an $(n+k)$-dimensional metric measure space $(\bar{M}, g)$ If $\bar{M}$ has pure curvature tensor and $\overline{\text{BiRic}} \geq 0$. Moreover, assume that $\overline{R_{kjih}}k^jT^i \leq 0$ for any antisymmetric tensor $T$, $M$ has flat normal bundle. If one of the following conditions holds,

\[(1) \frac{n^2}{3} \leq q, \]
\[(2) |A|^2 \leq \frac{n^2|H|^2}{C_{n,q}}, \]

then any harmonic $q$-form with finite $L^2$-energy on $M$ must be parallel.

**Remark 9.** When $\bar{M} = \mathbb{S}^{n+1}$ or $\mathbb{R}^{n+1}$, the condition for $\bar{M}$ holds. This theorem doesn’t hold for $\rho(\neq 2)$ harmonic $q$ form because of the lack of good Kato’s inequality.
Proof. By Gauss equation, we have
\[
q \sum R_{ij\omega_{i_2\ldots i_q}^j} - \frac{q(q-1)}{2} \sum R_{kjih\omega_{i_3\ldots i_q}^j} \omega_{i_3\ldots i_q}^{hi} + (|A|^2 + \overline{Ric(n)})|\omega|^{2p-2} \\
= q \sum \left( \overline{R}_{ikjkh} + nHA_{ij} - \sum_k A_{ik}A_{jk} \right) \omega_{i_2\ldots i_q}^i \omega_{ji_2\ldots i_q}^j \\
- \frac{q(q-1)}{2} \sum (\overline{R}_{kjhi} + A_{ki}A_{jh} - A_{kh}A_{ji}) \omega_{i_3\ldots i_q}^k \omega_{i_3\ldots i_q}^{hi} + q(|A|^2 + \overline{Ric(n)})|\omega|^{2p-2}.
\]
(5.9)
Since BiRic_f \geq 0, we have
\[
Ric(X, X) + Ric(n) - K(X, n) \geq 0.
\]
where \( X = \frac{\omega^*}{|\omega|^*} \). Thus, we have
\[
q \sum \overline{R}_{ikjkh} \omega_{i_2\ldots i_q}^i \omega_{ji_2\ldots i_q}^j + q\overline{Ric(n)}|\omega|^2 \geq 0.
\]
(5.10)
By[24], we have
\[
q \sum \left( nHA_{ij} - \sum_k A_{ik}A_{jk} \right) \omega_{i_2\ldots i_q}^i \omega_{ji_2\ldots i_q}^j \\
- |\omega|^{2p-4} \frac{q(q-1)}{2} \sum (A_{ki}A_{jh} - A_{kh}A_{ji}) \omega_{i_3\ldots i_q}^k \omega_{i_3\ldots i_q}^{hi} \\
\geq \left( \frac{1}{2} (n^2|H|^2 - C_{n,q}|A|^2) \right).
\]
(5.11)
Thus, we have
\[
q \sum \left( nHA_{ij} - \sum_k A_{ik}A_{jk} \right) \omega_{i_2\ldots i_q}^i \omega_{ji_2\ldots i_q}^j \\
- \frac{q(q-1)}{2} \sum (A_{ki}A_{jh} - A_{kh}A_{ji}) \omega_{i_3\ldots i_q}^k \omega_{i_3\ldots i_q}^{hi} + q|A|^2 \\
\geq \left( \frac{1}{2} (n^2|H|^2 - C_{n,q}|A|^2) + q|A|^2 \right) \geq 0.
\]
(5.12)
where in the last inequality, we used the conditions in the theorem.
By (5.12) and (5.10), we have
\[
\int_M |\nabla h|^2 |\omega|^2 dv_g \\
\geq \int_M |h|^2 \left( |\omega| \Delta |\omega| + (|A|^2 + \text{Ric}(n)) |\omega|^2 \right) dv_g \\
\geq \int_M |h|^2 \left( F(\omega) + \Phi q( |A|^2 + \text{Ric}(n)) |\omega|^2 \right) dv_g \\
\geq \int_M |h|^2 F(\omega) dv_g.
\]
(5.13)

By Kato inequality for harmonic $q$ form (c.f.[4]), we have
\[
K_q \int_M h^2 |\nabla| |\omega|^2 dv_g \leq \int_M |\nabla h|^2 |\omega|^2 dv_g
\]
(5.14)
where
\[
K_q = \begin{cases} 
\frac{1}{n-q} & \text{if } 2 \leq q \leq \frac{n}{2} \\
\frac{1}{q} & \text{if } \frac{n}{2} \leq q \leq n - 2,
\end{cases}
\]

Thus, we have
\[
K_q \int_M h^2 |\nabla| |\omega|^2 dv_g \leq \int_M |\nabla h|^2 |\omega|^2 dv_g
\]
(5.15)
Now it is standard to infer that $\omega$ is parallel. \(\Box\)

References

[1] Paul Baird and Sigmundur Gudmundsson. $p$-harmonic maps and minimal submanifolds. *Mathematische Annalen*, 294(1):611–624, 1992. 2

[2] Paul Baird and Sigmundur Gudmundsson. $p$-harmonic maps and minimal submanifolds. *Math. Ann.*, 294(4):611–624, 1992. 2

[3] Pierre Bérard. A note on Bochner type theorems for complete manifolds. *Manuscr. Math.*, 69(3):261–266, 1990. 6
[4] David MJ Calderbank, Paul Gauduchon, and Marc Herzlich. Refined kato inequalities and conformal weights in riemannian geometry. *Journal of Functional Analysis*, 173(1):214–255, 2000. 24, 26

[5] Xiangzhi Cao. Liouville type theorem about $p$-harmonic function and $p$-harmonic map with finite $L^q$-energy. *Chinese Ann. Math. Ser. B*, 38(5):1071–1076, 2017. 2, 9, 10, 11, 12, 13

[6] Xiaoli Chao, Aiying Hui, and Miaomiao Bai. Vanishing theorems for $p$-harmonic $\ell$-forms on Riemannian manifolds with a weighted Poincaré inequality. *Differ. Geom. Appl.*, 76:13, 2021. Id/No 101741. 3

[7] Xu Cheng. $L^2$ harmonic forms and stability of hypersurfaces with constant mean curvature. *Boletim da Sociedade Brasileira de Matemática-Bulletin/Brazilian Mathematical Society*, 31(2):225–239, 2000. 4, 5

[8] Xu Cheng. On constant mean curvature hypersurfaces with finite index. *Archiv der Mathematik*, 86(4):365–374, 2006. 4

[9] Xianzhe Dai and Guofang Wei. A heat kernel lower bound for integral Ricci curvature. *Mich. Math. J.*, 52(1):61–69, 2004. 3, 7

[10] Nguyen Thac Dung. $p$-harmonic $\ell$-forms on Riemannian manifolds with a weighted Poincaré inequality. *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, 150:138–150, 2017. 3

[11] Nguyen Thac Dung, Nguyen Van Duc, and Juncheol Pyo. Harmonic 1-forms on immersed hypersurfaces in a Riemannian manifold with weighted bi-Ricci curvature bounded from below. *J. Math. Anal. Appl.*, 484(1):29, 2020. Id/No 123693. 4, 5, 16, 18, 21

[12] Nguyen Thac Dung and Keomkyo Seo. $p$-harmonic functions and connectedness at infinity of complete submanifolds in a riemannian manifold. *Annali di Matematica Pura ed Applicata (1923-)*, 196(4):1489–1511, 2017. 2
[13] Nguyen Thac Dung and Pham Trong Tien. Vanishing properties of $p$-harmonic $\ell$-forms on Riemannian manifolds. *J. Korean Math. Soc.*, 55(5):1103–1129, 2018.

[14] Wenzhen Gan and Peng Zhu. $L^2$ harmonic 1-forms on minimal submanifolds in spheres. *Results in Mathematics*, 65(3-4):483–490, 2014.

[15] Yingbo Han. $p$-harmonic $l$-forms on complete noncompact submanifolds in sphere with flat normal bundle. *Bull. Braz. Math. Soc. (N.S.*), 49(1):107–122, 2018.

[16] Yingbo Han. Vanishing theorem for $p$-harmonic 1-forms on complete submanifolds in spheres. *Bull. Iran. Math. Soc.*, 44(3):659–671, 2018.

[17] Yingbo Han and Shuxiang Feng. A Liouville type theorem for $p$-harmonic functions on minimal submanifolds in $\mathbb{R}^{n+m}$. *Mat. Vesn.*, 65(4):494–498, 2013.

[18] Yingbo Han and Hong Pan. $L^p$ $p$-harmonic 1-forms on submanifolds in a Hadamard manifold. *Journal of Geometry and Physics*, 107:79–91, 2016.

[19] Yingbo Han, Qianyu Zhang, and Mingheng Liang. $L^p$ $p$-harmonic 1-forms on locally conformally flat Riemannian manifolds. *Kodai Math. J.*, 40(3):518–536, 2017.

[20] Shigeo Kawai. $p$-Harmonic Maps and Convex Functions. *Geometriae Dedicata*, 74(3):261–265, 1999.

[21] Haizhong Li. $L^2$ harmonic forms on a complete stable hypersurfaces with constant mean curvature. *Kodai Math. J.*, 21(1):1–9, 1998.
[22] Jin Tang Li. P-harmonic maps for submanifolds with positive Ricci curvature. *Xiamen Daxue Xuebao Ziran Kexue Ban*, 40(6):1191–1195, 2001.

[23] Yao Wen Li and Xiao Rong Zou. On the bi-Ricci curvature and some applications. *Houston journal of mathematics*, 34:467–481, 2008.

[24] Hezi Lin. On the structure of submanifolds in Euclidean space with flat normal bundle. *Result. Math.*, 68(3-4):313–329, 2015.

[25] Peter Lindqvist. *Notes on the p-Laplace equation*. Number 161. University of Jyväskylä, 2017.

[26] Yozo Matsushima. Vector bundle valued harmonic forms and immersions of Riemannian manifolds. *Osaka Journal of Mathematics*, 8(1):1–13, 1971.

[27] P. Petersen and G. Wei. Relative volume comparison with integral curvature bounds. *Geom. Funct. Anal.*, 7(6):1031–1045, 1997.

[28] Stefano Pigola, Marco Rigoli, and Alberto G Setti. Constancy of p-harmonic maps of finite q-energy into non-positively curved manifolds. *Mathematische Zeitschrift*, 258(2):347–362, 2008.

[29] Stefano Pigola, Marco Rigoli, and Alberto G Setti. Constancy of p-harmonic maps of finite q-energy into non-positively curved manifolds. *Mathematische Zeitschrift*, 258(2):347–362, 2008.

[30] Keomkyo Seo and Gabjin Yun. Liouville-type theorems for weighted p-harmonic 1-forms and weighted p-harmonic maps. *Pac. J. Math.*, 305(1):291–310, 2020.

[31] Ying Shen and Rugang Ye. On stable minimal surfaces in manifolds of positive bi-ricci curvatures. *Duke Mathematical Journal*, 85(1):109–116, 1996.
[32] Ying Shen and Rugang Ye. On the geometry and topology of manifolds of positive bi-ricci curvature. *arXiv preprint dg-ga/9708014*, 1997. 4

[33] Hiroshi Takeuchi. Stability and Liouville theorems of p-harmonic maps. *Japan. J. Math. (N.S.)*, 17(2):317–332, 1991. 2

[34] Shukichi Tanno. L2 harmonic forms and stability of minimal hypersurfaces. *Journal of the Mathematical Society of Japan*, 48(4):761–768, 1996. 4

[35] Peter Tolksdorf. Regularity for a more general class of quasilinear elliptic equations. *Journal of Differential Equations*, 51(1):126–150, 1984. 2

[36] Lili Wang and Guofang Wei. Local Sobolev constant estimate for integral Bakry-Émery Ricci curvature. *Pac. J. Math.*, 300(1):233–256, 2019. 3, 6, 8

[37] Qiaoling Wang. Harmonic maps and the topology of manifolds with positive spectrum and stable minimal hypersurfaces. *Publicaciones Matemàtiques*, pages 301–313, 2006. 4, 12

[38] Qiaoling Wang. Complete submanifolds in manifolds of partially non-negative curvature. *Annals of Global Analysis and Geometry*, 37(2):113–124, 2010. 2, 10

[39] Xiaodong Wang and Lei Zhang. Local gradient estimate for p-harmonic functions on riemannian manifolds. *Communications in analysis and geometry*, 19(4):759–772, 2011. 2

[40] Jia-Yong Wu. Comparison geometry for integral Bakry-Émery Ricci tensor bounds. *J. Geom. Anal.*, 29(1):828–867, 2019. 3, 6

[41] Shing Tung Yau. Some function-theoretic properties of complete Riemannian manifold and their applications to geometry. *Indiana University Mathematics Journal*, 25(7):659–670, 1976. 21
[42] Jian Feng Zhang and Yue Wang. A theorem of Liouville type for \( p \)-harmonic maps in weighted Riemannian manifolds. *Kodai Mathematical Journal*, 39(2):354–365, 2016.

[43] Xi Zhang. A note on \( p \)-harmonic 1-forms on complete manifolds. *Canadian mathematical bulletin*, 44(3):376–384, 2001.