MULTIPLICITY ESTIMATES: A MORSE-THEORETIC APPROACH

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Abstract. The problem of estimating the multiplicity of the zero of a polynomial when restricted to the trajectory of a non-singular polynomial vector field, at one or several points, has been considered by authors in several different fields. The two best (incomparable) estimates are due to Gabrielov and Nesterenko.

In this paper we present a refinement of Gabrielov’s method which simultaneously improves these two estimates. Moreover, we give a geometric description of the multiplicity function in terms certain naturally associated polar varieties, giving a topological explanation for an asymptotic phenomenon that was previously obtained by elimination theoretic methods in the works of Brownawell, Masser and Nesterenko. We also give estimates in terms of Newton polytopes, strongly generalizing the classical estimates.

1. Introduction

Consider a polynomial vector field $V$ of degree $\delta$ and a polynomial $P$ of degree $d$ on $\mathbb{C}^n$,

$$V = \sum_{i=1}^{n} Q_i \frac{\partial}{\partial x_i}, \quad Q_i \in \mathbb{C}[x_1, \ldots, x_n], \deg Q_i \leq \delta$$

$$P \in \mathbb{C}[x_1, \ldots, x_n], \deg P \leq d$$

(1)

For $p \in \mathbb{C}^n$ a non-singular point of $V$, let $\gamma_p$ denote the germ of the trajectory of $V$ through the point $p$. We define $\text{mult}_P^V$, the multiplicity of $P$ at the point $p$, to be the order of zero of $P$ along $\gamma_p$. Alternatively, we may think of $V$ as a general non-linear system of differential equations of the time variable $t$, the variables $x_1, \ldots, x_n$ as the dependent variables, and $\text{mult}_P^V$ as the order of zero of $P$ evaluated on a particular solution of the system.

A multiplicity estimate is an answer for the following question: For a given $P$ and $V$, how large can $\text{mult}_P^V$ be? It is usual for the answer to be given in terms of the dimension $n$ and the degrees $d, \delta$. More generally, given a finite set of points $p_1, \ldots, p_\nu$ one may ask for an upper bound for $\sum_{i} \text{mult}_{p_i}^V$ or for $\min_{i} \text{mult}_{p_i}^V$, depending on the geometry of the set.

1.1. Historical sketch. The problem of deriving multiplicity estimates of the types mentioned above has been considered in various areas of mathematics. Below we give a brief outline of some of the main contributions. Our presentation follows the development in each field separately, rather than in historical order.
In transcendental number theory, the subject began with Nesterenko’s treatment of the Siegel-Shidlovski theory of E-functions [17]. Nesterenko developed an algebraic technique for the estimation of multiplicities for functions satisfying a certain type of linear differential equations. Similar ideas were later successfully applied to non-linear systems by Brownawell [2, 5], by Brownawell and Masser [3, 4] and by Nesterenko [18]; these authors also gave estimates for the sum of multiplicities over an arbitrary finite set of points, which is of importance in transcendental number theory. The algebraic techniques developed in these papers were later applied to various more specific situations, notably to translation-invariant vector fields on group varieties and sets of points related to the group structure, leading to great progress in the field (see [15] for a survey).

In control theory, Risler [21] suggested the problem of multiplicity estimates in the study of nonholonomic control systems. Risler considered the planar case \( n = 2 \), obtained a multiplicity estimate, and used it to bound the degree of nonholonomy for planar control systems. Gabrielov and Risler extended this work and gave good multiplicity estimates for the case \( n = 3 \) in [9]. For arbitrary dimension, Gabrielov obtained a multiplicity estimate in [7], and subsequently developed a powerful technique involving Milnor fibres of certain deformations, leading to much sharper estimates in [8]. Khovanskii later simplified Gabrielov’s arguments considerably using the notion of integration of Euler characteristics. This simplification was later applied by Gabrielov and Khovanskii to establish multiplicity estimates in the multi-dimensional setting in [10].

In the qualitative theory of differential equations, Novikov and Yakovenko [20] have obtained multiplicity estimates in their study of abelian integrals, related to the infinitesimal Hilbert 16th problem. While not as sharp as the preceding estimates, these estimates remain valid when one considers the number of zeros in a small interval or prescribed length. Yomdin [26] has studied a similar problem in the context of bifurcations of zeros in analytic families, and obtained upper bounds using Gabrielov’s estimate.

1.2. The estimates of Nesterenko and Gabrielov. In this section give precise statements and some further discussion of the multiplicity estimates of Nesterenko and Gabrielov, which are the two best known estimates in our context. In summary, these two estimates are incomparable: Nesterenko’s estimate is sharp up to a multiplicative constant with respect to the degree \( d \), which is the main asymptotic considered in transcendental number theory, but doubly-exponential in \( n \); Gabrielov’s estimate on the other hand is not sharp with respect to \( d \), but exhibits an essentially optimal simply-exponential growth with respect to the dimension \( n \).

1.2.1. Nesterenko’s estimate. The estimates presented in this subsection are those of [18]. This work improved several previous results by Brownawell [2, 5] and by Brownawell and Masser [3, 4]. Brownawell and Masser’s principal idea was that while the multiplicity at a given point may be quite large, this cannot occur too frequently. If one sums over the multiplicities over several points, most points will contribute terms of lower order. Nesterenko establishes a more refined result following the same paradigm.

Nesterenko states his results in the projective context, for homogeneous vector fields. To make the comparison with the rest of our text transparent, we translate
his result to the affine context. The two formulations are easily seen to be equivalent (up to the precise values of constants).

Let $V, P$ be as in (1), and let $p_1, \ldots, p_\nu \in \mathbb{C}^n$ be non-singular points of $V$ which belong to the same trajectory of $V$ (that is, $\gamma_{p_i}$ can be obtained by analytic continuation from $\gamma_{p_j}$ for any $1 \leq i, j \leq \nu$), and assume that $P$ does not vanish identically on this trajectory. Let $\kappa$ denote the transcendence degree of this trajectory (i.e., the dimension of the smallest algebraic set containing the trajectory).

**Theorem ([18, Theorem 1]).** There exists a constant $C_\gamma$ depending only on the trajectory containing $p_1, \ldots, p_\nu$ such that

$$\sum_{i=1}^{\nu} \text{mult}_{p_i}^V P \leq C_\gamma \sum_{j=1}^{n} a_{\kappa-j}(C_\gamma d^j) d^j$$

where $a_j(T)$ denotes the maximum number of points among $p_1, \ldots, p_\nu$ lying in an irreducible variety of dimension $j$ and degree at most $T$ in $\mathbb{C}^n$. In particular $a_0(T) \equiv 1$.

The constant $C_\gamma$ is not explicitly worked out in [18], but from the proof one can determine that it grows doubly-exponentially with the dimension $n$. We note also that when $\kappa < n$, i.e. the trajectory is not completely transcendental, the constant $C_\gamma$ depends on algebraic complexity (for instance, the degree) of the Zariski closure of $\gamma$. In this sense, the estimate is not entirely explicit, since one cannot in general estimate the degree of this Zariski closure purely in terms of $n, d, \delta$. For a discussion and a comparison of Nesterenko’s result and our result in this context see Remark [18].

A few remarks are in order. If we restrict our attention to the case of a single point $p$ and make no special assumptions on $\kappa$, then the theorem states that $\text{mult}_{p}^V P \leq C' d^n$ where $C'$ is a constant depending only on $p$. Since the linear space of polynomials of degree $d$ has dimension of the order of $d^n$, this result is essentially the best possible up to a multiplicative constant.

One could naively estimate the sum of the multiplicities over $\nu$ different points by $\nu d^n$. However, Nesterenko’s result implies that the coefficient of the $d^n$ term is a constant independent of the number and position of the points $p_1, \ldots, p_\nu$. The coefficient of the next term, of order $d^{n-1}$, may already depend on the number and position of the points. However, the theorem essentially states that the number of contributions of this order is bounded by the number of points $p_i$ that could belong to an irreducible curve of degree $C_\gamma d^{n-1}$. Next we have a contributions of order $d^{n-2}$, whose number is bounded by the number of points $p_i$ that could belong to an irreducible surface of degree $C_\gamma d^{n-2}$, and so on.

As we have already seen, even for the case of a single point, the growth of $\text{mult}_p^V P$ with respect to $n$ is at least exponential in $n$. However, the dependence of the constant $C_\gamma$ on the dimension $n$ is doubly exponential. In this sense the result is possibly suboptimal, and as we shall see in §1.2.2 the multiplicity can in fact grow no faster than exponentially in $n$.

1.2.2. Gabrielov’s estimate. We turn now to Gabrielov’s estimate presented in [8]. In this work only the case of a single point was considered. Therefore let $V, P$ be
as in (1) and \( p \in \mathbb{C}^n \) a non-singular point of \( V \). Assume that \( P \) does not vanish identically on \( \gamma_p \) (the trajectory through \( p \)).

**Theorem (§ Theorem 2).** We have the upper bound

\[
\text{mult}_p V P \leq 2^{2n-1} \sum_{i=1}^{n} [d + (i-1)(\delta - 1)]^{2n} 
\]

The dependence of this estimate on \( n \) is simply-exponential, which as we have seen is essentially the best possible. However, considering \( n \) and \( \delta \) as fixed this estimate has order \( d^{2n} \), which is the square of the correct growth (as we know from §1.2.1).

1.3. **Overview of this paper.** In this paper we consider the problem of bounding the multiplicity at a point \( p \), and more generally, the sum of multiplicities over an arbitrary finite set of points.

Our approach is based on a refinement of Gabrielov’s deformation technique. Following Gabrielov’s ideas, we translate the problem of estimating multiplicities to the problem of estimating the Euler characteristics of Milnor fibers of certain deformations related to \( P \) and \( V \). More generally, we consider the problem of estimating the individual Betti numbers of Milnor fibers of general deformations (under a certain smoothness assumption).

Through classical techniques of polar varieties, we translate the problem of estimating Betti numbers to the study of some naturally associated algebraic cycles and their intersection numbers. One can then apply ideas from algebraic geometry to estimate the degrees of the cycles and, consequently, obtain upper bounds for their intersection numbers. Moreover, the algebraic cycles are defined globally and provide a clear geometric picture for the situation involving several points \( p_1, \ldots, p_\nu \).

As a result we obtain a multiplicity estimate which simultaneously improves the estimates of Nesterenko and of Gabrielov.

Since our description is given in terms of certain naturally defined varieties, it can be adapted to take into account additional geometric structure on the ambient space, the polynomial \( P \) and the vector field \( V \). To demonstrate this, we give multiplicity estimates depending on the volumes of the Newton polytopes of \( P \) and \( V \) in the context of the affine space \( \mathbb{C}^n \) and the torus group \( (\mathbb{C}^*)^n \), analogous to the Bernstein-Kushnirenko theorem (which is reviewed in §2.2). The classical estimates in terms of degrees follow immediately as special cases.

To present our result in the simplest context, we now give a simplified formulation of the multiplicity estimates in terms of the degrees \( d, \delta \) (in the notation of (1)). We begin with a definition.

**Definition 1.** Let \( M \) denote \( \mathbb{C}^n \) or \( (\mathbb{C}^*)^n \). For an irreducible variety \( V \subset M \), we define the function

\[
\mathcal{D}_V : M \to \mathbb{N} \\
\mathcal{D}_V(p) = \begin{cases} 
\deg V & p \in V \\
0 & \text{otherwise} 
\end{cases} 
\]

We extend this by linearity to define \( \mathcal{D}_C \) for an arbitrary algebraic cycle \( C \subset M \).

Our theorem may now be stated as follows.
Theorem. There exist algebraic cycles $\mathcal{M}^0(P), \ldots, \mathcal{M}^{n-1}(P) \subset \mathbb{C}^n$ where
\[
\dim \mathcal{M}^k(P) = k
\]
\[
\deg \mathcal{M}^k(P) < 2^n (d + (n - k - 1)(\delta - 1))^{n-k}
\]
such that for every $p \in \mathbb{C}^n$ where $V$ is non-singular and $\text{mult}_p^V P$ is finite,
\[
\text{mult}_p^V P \leq \sum_{k=0}^{n-1} D_{\mathcal{M}^k(P)}(p)
\] (5)

We thus see a picture very similar to the one given in Nesterenko’s estimate. In §5.2 we show how to derive Nesterenko’s result from ours.

We remark that in the full formulation of our result, the contribution of each cycle $\mathcal{M}^k(P)$ is not the degree $D_{\mathcal{M}^k(P)}$, but rather the order of intersection between $\mathcal{M}^k(P)$ and a certain special affine-linear space passing through $p$. While we do not fully investigate this in the present paper, in some contexts this extra information can lead to significantly stronger estimates than the naive one used in (5).

The contents of this paper are as follows. In §2 we discuss general preliminaries. In §3 we discuss Milnor fibers and their relation to multiplicity estimates. In §4 we give estimates for the Betti numbers of the Milnor fiber. In §5 we give the full formulation of our multiplicity estimates in various contexts. We show how our estimates improve those of Nesterenko, Gabrielov and Risler. Finally, in the appendix we discuss a general compactness property which is useful in establishing uniform algebraic semicontinuous bounds.

2. General preliminaries

In this section we discuss some preliminaries that shall be needed in the sequel. In §2.1 we review the theory of algebraic cycles and their intersection numbers. In §2.2 we review the notion of mixed volume of convex bodies and its relation to intersection theory on the torus $(\mathbb{C}^*)^n$ through the Bernstein-Kushnirenko theorem.

2.1. Algebraic cycles and intersection numbers. We introduce some basic results on algebraic cycles and their intersection theory. For the purposes of this paper we will assume that the ambient variety $M$ is given by $\mathbb{C}^n$ or $(\mathbb{C}^*)^n$ in the algebraic case, or by the germ of these varieties at a point in the analytic case (although the subject can be developed in far greater generality, see [6] for a canonical reference). We denote the coordinate ring of $M$ by $R$.

A $k$-cycle is a finite formal sum $\sum n_i[V_i]$ where $V_i \subset M$ are $k$-dimensional irreducible subvarieties of $M$ and $n_i$ are integers. A cycle is a (finite) sum of cycles of any dimension. In this paper, we shall deal exclusively with cycles with positive coefficients.

We say that two varieties $V, W \subset M$ intersect properly at a component $Z \subset V \cap W$ if $\text{codim} Z = \text{codim} V + \text{codim} W$. In this case there is a well defined intersection number $i(Z; V \cdot W; M)$. If $V, W$ intersect properly at every component of their intersection then there is a well defined intersection cycle
\[
V \cdot W = \sum_{Z \subset V \cap W} i(Z; V \cdot W; M)[Z],
\] (6)

This product can be extended by linearity to the product of arbitrary cycles, assuming that all intersections are proper.
We now describe the behavior of the intersection product with respect to continuous deformation. Let $T$ denote the germ of a non-singular curve at a point $t_0$, and consider a variety $V \subset M \times T$ which is flat over $T$. Then for $t \in T$ we have a well defined cycle $V_t := V \cap (M \times \{t\})$ (see [6 10.1]).

If $\dim V = 1$, then $V_t$ is a formal sum of points with positive multiplicities. Conservation of numbers implies that the multiplicity of the cycle $[p]$ in $V_{t_0}$ is given by the number of points in $V_t$ (with multiplicities) converging to $p$ as $t \to t_0$.

To generalize this to arbitrary intersections, we have the following continuity axiom [6 11.4.4.iii]. Consider another $W \subset M \times T$ which is flat over $T$, and suppose that $W_t$ meets $V_t$ properly for each $t \in T$. Then $V$ meets $W$ properly in $M \times T$ and

$$(V \cdot W)_t = V_t \cdot W_t.$$  

We remark that in [6] this property is stated axiomatically for the case where $W$ is a constant family $W_0 \times T$. To obtain the general case one considers the intersection of $V \times_T W \subset M \times M \times T$ and the diagonal $\Delta \times T \subset M \times M \times T$.

As a particular case of (7), when $V \cdot W$ is a curve we obtain a description of the multiplicity of $p$ in $V_{t_0} \cdot W_{t_0}$ as the number of points of $(V \cdot W)_t$ converging to $p$ as $t \to t_0$.

If $V = \sum n_t[p_t]$ is an algebraic cycle of dimension 0, then we define $\deg V := \sum n_t$. If $M$ is $\mathbb{C}^n$ or $(\mathbb{C}^*)^n$ and $V$ is an algebraic cycle of pure dimension $k$, then we define $\deg V$ to be $\deg V \cdot L$ where $L$ is a generic affine plane of codimension $k$. By the continuity of intersection numbers, the degree function is lower-semicontinuous on flat families.

2.2. The Bernstein-Kushnirenko Theorem. We give an overview of the notion of mixed volume and its relation to the geometry of the torus group $(\mathbb{C}^*)^n$, encapsulated by the Bernstein-Kushnirenko theorem. We follow the presentation of [13].

Recall that for $n$ convex bodies $\Delta_1, \ldots, \Delta_n$ in $\mathbb{R}^n$, their mixed volume is defined to be

$$V(\Delta_1, \ldots, \Delta_n) = \frac{\partial^n}{\partial \lambda_1 \cdots \partial \lambda_n} \operatorname{Vol}(\lambda_1 \Delta_1 + \cdots + \lambda_n \Delta_n)|_{\lambda_1 = \cdots = \lambda_n = 0}. \quad (8)$$

The mixed volume is symmetric and multilinear, and generates the volume function in the sense that $V(\Delta, \ldots, \Delta) = \operatorname{Vol}(\Delta)$. In fact, these properties completely determine the mixed volume function.

Given a Laurent polynomial $P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, we define its support $\supp P \subset \mathbb{Z}^n$ to be the set of exponents appearing with non-zero coefficients in $P$. For any set $A \subset \mathbb{Z}^n$ we denote by $\Delta_A$ the convex hull of $A$ (in $\mathbb{R}^n$). Finally, we let $\Delta(P) := \Delta_{\supp P}$.

To each nonempty set $A \subset \mathbb{Z}^n$ we associate the vector space of polynomials $P$ having $\supp P \subset A$.

**Theorem 1** ([13 1]). Let $A_1, \ldots, A_n \subset \mathbb{Z}^n$. Then for generic $P_1 \in L_{A_1}, \ldots, P_n \in L_{A_n}$ the system of equations $P_1 = \cdots = P_n = 0$ admits exactly $\mu$ solutions in $(\mathbb{C}^*)^n$, where

$$\mu = n! V(\Delta_{A_1}, \ldots, \Delta_{A_n}). \quad (9)$$

It follows by conservation of numbers that for any choice of $P_i \in L_{A_i}$, not necessarily generic, the quantity $\mu$ above is an upper bound for the number of isolated solutions of $P_1 = \cdots = P_n = 0$.

We record a simple consequence for the computation of mixed volumes.
Corollary 2. Suppose that $\mathbb{R}^n = L_1 \oplus L_2$ is an orthogonal decomposition, and that $\Delta_1, \ldots, \Delta_s \subset L_1$ and $\Delta_{s+1}, \ldots, \Delta_n \subset L_2$ are collections of convex bodies. Then
\[
n!V(\Delta_1, \ldots, \Delta_n) = [s!V(\Delta_1, \ldots, \Delta_s)] \cdot [(n-s)!V(\Delta_{s+1}, \ldots, \Delta_n)].
\] (10)

In particular, if $s \neq \dim L_1$ then $V(\Delta_1, \ldots, \Delta_n) = 0$.

Proof. We may after an orthonormal change of coordinates assume that $L_1$ is spanned by the $x_1, \ldots, x_s$ coordinates and $L_2$ is spanned by the $x_{s+1}, \ldots, x_n$ coordinates. We will prove the claim under the assumption that $\Delta_1, \ldots, \Delta_n$ are Newton polytopes, i.e. convex hulls of subsets of the lattice $\mathbb{Z}^n$. The general claim can be established by continuous approximation (although in this paper we will only use the claim in this more restrictive sense).

Under our assumption, the left hand side of (10) may be viewed as the number of zeros of a generic set of equations $P_1 = \cdots = P_n = 0$ with $P_i \in L_{\Delta_i}$. Since $P_1, \ldots, P_s$ and $P_{s+1}, \ldots, P_n$ involve disjoint sets of variables, it is clear that
\[
\{P_1 = \cdots = P_n = 0\} = \{P_1 = \cdots = P_s = 0\} \times \{P_{s+1} = \cdots = P_n = 0\}. \tag{11}
\]
The claim now follows by the Bernstein-Kushnirenko theorem. \qed

Let $\Delta_x := \Delta(1+x_1+\cdots+x_n)$ denote the standard simplex in the $x$-variables. For any convex body $\Delta$ and $j = 0, \ldots, n$ we define the $j$-th (simplicial) quermassintegral as
\[
W_j^s(\Delta) = V(\Delta_{n-j \text{ times}}, \underbrace{\Delta, \ldots, \Delta}_{\text{j times}}).
\] (12)

We note that it is customary to use the Euclidean ball in place of the standard simplex $\Delta_x$, but for our purposes the simplicial normalization is more convenient.

Finally we remark on the affine case.

Remark 3. Suppose that $\Delta_1, \ldots, \Delta_n \subset \mathbb{Z}^n_{>0}$ are convex co-ideals and let $c_1, \ldots, c_n \in \mathbb{C}^n$. Then
\[
\Delta(P_i) \subset \Delta_i \implies \Delta(P_i(x_1 + c_1, \ldots, x_n + c_n)) \subset \Delta_i \tag{13}
\]
In this case the Bernstein-Kushnirenko estimate holds even if one considers the number of solutions of a generic systems of equations with assigned Newton polyhedrons in $\mathbb{C}^n$. Indeed, after a generic translation one may assume that all solutions lie in $(\mathbb{C}^*)^n$ and apply the usual Bernstein-Kushnirenko theorem.

3. Milnor fibers and Multiplicities

In this section we recall a general notion of a Milnor fiber of a deformation (due to Lê), and its relation to the multiplicity of an analytic function restricted to the trajectory of an analytic vector field (due to Gabrielov).

In an effort to make the presentation self-contained, we have included sketches for the proofs of most results that we shall use in the sequel (with references for the original full proofs).

3.1. Lê’s Milnor fiber of a deformation. We begin with the definition due to Lê [23], extending the notion of a Milnor fiber to a general deformation of an analytic set. For simplicity we work in the ambient space $M = \mathbb{C}^N \times \mathbb{C}$, and denote the projections to the second factor by $e : M \to \mathbb{C}$. 

Definition 4. Let $X \subset M$ and $x \in X$ with $e(x) = 0$. Denote $X_\varepsilon := X \cap e^{-1}(\varepsilon)$, and suppose that $e|_X$ is flat in a neighborhood of $x$, i.e. $X_0$ is obtained as the limit of $X_\varepsilon$. We think of $X$ as a deformation of $X_0$.

Then for any sufficiently small $\delta > 0$ and $\delta \gg \varepsilon > 0$, the homotopy type of the set $B_\delta(x) \cap X_\varepsilon$ is independent of the choice of $\delta, \varepsilon$ and is called the Milnor fiber of $(X, x)$. Here $B_\delta(x)$ denotes the real Euclidean ball with respect to the standard metric.

We remark that much of the material in this section can be generalized to the case where $e$ is an arbitrary analytic function defined in a neighborhood of $x$. The assumption that $e$ is a separate coordinate slightly simplifies our presentation.

It is of importance for us that the Milnor fiber does not depend on the coordinate system used to construct the balls. More generally, one may also be interested in computing the Milnor fiber replacing the balls $B_\delta(x)$ by a family of polydiscs. Lê [22] defines the notion of privileged families of neighborhoods for this purpose.

The full definition is technical and goes beyond the scope of this paper, but for the purpose of our exposition it will suffice to specify one key property: if $\{P_\alpha\}$ is a privileged family of neighborhoods, then for any sufficiently small $\delta > 0$ there exists $\varepsilon' > 0$ such that for any $0 < \varepsilon < \varepsilon'$ the inclusion $P_\beta \cap X_\varepsilon \hookrightarrow P_\delta \cap X_\varepsilon$ is a homotopy equivalence. Lê shows that the family of balls and the family of polydiscs in sufficiently generic linear coordinates form privileged families.

With this definition we have the following standard fact.

Fact 5. The Milnor fiber of $X$ at $x$ is the same (up to homotopy equivalence) when computed using any privileged family.

Proof. Let $\{P_\alpha\}, \{Q_\alpha\}$ be two privileged families around $x$. Choose sufficiently small $P_1 \subset Q_1 \subset P_2 \subset Q_2$ from the two families. Then for any sufficiently small $\varepsilon \neq 0$ the inclusions

\begin{align*}
P_1 \cap X_\varepsilon &\hookrightarrow P_2 \cap X_\varepsilon \\
Q_1 \cap X_\varepsilon &\hookrightarrow Q_2 \cap X_\varepsilon
\end{align*}

are homotopy equivalences. It follows that the inclusion map

\begin{equation}
Q_1 \cap X_\varepsilon \hookrightarrow P_2 \cap X_\varepsilon
\end{equation}

admits a right inverse $R$ and a left inverse $L$. Then $L \simeq L \cdot (iR) \simeq R$, so $i$ is a homotopy equivalence. 

We now present a result of Lê [22] explaining how the Milnor fiber of a deformation is obtained from the Milnor fiber of a hyperplane section by gluing cells corresponding to certain critical points (cf. also [8, Proposition 2]). This may be seen as a local complex analog of classical Morse theory. We present only a special case which will suffice for our purposes.

Theorem 2. Let $(X, x)$ be a deformation, and assume that in a neighborhood of $x$, the fibers $X_\varepsilon, \varepsilon \neq 0$ are smooth. Let $\ell$ be an affine form satisfying $\ell(x) = 0$ and denote $Y := X \cap \ell^{-1}(0)$.

Then, for $\ell$ sufficiently generic, we have

1. In a sufficiently small neighborhood of $x$, the fibers $Y_\varepsilon, \varepsilon \neq 0$ are smooth.
2. In a sufficiently small neighborhood of $x$, $\ell|_{X_\varepsilon}$ admits only isolated critical points for any $\varepsilon \neq 0$. 

(3) The Milnor fiber of \((X, x)\) is obtained from the Milnor fiber of \((Y, x)\) by attaching \(\mu\) cells of dimension \(\dim \mathbb{C} X_\varepsilon\), where \(\mu\) is the number of critical points of \(|\ell|_{X, x}\) converging to \(x\) as \(\varepsilon \to 0\).

Proof. The first two claims follow by a Bertini type argument which we omit (see [24, Lemma 2.2 and Remark 2.3]). We continue with the demonstration of the third claim.

We may assume that \(\ell\) is just the first coordinate. Let \(\Phi = (\ell, \varepsilon) : X \to \mathbb{C}^2\). According to [24, Theorem 2.4] for sufficiently generic \(\ell\) there is a privileged family of polydiscs \(P_\alpha = D_\alpha \times P_\alpha'\) for \((X, x)\) such that

\[
F =: P_\alpha \cap \Phi^{-1}(D_\alpha \times \{\varepsilon\}) \text{ is the Milnor fiber of } (X, x)
\]

\[
F_0 := P_\alpha \cap \Phi^{-1}(\{0\} \times \{\varepsilon\}) \text{ is the Milnor fiber of } (Y, x)
\]

for sufficiently small \(P_\alpha\) and (even smaller) \(\varepsilon\). Now, since \(X_\varepsilon\) is smooth by assumption, it remains only to interpolate these two spaces by

\[
F_r = F \cap |\ell|^{-1}([0, r])
\]

where \(r\) runs from 0 to the radius of \(D_\alpha\). Since all critical points of \(\ell\) are simple by assumption, all critical points of \(|\ell|\) are Morse of index \(\dim \mathbb{C} X_\varepsilon\). By classical Morse theory, whenever \(r\) crosses the absolute value of a critical value of \(\ell|_F\), \(F_r^+\) is glued with a cell of dimension \(\dim \mathbb{C} X_\varepsilon\). This concludes the proof.

Theorem 2 allows one to compute a cellular decomposition for the Milnor fiber using induction on dimension. It motivates the following definitions. Let \((X, x)\) be a deformation. Assume that in a neighborhood of \(x\), the fiber \(X_\varepsilon, \varepsilon \neq 0\) is smooth. Fix generic functionals

\[
\ell = (\ell_1, \ldots, \ell_{\dim \mathbb{C} X_0}), \quad \ell_i \in M^*
\]

For \(k = 1, \ldots, \dim \mathbb{C} X_0\) we denote by \(L^k_\ell(x)\) the affine space

\[
L^k_\ell(x) := \ell_1^{-1}(\ell_1(x)) \cap \cdots \cap \ell_k^{-1}(\ell_k(x)).
\]

We note that under the generic assumption that \(e\) does not belong to the span of \(\ell\), the maps \(e : L^k_\ell(x) \to \mathbb{C}\) are flat. As usual we denote by \(L^k_\ell(x)_\varepsilon := L^k_\ell(x) \cap e^{-1}(\varepsilon)\) the corresponding fibers.

**Definition 6.** For \(k = 1, \ldots, \dim \mathbb{C} X_0 + 1\), we define the \(k\)-th polar variety of \(X\), denoted \(\Gamma^k_\ell(X)\), as follows

\[
\Gamma^k_\ell(X) = \text{Clo} \left[ \{ \partial \ell_1 \wedge \cdots \partial \ell_k \wedge \partial e|_{X} = 0 \} \setminus \{ e = 0 \} \right].
\]

In a neighborhood of \(x\) where all fibers \(X_\varepsilon, \varepsilon \neq 0\) are smooth, this may be stated as follows: we define \(\Gamma^k_\ell(X)\) as the locus where \(\partial \ell_1, \ldots, \partial \ell_k\) are linearly dependent on the tangent space of the fiber \(X_\varepsilon\) for \(\varepsilon \neq 0\), and complete this to a flat family over \(e = 0\).

For any \(\varepsilon\), we denote by \(\Gamma^k_\ell(X)_\varepsilon\) the analytic cycle in \(\mathbb{C}^N \times \{\varepsilon\}\) as defined in section 2.7.

Recall that an analytic stratification \(\{Z_\alpha\}\) of an analytic subset of \(X_0\) is said to satisfy Thom’s \(A_e\) condition if the following condition holds: for any sequence of points \(x_i \in X\) converging to a point \(x' \in Z_\alpha\), if the sequence of tangent spaces \(T_{x_i} X_\varepsilon(x_i)\) converges to a limit \(T\), then \(T_{x'} Z_\alpha \subset T\). Such stratifications always exists by a result of Hironaka [11].
The following two propositions establish the key properties of $L^k_\ell(x)$ and $\Gamma^k_\ell(X)$, and their relation to the cellular structure of the Milnor fiber. Polar varieties have been used for such purposes extensively in the literature (see [23][16] for a survey), and the ideas for the proofs are standard. However, we are not aware of suitable reference in this generality, and we therefore present full proofs.

**Proposition 7.** Let $(X,x)$ be a deformation, and assume that in a neighborhood of $x$, the fibers $X_\varepsilon, \varepsilon \neq 0$ are smooth. There exists a neighborhood of $x$ and generic $\ell$ such that for every $k = 1, \ldots, \dim_{\mathbb{C}} X_0 + 1$ and sufficiently small $\varepsilon$ (including zero) we have:

1. $\Gamma^k_\ell(X)_\varepsilon$ has pure dimension $k - 1$
2. $\Gamma^k_\ell(X)_\varepsilon$ intersects $L^{k-1}_\ell(x)_\varepsilon$ properly (i.e. at isolated points).

**Proof.** We describe the choice of generic $\ell$. Let $\{Z^1_\alpha\}$ be a stratification of $X_0$ satisfying Thom’s $A_\varepsilon$ condition. We may restrict to an open neighborhood of $x$ where the only zero-dimensional strata (if any) is $\{x\}$. Let $\ell_1$ be transversal to all the other strata in $\{Z^1_\alpha\}$.

Consider now a stratification $\{Z^2_\alpha\}$ of $X_0 \cap L^1_\ell(x)_0$ refining $\{Z^1_\alpha\}$. Once again, we may restrict to an open neighborhood of $x$ where the only zero-dimensional strata (if any) is $\{x\}$. Let $\ell_2$ be transversal to all the other strata in $\{Z^2_\alpha\}$.

Continuing in this fashion we obtain for $k = 1, \ldots, \dim_{\mathbb{C}} X_0 + 1$ stratifications $\{Z^k_\alpha\}$ of $X_0 \cap L^{k-1}_\ell(x)_0$, and functional $\ell_k$ transversal to all strata in $\{Z^k_\alpha\}$ except perhaps $\{x\}$. Note that these stratifications all satisfy Thom’s $A_\varepsilon$ condition (since the condition is preserved under refinement). Denote by $U$ the open neighborhood of $x$ in which all the stratifications were constructed.

We now proceed with the proof. First note that for $\varepsilon \neq 0$, $\Gamma^k_\ell(X)_\varepsilon$ is a determinantal variety given by

$$\Gamma^k_\ell(X)_\varepsilon = d\ell_1 \wedge \cdots \wedge d\ell_k|_{X_\varepsilon} = 0$$

and as such, each of its components has dimension at least $k - 1$. Since $\Gamma^k_\ell(X)_0$ is obtained by a flat limit, the same is true for it.

The other direction of (1), as well as (2), will be proved by continuity once we show that $\Gamma^k_\ell(X)_0$ intersects $L^{k-1}_\ell(x)_0$ properly. More specifically, we will show that in $U$ this intersection contains only $x$.

Assume to the contrary that

$$x' \in \Gamma^k_\ell(X)_0 \cap L^{k-1}_\ell(x)_0 \cap U, \quad x' \neq x.$$  

Since $\Gamma^k_\ell(X)_0$ is defined by a flat limit, there exists a sequence $x_i \in \Gamma^k_\ell(X)_{e(x_i)}$ with $e(x_i) \neq 0$ and $x_i \to x'$. By compactness of the Grassmannian we may assume that $T_{x_i}X_{e(x_i)}$ converges to a limit $T$.

For $p = 1, \ldots, k$ denote by $Z^k_\alpha$ the strata in $\{Z^k_\alpha\}$ containing $x'$. By construction there exists a vector $v_p \in T_{x'}Z^k_\alpha$ such that $d\ell_p(v_p) = 1$. Since $Z^k_\alpha$ is an analytic subset of $X_0 \cap L^{p-1}_\ell(x)_0$, we have $d\ell_q(v_p) = 0$ for $q < p$. It follows that the matrix $(d\ell_p(v_q))_{p,q=1..k}$ is upper triangular with determinant 1.

By Thom’s $A_\varepsilon$ condition for each of the stratifications $\{Z^k_\alpha\}$, we have that $v_1, \ldots, v_k \in T$. Thus eventually $d\ell_1, \ldots, d\ell_k$ become linearly independent on $T_{x_i}X_{e(x_i)}$, contradicting our assumption that $x_i \in \Gamma^k_\ell(X)_{e(x_i)}$. \hfill $\square$

Finally, we present a proposition expressing the cellular structure of a Milnor fiber in terms of the polar varieties $\Gamma^k_\ell(X)$ and their intersections with $L^k_\ell(x)$. 

...
Proposition 8. Let \((X, x)\) be a deformation, and assume that in a neighborhood of \(x\), the fibers \(X_e, e \neq 0\) are smooth of dimension \(d := \dim C X_e\). For any sufficiently generic \(\ell\), the Milnor fiber of \((X, x)\) admits a cellular structure where the number of \(k\)-cells, denoted \(c_k\), is given by
\[
c_{d+1-k} = i(x; \Gamma^{k}_\ell(X)_0 \cdot L^{k-1}_\ell(x)_0; \mathbb{C}^N).
\]
Thus, the \(k\)-th Betti number of the Milnor fiber is bounded by \(c_k\). In particular the \(k\)-th Betti number vanishes for \(k > d\).

Proof. Applying Theorem 2 inductively, we obtain a cellular decomposition for the Milnor fiber where \(c_{d+1-k}\) is equal to the number of critical points of \(\ell_{k}|_{X_e \cap L^{k-1}_\ell(x)}\) converging to \(x\) as \(e \to 0\), i.e. to the number of points in \(\Gamma^{k}_\ell(X)_e \cdot L^{k-1}_\ell(x)_e\) converging to \(x\) as \(e \to 0\). Since the intersection \(\Gamma^{k}_\ell(X)_0 \cdot L^{k-1}_\ell(x)_0\) is proper at \(x\) according to Proposition 7, we obtain (23) by the continuity of intersection numbers. \(\square\)

3.2. Milnor fibers and multiplicities. In this section we present the main ideas of [8], relating the multiplicity of an analytic function restricted to the trajectory of an analytic vector field to the Euler characteristics of the Milnor fibers of certain deformations.

Let \(p \in \mathbb{C}^n\). Let \(V\) be an analytic vector field and \(P\) an analytic function, both defined in a neighborhood of \(p\). As in [8, §3.1] we consider the ambient space \(M = \mathbb{C}^n \times \mathbb{C}\). Finally consider an analytic deformation \(P_e\) of \(P\), i.e.
\[
P_e(x_1, \ldots, x_n) \in O_{(u, p)}(e, x_1, \ldots, x_n) \quad P_0(x) \equiv P(x).
\]
Following Gabrielov, we define for every \(r \in \mathbb{N}\)
\[
X^r = \text{Clo} \left[ \{ P_e = \cdots = V^{r-1} P_e = 0 \} \setminus \{ e = 0 \} \right]
\]
where \(\text{Clo}\) denotes analytic closure. In other words, we define \(X^r\) by the vanishing of the first \(r\) derivatives outside \(e = 0\) and complete this variety as a flat family over \(e = 0\). We denote by \(F^r_p\) the Milnor fiber of \((X^r, p)\).

Gabrielov’s key insight is Theorem 3 expressing the multiplicity \(\text{mult}^V_P\) in terms of the Euler characteristics of the Milnor fibers defined above. The proof we present is based on an idea of Khovanskiî, and appeared in [8].

We remark that the presentation of this proof in [8] contains a small gap. A second, complete proof is also given in [8]. However, for the multi-dimensional generalization developed in [10] it is necessary to use the former proof. We therefore pause to present a lemma making the argument precise.

The following lemma was suggested to the author by David B. Massey. In [12], this lemma is presented in the general context of constructible sheaves, and is stated as a result concerning cohomology. We give below a simplified formulation in our more elementary context, and a proof sketch (adapted from [12]) in the homotopic category.

Lemma 9 ([12, Lemma 8.4.7]). In the notations of Definition 4, let \(\phi : X \to \mathbb{R}_{>0}\) be a nonnegative real analytic function. Suppose that \(\phi\) is proper in a neighborhood of \(x\) and that and \(\phi^{-1}(0) = \{x\}\).

Then the Milnor fiber of \((X, x)\) is given up to homotopy-equivalence by \(\phi^{-1}(B_k) \cap X_e\) for sufficiently small \(0 \neq e \ll \delta \ll 1\).
With $\phi$ given by the squared-distance to $x$ one obtains the usual expression for the Milnor fiber.

**Proof sketch.** Fix a stratification of $X$ refining a stratification of $X_0$, which satisfies the Whitney B condition as well as Thom’s $A_e$ condition.

By an argument of Bertini-Sard type [12, Lemma 8.4.7], one checks that $\phi$ has a discrete set of stratified critical values. In particular, for sufficiently small $\delta < \delta'$, $\phi$ has no stratified critical values between $\delta$ and $\delta'$. Moreover, using Thom’s $A_e$ condition one proves that for sufficiently small $\varepsilon$, the restriction of $\phi$ to $X_{\varepsilon}$ also has no critical values between $\delta$ and $\delta'$.

It follows from the stratified Morse lemma that there is a homotopy equivalence between $\phi^{-1}(B_\delta) \cap X_{\varepsilon}$ and $\phi^{-1}(B_{\delta'}) \cap X_{\varepsilon}$. One can now conclude the proof as in the proof of Fact 5. \qed

We are now ready to state Gabrielov’s main theorem.

**Theorem 3 ([8, Theorem 1]).** Let $p \in C^n$ be a non-singular point of $V$ and suppose that $\text{mult}_p P$ is finite. Then

$$\text{mult}_p P = \sum_{r=1}^{\infty} \chi(F^r)$$

(26)

**Proof suggested by Khovanskii.** We first note that since $\mu:=\text{mult}_p P$ is finite, it follows that $F^r = \emptyset$ for $k > \mu$, and hence the sum (26) is finite.

We may choose local analytic coordinates $(z_1, z')$ around $p$ such that $V = \frac{\partial}{\partial z_1}$. Denote by $\pi : C^n \to C^{n-1}$ the projection to the $z'$ coordinates. By assumption, $\pi|_{X^r_0}$ has ramification of multiplicity $\mu := \text{mult}_p P$ at the origin. Then one can choose a neighborhood of the form $U = D \times B$ of the origin such that the $\pi|_{U \cap X^r_0}$-fiber of any point in $B$ has exactly $\mu$ points counted with multiplicities. It also follows that the same is true for $\pi|_{U \cap X^r_0}$ for sufficiently small $\varepsilon$. We would essentially like to think of $U \cap X^r_0$ as representing the Milnor fiber $F^r_p$. This requires a small technical justification, as follows.

Let $\phi(z_1, z', e) = \|z'|^2 + \|e\|^2$. Then applying Lemma 9 for each $U \cap X^r_0$ and $\phi$, we have

$$F^r_p \simeq \phi^{-1}(\delta') \cap X^r_0 = X^r_0 \cap (D \times B_\delta) =: Y^r, \quad \text{where } \delta = \sqrt{\delta' - \varepsilon^2}$$

(27)

for sufficiently small $\varepsilon \ll \delta' \ll 1$. Fix such a pair.

We know that the $\pi|_{Y^r}$-fiber of any point $z' \in B_\delta$ contains $\mu$ points, counted with multiplicities. Since the points of $Y^r$ are exactly the points where $\pi$ has multiplicity $k + 1$, it follows by a Riemann-Hurwitz type counting argument that for any $z' \in B_\delta$,

$$\mu = \sum_{k=1}^{\infty} \#\{\pi|_{Y^r}^{-1}(z')\}.$$
Using the Fubini theorem for integration over Euler characteristic \([25]\), we obtain
\[
\mu = \int_{B^s} \mu \, d\chi = \int_{B^s} \sum_{r=1}^{\infty} \# \{ [\pi|_{Y^r}]^{-1}(z') \} \, d\chi(z') \\
= \sum_{r=1}^{\infty} \int_{B^s} \chi \left( [\pi|_{Y^r}]^{-1}(z') \right) \, d\chi(z') \\
= \sum_{r=1}^{\infty} \chi(Y^r) = \sum_{r=0}^{\infty} \chi(F^r_p).
\]  

(29)

The usefulness of Theorem \([8]\) becomes apparent in view of the following lemma, which guarantees the existence of sufficiently generic deformations. We omit the proof, which is a standard exercise in Sard type arguments, and refer the reader to \([8]\) for details. We say that \(X^r_\epsilon\) is effectively smooth at a point \(x\) if it is smooth, and moreover \(dP_\epsilon \wedge \cdots \wedge d(V^{r-1}P_\epsilon)|_{X^r_\epsilon}\) is non-zero at \(x\).

Lemma 10 (\([8]\) Lemma 1). Let \(\ell\) be a germ of an analytic function and suppose that \(V\ell(p) \neq 0\). Let \(P^e = c_0 + \cdots + c_n x^{n-1}\) where the coefficients \(c_i\) are chosen generically, and consider the deformation \(P^e_\epsilon(x) = P(x) + \epsilon P^e(x)\). Then there exists a neighborhood \(U\) of \(p\) such that for any sufficiently small \(0 < \epsilon \ll 1\), \(X^r_\epsilon\) is an effectively smooth \(n-r\) dimensional set in \(U\). In particular, \(X^r_\epsilon\) is empty in \(U\) for \(r > n\).

4. An estimate for the Betti numbers of the Milnor fiber

In this section we present an estimate for the Betti numbers of the Milnor fiber of a deformation, under a smoothness assumption. The estimate is expressed in terms of the geometry of the polar varieties.

In this section we consider the ambient manifold \(M = (\mathbb{C}^*)^n \times \mathbb{C}\) with coordinate ring \(R = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, \epsilon]\) or \(M = \mathbb{C}^n \times \mathbb{C}\) with coordinate ring \(R = \mathbb{C}[x_1, \ldots, x_n, \epsilon]\). Let \(P^1_\epsilon, \ldots, P^n_\epsilon \in R\), where as usual we think of \(\epsilon\) as a deformation parameter. Let \(\Delta \subset \mathbb{Z}^n\) denote the convex hull of \(\Delta(P^1_\epsilon), \ldots, \Delta(P^n_\epsilon)\) for a generic \(\epsilon\).

We denote
\[
X^r = \text{Clo} \left[ \{ P^1_\epsilon = P^2_\epsilon = \cdots = P^n_\epsilon = 0 \} \setminus \{ \epsilon = 0 \} \right]
\]

(30)

where \(\text{Clo}\) denotes analytic closure. In other words, we define \(X^r\) by the vanishing of \(P^1_\epsilon, \ldots, P^n_\epsilon\) outside \(\epsilon = 0\) and complete this variety as a flat family over \(\epsilon = 0\). We denote by \(F_\epsilon\) the Milnor fiber of \((X, p)\).

Let \(\Sigma(X^r)\) denote the set of points where the fiber \(X^r_\epsilon\) is not effectively smooth (completed as a flat family over \(\epsilon = 0\)), i.e.
\[
\Sigma(X^r) = \text{Clo} \left[ \{ dP^1_\epsilon \wedge \cdots \wedge dP^n_\epsilon \wedge d\epsilon = 0 \} \setminus \{ \epsilon = 0 \} \right]
\]

(31)

We will say that a point \(p \in X^r_\epsilon\) is good if \(p \notin \Sigma(X^r)\). Our goal is to estimate the Betti numbers \(b_\ell(F^r_\epsilon_p)\) at good points \(p\) in terms of the Newton polytope \(\Delta\). More specifically, we give an appropriate definition for globally defined polar varieties, whose degrees are bounded in terms of the Newton polytopes, and show that the geometry of these polar varieties controls the Betti numbers.
4.1. **The polar varieties.** In this section we keep the notation of [31]. Naturally, our objective is to obtain upper bounds on the Betti numbers of the Milnor fiber in terms of the polar varieties through Proposition 8. One could attempt to use the definition of polar varieties given in Definition 10 directly. However, in the global context this definition gives rise to certain degeneracies (for instance, where the sets $X^c_r$ are singular, and when the functionals $\ell$ are not sufficiently generic), making the polar varieties more difficult to study. Since we are interested primarily in the behavior of these varieties around good points, we opt to use a refined definition which agrees with Definition 6 in a neighborhood of such points while eliminating some of the more complicated degenerate behavior.

**Definition 11.** For $k = 1, \ldots, n-r+1$ and $\ell$ as in [18], we define the set $\tilde{\Gamma}_k^k(X^r)$ by

$$\tilde{\Gamma}_k^k(X^r) := C_k \left[ \Gamma_k^k(X^r) \setminus (\Sigma(X^r) \cup \Gamma_k^k(X^r)) \right]$$

where $C_k(A)$ denotes the union of the $k$-dimensional components of $A$ which are not contained in a fiber $e = \text{const}$. We define the refined polar variety, denoted $\tilde{\Gamma}_k^k(X^r)$, to be the Zariski closure of $\tilde{\Gamma}_k^k(X^r)$.

By definition, $\tilde{\Gamma}_k^k(X^r)$ is a $k$-dimensional flat family, with pure $k-1$-dimensional fibers $\Gamma_k^k(X^r)_e$. Let $p \in X_0$ be a good point and suppose that $\ell$ is sufficiently generic. Then by Proposition 7 in a neighborhood of $p$ the set $\Gamma_k^k(X^r)$ has pure dimension $k$ and the set $\Gamma_k^k(X^r)$ has pure dimension $k-1$. It follows that in a neighborhood of $p$, $\tilde{\Gamma}_k^k(X^r) = \Gamma_k^k(X^r).

We now consider the degree of the refined polar variety. We begin with the case of the torus $M_0 = (\mathbb{C}^*)^n$.

**Proposition 12.** For any $\varepsilon \in \mathbb{C}$, we have the bound

$$\deg \tilde{\Gamma}_k^k(X^r) \leq \binom{n}{r+k-1}n!W_k^s(\Delta + \Delta_{x})$$

where $\deg$ denotes degree in $(\mathbb{C}^*)^n$, $W^s$ denotes the quermassintegral, and $\Delta_{x}$ denotes the standard simplex in the $x$ variables.

**Proof.** We may assume without loss of generality that $\Delta$ contains the origin. Indeed, one can always translate $\Delta$ to achieve this by multiplying the equations $P^1, \ldots, P^r$ by a common monomial. This does not affect the set $X^r$ or $\Sigma(X^r)$ outside of the coordinate axes (which lie outside $(\mathbb{C}^*)^n$) and thus it is straightforward to check that it does not affect the refined polar variety $\tilde{\Gamma}_k^k(X^r)$.

By the lower semicontinuity of the degree function in flat families, it suffices to prove the claim for a generic fiber. Let $L$ be a generic affine plane of codimension $k-1$ in $(\mathbb{C}^*)^n$. Since the generic fiber $\tilde{\Gamma}_k^k(X^r)_e$ has pure dimension $k-1$ and the generic fiber $\tilde{\Gamma}_k^k(X^r)_e \setminus \Gamma_k^k(X^r)_e$ has strictly smaller dimension, we may assume that $L$ intersects $\Gamma_k^k(X^r)_e$ only in points of $\tilde{\Gamma}_k^k(X^r)_e$. Let $f_1, \ldots, f_{k-1}$ denote $k-1$ affine linear functionals defining $L$.

We now restrict attention to a particular generic fiber $e^{-1}(\varepsilon) \simeq (\mathbb{C}^*)^n$. All exterior derivatives computed below are taken with respect to this ambient space. At any point $p \in \tilde{\Gamma}_k^k(X^r)_e$, the differentials

$$dp^1_\varepsilon, \ldots, dp^r_\varepsilon, d\ell_1, \ldots, d\ell_{k-1}$$

are linearly independent, while the differentials

$$dp^1_\varepsilon, \ldots, dp^r_\varepsilon, d\ell_1, \ldots, d\ell_{k-1}, d\ell_k$$

(35)
are linearly dependent. Thus there exists one and only one linear dependence of the form
\[ d\ell_k = \lambda_1 dP^1_{\varepsilon} + \cdots + \lambda_r P^r_{\varepsilon} + \lambda_{r+1} d\ell_1 + \cdots + \lambda_{r+k-1} d\ell_{k-1} \]
with \(\lambda_1, \ldots, \lambda_{r+k-1} \in \mathbb{C}\) at \(p\).

In other words, each intersection between \(\tilde{\Gamma}_k^{\varepsilon}(X^r)\) and \(L\) corresponds to an isolated solution of the following system of equations
\[
P_j\varepsilon = 0 \quad j = 1, \ldots, r
\]
\[
\frac{\partial \ell_k}{\partial x_j} = \sum_{i=1}^{r} \lambda_i \frac{\partial P_i}{\partial x_j} + \sum_{i=r+1}^{k-1} \lambda_{r+i} \frac{\partial \ell_i}{\partial x_j} = 0 \quad j = 1, \ldots, n
\]
\[
f_j = 0 \quad j = 1, \ldots, k - 1
\]

Denote by \(\Delta_x\) (resp. \(\Delta_{\lambda}\)) the standard simplex in the \(x\) (resp. \(\lambda\)) variables. Then the system above has Newton polytopes bounded by
\[
\Delta^{r\text{ times}}, \Delta - \Delta_x + \Delta_{\lambda}, \Delta_x^{k-1\text{ times}}
\]
We now estimate the number of solutions of (37) by the Bernstein-Kushnirenko theorem. Since the Newton polytopes above are invariant under translation in the \(\lambda\) variables, we have that the number of solutions in \((\mathbb{C}^*)^n \times \mathbb{C}^{r+k-1}\) is bounded by the mixed volume
\[
(n + r + k - 1)!W\left( \Delta^{r\text{ times}}, \Delta - \Delta_x + \Delta_{\lambda}, \Delta_x^{k-1\text{ times}} \right).
\]
We expand this mixed volume by linearity. In the expansion, if the \(\Delta_{\lambda}\) term is not taken \(r + k - 1\) times out of the \(n\) appearances, then the mixed volume vanishes by Corollary 2. Thus, again by Corollary 2 the mixed volume is equal to
\[
\binom{n}{r+k-1} n! W\left( \Delta^{r\text{ times}}, \Delta - \Delta_x, \Delta_x^{k-1\text{ times}} \right)
\]
and since the mixed volume is invariant under translation and monotone with respect to each argument, we finally obtain that the number of solutions of (37) is bounded by
\[
\binom{n}{r+k-1} n! W\left( \Delta + \Delta_x, \Delta_x^{k-1\text{ times}} \right)
\]
as stated.

We move now to the case of the affine space \(M_0 = \mathbb{C}^n\). Suppose that \(\Delta \subset \mathbb{Z}_{\geq 0}^n\) is a convex co-linear. Then
\[
\Delta(P) \subset \Delta \quad \implies \quad \Delta(P_{x_i}) \subset \Delta \quad i = 1, \ldots, n
\]
In this case one can repeat the proof of Proposition 12 in combination with Remark 3 to obtain the following.

**Proposition 13.** Suppose that \(\Delta \subset \mathbb{Z}_{\geq 0}^n\) is a convex co-linear. Then for any \(\varepsilon \in \mathbb{C}\) we have the bound
\[
\deg \tilde{\Gamma}_k^{\varepsilon}(X^r)_\varepsilon \leq \binom{n}{r+k-1} n! W_{k-1}^*(\Delta)
\]
where \(\deg\) denotes degree in \(\mathbb{C}^n\) and \(W^*\) denotes the quermassintegral.
In particular, if \( P^{1}, \ldots, P^{r} \) are polynomials (with respect to \( x \)) of degrees bounded by \( d \), then
\[
\deg \Gamma_{\ell}^{k}(X^{r}) \leq \binom{n}{r+k-1} d^{n-k+1}.
\] (44)

4.2. Upper bounds for Betti numbers. In this subsection we present two upper bounds for the Betti numbers of the Milnor fiber. The first of these is given in terms of intersection numbers between the polar varieties \( \Gamma_{\ell}^{k}(X^{r}) \) and the affine spaces \( L_{\ell}^{k-1}(p) \).

**Theorem 4.** Let \( S \subset M_{0} \) be a finite collection of good points (i.e., points \( p \) such that \( X_{\varepsilon}^{r} \) is effectively smooth in a neighborhood of \( p \) for \( \varepsilon \neq 0 \)). Fix \( \ell \) sufficiently generic.

Then for \( k = 0, \ldots, n-r \) and for any good \( p \in S \), we have
\[
b_{k}(F_{p}^{r}) \leq i(p; \Gamma_{\ell}^{n-r-k+1}(X^{r})_0 \cdot L_{\ell}^{n-r-k}(p); M_0)
\] (45)

**Proof.** The statement follows by application of Proposition 8 after noting that \( \Gamma_{\ell}^{n-r-k+1}(X^{r}) \) agrees with the polar variety \( \Gamma_{\ell}^{n-r-k+1}(X^{r})_0 \) in a neighborhood of any good point \( p \).

Next, we give a bound that holds uniformly at all good points \( p \in M_{0} \).

**Theorem 5.** Fix \( \ell \) sufficiently generic. Then for \( k = 0, \ldots, n-r \) and for any good \( p \in M_{0} \), we have
\[
b_{k}(F_{p}^{r}) \leq \mathcal{D}_{\Gamma_{\ell}^{n-r-k+1}(X^{r})_0}(p)
\] (46)

**Proof.** The proof follows by application of the results from the appendix. Namely, consider function
\[
f: M_{0} \to \mathbb{N} \quad f_{k}(p) = \begin{cases} b_{k}(F_{p}^{r}) & \text{if } p \text{ is good} \\ 0 & \text{otherwise} \end{cases}
\] (47)

By Corollary 22 the function \( F_{\ell} := \mathcal{D}_{\Gamma_{\ell}^{n-r-k+1}(X^{r})_0} \) has uniformly bounded complexity independent of \( \ell \). By Proposition 13 there exists a finite set \( S \subset M_{0} \) such that for any \( \ell \), \( f|_{S} \leq F_{\ell}|_{S} \) implies \( f \leq F_{\ell} \).

Choose \( \ell \) sufficiently generic so that Theorem 3 applies for that set \( S \). We claim that \( f|_{S} \leq F_{\ell}|_{S} \). Indeed, the inequality is trivial for points \( p \in S \) which are not good, and for good points it follows from (45) and the simple observation
\[
i(p; \Gamma_{\ell}^{n-r-k+1}(X^{r})_0 \cdot L_{\ell}^{n-r-k}(p); M_0) \leq \mathcal{D}_{\Gamma_{\ell}^{n-r-k+1}(X^{r})_0}(p)
\] (48)

\[\square\]

5. Multiplicity estimates

In this section we turn to the subject of multiplicity estimates. Once again we consider the ambient manifold \( M = (\mathbb{C}^{*})^{n} \times \mathbb{C} \) with coordinate ring \( R = \mathbb{C}[x_{1}^{\pm 1}, \ldots, x_{n}^{\pm 1} ; e] \) or \( M = \mathbb{C}^{n} \times \mathbb{C} \) with coordinate ring \( R = \mathbb{C}[x_{1}, \ldots, x_{n}, e] \), where \( e \) is viewed as the parameter of a deformation.

Consider a Laurent vector field \( V \) and a Laurent polynomial \( P \),
\[
V = \sum_{i=1}^{n} Q_{i} \frac{\partial}{\partial x_{i}}, \quad \text{for } Q_{i} \in \mathbb{C}[x_{1}^{\pm 1}, \ldots, x_{n}^{\pm 1}],
\]
\[
\text{and } P \in \mathbb{C}[x_{1}^{\pm 1}, \ldots, x_{n}^{\pm 1}].
\] (49)
5.1. The multiplicity cycles. Recall the notations of \[3.2\] Let \(\lambda = (\ell, c)\) where \(c\) denotes the parameters defining the deformation \(P^c\) of \(P\) given in Lemma \[10\]. We will denote \(P^c\) by \(P^\ell\) to simplify the notation.

We start by defining a collection of algebraic cycles which play a key role in our multiplicity estimates.

**Definition 14.** For \(k = 0, \ldots, n - 1\) and \(\ell\) as in \[18\], we define the \(k\)-th multiplicity cycle of the deformation \(P^\lambda\), denoted \(\mathcal{M}_k(P^\lambda)\), to be the \(k\)-cycle in \(M_0\) given by

\[
\mathcal{M}_k(P^\lambda) := \sum_{r=1}^{n-k} \Gamma_{\ell}^{k+1}(X_r)_0
\]

(50)

The motivation for this definition becomes apparent in light of the following theorem, describing the behavior of the multiplicity function \(\text{mult}_V P\) in terms of the multiplicity cycles.

**Theorem 6.** Let \(S \subset M_0\) be a finite collection of points, and assume that for every \(p \in S\) the vector field \(V\) is non-singular and \(\text{mult}_V P < \infty\). Fix \(\lambda\) sufficiently generic.

Then for any \(p \in S\) we have

\[
\text{mult}_V P \leq \sum_{k=0}^{n-1} i(p; \mathcal{M}_k(P^\lambda) \cdot L_k^\ell(p); M_0)
\]

(51)

**Proof.** By Lemma \[10\] we may assume that each \(p \in S\) is a good point of the corresponding deformation \(P^\lambda\), and \(F_r^\ell(p)\) is empty for \(r > n\). Therefore, for any \(p \in S\) we have

\[
\text{mult}_V P = \sum_{r=1}^{\infty} \chi(F_r^\ell(p)) \leq \sum_{r=1}^{n} \chi(F_r^\ell(p)) \leq \sum_{r=1}^{n} \sum_{k=0}^{n-r} b_k(F_r^\ell(p))
\]

\[
\leq \sum_{r=1}^{n} \sum_{k=0}^{n-r} i(p; \Gamma_{\ell}^{n-r-k+1}(X_r)_0 \cdot L_k^{n-r-k}(p); M_0)
\]

\[
\leq \sum_{k=0}^{n-1} i(p; \mathcal{M}_k(P^\lambda) \cdot L_k^\ell(p); M_0)
\]

(52)

where (i) follows from Theorem \[3\] (ii) follows since \(F_r^\ell(p)\) is empty for \(r > n\); (iii) follows since the Euler characteristic is bounded by the sum of the Betti numbers, and the \(k\)-th Betti number of \(F_r^\ell(p)\) vanishes for \(k > n - r\); (iv) follows from Theorem \[4\] and (v) is a resummation. \(\square\)

Next, we give a bound that holds uniformly at all points \(p \in M_0\) where \(V\) is non-singular and \(\text{mult}_V P < \infty\).

**Theorem 7.** Fix \(\lambda\) sufficiently generic. Then for every point \(p \in M_0\) where \(V\) is non-singular and \(\text{mult}_V P < \infty\) we have

\[
\text{mult}_V P \leq \sum_{k=0}^{n-1} \mathcal{D}_{\mathcal{M}_k(P^\lambda)}(p)
\]

(53)
Proof. To show that for a sufficiently generic \( \lambda \) the bound \((53)\) holds uniformly over the points \( p \in M_0 \) where \( V \) is non-singular and the multiplicity is finite, we proceed as in the proof of Theorem 5. Consider the function

\[
f : M_0 \to \mathbb{N} \quad f(p) = \begin{cases} \text{mult}_p^V P & \text{if } V(P) \neq 0 \text{ and } \text{mult}_p^V P < \infty \\ 0 & \text{otherwise} \end{cases}
\]  

(54)

By Corollary 22 the function \( F_{\ell} \) has uniformly bounded complexity independent of \( \lambda \). By Proposition 19 there exists a finite set \( S \subset M_0 \) such that for any \( \lambda \), \( f|_S \leq F_{\ell}|_S \) implies \( f \leq F_{\ell} \).

Choose \( \lambda \) sufficiently generic so that Theorem 6 applies for the set \( S \). We claim that \( f|_S \leq F_{\ell}|_S \). Indeed, the inequality is trivial for points \( p \in S \) where \( V \) is singular or where the multiplicity is infinite, and for the remaining points it follows from \((51)\) and the simple observation

\[
i(p; M^k(P^\lambda) \cdot L^k_{\ell}(p); M_0) \leq D_{M^k(P^\lambda)}.
\]  

(55)

\[\square\]

Remark 15. In fact, since we are interested in upper bounds for the Euler characteristic, it would be reasonable to include in the definition of multiplicity cycles only those polar varieties that contribute Betti numbers of even dimension, or even include those that contribute Betti numbers of odd dimension with a negative sign. This would improve many of our multiplicity estimates roughly by a factor of two. We have avoided this in this paper in order to simplify the notation. However, see §5.2.3 for an illustration.

We now give estimates on the degrees of the multiplicity cycles in the torus and affine cases. We will assume for simplicity that \((n - 1)\Delta_x \subset \Delta(P)\) where \( \Delta_x \) denotes the standard simplex in the \( x \)-variables. Under this assumption we have \( \Delta(P^\lambda) = \Delta(P) \).

The following estimates are obtained in a straightforward manner from the corresponding propositions for polar varieties, namely Proposition 12 and Proposition 13, by noting that the equations defining \( X^r \) have Newton polygons contained in \( \Delta(P) + (r - 1)\Delta(V) \).

**Proposition 16.** Suppose that a translate of \((n - 1)\Delta_x \) is contained in \( \Delta(P) \). Then we have the bound

\[
\deg M^k(P^\lambda) \leq \sum_{r=1}^{n-k} \binom{n}{r+k} n! W_k^x(\Delta(P) + (r - 1)\Delta(V) + \Delta_x) < 2^n n! W_k^x(\Delta(P) + (n - k - 1)\Delta(V) + \Delta_x)
\]  

(56)

where \( \deg \) denotes degree in \( (\mathbb{C}^*)^n \), \( W^x \) denotes the quermassintegral, and \( \Delta_x \) denotes the standard simplex in the \( x \) variables.

**Proposition 17.** Suppose that \( \Delta \subset \mathbb{Z}^n_{\geq 0} \) is a convex co-ideal containing \((n - 1)\Delta_x \). Then we have the bound

\[
\deg M^k(P^\lambda) \leq \sum_{r=1}^{n-k} \binom{n}{r+k} n! W^x_k(\Delta(P) + (r - 1)\Delta(V)) < 2^n n! W^x_k(\Delta(P) + (n - k - 1)\Delta(V))
\]  

(57)

where \( \deg \) denotes degree in \( \mathbb{C}^n \), \( W^x \) denotes the quermassintegral.
In particular, if $P$ is a polynomial of degree $d \geq n - 1$ and $V$ is a polynomial vector field of degree $\delta$, then
\[
\deg M^k(P^\lambda) < 2^n (d + (n - k - 1)(\delta - 1))^{n-k}
\] (58)

5.2. Improving the estimates of Nesterenko, Gabrielov and Risler. In this section we show how Theorem 7 and Proposition 17 imply a strengthening of the results of Nesterenko [18], Gabrielov [8] and Gabrielov and Risler [9]. We therefore restrict attention to the case where $P, V$ given as in (1). We fix $\lambda$ sufficiently generic for the application of Theorem 7.

5.2.1. The case of a single point in arbitrary dimension. We assume for simplicity of the formulation that $d \geq 1$. If $p \in \mathbb{C}^n$ is a non-singular point of $V$ and $\text{mult}_p^V P < \infty$ then by Theorem 7 and Proposition 17 we have
\[
\text{mult}_p^V P \leq \sum_{k=0}^{n-1} \deg M^k(P^\lambda)(p) \leq 2^n \sum_{k=0}^{n-1} (d + (n - k - 1)(\delta - 1))^{n-k} \leq 2^n (d + (n - 1)(\delta - 1))^n
\] (59)
which improves the estimates of Nesterenko and Gabrielov for the case of a single point.

5.2.2. The case of a single point in $\mathbb{C}^3$. In [9] Gabrielov and Risler considered the case $n = 3$ in detail using a different deformation technique. Their estimate, which is the best estimate known for this particular case, is as follows
\[
\text{mult}_p^V P \leq d + 2d(d + \delta - 1)^2.
\] (60)
A naive application of Theorem 6 does not yield and improvement of this result. However, using the more refined approach indicated in Remark 15 one can still obtain an improvement using our method.

Assume for simplicity that $d \geq n - 1 = 2$ (the remaining case $d = 1$ can be treated separately, for instance by reduction of dimension; we leave the details for the reader). Then, in the notations of [3,2] we have three Milnor fibers $F_0^0, F_1^1$ and by Remark 15 we are interested in an upper bound for the sum of their even Betti numbers. Simple computations using the corresponding polar varieties give
\[
\begin{align*}
b_0(F^0_0) &\leq d \\
b_0(F^1_0) &\leq d(d + \delta - 1) \\
b_0(F^2_0) &\leq d(d + \delta - 1)(d + 2\delta - 2) \\
b_2(F^0_0) &\leq d(d - 1)^2
\end{align*}
\]
and accordingly,
\[
\text{mult}_p^V P \leq d \left[ 1 + (d - 1)^2 + (d + \delta - 1)(d + 2\delta - 1) \right]
\] (61)
and it is a simple exercise, left for the reader, to verify that this improves (60) for any $d, \delta$. 
5.2.3. The case of several points. Moving now to the case of several points, let \( p_1, \ldots, p_\nu \in \mathbb{C}^n \) be non-singular points of \( V \) and assume that \( \text{mult}_{p_i}^V P < \infty \). Recall the notations of §1.2.1. We consider first the case \( \kappa = n \).

Let \( Z \) denote any \( k \)-cycle in \( \mathbb{C}^n \) and write \( Z = Z_1 + \ldots + Z_q \) where each \( Z_i \) is a cycle supported on an irreducible variety (possibly with a coefficient greater than 1). Then

\[
\sum_{i=1}^\nu \mathcal{D}_Z(p_i) \leq \sum_{j=1}^q \mathcal{D}_{Z_j}(p_i) \leq \sum_{j=1}^q a(Z_j) \deg Z_j \tag{62}
\]

\[
\leq (\max a(Z_j)) \sum_{j=1}^n \deg Z_j = a(Z) \deg Z
\]

where \( a(Z_j) \) denotes the number of points \( p_i \) lying in \( Z_j \), and \( a(Z) \) denotes the maximal number of points \( p_i \) lying in one of the irreducible components of \( Z \).

We now proceed with the multiplicity estimate, again relying on Theorem 7 and Proposition 17

\[
\sum_{i=1}^\nu \text{mult}_{p_i}^V P \leq \sum_{i=1}^\nu \sum_{k=0}^{n-1} \mathcal{D}_{\mathcal{M}_k(P^\lambda)}(p_i) \leq \sum_{k=0}^{n-1} a(\mathcal{M}_k(P^\lambda)) \deg \mathcal{M}_k(P^\lambda) \tag{63}
\]

\[
\leq 2^n \sum_{k=0}^{n-1} a(\mathcal{M}_k(P^\lambda))(d + (n - k - 1)(\delta - 1))^{n-k}
\]

and noting that \( \mathcal{M}_k(P^\lambda) \) does indeed have degree of the order \( O(d^{n-k}) \) with respect to \( d \), we obtain Nesterenko’s estimate (with improved constants).

Finally, we consider the case \( \kappa < n \). That is, we now assume that all points \( p_i \) belong to a single trajectory \( \gamma \) which has transcendence degree \( \kappa \). Let \( Y \subset \mathbb{C}^n \) denote the algebraic closure of \( \gamma \). Then \( \dim Y = \kappa \) and \( Y \) is invariant under the flow of \( V \) (since it has a Zariski dense subset, namely \( \gamma \), which is invariant). Since the flow of \( V \) maps \( Y \) to itself and maps the ambient space \( \mathbb{C}^n \) biholomorphically to itself (whenever defined), and since the singular part of an analytic set is a holomorphic invariant, it follows that the singular part \( \text{Sing} V \) is invariant under the flow of \( V \) as well.

We claim that the points \( p_i \) belong to the smooth part of \( Y \). Indeed, suppose that some point \( p_i \) belongs to \( \text{Sing} Y \). Then since \( \text{Sing} Y \) is invariant under the flow of \( V \), it follows that the germ \( \gamma_{p_i} \) is contained in \( \text{Sing} Y \). Since we assume that all points \( p_i \) belong to a single trajectory \( \gamma \), by analytic permanence it follows that \( \gamma \subset \text{Sing} Y \), contradicting our assumption that \( Y \) is the Zariski closure of \( \gamma \).

One can now carry out all preceding computations in the ambient space \( Y \) instead of \( \mathbb{C}^n \): the only assumption which is needed is the smoothness of the ambient space at the points being considered. Naturally, in the estimates of the degrees of the corresponding multiplicity cycles, the degrees of the equations defining \( Y \) would play a role giving rise to existential constants as in Nesterenko’s result. However, these existential constants do not affect the asymptotic dependence on \( d \), which agrees with Nesterenko’s estimate. We omit the details of this computation.
Remark 18. In the case \( \kappa = n \), the constants appearing in our result, as well as Nesterenko’s, are explicit. In the case \( \kappa < n \) the constants, for both proofs, depend on the algebraic complexity (for instance the degree) of the Zariski closure \( Y \). This degree cannot in general be estimated in terms of \( n, d, \delta \), as illustrated by the vector field \( rx \partial_x + sy \partial_y \) which admits a trajectory \( \{ x^s = y^r \} \) of degree depending on the coefficients \( r, s \).

However, using our method one can obtain estimates with explicit constants — albeit involving terms of order up to \( d^n \) — even when \( \kappa < n \). Indeed, nowhere in the derivation of (63) did we use the assumption \( \kappa = n \). On the other hand, Nesterenko’s approach appears to depend in a more essential way on the assumption \( \kappa = n \), and it is not clear that it can be used to produce explicit bounds, even ones allowing terms of order \( d^n \), when \( \kappa < n \).

5.3. Concluding remarks and some directions for future research. Beyond the general type of multiplicity estimates considered in this paper, many different forms have been treated in the literature. It would be interesting to see if the methods used in this paper could be generalized to these contexts. We list a few examples below.

Many results have been obtained for the case when the ambient manifold is a commutative algebraic group, the vector field is an invariant field for the group, and the set of points is a “cube” of a specified dimension and length. For a survey of some of these results and their applications in transcendental number theory see [15].

Another possible generalization is for the case of analytic trajectories at singular points of the vector field \( V \). In [19], Nesterenko considers a singular vector field satisfying the additional “D-property”. Under this extra assumption, Nesterenko again obtains estimates which are sharp up to a multiplicative constant with respect to \( d \). This result and various generalizations also play an important role in transcendental number theory.

Finally, in [10] Gabrielov and Khovanskii consider multiplicity estimates in several dimensions. Specifically, they consider a tuple of commuting vector fields \( V_1, \ldots, V_m \) defining an integral manifold \( \mathcal{L} \) of dimension \( m \), and a tuple of \( m \) polynomials \( P_1, \ldots, P_m \). They give an estimate for the maximal multiplicity of an isolated common zero \( P_1 = \cdots = P_m = 0 \). Our method does not directly extend to this generality due to some technical difficulties (specifically, the literal analog of Lemma 10 fails), but it would be interesting to check whether similar ideas can be used to improve this result.

Appendix A. A compactness property for semicontinuous bounds

In this appendix we will assume for simplicity of the formulation that the ambient variety \( M \) is the affine space \( \mathbb{C}^n \) or the torus \( (\mathbb{C}^*)^n \), though the ideas can be carried out verbatim in a much more general context.

Recall that a function \( F : M \to \mathbb{N} \) is said to be (algebraic) upper semicontinuous if the sets \( F \geq n := F^{-1}([n, \infty)) \) are closed varieties for each \( n \in \mathbb{N} \). We will say that \( F \) has complexity bounded by \( D \) if moreover, all of these sets can be defined by equations of degree at most \( D \).

Proposition 19. Let \( D \in \mathbb{N} \) and \( f : M \to \mathbb{N} \) an arbitrary bounded function. Then there exists a finite set of points \( P \subset M \) such that for any upper semicontinuous
function $F$ of complexity bounded by $D$,

$$f|_P \leq F|_P \implies f \leq F.$$

(64)

**Proof.** Denote by $N$ an upper bound for $f$. Then $f \leq F$ if and only if $f_{\geq i} \subset F_{\geq i}$ for $i = 1, \ldots, N$. Thus it will suffice to construct a finite set $P_i \subset f_{\geq i}$ such that for any set $S$ of complexity bounded by $D$,

$$P_i \subset S \implies f_{\geq i} \subset S$$

(65)

and take $P = \bigcup_{i=1}^N P_i$.

Let $L$ denote the linear space of polynomials of degree bounded by $D$ on $M$. For any $p \in M$ let $\phi_p : L \to \mathbb{C}$ denote the functional of evaluation at $p$. Finally, for any set $P \subset M$ denote by $L_P \subset L$ the linear subspace of polynomials which vanish at every point of $P$.

We need to construct a finite set $P_i \subset f_{\geq i}$ with $L_{P_i} = L_{f_{\geq i}}$. This is clearly possible. Indeed, $L_{f_{\geq i}}$ is the kernel of the set of functionals $\{\phi_p : p \in f_{\geq i}\}$. Since $L_{f_{\geq i}}$ has finite codimension in $L$, one can choose a finite subset $P_i$ (in fact, of size equal to this codimension) of functionals whose kernel, $L_{P_i}$, agrees with $L_{f_{\geq i}}$. This concludes the proof. $\square$

The proofs of the following simple lemmas are left for the reader.

**Lemma 20.** Let $F_i, i = 1, \ldots, N$ be upper semicontinuous functions of complexity bounded by $D_i$. Then the function $\sum_i F_i$ is an upper semicontinuous function of complexity bounded by a constant $D'$ depending only on $N$ and $D_1, \ldots, D_N$.

**Lemma 21.** If $V \subset M$ is an irreducible variety of degree bounded by $d$, then $D_V$ is an upper semicontinuous function of complexity bounded by $d$.

For the proof of the second lemma it suffices to recall the standard fact that a variety of degree $d$ is cut out set-theoretically by equations of degree bounded by $d$. Finally, we have the following simple corollary.

**Corollary 22.** Let $C$ be an algebraic cycle (possibly of mixed dimension) of total degree bounded by $d$. Then $D_C$ is an upper semicontinuous function of total degree bounded by a constant $D$ depending only on $d$.

**Proof.** Indeed, $D_C$ is a sum of at most $\deg C$ upper semicontinuous functions, each of complexity bounded by $\deg C$. The statement follows by the preceding two lemmas. $\square$

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