Relativistic dynamics of domain wall in one-dimensional SQUID array

Munehiro Nishida¹, Yuushi Aoki¹ and Toshiyuki Fujii²

¹ Graduate School of Advanced Science of Matter, Hiroshima University, Higashi-Hiroshima, Japan.
² Department of Physics, Asahikawa Medical University, Asahikawa, Japan.

E-mail: mnishida@hiroshima-u.ac.jp

Abstract. We investigate the dynamics of a domain wall in a one-dimensional array of superconducting quantum interference device (SQUID) composed of three conventional Josephson junctions and a \(\pi\)-junction. The domain wall is formed between two domains with oppositely circulating current through the SQUID loop. It is shown that the SQUIDs in this array can be approximately described by a double sine-Gordon (DSG) model which obeys Einstein’s special theory of relativity. We conduct numerical simulations of a discrete DSG equation, and show that the domain wall propagates solitonically through the SQUID array and exhibits quasi-relativistic behavior, such as the Lorentz contraction and the relativistic time dilation, which agrees reasonably well with the predictions from a relativistic equation of motion of a particle, whose rest mass is extremely small compared to that of a single electron.

1. Introduction

A topological soliton excitation in a long Josephson junction, called fluxon or Josephson vortex, is known to behave like a relativistic particle and exhibits counterintuitive phenomena such as Lorentz contraction, i.e., length contraction when an object is travelling close to the speed of light [1, 2]. However, a single fluxon cannot exhibit another relativistic effect in the time domain, namely, time dilation where moving clocks run slowly, because it does not have its own clock. In our recent works, we have shown that some kind of solitonic excitation, such as a bound (half-)fluxon pair in a long superconductor-feffomagnet-superconductor (SFS) junction [3] or in a one-dimensional superconducting quantum interference device (SQUID) array [4], can be used in testing a relativistic time dilation, since it has an internal oscillation acting as its own clock. However, there seems to be some difficulties in preparing these systems with nowadays technology. In the case of the SFS junction, we need a junction that has high normal resistance and quite nonsinusoidal current-phase relation, simultaneously. In the case of the SQUID array, it is necessary to create almost completely equivalent two junctions in each SQUID, since there is an instability against the fluctuation of the relative phase difference between the junctions [5].

Here, we propose a domain wall in a one-dimensional SQUID array with the Möbius boundary condition as another candidate for a solitonic excitation that can be used in testing relativistic time dilation. In our proposed system, each SQUID is composed of three asymmetrically positioned conventional Josephson junctions (0-junctions) and a \(\pi\)-junction whose critical current is much larger than that of 0-junctions, as shown in Figure 1. The \(\pi\)-junction acts as a passive \(\pi\)-phase shifter, i.e., the phase difference across the \(\pi\)-junction is always kept \(\pi\) [6]. This means...
that the phase difference of the remaining 0-junctions of the SQUID becomes $-\pi$ or $\pi$, which corresponds to the state $|0\rangle$ with the clockwise circulating current or the state $|1\rangle$ with the counter-clockwise circulating current, just as in the persistent current qubit [7]. In a one-dimensional array of such SQUIDs, domain walls are formed between $|0\rangle$ and $|1\rangle$ domains.

Using a series of two identical Josephson junctions in the left arm of the SQUID, the current-phase relation for each SQUID acquires a double sine term [8], and thus, the phase differences of our SQUID array obey a discrete double sine-Gordon (DSG) equation [4]. Since the discrete DSG model approximately obeys Einstein’s special theory of relativity, it is expected that relativistic motion of the domain wall will be observed. Moreover, the domain wall corresponds to the small kink solution of the DSG equation, which has an internal oscillation mode, called shape mode [8]. This means the domain wall has its own clock.

In this paper, we investigate the dynamics of a domain wall in this system. We conduct numerical simulations and show that the domain wall has no instability and propagates solitonically through the SQUID array. It is shown that the domain wall exhibits quasi-relativistic behavior, such as the Lorentz contraction and the time dilation, which agrees reasonably well with the predictions from a relativistic equation of motion of a particle, whose rest mass is extremely small compared to that of a single electron.

2. 1D SQUID array with Möbius boundary condition

As in Ref. [4], we assume that the phase differences $\varphi^j$ of the two 0-junctions in the left arm $(j = l1, l2)$ of each SQUID and one 0-junction in the right arm $(j = r)$ can be adequately described by the resistively and capacitively shunted junction (RCSJ) model. We also assume that the two junctions in the left arm are identical and take $C^{l1} = C^{l2} \equiv 2C^l$, $R^{l1} = R^{l2} \equiv R^l/2$, and $J^{l1} = J^{l2} \equiv J^l$, where $R^j$, $C^j$, and $J^j$ are the resistance, capacitance, and critical current of junction $j$. Due to the existence of the $\pi$-shifter, the phase differences are connected via $\varphi^r = \varphi^{l1} + \varphi^{l2} \pm \pi$. Therefore, the total current through the SQUID, $I^s$, is expressed by the total phase difference $\varphi = \varphi^{l1} + \varphi^{l2}$ and the relative phase difference $\theta = \varphi^{l1} - \varphi^{l2}$ of the left arm as $I^s = \frac{hC^l}{2e} \varphi + \frac{h}{2eR^l} \theta + J^l \sin \left( \frac{\varphi^r}{2} \right)$, where $R = (1/2R^l + 1/R^r)^{-1}$, $C = C^l/2 + C^r$, and the dot denotes the time differentiation.

The current continuity in the left arm yields $\frac{hC^l}{2e} \dot{\varphi} + \frac{h}{2eR^l} \dot{\theta} + 2J^l \cos \left( \frac{\varphi^r}{2} \right) \sin \left( \frac{\theta}{2} \right) = 0$. This means that $\theta = 0$ becomes an unstable equilibrium when $\varphi$ exceeds the range of $-\pi < \varphi < \pi$. 

![Figure 1. Equivalent circuit of a SQUID array with Möbius boundary condition, where the cross denotes a conventional junction and the star denotes a $\pi$-junction.](image1)

![Figure 2. (a) Phase profile for a domain wall when $d = 0.5, \zeta = 1/\sqrt{2}$. (b) The “potential” and the eigenfunctions for the zero mode and the shape mode.](image2)
However, in our proposed domain-wall system, \( \varphi \) is always restricted in this range and there is no instability. We have confirmed with numerical simulations that we can safely assume \( \theta = 0 \).

Figure 1 shows the equivalent circuit of our proposed array of \( N \) SQUIDs. The fluxoid quantization condition for the \( i (\neq N) \)th mesh between SQUIDs is \( \varphi_i - \varphi_{i+1} = -\pi L (I^i - I^b) / \Phi_0 \), where \( L \) is the loop inductance of the mesh, and \( (I^i - I^b) / 2 \) is the mesh current circulating in each loop. The current conservation reads \( I^c + I^i = I^s + I^l_1 \) and \( I^c + I^l_1 = I^s + I^b \), where \( I^c \) is the external current. Assuming \( \theta = 0 \) in these relations, we obtain

\[
\frac{\partial^2 \varphi_i}{\partial t^2} + \alpha \frac{\partial \varphi_i}{\partial t} + \frac{2}{1 + 2\zeta} \left\{ \sin \left( \frac{\varphi_i}{2} \right) - \zeta \sin(\varphi_i) \right\} - \frac{\varphi_{i+1} + \varphi_{i-1} - 2\varphi_i}{d^2} = \gamma(t),
\]

where \( \alpha = t_0 / RC \), \( \zeta = J^c / J^s \), \( \gamma(t) = I^c(t) / I_0 \), and \( d = \sqrt{2\pi LI_0 / \Phi_0} \) with \( t_0 = \sqrt{\Phi_0 C / 2\pi I_0} \) being the unit of time, \( I_0 = J^c + J^s / 2 \) being the unit of current. Thus, the SQUID array obeys a discrete DSG equation with friction and external driving.

The Möbius boundary condition is derived from the fluxoid quantization condition for the \( N \)th mesh, \( \varphi_N + \varphi_1 = -\pi L (I_N - I_b) / \Phi_0 \), which means \( \varphi_{N+1} = -\varphi_1 \) or \( \varphi_0 = -\varphi_N \). With this condition, the \( |0 \rangle \) domain and the \( |1 \rangle \) domain can connect continuously through the \( N \)th mesh.

3. Small kink and shape mode

The parameter \( d \) can be considered the effective mesh size of the SQUID array normalized by the typical width of a single fluxon. In the limit of a small \( d \), i.e., the dense array limit, and when \( \alpha = \gamma = 0 \), Eq. (1) becomes a continuous DSG equation:

\[
\frac{d^2 \psi}{dx^2} - \frac{d^2 \psi}{dt^2} + \frac{2}{1 + 2\zeta} \left\{ \sin \left( \frac{\psi}{2} \right) - \zeta \sin(\psi) \right\} = 0,
\]

where the normalization unit of velocity is the speed of light in this system \( c = d_0 / d_0 = d_0 / \sqrt{LC} \) with \( d_0 \) being the actual width of the mesh. The DSG equation is known to have a small kink solution [8]:

\[
\varphi_K = 4 \tan^{-1} \left[ \sqrt{\frac{2 - 1}{2 + 1}} \tanh \left\{ \sqrt{\frac{2 - 1}{8}} x \right\} \right] \quad \text{with} \quad \varphi_K (\pm \infty) = \pm \varphi_0 = \pm 2 \cos^{-1}(1/2\zeta).
\]

Figure 2(a) shows the plot of the lowest energy configurations of \( \varphi_K \) for \( d = 0.5 \) and \( \zeta = 1 / \sqrt{2} \) as a function of \( x_i = d(i - N/2) \). In this condition, \( \varphi_0 = \pi / 2 \) and the small kink becomes \( \pi \)-kink. We see that the small kink solution well describes the phase differences in the domain wall of the SQUID array system.

The linearized oscillations about the small kink obey an equation of the form, \( -\frac{d^2 \psi}{dx^2} + V_K(x) \psi(x) = \omega^2 \psi(x) \), where \( \psi(x) = \varphi(x) - \varphi_K(x) \) and \( V_K = -\frac{1}{1 + 2\zeta} \left\{ \cos(\varphi_K / 2) - 2\zeta \cos(\varphi_K) \right\} \) [8]. This equation has two bound states, the zero mode (\( \omega = 0 \)) and the shape mode (\( \omega = \omega_s \)). The zero mode is nothing but the well-known “Goldstone” mode, which corresponds to the translational degree of freedom of the small kink. On the other hand, the shape mode can be viewed as an internal oscillation of the shape of the kink. Figure 2(b) shows the “potential” \( V_K(x) \) and the eigenfunctions for the zero mode and the shape mode. In the case of \( \pi \)-kink (\( \zeta = 1 / \sqrt{2} \)), the eigen angular frequency for the shape mode becomes \( \omega_s \approx 0.4885 \), which corresponds to the period of \( T_s \approx 12.86 \).

It is known that the dynamics of a solitonic excitation can be well-described by a so-called collective coordinate (CC). Using the same approach adopted in [9], we have derived a relativistic equation of motion for the velocity \( v(t) \) of the center-of-mass position of a domain wall as

\[
\frac{d}{dt} \left( \frac{M_0 v(t)}{\sqrt{1 - v^2(t)}} \right) = -\alpha M_0 v(t) - 2\varphi_0 \gamma(t),
\]

with the rest mass, \( M_0(\zeta) = \sqrt{\frac{32}{(\zeta + 2\zeta^2)}} \left\{ \sqrt{4\zeta^2 - 1} - \frac{2\zeta}{2} \right\} \).

Then, the terminal velocity \( v_\infty \) under the constant external current \( \gamma_0 \) is estimated as

\[
v_\infty = 2\varphi_0 \gamma_0 / \sqrt{\alpha^2 M_0^2 + 4\varphi_0^2 \gamma_0^2}.
\]

The typical value of the rest mass is estimated as \( M_0(1 / \sqrt{2}) \approx 0.93 \approx 10^{-36} \text{[kg]} \) for \( d = 0.5 \), using realistic parameters, \( C = 0.5 \text{[pF]} \), \( d_0 = 100 \text{[\mu m]} \) [5]. Thus, the rest mass of the domain wall is extremely small compared to that of a single electron.
4. Lorentz contraction and relativistic time dilation

![Figure 3](image1)

**Figure 3.** (a) Relation between the terminal velocity and the external current. The solid line is obtained by a collective coordinate approach. (b) Velocity dependence of the domain-wall width. The solid line denotes the relation of the Lorentz contraction.

![Figure 4](image2)

**Figure 4.** Amplitude of the shape mode as a function of the period of external alternating current and the velocity of the domain wall. The white dotted line denotes the relation of the relativistic time dilation.

We now show the results of numerical simulations of Eq. (1) for $\alpha = 0.01$, $d = 0.5$, $N = 256$ and $\zeta = 1/\sqrt{2}$ ($\pi$-kink), with $\gamma(t) = \gamma_0 \text{sign}(\varphi_N) + \gamma_1 \sin(2\pi \nu_e t)$. Here, “$\text{sign}(\varphi_N)$” is necessary because the domain wall ($\pi$-kink) is converted to the anti-domain wall ($-\pi$-kink) when passing through the $N$th mesh, and the induced force from the external current changes its sign.

Figure 3(a) shows the relation between the terminal velocity $v_\infty$ and the external current $\gamma_0$, when $\gamma_1 = 0$. We see that the results of the simulation agrees well with Eq. (2) obtained by a CC approach except near the speed of light. As we see from Figure 3(b), the width of the domain wall obeys the Lorentz contraction rule. When the velocity $v_\infty$ approaches to the speed of light, the width $r$ becomes comparable to the mesh size $d$ due to the Lorentz contraction, and the discreteness becomes effectively strong, which results in this discrepancy.

When $\gamma_1$ is switched on, resonance would occur when the frequency $\nu_1$ of the external current matches the frequency of the shape mode, and the amplitude of the oscillation becomes large. Thus, we can measure the frequency of the shape mode by detecting the variance of the width of the domain wall as a function of the frequency of the applied alternating current. Figure 4 shows the amplitude of the oscillation of $r$ as a function of the period of the external alternating current, $1/\nu_e$, and the average velocity $v$ of a pair after convergence to the terminal velocity. The amplitude is estimated from the Fourier spectra of $r(t)$. We can easily find the resonance peaks of the internal oscillation, which coincide with the (white) dotted curve, namely, the relation of the relativistic time dilation, $T = T_0/\sqrt{1 - v^2}$, with the predicted value, $T_0 \approx 12.86$.

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