Chirality Deconfinement Beyond the $C = 1$ Barrier of Two Dimensional Gravity

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Abstract

The characteristic novel features of strongly coupled gravity at the special values ($C_L = 7, 13, 19$) are reviewed in a simple manner using pictures as much as possible.

(Notes of lectures at the 1995 Cargese Meeting Low Dimensional Applications of Quantum Field Theory)

\footnote{Unité Propre du Centre National de la Recherche Scientifique, associée à l’École Normale Supérieure et à l’Université de Paris-Sud.}
1 Introduction

In recent times we have witnessed important developments in our understanding of two dimensional gravity in the weak coupling regime, especially in what concerns the dressing by gravity of matter with central charge smaller than one. Nevertheless, the popular approaches seem to be unable to overcome the so called $c = 1$ barrier, powerful and elegant as they may be. The point of these two lectures is to show, following refs. [1, 2, 3, 4] how the operator approach to Liouville theory which I started to develop long ago with A. Neveu, remains applicable in the strong coupling regime. The principle is as follows. In the weak coupling regime, the Liouville exponentials are expressed in terms of chiral vertex operators, themselves constructed from a free field in two dimensions. This is the quantum version of the well known classical Bäcklund transformation. Locality and closure of their OPA (operator product algebra) uniquely determines this chiral decomposition. In the strong coupling regime, this latter construction looses meaning since the OPA of the Liouville exponentials involves operators and/or highest weight states with complex Virasoro weights. Nevertheless the general operator-family of chiral components may still be used, when truncation theorems [2, 1, 4] apply, that is with central charges $C_L = 7, 13, 19$. Indeed for these values there exist subfamilies of the above chiral operators which form closed operator algebras, only involving real Virasoro weights. In the present operator approach they are used to construct a new set of local fields which replace the Liouville exponentials. Since both sets are constructed out of the same free Bäcklund fields, they may be considered as related by a new type of quantum Bäcklund transformation, that connect the weak and strong coupling regimes of two-dimensional gravity. In the present lecture notes, I summarise the basic features (deconfinement of chirality, new expression for the string susceptibility) of these new set of local fields that replace the Liouville exponentials, in the strong coupling regime. Moreover, the basic properties of the new topological models will be recalled, where the gravity part is in the strong coupling regime. The message will be that the derivation of the new features is very close to the previous weak-coupling one, once the new set of local physical fields is established.

2 Basic points about Liouville theory
2.1 The classical case

In order to set the stage, we recall the classical structure for the special case of the Liouville theory. We shall work with Euclidean coordinates $\sigma, \tau$. As a preparation for the quantum case, the classical action is defined as

$$S = \frac{1}{8\pi} \int d\sigma d\tau \left( \left( \frac{\partial \Phi}{\partial \sigma} \right)^2 + \left( \frac{\partial \Phi}{\partial \tau} \right)^2 + e^{2\sqrt{\gamma} \Phi} \right). \quad (2.1)$$

We use world sheet variables $\sigma$ and $\tau$, which are local coordinates such that the metric tensor takes the form $h_{ab} = \delta_{a,b} e^{2\Phi \sqrt{\gamma}}$. The complex structure is assumed to be such that the curves with constant $\sigma$ and $\tau$ are everywhere tangent to the local imaginary and real axis respectively. The action (2.1) corresponds to a conformal theory such that $\exp(2\sqrt{\gamma} \Phi) d\sigma d\tau$ is invariant. It is convenient to let

$$x_{\pm} = \sigma \mp i\tau, \quad \partial_{\pm} = \frac{1}{2} (\partial_{\sigma} \pm i\partial_{\tau}). \quad (2.2)$$

By minimizing the above action, one derives the Liouville equations

$$\frac{\partial^2 \Phi}{\partial \sigma^2} + \frac{\partial^2 \Phi}{\partial \tau^2} = \sqrt{\gamma} e^{2\sqrt{\gamma} \Phi}. \quad (2.3)$$

The chiral modes may be separated very simply using the fact that the field $\Phi(\sigma, \tau)$ satisfies the above equation if and only if

$$e^{-\sqrt{\gamma} \Phi} = i \sqrt{\gamma} \sum_{j=1,2} f_j(x_+) g_j(x_-); \quad x_{\pm} = \sigma \mp i\tau \quad (2.4)$$

where $f_j$ (resp. $g_j$), which are functions of a single variable, are solutions of the same Schrödinger equation (primes mean derivatives)

$$- f''_j + T(x_+) f_j = 0, \quad \text{resp.} \quad - g''_j + T(x_-) g_j.$$ \quad (2.5)

The solutions are normalized such that their Wronskians $f'_1 f_2 - f_1 f'_2$ and $g'_1 g_2 - g_1 g'_2$ are equal to one. The proof goes as follows.

1) First check that Eq. (2.4) is indeed solution. Taking the Laplacian of the logarithm of the right-hand side gives

$$\frac{\partial^2 \Phi}{\partial \sigma^2} + \frac{\partial^2 \Phi}{\partial \tau^2} \equiv 4 \partial_+ \partial_- \Phi = -4 \sqrt{\gamma} \left( \sum_{i=1,2} f_i g_i \right)^2$$

$^2$ The factor $i$ means that these solutions should be considered in Minkowsky space-time
where $\partial_{\pm} = (\partial/\partial \sigma \pm i \partial/\partial \tau)/2$. The numerator has been simplified by means of the Wronskian condition. This is equivalent to Eq. 2.3.

2) Conversely check that any solution of Eq. 2.3 may be put under the form Eq. 2.4. If Eq. 2.3 holds one deduces

$$\partial_{\pm} T^{(\pm)} = 0; \quad \text{with } T^{(\pm)} := e^{\sqrt{\gamma} \Phi} \partial_{\pm}^2 e^{-\sqrt{\gamma} \Phi}$$

$T^{(\pm)}$ are thus functions of a single variable. Next the equation involving $T^{(+)}$ may be rewritten as

$$(-\partial_{+}^2 + T^{(+)} e^{\Phi} = 0$$

with solution

$$e^{-\sqrt{\gamma} \Phi} = \frac{i \sqrt{\gamma}}{2} \sum_{j=1,2} f_j(x_+) g_j(x_-); \quad \text{with } -f_j'' + T^{(+)} f_j = 0$$

where the $g_j$ are arbitrary functions of $x_-$. Using the equation 2.6 that involves $T^{(-)}$, one finally derives the Schrödinger equation $-g_j'' + T^{(-)} g_j = 0$. Thus the theorem holds with $T = T^{(+)}$ and $\overline{T} = T^{(-)}$. One may deduce from Eq. 2.6 that the potentials of the two Schrödinger equations coincide with the two chiral components of the stress-energy tensor. Thus these equations are the classical equivalent of the Ward identities that ensure the decoupling of Virasoro null vectors. Next a Bäcklund transformation to free fields is easily obtained as follows. It is easy to verify explicitly that, since $f_1$ is a solution of Eq. 2.3, as second solution is given by $f_2(x) = f_1(x) \int^x dy f_1^{-2}(y)$. So we may introduce a field $q(x)$ by letting

$$f_1(x) = e^{q(x) \sqrt{\gamma}}, \quad f_2(x) = f_1(x) \int^x dx_1 e^{-2q(x_1) \sqrt{\gamma}},$$

It follows from the canonical Poisson brackets of Liouville theory that

$$\left\{ \vartheta(\sigma_1 - i\tau), \vartheta(\sigma_2 - i\tau) \right\}_{\text{PB}} = 2\pi \delta' (\sigma_1 - \sigma_2).$$

where $\vartheta = q'$. Doing the same construction with the functions $g$, one introduces another field $\vartheta$ with opposite chirality. Altogether $\vartheta$ and $\overline{\vartheta}$ are the two chiral components of the two dimensional free field derived by the quantum Bäcklund transformation. Clearly the differential equation Eq. 2.3 gives $T = p^2 + p'/\sqrt{\gamma}$, or equivalently $T = (q')^2 + \vartheta''/\sqrt{\gamma}$. The last expression coincides with $U_1$ Sugawara stress-energy tensor with a linear term. From the viewpoint of differential equations, these relations are simply
the well-known Riccati equations associated with the Schrödinger equation Eq.2.3. An
easy computation shows that $T$ satisfies the Poisson bracket Virasoro algebra with
$C_{\text{class}} = 3/\gamma$. We shall consider the typical situation of a cylinder with $0 \leq \sigma \leq \pi$, and
$-\infty \leq \tau \leq \infty$. After appropriate coordinate change, this may describe one handle of a
surface with arbitrary genus. Then $T$ is periodic in $\sigma$ with period $2\pi$, and the standard
Virasoro generators are given by $L_n = \int_0^{2\pi} d\sigma e^{-in\sigma} T(\sigma - i\tau)$. Under the corresponding
Poisson bracket action, it is easy to see that $f_1$, and $g_1$ are primary fields with weights
$-1/2, 0$ and $0, -1/2$ respectively, so that $f_1^{-2}$ has weight $1, 0$. Thus it is the classical
version of the screening operator in accordance with Eq.2.5. Classically, the Virasoro
weights are simply additive, so that the weight of $\exp(-J{\sqrt{\gamma}}\Phi) = (\exp(-\Phi{\sqrt{\gamma}}/2))^{2J}$ is
$-J, -J$. Therefore, we get $1, 1$ for $J = -1$ in agreement with the fact that the potential
term of the action Eq.2.1 is conformally invariant.

2.2 Quantization (outline)

The parameter $\gamma$ plays the role of $\hbar$. In this operator method, one starts from a
free quantum field with chiral components $\vartheta$ (and $\vartheta$) satisfying the quantum version
of Eq.2.9, that is, one replaces $i$ times Poisson brackets by commutators. The above
free-field expressions are extended to the quantum case by using normal ordering and
allowing for finite renormalizations of the classical constants. So we write $T \propto \vartheta^2 : + Q_2 \vartheta''$, which gives a central charge $C_L = 1 + 3Q^2$. $C_L$ is the new free parameter. The
chiral vertex operators are constructed from the quantum version of (powers of) Eq.2.8
with “powers” of $f_1$ defined by normal ordered exponentials of the type : $\exp(2J\alpha\vartheta)^\dagger$, with $\alpha$ specified as follows. One imposes that the choice $J = -1$ again gives an operator
of weight $1, 0$, which is the screening operator. There are two choices, the two well known
screening charges:

$$\alpha_{\pm} = \sqrt{\frac{C_L - 1}{12}} \pm \sqrt{\frac{C_L - 25}{12}}, \quad \alpha_{+}\alpha_{-} = 2. \quad (2.10)$$

With these choices, the fields with $J = 1/2$ satisfy a quantum version of the Schrödinger
equation Eq.2.3, which is equivalent to the Virasoro null-field decoupling equation at
level 2. Using its monodromy, and the associativity of the operator products —or,
more precisely the polynomial equations within the Moore Seiberg formalism[6]—, one
deduces[4, 8, 2] the OPA of the chiral vertex operators. A quantum group structure
emerges of the type $U_q(sl(2)) \otimes U_{\hat{q}}(sl(2))$, with

$$q = e^{i\hbar}, \quad \hat{q} = e^{i\hat{\hbar}}, \quad \hbar = \pi \frac{\alpha^2}{2}, \quad \hat{\hbar} = \pi \frac{\alpha^2}{2}. \quad (2.11)$$
The two quantum group parameters are related by
\[ \hbar \widehat{\hbar} = \pi^2, \quad \hbar + \widehat{\hbar} = \frac{C_L - 13}{6}. \] (2.12)

The first relation shows that they are, in a sense, dual pairs. The above symbol \( \odot \) has a special meaning which will be clarified as we go on. The Hilbert space of states is of course a direct sum of Virasora Verma modules. They are characterized by the eigenvalue of the free field rescaled zero-mode momentum \( \varpi \) defined by writing
\[ \vartheta(x) = q_0 + ip_0 x + i \sum_{n \neq 0} e^{-inx} p_n/n, \quad \varpi = \frac{2i}{\alpha_-} p_0. \] (2.13)

A Verma module is charaterized by the highest-weight eigenvalue
\[ \Delta(\varpi) = \frac{\hbar}{4\pi} (\varpi_0^2 - \varpi^2), \quad \varpi_0 = 1 + \frac{\pi}{\hbar} \] (2.14)
of \( L_0 \). For \( \varpi = \pm \varpi_0 \), \( \Delta \) vanishes. This describes the two \( \text{Sl}(2, C) \) invariant states. In general, the spectrum of \( \varpi \) eigenvalues is of the form
\[ \varpi_{J, \widehat{J}} = \varpi_0 + 2J^e. \] (2.15)

where \( J^e \) which is called the effective spin specifies the representations\(^3\) of \( U_q(sl(2)) \odot U^\ast_q(sl(2)) \). If \( J^e \) is generic, the corresponding Verma module is simply the bosonic Fock space of the free field \( \vartheta \). If \( J^e \) may be written as \( J^e = J + \frac{\pi}{\hbar} \widehat{J} \), with \( 2J^e \) non negative integers, are this Fock space involves null states and the irreducible Verma modules must be projected out. We shall assume that this is done if needed. In this latter case \( J \) and \( \widehat{J} \) are ordinary quantum group spins.

### 2.3 Chiral operator algebra

For continuous spins, \( J \) and \( \widehat{J} \) loose meaning in general, and only \( J^e \) makes sense. However, in our strong coupling discussion—where \( \frac{2}{\pi} \) is not rational—we shall only need to deal with the case where \( J^e \) takes the form \( J + \frac{\pi}{\hbar} \widehat{J} \), with \( J, \widehat{J} \) rational numbers (not necessarily positive integers). Then \( J \) and \( \widehat{J} \) may be defined uniquely from \( J^e \), so that we use them in our discussion. We shall try to display the structure of the

\(^3\) We refer to the original articles\(^7, 8, 2\) to explain this point which is not central in these lectures. For recent developments in this direction see refs.\(^3\).
result as imply as possible by drawing pictures. There will be a double line for each Verma module, with \( J \) and \( \hat{J} \) represented by a solid and a dashed line, respectively, Concerning the three-leg conformal blocks, the matrix elements of the chiral vertex operators (which are the chiral components of the Liouville exponentials) denoted\(^4\) by \( \tilde{V} \) are represented by three leg vertices. The operator product algebra (OPA) is thus pictured by diagrams with basic elements

![Diagram](image)

which correspond to

\[
< J^e_2 \{ \beta_2 \} | \tilde{V}^{J^e_1, \{ \beta_1 \}} (z) J^e_3, \{ \beta_3 \} >, \quad \sum_{\{ \beta \}} | J^e, \{ \beta \} > < J^e, \{ \beta' \} | < J^e, \{ \beta' \} | J^e, \{ \beta \} >.
\]

From now on we use the world sheet variable \( z = \exp(i x_+) \). The symbol \( \{ \beta \} \) stands for a multi-index that characterizes the descendent of the Verma module—or the primary field. In order to avoid clumsy drawings, we omit it from the diagrams. It is understood that it is summed over for each intermediate (double) line, as shown on the last equation above. On the contrary, for the effective spins, the summation over intermediate line should be performed only when indicated. Operatorially, this reflects the fact that the \( V^{(J^e)} \) operators possess an additional index which specifies the shift of \( J^e \) between bra and ket. More precisely, for continuous \( J^e \), one defines operators \( \tilde{V}^{(J^e_1), \{ \beta \}}_{m^e} \) with the condition that (\( \mathcal{Z} \) represents the set of non negative integers)

\[
J^e + m^e = \nu + \frac{\pi}{\hbar} \tilde{\nu}, \quad \nu, \tilde{\nu} \in \mathcal{Z},
\]

such that

\[
< J^e_2 \{ \beta_2 \} | \tilde{V}^{J^e_1, \{ \beta_1 \}}_{m^e_1} (z) J^e_3, \{ \beta_3 \} > = 0, \text{ unless } m^e_1 = J^e_3 - J^e_2
\]

\(^4\) The tilde is to distinguish from another normalisation (see ref.\([\text{F}]\)).
For instance, a four leg conformal block is drawn as

\[
\begin{array}{c}
(1) \quad (2) \\
(123) \quad (23) \quad (3)
\end{array} \quad \Leftrightarrow \quad < J_{123}^c, \{ \beta_{123} \} | \bar{V}_{J_{23}^c - J_{123}^c}(z_1) \bar{V}_{J_{32}^c - J_{23}^c}(z_2) | J_{3}^c, \{ \beta_{3} \} >
\]

(2.20)

These chiral vertex operators are closed by fusion and braiding. Let us illustrate this in the particular case where all the \( \hat{J} \)'s are zero. Using therefore simple lines one has the following representation. For fusion,

\[
\begin{array}{c}
(1) \\
(123)
\end{array} \quad (2) \\
\begin{array}{c}
(23) \quad (3)
\end{array} = \sum_{J_{12}} F_{J_{23}, J_{12}}[J_1, J_2, J_3] \times
\begin{array}{c}
(123) \\
(3)
\end{array}
\]

(2.21)

where the fusion matrix is given by

\[
F_{J_{23}, J_{12}}[J_1, J_2, J_3] = \{ J_1, J_2, J_3 | J_1, J_2 \}_q.
\]

(2.22)

On the right there appears the \( q \) 6j symbol. Similarly, the braiding is represented by

\[
\begin{array}{c}
(1) \quad (2) \\
(23)
\end{array} = \sum_{J_{13}} B_{J_{23}, J_{13}}^{\pm}[J_1, J_2, J_3] \times
\begin{array}{c}
(123) \\
(13)
\end{array}
\]

(2.23)

where the braiding matrix is related to the above by

\[
B_{J_{23}, J_{13}}^{\pm}[J_1, J_2, J_3] = e^{ \pm i \pi (\Delta_{J_{123}} + \Delta_{J_{2}} - \Delta_{J_{23}} - \Delta_{J_{13}})} F_{J_{23}, J_{13}}[J_1, J_2, J_3].
\]

(2.24)

In the general case the structure is similar, with the corresponding extended 6j symbols associated with our \( U_q(sl(2)) \otimes U_q(sl(2)) \), which are given by

\[
F_{J_{23}, J_{12}}[J_1^c, J_2^c, J_3^c] = \{ \{ J_1^c, J_2^c, J_3^c | J_{12}^c \} \} \{ \{ J_1^c, J_2^c, J_{23}^c \} \} \hat{q}.
\]

(2.25)
The double brace means 6j symbols with shifted entries (see ref. 2 for details). The symbol $\hat{J}^e$ is defined by $\hat{J}^e = \hat{J} + \frac{\hbar}{\pi} J \equiv \frac{\hbar}{\pi} J^e$. The normalization of the chiral vertex operators $\tilde{V}$ is fixed by the condition that their fusion and braiding matrices be exactly equal to q 6j symbols—not simply proportional as they would be with other choices. Then, their highest weight matrix element is given by

$$< J^e_2 | \tilde{V}^{(J^e_1)}(z) | J^e_3 > = g_{j^e_1, j^e_3}^{J^e_2}$$

(2.26)

where the $g$'s, which are called coupling constant, are not simply determined by the quantum group symmetry—they are not trigonometrical functions—but from the monodromy properties of the quantum version of Eq. 2.5, the null vector decoupling equation.

2.4 The weak coupling regime

2.4.1 The Liouville exponentials

We have to put the two chiralities together. The $x_-$ components—which we did not consider so far—are of course similar to the above. Their quantum numbers will be distinguished by a bar, so we have $\bar{\pi}, \bar{J}, \bar{\hat{J}}$ and so on. Graphically, the Verma modules are represented by gray lines in contrast with the black ones of the $x_+$ components. The Liouville exponentials may be determined as follows: one writes general chiral decompositions and imposes that they be local, with $\sigma$ and $\tau$ being space and time coordinates respectively. Remarkably, this determines them completely, and the result may be consistently restricted to Verma modules such that

$$J = \bar{J}, \quad \hat{J} = \bar{\hat{J}}$$

(2.27)

To represent this restriction, we introduce springs that link our lines as follows

```
  _________________________  
     |                    |   \J
     |                    |     \hat{J}
     |                    |       \bar{J} = J
     |                    |         \hat{J} = \hat{J}
```

(2.28)
Then the Liouville three point function is represented by

\[ \text{such that } = 0 \] (2.29)

The right-hand side drawing means that the Liouville exponentials applied to a state satisfying condition Eq.2.27 only give states satisfying the same condition. Operatorially, the Liouville exponentials are given by

\[ e^{-J^e \Phi(z, \bar{z})} = \sum_m \bar{V}_{m^e}(z) \bar{V}_{m^e}(\bar{z}). \] (2.30)

This chiral decomposition only involves \( \bar{V} \) and \( \bar{V} \) operators with the same \( J^e \) and \( m^e \), in accordance with the diagrams just written (recall Eq.2.19). It is the quantum version of the classical expression Eq.2.4 and of its higher spin generalizations. The properties of the quantum Liouville exponentials under fusion and braiding are deduced from the chiral ones summarized above, and are consequences of the orthogonality of the q 6j symbols. For the standard ones it reads

\[ \sum_{J_{23}} \left\{ J_1 J_2 | J_{12} \right\} \left\{ J_3 J_{23} | J_{123} \right\} = \delta_{J_{12} - K_{12}}. \] (2.31)

As a result, the braiding matrix of the Liouville exponentials are equal to one, so that they are mutually local, i.e.

\[ \sum_{J_{23}} (123) \] (2) (3)
They are moreover closed by fusion, following the diagram

As usual, the upper vertex on the right hand side is a book keeping device for descendents. To leading order, at short distance, the coefficient is determined by its expectation value between highest weight states. According to Eqs. 2.26-2.30 it follows that
\[
< J^e_2 | e^{-J^e_1 \alpha - \Phi(z, \bar{z})} | J^e_3 > = | g_{J^e_2 J^e_1 J^e_3} |^2.
\] (2.33)
and the contribution of \( \exp(-J^e_1 \alpha - \Phi) \) —and of its descendents—in the operator product of \( \exp(-J^e_1 \alpha - \Phi) \) with \( \exp(-J^e_2 \alpha - \Phi) \), is proportional to \( | g_{J^e_1 J^e_2 J^e_3} |^2 \). Thus the coupling constants determine the three point functions of the OPA, as expected. Concerning the braiding diagram, it explicitly means that
\[
e^{-J^e_1 \alpha - \Phi(z_1, \bar{z}_1)} e^{-J^e_2 \alpha - \Phi(z_2, \bar{z}_2)} = e^{-J^e_2 \alpha - \Phi(z_2, \bar{z}_2)} e^{-J^e_1 \alpha - \Phi(z_1, \bar{z}_1)},
\] (2.34)
where we are equal \( \tau \), so that \( z_1 = \exp(\tau + i\sigma_1) \), \( z_2 = \exp(\tau + i\sigma_2) \).

In general, applying a similar discussion to the higher point functions, one generates a set of diagrams where condition Eq. 2.27 is obeyed on every line. Since this has an obvious analogy with quark diagrams, we may say that chirality is confined.
2.4.2 The coupling to matter

One represents matter by another copy of the theory summarized above, now with central charge \( c = 26 - C_L \). One constructs local fields in analogy with Eq. (2.30):

\[
e^{- (J\alpha - \hat{J}\alpha')} \Phi' (z, \bar{z}) = \sum_{m, \hat{m}} \tilde{V}^m_J (z) \tilde{V}^{\hat{m}}_{\hat{J}} (\bar{z}).
\]

(2.35)

Symbols pertaining to matter are distinguished by a prime. In particular \( \Phi' (z, \bar{z}) \) is the matter field (it commutes with \( \Phi (z, \bar{z}) \)), and \( \alpha'_{\pm} \) are the matter screening charges

\[
\alpha'_{\pm} = \mp i \alpha_{\pm}.
\]

(2.36)

The correct dressing of these operators by gravity is achieved by considering the vertex operators

\[
\mathcal{W}^{j, \hat{j}} \equiv e^{-((-\hat{j} - 1)\alpha_\pm + J\alpha_\pm)\Phi - (J\alpha'_\pm + \hat{J}\alpha'_{\pm})\Phi'}
\]

(2.37)

which is an operator of weights \( \Delta = \tilde{\Delta} = 1 \). In particular for \( J = \hat{J} = 0 \), we get the cosmological term \( \exp (\alpha_\pm \Phi) \). The three-point function was computed in refs [11, 2]. The corresponding product of coupling constants gives the correct leg factors after drastic simplifications. Of course, the dressing is such that the Verma modules involved satisfy the appropriate BRST condition. Graphically, they may be represented as

\[
\text{Gravity} \vline \quad \text{Matter}
\]

In this drawing the straight line between the first and sixth lines indicates that the \( J \)'s satisfy the condition \( J + \hat{J}' + 1 = 0 \), as required by the dressing condition.

2.5 The strong coupling regime
2.5.1 The barrier

Eqs.2.10 for the screening charges are real for \( C_L > 25 \), pure imaginary for \( C_L < 1 \) and complex otherwise. The weights of the primary fields with spins \( J \tilde{J} \) are given by Kac’s formula

\[
\Delta(J, \tilde{J}) = \frac{C_L - 1}{24} - \frac{1}{24} \left[ (J + \tilde{J} + 1)\sqrt{C_L - 1} - (J - \tilde{J})\sqrt{C_L - 25} \right]^2, \tag{2.39}
\]

which, for real \( J \tilde{J} \), is real for \( C_L > 25 \), and \( C_L < 1 \), and complex in general otherwise. In the zero coupling limit \( \gamma \to 0 \), \( C_L \sim 3/\gamma \) blows up. Thus the region \( C_L > 25 \) is called the weak coupling regime of gravity, which is connected with the classical limit \( \gamma \to 0 \). There the above formula is real and the construction of the Liouville exponentials just summarized applies.

Can we go beyond \( C_L = 25 \)? At this moment it seems that one cannot make sense of operators with complex \( \Delta \). For \( 1 < C_L < 25 \), Eq.2.39 still gives real weights in two particular cases. First \( \Delta \) is real and negative if \( J = \tilde{J} \). In the type of picture we are using (see 2.28), this means that a spring is introduced between the two lines, but here they have the same chirality. Second, \( \Delta \) is real and positive if \( J + \tilde{J} + 1 = 0 \). This kind of condition (a sort of repulsive “interaction”) already appeared on the diagram 2.38. It is represented by a solid line. Thus the subspaces with real \( \Delta \)’s are represented graphically as follows.

\[
\leftrightarrow \quad \tilde{J} = J, \quad \Delta(J, J) \text{ real } < 0, \tag{2.40}
\]

\[
\leftrightarrow \quad \tilde{J} = -J - 1, \quad \Delta(J, -J - 1) \text{ real } > 0. \tag{2.40}
\]

Now we return to the Liouville exponential. Does it respect this reality conditions? The answer is no, since, in Eq.2.30, one sums over \( m^e \) and \( \overline{m}^e \) independently, so that the shift of \( J^e \) and \( \overline{J}^e \) are not correlated. Graphically, this is illustrated as follows

\[
\neq 0, \tag{2.41}
\]
and by a similar relation with the other condition. Operiotorially, this means that we may try to use Liouville exponentials with real weights (depicted by the additional springs on the vertical lines) but applied to states with real weights they give states with complex weights so that they loose meaning in the strong coupling regime.

2.5.2 The new set of chiral fields

The solution to this problem is as follows. The chiral operator algebra of the V fields has a natural extension to $C_L < 25$, albeit with complex weights. One constructs different types of local operators, where the conditions $\hat{J} = J$ and $\hat{\bar{J}} = \bar{J}$ are not imposed. They are built from the same chiral vertex operators as the Liouville exponentials, but one takes different combinations so that they respect one of the two reality conditions Eq.2.40. Following our graphical rules, their three point function are of the type

![Diagram](image)

The reality conditions on the Verma modules do not link the two chiralities any longer, so we may say that chirality is deconfined. The possibility of constructing these operators consistently is based on the truncation theorems that hold for special values of $C_L$, i.e. $C_L = 7, 13, 19$. One first constructs chiral operators which are closed by fusion and braiding. They are particular linear combinations of the chiral vertex operators drawn on figure 2.16. These so called called physical operators $\chi_{\pm}^{(J)}$ have three point functions of the type (for the $x_+$ components)

![Diagram](image)

which represent

$$< J_2^{e \pm \{\beta_2\}} | \chi_{\pm}^{J_3^{e \pm \{\beta_1\}}(z)} | J_3^{e \pm \{\beta_3\}} > ,$$

(2.42)
where

\[ J^{e\pm} = J(1 \mp \frac{\pi}{2\hbar}) + \frac{\pi}{2\hbar}(1 \pm 1) \quad (2.43) \]

A similar representation with gray lines holds of course for the \( x_- \) components. We do not discuss it explicitly.

Now is a good time to recall the truncation theorems which hold for

\[ C = 1 + 6(s + 2), \quad s = 0, \pm 1. \quad (2.44) \]

First define the physical Hilbert space

\[ \mathcal{H}^\pm_{\text{phys}} \equiv \bigoplus_{r=0}^{1} \bigoplus_{n=-\infty}^{\infty} \mathcal{H}_{r/(2\mp s) + n/2} \quad (2.45) \]

where \( \mathcal{H}^\pm_{Je} \) denotes the Verma modules with highest weights \( |J^{e\pm}> \). The tree leg conformal bloch of the \( \chi \) operators take the form

\[
\begin{align*}
(1) & \quad (2) \quad (3) \\
& \quad (2) & \quad (3) \\
& \quad (2) & \quad (3)
\end{align*}
\]

\[ = (-1)^{(2-s)(2J_3 + \nu(\nu+1)/2)} \]

\[
\begin{align*}
(1) & \quad (2) \quad (3) \\
& \quad (2) & \quad (3) \\
& \quad (2) & \quad (3)
\end{align*}
\]

\[ = (-1)^{(2+s)(2J_3 + \nu(\nu+1)/2)} \]

if \( \nu = J_1 + J_3 - J_2 \in \mathbb{Z}_+, \) and \( 2J_i \in \mathbb{Z}/(2 \mp s) \); and they are taken to vanish otherwise.

By construction the physical operators are restricted to \( \mathcal{H}^\pm_{\text{phys}} \). Denote their ensemble by \( \mathcal{A}^\pm_{\text{phys}} \). The basic properties of the special values Eq. 2.44 is the TRUNCATION THEOREM:

For \( C_L = 1 + 6(s + 2), \ s = 0, \pm 1, \) the above set \( \mathcal{A}^+_{\text{phys}} \) (resp. \( \mathcal{A}^-_{\text{phys}} \)) is closed by braiding and fusion.
Let us display, for instance the braiding properties. One has
\[
\sum_{J_{23}} \chi_{(J_{23})}^{(1)\pm(23)} = \sum_{J_{13}} (-1)^{\pm2i\pi(2-s)J_1 J_2} \chi_{(J_{13})}^{(2)\pm(23)}
\]
\[
\sum_{J_{23}} \chi_{(J_{23})}^{(1)\pm(23)} = \sum_{J_{13}} (-1)^{\pm2i\pi(2+s)J_1 J_2} \chi_{(J_{13})}^{(2)\pm(23)}
\]

Note an important difference with the braiding properties of the previous chiral components themselves (figure 2.23). On the left hand side of the last two drawings one has to sum over the intermediate $J_{23}$, contrary to 2.23. Thus the $\chi$ operators, contrary to the $V$ do not have a quantum number specifying the shift of $J^e$. As a matter of fact, the operatorial relation between $\chi$ and $V$ operators consistent with the three and four leg diagrams just displayed is as follows.

\[
\chi_{(J_1)}^{(J_1)\pm} \mathcal{P}_{\mathcal{H}_{J_3}^\pm} = \sum_{\nu=J_1+m \in \mathbb{Z}_+} (-1)^{(2\mp s)(2J_3+\nu+1)/2} V_{m^e \pm}^{(J_3 \pm)} \mathcal{P}_{\mathcal{H}_{J_3}^\pm}
\]

where $\mathcal{P}_{\mathcal{H}_J^\pm}$ is the projector on $\mathcal{H}_J^\pm$, and $m^e = m(1 \pm \frac{\pi}{h})$. Note that, on the contrary, the operators $V_{m^e \pm}$ themselves cannot be consistently restricted to either $\mathcal{H}_{\text{phys}}^+$ or $\mathcal{H}_{\text{phys}}^-$. Operatorially, the last two figures correspond to

\[
\chi_{(J_1)}^{(J_1)\pm} \chi_{(J_2)}^{(J_2)\pm} = e^{2i\pi\epsilon(2\mp s)J_1 J_2} \chi_{(J_2)}^{(J_1)\pm} \chi_{(J_1)}^{(J_2)\pm}
\]

where $\epsilon = \pm 1$ is fixed by the ordering of the operator on the left-hand side in the usual way. Next we construct local fields out of the chi fields. The braiding of the chi fields is a simple phase. From the spectrum of the $J$’s, it follows that this phase factor is of the form $\exp(i\pi N/2(2\mp s))$, where $N \in \mathbb{Z}$. Thus, we have parafermions. As shown in ref.4, simple products of the form $\chi_{(J)}^\pm \chi_{(\overline{J})}^\mp$, with $J - \overline{J} \in \mathbb{Z}$ are local. In such a product, the summations over $m$, and $\overline{m}$ are independent, while the summations over $m$, and $\overline{m}$ are correlated. Now we have a complete reversal of the weak coupling situation summarized by the drawing 2.29: the new fields preserve the reality condition, but do not preserve the equality between $J$ and $\overline{J}$ quantum numbers. Thus, as stressed in ref.2, in the strong coupling regime we observe a sort of deconfinement of chirality.
3 The Liouville string

One may consider two different problems. First, one may build a full-fledged string theory, by coupling, for instance, the above with $26 - C_L$ free fields $\vec{X}$. A typical string vertex is of the form $\exp(i \vec{k} \cdot \vec{X})^{(j)} \chi_{\pm}^{(j)}$, where $\vec{k}$, $J$, and $\bar{J}$ are related so that this is a 1,1 operator. Here obviously, the restriction to real weight is instrumental. Moreover, since one wants the representation of Virasoro algebra to be unitary, one only uses the chi+ fields. This line was already pursued with noticeable success in refs. [10]. However, the N-point functions seem to be beyond reach at present. Second a simpler problem seems to be tractable, namely, we may proceed as in the construction of topological models just recalled. We consider another copy of the present strongly coupled theory, with central charge $c = 26 - C_L$. Since this gives $c = 1 + 6(-s + 2)$, we are also at the special values, and the truncation theorems applies to matter as well. This "string theory" has no transverse degree of freedom, and is thus topological. The complete dressed vertex operator is now

$$V^{(J, \bar{J})} = \chi_{+}^{(j)} \chi_{-}^{(j)} \chi'_{-}^{(j)} \chi'_{-}^{(j)}$$

As in the weak coupling formula, operators relative to matter are distinguished by a prime. The definition of the $\chi$ is similar to the above, with an important difference. Clearly, the definition of $\chi_{+}$ is not symmetric between $\alpha_{+}$, and $\alpha_{-}$. The truncation theorems also holds if we interchange the two screening charges. We re-establish some symmetry between them by taking the other possible definition for $\chi_{+}$. Our results will then be invariant by complex conjugation provided we exchange $J$'s and $\bar{J}$'s. Thus left and right movers are interchanged, which seems to be a sensible requirement. The spectrum of Verma modules involved may be depicted graphically as follows:

$$\text{Gravity}$$

$$\text{Matter}$$

It is suggestive to compare with the corresponding drawing 2.38 of the weak coupling regime. The vertical links between the lines are of the same nature, but they are
distributed differently. A basic difference here is that the present drawing is not connected since there are no links between the two chiral components. Thus, chirality is deconfined. For $J = \bar{J} = 0$, we get the new cosmological term

$$V^{0,0} = \chi^{(0)}_+(z)\bar{\chi}^{(0)}_+(\bar{z}).$$

Thus the area element of the strong coupling regime is $\chi^{(0)}_+(z)\bar{\chi}^{(0)}_+(\bar{z})dzd\bar{z}$. It is factorized into a simple product of a single $z$ component by a $\bar{z}$ component. From this expression one may compute the string susceptibility using the operator version of the DDK argument developed in ref.\[11\] for the weak coupling regime. We refer to refs.\[3, 4\] for details. One finds

$$\gamma_{\text{str}} = (2 - s)/2. \quad (3.4)$$

The result is real for $c > 1$ ($C_L < 25$), contrary to the continuation of the weak-coupling equation $\gamma_{\text{str}} = 2 - Q/\alpha_-$. Explicitly one has

$$\begin{bmatrix}
s & c & C_L & \gamma_{\text{str}} \\
1 & 7 & 19 & 1/2 \\
0 & 13 & 13 & 1 \\
-1 & 19 & 7 & 3/2
\end{bmatrix}, \quad \begin{bmatrix}
s & c & C_L & \gamma_{\text{str}} \\
2 & 1 & 25 & 0 \\
-2 & 25 & 1 & 2
\end{bmatrix} \quad (3.5)$$

The last two are the extreme points of the strong coupling regime. The values at $c = 1$, and $c = 25$ agree with the weak-coupling formula. The result is always positive, contrary to the weak-coupling regime. At $c = 7$, we find the value $\gamma_{\text{str}} = 1/2$ of branched polymers.

Finally, we have computed the N-point functions with one incoming and N-1 outgoing legs, defined as follows. First in general\[7\] the two-point function of two $\tilde{V}$ fields with spins $J_1$, $\bar{J}_1$, and $J_2$, $\bar{J}_2$ vanishes unless $J_1 + J_2 + 1 = 0$, and $\bar{J} + \bar{J}_2 + 1 = 0$. Thus conjugation involves the transformation $J \rightarrow -J - 1$. Taking account of the exchange between $J$ and $\bar{J}$, due to complex conjugation, yields the following vertex operator for the conjugate representation:

$$V^{J,\bar{J}}_{\text{conj}} = \chi^{(J)}_+\bar{\chi}^{(\bar{J})}_+\chi^{(-J-1)}_-\bar{\chi}^{(-\bar{J}-1)}_-. \quad (3.6)$$

Thus we have computed the matrix elements $\left\langle V^{J_1,\bar{J}_1}_{\text{conj}}V^{J_2,\bar{J}_2}...V^{J_N,\bar{J}_N} \right\rangle$. The method is similar to the one developed in ref.\[12\], with a reshuffling of quantum numbers. In the weak coupling regime, left and right quantum numbers are kept equal, while the ones associated with different screening charge are chosen independently. In the
strong coupling regime, the situation is reversed: the reality condition ties the quantum numbers which differ by the screening charge, but the quantum numbers with different chiralities become independent. See refs. [3, 4] for details.

4 Concluding remarks

We should probably stress that no total conformal spin has been introduced. Indeed, although, the conformal spin is non zero for the gravity, and matter components of our vertex operators separately, the total weights for the left and right components are kept equal to one.

One may wonder why the present approach succeeds to break through the $c = 1$ barrier, in sharp contrast with the other ones. This may be traced to the fact that we first deal with the chiral components of gravity and matter separately before reconstructing the vertex operators. This is more painful than the matrix model approach which directly constructs the expectation values of the dressed matter operators. However, in this way we have a handle over the way the gravity quantum numbers are coupled, and so we may build up vertex operators which change the gravity chirality. This seems to be the key to the $c = 1$ problem, since this quantum number plays the role of order parameter.

Let us turn to a final remark. The redefinition of the cosmological term led us to modify the KPZ formula [13]. On the other hand, in standard studies of the matrix models or KP flows, one first derives $\gamma_{\text{str}}$ and deduces the value of the central charge by assuming that the KPZ formula holds. In this way of thinking, one would start from our formula Eq.3.4 and apply KPZ, which would lead to a different value of the central charge, say $d$. It is easy to see that for $c = 1 + 6(-s + 2)$ one gets $d = 1 - 6(2 - s)^2/2s$. This is the value of a 2, $s$ minimal model! What happens is that in terms of $d$, we have $\gamma_{\text{str}} = (d - 1 + \sqrt{(d - 1)(d - 25)})/12$, in contrast with the KPZ formula $(d - 1 - \sqrt{(d - 1)(d - 25)})/12$. Thus the strongly coupled topological theories may be given by another branch of $d < 1$ theories.

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