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ENERGY SPECTRUM OF ANYON IN THE COULOMB FIELD

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Abstract

Anyonic atom is considered as a two-dimensional system. Using some approximations we find the energy spectrum of the anyon in the Coulomb field. It is shown that the anyonic atom is stable.

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One of the interesting fundamental phenomenon which was observed in the last decades is the discovery of anyons, relativistic spinning particles in 2 + 1 dimensions. In contrast to three-dimensional space, indistinguishable quantum particles in two-dimensional space can, in general, have anomalous statistics [1-4]. These quasiparticles carry not only a charge $q$ also the magnetic flux $\Phi_0$.

During the last several years the (2 + 1)-dimensional physics has been an area of intense activity, mainly due to the application of anyons in realistic planar physics [5]. This has provided the impetus for constructing viable models of free anyons in analogy to the Dirac equation for spin $1/2$ particles [6-8]. However, many interesting properties of anyons, in particular, their gyromagnetic ratio $g$, can only be probed in the presence of electromagnetic interactions [9,10].

There is a problem to construct the relativistic wave equations for anyons interacting with the electromagnetic field in 2 + 1 dimensions. A new simple wave equation for an anyon was proposed in Ref. [9]. To derive this equation the simplectic framework for the induced representation of the Poincaré group for anyons, minimally coupled to an external electromagnetic field, has been used.

In the present paper, we have investigate the Coulomb interaction of anyons. Unlike Ref. [9], we define the spin operator to be $S_\mu = -Sp_\mu/m$, where $S$ is the fractional spin and $m$ is the rest mass of the anyon. Such redefinition has some advantages and allows to obtain an approximate analytic expression for the bound state spectrum of the two-body Coulomb problem in the presence of an anyonic vector potential.

We start with the following equation of motion for the anyon in 2 + 1 dimension:

$$\left(D^2 - \frac{2qS_i}{m}F_\mu D_\mu - m^2\right) \phi = 0,$$

(1)

where $q$ is the charge of the anyon, $D_\mu = \partial_\mu - iqA_\mu$, $F_\mu = \frac{1}{2}\epsilon_{\mu\nu\alpha}F_{\nu\alpha}$, $\epsilon_{\mu\nu\alpha}$ is the antisymmetric tensor ($\epsilon_{012} = 1$), $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the strength tensor, $A_\mu$ is the vector-potential...
of the external electromagnetic field. The equation of motion (1) is the relativistic and
gauge invariant equation like the Klein-Gordon equation. But here there is an additional
second term which gives the spin-orbit interaction. The gyromagnetic ratio of the anyon
in this scheme is \( g = 2 \) [9-11].

In two dimensional space, the vector potential \( A_\mu \) to be \( A_k = 0 \) \((k = 1, 2)\), \( A_0 = -Ze/r\), where \( r = (x_1^2 + x_2^2)^{1/2} \). We assume that the hydrogen-like atom consists of the
anyon of charge \( q \) and mass \( m \) and a nucleus of charge \( Ze \) and large mass \( M \gg m \). Then
Eq. (1) takes the form:

\[
\left[ \partial_t \partial_\kappa - \partial^2_\kappa + \frac{2i\xi Z}{r} \partial_t + \frac{\xi^2 Z^2}{r^2} - \frac{4\xi Z S}{mr^3} \left[ i(x_2 \partial_1 - x_1 \partial_2) \right] - m^2 \right] \psi(\vec{r}) = 0, \tag{2}
\]

where \( \xi = eq \) is the coupling constant.

By introducing the polar coordinates \( x_1 = r \cos \varphi \), \( x_2 = r \sin \varphi \) and replacing the
angular momentum operator \( \hat{J} = -i(x_2 \partial_1 - x_1 \partial_2) \) by \( \hat{J} = i \partial/\partial \varphi \) we may take solution
having the form

\[
\psi(r, \varphi) = e^{-iEt} \sum_{-\infty}^{\infty} e^{il\varphi} f_l(r), \tag{3}
\]

where \( l \) is the orbital quantum number and \( E \) is the energy of the anyon. Substituting
(3) into Eq. (2) we obtain

\[
\left[ \frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} \right) + E^2 - m^2 + \frac{2\xi EZ}{r} - \frac{l^2 - \xi^2 Z^2}{r^2} - \frac{4\xi ZSl}{mr^3} \right] f_l(r) = 0. \tag{4}
\]

Units are chosen such that \( \hbar = c = 1 \).

Thus the problem under consideration has reduced to the radial Schrödinger type
equation (4) which determines the energy eigenvalues of the anyonic atom. However, Eq.
(4) represents the so-called ”insolvable” problem. By ”solvable” models we mean those
models for which the eigenvalue problem can be reduced to a hypergeometric function by
a suitable transformation. Equation (4) is not reduced to the hypergeometric equation.
Thus a different approach is necessary if we are to find solution of Eq. (4).

One of the earliest and simplest methods of obtaining the approximate eigenvalues of
the one-dimensional problems is the WKB method. Proofs of varying degrees of rigor
have been advanced which demonstrate the exactness of the WKB quantization condition
[12]-[15]. It is well known that the standard WKB approximation in leading order in
\( \hbar \) always reproduces exact energy spectrum for the ”solvable” spherically symmetrical
potentials, \( V(r) \), if the Langer correction \( L^2 \to L^2 + \frac{1}{4} ; L^2 = l(l + 1) \) [12] \((l \) is the orbital
quantum number) in the centrifugal term of the radial Schrödinger equation has fulfilled.
One can easily to check that analogous Langer type correction of the form \( l^2 \to l^2 + \frac{1}{4} \)
in the leading order WKB quantization condition results in the exact energy eigenvalues for
the nonrelativistic two-dimension Coulomb problem [16]: \( E_n = -\frac{1}{2}\alpha^2 Z^2 m(n'+l + \frac{1}{2})^{-2} \).

Within the framework of the WKB method the solvable potentials mean those potential-
s for which the eigenvalue problem has two turning points. The radial problem (4) has
three turning points that represent a serious difficulty. To estimate the energy eigenvalues of the anyonic atom let us consider Eq. (4) as a quasi-solvable problem in the framework of the WKB method.

For this by writing \[ f_l = U_l(r) / \sqrt{r} \] we reduce Eq. (4) to the canonical form. Then making the Langer-type replacement, \( l^2 \to l^2 + \frac{1}{4} \), we obtain the second-order equation in the required form:

\[
\frac{d^2}{dr^2} + \frac{E^2 - m^2 + \frac{2\xi Z E}{r} - \frac{\lambda^2}{r^2} - \frac{4\xi Z S l'}{m r^3}}{U_l(r)} = 0, \tag{5}
\]

where \( \lambda^2 = l^2 - \xi^2 Z^2 \), \( l' = \left(l^2 + \frac{1}{4}\right)^{1/2} \), \( U_l(r) = f_l(r) \sqrt{r} \). Let us define the radial quantum number, \( n' \), to be a number of zeros of the radial wave function in the physical region, i.e., at \( r > 0 \). Turning points of the problem, \( r_1, r_2, r_3 \) (\( r_1 < r_2 < r_3 \)) are defined as roots of the cubic equation

\[
(E^2 - m^2)r^3 + 2\xi Z E r^2 - \lambda^2 r - 4\xi S Z l'm^{-1} = 0. \tag{6}
\]

Analysis of Eq. (6) shows that only two turning points, \( r_2, r_3 \), lie in the physical region, i.e., at \( r > 0 \), and the turning point \( r_1 \) lies in the nonphysical region \( r < 0 \). Therefore one can consider the leading order WKB quantization condition to be appropriate at the interval \( [r_2, r_3] \):

\[
\int_{r_2}^{r_3} \sqrt{E^2 - m^2 + \frac{2\xi Z E}{r} - \frac{\lambda^2}{r^2} - \frac{4\xi S Z l'}{m r^3}} dr = \pi (n' + \frac{1}{2}). \tag{7}
\]

To calculate integral (7) let us note that the term \( \sim r^{-3} \) in this integral is essential at very small \( r \), i.e., its contribution can be considered as a small perturbation at the interval \( [r_2, r_3] \). This means that the phase-space integral (7) can be represented at this interval in the form:

\[
\int_{r_2}^{r_3} \sqrt{k^2 + \frac{2\xi Z E}{r} - \frac{\lambda^2}{r^2}} dr - \int_{r_2}^{r_3} \frac{1}{\sqrt{k^2 r^2 + 2\xi Z E r - \lambda^2 r^2}} dr = \pi (n' + \frac{1}{2}), \tag{8}
\]

where the turning points \( r_2, r_3 \) are determined approximately from the equation \( k^2 r^2 + 2\xi Z E r - \lambda^2 = 0 \), \( k^2 = E^2 - m^2 \). Simple integration of (8) gives the equation for \( E \):

\[
\frac{\xi Z E}{\sqrt{-E^2 + m^2}} - \frac{2\xi^2 Z^2 S l' E}{\lambda^3} = n' + \frac{1}{2} + \lambda. \tag{9}
\]

This equation results in the fourth degree equation with respect to \( E \). However Eq. (9) can be simplified. For this note that the second term in the left-hand side of the Eq. (9) arising from the spin-orbit interaction is a small correction of the order \( \sim \xi^2 \) at the interval under consideration. Therefore, if we represent \( E = m + E' \), where \( E' \) is the kinetic energy which is of order \( \sim \xi^2 \) for the Coulomb potential, then this term can be written approximately in the form \( \sigma_l = 2\xi^2 Z^2 S l' \lambda^{-3} \). Now, the subsequent simple calculations result in the energy eigenvalues for the anyonic atom:
$$E_n = m \left[ 1 + \frac{\xi^2 Z^2}{\left(n' + \frac{1}{2} + \sqrt{l^2 - \xi^2 Z^2 + \sigma_l}\right)^2} \right]^{-\frac{1}{2}}.$$  \hspace{1cm} (10)

Let us analyze the result obtained. The formula (10) is analogous to the relativistic energy eigenvalues for the Coulomb problem obtained from the Klein-Gordon equation. The difference is in the structure of the principal quantum number which contains the additional quantity $\sigma_l$. If we write the principal quantum number in (10) with the accuracy up to $\xi^2$, $N \simeq n' + \frac{1}{2} + l + [2Sl^{-2}(l^2 + \frac{1}{4})^{1/2} - \frac{1}{2}\xi^2 Z^2 l^{-1}]$, then one can see that the additional term $\tilde{\sigma_l} = 2\xi^2 Z^2 Sl^{-3} (l^2 + \frac{1}{4})^{1/2}$ can be considered as the anyon fractional spin which is added to the angular momentum $l$. This term is the relativistic correction; in nonrelativistic approximation, the formula (10) results in the known exact result for the two-dimensional atom.

To check formula (10) we compare (see Table 1) the eigenvalues calculated for kinetic energy $E' = E - m$ with the corresponding numerical calculations. As an example, in these calculations, we have considered the case $S = 1/2$, $\xi = \alpha$ and $m = m_e$ (electron mass).

| $n'$ | $l$ | $E'_{\text{anat}}, \text{eV}$ | $E'_{\text{num}}, \text{eV}$ |
|------|-----|----------------------------|-----------------------------|
| 0    | 1   | −6.0464                    | −6.0467                     |
| 0    | 2   | −2.1769                    | −2.1769                     |
| 1    | 1   | −2.1768                    | −2.1770                     |
| 1    | 2   | −1.1107                    | −1.1106                     |
| 2    | 1   | −1.1106                    | −1.1106                     |
| 2    | 2   | −0.6719                    | −0.6718                     |

We see that formula (10) reproduces the exact eigenvalues to three-place accuracy.

To conclude let us summarize the obtained results. In this work, the anyonic atom have considered as a two-dimensional system. Using the semiclassical approximation we have found the energy eigenvalues of the atom. There have shown that the anyonic atom is stable and has the complex structure of the energy spectrum which is connected with the fractional spin of the anyon.

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