The space of all triples of projective lines of distinct intersections in $\mathbb{RP}^n$

Ali Berkay Yetişer

Abstract

We study the space of all triples of projective lines in $\mathbb{RP}^n$ such that any line in a triple intersects the two others at distinct points. We show that for $n = 2$ and $3$ these spaces are homotopically equivalent to the real complete flag variety $Flag(\mathbb{R}^n)$ for $n = 3$ and 4, respectively, and we explicitly calculate the integral homology of the corresponding spaces. We prove that for arbitrary $n$, this space is homotopy equivalent to $Flag(1, 2, 3, \mathbb{R}^{n+1})$, the variety of all partial flags of signature $(0, 1, 2, 3, n + 1)$ in an $(n + 1)$-dimensional vector space over $\mathbb{R}$.

Keywords: flag variety, homogeneous space, homotopy equivalence, projective configuration

1 Introduction

We wish to study the topology of the space $\mathcal{H}$ of all triples of projective lines in $\mathbb{RP}^3$ such that any line in a triple intersects the two others at distinct points. A triple of such lines spans a unique projective plane in $\mathbb{RP}^3$. We are also interested in the corresponding problem in $\mathbb{RP}^2$ because it will come up as the fibers of a fibration $\mathcal{H} \to \mathbb{RP}^3$, and the problem is more straightforward in $\mathbb{RP}^2$ because the corresponding matrices have full rank. Hence, we will first study the space $\mathcal{H}'$ of all triples of lines $(\ell_1, \ell_2, \ell_3)$ in $\mathbb{RP}^2$ with the same condition, that is, any line in a triple intersects the two others at distinct points. In Section 2, we will prove the following results.

Theorem 1. The space $\mathcal{H}'$ is homotopy equivalent to the quotient of $SU(2)$ by the free action of the dihedral group of order 8, $D_8$. 

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Theorem 2. The integral homology groups of $\mathcal{H}'$ are as follows.

\[ H_0(\mathcal{H}') \cong H_3(\mathcal{H}') \cong \mathbb{Z}, \quad H_1(\mathcal{H}') \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \]

and for any other subscript the homology is trivial.

In Section 3, we will prove that the space $\mathcal{H}$ is homotopically equivalent to a complete flag variety.

Theorem 3. The space $\mathcal{H}$ is homotopy equivalent to $\text{Flag}(\mathbb{R}^4)$.

We will explicitly calculate the integral homology groups of $\mathcal{H}$ using the theory of homogeneous spaces and the Serre spectral sequence associated to the fiber bundle $\pi : \mathcal{H} \to \mathbb{R}P^3$.

Theorem 4. The integral homology groups of $\mathcal{H} \cong \text{Flag}(\mathbb{R}^4)$ are as follows.

\[ H_0(\mathcal{H}) \cong H_6(\mathcal{H}) \cong \mathbb{Z}, \quad H_1(\mathcal{H}) \cong H_4(\mathcal{H}) \cong (\mathbb{Z}/2\mathbb{Z})^3, \]
\[ H_2(\mathcal{H}) \cong (\mathbb{Z}/2\mathbb{Z})^2, \quad H_3(\mathcal{H}) \cong \mathbb{Z}^2 \times (\mathbb{Z}/2\mathbb{Z})^2, \]

and for any other subscript homology is trivial.

In Section 4, we will generalize the homotopy equivalence between $\mathcal{H}$ and $\text{Flag}(\mathbb{R}^4)$ to a homotopy equivalence between the space of triples of projective lines of distinct intersections in $\mathbb{R}P^n$ and the partial flag variety $\text{Flag}(1, 2, 3, \mathbb{R}^{n+1})$.

Theorem 5. The space of all triples of projective lines in $\mathbb{R}P^n$ such that any line in a triple intersects the two others at distinct points is homotopy equivalent to the real partial flag variety $\text{Flag}(1, 2, 3, \mathbb{R}^{n+1})$ consisting of partial flags of signature $(0, 1, 2, 3, n + 1)$.

### 2 The space $\mathcal{H}'$

The projective lines $\ell_i \subset \mathbb{R}P^2$ correspond to planes passing through the origin in $\mathbb{R}^3$, $\sum_{j=1}^{3} a_{ij} x_j = 0$ and a plane passing through the origin in $\mathbb{R}^3$ can be described with a normal vector to the plane, whose components are the coefficients $a_{ij}$. The intersection condition, which says that any two planes intersect at a distinct line, is equivalent to saying that the three normal vectors are linearly independent which translates to the fact that the rows of
the matrix \( A = [a_{ij}] \) are linearly independent. Since the rows of the matrix \( A \) correspond to the normal vectors, if a normal vector is scaled by a non-zero scalar \( \lambda \in \mathbb{R} \) it describes the same plane passing through the origin. Hence the space \( \mathcal{H}' \) is the quotient \( GL(3, \mathbb{R})/\sim \) where the equivalence relation \( \sim \) is defined as follows. For \( A, B \in GL(3, \mathbb{R}) \), \( A \sim B \) if for any \( i = 1, 2, 3 \) there exists a non-zero scalar \( \lambda_i \in \mathbb{R} \) such that \( A_i = \lambda_i B_i \) where \( A_i \) is the \( i \)th row of the matrix \( A \) and similarly \( B_i \) is the \( i \)th row of the matrix \( B \). We may write this quotient space as \( GL(3, \mathbb{R})/T \) where \( T \) is the maximal torus in \( GL(3, \mathbb{R}) \) consisting of diagonal matrices.

**Theorem 1.** The space \( \mathcal{H}' \), that is, the quotient \( GL(3, \mathbb{R})/\sim = GL(3, \mathbb{R})/T \) is homotopy equivalent to the quotient of \( SU(2) \) by the free action of the dihedral group of order 8, \( D_8 \).

**Proof.** By using QR factorization we can write \( GL(3, \mathbb{R}) = O(3, \mathbb{R}) \cdot U \) where \( O(3, \mathbb{R}) \) is the subgroup of \( GL(3, \mathbb{R}) \) consisting of 3 by 3 orthogonal matrices with real entries and \( U \) is the subgroup of \( GL(3, \mathbb{R}) \) consisting of upper-triangular matrices with positive diagonal entries.

The deformation retraction of \( GL(3, \mathbb{R}) \) to \( O(3, \mathbb{R}) \) induces a deformation retraction of \( GL(3, \mathbb{R})/\sim = GL(3, \mathbb{R})/T \) to \( O(3, \mathbb{R})/\sim = O(3, \mathbb{R})/(T \cap O(3, \mathbb{R})) \). Note that \( T \cap O(3, \mathbb{R}) \) consists of diagonal matrices with entries \( \pm 1 \).

The subgroup \( O(3, \mathbb{R}) \) has two connected components and there is an element of \( T \cap O(3, \mathbb{R}) \) with negative determinant that can take an element of the connected component consisting of orthogonal matrices with determinant \( -1 \) to the other connected component \( SO(3, \mathbb{R}) \), so we see that \( O(3, \mathbb{R})/(T \cap O(3, \mathbb{R})) \) is diffeomorphic to \( SO(3, \mathbb{R})/(T \cap SO(3, \mathbb{R})) \) where \( T \cap SO(3, \mathbb{R}) \) is isomorphic to the Klein 4-group \( V_4 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \).

Now the universal covering group of \( SO(3, \mathbb{R}) \) is \( SU(2) \) with the kernel of the universal covering map \( \pi : SU(2) \rightarrow SO(3, \mathbb{R}) \) being \( Q := \ker \pi \cong \mathbb{Z}/2\mathbb{Z} \). The lifting of \( K := T \cap SO(3, \mathbb{R}) \) to the covering \( SU(2) \) is the nontrivial central extension \( K^* \) of \( K \) by the center of \( SU(2) \) which is isomorphic to the dihedral group of order 8, \( D_8 \), we have two short exact sequences

\[
1 \rightarrow Q(\cong \mathbb{Z}/2\mathbb{Z}) \rightarrow SU(2) \rightarrow SO(3) \rightarrow 1
\]

\[
1 \rightarrow Q(\cong \mathbb{Z}/2\mathbb{Z}) \rightarrow K^*(\cong D_8) \rightarrow K(\cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \rightarrow 1
\]

and \( Q \) and \( K^* \) are normal in \( SU(2) \), and since \( SU(2)/Q \cong SO(3, \mathbb{R}) \), we get \( SU(2)/K^* \cong SO(3, \mathbb{R})/K \cong \mathcal{H}' \).
By definition a flag in an $n$-dimensional vector space $V$ over a field $k$ is a sequence of subspaces of $V$, $0 = V_0 \subset V_1 \subset \cdots V_m = V$ such that $d_i = \dim V_i$ and $0 = d_0 < d_1 < \cdots < d_m = n$. A flag is called a complete flag if $d_i = i$ for all $i$. A standard flag associated to an ordered basis of $V$ is the flag in which $i$th subspace is spanned by the first $i$ basis vectors. Let us call the variety of complete flags in an $n$-dimensional vector space over real numbers, $\text{Flag}(\mathbb{R}^n)$.

Note that $\mathcal{H}'$ is a homogeneous space and it is homotopy equivalent to the variety of complete flags in a three-dimensional vector space over the real numbers, $\text{Flag}(\mathbb{R}^3)$. The general linear group $\text{GL}(3, \mathbb{R})$ acts transitively on the set of all complete flags in $\mathbb{R}^3$, and with respect to a fixed basis the stabilizer of the standard flag is the group of nonsingular lower triangular matrices $B_3$, and so the real complete flag variety is the quotient $\text{GL}(3, \mathbb{R})/B_3$ which is again homotopy equivalent to the space $\text{O}(3, \mathbb{R})/(T \cap \text{O}(3, \mathbb{R}))$ described in Theorem 1.

We will calculate the integral homology groups of $\mathcal{H}'$ using Theorem 1. Note that the homology of the corresponding real complete flag variety is calculated in [6], and so it constitutes a verification of the result of the next theorem.

**Theorem 2.** The integral homology groups of $\mathcal{H}' \cong \text{GL}(3, \mathbb{R})/\sim \cong \text{SU}(2)/K^*$, are as follows.

$$H_0(\mathcal{H}') \cong H_3(\mathcal{H}') \cong \mathbb{Z}, H_1(\mathcal{H}') \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$$

and for any other subscript the homology is trivial.

**Proof.** Let us compute the integral homology of $\text{SU}(2)/K^* \cong \text{SO}(3)/K$. Since the equivalence relation acts smoothly and freely on the smooth manifold $\text{SO}(3)$, the dimension is $\dim(\text{SO}(3, \mathbb{R})) - \dim K = 3 - 0 = 3$. The quotient is compact and connected because $\text{SO}(3)$ is compact and connected. For any $\gamma \in K$, the diffeomorphism $\gamma \cdot - : \text{SO}(3) \to \text{SO}(3)$ which is defined as $x \mapsto \gamma x$ is orientation-preserving, so $K$ is orientation-preserving, hence the quotient is orientable because $\text{SO}(3, \mathbb{R})$ is a closed, oriented, smooth manifold and $K$ is a discrete group acting freely and properly on $\text{SO}(3, \mathbb{R})$. Thus, $H_0(\mathcal{H}') \cong H_3(\mathcal{H}') \cong \mathbb{Z}$.

By the Hurewicz theorem $H_1(\text{SU}(2)/K^*)$ is isomorphic to the abelianization of $\pi_1(\text{SU}(2)/K^*) \cong K^* \cong D_8$ which is the Klein 4-group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Hence, $H_1(\mathcal{H}') \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
Finally, by the Poincaré duality $H_2(SU(2)/K^*)$ is isomorphic to $H^1(SU(2)/K^*)$. By the universal coefficients theorem for cohomology and since
\[ \text{Ext}^1(H_0(SU(2)/K^*), \mathbb{Z}) = \text{Ext}^1(\mathbb{Z}, \mathbb{Z}) \]
is trivial we have
\[ H_2(\mathcal{H}') \cong H_2(SU(2)/K^*) \cong H^1(SU(2)/K^*) \cong \text{Hom}(H_1(SU(2)/K^*), \mathbb{Z}) \cong 0. \]
\[ \square \]

We could have taken a possible shortcut in the calculation of the homology groups of $\mathcal{H}'$ by the fact that this space is homotopy equivalent to a fiber bundle over $\mathbb{RP}^2$ with fibers $\mathbb{RP}^1$ homeomorphic to $S^1$.

3 The space $\mathcal{H}$

Let us now consider the space $\mathcal{H}$ of all triples of lines in $\mathbb{RP}^3$ such that any line in a triple intersects the other two others at distinct points. We can homotopically identify $\mathcal{H}$ with the variety of complete flags in a four-dimensional vector space over the real numbers, $\text{Flag}(\mathbb{R}^4)$.

**Theorem 3.** The space $\mathcal{H}$ is homotopy equivalent to $\text{Flag}(\mathbb{R}^4)$.

**Proof.** Let $p = (\ell_1, \ell_2, \ell_3)$ be a point in $\mathcal{H}$. Each $\ell_i$ corresponds to a 2-subspace of $V = \mathbb{R}^4$, and $\ell_1 \cup \ell_2 \cup \ell_3$ is contained in some 3-subspace $V'$ of $V$. Let $e_i$ be a vector orthogonal to $\ell_i$ in $V'$, and let $V_1 = \langle e_1 \rangle, V_2 = \langle e_1, e_2 \rangle, V_3 = \langle e_1, e_2, e_3 \rangle$, and so $(0 = V_0 < V_1 < V_2 < V_3 < V)$ is a complete flag in $\mathbb{R}^4$ and we define $f : \mathcal{H} \to \text{Flag}(\mathbb{R}^4)$ to be the map that takes $p$ to this complete flag.

A complete flag $(0 = V_0 < V_1 < V_2 < V_3 < V = \mathbb{R}^4)$ has a unique orthonormal basis $(e_1, e_2, e_3, e_4)$ up to multiplying each basis element by a unit such that $0 = V_0 < V_1 = \langle e_1 \rangle < V_2 = \langle e_1, e_2 \rangle < V_3 = \langle e_1, e_2, e_3 \rangle < V = \langle e_1, e_2, e_3, e_4 \rangle = \mathbb{R}^4$. Let $\ell_i$ be the 2-subspace of $V_3$ that is orthogonal to $e_i$, note that each $\ell_i$ is also a 2-subspace of $V$ and so a triple $p = (\ell_1, \ell_2, \ell_3)$ is a point in $\mathcal{H}$, we define $g : \text{Flag}(\mathbb{R}^4) \to \mathcal{H}$ to be the map that takes a flag $(0 = V_0 < V_1 < V_2 < V_3 < V = \mathbb{R}^4)$ to this point $p$.

Note that both $f$ and $g$ are well-defined. Now $f \circ g : \text{Flag}(\mathbb{R}^4) \to \mathcal{H} \to \text{Flag}(\mathbb{R}^4)$ is the identity map on $\text{Flag}(\mathbb{R}^4)$ and the composition $g \circ f : \mathcal{H} \to \text{Flag}(\mathbb{R}^4)$ is the identity map on $\text{Flag}(\mathbb{R}^4)$.
Flag($\mathbb{R}^4$) $\to \mathcal{H}$ first maps a triple $(\ell_1, \ell_2, \ell_3)$ in the 3-subspace containing all $\ell_i$ to the corresponding complete flag $(0 = V_0 < V_1 = \langle e_1 \rangle < V_2 = \langle e_1, e_2 \rangle < V_3 = \langle e_1, e_2, e_3 \rangle < V = \mathbb{R}^4)$ which is then mapped to $(\ell'_1 = \ell_1, \ell'_2, \ell'_3)$ where $\ell'_i$ is orthogonal to $e'_i$ in $V_3$, $e'_2$ is $e_2$ rotated along along a vector in $V_3$ orthogonal to $V_2$, and similarly $e'_3$ can be obtained from $e_3$ via a rotation in $V_3$, so the composition $g \circ f$ is homotopic to the identity on $\mathcal{H}$. Hence, $\mathcal{H}$ and Flag($\mathbb{R}^4$) are homotopy equivalent.

Note that in the complex case there is a fibration

\[ \text{Flag}(\mathbb{C}^{n-1}) \to \text{Flag}(\mathbb{C}^n) \xrightarrow{p_{n-1}} \text{Gr}(n-1, \mathbb{C}^n) \cong \mathbb{CP}^{n-1} \]

where $p_{n-1}$ is a projection map, namely $p_k : \text{Flag}(\mathbb{C}^{n+k}) \to \text{Gr}(k, \mathbb{C}^{n+k})$ is defined by mapping a point $(0 < V_1 < \cdots < V_{n+k})$ in $\text{Flag}(\mathbb{C}^{n+k})$ to $V_k$, and the map $p$ induces an injection in cohomology $p^* : H^*(\text{Gr}(k, \mathbb{C}^{n+k})) \to H^*(\text{Flag}(\mathbb{C}^{n+k}))$ [5], and by a theorem of Borel we can write an isomorphism of the cohomology of Flag($\mathbb{C}^n$) with the coinvariant algebra.

In [6], Kocherlakota gives an algorithm for the calculation of integral homology groups of real flag manifolds, but in the spirit of Theorem 1 and Theorem 2, let us explicitly calculate the homology groups of $\mathcal{H} \cong \text{Flag}(\mathbb{R}^4)$.

**Theorem 4.** The integral homology groups of $\mathcal{H} \cong \text{Flag}(\mathbb{R}^4)$, are as follows.

\[
\begin{align*}
H_0(\mathcal{H}) &\cong H_6(\mathcal{H}) \cong \mathbb{Z}, H_1(\mathcal{H}) \cong H_4(\mathcal{H}) \cong (\mathbb{Z}/2\mathbb{Z})^3, \\
H_2(\mathcal{H}) &\cong (\mathbb{Z}/2\mathbb{Z})^2, H_3(\mathcal{H}) \cong \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2,
\end{align*}
\]

and for any other subscript homology is trivial.

**Proof.** By the transitive action of the general linear group on the space of complete flags in $\mathbb{R}^4$, we see that Flag($\mathbb{R}^4$) is GL($4, \mathbb{R}$)/$U$ where $U$ is the maximal closed torus consisting of upper triangular matrices. This space is homotopy equivalent to $O(4, \mathbb{R})/(T \cap O(4, \mathbb{R}))$ where $T \cap O(4, \mathbb{R})$ consists of matrices with $\pm 1$ on the main diagonal.

The Lie group $O(4, \mathbb{R})$ has two connected components and there is an element of $T \cap O(4, \mathbb{R})$ with negative determinant that can take an element of the connected component consisting of orthogonal matrices with determinant $-1$ to the other connected component $SO(4, \mathbb{R})$, hence, we have that $O(4, \mathbb{R})/(T \cap O(4, \mathbb{R}))$ is diffeomorphic to $SO(4, \mathbb{R})/(T \cap SO(4, \mathbb{R}))$ where...
$T \cap SO(4, \mathbb{R})$ is isomorphic to the elementary abelian group of order 8, $E_8$, also isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$.

The universal covering group of SO(4, $\mathbb{R}$) is $\text{Spin}(4) \cong SU(2) \times SU(2) \cong S^3 \times S^3$ with the kernel of the universal cover map $\pi : \text{Spin}(4) \to SO(4)$ being $Q := \ker \pi \cong \mathbb{Z}/2\mathbb{Z}$. Let $K^*$ be the lifting of $K := T \cap SO(4, \mathbb{R})$ to Spin(4), it is isomorphic to $D_8 \times \mathbb{Z}/2\mathbb{Z}$. We want to find the homology of $\text{Spin}(4)/K^* \cong SO(4, \mathbb{R})/K$. Since the defining equivalence relation acts smoothly and freely on Spin(4), the dimension of the quotient is $\dim SO(4) − \dim K^* = 4 \cdot 3/2 − 0 = 6$. Since SO(4) is compact and connected, the quotient is compact and connected. The quotient is also orientable because SO(4) is closed, orientable and the action of the discrete subgroup $K^*$ is orientation preserving. Thus, $H_0(\mathcal{H}) \cong H_6(\mathcal{H}) \cong \mathbb{Z}$.

Note that Spin(4) $\cong S^3 \times S^3$ has integral homology $H_0(\text{Spin}(4)) \cong H_6(\text{Spin}(4)) \cong \mathbb{Z}$, $H_3(\text{Spin}(4)) \cong \mathbb{Z}^2$ and other homology groups are trivial. By Hurewicz theorem, $H_1(\mathcal{H}) \cong H_1(\text{Spin}(4)/K^*)$ is isomorphic to the abelianization of $\pi_1(\text{Spin}(4)/K^*) \cong K^* \cong D_8 \times \mathbb{Z}/2\mathbb{Z}$ which is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$. By Poincaré duality $H_5(\mathcal{H})$ is isomorphic to $H^1(\mathcal{H})$, and by the universal coefficients theorem for cohomology we have an exact sequence

$$0 \to \text{Ext}^1(H_0(\mathcal{H}, \mathbb{Z}), \mathbb{Z}) \to H^1(\mathcal{H}, \mathbb{Z}) \to \text{Hom}(H_1(\mathcal{H}, \mathbb{Z}), \mathbb{Z}) \to 0$$

where $\text{Ext}^1(H_0(\mathcal{H}, \mathbb{Z}), \mathbb{Z})$ is trivial and $\text{Hom}(H_1(\mathcal{H}, \mathbb{Z}), \mathbb{Z})$ is also trivial, so $H_5(\mathcal{H}, \mathbb{Z}) \cong H^1(\mathcal{H}, \mathbb{Z})$ is trivial. Again by universal coefficients theorem for cohomology we have $\text{Ext}^1(H_4(\mathcal{H}, \mathbb{Z}), \mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^3$ and since $H_4(\mathcal{H})$ is finitely generated and since it has no free part, we must have $H_4(\mathcal{H}) \cong (\mathbb{Z}/2\mathbb{Z})^3$.

Let $\pi : \mathcal{H} \to \mathbb{RP}^3$ be defined as follows. Consider $\mathbb{RP}^3$ as the space of projective 2-planes. A triple $p = (\ell_1, \ell_2, \ell_3) \in \mathcal{H}$ maps to the unique plane defined by $p$, $\pi(p) \cong \mathbb{RP}^2$, the space $\mathcal{H}$ is a fiber bundle over $\mathbb{RP}^3$. A fiber of this bundle over a projective 2-plane, a point $A \cong \mathbb{RP}^2$ in $\mathbb{RP}^3$ is the space of all triples of lines in $A$ such that any line in a triple intersects the two others at distinct points, hence a fiber is precisely homeomorphic to $\mathcal{H}'$ whose homology we already calculated in Theorem 2.

Since the homology of the base space and the fiber are known, and since the base space $\mathbb{RP}^3$ is path-connected, admits the structure of a CW-complex, and $\pi_1(\mathbb{RP}^3)$ acts trivially on $H^*(\mathcal{H}', \mathbb{Z})$, there is a Serre spectral sequence of our fiber bundle $\pi : \mathcal{H} \to \mathbb{RP}^3$. The cohomology of $\mathcal{H}'$ is $H^0(\mathcal{H}', \mathbb{Z}) = H^3(\mathcal{H}', \mathbb{Z}) = \mathbb{Z}$, $H^2(\mathcal{H}', \mathbb{Z}) = V_1$ and any other cohomology group is trivial. The cohomology of $\mathbb{RP}^3$ is $H^0(\mathbb{RP}^3, \mathbb{Z}) = H^3(\mathbb{RP}^3, \mathbb{Z}) = \mathbb{Z}$, $H^2(\mathbb{RP}^3, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ and any other cohomology group is trivial.
Hence, by $E_{p,q}^2 = H^p(\mathbb{RP}^3, H^q(H', \mathbb{Z}))$, the second page $E_2$ of the Serre spectral sequence corresponding to our fiber bundle is as below.

\[
\begin{array}{cccc}
\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z} \\
V_4 & V_4 & V_4 & V_4 \\
0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}
\end{array}
\]

The only non-trivial differential is $d_{2}^{0,3} : E_{2}^{0,3} \to E_{2}^{2,2}$ and since it is a homomorphism from $\mathbb{Z}$ to $V_4$ it cannot be surjective, hence the index $(2, 2)$ survives in the $E_3$ page. By degree reasons $E_{\infty}$ degenerates to $E_3$ where every term in $E_3$ is same as that of $E_2$ because by the universal coefficients theorem for cohomology and the Poincaré duality, $\text{Ext}^1(H_3(H, \mathbb{Z}), \mathbb{Z}) \cong H_2(H, \mathbb{Z})$, so we must have $H_3(H, \mathbb{Z}) \cong H^3(H, \mathbb{Z}) \cong \mathbb{Z}^2 \times V_4 \cong \mathbb{Z}^2 \times (\mathbb{Z}/2\mathbb{Z})^2$ and $H_2(H, \mathbb{Z}) \cong H^4(H, \mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^2$. □

4 The space of triples of projective lines of distinct intersections in $\mathbb{RP}^n$

The proof of Theorem 3 can be generalized to the space of all triples of projective lines of distinct intersections in $\mathbb{RP}^n$.

Theorem 5. The space of all triples of projective lines in $\mathbb{RP}^n$ such that any line in a triple intersects the two others at distinct points is homotopy equivalent to the real partial flag variety $\text{Flag}(1, 2, 3, \mathbb{R}^{n+1})$ consisting of partial flags of signature $(0, 1, 2, 3, n + 1)$.

Proof. The difference with the proof of Theorem 3 is that we cannot complete a basis of a three-dimensional subspace of $V = \mathbb{R}^n$ uniquely to get a complete flag in $\text{Flag}(\mathbb{R}^{n+1})$, but instead given a triple of projective lines, we can construct a unique partial flag of signature $(0, 1, 2, 3, n + 1)$.

Let $\mathcal{M}$ be the space of all triples of projective lines of distinct intersections. Let $p = (\ell_1, \ell_2, \ell_3)$ be a point in $\mathcal{M}$. Each $\ell_i$ corresponds to a 2-subspace of $V = \mathbb{R}^{n+1}$, and $\ell_1 \cup \ell_2 \cup \ell_3$ is contained in some 3-subspace $V'$ of $V$. Let $e_i$ be a vector orthogonal to $\ell_i$ in $V'$, and let $V_1 = \langle e_1 \rangle$, $V_2 = \langle e_1, e_2 \rangle$, $V_3 = \langle e_1, e_2, e_3 \rangle$, and so $(0 = V_0 < V_1 < V_2 < V_3 < V)$ is a partial flag in $\mathbb{R}^{n+1}$ and we define $f : \mathcal{M} \to \text{Flag}(1, 2, 3, \mathbb{R}^{n+1})$ to be the map that takes $p$ to this partial flag.
Given a partial flag \((0 = V_0 < V_1 < V_2 < V_3 < V = \mathbb{R}^{n+1})\) with signature \((0, 1, 2, 3, n + 1)\), we can construct an orthonormal basis \((e_1, e_2, e_3)\) unique up to multiplying each basis element by a unit such that \(0 = V_0 < V_1 = \langle e_1 \rangle < V_2 = \langle e_1, e_2 \rangle < V_3 = \langle e_1, e_2, e_3 \rangle < V\). Let \(\ell_i\) be the 2-subspace of \(V_3\) which is orthogonal to \(e_i\). Note that each \(\ell_i\) is also a 2-subspace of \(V\) and so a triple \(p = (\ell_1, \ell_2, \ell_3)\) is a point in \(\mathcal{M}\). We define \(g : \text{Flag}(1, 2, 3, \mathbb{R}^4) \to \mathcal{M}\) to be the map that takes a flag \((0 = V_0 < V_1 < V_2 < V_3 < V = \mathbb{R}^{n+1})\) to this point \(p\). Both \(f\) and \(g\) are well-defined.

Now \(f \circ g : \text{Flag}(1, 2, 3, \mathbb{R}^4) \to \mathcal{M} \to \text{Flag}(1, 2, 3, \mathbb{R}^4)\) is the identity map on \(\text{Flag}(1, 2, 3, \mathbb{R}^4)\) and the composition \(g \circ f : \mathcal{M} \to \text{Flag}(1, 2, 3, \mathbb{R}^4) \to \mathcal{M}\) first maps a triple \((\ell_1, \ell_2, \ell_3)\) with respective orthogonal vectors \((e_1, e_2, e_3)\) in the 3-subspace containing all \(\ell_i\) to the corresponding complete flag \((0 = V_0 < V_1 = \langle e_1 \rangle < V_2 = \langle e_1, e_2 \rangle < V_3 = \langle e_1, e_2, e_3 \rangle < V)\) which is then mapped to \((\ell'_1 = \ell_1, \ell'_2, \ell'_3)\) where \(\ell'_i\) is orthogonal to \(e'_i\) in \(V_3\), and the orthonormal triple \((e'_1, e'_2, e'_3)\) is obtained from \((e_1, e_2, e_3)\) via some rotation, and so the composition \(g \circ f\) is homotopic to the identity on \(\mathcal{M}\). Hence, \(\mathcal{M}\) and \(\text{Flag}(1, 2, 3, \mathbb{R}^4)\) are homotopy equivalent.

Since there is an algorithm by Kocherlakota [6] to compute the integral homology of real flag manifolds, this particular projective configuration space is well-understood in terms of (co)homology.

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A.B. Yetişer, Department of Mathematics, National Research University Higher School of Economics, Moscow, Russia

E-mail address: ayetisher@edu.hse.ru, abyetiser@gmail.com