SUPEREXTENSION $n = (2, 2)$ OF THE COMPLEX LIOUVILLE EQUATION AND ITS SOLUTION $^a$

A. A. KAPUSTNIKOV
Department of Physics, Dnepropetrovsk University,
49050, Dnepropetrovsk, Ukraine
E-mail: kpstnko@creator.dp.ua

It is shown that the method of the nonlinear realization of local supersymmetry previously developed in framework of supergravity being applied to the $n = (2, 2)$ superconformal symmetry allows one to get the new form of the exactly solvable $n = (2, 2)$ super-Liouville equation. The general advantage of this version as compared with the conventional one is that its bosonic part includes the complex Liouville equation. We obtain the suitable supercovariant constraints imposed on the corresponding superfields which provide the set of the resulting system of component equations be the same as that in model of $N = 2$, $D = 4$ Green-Schwarz superstring. The general solution of this system is derived from the corresponding solution of the bosonic string equation.

Introduction

It is well-known that a doubly supersymmetric generalization of the geometrical approach to superstring leads in the case of $N = 2$, $D = 3$ Green-Schwarz superstring to the new version of the Liouville equation referring in literature as $n = (1, 1, 1)$. The latter as one can expected include the real Liouville equation in its bosonic part. The problem, however, arise when we try to extend this result on the case of $N = 2$, $D = 4$ superstring. It turns out that in this case the well-known form of the suitable super-Liouville equation is not relevant in virtue of absence of the complex Liouville equation in the corresponding bosonic part. Thus, the equation proposed in $^1$ can not be applied for description of the $N = 2$, $D = 4$ Green-Schwarz superstring, which as one know is reduced to ordinary complex Liouville equation when neglecting by all the fermionic component fields.

In this paper we would like to propose the new version of the $n = (2, 2)$ super-Liouville equation which appears to be agreement with the equations of motion of the $N = 2$, $D = 4$ Green-Schwarz superstring. Our approach is based on the method of the nonlinear realization of local supersymmetries developed by Ivanov and Kapustnikov in frame of supergravity. It will be shown that when applied to the $n = (2, 2)$ superconformal symmetry this method makes the possibility to impose the supercovariant constraints on the

---

$^a$Talk given at the XIV-th Max Born Symposium, Karpacz, Poland, September 21-25, 1999
superfields in such a way that all the unphysical degrees of freedom occur in
the original equation will appear removed from the residual set of the equa-
tion of motion. The latter amounts to the complex Liouville equation for the
bosonic worldsheet variable $\tilde{u}(\tilde{\xi}^{+}, \tilde{\xi}^{-})$ supplemented with two first order
free equations $\tilde{\partial}_{-} \lambda^{+}(\tilde{\xi}^{+}, \tilde{\xi}^{-}) = \tilde{\partial}_{+} \lambda^{-}(\tilde{\xi}^{+}, \tilde{\xi}^{-}) = 0$ for the fermions
of opposite chirality.

In Section 3 we present the general solution of this equation in terms of
the restricted Lorentz harmonic variables $^6$, which by a proper fashion extends
a corresponding bosonic string solution obtained in $^7$.

1 New version of the $n = (2, 2)$ super-Liouville equation

1.1 Linear realization

We begin with the linear realization of two copies of one dimensional supercon-
formal group acting separately on the light-cone complex coordinates of $N = 2,
D = 4$ superstring $C^{(2|2)} = (\xi^{++} = \xi^{++} + i\eta^{+}\bar{\eta}^{+}, \eta^{+}; \xi^{--} = \xi^{--} + i\eta^{-}\bar{\eta}^{-}, \eta^{-})$:

$$\begin{align*}
\xi^{\pm\pm'}_{L} &= \Lambda^{\pm\pm} - \bar{\eta}^{\pm} \bar{D}^{\pm}_{\pm} \Lambda^{\pm\pm} \\
\eta^{\pm'} &= \frac{i}{2} \bar{D}^{\pm}_{\pm} \Lambda^{\pm\pm} = \epsilon^{\pm}(\xi^{\pm\pm'}_{L}) + \eta^{\pm}(\xi^{\pm\pm'}_{L}) e^{i\rho^{(\pm\pm)}(\xi^{\pm\pm'}_{L})}, \\
a^{\pm}_{L}(\xi^{\pm\pm'}_{L}) &= \xi^{\pm\pm} + \eta^{\pm}(\xi^{\pm\pm'}_{L}) e^{i\rho^{(\pm\pm)}(\xi^{\pm\pm'}_{L})}, \\
g^{(\pm\pm)} &= \sqrt{1 + \partial_{\pm\pm} a^{\pm\pm} + i(\epsilon^{\pm} \partial_{\pm\pm} \epsilon^{\pm} + \bar{\tau}^{\pm} \partial_{\pm\pm} \bar{\tau}^{\pm}).}
\end{align*}$$

In Eq. (2) the general superfield (SF)

$$\begin{align*}
\Lambda^{\pm\pm}(\xi^{\pm\pm'}_{L}, \eta^{\pm'}, \bar{\eta}^{\pm}) = a^{\pm}_{L}(\xi^{\pm\pm'}_{L}) + 2i\eta^{\pm} \eta^{\pm'} g^{(\pm\pm)}(\xi^{\pm\pm'}_{L}) e^{i\rho^{(\pm\pm)}(\xi^{\pm\pm'}_{L})} + 2i\bar{\eta}^{\pm} \epsilon^{\pm}(\xi^{\pm\pm'}_{L}) e^{i\rho^{(\pm\pm)}(\xi^{\pm\pm'}_{L})},
\end{align*}$$

is composed out from parameters $\epsilon^{+}(\xi^{\pm\pm'}_{L}), \epsilon^{-}(\xi^{\pm\pm'}_{L})$ of local supertranslations;
two real parameters $a^{+\pm}(\xi^{\pm\pm'}_{L}), a^{-\pm}(\xi^{\pm\pm'}_{L})$ of D1-reparametrizations and two
real parameters $\rho^{+\pm}(\xi^{\pm\pm'}_{L}), \rho^{-\pm}(\xi^{\pm\pm'}_{L})$ describing local $U(1) \times U(1)$-rotations.
The spinor covariant derivatives are defined as

$$\begin{align*}
D_{\pm} &= \partial_{\pm} + 2i\eta^{\pm} \partial_{\pm\pm}, \\
\bar{D}_{\pm} &= \partial_{\pm}.
\end{align*}$$

It is worth to mention that since the parameters $\xi^{\pm\pm'}_{L}$ and $\eta^{\pm'}$ in Eqs. (2) are
subjected to the constraints

$$\begin{align*}
D_{\pm} \xi^{\pm\pm'}_{L} - 2i\bar{\eta}^{\pm\pm'} D_{\pm} \eta^{\pm'} = 0
\end{align*}$$

2
the flat spinor covariant derivatives are transformed homogeneously with respect to
\[ D_\pm = (D_\pm \eta^{\pm'})D'_\pm. \] (5)

Therefore, the following superconformal-covariant equation can be proposed as a natural candidate for the (2, 2) superextension of the corresponding (1, 1) super-Liouville equation
\[ D_-D_+ W = e^{2W} \Psi^-\Psi^+. \] (6)

In Eq. (6) one double-analytical SF
\[ W(\xi^\pm_L, \eta^\pm) = u(\xi^\pm_L) + \eta^+ \psi^- (\xi^\pm_L) + \eta^- \psi^+ (\xi^\pm_L) + \eta^+ \eta^- F(\xi^\pm_L), \] (7)
and two general SFs $\Psi_+(\xi^+, \eta^+, \eta^+\bar{\eta}^+)$, $\Psi_-(\xi^-, \eta^-, \eta^-\bar{\eta}^-)$, depending separately on the (2, 0) and (0, 2) light-cone variables, are introduced. In Eq. (6) is invariant under the following gauge transformations
\[ W'(\xi^\pm_L', \eta^\pm') = W(\xi^\pm_L) - \frac{1}{2} \ln(D_+\bar{\eta}^+) - \frac{1}{2} \ln(D_-\eta^-), \] (8)
\[ \Psi'_+(\xi^+, \eta^+\bar{\eta}^+) = (D_+\eta^+)^{-1}(D_+\bar{\eta}^+)\Psi_+(\xi^+, \eta^+\bar{\eta}^+), \]
\[ \Psi'_-(\xi^-, \eta^\bar{\eta}^-) = (D_-\eta^-)^{-1}(D_-\eta^\bar{\eta}^-)\Psi_-(\xi^-, \eta^\bar{\eta}^-). \]

Note that due to the nilpotence of the covariant derivatives ($D^2_\pm = 0$) the SFs $W$ and $\Psi_\pm$ included in the Eq. (6) appear restricted
\[ D_\pm \Psi_\pm + 2(D_\mp W)\Psi_\pm = 0. \] (9)

A particular property we shall encounter with here is, however, that the constraints can be solved explicitly in terms of the unrestricted FSs
\[ \Psi_+ = D_+ M + 2(D_+ W)M, \quad \Psi_- = D_- N + 2(D_- W)N. \] (10)

In Eq. (10) the general SFs
\[ M(\xi^\pm_L, \eta^+, \eta^+) = f(\xi^\pm_L) + \eta^+ \omega^- (\xi^\pm_L) + \eta^- \bar{\eta}^+ m^- (\xi^\pm_L), \]
\[ N(\xi^\pm_L, \eta^-, \eta^-) = g(\xi^\pm_L) + \eta^- \omega^+ (\xi^\pm_L) + \eta^+ \bar{\eta}^- n^+ (\xi^\pm_L), \]
\[ b\text{We omit temporarily the upper indices of SFs } \Psi \text{ and } M \text{ for the enlightening of formulas but we shall come back to them in Section 3.} \]
\[ c\text{It can be shown that in the case of chiral SFs } M(\xi^\pm_L, \eta^+), N(\xi^\pm_L, \eta^-) \text{ Eq. (10) is reduced to free one } D_+ D_- W = 0 \text{ for the SF } \bar{W} = W + \frac{1}{2} \ln(MN). \]
are supposed transform as a superconformal densities

\[ M'(\xi^+L, \eta^+L, \eta^+) = (\bar{D}_+\eta^+)M(\xi^+L, \eta^+, \eta^+), \]
\[ N'(\xi^-L, \eta^-L, \eta^-) = (D_-\eta^-)N(\xi^-L, \eta^-, \eta^-). \]

Although the component content of SFs \( W, M, N \) even upon the gauge fixing is still too large to be related with the \( N = 2, D = 4 \) superstring there is very important feature of Eq. (6). It contains complex Liouville equation in its bosonic part

\[ \partial_+ \partial_- u(\xi^\pm) = \frac{1}{4} e^{2u(\xi^\pm)} m^-(\xi^+L) m^+(\xi^-L) + ..., \]

where all the unessential terms in the r.h.s. are omitted. It is clear, however, that to be connected with the superstring theory the SFs we have considered here must be covariantly constrained. In the next Section we are going to show that the desirable constraints could be imposed in frame of the nonlinear realization of \( n = (2, 2) \) superconformal symmetry in which the original SFs becomes reducible.

1.2 Nonlinear realization

To see this let us suppose that the v.e.v. of the component fields \( m^-L(\xi^+L) \) and \( n^+L(\xi^-L) \) are not equal to zero and as consequence of this the local supersymmetry is actually spontaneously broken. In this case the fermionic components \( \chi^\pmL \) acquire the sense of the corresponding Goldstone fermions and one can exploit them for the singling out of the complex Liouville equation from the system (6) in a manifestly covariant manner. Indeed, it is well-known that in the models with spontaneously broken supersymmetry all the SFs becomes reducible. Their irreducible parts are transformed, however, universally with respect to the action of the original supergroups, as the linear representations of the underlying unbroken subgroups but with the parameters depending nonlinearly on the Goldstone fermions. It makes the possibility to impose generally on the SFs in question some absolutely covariant restrictions providing to remove out from the model under consideration undesirable degrees of freedom. Here we can to avail oneself of the opportunity to restrict the SFs enter the Eq. (6) with the help of this approach.

For the beginning let us derive the nonlinear realization of the superconformal symmetry in superspace. Following closely to the general method developed in we need firstly splits the general finite element of the group

\[ G(\zeta_L) \equiv \zeta'_L, \]
where \( \zeta_L = \{ \xi_L^{\pm \pm}, \eta^\pm \} \), onto the product of two successive transformations

\[
G(\zeta_L) = K(G_0(\zeta_L)).
\]  

(15)

In Eq. (15) the following standard notations are used. As before the \( G_0(\zeta_L) \) refer to the "primes" coordinates \( \zeta'_L \) but index zero means that they referring now only to the stability subgroup

\[
\xi_L^{\pm \pm'} = \xi_L^{\pm \pm} + a^{\pm \pm}(\xi_L^{\pm \pm}),
\]

\[
\eta^\pm' = \eta^\pm e^{i\rho^{\pm \pm}(\xi_L^{\pm \pm})} \sqrt{1 + \partial^{\pm \pm} a^{\pm \pm}}.
\]  

(16)

The latter include only the ordinary conformal transformations (parameters \( a^{\pm \pm}(\xi_L^{\pm \pm}) \)) supplemented with the local \( U(1) \times U(1) \)-rotations (parameters \( \rho^{\pm \pm}(\xi_L^{\pm \pm}) \)). Note, that the first multiplier in the decomposition (15) is easily recognized as the representatives of the left coset space \( G/G_0 \)

\[
K^{\pm \pm}(\zeta_L) = \xi_L^{\pm \pm} + i \lambda^\pm(\xi_L^{\pm \pm}) \tau^\pm(\xi_L^{\pm \pm})
\]

\[
+ 2i \eta^\pm \tau^\pm(\xi_L^{\pm \pm}) \sqrt{1 + i(\lambda^\pm \partial_{\pm \pm} \lambda^\pm + \tau^\pm \partial_{\pm \pm} \tau^\pm)},
\]

\[
K^{\pm}(\zeta_L) = \epsilon^\pm(\xi_L^{\pm \pm}) + \eta^\pm \sqrt{1 + i(\lambda^\pm \partial_{\pm \pm} \lambda^\pm + \tau^\pm \partial_{\pm \pm} \tau^\pm)}.
\]  

(17)

It deserves to mention that in the decomposition (15) the comultiplie rs \( K \) and \( G_0 \) are chosen in such a way that the irreduciblity constraint (3) is satisfied separately for both of them. The prescription for constructing the corresponding nonlinear realization is as follows. Let us identify the local parameters \( \epsilon^\pm(\xi_L^{\pm \pm}), \tau^\pm(\xi_L^{\pm \pm}) \) in (17) with the Goldstone fields \( \lambda^\pm(\xi_L^{\pm \pm}), \bar{\lambda}^\pm(\xi_L^{\pm \pm}) \)

\[
\tilde{K}^{\pm \pm}(\tilde{\zeta}_L) = \tilde{\xi}_L^{\pm \pm} + i \lambda^\pm(\tilde{\xi}_L^{\pm \pm}) \bar{\lambda}^\pm(\tilde{\xi}_L^{\pm \pm})
\]

\[
+ 2i \bar{\eta}^\pm \bar{\lambda}^\pm(\tilde{\xi}_L^{\pm \pm}) \sqrt{1 + i(\lambda^\pm \partial_{\pm \pm} \lambda^\pm + \bar{\lambda}^\pm \partial_{\pm \pm} \bar{\lambda}^\pm)},
\]

\[
\tilde{K}^{\pm}(\tilde{\zeta}_L) = \tilde{\epsilon}^\pm(\tilde{\xi}_L^{\pm \pm}) + \bar{\eta}^\pm \sqrt{1 + i(\lambda^\pm \partial_{\pm \pm} \lambda^\pm + \bar{\lambda}^\pm \partial_{\pm \pm} \bar{\lambda}^\pm)}
\]  

(18)

and take for \( \tilde{K}(\tilde{\zeta}_L) \) the transformation law associated to (17)

\[
G(\tilde{K}(\tilde{\zeta}_L)) = \tilde{K}'(G_0(\tilde{\zeta}_L)).
\]  

(19)

In Eq. (19) the newly introduced coordinates \( \tilde{\zeta}_L = \{ \tilde{\xi}_L^{\pm \pm}, \tilde{\eta}^\pm \} \) are transformed differently as compared with \( \zeta_L = \{ \xi_L^{\pm \pm}, \eta^\pm \} \) in (1). Indeed, in accordance with virtue of (15) all the parameters in (1) should be regarded as composite ones which are composed out from the parameters of transformations (16) and (17).
with (16) they change only under the vacuum stability subgroup

\begin{align}
\tilde{\xi}_L &\rightarrow \tilde{\xi}_L^\pm + \tilde{a}^{\pm\pm} (\tilde{\xi}_L^\pm), \\
\tilde{\eta}^{\pm'} &\rightarrow \tilde{\eta}^{\pm} e^{i\tilde{\rho}^{\pm\pm}(\tilde{\xi}_L^\pm)} \sqrt{1 + \tilde{\partial}_{\pm\pm} \tilde{a}^{\pm\pm}},
\end{align}

(20)

where the parameters \( \tilde{a}^{\pm\pm}(\tilde{\xi}_L^\pm) \) and \( \tilde{\rho}^{\pm\pm}(\tilde{\xi}_L^\pm) \) turns out to be dependent nonlinearly on the fields \( \lambda^\pm(\tilde{\xi}_L^\pm), \tilde{\lambda}^\pm(\tilde{\xi}_L^\pm) \). Eqs. (19) and (20) determine the transformation properties of the Goldstone fermions \( \lambda^\pm(\tilde{\xi}_L^\pm), \tilde{\lambda}^\pm(\tilde{\xi}_L^\pm) \) with respect to the nonlinear realization of the superconformal group \( G \) in coset space (18).

2 Splitting superspace and irreducible form of SFs

Up to now we have dealt with only formal prescription of construction of the nonlinear realization of superconformal group \( G \) without any relation of this procedure to the original equation (3). Nevertheless, there is the simple possibility to gain a more deeper insight into the model we started with if we compare two Eqs. (14) and (19). We find that \( \tilde{K} (\tilde{\zeta}_L) \) transform under \( G \) in precisely the same manner as the initial coordinates \( \zeta_L \) of superspace \( C^{(2|2)} \).

Thus we have the unique possibility to identify them

\begin{equation}
\zeta_L = \tilde{K} (\tilde{\zeta}_L).
\end{equation}

(21)

Eq. (21) establish the relationship between two forms of the realization of superconformal symmetries in superspace. One of the remarkable futures of the transformations (21) is that superspace of the nonlinear realization \( \tilde{C}^{(2|2)} = \tilde{\zeta}_L \) turns out to be completely "splitting" in virtue of the transformations (20) which are not mixed the bosonic and fermionic variables. Due to this very important fact the SFs of the nonlinear realization becomes actually reducible. Indeed, let us perform the change of variables (21) in the Eq. (3)

\begin{equation}
\tilde{D}_- \tilde{D}_+ \tilde{W} = e^{2\tilde{W}} \tilde{\Psi}_+ \tilde{\Psi}_-,
\end{equation}

(22)

where the SFs and covariant derivatives of the nonlinear realization (19), (20) and (21) are introduced

\begin{align}
W = \tilde{W} - \frac{1}{2} \ln(\tilde{D}_+ \tilde{\eta}^+ - \frac{1}{2} \ln(\tilde{D}_- \tilde{\eta}^-), \quad D_\pm = (\tilde{D}_\pm \tilde{\eta}^\pm)^{-1} \tilde{D}_\pm, \\
\tilde{\Psi}_+ = \tilde{D}_+ \tilde{M} + 2(\tilde{D}_+ \tilde{W}) \tilde{M}, \quad \tilde{\Psi}_- = \tilde{D}_- \tilde{N} + 2(\tilde{D}_- \tilde{W}) \tilde{N}, \\
M(\tilde{\xi}_L^{++}, \tilde{\eta}^+, \tilde{\eta}^+) = (\tilde{D}_+ \tilde{\eta}^+) \tilde{M}(\tilde{\xi}_L^{++}, \tilde{\eta}^+, \tilde{\eta}^+),
\end{align}

(23)
\[ N(\xi_L^-, \eta^-) = (D_- \eta^-) \tilde{N}(\xi_L^-, \eta^-) \]  
(25)
3 General solution

Let us consider shortly the problem of construction of general solution of the Eq. (6). It is well-known that the Virasoro constraints simplifying significantly the string equations of motion can generally be solved in terms of two copies (left- and right-moving) of the Lorentz harmonic variables parameterizing the compact coset spaces isomorphic to the \((D-2)\)-dimensional sphere

\[ S_{D-2} = \frac{SO(1, D-1)}{SO(1, 1) \times SO(D-2) \times K_{D-2}} \]  

(31)

Moreover, it was shown in [7] that from these variables the particular Lorentz covariant combinations can be formed which resolve generally the corresponding nonlinear \(\sigma\)-model equations of motion inspired by the bosonic strings in the geometrical approach [1]. By the construction the number of two copies of chiral variables parameterizing the coset space (31) is apparently enough to recover the \(2(D-2)\) physical degrees of freedom of \(D\)-dimensional bosonic strings. But in the case of superstrings these variables replaced by the world-sheet superfields must be properly restricted to provide the necessary balance between bosonic and fermionic degrees of freedom \((D-2)B = (D-2)F\).

In this Section we shall show that the suitable constraints can be achieved within the method of the nonlinear realization of superconformal symmetry developed in Section 1.2.

Proceeding from [7] one can check that the general solution of the Liouville Eq. (13) can be written in form

\[ e^{-2\hat{a}^{}_{\pm}}(\xi_{L}^{\pm}) = \frac{1}{2} \hat{r}_{m}^{++}(\xi_{L}^{-})\hat{\eta}_{m}^{++}(\xi_{L}^{+}), \]  

(32)

\[ \hat{m}_{m}^{--}(\xi_{L}^{++}) = \hat{m}_{m}^{--}(\xi_{L}^{-})\hat{\eta}_{m}^{++}(\xi_{L}^{+}), \]  

(33)

\[ \hat{\eta}_{m}^{++}(\xi_{L}^{-}) = \hat{\eta}_{m}^{--}(\xi_{L}^{+})\hat{\eta}_{m}^{++}(\xi_{L}^{+}), \]  

(34)

where the left(right)-moving Lorentz harmonics are normalized as follows

\[ \hat{I}_{m}^{++}\hat{I}_{m}^{++} = 0, \quad \hat{I}_{m}^{--}\hat{I}_{m}^{++} = 0, \quad \hat{I}_{m}^{--}\hat{I}_{m}^{--} = 0, \]  

(35)

\[ \hat{I}_{m}^{--}\hat{I}_{m}^{++} = 2, \quad \hat{I}_{m}^{--}\hat{I}_{m}^{--} = -1. \]  

(36)

Substituting these solutions into the Eqs. (29) and taking account of the expressions (25) one finds

\[ M \equiv M^{--}(\xi_{L}^{++}, \eta^{+}, \Pi^{+}) = (\hat{D}_{+}\Pi^{+})\hat{\eta}^{++}\hat{I}_{m}^{--}(\xi_{L}^{++})\hat{\eta}_{m}^{++}(\xi_{L}^{++}), \]  

(37)

\[ N \equiv M^{++}(\xi_{L}^{--}, \eta^{+}, \Pi^{+}) = (\hat{D}_{-}\Pi^{+})\hat{\eta}^{--}\hat{I}_{m}^{++}(\xi_{L}^{--})\hat{\eta}_{m}^{++}(\xi_{L}^{++}) \]  

(38)
Now comparing these SFs with explicit form of the general solution of the Eq. (6)

\[ e^{-2W(\xi_L^\pm, \eta^\pm)} = \frac{1}{2} r_{m}^{\pm} (\xi_L^-, \eta^-) l_{m}^{\pm} (\xi_L^+, \eta^+) , \]

\[ \Psi_+ (\xi_L^+, \eta^+, \eta^+) = \frac{1}{2i} \hat{l}_{m}^- (\xi_L^+, \eta^) D_{+} \bar{m} (\xi_L^+, \eta^+), \]

\[ \Psi_- (\xi_L^-, \eta^-, \eta^-) = \frac{1}{2i} \hat{r}_{m}^+ (\xi_L^-, \eta^-) D_- \bar{m} (\xi_L^-, \eta^-), \]

one can derives the following expressions for the corresponding SFs of linear realization

\[ M^{\pm\pm} = \Phi^{\pm\pm} \Omega^{\pm\pm}, \quad \overline{\mathcal{D}}^{\pm} \Phi^{\pm} = 0, \quad D_{\pm} \Omega^{\pm} = 1, \quad (38) \]

\[ \Phi^{\pm} = \overline{D}^{\pm} \bar{\eta}^{\pm}, \quad \Omega^{\pm} = (\overline{D}^{\pm} \eta^{\pm}) \bar{\eta}^{\pm}, \quad (39) \]

\[ l_{m}^{\pm, 0} (\xi_L^+, \eta^+) = \hat{l}_{m}^{\pm, 0} (\xi_L^+), \]

\[ r_{m}^{\pm, 0} (\xi_L^-, \eta^-) = \hat{r}_{m}^{\pm, 0} (\xi_L^-), \quad (40) \]

4 Conclusion

Thus, we have established that the \( n = (2, 2) \) generalization of complex Liouville equation appropriated to the \( N = 2, D = 4 \) superstring is given by the Eq. (6) in which the auxiliary SFs \( \Psi \) are subjected to the constraints (10) and (38).

Then the general solution of this equation can be given in terms of Lorentz harmonics (37) which in one’s turn are also restricted by the conditions (40).

Note, that in its own rights this fact actually means that the Eq. (6) proved to be exactly solvable as the corresponding bosonic string equation does, but unlike to the bosonic case the corresponding harmonic SFs becomes essentially restricted by the constraints (40) which provide the supersymmetric balance between bosonic and fermionic degrees of freedom. It is worth mentioning that the first constraint in Eqs. (38) implies that the SFs \( M \) are actually nilpotence \( M^2 = 0 \). From the theory of spontaneously broken supersymmetries we know that such a type of constraints leads directly to the nonlinear realizations of the underline symmetries in frame of which these constraints could be solved explicitly in terms of the corresponding Goldstone (super)fields. In the case under consideration we find the suitable manifestly supercovariant
solution \( \lambda \) in terms of the Goldstone fermions of the nonlinear realization of 
\( n = (2, 2) \) superconformal symmetry \( \lambda^{\pm} (\xi^{\pm}_{L}), \lambda^{\pm} (\xi^{\pm}_{L}) \).

We are convinced that this approach actually gives the universal way of 
deriving the equations of motion as well as their solutions for the superstrings 
in the cases of higher dimensions too, i.e. \( D = 6, 10 \). In particular, the 
\( N = 2, D = 6 \) superstring is expected should be described by the nonlinear 
realization of the \( n = (4, 4) \) supersymmetric WZNW \( \sigma \)-model in which \( W \) 
is replaced by the double-analytical SF \( q^{(1,1)} \) representing twisted multiplet in 
the harmonic \( (4, 4) \) superspace \( \mathbb{H}^{4,4} \).

We hope return to this question in a forthcoming publications.

Acknowledgments

It is a great pleasure for me to express grateful to I. Bandos, E. Ivanov, 
S. Krivonos, A. Pashnev and D. Sorokin for interest to this work and valu-
able discussions.

References

1. F. Lund and T. Regge Phys. Rev. D 14 1524 (1976);
   R. Omnes Nucl. Phys. B 149 269 (1979);
   B.M. Barbashev and V.V. Nesterenko Commun. Math. Phys. 78 499 (1981).
   A. Zheltukhin Sov. J. Nucl. Phys. (Yadern.Fiz.) 33 1723 (1981); Theor. 
   Mat. Phys. 52 73 (1982); Phys. Lett. B 116 147 (1982); Theor. Mat. 
   Phys. 56 230 (1983).
2. I. Bandos, P. Pasti, D. Sorokin, M. Tonin and D. Volkov, Nucl. Phys. B 
   446 79-119 (1995) [hep-th/9501113].
3. I. Bandos, D. Sorokin and D. Volkov, Phys. Lett. B 372 77-82 (1996).
4. E. Ivanov and S. Krivonos, J. Physics A17 L671 (1984).
5. E. Ivanov and A. Kapustnikov, Nucl. Phys. B 333 439 (1990);
6. A.S. Galperin, P.S. Howe and K.S. Stelle, Nucl. Phys. B 368 281 (1992).
7. I. Bandos, E. Ivanov, A. Kapustnikov and S. Ulano, J. Math. Phys. 40 
   5203-5223 (1999) [hep-th/9810038].
8. S. Coleman, J. Wess and B. Zumino, Phys. Rev. 177 2239 (1969);
   C. Callan, S. Coleman, J. Wess and B. Zumino, Phys. Rev. 177 2247 
   (1969);
   D. Volkov, J. Elem. Part. Atom. Nucl. 4 3 (1973);
   V. Ogievsky, in Proc. X Winter School of Theor. Physics (Wroclaw, 
   1974), vol. 1, p. 117.
9. D.B. Fairlie and C.A. Manogue, Phys. Rev. D 36 475-479 (1987); N. Berkovits, Phys. Lett. B 241 497-502 (1990).
10. E. Ivanov, A.A. Kapustnikov, J. Physics A11 2375 (1978); J. Physics G8 167 (1982); Int. J. Mod. Phys. A 7 2153 (1992).
11. M. Rocek, A.A. Tseytlin, ITP-SB-98-68, Imperial/TP/98-99/013, hep-th/9811232.
12. E. Ivanov, S. Krivonos and V. Leviant, Nucl. Phys. B 304 601 (1988); O. Gorovoy and E. Ivanov, Nucl. Phys. B 381 394 (1992); E. Ivanov and A. Sutulin, Nucl. Phys. B 432 246-280 (1994); Nucl. Phys. B 483 531 (1997)(E); E. Ivanov and A. Sutulin, Class. Quantum Grav. 14 843-863 (1997).