The Barnett vs. Landau levels in the rotating two-dimensional electron gas

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(Dated: March 17, 2015)

We investigate magnetization induced by mechanical rotation, known as the Barnett effect, in the two-dimensional electron gas. The energy eigenvalues of the rotating system (Barnett levels) are non-degenerate, thereby differ from Landau levels in the presence of a magnetic field. The magnetic response caused by the coupling of the Barnett gauge field to both the electron spin and orbital degree of freedom is found to be paramagnetic. Surprisingly, rotation does not cause any charge redistribution, i.e. in the two-dimensional electron gas centrifugal forces are quantum mechanically suppressed.

PACS numbers: 71.10.Ca, 75.20.−g, 62.25.−g, 73.22.−f

I. INTRODUCTION

A close relationship between ferromagnetism and angular momentum was first proposed by Ampère and later elaborated in detail by Weber. However, the so-called molecular current hypothesis remained unproven until 1915, when Barnett managed to induce a net magnetization in a demagnetized ferromagnetic body by mechanical rotation. Shortly thereafter, Einstein and de Haas observed a mechanical rotation induced by the change of magnetization of a suspended ferromagnetic body.

The Barnett effect – the reorientation of the electron spin induced by mechanical rotation – was discovered at the dawn of quantum mechanics and provided first evidence for an anomalous $g$-factor of the electron. The Barnett effect can be understood in terms of classical mechanics: A gyroscopic wheel aligns its angular momentum with the axis of an impressed rotation in order to minimize energy. If the system is charged, the rotation generates a magnetization that in the rotating frame is caused by the "Barnett gauge field" for electrons

$$B_g = -\frac{\omega}{\gamma},$$

where $\omega/\omega$ denotes the rotation axis, with rotation velocity $\omega = |\omega|$, and $\gamma = g|e|/(2m)$. Superseded by electron spin resonance to measure $g$-factors, gyromagnetic methods have been largely forgotten in the last decades. However, in recent years, the miniaturization of electric circuits and mechanical systems revived some interest and its application on a burgeoning field of spintronics. An Einstein-de Haas type of experiment was discussed in a mesoscopic cantilever with a ferromagnetic tip. The Barnett field has been measured for macroscopic samples but recently also for nuclear spins through nuclear magnetic resonance.

Previous theoretical studies on the Barnett effect in ferromagnets were based on the Landau-Lifshitz-Gilbert equation. The equivalence of Barnett and Einstein-de Haas effects was shown in a model system consisting of a suspended magnetic wire by invoking the Onsager reciprocity relations. Matsuo et al. derived the Pauli-Schrödinger equation in a rotating frame, using the covariant Dirac equation as a starting point. The resulting Hamiltonian contains a spin-orbit interaction term augmented by a term due to mechanical rotation and led to the prediction of mechanically generated spin currents in systems rotating at non-relativistic speeds. The amplification of spin-rotation coupling via interband mixing in solids and spin current generation in nonmagnetic metals and semiconductors by application of surface acoustic waves were predicted as well.

Even in the non-relativistic limit, rotation acts not only on the spin but also on the orbit. In classical mechanics, it generates the Coriolis and centrifugal forces that persist of course in quantum mechanics. The Coriolis force can again be expressed in the rotating frame by an effective magnetic field that couples to the orbital angular momentum. Much work has been done on the effects of rotation on atomic Bose-Einstein condensates. A Hall effect induced by centrifugal forces in a rotating Corbino disk configuration of the two-dimensional electron gas (2DEG) has been discussed.

Here we discuss the Barnett gauge field acting on the spin and orbital motion of charged particles. While the Barnett gauge field is equivalent to the conventional (Zeeman) magnetic field when acting on spins, this is not the case for the orbital momentum. We illustrate this by comparing the eigenstates of the rotating 2DEG ("Barnett levels"), with the Landau levels in
an external magnetic field. In contrast to the diamagnetism induced by a real magnetic field, the rotation-induced response of spin-less charges is found to be paramagnetic. Furthermore, the charge distribution of the 2DEG is inert with respect to rotation, so there is no rotational Hall effect in the 2DEG.

The paper is organized as follows. First we discuss the dynamics of electrons in a rotating frame in Sec. II. In Sec. III, Barnett level wave functions and eigenenergies are derived and the difference between Barnett and Landau levels is shown by a perturbative analysis. In Sec. IV, we compute the ensuing magnetization in a rotating frame and prove its paramagnetism, while rotation to leading order does not disturb the charge density distribution. In Sec. V, we estimate the magnitude of the Barnett effect and rotation induced magnetizations. We conclude with a summary and outlook in Sec. V.

II. ROTATING FRAME OF REFERENCE

The dynamics of electrons in a rotating frame is governed by the covariant Dirac equation, since Einstein’s principle of equivalence states that gravitation is locally indistinguishable from inertia due to acceleration of the reference frame. Nevertheless, condensed matter systems that have been subjected to rotation such as graphene flakes [22] and atomic gases [20] as well as astrophysical objects such as neutron stars [23] have radial velocities far below the speed of light. We therefore focus here on non-relativistic rotations of a quantum state \( |\psi\rangle \) by an angle \( \phi \) as represented by a unitary operator \( \hat{R} \),

\[
|\psi(\phi)\rangle = \hat{R}(\phi)|\psi(0)\rangle,
\]

Operators \( \hat{A} \) transform under rotation as

\[
\hat{A}(\phi) = \hat{R}(\phi)\hat{A}(0)\hat{R}^\dagger(\phi),
\]

such that the change of the coordinate system does not change the expectation value of a quantum state. The operator for rotation around the \( z \)-direction is

\[
\hat{R}(\phi) = e^{-\frac{i}{\hbar}J_z\phi},
\]

where the total angular momentum operator is \( \hat{J}_z \), if the rotating system itself is non-relativistic. Here \( \hat{J}_z = \hat{L}_z + \hat{s}_z \) is the sum of orbital \( \hat{L}_z \) and spin \( \hat{s}_z \) angular momentum operators.

Steady rotation implies a time-dependence that can be removed by a transformation into a rotating frame of reference. An orbital coordinate under a rigid rotation can be written \( \Theta(t) = \Theta_{rot} + \phi(t) \), where \( \phi(t) \) is the angle by which the system has been rotated and \( \Theta_{rot} \) denotes a constant value in the rotating frame of reference. Using Eqs. (2,3), the state and Hamiltonian in the inertial or laboratory frame are \( |\psi(\Theta)\rangle = \hat{R}(\phi)|\psi(\Theta_{rot})\rangle \) and \( \hat{H}(\Theta) = \hat{R}(\phi)\hat{H}_0(\Theta_{rot})\hat{R}^\dagger(\phi) \) respectively. The Schrödinger equation \( i\hbar|\dot{\psi}(\Theta)\rangle = \hat{H}(\Theta)|\psi(\Theta)\rangle \) then reads

\[
i\hbar \frac{d}{dt} \left( \hat{R}(\phi)|\psi(\Theta_{rot})\rangle \right) = \hat{R}(\phi)\hat{H}_0(\Theta_{rot})\hat{R}^\dagger(\phi)\hat{R}(\phi)|\psi(\Theta_{rot})\rangle. \tag{5}
\]

The left hand side becomes

\[
i\hbar \frac{d}{dt} \left( \hat{R}(\phi)|\psi(\Theta_{rot})\rangle \right) = i\hbar \left[ \frac{d}{dt} \left( \hat{R}(\phi) \right) |\psi(\Theta_{rot})\rangle + \hat{R}(\phi) \frac{d}{dt} |\psi(\Theta_{rot})\rangle \right] = i\hbar \left[ \frac{i}{\hbar} \hat{R}(\phi)\hat{J}_z\frac{d\phi}{dt} |\psi(\Theta_{rot})\rangle + \hat{R}(\phi) \frac{d}{dt} |\psi(\Theta_{rot})\rangle \right] = i\hbar \hat{R}(\phi) \frac{d}{dt} |\psi(\Theta_{rot})\rangle + \hat{R}(\phi)\hat{J}_z\frac{d\phi}{dt} |\psi(\Theta_{rot})\rangle. \tag{6}
\]

Similarly, the right hand side is \( \hat{R}(\phi)\hat{H}_0(\Theta_{rot})|\psi(\Theta_{rot})\rangle \). Applying \( \hat{R}^\dagger(\phi) \) from the left side, we arrive at the Schrödinger equation in the rotating frame

\[
i\hbar \frac{d}{dt} |\psi(\Theta_{rot})\rangle = \left( \hat{H}_0(\Theta_{rot}) - \hat{J}_z\frac{d\phi(t)}{dt} \right) |\psi(\Theta_{rot})\rangle. \tag{7}
\]

The total Hamiltonian \( \hat{H}_0 - \hat{J}_z\frac{d\phi(t)}{dt} \) is still time-dependent when the rotation velocity \( \omega = \frac{d\phi(t)}{dt} \) is, but we focus here on \( \dot{\omega} = 0 \).

When a system is invariant with respect to rotation around an axis, the Schrödinger equation is separable in the spatial coordinates: the motion in the direction of the rotation axis is unaffected by the rotation and the problem becomes effectively two-dimensional. We therefore consider here a circular disk of a two-dimensional free electron gas (2DEG) with radius \( R \).
in the $x$-$y$-plane that rotates around the $z$-axis with angular frequency $\omega$. In Section V, we will argue that the 2DEG behaves qualitatively different from the 3DEG under rotation. In the rotating frame of reference

$$\hat{H}_{\text{rot}} = -\frac{\hbar^2 \nabla^2}{2m} - \omega \hat{J}_z$$

(8)

and the Schrödinger equation in polar coordinates reads

$$\left[ -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \right) - \omega \left( \frac{\hbar}{i} \frac{\partial}{\partial \varphi} + \hat{s}_z \right) \right] \psi(r, \varphi, \sigma) = E\psi(r, \varphi, \sigma).$$

(9)

Since the operator $\hat{s}_z$ commutes with the Hamiltonian, we can separate the wave function into orbital and spin part as $\psi(r, \varphi, \sigma) = \psi(r, \varphi)|\sigma\rangle$ where $|\sigma\rangle = (\sigma \hbar/2)|\sigma\rangle$ denotes the spinor wave functions with $\hat{s}_z|\sigma\rangle = (\sigma \hbar/2)|\sigma\rangle$:

$$\left[ -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \right) - \omega \left( \frac{\hbar}{i} \frac{\partial}{\partial \varphi} + \frac{\hbar \sigma}{2} \right) \right] \psi(r, \varphi)|\sigma\rangle = E\psi(r, \varphi)|\sigma\rangle.$$  

(10)

$\psi(r, \varphi) = f(r)g(\varphi)$ should be single-valued, therefore $(\ln g)' = \frac{m}{n}$, where $n$ is an integer, leading to the Bessel equation

$$\left[ -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{n^2}{r^2} \right) - \hbar \omega \left( n + \frac{\sigma}{2} \right) \right] f_n(r)e^{in\varphi}|\sigma\rangle = E f_n(r)e^{in\varphi}|\sigma\rangle.$$  

(11)

$$r^2 f''(r) + r f'(r) + \left[ \frac{2m}{\hbar^2} \left( E + \hbar \omega \left( n + \frac{\sigma}{2} \right) \right) r^2 - n^2 \right] f_n(r) = 0,$$

(12)

solved by $f_n(r) = c_{nk}J_n(r\sqrt{\frac{2m}{\hbar^2} \left( E + \hbar \omega \left( n + \frac{\sigma}{2} \right) \right)})$, where $c_{nk}$ is a normalization constant and $J_n$ the $n$th-order Bessel function of the first kind with zeros at $x = j_{nk}$, where $k$ denotes the $k$th node. Requiring that the wave function is regular at the origin by $f(r = R) = 0$, we arrive at the eigenenergies of the rotating 2DEG or Barnett levels,

$$E_{nk\sigma} = \frac{\hbar^2 j_{nk}^2}{2mR^2} - \hbar \omega \left( n + \frac{\sigma}{2} \right)$$

(13)

with normalized wave functions,

$$\psi_{nk\sigma}(r, \varphi) = \frac{e^{in\varphi}}{\sqrt{\pi}R J_{n+1}(j_{nk})} J_n\left( j_{nk} \frac{r}{R} \right) |\sigma\rangle,$$

(14)

where $n$ are the angular momentum quantum numbers and $\sigma$ are the radial ones. This corresponds to the energy spectrum for spin-less, non-relativistic particles in the rotating frame with an external magnetic field. In contrast to the energies, the wave functions are not affected by rotation since the Hamiltonian (including boundary conditions) commutes with the rotation operator. Rotation lifts the degeneracy of states with opposite spin and orbital angular momentum with respect to the axis of rotation thereby causing the “Barnett” energy splitting $\Delta E_{nk\sigma} = \hbar \omega (n + \frac{\sigma}{2})$. To illustrate this effect, we compare the energy levels and Fermi surface for a stationary as well as rotating ($\omega = 10 \text{GHz}$) disk of a 2DEG with $R = 1 \mu\text{m}$ and $m = m_e$ in Fig. [1] where $m_e$ is the mass of the free electron. We observe that the rotation tilts the energy surface towards the positive direction of $n$ axis. The redistribution of states for a constant Fermi energy implies that rotation imparts angular momentum to the electrons. Each time a state with $(-n < 0, \sigma)$ leaves the Fermi sea, the state $(n, \sigma)$ enters it. The result is an overall nearly rigid rotation of the many-particle ground state with the impressed rotation of the lattice, without any deformation due to the centrifugal force.

It is instructive to compare Barnett with Landau levels, the eigenstates of 2DEG in a magnetic field $B = (0, 0, B_0)$. In the symmetric gauge $\mathbf{A} = (B \times r)/2$ the 2DEG Hamiltonian reads

$$\hat{H}_{\text{LL}} = \hat{H}_0 + \hat{H}_p = \frac{\mathbf{p}^2}{2m} + \frac{|e|}{2m} B_0 \left( \hat{L}_z + g_s \hat{s}_z \right) + \frac{|e|^2 B_0^2}{2m} \frac{r^2}{4}.$$  

(15)

The Landau levels are characterized by a discrete and highly degenerate energy spectrum $E_{n\mu}^{LL} = \hbar \omega_c (n + 1/2 + g_s \sigma/4)$, where $\omega_c = 2\mu_B B_0/\hbar$ is the cyclotron frequency and $\mu_B = |e|\hbar/(2m)$ the Bohr magneton. In the Barnett spectrum, on the other hand, each quantum number $n$ is associated with an infinite number of states with a non-degenerate continuum of states. We illustrate this difference between Barnett and Landau wave functions by perturbation theory when $\lambda = R^4 B_0^2 |e|^2/\hbar^2 \ll 1$
FIG. 1: (Color online) Barnett levels for a 2DEG as an energy surface, plotted as a function of quasicontinuous quantum numbers. The blue/red plane corresponds to the energy surface with/without rotation. The green line represents the Fermi surface at an energy of 2.5 meV.

$\lambda < 1$ for $R = 1 \mu m$ and $B_0 \lesssim 0.5 \text{ mT}$, where we treat $\hat{H}_p = |e|^2 B_0^2 \hat{\rho}^2 / (8m)$ as a perturbation to $\hat{H}_0$. The eigenstates of $H_0$ Eq. (14) have the energies

$$E^{(0)}_{nk\sigma} = \frac{\hbar^2 J^2_{|n|k}}{2mR^2} + \mu_B B_0 \left(n + \frac{g_s \sigma}{2}\right).$$

Using (14), the first-order correction to $E^{(0)}_{nk\sigma}$ is

$$E^{(1)}_{nk\sigma} = \langle \psi^{(0)}_{nk\sigma}(r, \varphi) | \hat{H}_p | \psi^{(0)}_{nk\sigma}(r, \varphi) \rangle = \frac{|e|^2 B_0^2}{2m} \frac{\langle \psi^{(0)}_{nk\sigma}(r, \varphi) | \hat{\rho}^2 | \psi^{(0)}_{nk\sigma}(r, \varphi) \rangle}{\langle \psi^{(0)}_{nk\sigma}(r, \varphi) | \hat{\rho}^2 | \psi^{(0)}_{nk\sigma}(r, \varphi) \rangle}$$

where, with $\tilde{r} = r/R$

$$\langle \psi^{(0)}_{nk\sigma}(r, \varphi) | \hat{\rho}^2 | \psi^{(0)}_{nk\sigma}(r, \varphi) \rangle = \frac{1}{\pi R^2} \int_0^{2\pi} e^{in\varphi} e^{-in\varphi} \int_0^R dr r^3 J^2_{|n|k} \left( \frac{\tilde{r}^2}{R^2} \right) \frac{J^2_{|n|k+1}}{J^2_{|n|k}} d\varphi$$

$$= \frac{2\pi R^4}{\pi R^2} \frac{J^2_{|n|k+1}}{J^2_{|n|k}} \int_0^1 \tilde{r}^3 J^2_{|n|k} \left( \frac{\tilde{r}^2}{R^2} \right) d\tilde{r}$$

$$E^{(1)}_{nk\sigma} = \frac{|e|^2 B_0^2 R^2}{4m} \frac{J^2_{|n|k+1}}{J^2_{|n|k}} \left( \frac{\tilde{r}^2}{R^2} \right) = \frac{R^2 m}{\hbar^2} (\mu_B B_0)^2 \kappa_{kk}.$$
introducing
\[
\kappa_{kq} = \frac{\int_0^1 d\hat{r} \hat{r} \cdot J_{|n|}(\hat{r}) J_{|q|}(\hat{r})}{J_{|n|+1}(\hat{r}) J_{|q|+1}(\hat{r})}.
\]  

(20)

The first-order correction to the wave functions \( \psi_{nks}^{(0)}(r, \varphi) \) \(^{14}\) is
\[
\psi_{nks}^{(1)}(r, \varphi) = \sum_{m\sigma \sigma_1} \frac{1}{E_{nks}^{(0)} - E_{ms\sigma_1}^{(0)}} \langle \psi_{ms\sigma_1}^{(0)} | \hat{H}_p | \psi_{nks}^{(0)} \rangle \psi_{ms\sigma_1}^{(0)}(r, \varphi)
\]
\[
= \sum_{m\sigma \sigma_1} \frac{\delta_{m,n} \delta_{\sigma,\sigma_1}}{E_{nks}^{(0)} - E_{ms\sigma_1}^{(0)}} \langle \psi_{ms\sigma_1}^{(0)} | \hat{H}_p | \psi_{nks}^{(0)} \rangle \psi_{ms\sigma_1}^{(0)}(r, \varphi)
\]
\[
= \sum_{s \neq k} \frac{1}{E_{nks}^{(0)} - E_{ns\sigma}^{(0)}} \langle \psi_{ns\sigma}^{(0)} | \hat{H}_p | \psi_{nks}^{(0)} \rangle \psi_{ns\sigma}^{(0)}(r, \varphi).
\]

(21)

With \(^{16}\)
\[
E_{nks}^{(0)} - E_{ns\sigma}^{(0)} = \frac{\hbar^2}{2mR^2} \left( j_{|n|k}^2 - j_{|n|s}^2 \right)
\]

(22)

and analogous to Eq. \(^{19}\)
\[
\langle \psi_{ns\sigma}^{(0)} | \hat{H}_p | \psi_{nks}^{(0)} \rangle = \frac{R^2 B_0^2 |e|^2}{4m} \kappa_{ks},
\]

(23)

the first-order correction reads
\[
\psi_{nks}^{(1)}(r, \varphi) = \sum_{s \neq k} \frac{2mR^2}{\hbar^2} \frac{1}{j_{|n|k}^2 - j_{|n|s}^2} \frac{R^2 B_0^2 |e|^2}{4m} \kappa_{ks} \frac{e^{i\varphi}}{\sqrt{\pi} R} \frac{1}{J_{|n|+1}(\hat{r})} \left( \frac{J_{|n|s} r}{R} \right) |\sigma\rangle
\]
\[
= \frac{e^{i\varphi}}{\sqrt{\pi} R} \sum_{s \neq k} \left( \frac{R^2 B_0 |e|}{\hbar} \right)^2 \frac{2}{j_{|n|k}^2 - j_{|n|s}^2} \frac{J_{|n|s} r}{2j_{|n|s} |\sigma\rangle}.
\]

(24)

This leads to the approximated Landau levels
\[
\psi_{nks}^{LL} \left( \frac{r}{R}, \varphi \right) = \psi_{nks}^{(0)}(r, \varphi) + \psi_{nks}^{(1)}(r, \varphi)
\]
\[
= \frac{e^{i\varphi}}{\sqrt{\pi} R} \left[ \frac{J_{|n|s} r}{J_{|n|+1}(\hat{r})} + \sum_{s \neq k} \left( \frac{R^2 B_0 |e|}{\hbar} \right)^2 \frac{J_{|n|s} r}{j_{|n|s}^2} \frac{1}{2j_{|n|s} |\sigma\rangle} \right]
\]

(25)

with energies
\[
E_{nks} = \frac{\hbar^2 J_{|n|s}^2}{2mR^2} + \mu_B B_0 \left( n + \frac{g_s \sigma}{2} \right) + \frac{R^2 m}{\hbar^2} (\mu_B B_0)^2 \kappa_{kk}.
\]

(26)

In a 2DEG with the Fermi energy \( E_F \gg \hbar^2/(2mR^2) \), states with \( n \gg 1 \) are occupied. The difference between the probability distributions Eqs. \(^{14}\) and \(^{25}\) is plotted for \( n = 100 \) in Fig. 2 for the four lowest energy states at weak magnetic fields with \( \lambda = 0.01 \) and \( R = 1 \) \( \mu m \). The diamagnetic term confines the electrons so their density near the edge is smaller for Landau than for Barnett levels; \( B \) and \( \omega \) act differently on the electrons: the magnetic field contracts the orbitals due to the absence of centrifugal forces.

### III. SPIN AND CHARGE REDISTRIBUTION BY ROTATION

We derived above that for a circular 2DEG the wave functions are not modified by an arbitrary rotation around a perpendicular axis. However, the physical properties of many-particle systems are modulated by the dependence of the eigenenergies on the angular velocity and the subsequent repopulation of states. In this section we investigate the changes in the ground state charge and spin distributions by an impressed rotation.
FIG. 2: (Color online) Difference between the probability distribution of the lowest energy eigenstates with $\langle L_z \rangle = 100 \hbar$ for a (rotating) 2DEG and a 2DEG with applied magnetic field.

The electron density change by the rotation is given by

$$\Delta \rho (r) = \sum_{nk} |\psi_{nk}(r)|^2 \sum_{\sigma} \left( f_{nk\sigma}^{(\omega)} - f_{nk\sigma}^{(0)} \right),$$  \hspace{1cm} (27)

where $f_{nk\sigma}^{(\omega)}$ and $f_{nk\sigma}^{(0)}$ are Fermi-Dirac distribution functions with and without rotation, respectively. From (14) the orbital densities are

$$|\psi_{nk}(r)|^2 = \frac{1}{\pi R^2 J^2_n} \frac{J^2_{n+k}}{R r}.$$  \hspace{1cm} (28)

When the Barnett splitting and thermal energy $k_B T$ is small compared to the kinetic energy of the electrons $E_{nk}^{(0)} = \hbar^2 j_{n+k}^2 / (2mR^2)$

$$f_{nk\sigma}^{(\omega)} - f_{nk\sigma}^{(0)} \approx \hbar \omega \left( n + \frac{\sigma}{2} \right) \delta \left( E_{nk}^{(0)} - E_F \right).$$  \hspace{1cm} (29)

Then the density is modulated as

$$\Delta \rho (r) = \hbar \omega \sum_{nk} |\psi_{nk}(r)|^2 \sum_{\sigma} \left( n + \frac{\sigma}{2} \right) \delta \left( E_{nk}^{(0)} - E_F \right) = 0$$  \hspace{1cm} (30)
since the summand is an odd function of \( n \). The absence of a charge polarization is at odds with the naive expectation that centrifugal forces charge-polarize the system (see Section V for a discussion) and also implies that there are no Hartree corrections to the Barnett level spectrum for charged particles. While derived here for small rotation frequencies, it is easily seen that \( \Delta \rho (r) = 0 \) is not a perturbative result, because the Barnett splitting is strictly linear, while the wave functions do not depend on the rotation at all.

The magnetization \( \langle m_z \rangle \), where \( \langle \ldots \rangle \) refers to grand-canonical ensemble averaging that is induced by an applied magnetic field \( B \) in the z-direction can be obtained from the grand canonical potential \( \Omega \) as \( \langle m_z \rangle = -(\partial \Omega / \partial B)_{T, \mu} \). We may derive the orbital magnetization by rotation analogously. The work needed to increase the rotation frequency by \( d\omega \) is given by

\[
\delta W = \langle (H(\omega + d\omega) - H(\omega)) \rangle = -\langle J_z \rangle d\omega,
\]

where in the last step we used the Hamiltonian in the rotating frame of reference, Eq. (8). The total angular and magnetic momenta are related by

\[
\langle m_z \rangle = -\gamma_J \langle J_z \rangle,
\]

where \( \gamma_J = g_J \mu_B / h \) with Bohr magneton \( \mu_B \) and \( g_J \) the Landé g-factor of the total angular momentum. Thus

\[
\delta W = \frac{1}{\gamma_J} \langle m_z \rangle d\omega
\]

and the complete differential of the grand canonical potential becomes

\[
d\Omega = -SdT - N d\mu + \frac{1}{\gamma_J} \langle m_z \rangle d\omega.
\]

Therefore,

\[
\langle m_z \rangle = \frac{\gamma_J}{\pi R^2} \frac{\partial \Omega}{\partial B}_{T, \mu}.
\]

We may now compute the Barnett area magnetization density, \( M_z = \langle m_z \rangle / \pi R^2 \), in the 2DEG with energy eigenvalues \( E_{nk\sigma} \) given by Eq. (13). Then,

\[
M_z = \frac{\gamma_J}{\pi R^2} \left( \frac{\partial \Omega}{\partial B}_{T, \mu} \right)_{T, \mu} = \gamma_J \frac{n}{\pi R^2} \frac{\partial}{\partial \omega} \ln \left( 1 + e^{-\beta(E_{nk\sigma} - \mu)} \right)_{T, \mu}
\]

\[
= -k_B T \gamma_J \frac{n}{\pi R^2} \sum_{nk\sigma} \frac{\partial}{\partial \omega} \ln \left( 1 + e^{-\beta(E_{nk\sigma} - \mu)} \right) = -\gamma_J \frac{h}{\pi R^2} \sum_{nk\sigma} \left( n + \frac{\sigma}{2} \right) f(E_{nk\sigma}),
\]

where \( f(E_{nk\sigma}) = f(\omega_{nk\sigma}) \). We evaluate the sum over states at low temperatures and small rotation frequencies, i.e., \( \hbar \omega n \ll \hbar^2 j_{n|k}^2 / (2m R^2) \), by a Taylor expansion of \( f \) around \( \hbar^2 j_{n|k}^2 / (2m R^2) \). Since the zeroth order term and terms proportional to \( \sigma n \) in the first order term drop out when summing over \( n \) and \( \sigma = \pm 1 \), we can write

\[
M_z \approx -\gamma_J \hbar \sum_{nk\sigma} \left( n + \frac{\sigma}{2} \right) \left[ f \left( \frac{\hbar^2 j_{n|k}^2}{2m R^2} \right) + \delta \left( \frac{\hbar^2 j_{n|k}^2}{2m R^2} - \mu \right) \hbar \omega \left( n + \frac{\sigma}{2} \right) \right]
\]

\[
= -\gamma_J \hbar \sum_{nk\sigma} \left( n + \frac{\sigma}{2} \right) f \left( \frac{\hbar^2 j_{n|k}^2}{2m R^2} \right) - \gamma_J \frac{h^2 \omega}{\pi R^2} \sum_{nk\sigma} \delta \left( \frac{\hbar^2 j_{n|k}^2}{2m R^2} - \mu \right) \sigma n
\]

\[
\approx \frac{\gamma_J}{\pi R^2} \sum_{nk\sigma} \delta \left( \frac{\hbar^2 j_{n|k}^2}{2m R^2} - \mu \right) \hbar^2 \omega \left( n^2 + \frac{1}{4} \right)
\]

\[
= \gamma_J M_z^{(1)}.
\]

Splitting the summation into

\[
M_z^{(1)} = -\gamma_J \hbar \sum_{nk\sigma} \delta \left( \frac{\hbar^2 j_{n|k}^2}{2m R^2} - \mu \right)
\]
Similarly, using the second one is proportional to the energy density of states per unit area of the 2DEG

$$D(E) = \frac{1}{\pi R^2} \sum_{nk\sigma} \delta \left( \frac{h^2 j_{nk}^2}{2mR^2} - E \right).$$

Moreover, the Bessel function zeros obey $|n| < j_{|n|1} < j_{|n|2} < \cdots < j_{|n|k}$ and for large $|n|

$$j_{n,1} \approx n + cn^{1/3} + O(n^{-1/3}),$$

where $c \approx 1.856^{24}$ Denoting the highest occupied angular momentum quantum number with $|n_{max}|$ and using $\mu \gg \hbar^2/(2mR^2)$ we find

$$|n_{max}| \approx \sqrt{\frac{2mR^2\mu}{\hbar^2}} \gg 1$$

and

$$\sum_{n=-\infty}^{\infty} \sum_{k=1}^{\infty} n^2 \sum_{k=1}^{\infty} \delta \left( \frac{h^2 j_{nk}^2}{2mR^2} - \mu \right) \approx 2 \sum_{n=0}^{\infty} n^2 \sum_{k=1}^{\infty} \frac{1}{2\sqrt{\mu}} \delta \left( \frac{h}{\sqrt{2mR^2}} j_{nk} - \sqrt{\mu} \right) = \sqrt{\frac{2mR^2}{\hbar^2\mu}} \sum_{n=0}^{\infty} n^2 \sum_{k=1}^{\infty} \delta \left( j_{nk} - \sqrt{\frac{2mR^2\mu}{\hbar^2}} \right).$$

The factor

$$\rho_n(x) = \sum_{k=1}^{\infty} \delta (j_{nk} - x)$$

is the density of zeros of the Bessel function of the first kind in the interval $[x, x + dx]$. For large $n^{28}$

$$\rho_n(x) \approx \frac{1}{\pi} \sqrt{1 - \frac{n^2}{x^2}}$$

and

$$\sqrt{\frac{2mR^2}{\hbar^2\mu}} \sum_{n=0}^{\infty} n^2 \frac{1}{2\sqrt{\mu}} \int_0^1 dx x^2 \sqrt{1 - x^2} = 1 \sqrt{\frac{mR^2}{\hbar^2}} \mu$$

using

$$\int_0^1 dx x^2 \sqrt{1 - x^2} = \lim_{N \to \infty} \sum_{i=0}^{N} x_i \sqrt{1 - x_i^2} = \lim_{N \to \infty} \sum_{i=0}^{N} \frac{1}{N} \left( \frac{i}{N} \right)^2 \sqrt{1 - \left( \frac{i}{N} \right)^2} \approx \frac{1}{n_{max}^2} \sum_{n=0}^{\infty} n^2 \sqrt{1 - \frac{n^2}{n_{max}^2}}.$$
and according to [40],

\[ M_z^{(2)} = -\frac{\gamma_J}{4\pi R^2} 2\hbar^2 \omega \frac{mR^2}{2\hbar^2} = -\frac{\gamma_J \omega m}{4\pi} \]  \hspace{1cm} (51)

Finally, at low temperatures,

\[ M_z = -\frac{\gamma_J \omega m}{4\pi} \left( 1 + \frac{2mR^2}{\hbar^2} \mu \right) = -\frac{\gamma_J \omega m}{4\pi} \left( 1 + \frac{2mR^2}{\hbar^2} E_F \right) \]

\[ = -\frac{\gamma_J \omega m}{4\pi} \left( 1 + \frac{2mR^2}{\hbar^2} \frac{\pi h^2 N}{m\pi R^2} \right) = -\frac{\gamma_J \omega m}{4\pi} (1 + 2N), \]  \hspace{1cm} (52)

where \( N \) is total number of electrons. The Barnett magnetization is thus paramagnetic, i.e., pointing in direction of the Barnett gauge field \(-\omega/\gamma_J\).

The first term in Eq. (52) stems from the coupling of the electron spin to the Barnett gauge field leading to Pauli paramagnetism. Defining the Pauli susceptibility by \( \chi_{\text{Pauli}} R_0 \),

\[ \chi_{\text{Pauli}} = \mu_B D(\mu) = \frac{1}{\pi} \eta \left( \frac{\mu B}{\hbar} \right)^2 \]  \hspace{1cm} (53)

and

\[ M_z |_{\text{Pauli}} = \chi_{\text{Pauli}} \left( \frac{g_J}{2} \right)^2 \left( \frac{-\omega}{\gamma_J} \right), \]  \hspace{1cm} (54)

where \( g_J = \gamma_J \hbar/\mu_B \).

Applying a magnetic field to an electron gas gives rise to Landau diamagnetism with susceptibility \( \chi_{\text{Landau}} = -\frac{\chi_{\text{Pauli}}}{3} \).

In the case of a rotating free electron gas the second term in Eq. (52) stems from the orbital motion of the electrons, which is also a paramagnetic response to the Barnett gauge field \(-\omega/\gamma_J\). The orbital contribution \( M_z |_{\text{orbital}} \) to the magnetization \( M_z \) of Eq. (52) can be written as \( M_z |_{\text{orbital}} = \chi_{\text{orbital}} (\omega/\gamma_J) \) with

\[ \chi_{\text{orbital}} = -\frac{3g_J^2 m \mu R^2}{2\hbar^2} \chi_{\text{Landau}} = \frac{g_J^2 m \mu R^2}{2\hbar^2} \chi_{\text{Pauli}}. \]  \hspace{1cm} (55)

IV. ORDER OF MAGNITUDE

The magnitude of the Barnett energy splitting is \( \Delta E_{n,\sigma} = \hbar \omega (n + \sigma/2) \), where at \( \omega = 1 \) MHz for electrons with \( m = m_e \), \( \hbar \omega = 7 \times 10^{-10} \text{eV} \), which is almost three orders of magnitude below the hyperfine splitting of hydrogen. Large (effective) masses and radii increase significance of the effect. The significance of the Barnett splitting in Eq. (13) can be also be expressed as the ratio of the the Barnett splitting and kinetic energy term

\[ \frac{\hbar \omega |n|}{\hbar^2 j_{\text{orbital}}} \ll \frac{mR^2 \omega}{\hbar |n|} = \frac{\eta}{|n|} \]  \hspace{1cm} (56)

since \(|n| < j_{|n|} \), implying that the Barnett splitting is a small correction to the eigenenergies as long as the dimensionless parameter \( \eta \ll 1 \). In a 2DEG with \( R = 1 \mu \text{m} \) and rotating at 1 MHz, \( \eta = 0.01 \). Nevertheless for other physical systems with large radii or massive particles, the Barnett splitting cannot necessarily be ignored. For neutron stars with \( R \approx 10 \text{ km} \) and \( \omega \approx 4500 \text{ Hz} \), the fastest spinning case observed so far, \( \eta = 7 \times 10^{-10} \). For fermionic atomic gases such as \(^{40}\)K in a trap with \( R = 1 \mu \text{m} \) and rotating at 1 MHz, \( \eta \approx 660 \). Note that these systems are not two-dimensional, however, and the orbital contribution to the rotation–induced paramagnetism is generated only by charged particles. For neutral systems, such as neutron gas, the paramagnetism comes from reorientation of the nucleonic spin, but charged impurities would generate large (Oersted) magnetic fields.

The Barnett gauge field \( B_k = -\omega/\gamma_J \) is also pretty small for rotational frequencies achievable in the laboratory. With \( \gamma_J / g_J \approx 10^{11} \) (Ts) \(^{-1} \) and for \( \omega = 1 \) MHz, \( B_k \approx 10 \mu \text{T} \). As a consequence, for \( R = 1 \mu \text{m} \) the achievable Barnett area magnetization density can be estimated to be

\[ |M_z^{(1)}| \approx \left( \frac{\omega}{1 \text{MHz}} \right) \left( \frac{m}{m_e} \right)^2 \left( \frac{\mu}{10 \text{meV}} \right) \cdot 10^{14} \frac{\mu B}{\text{m}^2} \]  \hspace{1cm} (57)
and

\[ |M_z^{(2)}| \approx \left( \frac{\omega}{1 \text{ MHz}} \right) \left( \frac{m}{m_e} \right) \times 10^9 \frac{\mu_B}{m^2}, \]

(58)

where \( m_e \) is the mass of the free electron. Therefore, for electrons and even more so for heavier (charged) particles, the orbital magnetization dominates.

**V. DISCUSSION AND CONCLUSIONS**

We explored the differences between the “Barnett gauge field” \( \mathbf{B}_s = -\omega/\gamma \) due to rotation and an applied external magnetic field for the two-dimensional electron gas. The “Barnett levels” in the rotating system differ from the Landau levels generated by an applied magnetic field. Whereas the latter are highly degenerate, the Barnett levels are not. In addition, applied magnetic fields induce a diamagnetic response, which leads – in contrast to rotation – to a contraction of the orbitals towards the edge of the disk.

We find that \( \mathbf{B}_s \) does not induce a charge polarization. This is counter intuitive, since in a classical rotating system the particles are forced to the edges by the centrifugal forces. We can understand the suppression of centrifugation as an effect of size quantization and Pauli principle. The gauge field operator commutes with the Hamiltonian, so that wave functions are not modified by the rotation; only the occupation numbers are affected. Under rotation the degeneracy of the orbital angular momentum states is broken and undergo a linear Zeeman type of splitting. But for each state that leaves the Fermi sea, another one enters it under rotation. The ground state of the rotating system is therefore a rigidly rotating electron gas with total angular momentum equal to that of the underlying lattice. This conclusion is not limited to perturbation theory, but holds for general \( \omega \). Electrons cannot be accelerated to the edges since those states are occupied, therefore forbidden for other electrons. While focusing here on a cylindrical well confinement potential, the theorem holds for any velocity-independent potential with axial symmetry. For bosons, the situation is quite different since particles tend to accumulate at the lowest energy state that with increasing rotation shifts to the edges, however.

The absence of effects of centrifugal forces on the 2DEG can be reconciled with the correspondence principle by realizing that two-dimensional systems are a mathematical idealization for particles in a finite perpendicular confinement potential. When only the lowest quantum-confined state is occupied, the system behaves in many respects two-dimensionally. However, when rotational velocities become sufficiently large, the Barnett splitting leads to the occupation of higher subband states. These have a relatively large angular momentum and electron density shifted to the edges. Higher subband occupation, i.e. three dimensionality, is required in order to centrifuge 2D electrons. Also, when the rotation axis is not normal to the 2DEG, deformation of the disk.

The absence of charge polarization implies that there is no rotational Hall effects in the 2DEG. Ref. [22] showed that the quantum Hall effect is not affected by the rotation, which is consistent with our conclusion that there is no Hall effect at zero external magnetic field.

The main observable effect of rotation of charged particle systems is the Oersted magnetic field induced by the moving charges in the laboratory frame. It is intuitively clear that these should be paramagnetic, i.e. the magnetization scales linearly with rotation velocity. Conventional 2DEG systems are globally electro-neutral with only small and system-specific deviations of the positive and negative charge distributions at the edges. The magnetic moments of the electrons and the underlying background cancel therefore to a large extent. The magnetic moment generated by the electrons can be measured by subjecting the 2DEG to an oscillating motion with frequency higher than the electron scattering relaxation rate. The effects predicted here are small and it is not clear whether they can be observed at all on Earth for macroscopic electronic systems. Analogues of the orbital Barnett effect might be observable in rapidly spinning charged atomic gases, molecules and nanoparticles, or astrophysical objects.

This work was supported by Grants-in-Aid for Scientific Research (Grant Nos. 25247056, 25220910, and 26103006) from the JSPS, the FOM (Stichting voor Fundamenteel Onderzoek der Materie), the ICC-IMR, EU-FET Grant InSpin 612759, and DFG Priority Programme 1538 “Spin-Caloric Transport” (BA 2954/2).

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