Gravitational Waveforms for Precessing, Quasi-circular Binaries via Multiple Scale Analysis and Uniform Asymptotics: The Near Spin Alignment Case

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We calculate analytical gravitational waveforms in the time- and frequency-domain for precessing quasi-circular binaries with spins of arbitrary magnitude, but nearly aligned with the orbital angular momentum. We first derive an analytical solution to the precession equations by expanding in the misalignment angle and using multiple scale analysis to separate timescales. We then use uniform asymptotic expansions to analytically Fourier transform the time-domain waveform, thus extending the stationary-phase approximation, which fails when precession is present. The resulting frequency-domain waveform family has a high overlap with numerical waveforms obtained by direct integration of the post-Newtonian equations of motion and discrete Fourier transformations. Such a waveform family lays the foundations for the accurate inclusion of spin precession effects in analytical gravitational waveforms, and thus, it can aid in the detection and parameter estimation of gravitational wave signals from the inspiral phase of precessing binary systems.

I. INTRODUCTION

Gravitational waves are a promising new tool for astrophysics, capable of bringing breakthroughs in our understanding of the Universe. Currently, an array of ground-based interferometers are undergoing upgrades that should lead to the first direct detection of gravitational waves within this decade [1, 2]. A project for a space-based interferometer [3] is also under way, and could be operational in the next decade.

The likelihood of detecting gravitational waves improves when reliable and efficient waveform models are available for use in the data analysis [4]. Systems with spins, unless they are exactly aligned or anti-aligned with the orbital angular momentum, will undergo a secular precession of the spins and of the orbital plane [5]. This precession will impact the waveforms, and it is important to take it into account to properly extract the spins of the binary components, as well as to break degeneracies between different system parameters [6, 7].

The purpose of this paper is to develop accurate analytical waveforms for spinning and precessing compact binary, quasi-circular inspirals. Currently, the most accurate waveforms are numerical, time-domain solutions of the PN equations [6, 7] that are then numerically Fourier transformed. Such numerical solutions, however, can be rather computationally expensive, especially when a large number of templates are needed, as is the case for spinning systems. We will here employ novel mathematical techniques (multiple scale analysis and uniform asymptotic expansions) [8] to produce analytic, Fourier-domain, waveform families that accurately reproduce their numerical counterparts.

A. Previous Waveforms for Spinning Binaries

Over the years, several groups have developed increasingly accurate waveforms for use in gravitational wave astrophysics. Post-Newtonian (PN), quasi-circular and eccentric inspiral waveforms for compact binaries have been obtained to high-order as an expansion in the orbital velocity [9]. The leading-order, spin-orbit and spin-spin contributions to the dynamics appear at 1.5PN and 2PN relative order respectively [10, 11]. These calculations have now been extended through 2.5PN [13, 15] and 3PN order [16] out to 3.5PN order [17, 20]. The spin-orbit contributions to the gravitational wave phase and amplitude are currently known to 3.5PN order [20, 22].

Presently, there are three special cases where purely analytic, inspiral waveform models exist for spinning binaries:

(i) **Aligned**: Systems where the spins are co-aligned or anti-aligned with the orbital angular momentum; will undergo a secular precession of the spins and of the orbital plane [5]. This precession will impact the waveforms, and it is important to take it into account to properly extract the spins of the binary components, as well as to break degeneracies between different system parameters [6, 7].

For case (i), the spins stay aligned (or anti-aligned) with the orbital angular momentum throughout the evolution and there is no precession. The waveforms here bear strong resemblance to non-spinning waveforms, but with a spin-corrected chirping (see e.g. [23, 24]). For cases (ii) and (iii), the system undergoes simple precession [1], i.e. the precession of the orbital and spin angular momenta about the total angular momentum with a single (evolving) precession frequency (see e.g. [5]).

1 For case (iii), the system undergoes simple precession if the 2PN spin-spin corrections to the spin dynamics are neglected.
The goals of the present work are the following:

(a) To develop a general formalism to perturbatively solve the time-domain PN evolution equations and obtain analytic time-domain waveforms for precessing quasi-circular inspirals;

(b) To develop a general formalism to perturbatively Fourier transform a precessing quasi-circular inspiral, time-domain waveform.

Goal (a) will be achieved through the technique of multiple scale expansions \([8]\), where one solves the evolution equations using the fact that \(t_{\text{orb}} \ll t_{\text{prec}} \ll t_{\gamma}\), where \(t_{\text{orb}}\), \(t_{\text{prec}}\), and \(t_{\gamma}\), are the orbital, precession and radiation-reaction timescales respectively. Goal (b) is reached through the technique of uniform asymptotic expansions \([8, 40]\), where one recasts the phase modulation induced by precession as a sum of Bessel functions that are then amenable to a formal SPA treatment. Both of these techniques have proven very successful in various areas, from quantum field theory to aerospace engineering.

Although these formalisms are general, we exemplify them here by generalizing case (i) to accommodate systems where the spins are only partially aligned with the orbital angular momentum. The accretion torques that drive the spin-orbit alignment are not expected to produce perfect alignment, so it is useful to have analytic waveforms that cover the more realistic, partial alignment case. Expanding the spin precession equations in the misalignment angle leads to a system of coupled harmonic oscillators that diagonalizes to yield two precession frequencies \(\omega_+\) and \(\omega_-\). Multiple scale analysis is then used to derive an analytic expression for the evolution of these precession frequencies as the black holes spiral together \([8]\).

Spin precession alters the phasing of the waveform, and causes the mapping between gravitational wave frequency and time to become multi-valued, rendering the standard SPA inapplicable. The SPA returns singular results at turning points in the time-frequency mapping, resulting in what are known as fold and cusp catastrophes in the optical literature \([41]\). We show that the singularities can be cured using uniform asymptotic expansions of the phase in terms of Bessel functions \([8, 40]\).

The final result is a family of fully analytic, approximate, time and frequency domain waveforms for spinning, precessing, quasi circular binaries with moderately misaligned spins. The frequency-domain waveform family can be constructed from the following recipe:

1. The waveform is mode-decomposed as

\[
\tilde{h}(f) = \sum_{n \geq 0} \sum_{k \in \mathbb{Z}} \sum_{m = \{-2, 2\}} \tilde{h}_{n, k, m}(f),
\]  

\[1\)
2. Each Fourier mode is given by Eq. (191) in terms of the carrier phase \( \phi_c \), the precession phases \( \phi_{p, \pm} \), the time-frequency mapping \( t = t(f) \), the mode-decomposed, time-domain amplitudes \( A_{n,k,m} \), and the additional constant, amplitude, and phase modulation corrections \( A_{0,n,k,m}, A_{\pm,n,k,m} \), and \( \phi_{\pm,n,k,m} \), respectively.

3. The carrier phase \( \phi_c \) and its second time derivative are given in Appendix A as a function of the orbital frequency.

4. The precession phases and their second time derivatives \( \phi_{p, \pm} \) are given in Appendix C as a function of the orbital frequency.

5. The time-frequency mapping \( t = t(f) \) is given as a function of the orbital frequency in Appendix A.

6. The mode-decomposed, time-domain amplitudes \( A_{n,k,m} \) are given as a function of the orbital frequency in Appendix E.

7. The constant, amplitude, and phase modulation corrections \( A_{0,n,k,m}, A_{\pm,n,k,m} \), and \( \phi_{\pm,n,k,m} \) are given as a function of the orbital frequency in Appendix D.

8. The orbital frequency is given in terms of the Fourier frequency in Eqs. (193)-(194).

We prove that these new waveforms are accurate (i.e. faithful) by comparing them to the waveforms obtained by numerically solving for the orbital evolution and discretely Fourier transforming. We find typical matches of 0.99-0.999, maximized only over time and phase of coalescence, when the misalignment angles do not exceed 25°.

The benefit of our approach is two-fold. On the one hand, we provide ready-to-use, analytic waveforms that are computationally inexpensive to produce. The computational cost of numerically solving for the Fourier transform of spinning and precessing systems is currently a roadblock in the data analysis of signals for advanced ground detectors. On the other hand, an analytical treatment provides insight into the physics of the problem. Our results analytically explain why the waveforms of spinning and precessing binaries are essentially simple harmonic oscillators, with a carrier band and side-bands induced by evolving precession frequencies [38]. Moreover, our results extend the recently-proposed kludge waveforms [38] to account for amplitude and additional phase corrections induced by precession, which cannot be captured by educated guesses from numerical relativity waveforms.

The general formalism presented here opens the door to several other applications. For instance, one can extend case (ii) to include the first-order correction in the ratio of the spins of the two bodies. In this case, our formalism can be viewed as a systematic extension of the simple-precessing treatment of Apostolatos, et al [5], which allows us to compute the time-domain waveforms to higher order in the ratio of the timescales of the problem. Moreover, our method allows for the correct analytical calculation of the Fourier-domain waveforms, which cannot be obtained via a standard SPA treatment, contrary to older claims [5]. Our formalism can also be applied to other systems, such as inspiraling binary neutron stars, where the magnitude of both spin angular momenta are much smaller than the orbital one.

C. Organization and Conventions

The remainder of this paper is organized as follows: Section II A reviews multiple scale analysis through selected examples, which we use later to solve the equations of precession analytically; Section II B discusses the SPA and uniform asymptotic expansions applied to the Fourier transform of oscillatory functions, where the mapping between time and frequency is multi-valued and the standard SPA fails; Section III derives an analytic formula for the evolution of the angular momenta in the case of nearly aligned spins; Section IV uses the results of the previous sections to derive an analytical gravitational waveform valid for nearly aligned spins; Section V compares our waveform to the results obtained by taking a discrete Fourier transform of the time series.

Throughout this article we use geometric units with \( G = c = 1 \). We also employ the following conventions:

- Three-dimensional vectors are written in boldface and unit vectors carry a hat over them, e.g. \( \mathbf{A} = (A_x, A_y, A_z) \), with norm \( |\mathbf{A}| = A \), and unit vector \( \hat{\mathbf{A}} = \mathbf{A}/A \).

- Three-dimensional matrices are written in mathematical boldface, e.g. \( \mathbf{M} \) and \( \mathbf{A} \).

- Total time derivatives are denoted with an overhead dot: \( \dot{f} = df/dt \).

- \( \omega \) is the angular frequency in a frame fixed to the orbital plane.

- \( L \) is the Newtonian orbital angular momentum 3-vector.

- \( S_A \) is the spin angular momentum 3-vector for component \( A \).

- \( m_A \) is the mass of component \( A \), and we assume \( m_1 \geq m_2 \).

- \( \mu = m_1m_2/M \) is the reduced mass.

- \( \nu = m_1m_2/M^2 \) is the symmetric mass ratio.

- \( \chi_A \equiv |S_A|/m_A^2 \) is the dimensionless spin parameter for component \( A \).

- \( \hat{\mathbf{N}} \) is a unit vector pointing from the observer to the source.
II. TECHNIQUES FROM ASYMPTOTIC ANALYSIS

A. A Primer on Multiple Scale Analysis

Multiple scale analysis is a powerful mathematical formalism that serves as the theoretical foundations of boundary-layer theory and the Wentzel-Kramers-Brillouin (WKB) approximation. In this section, we review some important features of this formalism, as they will be essential in the solution to the precession equations. We will mostly follow and summarize the treatment in Bender and Orszag [8].

Consider the non-linear oscillator ordinary differential equation \( \ddot{y} + y + \epsilon y^3 = 0 \), where \( y \) is a function of time \( t \), with initial conditions \((y(0), \dot{y}(0)) = (1, 0)\). If we attempted the series solution

\[
y(t) = \sum_{n=0}^{\infty} \epsilon^n y_n(t), \tag{2}
\]

assuming \( \epsilon \ll 1 \), and matched coefficients of the same order in \( \epsilon \), we would find the solution

\[
y(t) = \cos(t) + \epsilon \left[ 1 - \frac{1}{8} \cos t - \frac{3}{8} t \sin t \right] + O(\epsilon^2). \tag{3}
\]

Clearly, this series approximation diverges as \( t \to \infty \), but in fact, it becomes invalid much sooner, when \( 3 \epsilon t / 8 \approx 1 \). As we will show below, however, the exact solution to this differential equation remains perfectly bounded in the \( t \to \infty \) limit; a multiple-scale expansion treatment will allow us to find such a solution.

Let us then introduce a new variable \( \tau = \epsilon t \) that defines a long time scale, as \( \tau \) does not become negligible when \( t \sim 1/\epsilon \). In multiple scale analysis, we search for solutions that are functions of all timescales in the problem, in this case \( t \) and \( \tau \), treated as if they were independent variables. This, of course, is just a mathematical trick, since at the end of the day, we can replace \( \tau \) in favor of \( \epsilon t \) to obtain a solution that is only \( t \)-dependent. We then assume a perturbative ansatz of the form

\[
y(t) = \sum_{n=0}^{\infty} \epsilon^n Y_n(t, \tau). \tag{4}
\]

Taking the sum to \( n = 1 \), the non-linear oscillator equation leads to the following two evolution equations

\[
\frac{\partial^2 Y_0}{\partial \tau^2} + Y_0 = 0, \tag{5}
\]

\[
\frac{\partial^2 Y_1}{\partial \tau^2} + Y_1 = -Y_0^3 - 2\frac{\partial^2 Y_0}{\partial \tau \partial t}. \tag{6}
\]

Notice that the differential operator (the terms on the left-hand side of both equations) is always the same, an expected result in perturbation theory. The most general solution to Eq. [5] is \( Y_0 = A(\tau)e^{it} + A^*(\tau)e^{-it} \), where the star stands for complex conjugation. Inserting this solution into Eq. [6], we find

\[
\frac{\partial^2 Y_1}{\partial t^2} + Y_1 = e^{it} \left[ -3A^2A'^* - 2i\frac{\partial A}{\partial \tau} \right] - e^{3it}A^3 + c.c., \tag{7}
\]

where c.c. stands for the complex conjugate. The first term of the right-hand side in Eq. (7) is a solution to Eq. [5], and thus, it is it which induces a secular growth. We can eliminate this secular growth by requiring that the term inside square brackets on the right-hand side of Eq. (7) vanishes, which then leads to a differential equation for \( A(\tau) \), whose solution is

\[
A(\tau) = R(0)e^{i\theta(0) + 3i\tau^2/2}, \tag{8}
\]

where \( R(0) \) and \( \theta(0) \) are constants of integration. Using the initial conditions stipulated above and reassembling the full solution, we find

\[
y(t) = \cos \left[ t \left( 1 + \frac{3}{8} \epsilon t \right) \right] + O(\epsilon). \tag{9}
\]

Notice that this solution is bounded for all \( t \) and it is much more accurate than the series expansion in Eq. (3) for large \( t \).

An extra degree of complication arises when we consider differential equations with implicit functional dependence in the source. For example, let us consider the oscillator ordinary differential equation

\[
e^2 \frac{d^2 y}{dt^2} + \omega^2(\tau)y = 0, \tag{10}
\]

where again \( \tau = \epsilon t \) and try to solve it with multiple-scale analysis. Imposing the same expansion of the solution as in Eq. (4), Eq. (10) becomes

\[
\frac{\partial^2 Y_0}{\partial \tau^2} + \omega^2(\tau)Y_0 = 0, \tag{11}
\]

\[
\frac{\partial^2 Y_1}{\partial \tau^2} + \omega^2(\tau)Y_1 = -2\frac{\partial^2 Y_0}{\partial \tau \partial \tau}. \tag{12}
\]

The solution to the zeroth-order equation is now \( Y_0 = A(\tau)e^{i\omega(\tau)t} + A^*(\tau)e^{-i\omega(\tau)t} \), which when inserted into Eq. (12) leads to

\[
\frac{\partial^2 Y_1}{\partial \tau^2} + \omega^2(\tau)Y_1 = -2ie^{i\omega(\tau)t} \left[ \frac{\partial(A\omega)}{\partial \tau} + itA\omega^2 \frac{\partial \omega}{\partial \tau} \right] + c.c.. \tag{13}
\]

To eliminate secularity, we would want to set the term inside the square brackets on the right-hand side of Eq. (13) to zero, but due to the explicit appearance of \( t \), this would force \( A = 0 \). Multiple scale analysis fails if the long time scale is proportional to the short time scale and \( \omega \) frequency of oscillation is not a constant.

We can force the frequency to be constant by changing variables to \( T = f(t) = f(\tau/\epsilon) \), which then transforms Eq. (10) into

\[
\frac{d^2 y}{dT^2} + f''(t) \frac{dy}{dT} + \frac{\omega^2(\tau)}{f'(t)^2}y = 0, \tag{14}
\]
We can force the frequency oscillation to be constant by choosing
\[ T = f(t) = \int^t \omega(\epsilon s) ds = \frac{1}{\epsilon} \int^\tau \omega(s) ds, \] (15)
which then leads to
\[ \frac{d^2 y}{d\tau^2} + y + \epsilon \frac{\omega'(\tau)}{\omega^2(\tau)} \frac{dy}{d\tau} = 0. \] (16)

Now, this equation can be solved via multiple-scale analysis. Using the expansion in Eq. (4), the above equation leads to
\[ \frac{\partial^2 Y_0}{\partial \tau^2} + Y_0 = 0, \] (17)
\[ \frac{\partial^2 Y_1}{\partial \tau^2} + Y_1 = -\frac{\omega'(\tau)}{\omega^2(\tau)} \frac{\partial Y_0}{\partial \tau} - 2 \frac{\partial^2 Y_0}{\partial \omega \partial \tau}. \] (18)

The solution to the zeroth-order in \( \epsilon \) is the same as that of the non-linear oscillator, and with this, the first-order in \( \epsilon \) equation becomes
\[ \frac{\partial^2 Y_1}{\partial \tau^2} + Y_1 = -i e^{i \tau} \left[ \frac{2}{\omega} \frac{\partial A}{\partial \sigma} + \frac{\omega'(\tau)}{\omega^2(\tau)} A \right] + \text{c.c.}. \] (19)

This time we can eliminate the secularly growing terms by requiring that the terms inside square brackets in Eq. (19) vanish, which leads to a partial differential equation for \( A(\tau) \), whose solution is \( A(\tau) = A_0/\sqrt{\omega(\tau)} \). The full solution is then
\[ y(t) = \frac{A_0}{\sqrt{\omega(t)}} e^{\frac{i}{\epsilon} \int^t \omega(s) ds} + \text{c.c.}, \] (20)
which is the same as what one would have obtained through the WKB physical-optics approximation. We see then that multiple-scale analysis is a generic and powerful technique that, in certain cases, allows us to recover the WKB approximation, among others. Of course, the above examples employed only 2 scales, but multiple scale analysis is in principle valid given an arbitrary number of scales provided they satisfy a certain scale hierarchy.

### B. The Stationary Phase Approximation and Uniform Asymptotic Expansions

The leading-order gravitational wave signal from a quasi-circular binary inspiral can be expressed in the form
\[ h(t) = A(t)e^{-i \Phi(t)} \] (21)
where the amplitude \( A(t) \) and the phase \( \Phi(t) \) are slowly evolving functions of time. The full signal is given by a sum of such terms that form a harmonic series in the orbital frequency. The function \( h(t) \) oscillates on the orbital timescale, with an amplitude and frequency that evolve on the slower spin-precession and radiation-reaction timescales.

In gravitational wave data analysis, quantities of interest (such as the likelihood function) are usually calculated in the frequency domain, where the noise-autocorrelation function is assumed to take a simple form. Thus, we are faced with the task of Fourier transforming the waveform in Eq. (21).

\[ \tilde{h}(f) = \int A(t)e^{-i \Phi(t)} e^{2\pi if t} dt = \int A(t)e^{i \Phi(f, t)} dt. \] (22)

A direct numerical implementation using a fast Fourier transform algorithm is possible, but the computational cost can be high since the waveform needs to be sampled at a cadence set by the orbital period.

The quadratic SPA is the standard analytic approach to solve Eq. (22). At a given frequency \( f \), the integral is dominated by the contributions where the phase \( \phi(f, t) \) is a slowly-varying function of time. Away from this region, the integrand oscillates rapidly and contributes little. Defining the stationary phase points implicitly as the times \( t_{\text{SPA}} \) where \( \phi(f, t_{\text{SPA}}) = 0 \), or equivalently,
\[ \hat{\Phi}(t_{\text{SPA}}) = 2\pi f, \] (23)
the Fourier phase can be expanded as
\[ \phi(f, t) = \phi(f, t_{\text{SPA}}) + \frac{1}{2} \tilde{\phi}(f, t_{\text{SPA}})(t - t_{\text{SPA}})^2 + \ldots \] (24)

Given such an expansion, one can then analytically solve the generalized Fourier integral in Eq. (22) through a change of variables [8]
\[ \tilde{h}_{\text{SPA}}(f) = \left[ \frac{2}{\hat{\Phi}(t_{\text{SPA}})} \right]^{1/2} A(t_{\text{SPA}}) \Gamma(1/2) e^{i[2\pi ft_{\text{SPA}} - \Phi(t_{\text{SPA}}) - \sigma \pi/4]}, \] (25)
where \( \sigma = \text{sign}(\hat{\Phi}(t_{\text{SPA}})), \Gamma(\cdot) \) is the Gamma function and \( t_{\text{SPA}}(f) \) is understood as a function of frequency.

Several assumptions go into the solution of Eq. (25), which have been implicitly taken for granted in gravitational wave modeling. First, one assumes that there is a unique stationary phase time \( t_{\text{SPA}} \) for a given frequency \( f \), so that the time-frequency mapping \( t_{\text{SPA}}(f) \) is single valued for each harmonic. Second, one assumes that the expansion for the phase about the stationary point in Eq. (24) can be truncated at quadratic order, and that the amplitude \( A(t) \) can be replaced by the constant value \( A(t_{\text{SPA}}) \). When the mapping in Eq. (23) between frequency and time yields multiple stationary points, the full solution is given by summing up the contributions of the form (25) for all the stationary points. But when this mapping is not single valued, the SPA can lead to divergent results, i.e. \( \hat{\Phi}(t_{\text{SPA}}) \) can vanish and the amplitude can diverge.
The goal of uniform asymptotic expansions is to replace non-uniform expansions, like that of Eq. (25), by a new expansion that remains valid in a domain containing the singular point. A standard example is the Airy function uniformization of a fold catastrophe [11], which occurs when two stationary points coalesce and \( \dot{\phi}(t_{\text{SPA}}) = \Phi(t_{\text{SPA}}) = 0 \). At these catastrophe points, the stationary point is defined by the last equation and the Taylor expansion of the phase in Eq. (24) has to be continued to higher order. At cubic order, the integral in Eq. (22) yields an Airy function, and the cubic SPA is

\[
\hat{h}_{\text{SPA}}(f) = \left[ \frac{2}{\Phi(t_{\text{SPA}})} \right]^{1/3} A(t_{\text{SPA}}) 2\pi e^{i2\pi ft_{\text{SPA}} - \Phi(t_{\text{SPA}})} \left\{ -\sigma[2\pi f - \Phi(t_{\text{SPA}})] \left[ \frac{2}{\Phi(t_{\text{SPA}})} \right]^{1/3} \right\},
\]

where \( \sigma = \text{sign}[\Phi(t_{\text{SPA}})] \), and the amplitude and phase are evaluated at the singular point \( t_{\text{SPA}} \). The expression in Eq. (26) matches the numerical Fourier transform for frequencies \( f \) in the neighborhood of the critical point \( f = \Phi(t_{\text{SPA}})/2\pi \) where the phase is well approximated by a cubic Taylor expansion. In many instances, there is an overlap region where both approximations [Eqs. (25) and (26)] are valid simultaneously. In such cases, it is possible to construct a complete SPA waveform from a piecewise collection of the quadratic and cubic SPAs.

A completely different uniformization is required, however, for situations where the singular points become so dense that the Airy function and related techniques breakdown. A good example is when the phase has an oscillatory component, which is exactly the situation for precessing black hole binaries. One solution is to re-express the original waveform as the sum of simpler waveforms that each have a well-behaved SPA [10]. For example, if the GW phase can be written as the sum of a carrier phase and an oscillatory component:

\[
\Phi(t) = \Phi_C(t) + \alpha(t)\cos \beta(t)
\]

where \( \Phi_C(t) \), \( \alpha(t) \) and \( \beta(t) \) are monotonic functions of time, then

\[
h(t) = A(t)e^{-i\Phi_C(t)} \sum_{n=-\infty}^{\infty} (-i)^n J_n(\alpha(t))e^{-in\beta(t)}
\]

and

\[
\hat{h}_{\text{SPA}}(f) = \left[ \frac{2\pi}{\Phi_C(t_n) + n\beta(t_n)} \right]^{1/2} A(t_n) \times (-i)^n J_n(\alpha(t_n))e^{i2\pi ft - \Phi_C(t_n) - n\beta(t_n) - \sigma\pi/4},
\]

where \( \sigma = \text{sign}[\Phi_C(t_n) + n\beta(t_n)] \). Notice that there are now \( n \) different stationary points \( t_n \), defined by the stationary phase condition \( 2\pi f = \Phi_C(t_n) + n\beta(t_n) \). In Eq. (29), we have assumed that the individual contributions are non-singular: \( \Phi_C(t_n) + n\beta(t_n) \neq 0 \). If a singularity does occur in any of the terms, then the standard SPA for this term can be replaced by the Airy uniformization of Eq. (26). The rapid decay of the Bessel functions with increasing order \( |n| \) means that just a few terms are needed in the sum of Eq. (29) to obtain a good approximation to the full Fourier transform.

To illustrate the Bessel uniformization approach, let us consider a simple toy model that shares many of the features of the waveforms produced by spinning black hole binaries. Let us then consider the phase given by

\[
\Phi(t) = 2\pi \left[ f_0(t/T) + \frac{1}{2} \dot{f}_0(t/T)^2 + \alpha_0(t/T)\cos(\omega_0(t/T) + \frac{1}{2}\dot{\omega}_0(t/T)^2) \right],
\]

and an amplitude given by a Tukey tapered cosine window

\[
A(t) = \begin{cases} \frac{1}{2} \left[ 1 + \cos \left( \pi \left( \frac{t}{\kappa T} - 1 \right) \right) \right], & t \leq \kappa T \\ 1, & \kappa T < t < (1 - \kappa)T \\ \frac{1}{2} \left[ 1 + \cos \left( \pi \left( \frac{t - T}{\kappa T} + 1 \right) \right) \right], & t \geq (1 - \kappa)T. \end{cases}
\]

The Tukey window helps suppress spectral leakage in the numerical Fourier transform. Fig. 1 shows the amplitude of the Fourier transform computed three different ways: (i) using a numerical DFT; (ii) using the quadratic SPA; and (iii) using the Bessel function uniformization of the SPA summing up to \( |n| = 5 \). The parameters chosen were \( \{ T = 1, f_0 = 300, \dot{f}_0 = 900, \omega_0 = 30, \dot{\omega}_0 = 30, \alpha_0 = 0.4, \kappa = 0.2 \} \). The uniform asymptotic expansion provides a near perfect match to the numerical Fourier transform, while the quadratic SPA diverges at turning points of the frequency.

### III. SPIN AND ORBITAL ANGULAR MOMENTUM

In this section, we explore the evolution equations of the spin and orbital angular momentum vectors using techniques from multiple-scale analysis. We first develop the formalism of multiple-scale analysis as applicable to inspiraling binaries, and then apply it to systems where the spin angular momentum vectors are nearly aligned with the orbital angular momentum vector. Physically, this corresponds to the inspiral of binary BHs or binary NSs in a gas-rich environment, where the latter tends to align the spin and orbital angular momenta.

Such a system allows us to make several approximations that enable a perturbative analytic solution. First, we expand all quantities in the misalignment angle

\[
K = \sum_{n>0} K^{(n)}(t) e^n
\]
A circular inspiral in the center of mass frame can be written as

\[
\frac{\mathbf{a}_c}{\sigma} \equiv \frac{t_{\text{proc}}}{t_{\text{rr}}} \ll 1, \tag{33}
\]

where \(t_{\text{proc}}\) and \(t_{\text{rr}}\) are the precession and radiation-reaction timescales. This last expansion is justified for binaries in the PN (slow-motion/weak-gravity) regime, where all three characteristic timescales of the problem separate. The precession order counting parameter \(\sigma\) and the PN one \(c\) are not independent, but rather \(O(\sigma) = O(c^{-3}).\) Henceforth, a term of \(O(c^{-2.5})\) will be said to be of NPN order.

### A. Precession Equations

The spin and orbital angular momentum precession equations for an compact binary system in a quasi-circular inspiral in the center of mass frame can be written as

\[
\begin{align*}
\dot{\mathbf{S}}_1 &= \Omega_{LS_1} \mathbf{L} \times \mathbf{S}_1 + \Omega_{S_1 S_2} \mathbf{S}_2 \times \mathbf{S}_1, \tag{35} \\
\dot{\mathbf{S}}_2 &= \Omega_{LS_2} \mathbf{L} \times \mathbf{S}_2 + \Omega_{S_1 S_2} \mathbf{S}_1 \times \mathbf{S}_2, \tag{36} \\
\dot{\mathbf{L}} &= \frac{\Omega_{LS_1}}{L} \mathbf{S}_1 \times \dot{\mathbf{L}} + \frac{\Omega_{LS_2}}{L} \mathbf{S}_2 \times \dot{\mathbf{L}}, \tag{37} \\
\dot{\mathbf{L}} &= -\frac{1}{3} \frac{a_0}{M} (\mathbf{M} \omega)^{8/3} \left[ 1 + \sum_{n=2}^{\infty} a_n (\mathbf{M} \omega)^{n/3} \right] L, \tag{38}
\end{align*}
\]

where we have defined

\[
\begin{align*}
\Omega_{LS_1} &= \frac{\omega^2}{M} \left[ (2 + \frac{3m_2}{2m_1}) \mathbf{L} - \frac{3}{2} (\mathbf{L} \cdot \mathbf{S}_2) \right], \tag{39} \\
\Omega_{LS_2} &= \frac{\omega^2}{M} \left[ (2 + \frac{3m_1}{2m_2}) \mathbf{L} - \frac{3}{2} (\mathbf{L} \cdot \mathbf{S}_1) \right], \tag{40} \\
\Omega_{S_1 S_2} &= \frac{1}{2} \frac{a_2}{M}, \tag{41}
\end{align*}
\]

where \(m_{1,2}\) are the component masses, \(\omega = M^2 (\mu / L)^3\) is the orbital frequency of the system, with \(L\) the magnitude of the Newtonian orbital angular momentum \(\mathbf{L}\). The spin angular momentum of the 4th binary component is \(\mathbf{S}_A^\parallel\), while \(\hat{\mathbf{L}} = \mathbf{L} / L\) is an orbital angular momentum unit vector. All cross- and dot-products represent the standard (Euclidean) three-dimensional operations of vector calculus. The quantities \(a_n\) are functions of the symmetric mass ratio as well as \((\mathbf{S}_A \cdot \hat{\mathbf{L}})\) and \((\mathbf{S}_A \cdot \mathbf{S}_B)\), and they are explicitly given in Appendix A.

The evolution equations presented above are in principle valid to different PN orders. The evolution equation for \(L\) (or equivalently for \(\omega\)) is valid to \(N/2\) PN order, where here we choose \(N = 5\), i.e. we model the evolution of \(L\) to 2.5PN order. The evolution equations for \(\mathbf{S}_A\) and \(\mathbf{L}\), however, are only valid to first-subleading PN order, i.e. leading order in spin-orbit (1.5PN) and spin-spin (2PN) coupling. Since spin enters at 1.5PN order in the phase, the 2.5PN order corrections that we are leaving out in the evolution of \(\mathbf{S}_A\) would contribute at 4PN order. We are then allowed to use Newtonian expressions to map between \(\mathbf{L}\) and \(\omega\), and the norm of \(\mathbf{S}_A\) is conserved. Of course, one could extend the analysis in this paper by adding corrections to \(\Omega_{LS_1,2}\) and \(\Omega_{S_1 S_2}\), thus including sub-leading PN corrections to the evolution of \(\mathbf{S}_A\). But to be consistent in PN order counting, one would also have to include 4PN corrections to the evolution of \(\mathbf{L}\), which are currently unknown.

Let us now pick a frame and implement the near-alignment approximation. We choose \(\mathbf{z} = \mathbf{J}(t = 0)\), where \(\mathbf{J} = \mathbf{L} + \mathbf{S}_1 + \mathbf{S}_2\) is the total angular momentum. Since \(\mathbf{J}\) is not constant, we have to specify it at a given point as

\[\mathbf{J}(\mathbf{z}) = \mathbf{S}_1 + \mathbf{S}_2.\]
time for the frame to be inertial. In this frame, we can write
\[ K = K_x \hat{x} + K_y \hat{y}, \]
for any \( K = L, S_1, \) or \( S_2 \). In the near-alignment approximation, the components of \( K \) in this frame can then be expanded as
\[
K_x = \sum_{n \geq 0} K_x^{(2n)}(t) \epsilon^{2n},
\]
\[
K_y = \sum_{n \geq 0} K_y^{(2n+1)}(t) \epsilon^{2n+1},
\]
\[
K_z = \sum_{n \geq 0} K_z^{(2n+1)}(t) \epsilon^{2n+1}.
\]
The structure of the equations of motion ensures that odd-powers of \( \epsilon \) in \( K_x, j = x \) or \( y \), vanish. In this paper, we will take these sums only up to \( \mathcal{O}(\epsilon) \), but extending these results to higher-order is straightforward.

Before proceeding, let us make an important comment on the near-alignment approximation. Consider a binary system at early times, where the misalignment angle is \( \epsilon_0 \). As the binary evolves, the norm of the orbital angular momentum \( L \) shrinks by radiation-reaction. But since the norm of the spin angular momentum \( S_A \) is conserved, the misalignment angle will grow. This implies that our perturbation parameter is not a constant, but rather an increasing function of time. Thus, just like the PN approximation is expected to break in the late stages of inspiral because the orbital velocity increases, the misalignment approximation is also expected to break as \( \epsilon \) increases and the series becomes asymptotic.

### B. Analysis to \( \mathcal{O}(\epsilon^0) \)

Let us first focus on the precession equations to leading-order in \( \epsilon \). The \( x \)- and \( y \)-components of these equations are trivially satisfied. The \( z \)-component of the spin angular momentum equation requires that \( S_A^{(0)} \) be a constant, which can be obtained by demanding that \( \|S_A\| = m_A^2 \gamma_A \), and thus
\[
\|S_A\| = S_A^{(0)} + \mathcal{O}(\epsilon^2) = m_A^2 \gamma_A.
\]
We can use this property to solve for \( S_A^{(2n)} \) at all orders, given the lower order solutions for \( S_A^{(2n-1)}, j = x \) or \( y \).

The \( z \)-component of the orbital angular momentum evolution equation requires a bit more work. First, let us define the quantity
\[
\xi_0 \equiv \frac{\mu M}{L_z^{(0)}} = \mathcal{O}(\epsilon^{-1}),
\]
as a new PN quantity. This parameter is exactly the square-root of the frequency parameter often used in the literature \( x = (M\omega)^{2/3} = \mu^2 M^2 L^{-2} \) only when the spins and orbital angular momentum are aligned. Using this parameter, we can rewrite the \( \mathcal{O}(\epsilon^0) \) part of the evolution equation for the \( z \)-component of orbital angular momentum as
\[
\dot{\xi}_0 = \frac{1}{3} \frac{\alpha_0}{M} \xi_0^3 \left( 1 + \sum_{n=2}^{N} a_n \xi_0^n \right),
\]
where the spin-dependent part of the couplings were evaluated at leading-order in \( \epsilon \):
\[
\hat{L}^{(0)} = (0, 0, 1),
\]
\[
S_A^{(0)} = (0, 0, m_A^2 \gamma_A).
\]

Since all coefficients are constants, we can directly integrate Eq. (48) by Taylor expanding \( (\dot{\xi}_0)^{-1} \). After inverting the PN series and integrating, we obtain
\[
\xi_0(t) = \zeta \left[ 1 - \frac{a_2}{6} \xi^2 - \frac{a_3^{(0)}}{5} \xi^3 + \frac{5a_2^2 - 6a_4^{(0)}}{24} \xi^4 + \frac{9a_2a_3^{(0)} - 5a_5^{(0)}}{15} \xi^5 + \mathcal{O}(\xi^6) \right],
\]
with
\[
\zeta = \left( \frac{3M}{8a_0(t_{\text{coal}} - t)} \right)^{1/8}.
\]
The notation \( a_i^{(a)} \) means the part of \( a_i \) at \( \mathcal{O}(\epsilon^a) \), where recall that \( a_2 \) is a 1PN correction that is spin-independent. We have here kept terms up to 2.5PN order, while the spin-dependent couplings are included to leading-order in \( \epsilon \).

For convenience, let us also introduce a new quantity \( \xi \), which coincides with \( \xi_0 \) at \( \mathcal{O}(\epsilon^0) \) but differs at higher orders. This quantity is defined via
\[
\xi(t) = \zeta \left[ 1 - \frac{a_2}{6} \xi^2 - \frac{a_3(t = 0)}{5} \xi^3 + \frac{5a_2^2 - 6a_4(t = 0)}{24} \xi^4 + \frac{9a_2a_3(t = 0) - 5a_5(t = 0)}{15} \xi^5 + \mathcal{O}(\xi^6) \right],
\]
which extends Eq. (51) by using the full \( a_i \) coefficients, evaluated at \( t = 0 \), instead of their \( \mathcal{O}(\epsilon^0) \) parts \( a_i^{(0)} \). Therefore, the difference between \( \xi \) and \( \xi_0 \) is of \( \mathcal{O}(\xi^2) \), which implies
\[
L_z^{(0)} = \frac{\mu M}{\xi_0} = \frac{\mu M}{\xi} + \mathcal{O}(\xi^2).
\]
With such a definition, \( \xi \) coincides with \( x^{1/2} = (M\omega)^{1/3} \) when the scalar products between \( \hat{L}, S_1, \) and \( S_2 \) are time-independent. That is, in the near-alignment approximation, we can write
\[
\xi = (M\omega)^{1/3} + \mathcal{O}(\epsilon^2).
\]
Henceforth, we will use \( \xi \) as our independent variable.
C. Analysis to $O(\epsilon^4)$

Let us now look at the evolution equations to first-order in $\epsilon$. We can write them in matrix notation as

$$\frac{dW^{(1)}}{dt} = -\mathbb{M}W^{(1)} - aA W^{(1)},$$

$$\frac{dW^{(1)}}{dt} = \mathbb{M}W^{(1)} - aA W^{(1)},$$

where we have defined the vectors

$$W^{(1)}_1 = \begin{pmatrix} L^{(1)}_x \\ S^{(1)}_{1,x} \\ S^{(1)}_{2,x} \end{pmatrix}, \quad W^{(1)}_2 = \begin{pmatrix} L^{(1)}_y \\ S^{(1)}_{1,y} \\ S^{(1)}_{2,y} \end{pmatrix},$$

and the matrices

$$\mathbb{M} = \begin{pmatrix} (b+c) & -d & -e \\ -b & (d+f) & -g \\ -c & -f & (e+g) \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.\tag{58}$$

with

$$a = \frac{1}{3} \frac{a_0}{M} \xi(t)^8 \left(1 + \sum_{n=2}^{N} a_n \xi(t)^n\right),$$

$$b = \frac{\xi(t)^6}{M} \left[2 + \frac{3m_2}{2m_1}\right] m_1^2 \frac{M}{M^2} \chi_2 - \frac{3}{2} \xi(t) \nu \chi_1 \chi_2,$$

$$c = \frac{\xi(t)^6}{M} \left[2 + \frac{3m_1}{2m_2}\right] m_2^2 \frac{M}{M^2} \chi_2 - \frac{3}{2} \xi(t) \nu \chi_1 \chi_2,$$

$$d = \frac{\xi(t)^5}{M} \left[2 + \frac{3m_2}{2m_1}\right] \nu - \frac{3}{2} \xi(t) \frac{m_2^2}{M^2} \chi_2,$$

$$e = \frac{\xi(t)^5}{M} \left[2 + \frac{3m_1}{2m_2}\right] \nu - \frac{3}{2} \xi(t) \frac{m_1^2}{M^2} \chi_2,$$

$$f = \frac{1}{2} \frac{\xi(t)^6}{M} \frac{m_2^2}{M^2} \chi_2, \quad g = \frac{1}{2} \frac{\xi(t)^6}{M} \frac{m_2^2}{M^2} \chi_1.$$

The solution to the system in Eq. (58) can be obtained via a standard linear algebra approach. First, we diagonalize $\mathbb{M}$ via a similarity transformation in matrix $\mathbb{R}$, $\mathbb{R}^{-1}\mathbb{M}\mathbb{R} = \mathbb{D}$, where $\mathbb{D}$ is the diagonal matrix

$$\mathbb{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \omega_{\nu,+} & 0 \\ 0 & 0 & \omega_{\nu,-} \end{pmatrix},$$

with eigenvalues

$$\omega_{\nu,\pm} = \frac{1}{2} \left[b + c + d + e + f + g \pm \sqrt{(b-c+d-e+f-g)^2 + 4(c-f)(b-g)}\right].$$

The transformation matrix $\mathbb{R}$ is given explicitly in Appendix [B]. The first few terms of the PN expansion of $\omega_{\nu,\pm}$ are

$$\omega_{\nu,+} = \frac{1}{M} \left(2\nu + \frac{3}{2} \frac{m_1^2}{M^2}\right) \xi^5$$

$$\omega_{\nu,-} = \frac{1}{M} \left(2\nu + \frac{3}{2} \frac{m_2^2}{M^2}\right) \xi^5$$

$$+ \frac{1}{M} \left[\left(2 \frac{m_2^2}{M^2} + 3 \frac{1}{2}\right) \chi_2 - \frac{m_1^2}{M^2} \chi_1\right] \xi^6 + O(c^{-7}),$$

$$\omega_{\nu,-} = \frac{1}{M} \left[2\nu + \frac{3}{2} \frac{m_2^2}{M^2}\right] \xi^5$$

$$+ \frac{1}{M} \left[2 \frac{m_2^2}{M^2} + 3 \frac{1}{2}\right) \chi_1 - \frac{m_1^2}{M^2} \chi_2] \xi^6 + O(c^{-7}).$$

With this at hand, Eq. (56) can be transformed into

$$\frac{dQ^{(1)}}{dt} = -\mathbb{D} Q^{(1)}_2 - \mathbb{E} Q^{(1)}_1,$$

$$\frac{dQ^{(2)}}{dt} = -\mathbb{D} Q^{(1)}_1 - \mathbb{E} Q^{(2)}_1,$$

where we have defined the transformed $\mathbb{W}_i$, i.e. the eigenvectors or quasi-normal modes, via

$$Q^{(1)}_j \equiv \mathbb{R}^{-1} \mathbb{W}^{(1)}_j = \begin{pmatrix} Q^{(1)}_{0,0} \\ Q^{(1)}_{-1} \\ Q^{(1)}_{0,1} \end{pmatrix},$$

with $j = 1$ or $2$, and where the remainder matrix

$$\mathbb{E} = \mathbb{R}^{-1} \left(aA\mathbb{R} + \frac{d\mathbb{R}}{dt}\right).$$

The second term in the above equation is necessary because the rotation matrix $\mathbb{R}$ is not constant. We chose to normalize the eigenvectors such that

$$L^{(1)}_x = Q^{(1)}_{0,1} + Q^{(1)}_{-1} + Q^{(1)}_{1,},$$

$$L^{(1)}_y = Q^{(1)}_{0,2} + Q^{(1)}_{-2} + Q^{(1)}_{2},$$

as explained in Appendix [B].

We decouple the system in Eq. (70) by taking an extra time-derivative to obtain

$$\frac{d^2 Q^{(1)}}{dt^2} = -\mathbb{D}^2 Q^{(1)} + \left(\{\mathbb{D},\mathbb{E}\} - \frac{d\mathbb{D}}{dt}\right) Q^{(1)} + \left(\mathbb{E}^2 - \frac{d\mathbb{E}}{dt}\right) Q^{(1)},$$

$$\frac{d^2 Q^{(2)}}{dt^2} = -\mathbb{D}^2 Q^{(2)} + \left(\{\mathbb{D},\mathbb{E}\} - \frac{d\mathbb{D}}{dt}\right) Q^{(2)} + \left(\mathbb{E}^2 - \frac{d\mathbb{E}}{dt}\right) Q^{(2)},$$

where $\{A, B\} \equiv AB + BA$ denotes the matrix anticommutator. In what follows, we solve this system of equations via multiple scale analysis.

1. Separation of Scales

Inspection of Eqs. (75) and (76) reveals that this is a system of perturbed harmonic oscillators with time-dependent frequencies. Thus, the multiple scale analysis
methods presented in Sec. II A are well-suited to solve this problem. As explained in that section, however, we must first transform to a new independent variable such that the problem becomes that of a system of perturbed harmonic oscillators with constant frequencies.

Because the eigenvalues of the D matrix are generically not the same, i.e. \( \omega_{r,+} \neq \omega_{r,-} \), we are forced to introduce a different transformation for + and − modes. Changing variables \( t \to \phi_{r,\pm}(t) \) in the \( Q_{1,1}^{(1)} \) equation, Eq. (75) becomes

\[
\frac{dQ_{0,1}^{(1)}}{dt} = -\left[ E_1 Q_1^{(1)} \right] \cdot P_0, \quad (77)
\]

\[
\frac{d^2Q_{1,1}^{(1)}}{d\phi_{r,+}^2} = -\left( \frac{dt}{d\phi_{r,+}} \right)^2 \omega_{r,+}Q_{1,1}^{(1)} - \left( \frac{dt}{d\phi_{r,+}} \right)^2 \frac{d^2\phi_{r,+}}{dt^2} \omega_{r,+}Q_{1,1}^{(1)} + \left( \frac{dt}{d\phi_{r,+}} \right)^2 \left( (D, E) - \frac{dd}{dt} \right) Q_2^{(1)} \cdot P_+ + \left( \frac{dt}{d\phi_{r,+}} \right)^2 \left( E^2 - \frac{dE}{dt} \right) Q_1^{(1)} \cdot P_+ \quad (78)
\]

\[
\frac{d^2Q_{1,-1}^{(1)}}{d\phi_{r,-}^2} = -\left( \frac{dt}{d\phi_{r,-}} \right)^2 \omega_{r,-}Q_{1,-1}^{(1)} - \left( \frac{dt}{d\phi_{r,-}} \right)^2 \frac{d^2\phi_{r,-}}{dt^2} \omega_{r,-}Q_{1,-1}^{(1)} + \left( \frac{dt}{d\phi_{r,-}} \right)^2 \left( (D, E) - \frac{dd}{dt} \right) Q_2^{(1)} \cdot P_- + \left( \frac{dt}{d\phi_{r,-}} \right)^2 \left( E^2 - \frac{dE}{dt} \right) Q_1^{(1)} \cdot P_- \quad (79)
\]

where we introduced the projectors

\[
P_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad P_+ = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad P_- = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (80)
\]

We obtain similar equations for \( Q_{1,2}^{(1)} \) with \( i = 0, +, \) or −. Notice that we have not transformed the time coordinate for the zero-frequency mode. For the above equations to have a constant normal frequency, we must set

\[
\frac{d\phi_{r, \pm}}{dt} = \omega_{r, \pm}, \quad (81)
\]

modulo a proportionality constant, which we choose to be unity so that \( \phi_{r, \pm} \) are exactly the precession angles. With this rescaling of the independent variable, the differential system becomes

\[
\frac{dQ_{0,1}^{(1)}}{dt} = -\left[ E_1 Q_1^{(1)} \right] \cdot P_0, \quad (82)
\]

\[
\frac{d^2Q_{1,1}^{(1)}}{d\phi_{r,+}^2} = -Q_{1,1}^{(1)} - \frac{\omega_{r,+}}{\omega_{r,+}} \frac{dQ_{1,1}^{(1)}}{d\phi_{r,+}} + \frac{1}{\omega_{r,+}} \left( (D, E) - \frac{dd}{dt} \right) Q_2^{(1)} \cdot P_+ + \frac{1}{\omega_{r,+}} \left[ E^2 - \frac{dE}{dt} \right] Q_1^{(1)} \cdot P_+, \quad (83)
\]

\[
\frac{d^2Q_{1,-1}^{(1)}}{d\phi_{r,-}^2} = -Q_{1,-1}^{(1)} - \frac{\omega_{r,-}}{\omega_{r,-}} \frac{dQ_{1,-1}^{(1)}}{d\phi_{r,-}} + \frac{1}{\omega_{r,-}} \left( (D, E) - \frac{dd}{dt} \right) Q_2^{(1)} \cdot P_- + \frac{1}{\omega_{r,-}} \left[ E^2 - \frac{dE}{dt} \right] Q_1^{(1)} \cdot P_- \quad (84)
\]

Note that the source to these oscillators depends on \( t \), which must be in principle solved for through inversion of the solution to Eq. (81). We will here leave these expressions as implicit functions of \( \phi_{r, \pm} \).

Now, let us perform an expansion of the above differential equations in powers of \( \sigma \), which we recall is a book-keeping parameter of \( O(t_{rev}/t_\tau) \). In terms of the \( \xi \) variable, \( \sigma \) counts the powers in \( (\xi / \xi) \omega_{r, \pm} = a / \omega_{r, \pm} \), since \( \xi = a \xi \). The differential equations then become

\[
\frac{dQ_{0,1}^{(1)}}{dt} = -\sigma a \left[ R^{-1} \left( A R + \xi \frac{\partial R}{\partial \xi} \right) Q_1^{(1)} \right] \cdot P_0, \quad (85)
\]

\[
\frac{d^2Q_{1,1}^{(1)}}{d\phi_{r,+}^2} = -Q_{1,1}^{(1)} + \sigma a \frac{a^2}{\omega_{r,+}} \left[ F Q_2^{(1)} \right] \cdot P_+ + \sigma^2 a^2 \frac{a^2}{\omega_{r,+}^2} \left\{ \left( \frac{\xi}{\omega_{r,+}} \frac{\partial \omega_{r,+}}{\partial \xi} R^{-1} \right) - \left( A R + \xi \frac{\partial R}{\partial \xi} \right) \right\} G Q_1^{(1)} \cdot P_+, \quad (86)
\]

\[
\frac{d^2Q_{1,-1}^{(1)}}{d\phi_{r,-}^2} = -Q_{1,-1}^{(1)} + \sigma a \frac{a^2}{\omega_{r,-}^2} \left[ F Q_2^{(1)} \right] \cdot P_- + \sigma^2 a^2 \frac{a^2}{\omega_{r,-}^2} \left\{ \left( \frac{\xi}{\omega_{r,-}} \frac{\partial \omega_{r,-}}{\partial \xi} R^{-1} \right) - \left( A R + \xi \frac{\partial R}{\partial \xi} \right) \right\} G Q_1^{(1)} \cdot P_-, \quad (87)
\]

where we have defined the new matrices

\[
F = \left\{ D, R^{-1} \left( A R + \xi \frac{\partial R}{\partial \xi} \right) \right\}, \quad (88)
\]

\[
G = R^{-1} \left\{ A R + \xi \frac{\partial R}{\partial \xi} + \xi \frac{\partial R}{\partial \xi} R^{-1} A R + \xi^2 \frac{\partial R}{\partial \xi} - \left( \frac{\xi}{\omega_{r,+}} \frac{\partial \omega_{r,+}}{\partial \xi} \right) + \xi \frac{\partial R}{\partial \xi} R^{-1} A R \right\} \left\{ A R + \xi \frac{\partial R}{\partial \xi} \right\} \cdot P_+ \quad (89)
\]

\[
\left( A R + \xi \frac{\partial R}{\partial \xi} \right) - \xi \frac{\partial R}{\partial \xi} \right\} \cdot P_-. \quad (87)
\]

We can now proceed to the separation of timescales by introducing a new time variable \( \tau \) such that \( \tau / \phi_{r, \pm} = O(\sigma) \). In Sec. II A we used a linear relation between \( \tau \) and \( t \), i.e. \( \tau = \sigma t \). Although we could do the same here,
we find it more convenient to use the non-linear relation 
\( \dot{\sigma}/dt = \sigma a \), or in angle-variables \( \dot{\tau}/\dot{\phi}_\tau = \sigma a/\dot{\phi}_\tau \). Such a non-linear mapping between \( \tau, t \), and \( \phi_{\tau\pm} \) leads to a better match with numerical solutions because it allows for the ratio of timescales \( t_{\text{prec}}/t_\tau \) to vary as the inspiral proceeds. Of course, we can solve for \( \tau \) in terms of \( t \) via

\[ \tau = \sigma \int a \, dt \equiv \sigma \int \frac{1}{\xi} \, d\xi = \sigma \log \xi. \] (90)

We then postulate the series ansatz

\[ Q_j^{(1)}(t) = \sum_{n \geq 0} \sigma^n Q_j^{(1,n)}(\phi_{\tau\pm}, \tau), \] (91)

where \( j = 1 \) or 2. Recall here that the first superscript 1 reminds us that these are quantities of \( O(\epsilon) \), while the second superscript labels the orders in \( \sigma \). Thus, the quantity \( Q_j^{(m,n)} \) is of bivariate \( O(\epsilon^m, \sigma^n) \). With this ansatz, we convert the system of ordinary differential equations of Eqs. (85)-(87) into a system of partial differential equations (PDEs). In doing so, we transform the differential operators via

\[
\begin{align*}
\frac{d}{dt} &= \frac{\partial}{\partial t} + \sigma \frac{\partial}{\partial \tau}, \\
\frac{d^2}{dt^2} &= \frac{\partial^2}{\partial \tau^2} + 2\sigma \frac{\partial}{\partial \tau} + \sigma^2 \frac{\partial^2}{\partial \phi_{\tau\pm}^2} + \sigma^2 \frac{\partial^2}{\partial \tau^2} + \sigma^2 \frac{\partial^2}{\partial \phi_{\tau\pm}^2}
\end{align*}
\] (92)

and re-expand all quantities in \( \sigma \ll 1 \). In what follows, we solve the resulting system of PDEs order by order in \( \sigma \).

2. Solution to \( O(\epsilon^1, \sigma^0) \)

At lowest order in \( \sigma \), the system of PDEs becomes

\[
\begin{align*}
\frac{\partial Q_{0,j}^{(1,0)}}{\partial t} &= 0, \\
\frac{\partial^2 Q_{0,j}^{(1,0)}}{\partial \tau^2} &= \frac{\partial Q_{0,j}^{(1,0)}}{\partial \phi_{\tau\pm}^2} = -Q_{0,j}^{(1,0)}, \\
\frac{\partial^2 Q_{0,j}^{(1,0)}}{\partial \phi_{\tau\pm}^2} &= -Q_{0,j}^{(1,0)},
\end{align*}
\] (94)

where \( j = 1 \) or 2. Solving these equations and requiring that they satisfy the original first-order differential system of Eqs. (95), at leading order in \( \sigma \), we find

\[ Q_{0,j}^{(1,0)} = A_{0,j}^{(1,0)}(\tau), \] (95)

where

\[
\begin{align*}
Q_{1,1}^{(1,0)} &= A_{1,1}^{(1,0)}(\tau) \cos \phi_{\tau\pm} - A_{1,2}^{(1,0)}(\tau) \sin \phi_{\tau\pm}, \\
Q_{1,2}^{(1,0)} &= A_{1,2}^{(1,0)}(\tau) \cos \phi_{\tau\pm} + A_{1,1}^{(1,0)}(\tau) \sin \phi_{\tau\pm}.
\end{align*}
\] (96, 97)

with \( j = 1 \) or 2 as usual.

3. Solution to \( O(\epsilon^1, \sigma^1) \)

Let us now consider the differential system at \( O(\sigma^1) \). The zero-frequency equations are

\[
\frac{\partial Q_{0,j}^{(1,1)}}{\partial t} = -a \frac{\partial A_{0,j}^{(1,0)}}{\partial \tau} - a \left( Q_{1,j}^{(1,0)} + Q_{-1,j}^{(1,0)} \right) [1 + O(\epsilon^{-1})],
\] (98)

where \( j = 1 \) or 2. As we saw in Sec. [1A] we must require that secular terms do not appear, which then leads to the equation

\[
\frac{\partial A_{0,j}^{(1,0)}}{\partial \tau} + A_{0,j}^{(1,0)} = 0,
\] (99)

whose solution is

\[ A_{0,j}^{(1,0)}(\tau) = B_{0,j}^{(1,0)} e^{-\tau} = \frac{P_{0,j}^{(1,0)}}{\xi}. \] (100)

The terms depending on \( Q_{1,j}^{(1,0)} \) in Eq. (98) will induce an oscillatory term in \( Q_{0,j}^{(1,1)} \).

The non-zero frequency equations at \( O(\sigma^1) \) are

\[
\begin{align*}
\frac{\partial^2 Q_{0,j}^{(1,1)}}{\partial \tau^2} + Q_{0,j}^{(1,1)} &= -a \frac{\partial Q_{\pm,2}^{(1,0)}}{\partial \tau} + \frac{1}{\omega_{\tau\pm}} \left[ F_{Q_{\pm,2}^{(1,0)}} \cdot P_{\pm} \right], \\
\frac{\partial^2 Q_{0,j}^{(1,1)}}{\partial \phi_{\tau\pm}^2} + Q_{0,j}^{(1,1)} &= -a \frac{\partial Q_{\pm,1}^{(1,0)}}{\partial \tau} + \frac{1}{\omega_{\tau\pm}} \left[ F_{Q_{\pm,1}^{(1,0)}} \cdot P_{\pm} \right].
\end{align*}
\] (101, 102)

Expanding these equations, we find
To prevent secular terms, we must require that there are no source terms proportional to solutions of the homogeneous equation. This then imposes

\[
\frac{\partial A^{(1,0)}_{+,j}}{\partial \tau} + \frac{1}{2} \omega^{(1,0)}_{r,+} A^{(1,0)}_{+,j} = 0,
\]

\[
\frac{\partial A^{(1,0)}_{-,j}}{\partial \tau} + \frac{1}{2} \omega^{(1,0)}_{r,-} A^{(1,0)}_{-,j} = 0,
\]

where \( j = 1 \) or 2. The solutions to these equations are

\[
A^{(1,0)}_{\pm,j}(\tau) = B^{(1,0)}_{\pm,j} \exp \left[ -\frac{1}{2} \int \frac{\omega^{(1,0)}_{r,\pm}}{\omega^{(1,0)}_{r,\pm}} d\tau \right]
\]

\[
= B^{(1,0)}_{\pm,j} \exp \left[ -\frac{1}{2} \int \frac{\omega^{(1,0)}_{r,\pm}}{\omega^{(1,0)}_{r,\pm}} d\xi \right]
\]

\[
= B^{(1,0)}_{\pm,j} \left[ 1 + \mathcal{O}(c^{-1}) \right].
\]

where \( B^{(1,0)}_{\pm,j} \) are integration constants and we have used Eq. [109]. A proof of Eq. [109] is given in Appendix [3].

4. Precession phases

The precession angles \( \phi_{r,\pm} \) can be computed using Eq. (81) and (83), which leads to

\[
\phi_{r,\pm} = \int \omega_{r,\pm} dt = \int \frac{\omega_{r,\pm}}{a\xi} d\xi.
\]

Care must be exercised when computing this integral because \( \delta \omega_{p} = \omega_{r,+} - \omega_{r,-} \) satisfies

\[
\delta \omega_{p}^2 = \mathcal{O}(\delta m^2) \xi^{10} + \mathcal{O}(\delta m) \xi^{11}
\]

\[
\mathcal{O}(\delta m^0) \xi^{12} + \mathcal{O}(\delta m^0, \xi^{13}),
\]

where \( \delta m = (m_1 - m_2)/M \) is the dimensionless mass difference. Depending on the magnitude of \( \delta m \) relative to the magnitude of \( \xi \), the PN expansion will be somewhat different: if \( \delta m \gg \mathcal{O}(c^{-1}) \), \( \delta \omega_{p}^2 \sim \xi^{10} \); if \( \delta m \ll \mathcal{O}(c^{-1}) \), \( \delta \omega_{p}^2 \sim \xi^{12} \). Notice that this is not a problem in the PN treatment of non-spinning inspirals, as there \( \delta m \) does not appear in the controlling factor of the approximation.

In order to address this feature of the solution, let us separate the precession phases via \( \phi_{r,\pm} = \phi_{r,m} \pm \delta \phi_{r} \). The mean precession phase \( \phi_{r,m} \equiv (\phi_{r,+} + \phi_{r,-})/2 \) can be computed using standard PN methods:

\[
\phi_{r,m} = \frac{1}{2} \int \frac{d\xi}{a\xi} \frac{d\xi}{d\tau} dt
\]

\[
= \frac{1}{2} \int b + c + d + e + f + g d\xi
\]

\[
= -\frac{5}{128} \left[ \frac{8}{3} + \left( \frac{m_1}{m_2} + \frac{m_2}{m_1} \right) \right] \xi^{-3} \left[ 1 + \mathcal{O}(c^{-1}) \right].
\]

We can expand the integrand of this expression to any relevant PN order, and we provide higher-order PN terms in Appendix [4].

The calculation of the precession phase difference \( \delta \phi_{r} \equiv (\phi_{r,+} - \phi_{r,-})/2 \) must be studied in two different cases: \( \delta m \ll \mathcal{O}(c^{-1}) \) and \( \delta m \gg \mathcal{O}(c^{-1}) \). Since the PN parameter that controls the expansion in \( c^{-1} \) evolves with time, some systems might comply with the former case at some point in time, and to the latter case at another. Thus, we find it useful to consider a third case, i.e. when \( \delta m \sim \mathcal{O}(c^{-1}) \). As we will see below, this will allow us to use a uniform approximation depending only on the value of \( \delta m \).

Let us first focus on the \( \delta m \gg \mathcal{O}(c^{-1}) \) case. In this case, \( \delta m \) can be treated as a quantity of order unity, and
we can carry out a standard PN expansion:
\[
\delta \phi_{r,1} = \frac{1}{2} \int (\omega_{r,+} - \omega_{r,-}) \frac{dt}{d\xi} d\xi
\]
\[
= \frac{1}{2} \int \frac{1}{a_1^2} [(b - c + d - e + f - g)^2 + 4(c - f)(b - g)]^{1/2} d\xi
\]
\[
= \frac{15}{256} \left\{ - \frac{1}{3} \left( \frac{m_1}{m_2} - \frac{m_2}{m_1} \right) \xi^{-3} - \frac{1}{2} \left[ \chi_1 - \chi_2 + 2 \left( \frac{m_1}{m_2} \chi_1 - \frac{m_2}{m_1} \chi_2 \right) \right] \xi^{-2} + O(c) \right\}.
\]
(114)

Let us now concentrate on the case where \( \delta m \sim O(c^{-1}) \). In this case, we must treat \( \delta m \) as a quantity of the same order as the PN parameter \( \xi \). Doing so and PN expanding, we find
\[
\delta \phi_{r,2} = \frac{1}{2} \int (\omega_{r,+} - \omega_{r,-}) \frac{dt}{d\xi} d\xi
\]
\[
= \frac{1}{2} \int \frac{1}{a_1^2} [(b - c + d - e + f - g)^2 + 4(c - f)(b - g)]^{1/2} d\xi
\]
\[
= -\frac{5}{8192 \delta m^2 \xi^3} \left\{ T_0 \left[ 32 \delta m^2 - 12 (\chi_1 - \chi_2) \delta m \xi + (9 \chi_1^2 - 50 \chi_1 \chi_2 + 9 \chi_2^2) \xi^3 \right] + 144 \chi_1 \chi_2 (\chi_1 - \chi_2) \xi^3 \right\},
\]
(115)

where we have defined the quantity
\[
T_0 = \left[ 16 \delta m^2 - 24 (\chi_1 - \chi_2) \delta m \xi + (9 \chi_1^2 - 2 \chi_1 \chi_2 + 9 \chi_2^2) \xi^2 \right]^{1/2}.
\]
(116)

Finally, when \( \delta m \ll O(c^{-1}) \), we expand \( \delta \omega_p \) in \( \delta m \), but leave the \( \xi \) factors unexpanded:
\[
\delta \phi_{r,3} = \frac{1}{2} \int (\omega_{r,+} - \omega_{r,-}) \frac{dt}{d\xi} d\xi
\]
\[
= \frac{1}{2} \int \frac{1}{a_1^2} [(b - c + d - e + f - g)^2 + 4(c - f)(b - g)]^{1/2} d\xi
\]
\[
= -\frac{15}{512 \xi^2} \left\{ T_2 T_3 + \frac{20}{\sqrt{T_1}} \chi_1 \chi_2 (\chi_1 - \chi_2) \xi^2 \right\} \times \log \left( \frac{1}{\xi} \left( T_3 + \sqrt{T_1} T_2 \right) \right) + O(\delta m),
\]
(117)

where we have defined the quantities
\[
T_1 = 9 \chi_1^2 - 2 \chi_1 \chi_2 + 9 \chi_2^2,
\]
\[
T_2 = 9 \chi_1^2 - 2 \chi_1 \chi_2 + 9 \chi_2^2
\]
\[
- 8 \chi_1 \chi_2 (\chi_1 + \chi_2) \xi + 4 \chi_1^2 \chi_2^2 \xi^2 \right\}^{1/2}.
\]
(118)

\[T_3 = 9 \chi_1^2 - 2 \chi_1 \chi_2 + 9 \chi_2^2\]
(120)

In this way, we can reconstruct \( \phi_{r, \pm} \) by combining \( \phi_{r, m} \) in Eq. (113) with one of the \( \delta \phi_p \) in Eqs. (114), (115) and (117), depending on the magnitude of \( \delta m \). Later on, we will use the particular implementation described in Appendix [C] where we found useful, in practice, to use \( \delta \phi_p = \delta \phi_{r,1} \) for \( \delta m \geq 0.2 \), \( \delta \phi_p = \delta \phi_{r,2} \) for \( 10^{-5} \leq \delta m < 0.2 \), and \( \delta \phi_p = \delta \phi_{r,3} \) for \( \delta m < 10^{-5} \).

Different classes of compact binaries will, of course, have a different natural set of \( \delta m \). Neutron star binaries must have \( \delta m \in (0, 0.375) \), with typical values in \( \delta m \sim 0.08 \) for which \( \delta \phi_p = \phi_{r,2} \). Neutron star/black hole binaries can have \( \delta m \in (0.375, 0.96) \), with typical values in \( \delta m \sim 0.75 \) for which \( \delta \phi_p = \phi_{r,1} \). Black hole binaries detectable by advanced ground detectors with total mass less than \( 50 M_\odot \) must have \( \delta m \in (0, 0.82) \), with typical values in \( \delta m \lesssim 0.4 \) for the best gravitational wave candidates. In this case, one may have to switch between \( \delta \phi_p = \phi_{r,1} \) and \( \delta \phi_p = \phi_{r,2} \). The choice \( \delta \phi_p = \phi_{r,3} \) is only relevant for almost exactly equal mass, which has a very low probability of happening.

We argue in Sec. [V C] that the discontinuity in the solution \( \delta \phi_p \) at \( \delta m = 0.2 \) is of no concern due to high faithfulness between the two approximations at the boundary.

5. Solutions to higher order in \( \sigma \)

Going to higher order in \( \sigma \) is straightforward, except perhaps for the treatment of certain timescale mixing that will generically occur; see e.g. Eqs. (103-106) at higher order in \( \sigma \). In order to exemplify how a higher-order in \( \sigma \) calculation would proceed, let us return to Eq. (103). This equation is that of a harmonic oscillator with frequency \( \omega_{r,+} \), sourced by an oscillatory term of frequency \( \omega_{r,-} \). Let us transform the left-hand-side of this equation from \( \phi_{r,+} \) to \( t \):
\[
\frac{\partial^2 Q_{r,1}^{(1)}(t)}{\partial \phi_{r,+}^2} = \frac{1}{\omega_{r,+}^2} \frac{\partial^2 Q_{r,1}^{(1)}(t)}{\partial t^2} - \frac{1}{\omega_{r,+}^2} \frac{d \omega_{r,+}}{dt} \frac{\partial Q_{r,1}^{(1)}(t)}{\partial t}.
\]
(121)

The last term in this equation contains a \( d \omega_{r,+}/dt \) factor, which introduces an extra factor of \( \sigma \), and must therefore be kept to \( O(\sigma^2) \). Equation (103) then becomes
\[
\frac{\partial^2 Q_{r,1}^{(1)}(t)}{\partial t^2} + \omega_{r,+}^2 Q_{r,1}^{(1)}(t) = \left[ F_0 A_{0,2}^{(1,0)} \right. + \omega_{r,+}^2 Q_{r,1}^{(1)}(t)
\]
\[
+ \left. F_+ \left( A_{+,-}^{(1,0)} \cos \phi_{r,-} + A_{+,-}^{(1,0)} \sin \phi_{r,-} \right) \right],
\]
(122)

and its solution is
\[
Q_{r,1}^{(1)} = A_{+,1}^{(1,0)}(\psi_{r,+}) \cos \phi_{r,+} - A_{+,-}^{(1,0)}(\psi_{r,+}) \sin \phi_{r,+}
\]
\[
- \frac{a}{\omega_{r,+} - \omega_{r,-}} \left[ \frac{m_2^2 \chi_2}{m_1 (m_1 - m_2)} \xi + O(c^{-2}) \right]
\]
\[
\times \left( A_{+,1}^{(1,0)} \cos \phi_{r,-} + A_{+,-}^{(1,0)} \sin \phi_{r,-} \right).
\]
(123)
Notice that the above solution has a pole at $\omega_{r,+} = \omega_{r,-}$, because the terms oscillating with frequency $\omega_{r,-}$ drive a resonance in Eq. (122). In practice, however, this happens only when $\chi_2 = 0$ and at a single point in time. Such a limit, therefore, must be excluded from higher-order solutions.

We can now use this $O(\epsilon, \sigma)$ solution in the source to the $O(\epsilon, \sigma^2)$ differential equation and require that terms oscillating at frequency $\omega_{r,+}$ vanish so as not to produce secularly growing terms. This would then lead to differential equations for $A_{\sigma,\epsilon,1}^{(1)}(\psi_{r,+})$, just as we obtained for $A_{\sigma,\epsilon,0}^{(1)}(\psi_{r,+})$ at $O(\epsilon, \sigma)$. The solution to these equations would then lead to a solution to the precession equations at $O(\epsilon, \sigma)$. We will not carry out a higher-order development here.

D. Summary

Let us here collect all the pieces of the solution to $O(\epsilon, \sigma)$ obtained in the previous subsections. Using the initial conditions $\hat{J}(0) = \hat{z}$, we can write

$$W_j^{(1)}(t) = \begin{pmatrix} -S_{1,k}^{(1)}(t = 0) - S_{2,k}^{(1)}(t = 0) \\ S_{1,k}^{(1)}(t = 0) \\ S_{2,k}^{(1)}(t = 0) \end{pmatrix}, \quad (124)$$

where $j = 1$ and $k = x$, or $j = 2$ and $k = y$. Furthermore,

$$Q_j^{(1)}(t = 0) = \begin{pmatrix} B_{0,j}^{(1,0)} \\ B_{+,j}^{(1,0)} \\ B_{-,j}^{(1,0)} \end{pmatrix} + O(\sigma, \epsilon^{-1}). \quad (125)$$

and since

$$Q_j^{(1)}(t = 0) = \mathbb{R}^{-1}(t = 0) W_j^{(1)}(t = 0), \quad (126)$$

we find that

$$B_{0,j}^{(1,0)} = 0. \quad (127)$$

This happens because we chose $\hat{J} = \hat{J}(t = 0)$. Any equivalent but different choice would result in a nonvanishing $B_{0,j}^{(1,0)}$.

The solution for the orbital angular momentum is then

$$L_z = \frac{\mu M}{\xi} + O(\epsilon^2), \quad (128)$$

$$L_x = \epsilon \left( B_{+,1}^{(1,0)} \cos \phi_{r,+} - B_{+,2}^{(1,0)} \sin \phi_{r,+} + B_{-,1}^{(1,0)} \cos \phi_{r,-} - B_{-,2}^{(1,0)} \sin \phi_{r,-} \right) + O(\sigma, \epsilon^3), \quad (129)$$

$$L_y = \epsilon \left( B_{+,1}^{(1,0)} \cos \phi_{r,+} + B_{+,2}^{(1,0)} \sin \phi_{r,+} + B_{-,1}^{(1,0)} \cos \phi_{r,-} + B_{-,2}^{(1,0)} \sin \phi_{r,-} \right) + O(\sigma, \epsilon^3), \quad (130)$$

where $\xi$ is given by Eq. (52), the precession angles $\phi_{r,\pm}$ are shown in Sec. III C 1 and Appendix C, and $B_{+,j}^{(1,0)}$ are given in Appendix B.

The $z$-components of the spins are simply

$$S_{1,z} = m_1^2 \chi_1 + O(\epsilon^2), \quad (131)$$

$$S_{2,z} = m_2^2 \chi_2 + O(\epsilon^2), \quad (132)$$

and the $x$ and $y$-components are

$$S_{1,x} \quad \frac{2(g - b)}{b + c - (d - e) - (f + g) + \delta \omega_p} \left( B_{+,1}^{(1,0)} \cos \phi_{r,+} - B_{+,2}^{(1,0)} \sin \phi_{r,+} \right) + O(\sigma, \epsilon^3), \quad (133)$$

$$S_{1,y} = \epsilon \left[ \frac{2(g - b)}{b + c - (d - e) - (f + g) + \delta \omega_p} \left( B_{+,1}^{(1,0)} \cos \phi_{r,+} + B_{+,2}^{(1,0)} \sin \phi_{r,+} \right) + O(\sigma, \epsilon^3), \quad (134)$$

$$S_{2,x} = \epsilon \left[ \frac{2(f - c)}{b + c - (d - e) - (f + g) + \delta \omega_p} \left( B_{+,1}^{(1,0)} \cos \phi_{r,+} - B_{+,2}^{(1,0)} \sin \phi_{r,+} \right) + O(\sigma, \epsilon^3), \quad (135)$$

$$S_{2,y} = \epsilon \left[ \frac{2(f - c)}{b + c - (d - e) - (f + g) + \delta \omega_p} \left( B_{+,1}^{(1,0)} \cos \phi_{r,+} + B_{+,2}^{(1,0)} \sin \phi_{r,+} \right) + O(\sigma, \epsilon^3), \quad (136)$$

where $\delta \omega_p = \omega_{r,+} - \omega_{r,-}$. 

\[\]
E. Comparison with Simple Precession

Another physically relevant case where the equations of precession can be solved analytically is simple precession. This occurs when one of the spins vanishes or when the masses are equal, provided we neglect spin-spin interactions. In simple precession, the orbital and spin angular momentum vectors precess around the total angular momentum vector at exactly the same frequency. Let us begin to study simple precession by rewriting the evolution equations without spin-spin couplings:

\[
\dot{\mathbf{S}} = \frac{\omega^2}{M} \left( 2 + \frac{3\mu}{2m_M} \right) \mathbf{L} \times \dot{\mathbf{S}},
\]

\[
\dot{\mathbf{L}} = \frac{\omega^2}{M} \left[ \left( 2 + \frac{3\mu}{2m_M} \right) \mathbf{S} + \left( 2 + \frac{3\mu}{2m_M} \right) \mathbf{S} \right] \times \dot{\mathbf{L}},
\]

where \( \mathbf{L} \) as before is the Newtonian orbital angular momentum with norm \( L = \mu M^2 \omega^{-1/3} \).

If either spin vanishes or if the masses are equal, the derivative of the total spin vector \( \mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 \) is perpendicular to \( \mathbf{S} \). We can then rewrite the equations of precession for \( \mathbf{S} \) and \( \mathbf{L} \) as

\[
\dot{\mathbf{S}} = 0,
\]

\[
\dot{\mathbf{L}} = -\frac{1}{3} \frac{a_0}{M} \omega^{8/3} \left\{ 1 + \frac{1}{2} \sum_{n \geq 2} a_n \omega^{n/3} \right\} L,
\]

\[
\dot{\mathbf{S}} = \frac{\omega^2}{M} \left( 2 + \frac{3\mu}{2m_M} \right) \mathbf{J} \times \mathbf{S},
\]

\[
\dot{\mathbf{L}} = \frac{\omega^2}{M} \left( 2 + \frac{3\mu}{2m_M} \right) \mathbf{J} \times \mathbf{L},
\]

where \( \mathbf{J} = \mathbf{L} + \mathbf{S} \) is the total angular momentum vector, the vanishing spin, if any, is labelled by the subscript \( \nu \), while the non-vanishing one is labelled by the subscript \( \nu' \). We then see that in simple precession both \( \dot{\mathbf{S}} \) and \( \dot{\mathbf{L}} \) precess around \( \mathbf{J} \) at a frequency

\[
\omega_{\nu,sp} = \frac{\omega^2}{M} \left( 2 + \frac{3\mu}{2m_M} \right) J.
\]

If the spins are only slightly misaligned with the orbital angular momentum, we have to leading order in \( \epsilon \)

\[
J = \mathbf{L} + \mathbf{S}_1 + \mathbf{S}_2 = \frac{\mu M}{\xi} + m_{\nu'}^2 \chi_{\nu'} + m_{\nu}^2 \chi_{\nu},
\]

which then leads to

\[
M_{\nu,sp} = \left( 2 + \frac{3\mu}{2m_M} \right) \left[ \nu \xi^5 + \frac{1}{M^2} \left( m_{\nu'}^2 \chi_{\nu'} + m_{\nu}^2 \chi_{\nu} \right) \xi \right],
\]

where recall that \( \omega = \xi^3 / M \) and \( \xi \) was defined in Eq. \([53]\).

Let us now return to our results for the near-alignment, multiple-scale analysis calculation. In order to map our results to those of simple precession, we must neglect spin-spin interactions, which naturally vanish in the single spin case. This implies using the following relations:

\[
b = \frac{\xi(t)^6}{M^3} \left( 2 + \frac{3\mu}{2m_M} \right) m_1^2 \chi_1,
\]

\[
c = \frac{\xi(t)^6}{M^3} \left( 2 + \frac{3\mu}{2m_M} \right) m_2^2 \chi_2,
\]

\[
d = \frac{\xi(t)^5}{M} \left[ 2 + \frac{3\mu}{2m_M} \right] \nu,
\]

\[
e = \frac{\xi(t)^5}{M} \left[ 2 + \frac{3\mu}{2m_M} \right] \nu,
\]

\[
f = 0,
\]

\[
g = 0.
\]

Thus, in the equal-mass case we have

\[
M_\omega \chi_{\nu,+} = \frac{7}{8} \left( \xi^5 + \xi^6 (\chi_1 + \chi_2) \right),
\]

\[
M_\omega \chi_{\nu,-} = \frac{7}{8} \xi^5,
\]

\[
B_{+,j}^{(1,0)} = -S_{1,j} - S_{2,j},
\]

\[
B_{-j}^{(1,0)} = 0.
\]

On the other hand, in the case where one of the spins vanishes, \( \chi_\nu = 0 \), we get

\[
M_\omega \chi_{\nu,+} = \left( 2 + \frac{3\mu}{2m_M} \right) \left( \nu \xi^5 + m_{\nu'}^2 \chi_{\nu'} \xi^6 \right),
\]

\[
M_\omega \chi_{\nu,-} = \left( 2 + \frac{3\mu}{2m_M} \right) \nu \xi^5,
\]

\[
B_{+j}^{(1,0)} = -S_{\nu',j},
\]

\[
B_{-j}^{(1,0)} = 0,
\]

provided

\[
\xi \geq \xi_c \equiv \frac{3 \left( m_{\nu'}^2 - m_\nu^2 \right)}{(4m_{\nu'}^2 + 3m_\nu)}.
\]

In the complementary case, when \( \xi < \xi_c \), the results are the same modulo \( \omega_{\nu,+} \leftrightarrow \omega_{\nu,-} \) and \( A_{+j}^{(1,0)} \leftrightarrow A_{-j}^{(1,0)} \). We see then clearly that our results in the simple precession limit reproduce exactly the results of simple precession in the nearly aligned limit. That is, both in the equal-mass case or in the vanishing single spin case, \( \omega_{\nu,+} \) becomes equal to \( \omega_{\nu,sp} \), while the \( \omega_{\nu,-} \) mode is irrelevant because its amplitude vanishes.

An interesting transition occurs if \( \xi_c > 0 \): the only evolution frequency continuously switches between \( \omega_{\nu,+} \) and \( \omega_{\nu,-} \). This transition only occurs if the vanishing spin is \( \chi_2 \), because then \( m_\nu = m_2 \) and by the conventions used in this paper, the numerator of Eq. \([159]\) is positive, i.e. \( m_{\nu'}^2 - m_\nu^2 = m_2^2 - m_2^2 > 0 \). The transition occurs at a particular value in time, given by \( \xi = \xi_c \). At this time, however, \( \omega_{\nu,+} = \omega_{\nu,-} \), and thus, the transition is continuous.
IV. GRAVITATIONAL WAVES

The results of the previous sections can be used to derive a purely analytic time-domain waveform for precessing nearly aligned binaries. In the rest of this section, we will derive such a waveform.

A. Time-Domain Waveforms: Standard Representation

An impinging GW will induce the following response in an interferometer with perpendicular arms in the long wavelength approximation:

\[
h(t) = \sum_{n \geq 0} \left[ F_+ h_{n,+} + F_\times h_{n,\times} \right],
\]

\[
h_{n,+} = A_{n,+}(i_L) \cos n\phi + B_{n,+}(i_L) \sin n\phi,
\]

\[
h_{n,\times} = A_{n,\times}(i_L) \cos n\phi + B_{n,\times}(i_L) \sin n\phi,
\]

where \( n \in \mathbb{N} \) is the harmonic number, \( \cos i_L = \hat{L} \cdot \hat{N} \), and the antenna pattern functions are

\[
F_+(\theta_N, \phi_N, \psi_N) = \frac{1}{2} \left( 1 + \cos^2 \theta_N \right) \cos 2\phi_N \cos 2\psi_N
\]

\[- \cos \theta_N \sin 2\phi_N \sin 2\psi_N,
\]

\[
F_\times(\theta_N, \phi_N, \psi_N) = F_+(\theta_N, \phi_N, \psi_N - \pi/4),
\]

with \((\theta_N, \phi_N)\) the spherical angles that label the position of the binary in the detector frame, and \( \psi_N \) the polarization angle defined through

\[
\tan \psi_N = \frac{\hat{L} \cdot \hat{z} - (\hat{L} \cdot \hat{N})(\hat{z} \cdot \hat{N})}{\hat{N} \cdot (\hat{L} \times \hat{z})},
\]

where \( \hat{z} \) is the unit normal vector to the detector plane.

The time-domain GW phase can be decomposed into a carrier phase and a precession perturbation \( \phi = \phi_c + \delta \phi \). Defining the reference of the orbital phase in the orbital plane as \( \hat{L} \times \hat{N} \), the equation of motion for the orbital phase is

\[
\dot{\phi} = \dot{\phi}_c + \dot{\delta} \phi, \quad \dot{\phi}_c = \omega,
\]

\[
\delta \phi = \frac{1}{L} \frac{L \cdot \hat{N}}{L^2 - (L \cdot \hat{N})^2} \left( L \times \hat{N} \right) \cdot \dot{L}.
\]

The carrier \( \phi_c \) is a secular, non-precessing phase, while the perturbation \( \delta \phi \) models the precession of the orbital plane.

B. Time-Domain Waveforms: Fourier Representation

Before we can Fourier transform the GW response via uniform asymptotics, we need to first figure out the relative scale and variability of all relevant quantities. This is important as it will tell us which quantities can be safely left in the slowly-varying signal amplitude, and which ones have to be promoted to the rapidly-varying phase. The part of the amplitude that depends only on the sky location \((\theta_N, \phi_N)\) varies on the timescale of variation of the normal to the detector. For ground-based instruments, this is roughly \( t_{\text{obs}} \sim O(1 \text{ year}) \), much larger than the typical observation time of \( t_{\text{obs}} \sim O(100 \text{ s}) \). For space-based instruments, \( t_{\text{obs}} \sim O(1 \text{ year}) \), which is of the same order as the typical observation time, but bigger than the typical precession timescale \( t_{\text{prec}} \sim O(1 \text{ month}) \). This implies that it is safe to leave such terms in the slowly-varying signal amplitude.

The different phases, however, can vary on a much shorter timescale. Using the equations of motion for \( \phi \), one can show that

\[
\dot{\phi}_c \sim O(c^{-3}), \quad \ddot{\phi}_c \sim O(c^{-11}),
\]

\[
\delta \phi \sim O(c^{-6}), \quad \ddot{\delta} \phi \sim O(c^{-11}),
\]

while from Eq. 165 and \( \cos i_L = \hat{L} \cdot \hat{N} \), one finds

\[
\ddot{\psi}_N \sim O(c^{-6}), \quad \dddot{\psi}_N \sim O(c^{-11}),
\]

\[
i_L \sim O(c^{-6}), \quad \ddot{i}_L \sim O(c^{-11}).
\]

Clearly then, \( \dot{\phi}_c, \ddot{\phi}_c, \dot{\psi}_N \), and \( \ddot{i}_L \) are all of the same order, and thus, they must all be promoted to the rapidly-varying signal phase. The phase \( \phi \) in \( h_{n,+} \) can be put into an exponential via Euler's formula. Similarly, the polarization angle \( \psi_N \) can be included in the phase by rewriting the antenna pattern functions and the harmonic polarizations in Eqs. 161 and 162 via Euler's formula as

\[
F_+ = \frac{1}{2} (A_F + iB_F) e^{2i\psi_N} + \text{c.c.},
\]

\[
F_\times = \frac{1}{2} (B_F - iA_F) e^{2i\psi_N} + \text{c.c.},
\]

\[
h_{n,+} = \frac{1}{2} (A_{n,+} - iB_{n,+}) e^{i(n\phi_c + \delta \phi + k i_L + m\psi_N)} + \text{c.c.},
\]

\[
h_{n,\times} = \frac{1}{2} (A_{n,\times} - iB_{n,\times}) e^{i(n\phi_c + \delta \phi) + \text{c.c.}}.
\]

where we have defined the slowly-varying amplitudes

\[
A_F = \frac{1}{2} (1 + \cos^2 \theta_N) \cos 2\phi_N,
\]

\[
B_F = \cos \theta_N \sin 2\phi_N.
\]

Finally, the inclination angle \( i_L \) can also be included in the signal phase if the amplitudes \( A_{n,+}, B_{n,+}, A_{n,\times}, \) and \( B_{n,\times} \) are rewritten as Fourier series.

Combining all of these results, one can rewrite Eq. 160 as

\[
h(t) = \sum_{n \geq 0} \sum_{k \in \mathbb{Z}} \sum_{m=-2,2} h_{n,k,m}(t)
\]

\[
h_{n,k,m} = A_{n,k,m}(\theta_N, \phi_N) e^{i(n\phi_c + n\delta \phi + k i_L + m\psi_N)} + \text{c.c.},
\]
where the slowly-varying amplitudes $A_{n,k,m}$ can be computed from [44] and are also given explicitly at 2PN order in Appendix [E].

\section*{C. Preparing for a Uniform Asymptotic Expansion}

The time-domain waveform in Eq. (177) is almost ready for a uniform asymptotic treatment; the last step is to convert $\varphi_C + \delta \dot{\phi} + \psi_N + i_N$ into a phase of the form $\Phi_C + \alpha(t) \cos(\beta(t))$ as in Eq. (27). Recall that here $\alpha(t)$ varies on the radiation reaction timescale, while $\beta(t)$ varies on the precession timescale.

The carrier phase can be solved for using standard techniques as a function of the orbital frequency:

$$\phi_C = \int \frac{\xi^3}{M} dt = \int \frac{\xi^2}{Ma} d\xi = \phi_{\text{coal}} - \frac{3}{5a_0} \frac{\xi^{-5}}{1 - \frac{5}{3} \xi^2 - \frac{5}{2} a_0 \xi^3} - 5 (a_4 - \frac{1}{2} a_2) \xi^4 + 5 (a_5 - 2a_2a_3) \xi^5 \log \xi + \mathcal{O}(e^{-1}),$$

(178)

where we used $d\xi/dt = a\xi$ with $\xi = (M\omega)^{1/3} + \mathcal{O}(e^2)$ as in the previous section; recall that the $a_i$ coefficients are given in Appendix [A] and should be evaluated at $t = 0$.

In our implementation, in order to isolate the effects of spin precession, we artificially increase the order of the above equation to 6PN; that is, we keep terms in $\dot{\omega}$ up to relative $\mathcal{O}(e^{-5})$, but we also keep all induced terms up to relative $\mathcal{O}(e^{-12})$. The resulting PN series contains terms of relative $\mathcal{O}(e^{-6})$ and higher that will not match the expected result from full GR; yet, they provide a more accurate result for the integration of the truncated $\dot{\omega}$ equation relative to a numerical solution. The exact form of Eq. (178) that we used in our comparisons is given in Appendix [A].

In principle, the remaining phase terms can be rewritten in the desired form only if one first distinguishes between two complementary cases:

\begin{align*}
\text{case 1:} & \quad \dot{N}_x^2 + \dot{N}_y^2 \sim \mathcal{O}(e^0), \\
\text{case 2:} & \quad \dot{N}_x \lesssim \mathcal{O}(e), \quad \dot{N}_y \lesssim \mathcal{O}(e).
\end{align*}

Assuming case (i), the equation of motion for the correction to the orbital phase is

$$\delta \dot{\phi} = \epsilon \frac{\dot{N}_x}{1 - N_x^2} \left[ \dot{N}_x \frac{L_y^{(1)} - L_x^{(0)} L_y^{(1)}}{L_x^{(0)}} - \dot{N}_y \frac{L_x^{(0)} L_y^{(1)}}{L_x^{(0)}} \right] + \mathcal{O}(e^2)$$

and therefore

$$\delta \phi = \epsilon \frac{\dot{N}_x}{1 - N_x^2} \left[ \dot{N}_x \frac{L_y^{(1)}}{L_x^{(0)}} - \dot{N}_y \frac{L_x^{(1)}}{L_x^{(0)}} \right] + \mathcal{O}(e^2).$$

(179)

Similarly, in case (i) the inclination phase becomes

$$i_L = \text{arccos} \left( \frac{\dot{N}_z}{L_x^{(0)} \sqrt{N_x^2 + N_y^2}} \right) + \mathcal{O}(e^2).$$

(180)

and the polarization angle is

$$\psi_N = \arctan \left( \frac{\dot{z}_x - \dot{N}_x \dot{N} \cdot \dot{z}}{\dot{N}_y \dot{z}_x - \dot{N}_x \dot{z}_y} \right) + \mathcal{O}(e^2).$$

(181)

Case (ii) leads at first to different expressions, but when these are re-expanded in the PN approximation, assuming that $L_x^{(1)}/L_x^{(0)} \ll 1$, one recovers the above expressions. Furthermore, the expansions for $\psi_N$ also depend on whether $\dot{N}$ is nearly aligned with $\dot{N}$ or not. But as before, the results obtained when they are not aligned is recovered by re-expanding the nearly aligned result in a PN expansion. However, we expect our result not to yield a match to the numerical solutions as good when $\dot{N}$ or $\dot{z}$ are nearly aligned.

Using the results from Eqs. (128, 130), we can express $i_N$, $\psi_N$ and $\delta \phi$ in a Fourier series of the precession phases $\phi_{p,+}$ and $\phi_{p,-}$. That is, we can rewrite the phase modulation in Eq. (177) as
\[ n \delta \phi + k_i L + m \psi_N = A_{0,n,k,m} + nA_{\delta \phi,+} \cos(\phi_{P,+} + \phi^{(0)}_{\delta \phi,+}) + nA_{\delta \phi,-} \cos(\phi_{P,-} + \phi^{(0)}_{\delta \phi,-}) + kA_{i L,+} \cos(\phi_{P,+} + \phi^{(0)}_{i L,+}) \\
+ kA_{i L,-} \cos(\phi_{P,-} + \phi^{(0)}_{i L,-}) + mA_{\psi_N,+} \cos(\phi_{P,+} + \phi^{(0)}_{\psi_N,+}) + mA_{\psi_N,-} \cos(\phi_{P,-} + \phi^{(0)}_{\psi_N,-}) + O(\epsilon, \epsilon^{-1}) \]

where the amplitudes \( A_{0,n,k,m} \) and \( A_{n,\pm} \), and the phases \( \phi^{(0)}_{n,\pm} \) are given in Appendix \[ \text{D} \] while the harmonic amplitudes are given by

\[
A_{\pm,n,k,m} = \text{sign}(A_{c,\pm}) \sqrt{A_{c,\pm}^2 + A_{s,\pm}^2},
\]

\[
\phi_{\pm,n,k,m} = \arctan \left( \frac{A_{s,\pm}}{A_{c,\pm}} \right),
\]

\[
A_{c, \pm} = nA_{\delta \phi, \pm} \cos(\phi^{(0)}_{\delta \phi, \pm}) + kA_{i L, \pm} \cos(\phi^{(0)}_{i L, \pm}) + mA_{\psi_N, \pm} \cos(\phi^{(0)}_{\psi_N, \pm}),
\]

\[
A_{s, \pm} = nA_{\delta \phi, \pm} \sin(\phi^{(0)}_{\delta \phi, \pm}) + kA_{i L, \pm} \sin(\phi^{(0)}_{i L, \pm}) + mA_{\psi_N, \pm} \sin(\phi^{(0)}_{\psi_N, \pm}),
\]

This then puts the time-domain waveforms in the desired form to carry out a uniform asymptotic expansion.

Before proceeding, let us comment on the remainders of Eq. (183). In going from the left-hand side of this equation to the right-hand side, we have neglected terms of \( O(\epsilon) \) and terms of \( O(\epsilon^{-1}) \), when \( \mathcal{N} \) and \( \mathcal{J} \) are aligned. When these vectors are misaligned, the remainders are actually smaller, namely of \( O(\epsilon^2) \). We will see later on that the neglect of higher-order terms in \( \epsilon^{-1} \) is the dominant source of discrepancy between our analytic frequency-domain waveforms and the DFT of numerical time-series waveforms.

### D. Frequency-Domain Gravitational Waveform via Uniform Asymptotic Expansions

We are interested in the Fourier transform of the GW signal. Taking advantage of the linearity of the Fourier transform, Eq. (22) can be rewritten as

\[
\tilde{h}(f) = \sum_{n \geq 0} \sum_{k \in \mathbb{Z}} \sum_{m=-2,2} \tilde{h}_{n,k,m}(t),
\]

where the Fourier harmonic components are

\[
\tilde{h}_{n,k,m}(f) = \int A^{*}_{n,k,m} e^{i(2\pi f t - n \phi_C - n \delta \phi - k_i L - m \psi_N)} dt \\
+ \int A_{n,k,m} e^{i(2\pi f t + n \phi_C + n \delta \phi + k_i L + m \psi_N)} dt,
\]

and recall that the star denotes complex conjugation.

Our particular asymptotic uniformization requires that we transform the above integrands via

\[
e^{-i(n \delta \phi + k_i L + m \psi_N)} = e^{-iA_{0,n,k,m}} \sum_{\{k_+, k_-\} \in \mathbb{Z}^2} J_{k_+}(A_{+,n,k,m})J_{k_-}(A_{-,n,k,m}) \times e^{-i[k_+(\phi_{P,+} + \phi_{+,n,k,m} + \pi/2) + k_- (\phi_{P,-} + \phi_{-,n,k,m} + \pi/2)]},
\]

and similarly for the second term.

After this transformation, we can apply the SPA to compute the integrals in Eq. (189). Since \( \phi_C \gg \phi_{P,\pm} \), we can safely neglect the second term as it will only contribute to negative frequencies. We then obtain

\[
\tilde{h}_{n,k,m}(f) = \sum_{\{k_+, k_-\} \in \mathbb{Z}^2} \frac{2\pi}{\phi_C + k_+ \phi_{P,+} + k_- \phi_{P,-}} A^{*}_{n,k,m} J_{k_+}(A_{+,n,k,m})J_{k_-}(A_{-,n,k,m}) \times e^{i(2\pi f t - n \phi_C - A_{0,n,k,m} - k_+(\phi_{P,+} + \phi_{+,n,k,m} + \pi/2) - k_- (\phi_{P,-} + \phi_{-,n,k,m} + \pi/2) - \pi/4)},
\]

where all time dependent functions are evaluated at \( t = t_{\text{SPA}} \), defined via

\[
2\pi f = n \dot{\phi}_C(t_{\text{SPA}}) + k_+ \dot{\phi}_{P,+}(t_{\text{SPA}}) + k_- \dot{\phi}_{P,-}(t_{\text{SPA}}).
\]
We can invert the above equation to find
\[ \xi = u - \frac{1}{24n} [k_+(7 + 6\delta m - \delta m^2) + k_-(7 - 6\delta m - \delta m^2)] u^3 + \frac{1}{24n} [k[2(1 - \delta m)^2 \chi_2 - (7 + 8\delta m + \delta m^2) \chi_1] + k_[-2(1 + \delta m)^2 \chi_1 - (7 - 8\delta m + \delta m^2) \chi_2)] u^4 + O(u^5). \]  
(193)

where the dimensionless mass difference \( \delta m = (m_1 - m_2)/M \), and we have defined the reduced frequency parameter
\[ u = \left( \frac{2\pi f}{n} \right)^{1/3}, \]  
(194)
and integrate \( dt = \int (d\xi/dt)^{-1} \, dt \) to find
\[ t_{\text{SPA}} = t_{\text{coal}} - \frac{3M}{80} \xi^{-3} \left[ 1 - \frac{4a_2}{3} \xi^2 - \frac{8a_3}{5} \xi^3 + 2 \left( a_2^2 - a_4 \right) \xi^4 + \frac{8}{3} (2a_2 a_3 - a_5) \xi^5 + O(\xi^6) \right]. \]  
(195)

In our implementation, similar to Eq. [178], we chose to artificially increase the above equation to 6PN. The exact result can be found in Appendix A.

By inspecting the results of Sec. [IVC] we can see that \( A_{\pm,n,k,m} \sim O(c^{-1}) \), and therefore the Bessel functions \( J_{k_{\pm}}(A_{\pm,n,k,m}) \) will be rapidly suppressed for high values of \( k_{\pm} \). This suggests that only a few terms may be needed in the Bessel expansion to accurately approximate the Fourier transform of the time-domain signal.

V. WAVEFORM COMPARISONS

In this section, we study how well the analytic frequency-domain waveform calculated in the previous section compares to others presented in the literature. First, we compare the phase and amplitude of the waveforms against each other. Then, we use a faithfulness measure to carry out integrated comparisons, without maximizing over intrinsic parameters. We perform a Monte Carlo study over a variety of systems with different spin misalignments, positions in the sky and relative orientation with respect to the detector plane.

A. Comparison Preliminaries

1. Waveform Models

The waveforms we compare against each other are the following:

- **DFT**: The discrete Fourier transform of the numerically-calculated, time-domain response function, tapered by a Tukey window to remove spectral leakage. The time-domain response is constructed from Eq. [160], with all angular momenta and phases obtained numerically by solving the evolution equations in the time-domain.

- **UAA**: The fully-analytic, frequency-domain, uniform asymptotic approximate waveform of Sec. [IVD].

- **HSPA [6]**: A hybrid, semi-analytic, frequency-domain template, given by the non-precessing, spin-aligned SPA waveform with higher harmonics (un-restricted PN), where the spin couplings are promoted \textit{a posteriori} to functions of the frequency and the phase is enhanced by the precession correction \( \delta \phi \) obtained by numerically integrating Eq. [167]. All angular momenta are obtained by solving all evolution equations numerically in the time-domain, and then numerically inverting them to find \( S_{1,2} \) and \( L \) as a function of orbital frequency.

- **Aligned SPA**: A non-precessing, spin-aligned, frequency-domain waveform, computed in the SPA with higher harmonics (un-restricted PN).

The different waveforms described above have different advantages and disadvantages. Perhaps the most accurate one is the DFT family, where the only mis-modeling systematic is induced by numerical error, from the numerical solutions to the evolution equations and the DFT. Unfortunately, however, this is also the most computationally expensive family to evaluate and the one that provides the least analytical insight. The aligned SPA family contains the largest mis-modeling systematics, since it attempts to model the system as non-precessing, but it is also the cheapest to evaluate. The HSPA family is somewhere in between the DFT and aligned SPA families, being computationally less expensive than DFT, but containing some systematics due to the improper use of the stationary phase approximation. Moreover, although less expensive than DFT, the HSPA family is more expensive to evaluate than the analytical waveforms, since each template requires the numerical solution and inversion of the evolution equations.

Let us make an important clarification regarding the DFT family. In multiple scale analysis, one usually compares approximations to some exact answer to determine, for example, the region of validity and accuracy of the former. Here, however, we lack such an exact solution. The DFT is perhaps the closest quantity to an exact Fourier transform that we possess, but of course, it is not an exact solution, as numerical error is non-negligible and filtering has been employed to prevent spectral leakage. We have checked, however, that the DFT is robust upon changes to the Tukey filter and numerical resolution. Thus, we here adopt the DFT as “exact” and compare the different approximations to it.

Care must be exercised, however, when comparing analytical and numerical spinning waveforms. Even when
spins are exactly aligned with the orbital angular momentum, the analytical expansion of the carrier phase does not match the numerical integration of Eq. (178) to sufficiently high accuracy. Similarly, the analytic, perturbative inversion of the time-frequency relation, Eq. (195), is not sufficiently accurate relative to the numerical inversion. Therefore, to isolate spin precession effects, we will keep terms in Eqs. (178) and (195) up to 6PN order. This is sufficient to guarantee that any discrepancies in the compared waveforms arise due to spinning effects only. The exact relations we use in our comparisons are given in Appendix A.

2. Detector Models

The comparisons of response functions are, of course, sensitive to the particular detector considered. We here consider both a typical ground-based detector and a typical space-based detector, both in the long-wavelength approximation.

Different detectors will operate in different frequency bands, for different observation times, and they will lead to different relations between the detector frame and a fixed frame tied to the distant stars. The latter will impact the functional form of the angles $\theta_N$, $\phi_N$, and $\psi_N$ in Eqs. (163–165): for a typical ground-based detector, since the observation time is very short, we can approximate the angles $\theta_N$ and $\phi_N$ as constant; for a typical space-based detector, the observation time is not short, and thus, one must properly model the time-dependence of the angles. We here use an eLISA configuration [45], where a LISA-type configuration trails behind Earth at a rate of 7.5° per year.

The relation between the detector frame $(\hat{x}_{\text{det}}, \hat{y}_{\text{det}}, \hat{z}_{\text{det}})$ and a frame tied to the distant stars $(\hat{x}, \hat{y}, \hat{z})$ for the space-based detector is given by

$$\begin{align*}
\hat{x}_{\text{det}} &= \left(\frac{3}{4} - \frac{1}{4} \cos 2\Phi_{\text{eLISA}}(t)\right) \hat{x} - \frac{1}{4} \sin 2\Phi_{\text{eLISA}}(t) \hat{y} \\
&\quad + \sqrt{\frac{3}{2}} \cos \Phi_{\text{eLISA}}(t) \hat{z}, \\
\hat{y}_{\text{det}} &= -\frac{1}{4} \sin 2\Phi_{\text{eLISA}}(t) \hat{x} + \left(\frac{3}{4} + \frac{1}{4} \cos 2\Phi_{\text{eLISA}}(t)\right) \hat{y} \\
&\quad + \sqrt{\frac{3}{2}} \sin \Phi_{\text{eLISA}}(t) \hat{z}, \\
\hat{z}_{\text{det}} &= -\sqrt{3} \cos \Phi_{\text{eLISA}}(t) \hat{x} - \sqrt{\frac{3}{2}} \sin \Phi_{\text{eLISA}}(t) \hat{y} + \frac{1}{2} \hat{z},
\end{align*}$$

where the detector barycenter is located at $(\cos \Phi_{\text{eLISA}}(t), \sin \Phi_{\text{eLISA}}(t), 0)$ in the Solar System frame with $\Phi_{\text{eLISA}}(t) = \omega_{\text{eLISA}} t$ and $\omega_{\text{eLISA}} = 2\pi(352.5/360) \text{ yr}^{-1}$. In addition, we modify the carrier phase by adding the so-called Doppler term [46]

$$\phi_C \rightarrow \phi_C + \omega R \sin \theta_N \cos (\Phi_{\text{eLISA}}(t) - \phi_N),$$

due to the fact that the barycenter of the detector moves in the frame tied to the distant stars. Here, $R = 1 \text{ AU}$, and $\theta_N$ and $\phi_N$ are the spherical angles of $\vec{N}$ in the frame tied to the distant stars.

3. Comparison Measures

We use two distinct comparison measures:

- **Waveform Comparison.** A direct waveform amplitude and phase comparison as a function of GW frequency and PN expansion parameter.

- **Match Comparison.** An integrated overlap waveform comparison, with white noise and without maximizing over intrinsic parameters.

The waveform comparison consists of comparing the Fourier amplitudes (and the Fourier phases) computed with different waveform families against each other, as a function of the dimensionless PN parameter $x$ and the GW frequency in Hz. The dimensionless PN parameter $x$ corresponding to the frequency $f$ for harmonic $n$ is computed using the standard SPA relation $x^{3/2} = 2\pi M f / n$.

When comparing amplitudes and phases, we isolate spin precession effects by normalizing or subtracting by the controlling factors in the non-precessing case. The Fourier amplitude of the $n$-th harmonic is normalized by the amplitude pre-factor of the SPA:

$$A_0 = \sqrt{\frac{5\pi}{16}} \frac{M^2}{8D_L} \left(\frac{2\pi M f}{n}\right)^{-7/6},$$

(200)

The Fourier phase is subtracted from the non-precessing SPA phase:

$$\Psi_0 = 2\pi ft(f) - n\phi_C(f) - \frac{\pi}{4},$$

(201)

where $t(f)$ and $\phi_C(t(f))$ are obtained from the numerical inversion of $n\phi_C(t) = 2\pi f$ and from the numerical solution to the evolution equations, respectively.

The match comparison is carried out through the so-called *faithfulness*:

$$F_{\tilde{h}_1, \tilde{h}_2} \equiv \frac{\langle \tilde{h}_1 | \tilde{h}_2 \rangle}{\sqrt{\langle \tilde{h}_1 | \tilde{h}_1 \rangle \langle \tilde{h}_2 | \tilde{h}_2 \rangle}},$$

(202)

where $\tilde{h}_{1,2}$ are different Fourier-domain waveforms with the same physical parameters. We define the inner-product in the usual way:

$$\langle \tilde{h}_1 | \tilde{h}_2 \rangle \equiv 4\Re \int_{f_{\text{min}}}^{f_{\text{max}}} \frac{\tilde{h}_1 \tilde{h}_2^*}{S_n} df,$$

(203)

where $\Re[\cdot]$ the real part operator, $(f_{\text{min}}, f_{\text{max}})$ are the boundaries of the detector’s sensitivity band and $S_n$ is
the detector’s spectral noise density. For ground-based detectors, we choose \( f_{\text{min}} = 10 \) Hz and \( f_{\text{max}} = 10^3 \) Hz, with a maximum observation time of 1 hr. For space-based detectors, we choose \( f_{\text{min}} = 10^{-5} \) Hz and \( f_{\text{max}} = 1 \) Hz, with a maximum observation time of 2 yrs. In all cases, we also terminate the comparisons if the system reaches a separation of 6 times the total mass, i.e. the innermost stable circular orbit of a point particle in a Schwarzschild spacetime, prior to reaching the frequency \( f_{\text{max}} \). We here employ white noise, so \( S_n \) can be taken out of the integral and it cancels when computing the faithfulness parameter. We expect the fitting factor, maximized over physical parameters, to be in general higher when computed with colored noise than when computed with white noise. This is because colored noise has the effect of weighing one part of the frequency spectrum more heavily, and thus, the maximization occurs in a reduced frequency window, leading to a higher overlap.

The faithfulness parameter exists in the interval \([-1, 1]\) and it indicates how well waveforms agree with each other, with unity representing perfect agreement. The integrations required are carried out numerically, with errors of \( O(10^{-5}) \); thus, we consider \( F_{\text{h}_1h_2} = 0.9999 \) to be consistent with perfect agreement. A 98% fitting factor is sometimes argued to be “good enough” for detection purposes, but this of course depends on the detection tolerance chosen.

Both when using a waveform measure or a match measure, we will compare the UAA and aligned SPA models to the DFT model. That is, we will treat the DFT model as a reference waveform, say \( \hat{h}_2 \), and let \( \hat{h}_1 \) be the UAA, the aligned SPA or the HSPA waveform.

The comparisons we show below should be considered conservative because the faithfulness parameter is maximized only over the non-physical parameters \( t_{\text{coal}} \) (appearing in the Fourier phase of the analytical models through Eq. (195)) and \( \varphi_{\text{coal}} \) (appearing in the Fourier phase of the analytical models through Eq. (178)). All other parameters, including the physical ones, such as the total mass and mass ratio, are kept unchanged. Higher matches would be obtained by allowing all parameters to vary, as is done in parameter estimation.

### 4. Systems Considered

Different systems will be studied when using different comparison measures. When using the waveform measure, we will investigate the following four systems:

- **System A**: \( \delta m = 0.5, \alpha_0 = 57^\circ \),
- **System B**: \( \delta m = 0.1, \alpha_0 = 57^\circ \),
- **System C**: \( \delta m = 0.5, \alpha_0 = 23^\circ \),
- **System D**: \( \delta m = 0.1, \alpha_0 = 23^\circ \),

where \( \alpha_0 \) is the angle between the line of sight vector \( \hat{N} \) and the Newtonian orbital angular momentum vector \( \hat{L} \) at \( t = 0 \). For these four systems, we choose \((\chi_1, \chi_2) = (0.89, 0.77)\), and initial misalignment angles of 22° and 25°. When considering space-based detectors, we choose a total redshifted mass of \( M = 5 \times 10^6 M_\odot \) and a total observation time of \( T_{\text{obs}} = 2 \) yrs. When considering ground-based detectors, we choose a total mass of \( M = 20 M_\odot \) and a total observation time of \( T_{\text{obs}} = 100 \) secs. We have also investigated other systems, but the results presented will be representative. When computing a UAA waveform, we use for \( \delta \varphi_\nu \), Eq. (114) for Systems A and C, and Eq. (115) for Systems B and D (recall that for systems with small mass differences, different PN expansions are needed).

When using the match measure, we will perform a Monte-Carlo study over 200 points in parameter space for each type of detector, involving systems randomized over all waveform parameters. The misalignment angles will be set equal to each other, but they will be allowed to vary between 0° and 90°. All throughout, we consider typical systems for ground-based detectors with masses in \((5, 20) M_\odot\), and systems for space-based detectors with masses in \((10^5, 10^8) M_\odot\) and mass ratio \( m_1/m_2 \leq 10 \). The distribution of the spin magnitudes \( \chi_1 \) and \( \chi_2 \) is chosen to be flat in \([0, 1]\), and the distributions of unit vectors are chosen to be flat on the sphere.

### B. Match Comparison

Fig. 2 shows the median match and 1-σ deviations between DFT-UAA waveforms (red dotted curve) and DFT-aligned SPA waveforms (blue dashed curve), as a function of the misalignment angle \( \epsilon \) in degrees for ground-based systems (left panel) and space-based systems (right panel).

Several observations are due at this time. First, observe that the match for the aligned-SPA family is significantly worse than that of the UAA family, as soon as the system is even slightly misaligned. This is mostly due to the fact that the spin couplings in the phase evolution equation are greatly overestimated in the aligned-SPA model. Second, observe also that even for misalignment angles around 50° the UAA family achieves matches around 98% for half the systems considered. This is surprising given that UAA waveforms rely on an expansion around 98% for half the systems considered.

This is because the impact of the detector’s motion on the waveform, and in part because typical space-based systems spend more time in the detector band than ground-based ones, thus leading to more important phase discrepancies. Fourth, observe that the median and upper 1-σ match deviations are slightly better for space-based than for ground-based systems.

Fig. 3 shows a similar match calculation, but this time using the HSPA family of [6]. Observe that, for ground-based systems, these waveforms fail to provide a high median match for misalignments \( \epsilon \gtrsim 30^\circ \). Observe also
that similar poor behavior is observed for space-based systems, which have a lower 1-σ deviation that dips below 98% at roughly the same value of \( \epsilon \). Recall that such poor behavior is in spite of HSPA waveforms using the same numerical solution to the precession equations used to compute DFT waveforms. The poor behavior is because one of the requirements of the SPA used to derive HSPA waveforms (that the amplitude of the signal varies much more slowly than its phase) breaks down for highly misaligned systems. While the first time derivative of the phase is much larger than that of the amplitude, their second time derivatives are of the same order. One should keep in mind when comparing HSPA waveforms to UAA ones that the former require the numerical integration of the equations of precession, while the latter are fully analytic.

C. Discontinuity in the Solution to the Equations of Precession

One concern with the waveform family developed here is that the precession phase difference \( \delta \phi_p \) is a discontinuous function of the mass difference \( \delta m \). This quantity satisfies \( \delta \phi_p = \delta \phi_{p,1} \) as given by Eq. (114) if \( \delta m \geq 0.2 \), \( \delta \phi_p = \delta \phi_{p,2} \) as given by Eq. (115) if \( 10^{-5} \leq \delta m < 0.2 \) and \( \delta \phi_p = \delta \phi_{p,3} \) as given by Eq. (117) if \( \delta m < 10^{-5} \). Formally then, the waveform derivatives with respect to \( \delta m \) are ill-defined at the boundaries of the piecewise function.

Let us then investigate whether this discontinuity is a problem. To do so, we compute the match at \( \delta m = 0.2 \) between a waveform that uses \( \delta \phi_{p,2} \) and one that uses \( \delta \phi_{p,1} \). Fig. 4 shows cumulative distributions of faithfulness for ground-based (dotted red curve) and space-based (dashed blue curve) detections. Observe that the match is above 0.999 for over 95% of the systems investigated. This then implies that the formal discontinuity in
the waveform derivative with respect to \( \delta m \) at the boundary of the piecewise function should not affect parameter estimation.

Let us make several observations about these figures. First, recall that all phase quantities are here presented relative to the carrier, non-spinning phase of the corresponding system. Therefore, the \( \sim \mathcal{O}(10) \) oscillations in the phase plots (right panels) occur on a precession timescale, while in reality 850 and 2000 total GW cycles have elapsed for ground-based and space-based systems respectively. Second, observe that spin precession clearly induces modulations on the phase and amplitude respectively. Third, observe that all approximations agree on the frequency of these modulations but not on the amplitudes or overall trends, i.e. the troughs and valleys do occur roughly at the same values of \( x \) for all waveforms. Fourth, in the ground-based detectors for system C, the Fourier amplitude of the DFT shows peculiar features (e.g. at \( x \approx 0.06 \)) that are approximated by the UAA waveform. Thus, these features are not an artifact of the DFT, and we have checked that they are not induced by spectral leakage. Fifth, we can observe a spike in the DFT and UAA phase difference \( \Psi - \Psi_0 \) for space-based system A around \( x = 0.017 \) that is missed by the aligned SPA approximation. By inspecting the corresponding amplitude plots, we can see that these spikes correspond to moments when the amplitudes of the waveforms almost vanish, i.e. the detector is going through a node in the waveform power distribution. We can see that the UAA waveform reproduces this feature, and we checked that it agrees with the DFT when it is present. Sixth, the phase discrepancy between the aligned SPA and DFT models does not seem to be consistent from system to system. This is because the match is too small for the maximization method that we used to yield trustworthy results for \( \phi_\text{coal} \) and \( t_\text{coal} \) in the aligned SPA case. Seventh, the amplitudes are much better recovered by the UAA for systems A and B than C and D. This is because the precession modulation angles \( \delta \phi \) and \( \psi_N \) are worse approximations for the latter systems, as discussed in Sec. [IV C] and shown in Fig. [7].

We compare four waveform models in Fig. [4] using ground-based system C. Three of those models are based on a discrete Fourier transform, and the fourth one is the UAA model. The first DFT model, DFT1, is the one used in the rest of this section, constructed using the full numerical solution to the equations of motion. The second one, DFT2, is computed using the carrier orbital phase \( \phi_c(t) \) from Eq. (178), together with precession modulation phases \( \psi_N(t) \), and \( i_L(t) \) computed with the analytical solution for \( L(t) \) derived in Sec. [III] and using

\[
\delta \phi(t) = \hat{N}_z \arctan \left( \frac{L_z N_x - L_x}{L_z N_y - L_y} \right),
\]

an approximation valid for any degree of misalignment between \( \hat{N} \) and \( \hat{L} \). The third one, DFT3, is identical to DFT2 but for the precession modulation phases \( \delta \phi(t), \psi_N(t), \) and \( i_L(t) \), using those used to derive the UAA waveform, derived in Sec [IV C]. The top panel of Fig. [7] shows that the amplitude discrepancy between the DFT1 and the DFT2 models is much smaller than between the DFT1 and UAA models (Fig. [5], third plot from the top on the left panel), meaning that the main source of inaccuracy is not due to the inaccuracy in \( L(t) \). The bottom panel shows that the DFT3 amplitude is very well approximated by the UAA amplitude. The main source of amplitude discrepancy between the DFT and UAA models for systems C and D that can be observed in Figs. [4] and [6] is thus the Fourier decomposition derived in Sec. [IV C], which is less accurate when \( \hat{N} \) and \( \hat{L} \) are close to being aligned.

D. Waveform Measure

Fig. [5] and [6] compare the dominant \( \ell = 2 \), Fourier waveform amplitude (left panels) and phase (right panels) for Systems A through D as a function of the PN parameter \( x \) (bottom axis) and the GW frequency in Hz (top axis) for ground-based and space-based systems respectively. The solid black curves correspond to the DFT waveform, the red dotted curves to the UAA waveform and the blue dashed curves to the aligned-SPA waveforms. For reference, the total accumulated phase of the waveform, derived in Sec. [IV C]. The top panel of Fig. 7 shows that the amplitude discrepancy between the DFT1 and UAA models (Fig. 5, third plot from the top) is much smaller than between the DFT2 models is much smaller than between the DFT1 and UAA models (Fig. 5, third plot from the top) but for the precession modulation phases \( \delta \phi(t), \psi_N(t) \), and \( i_L(t) \), using those used to derive the UAA waveform, derived in Sec [IV C]. The top panel of Fig. 7 shows that the amplitude discrepancy between the DFT1 and the DFT2 models is much smaller than between the DFT1 and UAA models (Fig. [5], third plot from the top on the left panel), meaning that the main source of inaccuracy is not due to the inaccuracy in \( L(t) \). The bottom panel shows that the DFT3 amplitude is very well approximated by the UAA amplitude. The main source of amplitude discrepancy between the DFT and UAA models for systems C and D that can be observed in Figs. [4] and [6] is thus the Fourier decomposition derived in Sec. [IV C], which is less accurate when \( \hat{N} \) and \( \hat{L} \) are close to being aligned.

![Graph](image)

**FIG. 4:** Cumulative distributions of faithfulnesses between a waveform that uses \( \delta \phi_{N,1} \) and one that uses \( \delta \phi_{N,2} \) at \( \delta m = 0.2 \) for a ground-based (dotted red curve) and space-based (dashed blue curve) set of detections. Observe that for over 95% of the systems investigated, the match is higher than 0.999, implying the discontinuity would not have a serious effect in parameter estimation.
FIG. 5: Comparison of the Fourier amplitude (left) and phase (right) of the $\ell = 2$ waveform harmonic for ground-based systems as a function of the PN parameter $x = (\pi M f)^{2/3}$ (bottom axis) and as a function of the frequency in Hz (the top axis). The solid black curve corresponds to the DFT result and the dashed red curve to the UAA. The accumulated phase of the time-domain $\ell = 2$ harmonic for each system is $2\Delta\phi_{\text{orb}} \sim 850$ cycles. From top to bottom, we present results for Systems A, B, C and D.
FIG. 6: Same as Fig. 5 for space-based systems. The accumulated phase of the $\ell = 2$ harmonic is $2\Delta \phi_{\text{orb}} \sim 2000$ cycles.
FIG. 7: Comparison between DFT and UAA waveform amplitudes, with same parameters as the third row of Fig. 5 (ground-based system C). On the top, comparison between the amplitudes of two DFT waveforms, one constructed with the fully numerical solution to the equations of motion (black solid line, the DFT waveform that we used in the rest of this section), and the other constructed with the analytical solution for $L(t)$ derived in Sec. III, as well as $\phi_C(t)$ from Eq. (178) (dotted red line). At the bottom, comparison between the amplitudes of a DFT waveform constructed with the phases $\delta \phi(t)$, $\psi_N(t)$, and $i_L(t)$ derived in Sec. IV C as well as $\phi_C(t)$ from Eq. (178) (solid black line), and the UAA waveform (dotted red line).

VI. DISCUSSION

The coming enhancements of ground-based detectors will allow for the first direct detection of gravitational waves. In order to carry out efficient searches, one needs computationally cheap and accurate waveforms. Systems with spins will generically undergo precession, unless the spins are perfectly aligned with the orbital angular momentum. Precession will induce a drastic modification to the waveform, generating corrections in both the phase and amplitude. Such modifications cannot be captured by spin-aligned waveform families, as we demonstrate in this paper.

Binaries in the presence of gas, however, will tend to have spin vectors almost aligned with the orbital angular momentum vector [25, 26, 28], i.e. the misalignment angles should be small. Motivated by this, we have constructed a waveform family that captures faithfully the main features of GWs emitted by compact binaries with small spin-orbital angular momentum misalignment angles. One can think of this waveform family as a perturbation of the spin-aligned family, with corrections that model precession effects that enter both the waveform amplitude and phase.

The waveforms calculated here are purely analytical, constructed both in the time- and in the frequency-domain. Such analytical waveforms have several advantages. On the one hand, analytical waveforms are usually computationally more efficient to evaluate. Given the large number of templates needed for parameter estimation of spinning systems, computational efficiency is highly desirable. On the other hand, the analytic structure provides important physical insight into how each precession effect comes into play. Such insight can then be used to construct simpler, phenomenological waveforms, such as those recently constructed for binaries where one object is not spinning [38].

The mathematical methods used to construct these analytical waveforms had never been used in waveform modeling, to our knowledge. These methods, however, are very well-known in other fields, such as non-linear optics and aerodynamics. The first method is that of multiple scale analysis, amenable to problems with several timescales that separate. This method allows us to solve the precession equations analytically as an expansion in the ratio of the precession to the radiation-reaction timescale. The second method is that of uniform asymptotics, which allows us to construct a single asymptotic expansion to the solution of a given problem, instead of a series of asymptotic expansions in different regimes. This method is essential to cure the stationary phase approximation, which fails in the presence of precession due to the coalescence of stationary points.

Many other problems would benefit from the application of the mathematical methods implemented here. For example, one could study compact binary inspirals, where the spin angular momenta has a small magnitude (relative to the orbital angular momentum) but arbitrary orientation. This application would be complementary to the example studied here. The resulting analytic waveform can be thought of as a perturbation of the non-spinning SPA waveform. Similarly, one can study compact binaries where one component has arbitrary angular momentum, but the companion has a small spin with arbitrary orientation. This case would be intermediate between the one studied here and the one where both binary components have small spin. The resulting analytic waveform can be thought of as a perturbation of a simple precessing waveform. We are currently investigating both of these cases.
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Appendix A: Frequency evolution

The PN evolution equation for the orbital frequency for binaries on quasicircular orbits is given at 2.5PN order by [17] [28]

$$\dot{\omega} = \frac{a_0}{M^2} (M\omega)^{4/3} \left( 1 + \sum_{n \geq 2} a_n (M\omega)^{n/3} \right),$$  \hspace{1cm} (A1)

$$a_0 = \frac{96\nu}{5},$$  \hspace{1cm} (A2)

$$a_2 = -\left( \frac{743}{336} + \frac{11\nu}{4} \right),$$  \hspace{1cm} (A3)

$$a_3 = 4\pi - \beta_{1.5},$$  \hspace{1cm} (A4)

$$a_4 = \frac{40}{233} \left( \frac{136}{8} + \frac{59\nu^2}{18} - \sigma \right),$$  \hspace{1cm} (A5)

$$a_5 = -\left( \frac{4159}{672} + \frac{189}{8}\nu \right) \pi - \beta_{2.5},$$  \hspace{1cm} (A6)

$$\beta_{1.5} = \sum_{A=1}^2 \left( \frac{113}{12} + \frac{25\mu_M}{4m_A^2} \right) S_A \cdot \hat{L},$$  \hspace{1cm} (A7)

$$\beta_{2.5} = \sum_{A=1}^2 \left[ \frac{8209}{84} - \frac{281}{8}\nu \right] S_A \cdot \hat{L}.$$  \hspace{1cm} (A8)

Using $\xi^3 = M\omega$, we can express the carrier phase $\phi_C$ at 6PN order in terms of these couplings as

$$\phi_C = \int \frac{\xi^3}{M} dt = \int \frac{\xi^3}{M\xi} d\xi$$

$$= \phi_{\text{total}} - \xi^{-5} \left[ \phi_0 + \phi_2 \xi^2 + \phi_3 \xi^3 + \phi_4 \xi^4 + \phi_5 \xi^5 + \phi_6 \xi^6 + \phi_7 \xi^7 + \phi_8 \xi^8 + \phi_9 \xi^9 + \phi_{10} \xi^{10} + \phi_{11} \xi^{11} + \phi_{12} \xi^{12} \right],$$  \hspace{1cm} (A9)

$$\phi_0 = \frac{3}{5a_0},$$  \hspace{1cm} (A10)

$$\phi_2 = -\frac{a_2}{a_0},$$  \hspace{1cm} (A11)

$$\phi_3 = \frac{3a_3}{2a_0},$$  \hspace{1cm} (A12)

$$\phi_4 = \frac{3(a_4 - a_2^2)}{a_0},$$  \hspace{1cm} (A13)

$$\phi_5 = \frac{3(a_5 - 2a_3 a_2)}{a_0} \log \xi,$$  \hspace{1cm} (A14)

$$\phi_6 = \frac{3}{a_0} (2a_2 a_4 + a_3^2 - a_2^3),$$  \hspace{1cm} (A15)

$$\phi_7 = \frac{3}{2a_0} (2a_2 a_5 + 2a_3 a_4 - 3a_2^2 a_3),$$  \hspace{1cm} (A16)

$$\phi_8 = \frac{1}{a_0} (2a_3 a_5 + a_2^2 - 3a_2^2 a_4 - 3a_2 a_3 + a_2^4) - 6 \log \xi,$$  \hspace{1cm} (A17)

$$\phi_9 = \frac{3}{4a_0} (2a_4 a_5 - 3a_2^2 a_5 - 6a_2 a_3 a_4 - a_3^3 + 4a_2^3 a_3),$$  \hspace{1cm} (A18)

$$\phi_{10} = -\frac{3}{5a_0} \left( a_2^2 - 6a_2 a_3 a_5 - 3a_2^2 a_4 \right)$$

$$- 3a_2^2 a_4 + 4a_2^3 a_4 + 6a_2 a_3^2 - a_2^5 \right) + 3\nu \log \xi,$$  \hspace{1cm} (A19)

$$\phi_{11} = \frac{1}{2a_0} (6a_2 a_4 a_5 + 3a_3 a_5 - 4a_4 a_5)$$

$$+ 3a_2 a_4^2 - 12a_2 a_3 a_4 - 4a_2^2 a_3 + 5a_2 a_3),$$  \hspace{1cm} (A20)

$$\phi_{12} = \frac{3}{7a_0} (3a_2 a_5^2 + 6a_3 a_4 a_5 - 12a_2 a_4 a_5 + a_3 - 6a_2 a_4^2)$$

$$- 12a_2 a_3 a_4 - 5a_2 a_4 - a_4 + 10a_2^2 a_3 - a_2^5).$$  \hspace{1cm} (A21)

The factors proportional to $\log \xi$ in $\phi_8$ and $\phi_{10}$ come from the reabsorption of $\log(\omega/\omega_0)$ terms from the amplitude to the phase that would lead to an unphysical arbitrary amplitude factor [44].

We can also find a time-ordinal frequency relation using $\int dt = \int d\xi/\xi$:

$$t = t_{\text{total}} - \xi^{-8} \left[ t_0 + t_2 \xi^2 + t_3 \xi^3 + t_4 \xi^4 + t_5 \xi^5 + t_6 \xi^6 \right.$$

$$t_7 \xi^7 + t_8 \xi^8 + t_9 \xi^9 + t_{10} \xi^{10} + t_{11} \xi^{11} + t_{12} \xi^{12}],$$  \hspace{1cm} (A22)

$$t_0 = \frac{3}{8a_0},$$  \hspace{1cm} (A23)

$$t_2 = -\frac{a_2}{2a_0},$$  \hspace{1cm} (A24)

$$t_3 = -\frac{3a_3}{5a_0},$$  \hspace{1cm} (A25)

$$t_4 = -\frac{3(a_4 - a_2^2)}{4a_0},$$  \hspace{1cm} (A26)

$$t_5 = \frac{a_5 - 2a_2 a_3}{a_0},$$  \hspace{1cm} (A27)

$$t_6 = \frac{3}{2a_0} (2a_2 a_4 + a_3^2 - a_2^2),$$  \hspace{1cm} (A28)

$$t_7 = \frac{3}{a_0} (2a_2 a_5 + 2a_3 a_4 - 3a_2^2 a_3),$$  \hspace{1cm} (A29)

$$t_8 = -\frac{3}{a_0} (2a_3 a_5 + a_2^2 - 3a_2^2 a_4 - 3a_2 a_3^2 + a_2^4) \log \xi,$$  \hspace{1cm} (A30)
\[ t_9 = \frac{3}{a_0} (2a_2a_5 - 3a_2^2a_5 - 6a_2a_3a_4 - a_3^2 + 4a_0^2a_3), \]
\[ t_{10} = -\frac{3}{2a_0} (a_3^2 - 6a_2a_3a_5 - 3a_2a_3^2 - 3a_2a_4^2 + 4a_0^2a_4 + 6a_2a_3^2 - a_5^2), \]
\[ t_{11} = \frac{1}{a_0} (6a_2a_4a_5 + 3a_2^2a_5 - 4a_2a_4). \]

The amplitudes \( B_{i,j} \) with \( i = + \) or \( - \), and \( j = 1 \) or \( 2 \) can be computed using Eqs. \( \text{B1-B3} \) at \( t = 0 \), with \( Q_{i,j}^{(1)} \rightarrow B_{i,j}^{(1,0)} \) and \( L_{1,k}^{(1)} \rightarrow -S_{1,k} - S_{2,k} \). Using \( S_{j,k}(t=0) = m_j^2 \chi_j \hat{S}_{j,k,0} \), we get

\[ B_{1,1}^{(1,0)} = T_{+,1} \hat{S}_{1,x,0} + T_{+,2} \hat{S}_{2,x,0}, \]
\[ B_{1,2}^{(1,0)} = T_{+,1} \hat{S}_{1,y,0} + T_{+,2} \hat{S}_{2,y,0}, \]
\[ B_{0,1}^{(1,0)} = T_{-,1} \hat{S}_{1,x,0} + T_{-,2} \hat{S}_{2,x,0}, \]
\[ B_{0,2}^{(1,0)} = T_{-,1} \hat{S}_{1,y,0} + T_{-,2} \hat{S}_{2,y,0}, \]
\[ T_{+,1} = -\frac{(T_4 + T_2)T_5}{T_3}, \]
\[ T_{+,2} = T_{-,1} \hat{S}_{1,x,0} + T_{-,2} \hat{S}_{2,y,0}, \]
\[ T_{-,1} = \frac{(T_4 + T_2)T_5}{T_3}, \]
\[ T_{-,2} = T_{-,1} \hat{S}_{1,y,0} + T_{-,2} \hat{S}_{2,x,0}. \]
In this section, we give the exact expressions for the precession phases that we used in our implementation. \( \phi_{\nu, \pm} \) is calculated from \( \phi_{\nu, m} \pm \delta \phi_{\nu} \). We give them in terms of the \( a_i \) given in appendix A and \( \delta m = (m_1 - m_2)/M \).

The mean precession phase we used in our implementation is

\[
\phi_{\nu, m} = \frac{1}{2} \int (\omega_{\nu, +} + \omega_{\nu, -}) \frac{dt}{d\xi} d\xi = \frac{1}{2} \int \frac{b + c + d + e + f + g}{a} \, d\xi
\]

\[
= \phi_{\nu, m, 0} + \phi_{\nu, m, -3} \xi^{-3} + \phi_{\nu, m, -2} \xi^{-2} + \phi_{\nu, m, -1} \xi^{-1} + \phi_{\nu, m, 0} \log \xi + \phi_{\nu, m, 1} + \phi_{\nu, m, 2} + \phi_{\nu, m, 3},
\]

\[
\phi_{\nu, m, -3} = -\frac{7 - \delta m^2}{8a_0},
\]

\[
\phi_{\nu, m, -2} = -\frac{3}{32a_0} \left[ (5 + 4\delta m - \delta m^2)\chi_1 + (5 - 4\delta m - \delta m^2)\chi_2 \right],
\]

\[
\phi_{\nu, m, -1} = \frac{3}{8a_0} \left[ a_2(7 - \delta m^2) + 3(1 - \delta m^2)\chi_1\chi_2 \right].
\]
\[
\phi_r^{(0)} = -\frac{3}{16a_0} \{2a_3(7 - \delta m^2) + a_2[5 + 4\delta m - \delta m^2]_1 + (5 - 4\delta m - \delta m^2)_2\},
\]

\[
\phi_r^{(1)} = \frac{3}{16a_0} \{2(a_3^2 - a_4)(7 - \delta m^2) - a_3[5 + 4\delta m - \delta m^2]_1 + (5 - 4\delta m - \delta m^2)_2 + 6a_2(1 - \delta m^2)_1\},
\]

\[
\phi_r^{(2)} = \frac{3}{32a_0} \{(4a_2a_3 - 2a_5)(7 - \delta m^2) + (a_3^2 - a_4)[5 + 4\delta m - \delta m^2]_1 + (5 - 4\delta m - \delta m^2)_2 + 6a_3(1 - \delta m^2)_1\},
\]

\[
\phi_r^{(3)} = -\frac{1}{16a_0} \{(2a_3^2 - 2a_3^2 - 4a_2a_4)(7 - \delta m^2)
- (2a_2a_3 - a_5)[5 + 4\delta m - \delta m^2]_1 + (5 - 4\delta m - \delta m^2)_2 + 6(a_2^2 - a_4)(1 - \delta m^2)_1\},
\]

\[
\delta \phi_r \text{ for } \delta m \sim O(c^{-1}) \text{ is given by}
\]

\[
\delta \phi_{r,1} = \frac{1}{2} \int (\omega_{r,+} - \omega_{r,-}) \frac{dt}{d\xi} = \frac{1}{2} \int \sqrt{(b - c + d - e + f - g)^2 + 4(c - f)(b - g)} d\xi
= \delta \phi_{r,1,0} + \delta \phi_{r,1}^{(-3)}\xi^{-3} + \delta \phi_{r,1}^{(-2)}\xi^{-2} + \delta \phi_{r,1}^{(-1)}\xi^{-1} + \delta \phi_{r,1}^{(0)} \log \xi + \delta \phi_{r,1}^{(1)} \xi + \delta \phi_{r,1}^{(2)} \xi^2 + \delta \phi_{r,1}^{(3)} \xi^3,
\]

\[
\delta \phi_{r,2} = -\frac{9}{32a_0\delta m}[2a_2\delta m^2 - (1 - \delta m^2)_1\chi_1],
\]

\[
\delta \phi_{r,3} = \frac{9}{128a_0\delta m} \{8\delta m^5a_3(\chi_1 - \chi_2) + 32\delta m^4[a_2^2 - a_1 + a_3(\chi_1 + \chi_2)] + 24\delta m^3a_3(\chi_1 - \chi_2)
- [16\delta m^2a_2 - (1 + \delta m^2)(9 - \delta m^2)\chi_1^2 + 2(\delta m^4 - 12\delta m^2 + 11)\chi_1\chi_2 - (1 - \delta m^2)(9 - \delta m^2)\chi_2^2](1 - \delta m^2)_1\chi_2],
\]

\[
\delta \phi_{r,4} = -\frac{9}{1024a_0\delta m^2} \{32\delta m^6(a_2^2 - a_4)[(\chi_1 - \chi_2) + 128\delta m^5[a_5 - 2a_2a_3 + (a_2^2 - a_4)(\chi_1 + \chi_2)]
+ 96\delta m^4(a_2^2 - a_1)(\chi_1 - \chi_2) + (\delta m^6(\chi_1^2 - \chi_2^2)(\chi_1 + \chi_2) + 2\delta m^5(3\chi_1^2 + 2\chi_1\chi_2 + 3\chi_2^2)(\chi_1 + \chi_2)
- \delta m^4[16a_3 + 3(\chi_1 + \chi_2)\chi_1\chi_2)] + 4\delta m^3\{16a_3 + [8a_2 - 11(\chi_1 + \chi_2)\chi_1\chi_2)](\chi_1 + \chi_2)
- \delta m^2[48a_2 - 105\chi_1 - 226(\chi_1\chi_2 - 105\chi_1\chi_2)] + 25\delta m^4(45\chi_1^2 - 106\chi_1\chi_2 + 45\chi_2^2)(\chi_1 + \chi_2)
- 9(3\chi_1^2 - 10\chi_1\chi_2 + 3\chi_2^2)(\chi_1 - \chi_2))(1 - \delta m^2)_1\chi_2],
\]

\[
\delta \phi_{r,5} = \frac{3}{128a_0\delta m^3} \{8\delta m^3(a_2^2 - 2a_3)[(3 + 4\delta m + \delta m^2)\chi_1 - (3 - 4\delta m + \delta m^2)\chi_2] + (\delta m^4a_2(\chi_1 + \chi_2)^2
+ 2\delta m^3[2a_3 + a_2(\chi_1 + \chi_2)](\chi_1 - \chi_2) - 8\delta m^2[2(a_4 - a_2) + a_3(\chi_1 + \chi_2) + a_2(\chi_1^2 + 3\chi_1\chi_2 + \chi_2^2)]
- 6\delta m[2a_3 + 3a_2(\chi_1 + \chi_2)](\chi_1 - \chi_2) - a_2(9\chi_1^2 - 22\chi_1\chi_2 + 9\chi_2^2)(1 - \delta m^2)_1\chi_2],
\]

When \(\delta m \sim O(c^{-1})\), \(\delta \phi_r\) is given by

\[
\delta \phi_{r,6} = \frac{1}{2} \int (\omega_{r,+} - \omega_{r,-}) \frac{dt}{d\xi} = \frac{1}{2} \int \sqrt{(b - c + d - e + f - g)^2 + 4(c - f)(b - g)} d\xi
= \delta \phi_{r,6,0} + \frac{1}{T^3} \{\delta \phi_{r,6}^{(-3)}\xi^{-3} + \delta \phi_{r,6}^{(-2)}\xi^{-2} + \delta \phi_{r,6}^{(-1)}\xi^{-1} + \delta \phi_{r,6}^{(0)} \log \xi + \delta \phi_{r,6}^{(1)} \xi + \delta \phi_{r,6}^{(2)} \xi^2 + \delta \phi_{r,6}^{(3)} \xi^3 \}
+ \delta \phi_{r,6}^{(1)} \log \xi + \log [4\delta m - 3(\chi_1 - \chi_2)\xi + T],
\]

\[
T = [16\delta m^2 - 24\delta m(\chi_1 - \chi_2)\xi + (9\chi_1^2 - 2\chi_1\chi_2 + 9\chi_2^2)\xi^2]^{1/2},
\]

\[
\delta \phi_{r,6}^{(-3)} = -\frac{3}{16a_0},
\]
\[ \delta \phi_{r,2}^{(-2)} = \frac{3}{512 a_0 d m^4} [96 a_2 d m^2 + 9(1 + 4 d m + d m^2) \chi_2^2 - 10(5 - 3 d m^2) \chi_1 \chi_2 + 9(1 - 4 d m + d m^2) \chi_2^2], \]  
\[ \delta \phi_{r,2}^{(-1)} = \frac{9}{32 a_0 d m^2} [128 a_0 d m^2 (1 + 4 d m + d m^2) \chi_1 (1 - 4 d m + d m^2) \chi_2], \]  
\[ \delta \phi_{r,2}^{(0)} = \frac{3}{16 a_0 (9 \chi_1^2 - 2 \chi_1 \chi_2 + 9 \chi_2^2)} [3456 \chi_1^4 - 81 \chi_1^2 - 567 \chi_1 \chi_2 + 123 \chi_1^3 \chi_2 - 1530 \chi_1^3 \chi_2^3 + 1123 \chi_1 \chi_2^3 - 567 \chi_1^5 \chi_2 - 864 \chi_1^3 \chi_2^3 + 27 \chi_1^3 \chi_2 + 63 \chi_1 \chi_2^3 + 3 \chi_1^3 \chi_2 + 27 \chi_1 \chi_2^3 + 2 \chi_1^3 \chi_2 + 2 \chi_1 \chi_2^3, \]  
\[ \delta \phi_{r,2}^{(1)} = \frac{9}{32 a_0 (9 \chi_1^2 - 2 \chi_1 \chi_2 + 9 \chi_2^2)^2} [576 \chi_1^4 - 2781 \chi_1 \chi_2 + 5577 \chi_1^3 \chi_2 - 7870 \chi_1 \chi_2^3 + 5777 \chi_1^2 \chi_2^4 - 2781 \chi_1 \chi_2^5 + 243 \chi_1^4 \chi_2^2 - 144 \delta m^2 a_2 (81 \chi_1^6 - 567 \chi_1^3 \chi_2 + 123 \chi_1^3 \chi_2^3 - 1530 \chi_1 \chi_2^3 + 1123 \chi_1^3 \chi_2^3 - 567 \chi_1 \chi_2^3 - 864 \chi_1^3 \chi_2^3 + 27 \chi_1^3 \chi_2 + 63 \chi_1 \chi_2^3 + 3 \chi_1^3 \chi_2 + 27 \chi_1 \chi_2^3 + 2 \chi_1^3 \chi_2 + 2 \chi_1 \chi_2^3], \]  
\[ \delta \phi_{r,2}^{(2)} = \frac{9}{16 a_0 (9 \chi_1^2 - 2 \chi_1 \chi_2 + 9 \chi_2^2)^3} [2538 \chi_1^4 \chi_2 - 25065 \chi_1 \chi_2^3 + 53780 \chi_1^3 \chi_2^3 - 25065 \chi_1 \chi_2^3 + 53780 \chi_1^3 \chi_2^3 - 25065 \chi_1 \chi_2^3 - 5238 \chi_1^3 \chi_2^3 (\chi_1 - \chi_2) \chi_1 \chi_2 - 64 \delta m [45 \chi_1^2 - 34 \chi_1 \chi_2 + 45 \chi_1^2 \chi_2^3 (\chi_1 - \chi_2)] - 2 \delta m^2 a_2 (27 \chi_1^4 + 230 \chi_1^2 \chi_2 + 194 \chi_1^2 \chi_2^3 + 230 \chi_1 \chi_2^3 + 27 \chi_1^2 (\chi_1 - \chi_2) \chi_1 \chi_2 - 4 \delta m a_2 (9 \chi_1^2 - 2 \chi_1 \chi_2 + 9 \chi_2^2)^3 (\chi_1 - \chi_2)] - 2 \delta m [45 \chi_1^2 - 34 \chi_1 \chi_2 + 45 \chi_1^2 \chi_2^3 (\chi_1 - \chi_2)] - 2 \delta m a_2 (9 \chi_1^2 - 2 \chi_1 \chi_2 + 9 \chi_2^2)^3 (\chi_1 - \chi_2), \]  
When \( \delta m \ll \mathcal{O}(c^{-1}) \), \( \delta \phi_r \) is given by

\[ \delta \phi_{r,3} = \frac{1}{2} \int (\omega_{r,+} - \omega_{r,-}) \frac{dt}{d \xi} \frac{d \xi}{d t} = \frac{1}{2} \int \sqrt{(b - c - d + e - f - g)^2 + 4(c - f)} (b - g) \frac{d t}{d \xi}, \]  
\[ \delta \phi_{r,3}^{(0)} = T \left( \delta \phi_{r,3}^{(-3)} + \delta \phi_{r,3}^{(-2)} + \delta \phi_{r,3}^{(-1)} + \delta \phi_{r,3}^{(0)} + \delta \phi_{r,3}^{(1)} \right), \]  
\[ \delta \phi_{r,3}^{(-3)} = \frac{9 \delta m (\chi_1 - \chi_2)}{4 a_0 (9 \chi_1^2 - 2 \chi_1 \chi_2 + 9 \chi_2^2)}, \]  
\[ \delta \phi_{r,3}^{(-2)} = \frac{9 \delta m (27 \chi_1^2 - 26 \chi_1 \chi_2 + 27 \chi_2^2)}{8 a_0 (9 \chi_1^2 - 2 \chi_1 \chi_2 + 9 \chi_2^2)}, \]  
\[ \delta \phi_{r,3}^{(-1)} = \frac{9 (\chi_1 + \chi_2)}{8 a_0 (9 \chi_1^2 - 2 \chi_1 \chi_2 + 9 \chi_2^2)} + \frac{27 \delta m (\chi_1 - \chi_2)}{4 a_0 (9 \chi_1^2 - 2 \chi_1 \chi_2 + 9 \chi_2^2)}. \]
Note that despite appearances, \( \delta\phi \delta \dot{\chi} \frac{\chi_0}{32a_0\chi_1\chi_2} \) \( \left( T - \sqrt{9\chi_1^2 - 2\chi_1\chi_2 + 9\chi_2^2} \right) \).  

\begin{align}
\delta\phi^{(0)}_{\nu,3} &= -\frac{9a_2T}{16a_0} + \frac{9a_3\chi_1 + \chi_2 + 25m(\chi_1 - \chi_2)}{32a_0\chi_1\chi_2} \left( T - \sqrt{9\chi_1^2 - 2\chi_1\chi_2 + 9\chi_2^2} \right), \\
\delta\phi^{(1,1)}_{\nu,3} &= \frac{9a_3(5 + 16\delta m)\chi_1 - (5 - 16\delta m)\chi_2)(\chi_1 - \chi_2) - 8a_2[\chi_1 + \chi_2 + \delta m(\chi_1 - \chi_2)]\chi_1\chi_2}{64a_0\chi_1\chi_2}, \\
\delta\phi^{(0)}_{\nu,3} &= -\frac{9a_3}{32a_0}, \\
\delta\phi^{(1,2)}_{\nu,3} &= \frac{9}{16a_0(9\chi_1^2 - 2\chi_1\chi_2 + 9\chi_2^2)^{7/2}} \left( 9(9\chi_1^2 - 2\chi_1\chi_2 + 9\chi_2^2)^2(9\chi_1^2 - 2\chi_1\chi_2 + 9\chi_2^2)^2 - 10(\chi_1 - \chi_2)^2\chi_1^2\chi_2^2 \right) - \delta m[12a_3(9\chi_1^2 - 2\chi_1\chi_2 + 9\chi_2^2)^3(\chi_1 - \chi_2) - 36a_2(9\chi_1^2 - 2\chi_1\chi_2 + 9\chi_2^2)^2(\chi_1^2 - 2\chi_1\chi_2 + 3\chi_2^2)(\chi_1^2 - \chi_2^2)] \\
&- 8(81\chi_1^4 - 840\chi_1^3\chi_2 + 280\chi_1^2\chi_2^2 - 840\chi_1\chi_2^3 + 81\chi_2^4)(\chi_1^2 - \chi_2^2)^2\chi_1^2\chi_2^2}.  
\end{align}

Note that despite appearances, \( \delta\nu_{\nu,3} \) is regular in the limits \( \chi_1 \to 0 \) and \( \chi_2 \to 0 \). From \( \omega_{\nu,m}(\xi) \), \( \delta\omega_{\nu}(\xi) \) and \( \xi \), we can compute \( \phi_{\nu,\pm} = \omega_{\nu,\pm} = \omega_{\nu,\pm} \pm \delta\omega_{\nu} \). We get

\begin{align}
\omega_{\nu,m} &= \omega \left\{ \frac{5}{24} (7 - \delta m^2) \xi^2 + \frac{1}{8} \left[ (5 + 4\delta m - \delta m^2) \chi_1 + (5 - 4\delta m - \delta m^2) \chi_2 \right] \xi^3 - \frac{7}{8} (1 - \delta m^2) \chi_1 \chi_2 \xi^4 \right\}, \quad \delta\omega_{\nu} = \frac{\omega}{\delta\nu_{\nu}} \left\{ \frac{15\delta m^2}{16} \xi^7 + \frac{33\delta m}{64} \left\{ (3 + 4\delta m + \delta m^2) \chi_1 - (3 - 4\delta m + \delta m^2) \chi_2 \right\} \xi^8 \right. \\
&+ \frac{9}{128} \left\{ (3 + 4\delta m + \delta m^2)^2 \chi_1^2 - 2 (1 - \delta m^2) \chi_1 \chi_2 + (3 - 4\delta m + \delta m^2)^2 \chi_2^2 \right\} \xi^9 \right. \\
&+ \frac{39}{64} (1 - \delta m^2) \left\{ (1 + \delta m) \chi_1 + (1 - \delta m) \chi_2 \right\} \chi_1 \chi_2 \xi^{10} + \frac{21}{64} (1 + \delta m)^2 (1 - \delta m)^2 \chi_1^2 \chi_2^2 \xi^{11} \}. \end{align}

### Appendix D: Amplitude and Phase Modulations

Using \( \mathbf{V}_1 = \mathbf{\hat{N}} \times \mathbf{\hat{N}} \) and \( \mathbf{V}_2 = \mathbf{z} - \mathbf{\hat{N}}(\mathbf{\hat{N}} \cdot \mathbf{\hat{N}}) \), the amplitudes and phases of Eq. (183) are given by

\begin{align}
A_{0,n,k,m} &= k \arccos \hat{N}_z + m \arctan \frac{V_{1,z}}{V_{2,z}} \\
A_{\phi,\pm} &= \text{sign} \left( \hat{N}_x B^{(1,0)}_{\pm,1} - \hat{N}_y B^{(1,0)}_{\pm,2} \right) \left[ \frac{B^{(1,0)}_{\pm,1} + B^{(1,0)}_{\pm,2}}{N_x^2 + N_y^2} \right]^{1/2} \hat{N}_z, \\
\phi^{(0)}_{\phi,\pm} &= -\arctan \left( \frac{\hat{N}_x B^{(1,0)}_{\pm,1} + \hat{N}_y B^{(1,0)}_{\pm,2}}{\hat{N}_x B^{(1,0)}_{\pm,2} - \hat{N}_y B^{(1,0)}_{\pm,1}} \right), \\
A_{\chi,\pm} &= \text{sign} \left[ \left( B^{(1,0)}_{\pm,1} V_{2,x} + B^{(1,0)}_{\pm,2} V_{2,y} \right) - V_{2,z} \left( B^{(1,0)}_{\pm,1} V_{1,x} + B^{(1,0)}_{\pm,2} V_{1,y} \right) \right] \\
&\times \left\{ \left[ V_{1,z} \left( B^{(1,0)}_{\pm,1} V_{2,x} + B^{(1,0)}_{\pm,2} V_{2,y} \right) - V_{2,z} \left( B^{(1,0)}_{\pm,1} V_{1,x} + B^{(1,0)}_{\pm,2} V_{1,y} \right) \right]^2 \right. \\
&\left. + \left( V_{1,z} \left( B^{(1,0)}_{\pm,1} V_{2,y} - B^{(1,0)}_{\pm,2} V_{2,x} \right) - V_{2,z} \left( B^{(1,0)}_{\pm,1} V_{1,y} - B^{(1,0)}_{\pm,2} V_{1,x} \right) \right)^2 \right\}^{1/2} \left( V_{1,z}^2 + V_{2,z}^2 \right)^{-1} \frac{\xi}{\nu}, \\
\phi^{(0)}_{\chi,\pm} &= -\arctan \left[ \left( V_{1,z} \left( B^{(1,0)}_{\pm,1} V_{2,y} - B^{(1,0)}_{\pm,2} V_{2,x} \right) - V_{2,z} \left( B^{(1,0)}_{\pm,1} V_{1,y} - B^{(1,0)}_{\pm,2} V_{1,x} \right) \right] \\
&\times \left[ V_{1,z} \left( B^{(1,0)}_{\pm,1} V_{2,x} + B^{(1,0)}_{\pm,2} V_{2,y} \right) - V_{2,z} \left( B^{(1,0)}_{\pm,1} V_{1,x} + B^{(1,0)}_{\pm,2} V_{1,y} \right) \right]^{-1} \right) \cdot.
\end{align}
Using this, we can express
\[
A_{\pm,n,k,m} = \text{sign}(A_{c,\pm}) \sqrt{A_{c,\pm}^2 + A_{s,\pm}^2},
\]
(D8)
\[
\phi_{\pm,n,k,m} = \arctan \left( \frac{A_{c,\pm}}{A_{s,\pm}} \right),
\]
(D9)
\[
A_{c,\pm} = nA_{3\phi,\pm} \cos(\phi_{0,\pm}) + kA_{1L,\pm} \cos(\phi_{0,\pm}) + mA_{\psi,\pm} \cos(\phi_{0,\pm}),
\]
(D10)
\[
A_{s,\pm} = nA_{3\phi,\pm} \sin(\phi_{0,\pm}) + kA_{1L,\pm} \sin(\phi_{0,\pm}) + mA_{\psi,\pm} \sin(\phi_{0,\pm}),
\]
(D11)

**Appendix E: Mode-Decomposed, Time-Domain Amplitudes**

The waveforms are expressed as
\[
h(t) = \frac{G\mu\xi^2}{D_Lc^2} \sum_{n \geq 0} \sum_{k \in \mathbb{Z}} \sum_{m = -2, 2} A_{n,k,m}(\theta_N, \phi_N)e^{i[n\psi_c+n\delta\phi+k_1L+m\psi_N]} + \text{c.c.},
\]
(E1)
where \(\psi_c = \phi_c - (6 - 3\nu\xi^2)\xi^3 \log \xi\).

Defining
\[
A_{F} = \frac{1}{2} \left( 1 + \cos^2 \theta_N \right) \cos 2\phi_N,
\]
(E2)
\[
B_{F} = \cos \theta_N \sin 2\phi_N,
\]
(E3)
and using \(\delta m = (m_1 - m_2)/M\) the dimensionless mass difference, we can express the \(A_{n,k,m}\) at 2PN for \(n \neq 0\) as (see [44])
\[
A_{1,1,2} = -A_{1,-1,2} = -\delta m(B_{F} - iA_{F}) \left\{ \frac{21}{128} \xi - \left( \frac{575}{6144} - \frac{367\nu}{3072} \right) \xi^3 + \left[ \frac{21\pi}{128} + i \left( \frac{21 \log 2}{64} + \frac{51}{640} \right) \right] \xi^4 \right\},
\]
(E4)
\[
A_{1,2,2} = -A_{1,-2,2} = \delta m(B_{F} - iA_{F}) \left\{ \frac{3}{32} \xi - \left( \frac{121}{1536} - \frac{41\nu}{768} \right) \xi^3 + \left[ \frac{3\pi}{32} + i \left( \frac{3 \log 2}{16} + \frac{9}{160} \right) \right] \xi^4 \right\},
\]
(E5)
\[
A_{1,3,2} = -A_{1,-3,2} = -\delta m(B_{F} - iA_{F}) \left\{ \frac{1}{128} \xi - \left( \frac{79}{4096} + \frac{17\nu}{2048} \right) \xi^3 + \left[ \frac{\pi}{128} + i \left( \frac{2 \log 2}{64} + \frac{7}{640} \right) \right] \xi^4 \right\},
\]
(E6)
\[
A_{1,4,2} = -A_{1,-4,2} = \delta m(B_{F} - iA_{F}) \left( \frac{5}{3072} - \frac{5\nu}{1536} \right) \xi^3,
\]
(E7)
\[
A_{1,5,2} = -A_{1,-5,2} = -\delta m(B_{F} - iA_{F}) \left( \frac{1}{12288} - \frac{\nu}{6144} \right) \xi^3,
\]
(E8)
\[
A_{1,1,-2} = -A_{1,-1,-2} = \delta m(B_{F} + iA_{F}) \left\{ \frac{21}{128} \xi - \left( \frac{575}{6144} - \frac{367\nu}{3072} \right) \xi^3 + \left[ \frac{21\pi}{128} + i \left( \frac{21 \log 2}{64} + \frac{51}{640} \right) \right] \xi^4 \right\},
\]
(E9)
\[
A_{1,2,-2} = -A_{1,-2,-2} = \delta m(B_{F} + iA_{F}) \left\{ \frac{3}{32} \xi - \left( \frac{121}{1536} - \frac{41\nu}{768} \right) \xi^3 + \left[ \frac{3\pi}{32} + i \left( \frac{3 \log 2}{16} + \frac{9}{160} \right) \right] \xi^4 \right\},
\]
(E10)
\[
A_{1,3,-2} = -A_{1,-3,-2} = \delta m(B_{F} + iA_{F}) \left\{ \frac{1}{128} \xi - \left( \frac{79}{4096} + \frac{17\nu}{2048} \right) \xi^3 + \left[ \frac{\pi}{128} + i \left( \frac{2 \log 2}{64} + \frac{7}{640} \right) \right] \xi^4 \right\},
\]
(E11)
\[
A_{1,4,-2} = -A_{1,-4,-2} = \delta m(B_{F} + iA_{F}) \left( \frac{5}{3072} - \frac{5\nu}{1536} \right) \xi^3,
\]
(E12)
\[
A_{1,5,-2} = -A_{1,-5,-2} = \delta m(B_{F} + iA_{F}) \left( \frac{1}{12288} - \frac{\nu}{6144} \right) \xi^3,
\]
(E13)
\[
A_{2,0,2} = -(A_{F} + iB_{F}) \left[ \frac{3}{4} - \left( \frac{91}{48} - \frac{15\nu}{16} \right) \xi^2 + \frac{3\pi}{2} \xi^3 - \left( \frac{291}{256} + \frac{8137\nu}{2304} - \frac{257\nu^2}{256} \right) \xi^4 \right],
\]
(E14)
\[
A_{2,1,2} = A_{2,-1,2} = (A_{F} + iB_{F}) \left[ \frac{1}{2} - \left( \frac{7}{6} + \frac{\nu}{3} \right) \xi^2 + \xi^3 - \left( \frac{365}{384} + \frac{1861\nu}{1152} - \frac{103\nu^2}{384} \right) \xi^4 \right],
\]
(E15)
\[
A_{2,2,2} = A_{2,-2,2} = -(A_{F} + iB_{F}) \left[ \frac{1}{8} - \left( \frac{7}{48} + \frac{17\nu}{48} \right) \xi^2 + \frac{\pi}{4} \xi^3 - \left( \frac{8581}{15360} + \frac{7247\nu}{9216} - \frac{1541\nu^2}{3072} \right) \xi^4 \right],
\]
(E16)
\[ A_{2,3,2} = A_{2,-3,2} = (A_F + iB_F) \left[ \left( \frac{1}{12} - \frac{\nu}{4} \right) \xi^2 - \left( \frac{829}{3840} - \frac{1735\nu}{2304} + \frac{205\nu^2}{768} \right) \xi^4 \right], \] (E17)

\[ A_{2,4,2} = A_{2,-4,2} = -(A_F + iB_F) \left[ \left( \frac{1}{96} - \frac{\nu}{32} \right) \xi^2 - \left( \frac{55}{1536} - \frac{457\nu}{4608} - \frac{29\nu^2}{1536} \right) \xi^4 \right], \] (E18)

\[ A_{2,5,2} = A_{2,-5,2} = (A_F + iB_F) \left( \frac{1}{256} - \frac{5\nu}{256} + \frac{5\nu^2}{256} \right) \xi^4, \] (E19)

\[ A_{2,6,2} = A_{2,-6,2} = -(A_F + iB_F) \left( \frac{1}{3072} - \frac{5\nu}{3072} + \frac{5\nu^2}{3072} \right) \xi^4, \] (E20)

\[ A_{2,0,-2} = -(A_F - iB_F) \left[ \frac{3}{4} - \frac{91}{48} - \frac{15\nu}{16} \right] \xi^2 + \frac{3\pi^3}{2} \xi^3 - \left( \frac{291}{256} + \frac{8137\nu}{2304} + \frac{257\nu^2}{256} \right) \xi^4 \right], \] (E21)

\[ A_{2,1,-2} = A_{2,-1,2} = -(A_F - iB_F) \left[ \frac{1}{2} - \left( \frac{7}{6} - \frac{\nu}{3} \right) \xi^2 + \pi^3 - \left( \frac{365}{384} + \frac{1861\nu}{1152} - \frac{103\nu^2}{384} \right) \xi^4 \right], \] (E22)

\[ A_{2,2,-2} = A_{2,-2,2} = -(A_F - iB_F) \left[ \frac{1}{8} - \left( \frac{7}{48} + \frac{17}{48} \right) \xi^2 + \frac{\pi^3}{4} - \left( \frac{8581}{15360} + \frac{7247\nu}{9216} - \frac{1541\nu^2}{3072} \right) \xi^4 \right], \] (E23)

\[ A_{2,3,-2} = A_{2,-3,2} = -(A_F - iB_F) \left[ \left( \frac{1}{12} - \frac{\nu}{4} \right) \xi^2 - \left( \frac{829}{3840} + \frac{1735\nu}{2304} + \frac{205\nu^2}{768} \right) \xi^4 \right], \] (E24)

\[ A_{2,4,-2} = A_{2,-4,2} = -(A_F - iB_F) \left[ \left( \frac{1}{96} - \frac{\nu}{32} \right) \xi^2 - \left( \frac{55}{1536} + \frac{457\nu}{4608} - \frac{29\nu^2}{1536} \right) \xi^4 \right], \] (E25)

\[ A_{2,5,-2} = A_{2,-5,2} = -(A_F - iB_F) \left( \frac{1}{256} - \frac{5\nu}{256} + \frac{5\nu^2}{256} \right) \xi^4, \] (E26)

\[ A_{2,6,-2} = A_{2,-6,2} = -(A_F - iB_F) \left( \frac{1}{3072} - \frac{5\nu}{3072} + \frac{5\nu^2}{3072} \right) \xi^4, \] (E27)

\[ A_{3,1,2} = -A_{3,-1,2} = \delta m(B_F - iA_F) \left\{ \frac{45}{128} \xi - \left( \frac{5895}{4096} - \frac{1575\nu}{2048} \right) \xi^3 + \left[ \frac{135\pi}{128} - i \left( \frac{135 \log(3/2)}{64} - \frac{189}{128} \right) \right] \xi^4 \right\}, \] (E28)

\[ A_{3,2,2} = -A_{3,-2,2} = -\delta m(B_F - iA_F) \left\{ \frac{9}{32} \xi - \left( \frac{1071}{1024} - \frac{207\nu}{512} \right) \xi^3 + \left[ \frac{27\pi}{32} + i \left( \frac{27 \log(3/2)}{16} - \frac{189}{160} \right) \right] \xi^4 \right\}, \] (E29)

\[ A_{3,3,2} = -A_{3,-3,2} = -\delta m(B_F - iA_F) \left\{ \frac{9}{128} \xi - \left( \frac{1197}{8192} + \frac{531\nu}{4096} \right) \xi^3 + \left[ \frac{27\pi}{128} - i \left( \frac{27 \log(3/2)}{64} - \frac{189}{640} \right) \right] \xi^4 \right\}, \] (E30)

\[ A_{3,4,2} = -A_{3,-4,2} = -\delta m(B_F - iA_F) \left\{ \frac{135}{2048} - \frac{135\nu}{1024} \right\} \xi^3, \] (E31)

\[ A_{3,5,2} = -A_{3,-5,2} = -\delta m(B_F - iA_F) \left( \frac{81}{8192} - \frac{81\nu}{4096} \right) \xi^3, \] (E32)

\[ A_{3,1,-2} = -A_{3,-1,-2} = -\delta m(B_F + iA_F) \left\{ \frac{45}{128} \xi - \left( \frac{5895}{4096} - \frac{1575\nu}{2048} \right) \xi^3 + \left[ \frac{135\pi}{128} + i \left( \frac{135 \log(3/2)}{64} - \frac{189}{128} \right) \right] \xi^4 \right\}, \] (E33)

\[ A_{3,2,-2} = -A_{3,-2,-2} = -\delta m(B_F + iA_F) \left\{ \frac{9}{32} \xi - \left( \frac{1071}{1024} - \frac{207\nu}{512} \right) \xi^3 + \left[ \frac{27\pi}{32} - i \left( \frac{27 \log(3/2)}{16} - \frac{189}{160} \right) \right] \xi^4 \right\}, \] (E34)

\[ A_{3,3,-2} = -A_{3,-3,-2} = -\delta m(B_F + iA_F) \left\{ \frac{9}{128} \xi - \left( \frac{1197}{8192} + \frac{531\nu}{4096} \right) \xi^3 + \left[ \frac{27\pi}{128} + i \left( \frac{27 \log(3/2)}{64} - \frac{189}{640} \right) \right] \xi^4 \right\}, \] (E35)

\[ A_{3,4,-2} = -A_{3,-4,-2} = -\delta m(B_F + iA_F) \left( \frac{135}{2048} - \frac{135\nu}{1024} \right) \xi^3, \] (E36)

\[ A_{3,5,-2} = -A_{3,-5,-2} = -\delta m(B_F + iA_F) \left( \frac{81}{8192} - \frac{81\nu}{4096} \right) \xi^3, \] (E37)

\[ A_{4,0,2} = -(A_F + iB_F) \left[ \left( \frac{5}{12} - \frac{5\nu}{4} \right) \xi^2 - \left( \frac{89}{40} - \frac{571\nu}{72} + \frac{77\nu^2}{24} \right) \xi^4 \right], \] (E38)
\[ A_{4,1,2} = A_{4,1,2} = (A_F + iB_F) \left[ \left( \frac{1}{6} - \frac{\nu}{2} \right) \xi^2 - \left( \frac{49}{60} - \frac{101 \nu}{36} + \frac{11 \nu^2}{12} \right) \xi^4 \right], \]  
(E39)

\[ A_{4,2,2} = A_{4,2,2} = (A_F + iB_F) \left[ \left( \frac{1}{6} - \frac{\nu}{2} \right) \xi^2 - \left( \frac{119}{120} - \frac{265 \nu}{72} + \frac{43 \nu^2}{24} \right) \xi^4 \right], \]  
(E40)

\[ A_{4,3,2} = A_{4,3,2} = -(A_F + iB_F) \left[ \left( \frac{1}{6} - \frac{\nu}{2} \right) \xi^2 - \left( \frac{13}{15} - \frac{5 \nu}{18} + \frac{7 \nu^2}{6} \right) \xi^4 \right], \]  
(E41)

\[ A_{4,4,2} = A_{4,4,2} = (A_F + iB_F) \left[ \left( \frac{1}{24} - \frac{\nu}{8} \right) \xi^2 - \left( \frac{31}{240} - \frac{47 \nu}{144} - \frac{7 \nu^2}{48} \right) \xi^4 \right], \]  
(E42)

\[ A_{4,5,2} = A_{4,5,2} = -(A_F + iB_F) \left( \frac{1}{20} - \frac{\nu}{4} + \frac{\nu^2}{4} \right) \xi^4, \]  
(E43)

\[ A_{4,6,2} = A_{4,6,2} = (A_F + iB_F) \left( \frac{1}{120} - \frac{\nu}{24} + \frac{\nu^2}{24} \right) \xi^4, \]  
(E44)

\[ A_{4,0,2} = -(A_F - iB_F) \left[ \left( \frac{5}{12} - \frac{5 \nu}{4} \right) \xi^2 - \left( \frac{89}{40} - \frac{571 \nu}{72} + \frac{77 \nu^2}{24} \right) \xi^4 \right], \]  
(E45)

\[ A_{4,1,2} = A_{4,1,2} = -(A_F - iB_F) \left[ \left( \frac{1}{6} - \frac{\nu}{2} \right) \xi^2 - \left( \frac{49}{60} - \frac{101 \nu}{36} + \frac{11 \nu^2}{12} \right) \xi^4 \right], \]  
(E46)

\[ A_{4,2,2} = A_{4,2,2} = (A_F - iB_F) \left[ \left( \frac{1}{6} - \frac{\nu}{2} \right) \xi^2 - \left( \frac{119}{120} - \frac{265 \nu}{72} + \frac{43 \nu^2}{24} \right) \xi^4 \right], \]  
(E47)

\[ A_{4,3,2} = A_{4,3,2} = (A_F - iB_F) \left[ \left( \frac{1}{6} - \frac{\nu}{2} \right) \xi^2 - \left( \frac{13}{15} - \frac{5 \nu}{18} + \frac{7 \nu^2}{6} \right) \xi^4 \right], \]  
(E48)

\[ A_{4,4,2} = A_{4,4,2} = (A_F - iB_F) \left[ \left( \frac{1}{24} - \frac{\nu}{8} \right) \xi^2 - \left( \frac{31}{240} - \frac{47 \nu}{144} - \frac{7 \nu^2}{48} \right) \xi^4 \right], \]  
(E49)

\[ A_{4,5,2} = A_{4,5,2} = (A_F - iB_F) \left( \frac{1}{20} - \frac{\nu}{4} + \frac{\nu^2}{4} \right) \xi^4, \]  
(E50)

\[ A_{4,6,2} = A_{4,6,2} = (A_F - iB_F) \left( \frac{1}{120} - \frac{\nu}{24} + \frac{\nu^2}{24} \right) \xi^4, \]  
(E51)

\[ A_{5,1,2} = -A_{5,1,2} = \delta m (B_F - iA_F) \left( \frac{4375}{12288} - \frac{4375 \nu}{6144} \right) \xi^3, \]  
(E52)

\[ A_{5,2,2} = -A_{5,2,2} = \delta m (B_F - iA_F) \left( \frac{625}{3072} - \frac{625 \nu}{1536} \right) \xi^3, \]  
(E53)

\[ A_{5,3,2} = -A_{5,3,2} = \delta m (B_F - iA_F) \left( \frac{625}{8192} - \frac{625 \nu}{4096} \right) \xi^3, \]  
(E54)

\[ A_{5,4,2} = -A_{5,4,2} = \delta m (B_F - iA_F) \left( \frac{625}{6144} - \frac{625 \nu}{3072} \right) \xi^3, \]  
(E55)

\[ A_{5,5,2} = -A_{5,5,2} = \delta m (B_F - iA_F) \left( \frac{625}{24576} - \frac{625 \nu}{12288} \right) \xi^3, \]  
(E56)

\[ A_{5,1,2} = -A_{5,1,2} = \delta m (B_F + iA_F) \left( \frac{4375}{12288} - \frac{4375 \nu}{6144} \right) \xi^3, \]  
(E57)

\[ A_{5,2,2} = -A_{5,2,2} = \delta m (B_F + iA_F) \left( \frac{625}{3072} - \frac{625 \nu}{1536} \right) \xi^3, \]  
(E58)

\[ A_{5,3,2} = -A_{5,3,2} = \delta m (B_F + iA_F) \left( \frac{625}{8192} - \frac{625 \nu}{4096} \right) \xi^3, \]  
(E59)

\[ A_{5,4,2} = -A_{5,4,2} = \delta m (B_F + iA_F) \left( \frac{625}{6144} - \frac{625 \nu}{3072} \right) \xi^3, \]  
(E60)

\[ A_{5,5,2} = -A_{5,5,2} = \delta m (B_F + iA_F) \left( \frac{625}{24576} - \frac{625 \nu}{12288} \right) \xi^3, \]  
(E61)

\[ A_{6,0,2} = -(A_F + iB_F) \left( \frac{567}{1280} - \frac{567 \nu}{256} + \frac{567 \nu^2}{256} \right) \xi^4, \]  
(E62)
\[ A_{6,1,2} = A_{6,-1,2} = (A_F + iB_F) \left( \frac{81}{640} - \frac{81\nu}{128} + \frac{81\nu^2}{128} \right) \xi^4, \]  \hspace{1cm} (E63)

\[ A_{6,2,2} = A_{6,-2,2} = (A_F + iB_F) \left( \frac{1377}{5120} - \frac{1377\nu}{1024} + \frac{1377\nu^2}{1024} \right) \xi^4, \]  \hspace{1cm} (E64)

\[ A_{6,3,2} = A_{6,-3,2} = -(A_F + iB_F) \left( \frac{243}{1280} - \frac{243\nu}{256} + \frac{243\nu^2}{256} \right) \xi^4, \]  \hspace{1cm} (E65)

\[ A_{6,4,2} = A_{6,-4,2} = -(A_F + iB_F) \left( \frac{81}{1280} - \frac{81\nu}{256} + \frac{81\nu^2}{256} \right) \xi^4, \]  \hspace{1cm} (E66)

\[ A_{6,5,2} = A_{6,-5,2} = (A_F + iB_F) \left( \frac{81}{1280} - \frac{81\nu}{256} + \frac{81\nu^2}{256} \right) \xi^4, \]  \hspace{1cm} (E67)

\[ A_{6,6,2} = A_{6,-6,2} = -(A_F + iB_F) \left( \frac{81}{1280} - \frac{81\nu}{256} + \frac{81\nu^2}{256} \right) \xi^4, \]  \hspace{1cm} (E68)

\[ A_{6,0,-2} = -(A_F - iB_F) \left( \frac{567}{1280} - \frac{567\nu}{256} + \frac{567\nu^2}{256} \right) \xi^4, \]  \hspace{1cm} (E69)

\[ A_{6,1,-2} = A_{6,-1,-2} = -(A_F - iB_F) \left( \frac{81}{640} - \frac{81\nu}{128} + \frac{81\nu^2}{128} \right) \xi^4, \]  \hspace{1cm} (E70)

\[ A_{6,2,-2} = A_{6,-2,-2} = (A_F - iB_F) \left( \frac{1377}{5120} - \frac{1377\nu}{1024} + \frac{1377\nu^2}{1024} \right) \xi^4, \]  \hspace{1cm} (E71)

\[ A_{6,3,-2} = A_{6,-3,-2} = (A_F - iB_F) \left( \frac{243}{1280} - \frac{243\nu}{256} + \frac{243\nu^2}{256} \right) \xi^4, \]  \hspace{1cm} (E72)

\[ A_{6,4,-2} = A_{6,-4,-2} = -(A_F - iB_F) \left( \frac{81}{1280} - \frac{81\nu}{256} + \frac{81\nu^2}{256} \right) \xi^4, \]  \hspace{1cm} (E73)

\[ A_{6,5,-2} = A_{6,-5,-2} = -(A_F - iB_F) \left( \frac{81}{1280} - \frac{81\nu}{256} + \frac{81\nu^2}{256} \right) \xi^4, \]  \hspace{1cm} (E74)

\[ A_{6,6,-2} = A_{6,-6,-2} = -(A_F - iB_F) \left( \frac{81}{1280} - \frac{81\nu}{256} + \frac{81\nu^2}{256} \right) \xi^4, \]  \hspace{1cm} (E75)

\[ \nu = 2^{-1}, 2^{-2}, 2^{-3}, 2^{-4}, 2^{-5}, 2^{-6}, 2^{-7}. \]

\[ \frac{567}{1280} = \frac{567}{256} \xi^4. \]

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