PD Control at the Hopf Bifurcation Point of a Neuron System with Inertia and Delay

Shuo Shi¹, *, Huaifei Wang¹ and Min Xiao¹

¹College of Automation, Nanjing University of Posts and Telecommunications, Nanjing, China

*2312081035@qq.com

Abstract. In this paper, a two-neuron system with inertia and delay is proposed firstly. A PD controller is then applied to the system for the purpose of improving its dynamical performance. Through the mathematical transformation, we extend the system to a four-dimensional one with only time delays. With the help of the associated characteristic equation of the mathematical model, sufficient conditions for ensuring the system stability are proposed. Furthermore, with the time delay as the bifurcation parameter, relevant requirements for the generation of Hopf bifurcation are derived. Then a series of numerical simulations are carried out to justify the theoretical analysis and it is found that the application of PD control scheme helps to advance the bifurcation point dramatically through a slight adjustment of the controller parameters.

1. Introduction

A growing number of attention has been paid to the neuron system ever since the simplified neural network was initially proposed by Hopfield in [1] and it has been applied successfully in different engineering fields like pattern recognition, signal and image processing and optimization cryptography [2]. Biological experiments on the human brain have demonstrated that neurons interact with each other, displaying various dynamical phenomena, such as oscillations, bifurcation and chaos [3, 4, 5, 6]. Inertial neuron networks (INNs) were first proposed in [7] by Babcock and Westervelt. It is found in practical use that the intervention of the inertia parts helps with the unordered search of memory. INNs also bear a profound bioengineering background. For example, the squid axon can be described as a mathematical model containing a phenomenological inductance. Besides, in the semicircular canals of some organisms, the membrane of a hair cell can be modeled by the equivalent circuits consisting of inductance. Such networks with inductance are usually referred as inertial networks.

It is known that Hopf bifurcations serve as the strategy for coping with various dynamic properties of nonlinear systems. More detailed information about the performance of periodic solutions around the equilibrium can be obtained after the Hopf bifurcation analysis. Recently, bifurcation control has aroused tremendous concern for researchers from different disciplines. Ever since the initial work done by Ott et al. [8], diverse bifurcation control schemes have been proposed [9, 10, 11], and these control strategies have been successfully applied to different nonlinear systems. By contrast, although the dynamics of inertial neural networks have been discussed, the control of such systems is rarely covered. Therefore, in this paper, we attempt to apply PD control scheme to adjust the dynamical performance of neuron systems with inertia. For one thing, PD control has been extensively applied in the robotic manipulation, indicating its excellence in industrial engineering. For another thing, in terms of Hopf bifurcation and bifurcation control, complex networks under the PD controller have been investigated in...
recent years. Tang et al. [12] applied the fractional-order PD control to a congestion control system. Xiao et al. [13] addressed the problem of fractional-order PD control in a small-world networks and demonstrated that the onset of Hopf bifurcations can be altered via the proposed PD controller by setting proper control parameters. Motivated by the previous works, we adopt a PD controller to a two-neuron system with inertia and delay for the purpose of altering the system dynamics.

2. Model Description

Inspired by the work done in [14], we simplify the model and propose a two-neuron system with inertia and delays, which is described as follows:

\[ \begin{align*}
\dot{u}_1(t) &= -\alpha_1 u_1(t) + \nu_1 f(u_2(t - \tau)) + v_1 f(u_2(t - \tau)), \\
\dot{u}_2(t) &= -\alpha_2 u_2(t) + v_2 f(u_1(t - \tau)).
\end{align*} \]

(1)

where \(\alpha_1 > 0; \alpha_2 > 0\) indicate respectively the stabilities of internal neuron processes. \(v_1 < 0; v_2 < 0\) stand for the interaction parameters between two neurons. \(\tau \geq 0\) is the coupled time delay and \(f(\cdot)\) is the activation function.

It is obvious that system (1) has a unique equilibrium \((u_1, u_2) = (0, 0)\). Therefore, when adopting a PD controller to the system, it turns into

\[ \begin{align*}
\dot{u}_1(t) &= -\alpha_1 u_1(t) - \alpha_2 u_2(t) + v_1 f(u_2(t - \tau)) + K_p u_1(t) + K_d u_2(t), \\
\dot{u}_2(t) &= -\alpha_2 u_2(t) - \alpha_1 u_1(t) + v_2 f(u_1(t - \tau)) + K_p u_1(t) + K_d u_2(t).
\end{align*} \]

(2)

Then let \(u_1(t) = x_1(t), u_2(t) = x_2(t), \dot{u}_1(t) = x_3(t), \dot{u}_2(t) = x_4(t)\), system (2) turns to be four-dimensional described as follows:

\[ \begin{align*}
\dot{x}_1(t) &= x_2(t), \\
\dot{x}_2(t) &= -x_2(t) - \alpha_1 x_1(t) + v_1 f(x_3(t - \tau)) + K_p x_1(t) + K_d x_2(t), \\
\dot{x}_3(t) &= x_4(t), \\
\dot{x}_4(t) &= -x_4(t) - \alpha_2 x_3(t) + v_2 f(x_1(t - \tau)) + K_p x_3(t) + K_d x_2(t).
\end{align*} \]

(3)

3. Stability and Bifurcation Analysis

In this section, the stability will be studied for controlled system (3) and relevant conditions for the existence of Hopf bifurcation will be obtained.

Linearizing system (3) at the equilibrium leads to:

\[ \begin{align*}
\dot{x}_1(t) &= x_2(t), \\
\dot{x}_2(t) &= (K_p - \alpha_1)x_1(t) + (K_d - 1)x_2(t) + v_1 f'(0)x_3(t - \tau), \\
\dot{x}_3(t) &= x_4(t), \\
\dot{x}_4(t) &= (K_p - \alpha_2)x_3(t) + (K_d - 1)x_4(t) + v_2 f'(0)x_1(t - \tau).
\end{align*} \]

(4)

Therefore, the associated characteristic equation of system (17)

\[ \lambda^4 + \beta_4 \lambda^3 + \beta_3 \lambda^2 + \beta_2 \lambda + \beta_1 \lambda + \beta_0 e^{-\lambda \tau} + \beta_5 e^{-\lambda \tau} = 0, \]

(5)

Namely

\[ \lambda^4 + \beta_4 \lambda^3 + \beta_3 \lambda^2 + \beta_2 \lambda + \beta_1 \lambda + \beta_0 = 0, \]

(6)

where \(\beta_3 = 2(1 - K_d)\), \(\beta_4 = (1 - K_d)^2 + \alpha_1 + \alpha_2 - 2K_p\), \(\beta_5 = -v_1 f'(0), \beta_2 = -v_2 f'(0), \beta_0 = (a_1 - K_p)(a_2 - K_p)\).

Let \(\lambda = i\omega = \omega(\cos \omega \tau + i \sin \omega \tau)(\omega > 0)\) be a root of Eq. (6), then it can be obtained that
\[ \begin{align*}
\omega^4 - \beta_4 \omega^2 + \beta_3 &= -\beta_4 \cos 2\omega \tau, \\
-\beta_3 \omega^3 + \beta_2 \omega &= \beta_3 \sin 2\omega \tau.
\end{align*} \] (7)

Taking the square on both sides of the two equations in Eq. (7) and adding them up lead to
\[ \omega^8 + A_4 \omega^6 + A_2 \omega^4 + A_0 \omega^2 + A_4 = 0, \] (8)
in which \( A_4 = \beta_3^2 - 2 \beta_2 \beta_4, A_2 = \beta_2^2 + 2 \beta_4^2 - 2 \beta_2 \beta_3, A_3 = \beta_3^2 - 2 \beta_2 \beta_5, A_3 = \beta_3^2 - 2 \beta_4 \beta_2. \) Here we label \( M(\omega) = \omega^8 + A_4 \omega^6 + A_2 \omega^4 + A_0 \omega^2 + A_4 \) for further use.

It is obvious that if \( A_4 < 0, \) Eq. (8) owns no less than one positive root. From the first equation of Eq. (7), we can obtain that
\[ \tau_n^{(j)} = \frac{1}{2\omega_n} \left( \arccos \frac{-\omega_n^3 + \beta_2 \omega_n - \beta_3}{\beta_4} + 2j\pi \right), \] (9)
where \( n = 1; 2; 3; 4; j = 1; 2; \cdots. \)

Given that \( \tau = \tau_n^{(j)}, \) (7) owns a pair of purely imaginary roots \( \pm i\omega_n. \) It is also clear from (9) that the sequence \( \{\tau_n^{(j)}\}_{j=0} \) is monotone increasing and \( \lim_{j \to \infty} \tau_n^{(j)} = +\infty, \) \( n = 1, 2, 3, 4. \)

We then make the following assumption that
\[ \tau_0 = \tau_n^{(0)} = \min_{n=1,2,3,4} \{\tau_n^{(0)}\}, \quad \omega_0 = \omega_{\tau_0}. \] (10)

In order to establish of system (1) when \( \tau = 0 \) we put forward the following conditions.
\[ \begin{align*}
D_1 &= 4 > 0, \\
D_2 &= \begin{vmatrix} 2 & 1 \\ a_1 + a_2 & a_1 + a_2 + 1 \end{vmatrix} > 0, \\
D_3 &= \begin{vmatrix} 2 & 0 \\ a_1 + a_2 & a_1 + a_2 + 1 \end{vmatrix} > 0.
\end{align*} \] (11) (12) (13)

Lemma 1. If conditions of (11)-(13) are satisfied, then the positive equilibrium \((u_1; u_2)\) of the system (1) is asymptotically stable when \( \tau = 0. \)

Proof 1. If time delay and the PD controller are removed from the system, Eq. (17) turns to
\[ \lambda^4 + 2\lambda^3 + (a_1 + a_2 + 1)\lambda^2 + (a_1 + a_2)\lambda + a_1 a_2 - v_1 v_2 f^{\tau^2}(0) = 0. \] (14)

According to (11)-(13), it is obvious that the well-known Routh-Hurwits criterion is met in Eq. (14), indicating that all roots of Eq. (14) have negative real parts. Therefore, the positive equilibrium \((u_1; u_2)\) of the system (1) is asymptotically stable when \( \tau = 0. \) This ends the proof.

It is notable that the results in Lemma 1 are only sufficient conditions for guaranteeing the stability of system (1). In order to further probe into the the conditions for the generation of Hopf bifurcation, we give the following lemma.

Lemma 2. Let \( \lambda(\tau) = \gamma(\tau) + i\alpha(\tau) \) be the solution of Eq. (6) near \( \tau = \tau_j, \) which satisfies that \( \gamma(\tau_j) = 0, \alpha(\tau_j) = \omega_0, \) then the transversality condition puts that
\[ \text{Re} \left[ \frac{d\lambda}{d\tau} \right]_{\tau = \tau_j, \alpha = \omega_0} \neq 0, \] in which \( \omega_0 \) represents the critical frequency and \( \tau_0 \) the bifurcation point.

Proof 2. Differentiating Eq. (6) in terms of \( \tau \) on both sides leads to
\[ \frac{d\lambda}{d\tau} \left[ \begin{array}{c} 2 \lambda^2 - 3 \beta \lambda^2 - 2 \beta_i \lambda + \beta_i \\
2 \beta_i \lambda e^{-2\lambda \tau} - \frac{\tau}{\lambda} \end{array} \right] = 4 \lambda^3 + 3 \beta_i \lambda^2 + 2 \beta_i \lambda + \beta_i, \] (15)

Then we assume that \( \lambda = i\omega_0, \) and it can be obtained that
\[
Re \left[ \frac{d \lambda}{d \tau} \right]_{\tau=\tau_0}^{-1} = \frac{1}{2\beta_0} Re \left[ e^{\lambda \tau} \left( 4\lambda^3 + 3\beta_1\lambda^2 + 2\beta_2\lambda + \beta_3 \right) \right] = \frac{M'(\omega)}{4\beta_0\omega^2} > 0
\]

Therefore, the transversality condition holds. This completes the proof. Then we summarize the following theorem based on the discussion above.

Theorem 1 For system (2), the following results can be determined:

(i) The controlled system (2) is asymptotically stable at the equilibrium \((u_1, u_2)\) when \(\tau \in [0, \tau_0)\).

(ii) System (2) undergoes a Hopf bifurcation at the equilibrium when \(\tau = \tau_0\), i.e., and a class of periodic solutions bifurcate around the equilibrium when passes through \(\tau_0\).

4. Simulation Examples.

In this section, several numerical examples are illustrated to verify the efficiency and feasibility of the previous theoretical analysis.

Example 1 In this example, we propose a specific two-neuron inertial model without controller as that of system (1) in which \(a_1 = 1, a_2 = 1.2, v_1 = v_2 = -1\) and the activation function is selected as \(f(\cdot) = \tanh(\cdot)\). The time delay is chosen as the bifurcation parameter and the mathematical model can be described as follows:

\[
\begin{align*}
\dot{u}_1(t) &= -u_1(t) - u_2(t) - \tanh(u_2(t - \tau)), \\
\dot{u}_2(t) &= -u_2(t) - 1.2u_2(t) - \tanh(u_1(t - \tau)).
\end{align*}
\tag{16}
\]

It is known that the equilibrium point of system (16) is \((0,0)\). After careful calculation, we can obtain that the bifurcation point is \(\tau_0 = 1.464\). Figure 1 shows that when the selected \(\tau = 1.223 < \tau_0\), the uncontrolled system (1) remains asymptotically stable around the equilibrium.

As in the figure, \(x(t)\) stands for \(u_1(t)\), \(y(t)\) for \(u_1(t)\), \(z(t)\) for \(u_2(t)\), and \(s(t)\) for \(u_2(t)\). Whereas, when \(\tau = 1.664 > \tau_0\), system (1) turns unstable and the Hopf bifurcation is generated at the equilibrium subsequently, as displayed in Figure 2.

**Figure 1.** Waveform plot of system (1) with \(a_1 = 1, a_2 = 1.2, v_1 = v_2 = -1\) and \(f(\cdot) = \tanh(\cdot)\). The equilibrium is asymptotically stable when \(\tau = 1.223 < \tau_0 = 1.464\).

**Figure 2.** Waveform plot of system (1) with \(a_1 = 1, a_2 = 1.2, v_1 = v_2 = -1\) and \(f(\cdot) = \tanh(\cdot)\). A periodic oscillation bifurcate around the equilibrium when \(\tau = 1.664 > \tau_0 = 1.464\).
Example 2 In this example, we apply a PD controller to the delayed neuron system with inertia and the specific mathematical model is described as follows in which \( a_1 = 1 \), \( a_2 = 1.2 \), \( v_1 = v_2 = -1 \), \( K_p = -1.5 \), and \( K_d = -0.5 \).

\[
\begin{align*}
\dot{x}_1(t) &= x_3(t), \\
\dot{x}_2(t) &= -2.5x_1(t) - 0.5x_2(t) - \tanh(x_3(t - \tau)), \\
\dot{x}_3(t) &= x_4(t), \\
\dot{x}_4(t) &= -2.7x_3(t) - 0.5x_4(t) - \tanh(x_4(t - \tau)).
\end{align*}
\] (17)

After calculation, it can be obtained that the bifurcation point of (2) is \( \tau_0 = 0.615 \). Viewed from the three dimensional phase plot, it can be seen from Figure 3,4 that when \( \tau = 0.455 < \tau_0 \), the curve is bound to converge to the equilibrium in a long given time while when \( \tau = 0.723 > \tau_0 \), a limit circle will appear, implying the existence of Hopf bifurcation.

From the simulation results displayed above, it can be concluded that the application of PD controller can greatly advance the bifurcation point of the original system, which corroborates the efficiency and feasibility of PD controller.

Furthermore, we investigate into the relationship between controller parameters and bifurcation point and the results shown in Figure 5 and 6 indicate that both \( K_p \) and \( K_d \) remain monotonically decreasing with relevance to \( \tau_0 \), respectively.

![Figure 3](image_url)
**Figure 3.** Phase portraits of system (2): projection on \( x - y - z \) and \( x - y - s \), with \( a_1 = 1 \), \( a_2 = 1.2 \), \( v_1 = v_2 = -1 \), \( K_p = -1.5 \), \( K_d = 0.5 \) when \( \tau = 0.423 < \tau_0 = 0.615 \).

![Figure 4](image_url)
**Figure 4.** Phase portraits of system (2): projection on \( x - y - z \) and \( x - y - s \), with \( a_1 = 1 \), \( a_2 = 1.2 \), \( v_1 = v_2 = -1 \), \( K_p = -1.5 \), \( K_d = 0.5 \) when \( \tau = 0.723 > \tau_0 = 0.615 \).

![Figure 5](image_url)
**Figure 5.** Relationship between \( K_p \) and \( \tau_0 \).

![Figure 6](image_url)
**Figure 6.** Relationship between \( K_d \) and \( \tau_0 \).
5. Conclusion
In this paper, we have introduced a two-neuron system with inertia and delay. Then we have managed to employ a PD controller to interfere with the dynamical behavior of the neural system. A stability criterion has been obtained and relevant conditions for generating Hopf bifurcation has also been proposed. Moreover, it has been justified through numerical simulations that the bifurcation point can be advanced drastically by altering the controller parameters. Since the original is system two-dimensional, future work lies in that PD controller may be applied to high-dimensional neural networks with inertia and delays.

Acknowledgements
This work is supported in part by the National Natural Science Foundation of China (No. 61573194), the Natural Science Foundation of Jiangsu Province of China (No. BK20181389), the Key Project of Philosophy and Social Science Research in Colleges and Universities in Jiangsu Province (Grant No. 2018SJZDI142), Postgraduate Research & Practice Innovation Program of Jiangsu Province (No.KYCX18 0924

References
[1] Hopfield J J 1984 Proc. Natl. Acad. Sci. 81 3088
[2] Xie WD, Fu Z J and Xie W F 2016 Intell. Autom. Soft. Co. 22 111
[3] Xiao M, Zheng W and Cao J 2013 Neurocomputing 99 206
[4] Huang C, Meng Y, Cao J, Alsaedi A and Alsaadi F 2017 Chaos Solitons Fractals 100 31
[5] Xu W, Cao J, Xiao M, Ho DWC and Wen G 2015 IEEE Trans Cybern. 45 2224
[6] Xiao M, Zheng W and Cao J 2013 IEEE Trans Neural Netw. Learn. Syst. 24 118
[7] Babcock K and Westervelt R 1987 Physica D 28 305
[8] Ott E, Grebogi C and Yorke J A 1990 Phys. Rev. Lett. 64 1196
[9] Yu P and Chen G 2004 Int. J. Bifurcat. Chaos 14 1683
[10] Nguyen L H, Hong K S 2012 Phys. Lett. A 376 442
[11] Chen G, Moiola J L and Wang H O 2000 Int. J. Bifurcat. Chaos 10 511
[12] Tang Y H, Xiao M, Jiang G P, Lin J X, Cao J D and Zheng W X 2017 Nonlinear Dynam. 90 2185
[13] Xiao M, Zheng W X, Lin J X, Jiang G P, Zhao L D, Cao J D 2017 J. Franklin Inst. 354 7643
[14] Ge J and Xu J 2013 Sci. China Technol. Sci. 56 2299