The Holt-Klee condition for oriented matroids

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Abstract
Holt and Klee have recently shown that every (generic) LP orientation of the graph of a \(d\)-polytope satisfies a directed version of the \(d\)-connectivity property, i.e. there are \(d\) internally disjoint directed paths from a unique source to a unique sink. We introduce two new classes HK and HK* of oriented matroids (OMs) by enforcing this property and its dual interpretation in terms of line shellings, respectively. Both classes contain all representable OMs by the Holt-Klee theorem. While we give a construction of an infinite family of non-HK* OMs, it is not clear whether there exists any non-HK OM. This leads to a fundamental question as to whether the Holt-Klee theorem can be proven combinatorially by using the OM axioms only. Finally, we give the complete classification of OM(4, 8), the OMs of rank 4 on 8-element ground set with respect to the HK, HK*, Euclidean and Shannon properties. Our classification shows that there exists no non-HK OM in this class.

1 Introduction
Let \(P\) be a \(d\)-dimensional convex polytope (\(d\)-polytope) in \(\mathbb{R}^d\). We consider a linear program whose feasible region is \(P\) with a generic objective function \(f(x) = c^Tx\), i.e. \(f(u) \neq f(v)\) for any two distinct vertices \(u\) and \(v\). We orient each edge \((u, v)\) from \(u\) to \(v\) if and only if \(f(u) < f(v)\). The resulting orientation on the graph \(G(P)\) of \(P\) is known as an LP orientation, which represents the possible pivot operations of the simplex method to solve the linear program.

We call \(G(P)\) with an LP orientation an LP digraph.

Every LP digraph satisfies the following three properties: (1) acyclicity, i.e., there exists no directed cycle, (2) unique sink&source property [22], namely there exist a unique sink and a unique source, and (3) the Holt-Klee condition [11]. The Holt-Klee condition is a directed version of the \(d\)-connectivity property by Balinski [1]. i.e., there are \(d\) internally disjoint directed paths from a unique source to a unique sink.

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paths from the source to the sink. Particularly when $d = 3$, these three properties are also sufficient for an LP digraph \[14\].

For example, consider two orientations on the graph $G(C_3)$ of a 3-cube $C_3$ in Figure 2. While the minimum size of vertex cut sets is three in the left orientation, it is only two in the right orientation and in particular, every dipath from the source $v_1$ to the sink $v_8$ must go through either $v_2$ or $v_7$. Hence, the right digraph does not satisfy the Holt-Klee condition and thus is not an LP digraph.

![Figure 1: An LP digraph induced by $f(x)$](image)

![Figure 2: Two orientations satisfying and not satisfying the Holt-Klee condition.](image)

Our main objective is to understand how restrictive the Holt-Klee condition is. The original proof of the Holt-Klee theorem \[11\] relies heavily on geometric operations such as affine transformations and orthogonal projections, and it is not clear whether more combinatorial proof is possible or not. In particular, it is natural to ask whether this condition is valid for the oriented matroid program, which is a combinatorial abstraction of the linear program. This motivates us to study the Holt-Klee condition in the setting of oriented matroids. For this purpose, we introduce two new subclasses of oriented matroids using this condition, the class of $HK$ oriented matroids and the class of $HK^*$ oriented matroids.

Before defining the two subclasses, we set notations for oriented matroids. We assume that the reader is familiar with oriented matroids. The standard reference is \[2\]. An oriented matroid $M$ is defined as a pair $(E, \mathcal{L})$ of a finite ground set $E$ and the set $\mathcal{L}$ of covectors. By the topological representation theorem, every oriented matroid of rank $d + 1$ can be represented by a pseudosphere arrangement \[8\]. A pseudosphere arrangement is specified as a triple $\mathcal{A} = (E, S, \mathcal{D})$ of a finite ground set $E$, a $d$-dimensional unit sphere $S$ in $\mathbb{R}^{d+1}$, and a family $\mathcal{D} = \{s_e^+, s_e, s_e^-\} : e \in E$, where $s_e$ is a $(d-1)$-dimensional pseudosphere on $S$, $s_e^+$ is the positive side of $s_e$, and $s_e^-$ is the negative side of $s_e$, respectively. The location vector of a point $x \in S$ is the sign vector $\sigma(x) \in \{+, 0, -\}^E$ defined by $\sigma(x)_e = +$ if $x \in s_e^+$, $\sigma(x)_e = 0$ if $x \in s_e$, and $\sigma(x)_e = -$ if $x \in s_e^-$. A pseudosphere arrangement $\mathcal{A}$ is said to represent an oriented matroid $M = (E, \mathcal{L})$. 


if $\sigma(S) \cup \{0\} = \mathcal{L}$, where $\sigma(S) := \{\sigma(x) : x \in S\}$. For the sequel, we use $s_e$ as a topological representation of an element $e \in E$ of $M$. If all pseudospheres $\{s_e : e \in E\}$ are realized by $(d-1)$-dimensional linear spheres on $S$, i.e. the intersection of $S$ and a hyperplane through the origin, $M$ is said to be representable, and non-representable otherwise. We will define the HK property and the HK* property of oriented matroids to be a consistent generalization of the Holt-Klee condition, extended to oriented matroids.

First, we define the class of HK oriented matroids, which is based on the direct relation between the oriented matroid program and the Holt-Klee condition. An oriented matroid program is a triple $\pi = (M, g, f)$ where $M$ is an oriented matroid $M = (E, \mathcal{L})$, $g \in E$ is not a loop of $M$, and $f(\neq g) \in E$ is not a coloop of $M$. The feasible region $P_\pi$ of $(M, g, f)$ is $P_\pi := \{X \in \mathcal{L} : X_g = +, X_e \in \{+0\}$ for all $e \in E \setminus \{g,f\}\}$. The region $P_\pi$ is unbounded if it is nonempty and there exists $X \in \mathcal{L}$ such that $X_g = 0$ and $X_e = \{+0\}$ for all $e \in E \setminus \{g,f\}$, and bounded otherwise. We orient the graph of $(M, g, f)$, i.e., the 1-skeleton of the arrangement $\{s_e : e \in E \setminus \{f\}\}$ restricted to the positive side $s^+_g$ of the infinity $g$, so that each edge is oriented from the negative side $s^-_f$ of the objective $f$ toward the positive side $s^+_f$ of $f$ in [2, Definition 10.1.16]. The graph $G_\pi$ restricted to $P_\pi$ is denoted by $G^+_\pi$ and called an OMP digraph. The objective $f$ is generic in $P_\pi$ if there is no non-oriented edges in $G^+_\pi$. An oriented matroid program $\pi = (M, g, f)$ is called proper if $P_\pi$ is bounded, full dimensional (i.e. containing a tope) and $f$ is generic.

![Figure 3: An oriented matroid program $\pi = (M, g, f)$ and its OMP digraph $G^+_\pi$](image)

If $M$ is representable, its OMP digraph for any choice of $g$ and $f$ satisfies the three properties of LP digraphs [2, 9], (1) acyclicity, (2) unique sink&source property [22], and (3) the Holt-Klee condition [11], but the situation is different in general. Every OMP digraph satisfies the unique sink&source property [2, page 426], but there exists an OMP digraph with a directed cycle, see non-BOMs [9] and non-Euclidean OMs [2]. On the other hand, it is not known whether every OMP digraph $G^+_\pi$ satisfies the Holt-Klee condition. This motivates us to define the HK property for oriented matroids:

**Definition 1** A proper oriented matroid program $\pi = (M, g, f)$ is called HK if $G^+_\pi$ satisfies the Holt-Klee condition where the dimension $d$ is $r(M) - 1$, and non-HK otherwise.

**Definition 2** An oriented matroid $M = (E, \mathcal{L})$ is called HK if the oriented matroid program $\pi = (M, g, f)$ is HK for any two distinct elements $f, g \in E$ for which $\pi$ is proper and the same holds for any reorientations and any minors of $M$, and non-HK otherwise.
As we mentioned above, if $M$ is representable, every OMP digraph satisfies the Holt-Klee condition, hence we obtain the following proposition:

**Proposition 1**  
Every representable oriented matroid has the HK property.

For $r \leq 3$, every unique-sink unique-source orientation on the graph of $P_r$ satisfies the Holt-Klee condition. This observation gives the following proposition:

**Proposition 2**  
Every oriented matroid of rank $r$ has the HK property if $r \leq 3$.

Now we consider the dual interpretation of the HK condition which leads to the notion of HK* property. For this, we use the facet graph of a convex polytope with orientation induced by a line shelling ordering of facets, see [4] and [23, Theorem 8.11]. In the setting of oriented matroids, the role of a straight line in an arrangement of hyperplanes can be played by a coline.

What we shall obtain is a coline shelling, which is a special kind of what is known as tope graph shelling or pseudoline shelling [9], see [2, Section 4.3] for a more algebraic treatment.

A **coline fixation** is a pair $\omega = (M, T)$, where $M$ is an oriented matroid $M = (E, L)$ and $T \subseteq E$ is a coline of $M$. The associated **supercell** $P_\omega$ of $(M, T)$ is $P_\omega := \{ X \in L : X_e = \{0, +\} \text{ for all } e \in E \setminus T \}$. A vector $Z \in L$ is an interior point of $P_\omega$ if $Z^+ = E \setminus T$. For each element $f \in E \setminus T$, the subset of $P_\omega$ defined by $P_\omega(f) := \{ X \in P_\omega : X_f = 0, X^+ = E \setminus \{T \cup \{f\}\} \}$ is the face of $P_\omega$ induced by $f$. The facets of $P_\omega$ are the faces $P_\omega(f)$ that are maximal.

We say $T$ is generic in $M$ if there exists $Y \in L$ for every $f \in E \setminus T$ such that $Y^0 = T \cup \{f\}$ and $Y = E \setminus (T \cup \{f\})$. A coline fixation $\omega = (M, T)$ is called **proper** if $T$ is generic, there exists an interior point $Z \in P_\omega$ such that $Z^0 = T$, and all faces $P_\omega(f)$ ($f \in E \setminus T$) are facets.

When a coline fixation $\omega = (M, T)$ is proper, there is a unique linear ordering of the elements of $E \setminus T$ up to reversal: $e_1, e_2, \ldots, e_s$ ($s = |E \setminus T|$) such that for each $k = 1, \ldots, s$, the vector $V^k$ defined by $(V^k)^+ = \{ e_i : 1 \leq i \leq k - 1 \}$ and $(V^k)^0 = T \cup \{ e_k \}$ is a cocircuit of $M$. This ordering (unique up to reversal) is called the **coline shelling** induced by $\omega$, denoted by $CS_\omega$.

By the duality of the ranking of vertices in a convex polytope and the line shelling of the dual polytope, we define the facet graph with orientation induced by a shelling. Namely, we define the **shelling digraph** $SG_\omega$ of $\omega$ as follows: The set of vertices of $SG_\omega$ is $E \setminus T = \{ e_1, \ldots, e_s \}$, and there is an edge $(e_i, e_j)$ directed from $e_i$ to $e_j$ if and only if $i < j$ and the associated two facets $P_\omega(e_i)$ and $P_\omega(e_j)$ are adjacent, i.e., their intersection is maximal over all intersections of two distinct facets.

![Coline fixation and shelling digraph](image)

**Figure 4:** A coline fixation $\omega = (M, T)$ and its shelling digraph $SG_\omega$.

If $M$ is representable, the arrangement is realizable as a hyperplane arrangement, and thus every coline shelling $CS_\omega$ is realizable as a line shelling. By duality, the shelling digraph is an LP digraph [23]. Hence every shelling digraph of a proper coline fixation in a representable
oriented matroid satisfies (1) acyclicity, (2) unique sink\&source property [22], and (3) the Holt-Klee condition [11]. It is worthwhile to observe that even when $M$ is non-representable, every shelling digraph satisfies acyclicity and the unique sink\&source property [15]. We will show that not every shelling digraph $SG_\omega$ satisfies the Holt-Klee condition. To make our claim clear, it is important to define the HK* property of an oriented matroid $M$:

**Definition 3** A proper coline fixation $\omega = (M, T)$ is called HK* if $SG_\omega$ satisfies the Holt-Klee condition, and non-HK* otherwise.

**Definition 4** An oriented matroid $M = (E, \mathcal{L})$ is called HK* if the coline fixation $\omega = (M, T)$ is HK* for any coline $T \subset E$ such that $\omega$ is proper, and the same holds for any reorientations of $M$ and any minors of $M$, and non-HK* otherwise.

As we mentioned above, if $M$ is representable, every shelling digraph satisfies the Holt-Klee condition, hence we obtain the following proposition:

**Proposition 3** Every representable oriented matroid has the HK* property.

Likewise Proposition 2, we also obtain the following proposition:

**Proposition 4** Every oriented matroid of rank $r$ has the HK* property if $r \leq 3$.

By Definition 2 and Definition 4, every representable oriented matroid belongs to both the class of HK oriented matroids and the class of HK* oriented matroids. In other words, all non-HK and all non-HK* oriented matroids are non-representable. However, a non-representable oriented matroid is not necessarily non-HK or non-HK*. To understand these two classes more clearly, we look at the oriented matroids of rank 4 on an 8-element ground set. We denote by OM(4,8) the class of all (non-isomorphic) oriented matroids of rank 4 on an 8-element ground set. The class OM(4,8) is the smallest (with respect to the rank and at the same time the size of a ground set) that contain a non-representable oriented matroid. Finschi and Fukuda gave a complete enumeration of the oriented matroids in OM(4, 8) including non-uniform oriented matroids, and we utilize their list [5, 7, 6]. Note that an oriented matroid or rank $r$ is called *uniform* if every subset of cardinality $r$ is a basis, otherwise *non-uniform*.

The class OM(4,8) contains 2,628 uniform oriented matroids, and 24 out of them are non-representable [17, 8]. By a computer program we found 18 non-HK* oriented matroids while there exist no non-HK oriented matroids. Furthermore, the class OM(4,8) contains 178,844 non-uniform oriented matroids, and we found that 1,364 out of them have the non-HK* property and none has the non-HK property.

Finally, we show how one can use sensitive LP digraphs to construct an infinite family of non-HK* oriented matroids. The word “sensitive” means that some minor change makes the LP digraph violate the Holt-Klee condition. It is interesting to note, however, that we have not found any non-HK oriented matroid so far. If every oriented matroid is HK, then it implies that the Holt-Klee theorem may be provable in a purely combinatorial manner using the oriented matroid axioms only. We leave this question as an open problem.

## 2 Enumeration of non-HK and non-HK* oriented matroids

In this section, we give a classification of oriented matroids on an 8-element ground set $E$ of rank 4 in terms of HK and HK* properties. We also compare these two properties with the existing properties of representable oriented matroids.
| $|E|$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $r = 1$ | 1 | | | | | | | | | |
| $r = 2$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | |
| $r = 3$ | 1 | 2 | 4 | 17 | 143 | 4890 | 461053 | 95052532 | | |
| $r = 4$ | 1 | 3 | 12 | 206 | **181472** | | | | | |
| $r = 5$ | 1 | 4 | 25 | 181472 | | | | | | |
| $r = 6$ | 1 | 5 | 50 | 508321 | | | | | | |
| $r = 7$ | 1 | 6 | 91 | | | | | | | |
| $r = 8$ | 1 | 7 | 164 | | | | | | | |
| $r = 9$ | 1 | 8 | | | | | | | | |
| $r = 10$ | 1 | | | | | | | | | |

Table 1: The number of non-isomorphic oriented matroids on a ground set $E$ of rank $r$ \([5, 7]\)

For the classification, we use the database of oriented matroids by Finschi and Fukuda \([5, 7]\) that contains both all uniform and non-uniform oriented matroids. Table 2 shows the number of non-isomorphic oriented matroids. It enumerates the oriented matroids up to isomorphism of the associated big face lattices (see \([2\) Section 4.1]). Thus, in particular, two oriented matroids equivalent by a reorientation or by a permutation of the ground set are isomorphic. Since the HK and the HK* properties as well as representability are closed under such operations, the database is well suited for our purpose.

Denote by $\text{OM}(r, n)$ the class of non-isomorphic oriented matroids of rank $r$ on an $n$-element ground set. From Proposition 2 and Proposition 1 if $r \leq 3$, then $\text{OM}(r, n)$ contains no non-HK or non-HK* oriented matroid. Thus, to seek a non-HK or a non-HK* oriented matroid, the rank $r$ has to be at least four. Since every rank-4 oriented matroid is representable if $n \leq 7$ \([2\) Corollary 8.3.3], $\text{OM}(4,8)$ is the first candidate class that may contain a non-HK or non-HK* oriented matroid. In 181,472 oriented matroids of $\text{OM}(4,8)$, the number of uniform ones is 2,628 and the number of non-uniform ones is 178,844. Bokowski and Richter-Gebert showed that there exist 24 non-representable uniform oriented matroids among the 2,628 uniform oriented matroids \([5]\). Nakayama, Moriyama, Fukuda and Okamoto reconfirmed their result using biquadratic final polynomials with the rational arithmetic \([16]\). On the other hand, the number of all non-representable non-uniform oriented matroids is not known.

First, by our computer program, we enumerate the non-HK and the non-HK* oriented matroids in $\text{OM}(4,8)$. We found 18 non-HK* uniform oriented matroids out of 2,628 uniform oriented matroids, and 1,364 non-HK* non-uniform oriented matroids out of 178,844 non-uniform oriented matroids. On the other hand, there exist no non-HK oriented matroids in $\text{OM}(4,8)$.

Here, we present one non-HK* uniform oriented matroid $\text{IC}(8,4,2)$. Note that $\text{IC}(n, r, c)$ refers to the $c$-th oriented matroid of rank $r$ on the ground set $E$ such that $|E| = n$ in the catalog by Finschi and Fukuda \([7, 6]\). This OM is known as RS(8), constructed by Roudneff and Sturmfels \([18]\). Table 2 shows the chirotope representation of $\text{IC}(8,4,2)$. For example, the second column of the table indicates that the sign of the basis $\{1,2,3,5\}$ is +. It is clear from the table that the OM is uniform, because there is no quadruple taking the zero sign.
Table 2: The chirotopes of IC(8,4,2)

For a coline fixation $\omega = (IC(8,4,2), \{1,8\})$, the coline shelling $CS_\omega$ is the sequence 3, 2, 7, 6, 4, 5 (up to reversal). The coline shelling yields the following shelling digraph $SG_\omega$.

![Figure 5: The shelling digraph for a coline fixation $\omega = (IC(8,4,2), \{1,8\})$](image)

There are only two internally vertex-disjoint paths, since all dipaths from $s$ to $t$ must go through at least one of the vertices 2 or 4. Therefore, $IC(8,4,2)$ is a non-HK* oriented matroid.

Secondly, we enumerate the oriented matroids with the two known properties of representable oriented matroids, the Euclidean [9, 13] and Shannon properties [19, 20], in OM(4,8), and compare them with the non-HK and non-HK* properties. Let us recall the notion of Euclidean and Shannon matroids.

A oriented matroid program $\pi = (M, g, f)$ is called Euclidean if there exists no directed cycle in $G_\pi$, and non-Euclidean otherwise. An oriented matroid $M = (E, L)$ is called Euclidean if $\pi = (M, g, f)$ is Euclidean for any two distinct elements $f \neq g \in E$. By definition, the class is closed under reorientations and taking minors. Also, it is easy to see that every representable oriented matroid is Euclidean [9, 13].

The Shannon property is naturally defined by a theorem by Shannon [19, 20], stating that every representable oriented matroid has simplicial topes at least as many as twice the size of the ground set.

We found 18 non-Euclidean uniform oriented matroids, and they are the same as the 18 non-HK* uniform oriented matroids. Furthermore, we found 3,444 non-Euclidean non-uniform oriented matroids, and they properly include all 1,344 non-HK* non-uniform oriented matroids. On the other hand, there exists only one non-Shannon oriented matroid in OM(4,8), which is also non-HK* and furthermore non-Euclidean. This is known as RS(8) constructed by Roudneff and Sturmfels in [18]. Figure 6 summarizes the results. For more precise information, we suggest the reader to look at the web site

http://www-imai.is.s.u-tokyo.ac.jp/~nak-den/OMcatalog/index.html

which maintains the best of our knowledge.
### Oriented matroids of rank 4 on $E$ ($|E|=8$) = 181472

| Uniform | Non-uniform |
|---------|-------------|
| 2628    | 178844      |

- Non-representable uniform OMs: 24
- Non-representable non-uniform OMs: (θ of such OMs is not known.)
- Non-Euclidean: 3444
- Non-HK*: (Uniform) 18
- Non-HK*: (Non-uniform) 1364
- Non-Shannon: 1

![Figure 6: A Classification of OM(4, 8)](image)

### 3 Construction of an infinite family of non-HK* oriented matroids

In this section, we prove the following theorem.

**Theorem 5** For every $r \geq 4$ and every $n \geq 2r$, there exists a non-HK* oriented matroid of rank $r$ on the ground set $E$ such that $|E| = n$.

The essential idea of this proof is the notion of sensitive LP digraphs, which are special LP digraphs that can lose the Holt-Klee property only by one flip.

Let $P$ be a $d$-polytope in $\mathbb{R}^d$, $f(x) = c^T x$ a generic objective function, $s$ the vertex of $P$ attaining the smallest value of $f$, and $w$ the vertex attaining the second smallest value. Notice that $(s, w)$ is an edge of $P$. The quadruple $\gamma = (P, f, s, w)$ represents the LP digraph with two special vertices marked, which will be called a marked LP digraph.

**Definition 5** A marked LP digraph $\gamma = (P, f, s, w)$ is called a sensitive LP digraph if by reversing (flipping) the orientation of the edge $(s, w)$, the resulting digraph violates the Holt-Klee condition.

We observe that both (1) acyclicity and (2) unique sink&source property \[22\] remain satisfied after reversing the orientation of the edge $(s, w)$. A digraph satisfying the unique sink&source property is called a USO digraph. In the case of $d = 2$, any acyclic USO digraph satisfies the Holt-Klee condition. Hence there exist no sensitive LP digraphs. On the other hand, in $d \geq 3$, there exists an acyclic USO digraph not satisfying the Holt-Klee condition, as in Figure\[2\]. Based on the enumeration of combinatorial types of polytopes with respect to the dimension and the number of vertices by Finschi and Fukuda \[5, 7\], we have checked whether a 3-polytope with few vertices admits a sensitive LP digraph. When the number of vertices is less than six, all acyclic USO digraphs satisfy the Holt-Klee condition.
However, things are different if the number of vertices is equal to six. Among the seven types of 3-polytopes with six vertices in Figure 7, all (five polytopes) except for the leftmost two polytopes admit sensitive LP orientations, as shown in Figure 8. Thus, we have the following.

**Proposition 6** The sensitive LP digraphs $\gamma = (P, f, s, w)$ in Figure 8 are smallest with respect to the dimension of $P$ and the number of vertices of $P$.

By using sensitive LP digraphs, we construct an infinite family of non-HK* oriented matroids through the following three steps. First, we prove Theorem 7, which states that if a $d$-polytope with $n$ vertices admits a sensitive LP digraph, then there exists a non-HK* oriented matroid of rank $r = d+1$ on a $(d+n-1)$-element ground set. Second, we prove in Lemma 9 that if a 3-polytope $P$ with $n$ vertices admits a sensitive LP digraph, then a 3-polytope with $n+1$ vertices
induced by $P$ by a truncation, also admits a sensitive LP digraph. Therefore, combining them with Proposition 6, we obtain Proposition 11 which states that there exists a 3-polytope with $n$ vertices admitting a sensitive LP orientation for every $n \geq 6$. Third, we prove in Proposition 12 that if a $d$-polytope $P$ with $n$ vertices admits a sensitive LP digraph, a $(d+1)$-polytope with $n+1$ vertices obtained from $P$ by a pyramid construction also admits a sensitive LP digraph. Therefore, combining them all, we obtain Proposition 13 stating that there exists a $d$-polytope with $n$ vertices admitting a sensitive LP digraph for every $d \geq 3$ and $n \geq d+3$. Finally, we derive Theorem 5 from Theorem 7 and Proposition 13.

3.1 Construction of non-HK* OMs from representable OMs by a flipping

In this section, we prove the following theorem:

**Theorem 7** Suppose there is a $d$-polytope with $n$ vertices whose graph admits a sensitive LP orientation. Then there exists a non-HK* oriented matroid of rank $r = d+1$ on the ground set of size $|E| = n+d-1$.

**Proof.** Let $P$ be a $d$-polytope in $\mathbb{R}^d$, $F_i$'s (1 ≤ $i$ ≤ $n$) be the facets of $P$, and $H_i$ be the facet-supporting hyperplane of $F_i$. Here we take a line $L$ in general position through the interior of $P$.

First, we show that for any two adjacent facets $F_a$ and $F_b$, there exists a $d$-simplex in $\mathbb{R}^d$ such that $V_a = L \cap H_a$ and $V_b = L \cap H_b$ are its vertices, and the other $d-1$ vertices are on the relative interior of $F_a \cap F_b$, see Figure 9. In fact, we can take an arbitrary $(d-2)$-simplex $\Delta^{d-2}$ in the relative interior of $F_a \cap F_b$. The vertex $V_a = L \cap H_a$ is not on the $(d-2)$-dimensional flat $H_a \cap H_b$, and hence the convex hull of $V_a$ and $\Delta^{d-2}$ is a $(d-1)$-simplex contained in $H_a$. Similarly, the vertex $V_b = L \cap H_b$ is not on $H_a$, and thus the convex hull of $V_b$, $V_a$ and $\Delta^{d-2}$ is a $d$-simplex, which we denote by $\Delta^d$. Because the $d+1$ vertices of $\Delta^d$ are affinely independent, for every facet $G_j$ for 1 ≤ $j$ ≤ $d-1$ of $\Delta^{d-2}$, the $d-1$ vertices of $G_j$, $V_a$ and $V_b$ are also affinely independent in $\mathbb{R}^d$. Here, we denote by $T_j$ the hyperplane determined by the $d-1$ vertices of $G_j$, $V_a$ and $V_b$. Therefore, $H_a$, $H_b$ and $\{T_j : 1 \leq j \leq d-1\}$ are the supporting hyperplanes of the $d$-simplex $\Delta^d$, and $L$ is the intersection of $\{T_j : 1 \leq j \leq d-1\}$.

Now, suppose there is a $d$-polytope $Q$ with $n$ vertices admitting a sensitive LP orientation. By duality, this implies that we may suppose that the $d$-polytope $P$ above is a dual to $Q$ and that the line $L$ induces a shelling digraph isomorphic to the sensitive LP digraph. Particularly, we may suppose that $F_a$ and $F_b$ are the first (or the last) two facets of the line shelling of $P$ induced by $L$, see Figure 9. This implies that each of the hyperplanes $\{H_i : 1 \leq i \leq n\} \setminus \{H_a, H_b\}$ does not intersect with the $d$-simplex $\Delta^d$.

We claim that one can apply a flipping operation $10$ to the representable oriented matroid $M = (E, \mathcal{L})$ of the hyperplane arrangement $\mathcal{H}$ of $\{H_i : 1 \leq i \leq n\} \cup \{T_j : 1 \leq j \leq d-1\}$. Let us present the precise definition of $M$. For every $d$-dimensional hyperplane $h \in \mathcal{H}$, we take a hyperplane $h'$ in $\mathbb{R}^{d+1}$ containing the origin and a lifted copy $H \times \{1\}$ of $H$. $M$ is the representable oriented matroid of rank $d+1$ represented by the linear sphere arrangement on the $d$-dimensional unit sphere $S^{d+1}$ in $\mathbb{R}^{d+1}$ of $\{e_i = S^{d+1} \cap H'_i : 1 \leq i \leq n\} \cup \{e_{n+j} = S^{d+1} \cap T'_j : 1 \leq j \leq d-1\}$, i.e. $E = \{e_k : 1 \leq k \leq n+d-1\}$. The subset $\{e_k : n+1 \leq j \leq n+d-1\} \subset E$ is a coline of $M$ by construction. Let $\omega$ be the coline fixation $(M,T)$, where $T = \{e_{n+j} : 1 \leq j \leq d-1\}$. Then, its coline shelling $CS_\omega$ coincides with the line shelling of $P$ given by $L$. In particular, the first two elements of $CS_\omega$ are $e_a$ and $e_b$.

Exploiting the structure of the hyperplane arrangement $\mathcal{H}$, we apply a flipping operation $10$ to $M$. Namely, we may flip any element $e_k \in E' = \{e_a, e_b\} \cup \{e_{n+j} : 1 \leq j \leq d-1\}$ so that the
associated simplex tope is flipped over. The resulting oriented matroid does not depend on the choice of $e_k$ and is denoted by $M'$.

Now, we observe that $T$ remains a coline in $M'$ and the coline shelling $CS_{\omega'}$ induced by the fixation $\omega' = (M', T)$ differs from $CS_{\omega}$ only for the ordering of $e_a, e_b$. This means that the orientation of the edge $(e_a, e_b)$ in the shelling digraph $SG_\omega$ is reversed in $SG_{\omega'}$, and thus $SG_{\omega'}$ does not satisfy the Holt-Klee condition. This means that the oriented matroid $M'$ is non-HK*. It has rank $r = d + 1$ and $|E| = n + d - 1$. This completes the proof.

Using both Proposition 6 and Theorem 7 we have a theoretical proof for the fact we knew from our computational classification.

**Corollary 8** There exists a non-HK* oriented matroid of rank 4 on a 8-element ground set.

### 3.2 Construction of non-HK* OMs of rank 4 by a truncation

First, we define a truncated polytope, which is a key idea of this section.

**Definition 6** Let $P$ be a 3-polytope in $\mathbb{R}^3$ containing a simple vertex $v$ (i.e. a vertex $v$ with exactly 3 neighbors), $\{v_i : i = 1, 2, 3\}$ the vertices adjacent to $v$ and $\{u_j : j = 1, 2\}$ points in the relative interior of an edge $(v, v_j)$. By a truncated polytope $tr(P)$, we mean a 3-polytope $P \cap (H \cup H^+)$ where $H$ is the hyperplane determined by $u_1, u_2$ and $v_3$, and $H^+$ is the open halfspace of $H$ containing all vertices except $v$, see Figure 10.

![Figure 10: A truncated polytope](image)

**Lemma 9** Let $P$ be a 3-polytope in $\mathbb{R}^3$ containing a simple vertex $v$. If $P$ admits a sensitive LP digraph, a truncated polytope $tr(P)$ also admits a sensitive LP digraph.
Proof. Let \((P, f, s, w)\) be a sensitive LP digraph. A truncation operation with respect to a simple vertex \(v\) generates the following five new edges: \((v_1, u_1)\), \((v_2, u_2)\), \((v_3, u_1)\), \((v_3, u_2)\), and \((u_1, u_2)\). We show how to orient the five edges so that \(tr(P)\) also admits a sensitive LP digraph.

By the symmetry of \(v_1\) and \(v_2\), there are exactly six types of orientations of \((v, v_1)\), \((v, v_2)\), and \((v, v_3)\) with respect to outdegree and indegree of \(v\).

In the case of (a), (b) and (c), since no vertices of \(v\), \(v_1\), \(v_2\) and \(v_3\) are the global source \(s\), we only have to orient the five edges so that (1) acyclicity, (2) the unique sink&source property and (3) the Holt-Klee condition are satisfied, i.e., every facet of \(F_1\), \(F_2\), \(F_3\) and \(F_4\) in Figure 11 has a unique source and a unique sink, and the number of disjoint paths between \(v_1\), \(v_2\) and \(v_3\) remains unchanged. Then, the resulting orientation of \((v_1, u_1)\), \((v_2, u_2)\), \((v_3, u_1)\), \((v_3, u_2)\), and \((u_1, u_2)\) is classified into (a), (b) and (c) in Figure 13. On the other hand, in the case of (d) and (e), it is possible that \(v_2\) in (d) and \(v_3\) in (e) are the global source \(s\), and \(v\) in (d) and \(v_3\) in (e) are \(w\). In these cases, if the five edges are oriented as (d) and (e) in Figure 13 i.e., \(v_2\) in (d) and \(v_3\) in (e) are also \(s\), and \(u_2\) in (d) and \(u_1\) in (e) are \(w\), the three properties are satisfied and the sensitivity of an LP digraph remains unchanged. Otherwise, as well as (a), (b) and (c), we have only to orient the five edges such that the three properties are satisfied. Finally, in the case of (f), \(v\) is \(s\) and one of \(v_1\), \(v_2\) and \(v_3\) is \(w\). If the five edges are oriented as (f) in Figure 13 the three properties are also satisfied and the sensitivity of an LP digraph also remains unchanged. From the above, a truncated polytope \(tr(P)\) also admits a sensitive LP digraph. 

From Definition 6, a truncation operation generates two simple vertices \(u_1\) and \(u_2\) while one simple vertex \(v\) is removed. Hence we have the following.

**Remark 10** There exists at least one simple vertex in a truncated polytope.

Furthermore, because all five polytopes in Proposition 6 contain a simple vertex, we may apply a truncation operation to a 3-polytope successively. Thus, we obtain the main proposition of this section.
Figure 13: The orientation of $(v_1, u_1), (v_2, u_2), (v_3, u_1), (v_3, u_2),$ and $(u_1, u_2)$

**Proposition 11** There exists a 3-polytope with $n$ vertices which admits a sensitive LP digraph for every $n \geq 6$.

### 3.3 Construction of non-HK* OMs of higher ranks

In this section, we present a construction of a sensitive LP digraph starting from a sensitive LP digraph in one lower dimension.

Given a $d$-polytope $P$ in $\mathbb{R}^d$, its pyramid polytope $py(P, v)$ is a $(d+1)$-polytope in $\mathbb{R}^{d+1}$ which is the convex hull of $P \times \{0\}$ and a point $v \in \mathbb{R}^{d+1}$ not on the $d$-dimensional subspace containing $P$. A canonical choice is to set $v_{d+1} = 1$, see Figure 14.

![Figure 14: An intuitive image of the pyramid polytope in $\mathbb{R}^4$](image)

**Proposition 12** Let $P$ be a $d$-polytope in $\mathbb{R}^d$. If $P$ admits a sensitive LP digraph, a pyramid polytope $py(P, v)$, also admits a sensitive LP digraph.

**Proof.** Let $P$ be a $d$-polytope in $\mathbb{R}^d$ which admits a sensitive LP digraph. This means $d \geq 3$ and thus $P$ has at least four vertices. Let $\gamma = (P, f, s, w)$ be a sensitive LP digraph, where $f$ is a generic objective function and let $z$ be the vertex of $P$ attaining the third smallest objective value. Thus, $f(s) < f(w) < f(z)$.

Let $v \in \mathbb{R}^{d+1}$ be any point with $v_{d+1} = 1$, and consider the pyramid $Q = py(P, v)$. We shall construct an objective function $g$ for $Q$ which induces a sensitive LP orientation. Let $g$ be a natural extension of $f$: $g(y) = f(x) + c_{d+1}x_{d+1}$. Since $v$ is the only vertex of $Q$ with nonzero last
component, one can set $c_{d+1}$ in such a way that $g(S) < g(W) < g(V) < g(Z)$, where uppercase letters denote the same (lowercase) vectors lifted to $\mathbb{R}^{d+1}$: $S^T = (s^T, 0)$, $W^T = (w^T, 0)$, $Z^T = (z^T, 0)$, $V = v$, see Figure 15.

![Figure 15: A pyramid polytope and a generic function $g$](image)

We claim that the LP digraph $Q$ induced by $g$ is sensitive, or more precisely, the marked LP digraph $\gamma' = (Q, g, S, W)$ is a sensitive LP digraph. By the construction, $S$, $W$ and $T$ are the vertices attaining the smallest, the second smallest, and the largest $g$ value, respectively. The only difference between $\gamma$ and $\gamma'$ are the extra edges in $\gamma'$ incident with $V$. This means that the maximum number of dipaths from $W$ to $T$ in $\gamma'$ with the edge $(S, W)$ reversed is at most one more than the maximum number of dipaths from $w$ to $t$ in $\gamma$ with the edge $(s, w)$ reversed. Since $\gamma$ is sensitive, $\gamma'$ is sensitive as well.

Combining with Proposition 11, we can construct a $d$-polytope which admits a sensitive LP digraph successively. Thus, we have the following.

**Proposition 13** For each $d \geq 3$ and $n \geq d+3$, there exists a $d$-polytope with $n$ vertices which admits a sensitive LP digraph.

Finally, from Theorem 7 and Proposition 13 the main theorem of this section, Theorem 5, follows.

4 Concluding remarks

In this paper, we introduced two new classes HK and HK* of oriented matroids based on the Holt-Klee condition and its dual interpretation in terms of line shellings, respectively. In particular, the non-HK and non-HK* properties are certificates for non-representability.

We have shown that these two classes are distinct. While we gave a construction of an infinite family of non-HK* OMs using the notion of sensitive LP digraphs, it is not clear whether there exists any non-HK OM. This leads to a fundamental question as to whether the Holt-Klee theorem can be proven combinatorially by using the OM axioms only while the original proof in [11] relies heavily on geometric operations such as affine transformations and orthogonal projections.

To get a better understanding, we presented a classification of the oriented matroids of rank 4 on 8-element ground set with respect to the HK, HK*, Euclidean and Shannon properties. Our classification shows that there exists no non-HK OM in this class. This suggests us to try to prove the statement that every OM is HK. A successful trial would yield a purely combinatorial proof of the Holt-Klee theorem.
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