QUANTUM HOMOLOGY OF COMPACT CONVEX SYMPLECTIC MANIFOLDS

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ABSTRACT. We study the space of pseudo-holomorphic spheres in compact symplectic manifolds with convex boundary. We show that the theory of Gromov-Witten invariants can be extended to the class of semi-positive manifolds with convex boundary. This leads to a deformation of intersection products on the absolute and relative singular homologies. As a result, absolute and relative quantum homology algebras are defined analogously to the case of closed symplectic manifolds. In addition, we prove the Poincaré-Lefschetz duality for the absolute and relative quantum homology algebras.

1. Introduction and results.

1.1. History and motivation. The celebrated work of Gromov [7] opened the era of invariants of a symplectic manifold based on the counting of pseudo-holomorphic curves in it. The first example of such invariants, now called the Gromov invariant, was defined by Gromov itself in the same paper [7], which led to the proof of the famous Gromov’s non-squeezing theorem: The symplectic ball of radius \( r \) can not be symplectically embedded into the cylinder \( B_R^2 \times \mathbb{C}^n \) for \( R < r \). More subtle invariants, now known under the name of Gromov-Witten (GW) invariants, appeared later in physics as correlation functions in Wittens topological sigma models [17], and were put on a solid mathematical basis by Ruan and Tian in [14, 15]. See also [11, 12]. One of the consequences of GW invariants is the theory of quantum homology of a symplectic manifold, i.e. the deformation of the intersection product in its ordinary homology. Historically, these kinds of structures were considered in the case of closed (semi-positive) symplectic manifolds.

The purpose of this paper is to extend the theory of Gromov-Witten invariants and quantum homology to the class of semi-positive convex compact symplectic manifolds.

1.2. Setting. Consider a \( 2n \)-dimensional compact symplectic manifold \((M, \omega)\) with non-empty boundary \( \partial M \). Recall the following
Definition 1.1. (cf. [6, 9, 12]) The boundary $\partial M$ is called convex if there exists a Liouville vector field $X$ (i.e. $\mathcal{L}_X \omega = d(\omega) = \omega$), which is defined in the neighborhood of $\partial M$ and which is everywhere transverse to $\partial M$, pointing outwards; equivalently, there exists a 1-form $\alpha$ on $\partial M$ such that $d\alpha = \omega |_{\partial M}$ and such that $\alpha \wedge (d\alpha)^{n-1}$ is a volume form inducing the boundary orientation of $\partial M \subset M$. Therefore, $(\partial M, \ker \alpha)$ is a contact manifold, and that is why a convex boundary is also called of a contact type.

A compact symplectic manifold $(M, \omega)$ with non-empty boundary $\partial M$ is convex if $\partial M$ is convex.

A non-compact symplectic manifold $(M, \omega)$ is convex if there exists an increasing sequence of compact convex submanifolds $M_i \subset M$ exhausting $M$, that is,

$$M_1 \subset M_2 \subset \ldots \subset M_i \subset \ldots \subset M \quad \text{and} \quad \bigcup_i M_i = M.$$ 

Examples 1.

1. The standard $(\mathbb{R}^{2n}, \omega_0)$ exhausted by balls.
2. Cotangent $r$-ball bundles $(\mathbb{D}_r T^* X, \omega_{can})$.
3. Cotangent bundles $(T^* X, \omega_{can})$ over a closed base, exhausted by ball bundles, $T^* X = \bigcup_{r>0} \mathbb{D}_r T^* X$.
4. Stein manifolds $(M, J, f)$, where $(M, J)$ is an open complex manifold and $f : M \to \mathbb{R}$ is a smooth exhausting plurisubharmonic function without critical points off a compact subset of $M$. Here, ”exhausting” means that $f$ is proper and bounded from below, and ”plurisubharmonic” means that $\omega_f := -d(df \circ J)$ is a symplectic form with Levi form $\omega_f (v, Jv) > 0$ for all $0 \neq v \in TM$. Then the gradient $X = \nabla f$ with respect to the Kähler form $\omega_f \circ (id \times J)$ satisfies $\mathcal{L}_X \omega_f = d(\omega_f) = \omega_f$. Furthermore, if the critical points of $f$ are in, say, $\{f < 1\}$, then the manifold $M$ is exhausted by the exact convex symplectic manifolds $M_i = \{f \leq i\}$.
5. Symplectic blow-up of the above examples at finitely many interior points.

Recall that an almost complex structure on a smooth $2n$-dimensional manifold $M$ is a section $J$ of the bundle $\text{End} \ TM$ such that $J^2(x) = -1_{T_x M}$ for every $x \in M$. An almost complex structure $J$ on $M$ is called compatible with $\omega$ (or $\omega$-compatible) if $g_J := \omega \circ (1 \times J)$ defines a riemannian metric on $M$. Denote by $\mathcal{J}(M, \omega)$ the space of all $\omega$-compatible almost complex structures on $(M, \omega)$.

Given $(M, \omega)$ and $J \in \mathcal{J}(M, \omega)$, then $(TM, J)$ becomes a complex vector bundle and, as such, its first Chern class $c_1(TM, J, \omega) \in H^2(M; \mathbb{Z})$ is defined. Note that since the space $\mathcal{J}(M, \omega)$ is non-empty and contractible, (see [10, Proposition 4.1, (i)]), the class $c_1(TM, J, \omega)$ does not depend on $J \in \mathcal{J}(M, \omega)$, and we shall denote it just by $c_1(TM, \omega)$.

Let $H^2_S(M)$ be the image of the Hurewicz homomorphism $\pi_2(M) \to H_2(M, \mathbb{Z})$. The homomorphisms $c_1 : H^2_S(M) \to \mathbb{Z}$ and $\omega : H^2_S(M) \to \mathbb{R}$ are given by $c_1(A) := c_1(TM, \omega)(A)$ and $\omega(A) = [\omega](A)$ respectively. The minimal Chern number of a
symplectic manifold \((M, \omega)\) is the integer \(N > 0\), such that \(\text{Im}(c_1) = N \cdot \mathbb{Z}\). If \(c_1(A) = 0\) for every \(A \in H_2^S(M)\), then we define the minimal Chern number to be \(N = \infty\).

**Definition 1.2.** (cf. [8], [9]) A symplectic 2n-manifold \((M, \omega)\) is called semi-positive, if \(\omega(A) \leq 0\) for any \(A \in H_2^S(M)\) with \(3 - n \leq c_1(A) < 0\).

The semi-positivity can be characterized by the following

**Lemma 1.3.** (E.g. [8] Lemma 1.1.)
A symplectic 2n-manifold \((M, \omega)\) is semi-positive if and only if one of the following conditions is satisfied.

(i) \(\omega = \alpha c_1\), for some \(\alpha \geq 0\).
(ii) \(c_1(A) = 0\) for every \(A \in H_2^S(M)\).
(iii) The minimal Chern number \(N\) satisfies \(N \geq n - 2\).

### 1.3. Main results and the structure of the paper.
Throughout the paper we consider a semi-positive convex compact symplectic manifold \((M, \omega)\). We shall always work over the base field \(\mathbb{F}\), which is either \(\mathbb{Z}_2\) or \(\mathbb{Q}\).

In Section 2 we analyze the moduli spaces of pseudo-holomorphic spheres in \(M\). We show that due to the convexity and the semi-positivity, such moduli spaces are well defined and generically they are smooth orientable manifolds.

In Section 3 we develop the following notion of genus zero Gromov-Witten invariant relative to the boundary. Let us fix a finite number of pairwise distinct marked points \(z := (z_1, \ldots, z_m) \in (\mathbb{S}^2)^m\). For every integer \(p \in \{0, 1, \ldots, m\}\) we define the genus zero Gromov-Witten invariant \(GW_{A, p, m}\) relative to the boundary as a \(m\)-linear (over \(\mathbb{F}\)) map \(GW_{A, p, m} : H_*(M; \mathbb{F})^\times p \times H_*(M, \partial M; \mathbb{F})^{\times (m-p)} \to \mathbb{F}\), which roughly speaking counts the following geometric configurations.

If \(\mathbb{F} = \mathbb{Z}_2\), choose smooth cycles \(f_i : V_i \to M\) representing classes \(a_i \in H_*(M; \mathbb{Z}_2)\) for every \(i = 1, \ldots, p\), and choose relative smooth cycles \(f_j : (V_j, \partial V_j) \to (M, \partial M)\) representing classes \(a_j \in H_*(M, \partial M; \mathbb{Z}_2)\) for every \(j = p + 1, \ldots, m\), such that all the maps are in general position. Then \(GW_{A, p, m}(a_1, \ldots, a_m)\) counts the parity of \(J\)-holomorphic spheres in class \(A \in H_2^S(M)\), such that \(z_i\) is mapped to \(f_i(V_i)\) for every \(i = 1, \ldots, p\), and \(z_j\) is mapped to \(f_j(V_j \setminus \partial V_j)\) for every \(j = p + 1, \ldots, m\).

Now, let \(\mathbb{F} = \mathbb{Q}\). For non-zero \(a_i \in H_*(M; \mathbb{Q})\), \(i = 1, \ldots, p\), there exist non-zero \(r_i \in \mathbb{Q}\) and smooth cycles \(f_i : V_i \to M\) representing \(r_i a_i\), and for non-zero \(a_j \in H_*(M, \partial M; \mathbb{Q})\), \(j = p + 1, \ldots, m\), there exist non-zero \(r_j \in \mathbb{Q}\) and relative smooth cycles \(f_j : (V_j, \partial V_j) \to (M, \partial M)\) representing \(r_j a_j\), such that all the maps are in general position. Then the invariant \(GW_{A, p, m}(r_1 a_1, \ldots, r_m a_m)\) counts the algebraic number of \(J\)-holomorphic spheres in class \(A \in H_2^S(M)\), such that \(z_i\) is mapped to \(f_i(V_i)\) for every \(i = 1, \ldots, p\), and \(z_j\) is mapped to \(f_j(V_j \setminus \partial V_j)\) for every \(j = p + 1, \ldots, m\). Define \(GW_{A, p, m}(a_1, \ldots, a_m) := \frac{1}{r_1 \cdots r_m} GW_{A, p, m}(r_1 a_1, \ldots, r_m a_m)\).
In section 4 we use the Gromov-Witten invariants $GW_{A,p,m}$ to deform the classical intersection products $\bullet, i = 1, 2, 3$; cf. Definition 3.8. These deformations give rise to the absolute (that is $H_*(M; F) \otimes F \Lambda$) and the relative (that is $H_*(M, \partial M; F) \otimes F \Lambda$) quantum homology algebras of $(M, \omega)$, where $\Lambda$ is the appropriate Novikov ring. We prove the Poincaré-Lefschetz duality for the absolute and relative quantum homology algebras. Finally, we compute the quantum homology algebras for some manifolds $(M, \omega)$.

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2. J-holomorphic curves in semi-positive convex compact symplectic manifolds.

2.1. J-holomorphic curves near a convex boundary. Let $(M, \omega)$ be a $2n$-dimensional compact semi-positive convex symplectic manifold and $J \in J(M, \omega)$ is an $\omega$-compatible almost complex structure. In addition, let $(\Sigma, j_\Sigma)$ be a closed Riemann surface with its standard complex structure $j_\Sigma$.

Definition 2.1. A $J$-holomorphic curve is a smooth map $u : \Sigma \to M$ such that $du \circ j_\Sigma = J \circ du$.

The following important lemmas show that $J$-holomorphic spheres for a suitable $J$ are bounded away from the boundary, i.e., they lie in the complement of some open neighborhood of $\partial M$.

Lemma 2.2. (E.g. [12, Lemma 9.2.7].) Let $(M, \omega)$ be a $2n$-dimensional compact convex symplectic manifold and let $X$ be a Liouville vector field (see, Definition 1.1). Then there exists a pair $(f, J)$, where $J \in J(M, \omega)$ is an $\omega$-compatible almost complex structure and $f : M \to (-\infty, 0]$ is a proper smooth function such that

- For all $x \in \partial M$ and for all $v \in T_x \partial M$ we have $J(x)X(x) \in T_x \partial M$, $\omega(v, J(x)X(x)) = 0$, $\omega(X(x), J(x)X(x)) = 1$.
- $\partial M = f^{-1}(\{0\})$.
- There exists a collar neighborhood $W \cong (-\varepsilon, 0] \times \partial M$ of the boundary for some $\varepsilon > 0$, which depends on the Liouville vector field $X$ such that $\omega = -d(df \circ J)$ on $W$.

Such an almost complex structure $J$ is said to be adapted to the boundary or contact, and such a function $f$ is called plurisubharmonic. The set of $\omega$-compatible almost complex structures that are adapted to the boundary will be denoted by
According to the lemma 2.2 and [2, Remark 4.1.2], or [3, discussion on the page 106] we have the following

**Lemma 2.3.** Let $(M, \omega)$ be a 2n-dimensional compact convex symplectic manifold. Then the set $\mathcal{J}(M, \partial M, \omega)$ of $\omega$-compatible almost complex structures that are adapted to the boundary is non-empty and connected.

The next lemma (see also [12, Lemma 9.2.9]) and its corollaries are crucial for the study of $J$-holomorphic spheres. Let us denote by $\Delta := \partial_s^2 + \partial_t^2$ the standard Laplacian.

**Lemma 2.4.** Let $J \in \mathcal{J}(M, \omega)$, $\Omega \subseteq \mathbb{C}$ be an open set, $u : \Omega \rightarrow M$ be a $J$-holomorphic curve, and $f : M \rightarrow \mathbb{R}$ is a smooth function such that $\omega = -d(df \circ J)$ on a neighborhood of the image of $u$. Then

$$\Delta(f \circ u) = \omega(\partial_s u, J\partial_s u)$$

and hence $f \circ u$ is subharmonic, i.e. $\Delta(f \circ u) \geq 0$.

Let $(M, \omega)$ be a compact convex symplectic manifold. We have the following

**Corollary 2.5.** (See [12, Corollary 9.2.10].)

Let $(\Sigma, j)$ be a connected closed Riemann surface, $W \subset M$ be an open neighborhood of $\partial M$, and $u : \Sigma \rightarrow M$ be a smooth map, whose restriction to $u^{-1}(W)$ is $J$-holomorphic for some $J \in \mathcal{J}(M, \partial M, \omega)$. Then $u(\Sigma) \cap \partial M \neq \emptyset$ if and only if $u(\Sigma) \subset \partial M$.

**Corollary 2.6.** Let $W$ be an open neighborhood of $\partial M$, such that $\omega = -d(df \circ J)$ on $W$, where the pair $(f, J)$, satisfies the conditions of lemma 2.2. Then for any non-constant $J$-holomorphic sphere $u : S^2 \rightarrow M$ we have $u(S^2) \subset M \setminus W$.

On can also consider the solutions of the perturbed Cauchy-Riemann equation

$$\bar{\partial}_J(u) = \nu(u),$$

where an inhomogeneous term $\nu(u) \in \Omega^{0,1}_J(\Sigma, u^*TM)$ is Hamiltonian with compact support. Recall, if $(M, \omega)$ is a symplectic manifold and $H \in \mathcal{C}^\infty(M)$ is a smooth function on $M$, called a Hamiltonian, one defines the smooth vector field $X_H \in \Gamma(M, TM)$ on $M$ by the formula

$$\omega(X_H, \cdot) = -dH(\cdot).$$

Such a vector field is called a Hamiltonian vector field, and the space of Hamiltonian vector fields on $M$ will be denoted by $\Gamma_{\text{Ham}}(M, TM)$.

Let $(M, \omega)$ be a compact convex symplectic manifold. Denote by $\mathcal{C}^\infty_c(M) \subseteq \mathcal{C}^\infty(M)$ the space of compactly supported Hamiltonians, i.e.,

$$H \in \mathcal{C}^\infty_c(M) \Leftrightarrow \text{supp}(H) \subseteq \overset{\circ}{M} := M \setminus \partial M.$$
Suppose we have a 1-form
\[ H \in \Omega^1(\Sigma, C^\infty_c(M)) := \Gamma(\Sigma, \text{Hom}(T\Sigma, C^\infty_c(M))) \]
on \Sigma with values in the linear space \( C^\infty_c(M) \). Define the support of \( H \) to be
\[ \text{supp}(H) := \bigcup_{z \in \Sigma, \xi \in T_z\Sigma} \text{supp}(H(z)(\xi)). \]

Denote by
\[ \Omega^1_c(\Sigma, C^\infty_c(M)) \subseteq \Omega^1(\Sigma, C^\infty_c(M)) \]
the subspace of compactly supported 1-forms, i.e.,
\[ H \in \Omega^1_c(\Sigma, C^\infty_c(M)) \iff \text{supp}(H) \subseteq \mathring{M}. \]

Such a 1-form gives rise to the 1-form
\[ X_H \in \Omega^1(\Sigma, \Gamma\text{Ham}(M, TM)) := \Gamma(\Sigma, \text{Hom}(T\Sigma, \Gamma\text{Ham}(M, TM))) \]
on \Sigma with values in the linear space of Hamiltonian vector fields. It is given by
\[ T_z\Sigma \ni \xi \mapsto X_H(z)(\xi) := X_H(z)(\xi) \in \Gamma\text{Ham}(M, TM), \]
and its support satisfies
\[ \text{supp}(X_H) := \bigcup_{z \in \Sigma, \xi \in T_z\Sigma} \text{supp}(X_H(z)(\xi)) \subseteq \mathring{M}. \]

Given a smooth map \( u : \Sigma \to M \), denote by
\[ \mathcal{X}_{\mathcal{H}}(u) \in \Omega^1(\Sigma, u^*TM) := \Gamma(\Sigma, \text{Hom}(T\Sigma, u^*TM)) \]
the 1-form along \( u \) with values in the pullback tangent bundle. It is given by
\[ T_z\Sigma \ni \xi \mapsto \mathcal{X}_{\mathcal{H}}(u)(z)(\xi) := X_{\mathcal{H}(z)(\xi)}(u(z)) \in T_{u(z)}M. \]

Finally, fix \( J \in \mathcal{J}(M, \partial M, \omega) \) and define the inhomogeneous term \( \nu \) by
\[ \nu := - (\mathcal{X}_{\mathcal{H}})^{0,1} \quad \text{and} \quad \nu(u) := - (\mathcal{X}_{\mathcal{H}(u)})^{0,1}, \]
where \((\mathcal{X}_{\mathcal{H}})^{0,1}\) and \((\mathcal{X}_{\mathcal{H}(u)})^{0,1}\) denote the \( J \)-complex anti-linear parts of \( \mathcal{X}_{\mathcal{H}} \) and \( \mathcal{X}_{\mathcal{H}(u)} \) respectively. Then the perturbed Cauchy-Riemann equation has the form
\[ (1) \quad \overline{\partial}_{J,\mathcal{H}}(u) := \overline{\partial}_J(u) + (\mathcal{X}_{\mathcal{H}(u)})^{0,1} = 0 \]
A smooth map \( u : \Sigma \to M \) that satisfies Equation (1) will be called \((J, \mathcal{H})\)-holomorphic curve. Note that the form \( \overline{\partial}_{J,\mathcal{H}}(u) \) is the \( J \)-complex anti-linear part of the covariant derivative
\[ d_{\mathcal{H}}(u) := du + \mathcal{X}_{\mathcal{H}(u)}. \]

Since \( \overline{\partial}_{J,\mathcal{H}}(u) = \overline{\partial}_J(u) \) on \( M \setminus \text{supp}(\mathcal{H}) \) we have the following corollary.
Corollary 2.7. Let $W$ be an open neighborhood of $\partial M$, such that $\omega = -d(df \circ J)$ on $W$, where the pair $(f, J)$, satisfies the conditions of Lemma 2.2. Suppose that $H \in \Omega^1_c(\mathbb{S}^2, C^\infty_c(M))$ such that $W \subseteq M \setminus \text{supp}(H)$. Then for any non-constant $(J, H)$-holomorphic sphere $u : \mathbb{S}^2 \to M$ we have $u(\mathbb{S}^2) \subset M \setminus W$.

Due to Gromov, one can view a perturbed $(J, H)$-holomorphic curve $u : \Sigma \to M$ as a $(j_\Sigma, \tilde{J})$-holomorphic map (section)

$$\tilde{u} : \Sigma \to \tilde{M} := \Sigma \times M, \quad \tilde{u}(z) = (z, u(z)),$$

where the almost complex structure $\tilde{J}$ on $\tilde{M}$ is given by

$$\tilde{J} := \begin{pmatrix} j_\Sigma & 0 \\ -(X_H)^{0,1} \circ j_\Sigma & J \end{pmatrix}.$$

We have that the projection

$$\pi : (\tilde{M}, \tilde{J}) \to (\Sigma, j_\Sigma)$$

is $(\tilde{J}, j_\Sigma)$-holomorphic and for each $z \in \Sigma$, the fibre

$$\{z\} \times M, J \subseteq (\tilde{M}, \tilde{J})$$

is an almost complex submanifold. Moreover, there exits a symplectic form $\omega_\Sigma$ on $\Sigma$, such that $J$ is $\omega$-compatible if and only if $\tilde{J}$ is $(\omega_\Sigma \oplus \omega)$-compatible. We have seen in the Corollary 2.7 that for any compactly supported Hamiltonian perturbation $H$, a $(J, H)$-holomorphic curve lies in the complement of some open neighborhood of the boundary $\partial M$. Hence, the respective $(j_\Sigma, \tilde{J})$-holomorphic section lies in the complement of some open neighborhood of the boundary $\partial \tilde{M} = \Sigma \times \partial M$.

It follows from corollaries 2.5 and 2.6 that the Gromov compactness theorem holds for both non-perturbed and perturbed $J$-holomorphic curves. See [12, Chapter 4], [13, Section 6], [14, Section 3], [11, Theorem 4.1.1].

2.2. Moduli spaces of $J$-holomorphic spheres. Let $(M, \omega)$ be a $2n$-dimensional semi-positive compact convex symplectic manifold and let $J \in \mathcal{J}(M, \partial M, \omega)$. From the above discussion it follows that one can repeat verbatim the construction of moduli spaces of (perturbed) $J$-holomorphic spheres. The complete proves can be found in [12, Chapter 3]. Let us just restate the central results.

Recall that a smooth map $u : \mathbb{S}^2 = \mathbb{C}P^1 \to M$ is called simple if $u = w \circ \phi$ with meromorphic $\phi : \mathbb{C}P^1 \to \mathbb{C}P^1$ implies $\deg \phi = 1$. Given a class $A \in H^2_*(M)$, denote by $\mathcal{M}(A, J)$ the set of all $J$-holomorphic spheres, which represent the class $A$. Denote also by $\mathcal{M}_s(A, J) \subseteq \mathcal{M}(A, J)$ the subset of simple $J$-holomorphic spheres, which represent the class $A$.
Theorem 2.8. There exists a subset $\mathcal{J}_{\text{reg}} := \mathcal{J}_{\text{reg}}(A) \subset \mathcal{J}(M, \partial M, \omega)$ of the second Baire category in $C^\infty$-topology of regular almost complex structures, such that for any $J \in \mathcal{J}_{\text{reg}}$ the space $\mathcal{M}_s(A, J)$ is a smooth manifold of dimension
\[
\dim \mathcal{M}_s(A, J) = 2n + 2I_c(A).
\]
It carries a canonical orientation. For different regular almost complex structures, the moduli spaces are oriented cobordant.

It follows from [12, Proposition 2.5.1]) that the group of Möbius transformations $G := PSL_2(\mathbb{C})$ acts freely on $\mathcal{M}_s(A, J)$ by $\phi \cdot u := u \circ \phi^{-1}$ for every $\phi \in G$. Hence, for any $J \in \mathcal{J}_{\text{reg}}$ the quotient $\mathcal{C}_s(A, J) := \mathcal{M}_s(A, J)/G$ is a finite dimensional manifold of dimension
\[
\dim \mathcal{C}_s(A, J) = 2n + 2I_c(A) - 6.
\]
In particular, it is empty whenever $I_c(A) < 3 - n$.

In order to deal with multiply covered spheres one can consider moduli spaces of perturbed J-holomorphic spheres. In the closed case it was done by Y. Ruan in [14, Section 3]. A similar construction can be found in the monograph [12, Chapter 8]. We shall consider the Hamiltonian perturbations with compact support. For any almost complex structure $J \in \mathcal{J}(M, \partial M, \omega)$, any Hamiltonian perturbation with compact support $\mathcal{H} \in \Omega^1_c(S^2, C^\infty_c(M))$ and any class $A \in H^2_c(M)$, denote by $\mathcal{M}(A, J, \mathcal{H})$ the set of all $(J, \mathcal{H})$-holomorphic spheres in $M$, which represent the class $A$, i.e.
\[
\mathcal{M}(A, J, \mathcal{H}) = \{ u \in C^\infty(S^2, M) \mid \overline{\partial}_J u = 0, [u] = A \}.
\]

Theorem 2.9. (See [14, Theorems 3.2.2 and 3.2.8].) There exists a subset $\Omega^1_{c, \text{reg}}(S^2, C^\infty_c(M)) \subset \Omega^1_c(S^2, C^\infty_c(M))$ of the second category in the $C^\infty$-topology, such that for any $\mathcal{H} \in \Omega^1_{c, \text{reg}}(S^2, C^\infty_c(M))$ the moduli space $\mathcal{M}(A, J, \mathcal{H})$ is a smooth manifold of dimension
\[
\dim \mathcal{M}(A, J, \mathcal{H}) = 2n + 2I_c(A).
\]
It carries a canonical orientation. Moreover, for a generic (in the Baire categorical sense) path
\[
I := [0, 1] \ni t \mapsto (J_t, \mathcal{H}_t)
\]
the set
\[
\mathcal{K}(A, (J_t, \mathcal{H}_t)_{t \in I}) := \{(t, u) \in I \times C^\infty(S^2, \mathcal{M}) \mid u \in \mathcal{M}(A, J_t, \mathcal{H}_t)\}
\]
is a smooth oriented manifold of dimension
\[
\dim \mathcal{K}(A, (J_t, \mathcal{H}_t)_{t \in I}) = 2n + 2I_c(A) + 1
\]
and with boundary
\[
\partial \mathcal{K}(A, (J_t, \mathcal{H}_t)_{t \in I}) = \mathcal{M}(A, J_1, \mathcal{H}_1) \cup -\mathcal{M}(A, J_0, \mathcal{H}_0),
\]
where the minus sign denotes the reversed orientation.
Let us fix a finite number of pairwise distinct points \( z := (z_1, \ldots, z_m) \in (S^2)^m \). We can consider well-defined evaluation maps

\[
ev_{z,J} : \mathcal{M}_s(A, J) \to \left( \hat{M} \right)^m, \quad ev_{z,J}(u) := (u(z_1), \ldots, u(z_m)),
\]

and

\[
ev_{z,J,\mathcal{H}} : \mathcal{M}(A, J, \mathcal{H}) \to \left( \hat{M} \right)^m, \quad ev_{z,J,\mathcal{H}}(u) := (u(z_1), \ldots, u(z_m)).
\]

Given a smooth submanifold \( X \subseteq \hat{M} \) define two moduli spaces

\[
\mathcal{M}_s(A, J; z, X) := \{ u \in \mathcal{M}_s(A, J) | ev_{z,J}(u) \in X \}
\]

and

\[
\mathcal{M}(A, J, \mathcal{H}; z, X) := \{ u \in \mathcal{M}(A, J, \mathcal{H}) | ev_{z,J,\mathcal{H}}(u) \in X \}.
\]

**Theorem 2.10.** (E.g. [12, Theorem 3.4.1].) There exist sets \( J_{\text{reg}}(A, z, X) \) (resp. \( \Omega^1_{c,\text{reg}}(S^2, C^\infty_c(M), A, z, X) \)) of the second category in \( J(M, \partial M, \omega) \) (resp. \( \Omega^1_c(S^2, C^\infty_c(M)) \)), such that for any \( J \in J_{\text{reg}}(A, z, X) \) (resp. \( (J, \mathcal{H}) \in J(M, \partial M, \omega) \times \Omega^1_{c,\text{reg}}(\Sigma, C^\infty_c(M), A, z, X) \)) the moduli space \( \mathcal{M}_s(A, J; z, X) \) (resp. \( \mathcal{M}(A, J, \mathcal{H}; z, X) \)) is a finite dimensional smooth manifold of dimension

\[
\dim \mathcal{M}_s(A, J; z, X) = \dim \mathcal{M}(A, J, \mathcal{H}; z, X) = 2n + 2I_c(A) - \text{codim} X.
\]

3. Genus zero Gromov-Witten invariants relative to the boundary.

3.1. Interior pseudocycles and oriented bordisms. Let \( X \) be a smooth compact \( n \)-dimensional manifold with boundary \( \partial X \) or more generally compact manifold with corners. We shall deal only with the simplest case of manifolds with corners, namely the product of compact manifolds with boundary. As usual, \( \hat{X} := X \setminus \partial X \) denotes the interior of \( X \).

3.1.1. Interior pseudocycles.

**Definition 3.1.** An arbitrary subset \( B \subseteq \hat{X} \) is said to be of dimension at most \( d \) if it is contained in the image of a map \( g : W \to X \), which is defined on a manifold \( W \) whose components have dimension less than or equal to \( d \). In this case we write \( \dim B \leq d \).

**Definition 3.2.** A \( d \)-dimensional interior pseudocycle in \( X \) is a smooth map \( f : V \to X \) defined on an oriented \( d \)-dimensional manifold \( V \), such that

- \( f(V) \) is compact in \( X \),
- \( f(V) \varsubsetneq \hat{X} \),
- \( \dim \Omega_f \leq d - 2 \), where \( \Omega_f := \bigcap_{K \in V} f(V \setminus K) \) and \( K \subseteq V \) means that \( K \) is a compact subset of \( V \). This set is called the (omega) limit set of \( f \).
Two $d$-dimensional interior pseudocycles $f_0 : V_0 \to X$ and $f_1 : V_1 \to X$ are called (interiorly) bordant if there is a $(d+1)$-dimensional oriented manifold $W$ with boundary $\partial W = V_1 \cup (-V_0)$ and a smooth map $F : W \to X$ such that

$$\overline{F(W)} \subset X, \quad F|_{V_0} = f_0, \quad F|_{V_1} = f_1, \quad \dim \Omega_F \leq d - 1.$$ 

We have the following elementary properties of pseudocycles.

**Proposition 3.3.** Let $f : V \to X$ be a $d$-dimensional interior pseudocycle.

(i) A point $x$ lies in $\Omega f$ if and only if $x$ is the limit point of a sequence $\{f(v_n)\}_{n \in \mathbb{N}}$, where $\{v_n\}_{n \in \mathbb{N}}$ has no convergent subsequence.

(ii) The limit set $\Omega f$ is always compact.

(iii) If $V$ is the interior of a compact manifold $\overline{V}$ with boundary $\partial \overline{V}$ and $f$ extends to a continuous map $\overline{f} : \overline{V} \to X$ then $\Omega f = \overline{f}(\partial \overline{V})$.

### 3.1.2. Index of intersection.

Suppose that $X$ is a smooth compact manifold with boundary $\partial X$. Let $e : U \to X^m$ be an interior pseudocycle in $X^m$, for some $m \in \mathbb{N}$. In addition, let $f_i : V_i \to X$ be a smooth singular manifold in $X$ for $i = 1, \ldots, p$ and let $f_j : (V_j, \partial V_j) \to (X, \partial X)$ be a smooth singular manifold in $(X, \partial X)$, for $j = p + 1, \ldots, m$, where $p \in \{0, \ldots, m\}$. Then

$$f := \prod_{i=1}^{m} f_i : \prod_{i=1}^{m} V_i \to X^m$$

is a smooth map of manifolds with corners, such that $f\left(\partial \left(\prod_{i=1}^{m} V_i\right)\right) \subseteq \partial (X^m)$. In this case we would like to define the index of intersection $e \cdot \overline{f}$ of these two "cycles". So we need some notion of transversality.

**Definition 3.4.** Let $e : U \to X^m$ and $f : \prod_{i=1}^{m} V_i \to X^m$ be two maps as above. We say that they are transverse if

1. $\Omega e \cap f\left(\prod_{i=1}^{m} V_i\right) = \emptyset$,
2. $e(u) = f(v) =: x \in X^m \Rightarrow T_x X^m = \text{Im } d_u e + \text{Im } d_v f$.

We shall write classically $e \cap f$.

If $e \cap f$ then the set

$$\Delta_{e,f} := \left\{(u, v) \in U \times \prod_{i=1}^{m} V_i | e(u) = f(v)\right\} \subseteq U \times \prod_{i=1}^{m} V_i$$

is a compact (by property 1 of Definition 3.4) manifold of dimension $\dim U + \sum_{i=1}^{m} \dim V_i - m \dim X$ (by property 2 of Definition 3.4). In particular, this set is finite if $\dim U + \sum_{i=1}^{m} \dim V_i = m \dim X$. 

Theorem 3.5. Let $e : U \to X^m$ and $f : \prod_{i=1}^m V_i \to X^m$ be two maps as above, such that $\dim U + \sum_{i=1}^m \dim V_i = m \dim X$.

(i) There exists a set $\text{Diff}_{reg}(e, f) \subseteq (\text{Diff}(X, \partial X))^m$ of the second category such that $e \pitchfork \varphi \circ f$ for every $\varphi \in \text{Diff}_{reg}(e, f)$. Here, $\text{Diff}(X, \partial X)$ denotes the group of diffeomorphisms $\varphi : X \to X$, such that $\varphi|_{\partial X} = \text{id}_{\partial X}$.

(ii) If $e \pitchfork f$ then the set $\Delta_{e, f}$ is finite. In the non-oriented case, where the base field $\mathbb{F}$ is $\mathbb{Z}_2$, we define the $\mathbb{Z}_2$-valued index of intersection $e \cdot_{\mathbb{F}} f$ of $e$ and $f$ as
$$e \cdot_{\mathbb{F}} f = \# \Delta_{e, f} \pmod{2}.$$ 

In the oriented case, where the base field $\mathbb{F}$ is $\mathbb{Q}$, we define the $\mathbb{Z}$-valued index of intersection $e \cdot_{\mathbb{F}} f$ of $e$ and $f$ as
$$e \cdot_{\mathbb{F}} f = \sum_{(u,v) \in \Delta_{e, f}} I(u,v),$$
where $I(u,v)$ is the classical local intersection number of $e(U)$ and $f(V)$ at $e(u) = f(v)$.

(iii) The index of intersection $e \cdot_{\mathbb{F}} f$ depends only on the bordism classes of $e$ and of $(V_i, f_i)$ for $i = 1, \ldots, m$.

Proof. We consider only the oriented case. The non-oriented one is proved in a similar way.

The proof of statements (i) – (ii) verbatim repeats that in [12, Lemma 6.5.5]. For the statement (iii), suppose first that $e$ is interiorly bordant to the empty set, and $E : W \to X^m$ is a corresponding bordism. By the standard argument on the general position in differential topology this bordism can be chosen to be transversal to $f$ in the sense of Definition 3.4. Condition 1 in Definition 3.4 follows from the inequality
$$\dim \Omega_E \leq \dim U - 1$$
that implies
$$\dim \Omega_E + \sum_{i=1}^m \dim V_i - m \dim X \leq -1.$$ 
Hence, the set
$$\Delta_{E,f} = \left\{ (w, v) \in W \times \prod_{i=1}^m V_i \mid E(w) = f(v) \right\} \subseteq W \times \prod_{i=1}^m V_i$$
is a compact oriented 1-dimensional manifold with boundary $\partial \Delta_{E,f} = \Delta_{e,f}$. Thus $e \cdot_{\mathbb{F}} f = 0$. Secondly, suppose that $[V_1, f_1] = 0$ with a corresponding bordism $F_1 : W_1 \to X$ and put
$$F := F_1 \times f_2 \times \cdots \times f_m : W_1 \times V_2 \times \cdots \times V_m \to X^m.$$
Then $F$ is a smooth map of manifolds with corners, such that
\[
F \left( \partial(W_1 \times \prod_{i=2}^m V_i) \right) \subseteq \partial(X^m),
\]
and, as above, we can choose this bordism to be transversal to $e$ in the sense of Definition 3.4. Condition 1 in Definition 3.4 follows from the inequality
\[
\dim \Omega_e + \dim W_1 + \sum_{i=2}^m \dim V_i - m \dim X \leq -1.
\]
Hence, the set
\[
\Delta_{e,F} = \left\{ (u,v) \in U \times \left( W_1 \times \prod_{i=2}^m V_i \right) \mid e(w) = F(v) \right\} \subseteq U \times W_1 \times \prod_{i=2}^m V_i
\]
is a compact oriented 1-dimensional manifold with boundary $\partial \Delta_{e,F} = \Delta_{e,f}$. Thus $e \cdot_F f = 0$. The same argument shows that $e \cdot_F f$ does not depend on $(V_i, f_i)$ for $i = 2, \ldots, m$. \hfill \Box

As a consequence, every interior pseudocycle $e : U \to X^m$ determines a well defined $m$-linear map
\[
(2) \quad \hat{\Psi}_{e,p} : \Omega_\ast(X; \mathbb{F})^\times p \times \Omega_\ast(X, \partial X; \mathbb{F})^\times (m-p) \to \mathbb{F}
\]
by
\[
\hat{\Psi}_{e,p}([V_1, f_1], \ldots, [V_m, f_m]) = e \cdot_F (f_1 \times \cdots \times f_m),
\]
where $\Omega_\ast(X; \mathbb{F}), \Omega_\ast(X, \partial X; \mathbb{F})$ are the classical Thom bordism groups – see [4] and [16]. Note that the homomorphism $\hat{\Psi}_{e,p}$ depends only on the bordism class of the interior pseudocycle $e$.

Recall that the evaluation homomorphism
\[
\mu^{p,m} : \Omega_\ast(X; \mathbb{F})^\times p \times \Omega_\ast(X, \partial X; \mathbb{F})^\times (m-p) \to H_\ast(X; \mathbb{F})^\times p \times H_\ast(X, \partial X; \mathbb{F})^\times (m-p)
\]
given by
\[
\mu^{p,m}([V_1, f_1], \ldots, [V_m, f_m]) = \left( ((f_i)_\ast([V_i]))^p_{i=1}, ((f_j)_\ast([V_j, \partial V_j]))^m_{j=p+1} \right)
\]
is an epimorphism. In particular, the homomorphism $\hat{\Psi}_{e,p}$ descends to a well defined homomorphism
\[
\Psi_{e,p} : H_\ast(X; \mathbb{F})^\times p \times H_\ast(X, \partial X; \mathbb{F})^\times (m-p) \to \mathbb{F},
\]
which also depends only on the bordism class of the interior pseudocycle $e$.

3.2. Gromov-Witten pseudocycle. Let us fix a finite number of pairwise distinct marked points $z := (z_1, \ldots, z_m) \in (\mathbb{S}^2)^m$. In the following constructions we shall consider mainly the case of $m \in \{3, 4\}$.
3.2.1. **Case I - non-perturbed.** Let \( A \in H_2^S(M) \) and \( J \in \mathcal{J}(M, \partial M, \omega) \). We shall compactify the image

\[
\text{ev}_{z,J} \left( \mathcal{M}_s(A, J) \right) \subseteq \left( \overset{\circ}{M} \right)^m
\]

by adding all possible limits of points \( \text{ev}_{z,J}(u_n) \), where \((u_n)_{n \in \mathbb{N}}\) is any sequence in \( \mathcal{M}_s(A, J) \). By the Gromov compactness theorem the boundary of \( \text{ev}_{z,J} \left( \mathcal{M}_s(A, J) \right) \) will be formed by the \( \text{ev}_{z,J} \)-images of different cusp (reducible) curves. By the convexity it follows that the compactified space \( \text{ev}_{z,J} \left( \mathcal{M}_s(A, J) \right) \) lies in \( \left( \overset{\circ}{M} \right)^m \). Thus we can state the following theorem, which are completely analogous to the closed case. Recall that a class \( B \in H_2^S(M) \) is called \( J \)-effective if it can be represented by a \( J \)-holomorphic sphere. Assume that \( A \in H_2^S(M) \) satisfies the following conditions.

\( (A_3) \) In the case \( m = 3 \), i.e. the case of three marked points, we require that every \( J \)-effective class \( B \in H_2^S(M) \) has Chern number \( c_1(B) \geq 0 \). Moreover, the class \( A \) is not a multiple \( A = kB \) of a \( J \)-effective class \( B \in H_2^S(M) \) with \( k > 1 \) and \( c_1(B) = 0 \).

\( (A_4) \) In the case \( m = 4 \), i.e. the case of four marked points, we require that every \( J \)-effective class \( B \in H_2^S(M) \) has Chern number \( c_1(B) \geq 1 \). Moreover, the class \( A \) is not a multiple \( A = kB \) of a \( J \)-effective class \( B \in H_2^S(M) \) with \( k > 1 \) and \( c_1(B) = 1 \).

**Theorem 3.6.** (E.g. [11, Theorem 5.4.1].)

Let \((M, \omega)\) be a \( 2n \)-dimensional compact convex semi-positive symplectic manifold. Fix a class \( A \in H_2^S(M) \setminus \{0\} \) and fix pairwise distinct marked points

\[
z := (z_1, \ldots, z_m) \in (\mathbb{S}^2)^m,
\]

where \( m \in \{3,4\} \). Then there exists a set

\[
\mathcal{J}_{\text{reg}}(M, \partial M, \omega, A, z) \subseteq \mathcal{J}(M, \partial M, \omega)
\]

of the second category, such that

\( (i) \) \( \mathcal{J}_{\text{reg}}(M, \partial M, \omega, A, z) \subseteq \mathcal{J}_{\text{reg}}(A) \), see the notation of Theorem 2.8

\( (ii) \) if \( J \in \mathcal{J}_{\text{reg}}(M, \partial M, \omega, A, z) \) and the class \( A \) satisfies \((A_m)\), then

\[
\text{ev}_{z,J} : \mathcal{M}_s(A, J) \to M^m
\]

is an interior (oriented) pseudocycle of dimension \( 2n + 2c_1(A) \). Its bordism class is independent of \( J \) and \( z \).

3.2.2. **Case II - perturbed.** We should treat the case, where \( A \in H_2^S(M) \) does not satisfy conditions \((A_3), (A_4)\). There are several ways of dealing with this problem. One can use the argument by Y. Ruan in [14, Section 3], where he considers the perturbed Cauchy-Riemann equation \( \overline{\partial}_{J, \mathcal{H}}(u) = 0 \) that already has no multiply covered solutions. See also [11, Section 9.1] or [12, Chapter 8]. Like in Case I, we compactify the image \( \text{ev}_{z,J, \mathcal{H}}(\mathcal{M}(A, J, \mathcal{H})) \) by adding all possible limits of
points \( ev_{z,J,H}(u_n) \), where \((u_n)_{n \in \mathbb{N}} \) is any sequence in \( \mathcal{M}(A, J, H) \). It follows from the convexity that the compactified space \( ev_{z,J,H}(\mathcal{M}(A, J, H)) \) lies in \( \hat{M}^m \).

**Theorem 3.7.** (E.g. [14] Section 3] or [12] Items 8.5.1 – 8.5.4.)

Let \((M, \omega)\) be a \(2n\)-dimensional compact convex semi-positive symplectic manifold. Fix a class \(A \in H_2^S(M) \setminus \{0\}\) and fix pairwise distinct marked points \(z := (z_1, \ldots, z_m) \in (S^2)^m\), where \(m \in \{3, 4\}\). Then there exists a set

\[
\mathcal{J} \mathcal{H}_{reg}(M, \partial M, \omega, A, z) \subseteq \mathcal{J}(M, \partial M, \omega) \times \Omega^1_c(S^2, C^\infty_c(M))
\]

of the second category such that

(i) for every \((J, H) \in \mathcal{J} \mathcal{H}_{reg}(M, \partial M, \omega, A, z)\), we have that \(H \in \Omega^1_{c, reg}(S^2, C^\infty_c(M))\), see the notation of Theorem 2.7,

(ii) if \((J, H) \in \mathcal{J} \mathcal{H}_{reg}(M, \partial M, \omega, A, z)\)

\[
eq \text{ev}_{z,J,H} : \mathcal{M}(A, J, H) \to M^m
\]

is an interior (oriented) pseudocycle of dimension \(2n + 2c_1(A)\). Its bordism class is independent of \((J, H, z)\).

3.3. **Invariants** \(GW_{A,p,m}\). Following [14], [11], [12] we shall define two types of genus zero Gromov-Witten invariants. But first, let us recall the definition of intersection products from algebraic topology. Recall that \(\mathbb{F}\) is either \(\mathbb{Z}_2\) or \(\mathbb{Q}\).

**Definition 3.8.** **Homomorphisms**

\[
\begin{align*}
\bullet_1 : H_i(M; \mathbb{F}) \otimes H_j(M; \mathbb{F}) & \to H_{i+j-2n}(M; \mathbb{F}) \\
\bullet_2 : H_i(M; \mathbb{F}) \otimes H_j(M, \partial M; \mathbb{F}) & \to H_{i+j-2n}(M; \mathbb{F}) \\
\bullet_3 : H_i(M, \partial M; \mathbb{F}) \otimes H_j(M, \partial M; \mathbb{F}) & \to H_{i+j-2n}(M, \partial M; \mathbb{F})
\end{align*}
\]

given by

\[
\begin{align*}
a \bullet_1 b & := \text{PLD}_2(\text{PLD}_2^{-1}(b) \cup \text{PLD}_2^{-1}(a)) \\
a \bullet_2 b & := \text{PLD}_2(\text{PLD}_2^{-1}(b) \cup \text{PLD}_1^{-1}(a)) \\
a \bullet_3 b & := \text{PLD}_1(\text{PLD}_1^{-1}(b) \cup \text{PLD}_1^{-1}(a))
\end{align*}
\]

are called the intersection products in homology. Here

\[
\begin{align*}
\text{PLD}_1 : H^j(M; \mathbb{F}) & \to H_{2n-j}(M, \partial M; \mathbb{F}) \\
\text{PLD}_2 : H^j(M, \partial M; \mathbb{F}) & \to H_{2n-j}(M; \mathbb{F})
\end{align*}
\]

are the Poincaré-Lefschetz duality isomorphisms given by

\[
\text{PLD}_i(\alpha) := \alpha \cap [M, \partial M], \quad i = 1, 2,
\]

where \([M, \partial M]\) is the positive generator of \(H_{2n}(M, \partial M; \mathbb{F}) \cong \mathbb{F}\) – the relative fundamental class.
Theorem-Definition 3.9. (E.g. [14; 15; 11] Chapter 7; [12 Chapter 7].)

Let $(M, \omega)$ be a $2n$-dimensional compact convex semi-positive symplectic manifold, $m \in \{3, 4\}$ and $p \in \{0, 1, \ldots, m\}$. Fix pairwise distinct marked points

$$z := (z_1, \ldots, z_m) \in (\mathbb{S}^2)^m.$$

(i) Let $A \in H^2_{\text{rel}}(M) \setminus \{0\}$ and $(J, \mathcal{H}) \in J\mathcal{H}_{\text{reg}}(M, \partial M, \omega, A, z)$. Then the $m$-linear map

$$GW_{A,p,m} : H_*(M; \mathbb{F})^p \times H_*(M, \partial M; \mathbb{F})^{(m-p)} \to \mathbb{F}$$

given by

$$GW_{A,p,m}(a_1, \ldots, a_m) := \Psi^{\text{ev}}_{z, J, \mathcal{H}, p}(a_1, \ldots, a_m)$$

is well-defined and is independent of the pair $(J, \mathcal{H}) \in J\mathcal{H}_{\text{reg}}(M, \partial M, \omega, A, z)$ and of the tuple $z$ of fixed marked points.

(ii) If $A \in H^2_{\text{rel}}(M) \setminus \{0\}$ satisfies $(A_m)$ (see Section 3.2.1) then a pair $(J, \mathcal{H} = 0)$ with $J \in J_{\text{reg}}(M, \partial M, \omega, A, z)$ belongs to $J\mathcal{H}_{\text{reg}}(M, \partial M, \omega, A, z)$.

(iii) The Gromov-Witten invariant $GW_{A,p,m}$ depends only on the semi-positive deformation class of $\omega$.

Proof. Statements (i) and (ii) follow directly from Theorems 3.6, 3.7. The proof of (iii) repeats verbatim the proof of [15, Proposition 2.3]. □

Since for $A = 0$ the above evaluation maps are not interior pseudocycles, we define the Gromov-Witten invariants for these classes as follows.

Definition 3.10. Let $(M, \omega)$ be a $2n$-dimensional compact convex semi-positive symplectic manifold. Let $m \in \{3, 4\}$ and $p \in \{0, 1, \ldots, m\}$. If

$$\sum_{i=1}^{m} \deg(a_i) \neq 2n(m - 1),$$

define

$$GW_{0,p,m}(a_1, \ldots, a_m) := 0.$$ 

Otherwise, define

$$(7) \quad GW_{0,p,m}(a_1, \ldots, a_m) := \begin{cases} 
0 & \text{if } p = 0, \\
(\ldots((a_1 \ast_2 a_2) \ast_2 a_3)\ldots) \ast_2 a_m & \text{if } p = 1, \\
(\ldots((a_1 \ast_1 a_2) \ast_2 a_3)\ldots) \ast_2 a_m & \text{if } p = 2, \\
(\ldots((a_1 \ast_1 a_2) \ast_1 a_3)\ldots) \ast_2 a_m & \text{if } p = 3, \\
a_1 \ast_1 a_2 \ast_1 a_3 \ast_1 a_4 & \text{if } p = 4.
\end{cases}$$

We see that this definition correlates with the closed case and reflects the behavior of the relative intersection products.
The next properties follow immediately from the definition of the Gromov-Witten invariants, see \cite{12, Chapter 7}; \cite{14}; \cite{15}.

**Proposition 3.11.** Let \((M, \omega)\) be a 2n-dimensional compact convex semi-positive symplectic manifold.

(i) Let \(m \in \{3, 4\}\) and \(p \in \{0, 1, \ldots, m\}\). Following \cite{12} Section 7.5 for each permutation \(\sigma \in S_m\) and each monomial

\[a := (a_1, \ldots, a_m) \in H_*(M; \mathbb{F})^{\times p} \times H_*(M, \partial M; \mathbb{F})^{\times (m-p)}\]

denote by

\[\varepsilon := \varepsilon(\sigma; a) := (-1)^{\#\{i < j | \sigma(i) > \sigma(j), \deg(a_i) \cdot \deg(a_j) \in 2\mathbb{Z}+1\}}\]

the sign of the induced permutation on the classes of odd degree. Then

\[GW_{A,p,m}(\sigma_*a) = \varepsilon GW_{A,p,m}(a),\]

where \(\sigma_*a := a_{\sigma(1)}, \ldots, a_{\sigma(m)}\).

(ii) If \(m = 3, p \in \{0, 1, 2\}\) and \(A \in H^S_2(M) \setminus \{0\}\), then

\[GW_{A,p,3}(a_1, a_2, [M, \partial M]) = 0.\]

(iii) Let \(j_* : H_*(M; \mathbb{F}) \to H_*(M, \partial M; \mathbb{F})\) be the natural homomorphism, induced by the inclusion \((M, \emptyset) \hookrightarrow (M, \partial M)\). Let \(m \in \{3, 4\}, p \in \{1, \ldots, m\}\) and \(A \in H^S_2(M) \setminus \{0\}\). Then

\[GW_{A,p,m}(a_1, \ldots, a_m) = 0,\]

for any classes \(\{a_i\}_{i=1}^m\), such that \(j_*(a_1) = 0\).

3.4. **Calculation of Gromov-Witten invariants.** In the following examples we shall calculate the Gromov-Witten invariants over \(\mathbb{F} = \mathbb{Q}\) of the symplectic blow-up \((\widetilde{X} \cong X \# \mathbb{CP}^2, \sigma_\delta)\) of an interior point in some compact convex semi-positive symplectic manifolds \((X, \sigma)\), i.e. the symplectic blow up of size \(\delta\) that corresponds to a symplectic embedding

\[\psi : (\mathbb{B}^4(\delta) \subseteq \mathbb{C}^2, \omega_0) \to (X, \sigma),\]

see \cite{10} Section 7.1 or \cite{12} Section 9.3 for the precise definition of the symplectic blow-up \((\widetilde{X}, \sigma_\delta)\).

**Example 1.**

Let \((\mathbb{B}^4, \omega_\delta)\) be the symplectic blow-up at zero of the standard unit 4-ball in \((\mathbb{C}^2, \omega_0)\) of a small size \(0 < \delta << 1\). It is a 4-dimensional compact convex symplectic manifold equipped with an \(\omega\)-compatible almost complex structure \(\tilde{J}_0\), where \(\tilde{J}_0\) is the standard complex structure (multiplication by \(\sqrt{-1}\)) on \(\mathbb{C}^2\). Let \(E \in H_2(\mathbb{B}^4; \mathbb{C})\) be the class of the exceptional divisor. It can be realized by a \(\tilde{J}_0\)-holomorphic embedding \(\iota : \mathbb{CP}^1 \to \mathbb{B}^4\), i.e. \(E = [\iota(\mathbb{CP}^1)] = \iota_*[\mathbb{CP}^1] = \omega_\delta(E) = \delta\). Set
$C := \iota(\mathbb{CP}^1)$. A standard calculation shows that $c_1(E) = 1$. In particular, the minimal Chern number $N$ of $\mathbb{B}^4$ (see Section 1.2) equals 1 and the class $E$ satisfies condition $(A_3)$, (see Section 3.2.1). Note that

$$H_i(\mathbb{B}^4; \mathbb{Q}) \cong \begin{cases} \text{Span}_\mathbb{Q}([pt]) \cong \mathbb{Q} & , i = 0, \\ 0 & , i = 1, \\ \text{Span}_\mathbb{Q}(E) \cong \mathbb{Q} & , i = 2, \\ 0 & , i = 3, \\ 0 & , i = 4. \end{cases}$$

(11)

$$H_i(\mathbb{B}^4, \partial \mathbb{B}^4; \mathbb{Q}) \cong \begin{cases} 0 & , i = 0, \\ 0 & , i = 1, \\ \text{Span}_\mathbb{Q}(E^\vee) \cong \mathbb{Q} & , i = 2, \\ 0 & , i = 3, \\ \text{Span}_\mathbb{Q}(\mathbb{B}^4, \partial \mathbb{B}^4) \cong \mathbb{Q} & , i = 4. \end{cases}$$

(12)

By [12, Section 3.3 and Lemma 7.1.8], we can use the almost complex structure $\tilde{J}_0$ to calculate the invariants $GW_{E,p,3}$. By the non-negativity of intersection in 4-dimensional almost complex manifolds, cf. [12, Theorem 2.6.3], the moduli space $\mathcal{M}(E, \tilde{J}_0)$ consists of a single $\tilde{J}_0$-holomorphic curve up to a reparametrization, namely the exceptional sphere itself. Take $\{[pt], E\}$ and $\{E^\vee, [\mathbb{B}^4, \partial \mathbb{B}^4]\}$ to be homogeneous bases for $H_*(\mathbb{B}^4; \mathbb{C})$ and $H_*(\mathbb{B}^4, \partial \mathbb{B}^4; \mathbb{C})$ respectively. Then

$$j_*([pt]) = 0, E \cdot_1 E = -1, E \cdot_2 E^\vee = 1,$$

where $j_* : H_*(\mathbb{B}^4; \mathbb{Q}) \to H_*(\mathbb{B}^4, \partial \mathbb{B}^4; \mathbb{Q})$ is the natural homomorphism induced by the inclusion. Thus, the only non-zero Gromov-Witten invariants for the class $E$ and the basic homology classes are

$$GW_{0,1,3}(E, E^\vee, [\mathbb{B}^4, \partial \mathbb{B}^4]) = (E \cdot_2 E^\vee) \cdot_2 [\mathbb{B}^4, \partial \mathbb{B}^4] = 1,$$

$$GW_{0,2,3}(E, E, [\mathbb{B}^4, \partial \mathbb{B}^4]) = (E \cdot_1 E) \cdot_2 [\mathbb{B}^4, \partial \mathbb{B}^4] = -1,$$

$$GW_{E,0,3}(E^\vee, E^\vee, E^\vee) = (E \cdot_2 E^\vee)^3 = 1,$$

$$GW_{E,1,3}(E, E^\vee, E^\vee) = (E \cdot_2 E^\vee)^2(E \cdot_1 E) = -1,$$

$$GW_{E,2,3}(E, E, E^\vee) = (E \cdot_2 E^\vee)(E \cdot_1 E)^2 = 1,$$

$$GW_{E,3,3}(E, E, E) = (E \cdot_1 E)^3 = -1.$$
Example 2.
Let $\Sigma$ be a smooth closed surface of genus $g \geq 0$ and $G$ a fixed Riemannian metric on $\Sigma$. Consider the cotangent unit disk bundle $X := \mathbb{D}T^*\Sigma$ w.r.t. $G$ equipped with the canonical symplectic form $\omega_{can} := d\lambda_{can}$. The metric $G$ induces a horizontal-vertical splitting of $TT^*\Sigma \cong T\Sigma \oplus T^*\Sigma$ and as a result, it induces an $\omega_{can}$-compatible almost complex structure $J_G$ on $X$, which in the horizontal-vertical splitting takes the form

$$J_G = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let $(\tilde{X}, \omega_\delta)$ be the symplectic blow-up of $X$ of a sufficiently small size $0 < \delta << 1$, so that the corresponding symplectic $\delta$-ball lies in $\tilde{X}$ and doesn’t intersect the zero section $\Sigma$ of $X$. Then $(\tilde{X}, \omega_\delta)$ is a 4-dimensional compact convex symplectic manifold equipped with an $\omega_\delta$-compatible almost complex structure $\tilde{J}_G$. Let $E \in H_2(\tilde{X}; \mathbb{Q})$ be the class of the exceptional divisor. It is realized by a $\tilde{J}_G$-holomorphic embedding $\iota : \mathbb{CP}^1 \to \tilde{X}$ and $\omega_\delta(E) = \delta$. The minimal Chern number $N$ of $\tilde{X}$ equals 1, and the class $E$ satisfies condition $(A_3)$. We also have

$$H_i(\tilde{X}; \mathbb{Q}) \cong \begin{cases} \text{Span}_\mathbb{Q}(\{pt\}) \cong \mathbb{Q} , & i = 0, \\ \text{Span}_\mathbb{Q}(\{[a_1], \ldots, [a_g], [b_1], \ldots, [b_g]\}) \cong \mathbb{Q}^{2g} , & g = 0, i = 1, \\ \text{Span}_\mathbb{Q}(\{[a_1], \ldots, [a_g], [b_1], \ldots, [b_g]\}) \cong \mathbb{Q}^{2g} , & g \geq 1, i = 1, \\ 0 , & i = 2, \\ 0 , & i = 3, \\ \text{Span}_\mathbb{Q}(\{\tilde{X}, \partial \tilde{X}\}) \cong \mathbb{Q} , & i = 4. \end{cases}$$

Denote by $[F] \in H_2(X, \partial X; \mathbb{Q})$ the class of the fiber $F \cong \mathbb{D}^2$ of the fibration $X = \mathbb{D}T^*\Sigma \to \Sigma$. Hence, $[\Sigma]^\vee = [F]$, $[a_i]^\vee = [X|_{a_i}]$, $[b_i]^\vee = -[X|_{b_i}]$ for $i = 1, \ldots, g$. It follows that

$$H_i(\tilde{X}, \partial \tilde{X}; \mathbb{Q}) \cong \begin{cases} 0 , & i = 0, \\ 0 , & i = 1, \\ \text{Span}_\mathbb{Q}(\{[F]\} \oplus \text{Span}_\mathbb{Q}(E^\vee) \cong \mathbb{Q}^2 , & i = 2, \\ \text{Span}_\mathbb{Q}(\{[a_1]^\vee, \ldots, [a_g]^\vee, [b_1]^\vee, \ldots, [b_g]^\vee\}) \cong \mathbb{Q}^{2g} , & g = 0, i = 3, \\ \text{Span}_\mathbb{Q}(\{[\tilde{X}, \partial \tilde{X}]\}) \cong \mathbb{Q} , & i = 3, \\ \text{Span}_\mathbb{Q}(\{[a_1]^\vee, \ldots, [a_g]^\vee, [b_1]^\vee, \ldots, [b_g]^\vee\}) \cong \mathbb{Q}^{2g} , & g \geq 1, i = 4. \end{cases}$$
In addition, we have

\[ j_\ast([pt]) = 0, \]

\[ E \cdot_1 E = -1, \quad [\Sigma] \cdot_1 [\Sigma] = 2 - 2g, \quad E \cdot_2 E' = 1, \]

\[ \begin{align*}
[\Sigma] \cdot_1 E &= [F] \cdot_2 E = [a_i] \cdot_2 E = [b_i] \cdot_2 E = 0, \\
[a_i] \cdot_1 [b_j] &= [a_i] \cdot_1 [a_j] = [b_i] \cdot_1 [b_j] = [a_i] \cdot_2 [b_j]' = [b_i] \cdot_2 [a_j]' = 0, \\
[a_i] \cdot_2 [a_j]' &= [b_i] \cdot_2 [b_j]' = \delta_{ij}.
\end{align*} \]

By \cite{12} Section 3.3 and Lemma 7.1.8, we can use the almost complex structure \( \widetilde{J}_G \) to calculate the invariants \( GW_{E,p,3} \). By the non-negativity of intersection in 4-dimensional almost complex manifolds, cf. \cite{12} Theorem 2.6.3, we have that the moduli space \( \mathcal{M}(E, \widetilde{J}_G) \) consists of a single \( \widetilde{J}_G \)-holomorphic curve up to a reparametrization, namely, the exceptional sphere itself. By \cite{12} Theorem 2.6.4 and \cite{8} Proposition 2.3, we have that for all \( d \in \mathbb{Z} \setminus \{0, 1\} \) the moduli space \( \mathcal{M}_s(dE, J) \) is empty for any regular \( J \in \mathcal{J}_{reg}(dE) \). As a result, the only non-zero invariants on the basic classes are

\[ \begin{align*}
GW_{0,1,3}(a_i, [a_j]'', [\tilde{X}, \partial \tilde{X}]) &= [a_i] \cdot_2 [a_j]' = \delta_{ij}, \\
GW_{0,1,3}(b_i, [b_j]'', [\tilde{X}, \partial \tilde{X}]) &= [b_i] \cdot_2 [b_j]' = \delta_{ij}, \\
GW_{0,1,3}([\Sigma], [a_i]'', [b_j]'') &= ([\Sigma] \cdot_2 [a_i]'') \cdot_2 [b_j]' = [b_i] \cdot_2 [b_j]' = \delta_{ij}, \\
GW_{0,2,3}([\Sigma], [\Sigma], [\tilde{X}, \partial \tilde{X}]) &= [\Sigma] \cdot_1 [\Sigma] = 2 - 2g, \\
GW_{0,2,3}(E, E, [\tilde{X}, \partial \tilde{X}]) &= E \cdot_1 E = -1, \\
GW_{E,0,3}(E'', E'', E') &= (E \cdot_2 E')^3 = 1, \\
GW_{E,1,3}(E, E'', E') &= (E \cdot_2 E')^2 (E \cdot_1 E) = -1, \\
GW_{E,2,3}(E, E, E') &= (E \cdot_2 E') (E \cdot_1 E)^2 = 1, \\
GW_{E,3,3}(E, E, E) &= (E \cdot_1 E)^3 = -1.
\end{align*} \]

4. Quantum homology

Let \((M, \omega)\) be a 2n-dimensional compact convex semi-positive symplectic manifold. Following \cite{12} Chapter 11 we use the Gromov-Witten invariants \( GW_{A,p,m} \) to deform the classical intersection products \( \cdot_i, i = 1, 2, 3 \); cf. Definition 3.8. These deformations give rise to new ring structures on the absolute (that is \( H_\ast(M) \)) and the relative (that is \( H_\ast(M, \partial M) \)) homology algebras of \((M, \omega)\). From now on we shall consider the following Novikov ring \( \Lambda \). Let

\[ \Gamma := \Gamma(M, \omega) := \frac{H_2^S(M)}{\ker(c_1) \cap \ker(\omega)} \]
and let
\[ G := G(M, \omega) := \frac{1}{2} \omega \left( H^S_2(M) \right) \subseteq \mathbb{R} \]
be the subgroup of half-periods of the symplectic form \( \omega \) on spherical homology classes. Recall that \( \mathbb{F} \) denotes a base field, which in our case will be either \( \mathbb{Q} \) or \( \mathbb{Z}_2 \). Let \( s \) be a formal variable. Define the field \( \mathbb{K}_G \) of generalized Laurent series in \( s \) over \( \mathbb{F} \) of the form
\[ \mathbb{K}_G := \left\{ f(s) = \sum_{\alpha \in G} z_\alpha s^\alpha, z_\alpha \in \mathbb{F} \mid \#\{ \alpha > c | z_\alpha \neq 0 \} < \infty, \forall c \in \mathbb{R} \right\} \]

**Definition 4.1.** Let \( q \) be a formal variable. The *Novikov ring* \( \Lambda := \Lambda_G \) is the ring of polynomials in \( q, q^{-1} \) with coefficients in the field \( \mathbb{K}_G \), i.e.
\[ \Lambda := \Lambda_G := \mathbb{K}_G[q, q^{-1}] \]

We equip the ring \( \Lambda_G \) with the structure of a graded ring by setting \( \deg(s) = 0 \) and \( \deg(q) = 1 \). We shall denote by \( \Lambda_k \) the set of elements of \( \Lambda \) of degree \( k \). Note that \( \Lambda_0 = \mathbb{K}_G \).

**4.1. Quantum homology.** The quantum homology \( QH_*(M) \) and the relative quantum homology \( QH_*(M, \partial M) \) are defined as follows. As modules, they are graded modules over \( \Lambda \) defined by
\[ QH_*(M) := H_*(M; \mathbb{F}) \otimes_{\mathbb{F}} \Lambda, \quad QH_*(M, \partial M) := H_*(M, \partial M; \mathbb{F}) \otimes_{\mathbb{F}} \Lambda. \]

A grading on both modules is given by
\[ \deg(a \otimes zs^\alpha q^m) = \deg(a) + m, \]
and for \( k \in \mathbb{Z} \) the degree-\( k \) parts of \( QH_*(M) \) and of \( QH_*(M, \partial M) \) are given by
\[ QH_k(M) := \bigoplus_i H_i(M; \mathbb{F}) \otimes_{\mathbb{F}} \Lambda_{k-i}, \]
\[ QH_k(M, \partial M) := \bigoplus_i H_i(M, \partial M; \mathbb{F}) \otimes_{\mathbb{F}} \Lambda_{k-i} \]
respectively.

Next, we define the quantum products \( \ast_i, i = 1, 2, 3 \), which are the deformations of the classical intersection products \( \bullet_i, i = 1, 2, 3 \). Choose a homogeneous basis \( \{e_k\}_{k=1}^d \) of \( H_*(M; \mathbb{F}) \), such that \( e_1 = [pt] \in H_0(M; \mathbb{F}) \). Let \( \{e^\vee_k\}_{k=1}^d \) be the dual homogeneous basis of \( H_*(M, \partial M; \mathbb{F}) \) defined by
\[ \langle e_i, e^\vee_j \rangle = \delta_{ij}, \]
where \( \langle \cdot, \cdot \rangle \) is the Kronecker pairing.

**Definition 4.2.** Let \( A \in H^S_2(M) \) and let \([A] \in \Gamma \) be the image of \( A \) in \( \Gamma \).
A bilinear homomorphism of $\Lambda$-modules

\[ *_1 : \text{QH}_\ast(M) \times \text{QH}_\ast(M) \to \text{QH}_\ast(M) \]

is given as follows. If $a \in H_\ast(M; \mathbb{F})$ and $b \in H_\ast(M; \mathbb{F})$ then

\[ \text{QH}_{i+j-2n}(M) \ni a *_1 b : = \sum_{[A] \in \Gamma} (a *_1 b)_{[A]} \otimes s^{-\omega(A)} q^{-2c_1(A)}, \]

where $(a *_1 b)_{[A]} \in H_{i+j-2n+2c_1(A)}(M; \mathbb{F})$ is given by

\[ (a *_1 b)_{[A]} = \sum_{i=1}^d \sum_{A' \in [A]} \text{GW}_{A', 2, 3}(a, b, e^\vee_i) e_i. \]

We extend this definition by $\Lambda$-linearity to the whole $\text{QH}_\ast(M) \times \text{QH}_\ast(M)$.

(ii) A bilinear homomorphism of $\Lambda$-modules

\[ *_2 : \text{QH}_\ast(M) \times \text{QH}_\ast(M, \partial M) \to \text{QH}_\ast(M) \]

is given as follows. If $a \in H_\ast(M; \mathbb{F})$ and $b \in H_\ast(M, \partial M; \mathbb{F})$ then

\[ \text{QH}_{i+j-2n}(M) \ni a *_2 b : = \sum_{[A] \in \Gamma} (a *_2 b)_{[A]} \otimes s^{-\omega(A)} q^{-2c_1(A)}, \]

where $(a *_2 b)_{[A]} \in H_{i+j-2n+2c_1(A)}(M; \mathbb{F})$ is given by

\[ (a *_2 b)_{[A]} = \sum_{i=1}^d \sum_{A' \in [A]} \text{GW}_{A', 1, 3}(a, b, e^\vee_i) e_i. \]

We extend this definition by $\Lambda$-linearity to the whole $\text{QH}_\ast(M) \times \text{QH}_\ast(M, \partial M)$.

(iii) A bilinear homomorphism of $\Lambda$-modules

\[ *_3 : \text{QH}_\ast(M, \partial M) \times \text{QH}_\ast(M, \partial M) \to \text{QH}_\ast(M, \partial M) \]

is given as follows. If $a \in H_\ast(M, \partial M; \mathbb{F})$ and $b \in H_\ast(M, \partial M; \mathbb{F})$ then

\[ \text{QH}_{i+j-2n}(M, \partial M) \ni a *_3 b : = \sum_{[A] \in \Gamma} (a *_3 b)_{[A]} \otimes s^{-\omega(A)} q^{-2c_1(A)}, \]

where $(a *_3 b)_{[A]} \in H_{i+j-2n+2c_1(A)}(M, \partial M; \mathbb{F})$ is given by

\[ (a *_3 b)_{[A]} = \sum_{i=1}^d \sum_{A' \in [A]} \text{GW}_{A', 1, 3}(a, b, e_i) e^\vee_i. \]

We extend this definition by $\Lambda$-linearity to the whole $\text{QH}_\ast(M, \partial M) \times \text{QH}_\ast(M, \partial M)$.

Remark 4.3. By the Gromov compactness theorem, the sums $\sum_{A' \in [A]} \text{GW}_{A', 2, 3}(a, b, e^\vee_i)$, $\sum_{A' \in [A]} \text{GW}_{A', 1, 3}(a, b, e_i)$, $\sum_{A' \in [A]} \text{GW}_{A', 1, 3}(a, b, e^\vee_i)$, $\sum_{A' \in [A]} \text{GW}_{A', 1, 3}(a, b, e_i)$ are finite, see [12, Corollary 5.3.2].
The next theorem summarizes the main properties of these quantum products.

**Theorem 4.4.** (E.g. [12 Proposition 11.1.9].)  
(i) The quantum products $\ast_l, l = 1, 2, 3$, are distributive over addition and supercommutative in the sense that  
\[(a \otimes 1) \ast_l (b \otimes 1) = (-1)^{\deg(a) \deg(b)} (b \otimes 1) \ast_l (a \otimes 1)\]  
for $l = 1, 2, 3$ and for elements $a, b \in H_*(M; \mathbb{F}) \cup H_*(M, \partial M; \mathbb{F})$ of pure degree.  
(ii) The quantum products $\ast_l, l = 1, 2, 3$, commute with the action of $\Lambda$, i.e. for any $\lambda \in \Lambda$ and for any $a, b \in QH_*(M) \cup QH_*(M, \partial M)$  
\[\lambda(a \ast b) = (\lambda a) \ast_l b = a \ast_l (\lambda b) = (a \ast_l b)\lambda.\]  
(iii) The quantum products $\ast_l, l = 1, 2, 3$, are associative in the sense that  
\[(a \ast_l b) \ast_l c = a \ast_l (b \ast_l c), \forall a, b, c \in QH_*(M),\]  
\[a \ast_l b \ast_l c = a \ast_l (b \ast_l c), \forall a \in QH_*(M), \forall b, c \in QH_*(M, \partial M),\]  
\[(a \ast_l b) \ast_l c = a \ast_l (b \ast_l c), \forall b, c \in QH_*(M, \partial M),\]  
\[(a \ast_l b) \ast_l c = a \ast_l (b \ast_l c), \forall a, b, c \in QH_*(M, \partial M).\]  
(iv) The zero class term in the quantum products $\ast_l, l = 1, 2, 3$, is the classical intersection products $\bullet_l, l = 1, 2, 3$, i.e.  
\[(a \ast_l b)_{\Lambda=0} = a \bullet_l b \otimes 1, \forall a, b \in QH_*(M) \cup QH_*(M, \partial M).\]  
(v) The relative fundamental class $[M, \partial M] \otimes 1$ is the unit for $\ast_l, l = 2, 3$, i.e.  
\[a \ast_2 [M, \partial M] \otimes 1 = a, \forall a \in QH_*(M),\]  
\[a \ast_3 [M, \partial M] \otimes 1 = a, \forall a \in QH_*(M, \partial M).\]  

**Corollary 4.5.**  
(i) The triple $(QH_*(M), +, \ast_1)$ has the structure of a non-unital associative $\Lambda$-algebra.  
(ii) The triple $(QH_*(M, \partial M), +, \ast_3)$ has the structure of a unital associative $\Lambda$-algebra, where $[M, \partial M]$ is the multiplicative unit.  
(iii) The quantum product $\ast_2$ defines on $(QH_*(M), +, \ast_1)$ the structure of an associative algebra over the algebra $(QH_*(M, \partial M), +, \ast_3)$.  

Like in the closed case, we have different natural pairings. The $\mathbb{K}_G$-valued pairings are given by  
\[\Delta_1 : QH_k(M) \times QH_{2n-k}(M) \to \Lambda_0 = \mathbb{K}_G,\]  
\[\Delta_2 : QH_k(M) \times QH_{2n-k}(M, \partial M) \to \Lambda_0 = \mathbb{K}_G,\]  
\[\Delta_l (a, b) := \iota(a \ast_l b), \text{ for } l = 1, 2,\]  
where the map  
\[\iota : QH_0(M) = \bigoplus_{i} H_i(M; \mathbb{F}) \otimes_{\mathbb{F}} \Lambda_{-i} \to \mathbb{K}_G\]
sends \([pt] \otimes f_0(s) + \sum_{m=1}^{2n} a_m \otimes f_m(s)q^{-m}\) to \(f_0(s)\). The \(\mathbb{F}\)-valued pairings are given by

\[
\begin{align*}
\Pi_1 & : QH_k(M) \times QH_{2n-k}(M) \to \mathbb{F}, \\
\Pi_2 & : QH_k(M) \times QH_{2n-k}(M, \partial M) \to \mathbb{F}, \\
\Pi_l & = j \circ \Delta_l, \text{ for } l = 1, 2,
\end{align*}
\]

where the map \(j : \mathbb{K}_G \to \mathbb{F}\) sends \(f(s) = \sum_{\alpha} z_\alpha s^\alpha \in \mathbb{K}_G\) to \(z_0\).

**Proposition 4.6.** The pairings \(\Delta_l\) and \(\Pi_l\) for \(l = 1, 2\) satisfy

\[
\begin{align*}
\Delta_l(a, b) & = \Delta_2(a \ast_l b, [M, \partial M] \otimes 1), \\
\Pi_l(a, b) & = \Pi_2(a \ast_l b, [M, \partial M] \otimes 1),
\end{align*}
\]

for any quantum homology classes \(a \in QH_k(M), b \in QH_{2n-k}(M, \partial M)\). Moreover, the pairings \(\Delta_2\) and \(\Pi_2\) are non-degenerate. Since the quantum homology groups are finite-dimensional \(\mathbb{K}_G\)-vector spaces in each degree, it follows that the paring \(\Delta_2\) gives rise to Poincaré-Lefschetz duality over the field \(\mathbb{K}_G\).

**Proof.** The formulas (29) follow immediately from (26). Let us prove the non-degeneracy statement. If

\[
a = \sum_{i \geq 0} a_i \otimes f_i(s)q^{k-i}, \quad a_i \in H_i(M; \mathbb{F}), f_i(s) \in \mathbb{K}_G,
\]

and

\[
b = \sum_{j \geq 0} b_j \otimes g_j(s)q^{2n-k-j}, \quad b_j \in H_j(M, \partial M; \mathbb{F}), g_j(s) \in \mathbb{K}_G,
\]

then we have that

\[
\Delta_2(a, b) = \sum_{i \geq 0} (a_i \ast_2 b_{2n-i})f_i(s)g_{2n-i}(s).
\]

Suppose that \(a \neq 0\). Then \(\exists i \geq 0\) such that \(a_i \otimes f_i(s) \neq 0\). Since the intersection product \(\ast_2\) is non-degenerate, the Kronecker-dual element \(a_i^\vee \in H_{2n-i}(M, \partial M; \mathbb{F})\) is non-zero and \(a_i \ast_2 a_i^\vee \neq 0\). Consider the element \(0 \neq b := a_i^\vee \otimes (f_i(s))^{-1}q^{-k+i} \in QH_{2n-k}(M, \partial M)\). We get that

\[
\Delta_2(a, b) = (a_i \ast_2 a_i^\vee) \otimes 1 \neq 0
\]

and

\[
\Pi_2(a, b) = j((a_i \ast_2 a_i^\vee) \otimes 1) = a_i \ast_2 a_i^\vee \neq 0.
\]

We conclude that the pairings \(\Delta_2\) and \(\Pi_2\) are non-degenerate. \(\square\)

**Remark 4.7.** In general, the intersection product \(\ast_1\) is degenerate. If it is non-degenerate, the pairings \(\Delta_1\) and \(\Pi_1\) are also non-degenerate. It happens, for example, when \(\partial M\) is a homology sphere.
Remark 4.8. The Novikov ring $\Lambda$ and quantum homology groups $QH_*(M, \partial M)$, $QH_*(M)$ admit the following valuation. Define a valuation

$$\nu : \mathbb{K}_G \to G \cup \{-\infty\}$$

on the field $\mathbb{K}_G$ by

$$\begin{cases} 
\nu \left( f(s) = \sum_{\alpha \in G} z_{\alpha} s^\alpha \right) := \max\{\alpha | z_{\alpha} \neq 0\}, & f(s) \neq 0 \\
\nu(0) = -\infty. & 
\end{cases}$$

(30)

Extend $\nu$ to $\Lambda$ by

$$\nu(\lambda) := \max\{\alpha | p_{\alpha} \neq 0\},$$

where $\lambda$ is uniquely represented by

$$\lambda = \sum_{\alpha \in G} p_{\alpha} s^\alpha, \ p_{\alpha} \in \mathbb{F}[q, q^{-1}].$$

Note that for all $\lambda, \mu \in \Lambda$ we have

$$\nu(\lambda + \mu) \leq \max(\nu(\lambda), \nu(\mu)), \ \nu(\lambda \mu) = \nu(\lambda) + \nu(\mu), \ \nu(\lambda^{-1}) = -\nu(\lambda).$$

Now, any non-zero $a \in QH_*(M, \partial M)$ ($QH_*(M)$) can be uniquely written as $a = \sum_i a_i \otimes \lambda_i$, where $a_i \in H_*(M, \partial M; \mathbb{F}) (H_*(M; \mathbb{F}))$ and $\lambda_i \in \Lambda$. Define

$$\nu(a) := \max_i \{\nu(\lambda_i)\}.$$

4.2. Examples. In the following examples we consider the base field $\mathbb{F} = \mathbb{Q}$.

Example 3. Let $(M, \omega) := (\mathbb{P}^4, \omega_\delta)$ be the symplectic blow-up at the origin of the standard unit 4-ball in $(\mathbb{C}^2, \omega_0)$ of a small size $0 < \delta << 1$. Then $\{[pt], E\}$ is the integral homogeneous basis of $H_*(M; \mathbb{Q})$ and $\{E^\vee, [M, \partial M]\}$ is the integral homogeneous basis of $H_*(M, \partial M; \mathbb{Q})$. Since $\omega_\delta(E) = \delta > 0$ and $c_1(E) = 1$, it follows that $\Gamma = H^2_*(M) = \mathbb{Z}\langle E \rangle$ and $G = \frac{\delta}{2} \cdot \mathbb{Z}$, see (18) and (19) for the notation. From
we find the quantum products:

1. $[pt] *_{1} H_{s}(M;\mathbb{Q}) = 0$
   
   $E *_{1} E = -[pt] + E \otimes s^{-\delta}q^{-2}$

2. $[pt] *_{2} E^{\vee} = 0$
   
   $E *_{2} E^{\vee} = [pt] - E \otimes s^{-\delta}q^{-2}$

3. $E^{\vee} *_{3} E^{\vee} = -E^{\vee} \otimes s^{-\delta}q^{-2}$
   
   $[M,\partial M]$ is the $*_{3}$-unit

Example 4. Let $(M,\omega) := (\tilde{X}, \omega_{\delta})$ be the symplectic blow-up of the cotangent unit disk bundle $X := \mathbb{D} T^{*}\Sigma$, see Example 2. Then $\{[pt], \{a_{i}, b_{i}\}_{i=1}^{g}, [\Sigma], E\}$ is the integral homogeneous basis of $H_{s}(M;\mathbb{Q})$, and $\{[F], E^{\vee}, \{a_{i}^{\vee}, b_{i}^{\vee}\}_{i=1}^{g}, [M, \partial M]\}$ is the integral homogeneous basis of $H_{s}(M,\partial M;\mathbb{Q})$. Since $\omega_{\delta}(E) = \delta > 0, \omega_{\delta}([\Sigma]) = 0$ and $c_{1}(E) = 1, c_{1}([\Sigma]) = 0$, it follows that $\Gamma = \mathbb{Z}\langle E \rangle$ and $G = \delta \cdot \mathbb{Z}$, see (18) and (19) for the notation. From (17) we find the quantum products:

1. $[pt] *_{1} H_{s}(M;\mathbb{Q}) = 0$
   
   $a_{i} *_{1} a_{j} = a_{i} *_{1} b_{j} = b_{i} *_{1} b_{j} = 0$

   $[\Sigma] *_{1} a_{i} = [\Sigma] *_{1} b_{i} = [\Sigma] *_{1} E = 0$

   $[\Sigma] *_{1} [\Sigma] = (2 - 2g)[pt]$

   $E *_{1} a_{i} = E *_{1} b_{i} = 0$

   $E *_{1} E = -[pt] + E \otimes s^{-\delta}q^{-2}$

2. $z *_{2} w = z \bullet_{2} w$ for any basic $(z, w) \neq (E, E^{\vee})$

   $E *_{2} E^{\vee} = [pt] - E \otimes s^{-\delta}q^{-2}$

   $z *_{2} [M, \partial M] = z, \forall z \in H_{s}(M;\mathbb{Q})$
\[ [F] \ast_3 [F] = [F] \ast_3 F^\vee = [F] \ast_3 a_i^\vee = [F] \ast_3 b_i^\vee = 0 \]
\[ E^\vee \ast_3 a_i^\vee = E^\vee \ast_3 b_i^\vee = 0 \]
\[ E^\vee \ast_3 E = -E^\vee \otimes s^{-2} q^{-2} \]
\[ a_i^\vee \ast_3 a_j^\vee = a_i^\vee \ast_3 b_j^\vee = b_i^\vee \ast_3 b_j^\vee = 0 \]
\[ [M, \partial M] \text{ is the } \ast_3\text{-unit} \]

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