Abstract

The main goal of this note is to prove the following theorem. If $A_n$ is a sequence of measurable sets in a $\sigma$-finite measure space $(X,\mathcal{A},\mu)$ that covers $\mu$-a.e. $x \in X$ infinitely many times, then there exists a sequence of integers $n_i$ of density zero so that $A_{n_i}$ still covers $\mu$-a.e. $x \in X$ infinitely many times. The proof is a probabilistic construction.

As an application we give a simple direct proof of the known theorem that the ideal of density zero subsets of the natural numbers is random-indestructible, that is, random forcing does not add a co-infinite set of naturals that almost contains every ground model density zero set. This answers a question of B. Farkas.

1 Introduction

Maximal almost disjoint (MAD) families of subsets of the naturals play a central role in set theory. (Two sets are almost disjoint if their intersection is finite.)
A fundamental question is whether MAD families remain maximal in forcing extensions. This is often studied in a little more generality as follows. For a MAD family $\mathcal{M}$ let $\mathcal{I}_\mathcal{M}$ be the ideal of sets that can be almost contained in a finite union of members of $\mathcal{M}$. (Almost contained means that only finitely many elements are not contained.) Then it is easy to see that $\mathcal{M}$ remains MAD in a forcing extension if and only if there is no co-infinite set of naturals in the extension that almost contains every (ground model) member of $\mathcal{I}_\mathcal{M}$. Hence the following definition is natural.

**Definition 1.1** An ideal $\mathcal{I}$ of subsets of the naturals is called *tall* if there is no co-infinite set that almost contains every member of $\mathcal{I}$. Let $\mathcal{I}$ be a tall ideal and $P$ be a forcing notion. We say that $\mathcal{I}$ is *$P$-indestructible* if $\mathcal{I}$ remains tall after forcing with $P$.

This notion is thoroughly investigated for various well-known ideals and forcing notions, for instance Hernández-Hernández and Hrušák proved that the ideal of density zero subsets (see Definition 2.1) of the natural numbers is random-indestructible. (Indeed, just combine [3, Thm 3.14], which is a result of Brendle and Yatabe, and [3, Thm 3.4].) B. Farkas asked if there is a simple and direct proof of this fact. In this note we provide such a proof.

This proof actually led us to a covering theorem (Thm. 2.5) which we find very interesting in its own right from the measure theory point of view. First we prove this theorem in Section 2 by a probabilistic argument, then we apply it in Section 3 to reprove that the density zero ideal is random-indestructible (Corollary 3.3), and finally we pose some problems in Section 4.

## 2 A covering theorem

Cardinality of a set $A$ is denoted by $|A|$.

**Definition 2.1** A set $A \subseteq \mathbb{N}$ is of *density zero* if $\lim_{n \to \infty} \frac{|A \cap \{0, \ldots, n-1\}|}{n} = 0$. The ideal of density zero sets is denoted by $\mathcal{Z}$.

$A \subset^* B$ means that $B$ almost contains $A$, that is, $A \setminus B$ is finite. The following is well-known.

**Fact 2.2** $\mathcal{Z}$ is a P-ideal, that is, for every sequence $Z_n \in \mathcal{Z}$ there exists $Z \in \mathcal{Z}$ so that $Z_n \subset^* Z$ for every $n \in \mathbb{N}$.

**Lemma 2.3** Let $(X, \mathcal{A}, \mu)$ be a measure space of $\sigma$-finite measure, and let $\{A_n\}_{n \in \mathbb{N}}$ be a sequence of measurable sets. Suppose that there exists $0 = N_0 < N_1 < N_2 < \ldots$ so that $A_{N_{k-1}}, \ldots, A_{N_{k-1}}$ is a cover of $X$ for every $k \in \mathbb{N}^+$, and also that $k$ divides $N_k - N_{k-1}$ for every $k \in \mathbb{N}^+$. Then there exists a set $Z \in \mathcal{Z}$ so that $\{A_n\}_{n \in Z}$ covers $\mu$-a.e. every $x \in X$ infinitely many times.
It is easy to see that $Z \in \mathcal{Z}$. Hence it suffices to show that with probability 1 $\mu$-a.e. $x \in X$ is covered infinitely many times by $\{A_n\}_{n \in \mathbb{Z}}$.

Let us now fix an $x \in X$. Let $E_k$ be the event $\{x \in \bigcup_{n \in W^k_{\xi_k}} A_n\}$, that is, $x$ is covered by the set chosen in the $k^{th}$ block. As the $k^{th}$ block is a cover of $X$, $Pr(E_k) \geq \frac{1}{k}$, so $\sum_{k \in \mathbb{N}^+} Pr(E_k) = \infty$. Moreover, the events $\{E_k\}_{k \in \mathbb{N}^+}$ are independent. Hence by the second Borel-Cantelli Lemma, $\sum_{k \in \mathbb{N}^+} Pr(E_k) > 1$. So every fixed $x$ is covered infinitely many times with probability 1, but then by the Fubini theorem with probability 1 $\mu$-a.e. $x$ is covered infinitely many times, and we are done. (To be more precise, let $(\Omega, \mathcal{S}, Pr)$ be the probability measure space, then $Z(\omega) = \bigcup_{k \in \mathbb{N}^+} W^k_{\xi_k(\omega)}$. Since the sets $\{(x, \omega) : x \in A_n\}$ and $\{(x, \omega) : \xi_k(\omega) = n\}$ are clearly $\mathcal{A} \times \mathcal{S}$-measurable, it is straightforward to show that $\{(x, \omega) : x \text{ is covered infinitely many times by } \{A_n\}_{n \in \mathbb{Z}}\} \subset X \times \Omega$ is $\mathcal{A} \times \mathcal{S}$-measurable, and hence Fubini applies.) \hfill \Box

**Lemma 2.4** Let $(X, \mathcal{A}, \mu)$ be a measure space of finite measure, and let $\{A_n\}_{n \in \mathbb{N}}$ be a sequence of measurable sets that covers $\mu$-a.e. every $x \in X$ infinitely many times. Then there exists a set $Z \in \mathcal{Z}$ so that $\{A_n\}_{n \in \mathbb{Z}}$ still covers $\mu$-a.e. every $x \in X$ infinitely many times.

**Proof.** Let $\varepsilon > 0$ be arbitrary and set $N_0 = 0$. By the continuity of measures, there exists $N_1$ so that $\mu(X \setminus (A_{N_0} \cup \ldots \cup A_{N_1-1})) \leq \frac{\varepsilon}{2}$. Since $\{A_n\}_{n \geq N_1}$ still covers $\mu$-a.e. $x \in X$ infinitely many times, we can continue this procedure, and recursively define $0 = N_0 < N_1 < N_2 < \ldots$ so that $\mu(X \setminus (A_{N_{k-1}} \cup \ldots \cup A_{N_k-1})) \leq \frac{\varepsilon}{2^k}$ for every $k \in \mathbb{N}^+$. We can also assume (by choosing larger $N_k$’s at each step) that $k$ divides $N_k - N_{k-1}$ for every $k \in \mathbb{N}^+$.

Let $X_\varepsilon = \cap_{k \in \mathbb{N}^+} (A_{N_{k-1}} \cup \ldots \cup A_{N_k-1})$, then $\mu(X \setminus X_\varepsilon) \leq \varepsilon$. Let us restrict $\mathcal{A}$, the $A_n$’s and $\mu$ to $X_\varepsilon$, and apply the previous lemma with this setup to obtain $Z_\varepsilon$.

Let us now consider $\varepsilon = 1, \frac{1}{2}, \frac{1}{3}, \ldots$, then for every $m \in \mathbb{N}^+$ every $x \in \bigcup_{n \in \mathbb{Z}} X_m$ is covered infinitely many times by $\{A_n\}_{n \in \mathbb{Z}}$. Since $Z$ is a $\mathcal{P}$-ideal, there exists a $Z \in \mathcal{Z}$ such that $Z \cap X_m \subset \subset Z$ for every $m$. Hence for every $m \in \mathbb{N}^+$ every $x \in X_m$ is covered infinitely many times by $\{A_n\}_{n \in \mathbb{Z}}$. But then we are done, since $\mu$-a.e. $x \in X$ is in $\bigcup_m X_m$. \hfill \Box

**Theorem 2.5** Let $(X, \mathcal{A}, \mu)$ be a measure space of $\sigma$-finite measure, and let $\{A_n\}_{n \in \mathbb{N}}$ be a sequence of measurable sets that covers $\mu$-a.e. every $x \in X$ infinitely many times. Then there exists a set $Z \subset \mathbb{N}$ of density zero so that $\{A_n\}_{n \in \mathbb{Z}}$ still covers $\mu$-a.e. every $x \in X$ infinitely many times.
Proof. Write $X = \bigcup X_m$, where each $X_m$ is of finite measure. For each $X_m$ obtain $Z_m$ by the previous lemma. Then a $Z \in Z$ such that $Z_m \subset^* Z$ for every $m$ clearly works. □

The following example shows that the purely topological analogue of Theorem 2.5 is false.

Example 2.6 There exists a sequence $U_n$ of clopen sets covering every point of the Cantor space infinitely many times so that for every $Z \in Z$ there exists a point covered only finitely many times by $\{U_n : n \in Z\}$.

Proof. By an easy recursion we can define a sequence $U_n$ of clopen subsets of the Cantor set $C$ and a sequence of naturals $0 = N_0 < N_1 < \ldots$ with the following properties.

1. $U_{N_{k-1}}, \ldots, U_{N_k-1}$ (called a ‘block’) is a disjoint cover of $C$,
2. every block is a refinement of the previous one,
3. if $U_n$ is in the $k^{th}$ block and is partitioned into $U_t, \ldots, U_s$ in the $k+1^{st}$ block (called the ‘immediate successors of $U_n$’) then $s \geq 2t$.

Let $Z \in Z$ be given, and let $n_0$ be so that $\frac{|Z \cap \{0, \ldots, n-1\}|}{n} < \frac{1}{2}$ for every $n \geq n_0$. By 3. $\{U_n : n \in Z\}$ cannot contain all immediate successors of any $U_m$ above $n_0$. Therefore, starting at a far enough block, we can recursively pick a $U_{n_i}$ from each block so that $n_i \notin Z$ for every $i$, and $\{U_{n_i} : i \in \mathbb{N}\}$ is a nested sequence of clopen sets. But then the intersection of this sequence is only covered finitely many times by $\{U_n : n \in Z\}$. □

Remark 2.7 We can ‘embed’ this example into any topological space containing a copy of the Cantor set (e.g. to any uncountable Polish space) by just adding the complement of the Cantor set to all $U_n$’s. Of course, the new $U_n$’s will only be open, not clopen.

3 An application: The density zero ideal is random-indestructible

In this section we give a simple and direct proof of the random-indestructibility of $Z$, which was first proved in [3].

$[\mathbb{N}]^\omega$ denotes the set of infinite subsets of $\mathbb{N}$. Since it can be identified with a $G_\delta$ subspace of $2^\omega$ in the natural way, it carries a Polish space topology where the sub-basic open sets are the sets of the form $[n] = \{A \in [\mathbb{N}]^\omega : n \in A\}$ and their complements. Let $\lambda$ denote Lebesgue measure.

Lemma 3.1 For every Borel function $f : \mathbb{R} \to [\mathbb{N}]^\omega$ there exists a set $Z \in Z$ such that $f(x) \cap Z$ is infinite for $\lambda$-a.e. $x \in \mathbb{R}$. 

4
Proof. Let \( A_n = f^{-1}([n]) \), then \( A_n \) is clearly Borel, hence Lebesgue measurable. For every \( x \in \mathbb{R} \)

\[
x \in A_n \iff x \in f^{-1}([n]) \iff f(x) \in [n] \iff n \in f(x).
\]  

(3.1)

Since every \( f(x) \) is infinite, (3.1) yields that every \( x \in \mathbb{R} \) is covered by infinitely many \( A_n \)'s. By Theorem 2.5 there exists a \( Z \in \mathcal{Z} \) such that for \( \lambda \)-a.e. \( x \in \mathbb{R} \) we have \( x \in A_n \) for infinitely many \( n \in Z \). But then by (3.1) for \( \lambda \)-a.e. \( x \in \mathbb{R} \) we have \( n \in f(x) \) for infinitely many \( n \in Z \), so \( f(x) \cap Z \) is infinite.

Recall that random forcing is \( B = \{ p \subseteq \mathbb{R} : p \) is Borel, \( \lambda(p) > 0 \} \) ordered by inclusion. The random real \( r \) is defined by \( \{ r \} = \cap_{p \in G} p \), where \( G \) is the generic filter. For the terminology and basic facts concerning random forcing consult e.g. [5], [4], [1], or [6]. In particular, we will assume familiarity with coding of Borel sets and functions, and will freely use the same symbol for all versions of a Borel set or function. The following fact is well-known and easy to prove.

Fact 3.2 Let \( B \subseteq \mathbb{R} \) be Borel. Then \( p \vdash "r \in B" \) iff \( \lambda(p \setminus B) = 0 \).

Corollary 3.3 The ideal of density zero subsets of the natural numbers is random-indestructible, that is, random forcing does not add a co-infinite set of naturals that almost contains every ground model density zero set.

Proof. For a Borel function \( f : \mathbb{R} \to [\mathbb{N}]^\omega \) and a set \( Z \in \mathcal{Z} \) let

\[ B_{f,Z} = \{ x \in \mathbb{R} : f(x) \cap Z \text{ is infinite} \} , \]

then by the previous lemma for every \( f \) there is a \( Z \) so that \( B_{f,Z} \) is of full measure. By Fact 3.2 for every \( f \) there is a \( Z \) so that \( 1_B \vdash "f(r) \cap Z \text{ is infinite}" \). Hence for every \( f \) \( 1_B \vdash "\exists Z \in \mathcal{Z} \cap V \text{ so that } f(r) \cap Z \text{ is infinite}" \). But every \( y \in [\mathbb{N}]^\omega \cap V[r] \) is of the form \( f(r) \) for some ground model (coded) Borel function \( f : \mathbb{R} \to [\mathbb{N}]^\omega \), so we obtain that for every \( y \in [\mathbb{N}]^\omega \cap V[r] \) \( 1_B \vdash "\exists Z \in \mathcal{Z} \cap V \text{ so that } y \cap Z \text{ is infinite}" \). Therefore \( 1_B \vdash "\forall y \in [\mathbb{N}]^\omega \exists Z \in \mathcal{Z} \cap V \text{ so that } y \cap Z \text{ is infinite}" \), and setting \( x = \mathbb{N} \setminus y \) yields \( 1_B \vdash "\forall x \subseteq \omega \text{ co-infinite } \exists Z \in \mathcal{Z} \cap V \text{ so that } Z \nsubseteq x" \), so we are done.

Remark 3.4 Clearly, \( Z \) is also \( \mathbb{B}(\kappa) \)-indestructible, since every new real is already added by sub-poset isomorphic to \( \mathbb{B} \). (\( \mathbb{B}(\kappa) \) is the usual poset for adding \( \kappa \) many random reals by the measure algebra on \( 2^\kappa \).)

Remark 3.5 The referee of this paper has pointed out that all arguments of the paper can actually be carried out in the axiom system \( ZF + DC \). (\( ZF \) is the usual Zermelo-Fraenkel axiom system without the Axiom of Choice, and \( DC \) is the Axiom of Dependent Choice.) Hence Corollary 3.3 actually applies to forcing over a model of \( ZF + DC \) as well.
4 Problems

There are numerous natural directions in which one can ask questions in light of Corollary 3.3 and Theorem 2.5. As for the former one, one can consult e.g. [2] and the references therein. As for the latter one, it would be interesting to investigate what happens if we replace the density zero ideal by another well-known one, or if we replace the measure setup by the Baire category analogue, or if we consider non-negative functions (summing up to infinity a.e.) instead of sets, or even if we consider $\kappa$-fold covers and ideals on $\kappa$.

Acknowledgment The author is indebted to Richárd Balka for some helpful discussions and to an anonymous referee for suggesting Remark 3.5.

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