A note on superposition of two unknown states using Deutsch CTC model

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(Dated: March 19, 2018)

In a recent work, authors prove a yet another no-go theorem that forbids the existence of a universal probabilistic quantum protocol producing a superposition of two unknown quantum states. In this short note, we show that in the presence of closed time like curves, one can indeed create superposition of unknown quantum states and evade the no-go result.

PACS numbers: Valid PACS appear here

I. INTRODUCTION

In the past two decades the quantum information theory played an important role in achieving a huge range of information processing tasks which are still impossible to achieve with the current set of technologies available in the classical world [1–18]. At the same time there are certain tasks in the classical world which are impossible to execute with the quantum resources. These impossible operations are termed as no-go theorems of quantum information theory [19–27] and indeed they play a very crucial role in the security and privacy aspect of the quantum technology. A good example in this context is the no-cloning theorem, which states that non-orthogonal quantum states cannot be cloned which serves as an underlying reason for the existence of secure quantum cryptography. Quantum information provides different computational resources than classical information. In a pioneering work by Alvarez-Rodriguez et al [28], it was demonstrated that there is no unitary protocol which would allow to add quantum states belonging to different Hilbert space. This analysis was further extended recently where researchers articulate the fact that in spite superposition being an intriguing phenomenon of quantum physics, it is impossible to create an arbitrary superposition of unknown quantum states [29]. Circumventing the said no-go is a challenging problem and is the main motivation of the present letter.

Further investigations revealed that the presence of closed time like curves can significantly affect the computational and other abilities of a system [48–50]. These include factorization of composite numbers efficiently with the help of a classical computer [48] and ability to solve NP-complete problems [49]. Brun et al. [51] have shown that with the access to CTCs, it is possible to perfectly distinguish non orthogonal quantum states, having wide range of implications for the security of quantum cryptography. In another work [52], the information flow of quantum states interacting with closed time like curves was investigated. Few years back, it has been shown that the presence of CTC has implications for purification of mixed states [53], and in making non local no signaling boxes to signaling boxes [54]. Recently, it was demonstrated that teleportation of quantum information, even in its approximate version, from a CR region to a CTC region is disallowed [55]. In a paper [56], Bennett et al. have argued against the Deutsch model (D-CTC) and opined for revisiting the implications obtained by assuming the existence of CTCs as described by Deutsch model.

Qubits having a closed time like world line can give rise to various paradoxes. A predominant one of them is the grandfather paradox. However these paradoxes can be avoided by using the self consistency condition of the D-CTC model. The Deutsch self-consistency conditions have two components to it: one qubit from the chronology respecting region (CR) and another qubit having a word line like a closed time like curve which we will refer as CTC qubit. This condition demands the initial density matrix of a CTC system must be equal to its output density matrix after it has interacted with a chronology respecting system CR under a unitary operation $U$, 

$$\rho_{\text{CTC}} = \text{Tr}_{\text{CR}}\{U(\rho_{\text{CR}} \otimes \rho_{\text{CTC}})U^\dagger\}, \quad (1)$$

$$\rho_{\text{out}} = \text{Tr}_{\text{CTC}}\{U(\rho_{\text{CR}} \otimes \rho_{\text{CTC}})U^\dagger\}. \quad (2)$$

In Eq. (1) $\rho_{\text{CTC}}$ stands for the density matrix of the CTC system before interaction and the right hand side of the equation gives the partial density matrix of the CTC system after interaction. In Eq.2 $\rho_{\text{out}}$ gives the density
matrix of the chronology respecting system (CR) after interaction, whose initial density matrix $\rho_{CR}$. Let us reiterate that although CTC’s are problematic in Einstein’s theory of relativity due to violation of principle of causality, the problem is circumvented in quantum framework. As demonstrated in [57–60] the effect of CTC’s can be mimicked using classical and quantum simulation. For instance in Ref.[59], the authors succeeded in experimentally simulating the nonlinear behavior of a qubit interacting in a unitary fashion with an older version of itself. The presence of CTC’s, in particular, allowed them to discriminate the non-orthogonal states which is otherwise impossible. They also examined the other no-go results leaving aside the problem of superposition of unknown quantum states.

In this work we show that if we have access to a closed time like curve satisfying Deutsch kinematic conditions then we can indeed design an unitary operator which will be able to create a superposition of two unknown quantum states.

According to a recent no go theorem [28, 29], given two unknown quantum states $|\phi_1\rangle$ and $|\phi_2\rangle$ it is not possible to create the state $|\phi\rangle$ where $|\phi\rangle = \gamma^{-1}(|\alpha|\phi_1 + |\beta|\phi_2)$, where $\gamma$ is the normalizing factor and $\alpha, \beta$ are given complex numbers. A probabilistic protocol is also given, to create superposition of two unknown states where the class of input states for which superposition is to be created is given along with information from which superposition has to be generated. But with the assistance of Deutsch CTC we can create superposition of two unknown states deterministically, correspondings to fixed complex numbers $\alpha$ and $\beta$, if the set $\{|\psi_j\rangle\}$ from which the two unknown states are taken is known beforehand. The method follows directly from the proof of distinguishability of non-orthogonal states with under Deutsch CTC. As shown by Brun et al.[51] if $\{|\psi_j\rangle\}_{j=0}^{N-1}$ is a set of $N$ distinct states in a $N$ dimensional space, then by using Deutsch CTC we can implement the mapping $|j\rangle |\psi_j\rangle \rightarrow |j\rangle |\psi_j\rangle$, where $|j\rangle$ forms an orthonormal basis for the $N$ dimensional space. The unitary operation they used to carry out this transformation is a SWAP operation followed by a controlled unitary operation from chronology respecting system to the CTC system given by,

$$U = \sum_{k=0}^{N-1} |k\rangle\langle k| \otimes U_k,$$

where $U_k$ are unitary operations that satisfy the following conditions: (1) $U_k|\psi_k\rangle = |k\rangle$ for $0 \leq k < N$ and (2) $\langle j|U_k|\psi_j\rangle \neq 0$ for $0 \leq j, k < N$. The latter conditions come from constraint of unique solution to the Deutsch self-consistency condition. Brun et al. showed that it is always possible to construct unitary operations $U_k$ satisfying constraints (1) and (2) and gave a method for the same. It can be check that if initially the chronologically respecting system is in state $|\psi_j\rangle$ then after interacting it with the CTC system under the unitary operation given by Eq. (3) the final state of both CR and CTC system is

$$\rho_{out} = \rho_{CTC} = |j\rangle\langle j|$$

$$\langle \psi_j | \psi_j \rangle \otimes \rho_{CTC} = |\psi_j\rangle\langle \psi_j| \otimes |j\rangle\langle j|$$

$$\rightarrow (\text{SWAP}) \rightarrow |j\rangle\langle j| \otimes |\psi_j\rangle$$

$$\rightarrow (U) \rightarrow |j\rangle\langle j| \otimes |\psi_j\rangle$$

Here we also follow the similar setup. Let $|\phi_1\rangle = |\psi_m\rangle$ and $|\phi_2\rangle = |\psi_n\rangle$ be two unknown states from the set $\{|\psi_j\rangle\}_{j=0}^{N-1}$ for which we wish to create the superimposition $|\phi\rangle = \gamma^{-1}(|\alpha|\phi_1 + |\beta|\phi_2)$ where $\gamma$ is the normalizing factor and $\alpha, \beta$ are given complex numbers. For this we require two CTC systems, for each of these states $|\phi_1\rangle$ and $|\phi_2\rangle$. To do so, we interact both the states with separate CTC systems under unitary given by Eq.3. Let the states of chronological respecting systems in both the cases after interaction with their respective CTC systems be $\rho_{out_1} = |m\rangle\langle m|$ and $\rho_{out_2} = |n\rangle\langle n|$ respectively. Let $U'$ be a unitary defined as

$$U' = \sum_{i,j=0}^{N-1} |i\rangle\langle i| \otimes |j\rangle\langle j| \otimes U_{\alpha,\beta}^{i,j}$$

where $U_{\alpha,\beta}^{i,j}$ are unitary operations for $0 \leq i, j < N$, such that,

$$U_{\alpha,\beta}^{i,j} |0\rangle = |\omega\rangle^{i,j}_{\alpha,\beta} = \gamma^{-1}(|\alpha|\psi_i + |\beta|\psi_j)$$

for some fixed state $|0\rangle$. Such unitary operations $U_{\alpha,\beta}^{i,j}$ can always be constructed by Gram Schmidt process on the set $S = [|\omega\rangle^{i,j}_{\alpha,\beta} \cup \{|\psi_j\rangle\}_{j=0}^{N-1}$ with the first element for the process being $|\omega\rangle^{i,j}_{\alpha,\beta}$. If $S$ does not contain $N$ linearly independent states, the orthonormal states obtained by the process can always be extended. If the input states are the same that is $|\phi_1\rangle = |\phi_2\rangle$ then the desired superposition is same as the input states. So for simplicity $U_{\alpha,\beta}^i = U_{\alpha,\beta}^{-1}P_i$ where $P_i$ is a permutation unitary such that $P_i|0\rangle = |i\rangle$ and $U_i^{-1}$ is the inverse of the unitary $U_i$ given by $U_k$ in Eq (3). When the unitary $U'$ defined by Eq (5) is applied on $\rho_{out_1} \otimes \rho_{out_2} \otimes |0\rangle\langle 0|$ (where $|0\rangle$ is the fixed ancilla state defined above ) then the desired superimposition of $|\phi_1\rangle$ for $|\phi_2\rangle$ for the given complex numbers $\alpha, \beta$ is obtained on the ancilla system.

$$U' (\rho_{out_1} \otimes \rho_{out_2} \otimes |0\rangle\langle 0|) = U' (|m\rangle\langle m| \otimes |n\rangle\langle n| \otimes |0\rangle\langle 0|)$$

$$= |m\rangle\langle m| \otimes |n\rangle\langle n| \otimes U_{\alpha,\beta}^{i,j}|0\rangle\langle 0|$$

$$= |m\rangle\langle m| \otimes |n\rangle\langle n| \otimes |\omega_{\alpha,\beta}^{i,j}\rangle\langle \omega_{\alpha,\beta}^{i,j}|$$

Example: Now consider an example where $N = 2$ and the given set of distinct states is $\{0\rangle, -\rangle\}$. And let $\alpha, \beta$ be the given complex numbers. In this case the unitary given by Eq. (3) reduces to $U = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes H$ where $I$ and $H$ are identity and Hadamard operators.
And the unitary operators $U^{i,j}_{\alpha,\beta}$ reduce to

$$U^{0,0}_{\alpha,\beta} = I$$

$$U^{0,1}_{\alpha,\beta} = \frac{1}{\gamma_1} \left[ \begin{array}{cc} \alpha + \frac{\beta}{\sqrt{2}} & \beta \frac{\alpha^*}{\sqrt{2}} \\ -\frac{\beta^*}{\sqrt{2}} & \alpha^* + \frac{\beta^*}{\sqrt{2}} \end{array} \right]$$

$$U^{1,0}_{\alpha,\beta} = \frac{1}{\gamma_2} \left[ \begin{array}{cc} \beta + \frac{\alpha}{\sqrt{2}} & \alpha \frac{\beta^*}{\sqrt{2}} \\ -\frac{\alpha^*}{\sqrt{2}} & \beta^* + \frac{\alpha^*}{\sqrt{2}} \end{array} \right]$$

$$U^{1,1}_{\alpha,\beta} = HX,$$

where $\gamma_1 = ((\alpha + \frac{\beta}{\sqrt{2}})^2 + \frac{\beta^2}{2})^{\frac{1}{2}}$ and $\gamma_2 = ((\beta + \frac{\alpha}{\sqrt{2}})^2 + \frac{\alpha^2}{2})^{\frac{1}{2}}$ are normalizing factors and $X$ is the phase flip operator. Using Eq’s. (5) and (7), it can be checked that for values of $i,j$ the desired superposition is created.

In this letter, we have shown that creating superposition of an unknown state is possible in causality respecting regions provided we allow the interaction with a closed time like curve. This once again shows the enormous power of closed time like curves in making things possible which are otherwise impossible in the chronology respecting regions. It would be interesting to see how by mimicking the effect of CTC by classical and quantum simulation [59] we can create a superposition of two unknown quantum states. Simultaneously it will be fascinating to see whether in such a situation if we are not able to get perfect superposition, then can we get a fidelity more than the fidelity obtained in case of an approximate quantum adder [28]. Both of these questions are very interesting and we defer it to our future investigations.

II. ACKNOWLEDGMENT

We are indebted to A. K. Pati for suggesting the problem and for his constant help and encouragement thereafter.

[1] A. Ekert, Phys. Rev. Lett. 67, 661 (1991).
[2] C. H. Bennett and G. Brassard, Proceedings of IEEE International Conference on Computers, System and Signal Processing, Bangalore, India, pp.175-179 (1984).
[3] P. W. Shor and J. Preskill, Phys. Rev. Lett. 85, 441 (2000).
[4] M. Hillery, V. Buzek and A. Berthiaume, Phys. Rev. A 59, 1829 (1999).
[5] S. K. Sazim, I. Chakrabarty, C. Var- narasa and K. Sri- nathan, arXiv:1311.5378.
[6] S. Adhikari, I. Chakrabarty and P. Agrawal, Quant. Inf. Comp 12, 0253 (2012).
[7] Sk. Sazim, I. Chakrabarty, Eur. Phys. J. D 67, 174 (2013).
[8] S. Adhikari, I. Chakrabarty and B. S. Choudhary, J. Phys. A 39, 8439 (2006).
[9] M. K. Shukla, I. Chakrabarty, S. Chatterjee, arXiv: 1511.05796.
[10] S. Chatterjee, S. Sazim, I. Chakrabarty, Phys. Rev. A 93, 042309 (2016).
[11] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres and W. K. Wootters, Phys. Rev. Lett. 70, 1895 (1993).
[45] C. H. Bennett, in Proceedings of QUPON, Wien, 2005, http://www.research.ibm.com/people/b/bennetc/
[46] G. Svetlichny, International Journal of Theoretical Physics, 50, 3903 (2011).
[47] S. Lloyd, L. Maccone, R. Garcia-Patron, V. Giovannetti, Y. Shikano, S. Pirandola, L. A. Rozema, A. Darabi, Y. Soudagar, L. K. Shalm, and A. M. Steinberg, Phys. Rev. Lett. 106, 040403.
[48] T. A. Brun, Found. Phys. Lett. 16, 245 (2003).
[49] D. Bacon, Phys. Rev. A 70, 032309 (2004).
[50] S. Aaronson and J. Watrous, Proc. R. Soc. A 465, 631 (2009).
[51] T. A. Brun, J. Harrington, and M. M. Wilde, Phys. Rev. Lett. 102, 210402 (2009).
[52] T. C. Ralph and C. R. Myers, Phys Rev A 82, 062330 (2010).
[53] A. K. Pati, I. Chakrabarty, and P. Agrawal, Phys. Rev. A 84, 062325 (2011).
[54] I. Chakrabarty, T. Pramanik, A. K. Pati, and P. Agrawal, Quantum Information and Computation 14, 1251 (2014).
[55] Asutosh Kumar, Indranil Chakrabarty, Arun Kumar Pati, Aditi Sen(De) , Ujjwal Sen, arXiv: 1511.08560.
[56] C. H. Bennett, D. Leung, G. Smith, and J. A. Smolin, Phys. Rev. Lett. 103, 170502 (2009).
[57] J. Casanova et al., Phys. Rev. X. 1, 021018 (2011).
[58] U. Alvarez-Rodriguez et al, Phys. Rev. Lett. 111, 090503 (2013).
[59] M. Ringbauer et al., Nature Communications 5, 4145 (2014).
[60] X. Zhang et al., Nat. Commun. 6, 7917 (2015).