On the equivalence between four versions of thermostatistics based on strongly pseudo-additive entropies

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Abstract

The class of strongly pseudo-additive entropies, which can be represented as an increasing continuous transformation of Shannon and Rényi entropies, have intensively been studied in previous decades. Although their mathematical structure has thoroughly been explored and established by generalized Shannon-Khinchin axioms, the analysis of their thermostatistical properties have mostly been limited to special cases which belong to two parameter Sharma-Mittal entropy class, such as Tsallis, Rényi and Gaussian entropies.

In this paper we present a general analysis of the strongly pseudo-additive entropies thermostatistics by taking into account both linear and escort constraints on internal energy. We develop two types of dualities between the thermostatistics formalisms. By the first one, the formalism of Rényi entropy is transformed in the formalism of SPA entropy under general energy constraint and, by the second one, the generalized thermostatistics which corresponds to the linear constraint is transformed into the one which corresponds to the escort constraint. Thus, we establish the equivalence between four different thermostatistics formalisms based on Rényi and SPA entropies coupled with linear and escort constraints and we provide the transformation formulas. In this way we obtain a general framework which is applicable to the wide class of entropies and constraints previously discussed in the literature.

As an example, we rederive maximum entropy distributions for Sharma-Mittal entropy and we establish new relationships between the corresponding thermodynamic potentials. We obtain, as special cases, previously developed expressions for maximum entropy distributions and thermodynamic quantities for Tsallis, Rényi and Gaussian entropies. In addition, the results are applied for derivation of thermostatistical relationships for supra-extensive entropy, which has not previously been considered.

Keywords: Strongly pseudo-additive entropy, Legendre structure, generalized thermodynamics, maximum entropy, energy constraint, Sharma-Mittal entropy, supra extensive entropy

1. Introduction

The class of strongly pseudo-additive (SPA) entropies is the most general class of entropies which satisfies generalized Shannon-Khinchin axioms [1]. SPA entropies can be represented as an increasing continuous transformation of Shannon and Rényi entropies and some well known generalized entropies, such as Sharma-Mittal, Tsallis and Gaussian ones, belonging to this class. In the past, the SPA entropies have widely been explored in the context of statistics [2, 3], computer sciences [4] and quantum mechanics [5], and their ubiquitous mathematical structure has been recognized in the context of formal group theory where they are also known as Z-entropies [6]. A special importance of SPA entropies has been recognized...
in thermostatistics based on generalized maximum entropy principle [7], [8], which is the basis of non-extensive statistical mechanics [9], [10], [11], [12].

In accordance to the generalized maximum entropy principle, the equilibrium state of a system is determined by the configuration for which the generalized entropy attains the maximum value, subject to appropriately chosen mean energy constraint, which ensures that the energy average of the system is fixed to a predefined value. The average is usually defined as a linear expectation with respect to the original distribution (referred as the first choice constraint), or with respect to the escort distribution (the third choice constraint) [9]. Thus, Rényi entropy is analyzed in [10], Tsallis entropy in [9], Sharma-Mittal entropy in [11] and Gaussian entropy in [12], were the maximum entropy distributions are derived and it is shown that the corresponding thermodynamic quantities satisfy Legendre structure. On the other hand, although the Legendre structure is satisfied in the general case of entropies and constraints [13], [14], [15], detailed and general analysis of thermostatistics which is based on SPA entropies still has not been conducted.

In this paper we consider generalized thermostatistics for SPA entropies by taking into account linear and escort constraints. We develop a generalized entropy duality principle by which the thermostatistics formalism of Rényi entropy is transformed in the thermostatistics formalism of SPA entropy, we establish the conditions for the transformation which preserve Legendre structure of thermostatistics and we provide the transformation formulas. In addition, we derive the linear to escort energy constraints duality for SPA entropy, by which the generalized thermostatistics which corresponds to the linear constraints can be transformed into the one which corresponds to the escort constraint. Thus, we establish the equivalence between four different thermostatistic formalisms based on Rényi and SPA entropies coupled with linear and escort constraints. In this way we obtain a general framework which is applicable to the wide class of entropies and constraints previously discussed in the literature.

As an example, we rederive maximum entropy distributions for Sharma-Mittal entropy which is coupled with linear [16], [17] and escort constraints [11] and we establish new relationships between the corresponding thermodynamic quantities. We obtain, as special cases, previously developed expressions for maximum entropy distributions and thermodynamic quantities for Tsallis entropy [9], [18], [19], Rényi entropy [20], [10] and Gaussian entropy [12]. In addition, the results are applied for derivation of thermostatistical relationships for supra-extensive entropy [21], which has not previously been considered.

The paper is organized as follows. In Section 2 we present basic facts about SPA entropies, with a special attention to Sharma-Mittal and supra-extensive entropies. In Section 3 we introduce the generalized thermostatistical relationships in the case of generalized entropies. These relationships are further used in Section 4 for the derivation of the Rényi-SPA duality. The entropy constraint duality is considered in Section 5 and the results are combined in Section 6 where we present the equivalence between four different versions of thermostatistics. In Section 7 we derive the thermostatistical relationships among Sharma-Mittal, supra-extensive and Rényi entropies. Concluding remarks are given in Section 8.

2. Preliminaries

2.1. Generalized logarithm, exponential and pseudo-addition

Let the sets of positive and nonnegative real numbers be denoted with $\mathbb{R}^+$ and $\mathbb{R}_0^+$, respectively, and let $h : \mathbb{R} \to \mathbb{R}$ be an increasing continuous (hence invertible) function such that $h(0) = 0$. The generalized logarithm $\log_h : \mathbb{R}_0^+ \to \mathbb{R}$ and its inverse, the generalized exponential, are defined as:

$$\log_h(u) = h(\ln u) \quad \text{and} \quad \exp(v) = \log^{-1}(v) = e^{h^{-1}(v)}.$$  \hspace{1cm} (1)

The pseudo-addition operation $\oplus$ is generated using $h$ as [22]

$$h(x + y) = h(x) \oplus h(y); \quad x, y \in \mathbb{R}. \hspace{1cm} (2)$$

which can be rewritten in terms of the generalized logarithm by settings $x = \log u$ and $y = \log v$ so that

$$\log(u \cdot v) = \log(u) \oplus \log(v); \quad u, v \in \mathbb{R}_+.$$  \hspace{1cm} (3)
An important type of the mapping $h$ is $h_q : \mathbb{R} \rightarrow \mathbb{R}$ defined in

$$h_q(x) = \begin{cases} x, & \text{for } q = 1 \\ \frac{e^{(1-q)x} - 1}{1-q}, & \text{for } q \neq 1 \end{cases}$$

and its inverse is given in

$$h_q^{-1}(x) = \begin{cases} x, & \text{for } q = 1 \\ \frac{1}{1-q} \ln((1-q)x + 1), & \text{for } q \neq 1 \end{cases} \tag{5}$$

with $q > 0$. By the setting $h = h_q$, the generalized logarithm and exponential [1] reduce to the Tsallis $q$-logarithm, defined in

$$\text{Log}_q = \begin{cases} \log x, & \text{for } q = 1 \\ x^{(1-q) - 1} \frac{1}{1-q}, & \text{for } q \neq 1 \end{cases} \tag{6}$$

and $q$-exponential, defined in

$$\text{Exp}_q(y) = \begin{cases} e^y, & \text{for } q = 1 \\ \frac{(1 + (1-q)y)^{\frac{1}{1-q}}}{(1-q)y} & \text{for } q \neq 1 \end{cases} \tag{7}$$

It holds that $h_q(x + y) = h_q(x) \oplus_q h(y)$, where $\oplus_q$ is $q$-addition [23], [24].

2.2. Strongly pseudo-additive entropy

Let the set of all $n$-dimensional distributions be denoted in

$$\Lambda_n = \left\{ (p_1, \ldots, p_n) \mid p_k \in \mathbb{R}_0^+, \sum_{k=1}^n p_k = 1 \right\}; \quad n > 1. \tag{8}$$

Rényi entropy is a function $R_\alpha : \Lambda_n \rightarrow \mathbb{R}$ defined in

$$R_\alpha(P) = \begin{cases} -\sum_{k=1}^n p_k \log(p_k), & \text{for } \alpha = 1 \\ \frac{1}{1-\alpha} \log \left( \sum_{k=1}^n p_k^\alpha \right), & \text{for } \alpha \neq 1 \end{cases} \tag{9}$$

with $\alpha > 0$. Notably, it can be derived as the unique function that satisfies generalized Shannon-Khinchin (SK) axioms, which state that the entropy should be continuous, maximized for uniform distribution, expandable, and strongly additive. The latest property states that the entropy of a joint system can be represented as sum of the entropy of one system and the (generalized) conditional entropy of another, with respect to the first one [25].

An important generalization of Rényi entropies is the class of strongly pseudo-additive (SPA) entropies $H_\alpha$ which can be represented as the $h$ transformation of Rényi entropy [11]:

$$H_\alpha(P) = h(R_\alpha(P)). \tag{10}$$

The class of SPA entropies can uniquely be derived from the generalized Shannon-Khinchin axioms if the strong additivity is replaced with the more general property, the strong pseudo-additivity. More
specifically, let $P = (p_1, \ldots, p_n) \in \Delta_n$, $PQ = (r_{11}, r_{12}, \ldots, r_{nm}) \in \Delta_{nm}$, $n, m \in \mathbb{N}$, $n > 1$ such that $p_i = \sum_{j=1}^{m} r_{ij}$ and $Q_i = (q_{i1}, q_{i2}, \ldots, q_{im}) \in \Delta_{nm}$, where $q_{ij} = r_{ij}/p_i$ and $q \in \mathbb{R}_+$ is some fixed parameter. Then, the strong pseudo-additivity states that \[ H_a(PQ) = H_a(P) \oplus H_a(Q|P), \] where
\[ H_a(Q|P) = f^{-1} \left( \sum_{i=1}^{n} p_i^{(a)} f(H_a(Q_i)) \right), \] (12)

$f$ is an invertible continuous function, $\alpha > 0$ is given parameter and the $\alpha$-escort distribution $P^{(\alpha)} \in \Delta_n$ of the distribution $P \in \Delta_n$ is given in
\[ p_i^{(\alpha)} = (P_1^{(\alpha)}, \ldots, P_n^{(\alpha)}); \quad p_k^{(\alpha)} = \frac{P_k^{\alpha}}{\sum_{i=1}^{n} P_i^{\alpha}}, \quad k = 1, \ldots, n. \] (13)

Obviously, Rényi entropy is a special case of SPA entropies which is obtained if $h = 1$, where 1 stands for the identity function. In the remaining part of the section we review some SPA entropies which were previously considered in statistical physics.

2.2.1. Sharma-Mittal entropy

An important special case of SPA entropies is the Sharma-Mittal entropy \cite{26, 27} which is obtained for $h = h_q$, where $h_q$ is defined in \cite{4}. Thus, the Sharma-Mittal entropy is given by
\[ SM_{\alpha,q}(P) = h_q \left( R_\alpha(P) \right) = \frac{1}{1-q} \left( e^{(1-q)R_\alpha(P)} - 1 \right) \] (14)

and it reduces to the Shannon entropy \cite{28}, for $q = \alpha = 1$, to the Tsallis entropy, for $\alpha = q \neq 1$, to the Gaussian entropy \cite{27}, for $q \neq 1, \alpha = 1$, and to the Rényi entropy \cite{29}, for $q = 1, \alpha \neq 1$. Note that in the case of the mapping $h_q$, it holds that $h_q(x + y) = h_q(x) \oplus_q h(y)$, so that Sharma-Mittal entropy is strongly $q$-additive and it follows the $\phi_q$ decomposition rule \cite{4}.

2.2.2. Supra-extensive entropy

Another important special case of SPA entropies is the supra-extensive entropy \cite{21} which is obtained for $h = s_{\alpha,r}$ where
\[ s_{\alpha,r}(x) = h_\alpha \left( h_r^{-1}(x) \right) = \frac{1}{1-\alpha} \left( (1 + (1-r)x)_{\frac{1}{\alpha}} - 1 \right). \] (15)

Thus the supra-extensive entropy has the form:
\[ SE_{\alpha,r}(P) = s_{\alpha,r} \left( R_\alpha(P) \right) = \frac{1}{1-\alpha} \left( ((1-r)R_\alpha(P) + 1)_{\frac{1}{\alpha}} - 1 \right) \] (16)
and reduces to Rényi entropy for $\alpha = r$ and to Tsallis entropy for $r = 1$.

3. Legendre structure of SPA entropy thermodynamics

Let the energy levels of $n$-states system ($n \in \mathbb{N}$) be represented by the vector $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ and let the probability distribution on states be denoted with $P = (p_1, \ldots, p_n) \in \Delta_n$. The equilibrium distribution $\hat{P} \in \Delta_n$ of a system is determined by the maximum-entropy (ME) principle, as a state for which the entropy
$H_\alpha$ attains the maximum value, subject to the internal energy constraint $U = u(P, E)$, where $u(P, E)$ is an appropriately chosen regular function, which defines the mean value of $E$,

$$\hat{P} = \arg\max_{P \in \Delta_n} \{H_\alpha(P) \mid u(P, E) = U\}, \quad (17)$$

and the maximum entropy value is given by

$$\hat{H}_\alpha = H_\alpha(\hat{P}). \quad (18)$$

Note that the equilibrium distribution (17) and the maximum entropy (18) depend on the choice of the constraint function $u(P, E)$ as well as on internal energy $U$. However, in the following paragraphs we implicitly assume that the function $u(P, E)$ is fixed, so that we will consider $\hat{P}$ and $\hat{H}_\alpha$ as functions of $U$ only and we study the generalized thermostatistics based on the equilibrium entropy (18).

In this paper, we limit our discussion to the systems for which the generalized beta parameter (coldness), defined in

$$\beta_\alpha = \frac{\partial \hat{H}_\alpha}{\partial U}, \quad (19)$$

is a nonzero, continuously differentiable monotonic function of $U$, with the continuous first derivative for all $\alpha > 0$, and we denote the conditions with LSH, i.e.

$$\text{LSH: } \beta_\alpha \in C^1, \quad \beta_\alpha \neq 0 \quad \text{and} \quad \frac{\partial \beta_\alpha}{\partial U} \neq 0, \quad (20)$$

where $C^1$ denotes the space of continuously differentiable functions. Due to the monotonicity of $\beta_\alpha$, $U$ can be uniquely represented as a function of $\beta_\alpha$. Therefore, the generalized entropy (18), which is a function of $U$, can be represented in terms of $\beta_\alpha$ by the substitution $U = U(\beta_\alpha)$. With this settings, we can define generalized log partition function as Legendre transform of generalized entropy, like in the Boltzman-Gibbs thermostatistics:

$$\ln Z_\alpha = \hat{H}_\alpha - \beta_\alpha U \quad (21)$$

In addition, standard relationships also holds in this case

$$\frac{\partial \hat{H}_\alpha}{\partial \beta_\alpha} = \beta_\alpha \frac{\partial U}{\partial \beta_\alpha} \quad \text{and} \quad U = -\frac{\partial \ln Z_\alpha}{\partial \beta_\alpha}, \quad (22)$$

and generalized free energy can be defined (as a function of $\beta_\alpha$) with:

$$F_\alpha = U - \frac{1}{\beta_\alpha} \hat{H}_\alpha = -\frac{1}{\beta_\alpha} \ln Z_\alpha, \quad (23)$$

where the last equality follows from (21). Note that, using (22), we can derive the standard inverse relationship:

$$\hat{H}_\alpha = \beta_\alpha^2 \frac{\partial F_\alpha}{\partial \beta_\alpha}. \quad (24)$$

Moreover, the generalized specific heat capacity can be defined as

$$C_\alpha = -\beta_\alpha \frac{\partial \hat{H}_\alpha}{\partial \beta_\alpha} = -\beta_\alpha^2 \frac{\partial U}{\partial \beta_\alpha}. \quad (25)$$
where the righthand side equalities follow from (22). The equations (19)-(24) describe the Legendre structure of thermodynamics which is based on SPA entropies and is valid for any \( h \) and any \( u \). In the following sections we draw the relationships between the thermodynamics which are derived for \( h = 1 \) (Rényi entropy) and an arbitrary \( h \) (SPA entropy), for two most commonly used types of constraints (linear and escort ones).

**Remark 3.1.** Due to the continuity of \( \beta_\alpha \) and its derivative per \( U \), the LSH conditions (20) can be rewritten as

\[
\beta_\alpha \neq 0 \quad \text{and} \quad C_\alpha \neq 0,
\]

with \( C_\alpha \) being continuous. The LSH conditions ensure that the entropy is monotonic and strictly convex or concave function of internal energy which implies the uniqueness of its Legendre transform in (21). From physical point of view, the conditions should be more restrictive to ensure some desirable properties such as thermodynamic stability [30], [16]. However, the general discussion from the paper will also hold with these more restrictive conditions.

### 4. Duality between Rényi and SPA entropies (R-SPA duality)

Like in the previous section, we consider \( n \)-states system \((n \in \mathbb{N})\) which is represented by the vector \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \) and optimization procedure (17) under the general constraint \( u(P,E) = U \), this time for Rényi entropy (the case \( h = 1 \)). Thus, we consider the optimal distribution:

\[
\hat{P}_\alpha = \arg\max_{P \in \Delta_n} \left\{ R_\alpha(P) \right\} \quad \text{subject to} \quad u(P,E) = U,
\]

for which we indicate the maximum entropy

\[
\hat{R}_\alpha = R_\alpha(\hat{P}_\alpha).
\]

Note that the optimal distribution in the case of SPA entropies

\[
\hat{P}_\alpha = \arg\max_{P \in \Delta_n} \left\{ H_\alpha(P) \right\} \quad \text{subject to} \quad u(P,E) = U,
\]

has the same expression as obtained in (27) since \( H_\alpha = h(R_\alpha) \) and \( h \) is increasing, so that the general relationship between SPA and Rényi entropy (14) also holds in the case of equilibrium and the maximum SPA entropy \( \hat{H}_\alpha = H_\alpha(\hat{P}_\alpha) \) can be expressed in terms of the maximum Rényi entropy (28) and vice versa as

\[
\hat{H}_\alpha = h(\hat{R}_\alpha) \quad \Leftrightarrow \quad \hat{R}_\alpha = h^{-1}(\hat{H}_\alpha).
\]

Following the discussion in Section 3 (by the setting \( h = 1 \)), we are able to define all thermodynamic quantities that corresponds to Rényi entropy and, for notational convenience, we omit the bars and denote them as \( \beta_\alpha, T_\alpha, F_\alpha \) and \( C_\alpha \). In order to ensure the Legendre structure of Rényi thermodynamics we also assume LSH conditions, which are in the case of Rényi entropy denoted with LSR and are given in

\[
\text{LSR:} \quad \beta_\alpha \in C^1, \quad \beta_\alpha \neq 0 \quad \text{and} \quad \frac{\partial \beta_\alpha}{\partial U} \neq 0.
\]

Two questions that naturally arises at this point are:

1. What are the choices of function \( h \) for which the Legendre structure of thermodynamics preserved under the transformation from the Rényi to the SPA formalism and vice versa?
2. What are the transformation formulas for thermodynamic potentials from the Rényi to the SPA formalism and vice versa?
To answer the questions we first derive additional conditions under which the LSH conditions are satisfied, if LSR conditions are assumed. We relate the SPA coldness \( \overline{\beta}_\alpha \) and Rényi coldness \( \beta_\alpha \) by taking the partial derivative of (30) over \( U \) and using the chain rule:

\[
\overline{\beta}_\alpha = h'(\hat{R}_\alpha) \beta_\alpha.
\] (32)

Obviously, the LSH condition, \( \overline{\beta}_\alpha \neq 0 \) directly follows from \( \beta_\alpha \neq 0 \) since the function \( h \) is increasing. However, the second one, i.e. the monotonicity of \( \beta_\alpha \),

\[
\frac{\partial \beta_\alpha}{\partial U} = \frac{\partial \beta_\alpha}{\partial \beta_\alpha} \neq 0,
\] (33)

is satisfied if and only if \( \frac{\partial \beta_\alpha}{\partial \beta_\alpha} \) is continuous and non-zero,

\[
\frac{\partial \beta_\alpha}{\partial \beta_\alpha} = h''(\hat{R}_\alpha) \beta_\alpha + h'(\hat{R}_\alpha) C_\alpha \neq 0,
\] (34)

where the Rényi heat capacity is expressed using the relationship (25),

\[
C_\alpha = -\beta_\alpha \frac{\partial \hat{R}_\alpha}{\partial \beta_\alpha}.
\] (35)

Therefore, if LSR conditions are assumed, LSH conditions also hold, providing that the additional condition (denoted with HC) is satisfied:

\[
HC: \ h'(\hat{R}_\alpha) \neq h''(\hat{R}_\alpha) C_\alpha,
\] (36)

which can be denoted in

\[
LSR + HC \Rightarrow LSH.
\] (37)

In addition, since \( \frac{\partial \beta_\alpha}{\partial \beta_\alpha} \) and \( \frac{\partial \beta_\alpha}{\partial \beta_\alpha} \) are continuous and non-zero simultaneously, the opposite direction can also be derived,

\[
LSH + HC \Rightarrow LSR.
\] (38)

In other words, if HC conditions are assumed, LSH and LSR conditions are equivalent and Legendre structure of the Rényi and SPA formalisms is simultaneously preserved.

Remark 4.1. In general, the satisfiability of the HC condition depends on the choice of the function \( h \), value of Rényi entropy \( \hat{R}_\alpha \) and the heat capacity \( C_\alpha \) and, consequently, on the choice of \( \epsilon \) and \( \alpha \), as well as on the choice of the internal energy constraint \( u(P,E) \). Therefore, in particular cases, it is possible to specify further the properties of \( h \) that ensure the inequality (35) and the equivalence of LSH and LSR conditions. As examples, we consider the following cases.

1. In the case of positive Rényi heat capacity, which usually appears as the requirement for thermodynamic stability [30], [18], Legendre structures of the Rényi and the SPA entropy \( h(\hat{R}_\alpha) \) formalisms are preserved simultaneously for any concave function \( h \) (concavity of \( h \) is a sufficient condition).

2. In the case of \( C_\alpha = 1 \), which appears in 1-dimensional classical harmonic oscillator with two particles [13], concavity of \( h \) is too restrictive. Here, the HC condition has the simple form \( h'(x) \neq h''(x) \). By taking into account the definition of \( h \) from the section 2.1 we can easily obtain that HC requirement is satisfied for any choice of increasing continuous function \( h \) such that \( h(0) = 0 \), which differs from \( c \cdot h_0 \) \((c > 0 \) and \( h_0 \) is defined by the setting \( q = 0 \) in (4)). Note that the case of \( h = c \cdot h_0 \) corresponds to (the multiple of) Sharma-Mittal entropy [14], SM\( _{\alpha,0} \).
3. Oppositely to the case 1, if the Rényi heat capacity is negative, which appears in systems with longrange correlations \[31\], \[30\], Legendre structures is preserved simultaneously for Rényi and SPA thermostatistics for any convex function \(h\) (convexity of \(h\) is a sufficient condition).

In the following discussions, we assume that the HC condition is satisfied, so that LSR and LSH are equivalent and we derive the transformation formulas from one to another formalism.

The Rényi and SPA log partition functions are related using the equation (21), once for the SPA, and once for the Rényi entropy, and by eliminating \(U\) from the equations, so that

\[
\frac{\hat{H}_\alpha - \ln \hat{Z}_\alpha}{\hat{\beta}_\alpha} = \frac{\hat{R}_\alpha - \ln Z_\alpha}{\hat{\beta}_\alpha}.
\]  

(39)

Thus, we obtain

\[
\ln \hat{Z}_\alpha = h(\hat{R}_\alpha) - h'(\hat{R}_\alpha)(\hat{R}_\alpha - \ln Z_\alpha),
\]  

(40)

and, vice versa,

\[
\ln \hat{Z}_\alpha = g(\hat{H}_\alpha) - g'(\hat{H}_\alpha)(\hat{H}_\alpha - \ln \hat{Z}_\alpha),
\]  

(41)

where we set up \(g = h^{-1}\) and used \(30\) and \(32\). The inverse relationships for the free energy can be straightforwardly computed in the same manner, so that

\[
\tilde{F}_\alpha = -\frac{1}{\hat{\beta}_\alpha h'(\hat{R}_\alpha)} \left( \frac{\hat{R}_\alpha}{\hat{\beta}_\alpha} + F_\alpha \right)
\]  

(42)

and, vice versa,

\[
F_\alpha = -\frac{1}{\hat{\beta}_\alpha g'(\hat{H}_\alpha)} \left( \frac{\hat{H}_\alpha}{\hat{\beta}_\alpha} + \tilde{F}_\alpha \right).
\]  

(43)

In addition, by usage of chain rule and inverse function derivative formula, and taking into account the equations \(30\), \(32\), \(34\) and \(35\), we obtain the connection between the corresponding specific heats:

\[
C_\alpha = -\hat{\beta}_\alpha \frac{\partial \hat{H}_\alpha}{\partial \hat{\beta}_\alpha} = -\hat{\beta}_\alpha \frac{\partial \hat{R}_\alpha}{\partial \hat{\beta}_\alpha} \frac{\partial \hat{\beta}_\alpha}{\partial \hat{\beta}_\alpha} = \frac{h'(\hat{R}_\alpha)^2 C_\alpha}{h'(\hat{R}_\alpha) - h''(\hat{R}_\alpha) C_\alpha},
\]  

(44)

or equivalently

\[
(C_\alpha)^{-1} = \frac{h''(\hat{R}_\alpha)}{h'(\hat{R}_\alpha)} + h'(\hat{R}_\alpha)(C_\alpha)^{-1}.
\]  

(45)

Note that the HC condition \(36\) ensures that \(C_\alpha\) and \(\overline{C}_\alpha\) are finite and non-zero simultaneously.

5. Entropy-constraints duality for SPA entropies (E-C duality)

The discussion so far is general and applicable to any type of constraint \(U = u(P, E)\), where \(u(P, E)\) is an appropriately chosen regular function. In this section, we specify the discussion to two most important cases which are considered in statistical mechanics \[9\]:
1. Linear expectation, also referred as the first choice of the constraint: \( u(P, E) = \sum_{i=1}^{n} p_i E_i \), which generate the maximum entropy distribution

\[
\hat{P}_a^L = \left( \hat{p}_1^L(a), \ldots, \hat{p}_n^L(a) \right)
= \arg \max_{P \in \Delta_n} \left\{ H_a(P) \mid \sum_i p_i E_i = U \right\}
\tag{46}
\]

and the equilibrium entropy

\[
\hat{H}_a^L = H_a \left( \hat{P}_a^L \right).
\tag{47}
\]

2. Escort expectation, also referred as the third choice of the constraint: \( u(P, E) = \sum_{i=1}^{n} p_i^{(a)} E_i \), where the \( \alpha \)-escort distribution is defined in \( \ref{13} \). The maximum entropy distribution for the \( \alpha \)-escort constraint is given in

\[
\hat{P}_a^E = \left( \hat{p}_1^E(a), \ldots, \hat{p}_n^E(a) \right)
= \arg \max_{P \in \Delta_n} \left\{ H_a(P) \mid \sum_i p_i^{(a)} E_i = U \right\}
\tag{48}
\]

and the equilibrium entropy is

\[
\hat{H}_a^E = H_a \left( \hat{P}_a^E \right).
\tag{49}
\]

The relationships among the quantities defined using linear and escort constraints can be obtained using entropy-constraint (E-C) duality principle, which can be stated as follows. From equation \( \ref{13} \) it is easy to find the inverse relationship between a distribution \( P \in \Delta_n \) and its escort distribution \( P^{(a)} \in \Delta_n \), which is given in

\[
P = \left( p^{(a)} \right)^{(a^{-1})},
\tag{50}
\]

so that there exists one to one correspondence between two of them, and the optimal escort distribution can be obtained if \( \left( \hat{p}_a^{(a)} \right) \) is searched for in \( \ref{48} \), instead of \( \hat{P}_a^E \). In addition, it is straightforward to show that

\[
H_{a^{-1}} \left( p^{(a)} \right) = H_a \left( P \right) \Leftrightarrow H_a \left( p^{(a^{-1})} \right) = H_{a^{-1}} \left( P \right)
\tag{51}
\]

with \( a > 0 \) and the equation \( \ref{48} \) can be converted into the equivalent one:

\[
\left( \hat{p}_a^{(a)} \right) = \arg \max_{P^{(a)} \in \Delta_n} \left\{ H_a(P) \mid \sum_i p_i^{(a)} E_i = U \right\}
= \arg \max_{P^{(a)} \in \Delta_n} \left\{ H_{a^{-1}}(P^{(a)}) \mid \sum_i p_i^{(a)} E_i = U \right\}.
\tag{52}
\]

After formal substitution \( P^{(a)} \rightarrow P \) into second equality of \( \ref{52} \), we obtain the form \( \ref{46} \) for ME problem of \( H_{a^{-1}} \) with the first choice constraint which has the solution \( \hat{P}_{a^{-1}}^L \). Therefore, we obtain \( \hat{P}_{a^{-1}}^L = \left( \hat{P}_a^L \right)^{(a^{-1})} \), or by taking into account the relationship \( \ref{50} \),

\[
\hat{P}_a^E = \left( \hat{P}_{a^{-1}}^L \right)^{(a^{-1})}.
\tag{53}
\]
Finally if (53) is combined with (51) we obtain the basic equation of E-C duality \( H_\alpha (p^x) = H_{\alpha-1} (p^{x-1}) \), i.e. \( \hat{H}_\alpha^E = \hat{H}_{\alpha-1}^E \). Note that the E-C duality hold for any \( \alpha > 0 \) and any choice of \( h \) (including R\'enyi case \( h = 1 \)), providing that the conditions (20) are satisfied. On the other hand, if the conditions are satisfied for \( \hat{H}_\alpha^E \), they are also satisfied for \( H_{\alpha}^E \) so that the Legendre structure is preserved in both the cases. Therefore, the dualities between all other thermostatistical quantities also hold, so that we get the complete list of the E-C duality relationships:

\[
\begin{align*}
\hat{p}_a^E & = \left( \hat{p}_a^L \right)^{(\alpha-1)}, & \hat{E}_a^E & = \hat{E}_a^L, & \hat{H}_a^E & = \hat{H}_a^L, \\
\hat{C}_a^E & = \hat{C}_a^{L-1}, & \hat{F}_a^E & = \hat{F}_a^{L-1}, & \hat{\beta}_a^E & = \hat{\beta}_a^{L-1}, & \hat{Z}_a^E & = \hat{Z}_a^{L-1}.
\end{align*}
\]

(54)

6. On the equivalence between four versions of generalized thermostatistics

If SPA and R\'enyi entropies are combined with two different types of constrains, four different thermostatistics (TS) can be defined:

- \((R_\alpha, u_L)\)-TS, which is based on R\'enyi entropy with linear constraints,
- \((R_\alpha, u_E)\)-TS, which is based on R\'enyi entropy with escort constraints,
- \((H_\alpha, u_L)\)-TS, which is based on SPA entropy with linear constraints, and
- \((H_\alpha, u_E)\)-TS, which is based on SPA entropy with escort constraints.

The formalisms are related by R-SPA duality derived in Section 4 and E-C duality derived in Section 5. Thus, if we have computed the thermostatistical quantities for \((R_\alpha, u_L)\) thermostatistics, we are able to compute the quantities for \((H_\alpha, u_L)\) thermostatistics using the R-SPA duality, and the quantities for \((R_\alpha, u_E)\) thermostatistics using the E-C duality. Similarly, \((H_\alpha, u_E)\) thermostatistics can be derived from either \((H_\alpha, u_L)\) thermostatistics or \((R_\alpha, u_E)\) thermostatistics. Note that all of the transformations derived in Sections 4 and 5 are invertible, so that the derivation path also exists from \((H_\alpha, u_E)\) to \((R_\alpha, u_L)\) thermostatistics. Thus, starting from any of these thermostatistics we can derive anothers, so that the formalisms are equivalent as represented in Figure 1.

The discussion will be illustrated by considering the maximum entropy distribution \((R_\alpha, u_L)\) thermostatistics which can be derived using Lagrangian optimization [20]:

\[
\hat{p}_i^L(\alpha) = e^{-\hat{K}_i} \left[ 1 - \frac{1}{\alpha} \hat{p}_i^L \Delta E_i \right]^{\frac{1}{\alpha}},
\]

(55)

where

\[
\Delta E_i = E_i - U
\]

(56)

for \( i = 1, \ldots, n \). As we noted before, \( H_\alpha \) and \( R_\alpha \) are optimized by the same maximum entropy distribution (27) although the generalized coldness \( \hat{p}_a^L \) depends on \( h \) and is related to \( \hat{p}_a^E \) in accordance to equation (32). Thus, ME distribution can be represented in the form

\[
\hat{p}_i^L(\alpha) = e^{-\hat{K}_i} \left[ 1 - \frac{1}{\alpha} \hat{p}_i^L \frac{\Delta E_i}{h'(\hat{K}_i^E)} \right]^{\frac{1}{\alpha}},
\]

(57)

for \( i = 1, \ldots, n \), which follows left-hand line of Figure 1 and corresponds to \((H_\alpha, u_L)\)-TS.
The optimal distribution which corresponds to \((R, u_L)\) thermostatistics can be derived following the upper line of the Figure 1. From the equalities (54), we obtain
\[
\hat{p}_i^L(\alpha^{-1}) = e^{-R_{\alpha}^L} \left[ 1 - \frac{\alpha^{-1} - 1}{\alpha^{-1}} \beta_{\alpha}^L \Delta E_i \right]^{\frac{1}{\alpha-1}}
\]
\[
= e^{-R_{E}^L} \left[ 1 - (1 - \alpha) \beta_{\alpha}^E \Delta E_i \right]^{\frac{1}{\alpha-1}}
\] (58)
Since \(\sum_i \hat{p}_i^L(\alpha^{-1})^{\alpha-1} = e^{(1-\alpha^{-1})R_{\alpha}^L} = e^{(1-\alpha^{-1})R_{E}^L}\), the equality (53), can be written as:
\[
\hat{p}_i^E(\alpha) = \frac{\hat{p}_i^L(\alpha^{-1})^{\alpha-1}}{\sum_i \hat{p}_i^L(\alpha^{-1})^{\alpha-1}} = e^{-R_{E}^L} \left[ 1 - (1 - \alpha) \beta_{\alpha}^E \Delta E_i \right]^{\frac{1}{\alpha-1}},
\] (59)
for \(i = 1, \ldots, n\). Finally, the optimal distribution which corresponds to \((H, u_E)\) thermostatistics can be derived in similar manner, following lower or right-hand line in the Figure 1 so that we obtain:
\[
\hat{p}_i^E(\alpha) = e^{-R_{E}^L} \left[ 1 - (1 - \alpha) \frac{\beta_{\alpha}^E \Delta E_i}{h'(\hat{R}_{\alpha})} \right]^{\frac{1}{\alpha-1}},
\] (60)
for \(i = 1, \ldots, n\).

7. Applications

In this section, we illustrate the previous discussion for Sharma-Mittal and supra-extensive thermostatistics and derive their relationships to Rényi one.

7.1. Generalized thermostatistics for Sharma-Mittal entropy

Sharma-Mittal entropy \(\widehat{SM}_{\alpha,q} = h_q(\hat{R}_\alpha)\) is obtained from the equation (14) by the setting \(h = h_q\), where \(h_q\) is given in (4). In this case we can compute
\[
h_q'(\hat{R}_\alpha) = e^{(1-q)\hat{R}_\alpha} = 1 + (1 - q) \widehat{SM}_{\alpha,q}.
\] (61)
As mentioned before, the generalized coldness \(\beta_{\alpha}^E\) depends on the parameter \(\alpha\) and on the mapping \(h\). In the case of the mapping \(h_q\), the dependence is translated into dependence on the pair of the parameters \(\alpha\) and \(q\).
and $q$. Thus, Sharma-Mittal coldness will be denoted as $\beta_{\alpha,q}$ and can be related to the Rényi coldness $\beta_{\alpha}$ for an arbitrary energy constraint $u(P,E) = U$, using the equations (32):

$$\beta_{\alpha,q} = \left(1 + (1 - q)\tilde{S}M_{\alpha,q}\right)\beta_{\alpha}.$$  

(62)

In addition, we have

$$h''(\tilde{R}_\alpha) = (1 - q) e^{(1-q)\tilde{R}} = (1 - q) \left(1 + (1 - q)\tilde{S}M_{\alpha,q}\right)$$  

(63)

with

$$\frac{h'_q(\tilde{R}_\alpha)}{h''(\tilde{R}_\alpha)} = \frac{1}{1 - q},$$  

(64)

so that the HC condition (64), which preserve Legendre structure of Sharma-Mittal thermostatistics, has the form:

$$C_{\alpha,q} = \frac{1}{1 - q}. $$

We can relate Sharma-Mittal heat capacity $C_{\alpha,q}$ with Rényi heat capacity $C_{\alpha}$ using the equation (45):

$$(C_{\alpha})^{-1} = (1 - q) + \left(1 + (1 - q)\tilde{S}M_{\alpha,q}\right)(C_{\alpha,q})^{-1}$$

$$= (1 - q) + e^{1-q}\tilde{R}_\alpha \left(C_{\alpha,q}\right)^{-1}. $$

(65)

Thus, we have derived new relationships which reduce to previously derived special cases of Tsallis entropy [19] (for $\alpha = q$) and Gaussian entropy [12] (for $\alpha = 1$).

The maximum entropy distributions which correspond to Sharma-Mittal entropy can be obtained by the substitution of the expression (61) in the expressions (57) and (60). Therefore, for the first choice theromstatistics we rederive the ME forms from [16] and [17]:

$$p^L_i(\alpha) = e^{-\tilde{R}_\alpha} \left[1 - \frac{\alpha - 1}{\alpha} \frac{\beta_{\alpha,q}^L}{e^{1-q}\tilde{R}_\alpha} \Delta E_i\right]^+, $$

(66)

for $i = 1, \ldots, n$, and for the third choice theromstatistics we obtain the ME form derived [11]:

$$p^E_i(\alpha) = e^{-\tilde{R}_\alpha} \left[1 - (1 - \alpha) \frac{\beta_{\alpha,q}^E}{e^{1-q}\tilde{R}_\alpha} \Delta E_i\right]^+, $$

(67)

for $i = 1, \ldots, n$. In addition, by special choices of the parameters, the ME distributions (66) and (67) reduces to the previously derived ME distributions for Shannon entropy ($\alpha = q = 1$), Rényi entropy ($q = 1$) [10], Tsallis entropy ($\alpha = q$) [9] and Gaussian entropy ($\alpha = 1$) [12].

### 7.2. Generalized thermostatistics for supra-extensive entropy

The presented framework can also be used for the derivation of thermostatistical relationships for the supra-extensive entropy $\tilde{S}E_{\alpha,r} = h_q(\tilde{R}_\alpha)$ introduced in [21], which is obtained if the SPA entropy [10] is defined for $h = s_{\alpha,r}$, where $s_{\alpha,r}$ is given in (15). For the sake of simplicity, we will keep the same notation for generalized coldness and heat capacity as in the previous section. The first derivative of $s_{\alpha,r}$ can be expressed as

$$s'_{\alpha,r}(\tilde{R}_\alpha) = \left((1 - r) \tilde{R}_\alpha + 1\right)^{-1}$$  

(68)

For $i = 1, \ldots, n$. In addition, by special choices of the parameters, the ME distributions (66) and (67) reduces to the previously derived ME distributions for Shannon entropy ($\alpha = q = 1$), Rényi entropy ($q = 1$) [10], Tsallis entropy ($\alpha = q$) [9] and Gaussian entropy ($\alpha = 1$) [12].

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$$s'_{\alpha,r}(\tilde{R}_\alpha) = \left((1 - r) \tilde{R}_\alpha + 1\right)^{-1}$$  

(68)
so that the relationships among the supra-extensive coldness $\beta_{s,r}$ and the Rényi coldness $\beta_{r}$, for an arbitrary energy constraint $U = u(P,E)$, can be derived using the equations (62):

$$\beta_{s,r} = \left((1 - r) \hat{R}_a + 1\right)^{-1} \beta_r.$$  \hspace{1cm} (69)

In addition, the second derivative of $s_{s,r}$ can be expressed as

$$s''_{s,r}(\hat{R}_a) = (r - \alpha)\left((1 - r) \hat{R}_a + 1\right)^{-1}$$  \hspace{1cm} (70)

with

$$\frac{s''_{s,r}(\hat{R}_a)}{s''_{s,r}(\hat{R}_a)} = \frac{(1 - r) \hat{R}_a + 1}{r - \alpha},$$  \hspace{1cm} (71)

so that the HC condition (56), which preserve Legendre structure of Sharma-Mittal thermostatistics, has the form:

$$C_{\alpha} \neq \frac{(1 - r) \hat{R}_a + 1}{r - \alpha}.$$  

The supra-extensive heat capacity $C_{s,r}$ and the Rényi heat capacity $C_{r}$ are related using equation (45):

$$(C_{s,r})^{-1} = \frac{r - \alpha}{(1 - r) \hat{R}_a + 1} + \left((1 - r) \hat{R}_a + 1\right)^{-1} (C_{s,r})^{-1}.$$  \hspace{1cm} (72)

The maximum entropy distributions which correspond to supra-extensive entropy can be obtained by the substitution of the expressions (58) in the expressions (57) and (60). Thus, for the first choice theromstatistics we obtain

$$\hat{P}_i^L(\alpha) = e^{-K_i\left[1 - \frac{\alpha - 1}{\alpha} \left((1 - r) \hat{R}_a + 1\right)^{-1} \beta_{s,r}^L \Delta E\right]}_+,$$  \hspace{1cm} (73)

for $i = 1, \ldots, n$, and for the third choice theromstatistics we obtain

$$\hat{P}_i^E(\alpha) = e^{-K_i\left[1 - (1 - \alpha) \left((1 - r) \hat{R}_a + 1\right)^{-1} \beta_{s,r}^E \Delta E\right]}_+,$$  \hspace{1cm} (74)

for $i = 1, \ldots, n$. Note that all the expressions reduces to Rényi case for $r = \alpha$, and to the Tsallis case for $r = 1$. To the best of our knowledge, these relationships have not previously been derived in the literature and can be used as a base for future applications of supra-extensive entropy.

8. Conclusion

In this paper we considered generalized maximum entropy thermostatistics, under general type of internal energy constraints, for the class of strongly pseudo-additive (SPA) entropies which can be represented as an increasing continuous transformation $h$ of Rényi entropy. We developed a SPA-Rényi entropy duality principle by which the thermostatistics formalism of Rényi entropy $\hat{R}_a$ is transformed in the thermostatistics formalism of SPA entropy $h(\hat{R}_a)$ and established the conditions for the function $h$ which preserve Legendre structure of thermodynamics when passing from one to another formalism. We considered the question what are the choices of function $h$ for which the Legendre structure is preserved and we shown that, in general, in the case of positive Rényi heat capacity, concavity of the function $h$ represents a sufficient condition for the equivalence. In special cases of the Rényi heat capacity equals to unity, the formalisms are shown to be equivalent for any choice of $h$, excluding the case that corresponds to Sharma-Mittal entropy $[11]$ with the non-extenivity parameter $q = 0$. In addition, we derived general entropy-constraint duality, by which
the SPA (and Rényi) thermostatistics which corresponds to the linear constraints can be transformed into the one which corresponds to the escort constraint.

Thus, we established the equivalence between four different thermostatistics formalisms based on Rényi and SPA entropies coupled with linear and escort constraints. Using the equivalence, we provided the transformation formulas from one to another formalisms, we derived corresponding maximum entropy distributions, and we established new relationships between the corresponding thermodynamic potentials and temperatures. In this way we obtained a consistent framework that unifies and generalizes the results for wide class of entropies and constraints previously discussed in the literature, which is illustrated with several examples. As a special case, we derived the expressions for maximum entropy distributions and thermodynamic quantities for Sharma-Mittal entropy class \[11\], which includes previously derived expressions for Tsallis, Rényi and Gaussian entropies. In addition, the results are applied for the derivation of maximum entropy distributions and thermostatistical relationships for supra-extensive entropy \[21\], which have not been considered so far. Presented framework could also be applied for thermostatistical analysis of another SPA entropies not considered here, such as Bekenstein-Hawking entropy \[32\] and super-exponential entropy \[33\], as well as for more general group entropies \[34\], \[22\] \[35\], \[36\], \[37\], \[38\]. The framework can particularly be useful for the analysis of thermodynamic stability of SPA entropies \[18\], \[16\] which will be discussed elsewhere.

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