Dynamics in the Schwarzschild Isosceles Three Body Problem

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Abstract The Schwarzschild potential, defined as $U(r) = -A/r - B/r^3$, where $r$ is the relative distance between two mass points and $A, B > 0$, models astrophysical and stellar dynamics systems in a classical context. In this paper we present a qualitative study of a three mass point system with mutual Schwarzschild interaction where the motion is restricted to isosceles configurations at all times. We retrieve the relative equilibria and provide the energy–momentum diagram. We further employ appropriate regularization transformations to analyze the behavior of the flow near triple collision. We emphasize the distinct features of the Schwarzschild model when compared to its Newtonian counterpart. We prove that, in contrast to the Newtonian case, on any level of energy the measure of the set on initial conditions leading to triple collision is positive. Further, whereas in the Newtonian problem triple collision is asymptotically reached only for zero angular momentum, in the Schwarzschild problem the triple collision is possible for nonzero total angular momenta (e.g., when two of the mass points spin infinitely many times around the center of mass). This phenomenon is known in celestial mechanics as the black-hole effect and is understood as an analog

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in the classical context of behavior near a Schwarzschild black hole. Also, while in the Newtonian problem all triple collision orbits are necessarily homothetic, in the Schwarzschild problem this is not necessarily true. In fact, in the Schwarzschild problem there exist triple collision orbits that are neither homothetic nor homographic.

**Keywords**  Celestial mechanics · Isosceles three-body problem · Schwarzschild model · Singularities · Triple collision

1 Introduction

In 1916 Schwarzschild Schwarzschild (2008) gave a solution to Einstein’s field equations that describes the gravitational field of an uncharged spherical nonrotating mass. It is known that the Schwarzschild metric leads – via a canonical formalism that transposes the relativistic problem into the realm of celestial mechanics (see Eddington 1923) – to a Binet-type equation, which describes motion as governed by a force originating in a potential of the form

\[ U(r) = -\frac{A}{r} - \frac{B}{r^3}, \tag{1} \]

where \( r \) is the relative distance between two mass points and \( A, B > 0 \). The aforementioned potential, which we call the Schwarzschild potential, was brought to the attention of the dynamics and celestial mechanics communities by Mioc et al. in Mioc and Stavinschi (1997) and Stoica and Mioc (1997). Classical dynamics in the Schwarzschild potential has interesting features that are quite distinct when compared to their Newtonian counterparts. In particular, collisions may appear at nonzero angular momenta, giving rise to a so-called black-hole effect, where a particle “falls” into a Schwarzschild source field while spinning infinitely many times around it. This is in contrast to the Newtonian \( N \)-body problem, where collision is possible only if the total angular momentum is zero.

The black-hole effect was introduced in celestial mechanics by Diacu et al. in Diacu et al. (1995), Delgado et al. (1996), and Diacu et al. (2000) (see also Stoica 2000). For the Schwarzschild one-body problem, the existence of the black-hole effect was proven analytically in Stoica and Mioc (1997) by employing a technique due to Mcgehee (1981) for the regularization of the vector field’s singularity at collision. Since then, various studies concerning Schwarzschild one- and two-body problems have appeared (see Mioc et al. 2005, 2003; Stoica 2000; Valls 2010). Recently Campanelli et al. (2008) have simulated numerically a three-black-hole system in the relativistic context and observed that total collision is reached via spiral trajectories. One of the results of the present paper is proving that the same kind of dynamics is feasible in a classical mechanics model.

We consider a particular case of the three-body problem in which two equal point masses \( m_1 = m_2 = M \) are confined to a horizontal plane, symmetrically disposed with respect to their common center of mass \( O \), and a third point mass \( m \) is allowed to move only on the vertical axis perpendicular to the plane of masses \( M \) through point \( O \). At any time, the configuration formed by the three mass points is that of an isosceles triangle (possibly degenerated to a segment), and the only rotations allowed
are with respect to the vertical axis on which \( m \) lies. The dynamics is given, after taking into account the angular momentum conservation, by a two-degree-of-freedom Hamiltonian system. It can be shown that for a three mass point system with two equal masses and with rotationally invariant interactions, isosceles motions form a nontrivial invariant manifold of the three mass point system phase space. We call the constrained three-body problem described above and with mutual Schwarzschild interaction the “isosceles Schwarzschild problem”.

Isoceles three-body problems are often considered case studies for the more complicated dynamics of the \( N \)-body problem. One of the main references is a study by Devaney (1980), where the author employs McGehee’s technique (McGehee 1974) and presents a qualitative description of the planar isosceles three-body problem (i.e., the isosceles three-body problem with zero angular momentum), with an emphasis on orbits that begin or end in a triple collision. In Moeckel (1984) the author uses geometrical methods to construct an invariant set containing a variety of periodic orbits that exhibit close approaches to triple collision and wild changes of configuration. He also finds heteroclinic connections between these periodic orbits, as well as oscillation and capture orbits. Simo and Martinez (1988) apply analytical and numerical tools to study homoclinic and heteroclinic orbits that connect triple collisions to infinity and use homothetic solutions to obtain a characterization of the orbits that pass near triple collision. Elbialy (1989) uses the Euclidean norm (rather than the kinetic mass matrix norm) to perform a McGehee-type change of coordinates to discuss flow behavior near the collision manifold as the ratio between \( m/M \rightarrow 0 \). In a recent study, Mitsuru and Kazuyuki (2009) deduce numerically the existence of infinite families of relative periodic orbits.

The first part of our study follows a classical methodology: we write the Hamiltonian, use the rotational symmetry to reduce the system, and obtain the reduced Hamiltonian as a sum of the kinetic energy and the reduced (amended) potential. The internal parameters are the energy \( h \) and the total angular momentum \( C \). We retrieve the relative equilibria, that is, solutions where the three mass points are steadily rotating about their common center of mass, and discuss their stability modulo rotations.

Recall that in the Newtonian three-body problem there are two classes of relative equilibria (up to a homothety): the Lagrangian class, where the three mass points form an equilateral triangle and the rotation axis is perpendicular to the plane determined by the mass points, and the Eulerian class, where the mass points are in a collinear configuration and the axis of rotation is perpendicular to the line of the mass points at the center of mass. For certain mass ratios the Lagrangian relative equilibria are linearly stable, whereas the Eulerian relative equilibria are always unstable. In the isosceles problem, since the axis of rotation is perpendicular to the line of the equal masses at their center of mass, one retrieves only the Eulerian relative equilibria, which are still unstable, even when considered only in the invariant manifold of isosceles motions. In the Schwarzschild isosceles problem we find no relative equilibria for small angular momenta. For angular momenta larger than a critical value, we find two distinct Eulerian relative equilibria: one unstable, similar to the Newtonian case, and one stable modulo rotations about the vertical axis. This is the case even if \( B \) is small, that is, when the Schwarzschild problem may be considered as a perturbation of the Newtonian model.
The second part of our study focuses on motions near collisions when $m << M$. A similar analysis may be performed for general ratios $m/M$, which we expect would lead to similar conclusions, but we leave this for a future study. Using a McGehee-type transformation (similar as in Devaney 1980) we introduce coordinates that regularize the flow at triple collisions. The triple collision singularity appears as a manifold pasted into the phase space for all levels of energy and angular momentum. This manifold contains fictitious dynamics that is used to draw conclusions about the motions near, or leading to, triple collisions. In our case, the triple collision manifold is not closed and has two edges, which correspond to triple collisions attained through motions in which the equal masses are in a black-hole-type binary collision and the third mass $m$ is on one side of the vertical axis (Fig. 4). The condition $m << M$ ensures the presence of six equilibria on the triple collision manifold, three corresponding to orbits that begin on the manifold, called ejection orbits, and three that end on it, called collision orbits. We analyze the flow on the triple collision manifold, including the stability of equilibria, and deduce that on every energy level the set of initial conditions ejecting from or leading to a triple collision has positive Lebesgue measure. We also find a condition over the set of parameters so that certain orbit connections are satisfied.

It is important to remark that one of the main differences between the Newtonian and Schwarzschild isosceles problem at triple collisions is that in the latter the binary collisions do not regularize as elastic bounces. This is not surprising, in light of previous results on the two-body collision in the problem with nongravitational interactions (see Mcgehee 1981 and Stoica 2000).

It would be interesting to adapt the McGehee-type transformation used by Elbialy (1989) to the context of the Schwarzschild problem. We believe that this would allow for an analysis of the flow at collision for both $m/M \neq 0$ and $m/M = 0$ (i.e., $m = 0$) and that, similarly to the Newtonian case, the collision manifold will be shown to have different topologies for nonzero and zero mass ratios. However, taking into account the remark in the paragraph preceding, it is hard to predict whether other similarities would occur. We defer this problem to a future investigation.

We continue by studying homographic solutions and, implicitly, central configurations. By definition, a homographic solution for a $N$ mass-point system is a solution along which the geometric configuration of the mass points is similar to the initial geometric configuration. There are two extreme cases of homographic motion: if the motion of the mass points is a steady rotation, then the solution is in fact a relative equilibrium; and if the mass points evolve on straight lines through the common center of mass, then the solution is called homothetic. The geometric configuration (within its similarity class) of the points of a homographic solution is called a central configuration if at all times the position vectors are parallel to the acceleration vectors. It can be shown that, for rotationally invariant potentials, there are only two instances where central configurations are possible: either the mass points are in a relative equilibrium, or they are homothetic. Central configurations play an important role in understanding $N$-body systems. Excellent discussion on this subject can be found in Saari (1980) and 2005. For the Newtonian problem, the mass points tend to such central configurations as they approach total collisions; it is worth mentioning that an outstanding open
Table 1  Newtonian versus Schwarzschild dynamics in isosceles problem

| Newtonian isosceles problem | Schwarzschild isosceles problem |
|----------------------------|---------------------------------|
| There is one collinear (Eulerian) relative equilibrium (up to a permutation of the equal masses), which is unstable. | There are two collinear (Eulerian) relative equilibria (up to a permutation of the equal masses), one stable (modulo rotations) and one unstable. |
| On every level of energy, the set of initial conditions leading to a triple collision has zero Lebesgue measure. | On every level of energy, the set of initial conditions leading to a triple collision has positive Lebesgue measure. |
| Triple collision is possible only when \( C = 0 \). | Triple collision is possible for all \( C \). For \( C \neq 0 \), the equal masses display black-hole-type motion. |
| All triple collision orbits are homothetic. | There are triple collision orbits that are not homothetic (also, there are triple collision orbits that are not homographic.) |
| All asymptotic geometric configurations at collision are central configurations. | There are asymptotic geometric configurations at triple collision that are not central configurations. This is true for both \( C = 0 \) and \( C \neq 0 \) cases. |

problem is the finiteness of central configurations in the Newtonian \( N \)-body problem (see Smale 1998).

We show that for the Schwarzschild isosceles problem, homographic motions are confined to the horizontal plane and with \( m \) resting at \( O \) for all times. In particular, the only central configurations are given by the Eulerian relative equilibria and the (collinear) homothetic motions. This is in agreement with the analysis of central configurations for the three-body problem with a generalized Schwarzschild interaction presented by Arredondo et al. in Arredondo and Perez-Chavela (2013).

Finally, we analyze motions near triple collisions. We discuss the asymptotic (limiting) geometric configurations of the solutions corresponding to the ejection/collision orbits as they depart from or tend to triple collision. In the Newtonian problem, such limiting configurations are associated to central configurations as the solutions corresponding to the ejection/collision orbits are homothetic. We prove that this is not the case in the generic Schwarzschild problem. [The nongeneric case is given by condition (35); see also Sect. 4.1.] Moreover, there are limiting triangular configurations that are not even associated to homographic solutions. To our knowledge, this is the first time when such nonhomographic configurations are observed to be limiting configurations at triple collision. We further remark that on any energy level, the set of initial conditions leading to a triple collision with a limiting geometric configuration of a homographic solution is of positive Lebesgue measure. These solutions correspond to motions starting/ending in triple collision, where the mass \( m \) on the vertical axis crosses the horizontal plane infinitely many times before collision (i.e., \( m \) oscillates about the center of mass of the binary equal mass system). The set of initial conditions leading to triple collision with a nondegenerate nonhomographic triangular limiting configuration is of zero Lebesgue measure. In all cases, whenever the angular momentum is not zero, collisions are attained while the equal masses perform black-hole-type motions. We end by proving that for negative energies there is an open set of initial conditions for which solutions end in double (i.e., the collision of the equal masses while \( m \) is above or below the horizontal plane) or triple collisions with \( m \) crossing the horizontal plane a finite number of times.
Our study emphasizes the differences between the Newtonian and the Schwarzschild model. Our findings are summarized in Table 1. As mentioned earlier, one of the most important dissimilarities concerns the approach to total collapse. Besides the presence of the black-hole effect, the Schwarzschild problem displays asymptotic total collision trajectories that are not homographic; in particular, this implies that there exist asymptotic geometric configurations at triple collision that are not central configurations.

To put these conclusions in context, we note that the presence of the so-called strong-force \(-B/r^3\) term in the Schwarzschild potential, which dominates at small distances, leads to the expectation that black-hole effects near collisions are present (see also Stoica 2000); initially, the main goal of this study was to prove this for three mass point interactions. The existence of the nonhomographic triple collision orbits is due to the nonhomogeneity of the potential because it is given by a sum of two homogeneous terms that are taken in a “generic” position (Sect. 3.2). Related work was performed on the three-body problem with quasihomogeneous interactions, a generalization of the Schwarzschild potential of the form \(-A/r^a - B/r^b\), \(1 \leq a < b\). Diacu (1996) found that in the quasihomogeneous three-body problem, the set of collision orbits form asymptotically quasicentral configurations, that is, geometric configurations of orbits that are homographic only with respect to the term \(-B/r^b\) of the potential. Diacu et al. (2006) studied the so-called simultaneous central configuration for quasihomogeneous interactions; these are central configurations arising in a nongeneric case [see Eq. (35)], which we do not consider here. Perez-Chavela and Vela-Arevalo (1998) studied the collinear quasihomogeneous three-body problem (it is assumed that there is no rotation) and proved that the set of initial conditions leading to a triple collision has positive Lebesgue measure. In the same paper the authors show that there are triple collision orbits that are not asymptotic to a central configuration, but these orbits do not originate/terminate in a fixed point on the collision manifold. (They involve infinitely many double collisions of a pair of outer masses.)

The paper is organized as follows. In Sect. 2 we describe the Schwarzschild isosceles problem and reduce the system to a two-degree-of-freedom Hamiltonian system. We further study the relative equilibria and their stability and provide the energy–momentum bifurcation diagram. In Sect. 3 we introduce new coordinates to regularize singularities due to double and triple collisions, define the triple collision manifold, and describe the flow behavior on it. In Sect. 4 we study homographic solutions. Finally, in Sect. 5 we analyze the triple collision/ejection orbits.

## 2 Schwarzschild Isosceles Problem

Consider three point masses with masses \(m_1 = m_2 = M\) and \(m_3 = m\) interacting mutually via a Schwarzschild-type potential. Let \(\mathbf{r}_1\) and \(\mathbf{r}_2\) be the position vectors in Jacobi coordinates, that is, \(\mathbf{r}_1\) is the vector from a particle of mass \(m_1\) to a particle of mass \(m_2\) and \(\mathbf{r}_2\) is the vector from the center of mass of the first two particles to a particle with mass \(m_3\). The associated momenta are \(\mathbf{p}_1\) and \(\mathbf{p}_2\). In these coordinates the respective Hamiltonian is given by
\[ H = \frac{1}{m} p_1^2 + \frac{2M + m}{4Mm} p_2^2 + U_{12}(|r_1|) + U_{13}\left(\frac{1}{2} r_2 + \frac{1}{2} r_1\right) + U_{23}\left(\frac{1}{2} r_2 - \frac{1}{2} r_1\right), \]

(2)

where the \( U_{ij}, i, j = 1, 2, 3, i \neq j \), are Schwarzschild-type potentials. The system is invariant under the diagonal action of the \( SO(3) \) group of spatial rotations on the configuration space \( \mathbb{R}^3 \times \mathbb{R}^3 \setminus \{(r_1, r_2) | r_1 \neq 0\} \), which leads to the conservation of the level sets of the momentum map

\[ J(r_1, r_2, p_1, p_2) = r_1 \times p_1 + r_2 \times p_2. \]

In our modeling, the equal point masses \( M \) are confined to a horizontal plane and are symmetrically disposed with respect to their common center of mass \( O \), and \( m \) is allowed to move on the vertical axis perpendicular to the \( xy \) plane in \( O \). For motions with zero angular momentum, the three masses lie in their initial plane for all times, whereas for motions with nonzero angular momentum the masses \( M \) are rotating about the vertical axis, on which \( m \) lies. The motion is described by a Hamiltonian system that in coordinates \( r_1 = (x_1, y_1, 0), r_2 = (0, 0, z_2) \) and momenta \( p_1 = (p_{x_1}, p_{y_1}, 0), p_2 = (0, 0, p_{z_2}) \), respectively, is

\[ H : \left(\mathbb{R}^3 \setminus \{(x_1, y_1, z_2) | x_1^2 + y_1^2 = 0\}\right) \times \mathbb{R}^3 \rightarrow \mathbb{R} \]

\[ H(x_1, y_1, z_2, p_{x_1}, p_{y_1}, p_{z_2}) = \frac{1}{M} (p_{x_1}^2 + p_{y_1}^2) + \frac{2M + m}{4Mm} p_{z_2}^2 + U(x_1, y_1, z_2), \]

(3)

where the potential has the form

\[ U(x_1, y_1, z_2) = -\frac{A}{\sqrt{x_1^2 + y_1^2}} - \frac{B}{\sqrt{(x_1^2 + y_1^2)^3}} - \frac{4A_1}{\sqrt{x_1^2 + y_1^2 + 4z_2^2}} - \frac{16B_1}{\sqrt{(x_1^2 + y_1^2 + 4z_2^2)^3}}. \]

(4)

The angular momentum integral is given by

\[ C = x_1 p_{x_2} - x_2 p_{x_1}. \]

(5)

Remark 1 In the isosceles Schwarzschild problem there are six (external) parameters: \( M, m \) and \( A, A_1, B, \) and \( B_1 \). Since without loss of generality we could take one of the parameters to be 1 (e.g., one of the masses), there are five independent parameters.

It is convenient to pass to cylindrical coordinates \((x_1, y_1, z_2, p_{x_1}, p_{y_1}, p_{z_2}) \rightarrow (R, \phi, z, P_R, P_\phi, P_z)\) given by the change of coordinates

\[ x_1 = R \cos \phi, \quad y_1 = R \sin \phi, \quad z_2 = z, \]
and its associated (canonical) transformation of the momenta. The Hamiltonian becomes

$$H(R, \phi, z, P_R, P_\phi, P_z) = \frac{1}{M} \left( \frac{P_R^2}{R^2} + \frac{P_\phi^2}{R^2} \right) + \frac{2M+m}{4Mm} P_z^2 + U(R, z),$$

with

$$U(R, z) = -\frac{A}{R} - \frac{B}{R^3} - \frac{4A_1}{\sqrt{R^2 + 4z^2}} - \frac{16B_1}{(R^2 + 4z^2)^{3/2}}.$$  \hspace{1cm} (7)

The equations of motion for the variables $(\phi, P_\phi)$ are

$$\dot{\phi} = \frac{\partial H}{\partial P_\phi} = \frac{2P_\phi}{MR^2}, \quad \dot{P}_\phi = -\frac{\partial H}{\partial \phi} = 0,$$

leading to the explicit equation of the angular momentum conservation

$$P_\phi(t) = \text{const.} =: C.$$  \hspace{1cm} (8)

Using the preceding equation, we obtain a two-degree-of-freedom Hamiltonian system determined by the reduced Hamiltonian

$$H_{\text{red}}(R, z, P_R, P_z; C) := \frac{1}{M} \left( \frac{P_R^2}{R^2} + \frac{C^2}{R^2} \right) + \frac{2M+m}{4Mm} P_z^2 + U(R, z).$$

that is, a system of the form “kinetic + potential”:

$$H_{\text{red}}(R, z, P_R, P_z; C) = \frac{1}{2} (P_R \ P_z) \begin{pmatrix} \frac{2}{M} & 0 \\ 0 & \frac{2M+m}{2Mm} \end{pmatrix} \begin{pmatrix} P_R \\ P_z \end{pmatrix} + U_{\text{eff}}(R, z),$$  \hspace{1cm} (9)

with the effective (or amended) potential given by

$$U_{\text{eff}}(R, z; C) := \frac{C^2}{MR^2} + U(R, z) = \frac{C^2}{MR^2} - \frac{A}{R} - \frac{B}{R^3}$$

$$- \frac{4A_1}{(R^2 + 4z^2)^{1/2}} - \frac{16B_1}{(R^2 + 4z^2)^{3/2}}.$$  \hspace{1cm} (10)

The equations of motion are

$$\dot{R} = \frac{2}{M} P_R, \quad \dot{P}_R = -\frac{\partial H}{\partial R} = -\left( \frac{2C^2}{MR^3} + \frac{A}{R^2} + \frac{3B}{R^4} + \frac{4A_1 R}{(R^2 + 4z^2)^{3/2}} \right) + \frac{3 \cdot 16B_1 R}{(R^2 + 4z^2)^{5/2}},$$

$$\dot{z} = \frac{2M+m}{2Mm} P_z, \quad \dot{P}_z = -\frac{\partial H}{\partial z} = -\left( \frac{4A_1}{(R^2 + 4z^2)^{3/2}} + \frac{3 \cdot 16B_1}{(R^2 + 4z^2)^{5/2}} \right) 4z.$$
Along any integral solution, the energy is conserved:

$$H_{\text{red}}(R(t), z(t), P_R(t), P_z(t); C) = \text{const.} = h.$$  \hfill (11)

**Remark 2** The submanifold

$$\{(R, z, P_R, P_z) \in (0, \infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} | z = 0, P_z = 0\}$$  \hfill (12)

is invariant. Physically, this submanifold contains planar motions, with the two masses \(M\) symmetrically disposed with respect to their midpoint \(O\) where \(m\) rests at all times. The motion on this manifold is the subject of Sect. 4.2.

### 2.1 Relative Equilibria

Following the classical methodology, for nonzero angular momenta, the equilibria of (9) are in fact relative equilibria, that is, dynamical solutions that are also one-parameter orbits of the symmetry group. In our case, relative equilibria correspond to trajectories where the mass points \(M\) are steadily rotating about the vertical \(z\) axis. Note that since \(m\) lies on the \(z\) axis, it does not “feel” such rotations.

**Proposition 1** Consider the spatial isosceles Schwarszchild three-body problem, and let \(C\) be the magnitude of the angular momentum. Without loss of generality let us consider \(C > 0\). (For \(C < 0\) the same results are obtained, but where the spin of the angular momentum is reversed.) Use the notation

$$\alpha := M(A + 4A_1), \quad \beta := M(B + 16B_1),$$  \hfill (13)

and let

$$C_0 := \frac{4}{3} \sqrt{\alpha \beta}.$$  \hfill (14)

Then:

1. If \(C < C_0\), then there are no relative equilibria.
2. If \(C = C_0\), then there is one relative equilibrium, and it is of collinear configuration with the equal point masses situated at

$$R_0 = \frac{C^2}{\alpha}.$$  \hfill (15)

This relative equilibrium is of degenerate stability, having a zero pair of eigenvalues.
3. If \(C > C_0\), then there are two relative equilibria, both of collinear configuration with the equal point masses situated at \((R, z) = (R_i, 0), i = 1, 2\) where

$$R_1 = \frac{C^2 + \sqrt{C^4 - C_0^4}}{\alpha} \quad \text{and} \quad R_2 = \frac{C^2 - \sqrt{C^4 - C_0^4}}{\alpha}.$$  \hfill (16)
The relative equilibrium \((R_1, 0)\) is nonlinearly stable modulo rotations, whereas \((R_2, 0)\) is unstable.

**Proof** The relative equilibria of the system given by the Hamiltonian (9) correspond to the critical points of the effective potential (10):

\[
\frac{\partial U_{\text{eff}}}{\partial R} = \frac{\partial U_{\text{eff}}}{\partial z} = 0.
\]  
(17)

For \(C \geq C_0 = \sqrt[3]{3\alpha\beta}\) we find solutions with \(z = 0\) and \(R = R_{1,2}\), given by the roots of

\[
\alpha R^2 - 2C^2 R + 3\beta = 0.
\]  
(18)

We have

\[
R_{1,2} = \frac{C^2 \pm \sqrt{C^4 - 3\alpha\beta}}{\alpha} = \frac{C^2 \pm \sqrt{C^4 - C_0^4}}{\alpha}.
\]  
(19)

For \(C > C_0\), the nonlinear stability of the relative equilibria may be established by calculating \(D^2 U_{\text{eff}}\) at \((R_{1,2}, 0)\). More precisely, if \(D^2 U_{\text{eff}}\) at one of the relative equilibria is positive definite, then the particular relative equilibrium is nonlinear stable modulo rotations (see Arnold 1989; Meyer et al. 2009). We have

\[
D^2 U_{\text{eff}}|_{z=0} = \begin{pmatrix}
\frac{6C^2}{MR^4} - \frac{2(A + 4A_1)}{R^3} - \frac{12(B + 16B_1)}{R^5} & 0 \\
0 & \frac{16A_1}{R^3} + \frac{192B_1}{R^5}
\end{pmatrix}.
\]

Now we check the positive definiteness of \(D^2 U_{\text{eff}}\) at \((R_1, 0)\) and \((R_2, 0)\). For this we must analyze the behavior of the first entry in the preceding matrix. Let us define

\[
f(R) = -2\alpha R^2 + 6C^2 R - 12\beta
\]  
(20)

and note that

\[
\frac{6C^2}{MR^4} - \frac{2(A + 4A_1)}{R^3} - \frac{12(B + 16B_1)}{R^5} = \frac{f(R)}{MR^5}.
\]

From Eq. (18) we have that any of the roots \(R = R_{1,2}\) verifies

\[
R^2 = \frac{2C^2 R - 3\beta}{\alpha}.
\]

Substituting the preceding equation into \(f(R)\), after some calculations we have

\[
f(R_{1,2}) = \frac{2C^2(C^2 \pm \sqrt{C^4 - 3\alpha\beta}) - 6\alpha\beta}{\alpha}.
\]  
(21)
At $R_1$ we have
\[
f(R_1) = \frac{2(C^4 - 3\alpha\beta)}{\alpha} + \frac{2C^2\sqrt{C^4 - 3\alpha\beta}}{\alpha} > 0,
\]
and so $D_2U_{\text{eff}}|_{z=0,R=R_1}$ is positive definite i.e., the relative equilibrium $(R_1, 0)$ is nonlinear stable.

At $(R_2, 0)$, the Hessian matrix is indefinite, and so it does not give information about its stability. We then calculate the spectral stability of $(R_2, 0)$ by computing the eigenvalues of the matrix linearization
\[
L = \begin{pmatrix}
0 & 0 & \frac{2}{M} & 0 & 0 \\
0 & 0 & \frac{2}{M} & 0 & 2Mm \\
\frac{\partial^2 U_{\text{eff}}}{\partial R^2} & \frac{\partial^2 U_{\text{eff}}}{\partial R \partial z} & 0 & 0 & \frac{2}{M} \\
\frac{\partial^2 U_{\text{eff}}}{\partial R \partial z} & \frac{\partial^2 U_{\text{eff}}}{\partial z^2} & 0 & 0 & 0 \\
\frac{f(R_2)}{MR_2^2} & 0 & \frac{2}{M} & 0 & 2Mm \\
0 & -\left(\frac{16A_1}{R_2^3} + \frac{192B_1}{R_2^5}\right) & 0 & 0 & 0
\end{pmatrix}.
\]

At $z = 0$ these correspond to
\[
\lambda_{1,2} = \pm 4i \sqrt{\frac{2M + m}{2Mm} \left(\frac{16A_1}{R_2^3} + \frac{192B_1}{R_2^5}\right)}
\]
and
\[
\lambda_{3,4} = \pm \sqrt{\frac{2}{M} \left(\frac{f(R_2)}{MR_2^2}\right)} = \pm \frac{2}{M} \sqrt{\frac{1}{R_2^5} (3\beta - C^2 R_2)}.
\]
The eigenvalues $\lambda_{1,2}$ are purely imaginary. A direct calculation shows that $3\beta - C^2 R_2 > 0$, and so $\lambda_{3,4}$ are real. In conclusion, the relative equilibrium $(R_2, 0)$ is unstable.

2.2 Energy–Momentum Diagram

The energy–momentum diagram provides the location of the relative equilibria in the $(h, C)$ parameter space. As is known (see Smale 1970), this is the set of points where the topology of the phase space changes.
In our case, the energy–momentum curve is determined by eliminating $R_i$, $i = 1, 2$, as given by formula (16), from the energy relation at a relative equilibrium (where $P_R = P_z = 0$ and $z = 0$),

$$\frac{C^2}{MR_i^2} - \frac{A + 4A_1}{R_i} - \frac{B + 16B_1}{R_i^3} = h,$$

or using the notation (13)

$$\frac{C^2}{R_i^2} - \frac{\alpha}{R_i} - \frac{\beta}{R_i^3} = h.$$

On this curve there are two points where the relative equilibria curves intersect. The momentum $C$ of these points is given by the equation

$$\frac{C^2}{R_1^2} - \frac{\alpha}{R_1} - \frac{\beta}{R_1^3} = \frac{C^2}{R_2^2} - \frac{\alpha}{R_2} - \frac{\beta}{R_2^3},$$

which we can rewrite as

$$\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\left[ C^2 \left(\frac{1}{R_1} + \frac{1}{R_2}\right) - \alpha - \beta \left(\frac{1}{R_1} + \frac{1}{R_1R_2} + \frac{1}{R_2^2}\right)\right] = 0.$$

An immediate solution is $C = C_0$, where $R_1 = R_2 = R_0$. A second solution is given by the equation

$$C^2 \left(\frac{1}{R_1} + \frac{1}{R_2}\right) - \alpha - \beta \left(\frac{1}{R_1^2} + \frac{1}{R_1R_2} + \frac{1}{R_2^2}\right) = 0,$$

which, after substituting the formulae (16) for $R_1, R_2$, leads to $C = \sqrt[3]{3\alpha\beta} = C_0$. Thus, we deduce that the energy–momentum curve has no self-intersections regardless of the choice of parameters.

A generic graph of the energy–momentum map is presented in Fig. 1.

**Remark 3** Recall that in the Newtonian case (i.e., when $B = B_1 = 0$) the isosceles problem displays collinear (Eulerian) relative equilibria, which are unstable. We observe that in the presence of the inverse cubic terms there are two families of collinear relative equilibria and that one of these is nonlinearly stable. This is the case even if $B$ and $B_1$ are small, and so the inverse cubic terms can be thought of as a perturbation of the Newtonian problem.

### 3 Triple Collision Manifold

In this section we regularize the equations of motion of the isosceles Schwarzschild three-body problem so that the dynamics at triple and double collisions appears on
a fictitious collision invariant manifold. We further discuss the orbit behavior on the collision manifold. Henceforth, unless otherwise stated, we assume that $M \gg m$.

3.1 New Coordinates

To start the study of the dynamics near singularities (i.e., collisions), it is convenient to transform the system associated to the Hamiltonian (9), so that the singularities are regularized. For this we follow closely the McGehee technique as used in the Newtonian isosceles problem by Devaney (see Devaney 1980). Using the notation

$$x := \begin{pmatrix} R \\ z \end{pmatrix}, \quad p := \begin{pmatrix} P_R \\ P_z \end{pmatrix}, \quad \text{and} \quad T = \begin{pmatrix} M^2 & 0 \\ 0 & \frac{2Mm}{2M+m} \end{pmatrix},$$

we introduce the coordinates $(r, v, s, u)$ defined by

$$r = \sqrt{x^T T x}, \quad v = r^{\frac{3}{2}} (s \cdot p),$$

$$s = \frac{x}{r}, \quad u = r^{\frac{3}{2}} (T^{-1} p - (s \cdot p) s).$$

Note that $r = 0$ corresponds to $R = z = 0$, i.e., to the triple collision of the bodies. One may verify that in the new coordinates we have that $s^T T s = 1$ and $s^T T u = 0$. The equations of motion read as follows:

$$\dot{r} = \frac{v}{r^{\frac{3}{2}}},$$

$$\dot{v} = \frac{3}{2} \frac{v^2}{r^{\frac{5}{2}}} + \frac{u^T T u}{r^{\frac{5}{2}}} + \frac{1}{r^{\frac{3}{2}}} \frac{2C^2}{Ms_1^2} - \frac{1}{r^{\frac{1}{2}}} V(s) - \frac{3}{r^{\frac{5}{2}}} W(s),$$
\[ \dot{s} = \frac{u}{r^2}, \]
\[ \ddot{u} = \frac{1}{2} \frac{v}{r^2} u + \left( -\frac{\mathbf{u}' \mathbf{u}}{r^2} - \frac{2C^2}{MS_1^2 r^2} + \frac{1}{r^2} V(s) + \frac{3}{r^2} W(s) \right) s \]
\[ + \frac{1}{r^2} \left( \frac{2}{M} \frac{\partial V}{\partial s_1} + \frac{1}{r^2} \left( \frac{\partial}{\partial s_1} \left( -\frac{2C^2}{M^2 s_1} \right) \right) \right) + \frac{1}{r^2} \left( \frac{2}{2M + m} \frac{\partial W}{\partial s_1} \right), \]
where
\[ V(s) = \frac{A}{s_1^2} + \frac{4A_1}{(s_1^2 + 4s_2^2)^{1/2}} \quad \text{and} \quad W(s) = \frac{B}{s_1^3} + \frac{16B_1}{(s_1^2 + 4s_2^2)^{3/2}}. \]

We further introduce the change of coordinates given by
\[ s = \sqrt{(T^{-1})(\cos \theta, \sin \theta)} \quad \text{and} \quad u = u \sqrt{(T^{-1})(-\sin \theta, \cos \theta)}, \]
where \(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\), so that the boundaries \( \theta = \pm \frac{\pi}{2} \) correspond in the original coordinates to \( R = 0 \), that is, to double collisions of the masses \( M \). More precisely, at \( \theta = \pi/2 \) we have \( R = 0 \) and \( z > 0 \), whereas at \( \theta = -\pi/2 \), \( R = 0 \) and \( z < 0 \). One may easily verify that \( \mathbf{u}' \mathbf{T} \mathbf{u} = u^2 \) and \( \ddot{u} = (\ddot{u}/u)u - u \dot{\theta} \dot{s} \). Using the notation
\[ \mu := \frac{2M + m}{m} \quad \text{(25)} \]
and applying the time reparametrization \( dt = r^5 d\tau \), we obtain the system
\[ r' = rv, \]
\[ v' = \frac{3}{2} v^2 + u^2 + \frac{C^2}{\cos^2 \theta} r - r^2 V(\theta) - 3W(\theta), \quad \text{(26)} \]
\[ \theta' = u, \]
\[ u' = \frac{1}{2} uv - C^2 \frac{\sin \theta}{\cos^2 \theta} r + r^2 \frac{\partial V(\theta)}{\partial \theta} + \frac{\partial W(\theta)}{\partial \theta}, \]
where
\[ V(\theta) = \left( \frac{M}{2} \right)^{1/2} \left( \frac{A}{\cos \theta} + \frac{4A_1}{(\cos^2 \theta + \mu \sin^2 \theta)^{1/2}} \right), \quad \text{(27)} \]
\[ W(\theta) = \left( \frac{M}{2} \right)^{3/2} \left( \frac{B}{\cos^3 \theta} + \frac{16B_1}{(\cos^2 \theta + \mu \sin^2 \theta)^{3/2}} \right). \quad \text{(28)} \]
In the new coordinates the energy integral is given by

\[ hr^3 = \frac{1}{2} \left( u^2 + v^2 \right) + \frac{C^2}{2\cos^2 \theta} r - r^2 V(\theta) - W(\theta). \] (29)

3.2 Potential Functions \( V(\theta) \) and \( W(\theta) \)

Recall that our study considers the case \( M \gg m \) and that we introduced \( \mu := \frac{2M+m}{m} \).

In particular, we have

\[ \mu = 1 + \frac{2M}{m} \gg 1. \] (30)

In addition, we assume \( \mu \) is sufficiently large so that

\[ \mu > 1 + \frac{A}{4A_1}. \] (31)

A direct calculation shows that in this case, \( V(\theta) \) has three critical points at \( \theta_0 = 0 \) and \( \theta = \pm \theta_v \), where

\[ \cos \theta_v = \sqrt{\frac{\mu}{(\mu - 1) + (\mu - 1)^{2/3} \left( \frac{4A_1}{A} \right)^{2/3}}}. \] (32)

Likewise, assuming that

\[ \mu > 1 + \frac{B}{16B_1}, \] (33)

it follows that the function \( W(\theta) \) has three critical points at \( \theta_0 = 0 \) and \( \theta = \pm \theta_w \), where

\[ \cos \theta_w = \sqrt{\frac{\mu}{(\mu - 1) + (\mu - 1)^{2/5} \left( \frac{16B_1}{B} \right)^{2/5}}}. \] (34)

Comparing the expressions of the nonzero critical points of \( V(\theta) \) and \( W(\theta) \), we deduce that in a generic situation these points do not coincide (Fig. 2). The generic case corresponds to the condition

\[ (\mu - 1)^{4/15} \left( \frac{4A_1}{A} \right)^{2/3} \neq \left( \frac{16B_1}{B} \right)^{2/5}. \] (35)

Henceforth, unless otherwise stated, we assume that (30), (31), (33), and (35) are satisfied.

3.3 Regularized Equations of Motion and Triple Collision Manifold

The system (26) is analytic for \( (r, v, \theta, u) \in [0, \infty) \times \mathbb{R} \times (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R} \), and thus orbits at the triple collision \( r = 0 \) are now well defined. To regularize the equations of
motion at double collisions, i.e., at points with \( \theta = \pm \pi/2 \), we make the substitutions

\[
U(\theta) = W(\theta) \cos^3 \theta, \quad w = \frac{\cos^3 \theta}{\sqrt{U(\theta)}} u, \quad (36)
\]

and introduce a new time parametrization given by \( \frac{\cos^3 \theta}{\sqrt{U(\theta)}} = \frac{d\tau}{d\sigma} \).

Note that the function \( U(\theta) > 0 \) for all \( \theta \in [-\pi/2, \pi/2] \) and that \( U(\pm\pi/2) = (M/2)^{3/2} B > 0 \) (Fig. 3). With these transformations system (26) becomes
\[ r' = \frac{\cos^3 \theta}{\sqrt{U(\theta)}} r v, \]
\[ v' = \frac{\cos^3 \theta}{\sqrt{U(\theta)}} \left( \frac{3}{2} v^2 + \frac{U(\theta)}{\cos^6 \theta} w^2 - r^2 V(\theta) - 3 \frac{U(\theta)}{\cos^3 \theta} + \frac{C^2 r}{\cos^2 \theta} \right), \] (37)
\[ \theta' = w, \]
\[ w' = \frac{1}{2} w \frac{\cos^3 \theta}{\sqrt{U(\theta)}} + r^2 V'(\theta) \frac{\cos^6 \theta}{U(\theta)} + \frac{U'(\theta)}{U(\theta)} \left( \cos^3 \theta - \frac{w^2}{2} \right) \]
\[ + 3 \sin \theta \cos^2 \theta - \frac{\sin \theta \cos^3 \theta}{U(\theta)} C^2 r. \]

where the derivation is with respect to the new time \( \tau \), and the energy relation is

\[ 2 h r^3 \cos^6 \theta = U(\theta) w^2 + \left( v^2 \cos^3 \theta - 2 U(\theta) \right) \cos^3 \theta \]
\[ + \left( C^2 - 2 r V(\theta) \cos^2 \theta \right) r \cos^4 \theta. \] (38)

Finally, using the energy relation, we substitute the term containing the angular momentum \( C \) into the \( v' \) equation and obtain

\[ r' = r v \frac{\cos^3 \theta}{\sqrt{U(\theta)}}, \]
\[ v' = \left( \frac{\cos^3 \theta}{2 \sqrt{U(\theta)}} v^2 - \sqrt{U(\theta)} \right) + r^2 \left( 2 h r + V(\theta) \right) \frac{\cos^3 \theta}{\sqrt{U(\theta)}}, \] (39)
\[ \theta' = w, \]
\[ w' = \frac{1}{2} w \frac{\cos^3 \theta}{\sqrt{U(\theta)}} + r^2 V'(\theta) \frac{\cos^6 \theta}{U(\theta)} + \frac{U'(\theta)}{U(\theta)} \left( \cos^3 \theta - \frac{w^2}{2} \right) \]
\[ + 3 \sin \theta \cos^2 \theta - \frac{C^2 r \sin \theta \cos^3 \theta}{U(\theta)}. \]

The vector field (39) is analytic on \([0, \infty) \times \mathbb{R} \times \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \times \mathbb{R} \), and thus the flow is well defined everywhere on its domain, including the points corresponding to triple \((r = 0)\) and double \((\theta = \pm \pi/2)\) collisions. The restriction of the energy relation (38) to \( r = 0 \),

\[ \Delta := \left\{ (r, v, \theta, w) \in [0, \infty) \times \mathbb{R} \times \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \times \mathbb{R} \mid r = 0, w^2 + \frac{\cos^6 \theta}{U(\theta)} v^2 = 2 \cos^3 \theta \right\}, \] (40)

defines a fictitious invariant manifold, called the \textit{triple collision manifold}, pasted into the phase space for any level of energy and angular momentum. By continuity with respect to the initial data, the flow on \( \Delta \) provides information about the orbits that pass close to collision. The triple collision manifold is depicted in Fig. 4. It is a symmetric surface with respect to the horizontal plane \((\theta, w)\) and the vertical plane \((v, w)\), which
has on the top and bottom the profile of the function $W(\theta)$. The vector field on the collision manifold $\Delta$ is obtained by setting $r = 0$ in system (39) and is given by

$$
\begin{align*}
v' &= \left( \frac{\cos^3 \theta}{2 \sqrt{U(\theta)}} v^2 - \sqrt{U(\theta)} \right), \\
\theta' &= w, \\
w' &= \frac{1}{2} v w \cos^3 \theta \sqrt{U(\theta)} + \frac{U'(\theta)}{U(\theta)} \left( \cos^3 \theta - \frac{w^2}{2} \right) + 3 \sin \theta \cos^2 \theta.
\end{align*}
$$

Recall that a vector field is called gradient-like with respect to a function $f$ if $f$ increases along all nonequilibrium orbits. We have the following proposition.

**Proposition 2** The flow over the collision manifold is gradient-like with respect to the coordinate $-v$.

**Proof** On the collision manifold $\Delta$ we have

$$
w^2 + \frac{\cos^6 \theta}{U(\theta)} v^2 = 2 \cos^3 \theta.
$$

Substituting $v^2$ in the expression of $v'$ into system (41) we obtain

$$
v' = -\frac{\sqrt{U(\theta)}}{2 \cos^3 \theta} w^2,
$$

and so $v' < 0$. $\square$

**Proposition 3** (Double collision manifolds) For each $r_0 > 0$ the set

$$
B_{\pm}(r_0) := \left\{ (r, v, \theta, w) \in [0, \infty) \times \mathbb{R} \times \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \times \mathbb{R} \mid r = r_0, \theta = \pm \frac{\pi}{2}, w = 0 \right\}
$$

is an invariant submanifold of the flow of system (39) on which the flow is gradient-like with respect to the coordinate $-v$.

**Proof** From the equations of motion (39), for $\theta = \pm \pi/2$ we have

$$
\begin{align*}
r' &= 0, \\
v' &= -\sqrt{U(\pm \pi/2)} < 0, \\
\theta' &= w, \\
w' &= 0,
\end{align*}
$$

where we take into account that $U'(\pm \pi/2) = 0$. From the energy relation (38) we also have that $U(\pm \pi/2) w^2 = 0$, whence $w \equiv 0$, and so $\theta = \pm \pi/2$ are invariant. $\square$
Remark 4 Physically, motions ending in $B_{\pm}(r_0)$ correspond to the double collision of the masses $M$ while $m$ is located on the vertical axis at $z = \pm r_0 \sqrt{\frac{2M+m}{2Mm}}$ for $\theta = \pm \pi/2$. We call $B_{\pm}(r_0)$ a double collision with $m$ at distance $r_0$.

Remark 5 Asymptotic solutions to $B_{+}(0)$ approach a triple collision via configurations with equal masses close to a double collision and $m$ on the same side of the vertical axis.

As a consequence of the previous propositions, the flow on the collision manifold consists in curves that flow down and either approach asymptotically the invariant sets $B_{\pm}(0)$ or end in one of the equilibrium points (Fig. 4).

3.4 Collision Manifold Equilibria

The equilibria of (39) at points where $r \neq 0$ are equilibria of the flow in the rotational system, and they were examined in Sect. 2.1, Proposition 1.

The flow on the collision manifold accepts fictitious equilibria, which will play an important role in understanding the orbit behavior near singularities. We have the equilibrium points (Fig. 4)

$$Q := (0, \sqrt{2W(0)}, 0, 0), \quad Q^* := (0, -\sqrt{2W(0)}, 0, 0)$$

and

$$E_{\pm} := (0, \sqrt{2W(\theta_w)}, \pm \theta_w, 0), \quad E_{\pm}^* := (0, -\sqrt{2W(\theta_w)}, \pm \theta_w, 0).$$

**Proposition 4** Consider the spatial Schwarzschild isosceles three-body problem with parameters such that (30), (31), (33), and (35) are satisfied. Let $h$ be fixed. Then on
the collision manifold there are the following equilibrium points:

\[ Q := (0, \sqrt{2}W(0), 0, 0), \quad Q^* := (0, -\sqrt{2}W(0), 0, 0) \]

and

\[ E_{\pm} := (0, \sqrt{2}W(\theta_w), \pm \theta_w, 0), \quad E^*_{\pm} := (0, -\sqrt{2}W(\theta_w), \pm \theta_w, 0). \]

Furthermore, the equilibrium \( Q \) is a spiral source with

\[ \text{dim} \mathcal{W}_u(Q) = 3, \]

while the equilibrium \( Q^* \) is a spiral sink with

\[ \text{dim} \mathcal{W}_s(Q^*) = 3. \]

The equilibria \( E_{\pm} \) and \( E^*_{\pm} \) are saddles with

\[ \text{dim} \mathcal{W}_u(E_{\pm}) = 2, \quad \text{dim} \mathcal{W}_s(E_{\pm}) = 1, \]

\[ \text{dim} \mathcal{W}_u(E^*_{\pm}) = 2, \quad \text{dim} \mathcal{W}_s(E^*_{\pm}) = 1. \]

Proof The preceding points are equilibria by direct verification in Eq. (41). Let \( \theta_c \in \{0, -\theta_w, \theta_w\} \). The Jacobian matrix of system (37) evaluated at the equilibrium point \((0, \pm \sqrt{2}W(\theta_c), \theta_c, 0)\) is

\[
J = \begin{pmatrix}
\pm \sqrt{2} \cos^3 \theta_c & 0 & 0 & 0 \\
0 & \pm \sqrt{2} \cos^3 \theta_c & 0 & 0 \\
0 & 0 & 0 & 1 \\
-\sin \theta_c W(\theta_c) C^2 & 0 & \frac{W''(\theta_c)}{W(\theta_c)} \cos^3 \theta_c & \pm \sqrt{\cos^3 \theta_c} \\
\end{pmatrix}
\]  \hspace{1cm} (45)

From the energy relation (38) the level of energy \( h \) is given by

\[
F(r, v, \theta, w) := -2hr^3 \cos^6 \theta + U(\theta) w^2 + \left(v^2 \cos^3 \theta - 2U(\theta)\right) \cos^3 \theta \\
+ \left(C^2 - 2r V(\theta) \cos^2 \theta\right) r \cos^4 \theta = 0.
\]  \hspace{1cm} (46)

The tangent space of this manifold at an equilibrium point \( P \in \{Q, Q^*, E_{\pm}, E^*_{\pm}\} \) is

\[
T_PF = \{(\rho_1, \rho_2, \rho_3, \rho_4) \mid \nabla F\big|_P \cdot (\rho_1, \rho_2, \rho_3, \rho_4) = 0\}
\]

\[
= \{(\rho_1, \rho_2, \rho_3, \rho_4) \mid (C^2 \cos^4 \theta_c)\rho_1 \pm \left(2\sqrt{2}W(\theta_w) \cos^6 \theta_c\right) \rho_2 = 0\}.
\]

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If the angular momentum is zero, i.e., if \( C = 0 \), then we have \( T_p F = \{ (\rho_1, \rho_2, \rho_3, \rho_4) \mid \rho_2 = 0 \} \). The linear part of the vector field (37) restricted to the tangent space is given by

\[
\tilde{J} = \begin{pmatrix}
\pm \sqrt{2} \cos^3 \theta_c & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & \frac{W''(\theta_c)}{W(\theta_c)} \cos^3 \theta_c & \pm \sqrt{\frac{\cos^3 \theta_c}{2}} \\
\end{pmatrix},
\]

so a basis for \( T_p F \) is given by the vectors \( \xi_1 = (1, 0, 0, 0) \), \( \xi_3 = (0, 0, 1, 0) \), and \( \xi_4 = (0, 0, 0, 1) \). A representative of \( \tilde{J} \) in this basis is

\[
\begin{pmatrix}
\pm \sqrt{2} \cos^3 \theta_c & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & \frac{W''(\theta_c)}{W(\theta_c)} \cos^3 \theta_c & \pm \sqrt{\frac{\cos^3 \theta_c}{2}} \\
\end{pmatrix},
\]

From here it follows that for \( P \in \{ Q, E_\pm \} \) we have that \( \xi_1 \) is an eigenvector with an eigenvalue \( \lambda_r := \sqrt{2} \cos^3 \theta_c \). For \( P \in \{ Q^*, E_{\pm}^* \} \) one of the eigenvalues is given by \( \lambda_r := -\sqrt{2} \cos^3 \theta_c \). The other eigenvalues are roots of

\[
\lambda^2 + \sqrt{\frac{\cos^3 \theta_c}{2}} \lambda - \frac{W''(\theta_c)}{W(\theta_c)} \cos^3 \theta_c = 0,
\]

with eigenvectors of the form

\[
\begin{pmatrix}
0 \\
1 \\
\lambda_1 \\
\end{pmatrix}, \quad \begin{pmatrix}
0 \\
1 \\
\lambda_2 \\
\end{pmatrix}.
\]

The eigenvalues at the equilibrium \( Q \) (where \( \theta_c = 0 \) and \( v = \sqrt{2W(0)} \)) are

\[
\lambda_{1,2} = \frac{1}{2} \left( \frac{\sqrt{2}}{2} \pm \sqrt{\frac{25B + 16B_1(1 - 24(\mu - 1))}{2(B + 16B_1)}} \right),
\]

and at \( Q^* \) (where \( \theta_c = 0 \) and \( v = -\sqrt{2W(0)} \))

\[
\lambda_{1,2} = \frac{1}{2} \left( -\frac{\sqrt{2}}{2} \pm \sqrt{\frac{25B + 16B_1(1 - 24(\mu - 1))}{2(B + 16B_1)}} \right).
\]

Given that \( \mu >> 1 \), the quantity under the root is negative. It follows that \( Q \) is a spiral source and \( Q^* \) is a spiral sink. For \( E_{\pm} \) (where \( \theta_c = \pm \theta_w \) and \( v = \sqrt{2W(\theta_w)} \))
the other two eigenvalues are of the form

\[ \lambda_{1,2} = \frac{1}{2} \left( + \sqrt{\frac{\cos^3 \theta_w}{2}} \pm \sqrt{\frac{\cos^3 \theta_w}{2} + \frac{4W''(\theta_w) \cos^3 \theta_w}{W(\theta_c)}} \right). \]

Since \( W''(\theta_w) > 0 \), these points are saddles. Similarly, for \( E_{\pm}^* \) (where \( \theta_c = \pm \theta_w \) and \( v = -\sqrt{2W(\theta_w)} \)) we have

\[ \lambda_{1,2} = \frac{1}{2} \left( - \sqrt{\frac{\cos^3 \theta_w}{2}} \pm \sqrt{\frac{\cos^3 \theta_w}{2} + \frac{4W''(\theta_w) \cos^3 \theta_w}{W(\theta_w)}} \right), \]

and so \( E_{\pm}^* \) are saddles, too.

If the angular momentum is nonzero, i.e., \( C \neq 0 \), then a basis for \( T_P F \) is given by \( \xi_1 = (\pm 2\sqrt{2W(\theta_w)} \cos^6 \theta_c, C^2 \cos \theta_c, 0, 0) \), \( \xi_3 = (0, 0, 1, 0) \), and \( \xi_4 = (0, 0, 0, 1) \).

A representative of \( \bar{J} \) in the \( \{\xi_1, \xi_3, \xi_4\} \) basis is of the form

\[
\left( \begin{array}{ccc}
\pm\sqrt{2} \cos^3 \theta_c & 0 & 0 \\
0 & 0 & 1 \\
0 & U''(\theta_c) \cos^3 \theta_c & \frac{\cos^3 \theta_c}{\sqrt{U(\theta_c)}} \\
\end{array} \right)
\]

and the rest of the proof is identical to that for the case \( C = 0 \). \( \square \)

**Corollary 1** On any energy level the set of initial conditions leading to a triple collision is of positive Lebesgue measure.

**Corollary 2** For the flow restricted to the collision manifold the equilibrium \( Q \) is a spiral source with

\[ \dim \mathcal{W}_t(Q) = 2, \]

the equilibrium \( Q^* \) is a spiral sink with

\[ \dim \mathcal{W}_s(Q^*) = 2, \]

and the equilibria \( E_{\pm} \) and \( E_{\pm}^* \) are saddles with

\[ \dim \mathcal{W}_s(E_{\pm}) = 1, \quad \dim \mathcal{W}_u(E_{\pm}) = 1, \]

\[ \dim \mathcal{W}_s(E_{\pm}^*) = 1, \quad \dim \mathcal{W}_u(E_{\pm}^*) = 1. \]
3.5 Orbit Behavior on Triple Collision Manifold

On the triple collision manifold the flow is gradient-like with respect to the coordinate \(-v\) and has six equilibria, three in the half-space \(v > 0\) and three in the half-space \(v < 0\), symmetrically disposed with respect to the plane \(v = 0\). Given the symmetry 
\[
\theta'(v, \theta, w) = -\theta'(v, -\theta, -w) \quad \text{and} \quad w'(v, \theta, w) = -w'(v, -\theta, -w),
\]
it is sufficient to analyze the flow on the half-space \(\{(v, \theta, w) \in \Delta \mid w > 0\}\). Given the symmetry
\[
\theta'(v, \theta, w) = -\theta'(v, -\theta, -w) \quad \text{and} \quad w'(v, \theta, w) = -w'(v, -\theta, -w),
\]
it is sufficient to analyze the flow on the half-space \(\{(v, \theta, w) \in \Delta \mid w > 0\}\). It is also useful to note that the vector field is invariant under \((v(-\tau), \theta(-\tau), w(-\tau)) \rightarrow (-v(\tau), -\theta(\tau), w(\tau))\).

On the triple collision manifold the equilibrium \(Q\) has a two-dimensional unstable manifold. All orbits emerging from \(Q\) flow down on \(\Delta\) above \(W_u(E^-)\). Looking at \(E^-\), the branch \(W_u(E^-)|_{\{w > 0\}}\) either ends in \(Q^*\) or \(E^*\) and so coincides with \(W_s(E^*)|_{\{w > 0\}}\) or falls in the basin of \(B_+(0)\).

In what follows we give sufficient conditions so that \(W_u(E^-)|_{\{w > 0\}}\) ends in the basin of \(B_+(0)\). This will imply that all orbits emerging from \(Q\) (except for the two ending in \(E^\pm\)) flow into \(B_+(0)\). Also, we show that \(W_u(E^+)|_{\{w > 0\}}\) flows into \(B_+(0)\). This case is represented in Fig. 4.

The flow on the half-space \(w > 0\) may be obtained by substituting \(w\) into the \(\theta'\) equation of system (41) with its expression as defined on the collision manifold Eq. (40). Thus, after rearranging the equation for \(v'\) in (41), it is given by
\[
v' = -\sqrt{U(\theta)} \left( 1 - \frac{\cos^3 \theta}{2U(\theta)} v^2 \right), \quad (50)
\]
\[
\theta' = \sqrt{2 \cos^3 \theta} \left( 1 - \frac{\cos^3 \theta}{2U(\theta)} v^2 \right). \quad (51)
\]

Since \(\theta\) is increasing, for \(\theta \in (-\pi/2, \pi/2)\), we may divide the two equations and obtain the nonautonomous differential equation
\[
\frac{dv}{d\theta} = -\frac{1}{\sqrt{2}} \sqrt{W(\theta) - \frac{1}{2} v^2}, \quad (52)
\]
where we used that \(U(\theta) = W(\theta) \cos^3 \theta\) [see (36)]. The preceding equation has a smooth vector field on the domain
\[
D := \left\{ (\theta, v) : |\theta| < \frac{\pi}{2}, |v| < \sqrt{2W(\theta)} \right\}. \quad (53)
\]

In addition, it is symmetric under \(\theta \rightarrow -\theta\) and \(v \rightarrow -v\). Thus, whenever \(v(\theta)\) is a solution, so is \(-v(-\theta)\). The invariant manifold \(W_u(E^-)|_{\{w > 0\}}\) corresponds to the solution \(\tilde{v}(\theta)\) of (52), which satisfies
\[
\lim_{\theta \rightarrow -\theta_w} \tilde{v}(\theta) = \sqrt{2W(-\theta_w)} = \sqrt{2W(\theta_w)}.
\]

We denote by \(v_1(\theta)\) the integral curve of (52), which passes through zero at \(\theta = 0\), i.e., \(v_1(0) = 0\). In what follows, we will determine the parameter values for which
Fig. 5 Domain $D$ of ordinary differential equation (52) is bounded by $|\theta| < \pi/2$ and $|v| < \sqrt{2W(\theta)}$. The solution $v_1(\theta)$ passes through $(0, 0)$, where its slope is above that of the segment $E_-, E^w_+$. The solution $\tilde{v}(\theta)$, which asymptotically starts in $E_-$, is always above $v_1(\theta)$.

$v_1(\theta_w) > -\sqrt{2W(0)}$. Then we will show that the integral curve $\tilde{v}(\theta)$ is above the integral curve $v_1(\theta)$ for all $\theta > -\theta_w$. This will imply that $\tilde{v}(\theta_w) > -\sqrt{2W(\theta_w)}$, and so, given that $dv/d\theta < 0$, $\tilde{v}(\theta)$ must tend to $-\infty$ as $\theta \to \pi/2$. In particular, we will obtain that $W_u(E_-)|_{w>0} = B_+(0)$.

We start by observing that since

$$\frac{d^2v}{d\theta^2} = -\frac{1}{2\sqrt{2}} \frac{1}{\sqrt{W(\theta) - \frac{v^2}{2}}} \left( W'(\theta) - \frac{dv}{d\theta} \right)$$

$$= -\frac{1}{2\sqrt{2}} \frac{1}{\sqrt{W(\theta) - \frac{v^2}{2}}} \left( W'(\theta) + \frac{v}{\sqrt{2}} \sqrt{W(\theta) - \frac{v^2}{2}} \right),$$

we have that

$$\begin{cases} 
\frac{d^2v}{d\theta^2} < 0 & \text{if } (\theta, v) \in \left\{ (\theta, v) \mid \theta \in (-\theta_w, 0), v \in \left(0, \sqrt{2W(\theta)}\right) \right\}, \\
\frac{d^2v}{d\theta^2} > 0 & \text{if } (\theta, v) \in \left\{ (\theta, v) \mid \theta \in (0, \theta_w), v \in \left(-\sqrt{2W(\theta)}, 0\right) \right\}. 
\end{cases}$$

In other words, any integral curve of (52) is concave down in the upper left quadrant of $D$ and concave up in the lower right quadrant of $D$ (Fig. 5).

**Lemma 1** In the preceding context, if

$$\sqrt{\frac{W(0)}{2}} \leq \frac{\sqrt{2W(\theta_w)}}{\theta_w},$$

then $\tilde{v}(\theta)$ is well defined for all $\theta \in (-\theta_w, \pi/2)$ and $\lim_{\theta \to \pi/2} \tilde{v}(\theta) = -\infty$.

**Proof** Consider (52) and its solution, which passes through $(\theta, v) = (0, 0)$, which we denoted by $v_1(\theta)$. By (55), $v_1(\theta)$ is concave up for $\theta > 0$. It follows that if
\[ \frac{d v_1}{d \theta} \bigg|_{\theta=0} = -\sqrt{\frac{W(0)}{2}} \geq m, \]

where \( m \) is the slope of the segment joining \( E_- \) to \( E_+^* \), then there is \( \varepsilon_0 > 0 \) such that \( v_1(\theta_w) = -\sqrt{2W(\theta_w)} + \varepsilon_0 \). The inequality \( \frac{d v_1}{d \theta} \bigg|_{\theta=0} = -\sqrt{\frac{W(0)}{2}} \geq m \) is insured by the hypothesis condition (56). Since \( v_1(\theta) \) is decreasing all along, it follows that \( v_1(\theta) \) tends to \( B_{\pm}^* \), that is, \( \lim_{\theta \to \pi/2} v_1(\theta) = -\infty \). Given that \( \tilde{v}(\theta) \) is decreasing, \( \tilde{v}(\theta) > v_1(\theta) \) for all \( \theta \) (we observe that \( \tilde{v}(\theta) \) and \( v_1(\theta) \) cannot cross as a consequence of the existence and uniqueness of ordinary differential equation solutions), and \( \lim_{\theta \to \pi/2} v_1(\theta) = -\infty \), we must have \( \lim_{\theta \to \pi/2} \tilde{v}(\theta) = -\infty \).  

**Remark 6** The set of parameters for which (56) is satisfied is nonempty. Indeed, after some computations (sketched subsequently), condition (56) is equivalent to

\[
\cos^2 \left( \frac{1}{1 + \gamma} \left( 1 - \frac{1}{\mu} \right)^{3/4} \left( 1 + \frac{\gamma^{2/5}}{(\mu - 1)^{3/5}} \right)^{5/4} \right) \leq \frac{1}{1 + \gamma} \left( 1 + \frac{\gamma^{2/5}}{(\mu - 1)^{3/5}} \right),
\]

where \( \gamma := \frac{16B_1}{B} \). It can be verified (at least numerically) that for a fixed \( \mu > 1 \) there are values of \( \gamma \) that satisfy (33) and (57). Note that condition (57) is independent of the angular momentum and that for \( \mu \to \infty \), at the limit it becomes \( \cos^2 \left( \sqrt{1/(1 + \gamma)} \right) \leq 1 \).

To see that (56) is equivalent to (57), first we substitute definition (34) for \( \theta_w \) in the expression of \( W(\theta_w) \) and calculate \( W(\theta_w) \), as well as \( W(0) \). We obtain

\[
\theta_w^2 \leq \frac{4W(\theta_w)}{W(0)} = \frac{4\left( (\mu - 1) + (\mu - 1)^{2/5} \gamma^{2/5} \right)^{3/2} \mu^{-3/2} \left( 1 + \gamma^{2/5} (\mu - 1)^{-3/5} \right)}{1 + \gamma}.
\]

After some algebra, the preceding inequality becomes

\[
\theta_w^2 \leq \frac{4}{1 + \gamma} \left( 1 - \frac{1}{\mu} \right)^{3/2} \left( 1 + \frac{\gamma^{2/5}}{(\mu - 1)^{3/5}} \right)^{5/2},
\]

which, given that \( \theta_w \in (0, \pi/2) \), is equivalent to

\[
\cos^2 \theta_w \geq \cos^2 \sqrt{\frac{4}{1 + \gamma} \left( 1 - \frac{1}{\mu} \right)^{3/2} \left( 1 + \frac{\gamma^{2/5}}{(\mu - 1)^{3/5}} \right)^{5/2}}.
\]

After using again (34) and some algebra, the preceding relation can be written as (57).

**Corollary 3** If (57) is fulfilled, then on the triple collision manifold \( W_u(E_-) \big|_{w>0} = B_{\pm}^* \).

**Corollary 4** If (57) is fulfilled, then on the triple collision manifold all orbits emerging from \( Q \) end in \( B_{\pm} \), except for two, which end in \( E_{\pm} \).
4 Aspects of Global Flow

In this section we discuss homothetic solutions (defined subsequently) and use the properties of the flow on the collision manifold to analyze the orbit behavior near double and triple collisions.

4.1 Homographic Solutions

By definition, a homographic solution for an \( N \) mass point system is a solution along which the geometric configuration of the mass points is similar to the initial geometric configuration. If the motion of the mass points is a uniform rotation of the initial configuration (which stays rigid all along), then the solution is in fact a relative equilibrium. If the mass points evolve on straight lines while forming a configuration similar to the initial configuration, then the solution is called homothetic. In general, a homographic solution is a superposition of a dilation and a rotation of the initial geometric configuration of the system. Denoting by \( q := (q_1(t), q_2(t), \ldots, q_N(t)) \in \mathbb{R}^{3N} \) the trajectories of the mass points \( m_i \), it has \( q_i(t) = \varphi(t)\Omega(t)a_i, \ i = 1, 2, \ldots, N, \) for some scalar function \( \varphi(t) \), with \( \text{Im} \varphi \in \mathbb{R} \setminus \{0\} \), some path \( \Omega(t) \in SO(3) \), and some fixed nonzero vector \( (a_1, a_2, \ldots, a_N) \in \mathbb{R}^{3N} \). If the geometric configuration (up to dilations and rotations) of the system is such that at all times the position vectors are parallel to the acceleration vectors, then it is called a central configuration.

For the isosceles three-body problem in Jacobi coordinates \( r_1 = (x_1, y_1, 0) \) and \( r_2 = (0, 0, z_2) \) homographic solutions take the form

\[
\begin{align*}
    r_1(t) &= \varphi(t)\Omega(t)a_1 \\
    r_2(t) &= \varphi(t)a_2
\end{align*}
\]

for some scalar function \( \varphi(t) \), with \( \text{Im} \varphi \in \mathbb{R} \setminus \{0\} \), some rotation \( \Omega(t) \) about the vertical \( OZ \) axis, and a configuration given by the fixed vectors \( a_1 = (a_{1x}, a_{1y}, 0) \) and \( a_2 = (0, 0, a_{2z}) \), where \( (a_1, a_2) \neq (0, 0) \). Use the notation

\[
\Omega(t) = \begin{bmatrix}
\cos \psi(t) & -\sin \psi(t) & 0 \\
\sin \psi(t) & \cos \psi(t) & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

In cylindrical coordinates \( (R, \phi) \), Eqs. (58) are equivalent to

\[
\begin{align*}
    R(t) \cos \phi(t) &= \varphi(t) \left( a_{1x} \cos \psi(t) - a_{1y} \sin \psi(t) \right) \\
    R(t) \sin \phi(t) &= \varphi(t) \left( a_{1x} \sin \psi(t) + a_{1y} \cos \psi(t) \right) \\
    z(t) &= \varphi(t)a_{2z},
\end{align*}
\]

and so

\[
R^2(t) = \varphi^2(t)(a_{1x}^2 + a_{1y}^2) \quad \text{and} \quad z(t) = \varphi(t)a_{2z}.
\]
Passing now to the \((r, \theta)\) coordinates of Eq. (26), we obtain that homographic solutions satisfy

\[
\sqrt{\frac{2}{M}} r(\tau) \cos \theta(\tau) = \varphi(\tau) \sqrt{a_1^2 + d_1^2}, \quad \sqrt{\frac{2M + m}{2Mm}} r(\tau) \sin \theta(\tau) = \varphi(\tau) a_2 z,
\]

where

\[
r(\tau) \neq 0 \text{ for any } \tau.
\]

From the equation of motion of (39), note that if there is a \(\tau_0\) such that \(r(\tau_0) = 0\), then \(r(\tau) \equiv 0\) for all \(\tau\).

If there is a \(\bar{\tau}\) such that \(\theta(\bar{\tau}) = \pi/2\) or \(\theta(\bar{\tau}) = -\pi/2\), then, from (59), at this value we must have \(\varphi(\bar{\tau}) = 0\), which contradicts the fact that \(\text{Im } \varphi \notin \mathbb{R}\{0\}\). Thus, \(\theta(\tau) \neq \pm \pi/2\) for all \(\tau\), and so we can divide Eq. (59) and obtain that the homographic solutions must have \(\tan \theta(\tau)\) constant or, equivalently,

\[
\theta(\tau) = \text{const.} = \theta(0) =: \theta_0.
\]

Given that \(r(\tau) \neq 0\) and \(\theta(\tau) \neq \pm \pi/2\) for all \(\tau\), we can work on system (26) rather than on (39). For the reader’s convenience we rewrite (26):

\[
r' = rv, \quad v' = \frac{3}{2} v^2 + \frac{C^2}{\cos^2 \theta} r - r^2 V(\theta) - 3W(\theta), \quad \theta' = u, \quad u' = \frac{1}{2} uv - \frac{C^2 \sin \theta}{\cos^3 \theta} r + r^2 \frac{\partial V(\theta)}{\partial \theta} + \frac{\partial W(\theta)}{\partial \theta},
\]

with the energy relation

\[
h r^3 = \frac{1}{2} \left( u^2 + v^2 \right) + \frac{C^2}{2 \cos^2 \theta} r - r^2 V(\theta) - W(\theta).
\]

Thus, a homographic solution is a solution of (62) subject to (60) and (61). Now, given (61), using the third equation of (62) we must have \(\theta'(\tau) = u(\tau) = 0\) for all \(\tau\).

Let the initial condition of a homographic solution be \((r_0, v_0, \theta_0, 0)\). Then \((r(\tau), v(\tau))\) must satisfy

\[
r' = rv \quad \text{(64)}
\]

\[
v' = \frac{3}{2} v^2 + \frac{C^2}{\cos^2 \theta_0} r - r^2 V(\theta_0) - 3W(\theta_0), \quad 0 = -C^2 \frac{\sin \theta_0}{\cos^3 \theta_0} r + r^2 V'(\theta_0) + W'(\theta_0), \quad \theta' = u.
\]
together with the energy constraint

\[ hr^3 = \frac{1}{2} v^2 + \frac{C^2}{2 \cos^2 \theta_0} r - r^2 V(\theta_0) - W(\theta_0). \]  

(67)

To solve system (64)–(67), we distinguish the following three cases: (1) \( \theta_0 = 0 \); (2) \( \theta_0 = \pm \theta_v \), i.e., \( \theta_0 \) is a nonzero critical point of \( V(\theta) \); and (3) \( \theta_0 \in (-\pi/2, \pi/2) \setminus \{0, \pm \theta_v\} \).

1. If \( \theta_0 = 0 \), since \( V'(0) = W'(0) = 0 \), then Eq. (66) is identically satisfied. It remains to solve

\[ r' = rv, \]  
\[ v' = \frac{3}{2} v^2 + C^2 r - r^2 V(0) - 3W(0), \]  

with

\[ hr^3 = \frac{1}{2} v^2 + \frac{C^2}{2} r - r^2 V(0) - W(0). \]  

(70)

The solutions of this system describe planar motions that, in the original coordinates, take place on the invariant manifold (12) of planar motions. We analyze this case in detail in the next section.

2. If \( \theta_0 = \theta_v \), then from (66) it follows that

\[ -C^2 \sin \theta_v \cos^3 \theta_v r + W'(\theta_v) = 0. \]  

(71)

(a) If \( C = 0 \), then (71) becomes

\[ W'(\theta_v) = 0, \]

and so (71) is satisfied only if \( \theta_v \) is a critical point of \( W \) as well. This is a nongeneric situation that, as was mentioned in Sect. 3.2, is not considered here.

(b) If \( C \neq 0 \), then from (71) it follows that \( r(\tau) = \text{const.} := r_0 \) for all \( \tau \), and so \( r'(\tau) \equiv 0 \). Using (64) it follows that \( v \equiv 0 \). Further, using (65) we must have

\[ r_0^2 V(\theta_v) - \left( \frac{C^2}{\cos^2 \theta_v} \right) r_0 + 3W(\theta_v) = 0. \]  

(72)

Since \( \theta_v \neq 0 \) and \( r \equiv r_0 \), from (71) we must have

\[ C^2 = \frac{W'(\theta_v) \cos^3 \theta_v}{r_0 \sin \theta_v}. \]
Substituting $C^2$ as previously into (72), after some calculations we obtain
\[
 r_0^2 = \frac{W(\theta_v)}{V(\theta_v)} \frac{\cos \theta_v}{\sin \theta_v} \left( \frac{W'(\theta_v)}{W(\theta_v)} - 3 \frac{\sin \theta_v}{\cos \theta_v} \right). \tag{73}
\]

A tedious but straightforward calculation shows that $\frac{W'(\theta_v)}{W(\theta_v)} - 3 \frac{\sin \theta_v}{\cos \theta_v} < 0$ for $\mu > 1$. Since all of the other terms on the right-hand side are strictly positive, we obtain that $r_0^2 < 0$, which is a contradiction.

(3) $\theta_0 \in (-\pi/2, \pi/2) \setminus \{0, \pm \theta_v\}$. In this case, using (66) we deduce that $r$ is constant. Then, since $r' \equiv 0$, and so $v \equiv 0$, Eq. (65) can be written as
\[
 r^2 V(\theta_0) - \frac{C^2}{\cos^2 \theta_0} r + 3W(\theta_0) = 0. \tag{74}
\]

A necessary and sufficient condition for system (66)–(74) to have solutions is that the coefficients of $r^2$ and $r$ and the free term in the two equations coincide, that is,
\[
 \frac{V'(\theta_0)}{V(\theta_0)} = \tan \theta_0 = \frac{W'(\theta_0)}{3W(\theta_0)}.
\]

The first equality leads to the equation
\[
 \mu \cos \theta_0 (\mu - \cos^2 \theta_0) = 0,
\]
which, since $\mu > 1$, has no solutions.

In conclusion, the only homographic solutions in the Schwarzschild isosceles problem are described by the solutions of system (68)–(70). This is the subject of the next section.

4.2 Planar Motions

For the isosceles problem, due to the symmetry, planar motions are homographic solutions. In our initial setting, these planar motions take place on the plane $z \equiv 0$ and are mentioned in Remark 2. In $(r, v, \theta, u)$ coordinates, planar motions take place on the invariant manifold
\[
 \mathcal{P} := \{(r, v, \theta, u) \mid \theta = 0, u = 0\} \tag{75}
\]
and are described as given by system (68)–(70). This system is a one-degree-of-freedom Hamiltonian system, and thus it is possible to do a full qualitative analysis of the phase space. The relative equilibria calculated in Sect. 2.1 emerge as equilibria of (68)–(69) (scaled by a positive factor). Indeed, due to the attractive nature of the forces, all relative equilibria belong to the plane $\{z = 0\}$, i.e., to the invariant manifold.
A direct calculation shows that $C_0$, the critical value of the angular momentum found in Proposition 1, can be written as

$$ C_0 = \sqrt[3]{12 V(0) W(0)}, $$

and we have:

1. For $C < C_0$, system (68)–(69) has no equilibria with $r \neq 0$;
2. For $C = C_0$, system (68)–(69) has a degenerate equilibrium

$$ r_0 := \left( \frac{C^2}{2 V(0)}, 0 \right). $$

3. For $C > C_0$, system (68)–(69) has two equilibria, with $r \neq 0$ located at

$$ r_1 := \left( \frac{C^2 - \sqrt{C^4 - C_0^4}}{2 V(0)}, 0 \right), \quad r_2 := \left( \frac{C^2 + \sqrt{C^4 - C_0^4}}{2 V(0)}, 0 \right). $$

The equilibrium $r_1$ is a saddle, whereas $r_2$ is a center.

Another class of equilibria is given by $(r, v) = (0, \pm \sqrt{2W(0)})$. These equilibria are independent of the angular momentum level and mark the intersection of the triple collision manifold $\Delta$ with $\mathcal{P}$; in Fig. 4 they correspond to the points $Q$ and $Q^*$, respectively.

We are ready to sketch the phase space of (68)–(69). Using the energy integral (70), we have

$$ v = \pm \sqrt{2h r^3 + 2V(0)r^2 - C^2 r + 2W(0)} $$

and deduce the classification given below.

(A) For $0 \leq C < C_0$ the phase curves are sketched in Fig. 6. For $C = 0$ all motions are rectilinear, with $m$ as the midpoints of the masses $M$. For $0 < C < C_0$ the mass points $M$ spin around the center $m$. We have two subcases given by $h < 0$ and $h \geq 0$.

(a) For $h < 0$, all orbits are bounded. They eject from a triple collision and end in a triple collision. For $C = 0$, the mass points $M$ eject/collide linearly from/into $m$. For $0 < C < C_0$ the dynamics is given by black-hole-type orbits, where the mass points $M$ spin infinitely many times after ejecting from, and before colliding into, the third mass.

(b) For $h \geq 0$, all orbits are unbounded, starting from a triple collision and tending asymptotically to infinity. The asymptotic escape velocities at infinity are zero and strictly positive for $h = 0$ and $h > 0$, respectively.

(B) For $C = C_0$, the phase space is similar to the case where $0 \leq C < C_0$, except that there is a critical energy level $h = h_{cr} < 0$ for which a degenerate relative equilibrium $(r_0, 0)$ lies on the associated phase curve.

(C) For $C > C_0$, we distinguish again the cases $h < 0$ and $h \geq 0$ (Fig. 7).
Fig. 6 Curve solution of Eq. (77) for a fixed angular momentum $0 \leq C < C_0$. For $h < 0$ all the orbits are bounded; for $h = 0$ the orbits are parabolas and for $h > 0$ cubic functions. The plot is generated for values $M = 1$, $A = A_1 = 1$, $B = B_1 = 0.2$, and $C = C_0 - 0.5 \simeq 2.67 - 0.5$. The curves sketched have (starting with the curve from the top and going down) $h = 1$ (blue), $h = 0$ (green), $h = -1$ (red), and $h = -5$ (red).

(a) For $h < 0$ all orbits are bounded. A large set of orbits ejects from a triple collision and ends in a triple collision. Another set of orbits, bounded but of the noncollisional type, is given by those surrounding the center equilibrium $r_2$. There is also a homoclinic orbit that joins $r_1$ to itself.

b) For $h \geq 0$ all orbits are unbounded.

5 Collision/Ejection Orbits

In this section, we describe the orbit behavior near a triple collision by employing the information gathered on the behavior of flow on the collision manifold and the homographic solutions.

Consider a solution of the isosceles Schwarzschild problem that asymptotically tends to a triple collision. (Analogous reasoning can be followed for a solution that asymptotically starts in a triple collision.) Such a solution must tend asymptotically to the triple collision manifold $\Delta$ and, in particular, given that the flow is gradient-like on $\Delta$, to either of the equilibria $Q^*$, $E^*_{\pm}$ or to one of the edges $B_{\pm}$.

Consider the collision orbit to $Q^*$. In the original coordinates, this means that

$$\frac{z}{R} = \sqrt{\frac{2M + m}{2Mm}} \frac{\sin \theta}{\sqrt{\frac{2}{M} \cos \theta}} \to 0,$$
so the masses tend to a coplanar configuration as they approach collision. By Sect. 4.1, any planar motion is homographic. Furthermore, the geometric configuration of such a motion is a central configuration when either the mass points are in a relative equilibrium or the angular momentum is zero (in which case the solution is homothetic). Thus, the limiting configuration of any triple collision orbit ending in $Q^*$ is the geometric configuration (no necessarily central) of a homographic solution. Note that for Newtonian interactions (see Devaney 1980) and, more generally, for interactions given by a homogenous potential, all limiting configurations at a triple collision are central configurations.

If the triple collision orbit ends in $E^*_- (or E^*_+)$, then the limiting configuration of the mass points forms a (nondegenerate) isosceles triangle such that, in the original coordinates,

$$\frac{z}{R} \to \tan \theta_w.$$

As a consequence of the analysis of Sect. 4.1, no homographic solutions have geometric configurations of this form. Thus, the limiting configurations of solutions that asymptotically tend to $E^*_- or E^*_+$ are not configurations associated to homographic solutions. To our knowledge, this is the first time when such so-called nonhomographic configurations are observed to be limiting configurations at a triple collision.
Given the dimensions of the stable and unstable manifolds of $Q^*$ and $E_\pm^*$ in Sect. 3.4, we deduced that on any energy level, the set of initial conditions leading to a triple collision is of positive Lebesgue measure (Corollary 1). By the preceding remarks, we can improve this result as follows.

**Proposition 5** Consider the isosceles Schwarzschild problem. Then, on any energy level, the set of initial conditions leading to a triple collision with a limiting geometric configuration of a homographic solution is of positive Lebesgue measure.

**Proposition 6** Consider the isosceles Schwarzschild problem. Then, on any energy level, the set of initial conditions leading to a triple collision with a nondegenerate triangular limiting configuration is of zero Lebesgue measure.

Having fixed $h$, given that $\dim W_u(Q) = \dim W_s(Q^*) = 3$, most orbits that pass close to Q (or $Q^*$) are in fact triple ejection–triple collision orbits that start in Q and end in $Q^*$; among these we note the unique planar (homographic) orbit that joins Q and $Q^*$. This conclusion is valid for zero and nonzero angular momenta alike. If the momentum is zero, then the motion takes place in the vertical plane of the initial configuration, and all the masses are simply falling into O, the midpoint of the equal masses. If the momentum is nonzero, then the masses approach the triple collision following a scenario where the equal masses are on a black-hole-type trajectory, spinning infinitely many times around their midpoint O, while $m$ oscillates about O on the vertical axis with decreasing amplitude.

Recall that the dynamics in the full space is given by system (39). We are also able to prove the following proposition.

**Proposition 7** Let $h < 0$ be fixed. Then any solution with an initial condition $(r_0, v_0, \theta_0, w_0)$ such that

$$2r_0^2 \tilde{V}(0) < \frac{C^2}{2} \quad \text{and} \quad v_0 < 0,$$

where

$$\tilde{V}(\theta) := V(\theta) \cos \theta,$$

tends either to a double collision manifold $B_\pm(r)$ with $0 < r < r_0$ or to a triple collision manifold (including the sets $B_\pm(0)$).

**Proof** In system (37), in the equation for $v'$, we substitute the term $3/2v^2$ in terms of $r$, $\theta$, and $w$ using the energy relation (38) and obtain

$$v'(\tau) = -\frac{U(\theta)}{2\cos^3 \theta} w^2 + \frac{3h \cos^3 \theta}{\sqrt{U(\theta)}} r^3 + \frac{r}{\sqrt{U(\theta)}} \left(2r^2 V(\theta) \cos^2 \theta - \frac{C^2}{2}\right).$$

Note that $\cos \theta > 0$ for all $\theta \in (-\pi/2, \pi/2)$, and since $U(\theta) > 0$ and $h < 0$, the coefficients of $w^2$ and $r^3$ are negative at all times.
The function $V(\theta) \cos^2 \theta = \tilde{V}(\theta) \cos \theta$ is positive and bounded with

$$0 < V(\theta) \cos^2 \theta \leq \tilde{V}(0) \quad \text{for all} \quad \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Consider a solution with $(r_0, v_0, \theta_0, w_0)$ such that (78) is true. Then, from the first equation of (37),

$$r' = \frac{\cos^3 \theta}{\sqrt{U(\theta)}} rv,$$

$r_0' < 0$, and so $r$ is decreasing. Also,

$$v' = -\frac{U(\theta_0)}{2 \cos^3 \theta_0} w_0^2 + \frac{3h \cos^3 \theta_0}{\sqrt{U(\theta_0)}} r_0^3 + \frac{r_0}{\sqrt{U(\theta_0)}} \left(2r_0^2 V(\theta_0) \cos^2 \theta_0 - \frac{C^2}{2}\right)$$

$$< -\frac{U(\theta_0)}{2 \cos^3 \theta_0} w_0^2 + \frac{3h \cos^3 \theta_0}{\sqrt{U(\theta_0)}} r_0^3 + \frac{r_0}{\sqrt{U(\theta_0)}} \left(2r_0^2 \tilde{V}(0) - \frac{C^2}{2}\right) < 0.$$

Thus, $v$ is decreasing. At some time $\tau > 0$ later, $v(\tau) < v_0 < 0$, and so, since $v$ is negative, $r$ will decrease. We have $r(\tau) < r_0$ and

$$v' = -\frac{U(\theta)}{2 \cos^3 \theta} w^2 + \frac{3h \cos^3 \theta}{\sqrt{U(\theta)}} r^3 + \frac{r}{\sqrt{U(\theta)}} \left(2r^2 V(\theta) \cos^2 \theta - \frac{C^2}{2}\right)$$

$$< -\frac{U(\theta)}{2 \cos^3 \theta} w^2 + \frac{3h \cos^3 \theta}{\sqrt{U(\theta)}} r^3 + \frac{r}{\sqrt{U(\theta)}} \left(2r_0^2 \tilde{V}(0) - \frac{C^2}{2}\right) < 0.$$

Thus, at $\tau$ we have fulfilled again conditions (78), and the solution will continue to have $r$ decreasing and $v'$ negative for all $\tau$. It follows that the solution must tend either to one of the $B_{\pm}(r)$ for some $r$ fixed, $r < r_0$, or to the triple collision manifold. □

**Remark 7** In terms of the parameters, the value of $\tilde{V}(0)$ is given by

$$\tilde{V}(0) = \left(\frac{M}{2}\right)^{1/2} A.$$

**Remark 8** If a solution tends to a $B_{\pm}(r)$, with $r > 0$, then, starting with a $\tau$ large enough, the mass $m$ must remain on the positive or negative side of the vertical axis as $\theta \to \pi/2$ or $\theta \to -\pi/2$, respectively. In particular, $m$ crosses the horizontal plane a finite number of times.

**Corollary 5** Let $h < 0$ be fixed. Then any solution with an initial condition $(r_0, v_0, \theta_0, w_0)$, such that

$$2r_0^2 \tilde{V}(0) < \frac{C^2}{2} \quad \text{and} \quad v_0 < -\sqrt{2W(0)},$$

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tends to one of the $B_{\pm}(r)$, with $0 \leq r < r_0$. If the limit is $B_{\pm}(0)$, then the triple collision is reached after $m$ crosses the horizontal plane a finite number of times.

**Proof** Since $v_0 < -\sqrt{2W(0)}$ and $v$ is decreasing, the motion ends in one of the $B_{\pm}(r)$, with $0 \leq r < r_0$. Further, note that for $r$ small, the only oscillatory motions with $m$ crossing the horizontal plane (or, equivalently, with $\theta$ changing its sign) are near $Q$ and $Q^*$, where $v$ is near $\pm \sqrt{2W(0)}$. But $v_0 < -\sqrt{2W(0)}$ and $v$ is decreasing. In particular, the only possible motion tending to the collision manifold subset $B_{\pm}(0)$, with $v$ much smaller than $-\sqrt{2W(0)}$, must have either $\theta \to \pi/2$ or $\theta \to -\pi/2$, and so for $\tau$ large enough, oscillations are no longer possible. \hfill \Box

**Remark 9** In the conditions of Proposition 7, if $C \neq 0$, then the equal masses collide following a black-hole-type trajectory.

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**References**

Arnold, V.: Mathematical Methods in Classical Mechanics, Nauka, Graduate Texts in Mathematics, vol. 60, 2nd edn. Springer, New York (1989)

Arredondo, J., Perez-Chavela, E.: Central configurations in the Schwarzschild three body problem. Qual. Theory. Dyn. Syst. 12(1), 183–206 (2013)

Campanelli, M., Lousto, C., Zlochower, Y.: Close encounters of three black holes. Phys. Rev. D 77(10), 101501–101506 (2008)

Delgado, J., Diacu, F., Lacombe, E., Mingarelli, A., Mioc, V., Perez-Chavela, E., Stoica, C.: The global flow of the Manev problem. J. Math. Phys. 37, 2748–2761 (1996)

Devaney, R.: Collision in the planar isosceles three body problem. Invent. Math. 60, 249–267 (1980)

Diacu, F.: Near-collision dynamics for particle systems with quasihomogeneous potentials. J. Differ. Equ. 128, 58–77 (1996)

Diacu, F., Perez-Chavela, E., Santoprete, M.: Central configurations and total collisions for quasihomogeneous for n-body problems. Nonlinear Anal. 65(7), 1425–1439 (2006)

Diacu, F., Mingarelli, A., Mioc, V., Stoica, C.: The Manev two-body problem: quantitative and qualitative theory, dynamical systems and applications. World Sci. Ser. Appl. Anal. 4, 213–227 (1995)

Diacu, F., Mioc, V., Stoica, C.: Phase-space structure and regularization of Manev-type problems. Nonlinear Anal. 41, 1029–1055 (2000)

Eddington, A.S.: Mathematical Theory of Relativity. Cambridge University Press, Cambridge (1923)

Elbialy, M.: Triple collisions in the isosceles three body problem with small mass ratio. J. Appl. Math. (ZAMP) 40, 645–664 (1989)

McGehee, R.: Triple collision in the collinear three-body problem. Invent. Math. 27, 191–227 (1974)

Mcgehee, R.: Double collisions for a classical particle system with nongravitational interactions. Comment. Math. Helv. 56(4), 524–557 (1981)

Meyer, K., Hall, G., Offin, D.: Introduction to Hamiltonian Dynamical Systems and the N-Body Problem, 2nd edn. Springer, New York (2009)

Mioc, V., Anisiu, M.C., Barbosu, M.: Symmetric periodic orbits in the anisotropic Schwarzschild-type problem. Celest. Mech. Dyn. Astron. 91(3–4), 269–285 (2005)

Mioc, V., Perez-Chavela, E., Stavinschi, M.: The anisotropic Schwarzschild-type problem, main features. Celest. Mech. Dyn. Astron. 86(1), 81–106 (2003)

Mioc, V., Stavinschi, M.: The Schwarzschild problem: a model for the motion in the solar system. Bull. Astron. Belgrade 156, 21–26 (1997)
Mitsuru, S., Kazuyuki, Y.: Heteroclinic connections between triple collisions and relative periodic orbits in the isosceles three-body problem. Nonlinearity 22, 2377–2403 (2009)
Moeckel, R.: Heteroclinic phenomena in the isosceles three body problem. SIAM J. Math. Anal. 15(5), 857–976 (1984)
Perez-Chavela, E., Vela-Arevalo, L.: Triple collision in the quasi-homogeneous collinear three-body problem. J. Differ. Equ. 148, 186–211 (1998)
Saari, D.: On the role and properties of n-body central configurations. Celest. Mech. 21, 9–20 (1980)
Saari, D.: Collisions, Rings and Other Newtonian N-Body Problems, Regional Conferences Series in Mathematics. American Mathematics Society, Washington (2005)
Schwarzschild, K.: On the gravitational field of a point-mass, according to Einstein’s theory. Abraham Zelmanov J. 1, 10–19 (2008)
Simo, C., Martinez, R.: Qualitative study of the planar isosceles three-body problem. Celest. Mech. 41, 179–251 (1988)
Smale, S.: Topology and mechanics I. Invent. Math. 10, 305–331 (1970)
Smale, S.: Mathematical problems for the next century. Math. Intell. 20, 7–15 (1998)
Stoica, C., Mioc, V.: The Schwarzschild problem in astrophysics. Astrophys. Space Sci. 249, 161–173 (1997)
Stoica, C.: Particle systems with quasihomogeneous interaction. PhD Thesis, University of Victoria (2000).
Valls, C.: On the anisotropic potentials of Manev–Schwarzschild type. Int. J. Bifurc. Chaos Appl. Sci. Eng. 20(4), 1233–1243 (2010)