Quantization of classical integrable systems
Part I: quasi-integrable quantum systems

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Abstract

We propose in this work a concept of integrability for quantum systems, which corresponds to the concept of noncommutative integrability for systems in classical mechanics. We determine a condition for quantum operators which can be a suitable replacement for the condition of functional independence for functions on the classical phase space. This condition is based on the properties of the main parts of the operators with respect to the momenta. We are led in this way to the definition of what we call a “quasi-integrable quantum system”. This concept will be further developed in a series of following papers.

1 Introduction

The theory of quantum systems is similar under many respects to the theory of classical systems. Although there exist some intrinsic differences, many important quantities, such as energy, momentum, angular momentum, potential energy and so on, which play a fundamental role in classical mechanics, are also usefully employed in quantum mechanics after some modifications. To quantities of this type, in the classical case there correspond functions on the phase space of the system. The modification for the quantum case is the following. The state of the system is no longer described by a point in phase space, but by a function defined on configuration space, called “wave function”. Furthermore, by means of some recipe, one associates with a classical function on phase space a linear differential operator acting on the wave function of the system. This correspondence between classical functions and quantum operators is called quantization. One takes as the analogue of classical Poisson brackets between functions the commutators of the corresponding linear operators [1, 2].

One of the main concepts of classical mechanics is that of integrable system. There are several equivalent definitions. In our context the following definition of hamiltonian integrable system will be suitable [3]. Let us assume that the phase space $M^{2n}$ is a 2n-dimensional symplectic manifold, where $n$ is called number of degrees of freedom. We say that the system is integrable if there exists a set of functionally independent real functions $F = (F_1, \ldots, F_k; F_{k+1}, \ldots, F_{2n-k})$ defined on the phase space, with $1 \leq k \leq n$, and this set has the following
The first \( k \) functions are in involution with all functions of the set \( F \), and \( H \) is a function of the \( k \) “central” functions \( F_1, \ldots, F_k \), i.e.

\[
\{ F_i, F_j \} = 0, \quad i = 1, \ldots, k, \quad j = 1, \ldots, 2n - k, \quad (1.1)
\]

\[
H = f(F_1, \ldots, F_k). \quad (1.2)
\]

Here \( \{ , \} \) denote Poisson brackets, and \( f \) is an arbitrary real function of \( k \) variables. The most interesting case is the compact case, i.e., the case in which some connected components of the preimage \( F^{-1}(b) \) of each point \( b \) of the linear space \( \mathbb{R}^{2n-k} \) is a compact set. This case is investigated in detail in [3]. The definition of integrability given above is a generalization of the more usual one based on the well-known Liouville–Arnold theorem [4], which in the above notation corresponds to the case \( k = n \). For \( k = n \) all functions of the set \( F \) are pairwise in involution, which is not true in the cases with \( k < n \). For this reason, the generalized concept of integrability adopted here (see also [5, 6]) is sometimes called “noncommutative integrability”. Some important ideas, related with this type of integrability, can be extended in a useful way also to significant classes of nonintegrable systems [7, 8].

The integrability in the classical case is based on two main conditions: involution and functional independence. The latter means that the differentials of the functions of the set \( F \) are linearly independent at almost all points of the phase space \( M^{2n} \). In section 2 of the present paper some well-known examples of classical integrable systems are reviewed, together with their corresponding quantized systems.

The main aim of section 3 is the construction of an abstract definition of integrable quantum system, and the determination of necessary conditions for integrability. In the literature these issues have been approached in many different ways (see for example [9, 10, 11, 12, 13, 14, 15, 16]). We shall adopt a definition of integrable quantum system which is the direct analogue of the definition of integrable classical system given above. Instead of functions one considers linear differential operators of \( n \) variables \( x = (x_1, \ldots, x_n) \). To the product of functions there corresponds the composition of operators. In the definition of integrable system, the condition of involution is obviously translated to the quantum case by changing the classical Poisson brackets into commutators. However, the condition of functional independence is apparently translated in a less trivial way [14, 17, 18, 19, 20, 21]. We have made an attempt to understand the meaning of this condition in quantum mechanics. To this end, we have started from the concept of algebraic dependence of a set of operators. In order to formulate this condition in a way which is suitable for quantum mechanics, we make use of the concept of “main part” of an operator, i.e., its component of highest degree in the momenta. Let us consider a quantum situation analogous to (1.1)–(1.2). It means that a set of operators \( \mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_k; \mathcal{F}_{k+1}, \ldots, \mathcal{F}_{2n-k}) \) satisfies the commutation relations \( [\mathcal{F}_i, \mathcal{F}_j] = 0 \) for \( i = 1, \ldots, k, \; j = 1, \ldots, 2n - k \), and the Hamiltonian operator \( \mathcal{H} \) of a quantum system is some function of \( \mathcal{F}_1, \ldots, \mathcal{F}_k \). We obviously assume that, in case of algebraic dependence of the set \( \mathcal{F} \), these conditions do not ensure the integrability of the considered quantum system. We are led in this way to introduce the notion of “quasi-integrability” of a quantum system. It is based on the simple concept of “quasi-independence” of the set of operators \( \mathcal{F} \), which is expressed as a property of the main parts of these operators. We prove that the condition of quasi-independence is sufficient to
exclude algebraic dependence.

This paper is the first of a series of four, which are devoted to integrable systems in classical and quantum mechanics. In the second paper of this series (Part II) we will discuss about the mathematical basis of the quantization by symmetrization. Then, in Parts III and IV some classes of concrete integrable classical and quantum systems will be described.

The problem of the correspondence between classical and quantum mechanics has been discussed for a long time and appeared even before the construction of modern quantum mechanics. The present series of works represents an attempt to consider this correspondence from a general perspective, with a particular emphasis on the importance of the concept of noncommutatively integrable system.

2 Simplest examples of correspondence

Let us consider a few examples of integrable classical systems and their corresponding quantum systems.

2.1 The Kepler system

Let $(x, y, z)$ and $(r, \theta, \phi)$ respectively denote the cartesian and polar coordinates in three-dimensional space. The classical Hamiltonian function for the Kepler system is

$$H = \frac{p_x^2 + p_y^2 + p_z^2}{2} - \frac{\alpha}{r} = \frac{1}{2} \left( \frac{p_r^2}{r^2} + \frac{p_\theta^2}{r^2 \sin^2 \theta} + \frac{p_\phi^2}{r^2 \sin^2 \theta} \right) - \frac{\alpha}{r},$$

where $\alpha$ is a real parameter. The configuration space $K$ is the euclidean three-dimensional space without zero: $\mathbb{R}^3 \setminus \{0\}$, and the phase space $M = M^6$ is the cotangent space $T^* K \sim \mathbb{R}^6 \setminus \mathbb{R}^3$ to $K$. The corresponding quantum Hamiltonian operator on $K = \mathbb{R}^3$ is

$$\hat{H} = \frac{\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2}{2} - \frac{\alpha}{r} = \frac{1}{r^2} \left( \hat{p}_r^2 + \frac{2i}{r} \hat{p}_r + \frac{1}{r^2} \hat{p}_\theta^2 - \frac{i \cos \theta}{r^2 \sin \theta} \hat{p}_\theta + \frac{1}{r^2 \sin^2 \theta} \hat{p}_\phi^2 \right) - \frac{\alpha}{r},$$

where $\hat{p}_x = -i \hbar \partial / \partial x$, $\hat{p}_r = -i \hbar \partial / \partial r$, $\hat{p}_\theta = -i \hbar \partial / \partial \theta$ and so on, with $i = \sqrt{-1}$.

The classical system has three integrals of motion which are the components of the angular momentum vector $M = x \times p$. This implies that the orbit is planar. Another integral is for example the angle $\psi$ between the longer axis of the elliptical orbit in configuration space and a fixed axis in the plane of the orbit. We thus obtain the integrable set of functions $(F_1; F_2, F_3, F_4) = (H; M_1, M_2, M_3, \psi)$. For this system we then have $n = 3$ and $k = 1$. For the quantum case the Hamiltonian operator $\hat{H}$ of the system commutes with the vector operator of angular momentum $\hat{M} = \hat{x} \times \hat{p}$. Furthermore, there exists a quantum operator which commutes with $\hat{H}$ and corresponds to the third classical integral $\psi$. Therefore, also this integrable quantum system has $k = 1$.

This is related to the fact that the set of eigenvalues has one natural index, and all of these eigenvalues (except the fundamental one) are degenerate.
2.2 Rotations of a free rigid body

For this system one has $K = SO(3)$, $M^0 = T^*SO(3)$. The hamiltonian function $H$ of the system has the form

$$H = \frac{1}{2} \left( \frac{\Gamma_1^2}{I_1} + \frac{\Gamma_2^2}{I_2} + \frac{\Gamma_3^2}{I_3} \right),$$

where $I_1, I_2, I_3$ are the principal moments of inertia of the rigid body, and $\Gamma_1, \Gamma_2, \Gamma_3$ are the components of the angular momentum vector along the principal axes of the body (see for example [1][4]). The system has the integrals $M_x, M_y, M_z$, which are the components of the angular momentum along the axes of a fixed inertial frame $(x, y, z)$. If $SO(3)$ is parametrized by means of the usual Euler angles $(\phi, \theta, \psi)$, we can write

$$\Gamma_1 = \frac{\sin \psi}{\sin \theta} p_\phi + \cos \psi p_\theta - \cot \theta \sin \psi p_\psi,$$

$$\Gamma_2 = \frac{\cos \psi}{\sin \theta} p_\phi - \sin \psi p_\theta - \cot \theta \cos \psi p_\psi,$$

$$\Gamma_3 = p_\psi,$$

and

$$M_x = -\sin \phi \cot \theta p_\phi + \cos \phi p_\theta + \frac{\sin \phi}{\sin \theta} p_\psi,$$

$$M_y = \cos \phi \cot \theta p_\phi + \sin \phi p_\theta - \frac{\cos \phi}{\sin \theta} p_\psi,$$

$$M_z = p_\phi.$$

The system is integrable with $n = 3$ and $k = 2$, because almost everywhere in $M^0$ a basis of the algebra of first integrals is given by the set of functionally independent functions $(F_1, F_2; F_3, F_4) = (H, M^2; M_x, M_y)$. The corresponding quantum system is obtained by taking as hamiltonian operator

$$\hat{H} = \frac{1}{2} \left( \frac{\hat{\Gamma}_1^2}{I_1} + \frac{\hat{\Gamma}_2^2}{I_2} + \frac{\hat{\Gamma}_3^2}{I_3} \right),$$

where

$$\hat{\Gamma}_1 = \frac{\sin \psi}{\sin \theta} \hat{p}_\phi + \cos \psi \hat{p}_\theta - \cot \theta \sin \psi \hat{p}_\psi,$$

$$\hat{\Gamma}_2 = \frac{\cos \psi}{\sin \theta} \hat{p}_\phi - \sin \psi \hat{p}_\theta - \cot \theta \cos \psi \hat{p}_\psi,$$

$$\hat{\Gamma}_3 = \hat{p}_\psi,$$

and $\hat{p}_\phi = -i\hbar \partial/\partial \phi$, etc.

2.3 Symmetrical top in a gravitational field

The configuration and phase spaces for a rigid body in the presence of a gravitational field are the same as for a free one. However, the corresponding hamiltonian system is integrable only if the rigid body has a symmetry axis. We have
in this case \( n = k = 3 \), and a basis of the algebra of first integrals is given by the set of functionally independent functions \( (F_1, F_2, F_3) = (H, M_2, \Gamma_3) \). Here \( M_3 \) is the component of the angular momentum along the coordinate axis \( z \) parallel to the direction of the gravitational field, while \( \Gamma_3 \) is the component of the angular momentum along the symmetry axis of the body. The quantization is directly obtained as in the previous case.

### 3 Quasi-integrable quantum systems

#### 3.1 Local and global independence of differential operators of finite order

Let \( \hat{p}_i \) denote the differential operator \( \hat{p}_i := \partial/\partial x_i, i = 1, \ldots, n \). We denote the operator of multiplication by a function \( f(x) \) with the same symbol \( f(x) \). We consider functions \( f \in C^\infty(K) \), where \( K \) is an open subset of \( \mathbb{R}^n \). Typically \( \mathbb{R}^n \setminus K \) is a “thin” subset. For example, for the Kepler problem we have \( K = \mathbb{R}^3 \setminus \{0\} \), and for the two-body problem \( K = \mathbb{R}^6 \setminus \{x = y\} \), where \( x = (x_1, x_2, x_3) \), \( y = (y_1, y_2, y_3) \) are the position vectors of the two bodies in three-dimensional space, and \( (x, y) \in \mathbb{R}^3 \). Let us consider all the linear operators which are finite sums of finite compositions of differential operators \( p_i, i = 1, \ldots, n \), and arbitrary multiplication operators \( f(x) \). The algebra of such operators we denote by \( \mathcal{O} = \mathcal{O}_K \). Note that we are here considering only real operators.

Let us consider a linear differential operator \( \mathcal{F} \) on \( K \subset \mathbb{R}^n \) of the form

\[
\mathcal{F} = \sum_{|\alpha| \leq m} A_\alpha(x) \hat{p}^\alpha, \tag{3.1}
\]

where \( m \in \mathbb{Z}_+ \), \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n \), \( \mathbb{Z}_+ := \{0, 1, 2, \ldots\} \), and \( |\alpha| := \alpha_1 + \ldots + \alpha_n \). We have equivalently \( \mathbb{Z}_+^n := \mathbb{Z}^n \cap \mathbb{R}_+^n \), \( \mathbb{R}_+ := \{\alpha \in \mathbb{R}^n : \alpha_i \geq 0 \text{ for } i = 1, \ldots, n\} \), where \( \mathbb{Z}^n \) is the subsets of vectors of \( \mathbb{R}^n \) with integer components. Here \( \hat{p}^\alpha := \partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n} \). We suppose that \( A_\alpha(x) \in C^\infty(K) \) for any \( \alpha \), that is \( A_\alpha(x) \) is an infinitely differentiable function on \( K \), \( A_\alpha : K \to \mathbb{R} \).

**Definition 3.1.** The operators of the algebra \( \mathcal{O}_K \) we call operators of class \( \mathcal{O} \). We say that two operators \( \mathcal{A} \) and \( \mathcal{B} \) of class \( \mathcal{O} \) are equal to each other if \( \mathcal{A}\psi = \mathcal{B}\psi \) for any function \( \psi \in C^\infty(K) \).

**Proposition 3.1.** An operator \( \mathcal{F} \) of the form (3.1) is an operator of class \( \mathcal{O} \). Conversely, any operator of class \( \mathcal{O} \) can be represented in a unique way in the form (3.1).

**Proof.** The first part of the proposition is obvious. The fact that an operator of class \( \mathcal{O} \) can be always represented in the form (3.1) follows easily from the relation

\[
\hat{p}_i f(x) = f(x) \hat{p}_i + \frac{\partial}{\partial x_i} f(x), \tag{3.2}
\]

which is a consequence of Leibniz rule of differentiation. In order to prove that the representation of type (3.1) is unique, let us consider the family of functions \( \psi = e^{\lambda x}, \lambda \in \mathbb{R}^n, \lambda x := \lambda_1 x_1 + \cdots + \lambda_n x_n \). It follows from (3.1) that

\[
\mathcal{F} e^{\lambda x} = \sum_{|\alpha| \leq m} A_\alpha(x) \lambda^\alpha e^{\lambda x}. \tag{3.3}
\]
where $\lambda^\alpha := \lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n}$. According to definition 3.1, $F = 0$ implies $F e^{\lambda x} = 0$ for every $\lambda \in \mathbb{R}^n$ and every $x \in K$. Then from (3.3) we derive that $F = 0$ if and only if $A_\alpha(x) = 0$ for every $\alpha \in \mathbb{Z}_+^n$ and every $x \in K$. This obviously implies that a representation of the form (3.1) is unique for any operator of class $\mathcal{O}$. 

**Definition 3.2.** We call formula (3.1) the *standard representation* of the operator $F$. The function $F : K \times \mathbb{R}^n \to \mathbb{R}$ defined as

$$F(x, p) = \sum_{|\alpha| \leq m} A_\alpha(x) p^\alpha,$$  

where $p^\alpha := p_1^{\alpha_1} \cdots p_n^{\alpha_n}$, is called the *symbol* of the operator $F$. We shall also use the notation $F^{\text{ymb}}$ for $F$, that is $F^{\text{ymb}} = F$. It is a polynomial function of order $\leq m$ in the variables $p = (p_1, \ldots, p_n)$.

Conversely, given a function $F : K \times \mathbb{R}^n \to \mathbb{R}$ of the form (3.4), we say that the differential operator $F$ on $K \subset \mathbb{R}^n$ defined by (3.1) is the *standard quantization* of $F$.

**Definition 3.3.** Given an operator $F$ of class $\mathcal{O}$ having the standard representation (3.1), we call the *homogeneous part of order $g \in \mathbb{Z}_+^n$* of the operator $F$ the operator

$$H_g(F) := \begin{cases} \sum_{|\alpha| = g} A_\alpha(x) \hat{p}^\alpha & \text{if } 0 \leq g \leq m \\ 0 & \text{if } g > m. \end{cases}$$

We define in a similar way the homogeneous part $H_g(F)$ of order $g$ of the symbol $F = F^{\text{ymb}}$. Therefore, if $F = F^{\text{ymb}}$ is the symbol of the operator $F$, we have $H_g(F) = (H_g(F))^{\text{ymb}}$.

If $F \neq 0$, let $m$ be maximum nonnegative integer such that $H_m(F) \neq 0$. We call the operator

$$MF := H_m(F)$$

the *main part* (with respect to $\hat{p}$) of the operator $F$. Similarly, we call the function

$$MF := H_m(F)$$

the *main part* (with respect to $p$) of the symbol $F$. We call $m$ the *order* of the operator $F$ and we write $\text{ord } F = m$. If $F = 0$, then we define $MF := 0$ and $MF := 0$.

Note that $MF$ is the symbol of the operator $MF$, i.e., the main part of the symbol of an operator of class $\mathcal{O}$ coincides with the symbol of its main part.

The following proposition can be easily proved by making use of the identity (3.2).

**Proposition 3.2.** If $A = \sum_{|\alpha| \leq m_1} A_\alpha(x) \hat{p}^\alpha$, $B = \sum_{|\alpha| \leq m_2} B_\alpha(x) \hat{p}^\alpha$ and $a, b \in \mathbb{R}$, then

$$(aA + bB)^{\text{ymb}} = aA + bB$$

$$(AB)^{\text{ymb}} = \sum_{0 \leq |\alpha| \leq m_1} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} A}{\partial p^\alpha} \frac{\partial^{|\alpha|} B}{\partial x^\alpha}.$$
where $A = A^{smb}$, $B = B^{smb}$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $\partial x^\alpha = \partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$. We have for any nonvanishing term of this sum

$$\text{ord} \frac{1}{\alpha!} \frac{\partial^{\alpha|} A \partial^{\alpha|} B}{\partial x^\alpha} \leq m_1 + m_2 - |\alpha|.$$ 

Let us consider a set of operators $F_1, \ldots, F_r$ of class $O_K$. Let $MF_1, \ldots, MF_r$ be the main parts of the symbols $F_1, \ldots, F_r$ of these operators.

**Definition 3.4 (quasi-independence).** If the differentials $d(MF_1), \ldots, d(MF_r)$ are linearly independent at a point $(x, p) \in K \times \mathbb{R}^n_p$, we will say that the operators $F_1, \ldots, F_r$ are quasi-independent at that point. Moreover, if the differentials $d(MF_1), \ldots, d(MF_r)$ are linearly independent at each point of $K \times \mathbb{R}^n_p$, we will say that the operators $F_1, \ldots, F_r$ are globally quasi-independent.

For example, the operators $(x_1, \ldots, x_n, \hat{p}_1, \ldots, \hat{p}_n)$ are globally quasi-independent. We call this set the standard set of operators.

Similarly, if the differentials $d(MF_1), \ldots, d(MF_r)$ of the main parts of $r$ functions ($F_1, \ldots, F_r$) are linearly independent at a point $(x, p) \in K \times \mathbb{R}^n_p$, we will say that these functions are quasi-independent at that point. If the differentials $d(MF_1), \ldots, d(MF_r)$ are linearly independent at almost each point of $K \times \mathbb{R}^n_p$, we will say that these functions are globally quasi-independent.

The motivation of this definition of quasi-independence lies in its connection with the property of dependence, which for a set of operators will be formulated in section 3.3. There we shall show in fact that the property of quasi-independence defined above is a sufficient condition to exclude dependence.

### 3.2 Noncommutative polynomials

In order to formulate a rigorous definition of “dependence” for a set of operators, which in general do not commute with each other, we need first of all to introduce the abstract notion of “noncommutative polynomial”. In this section we will then define the quasi-homogeneous parts and the main part of a noncommutative polynomial with given weights and with respect to a given set of operators. We will also give the definition of the homogeneous parts of an operator.

Let us consider a set of real (commuting) variables $G = (G_1, \ldots, G_l)$ and a set of “noncommutative symbols” $F = (F_1, \ldots, F_r)$. At a later stage in this section we shall identify the commuting variables $G$ with real multiplication operators, and the noncommutative symbols $F$ with differential operators of class $O$. However, we want here to establish formal properties of noncommutative polynomials, which are independent of the identification of the abstract variables $G$ and $F$ on which the polynomials depend. Therefore, we shall here simply postulate that formal operations of addition and multiplication are defined among these variables, with the same properties of associativity and distributivity which are valid for the corresponding operations among operators. In particular, we shall postulate that multiplication is commutative between any two variables $G$, and noncommutative between any two variables $F$ or a variable $G$ and a variable $F$. Moreover, a commutative multiplication is defined between scalar numbers and variables $F$ or $G$. By making use of these abstract operations, it is possible to define the formal algebra of noncommutative polynomials in the following way.
Definition 3.5. If $\beta = (\beta_1, \ldots, \beta_q)$ is a finite sequence of indexes, with $q \in \mathbb{N}$ and $\beta_i \in \{1, \ldots, r\}$ for $i = 1, \ldots, q$, we call noncommutative monomial (with respect to $F_1, \ldots, F_r$) associated with $\beta$ a formal product

$$M_\beta = M_\beta(G, F) = Z_0 F_{\beta_1} Z_1 F_{\beta_2} \cdots Z_{q-1} F_{\beta_q} Z_q,$$

where $Z_i = Z_i(G)$ is an arbitrary usual function of class $C^\infty$ of the commuting variables $G_1, \ldots, G_l$ for $i = 0, \ldots, q$. Note that some (or all) functions $Z_i$ can be constants, for example $Z_0 = 1$. In particular, if $l = 0$ then monomials have the form $M_\beta(F) = c F_{\beta_1} F_{\beta_2} \cdots F_{\beta_q}$, where $c$ is a constant. A function $M_\beta = Z(G)$ independent of $F$ can be considered as a noncommutative monomial associated with $\beta = \emptyset$.

We consider that the monomial $M_\beta$ is (identically) zero if and only if there exists $j \in \{0, \ldots, q\}$ such that $Z_j$ is the zero function. The multiplication of a monomial by a scalar and the product of two monomials are defined in an obvious way. In particular, the product of two monomials respectively associated with $\beta = (\beta_1, \ldots, \beta_q)$ and $\gamma = (\gamma_1, \ldots, \gamma_p)$ provides a monomial associated with $(\beta_1, \ldots, \beta_q, \gamma_1, \ldots, \gamma_p)$. This product is associative and noncommutative.

We further introduce the formal sum of noncommutative monomials, which is assumed to be associative and commutative.

Definition 3.6. We call noncommutative polynomial a formal sum of noncommutative monomials. The set of such polynomials we denote equivalently as $S_N = S_N^s = S_N^{s,r} = S_N^{l,r}[G_1, \ldots, G_l, F_1, \ldots, F_r]$, where $s = l + r$. We postulate that the operations of sum, multiplication by a scalar and product among noncommutative polynomials enjoy all the usual formal properties which make $S_N^s$ a noncommutative algebra. We call noncommutative polynomial associated with $\beta$ a formal sum of monomials all associated with the same $\beta$, that is an expression

$$S_\beta = S_\beta(G, F) = \sum_{i \in I} M_{\beta,i},$$

where $I$ is a finite set of indexes and $M_{\beta,i}$ is a noncommutative monomial associated with $\beta$ for all $i \in I$.

We consider that the polynomial (3.6) is (identically) zero if the equality $\sum_i M_{\beta,i} = 0$ follows from the formal properties of the algebraic operations. Obviously, two polynomials are considered to be equal to each other if and only if their difference is zero. For example, we have $Z_1 F_1 Z_2 F_2 + Y_1 F_1 Y_2 F_2 = X_1 F_1 Z_2 F_2 + Y_1 F_1 X_2 F_2$ if $X_1(G) = Z_1(G) - Y_1(G)$ and $X_2(G) = Z_2(G) + Y_2(G) \forall G \in \mathbb{R}$.

For $Q \in \mathbb{Z}_+$, let $B_Q$ denote the set of all finite sequences $\beta = (\beta_1, \ldots, \beta_q)$, with $0 \leq q \leq Q$ and $\beta_i \in \{1, \ldots, r\}$ for $i = 1, \ldots, q$. A generic noncommutative polynomial can then be expressed in the form

$$S = \sum_{\beta \in B_Q} S_\beta,$$

where $S_\beta$ is a noncommutative polynomial associated with $\beta$ for each $\beta \in B_Q$. We consider that $S$ is (identically) zero if and only if $S_\beta = 0$ for all $\beta \in B_Q$. 

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Definition 3.7. With any noncommutative polynomial $S(G,F)$ we associate the commutative polynomial $S_C(G,F)$ which is obtained by considering all variables $G_1,\ldots,G_l,F_1,\ldots,F_r$ as commuting variables, and then operating the reduction of analogous terms, i.e., monomials of $F_1,\ldots,F_r$ which differ from each other only at most in the coefficients. (These coefficients are functions of $G_1,\ldots,G_l$, or constants if $l=0$.) We call the transformation $T: S \mapsto S_C$ the abelianization of $S$.

Definition 3.8. Let $w = (w_1,\ldots,w_r)$ be a given set of natural numbers. If the noncommutative polynomial $S \in S^{l,r}_N$ is represented in the form (3.7), let us consider for $d \in \mathbb{Z}^+$ the subset of $B_Q$: $d,w := \{ \beta = (\beta_1,\ldots,\beta_q) \in B_Q : \sum_{j=1}^{q} w_{\beta_j} = d \}$.

We will call the polynomial $C_d = C_{d,w}(S) := \sum_{\beta \in L_{d,w}} S_{\beta}$ the quasi-homogeneous part of degree $d$ with weights $w$ of the polynomial $S$. If $S = C_{d,w}(S) \neq 0$, then $S$ is called a quasi-homogeneous polynomial of degree $d$ with weights $w$. If $S \neq 0$, we define the degree $\deg_w S$ of the polynomial (3.7) with weights $w$ the maximum $d \in \mathbb{Z}^+$ such that $C_{d,w}(S) \neq 0$. We define the main part $M_{\mathcal{F}}S$ of the polynomial $S$ with weights $w$ as

$$M_{\mathcal{F}}S := C_{\bar{d},w}(S),$$

where $\bar{d} = \deg_w S$. If $S = 0$, then we define $M_{\mathcal{F}}S := 0$.

We define the main part $M_{\mathcal{F}}S$ of the polynomial $S$ with respect to the set $\mathcal{F} = (F_1,\ldots,F_r)$ of operators of class $\mathcal{O}$ as

$$M_{\mathcal{F}}S = M_{\mathcal{W}}S,$$

where $w = (w_1,\ldots,w_s)$, $w_i = \ord F_i$ for $i = 1,\ldots,r$.

We will also write $\deg_{\mathcal{F}} S := \deg_{w_i} S = \bar{d}$, and we say that $\bar{d}$ is the degree of $S$ with respect to $\mathcal{F}$. If $\mathcal{W} = (\mathcal{G},\mathcal{F})$, where $\mathcal{G} = (G_1,\ldots,G_l)$ is a set of multiplication operators (i.e., a set of operators of class $\mathcal{O}$ such that $\ord G_i = 0 \forall i = 1,\ldots,l$), we shall also write $M_{\mathcal{W}}S = M_{\mathcal{F}}S$.

Lemma 3.3. Let $S$ be a noncommutative polynomial. If $S_C \neq 0$ then $S \neq 0$. If $S$ is a quasi-homogeneous polynomial of degree $d$ with weights $w$, then either $S_C = 0$ or $S_C$ is also a quasi-homogeneous polynomial of degree $d$ with weights $w$.

Using this obvious lemma and proposition 3.2 it is easy to prove the following proposition.

Proposition 3.4. Let $\mathcal{W} = (\mathcal{G},\mathcal{F})$ be a set of operators of class $\mathcal{O}$, where $\mathcal{G} = (G_1,\ldots,G_l)$, $\ord G_i = 0 \forall i = 1,\ldots,l$, and $\mathcal{F} = (F_1,\ldots,F_r)$. Let $G$ be the symbol
of $G$, and $MF$ the symbol of the main parts of $F$. Let $S \in S_{l,r}^N$ be a noncommutative polynomial such that $\deg_W S = \bar{d}$. Then $S(G,F) := S(G,F)|_{G=G,F=F}$ is an operator of class $O$, and

$$(H_{\bar{d}}(S(G,F)))^{\text{amb}} = (M_W S)_C(G, MF).$$ (3.8)

The above formula means that, if the noncommutative polynomial $S$ has degree $\bar{d}$ with respect to $W$, then the symbol of the homogeneous part of order $\bar{d}$ of $S(G,F)$ is obtained by taking the symbols $(G, MF)$ as the variables of the abelianization of the main part of $S$ with respect to $W$. Note that the two members of equality (3.8) may be zero. When they are nonzero, they represent the symbol of the main part of the operator $S(G,F)$.

### 3.3 Dependent sets of operators

In this section we define the concept of dependence for a set of operators. There seems to exist no definition as general and natural as that for the functional dependence of a set of functions. Our idea is here to consider as dependent all sets of operators for which there exists a correlation, that is a nontrivial operator function which, when its arguments are replaced by the operators of the set, vanishes identically in a neighborhood of a given point of configuration space. However, when we are dealing with operators, the class of available functions in our general scheme is restricted to noncommutative polynomials. Hence such a definition of dependence would appear too restrictive. We shall therefore adopt a different definition, which offers a higher level of generality (see definition 3.12). We will prove that a dependent set defined in this way cannot be quasi-independent (see theorem 3.19).

We begin by giving the definition of “regular correlation” among the class $O$ operators of the set $W = (G_1, \ldots, G_l, F_1, \ldots, F_r)$, with $\text{ord} G_i = 0$ for $i = 1, \ldots, l$, and $\text{ord} F_j \geq 1$ for $j = 1, \ldots, r$. Let $MW : K \times \mathbb{R}^n \to \mathbb{R}^{l+r}$ denote the vector function $(G_1, \ldots, G_l, MF_1, \ldots, MF_r)$ defined by the symbols of the main parts of these operators. Let us consider a point $(\bar{x}, \bar{p}) \in K \times \mathbb{R}^n$ and its image $\bar{W} := MW(\bar{x}, \bar{p}) \in \mathbb{R}^{l+r}$ with respect to the function $MW$. Let us suppose that there exists a nonvanishing noncommutative polynomial $S = S(G,F) \in S_{l,r}^N$ with the following two properties:

1. The differential of the abelianization $(M_W S)_C$ of the main part $M_W S$ of the noncommutative polynomial $S$ with respect to $W$ is nonzero at $\bar{W}$, that is

   $$d((M_W S)_C(\bar{W})) \neq 0.$$

2. $S(G,F) := S(G,F)|_{G=G,F=F}$ is the zero operator in a neighborhood $H \subset K$ of $\bar{x}$, that is

   $$S(G,F)\psi(\bar{x}) = 0$$ (3.10)

   $\forall \psi \in C^\infty(H)$ and $\forall x \in H$.

Note that, according to propositions (3.1) and (3.4), property 2 implies that $(M_W S)_C(\bar{W}) = 0$.

**Definition 3.9.** We say that a noncommutative polynomial $S \in S_{l,r}^N$ with the properties 1 and 2 is a regular correlation among the operators $W$ at the point...
and $W$. Particular, any set of operators clearly satisfies the conditions 1 and 2 of definition 3.10. It follows that, in a point $(x, p) \in K \times \mathbb{R}^n_p$, if condition (3.9) were not required, the noncommutative polynomial $S(G, F) = GF - FG + 1$ would be a global regular correlation between the operators $G$ and $F$, but this would be in contradiction with the fact that these operators are quasi-independent according to definition 3.4.

As we have already explained, our aim is to find a definition of dependence which includes (but is not restricted to) the case in which there exists a regular correlation among the operators of a set. As a first step in this direction, we now introduce the concept of dependence of a set of operators on another set. To this purpose, let us consider two sets of operators $W = (W_1, \ldots, W_s)$ and $Y = (Y_1, \ldots, Y_r)$ of class $O$. We can suppose in general that ord $W_i = 0$ and ord $Y_j = 0$ for $i = 1, \ldots, s'$ and $j = 1, \ldots, r'$, where $0 \leq s' \leq s$ and $0 \leq r' \leq r$. Let $(MW, MY) : K \times \mathbb{R}^n_p \to \mathbb{R}^{n+r}$ be the vector function defined by the symbols of the main parts of these operators, and $(W, Y) := (MW, MY)(\bar{x}, \bar{p}) \in \mathbb{R}^{n+r}$ the image of the point $(\bar{x}, \bar{p})$ with respect to this function. Let us consider the class of noncommutative polynomials $S^+_N = S^+_N$, where $l = s' + r'$ and $m = s + r - l$. Let us suppose that there exists a set of noncommutative polynomials $S = (S_1, \ldots, S_s)$ of this class with the following two properties:

1. Let $M(W, Y)S$ be the set of main parts $M(W, Y)S_i$ of the components $S_i$ of the vector $S$, with $i = 1, \ldots, s$, and let $(M(W, Y)S)_C$ be the vector of their abelianizations. Then

$$
\det \frac{\partial (M(W, Y)S)_C}{\partial W}(W, Y) \neq 0,
$$

where on the left-hand side we have the determinant of the $s \times s$ Jacobi matrix with respect to $W$ of the vector function $(M(W, Y)S)_C$ of the $s+r$ variables $(W, Y)$.

2. After substitution of the $s + r$ operators $(W, Y)$ to the variables of the polynomial $S$ we obtain the zero operator in a neighborhood $H \subset K$ of $\bar{x}$: $S(W, Y) = 0$ in $H$.

Owing to propositions (3.1) and (3.4), property 2 clearly implies that $(M(W, Y)S)_C(W, Y) = 0$. Note also that the $S_i$ are correlations at $(\bar{x}, \bar{p})$ among the operators $(W, Y)$ in the sense of definition 3.9.

**Definition 3.10.** In this case we say that the set of operators $W$ is algebraically dependent on the set of operators $Y$ at the point $(\bar{x}, \bar{p})$.

**Remark 3.1.** Obviously, if the set of operators $W$ is (properly or improperly) contained in the set $Y$, i.e., $W \subseteq Y$, then $W$ is algebraically dependent on $Y$ at any point $(x, p) \in K \times \mathbb{R}^n_p$. In fact, suppose that $s \leq r$ and $W_i = Y_i \forall i = 1, \ldots, s$. Then the vector of polynomials $S_i(W, Y) = W_i - Y_i$ for $i = 1, \ldots, s$ clearly satisfies the conditions 1 and 2 of definition 3.10. It follows that, in particular, any set of operators $W$ is algebraically dependent on itself at any point $(x, p) \in K \times \mathbb{R}^n_p$. It is also immediate to see that, if $W = (W_1, \ldots, W_s)$ and $W' = (W_1, \ldots, W_s, W_{s+1})$, where $W_{s+1} = W_j$ for some $j$ with $1 \leq j \leq s$, then $W$ is algebraically dependent on $W'$. However, this is not the case, as we shall see in the next section, if the set of operators $W$ is algebraically dependent on itself at the point $(\bar{x}, \bar{p})$.
then \( W' \) is algebraically dependent on \( W \) and \( W \) is algebraically dependent on \( W' \).

A particularly simple type of algebraic dependence occurs when (some powers of) the operators of the set \( W \) can be explicitly expressed as noncommutative polynomial functions of the operators of the set \( Y \). This is stated in more precise terms by the following proposition.

**Proposition 3.5.** Let us consider the set of operators \( W = (W_1, \ldots, W_s) \). Let us suppose that there exists another set of operators \( Y = (Y_1, \ldots, Y_r) \) such that

\[
W_i^j = S_i(Y),
\]

where \( j_i \in \mathbb{N} \) (for example \( j_i = 1 \)) and \( S_i(Y) \) is a noncommutative polynomial of class \( S_N^r \) such that

\[
\text{deg}_Y S_i = j_i \text{ ord}_Y W_i \quad \forall i = 1, \ldots, s.
\]

Then \( W \) is algebraically dependent on \( Y \) at all points \((x, p)\) such that \( MW_i(x, p) \neq 0 \forall i \in J \), where \( J := \{i \in \{1, \ldots, s\} : j_i > 1\} \). In particular, \( W \) is algebraically dependent on \( Y \) in all \( K \times \mathbb{R}_n^p \) if \( j_i = 1 \forall i \).

**Proof.** Let \( S_i(W, Y) := W_i^j - S_i(Y) \) denote the given correlations among the operators \( W \) and \( Y \). The element with pair of indexes \((i, h)\) of the Jacobi matrix for the abelianization of the main part of \( S \) is

\[
\frac{\partial}{\partial W_h}(M(W, Y)S_i)|_C = \delta_{ih} j_i W_i^{j_i - 1}.
\]

Hence this Jacobi matrix is diagonal (in the previous formula \( \delta_{ih} \) is the Kronecker symbol, such that \( \delta_{ii} = 1, \delta_{ii} = 0 \) for \( i \neq h \)) and nondegenerate for \( W = MW(x, p) \) if \( MW_i(x, p) \neq 0 \forall i \in J \). Therefore at such points \((x, p)\) both conditions of definition (3.10) are satisfied.

Note that condition (3.12) is necessary for the validity of formula (3.13). In fact, if this condition is not true for some \( i \), then clearly \( \text{ord}_Y W_i^j < \text{deg}_Y S_i(Y) \) for this \( i \), so that \( M(W, Y)S_i = M(Y)S_i \). It follows that all the elements of the \( i \)-th row of the Jacobi matrix for \( (M(W, Y)S)_C \) are identically zero. Hence the determinant of this matrix is also identically zero.

**Proposition 3.6.** Let \( W = (W_1, \ldots, W_s) \) be a set of operators of class \( O_K \). Then \( W \) is algebraically dependent on \((x, \hat{p})\) in all \( K \times \mathbb{R}_n^p \), where \((x, \hat{p}) = (x_1, \ldots, x_n, \hat{p}_1, \ldots, \hat{p}_n)\) is the standard set of operators.

**Proof.** Let

\[
W_j = \sum_{|\alpha| \leq m_j} A_{j,\alpha}(x)\hat{p}^\alpha, \quad j = 1, \ldots, s,
\]

be the standard representation of the operators \( W_j \) in \( O \), see definition (5.2). We can rewrite these equalities as

\[
W_j = S_j(x, \hat{p}), \quad j = 1, \ldots, s,
\]

where \( S_j := \sum_{|\alpha| \leq m_j} A_{j,\alpha}(X)P^\alpha \) is a noncommutative polynomial of class \( S_N^{n_n} \) for \( j = 1, \ldots, s \). Clearly

\[
\text{ord}_W W_j = \text{deg}_{(x, \hat{p})} S_j \quad \forall j = 1, \ldots, s.
\]

The thesis then follows from proposition (3.5).
Proposition 3.7. If $\mathcal{Y}^{(1)}$ is algebraically dependent on $\mathcal{Y}^{(2)}$ and $\mathcal{W}^{(2)}$ is algebraically dependent on $\mathcal{Y}^{(2)}$ at $(\bar{x}, \bar{p})$, then at that point $\mathcal{W}$ is algebraically dependent on $\mathcal{Y}$, where $\mathcal{W} := \mathcal{W}^{(1)} \cup \mathcal{W}^{(2)}$ and $\mathcal{Y} := \mathcal{Y}^{(1)} \cup \mathcal{Y}^{(2)}$.

Proof. If $\mathcal{W}^{(i)} = (W^{(i)}_1, \ldots, W^{(i)}_n)$ and $\mathcal{Y}^{(i)} = (\mathcal{Y}^{(i)}_1, \ldots, \mathcal{Y}^{(i)}_r)$, for $i = 1, 2$, according to the hypotheses there exist two sets of noncommutative polynomials $S^{(i)} = (S^{(i)}_1, \ldots, S^{(i)}_s)$ of class $S_N^{s_1 + r_1}$ such that

$$\det \frac{\partial (M_{(\mathcal{W}^{(i)}, \mathcal{Y}^{(i)})} S^{(i)})}{\partial W^{(i)}}(W^{(i)}, \mathcal{Y}^{(i)}) \neq 0$$

and

$$S^{(i)}(W^{(i)}, \mathcal{Y}^{(i)}) = 0 \quad \text{in} \quad H_i,$$

where $(W^{(i)}, \mathcal{Y}^{(i)}) := (MW^{(i)}, MY^{(i)})(\bar{x}, \bar{p}) \in \mathbb{R}^{s_i + r_i}$ and $H_i \subset K$ is an open neighborhood of $\bar{x}$. Then for the set

$$S = (S^{(1)}_1, \ldots, S^{(1)}_s, S^{(2)}_1, \ldots, S^{(2)}_s)$$

of $s$ noncommutative polynomials of class $S_N^{s+r}$, where $s = s_1 + s_2$ and $r = r_1 + r_2$, we have

$$\det \frac{\partial (M_{(W, Y)}) S}{\partial W}(W, \mathcal{Y}) = \det \frac{\partial (M_{(W^{(1)}, Y^{(1)})} S^{(1)})}{\partial W^{(1)}}(W^{(1)}, \mathcal{Y}^{(1)}) \times \det \frac{\partial (M_{(W^{(2)}, Y^{(2)})} S^{(2)})}{\partial W^{(2)}}(W^{(2)}, \mathcal{Y}^{(2)}) \neq 0$$

and

$$S(W, \mathcal{Y}) = 0 \quad \text{in} \quad H = H_1 \cap H_2,$$

where

$$\mathcal{W} = (W^{(1)}_1, \ldots, W^{(1)}_n, W^{(2)}_1, \ldots, W^{(2)}_n)$$
$$\mathcal{Y} = (\mathcal{Y}^{(1)}_1, \ldots, \mathcal{Y}^{(1)}_r, \mathcal{Y}^{(2)}_1, \ldots, \mathcal{Y}^{(2)}_r)$$

$$(\bar{W}, \bar{Y}) = (MW, MY)(\bar{x}, \bar{p}) \in \mathbb{R}^{s+r}.$$ 

This means that $\mathcal{W}$ is algebraically dependent on $\mathcal{Y}$.

The following important proposition says that, if the set $\mathcal{W}$ is algebraically dependent on the set $\mathcal{Y}$ at a point of $K \times \mathbb{R}^n_p$, then the same relation of algebraic dependence also holds in a full neighborhood of this point. Furthermore, in this neighborhood the symbols $MW$ of the main parts of $\mathcal{W}$ are functionally dependent on the symbols $MY$ of the main parts of $\mathcal{Y}$.

Proposition 3.8. Let $\mathcal{W} = (W_1, \ldots, W_n)$ and $\mathcal{Y} = (Y_1, \ldots, Y_r)$ be two sets of operators of class $\mathcal{O}_K$, such that $\mathcal{W}$ is algebraically dependent on $\mathcal{Y}$ at $(\bar{x}, \bar{p}) \in K \times \mathbb{R}^n_p$. Then there exists a neighborhood $\Omega \subset \mathbb{R}^n$ of $\bar{Y} = MY(\bar{x}, \bar{p})$, and a function $f : \Omega \to \mathbb{R}^s$, $f \in C^\infty(\Omega)$, such that $MW(x, p) = f(MY(x, p)) \forall (x, p) \in \Omega$.

i. The main parts $MW$ of the symbols of $\mathcal{W}$ are functions of the main parts $MY$ of the symbols of $\mathcal{Y}$ in $\Omega$. More precisely, there exist a neighborhood $\Omega \subset \mathbb{R}^s$ of $\bar{Y} = MY(\bar{x}, \bar{p})$, and a function $f : \Omega \to \mathbb{R}^s$, $f \in C^\infty(\Omega)$, such that $MW(x, p) = f(MY(x, p)) \forall (x, p) \in \Omega$. 

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\( W \) is algebraically dependent on \( Y \) at all points of \( O \).

Proof. Let \( S = (S_1, \ldots, S_s) \) be the vector of noncommutative polynomials of class \( S^{s+r}_N \) satisfying conditions 1 and 2 of definition 3.10. Since, by condition 2, \( S_i(W, Y) \) is the zero operator in a neighborhood \( H \subset K \) of \( \bar{x} \) for all \( i = 1, \ldots, s \), its homogeneous part \( H_{d_i}(S_i(W, Y)) \) of order \( d_i \) is also obviously zero in \( H \), where \( d_i = \deg_{(W, Y)} S_i \). Therefore, using proposition 3.4 we obtain

\[
\tilde{S}((MW, MY)(x, p)) = 0 \quad \forall (x, p) \in H \times \mathbb{R}^n_p, \tag{3.14}
\]

where we have introduced the vector function \( \tilde{S} := (M_{(W, Y)} S)_C, \tilde{S} : \mathbb{R}^{s+r} \rightarrow \mathbb{R}^s \). This implies in particular that \( \tilde{S}(\bar{W}, \bar{Y}) = 0 \). According to our definition of noncommutative polynomial, we know that \( \tilde{S} \in C^\infty(\mathbb{R}^{s+r}) \). Taking also into account condition 1 of definition 3.10, it follows from the theorem on implicit functions that there exist a neighborhood \( \Omega \subset \mathbb{R}^r \) of \( \bar{Y} \), a neighborhood \( \Omega' \subset \mathbb{R}^s \) of \( \bar{W} \), and a function \( f : \Omega \rightarrow \Omega', f \in C^\infty(\Omega) \), with the following properties:

a. \( f(\bar{Y}) = \bar{W} \).

b. \( \tilde{S}(f(Y), Y) = 0 \quad \forall Y \in \Omega \).

c. For all \( Y \in \Omega \), \( W = f(Y) \) is the only solution \( W \in \Omega' \) of the equation \( \tilde{S}(W, Y) = 0 \).

d. If \( \partial \tilde{S}/\partial W \) and \( \partial \tilde{S}/\partial Y \) denote the Jacobi matrices of the vector function \( \tilde{S} \) with respect to variables \( W \) and \( Y \) respectively, then

\[
\det \frac{\partial \tilde{S}}{\partial W}(W, Y) \neq 0 \quad \forall (W, Y) \in \Omega \times \Omega'. \tag{3.15}
\]

and

\[
df(Y) = - \left( \frac{\partial \tilde{S}}{\partial W}(f(Y), Y) \right)^{-1} \left( \frac{\partial \tilde{S}}{\partial Y}(f(Y), Y) \right) \quad \forall Y \in \Omega. \tag{3.16}
\]

According to our definition of operator of class \( O_K \), we know that \( (MW, MY) \in C^\infty(K \times \mathbb{R}^n_p) \). Therefore there exists a neighborhood \( \tilde{O} \subset K \times \mathbb{R}^n_p \) of \( (\bar{x}, \bar{p}) \) such that

\[
(MW, MY)(x, p) \in \Omega \times \Omega' \quad \forall (x, p) \in \tilde{O}. \tag{3.17}
\]

The set \( O := \tilde{O} \cap (H \times \mathbb{R}^n_p) \) is also a neighborhood of \( (\bar{x}, \bar{p}) \). From (3.14) and from property \( \Box \) of function \( f \), it then follows that \( MW(x, p) = f(MY(x, p)) \forall (x, p) \in O \). Furthermore, from (3.15) and (3.17) it follows that

\[
\det \frac{\partial \tilde{S}}{\partial W}((MW, MY)(x, p)) \neq 0 \quad \forall (x, p) \in O. \]

Therefore, according to definition 3.10 \( W \) is algebraically dependent on \( Y \) at all points of \( O \).

The relation of dependence between sets of operators, given by definition 3.10, does not automatically enjoy the transitive property. For this reason, we will now introduce another relation which extends that of algebraic dependence in such a way to insure transitivity.
Definition 3.11. Let us suppose that there exist \( q \) finite sets \( \mathcal{Z}^{(1)}, \ldots, \mathcal{Z}^{(q)} \) of operators of class \( \mathcal{O} \), such that \( \mathcal{W} \) is algebraically dependent on \( \mathcal{Z}^{(1)} \) at the point \( (\bar{x}, \bar{p}) \), \( \mathcal{Z}^{(i)} \) is algebraically dependent on \( \mathcal{Z}^{(i+1)} \) at \( (\bar{x}, \bar{p}) \) for \( i = 1, \ldots, q - 1 \), and \( \mathcal{Z}^{(q)} \) is algebraically dependent at \( (\bar{x}, \bar{p}) \) on \( \mathcal{Y} \). In this case we say that the set of operators \( \mathcal{Y} \) algebraically contains the set of operators \( \mathcal{W} \) at the point \( (\bar{x}, \bar{p}) \) and we write \( \mathcal{Y} \supseteq \mathcal{W} \). We also say that the set \( \mathcal{W} \) is algebraically contained in the set \( \mathcal{Y} \) at \( (\bar{x}, \bar{p}) \) and we write \( \mathcal{W} \subseteq \mathcal{Y} \).

Remark 3.2. Obviously, if \( \mathcal{W} \) is algebraically dependent on \( \mathcal{Y} \) at a point \((x, p)\), then \( \mathcal{W} \subseteq \mathcal{Y} \) at that point. In particular, \( \mathcal{W} \subseteq \mathcal{Y} \) at any point \((x, p) \in K \times \mathbb{R}^n\) for any set of operators \( \mathcal{W} \). Furthermore, if \( \mathcal{W} \subseteq \mathcal{Y} \) and \( \mathcal{Y} \subseteq \mathcal{Z} \) at a point \((x, p)\), then \( \mathcal{W} \subseteq \mathcal{Z} \) at that point.

The following proposition can be deduced from proposition 3.7.

Proposition 3.9. If \( \mathcal{W}^{(1)} \subseteq \mathcal{Y}^{(1)} \) and \( \mathcal{W}^{(2)} \subseteq \mathcal{Y}^{(2)} \) at \((\bar{x}, \bar{p})\), then at that point \( \mathcal{W} \subseteq \mathcal{Y} \), where \( \mathcal{W} := \mathcal{W}^{(1)} \cup \mathcal{W}^{(2)} \) and \( \mathcal{Y} := \mathcal{Y}^{(1)} \cup \mathcal{Y}^{(2)} \).

Corollary 3.10. If \( \mathcal{W}^{(1)} \subseteq \mathcal{Y} \) and \( \mathcal{W}^{(2)} \subseteq \mathcal{Y} \) at \((\bar{x}, \bar{p})\), then at that point \( \mathcal{W} \subseteq \mathcal{Y} \), where \( \mathcal{W} := \mathcal{W}^{(1)} \cup \mathcal{W}^{(2)} \).

Proposition 3.11. Let \( \mathcal{W} = (W_1, \ldots, W_s) \), \( \mathcal{V} = (V_1, \ldots, V_r) \) and \( \mathcal{Y} = (Y_1, \ldots, Y_r) \) be three sets of operators of class \( \mathcal{O} \). Let us consider a point \((\bar{x}, \bar{p}) \in K \times \mathbb{R}^n\), and its image \((\bar{W}, \bar{V}, \bar{Y}) = (MW, MV, MY)(\bar{x}, \bar{p}) \in \mathbb{R}^{s+r+r} \) with respect to the symbols of the main parts of \( \mathcal{W} \), \( \mathcal{V} \), \( \mathcal{Y} \). Let us suppose that there exists a set \( S = (S_1, \ldots, S_{s+r}) \) of \( s + r \) noncommutative polynomials of class \( S^{s+r+1} \), such that

\[
\det \frac{\partial}{\partial(W,V)}(M_{(W,V,Y)}S)C(W,V,Y) \neq 0 ,
\]

and \( S(W,V,Y) = 0 \) in a neighborhood \( H \subset K \) of \((\bar{x}, \bar{p})\), that is

\[
S(W,V,Y)\psi(x) = 0
\]

for all \( \psi \in C^\infty(H) \) and all \( x \in H \). Then \( \mathcal{W} \subseteq \mathcal{Y} \) at \((\bar{x}, \bar{p})\).

Proof. Let us consider the set of \( s + r \) operators \( \mathcal{U} := (W_1, \ldots, W_s, V_1, \ldots, V_r) \). Since \( S(\mathcal{U}, \mathcal{Y}) = 0 \) in \( H \) and

\[
\det \frac{\partial}{\partial U}(M_{(U,Y)}S)C(\bar{U}, \bar{Y}) \neq 0 ,
\]

where \( \bar{U} = (\bar{W}, \bar{V}) \), we have that \( \mathcal{U} \) is algebraically dependent on \( \mathcal{Y} \) at \((\bar{x}, \bar{p})\). Moreover, considering the vector of correlations

\[
T_i(W, U) = W_i - U_i , \quad i = 1, \ldots, s ,
\]

we have that \( \mathcal{W} \) is algebraically dependent on \( \mathcal{U} \) everywhere. Therefore \( \mathcal{W} \subseteq \mathcal{Y} \) at \((\bar{x}, \bar{p})\) according to definition 3.11.

We are now going to show that the statement of proposition 3.8 remains valid if the relation of algebraic dependence is replaced by the one specified by definition 3.11.

Proposition 3.12. Let \( \mathcal{W} = (W_1, \ldots, W_s) \) and \( \mathcal{Y} = (Y_1, \ldots, Y_r) \) be two sets of operators of class \( \mathcal{O}_K \), such that \( \mathcal{W} \subseteq \mathcal{Y} \) at \((\bar{x}, \bar{p}) \in K \times \mathbb{R}^n\). Then there exists a neighborhood \( U \subset K \times \mathbb{R}^n \) of \((\bar{x}, \bar{p})\) such that:
Let us now formulate a definition of “regular dependence” for the operators of the set $W = (W_1, \ldots, W_s)$ at a point $(\bar{x}, \bar{p}) \in K \times \mathbb{R}_p^n$.

**Definition 3.12 (regular dependence).** Let us consider a set $W$ of $s$ operators of class $O$. Let us suppose that there exists a set $\mathcal{Y} = (Y_1, \ldots, Y_r)$ of operators of class $O$, with $r < s$, such that $W \subseteq \mathcal{Y}$ at the point $(\bar{x}, \bar{p}) \in K \times \mathbb{R}_p^n$. In this case we say that the operators of $W$ are regularly dependent at $(\bar{x}, \bar{p})$. If the operators $W$ are regularly dependent almost everywhere in $K \times \mathbb{R}_p^n$, we say that they are globally dependent in $K \times \mathbb{R}_p^n$. In this paper, almost everywhere means in a subset such that the closure of its complement has zero measure.

**Definition 3.13.** We define the rank of the set of operators $W$ at $(x, p)$ as the minimum integer $r$ such that there exists another set $\mathcal{Y} = (Y_1, \ldots, Y_r)$ with $W \subseteq \mathcal{Y}$ at $(x, p)$. We denote this rank as $r = \text{rank} W(x, p)$.

If $W = (W_1, \ldots, W_s)$, then obviously $\text{rank} W(x, p) \leq s$ at any point $(x, p) \in K \times \mathbb{R}_p^n$. The operators $W$ are regularly dependent at $(x, p)$ if and only if $\text{rank} W(x, p) < s$. It is also evident that, if $W$ and $\mathcal{Y}$ are two sets of operators such that $W \subseteq \mathcal{Y}$ at $(x, p)$, then $\text{rank} W(x, p) \leq \text{rank} \mathcal{Y}(x, p)$. The following proposition easily follows from statement [ii] of proposition 3.12.

**Proposition 3.13.** If $\text{rank} W(\bar{x}, \bar{p}) = r$, then there exists a neighborhood $O \subset K \times \mathbb{R}_p^n$ of $(\bar{x}, \bar{p})$ such that $\text{rank} W(x, p) \leq r \forall (x, p) \in O$. In particular, if $W$ is regularly dependent at $(\bar{x}, \bar{p})$, then $W$ is also regularly dependent at all points of a neighborhood of $(\bar{x}, \bar{p})$.

The following proposition is an immediate consequence of propositions 3.9.

**Proposition 3.14.** If $s > 2n$, then any set $W = (W_1, \ldots, W_s)$ of operators of class $O$ is globally dependent. Equivalently, any $2n + 1$ such operators $(W_1, \ldots, W_{2n+1})$ are globally dependent.

The next proposition shows that definition 3.12 of regular dependence indeed includes as a particular case the existence of a regular correlation among the operators of a set, in the sense of definition 3.11.

**Proposition 3.15.** If there exists a regular correlation at $(\bar{x}, \bar{p})$ among the operators $W = (W_1, \ldots, W_s)$, then the operators $W$ are regularly dependent at $(\bar{x}, \bar{p})$. 

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References [16]
Proof. If the correlation is represented by the noncommutative polynomial \( S \in \mathcal{S}_n \), and \( \bar{W} := MW(\bar{x}, \bar{p}) \in \mathbb{R}^s \), it follows from condition 1 of definition \( 3.9 \) that there exists at least one \( j \in \{1, \ldots, s\} \) such that \( (\partial/\partial W_j)(\mathcal{M}_W S)_C(\bar{W}) \neq 0 \). If necessary, let us rearrange the order of the operators \( W \) so that

\[
\frac{\partial}{\partial W_i}(\mathcal{M}_W S)_C(\bar{W}) \neq 0 .
\]  

(3.18)

Let us take \( \mathcal{Y} = (\mathcal{Y}_1, \ldots, \mathcal{Y}_{s-1}) := (\mathcal{W}_1, \ldots, \mathcal{W}_{s-1}) \) and consider the set of \( s \) correlations

\[
S_i(W, Y) = W_i - Y_i , \quad i = 1, \ldots, s - 1,
\]

\[
S_s(W, Y) = S(W) .
\]

If \( \bar{Y} := MY(\bar{x}, \bar{p}) \), it is easy to see that

\[
\det \frac{\partial (\mathcal{M}_W Y)_C}{\partial W}(\bar{W}, \bar{Y}) = \frac{\partial}{\partial W}(\mathcal{M}_W S)_C(\bar{W}) \neq 0 .
\]

Hence \( W \) is algebraically dependent on \( \mathcal{Y} \) at \( (\bar{x}, \bar{p}) \), according to definition \( 3.10 \). This implies that \( W \) is regularly dependent at \( (\bar{x}, \bar{p}) \).

Definition 3.14. If \( \mathcal{W} = (\mathcal{W}_1, \ldots, \mathcal{W}_s) \) is a set of operators of class \( \mathcal{O} \), let us denote with \( r_{\mathcal{W}}(x, p) \) the dimension of the linear space generated by the differentials \( (dM_{\mathcal{W}1}, \ldots, dM_{\mathcal{W}s}) \) of the symbols of the main parts of the operators \( \mathcal{W} \) at the point \((x, p) \in K \times \mathbb{R}_p^n \). We have equivalently \( r_{\mathcal{W}}(x, p) := \text{rank} \bar{W}(x, p) \), where \( \bar{W} \) is the \( s \times 2n \) Jacobi matrix of the function \( MW \). We call \( r_{\mathcal{W}}(x, p) \) the main dimension of the set \( \mathcal{W} \) at the point \((x, p) \).

Clearly \( r_{\mathcal{W}}(x, p) \leq s \), and the operators \( \mathcal{W} \) are quasi-independent at \((x, p) \) if and only if \( r_{\mathcal{W}}(x, p) = s \) (see definition \( 3.4 \)).

Proposition 3.16. If \( r_{\mathcal{W}}(\bar{x}, \bar{p}) = r \), then there exists a neighborhood \( O \subset K \times \mathbb{R}_p^n \) of \((\bar{x}, \bar{p}) \) such that \( r_{\mathcal{W}}(x, p) \geq r \ \forall (x, p) \in O \). In particular, if \( \mathcal{W} \) is quasi-independent at \((\bar{x}, \bar{p}) \), then \( \mathcal{W} \) is also quasi-independent at all points of a neighborhood of \((\bar{x}, \bar{p}) \).

Proof. The thesis easily follows from the regularity of the symbols of the operators of class \( \mathcal{O} \).

The following proposition is an immediate consequence of statement \( a \) of proposition \( 3.12 \).

Proposition 3.17. Let \( \mathcal{W} = (\mathcal{W}_1, \ldots, \mathcal{W}_s) \) and \( \mathcal{Y} = (\mathcal{Y}_1, \ldots, \mathcal{Y}_r) \) be two sets of operators of class \( \mathcal{O} \), such that \( \mathcal{W} \subseteq \mathcal{Y} \) at a point \((\bar{x}, \bar{p}) \in K \times \mathbb{R}_p^n \). Then the vectors \( dM_{\mathcal{W}i}(\bar{x}, \bar{p}) \), for \( i = 1, \ldots, s \), are linearly dependent on the vectors \( (dM_{\mathcal{Y}1}(\bar{x}, \bar{p}), \ldots, dM_{\mathcal{Y}r}(\bar{x}, \bar{p})) \). Therefore \( r_{\mathcal{W}}(\bar{x}, \bar{p}) \leq r_{\mathcal{Y}}(\bar{x}, \bar{p}) \).

According to statement \( a \) of proposition \( 3.12 \), the hypothesis of the preceding proposition actually implies that \( \mathcal{W} \subseteq \mathcal{Y} \) at all points of a neighborhood \( O \subset K \times \mathbb{R}_p^n \) of \((\bar{x}, \bar{p}) \). We have therefore \( r_{\mathcal{W}}(x, p) \leq r_{\mathcal{Y}}(x, p) \ \forall (x, p) \in O \).

Corollary 3.18. Let \( \mathcal{W} = (\mathcal{W}_1, \ldots, \mathcal{W}_s) \) be a set of operators of class \( \mathcal{O} \). Then \( r_{\mathcal{W}}(x, p) \leq \text{rank} \mathcal{W}(x, p) \ \forall (x, p) \in K \times \mathbb{R}_p^n \).
Proof. If rank \( W(x, p) = r \) at \((x, p) \in K \times \mathbb{R}_p^n\), let \( \mathcal{Y} = (\mathcal{Y}_1, \ldots, \mathcal{Y}_r) \) be a set of operators such that \( W \subseteq \mathcal{Y} \) at \((x, p)\). Then proposition 3.17 implies that 
\[
 r_W(x, p) \leq r_{\mathcal{Y}}(x, p) \leq r.
\]

Using proposition 3.13 and corollary 3.18 one immediately obtains the theorem which relates the property of regular dependence with that of quasi-independence of a set of operators.

**Theorem 3.19.** If the operators \( W \) of class \( O \) are regularly dependent at a point \((\bar{x}, \bar{p}) \in K \times \mathbb{R}_p^n\), then there exists a neigbourhood \( O \subset K \times \mathbb{R}_p^n \) of \((\bar{x}, \bar{p})\), such that they are not quasi-independent at any point of \( O \). In particular, they are not globally quasi-independent.

**Corollary 3.20.** Let us consider the set of \( 2n \) operators \((x_1, \ldots, x_n, \hat{p}_1, \ldots, \hat{p}_n)\). These operators are quasi-independent. Therefore they are not regularly dependent at any point \((x, p)\) and are not globally dependent. The same is true for any subset of this set of operators.

**Proposition 3.21.** A set \( \mathcal{Y} = (\mathcal{Y}_1, \ldots, \mathcal{Y}_{2n}) \) of operators of class \( O_K \) is quasi-independent at a point \((\bar{x}, \bar{p}) \in K \times \mathbb{R}_p^n\) if and only if \( \mathcal{Y} \supseteq (x_1, \ldots, x_n, \hat{p}_1, \ldots, \hat{p}_n) \) at that point.

**Proof.** Let 
\[
 \mathcal{Y}_j = \sum_{|\alpha| \leq m_j} A_{j,\alpha}(x) \hat{p}^\alpha, \quad j = 1, \ldots, 2n
\]
be the standard representation of the operators \( \mathcal{Y}_j \in O_K \), see definition 3.2. Consider the \( 2n \) noncommutative polynomials of class \( S_N^{1n} \)
\[
 S_j(Y, X, P) = Y_j - \sum_{|\alpha| \leq m_j} A_{j,\alpha}(X) P^\alpha, \quad j = 1, \ldots, 2n.
\]
We have
\[
 (M_j(\mathcal{Y}, x, \bar{p}) S_j) C(Y, X, P) = Y_j - M_{\mathcal{Y}_j}(X, P),
\]
where
\[
 M_{\mathcal{Y}_j}(X, P) = \sum_{|\alpha| = m_j} A_{j,\alpha}(X) P^\alpha
\]
is the symbol of the main part of \( \mathcal{Y}_j \) according to definition 3.3. We see therefore that, if \( \mathcal{Y} \) is a quasi-independent set at \((\bar{x}, \bar{p})\), then the hypotheses of definition 3.10 are satisfied with \( W = (x_1, \ldots, x_n, \hat{p}_1, \ldots, \hat{p}_n) \). Hence \((x_1, \ldots, x_n, \hat{p}_1, \ldots, \hat{p}_n)\) is algebraically dependent on \( \mathcal{Y} \) at \((\bar{x}, \bar{p})\).

Viceversa, if \( \mathcal{Y} \supseteq (x_1, \ldots, x_n, \hat{p}_1, \ldots, \hat{p}_n) \) at \((\bar{x}, \bar{p})\), then from proposition 3.17 we have that \( r_\mathcal{Y}(\bar{x}, \bar{p}) \geq 2n \), so that the set \( \mathcal{Y} \) is quasi-independent at \((\bar{x}, \bar{p})\).

**Remark 3.3.** We have considered two main local conditions on a set of operators: quasi-independence and regular dependence. A justification for their names is provided by theorem 3.19 which shows that these conditions are mutually incompatible. One can however still ask if these conditions embrace all possible cases. Not surprisingly, the answer to this question is negative. We prove in fact in Appendix A that there exist sets of operators which are neither quasi-independent nor regularly dependent. This suggests that it might be possible to find improvements on the definitions of dependence and independence for sets of operators, so as to reduce or even suppress the gap between the two conditions. The achievement of such a goal is left for future investigations.
3.4 Integrable set of operators and integrable operator

In this section we give the definition of integrable operator $\mathcal{H} \in \mathcal{O}$: it is based on the concept of integrable set of operators of class $\mathcal{O}$. We give also a non-formal definition of quantum system and integrable quantum system. These definitions will be often employed in the following papers of this series. They are motivated by their correspondence with the analogous concepts of classical mechanics, which will be now briefly recalled (see reference [3]).

**Definition 3.15.** Let $A$ and $B$ be two differentiable functions defined on the $2n$-dimensional symplectic manifold $K \times \mathbb{R}^n_p$. Their Poisson bracket $\{ A, B \}$ is the function defined as

$$\{ A, B \} := \sum_{i=1}^{n} \left( \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial x_i} - \frac{\partial B}{\partial p_i} \frac{\partial A}{\partial x_i} \right).$$

(3.19)

We recall that a set of functions $V = (V_1, \ldots, V_l)$, $V_i : M \to \mathbb{R}$, $i = 1, \ldots, l$, is said to be **functionally independent** if the differentials $dV_1, \ldots, dV_l$ are linearly independent almost everywhere in $M$.

**Proposition 3.22.** Let $(V_1, \ldots, V_l)$ and $(W_1, \ldots, W_s)$ be two sets of functionally independent functions on a $2n$-dimensional symplectic manifold, such that $\{ V_i, W_k \} = 0$ for $i = 1, \ldots, l$ and $k = 1, \ldots, s$. Then $l + s \leq 2n$.

**Definition 3.16.** A set $V = (V_1, \ldots, V_k; V_{k+1}, \ldots, V_{2n-k})$ of functionally independent functions on a $2n$-dimensional symplectic manifold $M$ is said to be a (classically) integrable set on $M$, with $k$ central functions $V_1, \ldots, V_k$, if $\{ V_i, V_j \} = 0$ everywhere in $M$ for $i = 1, \ldots, k$ and $j = 1, \ldots, 2n - k$. If the hamiltonian $H$ of a dynamical system is in involution with all functions of the set $V$, we say that the system is **globally integrable with $k$ central functions** and with integrable set of invariants $V$. In this case it is easy to see that $H$ is locally dependent on the set of central functions, i.e., $H = f(V_1, \ldots, V_k)$.

The quantum equivalent of the preceding definition is contained in the two following ones.

**Definition 3.17 (quasi-integrable operator).** Let us consider the set of operators of class $\mathcal{O}_K$

$$\mathcal{W} = (W_1, \ldots, W_k; W_{k+1}, \ldots, W_{2n-k}),$$

(3.20)

with $0 < k \leq n$ (some of these operators may be functions only of $x$). Let these operators satisfy the following two conditions:

1. The $2n - k$ operators $W_1, \ldots, W_{2n-k}$ are quasi-independent at the point $(x, p) \in K \times \mathbb{R}^n_p$;

2. $[W_i, W_j] = 0$ for $i = 1, \ldots, k$ and $j = 1, \ldots, 2n - k$.

We then say that the operators $W_1, \ldots, W_{2n-k}$ are an **integrable set of operators at the point $(x, p)$ with $k$ central operators $W_1, \ldots, W_k$**.

If in condition 1, instead of independence at a point $(x, p)$, we have global quasi-independence in the full phase space $K \times \mathbb{R}^n_p$, then we say that this set is a **(globally) integrable set of operators on $K \times \mathbb{R}^n_p$**.
Let $\mathcal{H}$ be an operator of class $\mathcal{O}_K$ and let us suppose that the set $(\mathcal{H}, W_1, \ldots, W_k)$ is globally dependent, where $W_1, \ldots, W_k$ are central operators of an integrable set of operators $W$ at a point $(x, p)$. For example, $\mathcal{H} = S(W_1, \ldots, W_k)$, where $S$ is an arbitrary usual polynomial of $k$ variables, that is $S \in S^k$. We suppose also that $[\mathcal{H}, W_i] = 0$ for each $i = 1, \ldots, 2n - k$. In this case, we will say that $\mathcal{H}$ is a quasi-integrable operator with $k$ central operators at the point $(x, p)$. If the set $W$ is globally independent, then we will say that $\mathcal{H}$ is a (globally) quasi-integrable operator with $k$ central operators.

**Definition 3.18 (quasi-integrable quantum system).** Let $\mathcal{H}$ be an operator of class $\mathcal{O}_K$. Let some solutions of the equation $(\mathcal{H} - \lambda)\psi = 0$ describe phenomena of the microscopic world. By this we mean that they are (similar to) wave functions either of real microscopic systems, or of useful approximated models of these systems. In this case we will say that $\mathcal{H}$ defines a (scalar) quantum system of general type on the configuration space $K$ and $\mathcal{H}$ is the hamiltonian operator of the system. If the hamiltonian $\mathcal{H}$ is a quasi-integrable operator with $k$ central operators, we will say that the quantum system is quasi-integrable with $k$ central operators.

The symbol of a commutator can be calculated by means of the following lemma, which is an immediate consequence of proposition 3.2.

**Lemma 3.23.** If $A = \sum_{|\alpha| \leq m_1} A_\alpha(x)\hat{p}^\alpha$ and $B = \sum_{|\alpha| \leq m_2} B_\alpha(x)\hat{p}^\alpha$, then

$$(A, B)_{\text{sym}} = \sum_{1 \leq |\alpha| \leq m} \frac{1}{\alpha!} \left( \frac{\partial^{\alpha} A}{\partial \hat{p}^\alpha} \frac{\partial^{\alpha} B}{\partial x^\alpha} - \frac{\partial^{\alpha} B}{\partial \hat{p}^\alpha} \frac{\partial^{\alpha} A}{\partial x^\alpha} \right),$$

where $A = A_{\text{sym}}$, $B = B_{\text{sym}}$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $m := \max(m_1, m_2)$. We have for any nonvanishing term of this sum

$$\text{ord} \frac{1}{\alpha!} \left( \frac{\partial^{\alpha} A}{\partial \hat{p}^\alpha} \frac{\partial^{\alpha} B}{\partial x^\alpha} - \frac{\partial^{\alpha} B}{\partial \hat{p}^\alpha} \frac{\partial^{\alpha} A}{\partial x^\alpha} \right) \leq m_1 + m_2 - |\alpha| .$$

By making use of the preceding lemma one can easily prove the following one, which establishes an important connection between the symbol of the commutator of two operators, and the Poisson bracket between the main parts of the respective symbols.

**Lemma 3.24.** Let $\text{ord} A = k$, $\text{ord} B = l$. Then $\text{ord} [A, B] \leq g$, where $g := k + l - 1$. Moreover, the homogeneous parts of order $g$ of the commutator and of the Poisson bracket are connected by the relation

$$(H_g([A, B])_{\text{sym}} = H_g([A, B]) = H_g(\{MA, MB\}) = \{MA, MB\} . \quad (3.21)$$

Here $A = A_{\text{sym}}$, $B = B_{\text{sym}}$, and $H_g$ denotes the homogeneous part of order $g$. In particular, if $\text{ord} [A, B] = g$, then

$$(M[A, B])_{\text{sym}} = \{MA, MB\} .$$

In these formulas, $M[A, B]$ denotes the main part of the operator $[A, B]$, whereas $MA$ and $MB$ denote the main parts of symbols $A$ and $B$ respectively.
**Proposition 3.25.** Let \( W = (W_1, \ldots, W_k; W_{k+1}, \ldots, W_{2n-k}) \) be an integrable set of operators on \( K \), and let \( MW = (MW_1, \ldots, MW_k; MW_{k+1}, \ldots, MW_{2n-k}) \) denote the set of the main parts of their symbols, \( W_i = W_i^{\text{smh}} \) for \( i = 1, \ldots, 2n - k \). Then \( MW \) is a classically integrable set on \( K \times \mathbb{R}^n \).

**Proof.** According to definitions \( 3.17 \) and \( 3.4 \) the set \( MW = (MW_1, \ldots, MW_k; MW_{k+1}, \ldots, MW_{2n-k}) \) is functionally independent on \( M^{2n} = K \times \mathbb{R}^n \). Moreover, applying formula \( 3.21 \) we obtain that \( \{MW_i, MW_j\} = 0 \) everywhere in \( K \times \mathbb{R}^n \) for \( i = 1, \ldots, k, j = 1, \ldots, 2n - k \).

The following proposition represents the quantum analogue of proposition \( 3.22 \).

**Proposition 3.26.** Let \( (V_1, \ldots, V_l) \) and \( (W_1, \ldots, W_s) \) be two sets of quasi-independent operators, such that \( [V_i, W_k] = 0 \) for \( i = 1, \ldots, l, k = 1, \ldots, s \). Then \( l + s \leq 2n \).

**Proof.** According to definition \( 3.4 \) of quasi-independence, the sets \( (MV_1, \ldots, MV_l) \) and \( (MW_1, \ldots, MW_s) \) of the main parts of the symbols of the given operators are functionally independent. Moreover, it follows from lemma \( 3.24 \) that \( \{MV_i, MW_k\} = 0 \) for \( i = 1, \ldots, l, k = 1, \ldots, s \). The thesis then follows from proposition \( 3.22 \).

**Corollary 3.27.** Let an operator \( W_0 \) commute with all the \( 2n - k \) operators of the integrable set \( W \), see formula \( 3.20 \). Then the operators \( (W_0, W_1, \ldots, W_k) \) are not quasi-independent. Therefore, in definition \( 3.17 \) \( k \) is the maximal number of operators of the set \( W \) which commute with all operators of this set \( W \).

The last corollary motivates the definition \( 3.17 \) of central operators.

There exists some trick which usually allows one to obtain the property of quasi-independence for a set of commuting operators. Let \( A = \sum_{|\alpha| \leq m} A_{\alpha}(x) \hat{\beta}^\alpha \) be an operator of class \( \mathcal{O} = \mathcal{O}_n \), i.e., \( x = (x_1, \ldots, x_n) \). Let us associate with this operator the operator \( B = \Phi_A, B \in \mathcal{O}_{n+1} \), where \( B \equiv \sum_{|\beta| = m} B_{\beta}(x) \hat{\beta}^\beta \), \( x = (x_0, x_1, \ldots, x_n) \), \( \hat{\beta} := (\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_n) \). Here \( B_{\beta}(x) := A_{\alpha}(x), \alpha = \alpha(\beta) := (\alpha_1, \ldots, \alpha_n) \), with \( \alpha_i = \beta_i, i = 1, \ldots, n \). This embedding \( A \mapsto \Phi_A \) maps any operator \( A \) of class \( \mathcal{O}_n \) to an operator \( B = \Phi_A \) of class \( \mathcal{O}_{n+1} \), homogeneous with respect to \( \hat{p} \), whose coefficients are independent of \( x_0 \). Note that we have \( B = M \hat{B} \). It is easy to check that the following propositions are true.

**Proposition 3.28.** For any operators \( A, B \in \mathcal{O}_n \), we have \( [\Phi_A, \Phi_B] = \Phi_{[A,B]} \) and \( [\Phi_A, \hat{p}_0] = 0 \).

**Proposition 3.29.** Let the set of \( k \) functions \( (F_1, \ldots, F_k) \) be locally independent, where \( F_i \) is the symbol of an operator \( \mathcal{F}_i \) of class \( \mathcal{O}_n \) for \( i = 1, \ldots, k \). Then the set of \( k + 1 \) operators \( (\hat{p}_0, \Phi_{\mathcal{F}_1}, \ldots, \Phi_{\mathcal{F}_k}) \) is locally quasi-independent.

**Corollary 3.30.** Let \( (F_1, \ldots, F_k; F_{k+1}, \ldots, F_{2n-k}) \) be a set of operators of class \( \mathcal{O}_n \) such that \( [F_i, F_j] = 0 \) for \( i = 1, \ldots, k, j = 1, \ldots, 2n - k \). Let the set \( (F_1, \ldots, F_{2n-k}) \) of the symbols of these operators be globally functionally independent, i.e., their differentials are linearly independent almost everywhere. Then the set of operators \( (\hat{p}^0, \Phi_{\mathcal{F}_1}, \ldots, \Phi_{\mathcal{F}_k}; \Phi_{\mathcal{F}_{k+1}}, \ldots, \Phi_{\mathcal{F}_{2n-k}}) \), where \( \Phi_{\mathcal{F}_i} = \Phi_{\mathcal{F}_i} \) for \( i = 1, \ldots, 2n - k \), is a quasi-integrable set of operators.
Note that the eigenvalue equation \((A - \lambda)u = 0\) for the original operator \(A\), where \(u = u(x_1, \ldots, x_n)\), takes the form \((\Phi_A - \lambda \hat{p}_0)U = 0\) for the operator \(\Phi_A\), where \(U = U(x_0, x_1, \ldots, x_n)\). To the eigenfunction \(u(x)\) there corresponds the eigenfunction of special form \(U(x_0, x) = e^{x_0}u(x)\). Note also that the symbol \(F = \sum |\alpha| \leq m A_\alpha(x)p^\alpha\) of an operator \(F\) contains more monomials than its main part \(MF = \sum |\alpha| = m A_\alpha(x)p^\alpha\). Hence we can expect that the probability of not being quasi-independent is essentially lower for the set of operators \((\hat{p}_0, \Phi_1, \ldots, \Phi_{2n-k})\) than for the set \((F_1, \ldots, F_{2n-k})\).

**Remark 3.4.** The main object of the investigations of the present article is quantum mechanics. We underline however that the class of operators which we have been considering in this section includes arbitrary (scalar) linear differential operators which may also be used in other domains of mathematical physics.

An example is given by the operator
\[
\mathcal{H} = \frac{\partial}{\partial t} - \Delta_\xi + U(\xi),
\]
which corresponds to the heat equation
\[
\frac{\partial}{\partial t} \psi = \Delta_\xi \psi - U(\xi)\psi,
\]
where \(\xi = (\xi_1, \ldots, \xi_n)\), \(\Delta_\xi\) is the Laplace operator on \(\xi\), and \(\psi = \psi(t, \xi)\). If we take \(x = (t, \xi)\), then we obtain that the operator \(\mathcal{H}\) of \(n + 1\) variables \(x\) belongs to the class \(\mathcal{O}\) considered above. In case \(U(\xi) = U_1(\xi_1) + \cdots + U_n(\xi_n)\), the operator \(\mathcal{H}\) is an operator with separable variables. Moreover, it is obvious that \(\mathcal{H}\) is a quasi-integrable operator in the straight Liouville sense, that is with \(k = n + 1\) (see definition 3.18).

Any (scalar) quantum system is apparently described by special solutions of a linear differential equation with nonconstant real coefficients. For example, consider the Schrödinger equation
\[
-i \frac{\partial}{\partial t} \psi = \Delta_\xi \psi - U(\xi)\psi,
\]
with \(i = \sqrt{-1}\). In order to obtain an operator of class \(\mathcal{O}\) it must be converted into the heat equation
\[
\frac{\partial}{\partial \tau} \chi = \Delta_\xi \chi - U(\xi)\chi,
\]
with \(\chi = \chi(\tau, \xi)\). In quantum applications, we are interested in complex valued solutions of type \(\chi = \chi(\tau, \xi)\) with \(\tau = it\), where \(t \in \mathbb{R}, \xi \in \mathbb{R}^n\). In this case the functions
\[
\psi(t, \xi) = \chi(it, \xi)
\]
are wave functions of the quantum system which is described by the given Schrödinger equation. If we are interested in the diffusion of heat, we consider instead real valued solutions \(\chi(\tau, \xi)\) in the real variables \((\tau, \xi)\) of the heat equation. Note finally that we do not consider in this paper operators with small parameters. In particular, in our investigation, we take the Planck constant \(\hbar\) equal to 1 in the operators of quantum mechanics.
We have already observed that we do not know of any general definition of independence for a set of operators, which fully correspond to the notion of independence for a set of functions on a symplectic manifold. Nevertheless, for concrete cases connected with applications, the condition of quasi-independence is usually helpful to distinguish between integrable and nonintegrable quantum systems. Indeed, this fact is true for the familiar examples of quantum systems considered in section 2, as the following proposition shows.

**Proposition 3.31.** The Hamiltonian operators \( \hat{H} \) of the quantum systems of the three examples 2.1–2.3 are quasi-integrable, with central operators which are indicated in these examples.

**Remark 3.5.** The system with Hamiltonian

\[
\hat{H} = \sum_{i=1}^{n} \omega_i \left( \frac{\hat{p}_i^2 + x_i^2}{2} \right)
\]

is called \( n \)-dimensional quantum harmonic oscillator. This system is obviously quasi-integrable with \( k = n \), because the set of \( n \) operators

\[
\frac{\hat{p}_1^2 + x_1^2}{2}, \ldots, \frac{\hat{p}_n^2 + x_n^2}{2}
\]

is quasi-independent. Let us consider the case of complete resonance, which means that the vector of frequencies \( \omega = (\omega_1, \ldots, \omega_n) \) is proportional to a vector with integer components: \( \omega = c h \), where \( c \in \mathbb{R} \), \( \omega_1 \omega_2 \cdots \omega_n \neq 0 \), \( h = (h_1, \ldots, h_n) \in \mathbb{Z}^n \). In Part IV it will be shown that in this case the quantum system is (strongly) integrable with \( k = 1 \), like the corresponding classical system. In other words, the Hamiltonian \( \hat{H} \) of the system commutes with \( 2n - 1 \) quasi-independent operators. The proof of this fact is not based on the separation of variables. For the construction of such an integrable system of operators we will use the operation of “symmetrization” of the corresponding classical functions. This operation plays the main role in the construction of quantum integrable systems starting from classical integrable systems (quantization). This construction is described in Part II.

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**A Implications of regular dependence**

Theorem 3.19 asserts that a set of regularly dependent operators is not quasi-independent. It has been proved using the properties of the main parts of a set of operators, which one can derive from the condition of regular dependence. However, we want to show in this appendix that regular dependence has also relevant implications on the properties of the homogeneous parts of lower order.

Formula (3.8) gives an expression for the symbol of the homogeneous part of order \( \bar{d} \) of the operator \( S(\mathcal{G}, \mathcal{F}) \), where \( S \) is a noncommutative polynomial such that \( \deg(\mathcal{X}, \mathcal{G}) S = \bar{d} \). In order to write an expression also for the symbol of the
homogeneous part of order $d - 1$, it is useful to introduce preliminarily a short notation to indicate the “second-main part” of an operator or a polynomial.

**Definition A.1.** Given an operator $F$ of class $\mathcal{O}$, such that $\text{ord } F = m > 0$, we call second-main part of $F$ the homogeneous part of order $m - 1$ of $F$, see definition 3.3. We denote this part with the symbol $M' F$:

$$M' F := H_{m - 1}(F).$$

Correspondingly, we call second-main part of the symbol $F$ the function

$$M' F := H_{m - 1}(F) = (M' F)_{\text{smb}}.$$

If $\text{ord } F = 0$, we define $M' F := 0$ and $M' F := 0$.

Note that, at variance with the main part, the second-main part may be zero also for a nonvanishing operator of arbitrary order. For example, if $n = 1, F = x^p + \hat{p} + \cos x$, then $M F = x^p$, $M' F = 0$.

**Definition A.2.** Let $S \in S^l_r$ be a noncommutative polynomial, and $w = (w_1, \ldots, w_r)$ a set of integers. If $\text{deg}_w S = d > 0$ (see definition 3.8), we call second-main part of $S$ with weights $w$ the quasi-homogeneous part of degree $d - 1$ of $S$. We denote this part with the symbol $M'_{w'} S$:

$$M'_{w'} S := C_{d - 1, w}(S).$$

If $\text{deg}_w S = 0$, we define $M'_{w'} S := 0$.

If $F = (F_1, \ldots, F_r)$ is a set of operators of class $\mathcal{O}$, then we call second-main part of $F$ with respect to $F$ the polynomial $M'_F S := M'_{w'} S$, where $w_i = \text{ord } F_i \forall i = 1, \ldots, r$.

**Definition A.3.** Let $A$ and $B$ be two differentiable functions defined on the $2n$-dimensional symplectic manifold $K \times \mathbb{R}^r$. Their Poisson semi-bracket $\{A, B\}^+$ is the function defined as

$$\{A, B\}^+ := \sum_{i=1}^{n} \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial x_i}.$$

**Proposition A.1.** Let $W = (F, G)$ and $S \in S^l_r$ be as in the hypotheses of proposition 3.4. Let $\tilde{S} = (M_W S)_C$ and $\tilde{S}' = (M'_W S)_C$ denote the abelianizations of the main and second-main parts, respectively, of polynomial $S$ with respect to $W$. Then

$$(H_{d-1}(S(G, F)))_{\text{smb}} = \tilde{S}'(G, M F) + \sum_{i=1}^r \frac{\partial \tilde{S}}{\partial F_i}(G, M F) M' F_i + \sum_{i,j} A_{ij}(G, M F) \{M F_i, M F_j\}^+ + \sum_{i,h} B_{ih}(G, M F) \{M F_i, MG_h\}^+. \quad (A.2)$$

In the above formula, $A_{ij}$ and $B_{ih}$ are functions of $l + r$ real variables, which one can easily construct from the expression of $M_W S$. For instance, if $M_W S$
is the noncommutative monomial represented in formula (3.3), then
\[
\sum_{i,j} A_{ij}(G,MF){MF}_i,MF_j^+ = \sum_{h<k} (M_\beta)_C(G,MF) \frac{MF_{\beta_h}MF_{\beta_k}}{MF_{\beta_h}MF_{\beta_k}} {MF_{\beta_h}^+}_i {MF_{\beta_k}^+}_j,
\]
\[
\sum_{i,h} B_{ih}(G,MF){MF}_i,MG_h^+ = \sum_{h \leq k} (M_\beta)_C(G,MF) \sum_{j=1}^r \frac{\partial Z_k}{\partial G_j} {MF_{\beta_h}^+}_i {MG_{\beta_i}^+}_j,
\]
where \((M_\beta)_C\) denotes the abelianization of monomial \(M_\beta\). When \(M_\beta S\) is a generic quasi-homogeneous noncommutative polynomial, the expressions of \(A_{ij}\) and \(B_{ih}\) follow by linearity from the formulas written above.

We can now strengthen proposition 3.17 in the following way:

**Proposition A.2.** Let \(W = (W_1,\ldots,W_s)\) and \(Y = (Y_1,\ldots,Y_t)\) be two sets of operators of class \(O\), such that \(W \subseteq Y\) at a point \((\bar{x}, \bar{p})\) ∈ \(\mathbb{R}^n_p\). Then there exists a set of real numbers dependent on \((\bar{x}, \bar{p})\), which we call \(a_{ih}, b_{ih}, c_{ihk}\) and \(d_{ijk}\), with \(i, j = 1, \ldots, s\) and \(h, k = 1, \ldots, r\), such that at the point \((\bar{x}, \bar{p})\) the differentials of the functions \(MW_i, M'W_i, W_i^+ := \{MW_i,MW_j\}^+\) can be expressed as linear combinations of the differentials of the functions \(MY_h, Y_h^+ := \{MY_h,MY_k\}^+\) according to the relations

\[
d(MW_i) = \sum_{h=1}^r a_{ih}d(MY_h), \quad (A.3)
\]
\[
d(M'W_i) = \sum_{h=1}^r (a_{ih}d(M'Y_h) + b_{ih}d(MY_h)) + \sum_{h,k=1}^r c_{ihk}dY_h^+, \quad (A.4)
\]
\[
dW_i^+ = \sum_{h,k=1}^r a_{ih}a_{jh}dY_h^+ + \sum_{h=1}^r d_{ijk}d(MY_h) \quad (A.5)
\]
for \(i, j = 1, \ldots, s\). Note that the same coefficients \(a_{ih}\) appear in all three relations.

**Proof.** It is straightforward to verify that, if the sets \(W\) and \(Y\) satisfy relations (A.3) - (A.5) at \((\bar{x}, \bar{p})\), and at the same point the sets \(\bar{Y}\) and \(\bar{Y}'\) (in this order) satisfy relations of the same type (in general with other coefficients \(a', b', c'\) and \(d'\) in place of \(a, b, c, d\)), then still other relations of the same type hold at \((\bar{x}, \bar{p})\) between the sets \(\bar{W}\) and \(\bar{Y}'\). Taking into account this fact, it is immediate to see that it is enough to prove the proposition in the hypothesis that \(W\) is algebraically dependent on \(\bar{Y}\).

Let then \(S = (S_1,\ldots,S_s)\) be a set of polynomials of class \(S_N^{s+r}\) as in definition 5.10 and consider the two sets of functions \(\bar{S} = (\mathcal{M}(W,Y)S)_C, \bar{S}' = (\mathcal{M}'(W,Y)S)_C\). For all \((x,p)\) ∈ \(H \times \mathbb{R}_p^n\) we have

\[
0 = (H_{\tilde{d}_i}(S_i(W,Y)))^s_{\text{sub}} = \bar{S}_i(MW,MY)^s_{\text{sub}} \quad \forall i = 1,\ldots,s,
\]
where \(\tilde{d}_i = \deg_0(S_i(W,Y))\). From this, one can deduce that there exists a neighborhood \(O \subset H \times \mathbb{R}_p^n\) of \((x,\bar{p})\) and a vector function \(f\), such that \(MW(x,\bar{p}) = f(MY(x,\bar{p})) \forall (x,p) \in O\), see proposition 3.3. We also have

\[
d(MW_i)(x,p) = \sum_{h=1}^r \bar{a}_{ih}(MY)d(MY_h)(x,p) \quad \forall (x,p) \in O, \quad (A.6)
\]

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where $\tilde{a}_{ih}$, $i = 1, \ldots, s$, $h = 1, \ldots, r$, are the elements of the matrix

$$
\tilde{A}(MY) := -\left( \frac{\partial \tilde{S}}{\partial W}(f(MY), MY) \right)^{-1} \left( \frac{\partial \tilde{S}}{\partial Y}(f(MY), MY) \right).
$$

For $(x, p) = (\bar{x}, \bar{p})$ equality (A.6) coincides with (A.3), with

$$
a_{ih} = \tilde{a}_{ih}(MY(\bar{x}, \bar{p})) = -\sum_{j=1}^{s} U_{ij} \left( \frac{\partial \tilde{S}}{\partial Y}(\bar{W}, \bar{Y}) \right)_{jh}.
$$

In the above equality we introduced the matrix

$$
U := \left( \frac{\partial \tilde{S}}{\partial W}(\bar{W}, \bar{Y}) \right)^{-1},
$$

where $\bar{W} := MW(\bar{x}, \bar{p})$, $\bar{Y} := MY(\bar{x}, \bar{p})$.

Using proposition A.1 we also have at all points of $H \times \mathbb{R}^n_p$

$$
0 = (H_{d-1}(S_i(W, Y)))^{mb}
= \tilde{S}_i(MW, MY) + \sum_{k=1}^{s} \frac{\partial \tilde{S}_i}{\partial W_k}(MW, MY)M'W_k
+ \sum_{k=1}^{r} \frac{\partial \tilde{S}_i}{\partial Y_k}(MW, MY)M'Y_k + \sum_{h,k=1}^{s} A_{i,hk}(MW, MY)\{MW_h, MW_k\}^+
+ \sum_{h,k=1}^{s} \left[ B_{i,hk}(MW, MY)\{MY_h, MW_k\}^+ \right.
+ C_{i,hk}(MW, MY)\{MY_h, MY_k\}^+
+ \left. D_{i,hk}(MW, MY)\{MY_h, MY_k\}^+ \right] \quad \forall i = 1, \ldots, s,
$$

where $A_{i,hk}$, $B_{i,hk}$, $C_{i,hk}$, and $D_{i,hk}$ are given functions of $s + r$ variables. From definition (A.3) and formula (A.6) it follows that

$$
W_{hk}^+ := \{MW_h, MW_k\}^+ = \sum_{h',k'=1}^{r} \tilde{a}_{hh'\bar{a}_{kk'}Y_{h'k'}^+},
$$

$$
\{MW_h, MY_k\}^+ = \sum_{h'=1}^{r} \tilde{a}_{hh'}Y_{h'k}^+,
$$

$$
\{MY_k, MW_h\}^+ = \sum_{h'=1}^{r} \tilde{a}_{kh'}Y_{h'k}^+,
$$

where $Y_{hk}^+ := \{MY_h, MY_k\}^+$. We can therefore rewrite (A.8) as

$$
\tilde{S}_i(MW, MY) + \sum_{k=1}^{s} \frac{\partial \tilde{S}_i}{\partial W_k}(MW, MY)M'W_k
+ \sum_{k=1}^{r} \frac{\partial \tilde{S}_i}{\partial Y_k}(MW, MY)M'Y_k + \sum_{h,k=1}^{s} D_{i,hk}(MW, MY)Y_{hk}^+ = 0,
$$

(A.10)
where $MW = f(MY)$ and

$$D_{i,hk} = \sum_{h',k'=1}^r \tilde{a}_{h'k'} \overline{a}_{h'k'} A_{i,h'k'} + \sum_{h'=1}^r (\tilde{a}_{h'k'} B_{i,h'k} + \tilde{\alpha}_{h'k'} C_{i,h'k}) + D_{i,hk}.$$ 

By calculating the differential of (A.10) we get

$$\sum_{k=1}^s \frac{\partial \tilde{S}_i}{\partial W_k}(f(MY), MY)d(MW_k) + \sum_{k=1}^r \frac{\partial \tilde{S}_i}{\partial Y_k}(f(MY), MY)d(MY_k) + \sum_{h,k=1}^r D_{i,hk}(f(MY), MY)dY_{hk}^+ + \sum_{h=1}^r G_{i,h}(x,p)d(MY_h) = 0,$$

where $G_{i,h}(x,p)$ are given functions. For $(x,p) = (\bar{x}, \bar{p})$ the above equality implies (A.4), with

$$b_{ih} = -\sum_{j=1}^s U_{ij} G_{j,h}(\bar{x}, \bar{p}),$$

$$c_{ikh} = -\sum_{j=1}^s U_{ij} D_{j,hk}(\bar{x}, \bar{p}),$$

while $a_{ih}$ is still given by (A.2).

Finally, formula (A.5) is obtained in a similar way, by differentiating (A.9) at $(\bar{x}, \bar{y})$.

Let $r_W(x,p)$ denote the main dimension of a set of operators $W = (W_1, \ldots, W_s)$ at the point $(x,p) \in K \times \mathbb{R}_p^m$, see definition 3.14. According to theorem 3.19, the inequality $r_W(x,p) < s$ is a necessary condition for the regular dependence of $W$ at $(x,p)$. Let us suppose that $r_W(x,p) = s - 1$. This implies that there exists a nonvanishing vector $\gamma = \gamma(x,p) \in \mathbb{R}^s$, univocally determined apart from a multiplicative scalar coefficient, such that

$$\sum_{i=1}^s \gamma_i d(MW_i) = 0. \quad (A.11)$$

The following proposition provides an additional necessary condition for regular dependence, which involves the second-main parts ($MW_1, \ldots, MW_s$) of the symbols of the operators $W$, together with their main parts ($MW_1, \ldots, MW_s$) and the Poisson semi-brackets $W_{ij}^+ := \{MW_i, MW_j\}^+$.

**Proposition A.3.** Let $W = (W_1, \ldots, W_s)$ be a set of operators of class $O_K$, such that $r_W(\bar{x}, \bar{p}) = s - 1$, where $(\bar{x}, \bar{p}) \in K \times \mathbb{R}_p^m$. If the set $W$ is regularly dependent at $(\bar{x}, \bar{p})$, then there exists a neighborhood $O \subset K \times \mathbb{R}_p^m$ of $(\bar{x}, \bar{p})$ such that $\text{rank} W(x,p) = r_W(x,p) = s - 1$ for all $(x,p) \in O$. Furthermore, at all points of $O$ the cotangent vector

$$v = \sum_{i=1}^s \gamma_i d(MW_i),$$

where $\gamma = \gamma(x,p) \neq 0$ satisfies relation (A.11), belongs to the linear subspace $L \subseteq T^*\mathbb{R}_p^m$ spanned by the cotangent vectors $d(MW_i)$ and $dW_{ij}$ for $i,j = 1, \ldots, s$.  

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Proof. If the set \( W \) is regularly dependent at \((\bar{x}, \bar{p})\), then from \( r_W(\bar{x}, \bar{p}) = s - 1 \) and from corollary \( \text{(3.18)} \) it follows that \( \text{rank} \, W(\bar{x}, \bar{p}) = s - 1 \). Moreover, according to proposition \( \text{(3.16)} \) there exists a neighborhood \( O' \subseteq K \times \mathbb{R}^n_p \) of \((\bar{x}, \bar{p})\), such that \( \text{rank} \, W(x, p) \leq s - 1 \, \forall \, (x, p) \in O' \). On the other hand, from \( r_W(\bar{x}, \bar{p}) = s - 1 \) and from proposition \( \text{(3.16)} \) it follows that there exists another neighborhood \( O'' \subseteq K \times \mathbb{R}^n_p \) of \((\bar{x}, \bar{p})\), such that \( r_W(x, p) \geq s - 1 \, \forall \, (x, p) \in O'' \). Therefore, applying again corollary \( \text{(3.18)} \) we have that \( r_W(x, p) = \text{rank} \, W(x, p) = s - 1 \, \forall \, (x, p) \in O' \cap O'' \).

Let \( \mathcal{Y} = (\mathcal{Y}_1, \ldots, \mathcal{Y}_{s-1}) \) be a set of operators of class \( \mathcal{O}_K \), such that \( W \subseteq \mathcal{Y} \) at the point \((x, p) \in O \). We can then rewrite formula \((A.3)\) of proposition \(A.2\) as
\[
d(W_i) = \sum_{h=1}^{s-1} \alpha_{ih} d(MY_h), \quad i = 1, \ldots, s.
\]
We can always assume that the operators of set \( W \) have been ordered in such a way that the differentials \((d(W_i), \ldots, d(W_{s-1}))\) are linearly independent. If we consider the square matrix
\[
\bar{A} := (\alpha_{ih}, i, h = 1, \ldots, s - 1),
\]
it then follows from \((A.12)\) that \( \det \bar{A} \neq 0 \). We can thus write
\[
d(MY_i) = \sum_{h=1}^{s-1} \alpha_{ih} d(W_h), \quad i = 1, \ldots, s - 1, \tag{A.13}
\]
where
\[
\bar{A}^{-1} = (\alpha_{ih}, i, h = 1, \ldots, s - 1).
\]
Using formula \((A.3)\), with \( r = s - 1 \), we then obtain
\[
dY_{ij}^+ = \sum_{h,k=1}^{s-1} \alpha_{ih}\alpha_{jk} \left( dW_{hk}^+ - \sum_{l,m=1}^{s-1} d_{hkl}\alpha_{lm} d(MW_m) \right) \tag{A.14}
\]
for \( i, j = 1, \ldots, s - 1 \). Formulas \((A.13)-(A.14)\) imply that \( d(MY_i) \in L \) and \( dY_{ij}^+ \in L \, \forall \, i, j = 1, \ldots, s - 1 \).

It follows from proposition \(\text{(3.17)}\) that \( \mathcal{Y} \) is a quasi-independent set at \((x, p)\). Hence, equalities \((A.11)\) and \((A.12)\) imply that \( \sum_{i=1}^s \gamma_i a_{ih} = 0 \, \forall \, h = 1, \ldots, s - 1 \). If we multiply by \( \gamma_i \) both members of formula \((A.11)\), with \( r = s - 1 \), and then sum over \( i \) from 1 to \( s \), we thus obtain
\[
u = \sum_{i=1}^s \gamma_i \left( \sum_{h=1}^{s-1} b_{ih} d(MY_h) + \sum_{h,k=1}^{s-1} c_{hkh} dY_{h}^+ \right),
\]
whence \( v \in L \).

It is easy to construct examples of sets of operators \( W = (W_1, \ldots, W_s) \) such that \( r_W(\bar{x}, \bar{p}) = s - 1 \), which do not satisfy the necessary condition for regular dependence expressed by proposition \(A.3\). Consider for instance the case \( n = 1 \), \( s = 2 \), \( W = (W_1, W_2) \), with
\[
W_1 = \sum_{i=0}^l f_i(x) p^i, \quad W_2 = \sum_{i=0}^m g_i(x) p^i,
\]
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where $l, m \in \mathbb{N}$. Suppose that $f_l(x) \equiv g_m(x) \equiv 1$. Then $MW_1 = p^l$, $MW_2 = p^m$, so that $d(MW_1) = lp^{l-1}dp$ and $d(MW_2) = mp^{m-1}dp$. It follows that $r_{MW}(x, p) = 1$ for all $(x, p)$ such that $p \neq 0$, and $\gamma_1 d(MW_1) + \gamma_2 d(MW_2) = 0$, where $\gamma_1 = mp^m$, $\gamma_2 = -lp^l$. Let us consider the covector

$$v = \gamma_1 d(M'W_1) + \gamma_2 d(M'W_2)$$

$$= p^{l+m-1}[mf_{l-1}(x) - lg_{m-1}(x)]dx$$

$$+ p^{l+m-2}[m(l-1)f_{l-1}(x) - l(m-1)g_{m-1}(x)]dp.$$  \hfill (A.15)

Since $W_{ij}^+ \equiv 0$ for $i, j = 1, 2$, subspace $L \subseteq T^*\mathbb{R}^2_{xp}$ of proposition A.3 is the subspace of covectors which are multiple of $dp$. Therefore a necessary condition, in order for the set $W$ to be regularly dependent at $(\bar{x}, \bar{p})$, with $\bar{p} \neq 0$, is that the coefficient of $dx$, on the right-hand side of (A.15), vanishes for all $x$ belonging to an open neighborhood of $\bar{x}$. This is equivalent to the condition that in this neighborhood $mf_{l-1}(x) - lg_{m-1}(x) = 0$, or

$$mf_{l-1}(x) - lg_{m-1}(x) = c,$$ \hfill (A.16)

where $c$ is a constant. It follows that, if the functions $f_{l-1}(x)$ and $g_{m-1}(x)$ do not satisfy the above relation, then the set $W$ is neither quasi-independent nor regularly dependent.

Let us consider the case $l = m = 1$, $W_1 = \hat{p} + f(x)$, $W_2 = \hat{p} + g(x)$. Then necessary condition (A.16) for regular dependence becomes $f(x) = g(x) + c$. In this particularly simple case, this condition is also sufficient. When it is satisfied, we have in fact that the set $W = (W_1, W_2)$ satisfies the regular correlation $W_1 - W_2 - c = 0$. Hence, according to proposition 3.15 the set is regularly dependent.

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