The Schur Algorithm in Terms of System Realizations

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Dedicated to Moshe Livšic: A great man, thinker, philosopher and mathematician.

Abstract. The main goal of this paper is to demonstrate the usefulness of certain ideas from System Theory in the study of problems from complex analysis. With this paper, we also aim to encourage analysts, who might not be familiar with System Theory, colligations or operator models to take a closer look at these topics. For this reason, we present a short introduction to the necessary background. The method of system realizations of analytic functions often provides new insights into and interpretations of results relating to the objects under consideration. In this paper we will use a well-studied topic from classical analysis as an example. More precisely, we will look at the classical Schur algorithm from the perspective of System Theory. We will confine our considerations to rational inner functions. This will allow us to avoid questions involving limits and will enable us to concentrate on the algebraic aspects of the problem at hand. Given a non-negative integer \( n \), we describe all system realizations of a given rational inner function of degree \( n \) in terms of an appropriately constructed equivalence relation in the set of all unitary \((n+1)\times(n+1)\)-matrices. The concept of Redheffer coupling of colligations gives us the possibility to choose a particular representative from each equivalence class. The Schur algorithm for a rational inner function is, consequently, described in terms of the state space representation.

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NOTATION:
\( \mathbb{T} \) is the unit circle in the complex plane: \( \mathbb{T} = \{ t \in \mathbb{C} : |t| = 1 \} \)
\( \mathbb{D} \) is the unit disc in the complex plane: \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \)
\( \mathbb{D}^- \) is the exterior of the unit circle: \( \mathbb{D}^- = \{ z : 1 < |z| \leq \infty \} \).
\( \mathcal{M}_{p \times q} \) is the set of all \( p \times q \) (\( p \) rows, \( q \) columns) matrices with complex entries.
\( I_n \) - the identity \( n \times n \) matrix.

Table of Contents.

0. Introduction.
1. Rational Inner Functions.
2. The Schur Algorithm.
3. The System Representation of a Rational Inner Function.
4. Coupled Systems and The Schur Transfopmation: Input-Output Mappings.
5. The Redheffer Coupling of Unitary Colligations.
6. The Inverse Schur Transformation and Redheffer Couplings of Colligations.
7. One Step of the Schur Algorithm, Expressed in the Language of Colligations.
8. Hessenberg Matrices. The Householder Algorithm.
9. The Schur Algorithm in Terms of System Representations.
10. An Expression for the Colligation Matrix in Terms of the Schur Parameters.
11. On Work Related to System Theoretic Interpretations of the Schur Algorithm
12. Appendix: System Realizations of Inner Rational Functions.

0. Introduction

Up until the 1960s System Theory suggested that a system be considered only in terms of its input and output. A system was treated as a ‘black box’ with input and output terminals. Associated with each system was an ‘input-output’ mapping, considered to be of primary importance to the theory at the time. This approach, however, did not take the internal state of the system into account. It is to be assumed that an input signal will, in some way, influence the internal state of a system. Nevertheless, there was little discussion of the relationship between input and the inner state of a system until the introduction of State Space System Theory. This theory not only incorporated input and output spaces, serving, respectively, as ‘domains’ for input and output ‘signals’, but also a ‘state space’. This state space was introduced to describe the interior state of the system.

State Space System Theory (both the linear and general non-linear variations of the theory) was developed in the early 1960s. Two names closely associated with the early development of this theory are those of R. Kalman and M.S. Livshitz. Kalman’s first publications pertaining to State Space System Theory include [Kal1, Kal2, Kal3]. The monograph [KFA] summarizes these papers, among others. R. Kalman’s approach to State Space System Theory was from the perspective of Control Theory. This approach suggested that the questions of a system’s controllability and observability be given the most attention. Control Theory does not, however, put much emphasis on energy relations and, as a result, Kalman’s work does not address the subject of energy balance relations (Kalman’s approach to System Theory was abstract. He develops the theory over arbitrary fields, not specifically over the field of complex numbers). In Kalman’s theory, one
first starts from the input-output behavior (i.e. transfer function) and then constructs the state operator. In Livshitz’s theory, the reverse approach is used: The characteristic function (which is the analogue of the transfer function) is produced from the main operator (which is the analogue of the state operator). Kalman’s theory is mainly finite-dimensional and affine, whereas Livshitz’s theory is mainly infinite-dimensional and metric. It took some decades before the connections between these two theories were discovered in the 1970s. Among others, Dewilde [Dew1, Dew2] and Helton [He1, He2, He3] produced much of the work leading to this discovery. The connections between the two approaches were made explicit in the monograph [BGK].

M. S. Livshitz, a pioneer in the theory of non-self-adjoint operators, chose to approach State Space Theory from the perspective of Operator Theory. For a particular class of non-self-adjoint operators, Livshitz was able to associate each operator of this class with an analytic function in the upper half-plane or unit disc. These analytic functions were called ‘characteristic functions’. Livshitz was, furthermore, able to determine a correspondence between the invariant subspaces of a linear operator and the factors of its characteristic function (See [Liv3] and references within [Liv3]). Using the framework provided for by these results for characteristic functions, Livshitz constructed triangular models of non-self-adjoint operators (Triangular models were later partially supplanted by functional models. See [SzNFo]). Following this, Livshitz focused on questions in both mathematics and physics. Oscillation and wave propagation problems in linear isolated systems are related to self-adjoint operators. In the mid-1950s M.S. Livshitz began to look for a physical example to which his theory of non-self-adjoint operators could be applied. This lead him to consider a number of concrete linear systems. These systems were not isolated systems, but were such that they allowed for the exchange of energy with the ‘external world’. The model of the dynamical behavior of a system of this type makes use of an operator and this ‘principal’ operator is, in general, non-self-adjoint. The energy exchange of the system is reflected in the non-self-adjointness of the operator. Livshitz worked on problems involving the scattering of elementary particles (See [Liv4], [Liv5], [BrLi]), problems in electrical networks (See [LiFl]) and questions dealing with wave propagation in wave-guides (See [Liv6]). It was at this juncture that the notion of an ‘operator colligation’ (also common are the terms ‘operator node’ and ‘operator cluster’) was introduced to provide further clarity. An operator colligation consists of the aforementioned ‘principal’ operator, but also ‘channel’ spaces and ‘channel’ operators, of which the latter two objects describe the non-self-adjointness of the ‘principal’ operator. The introduction of this concept allowed a characteristic function to be associated with an operator colligation, as opposed to its respective ‘principal’ operator (See [BrLi], [Br], [LiYa] and references therein). At much the same time, the concept of an ‘open system’ was then being established (What Livshitz then referred to as an ‘open system’ was, in essence, what is now known as a stationary linear dynamical system). Livshitz first introduced the notion of an ‘open system’ in his influential paper, [Liv9] (See Definition 1 on p. 1002 of the original Russian paper [Liv7] or
p. ?? of the English translation in the present volume). To each system there is an associated colligation and in [Liv7] it is shown that a system’s transfer operator coincides with the characteristic function of the system’s colligation. [Liv8] introduces the operation of coupling open systems as well as the concept of closing coupling channels. [Liv8] furthermore introduces the ‘kymological resolution’ of an open system, i.e. the resolution of this system into a chain of simpler coupled open systems. These simpler systems correspond to the invariant subspaces of the ‘inner-state’ operator of the original open system. To emphasize that the notion of an open system is closely related to oscillations and to wave-propagation processes, Livshitz uses the terminology ‘kymological’, ‘kymmer’ and ‘kymmery’, derived from the Greek word ‘κυµα’, meaning ‘wave’. We quote from page 15 of the English translation of [Liv9] and mention that: ”the appropriate representation of an open system, transforming a known input into a known output, depends on which are known and unknown variables, so that the concept of an open system is ‘physico-logical’ rather than purely physical in nature.”

The relevant theory of open systems and operator colligations, as developed by Livshitz and other mathematicians, is presented in the monographs [Liv9], [LiYa] and [Br]. Chapter 2 of the monograph [Liv9] deals with the details of the kymological resolution of open systems (a concept of which much use is made in the following). A detailed presentation of Scattering Theory for linear stationary dynamical systems (with an emphasis on applications to the Wave Equation in \( \mathbb{R}^n \)) can be found in [LaPh].

General State Space System Theory, as developed by R. Kalman and M. S. Livshitz provides us with the proper setting and the necessary language for the further study of physical systems and various aspects of Control Theory. Despite the fact that State Space System Theory does not immediately lead to a solution of the initial physical or control problem, it does lead to some interesting related questions (mostly analytic). It should, furthermore, be noted that general State Space Theory’s importance extends beyond its significance within Control Theory and when applied to physical systems. M. S. Livshitz was very likely the first to understand that this theory had wide-reaching applications within mathematics, e.g. in Complex Analysis.

Analytic functions can be represented or specified in many ways, e.g. as Taylor-series, by decomposition into continuous fractions, or via representations as Cauchy or Fourier integrals. In the early half of the 1970s an additional method for representing an analytic function was introduced, namely the method of ‘system realization’. This theory has its origins in Synthesis Theory for linear electrical networks, the theory of linear control systems and the theory of operator colligations (and associated characteristic functions). M. S. Livshitz established the Theory of System Realizations and L. A. Sakhnovich, a former Ph.D. student of Livshitz’s, later made further important progress in the theory (See [Sakh1] and also [Sakh2] for a more detailed presentation of these results). L. A. Sakhnovich studied the spectral factorization of a given rational matrix-function \( R \), where both \( R \) and the
inverse function $R^{-1}$ are transfer functions corresponding to linear systems (operator colligations). Unfortunately, the paper [Sakh1] did not garner the attention it deserved at the time. L. A. Sakhnovich’s factorization theorem is a predecessor to a fundamental result due to Bart/Gohberg/Kaashoek/van Dooren [BGKV], which was remembered as Theorem 2 in the Editorial Introduction to [CWHF], where one can also find a detailed account of the history of the state space factorization theorem.

Our goal is not to provide a comprehensive survey of the history of System Theory, so that we have focused on the period leading up to the mid-1970s (with particular emphasis on the contributions of M. S. Livshitz and his co-workers). His work on open systems was unknown in the western world until his monograph [Liv9] was translated in 1973. His fundamental papers [Liv7] and [Liv8] remained untranslated up until this memorial volume.

The subsequent development of the Theory of System Realizations is generally associated with the name I. Gohberg, who produced and inspired much in the way of new work and results for this theory and its applications. As a result, the theory experienced a period of accelerated growth, beginning in the late 1970s. Published in 1979, the monograph [BGK] dealt with general factorizations of a rational matrix-functions as well as with the Wiener-Hopf factorization of rational matrix-function, where, in both cases, this function is a transfer function for a linear system (operator colligation).

I. Gohberg and his co-workers have shown that State Space Theory has a much wider range and goes far beyond System Theory and the theory of operator colligations. We list a few topics to which State Space Theory can be applied:

1. Methods of factorization of matrix- and operator-valued functions; solutions of Wiener-Hopf and singular integral equations.
2. Interpolation in the complex plane and generalizations.
3. Limit formulas of Akhiezer/Kac/Widom type.
4. Projection methods, Bezoutiants, resultants.
5. Inverse problems.

The monograph [BGR] offers a detailed discussion of interpolation problems and many other questions. Matrix-function factorization is a tool applied in discussions of many other problems as well, e.g. in the theory of inverse problems for differential equations and also in prediction theory for stationary stochastic processes. If a matrix-function is rational, then this factorization can be attained using system realizations. These system realizations, in turn, play a certain role in the solution of the original problem (See, for example, [AG]). The Theory of Isoprincipal Deformations of Rational Matrix-Functions (which is, in particular, a useful tool for investigating rational solutions of Schlesinger systems) is formulated in terms of the Theory of System Realizations (See [KaVo1] and [KaVo2]. For our purposes, the theory developed in [Ka] is most relevant). The current state of System Theory, as a branch of pure mathematics, is presented in [Nik].

1 N.b. There is now an extended version of this monograph. See [BGKR].
In the present paper we show how the Schur algorithm for contractive holomorphic functions in the unit disc can be described in terms of system realizations. In the following, we consider only rational inner functions, which allows us to avoid questions involving limits and enables us to concentrate on the algebraic aspects of the problem at hand. At first glance the formulas here presented might seem rather complicated and, to some degree, less than intuitive. This is, however, from the perspective of System Theory, not the case. The aforementioned formulas serve as the function-theoretical counterpart to Livshitz’s kymological resolution as applied to the system (represented by the original inner function) corresponding to the cascade coupling, i.e., the Redheffer coupling, of open systems. The elementary open systems, which make up this cascade (or chain) correspond to the steps of the Schur algorithm.

This paper is organized as follows. In Section 1, we state some facts relating to rational inner functions. In Section 2, we discuss some aspects of the classical Schur algorithm. Section 3 is devoted to a short introduction to operator colligations and their characteristic functions, where particular attention is paid to finite-dimensional unitary colligations. The characteristic functions of finite-dimensional unitary colligations are shown to be rational inner matrix-functions (See Theorem 3.5). Theorem 3.6 shows that an arbitrary rational inner matrix-function can, on the other hand, be realized as a characteristic function of a finite-dimensional minimal unitary colligation. The scalar rational inner functions of degree \( n \) are just the finite Blaschke products of \( n \) elementary Blaschke factors. The essential facts on the realization of scalar inner rational functions of degree \( n \) as characteristic functions of minimal unitary colligations are summarized in Theorem 3.10. These minimal unitary colligations can be equivalently described by equivalence classes of minimal unitary \((n+1) \times (n+1)\)-matrices. A proof for Theorem 3.10 can be found in the Appendix at the end of the paper.

The main objective of this paper can be described as follows. The application of the Schur algorithm to a given rational inner function \( s(z) \) of degree \( n \) produces a sequence \( s_k(z) \), \( k = 0, 1, \ldots, n \) of rational inner functions with \( s_0(z) = s(z) \) and \( \deg s_k(z) = n - k \). In particular, the function \( s_n(z) \) is constant with unimodular value. In Section 3 it will be shown that each of the functions \( s_k(z) \) admits a system representation

\[
s_k(z) = A_k + zB_k (I - zD_k)^{-1} C_k
\]

in terms of the blocks of some minimal unitary matrix \( U_k \in \mathbb{M}_{(n+1) \times (n+1)} \),

\[
U_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix}.
\]

We assume that \( U_0 \) is given. The goal is to recursively produce the sequence matrices \( U_k \). In other words, the steps of the Schur algorithm have to be described in terms of the state space representation. Since the unitary matrices \( U_k \) are defined only up to an equivalence relation, we have to find corresponding operations for the arithmetic of these equivalence classes.
In Section 4 we discuss the means by which the linear-fractional transformation associated with the Schur algorithm can be described in terms of the input–output mapping of linear systems. The Redheffer coupling of linear systems will be introduced as a useful tool in these considerations.

In Section 5, the Redheffer coupling of linear systems will be translated into the language of unitary systems.

In Section 6, we apply the concept of Redheffer couplings of colligations to the linear-fractional transformation associated with the inverse of the Schur algorithm. In so doing, we will describe the ‘degrees of freedom’ of unitary equivalence. A closer look shows us that amongst all the unitary matrices which realize the desired system realization, there are some distinguished by the fact that they are, in a sense, associated with the concept of Redheffer coupling.

In Section 7, the basic step of the Schur algorithm will be described in the language of colligations. This requires that we solve a particular equation for unitary matrices, suggested by the results of Section 6. The solution to this matrix equation is given in Theorem 7.1. Together with Lemma 7.2, Theorem 7.3 describes the basic step of the Schur algorithm in terms of system representations.

The investigations of Section 8 show that a certain normalization procedure has to be performed at every step of the Schur algorithm if the Schur algorithm is to be dealt with in the language of system realizations. We consider the degrees of freedom for this normalization procedure. It turns out that we can use these degrees of freedom to make the normalization procedure a one-time-procedure, so that it might be dealt with during preprocessing for further step-by-step recurrence. A one-time-normalization of this kind is related to the reduction of the ‘initial’ colligation matrix to the lower Hessenberg matrix.

In Section 9, we will be well-positioned to present the Schur algorithm in terms of unitary colligations representing the appropriate functions.

In Section 10 we express the colligation matrix in terms of the Schur parameters.

In the final section (Section 11) we discuss some connections between the present work and other work relating to the Schur algorithm as expressed in terms of system realizations. In particular, we discuss the results presented in Alpay/Azizov/Dijksma/Langer [AADL] and Killip/Nenciu [KNe].

1. Rational Inner Functions

We say that a function \( s : \mathbb{D} \rightarrow \mathbb{C} \), where \( s \) is holomorphic in \( \mathbb{D} \), is contractive if

\[
|s(z)| \leq 1 \quad \text{for every } z \in \mathbb{D}.
\]

A contractive function \( s \) is called an inner function if

\[
|s(t)| = 1 \quad \text{for every } t \in \mathbb{T}.
\]
In the following we consider rational inner functions, so that \( s(t) \) is defined for every \( t \in \mathbb{T} \).

A rational function is representable as a quotient of irreducible polynomials and we call the order of the highest-degree polynomial the \textit{degree} of the rational function.

If a rational function \( s \) is an inner function, then the degree of its numerator and the degree of its denominator are equal.

An inner rational function \( s \) is representable as a finite Blaschke product, i.e. in the form
\[
s(z) = c \prod_{1 \leq k \leq n} \frac{z_k - z}{1 - \overline{z_k} z}.
\]  
(1.1)

\( z_1, \ldots, z_n \) are points in \( \mathbb{D} \), or, in other words, complex numbers satisfying the condition
\[
|z_1| < 1, \ldots, |z_n| < 1,
\]  
(1.2)
c is a unimodular complex number, i.e.
\[
|c| = 1.
\]  
(1.3)

Conversely, given complex numbers \( z_1, \ldots, z_n \) and \( c \) satisfying the conditions (1.2) and (1.3), respectively, the function \( s \) in (1.1) is an inner rational function of degree \( n \).

The number \( c \) and the set \( \{z_1, \ldots, z_n\} \) are uniquely defined by the inner function \( s \) (the \textit{sequence} of numbers \( (z_1, \ldots, z_n) \)) up to permutation.

The notions of contractive and inner functions can also be defined for matrix-functions:

We say that a matrix-function \( S : \mathbb{D} \rightarrow \mathbb{M}_{p \times p} \), where \( S \) is holomorphic in \( \mathbb{D} \), is \textit{contractive} if
\[
I_p - S^*(z)S(z) \geq 0 \text{ for every } z \in \mathbb{D}.
\]
A contractive matrix-function \( S : \mathbb{D} \rightarrow \mathbb{M}_{p \times p} \), is called an \textit{inner function} if \(^2\)
\[
I_p - S^*(t)S(t) = 0 \text{ for almost every } t \in \mathbb{T}.
\]

2. The Schur Algorithm

In this section, we present a short introduction to the classical Shur algorithm, which orginated in Issai Schur’s renowned paper, [Sch]. In so doing, we will mainly emphasize those aspects of the Schur algorithm, which are essential for this paper.

For comprehensive treatments of the Schur algorithm and its matricial generalizations, we refer the reader to [BFK1], [BFK2], [Con2], [DFK], [S:Meth] and the references therein.

Let \( s(z) \) be a contractive holomorphic function in \( \mathbb{D} \) and
\[
s_0 = s(0).
\]  
(2.1)

\(^2\)For a contractive holomorphic function \( S \) in \( \mathbb{D} \), the boundary values \( S(t) \) exist for almost every \( t \in \mathbb{T} \) (with respect to the Lebesgue measure).
Then $|s_0| \leq 1$, where $|s_0| = 1$ only if $s(z) \equiv s_0$. If $|s_0| < 1$, then the function

$$\omega(z) = \frac{1}{z} \frac{s(z) - s_0}{1 - s(z)\overline{s_0}}$$

(2.2)

is well-defined. Moreover, it is contractive holomorphic in $D$. The function $s(z)$ can be expressed in terms of these $\omega(z)$ and $s_0$:

$$s(z) = \frac{s_0 + z\omega(z)}{1 + z\overline{s_0}\omega(z)}.$$  

(2.3)

If the function $s(z)$ is an inner function, then $\omega(z)$ is also an inner function. If $s(z)$ is an inner rational function of degree $n$, then $\omega(z)$ is an inner rational function of degree $n - 1$.

Conversely, if $\omega(z)$ is an arbitrary contractive holomorphic function in $D$ and $s_0$ is an arbitrary complex number satisfying the condition $|s_0| < 1$, then the expression on the right-hand side of (2.2) defines the function $s(z)$, which is holomorphic and contractive in $D$. Furthermore, if $\omega(z)$ is an inner function, then $s(z)$ is an inner function as well.

**DEFINITION 2.1.**

I. We call the transformation $s(z) \mapsto \omega(z)$, defined by (2.2), where $s_0 = s(0)$, the (direct) Schur transformation.

II. We call the transformation $\omega(z) \mapsto s(z)$, defined by (2.3), where $s_0$ is a given complex number, the inverse Schur transformation.

The correspondence $s(z) \iff (s(0), \omega(z))$ describes the elementary step of the Schur algorithm.

The Schur algorithm is applied to a holomorphic function $s(z)$, which is contractive in $D$. This algorithm inductively produces the sequence (finite or infinite) of contractive holomorphic functions $s_k(z)$ in $D$ and contractive numbers $s_k = s_k(0)$, $k = 0, 1, 2, \ldots$. The algorithm terminates only if $s(z)$ is a rational inner function. Starting from $s(z)$, we define

$$s_0(z) = s(z), \quad s_0 = s_0(0).$$

If the functions $s_i(z)$, $i = 0, 1, \ldots, k$ are already constructed and $|s_k(0)| < 1$, then we construct the function $s_{k+1}(z)$ as follows:

$$s_{k+1}(z) = \frac{1}{z} \frac{s_k - s_k(z)}{1 - s_k(z)s_k}, \quad s_{k+1} = s_{k+1}(0).$$

(2.4)

If $s(z)$ is not a rational inner function, then the algorithm does not terminate: On the $k$-th step we obtain the function $s_k(z)$, for which $|s_k(0)| < 1$, so that we can construct the function $s_{k+1}(z)$ and still have $|s_{k+1}(0)| < 1$.

If $s(z)$ is a rational inner function of degree $n$, then we can define the functions $s_i(z)$ for $i = 0, 1, \ldots, n$ such that

$$\deg s_i(z) = n - i, \quad i = 0, 1, \ldots, n.$$
The numbers \( s_i = s_i(0) \) satisfy the conditions
\[
|s_i| < 1, \quad i = 0, 1, \ldots, n - 1.
\]
However, in this case
\[
|s_n| = 1, \quad s_n(z) \equiv s_n.
\]
So, for \( k = n \) the numerator and the denominator of the expression on the right-hand side of (2.4) vanish identically. The function \( s_{n+1}(z) \) is thus not defined and the Schur algorithm terminates.

The numbers \( s_k = s_k(0) \) are called the Schur parameters of the function \( s(z) \).

If \( s(z) \) is not an inner rational function, then the sequence of its Schur parameters is infinite and these parameters \( s_k \) satisfy the inequality \( |s_k| < 1 \) for all \( k : 0 \leq k < \infty \). If \( s(z) \) is an inner rational function with \( \text{deg}
 s(z) = n \), then its Schur parameters \( s_k \) are defined only for \( k = 0, 1, \ldots, n \) and
\[
|s_k| < 1, \quad k = 0, 1, \ldots, n - 1, \quad |s_n| = 1. \tag{2.5}
\]

Conversely, given complex numbers \( s_0, s_1, \ldots, s_n \) satisfying the conditions (2.5), one can construct the inner rational function of degree \( n \), having Schur parameters \( s_0, s_1, \ldots, s_n \). This function \( s(z) \) can be constructed inductively: First, we set
\[
s_n(z) \equiv s_n.
\]
If the functions \( s_i(z) \) for \( i = n, n - 1, \ldots, k \) are already constructed, then we set
\[
s_{k-1}(z) = \frac{s_{k-1} + z s_k(z)}{1 + z s_{k-1} s_k(z)}.
\]
In the final step we construct the function \( s_0(z) \) and set
\[
s(z) = s_0(z).
\]
Thus, there exists a one-to-one correspondence between rational inner functions of degree \( n \) and sequences of complex numbers \( \{s_k\}_{0 \leq k \leq n} \) satisfying the conditions (2.5).

3. The System Representation of a Rational Inner Function.

Contractive holomorphic functions appear in several roles. In particular, such functions appear in Operator Theory as the characteristic functions of operator colligations. The notion of an operator colligation is closely related to that of a linear stationary dynamical system. There is a correspondence between the theory of operator colligations and the theory of linear stationary dynamical systems. The concepts and results of one theory can be translated into the language of the other. There are interesting connections to be made between these theories. Definitions and constructions, which are well-motivated and natural in the framework of one theory may look artificial in the framework of the other. In particular, the notion
of the characteristic function of a colligation and of the coupling of colligations are more transparent in the language of System Theory.

In this section, the term ‘operator’ means ‘continuous linear operator’.

**DEFINITION 3.1.** Let \( \mathcal{H}, \mathcal{E}^{in}, \mathcal{E}^{out} \) be Hilbert spaces and \( U \) be an operator:

\[
U : \mathcal{E}^{in} \oplus \mathcal{H} \rightarrow \mathcal{E}^{out} \oplus \mathcal{H},
\]

(3.1)

Let

\[
U = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\]

(3.2)

be the block decomposition of the operator \( U \), corresponding to (3.1):

\[
A : \mathcal{E}^{in} \rightarrow \mathcal{E}^{out}, \quad B : \mathcal{H} \rightarrow \mathcal{E}^{out}, \quad C : \mathcal{E}^{in} \rightarrow \mathcal{H}, \quad D : \mathcal{H} \rightarrow \mathcal{H}.
\]

(3.3)

The quadruple \( (\mathcal{E}^{in}, \mathcal{E}^{out}, \mathcal{H}, U) \) is called an operator colligation. \( \mathcal{E}^{in} \) and \( \mathcal{E}^{out} \) are, respectively, the input and output spaces of the colligation. We call \( \mathcal{H} \) the state space of the colligation and \( A \) the exterior operator. We call \( B \) and \( C \) channel operators, while \( D \) is referred to as the principal operator of the colligation. Finally, we call \( U \) the colligation operator.

If the input and the output spaces \( \mathcal{E}^{in} \) and \( \mathcal{E}^{out} \) coincide: \( \mathcal{E}^{in} = \mathcal{E}^{out} = \mathcal{E} \), we call the space \( \mathcal{E} \) the exterior space of the colligation and denote the colligation by the triple \( (\mathcal{E}, \mathcal{H}, U) \).

**DEFINITION 3.2.** Let \( (\mathcal{E}^{in}, \mathcal{E}^{out}, \mathcal{H}, U) \) be an operator colligation.

The operator-function

\[
S(z) = A + zB(I_{\mathcal{H}} - zD)^{-1}C
\]

(3.4)

is called the characteristic function of the colligation \( (\mathcal{E}^{in}, \mathcal{E}^{out}, \mathcal{H}, U) \).

The function \( S(z) \) is defined for the \( z \in \mathbb{C} \) where the operator \( (I_{\mathcal{H}} - zD)^{-1} \) exists. The values of \( S \) are operators acting from \( \mathcal{E}^{in} \) into \( \mathcal{E}^{out} \).

**REMARK 3.1.** The function \( S(z) \) is defined and holomorphic in some neighborhood of the point \( z = 0 \). Furthermore, \( S(0) = A \).

The notion of a colligation’s characteristic function draws on the framework of the theory of linear stationary dynamical systems (LSDS). (The theory of open systems, in the terminology of M.S.Livšic). The theory of LSDS, which we are dealing with is not a ‘black box theory’, where only the input signals, output signals and the mapping ‘input → output’ are considered. The theory of LSDS also takes ‘interior states’ of the system into account. The input and output signals are described (in the discrete time case, where the index \( k \) serves as time) by sequences \( \{\varphi_k\}_{0 \leq k < \infty} \) and \( \{\psi_k\}_{0 \leq k < \infty} \) of vectors belonging to some Hilbert spaces \( \mathcal{E}^{in} \) and \( \mathcal{E}^{out} \) (the input and the output spaces of the system). The ‘interior states’ are described by vectors \( h \) of a Hilbert space \( \mathcal{H} \), called the state space of the system.
The dynamics of a linear stationary system is described by the linear equations
\[
\begin{bmatrix}
\psi_k \\
h_{k+1}
\end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \varphi_k \\
h_k
\end{bmatrix}, \quad k = 0, 1, 2, \ldots , \tag{3.5}
\]
where the operators \(A, B, C, D\) do not depend on \(k\) (‘time’) and are defined in (3.3).

It is natural to consider the four operators \(A, B, C, D\) as blocks of the ‘unified’ operator, say \(U\) as in (3.2), from the space \(E^{in} \oplus H\) into the space \(E^{out} \oplus H\). The operator colligation \((E^{in}, E^{out}, H, U)\) then corresponds to the LSDS (3.5), (3.3).

Given the sequence \(\{\varphi_k\}_{0 \leq k \leq m}\) and the initial value \(h_0\), the system (3.5) uniquely determines the sequences \(\{\psi_k\}_{0 \leq k \leq m}\) and \(\{h_k\}_{0 \leq k \leq m+1}\). In the case \(h_0 = 0\),
\[
\psi_0 = A\varphi_0, \quad \psi_m = A\varphi_m + \sum_{1 \leq k \leq m-1} BD^k C \varphi_{m-k-1}, \quad m \geq 1. \tag{3.6}
\]
The relation (3.6) can be considered as the description of the evolution of the LSDS (3.5) in the time domain. The description of the evolution is, however, especially transparent in the frequency domain. Since the considered sequences are unilateral, the Fourier transforms of these sequences are the (formal) power series
\[
\varphi(z) = \sum_{0 \leq k < \infty} \varphi_k z^k, \quad \psi(z) = \sum_{0 \leq k < \infty} \psi_k z^k, \quad h(z) = \sum_{0 \leq k < \infty} h_k z^k. \tag{3.7}
\]
The complex variable \(z\) can be interpreted as the frequency. Under the extra assumption that \(h_0 = 0\) we can rewrite (3.5) in terms of the Fourier representations:
\[
\begin{bmatrix}
\psi(z) \\
z^{-1}h(z)
\end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \varphi(z) \\
h(z)
\end{bmatrix}. \tag{3.8}
\]
From (3.8) we obtain
\[
\psi(z) = A\varphi(z) + Bh(z), \tag{3.9a}
\]
\[
h(z) = z(I - zD)^{-1}C\varphi(z). \tag{3.9b}
\]
Eliminating \(h(z)\), we get
\[
\psi(z) = S(z)\varphi(z), \tag{3.10}
\]
where \(S(z)\) is expressed in terms of the matrix \(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\) as in (3.4):
\[
S(z) = A + zB(I - zD)^{-1}C.
\]
The operator function \(S(z)\) describes the input-output mapping corresponding to LSDS (3.5).

**DEFINITION 3.3.** In the framework of System Theory, the function \(S(z)\) in (3.10) is called the **transfer matrix** of the LSDS (3.5).

In the theory of operator colligations the operator function \(S(z)\) is called the **characteristic function**, while in the theory of LSDS it is called the **transfer function**. This notion, however, makes more sense in the theory of LSDS. Along
with the input-output mapping described by the transfer function $S(z)$, the input-state mapping:

$$\varphi(z) \to h(z), \quad \text{where} \quad h(z) = z(I - zD)^{-1}C\varphi(z),$$

is also naturally related to the system (3.5).

If the dimensions $\dim \mathcal{E}^\text{in}$ and $\dim \mathcal{E}^\text{out}$ of the input and output spaces are finite, then, choosing bases in $\mathcal{E}^\text{in}$ and $\mathcal{E}^\text{out}$, we can consider $S(z)$ as a matrix-valued function. If, moreover, the dimension $\dim \mathcal{H}$ of the state space is finite, then $S(z)$ is a rational matrix-function.

**DEFINITION 3.4.** The colligation $(\mathcal{E}^\text{in}, \mathcal{E}^\text{out}, \mathcal{H}, U)$ is said to be finite-dimensional if $\dim \mathcal{E}^\text{in} < \infty$, $\dim \mathcal{E}^\text{out} < \infty$ and $\dim \mathcal{H} < \infty$.

The dimension $\dim \mathcal{H}$ of the state space of the finite-dimensional colligation $(\mathcal{E}^\text{in}, \mathcal{E}^\text{out}, \mathcal{H}, U)$ is related to the degree of its characteristic function. Here we use the notion of the McMillan degree of a rational matrix-valued function as it is defined in [McM]. The notion of the degree of a rational matrix-function is discussed in [DuHa] and [Kal4]. See also [BGK]. In the case when $\dim \mathcal{E} = 1$, i.e. in the case when the considered rational function is scalar (or $\mathbb{C}$-valued), the McMillan degree of this function coincides with its ‘standard’ degree.

To precisely formulate how the dimension of the state space $\mathcal{H}$ and the degree of the characteristic function $S(z)$ are related, we need to introduce the notion of a minimal colligation $(\mathcal{E}^\text{in}, \mathcal{E}^\text{out}, \mathcal{H}, U)$.

**DEFINITION 3.5.** Let $(\mathcal{E}^\text{in}, \mathcal{E}^\text{out}, \mathcal{H}, U)$ be a colligation. We define the following subspaces of the state space $\mathcal{H}$:

$$\mathcal{H}^c = \text{clos} \left( \bigvee_{0 \leq k < \infty} (D^k C)^{\mathcal{E}^\text{in}} \right), \quad \mathcal{H}^o = \text{clos} \left( \bigvee_{0 \leq k < \infty} (D^k B^*)^{\mathcal{E}^\text{out}} \right),$$

where $\bigvee M$ denotes the linear hull of the vectors $f_k$ and $\text{clos}(M)$ denotes the closure of the set $M$.

The subspaces $\mathcal{H}^c$ and $\mathcal{H}^o$ are, respectively, called the controllability and observability subspaces of the colligation $(\mathcal{E}^\text{in}, \mathcal{E}^\text{out}, \mathcal{H}, U)$.

**REMARK 3.2.** If the state space $\mathcal{H}$ is finite-dimensional, say $\dim \mathcal{H} = n < \infty$, then it is enough to restrict our considerations in (3.11) to the linear hull of the vectors $(D^k C)^{\mathcal{E}^\text{in}}$ and $(D^k B^*)^{\mathcal{E}^\text{out}}$ with $k < n$. In this case there is no need to make use of the closure in (3.11).

**DEFINITION 3.6.** We say that a colligation $(\mathcal{E}^\text{in}, \mathcal{E}^\text{out}, \mathcal{H}, U)$ is controllable if $\mathcal{H}^c = \mathcal{H}$ and observable if $\mathcal{H}^o = \mathcal{H}$.

We say that a colligation is simple if the sum of the controllability and the observability subspaces is dense in the state space, i.e. if

$$\text{clos}(\mathcal{H}^c + \mathcal{H}^o) = \mathcal{H}.$$
We say that a colligation \((\mathcal{E}^{\text{in}}, \mathcal{E}^{\text{out}}, \mathcal{H}, U)\) is **minimal** if it is both controllable and observable, i.e. if

\[
\mathcal{H}^{\text{c}} = \mathcal{H} \quad \text{and} \quad \mathcal{H}^{\text{o}} = \mathcal{H}.
\]

**THEOREM 3.1.** Let \((\mathcal{E}^{\text{in}}, \mathcal{E}^{\text{out}}, \mathcal{H}, U)\) be a finite-dimensional colligation and let \(S(z)\) be the characteristic function of this colligation.

\(S(z)\) is then a rational matrix-function, which is holomorphic at \(z = 0\) and such that

\[
\deg S \leq \dim \mathcal{H} \tag{3.12}
\]

Equality holds in (3.12) if and only if the colligation \((\mathcal{E}^{\text{in}}, \mathcal{E}^{\text{out}}, \mathcal{H}, U)\) is minimal.

**THEOREM 3.2.** Let \(\mathcal{E}_1\) and \(\mathcal{E}_2\) be finite-dimensional spaces and let \(S(z)\) be a rational function, whose values are operators acting from \(\mathcal{E}_1\) to \(\mathcal{E}_2\) and which is holomorphic at the point \(z = 0\).

There then exists a finite-dimensional minimal operator colligation \((\mathcal{E}_1^{\text{in}}, \mathcal{E}_1^{\text{out}}, \mathcal{H}, U)\) and \((\mathcal{E}_2^{\text{in}}, \mathcal{E}_2^{\text{out}}, \mathcal{H}, U)\), with \(\mathcal{E}_1^{\text{in}} = \mathcal{E}_1\) and \(\mathcal{E}_2^{\text{out}} = \mathcal{E}_2\), whose characteristic function \(S_U(z) = A + zB(I - zD)^{-1}C\) coincides with the original function \(S(z)\). In other words, \(S\) can be expressed in the form (3.4).

**DEFINITION 3.7.** The representation of a given function \(S(z)\) as a characteristic function of an operator colligation is called the **state space representation** of \(S(z)\) or the **state space realization** of \(S(z)\). If the representative operator colligation is minimal, then we say that the state space realization of \(S(z)\) is **minimal**.

Let us discuss the uniqueness of the state space representation.

**DEFINITION 3.8.** Let \((\mathcal{E}_1^{\text{in}}, \mathcal{E}_1^{\text{out}}, \mathcal{H}_1, U_1)\) and \((\mathcal{E}_2^{\text{in}}, \mathcal{E}_2^{\text{out}}, \mathcal{H}_2, U_2)\) be two operator colligations:

\[
U_1 = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}, \quad U_2 = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}, \tag{3.13}
\]

where

\[
A_i : \mathcal{E}_i^{\text{in}} \to \mathcal{E}_i^{\text{out}}, \quad B_i : \mathcal{H}_i \to \mathcal{E}_i^{\text{out}}, \quad C_i : \mathcal{E}_i^{\text{in}} \to \mathcal{H}_i, \quad D_i : \mathcal{H}_i \to \mathcal{H}_i, \quad i = 1, 2. \tag{3.14}
\]

We consider the colligations \((\mathcal{E}_1^{\text{in}}, \mathcal{E}_1^{\text{out}}, \mathcal{H}_1, U_1)\) and \((\mathcal{E}_2^{\text{in}}, \mathcal{E}_2^{\text{out}}, \mathcal{H}_2, U_2)\) to be equivalent if invertible operators \(E^{\text{in}}, E^{\text{out}}\) and \(V\) exist, such that the intertwining relation

\[
E^{\text{in}} : \mathcal{E}_2^{\text{in}} \to \mathcal{E}_1^{\text{in}}, \quad E^{\text{out}} : \mathcal{E}_2^{\text{out}} \to \mathcal{E}_1^{\text{out}}, \quad V : \mathcal{H}_2 = \mathcal{H}_1, \tag{3.15}
\]

exist, such that the intertwining relation

\[
\begin{bmatrix} E^{\text{out}} & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} E^{\text{in}} & 0 \\ 0 & V \end{bmatrix} \tag{3.16}
\]

holds.
Clearly, given two equivalent operator colligations, one of these colligations is controllable, observable, simple or minimal if and only if the other colligation possesses the same respective property.

The following result is evident:

**Theorem 3.3.** Let \((\mathcal{E}_{1}^{in}, \mathcal{E}_{1}^{out}, \mathcal{H}_{1}, U_{1})\) and \((\mathcal{E}_{2}^{in}, \mathcal{E}_{2}^{out}, \mathcal{H}_{2}, U_{2})\) be operator colligations. Assume that these colligations are equivalent, i.e. that the intertwining relation (3.16) holds with some invertible operators \(E_{in}, E_{out}\) and \(V\).

Then the characteristic functions \(S_{1}(z)\) and \(S_{2}(z)\) of these colligations, \(S_{i}(z) = A_{i} + zB_{i}(I - zD_{i})^{-1}C_{i}, \quad i = 1, 2,\) (3.17)
satisfy the intertwining relation:

\[E_{out}S_{2}(z) = S_{1}(z)E_{in},\] (3.18)

for all \(z\) where \(S_{1}\) and \(S_{2}\) are defined.

Under the extra assumptions that the colligations are minimal and finite-dimensional we can show that for Theorem 3.3 the converse assertion also holds.

**Theorem 3.4.** Let \((\mathcal{E}_{1}^{in}, \mathcal{E}_{1}^{out}, \mathcal{H}_{1}, U_{1})\) and \((\mathcal{E}_{2}^{in}, \mathcal{E}_{2}^{out}, \mathcal{H}_{2}, U_{2})\) be finite-dimensional operator colligations. Let \(S_{1}(z)\) and \(S_{2}(z)\), (3.17), be the characteristic functions of these colligations. We make the following assumptions:

1. The functions \(S_{1}(z)\) and \(S_{2}(z)\) satisfy the intertwining relation (3.18) for all \(z\) small enough, where \(E_{in} : \mathcal{E}_{2}^{in} \rightarrow \mathcal{E}_{1}^{in}\) and \(E_{out} : \mathcal{E}_{2}^{out} \rightarrow \mathcal{E}_{1}^{out}\) are some invertible operators.
2. The colligations \((\mathcal{E}_{1}^{in}, \mathcal{E}_{1}^{out}, \mathcal{H}_{1}, U_{1})\) and \((\mathcal{E}_{2}^{in}, \mathcal{E}_{2}^{out}, \mathcal{H}_{2}, U_{2})\) are minimal.

These colligations are then equivalent, i.e. there exists an invertible operator \(V : \mathcal{H}_{2} \rightarrow \mathcal{H}_{1}\) such that the intertwining relation (3.16) holds.

Up to this point, we have not taken advantage of any scalar products that may be defined in the input, output and state spaces. From this point forward, we will focus more on these scalar products and the benefits they bring when we have them at our disposal. In what follows, we consider rational inner functions. Operator colligations representing such functions are unitary, finite-dimensional operator colligations.

For convenience, we recall the definition of a unitary operator:

Let \(\mathcal{L}_{1}\) and \(\mathcal{L}_{2}\) be Hilbert spaces and \(T : \mathcal{L}_{1} \rightarrow \mathcal{L}_{2}\) be an operator. We say that \(T\) is unitary if it satisfies the following two conditions:

a) \(T\) preserves the scalar product, i.e.

\[\langle Tx, Ty \rangle_{\mathcal{L}_{2}} = \langle x, y \rangle_{\mathcal{L}_{1}}, \quad \forall x \in \mathcal{L}_{1}, y \in \mathcal{L}_{1}.\]

b) \(T\) maps \(\mathcal{L}_{1}\) onto \(\mathcal{L}_{2}\), i.e. \(T\) is invertible.
The unitarity property of a linear operator $T$ can also be characterized as follows:

$$T^* T = I_{\mathcal{L}_1}, \quad TT^* = I_{\mathcal{L}_2}.$$  

**DEFINITION 3.9.** Let $(\mathcal{E}^\text{in}, \mathcal{E}^\text{out}, \mathcal{H}, U)$, (3.2) - (3.3), be an operator colligation. We call $(\mathcal{E}^\text{in}, \mathcal{E}^\text{out}, \mathcal{H}, U)$ a unitary colligation if the colligation operator $U$ is a unitary operator, i.e. if

$$U^* U = I_{\mathcal{E}^\text{in} \oplus \mathcal{H}}, \quad U U^* = I_{\mathcal{E}^\text{out} \oplus \mathcal{H}}.$$  

**DEFINITION 3.10.** Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be finite-dimensional Hilbert spaces and let $S(z)$ be a rational function whose values are operators acting from $\mathcal{E}_1$ to $\mathcal{E}_2$.

The matrix-function $S$ is called an inner function if its values $S(z)$ are contractive operators for $z \in \mathbb{D}$ and unitary operators for $t \in \mathbb{T}$, i.e. if the conditions

$$I_{\mathcal{E}_1} - S^*(z)S(z) \geq 0, \quad I_{\mathcal{E}_2} - S(z)S^*(z) \geq 0, \quad \text{for } z \in \mathbb{D}, \quad (3.20a)$$

$$I_{\mathcal{E}_1} - S^*(t)S(t) = 0, \quad I_{\mathcal{E}_2} - S(t)S^*(t) = 0, \quad \text{for } t \in \mathbb{T}. \quad (3.20b)$$

hold. (In particular, $S$ has no singularities in $\mathbb{D} \cup \mathbb{T}$.)

**REMARK 3.3.** Since unitary operators are invertible, $\mathcal{E}_1 - \mathcal{E}_2$ inner functions exist only if $\dim \mathcal{E}_1 = \dim \mathcal{E}_2$.

**THEOREM 3.5.** Let $(\mathcal{E}^\text{in}, \mathcal{E}^\text{out}, \mathcal{H}, U)$, (3.2) - (3.3), be a finite-dimensional unitary colligation and $S(z)$, (3.19), be its characteristic function.

Then the function $S(z)$ is a rational inner function.

**PROOF.** The proof of this lemma is based on identity (3.3), where $h(z)$ is expressed in terms of $\varphi(z)$ as in (3.9b). Let $z$ and $\zeta$ be such that the operators $I - zD$ and $I - \zeta D$ are invertible (These operators are invertible if $z \in \mathbb{D}, \zeta \in \mathbb{D}$). Also, since the spectrum of the operator $D$ is a finite set, the operators $I - zD$ and $I - \zeta D$ are invertible for all but finitely many $z \in \mathbb{T}, \zeta \in \mathbb{T}$.) Because the operator $U$ is unitary, (3.3) yields

$$\langle \psi(z), \psi(\zeta) \rangle_{\mathcal{E}^\text{out}} + (z\bar{\zeta})^{-1} \langle h(z), h(\zeta) \rangle_{\mathcal{H}} = \langle \varphi(z), \varphi(\zeta) \rangle_{\mathcal{E}} + \langle h(z), h(\zeta) \rangle_{\mathcal{H}},$$

or

$$\langle \varphi(z), \varphi(\zeta) \rangle_{\mathcal{E}^\text{in}} - \langle S(z)\varphi(z), S(\zeta)\varphi(\zeta) \rangle_{\mathcal{E}^\text{out}} =$$

$$= (1 - z\bar{\zeta}) \langle (I - zA)^{-1} C \varphi(z), (I - \zeta A)^{-1} C \varphi(\zeta) \rangle_{\mathcal{H}}. \quad (3.21)$$

In particular, taking $\varphi(z) \equiv \varphi'$ and $\varphi(\zeta) \equiv \varphi''$, where $\varphi'$ and $\varphi''$ are arbitrary vectors in $\mathcal{E}^\text{in}$, we obtain the equality

$$\frac{I_{\mathcal{E}^\text{in}} - S^*(\zeta)S(z)}{1 - \zeta z} = C^*(I - \zeta D^*)^{-1}(I - zD)^{-1}C. \quad (3.22)$$

In the same way we obtain the equality

$$\frac{I_{\mathcal{E}^\text{out}} - S(z)S^*(\zeta)}{1 - z\zeta} = B(I - zD)^{-1}(I - \zeta D^*)^{-1}B^*. \quad (3.23)$$
Using the identity \( \frac{(I - \zeta D)^{-1} - z(I - zD)^{-1}}{\zeta - z} \), we obtain

\[
S(\zeta) - S(z) = B(I - \zeta D)^{-1}(I - zD)^{-1}C,
\]

(3.24)

and

\[
S^*(\zeta) - S^*(z) = C^*(I - \overline{\zeta} D^*)^{-1}(I - \overline{z} D^*)^{-1}B^*,
\]

(3.25)

To get (3.20), we let \( \zeta = z \) in (3.22) - (3.23):

\[
I_{E_{\text{in}}} - S^*(z)S(z) = (1 - |z|^2)C^*(I - \overline{\zeta} A^*)^{-1}(I - zA)^{-1}C,
\]

(3.26a)

\[
I_{E_{\text{out}}} - S(z)S^*(z) = (1 - |z|^2)B(I - z\overline{A})^{-1}(I - \overline{\zeta} A^*)^{-1}B^*.
\]

(3.26b)

The inequalities (3.20a) follow from equalities (3.26), which hold for all \( z \in \mathbb{D} \). The equalities (3.26) furthermore hold for all but finitely many \( z \in \mathbb{T} \). Thus, the rational function \( S(z) \) is bounded in \( \mathbb{T} \), except on a finite set. \( S \) therefore has no singularities in \( \mathbb{T} \) and takes unitary values there.

The following theorem serves as a ‘unitary’ counterpart to Theorem 3.2.

**THEOREM 3.6.** Let \( S(z) \) be a rational inner function whose values are operators acting from \( \mathcal{E}_1 \) into \( \mathcal{E}_2 \), where \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) are finite-dimensional Hilbert spaces.

Then there exists a finite-dimensional, minimal, unitary operator colligation \( (\mathcal{E}_{\text{in}}^1, \mathcal{E}_{\text{out}}^1, \mathcal{H}, U), \) \((3.1) - (3.2) - (3.3)\), with \( \mathcal{E}_{\text{in}} = \mathcal{E}_1 \) and \( \mathcal{E}_{\text{out}} = \mathcal{E}_2 \), whose characteristic function \( S_U(z) = A + zB(I - zD)^{-1}C \) coincides with the original function \( S(z) \). In other words, the function \( S \) is representable in the form (3.4).

**DEFINITION 3.11.** Let \( (\mathcal{E}_{\text{in}}^1, \mathcal{E}_{\text{out}}^1, \mathcal{H}_1, U_1) \) and \( (\mathcal{E}_{\text{in}}^2, \mathcal{E}_{\text{out}}^2, \mathcal{H}_2, U_2) \) be operator colligations, \((3.13) - (3.14)\). If these colligations are equivalent (i.e. if they satisfy the intertwining relation (3.10) - (3.15)) and each of the operators \( E_{\text{in}}, E_{\text{out}}, V \) is unitary, we say that \( (\mathcal{E}_{\text{in}}^1, \mathcal{E}_{\text{out}}^1, \mathcal{H}_1, U_1) \) and \( (\mathcal{E}_{\text{in}}^2, \mathcal{E}_{\text{out}}^2, \mathcal{H}_2, U_2) \) are unitarily equivalent.

Clearly, if two operator colligations are unitarily equivalent and one of these colligations is unitary, then the second colligation is also unitary.

The following theorem provides us with a ‘unitary’ counterpart to Theorem 3.3.

**THEOREM 3.7.** Let \( (\mathcal{E}_{\text{in}}^1, \mathcal{E}_{\text{out}}^1, \mathcal{H}_1, U_1) \) and \( (\mathcal{E}_{\text{in}}^2, \mathcal{E}_{\text{out}}^2, \mathcal{H}_2, U_2) \) be unitary colligations, \((3.13)\). Furthermore, let these colligations be unitarily equivalent, i.e. suppose that the intertwining relation (3.16) holds for some unitary operators \( E_{\text{in}}, E_{\text{out}} \) and \( V \).

The respective characteristic functions \( S_1(z) \) and \( S_2(z) \) of these colligations, \((3.17)\), then satisfy the intertwining relation (3.18) with these very same unitary operators \( E_{\text{in}} \) and \( E_{\text{out}} \).
If we, furthermore, assume that both unitary colligations are simple, we can show that the converse to Theorem 3.7 also holds.

The next theorem serves as a ‘unitary’ counterpart to Theorem 3.4.

**THEOREM 3.8.** Let \((E_1^\text{in}, E_1^\text{out}, \mathcal{H}_1, U_1)\) and \((E_2^\text{in}, E_2^\text{out}, \mathcal{H}_2, U_2)\) be finite-dimensional unitary operator colligations. Let \(S_1(z)\) and \(S_2(z)\), be the characteristic functions of \((E_1^\text{in}, E_1^\text{out}, \mathcal{H}_1, U_1)\) and \((E_2^\text{in}, E_2^\text{out}, \mathcal{H}_2, U_2)\), respectively. We now make the following assumptions:

1. The functions \(S_1(z)\) and \(S_2(z)\) satisfy the intertwining relation (3.18) for \(z \in \mathbb{D}\), where \(E_1^\text{in} : E_1^\text{in} \to E_1^\text{in}\), \(E_2^\text{out} : E_2^\text{out} \to E_2^\text{out}\) are some unitary operators.
2. The colligations \((E_1^\text{in}, E_1^\text{out}, \mathcal{H}_1, U_1)\) and \((E_2^\text{in}, E_2^\text{out}, \mathcal{H}_2, U_2)\) are simple.

The colligations \((E_1^\text{in}, E_1^\text{out}, \mathcal{H}_1, U_1)\) and \((E_2^\text{in}, E_2^\text{out}, \mathcal{H}_2, U_2)\) are then unitarily equivalent, i.e. there exists a unitarily operator \(V : \mathcal{H}_2 \to \mathcal{H}_1\) such that the intertwining relation (3.10) holds.

Let us compare the assumptions of Theorems 3.4 and 3.8. In Theorem 3.4 we assume that the colligations \((E_i^\text{in}, E_i^\text{out}, \mathcal{H}_i, U_i)\), \(i = 1, 2\), are minimal, however it is not assumed that these colligations are unitary. In Theorem 3.8 we assume that the colligations are unitary and simple, but we do not explicitly assume that these colligations are minimal, because they are, in fact, already minimal.

**THEOREM 3.9.** Let \((E^\text{in}, E^\text{out}, \mathcal{H}, U)\) be a finite-dimensional, unitary operator colligation. The following statements are then equivalent:

1. The colligation is simple.
2. The colligation is minimal.
3. The colligation is controllable.
4. The colligation is observable.

In what follows we deal only with scalar-valued inner functions \(S(z)\), i.e. with functions whose values are complex numbers. The input space \(E^\text{in}\) and the output space \(E^\text{out}\) of the unitary colligation \((E^\text{in}, E^\text{out}, \mathcal{H}, U)\) representing this \(S(z)\) can be identified with the space \(\mathbb{C}^n = E^\text{out} = \mathcal{H}\). The finite-dimensional state space \(\mathcal{H}\), with \(\dim \mathcal{H} = n\), can be identified with the space \(\mathbb{C}^n\) (with the standard scalar product): \(\mathcal{H} = \mathbb{C}^n\). With these conventions in place, the orthogonal sums \(E^\text{in} \oplus \mathcal{H}\) and \(E^\text{out} \oplus \mathcal{H}\) can be identified naturally with the space \(\mathbb{C}^n\).

We note that \(\mathbb{C}^n \oplus \mathbb{C}^n\) represents a canonical decomposition of the space \(\mathbb{C}^{n+1}\) into an orthogonal sum. We consider the space \(\mathbb{C}^{n+1}\) as the set \(\mathfrak{M}_{(n+1) \times 1}\) of all \((n+1)\)-column-vectors, along with the standard linear operations and scalar product:

\[
\langle f, g \rangle = g^* f, \quad f, g \in \mathfrak{M}_{(n+1) \times 1}, \tag{3.27}
\]

where the asterisk * denotes Hermitian conjugation.

A unitary operator, \(U\), acting in \(\mathbb{C}^{n+1}\) is described by a unitary \((1+n) \times (1+n)\)-matrix, which will also be denoted by \(U\). \(U\) maps the column-vector \(f\) to the column-vector \(Uf\), where \(Uf\) is the usual matrix product. The decomposition
The Schur Algorithm in Terms of System Realizations

\( C^{n+1} = \mathbb{C} \oplus \mathbb{C}^{n} \) of the space \( C^{n+1} \) suggest that we consider the following block-matrix decomposition of \( U \):

\[
U = \begin{bmatrix} A & B \\ C & D \end{bmatrix},
\]

\( A \in \mathbb{M}_{1 \times 1}, B \in \mathbb{M}_{1 \times n}, C \in \mathbb{M}_{n \times 1}, D \in \mathbb{M}_{n \times n} \).

(3.28a)

The matrix entries are considered as operators:

\( A : E \rightarrow E, \ B : H \rightarrow E, \ C : E \rightarrow H, \ D : H \rightarrow H. \)

(3.29)

where 

\( E = \mathbb{M}_{1 \times 1} (= \mathbb{C}), \ H = \mathbb{M}_{n \times 1} (= \mathbb{C}^{n}). \)

(3.30)

**DEFINITION 3.12.** Given a unitary matrix \( U \in \mathbb{M}_{(n+1) \times (n+1)} \) with block decomposition (3.28), we associate the unitary colligation \((E, H, U)\) with \( U \). The exterior space \( E \) and the state space \( H \) of this colligation are as in (3.30), where the spaces \( \mathbb{C} \) and \( \mathbb{C}^{n} \) have the standard scalar products. The exterior, principal and channel operators \( A, D, B, C \) correspond to the block-matrix entries in (3.28) and satisfy (3.29).

We call this colligation the **unitary colligation associated with the unitary matrix** \( U \).

Given two unitary colligations associated with unitary matrices \( U' \) and \( U'' \), how do we express that these colligations are unitarily equivalent? The exterior spaces of both colligations are ‘copies’ of the same space \( \mathbb{C} \). To identify the exterior spaces \( \mathbb{C} \) of two different colligations, we should specify the unitary operators \( E_{\text{in}} \) and \( E_{\text{out}} \) for the two copies of \( \mathbb{C} \) (These operators, \( E_{\text{in}} \) and \( E_{\text{out}} \), appear in (3.15) and in the intertwining relations (3.16) and (3.18).) We can naturally choose these identification operators as the identity operators, i.e. such that each of operators \( E_{\text{in}} \) and \( E_{\text{out}} \) is represented by the \( 1 \times 1 \)-matrix whose (unique) entry is the number 1 (Such operators can be represented by \( 1 \times 1 \)-matrices, where the matrices corresponding to \( E_{\text{in}} \) and \( E_{\text{out}} \) consist, respectively, of an arbitrary number \( \nu_{\text{in}} \) and \( \nu_{\text{out}} \) with \( |\nu_{\text{in}}| = 1 \) and \( |\nu_{\text{out}}| = 1 \).)

With this convention in place, the unitary equivalence of the colligations associated with the block-matrices

\[
U' = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \in \mathbb{M}_{(n+1) \times (n+1)} \quad \text{and} \quad U'' = \begin{bmatrix} A'' & B'' \\ C'' & D'' \end{bmatrix} \in \mathbb{M}_{(n+1) \times (n+1)}
\]

(3.31)

means that these matrices satisfy the intertwining relation:

\[
\begin{bmatrix} 1 & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} A'' & B'' \\ C'' & D'' \end{bmatrix} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & V \end{bmatrix}
\]

(3.32)

where \( V \in \mathbb{M}_{n \times n} \) is a unitary matrix. The equality (3.18) then becomes:

\( S_{1}(z) = S_{2}(z) \).
DEFINITION 3.13. We say that the unitary matrices $U' \in \mathcal{M}_{(n+1) \times (n+1)}$ and $U'' \in \mathcal{M}_{(n+1) \times (n+1)}$, are equivalent if there exists a unitary matrix $V \in \mathcal{M}_{n \times n}$ such that the intertwining relation (3.32) holds.

Let $U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{M}_{(n+1) \times (n+1)}$. We now consider the following matrices associated with the unitary matrix $U$:

\[
\mathcal{E}(U) = \begin{bmatrix} C & DC & \ldots & D^{n-1}C \end{bmatrix}, \quad \mathcal{E}(U) \in \mathcal{M}_{n \times n},
\]

\[
\mathcal{B}(U) = \begin{bmatrix} B^* & D^*B^* & \ldots & (D^*)^{n-1}B^* \end{bmatrix}, \quad \mathcal{B}(U) \in \mathcal{M}_{n \times n},
\]

and

\[
\mathcal{S}(U) = \begin{bmatrix} C & DC & \ldots & D^{n-1}C, B^* & D^*B^* & \ldots & (D^*)^{n-1}B^* \end{bmatrix}, \quad \mathcal{S}(U) \in \mathcal{M}_{n \times 2n}.
\]

If the unitary colligation associated with the matrix $U$ is controllable, observable or simple, this means that the matrix (3.33a), (3.33b) or (3.33c) is, respectively, of rank $n$.

REMARK 3.4. If one of the matrices (3.33) has rank $n$, then its columns (considered as vectors in $\mathbb{C}^n = \mathcal{M}_{n \times 1}$) generate the whole space. The columns of these matrices are of the form $D^kC$ or $(D^*)^kB^*$, where $k$ takes values in the interval $[0, \ldots, (n-1)]$. It is possible to consider matrices of this kind for $k$ over a larger interval. Extending the interval $[0, \ldots, (n-1)]$ does not, however, lead to an increase in rank for these matrices: The Cayley-Hamilton Theorem tells us that the column-vectors, $D^kC$ and $(D^*)^kB^*$ with $k \geq n$, are, respectively, linear combinations of the column-vectors $D^kC$ and $(D^*)^kB^*$ with $k \in [0, \ldots, (n-1)]$.

DEFINITION 3.14. We say that a unitary matrix $U \in \mathcal{M}_{(n+1) \times (n+1)}$, expressed using the block-decomposition in (3.28), is controllable if rank $\mathcal{E}(U) = n$, observable if rank $\mathcal{B}(U) = n$ and simple if rank $\mathcal{S}(U) = n$. If the matrix $U$ is both controllable and observable, we say that it is minimal.

(We note that any one of the matrices (3.33) is of rank $n$ if and only if the other two have rank $n$. See Theorem (3.9))

The results of this section on the state space representation of scalar (i.e. complex-valued) rational inner functions can be summarized in the following way:

THEOREM 3.10. (Rational Inner Functions $\Longleftrightarrow$ Equivalence Classes of Unitary Matrices)

1. Let $S(z)$ be an inner rational function of degree $n$. Then $S(z)$ can be represented in the form:

$$S(z) = A + zB(I_n - zD)^{-1}C,$$

where $A, B, C, D$ are blocks of some unitary minimal matrix $U$, $U \in \mathcal{M}_{(n+1) \times (n+1)}$, (3.28).
2. Let $U \in \mathbb{M}_{(n+1) \times (n+1)}$ be a unitary matrix with block-decomposition (3.28) and let the function $S(z)$ be defined in terms of $U$ by (3.34). Then the function $S(z)$ is a rational inner function with $\deg S \leq n$. If the matrix $U$ is minimal, then $\deg S = n$.

3. Let $U' \in \mathbb{M}_{(n+1) \times (n+1)}$ and $U'' \in \mathbb{M}_{(n+1) \times (n+1)}$ be unitary matrices with block-decomposition (3.31) and let $S'(z)$ and $S''(z)$ be the functions defined in terms of $U'$ and $U''$ by:

$$S'(z) = A' + zB'(I_n - zD')^{-1}C', \quad S''(z) = A'' + zB'(I_n - zD'')^{-1}C''. \quad (3.35)$$

If the matrices $U'$ and $U''$ are equivalent, then $S'(z) \equiv S''(z)$. If $S'(z) \equiv S''(z)$ and the matrices $U'$ and $U''$ are minimal, then $U'$ and $U''$ are equivalent.

The substance of this theorem can be summarized as follows:

- There exists a one-to-one correspondence between the set of all rational inner functions of degree $\leq n$ and the set of all equivalence classes of unitary matrices in $\mathbb{M}_{(n+1) \times (n+1)}$.
- This correspondence can be expressed as a mapping from the set of all rational inner functions of degree $n$ onto the set of all equivalence classes of minimal unitary matrices in $\mathbb{M}_{(n+1) \times (n+1)}$.

For a proof of Theorem 3.10 see the Appendix at the end of this paper.

The Main Objective of This Paper.

Applying the Schur algorithm to a given rational inner function $s(z)$ of degree $n$ produces the sequence $s_k(z)$, $k = 0, 1, \ldots, n$, of rational inner functions with $s_0(z) = s(z)$ and $\deg s_k(z) = n - i$. In particular, $s_n(z) = s_n$ is a unitary constant. According to what was stated in Section 3 each of the functions $s_k(z)$ admits a system representation,

$$s_k(z) = A_k + zB_k(I_n - zD_k)^{-1}C_k, \quad (3.36)$$

in terms of the blocks of some minimal unitary matrix $U_k \in \mathbb{M}_{(1+n-k) \times (1+n-k)}$:

$$U_k = \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix}. \quad (3.37)$$

We assume that from the very beginning, the given inner rational function $s(z) = s_0(z)$ is determined in terms of its state space representation, so that the matrix $U_0$ is given. The goal is to recursively produce the sequence of matrices $U_k$ representing the functions $s_k(z)$, $k = 1, 2, \ldots, n$. The matrix $U_{k+1}$, representing the function $s_{k+1}(z)$, is thus constructed from the matrix $U_k$, representing the function $s_k(z)$. In other words, the steps (2.3) of the Schur algorithm must be described in terms of the state space representation (3.36).
It should be noted that the unitary matrices in the system representation of a rational inner function are determined only up to the equivalence
\[
\begin{bmatrix}
A_k & B_k \\
C_k & D_k
\end{bmatrix} \sim
\begin{bmatrix}
1 & 0 \\
0 & V_k^{-1}
\end{bmatrix}
\begin{bmatrix}
A_k & B_k \\
C_k & D_k
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & V_k
\end{bmatrix},
\tag{3.38}
\]
where $V_k$ is an arbitrary unitary $k \times k$ matrix. So we have to find a rule for constructing a matrix $U_{k+1}$, which belongs to the equivalence class of matrices representing the function $s_{k+1}(z)$, from an arbitrary element $U_k$ of the equivalence class of matrices representing the function $s_k(z)$.

The Schur algorithm in the framework of system representations is described in Section 9.

**Historical Remarks.** The definition of a characteristic function was developed gradually, starting from the pioneering works of M.S.Livshitz. The first definition appeared in [Liv1] (for operators for which $I - T^*T$ and $I - TT^*$ have rank one) and in [Liv2] (for the case that these operators have finite rank). M.S.Livshitz and those working in the same field, subsequently turned their attention to bounded operators $T$ for which $T - T^*$ is of finite rank or at least of finite trace. For these operators $T$, a characteristic function was defined in an analogous way and by means of this function, a wide-reaching theory for these operators developed. In particular, triangular models of non-self-adjoint operators were introduced. See [Liv3, BrLi, Br]. In the course of the evolution of the concept of characteristic functions, it became clear that it was advantageous to consider, not just non-self-adjoint operators, but also more general objects: operator nodes (or operator colligations). The notion of an operator colligation was prompted by physical applications of the Livshitz theory of non-selfadjoint operators. (See [BrLi, Liv9] and references there.)

B. Sz. Nagy and C. Foias used a different approach to characteristic functions in 1962. Their work involved harmonic analysis of the unitary dilation of the contractive operator $T$. Moreover, they simultaneously obtained a functional model of $T$ depending explicitly and exclusively on the characteristic function of $T$. See [SzNFo, especially Chapter VI] and references therein.

The version of operator colligations, which appears in Definition 3.1 goes back to a remark of M.G. Krein to the work [BrSV1]. In [BrSV1], the notion of a contractive operator colligation (node) was defined as the collection of Hilbert spaces $\mathcal{H}, \mathcal{F}, \mathcal{G}$ and operators
\[
T_0 : \mathcal{G} \to \mathcal{F}, \quad F : \mathcal{F} \to \mathcal{H}, \quad G : \mathcal{G} \to \mathcal{H}, \quad T : \mathcal{H} \to \mathcal{H},
\tag{3.39}
\]
satisfying the conditions
\[
I - TT^* = FF^*, \quad I - T^*T = GG^*, \quad I - T_0 T_0^* = F^* F, \quad I - T_0^* T_0 = G^* G, \quad TG = FT_0,
\tag{3.40}
\]
The results presented in the paper [BrSV1] were reported on in a seminar of Krein’s in Odessa. In the remark to this talk, M.G. Krein noticed that the conditions (3.39)-(3.40) mean that the block-operator
\[
\begin{bmatrix}
T^* \\
-F & T
\end{bmatrix} : \begin{bmatrix}
\mathcal{F} \\
\mathcal{H}
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{G} \\
\mathcal{H}
\end{bmatrix},
\]
acting in the appropriate orthogonal sums of Hilbert spaces, is a unitary operator. Starting from this remark of M.G. Krein’s, mathematicians belonging to the Odessa school as well as other mathematicians, defined the operator colligation as the block operator acting from the direct sum \begin{bmatrix}
\text{input space} \\
\text{state space}
\end{bmatrix} into the direct sum \begin{bmatrix}
\text{output space} \\
\text{state space}
\end{bmatrix}. If the spaces have scalar products and the block operator is a unitary operator with respect to this product, then the operator colligation is called an unitary colligation.

It should be mentioned that the paper [BrSV1] has connections to the theory of functional models of contractive operators developed in [SzNFo]. The definition (3.4) of the characteristic function of the colligation (3.2) - (3.3) agrees with the definition of the characteristic function in [BrSV1].

The notions of controllability and observability (and minimality) in the setting of State Space Theory were introduced by R. Kalman in [Kal1]. The study of controllability and observability of composite systems was first dealt with in [Gil]. Under other names, the notion of controllability also appears in the Livshitz theory of open systems. See the notions of the simple system and of the complementary component in section 1.3 of [Liv9]. (See pages 36 - 37 of the Russian original, or pages 27-29 of the English translation.)

The fact that every rational matrix-function \( S \) can be realized as the transfer function of some minimal stationary linear system (which here appears as Theorem 3.2), the uniqueness of the state space representation (Theorem (3.4)) and the equality \( \dim \mathcal{H} = \deg S \) were all established by R. Kalman in a very general setting. These results, as well as many other results, can be found in Chapter 10 of [KFa]. See also Chapter 1 of [Fuh].

Some algorithms for the system realization of a given rational function were proposed by R. Kalman and his collaborators. (See Chapter 10 of the monograph [KFa] and references there.) R. Kalman did not consider questions related to the realization of contractive or inner matrix-functions: He developed system theory over arbitrary fields rather than over the field of complex numbers.

An excellent (and short!) presentation of the state space approach to the problems of minimal realization and factorization of rational functions can be found in [Kaa].

Realizations of contractive or inner rational matrix-functions (rational and more general) were later considered in the framework of the SzNagy-Foias model for contractive operators. These and also more general results can be found in many publications now. For convenience, we present some basic facts on system
realizations for inner rational functions (scalar) in the Appendix to the present paper.

The state space description of the composite system, which is formed by the cascade (or Redheffer) coupling of several state space systems, was dealt with in [HeBa] in more generality. We make use of these results, but prefer to derive them independently of [HeBa] in the form and in the generality which is most suitable for our goal.

4. Coupled Systems and The Schur Transformation : Input-Output Mappings.

To describe the Schur algorithm using system representations, we must first consider how the linear-fractional Schur transformation

\[ \omega(z) \mapsto s(z), \quad s(z) = \frac{s_0 + z\omega(z)}{1 + z\omega(z)s_0} \quad (s_0 \text{ is a complex number, } |s_0| < 1) \]  

(4.1)
can be described in terms of the input-output mappings of linear systems. The linear-fractional transform (4.1) is of the form

\[ s(z) = \frac{w_{11}(z)\omega(z) + w_{12}(z)}{w_{21}(z)\omega(z) + w_{22}(z)}. \]  

(4.2)

This form of a linear-fractional transform is the most familiar to the classical analyst. In the theory of unitary operator colligations, the Redheffer form for linear-fractional transforms, i.e.

\[ s(z) = s_{11}(z) + s_{12}(z)\omega(z)(I - s_{22}(z)\omega(z))^{-1}s_{21}(z). \]  

(4.3)
is often more convenient. Every linear-fractional transformation of the form (4.2) can be rewritten in the Redheffer form (4.3), but not every transformation in Redheffer form can be expressed in linear-fractional form.

The matrix \( W(z) = \begin{bmatrix} w_{11}(z) & w_{12}(z) \\ w_{21}(z) & w_{22}(z) \end{bmatrix} \) for the transformation (4.1) and (4.2) (under the appropriate normalization\(^4\)) is

\[ W(z) = (1 - |s_0|^2)^{-1/2} \begin{bmatrix} z & s_0 \\ \bar{z}s_0 & 1 \end{bmatrix}. \]  

(4.4)

\( W(z) \) in (4.4) is not an inner matrix but it is a \( j \)-inner matrix:

\[ j - W^*(z)jW(z) \geq 0, \quad z \in \mathbb{D}, \quad j - W^*(t)jW(t) = 0, \quad t \in \mathbb{T}, \]

where \( j = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \).

\(^3\)Raymond Redheffer (1921-2005) was a US mathematician working at UCLA.

\(^4\)The matrix of the linear-fractional transform (4.2) is determined only up to the proportionality \( W(z) \mapsto \lambda(z)W(z), \) where \( \lambda \in \mathbb{C} \setminus \{0\}. \)
Let us express the fractional-linear transformation \((4.1)\) in the Redheffer form \((4.3)\), where the \(2 \times 2\)-matrix-function \(S(z) = \begin{bmatrix} s_{11}(z) & s_{12}(z) \\ s_{21}(z) & s_{22}(z) \end{bmatrix}\) is:

\[
S(z) = \begin{bmatrix} s_0 & z \left(1 - |s_0|^2\right)^{1/2} \\ (1 - |s_0|^2)^{1/2} & -z s_0 \end{bmatrix}
\]

(4.5)

Unlike \(W(z), (4.4)\), the matrix-function \(S(z), (4.5)\), is an \emph{inner} function.

The transformation in the Redheffer form \((4.3)\) admits an interpretation in System Theory. We discuss this in more generality than is needed for our considerations, which are centered on the linear-fractional Schur transformation.

Suppose that LSDS\(^I\) and LSDS\(^II\) are two linear stationary dynamical systems. In this section, we focus on the input-output mapping and do not touch on considerations related to state spaces.

Let \(S(z) : \mathcal{E}^I \rightarrow \mathcal{E}^I\) be the transfer matrix-function of the system LSDS\(^I\). Furthermore, let

\[
\psi(z) = S(z) \varphi(z)
\]

be the input-output mapping corresponding to the system LSDS\(^I\), where \(\varphi(z) : \mathbb{D} \rightarrow \mathcal{E}^I\) is the input signal and \(\psi(z) : \mathbb{D} \rightarrow \mathcal{E}^I\) is the output signal. Suppose now that the exterior space \(\mathcal{E}^I\) of the system LSDS\(^I\) is the orthogonal sum of the subspaces \(\mathcal{E}^I_1\) and \(\mathcal{E}^I_2\):

\[
\mathcal{E}^I = \mathcal{E}^I_1 \oplus \mathcal{E}^I_2.
\]

(4.6)

Equation (4.6) suggests that the input and output signals be decomposed as follows:

\[
\varphi(z) = \begin{bmatrix} \varphi_1(z) \\ \varphi_2(z) \end{bmatrix}, \quad \psi(z) = \begin{bmatrix} \psi_1(z) \\ \psi_2(z) \end{bmatrix},
\]

(4.7)

And furthermore that the matrix \(S(z)\) be decomposed accordingly:

\[
S(z) = \begin{bmatrix} s_{11}(z) & s_{12}(z) \\ s_{21}(z) & s_{22}(z) \end{bmatrix},
\]

(4.8)

So that

\[
\begin{bmatrix} \psi_1(z) \\ \psi_2(z) \end{bmatrix} = \begin{bmatrix} s_{11}(z) & s_{12}(z) \\ s_{21}(z) & s_{22}(z) \end{bmatrix} \begin{bmatrix} \varphi_1(z) \\ \varphi_2(z) \end{bmatrix}.
\]

(4.9)

The system LSDS\(^I\) can be considered as a linear stationary dynamical system with two input channels, corresponding to the input signals \(\varphi_1(z)\) and \(\varphi_2(z)\), and two output channels, corresponding to the output signals \(\psi_1(z)\) and \(\psi_2(z)\):

\[
\begin{array}{c}
\varphi_1 \\
\varphi_2 \\
\text{LSDS}^I \\
\psi_1 \\
\psi_2
\end{array}
\]

Figure 1
Let
\[ \tau(z) = \omega(z)\sigma(z) \] (4.10)
be the input-output mapping corresponding to the system LSDS\textsuperscript{II}, where \( \sigma(z) : \mathbb{D} \rightarrow \mathcal{E} \) is the input signal and \( \tau(z) : \mathbb{D} \rightarrow \mathcal{E} \) is the output signal. The system LSDS\textsuperscript{II} can be considered as a linear stationary dynamical system with one input channel, corresponding to the input signal \( \sigma(z) \), and one output channel, corresponding to the output signal \( \tau(z) \):

![Figure 2](image)

Suppose now that
\[ \mathcal{E}_2^I = \mathcal{E} \] (4.11)
This allows us to ‘link’ the systems LSDS\textsuperscript{I} and LSDS\textsuperscript{II}. We connect the output channel of the system LSDS\textsuperscript{II} with the second LSDS\textsuperscript{I} input channel and the LSDS\textsuperscript{II} input channel with the second LSDS\textsuperscript{I} output channel, as shown in Figure 3.

![Figure 3](image)

The resulting linear stationary dynamical system LSDS has exterior space \( \mathcal{E}_1^I \), input signal \( \varphi_1(z) \) and output signal \( \psi_1(z) \). The output signal \( \psi_1(z) \) is linearly dependent on the input signal \( \varphi_1(z) \):
\[ \psi_1(z) = s(z)\varphi_1(z) \] (4.12)
where \( s(z) \) is the transfer function for LSDS.

We call LSDS the Redheffer coupling of the systems LSDS\textsuperscript{I} and LSDS\textsuperscript{II}. We now look to express \( s(z) \) in terms of \( S(z) \) and \( \omega(z) \). The above-described connection between the systems LSDS\textsuperscript{I} and LSDS\textsuperscript{II} can be formally expressed by means of the constraints
\[ \varphi_2(z) = \tau(z), \quad \psi_2(z) = \sigma(z) \] (4.13)
Eliminating \( \varphi_2(z) \), \( \psi_2(z) \), \( \sigma(z) \), \( \tau(z) \) from the system of linear equations (4.9), (4.10) and (4.13), we obtain the equation (4.12), where \( s(z) \) has the form

\[
s(z) = s_{11}(z) + s_{12}(z)\omega(z)(I - s_{22}(z)\omega(z))^{-1}s_{21}(z).
\]

We now turn our attention to the ‘energy relation’ associated with the linear fractional transformation (4.14): \( \omega(z) \rightarrow s(z) \).

Equation (4.9) yields,

\[
\varphi^*_1\varphi_1 + \varphi^*_2\varphi_2 - \psi^*_1\psi_1 - \psi^*_2\psi_2 = \begin{bmatrix} \varphi^*_1 & \varphi^*_2 \end{bmatrix} (I - S^*S) \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}.
\]

Making the substitutions \( \psi_1 = s\varphi_1 \), \( \psi_2 = \sigma \) and \( \varphi_2 = \omega\sigma \), we obtain

\[
\varphi^*_1(1 - s^*s)\varphi_1 = \begin{bmatrix} \varphi^*_1 & \varphi^*_2 \end{bmatrix} (I - S^*S) \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} + \sigma^*(1 - \omega^*\omega)\sigma,
\]

where

\[
\sigma = (1 - s_{22}\omega)^{-1}s_{21}\varphi_1, \quad \varphi_2 = \omega(1 - s_{22}\omega)^{-1}s_{21}\varphi_1.
\]

It follows from equation (4.15) that if \( I - S^*S \geq 0 \) and \( 1 - \omega^*\omega \geq 0 \), then \( 1 - s^*s \geq 0 \). If \( I - S^*S = 0 \) and \( 1 - \omega^*\omega = 0 \), then \( 1 - s^*s = 0 \). In particular, this brings us to:

**THEOREM 4.1.** Let \( S(z) \) and \( \omega(z) \) be rational inner matrix-functions. Furthermore, let \( s(z) \) be given by the Redheffer linear-fractional transform (4.14). Then \( s(z) \) is a rational inner matrix-function.

We note that the linear-fractional transform, in its classical form (4.2), is related to another kind of coupling. The relevant connection is shown in Figure 3.

![Figure 3](image-url)
output channel of LSDS (Shown in Figure 3.) Let \( W(z) = \begin{bmatrix} w_{11}(z) & w_{12}(z) \\ w_{21}(z) & w_{22}(z) \end{bmatrix} \) be the transfer matrix for LSDS\(^I\) and \( \omega(z) \) be the transfer matrix for LSDS\(^II\):
\[
\begin{bmatrix} \psi_1(z) \\ \psi_2(z) \end{bmatrix} = \begin{bmatrix} w_{11}(z) & w_{12}(z) \\ w_{21}(z) & w_{22}(z) \end{bmatrix} \begin{bmatrix} \varphi_1(z) \\ \varphi_2(z) \end{bmatrix}, \quad \tau(z) = \omega(z)\sigma(z).
\]
The link between the systems LSDS\(^I\) and LSDS\(^II\), shown in Figure 3, is described by the constraints
\[
\sigma(z) = \varphi_2(z), \quad \tau(z) = \varphi_1(z).
\]
In which the input and the output signals of the system LSDS are denoted by \( \varphi(z) \) and \( \psi(z) \), respectively:
\[
\varphi(z) = \psi_2(z), \quad \psi(z) = \psi_1(z),
\]
so that:
\[
\psi(z) = s(z)\varphi(z),
\]
where
\[
s(z) = (w_{11}(z)\omega(z) + w_{12}(z)) \cdot (w_{21}(z)\omega(z) + w_{22}(z))^{-1}.
\]

**Historical Remark.** The coupling of input-output systems having four terminals, considered in this section (See Figures 1-3), is sometimes called *cascade coupling*. This kind of coupling (as well as related mathematical questions) was investigated by R. Redheffer in [Red1] - [Red5]. Because of this, we use the name *Redheffer coupling*. Redheffer did not consider questions related to cascade coupling of state space linear systems. These questions were later addressed in [HeBa] (Without any reference to Redheffer.)

The results presented in [HeBa] are more general than here needed. We have tailored our approach to the theory of Redheffer coupling in the next two sections to fit our needs.

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**5. The Redheffer Coupling of Unitary Colligations.**

As rational inner functions, \( S(z), \omega(z) \) and \( s(z) \) admit system representations as characteristic functions of the unitary operator colligations with colligation operators \( U^I, U^{II} \) and \( U \), respectively. We now turn to the question of how we might express \( U \) in terms of the operators \( U^I \) and \( U^{II} \).

Our approach to this problem will be more general than is here called for, our goal being to describe the colligations related to Schur transformations. We assume that the unitary colligations corresponding to the systems LSDS\(^I\) and LSDS\(^II\) are given. We look to obtain the unitary colligation corresponding to the system LSDS, the Redheffer coupling of the systems LSDS\(^I\) and LSDS\(^II\). The system LSDS\(^I\) is not assumed to be related to the Schur transformation. LSDS\(^I\) and LSDS\(^II\) can be generic systems. The only condition imposed on these systems is that the exterior space \( E^{II} \) of the system LSDS\(^II\) is identified with the subspace \( E^I_1 \) of the exterior space \( E^I \) belonging to LSDS\(^I\). To avoid technical complications
we assume that the exterior and state spaces of the systems LDS$_I$ and LDS$_{II}$
are finite-dimensional.

To simplify the notation, we denote the matrix entries of the colligation
operator $U^I$, corresponding to the system LDS$_I$, as follows

$$U^I = \begin{bmatrix}
a_{11} & a_{12} & b_1 \\
a_{21} & a_{22} & b_2 \\
c_1 & c_2 & d
\end{bmatrix}, \quad (5.1)$$

where

$$a_{p,q} : \mathcal{E}^I_p \rightarrow \mathcal{E}^I_q, \quad b_q : \mathcal{H}^I \rightarrow \mathcal{E}^I_q, \quad c_p : \mathcal{E}^I_p \rightarrow \mathcal{H}^I, \quad d : \mathcal{H}^I \rightarrow \mathcal{H}^I.$$ 

The matrix entries for the colligation operator $U^{II}$, corresponding to the system
LDS$_{II}$, are denoted as follows:

$$U^{II} = \begin{bmatrix}
\alpha \\
\gamma
\end{bmatrix}, \quad (5.2)$$

where

$$\alpha : \mathcal{E}^{II} \rightarrow \mathcal{E}^{II}, \quad \beta : \mathcal{H}^{II} \rightarrow \mathcal{E}^{II}, \quad \gamma : \mathcal{E}^{II} \rightarrow \mathcal{H}^{II}, \quad \delta : \mathcal{H}^{II} \rightarrow \mathcal{H}^{II}.$$ 

The linear equations describing the dynamics of the system LDS$_I$ are

$$\begin{bmatrix}
\psi_1(z) \\
\psi_2(z) \\
z^{-1} h(z)
\end{bmatrix} = \begin{bmatrix}
a_{11} & a_{12} & b_1 \\
a_{21} & a_{22} & b_2 \\
c_1 & c_2 & d
\end{bmatrix} \begin{bmatrix}
\varphi_1(z) \\
\varphi_2(z) \\
h(z)
\end{bmatrix}, \quad (5.3)$$

where

$$\varphi(z) = \begin{bmatrix}
\varphi_1(z) \\
\varphi_2(z)
\end{bmatrix}, \quad \psi(z) = \begin{bmatrix}
\psi_1(z) \\
\psi_2(z)
\end{bmatrix} \quad \text{and} \quad h(z)$$

are, respectively, the input signal, the output signal and the inner state signal
corresponding to the system LDS$_I$.

The linear equations describing the dynamics of the system LDS$_{II}$ are

$$\begin{bmatrix}
\tau(z) \\
z^{-1} l(z)
\end{bmatrix} = \begin{bmatrix}
\alpha & \beta \\
\gamma & \delta
\end{bmatrix} \begin{bmatrix}
\sigma(z) \\
l(z)
\end{bmatrix}, \quad (5.4)$$

where $\sigma(z), \tau(z)$ and $l(z)$ are, respectively, the input signal, output signal and the
interior state signal corresponding to the system LDS$_{II}$.

The constraints

$$\tau(z) = \varphi_2(z), \quad \sigma(z) = \psi_2(z) \quad (5.5)$$
correspond to the Redheffer coupling of the systems LDS$_I$ and LDS$_{II}$. 
We now aim to eliminate the variables $\varphi_2(z), \psi_2(z), \sigma(z), \tau(z)$ from the systems (5.3), (5.4), (5.5). To this end, we substitute the expressions $\alpha \sigma(z) + \beta l(z)$ and $\sigma(z)$ for the variables $\varphi_2(z)$ and $\psi_2(z)$ into the equation
\[
\psi_2(z) = a_{21} \varphi_1(z) + a_{22} \varphi_2(z) + \beta h(z).
\]
With this we can express $\sigma(z)$ in terms of $\varphi_1(z), h(z)$ and $l(z)$:
\[
\sigma(z) = (1 - a_{22} \alpha)^{-1} a_{21} \varphi_1(z) + (1 - a_{22} \alpha)^{-1} b_2 h(z) + (1 - a_{22} \alpha)^{-1} \beta l(z). \tag{5.6}
\]
Substituting this expressions for $\sigma$ into (5.3), (5.4), (5.5), we obtain
\[
\begin{bmatrix}
\psi_1(z) \\
\psi(z) \\
\psi(z)
\end{bmatrix} =
\begin{bmatrix}
A & B_1 & B_2 \\
C_1 & D_{11} & D_{12} \\
C_2 & D_{21} & D_{22}
\end{bmatrix}
\begin{bmatrix}
\varphi_1(z) \\
h(z) \\
l(z)
\end{bmatrix}, \tag{5.7}
\]
where
\[
A : \mathcal{E}_1 \to \mathcal{E}_1, \quad B_1 : \mathcal{H}^1 \to \mathcal{E}_1, \quad B_2 : \mathcal{H}^1 \to \mathcal{E}_1,
\]
\[
C_1 : \mathcal{E}_1 \to \mathcal{H}^1, \quad C_2 : \mathcal{H}^1 \to \mathcal{E}_1,
\]
\[
D_{11} : \mathcal{H}^1 \to \mathcal{H}^1, \quad D_{12} : \mathcal{H}^1 \to \mathcal{H}^1, \quad D_{21} : \mathcal{H}^1 \to \mathcal{H}^1, \quad D_{22} : \mathcal{H}^1 \to \mathcal{H}^1.
\]
The matrix
\[
U =
\begin{bmatrix}
A & B_1 & B_2 \\
C_1 & D_{11} & D_{12} \\
C_2 & D_{21} & D_{22}
\end{bmatrix} \tag{5.8}
\]
can be expressed using the entries of the matrices $U_1$, (5.1), and $U_2$, (5.2), as follows:
\[
U =
\begin{bmatrix}
a_{11} & b_1 & a_{12} \beta \\
0 & 0 & \beta \gamma
\end{bmatrix}
+ \begin{bmatrix}
a_{12} \alpha \\
0 & 0 & \beta \gamma
\end{bmatrix}
= \begin{bmatrix}
1 - a_{22} \alpha \\
0 & 0 & \beta \gamma
\end{bmatrix}
\cdot \begin{bmatrix}
a_{21} & b_2 & a_{22} \beta
\end{bmatrix}. \tag{5.9}
\]
The operator $U$ is called the Redheffer product of the operators $U_1$ and $U_2$.

We again turn our attention to the ‘energy relation’ associated with the operators $U_1$, $U_2$ and $U$. Suppose that $U_1$ and $U_2$ are unitary. Let $\varphi_1 \in \mathcal{E}_1, \varphi_2 \in \mathcal{E}_1, h \in \mathcal{H}^1, \sigma \in \mathcal{E}_1$ and $l \in \mathcal{H}_1$ be arbitrary vectors. If $\psi_1 \in \mathcal{E}_1, \psi_2 \in \mathcal{E}_1, k \in \mathcal{H}^1, \tau \in \mathcal{H}^1$ and $m \in \mathcal{H}_1$ are defined by the equalities
\[
\begin{bmatrix}
\psi_1 \\
\psi_2 \\
k
\end{bmatrix} = U_1 \begin{bmatrix}
\varphi_1 \\
\varphi_2 \\
h
\end{bmatrix}, \quad \begin{bmatrix}
\tau \\
m
\end{bmatrix} = U_2 \begin{bmatrix}
\sigma \\
l
\end{bmatrix},
\]
then
\[
||\psi_1||^2 + ||\psi_2||^2 + ||k||^2 = ||\varphi_1||^2 + ||\varphi_2||^2 + ||h||^2, \tag{5.10}
\]
and
\[
||\tau||^2 + ||m||^2 = ||\sigma||^2 + ||l||^2. \tag{5.11}
\]
For arbitrary $\varphi_1, h, l$ and

$$\sigma = (1 - a_{22}\alpha)^{-1} a_{21} \varphi_1 + (1 - a_{22}\alpha)^{-1} b_2 + (1 - a_{22}\alpha)^{-1} \beta,$$  
(5.12)

$$\varphi_2 = \alpha((1 - a_{22}\alpha)^{-1} a_{21} \varphi_1 + (1 - a_{22}\alpha)^{-1} b_2 + (1 - a_{22}\alpha)^{-1} \beta) + \beta l,$$  
(5.13)

it follows that

$$\psi_2 = \sigma, \quad \tau = \varphi_2,$$

and

$$||\psi_1||^2 + ||l||^2 + ||m||^2 = ||\varphi_1||^2 + ||h||^2 + ||l||^2.$$  
(5.14)

According to the definition of the operator $U$,

$$\begin{bmatrix} \psi_1 \\ k \\ m \end{bmatrix} = U \begin{bmatrix} \varphi_1 \\ h \\ l \end{bmatrix}.$$  
(5.15)

Since $\varphi_1, h, l$ are arbitrary, equality (5.14) means that $U$ is unitary. This operator, partitioned into blocks according to (5.8), is related to the unitary colligation $(\mathcal{E}, \mathcal{H}, U)$, where $\mathcal{E} = \mathcal{E}_I$, $\mathcal{H} = \mathcal{H}_I \oplus \mathcal{H}_II$.

**Definition 5.1.** The colligation $(\mathcal{E}, \mathcal{H}, U)$ is called the Redheffer coupling of the colligations $(\mathcal{E}_I, \mathcal{H}_I, U^I)$ and $(\mathcal{E}_II, \mathcal{H}_II, U^II)$.

**Theorem 5.1.** Let $S(z) = \begin{bmatrix} s_{11}(z) & s_{12}(z) \\ s_{21}(z) & s_{22}(z) \end{bmatrix}$, $\omega(z)$ and $s(z)$ be the characteristic functions of the colligations $(\mathcal{E}_I, \mathcal{H}_I, U^I)$, $(\mathcal{E}_II, \mathcal{H}_II, U^II)$ and their Redheffer coupling $(\mathcal{E}, \mathcal{H}, U)$, respectively:

$$\begin{bmatrix} s_{11}(z) & s_{12}(z) \\ s_{21}(z) & s_{22}(z) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + z \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} (I - zd)^{-1} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$  
(5.16)

$$\omega(z) = \alpha + z\beta(1 - zd)^{-1}\gamma,$$  
(5.17)

$$s(z) = A + z \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \left( \begin{bmatrix} I_{H_I} & 0 \\ 0 & I_{H_{II}} \end{bmatrix} - z \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \right)^{-1} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}.$$  
(5.18)

(The notation for the entries of the matrices $U^I, U^II$ and $U$ is taken from $(5.1), (5.2)$ and $(5.8)$, respectively.)

Then

$$s(z) = s_{11}(z) + s_{12}(z)\omega(z)(I - s_{22}(z)\omega(z))^{-1}s_{21}(z).$$  
(5.19)
We now focus again on the linear-fractional transformation (4.1) in the Redheffer form (4.3), where $\omega(z)$ is a rational inner matrix-function of degree $n - 1$, so that $s(z)$ is a rational inner matrix-function of degree $n$.

The function $S(z)$, which appears in (4.5) is a rational inner function. It is a characteristic matrix-function for the operator colligation $(E^I, H^I, U^I)$, which we now describe.

The outer space $E^I$ is two-dimensional. We identify $E^I$ with $\mathbb{C}^2$. The space $E^I$ is considered as the orthogonal sum $E^I = E^I_1 \oplus E^I_2$, where $E^I_1$ is identified with $\mathbb{C}$ and $E^I_2$ is identified with $\mathbb{C}$. The orthogonal decomposition $E^I = E^I_1 \oplus E^I_2$ is thus the canonical decomposition $\mathbb{C}^2 = \mathbb{C} \oplus \mathbb{C}$. The inner space $H^I$ is one-dimensional. We identify $H^I$ with $\mathbb{C}$. The colligation operator $U^I$ is defined by the unitary $3 \times 3 = (2 + 1) \times (2 + 1)$-matrix considered as an operator acting in $\mathbb{C}^3 = \mathbb{C}^2 \oplus \mathbb{C}$:

$$U^I = \begin{bmatrix} A^I & B^I \\ C^I & D^I \end{bmatrix}$$

with

$$A^I = \begin{bmatrix} s_0 & 0 \\ (1 - |s_0|^2)^{1/2} & 0 \end{bmatrix}, \quad B^I = \begin{bmatrix} (1 - |s_0|^2)^{1/2} \\ -\bar{s}_0, \end{bmatrix}.$$

$$C^I = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D^I = \begin{bmatrix} 0 \end{bmatrix}.$$

The characteristic function of the colligation $(E^I, H^I, U^I)$ is the matrix-function $S(z)$ of the form (4.5):

$$\begin{bmatrix} s_0 & z(1 - |s_0|^2)^{1/2} \\ (1 - |s_0|^2)^{1/2} & -z\bar{s}_0 \end{bmatrix} = A^I + zB^I(I - zD^I)^{-1}C^I.$$

The rational inner function $\omega(z)$ of degree $n - 1$ is the characteristic function of the colligation $(E^{II}, H^{II}, U^{II})$. The outer space $E^{II}$ is one-dimensional and is identified with $\mathbb{C}$ and the inner space $H^{II}$ is $(n - 1)$-dimensional and is identified with $\mathbb{C}^{n-1}$. The colligation operator $U^{II}$ thus acts in $\mathbb{C}^n = \mathbb{C} \oplus \mathbb{C}^{n-1}$. We identify the operator $U^{II}$ with its matrix in the canonical basis of $\mathbb{C}^n$:

$$U^{II} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix},$$

where

$$\alpha \in \mathbb{M}_{1 \times 1}, \ \beta \in \mathbb{M}_{1 \times (n-1)}, \ \gamma \in \mathbb{M}_{(n-1) \times 1}, \ \delta \in \mathbb{M}_{(n-1) \times (n-1)}.$$

$\alpha$ is simply a complex number. The matrix $U^{II}$ is unitary. The system representation of the function $\omega(z)$ is given by:

$$\omega(z) = \alpha + \beta(1 - z\delta)^{-1}\gamma.$$

(6.4)
In particular,
\[ \omega(0) = \alpha. \] (6.5)

The function
\[ s(z) = \frac{s_0 + z\omega(z)}{1 + z\omega(z)s_0}, \] (6.6)
written as a Redheffer fractional-linear transform, takes the form:
\[ s(z) = s_0 + z(1 - |s_0|^2)^{1/2} \omega(z)(1 + z\omega(z)s_0)^{-1}(1 - |s_0|^2)^{1/2}, \] (6.7)
and admits a system realization by means of the operator colligation \((E, H, U)\), where \((E, H, U)\) is the Redheffer coupling of the colligations \((E^1, H^1, U^1)\), representing the function \(S(z)\), and \((E^{11}, H^{11}, U^{11})\), representing the function \(\omega(z)\).

Clearly, \(E = C\) and \(H = C^n\). \(U\) is the Redheffer coupling of the matrices \(U^1\) and \(U^{11}\).

Applying formula (5.9) to \(U^1\) and \(U^{11}\), we obtain:
\[ U = \begin{bmatrix} s_0 & (1 - |s_0|^2)^{1/2} & 0_{1 \times (n-1)} \\ \alpha (1 - |s_0|^2)^{1/2} & -\alpha s_0 & \beta \\ \gamma (1 - |s_0|^2)^{1/2} & -\gamma s_0 & \delta \end{bmatrix}, \] (6.8)
so that \(U\) takes the form:
\[ U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \] (6.9)
where
\[ A = s_0, \quad B = \begin{bmatrix} (1 - |s_0|^2)^{1/2} & 0_{1 \times (n-1)} \end{bmatrix}, \]
\[ C = \begin{bmatrix} \alpha (1 - |s_0|^2)^{1/2} \\ \gamma (1 - |s_0|^2)^{1/2} \end{bmatrix}, \quad D = \begin{bmatrix} -\alpha s_0 & \beta \\ -\gamma s_0 & \delta \end{bmatrix}, \]
\[ A \in \mathbb{M}_{1 \times 1}, \quad B \in \mathbb{M}_{1 \times n}, \quad C \in \mathbb{M}_{(n-1) \times 1}, \quad D \in \mathbb{M}_{n \times n}. \] (6.10)

Clearly, \(U\) in (6.8)-(6.9) can be expressed as follows:
\[ U = \begin{bmatrix} 1 & 0 & 0_{1 \times (n-1)} \\ 0 & \alpha & \beta \\ 0_{(n-1) \times 1} & \gamma & \delta \end{bmatrix} \begin{bmatrix} s_0 & (1 - |s_0|^2)^{1/2} & 0_{1 \times (n-1)} \\ (1 - |s_0|^2)^{1/2} & -s_0 & 0_{1 \times (n-1)} \\ 0_{(n-1) \times 1} & 0_{(n-1) \times 1} & 1_{(n-1) \times (n-1)} \end{bmatrix}, \] (6.11)

Applying Theorem 5.1 to the Redheffer coupling of the colligations \(U^1\), (6.1), and \(U^{11}\), (6.1), yields:

**THEOREM 6.1.** Let \(\omega(z)\) be a rational inner matrix-function of degree \(n - 1\) and let
\[ \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \quad \alpha \in \mathbb{M}_{1 \times 1}, \quad \beta \in \mathbb{M}_{1 \times (n-1)}, \quad \gamma \in \mathbb{M}_{(n-1) \times 1}, \quad \delta \in \mathbb{M}_{(n-1) \times (n-1)}; \] (6.12)
be a unitary matrix so that the system representation (6.4) for \( \omega(z) \) holds. Let \( s_0 \) be a complex number with \( |s_0| < 1 \). Let the function \( s(z) \) be defined as the inverse Schur transform (6.6) (using \( s_0 \) and \( \omega(z) \)) and let the matrix \( U \),

\[
U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad A \in \mathcal{M}_{1\times1}, \ B \in \mathcal{M}_{1\times(n)}, \ C \in \mathcal{M}_{(n)\times1}, \ D \in \mathcal{M}_{(n)\times(n)},
\]

be defined by equation (6.11).

\( U \) is then unitary and yields the system representation of \( s(z) \):

\[
s(z) = A + zB(I - zD)^{-1}C.
\]

(6.14)

Unitary Equivalence Freedom.

The same function \( s(z) \), for which we earlier found a representation using the matrix \( U \) in (6.11), can also be represented with the help of a matrix having the form:

\[
UV = \begin{bmatrix} 1 & 0 \\ 0 & V^* \end{bmatrix} U \begin{bmatrix} 1 & 0 \\ 0 & V \end{bmatrix},
\]

where \( V \in \mathcal{M}_{n\times n} \) is a unitary matrix.

The matrix representing \( s(z) \) and which, furthermore, appears as the Redheffer coupling matrix for the matrices representing \( S(z) \) and \( \omega(z) \), can be considered to have fewer ‘degrees of freedom’ than matrices of the form (6.15). The degree of freedom for the Redheffer coupling matrix is derived from this same property in the Redheffer coupled matrices. The more general form of the matrix, which represents the \( 2 \times 2 \)-matrix-function \( S(z) \), is the ‘transformed’ matrix:

\[
U^{1,\varepsilon} = \begin{bmatrix} 1_{2\times2} & 0_{2\times1} \\ 0_{1\times2} & \varepsilon \end{bmatrix} U^{1} \begin{bmatrix} 1_{2\times2} & 0_{2\times1} \\ 0_{1\times2} & \varepsilon \end{bmatrix},
\]

(6.16)

i.e.

\[
U^{1,\varepsilon} = \begin{bmatrix} A^1 & B^{1\varepsilon} \\ C^{1} & D^{1} \end{bmatrix},
\]

(6.17)

where \( U^1 \) is the matrix from (6.1) and \( \varepsilon \) is an arbitrary unimodular complex number. A more general form of the colligation matrix representing the function \( \omega(z) \) is given by:

\[
U^{11,\nu} = \begin{bmatrix} 1 & 0_{1\times(n-1)} \\ 0_{(n-1)\times1} & \nu^* \end{bmatrix} U^{11} \begin{bmatrix} 1 & 0_{1\times(n-1)} \\ 0_{(n-1)\times1} & \nu \end{bmatrix},
\]

(6.18)

i.e.

\[
U^{11,\nu} = \begin{bmatrix} \alpha & \beta \nu \\ \gamma^\nu & \delta \nu \end{bmatrix},
\]

(6.19)

where \( U^{11}, (6.3) \), is some \( n \times n \) unitary colligation matrix representing the function \( \omega(z) \),

\[
\beta^\nu = \beta \nu, \quad \gamma^\nu = v^* \gamma, \quad \delta^\nu = v^* \delta \nu,
\]

(6.20)
and $v$ is an arbitrary unitary $(n-1) \times (n-1)$-matrix. Applying formula (5.9) to
the matrices $U^{1,\varepsilon}$ and $U^{II,v}$, we obtain the Redheffer coupling matrix:

$$U^{\varepsilon,v} = \begin{bmatrix}
    s_0 & \varepsilon (1 - |s_0|^2)^{1/2} & 0_{1 \times (n-1)} \\
    \alpha \varepsilon (1 - |s_0|^2)^{1/2} & -\alpha s_0 & \varepsilon \beta^v \\
    \gamma^v (1 - |s_0|^2)^{1/2} & -\gamma \varepsilon s_0 & \delta^v \\
\end{bmatrix}. \quad (6.21)$$

Clearly,

$$U^{\varepsilon,v} = \begin{bmatrix}
    1 & 0 & 0_{1 \times (n-1)} \\
    0 & \alpha & \varepsilon \beta^v \\
    0_{(n-1) \times 1} & \gamma^v \varepsilon & \delta^v \\
\end{bmatrix} \times
\begin{bmatrix}
    s_0 & \varepsilon (1 - |s_0|^2)^{1/2} & 0_{1 \times (n-1)} \\
    \varepsilon (1 - |s_0|^2)^{1/2} & -\varepsilon s_0 & 0_{1 \times (n-1)} \\
    0_{(n-1) \times 1} & 0_{(n-1) \times 1} & 1_{(n-1) \times (n-1)} \\
\end{bmatrix}, \quad (6.22)$$

and finally

$$U^{\varepsilon,v} = \begin{bmatrix}
    1 & 0_{n \times n} \\
    0_{n \times n} & V_{\varepsilon,v}^* \\
\end{bmatrix} U \begin{bmatrix}
    1 & 0_{n \times n} \\
    0_{n \times n} & V_{\varepsilon,v} \\
\end{bmatrix}, \quad (6.23)$$

where

$$V_{\varepsilon,v} = \begin{bmatrix}
    \varepsilon & 0_{1 \times (n-1)} \\
    0_{(n-1) \times 1} & v \\
\end{bmatrix}, \quad (6.24)$$

$\varepsilon$ is an arbitrary unimodular complex number and $v$ is an arbitrary
unitary $(n-1) \times (n-1)$-matrix ($\varepsilon$ and $v$ are the same as in (6.18)).

Comparing formulas (6.15) and (6.23)-(6.24), we see that the matrices $U^{\varepsilon,v}$
which come from Redheffer coupling of the matrices representing $S(z)$ and $\omega(z)$
are special. The distinguishing feature of the matrices $U^{\varepsilon,v}$ can be summarized as
follows:

\textit{Among all of the $(n+1) \times (n+1)$-matrices $U^V = \begin{bmatrix}
    s_0 & B^V \\
    C^V & D^V \\
\end{bmatrix}$ of the form
(6.15), it is precisely those for which the block-matrix entry $B^V$ takes the form
$$B^V = \begin{bmatrix}
    \varepsilon (1 - |s_0|^2)^{1/2} & 0_{1 \times (n-1)} \\
\end{bmatrix},$$
where $\varepsilon$ is an arbitrary unimodular complex number, that can be expressed as
in (6.23)-(6.24).}

### 7. One Step of the Schur Algorithm, Expressed in the Language of Colligations.

The results from Section 6 can be summarized as follows: Starting from the unitary
$n \times n$-matrix
$$\begin{bmatrix}
    \alpha & \beta \\
    \gamma & \delta \\
\end{bmatrix}$$
representing a given inner rational matrix-function $\omega(z)$ of
degree \( n \),
\[
\omega(z) = \alpha + z\beta(I - z\delta)^{-1}\gamma,
\]
we constructed the unitary \((n + 1) \times (n + 1)\)-matrix
\[
\begin{bmatrix}
s_0 & B \\
C & D
\end{bmatrix}
\]
representing the function \( s(z) \):
\[
s(z) = s_0 + zB(I - zD)^{-1}C,
\]
where \( s(z) \) is the inverse Schur transform \((6.6)\).

Our goal is not, however, to determine \( s(z) \) from \( \omega(z) \), but instead to start with \( s(z) \) and determine \( \omega(z) \). We look to describe a step of the Schur algorithm when applied to a rational inner function \( s(z) \),
\[
s(z) \rightarrow \omega(z) = \frac{1}{z} \frac{s(z) - s_0}{1 - s(z)s_0}, \quad s_0 = s(0),
\]
in terms of system representations. In other words, we would like to find the unitary matrix
\[
\begin{bmatrix}
\alpha & \beta \\
\gamma & \delta
\end{bmatrix}
\]
representing \( \omega(z) \), starting from the matrix \( U \) representing the function \( s(z) \).

Equation \((6.11)\) serves as a heuristic argument. Until now, \( \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \) was given and \( U \) was unknown. Now we assume that the unitary matrix \( U \) is given and that the matrix \( \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \) is unknown. We consider \((6.11)\) as an equation with respect to the matrix \( \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \) and \( U \) as given. Because the second factor on the right-hand side of \((6.11)\) is a unitary matrix, the solution matrix \( \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \) (if it exists) for equation \((6.11)\) is also a unitary matrix.

For a general unitary matrix \( U \), equation \((6.11)\) has no solution with respect to the matrix \( \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \). We know that the block-matrix entry \( B \) in \( U = \begin{bmatrix} s_0 & B \\ C & D \end{bmatrix} \) (\( U \) as in \((6.11)\)) is necessarily of the form
\[
B = \begin{bmatrix}
(1 - |s_0|^2)^{1/2} & 0_{1 \times (n-1)} \\
0 & 1_{(n-1) \times 1}
\end{bmatrix}.
\]
Since the characteristic functions of unitarily equivalent colligations coincide, it is enough to find a solution for \((6.11)\) with \( U \) replaced by some matrix \( U^V \) of the form \((6.15)\):
\[
U^V = \begin{bmatrix}
1 & 0 & 0_{1 \times (n-1)} \\
0 & \alpha & \beta \\
0_{(n-1) \times 1} & \gamma & \delta
\end{bmatrix}
\begin{bmatrix}
s_0 & (1 - |s_0|^2)^{1/2} & 0_{1 \times (n-1)} \\
(1 - |s_0|^2)^{1/2} & -s_0 & 0_{1 \times (n-1)} \\
0_{(n-1) \times 1} & 0_{(n-1) \times 1} & 1_{(n-1) \times (n-1)}
\end{bmatrix},
\]
\[(7.1)\]
LEMMA 7.1. Given a unitary \((n+1) \times (n+1)\)-matrix \(U\), the unitary \(n \times n\)-matrix \(V\) can be chosen such that equation (7.1) has a solution.

LEMMA 7.2. Given a unitary \((n+1) \times (n+1)\)-matrix \(U\):

\[
U = \begin{bmatrix} s_0 & B \\ C & D \end{bmatrix},
\]

(7.2)

we can find a unitary \(n \times n\)-matrix \(V_0\) such that \(U V_0\), given by

\[
U V_0 = \begin{bmatrix} 1 & 0_{n \times n} \\ 0_{n \times n} & V_0^* \end{bmatrix} U \begin{bmatrix} 1 & 0_{n \times n} \\ 0_{n \times n} & V_0 \end{bmatrix},
\]

takes the form \(U V_0 = U^0\), where

\[
U^0 = \begin{bmatrix} s_0 & B_0 \\ C_0 & D_0 \end{bmatrix},
\]

(7.3)

and the block-matrix entry \(B_0 \in \mathcal{M}_{1 \times n}\) is

\[
B_0 = [(1 - |s_0|^2)^{1/2} \cdots 0_{1 \times (n-1)}].
\]

(7.4)

PROOF. The row-vectors \(B\) and \(B_0\) satisfy the condition

\[
B B^* = B_0 B_0^* \quad (= (1 - |s_0|^2))
\]

The equality \(B_0 B_0^* = 1 - |s_0|^2\) follows from the definition of \(B_0\), (7.4). The equality \(s_0 s_0^* + B B^* = 1\) holds, since the matrix \(U\), (7.2), is unitary. Applying Lemma 5.1 to the row-vectors \(B\) and \(B_0\), we find the unitary \(n \times n\)-matrix \(V_0\) such that \(BV_0 = B_0\). For every such choice of \(V_0\), the matrix \(U V_0\) has the form (7.3)-(7.4).

REMARK 7.1. If \(n > 1\), the matrices \(U^0\) and \(V_0\) are not uniquely defined. The row-vector \(B\) of any matrix \(U V\) with \(V\) of the form \(V = V_0 \begin{bmatrix} 1 & 0 \end{bmatrix}\), where \(v\) is an arbitrary unitary \((n-1) \times (n-1)\)-matrix is also of the form (7.4).

THEOREM 7.1. Given a unitary \((n+1) \times (n+1)\)-matrix of the form

\[
U^0 = \begin{bmatrix} s_0 & (1 - |s_0|^2)^{1/2} & 0_{1 \times (n-1)} \\ c_1 & d_{11} & d_{12} \\ c_2 & d_{21} & d_{22} \end{bmatrix},
\]

(7.5)

where \(s_0 \in \mathbb{C}, \ |s_0| \leq 1, \ n \geq 2, \)

\[
c_1 \in \mathcal{M}_{1 \times 1}, \ d_{11} \in \mathcal{M}_{1 \times 1}, \ d_{12} \in \mathcal{M}_{1 \times (n-1)}, \c_2 \in \mathcal{M}_{(n-1) \times 1}, \ d_{21} \in \mathcal{M}_{(n-1) \times 1}, \ d_{22} \in \mathcal{M}_{(n-1) \times (n-1)};
\]

the equation

\[
U^0 = \begin{bmatrix} 1 & 0 & 0_{1 \times (n-1)} \\ 0 & \alpha & \beta \\ 0_{(n-1) \times 1} & \gamma & \delta \end{bmatrix} \begin{bmatrix} s_0 & (1 - |s_0|^2)^{1/2} & 0_{1 \times (n-1)} \\ (1 - |s_0|^2)^{1/2} & -s_0 & 0_{1 \times (n-1)} \\ 0_{(n-1) \times 1} & 0_{(n-1) \times 1} & 1_{(n-1) \times (n-1)} \end{bmatrix},
\]

(7.6)
where 
\[ \alpha \in \mathcal{M}_{1 \times 1}, \quad \beta \in \mathcal{M}_{1 \times (n-1)}, \quad \gamma \in \mathcal{M}_{(n-1) \times 1}, \quad \delta \in \mathcal{M}_{(n-1) \times (n-1)}, \]
has a solution with respect to the matrix
\[ U^1 = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}. \]  
(7.7)

The solution of this equation can be expressed as
\[ \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} -d_{11} \beta_0 + c_1 (1 - |\beta_0|^2)^{1/2} & d_{12} \\ -d_{21} \beta_0 + c_2 (1 - |\beta_0|^2)^{1/2} & d_{22} \end{bmatrix} \]  
(7.8)

**PROOF of THEOREM 7.1.** We consider equation (7.6) in further detail. If this equation is solvable, then
\[ \begin{bmatrix} \beta_0 \\ 1 - |\beta_0|^2 \end{bmatrix} \begin{bmatrix} (1 - |\beta_0|^2)^{1/2} \alpha \\ (1 - |\beta_0|^2)^{1/2} \beta \end{bmatrix} \times \begin{bmatrix} (1 - |\beta_0|^2)^{1/2} \gamma \\ (1 - |\beta_0|^2)^{1/2} \delta \end{bmatrix} = \begin{bmatrix} \gamma \\ \delta \end{bmatrix} \]  
(7.9)

Multiplying the matrices on the left-hand side of (7.9), we see that their product is of the form \[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & \gamma & \delta \end{bmatrix} \]. Since the matrix \( \bar{U}^0 \), (7.5), is unitary, the scalar product of its different rows vanishes. The fact that the first row of this matrix is orthogonal to each other row can be expressed as
\[ \begin{bmatrix} \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} \gamma \\ \delta \end{bmatrix} \]  
(7.10)

The latter equalities mean that the product of the matrices on the left-hand side of (7.9) takes the form \[ \begin{bmatrix} 1 & 0 \\ 0 & \alpha & \beta \\ 0 & \gamma & \delta \end{bmatrix} \]. Thus, the product of the matrices on the left-hand side of (7.9) has the desired form \[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & \gamma & \delta \end{bmatrix} \]. Multiplying out the matrices in (7.9), we obtain (7.8). Q.E.D.

**REMARK 7.2.** In view of (7.10), the solution \[ \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \] of equation (7.6) can also be written as:
\[ \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} (1 - |\beta_0|^2)^{-1/2} & d_{12} \\ (1 - |\beta_0|^2)^{-1/2} & d_{22} \end{bmatrix} \]  
if \( |\beta_0| < 1 \),  
(7.11)
and
\[
\begin{bmatrix}
\alpha & \beta \\
\gamma & \delta
\end{bmatrix} = \begin{bmatrix}
-(s_0)^{-1}d_{11} & d_{12} \\
-(s_0)^{-1}d_{21} & d_{22}
\end{bmatrix} \text{ if } s_0 \neq 0.
\tag{7.12}
\]

**REMARK 7.3.** If \( n = 1 \), then there is no room for the matrices \( d_{12}, d_{21}, d_{22} \) and \( \beta, \gamma, \delta \). In this case \( U^0 \), (7.5), should be replaced by the matrix:
\[
U^0 = \begin{bmatrix}
s_0 & (1 - |s_0|^2)^{1/2} \\
c_1 & d_{11}
\end{bmatrix},
\tag{7.13}
\]
where \( s_0 \in \mathbb{C} \) with \( |s_0| \leq 1 \),
\[
c_1 \in \mathfrak{M}_{1 \times 1}, \quad d_{11} \in \mathfrak{M}_{1 \times 1}.
\]
and the matrix \( U^1 \), (7.7), should be replaced with: matrix \( U^1 \)
\[
U^1 = [\alpha],
\tag{7.14}
\]
where
\[
\alpha \in \mathfrak{M}_{1 \times 1}.
\]
Equation (7.10) takes the form
\[
U^0 = \begin{bmatrix}
1 & 0 \\
0 & \alpha
\end{bmatrix} \begin{bmatrix}
s_0 & (1 - |s_0|^2)^{1/2} \\
(1 - |s_0|^2)^{1/2} & -\overline{s_0}
\end{bmatrix}.
\tag{7.15}
\]
The solution of this equation can be expressed as
\[
[\alpha] = [-d_{11}s_0 + c_1(1 - |s_0|^2)^{1/2}],
\tag{7.16}
\]
as well as in the forms:
\[
[\alpha] = [(1 - |s_0|^2)^{-1/2}c_1] \text{ if } |s_0| < 1,
\tag{7.17}
\]
and
\[
[\alpha] = [-\overline{s_0})^{-1}d_{11}] \text{ if } s_0 \neq 0.
\tag{7.18}
\]

Since both factors on the right-hand side of (7.16) are unitary matrices, we have that \( U^1 \) is also a unitary matrix. The matrix \( U^0 \) in (7.5) can be considered as a matrix of the unitary colligation \((E^0, H^0, U^0)\) with outer space \( E^0 = \mathbb{C} \) and with inner space \( H^0 = \mathbb{C}^n \). The matrix \( U^1 \) in (7.7) can, in turn, be considered as a matrix of the unitary colligation \((E^1, H^1, U^1)\) with outer space \( E^1 = \mathbb{C} \) and with inner space \( H^1 = \mathbb{C}^{n-1} \).

**LEMMA 7.3.**

I. If the colligation \((E^0, H^0, U^0)\) is controllable, then the colligation \((E^1, H^1, U^1)\) is also controllable.

II. If the colligation \((E^0, H^0, U^0)\) is observable, then the colligation \((E^1, H^1, U^1)\) is also observable.
**Proof.** Without loss of generality, we assume that $|s_0| < 1$. Otherwise the colligation $(E^0, H^0, U^0)$ cannot be neither controllable nor observable. Our reasoning is based on the equalities
\[
\begin{bmatrix}
d_{11} & d_{12} \\
d_{21} & d_{22}
\end{bmatrix} = \begin{bmatrix}
(\overline{-s_0}) \alpha & \beta \\
(\overline{-s_0}) \gamma & \delta
\end{bmatrix},
\]
(7.19)
and
\[
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix} = (1 - |s_0|^2)^{1/2} \begin{bmatrix}
\alpha \\
\gamma
\end{bmatrix}.
\]
(7.20)

**Proof of the Statement I.** The condition that the colligation $(E^0, H^0, U^0)$ be controllable means that
\[
\bigvee_{0 \leq k} \begin{bmatrix}
d_{11} & d_{12} \\
d_{21} & d_{22}
\end{bmatrix}^{k} \begin{bmatrix}
c_1 \\
c_2
\end{bmatrix} = M_{n \times 1} = \begin{bmatrix}
M_{1 \times 1} \\
M_{(n-1) \times 1}
\end{bmatrix}.
\]
(7.21)
And the controllability of the colligation $(E^1, H^1, U^1)$ can be expressed as:
\[
\bigvee_{0 \leq k} \delta^{k} \gamma = M_{(n-1) \times 1}.
\]
(7.22)
We look to show that (7.22) follows from (7.21). In view of (7.19) and (7.20), we can express (7.21) as:
\[
\bigvee_{0 \leq k} \begin{bmatrix}
(\overline{-s_0}) \alpha & \beta \\
(\overline{-s_0}) \gamma & \delta
\end{bmatrix}^{k} \begin{bmatrix}
\alpha \\
\gamma
\end{bmatrix} = M_{n \times 1} = \begin{bmatrix}
M_{1 \times 1} \\
M_{(n-1) \times 1}
\end{bmatrix}.
\]
(7.23)
Let
\[
\begin{bmatrix}
(\overline{-s_0}) \alpha & \beta \\
(\overline{-s_0}) \gamma & \delta
\end{bmatrix}^{k} \begin{bmatrix}
\alpha \\
\gamma
\end{bmatrix} = \begin{bmatrix}
* \\
f_k
\end{bmatrix}, \quad k = 0, 1, 2, \ldots,
\]
(7.24)
where $f_k \in M_{(n-1) \times 1}$. In view of (7.23),
\[
\bigvee_{0 \leq k} f_k = M_{(n-1) \times 1}.
\]
(7.25)
Clearly, we have that for every $k = 0, 1, 2, \ldots$
\[
f_k = \xi_{0,k} \delta^0 \gamma + \cdots + \xi_{k-1,k} \delta^{k-1} \gamma + \delta^k \gamma,
\]
(7.26)
where $\xi_{j,k}, 0 \leq j \leq k - 1, are some complex numbers. Therefore,
\[
\bigvee_{0 \leq k} f_k = \bigvee_{0 \leq k} \delta^k \gamma.
\]
We have thus proved Statement I.
Proof of the Statement II. The condition that the colligation be observable \((\mathcal{E}^0, \mathcal{H}^0, U^0)\) can be written as:
\[
\bigvee_{0 \leq k} \begin{bmatrix} 1 - |s_0|^2 \end{bmatrix}^{1/2} \begin{bmatrix} 0_{1 \times (n-1)} \\
\begin{bmatrix} d_{11} & d_{12} \\
\end{bmatrix} & d_{22}
\end{bmatrix}^k = \mathcal{M}_{1 \times n} \left( = \begin{bmatrix} \mathcal{M}_{1 \times 1} & \mathcal{M}_{1 \times (n-1)} \end{bmatrix} \right).
\]
(7.27)

And the observability of the colligation \((\mathcal{E}^1, \mathcal{H}^1, U^1)\) can be expressed as:
\[
\bigvee_{0 \leq k} \beta \delta^k = \mathcal{M}_{1 \times (n-1)}.
\]
(7.28)

We aim to show that (7.28) follows from the formulas (7.27), (7.8) and (7.10). In view of (7.19), we can express (7.27) as follows:
\[
\bigvee_{0 \leq k} \begin{bmatrix} 1 & 0_{1 \times (n-1)} \\
\begin{bmatrix} (-s_0)\alpha & \beta \\
\end{bmatrix} & \begin{bmatrix} (-s_0)\beta & \delta \\
\end{bmatrix}
\end{bmatrix}^k = \mathcal{M}_{1 \times n} \left( = \begin{bmatrix} \mathcal{M}_{1 \times 1} & \mathcal{M}_{1 \times (n-1)} \end{bmatrix} \right).
\]
(7.29)

Let
\[
\begin{bmatrix} 1 & 0_{1 \times (n-1)} \\
\begin{bmatrix} (-s_0)\alpha & \beta \\
\end{bmatrix} & \begin{bmatrix} (-s_0)\beta & \delta \\
\end{bmatrix}
\end{bmatrix}^k = [g_k], \quad k = 0, 1, 2 \ldots,
\]
(7.30)

where \(g_k \in \mathcal{M}_{1 \times (n-1)}\). In view of (7.29), we have that
\[
\bigvee_{0 \leq k} [g_k] = \mathcal{M}_{1 \times (n-1)}.
\]
(7.31)

Clearly, \(g_0 = 0_{1 \times (n-1)}\) and for every \(k = 0, 1, 2 \ldots\)
\[
g_{k+1} = \eta_{0,k} \beta \delta^0 + \cdots + \eta_{k-1,k} \beta \delta^{k-1} + \beta \delta^k,
\]
(7.32)

where \(\eta_{j,k}, 0 \leq j \leq k - 2\), are some complex numbers. Therefore,
\[
\bigvee_{0 \leq k} [g_k] = \bigvee_{0 \leq k} \beta \delta^k.
\]

We have thus proved Statement II. Q.E.D.

The following Lemma is an immediate consequence of Lemma 7.3

**Lemma 7.4.** Let \(s_0 \in \mathbb{C}\) with \(|s_0| < 1\) and \(U^0\) be a unitary \((n + 1) \times (n + 1)\)-matrix of the form (7.5). Suppose that the \(n \times n\)-matrix \(U^1\), (7.7), is related to the matrix \(U^0\) by equation (7.6). Let \((\mathcal{E}^0, \mathcal{H}^0, U^0)\) and \((\mathcal{E}^1, \mathcal{H}^1, U^1)\) be the above-described operator colligations related to the matrices \(U^0\) and \(U^1\). If the colligation \((\mathcal{E}^0, \mathcal{H}^0, U^0)\) is minimal, then the colligation \((\mathcal{E}^1, \mathcal{H}^1, U^1)\) is also minimal.

**Theorem 7.2.** Let \(s(z)\) be a rational inner matrix-function of degree \(n > 1\) \((s(z)\) is thus non-constant and \(|s_0| < 1\), where \(s_0 = s(0)\)) and let
\[
\omega(z) = \frac{1}{z} \cdot \frac{s(z) - s_0}{1 - s(z) \overline{s_0}}, \quad s_0 = s(0),
\]
(7.33)
be the Schur transformation of the function \( s(z) \).

Let the unitary matrix \( U \),
\[
U = \begin{bmatrix}
s_0 & B_0 \\
C_0 & D_0
\end{bmatrix}, \quad B_0 \in \mathcal{M}_{1 \times n}, \ C_0 \in \mathcal{M}_{n \times 1}, \ D_0 \in \mathcal{M}_{n \times n},
\]  
(7.34)

which yields the minimal system representation
\[
s(z) = s_0 + zB_0(I - zD_0)^{-1}C_0,
\]  
(7.35)

have row \( B_0 \) of the special form
\[
B_0 = \begin{bmatrix} b_0 & 0_{1 \times (n-1)} \end{bmatrix}, \quad b > 0.
\]  
(7.36)

Then the function \( \omega(z) \) admits the system representation
\[
\omega(z) = \alpha + z\beta(I - z\delta)^{-1}\gamma,
\]  
(7.37)

where the unitary \( n \times n \)-matrix 
\[
\begin{bmatrix}
\alpha & \beta \\
\gamma & \delta
\end{bmatrix},
\]
\( \alpha \in \mathcal{M}_{1 \times 1}, \ \beta \in \mathcal{M}_{1 \times (n-1)}, \ \gamma \in \mathcal{M}_{(n-1) \times 1}, \ \delta \in \mathcal{M}_{(n-1) \times (n-1)}, \)
can be determined from the matrix \( U_0 \) using
\[
\begin{bmatrix}
\alpha & \beta \\
\gamma & \delta
\end{bmatrix} = \begin{bmatrix}
(1 - |s_0|^2)^{-1/2}c_1 & d_{12} \\
(1 - |s_0|^2)^{-1/2}c_2 & d_{22}
\end{bmatrix},
\]  
(7.38)

where \( c_j \) and \( d_{jk} \) are the block-matrix entries of the block-matrix decompositions
\[
C_0 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \quad D_0 = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix},
\]  
(7.39)

\( c_1 \in \mathcal{M}_{1 \times 1}, \ c_2 \in \mathcal{M}_{(n-1) \times 1}, \)
\( D_{11} \in \mathcal{M}_{1 \times 1}, \ D_{12} \in \mathcal{M}_{1 \times (n-1)}, \ D_{21} \in \mathcal{M}_{(n-1) \times 1}, \ D_{22} \in \mathcal{M}_{(n-1) \times (n-1)}. \)

The unitary colligation associated with the matrix \( \tilde{U}^1 \) is minimal.

**PROOF.** The matrix \( U^0 \), \( (7.34) \), the matrix 
\[
\begin{bmatrix}
\alpha & \beta \\
\gamma & \delta
\end{bmatrix}, \quad (7.38),
\]
and the number \( s_0 \) are related by equation \( (6.11) \). According to Theorem \( 6.1 \) the function \( \omega(z) \), defined by \( (7.37) \), and the function \( s(z) \) are related by the equality \( (6.6) \). Q.E.D.

Theorem \( 7.2 \) together with Lemma \( 7.2 \) describe a step of the Schur algorithm in terms of system representations. Before applying the direct Schur transform \( (7.33) \), which is a step of the Schur algorithm, we should first ‘normalize’ the colligation matrix \( U \) representing the ‘initial’ function \( s(z) \). This normalization starts with the matrix \( U \), from which we determine the unitarily equivalent matrix \( U^0 \), \( (7.5) \), whose row \( B^0 \) is of the special form \( (7.3) \). We then aim to solve the equation \( (7.6) \) with respect to the matrix \( \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \). The solution of \( (7.6) \) is given by \( (7.38) \). The unitary matrix \( \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \) yields the system representation of the function \( \omega(z) \).
It should be emphasized that, in general, the matrix \[
\begin{bmatrix}
\alpha & \beta \\
\gamma & \delta
\end{bmatrix}
\] is not normalized, i.e. its row \(\beta\) is not of the form \(\beta = [\ast \ 0_{1 \times (n-2)}]\). To perform the next step of the Schur algorithm, we must therefore ‘normalize’ the matrix \(\begin{bmatrix}
\alpha & \beta \\
\gamma & \delta
\end{bmatrix}\), obtaining the ‘normalized’ form \(U^1\). We then have to solve the equation of the form (7.3), where \(U^0\) is replaced by \(U^1\), etc. The normalization procedure must therefore be performed at every step of the Schur algorithm. This normalization procedure is, however, not quite unique. It has some degrees of freedom (See Remark 7.1.) It turns out that we can use these degrees of freedom to make the normalization procedure a one-time procedure, so that it might be dealt with during preprocessing for the further step-by-step recurrence. In further processing there is then no need for normalization and one only has to solve the recurrent chain of equations of the form (7.6). A one-time normalization of this kind is related to the reduction of the ‘initial’ colligation matrix to the lower Hessenberg form.

8. Hessenberg Matrices.

The Householder Algorithm.

Roughly speaking, the lower (upper) Hessenberg matrix, is a matrix which is almost lower (upper) triangular. The precise definition is:

**DEFINITION 8.1.**
1. We say that a square matrix \(H\) is a **lower Hessenberg matrix** if it has zero-entries above the first superdiagonal. If \(H = ||h_{jk}||_{0 \leq j, k \leq n}\), then \(H\) is lower Hessenberg matrix if \(h_{jk} = 0\) for \(k > j + 1, 0 \leq j \leq n - 1\).
2. We say that a lower Hessenberg matrix \(H = ||h_{jk}||_{0 \leq j, k \leq n}\) is **special** if all entries of its first superdiagonal are non-negative: \(h_{jj+1} \geq 0, 0 \leq j \leq n - 1\).
3. We say that a Hessenberg matrix \(H = ||h_{jk}||_{0 \leq j, k \leq n}\) is **HL-non-singular** if all entries of its first superdiagonal are non-zero: \(h_{j,j+1} \neq 0, 0 \leq j \leq n - 1\).

The definition of an upper Hessenberg matrix, special upper Hessenberg matrix and non-singular upper Hessenberg matrix is similar to Definition 8.1.

**DEFINITION 8.2.**
1. We say that a square matrix \(H\) is an **upper Hessenberg matrix** if it has zero-entries below the first subdiagonal. If \(H = ||h_{jk}||_{0 \leq j, k \leq n}\), then \(H\) is upper Hessenberg matrix if \(h_{jk} = 0\) for \(k < j - 1, 0 \leq j \leq n - 1\).
2. We say that an upper Hessenberg matrix \(H = ||h_{jk}||_{0 \leq j, k \leq n}\) is **special** if all entries of its first subdiagonal are non-negative: \(h_{j,j-1} \geq 0, 1 \leq j \leq n\).
3. We say that an upper Hessenberg matrix \(H = ||h_{jk}||_{0 \leq j, k \leq n-1}\) is **HU-non-singular** if all entries of its first subdiagonal are non-zero: \(h_{j,j-1} \neq 0, 1 \leq j \leq n\).

Hessenberg matrices were investigated by Karl Hessenberg (1904-1959), a German engineer whose dissertation dealt with the computation of eigenvalues and eigenvectors of linear operators.
THEOREM 8.1.
1. Given an \((n+1) \times (n+1)\)-matrix \(M = |M_{j,k}|_{0 \leq j, k \leq n}\), there exists a unitary \(n \times n\)-matrix \(V\) such that the matrix \(H^L\),

\[
H^L = \begin{bmatrix} 1 & 0_{1 \times n} \\ 0_{1 \times n} & V^* \end{bmatrix} M \begin{bmatrix} 1 & 0_{1 \times n} \\ 0_{n \times 1} & V \end{bmatrix}
\]  

(8.1)

is a special lower Hessenberg matrix.

2. If the matrix \(M\) is HL-non-singular, then both matrices \(H^L\) and \(V\) are uniquely determined. From the equalities

\[
H^L_j = \begin{bmatrix} 1 & 0_{1 \times n} \\ 0_{n \times 1} & V_j^* \end{bmatrix} M \begin{bmatrix} 1 & 0_{1 \times n} \\ 0_{n \times 1} & V_j \end{bmatrix}, \quad j = 1, 2,
\]

(8.2)

where \(H^L_1\) and \(H^L_2\) are special upper Hessenberg matrices, \(V_1\) and \(V_2\) are unitary matrices and the Hessenberg matrix \(H^L_1\) is HL-non-singular, it follows that \(H^L_2 = H^L_1\) and \(V_2 = V_1\).

DEFINITION 8.3. Given a square matrix \(M\), a lower Hessenberg matrix \(H^L\) to which \(M\) can be reduced, (8.1), is called a lower Hessenberg form of the matrix \(M\).

THEOREM 8.2.
1. Given an \((n+1) \times (n+1)\)-matrix \(M = |M_{j,k}|_{0 \leq j, k \leq n}\), there exists a unitary \(n \times n\)-matrix \(V\) such that the matrix \(H^U\),

\[
H^U = \begin{bmatrix} 1 & 0_{1 \times n} \\ 0_{n \times 1} & V^* \end{bmatrix} M \begin{bmatrix} 1 & 0_{1 \times n} \\ 0_{n \times 1} & V \end{bmatrix}
\]

(8.3)

is a special upper Hessenberg matrix.

2. If the matrix \(M\) is HU-non-singular, then both matrices \(H^U\) and \(V\) are uniquely determined. From the equalities

\[
H^U_j = \begin{bmatrix} 1 & 0_{1 \times n} \\ 0_{n \times 1} & V_j^* \end{bmatrix} M \begin{bmatrix} 1 & 0_{1 \times n} \\ 0_{n \times 1} & V_j \end{bmatrix}, \quad j = 1, 2,
\]

(8.4)

where \(H^U_1\) and \(H^U_2\) are upper Hessenberg matrices, \(V_1\) and \(V_2\) are unitary matrices and the Hessenberg matrix \(H^U_1\) is HU-non-singular, it follows that \(H^U_2 = H^U_1\) and \(V_2 = V_1\).

DEFINITION 8.4. Given a square matrix \(M\), an upper Hessenberg matrix \(H^U\) to which \(M\) can be reduced, (8.3), is called an upper Hessenberg form of the matrix \(M\).

THEOREM 8.3. Let \(U\) be an \((n+1) \times (n+1)\)-unitary matrix.
1. The unitary colligation associated with the matrix \(U\) is observable if and only if lower Hessenberg form of \(U\) is HL-non-singular.
2. The unitary colligation associated with the matrix \(U\) is controllable if and only if the upper Hessenberg form of \(U\) is HU-non-singular.
COROLLARY 8.1. According to Theorem 3.9, the finite-dimensional unitary coligation is observable if and only if it is controllable. Thus, for a unitary matrix $U$, the lower Hessenberg form of $U$ is $HL$-nonsingular if and only if the upper Hessenberg form of $U$ is $HU$-nonsingular.

LEMMA 8.1. Given two row-vectors $B' = [b'_1 b'_2 \ldots b'_n] \in \mathbb{M}_{1 \times n}$ and $B'' = [b''_1 b''_2 \ldots b''_n] \in \mathbb{M}_{1 \times n}$ having same norm:

$$BB'^* = B''B''^*,$$

(8.5)

there exists a unitary $n \times n$-matrix $V$ such that

$$B'V = B''.$$

(8.6)

PROOF of LEMMA 8.1. We first consider the question in a more general setting. Assume that $\mathcal{H}$ is a complex Hilbert space with scalar product $\langle u, v \rangle$, where $\langle u, v \rangle$ is linear with respect to the argument $u$ and antilinear with respect to $v$. Let $x \in \mathcal{H}$ and $y \in \mathcal{H}$ be two vectors such that $\langle x, x \rangle = \langle y, y \rangle \neq 0$. Let $\|u\|$ denote the norm of the vector $u$: $\|u\| = \langle u, u \rangle^{1/2}$. Given two vectors $x \in \mathcal{H}, y \in \mathcal{H}$ such that $\|x\| = \|y\| \neq 0$, our goal is to construct a unitary operator $V : \mathcal{H} \rightarrow \mathcal{H}$ such that $Vx = y$. If the vector $y$ is proportional to the vector $x$: $x = \lambda y$ for some $\lambda \in \mathbb{C}$, we put $Vz = \lambda z \forall z \in \mathbb{C}$. This operator is unitary: $|\lambda| = 1$, because $\|x\| = \|y\| \neq 0$.

If the vectors $x$ and $y$ are not proportional, we choose $\lambda \in \mathbb{C}, |\lambda| = 1$ such that $\lambda \langle x, y \rangle \geq 0$. (If $\langle x, y \rangle \neq 0$, then this $\lambda$ is unique. If $\langle x, y \rangle = 0$, we can choose arbitrary $\lambda$ with $|\lambda| = 1$.) Let

$$Vz = \lambda z - 2\langle z, x - \lambda y \rangle \langle x - \lambda y, x - y \rangle^{-1/2} (\lambda x - y) \forall z \in \mathcal{H}.$$  

(8.7)

The vectors

$$e_1 = x + \lambda y \quad \text{and} \quad e_2 = x - \lambda y$$

are non-zero ($x$ and $y$ are not proportional to one another) and orthogonal:

$$\langle e_1, e_2 \rangle = 0,$$

(8.8)

because

$$\langle x + \lambda y, x - \lambda y \rangle = \langle x, x \rangle - \overline{\lambda} \lambda \langle y, y \rangle + \overline{\lambda} \lambda \langle x, y \rangle - \lambda \langle x, y \rangle$$

and $\langle x, x \rangle = \langle y, y \rangle, \overline{\lambda} \lambda = 1, \lambda \langle x, y \rangle = \overline{\lambda} \lambda \langle y, x \rangle$. From (8.7) and (8.8) it follows that

$$Ve_1 = \lambda e_1.$$  

(8.9)

From (8.7) it follows that

$$Ve_2 = -\lambda e_2,$$

(8.10)

and

$$Vz = \lambda z \forall z \in \mathcal{H}: \langle z, e_1 \rangle = 0, \langle z, e_2 \rangle = 0.$$  

Therefore the operator $V$ is unitary. Since

$$x = \frac{1}{2}(e_1 + e_2), \quad y = \frac{\lambda}{2}(e_1 - e_2)$$

we have

$$Vx = \lambda y.$$  

(8.11)
from (8.9) and (8.10) it follows that $Vx = y$.

Let us turn to the proof of the statement of Lemma 8.1. Let $H$ be the set of all $n$-row-vectors with complex entries (in other words, $H = \mathbb{C}^{1 \times n}$) and with the following scalar product: if $u = [u_1, \ldots, u_n]$ and $v = [v_1, \ldots, v_n]$ are vectors in $H$, then their scalar product $\langle u, v \rangle$ is defined as

$$\langle u, v \rangle = uv^*$$

where $v^*$ is the Hermitian conjugate of the row-vector $v$. If $H$ is some $n \times n$-matrix, then it generates an operator in $H$. This operator maps the row-vector $u$ to the row-vector $uH$, where $uH$ is the product of the matrices $u$ and $H$. This operator is unitary if and only if $H$ is unitary.

In the notation of Lemma 8.1:

$$x = B' = [b'_1, b'_2, \ldots, b'_n], \quad y = B'' = [b''_1, b''_2, \ldots, b''_n].$$

Thus the matrix $V$ corresponding to the operator (8.7) takes the form

$$V = ||v_{jk}||_{1 \leq j, k \leq n}, \quad (8.12)$$

where

$$v_{jk} = \lambda \delta_{jk} - 2(\overline{b'_j} - \lambda b''_j)(B' - \overline{B''}, B' - \overline{B''})^{-1}(\lambda b'_k - b''_k). \quad (8.13)$$

and $\lambda$ is such that

$$\lambda B'(B'')^* \geq 0, \quad |\lambda| = 1.$$

$\delta_{jk}$ is the Kronecker symbol. Q.E.D.

**REMARK 8.1.** In the case when the rows $B'$ and $B''$ are real, the matrix $V$, (8.12) - (8.13), is also real. In this case matrices of the form (8.12) - (8.13) are known as Householder reflection matrices. Householder reflection matrices and the Householder Algorithm (which is based on matrices of this type) are widely used in numerical linear algebra. See [Wil], [Str], [GolV] and [Hou].

**REMARK 8.2.** A unitary matrix $V$ satisfying the condition (8.6) is not unique. The process of constructing such matrices (8.12) - (8.13) is constructive.

We will apply Lemma 8.1 to the following special situation: Let $B' \neq 0$ be an arbitrary $1 \times n$-column and $B''$ be of the special form $B'' = |b''_1 \ldots 0_{1\times(n-1)}|$, where $b'' > 0$ and thus $b'' = (B'(B'')^*)^{1/2}$. For these $B', B''$, the first column of the unitary matrix $V$ satisfying (8.6) is uniquely determined:

$$v_{1j} = \overline{b'_j}(B'(B'')^*)^{-1/2}, \quad 1 \leq j \leq n.$$

The construction of the desired matrix $V$ is thus reduced to the following problem: Given the first column of an $n \times n$-matrix, one needs to extend this column to a full unitary matrix. The Householder reflection procedure is one way of doing this.

We use the Householder reflection matrices to reduce an arbitrary matrix to a Hessenberg matrix.
Lemma 8.1 ensures that this choice is possible. We then define the matrix
\[
V^0 = \begin{bmatrix} 1 & 0_{1 \times n} \end{bmatrix} M^0 \begin{bmatrix} 1 & 0_{1 \times n} \end{bmatrix},
\]
where
\[
m_{1,1}^0 \geq 0, m_{0,k}^0 = 0, 2 \leq k \leq n.
\] (8.16)

Applying Lemma 8.1 we choose the unitary matrix \(V_1 \in \mathbb{M}_{n,n}\) such that
\[
[m_{0,1}^0, m_{0,2}^0, \ldots, m_{0,n}^0] V_1 = [m_{1,1}^0, m_{1,2}^0, \ldots, m_{1,n}^0],
\] (8.15)

where
\[
V_1 = \begin{bmatrix} I_l \end{bmatrix} \begin{bmatrix} V^0 & 0_{l \times (n-l)} \end{bmatrix} \begin{bmatrix} 0_{l \times (n-l)} & V_1^{-1} \end{bmatrix},
\]
and let \(m_{j,k}^1\) denote the entries of the matrix \(M^1\):
\[
M^1 = ||m_{j,k}^1||_{0 \leq j, k \leq n}
\] (8.18)

Clearly,
\[
m_{0,0}^1 = m_{0,0}^0.
\] (8.19)

We continue this procedure inductively. We next turn to the inductive step from \(l\) to \(l+1\).

Suppose that the matrices \(M^p \in \mathbb{M}_{n+1,n+1}\) and \(V_p \in \mathbb{M}_{n-p+1,n-p+1}\) with \(0 \leq p \leq l\) are already known and that the following condition for the entries of the matrix \(M^p\),
\[
M^p = ||m_{j,k}^p||_{0 \leq j, k \leq n},
\] (8.20)

are satisfied:
\[
m_{j,j+1}^p \geq 0, m_{j,k}^p = 0, j + 2 \leq k \leq n, j = 0, 1, \ldots, p-1.
\] (8.21)

The matrices \(V_p\), \(1 \leq p \leq l\) are unitary and we have
\[
M^p = \begin{bmatrix} I_p & 0_{p \times (n-p+1)} \\ 0_{(n-p+1) \times p} & V_p^* \end{bmatrix} M^{p-1} \begin{bmatrix} I_p & 0_{p \times (n-p+1)} \\ 0_{(n-p+1) \times p} & V_p \end{bmatrix},
\] (8.22)

for every \(p \leq l\).

We choose the unitary \((n-l) \times (n-l)\)-matrix \(V_{l+1}\) such that
\[
[m_{l,l+1}^l, m_{l,l+2}^l, \ldots, m_{l,n}^l] V_{l+1} = [m_{l,l+1}^{l+1}, m_{l,l+2}^{l+1}, \ldots, m_{l,n}^{l+1}],
\] (8.23)

where
\[
m_{l,l+1}^{l+1} \geq 0, m_{l,k}^{l+1} = 0, l + 2 \leq k \leq n.
\] (8.24)

Lemma 8.1 ensures that this choice is possible. We then define the matrix \(M^{l+1}\),
\[
M^{l+1} = ||m_{j,k}^{l+1}||_{0 \leq j, k \leq n}
\] (8.25)
as
\[
M^{l+1} = \begin{bmatrix}
I_{l+1} & 0_{l+1 \times (n-l)} \\
0_{(n-l) \times (l+1)} & V_{l+1}^* \\
\end{bmatrix} M^l \begin{bmatrix}
I_{l+1} & 0_{(l+1) \times (n-l)} \\
0_{(n-l) \times (l+1)} & V_{l+1} \\
\end{bmatrix}.
\] (8.26)

The entries of the matrix \(M^{l+1}\) satisfy the condition
\[
m^{l+1}_{j,j+1} \geq 0, \quad m^{l+1}_{j,k} = 0, \quad j = 0, 1, \ldots, l, \quad j + 2 \leq k \leq n.
\] (8.27)

For \(j = l\), condition (8.24) holds in view of (8.23) (Ensuring this was our goal in choosing the matrix \(V^l_{l+1}\) as we did.)

For \(0 \leq j \leq l - 1\), condition (8.24) holds, because going from the matrix \(M^l\) to the matrix \(M^{l+1}\) we do not change the rows with indices \(j: 0 \leq j \leq l - 1\):
\[
m^{l+1}_{j,k} = m^l_{j,k}, \quad 0 \leq j \leq l - 1, \quad 0 \leq k \leq n.
\] (8.28)

The equality (8.28) holds, firstly because the identity matrix of size \(l + 1\) is the left upper corner of the block-matrix
\[
\begin{bmatrix}
I_{l+1} & 0_{l+1 \times (n-l)} \\
0_{(n-l) \times (l+1)} & V_{l+1} \\
\end{bmatrix}
\] and secondly, because
\[
m^l_{j,k} = 0 \quad \forall j, k: 0 \leq j \leq l - 1, \quad l + 1 \leq k \leq n
\] (The latter is a consequence of the induction hypothesis (8.21) for \(p = l - 1\).)

The inductive process finishes when we construct the matrix \(M_n = M^{l+1}\) for \(l = n - 1\).

The matrix \(V\) satisfying (8.1) appears as the product
\[
V = V_1 \cdot \begin{bmatrix}
I_1 & 0_{1 \times (n-1)} \\
0_{(n-2) \times 2} & V_2 \\
\end{bmatrix} \cdot \begin{bmatrix}
I_2 & 0_{2 \times (n-2)} \\
0_{(n-2) \times 2} & V_3 \\
\end{bmatrix} \cdots \begin{bmatrix}
I_{n-2} & 0_{(n-2) \times 2} \\
0_{(n-2) \times 2} & V_{n-1} \\
\end{bmatrix}.
\] (8.29)

According to the above construction, the entries of the matrix \(H = \begin{bmatrix} h_{j,k} \end{bmatrix}_{0 \leq j, k \leq n}\), satisfy:
\[
h_{j,k} = m^{j+1}_{j,k}, \quad j \leq k \leq n,
\] (8.30)

and thus we have:
\[
h_{j,j+1} = m^{j+1}_{j,j+1} \geq 0, \quad h_{j,k} = 0, \quad j + 2 \leq k \leq n,
\] (8.31)
Q.E.D.

The reduction of matrices to the Hessenberg form is a tool often applied in numerical linear algebra as a preliminary step for further numerical algorithms. See [Wil], [Str], [GolV] and other sources in numerical linear algebra.

The Householder algorithm is implemented in the programming system MATLAB. The MATLAB command \(H = \text{hess}(A)\) reduces the matrix \(A\) to the upper Hessenberg form \(H\).

In the next section we discuss the Schur algorithm for rational inner functions in terms of the unitary colligation for the system representation of this function.
Reducing the colligation matrix to the upper Hessenberg form is a preliminary step for further developing the Schur algorithm in terms of system representations.

REMARK 8.3. In [KiNe], the Householder algorithm and the Hessenberg form for unitary matrices are used to study the probability measures associated with finite Blaschke products via Cayley transform.

9. The Schur Algorithm in Terms of System Representations.

We have now finished all necessary preparations and we are well positioned to present the Schur algorithm in terms of unitary colligations representing the appropriate functions.

Let \( s(z) \) be a rational inner matrix-function of degree \( n > 0 \) (\( s(z) \) is thus non-constant and \( |s_0| < 1 \), where \( s_0 = s(0) \)) and let

\[
s_0(z) = s(z), \quad s_k(z), \quad k = 1, 2, \ldots, n,
\]

be the sequence of rational inner functions constructed according to (2.4) (\( \deg s_k(z) = n - k \), so that \( s_n(z) = s_n \) is a unitary constant.)

Let

\[
s(z) = A + zB(I_n - zD)^{-1}C
\]

be the system representation of \( s(z) \), where

\[
U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}
\]

is the matrix of the minimal unitary colligation representing \( s(z) \):

\[
A \in \mathbb{M}_{1 \times 1}, \quad B \in \mathbb{M}_{1 \times n}, \quad C \in \mathbb{M}_{n \times 1}, \quad D \in \mathbb{M}_{n \times n}. \quad \text{(So, } A = s_0.)
\]

We first reduce \( U \) to the lower Hessenberg form. Let \( V \) be a unitary \( n \times n \)-matrix such that the matrix \( U^0 \) (also unitary):

\[
U^0 = \begin{bmatrix} 1 & 0_{1 \times n} \\ 0_{n \times 1} & V^* \end{bmatrix} U \begin{bmatrix} 1 & 0_{1 \times n} \\ 0_{n \times 1} & V \end{bmatrix}, \quad j = 1, 2,
\]

is an upper Hessenberg matrix. The block entries of the matrix

\[
U^0 = \begin{bmatrix} A^0 & B^0 \\ C^0 & D^0 \end{bmatrix}
\]

are:

\[
A^0 \in \mathbb{M}_{1 \times 1}, \quad B^0 \in \mathbb{M}_{1 \times n}, \quad C^0 \in \mathbb{M}_{n \times 1}, \quad D^0 \in \mathbb{M}_{n \times n}. \quad \text{(So, } A^0 = A = s_0.)
\]

The unitary colligations associated with the matrices \( U \) and \( U^0 \) are unitarily equivalent. The unitary colligation associated with the unitary matrix \( U^0 \) is therefore minimal and represents the function \( s_0(z) = s(z) \):

\[
s_0(z) = A^0 + zB^0(I_n - zD^0)^{-1}C^0.
\]

Inductively, we construct the sequence \( U^p, \ p = 0, 1, \ldots, n - 1 \) of unitary upper Hessenberg matrices such that the unitary colligation associated with the matrix
$U^p$ is minimal and represents the function $s_p(z)$, which appears in the $p$-th step of the Schur algorithm.

For $p = 0$, the representation in (9.5) holds. We consider the step from $p$ to $p + 1$.

Suppose that $U^p$, $0 \leq p < (n - 1)$ is a unitary lower $HL$-non-singular $(n - p + 1) \times (n - p + 1)$ Hessenberg matrix with the block-matrix decomposition:

$$U^p = \begin{bmatrix} A^p & B^p \\ C^p & D^p \end{bmatrix}$$

(9.6)

where

$$A^p \in \mathfrak{M}_{1 \times 1}, \ B^p \in \mathfrak{M}_{1 \times (n-p)}, \ C^p \in \mathfrak{M}_{(n-p) \times 1}, \ D^p \in \mathfrak{M}_{(n-p) \times (n-p)}.$$

The unitary colligation associated with the matrix $U^p$ is minimal and represents the function $s_p(z)$, which appears in the $p$-th step of the Schur algorithm:

$$s_p(z) = A^p + zB^p(I_{n-p} - zD^p)^{-1}C^p.$$  

(9.7)

Let

$$C^p = \begin{bmatrix} C_{1}^p \\ C_{2}^p \end{bmatrix} \quad D^p = \begin{bmatrix} D_{11}^p & D_{12}^p \\ D_{21}^p & D_{22}^p \end{bmatrix},$$

(9.8)

be the more refined block matrix decomposition of the block-matrix entries $C^p$ and $D^p$:

$$C_{1}^p \in \mathfrak{M}_{1 \times 1}, \ C_{2}^p \in \mathfrak{M}_{(n-1-p) \times 1}, \ D_{11}^p \in \mathfrak{M}_{1 \times 1}, \ D_{12}^p \in \mathfrak{M}_{1 \times (n-1-p)}, \ D_{21}^p \in \mathfrak{M}_{(n-1-p) \times 1}, \ D_{22}^p \in \mathfrak{M}_{(n-1-p) \times (n-1-p)}.$$

Since $U^p$ is an upper Hessenberg matrix and also an $HU$-non-singular matrix, we have that $B^p \neq 0$. Because $U^p$ is also unitary, it follows that $|A^p| < 1$, i.e. that

$$|s_p| < 1 \quad \text{where} \quad s_p = s_p(0).$$

(9.9)

The row $B^p$ is of the form

$$B^p = [(1 - |s_p|^2)^{1/2}, 0_{1 \times (n-p-1)}]$$

(9.10)

We construct the $(n - p) \times (n - p)$-matrix $U^{p+1}$:

$$U^{p+1} = \begin{bmatrix} A^{p+1} & B^{p+1} \\ C^{p+1} & D^{p+1} \end{bmatrix},$$

(9.11)

$$A^{p+1} \in \mathfrak{M}_{1 \times 1}, B^{p+1} \in \mathfrak{M}_{1 \times (n-p-1)}, C^{p+1} \in \mathfrak{M}_{(n-p-1) \times 1}, D^{p+1} \in \mathfrak{M}_{(n-p-1) \times (n-p-1)},$$

where

$$\begin{bmatrix} A^{p+1} & B^{p+1} \\ C^{p+1} & D^{p+1} \end{bmatrix} \quad \text{def} \quad \begin{bmatrix} (1 - |s_p|^2)^{-1/2} C_{1}^p & D_{12}^p \\ (1 - |s_p|^2)^{-1/2} C_{2}^p & D_{22}^p \end{bmatrix}.$$ 

(9.12)

To obtain the matrix $U^p$ from $U^{p+1}$, one should delete the left column and the upper row of the matrix $U^{p+1}$ and then recalculate the first column of the resulting matrix. The matrix $U^{p+1}$ is then an upper Hessenberg matrix. The matrix $U^{p+1}$ is $HL$-non-degenerate, because $U^p$ is $HL$-non-degenerate and because the first superdiagonal of the matrix $U^{p+1}$ is a subset of the first superdiagonal of the matrix $U^p$. According to Theorem 7.2 (which can be applied to the matrix $U^p$ in
view of (9.10), the matrix $U^{p+1}$ is unitary and the unitary colligation associated with $U^{p+1}$ represents the function $s_{p+1}(z)$ appearing in the $p+1$-th step of the Schur algorithm:

$$s_{p+1}(z) = A^{p+1} + zB^{p+1}(I_{n-p-1} - zD^{p+1})^{-1}C^{p+1}.$$  

(9.13)

These considerations do not directly apply when $p = n - 1$. In this case, there is no room for $B^n, C^n, D^n$. However, we can construct ‘part’ of the matrix (9.12):

$$A^n = (1 - |s_{n-1}|^2)^{-1/2}C^{n-1}.$$  

(9.14)

(See Remark 7.3.) The $1 \times 1$-matrix $A^n$ is unitary, hence it is a unitary constant. Clearly, $A^n = s_n$, where $s_n$ is the $n$-th Schur parameter. This completes the description of the Schur algorithm for inner rational matrix-functions in terms of system representations. Q.E.D.

**REMARK 9.1.** It is particularly easy to determine the sequence $\{D^p\}_{p=0,1,\ldots,n}$ of matrices representing the inner operators of the unitary colligations associated with the colligation matrices $U^p$. The matrix $D^p$ makes up the $(n-p) \times (n-p)$ lower-right corner of the matrix $D^0$. The inner rational matrix-function $s(z)$ is the ratio of two polynomials:

$$s_p(z) = c_p \frac{z^{n-p}X_p(1/z)}{X_p(z)}, \quad \deg X_p(z) = n - p, \; X_p(0) = 1 \; \frac{|c_p|}{1}.$$  

(9.15)

Clearly,

$$X_p(z) = \det(I_{n-p} - zD^p), \quad z^{n-p}X_p(1/z) = \det(zI_{n-p} - (D^p)^*),$$  

(9.16a)

thus

$$s_p(z) = c_p \det\left(\left(zI_{n-p} - (D^p)^*\right)(I_{n-p} - zD^p)^{-1}\right).$$  

(9.16b)

10. **An Expression for the Colligation Matrix in Terms of the Schur Parameters.**

Let $s(z)$ be a rational inner matrix-function of degree $n$. Let $s_p(z), p = 0, 1, \ldots, n$ be the sequence of rational inner functions produced by the Schur algorithm from the function $s(z)$, as described in (2.4), $\deg s_p(z) = n - p$. Let $U^p$, (9.6), be the colligation matrix of the minimal unitary colligation, which yields the system representation (9.7) of the function $s_p$. Among all unitary $(n-p+1) \times (n-p+1)$-matrices representing the function $s_p$ we choose a lower Hessenberg matrix $U^p$. Such a matrix $U^p$ exists and is unique.
The equality (7.6), where $U^p$ is taken as the matrix $U^0$ and $U^{p+1}$ is taken as the matrix $[\alpha \beta \\ \gamma \delta]$ takes the form

$$U^p = \begin{bmatrix}
1 & 0_{1 \times 1} & 0_{1 \times (n-1-p)} \\
0_{1 \times 1} & 0_{(n-1-p) \times 1} & U^{p+1} \\
0_{(n-1-p) \times 1} & s_p & (1 - |s_p|^2)^{1/2} \\
\end{bmatrix} \cdot \begin{bmatrix}
(1 - |s_p|^2)^{1/2} & 0_{1 \times (n-1-p)} \\
-\overline{s_p} & 0_{1 \times (n-1-p)} \\
0_{(n-1-p) \times 1} & I_{(n-1-p) \times (n-1-p)} \\
\end{bmatrix}, \quad p = 0, 1, \ldots, n-2.$$

The latter formula can be rewritten in the equivalent but more convenient form:

$$\begin{bmatrix}
I_p & 0_{p \times (n-p+1)} \\
0_{(n-p+1) \times p} & U^p \\
\end{bmatrix} = \begin{bmatrix}
I_{p+1} & 0_{(p+1) \times (n-p)} \\
0_{(n-p) \times (p+1)} & U^{p+1} \\
\end{bmatrix}.$$

$$\begin{bmatrix}
I_p & 0_{p \times 2} & 0_{p \times (n-1-p)} \\
0_{2 \times p} & s_p & (1 - |s_p|^2)^{1/2} \\
0_{(n-p-1) \times p} & 0_{(n-p-1) \times 2} & I_{n-1-p} \\
\end{bmatrix} \cdot \begin{bmatrix}
(1 - |s_p|^2)^{1/2} & 0_{2 \times (n-p-1)} \\
-\overline{s_p} & 0_{2 \times (n-p-1)} \\
0_{2 \times (n-p-1)} & I_{n-1-p} \\
\end{bmatrix}.$$

For $p = 0$, the matrix on the left-hand side of (10.1) takes the form $[U^0]$. For $p = n-1$, the matrix $U^{p+1}$ takes the form $U^n = s_n$ and the second factor on the right-hand side of (10.1) takes the form

$$\begin{bmatrix}
I_{n-2} & 0_{(n-2) \times 2} \\
0_{2 \times (n-2)} & s_{n-1} & (1 - |s_{n-1}|^2)^{1/2} \\
\end{bmatrix} \cdot \begin{bmatrix}
\end{bmatrix}.$$
From (10.1) it follows that

\[
U^0 = \prod_{0 \leq p \leq n-1} \begin{bmatrix}
I_p & 0_{p \times 2} & 0_{p \times (n-1-p)} \\
0_{2 \times p} & s_p & (1 - |s_p|^2)^{1/2} \\
0_{(n-p-1) \times p} & 0_{(n-p-1) \times 2} & I_{n-1-p}
\end{bmatrix}.
\]

(10.2)

Multiplying the matrices in (10.2), we obtain an expression for the entries of the matrix \(U^0\), which gives us the system representation of the function \(s(z)\) in terms of the Schur parameters of \(s(z)\):

\[
U^0 = ||u^0_{j,k}||_{0 \leq j, k \leq n},
\]

(10.3)

where

\[
u^0_{j,k} = \begin{cases}
s_0, & j = 0, k = 0, \\
s_j \Delta_{j-1} \Delta_{j-2} \cdot \cdot \cdot \Delta_1 \Delta_0, & 1 \leq j \leq n, k = 0, \\
-s_j \Delta_{j-1} \Delta_{j-2} \cdot \cdot \cdot \Delta_k s_{k-1}, & 1 \leq j \leq n, 1 \leq k \leq j, \\
\Delta_j, & 0 \leq j \leq n-1, k = j + 1, \\
0, & 0 \leq j < n-1, j + 1 < k \leq n,
\end{cases}
\]

(10.4)

with

\[
\Delta_j = (1 - |s_j|^2)^{1/2}.
\]

(10.5)

One can, in the same way, obtain expressions for the matrices \(U^j\) of the unitary colligations representing the functions \(s_j, 1 \leq j \leq n\).

It should be mentioned that a matrix of the form \(10.3, 10.4\) appeared in the paper \cite{Ger, formula (66′)} and was then rediscovered a number of times. See \cite{Grg, Con1, Con2, Section 2.5}, \cite{Lep, Sim, Chapter 4}, \cite{Dub, Theorem 2.17}.

11. On Work Related to System Theoretic Interpretations of the Schur Algorithm

In this section we discuss the connections between the present work and other work relating to the Schur algorithm as expressed in terms of system realizations. In particular, we discuss the results presented in \cite{AADL} and in \cite{KiNe}.

The paper \cite{AADL} deals with functions of the class \(S_\kappa\), i.e., with the functions \(s(z)\) meromorphic in the unit disc and possessing the properties:

1). For every \(N\) and for all points \(z_1, \ldots, z_N \in \mathbb{D}\) which are holomorphicity points for \(s\), the matrix \(||K(z_p, z_q)||_{1 \leq p, q \leq N}, K(z, \zeta) = \frac{1-s(z)\zeta}{1-z\zeta}\), does not have more than \(\kappa\) negative squares.
2). There exists an $N$ and points $z_1, \ldots, z_N \in \mathbb{D}$ such that this matrix has precisely $\kappa$ negative squares.

One of the goals of the paper [AADL] is to discuss the Schur algorithm for functions from the class $S_\kappa$ in terms of system realizations. In particular, the results of [AADL] are applicable to the special case $\kappa = 0$, in which they can be simplified.

In our considerations on the algebraic structure of a step of the Schur algorithm we will, for the sake of simplicity, restrict ourselves to finite-dimensional systems, which correspond to rational inner functions (of, say, degree $n$). We now describe the relevant result from [AADL], adopting the notation used there (to make the comparison with the results presented in our paper easier). In [AADL] the function $s(z)$ is given by

$$s(z) = s_0 + zB(I - zD)^{-1}C,$$

(11.1)

where $B$, $C$, $D$ are entries of a unitary matrix $U$,

$$U = \begin{bmatrix} s_0 & B \\ C & D \end{bmatrix}, \quad B \in \mathbb{M}_{1 \times n}, C \in \mathbb{M}_{n \times 1}, D \in \mathbb{M}_{n \times n}$$

(11.2)

It is not explicitly assumed from the very beginning that the entry $B$ of the matrix $U$ has the special form (7.36). The matrix $U$ appears as the matrix $V$ (1.2), in [AADL]. Our notation corresponds to that of [AADL] as follows: The objects, which appear as $\gamma$, $v$, $u$, $T$ in formula (1.2) of [AADL] are $s_0$, $B^*$, $C$, $D$ in our formulas (7.34)-(7.35). The state space which is denoted by $K$ in (1.2) of [AADL] is the space $\mathcal{H} = \mathbb{M}_{n \times 1}$ (= $\mathbb{C}^n$) in our paper.

Let $s_1(z)$ be the Schur transform of the function $s(z)$,

$$s_1(z) = \frac{1}{z} \cdot \frac{s(z) - s_0}{1 - s(z)s_0}, \quad s_0 = s(0)$$

(11.3)

(or (7.33) in our paper). According to [AADL], $s_1(z)$ is representable in the form

$$s_1(z) = \alpha + z\beta(I - z\delta)^{-1}\gamma,$$

(11.4)

with

$$\alpha = \frac{1}{1 - |s_0|^2}BC, \quad \beta = \frac{1}{\sqrt{1 - |s_0|^2}}BDP,$$

$$\gamma = \frac{1}{\sqrt{1 - |s_0|^2}}PC, \quad \delta = PDP,$$

(11.5)

where $P$ is the matrix of the orthogonal projector onto the orthogonal complement of the vector $B^*$ in $\mathcal{H}$, i.e.

$$P \in \mathbb{M}_{n \times n}, \text{rank } P = n - 1, \ P^2 = P, \ P = P^*, \ BP = 0.$$ 

(Formulas (11.5) are the formulas for the entries of the matrix $V_1$ which appear on page 11 of [AADL].) If we would like to represent the image space $PH$ as the space $\mathcal{H}_1 = \mathbb{M}_{(n-1) \times 1}$ (= $\mathbb{C}^{n-1}$), $\mathcal{H} = \mathbb{C} \oplus \mathcal{H}_1$, that is, if we would like the matrix $S_0$ is the class of contractive functions holomorphic in the unit disc.
of the projector \( P \) to be of the form \( P = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-1} \end{bmatrix} \), then we have to replace the original matrix \( U \) with the matrix
\[
U^0 = \begin{bmatrix} 1 & 0_n \times n \\ 0_n \times n & V^* \end{bmatrix} U \begin{bmatrix} 1 & 0_n \times n \\ 0_n \times n & V \end{bmatrix},
\]
\[
U^0 = \begin{bmatrix} s_0 & B_0 \\ C_0 & D_0 \end{bmatrix}
\]
(11.6)
where \( V \in \mathbb{M}_{n \times n} \) is a unitary matrix such that
\[
V^* P V = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-1} \end{bmatrix}.
\]
(11.7)
The condition \( BP = 0 \) implies the condition \( B_0 \begin{bmatrix} 1 & 0 \\ 0 & I_{n-1} \end{bmatrix} = 0 \). The last equality means that \( B_0 \) is of the form \( B_0 = \begin{bmatrix} b & 0_{1 \times (n-1)} \end{bmatrix} \). Since the matrix \( U^0 \) is unitary, we have \( |s_0|^2 + B_0B_0^* = 1 \). Therefore, \( B_0 \) must be of the form
\[
B_0 = \begin{bmatrix} \delta(1 - |s_0|^2)^{1/2} & 0_{1 \times (n-1)} \end{bmatrix},
\]
where \( \delta \) is a unimodular complex number. The unitary matrix \( V \) from (11.6) is not unique: In this case, the degrees of freedom are clear, when we consider the replacement \( V \rightarrow V \cdot \begin{bmatrix} \varepsilon & 0 \\ 0 & v \end{bmatrix} \), where \( \varepsilon \) is an arbitrary unimodular complex number and \( v, v \in \mathbb{M}_{(n-1) \times (n-1)} \) are unitary matrices. Choosing the number \( \varepsilon \) appropriately, we can ensure that \( B_0 \) is of the form
\[
B_0 = \begin{bmatrix} (1 - |s_0|^2)^{1/2} & 0_{1 \times (n-1)} \end{bmatrix}.
\]
(11.8)
Let us decompose the matrices \( C_0, D_0 \), which appear as the entries of the matrix \( U_0 \) from (11.5):
\[
C_0 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \quad D_0 = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix},
\]
(11.9)
\[
c_1 \in \mathbb{M}_{1 \times 1}, \quad c_2 \in \mathbb{M}_{(n-1) \times 1},
\]
\[
D_{11} \in \mathbb{M}_{1 \times 1}, \quad D_{12} \in \mathbb{M}_{1 \times (n-1)}, \quad D_{21} \in \mathbb{M}_{(n-1) \times 1}, \quad D_{22} \in \mathbb{M}_{(n-1) \times (n-1)}.
\]
The equalities (11.5) (where \( B, C, D \) are replaced by \( B_0, C_0, D_0 \)) now take the form
\[
\alpha = \frac{1}{\sqrt{1 - |s_0|^2}} c_1, \quad \beta = d_{12},
\]
\[
\gamma = \frac{1}{\sqrt{1 - |s_0|^2}} c_2, \quad \delta = d_{22},
\]
(11.10)
Thus, the matrix
\[
U^1 = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix},
\]
(11.11)
from [AADL], whose entries appear in the representation (11.4) of the function \( s_1(z) \) is the same matrix which appears in our Theorem 7.2 as the matrix (7.38).
The matrix $V_1$ from page 11 of [AADL] can be considered as a coordinate-free expression for the colligation matrix representing the function $s_1(z)$. The difference between our work and the work [AADL] is not in the results but in the methods. The reason for choosing the expression for the colligation matrix $U_1$ given in [AADL] is not fully explained. The facts that the matrix $U_1$ is unitary and that the matrix $U_1$ represents the Schur transform $s_1$ of the function $s$ are obtained as the result of a long chain of formal calculations. These calculations come across as somewhat contrived and do not serve to further our understanding of the subject at hand.

The state system approach is much more transparent. The fact that the matrix $U_1$ is unitary is an immediate consequence of our formula (7.6). The fact that the matrix $U_1$ represents the function $s_1$ is a consequence of the interpretation of the linear fractional transform (6.6)-(6.7) in terms of the Redheffer coupling of the appropriate colligation.

The paper [KiNe] can also be considered as relevant to our paper. In [KiNe] the system representation of Schur functions is not considered at all. Nevertheless, in this work the Householder algorithm is used to calculate the sequence of numbers, which can be identified with the Schur parameters of the rational inner function naturally related to the appropriate unitary matrix. Namely, given a unitary matrix $U \in \mathbb{M}_{(n+1) \times (n+1)}$, the measure $\mu$ on the unit circle is related to $U$ in the following way: $\mu(dt) = (E(dt)e_1, e_1)$, where $E(dt)$ is the spectral measure of the matrix $U$ and $e_1 = (1, 0, \ldots, 0)^T \in \mathbb{M}_{(n+1) \times 1}$. It is assumed that $e_1$ is a cyclic vector of $U$. The following equality holds:

$$e_1^* I + zU I - zU e_1 = \int_T \frac{1 + zt}{1 - zt} \mu(dt) \quad (11.12)$$

The measure $\mu$ generates the (finite) sequence of polynomials orthogonal on the unit circle. These orthogonal polynomials ($\Phi_k$ is monic of degree $k$) satisfy the recurrence relations

$$\Phi_{k+1}(z) = z\Phi_k(z) - s_k \Phi_k^*(z) \quad (11.13)$$

$$\Phi_{k+1}^*(z) = z\Phi_k^*(z) - s_k z \Phi_k(z) \quad (11.14)$$

where $s_k$, $k = 0, 1, \ldots, n$ are some recurrence coefficients. There are many different names for these coefficients. Recently dubbed ‘Verblunsky parameters’ by Barry Simon in [Sim]. On the other hand, the function in (11.12), which we denote by $p(z)$ is holomorphic in the unit disc $\mathbb{D}$ and has the following properties.

$$p(0) = 1, \quad p(z) + \overline{p(z)} \geq 0 \quad (z \in \mathbb{D}).$$

Therefore $p(z)$ is representable in the form

$$p(z) = \frac{1 + zs(z)}{1 - zs(z)}, \quad (11.15)$$

where $s(z)$ is a function holomorphic and contractive in $\mathbb{D}$. Ya. L. Geronimus established that the Verblunsky coefficients $s(z)$ in the recurrence relations (11.13)
- (11.14) are also the Schur parameters of the functions $s(z)$, which appear in (11.15). From (11.12) and (11.15) it follows that

$$e_1 I + zU = \frac{1 + zs(z)}{1 - zs(z)}.$$  

(11.16)

In Lemma 3.2 of [KiNe], the following method for finding Schur (=Verblunsky) parameters was proposed: First, the given unitary matrix $U$ should be converted to Hessenberg form:

$$U^0 = \begin{bmatrix} 1 & 0_{1 \times n} \\ V^* & V \end{bmatrix} U \begin{bmatrix} 1 & 0_{1 \times n} \\ V^* & V \end{bmatrix},$$  

(11.17)

In [KiNe] it is claimed that the entries of the (lower Hessenberg) matrix $U^0$ are of the form (10.3)-(10.5), from which the Schur-Verblunsky parameters $s_k$ can be found. However, it follows from (11.16) that

$$s(z) = A + sB(I - zD)^{-1}C,$$  

(11.18)

where

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$  

(11.19)

$A \in \mathbb{M}_{1 \times 1}$, $B \in \mathbb{M}_{1 \times n}$, $C \in \mathbb{M}_{n \times 1}$, $D \in \mathbb{M}_{n \times n}$.

Thus the formula (11.18) can be interpreted as the system representation of the function $s(z)$. The formula (11.18), where $s(z)$ is defined by (11.16) from $U$ was unfamiliar to us, but we do not think that this formula is new.

In his forthcoming paper [Arl] Yu. M. Arlinskii studied a related question for operator-valued Schur functions $\Theta$ acting between separable Hilbert spaces. These investigations correspond to the operator generalization of the classical Schur algorithm which is due to Constantinescu (see Section 1.3 in [BC]). Yu. M. Arlinskii presents a construction of conservative and simple realizations of the Schur algorithm iterates $\Theta_n$ of $\Theta$ by means of the conservative and simple realization of $\Theta$.

Appendix:

System Realizations of Inner Rational Functions.

We prove that every complex-valued (i.e. scalar) inner rational function of degree $n$ can be represented as the characteristic function of the minimal unitary colligation associated with some unitary matrix $U \in \mathbb{M}_{(n+1) \times (n+1)}$. Let us denote a given rational inner function by $S$. The operator colligation whose characteristic function is $S$ will be constructed as the ‘left shift’ operator in the appropriate space of analytic functions constructed from $S$. A similar construction appears in a paper by B.SzNagy-C.Foias. See [SzNFo, Chapter VI]. The construction of B.SzNagy-C.Foias was adapted to unitary colligations in [BrSv2].
1. The space $K_S$. The most important part of our construction is the Hilbert space $K_S$ of rational functions. We consider $S$ as a function defined on the unit circle $T$, i.e. $S : T \rightarrow T$. As usual, $L^2 = \{ x : T \rightarrow C, \|x\| < \infty \}$, where $\|x\|^2 = \langle x, x \rangle$

and

$$\langle x, y \rangle = \int_T x(t) \overline{y(t)} \ m(dt),$$

$m(dt)$ is the normalized Lebesgue measure on $T$. Let $H^2_+$ and $H^2_-$ be the Hardy subspaces of the space $L^2$:

$$H^2_+ = \{ x \in L^2 : \langle x(t), t^k \rangle = 0, \ k = -1, -2, \ldots \}.$$

$$H^2_- = \{ x \in L^2 : \langle x(t), t^k \rangle = 0, \ k = 0, 1, 2, \ldots \}.$$

Clearly, $L^2 = H^2_+ \oplus H^2_-$. It is also convenient to consider the functions from $H^2_+$ and from $H^2_-$ as functions holomorphic in $D$ and in $D^-$, respectively. In particular, the evaluation $f \rightarrow f(0)$ is defined for every $f$ in $H^2_+$ and $f(\infty) = 0$ for every $f$ in $H^2_-$. The space $K_S$ is defined as

$$K_S = H^2_+ \ominus S \ H^2_+,$$

where $S H^2_+ = \{ S(t) \ h(t) : h \in H^2_+ \}$. Another description of the space $K_S$ is:

$$K_S = \{ x \in L^2 : x \in H^2_+, xS^{-1} \in H^2_- \}.$$  \hspace{1cm} \text{(A.2)}

It can be shown that the space $K_S$ consists of rational functions whose poles are contained in the set of poles of the function $S$ and that $\dim K_S = \deg S$. If all zeros $z_k$ of $S$ are simple (see (1.1)), then the space $K_S$ is generated by the functions $\{ (1 - t^{-i z_k})^{-1} \}_{1 \leq k \leq n}$. If $S$ has non-simple zeros, the modification of this statement is clear. The space $K_S$ is a reproducing kernel Hilbert space. If $f \in K_S$, then

$$f(z) = \langle f(t), K(t, z) \rangle,$$

where the reproducing kernel $K(t, z)$ is:

$$K(t, z) = \frac{1 - S(t) \overline{S(z)}}{1 - \overline{t} z}.$$  \hspace{1cm} \text{(A.4)}

2. The left shift operator. The left shift operator $T$ is defined as

$$T(f)(t) = (f(t) - f(0)e(t)) \cdot t^{-1} \text{ for } f \in H^2_+.$$  \hspace{1cm} \text{(A.5)}

where

$$e(t) = 1 \ \forall t \in T.$$  \hspace{1cm} \text{(A.6)}

This operator is contractive:

$$\|Tf\|^2 = ||f||^2 - |f(0)|^2 \ \forall f \in H^2_+.$$  \hspace{1cm} \text{(A.7)}
The space $K_S$, considered as a subspace of $H^2_+$, is an invariant subspace of the left shift operator $T$. This is evident from the description (A.2) of the space $K_S$.

3. The construction of the unitary colligation $U$. The unitary colligation $(\mathcal{E}, \mathcal{H}, U)$ (see Definition 3.1) is defined as follows: Let the state space $\mathcal{H}$ be the space $K_S$ and let the principal operator $D$ be the left shift operator $T$, restricted to $K_S$: $H = K_S$, $Df(t) = (f(t) - f(0)\epsilon(t)) \cdot t^{-1} \forall f \in \mathcal{H}$. (A.8)

The equality $\|Df\|^2 + |f(0)|^2 = \|f\|^2$, together with the requirement that the colligation operator $U$, (3.1)-(3.2), be unitary, prompts us to define the exterior space $\mathcal{E}$ and the channel operator $B : \mathcal{H} \to \mathcal{E}$ as follows:

Let $\mathcal{E}$ be a one-dimensional Hilbert space which is identified with the vector space $\mathbb{C}$ over the field $\mathbb{C}$ of scalars. We choose the number $\beta = 1$ as a basis vector in $\mathbb{C}$ and will denote this basis vector by $\underline{1}$. Every number $\epsilon \in \mathbb{C}$, considered as an element of the vector space $\mathbb{C}$, can be presented as $\epsilon \underline{1}$, where the factor in front of $\underline{1}$ is the same number $\epsilon$, but considered as an element of the field of scalars $\mathbb{C}$.

The channel operator $B$ is:

$$(Bf)(t) = f(0)\underline{1}, \forall f \in \mathcal{H}. \quad (A.9)$$

Equation (A.7) ensures that

$$\|Bf\|^2 + \|Df\|^2 = \|f\|^2, \forall f \in \mathcal{H}. \quad (A.8)$$

$f(0)$, which appears in (A.8) and (A.9), can be represented using the reproducing kernel (A.3)-(A.4). Let $k(t) = 1 - s(t)\overline{s(0)}$, $(= K(t,0))$. (A.10)

Then

$$Bf = \langle f, k \rangle \underline{1}. \quad (A.11)$$

The operator $A : \mathcal{E} \to \mathcal{E}$ (as is the case for every operator in $\mathcal{E} : \dim \mathcal{E} = 1$) is of the form

$$A\epsilon = \alpha \langle \epsilon, \underline{1} \rangle \underline{1}, \epsilon \in \mathcal{E},$$

where $\alpha \in \mathbb{C}$. Since the vector $\underline{1}$, which generates $\mathcal{E}$, is orthogonal to $\mathcal{H}$ in the orthogonal sum $\mathcal{E} \oplus \mathcal{H}$, the unitary property of $U$ implies that

$$\langle Bf, \underline{1} \rangle + \langle Df, C\underline{1} \rangle = 0 \forall f \in \mathcal{H}. \quad (A.12)$$

Therefore

$$\alpha \langle Bf, \underline{1} \rangle + \langle Df, C\underline{1} \rangle = 0 \forall f \in \mathcal{H}. \quad (A.12)$$

Let us denote

$$C\underline{1} = l, l \in \mathcal{H}.$$ 

Equation (A.12) means that

$$\overline{\pi}(f, k) + \langle Df, l \rangle = 0, \forall f \in \mathcal{H}.$$
Thus, one should take
\[ l = -\alpha (D^*)^{-1} k, \]  
(A.13)
where \( D^* \) is the adjoint to the operator \( D \), with respect to the scalar product \( \langle , \rangle \).

We now look to determine the operator \( D^* \). The equality \( \langle Df, g \rangle = \langle f, D^* g \rangle \) means that
\[ \langle (f(t) - f(0)e(t)) t^{-1}, g(t) \rangle = \langle f(t), (D^* g)(t) \rangle. \]
The last equality implies that
\[ (D^* g)(t) = P(tg(t)), \quad \forall g \in K_S \]  
(A.14)
where \( P \) is the orthogonal projector from \( L^2 \) onto \( K_S \). Clearly,
\[ \langle h(t), S(t) \rangle = 0 \quad \forall h \in K_S, \]
and
\[ tg(t) - \langle tg(t), S(t) \rangle S(t) \in K_S \quad \forall g \in K_S. \]
Therefore,
\[ P(tg(t)) = tg(t) - \langle tg(t), S(t) \rangle S(t) \quad \forall g \in K_S, \]
that is
\[ (D^* g)(t) = tg(t) - \langle tg(t), S(t) \rangle S(t) \quad \forall g \in H. \]  
(A.15)
From (A.13), we obtain
\[ l(t) = \frac{\alpha}{S(0)} \frac{S(t) - S(0)e(t)}{t}. \]
\[ \| A e \|^2 + \| C e \|^2 = 1 \] gives us \( |\alpha| = |S(0)| \). We choose
\[ \alpha = S(0) \]  
(Later we see that this is the only possible choice for \( \alpha \).) We set
\[ l(t) = \frac{S(t) - S(0)e(t)}{t}. \]  
(A.16)
(In intermediate steps we assumed that \( S(0) \neq 0 \), but this does not appear in the final expression \( l(t). \) Thus,
\[ A e = S(0) (e, \mathbb{1}) \mathbb{1}, \quad B f = \langle f, k \rangle \mathbb{1}, \quad C e = \langle e, \mathbb{1} \rangle l, \]
\[ (D f)(t) = (f - \langle f, k \rangle e(t)) t^{-1} \quad \forall e \in \mathcal{E}, f \in \mathcal{H}. \]  
(A.17)
or
\[ A e = \varepsilon S(0) \mathbb{1}, \quad B f = f(0) \mathbb{1}, \quad (C e) = \varepsilon l(t), \]
\[ (D f)(t) = (f - f(0)e(t)) t^{-1} \quad \forall e = \varepsilon \mathbb{1} \in \mathcal{E}, f \in \mathcal{H}. \]  
(A.18)
From (A.18) it follows that the block-operator
\[ U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \]  
(A.19)
is unitary (After the block $D$ was chosen, the other blocks $A, B, C$ were chosen to ensure that $U$ be a unitary operator.) The characteristic function $S_U(z)$,

$$S_U(z) = A + zB(I - zD)^{-1}C,$$  

(A.20)

of the colligation $U$ coincides with the original rational inner function $S(z)$. This can be checked by direct calculation of $S_U(z)$ using the expression (A.18) for blocks of the colligation operator $U$. The expression for the operator $(I - zD)^{-1}$, which is needed for this calculation, is

$$((I - zD)^{-1}f)(t) = \frac{tf(t) - zf(z)}{t - z}, \quad \forall f \in \mathcal{H}. \quad \text{(A.21)}$$

In what follows we also need the expression for the operator $(I - zD^*)^{-1}$:

$$((I - zD^*)^{-1}f)(t) = \frac{f(t) - (fS^{-1})(z^{-1})S(t)}{1 - tz}, \quad \forall f \in \mathcal{H}. \quad \text{(A.22)}$$

(Since the function $fS^{-1}$ belongs to $H^2$, the evaluation $fS^{-1} \to (fS^{-1})(z^{-1})$ is defined for $z \in \mathbb{D}$.)

Choosing an orthogonal basis in the ($n$-dimensional) Hilbert space $K_S$, we realize that the unitary operator $U$, (A.18)-(A.19), originally constructed as an operator acting in a functional space, is a matrix operator acting in $\mathbb{C}^{n+1} = \mathbb{C} \oplus \mathbb{C}^n$.

**DEFINITION 11.1.** The colligation (A.18)-(A.19) is called the model unitary colligation constructed from the rational inner function $S$.

4. Minimality of the model unitary colligation $(\mathcal{E}, \mathcal{H}, U)$. We look to prove that the model colligation $U$, (A.18)-(A.19), is controllable and observable. In view of the expression for the channel operator $C$ (one-dimensional), controllability of $U$ can be formulated as follows:

The set of vectors $\{(I - zD)^{-1}l\}_{z \in \mathbb{D}}$ generates the space $K_S$. \quad \text{(A.23)}

From (A.16) and (A.21) it follows that

$$((I - zD)^{-1}l)(t) = \frac{S(t) - S(z)}{t - z}.$$ 

Let $f \in L^2$ be such that

$$\int_{\mathbb{T}} \frac{S(t) - S(z)}{t - z} \overline{f(t)} m(dt) = 0 \quad \forall z \in \mathbb{D} \quad \text{(A.24)}$$

If $f \in H^2$, then $\int_{\mathbb{T}} \frac{f(t)}{t - z} m(dt) = 0 \quad \forall z \in \mathbb{D}$, hence, $\int_{\mathbb{T}} \frac{S(t)f(t)}{t - z} m(dt) = 0 \quad \forall z \in \mathbb{D}$.

The last equality implies that $f(t)S^{-1}(t) \in H^2$. (Here we use that $S(t) = S^{-1}(t)$ for $t \in \mathbb{T}$.) If also $f(t)S^{-1}(t) \in H^2$, then $f(t)S^{-1}(t) \equiv 0$ and $f \equiv 0$. Therefore, if the condition (A.24) holds for some $f \in K_S$, then $f \equiv 0$. Controllability of the colligation $(\mathcal{E}, \mathcal{H}, U)$ is thus proved.
Observability of this colligation can be proved analogously. According to (A.11), \( B f = (f,e)k \). Therefore the observability criterion is reduced to the statement:

The set of vectors \( \{(I - zD^*)^{-1}k\}_{z \in \mathbb{D}} \) generates the space \( K_S \). (A.25)

Using expressions (A.22) and (A.10), we obtain:

\[
\left((I - zD^*)^{-1}k\right)(t) = \frac{1 - S(t) S^{-1}(z^{-1})}{1 - tz}.
\]

Let \( f \in L^2 \) is such that

\[
\int_T \frac{1 - S(t) S^{-1}(z^{-1})}{1 - tz} f(t) m(dt) = 0 \quad \forall z \in \mathbb{D}.
\] (A.26)

If \( f(t)S^{-1}(t) \in H^2 \), then \( \int_T \frac{S(t)T(t)}{1 - tz} = 0 \), hence \( \int_T \frac{T(t)}{1 - tz} m(dt) = 0 \ \forall z \in \mathbb{D} \) and \( f(t) \in H^2 \). If also \( f \in H^2 \), then \( f \equiv 0 \). Therefore if the condition (A.26) holds for some \( f \in K_S \), then \( f \equiv 0 \).

5. Uniqueness of simple realization. The uniqueness of the minimal realization is, in fact, a version of a result by M.S. Livshitz, which, in the language of M.S. Livshitz, claims that the characteristic function uniquely determines (up to unitary equivalence) the operator colligation without complementary component.

Let \( U_1 = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \) and \( U_2 = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \) be two unitary matrices divided into blocks,

\[
A_j \in \mathbb{M}_{1 \times 1}, \quad B_j \in \mathbb{M}_{1 \times n_j}, \quad C_j \in \mathbb{M}_{n_j \times 1}, \quad D_j \in \mathbb{M}_{n_j \times n_j},
\]

where \( n_j, j = 1, 2 \), are natural numbers. (A.27)

We do not assume that \( n_1 = n_2 \).

Let

\[
S_i(z) = A_i + B_i (I - zD_i)^{-1}C_i, \quad i = 1, 2,
\]

be the characteristic functions of the unitary colligations associated with the matrices \( U_1 \) and \( U_2 \) respectively. Suppose that

1. The characteristic functions are equal.

\[
S_1(z) = S_2(z). \tag{A.28}
\]

2. Each of the matrices \( U_1 \) and \( U_2 \) is simple in the sense of Definition 3.14.

We prove that under these assumptions the matrices \( U_1 \) and \( U_2 \) are equivalent in the sense of Definition 3.13, in particular, \( n_1 = n_2 \).

To prove this, we have to first of all construct a unitary mapping \( V \) of the space \( \mathbb{C}^{n_2} \) onto the space \( \mathbb{C}^{n_1} \). Assume that the matrices \( U_1 \) and \( U_2 \) are simple. Let us consider the vectors \( f_j^k, g_j^l \in \mathbb{C}^{n_j} (= \mathbb{M}_{n_j \times 1}), j = 1, 2, 0 \leq k, l \):

\[
f_j^k = D_j^kC_j, \quad g_j^l = (D_j^*)^lB_j^*, \quad j = 1, 2, \quad 0 \leq k, l. \tag{A.29}
\]
By the assumption, for each \( j = 1, 2 \), the vectors \( f_j^k, g_j^k \), \( 0 \leq k \leq \max(n_1, n_2) \) generate the space \( \mathbb{C}^{n_j} \). The equality (A.28) implies
\[
A_1 = A_2, \quad (A.30)
\]
and the equalities
\[
B_1 D_1^p C_1 = B_2 D_2^p C_2, \quad 0 \leq p, \quad (A.31)
\]
or
\[
B_1 D_1^k D_2^l C_1 = B_2 D_2^k D_2^l C_2, \quad 0 \leq k, l.
\]
The latter equalities can be interpreted as
\[
\langle f_1^k, g_1^l \rangle_{\mathbb{C}^{n_1}} = \langle f_2^k, g_2^l \rangle_{\mathbb{C}^{n_2}}, \quad \forall k, l : 0 \leq k, 0 \leq l. \quad (A.32a)
\]
Moreover, the equalities (A.28) imply that
\[
1 - S_1^*(\zeta) S_1(z) = 1 - S_2^*(\zeta) S_2(z), \quad 1 - S_1(z) S_1^*(\zeta) = 1 - S_2(z) S_2^*(\zeta).
\]
In view of (3.22), (3.23), the latter equalities imply that
\[
C_1^*(D_1^q D_1^p) C_1 = C_2^*(D_2^q D_2^p) C_2, \quad \text{and} \quad B_1 D_1^q (D_1^p) C_1 = B_2 D_2^q (D_2^p) C_2,
\]
\[0 \leq p, q.\]
This can, in turn, be interpreted as
\[
\langle f_1^p, f_1^q \rangle_{\mathbb{C}^{n_1}} = \langle f_2^p, f_2^q \rangle_{\mathbb{C}^{n_2}}, \quad \text{and} \quad \langle g_1^p, g_1^q \rangle_{\mathbb{C}^{n_1}} = \langle g_2^p, g_2^q \rangle_{\mathbb{C}^{n_2}},
\]
\[\forall p, q : 0 \leq p, 0 \leq q. \quad (A.32b)
\]
From (A.32), it follows that for arbitrary \( \alpha_k, \beta_l \) (such that only finitely many of them differ from zero),
\[
\| \sum \alpha_k f_1^k + \sum \beta_l g_1^l \|_{\mathbb{C}^{n_1}} = \| \sum \alpha_k f_2^k + \sum \beta_l g_2^l \|_{\mathbb{C}^{n_2}}. \quad (A.33)
\]
Let us define the operator \( V : \mathbb{C}^{n_2} \to \mathbb{C}^{n_1} \) first as
\[
V f_2^k = f_1^k, \quad V g_2^l = g_1^l, \quad \forall \ k \geq 0, \ l \geq 0, \quad (A.34a)
\]
and then extend this operator by linearity to all vector columns \( h \in \mathbb{C}^{n_2} \) representable as a finite linear combination of the form \( h = \sum \alpha_k f_2^k + \sum \beta_l g_2^l \). Thus,
\[
V(\sum \alpha_k f_2^k + \sum \beta_l g_2^l) = \sum \alpha_k f_1^k + \sum \beta_l g_1^l. \quad (A.34b)
\]
If some \( h \in \mathbb{C}^{n_2} \) admits two different representations, say
\[
h = \sum \alpha_k f_2^k + \sum \beta_l g_2^l, \quad \text{and} \quad h = \sum \alpha'_k f_2^k + \sum \beta'_l g_2^l,
\]
then \( Vh \) also admits two different representations:
\[
Vh = \sum \alpha_k f_1^k + \sum \beta_l g_1^l, \quad \text{and} \quad Vh = \sum \alpha'_k f_1^k + \sum \beta'_l g_1^l.
\]
However, since \( \sum \alpha_k f_2^k + \sum \beta_l g_2^l = 0 \), where \( \alpha_k = \alpha'_k - \alpha_k, \beta_l = \beta'_l - \beta_l \), the equality (A.33) implies that \( \sum \alpha_k f_1^k + \sum \beta_l g_1^l = 0 \), i.e.
\[
\sum \alpha_k f_1^k + \sum \beta_l g_1^l = \sum \alpha'_k f_1^k + \sum \beta'_l g_1^l.
\]
The definition (A.34) of $V$ is thus non-contradictory.

The operator $V$ is defined on the linear hull of all vectors $\{f^k, g^l\}_{k,l}$ and isometrically maps its definition domain onto the linear hull of all vectors $\{f^k, g^l\}_{k,l}$. If both the matrices $U^2$, $U^1$ are simple, then these linear hulls are the whole spaces $\mathbb{C}^{n_2}$ and $\mathbb{C}^{n_1}$, respectively. In this case $n_1 = n_2 (\equiv n)$ and

$$V^*V = I_n, \quad VV^* = I_n$$

(A.35)

We now prove the intertwining relation

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & V \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix},$$

which can be rewritten as follows:

$$C_1 = VC_2, \quad B_2 = B_1V,$$

(A.36)

and

$$VD_2 = D_1V.$$  

(A.37)

The first of the equalities (A.36) corresponds to the first of the equalities (A.29) for $k = 0$ (See (A.34a) for $k = 0$.) The second of the equalities (A.36) relates to the second of equality in (A.29) for $l = 0$ (See (A.35).)

To check the splitting relation (A.37), it is enough to check that

$$VD_2f^k = D_1Vf^k \quad \text{for } \forall k \geq 0,$$

(A.38a)

and

$$VD_2g^l = D_1Vg^l \quad \text{for } \forall l \geq 0.$$  

(A.38b)

The equality (A.38a) is an obvious consequence of the definitions of the operator $V$ and vectors $f^k_j$. Indeed, $D_2f^k_2 = f^{k+1}_2$, $Vf^{k+1}_2 = f^{k+1}_1$. On the other hand, $Vf^k_1 = f^k_1$, $D_1f^k_1 = f^{k+1}_1$. Therefore, (A.38a) holds.

Our approach to checking the condition (A.38b) will be different in the two cases $l = 0$ and $l > 0$. For $l = 0$ the equality (A.38a) takes the form $VD_2B^*_2 = D_1VB^*_2$. (A.34) for $l = 0$ means that $Vb^*_2 = b^*_1$, so we should check that $VD_2b^*_2 = D_1b^*_1$. Since $D_1b^*_j = -C_jA^*_j$, $j = 1, 2$, the last equality is equivalent to $VC_2A^*_2 = C_1A^*_1$. The latter equation is a consequence of the first of the equalities (A.36). Thus, (A.38b) holds for $l = 0$.

We check condition (A.38b) for $l > 0$. Since the matrices $U_j$ are unitary, we have that $D_jD^*_j = I - C_jC^*_j$. Thus, (A.38b) is equivalent to

$$V(I - C_2C^*_2)(D^*_2)^{l-1}B^*_2 = (I - C_1C^*_1)(D^*_1)^{l-1}B^*_1.$$  

This equation is a consequence of the following three equalities:

$$V(D^*_2)^{l-1}B^*_2 = (D^*_1)^{l-1}B^*_1,$$

(A.39a)

$$VC_2 = C_1,$$

(A.39b)

and

$$C^*_2(D^*_2)^{l-1}B^*_2 = C^*_1(D^*_1)^{l-1}B^*_1.$$

(A.39c)
(A.39a) holds, because it can be written as $V g_2^{l-1} = g_1^{l-1}$, which is part of the definition (A.34) of the operator $V$. (A.39b) has already been checked: This is the first of the relations (A.36). (A.39c) is the same as (A.31) for $p = l - 1$. The condition (A.38b) has also been checked for $l > 0$.

6. Simple realization is minimal. Let $U \in M_{(1+n) \times (1+n)}$ be a simple unitary matrix. The matrix $U$ is then minimal. Indeed, let $S(z)$ be the characteristic function of the unitary colligation associated with $U$. $S(z)$ is a rational inner function, $\deg S \leq n$. Let $(C, K_S, T)$ be the model colligation constructed from this $S$. We established that the model unitary colligation is minimal (in particular, simple) and that its characteristic function is the function $S$, from which it was constructed. Both colligations (the original colligation and the model colligation) have the same characteristic function and both are simple. Hence, these colligations are equivalent. Since the model colligation is minimal, the original colligation is also minimal.

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