Study on linearized inversion of surface wave dispersion for estimating near-surface shear wave velocity

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Abstract. Near-surface shear wave velocity is very important for geophysical and geotechnical engineering. The use of surface wave for characterization of the near-surface subsurface involves three steps: acquisition, extracting dispersion curve, and inversion method to generate near-surface shear wave velocity. In this study, we explored a linear inversion technique to see the effect of some parameters such as the initial model and the damping factor on the inversion results. We used the Levenberg-Marquardt algorithm and written in Matlab. Synthetic data used three-layer earth model with assumed homogeneous, isotropic, flat, and half-space. Synthetic examples demonstrated that inverted result depends on the initial model. Based on our results, we have ranged from 30% to 120% of the true model in providing the initial model. It is found that there is an asymmetry between the range of the initial model which is larger and smaller than the true model. This is due to the non-linearity of the differential equation and the solution of the Jacobian matrix using numerical approach. There are several suggestions for solving this problem: (1) using an analytical approach, (2) higher order deductions when using numerical approaches. In term damping factor, our result show range between $10^{-2}$ to 1 give a stable inversion.

1. Introduction
Rayleigh waves is a surface wave that travels along a free surface, such as the earth-air interface [1]. This wave is the result of interfering P- and S waves so the particle motion of Rayleigh waves moving from left to right is elliptical in a counter-clockwise (retrograde) direction. In the seismic exploration, the Rayleigh often called ground roll is a coherent noise masking useful signals and must be removed through the filtering process. In the last decades, the use of Rayleigh waves for the characterization of the shallow subsurface has become of growing interest to geophysicist and geotechnical engineers. Utilization of Rayleigh wave has a big advantage because non-destructive and can be performed quickly and cheaply than traditional borehole methods. The most important step in surface wave method is inverting Rayleigh wave phase velocity. The S-wave velocity profile, a function of depth, can be derived from inverting the phase velocity of the Rayleigh waves. Several inversion techniques have been proposed for inverting Rayleigh wave phase velocity, either a local search [2,3] or a global search [4,5,6]. The local search or linear inversion technique is more widely used than global optimization method because it can be considered as an acceptable and computationally effective when some robust a priori

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information about the subsurface is available and a good starting model can thus be established that is sufficiently close to the real solution.

In this study, we explored linear inversion for estimating near-surface shear wave velocity using the Levenberg-Marquardt algorithm [3]. The algorithm applied in the synthetic data. We want to know the effect of variation the initial model and the damping factor on the inversion results.

2. Methodology
The method in this study divided into two sections, the first one is forward modelling of dispersion curve of Rayleigh wave and the second one is inversion method for estimating near-surface shear wave velocity. We inverted a dispersion curve which obtained from forwarding modelling process.

2.1 Forward Modeling of Dispersion Curve
Forward modeling of dispersion curve of Rayleigh wave is obtained by solving the eigenvalue and eigenvector problems in the differential equation [7]:

$$\frac{d}{dz} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} = \begin{pmatrix} 0 & k & \mu^{-1}(z) & 0 \\ -k\lambda(z)\lambda(z) + 2\mu(z)^{-1} & 0 & 0 & [\lambda(z) + 2\mu(z)]^{-1} \\ k^2\xi(z) - \omega^2\rho(z) & 0 & 0 & k\lambda(z)[\lambda(z) + 2\mu(z)]^{-1} \\ -\omega^2\rho(z) & -k & 0 & 0 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix}$$

(1)

where \(\xi(z) = \frac{4\mu(z)\lambda(z)}{\lambda(z) + 2\mu(z)}\) is Lame constants, \(k\) is wave number, \(\omega\) is angular frequency, \(z\) is depth function, and \(r_1, r_2, r_3,\) and \(r_4\) is stress-motion vectors. Many methods have been proposed to solve a differential problem in equation 1. They can be categorized into propagator matrix and numerical methods. In this study, we used the mat_disperse program to solve the eigenvalue and the eigenvector problems. This program based on propagator matrix method, specifically reflection-transmission (R/T) method [8]. This program requires at least four earth parameters ie \(V_p, V_s, \rho,\) and \(H\) for each layer and frequency range. The output of this program is the phase velocity of the Rayleigh wave for each frequency.

2.2 Inversion Modelling
In this study, we focused on linear inversion technique. We used Levenberg-Marquardt algorithm. The formulation for Levenberg-Marquardt written as:

$$m_{n+1} = m_n + \left[J^T J_n + \varepsilon^2 I\right]^{-1} J_n^T (d - g(m_n))$$

(2)

where \(m_{n+1}\) and \(m_n\) is model parameters at iteration \(n + 1\) and \(n, J\) is Jacobian matrix, \(\varepsilon\) is weighted, \(I\) is identity matrix, \(d\) is observation data, \(g(m_n)\) is the calculation data. Jacobian matrix solved using numerical approach (finite difference method):

$$\frac{\partial Vr_i}{\partial m_j} \approx \frac{Vr_i(m_j + \Delta m_j) - Vr_i(m_j)}{\Delta m_j}$$

(3)

\(Vr_i\) is phase velocity of Rayleigh wave at \(i\)-th sample, \(m_j\) is model parameters, and \(\Delta m_j\) is perturbation of model parameters. In this section, we vary four parameters: the initial model, damping factor, number of dispersion picks, and frequency density of dispersion picks.

3. Forward Modeling Test
The dispersion curve of Rayleigh wave depends on four earth parameters: compressional wave velocity \((V_p)\), shear wave velocity \((V_s)\), density \((\rho)\), and thickness \((H)\) [3]. Synthetic data used three-layers earth model with assumed homogeneous, isotropic, flat, and half-space. This model is normal dispersive. To test forward modelling, we used earth model parameters each layer in table 1:
Table 1. Earth model parameters

| Layer number | Vp (m/s) | Vs (m/s) | \( \rho \) (kg/m\(^3\)) | H (m) |
|--------------|----------|----------|--------------------------|-------|
| 1            | 600      | 350      | 1800                     | 5     |
| 2            | 700      | 400      | 1800                     | 10    |
| 3            | 800      | 450      | 1800                     | \( \infty \) |

The dispersion curve of model in table 1 can be seen in the figure below:

Figure 1. Dispersion curve of Rayleigh wave.

The frequency used is in the range 0.1 to 100 Hz with 50 sample points. From the result, it is seen that the curve decays with asymptotic at 70 Hz. Maximum and minimum phase velocity is 414 m/s and 321 m/s, respectively. These values are about 90% of the maximum and minimum of shear velocity. The dispersion curve generated from this process will serve as an observational data on the inversion process.

4. Inversion Test
As mentioned before, the dispersion curve depends on four parameters. In the inversion process, we only used Vs and H as model parameters. This is because the two parameters are more sensitive than two other parameters [9]. Vp and density we considered to be a priori information. Vs and thickness in table 1 as the true model in inversion process, then we vary the initial model with the value larger and smaller than the true model. The results are presented in the table below:

Table 2. The influence of the initial model on the inversion result

| Initial Model (% of true model) | Result   | Initial Model (% of true model) | Result   |
|---------------------------------|----------|---------------------------------|----------|
| 75                              | Convergent| 110                             | Convergent|
| 50                              | Convergent| 120                             | Convergent|
| 30                              | Convergent| 125                             | No       |
| 25                              | No       |                                  |          |
The table above shows the difference when the initial model larger and smaller than the true model, as if there is asymmetry. When we give a smaller initial model of the true model, we have the opportunity to give even smaller values of the true model to produce convergent inversion results. On the contrary when we give larger initial model of the true model, we get smaller range. Before we hypothesize, it should give a symmetrical result. We investigated this issue by changing the value of perturbation (1%, 2%, 5%, and 10% of model parameter), did not give significant results to the inversion result. We also try to change finite difference techniques (forward, backward, and central), given less change anything. We suspect this issue is due to the limitations of the method for solving the Jacobian matrix. In this study, we used numerical approach (finite difference method). When we used finite difference method to solve first-order differential equations, the higher order is cut. There may be information loss that causes this asymmetry. Besides, the nonlinearity of the differential equation is thought to be the cause. Inversion results when initial model 30% and 120% of true model can be seen in figures 2 and 3 below.

![Figure 2. Inversion result when initial model 30% of true model](image1)

![Figure 3. Inversion result when initial model 120% of true model](image2)

Figure 2 show inversion result when we give initial model 30% of the true model. The blue line is a true model, the red line is an initial model, and the black line is inversion model. In this case, true model is 350 m/s, 400 m/s, 450 m/s for shear waves and 5 m and 10 m for thickness layers. We give 30% of the true model, 105 m/s, 120 m/s, and 135 m/s for shear wave and 1.5 m and 3 m for thickness layers, respectively. In figure 2, we can see that black line coincides with the blue line. It shows the inversion model fit with the true model. Same as before, when we give initial model 120% of the true model, inversion model show fit with the true model. If we give the initial model out of the range, inversion to be unstable and not convergent. It is a limitation of linear inversion, depend on the initial model. When we talk about real data, we do not know the true model. To solve that issue, we can see the dispersion curve data. Maximum the phase velocity can be used as the initial model of the shear wave velocity for the deepest layer, while minimum phase velocity for the top layer. We can limit the velocity range of initial model by looking at the maximum and minimum the phase velocity in the dispersion curve. For the middle layers, we can see the slope of the curve. Steep curves associate with the large velocity differences from one layer to another, vice versa. Another procedure [3] to determine initial Vs using the following formula

\[ V_{s1} = \frac{C_r(\text{high})}{\beta} \]  

\[ V_{sn} = \frac{C_r(\text{low})}{\beta} \]  

\[ V_{sn} = \frac{C_r(f \ell)}{\beta} \]
where $\beta$ is a constant ranging from 0.874 to 0.955. Equation (4) for the first layer, equation (5) for the half space, and equation (6) for the layers between the first layer and half-space.

The use of term damping is associated with the process of dampening instability that may arise during the inversion process. The damping factor is usually determined by trial and error or by using a trade-off curve. In this section, we vary the damping factor from $10^{-5}$ to $10^5$. Our result show damping with range between $10^{-2}$ to 1 give a stable inversion.

5. Conclusion

The initial model selection is very influential on the inversion process, especially in linear inversion. Based on our results, we have ranged from 30% to 120% of the true model in providing the initial model. It is found that there is an asymmetry between the range of the initial model which is larger and smaller than the true model. This is due to the non-linearity of the differential equation and the solution of the Jacobian matrix using numerical approach. There are several suggestions for solving this problem: (1) using an analytical approach, (2) higher order deductions when using numerical approaches. In term damping factor, our result show range between $10^{-2}$ to 1 give a stable inversion.

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