HARMONIC ANALYSIS ON QUANTUM TORI

ZEQIAN CHEN, QUANHUA XU, AND ZHI YIN

Abstract. This paper is devoted to the study of harmonic analysis on quantum tori. We consider several summation methods on these tori, including the square Fejér means, square and circular Poisson means, and Bochner-Riesz means. We first establish the maximal inequalities for these means, then obtain the corresponding pointwise convergence theorems. In particular, we prove the noncommutative analogue of the classical Stein theorem on Bochner-Riesz means. The second part of the paper deals with Fourier multipliers on quantum tori. We prove that the completely bounded $L_p$ Fourier multipliers on a quantum torus are exactly those on the classical torus of the same dimension. Finally, we present the Littlewood-Paley theory associated with the circular Poisson semigroup on quantum tori. We show that the Hardy spaces in this setting possess the usual properties of Hardy spaces, as one can expect. These include the quantum torus analogue of Fefferman’s $H^1$-BMO duality theorem and interpolation theorems. Our analysis is based on the recent developments of noncommutative martingale/ergodic inequalities and Littlewood-Paley-Stein theory.

Contents

1. Introduction 1
2. Preliminaries 3
  2.1. Noncommutative $L_p$ spaces 3
  2.2. Quantum tori 4
  2.3. Transference 4
3. Mean Convergence 5
4. Maximal inequalities 7
5. Pointwise convergence 14
6. Bochner-Riesz means 15
7. Fourier multipliers 21
8. Hardy spaces 26
References 36

1. Introduction

The subject of this paper follows the current line of investigation on noncommutative harmonic analysis. This topic has many interactions with other fields such as operator spaces, quantum probability, operator algebras, and of course, harmonic analysis. The aspect we are interested in is particularly related to the recent developments of noncommutative martingale/ergodic inequalities and Littlewood-Paley-Stein theory for quantum Markov semigroups. Motivated by operator spaces and by using tools from this theory, many classical martingale and ergodic inequalities have been successfully transferred to the noncommutative setting (see, for instance, [37, 13, 22, 23, 39, 40, 32, 2, 3, 33]). These inequalities of quantum probabilistic nature have, in return, applications to operator space theory (cf., e.g. [36, 14, 19, 20, 21, 49, 54]). Closely related to that, harmonic analysis on quantum semigroups has started to be developed in the last years. This first period

2000 Mathematics Subject Classification: Primary: 46L50, 46L07. Secondary: 58L34, 43A55.

Key words: Quantum tori, Fourier series, square Fejér means, square and circular Poisson means, Bochner-Riesz means, maximal inequalities, pointwise convergence, completely bounded Fourier multipliers, Hardy and BMO spaces, Littlewood-Paley theory.

Z. Chen is partially supported by NSFC grant No. 11171338.
Q. Xu and Z. Yin are partially supported by ANR-2011-BS01-008-01.
of development of the noncommutative Littlewood-Paley-Stein theory deals with square function inequalities, $H_1$-BMO duality and Riesz transforms (cf. [15, 27, 28, 16, 17]). One can also include in this topic the very fresh promising direction of research on the Calderón-Zygmund singular integral operators in the noncommutative setting (cf. [31, 29, 13]). The concern of the present paper is directly linked to this last direction. Our objective is to develop harmonic analysis on quantum tori.

Quantum or noncommutative tori are fundamental examples in operator algebras and probably the most accessible interesting class of objects of study in noncommutative geometry (cf. [5, 6]). There exist extensive works on them (see, for instance, the survey paper by Rieffel [41] for those before the 1990’s). We refer to [7, 9, 40] for some illustrations of more recent developments on this topic.

We now recall the definition of quantum tori. Let $d \geq 2$ and $\theta = (\theta_{kj})$ be a real skew-symmetric $d \times d$-matrix. The $d$-dimensional noncommutative torus $A_\theta$ is the universal $C^*$-algebra generated by $d$ unitary operators $U_1, \ldots, U_d$ satisfying the following commutation relation

$$U_k U_j = e^{2\pi i \theta_{kj}} U_j U_k, \quad j, k = 1, \ldots, d.$$ 

There exists a faithful tracial state $\tau$ on $A_\theta$. Let $T^d_\theta$ be the von Neumann algebra in the GNS representation of $\tau$. $T^d_\theta$ is called the quantum $d$-torus associated with $\theta$. Note that if $\theta = 0$, then $A_\theta = C(T^d)$ and $T^d_0 = L_\infty(T^d)$, where $T^d$ is the usual $d$-torus. So a quantum $d$-torus is a deformation of the usual $d$-torus. It is thus natural to expect that $T^d_\theta$ shares many properties with $T^d$. This is indeed the case for differential geometry, as shown by the works of Connes and his collaborators. However, little is done regarding analysis. To our best knowledge, up to now, only the mean convergence theorem of quantum Fourier series by the square Fejér summation was proved at the $C^*$-algebra level (cf. [47, 18]), and on the other hand, the quantum torus analogue of Sobolev inequalities was obtained only in the Hilbert, i.e., $L_2$ space case (cf. [42]). The reason of this lack of development of analysis might be explained by numerous difficulties one may encounter when dealing with noncommutative $L_\rho$-spaces, since these spaces come up unavoidably if one wishes to do analysis on quantum tori. For instance, the usual way of proving pointwise convergence theorems is to pass through the corresponding maximal inequalities. But the study of maximal inequalities is one of the most delicate and difficult parts in noncommutative analysis.

This paper is the first one of a long project that intends to develop analysis on quantum tori and more generally on twisted crossed products by amenable groups. Our aim here is to study some important aspects of harmonic analysis on $T^d_\theta$. The subject that we address is three-fold:

i) **Convergence of Fourier series.** We consider several summation methods on $T^d_\theta$, including the square Fejér means, square and circular Poisson means, and Bochner-Riesz means. We first establish the maximal inequalities for them and then obtain the corresponding pointwise convergence theorems. This part heavily relies on the theory of noncommutative martingale and ergodic inequalities.

ii) **Fourier multipliers.** The right framework for our study of Fourier multipliers is operator space theory. We show that for $1 \leq p \leq \infty$ the completely bounded $L_p$ Fourier multipliers on $T^d_\theta$ coincide with those on $T^d$.

iii) **Hardy and BMO spaces.** Based on the recent development of the noncommutative Littlewood-Paley-Stein theory and the operator-valued harmonic analysis, we define Hardy and BMO spaces on $T^d_\theta$ via the circular Poisson semigroup. We show that the properties of Hardy spaces in the classical case remain true in the quantum setting. In particular, we get the $H_1$-BMO duality theorem.

One main strategy for approaching these problems is to transfer them to the corresponding ones in the case of operator-valued functions on the classical tori, and then to use existing results in the latter case or adapt classical arguments. Due to the noncommutativity of operator product, substantial difficulties arise in our arguments, like usually in noncommutative analysis. One of the subtlest parts of our arguments is the proof of the weak type $(1, 1)$ maximal inequalities for the square Fejér and Poisson means because of their multiple-parameter nature. This is the first time that noncommutative weak type $(1, 1)$ maximal inequalities are considered for mappings of this nature. Another intricate part concerns the analogue for $T^d_\theta$ of the classical Stein theorem on Bochner-Riesz means. The proof of the corresponding maximal inequalities is quite technical too.
Our study of Hardy spaces via the Littlewood-Paley theory necessitates a very careful analysis of various BMO-norms and square functions. The difficulty of this study is partly explained by the lack of an explicit handy formula of the circular Poisson kernel on $\mathbb{T}^d$ for $d \geq 2$.

We end this introduction with a brief description of the organization of the paper. In Section 2 we present some preliminaries and notation on quantum tori. This section also introduces our transference method. The simple section 3 defines the summation methods studied in the paper and deals with the mean convergence of quantum Fourier series by them. Section 4 is devoted to the maximal inequalities associated to these summation methods. Their proofs depend, via transference, on some general maximal inequalities for operator-valued functions on $\mathbb{R}^d$ (or $\mathbb{T}^d$) that are of interest for their own right. These maximal inequalities are then applied in Section 5 to obtain the corresponding pointwise convergence theorems. Section 6 deals with the Bochner-Riesz means. The main theorem there is the quantum analogue of Stein’s classical theorem. The difficult part is the type $(p,p)$ maximal inequalities for these means. In Section 7 we discuss $L_p$ Fourier multipliers on $\mathbb{T}^d$. We show that a Fourier multiplier is completely bounded on the noncommutative $L_p$-space associated to $\mathbb{T}^d$ if it is completely bounded on $L_p(\mathbb{T}^d)$. In this case, the two completely bounded norms are equal. Finally, in Section 8 we present the Littlewood-Paley theory on $\mathbb{T}^d$ and define the associated Hardy and BMO spaces using the circular Poisson semigroup, and show that they possess all expected properties of the usual Hardy spaces on $\mathbb{R}^d$. Our approach is to transfer this theory to the operator-valued case on $\mathbb{T}^d$ and to use Mei’s arguments in [27] for the $\mathbb{R}^d$ setting. Since the geometry of $\mathbb{T}^d$ and the circular Poisson kernel are less handy than those of $\mathbb{R}^d$, we cannot directly apply Mei’s results to our case. However, considering functions on $\mathbb{T}^d$ as periodic functions on $\mathbb{R}^d$, we can still reduce most of our problems to the corresponding ones on periodic functions on $\mathbb{R}^d$, then adapt Mei’s argument to the periodic case. A good part of this section is devoted to the study of several BMO-norms and square functions naturally appearing in this periodization procedure.

2. Preliminaries

2.1. Noncommutative $L_p$ spaces. Let $\mathcal{M}$ be a von Neumann algebra and $\mathcal{M}_+$ its positive part. Recall that a trace on $\mathcal{M}$ is a map $\tau : \mathcal{M}_+ \rightarrow [0,\infty]$ satisfying:

i) $\tau(x+y) = \tau(x) + \tau(y)$ for arbitrary $x,y \in \mathcal{M}_+$;

ii) $\tau(\lambda x) = \lambda \tau(x)$ for any $\lambda \in [0,\infty)$ and $x \in \mathcal{M}_+$;

iii) $\tau(x^*x) = \tau(xx^*)$ for all $x \in \mathcal{M}$.

$\tau$ is said to be normal if $\sup_x \tau(x_n) = \tau(\sup_x x_n)$ for any bounded increasing net $(x_n)$ in $\mathcal{M}_+$, semifinite if for each $x \in \mathcal{M}_+ \setminus \{0\}$ there is a nonzero $y \in \mathcal{M}_+$ such that $y \leq x$ and $\tau(y) < \infty$, and faithful if for each $x \in \mathcal{M}_+ \setminus \{0\}$, $\tau(x) > 0$. A von Neumann algebra $\mathcal{M}$ is called semifinite if it admits a normal semifinite faithful trace $\tau$. We refer to [15] for theory of von Neumann algebras. Throughout this paper, $\mathcal{M}$ will always denote a semifinite von Neumann algebra equipped with a normal semifinite faithful trace $\tau$.

Denote by $S_+$ the set of all $x \in \mathcal{M}_+$ such that $\tau(\text{supp}(x)) < \infty$, where $\text{supp}(x)$ is the support of $x$ which is defined as the least projection $e$ in $\mathcal{M}$ such that $ex = x$ or equivalently $xe = x$. Let $S$ be the linear span of $S_+$. Then $S$ is a $*$-subalgebra of $\mathcal{M}$ which is $w^*$-dense in $\mathcal{M}$. Moreover, for each $0 < p < \infty$, $x \in S$ implies $|x|^p \in S_+$ (and so $\tau(|x|^p) < \infty$), where $|x| = (x^*x)^{1/2}$ is the modulus of $x$. Now, we define $\|x\|_p = [\tau(|x|^p)]^{1/p}$ for all $x \in S$. One can show that $\| \cdot \|_p$ is a norm on $S$ if $1 \leq p < \infty$, and a quasi-norm (more precisely, $p$-norm) if $0 < p < 1$. The completion of $(S,\| \cdot \|_p)$ is denoted by $L_p(M,\tau)$ or simply by $L_p(M)$. This is the noncommutative $L_p$-space associated with $(M,\tau)$. The elements of $L_p(M)$ can be described by densely defined closed operators measurable with respect to $(M,\tau)$, like in the commutative case. For convenience, we set $L_\infty(M) = M$ equipped with the operator norm. The trace $\tau$ can be extended to a linear functional on $S$, still denoted by $\tau$. Since $|\tau(x)| \leq \|x\|_1$ for all $x \in S$, $\tau$ further extends to a continuous functional on $L_1(M)$.

Let $0 < r,p,q \leq \infty$ be such that $1/r = 1/p + 1/q$. If $x \in L_p(M), y \in L_q(M)$ then $xy \in L_r(M)$ and the following Hölder inequality holds:

$$\|xy\|_r \leq \|x\|_p\|y\|_q.$$
In particular, if \( r = 1 \), \(|\tau(xy)| \leq \|xy\|_1 \leq \|x\|_p \|y\|_q \) for arbitrary \( x \in L_p(M) \) and \( y \in L_q(M) \). This defines a natural duality between \( L_p(M) \) and \( L_q(M) \) : \( (x,y) = \tau(xy) \). For any \( 1 \leq p < \infty \) we have \( L_p(M)^* = L_q(M) \) isometrically. Thus, \( L_1(M) \) is the predual \( M_\ast \) of \( M \), and \( L_q(M) \) is reflexive for \( 1 < p < \infty \). We refer to [33] for more information on noncommutative \( L_p \)-spaces.

2.2. Quantum tori. Let \( d \geq 2 \) and \( \theta = (\theta_{kj}) \) be a real skew symmetric \( d \times d \)-matrix. The associated \( d \)-dimensional noncommutative torus \( A_\theta \) is the universal \( C^\ast \)-algebra generated by \( d \) unitary operators \( U_1, \ldots , U_d \) satisfying the following commutation relation

\[
U_k U_j = e^{2\pi i \theta_{kj}} U_j U_k, \quad j, k = 1, \ldots , d.
\]

We will use standard notation from multiple Fourier series. Let \( U = (U_1, \cdots , U_d) \). For \( m = (m_1, \cdots , m_d) \in \mathbb{Z}^d \) we define

\[
U^m = U_1^{m_1} \cdots U_d^{m_d}.
\]

A polynomial in \( U \) is a finite sum

\[
x = \sum_{m \in \mathbb{Z}^d} \alpha_m U^m \quad \text{with} \quad \alpha_m \in \mathbb{C},
\]

that is, \( \alpha_m = 0 \) for all but finite indices \( m \in \mathbb{Z}^d \). The involution algebra \( \mathcal{P}_\theta \) of such all polynomials is dense in \( A_\theta \). For any polynomial \( x \) as above we define

\[
\tau(x) = \alpha_0,
\]

where \( \theta = (0, \cdots, 0) \). Then, \( \tau \) extends to a faithful tracial state on \( A_\theta \). Let \( T^d_\theta \) be the \( w^\ast \)-closure of \( A_\theta \) in the GNS representation of \( \tau \). This is our \( d \)-dimensional quantum torus. The state \( \tau \) extends to a normal faithful tracial state on \( T^d_\theta \) that will be denoted again by \( \tau \). Recall that the von Neumann algebra \( T^{\mathbb{N}}_\theta \) is hyperfinite.

Since \( \tau \) is a state, \( L_q(T^d_\theta) \subset L_p(T^d_\theta) \) for any \( 0 < p < q \leq \infty \). Any \( x \in L_1(T^d_\theta) \) admits a formal Fourier series:

\[
x \sim \sum_{m \in \mathbb{Z}^d} \hat{x}(m) U^m,
\]

where

\[
\hat{x}(m) = \langle \tau(U^m \ast x) \rangle, \quad m \in \mathbb{Z}^d
\]

are the Fourier coefficients of \( x \). \( x \) is, of course, uniquely determined by its Fourier series.

2.3. Transference. We denote the usual \( d \)-torus by \( T^d \):

\[
T^d = \{(z_1, \ldots , z_d) : |z_j| = 1, z_j \in \mathbb{C}, 1 \leq j \leq d\}.
\]

\( T^d \) is equipped with the usual topology and group law multiplication, that is,

\[
z \cdot w = (z_1, \ldots , z_d) \cdot (w_1, \ldots , w_d) = (z_1 w_1, \ldots , z_d w_d).
\]

For any \( m \in \mathbb{Z}^d \) and \( z = (z_1, \ldots , z_d) \in T^d \) let

\[
z^m = z_1^{m_1} \cdots z_d^{m_d}.
\]

We will need the tensor von Neumann algebra \( \mathcal{N}_\theta = L_\infty(T^d) \tilde{\otimes} T^d_\theta \), equipped with the tensor trace \( \nu = \int dm \otimes \tau \), where \( dm \) is normalized Haar measure on \( T^d \). Note that for every \( 0 < p < \infty \),

\[
L_p(\mathcal{N}_\theta, \nu) \cong L_p(T^d; L_p(T^d_\theta)).
\]

The space on the right hand side is the space of Bochner \( p \)-integrable functions from \( T^d \) to \( L_p(T^d_\theta) \). Accordingly, let \( C(T^d, A_\theta) \) denote the \( C^\ast \)-algebra of continuous functions from \( T^d \) to \( A_\theta \). For each \( z \in T^d \), define \( \pi_z \) to be the isomorphism of \( T^d_\theta \) determined by

\[
\pi_z(U^m) = z^m U^m = z_1^{m_1} \cdots z_d^{m_d} U_1^{m_1} \cdots U_d^{m_d}.
\]

It is clear that \( \pi_z \) is trace preserving, so extends to an isometry on \( L_p(T^d_\theta) \) for every \( 0 < p < \infty \). Thus we have

\[
\|\pi_z(x)\|_p = \|x\|_p, \quad x \in L_p(T^d_\theta), \quad 0 < p \leq \infty.
\]

Proposition 2.1. For any \( x \in L_p(T^d_\theta) \) the function \( \tilde{x} : z \mapsto \pi_z(x) \) is continuous from \( T^d \) to \( L_p(T^d_\theta) \) (with respect to the \( w^\ast \)-topology for \( p = \infty \)). If \( x \in A_\theta \), it is continuous from \( T^d \) to \( A_\theta \).
Proof. Consider first the case $0 < p < \infty$. Let $x \in L_p(T_d^\theta)$. Since $\mathcal{P}_\theta$ is dense in $L_p(T_d^\theta)$, for arbitrary $\varepsilon > 0$ there is $x_0 \in \mathcal{P}_\theta$ such that $\|x - x_0\|_p < \varepsilon$. Clearly, $\pi_z(x_0)$ is a polynomial in $\mathcal{U}$ of the same degree as $x_0$. Thus, $z \mapsto \pi_z(x_0)$ is continuous from $T_d^\theta$ into $L_p(T_d^\theta)$. We then deduce the desired continuity of $\tilde{x}$. The same argument works equally for $\mathcal{A}_\theta$. The case of $p = \infty$ follows from that of $p = 1$ by duality. \hfill $\square$

The previous result in the case of $p = \infty$ implies, in particular, that the map $x \mapsto \tilde{x}$ establishes an isomorphism from $T_d^\theta$ into $\mathcal{N}_\theta$. It is also clear that this isomorphism is trace preserving. Thus we get the following

**Corollary 2.2.** i) Let $0 < p < \infty$. If $x \in L_p(T_d^\theta)$, then $\tilde{x} \in L_p(\mathcal{N}_\theta)$ and $\|\tilde{x}\|_p = \|x\|_p$, that is, $x \mapsto \tilde{x}$ is an isometric embedding from $L_p(T_d^\theta)$ into $L_p(\mathcal{N}_\theta)$. Moreover, this map is also an isomorphism from $\mathcal{A}_\theta$ into $C(T_d^\theta; \mathcal{A}_\theta)$.

ii) Let $T_d^\theta = \{\tilde{x} : x \in T_d^\theta\}$. Then $T_d^\theta$ is a von Neumann subalgebra of $\mathcal{N}_\theta$ and the associated conditional expectation is given by

$$E(f)(z) = \pi_z\left(\int_{T_d^\theta} \pi_\theta[f(w)]dm(w)\right), \quad z \in T_d^\theta, \ f \in \mathcal{N}_\theta.$$  

Moreover, $E$ extends to a contractive projection from $L_p(\mathcal{N}_\theta)$ onto $L_p(T_d^\theta)$ for $1 \leq p \leq \infty$.

iii) $L_p(T_d^\theta)$ is isometric to $L_p(T_d^\theta)$ for every $0 < p \leq \infty$.

Our transference method consists in the following procedure:

$x \in L_p(T_d^\theta) \mapsto \tilde{x} \in L_p(T_d^\theta) \subset L_p(\mathcal{N}_\theta)$.

This allows us to work in $L_p(\mathcal{N}_\theta)$. Then, in order to return back to $L_p(T_d^\theta) \cong L_p(T_d^\theta)$, we apply the conditional expectation $E$ to elements in $L_p(\mathcal{N}_\theta)$.

### 3. Mean Convergence

We begin with the mean convergence of Fourier series defined on quantum tori for an illustration of the transference method described in the previous section. This section also introduces the summation methods studied throughout the paper. They are the following:

- **The square Fejér mean**

$$F_N[x] = \sum_{m \in \mathbb{Z}^d, |m|_\infty \leq N} \left(1 - \frac{|m|}{N + 1}\right) \ldots \left(1 - \frac{|m|}{N + 1}\right) \hat{x}(m)U^m, \quad N \geq 0.$$  

- **The square Poisson mean**

$$P_r[x] = \sum_{m \in \mathbb{Z}^d} \hat{x}(m)r^{|m|_1}U^m, \quad 0 \leq r < 1.$$  

- **The circular Poisson mean**

$$P_r[x] = \sum_{m \in \mathbb{Z}^d} \hat{x}(m)r^{|m|_2}U^m, \quad 0 \leq r < 1.$$  

- Let $\Phi$ be a continuous function on $\mathbb{R}^d$ with $\Phi(0) = 1$. Define

$$\Phi^\varepsilon[x] = \sum_{m \in \mathbb{Z}^d} \Phi(\varepsilon|m|)\hat{x}(m)U^m, \quad \varepsilon > 0.$$  

We will always impose the following condition to $\Phi$:

\begin{equation}
\left\{
\begin{array}{ll}
\Phi(s) = \varphi(s) & \text{with } \int_{\mathbb{R}^d} \varphi(s)ds = 1;

|\Phi(s)| + |\varphi(s)| \leq A(1 + |s|)^{-d-\delta}, & \forall s \in \mathbb{R}^d,
\end{array}
\right.
\end{equation}

for some $A, \delta > 0$ (cf. [44, p. 253]).

In the above, $x \in L_1(T_d^\theta)$ has its Fourier series expansion: $x \sim \sum_{m \in \mathbb{Z}^d} \hat{x}(m)U^m$, and for $m \in \mathbb{Z}^d$

$$|m|_p = \left(\sum_{j=1}^d |m_j|^p\right)^{1/p}.$$  


with the usual modification for \( p = \infty \).

The last summation method contains two special important examples of the function \( \Phi \). The first one is
\[
\Phi(s) = e^{-2\pi |s|} \quad \text{and} \quad \varphi(s) = c_d (1 + |s|^2)^{-(d+1)/2}, \quad \forall s \in \mathbb{R}^d,
\]
where we have used the standard notation in harmonic analysis that \( |s| = |s|_2 \) denotes the Euclidean norm of \( \mathbb{R}^d \). In this case,
\[
\Phi^\varepsilon[x] = \sum_{m \in \mathbb{Z}^d} e^{-2\pi |m|_2 \varepsilon} \hat{x}(m) U^m.
\]
This is the circular Poisson integral \( P_{\varepsilon}[x] \) of \( x \) with \( r = e^{-2\pi\varepsilon} \).

The second example arises when \( \alpha > (d - 1)/2 \) in the following definition
\[
\Phi(s) = \begin{cases} 
(1 - |s|^2)^\alpha & \text{if } |s| < 1, \\
0 & \text{if } |s| \geq 1.
\end{cases}
\]
It is well known that
\[
\varphi(s) = \hat{\Phi}(s) = \frac{\Gamma(\alpha + 1) J_{\frac{d}{2} + \alpha} (2\pi |s|)}{\pi^{|s|_2^{d + \alpha}}}, \quad \forall s \in \mathbb{R}^d \setminus \{0\},
\]
where \( J_{\lambda} \) is the Bessel function of order \( \lambda \). In this case we obtain the Bochner-Riesz mean of order \( \alpha \) on the quantum torus:
\[
B_R^{\alpha}[x] = \sum_{|m|_2 \leq R} \left( 1 - \frac{|m|^2}{R^2} \right)^\alpha \hat{x}(m) U^m.
\]

A fundamental problem is in which sense the above means of the operator \( x \) converge back to \( x \). This problem is partly investigated in this section. Indeed, we have the following mean convergence theorem.

**Proposition 3.1.** Let \( 1 \leq p < \infty \) and \( x \in L_p(\mathbb{T}_d^d) \). Then \( F_N[x] \) converges to \( x \) in \( L_p(\mathbb{T}_d^d) \) as \( N \to \infty \). The same convergence holds for \( P_{\varepsilon}[x], P_{r}[x] \) as \( r \to 1 \) and \( \Phi^\varepsilon[x] \) as \( \varepsilon \to 0 \). Moreover, for \( p = \infty \) these limits hold for any \( x \in \mathcal{A}_\theta \).

The proof can be done either by imitating the classical proofs (cf. [44]), or using the transference argument. The second method is more elegant and simpler. The corresponding results in \( L_p(\mathbb{N}_0) \) are simple and well-known when one writes \( L_p(\mathbb{N}_0) = L_p(\mathbb{T}_d^d; L_p(\mathbb{T}_d^d)) \), which reduces the mean convergence in \( L_p(\mathbb{T}_d^d) \) to the corresponding one in the vector-valued case on the usual torus \( \mathbb{T}_d^d \).

As all these summation methods in the vector-valued case are given by approximation identities, it is better to state and prove first a general convergence theorem for convolution operators by an approximation identity in \( L_p(\mathbb{T}_d^d; X) \), where \( X \) is a Banach space. Here \( L_p(\mathbb{T}_d^d; X) \) denotes the \( L_p \)-space of Bochner \( p \)-integrable functions from \( \mathbb{T}_d^d \) to \( X \).

Let \( \Lambda \) be a directed set. An approximation identity on the multiplication group \( \mathbb{T}_d^d \) (as \( \lambda \to \lambda_0 \)) is a family of functions \( \{\varphi_\lambda\}_{\lambda \in \Lambda} \) in \( L_1(\mathbb{T}_d^d) \) verifying the following three conditions:

i) \( \int_{\mathbb{T}_d^d} \varphi_\lambda(z) dm(z) = 1 \) for all \( \lambda \in \Lambda \).

ii) \( \sup_{\lambda \in \Lambda} \|\varphi_\lambda\|_1 < \infty \).

iii) For any neighborhood \( V \) of the identity \( 1, \ldots, 1 \) of the group \( \mathbb{T}_d^d \) we have
\[
\int_{\mathbb{T}_d^d \setminus V} |\varphi_\lambda| dm(z) \to 0 \quad \text{as} \quad \lambda \to \lambda_0.
\]

Recall that for \( N \geq 0 \) an integer, the square Fejér kernel on \( \mathbb{T}_d^d \) is
\[
F_N(z) = \sum_{m \in \mathbb{Z}^d, |m|_\infty \leq N} \left( 1 - \frac{|m|}{N + 1} \right) \cdots \left( 1 - \frac{|m_d|}{N + 1} \right) z^m.
\]

For \( 0 \leq r < 1 \), the square and circular Poisson kernels are respectively
\[
P_r(z) = \sum_{m \in \mathbb{Z}^d} r^{|m|_1} z^m \quad \text{and} \quad \mathbb{P}_r(z) = \sum_{m \in \mathbb{Z}^d} r^{|m|_2} z^m.
\]
It is well known that \((F_N)_{N \geq 1}, \,(P_r)_{0 \leq r < 1}\) and \((\overline{P}_r)_{0 \leq r < 1}\) are all approximation identities on \(\mathbb{T}^d\). Also, if we write \(\Phi(s) = \Phi(\varepsilon s)\), then \(\Phi = \hat{\varphi}_s\) with \(\varphi_s(s) = \frac{1}{2\pi} \varphi(s)\) for \(s \in \mathbb{R}^d\). Let
\[
K_\varepsilon(s) = \inf_{m \in \mathbb{Z}^d} \varphi_s(s + m), \quad s \in \mathbb{R}^d.
\]
\(K_\varepsilon\) is periodic, so can be viewed as a function on \(\mathbb{T}^d\). Then by (3.1) it can be proved that \((K_\varepsilon)_{\varepsilon > 0}\)
is an approximation identity on \(\mathbb{T}^d\) such that
\[
(K_\varepsilon * f)(z) = \sum_{m \in \mathbb{Z}^d} \Phi(\varepsilon m) f(m) z^m, \quad f \sim \sum_{m \in \mathbb{Z}^d} f(m) z^m
\]
(see the proof of Theorem VII.2.11 in [44]).

Let \(X\) be a Banach space and let \(1 \leq p \leq \infty\). Suppose that \((\varphi_\lambda)_{\lambda \in \Lambda}\) is an approximation identity on \(\mathbb{T}^d\). For any \(f \in L_p(\mathbb{T}^d; X)\) we define the convolution \(\varphi_\lambda * f\) by
\[
(\varphi_\lambda * f)(z) = \int_{\mathbb{T}^d} f(w) \varphi_\lambda(\overline{w} \cdot z) dm(w), \quad \forall \, z \in \mathbb{T}^d.
\]
Then for any \(f \in L_p(\mathbb{T}^d; X)\) we have \(\varphi_\lambda * f \in L_p(\mathbb{T}^d; X)\) and
\[
\|\varphi_\lambda * f\|_p \leq \|f\|_p \|\varphi_\lambda\|_1.
\]
The following vector-valued result is well-known. The proof in the scalar case (cf. e.g. [11, Theorem 1.2.19]) is valid as well in the vector-valued setting without any change. \(C(\mathbb{T}^d; X)\) denotes the space of continuous functions from \(\mathbb{T}^d\) to \(X\), equipped with the uniform norm.

**Proposition 3.2.** Let \(X\) be a Banach space and let \(1 \leq p < \infty\). Let \((\varphi_\lambda)_{\lambda \in \Lambda}\) be an approximation identity on \(\mathbb{T}^d\). If \(f \in L_p(\mathbb{T}^d; X)\), then
\[
\|\varphi_\lambda * f - f\|_p \to 0 \quad \text{as} \quad \lambda \to \lambda_0.
\]
Moreover, when \(p = \infty\) the above limit holds for any \(f \in C(\mathbb{T}^d; X)\).

It is now clear that Proposition 3.1 immediately follows from Proposition 3.2 via the transference method.

### 4. Maximal inequalities

In this section, we present the maximal inequalities of the summation methods of Fourier series defined previously. These inequalities will be used for the pointwise convergence in the next section. We first recall the definition of the noncommutative maximal norm introduced by Pisier [34] and Junge [13]. Let \(\mathcal{M}\) be a von Neumann algebra equipped with a normal semifinite faithful trace \(\tau\). Let \(1 \leq p \leq \infty\). We define \(L_p(\mathcal{M}; \ell_\infty)\) to be the space of all sequences \(x = (x_n)_{n \geq 1}\) in \(L_p(\mathcal{M})\) which admit a factorization of the following form: there exist \(a, b \in L_{2p}(\mathcal{M})\) and a bounded sequence \(y = (y_n)\) in \(L_\infty(\mathcal{M})\) such that
\[
x_n = ay_n b, \quad \forall \, n \geq 1.
\]
The norm of \(x\) in \(L_p(\mathcal{M}; \ell_\infty)\) is given by
\[
\|x\|_{L_p(\mathcal{M}, \ell_\infty)} = \inf \left\{ \|a\|_{2p} \sup_{n \geq 1} \|y_n\|_\infty \|b\|_{2p} \right\},
\]
where the infimum runs over all factorizations of \(x\) as above.

We will follow the convention adopted in [23] that \(\|x\|_{L_p(\mathcal{M}, \ell_\infty)}\) is denoted by \(\|\sup_n^+ x_n\|_p\). We should warn the reader that \(\|\sup_n^+ x_n\|_p\) is just a notation since \(\sup_n x_n\) does not make any sense in the noncommutative setting. We find, however, that \(\|\sup_n^+ x_n\|_p\) is more intuitive than \(\|x\|_{L_p(\mathcal{M}, \ell_\infty)}\). The introduction of this notation is partly justified by the following remark.

**Remark 4.1.** Let \(x = (x_n)\) be a sequence of selfadjoint operators in \(L_p(\mathcal{M})\). Then \(x \in L_p(\mathcal{M}; \ell_\infty)\) iff there exists a positive element \(a \in L_p(\mathcal{M})\) such that \(-a \leq x_n \leq a\) for all \(n \geq 1\). In this case we have
\[
\left\|\sup_{n \geq 1}^+ x_n\right\|_p = \inf \left\{ \|a\|_p : a \in L_p(\mathcal{M}), \quad -a \leq x_n \leq a, \quad \forall \, n \geq 1 \right\}.
\]
More generally, if $\Lambda$ is any index set, we define $L_p(\mathcal{M}; \ell_\infty(\Lambda))$ as the space of all $x = (x_\lambda)_{\lambda \in \Lambda}$ in $L_p(\mathcal{M})$ that can be factorized as

$$x_\lambda = a_\lambda b \quad \text{with} \quad a, b \in L_{2p}(\mathcal{M}), \quad y_\lambda \in L_{\infty}(\mathcal{M}), \quad \sup_\lambda \|y_\lambda\|_\infty < \infty.$$ 

The norm of $L_p(\mathcal{M}; \ell_\infty(\Lambda))$ is defined by

$$\|\sup_{\lambda \in \Lambda}^+ x_\lambda\|_p = \inf_{x_\lambda = a_\lambda b} \left\{ \|a\|_{2p} \sup_{\lambda \in \Lambda} \|y_\lambda\|_\infty \|b\|_{2p} \right\}. $$

It is shown in [23] that $x \in L_p(\mathcal{M}; \ell_\infty(\Lambda))$ iff

$$\sup_{\lambda \in J} \{\|\sup_{\lambda \in \Lambda}^+ x_\lambda\|_p : J \subset \Lambda, J \text{ finite} \} < \infty.$$ 

In this case, $\|\sup_{\lambda \in \Lambda}^+ x_\lambda\|_p$ is equal to the above supremum.

The following is the main theorem of this section.

**Theorem 4.2.** i) Let $x \in L_1(T^d_\theta)$. Then for any $\alpha > 0$ there exists a projection $e \in T^d_\theta$ such that

$$\sup_{N \geq 0} \|e F_N[x] e\|_{\infty} \leq \alpha \quad \text{and} \quad \tau(e^{\lambda}) \leq C_d \frac{\|x\|_1}{\alpha}.$$ 

ii) Let $1 < p \leq \infty$. Then

$$\sup_{N \geq 0} \|\sup_{\lambda \in \Lambda}^+ F_N[x]\|_p \leq C_d \frac{p^2}{(p-1)^2} \|x\|_p, \quad \forall \ x \in L_p(T^d_\theta).$$

Both statements hold for the three other summation methods $P_r$, $P_r$ and $\Phi$. In the case of $\Phi$, the constant $C_d$ also depends on the two constants in [5,1].

In the terminology of [23], we can rephrase parts i) and ii) as that the map $x \mapsto (F_N[x])_{N \geq 0}$ is of weak type $(1,1)$ and of type $(p,p)$, respectively. Before proceeding to the proof of the theorem, we point out that its part concerning the circular Poisson mean $P_r$ can be easily deduced from [23]. This is due to the fact that $(P_r)_{0 \leq r \leq 1}$ is a symmetric diffusion semigroup on $T^d_\theta$. Let us show this latter statement. Define

$$\delta_j(U_j) = 2\pi i U_j, \quad \delta_j(U_k) = 0, \quad k \neq j$$

(cf. [5]). These operators $\delta_j$ commute with the involution of $T^d_\theta$ and play the role of the partial derivatives $\frac{\partial}{\partial x_j}$ on the classical $d$-torus. Let $\Delta = \sum_{j=1}^d \delta_j^2$. Then $\Delta$ is a negative operator on $L_2(T^d_\theta)$ and its spectrum consists of the numbers $-4\pi^2 |m|^2$, $m \in \mathbb{Z}^d$. For any $\lambda > 0$, we have

$$\|(|\lambda - \Delta|^{-1})\| \leq \sup_{z \in \sigma(-\Delta)} \frac{1}{|\lambda + z|} \leq \frac{1}{\lambda}.$$ 

Then by the Hille-Yosida theorem, $\Delta$ is the infinitesimal generator of a semigroup of contractions on $L_2(T^d_\theta)$. Denote this semigroup by $(T_t)$. Then $T_t = \exp(t\Delta)$. It is easy to check that $(T_t)$ satisfies the following properties:

i) $T_t$ is a contraction on $T^d_\theta$: $\|T_t x\| \leq \|x\|$ for all $x \in T^d_\theta$;

ii) $T_t$ is positive: $T_t x \geq 0$ if $x \geq 0$;

iii) $\tau \circ T_t = \tau \circ (T_t x) = \tau(x)$ for all $x \in T^d_\theta$;

iv) $T_t$ is symmetric relative to $\tau: \tau(T_t(y)^* x) = \tau(y^* T_t(x))$ for all $x,y \in L_2(T^d_\theta)$.

Then $(T_t)$ extends to a semigroup of contractions on $L_p(T^d_\theta)$ for every $1 \leq p \leq \infty$. This is the heat semigroup of $T^d_\theta$. The circular Poisson means $P_r[x]$ is exactly the Poisson semigroup subordinated to $T_t$, where $r = e^{-2\pi t}$. Then by [23], we get the part of Theorem 4.2 concerning the circular Poisson means.

The previous argument does not apply to the three other means. However, we can get the type $(p,p)$ inequality for $F_N$ and $P_r$ again from [23] but not with the right estimate on the constant $C_p$. Indeed, the square Poisson mean $P_r$ is the restriction to the diagonal $(r, \ldots, r)$ of the following multiple parameter semigroup $P_{(r_1, \ldots, r_d)}$:

$$P_{(r_1, \ldots, r_d)}[x] = \sum_{m \in \mathbb{Z}^d} \hat{x}(m) r_1^{m_1} \cdots r_d^{m_d} U^m.$$
Let maximal inequality for that and require the following two theorems which are of interest for their own right. Recall that $M$ denotes a von Neumann algebra with a normal semifinite faithful trace $\tau$. $L_\infty(\mathbb{R}^d)\otimes M$ is equipped with the tensor trace $\nu = dx \otimes \tau$, where $dx$ is Lebesgue measure on $\mathbb{R}^d$.

**Theorem 4.3.** Let $\varphi$ be an integrable function on $\mathbb{R}^d$ such that $|\varphi|$ is radial and radially decreasing. Let $\varphi_\varepsilon(s) = \frac{1}{\varepsilon^d} \varphi(\frac{s}{\varepsilon})$ for $s \in \mathbb{R}^d$ and $\varepsilon > 0$.

i) Let $f \in L_1(\mathbb{R}^d; L_1(M))$. Then for any $\alpha > 0$ there exists a projection $e \in L_\infty(\mathbb{R}^d)\otimes M$ such that

$$\sup_{\varepsilon > 0} \|e(\varphi_\varepsilon * f)e\|_\infty \leq \alpha \quad \text{and} \quad \nu(e^{-1}) \leq C_d \|\varphi\|_1 \frac{\|f\|_1}{\alpha}.$$  

ii) Let $1 < p \leq \infty$. Then

$$\|\sup_{\varepsilon > 0} \varphi_\varepsilon * f\|_p \leq C_d \|\varphi\|_1 \frac{p^d}{(p-1)^2} \|f\|_p, \quad \forall f \in L_p(\mathbb{R}^d; L_p(M)).$$

**Proof.** Let $f \in L_1(\mathbb{R}^d; L_1(M))$. Without loss of generality, we assume that $f$ is positive. On the other hand, it is easy to reduce to the case where $\varphi$ is positive too. Indeed, decomposing $\varphi$ into its real and imaginary parts, we need only to consider each part separately. Since $f \geq 0$, we have

$$\text{Re}(\varphi_\varepsilon) * f \leq |\text{Re}(\varphi_\varepsilon)| * f \leq |\varphi_\varepsilon| * f.$$  

This gives the announced reduction. Thus in the sequel we assume that $\varphi \geq 0$. First take $\varphi$ to be of the form $\varphi = \sum_k \alpha_k \mathbb{1}_{B_k}$ (a finite sum), where $B_k$ are balls of center 0 and $\alpha_k \geq 0$. Then

$$\varphi_\varepsilon * f(s) = \sum_k \alpha_k (\mathbb{1}_{B_k})_\varepsilon * f(s) = \sum_k \alpha_k |B_k| M_{\mathbb{1}_{B_k}}(f)(s),$$

where $M_B(f)(s) = \frac{1}{|B|} \int_B f(s-t)dt$ for any ball $B$ centered at 0. We now appeal to Mei’s noncommutative Hardy-Littlewood maximal weak type $(1,1)$ inequality [27]: For any $\alpha > 0$ there exists a projection $e \in L_\infty(\mathbb{R}^d)\otimes M$ such that

$$\nu(e^{-1}) \leq C_d \frac{\|f\|_1}{\alpha} \quad \text{and} \quad \|eM_B(f)e\|_\infty \leq \alpha, \quad \forall \text{ ball } B \text{ centered at 0}.$$  

We then deduce that

$$\|e(\varphi_\varepsilon * f)e\|_\infty \leq C_d \sum_k \alpha_k |B_k| \alpha = C_d \|\varphi\|_1 \alpha, \quad \forall \varepsilon > 0.$$  

For a general positive $\varphi$, choose an increasing sequence $(\varphi^{(n)})$ of functions of the previous form such that $\varphi^{(n)}$ converges to $\varphi$ pointwise. Then for any $\alpha > 0$, there exists a projection $e_n \in L_\infty(\mathbb{R}^d)\otimes M$ such that

$$\nu(e_n^{-1}) \leq \frac{\|f\|_1}{\alpha} \quad \text{and} \quad \|e_n(\varphi^{(n)}_\varepsilon * f)e_n\|_\infty \leq \alpha, \quad \forall \varepsilon > 0.$$  

Let $a$ be a $w^*\text{-accumulation point of } e_n$. Note that

$$(\varphi^{(n)}_\varepsilon * f)^{\frac{1}{2}} e_n - (\varphi_\varepsilon * f)^{\frac{1}{2}} a = ((\varphi^{(n)}_\varepsilon * f)^{\frac{1}{2}} - (\varphi_\varepsilon * f)^{\frac{1}{2}}) e_n + (\varphi_\varepsilon * f)^{\frac{1}{2}} (e_n - a).$$

Since $\varphi^{(n)} \to \varphi$ increasingly, then $(\varphi^{(n)}_\varepsilon * f)^{\frac{1}{2}}$ strongly converges to $(\varphi_\varepsilon * f)^{\frac{1}{2}}$. Hence $(\varphi^{(n)}_\varepsilon * f)^{\frac{1}{2}} e_n$ weakly converges to $(\varphi_\varepsilon * f)^{\frac{1}{2}} a$. Then we deduce

$$\nu(1-a) \leq \frac{\|f\|_1}{\alpha} \quad \text{and} \quad \|(\varphi_\varepsilon * f)^{\frac{1}{2}} a\|_\infty \leq \liminf_n \|((\varphi^{(n)}_\varepsilon * f)^{\frac{1}{2}} e_n\|_\infty \leq \frac{\alpha}{2}.$$  

Let $e = \mathbb{1}_{[0,1]}(a)$, the spectral projection of $a$ corresponding to the interval $[\frac{1}{2},1]$. Note that $1-e = \mathbb{1}_{[0,\frac{1}{2}]}(1-a)$. Then $\frac{1}{2}(1-e) \leq 1-a$, which implies that $\frac{1}{2} \nu(1-e) \leq \nu(1-a)$. Moreover, letting $g(r) = \frac{1}{2} \mathbb{1}_{[0,\frac{1}{2}]}(r)$, $r \in (0,1]$, we have $e = eg(a) a$ and

$$e(\varphi_\varepsilon * f)e = eg(a)[a(\varphi_\varepsilon * f)a]e g(a).$$
By symmetry, it suffices to consider one of these regions, say the one where
permutation of the set $\nu \geq 0$. This proof is much more involved than the previous one. Again, we can assume that
all functions $\varphi$ are positive. It suffices to show the weak type $(1, 1)$ inequality. Fix a positive
$f \in L_1(\mathbb{R}^d; L_1(M))$. Let $I'_0 = [-1, 1]$ and $I'_k = \{t \in \mathbb{R} : 2^{k-1} < |t| \leq 2^k\}$ for $k = 1, 2, \ldots$. Also, let $I_k = [-2^k, 2^k]$. Split $\mathbb{R}^d$ into $d!$ regions of the form $|t_j| \geq \cdots \geq |t_d|$, where $\{t_1, \ldots, t_d\}$ is a permutation of the set $\{1, \ldots, d\}$. Then

$$
\varphi \ast f(s) = \sum_{|t_1| \geq \cdots \geq |t_d|} \int_{|t_j| \geq \cdots \geq |t_d|} \varphi(t) f(s - \varepsilon t) dt.
$$

By symmetry, it suffices to consider one of these regions, say the one where $|y_1| \geq \cdots \geq |y_d|$. Let

$$
F_k(s) = \int_{|t_1| \geq \cdots \geq |t_d|} \varphi(t) f(s - \varepsilon t) dt, \quad s = (s_1, \ldots, s_d) \in \mathbb{R}^d.
$$

We must show that for any $\alpha > 0$ there exists a projection $e \in L_\infty(\mathbb{R}^d; \mathbb{M})$ such that

$$
\nu(e^+ \leq \frac{\|f\|_1}{\alpha} \quad \text{and} \quad \|eF_k e\|_\infty \leq \alpha.
$$

Using the assumption on $\varphi$ and by change of variables, we have

$$
F_k(s) \leq \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{k_1} \cdots \sum_{k_d=0}^{k_{d-1}} \int_{I_{k_1}} \int_{I_{k_2}} \cdots \int_{I_{k_d}} \varphi(t) f(s - \varepsilon t) dt
$$

$$
\leq \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{k_1} \cdots \sum_{k_d=0}^{k_{d-1}} 2^{-k_1(1+\delta)} \cdots \int_{I_{k_1}} \int_{I_{k_2}} \cdots \int_{I_{k_d}} f(s - \varepsilon t) dt
$$

$$
\leq \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{k_1} \cdots \sum_{k_d=0}^{k_{d-1}} 2^{-k_1(1+\delta)} \cdots \int_{I_{k_1}} \int_{I_{k_2}} \cdots \int_{I_{k_d}} f(s - \varepsilon t) dt
$$

$$
\leq \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{k_1} \cdots \sum_{k_d=0}^{k_{d-1}} 2^{-(k_1+\cdots+k_d)} \frac{1}{|I_{k_1}|} \int_{I_{k_1}} f(s_1 - \varepsilon t_1, s_2 - 2^{k_2-k_1} \varepsilon t_2, \ldots, s_d - 2^{k_d-k_1} \varepsilon t_d) dt.
$$

Given a function $g \in L_1(\mathbb{R}^d; L_1(M))$ and a cube $Q \subset \mathbb{R}^d$ centered at 0 and with sides parallel to
the axes put

$$
M_Q(g)(s) = \frac{1}{|Q|} \int_Q g(s-t) dt, \quad s \in \mathbb{R}^d.
$$
Note that this average function appeared already in the proof of Theorem 4.3 but with balls instead of cubes. For any fixed \( k = (k_1, \ldots, k_d) \) with \( k_1 \geq k_2 \geq \cdots \geq k_d \) let
\[
f_k(z_1, z_2, \cdots, z_d) = f(z_1, 2^{k_2-k_1}z_2, \cdots, 2^{k_d-k_1}z_d).
\]
Then
\[
\frac{1}{|I|} \int_{I_k} f(s_1-\varepsilon t_1, s_2-2^{k_2-k_1}\varepsilon t_2, \cdots, s_d-2^{k_d-k_1}\varepsilon t_d) dt = M_{\varepsilon I_k}^T(f_k)(s_1, 2^{k_1-k_2}s_2, \cdots, 2^{k_1-k_d}s_d).
\]
Thus
\[
F_\varepsilon(s) \lesssim \sum_{k_1=0}^\infty \sum_{k_2=0}^\infty \cdots \sum_{k_d=0}^{k_{d-1}} 2^{-(k_1+\cdots+k_d)\delta} M_{\varepsilon I_k}^T(f_k)(s_1, 2^{k_1-k_2}s_2, \cdots, 2^{k_1-k_d}s_d).
\]
(4.2) Now we use again Mei’s noncommutative Hardy-Littlewood maximal weak type \((1,1)\) inequality which remains true with balls replaced by cubes. For any \( \alpha_k > 0 \), there exits a projection \( e_k \) in \( L_\infty(\mathbb{R}^d) \otimes M \) such that
\[
\nu(e_\varepsilon^+) \leq C_d \frac{\|f_k\|_1}{\alpha_k} \quad \text{and} \quad \|e_k M_{\varepsilon I_k}^T(f_k)e_k\|_\infty \leq \alpha_k, \quad \forall \varepsilon > 0.
\]
(4.3) Let \( T \) be the mapping
\[
(s_1, s_2, \cdots, s_d) \mapsto (s_1, 2^{k_1-k_2}s_2, \cdots, 2^{k_1-k_d}s_d).
\]
\( T \) is a homeomorphism of \( \mathbb{R}^d \), so induces an isomorphism of \( L_\infty(\mathbb{R}^d) \otimes M \), still denoted by \( T \). Then for any \( g \in L_\infty(\mathbb{R}^d) \otimes M \), we have
\[
\int \tau(T(g))(s) ds = \int \tau(g \circ T(s)) ds = 2^{k_2-k_1} \cdots 2^{k_d-k_1} \int \tau(g(s)) ds.
\]
Let \( \tilde{e}_k = T(e_k) \). Then \( \tilde{e}_k \) is a projection and
\[
\nu(\tilde{e}_k^+) = 2^{k_2-k_1} \cdots 2^{k_d-k_1} \nu(e_\varepsilon^+).
\]
(4.4) On the other hand,
\[
M_{\varepsilon I_k}^T(f_k)(s_1, 2^{k_1-k_2}s_2, \cdots, 2^{k_1-k_d}s_d) = T(M_{\varepsilon I_k}^T(f_k))(s_1, s_2, \cdots, s_d)
\]
and
\[
T(e_k M_{\varepsilon I_k}^T(f_k)e_k) = \tilde{e}_k M_{\varepsilon I_k}^T(f_k)(s_1, 2^{k_1-k_2}, \cdots, 2^{k_1-k_d}) \tilde{e}_k.
\]
Therefore, by (4.3)
\[
\|e_k M_{\varepsilon I_k}^T(f_k)e_k\|_\infty = \|e_k M_{\varepsilon I_k}^T(f_k)e_k\|_\infty \leq \alpha_k, \quad \forall \varepsilon > 0.
\]
(4.5) Let \( \alpha > 0 \). For each \( k \) with \( k_1 \geq k_2 \geq \cdots \geq k_d \) we choose
\[
\alpha_k = \alpha 2^{k_1\delta/(2d)} 2^{k_2\delta(1-1/(2d))} \cdots 2^{k_d\delta(1-1/(2d))}.
\]
Then
\[
2^{-(k_1+\cdots+k_d)\delta} \alpha_k = \alpha 2^{-k_1\delta/2} 2^{-n_1/2d} \cdots 2^{-n_d/2d},
\]
where \( n_1 = k_1 - k_2, \ldots, n_d = k_1 - k_d \). Note that all \( n_j \) are nonnegative integers. Finally, let \( e = \bigwedge_k \tilde{e}_k \). Then \( e \) is a projection in \( L_\infty(\mathbb{R}^d) \otimes M \), and by (4.3), (4.3), the definition of \( f_k \) and the
As in the proof of Theorem 4.3, we then see that

\[ \|eF_x e\|_\infty \leq \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{k_1-1} \cdots \sum_{k_d=0}^{k_{d-1}-1} 2^{-(k_1+\cdots+k_d)\delta} \|\hat{e} M_{\theta}^{\infty} (f_k)(\cdot, 2^{k_1-k_2}, \ldots, 2^{k_1-k_d})\hat{e}\|_\infty \]

\[ \leq C_d \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{k_1-1} \cdots \sum_{k_d=0}^{k_{d-1}-1} 2^{-(k_1+\cdots+k_d)\delta} \alpha_k \]

\[ \leq \alpha \sum_{k_1 \geq 0} \sum_{k_2 \geq 0} \cdots \sum_{k_d \geq 0} 2^{-k_1 \delta/2} 2^{-n_2/(2d)} \cdots 2^{-n_d/(2d)} \lesssim \alpha. \]

Thus we get the desired estimate (4.11), so finish the proof of the theorem. \( \square \)

We also require the following lemma for the proof of Theorem 4.2.

**Lemma 4.5.** Let \( \mathcal{N} \) be a \( w^* \)-closed involutive subalgebra of \( \mathcal{M} \) that is the image of a normal conditional expectation \( \mathcal{E} \). Let \( (x_n) \) be a sequence of positive operators in \( L_1(\mathcal{N}) \). Assume that for any \( \alpha > 0 \) there exists a projection \( e \in \mathcal{M} \) such that

\[ \|e x_n e\|_\infty \leq \alpha \quad \text{and} \quad \tau(e^+) \leq \frac{C}{\alpha}. \]

Then there exists a projection \( e \in \mathcal{N} \) such that

\[ \|e x_n e\|_\infty \leq 4\alpha \quad \text{and} \quad \tau(e^+) \leq \frac{2C}{\alpha}. \]

**Proof.** Let \( a = \mathcal{E}(e) \). Then \( a \in \mathcal{N} \) and

\[ \|a x_n^{1/2}\|_\infty = \|e(\hat{e} x_n^{1/2})\|_\infty \leq \alpha^{1/2}. \]

As in the proof of Theorem 4.3, we then see that \( e = I_{[1/2, 1]}(a) \) is the desired projection in \( \mathcal{N} \). \( \square \)

**Proof of Theorem 4.3** We will identify the \( d \)-torus \( \mathbb{T}^d \) with the cube \( I^d = [0, 1]^d \subset \mathbb{R}^d \) (with \( I = [0, 1] \)) via \((e^{2\pi i x_1}, \ldots, e^{2\pi i x_d}) \leftrightarrow (s_1, \ldots, s_d)\). Accordingly, \( \mathcal{N}_\theta = L_\infty(I^d \otimes \mathbb{R}^d) \) is viewed as a subalgebra of \( \mathcal{M}_\theta = L_\infty(\mathbb{R}^d \otimes \mathbb{R}^d) \). The associated conditional expectation is just the multiplication by the indicator function \( \mathbb{T}^d \), viewed as \( \mathbb{R}^d \). Thus \( \mathcal{N}_\theta \) becomes a subalgebra of \( \mathcal{M}_\theta \) too. The corresponding conditional expectation is \( \mathbb{T}^d \cdot \mathcal{E} \), where \( \mathcal{E} \) is the conditional expectation from \( \mathcal{N}_\theta \) to \( \mathbb{T}^d \) given by Corollary 2.2.

Now let us show the weak type \((1, 1)\) inequality for the Fejér means. Recall that \( F_N \) is the Fejér kernel on \( \mathbb{T}^d \) given by (3.2) and that

\[ F_N(s_1, \ldots, s_d) = G_N(s_1) \cdots G_N(s_d), \]

where \( G_N \) is the 1-dimensional Fejér kernel. It is a well-known elementary fact that

\[ G_N(s) \leq \frac{\pi^2}{2} \frac{N + 1}{(N + 1)^2|s|^2}. \]
Thus
\[ F_N(s_1, \ldots, s_d) \lesssim \frac{1}{\varepsilon} \eta(s_1) \cdots \eta(s_d) = \eta(s_1) \cdots \eta(s_d), \]
where \( \eta(s) = (1 + |s|^2)^{-1} \) and \( \varepsilon = (N + 1)^{-1} \). Let \( x \in L_1(T^d_\theta) \). Writing \( x \) as a linear combination of four positive elements, we can assume \( x \geq 0 \). Using transference, we have that \( \tilde{x} \in L_1(T^d_\theta) \subset L_1(N_\theta) \) and
\[
\widehat{F_N[x]}(s_1, \ldots, s_d) = F_N \ast \tilde{x}(s_1, \ldots, s_d)
= \int_{t^d} F_N(s_1 - t_1, \ldots, s_d - t_d) \tilde{x}(t_1, \ldots, t_d) dt
= \int_{\mathbb{R}^d} F_N(s_1 - t_1, \ldots, s_d - t_d) \mathbb{1}_{\varepsilon^d}(t_1, \ldots, t_d) \tilde{x}(t_1, \ldots, t_d) dt.
\]
Therefore, we are in a situation of applying Theorem 4.4 so for any \( \alpha > 0 \) there exists a projection \( \tilde{e} \in M_\theta \) such that
\[
\sup_N \| \tilde{e} F_N[x] \tilde{e} \|_\infty \leq \alpha \quad \text{and} \quad \nu(\tilde{e}^\perp) \lesssim \frac{\| \tilde{e} F_N[x] \|_{L_1(N_\theta)}}{\alpha} = \frac{\| x \|_1}{\alpha}.
\]
Since \( x \geq 0, \widehat{F_N[x]} \geq 0 \) for every \( N \). Thus by Lemma 4.3, we get the desired weak type \((1, 1)\) inequality for \( F_N \). Similarly, we show the type \((p, p)\) inequality. The same argument works equally for the square Poisson means \( P_r \).

It remains to show the part of the theorem concerning \( \Phi^\varepsilon \) (which contains the circular Poisson mean \( P_r \) as a special case). We will use the convolution formula (3.4). Note that for maximal inequalities on \( \Phi^\varepsilon \) we do not need all conditions on \( \Phi \) and \( \varepsilon \) in (3.4). What we really need here is the last growth assumption on \( \varphi \) there:
\[
|\varphi(s)| \leq \frac{A}{(1 + |s|)^{d+\delta}}, \quad s \in \mathbb{R}^d.
\]
Then like in the proof of Theorem 4.3 we can assume that \( \varphi \) is nonnegative. In this case the kernel \( K_\varepsilon \) is nonnegative too. Moreover, replacing \( \varphi \) by the function on the right hand side above, we can further suppose that \( \varphi \) satisfies the assumption of Theorem 4.3. Now let \( x \in L_1(T^d_\theta) \). Without loss of generality, assume again \( x \geq 0 \). By (3.4), for \( s = (s_1, \ldots, s_d) \in \mathbb{R}^d \) we have
\[
\overline{\Phi^\varepsilon[x]}(s) = \int_{\mathbb{Z}^d} K_\varepsilon(s - t) \tilde{x}(t) dt
= \sum_{m \in \mathbb{Z}^d} \int_{\mathbb{Z}^d} \varphi_\varepsilon(s - t + m) \tilde{x}(t) dt
= \int_{\mathbb{Z}^d} \varphi_\varepsilon(s - t) \tilde{x}(t) dt + \sum_{m \neq 0} \int_{\mathbb{Z}^d} \varphi_\varepsilon(s - t + m) \tilde{x}(t) dt.
\]
The first term on the right can be dealt with in the same way as before for \( F_N \):
\[
\int_{\mathbb{Z}^d} \varphi_\varepsilon(s - t) \tilde{x}(t) dt = \int_{\mathbb{R}^d} \varphi_\varepsilon(s - t) \mathbb{1}_{\varepsilon^d}(t) \tilde{x}(t) dt.
\]
Then by Theorem 4.3 for any \( \alpha > 0 \) there exists a projection \( \tilde{e}_1 \in M_\theta \) such that
\[
\nu(\tilde{e}_1^\perp) \lesssim \frac{\| x \|_1}{\alpha} \quad \text{and} \quad \| \tilde{e}_1 \[ \int_{\mathbb{Z}^d} \varphi_\varepsilon(\cdot - t) \tilde{x}(t) dt \|_\infty \leq \alpha, \quad \forall \varepsilon > 0.
\]
On the other hand, for \( s, t \in \mathbb{Z}^d \) and \( m \neq 0 \) we have
\[
\varphi_\varepsilon(s - t + m) \lesssim \frac{1}{\varepsilon^d} (1 + \frac{|m|}{\varepsilon})^{-d-\delta}.
\]
Note that
\[
\sum_{m \neq 0} \frac{1}{\varepsilon^d} (1 + \frac{|m|}{\varepsilon})^{-d-\delta} \approx \frac{1}{\varepsilon^d} \sum_{1 \leq |m| \leq \varepsilon} + \varepsilon^{d} \sum_{\varepsilon < |m|} \frac{1}{|m|^{d+\delta}} \lesssim 1.
\]
Hence (recalling that $x \geq 0$),
\[ \sum_{m \neq 0} \int_{\mathbb{R}^d} \varphi_{e}(s-t+m) \hat{x}(t) dt \lesssim \sum_{m \neq 0} e^{-\frac{1}{\varepsilon}} (1 + \frac{|m|}{\varepsilon})^{-d-\delta} \int_{\mathbb{R}^d} \hat{x}(t) dt \lesssim \int_{\mathbb{R}^d} \hat{x}(t) dt. \]
The last integral is an operator in $L_1(\mathbb{T}_d)$ and its $L_1$-norm is less than or equal to that of $x$. Thus there exists a projection $\tilde{e}_2 \in \mathbb{T}_d$ such that
\[ \nu(\tilde{e}_2^\perp) \lesssim \frac{\|x\|_1}{\alpha} \quad \text{and} \quad \|	ilde{e}_2[\int_{\mathbb{R}^d} \hat{x}(t) dt]\|_\infty \leq \alpha. \]

Let $\tilde{e} = \tilde{e}_1 \lor \tilde{e}_2$. Then $\tilde{e}$ is a projection in $\mathcal{M}_\theta$, and combining the preceding two parts we get
\[ \nu(\tilde{e}^\perp) \lesssim \frac{\|x\|_1}{\alpha} \quad \text{and} \quad \|	ilde{e} \Phi^x [x] \tilde{e}\|_\infty \leq \alpha, \quad \forall \varepsilon > 0. \]

We then deduce the weak type $(1, 1)$ inequality for $\Phi^x$ thanks to Lemma 14.3. The type $(p, p)$ inequality is proved similarly. Therefore, the proof of Theorem 12 is complete. \hfill \Box

5. Pointwise convergence

In this section we apply the maximal inequalities proved in the previous section to study the pointwise convergence of Fourier series on quantum tori. To this end we first need an appropriate analogue for the noncommutative setting of the usual almost everywhere convergence. This is the almost uniform convergence introduced by Lance [26].

Let $\{x_\lambda\}_{\lambda \in \Lambda}$ be a family of elements in $L_p(\mathcal{M})$. Recall that $\{x_\lambda\}_{\lambda \in \Lambda}$ is said to converge almost uniformly to $x$, abbreviated as $x_\lambda \overset{a.u.}{\rightarrow} x$, if for every $\varepsilon > 0$ there exists a projection $e \in \mathcal{M}$ such that
\[ \tau(1-e) < \varepsilon \quad \text{and} \quad \lim_{\lambda} \|(x_\lambda - x)e\|_\infty = 0. \]

Also, $\{x_\lambda\}_{\lambda \in \Lambda}$ is said to converge bilaterally almost uniformly to $x$, abbreviated as $x_\lambda \overset{b.a.u.}{\rightarrow} x$, if the limit above is replaced by
\[ \lim_{\lambda} \|e(x_\lambda - x)e\|_\infty = 0. \]

In the commutative case, both convergences are equivalent to the usual almost everywhere convergence thanks to Egorov's theorem. However, they are different in the noncommutative setting.

**Theorem 5.1.** Let $1 \leq p \leq \infty$ and $x \in L_p(\mathbb{T}_d)$. Then $F_N[x] \overset{b.a.u.}{\rightarrow} x$ as $N \rightarrow \infty$. Moreover, for $2 \leq p \leq \infty$ the b.a.u. convergence can be strengthened to a.u. convergence.

Similar statements hold for the two Poisson means $P_r$, $\mathbb{P}_r$ as $r \to \infty$ as well as for the mean $\Phi^x$ as $\varepsilon \to 0$.

**Proof.** Let $x \in L_1(\mathbb{T}_d)$ and $\varepsilon > 0$. Let $(\varepsilon_m)$ and $(\delta_m)$ be two sequences of small positive numbers. Then for each $m \geq 1$ choose $y_m \in \mathcal{A}_0$ such that $\|x - y_m\|_1 \leq \delta_m$. Let $z_m = x - y_m$, so $x = y_m + z_m$. Applying Theorem 4.2 to each $z_m$, we find a projection $e_m$ such that
\[ \sup_N \|e_m F_N[z_m]e_m\|_\infty \leq \varepsilon_m \quad \text{and} \quad \tau(e_m) \leq C \|z_m\|_1 \varepsilon_m^{-1} \leq C \delta_m \varepsilon_m^{-1}. \]

The first inequality implies that $\|e_m z_m e_m\|_\infty \leq \varepsilon_m$.

Let $e = \bigwedge_m e_m$. Then
\[ \tau(e^\perp) \leq C \sum_m \delta_m \varepsilon_m^{-1} < \varepsilon \]
provided $\varepsilon_m$ and $\delta_m$ are appropriately chosen. On the other hand,
\begin{align*}
\|e(F_N[x] - x)e\|_\infty &\leq \|e(F_N[y_m] - y_m)e\|_\infty + \|eF_N[z_m]e\|_\infty + \|ez_m e\|_\infty \\
&\leq \|F_N[y_m] - y_m\|_\infty + 2 \varepsilon_m.
\end{align*}

By Proposition 3.1
\[ \lim_{N \to \infty} \|F_N[y_m] - y_m\|_\infty = 0. \]
for $y_m \in \mathcal{A}_\theta$. It then follows that
\[
\limsup_{N \to \infty} \|e(F_N[x] - x)e\|_\infty \leq 2\varepsilon_m.
\]
Whence $\lim_{N \to \infty} \|e(F_N[x] - x)e\|_\infty = 0$. Therefore, $F_N[x]$ converges to $x$ b.a.u. The b.a.u. convergence statements for the other summation methods are proved exactly in the same way.

Let us turn to the a.u. convergence. Let $x \in L^2_2(\mathbb{T}_\theta^d)$ and $\epsilon > 0$. We can assume $x$ selfadjoint. As in the preceding argument, let $x = y_m + z_m$ with $y_m \in \mathcal{A}_\theta$ and $\|z_m\|_2 \leq \delta_m$. Both $y_m$ and $z_m$ can be chosen selfadjoint. Now applying Theorem 4.2 to $y_m^2$, we find a projection $e_m$ such that
\[
\sup_N \|e_m F_N[z_m^2]e_m\|_\infty \leq \varepsilon_m \quad \text{and} \quad \tau(e_m) \leq C \varepsilon_m^{-1} \delta_m^{-1}.
\]
Since the map $z \mapsto F_N[z]$ is positive, by Kadison’s Cauchy-Schwarz inequality [25], we have
\[
(F_N[z_m])^2 \leq F_N[z_m^2].
\]
Thus
\[
(5.1) \quad \|e_m F_N[z_m]e_m\|_\infty \leq \|e_m F_N[z_m^2]e_m\|_\infty \leq \varepsilon_m.
\]
Let $e = \bigvee_m e_m$. Then $\tau(e) \leq \epsilon$ for appropriate $\varepsilon_m$ and $\delta_m$ and $\lim_N \|F_N[x] - x\|_\infty = 0$. Therefore, $F_N[x] \xrightarrow{a.u} x$. The proof of the corresponding statements for $F_N$ and $F_\theta$ is the same.

However, a minor extra argument is required for the mean $\Phi^\varepsilon$ because the map $z \mapsto \Phi^\varepsilon[z]$ is not positive in general. So we cannot apply directly Kadison’s inequality to this map. But what is really missing is the one-sided weak type $(1,1)$ maximal inequality [5.1] for $\Phi^\varepsilon$ instead of $F_N$. In order to show this latter inequality, we can assume, as in the proof of Theorem 4.2, that $\varphi$ is nonnegative. Then the kernel $K_\varepsilon$ in (5.1) is nonnegative too. Thus the map $z \mapsto K_\varepsilon \ast \hat{z}$ is positive, so we can apply Kadison’s inequality to this map. Then as before for $F_N$, we get the desired inequality (5.1) with $F_N$ replaced by $\Phi^\varepsilon$, and then deduce that $\Phi^\varepsilon[x] \xrightarrow{a.u} x$ as $\varepsilon \to 0$. Therefore, the theorem is completely proved.

6. Bochner-Riesz means

As pointed out in section 8 when $\alpha > (d - 1)/2$, the function $\Phi$ and $\varphi$ associated with the Bochner-Riesz mean satisfy (5.1). Therefore, by Proposition 5.1 Theorems 4.2 and 5.1 we get the following

Proposition 6.1. Let $\alpha > (d - 1)/2$ and $x \in L_\mu(\mathbb{T}_\theta^d)$ with $1 \leq p < \infty$. Then
i) \[
\lim_{R \to \infty} \|B_{R}^\alpha[x]\| = x \text{ in } L_\mu(\mathbb{T}_\theta^d) \quad \text{(relative to the } w^*\text{-topology for } p = \infty).
\]
ii) \[
\sup_{R > 0}\|B_{R}^\alpha[x]\| \lesssim \|x\|_p \text{ for } p > 1.
\]
iii) \[
B_{R}^\alpha[x] \xrightarrow{b.a.u} x \text{ as } R \to \infty.
\]

If $\alpha$ is below the critical index $(d - 1)/2$, the above results usually fail even in the scalar case, see for example [44, VII.4]. However, we have the following theorem, i.e., Theorem 6.2 which is the noncommutative analogue of Stein’s theorem [43] (see also [44] VII.5).

Theorem 6.2. Let $1 < p < \infty$ and $\alpha > (d - 1)/2 - 1/p$. Then for any $x \in L_\mu(\mathbb{T}_\theta^d)$

\begin{enumerate}
\item \[
\sup_{R > 0}\|B_{R}^\alpha[x]\| \lesssim \|x\|_p \text{ with the relevant constant depends only } p, d \text{ and } \alpha.
\]
\item \[
\lim_{R \to \infty} \|B_{R}^\alpha[x]\| = x \text{ in } L_\mu(\mathbb{T}_\theta^d).
\]
\item \[
B_{R}^\alpha[x] \xrightarrow{b.a.u} x \text{ as } R \to \infty.
\]
\end{enumerate}

Proof. The hard part of the theorem is the maximal inequality i). Assuming this part, it is easy to show the two others. Indeed, i) implies that for any $R > 0$
\[
\|B_{R}^\alpha[x]\| \lesssim \|x\|_p \quad \forall x \in L_\mu(\mathbb{T}_\theta^d).
\]
Whence
\[
\sup_{R > 0}\|B_{R}^\alpha\|_{L_\mu \to L_\mu} < \infty.
\]
Together with the density of polynomials in \( L_p(\mathbb{T}_d^q) \), this implies the mean convergence in ii). The pointwise convergence iii) can be proved as Theorem 5.1. The only thing to note is the fact that the type \((p, p)\) maximal inequality in i) implies the corresponding weak type \((p, p)\) inequality. The details are left to the reader.

The remainder of this section is devoted to the proof of i). We will follow the pattern set up by Stein in the classical setting. The proof is quite technical and complicated, but essentially everything is based on two main ideas: estimate maximal function and square function by duality and interpolation.

We will frequently use the duality between \( L_{p'}(\mathbb{T}_d^q; \ell_1) \) and \( L_p(\mathbb{T}_d^q; \ell_\infty) \) \((p'\) being the conjugate index of \(p\)). For the convenience of the reader we recall this duality. \( L_{p'}(\mathbb{T}_d^q; \ell_1) \) is defined to be the space of all sequences \( y = (y_n) \) in \( L_{p'}(\mathbb{T}_d^q) \) which can be decomposed as

\[
y_n = \sum_{k \geq 1} u_{kn} v_{kn}, \quad \forall n \geq 1
\]

for two families \((u_{kn})_{k,n \geq 1}\) and \((v_{kn})_{k,n \geq 1}\) in \( L_{2p'}((\mathbb{T}_d^q))\) such that

\[
\sum_{k,n \geq 1} u_{kn}^* u_{kn} \leq \sum_{k,n \geq 1} v_{kn}^* v_{kn} < \infty.
\]

\( L_{p'}(\mathbb{T}_d^q; \ell_1) \) is equipped with the norm

\[
\|y\|_{L_{p'}(\mathbb{T}_d^q; \ell_1)} = \inf \left\{ \sum_{k,n \geq 1} u_{kn}^* u_{kn}^{1/2} \right\} \sum_{k,n \geq 1} v_{kn}^* v_{kn}^{1/2},
\]

where the infimum runs over all decompositions of \( y \) as above. It is easy to see that if \( y_n \geq 0 \) for all \( n \), then \( (y_n) \in L_{p'}(\mathbb{T}_d^q; \ell_1) \) iff \( \sum_n y_n \in L_{p'}(\mathbb{T}_d^q) \). In this case, we have

\[
\|y\|_{L_{p'}(\mathbb{T}_d^q; \ell_1)} = \|\sum_n y_n\|_{p'}.
\]

Let \( 1 \leq p' < \infty \). Then the dual space of \( L_{p'}(\mathbb{T}_d^q; \ell_1) \) is \( L_p(\mathbb{T}_d^q; \ell_\infty) \). The duality bracket is given by

\[
\langle x, y \rangle = \sum_n \tau(x_n y_n), \quad x = (x_n) \in L_p(\mathbb{T}_d^q; \ell_\infty), \quad y = (y_n) \in L_{p'}(\mathbb{T}_d^q; \ell_1).
\]

We refer to [13] and [23] for more information.

For clarity we divide the proof of i) into three steps.

**Step 1.** If \( \alpha \in \mathbb{C} \) and \( \text{Re}(\alpha) > \frac{d-1}{2} \), then for \( 1 < p \leq \infty \),

\[
\|\sup_{R>0} B^\alpha_R[x]\|_p \lesssim \|x\|_p, \quad \forall x \in L_p(\mathbb{T}_d^q).
\]

To this end, choose \( \delta > 0 \) and \( \beta \in \mathbb{C} \) such that \( \text{Re}(\alpha) > \delta > \frac{d-1}{2} \) and \( \alpha = \delta + \beta \). We have the following identity

\[
B^\alpha_R = C_{\beta, \delta} R^{-2\alpha} \int_0^R (R^2 - t^2)^{\beta-1} l^{2\delta+1} B^\delta t^\beta dt,
\]

where \( C_{\beta, \delta} = 2 \Gamma(\beta + \delta + 1)/[\Gamma(\delta + 1) \Gamma(\beta)] \). Let \((R_n)\) be a sequence in \((0, \infty)\) and \((y_n)\) an element in the unit ball of \( L_p(\mathbb{T}_d^q; \ell_1) \). Then, for any \( x \in L_p(\mathbb{T}_d^q) \) we have

\[
\left| \tau \left( \sum_n B^\alpha_{R_n} [x] y_n \right) \right| = |C_{\beta, \delta}| \left| \sum_n R_n^{-2\alpha} \int_0^{R_n} (R_n^2 - t^2)^{\beta-1} l^{2\delta+1} \tau(B^\delta t^\beta [x] y_n) dt \right|
\]

\[
\leq |C_{\beta, \delta}| \int_0^1 |(1 - t^2)^{\beta-1} l^{2\delta+1}| \left| \tau \left( \sum_n B^\delta_{R_n} [x] y_n \right) \right| dt
\]

\[
\leq |C_{\beta, \delta}| \int_0^1 |(1 - t^2)^{\beta-1} l^{2\delta+1} | dt \sup_{R>0} \|B^\delta_R[x]\|_p
\]

\[
\lesssim \|x\|_p,
\]

where we have used Proposition 6.1 ii) in the last inequality and the fact that

\[
\int_0^1 |(1 - t^2)^{\beta-1} l^{2\delta+1} | dt = \int_0^1 (1 - t^2)^{\text{Re}(\beta)-1} l^{2\delta+1} dt < \infty
\]
since \( \text{Re} (\beta) = \text{Re} (\alpha) - \delta > 0 \) and \( \delta > 0 \). By duality we then deduce the desired maximal inequality.

**Step 2.** If \( \alpha > 0 \), then

\[
(6.2) \quad \| \sup_{R > 0} B_R^\alpha [x] \|_2 \lesssim \| x \|_2, \quad \forall \ x \in L_2 (T_\theta^d).
\]

We first consider the case of \( \alpha > 1/2 \). Choose \( \beta > 1 \) such that \( \alpha = \beta + \delta \) with \( \delta > -1/2 \). By (6.1)

\[
B^{\beta + \delta} = - C_{\beta, \delta} R^{-2(\beta + \delta)} \int_0^R \left( \int_0^t B^\delta_r dr \right) \left( (R^2 - t^2)^{\beta - 1} t^{2\delta + 1} \right) dt
\]

\[
= C_{\beta, \delta} \int_0^1 \varphi(t) M^\delta_R dt,
\]

where

\[
M^\delta_r = \frac{1}{t} \int_0^t B^\delta_x dr \quad \text{and} \quad \varphi(t) = 2(\beta - 1)(1 - t^2)^{\beta - 2} t^{2\delta + 3} - (2\delta + 1)(1 - t^2)^{\beta - 1} t^{2\delta + 1}.
\]

Note that \( \int_0^1 \| \varphi(t) \| dt < \infty \). We will use the following fact that for any \( (x_n) \in L_2 (T_\theta^d; \ell_\infty) \) one has

\[
\| \sup_{n} x_n \|_2 \approx \sup \left\{ \left\| \sum_n \tau (x_n y_n) \right\| : y_n \in L_2^+ (T_\delta^d), \| \sum y_n \|_2 \leq 1 \right\}
\]

with universal equivalence constants (see [13, 23]). In what follows, we fix \( x \in L_2 (T_\theta^d) \) and always assume that \( (R_n) \) is a sequence in \( (0, \infty) \) and \( (y_n) \) a sequence of positive elements in \( L_2 (T_\theta^d) \) with \( \| \sum y_n \|_2 \leq 1 \). Since

\[
\left| \tau \left( \sum_n B_{R_n}^\delta (x) y_n \right) \right| = |C_{\beta, \delta}| \left| \tau \left( \sum_n \left( \int_0^1 \varphi(t) M^\delta_{R_n,t} (x) dt \right) y_n \right) \right|
\]

\[
\leq |C_{\beta, \delta}| \int_0^1 \| \varphi(t) \| \left| \tau \left( \sum_n M^\delta_{R_n,t} (x) y_n \right) \right| dt
\]

\[
\lesssim \| \sup_{R > 0} M^\delta_R (x) \|_2 \int_0^1 \| \varphi(t) \| dt,
\]

where we have used duality in the last inequality. We then deduce that

\[
\| \sup_{R > 0} B_R^\alpha [x] \|_2 \lesssim \| \sup_{R > 0} M^\delta_R (x) \|_2.
\]

Now we must show that

\[
(6.3) \quad \| \sup_{R > 0} M^\delta_R (x) \|_2 \lesssim \| x \|_2 \quad \text{if} \quad \delta > -1/2.
\]

To this end, we again use duality. We have

\[
\left| \tau \left( \sum_n M_{R_n}^\delta (x) y_n \right) \right| \leq \left| \tau \left( \sum_n M_{R_n}^{\delta + 1} (x) y_n \right) \right| + \left| \tau \left( \sum_n \left[ M_{R_n}^{\delta + 1} (x) - M_{R_n}^\delta (x) \right] y_n \right) \right|
\]

\[
\leq \| \sup_{R > 0} M_{R_n}^{\delta + 1} (x) \|_2 + \left| \tau \left( \sum_n \left[ G_{R_n}^\delta (x) - G_{R_n}^\delta \right] y_n \right) \right|
\]

where \( G_{R_n}^\delta (x) = M_{R_n}^{\delta + 1} (x) - M_{R_n}^\delta (x) \). Using the following elementary inequality

\[
| \tau (a b) |^2 \leq | \tau (a | b \rangle | \tau (| a^* b \rangle |, \quad \forall \ a, b \in T_\theta^d \text{ with } b \geq 0,
\]

we have

\[
\left| \tau \left( \sum_n \left[ G_{R_n}^\delta (x) y_n \right] \right) \right|^2 \leq \tau \left( \sum_n \left| G_{R_n}^\delta (x) \right| y_n \right) \cdot \left. \sum \right| G_{R_n}^\delta (x)^* | y_n \right).
\]

Note that

\[
G_R^\delta (x) = \left| \frac{1}{R} \int_0^R \left[ B^{\delta + 1}_r [x] - B^\delta_r [x] \right] dr \right|
\]

\[
\leq \left( \int_0^R \left| B^{\delta + 1}_r [x] - B^\delta_r [x] \right|^2 dr \frac{1}{R} \right)^{1/2} \leq G^\delta (x),
\]
where
\[ G^\delta(x) = \left( \int_0^\infty |B_{t \tau}^{\delta+1}[x] - B_{t \tau}^{\delta}[x]|^2 \frac{dr}{r} \right)^{1/2}. \]

It then follows that
\[ \tau \left( \sum_n |G_{R_n}^\delta(x)|y_n \right) \leq \tau(G^\delta(x) \sum_n y_n) \leq \|G^\delta(x)\|_2 \|\sum_n y_n\|_2 \leq \|G^\delta(x)\|_2. \]

Similarly,
\[ \tau \left( \sum_n |G_{R_n}^\delta(x)^*|y_n \right) \leq \|G^\delta_*(x)\|_2, \]

where
\[ G^\delta_*(x) = \left( \int_0^\infty |(B_{t \tau}^{\delta+1}[x] - B_{t \tau}^{\delta}[x])^*|^2 \frac{dr}{r} \right)^{1/2}. \]

Combining the preceding inequalities, we obtain
\[ \|\sup_{R>0} M^\delta_R(x)\|_2 \leq \|\sup_{R>0} M^{\delta+1}_R(x)\|_2 + \|G^\delta(x)\|_2 \|G^\delta_*(x)\|_2^{1/2}. \]

We now claim that
\[ \max (\|G^\delta(x)\|_2, \|G^\delta_*(x)\|_2) \lesssim \|x\|_2, \quad \text{if} \quad \delta > -1/2. \]

Indeed, by Parseval’s identity we have
\[
\|G^\delta(x)\|_2^2 = \int_0^\infty \tau(|B_{t \tau}^{\delta+1}[x] - B_{t \tau}^{\delta}[x]|^2) \frac{dr}{r} \\
= \int_0^\infty \sum_{|m|_2 \leq R} \left| \left( 1 - \frac{|m|^2}{\tau^2} \right)^{\delta+1} - \left( 1 - \frac{|m|^2}{\tau^2} \right)^\delta \right|^2 |\hat{x}(m)|^2 \frac{dr}{r} \\
= \sum_{m \neq 0} |\hat{x}(m)|^2 \int_{|m|_2}^\infty \frac{|m|^2}{\tau^2} \left( 1 - \frac{|m|^2}{\tau^2} \right)^\delta \frac{dr}{r} \\
\lesssim \|x\|_2^2
\]

because the integral
\[
\int_{|m|_2}^\infty \frac{|m|^2}{\tau^2} \left( 1 - \frac{|m|^2}{\tau^2} \right)^\delta \frac{dr}{r} = \int_1^\infty r^{-5\delta}(1 - r^{-2})^\delta dr < \infty
\]

if \( \delta > -1/2 \). In the same way, we have
\[ \|G^\delta_*(x)\|_2 \lesssim \|x\|_2. \]

Hence our claim is proved. Consequently,
\[ \|\sup_{R>0} M^\delta_R(x)\|_2 \lesssim \|\sup_{R>0} M^{\delta+1}_R(x)\|_2 + \|x\|_2. \]

Then by iteration, for any positive integer \( k \) we have
\[ \|\sup_{R>0} M^\delta_R(x)\|_2 \lesssim \|\sup_{R>0} M^{\delta+k}_R(x)\|_2 + \|x\|_2. \]

Now, if we choose \( k \) such that \( \delta + k > (d-1)/2 \), then using Step 1, we have
\[ \|\sup_{R>0} M^{\delta+k}_R(x)\|_2 \lesssim \|\sup_{R>0} B^{\delta+k}_R[x]\|_2 \lesssim \|x\|_2. \]

Therefore, we deduce (6.3), and hence (6.2) provided \( \alpha > 1/2 \).
We now deal with the general case of $\alpha > 0$. Choose $\beta > 1/2$ and $\delta > -1/2$ so that $\alpha = \beta + \delta$. Then by (6.1)

\[ B_{R}^{\beta+\delta} - \frac{C_{\beta,\delta}R^{2-2(\beta+\delta)}}{C_{\beta,\delta+1}} = C_{\beta,\delta}R^{-2(\beta+\delta)} \left[ \int_{0}^{R} (R^2 - \ell^2)^{\beta-1} \ell^{2\delta+1} B_{t}^{\delta+1} dt \right] 
- R^{-2} \int_{0}^{R} (R^2 - t^2)^{\beta-1} t^{2(\delta+1)+1} B_{t}^{\delta+1} dt \]
\[ = C_{\beta,\delta}R^{-2(\beta+\delta)} \left[ \int_{0}^{R} (R^2 - t^2)^{\beta-1} t^{2\delta+1} (B_{t}^{\delta} - B_{t}^{\delta+1}) dt \right] 
+ \int_{0}^{R} (R^2 - t^2)^{\beta-1} t^{2\delta+1} (1 - R^{-2}t^2) B_{t}^{\delta+1} dt \]

\[ \triangleq I_R + II_R. \]

We first estimate $I_R$. By the argument already used above

\[ |\tau(\sum_{n} I_{Rn}(x)|y_n)|^2 \leq \tau(\sum_{n} |I_{Rn}(x)|y_n) \tau(\sum_{n} |I_{Rn}(x)^*|y_n). \]

However,

\[ |I_R(x)| = |C_{\beta,\delta}R^{-2(\beta+\delta)}| \int_{0}^{R} (R^2 - t^2)^{\beta-1} t^{2\delta+1} (B_{t}^{\delta+1}[x] - B_{t}^{\delta}[x]) dt \]
\[ \leq |C_{\beta,\delta}R^{-2(\beta+\delta)}| \int_{0}^{R} \left| (R^2 - t^2)^{\beta-1} t^{2\delta+1} \right|^2 dt \]
\[ \leq R^{1/2}R^{-1/2} \left( \int_{0}^{R} \left| B_{t}^{\delta+1}[x] - B_{t}^{\delta}[x] \right|^2 dt \right)^{1/2} \]
\[ \lesssim G^{\delta}(x) \]

because the integral

\[ R^{1-4(\beta+\delta)} \int_{0}^{R} \left| (R^2 - t^2)^{\beta-1} t^{2\delta+1} \right|^2 dt = \int_{0}^{1} (1 - t^2)^{\beta-1} t^{2\delta+1} dt < \infty \]

when $\beta > 1/2$. Similarly,

\[ |I_R(x)^*| \lesssim G^{\delta}(x). \]

Hence, we deduce

\[ \| \sup_{R>0} I_{R}(x) \|_{2} \lesssim \| G^{\delta}(x) \|_{2}^{1/2} \| G^{\delta}(x) \|_{2}^{1/2} \lesssim \| x \|_{2}. \]

Next, we estimate the second term $II_R$. Since

\[ II_R = C_{\beta,\delta}R^{-2(\beta+\delta)} \int_{0}^{R} (R^2 - t^2)^{\beta-1} t^{2\delta+1} (1 - R^{-2}t^2) B_{t}^{\delta+1} dt \]
\[ = C_{\beta,\delta}R^{-2(\beta+\delta)-2} \int_{0}^{R} (R^2 - t^2)^{\beta-1} t^{2\delta+1} B_{t}^{\delta+1} dt \]

and $\beta > 1/2$, $II_R$ can be dealt with as $B_{R}^{\alpha}$ in the case of $\alpha > 1/2$. So we conclude that

\[ \| \sup_{R>0} I_{R}(x) \|_{2} \lesssim \| x \|_{2}. \]

Therefore, we have finally arrived at

\[ \| \sup_{R>0} B_{R}^{\beta+\delta}(x) \|_{2} \leq \left| \frac{C_{\beta,\delta}}{C_{\beta,\delta+1}} \right| \| \sup_{R>0} B_{R}^{\beta+\delta+1}[x] \|_{2} \]
\[ + \| \sup_{R>0} I_{R}(x) \|_{2} + \| \sup_{R>0} II_R[x] \|_{2} \]
\[ \lesssim \| x \|_{2}. \]

This completes the proof of Step 2.

Step 3. When $p$ is near 1 or $\infty$, the announced result is in fact already contained in Step 1. Moreover, Step 2 gives the desired inequality in the special case of $p = 2$. The general case can be
Therefore, by the maximum principle we get
\begin{equation}
\|\sup_{R>0} B_R^n[x]\|_2 \lesssim \|x\|_2, \quad \alpha \in \mathbb{C}, \quad \text{Re}(\alpha) > 0.
\end{equation}

This can be reduced to the case of \( \alpha > 0 \) by using the argument in Step 1. We omit the details.

Let \( x \in L_p(\mathbb{T}_d^d) \) with \( \|x\|_p < 1 \) and \( y = (y_n) \) be a finite sequence in \( L_p(\mathbb{T}_d^d) \) with \( \|y\|_{L_p(\mathbb{T}_d^d, \ell_1)} < 1 \). Assume first that \( p < 2 \). For any fixed \( \alpha > (d-1)(1/p - 1/2) \) we can always choose \( p_1 > 1, \alpha_0 > 0 \) and \( \alpha_1 > (d-1)/2 \) such that
\[ \alpha = (1-t)\alpha_0 + t\alpha_1 \quad \text{and} \quad \frac{1}{p} = \frac{1-t}{2} + \frac{t}{p_1} \]
for some \( 0 < t < 1 \). Define
\[ f(z) = u|x|^z \frac{z^{1+\alpha}}{1+z}, \quad z \in \mathbb{C}, \]
where \( x = u|x| \) is the polar decomposition of \( x \). On the other hand, by Proposition 2.5 of [23], there is a function \( g = (g_n) \) continuous on the strip \( \{ z \in \mathbb{C} : 0 \leq \text{Re}(z) \leq 1 \} \) and analytic in the interior such that \( g(t) = y \) and
\[ \sup_{x \in \mathbb{R}} \left\{ \|g(is)\|_{L_2(\mathbb{T}_d^d, \ell_1)}, \|g(1+is)\|_{L_2(\mathbb{T}_d^d, \ell_1)} \right\} < 1. \]

Fix a sequence \((R_n) \subset (0, \infty)\) and \( \delta > 0 \). We define
\[ F(z) = \exp \left( \delta(z^2 - t^2) \right) \sum_n \tau(B_{R_n}^{(1-z)\alpha_0 + z\alpha_1} f(z))^n. \]

\( F \) is a function analytic in the open strip \( \{ z \in \mathbb{C} : 0 < \text{Re}(z) < 1 \} \). By (6.4), for any \( s \in \mathbb{R} \) we have
\[ |F(is)| \leq \exp \left( -\delta(s^2 + t^2) \right) \|B_{R_n}^{(1+is)(\alpha_1 - \alpha_0)}(f(is))\|_n \|g(is)\|_{L_2(\mathbb{T}_d^d, \ell_1)} \lesssim \|f(is)\|_2 \lesssim 1. \]

Similarly, by Step 1 we have
\[ |F(1+is)| \lesssim 1. \]

Therefore, by the maximum principle we get \( |F(t)| \lesssim 1 \) i.e.,
\[ |\tau(\sum_n B_{R_n}^n [x] y_n)| \lesssim 1 \]
if \( \|x\|_{L_p(M_N)} < 1 \). Then by duality and homogeneity, we deduce that
\[ \|\sup_{R>0} B_R^n[x]\|_p \lesssim \|x\|_p, \quad \forall x \in L_p(\mathbb{T}_d^d). \]

The argument for the case of \( p > 2 \) is similar once we begin by setting \( p_1 = \infty \). Thus the proof of Theorem 6.2 is complete. \( \square \)

**Remark 6.3.** The previous proof gives a slightly more general result by allowing \( \alpha \) to be complex. Namely, Theorem 6.2 remains true under the assumption that \( \text{Re}(\alpha) > (d-1)(1/p - 1/2) \) with \( \alpha \in \mathbb{C} \) and \( 1 < p < \infty \).

**Remark 6.4.** Let \( \mathcal{M} \) be a semifinite von Neumann algebra. Then Theorem 6.2 admits the following analogue for the algebra \( \mathbb{T}_d^d \mathcal{M} \) with the same proof: Let \( 1 < p \leq \infty \) and \( \text{Re}(\alpha) > (d-1)\left|\frac{1}{2} - \frac{1}{p}\right| \). Then
\[ \|\sup_{R>0} B_R^n[f]\|_p \lesssim \|f\|_p, \quad \forall f \in L_p(\mathbb{T}_d^d; L_p(\mathcal{M})). \]

Moreover, \( B_R^n[f] \) converges b.a.u. to \( f \) as \( R \to \infty \). Here
\[ B_R^n[f] = \sum_{|m|_2 \leq R} \left( 1 - \frac{|m|^2}{R^2} \right)^\alpha \hat{f}(m) z^m \]
for \( f \in L_p(\mathbb{T}_d^d; L_p(\mathcal{M})) \) with Fourier series expansion
\[ f \sim \sum_{m \in \mathbb{Z}^d} \hat{f}(m) z^m. \]
7. Fourier multipliers

It is our intention in this section to study Fourier multipliers in the quantum $d$-torus $\mathbb{T}_d^d$. We will compare (completely) bounded $L_p$ Fourier multipliers with those in the usual $d$-torus $\mathbb{T}^d$. The right framework for this investigation is the category of operator spaces.

We now recall some standard operator space notions and refer the reader to [8] and [35] for more information. A (concrete) operator space is a closed subspace $E$ of $B(H)$ for some Hilbert space $H$. Then $E$ inherits the matricial structure of $B(H)$ via the embedding $M_n(E) \subset M_n(B(H))$. More precisely, let $M_n(E)$ denote the space of $n \times n$ matrices with entries in $E$, equipped with the norm induced by $B(\ell_2^n(H))$. An abstract matricial norm characterization of operator spaces was given by Ruan. The morphisms in the category of operator spaces are completely bounded maps. Let $H,K$ be two Hilbert spaces. Suppose that $E \subset B(H)$ and $F \subset B(K)$ are two operator spaces. A map $u : E \to F$ is called completely bounded (in short c.b.) if

$$\sup_n \|\text{id}_{M_n} \otimes u\|_{M_n(E) \to M_n(F)} < \infty,$$

and the c.b. norm $\|u\|_{\text{cb}}$ is defined to be the above supremum. We denote by $\text{CB}(E,F)$ the space of all c.b. maps from $E$ to $F$, equipped with the norm $\| \cdot \|_{\text{cb}}$. This is a Banach space.

For an operator space $E$ there exists a natural matricial structure on the Banach dual $E^*$ of $E$ so that $E^*$ becomes an operator space too. The norm of $M_n(E^*)$ is that of $\text{CB}(E \otimes M_n, M_n(\mathbb{C}))$. This is usually called the standard dual of $E$. We will simply say the dual of $E$ since only standard duals are used in the sequel.

We will need the natural operator space structure on noncommutative $L_p$-spaces introduced by Pisier. Let $\mathcal{M}$ be a (semifinite) von Neumann algebra on a Hilbert space $H$. Then the embedding $\mathcal{M} \subset B(H)$ gives to $\mathcal{M}$ an operator space structure. To equip $L_1(\mathcal{M})$ with an operator space structure, we view $L_1(\mathcal{M})$ as the predual of the opposite algebra $\mathcal{M}^{\text{op}}$ instead of $\mathcal{M}$ itself. In this way, $L_1(\mathcal{M})$ becomes a subspace of the dual operator space of $\mathcal{M}^{\text{op}}$. This is the natural operator space structure of $L_1(\mathcal{M})$. Then for any $1 < p < \infty$ the operator space structure of $L_p(\mathcal{M})$ is defined via the complex interpolation formula $L_p(\mathcal{M}) = (L_\infty(\mathcal{M}), L_1(\mathcal{M}))(1/p,1/p)$. We refer the reader to [34] [35] for more details.

We will use the following fundamental property of c.b. maps between two noncommutative $L_p$-spaces due to Pisier [34]. Let $\mathcal{N}$ be another (semifinite) von Neumann algebra. Then a map $u : L_p(\mathcal{M}) \to L_p(\mathcal{N})$ is c.b. iff $\text{id}_{S_p} \otimes u : L_p(B(\ell_2)\overline{\otimes} \mathcal{M}) \to L_p(B(\ell_2)\overline{\otimes} \mathcal{N})$ is bounded. In this case,

$$\|u\|_{\text{cb}} = \|\text{id}_{S_p} \otimes u : L_p(\mathcal{M}) \to L_p(\mathcal{N})\|.$$

Here $S_p$ denotes the Schatten $p$-class, namely, the noncommutative $L_p$-space associated to $B(\ell_2)$ equipped with the usual trace. The readers who are not very familiar with operator space theory can take this property as the definition of c.b. maps between noncommutative $L_p$-spaces.

Now we turn to Fourier multipliers on quantum tori. Let $\varphi = (\varphi_m)_{m \in \mathbb{Z}^d} \subset \mathbb{C}$. We define $T_\varphi$ by

$$T_\varphi x(m) = \varphi_m \hat{x}(m), \quad \forall m \in \mathbb{Z}^d,$$

for any polynomial $x \in \mathcal{P}_\theta$. We call $\varphi$ a bounded $L_p$ multiplier (resp. c.b. $L_p$ multiplier) on the quantum torus $\mathbb{T}_\theta^d$ if $T_\varphi$ extends to a bounded (resp. c.b.) map on $L_p(T_\theta^d)$. The space of all $L_p$ multipliers (resp. c.b. $L_p$ multipliers) on $\mathbb{T}_\theta^d$ is denoted by $M(L_p(T_\theta^d))$ (resp. $M_{\text{cb}}(L_p(T_\theta^d))$), equipped with the natural norm (resp. c.b. norm). When $\theta = 0$, we recover the Fourier multipliers on the usual $d$-torus $\mathbb{T}^d$. The corresponding multiplier spaces are denoted by $M(L_p(\mathbb{T}^d))$ and $M_{\text{cb}}(L_p(\mathbb{T}^d))$, respectively.

The following remark summarizes some easily checked basic properties of quantum Fourier multipliers. We only state them for c.b. case, although all of them are equally valid for bounded multipliers.

**Remark 7.1.** Let $1 \leq p, p' \leq \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$.

i) $M_{\text{cb}}(L_p(T_\theta^d))$ is a Banach algebra under pointwise multiplication.

ii) $M_{\text{cb}}(L_p(T_\theta^d)) = M_{\text{cb}}(L_\infty(\mathbb{T}^d))$.

iii) $M_{\text{cb}}(L_2(T_\theta^d)) \subset M_{\text{cb}}(L_p(T_\theta^d))$, a contractive inclusion for $2 \leq p \leq q \leq \infty$.

iv) $M_{\text{cb}}(L_2(T_\theta^d)) = M(L_2(\mathbb{T}^d)) = L_\infty(\mathbb{Z}^d)$ with equal norms.
It is well-known that in the classical case Fourier multipliers are closely related to Schur multipliers. We will exploit such a relation in the quantum case too. To this end we first recall the definition of Schur multipliers. Let $\Lambda$ be an index set. The elements of $B(\ell_2(\Lambda))$ are represented by infinite matrices in the canonical basis of $\ell_2(\Lambda)$. A complex function $\psi = (\psi_{ab})$ on $\Lambda \times \Lambda$ (or matrix indexed by $\Lambda$) is called a bounded Schur multiplier on $B(\ell_2(\Lambda))$ if for every operator $a = (a_{st}) \in B(\ell_2(\Lambda))$, the matrix $(\psi_{st}a_{st})$ represents a bounded operator on $\ell_2(\Lambda)$. We then denote $M_\psi a = (\psi_{st}a_{st})$. In this case, $M_\psi$ is necessarily bounded on $B(\ell_2(\Lambda))$. More generally, for $1 \leq p \leq \infty$, if $M_\psi$ induces a bounded map on the Schatten $p$-class $S_p(\ell_2(\Lambda))$ based on $\ell_2(\Lambda)$, we call $\psi$ a bounded Schur multiplier on $S_p(\ell_2(\Lambda))$. Similarly, we define the completely boundedness of $M_\psi$.

Fourier and Schur multipliers are linked together via Toeplitz matrices. As usual, we represent $T_d^\phi$ as a von Neumann algebra on $L_2(T_d^\phi)$ by left multiplication. For every $x \in T_d^\phi$, let $[x]$ denote the representation matrix of $x$ on $\ell_2(\mathbb{Z}^d)$ in the orthonormal basis $(U^m)_{m \in \mathbb{Z}^d}$. Namely,

$$[x] = \left( (xU^m, U^m) \right)_{m,n \in \mathbb{Z}^d}.$$  

Let $\tilde{\theta}$ be the following $d \times d$-matrix deduced from the skew symmetric matrix $\theta$:

$$\tilde{\theta} = -2\pi \begin{pmatrix} 0 & \theta_{12} & \theta_{13} & & \theta_{1d} \\ 0 & 0 & \theta_{23} & & \theta_{2d} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \theta_{d-1,d} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$  

Then by the commutation relation (2.1), we have

$$xU^n = \sum_k \hat{x}(k)U^kU^n = \sum_k \hat{x}(k)U_1^{k_1} \cdots U_d^{k_d}U_1^{n_1} \cdots U_d^{n_d} = \sum_k \hat{x}(k) e^{i\tilde{\theta}k^t} U^{k+n},$$

where $n = (n_1, \ldots, n_d)$, $k^t$ is the transpose of $k = (k_1, \ldots, k_d)$ and $n\tilde{\theta}k^t$ denotes the matrix product. Thus

$$[x] = \left( \hat{x}(m-n)e^{i\tilde{\theta}(m-n)^t} \right)_{m,n \in \mathbb{Z}^d}.$$  

If $\theta = 0$, $[x]$ is a Toeplitz matrix. In the general case, $[x]$ is a twisted Toeplitz matrix.

For $\phi = (\phi_m)_{m \in \mathbb{Z}^d} \in \ell_\infty(\mathbb{Z}^d)$, we have

$$[T_\phi x] = \left( \phi_{m-n}\hat{x}(m-n)e^{i\tilde{\theta}(m-n)^t} \right)_{m,n \in \mathbb{Z}^d} = M_\phi([x]),$$

where $\tilde{\phi}_m = \phi_{m-n}$. This is the link between the Fourier and Schur multipliers associated to $\phi$. This link remains valid for operators $x$ in $B(\ell_2(\mathbb{Z}^d))$. In this case, the entries of the twisted Toeplitz matrix $[x]$ are operators in $B(\ell_2)$. To illustrate the usefulness of the relationship above, let us show the following simple result.

**Proposition 7.2.** We have

$$M_{cb}(T_d^\phi) = M_{cb}(L_\infty(T_d^\phi)) = M(L_\infty(T_d^\phi)) \quad \text{with equal norms.}$$

**Proof.** The argument below is standard. Let $\Gamma_\infty$ denote the subspace of $B(\ell_2(\mathbb{Z}^d))$ consisting of all twisted Toeplitz matrices of the form (7.1). By the preceding discussion, for any $x \in T_d^\phi$ we have

$$\|T_\phi(x)\|_\infty = \|T_\phi(x)\|_{B(L_2(T_d^\phi))} = \|[T_\phi(x)]\|_{B(\ell_2(\mathbb{Z}^d))} = \|M_\phi[x]\|_{B(\ell_2(\mathbb{Z}^d))}.$$  

Consequently,

$$T_\phi \text{ is bounded on } T_d^\phi \iff M_\phi|_{\Gamma_\infty} : \Gamma_\infty \rightarrow \Gamma_\infty \text{ is bounded.}$$

Moreover, in this case,

$$\|T_\phi\| = \|M_\phi|_{\Gamma_\infty}\|.$$  

Considering the vector-valued case where $x \in B(\ell_2(\mathbb{Z}^d))$, we get the c.b. analogue of the above equivalence:

$$T_\phi \text{ is c.b. on } T_d^\phi \iff M_\phi|_{\Gamma_\infty} \text{ is c.b. on } \Gamma_\infty \quad \text{and} \quad \|T_\phi\|_{cb} = \|M_\phi|_{\Gamma_\infty}\|_{cb}.$$  

Thus, if $M_\phi$ is c.b. on $B(\ell_2(\mathbb{Z}^d))$, then $M_\phi|_{\Gamma_\infty}$ is c.b. on $\Gamma_\infty$, so is $T_\phi$ on $T_d^\phi$. 


Conversely, suppose \( \phi \in M_{cb}(\mathbb{T}_d^d) \). Let \( V = \text{diag}(\cdots, U^n, \cdots) \) be the diagonal matrix with diagonal entries \( (U^n)_{n \in \mathbb{Z}^d} \). \( V \) is a unitary operator in \( \mathcal{B}(\ell_2(\mathbb{Z}^d) \otimes \mathbb{T}_d^d) \). For any \( a = (a_{mn})_{m,n \in \mathbb{Z}^d} \in \mathcal{B}(\ell_2(\mathbb{Z}^d)) \), let \( x = V(a \otimes 1_{\mathbb{T}_d^d})V^* \in \mathcal{B}(\ell_2(\mathbb{Z}^d) \otimes \mathbb{T}_d^d) \), where \( 1_{\mathbb{T}_d^d} \) denotes the unit of \( \mathbb{T}_d^d \). Then \( x = \sum_{m,n \in \mathbb{Z}^d} a_{mn}e_{mn} \otimes U^mU^{-n} = \sum_{m,n \in \mathbb{Z}^d} a_{mn}e_{mn} \otimes e^{-\imath \theta mn}U^mU^{-n} \), where \( (e_{mn}) \) are the canonical matrix units of \( \mathcal{B}(\ell_2(\mathbb{Z}^d)) \). Since \( V \) is unitary, we have
\[
|\|x\|_{\mathcal{B}(\ell_2(\mathbb{Z}^d) \otimes \mathbb{T}_d^d)} = |\|a\|_{\mathcal{B}(\ell_2(\mathbb{Z}^d))}.
\]
On the other hand,
\[
|\|x\|_{\mathcal{B}(\ell_2(\mathbb{Z}^d) \otimes \mathbb{T}_d^d)} = |\|a\|_{\mathcal{B}(\ell_2(\mathbb{Z}^d))}.
\]
Therefore, \( \phi \) is a bounded Schur multiplier on \( \mathcal{B}(\ell_2(\mathbb{T}_d^d)) \). Considering matrices \( a = (a_{mn})_{m,n \in \mathbb{Z}^d} \) with entries in \( \mathcal{B}(\ell_2) \), i.e., \( a = (a_{mn})_{m,n \in \mathbb{Z}^d} \in \mathcal{B}(\ell_2(\mathbb{T}_d^d)) \), we show in the same way that \( M_\phi \) is c.b. on \( \mathcal{B}(\ell_2(\mathbb{T}_d^d)) \), so \( \phi \) is a c.b. Schur multiplier on \( \mathcal{B}(\ell_2(\mathbb{Z}^d)) \) and \( \|M_\phi\|_{cb} \leq \|T_\phi\|_{cb} \).

In summary, we have proved that
\[
T_\phi \text{ is c.b. on } \mathbb{T}_d^d \iff M_\phi \text{ is c.b. on } \mathcal{B}(\ell_2(\mathbb{T}_d^d)).
\]
Applying this result to the commutative case \( (\theta = 0) \), we get that
\[
T_\phi \text{ is c.b. on } L_\infty(\mathbb{T}_d^d) \iff M_\phi \text{ is c.b. on } \mathcal{B}(\ell_2(\mathbb{T}_d^d)) \).
\]
Therefore,
\[
M_{cb}(\mathbb{T}_d^d) = M_{cb}(L_\infty(\mathbb{T}_d^d)) \quad \text{with equal norms.}
\]
However, it is well known that a Fourier multiplier \( \phi \) is bounded on \( L_\infty(\mathbb{T}_d^d) \) iff it is the Fourier transform of a bounded Borel measure \( \mu \) on \( \mathbb{T}_d^d \). In this case, \( T_\phi \) is the convolution operator by \( \mu \) and its norm is equal to \( \|\mu\| \). Then it is easy to check that \( T_\phi \) c.b. on \( L_\infty(\mathbb{T}_d^d) \). Thus
\[
M_{cb}(L_\infty(\mathbb{T}_d^d)) = M(L_\infty(\mathbb{T}_d^d)) \quad \text{with equal norms.}
\]
Combining the preceding results, we deduce the announced assertion.

The main result of this section is the following theorem, which extends the first equality in the previous proposition to all \( 1 \leq p \leq \infty \). We point out that the inclusion \( M_{cb}(L_p(\mathbb{T}_d^d)) \subset M_{cb}(L_p(\mathbb{T}_d^d)) \) was proved independently by Junge, Mei and Parcet [13].

**Theorem 7.3.** Let \( 1 < p < \infty \). Then \( M_{cb}(L_p(\mathbb{T}_d^d)) = M_{cb}(L_p(\mathbb{T}_d^d)) \) with equal norms.

**Proof.** The inclusion \( M_{cb}(L_p(\mathbb{T}_d^d)) \subset M_{cb}(L_p(\mathbb{T}_d^d)) \) can be easily proved by transference. Indeed, let \( \phi \in M_{cb}(L_p(\mathbb{T}_d^d)) \), and let \( x \in L_p(\mathcal{B}(\ell_2(\mathbb{T}_d^d))) \) be a polynomial in \( U \):
\[
x = \sum_{m \in \mathbb{Z}^d} \hat{x}(m) \otimes U^m,
\]
where only a finite number of coefficients \( \hat{x}(m) \) are nonzero operators in \( S_p \). Let
\[
\tilde{x}(z) = \sum_{m \in \mathbb{Z}^d} \hat{x}(m) \otimes U^mz^m, \quad z \in \mathbb{T}_d^d.
\]
Then \( \tilde{x} \in L_p(\mathbb{T}_d^d; L_p(\mathcal{B}(\ell_2(\mathbb{T}_d^d)))) \) and
\[
T_\phi(\tilde{x}) = \tilde{T}_\phi(\tilde{x}),
\]
where the first \( T_\phi \) is viewed as a multiplier on \( \mathbb{T}_d^d \) and the second on \( \mathbb{T}_d^d \). Recall that \( \mathbb{T}_d^d \) is hyperfinite, so the algebra \( \mathcal{B}(\ell_2(\mathbb{T}_d^d)) \) can be approximated by matrix algebras. Therefore, the complete boundedness of \( T_\phi \) on \( L_p(\mathbb{T}_d^d) \) implies
\[
|\|T_\phi(x)\|_{L_p(\mathbb{T}_d^d)}| \leq \|\phi\|_{M_{cb}(L_p(\mathbb{T}_d^d))} \|\tilde{x}\|_{L_p(\mathbb{T}_d^d)} \|T_\phi(\tilde{x})\|_{L_p(\mathbb{T}_d^d)}.
\]
However, by Corollary 2.2,
\[ \|T_\phi(x)\|_{L_p(T^{d^c}, L_p(\mathcal{B}(\mathcal{L}_p(\mathcal{T}_p^2))) = \|T_\phi(x)\|_{L_p(\mathcal{B}(\mathcal{L}_p(\mathcal{T}_p^2)))} \]
and
\[ \|\tilde{x}\|_{L_p(T^{d^c}, L_p(\mathcal{B}(\mathcal{L}_p(\mathcal{T}_p^2))) = \|x\|_{L_p(\mathcal{B}(\mathcal{L}_p(\mathcal{T}_p^2)))}. \]
Thus
\[ \|T_\phi(x)\|_p \leq \|\phi\|_{M_{cb}(L_p(\mathcal{T}_p^2)))} \|x\|_p. \]
Whence \( T_\phi \) is c.b., so \( \phi \in M_{cb}(L_p(\mathcal{T}_p^2))) \) and \( \|\phi\|_{M_{cb}(L_p(\mathcal{T}_p^2)))} \leq \|\phi\|_{M_{cb}(L_p(\mathcal{T}_p^2))). \)

For the converse inclusion, note that the argument in the second part of the proof of Proposition 7.2 works equally at the level of \( L_p \)-spaces. Thus we get that
\[ T_\phi \text{ is c.b. on } L_p(\mathcal{T}_p^2) \implies M_\phi \text{ is c.b. on } S_p(\ell_2(\mathbb{Z}^d)). \]

Then using Neuwirth and Ricard's transference theorem \[30], we deduce that \( T_\phi \) is c.b. on \( L_p(\mathcal{T}_p^d) \), so \( M_{cb}(L_p(\mathcal{T}_p^d))) \subset M_{cb}(L_p(\mathcal{T}_p^d))) \) contractively.

However, for reason of completeness, we include a self-contained proof in the spirit of the proof of Proposition 7.2 by adapting Neuwirth and Ricard's argument to the present setting of twisted Toeplitz matrices. Moreover, this proof does not need the first part above. Let
\[ (\mathbb{Z}_N) \text{ is a Følner sequence of } \mathbb{Z}^d, \text{ that is, } \lim_{N \to \infty} \frac{|Z_N \Delta (Z_N + n)|}{|Z_N|} = 0, \quad \forall n \in \mathbb{Z}^d. \]

Define two maps \( A_N \) and \( B_N \) as follows:
\[ A_N : \ell_2^d \rightarrow B(\ell_2^{|Z_N|}) \text{ with } x \mapsto P_N([x]), \]
where \( P_N : B(\ell_2(\mathbb{Z}^d)) \rightarrow B(\ell_2^{|Z_N|}) \) with \((a_{mn}) \mapsto (a_{mn})_{m,n \in \mathbb{Z}_N} \). And
\[ B_N : B(\ell_2^{|Z_N|}) \rightarrow \ell_2^d \text{ with } e_{mn} \mapsto \frac{1}{|Z_N|} e^{-in\delta(m-n)'} \sum_{m,n \in \mathbb{Z}_N} U^{m-n} e^{-in\delta(m-n)'} e^{in\delta(m-n)'} . \]

Here \( B(\ell_2^{|Z_N|}) \) is endowed with the normalized trace. It is easy to check that both \( A_N, B_N \) are unital, completely positive and trace preserving. Consequently, \( A_N \) extends to a complete contraction from \( L_p(\mathcal{T}_p^d)) \) into \( L_p(B(\ell_2^{|Z_N|})) \), while \( B_N \) a complete contraction from \( L_p(B(\ell_2^{|Z_N|})) \) into \( L_p(\mathcal{T}_p^d)). \)

We now claim that \( \lim_{N \to \infty} B_N \circ A_N(x) = x \) in \( L_p(\mathcal{T}_p^d)) \) for any \( x \in L_p(\mathcal{T}_p^d)) \). It suffices to consider a monomial \( x = U^k \). Then
\[ A_N(U^k) = (e^{in\delta(m-n)'} u_{m,n})_{m,n \in \mathbb{Z}_N, m-n = k}, \]
which implies
\[ B_N \circ A_N(U^k) = \frac{1}{|Z_N|} \sum_{m,n \in \mathbb{Z}_N, m-n = k} U^{m-n} e^{-in\delta(m-n)'} e^{in\delta(m-n)'} U^k. \]

Then by the Følner property of \( Z_N \), we deduce that \( \lim_{N \to \infty} B_N \circ A_N(U^k) = U^k \) in \( L_p(\mathcal{T}_p^d)). \) So the claim is proved.

Now assume that the Schur multiplier \( M_\phi \) is c.b. on \( S_p(\ell_2(\mathbb{Z}^d)). \) We want to prove that \( T_\phi \) is c.b. on \( L_p(\mathcal{T}_p^d)) \).
\[ \|\text{id} \otimes T_\phi(x)\|_{L_p(B(\ell_2) \overline{\otimes} \mathcal{T}_p^d)) = \lim_{N} \|\text{id} \otimes B_N \circ (\text{id} \otimes A_N)(\text{id} \otimes T_\phi(x))\|_{L_p(B(\ell_2) \overline{\otimes} \mathcal{T}_p^d))}. \]

Using (2.2), we see that \( \text{id} \otimes A_N(\text{id} \otimes T_\phi(x)) = \text{id} \otimes M_\phi(\text{id} \otimes A_N(x)). \) Thus
\[ \|\text{id} \otimes T_\phi(x)\|_{L_p(B(\ell_2) \overline{\otimes} \mathcal{T}_p^d)) \leq \lim_{N} \|\text{id} \otimes M_\phi(\text{id} \otimes A_N(x))\|_{L_p(B(\ell_2) \overline{\otimes} \mathcal{T}_p^d))} \leq \lim_{N} \|M_\phi\|_{cb} \|\text{id} \otimes A_N(x)\|_{L_p(B(\ell_2) \overline{\otimes} \mathcal{T}_p^d))} \leq \|M_\phi\|_{cb} \|x\|_{L_p(B(\ell_2) \overline{\otimes} \mathcal{T}_p^d))}. \]
This implies that $T_\phi$ is c.b. on $L_p(T^d_\theta)$ and $\|T_\phi\|_{cb} \leq \|M_\phi\|_{cb}$, as desired.

In summary, we have proved that

$$T_\phi \text{ is c.b. on } L_p(T^d_\theta) \iff M_\phi \text{ is c.b. on } S_p(\ell_2(\mathbb{Z}^d)).$$

Applying this result to the case of $\theta = 0$, we get that

$$T_\phi \text{ is c.b. on } L_p(T^d) \iff M_\phi \text{ is c.b. on } S_p(\ell_2(\mathbb{Z}^d)).$$

Therefore,

$$M_{cb}(L_p(T^d_\theta)) = M_{cb}(L_p(T^d)) \text{ with equal norms.}$$

Thus the theorem is proved.

\begin{remark}
The preceding proof shows that $\phi$ is a c.b. Fourier multiplier on $L_p(T^d_\theta)$ iff $\tilde{\phi}$ is a c.b. Schur multiplier on $S_p(\ell_2(\mathbb{Z}^d))$. This is the extension of Newirth and Ricard’s transference result to twisted Toeplitz matrices. We will pursue this subject elsewhere for more general groups.
\end{remark}

\begin{remark}
It would be interesting to study thin sets on $T^d_\theta$, for instance, $\Lambda(p)$-sets and Sidon sets. At the level of complete boundedness, Theorem 7.3 shows that the $\Lambda(\infty)$-sets are exactly those on $T^d$. We refer to Harcharras' thesis [12] for related results.
\end{remark}

\begin{problem}
Let $2 < p \leq \infty$. Does one have

$$M(L_p(T^d_\theta)) = M(L_p(T^d)) ?$$

We conjecture that the answer would be negative. Indeed, it is negative in the case of $p = \infty$ if one allows the number of generators to be infinite, as shown by the following remark that is communicated to us by Eric Ricard.

\begin{remark}
Let $\theta = (\theta_{kj})$ be the infinite skew matrix such that $\theta_{kj} = 1/2$ for all $k < j$. Let $T^\infty_\theta$ be the associated quantum torus. Now the generators of $T^\infty_\theta$ is a sequence $U = (U_1, U_2, \cdots)$ of anticommuting unitary operators:

$$U_k U_j = -U_j U_k, \quad \forall k \neq j.$$ 

Let $\phi$ be the indicator function of the subset $\Lambda = \{e_k : k \geq 1\}$ of $\mathbb{Z}^\infty$, where $e_k$ is the element of $\mathbb{Z}^\infty$ whose coordinates all vanish except the one on the k-th position which is equal to 1. Then $\phi \in M(L_\infty(T^\infty_\theta))$ but $\phi \not\in M(L_\infty(T^\infty)).$

Let us check this remark. Let $\alpha = (\alpha_k) \subset \mathbb{C}$ be a finite sequence and set

$$x = \sum_k \alpha_k U_k.$$ 

Then by the anticommuting relation we have

$$x^* x + xx^* = 2 \sum_k |\alpha_k|^2 + \sum_{j \neq k} \alpha_j \alpha_k (U_j^* U_k + U_k U_j^*) = 2 \sum_k |\alpha_k|^2.$$

It then follows that

$$\|x\|_\infty \leq \sqrt{2} \|\alpha\|_2.$$ 

On the other hand, it is clear that

$$\|x\|_\infty \geq \|x\|_2 \geq \|\alpha\|_2.$$ 

We then deduce that for any $\alpha = (\alpha_k) \subset \mathbb{C}$ the series $\sum_k \alpha_k U_k$ converges in $T^\infty_\theta$ iff $\alpha \in \ell_2$. In this case, we have

$$\|\alpha\|_2 \leq \|\sum_k \alpha_k U_k\|_\infty \leq \sqrt{2} \|\alpha\|_2.$$ 

This clearly implies that $\phi$ is a bounded $L_\infty$ multiplier on $T^\infty_\theta$. However, $\phi$ is not a bounded $L_\infty$ multiplier on $T^\infty$. Otherwise, the closed subspace of $L_\infty(T^\infty)$ generated by the generators $(z_1, z_2, \cdots)$ would be complemented in $L_\infty(T^\infty)$. But this subspace is isometric to $\ell_1$. It is well known that $\ell_1$ cannot be isomorphic to a complemented subspace of an $L_\infty$-space. This contradiction yields that $\phi \not\in M(L_\infty(T^\infty))$. This example also shows that

$$M_{cb}(L_\infty(T^\infty_\theta)) \subset \not\subset M(L_\infty(T^\infty)),$$
in contrast with equality (7.3) in the commutative case.

We end this section by showing the equality $M_{cb}(L_p(T^d_0)) = M_{cb}(L_p(T^d))$ in Theorem 7.3 holds completely isometrically. To this end we first need to equip these spaces with an operator space structure. Recall that for two operator spaces $E$ and $F$ the space $CB(E,F)$ has a natural operator space structure by setting $M_n(CB(E,F)) = CB(E,M_n(F))$. Then $M_{cb}(L_p(T^d_0))$ inherits the operator space structure of $CB(L_p(T^d_0), L_p(T^d_0))$. Let $TM_{cb}(S_p(ℓ_2(ℤ^d)))$ be the subspace of all c.b. Schur multipliers $ψ$ on $S_p(ℓ_2(ℤ^d))$ which are of the Toeplitz form, i.e., $ψ_{mn} = φ_{m-n}$ for some $φ$. $TM_{cb}(S_p(ℓ_2(ℤ^d)))$ is also an operator space via $TM_{cb}(S_p(ℓ_2(ℤ^d))) ⊂ CB(S_p(ℓ_2(ℤ^d)), S_p(ℓ_2(ℤ^d)))$.

**Proposition 7.8.** Let $1 ≤ p ≤ ∞$. Then

$$M_{cb}(L_p(T^d_0)) = M_{cb}(L_p(T^d)) \cong TM_{cb}(S_p(ℓ_2(ℤ^d)))$$

completely isometrically, where the last identification is realized by $φ \in M_{cb}(L_p(T^d)) \leftrightarrow \hat{φ} \in TM_{cb}(S_p(ℓ_2(ℤ^d)))$ with $φ_{mn} = φ_{m-n}$. 

**Proof.** We require the following elementary fact: Let $M$ be a von Neumann algebra and $u$ a unitary operator in $M_n(M)$. Then for any $x \in M_n(L_p(M))$

$$\|uxu^*\|_{M_n(L_p(M))} = \|x\|_{M_n(L_p(M))}.$$ 

Indeed, this is obvious for $p = ∞$. Then by duality, it is also true for $p = 1$. Finally, by interpolation, we deduce this equality for any $1 < p < ∞$. Armed with this fact, we can modify the proof of Theorem 7.3 to get the announced assertion. The details are left to the reader. □

### 8. HARDY SPACES

There exist several ways to define Hardy spaces on quantum tori. The resulting spaces may be different. The approach that we adopt in this section is based on the Littlewood-Paley theory and real variable method in Fourier analysis. Our Hardy spaces are defined by square functions in terms of the circular Poisson semigroup $P_r$. This allows us to use the recent developments of operator-valued harmonic analysis and noncommutative Littlewood-Paley-Stein theory.

For any $x \in T^d_0$ define

$$H_c(x) = \left( \int_0^1 \left| \frac{d}{dr} P_r[x] \right|^2 (1-r)dr \right)^{1/2}.$$ 

For $1 ≤ p < ∞$ let

$$\|x\|_{H_p} = |\hat{x}(0)| + \|H_c(x)\|_{L_p(T^d_0)}.$$ 

This is a norm on $T^d_0$ (cf. e.g. [15]). We define the column Hardy space $H_c^c(T^d_0)$ as the completion of $T^d_0$ with respect to this norm. The row Hardy space $H^r_p(T^d_0)$ is defined to be the space of all $x$ such that $x^* \in H^r_p(T^d_0)$ equipped with the natural norm. The mixture Hardy spaces are defined as follows: If $1 ≤ p < 2,$

$$H_p(T^d_0) = H_c^c(T^d_0) + H^r_p(T^d_0)$$

equipped with the sum norm

$$\|x\|_{H_p} = \inf \{ \|a\|_{H_p} + \|b\|_{H_p} : x = a + b, a \in H_c^c(T^d_0), b \in H^r_p(T^d_0) \},$$

and if $2 ≤ p < ∞,$

$$H_p(T^d_0) = H_c^c(T^d_0) \cap H^r_p(T^d_0)$$

equipped with the intersection norm

$$\|x\|_{H_p} = \max \{ \|x\|_{H_p}, \|x\|_{H_p} \}.$$ 

We will also study the BMO spaces over $T^d_0$. Set

$$\text{BMO}^r(T^d_0) = \{ x \in L_2(T^d_0) : \sup_r \| P_r[|x - P_r[x]|^2] \|_∞ < ∞ \}$$

equipped with the norm

$$\|x\|_{\text{BMO}^r} = \max \{ |\hat{x}(0)|, \sup_r \| P_r[|x - P_r[x]|^2] \|^{1/2}_∞ \}.$$
BMO′(T^d_θ) is defined as the space of all x such that x∗ ∈ BMO′(T^d_θ) with the norm ∥x∥_{BMO′} = ∥x∗∥_{BMO′}. The mixture BMO(T^d_θ) is the intersection of these two spaces:
\[ \text{BMO}(T^d_θ) = \text{BMO}′(T^d_θ) \cap \text{BMO}′(T^d_θ) \]
with the intersection norm.

The above definitions are motivated by Hardy spaces of noncommutative martingales ([22, 37]) and of quantum Markov semigroups ([15, 17, 27]). The main results of this section are summarized in the following statement which shows that the Hardy spaces on T^d_θ possess the properties of the usual Hardy spaces, as expected.

**Theorem 8.1.**

i) Let 1 < p < ∞. Then H_p(T^d_θ) = L_p(T^d_θ) with equivalent norms.

ii) The dual space of H^1(T^d_θ) is equal to BMO′(T^d_θ) with equivalent norms via the duality bracket
\[ \langle x, y \rangle = \tau(xy^*), \quad x \in L_2(T^d_θ), \ y \in \text{BMO}′(T^d_θ). \]
The same assertion holds for the row and mixture spaces too.

iii) Let 1 < p < ∞. Then
\[ (\text{BMO}′(T^d_θ), H^1(T^d_θ))_{1/p} = H^1_p(T^d_θ), \ H^1(T^d_θ))_{1/p}, \]
with equivalent norms, where \((\cdot, \cdot)_1/p\) and \((\cdot, \cdot)_{1/p,p}\) denote respectively the complex and real interpolation functors.

iv) Let 1 < p < ∞ and X_0 ∈ {BMO(T^d_θ), L_∞(T^d_θ)}, X_1 ∈ \{H^1(T^d_θ), L_1(T^d_θ)\}. Then
\[ (X_0, X_1)_{1/p} = L_p(T^d_θ) = (X_0, X_1)_{1/p,p} \]
with equivalent norms.

Some parts of this theorem can be deduced from existing results in literature. This is the case of i) and the complex interpolation equality (BMO(T^d_θ), L_1(T^d_θ))_{1/p} = L_p(T^d_θ) in iv). Let us explain these two points.

According to the discussion following Theorem 4.2, the circular Poisson semigroup \((P_r)_{0≤r<1}\) on T^d_θ is a noncommutative symmetric diffusion semigroup in the sense of [23]. We claim that \((P_r)_{0≤r<1}\) admits a Markov dilation (as well as a Rota dilation) in the sense of [13]. Indeed, considering the von Neumann subalgebra \(T^d_θ\) of \(L_∞(T^d)\otimes T^d_θ\), which is the image of \(T^d_θ\) under the map \(x \mapsto \tilde{x}\), we see that the circular Poisson semigroup on the usual torus T^d extends to a semigroup by tensoring with id_{T^d}. By a slight abuse of notation, we will also use \((P_r)_{0≤r<1}\) to denote the circular Poisson semigroup on the usual torus T^d. It is clear that \(P_r \otimes id_{T^d} \tilde{x} = P_r \tilde{x}\) for any \(x \in T^d_θ\). Since every symmetric diffusion semigroup on a commutative von Neumann algebra can be dilated to a Markov unitary group as well as an inverse martingale, \((P_r \otimes id_{T^d})_{0≤r<1}\) admits a Markov/Rota dilation, so does its restriction to \(T^d_θ\). Our claim then follows. Therefore, the semigroup \((P_r)_{0≤r<1}\) on T^d_θ satisfies the assumption of [13] which insures the existence of an associated \(H_∞\)-functional calculus. Thus by [13, Theorem 7.6], we get i). On the other hand, the interpolation theorem of [17] yields \((\text{BMO}(T^d_θ), L_1(T^d_θ))_{1/p} = L_p(T^d_θ)\). We also point out that the duality result in part ii) could be deduced from a work in progress of Avsec and Mei [11].

To prove the remaining parts of Theorem 8.1 we will use transference to reduce the problem to the corresponding one on N_θ and then use Mei’s results [27]. An advantage of this proof is that it also provides an alternative (more elementary) approach to the two parts already considered in the previous paragraph. Recall that the framework of [27] is the Euclidean space \(\mathbb{R}^d\), and the Hardy spaces there are defined by using the Poisson semigroup on \(\mathbb{R}^d\). The geometry of \(\mathbb{R}^d\) is simpler than T^d_θ. But what really renders matters more handy in \(\mathbb{R}^d\) is the explicit compact formula of the Poisson kernel (or its growth estimates). The situation for T^d_θ is harder. Although it is claimed in [27] as remarks that all results there hold equally with essentially the same proofs in the d-torus setting, this claim is clearly true for T thanks to the explicit simple formula of the Poisson kernel of T. However, it would not be so transparent whenever \(d ≥ 2\). As a byproduct of our proof below of Theorem 8.1 we remedy this situation, which constitutes another advantage of our approach via transference. Finally, it seems that even in the scalar case there does not exist published references on Hardy space theory on T^d for \(d ≥ 2\) via the Littlewood-Paley theory, although this theory is
Lemma 8.2. The analogue of the usual Garsia norm. This lemma is a special case of [17, Theorem 2.9].

Theorem 8.1. We start with the BMO space. Let

\[ \sup_{r} \| f - P_r f \|^{2} \]

be easily checked by (8.7) below. Then we deduce that

\[ \| f \|_{\text{BMO}} = \max\left\{ \| \hat{f}(0) \|_{\infty}, \sup_{r} \| P_r[f - P_r f]^{2} \|_{\infty}^{1/2} \right\} . \]

Here the first \( L_{\infty} \)-norm is the one of \( M \) and the second that of \( L_{\infty}(T^{d}) \circ \text{M} \).

Theorem 8.1. We start with the BMO space. Let

\[ \text{BMO}^{(d)}(T^{d}; M) = \{ f \in L_{2}(T^{d}; L_{2}(M)) : \sup_{r} \| f - P_r f \|^{2} \|_{\infty} < \infty \}, \]

equipped with the norm

\[ \| f \|_{\text{BMO}} = \max\left\{ \| \hat{f}(0) \|_{\infty}, \sup_{r} \| P_r[f - P_r f]^{2} \|_{\infty}^{1/2} \right\} . \]

Lemma 8.2. For any \( f \in L_{2}(T^{d}; L_{2}(M)) \) we have

(8.1) \[ \sup_{r} \| P_{r}[f^{2}] - |P_{r}[f]|^{2} \|_{\infty} \approx \sup_{r} \| P_{r}[f^{2}] - |P_{r}[f]|^{2} \|_{\infty} \]

with universal equivalence constants.

Proof. First note that

\[ P_{r}[f^{2}](z) - |P_{r}[f]|^{2}(z) = P_{r}[f - P_{r}[f](z)]^{2}(z), \quad \forall z \in T^{d}. \]

Thus

(8.2) \[ \sup_{0 \leq r < 1} \left\| P_{r}[f^{2}] - |P_{r}[f]|^{2} \right\|_{\infty} = \sup_{0 \leq r < 1} \sup_{z \in T^{d}} \left\| P_{r}[f - P_{r}[f](z)]^{2}(z) \right\|_{M}. \]

The right hand side is exactly the analogue of the usual Garsia norm (cf. [10 Corollary VI.2.4]).

For any fixed \( r \) and \( z \) we have

\[ \left\| P_{r}[f - P_{r}[f](z)]^{2}(z) \right\|_{M}^{1/2} \leq \left\| P_{r}[f - P_{r}[f](z)]^{2}(z) \right\|_{M}^{1/2} + \left\| P_{r}[P_{r}[f - P_{r}[f](z)]^{2}(z) \right\|_{M}^{1/2}. \]

By Kadison’s Cauchy-Schwarz inequality,

\[ P_{r}[P_{r}[f - P_{r}[f](z)]^{2}(z) \leq P_{r} \left| f - P_{r}[f](z) \right|^{2}. \]

On the other hand, since \( P_{r} \) is subordinated to the heat semigroup on \( T^{d} \), by the subordination formula, one has \( P_{r}[g] \leq 2P_{r}[g] \) for positive \( g \in L_{1}(T^{d}; L_{1}(M)) \). Alternatively, this inequality can be easily checked by [8.7] below. Then we deduce that

\[ \sup_{r} \sup_{z} \left\| P_{r}[f - P_{r}[f](z)]^{2}(z) \right\|_{M}^{1/2} \leq (1 + \sqrt{2}) \sup_{r} \sup_{z} \left\| P_{r}[f - P_{r}[f](z)]^{2}(z) \right\|_{M}^{1/2}. \]

This is the upper estimate of (8.1).
The converse inequality is harder. Fix \( f \in L^2(T^d; L^2(\mathcal{M})) \). By triangle inequality, we have

\[
\|P_{r}[|f|^2] - |P_{r}[f]|^2\|_{\infty}^{1/2} \leq \|P_{r}[|f - P_{r}[f]|^2] - |P_{r}[f - P_{r}[f]]|^2\|_{\infty}^{1/2} \\
+ \|P_{r}[|P_{r}[f]|^2] - |P_{r}[P_{r}[f]]|^2\|_{\infty}^{1/2} \\
\leq \|P_{r}[|f - P_{r}[f]|^2]\|_{\infty}^{1/2} \\
+ \|P_{r}[|P_{r}[f]|^2] - |P_{r}[P_{r}[f]]|^2\|_{\infty}^{1/2}.
\]

Assuming for the moment the following inequality

(8.3) \[2P_{r}[|P_{r}[f]|^2] \leq P_{r^2}[|f|^2] + |P_{r^2}[f]|^2,\]

we get

\[2(P_{r}[|P_{r}[f]|^2] - |P_{r^2}[f]|^2) \leq P_{r^2}[|f|^2] - |P_{r^2}[f]|^2.\]

Combining the preceding inequalities, we then deduce that

\[\sup_{r} \|P_{r}[|f|^2] - |P_{r}[f]|^2\|_{\infty}^{1/2} \leq \sup_{r} \|P_{r}[|f - P_{r}[f]|^2]\|_{\infty}^{1/2} + \frac{1}{\sqrt{2}} \sup_{r} \|P_{r}[|f|^2] - |P_{r}[f]|^2\|_{\infty}^{1/2}.
\]

Whence the lower estimate of (8.2) with \( 2 + \sqrt{2} \) as constant.

It remains to prove (8.3). To this end, it is more convenient to work with \( Q_{\varepsilon} = P_{r} \) for \( r = e^{-2\pi \varepsilon} \).

Then we must show

(8.4) \[Q_{\varepsilon}[|Q_{\varepsilon}[f]|^2] - |Q_{2\varepsilon}[f]|^2 \leq Q_{2\varepsilon}[|f|^2] - Q_{\varepsilon}[|Q_{\varepsilon}[f]|^2], \quad \forall \varepsilon > 0.
\]

Let us write

\[Q_{\varepsilon}[|Q_{\varepsilon}[f]|^2] - |Q_{2\varepsilon}[f]|^2 = -\int_{0}^{\varepsilon} \frac{d}{dt} Q_{\varepsilon-t}[Q_{\varepsilon+t}[f]]^{2}dt.
\]

Let \( A \) be the negative generator of \( Q_{\varepsilon} \): \( Q_{\varepsilon} = e^{-\varepsilon A} \). Then

\[
\frac{d}{dt} Q_{\varepsilon-t}[Q_{\varepsilon+t}[f]] = AQ_{\varepsilon-t}[Q_{\varepsilon+t}[f]] \\
- Q_{\varepsilon-t}[(AQ_{\varepsilon+t}[f])^*(Q_{\varepsilon+t}[f])] + (Q_{\varepsilon+t}[f])^*(AQ_{\varepsilon+t}[f]).
\]

For \( s > 0 \) let

\[F_s(g) = -AQ_{s}[|Q_{s}[g]|^2] + Q_{s}[(AQ_{s}[g])^*(Q_{s}[g]) + (Q_{s}[g])^*(AQ_{s}[g])].
\]

Then for \( g = Q_{\varepsilon+t}[f] \) we have

(8.5) \[Q_{\varepsilon}[|Q_{\varepsilon}[f]|^2] - |Q_{2\varepsilon}[f]|^2 = \lim_{s \to 0} \int_{0}^{\varepsilon} Q_{\varepsilon-t}[F_{s}(g)]dt.
\]

It is easy to check that \( \lim_{s \to \infty} F_{s}(g) = 0 \) (one can use, for instance, (8.7) below). Then

(8.6) \[F_{s}(g) = -\int_{s}^{\infty} \frac{d}{du} F_{u}(g)du.
\]

Elementary calculations lead to

\[
\frac{d}{du} F_{u}(g) = A^2 Q_{u}[|Q_{u}[g]|^2] - Q_{u}[(A^2 Q_{u}[g])^*(Q_{u}[g]) + (Q_{u}[g])^*(A^2 Q_{u}[g])] - 2Q_{u}[|AQ_{u}[g]|^2] \\
= Q_{u}[A^2|Q_{u}[g]|^2 - (A^2 Q_{u}[g] + (Q_{u}[g]^*(A^2 Q_{u}[g]) - (Q_{u}[g])^*(A^2 Q_{u}[g]) - 2|A Q_{u}[g]|^2).
\]

Note that

\[A = 2\pi \sqrt{-\Delta},
\]

where \( \Delta \) is the Laplacian of \( T^d \):

\[\Delta = \sum_{k=1}^{d} \frac{\partial^2}{\partial z_k^2}.
\]

So \( A^2 = -4\pi^2 \Delta \) and

\[A^2|Q_{u}[g]|^2 = (A^2 Q_{u}[g])^*(Q_{u}[g]) + (Q_{u}[g])^*(A^2 Q_{u}[g]) - 8\pi^2 \sum_{k=1}^{d} \left| \frac{\partial}{\partial z_k} Q_{u}[g] \right|^2.
\]
Therefore,
\[ \frac{d}{dt} F_u(g) = -8\pi^2 \sum_{k=1}^d Q_u \left[ \frac{\partial}{\partial z_k} Q_u [g] \right]^2 - 2Q_u [A Q_u [g]]^2. \]
Recall that \( g = Q_{e^2} f \). By Kadison’s Cauchy-Schwarz inequality and using the above equality twice, we obtain
\[ -\frac{d}{dt} F_u(g) \leq Q_{\varepsilon} \left[ 8\pi^2 \sum_{k=1}^d Q_u \left[ \frac{\partial}{\partial z_k} Q_u [h] \right]^2 + 2Q_u [A Q_u [h]]^2 \right] \leq -Q_{\varepsilon} \left[ \frac{d}{dt} F_u (h) \right], \]
where \( h = Q_{e^2} f \). Thus by (8.6),
\[ F_u(g) \leq Q_{\varepsilon} [F_u(h)]. \]
Hence by (8.5) and inverting the procedure leading to (8.5), we obtain
\[ \frac{\partial}{\partial t} \frac{Q_{\varepsilon} [f]}{\varepsilon} \leq \lim_{s \to 0} \int_0^t \frac{Q_{2\varepsilon} f (h) dt}{\varepsilon} = -\int_0^t \frac{\partial}{\partial t} Q_{2\varepsilon} \left[ Q_{\varepsilon} [f] \right]^2 dt = Q_{2\varepsilon} [f]^2 - Q_{\varepsilon} [f]^2. \]
This yields (8.7), and (8.8) too. Thus the lemma is proved.

Although this is not really necessary, it is more convenient to work with the cube \( I^d = [0, 1]^d \) instead of \( \mathbb{T}^d \). Another reason is that the case of \( I^d \) is closer to that of \( \mathbb{R}^d \). So we will identify \( \mathbb{T}^d \) with \( I^d \), as in the proof of Theorem 1.2. The addition in \( I^d \) is modulo 1 coordinatewise, which corresponds to the multiplication in \( \mathbb{T}^d \) under the identification \( (e^{2\pi i s_1}, \ldots, e^{2\pi i s_d}) \leftrightarrow (s_1, \ldots, s_d) \). Accordingly, functions on \( \mathbb{T}^d \) and \( I^d \) are identified too. Thus \( L_p (\mathbb{T}^d; L_p (M)) = L_p (I^d; L_p (M)) \).

We will use the following Poisson summation formula (see [14, Corollary VII.2.6]):
\[ P_r(z) = \sum_{m \in \mathbb{Z}^d} \varphi_{\varepsilon} (s + m) \quad \text{with} \quad z = (e^{2\pi i s_1}, \ldots, e^{2\pi i s_d}) \quad \text{and} \quad r = e^{-2\pi i z}, \]
where \( \varphi_{\varepsilon} \) is the Poisson kernel on \( \mathbb{R}^d 
\[ \varphi_{\varepsilon} (s) = \frac{e}{(e^2 + |s|^2)^{(d+1)/2}}, \quad s = (s_1, \ldots, s_d) \in \mathbb{R}^d. \]
In the sequel, we will always assume that \( z \) and \( s, r \) and \( \varepsilon \) are related as in (8.7). Let
\[ (8.8) \quad Q_{\varepsilon} (s) = \sum_{m \in \mathbb{Z}^d} \varphi_{\varepsilon} (s + m), \quad s \in I^d. \]
This notation is consistent with that introduced during the proof of Lemma 8.2 since
\[ (8.9) \quad P_r[f](z) = Q_{\varepsilon} [f](s) = Q_{\varepsilon} * f (s) = \int_{I^d} Q_{\varepsilon} (s-t) f(t) dt. \]
An interval of \( I \) is either a subinterval of \( I \) or a union \([b, 1] \cup [0, a] \) with \( 0 < a < b < 1 \). The latter union is the interval \([b-1, a] \) by the addition modulo 1 of \( I \). So the intervals of \( I \) correspond exactly to the arcs of \( T \). A cube of \( I^d \) is a product of \( d \) intervals. For \( f \in L_1(I^d; L_1(M)) \) and a cube \( Q \subset I^d \) let
\[ f_Q = \frac{1}{|Q|} \int_Q f ds, \]
where \( |Q| \) denotes the volume of \( Q \). Then we define \( \text{BMO}^e(I^d; M) \) as the space of all \( f \in L_2(I^d; L_2(M)) \) such that
\[ \sup_{Q \subset I^d \text{cube}} \frac{1}{|Q|} \int_Q \left| f - f_Q \right|^2 ds \leq \infty, \]
equipped with the norm
\[ \| f \|_{\text{BMO}} = \max \{ \| f_{Q} \|_\infty, \quad \sup_{Q \subset I^d \text{cube}} \| 1 \int_Q |f - f_Q|^2 ds \|_\infty^{1/2} \}. \]
Here \( \| \|_\infty \) denotes, of course, the norm of \( M \).

**Lemma 8.3.** \( \text{BMO}^e(\mathbb{T}^d; M) = \text{BMO}^e(I^d; M) \) with equivalent norms.
Proof. Fix \( f \in L_2(T^d; L_2(M)) \). Without loss of generality, assume that \( f(0) = f_{\zeta} = 0 \). By Lemma 6.2 and 6.4, we need to show

\[
\sup_{\varepsilon > 0} \sup_{s \in \mathbb{Z}^d} \| Q_{\varepsilon} |f - Q_{\varepsilon} f(s)|^2 \|_{\infty} \leq \sup_{Q \in \text{cube}} \frac{1}{|Q|} \int_Q |f - f_Q|^2 \, dt \|_{\infty}.
\]

Let \( Q \) be a cube of \( I^d \). Let \( s \) and \( \varepsilon \) be the center and half of the side length of \( Q \), respectively. It is clear that

\[
\frac{1}{|Q|} \mathbf{1}_Q(t) \leq C_d \varphi_{\varepsilon}(s - t) \leq C_d \varphi_{\varepsilon}(s - t).
\]

Thus

\[
\frac{1}{|Q|} \int_Q |f(t) - Q_{\varepsilon} f(t)|^2 \, dt \leq C_d \varphi_{\varepsilon} \| |f - Q_{\varepsilon} f(s)|^2 \|_{L^2}.
\]

Then

\[
\frac{1}{|Q|} \int_Q |f(t) - f_Q|^2 \, dt \leq \frac{1}{|Q|} \int_Q |f(t) - Q_{\varepsilon} f(t)|^2 \, dt \leq 4C_d \varphi_{\varepsilon} \| |f - Q_{\varepsilon} f(s)|^2 \|_{L^2}.
\]

This yields one inequality of (8.10).

To show the converse inequality fix \( s \in I^d \) and \( \varepsilon > 0 \). Consider first the case \( \varepsilon \geq 1/2 \). Then \( Q_{\varepsilon}(t) \approx 1 \) for any \( t \in I^d \). It follows that

\[
Q_{\varepsilon} |f - Q_{\varepsilon} f(s)|^2 \|_{L^2} \leq \int_{I^d} |f - Q_{\varepsilon} f(s)|^2 \leq \int_{I^d} |f|^2.
\]

Whence

\[
\| Q_{\varepsilon} |f - Q_{\varepsilon} f(s)|^2 \|_{L^2} \leq \| \| f \|_{L^2}^2 \|_{L^2} \| f \|_{BMO(I^d)}^2.
\]

Now assume \( \varepsilon < 1/2 \). By the proof of Theorem 4.2 for any \( t \in I^d \)

\[
\sum_{m \neq 0} \varphi_{\varepsilon}(t + m) \leq \varepsilon < \varphi_{\varepsilon}(t).
\]

Consequently,

\[
Q_{\varepsilon} |f - Q_{\varepsilon} f(s)|^2 \|_{L^2} \leq \int_{I^d} \varphi_{\varepsilon}(s - t) f(t) - Q_{\varepsilon} f(t) |^2 \, dt.
\]

Let \( Q = \{ t \in I^d : |t - s| \leq \varepsilon \} \) and \( Q_k = \{ t \in I^d : |t - s| \leq 2^{k+1} \varepsilon \} \). Then

\[
\int_{I^d} \varphi_{\varepsilon}(s - t) f(t) - f_Q |^2 \, dt = \int_{Q_k} \varphi_{\varepsilon}(s - t) f(t) - f_Q |^2 \, dt + \sum_{k \geq 0} \int_{I^d} \varphi_{\varepsilon}(s - t) f(t) - f_Q |^2 \, dt.
\]

The above sums on \( k \) are in fact finite sums. By triangle inequality (with \( Q_{-1} = Q \)),

\[
\frac{1}{|Q_k|} \int_{Q_k} |f - f_Q|^2 \|_{L^2}^{1/2} \leq \frac{1}{|Q_k|} \int_{Q_k} |f - f_{Q_{k+1}}|^2 \|_{L^2}^{1/2} + \sum_{j=0}^k \| f_{Q_j} - f_{Q_{j+1}} \|_{L^2}.
\]

However,

\[
\| f_{Q_j} - f_{Q_{j+1}} \|_{L^2} \leq \frac{1}{|Q_{j+1}|} \int_{Q_{j+1}} |f - f_{Q_{j+1}}|^2 \|_{L^2} \leq 2^d \| f \|_{BMO(I^d)}^2 \| f \|_{BMO(I^d)}^2.
\]

Combining the preceding inequalities, we obtain

\[
\| Q_{\varepsilon} |f - Q_{\varepsilon} f|^2 \|_{L^2} \leq \sum_{k \geq 0} \frac{k+1}{2^k} \| f \|_{BMO(I^d)}^2 \| f \|_{BMO(I^d)}^2.
\]

Finally,

\[
\| Q_{\varepsilon} |f - Q_{\varepsilon} f|^2 \|_{L^2} \leq 2 \| Q_{\varepsilon} |f - Q_{\varepsilon} f|^2 \|_{L^2} \leq 2 \| Q_{\varepsilon} |f - Q_{\varepsilon} f|^2 \|_{L^2} \| f \|_{BMO(I^d)}^2.
\]
This implies the missing inequality of (8.10). □

**Remark 8.4.** The previous proof shows implicitly that the supremum on $\varepsilon$ in (8.10) can be restricted to $0 < \varepsilon < 1$. In fact, only small values of $\varepsilon$ are important for this supremum. Accordingly, only values of $r$ close to 1 matter in the two suprema in (8.11). This property can be also verified by the argument in the proof of Lemma 8.3 below.

Functions on $T^d$ are 1-periodic functions on $R^d$, or equivalently, functions on $l^d$ can be extended to 1-periodic functions to $R^d$. For a function $f$ on $T^d$ (or $l^d$) $\hat{f}$ will denote the corresponding 1-periodic function on $R^d$. Then (8.11) implies that $Q_\varepsilon[f]$ is equal to the Poisson integral of $\hat{f}$ on $R^d$ that will be denoted by $\varphi_\varepsilon[\hat{f}]$. Let us record this useful fact here for later reference:

(8.11) \[ Q_\varepsilon[f] = \varphi_\varepsilon[\hat{f}] = \varphi_\varepsilon * \hat{f} \quad \text{on} \quad l^d. \]

Recall that $\text{BMO}^c(R^d; M)$ is defined as the space of all locally square integrable functions $\psi$ from $R^d$ to $L_2(M)$ such that

$$\|\psi\|_{\text{BMO}^c} = \max \{ \|\psi_\varepsilon\|_\infty, \sup_{Q \subset R^d} \|\int_Q (\psi - \psi_\varepsilon) \|^2 \|ds\|_\infty \}. $$

The following lemma shows that the map $f \mapsto \hat{f}$ establishes an isomorphic embedding of $\text{BMO}^c(T^d; M)$ into $\text{BMO}^c(R^d; M)$.

**Lemma 8.5.** For any $f \in \text{BMO}^c(T^d; M)$ we have

$$\|f\|_{\text{BMO}^c(T^d; M)} \approx \|\hat{f}\|_{\text{BMO}^c(R^d; M)}$$

with equivalence constants depending only on $d$.

**Proof.** Let $f \in L_2(T^d; L_2(M))$ with $f_\varepsilon = 0$. By (8.10) and (8.11), we have

$$\|f\|_{\text{BMO}^c(T^d; M)}^2 \approx \sup_{\varepsilon > 0} \sup_{s \in R^d} \|\varphi_\varepsilon[\hat{f} - \varphi_\varepsilon[s]\]s\|_\infty. $$

Then the proof of Lemma 8.3 shows that the right hand side above is equivalent to $\|\hat{f}\|_{\text{BMO}^c(R^d; M)}^2$. Alternately, one can directly prove that the supremum on the right hand side in (8.10) is equivalent to $\|\hat{f}\|_{\text{BMO}^c(R^d; M)}^2$. Namely,

$$\sup_{Q \subset R^d} \|\int_Q (\hat{f} - \hat{f}_Q)^2 ds\|_\infty \approx \sup_{Q \subset R^d} \|\int_Q (\hat{f} - \hat{f}_Q)^2 ds\|_\infty. $$

Indeed, let $Q$ be a cube in $R^d$. If $|Q| \leq 1$, then by the definition of cubes in $l^d$ and the periodicity of $\hat{f}$, $\hat{Q}$ can be considered as a cube in $l^d$. So assume $|Q| > 1$. Take another cube $R$ such that $Q \subset R$, $|R| \leq 2^d|Q|$ and the side length of $R$ is an integer $k$. Then $R$ is a union of $k^d$ cubes of side length 1. Thus by the periodicity of $\hat{f}$

$$\frac{1}{|Q|} \int_Q (\hat{f} - \hat{f}_Q)^2 ds \leq \frac{4}{|Q|} \int_Q (\hat{f}^2 ds \leq \frac{2^{d+2}}{|R|} \int_R (\hat{f}^2 ds = 2^{d+2} \int_{l^d} (\hat{f}^2 ds. $$

Therefore, we get the desired equivalence. □

Now we turn to the discussion of Hardy spaces. Let $1 \leq p < \infty$. For $f \in L_\infty(T^d; l^d; M)$ define

(8.12) \[ G_\varepsilon(f)(z) = \left( \int_0^1 \left| \frac{d}{dr} p_\varepsilon(f)(z) \right|^2 (1 - r) dr \right)^{1/2}, \quad z \in T^d \]

and

$$\|f\|_{H^p} = \|\hat{f}(0)\|_p + \|G_\varepsilon(f)\|_p. $$

Here the first $L_p$-norm is the one of $L_p(M)$ and the second that of $L_p(T^d; l_p(M))$. Completing $L_\infty(T^d; l^d; M)$ under the norm $\|f\|_{H^p}$, we get $H^p(T^d; M)$. Like in the BMO case, we wish to reduce these Hardy spaces to those on $l^d$. Using the kernel $Q_\varepsilon$ in (8.8), for $f \in L_\infty(l^d; l^d; M)$ let

(8.13) \[ \tilde{G}_\varepsilon(f)(s) = \left( \int_0^\infty \left| \frac{d}{ds} Q_\varepsilon(f)(s) \right|^2 ds \right)^{1/2}, \quad s \in l^d. \]
Let \( \tilde{f} \) be the periodic extension of \( f \) to \( \mathbb{R}^d \). Let \( \tilde{G}_c(\tilde{f}) \) be the \( g \)-function of \( \tilde{f} \) defined by the Poisson kernel \( \varphi_c \):

\[
\tilde{G}_c(\tilde{f})(s) = \left( \int_0^\infty \left| \frac{d}{dx} \varphi_c[f](s) \right|^2 dx \right)^{1/2}, \quad s \in \mathbb{R}^d.
\]

Thanks to (8.11), we have

\[
\tilde{G}_c(f) = \tilde{G}_c(\tilde{f}) \quad \text{on } \mathbb{R}^d.
\]

Thus \( \tilde{G}_c(\tilde{f}) \) is the periodic extension to \( \mathbb{R}^d \) of \( \tilde{G}_c(f) \). Let

\[
\|f\|_{H^p} = \|f_e\|_p + \|\tilde{G}_c(f)\|_p.
\]

Here the first \( L_p \)-norm is the one of \( L_p(\mathcal{M}) \) and the second that of \( L_p(\mathbb{R}^d; L_p(\mathcal{M})) \). Define \( H^p_p(\mathbb{R}^d; \mathcal{M}) \) to be the completion of \( (L_\infty(\mathbb{R}^d) \otimes \mathcal{M}, \| \cdot \|_{H^p_p}) \).

**Lemma 8.6.** Let \( 1 \leq p < \infty \). Then \( H^p_p(\mathbb{T}^d; \mathcal{M}) = H^p_p(\mathbb{R}^d; \mathcal{M}) \) with equivalent norms.

**Proof.** We first show that in the definition of the Littlewood-Paley function \( G_c(f) \) in (8.12) only values of \( r \) close to 1 matter. More precisely, for any \( 0 < r_0 < 1 \) setting

\[
G_{c,r_0}(f)(z) = \left( \int_{r_0}^1 \left| \frac{d}{dr} \mathbb{P}_r[f](z) \right|^2 (1 - r)dr \right)^{1/2},
\]

we have

\[
\|G_c(f)\|_p \approx \|G_{c,r_0}(f)\|_p,
\]

where the equivalence constants depend only on \( d \) and \( r_0 \). To this end take any \( 0 \leq r_0 < r_1 < 1 \) and let

\[
G'_c(f)(z) = \left( \int_{r_0}^{r_1} \left| \frac{d}{dr} \mathbb{P}_r[f](z) \right|^2 (1 - r)dr \right)^{1/2}.
\]

We claim that

\[
\|G'_c(f)\|_p \approx \sup_{n \in \mathbb{Z}^d, n \not= \mathbf{0}} \|\hat{f}(n)\|_p.
\]

Writing the Fourier series expansion of \( \mathbb{P}_r[f] \):

\[
\mathbb{P}_r[f](z) = \sum_{n \in \mathbb{Z}^d} r^{|n|} \hat{f}(n) z^n,
\]

we have

\[
\frac{d}{dr} \mathbb{P}_r[f](z) = \sum_{n \in \mathbb{Z}^d, n \not= \mathbf{0}} |n| r^{|n|-1} \hat{f}(n) z^n.
\]

We then easily get the upper estimate of (8.10). To show the lower one, for \( n \in \mathbb{Z}^d, n \not= \mathbf{0} \) we have

\[
|n| r^{|n|-1} \hat{f}(n) z^n = \int_{\mathbb{T}^d} \frac{d}{dr} \mathbb{P}_r[f](z \cdot w) w^{-n} dm(w).
\]

Let \( H = L_2((r_0, r_1); (1 - r)dr) \). It is clear that for any \( z \in \mathbb{T}^d \) we have

\[
\left( \int_{r_0}^{r_1} |n| r^{|n|-1} \hat{f}(n) z^n |^2 (1 - r)dr \right)^{1/2} \approx \|\hat{f}(n)\|.
\]

Then by the triangle inequality in the column \( L_p \)-space \( L_p(L_\infty(\mathbb{T}^d) \otimes \mathcal{M}; H^c) \), we deduce

\[
\|\hat{f}(n)\|_{L_p(\mathcal{M})} \lesssim \left( \int_{\mathbb{T}^d} \|G'_c(f)(z \cdot w)\|^p_{L_p(\mathcal{M})} dm(z) \right)^{1/p} dm(w) = \|G'_c(f)\|_{L_p(\mathbb{T}^d; L_p(\mathcal{M}))}.
\]

Thus the claim is proved. Using (8.10) twice, we get

\[
\|G_c(f)\|_p \leq \| \left( \int_0^1 \left| \frac{d}{dr} \mathbb{P}_r[f] \right|^2 (1 - r)dr \right)^{1/2} \|_p + \|G_{c,r_0}(f)\|_p
\]

\[
\lesssim \sup_{n \in \mathbb{Z}^d, n \not= \mathbf{0}} \|\hat{f}(n)\|_p + \|G_{c,r_0}(f)\|_p \lesssim \|G_{c,r_0}(f)\|_p.
\]
Similarly, we show that only small values of \( \varepsilon \) matter in (8.13) and (8.14). Namely, for \( 0 < \varepsilon_0 < \infty \) letting

\[
\tilde{G}_{c,\varepsilon_0}(f)(s) = \left( \int_0^{\varepsilon_0} \left| \frac{d}{d\varepsilon} \mathcal{Q}_\varepsilon[f](s) \right|^2 \varepsilon d\varepsilon \right)^{1/2}, \quad s \in \mathbb{R}^d,
\]

we have

\[
\|\tilde{G}_c(f)\|_p \approx \|\tilde{G}_{c,\varepsilon_0}(f)\|_p.
\]

Now it is easy to finish the proof of the lemma. Indeed, using the change of variables \( r = e^{-2\pi \varepsilon} \), we get

\[
G_{c,r_0}(f)(z) = \frac{1}{2\pi} \left( \int_0^{\varepsilon_0} \left| \frac{d}{d\varepsilon} \mathcal{Q}_\varepsilon[f](s) \right|^2 e^{2\pi\varepsilon} (1 - e^{-2\pi\varepsilon}) d\varepsilon \right)^{1/2} \approx \left( \int_0^{\varepsilon_0} \left| \frac{d}{d\varepsilon} \mathcal{Q}_\varepsilon[f](s) \right|^2 d\varepsilon \right)^{1/2} = \tilde{G}_{c,\varepsilon_0}(f)(s).
\]

Together with the previous equivalences, this implies the desired assertion.

We will also need the Lusin area integral function. For \( \alpha > 1 \) and \( z \in \mathbb{T}^d \), let \( \Delta_\alpha(z) \) be the Stoltz domain with vertex \( z \) and aperture \( \alpha \) (recalling that \( |w| \) denotes the Euclidean norm):

\[
\Delta_\alpha(z) = \{ w \in \mathbb{C}^d : |z - w| \leq \alpha (1 - |w|) \}.
\]

For \( f \in L_\infty(\mathbb{T}^d) \otimes \mathcal{M} \) define

\[
S_c^\alpha(f)(z) = \left( \int_{\Delta_\alpha(z)} \left| \frac{d}{dt} \mathcal{P}_r[f](rw) \right|^2 \frac{dm(w)dr}{(1 - r)^{d-1}} \right)^{1/2}, \quad z \in \mathbb{T}^d,
\]

where the integral is taken over \( \Delta_\alpha(z) \) with respect to \( rw \in \Delta_\alpha(z) \) with \( 0 \leq r < 1 \) and \( w \in \mathbb{T}^d \) (recalling that \( dm \) is Haar measure of \( \mathbb{T}^d \)).

Like for the \( g \)-function, we will transfer \( S_c^\alpha(f) \) to the usual area integral function on \( \mathbb{R}^d \). For \( \beta > 0 \) and \( s \in \mathbb{R}^d \) let

\[
\Gamma_\beta(s) = \{(t, \varepsilon) \in \mathbb{R}^d \times \mathbb{R}_+ : |t - s| \leq \beta \varepsilon \}.
\]

Let \( f \in L_\infty(\mathbb{R}^d) \otimes \mathcal{M} \) and \( \tilde{f} \) be its periodic extension to \( \mathbb{R}^d \). Define

\[
\tilde{S}_c^\beta(f)(s) = \left( \int_{\Gamma_\beta(s)} \left| \frac{d}{d\varepsilon} \mathcal{Q}_\varepsilon[f](t) \right|^2 \frac{dt d\varepsilon}{\varepsilon^{d-1}} \right)^{1/2}, \quad s \in \mathbb{R}^d.
\]

The following is the analogue of Lemma 8.6 for the Lusin square functions.

**Lemma 8.7.** Let \( \alpha > 1 \) and \( \beta > 0 \). Let \( f \in L_\infty(\mathbb{T}^d) \otimes \mathcal{M} \). Then

\[
\|S_c^\alpha(f)\|_{L_p(\mathbb{T}^d; L_p(\mathcal{M}))} \approx \|\tilde{S}_c^\beta(f)\|_{L_p(\mathbb{R}^d; L_p(\mathcal{M}))}
\]

with equivalence constants depending only on \( d \) and \( \alpha, \beta \). Moreover, the norms above are independent of \( \alpha \) and \( \beta \) up to equivalence.

**Proof.** This proof is similar to that of Lemma 8.6. For \( 0 < r_0 < 1 \) we introduce the truncated Stoltz domain:

\[
\Delta_{\alpha,r_0}(z) = \{ w \in \mathbb{C}^d : |z - w| \leq \alpha (1 - |w|), r_0 < |w| < 1 \}.
\]

Also for \( \varepsilon_0 > 0 \) set

\[
\Gamma_{\beta,\varepsilon_0}(s) = \{(t, \varepsilon) \in \mathbb{R}^d \times \mathbb{R}_+ : |t - s| \leq \beta \varepsilon, \varepsilon \leq \varepsilon_0 \}.
\]

The corresponding truncated square functions are

\[
S_{c,r_0}^\alpha(f)(z) = \left( \int_{\Delta_{\alpha,r_0}(z)} \left| \frac{d}{dt} \mathcal{P}_r[f](rw) \right|^2 \frac{dm(w)dr}{(1 - r)^{d-1}} \right)^{1/2}
\]

and

\[
\tilde{S}_{c,\varepsilon_0}^\beta(f)(s) = \left( \int_{\Gamma_{\beta,\varepsilon_0}(s)} \left| \frac{d}{d\varepsilon} \mathcal{Q}_\varepsilon[f](t) \right|^2 \frac{dt d\varepsilon}{\varepsilon^{d-1}} \right)^{1/2}.
\]

Then by the reasoning in the proof of Lemma 8.6, we have

\[
\|S_c^\alpha(f)\|_p \approx \|S_{c,r_0}^\alpha(f)\|_p.
\]
and a similar equivalence for $\tilde{S}_c^\beta(f)$. On the other hand, it is easy to see that for any $\alpha > 1$ and $0 < r_0 < 1$ there exist $\beta_1, \beta_2 > 0$ and $\varepsilon_1, \varepsilon_2 > 0$ such that under the change of variables $r = e^{-2\pi \varepsilon}$ and $w = e^{-2\pi \varepsilon t}$

$$\Gamma_{\beta_1, \varepsilon_1}(s) \subset \Delta_{\alpha, r_0}(z) \subset \Gamma_{\beta_2, \varepsilon_2}(s), \quad \forall z = e^{-2\pi \iota s} \in \mathbb{T}^d.$$  

Conversely, every truncated cone $\Gamma_{\beta, \varepsilon}(s)$ is located between two truncated Stoltz domains. Then the argument at the end of the proof of Lemma 8.6 implies

$$\tilde{S}_{c,\varepsilon_1}^\alpha(f)(s) \lesssim S_{c,\varepsilon_0}^\alpha(f)(z) \lesssim \tilde{S}_{c,\varepsilon_2}^\beta(f)(s);$$

whence

$$\|\tilde{S}_{c,\varepsilon_1}^\alpha(f)\|_{L_p([t_0, t_1]; L_p(M))} \lesssim \|S_{c,\varepsilon_0}^\alpha(f)\|_{L_p([t_0, t_1]; L_p(M))} \lesssim \|\tilde{S}_{c,\varepsilon_2}^\beta(f)\|_{L_p([t_0, t_1]; L_p(M))}.$$

However, standard arguments in harmonic analysis show that

$$\|\tilde{S}_c^\beta(f)\|_{L_p([t_0, t_1]; L_p(M))} \approx \|\tilde{S}_c^\beta(f)\|_{L_p([t_0, t_1]; L_p(M))},$$

where the equivalence constants depend on $d$ and $\beta_1, \beta_2$ (cf. e.g., [3]). Therefore, we deduce the first equivalence assertion of the lemma. The second part then follows too. 

Now we can show that the results of [27] remain valid for $\mathbb{T}^d$ too. We state only those relevant to Theorem 8.8. In the following statement, the row and mixture Hardy/BMO spaces are defined in the usual way, and $S_c(f) = S_c^\alpha(f), \tilde{S}_c(f) = \tilde{S}_c^\beta(f)$.

**Theorem 8.8.**

i) The dual space of $H_1^r(\mathbb{T}^d; M)$ coincides isomorphically with $\text{BMO}^r(\mathbb{T}^d; M)$ with the natural duality bracket. The same assertion holds for the row and mixture spaces.

ii) Let $1 \leq p < \infty$. Then for any $f \in L_\infty(\mathbb{T}^d; \mathcal{M})$

$$\|G_c(f)\|_p \approx \|S_c(f)\|_p$$

with relevant constants depending only on $d$. Consequently, the two square functions $G_c$ and $S_c$ define a same Hardy space.

iii) Let $1 < p < \infty$. Then $H_p(\mathbb{T}^d; M) = L_p(\mathbb{T}^d; L_p(M))$ with equivalent norms.

iv) Let $1 < p < \infty$. Then

$$(\text{BMO}^r(\mathbb{T}^d; M), H_1^r(\mathbb{T}^d; M))_{1/p} = H_p(\mathbb{T}^d; M) = (\text{BMO}^r(\mathbb{T}^d; M), H_1^r(\mathbb{T}^d; M))_{1/p,p}.$$

v) Let $X_0 \in \{\text{BMO}(\mathbb{T}^d; M), L_\infty(\mathbb{T}^d; L_p(M))\}, X_1 \in \{H_1(\mathbb{T}^d; M), L_1(\mathbb{T}^d; M)\}$. Then for any $1 < p < \infty$

$$(X_0, X_1)_{1/p} = L_p(\mathbb{T}^d; M) = (X_0, X_1)_{1/p,p}.$$

**Proof.** By the identification $\mathbb{T}^d \cong \mathbb{R}^d$ and Lemmas 8.3 and 8.4 it suffices to prove this theorem with $\mathbb{R}^d$ instead of $\mathbb{T}^d$. The geometry of $\mathbb{R}^d$ is closer to that of $\mathbb{R}^d$. Therefore, what makes our arguments parallel to those of [27] is the use of periodic functions. This periodization puts the arguments of [27] directly at our disposal. For any function $f$ on $\mathbb{R}^d$ with periodic extension $\tilde{f}$ to $\mathbb{R}^d$, by 8.19 and 8.21, we have

$$\tilde{G}_c(f) = \tilde{G}_c(\tilde{f}) \quad \text{and} \quad \tilde{S}_c(f) = \tilde{S}_c(\tilde{f}) \quad \text{on} \quad \mathbb{R}^d.$$  

Note that the two square functions on the right are exactly those introduced in [27] by using the Poisson kernel on $\mathbb{R}^d$. The only difference compared with [27] is that the $L_p$-norm of these square functions are now taken in $L_p([0, T]; L_p(M))$ instead of $L_p([0, T]; L_p(M))$. In other words, the integral is now taken on $[0, T]$ instead of $\mathbb{R}^d$. On the other hand, by Lemmas 8.3 and 8.4 the map $f \mapsto \tilde{f}$ is an isomorphic embedding of $\text{BMO}^r(\mathbb{R}^d; M)$ into $\text{BMO}^r(\mathbb{R}^d; M)$. It is now easy to see that the proof of [27] Theorem 2.4] is valid for periodic functions and integration on $\mathbb{R}^d$. Hence, we get the duality result in part i) and the equivalence for $p = 1$ in part ii). In the same way, we prove the periodic analogue of [27] Theorem 4.4], which implies the remaining case of ii). The reduction to dyadic martingales of [27] is clearly available in our present setting. The dyadic decomposition is now made in $\mathbb{R}^d$ (or equivalently, $\mathbb{T}^d$). In this way, we reduce parts iii)-v) to the martingale case as in [27]. The verification of all details is, however, tedious and lengthy, so it is more reasonable to skip it here. 

\[\square\]
Remark 8.9. It is stated in the final remark of [27, Chapter 2] that the relevant constants in part i) above for $\mathbb{R}^d$ are independent of $d$. This does not seem true. In fact, all constants appearing in Theorem 8.8 depend on $d$ (and on $p$ too), except those in part iii) since the semigroup argument described in the paragraph following Theorem 8.1 yields equivalence constants depending only on $p$. The same comment applies to Theorem 8.1 too. However, the constants there are independent of the given skew matrix $\theta$.

Remark 8.10. The $H_1$-BMO duality in Theorem 8.8 and Lemma 8.3 imply that $H^c_1(\mathbb{T}^d, \mathcal{M})$ admits an atomic decomposition like in the case of $\mathbb{R}^d$. We refer the interested reader to [27] for more details.

Armed with Theorem 8.8 and transference, it is easy to prove Theorem 8.1. To this end we still require the following simple lemma.

Lemma 8.11. The map $x \mapsto \tilde{x}$ in Corollary 2.2 extends to an isometric embedding from $H^c_1(\mathbb{T}^d, \mathcal{M})$ into $H^c_p(\mathbb{T}^d, \mathbb{T}^d)$ for any $1 \leq p < \infty$ and from $\text{BMO}_c(\mathbb{T}^d, \mathcal{M})$ into $\text{BMO}_c(\mathbb{T}^d, \mathbb{T}^d)$. Moreover, the images of this embedding are 1-complemented in their respective spaces.

Proof. The first part follows immediately from the identity $\mathbb{E}_x[\tilde{x}] = \mathbb{E}_x[x]$ for any $x \in L^1(\mathbb{T}^d)$. Identifying $\mathbb{T}^d_\theta$ with $\mathbb{T}^d$, we see that the conditional expectation $E$ from $H^c_p(\mathbb{T}^d, \mathbb{T}^d)$ onto $\mathbb{T}^d_\theta$ extends to a contractive projection from $H^c_p(\mathbb{T}^d; \mathbb{T}^d_\theta)$ onto $H^c_1(\mathbb{T}^d; \mathbb{T}^d_\theta)$ and from $\text{BMO}_c(\mathbb{T}^d; \mathbb{T}^d_\theta)$ onto $\text{BMO}_c(\mathbb{T}^d_\theta)$. This yields the second part. \hfill $\Box$

The proof of Theorem 8.1. It is now clear that Theorem 8.1 follows immediately from Theorem 8.8 (with $\mathcal{M} = \mathbb{T}^d_\theta$) and Lemma 8.11.

Remark 8.12. Since $\mathbb{T}^d_\theta \subset \text{BMO}(\mathbb{T}^d_\theta)$, part ii) of Theorem 8.1 implies that $H^c_1(\mathbb{T}^d_\theta) \subset L^1(\mathbb{T}^d_\theta)$ and $\|x\|_1 \leq C\|x\|_{H^c_1}, \ \forall \ x \in H^c_1(\mathbb{T}^d_\theta)$.

Acknowledgements. We are grateful to Tao Mei and Eric Ricard for useful discussions.

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Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, 30 West Strict, Xiao-Hong-Shan, Wuhan 430071, China

E-mail address: zqchen@mail.wipm.ac.cn

School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China and Laboratoire de Mathématiques, Université de Franche-Comté, 25030 Besançon Cedex, France

E-mail address: qxu@univ-fcomte.fr

School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China and Laboratoire de Mathématiques, Université de Franche-Comté, 25030 Besançon Cedex, France

E-mail address: hustycinzi@163.com