Purification of Single Qubits and Reconstruction from Post- Measurement State

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Purification of mixed states in Quantum Mechanics, by which we mean the transformation into pure states, has been viewed as an Operation in the sense of Kraus et al and explicit Kraus Operators [3–5] have been constructed for two separate purification protocols. The first one, initially due to Schrodinger [6] and subsequently elaborated by Sudarshan et al [7], is based on the preservation of probabilities. We have constructed a second protocol here based on optimization of fidelities. Both purification protocols have been implemented on a single qubit in an attempt to improve the fidelity of the purified post measurement state of the qubit with the initial pure state. We have considered both complete and partial measurements and have established bounds and inequalities for various fidelities. We show that our purification protocol leads to better state reconstruction, most explicitly so, when partial measurements are made.

I. KRAUS FORMALISM

Kraus et al have given an extensive formalism to study all possible changes of quantum states (a general quantum state can be described by a density matrix that can be pure or mixed). The most remarkable features of this formalism are an intrinsic algebraic structure from the mathematical point of view as well as the physically striking result that any quantum state can be changed to any other quantum state through the so-called Operations. An Operation $O$ is defined as follows: Consider a quantum system in the state $\rho_{\text{sys}}$ with a Hilbert Space $\mathcal{H}$ which is coupled to another quantum system, often called the environment, in the state $\rho_{E}$ and which has a state space $\mathcal{H}_{E}$. The system and the environment interact through a Unitary Evolution $U$ which acts on the total Hilbert Space $\mathcal{H} \otimes \mathcal{H}_{E}$. Now some property of the environment is selectively measured by a projection operator $Q_{E}$ so that the state becomes:

$$\hat{\rho} = (I \otimes Q_{E})U(\rho_{\text{sys}} \otimes \rho_{E})U^{\dagger}(I \otimes Q_{E}) \quad (1)$$

The system is then considered as an isolated system described by the reduced density matrix,

$$\hat{\rho}_{\text{sys}} = \text{Tr}_{E}\hat{\rho} \quad (2)$$

where the trace is taken over all possible states of the environment. The resulting state change $O : \rho_{\text{sys}} \rightarrow \hat{\rho}_{\text{sys}}$ is called an Operation. According to the Kraus formalism, this operation can be represented in terms of Kraus operators $A_{k}$ acting on the state space of the system such that

$$\hat{\rho}_{\text{sys}} = \sum_{k \in K} A_{k}\rho_{\text{sys}}A_{k}^{\dagger} \quad (3)$$

As the measurement $Q_{E}$ is selective, the $A_{k}$ operators satisfy the trace non-increasing condition

$$\sum_{k \in K} A_{k}A_{k}^{\dagger} \leq I \quad (4)$$

where $K$ is some indexing set. The operators $A_{k}$ are defined by

$$(f, A_{k}g) = ((f \otimes f_{k}^{E}), U(g \otimes g_{k}^{E})) \quad (5)$$

where $f, g$ arbitrary vectors in the state space of the system, $\{ f_{k}^{E} \}_{k \in K}$ form an orthonormal basis of $Q_{E}\mathcal{H}_{E}$ and $g_{k}^{E}$ is the pure state in which the environment can be assumed to have started in.

As remarked earlier, operations can connect any given pair of density matrices $\{ \rho_{1}, \rho_{2} \}$. In particular, one can go from an initial mixed state to a pure state. This may sound counter-intuitive as the process of going from an initially pure state to a mixed state, as happens in quantum measurement, is seen as an irreversible step with an increase of (von Neumann)entropy. In this sense, the reverse process of going from mixed state to a pure state may seem impossible. But, as is clear from the Kraus formalism, this cannot happen in any isolated system but needs coupling to another system. In classical thermodynamics also, the entropy of a sub-system can always decrease without any violation of the second law.

The process of going from a mixed state to a pure state is called Purification and was allegedly first considered by Schrodinger [6]. There is a vast literature on this topic [8]. In the current literature ‘Purification’ is often understood to be the process of associating a suitable pure state of a larger system whose reduced density matrix is the mixed state one started with, but for us purification of a mixed state is any protocol that produces a pure state from it.
II. KRAUS OPERATORS FOR QUBITS

Though only a selective measurement and one Unitary transformation was considered in arriving at equation (3), it is straightforward to generalize to any type of measurement and any Unitary transformation in different combinations. From now onwards, we shall relax the condition of selectivity in measurements and consider all possible outcomes for measurements (we only consider projective measurements here). We shall also be restricting ourselves to 2-level systems (qubits) only. Then one needs two Kraus operators for a general operation.

Any pair of operators,

\[ A_0 = (\alpha|0\rangle + \beta|1\rangle)\langle 0| \quad A_1 = (\alpha|0\rangle + \beta|1\rangle)\langle 1| \]  \hspace{1cm} (6)

satisfy

\[ A_0^\dagger A_0 + A_1^\dagger A_1 = \mathbf{I} \]  \hspace{1cm} (7)

where \( \alpha \) and \( \beta \) satisfy \( |\alpha|^2 + |\beta|^2 = 1 \). For any arbitrary density matrix \( \rho_{in} \), these operators produce

\[ \rho_{out} = A_0 \rho A_0^\dagger + A_1 \rho A_1^\dagger = \begin{pmatrix} |\alpha|^2 & \alpha \beta^* \\ \alpha^* \beta & |\beta|^2 \end{pmatrix} \]  \hspace{1cm} (8)

Clearly, \( \rho_{out} \) is a pure state and it is independent of the initial state \( \rho_{in} \). By eqn(5) the Kraus operators here are of the form,

\[ A_0 = \langle 0_E|U|0_E \rangle \quad A_1 = \langle 1_E|U|0_E \rangle \]  \hspace{1cm} (9)

where the environment is assumed to start in the pure state \( |0_E \rangle \). It is straightforward to check that the unitary operator \( U \) that generates the Kraus operators (6) for this operation is:

\[ U = ((\alpha|0\rangle + \beta|1\rangle)\langle 0|) \otimes |0_E\rangle\langle 0_E| \]

\[ + ((\alpha|0\rangle + \beta|1\rangle)\langle 1|) \otimes |1_E\rangle\langle 0_E| \]

\[ + (\alpha^*|1\rangle\langle 0| - \beta^*|0\rangle\langle 0|) \otimes |0_E\rangle\langle 1_E| \]

\[ + (\alpha^*|1\rangle\langle 1| - \beta^*|0\rangle\langle 1|) \otimes |1_E\rangle\langle 1_E| \]  \hspace{1cm} (10)

III. PURIFICATION PROTOCOL - A

Consider some density matrix which is a mixture of two orthogonal states

\[ \rho' = p_1 \rho_1 + p_2 \rho_2 \]  \hspace{1cm} (11)

where, \( p_1^2 = p_1, p_2^2 = p_2, tr(\rho_1 \rho_2) = 0, tr \rho_1 = tr \rho_2 = 1 \).

The Purification Protocol discussed here is based on the principle of preservation of probabilities. In [7] this was taken to mean that the overlap of \( \rho' \) with \( \rho_{1,2} \) is \( p_{1,2} \). This was first discussed by Schrodinger [6] and later elaborated by Sudarshan et al. Then this Purification Protocol leads to the family of pure states:

\[ \rho = p_1 \rho_1 + p_2 \rho_2 + \frac{\rho_1 \Pi \rho_2 + \rho_2 \Pi \rho_1}{\sqrt{tr(\rho_1 \Pi) tr(\rho_2 \Pi)}} \]  \hspace{1cm} (12)

where \( \Pi \) is a projection which is not orthogonal to either \( \rho_1 \) or \( \rho_2 \).

If \( \rho_1 = |0\rangle\langle 0|, \rho_2 = |1\rangle\langle 1| \) and \( \Pi \) is of the form

\[ \Pi = (\mu|0\rangle + \nu|1\rangle)\langle 0| \mu^* + \langle 1| \nu^* \]  \hspace{1cm} (13)

\( (\mu, \nu \neq 0 \) since \( \Pi \) is not orthogonal to either \( |0\rangle \langle 0| \) or \( |1\rangle \langle 1| \) then the purified state is given by

\[ \rho^\mathbb{A} = p_1 |0\rangle\langle 0| + p_2 |1\rangle\langle 1| + \frac{\rho_1 \rho_2 (e^{i\theta} |0\rangle\langle 1| + h.c)}{\sqrt{p_1 \rho_2}} \]  \hspace{1cm} (14)

where \( \phi \) is the phase of \( \mu \nu^* \) and h.c stands for Hermitian conjugate. The reason that only this phase appears in the purified \( \rho \) is that preservation of probabilities leaves only a phase left unspecified in a pure state. Different choices of \( \phi \) lead to different purified states. There is no principle that selects a particular value of \( \phi \). The Kraus operators which generate this operation are of the form (6):

\[ A_0 = \sqrt{p_1 e^{i\theta}} |0\rangle \langle 0| + \sqrt{p_2} |1\rangle \langle 1| \]

\[ A_1 = \sqrt{p_1 e^{i\theta}} |0\rangle \langle 1| + \sqrt{p_2} |1\rangle \langle 0| \]  \hspace{1cm} (15)

The Unitary transformation \( U \) generating these operators is of the form (10) with the substitutions: \( \alpha = \sqrt{p_1 e^{i\theta}} \) and \( \beta = \sqrt{p_2} \).

IV. PURIFICATION PROTOCOL - B

As shown in Sec. 2, any state (Pure or Mixed),

\[ \rho = \begin{pmatrix} a & p \\ p^* & 1 - a \end{pmatrix} \]  \hspace{1cm} (16)

can be purified to

\[ \rho_{out} = O(\rho) = \begin{pmatrix} |\alpha|^2 & \alpha \beta^* \\ \alpha^* \beta & |\beta|^2 \end{pmatrix} \]  \hspace{1cm} (17)

by using the Kraus operators of eqn(6). At this stage, \( \rho_{out} \) can be any state of the system and the purification scheme is too general. Now we adopt a principle different from the one followed in Protocol A to be able to fix the purified state. The Principle we adopt is that the purified state must have maximal overlap with the mixed state we started with. If we use the formula \( tr(\rho_1 \rho_2) \) for the distance between any two states \( (\rho_1, \rho_2) \), our principle is also equivalent to demanding that the purified state be as close as possible to the mixed state. The overlap between \( \rho \) and \( \rho_{out} \) is \( F = tr(\rho \rho_{out}) \). Thus,

\[ F = a |\alpha|^2 + p \alpha^* \beta + p^* \alpha \beta^* + (1 - a) |\beta|^2 \]  \hspace{1cm} (18)

Letting \( |\alpha|^2 = \tilde{p}, |\beta|^2 = 1 - \tilde{p}, \alpha \beta^* = \sqrt{\tilde{p}(1 - \tilde{p})} e^{-i\theta} \) and \( p = |p| e^{i\phi} \) one has
\[ F = a \hat{p} + (1 - a)(1 - \hat{\rho}) + 2|p|\sqrt{\hat{\rho}(1 - \hat{\rho})} \cos(\theta + \phi) \]  

(19)

The condition on the phase \( \theta \) for maximizing \( F \) is \( \theta = -\phi \). Then,

\[ F(\hat{\rho}) = a \hat{p} + (1 - a)(1 - \hat{\rho}) + 2|p|\sqrt{\hat{\rho}(1 - \hat{\rho})} \]  

(20)

which must be maximised w.r.t. \( \hat{\rho} \):

\[ F'(\hat{\rho}) = 2a - 1 + \frac{|p|(1 - 2\hat{\rho})}{\sqrt{\hat{\rho}(1 - \hat{\rho})}} = 0 \]  

(21)

It can be shown that \( F_{\text{max}} \) occurs for

\[ \hat{\rho} = (1/2)(1 - \frac{(1 - 2a)}{(4|p|^2 + (1 - 2a)^2)^{1/2}}) \]  

(22)

The case \( p = 0 \) needs to be handled separately. In that case

\[ F = a \hat{p} + (1 - a)(1 - \hat{\rho}) \]  

(23)

If \( a > 1/2 \) then \( \hat{\rho} = 1 \) gives \( F = a > 1/2 \) and \( \rho_{\text{out}} = |0\rangle\langle 0| \). However, if \( a < 1/2 \) then \( \hat{\rho} = 0 \) gives \( F = 1 - a > 1/2 \) and \( \rho_{\text{out}} = |1\rangle\langle 1| \). Finally if \( a = 1/2 \), then \( F = 1/2 \) for both the above \( \rho_{\text{out}}'s \) and no unique pure state can be picked.

V. MEASUREMENT OF SPIN AND IMPROVEMENT IN FIDELITY OF POST-MEASUREMENT STATE WITH INITIAL STATE BY PURIFICATION

An ensemble (\( N \) copies-\( N \) very large) of Spin - 1/2 particles is made. This ensemble is divided into 3 equal sub-ensembles and measurements are made of \( S_z \), \( S_y \) and \( S_x \) on the respective sub-ensembles. Let \( p_i \) be the probability for the outcome \( |\pm\rangle_x \), \( p_2 \) for \( |\pm\rangle_y \) and \( p_3 \) for \( |\pm\rangle_z \).

Let the basis for the 2-Dimensional Hilbert space be the eigen-vectors of \( \sigma_z \), \( |+\rangle_z \equiv (1, 0) \) and \( |-\rangle_z \equiv (0, 1) \). In this basis

\[ |\pm\rangle_x = (1/\sqrt{2}) \left( \begin{array}{c} 1 \\ \pm 1 \end{array} \right) \]  

(24)

Now, the three post-measurement density matrices are respectively:

\[ \rho_1 = p_1 |+\rangle_z \langle +| + (1 - p_1) |-\rangle_z \langle -| \]

\[ \rho_2 = p_2 |+\rangle_y \langle +| + (1 - p_2) |-\rangle_y \langle -| \]

\[ \rho_3 = p_3 |+\rangle_x \langle +| + (1 - p_3) |-\rangle_x \langle -| \]  

(25)

Now one takes an equal weightage of the three post-measurement density matrices to give \( \rho_{\text{msmt}} = (1/3)(\rho_1 + \rho_2 + \rho_3) \). Therefore,

\[ \rho_{\text{msmt}} = \frac{1}{6} \left( \begin{array}{cc} 2p_1 + 2 & (2p_3 - 1) + i(1 - 2p_2) \\ (2p_3 - 1) - i(1 - 2p_2) & 4 - 2p_1 \end{array} \right) \]  

(26)

Since this a complete measurement, the initial density matrix can be determined and is:

\[ \rho_{\text{ini}} = \left( \begin{array}{cc} p_1 & \frac{(2p_3 - 1) + i(1 - 2p_2)}{2} \\ \frac{(2p_3 - 1) - i(1 - 2p_2)}{2} & 1 - p_1 \end{array} \right) \]  

(27)

Clearly, the relation between \( \rho_{\text{msmt}} \) and \( \rho_{\text{ini}} \) is,

\[ \rho_{\text{msmt}} = (1/3)(\mathbf{I} + \rho_{\text{ini}}) \]  

(28)

In a separate publication we have established a result analogous to eqn(28) for arbitrary systems with finite dim \( \mathcal{H} \) [9]. The Fidelity of \( \rho_{\text{msmt}} \) with the initial state is:

\[ F(\rho_{\text{msmt}}, \rho_{\text{ini}}) = \text{tr}(\rho_{\text{msmt}} \rho_{\text{ini}}) = 2/3 \]  

(29)

independent of \( \rho_{\text{ini}} \). Since, \( \rho_{\text{ini}} \) is pure its eigen-values are 0, 1. Hence, the eigenvalues of \( \rho_{\text{msmt}} \) are, from (28), 1/3 and 2/3. Therefore \( \rho_{\text{msmt}} \) can be written as,

\[ \rho_{\text{msmt}} = \frac{2}{3}|l\rangle\langle l| + \frac{1}{3}|s\rangle\langle s| \]  

(30)

Substituting in eqn(28) and the completeness relation

\[ \mathbf{I} = |l\rangle\langle l| + |s\rangle\langle s| \]  

(31)

one finds

\[ \rho_{\text{ini}} = |l\rangle\langle l| \]  

(32)

Therefore, we have established that the eigenvector corresponding to the largest eigenvalue is the initial state. The Purification of

\[ \rho_{\text{msmt}} = (2/3)|l\rangle\langle l| + (1/3)|s\rangle\langle s| \]  

(33)

by Protocol - A is

\[ \rho_{\text{msmt}}^{(A)} = (2/3)|l\rangle\langle l| + (1/3)|s\rangle\langle s| + \sqrt{2}/3(e^{i\phi}|l\rangle\langle s| + e^{-i\phi}|s\rangle\langle l|) \]  

(34)

Now, the fidelity is,

\[ F(\rho_{\text{msmt}}^{(A)}, \rho_{\text{ini}}) = \langle l|\rho_{\text{msmt}}^{(A)}|l\rangle = 2/3 \]  

(35)

Comparing this result with (29) the Purification protocol - A does not improve the fidelity of the purified post-measurement state with the initial state for complete orthogonal measurements.

However, one could have taken a larger interpretation of protocol-A which would preserve all the \( p_1, p_2, p_3 \) in which case the purified state has to coincide with the initial state leading to a fidelity of unity.
However, protocol-B offers a different insight into the problem of complete and partial measurements. In fact, it is shown here that, for complete measurements, the initial or the pre-measurement state is the "closest" pure state (by the purification protocol-B) to the resultant post-measurement state (by taking an equal weightage of the three post-measurement states obtained from the three measurements).

Now, the purification of $\rho_{msmt}$ by protocol-B is

$$
\rho_{msmt}^{(B)} = \left( \frac{\tilde{\rho}}{\sqrt{\tilde{\rho}(1-\tilde{\rho})}e^{-i\phi}} \sqrt{\tilde{\rho}(1-\tilde{\rho})} e^{i\phi} \right) \tag{36}
$$

Here, $a = (p_1 + 1)/3$ and $p = \frac{(2p_1-1)+(1-2p_2)}{6}$. Therefore,

$$
|p| = \sqrt{\frac{(2p_1-1)^2+(1-2p_2)^2}{6}} \sin(\phi) = \sqrt{\frac{(2p_1-1)^2+(1-2p_2)^2}{6}},
$$

For complete measurements, $p_1$, $p_2$ and $p_3$ are related by,

$$
(2p_1 - 1)^2 + (2p_2 - 1)^2 + (2p_3 - 1)^2 = 1 \tag{37}
$$

It is easy to verify after some algebra that

$$
\Rightarrow \tilde{\rho} = p_1: \sqrt{\frac{(2p_1 - 1)^2 - (1-2p_2)^2}{2}} = \frac{(2p_1 - 1) + i(1-2p_2)}{2} \tag{38}
$$

In other words, $\rho_{msmt}^{(B)} = \rho_{ini}$. *The initial state is the closest pure state by Purification Protocol-B, in fidelity, to the mixed state $\rho_{msmt}$*. This also means that protocol-B has purified the post-measurement state to maximum fidelity with the initial state.

### A. Partial Measurements

The real issue is in the context of partial measurements where the initial state can not be unambiguously reconstructed. In this section we establish the following two results: (i) the purified state under protocol-B always has a greater fidelity with the pre-measurement state than does the post-measurement state, (ii) the fidelity of the purified state under protocol-B with the initial state is always greater than that of the purified state under protocol-A (in an unbiased average sense as protocol-A does not favour any single pure state) except in some singular cases where the fidelities are the same. Thus protocol-B is the better when trying to reconstruct the initial state from the post-measurement state.

Suppose two measurements are made. Let $p_1$ and $p_2 \equiv$ measurement results $|+\rangle_z$ and $|+\rangle_y$ respectively. Then post-measurement state,

$$
\phi_{msmt}^{(B)} = \left( \begin{array}{c}
\frac{(2p_1 + 1)/4}{i(1-2p_2)/4} \\
-i(1-2p_2)/4
\end{array} \right) \tag{39}
$$

Let, the initial state be $\psi = \alpha|+\rangle_z + \beta|\rangle_z$. Therefore, $p_1 = |\alpha|^2$, $p_2 = |\langle \psi |+\rangle|^2$ which is equivalent to $\beta^*\alpha = i(2p_2 - 1)$. From these relations we can compute the fidelity:

$$
F_1(\phi_{msmt}, \psi) = \frac{1}{4}((2p_1 - 1)^2 + (2p_2 - 1)^2 + 2) \tag{40}
$$

To purify the state by protocol-A, we can adopt the following procedure: We know that the initial state density matrix $\rho_{ini} = |\psi\rangle\langle\psi|$ is of the form,

$$
\rho_{ini} = I/2 + \langle S_x \rangle \sigma_x + \langle S_y \rangle \sigma_y + \langle S_z \rangle \sigma_z
$$

and the relation is,

$$
\langle S_x \rangle^2 + \langle S_y \rangle^2 + \langle S_z \rangle^2 = 1/4
$$

Therefore,

$$
\langle S_x \rangle = \pm \frac{1}{\sqrt{2}} \left( \frac{(2p_2 - 1)^2 - (2p_1 - 1)^2}{4} \right) \tag{41}
$$

where we use the relations $\langle S_x \rangle = (2p_1 - 1)/2$ and $\langle S_y \rangle = (2p_2 - 1)/2$. One of the signs for $\langle S_x \rangle$ gives the initial state so that the fidelity is 1 whereas the other choice gives some other state with a different fidelity. For argument’s sake we assume that the choice of positive root gives the initial state. Then,

$$
F_{2,a}(\rho_{ini}, \phi^{(A)}_{msmt,(S_x)+}) = 1 \tag{42}
$$

The fidelity when the negative root for $\langle S_x \rangle$ is chosen is,

$$
F_{2,b}(\rho_{ini}, \phi^{(A)}_{msmt,(S_x)-}) = \frac{1}{2} + 2\langle S_x \rangle - \langle S_x \rangle^2 + 2\langle S_y \rangle^2 + 2\langle S_z \rangle^2
$$

$$
= 1 - 4\langle S_z \rangle^2 \tag{43}
$$

The average fidelity is,

$$
F_{2,av} = 1 - 2\langle S_z \rangle^2 \tag{44}
$$

Now, purification by protocol-B gives,

$$
\phi_{msmt}^{(B)} = \left( \begin{array}{c}
e^{-i\phi} \sqrt{\frac{1}{p_1}} \\
e^{i\phi} \sqrt{\frac{1}{\tilde{\rho}(1-\tilde{\rho})}} \end{array} \right) \tag{45}
$$

By comparing eqns. (39) and (16) we get, $a = (2p_1 + 1)/4$ and $p = i(1-2p_2)/4 \Rightarrow |p| = |\langle S_x \rangle|/4$ and $e^{i\phi} = \pm i$ depending on the sign of $A_2$. Introducing the notation $A_1 = 2p_1 - 1$, $A_2 = 2p_2 - 1$, we have

$$
(1 - 2a) = -A_1/2; \quad 4|p|^2 + (1 - 2a)^2 = (1/4)(A_1^2 + A_2^2) \tag{46}
$$
Using the relations for $|\alpha|^2$, $p_1$ and $p_2$

\[
\mathcal{F}_3(\phi_{\text{msmt}}^{(B)}, \psi) = \langle \psi | \phi_{\text{msmt}}^{(B)} | \psi \rangle = \frac{1}{2} \left[ 1 + (A_1^2 + A_2^2)^{1/2} \right] + \frac{1}{2} \left[ (A_1^2 + A_2^2)^{1/2} - (A_1^2 + A_2^2) \right]
\]

Then the Fidelity $\mathcal{F}_3$ is:

\[
\mathcal{F}_3 = \frac{1}{2} \left[ 1 + (A_1^2 + A_2^2)^{1/2} \right]
\]

Therefore,

\[
\mathcal{F}_3 - \mathcal{F}_1 = \frac{1}{4} \left[ 2(A_1^2 + A_2^2)^{1/2} - (A_1^2 + A_2^2) \right] \\
\geq 0
\]

In other words,

\[
\mathcal{F}_3 \geq \mathcal{F}_1
\]

This means that for a partial measurement where only two components of spin are measured, the fidelity can always be improved over $\mathcal{F}_1(\phi_{\text{msmt}}, \rho_{\text{ini}})$ by purification Protocol - B. Furthermore,

\[
\mathcal{F}_3(\phi_{\text{msmt}}^{(B)}, \psi) = \frac{1}{2} \left[ 1 + (1 - 4\langle S_z \rangle^2) \right]
\]

Now from (44),

\[
\mathcal{F}_{2,av} = \frac{1}{2} \left( 1 + 1 - 4\langle S_z \rangle^2 \right)
\]

Hence,

\[
2\mathcal{F}_{2,av} - 1 = 1 - 4\langle S_z \rangle^2
\]

and this leads to

\[
2\mathcal{F}_3 - 1 = (2\mathcal{F}_{2,av} - 1)^{1/2}
\]

Clearly since $(2\mathcal{F}_{2,av} - 1) \leq 1$ we have,

\[
\mathcal{F}_3 \geq \mathcal{F}_{2,av}
\]

Clearly, Purification by Protocol - B reconstructs the state with better fidelity than does purification by protocol - A on the average.

When only one component of spin is measured, say, $S_z$ then we have the $p = 0$ case as has been worked out in the last part of section 4. Here,

\[
\chi_{\text{msmt}} = p_1 |+\rangle_{zz} \langle + | + (1 - p_1) | - \rangle_{zz} \langle - |
\]

Again, the initial state is of the form: $\psi = \alpha |+\rangle_z + \beta | - \rangle_z$. Now, we know only that $|\alpha|^2 = p_1$. The Fidelity of the post-measurement state with the initial state is:

\[
\mathcal{F}_4(\chi_{\text{msmt}}, |\psi\rangle \langle \psi|) = \langle \psi | \chi_{\text{msmt}} | \psi \rangle = p_1^2 + (1 - p_1)^2
\]

Now, if we purify the $\chi_{\text{msmt}}$ by protocol - A, then

\[
\chi_{\text{msmt}}^{(A)} = \frac{p_1}{\sqrt{p_1(1 - p_1)}} e^{-i\phi} \sqrt{p_1(1 - p_1)} e^{i\phi}
\]

The $\rho_{\text{ini}} = |\psi\rangle \langle \psi|$ is of the form:

\[
\rho_{\text{ini}} = \frac{p_1}{\sqrt{p_1(1 - p_1)}} e^{-i\phi} \sqrt{p_1(1 - p_1)} e^{i\phi}
\]

where, $p_1$ is known from measurement, but the phase $\theta$ cannot be determined. All values of $\theta$ should be considered equally likely. Therefore,

\[
\mathcal{F}_5(\chi_{\text{msmt}}, \rho_{\text{ini}}) = tr(\chi_{\text{msmt}} \rho_{\text{ini}}) = p_1^2 + (1 - p_1)^2 \cos(\theta + \phi)
\]

The average Fidelity with equal weightage for all $\theta$ is,

\[
\mathcal{F}_{5,av} = p_1^2 + (1 - p_1)^2
\]

Now, if $p_1 \geq 1/2$, then the purified state according to protocol - B is,

\[
\chi_{\text{msmt}}^{(B)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

Then,

\[
\mathcal{F}_6(\chi_{\text{msmt}}, |\psi\rangle \langle \psi|) = tr(\chi_{\text{msmt}} |\psi\rangle \langle \psi|) = p_1
\]

Since, $p_1 \geq 1/2$, it can be verified that

\[
\mathcal{F}_6(\chi_{\text{msmt}}^{(B)}, |\psi\rangle \langle \psi|) \geq \mathcal{F}_4(\chi_{\text{msmt}}^{(B)}, |\psi\rangle \langle \psi|)
\]

It can be verified that even for $p_1 < 1/2$ protocol - B always leads to an improvement in fidelity.

Also, since $\mathcal{F}_{5,av} = \mathcal{F}_4$, we have the relation

\[
\mathcal{F}_6(\chi_{\text{msmt}}^{(B)}, |\psi\rangle \langle \psi|) \geq \mathcal{F}_{5,av}(\chi_{\text{msmt}}^{(A)}, \rho_{\text{ini}})
\]

Thus, the fidelity offered by protocol - B is better than the average fidelity offered by protocol - A.
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[3] K.-E. Hellwig and K. Kraus, Commun. Math. Phys., 11(1969), 214.
[4] K.-E. Hellwig and K. Kraus, Commun. Math. Phys., 16(1970), 142.
[5] K. Kraus, Ann. of Phys., 64 (1971), 311.
[6] E. Schrödinger, Discussion of Probability Relations Between Separated Systems, Proc. Cambridge Phil. Soc., 31 555, (1935).
[7] E.C.G. Sudarshan, “Quantum Computers and Computing” V.A. No.1, 2003.
[8] J. Fiurasek, quant-ph/0403165; M. ricci et al, quant-ph/0403118; Kurt Jacobs, Phys. Rev. A68,054302(BR)(2003); A. Bassi and G.C. Ghirardi, Phys. Lett. A309(2003)24; A.S. Holevo, quant-ph/0204077; T.A. Brun, C.M. Caves, R. Schack, Phys. Rev. A63, 042309(2001); H. Mack et al, Phys. Rev. A62, 042301(2000); J.I. Cirac, A.K. Ekert and C. Macchiavello, Phys.Rev.Lett 82 (1999) 4344; J. Dittmann and A. Uhlmann, J. Math.Phys. 40(1999)3246; V. Vedral and M.B. Plenio, Phys.Rev. A57(1998)1619; S. Bose, V. Vedral and P.L. Knight, Phys. Rev. A60(1999) 194.
[9] N.D. Hari Dass and Chirag Dhara, quant-ph/0406169.
[10] R. Jozsa, Journal of Modern Optics, 41 No. 12, 2315.
[11] A. Peres, Quantum Theory: Concepts and Methods, Kluwer Academic Publishers, 1993.
[12] M.A. Nielsen and I.A. Chuang, Quantum Computation and Quantum Information, Cambridge University Press, 2002