Krein C*-modules

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Abstract

We introduce a notion of Krein C*-module over a C*-algebra and more generally over a Krein C*-algebra. Some properties of Krein C*-modules and their categories are investigated.

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Contents

1 Introduction 1

2 Krein C*-Algebras 2

3 Krein C*-modules over C*-algebras 3

4 Krein C*-modules over Krein C*-algebras 7

5 Categories of Krein C*-modules 10

6 Outlook 12

1 Introduction

Vector spaces with an indefinite inner product started to appear in physics with the work on special relativistic space-time by H.Minkowski \(^M\) and were later used for the first time in quantum field theory by P.Dirac \(^D\) and W.Pauli \(^P\), but their first mathematical discussion was provided by L.Pontrjagin \(^P\) and since then they have been an object of study mainly of the Russian school. Krein spaces, i.e. complete vector spaces equipped with an indefinite inner product, were formally defined by Ju.Ginzburg \(^G\) and in their present form by E.Scheibe \(^S\). Their properties have been investigated by several mathematicians such as I.Iohvidov, H.Langer, R.Phillips, M.Naǐmark, M.Krein

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and his school and have been extensively used in quantum field theory via the Gupta-Bleuler formalism in quantum electrodynamics.

They have been reconsidered in quantum field theory by K.Kawamura who also proposed axioms for Krein C*-algebras (involutive algebras of bounded linear operators on a Krein space). Krein spaces also appeared prominently in the definition of semi-Riemannian spectral triples in non-commutative geometry, by A.Strohmaier, M.Paschke, A.Sitarz and more recently by K.van den Dungen, M.Paschke, A.Rennie.

Hilbert C*-modules (complete modules over a C*-algebra with a C*-algebra-valued positive inner product) are a generalization of Hilbert spaces where the field of complex numbers is substituted by a general C*-algebra. They were first introduced in 1953 by I.Kaplanski in the case of commutative unital C*-algebras. Between 1972 and 1974 W.Paschke and M.Rieffel extended the theory to the case of modules over arbitrary C*-algebras and after that the subject grew and spread rapidly.

The purpose of this paper is to introduce an extremely elementary notion of Krein C*-module over a Krein C*-algebra, generalizing to the module case the usual decomposability condition of a Krein space into its “positive” and “negative” subspaces, and closely following the definition of Krein C*-module over a C*-algebra elaborated in S.Kaewumpai.

In practice, such “decomposable” kind of Krein modules admit a non-canonical splitting as direct sums of Hilbert C*-modules, with opposite signature that, when we allow the algebra to be a Krein C*-algebra, can be chosen to be “compatible” with one of the fundamental symmetry automorphisms of the algebra.

We first recall in section 2 (a variant of) the definition of Krein C*-algebra by K.Kawamura and, in section 3, for the benefit of the reader, we reproduce in some detail the key definitions and proofs of the main results on Krein C*-modules over C*-algebras that were developed in S.Kaewumpai’s thesis.

In section 4 we further extend the previous definition to cover the case of Krein C*-modules over Krein C*-algebras. In the subsequent section we expand the notion of tensor product of Krein C*-modules over C*-algebras, formulated in R.Tanadkithirun in order to cover our more general situation and we discuss some of the properties of the categories of modules and bimodules so obtained.

Several examples illustrating the scope of the definitions are presented. Of particular interest are the possible applications to the spectral geometry of semi-Riemannian manifolds and their non-commutative counterparts.

As it can be clearly appreciated by these geometric examples of modules of vector fields over the Clifford algebra of a semi-Riemannian manifold, the notion of Krein C*-module that is contained here is very specific and corresponds to the special case of tangent bundles admitting a global decomposition as Whitney orthogonal sums of positive and negative definite Hermitian vector sub-bundles i.e. semi-Riemannian manifolds that are time-orientable and space-orientable. More general notions of Krein C*-modules are necessary to deal with cases where such global “splitting” is not available, but for now we do not enter this interesting discussion that will very likely also require modifications in the definition of Krein C*-algebra.

2 Krein C*-Algebras

The following is a variation of the definition of Krein C*-algebra introduced by K.Kawamura.

Definition 2.1. A Krein C*-algebra is an involutive complete complex topological algebra (i.e. a complete complex topological vector space with a bilinear continuous product and a continuous involution) that admits at least one fundamental symmetry i.e. an involutive automorphism
\( \alpha : A \to A \) with \( \alpha \circ \alpha = \iota_A \) and one Banach algebra norm \( \| \cdot \|_\alpha \) (inducing the given topology) such that \( \|\alpha(a^*)a\|_\alpha = \|a\|^2_\alpha \), for all \( a \in A \).

**Proposition 2.2** (K. Kawamura, Example 2.4, Section 2.3). The set \( \mathcal{B}(K) \) of linear continuous operators on a Kreın space \( K \) is a Kreın C*-algebra. Every fundamental symmetry \( J \) of a Kreın space \( K \) is associated to a fundamental symmetry \( a \mapsto JaJ \), \( a \in \mathcal{B}(K) \) of the Kreın C*-algebra \( \mathcal{B}(K) \).

**Remark 2.3.** Note that although, contrary to K.Kawamura, we assume the existence of a given topology, we do not fix a priori any Banach norm on the Kreın C*-algebra, so that several topologically equivalent Banach norms can exist. Specifically, for every fundamental symmetry \( \alpha \) there is a unique topologically equivalent Banach norm \( \| \cdot \|_\alpha \) making \( A \) a C*-algebra, denoted by \( A^\alpha \), that coincides with \( A \) as a complex algebra and whose involution is given by \( x^\dagger := \alpha(x^*) \), for all \( x \in A \).

For example, different fundamental symmetries of a Kreın space \( K \), induce operator norms on the Kreın C*-algebra \( \mathcal{B}(K) \) of bounded linear operators on \( K \) that do not coincide, although they are topologically equivalent.

In the subsequent sections, we will often use the notation \( A_+ \) for the **even part** of the Kreın C*-algebra \( A \) under a fundamental symmetry \( \alpha \), i.e. the C*-algebra of elements such that \( \alpha(x) = x \); and similarly the notation \( A_- \) for the **odd part** of the Kreın C*-algebra \( A \) under a fundamental symmetry \( \alpha \), i.e. the Hilbert C*-module, over \( A_+ \), of elements such that \( \alpha(x) = -x \).

Later on we will see natural situations motivated from semi-Riemannian geometry that seem to require further generalization of the definition of Kreın C*-algebra, but for this work we will mostly limit our consideration to the definition above.

### 3 Kreın C*-modules over C*-algebras

In this section, we recall some basic material on unital Kreın C*-modules over unital C*-algebras that was developed in S.Kaewumpai’s Master thesis [Ka]. The material naturally covers, as a special case, the situation of Kreın spaces (that are Kreın C*-modules over the C*-algebra \( C_c(G) \)) and will be further generalized in the subsequent section where we will consider modules over Kreın C*-algebras.

Recall that, given a unital right module \( E_A \) over a unital C*-algebra \( A \), an \( A \)-valued **Hermitian inner product** is a map \( \langle \cdot | \cdot \rangle : E \times E \to A \) such that, for all \( x, y, z \in E, a \in A \):

\[
\langle z | x + y \rangle = \langle z | x \rangle + \langle z | y \rangle, \quad \langle z | xa \rangle = \langle z | x \rangle a, \quad \langle x | y \rangle^* = \langle y | x \rangle.
\]

In the case of unital left modules \( E \), the second property above is substituted with \( \langle ax | z \rangle = a \langle x | z \rangle \). Whenever confusion might arise, we will denote an inner product on \( E_A \) by \( \langle \cdot | \cdot \rangle_E \) and an inner product on \( A \) by \( \langle \cdot | \cdot \rangle \). The direct sum \( E \oplus F \) of two right (left) unital modules, over the unital C*-algebra \( A \), equipped with the inner product defined by \( \langle x_1 \oplus y_1 | x_2 \oplus y_2 \rangle_{E \oplus F} := \langle x_1 | x_2 \rangle_E + \langle y_1 | y_2 \rangle_F \), for all \( x_1, x_2 \in E \) and \( y_1, y_2 \in F \), is a right (left) unital module over \( A \) called **orthogonal direct sum** of \( E_A \) and \( F_A \).

A Hermitian inner product is **positive** if \( \langle x | x \rangle \in A_\geq := \{ a^*a | a \in A \} \), for all \( x \in E \), where \( A_\geq \) denotes the positive part of the unital C*-algebra \( A \). An inner product is **non-degenerate** if \( \langle x | x \rangle = 0 \Rightarrow x = 0 \). A unital right (left) **Hilbert C*-module** \( E \) over the unital C*-algebra \( A \) is a unital right (left) module on the unital C*-algebra \( A \), that is equipped with an \( A \)-valued positive non-degenerated inner product, making it a complete metric space with respect to the norm \( \|x\| := \sqrt{\langle x | x \rangle} \).

The family \( \mathcal{L}(E_A) \) of \( A \)-linear operators on the right (left) unital \( A \)-module \( E_A \) is a complex algebra with multiplication given by composition of maps. If the modules \( E_A \) and \( F_A \) are equipped with \( A \)-valued inner products, we say that a map \( T : E \to F \) is **adjointable** if there exists another map
The operators $J$ are two fundamental symmetries of $\mathcal{K}$ with associated fundamental decompositions $\mathcal{K} = \mathcal{K}^J_{+} \oplus \mathcal{K}^J_{-}$.

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For all the details of these standard results on Hilbert $C^*$-modules, the reader can consult N.Landsman [L, L1], E.Lance [La], N.Wegge-Olsen [WO] or S.Kaewunpai [Ka].
and \( \mathcal{K} = \mathcal{K}^j_+ \oplus \mathcal{K}^j_2 \), there are two \( \mathcal{A} \)-linear continuous bijective maps of Hilbert \( \mathcal{A} \)-C*-modules \( T_+^{j_1,j_2} : \mathcal{K}^j_+ \to \mathcal{K}^j_+ \) and \( T_-^{j_1,j_2} : -\mathcal{K}^j_+ \to -\mathcal{K}^j_+ \), given by \( T_+^{j_1,j_2}(x_1^j) := x_2^j \) and \( T_-^{j_1,j_2}(x_1^j) := x_2^j \), where \( x = x_1^j + x_2^j \) and \( x = x_1^j + x_2^j \) are the direct sum decompositions of \( x \in \mathcal{K}_A \) in each of the fundamental decompositions.

**Proof.** The map \( T_+^{j_1,j_2} : \mathcal{K}^j_+ \to \mathcal{K}^j_+ \) is adjointable with adjoint given by the map

\[
T_+^{j_1,j_2} : \mathcal{K}^j_+ \to \mathcal{K}^j_+ , \quad \text{defined by} \quad T_+^{j_1,j_2}(y) := y_1^j , \quad \text{for} \quad y \in \mathcal{K}^j_+ ,
\]

since \( (T_+^{j_1,j_2}(x) \mid y)_{\mathcal{K}^j_1} = (x_1^j \mid y_1^j)_{\mathcal{K}^j_1} = (x \mid y)_K = (x_1^j \mid y)_{\mathcal{K}^j_1} = (x_1^j \mid y_1^j)_{\mathcal{K}^j_1} \), for all \( x \in \mathcal{K}^j_+ \) and \( y \in \mathcal{K}^j_+ \). In exactly the same way we have that \( T_-^{j_1,j_2} : \mathcal{K}^j_+ \to \mathcal{K}^j_+ \) is adjointable with adjoint given by the map \( T_-^{j_1,j_2} : \mathcal{K}^j_2 \to \mathcal{K}^j_1 \) defined by \( T_-^{j_1,j_2}(y) := y_1^j \), for all \( y \in \mathcal{K}^j_2 \). Since an adjointable operator between Hilbert C*-modules (and also between anti-Hilbert C*-modules) is necessarily continuous, both \( T_+^{j_1,j_2}, T_-^{j_1,j_2} \) and their adjoints \( T_+^{j_1,j_2}, T_-^{j_1,j_2} \) are continuous \( \mathcal{A} \)-linear maps.

If \( x, y \in \mathcal{K}^j_+ \) and \( T_+^{j_1,j_2}(x) = T_+^{j_1,j_2}(y) \), we have \( x_1^j - y_1^j = x - y = x_2^j - y_2^j \) and since we have \( 0 \leq (x - y \mid x - y)_{\mathcal{K}^j_1} = (x - y \mid x - y)_{\mathcal{K}^j_1} \leq 0 \), via the non-degeneracy of the Kreın C*-module \( \mathcal{K}^j_+ \), we obtain the injectivity of \( T_+^{j_1,j_2} \).

Let \( (x_n) \) be a sequence in \( \mathcal{K}^j_+ \) such that \( T_+^{j_1,j_2}(x_n) \) converges to a given point \( z \in \overline{\text{Im}(T_+^{j_1,j_2})} \subset \mathcal{K}^j_+ \) in the submodules. Since \( (\|x_n - x_m\|^2_{\mathcal{K}^j_+})_{m \in \mathbb{N}} \leq 0 \), we have

\[
\|x_n - x_m\|^2_{\mathcal{K}^j_+} \leq \|T_+^{j_1,j_2}(x_n - x_m)\|^2_{\mathcal{K}^j_+},
\]

so that \( (x_n) \), being a Cauchy sequence in the Hilbert C*-module \( \mathcal{K}^j_+ \), converges to a point \( x_0 \in \mathcal{K}^j_+ \). By continuity of \( T_+^{j_1,j_2} \), we get \( z = \lim_{n \to \infty} T_+^{j_1,j_2}(x_n) = T_+^{j_1,j_2}(x_0) \in \text{Im}(T_+^{j_1,j_2}) \), that provides the closure of the range of \( T_+^{j_1,j_2} \).

The fact that \( T_+^{j_1,j_2} : \mathcal{K}^j_+ \to \mathcal{K}^j_+ \) is an adjointable map between Hilbert C*-modules with closed range is equivalent (see for example N.Wegge-Olsen [20] corollary 15.3.9]) to the complementability of the submodule \( \text{Im}(T_+^{j_1,j_2}) \subset \mathcal{K}^j_+ \).

If \( y \in \text{Im}(T_+^{j_1,j_2}) \subset \mathcal{K}^j_2 \), for all \( x \in \mathcal{K}^j_+ \), we get \( (T_+^{j_1,j_2}(y) \mid x)_{\mathcal{K}^j_1} = (y \mid T_+^{j_1,j_2}(x))_{\mathcal{K}^j_2} = 0 \) and hence \( y \in \text{Ker}(T_+^{j_1,j_2}) \). Since \( T_+^{j_1,j_2} \) is injective, we have \( \text{Im}(T_+^{j_1,j_2}) \downarrow = \{0\} \) and since \( \text{Im}(T_+^{j_1,j_2}) \) is complementable, from \( \mathcal{K}^j_2 = \overline{\text{Im}(T_+^{j_1,j_2}) \downarrow} = \text{Im}(T_+^{j_1,j_2}) \), we obtain the surjectivity of \( T_+^{j_1,j_2} \).

**Theorem 3.6.** If \( \mathcal{K}_A \) is a unital right (left) Kreın C*-module, over \( \mathcal{A} \), with two fundamental symmetries \( J_1 \) and \( J_2 \) with associated direct sum decompositions \( \mathcal{K} = \mathcal{K}^j_+ \oplus \mathcal{K}^j_1 = \mathcal{K}^j_2 \oplus \mathcal{K}^j_2 \), then the Hilbert C*-modules \( |\mathcal{K}|_1 := \mathcal{K}^j_1 \oplus \mathcal{K}^j_1 \) and \( |\mathcal{K}|_2 := \mathcal{K}^j_2 \oplus \mathcal{K}^j_2 \) have equivalent norms. Hence on \( \mathcal{K}_A \) there is a natural topology called the strong topology.

**Proof.** By theorem [20], we have two bijective \( \mathcal{A} \)-linear adjointable (hence continuous) maps

\[
T_+^{j_1,j_1} : \mathcal{K}^j_+ \to \mathcal{K}^j_2 , \quad T_-^{j_1,j_1} : -\mathcal{K}^j_+ \to -\mathcal{K}^j_2 ,
\]

It follows immediately that their direct sum map \( T_+^{j_1,j_1} \oplus T_-^{j_1,j_1} : |\mathcal{K}|_1 \to |\mathcal{K}|_2 \) is a bijective linear continuous map with continuous inverse and hence as Banach spaces \( |\mathcal{K}|_1 \) and \( |\mathcal{K}|_2 \) have equivalent norms.
Proposition 3.7. Let $\mathcal{K}_A = \mathcal{K}_A^j \oplus \mathcal{K}_A^-$ be the fundamental decomposition of the right (left) unital Kreîn $C^*$-module $\mathcal{K}_A$ associated to the fundamental symmetry $J$ and with associated Hilbert $C^*$-module $|K_jA|$. The operator $T : \mathcal{K}_A \rightarrow \mathcal{K}_A$ is an adjointable operator in $\mathcal{K}_A$ if and only if it is adjointable in $|K_jA|$. The adjoint $^*T$ of $T$ in the Kreîn $C^*$-module $\mathcal{K}_A$ and the adjoint $T^\dagger_j$ of $T$ in the Hilbert $C^*$-module $|K_jA|$ are related by the following formulas:

$$T^\dagger_J = J \circ T^* \circ J, \quad T^* = J \circ T^\dagger_j \circ J.$$  

The family $\mathcal{B}(\mathcal{K}_A)$ of adjointable operators in the unital right (left) Kreîn $C^*$-module $\mathcal{K}_A$ coincides as a set with the $C^*$-algebra $\mathcal{B}(|K_jA|)$ of adjointable operators in the unital right (left) Hilbert $C^*$-module $|K_jA|$.

Proof. If $T$ is adjointable in the Kreîn $C^*$-module $\mathcal{K}_A$ with adjoint $^*T$, we have, for all $x, y \in \mathcal{K}_A$,  

$$\langle x | T(y) \rangle_K = \langle T^*(x) | y \rangle_K \text{ and so } \langle J(x) | T^\dagger(y) \rangle_{|K_jA|} = \langle J(T^*(x)) | y \rangle_{|K_jA|} \text{ and taking } x := J(z) \text{ we get } \langle z | T(y) \rangle_{|K_jA|} = \langle J \circ T^* \circ J(z) | y \rangle_{|K_jA|} \text{ that gives the adjointability of } T \text{ in } |K_jA| \text{ with adjoint } T^\dagger_J = J \circ T^* \circ J. \text{ Following the same passages in the reverse order establishes the equivalence of the notions of adjointability in } \mathcal{K}_A \text{ and in } |K_jA|. \qedhere$$

Theorem 3.8. The algebra $\mathcal{B}(\mathcal{K}_A)$ of adjointable endomorphisms of a unital right (left) Kreîn $C^*$-module $\mathcal{K}_A$ is a Kreîn-$C^*$-algebra.

Proof. By the previous proposition, we see that $\mathcal{B}(\mathcal{K}_A) = \mathcal{B}(|K_jA|)$ as sets and also as unital associative algebras, since the operations of addition and scalar multiplication are the same. Taking on $\mathcal{B}(\mathcal{K}_A)$ the operator norm $\| \cdot \|_J$ defined in the $C^*$-algebra $\mathcal{B}(|K_jA|)$ of the unital Hilbert $C^*$-module $|K_jA|$, we see that $\mathcal{B}(\mathcal{K}_A)$ is a Banach space. The involution on the algebra $\mathcal{B}(\mathcal{K}_A)$ is obtained via the Kreîn $C^*$-module adjoint $T \mapsto T^*$. In order to complete the proof that $\mathcal{B}(\mathcal{K}_A)$ is a Kreîn $C^*$-algebra, we need to provide an involutive automorphism $\alpha : \mathcal{B}(\mathcal{K}_A) \rightarrow \mathcal{B}(\mathcal{K}_A)$ such that $\|\alpha(T^*) \circ T\|_J = \|T\|_J^2$ for all $T \in \mathcal{B}(\mathcal{K}_A)$. For this purpose, for all $T \in \mathcal{B}(\mathcal{K}_A)$, we define $\alpha(T) := J \circ T \circ J$ and, using the $C^*$-property of $\mathcal{B}(|K_jA|)$, verify that $\|\alpha(T^*) \circ T\|_J = \|(J \circ T \circ J) \circ T\|_J = \|T^\dagger_J \circ T\|_J = \|T\|_J^2. \qedhere$

Proposition 3.9. Let $\mathcal{K}_A$ be a unital right (left) Kreîn $C^*$-module. Any two fundamental symmetries $J_1, J_2 \in \mathcal{B}(\mathcal{K}_A)$ of $\mathcal{K}$ are unitarily equivalent.

Proof. With the notation used in theorem 3.4 consider the adjointable operator $U := T_{J_2J_1}^* \oplus T_{J_1J_2}^*$ and note that $U \circ J_1 = J_2 \circ U$ with $U^* = U^{-1}. \qedhere$

Example 3.10. Every Kreîn-space is a Kreîn-$C^*$-module over the $C^*$-algebra $\mathbb{C}$, the complexification $M_{\mathbb{C}} := M \otimes_{\mathbb{R}} \mathbb{C}$ of Minkowski space in special relativity being probably the most important example. \qedhere

Example 3.11. Let $M$ be a semi-Riemannian manifold that for now (although this makes the situation less interesting for applications to physics) we suppose to be compact. If the manifold is also supposed to be space-orientable and time-orientable (for a specific example, consider $\mathbb{T}^2$ with the indefinite metric coming from the product of a copy of $\mathbb{R}$ with positive metric and another copy with negative metric), the module $\Gamma(T(M))$ of continuous sections of its tangent bundle $T(M)$ is a unital Kreîn $C^*$-module over the unital $C^*$-algebra $C(M)$ of continuous functions.

Note that, when a semi-Riemannian manifold is not space-orientable and time-orientable, although each fiber of the tangent bundle $T(M)$ is still a Kreîn space, the module $\Gamma(T(M))$ of continuous vector fields, fails to be a Kreîn $C^*$-module over the $C^*$-algebra $C(M)$, because in this case $\Gamma(T(M))$ does not admit a global splitting as a direct sum of a Hilbert and an anti-Hilbert $C^*$-module (equivalently the tangent bundle $T(M)$ is not a Whitney direct sum of a positive definite and a negative definite sub-bundle). As a consequence, this very interesting kind of complete semi-definite Hilbert $C^*$-modules do not admit a globally defined fundamental symmetry!
4 Kreın C*-modules over Kreın C*-algebras

Definition 4.1. A left Kreın C*-module over a Kreın C*-algebra $A$ is a complete topological vector space that is also a left (unital) module $K$ over $A$ with the following properties:

- $K$ is equipped with an $A$-valued inner product $\langle \cdot \mid \cdot \rangle : K \times K \to A$ such that
  \[ A(x + y \mid z) = A(x \mid z) + A(y \mid z), \quad \forall x, y, z \in K, \]
  \[ A(ax \mid y) = a \cdot A(x \mid y), \quad \forall x, y \in K, \forall a \in A, \]
  \[ A(x \mid y)^* = A(y \mid x), \quad \forall x, y \in K, \]
  \[ \forall y \in K, \quad A(x \mid y) = 0 \Rightarrow x = 0, \]

- there exists a fundamental symmetry $J_A : K \to K$ such that $J \circ J = 1_{K}$,
  \[ J_A(x + y) = J_A(x) + J_A(y), \quad \forall x, y \in K, \]
  \[ J_A(a \cdot x) = a \cdot J_A(x), \quad \forall x \in K, \forall a \in A, \]
  \[ \alpha(A(x \mid y)) = A(J_A(x) \mid J_A(y)), \quad \forall x, y \in K, \]

- for the given choice of fundamental symmetries $\alpha$ on $A$ and $J_A$ on $K$, we have that the map $(x, y) \mapsto A(x \mid J_A(y))$ gives to $K$ the structure of a Hilbert C*-module over the C*-algebra $A_{\alpha}$ whose norm induces the original topology of $K$.

In a perfectly similar way, there is a definition of right Kreın C*-module $K$ over a Kreın C*-algebra $B$, where the $B$-valued inner product $\langle \cdot \mid \cdot \rangle : K \times K \to B$ satisfies $\langle x \mid yb \rangle_B = \langle x \mid y \rangle_B \cdot b$, for all $b \in B$ and all $x, y \in K$; the fundamental symmetry $J_B$ satisfies $J_B(x \cdot b) = J_B(x) \cdot \beta(b)$, for all $x \in K$ and $b \in B$; and where all the other properties remain essentially unchanged.

The following definition of bimodule, whenever we further impose the additional fullness requirements $A = A(K \mid K) := \text{span}(A(x \mid y) \mid x, y \in K)$ and $B = (K \mid K)_B := \text{span}(\langle x \mid y \rangle_B \mid x, y \in K)$, extends to the Kreın C*-algebras context the usual notion of imprimitivity Hilbert C*-bimodule that provides the well-known (strong) Morita equivalence of C*-algebras (see also remark 5.7).

Definition 4.2. A Kreın C*-bimodule $A_{KB}$ is a left Kreın C*-module over $A$ that is the same time a right Kreın C*-module over $B$ with additional properties: $(a \cdot x) \cdot b = a \cdot (x \cdot b)$, $J_A = J_B$ and with right and left inner products related via: $A(x \mid y)x = x(y \mid x)_B$, for all $x, y \in K$.

The compatibility condition requested above on the inner products assures that the induced left and right norms on the Hilbert C*-module $K$ coincide.

Remark 4.3. In our definitions the two auxiliary $A_{\alpha}$-valued Hilbert C*-module inner products
  \[ (x \mid y)_{A_{\alpha}} := (x \mid J_A(y))_{A_{\alpha}}, \quad J_A(x \mid y)_{A_{\alpha}} := (J_A(x) \mid y)_{A_{\alpha}}, \]

can be used in place of each other since they are related by the isomorphism $\alpha$ of the C*-algebra $A_{\alpha}$:
  \[ \alpha((J_A(x) \mid y)_{A_{\alpha}}) = (x \mid J_A(y))_{A_{\alpha}}. \]

Remark 4.4. Note that $K_B$ becomes naturally a Kreın C*-module over the C*-algebra $B$, when equipped with the even part of the original inner product i.e. taking $\frac{1}{2}(x \mid y)_B + \frac{1}{2}(J(x) \mid J(y))_B$ as inner product on $K$; however with this new inner product the submodules $K_+$ and $K_-$ are orthogonal (as in our original definition of Kreın C*-module over a C*-algebra) contrary to the general situation here, where the obstruction to the orthogonality is measured by the odd part of the original inner product $\frac{1}{2}(J(x) \mid y)_B + \frac{1}{2}(x \mid J(y))_B$ that is always a non-degenerate anti-Hermitian product on $K$. 


with values in \( \mathcal{B} \). Furthermore, although in some situations (for example in finite dimensional cases) the product topology induced by the decomposition \( \mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_- \) (that is actually the topology induced by the even part of the inner product, since the two inner products coincide, modulo sign, on their restriction to \( \mathcal{K}_+ \) and \( \mathcal{K}_- \) respectively) is the same as the original topology of \( \mathcal{K} \), we still suspect that in general this might fail.

A significant difference from the case of Kreı̇n C*-modules over C*-algebras is that the fundamental symmetry \( J_\mathcal{K} \) is in general not self-adjoint (and in general not even adjointable, as can be seen in the case of example 4.7 below, where adjointability happens only in the case of \( \alpha \) equal to the identity i.e. when \( \mathcal{A} \) is already a C*-algebra) with respect to the original Kreı̇n inner product as suggested by the general lack of orthogonality between the even and odd submodules \( \mathcal{K}_+ \) and \( \mathcal{K}_- \).

**Example 4.5.** Every Kreı̇n C*-module \( \mathcal{K} \) over a C*-algebra \( \mathcal{A} \), as defined in the previous section, is a special case of our new definition as results by taking \( \alpha \) to be the identity isomorphism of \( \mathcal{A} \).

Actually, whenever the Kreı̇n C*-algebra \( \mathcal{A} \) is a C*-algebra and \( \alpha \) is trivial, we reduce to the definition of the last section: the submodules \( \mathcal{K}_+ \) and \( \mathcal{K}_- \) are orthogonal and the fundamental symmetry \( J_\mathcal{K} \) is Hermitian. The new definition of Kreı̇n module over a Kreı̇n C*-algebra even allows for a possible choice of a nontrivial \( \alpha \) also in the case of a C*-algebra \( \mathcal{A} \) and in this situation we obtain a Kreı̇n C*-module over a C*-algebra where the two submodules \( \mathcal{K}_+ \) and \( \mathcal{K}_- \) fail to be orthogonal and \( J_\mathcal{A} \) is not Hermitian.

**Example 4.6.** Let \( \mathcal{K}_\mathcal{A} \) be a right Kreı̇n C*-module over the C*-algebra \( \mathcal{A} \) and let \( \mathcal{B}(\mathcal{K}_\mathcal{A}) \) be the Kreı̇n C*-algebra of adjointable operators on \( \mathcal{K}_\mathcal{A} \), then \( \mathcal{B}(\mathcal{K}_\mathcal{A}) \) is a left Kreı̇n C*-module over the Kreı̇n C*-algebra \( \mathcal{B}(\mathcal{K}_\mathcal{A}) \) with left inner product defined by \( \mathcal{B}(\mathcal{K}_\mathcal{A})(x | y) := \Theta_{x,y} \), where \( \Theta_{x,y}(z) := x \cdot (y | z)_\mathcal{A} \), for all \( x, y, z \in \mathcal{K}_\mathcal{A} \).

The bimodule \( \mathcal{B}(\mathcal{K}_\mathcal{A}) \) is actually a Kreı̇n C*-bimodule such that \( \mathcal{B}(\mathcal{K}_\mathcal{A})(x | y)z = x(y | z)_\mathcal{A} \) for all \( x, y, z \in \mathcal{K}_\mathcal{A} \).

**Example 4.7.** Let \( \mathcal{A} \) be a Kreı̇n C*-algebra. Then \( \mathcal{A} \mathcal{A} \) and \( \mathcal{A}^* \mathcal{A} \) are both Kreı̇n C*-algebras with inner products given by \( (x | y)_\mathcal{A} := x^*y \) and \( \mathcal{A}^*(x | y) := xy^* \), furthermore \( \mathcal{A} \mathcal{A} \mathcal{A}^* \) is a Kreı̇n C*-bimodule.

**Example 4.8.** Let \( K_1 \) and \( K_2 \) be two Kreı̇n spaces.

The space \( \mathcal{B}(K_1) \mathcal{B}(K_2) \mathcal{B}(K_1) \) of linear continuous maps between them is a Kreı̇n C*-bimodule with the left/right actions given by the usual compositions of linear operators and inner products given respectively by \( (T | S)\mathcal{B}(K_1) := T^* \circ S \) and \( \mathcal{B}(K_2)(T | S) := T \circ S^* \).

**Example 4.9.** Following the definitions provided in [BR], let \( A, B \in \mathcal{O}_\mathcal{K} \) be two objects in a Kreı̇n C*-category \( \mathcal{K} \). Then \( \mathcal{O}_{\mathcal{K}}(A, B) \) is a Kreı̇n C*-bimodule over the Kreı̇n C*-algebras \( \mathcal{O}_{\mathcal{K}}(A) \) on the left and \( \mathcal{O}_{\mathcal{K}}(B) \) on the right.

**Example 4.10.** Let \( \mathcal{M} \) be Minkowski space (or more generally any real vector space equipped with semi-definite inner product); let \( \Lambda^C(\mathcal{M}) \) denote the space of complex-valued antisymmetric forms on \( \mathcal{M} \) (the complexified Grassmann algebra of \( \mathcal{M} \)) and let \( \text{Cl}(\mathcal{M}) \) denote the complexified Clifford algebra of \( \mathcal{M} \).

Note that for every fundamental decomposition of \( \mathcal{M} = \mathcal{M}_+ \oplus \mathcal{M}_- \), we have for the Grassmann algebras the decomposition \( \Lambda^C(\mathcal{M}) := \Lambda^C(\mathcal{M}_+) \otimes \Lambda^C(\mathcal{M}_-) \) and similarly, for the Clifford algebras, \( \text{Cl}(\mathcal{M}) = \text{Cl}(\mathcal{M}_+) \otimes \text{Cl}(\mathcal{M}_-) \), where (if we work in the category of associative algebras) \( \otimes \) denotes the \( \mathbb{Z}_2 \)-graded tensor product.

Note that the (underlying complex vector space of the) Grassmann algebra \( \Lambda^C(\mathcal{M}) \) is naturally a Kreı̇n space with the semi-definite inner product induced by universal factorization property via \( (\omega_1 \wedge \cdots \wedge \omega_n | \phi_1 \wedge \cdots \wedge \phi_n) := \det[(\omega_i | \phi_j)] \) on each of the summands in \( \bigoplus_{q=0}^{\dim \mathcal{M}} \Lambda_q^C(\mathcal{M}) = \Lambda^C(\mathcal{M}) \), where \( (\omega_i | \phi_j) \in \mathbb{C} \) denotes the Minkowski inner product on complexified covectors \( \omega_i, \phi_j \in \Lambda_q^C(\mathcal{M}) \).
The Clifford algebra $\mathbb{C}$ has a natural structure of Krein C*-algebra as a sub-algebra of the Krein C*-algebra $\mathcal{B}(\Lambda^C(M))$: every defining universal property of the Clifford algebra that, under the linear isomorphism $\mathcal{Cl}(M) \simeq \Lambda^C(M)$, coincides with the (second quantized) fundamental symmetry $J_{\Lambda_C(M)} := \bigoplus_{n=0}^{\infty} J_{\Lambda_C(M)^n}$ of the Krein space $\Lambda^C(M)$. It follows that the Krein space $\Lambda^C(M)$ is a left Krein module over the Krein C*-algebra $\mathcal{Cl}(M)$.

The (underlying vector space of) the Grassmann algebra $\Lambda^C(M)$ also becomes a Krein C*-bimodule over $\mathcal{Cl}(M)$ via Clifford left and right actions and with the inner products induced via the linear isomorphism $\Lambda^C(M) \simeq \mathcal{Cl}(M)$ by the standard Krein C*-bimodule structure of the Krein C*-algebra $\mathcal{Cl}(M)$ over itself.

As described in more detail in H.Baum [B8] (see also A.Strohmaier [SG] section 5.1] and K.Van Den Dungen-M.Paschke-A.Rennie [DPR section 3.3.1]) the module $S(M)$ of (Dirac) spinors is a Krein space, whose fundamental symmetries are proportional to the product of the operators of Clifford multiplication by all the vectors in an orthonormal basis for the timelike summand of a fundamental decomposition $M = M_+ \oplus M_-$. The space $S(M)$ (for $M$ even-dimensional) becomes a left Krein C*-module over the Krein C*-algebra $\mathcal{Cl}(M)$ with the inner product induced by the linear isomorphisms $S(M) \otimes S(M)^* \simeq \Lambda^C(M) \simeq \mathcal{Cl}(M)$ and $\mathcal{Cl}(M)S(M)$ is a Krein C*-bimodule that, with the terminology introduced in remark 5.7 is an example of Morita-Krein equivalence C*-bimodule.

Example 4.11. Let $M$ be a (compact) semi-Riemannian space-orientable and time-orientable manifold. As already described in example 3.11 the module $\Gamma(T(M))$ of its continuous vector fields is a (unital) Krein C*-bimodule over the (unital) C*-algebra $\mathcal{C}(M)$. The algebra $\Gamma(\mathcal{Cl}(M))$ of continuous section of the complexified Clifford bundle $\mathcal{Cl}(T(M))$ of $M$ is a (unital) Krein C*-algebra and the module $\Gamma(\Lambda^*(M))$ of continuous sections of the complexified Grassmann bundle $\Lambda^*(M)$ of $M$ is a (unital) Krein C*-bimodule over the Krein C*-algebra $\Gamma(\mathcal{Cl}(M))$. The case of spinorial manifolds is described in example 5.9.

Remark 4.12. Note that, in the previous example, if the manifold $M$ is not time-orientable and space-orientable, the algebra $\Gamma(\mathcal{Cl}(M))$ (although being a nice involutive complete topological algebra) does not admit a globally defined fundamental symmetry and so does not fit into the current definition of Krein C*-algebra!

This clearly indicates that the environment of Krein C*-algebras and Krein C*-modules that we have developed here is insufficient to deal with a general axiomatization of “complete semi-definite C*-algebras and C*-modules over them”.

We pass now to briefly examine the main properties of the algebras of adjointable operators on Krein C*-modules over Krein C*-algebras.

Definition 4.13. Let $\mathcal{K}_A$ be a Krein C*-module over the Krein C*-algebra $A$. A map $T : \mathcal{K} \to \mathcal{K}$ is said to be adjointable if there exists another map $T^* : \mathcal{K} \to \mathcal{K}$ such that $(T(x) \mid y)_A = (x \mid T^*(y))_A$, for all $x, y \in \mathcal{K}$. The family of such adjointable maps is denoted by $\mathcal{B}(\mathcal{K}_A)$.

Remark 4.14. As usual, the adjointable maps are already $A$-linear and continuous and the adjoint $T^*$ is unique. The set $\mathcal{B}(\mathcal{K}_A)$ is a vector space and an associative unital algebra under composition, furthermore the map $* : T \mapsto T^*$ is involutive, antimultiplicative and conjugate $C$-linear so that $\mathcal{B}(\mathcal{K}_A)$ is a complex associative unital *-algebra.

Proposition 4.15. A map $T : \mathcal{K}_A \to \mathcal{K}_A$ is adjointable with respect to the inner product $(\cdot \mid \cdot)_A$ if and only if the map $T$ is adjointable for the Hilbert C*-module $\mathcal{K}_A$ with the auxiliary inner product

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On the usual Minkowski space $\mathbb{M}^4$, the Krein space $S(\mathbb{M}^4)$ has signature $(2,2)$ and the fundamental symmetries are just the Dirac $\gamma^0$ operators.
Similarly, \((\cdot | \cdot)^J A\). As a consequence, the associative unital algebra \(B(K_A)\) coincides with the associative unital algebra \(B(K_{A^\omega})\). The relation between the adjoint \(T^*\) of \(T\) in \(B(K_A)\) and the adjoint \(T^{\dagger\alpha}\) of \(T\) in the C*-algebra \(B(K_{A^\alpha})\) is given by:

\[ T^{\dagger\alpha} = J_A \circ T^* \circ J_A, \quad T^* = J_A \circ T^{\dagger\alpha} \circ J_A \]

**Proof.** Suppose that \((T(x) | y)_A = (y | T^*(y))_A\). The following calculation

\[ (T(x) | y)_A^{J A\alpha} = (T(x) | J_A(y))_A = (x | T^* J_A(y))_A \]

\[ = (x | J_A J_A T^* J_A(y))_A = (x | J_A T^* J_A(y))_A^{J A\alpha}, \]

assures that the adjointability in \(B(K_A)\) implies the adjointability in \(B(K_{A^\alpha})\) and the first formula relating the adjoints.

Suppose now that \((T(x) | y)_A^{J A\alpha} = (x | T^{\dagger\alpha}(y))_A^{J A\alpha}\) i.e. \((T(x) | J_A(y))_A = (x | J_A T^{\dagger\alpha}(y))_A\) and choosing \(y = J_A(y')\) for an arbitrary \(y' \in K\), we obtain \((T(x) | (y'))_A = (x | J_A T^{\dagger\alpha} J_A(y'))_A\) that assures the reverse and the second adjointability formula. \(\square\)

Although we know that in general \(J_A\) is not an adjointable operator, we still have the following fundamental symmetry of \(B(K_A)\):

**Proposition 4.16.** If \(T\) is adjointable in \(B(K_A)\), also the new operator \(J_A \circ T \circ J_A\) is adjointable in \(B(K_A)\) and the map

\[ \alpha_{J_A} : T \mapsto J_A \circ T \circ J_A \]

is a \(*\)-isomorphism of the involutive algebra \(B(K_A)\) of adjointable operators.

**Proof.** Since, for all \(T \in B(K_A)\), \((T^*)^* = T\), we obtain \(J_A (J_A T^{\dagger\alpha} J_A)^{\dagger\alpha} J_A = T\) or equivalently \((J_A T^{\dagger\alpha} J_A)^{\dagger\alpha} = J_A T J_A\). Since \((S^{\dagger\alpha})^{\dagger\alpha} = S\), we get \((J_A T^{\dagger\alpha} J_A) = (J_A T J_A)^{\dagger\alpha}\) i.e. \(\alpha_{J_A}\) is a \(*\)-isomorphism of the C*-algebra \(B(K_{A^\alpha})\). Similarly from \((S^{\dagger\alpha})^{\dagger\alpha} = S\), we get \(J_A (J_A S^* J_A)^* J_A = S\) or equivalently \((J_A S^* J_A)^* = J_A S J_A\) and hence \(J_A S^* J_A = (J_A S J_A)^*\) i.e. \(\alpha_{J_A}\) is a \(*\)-isomorphism of the involutive algebra \(B(K_A)\). \(\square\)

**Theorem 4.17.** The algebra \(B(K_A)\) of adjointable operators of a Krein C*-module over a Krein C*-algebra \(A\) is a Krein C*-algebra.

**Proof.** The \(*\)-isomorphism \(\alpha_{J_A}\) of the involutive algebra \(B(K_A)\), defined in the previous proposition, satisfies the C*-property \(\|\alpha_{J_A}(T^*)T\|\alpha = \|T\|_A^2\) with respect to the norm of the C*-algebra \(B(K_{A^\alpha})\). \(\square\)

## 5 Categories of Krein C*-modules

The following proposition, whose proof is self-evident, provides the most elementary category of morphisms of Krein C*-algebras that naturally contains, as a full subcategory, the category of unital \(*\)-homomorphisms of unital C*-algebras.

**Proposition 5.1.** There is a category \(\mathcal{A}\) whose objects are unital Krein C*-algebras \(A, B, \ldots\); whose arrows \(\phi : A \to B\) are unital Krein \(*\)-homomorphisms i.e. unital \(*\)-homomorphisms of involutive unital algebras \(\phi : A \to B\) such that there exist at least a fundamental symmetry of \(\alpha\) of \(A\) and \(\beta\) of \(B\) such that \(\phi \circ \alpha = \beta \circ \phi\); and composition is the usual composition of functions.

We now define the Krein C*-analogue of the well-known notion of C*-correspondence.
Definition 5.2. A left Krein C*-correspondence from the unital Krein C*-algebra $\mathcal{B}$ to unital the Krein C*-algebra $\mathcal{A}$ is unital left Krein C*-module $\mathcal{A}M$ over the Krein C*-algebra $\mathcal{A}$ equipped with a morphism of unital Krein C*-algebras from $\mathcal{B}$ to the unital Krein C*-algebra $\mathcal{B}(\mathcal{A}M)$ of adjointable operators on the Krein module $\mathcal{A}M$ such that $x \cdot \beta(b) = J_A(J_A(x) \cdot b)$.

A right Krein C*-correspondence from $\mathcal{B}$ to $\mathcal{A}$ is similarly defined as a unital right Krein C*-module $\mathcal{B}N$ over $\mathcal{B}$ equipped with a morphism of unital Krein C*-algebras from $\mathcal{A}$ to $\mathcal{B}(\mathcal{B}N)$ such that $\alpha(a) \cdot x = J_B(a \cdot (J_B x))$.

A morphism of Krein C*-correspondences is a map $\Phi: \mathcal{A}M \to \mathcal{A}N$ between right (respectively left) Krein C*-correspondences such that

$$\Phi(a \cdot x \cdot b) = a \cdot \Phi(x) \cdot b, \quad \forall a \in \mathcal{A}, b \in \mathcal{B}, x \in \mathcal{M}.$$ 

Remark 5.3. The previous definition entails that a left Krein C*-correspondence is actually a unital bimodule $\mathcal{A}M$ over the the unital Krein C*-algebras $\mathcal{A}$ and $\mathcal{B}$ (with $\mathcal{A}$-valued inner product) such that there exists at least one fundamental symmetry $J$ of $\mathcal{A}M$ and fundamental symmetries $\alpha$ of $\mathcal{A}$, $\beta$ of $\mathcal{B}$ that satisfy the compatibility condition $J(axb) = \alpha(a)J(x)\beta(b)$, for all $x \in \mathcal{M}$, $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Clearly we have categories of morphisms of right (respectively) Krein C*-correspondences under the usual composition of morphisms.

The following definition and theorems incorporate (and generalize to the case of Krein C*-modules over Krein C*-algebras) the notion of tensor product of Krein spaces and Krein C*-modules over C*-algebras developed in R.Tanadkithirum’s senior undergraduate project [7].

Definition 5.4. The internal tensor product of two right Krein C*-correspondences $\mathcal{A}M$ and $\mathcal{B}N$ is defined as a left $\mathcal{A}$-linear right $\mathcal{C}$-linear and $\mathcal{B}$-balanced map $\otimes: \mathcal{A}M \times \mathcal{B}N \to \mathcal{A}T$ with values into a right Krein C*-correspondence $\mathcal{A}T$ from $\mathcal{A}$ to $\mathcal{B}$ such that the following universal factorization property is satisfied:

for every left $\mathcal{A}$-linear right $\mathcal{C}$-linear and $\mathcal{B}$-balanced function $\phi: \mathcal{M} \times \mathcal{N} \to \Omega$ with values into a Krein C*-correspondence $\mathcal{A}Q$ from $\mathcal{A}$ to $\mathcal{C}$, there exists a unique morphism $\Phi: \mathcal{A}Q \to \mathcal{A}T$ such that $\Phi \circ \otimes = \phi$.

Theorem 5.5. Tensor products of right Krein C*-correspondences exist and are unique up to isomorphism in the category of morphisms of Krein C*-correspondences.

Proof. The unicity up to isomorphism is a standard consequence of a definition via universal factorization properties. For the proof of existence, consider the fundamental decompositions $\mathcal{M} = \mathcal{M}_+ \oplus \mathcal{M}_-$ and $\mathcal{N} = \mathcal{N}_+ \oplus \mathcal{N}_-$ induced by a pair of fundamental symmetries $J_\mathcal{M}$ and $J_\mathcal{N}$ that are compatible with three fundamental symmetries $\alpha$ of $\mathcal{A}$, $\beta$ of $\mathcal{B}$ and $\gamma$ of $\mathcal{C}$.

Using the algebraic tensor product of bimodules and the canonical isomorphism of bimodules

$$(\mathcal{M}_+ \otimes \mathcal{M}_-) \otimes \mathcal{B} (\mathcal{N}_+ \otimes \mathcal{N}_-) \simeq (\mathcal{M}_+ \otimes \mathcal{M}_- \otimes \mathcal{N}_+ \otimes \mathcal{N}_-) \oplus (\mathcal{M}_+ \otimes \mathcal{N}_- \otimes \mathcal{M}_- \otimes \mathcal{N}_+), \quad (5.1)$$

we have that $J_\mathcal{M} \otimes \mathcal{M}_+ \otimes \mathcal{N}_+ \otimes \mathcal{N}_-$ implies the previous decomposition and is compatible with the left action of $\mathcal{A}$ and the right action of $\mathcal{C}$.

By universal factorization property (two times), we can define on $\mathcal{M} \otimes \mathcal{B} \mathcal{N}$ a unique $\mathcal{C}$-valued inner product such that, for all $x_1, x_2 \in \mathcal{M}$ and $y_1, y_2 \in \mathcal{N}$,

$$\langle x_1 \otimes \mathcal{B} y_1 \mid x_2 \otimes \mathcal{B} y_2 \rangle_{\mathcal{M} \otimes \mathcal{B} \mathcal{N}} := \langle y_1 \mid (x_1 \mid x_2 \rangle_{\mathcal{B}}^\mathcal{M} \cdot \langle y_2 \rangle_{\mathcal{B}}^\mathcal{N}.$$ 

The following property holds for this inner product

$$\gamma((x_1 \otimes \mathcal{B} y_1 \mid x_2 \otimes \mathcal{B} y_2)_{\mathcal{C}}) = ((J_\mathcal{M} \otimes \mathcal{B} J_\mathcal{N})(x_1 \otimes \mathcal{B} y_1) \mid (J_\mathcal{M} \otimes \mathcal{B} J_\mathcal{N})(x_2 \otimes \mathcal{B} y_2))_{\mathcal{C}}.$$ 

11
and the algebra \( \mathcal{A} \) acts by adjointable operators on the left. The inner product so defined on \( \mathcal{M} \otimes_{\mathcal{B}} \mathcal{N} \) induces on each one of the direct summands of the even part of the decomposition in formula (5.1) the structure of Hilbert C*-module and, for the summands of the odd part, the structure of anti-Hilbert C*-module over the same C*-algebra \((\mathcal{C}, \mathcal{A})\).

**Theorem 5.6.** There is a weak category \( \mathfrak{M} \) whose objects are unital Kre˘ın C*-algebras \( \mathcal{A}, \mathcal{B}, \ldots; \) whose arrows are right Kre˘ın C*-correspondences; and whose composition is obtained by internal tensor product of Kre˘ın C*-correspondences. In a totally similar way, we have a weak category \( \mathfrak{M} \) of left Kre˘ın C*-correspondences under internal tensor product.

**Proof.** The associativity of composition modulo isomorphism is assured using the universal factorization property. The (weak) identities are given the Kre˘ın C*-algebras \( \mathcal{A} \) with their usual functional composition as composition over 1-arrows and their internal tensor product as composition over objects.

This pair of weak 2-categories is the “Kre˘ın counterpart” to the usual 2-categories of right and left Kre˘ın C*-correspondences with their usual functional composition as composition over 1-arrows and their internal tensor product as composition over objects.

The previous categories (exactly as their C*-counterparts) are not equipped with involutions: the contragredient of a right correspondence is a left correspondence, but usually not another right correspondence\(^5\). More interesting notions of “bivariant” Kre˘ın C*-bimodules will be developed elsewhere.

**Example 5.9.** Whenever the time-orientable space-orientable (compact) semi-Riemannian even-dimensional manifold \( M \) admits a spinorial structure, or more generally a spin\(^*\) structure, (see details in H.Baum [Ba]) the family \( \Gamma(S(M)) \) of continuous section of a given complex spinor bundle \( S(M) \) becomes a Kre˘ın-Morita equivalence Kre˘ın C*-bimodule between \( C(M) \) (on the right) and the Kre˘ın C*-algebra \( \Gamma(\text{Cl}(M)) \) on the left\(^6\). Its contragredient Kre˘ın C*-bimodule \( \Gamma(S(M))^* \) is isomorphic to the Kre˘ın C*-bimodule of sections of the dual spinor bundle \( S(M)^* \) and we have \( \Gamma(S(M)) \otimes_{C(M)} \Gamma(S(M))^* \simeq \Gamma(\text{Cl}(M))^* \) as tensor product of Kre˘ın C*-bimodules.

### 6 Outlook

The discussions of duality and of spectral theory, via suitable “Kre˘ın bundles”, for some “commutative” subclasses of the Kre˘ın C*-modules here defined, will be dealt with in future works.\(^9\)

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\(^5\)For additional details on the Morita-Rieffel categories and strong Morita equivalence see for example the review sections in [BCL] and the references therein.

\(^6\)Recall that, given a bimodule \( \mathcal{X}_\mathcal{B} \), over involutive algebras \( \mathcal{A}, \mathcal{B} \), its contragredient bimodule \( \mathcal{Y}_\mathcal{A} \) is the same Abelian group \( \mathcal{X} := \mathcal{X}_\mathcal{A} \) with left/right actions defined via \( a \cdot \gamma := a^\gamma \), for all \( a \in \mathcal{A}, \gamma \in \mathcal{X} \). For a right Kre˘ın C*-correspondence \( \mathcal{X}_\mathcal{B} \) with right inner product \( (x \mid y)_\mathcal{B} \), for \( x, y \in \mathcal{X} \), the contragredient \( \mathcal{Y}_\mathcal{A} \) is naturally a left Kre˘ın C*-correspondence \( \mathcal{Y}_\mathcal{A} := (x \mid y)_\mathcal{A} \), for all \( \pi, \gamma \in \mathcal{X} \). A similar statement holds for left correspondences. A similar statement holds in the odd-dimensional case if the Clifford algebra \( \Gamma(\text{Cl}(M)) \) is replaced by its even part \( \Gamma(\text{Cl}(M))^* \); see [CGR] section 9.2.

\(^9\)See anyway [BHR] for some elementary results in the case of commutative Kre˘ın C*-algebras.
The notion of Kreĭn C*-module over a Kreĭn C*-algebra that we presented here, although interesting as a first step to explore some of the issues in semi-definite situations, is still too elementary to be fully useful for general applications to non-commutative spectral geometry, at least whenever the semi-Riemannian geometry involved presents topological obstructions to orientability, either in spacelike or in timelike sense (or both). Since global fundamental symmetries in Kreĭn C*-algebras are remnants of the fundamental decompositions of the Kreĭn spaces on which they are faithfully represented, their existence in situations coming from semi-Riemannian geometry seems to be a consequence of such global topological conditions of orientability and it is likely that a more general definition of a complete semi-definite analog of C*-algebras might be necessary to deal with such cases. A possible line of attack would be to define semi-definite modules that are direct summand submodules of our “free-splitting” Kreĭn modules over a C*-algebra (eliminating the topological obstruction on orientability via “embedding” into a wider environment exactly as we usually do in the case of projective but non-free modules) and redefine Kreĭn C*-algebras as compressions of the “free-splitting” Kawamura case. We might explore these and other possibilities in subsequent work.

A more immediately achievable important goal (especially in view of applications to examples of semi-Riemannian geometries related to relativistic physics) is the removal of the unitality (compactness) requirements in the definitions of Kreĭn C*-modules and Kreĭn C*-algebras.

Our main long-term interest is to formulate notions of semi-definite involutive operator algebraic environments that are suitable, as a (topological) background, for the development of non-commutative geometry and spectral triples in a completely general semi-definite situation.

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