The Brylinski Filtration for Affine Kac-Moody Algebras and Representations of $\mathcal{W}$-algebras

Suresh Govindarajan$^1$ · Sachin S. Sharma$^2$ · Sankaran Viswanath$^3$

Received: 8 March 2021 / Accepted: 3 October 2021 / Published online: 19 October 2021
© The Author(s), under exclusive licence to Springer Nature B.V. 2021

Abstract

We study the Brylinski filtration induced by a principal Heisenberg subalgebra of an affine Kac-Moody algebra $\mathfrak{g}$, a notion first introduced by Slofstra. The associated graded space of this filtration on dominant weight spaces of integrable highest weight modules of $\mathfrak{g}$ has Hilbert series coinciding with Lusztig’s $t$-analog of weight multiplicities. For the level 1 vacuum module $L(\Lambda_0)$ of affine Kac-Moody algebras of type $A$, we show that the Brylinski filtration may be most naturally understood in terms of representations of the corresponding $\mathcal{W}$-algebra. We show that the sum of dominant weight spaces of $L(\Lambda_0)$ in the principal vertex operator realization forms an irreducible Verma module of $\mathcal{W}$ and that the Brylinski filtration is induced by the Poincaré-Birkhoff-Witt basis of this module. This explicitly determines the subspaces of the Brylinski filtration. Our basis may be viewed as the analog of Feigin-Frenkel’s basis of $\mathcal{W}$ for the $\mathcal{W}$-action on the principal rather than on the homogeneous realization of $L(\Lambda_0)$.

Keywords Brylinski filtration · $\mathcal{W}$-algebras · Principal Heisenberg algebra · Level 1 vacuum module · Affine Kac-Moody algebras

Mathematics Subject Classification (2010) 17B67 · 17B69

Presented by: Anne Moreau

Sankaran Viswanath
svis@imsc.res.in

Suresh Govindarajan
suresh@physics.iitm.ac.in

Sachin S. Sharma
sachinsh@iitk.ac.in

$^1$ Department of Physics, Indian Institute of Technology Madras, Chennai, India
$^2$ Department of Mathematics and Statistics, Indian Institute of Technology, Kanpur, India
$^3$ The Institute of Mathematical Sciences, HBNI, Chennai, India
1 Introduction

We assume throughout that the ground field is $\mathbb{C}$. Let $\mathfrak{g}$ denote a symmetrizable Kac-Moody algebra. Let $L(\lambda)$ be an integrable highest weight representation of $\mathfrak{g}$ and $\mu$ a dominant weight of $L(\lambda)$. Lusztig’s $t$-analog of weight multiplicity is defined to be the polynomial [27]:

$$m^\lambda_\mu(t) = \sum_{w \in W} \varepsilon(w) K(w(\lambda + \rho) - (\mu + \rho); t),$$

(1.1)

where $\varepsilon$ is the sign character of the Weyl group $W$ of $\mathfrak{g}$, $\rho$ is the Weyl vector and $K$ is the $t$-analog of Kostant’s partition function defined by:

$$\sum_{\beta \in Q_+} K(\beta; t) e^\beta = \prod_{\alpha \in \Delta_+} (1 - te^\alpha)^{-m_\alpha}.$$

Here $\Delta_+$ is the set of positive roots, $Q_+ = \mathbb{Z}_{\geq 0}(\Delta_+)$ and $m_\alpha$ is the multiplicity of the root $\alpha$. At $t = 1$, $m^\lambda_\mu(t)$ reduces to the weight multiplicity $\dim L(\lambda|_\mu)$.

When $\mathfrak{g}$ is finite-dimensional, it is a classical fact that $m^\lambda_\mu(t)$ has non-negative integral coefficients. A principal nilpotent element of $\mathfrak{g}$ induces a filtration on $L(\lambda)$, called the Brylinski-Kostant filtration. The associated graded space of its restriction to $L(\lambda|_\mu)$ has Hilbert-Poincaré series $m^\lambda_\mu(t)$ [6, 7, 18].

For $\mathfrak{g}$ of affine type, it was conjectured by Braverman-Finkelberg [5] that the analogous result holds, i.e., that the associated graded space of the principal nilpotent filtration on $L(\lambda|_\mu)$ has Hilbert-Poincaré series $m^\lambda_\mu(t)$. Slofstra [28] showed that this was false, but that it becomes true if the principal nilpotent filtration is replaced by the principal Heisenberg filtration [28, Theorem 2.2]. The latter is the filtration of the weight spaces induced by the positive part of the principal Heisenberg subalgebra of $\mathfrak{g}$ (see Section 5.1 below); we shall call this the Brylinski filtration, following Slofstra.

There is a second interpretation of the polynomials $m^\lambda_\mu(t)$. In the finite-dimensional case, they coincide with the Kostka-Foulkes polynomials $K_{\lambda, \mu}(t)$, the coefficients that occur when the character of $L(\lambda)$ is expressed in the basis of the Hall-Littlewood polynomials $P_\mu(t)$ [17, 25]. This result was generalized to all symmetrizable Kac-Moody algebras in [29, Theorem 1]. In particular, when $\mathfrak{g}$ is a simply-laced affine Kac-Moody algebra, this interpretation enables a closed-form computation of the $m^\lambda_\mu(t)$ for the level 1 vacuum module (basic representation) $L(\Lambda_0)$ of $\mathfrak{g}$. This takes the generating function form [29, Corollary 2]:

$$\sum_{n \geq 0} m_{\Lambda_0}^{\Lambda_0 - n\delta}(t) q^n = \prod_{k=1}^{\ell} \prod_{n=1}^{\infty} (1 - t^{d_k} q^n)^{-1},$$

(1.2)

where $d_k$ ($1 \leq k \leq \ell$) are the degrees of the underlying finite-dimensional simple Lie algebra $\mathfrak{g}$, $\Lambda_0$ is the dominant weight of $\mathfrak{g}$ of level 1 which vanishes on the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and $\delta$ is the null root of $\mathfrak{g}$. As shown in [29], Eq. 1.2 is essentially equivalent to the $q$-Macdonald-Mehta constant term identity [8, Theorem 5.3] proved by Cherednik via Double Affine Hecke Algebra (DAHA) techniques. We recall here that the degrees $d_k$ are nothing but the exponents of $\mathfrak{g}$ plus one.

Springer
Consider the subspace \( Z = L(\Lambda_0) \bar{\mathfrak{g}} \) of \( \mathfrak{g} \)-invariants of \( L(\Lambda_0) \). It admits the ("energy") grading:

\[
Z = \bigoplus_{n \geq 0} Z_n := \bigoplus_{n \geq 0} L(\Lambda_0)_{\Lambda_0 - n \delta}.
\]

We restrict the Brylinski filtration to \( Z \); the associated graded space \( \text{gr} Z \) is then a bi-graded vector space. From Slofstra’s result mentioned above [28, Theorem 2.2] and Eq. 1.2, we obtain its two-variable Hilbert-Poincaré series:

\[
H(\text{gr} Z; t, q) := \sum_{n \geq 0} \sum_{i \geq 0} \dim \left( F_i^i Z_n / F_i^{i-1} Z_n \right) t^i q^n = \prod_{k=1}^{\ell} \prod_{n=1}^{\infty} (1 - t^{d_k} q^n)^{-1}, \tag{1.3}
\]

where \( F_i^i L(\Lambda_0) \) denotes the \( i^{th} \) subspace in the Brylinski filtration (Section 5.1) of \( L(\Lambda_0) \) and \( F_i^i U = U \cap F_i^i L(\Lambda_0) \) for a subspace \( U \) of \( L(\Lambda_0) \). We note that the \( t \)-degree arises from the Brylinski filtration while the \( q \)-degree is the energy grading.

Given a subspace \( U \) of \( L(\Lambda_0) \), a basis \( \mathcal{B} \) of \( U \) is said to be **Brylinski-compatible** if for all \( d \geq 0 \), \( \mathcal{B} \cap F_d U \) is a basis of \( F_d U \); in other words if the image of \( \mathcal{B} \) in \( \text{gr} U \) is a basis of \( \text{gr} U \). The main result of this paper is the construction of a natural Brylinski-compatible basis of \( Z \), for the untwisted affine Lie algebras of type \( A \).

To state our main theorem, we briefly recall the relevant facts about \( \mathcal{W} \)-algebras in general. Let \( \mathfrak{g} \) denote a simply-laced affine Lie algebra, \( \bar{\mathfrak{g}} \) the underlying finite-dimensional simple Lie algebra and \( \mathcal{W} = \mathcal{W}(\bar{\mathfrak{g}}) \) the corresponding \( \mathcal{W} \)-algebra. This is a vertex algebra which can be realized as a subalgebra of the lattice vertex algebra \( V_Q \), where \( Q \) is the root lattice of \( \bar{\mathfrak{g}} \). Let \( d_1 \leq d_2 \leq \cdots \leq d_r \) be the list of degrees of \( \bar{\mathfrak{g}} \) as in Eq. 1.2. The following theorem, due to Feigin and Frenkel [11], states an important fact about \( \mathcal{W} \)-algebras.

**Theorem 1** (Feigin-Frenkel) There exist elements \( \omega^i \in \mathcal{W} \) of degree \( d_i \) such that \( \mathcal{W} \) is freely generated by \( \omega^1, \omega^2, \ldots, \omega^\ell \).

For \( \mathfrak{g} = A_\ell^{(1)} \) \((\ell \geq 1)\), the degrees \( d_i \) are \( 2, 3, \ldots, \ell + 1 \). The field \( Y(\omega^i, z) \) has conformal weight \( d_i \) and we write:

\[
Y(\omega^i, z) = \sum_{n \in \mathbb{Z}} \omega^i z^{-n-d_i}.
\]

Here \( \omega^1 \) is the conformal vector of \( \mathcal{W} \), which corresponds to the Virasoro field. Let \( \ket{0} \) denote the vacuum vector of \( \mathcal{W} \). Theorem 1 states that the following is a basis of \( \mathcal{W} \) (in PBW fashion):

\[
\omega_{k_1}^{p_1} \omega_{k_2}^{p_2} \cdots \omega_{k_r}^{p_r} \ket{0} \tag{1.4}
\]

with (i) \( r \geq 0 \), (ii) \( \ell \geq p_1 \geq p_2 \geq \cdots \geq p_r \geq 1 \), (iii) \( k_j \leq -d_{p_j} \) for all \( j \) and (iv) if \( p_i = p_{i+1} \), then \( k_i \leq k_{i+1} \).

The level 1 vacuum module \( L(\Lambda_0) \) of \( \mathfrak{g} \) has many realizations in terms of vertex operators, one for each conjugacy class of the Weyl group \( W(\bar{\mathfrak{g}}) \) of \( \bar{\mathfrak{g}} \) [23]. The principal realization of \( L(\Lambda_0) \) corresponds to the conjugacy class of a Coxeter element \( \sigma \in W(\bar{\mathfrak{g}}) \). This realization endows \( L(\Lambda_0) \) with the structure of a \( \sigma \)-twisted representation of the lattice vertex algebra \( V_Q \). When restricted to the vertex subalgebra \( \mathcal{W} \subset V_Q \), this representation becomes untwisted, since \( \mathcal{W} \) is pointwise fixed by \( \sigma \) (and indeed by every Weyl group element) [3]. The following is our main theorem.
Theorem 2 Let $\mathfrak{g} = A_\ell^{(1)}$. Consider the principal realization of $L(\Lambda_0)$ as a $\mathcal{W}$-module and let $v_{\Lambda_0}$ denote a highest weight vector of $L(\Lambda_0)$.

(a) The $\mathcal{W}$-submodule of $L(\Lambda_0)$ generated by $v_{\Lambda_0}$ is exactly the space $Z = L(\Lambda_0)^{\mathfrak{h}}$.

(b) $Z$ is isomorphic to a Verma module of $\mathcal{W}$, and is irreducible.

(c) The following vectors form a Brylinski-compatible basis of $Z$:

$$
\begin{align*}
\omega_{k_1}^{p_1}(\sigma) \omega_{k_2}^{p_2}(\sigma) \cdots \omega_{k_r}^{p_r}(\sigma) v_{\Lambda_0},
\end{align*}
$$

where (i) $r \geq 0$ (ii) $\ell \geq p_1 \geq p_2 \geq \cdots \geq p_r \geq 1$ (iii) $k_j \leq -1$ for all $j$ and (iv) if $p_i = p_{i+1}$, then $k_i \leq k_{i+1}$.

We note that the modes $\omega_{n}^i(\sigma)$ in Eq. 1.5 now refer to the action of $\omega^i \in \mathcal{W}$ on the principal realization of $L(\Lambda_0)$:

$$
Y_{L(\Lambda_0)}(\omega^i, z) = \sum_{n \in \mathbb{Z}} \omega_{n}^i(\sigma) z^{-n-d_i}.
$$

Unlike the Feigin-Frenkel basis Eq. 1.4 which involves only the modes $k_j \leq -d_p$, here we have $k_j \leq -1$ for all $j$. We may view the Feigin-Frenkel basis as the analog of Eq. 1.5 for the homogeneous realization of $L(\Lambda_0)$, with $\mathfrak{h}$-invariants replaced by $\mathfrak{g}$-invariants (see Remark 6). As a corollary to Theorem 2, we obtain an explicit description of the subspaces of the Brylinski filtration:

**Corollary 3** Let $n, d \geq 0$. The subspace $F^d Z_n$ has a basis given by the vectors in Eq. 1.5 satisfying $\sum_{i=1}^r d_{p_i} \leq d$ and $\sum_{i=1}^r k_i = -n$.

Verma modules of $\mathcal{W}$-algebras are generically irreducible [1]. We show that the highest weight of the $\mathcal{W}$-module $Z$ is a generic weight (see Proposition 12) indirectly, using results of Frenkel-Kac-Radul-Wang [12] which relate representations of $\mathcal{W}$-algebras in type $A$ to those of $\hat{D}$, the universal central extension of the Lie algebra of regular differential operators on the circle. We however expect Theorem 2 to hold for all simply-laced affine algebras.

We remark that the representation of $\mathcal{W}$ obtained by restricting the principal realization of $L(\Lambda_0)$ has been previously considered by Bakalov-Milanov [3] and other authors (see references in [3]) for its role in the study of singularities. Representations of $\mathcal{W}$-algebras obtained by restricting more general twisted representations of affine Kac-Moody algebras have also been recently studied via heuristic character formulas in [4]. However, to the best of our knowledge, a rigorous proof of Theorem 2(b) does not appear in the literature. Further, the connection to Cherednik’s $q$-Macdonald-Mehta constant term identity in type $A$ provided by Theorem 2(c) does not seem to have been noticed before.

The paper is organized as follows. Sections 2–4 contain background and establish notation on lattice vertex algebras and their twisted modules. A reader acquainted with these notions can skip directly to Section 5, which describes the Brylinski filtration on $L(\Lambda_0)$ in the principal realization. Section 6 recalls the main facts about $\mathcal{W}$-algebras and their Verma modules. Sections 7–8 culminate in proving that $Z$ is an irreducible Verma module of $\mathcal{W}$. Finally, Section 9 completes the proof of Theorem 2 by showing that the natural PBW basis of this Verma module is in fact compatible with the Brylinski filtration.

# 2 Vertex algebras, representations

We first recall the primary notions about vertex algebras and fix notation [2, 3, 21].
2.1 Vertex algebras

A vertex algebra is a vector space $V$, with a vector $|0\rangle \in V$ (vacuum vector) and a map $Y : V \rightarrow \text{End} V[[z, z^{-1}]]$ (state-field correspondence):

$$Y(a, z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}$$

satisfying the following axioms [21, Prop 4.8(b)]:

(a) $Y(a, z)b$ is a Laurent series in $z$ for all $a, b \in V$

(b) $Y(|0\rangle, z) = \text{id}_V$

(c) $a(-1) |0\rangle = a$ for all $a \in V$

(d) the modes satisfy the Borcherds identity:

$$\sum_{j=0}^{\infty} (-1)^j \binom{n}{j} (a(m+n-j)(b(k+j)c) - (-1)^n b(k+n-j)(a(m+j)c)) = \sum_{j=0}^{\infty} \binom{m}{j} (a(n+j)b)(k+m-j)c$$

for all $a, b, c \in V$ and $m, n, k \in \mathbb{Z}$.

2.2 Strongly generating subset

We recall that a vertex algebra $V$ is said to be strongly generated by a subset $X \subset V$ if $V$ is spanned by the vectors obtained by repeated actions of the negative modes of elements of $X$ on the vacuum $|0\rangle$. In other words:

$$V = \text{span} \{ x_1^{k_1}(t_1) \cdots x_r^{k_r}(t_r) |0\rangle : r \geq 0, x^i \in X, k_i < 0 \}.$$

2.3 The Heisenberg vertex algebra

Let $\hat{h}$ be a finite-dimensional vector space with a symmetric nondegenerate bilinear form $(\cdot | \cdot)$. We denote the affinization$^1$ of $h$ by $\hat{h} = h \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$, where $K$ is the central element. Letting $ht^j$ denote the element $h \otimes t^j$, the Lie bracket is defined by $[ht^j, ht^k] = \delta_{j+k,0} (ht^j) - \delta_{j-k,0} (ht^k) K$, for $h, t \in \hat{h}$ and $j, k \in \mathbb{Z}$. Thus $\hat{h}$ is isomorphic to a Heisenberg Lie algebra. Let

$$\mathcal{F} = \text{Ind} \hat{h} \oplus \mathbb{C}K = U\hat{h} \otimes U(\hat{h} \oplus \mathbb{C}K)$$

be the highest weight irreducible representation of $\hat{h}$, where $\hat{h}^+ := h \otimes \mathbb{C}[t]$ acts trivially on $\mathbb{C}$ and $K$ acts as 1. This is called the Fock space of $\hat{h}$. We may identify $\mathcal{F}$ (as a vector space) with $S\hat{h}^-$, where $\hat{h}^- := h \otimes t^{-1}\mathbb{C}[t^{-1}]$. We have a grading: $\mathcal{F} = \bigoplus_{p \geq 0} \mathcal{F}^{|p|}$, where $\mathcal{F}^{|p|}$ is spanned by $(\hat{h} \otimes t^{j_1}) \cdots (\hat{h} \otimes t^{j_r})$ with $j_i < 0, \sum j_i = -p$. The space $\mathcal{F}$ forms a vertex algebra, called the Heisenberg vertex algebra. It is strongly generated by $\hat{h} \otimes t^{-1}$, with state-field correspondence determined by:

$$Y(ht^{-1}, z) = \sum_{n \in \mathbb{Z}} (ht^n) z^{-n-1} \text{ for } h \in \hat{h}.$$

$^1$The $t$ which occurs here is not to be confused with the indeterminate $t$ of Eq. 1.1. Since the two don’t appear together in this paper, we permit ourselves this mild notational conflict so as to adhere to standard notations as much as possible.
This vertex algebra has a conformal vector [21, Prop. 3.5]:

\[
\omega = \frac{1}{2} \sum_{i=1}^{\dim \mathfrak{h}} a^i (-1) b^i (-1) |0\rangle,
\]

(2.3)

where \{a^i\} and \{b^i\} are bases of \(\mathfrak{h}\) such that \(a^i | b^j\) = \(\delta_{ij}\) and the vacuum vector \(|0\rangle\) is the highest weight vector \(1 \otimes 1\) of \(\mathcal{F}\). The central charge of the Virasoro field \(Y(\omega, z) = \sum_{n \in \mathbb{Z}} \omega_n z^{-n-2}\) is \(\dim \mathfrak{h}\).

2.4 Representations

A representation of the vertex algebra \(V\) (or \(V\)-module) is a vector space \(M\) together with fields \(Y_M(a, z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}\) for each \(a \in V, m \in M\). The axioms are (i) \(Y_M(|0\rangle, z) = \text{id}_M\), (ii) \(Y_M(a, z) m\) is a Laurent series in \(z\) for each \(a \in V, m \in M\), and (iii) the Borcherds identity Eq. 2.1 holds for all \(a, b \in V, c \in M\) and all integers \(m, n, k\).

2.5 The oscillator representations of \(\mathcal{F}\)

For \(\lambda \in \mathfrak{h}^*\), let \(C_\lambda\) be the one-dimensional \((\hat{\mathfrak{h}}^+ \otimes \mathbb{C}K)\)-module on which \(\mathfrak{h} \otimes \mathbb{C}[t] \) acts trivially, \(\mathfrak{h} \otimes 1\) acts as \(\lambda(h)\) for all \(h \in \mathfrak{h}\) and \(K\) acts as 1. Let

\[\pi_\lambda = \text{Ind}_{\hat{\mathfrak{h}}^+ \otimes \mathbb{C}K}^{\hat{\mathfrak{h}}} C_\lambda.\]

This is a highest weight irreducible representation of \(\hat{\mathfrak{h}}\), called the oscillator representation. We let \(|\lambda\rangle\) denote the highest weight vector \(1 \otimes 1\) of \(\pi_\lambda\). The space \(\pi_\lambda\) is also a representation of the vertex algebra \(\mathcal{F}\), with the fields given by the same formula as in Eq. 2.2, but now with the \(ht^n\) interpreted as operators on \(\pi_\lambda\). The conformal vector \(\omega \in \mathcal{F}\) induces a grading on \(\pi_\lambda\); more precisely, we have \(\pi_\lambda = \bigoplus_{k \in \mathbb{C}} \pi_\lambda^{[k]}\), where \(\pi_\lambda^{[k]} = \{x \in \pi_\lambda : \omega_0(x) = kx\}\). For instance, one has \(\omega_0(\lambda) = (|\lambda|/2)|\lambda\rangle\). The character of \(\pi_\lambda\) is defined to be \(\text{tr}_{\pi_\lambda} q^{\omega_0}\) and is given by [24, (3.4)]:

\[\text{tr}_{\pi_\lambda} (q^{\omega_0}) = \frac{q^{(|\lambda|/2)}}{\varphi(q)^{\dim \mathfrak{h}}},\]

where \(\varphi(q) = \prod_{n \geq 1} (1 - q^n)^{-1}\) is Euler’s \(\varphi\)-function.

2.6 Twisted representations

Let \(\sigma\) be a finite order automorphism of a vertex algebra \(V\), of order \(d\) say. Then \(\sigma\) is diagonalizable, with eigenvalues \(\eta^j\), where \(\eta = e^{2\pi i/d}\) and \(j \in \mathbb{Z}\). A \(\sigma\)-twisted representation of \(V\) is a vector space \(M\) together with fields \(Y_M(a, z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}\) for each \(a \in V\). The axioms are now [2, Remark 3.1], [3, §3.1]:

(i) \(Y_M(|0\rangle, z) = \text{id}_M\) (ii) \(Y_M(a, z) m\) is a Laurent series in \(z^{1/d}\) for each \(a \in V, m \in M\)

\(\odot\) Springer
and (iii) the Borcherd’s identity Eq. 2.1 holds for all \( a, b \in V, \ c \in M \) and for all \( n \in \mathbb{Z}, \ m, k \in \frac{1}{d}\mathbb{Z} \).

### 2.7 The twisted Heisenberg Fock space

Let \( \mathfrak{h} \) be a finite-dimensional vector space with a symmetric nondegenerate bilinear form \((\cdot | \cdot)\) and let \( \varphi \) be a linear automorphism of \( \mathfrak{h} \) of order \( m \), preserving \((\cdot | \cdot)\). This induces an automorphism (also denoted \( \varphi \)) of the vertex algebra \( \mathcal{F} \) of Section 2.3. One can extend \( \varphi \) to an operator on the space \( \mathfrak{h} \otimes \mathbb{C}[t^{1/m}, t^{-1/m}] \oplus \mathbb{C}K \) via:

\[
\varphi(h \otimes t^p) = \varphi(h) \otimes e^{2\pi i p} t^p, \quad \varphi(K) = K \quad \text{for all } h \in \mathfrak{h}, \ p \in \frac{1}{m}\mathbb{Z}.
\]

The \( \varphi \)-twisted Heisenberg algebra \( \mathfrak{h}_\varphi \) is defined to be the set of fixed points of \( \varphi \) [3, §3.3]. It has a basis comprising the elements \( K \) and \( ht^p := h \otimes t^p \), where \( h \in \mathfrak{h} \), \( mp \in \mathbb{Z} \) such that \( \varphi(h) = e^{-2\pi i p} h \) and the Lie bracket is: \([at^p, bt^q] = p \delta_{p,-q} (a | b) K \) with \( K \) central. Let \( \mathfrak{h}_\varphi^+ \) (respectively \( \mathfrak{h}_\varphi^- \)) denote the span of the elements \( ht^p \) of the above form for \( p \geq 0 \) (respectively \( p < 0 \)). As in Section 2.3, we define the \( \varphi \)-twisted Fock space

\[
\mathcal{F}_\varphi = \text{Ind}_{\mathfrak{h}_\varphi^-}^{\mathfrak{h}_\varphi^+} \mathbb{C},
\]

where \( \mathfrak{h}_\varphi^- \) acts trivially on \( \mathbb{C} \) and \( K \) acts as 1. Then \( \mathcal{F}_\varphi \cong S \mathfrak{h}_\varphi^- \) (as vector spaces) becomes a \( \varphi \)-twisted representation of \( \mathcal{F} \) [3, §3.3]. For \( h \in \mathfrak{h} \) with \( \varphi(h) = e^{2\pi i p} h \), we have

\[
\sum_{n \in p + \mathbb{Z}} (ht^n) z^{-n-1}.
\]

### 2.8 Product identity

For later use, we record the following “product identity” of Bakalov-Milanov [3, Proposition 3.2] for twisted representations, rewritten in terms of modes:

**Proposition 4** (Bakalov-Milanov) Let \( V \) be a vertex algebra, \( \sigma \) an automorphism of \( V \) of finite order \( d \) and \( M \) a \( \sigma \)-twisted \( V \)-module. Let \( a, b \in V \) and \( N_{ab} \) be a non-negative integer such that \( a(k)b = 0 \) for all \( k \geq N_{ab} \). Fix \( r \in \frac{1}{d}\mathbb{Z} \), \( n \in \mathbb{Z} \). Then for any \( c \in M \), we have:

\[
(a(n)b)_c^{(r)} = \sum_{p,q \in \frac{1}{d}\mathbb{Z}} \kappa(p) a(p)b(q)c,
\]

where \( \kappa(p) \) is a scalar, given by \( \kappa(p) = \sum_{m=0}^{N} (-1)^{N-m} \binom{N}{m} \binom{m-p-1}{N-n-1} \) with \( N = \max(N_{ab}, n + 1) \).
3 The root lattice vertex algebra and its automorphisms

3.1 The lattice vertex algebra $V_Q$

We follow [3, 10, 21]. Let $\mathfrak{g}$ denote a finite-dimensional simple Lie algebra over $\mathbb{C}$. We further assume $\mathfrak{g}$ is simply-laced, i.e., of types $A, D$ or $E$. Let $\Delta$ denote the set of roots, $Q = \mathbb{Z}\Delta$ denote the root lattice and $\overline{\mathfrak{g}} = \mathbb{C} \otimes_{\mathbb{Z}} Q$ the Cartan subalgebra of $\mathfrak{g}$. We assume $(\cdot | \cdot)$ is the Killing form normalized such that $(\alpha | \alpha) = 2$ for all roots $\alpha$. As in Section 2.3, we consider the Heisenberg Lie algebra $\hat{\mathfrak{h}} = \mathfrak{h}[t, t^{-1}] \oplus \mathbb{C} K$ and let $\mathcal{F}$ denote its Fock space.

Let $\mathbb{C}_{e[Q]}$ denote the twisted group algebra of $Q$ with multiplication $e^\alpha e^\beta = \varepsilon(\alpha, \beta) e^{\alpha+\beta}$, where $\varepsilon : Q \times Q \to \{\pm 1\}$ is a bimultiplicative cocycle satisfying $\varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha) = (-1)^{(\alpha | \beta)}$. The lattice vertex algebra $V_Q$ is defined to be the space

$$V_Q = \mathcal{F} \otimes_{\mathbb{C}} \mathbb{C}_{e[Q]}.$$ 

This can be made into an $\hat{\mathfrak{h}}$-module by declaring each $1 \otimes e^\beta$, $\beta \in Q$ to be a highest weight vector, of highest weight $\beta$, i.e., $ht^k (1 \otimes e^\beta) = \delta_{k,0} \langle h, \beta \rangle (1 \otimes e^\beta)$ for $k \geq 0$, $h \in \overline{\mathfrak{h}}$. We let $h_n$ denote the operator $ht^n \otimes \text{id}$ on $\mathcal{F} \otimes_{\mathbb{C}} \mathbb{C}_{e[Q]}$. We have the state-field correspondences:

$$Y(ht^{-1} \otimes 1, z) = \sum_{n \in \mathbb{Z}} h_n z^{-n-1},$$

$$Y(1 \otimes e^\alpha, z) = e^\alpha z^{\alpha_0} \exp \left( -\sum_{n<0} \frac{z^{-n}}{n} \alpha_n \right) \exp \left( -\sum_{n>0} \frac{z^{-n}}{n} \alpha_n \right),$$

where $h \in \overline{\mathfrak{h}}$, $\alpha \in Q$. Here, $e^\alpha$ denotes the left multiplication operator by $1 \otimes e^\alpha$, the operator $z^{\alpha_0}$ acts as $z^{\alpha_0} (\zeta \otimes e^\beta) = z^{(\alpha | \beta)} (\zeta \otimes e^\beta)$ for $\zeta \in \mathcal{F}$ and we identify $\alpha \in \overline{\mathfrak{h}}^*$ with its dual in $\overline{\mathfrak{h}}$ under the normalized Killing form. The vacuum vector is $|0\rangle = 1 \otimes e^0$. The elements $(ht^{-1} \otimes 1 : h \in \overline{\mathfrak{h}})$ and $\{1 \otimes e^\alpha : \alpha \in \Delta\}$ strongly generate $V_Q$. It is clear that $\mathcal{F}$ is a vertex subalgebra of $V_Q$.

3.2 Conformal vector, grading

The conformal vector of $V_Q$ is given by:

$$\omega = \frac{1}{2} \sum_{i=1}^{\dim \overline{\mathfrak{h}}} a^i_{-1} b^i_{-1} |0\rangle,$$

where $\{a_i^j\}$ and $\{b_i^j\}$ are bases of $\overline{\mathfrak{h}}$ such that $(a^i | b^j) = \delta_{ij}$. Let the corresponding Virasoro field be $L(z) = Y(\omega, z) = \sum_n L_n z^{-n-2}$ and define $V_Q^{[d]} := \{v \in V_Q : L_0 v = dv \}$.

This defines a vertex algebra grading $V_Q = \bigoplus_{d \in \mathbb{Z}_+} V_Q^{[d]}$, with $ht^{-n} \otimes 1 \in V_Q^{[n]}$ and $1 \otimes e^\alpha \in V_Q^{[(\alpha | \alpha)/2]}$ for all $n \geq 0$ and $\alpha \in Q$. In particular, $V_Q^{[1]}$ is spanned by the elements $ht^{-1} \otimes 1$ and $1 \otimes e^\alpha$ for $h \in \overline{\mathfrak{h}}$, $\alpha \in \Delta$. Hence $V_Q^{[1]}$ strongly generates $V_Q$.

3.3 Derivations and automorphisms

We follow [10]. An automorphism of the lattice vertex algebra $V_Q$ is a vector space isomorphism $\varphi : V_Q \to V_Q$ that satisfies $\varphi(\omega) = \omega$ and $\varphi(a_{(n)} b) = \varphi(a)_{(n)} \varphi(b)$ for all $a, b \in V_Q$. A derivation is an operator $\delta : V_Q \to V_Q$ that satisfies $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in V_Q$. An automorphism $\varphi$ is inner if it is of the form $\varphi = \psi_1 \circ \psi_2$ for some $\psi_1, \psi_2 \in GL(V_Q)$.
a, b ∈ V_Q, n ∈ ℤ. A derivation D : V_Q → V_Q is a linear map that satisfies Dω = 0 and 
D(a(n)b) = (Da)(n)b + a(n)Db for all a, b ∈ V_Q, n ∈ ℤ. If D is a derivation, then exp D 
defines an automorphism of V_Q. An automorphism fixes the vacuum vector |0⟩, while a 
derivation annihilates it. Further, derivations and automorphisms preserve the grading of 
V_Q, in particular they preserve the space V_Q^{[1]}. Since this space strongly generates V_Q, a 
derivation or automorphism is uniquely determined by its action on V_Q^{[1]}.

3.4 Inner automorphisms

The space V_Q^{[1]} becomes a Lie algebra under the bracket [a, b] = a(0)b for a, b ∈ V_Q^{[1]} . It is 
isomorphic to g, under an isomorphism mapping (ht⁻¹ ⊗ 1) to h and (1 ⊗ eα) to an element 
of the root space gα, for each h ∈ ℂ, α ∈ Δ. We identify V_Q with g via this isomorphism. 

Given X ∈ V_Q^{[1]}, the map X(0) is a derivation of V_Q. Restricted to V_Q^{[1]}, this coincides 
with the adjoint action of X on V_Q^{[1]} viewed as a Lie algebra. We let Inn V_Q denote the group 
of inner automorphisms of V_Q. This is the subgroup of Aut V_Q generated by {exp X(0) : 
X ∈ V_Q^{[1]}}. Further, since V_Q^{[1]} strongly generates V_Q, it is clear that Inn V_Q is isomorphic 
to the group Inn g = (exp(ad X) : X ∈ g) of inner automorphisms of g [10, §2].

Given ϕ ∈ Inn g, say ϕ = exp(ad X¹) exp(ad X²) . . . exp(ad X^p) with X^i ∈ g ⊆ V_Q^{[1]}, we 
denote by ˜ϕ ∈ Inn V_Q its unique lift to an automorphism of V_Q:

\[ \tilde{\varphi} = \exp X^1(0) \exp X^2(0) \cdots \exp X^p(0). \] (3.4)

It is clear that if ϕ has (finite) order d, then so does ˜ϕ.

4 Twisted modules of the lattice vertex algebra as g-modules

In this section, let g denote a simply-laced affine Lie algebra. Let ˜g denote the underlying 
finte-dimensional simple Lie algebra with root lattice Q and V_Q the corresponding lattice 
vertex algebra.

4.1 Twisted realization of g

Let m ≥ 1 and consider the Lie algebra ˜g ⊗ ℂ[t¹/m, t⁻¹/m] ⊕ K ⊕ ℂd. Here K is central 
and the other Lie brackets are given by [a ⊗ t^p, b ⊗ t^q] = [a, b] ⊗ t^{p+q} + pδ_{p,-q}(a | b)K, [d, a ⊗ 
t^p] = p(a ⊗ t^p) for a, b ∈ ˜g and p, q ∈ (1/m)ℤ.

Now let ϕ be an inner automorphism of ˜g of order m. Extend the action of ϕ to the space 
˜g ⊗ ℂ[t¹/m, t⁻¹/m] ⊕ K ⊕ ℂd via:

\[ \varphi(x ⊗ t^p) = \varphi(x) ⊗ e^{2\pi ip} t^p, \quad \varphi(K) = K, \quad \varphi(d) = d \quad \text{for all } x ∈ ˜g, p ∈ \frac{1}{m} \mathbb{Z}. \]

Define g[ϕ] to be the set of fixed points of ϕ. This is a Lie subalgebra and has a basis 
comprising the elements K, d and a ⊗ t^p, where a ∈ ˜g, mp ∈ ℤ such that ϕ(a) = e⁻²πipa. 
Since ϕ is an inner automorphism of ˜g, the Lie algebra g[ϕ] is isomorphic to the affine Lie 
algebra g [20, Theorem 8.5].
4.2 Twisted modules of $V_Q$

Let $\bar{g}$ denote the unique lift of $\varphi$ to an automorphism of $V_Q$ as in Eq. 3.4. Suppose $M$ is a $\bar{g}$-twisted $V_Q$-module. Let $Y_M(a, z) = \sum_{p \in (1/m)\mathbb{Z}} a(p) z^{-p-1}$ be the corresponding fields. Then $M$ becomes a $g[\varphi]$-module via the action:

$$a \otimes t^p \mapsto a(p), \quad K \mapsto 1, \quad d \mapsto -\omega_0,$$

where $p \in \frac{1}{m} \mathbb{Z}$, $\varphi(a) = e^{-2\pi i p} a$ and $\omega$ is the conformal vector of $V_Q$ [26].

4.3 The automorphisms $\sigma, \zeta$ and $\psi$

In what follows, we consider two specific automorphisms of $\bar{g}$. Let $\sigma$ denote a Coxeter element of $\bar{g}$. It is the element of $GL(h)$ given by the product of all simple reflections (in some order). It has order $h$, the Coxeter number of $\bar{g}$. Since $\sigma$ is a Weyl group element, we may lift it to an inner automorphism of $\bar{g}$ of order $h$. We also denote this automorphism as $\sigma$. We have another finite order inner automorphism of $\bar{g}$, defined by:

$$\zeta = \exp \left( \frac{2\pi i \rho^\vee}{h} \right) = \text{Ad} \left( \exp \frac{2\pi i \rho^\vee}{h} \right),$$

where $\rho^\vee \in \bar{h}$ is the unique element such that $\alpha_i(\rho^\vee) = 1$ for all simple roots $\alpha_i$ of $\bar{g}$. It is well-known [19, Prop. 3.4, Remark (e)] that the automorphisms $\sigma$ and $\zeta$ are conjugate under an inner automorphism $\psi$ of $\bar{g}$, i.e., $\psi \sigma \psi^{-1} = \zeta$. Since they are all inner, $\sigma, \zeta, \psi$ lift to automorphisms $\bar{\sigma}, \bar{\zeta}, \bar{\psi}$ of $V_Q$ via Eq. 3.4. The automorphisms $\bar{\sigma}$ and $\bar{\zeta}$ also have order $h$. Let $u = (h_1 t^{j_1})(h_2 t^{j_2}) \cdots (h_k t^{j_k}) \in \bar{F}$ with $h_i \in \bar{h}$ and $j_i < 0$ for $1 \leq i \leq k$ (see Section 2.3). The action of $\bar{\sigma}$ and $\bar{\zeta}$ on $u \otimes e^\alpha \in V_Q$ is given by [3, 9]:

$$\bar{\sigma}(u \otimes e^\alpha) = (\sigma(h_1) t^{j_1}) (\sigma(h_2) t^{j_2}) \cdots (\sigma(h_k) t^{j_k}) \otimes e^{\sigma \alpha},$$

$$\bar{\zeta}(u \otimes e^\alpha) = e^{2\pi i \alpha(\rho^\vee)/h} (u \otimes e^\alpha).$$

Finally, we observe that since $\psi \sigma \psi^{-1} = \zeta$, the automorphisms $\bar{\psi} \bar{\sigma} \bar{\psi}^{-1}$ and $\bar{\zeta}$ agree on $V_Q^{[1]}$. Since $V_Q^{[1]}$ strongly generates $V_Q$, we obtain $\bar{\psi} \bar{\sigma} \bar{\psi}^{-1} = \bar{\zeta}$ on $V_Q$.

4.4 The $\sigma$-twisted realization of the vacuum module

There is a $\bar{\sigma}$-twisted representation $M_\sigma$ of the lattice vertex algebra $V_Q$ such that the associated representation of $g \cong g[\sigma]$ is isomorphic to the vacuum module $L(\Lambda_0)$. This is called the principal vertex operator realization [22]. The linear automorphism $\sigma$ of $\bar{h}$ leaves the form on $\bar{h}$ invariant. We have

$$M_\sigma = \bar{F}_\sigma,$$

the corresponding $\sigma$-twisted Fock space obtained by taking $\varphi = \sigma$ in Eq. 2.4. The action of $\bar{F} \subset V_Q$ on $M_\sigma$ given by Eq. 2.5 extends to an action of all of $V_Q$ [3, (3.11)].

4.5 The $\zeta$-twisted realization of the vacuum module

Similarly, there is a $\bar{\zeta}$-twisted representation $M_\zeta$ of $V_Q$ which is isomorphic to $L(\Lambda_0)$ as a $g \cong g[\zeta]$-module. This is given by $M_\zeta = V_{Q+\rho/h}$, i.e., $M_\zeta$ coincides with $V_Q = \bar{F} \otimes_{\mathbb{C}} C_{V_Q}$ as a vector space, with the action of $V_Q$ on $M_\zeta$ given by the $\rho/h$-shifted versions of the vertex operators in Eqs. 3.1 and 3.2. To define these, we let the operators $h_n$, $n \neq 0$ and
e^\alpha, \alpha \in Q act on V_Q in the same manner as in Section 3.1, and redefine the actions of h_0 and \varepsilon^\alpha_0 as follows:

\[ h_0(u \otimes e^\beta) = (\beta + \rho/h)(h)(u \otimes e^\beta) \]  
(4.1)

\[ \varepsilon^\alpha_0(u \otimes e^\beta) = (\alpha | \beta + \rho/h)(u \otimes e^\beta) \]  
(4.2)

for all h \in \widehat{h}, \alpha, \beta \in Q, u \in \mathcal{F}. With these definitions, the vertex operators in Eqs. 3.1 and 3.2 make \( M_\zeta \) into a \( \tilde{\zeta} \)-twisted \( V_Q \)-module \(^2\). Viewed as a \( \mathfrak{g} = \mathfrak{g}[\zeta] \)-module, it is isomorphic to a level 1 irreducible highest weight representation \(^2\). That the highest weight is \( \Lambda_0 \) can be readily seen from the definition of the shifted vertex operators and the isomorphism \( \mathfrak{g} \sim \mathfrak{g}[\zeta] \).

### 4.6 Isomorphism of the two realizations of the vacuum module

Now \( M_\sigma \cong L(\Lambda_0) \) as a \( \mathfrak{g}[\sigma] \)-module and \( M_\zeta \cong L(\Lambda_0) \) as a \( \mathfrak{g}[\zeta] \)-module. In other words \( M_\sigma \) and \( M_\zeta \) are isomorphic as \( \mathfrak{g} \cong \mathfrak{g}[\sigma] \cong \mathfrak{g}[\zeta] \) modules. From the discussion of Sections 4.1-Section 4.3, this implies that there is a map (cf., \(^{14}\), Theorem 4.5.2 for the \( \hat{\mathfrak{sl}}_2 \)-case):

\[ \Psi : M_\sigma \rightarrow M_\zeta \]  
(4.3)

Here \( \bullet_\sigma \) and \( \bullet_\zeta \) indicate the \( V_Q \)-actions on \( M_\sigma \) and \( M_\zeta \) respectively. Since \( V_Q^{[1]} \) strongly generates \( V_Q \), Eq. 4.3 holds for all \( X \in V_Q \).

Equivalently:

\[ \Psi Y_{M_\sigma}(X; z) \Psi^{-1} = Y_{M_\zeta}(\tilde{\psi}(X); z) \]  
(4.5)

5 The Brylinski filtration on \( L(\Lambda_0) \)

### 5.1 Definition

Let \( \mathfrak{g} \) denote a simply-laced affine Lie algebra, with standard Chevalley generators \( e_i, f_i, \alpha_i^\vee \) \((0 \leq i \leq \ell)\), canonical central element \( K \) and derivation \( d \). The principal gradation of \( \mathfrak{g} \) is defined by setting \( \deg e_i = 1 = -\deg f_i \) and \( \deg \alpha_i^\vee = \deg K = \deg d = 0 \). The principal Heisenberg subalgebra \( \mathfrak{s} \) of \( \mathfrak{g} \) is defined as:

\[ \mathfrak{s} = \{ x \in \mathfrak{g} : [x, e] \in \mathbb{C}K \}, \]

where \( e = \sum_{i=0}^\ell e_i \). It is isomorphic to an infinite-dimensional Heisenberg Lie algebra \(^{20}\), Lemma 14.4]. Since \( e \) is a homogeneous element, it follows that \( \mathfrak{s} \) is graded with respect to the principal gradation of \( \mathfrak{g} \): \( \mathfrak{s} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{s}_j \) with \( \mathfrak{s}_0 = \mathbb{C}K \). We set \( \mathfrak{s}^+ \) (respectively \( \mathfrak{s}^- \)) to be \( \bigoplus_{j > 0} \mathfrak{s}_j \) (respectively \( \bigoplus_{j < 0} \mathfrak{s}_j \)). The multiset comprising the elements \( j \neq 0 \) each repeated \( \dim \mathfrak{s}_j \) times is called the multiset of exponents of the affine Lie algebra \( \mathfrak{g} \) \(^{20}\), Chapter 14, Table E].

\(^2\)There is a sign error in \(^{9}\), Theorem 3.4 (1) (it should be \( V_{L-\beta} \)) which is corrected in part (2) of that theorem. Their convention on twisted modules and our (now standard) convention differ by a sign.
The subspaces \( F^i L(\Lambda_0) \) of the Brylinski filtration [28] are defined (for \( i \geq -1 \)) by:
\[
F^i L(\Lambda_0) = \{ v \in L(\Lambda_0) : x^{i+1} v = 0 \text{ for all } x \in s^+ \}.
\]
Since \( s^+ \) is abelian, we have \( v \in F^i L(\Lambda_0) \) iff
\[
x_1 x_2 \cdots x_{i+1} v = 0 \text{ for all tuples } (x_1, \cdots, x_{i+1}) \text{ of elements from } s^+.
\] (5.1)

### 5.2 In the principal vertex operator realization

It is easiest to describe the action of \( s \) on \( L(\Lambda_0) \) in the principal vertex operator realization \( M_\sigma \) of Section 4.4. We identify \( L(\Lambda_0) \) with the space \( M_\sigma \) via the isomorphism of Section 4.4. Then, the action of \( s \) on \( M_\sigma \) is given by the modes of the fields:

\[
Y_{M_\sigma}(ht^{-1}, z) = \sum_{j \in \mathbb{Z}} h(j)(\sigma) z^{-j-1},
\]

where \( h \in \mathfrak{h} \). In particular, the image of \( s^+ \) (respectively \( s^- \)) under the natural map \( g \to \mathbb{End} M_\sigma \) is the span of \( \{ h(j)(\sigma) : h \in \mathfrak{h}, j > 0 \} \) (respectively \( \{ h(j)(\sigma) : h \in \mathfrak{h}, j < 0 \} \)). We recall that the Coxeter element \( \sigma \) acts on \( \mathfrak{h} \setminus \{0\} \) without fixed points, so \( h(j)(\sigma) = 0 \) for all \( j \in \mathbb{Z}, h \in \mathfrak{h} \).

We regard the Brylinski filtration as being defined on \( M_\sigma \). Now \( M_\sigma \) is a \( \sigma \)-twisted \( V_\mathcal{Q} \) module, and since \( \mathcal{F} \) is a \( \sigma \)-invariant vertex subalgebra of \( V_\mathcal{Q} \), we may view \( M_\sigma \) as a \( \sigma \)-twisted \( \mathcal{F} \)-module. We recall from Section 2.3 that \( \mathcal{F} = \bigoplus_{d \geq 0} \mathcal{F}^d \), with \( \mathcal{F}^1 = \mathfrak{h} \otimes t^{-1} \) identified with \( \mathfrak{h} \). The following lemma concerns the action of \( \mathcal{F} \) on the subspaces \( F^i M_\sigma \) of the Brylinski filtration on \( M_\sigma \).

**Lemma 5** With notation as above:

1. Let \( h \in \mathfrak{h}, j \in \mathbb{Z} \) and \( i \geq 0 \).
   
   (a) If \( j > 0 \), then \( h(j)(\sigma) \) maps \( F^i M_\sigma \) to \( F^{i-1} M_\sigma \).
   
   (b) If \( j < 0 \), then \( h(j)(\sigma) \) maps \( F^i M_\sigma \) to \( F^{i+1} M_\sigma \).

2. Let \( d \geq 0, j \in \mathbb{Z} \) and \( i \geq 0 \). If \( X \in \mathcal{F}^d \), then \( X(j)(\sigma) \) maps \( F^i M_\sigma \) to \( F^{i+d} M_\sigma \), where \( X(j)(\sigma) \) are the modes of the field \( Y_{M_\sigma}(X, z) \).

**Proof** It is clear that 1(a) follows directly from the definition Eq. 5.1. For 1(b), we proceed by induction on \( i \), starting with \( i = -1 \) (where it holds trivially). Let \( h' \in \mathfrak{h}, p \in \mathbb{Z} \) with \( p > 0 \) and \( v \in F^i M_\sigma, i \geq 0 \). We have [3, §3.3.3]:

\[
h'(\sigma) h(j)(\sigma) v = h(j)(\sigma) h'(\sigma) v + p \delta_{p-j} (h' | h) v,
\]

where \((\cdot | \cdot)\) is the standard invariant form on \( \mathfrak{h} \). By 1(a), \( h'(\sigma) v \in F^{i-1} M_\sigma \). The induction hypothesis implies that \( h(j)(\sigma) h'(\sigma) v \in F^i M_\sigma \). Thus the right hand side of Eq. 5.2 is in \( F^i M_\sigma \). Equation 5.1 completes the proof.

For (2), we recall from Section 2.3 that in the vertex algebra \( \mathcal{F} \), each \( X \in \mathcal{F}^d \) is a linear combination of terms of the form \( (h^1 \otimes t^{j_1}) \cdots (h^k \otimes t^{j_k}) = h_{(j_1)}^1 \cdots h_{(j_k)}^k |0\) with \( h^i \in \mathfrak{h}, j_i < 0, \sum j_i = -d \) and where \( |0\) denotes the vacuum vector of \( \mathcal{F} \). In particular, this implies \( k \leq d \). An iterated application of the product identity Eq. 2.6 implies that each mode \( X(j)(\sigma) \) is a linear combination of terms of the form:

\[
h_{(p_1)}^1 h_{(p_2)}^2 \cdots h_{(p_k)}^k (\sigma) \in \mathbb{End} M_\sigma.
\]
with \( p_i \in \frac{1}{p} \mathbb{Z} \). The assertion now follows from part (1) and the observation that \( k \leq d \). \( \square \)

6 The \( \mathcal{W} \)-algebra

We freely use the notations of the preceding sections.

6.1 Definition

Let \( \mathfrak{g} \) be a simply-laced affine Lie algebra. The \( \mathcal{W} \)-algebra associated to \( \mathfrak{g} \) is the vertex subalgebra of the lattice vertex algebra \( V_Q \) defined by:

\[
\mathcal{W} := \bigcap_{X \in V_Q^{[1]}} \ker_{V_Q} X(0). \quad (6.1)
\]

The conformal vector \( \omega \) of \( \mathcal{W} \) coincides with that of \( V_Q \) and is given by Eq. 3.3 [3, §2.4]. It induces a \( \mathbb{Z}_+ \)-grading on \( \mathcal{W} \).

Let \( d_1 \leq d_2 \leq \cdots \leq d_\ell \) be the list of degrees (exponents plus one) of \( \mathfrak{g} \). We recall from the introduction (Theorem 1) that there exist elements \( \omega^i \in \mathcal{W} \) of degree \( d_i \) such that \( \mathcal{W} \) is freely generated by \( \omega^1, \omega^2, \ldots, \omega^\ell \). Further, \( d_1 = 2 \) and the conformal vector \( \omega \) is precisely \( \omega^1 \).

Remark 6. We recall from Section 3.4 that \( V_Q^{[1]} \cong \mathfrak{g} \) as Lie algebras, with \( X(0) \) corresponding to \( \text{ad} X \). We can identify \( V_Q \) with the homogeneous vertex operator realization of \( L(\Lambda_0) \) [16]. It follows from Eq. 6.1 that \( \mathcal{W} \) maps to the subspace \( L(\Lambda_0) \mathfrak{g} \) of \( \mathfrak{g} \)-invariants under this identification (cf. [3, Remark 2.1]).

6.2 Twisted \( V_Q \)-modules restrict to ordinary \( \mathcal{W} \)-modules

Since \( \bigcap_{h \in \mathfrak{h}} \ker_{V_Q} h(0) \) is the Heisenberg vertex algebra \( \mathcal{F} \subset V_Q \), it follows that \( \mathcal{W} \) is a vertex subalgebra of \( \mathcal{F} \). In addition, the \( \mathcal{W} \)-algebra is pointwise fixed by any inner automorphism of \( V_Q \). This is because an inner automorphism (Section 3.4) is a product of automorphisms of the form \( \exp X(0) \) for \( X \in \mathfrak{g} = V_Q^{[1]} \). Since each \( X(0) \) annihilates \( \mathcal{W} \), we have \( \exp X(0) \) acts as the identity operator on \( \mathcal{W} \).

In particular, given a \( \varphi \)-twisted representation \( M \) of \( V_Q \), where \( \varphi \) is a finite order inner automorphism of \( V_Q \), its restriction to \( \mathcal{W} \) defines an ordinary (untwisted) representation of \( \mathcal{W} \) on \( M \). Now applying this to Eq. 4.4, we conclude \( \tilde{\psi}(X) = X \) for \( X \in \mathcal{W} \), and hence that:

\[
\tilde{\Psi}(X(k)_\varphi \bullet \sigma \nu) = X(k)_\varphi \bullet \xi \tilde{\Psi}(\nu)
\]

for \( X \in \mathcal{W} \), \( k \in \mathbb{Z} \) and \( \nu \in M_\sigma \). We have thus proved:

Proposition 7. The map \( \tilde{\Psi} : M_\sigma \to M_\xi \) is an isomorphism of \( \mathcal{W} \)-modules.

6.3 The Zhu algebra of \( \mathcal{W} \)

We recall the definition of the Zhu algebra \( \mathcal{Z}_h(\mathcal{W}) \) [30]. Given \( a \in \mathcal{W}^{[d]} \), we let \( \deg a := d \) and write \( Y(a, z) = \sum_n a_n z^{-n-d} \). Consider the subspace \( O(\mathcal{W}) \) spanned by the elements
of the form $\sum_{p=0}^{\deg a} \binom{\deg a}{p} a_{-p} b$ for homogeneous elements $a$, $b$ of $\mathcal{W}$. The Zhu algebra is the quotient $\mathcal{Z}(\mathcal{W}) = \mathcal{W} / O(\mathcal{W})$ equipped with the associative multiplication:

$$a \ast b = \sum_{p=0}^{\deg a} \binom{\deg a}{p} a_{-p} b \pmod{O(\mathcal{W})}.$$ 

Let $M = \bigoplus_{c \in C} M_c$ be a graded $\mathcal{W}$-module and let $M_{top}$ denote the sum of the nonzero homogeneous subspaces $M_c$ for which $M_{c+n} = 0$ for all $n > 0$ [1, §3.12]. Then $M_{top}$ is a $\mathcal{Z}(\mathcal{W})$-module with the action:

$$a \cdot m = a_0 m,$$

where $m \in M_{top}$ and $a$ is the image in $\mathcal{Z}(\mathcal{W})$ of a homogeneous element of $\mathcal{W}$.

It is well-known that $\mathcal{Z}(\mathcal{W})$ is isomorphic to $Z(U\overline{\mathfrak{g}})$, the center of the universal enveloping algebra of $\overline{\mathfrak{g}}$ [1, 13]. We identify $\mathcal{Z}(\mathcal{W})$ with $Z(U\overline{\mathfrak{g}})$ using the isomorphism of Arakawa [1, Theorem 4.16.3 (ii)]. With this identification, the one-dimensional representations of $\mathcal{Z}(\mathcal{W})$ are given by the central characters:

$$\gamma_\lambda : Z(U\overline{\mathfrak{g}}) \to \mathbb{C}$$

for $\lambda \in \overline{\mathfrak{h}}^*$; each $z \in Z(U\overline{\mathfrak{g}})$ acts as the scalar $\gamma_\lambda(z)$ on the Verma module $M(\lambda)$ of $\overline{\mathfrak{g}}$ with highest weight $\lambda$.

### 6.4 Verma modules of $\mathcal{W}$, character, PBW basis

We point the reader to Arakawa [1, §5.1] for the full definition of Verma modules of $\mathcal{W}$-algebras. Here, we will content ourselves with recalling their essential properties.

Given a central character $\gamma_\lambda$ of $U\overline{\mathfrak{g}}$, the Verma module $M(\gamma_\lambda)$ is a graded $\mathcal{W}$-module, with $M(\gamma_\lambda)_{top} = \mathbb{C} |\gamma_\lambda\rangle$, where $|\gamma_\lambda\rangle$ is a cyclic vector of $M(\gamma_\lambda)$. The Zhu algebra $\mathcal{Z}(\mathcal{W})$ acts on $|\gamma_\lambda\rangle$ by

$$z |\gamma_\lambda\rangle = \gamma_\lambda(z) |\gamma_\lambda\rangle \quad \text{for all } z \in \mathcal{Z}(\mathcal{W}).$$

Further, given a graded $\mathcal{W}$-module $M$ and a nonzero vector $m \in M_{top}$ such that $zm = \gamma_\lambda(z)m$ for all $z \in \mathcal{Z}(\mathcal{W})$, there exists a unique $\mathcal{W}$-homomorphism $\psi : M(\gamma_\lambda) \to M$ sending $|\gamma_\lambda\rangle \mapsto m$. If $M$ is a $\mathcal{W}$-module on which $\omega_0$ acts semisimply with finite-dimensional eigenspaces, we define its character by $\chi M = tr_M(q^{\omega_0})$. Our definition differs from that of [1, §6.5] by a normalization factor. Correcting for this, the character of a Verma module is given by [1, Proposition 5.6.6]:

$$\chi M(\gamma_\lambda) = \frac{q^{[\lambda+\rho]_F^2/2}}{\varphi(q)^{\ell}}, \quad (6.2)$$

where $\rho$ is the Weyl vector of $\overline{\mathfrak{g}}$ and $\varphi = \prod_{n \geq 1} (1 - q^n)$ is Euler’s function.

Further, the Verma module $M(\gamma_\lambda)$ has a PBW type basis. More precisely, as established by Arakawa [1, Prop 5.1.1], there exist filtrations $F^*\mathcal{W}$ on $\mathcal{W}$ and $F^*M(\gamma_\lambda)$ on the Verma module such that $\text{gr}^F M(\gamma_\lambda)$ is isomorphic to the polynomial algebra $\mathbb{C}[\omega^p_k : 1 \leq p \leq \ell, k \leq -1]$ as a $\text{gr}^F \mathcal{W}$-module. In particular, consider the following elements of $M(\gamma_\lambda)$:

$$\omega_{k_1}^{p_1} \omega_{k_2}^{p_2} \cdots \omega_{k_r}^{p_r} |\gamma_\lambda\rangle, \quad (6.3)$$

---

3 Our $\mathcal{W}$ corresponds to $\mathcal{W}_1$ in the notation of [12]. In Arakawa’s notation [1], this corresponds to $k + h^\vee = 1$, where $h^\vee$ is the dual Coxeter number. In this case, the eigenvalue of $\omega_0$ (where $\omega = \omega_1$ is the conformal vector of $\mathcal{W}$) on $|\gamma_\lambda\rangle$ is $\Delta_{\mu} = (|\mu| + 1)^2/2$ as per [1, (286)]. This is the top degree of $M(\gamma_\lambda)$.

4 Arakawa [1] defines it as $q^{-c(k)/24} \text{tr}_M(q^{\omega_0})$. For $k + h^\vee = 1$, $c(k) = \ell$. 

© Springer
where (i) \( r \geq 0 \) (ii) \( \ell \geq p_1 \geq p_2 \geq \cdots \geq p_r \geq 1 \) (iii) \( k_j \leq -1 \) for all \( j \) and (iv) if \( p_i = p_{i+1} \), then \( k_i \leq k_{i+1} \). Proposition 5.1.1 of [1] implies that

\[
\{ \text{gr}^F v : v \text{ is of the form (6.3)} \}
\]

is a basis of \( \text{gr}^F M(\gamma_\lambda) \). This in turn implies that the vectors of the form Eq. 6.3 form a basis of \( M(\gamma_\lambda) \).

7 The space \( Z \) and \( M_\zeta \)

Consider the subspace \( Z := L(\Lambda_0)^\gamma_0 \) of \( \gamma \)-invariants of \( L(\Lambda_0) \):

\[
Z = \bigoplus_{n \geq 0} Z_n = \bigoplus_{n \geq 0} L(\Lambda_0)\Lambda_0 - n\delta.
\]

Now consider the \( \zeta \)-twisted module \( M_\zeta \) of \( V_Q \). Under the \( g \)-module isomorphism \( L(\Lambda_0) \to M_\zeta \) of Section 4.5, the space \( Z \) maps to \( F \otimes e(\rho/\mathfrak{h}) \). Since \( \tilde{\zeta} \) clearly fixes every element of the Heisenberg vertex subalgebra \( \mathcal{F} \) of \( V_Q \), the restriction of \( M_\zeta \) to \( \mathcal{F} \) is an ordinary (untwisted) representation of \( \mathcal{F} \). It is clear from Eq. 4.1 that the subspace \( Z = \mathcal{F} \otimes e(\rho/\mathfrak{h}) \subset M_\zeta \) is \( \mathcal{F} \)-invariant. For \( u \in \mathcal{F}[p] \), let the corresponding field be denoted (renumbered with conformal weight) \( Y_{M_\zeta}(u, z) = \sum_{n \in \mathbb{Z}} u_n z^{-n-p} \). We then have

\[
u_{-k}(Z_m) \subset Z_{m+k} \quad \text{for all } k, m \in \mathbb{Z}.
\]

(7.1)

Viewed as a representation of the Heisenberg Lie algebra \( \mathfrak{h} \otimes \mathbb{C}[z, z^{-1}] \oplus \mathbb{C}c \), the space \( Z \) is isomorphic to the irreducible highest weight representation with highest weight \( \rho/\mathfrak{h} \).

8 \( Z \) is an irreducible \( \mathcal{W} \)-module

8.1 \( \mathcal{F} \)-modules and \( \mathcal{W} \)-modules

In the remainder of the paper, we consider \( g = A(1)_\ell \). Let \( \mathcal{W} \) be the \( \mathcal{W} \)-algebra of \( g \); it is a vertex subalgebra of \( \mathcal{F} \). Let \( \lambda \in \mathfrak{h}^* \) and let \( \pi_\lambda \) denote the corresponding irreducible representation of \( \mathcal{F} \), with a highest weight vector \( |\lambda\rangle \) (Section 2.5). We restrict \( \pi_\lambda \) to a representation of \( \mathcal{W} \).

Proposition 8 Let \( \lambda \in \mathfrak{h}^* \) and consider \( \pi_\lambda \) as a \( \mathcal{W} \)-module as above. If \( \pi_\lambda \) is irreducible (as a \( \mathcal{W} \)-module), then it is isomorphic to a Verma module of \( \mathcal{W} \).

Proof Firstly, \( \pi_\lambda \) is a graded \( \mathcal{W} \)-module with \( (\pi_\lambda)_\text{top} = \mathbb{C}|\lambda\rangle \). By Section 6.4, this space is a one-dimensional \( \mathfrak{z}\mathcal{h}(\mathcal{W}) \)-module, and is thus given by a central character of \( U\mathfrak{g} \). In other words, there exists \( \mu \in \mathfrak{h}^* \) such that \( z|\lambda\rangle = \gamma_\mu(z)|\lambda\rangle \) for all \( z \in \mathfrak{z}(\mathcal{W}) \cong Z(U\mathfrak{g}) \). Since \( |\lambda\rangle \) is of top degree, we also have \( \omega_\mu^n|\lambda\rangle = 0 \) for all \( 1 \leq p \leq \ell \) and \( n > 0 \).

Again by Section 6.4, there is a homomorphism of \( \mathcal{W} \)-modules:

\[
\phi : M(\gamma_\mu) \to \pi_\lambda \quad \text{sending } 1 \to |\lambda\rangle.
\]

The hypothesis that \( \pi_\lambda \) is irreducible implies that \( \phi \) is surjective. We now compare the characters of these two modules. By Eq. 6.2, we have \( \text{ch} M(\gamma_\mu) = \frac{q^{\mu + \rho/2}}{q(q)^{\ell/2}} \). Let \( \text{ch} \pi_\lambda =
tr_{π_λ}(q^{ω_0}). Now \( \mathcal{W} \) is a vertex subalgebra of \( \mathcal{F} \) and their conformal vectors coincide (and are given by Eq. 3.3). We recall from Section 2.5 that the character

\[
tr_{π_λ}(q^{ω_0}) = \frac{q^{|λ|^2/2}}{φ(q)^ℓ}.
\]

Since \( ω_0 1 = |μ + ρ|^2 1 \) and \( ω_0 |λ⟩ = |λ|^2 |λ⟩ \), we conclude \( |μ + ρ|^2 = |λ|^2 \) and hence that \( chM(γμ) = chπ_λ \). This proves that \( φ \) is an isomorphism. □

We have the following key theorem.

**Theorem 9** Let \( λ \in \mathfrak{h}^* \) be such that \( λ(α^∨) \notin \mathbb{Z} \) for all roots \( α \) of \( \mathfrak{g} \). Then \( π_λ \) is an irreducible \( \mathcal{W} \)-module.

For the proof, we shall use the main theorem of [12] which relates representations of the Lie algebra \( \mathcal{W}_{1+∞} \) with those of the \( \mathcal{W} \)-algebra of \( \mathfrak{gl}_ℓ + 1 \mathbb{C} \). We recall the relevant notations and results in the next two subsections.

### 8.2 The \( \mathcal{W} \)-algebra of \( \mathfrak{gl}_ℓ + 1 \)

The vertex algebra \( \mathcal{W}(\mathfrak{gl}_ℓ + 1) \) is defined analogously to \( \mathcal{W}(\mathfrak{sl}_ℓ + 1) \). Let \( \tilde{h}(\mathfrak{gl}_ℓ + 1) \) denote the set of diagonal matrices in \( \mathfrak{gl}_ℓ + 1 \mathbb{C} \) and let \( ε_i \in \tilde{h}(\mathfrak{gl}_ℓ + 1)^* \) be defined by \( ε_i(H) = H_{ii} \) for each diagonal matrix \( H \). Consider the integral lattice \( \tilde{Q} = \sum_i \mathbb{Z} ε_i \) with bilinear form \( (ε_i|ε_j) = δ_{ij} \). The lattice vertex algebra \( V_{\tilde{Q}} \) is defined as in Section 3.1:

\[
V_{\tilde{Q}} = \mathcal{F} \otimes_\mathbb{C} C_{ε(\tilde{Q})},
\]

where the Fock space \( \mathcal{F} \) is the symmetric algebra on the space \( \sum_{j<0} \tilde{h}(\mathfrak{gl}_ℓ + 1) \otimes t^j \) and \( ε \) is a certain bimultiplicative cocycle on \( \tilde{Q} \). The state-field correspondence is also analogous to that in Section 3.1 (see [21, Theorem 5.5] for details). The lattice vertex algebra is now \( 1/2 \mathbb{Z} \)-graded, with the grade 1 piece \( V_{\tilde{Q}}^{[1]} \) being spanned by \( h t^{-1} \otimes 1 \) and \( 1 \otimes e^α \) for \( h ∈ \tilde{h}(\mathfrak{gl}_ℓ + 1), α ∈ Δ \), where \( Δ = \{ ε_i - ε_j : i ≠ j \} \). We identify \( V_{\tilde{Q}}^{[1]} \) with \( \mathfrak{gl}_ℓ + 1 \).

Define \( \mathcal{W}(\mathfrak{gl}_ℓ + 1) \) to be the vertex subalgebra of \( V_{\tilde{Q}} \) given by:

\[
\mathcal{W}(\mathfrak{gl}_ℓ + 1) := \bigcap_{X ∈ V_{\tilde{Q}}^{[1]}} ker_{V_{\tilde{Q}}} X(0).
\]

There is a one-parameter family of conformal vectors of \( \mathcal{W}(\mathfrak{gl}_ℓ + 1) \) defined by ([12, §4]):

\[
ω_a = a I(-2)|0⟩ + \frac{1}{2} \sum_{i=1}^{ℓ+1} u_i(-1)u^i(-1)|0⟩,
\]

where \( a ∈ \mathbb{C}, \{ u_i \}, \{ u^i \} \) are dual bases of \( \tilde{h}(\mathfrak{gl}_ℓ + 1) \) with respect to the form \( (\cdot|\cdot) \), \( I \) is the identity matrix, \( |0⟩ = 1 \otimes 1 \) is the vacuum vector of \( V_{\tilde{Q}} \) and \( h(k) \) is the operator \( ht^k \otimes id \) on \( V_{\tilde{Q}} \) for \( h ∈ \tilde{h}(\mathfrak{gl}_ℓ + 1), k ∈ \mathbb{Z} \).
8.3 $\mathfrak{gl}_{\ell+1}$ vs $\mathfrak{sl}_{\ell+1}$

Since $\tilde{\mathfrak{h}}(\mathfrak{gl}_{\ell+1}) = \mathfrak{h}(\mathfrak{sl}_{\ell+1}) \oplus \mathbb{C} I$, where $I$ is the identity matrix, the corresponding Fock spaces are related by: $\tilde{\mathcal{F}} = \mathcal{F} \otimes \mathcal{F}_1$ (tensor product of vertex algebras), where $\mathcal{F}_1$ is the symmetric algebra on $\sum_{j<0} I \otimes t^j$. It is easily observed [12, Remark 4.2] that

$$\mathcal{W}(\mathfrak{gl}_{\ell+1}) \cong \mathcal{W}(\mathfrak{sl}_{\ell+1}) \otimes \mathcal{F}_1 \subset \mathcal{F} \otimes \mathcal{F}_1.$$

Given $\lambda \in \mathfrak{h}(\mathfrak{sl}_{\ell+1})^*$, we extend it to $\tilde{\lambda} \in \mathfrak{h}(\mathfrak{gl}_{\ell+1})^*$ by defining $\tilde{\lambda}(I) = 0$. Let $\pi_\lambda$, $\pi_\tilde{\lambda}$ denote the corresponding irreducible representations of $\mathcal{F}$ and $\mathcal{F}$ as well. Viewing $\mathcal{F}_1$ as a module over itself, we have:

$$\pi_\tilde{\lambda} \cong \pi_\lambda \otimes \mathcal{F}_1$$

as $\mathcal{F} \otimes \mathcal{F}_1$-modules, and hence by restriction as $\mathcal{W}(\mathfrak{sl}_{\ell+1}) \otimes \mathcal{F}_1 \cong \mathcal{W}(\mathfrak{gl}_{\ell+1})$-modules.

Now $\mathcal{F}_1$ is an irreducible module over itself. Hence, by [15, Proposition 4.7.2] (whose mild technical conditions hold in this case), we conclude that $\pi_\tilde{\lambda}$ is an irreducible $\mathcal{W}(\mathfrak{gl}_{\ell+1})$-module if and only if $\pi_\lambda$ is an irreducible $\mathcal{W}(\mathfrak{sl}_{\ell+1})$-module.

8.4 The Lie algebra $\mathcal{W}_{1+\infty}$

In this subsection, we recall the definition and key properties of the Lie algebra $\mathcal{W}_{1+\infty}$, following [12]. Let $\mathcal{D} \subset \text{End}_\mathbb{C} \mathbb{C}[t, t^{-1}]$ denote the Lie algebra of regular differential operators on $\mathbb{C}^\times$, with the usual bracket. Each of the following collections forms a basis of $\mathcal{D}$:

1. $J_k^\ell = -t^{\ell+k}(\partial_t)^k$ with $k, \ell \in \mathbb{Z}$, $k \geq 0$,
2. $L_k^\ell = -t^{\ell}(\partial_t)^k$ with $k, \ell \in \mathbb{Z}$, $k \geq 0$,

where $\partial_t = \frac{\partial}{\partial t}$. This Lie algebra has a $\mathbb{C}$-valued 2-cocyle $\psi$ given by:

$$\psi(f(t)\partial_t^m, g(t)\partial_t^n) = \frac{m! n!}{(m+n+1)!} \text{Res} (\partial_t^{n+1} f(t)) (\partial_t^m g(t)),$$

where as usual, for a Laurent polynomial $f \in \mathbb{C}[t, t^{-1}]$, Res $f(t)$ denotes the coefficient of $t^{-1}$ in $f(t)$. We let $\mathcal{W}_{1+\infty} = \mathcal{D} \oplus \mathbb{C} C$ be the one-dimensional central extension of $\mathcal{D}$ defined by the cocyle $\psi$, i.e., with Lie bracket defined by $[X, Y] := [X, Y]_{\mathcal{D}} + \psi(X, Y) C$ for all $X, Y \in \mathcal{D}$.

Consider the Lie subalgebra $\mathcal{P} = \text{span}\{J_k^\ell : \ell + k \geq 0\}$ of $\mathcal{D}$ and let $\hat{\mathcal{P}} = \mathcal{P} \oplus \mathbb{C} C \subset \mathcal{W}_{1+\infty}$. Given $c \in \mathbb{C}$, we form the induced $\mathcal{W}_{1+\infty}$-module

$$M_c = U \mathcal{W}_{1+\infty} \otimes_U \mathcal{P} \mathbb{C},$$

where $C$ acts as $c$ on the one-dimensional space $\mathbb{C}$ and $\mathcal{P}$ acts as zero. The module $M_c$ admits a unique irreducible quotient $V_c$. We recall that $V_c$ is a vertex algebra and that $V_c$-modules may be canonically viewed as $\mathcal{W}_{1+\infty}$-modules on which $C$ acts by the scalar $c$. Further, $V_c$ has a one-parameter family of conformal vectors [12, Theorem 3.1]:

$$\omega(\beta) = (J_1^1 - \beta J_0^0)|0\rangle,$$

where $|0\rangle$ is the image of $1 \otimes 1$ in $V_c$. The central charge of the corresponding Virasoro field is $-(12\beta^2 - 12\beta + 2)c$. Now, the key fact of relevance to us is the following [12, Theorem 5.1]:

$\mathbb{C}$ Springer
Proposition 10 (Frenkel-Kac-Radul-Wang) There is an isomorphism of vertex algebras $V_{\ell+1} \to \mathcal{W}(\mathfrak{gl}_{\ell+1})$ which maps the conformal vectors $\omega(\beta) \mapsto \omega_{1/2-\beta}$ (Eqs. 8.1 and 8.2).

Hence, any representation of the vertex algebra $\mathcal{W}(\mathfrak{gl}_{\ell+1})$ can be canonically lifted to a representation of $V_{\ell+1}$, or equivalently, to a representation of the Lie algebra $\mathcal{W}_{1+\infty}$ with central charge $\ell + 1$.

Given $\gamma \in \bar{h}(\mathfrak{gl}_{\ell+1})^*$, consider the representation $\pi_\gamma$ of $\mathcal{W}(\mathfrak{gl}_{\ell+1})$ as in Section 8.3. Let $|\gamma\rangle$ denote its highest weight vector. Let $U(\gamma)$ denote the $\mathcal{W}(\mathfrak{gl}_{\ell+1})$-submodule of $\pi_\gamma$ generated by $|\gamma\rangle$. This is a cyclic, graded module and therefore has a unique irreducible quotient $V(\gamma)$. The following is one of the main results of [12]:

Proposition 11 ([12, Prop 5.1]) The lifting of $V(\gamma)$ to a $\mathcal{W}_{1+\infty}$-module is isomorphic to the primitive $\mathcal{W}_{1+\infty}$-module with exponents $\gamma(E_{ii}) : 1 \leq i \leq \ell + 1$.

We refer the reader to [12] for the definition of primitive $\mathcal{W}_{1+\infty}$-modules and exponents. The character of such modules was also determined by Frenkel-Kac-Radul-Wang. We now state this result in the special case of interest to us.

Proposition 12 Let $s_i$ ($1 \leq i \leq \ell + 1$) be complex numbers such that $s_i \not\equiv s_j \pmod{\mathbb{Z}}$ for all $i \neq j$. Then the character of the primitive $\mathcal{W}_{1+\infty}$-module $M$ with exponents $\{s_i\}$ is given by:

$$\text{tr}_M(q^{L_0^1}) = \frac{q^{\sum_i s_i(s_i-1)/2}}{\varphi(q)^{\ell+1}}.$$  (8.3)

This follows from Proposition 2.1 and equation (2.5) of [12].

8.5 Proof of Theorem 9

We use the notations introduced in the above subsections. We have by Section 8.3 that $\pi_\lambda$ is an irreducible $\mathcal{W}(\mathfrak{sl}_{\ell+1})$-module iff $\pi_{\lambda,\gamma}$ is an irreducible $\mathcal{W}(\mathfrak{gl}_{\ell+1})$-module. Since $V(\lambda,\gamma)$ is a subquotient of $\pi_{\lambda,\gamma}$, the latter assertion is equivalent to the assertion that $\pi_{\lambda,\gamma}$ and $V(\lambda,\gamma)$ have the same character, i.e.,

$$\text{tr}_{\pi_{\lambda,\gamma}}(q^{L_0^1}) = \text{tr}_{V(\lambda,\gamma)}(q^{L_0^1}).$$

Define $s_i = \bar{\lambda}(E_{ii})$. The hypothesis that $\lambda(\alpha^\vee) \not\in \mathbb{Z}$ for all positive roots $\alpha$ of $\mathfrak{sl}_{\ell+1}\mathbb{C}$ implies that $s_i - s_j \not\in \mathbb{Z}$ for all $i \neq j$. Propositions 11 and 12 then imply that the character of $V(\lambda)$ is given by the right hand side of Eq. 8.3.

To compute the character of $\pi_{\lambda,\gamma}$, we need to identify $L_0^1 \in \mathcal{W}_{1+\infty}$ with the appropriate conformal vector $\omega_\alpha$ of $\mathcal{W}(\mathfrak{gl}_{\ell+1})$. We recall from Proposition 10 that under the isomorphism of vertex algebras $V_{\ell+1} \simeq \mathcal{W}(\mathfrak{gl}_{\ell+1})$, their conformal vectors correspond thus:

$$\omega(\beta) \mapsto \omega_{1/2-\beta}.$$
Taking $\beta = 0$, we obtain $\omega(0) \mapsto \omega_{1/2}$, and in particular, their zeroth modes correspond to each other. We have $\omega(0)_{0} = L_{0}^{1}$ and
\[
(\omega_{1/2})_{0}(\tilde{\lambda}) = \frac{\tilde{\lambda}(I)}{2} - \frac{\tilde{\lambda}(I)}{2} = \sum_{i} s_{i}(s_{i} - 1)/2.
\]
So, by Section 2.5, the character of $\pi_{\tilde{\lambda}}$ becomes:
\[
\text{tr}_{\pi_{\tilde{\lambda}}}(q^{L_{0}^{1}}) = \text{tr}_{\pi_{\tilde{\lambda}}}(q^{(\omega_{1/2})_{0}}) = q^{\sum_{i} s_{i}(s_{i} - 1)/2} \frac{\varphi(q)^{\ell+1}}{\varphi(q)^{\ell+1}}.
\]
This matches up correctly with the expression in Eq. 8.3, thereby completing the proof.

8.6 A basis of $\mathbb{Z}$

Let $\mathcal{W} := \mathcal{W}(\mathfrak{sl}_{\ell+1})$ and regard the space $\mathbb{Z}$ as a $\mathcal{W}$-module as before.

**Proposition 13** $\mathbb{Z}$ is isomorphic to a Verma module of $\mathcal{W}$. Further, the set of vectors in Eq. 1.5 forms a basis of $\mathbb{Z}$.

**Proof** The first part follows by applying Proposition 8 and Theorem 9 to the subspace $\mathbb{Z}$ identified with $\mathcal{F} \otimes e(\rho/h) \subset M_{\ell}$. The element $\lambda$ in this case is just $\rho/h$ which clearly satisfies the hypothesis of Theorem 9. For the second part, observe that the set of vectors in Eq. 1.5 is now exactly the PBW basis of the Verma module $\mathbb{Z} (6.3)$.

9 Brylinski-compatibility of the basis of $\mathbb{Z}$

We now complete the proof of Theorem 2. We begin with the following simple lemma.

**Lemma 14** For $1 \leq p \leq \ell$ and all $k \in \mathbb{Z}$, $\omega_{k}^{p}(\sigma)$ maps $F^{i}L(\Lambda_{0})$ to $F^{i+d_{p}}L(\Lambda_{0})$.

**Proof** Since $\omega^{p}$ lies in $\mathcal{F}[d_{p}]$, this is a direct consequence of Lemma 5.

9.1 Generalities on filtrations and bases

Let $M$ be a finite-dimensional vector space with a filtration $M_{0} \subset M_{1} \subset M_{2} \cdots$ such that $\bigcup M_{i} = M$. We consider $\text{gr} M = \bigoplus_{k \geq 0} M_{k}/M_{k-1}$, where $M_{-1} := 0$. For $v \in M$, we define $\text{gr} v \in \text{gr} M$ to be the image of $v$ under the projection $M_{k} \rightarrow M_{k}/M_{k-1}$, where $k \geq 0$ is minimal such that $v \in M_{k}$.

**Definition 15** A basis $B$ of $M$ is said to be compatible with the filtration $\{M_{i}\}$ if $B \cap M_{i}$ is a basis of $M_{i}$ for all $i$, or equivalently, if $\{\text{gr} v : v \in B\}$ is a basis of $\text{gr} M$.

It is elementary to check the equivalence of the two descriptions in the definition above.
Lemma 16 Let \( \{M_i\}_{i=0}^m \) be a filtration of \( M \) as above. Suppose \( B \) is a basis of \( M \) and \( \{B^i\}_{i=0}^m \) is a collection of pairwise disjoint subsets of \( B \) such that \( \bigcup B^i = B \). Suppose further that for all \( i \geq 0 \):

1. \( B^i \subset M_i \),
2. \( |B^i| = \dim (M_i/M_{i-1}) \).

Then \( B \) is compatible with the filtration \( \{M_i\} \).

Proof A repeated application of Eq. 16 above shows \( \dim M_i = \sum_{j \leq i} |B^j| \). Since \( \bigcup_{j \leq i} B^j \subset M_i \) is a linearly independent set which has cardinality \( \dim M_i \), it must be a basis of \( M_i \). The linear independence of \( B \) now implies that \( B \cap M_i \) is precisely \( \bigcup_{j \leq i} B^j \subset M_i \). □

9.2 Proof of Theorem 2

We now prove Theorem 2. Let \( \mathcal{B} \) denote the set comprising the vectors in Eq. 1.5. By Proposition 13, \( \mathcal{B} \) is a basis of \( Z \). Let \( \mathcal{B}_n = \mathcal{B} \cap Z_n \). Since Eq. 7.1 implies that \( \omega_{-k}^p (Z_m) \subset Z_{m+k} \) for \( 1 \leq p \leq \ell \) and all \( k, m \geq 0 \), the set \( \mathcal{B}_n \) consists of vectors of the form (1.5) with \( \sum_i k_i = -n \). In particular, this implies that \( |\mathcal{B}_n| = p_\ell(n) \), the number of partitions of \( n \) into parts of \( \ell \) colours. Now, it is well-known that this number is also the dimension of \( Z_n \) [20, Remark 12.13], i.e., \( |\mathcal{B}_n| = \dim Z_n = p_\ell(n) \). So \( \mathcal{B}_n \) forms a basis of \( Z_n \) for each \( n \).

We now claim that \( \mathcal{B}_n \) is compatible with the Brylinski filtration \( \{F^i Z_n\} \) on \( Z_n \). Consider a vector \( v \in \mathcal{B}_n \). It has the form

\[
v = \omega_{k_1}^{p_1} (\sigma) \omega_{k_2}^{p_2} (\sigma) \cdots \omega_{k_r}^{p_r} (\sigma) v_{\Lambda_0}
\]

satisfying the conditions in Eq. 1.5 and with \( \sum_{i=1}^r k_i = -n \). Since \( v_{\Lambda_0} \) is in \( F^0 L(\Lambda_0) \), it follows from Lemma 14 that \( v \in F^d L(\Lambda_0) \cap Z_n = F^d Z_n \), where \( d = \sum_{i=1}^r d_{p_i} \). For each \( d \geq 0 \), define \( \mathcal{B}_n^d \subset F^d Z_n \) to be the set of vectors in Eq. 1.5 satisfying \( \sum_{i=1}^r d_{p_i} = d \) and \( \sum_{i=1}^r k_i = -n \). It is clear from this definition that \( |\mathcal{B}_n^d| \) is the coefficient of \( t^d q^n \) in the product:

\[
\prod_{k=1}^\ell \prod_{j=1}^\infty (1 - t^k q^j)^{-1}.
\]

From Eq. 1.3, this is the same as \( \dim (F^d Z_n / F^{d-1} Z_n) \). An appeal to Lemma 16 (with \( M = Z_n, B = \mathcal{B}_n, B^i = \mathcal{B}_n^i \)) finishes the proof. Corollary 3 follows easily.

Acknowledgements It is a pleasure to thank Bojko Bakalov for useful discussions and pointers to relevant literature at an early stage of this work. The authors also thank an anonymous referee whose comments helped improve the exposition. Sachin Sharma also thanks the Institute of Mathematical Sciences, where parts of this work were done, for its excellent hospitality.

Funding Partial financial support was received from the Department of Science and Technology, Government of India through grants EMR/2016/001997 (SG) and MTR/2019/000071 (SV), from the Department of Atomic Energy, Government of India under a XII plan project (SV) and from the Indian Institute of Technology Kanpur through a faculty initiation grant (SS).

Data Availability Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.
References

1. Arakawa, Tomoyuki.: Representation theory of \(W\)-algebras. Invent. Math. 169(2), 219–320 (2007)
2. Bakalov, B., Kac, Victor.G.: Twisted modules over lattice vertex algebras. In: Lie Theory and Its Applications in Physics V, pp. 3–26. World Sci. Publ., River Edge (2004)
3. Bakalov, B., Milanov, T.: \(W\)-constraints for the total descendant potential of a simple singularity. Compos. Math. 149(5), 840–888 (2013)
4. Bershtein, M., Gavrylenko, Pavlo., Marshakov, Andrei.: Twist-field representations of \(W\)-algebras, exact conformal blocks and character identities. J. High Energ. Phys. 108 (2018)
5. Braverman, Alexander., Finkelberg, Michael.: Pursuing the double affine Grassmannian I: Transversal slices via instantons on \(A_2\)-singularities. Duke. Math. J. 152(2), 175–206 (2010)
6. Broer, Bram.: Line bundles on the cotangent bundle of the flag variety. Invent. Math. 113(1), 1–20 (1993)
7. Brylinski, R.-K.: Limits of weight spaces, Lusztig’s \(q\)-analogues, and fiberings of adjoint orbits. J. Am. Math. Soc. 2(3), 517–533 (1993)
8. Cherednik, Ivan.: Difference Macdonald-Mehta conjecture. Internat. Math. Res. Notices (10):449–467 (1997)
9. Dong, Chongying., Mason, Geoffrey.: Nonabelian orbifolds and the Boson-Fermion correspondence. Comm. Math. Phys. 163(3), 523–559 (1994)
10. Dong, Chongying., Nagatomo, Kiyokazu.: Automorphism groups and twisted modules for lattice vertex operator algebras. In: Recent developments in quantum affine algebras and related topics (Raleigh, NC, 1998), volume 248 of Contemp. Math., pp. 117–133. Amer. Math. Soc., Providence (1999)
11. Feigin, Boris., Frenkel, Edward.: Integrals of motion and quantum groups. In: Integrable systems and quantum groups (Montecatini Terme, 1993), volume 1620 of Lecture Notes in Math., pp. 349–418. Springer, Berlin (1996)
12. Frenkel, Edward., Kac, Victor., Radul, Andrei., Wang, Weiqiang.: \(W_{1+\infty}\) and \(W(g_{\lambda})\) with central charge \(\lambda\). Comm. Math. Phys. 170(2), 337–357 (1995)
13. Frenkel, Edward., Kac, Victor., Wakimoto, Minoru.: Characters and fusion rules for \(W\)-algebras via quantized Drinfeld-Sokolov reduction. Comm. Math. Phys. 147(2), 295–328 (1992)
14. Frenkel, Igor., Lepowsky, James., Meurman, Arne.: Vertex operator algebras and the Monster, volume 134 of Pure and Applied Mathematics. Academic Press, Inc, Boston (1988)
15. Frenkel, Igor.B., Huang, Yi-Zhi., Lepowsky, James.: On axiomatic approaches to vertex operator algebras and modules. Mem. Amer. Math. Soc. 104(494), xi+64 (1993)
16. Frenkel, Igor.B., Kac, Victor.G.: Basic representations of affine Lie algebras and dual resonance models. Invent. Math. 62(1), 23–66 (1980/81)
17. Gupta, Ranees.K.: Characters and the \(q\)-analog of weight multiplicity. J. London Math. Soc. (2) 36(1), 68–76 (1987)
18. Joseph, Anthony., Letzter, Gaile., Zelikson, S.: On the Brylinski-Kostant filtration. J. Amer. Math. Soc. 13(4), 945–970 (2000)
19. Kac, Victor.G., algebras, Infinite-dimensional.: Dedekind’s \(\eta\)-function, classical Möbius function and the very strange formula. Adv. in Math. 30(2), 85–136 (1978)
20. Kac, Victor.G.: Infinite Dimensional Lie Algebras, 3rd edn. Cambridge University Press, Cambridge (1990)
21. Kac, Victor.G.: Vertex Algebras for Beginners, volume 10 of University Lecture Series, 2nd edn. American Mathematical Society, Providence (1998)
22. Kac, Victor.G., Kazhdan, David.A., Lepowsky, James., Wilson, Robert.L.: Realization of the basic representations of the Euclidean Lie algebras. Adv. in Math. 42(1), 83–112 (1981)
23. Kac, Victor.G., Dale, H.: Peterson, 112 constructions of the basic representation of the loop group of \(E_8\). In: Symposium on Anomalies, Geometry, Topology (Chicago 1985), pp. 276–298. World Sci. Publ., Singapore (1985)
24. Kac, Victor.G., Raina, Ashok.K.: Bombay Lectures on Highest Weight Representations of Infinite-Dimensional Lie Algebras, volume 2 of Advanced Series in Mathematical Physics. World Scientific Publishing Co., Inc., Teaneck (1987)
25. Kato, S.: Spherical functions and a \(q\)-analogue of Kostant’s weight multiplicity formula. Invent. Math. 66(3), 461–468 (1982)
26. Lepowsky, James.: Calculus of twisted vertex operators. Proc. Nat. Acad. Sci. U.S.A. 82(24), 8295–8299 (1985)
27. Lusztig, George.: Singularities, character formulas, and a $q$-analog of weight multiplicities. In: Analysis and topology on singular spaces, II, III (Luminy, 1981), volume 101 of Astérisque, pp. 208–229. Soc. Math. France, Paris (1983)

28. Slofstra, W.: A Brylinski filtration for affine Kac–Moody algebras. Adv. Math. 229(2), 968–983 (2012)

29. Viswanath, Shankaran.: Kostka-Foulkes polynomials for symmetrizable Kac-Moody algebras. Sém. Lothar. Combin., 58:Art B58f (2008)

30. Zhu, Yongchang.: Modular invariance of characters of vertex operator algebras. J. Amer. Math. Soc. 9(1), 237–302 (1996)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.