Parametrized numerical scheme for the Einstein equations

Hidetomo Hoshino\textsuperscript{1}\textsuperscript{*}, Kei Satoh\textsuperscript{1} and Gen Yoneda\textsuperscript{1}

\textsuperscript{1} Graduate School of Fundamental Science and Engineering, Waseda University, 3-4-1 Okubo, Shinjuku-ku, Tokyo 169-8555, Japan

\textsuperscript{*}Corresponding author: rockfish3141@toki.waseda.jp

Received November 12, 2020, Accepted February 8, 2021

Abstract

In astrophysics and astronomy, it is necessary to solve numerically and accurately the Einstein equations, which are 2nd-order partial differential equations for a metric. We propose a method of estimating a numerical scheme in terms of constraints, and we also demonstrate that a numerical scheme with parameters makes it possible to perform a numerical calculation with less constraint violation.

Keywords numerical relativity, Einstein equation, constraint-preserving numerical method

Research Activity Group Scientific Computation and Numerical Analysis

1. Introduction

To consider gravitational collapse [1] or predict the waveforms of gravitational waves [2], the Einstein equations must be solved. They are, however, nonlinear simultaneous partial differential equations. They cannot be solved in general unless symmetry is imposed. Therefore, they must be solved numerically.

The Einstein equations are written as

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu} \quad (\mu, \nu = 0, 1, 2, 3), \]

where \( g_{\mu\nu} \) is the metric, \( R_{\mu\nu} \) is the Ricci tensor, \( R \) is the scalar curvature in spacetime, and \( T_{\mu\nu} \) is the energy-momentum tensor. In these equations, the time and space are treated equivalently; and thus in performing numerical calculations, spacetime must be split into time and space (3+1 decomposition). One of the most famous formulations for numerical calculations of the Einstein equations is the Arnowitt-Desser-Misner (ADM) formulation [3–5], in which the Einstein equations are split into time evolution equations and constraint equations.

In solving the Einstein equations numerically, numerical schemes such as the iterated Crank–Nicolson method (ICN) and the classical Runge–Kutta method are used. However, such common numerical schemes are not always appropriate for solving the Einstein equations, because the values of the constraints increase.

So far, the numerical stability of the Einstein equations has been studied by considering the constraint amplification factor (CAF) [5–6]. Constraint-preserving schemes that enable the numerical calculations without the violation of constraints are also considered under certain conditions [7]. In general, however, it is difficult to construct a constraint-preserving scheme for the Einstein equations. We study numerical schemes for the Einstein equations in terms of constraints. To estimate to what extent the constraint values are violated, we propose the idea of the "constraint’s order of accuracy (COA)". Using COA, we estimate some schemes often used in numerical relativity.

We show that a numerical scheme with suitable parameters is useful in terms of COA in discretizing differential equations with constraints.

2. Constraint’s order of accuracy (COA)

2.1 Definition of COA

Numerical schemes are often estimated on the basis of how the numerical solution reproduces the exact solution. When an ODE \( \dot{u} = f(u) \) is given, consider the calculation from the time \( t_n \) to \( t_{n+1} = t_n + \Delta t \). Calculate the difference between a numerical solution \( u_{n+1} \) obtained by a numerical scheme and \( u(t+\Delta t) \) obtained by the Taylor series, where

\[
\tilde{u}(t+\Delta t) = \sum_{k=0}^{\infty} \frac{u^{(k)}(t)}{k!} \Delta t^k
\]

\[ = u(t) + u'(t)\Delta t + \frac{u''(t)}{2} (\Delta t)^2 + \ldots. \]

If the difference is

\[
\frac{u_{n+1} - \tilde{u}(t + \Delta t)}{\Delta t} = O(\Delta t^p),
\]

then we define the "evolution’s order of accuracy (EOA)" of its numerical scheme equal to \( p \). We introduced EOA to distinguish COA, as shown later. For example, the values of EOA of the explicit Euler method and the classical Runge–Kutta method are equal to 1 and 4, respectively.

The Einstein equations, however, have constraints. We must estimate numerical schemes in terms of not only the reproducibility of the exact solution but also the preservation of the constraints.

Let a constraint \( C(t) \) satisfies \( C(t) = 0 \) and \( \dot{C}(t) = 0 \). If \( C(t) \) satisfies

\[
\frac{C_{n+1} - C_n}{\Delta t} = O(\Delta t^p) \tag{1}
\]

in using a numerical scheme, where \( n \) represents the cur-
rent time-step, then we say the COA of the numerical scheme equals \( p \).

In numerical relativity, the constraints are first-class constraints that satisfy \( \dot{C} = a(t)C \). In this case, COA is defined by

\[
\frac{C_{n+1} - C_n}{\Delta t} - a_n C_n = O(\Delta t^p)
\]

instead of (1).

2.2 Example of COA

We take simple harmonic motion as an example of COA. The time evolution equations are given as

\[
\begin{align*}
\dot{x} &= v \\
\dot{v} &= -x.
\end{align*}
\]

The constraint equation, which means conservation of energy, is given as

\[
C(t) = x^2 + v^2 - 1 = 0.
\]

We investigated the EOA and COA of some numerical schemes for simple harmonic motion.

The results in Table 1 are confirmed exactly and numerically.

For example, if we use the Heun method whose EOA equals 2, the difference in the constraint in simple harmonic motion is

\[
\frac{C_{n+1} - C_n}{\Delta t} = \frac{1}{4}(v_n^2 + x_n^2)\Delta t^3 + O(\Delta t^4),
\]

where \( n \) represents the current time step. Thus, the COA of the Heun method in simple harmonic motion is 3 exactly. Fig. 1 shows this numerically.

As you can see from this example, EOA and COA are not the same in general. Note that EOA depends on only the numerical scheme, whereas COA depends on the numerical scheme, the time evolution equations, and the constraint equations.

3. Numerical scheme with parameters

One of the most commonly used numerical schemes in numerical relativity is the ICN because of its numerical stability [8]. There is a numerical scheme called the geometric-averaging \( \theta \)-ICN method (GA \( \theta \)-ICN), which gives parameters to the ICN [9]. For example, \( \tilde{u} = f(u) \) is discretized by the GA \( \theta \)-ICN as follows.

\[
\begin{align*}
\text{step 1:} & \quad (1) \tilde{u}_{n+1} = u_n + f(u_n)\Delta t \\
\text{step 2:} & \quad (1) \bar{\pi}_{n+\theta_1} = \theta_1 (1) \tilde{u}_{n+1} + (1 - \theta_1) u_n. \quad (2) \\
\text{step 3:} & \quad (2) \tilde{u}_{n+2\theta_1} = u_n + 2\theta_1 f((1) \bar{\pi}_{n+\theta_1}) \Delta t.
\end{align*}
\]

If we set \( \theta_{x_1} = \phi_{x_1} = 1/4 \), the \( \Delta t^2 \) and \( \Delta t^3 \) terms on the right-hand side (underlined parts) disappear. With the GA \( \theta \)-ICN, the COA of simple harmonic motion is 5 (the \( \Delta t^4 \) term disappears unintentionally). Fig. 2 shows numerically that the COA is 5. In the case of the ICN (\( \theta_{x_1} = \phi_{x_1} = 1/2 \)), it is easy to confirm that the COA of simple harmonic motion is 3. Therefore, we can confirm that the GA \( \theta \)-ICN, which is one of the parameterized numerical schemes, may improve COA.

---

Table 1. EOA and COA of each scheme for simple harmonic motion. Euler, Heun, AB2, AB3, and RK4, mean the explicit Euler method, the Heun method, the two-step Adams–Bashforth method, the three-step Adams–Bashforth method, and the classical Runge–Kutta method, respectively.

| Scheme | EOA | COA |
|--------|-----|-----|
| Euler  | 1   | 2   |
| Heun   | 2   | 3   |
| AB2    | 3   | 4   |
| AB3    | 3   | 5   |
| RK4    | 4   | 4   |
4. Parametrized numerical scheme for numerical relativity

Adopting the Kasner ODE test solution, which is one of the exact solutions of Einstein equations,
\[ ds^2 = -dt^2 + t^{-2} dx^2 + t^2 dy^2 + t^2 dz^2 \]
as the initial condition, we set a metric
\[ ds^2 = -dt^2 + g(t) dx^2 + h(t)(dy^2 + dz^2). \]
Using this metric, we can write the time evolution equations and the constraint equation by the ADM formulation, which is one of the most famous formulations for numerical relativity. The time evolution equations are given as
\[
\begin{aligned}
\dot{g}(t) &= -2k \\
\dot{h}(t) &= -2l \\
\dot{k}(t) &= k \left( \frac{2l}{g} - \frac{k}{l} \right) \\
\dot{l}(t) &= \frac{kl}{g},
\end{aligned}
\]
\[ k(t) \text{ and } l(t) \text{ are derived from the extrinsic curvature, which corresponds to velocity in Newtonian mechanics. The constraint equation is given as} \]
\[ \mathcal{H} = \frac{2l(gl + 2hk)}{gh^2} = 0. \]

We call this numerical exercise the Kasner test. First, we investigated the COA of the ICN. Fig. 3 shows numerically that the COA of the ICN is 2. The difference in the constraint minus the term proportional to the constraint. At this time, \( \Delta t^3 \) and \( \Delta t^4 \) terms disappear. To make COA 5, we must introduce three equations such as \( \theta_{h2} = 1/(\theta_{h1}) \). One parameter improves COA by 1 compared with the ICN; hence, three free parameters improve COA by 3; then, COA becomes 2+3=5.

To obtain the values for the parameters, we calculated the difference of the constraint minus the term proportional to the constraint. At this time, \( \Delta t^3 \) and \( \Delta t^4 \) terms disappear. To make COA 5, we must eliminate the terms \( \Delta t^3, \Delta t^4 \). To do that, we get three equations that \( \theta_{h1}, \theta_{k1} \text{ and } \theta_{l1} \text{ should satisfy}. \)

From the condition that the term \( \Delta t^2 \) is eliminated, the following equation for \( \theta_{h1}, \theta_{k1} \) and \( \theta_{l1} \) can be obtained.
\[ 80\theta_{h1} + 125\theta_{k1} - 35\theta_{l1} - 76 = 0. \]
we can get
\[
640\theta_{k1}^2 + 80\theta_{k1}(10\theta_{l1} - 2\theta_{l1} + 5) + 250\theta_{k1}^2 \\
+ \theta_{l1}(535 - 190\theta_{l1}) - 345\theta_{l1} - 846 = 0 .
\] (7)

Finally, from the condition of eliminating the term \( \Delta t^4 \), we can obtain a more complicated equation,
\[
2560\theta_{k1}^3 + 64\theta_{k1}(50\theta_{k1} - 10\theta_{l1} + 29) \\
+ 169\theta_{l1}(\theta_{l1}(130 - 50\theta_{l1}) - 44\theta_{l1} + 59) \\
- 125\theta_{k1}^2 + 295\theta_{k1}(8 - 3\theta_{l1}) \\
+ 4\theta_{l1}(27\theta_{l1} - 346) - 3851 = 0.
\] (8)

Solving the three equations (6)–(8), we obtain the following values for the parameters \( \theta_{k1}, \theta_{l1} \), and \( \theta_{l1} \).
\[
\begin{aligned}
\theta_{k1} &= 0.804796569 \\
\theta_{l1} &= 0.0969820075 \\
\theta_{l1} &= 0.014470756 .
\end{aligned}
\]

If we free all four parameters, we cannot obtain the parameters between 0 and 1. In this case, therefore, introducing three free parameters is optimal for solving, while (4) keeps a higher COA. Note that EOA values of the ICN and the GA \( \theta \)-ICN are both 2.

Table 2 shows EOA and COA of some numerical schemes of the Kasner test. GA \( \theta \)-ICN has the highest COA.

5. Summary

When differential equations with constraints, such as the Einstein equations, are discretized by numerical schemes, their errors must be evaluated in terms of both the evolution’s order of accuracy (EOA) and the constraint’s order of accuracy (COA). We confirmed that these two are not the same in general from the example of simple harmonic motion.

The iterated Crank–Nicholson method (ICN) is one of the most famous numerical schemes for the Einstein equations but the geometric-averaging \( \theta \)-ICN method (GA \( \theta \)-ICN), which is one of the parametrized numerical schemes, is more useful in terms of COA. We demonstrated that numerical schemes with parameters can improve COA by finding the adequate parameters and performing the numerical experiments.

In this letter, we considered only the ODE cases so they involve simple situations. In future work, we would like to solve PDEs which represent a physical phenomenon.

Acknowledgments

G.Y. was partially supported by JSPS KAKENHI Grant Number 20K03740 and a Waseda University Grant for Special Research Projects 2020C-199.

References

[1] M. Shibata, Collapse of Rotating Supramassive Neutron Stars to Black Holes: Fully General Relativistic Simulations, Astrophys. J., 595 (2003), 992–999.
[2] B. P. Abbott et al. (LIGO Scientific Collaboration), Observation of Gravitational Waves from a Binary Black Hole Merger, Phys. Rev. Lett., 116 (2016), 061102.
[3] R. Arnowitt, S. Deser and C. W. Misner, Republication of: The dynamics of general relativity, Gen. Relativ. Gravit., 40 (2008), 1997–2027.
[4] L. Smarr and J. W. York, Jr., Kinematical conditions in the construction of spacetime, Phys. Rev. D, 17 (1978), 2529.
[5] R. Urakawa, T. Tsuchiya and G. Yoneda, Analyzing time evolution of constraint equations of Einstein’s equation, JSIAM Lett., 11 (2019), 21–24.
[6] T. Tsuchiya, G. Yoneda and H. Shinkai, Constraint Propagation of \( C^2 \)-adjusted formulation: Another recipe for robust ADM evolution system, Phys. Rev. D, 83 (2011), 064032.
[7] T. Tsuchiya and G. Yoneda, Constructing of constraint preserving scheme for Einstein equations, JSIAM Lett., 9 (2017), 57–60.
[8] S. A. Teukolsky, Stability of the iterated Crank-Nicholson method in numerical relativity, Phys. Rev. D, 61 (2000), 087501.
[9] Q. Tran and J. Liu, Modified iterated Crank-Nicolson method with improved accuracy, arXiv:1608.01344 [math.NA].