Automorphisms of Thurston’s Space of Measured Laminations
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The purpose of the note is to give a characterization of the action of the mapping class group on Thurston’s space of measured laminations.

1. We begin with some abstract definitions. Suppose $X$ is a topological space and $\mathcal{F}$ is a collection of real valued (or complex valued) functions on $X$. We say that $\mathcal{F}$ defines an $\mathcal{F}$-structure $(X, \mathcal{F})$ on $X$ if the topology on $X$ is the weakest topology so that each element in $\mathcal{F}$ is continuous, i.e., the collection $\{f^{-1}(U) | U \text{ open in } \mathbb{R}, f \in \mathcal{F}\}$ forms a subbasis for the topology on $X$. For instance, take a smooth manifold $X$ and let $\mathcal{F}$ be the set of all smooth functions on $X$. Then $(X, \mathcal{F})$ is the smooth structure on $X$. An automorphism of a structure $(X, \mathcal{F})$ is a self-homeomorphism $\phi$ of $X$ so that $\phi^*(\mathcal{F}) = \mathcal{F}$ where $\phi^*(\mathcal{F}) = \{f \circ \phi | f \in \mathcal{F}\}$.

2. Suppose now that $\Sigma = \Sigma_{g,r}$ is a compact orientable surface of genus $g$ with $r$ many boundary components ($r \geq 0$). Let $S(\Sigma)$ be the set of isotopy classes of homotopically non-trivial, not boundary parallel, unoriented simple loops in $\Sigma$. Given $\alpha$ and $\beta$ in $\Sigma$, their geometric intersection number, denoted by $I(\alpha, \beta)$, is the minimal number of intersection points between their representatives, i.e., $I(\alpha, \beta) = \min \{|a \cap b| | a \in \alpha, b \in \beta\}$. Thurston’s space of (compactly supported) measured laminations on $\Sigma$, denoted by $ML(\Sigma)$, is defined as follows. Given $\alpha \in S(\Sigma)$, let $I_\alpha$ be the function defined on $S(\Sigma)$ sending $\beta$ to $I(\alpha, \beta)$. The space $ML(\Sigma)$ is the closure of $Q_{\geq 0}\{I_\alpha | \alpha \in S(\Sigma)\}$ in $\mathbb{R}^{S(\Sigma)}$ under the product topology. Thurston showed that $ML(\Sigma)$ is homeomorphic to the Euclidean space $\mathbb{R}^{6g-6+2r}$ and the intersection pairing $I : S(\Sigma) \times S(\Sigma) \to \mathbb{R}$ extends to a continuous pairing $I : ML(\Sigma) \times ML(\Sigma) \to \mathbb{R}$ so that $I(k_1 \alpha, k_2 \beta) = k_1 \cdot k_2 \cdot I(\alpha, \beta)$ for $k_1, k_2 \in \mathbb{R}_{>0}$. (See [Bo], [FLP], [Re], [Th] and others for a proof.) In particular, for each $\alpha$ in $S(\Sigma)$, the map $I_\alpha$ from $ML(\Sigma)$ to $\mathbb{R}$ sending $m$ to $I(\alpha, m)$ is continuous and the collection $\mathcal{F} = \{I_\alpha | \alpha \in S(\Sigma)\}$ forms an $\mathcal{F}$-structure on $ML(\Sigma)$. According to [Th], the structure is called the piecewise integral linear structure on $ML(\Sigma)$. See also [Lu1].

Our result is the following.

**Theorem 1.** Suppose $\Sigma$ is a compact surface with or without boundary whose Euler characteristic is negative. Then any automorphism of the piecewise integral linear structure on the space of measured laminations $ML(\Sigma)$ is induced by a self-homeomorphism of the surface.

3. Proof of theorem 1.

Let $\phi$ be an automorphism of $(ML(\Sigma), \mathcal{F})$. Then $\phi$ induces a bijection $\psi$ of $S(\Sigma)$ by the equation $I_\alpha \circ \phi = I_{\psi(\alpha)}$.

We shall first show that $\psi$ is induced by a self-homeomorphism of the surface. To this end, let us recall that two classes $\alpha$ and $\beta$ in $S(\Sigma)$ are called disjoint, denoted by $\alpha \cap \beta = \emptyset$, if $\alpha \neq \beta$ and $I(\alpha, \beta) = 0$. By counting dimension of $I^{-1}_\alpha(0)$, we shall prove that $\psi$ preserves the disjoint relation on $S(\Sigma)$. Now by a result on the automorphism of $(S(\Sigma), \cap)$ (the automorphisms of the curve complex, [Iv], [Ko], [Lu2]), we see that $\psi$ is induced by a self-homeomorphism of the surface.

Given $\alpha$ in $S(\Sigma)$, let $Z_\alpha = I^{-1}_\alpha(0) \subset ML(\Sigma)$. By using the Dehn-Thurston coordinate associated to a 3-holed sphere decomposition of the surface so that $\alpha$ is a decomposing loop
We see that the dimension \( \dim(Z_{\alpha}) = \dim(ML(\Sigma)) - 1 \) (only the intersection coordinate with \( \alpha \) vanishes).

**Lemma 2.** Two elements \( \alpha \) and \( \beta \) in \( S(\Sigma) \) are disjoint if and only if \( \dim(Z_{\alpha} \cap Z_{\beta}) = \dim ML(\Sigma) - 2 \).

**Corollary 3.** The bijection \( \psi \) from \( S(\Sigma) \) to \( S(\Sigma) \) preserves the disjointness.

Indeed, the equation \( I_{\alpha} \circ \phi = I_{\psi(\alpha)} \) shows that \( \phi^{-1}(Z_{\alpha}) = Z_{\psi(\alpha)} \).

**Proof of Lemma 2.** We may assume that there exist disjoint elements in \( S(\Sigma) \), i.e., \( \dim(ML(\Sigma)) \geq 4 \). Clearly, if \( \alpha \) is disjoint from \( \beta \), then \( \dim(Z_{\alpha} \cap Z_{\beta}) = \dim(ML(\Sigma)) - 2 \). This can be seen by considering the Dehn-Thurston coordinate associated to a 3-holed sphere decomposition so that both \( \alpha \) and \( \beta \) are decomposing loops. We now prove that if \( \alpha \cap \beta \neq \emptyset \), then \( \dim(Z_{\alpha} \cap Z_{\beta}) \leq \dim ML(\Sigma) - 3 \).

To see this, take \( a \in \alpha \) and \( b \in \beta \) so that \( |a \cap b| = I(\alpha, \beta) > 0 \). Let \( N \) be a small regular neighborhood of \( a \cup b \). If \( N \) has null homotopic boundary components in \( \Sigma \), add the disc bounded by the boundary component to \( N \). As a result, we obtain a connected subsurface \( \Sigma' \) whose boundary components are essential in \( \Sigma \). Since \( \alpha \cap \beta \neq \emptyset \), the Euler characteristic of \( \Sigma' \) is negative and \( \Sigma' \neq \Sigma_{0,3} \), i.e., \( \dim(ML(\Sigma')) \geq 2 \). Furthermore, \( \alpha \) and \( \beta \) form a surface filling pair in \( \Sigma' \), i.e., \( I(\alpha, m) + I(\beta, m) > 0 \) for all \( m \in ML(\Sigma') \). This implies that if \( m \in ML(\Sigma) \) so that \( I(m, \alpha) + I(m, \beta) = 0 \), then \( m \) is supported in \( \Sigma - \Sigma' \), i.e., there exist \( m' \in ML(\Sigma - \Sigma') \) and some boundary components of \( \alpha_{1}, ..., \alpha_{n} \) of \( \Sigma' \) so that \( m \) is the disjoint union \( m' \alpha_{k_{1}}^{1} \cdots \alpha_{k_{n}}^{n} \), where \( k_{i} \in \mathbb{R}_{\geq 0} \). If \( \Sigma - \Sigma' \) consists of annuli, then clearly \( Z_{\alpha} \cap Z_{\beta} = \{0\} \). The result follows. If otherwise, choose a 3-holed sphere decomposition of \( \Sigma' \) and extend it to a 3-holed sphere decomposition of \( \Sigma \). For each isotopy class \( \gamma \) of a boundary component of \( \Sigma' \), we have \( I(m, \gamma) = 0 \) for all \( m \in Z_{\alpha} \cap Z_{\beta} \). Thus, by counting the Dehn-Thurston coordinates associated to the 3-holed sphere decomposition, we obtain \( \dim(Z_{\alpha} \cap Z_{\beta}) \leq \dim(ML(\Sigma)) - \dim(ML(\Sigma')) - 1 \leq \dim(ML(\Sigma)) - 3 \). □

Now if \( \dim ML(\Sigma) \geq 4 \) and \( \Sigma \neq \Sigma_{1,2} \), then by theorem 1(a) of [Lu2] (see also [Iv], [Ko]) there exists a self-homeomorphism \( f \) of \( \Sigma \) so that \( f_{*}^{-1}(\alpha) = \psi(\alpha) \) for all \( \alpha \in S(\Sigma) \). In particular, \( I_{\alpha} \circ \phi = I_{\alpha} \circ f_{*} \). Since the map from \( ML(\Sigma) \) to \( \mathbb{R}^{S(\Sigma)}_{\geq 0} \) sending \( m \) to \( (I_{\alpha}(m))_{\alpha \in S(\Sigma)} \) is an embedding, we obtain \( \phi = f_{*} \).

It remains to deal with the surfaces \( \Sigma = \Sigma_{1,2}, \Sigma_{1,1} \) or \( \Sigma_{0,4} \). For surface \( \Sigma_{1,2} \), we shall prove that \( \psi : S(\Sigma_{1,2}) \rightarrow S(\Sigma_{1,2}) \) preserves the classes represented by separating simple loops. Assume this, then theorem 1(b) of [Lu2] shows that \( \psi \) is induced by a self-homeomorphism of the surface. Thus, the above argument goes through.

Suppose otherwise that \( \psi \) sends a separating class to a non-separating class. We shall derive a contradiction by relating \( ML(\Sigma_{1,2}) \) to \( ML(\Sigma_{0,5}) \). Let \( \tau \) be an hyper-elliptic involution of \( \Sigma_{1,2} \) with four fixed points so that the quotient space \( \Sigma_{1,2}/\tau \) is the disc \( D^2 \) with four branch points. It is known by the work of Birman [Bi] and Viro [Vi] that \( \tau(s) \) is isotopic to \( s \) for each simple loop \( s \) not homotopic into \( \partial \Sigma_{1,2} \). In particular, we obtain \( \tau_{*}(m) = m \) for all \( m \in ML(\Sigma_{1,2}) \). Let \( \pi : \Sigma_{1,2} \rightarrow D^2 \) be the quotient map. Consider \( \Sigma_{0,5} \) as the disc \( D^2 \) with a regular neighborhood \( N(B) \) of the the branched point set \( B \) removed, i.e., \( \Sigma_{0,5} = D - int(N(B)) \).

Define \( p : S(\Sigma_{0,5}) \rightarrow ML(\Sigma_{1,2}) \) by sending the isotopy class \( [a] \) to the measured lamination
Note that if $\pi^{-1}(a)$ is connected, then it is a separating loop and if $\pi^{-1}(a)$ is not connected, then it is a union of two parallel copies of a non-separating simple loops. This map $p$ extends to a homeomorphism, still denoted by $p$, from $ML(\Sigma_{0,5})$ to $ML(\Sigma_{1,2})$ so that $I(p(m_1), p(m_2)) = 2I(m_1, m_2)$ for all $m_1, m_2 \in ML(\Sigma_{0,5})$.

Now consider the homeomorphism $\phi' : ML(\Sigma_{0,5}) \to ML(\Sigma_{0,5})$ given by $p^{-1} \phi p$. Since $I_{\alpha} \phi = I_{\psi(\alpha)}$ for all $\alpha \in S(\Sigma_{1,2})$, we obtain $\lambda I_{\alpha} \circ \phi' = I_{\psi'(\alpha)}$ for all $\alpha \in S(\Sigma_{0,5})$ where $\psi' : S(\Sigma_{0,5}) \to S(\Sigma_{0,5})$ is a bijection and $\lambda = 1$ or $1/2$ or $2$ depending on the components of $\pi^{-1}(\alpha)$ and $\pi^{-1}(\phi'(\alpha))$ being separating or not. By the assumption that $\psi$ sends some non-separating simple loops to separating ones, the function $\lambda$ is not a constant. Due to the equation $\lambda I_{\alpha} \circ \phi' = I_{\psi'(\alpha)}$, the map $\phi'$ preserves the set $\{ Z_{\alpha} \mid \alpha \in S(\Sigma_{0,5}) \}$. By lemma 2, we see that $\psi'$ preserves the disjointness. Thus $\psi'$ is induced by a self-homeomorphism $h$ of $\Sigma_{0,5}$. In particular we obtain $\lambda I_{\alpha} \circ \phi' = I_{\alpha} \circ h$ for all $\alpha$. Since the set of rational multiples of $S(\Sigma_{0,5})$ is dense in $ML(\Sigma_{0,5})$ and both $\phi'$ and $h$ are homogeneous, it follows that $\phi' = kh$ for some fixed constant $k = 1$ or $1/2$ or $2$. This contradicts the assumption that $\lambda$ is not a constant.

Finally, we show that any automorphism of $(ML(\Sigma_{1,1}), F)$ and $(ML(\Sigma_{0,4}), F)$ is induced by a surface homeomorphism. Since the structures $(ML(\Sigma_{1,1}), F)$ and $(ML(\Sigma_{0,4}), F)$ are isomorphic, we shall deal with the case $\Sigma_{1,1}$ only.

Let us first identify both $ML(\Sigma_{1,1})$ and $S(\Sigma_{1,1})$ with the first homology groups. Let $i : S(\Sigma_{1,1}) \to H_1(\Sigma_{1,1}, \mathbb{Z})/\pm 1$ be the natural map sending an isotopy class to the corresponding homology classes. It is well known that the map is a bijection from $S(\Sigma_{1,1})$ to $\mathbb{P}/\pm 1$ where $\mathbb{P}$ is the set of primitive elements in $H_1(\Sigma_{1,1}, \mathbb{Z}) \cong \mathbb{Z}^2$. Furthermore, by taking a $\mathbb{Z}$-basis for $H_1(\Sigma_{1,1}, \mathbb{Z})$, each $i(\alpha)$ can be written as $\pm(a, b)$ where $a, b$ are relatively prime integers. Under this identification, the intersection number $I(\alpha_1, \alpha_2) = |a_1b_2 - a_2b_1|$ where $a_i = \pm(a_i, b_i)$. In particular, this shows that $ML(\Sigma_{1,1})$ can be naturally identified with $H_1(\Sigma_{1,1}, \mathbb{R})/\pm 1 \cong \mathbb{R}^2/\pm 1$ so that the above intersection number formula still holds.

The action of self-homeomorphisms on $ML(\Sigma_{1,1})$ is induced by the $GL(2, \mathbb{Z})$ action on $\mathbb{R}^2/\pm 1$. Thus, it remains to show that if $\phi = (\phi_1, \phi_2) : \mathbb{R}^2/\pm 1 \to \mathbb{R}^2/\pm 1$ is a self-homeomorphism so that for each pair of relative prime integers $(a, b) \in \mathbb{P}$ there exists a new pair $(a', b') \in \mathbb{P}$ satisfying $|a\phi_1(x, y) - b\phi_2(x, y)| = |a'x - b'y|$ for all $(x, y) \in \mathbb{R}^2$, then $\phi$ is induced by an element in $GL(2, \mathbb{Z})$. By taking $(a, b)$ to be $(1, 0)$ and $(0, 1)$, we see that $|\phi_1(x, y)| = |a_1x + b_1y|$ and $|\phi_2(x, y)| = |a_2x + b_2y|$. Since $\phi$ is a homeomorphism, $a_1b_2 - a_2b_1 \neq 0$. The goal is to show that $a_1b_2 - a_2b_1 = \pm 1$. Since both $\phi_1, \phi_2$ are continuous and $|\phi_1(x, y) \pm \phi_2(x, y)|$ is of the form $|ax + by|$, it follows that $\phi_1(x, y) = \pm(a_i, x + b_i, y)$ for $i = 1, 2$. Now if $|a_1b_2 - a_2b_1| \geq 2$, then one can find $(a, b) \in \mathbb{P}$ so that $a\phi_1 - b\phi_2$ is of the form $|cx + dy|$ where $c$ and $d$ have a common non-trivial divisor. This contradicts the assumption. \(\square\)

4. One consequence of the proof of theorem 1 is the following characterization of the action of the mapping class group on the projectivized measured lamination space $PML(\Sigma) = ML(\Sigma) - \{0\}/\mathbb{R}_{>0}$.

**Theorem 4.** (Automorphisms of the projective measured lamination spaces) Suppose $\Sigma$ is a compact orientable surface so that $\text{dim}(ML(\Sigma)) \geq 2$ and $\Sigma \neq \Sigma_{1,2}$. For each $\alpha \in S(\Sigma)$, let $P_\alpha$ be the image of $\{m \in ML(\Sigma) - \{0\} \mid I(m, \alpha) = 0\}$ in $PML(\Sigma)$. If $\phi$ is a self-homeomorphism
of the projective measured lamination space \( PML(\Sigma) \) preserving the collection \( \{ P_\alpha | \alpha \in S(\Sigma) \} \), then \( \phi \) is induced by a self-homeomorphism of the surface.

**Proof.** By lemma 2 and the result on the automorphism of the curve complex, we see that there exists a self-homeomorphism \( f \) of the surface so that \( f_*\phi^{-1} : PML(\Sigma) \to PML(\Sigma) \) sends each \( P_\alpha \) to \( P_\alpha \). The image of \( \mathcal{P}(\alpha) \) of \( \alpha \in S(\Sigma) \) in \( PML(\Sigma) \) can be expressed as a finite intersection \( P_{\alpha_1} \cap P_{\alpha_2} \cap \ldots \cap P_{\alpha_k} \). Thus \( f_*\phi^{-1} \) is the identity map on the set \( \mathcal{P}(\alpha) \) is dense in \( PML(\Sigma) \), it follows that \( \phi = f_* \). \( \square \)

5. **Remark.** The theorem is valid for \( \Sigma_{1,2} \) if we assume that the self-homeomorphism \( \phi \) preserves the subset \( \{ P_\alpha | \alpha \text{ is a separating class} \} \). Otherwise, it is false. See [Lu2].

6. Similar automorphism results hold for the Teichmuller space and \( SL(2, \mathbb{R}) \) characters. For simplicity, we state the result for the Teichmuller space. The proof is essentially the same as above and will be omitted. Let \( T(\Sigma) \) be the space of isotopy classes of hyperbolic metrics with cusp ends on \( int(\Sigma) \). For each \( \alpha \in S(\Sigma) \), let \( l_\alpha : T(\Sigma) \to \mathbb{R} \) be the geodesic length function sending a metric \( m \) to the length of \( m \)-geodesic in \( \alpha \). The work of Fricke-Klein [FK] shows that the collection \( \{ l_\alpha | \alpha \in S(\Sigma) \} \) forms an \( F \)-structure on the Teichmuller space.

**Theorem 5.** Suppose \( \Sigma \) is a compact surface of negative Euler characteristic. Then any automorphism of \((T(\Sigma), F)\) is induced by a self-homeomorphism of the surface.

The key step in the proof is to show a result similar to lemma 2. In this case, it is the Margulis lemma that \( \alpha \cap \beta = \emptyset \) if and only if \( \inf \{ l_\alpha + l_\beta \} = 0 \) on \( T(\Sigma) \).

7. Currently, we are unable to solve the automorphism problem for the variety of characters of \( SL(2, \mathbb{C}) \) representations of a closed surface group with respect to the structure of the trace functions \( \{ tr_\alpha | \alpha \in S(\Sigma) \} \). Here \( tr_\alpha \) sends a character \( \chi \) to \( \chi(\alpha) \). See [CS] for an introduction to the subject. The main difficulty is due to the lacking of intrinsic characterization of disjointness \( \alpha \cap \beta = \emptyset \) in terms of the trace functions \( tr_\alpha \) and \( tr_\beta \).

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