Geometry and topology of the Kerr photon region in the phase space

Carla Cederbaum\textsuperscript{1} and Sophia Jahns\textsuperscript{1}

\textsuperscript{1}Department of Mathematics, University of Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany

1 Abstract

We study the set of trapped photons of a subcritical ($a < M$) Kerr spacetime as a subset of the phase space. First, we present an explicit proof that the photons of constant Boyer–Lindquist coordinate radius are the only photons in the Kerr exterior region that are trapped in the sense that they stay away both from the horizon and from spacelike infinity.

We then proceed to identify the set of trapped photons as a subset of the (co-)tangent bundle of the subcritical Kerr spacetime. We give a new proof showing that this set is a smooth 5-dimensional submanifold of the (co-)tangent bundle with topology $SO(3) \times \mathbb{R}^2$ using results about the classification of 3-manifolds and of Seifert fiber spaces.

Both results are covered by the rigorous analysis of Dyatlov \cite{5}; however, the methods we use are very different and shed new light on the results and possible applications.

2 Introduction

It is well-known that in the Schwarzschild spacetime with positive mass $M$, there is a family of photons that orbit the center of the spacetime at the fixed radius $r = 3M$ (in Schwarzschild coordinates), and the surface $\{r = 3M\}$ which they fill is totally umbilical and called the photon sphere.

A similar phenomenon is known in the exterior of the subcritical Kerr spacetime: there is a 2-parameter family of photons which stay at fixed coordinate radius \cite{21}. We give a thorough, explicit proof that these are the only trapped photons in this spacetime family (Proposition\textsuperscript{2}). This fact is often tacitly assumed to be well-known and indeed follows from Dyatlov’s rigorous analysis \cite{5}; whereas our proof explicitly demonstrates that the conventional computations (see e.g. \cite{21}) and heuristic arguments indeed do not overlook any trapped photons.

Unlike in the Schwarzschild situation, the spacetime region in a subcritical Kerr spacetime where trapped photons are found (the region accessible to trapping or the photon region) is not a submanifold (with or without boundary) of the spacetime.
Therefore, the spacetime itself seems not the right setting to investigate geometric properties of photon regions; instead, the (co-)tangent bundle, where the Kerr photon region will be seen to be a submanifold, is a much better setting for further investigation of its geometry. Of these two bundles, the cotangent bundle seems a more appropriate structure to understand photon regions, since the natural symplectic form it carries allows to understand constants of motion as symmetries of the phase space (see e.g. [19]); and the analysis of the constants of motion plays an important role in understanding trapped light.

For the Schwarzschild spacetime of positive mass, a direct computation shows that the set of trapped photons in the cotangent bundle is a submanifold of topology $SO(3) \times \mathbb{R}^2$. The fact that the photon region in the Kerr phase space is a manifold of topology $SO(3) \times \mathbb{R}^2$ as well was shown earlier as a consequence of more general results by Dyatlov in [5], where the implicit function theorem was used on a family of slowly rotating Kerr spacetimes considering them as perturbations of the Schwarzschild spacetime. Our proof uses a different, more direct approach, which does not rely on knowledge about the Schwarzschild case and might help gain better insights into why the set of trapped photons in the Kerr phase space exhibits the properties in question. Knowing this alternative way to determine the topology might also be useful in possibly proving a uniqueness theorem for asymptotically flat, stationary, rotating, vacuum spacetimes with a photon region, in the spirit of e.g. [1, 3], see below.

Other than being interesting in its own right, understanding the geometry of photon regions is also relevant in a variety of contexts: light trapping phenomena play a role in the analysis of stability of black holes (see e.g. the lecture notes [4]), as well as in gravitational lensing (see e.g. the review [14]). Of current interest is also the role that photon regions play in the understanding of black hole shadows (see e.g. [7]).

For static spacetimes, trapping of light has turned out to be a useful tool to prove uniqueness theorems; a static, asymptotically flat vacuum spacetime with a photon sphere as its inner boundary has to be a Schwarzschild spacetime [1, 3]; analogous results holds for some cases with matter [2, 22, 23, 24], and for one case even perturbative uniqueness was established [25], see also [17, 18].

Our article is organized as follows: In Section 3 we recall some well-known facts about the Kerr family and introduce some notation. In Section 4, we give a precise definition of trapping of light for stationary spacetimes and show that only photons of constant Boyer–Lindquist radius can be trapped in a subcritical Kerr spacetime. As a consequence of this, there are no null geodesically complete timelike umbilical hypersurfaces in the domain of outer communication in subcritical Kerr (Corollary 3).

The subsequent Section 5 explains how the set of trapped light can be understood as a subset of the (co-)tangent bundle; we prove that the photon region in the Kerr (co-)tangent bundle is a submanifold (Theorem 9). To this end, we make use of the characterization of trapped photons in terms of constants of motion and apply the implicit function theorem twice. Theorem 10 shows that the photon region in the phase space of a subcritical Kerr spacetime has topology $SO(3) \times \mathbb{R}^2$. This is done by calculating its fundamental group via the Seifert–van Kampen theorem and using the classification of 3-manifolds and a result about Seifert fiber spaces.
3 Basic Facts and Notation

We describe the Kerr spacetime of mass $M$ and angular momentum $a$ in Boyer–Lindquist coordinates $(t, r, \vartheta, \varphi)$ with $t \in \mathbb{R}$, radius $r > 0$ suitably large, latitude $0 \leq \vartheta \leq \pi$, and longitude $0 \leq \varphi \leq 2\pi$. We use the following abbreviations:

\[ S := \sin \vartheta \]
\[ C := \cos \vartheta \]
\[ \rho^2 := r^2 + a^2 \cos^2 \vartheta \]
\[ \Delta := r^2 - 2Mr + a^2 \]
\[ \mathcal{A} := (r^2 + a^2)^2 - \Delta a^2 \sin^2 \vartheta. \]

The metric of the Kerr spacetime in Boyer–Lindquist coordinates is given by

\[- \left(1 - \frac{2Mr}{\rho^2}\right) dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\vartheta^2 - \frac{4Mr\rho^2S^2}{\rho^2} dt d\varphi + \frac{\mathcal{A}}{\rho^4} S^2 d\varphi^2.\]

The domain of outer communication (DOC) is the spacetime patch where $r > M + \sqrt{M^2 - a^2}$. The Boyer–Lindquist coordinates cover the entire DOC, except the axis \( \{S = 0\} \). The breakdown of the metric in this form is a mere coordinate artefact; it can be extended smoothly over the axis, and so can the Killing vector fields $\partial_t$ and $\partial_\varphi$ (see [11]). We treat the subcritical case $|a| < M$. In this case, the horizon \( \{\Delta = 0\} \) has two connected components, and its outer component \( \{r = M + \sqrt{M^2 - a^2}\} \) is the boundary of the DOC.

Furthermore, we assume that $a \geq 0$. This is no restriction of generality, since a sign change of $a$ just means a change of the direction of rotation. If $a = 0$, we recover the Schwarzschild metric.

The Kerr spacetime will be denoted by \((K, g)\).

It is well-known that the motion of regular (that is, not entirely contained in the axis) photons in the DOC of Kerr is governed by the following equations of motion (see e.g. [11]):

\[ \Delta \rho^2 \dot{t} = \mathcal{A} E - 2MraL, \]  \( \rho^4 r^2 = E^2 r^4 + (a^2 E^2 - L^2 - \Theta) r^2 + 2M((aE - L)^2 + \Theta) r - a^2 \Theta =: R(r), \]
\[ \rho^4 \dot{\vartheta}^2 = \Theta - \left(\frac{L^2}{\mathcal{S}^2} - E^2 a^2\right) C^2 =: \Theta(\vartheta), \]
\[ \Delta \rho^2 \dot{\varphi} = 2MraE - \left(\rho^2 - 2Mr\right) \frac{L}{\mathcal{S}^2}. \]

The dot denotes the derivative with respect to the affine parameter.

The quantity $E = -\langle \dot{\gamma}, \partial_t \rangle$ is the energy of a geodesic $\gamma$, $L = \langle \dot{\gamma}, \partial_\varphi \rangle$ its angular momentum, and $\Theta$ its Carter constant, given as $T(\dot{\gamma}, \dot{\gamma})$, where $T$ is the Killing tensor given by $T^{\mu\nu} = \frac{1}{2} g^{\mu\nu} + 2\rho^2 \mathcal{T}(\mu, \nu)$ (where $l^\mu = \frac{1}{2} (r^2 + a^2, 1, 0, a)$ and $n^\nu = \frac{1}{2\rho^2} (r^2 + a^2, -\Delta, 0, a)$ are the principal null directions).

For dealing with general geodesics, it is useful to further introduce the quantity $q := g(\dot{\gamma}, \dot{\gamma})$, which equals zero for null geodesics (photons).
These so-called \textit{constants of motion} \( q, E, L, \Omega \) are constant along each geodesic, since they are derived from Killing vectors and Killing tensors, respectively. Note that they extend over the axis, since the generating tensors do so (see [11]).

Wherever \( \partial_t \) is timelike, all photons have positive energy \( E > 0 \). In case \( E \neq 0 \), we can rewrite the so-called \textit{R-equation} (2) and \textit{\( \Theta \)-equation} (3) as the \textit{scaled R-equation} and the \textit{scaled \( \Theta \)-equation}, respectively:

\[
\left( \frac{\rho^2}{E} \right)^2 \ddot{r}^2 = r^4 + (a^2 - \Phi^2 - Q)r^2 + 2M((a - \Phi)^2 + Q)r - a^2Q \left( = \frac{R}{E^2} \right), \tag{5}
\]

\[
\left( \frac{\rho^2}{E} \right)^2 \dot{\Theta}^2 = Q - (Q + \Phi^2 - a^2)C^2 - a^2C^4 \left( = \frac{\Theta}{E} \right), \tag{6}
\]

with the \textit{conserved quotients} \( \Phi := \frac{\rho}{E} \), and \( Q := \frac{\Theta}{E^2} \) (see [21]).

In the Schwarzschild case \( a = 0 \), the photons in the DOC with constant \( r \)-coordinate are precisely those that fulfill \( r(s) = 3M \) and \( \dot{r}(s) = 0 \) for all parameter values \( s \in \mathbb{R} \). The set \( \{ r = 3M \} \) in Schwarzschild is called the \textit{photon sphere}.

It was shown in [21] that in the DOC of a subcritical Kerr spacetime with \( a \neq 0 \) all photons with constant \( r \)-coordinate (\textit{spherical photon orbits}) belong to the one-parameter class of solutions of \( R(r) = \frac{\partial R(r)}{\partial r} = 0 \) given by

\[
\Phi_{\text{trap}}(r) := -\frac{r^3 - 3Mr^2 + a^2r + a^2M}{a(r - M)}, \tag{7}
\]

\[
Q_{\text{trap}}(r) := -\frac{r^3(r^3 - 6Mr^2 + 9M^2r - 4a^2M)}{a^2(r - M)^2} \tag{8}
\]

(see [2] and [5]), where \( r \) ranges from

\[
\hat{r}_1 := 2M(1 + \cos(2/3 \arccos(-a/M)))
\]

to

\[
\hat{r}_2 := 2M(1 + \cos(2/3 \arccos(a/M))).
\]

The corotating (counterrotating) orbits (i.e., those with positive (negative) angular momentum \( L \)) are located inside (outside) the hypersurface \( \{ r = r_m \} \), where

\[
r_m := M + 2\sqrt{M^2 - \frac{1}{3}a^2 \cos \left( \frac{1}{3} \arccos \frac{M(M^2 - a^2)}{M^2 - \frac{1}{3}a^2} \right)}, \tag{9}
\]

while those at the intermediate radius \( r_m \) have vanishing angular momentum \( L \).

The function \( \Phi_{\text{trap}} : [\hat{r}_1, \hat{r}_2] \rightarrow \mathbb{R} \) is strictly monotonically decreasing and has only one zero, located at \( r_m \), while \( Q_{\text{trap}} : [\hat{r}_1, \hat{r}_2] \rightarrow \mathbb{R}_{\geq 0} \) has its zeros at \( \hat{r}_1 \) and \( \hat{r}_2 \), is strictly increasing on \([\hat{r}_1, 3M]\), and strictly decreasing on \([3M, \hat{r}_2]\). The quantity \( \Phi_{\text{trap}} \) determines the maximal latitude for trapped photons. In particular, trapped photons with radius \( r \) can only reach the the axis \( \{ S^2 = 0 \} \) if \( \Phi_{\text{trap}}(r) = 0 \).

For these facts and more information about \textit{spherical photon orbits} (i.e., those of constant radius) in the Kerr spacetime, see [21].
4 Trapped Photons in the Kerr spacetime

It is natural to ask whether there are more photons than those of constant radial coordinate which are “trapped” around the Kerr center, without ever falling through the horizon or escaping to spatial infinity. Before answering this question, we first need a more precise notion of what it means to be trapped.

We suggest the following definition for the rather general case of stationary spacetimes:

**Definition 1.** A photon in the DOC of a stationary spacetime is called trapped if its orbit in the quotient of the DOC under the action of the stationary Killing vector field $\partial_t$ is contained in a compact set.

Thus, in the Kerr spacetime, a photon is trapped if and only if the range of its radial coordinate is a relatively compact set contained in $(M + \sqrt{M^2 - a^2}, \infty)$.

**Proposition 2.** The spherical photons are the only trapped photons in the DOC of a subcritical $(0 < a < M)$ Kerr spacetime.

*Proof.* Let $\gamma = (t, r, \theta, \varphi)$ be a trapped photon of non-constant radial coordinate in the DOC of a subcritical Kerr spacetime. We will first treat the case that $\gamma$ does not intersect the axis $\{S^2 = 0\}$.

We focus on the subcase that the photon has nonvanishing energy $E \neq 0$ and investigate the right-hand side $R(r)$ of Equation (2). Since its left-hand side is manifestly non-negative, $\gamma$ has to lie in a region of the Kerr space-time where $R(r) \geq 0$.

Since the range of the trapped photon’s radial coordinate is a relatively compact set contained in $(M + \sqrt{M^2 - a^2}, \infty)$, there must be two values $M + \sqrt{M^2 - a^2} < r_1 < r_2 < \infty$ such that the photon either turns around when reaching $r_i$ (that is, the sign of $\dot{r}$ changes at some $s_i$ with $r(s_i) = r_i$) or asymptotically approaches $r_i$ (that is, $\lim_{s \to \infty} r(s) = r_i$ or $\lim_{s \to -\infty} r(s) = r_i$).

As a consequence, for trapping with non-constant radial component, we need two zeros $r_i$ ($i = 1, 2$) of $R(r)$ that lie in the interval $(M + \sqrt{M^2 - a^2}, \infty)$, with $R(r) \geq 0$ between these zeros.

Thus, on the search for trapped photons of non-constant coordinate radius, analyzing the polynomial $R(r)$ of degree 4 with leading coefficient $E^2 > 0$, one would have to find constants of motion such that all roots of $R$ are real, and (writing $r_0 \leq r_1 \leq r_2 \leq r_3$ for these roots) such that $r_1, r_2, r_3$ are outside the outer horizon component $\{r = M + \sqrt{M^2 - a^2}\}$, and $r_1$ and $r_2$ do not coincide, see Figure 1.

Since we assumed that $E \neq 0$, we may work with the scaled $R$-equation (5). We see immediately that the coefficient of $r^2$ has to be negative to allow for the constellation of roots we need (since otherwise $\frac{d}{dr} \left( \frac{R}{E^2} \right)$ would be monotonous and $\frac{R}{E^2}$ convex), that is,

$$a^2 - \Phi^2 - Q < 0. \quad (10)$$

On the other hand, since the left-hand side of the scaled $\Theta$-equation is manifestly non-negative, choosing a non-positive $Q \leq 0$ forces $a^2 - \Phi^2 - Q \geq 0$, contradiction (10). We have thus ruled out the case $Q \leq 0$ for trapped light and may focus on the case $Q > 0$. 

Since $Q > 0$ implies $\frac{R(0)}{E^2} < 0$, a necessary condition for the second root $r_1$ of $\frac{R(r)}{E^2}$ to be outside the outer horizon component is $\frac{R(M)}{E^2} < 0$. But on the other hand, we can estimate (using $Q > 0$ and $a < M$):

$$\frac{R(M)}{E^2} = M^2((\Phi - 2a)^2 + M^2 - a^2) + M^2Q - a^2Q \geq M^2(\Phi - 2a)^2 \geq 0,$$

thus ruling out the existence of trapped light with $r \neq \text{const.}$ outside the axis with nonvanishing energy $E$.

In the subcase of vanishing energy $E = 0$, the $R$-equation reduces to

$$\rho^4r^2 = -(L^2 + \Omega)r^2 + 2M(L^2 + \Omega)r - a^2\Omega.$$

The roots of the right-hand side of this equation are given by

$$r_{1,2} = M \pm \sqrt{M^2 - \frac{a^2\Omega}{L^2 + \Omega}},$$

and at least one of them (if real) is smaller than the radius of the outer horizon component, $M + \sqrt{M^2 - a^2}$.

Summing up, so far we have shown that there are no trapped photons contained in $K \setminus \{S^2 = 0\}$ with nonconstant radial component.

We need to treat the case of photons that are contained in the axis $\{S^2 = 0\}$ separately. The axis is a 2-dimensional, totally geodesic submanifold with line element

$$-(1 - 2Mr/\rho^2)dt^2 + (\rho^2/\Delta)dr^2,$$

so every photon contained in the axis has to fulfill

$$-(1 - 2Mr/\rho^2)\dot{t}^2 + (\rho^2/\Delta)\dot{r}^2 = 0.$$

For trapped photons, $\dot{r}$ has to vanish at some parameter value (at least asymptotically), while $\dot{t} \neq 0$ cannot approach 0. This gives $(1 - 2Mr/\rho^2) = 0$, that is, $\Delta = 0$, which means that the photon is not in the DOC.
The case of photons that cross the axis but are not entirely contained in it can be treated like the off-axis case: the $R$-equation is also valid on the axis, and the conditions on the conserved quotients that were derived from the scaled $\Theta$-equation still would have to be fulfilled, since the conserved quotients can be calculated off-axis.

The statement of Proposition 2 was also discussed in [13], using different techniques, and follows from [5].

We can use Proposition 2 to show nonexistence of certain umbilical hypersurfaces in the DOC. We remind the reader that a semi-Riemannian manifold is called null geodesically complete if every inextendible null geodesic is defined on all of $\mathbb{R}$.

**Corollary 3.** There are no null geodesically complete, timelike umbilical hypersurfaces in the DOC of the subcritical Kerr spacetime (with $a \neq 0$) whose quotient under the stationary Killing vector field $\partial_t$ is contained in a compact set. In particular, none of the cylinders $\{r = \text{const.}\}$ are umbilical.

**Proof.** Assume towards a contradiction that there is such a hypersurface $\mathcal{S}$. By Proposition 3 in [15], a timelike hypersurface in a Lorentzian manifold is umbilical if and only if every null geodesic that is initially tangent to it remains tangent throughout. Since by the assumption the quotient of $\mathcal{S}$ after factoring out $\partial_t$ is relatively compact in the quotient of the DOC, every photon that stays on $\mathcal{S}$ has the property that the range of its radial coordinate is a relatively compact set contained in $(M + \sqrt{M^2 - a^2}, \infty)$. These two facts combine to the conclusion that every photon that is initially tangent to $\mathcal{S}$ is trapped.

By Proposition 2, the only trapped photons are those of constant radial coordinate, hence for any given radius where trapped photons exist, there is only one choice for their conserved quotients $\Phi$ and $Q$ (namely, the values for $\Phi_{\text{trap}}$ and $Q_{\text{trap}}$ from Equations (7) and (8)). We are free to choose a positive value of $E$ for a trapped photon; that is, there is one degree of freedom for choosing the constants of motion for a trapped photon at a given point. These constants of motion determine the off-axis trapped photons completely, possible up to a sign choice for $\dot{\vartheta}$. That is, at every point in the Kerr DOC, there is (at most) one degree of freedom for choosing an initial direction for a trapped photon.

On the other hand, at every point of $\mathcal{S}$, the cone of lightrays tangent to $\mathcal{S}$ is 2-dimensional, which means that there are two degrees of freedom for choosing trapped photons at a given point of $\mathcal{S}$. This gives a contradiction, proving that there is no such $\mathcal{S}$. \qed

**5 The Kerr Photon Region as a Submanifold of the (Co-)tangent Bundle**

We call the region in the Kerr spacetime where one can find trapped photons the region accessible to trapping. In contrast to the Schwarzschild case, in the subcritical Kerr spacetime the region accessible to trapping is not a submanifold (with our without

---

1 In Schwarzschild, the statement is no longer true, as is demonstrated by the striking counterexample of the photon sphere $\{r = 3M\}$. The proof of Corollary 3 cannot be imitated in the $a = 0$ case since there we do not have Equations 7 and 8 at our disposition.
boundary) of the spacetime; it can be imagined as the region covered by a crescent moon that rotates about an axis through its pointy ends, see Figure 2.

Figure 2: A $t = \text{const.}$ slice of the photon region in the Kerr spacetime.

We will identify the set of trapped photons in any spacetime with a subset of the (co-)tangent bundle. In the subcritical Kerr spacetime, it turns out that this subset of the (co-)tangent bundle is a much nicer geometrical object than the region accessible to trapping.

One can identify geodesics in any pseudo-Riemannian manifold $(M, g)$ with points in the tangent bundle $TM$ in the following way: The natural map

$$\{\text{geodesics in } M\} \to TM$$

given by

$$\gamma \mapsto (\gamma(0), \dot{\gamma}(0))$$

is a bijection by uniqueness of geodesics. The canonical isomorphism $TM \to T^*M$ now gives a similar identification of geodesics with points in $T^*M$.

We only consider future-directed trapped photons; by reversing the direction of time (using the symmetry $t \mapsto -t$) one can extend all statements to past directed photons.

The task in the present section is to show that the photon region in the tangent bundle of the subcritical Kerr spacetime $P' \subseteq TK$ and the photon region in the cotangent bundle $P \subseteq T^*K$ (that is, the image of $P'$ under the canonical isomorphism $TM \to T^*M$) are smooth submanifolds of $TK$ and $T^*K$, respectively. Of course, statements about the submanifold structure and the topology of $P'$ and $P$ imply each other immediately, but it will turn out to be useful to work in both settings.

**Remark 4.** If $(M, g)$ is any spacetime, the above identification allows us to view the set of photons in $M$ as

$$\{(m, p) \in TM : g_m(p, p) = 0\}.$$

As the preimage of 0 under $(m, p) \mapsto g_m(p, p)$, this set is a submanifold of $TM$ (and as regular as the spacetime itself), since the differential of this map contains the components of $2g_m(\cdot, p)$ as its last $n$ matrix components, and this never vanishes by non-degeneracy of $g$.

We can naturally extend all functions that were defined on the DOC of Kerr to the tangent bundle (of the DOC) of the Kerr spacetime, so that now in particular the constants of motion and the conserved quotients, but also $\Phi_{\text{trap}}$ and $Q_{\text{trap}}$ are functions
on $TK$. Slightly abusing notation, we will use the same letters to denote these new functions. This remark applies—mutatis mutandis—also to $T^*K$.

The region accessible to trapping intersects the axis in only two points; justified by the equatorial symmetry $\vartheta \mapsto -\vartheta$, we will call one of them the North Pole and the other one the South Pole, and we will refer to the two connected components of the region accessible to trapping without the equatorial plane $\{S^2 = 1\}$ as the Northern and the Southern Hemisphere.

**Remark 5.** As it turns out, in order to obtain later the topology of $P'$ and $P$ we will need to construct smooth bundle charts for a spatial slice of the image of $P$ under the canonical isomorphism $TK \rightarrow T^*K$. Nonetheless, this only proves that $P$ is a manifold, not that it is a submanifold of $T^*K$. Even if one is only interested in the manifold structure of $P'$ and $P$, not their submanifold structure, the proof of the submanifold property has advantages over the explicit construction of charts, since the latter will turn out to be quite technical.

On $TK = K \times \mathbb{R}^4$, we use coordinates $(t, r, \vartheta, \varphi, p^0, p^1, p^2, p^3)$, where $(p^0, p^1, p^2, p^3)$ are the components of the contravariant tangent vector in the same coordinate basis.

**Remark 6.** In the Schwarzschild case $a = 0$, it is easy to see that the photon sphere lifts to a submanifold of the tangent bundle of Schwarzschild: this photon region consists precisely of all photons of the form $(t, 3M, \vartheta, \varphi, p^0, 0, p^2, p^3)$ and is thus in particular a submanifold of the space of all photons.

We shall now consider the situation in the subcritical Kerr spacetime with $a \neq 0$.

**Proposition 7.** The Kerr photon region in the (co-)tangent bundle $TK$ ($T^*K$) of a subcritical Kerr spacetime is a smooth submanifold when the axis $\{S^2 = 0\}$ is deleted; that is, $P' \setminus \{S^2 = 0\}$ is a smooth submanifold of $TK$ (and $P \setminus \{S^2 = 0\}$ is a smooth submanifold of $T^*K$).

**Proof.** We will prove the statement for $P' \subseteq TK$.

By the equations of motion and the conditions for trapped photons (see [11] and [21]), a Kerr geodesic $\gamma = (t, r, \vartheta, \varphi, p^0, p^1, p^2, p^3)$ is a trapped photon if and only if

\begin{align}
    p^0 &= \frac{E}{\Delta p^2} (\omega - 2Mra\Phi_{\text{trap}}), \\
    p^1 &= 0, \\
    (p^2)^2 &= \frac{E^2}{\rho^4} \left( Q_{\text{trap}} - \left( \frac{\Phi_{\text{trap}}^2}{S^2} - a^2 \right) C^2 \right), \\
    p^3 &= \frac{E}{\Delta p^2} \left( 2Mra + (p^2 - 2Mr) \frac{\Phi_{\text{trap}}}{S^2} \right),
\end{align}

for some energy $E = -g_{00}p^0 - g_{03}p^3 > 0$, where $Q_{\text{trap}}$ and $\Phi_{\text{trap}}$ are the functions of $r$ which give the trapping conditions from [21] stated in Equations (7) and (8).

Note that these equations are invariant under the scaling

$$(E, p^0, p^1, p^2, p^3) \mapsto \lambda (E, p^0, p^1, p^2, p^3)$$

with $\lambda > 0$, which allows us to get rid of one of the above equations:
Solving Equation (11) for \( e := E(r, \vartheta, p^0) = \frac{p^0 \Delta \rho^2}{\sqrt{r} - 2Mr \Phi_{\text{trap}}} \) and plugging it into Equations (12)–(14) yields

\[
p^1 = 0 \tag{15}
\]
\[
(p^2)^2 = \frac{e^2}{\rho^4} \left( Q_{\text{trap}} - \left( \frac{\Phi_{\text{trap}}^2}{S^2} - a^2 \right) C^2 \right), \tag{16}
\]
\[
p^3 = \frac{e}{\Delta \rho^2} \left( 2Mr + (\rho^2 - 2Mr) \frac{\Phi_{\text{trap}}}{S^2} \right). \tag{17}
\]

Hence, the off-axis photon region in the tangent bundle is the preimage of 0 under the following smooth function \( f = (f_1, f_2, f_3) : TK \setminus \{S^2 = 0\} \to \mathbb{R} \),

\[
f_1 := p^1, \tag{18}
\]
\[
f_2 := (p^2)^2 - \frac{e^2}{\rho^4} \left( Q_{\text{trap}} - \left( \frac{\Phi_{\text{trap}}^2}{S^2} - a^2 \right) C^2 \right), \tag{19}
\]
\[
f_3 := p^3 - \frac{e}{\Delta \rho^2} \left( 2Mr + (\rho^2 - 2Mr) \frac{\Phi_{\text{trap}}}{S^2} \right). \tag{20}
\]

We will show that the differential of \( f \) has full rank, so that we can use the submersion theorem.

Some partial derivatives of \( f \) are:

\[
\frac{\partial f_1}{\partial p^1} = 1 \quad \frac{\partial f_1}{\partial p^3} = 0 \quad \frac{\partial f_2}{\partial p^1} = 0 \quad \frac{\partial f_2}{\partial p^2} = 2p^2 \quad \frac{\partial f_2}{\partial p^3} = 0
\]
\[
\frac{\partial f_2}{\partial r} = \frac{\partial}{\partial r} \left( \frac{e^2}{\rho^4} \left( Q_{\text{trap}} - \left( \frac{\Phi_{\text{trap}}^2}{S^2} - a^2 \right) C^2 \right) \right) + \frac{e^2}{\rho^4} \left( \frac{\partial Q_{\text{trap}}}{\partial r} - \left( \frac{\Phi_{\text{trap}}^2}{S^2} - a^2 \right) \frac{\partial C^2}{\partial r} \right)
\]
\[
\frac{\partial f_2}{\partial \vartheta} = \frac{\partial}{\partial \vartheta} \left( \frac{e^2}{\rho^4} \left( Q_{\text{trap}} - \left( \frac{\Phi_{\text{trap}}^2}{S^2} - a^2 \right) C^2 \right) \right) - \frac{e^2}{\rho^4} 2C \left( \frac{\Phi_{\text{trap}}^2}{S^3} + \left( \frac{\Phi_{\text{trap}}^2}{S^2} - a^2 \right) S \right)
\]
\[
\frac{\partial f_3}{\partial p^3} = 1
\]

By the above formulas for the partial derivatives of \( f_1 \) and \( f_3 \) we see that the differential of \( f \) has at least rank 2. In order to show that \( f \) is indeed submersive, we assume (towards a contradiction) that the differential of \( f \) has rank 2 at some point of the off-axis photon region \( P' \setminus \{S^2 = 0\} \) in \( TK \). Then \( \frac{\partial f_2}{\partial p^2}, \frac{\partial f_2}{\partial r}, \) and \( \frac{\partial f_2}{\partial \vartheta} \) vanish at this point; and in particular

\[
\frac{\partial f_2}{\partial p^2} = 2p^2 = 0
\]
implies
\[
\left( Q_{\text{trap}} - \left( \frac{\Phi_{\text{trap}}^2}{S^2} - a^2 \right) C^2 \right) = 0 \tag{21}
\]
because of Equation (19) and the fact that $e$ is non-zero at the photon region in $TK$. Hence, $\frac{\partial f_2}{\partial r} = 0$ and $\frac{\partial f_2}{\partial \vartheta} = 0$ reduce to
\[
\frac{\partial Q_{\text{trap}}}{\partial r} - \frac{\partial \Phi_{\text{trap}}^2}{\partial r} \frac{C^2}{S^2} = 0,
\tag{22}
\]
and
\[
2C \left( \frac{\Phi_{\text{trap}}^2}{S^3} + \left( \frac{\Phi_{\text{trap}}^2}{S^2} - a^2 \right) S \right) = 0.
\tag{23}
\]
In the case $C \neq 0$, combining Equations (21) and (23) yields
\[
\Phi_{\text{trap}}^2 + Q_{\text{trap}} \frac{S^4}{C^2} = 0.
\]
But since $Q_{\text{trap}} \geq 0$ in the region accessible to trapping, this implies that $\Phi_{\text{trap}}^2$ and $Q_{\text{trap}}$ vanish for the same value of $r$, which cannot happen (see Section 3).

In the case $C = 0$, Equations (21) and (22) give $Q_{\text{trap}} = 0$ and $\frac{\partial Q_{\text{trap}}}{\partial r} = 0$, which also cannot happen for the same radius $r$, see Section 3.

We have thus derived a contradiction from the assumption that the differential of $f$ does not have full rank somewhere on the off-axis photon region.

Thus, 0 is a regular value of $f$, and we may conclude by the submersion theorem that the photon region of the Kerr spacetime with deleted axis is a submanifold of the tangent bundle $TK$.

Due to the coordinate failure, we have to deal with the axis separately. To do this, we will once again use the submersion theorem, this time applying it to a different function $TK \to \mathbb{R}^3$.

**Proposition 8.** For a subcritical Kerr spacetime $K$, there is a neighborhood of the North (resp. South) Pole of the photon region $P'$ which is a smooth submanifold of $TK$ (or equivalently, a neighborhood of the North (resp. South) Pole of the photon region $P$ which is a smooth submanifold of $T^*K$).

**Proof.** We consider the function $h : TK \to \mathbb{R}^3$ given by
\[
h = (h_1, h_2, h_3) := (q, Q - E^2 \cdot Q_{\text{trap}}, L - E \cdot \Phi_{\text{trap}}),
\]
where $Q_{\text{trap}}, \Phi_{\text{trap}}$ are again the functions given in Equations (7) and (8).

Clearly, the upper hemisphere of the photon region can be described as
\[
\{(m, p) \in TK : p^0((m, p)) > 0, h((m, p)) = 0\}.
\]

The task is to show that $h$ is a submersion in a neighborhood of $P' \cap \{S^2 = 0\}$ in
For the calculations near the axis, we change coordinates

\[(t, r, \vartheta, \varphi) \mapsto (t, r, x := r \cos \varphi \sin \vartheta, y := r \sin \varphi \sin \vartheta).\]

We can work with the new coordinates on each hemisphere; for simplicity let us only consider the Northern Hemisphere, which suffices by the equatorial symmetry. Since also \(p^2, p^3\) fail on the axis, we change the \(p^i\) accordingly \((p^0, p^1, p^2, p^3) \mapsto (\tilde{p}^0, \tilde{p}^1, \tilde{p}^2, \tilde{p}^3)\), where \(\tilde{p}^0, \tilde{p}^1, \tilde{p}^2, \tilde{p}^3\) are the natural coordinates adapted to our new coordinate system on the Northern Hemisphere of \(K\), i.e.,

\[
p^0 = \tilde{p}^0, \quad p^1 = \tilde{p}^1, \quad p^2 = -\frac{(x^2 + y^2)^{\frac{1}{2}}}{z} \tilde{p}^1 + \frac{1}{z(x^2 + y^2)^{\frac{1}{2}}} (x \tilde{p}^2 + y \tilde{p}^3), \quad p^3 = \frac{1}{x^2 + y^2} (-y \tilde{p}^2 + x \tilde{p}^3),
\]

using the abbreviation \(z := \sqrt{r^2 - x^2 - y^2}\).

We note that

\[
p^2 = \mathcal{O}(1) \quad \frac{\partial p^2}{\partial \tilde{p}^2} = \mathcal{O}(1) \quad \frac{\partial p^2}{\partial \tilde{p}^3} = \mathcal{O}(1) \quad p^3 = \mathcal{O}(S^{-1}) \quad \frac{\partial p^3}{\partial \tilde{p}^2} = \mathcal{O}(S^{-1}) \quad \frac{\partial p^3}{\partial \tilde{p}^3} = \mathcal{O}(S^{-1})
\]
as \(S \to 0\).

Furthermore, the metric components with respect to the new coordinates are given on the axis as

\[
\tilde{g}_{\alpha\beta} = diag \left( -\frac{\Delta}{r^2 + a^2}, \frac{r^2 + a^2}{\Delta}, \frac{r^2 + a^2}{r^2}, \frac{r^2 + a^2}{r^2} \right).
\]

We calculate that on the axis

\[
\frac{\partial q}{\partial \tilde{p}^0} = -2\frac{\Delta}{r^2 + a^2} \tilde{p}^0, \quad \frac{\partial q}{\partial \tilde{p}^1} = 2\frac{r^2 + a^2}{r^2} \tilde{p}^1, \quad \frac{\partial q}{\partial \tilde{p}^2} = 2\frac{r^2 + a^2}{r^2} \tilde{p}^2, \quad \frac{\partial q}{\partial \tilde{p}^3} = 2\frac{r^2 + a^2}{r^2} \tilde{p}^3.
\]

For a function \(G : TK \to \mathbb{R}\), we will use the notation \(G = \mathcal{O}_{1, \tilde{p}}(S)\) to subsume \(G = \mathcal{O}(S)\) and \(\frac{\partial G}{\partial \tilde{p}^\alpha} = \mathcal{O}(S)\) for all \(0 \leq \alpha \leq 3\).

To calculate some partial derivatives of the Carter constant

\[
\mathcal{Q} = r^2 q + \frac{1}{\Delta} \left( (r^2 + a^2)^2 (p_0)^2 - \Delta^2 (p_1)^2 + a^2 (p_3)^2 + 2(r^2 + a^2)ap_0 p_3 \right)
\]
on the axis, we first note that \(p_0 = \left(-1 + \frac{2Mr}{\rho^2}\right) \tilde{p}^0 + \mathcal{O}_{1, \tilde{p}}(S), p_1 = \frac{\rho^2}{\Delta} \tilde{p}^1,\) and \(p_3 = \mathcal{O}_{1, \tilde{p}}(S)\) as \(S \to 0\).
This gives
\[ \mathcal{Q} = r^2 q + \frac{1}{\Delta} \left( (r^2 + a^2)^2 \left( -1 + \frac{2Mr}{\rho^2} \right)^2 (\vec{p}^0)^2 - \rho^4 (\vec{p}^3)^2 \right) + \mathcal{O}_{1,\vec{p}}(S) \]
as \( S \to 0 \), and hence
\[ \left. \frac{\partial \mathcal{Q}}{\partial \vec{p}^0} \right|_{S^2=0} = r^2 \left. \frac{\partial q}{\partial \vec{p}^0} \right|_{S^2=0} + 2\Delta \bar{p}^0, \]
\[ \left. \frac{\partial \mathcal{Q}}{\partial \vec{p}^\alpha} \right|_{S^2=0} = r^2 \left. \frac{\partial q}{\partial \vec{p}^\alpha} \right|_{S^2=0} \quad \text{for } \alpha = 2, 3. \]

Similarly, we calculate
\[ L = \langle \partial_v, \dot{\gamma} \rangle = -\frac{2MraS^2}{\rho^2} \vec{p}^0 + \left( r^2 + a^2 + \frac{2Mra^2S^2}{\rho^2} \right) S^2 \vec{p}^3 \]
\[ = -\frac{2MraS^2}{\rho^2} \vec{p}^0 + \frac{1}{r^2} \left( r^2 + a^2 + \frac{2Mra^2S^2}{\rho^2} \right) \left( -y\vec{p}^2 + x\vec{p}^3 \right), \]
and get, using \( \frac{\partial S^2}{\partial x} = \mathcal{O}(S) \) and \( \frac{\partial S^2}{\partial y} = \mathcal{O}(S) \):
\[ \left. \frac{\partial L}{\partial \vec{p}^\alpha} \right|_{S^2=0} = 0 \quad \forall \ 0 \leq \alpha \leq 3, \]
\[ \left. \frac{\partial L}{\partial x} \right|_{S^2=0} = \frac{r^2 + a^2}{r^2} \vec{p}^3, \]
\[ \left. \frac{\partial L}{\partial y} \right|_{S^2=0} = -\frac{r^2 + a^2}{r^2} \vec{p}^2. \]

Finally,
\[ E = \langle \partial_t, \dot{\gamma} \rangle = \left( -1 + \frac{2Mr}{\rho^2} \right) \vec{p}^0 + \frac{2Ma}{r^2} \left( -y\vec{p}^2 + x\vec{p}^3 \right), \]
so that
\[ \left. \frac{\partial E}{\partial \vec{p}^0} \right|_{S^2=0} = -\frac{\Delta}{r^2 + a^2}, \]
\[ \left. \frac{\partial E}{\partial \vec{p}^\alpha} \right|_{S^2=0} = 0 \quad \forall \ 1 \leq \alpha \leq 3. \]

Making use of the previous calculations, we see that the differential of
\[ h = (q, \mathcal{Q} - E^2 \cdot Q_{\text{trap}}, L - E \cdot \Phi_{\text{trap}}) \]
in terms of coordinates \((r, x, y, \vec{p}^a)\) at an axis point \( \{S^2 = 0\} \) of the photon region contains the submatrix
\[ \left( \begin{array}{cccc}
\frac{\partial h_1}{\partial \vec{p}^0} & \frac{\partial h_1}{\partial \vec{p}^2} & \frac{\partial h_1}{\partial \vec{p}^3} & \frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y}
\end{array} \right) \bigg|_{S^2=0} \]
Proof. Since the photon region in any member of the Kerr family is invariant under time translation $\partial_t$ and under rescaling of energy $(E, p^0, p^1, p^2, p^3) \mapsto \lambda (E, p^0, p^1, p^2, p^3)$, $\lambda > 0$, we may therefore restrict our attention to a 6-dimensional slice

$$\{ t = 0, p_0(= -E) = -1 \}$$

from Propositions \[8\] and \[7\], we immediately get the following:

**Theorem 9.** For a subcritical Kerr spacetime $K$, the photon region $P'$ in $TK$ and the photon region $P$ in $T^*K$ are smooth submanifolds.

### 6 The Topology of the Kerr Photon Region in the (Co-)tangent Bundle

We now turn our attention to the topology of $P'$ and $P$. From now on, it will be more useful to work with $P \subseteq T^*K$. We will prove the following theorem towards the end of this section and first show important lemmata and propositions to be used in the proof. The claim of Theorem \[10\] follows from Dyatlov’s implicit function theorem argument in Section 3 of \[5\].

**Theorem 10.** The Kerr photon region and in particular the Schwarzschild photon sphere in the (co-)tangent bundle have topology $SO(3) \times \mathbb{R}^2$.

**Proof.** Since the photon region in any member of the Kerr family is invariant under time translation $\partial_t$ and under rescaling of energy $(E, p^0, p^1, p^2, p^3) \mapsto \lambda (E, p^0, p^1, p^2, p^3)$, $\lambda > 0$, we may therefore restrict our attention to a 6-dimensional slice

$$\{ t = 0, p_0(= -E) = -1 \}$$
in the cotangent bundle and show that the photon region in this slice, 
\[ P_0 := P \cap \{ t = 0, p_0 = -E = -1 \}, \]
has topology \( SO(3) \).

In the Schwarzschild case, we see immediately that \( P_0 \) is the bundle of tangent 1-spheres to the 2-sphere \( \{ r = 3M \} \). Such a sphere bundle is, of course, topologically just \( T^1S^2 \), and it is well-known that \( T^1S^2 \) is homeomorphic to \( SO(3) \).

The remainder of this section is dedicated to the proof of the rotating Kerr case. In order to construct explicit bundle charts of \( P_0 \), we first prove the following

**Lemma 11.** Consider a Kerr spacetime with \( 0 < a < M \).

1. There are smooth functions \( r_{\text{min}}, r_{\text{max}} : (0, \pi) \to \mathbb{R} \) that give, for every \( \vartheta \in (0, \pi) \), the minimal and maximal Boyer–Lindquist radii where trapped photons with latitude \( \vartheta \) may be located.

2. There is a smooth function \( \tau : (0, \pi) \times [-1, 1] \to \mathbb{R} \) with
   \[ \tau(\vartheta, -1) = r_{\text{min}} \]
   \[ \tau(\vartheta, 0) = r_m, \text{ and} \]
   \[ \tau(\vartheta, 1) = r_{\text{max}} \]
   for all \( \vartheta \in (0, \pi) \), and with \( \tau \left( \frac{\pi}{2} + \vartheta, \cdot \right) = \tau \left( \frac{\pi}{2} - \vartheta, \cdot \right) \) for all \( \vartheta \in (0, \frac{\pi}{2}) \),
   (where \( r_m \) is given by Formula (9)), and with the additional property that \( \tau(\vartheta, \cdot) \) has a smooth inverse for every \( \vartheta \in (0, \pi) \).

The function \( \tau(\vartheta, \cdot) \) parametrizes the radial width of the crescent moon in Figure 2.

**Proof.**

1. The boundary of the region accessible to trapping in the spacetime is exactly where trapped photons have vanishing \( \vartheta \)-motion (\( \dot{\vartheta} = 0 \)); hence, the wanted functions \( r_{\text{min}}, r_{\text{max}} \) are given implicitly by the requirements that the right-hand side of the \( \Theta \)-equation (3) with \( Q = Q_{\text{trap}}(r) \) and \( \Phi = \Phi_{\text{trap}}(r) \) plugged in vanishes:
   \[ 0 = Q_{\text{trap}}(r_{\text{min}}) - (\Phi_{\text{trap}}(r_{\text{min}}) - a^2) \cos^2(\vartheta), \]
   \[ 0 = Q_{\text{trap}}(r_{\text{max}}) - (\Phi_{\text{trap}}(r_{\text{max}}) - a^2) \cos^2(\vartheta) \]
   for all \( \vartheta \in (0, \pi) \), and the additional conditions that \( r_{\text{min}} < r_m < r_{\text{max}} \) and that \( r_{\text{min}}(\vartheta), r_{\text{max}}(\vartheta) \) are the the zeros of \( \Theta(\cdot, \vartheta) = 0 \) that are closest to \( r_m(\vartheta) \) for each \( \vartheta \).

   Thus, \( r_{\text{min}} \) and \( r_{\text{max}} \) are well-defined by elementary properties of the region accessible to trapping, and smoothness follows from the smooth dependence of \( \Theta \) on \( r \) and \( \vartheta \).

2. This follows directly from the first part of the lemma, for example by explicitly defining
   \[ \tau(\vartheta, s) := \begin{cases} 
   (r_m(\vartheta) - r_{\text{min}}(\vartheta)) s + r_m(\vartheta) & \text{for } s < 0, \\
   r_m(\vartheta) \left( \frac{r_{\text{max}}(\vartheta) + r_{\text{min}}(\vartheta) - r_m(\vartheta)}{r_m(\vartheta)} \right)^s + (r_m(\vartheta) - r_{\text{min}}(\vartheta)) s & \text{for } s \geq 0. 
   \end{cases} \]
For the following lemma, we fix some notation and interpret the product $S^1 \times S^1$ as follows: let the first factor $S^1$ be parametrized by $\varphi \in [-\pi, \pi)$, and the second factor $S^1$ be viewed as $[-1, 1] \times \{ -1, 1 \} / \sim_{S^1}$, where $\sim_{S^1}$ is the equivalence relation on $[-1, 1] \times \{ -1, 1 \}$ generated by identifying $(\pm 1, 1)$ with $(\pm 1, -1)$. We thus denote elements of $S^1 \times S^1$ in the form $(\varphi, s, \varsigma)$.

Recall that we write $P_0 = P \cap \{ t = 0, p_0 = -1 \}$.

**Lemma 12.** There are functions $p_2, p_3 : (0, \pi) \times S^1 \times S^1 \to \mathbb{R}$ such that the map

$$H : (0, \pi) \times S^1 \times S^1 \to P_0 \setminus \{ S^2 = 0 \},$$

given in Boyer–Lindquist coordinates in the form

$$(\vartheta, \varphi, s, \varsigma) \mapsto H(\vartheta, \varphi, s, \varsigma) = (\varpi(\vartheta, s), \varphi, 0, p_2(\vartheta, \varphi, s, \varsigma), p_3(\vartheta, \varphi, s, \varsigma))$$

is a diffeomorphism, where $\varpi$ is defined as in Lemma 11 (and hence only depends on $|\vartheta - \frac{\pi}{2}|$ and $s$).

**Proof.** We define functions $p_2, p_3 : (0, \pi) \times S^1 \times S^1 \to \mathbb{R}$ as the solutions of Equations (16) and (17) with $\varpi(\vartheta, s)$ plugged in for $r$ and $E = 1$ plugged in for $e$, and with the additional requirement that $\text{sgn} \rho^2(\vartheta, \varphi, s, \varsigma) = \varsigma$. In other words,

\[ p_2(\vartheta, \varphi, s, \varsigma) = \varsigma \left( \frac{1}{\rho^2(\varpi, \varphi)} \cdot \left( Q_{\text{trap}}(\varpi) - \left( \frac{\Phi_{\text{trap}}(\varpi)}{S^2} - a^2 \right) C^2 \right) \right)^{\frac{1}{2}}, \]

\[ p_3(\vartheta, \varphi, s, \varsigma) = \frac{1}{\Delta(\varpi) \rho^2(\varpi, \varphi)} \cdot \left( 2M\varpi a + \left( \rho^2(\varpi, \varphi) - 2M\varpi \right) \frac{\Phi_{\text{trap}}(\varpi)}{S^2} \right), \]

where we use the shorthand $\varpi = \varpi(\vartheta, s)$.

Both $p_2$ and $p_3$ are smooth on $(0, \pi) \times S^1 \times S^1$, and they are of course, as the notation suggests, meant to be used as 7-th and 8-th coordinate in $TK$.

Then, a function $p_0$ is defined as the unique positive solution to

$$g \left((p^0, 0, p^2, p^3), (p^0, 0, p^2, p^3)\right) = 0;$$

it is smooth by smoothness of the metric $g$.

The functions $p_2, p_3$ from the present lemma are obtained by type-changing the vector $(p^0, 0, p^2, p^3) \in TK$.

Bijectivity of $H$ is clear by the construction, and smoothness of the inverse map $H^{-1}$ follows directly from the fact that for every $\vartheta \in (0, \pi)$, the radial width function $\varpi(\vartheta, \cdot)$ has a smooth inverse.

**Lemma 13.** Let $0 < \theta_0 < \pi$ and $I_{\theta_0} := (\Phi_{\text{trap}}(r_{\text{min}}(\theta_0)), \Phi_{\text{trap}}(r_{\text{max}}(\theta_0)))$. There is a unique smooth function $\bar{p}_2 : I_{\theta_0} \to \mathbb{R}_{\geq 0}, p_3 \mapsto \bar{p}_2(p_3)$ such that there is an element of $P_0$ with latitude $\theta_0$, angular momentum $p_3$, and third covariant Boyer–Lindquist coordinate $p_2 = \bar{p}_2(p_3)$.

If $\sin^2 \theta_0$ is small enough, $\bar{p}_2$ is concave.
Proof. Recall that the covariant Boyer–Lindquist coordinate $p_3$ coincides with the scaled angular momentum $\Phi = \Phi_{\text{trap}}$ for every photon in $P_0$. Since $\Phi_{\text{trap}}$ is strictly decreasing in $r$ on the photon region, we may regard $r$ as a function of $p_3$ and write $r(p_3)$ for the solution of $p_3 = \Phi_{\text{trap}}(r)$.

Similarly to the treatment of the functions $p^2, p^3$ in the proof of Lemma 12, we use Equation (6) to see that $\bar{p}_2 > 0$ is given by

$$(\bar{p}_2)^2 = Q_{\text{trap}}(r(p_3)) - (p_3)^2 C^2 S^2 - a^2 C^4 S^2.$$

Therefore,

$$\frac{d^2}{d(p_3)^2} (\bar{p}_2(r))^2 = \frac{d^2}{d(p_3)^2} Q_{\text{trap}}(r(p_3)) - 2 C^2 S^2.$$

Since $\frac{d^2}{d(p_3)^2} Q_{\text{trap}}(p_3)$ is bounded in a neighborhood of the photon region and $\frac{C^2}{S^2}$ approaches infinity at the poles, $\frac{d^2}{d(p_3)^2} (\bar{p}_2(r))^2$ is negative in a punctured neighborhood of any pole. Hence, $(\bar{p}_2)^2$ and also $\bar{p}_2$ are concave if $\sin^2 \vartheta_0$ is chosen sufficiently small.

Lemma 14. There is a $0 < \varepsilon < 1$ such that the neighborhood $P_0 \cap \{\sin^2 \vartheta < \varepsilon\}$ of the North and South Pole in of the photon region in the phase space slice $\{t = 0, p_0 = -1\}$ of a subcritical Kerr spacetime is is diffeomorphic to $(S^2 \cap \{\sin^2 \vartheta < \varepsilon\}) \times \mathbb{S}^1$ via

$$\Psi : P_0 \cap \{\sin^2 \vartheta < \varepsilon\} \to (S^2 \cap \{\sin^2 \vartheta < \varepsilon\}) \times \mathbb{S}^1,$$

where $\Psi$ is given in the coordinates of the proof of Proposition 7 by

$$\Psi(0, r, \vartheta, \varphi, -1, 0, \bar{p}_2, \bar{p}_3) := (\vartheta, \varphi, \text{atan}2(\bar{p}_2, \bar{p}_3)).$$

(A definition of atan2 is given in the proof below.)

Proof. We now view the sphere $\mathbb{S}^1$ as the quotient of the real line that is obtained by factoring out the equivalence relation generated by $u \sim u + 2\pi$ and denote its elements by $[u] := \{u + 2\pi k : k \in \mathbb{Z}\}$.

Note that using $(\vartheta, \varphi)$ also on the poles causes no problem, even though they are not coordinate functions at the poles.

As usual, $\text{atan}2 : \mathbb{R}^2 \setminus \{(0, 0)\} \to (-\pi, \pi)$ gives the angle between the positive $\bar{p}_2$-axis and $(\bar{p}_2, \bar{p}_3)$ (keeping track of its sign) and is defined by

$$\text{atan}2(v, u) = \begin{cases} 
\arctan \left( \frac{v}{u} \right) & \text{for } u > 0, \\
\arctan \left( \frac{v}{u} \right) + \text{sgn } v \cdot \pi & \text{for } u < 0, \\
\text{sgn } v \cdot \frac{\pi}{2} & \text{for } u = 0.
\end{cases}$$

Let $\varepsilon > 0$ be such that the function $\bar{p}_2(\vartheta, \cdot)$ from Lemma 13 is concave for every $\vartheta$ with $\sin^2 \vartheta < \varepsilon$.

The map $\Psi$ as given in the claim is well-defined and smooth. We write $\Psi_{\text{North}}$ and $\Psi_{\text{South}}$ for the restrictions of $\Psi$ to the Northern and Southern pole caps $P_0 \cap \{\sin^2 \vartheta < \varepsilon, \vartheta < (>) \frac{\pi}{2}\}$. Since the union of the pole caps is disjoint, it suffices to show that $\Psi_{\text{North}}$ and $\Psi_{\text{South}}$ are diffeomorphisms.

Focusing on $\Psi_{\text{North}}$, we prove that $\Psi_{\text{North}}$ is a bijection by arguing that for fixed...
\((\vartheta_0, \varphi_0)\), the map 
\[ \Psi_{\text{North}}(0, r, \vartheta_0, \varphi_0, -1, 0, \cdot, \cdot) : P_0 \cap \{ \vartheta = \vartheta_0, \varphi = \varphi_0 \} \to \{ \vartheta_0, \varphi_0 \} \times S^1 \]
is bijective:

First we fix a \((\vartheta_0, \varphi_0) \in S^2\) with \(\vartheta_0 < \pi, \sin^2 \vartheta_0 < \varepsilon\).
We need to show that \([\text{atan2}]\) maps the set

\[ \{(\tilde{p}_2, \tilde{p}_3) \in \mathbb{R}^2 : \exists \ r \text{ such that } (0, r, \vartheta_0, \varphi_0, -1, 0, \tilde{p}_2, \tilde{p}_3) \text{ is a trapped photon}\} \]
bijectionally to \(S^1\). To this end, we change from the covariant coordinates \(\tilde{p}_2, \tilde{p}_3\) to the ones that belong to Boyer–Lindquist coordinates, \(p_2\) and \(p_3\). The set

\[ \{(p_2, p_3) \in \mathbb{R}^2 : \exists \ r \text{ such that } (0, r, \vartheta_0, \varphi_0, -1, 0, p_2, p_3) \text{ is a trapped photon}\} \]

bounds a convex region around the origin in \(\mathbb{R}^2\), since it can be piecewise parametrized by \(p_3 \mapsto (\pm \tilde{p}_2(p_3), p_3)\), where \(\tilde{p}_2\) is the concave function from Lemma \(13\). Its image under the linear bijection

\[ \mathbb{R}^2 \ni (p_2, p_3) \mapsto (\tan \vartheta_0 \cdot (\cos \varphi_0 p_2 - \sin \varphi_0 p_3), -\sin \varphi_0 p_2 + \cos \varphi_0 p_3) \in \mathbb{R}^2 \]
is also a convex set around the origin, which means that \([\text{atan2}]\) maps it bijectively to \(S^1\).

On the other hand, one can easily check that

\[ \text{atan2}(\tilde{p}_2, \tilde{p}_3) = \text{atan2}(\tan \vartheta_0 \cdot (\cos \varphi_0 p_2 - \sin \varphi_0 p_3), -\sin \varphi_0 p_2 + \cos \varphi_0 p_3) , \]

since \(\frac{\tilde{p}_2}{\tilde{p}_3}\) can be rewritten in terms of \(p_2\) and \(p_3\) as

\[ \frac{\tilde{p}_2}{\tilde{p}_3} = \tan \vartheta_0 \cdot \frac{\cos \varphi_0 p_2 - \sin \varphi_0 p_3}{-\sin \varphi_0 p_2 + \cos \varphi_0 p_3} \]

if \(\tilde{p}_3 \neq 0\), and \(\text{sgn} \tilde{p}_0 = \text{sgn} (\cos \varphi_0 p_2 - \sin \varphi_0 p_3)\).

This proves that \(\Psi_{\text{North}}(0, r, \vartheta_0, \varphi_0, -1, 0, \cdot, \cdot)\) is bijective for every \((\vartheta_0, \varphi_0) \in S^2\) with \(\vartheta_0 < \pi, \sin^2 \vartheta_0 < \varepsilon\).

Now we show that \(\Psi_{\text{North}}|_{\{S^2=0\}}\) is a bijection onto \(\{S^2 = 0\} \times S^1\):

At the North Pole \(\{S^2 = 0\}\) of the region accessible to trapping, every lightlike point in the phase space with \(\tilde{p}_1 = 0\) is a trapped photon, and it is obvious by the rotational symmetry of the Kerr spacetime that the set \(\{(\tilde{p}_2, \tilde{p}_3) : (-1, 0, \tilde{p}_2, \tilde{p}_3) \text{ is a photon}\}\) is a circle around the origin in the \(\tilde{p}_2-\tilde{p}_3\)-plane. As before, it is mapped bijectively under \([\text{atan2}]\) onto \(S^1\).

We have now seen that \(\Psi_{\text{North}}\) is bijective. A straightforward calculation yields that the differential of \(\Psi_{\text{North}}\) has full rank everywhere, so that by the implicit function theorem \(\Psi_{\text{North}}\) is a diffeomorphism, which proves the claim.

\[
\square
\]

In order to determine the topology of \(P_0\) (and thereby the topology of \(P\)), we will endow \(P_0\) with a Seifert fibration and calculate its first fundamental group.

First, we will use the maps \(H\) and \(\Psi\) that were constructed in the Lemmata \([12]\) and \([14]\) to conclude:
Proposition 15. The Kerr photon region slice $P_0$ admits a Seifert fibration without exceptional fibers.

Recall that a trivially fibered torus is the standard decomposition of a solid torus $B(0) \times \mathbb{S}^1$ into fibers $\mathbb{S}^1$, which are the preimages of points in $B(0)$ under the canonical projection $\text{proj}: B(0) \times \mathbb{S}^1 \to B(0)$. Choosing a coprime pair of integers $(p, q)$ with $q > 0$, a standard fibered torus is obtained from a trivially fibered torus by slicing it open along $B(0)$, rotating one of the ends by $2\pi q/p$, and gluing the two ends back together. Its central fiber $\text{proj}^{-1}(0)$ is called ordinary if $q = 1$, and exceptional otherwise. A Seifert fibration is a decomposition of a 3-manifold in circles $\mathbb{S}^1$ such that each fiber $\mathbb{S}^1$ has a tubular neighborhood which is isomorphic to a standard fibered torus. For more information about Seifert manifolds, the reader to consult e.g. [12] [16].

Proof of Proposition 15 Each point in $P_0$ has a neighborhood that is homeomorphic to a solid torus $B(0) \times \mathbb{S}^1$ via an appropriate restriction of the map $H$ from Lemma 12 or of the map $\Psi$ from Lemma 14. These maps carry fibers of $\pi_B \in \pi_1$ to fibers of the canonical projection $\text{proj}: B(0) \times \mathbb{S}^1 \to B(0)$; in other words, every fiber in $P_0$ has a neighborhood that is an ordinary standard fibered torus: the given Seifert fibration of $P_0$ has no exceptional fibers.

Proposition 16. The Kerr photon region slice $P_0$ has first fundamental group $\mathbb{Z}_2$.

Proof. The diffeomorphisms $H$ from Lemma 12 and $\Psi$ from Lemma 14 will be used in what follows. Recall (from the proof of Lemma 8) the covariant coordinates $\tilde{p}_i$ that are naturally associated with the coordinates $(t, r, x = r \cos \varphi \sin \vartheta, y = r \sin \varphi \sin \vartheta)$.

While $U_{\text{North}} \approx B_1(0) \times \mathbb{S}^1$ and $U_{\text{South}} \approx B_1(0) \times \mathbb{S}^1$ both have $\mathbb{Z}$ as their first fundamental group, $U_{\text{Eq}} \approx (-\pi, \pi) \times \mathbb{S}^1 \times \mathbb{S}^1$ has first fundamental group $\mathbb{Z} \times \mathbb{Z}$. Moreover, $\pi_1(U_{\text{Eq}} \cap U_{\text{North}}) = \pi_1(U_{\text{Eq}} \cap U_{\text{South}}) = \mathbb{Z} \times \mathbb{Z}$.

Let $\frac{\pi}{2} < \vartheta_0 < \pi$ be such that every point of $P_0$ with latitude $\vartheta_0$ is in the domain of $\Psi_{\text{North}}$. We define homotopies of closed paths $\gamma_1, \gamma_2 : \mathbb{S}^1 \times I \to P_0$ by

\[
\gamma_{1,\lambda}(\varphi) := H(\vartheta_0 - \lambda(2\vartheta_0 - \pi), \varphi, 0, -1),
\]

\[
\gamma_{2,\lambda}(s, \varsigma) := H(\vartheta_0 - \lambda(2\vartheta_0 - \pi), 0, s, \varsigma).
\]

By construction, the paths $\gamma_{1,0}$ and $\gamma_{2,0}$ are representatives of a set of generators for $\pi_1(U_{\text{Eq}} \cap U_{\text{North}})$, and the analogous statement holds for $\gamma_{1,1}$ and $\gamma_{2,1}$ in $\pi_1(U_{\text{Eq}} \cap U_{\text{South}})$, as well as for $\gamma_{1,\lambda}$ and $\gamma_{2,\lambda}$ in $\pi_1(U_{\text{Eq}})$ (for every $\lambda$).

We will now determine what elements in $\pi_1(U_{\text{North}})$ the paths $\gamma_{i,0}$ represent and what elements in $\pi_1(U_{\text{South}})$ the paths $\gamma_{i,1}$ represent (for $i = 1, 2$).

Note that by construction of $H$, the image of each $\gamma_{1,\lambda}$ consist of points with $r = r_m$ and hence with $p_3 = 0$.

This makes it easy to calculate—using the standard transformations for the coordinates of the phase part of $\mathbb{C}^*$—that for points in the image of a path $\gamma_{1,\lambda}$ with $\lambda \neq \frac{1}{2}$:

\[
\tilde{p}_2 \circ \gamma_{1,\lambda}(\varphi) = \frac{p_2}{r \cos \vartheta} \cos \varphi
\]

\[
\tilde{p}_3 \circ \gamma_{1,\lambda}(\varphi) = -\frac{p_2}{r \sin \vartheta} \sin \varphi.
\]
where we use $\vartheta$ as shorthand for $\vartheta(\lambda) = \vartheta_0 - \lambda(2\vartheta_0 - \pi)$ and $p_2$ as shorthand for $p_2 \circ \gamma_{1,\lambda}(\varphi)$.

Clearly, $[p_2] \circ \gamma_{1,\lambda}(\varphi)$ can be calculated from $p_3 \circ \gamma_{1,\lambda} = 0$, $r \circ \gamma_{1,\lambda} = r_m$, and the modulus of the latitude, $|\vartheta - \frac{\pi}{2}|$. Since we have chosen negative sign of $p_2$ for all points in the range of $\gamma_{1,\lambda}$, even $p_2 \circ \gamma_{1,\lambda}(\varphi)$ only depends on $|\vartheta - \frac{\pi}{2}|$.

This allows us to simplify (24)–(25) to

$$\tilde{p}_2 \circ \gamma_{1,0}(\varphi) = A \cos \varphi,$$

$$\tilde{p}_2 \circ \gamma_{1,1}(\varphi) = -A \cos \varphi,$$

$$\tilde{p}_3 \circ \gamma_{1,\lambda}(\varphi) = B \sin \varphi \text{ for } \lambda = 0, 1$$

for positive constants $A, B$.

We see from these formulas that the map $P_3 \circ \Psi \circ \gamma_{1,0} : S^1 \to S^1$ (where $P_3$ is the projection onto the third component), given by $[\text{atan}2(\tilde{p}_2 \circ \gamma_{1,0}, \tilde{p}_3 \circ \gamma_{1,0})]$, has index 1 and is hence homotopy to the identity map on $S^1$.

Similarly, $P_3 \circ \Psi \circ \gamma_{1,1} : S^1 \to S^1$ is of index $-1$ and can be thought of as the map $\text{id} : S^1 \to S^1$.

We may thus note for later use Fact 1: the path $\Psi \circ \gamma_{1,0}$ is equivalent to $\varphi \mapsto (\vartheta_0, 0, \varphi)$ in $\pi_1(U_{\text{North}} \cap U_{\text{Eq}})$, and the path $\Psi \circ \gamma_{1,1}$ is equivalent to $\varphi \mapsto (\pi - \vartheta_0, 0, -\varphi)$ in $\pi_1(U_{\text{South}} \cap U_{\text{Eq}})$.

Similarly as we just did for $\gamma_{1,\lambda}$, we now calculate the components $\tilde{p}_2$ and $\tilde{p}_2$ on the orbits of $\gamma_{2,\lambda}$ for $\lambda \neq \frac{1}{2}$:

$$\tilde{p}_2 \circ \gamma_{2,\lambda}(s, \varsigma) = \frac{1}{r \cos \vartheta} p_2,$$

$$\tilde{p}_3 \circ \gamma_{2,\lambda}(s, \varsigma) = \frac{1}{r \sin \vartheta} p_3.$$

Here, $p_2$ and $p_3$ depend on $\vartheta(\lambda) = \vartheta_0 - \lambda(2\vartheta_0 - \pi)$ and $(s, \varsigma)$. Actually, $p_2$ and $p_3$ are independent of the sign of $\vartheta - \frac{\pi}{2}$, since the function $\tau$ used in the construction of $H$ is.

This allows us to write

$$\tilde{p}_2 \circ \gamma_{2,0}(s, \varsigma) = -\tilde{A} p_2,$$  \hspace{1cm} (26)

$$\tilde{p}_2 \circ \gamma_{2,1}(s, \varsigma) = \tilde{A} p_2,$$  \hspace{1cm} (27)

$$\tilde{p}_3 \circ \gamma_{2,\lambda}(s, \varsigma) = \tilde{B} p_3 \text{ for } \lambda = 0, 1$$  \hspace{1cm} (28)

for positive constants $\tilde{A}, \tilde{B}$.

We already know that $\gamma_{2,0}$ represents one of the generators of $\pi_1(U_{\text{Eq}} \cap U_{\text{North}})$, and that $\gamma_{2,1}$ represents one of the generators of $\pi_1(U_{\text{Eq}} \cap U_{\text{South}})$. Choosing an orientation for $S^1 \equiv [-1, 1] \times \{0\} / \sim$, we can decide that $P_3 \circ \Psi \circ \gamma_{2,\lambda} : S^1 \to S^1$ has index 1. Then $P_3 \circ \Psi \circ \gamma_{2,\lambda} : S^1 \to S^1$ has index $-1$. We can now state Fact 2: the path $\Psi \circ \gamma_{2,0}$ is equivalent to $\varphi \mapsto (\vartheta_0, 0, \varphi)$ in $\pi_1(U_{\text{North}} \cap U_{\text{Eq}})$, and the path $\Psi \circ \gamma_{2,1}$ is equivalent to $\varphi \mapsto (\pi - \vartheta_0, 0, -\varphi)$ in $\pi_1(U_{\text{South}} \cap U_{\text{Eq}})$.

In the last step of the proof, we finally calculate $\pi_1(P_0)$ using the Seifert–van Kampen theorem. Combining Fact 1 and Fact 2, we note that in $\pi_1(U_{\text{North}} \cap U_{\text{Eq}})$,

$$[\gamma_{1,0}]_{\pi_1(U_{\text{North}} \cap U_{\text{Eq}})} = [\gamma_{2,0}]_{\pi_1(U_{\text{North}} \cap U_{\text{Eq}})}.$$

20
and in $\pi_1(U_{\text{South}} \cap U_{\text{Eq}})$, that
\[
[\gamma_1,1]_{\pi_1(U_{\text{South}} \cap U_{\text{Eq}})} = [\gamma_2,1]_{\pi_1(U_{\text{South}} \cap U_{\text{Eq}})}.
\]

We apply the Seifert–van Kampen theorem two times; first to $U_{\text{North}}$ and $U_{\text{Eq}}$, then to $U_{\text{North}} \cup U_{\text{Eq}}$ and $U_{\text{South}}$.

For the sake of readability, the homomorphisms of the different fundamental groups induced by the various inclusion maps are notationally omitted.

Using the group presentations
\[
\pi_1(U_{\text{North}}) = \langle [\gamma_1,0]_{\pi_1(U_{\text{North}})} : [\gamma_2,0]_{\pi_1(U_{\text{North}})} \rangle
\]
and
\[
\pi_1(U_{\text{Eq}}) = \langle [\gamma_1,0]_{\pi_1(U_{\text{Eq}})} : [\gamma_2,0]_{\pi_1(U_{\text{Eq}})} \rangle,
\]
one obtains $\pi_1(U_{\text{Eq}} \cup U_{\text{North}})$ by factoring the identifications
\[
[\gamma_1,0]_{\pi_1(U_{\text{North}})} = [\gamma_2,0]_{\pi_1(U_{\text{Eq}})};
\]
\[
[\gamma_1,0]_{\pi_1(U_{\text{North}})} = [\gamma_1,0]_{\pi_1(U_{\text{Eq}})}
\]
out of the group
\[
\langle [\gamma_1,0]_{\pi_1(U_{\text{North}})} : [\gamma_2,0]_{\pi_1(U_{\text{North}})} : [\gamma_1,0]_{\pi_1(U_{\text{Eq}})} : [\gamma_2,0]_{\pi_1(U_{\text{Eq}})} \rangle.
\]

Now, $\pi_1(P_0) = \pi_1((U_{\text{Eq}} \cup U_{\text{North}}) \cup U_{\text{South}})$ can be calculated by factoring the identifications
\[
[\gamma_1,1]_{\pi_1(U_{\text{South}})} = [\gamma_2,0]_{\pi_1(U_{\text{Eq}})};
\]
\[
[\gamma_1,1]_{\pi_1(U_{\text{South}})} = [\gamma_1,0]_{\pi_1(U_{\text{Eq}})}
\]
(which come from Facts 1 and 2) out of the free product of $\pi_1(U_{\text{Eq}} \cup U_{\text{North}})$ with $\pi_1(U_{\text{South}})$.

Hence, $\pi_1(P_0)$ is the quotient of $\langle [\gamma_1,0]_{\pi_1(U_{\text{Eq}})} : [\gamma_2,0]_{\pi_1(U_{\text{Eq}})} : [\gamma_1,0]_{\pi_1(U_{\text{North}})} : [\gamma_1,1]_{\pi_1(U_{\text{South}})} \rangle$ after factoring out the identifications
\[
[\gamma_1,0]_{\pi_1(U_{\text{North}})} = [\gamma_2,0]_{\pi_1(U_{\text{Eq}})}^{-1};
\]
\[
[\gamma_1,0]_{\pi_1(U_{\text{North}})} = [\gamma_1,0]_{\pi_1(U_{\text{Eq}})};
\]
\[
[\gamma_1,1]_{\pi_1(U_{\text{North}})} = [\gamma_2,0]_{\pi_1(U_{\text{Eq}})};
\]
\[
[\gamma_1,1]_{\pi_1(U_{\text{South}})} = [\gamma_1,0]_{\pi_1(U_{\text{Eq}})}.
\]
The quotient we are left with is the group $\mathbb{Z}_2$. 

We are now going to combine the fact that $P_0$ admits a Seifert fibration without exceptional fibers (Proposition 15) with the fact that $P_0$ has first fundamental group $\mathbb{Z}_2$ (Proposition 16) in a very short argument to conclude that the photon region in the phase space of a subcritical Kerr spacetime has topology $SO(3)$.
Proof of Theorem 10. Since \( P_0 \) is a closed 3-dimensional manifold with \( \pi_1(P_0) = \mathbb{Z}_2 \), it must be homeomorphic to one of the lens spaces \( L(2;0) \) or \( L(2;1) \) (by general classification results for 3-manifolds, see for example \cite{10}). On the other hand, it is known (and follows, for example, as a special case of Theorem 4.8 in \cite{9}) that \( L(2;0) \) admits no Seifert fibration with less than 2 exceptional fibers. But Proposition 15 states that \( P_0 \) is Seifert fibered without exceptional fibers. Hence, the only remaining possibility for the topology of \( P_0 \) is \( P_0 \approx L(2;1) \approx SO(3) \). Recalling how \( P_0 \) was obtained as a slice \( P \cap \{ t = 0, p_0 = -1 \} \) of the photon region in the phase space, this proves Theorem 10.

7 Outlook

We have described how the phenomenon of trapping of light in subcritical Kerr spacetimes can be better understood in the framework of the cotangent bundle and have characterized the set of (future) trapped photons as a submanifold of \( T^*K \) with topology \( SO(3) \times \mathbb{R}^2 \). It is natural to ask if analogous results can be obtained with similar methods for other stationary spacetimes, possibly involving a cosmological constant. Note that in \cite{5}, the implicit function argument for the computation of the topology of the photon sphere in the phase space is also applied to the Kerr–de Sitter spacetime.

Null geodesics of constant coordinate radius in (various subfamilies of) the Plebański–Demiański class of metrics have been studied in \cite{8, 9}. To our knowledge, it has not yet been rigorously investigated whether these photon orbits are the only trapped photons in the respective spacetimes, which would be a crucial first step for a geometric understanding of the photon region. Results for spherical photons in the Plebański–Demiański spacetime family show a very similar spacetime picture of trapping as in the Kerr family (\cite{8, 9}); one might also hope to prove in this setting that the photon region in the phase space is a submanifold with topology \( SO(3) \times \mathbb{R}^2 \), but of course, the calculations would become much more involved in this generalized setting.

We can also ask similar questions about the photon region in stationary spacetimes of higher dimensions. By the symmetry of a Schwarzschild–Tangherlini black hole \cite{20} of dimension \( n+1 \) and mass \( M \), it is known that trapping of light occurs at a fixed radius \( r = nM \) for every photon that is initially tangent to the hypersurface \( \{ r = nM \} \). Hence (by arguments just like in the 4-dimensional Schwarzschild case), the photon region in the phase space can be expected to be a submanifold with the topology of a tangent unit sphere bundle \( T^1S^{n-1} \), times \( \mathbb{R}^2 \).

It seems reasonable to conjecture that for Myers–Perry spacetimes of dimension \( n+1 \), the photon region in the phase space has the same topology and bundle structure as the one of the Schwarzschild–Tangherlini solution of dimension \( n+1 \) (whether the various rotation parameters coincide or not), since letting the rotational parameters tend to zero should not cause jumps in the topology, just as in the 3 + 1-dimensional Kerr case. This can, however, not be proved by similar methods as the ones we used for the 4-dimensional case, since our proof relies on the classification of 3-manifolds via their fundamental groups, and there is no similar classification available for higher dimensions.

The presented way to determine the topology of the Kerr photon region in phase space might turn out to be useful in proving a uniqueness theorem for asymptotically flat, stationary, vacuum spacetimes with a photon region inner boundary, in the spirit
of the static results in this direction discussed in the introduction.

8 Acknowledgements

We would like to thank Pieter Blue and András Vasy for useful comments. Furthermore, we thank Oliver Schön for generating the figures for this article. This work is supported by the Institutional Strategy of the University of Tübingen (Deutsche Forschungsgemeinschaft, ZUK 63).

References

[1] Carla Cederbaum. Uniqueness of photon spheres in static vacuum asymptotically flat spacetimes. In Complex Analysis & Dynamical Systems IV, volume 667 of Contemp. Math., pages 86–99. AMS, 2015.

[2] Carla Cederbaum and Gregory J. Galloway. Uniqueness of photon spheres in electro-vacuum spacetimes. Class. Quantum Grav., 33(7):075006, 2016.

[3] Carla Cederbaum and Gregory J. Galloway. Uniqueness of photon spheres via positive mass rigidity. Commun. Anal. Geom., 25(2):303–320, 2017.

[4] Mihalis Dafermos and Igor Rodnianski. Lectures on black holes and linear waves. arXiv:0811.0354 [gr-qc], 2008.

[5] Semyon Dyatlov. Asymptotics of linear waves and resonances with applications to black holes. Communications in Mathematical Physics, 335(3):1445–1485, May 2015.

[6] Hansjörg Geiges and Christian Lange. Seifert fibrations of lens spaces. arXiv:gr-qc/0512066.

[7] Arne Grenzebach. The Shadow of Black Holes. An analytic description. Springer, Berlin, 2016.

[8] Arne Grenzebach, Volker Perlick, and Claus Lämmerzahl. Photon regions and shadows of Kerr-Newman–NUT black holes with a cosmological constant. Phys. Rev., D89(12):124004, 2014.

[9] Arne Grenzebach, Volker Perlick, and Claus Lämmerzahl. Photon regions and shadows of accelerated black holes. International Journal of Modern Physics D, 24:1542024, June 2015.

[10] Allen Hatcher. Notes on basic 3-manifold topology. http://pi.math.cornell.edu/~hatcher/3M/3Mdownloads.html 2007. last accessed 06/18/2018.

[11] Barrett O’Neill. The Geometry of Kerr Black Holes. Dover Publications, Mineola, New York, 2014.

[12] Peter Orlik. Seifert manifolds, volume 291 of Lecture notes in mathematics. Springer, Berlin, Heidelberg, New York, 1972.
[13] Claudio Paganini, Blazej Ruba, and Marius A. Oancea. Characterization of null geodesics on Kerr spacetimes. arXiv:1611.06927 [gr-qc], 2016.

[14] Volker Perlick. Gravitational lensing from a spacetime perspective. Living Reviews in Relativity, 7(1):9, Sep 2004.

[15] Volker Perlick. On totally umbilic submanifolds of semi-Riemannian manifolds. Nonlinear Analysis, 63(5-7), 2005. arXiv:gr-qc/0512066.

[16] Peter Scott. The geometries of 3-manifolds. Bulletin of the London Mathematical Society, 15(5):401–487, 1983.

[17] Andrey A. Shoom. Metamorphoses of a photon sphere. Phys. Rev. D, 96:084056, Oct 2017.

[18] Andrey A. Shoom, Cole Walsh, and Ivan Booth. Geodesic motion around a distorted static black hole. Phys. Rev. D, 93:064019, Mar 2016.

[19] Paul Sommers. On Killing tensors and constants of motion. Journal of Mathematical Physics, 14(6):787–790, 1973.

[20] Frank R. Tangherlini. Schwarzschild field in n dimensions and the dimensionality of space problem. Nuovo Cim., 27:636–651, 1963.

[21] Edward Teo. Spherical photon orbits around a Kerr black hole. General Relativity and Gravitation, 35(11):1909–1926, 2003.

[22] Stoytcho Yazadjiev and Boian Lazov. Uniqueness of the static Einstein–Maxwell spacetimes with a photon sphere. Class. Quantum Grav., 32(16):165021, 2015.

[23] Stoytcho S. Yazadjiev. Uniqueness of the static spacetimes with a photon sphere in Einstein-scalar field theory. Phys. Rev. D, 91(12):123013, 2015.

[24] Stoytcho S. Yazadjiev and Boian Lazov. Classification of the static and asymptotically flat Einstein-Maxwell-dilaton spacetimes with a photon sphere. Phys. Rev. D, 93(8):083002, 11, 2016.

[25] Hirotaka Yoshino. Uniqueness of static photon surfaces: Perturbative approach. Phys. Rev. D, 95:044047, Feb 2017.