Differential operators and reflection group of type $B_n$

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Abstract.

In this note, we study the polynomial representation of the quantum Olshanetsky-Perelomov system for a finite reflection group $W$ of type $B_n$. We endowed the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ with a structure of module over the Weyl algebra associated with the ring $\mathbb{C}[x_1, \ldots, x_n]^W$ of invariant polynomials under a reflections group $W$ of type $B_n$. Then we study the polynomials representation of the ring of invariant differential operators under the reflections group $W$. We make use of the theory of representation of groups namely the higher Specht polynomials associated with the reflection group $W$ to yield a decomposition of that structure by providing explicitly the generators of its simple components.

1. Introduction

The History of the Weyl algebra begins with the birth of quantum mechanics. The theory of groups has played a major role in the discovery of the general laws of quantum theory. It is not surprising that concepts arising in the theory of group find their applications in physics. As Hermann Weyl has said in [15], there exists a plainly discernible parallelism between the more recent developments of mathematics and physics. Also the theory of group representations is one of the best example of interaction between physics and pure mathematics. With this in mind we study the polynomials representation of the Weyl algebra with special consideration for physical problems. Since symmetries are relevant in physics, our attention has been drawn particularly to reflection groups $W(B_n)$ of type $B_n$ and its representation in connection to rational Olshanetsky-Perelomov operators. In what follows we try to understand the actions of invariant differential operators on the polynomial ring through the representation theory of reflection groups of type $B_n$. We show that the polynomial representation of the rational Olshanetsky-Perelomov systems for $W$ is related to the representation theory of $W$. In fact we have studied the polynomials representation of the rational quantum Calogero-Moser system in [12] and similar topic in [13, 11]. Since the rational Olshanetsky-Perelomov systems is a generalization of rationa, quantum Calogero-Moser system, we extend the results obtained in [12]to the rational Olshanetsky-Perelomov systems for reflection group of type $B_n$. 
2. Preliminaries and motivations

2.1. Reflection groups and root systems

We recall some basics facts about real reflection groups of type $B_n$ (see [8] for more details).

Let $\mathfrak{h}$ be a (real) euclidean space endowed with a positive definite symmetric linear form $(\cdot, \cdot)$. A reflection is a bilinear operator $s_\alpha$ on $\mathfrak{h}$ which sends some nonzero vector $\alpha$ to its negative while fixing pointwise the hyperplane $H_\alpha$ orthogonal to $\alpha$. We may write $s = s_\alpha$ and there is a simple formula:

$$s_\alpha = \lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}, \quad \forall \lambda \in \mathfrak{h}.$$  

The operator $s_\alpha$ is an element of order 2 in the group $O(\mathfrak{h})$ of all orthogonal transformations of $\mathfrak{h}$.

A finite group generated by reflections is called a reflection group, such group belongs to $O(\mathfrak{h})$. Let the symmetric group $S_n$ acts on $\mathfrak{h} = \mathbb{R}^n$ by permuting the standard basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$.

The transposition $(i,j)$ acts as a reflection, sending $\epsilon_i - \epsilon_j$ to its negative and fixing pointwise the orthogonal complement. It is clear that $S_n$ is a reflection group. Other reflections can be made by sending an $\epsilon_i$ to its negative and fixing all other $\epsilon_j$. Theses sign changes generate a group of order $2^n$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$, which intersects $S_n$ trivially and is normalized by $S_n$. Thus the semidirect product of $S_n$ and the group of sign changes yields a reflection group $W = (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$, called the real reflection group of type $B_n$, denoted by $W(B_n)$.

A root system $R$ associated to a reflection group $W$ is a finite set of nonzero vectors in $\mathfrak{h}$ satisfying the five following conditions: 1) $R$ spans $\mathfrak{h}$ as a vector space; 2) $R \cap R\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in R$; 3) $s_\alpha(R) = R$ for all $\alpha \in R$; 4) for any $\alpha, \beta \in R$, $(\alpha, \beta) \in \mathbb{Z}$ and 5) $W$ is generated by all reflections $s_\alpha$, $\alpha \in R$.

Let $\epsilon_1, \ldots, \epsilon_n$ the standard basis for $\mathfrak{h}$. The set corresponding set of roots $R$ associated with $W(B_n)$ consists of $2n$ sort roots $\pm \epsilon_i$ and and $2n(n - 1)$ roots $\pm \epsilon_i \pm \epsilon (i < j)$ [8].

2.2. Quantum Olshanetsky-Perelomov Hamiltonian

Let $W$ be a real reflection group, $R$ the system of roots associated to $W$ and $S = \{s_\alpha | \alpha \in R\}$ the set of reflections. Clearly, $W$ acts on $S$ by conjugate. Let $c : S \rightarrow \mathbb{C}$ be a conjugate invariant function.

The quantum Olshanetsky-Perelomov Hamiltonian attached to $W$ is the second order differential operator

$$H := \Delta_\mathfrak{h} - \sum_{s \in S} c_s (c_s + 1) (\alpha_s, \alpha_s) \alpha_s^2,$$

where $\Delta_\mathfrak{h}$ is the Laplace operator on $\mathfrak{h}$.

It turns out that the system defined by the Olshanetsky-Perelomov operator $H$ is completely integrable. Namely, we have the following theorem.

**Theorem 2.1.** [14] There exist differential operators $L_j$ on $\mathfrak{h}$ with rational coefficient and symbols $P_j$ such that $L_j$ are homogeneous (of degree $-d_j$), $L_1 = H$, and $[L_j, L_k] = 0$, $\forall j,k$.

This Theorem is obviously a generalization of [12, Theorem 2.1].

Let $G \subset \text{GL}(\mathfrak{h})$ be a finite subgroup. Let $S$ be the set of reflections in a $W$. For any reflection $s \in S$, let $\lambda_s$ be the eigenvalue of $s$ on $\alpha_s \in \mathfrak{h}^*$ (i.e. $s\alpha_s = \lambda_s \alpha_s$), and let $\alpha_s^\vee \in \mathfrak{h}$ be an eigenvector such that $s\alpha_s^\vee = \lambda_s^{-1} \alpha_s^\vee$. We normalize them in such a way that $\langle \alpha_s, \alpha_s^\vee \rangle = 2$. 


Let \( \mathfrak{a} \in \mathfrak{h} \), The **Dunkl-Opdam operator** \( D_a \) on \( \mathbb{C}(\mathfrak{h}) \) is defined by the formula

\[
D_a = \partial_a - \sum_{s \in S} \frac{2c_s a_s(a)}{(1-\lambda_s)\alpha_s} (1-s)
\]

where \( \partial_a \) denotes the partial derivatives in the direction of \( a \). Clearly, \( D_a \in \mathbb{C}[W] \ltimes \mathcal{D}(\mathfrak{h}_{\text{reg}}) \), where \( \mathfrak{h}_{\text{reg}} \) is the set of regular points of \( \mathfrak{h} \) and \( \mathcal{D}(\mathfrak{h}_{\text{reg}}) \) denotes the algebra of differential operators on \( \mathfrak{h}_{\text{reg}} \). The definition of the Dunkl-Opdam operator was made by Dunkl and Opdam for reflection groups [6].

For any element \( B \in \mathbb{C}[W] \ltimes \mathcal{D}(\mathfrak{h}_{\text{reg}}) \), define \( m(B) \) to be the differential operator \( \mathbb{C}(\mathfrak{h})^W \to \mathbb{C}(\mathfrak{h}) \), defined by \( B \). That is, if \( B = \sum_{g \in W} B_g \), \( B_g \in \mathcal{D}(\mathfrak{h}_{\text{reg}}) \), then \( m(B) = \sum_{g \in W} B_g \). It is clear that if \( B \) is \( W \)-invariant, then for all \( A \in \mathbb{C}[W] \ltimes \mathcal{D}(\mathfrak{h}_{\text{reg}}) \),

\[
m(AB) = m(A)m(B).
\]

Let \( S\mathfrak{h} \) be symmetric algebra of \( \mathfrak{h} \) and \((S\mathfrak{h})^W \) \( = \{ x \in S\mathfrak{h} \mid wx = x, \forall w \in W \} \). Let us recall that by the Chevalley-Shepard-Todd theorem, the algebra \((S\mathfrak{h})^W \) is free [4]. Let \( P_1, \ldots, P_r \) be homogeneous generators of \((S\mathfrak{h})^W \). Then we have

\[
L_j = m(P_j(D_{y_1}, \ldots, D_{y_n})).
\]

It is now clear that the differential operators \( H \) and the \( L_j \) are invariant under the reflection group \( W \) so they belong a localization of the ring of invariant differential operators under the symmetric group.

Then a way of understanding the actions of the Olshanetsky-Perelomov operator \( H \) on polynomials is to study the polynomials representation of the ring of differential operators (that are invariant under the reflection group \( W \)) localized at \( \Delta^2 \) where \( \Delta = \prod_{s \in S} (\alpha_s) \) is the polynomial discriminant.

### 2.3. Higher Specht Polynomials for Reflections group \( G(r,p) \)

In this subsection we recall some general facts about the representation of wreath product \( G(r,n) \). Let \( S_n \) be the group of permutations of the set of variables \( \{x_1, \ldots, x_n\} \) and \( \mathbb{Z}/r\mathbb{Z} \) be the cyclic group of order \( r \) which acts on the \( x_i \) by a primitive \( r \)-th root of unity.

The wreath product \( G(r,n) \) is the semi-direct product of \((\mathbb{Z}/r\mathbb{Z})^n \) with \( S_n \), written as \((\mathbb{Z}/r\mathbb{Z})^n \ltimes S_n \), where \((\mathbb{Z}/r\mathbb{Z})^n \) is the direct product of \( n \) copies of \( \mathbb{Z}/r\mathbb{Z} \) and \( \sigma \in S_n \). Let \( \xi \) be a primitive \( r \)-th root of 1. \((\mathbb{Z}/r\mathbb{Z})^n \ltimes S_n \) = \{ \( (\xi^{i_1}, \ldots, \xi^{i_n}; \sigma) \mid i_k \in \mathbb{N}, \sigma \in S_n \) \}, whose product is given by

\[
(\xi^{i_1}, \ldots, \xi^{i_n}; \sigma)(\xi^{j_1}, \ldots, \xi^{j_n}; \pi) = (\xi^{i_1+j_n-1(1)}, \ldots, \xi^{i_n+j_n-1(n)}; \sigma\pi).
\]

Let \( \mathcal{O}_X = \mathbb{C}[x_1, \ldots, x_n] \) be the ring of polynomials in \( n \) indeterminates on which the group \( G(r,n) \) acts as follows:

\[
((\xi^{i_1}, \ldots, \xi^{i_n}; \sigma)f) = f(\xi^{i_1(x_1)}x_{\sigma(1)}, \ldots, \xi^{i_n(x_n)}x_{\sigma(n)}; \sigma),
\]

for \( f \in \mathcal{O}_X \) and \( (\xi^{i_1}, \ldots, \xi^{i_n}; \sigma) \in G(r,n) \). It is known that the fundamental invariants under this actions are given by the elementary symmetric functions \( e_j(x_1^r, \ldots, x_n^r), \ 1 \leq j \leq n. \) Let \( J_+ \)
be the ideal of $O_X$ generated by these fundamental invariants and $\Lambda = O_X/J$ be the quotient ring. It is also known that $G(r, n)$-module $\Lambda$ is isomorphic to the group ring $\mathbb{C}[G(r, n)]$, namely the left regular representation. A description of all irreducible components of $R$ is known in [1], in terms of what is called "higher Specht polynomials". The irreducible representation of $G(r, n)$ are parametrized by the $r$-tuple of Young diagrams $(\lambda^1, \ldots, \lambda^r)$ with $|\lambda^1| + \cdots + |\lambda^r| = n$. Let $P_{r,n}$ be the set of $r$-tuples of Young diagrams $\lambda = (\lambda^1, \ldots, \lambda^r)$ with $|\lambda^1| + \cdots + |\lambda^r| = n$. By filling each cell with a positive integer in such a way that every $j$ ($1 \leq j \leq n$) occurs once, we obtain an $r$-tableau $T = (T^1, \ldots, T^r)$ of shape $\lambda = (\lambda^1, \ldots, \lambda^r)$. When the number $k$ occurs in the component $T^i$, we write $k \in T^i$. The set of $r$-tableaux of shape $\lambda$ is denoted by $\text{Tab}(\lambda)$. An $r$-tableau $T = (T^1, \ldots, T^r)$ is said to be standard if the numbers are increasing on each column and each row of $T^\nu$ ($0 \leq \nu \leq r$). The set of $r$-standard tableaux of shape $\lambda$ is denoted by $\text{STab}(\lambda)$.

Let $S = (S^1, \ldots, S^r) \in \text{STab}(\lambda)$. We associate a word $w(S)$ in the following way. First read each column of the component $S^2$ from the bottom to the top starting from the left. We continue this procedure for the tableau $S^2$ and so on. For word $w(S)$ we define index $i(w(S))$ inductively as follows. The number 1 in the word $W(S)$ has the index $i(1) = 0$. If the number $k$ has index $i(k) = p$ and the number has number $k + 1$ is sitting on the left (resp. right) of $k$, then $k + 1$ has index $p + 1$ (resp. $p$). Finally, assigning the index to the corresponding cell, we get a shape $\lambda = (\lambda^1, \ldots, \lambda^r)$, each cell filled with a nonnegative integer, which is denoted by $i(S) = (i(S)^1, \ldots, i(S)^r)$. Let $T = (T^1, \ldots, T^r)$ be an $r$-tableau of shape $\lambda$. For each component $T^\nu$ ($1 \leq \nu \leq r$), The Young symmetrizer $e_{T^\nu}$ of $T^3$ is defined by

$$
e_{T^\nu} = \frac{1}{\alpha_{T^\nu}} \sum_{\sigma \in R(T^\nu)} \sum_{\tau \in C(T^\nu)} \text{sgn}(\tau) \tau \sigma,$$

where $\alpha_{T^\nu}$ is the product of the hook lengths for the shape $\lambda^i$, $R(T)$ and $C(T)$ are the row-stabilizer and column-stabilizer of $T^i$ respectively. For $S \in \text{STab}(\lambda)$ and $T \in \text{Tab}(\lambda)$, Ariki, Terasoma and Yamada in [1] defined the higher Specht polynomial for $G(r, n)$ by

$$F^S_T = \prod_{\nu=1}^{r} \left( e_{T^\nu}(x_{T^\nu}^{(r(S)^\nu)}) \prod_{k \in T^\nu} x_k^\nu \right),$$

where $x_{T^\nu}^{(r(S)^\nu)} = \prod_{k \in T^\nu} x_k^{i(k)}$.

The following is the fundamental result in [1] on the higher Specht polynomials for $G(r, n)$.

**Theorem 2.3.** (i) The space $V^S_T(\lambda) = \sum_{T \in \text{STab}(\lambda)} \mathbb{C}F^S_T$ affords an irreducible representation of the reflection group $G(r, n)$.

(ii) For $S_1 \in \text{STab}(\lambda)$ and $S_2 \in \text{STab}(\mu)$, the representation $V^S_{S_1}(\lambda)$ and $V^S_{S_2}(\mu)$ are isomorphic if and only if $S_1$ and $S_2$ has the same shape, i.e. $\lambda = \mu$.

(iii) We have the irreducible decomposition

$$\mathbb{C}[G(r, n)] = \bigoplus_{\lambda \in P_{r,n}} \bigoplus_{S \in \text{STab}(\lambda)} V_S(\lambda)$$

as representation of $G(r, n)$.

**Theorem 2.4.** The higher Specht polynomials in $F = \{ F^S_T : S, T \in \text{STab}(\lambda), \lambda \vdash n \}$ form a basis of the $\mathbb{C}[x_1, \ldots, x_n]^{G(r, n)}$-module $\mathbb{C}[x_1, \ldots, x_n]$.
3. Decomposition Theorem

In this section we establish a decomposition theorem of the polynomial ring in \( n \) indeterminates as a module over the ring of differential operators.

We are interested in study the actions of the invariant differential operators under the real reflection group \( W = W(B_n) \) of type \( B_n \). We know that \( W(B_n) = (\mathbb{Z} / r \mathbb{Z})^n \rtimes S_n \). Let \( \epsilon_1, \ldots, \epsilon_n \) the standard basis for \( \mathfrak{h} \). The set corresponding set of roots \( R \subset \mathfrak{h}^* \) consists of \( 2n \) sort roots \( \pm \epsilon_i \) and and \( 2n(n - 1) \) roots \( \pm \epsilon_i \pm \epsilon (i < j) \). Then the polynomial discriminant \( \Delta = \prod_{s \in S} (\alpha_s) = 2^n n! x_1 \cdots x_n \prod_{1 \leq i < j \leq n} (x_j^2 - x_i^2) \).

3.1. Actions description

As we want to study the polynomials representation of a ring of invariant differential operators, localized at \( \Delta^2 \). It is convenient to precisely describe the action of that ring of invariant differential operators on the polynomials ring.

Let \( \mathcal{D}_X = \mathbb{C}\langle x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \rangle \) be the ring of differential operators associated with the polynomial ring \( \mathcal{O}_X = \mathbb{C}[x_1, \ldots, x_n] \), and \( \mathcal{O}_Y = \mathbb{C}[x_1, \ldots, x_n]^W = \mathbb{C}[y_1, \ldots, y_n] \) be the ring of invariant under the real reflection group \( W \) where

\[
y_j = \sum_{i=1}^{n} x_i^{2j} \text{ for } j = 1, \ldots, n,
\]

We denote by \( \mathcal{D}_Y = \mathbb{C}\langle y_1, \ldots, y_n, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n} \rangle \) the ring of differential operators associated with \( \mathcal{O}_Y = \mathbb{C}[y_1, \ldots, y_n] \). By \( [10] \), \( \mathcal{D}_Y \) is the ring of differential operators invariant under the actions of the real reflection group \( W \). \( \mathcal{O}_X \) is not clearly a \( \mathcal{D}_Y \)-module, we need to describe the actions of \( \mathcal{D}_Y \) on \( \mathcal{O}_X \). By localization, \( \mathcal{O}_X \) is turned into a \( \mathcal{D}_Y \)-module, as the following lemma states.

**Notations** We adopt the following notations

\[
\tilde{\mathcal{O}}_X := \mathbb{C}[x_1, \ldots, x_n, \Delta^{-1}], \quad \tilde{\mathcal{O}}_Y := \mathbb{C}[y_1, \ldots, y_n, \Delta^{-2}], \quad \tilde{\mathcal{D}}_Y := \mathbb{C}[y_1, \ldots, y_n, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}, \Delta^{-2}].
\]

**Lemma 3.1.** \( \tilde{\mathcal{O}}_X \) is a \( \tilde{\mathcal{D}}_Y \)-module.

**Proof.** Let us make clear the action of \( \tilde{\mathcal{D}}_Y \) on \( \tilde{\mathcal{O}}_X \).

We have \( y_j = \sum_{i=1}^{n} x_i^{2j} \), \( j = 1, \ldots, n \), hence \( \frac{\partial}{\partial x_i} = \sum_{j=1}^{n} 2j x_i^{2j-1} \frac{\partial}{\partial y_j}, \) \( i = 1, \ldots, n \). Let \( A = (x_i^{2j-1})_{1 \leq i, j \leq n} \) so that \( \det(A) = \Delta \). We get the following equation

\[
\left( \begin{array}{c}
\frac{\partial}{\partial x_n} \\
\vdots \\
\frac{\partial}{\partial x_1}
\end{array} \right) = A \left( \begin{array}{c}
\frac{\partial}{\partial y_1} \\
\vdots \\
\frac{\partial}{\partial y_n}
\end{array} \right).
\]

Since \( \Delta \neq 0 \), it follows that

\[
\left( \begin{array}{c}
\frac{\partial}{\partial y_1} \\
\vdots \\
\frac{\partial}{\partial y_n}
\end{array} \right) = A^{-1} \left( \begin{array}{c}
\frac{\partial}{\partial x_n} \\
\vdots \\
\frac{\partial}{\partial x_1}
\end{array} \right)
\]

and it is clear that \( \tilde{\mathcal{O}}_X \) is a \( \tilde{\mathcal{D}}_Y \)-module. \( \Box \)
Is $\tilde{O}_X$ a $\tilde{D}_Y$-semisimple module? If yes what are the simple components of $\tilde{O}_X$ as $\tilde{D}_Y$-module and their multiplicities?

3.2. Simple components and their multiplicities

In this paragraph, we state our first main result. We use the representation theory of the real reflection group $W$ to yield results on modules over the ring of differential operators. It is well-known that

$$O_X = \mathbb{C}[W] \otimes O_Y$$

as $O_Y$-modules.

Let us consider the multiplicative closed set $S = \{\Delta^k\}_{k \in \mathbb{N}} \subset O_X$. It follows that:

$$S^{-1}O_X = \mathbb{C}[W] \otimes S^{-1}O_Y$$

as $S^{-1}O_Y$-modules.

where $S^{-1}O_X$ and $S^{-1}O_Y$ are the localization of $O_X$ and $O_Y$ at $S$ respectively. But $S^{-1}O_X = \tilde{O}_X$ and $S^{-1}O_Y = \tilde{O}_Y$, whereby we get

$$\tilde{O}_X = \mathbb{C}[W] \otimes \tilde{O}_Y$$

as $\mathbb{C}[W]$-modules.

**Lemma 3.2.** There exists an injective map

$$\mathbb{C}[W] \hookrightarrow \text{Hom}_C(\tilde{O}_X, \tilde{O}_X).$$

*Proof.* The $\mathbb{C}[W]$-module $\mathbb{C}[W]$ acts on itself by multiplication, and this multiplication yields an injective map $\mathbb{C}[W] \hookrightarrow \text{Hom}_C(\mathbb{C}[W], \mathbb{C}[W])$. Since $\tilde{O}_Y$ is invariant under this action of $\mathbb{C}[W]$, we get the expected injective map. \hfill $\Box$

**Proposition 3.3.** There exists an injective map

$$\mathbb{C}[W] \hookrightarrow \text{Hom}_{\tilde{D}_Y}(\tilde{O}_X, \tilde{O}_X).$$

*Proof.* Since $\tilde{D}_Y = \mathbb{C}(y_1, \ldots, y_n, \partial y_1, \ldots, \partial y_n, \Delta^{-2})$, we only need to show that every element of $\mathbb{C}[W]$ commute with $y_1, \ldots, y_n, \partial y_1, \ldots, \partial y_n$.

- It is clear that every element of $\mathbb{C}[W]$ commute with $y_i$, $i = 1, \ldots, n$.
- Let us show that every element of $\mathbb{C}[W]$ commute with $\partial y_i$, $i = 1, \ldots, n$. Let $D$ be a derivation on the field $K = \mathbb{C}(y_1, \ldots, y_n)$ of fractions of $O_Y$, then $(K, D)$ is a differential field. Let $L = \mathbb{C}(x_1, \ldots, x_n)$ be the field of fractions of $\tilde{O}_X$. We have that $K = L^W$ is the fixed field and $L$ is an Galois extension of $K$, with Galois group $W$. Then by [3, Théorème 6.2.6] there exists a unique derivation on $L$ which extends $D$, then $(L, D)$ is also a differential ring. In this way, $\sigma^{-1}D\sigma = D$ for every $\sigma \in W$. Therefore $\sigma D = D\sigma$ and $\sigma$ commute with $D$. \hfill $\Box$

**Corollary 3.4.**

$$\mathbb{C}[W] \cong \text{Hom}_{\tilde{D}_Y}(\tilde{O}_X, \tilde{O}_X)$$

*Proof.* see [11, Corollary 26 ]
Before we state our first main result, let us recall some facts.
By Maschke’s Theorem [9, Chap XVIII], we know that $\mathbb{C}[W]$ is a semi-simple ring, and

$$\mathbb{C}[W] = \bigoplus_{\lambda \in P_{2,n}} R_\lambda,$$

where $P_{2,n}$ be the set of 2-tuples of Young diagrams $\lambda = (\lambda^1, \lambda^2)$ with $|\lambda^1| + |\lambda^2| = n$ and $R_\lambda$ are simple rings. We have the following corresponding decomposition of the identity element of $\mathbb{C}[W]$: 

$$1 = \sum_{\lambda \in \mathcal{P}_{2,n}} r_\lambda,$$

where $r_\lambda$ is the identity element of $R_\lambda$, $r_\lambda^2 = 1$ and $r_\lambda r_\mu = 0$ if $\lambda \neq \mu$, the set $\{r_\lambda\}_{\lambda \in \mathcal{P}(2,n)}$ is the set of primitive central idempotents of $\mathbb{C}[W]$. Let $n$ be a positive integer, $\lambda$ be a 2-tuples of Young diagram of $n$, Let $\text{Tab}(n) = \bigcup_{\lambda \in \mathcal{P}_{2,n}} \text{Tab}(\lambda)$ and $\text{STab}(n) = \bigcup_{\lambda \in \mathcal{P}_{2,n}} \text{STab}(\lambda)$.

**Theorem 3.5.** For every primitive idempotent $e_i \in \mathbb{C}[W]$.

(i) $e_i \mathcal{O}_X$ is a nontrivial $\mathcal{D}_Y$-submodule of $\mathcal{O}_X$.
(ii) The $\mathcal{D}_Y$-module $e_i \mathcal{O}_X$ is simple,
(iii) There exist $\lambda \in \mathcal{P}_{2,n}$ and a Higher Specht polynomial $F^S_T$ with $(S, T \in \text{STab}(\lambda))$ such that $e_i \mathcal{O}_X = \mathcal{D}_Y F^S_T$.

**Proof.**

(i) Let $e_i \in \mathbb{C}[S_n]$ be a primitive idempotent, We know that $\mathbb{C}[W]e_i$ is a $W$-irreducible representation. Theorem 2.3 states that there is $\lambda \in \mathcal{P}_{2,n}$ and $S \in \text{STab}(\lambda)$ such that $\mathbb{C}[W]e_i \cong V_S(\lambda)$, and $V_S(\lambda) \subset \mathcal{O}_X$. By [2, Chap III, §4, Theorem 3.9], we have $e_i \mathbb{C}[W]e_i \cong \mathbb{C}e_i \neq \{0\}$. $\{0\} \neq e_i V_S(\lambda) \subset e_i \mathcal{O}_X$. Since $e_i$ commute with every element of $\mathcal{D}_Y$ and $\mathcal{O}_X$ is a $\mathcal{D}_Y$-module, it follows that $e_i \mathcal{O}_X$ is a nontrivial $\mathcal{D}_Y$-module.

In fact $V_S(\lambda)$ is a cyclic $\mathbb{C}[W]$-module, i.e., there $T, S \in \text{Tab}(\lambda)$ and a higher Specht polynomial $F^S_T$ such that $V_S(\lambda) = \mathbb{C}[W]F^S_T$, so that $\mathbb{C}[W]e_i \cong V_S(\lambda)F^S_T$. Then by identify one of $e_i$ with $F^S_T$, we may say that $e_i F^S_T$ is a scalar multiple of $F^S_T$.

(ii) Assume that $1 = \sum_{i=1}^s e_i$ where the $\{e_i\}_{1 \leq i \leq s}$ is the set of primitive idempotents of $\mathbb{C}[W]$, then $\mathcal{O}_X = \sum_{i=1}^s e_i \mathcal{O}_X$. Let $m \in e_i \mathcal{O}_X \cap e_j \mathcal{O}_X$ with $i \neq j$ so that $m = e_i m_i$ and $m = e_j m_j$, but $e_i e_j = 0$ then $e_i m = e_i e_j m = 0$ hence $m = 0$. Therefore $\mathcal{O}_X = \bigoplus_{i=1}^s e_i \mathcal{O}_X$ and we get:

$$\text{Hom}_{\mathcal{D}_Y}(\mathcal{O}_X, \mathcal{O}_X) \cong \bigoplus_{i,j=1}^s \text{Hom}_{\mathcal{D}_Y}(e_i \mathcal{O}_X, e_j \mathcal{O}_X),$$

by Corollary 3.4 we know that:

$$\mathbb{C}[S_n] \cong \bigoplus_{i,j=1}^s \text{Hom}_{\mathcal{D}_Y}(e_i \mathcal{O}_X, e_j \mathcal{O}_X).$$

For every $\lambda \in \mathcal{P}_{2,n}$, we pick a unique irreducible representation $V_S(\lambda)$ for a certain $S \in \text{STab}(\lambda)$ which we denote by $V(\lambda) := V_S(\lambda)$.

We also have, by [7, Proposition 3.29], that

$$\mathbb{C}[W] \cong \bigoplus_{\lambda \in \mathcal{P}_{2,n}} \text{End}_{\mathbb{C}}(V(\lambda)).$$


But by the Wedderburn decomposition Theorem \[2, \text{Chap II, §4, Theorem 4.2}\] we also know that

\[ C[W] = \bigoplus_{\lambda \in P_{2,n}} r_{\lambda} C[W] \text{ and } r_{\lambda} C[W] \cong \text{Mat}_{f_{\lambda}}(C) \cong \text{End}_{C}(C_{f_{\lambda}}) \]

where \( f_{\lambda} = \dim_{C} V(\lambda) \). We recall that each standard tableau \( T_i \) is associated with an idempotent \( e_i \).

Let us show that \( C[W] \cong \bigoplus_{\lambda \in P_{2,n}} \left( \bigoplus_{T_i, T_j \in \text{STab}(\lambda)} \text{Hom}_{D_Y}(e_i \mathcal{O}_X, e_j \mathcal{O}_X) \right) \).

Let \( x \) be an element of \( C[W] \) and \( r_{\lambda} \) the primitive central idempotent with \( \lambda \in P_{2,n} \). Then \( x \) induces an \( D_Y \)-homomorphism \( \mathcal{O}_X \to \mathcal{O}_X, m \mapsto x \cdot m \). Since \( r_{\lambda} \) is in the centre of \( C[W] \), \( x \cdot r_{\lambda} \mathcal{O}_X = (x \cdot r_{\lambda}) \mathcal{O}_X \subseteq r_{\lambda} \mathcal{O}_X \), which means \( x \in \bigoplus_{\lambda \in P_{2,n}} \text{Hom}_{D_Y}(r_{\lambda} \mathcal{O}_X, r_{\lambda} \mathcal{O}_X) \). Then \( \text{Hom}_{D_Y}(e_i \mathcal{O}_X, e_j \mathcal{O}_X) = \{0\} \) if \( T_i \in \text{STab}(\lambda_i), T_j \in \text{STab}(\lambda_j) \) and \( \lambda_i \neq \lambda_j \). We get that

\[ \text{Hom}_{D_Y}(\mathcal{O}_X, \mathcal{O}_X) \cong \bigoplus_{\lambda \in P_{2,n}} \left( \bigoplus_{T_i, T_j \in \text{STab}(\lambda)} \text{Hom}_{D_Y}(e_i \mathcal{O}_X, e_j \mathcal{O}_X) \right). \]

The number of direct factors in the sum \( \bigoplus_{T_i, T_j \in \text{STab}(\lambda)} \text{Hom}_{D_Y}(e_i \mathcal{O}_X, e_j \mathcal{O}_X) \) is \((f_{\lambda})^2\).

Let us show that \( \text{Hom}_{D_Y}(e_i \mathcal{O}_X, e_j \mathcal{O}_X) \cong C \) if \( T_i, T_j \in \text{Tab}(\lambda) \). Consider the following commutative diagram:

\[
\begin{align*}
  & C[W] \quad \phi \quad \text{Hom}_{D_Y}(\mathcal{O}_X, \mathcal{O}_X) \\
\downarrow \quad \alpha_{\lambda} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \downarrow \quad \beta_{\lambda} \\
  r_{\lambda} C[W] \quad \psi \quad \quad \text{Hom}_{D_Y}(r_{\lambda} \mathcal{O}_X, r_{\lambda} \mathcal{O}_X)
\end{align*}
\]

where \( \beta_{\lambda} : \bigoplus_{\mu \in P_{2,n}} \text{Hom}_{D_Y}(r_{\mu} \mathcal{O}_X, r_{\mu} \mathcal{O}_X) \to \text{Hom}_{D_Y}(r_{\lambda} \mathcal{O}_X, r_{\lambda} \mathcal{O}_X) \) and \( \alpha_{\lambda} : \bigoplus_{\mu \in P_{2,n}} r_{\mu} C[W] \to r_{\lambda} C[W] \) are canonical projections et \( \phi \) is the isomorphism of

Corollary 3.4. It follows that \( \psi \) is an isomorphism hence \( r_{\lambda} C[W] \cong \text{Hom}_{D_Y}(r_{\lambda} \mathcal{O}_X, r_{\lambda} \mathcal{O}_X) \).

Now we identify \( r_{\lambda} C[W] \) with either the set \( \text{Mat}_{f_{\lambda}}(C) \) of square matrices of order \( f_{\lambda} \) with coefficients in \( C \) either with \( \text{End}_{C}(C_{f_{\lambda}}) \).

Let \( E_{i,j} \) be the square matrix of order \( f_{\lambda} \) with 1 at the position \((i, j)\) and 0 elsewhere and \( E_{i,i} = E_{i,i} \), then we identify the idempotent \( e_i \in r_{\lambda} C[W] \) to \( E_{i,i} \) in \( \text{Mat}_{f_{\lambda}}(C) \). Let \( B = (a_{i,j}) \in \text{Mat}_{f_{\lambda}}(C) \) we get \( B = \sum_{i,j} a_{i,j} E_{i,j} = \sum_{i,j} E_{i}BE_{j} \), in fact \( E_{i}BE_{j} \) is the matrix with \( a_{i,j} \) in the position \((i, j)\) and 0 elsewhere, if \( R = \text{Mat}_{f_{\lambda}}(C) \) we get that \( E_{i}RE_{j} \cong C \).

This isomorphism \( \psi \) implies that \( \bigoplus_{T_i, T_j \in \text{STab}(\lambda)} E_{i}RE_{j} \cong \bigoplus_{T_i, T_j \in \text{STab}(\lambda)} \text{Hom}_{D_Y}(e_i \mathcal{O}_X, e_j \mathcal{O}_X) \);

the restriction of \( \psi \) to \( E_{i}RE_{j} \) yields a map \( E_{i}RE_{j} \to \text{Hom}_{D_Y}(e_i \mathcal{O}_X, e_j \mathcal{O}_X) \) and this map is surjective, moreover we have \( E_{i}RE_{j} \cong \text{Hom}_{D_Y}(e_i \mathcal{O}_X, e_j \mathcal{O}_X) \). Therefore \( \text{Hom}_{D_Y}(e_i \mathcal{O}_X, e_j \mathcal{O}_X) \cong C \). Let us assume that \( e_i \mathcal{O}_X \) is not simple \( D_Y \)-module, then \( e_i \mathcal{O}_X \) may be written as \( e_i \mathcal{O}_X = \oplus_{j \in J} N_j \) where the \( N_j \) are simple \( D_Y \)-modules and \(|J| > 1\). It
follows that $\dim_{\mathbb{C}}(\text{Hom}_{D_Y}(e_i, \mathcal{O}_X, e_i, \mathcal{O}_X)) \geq |J|$ but $\text{Hom}_{D_Y}(e_i, \mathcal{O}_X, e_i, \mathcal{O}_X) \cong \mathbb{C}$ so we obtain that $J = 1$, which necessarily implies that $e_i \mathcal{O}_X$ is a simple $D_Y$-module.

(iii) By the the proof (i) there exist a higher Specht polynomial $F^S_T \in e_i \mathcal{O}_X$, $\lambda \vdash n$ such that $e_i \mathcal{O}_X = D_Y F^S_T$.

Corollary 3.6. With the above notations, $e_i \mathcal{O}_X \cong D_Y e_j \mathcal{O}_X$ if $T_i$ and $T_j$ have the same shape i.e., if there is a partition $\lambda \in \mathcal{P}_{2,n}$ such that $T_i, T_j \in \text{STab}(\lambda)$.

Proof. The $D_Y$-modules $e_i \mathcal{O}_X$ are simple and $\text{Hom}_{D_Y}(e_i \mathcal{O}_X, e_j \mathcal{O}_X) \cong \mathbb{C}$ whenever there exists a partition $\lambda \in \mathcal{P}_{2,n}$ such that $T_i, T_j \in \text{STab}(\lambda)$. Since $\text{Hom}_{D_Y}(e_i \mathcal{O}_X, e_j \mathcal{O}_X) \neq \{0\}$, we conclude by using the Schur lemma.

Proposition 3.7. Let $\lambda \in \mathcal{P}_{2,n}$, $T \in \text{STab}(\lambda)$, and $e$ the primitive idempotent associated to $T$, denote by $F^S_T := F^S_T$, be the corresponding higher Specht polynomial (for some $S \in \text{STab}(\lambda)$, in Theorem 3.5, such that $e \mathcal{O}_X = D_Y F^S_T$ then we have:

(i) 
\[ \mathcal{O}_X = \bigoplus_{T \in \text{STab}(n)} D_Y F^S_T = \bigoplus_{\lambda \in \mathcal{P}_{2,n}} \left( \bigoplus_{T \in \text{STab}(\lambda)} D_Y F^S_T \right); \]  
(3.1)

(ii) For each $\lambda \in \mathcal{P}_{2,n}$ fix a 2-tableau $T^* \in \text{STab}(\lambda)$, then
\[ \mathcal{O}_X = \bigoplus_{\lambda \in \mathcal{P}_{2,n}} f^\lambda D_Y F^S_{T^*} \]  
\[ \text{where } f^\lambda = \dim_{\mathbb{C}}(V(\lambda)) \]  

Proof. We have by the proof of Theorem 3.5 that
\[ \mathcal{O}_X = \oplus e_i \mathcal{O}_X \]
and the $e_i \mathcal{O}_X$ are simple $\mathcal{D}_Y$-modules. Since to each primitive idempotent $e_i$ corresponds a 2-diagram $\lambda_i \in \mathcal{P}_{2,n}$ and a tableau $T_i \in \text{STab}(\lambda_i)$ such that $e_i \mathcal{O}_X = D_Y F^S_{T_i}$ then $\mathcal{O}_X = \oplus_{T \in \text{STab}(n)} D_Y F^S_T$. By Cororary 3.6, $D_Y F^S_{T_i} \cong D_Y F^S_{T_j}$ if $T_i, T_j \in \text{STab}(\lambda)$ and so we have $f^\lambda$ isomorphic copies of $D_Y F^S_{T^*}$ in the direct sum (3.1).

We get in Proposition 3.7 a composition of the polynomial ring as a $D_Y$-module into irreducible $D_Y$ modules generated by the higher Specht polynomials.

4. Using correspondence between $G$-representations and $D$-modules

In this section we use an equivalence of categories between group representations and modules over differential ring to yield a better version of decomposition the polynomial ring as $D_Y$-modules.

Lemma 4.1. Let $V(\lambda)$ be an irreducible module associated to the 2-Young diagram $\lambda \in \mathcal{P}_{2,n}$ and $e_T$ the primitive idempotent associated to a 2-standard tableau $T \in \text{STab}(\lambda)$. Then $e_T(V(\lambda)) = \{e_T(m) | m \in V(\lambda)\}$ is a one dimensional $\mathbb{C}$-vector space.
Proof. In fact we have
\[
e_T V(\lambda) \cong e_T \mathbb{C}[W] e_T \\
\cong e_T \mathbb{C}[W] e_T \\
\cong \mathbb{C} e_T \text{ by } [2, \text{ Chap III, §4, Theorem 3.9 }]
\]

Recall that if \( M \) is a semi-simple module over a ring \( R \), and \( N \) is simple \( R \)-module, then the isotopic component of \( M \) associated to \( N \) is the sum \( \sum N' \subset M \) of all \( N' \subset M \) such that \( N' \cong N \).

Lemma 4.2. Let \( V(\lambda) \) be an irreducible module associated to the 2-Young diagram \( \lambda \in \mathcal{P}_{2,n} \) and \( M := \bar{O}_X \) and \( M^\lambda \) the isotopic component of \( M \) (as \( \bar{O}_Y \)-module) associated to \( V(\lambda) \). Then \( M^\lambda \) is \( \bar{D}_Y \)-module.

Proof. We only have to proof that \( \bar{D}_Y \cdot M^\lambda \subset M^\lambda \). Let \( D \in \bar{D}_Y \) and \( N \) be a \( \mathbb{C}[W] \)-module isomorphic to \( V(\lambda) \), since \( D \) commute with the elements of the group algebra \( \mathbb{C}[W] \), \( D \) is an \( \mathbb{C}[W] \)-homomorphism from \( N \) into \( D(N) \). Then by virtue of the Schur lemma \( D(N) = 0 \) or \( D(N) \cong N \) as a \( \mathbb{C}[W] \)-module, and \( D(N) \subset M^\lambda \). Hence \( \bar{D}_Y \cdot M^\lambda \subset M^\lambda \).

Lemma 4.3. For a 2-Young diagram \( \lambda \in \mathcal{P}_{2,n} \), and \( e_T \) the primitive idempotent associated to a 2-standard tableau \( T \in S\text{Tab}(\lambda) \), then \( e_T(M^\lambda) \) is \( \bar{D}_Y \)-module.

Proof. Let \( D \in \bar{D}_Y \), we have \( D(e_T(M^\lambda)) = e_T(D(M^\lambda)) \subset e_T(M^\lambda) \), so \( e_T(M^\lambda) \) is \( \bar{D}_Y \)-module.

Before we state our second main result, let us recall the correspondence between group representations and modules over differential ring that will be useful in what follows[11, Paragraph 2.4].

The correspondence between \( G \)-representations and \( D \)-modules Let \( L \) and \( K \) be two fields, say that a \( T_{K/k} \)-module \( M \) is \( L \)-trivial if \( L \otimes_K M \cong L^n \) as \( T_{L/k} \)-modules. Denote by \( \text{Mod}^L(T_{K/k}) \) the full subcategory of finitely generated \( T_{K/k} \)-modules that are \( L \)-trivial. It is immediate that it is closed under taking submodules and quotient modules. Using the lifting \( \phi : L \) may be thought of as a \( T_{K/k} \)-module. If \( G \) is a finite group let \( \text{Mod}(k[G]) \) be the category of finite-dimensional representations of \( k[G] \). Let now \( k \rightarrow K \rightarrow L \) be a tower of fields such that \( K = L^G \). Note that the action of \( T_{K/k} \) commutes with the action of \( G \). If \( V \) is a \( k[G] \)-module, \( L \otimes_k V \) is a \( T_{K/k} \)-module by \( D(l \otimes v) = D(l) \otimes v \), \( D \in T_{K/k} \), and \( (L \otimes_k V)^G \) is a \( T_{K/k} \)-submodule.

Proposition 4.4. The functor
\[
\nabla : \text{Mod}(k[G]) \rightarrow \text{Mod}(T_{K/k}), \quad V \mapsto (L \otimes_k V)^G
\]
is fully faithful, and defines an equivalence of categories
\[
\text{Mod}(k[G]) \rightarrow \text{Mod}^L(T_{K/k}).
\]
The quasi-inverse of \( \nabla \) is the functor
\[
\text{Loc} : \text{Mod}^L(T_{K/k}) \rightarrow \text{Mod}(k[G]), \quad \text{Loc}(M) = (L \otimes_K M)^{\phi(T_{K/k})}.
\]
Proof. see [11, Proposition 2.4]
In the following proposition we take $G = W$, $K := \mathbb{C}(y_1, \ldots, y_n)$ the field of fractions of $\mathcal{O}_Y$ and $L = \mathbb{C}(x_1, \ldots, x_n)$ the field of fractions of $\mathcal{O}_X$ so that $K = L W$. It is clear that $L$ is a Galois extension of $K$ with Galois $W$.

**Proposition 4.5.** For a 2-Young diagram $\lambda \in \mathcal{P}_{2,n}$ and $e_T$ the primitive idempotent associated to a 2-standard tableau $T \in \text{STab}(\lambda)$, set $M_T := e_T \hat{O}_X$. Then we have:

(i) $M_T = \nabla(V(\lambda))$

(ii) $M_T = e_T(M^\lambda)$ is simple $\hat{D}_Y$-module;

(iii) $M^\lambda = \bigoplus_{T \in \text{STab}(\lambda)} e_T(M^\lambda)$.

*Proof.* (i) Let us consider the right $\mathbb{C}[W]$-module $V = e_T \mathbb{C}[W]$ where $T \in \text{STab}(\lambda)$. This is the image of $\mathbb{C}[W]$ by right multiplication map $e_T : \mathbb{C}[W] \to \mathbb{C}[W]$. By [11, Example 2.5], we may turn this map into a left multiplication $\mathbb{C}[W]^r \to \mathbb{C}[W]^r$ and get an image which is isomorphic to $V(\lambda)$. We get an induced map

$$\nabla(\mathbb{C}[W]^r) \to \nabla(V(\lambda)) \subset \nabla(\mathbb{C}[W]^r),$$

which is a multiplication by $e_T$ according to [11, Example 2.5]. Then $\nabla(V(\lambda))$ is egal to $e_T \hat{O}_X = M_T$.

(ii) Since $V(\lambda)$ is a simple $\mathbb{C}[W]$-module, $\nabla(V(\lambda))$ is also a simple $\hat{D}_Y$-module.

(iii) follows from the fact that $1 = \sum_{T \in \text{STab}(\lambda)} e_T$ and $e_T(M^\lambda) = 0$ if $T \notin \text{STab}(\lambda)$.

\[\square\]

**Theorem 4.6.** Let $T$ be a 2-standard tableau of shape $\lambda$ where $\lambda \in \mathcal{P}_{2,n}$ and $M_T$ be as in the above proposition. Then

(i) $M_T = \bigoplus_{S \in \text{STab}(\lambda)} \hat{O}_Y F_T^S$ as $\hat{D}_Y$-module,

(ii) $\hat{O}_X = \bigoplus_{\lambda \vdash n} \left( \bigoplus_{S,T \in \text{STab}(\lambda)} \hat{O}_Y F_T^S \right)$ as a $\hat{D}_Y$-module.

*Proof.* (i) For $S \in \text{STab}(\lambda)$ We know by Theorem 2.3 that the polynomial $F_T^S$ generate a $\mathbb{C}[W]$-module inside $\hat{O}_X$ which is isomorphic to $V(\lambda)$. Then $F_T^S \in M^\lambda$ and $M^\lambda = \bigoplus_{S,T \in \text{STab}(\lambda)} \mathbb{C}[S_n] F_T^S \hat{O}_Y$ by Theorem 2.4. Moreover $e_T(F_T^S) = c F_T^S, c \in \mathbb{C}$ and by Lemma 4.1 $e_T(\mathbb{C}[W] F_T^S) = \mathbb{C} F_T^S$. Hence $M_T = e_T(M^\lambda) = \bigoplus_{S \in \text{STab}(\lambda)} \hat{O}_Y F_T^S$.

(ii) follows from Proposition 3.7.

\[\square\]

**Theorem 4.7.** Let $\lambda \in \mathcal{P}_{2,n}$ and $D \in \hat{D}_Y$ such that $D(F_T^S) \neq 0$ for a higher Specht polynomial $F_T^S$ with $S,T \in \text{STab}(\lambda)$. Then the image of the $\mathbb{C}[W]$-module $V_S(\lambda)$ by $D$ is an $\mathbb{C}[W]$-module isomorphic to $V_S(\lambda)$. In others words, the actions of the differential operators of $\hat{D}_Y$ on the higher Specht polynomials generate isomorphic copies of the corresponding irreducible $\mathbb{C}[W]$-module.
Proof. Let $\lambda \in \mathcal{P}_{d,n}, D \in \tilde{D}_Y$ such that $D(F^S_T) \neq 0$ for $S, T \in STab(\lambda)$ and set $W^D_S(\lambda) = D(V_S(\lambda))$ the image of the module $V_S(\lambda)$ under the map $D$. By Theorem 2.3, the $\mathbb{C}$-vector space $V_S(\lambda)$ is equipped with a basis $F = \{F^S_T; T \in STab(\lambda)\}$, then $W^D_S(\lambda)$ is the vector space spanned by the set $\{D(F^S_T); T \in STab(\lambda)\}$. The elements of $\{F^S_T; T \in STab(\lambda)\}$ are linearly independent over $\mathbb{C}$, otherwise the direct sums in Proposition 3.7 cannot hold not possible. It follows that the elements $\{D(F^S_T); T \in STab(\lambda)\}$ are linear independent over $\mathbb{C}$. Hence $\{D(F^S_T); T \in STab(\lambda)\}$ is a basis of $W^D_S(\lambda)$ over $\mathbb{C}$. Since $D$ commute with elements of $\mathbb{C}[W]$, $W^D_S(\lambda)$ is an $\mathbb{C}[W]$-module isomorphic to $V_S(\lambda)$. □

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References
[1] Ariki, S. Terasoma, T. and Yamada, H., Higher Specht polynomials, Hiroshima Math, 27 (1997), no. 1, 177-188
[2] Boerner,H. Representations of Groups with special Consideration for the Needs of Modern Physics, North-Holland, New York, NY, (1970).
[3] Chambert-Loir, A. Algèbre corporelle, Les éditions de l’Ecole polytechnique (2005).
[4] Chevally, C., Invariants of finite groups generated by reflections. Amer. J. of Math. 67 77 (4): 778-782, (1955).
[5] Dunkl, C. Differential-difference operators associated to reflection groups. Trans. Amer. Math. Soc. 311, 167-183, (1989).
[6] Dunkl, F.C.and Opdam, E.M. Dunkl operators for complex reflections groups, Proc. London. Soc (3) 86 (2003),no. 1, 70-108. MR MR 1971464 ( 2004d:20040)
[7] Fulton, W. and Harris, J. Representation Theory. A First Course, Graduate Texts in Mathematics, 129. Readings in Mathematics. Springer-Verlag, New York, (1991).
[8] Humphreys, J.E., Reflection Groups and Coxeter Groups. Cambridge Studies in Advanced, Mathematics, 29 ‘Cambridge University Press’ (1990).
[9] Lang, S, Algebra. Revisited third edition. Graduate Texts in Mathematics, 211. Springer-Verlag, New York, 2002.
[10] Levasseur, T., and Stafford,J.T. Invariant differential operators and a homomorphism of Harish-Chandra, J.Amer. Math. Soc. 8 (1995), 365-372. MR 95g:22029
[11] Nonkané, I., Specht polynomials and modules over the Weyl algebra. Afr. Mat, 30, Issue 11 pp 279-290 (2019).
[12] Nonkané, I., Differential operators and the symmetric groups, Journal of Physics: Conference Series 1730(1):012129, (2021).
[13] Nonkané, I., Representation Theory of Groups and D-Modules, International Journal of Mathematics and Mathematical Sciences, vol. 2021, Article ID 6613869, 7 pages, (2021).
[14] Olshanetsky, M.A. and Prerelomov, A.M., Quantum integrable systems related to Lie algebras. Phys. Rep. 94 (1983), no. 6, 313-404
[15] Weyl, Hermann, The theory of groups and quantum mechanics, Dover, New York.