On the BRST Operator Structure of the $N = 2$ String

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ABSTRACT

The BRST operator cohomology of $N = 2$ 2d supergravity coupled to matter is presented. Descent equations for primary superfields of the matter sector are derived. We find one copy of the cohomology at ghost number one, two independent copies at ghost number two, and conjecture that there is a copy at ghost number three. The $N = 2$ string has a twisted $N = 4$ superconformal symmetry generated by the $N = 2$ superstress tensor, the BRST supercurrent, the antighost superfield, and the ghost number supercurrent.

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1 Introduction

The BRST cohomology of a string theory describes the physical states of the theory (for a review, see [1]); moreover, the explicit understanding of the cohomology in a covariant gauge has been crucial for the construction of both \( N = 0 \) and \( N = 1 \) string field theories. In nontrivial backgrounds, the BRST cohomology has revealed a deep structure related to the underlying symmetries of string theory (for example, see [2]).

The \( N = 2 \) string is a relatively simple string theory (for a review, see for example [3]). It might therefore be practical to first understand aspects of the \( N = 2 \) string (for example, underlying symmetries are studied in [4]), and to use the insight gained to learn about more realistic, and more complicated, string theories.

In general, there is an equivalence between the BRST cohomology on states in Fock space, that is states annihilated by the BRST charge \( Q \) that are not themselves \( Q \) of another state, and BRST cohomology on operators, that is operators that (super)commute with \( Q \) but are not themselves (super)commutators with \( Q \). However, only operator cohomology can be conveniently described in superspace.

The state cohomology for the \( N = 2 \) string in \((2,2)\)-dimensional flat Minkowski space was constructed in [7]. Here we consider the operator \( N = 2 \) BRST cohomology in an arbitrary background.

More generally, in this paper we explore aspects of the BRST operator structure of the \( N = 2 \) string. We present the operator cohomology in superspace, and derive descent equations for primary superfields. We also find that the \( N = 2 \) string has a twisted critical \( \mathcal{N} = 4 \) superconformal symmetry generated by the \( N = 2 \) superstress tensor, the BRST supercurrent, the antighost superfield, and the ghost number supercurrent.

The paper is organized as follows: In section 2, we review BRST operator cohomology for \( \mathcal{N} = 0,1 \) strings. In section 3, we derive analogous

\(^3\)There are certain special operators, such as the picture changing operator for \( N = 1,2 \) and the instanton number operator for \( N = 2 \) that have not been constructed in superspace, and that we will not discuss.
formulae for $N = 2$ strings: given an $N = 2$ primary with vanishing dimension and charge, $h = q = 0$, we construct corresponding operators in the BRST cohomology, one at ghost number $G = 1$, two independent operators at $G = 2$, and a fourth operator at $G = 3$ that we conjecture is physical; all are described in superspace, without bosonization, etc. We derive the $N = 2$ descent equations, which explain how the operators arise. We find that a certain conjugation relation plays the role that is played by derivatives for the $N = 0, 1$ cases. We also consider the explicit example of the $N = 2$ string in a toroidal background, and find the usual discrete states. In section 4, we show how the superstress tensor $T$, the BRST supercurrent $j$, the antighost superfield $B$, and the ghost number supercurrent $BC$ form a twisted $N = 4$ algebra. In section 5 we end with some discussion and comments. Finally, in the Appendix, we give some computational details.

2 Review of BRST Cohomology for $N = 0, 1$ strings

We first review the BRST cohomology of the $N = 0$ bosonic string (see ref. [1]). The stress tensor $T(z) = T^{\Phi} + T^{bc}$ combines the matter sector ($T^{\Phi}$) with the reparametrization $b, c$ ghost system ($T^{bc}$). The stress tensor acts on a primary field of the matter sector $\Phi$ with dimension $h$ via the operator product expansion (OPE)

$$T^{\Phi}_1 \Phi_2 \sim \frac{h}{z_{12}^2} \Phi_2 + \frac{1}{z_{12}} (\partial \Phi)_2 + \ldots,$$

where the dots in the OPE indicate nonsingular terms, and we use the notation $f_1 \equiv f(z_1)$, and $z_{12} = z_1 - z_2$. The ghosts $c$ and $b$ have dimensions $-1$ and $2$ with respect to the ghosts stress tensor $T^{bc} = c \partial b + 2(\partial c)b$. They satisfy

$$c_1 b_2 \sim b_1 c_2 \sim \frac{1}{z_{12}} + \ldots$$

The $N = 0$ BRST current is

$$j = c(T^{\Phi} + \frac{1}{2} T^{bc}) + \frac{3}{2} \partial^2 c,$$
where the total derivative is chosen such that $j$ is a primary field with $h = 1$. At criticality, the nilpotent BRST charge

$$Q = \frac{1}{2\pi i} \oint dz \, j(z)$$

acts on a field $O$ by

$$Q(O) = [Q, O(z_2)] \equiv \frac{1}{2\pi i} \oint_{C_{z_2}} dz_1 \, j(z_1) O(z_2),$$

where the contour $C_{z_2}$ surrounds $z_2$ once.

Next we discuss BRST cohomology classes. For a primary field of the matter sector $\Phi$ with $h = 1$, the fields

$$O_{G=1} = c\Phi, \quad O_{G=2} = c\partial c\Phi$$

are BRST invariant, namely, $Q(O_{G=1}) = Q(O_{G=2}) = 0$. The operators $O_{G=1}$ and $O_{G=2}$ are not $Q$-anticommutators, and therefore, they represent cohomology classes. The label $G$ in (2.6) describes the ghost number of the operator $O$ (with the convention $G(c) = 1, G(b) = -1$). The two operators in (2.6) generate the same physical state in different ghost vacua.

An element of the BRST cohomology $O_{G=1}$ is related to its corresponding primary field $\Phi$ by the descent equation

$$Q(\Phi) = \partial O_{G=1}. \quad (2.7)$$

Moreover, $O_{G=1}$ and $O_{G=2}$ are related via the equation

$$Q(c\Phi_h) = (1 - h)c\partial c\Phi_h,$$

where $\Phi_h$ is off-shell when $h \neq 1$. Using equation (2.8) (and continuity in $h$) in the on-shell limit $h \to 1$, one finds that $O_{G=2}$ is physical. $O_{G=2}$ also satisfies the descent equation

$$Q(\partial c\Phi) = \partial O_{G=2}. \quad (2.9)$$

The critical bosonic string can (often) be regarded as a topological conformal field theory [8]. The OPE’s of the operators $T, b, j$ and $J = bc$ are
‘almost’ closed on a twisted $N = 2$ superconformal algebra (closing the algebra requires two more generators: $c$ and $c \partial c$). The untwisted $N = 2$ superconformal algebra is given by twisting $T$ to $T + \frac{1}{2} \partial J$. This concludes our discussion of the $N = 0$ case.

Next we discuss the BRST cohomology of the $N = 1$ superstring in the NS sector. We first set the notation [5]. We define

$$z_{12} = z_1 - z_2 - \theta_1 \theta_2,$$
$$\theta_{12} = \theta_1 - \theta_2,$$  \hspace{1cm} (2.10)

and the superderivative

$$D = \frac{\partial}{\partial \theta} + \theta \partial, \hspace{0.5cm} D^2 = \partial,$$  \hspace{1cm} (2.11)

where $\partial \equiv \frac{\partial}{\partial z}$. A function $f_1 \equiv f(z_1, \theta_1)$ can be expanded around $z_2, \theta_2$:

$$f_1 = f_2 + \theta_{12} (Df)_2 + z_{12} (\partial f)_2 + \theta_{12} z_{12} (D \partial f)_2 + ...$$  \hspace{1cm} (2.12)

An $N = 1$ stress tensor $T = G + \theta T_B$, where $T_B$ is the bosonic stress tensor, obeys the $N = 1$ superconformal algebra expressed via the operator product expansion

$$T_{12} T_2 \sim \frac{3}{8} c \frac{z_{12}}{z_{12}^3} T_2 + \frac{3}{2} \theta_1 \theta_{12} T_2 + \frac{1}{2} \frac{z_{12}}{z_{12}^2} (DT)_2 + \frac{\theta_{12}}{z_{12}} (\partial T)_2 + ...$$  \hspace{1cm} (2.13)

The $N = 1$ 2d theory combines a matter sector and a ghost sector. The $N = 1$ energy-momentum tensor of the matter sector $T^\Phi$ acts on a primary superfield of the matter sector $\Phi$ with dimension $h$ as

$$T^\Phi_1 \Phi_2 \sim \frac{h \theta_{12}}{z_{12}^2} \Phi_2 + \frac{1}{2} \frac{z_{12}}{z_{12}^2} (D \Phi)_2 + \frac{\theta_{12}}{z_{12}} (\partial \Phi)_2 + ...$$  \hspace{1cm} (2.14)

The superghosts $C$ and $B$ have dimensions $-1$ and $3/2$ with respect to the superghost stress tensor $T^{BC} = -C(\partial B) + \frac{1}{2} DC=DB - \frac{3}{2} (\partial C)B$. They satisfy

$$C_1 B_2 \sim B_1 C_2 \sim \frac{\theta_{12}}{z_{12}} + ...$$  \hspace{1cm} (2.15)

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4. If the background has a scalar field, then it is possible to add total derivative terms to $j$ and $J$, so that the modified generators $T, b, j, J$ form a twisted $N = 2$ algebra which closes without adding more generators [3].
The N=1 BRST current is
\[ j = C(T^\Phi + \frac{1}{2}T^{BC}) - \frac{3}{4}D(C(DC)B), \]  
(2.16)

The total derivative term in (2.16) is chosen such that \( j \) is a primary superfield with \( h = 1/2 \). At criticality, the nilpotent BRST charge
\[ Q = \frac{1}{2\pi i} \oint dzd\theta \ j(z,\theta), \]  
(2.17)

acts on a superfield \( O \) by
\[ Q(O) = [Q, O(z_2,\theta_2)] = \frac{1}{2\pi i} \oint_{C_{z_2}} dz_1d\theta_1 \ j(z_1,\theta_1)O(z_2,\theta_2), \]  
(2.18)

where the contour \( C_{z_2} \) surrounds \( z_2 \) once.

Next we describe BRST cohomology classes. For a primary superfield of the matter sector \( \Phi \) of dimension \( h = 1/2 \), the combinations
\[ O_{G=1} = CD\Phi - \frac{1}{2}(DC)\Phi \]  
(2.19)

and
\[ O_{G=2} = CD(\partial C\Phi) - \frac{1}{2}(DC)(\partial C\Phi) \]  
(2.20)

are BRST invariant, namely, \( Q(O_{G=1}) = Q(O_{G=2}) = 0 \). The operators \( O \) are not \( Q \)-(anti)commutators, and therefore, they represent cohomology classes. The label \( G \) in (2.19, 2.20) describes the ghost number of the operator \( O \) (with the convention \( G(C) = 1 \), \( G(B) = -1 \)). Unlike the \( N = 0 \) case, \( O_{G=1} \) and \( O_{G=2} \) differ not only by a choice of their ghost vacua, but also by a picture changing operation \([5],[10] \).

The superfields \( \Phi \) and \( O_{G=1} \) satisfy the descent equation
\[ Q(\Phi) = DO_{G=1}. \]  
(2.21)

In terms of components one gets
\[ Q(\Phi_0) = O_1, \quad Q(\Phi_1) = \partial O_0, \]  
(2.22)
where \( \Phi = \Phi_0 + \theta \Phi_1 \), and \( O_{G=1} = O_0 + \theta O_1 \), and thus \( O_0 \) is the only physical component amongst the components of \( O_{G=1} \). Moreover, \( O_{G=1} \) and \( O_{G=2} \) are related via the equation

\[
Q(O^h_{G=1}) = \left( \frac{1}{2} - h \right) O^h_{G=2}, \tag{2.23}
\]

where \( O^h \) is constructed from an off-shell \( \Phi_h \) when \( h \neq 1/2 \). Using equation (2.23) (and continuity in \( h \)) in the on-shell limit \( h \to 1/2 \), one finds that \( O_{G=2} \) is physical. \( O_{G=2} \) also satisfies the descent equation

\[
Q(\partial C \Phi) = D O_{G=2}. \tag{2.24}
\]

This concludes the review of the \( N = 1 \) BRST case. In the next section we discuss the \( N = 2 \) BRST case in \( N = 2 \) superspace, the BRST invariant operators, and the descent equation.

## 3 BRST Cohomology of the \( N = 2 \) String

We first set the notation. We define

\[
\begin{align*}
    z_{12} &= z_1 - z_2 - \frac{1}{2} (\theta_1^+ \theta_2^- + \theta_1^- \theta_2^+), \\
    \theta_{12}^\pm &= \theta_1^\pm - \theta_2^\pm,
\end{align*}
\]

and the spinor derivatives

\[
D_+ = \frac{\partial}{\partial \theta^+} + \frac{1}{2} \theta^- \partial, \quad D_- = \frac{\partial}{\partial \theta^-} + \frac{1}{2} \theta^+ \partial, \quad \{D_+, D_-\} = \partial,
\]

\[
D_{\pm 1} z_{12} = D_{\pm 2} z_{12} = \frac{1}{2} \theta_{12}^\mp,
\]

where \( \partial \equiv \frac{\partial}{\partial z} \). A function \( f_1 \equiv f(z_1, \theta_1^+, \theta_1^-) \) can be expanded around \( z_2, \theta_2^+, \theta_2^- \):

\[
f_1 = f_2 + \theta_{12}^- (D_- f)_2 + \theta_{12}^+ (D_+ f)_2 + z_{12} (\partial f)_2 + \frac{1}{2} \theta_{12}^\mp \theta_{12}^- ([D_-, D_+] f)_2 \\
+ z_{12} \theta_{12}^+ (D_- \partial f)_2 + z_{12} \theta_{12}^- (D_+ \partial f)_2 + \frac{1}{2} z_{12} \theta_{12}^\mp \theta_{12}^- (\partial [D_-, D_+] f)_2 + \ldots \tag{3.3}
\]
We define an operator $\Delta^{h,q}$ to be

$$\Delta^{h,q} = h \frac{\theta^+_{12} \theta^-_{12}}{z_{12}^2} + \frac{\theta^+_{12}}{z_{12}} D_+ - \frac{\theta^-_{12}}{z_{12}} D_- + \frac{\theta^+_{12} \theta^-_{12}}{z_{12}} \partial - \frac{q}{z_{12}},$$

(3.4)

and for simplicity we denote

$$\Delta^h \equiv \Delta^{h,0}. \quad (3.5)$$

An $N = 2$ stress tensor $T = J + \theta^+ G_+ + \theta^- G_- + \theta^+ \theta^- T_B$, where $T_B$ is the bosonic stress tensor, obeys the $N = 2$ superconformal algebra expressed via the OPE

$$T_1 T_2 \sim \frac{1}{3} c \frac{1}{z_{12}^2} + \Delta^1 T_2 + ..., \quad (3.6)$$

where $\Delta^1$ is defined in (3.5,3.4), and $c$ is the central charge. The $N = 2$ $2d$ theory combines a matter sector with a ghost sector. The $N = 2$ superstress tensor of the matter sector $T^\Phi$ acts on a primary superfield of the matter sector $\Phi_{h,q}$ with dimension $h$ and $U(1)$-charge $q$ as

$$T^\Phi_1(\Phi_{h,q})_2 \sim \Delta^{h,q}(\Phi_{h,q})_2 + ... . \quad (3.7)$$

The $N = 2$ superghosts $C$ and $B$ have components

$$C = c + \theta^+ \gamma_+ + \theta^- \gamma_- + \theta^+ \theta^- \xi,$$

$$B = \eta + \theta^+ \beta_+ - \theta^- \beta_- + \theta^+ \theta^- b,$$

(3.8)

and satisfy

$$C_1 B_2 \sim B_1 C_2 \sim \frac{\theta^+_{12} \theta^-_{12}}{z_{12}} + ... . \quad (3.9)$$

The superstress tensor of the ghost system is

$$T^{BC} = \partial(CB) - (D_+ CD_- B + D_- CD_+ B) \quad (3.10)$$

The superghosts $C$ and $B$ have dimension and charge $h(C) = -1$, $q(C) = 0$, $h(B) = 1$, $q(B) = 0$, with respect to $T^{BC}$. The total stress tensor of an $N = 2$ string is

$$T = T^\Phi + T^{BC}. \quad (3.11)$$
At criticality, the total central charge is 0, which implies that the central charge of the matter sector is $c = 6$.

The $N = 2$ BRST current is

$$ j = C(T \Phi + \frac{1}{2} T^{BC}) - \frac{1}{2} [D_-(CD_+(BC)) + D_+(CD_-(BC))] $$

$$ = CT + B(D_+CD_- - C\partial C). \quad (3.12) $$

The total derivative terms in $(3.12)$ are chosen such that $j$ is a primary superfield of $T$ in $(3.11)$ with $h = q = 0$. As for the $N = 1$ case in $(2.17)$, at criticality, the nilpotent $N = 2$ BRST charge

$$ Q = \frac{1}{2\pi i} \oint dz d\theta^+ d\theta^- j(z, \theta^+, \theta^-), \quad (3.13) $$

acts on a superfield $O$ by

$$ Q(O) = [Q, O(z_2, \theta^+_2, \theta^-_2)] \equiv \frac{1}{2\pi i} \oint_{C_{z_2}} dz_1 d\theta^+_1 d\theta^-_1 j(z_1, \theta^+_1, \theta^-_1) O(z_2, \theta^+_2, \theta^-_2), $$

where the contour $C_{z_2}$ surrounds $z_2$ once.

Next we describe the BRST cohomology classes. First we define

$$ O(\Psi) \equiv D_-(CD_+ \Psi) - D_+(CD_- \Psi). \quad (3.15) $$

Then for a primary superfield of the matter sector with $h = q = 0$, $\Phi \equiv \Phi_{0,0}$, the combinations

$$ O_{G=1} = O(\Phi), $$

$$ O_{G=2} = O(\partial C \Phi), $$

$$ \tilde{O}_{G=2} = \frac{1}{2} O([D_-, D_+] \Phi), $$

$$ O_{G=3} = \frac{1}{2} O(\partial C[D_-, D_+] \Phi), $$

are BRST invariant, namely, $Q(O_{G=1}) = Q(O_{G=2}) = Q(\tilde{O}_{G=2}) = Q(O_{G=3}) = 0$. The operators $O_{G=1}$, $O_{G=2}$, and $\tilde{O}_{G=2}$ are not $Q$-(anti)commutators, and therefore, represent cohomology classes. We conjecture that $O_{G=3}$ is $Q$-nontrivial as well. The label $G$ in $(3.16)$-$(3.19)$ describes the ghost number of the operator $O$ (with the convention $G(C) = 1$, $G(B) = -1$).
We have described the BRST cohomology in particular pictures; indeed, as in the $N = 1$ case, the operators $O_{G=1}$, $O_{G=2}$, $O_{G=2}$, and $O_{G=3}$ not only correspond to different ghost vacua, but also differ by picture changing operations.

The proof that $O_{G=1}$ is BRST invariant but is not a $Q$-commutator is straightforward. It helps to use the relations

$$Q(C) = C\partial C - D_+ CD_- C,$$

$$Q(\Phi) = -D_-(CD_+ \Phi) - D_+(CD_- \Phi),$$

and

$$Q(f(C)g(\Phi)) = Q(f(C))g(\Phi) + (-)^f f(C)Q(g(\Phi)),$$

where $f$ and $g$ are arbitrary functions of $C$ and $\Phi$ and their derivatives, respectively; the last relation follows because the structure of $Q$ is such that there are no double contractions with $f$ and $g$.

Let us denote

$$Y_\pm \equiv CD_\pm \Phi.$$  \hfill (3.23)

Equation (3.21) can be interpreted as the descent equation

$$Q(\Phi) = -D_- Y_+ - D_+ Y_- \equiv -* O_{G=1}.$$ \hfill (3.24)

The $*$ operation in (3.24) changes $D_- Y_+ \pm D_+ Y_-$ to $D_- Y_+ \mp D_+ Y_-$. The descent equation (3.24) tells us that the symmetric combination $*O_{G=1} = D_- Y_+ + D_+ Y_-$ is a $Q$-commutator of a primary superfield. However, for any $Y_\pm$ that obeys $Q(D_- Y_+ + D_+ Y_-) = 0$, one has $Q(Y_\pm) = D_\pm Y_\pm$, and hence the antisymmetric combination $*(D_- Y_+ + D_+ Y_-) = D_- Y_+ - D_+ Y_-$ is also annihilated by $Q$. Consequently $O_{G=1}$ is $Q$-closed. This is analogous to the $N = 0$ ($N = 1$) case, where the descent equation (2.7) ((2.21)) relates a (super)derivative of an element in the BRST cohomology to the $Q$-(anti)commutator of its corresponding primary (super)field; here the $*$ operation plays the role of the derivative.

Furthermore, generically $O_{G=1}$ cannot be $Q$-exact, since there is no natural candidate operator $\Psi$ for $Q(\Psi)$ of the correct ghost number and dimension.
other than \( \Phi \) itself; also, on physical grounds, since \( \Phi \) is an \( h = q = 0 \) primary, it creates a physical state, and hence \( O_{G=1} \) should not be trivial. Finally, we have explicitly checked that there are no further operators bilinear in \( C \) and \( \Phi \) in the cohomology.

Expanding the descent equation (3.24) in components one finds

\[
Q(\Phi_1) = \frac{1}{2} \partial O_0, \quad (3.25)
\]

\[
Q(\Phi_{\pm}) = \pm O_{\pm}, \quad (3.26)
\]

and

\[
Q(\partial \Phi_0) = -2O_1, \quad (3.27)
\]

where \( \Phi = \Phi_0 + \theta^+ \Phi_+ + \theta^- \Phi_- + \theta^+ \theta^- \Phi_1 \) and \( O_{G=1} = O_0 + \theta^+ O_+ + \theta^- O_- + \theta^+ \theta^- O_1 \) with

\[
\begin{align*}
O_0 &= \gamma_- \Phi_+ - \gamma_+ \Phi_- - 2c\Phi_1 \\
O_+ &= -\xi + \frac{1}{2} \partial c - \gamma_+ \Phi_1 - \frac{1}{2} \gamma_+ \partial \Phi_0 + c \partial \Phi_+ \\
O_- &= -\xi - \frac{1}{2} \partial c + \gamma_- \Phi_1 - \frac{1}{2} \gamma_- \partial \Phi_0 + c \partial \Phi_- \\
O_1 &= \frac{1}{2} \partial (\gamma_- \Phi_+ + \gamma_+ \Phi_- - c \partial \Phi_0). \quad (3.28)
\end{align*}
\]

From eqs. (3.24,3.27) we learn that \( O_0 \) is the only physical operator amongst the components of the superfield \( O_{G=1} \); it is referred to as the \((0,0)\)-picture physical operator (see [11] for a discussion of picture changing for the \( N = 2 \) string). This concludes the discussion of BRST cohomology at ghost number one.

To prove that \( O_{G=2} \) in (3.17) is BRST invariant, we define \( O_{G=2}^h \) by the relation

\[
Q(O_{G=1}^h) = -hO_{G=2}^h, \quad (3.29)
\]

where \( O_{G=1}^h \) is defined by replacing \( \Phi \) in \( O_{G=1} \) with an off-shell primary superfield with zero \( U(1) \)-charge: \( \Phi_h \equiv \Phi_{h,0} \). Equation (3.29) is similar to the \( N = 0 \) case in eq. (2.8), and to the \( N = 1 \) case in eq. (2.23), where \( O_{G=2} \) is related to a \( Q \)-anticommutator of \( O_{G=1} \) at the limit \( h \to 1 \) and \( h \to 1/2 \).
respectively; here we will find that $O_{G=2}$ in eq. (3.17) is the limit $h \to 0$ of $O_{G=2}^h$, and thus is $Q$-closed.

As a trick for computing $O_{G=2}^h$ we define the operator $Q_h$ by the relations

$$Q(\Phi_h) = - \star O_{G=1}^h + hQ_h(\Phi_h), \quad Q_h(C) = 0. \quad (3.30)$$

Note that $\star O_{G=1}^h$ has the same functional form as $\star O_{G=1}$ in (3.24) with $\Phi_h$ instead of $\Phi$, and therefore has no explicit $h$-dependence. Equation (3.30) implies that the operator $Q_h$ acts on $\Phi_h$ and $C$ by

$$Q_h(\Phi_h) = \partial C\Phi_h, \quad Q_h(C) = 0. \quad (3.31)$$

Because there is no explicit $h$ dependence in $O_{G=1}^h$, and since $Q(O_{G=1}) = 0$, one gets

$$Q(O_{G=1}^h) = hQ_h(O_{G=1}^h), \quad (3.32)$$

and hence from (3.29) it follows that

$$Q_h(O_{G=1}^h) = -O_{G=2}^h. \quad (3.33)$$

This holds in the limit $h \to 0$ as well, and thus, using eq. (3.31) one recovers $O_{G=2}$ in eq. (3.17).

To verify that $O_{G=2}$ is not $Q$-exact we have checked that acting with $Q$ on the 6 independent bilinear terms in $C$ and $\Phi$, it is impossible to get $O_{G=2}$ (see Appendix). We can also show that $\star O_{G=2}$ is $Q$-exact, which implies that we can construct $O_{G=2}$ from a descent equation

$$Q(\partial C\Phi) = - \star O_{G=2}. \quad (3.34)$$

This is a consequence of the equalities

$$\star O_{G=2}^h = -Q_h(\star O_{G=1}^h) = Q_h(Q(\Phi_h)) = -Q(Q_h(\Phi_h)) = -Q(\partial C\Phi_h), \quad (3.35)$$

where the first equality follows from (3.33) and because $\star$ commutes with $Q_h$ (recall that the $\star$ operation is defined by $\star(D_+Y_+ \pm D_+Y_-) = D_+Y_+ \mp D_+Y_-$). The second equality follows from eq. (3.30) and because $Q^2 = 0$ for all $h$ implies $Q_h^2 = 0$. The third equality follows because $Q^2 = 0$ for all $h$ implies $\{Q, Q_h\} = 0$, and the last equality follows from eq. (3.31).
To prove that $\tilde{O}_{G=2}$ in (3.18) is BRST invariant, we proceed in precisely the same way as for $O_{G=2}$, except that we go off-shell by continuing $q$ rather than $h$ away from zero. We find that

$$Q_q(O_{G=1}^q) = -\tilde{\tilde{O}}_{G=2}^q$$

(3.36)

where

$$Q_q(\Phi_q) = \frac{1}{2}[D_-, D_+]C \Phi_q, \quad Q_q(C) = 0,$$

(3.37)

and $\Phi_q \equiv \Phi_{0,q}$ is a zero dimension off-shell primary superfield. This gives (3.18). Again, we have explicitly checked that $\tilde{O}_{G=2}$ is not $Q$-exact and is not in the same cohomology class as $O_{G=2}$ (see Appendix). Finally, it again follows that

$$\frac{1}{2}Q([D_-, D_+]C\Phi) = -\ast \tilde{O}_{G=2}$$

(3.38)

and hence that $\tilde{O}_{G=2}$ can be derived from a descent equation. This completes the discussion of BRST cohomology at ghost number 2.

The conjectured BRST cohomology at $G = 3$ is found analogously, either from

$$O_{G=3} = Q_h(Q_q(O_{G=1}))$$

(3.39)

or from the descent equation

$$\frac{1}{2}Q(\partial C[D_-, D_+]C\Phi) = -\ast O_{G=3}.$$  

(3.40)

We have not explicitly verified that $O_{G=3}$ is not $Q$-exact.

Next we discuss an example: the $N = 2$ string in toroidal background. In terms of $N = 2$ chiral superfields

$$X^i(Z, \bar{Z}; \theta^-, \bar{\theta}^-) = X^i_L(Z, \theta^-) + X^i_R(\bar{Z}, \bar{\theta}^-)$$

$$= x^i(Z, \bar{Z}) + \psi^i_L(Z, \bar{Z})\theta^- + \psi^i_R(Z, \bar{Z})\bar{\theta}^- + F^i(Z, \bar{Z})\theta^-\bar{\theta}^-$$

$$Z = z - \theta^+\bar{\theta}^-,$$

(3.41)

(where $i = s, t$ denote complex spacelike and timelike components, and bars denote complex conjugation), the vertex operators to create a mode with Lorentzian momentum $(p_L, p_R)$ is

$$\Phi_p(X, \bar{X}) = e^{i(p_L X_L + p_L \bar{X}_L + p_R X_R + p_R \bar{X}_R)},$$

(3.42)
where $p_L$ and $p_R$ are complex 2-momentum, and $(p_L, p_R)$ is in the even-self-dual lattice $\Gamma^{4,4}$ (see [4] for more details). The antichiral superfield $\bar{X}$ in (3.42) is obtained from $X$ by a complex conjugation, together with $\theta^- \leftrightarrow \theta^+$. The on-shell condition is $p_L \cdot \bar{p}_L = p_R \cdot \bar{p}_R = 0$. With this condition, $\Phi_p(X_L, \bar{X}_L)$ is a primary superfield with $h = q = 0$, and it generates the BRST elements given in equations (3.16)-(3.19).

In addition to these cohomology classes, there are elements of the BRST cohomology corresponding to discrete states (that appear only at momentum $p = 0$). For a holomorphic chiral superfield $X_L$, one finds the discrete BRST invariant operator

$$O_{\text{discrete}} = D_+(CD_-X_L),$$

while for a holomorphic antichiral superfield $\bar{X}_L$ one finds the discrete operator

$$\bar{O}_{\text{discrete}} = D_-(CD_+ \bar{X}_L).$$

(This can be shown by inserting $\Phi = X_L$ or $\Phi = \bar{X}_L$ into eq. (3.16), and by using the property of (anti)chiral superfields: $D_+X = 0$ ($D_-\bar{X} = 0$)). The discrete states corresponding to these cohomology classes are formed from off-shell states by taking an appropriate limit $p \to 0$ (see ref. [4] for details).

There are also analogous discrete states at $G = 2$; these are

$$D_+(CD_-D_+CD_-X_L), \quad D_-(CD_+D_-CD_+\bar{X}_L);$$

relevant identities for finding these can be found in the Appendix.

4 The twisted $N = 4$ Symmetry of the $N = 2$ String

In ref. [12] it was shown that $N = 2$ string theory gives a realization of an $N = 2$ superfield extension of the topological conformal algebra. In this section we untwist the $N = 2$ topological algebra to get an $SU(2) \times SU(2) \times U(1)$ $N = 4$ superconformal current algebra.

5 A twisting of $N = 4$ superconformal field theories to topological field theories with $N = 2$ supersymmetry was also considered in ref. [13].
We introduce the following $N = 2$ superfields
\begin{align}
J &= BC, \\
G &= j, \\
\bar{G} &= B,
\end{align}
(4.1) (4.2) (4.3)
\begin{equation}
T_a = T + a\partial J.
\end{equation}
(4.4)
The superfield $J$ in (4.1) is the ghost number current, $G$ in (4.2) is the BRST current defined in (3.12), and $\bar{G}$ in (4.3) is the superghost $B$. The stress tensor $T_a$ is a twist of the ghost+matter stress tensor $T$ defined in (3.11). The central charge of $T$ is $c = 0$, the dimensions of $J$, $G$, $\bar{G}$ and $T_a$ with respect to $T$ are $h = 0, 0, 1$ and $1$, respectively, and their charge is $q = 0$.

It is notable [12] that the BRST charge in (3.13) acts on $B$ and $J$ by
\begin{equation}
Q(B) = T, \quad Q(J) = j.
\end{equation}
(4.5)

With the operator $\Delta^h$ defined in (3.5, 3.4), one finds that $J, G, \bar{G}, T_a$ obey the algebra
\begin{align}
T_a \cdot T_a &\sim \Delta^1 T_a + ..., & J \cdot G &\sim -\frac{\theta^+\theta^-}{z}G + ..., & J \cdot \bar{G} &\sim \frac{\theta^+\theta^-}{z}\bar{G} + ..., \\
T_a \cdot J &\sim \Delta^0 J + ..., & T_a \cdot G &\sim \Delta a G + ..., & T_a \cdot \bar{G} &\sim \Delta^{1-a}\bar{G} + ..., \\
G \cdot \bar{G} &\sim \frac{\theta^+\theta^-}{z}(T_a + a\partial J) + \frac{\theta^+}{z}D_+ - \frac{\theta^-}{z}D_- J + ..., & G \cdot \bar{G} &\sim \frac{\theta^+\theta^-}{z}(T_a + (1-a)\partial J) + \frac{\theta^+}{z}D_+ - \frac{\theta^-}{z}D_- J + ..., & J \cdot J &\sim G \cdot G \sim \bar{G} \cdot \bar{G} \sim 0.
\end{align}
(4.6) (4.7) (4.8) (4.9) (4.10) (4.11) (4.12) (4.13)
In (4.6)-(4.13) it is understood that the first operator in the product is at the point $(z_1, \theta^1)$ in superspace, while the second operator in the product and
the operator on the right-hand side are at the point \((z_2, \theta^{\pm}_{z_2})\). We have also used the shorthand notation: \(\theta^{\pm} \equiv \theta^{\pm}_{z_2}, \quad z \equiv z_{12}\). From eq. (4.8) we learn that \(T_a\) is a stress tensor with central charge \(c_a = 0\) for any twist parameter \(a\). Eq. (4.7) means that \(J\) is a primary superfield of \(T_a\) with \(h_a(J) = 0\). Eqs. (4.8) and (4.9) mean that \(h_a(G) = a\), and \(h_a(\bar{G}) = 1 - a\) with respect to \(T_a\), namely, there is a spectral flow under the twist \(T \to T_a\).

We now arrive to the important observation of this section. At \(a = 0\), the algebra (4.6)-(4.13) is the \(N = 2\) superfield extension of the topological conformal algebra in [12], namely, the algebra generated by \(T, J, B\) and \(j\). This algebra is twisted to the standard \(SU(2) \times SU(2) \times U(1)\) \(N = 4\) superconformal algebra, with central charge \(c = 0\), when \(a = 1/2\). Explicitly, we identify

\[
T = -iT_{a=1/2}, \quad J = -\frac{1}{2}J, \quad A = \frac{i}{2}(G + \bar{G}), \quad B = \frac{1}{2}(G - \bar{G}),
\]

(4.14)

where \(T\) is a super stress tensor of conformal spin 1, \(A\) and \(B\) are two spin-1/2 superfields, and \(I\) is a spin-0 superfield. It now follows that the \(N = 2\) superfields \(T, I, A, B\) obey the \(N = 4\) superconformal algebra appearing in eq. (23) of ref. [13] (with \(c_1 = c_2 = c_k = \alpha = 0\) in the notation of [13]), namely

\[
T \cdot T \sim -i\Delta^1T + ..., \quad (4.15)
\]

\[
T \cdot I \sim -i\Delta^0I + ..., \quad (4.16)
\]

\[
T \cdot A \sim -i\Delta^{1/2}A + ..., \quad T \cdot B \sim -i\Delta^{1/2}B + ..., \quad (4.17)
\]

\[
I \cdot A \sim \frac{i}{2} \frac{\theta^{+} \theta^{-}}{z} B + ..., \quad I \cdot B \sim \frac{i}{2} \frac{\theta^{+} \theta^{-}}{z} A + ..., \quad (4.18)
\]

\[
A \cdot A \sim B \cdot B \sim \frac{i}{2} \frac{\theta^{+} \theta^{-}}{z} T + ..., \quad (4.19)
\]

\[
A \cdot B \sim -i\Delta^0I + \frac{i}{2} \frac{\theta^{+} \theta^{-}}{z} \partial I + ..., \quad (4.20)
\]

\[
I \cdot I \sim 0. \quad (4.21)
\]
This algebra is the so-called ‘large’ \( N = 4 \) superconformal algebra \([14]\) with \( c = \alpha = 0 \), where \( c \) is the central charge, and \( \alpha \) is a measure for the asymmetry between the two affine \( SU(2) \) sub-algebras. It has 16 generators (the four components of the four superfields in (4.14)), which are: the spin-2 stress tensor \((\tilde{T}_1)\), 4 spin-3/2 supercurrents \((\tilde{T}_\pm, A_1, B_1)\), 7 spin-1 currents \((\tilde{T}_0, \ A_\pm, B_\pm, I_1, \partial I_0)\) generating the affine extension of \( SU(2) \times SU(2) \times U(1) \), and 4 spin-1/2 currents \((A_0, B_0, \partial I_\pm)\). (Note that the spin-0 field \( I_0 \) enters the right-hand side of the algebra (4.15)-(4.21) through its derivatives only.)

The twist described in this section resembles the twist of an \( N = 2 \) superconformal algebra to a topological conformal algebra. In the latter case, one defines \( T_a = T + a\partial J \) in terms of the \( N = 2 \) stress tensor \( T \) and \( U(1) \) current \( J \). It then follows that \( T_a \) is a stress tensor with central charge \( c_a = c_0(1-4a^2) \), and at \( a = 1/2 \) one recovers the \( N = 0 \) topological conformal algebra. In this section, we have shown that an \( SU(2) \times SU(2) \times U(1) \ N = 4 \) superconformal algebra can be twisted into an \( N = 2 \) topological conformal algebra.

Moreover, the twist of the \( N = 2 \) critical string algebra, regarded as a topological superconformal algebra, into an \( N = 4 \) superconformal algebra, is similar to the bosonic string regarded as a topological conformal field theory, that is ‘almost’ a twist of an \( N = 2 \) superconformal algebra (see section 2).\[7\]

In the next section we show that the algebra of \( any \ N \geq 2 \) ‘critical string’, regarded as a topological superconformal algebra, can be twisted into an \( N+2 \) superconformal algebra.\[8\]

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\( ^6 \) A general \( SU(2) \times SU(2) \times U(1) \ N = 4 \) superconformal algebra has a free parameter \( \alpha \) measuring the asymmetry between the two affine \( SU(2) \) sub-algebras. This algebra is sometimes referred to as the \( o(4) \ N = 4 \) algebra. When \( \alpha = 1/2 \) it becomes the \( SU(2) \ N = 4 \) algebra. The ‘large’ \( N = 4 \) symmetry algebra of the \( N = 2 \) string presented here has \( \alpha = 0 \).

\( ^7 \) During the years that we have been writing up our results, it was shown that the \( N = 1 \) superstring has an analogous \( N = 3 \) twisted supersymmetry \([13]\). As in the bosonic string case, the only requirement is that the combined supermatter and supergravity system has a \( U(1) \) supercurrent that can be used to improve the BRST current.

\( ^8 \) We thank Nathan Berkovitz for pointing this out and explaining it to us.
5 Comments and Discussion

In this section we discuss the results presented in this work, and make a few comments.

In section 4 we have shown that $N = 2$ strings with total ghost+matter central charge $c = 0$ have an $N = 4$ twisted symmetry.\[ It is remarkable that unlike the $N = 2$ superconformal structure of the $N = 0$ string, and the $N = 3$ superconformal structure of the $N = 1$ superstring, the $N = 4$ superconformal structure of the $N = 2$ string does not involve an improvement of the BRST current. While the combined supermatter and supergravity stress tensor is twisted, the BRST supercurrent $j$ and ghost number supercurrent $J$ are not modified. This occurs because the $N = 2$ ghost number current is not anomalous, and hence needs no improvement.

Iteratively, one can twist the algebra of \textit{any} $N > 2$ superconformally invariant $2d$ system into an $N + 2$ superconformal algebra. One regards the $N > 2$ system as a critical matter system, whether or not it already has ghosts (e.g., the $N = 4$ system constructed in section 4) and adds (new) ghosts and antighosts. Since for $N > 2$ the ghost system has $c = 0$, the new system with the ghosts remains critical. In particular, for $N \geq 2$ the ghost number current $J$ is not anomalous.\[ Therefore, by twisting the stress tensor $T \rightarrow T + \frac{1}{2} \partial J$, the BRST current and the gravitational antighost can be brought to dimensions $3/2$, and they become the two new supersymmetry currents. The remaining lower dimensional currents of the algebra can then be generated by the OPE’s of the supercurrents.

Returning to the $N = 2$ string, we note that the $N = 2$ descent equations have the novel feature that instead of involving derivatives, they involve the operation of interchanging the real and imaginary parts of an operator (with respect to complex conjugation defined on a Lorentz signature worldsheet).

Finally we comment on some open issues. The relation between the ghost

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\[9\] This is not to be confused with the \textit{physical} $N = 4$ supersymmetry proposed in ref.\[17\].

\[10\] $J \cdot J \sim 0$ because $J$ is a sum of commuting $N = 2$ currents each of which is not anomalous.
number one and two (and three) BRST classes in the $N = 1, 2$ strings is not completely clarified in this work. As was mentioned in the text, they are related by a change of the ghost vacuum as well as some picture changing operations, but the precise relation is not presented here. Furthermore, the full cohomology at all ghost numbers has not been found.

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A Appendix

In this appendix we briefly summarize some of our computations, and demonstrate that $O_{G=2}$ and $\tilde{O}_{G=2}$ in (3.17,3.18) are both cohomologically nontrivial and inequivalent.

We first evaluate $Q$ on a basis of dimension zero bilinears in $C$ and $\Phi$ using (3.20)-(3.22). Because $Q(O_{G=1}) = Q(*O_{G=1}) = 0$ implies $Q(D_{\pm}CD_{\mp} \Phi) = Q(CD_{\pm}D_{\mp} \Phi)$, we only need the following:

\begin{align}
Q(D_+ D_- C \Phi) &= (CD_+ D_- \partial C - D_+ CD_- \partial C) \Phi \\
&\quad + D_+ D_- C(D_+ CD_- \Phi + D_- CD_+ \Phi - C \partial \Phi), \\
Q(D_- D_+ C \Phi) &= (CD_- D_+ \partial C - D_- CD_+ \partial C) \Phi \\
&\quad + D_- D_+ C(D_- CD_+ \Phi + D_+ CD_- \Phi - C \partial \Phi), \quad (A.1) \\
Q(CD_+ D_+ \Phi) &= C(D_+ \partial C D_+ \Phi + D_+ CD_+ \Phi) - D_+ CD_+ D_+ \Phi, \\
Q(CD_- D_+ \Phi) &= C(D_- \partial C D_+ \Phi + D_- CD_+ \Phi) - D_- CD_+ D_+ \Phi, \\
\end{align}

From the definitions (3.17) and (3.18) it is easy to show that

\begin{align}
\tilde{O}_{G=2} &= \frac{1}{2} * O_{G=2} + D_-(C(D_+ D_- C)D_+ \Phi) + D_+(C(D_- D_+ C)D_- \Phi), \\
\frac{1}{2} O_{G=2} &= *\tilde{O}_{G=2} - D_-(C(D_+ D_- C)D_+ \Phi) + D_+(C(D_- D_+ C)D_- \Phi). \quad (A.2)
\end{align}

Thus $O_{G=2}$ and $\tilde{O}_{G=2}$ are cohomologically equivalent to terms without undifferentiated $\Phi$’s, and hence cannot possibly be expressible in terms of anything involving (A.1). It is then straightforward to show that no linear combination is expressible as a linear combination of (A.2). This proves that $O_{G=2}$ and $\tilde{O}_{G=2}$ are generically cohomologically nontrivial and inequivalent. Eqs. (A.3) also are useful for finding the $G = 2$ discrete states (3.43).
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