Josephson coupling between superconducting islands on single- and bi-layer graphene

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Abstract
We study the Josephson coupling of superconducting (SC) islands through the surface of single-layer graphene (SLG) and bilayer graphene (BLG) in the long-junction regime, as a function of the distance between the grains, temperature, chemical potential and external (transverse) gate-voltage. For SLG, we provide a comparison with existing literature. The proximity effect is analyzed through a Matsubara Green’s function approach. This represents the first step in a discussion of the conditions for the onset of a granular superconductivity within the film, made possible by Josephson currents flowing between superconductors. To ensure phase coherence over the 2D sample, a random spatial distribution can be assumed for the SC islands on the SLG sheet (or intercalating the BLG sheets). The tunable gate-voltage-induced band gap of BLG affects the asymptotic decay of the Josephson coupling-distance characteristic for each pair of SC islands in the sample, which results in a qualitatively strong field dependence of the relation between Berezinskii–Kosterlitz–Thouless transition critical temperature and gate voltage.

Keywords: graphene, bilayer graphene, Josephson coupling, supercurrent, long-junction regime

1. Introduction
The recent discovery of graphene [1, 2], a single sheet of carbon atoms, has naturally raised a question of superconductivity in this material [3] and its derivatives. This interest has been further stimulated by the recent observation of the proximity effect on graphene [4, 5]. We exhibit the potential for developing new superconducting (SC) devices starting with graphene as a basis material. Graphite, though not a superconductor in itself, can be made SC by intercalating certain dopants into its structure [6]. Among graphite-based materials which are known to superconduct, the alkali metal-graphite intercalation compounds [7] have been widely investigated. The most easily fabricated among them is the C₆K system [8] which exhibits a transition temperature $T_c = 0.14$ K [7], while higher pressure allows higher alkali metal concentration to be reached, so that the corresponding $T_c$ can increase up to 5 K in C₂ Na [9]. On the other hand, Weller et al have shown that, at ambient conditions, the intercalated compounds C₆Yb and CaC₆ exhibit superconductivity with transition temperatures $T_c = 6.5$ K and 11.5 K respectively [10].

It has been suggested [11] that Ca-intercalated bilayer graphene (BLG) should be a superconductor with a critical temperature comparable to that of the 3D compound CaC₆ [8, 10, 12, 13]. This prediction is based on a combination of the linear augmented wave method for band structure calculations [14] and density functional theory.

Experiments based on metal-graphene hybrid composites have allowed the tuning of a proximity effect induced on graphene by SC nanoparticles deposited on top of it. Decoration with tin clusters, separated by an average spacing...
much smaller than the mean free path and the SC coherence length \[\xi\], has been shown to induce a gate-tunable Berezinský–Kosterlitz–Thouless (BKT) transition on micron-scaled exfoliated graphene samples. More recently, full electrical control of the superconductivity has been experimentally achieved for a centimeter-scale graphene sheet \[16\] on whose surface an array of tin nanoparticles was placed. Unlike the most common case of proximity coupling with the gate electrodes \[4\], in these two setups the proximity effect was generated by coupling the graphene surface to a 2D network of SC clusters \[17\]. Hence, the resulting hybrid systems globally behave as granular superconductors with universal transition threshold and Cooper pairs are localized in the insulating phase \[16\]. These experiments provide support for the emergence of graphene as a backbone material for designing new superconductors. Transport and electronic properties of graphene-based junctions continue to be subject of considerable attention \[18–23\]. The proximity effect in graphene has inspired several possible applications, such as valley sensors \[24\], spin current filters \[25\] and current switches \[26, 27\]. These are promising nanotechnology directions where graphene-inspired materials are particularly useful.

Motivated by this scenario, we consider the proximity effect in single-layer graphene (SLG) and BLG under various conditions of temperature, chemical potential, transverse electric field.

In a complementary fashion with respect to the calculation of supercurrent in terms of Andreev reflection at the metal-superconductor interface \[28\], we discuss the Josephson effect in the (opposite) dilute granular regime, i.e. when the distance \(r\) between the SC adsorbates is much larger than the SC coherence length \(\xi\) (long-junction regime) and their own width \(W\), such that the 2D nature of SLG/BLG has to be taken in account. In this regime the supercurrent is well described in terms of Cooper pairs tunneling through the SLG/BLG junction \[29\]. The considered setups are schematically represented in figure 1, where the entire carbon scaffold, in principle, behaves as a junction connecting SC impurities. The Josephson current obtained in the SLG case at zero temperature endorses the result relevant to a clean undoped sample presented in \[30, 31\], and reproduces also for the 2D case the coincidence between the supercurrent distance-decay for clean undoped SLG and disordered normal metals. This is identical to the 1D case in the short junction regime \[28, 32\]. For both of these physical systems, the \(\sim 1/r\) distance-decay of the supercurrent for 1D junctions \[28\] is replaced by a faster \(\sim 1/r^4\) distance-decay for 2D Cooper pair propagation. We show that finite doping modifies this to an asymptotic \(\sim 1/r^4\) decay, for distances much larger than the critical length \(\hbar v_F/\mu\). The temperature effect on the supercurrent is computed, and the asymptotic exponential decay \(\sim r^{-4}e^{-2\pi r^4\hbar v_F/\hbar}\) is derived for distances \(r\) much larger than the thermal length \(r_T\). The supercurrent across BLG is also discussed, resulting into a \(\sim 1/r^3\) distance-decay at zero temperature, and an asymptotic exponential decay \(\sim r^{-3}e^{-4\pi r^3/\hbar}\) for distances much larger than the thermal length associated to the dispersion relation of BLG low-energy band. Finally we provide a comparison with former results in the literature for SLG, while the asymptotic of the Josephson current through BLG is analytically derived.

We point out that decoration of graphene by means of extended SC mesoscopic grains is in principle not required to generate SC. In fact, even an arrangement of local vibrational impurities is able to induce negative-\(U\) centers in the carbon structure, as a result of the coupling between the local vibrational modes and the surrounding lattice \[33\]. The fermionic state of the induced negative-\(U\) center can be empty, singly or doubly occupied with finite probability, and this suggests the formation of a local Cooper pair near each vibrational impurity \[33\]. It should, therefore, be possible to effectively produce a local SC order parameter by simply placing pointwise vibrational impurities on top of the graphene sheet.

The discussion presented in this work is twofold. First, we investigate the proximity effect for its intrinsic interest, motivated also by practical proposals specific to graphene junctions (which have been listed above). Second, we predict features of the BKT phase transition for lattice-adsorbate

Figure 1. Cooper pair tunneling (dashed black lines) between superconducting grain impurities (blue spots), through \(1(a)\) single-layer graphene and \(1(b)\) bilayer graphene. In both cases, the entire carbon scaffold plays the role of Josephson junction connecting each couple of SC grains. For convenience of graphical representation, tunnelings are represented here only up to an arbitrary spatial range.
graphene-based composites, as a function of the number density of lodged SC grain impurities, and the other parameters already mentioned. The study of such a phase transition is left for later research. The paper is structured as follows: the supercurrent across a long Josephson junction is expressed in terms of the junction electron propagator in section 2; respectively for SLG and BLG, sections 3 and 4 analyse the supercurrent radial dependence, considering the effect of doping and transverse gate-voltage, both at vanishing and finite temperature; in section 5 the results of section 3 are compared to the literature, while in section 6 we draw our conclusions; next section contains our acknowledgments, and the appendix specializes to our case a discussion of the asymptotic behavior of the Meijer functions.

2. Josephson current

In a continuum approximation model, the Hamiltonian for the system of graphene and SC contacts can be written as [30]

$$H = v_F \int d^2r \psi^{(a)}(r) \tilde{\sigma}^{(a)}(r) \cdot \frac{\partial \psi^{(a)}(r)}{\partial r} + \sum_{j=1,2} \int_0^W dy \; \psi_j^{(1)}(x_j, y) \psi_j^{(2)}(x_j, y) + \text{h.c.,}$$

(1)

where $\tilde{\sigma}^{(a)}$, $a = 1, 2$ are two sets of Pauli matrices, an implicit sum over index $a$ denotes the sum over valleys $K, -K$ in the Brillouin zone of graphene, $\psi_{S1}, \psi_{S2}$ are the electron fields in the SC electrodes, and $\psi_j^{(a)}$ are the electron spinor fields in graphene. In the long junction regime of interest, the time of propagation of the Cooper pairs between the contacts is much larger than $1/\Delta$ (i.e., the distance $r$ will be much larger than the SC coherence length $\xi$). The dominant transport process will be the Cooper pair tunneling from the contact $S1$ to the 2D layer, and then from the latter to $S2$. Under the assumption of large $\Delta$, we can write the approximation [30]

$$\langle \psi_{j,\sigma}(x_j, y; -i\tau_1) \psi_{j,-\sigma}(x_j, y; -i\tau_2) \rangle \approx e^{i\chi_N} \rho \delta(\tau_1 - \tau_2),$$

(2)

where the statistical averages at temperature $T$ are taken over ordered products with respect to the imaginary time $\tau$. The Josephson current $I$ can be computed as the free energy’s derivative with respect to $\chi = \chi_1 - \chi_2$ (with $\chi_1, \chi_2$ being the phases of the order parameters in $S1$, $S2$).

We begin by writing the expression for the critical Josephson current through a long junction ($r \gg \xi$), in terms of the electron propagator of the junction and the parameters of the system. Let $\bar{G}$ denote the electron propagator in momentum-energy coordinates, $\mathcal{F}^{-1}(\omega)$ the inverse and direct Fourier transform with respect to the set of variables ‘$\star$’. Disregarding the constant factor $2e v_F^2 W^2 (\rho t^2 W / \hbar v_F)^2$ ($W \ll \xi$) is the width of the superconductors, $\rho$ is the normal density of electron states (measured in erg⁻¹), $\rho t^2 W / \hbar v_F$ is a dimensionless constant, $t$ the inter atomic hopping integral (measured in erg), and $v_F$ the Fermi velocity), the supercurrent $I(\bar{r})$ in the long-junction regime $r \gg \xi$ is dominated [29–31] by its tunneling term:

$$\mathcal{F}^{-1}(\omega) i \text{Tr} \bar{G}^{(\omega)} \star \bar{G}(\omega)_{\nu=0} = i \text{Tr} \bar{G}(\omega) \star \bar{G}(\omega)_{\nu=0}.$$  

(3)

Here, $\bar{G}$ denotes the electron propagator in space-energy coordinates, the trace is performed by projecting over the singlet state of the Cooper pairs (see later sections for SLG and BLG cases), and $\star$ denotes the continuous/discrete convolution with respect to the set of variables ‘$\star$’. Therefore the zero/finite temperature supercurrent is ($\hbar = 1$)

$$I(\bar{r}, T = 0) \propto i \text{Tr} \int_{-i\infty}^{i\infty} \frac{d\omega}{2\pi} \bar{G}(\bar{r}, \omega) \cdot \bar{G}(\bar{r}, -\omega),$$

(4a)

$$I(\bar{r}, T) \propto -k_B T \sum_{\bar{n} = -\infty}^{\infty} \text{Tr} \bar{G}(\bar{r}, i\bar{n}) \star \bar{G}(\bar{r}, -i\bar{n}).$$

(4b)

Here $\bar{r}$ denotes the (vector) distance between the two SC dots under consideration. Formula (4b) reproduces formula (4a) in the zero temperature limit. We remark that this approach to the study of the Cooper pair propagation through SLG and BLG surfaces does not require any cutoff prescription. In what follows, we focus on the critical current, i.e. the maximal value of the supercurrent as a function of the phases difference between the order parameters of the two islands. In the long junction regime, we expect a contribution from higher harmonics to the current–phase relation at low temperature: the sinusoidal form about $T \sim T_c$ would evolve [34] into a saw-tooth dependence at low temperatures.

3. Single-layer graphene

3.1. Zero temperature

For $T \rightarrow 0$, the calculation is automatically regularized by taking the zero temperature limit of the Matsubara summation. Now we write down the propagator for SLG in the desired coordinates

$$\bar{G}(\bar{r}, i\omega) = \left[ \int d^2k \; e^{i\bar{k} \cdot \bar{r}} \; \frac{-i\omega}{\omega^2 + v_F^2 k^2} \right] 1$$

$$- \tilde{\sigma} \cdot \left[ \int d^2k \; e^{i\bar{k} \cdot \bar{r}} \; \frac{\hbar v_F k}{\omega^2 + v_F^2 k^2} \right]$$

$$=- 2\pi i v_F^2 \left[ \omega K_0 \left( \frac{\sqrt{\omega^2 r}}{v_F} \right) 1 + \sqrt{\omega^2} K_1 \left( \frac{\sqrt{\omega^2 r}}{v_F} \right) \tilde{\sigma} \cdot \hat{r} \right],$$

(5)
where $K_0(x)$ is the modified Bessel function of the second kind, $I$ is the $2 \times 2$ identity matrix in each sublattice space. We calculate the trace by using the notation:

$$G(\vec{r}, i\omega) = g_1 \mathbf{1} + g_2 \hat{d} \cdot \hat{r},$$

where

$$g_1 \equiv -2\pi i v_F^2 \omega K_0 \left( \frac{\sqrt{\omega^2 - r^2}}{v_F} \right),$$

$$g_2 \equiv -2\pi i v_F^2 \omega^2 K_1 \left( \frac{\sqrt{\omega^2 - r^2}}{v_F} \right).$$

Moreover, we define $\tilde{g}(\omega) \equiv g(-\omega)$, $i = 1, 2$. The initial and final singlet states (where $\epsilon_{0} \beta$ is the antisymmetric tensor of rank 2) of the Cooper pairs requires:

$$\text{Tr(subelectrons)} \tilde{G}(\vec{r}, i\omega_n) \cdot \tilde{G}(\vec{r}, -i\omega_n) = \epsilon_{0,\beta} (g_1 \mathbf{1}_{\beta\beta} + g_2 (\hat{d} \cdot \hat{r})_{\beta\beta}) \times \epsilon_{\mu\nu}$$

$$= 2(g_1 \tilde{g}_1 + g_2 \tilde{g}_2),$$

therefore the $\text{Tr}$ symbol acts as a simple overall factor of 2. As a consequence, the integration along the imaginary $i\omega_0$-axis gives from (4a):

$$I(\vec{r}) \sim \text{Tr} \int_{-i\infty}^{i\infty} \frac{d(i\omega_0)}{2\pi} \tilde{G}(\vec{r}, i\omega_0) \cdot \tilde{G}(\vec{r}, -i\omega_0)$$

$$= \frac{4}{v_F^3} \int_{0}^{\infty} \frac{dx}{2\pi} x^2 \left[ -4\pi^2 (K_0(x))^2 + 4\pi^2 (K_1(x))^2 \right]$$

$$= \frac{\pi^3}{(2v_F^3)} \frac{1}{r^3},$$

which agrees with the distance decay of the supercurrent described in [30, 31], up to a constant factor.

3.2. Finite temperature

For finite temperatures we use equation (4b), which yields

$$I(\vec{r}, T) \sim 16 \pi^4 v_F^2 (k_B T)^3 \sum_{n=0}^{\infty} (2n + 1)^2$$

$$\times [-K_0((2n+1)\pi x)]^2 + (K_1((2n+1)\pi x))]^2,$$

$$\int_{0}^{\infty} \frac{dx}{2\pi} x^2 \left[ -4\pi^2 (K_0(x))^2 + 4\pi^2 (K_1(x))^2 \right]$$

$$= \frac{\pi^3}{(2v_F^3)} \frac{1}{r^3},$$

where $x \equiv r k_B T/\hbar v_F^2 \gg 1$ for distances much larger than the thermal length $r \equiv \hbar v_F/k_B T$. For large distances $r \gg r_T$, the supercurrent can be expressed in analytical form as

$$I(\vec{r}, T) \approx \frac{8 \pi^3 k_B T}{v_F^2} e^{-2\pi x}, \quad r \gg r_T (i.e. x \gg 1).$$

The asymptotic decay (far beyond the thermal length) is exponential, which is different from the quintic decay in [31], equation (20) (see discussion in appendix).

For arbitrary distance $r$, we show the behavior of $I(r)$ in figure 2, obtained by numerical evaluation of the sum (9) for different temperatures. From visual inspection, the knees of the curves in figure 2 happen to be located very roughly at respective crossover-distances ranging from 3 to 5 times shorter than the critical thermal length $r_T$ for each temperature. Beyond this critical length $r_T$, the cubic distance-decay (8) gradually veers towards the exponential decay (10).

3.3. Finite doping

Doping of graphene generates a shift of the Fermi level from the neutrality point. We can introduce this shift in our approach by adding a finite chemical potential $\mu$. The corresponding Matsubara propagator for each Dirac electron is accordingly modified as follows

$$\tilde{G}(\vec{r}, i\omega_n) = \left[ \int \frac{d^2 k}{\omega_n - (\vec{\sigma} \cdot v_F \vec{k} - \mu)} \right] = \tilde{G}^{(\mu=0)}(\vec{r}, i\omega_n),$$

$$\tilde{G}(\vec{r}, -i\omega_n) = \left[ \int \frac{d^2 k}{\omega_n - (\vec{\sigma} \cdot v_F \vec{k} - \mu)} \right]$$

$$= \tilde{G}^{(\mu=0)}(\vec{r}, i\omega_n),$$

where we introduce the notation $\omega' \equiv \omega_n - i\epsilon_n$, $\omega'' \equiv -\omega_n - i\mu$. For any value of $\mu$, the square root $\sqrt{\omega'^2} = \pm \omega'$, $\sqrt{\omega''^2} = \mp \omega''$; upper/lower signs refer to positive/negative $\omega_n$. Proceeding as in equation (8), the supercurrent through doped graphene can, at zero temperature, be written in the following way (ω0 is the continuous variable version, for $T=0$, of the discrete variable $\omega_n$):

$$I(r)$$

Figure 2. Radial dependence of the supercurrent $I(r)$ through graphene for different temperatures, from red (top) to purple (bottom) $T = [0.140$ (red), $0.43$ (orange), $1.4$ (yellow), $4.3$ (green), $14$ (cyan), $43$ (blue), $140$ (purple)] K. Here the distance $r$ is expressed in nm and $I(r)$ in units of $(\rho^2 W/\pi v_F^3) \times 11.4$ nA $\times W^2$(nm$^2$) $\approx 24.4 \mu$A, where the dimensionless combination $\rho^2 W/\pi v_F$ is the relative conductance of each junction, $W$ (nm) the junction width expressed in nanometers, the Fermi velocity has been approximated as $v_F = 10^6$ m s$^{-1}$, and the numerical value of the unit refers to the case of niobium impurities of width $W = 50$ nm.
This result is qualitatively different from the inverse quadratic decay of the supercurrent derived in [30, 31] under the similar assumptions. In fact, the decay described here shows a different asymptotic power law, which does not include oscillations with the distance $r$. The crossover between the short distance cubic decay and the long-distance quartic decay is indicated in figure 3, where the set of curves corresponds to a wide range for the possible values of the chemical potential $\mu$. The knees of the curves with different $\mu$ sit roughly about their respective crossover-distances $r_\mu$.

There are a number of numerical works on Josephson junctions relevant to setups different from the one considered here, which display different current-dependences on the junction length at finite doping. We list a few examples. [35, 36] focus on the dependence of the current on the junction length in the short junction regime. [37] does not deal with an infinite plane geometry, while rather with a strip geometry featuring armchair or zigzag edges. The authors of [38] take into account intervalley tunneling, in contrast to previous papers (including [30, 31]), which can affect the result in presence of edges. On the contrary, in our analysis we consider an infinite 2D plane geometry, and the SC dots are much longer and wider than the lattice constant [39]. Aims to establish a new kind a more general proximity model, able to reveal features of the proximity effect which cannot be captured by the so-called ‘conventional model’ consisting of the Dirac Hamiltonian and an energy-independent pair potential. As well as each of the above mentioned papers about different setups, our paper also does not consider this recent generalization.

For finite temperature, we use equations (4b) and (11a) to obtain the supercurrent decay given by

\[
 I^\mu(\vec{r}) \propto -16 \pi^4 v_F^{-1} (k_B T)^3 \sum_{n=0}^{\infty} \left\{ (2n + 1)^2 + \left( \frac{y}{\pi x} \right)^2 \right\} \\
 \times \left[ |K_0((2n + 1) \pi x - iy)|^2 - |K_1((2n + 1) \pi x - iy)|^2 \right],
\]
where \( x \equiv r k_B T/(\hbar v_F) = r r_T \), and \( y \equiv r \mu_\text{F}/(\hbar v_F) = r r_\mu \).

We plot the characteristics of \( I^\mu(\vec{r}) \) in figure 4. We notice from the plots that the radial dependence of the supercurrent for finite temperature and doping is similar to the one we obtain for zero doping, see figure 2. Far beyond the thermal length (i.e. for \( x \gg 1/4\pi \)), the asymptotic expansion for the modified Bessel function gives the result

\[
I^\mu(\vec{r}) \approx \frac{8 \pi^3 k_B T}{v_F r^2} e^{-2 \pi x}, \quad r \gg r_T/(4\pi). \tag{16}
\]

This expression reduces to the one given in equation (10) in the non-degenerate limit \( \mu \ll \pi k_B T \), and to the expression

\[
I^\mu(\vec{r}) \approx \frac{8 \pi^3 (k_B T)^2}{\mu v_F r^2} e^{-2 \pi x}, \quad r, r_\mu \gg r_T/\pi \tag{17}
\]

in the strongly-degenerate limit \( \mu \gg \pi k_B T \). Visual inspection reveals that the knees of the curves in figure 4 sit about cross-over distances roughly 3 times shorter than the respective \( r_T \)’s.

4. Bilayer graphene

BLG can be modeled as a pair of hexagonal graphene lattices with inequivalent sites A1, B1 and A2, B2 on the bottom and top graphene sheets (see below about the vertical stacking of the sheets). The Hamiltonian can be written as a tight-binding model, starting from the tight-binding model for graphite [40]. In-plane hopping is parameterized by coupling \( \gamma_{A1B1} = \gamma_{A2B2} \equiv \gamma_0 \), corresponding to an in-plane velocity \( v = \sqrt{3}/2 a \gamma_0 / \hbar \), being \( a \) the lattice constant. For the inter-layer coupling between the vertically-aligned orbitals A2, B1 we write \( \gamma_{A1B2} \equiv \gamma_1 \). Let then the weaker inter-layer coupling be \( \gamma_{A2B1} \equiv \gamma_3 \), which corresponds to the effective velocity \( v_3 = \sqrt{3}/2 a \gamma_3 / \hbar \). BLG Hamiltonian \( \mathbf{H} \) [41] near the points \( K, K' \) of the valleys reads, respectively in the basis \( (\psi_{A1}, \psi_{B2}, \psi_{A2}, \psi_{B1}) \) and \( (\psi_{B2}, \psi_{A1}, \psi_{B1}, \psi_{A2}) \):

\[
\mathbf{H} = \begin{pmatrix}
-\frac{1}{2} u & v & 0 & v \\
-v & 0 & v & 0 \\
v & v & 0 & v \\
0 & v & v & 0
\end{pmatrix} + \begin{pmatrix}
\frac{1}{2} u & v \\
v & 0 & v \\
0 & v & 0 \\
\xi - v \\
\end{pmatrix}, \tag{18}
\]

where \( \pi = p_x + ip_y \), \( \pi^\dagger = p_x - ip_y \), \( \vec{p} = (p_x, p_y) \) is the momentum if \( K \) is taken as the origin of the momentum coordinate system, \( \xi = +1, -1 \) for \( K, K' \), and \( u = v_2 - v_1 \) is the on-site energy asymmetry between the two layers [43].

We assume the Bernal stacking (A2–B1) for the two coupled graphene sheets; in this configuration, the B1 site lies directly below an A2 site in the upper layer, while sites A1 and B2 do not match vertically any site in the opposite layer. We limit ourselves to the low-energy approximation in which our model reduces to a \( 2 \times 2 \) matrix formulation. We denote the effective Hamiltonian by \( \hat{H}_2 \), around the two valleys \( K (\xi = +1) \) and \( K' (\xi = -1) \). In addition, we employ the two simplifying approximations:

1. trigonal warping \( v_3 = 0 \),
2. \( v_F k / \gamma_1 \ll 1 \) in the significant domain of integration over momenta.

Under those assumptions we obtain

\[
\hat{H}_2(\vec{k}) = \hat{\Sigma} \cdot \hat{A}_\xi(\vec{k}), \tag{19a}
\]

\[
\hat{A}_\xi(\vec{k}) \equiv \left\{-\frac{k^2}{2m} \cos(2\theta_\xi), -\frac{k^2}{2m} \sin(2\theta_\xi), -\frac{\mu}{2}\right\}, \tag{19b}
\]

where \( m \approx 0.017 \text{meV}\text{c}^{-2} \) denotes the effective particle mass for low-energy bands of BLG [42, 43], and \( u \) the magnitude of the voltage associated to the electric field transverse to the BLG plane. The corresponding Green’s function in momentum space can be written

\[
G_{\text{BLG}}(\vec{k}, i\omega) = \frac{i\omega + \hat{H}_2(\vec{k})}{-\omega^2 - (\hat{H}_2(\vec{k}))^2} = \frac{-i\omega - \hat{\Sigma} \cdot \hat{A}_\xi(\vec{k})}{\omega^2 + |\hat{A}_\xi(\vec{k})|^2}, \tag{20}
\]

from which we derive the real space GF

\[
\tilde{G}_{\text{BLG}}(\vec{r}, i\omega) = \int d^2 k \ e^{i\vec{r}\vec{k}} e^{i\vec{r}\vec{k}} \left( \omega^2 + \left(\frac{k^2}{2m} + \frac{u^2}{4}\right)^{-1} \right) \times \left\{ -i\omega \hat{1} - \hat{\Sigma} \cdot \left\{-\frac{k^2}{2m} \cos(2\theta_\xi), -\frac{k^2}{2m} \sin(2\theta_\xi), -\frac{\mu}{2}\right\} \right\}
\]

\[
= -2\pi \left\{ \left\{ i\omega \hat{1} + \frac{\mu}{2} \hat{\Sigma}_3 \right\} f_1 + (\hat{\Sigma} \cdot \hat{r}) f_2 \right\}, \tag{21}
\]

where

\[
f_1 = \sqrt{\frac{m^2}{u^2 + 4\omega^2}} \mathcal{G}^0_{1 \to 0} \left( -0.5 \right) \left( \frac{1}{256} m^2 r^4 (u^2 + 4\omega^2) \right), \tag{22a}
\]

\[
f_2 = \sqrt{\frac{m^2}{2}} \mathcal{G}^0_{1 \to 0} \left( -0.5 \right) \left( \frac{1}{256} m^2 r^4 (u^2 + 4\omega^2) \right). \tag{22b}
\]

Here, we have used the traditional notation

\[
G_{p,q}^{m,n}(a_1, \ldots, a_p, b_1, \ldots, b_q) \mid z \rangle = G_{p,q}^{m,n}(a_p \mid b_q) \mid z \rangle \tag{23}
\]

for the Meijer function \( G \). Below, we present the details for the components \( f_1 \) and \( f_2 \) of the BLG matrix propagator.

4.1. Zero temperature

The supercurrent depends on the product \( \tilde{G}_{\text{BLG}}(\vec{r}, i\omega) \cdot \tilde{G}_G(\vec{r}, -i\omega) \). By tracing over the singlet state Cooper pairs of the BLG we obtain

\[
\mathbf{T}_{\text{sublattice}} \tilde{G}_{\text{BLG}}(\vec{r}, i\omega) \cdot \tilde{G}_G(\vec{r}, -i\omega) = \epsilon_{\alpha\beta} (\hat{g}_1 \hat{g}^\dagger_\mu \hat{g}_2 \hat{g}^\dagger_\mu + g_2^\dagger \hat{g}_\mu \hat{g}_\mu + g_3^\dagger \hat{g}_\mu \hat{g}_\mu) \times (\hat{g}_1 \hat{g}^\dagger_\mu + g_2^\dagger \hat{g}_\mu \hat{g}_\mu + g_3^\dagger \hat{g}_\mu \hat{g}_\mu) \epsilon_{\mu\nu} \times 2(g_1 \hat{g}_1 + g_2 \hat{g}_2 + g_3 \hat{g}_3). \tag{24}
\]
where \( g_i(\omega) \equiv g_i(-\omega) \), \( i = 1, 2, 3 \),
\[
g_1 \equiv -2\pi i f_1, \quad g_2 \equiv -2\pi f_2, \quad g_3 \equiv -\pi \xi f_1
\]
and \( f_1, f_2 \) are defined in equations (22a) and (22b). As in the SLG, the Tr effectively generates a factor of 2. Inserting this result into the zero temperature current, equation (4a), we obtain:
\[
I(\vec{r})\rvert_{T=0} \propto iTr \int_{-\infty}^{\infty} \frac{d(\omega_0)}{2\pi} G^{\mu=0}_{BLG} (\vec{r}, i\omega_0) \cdot G^{\mu=0}_{BLG} (\vec{r}, -i\omega_0)
= -8\pi^2 m \frac{\mu}{r^2}
\]
in absence of external field (\( \mu = 0 \)). In the following discussion we shall derive this quadratic decay of the supercurrent displayed in the last line of equation (25). This qualitatively different characteristic compared to the cubic decay for SLG is striking and we return to the physical reason for this below.

First, the real space GFs expressed in equation (21), can be rewritten as follows (where \( \tilde{s} \equiv i\vec{k}, \Omega \equiv 2mr^2\omega \ ))
\[
\tilde{G}_{\text{BLG}}(\vec{r}, \omega) = -2im\Omega \begin{bmatrix} \frac{1}{\Omega^2 + s^2 + (\mu m r^2)^2} \\
-2m\vec{s} \cdot \int d^2s \ e^{i\vec{s} \cdot \vec{r}} \frac{1}{\Omega^2 + s^2 + (\mu m r^2)^2} \end{bmatrix} = m [g_1(\Omega, \mu m r^2) I + \tilde{g}_2(\Omega, \mu m r^2) \cdot \vec{\sigma}],
\]
in obvious notation. Inserting this expression into the zero temperature current, equation (4a), we obtain
\[
I(\vec{r}) \propto m^2 \int_{-\infty}^{\infty} d\omega_0 \left[ g_1^2(\Omega, \mu m r^2) + g_2^2(\Omega, \mu m r^2) \right]
= \frac{m}{\pi} \tilde{g}_2(\mu m r^2), \quad (27a)
\]
hence, demonstrating the quadratic decay of \( I(\vec{r}) \) when an external field is applied. Physically this means that the low-energy parabolic dispersion relation of BLG automatically results in a quadratic radial decay of the supercurrent.

### 4.2. Finite temperature

We proceed by investigating the properties of the supercurrent for finite temperature. As we cannot analytically sum over the Matsubara frequencies for arbitrary distance, we approximate the supercurrent between two SC islands by truncating the sum (4b). Using the property of Fermion statistics \( \omega_n = (2n + 1)\pi k_B T \), only odd counting occurs in the infinite double sided sum in (4b), allowing a collapse of the latter into exactly a single sided sum; therefore for intermediate distances the supercurrent can be written as
\[
I(\vec{r}, T) \propto -4k_B T \sum_{n=0}^{\infty} Tr \tilde{G}^{\nu=0}_{BLG}(\vec{r}, \omega_n) \tilde{G}^{\nu=0}_{BLG}(\vec{r}, -\omega_n)
= -16\pi^2 k_B T \sum_{n=0}^{\infty} \left( \frac{\omega_n^2 + \mu^2}{4} \right) f_1^2(\omega_n) + f_2^2(\omega_n).
\]
In figure 5 we plot the radial dependence of \( I(r) \), which is obtained by calculating the series in equation (28) numerically (notations in (22a), (22b) are used). We introduce the notation
\[
\lambda_T^{(\mu)} \equiv \frac{\hbar}{\sqrt{m(u^2 + 4\pi^2 k_B T^2)^{1/2}}},
\]
where \( \lambda_T \equiv \lambda_T^{(\mu)} \rvert_{\mu=0} \) corresponding to the thermal de Broglie electron wavelength. For large distances \( r \gg (2/\pi) \lambda_T^{(\mu)} \), such that the dimensionless argument of the Meijer functions \( z = \frac{1}{2\pi} m^2 r^4 (u^2 + 4\omega^2) \) in equations (22a), (22b) is large for \( \omega = \omega_n = \pi k_B T \), the evaluation of (28) can be traced back to the asymptotic expansion of the Meijer function [44]. This expansion is not automatically workable by commonly used computer algebra systems and, therefore, we adapt the approach discussed in [44], which is briefly presented in appendix.

First, we write the asymptotic behaviors
\[
G_0^3 \left( 0, \frac{3}{4} \right) \left( \frac{z}{\sqrt{2}} \right) \sim c_1 z^{-1/8} \times \text{Re} \left[ e^{-2\sqrt{2}(1+i)z^{1/4} + i\phi} \right],
\]
\[
G_0^3 \left( 0, \frac{3}{4} - \frac{1}{2} \right) \left( \frac{z}{\sqrt{2}} \right) \sim c_2 z^{-1/8} \times \text{Re} \left[ e^{-2\sqrt{2}(1+i)z^{1/4} + i\phi} \right].
\]
are expressed in terms of the dimensionless combination \( f_i^2 \) for large real positive and rescaled by the prefactor Figure 6.

Figure 6. Distance-dependence of the components \( f_1, f_2 \) (plotted in arbitrary units) for the Matsubara (matrix) propagator of BLG for vanishing electric field \((u=0)\). (a) and (b): \( f_1, f_2 \) are expressed in terms of the dimensionless combination \( f_i^2 \equiv r / m (u^2 + 4w^2) / 4 \) and rescaled by the prefactor \( z^{1/2} \), \( z^{1/2} \). (c) \( f_1 \) (left panel) and \( f_2 \) (right panel) are represented at Matsubara frequency \( \hbar \omega_0 = 10 \) meV as functions of the distance \( r \) in units of nm.

for large real positive \( z \), \( \phi_1 \), \( \phi_2 \), constant real phases, and \( c_1 \approx c_2 \approx 2.6 \) (see formulas (47), (51) in appendix). Visual inspection reveals that the knees of the curves in figure 5 are located at distances roughly 6 times shorter than the respective thermal wavelengths \( \lambda_T \). In figures 6(a), (b) we plot a rescaling of the components \( f_1, f_2 \) of the BLG Matsubara propagator, properly chosen in order to offset their asymptotic decays. Such rescaling is clear from equations (30), (31), (22a), (22b). In figure 6(c) the same components \( f_1, f_2 \) are plotted without any exponential rescaling, for a fixed value of the Matsubara frequency \( \hbar \omega_0 = 10 \) meV and restricted to a radial range of the order of few tenths of nm.

At this point, for large distances \( r \gg (2/\pi) \lambda_T^{(0)} \), the supercurrent can be written as

\[
I(\vec{r}, T) \sim k_B T \sum_{n=0}^{\infty} \left[ w_n^2 + w_n^2 \right] f_1^2 (\omega_n) + f_2^2 (\omega_n). \tag{32}
\]

\[
\sim k_B T \left[ w_0^2 + w_0^2 \right] f_1^2 (\omega_0) + f_2^2 (\omega_0). \tag{33}
\]

\[
\sim \frac{k_B T m^2}{\hbar^2} \left[ G_0^3 0 4 \left( \begin{array}{c} 0 \\ 0.4, 1, 1 \end{array} \right) \right]^2 \\
+ \left[ G_0^3 0 \left( \begin{array}{c} 0 \\ 0.4, 1, 1 \end{array} \right) \right]^2. \tag{34}
\]

up to a constant factor, where we also recall that \( \omega_n = (2n + 1) \pi k_B T \). It follows that

\[
I(\vec{r}, T) \approx \frac{8 (c_1^2 + c_2^2) \lambda_T^{(0)} \hbar}{k_B T \lambda_T^{(0)}} e^{-4 \pi r / \lambda_T^{(0)}}. \tag{35}
\]

For the symmetric case \( u = 0 \), in particular:

\[
I(\vec{r}, T)_{u=0} \approx \frac{8 (c_1^2 + c_2^2) \lambda_T^{(0)} \hbar}{k_B T \lambda_T^{(0)}} e^{-4 \pi r / \lambda_T^{(0)}}. \tag{36}
\]

The length \( (2/\pi) \lambda_T^{(0)} \) is the reference distance scale for the validity of the approximations (35) and (36); this length may substantially exceed the SC coherence lengths of typical metals, which is the distance regime of interest in this paper, therefore either the large distance case (35) and the general case (34) may comply with a description of the proximity effect in terms of Cooper pair propagation.

### 5. Comparison with former results

In [31], the supercurrent through graphene is evaluated as the Fourier transform:

\[
I(\vec{r}, T) \sim \int_0^\infty \frac{dk}{2\pi} J_0(\lambda_T) D(\vec{k}, \omega = 0) e^{-k/k_c}, \tag{37}
\]

where \( k = |\vec{k}| \), \( J_0 \) is the zero order Bessel function of the first kind, \( k_c \) is a regularizing short distance cutoff which does not affect the behavior of the critical current in the limit of large \( r \), and the Cooper pair propagator is

\[
D(\vec{k}, \omega = 0) = i \text{Tr} \int \frac{d\omega_q}{2\pi} \frac{d^2 q}{(2\pi)^2} G^{(0)}(\vec{q} + \vec{k}, \omega_q) G^{(-a)}(\vec{q}, \omega_q). \tag{38}
\]

The high-temperature (or long-distance) limit of this propagator is approximated in formula (17) of the same reference, which looks like

\[
D(\vec{k}, \omega = 0) \sim \frac{\log 2}{\pi v_F} k_B T - \frac{1}{16\pi k_B T}, \quad k_B T \gg v_F k. \tag{39}
\]

We draw attention to two points. First, in [31] it is claimed that for very large \( T \) the second term in (39) dictates the long-distance decay of the critical current, while we claim that the first summand is evidently the dominant one, which results in a cubic (rather than quintic) long-distance decay. Second, contrary to the approach adopted in [31], the long-distance behavior of a 2D Fourier transform (here, the supercurrent) of a given function, in general, cannot be determined by Fourier transforming its small-momentum limit expression. A counterexample in support of this statement is given by considering the Fourier transform of the 2D Klein–Gordon
propagator

\[
\int \frac{d^2k}{(2\pi)^2} e^{ikr} \frac{1}{k^2 + (mc)^2} = K_0(mc r / \hbar)^{large r} \times \sqrt{\frac{h}{4mc r}} e^{-mc r / \hbar},
\]

while instead the Fourier transform of its small-momentum approximation \((mc r / \hbar)^2\) is

\[
\int \frac{d^2k}{(2\pi)^2} e^{ikr} \frac{1}{k^2 + (mc)^2} \propto \delta^{(2)}(\vec{r}) = \frac{\delta(r)}{\pi r},
\]

where the two-dimensional Dirac delta function has been rewritten in polar coordinates \([30, 31]\) would not reproduce the asymptotic exponential decay of the \(F\)-transformed full propagator \((40)\):

\[
\int \frac{d^2k}{(2\pi)^2} e^{ikr} r^{-k/r} \frac{1}{m^2} = \frac{k_c}{m^2 \sqrt{1 + k_c^2 r^2}} \frac{large r}{r}. \quad (42)
\]

The counterexample outlined in equations \((40)-(42)\), as well as the mismatch between the exponential \((10)\) and power-law decays claimed in \((31),\) indicate that arguments are based on the specific profile of the electron dispersion relation and not on the physical conditions in the realistic situation. In fact, both examples illustrate how the long distance decay of the supercurrent is not necessarily determined by the small-momentum behavior of the electron Green function.

Physically, we expect an exponential rather than power law decay of the supercurrent when the distance significantly exceeds the thermal length, because the latter represents the characteristic decay length of the electron propagator (compare with formula \((5)\)). For smaller distances \((r \gg r_c)\), the imaginary-time propagator at distance \(r\) takes a significant value for several different Matsubara energies, resulting in particular in a cubic current decay when such energies are extremely close to each other with respect to the energy scale associated with the distance \(r\) (i.e. \(r \ll r_c\)). Conversely, for \(r \gg r_c\) the supercurrent is dominated by the lowest Matsubara energy, allowing the summation over \(n\) to be approximated by only its term \(n = 0\) in \((33)\), so that the associated electron propagator features an exponential distance-decay of the propagator due to the asymptotic decay of the Bessel functions \(K_0, K_1\) with respect to their dimensionless arguments.

6. Conclusions

We have considered the Josephson effect across single layer and BLG in the dilute long junction regime, i.e. when the distance \(r\) between the SC islands is much larger than both their width \(W\) and the SC coherence length \(\xi\) of the islands. Therefore the 2D geometry of SLG/BLG is pivotal to our discussion. In this regime, the supercurrent is well described in terms of Cooper pairs tunneling between the SC islands where the SLG and BLG act as a junction link. The SC proximity effect has been studied as a function of the distance between the SC islands, for different temperatures, chemical potential (doping), and transverse electric fields. In terms of these variables, the behavior of the junction can be categorized into different regimes; in fact temperature, chemical potential and, in the BLG case, the interlayer gate voltage set respective characteristic lengths \(r_{\text{T}}, r_{\text{BLG}}\) for \(h v_{\text{F}} / k_{\text{B}} T, h / \sqrt{\mu m}, h / \sqrt{\mu m}\), such that the Cooper pairs’ tunneling is basically unaffected by each parameter for distances much shorter than the respective characteristic length. The aforementioned regimes for the proximity effect are therefore defined in terms of the relation between the lengths \(r, r_c, r_m\) (and also \(r_u\) for BLG). Later on, we summarize the the most evident signatures of these regimes.

Besides to the intrinsic interest of transport properties for graphene-based materials, the present study is motivated by several potential applications of proximity effect, see section 1. Possible applications include graphene-based Josephson electronics and graphene-superconductor heterostructures. On the other hand, this work can be seen as a theoretical starting point for the analysis of the BKT phase transition for impurities/ graphene composites, where the density of SC grain impurities and other adjustable variables such as temperature, doping, transverse electric field, mechanically-induced (zig-zag and/or armchair) strain \([46]\) in the system are treated as control parameters. Indeed, the BKT critical temperature \(T_{\text{c}}\) is roughly on the order of the Josephson energy, whose characteristic scale \(E_j\) is proportional to the critical supercurrent \(I_c\) (subject of the present investigation); in other words, \(T_{\text{c}} \approx E_j(T_{\text{c}}) \equiv (\Phi_0/2\pi) I_c(T_{\text{c}})\), where \(\Phi_0 \approx h/(2e)\) is the magnetic flux quantum. This rough relationship represents a stylized version of the determining condition for \(T_{\text{c}}\) resulting from a careful evaluation of the many-body Josephson coupling energy.

The vanishing density of states, characterizing graphene near its charge neutrality point, results in a particularly strong radial decay of the supercurrent. In fact, we confirm the previous result of cubic decay \([30]\) predicted for pristine SLG at zero temperature. This characteristic resembles the behavior of the Cooper pair propagation through a disordered normal metal \([47]\). An analogous correspondence, between SLG junctions and disordered metal junctions, also holds for the 1D case in the opposite short junction regime \([28, 32]\). We point out that qualitative similarities between ballistic transport in graphene and diffusive metals also extend to the long-distance regime. In both physical systems the critical current at distances much larger than the thermal length follows an exponential decay, which supersedes the short-distance power-law decay. For undoped SLG we found in \((10)\) an asymptotic decay \(~r^{-2m e^{-2\pi m k_{\text{B}} T}}\), beyond the thermal length \(r_T\) (of the order of \(~h v_{\text{F}} / k_{\text{B}} T\)).

By shifting the Fermi level away from the neutrality point, the supercurrent becomes damped. This is encoded in our formulation by the introduction of a finite chemical potential, corresponding for instance to the effect of a gate-voltage or a doping procedure. For a SLG sheet, the corresponding finite density of states at the Fermi level induces a drop in the supercurrent radial decay, leading to a new regime far beyond the critical length \(h v_{\text{F}} / \mu\). The finite charge density begins to have an influence at about this length scale, with a smooth transition from the \(~1/r^3\) behavior, described in formula...
Here, a comparison between our 2D geometry and the physics of 1D junctions (carbon nanotubes) is in order. In the case of long and narrow junctions the current can be described in terms of 1D propagation of the Cooper pairs, decaying as $\propto 1/L$ ($L$ is the length of the 1D junction) [29, 48]. Additionally, for carbon nanotubes the Coulomb interaction affects the supercurrent behavior as described in [49, 50] because of a strong power-law suppression of the density of states, which turns out to be marginal for graphene [51, 52].

Finally, we considered the supercurrent across BLG (firstly in the absence of inter-layer gate voltage $\phi$), for which we found a quadratic radial decay in (25), $\sim 1/r^2$, at zero temperature. This characteristics reflects the quadratic low-energy dispersion of BLG band structure. For finite temperatures, we retained the asymptotic exponential distance-decay $r^{-1} e^{-4\alpha r/\lambda_{x}}$, beyond the thermal length, similarly as for SLG. The effect of a transverse electric field on the supercurrent, and its radial dependence, was considered in analytical terms. We found in (27a) that $I(r, T = 0)$ is proportional to $r^{-2}$ times a function of $(ur^2)$ only, which generalizes the simpler decay $\sim r^{-2}$ typical of the aforementioned non-gated case $u = 0$. Similarly, at finite temperature and large distance $r \gg (2/\pi) \lambda_{x}^{T}$, $\lambda_{x}^{T} \equiv h/[m l^2/2 (u^2 + 4\pi^2 k_{F}^2 T)^{1/4}]$, the asymptotic distance-decay $r^{-1} e^{-4\alpha r/\lambda_{x}^{T}}$ (see (35)) is found to occur as a generalization of the distance-decay in absence of transverse field. The conclusions outlined in this work do not immediately apply to experimental setups characterized by spatial separation of the islands approaching or comparable to the island dimensions. A discussion of this case requires a careful account of the two-dimensional field configuration of the supercurrent density within the region comprised between the island boundaries, which we intend to carry out in a separate investigation.

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Appendix

Asymptotic supercurrent through BLG

We rewrite (22a), (22b) as

$$ f_1 = \frac{m}{\sqrt{u^2 + 4\omega^2}} \frac{G^3}{4} \left[ \begin{array}{c} - \frac{2}{0.5^2 + 0} \\ \frac{1}{0.5} \end{array} \right] z. $$

$$ f_2 = \frac{m}{2} G^3 \left[ \begin{array}{c} 0 \\ \frac{1}{0.5} \end{array} \right] z. $$

where $z \equiv \frac{1}{256} m r^4 (u^2 + 4\omega^2)$ and the ‘$-$’ symbol in the top row of parameters denotes conventionally the absence of $a_j$ parameters (being $j = 1, \ldots, p$), namely $p = 0$. Since in both $f_1$ and $f_2$ cases there are no $a_j$ parameters ($j = 1, \ldots, p$) in the Meijer functions, the hypotheses (1.1) of [44] are automatically fulfilled:

$$ 0 \leq m \leq q, \quad 0 \leq n \leq p, \quad a_{j} - b_{k} \approx a_{j} = 0, \quad j = 1, \ldots, p; \quad k = 1, \ldots, q, \quad a_{j} - a_{k} \approx a_{j} = 0, \quad j, k = 1, \ldots, p; \quad j \neq k. $$

(44)

We are in the $q > p$ case, and define $\nu \equiv q - p = 4$, $\mu \equiv q - m - n = 1$. Theorem 3 of [44] guarantees that, provided $\nu \geq 1$, and called $(r, s)$ each pair of integer numbers fulfilling

$$ |\arg z + \pi (\mu + 1 - 2r)| < \pi (\nu/2 + 1), $$

$$ |\arg z + \pi (\mu + 2 - 2s - 2h)| < \pi (\nu + \min(1, \nu/2)), $$

$$ h = 1, \ldots, \nu, $$

(45)

if the sector

$$ \pi (\nu - \mu - 2 + \max[2r - \frac{\nu}{2}, 0]) < \arg z $$

$$ \arg z < \pi (\nu/2 - \mu + \min[2r - \frac{\nu}{2}, 0]) $$

(46)

is not empty, then suitable constants $C_j(r, s), D_j(r, s)$ exist, such that:

$$ G^{m,n}_{p,q} = \sum_{j=1}^{p} C_j(r, s) L_j(z)^{\exp((\mu+1-2r))} $$

$$ + \sum_{h=1}^{\nu} D_h(r, s) G(z)^{\exp((\mu+2-2s-2h))}, $$

(47)

this expansion is referred to as the $(r, s)$ expansion for $G^{m,n}_{p,q}(z)$, and $L_j$ and $G$ stand for the functions defined below:

$$ L_j(w) \equiv G^q_p \frac{1}{q} \left[ \begin{array}{c} a_{j_1}, a_{j_2}, \ldots, a_{j_{p-1}}, a_{j_{p-1}+1}, \ldots, a_{j_{p-1}+1} \\ b_{j_1}, \ldots b_{j_{p-1}} \end{array} \right] w $$

$$ \sim \frac{\Gamma(1 + b_{j} - a_{j})}{\Gamma(1 + a_{p} - a_{j})} w^{-1 + a_{j}} $$

$$ \times q + 1 \Gamma_p \left[ 1 + \frac{1}{1 + a_{p} - a_{j}; \frac{1}{w}} \right] $$

$$ j = 1, \ldots, p, \quad w \to \infty, \quad \arg w < \pi (\nu/2 + 1). $$

(48)
where the following notations are intended
\[
\Gamma_n(c_p - t) \equiv \prod_{k=n+1}^p \Gamma(c_k - t), \quad \Gamma(c_M - t) = \prod_{k=M}^n \Gamma(c_k - t)
\]
\[
\equiv \Gamma_0(c_M - t), \quad \prod_{k=0}^q F_n^{(a_p)} \left[ z \right] \equiv \sum_{k=\infty}^{\infty} \Gamma(a_p + k) \Gamma(b_q + k) \frac{z^k}{k!}
\]
(49)

(upper case symbols \(P, Q, M\) are not parameters’ indexes in the same way of \(p, q, M\), but refer rather to ‘vectors of parameters’ indexes’, i.e. shorthands which turn out to be convenient in equations (48) and (49));
\[
G(w) \equiv \sum_{j=0}^{q} \lambda_{a_p} \cdots \lambda_{a_p} w \sim \left( \frac{2\pi)^{\nu-1}}{\nu} \right)^{1/2} \times e^{-\nu w^2} \sum_{j=0}^{\infty} K_j w^{\nu - 2j}, w \to \infty, \quad \text{arg } w < \pi(\nu + \min(1, \nu/2)),
\]
(50)

where
\[
\nu = \frac{1 - \nu}{2} + B_1 - A_1, \quad K_0 = 1,
\]
\[
K_j = A_2 - B_2 + \frac{B_1 - A_1}{2
\nu} \left[ \nu(A_1 + B_1) + A_1 - B_1 \right] + \frac{1 - \nu^2}{24}\nu,
\]
\[
\prod_{j=1}^{p} (x + a_j) = \sum_{j=0}^{p} A_j x^{p-j}, \quad \prod_{j=1}^{q} (x + b_j) = \sum_{j=0}^{q} B_j x^{q-j}
\]
(51)
and the remaining \(K_j\) are polynomials in \(A_j, B_j\) independent of \(w\). We do not provide expressions for the subsequent \(K_j, j = 2, 3, \ldots, \infty\), but stress that \(G(w)\) is dominated at large distances by the terms corresponding to \(K_0\) and \(K_1\) in view of the asymptotic expansion in \(w\) presented in (50). Accordingly, coefficients \(K_j, j = 2, 3, \ldots, \infty\) may be neglected in the description of the large-distance behavior of our system. Both Meijer functions featured in (43) fulfill the constraints referred to in (44) and (46) and required for the application of theorem 3 of [44], therefore the expansion (47) holds.

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