Cosmic ray diffusive acceleration at shock waves with finite upstream and downstream escape boundaries

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Abstract

In the present paper we discuss the modifications introduced into the first-order Fermi shock acceleration process due to a finite extent of diffusive regions near the shock or due to boundary conditions leading to an increased particle escape upstream and/or downstream the shock. In the considered simple example of the planar shock wave we idealize the escape phenomenon by imposing a particle escape boundary at some distance from the shock. Presence of such a boundary (or boundaries) leads to coupled steepening of the accelerated particle spectrum and decreasing of the acceleration time scale. It allows for a semi-quantitative evaluation and, in some specific cases, also for modelling of the observed steep particle spectra as a result of the first-order Fermi shock acceleration. We also note that the particles close to the upper energy cut-off are younger than the estimate based on the respective acceleration time scale. In Appendix A we present a new time-dependent solution for infinite diffusive regions near the shock allowing for different constant diffusion coefficients upstream and downstream the shock.

Keywords: cosmic rays – Fermi acceleration – acceleration time scale – shock waves

1 Introduction

In the test particle approximation, the first-order Fermi shock acceleration with infinitely extended diffusive regions upstream and downstream the shock leads
to a power-law particle spectrum with the spectral index

$$\alpha = \frac{3R}{R-1},$$

and the acceleration time scale

$$T_{acc} = \frac{3}{U_1 - U_2} \left( \frac{\kappa_1}{U_1} + \frac{\kappa_2}{U_2} \right),$$

where the index "1" ("2") indicates respectively the upstream (downstream) quantity, $U$ is the shock velocity, $R \equiv U_1/U_2$, $\kappa$ is the spatial diffusion coefficient. For a review of the results referring to the diffusive acceleration mechanism one should consult some of the numerous review papers (e.g. Drury 1983; Blandford & Eichler 1987; Berezhko et al. 1988; Jones & Ellison 1991).

In discussions of the acceleration process, the background conditions are usually considered with particle diffusion coefficients changing at most moderately with the distance from the shock. As a result, there are infinite diffusive regions for cosmic ray particles. Then, particles are removed from the acceleration region near the shock only due to advection with the general plasma flow far downstream. However, if the waves responsible for particle scattering are created due to the process of cosmic ray streaming instability upstream the shock, e.g., the finite amplitude waves resonant with particles near the spectrum cut-off energy could be created only close to the shock within a finite time available for the acceleration process. The analogous situation is the case if the shock propagates through the finite volume of turbulent plasma. Then, the energetic particles diffusing far from the shock will encounter the conditions enabling them to permanently escape from the shock. A finite extent of the shock wave to the sides and some particular boundary conditions may also allow for such escape. The situation can be qualitatively modelled by introducing the upstream (Berezhko et al. 1988) and/or downstream boundary for the energetic particle escape.

The conditions mentioned above are sometimes discovered in analysing in situ measurements near heliospheric shock waves. Then, an attempt to analyse the data within the standard approach, involving equations (1.1-2) to describe the first-order Fermi acceleration, may fail. For example, basing on such an analysis Bialk & Dröge (1993) rejected the possibility of the first-order acceleration at the considered shock wave and suggested the second-order acceleration downstream the shock to play a role, instead. Unfortunately, to date there is no theory available describing effects of the enhanced particle escape at the acceleration process and discussions of the 'difficult' cases have to be based on numerical methods. As a step forward, in the present paper we develop a simple analytic theory allowing for evaluation of measurements in terms of the diffusive length scales for energetic particles. It can become a starting point for more elaborate computations or numerical modelling. The theory describes modifications introduced into the acceleration process due to finite extent of diffusive
regions near the shock (Figure 1). Below, in Section 2 we discuss a simple time-dependent solution of the diffusion equation for cosmic ray particles at the shock with the upstream and/or the downstream escape boundary. To obtain analytic solutions, we consider a simple case with constant diffusion coefficients, leading in the stationary situation to power-law cosmic ray spectra. It enables us to discuss the relation between the spectrum inclination and other parameters of the acceleration process. In Section 3, we derive the particle spectral index and the acceleration time scale as a function of the boundary distance from the shock. The introduced particle sinks at escape boundaries lead to steepening of the spectrum accompanied by a substantial – a factor of two or three for reasonable spectral indices – decrease of the acceleration time scale. We discuss the dependence of these quantities on the upstream and downstream boundary distance from the shock. Finally, in Section 4, we shortly summarize the results. The presented theory allows one to interpret the above mentioned Bialk & Dröge (1993) data in terms of the first-order acceleration. We also note that the particles close to the upper spectrum energy cut-off are younger than the estimate based on the respective acceleration time scale valid at infinite times. The new time-dependent solution for the particle distribution function in the case of infinite boundary distance is presented in Appendix A. It is a generalization of the Toptygin (1980) solution for the case of different constant diffusion coefficients upstream and downstream the shock.
2 On solution of the time-dependent problem

The problem of time-dependent solution of the diffusion equation for cosmic ray particles accelerated at the shock has been discussed by Drury (1983, 1991). Below, we will follow the approach described in the former of the mentioned papers. In order to produce power-law particle momentum spectra with the finite extent of free escape boundaries one must restrict considerations to the momentum-independent diffusion coefficient. As in such conditions the spatial dependence of $\kappa$ does not lead to any qualitative changes of the acceleration process, we consider the most simple situation with the diffusion coefficient spatially constant, separately upstream and downstream. As mentioned previously, we consider the particle acceleration process in the test-particle approximation at the non-modified shock structure, i.e. with flow velocities constant outside the shock.

Let us consider a planar shock wave propagating along the $x$-axis of the reference frame, with the velocity in the upstream plasma frame directed toward decreasing $x$ values. In the text below we will refer to the spatial co-ordinates in the shock rest frame, where the co-ordinate $x$ upstream the shock is negative. We are going to solve the diffusion equation for the isotropic part of the cosmic ray distribution $f = f(t, x, p)$:

$$\frac{\partial f}{\partial t} + U \frac{\partial f}{\partial x} - \kappa \frac{\partial^2 f}{\partial x^2} = Q,$$

where $i = 1$ or 2 and the mono-energetic source function is taken as

$$Q(t, x, p) = Q_0 \delta(x) \delta(p - p_0)H(t),$$

where $Q_0$ is a constant, $\delta$ is the Dirac delta function and $H$ is the Heaviside step function. We look for the solutions satisfying the upstream ($x = -L_1$) and downstream ($x = L_2$) free escape boundary conditions

$$f(t, -L_1, p) = 0 = f(t, L_2, p),$$

and the matching conditions at the shock ($x = 0$):

$$[f] = 0,$$

$$\left[ \kappa \frac{\partial f}{\partial x} + \frac{U}{3} p \frac{\partial f}{\partial p} \right] = -Q_0 \delta(p - p_0)H(t),$$

where the square brackets denote the differences between the upstream and downstream values of the given quantity across the shock. Following a standard approach described by Drury (1983) we start with the Laplace transform with respect to time of the distribution $f$. 


\[ g(s, x, p) = \int_0^{\infty} e^{-st} f(t, x, p) dt \quad , \] (2.6)

leading to the following form of Equation (2.1) at \( p > p_0 \):

\[ sg + U_i \frac{\partial g}{\partial x} - \kappa_i \frac{\partial^2 g}{\partial x^2} = 0 \quad . \] (2.7)

The solutions to this linear equation have the form \( g(x) = C_+ \exp(\beta_{i,+} x) + C_- \exp(\beta_{i,-} x) \) \( (i = 1, 2; C_{\pm} = \text{const}) \). With the definition \( \tau_i \equiv 4\kappa_i / U_i^2 \), the exponents \( \beta_i \) are given as

\[ \beta_{i,\pm} = \frac{2}{\tau_i U_i} (1 \mp \sqrt{1 + \tau_i s}) \quad . \] (2.8)

Boundary conditions (2.3) for the function \( g(s, x, p) \) are \( g(s, -L_1, p) = 0 = g(s, L_2, p) \) and the upstream and downstream solutions of Equation (2.7) can be written as

\[ g_1(s, x, p) = C_1(s, p) \left[ e^{\beta_{1,-} x} - e^{-\frac{4\sqrt{1 + \tau_1 s}}{\tau_1} L_1 e^{\beta_{1,+} x}} \right] \quad , \] (2.9)

\[ g_2(s, x, p) = C_2(s, p) \left[ e^{\beta_{2,+} x} - e^{-\frac{4\sqrt{1 + \tau_2 s}}{\tau_2} L_2 e^{\beta_{2,-} x}} \right] \quad . \] (2.10)

One may note that, contrary to the solutions with no escape boundary, the present ones make use of both the indices \( \beta_+ \) and \( \beta_- \). The matching conditions at the shock for the function \( g(s, x, p) \) are derived from Equations (2.4,5) as

\[ [g] = 0 \quad , \] (2.11)

\[ \left[ \kappa \frac{\partial g}{\partial x} + \frac{U_i}{3} \frac{\partial g}{\partial p} \right] = -\frac{1}{s} Q_0 \delta(p - p_0) \quad . \] (2.12)

By imposing the condition (2.11) at the solutions (2.9,10) we obtain

\[ g_1(s, x, p) = g_0(s, p) \frac{e^{\beta_{1,-} x} - e^{-\frac{4\sqrt{1 + \tau_1 s}}{\tau_1} L_1 e^{\beta_{1,+} x}}}{1 - e^{-\frac{4\sqrt{1 + \tau_1 s}}{\tau_1} L_1}} \quad , \] (2.13)

\[ g_2(s, x, p) = g_0(s, p) \frac{e^{\beta_{2,+} x} - e^{-\frac{4\sqrt{1 + \tau_2 s}}{\tau_2} L_2 e^{\beta_{2,-} x}}}{1 - e^{-\frac{4\sqrt{1 + \tau_2 s}}{\tau_2} L_2}} \quad , \] (2.14)

where \( g_0(s, p) \equiv g(s, 0, p) \) is the Laplace transform of the distribution function at the shock. Its functional form can be derived with the use of the condition (2.12) and Equation (2.8) as
\[ g_0(s, p) = \frac{3Q_0}{(U_1 - U_2) s} \left( \frac{p_0}{p} \right)^{\alpha(s)} H(p - p_0) \quad , \quad (2.15) \]

where

\[ \alpha(s) = \frac{3R}{2(R - 1)} \left\{ 1 + \sqrt{1 + \tau_1 s} \coth \left( \frac{2L_1 \sqrt{1 + \tau_1 s}}{\tau_1 U_1} \right) + \frac{1}{R} \left[ \sqrt{1 + \tau_2 s} \coth \left( \frac{2L_2 \sqrt{1 + \tau_2 s}}{\tau_2 U_2} \right) - 1 \right] \right\} . \quad (2.16) \]

The distribution \( f(t, x, p) \) can be formally derived by inverting the respective upstream and downstream transforms (2.13) or (2.14):

\[ f_j(t, x, p) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} e^{st} g_j(s, x, p) \, ds \quad (j = 1, 2) \quad , \quad (2.17) \]

where the path of integration lies to the right of all singularities of the integrand. The result of such inversion for a particular case of \( L_1 = L_2 = \infty \), and for constant diffusion coefficients which can be chosen independently upstream and downstream and not necessarily be equal, \( \kappa_1 \neq \kappa_2 \), we present in Appendix A. However, even without actually performing the integration one is able to extract from Equations (2.13-15) quite a lot of information (Drury 1983). The asymptotic behaviour at large times is obtained by looking just at the contribution of the rightmost singularity of the integrand, here a simple pole at \( s = 0 \).

It gives the power-law steady state spectrum at the shock, \( f(\infty, 0, p) \equiv f_0(p) \), with the spectral index \( \alpha_0 \equiv \alpha(0) \):

\[ \alpha_0 = \frac{3R}{R - 1} \left\{ \frac{1}{1 - e^{-\frac{U_1 L_1}{\kappa_1}}} + \frac{1}{R} \frac{e^{-\frac{U_2 L_2}{\kappa_2}}}{1 - e^{-\frac{U_2 L_2}{\kappa_2}}} \right\} . \quad (2.18) \]

The first term in the product at the right-hand side represents the spectral index \( \alpha \) for the shock with infinite diffusive regions (cf. Equation 1.1), and the two parts of the second term describe modifications due to particle escape through the upstream and downstream boundaries.

With the use of Equations (2.15,16,18), in the solution (2.17) taken at the shock \( (x = 0) \) one can separate a part representing the limiting stationary solution \( f_0(p) \) multiplied by a factor describing the time dependence of the full solution:

\[ f_0(t, p) = f_0(p) \cdot \int_{0}^{t} \psi(t') dt' \quad , \quad (2.19) \]

where, with the notation \( \Delta \alpha(s) \equiv \alpha(s) - \alpha_0 \),
\[ \psi(t) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} e^{st} \left( \frac{p_0}{p} \right)^{\Delta\alpha(s)} \, ds . \] (2.20)

One can readily show that the integral
\[ \int_0^\infty \psi(t) \, dt = 1 . \] (2.21)

If we denote
\[ h(s) = \exp \left[ \Delta\alpha(s) \ln \left( \frac{p}{p_0} \right) \right] \] (2.22)
the mean acceleration time from \( p_0 \) to \( p \) is given as
\[ T_{\text{acc}}(p, p_0) \equiv \int_0^\infty t \psi(t) \, dt = \frac{dh}{ds}(s = 0) . \] (2.23)

With Equations (2.16, 18.22) and assuming \( \ln(p/p_0) = 1 \) the above formula yields the acceleration time scale
\[ T_{\text{acc}} = \frac{3R}{R - 1} \left\{ \frac{\kappa_1}{U_1^2} \coth \left( \frac{U_1 L_1}{2\kappa_1} \right) - \frac{L_1}{2U_1 \sinh^2 \left( \frac{U_1 L_1}{2\kappa_1} \right)} + \frac{\kappa_2}{RU_2^2} \coth \left( \frac{U_2 L_2}{2\kappa_2} \right) - \frac{L_2}{2RU_2 \sinh^2 \left( \frac{U_2 L_2}{2\kappa_2} \right)} \right\} . \] (2.24)

For \( L_i \to \infty \) (i = 1, 2) the expressions (2.18,24) reduce to the standard formulae (1.1,2).

### 3 The acceleration process with free escape boundaries

Let us consider variations of the particle acceleration time scale (2.24) and the spectral index (2.18) with the escape boundary distance. For the discussion, we choose the simplest conditions with \( \kappa_1 = \kappa_2 \) and either \( L_1 = L_2 = L \), or one \( L_i = \infty \), but the generalization to more general conditions is a straightforward one. Below, a boundary distance is expressed in the units of the respective diffusive length scale, \( L_{\text{diff},i} = \kappa_i/U_i \) (i = 1, 2).

In Figure 2, we present the acceleration time scale variations due to changing the escape boundary distance. A substantial decrease of this time scale at small \( L \) should be noted. The reason for that is clear. The particles which could diffuse for a long time in the case with escape boundaries in infinity, for finite \( L \) will probably disappear from the acceleration process as their chance to wander far
Figure 2: The dependence of the characteristic acceleration time, $T_{\text{acc}}$, normalized to the acceleration time scale at the shock with infinite diffusive regions, $T_{\text{acc},0}$, versus the distance $L$ of the free escape boundary (given in units of the respective diffusive scale $\kappa_i/U_i$, $i = 1$ or 2). Full lines show the relations for the case of equally distant upstream and downstream escape boundary ($L_1 = L_2 = L$) with different compressions: $R = 4$ (lower line), $R = 3$ (intermediate line) and $R = 2$ (upper line). The lines with long dashes represent the above relation for downstream particle escape ($L_1 = \infty$, $L_2 = L$) and those with short dashes for upstream particle escape ($L_1 \equiv L$, $L_2 = \infty$). For picture clarity, in the last two cases we give curves for our limiting compressions, the lower curve for $R = 4$ and the upper curve for $R = 2$. 
Figure 3: The dependence of the stationary particle spectral index $\alpha_0$ versus the distance $L$ of the free escape boundary (given in units of the respective diffusive scale $\kappa_i/U_i$, $i = 1$ or 2). The meaning of different curves is the same as at Figure 2. Here, the shock compression values are given near the respective curves.

Figure 4: The dependence of the characteristic acceleration time ratio $T_{acc}/T_{acc,0}$ (cf. Figure 2) versus the spectral index $\alpha_0$ (cf. Figure 3). The meaning of different curves is the same as at Figure 2, the shock compression values are given near the respective curves.
off the shock is higher. So, with small $L$, the spectrum is set up only by particles diffusing very close to the shock with a short mean time between the successive shock crossings, resulting in the short acceleration time. The dashed lines show how the acceleration time is affected by placing the escape boundary upstream only or downstream only. Here and in the figures below, one may note an apparent asymmetric influence of the upstream and the downstream boundary. The difference results from the fact that the mean time between successive particle interactions with the shock is related to the diffusive scale different on both sides of the shock. In Figure 3, variations of the stationary spectral index $\alpha_0$ are presented. The expected steepening of the spectrum with decreasing $L$ due to the accompanied increased particle escape is presented. For larger $L$, the curves quickly converge to the limiting values $3R/(R-1)$. Imposing the escape boundary downstream only has a much smaller influence on the spectral index than the upstream boundary at the same distance (in our units of $L_{\text{diff}}$) from the shock. This asymmetry can be easily understood by comparing the mean times for particles at the shock to reach the escape boundaries. For small $L_1$ ($i = 1 \text{ or } 2$), when the particle diffusive streaming dominates over advection, this time scale can be estimated as $t_i \approx L_i^2/\kappa_i$. For the assumed equal values for upstream and downstream diffusion coefficients the respective diffusive distances relate as $L_{\text{diff},2} = R L_{\text{diff},1}$ and $t_2 = R^2 t_1$. So, the escape through the upstream boundary will be a factor $t_2/t_1 = R^2$ greater than the downstream escape, and thus it will be mostly responsible for increasing the spectrum inclination. For the same reason the downstream boundary will influence the acceleration time scale to the higher degree. For larger $L_i$ particle advection with the general plasma flow modify these estimates, but our conclusions are still true. Always the boundary allowing for quick particle escape preferentially acts to increase the spectrum inclination, while the other one will provides the lower limit for the acceleration time scale.

A comparison of the spectrum and the acceleration time scale at various $L$ is presented at Figure 4. Along the curves, the parameter $L$ decreases to the right, i.e. with increasing $\alpha$. One may note that steepest parts of the curves are to the left, near the limiting spectral index. For all the considered shock compression values it is possible to decrease $T_{\text{acc}}$ by about two times (more for larger $R$, or if the particle escape boundary occurs only downstream) when the spectral index steepens on $\Delta \alpha \approx 1.0$.

4 Discussion

The presented results allow one to model a wider range of cosmic ray spectra as the output of the first-order Fermi acceleration at shock waves, including distributions with a spectral index steeper than the value given by Equation (1.1). One should note that accelerating particles with a steep spectrum due to finite escape boundaries allows to obtain spectra extended to somewhat higher ener-
gies, due to decrease of the acceleration time scale with respect to the one given in Equation (1.2). Another point to be mentioned here is an apparent asymmetry of the acceleration process with respect to imposing the escape boundary upstream or downstream the shock. We would like to note, however, that the respective influence of these boundaries discussed in the previous section can change if the downstream particle diffusion coefficient is much smaller than the upstream one. In general, the boundary allowing for quick particle escape will always preferentially act to increase the particle spectrum inclination, while the other one will limit in a higher degree the acceleration time scale. Let us also note the fact that a downstream boundary further away than approximately two diffusive length scales (cf. Figures 3) has almost no influence on the particle spectral index at the shock. Therefore physical processes and/or conditions behind that distance are not expected to modify the particle distribution at the shock.

The realistic conditions near astrophysical shocks are expected to involve the diffusion coefficients depending on particle momentum. The same will hold for the particle escape probability, defined by the boundary distance in our model. Therefore, the present results can be used only to a rather general evaluation of the acceleration conditions and compatibility of the observed (possibly non-power-law) spectra and time scales with the shock dynamics. A detailed modelling requires numerical methods and not-frequently available information about the local physical conditions and the boundary conditions.

Another fact of interest for modelling the generation of highest energy particles near the spectrum cut-off should be mentioned in this place. The first such particles to appear at the shock are those which have not spent too much time diffusing far from the shock, in analogy to the case with escape boundaries. Therefore, at a given energy, the time for these particles to appear can be substantially shorter than the respective acceleration time scale given in Equation (1.2) or (2.24), the one valid for the steady state particle spectrum. This fact is visible in analytic time-dependent solutions (e.g. Drury 1991; our solution in Appendix A).

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Appendix A: Inversion of Equation (2.17) for $L_1 = L_2 = \infty$

In the case of no escape boundaries $L_1 = L_2 = \infty$ we obtain from Equations (2.13-16)

$$g_1(s, x, p) = G_1(s)G_2(s) \quad , \quad g_2(s, x, p) = G_3(s)G_4(s) \quad , \quad (A1)$$

with

$$G_1(s) = \frac{3Q_0}{U_2(R-1)} \exp\left(\frac{2x}{\tau_1 U_1}\right) \left(\frac{p}{p_0}\right)^{-3/2} H(p - p_0) \frac{\exp(A_1 \sqrt{1 + \tau_1 s})}{s} \quad , \quad (A2)$$

$$G_2(s) = \exp(-A_2 \sqrt{1 + \tau_2 s}) \quad , \quad (A3)$$

$$G_3(s) = \frac{3Q_0}{U_2(R-1)} \exp\left(\frac{2x}{\tau_2 U_2}\right) \left(\frac{p}{p_0}\right)^{-3/2} H(p - p_0) \frac{\exp(A_3 \sqrt{1 + \tau_2 s})}{s} \quad , \quad (A4)$$

$$G_4(s) = \exp(-A_4 \sqrt{1 + \tau_1 s}) \quad , \quad (A5)$$

where

$$A_1 = \frac{3R}{2(R-1)} \ln\frac{p}{p_0} - \frac{2x}{\tau_1 U_1} \quad , \quad (A6)$$

$$A_2 = \frac{3}{2(R-1)} \ln\frac{p}{p_0} \quad , \quad (A7)$$

$$A_3 = \frac{3}{2(R-1)} \ln\frac{p}{p_0} + \frac{2x}{\tau_2 U_2} \quad , \quad (A8)$$

$$A_4 = \frac{3R}{2(R-1)} \ln\frac{p}{p_0} \quad . \quad (A7)$$

The upstream and downstream distribution functions then are

$$f_1(t, x, p) = \int_0^t du F_2(u) F_1(t - u) \quad (A9a)$$

and
\[ f_2(t, x, p) = \int_0^t du F_4(u) F_3(t - u) \text{ ,} \quad (A9b) \]

with the respective Laplace inverse transforms

\[
F_1(t) = \mathcal{L}^{-1}\left( G_1 \right) = \frac{3Q_0}{2U_2(R-1)} \left( \frac{p}{p_0} \right) \frac{3R}{4(R-1)} H(p - p_0) \\
\left\{ \text{erfc}\left( \frac{3R}{4(R-1)} \ln(p/p_0) - \frac{x}{\tau_1 U_1} \right) \sqrt{\tau_1/t} + \sqrt{t/\tau_1} \right\} + \left( \frac{p}{p_0} \right) \frac{3R}{4(R-1)} \exp\left( \frac{x}{\tau_1 U_1} \right) \text{erfc}\left( \frac{3R}{4(R-1)} \ln(p/p_0) - \frac{x}{\tau_1 U_1} \right) \sqrt{\tau_1/t} \\
- \sqrt{t/\tau_1} \right\} \text{ ,} \quad (A10a) \\
F_2(t) = \mathcal{L}^{-1}\left( G_2 \right) = \\
\frac{3\sqrt{\tau_2} \ln(p/p_0)}{4\sqrt{\pi}(R-1)t^{3/2}} \exp\left( -\frac{t}{\tau_2} - \frac{9\tau_2 (\ln(p/p_0))^2}{16(R-1)^2} \right) \text{ ,} \quad (A10b) \\
F_3(t) = \mathcal{L}^{-1}\left( G_3 \right) = \frac{3Q_0}{2U_2(R-1)} \left( \frac{p}{p_0} \right) \frac{3R}{4(R-1)} H(p - p_0) \\
\left\{ \left( \frac{p}{p_0} \right) \frac{3R}{4(R-1)} \exp\left( \frac{3R}{4(R-1)} \ln(p/p_0) + \frac{x}{\tau_2 U_2} \right) \sqrt{\tau_2/t} + \sqrt{t/\tau_2} \right\} + \exp\left( \frac{x}{\tau_2 U_2} \right) \text{erfc}\left( \frac{3R}{4(R-1)} \ln(p/p_0) + \frac{x}{\tau_2 U_2} \right) \sqrt{\tau_2/t} + \sqrt{t/\tau_2} \right\} \text{ ,} \quad (A10c) \\
F_4(t) = \mathcal{L}^{-1}\left( G_4 \right) = \\
\frac{3\sqrt{\tau_4 R} \ln(p/p_0)}{4\sqrt{\pi}(R-1)t^{3/2}} \exp\left( -\frac{t}{\tau_1} - \frac{9\tau_1 R^2 (\ln(p/p_0))^2}{16(R-1)^2} \right) \text{ .} \quad (A10d)
\]

In deriving the four Laplace inversions (A10) we use the tables of Oberhettinger and Badii (1973). Equations (A9) and (A10) represent the full solutions.

At the shock \((x = 0)\) we obtain
After infinite time Equation (A11) approaches the steady-state solution

\[
f_0(p, t \to \infty) = \frac{9Q_0 \sqrt{\tau_2} \ln(p/p_0)}{8\sqrt{\pi U_2(R-1)^2}} \left( \frac{p}{p_0} \right)^{\frac{3}{2(R-1)}} - \frac{3}{2(R-1)}
\]

\[
H(p - p_0) \int_0^t du u^{-3/2} \exp \left( \frac{u}{\tau_2} - \frac{9\tau_2(\ln(p/p_0))^2}{16(R-1)^2u} \right)
\]

\[
\left\{ \text{erfc} \left( \frac{3}{4(R-1)} \ln(p/p_0) \right) \sqrt{\tau_1/(t - u)} + (p/p_0)^{-\frac{3}{2(R-1)}} \text{erfc} \left( \frac{3R}{4(R-1)} \ln(p/p_0) \right) \sqrt{\tau_1/(t - u)} - \sqrt{(t - u)/\tau_1} \right\}
\]

\[
= \frac{3Q_0 H(p - p_0)}{U_2(R - 1)} \left( \frac{p}{p_0} \right)^{-\frac{3R}{2(R-1)}}
\]

where we used the integral

\[
\int_0^\infty dx x^{-3/2} \exp \left( -x - \frac{z^2}{4x} \right) = 2^{3/2} K_{1/2}(z) z^{-1/2} = 2\sqrt{\pi} e^{-z^2}/z
\]

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\[
\begin{array}{ccc}
\text{u.f.e.b.} & \text{shock} & \text{d.f.e.b.} \\
\uparrow & \uparrow & \uparrow \\
U_1 & U_2 & \\
\text{x} = -L_1 & \text{x} = 0 & \text{x} = L_2 \\
\end{array}
\]
