The Chess Board Independent Domatic Number Of Queen Graph

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Abstract: The independent domatic queen number of a graph \(Q_n\) is the maximum number of pairwise disjoint minimum independent queen dominating sets of \(P_n\) and it is denoted by \(i_d(Q_n)\) while maximum independent domatic queen number is denoted by \(I_d(Q_n)\). We discuss about in this paper the independent (maximum independent) domatic number of queen graph \(Q_n\) on \(n \times n\) chess board.

Keywords: Independent domatic queen number, maximum Independent domatic queen number, Queen number, Independent queen number

INTRODUCTION

Definition 1.1 [1,4]: A graph \(P_n\) is obtained by the chess piece \(P\) moving on an \(n \times n\) chess board and taking all the \(n^2\) squares of the board as vertices and two vertices are adjacent if the piece \(P\) placing on the one of the squares is able to move directly to the other. For example the Rook’s graph \(R\) on an \(n \times n\) chess board and squares are adjacent if they are on the same line in which placing on an one of the square (row and column)[1].

Definition 1.2 [1,4]: The domination number \(\gamma(G)\) of a graph \(G = (V, E)\) is the smallest cardinality of a subset \(D\) of \(V\) such that each vertex of \(V - D\) is adjacent to at least one vertex of \(D\). Obviously \(\gamma(G) \leq i(G)\) for any graph \(G\). The determination of \(\gamma(Q_n)\), which is the minimum number of queens required to cover the entire \(n \times n\) chess board and it is called by queen number.

Definition 1.3[1,4]: The minimum independent queen number \(i(Q_n)\) of \(Q_n\), the minimum number of queens which covers all the squares \(n^2\) of chess board, with the additional condition that no two queens attack each other. The maximum number of maximal independent set is denoted by \(\beta(Q_n)\) while minimum number of maximal independent set is denoted by \(i(Q_n)\).

Definition 1.4[1,3]: A domatic partition of \(G\) is a partition of \(V(G)\) into classes that are pairwise disjoint sets. The domatic number of \(G\) is the maximum cardinality of a domatic partition of \(G\) and it is denoted by \(d(G)\). The domatic number was introduced by Cockayne and Hedetniemi[4].

Theorem 1.1 (Welch)[2]: Let \(n = 3q + r\), where \(0 \leq r < 3\). Then
\[\gamma(Q_n) \leq 2q + r.\]
THE CHESS BOARD INDEPENDENCE DOMATIC NUMBER OF QUEEN GRAPH

Theorem 2.1: Let \( n = 3q + r \), where \( 0 \leq r \leq 2 \). Then \( i(Q_n) \leq 2q + r \).

Proof: Case(i) Suppose \( n = 3q \). The \( n \times n \) board is divided into 9 \( q \times q \) sub-boards (numbered 1 through 9 in figure 1). Queens are placed by each column (rows) has \( q - 1 \) queens as shown in figure 1. It is easily seen that these \( 2q \) or \( 3q - 3 \) queens cover the entire \( n \times n \) board.

Case(ii) Suppose \( n = 3q + r \), where \( r = 1 \) or 2, then consider the configuration of figure 1 augmented with \( r \) extra columns (rows) added on the right (bottom) and place extra queen(s) at position(s) \( \{(3q + i, 3q + i) \mid i = 1, 2\} \). This covering by \( 2q + r \) queens completes the proof.

Figure 1. Example Show \( i(Q_{3q}) = 2q \).

Definition 2.2: The independent domatic queen number of a graph \( Q_n \) is the maximum number of pairwise disjoint minimum independent queen dominating sets of \( P_n \) and it is donoted by \( \text{id}(Q_n) \) while maximum independent domatic queen number is denoted by \( \text{Id}(Q_n) \).

Theorem 2.3: If the maximum independent queen number of \( Q_n \) is \( n \) (except containing the middle square), then the number of disjoint maximum independent queen set is only 2.

Proof: Let \( S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \) be a maximum independent queen set. Suppose that the \( n \times n \) chess board rotate 180° and again naming all squares and repointing the elements of \( S \). Cosider the set \( S' \) whose elements are repointed of the elements of \( S \). Therefore, \( S \) and \( S' \) are disjoint. Clealy no sets other than \( S \) and \( S' \) are maximum independent queen set with cardinality \( n \).

Note 2.4: The maximum independent queen number of \( Q_n \) is need not be equal to \( n \).

For figure 2, \( 6 \times 6 \) chess board \( \beta (Q_6) = 5 \).
Example 2.5: In figure 3, $S = \{a_5, b_3, c_1, d_4, e_2\}$ is a maximum independent dominating set, by the theorem 2.4, there exists $S' = \{a_4, b_2, c_5, d_3, e_1\}$. The maximum independent domatic number of $Q_5$, $Id(Q_5) = 2$.

Theorem 2.6: The independent domatic number of queen graph $Q_n$ is less than or equal to $n$. (i.e) $Id(Q_n) \leq n$.

Proof: Suppose that the independent domatic number of $Q_n$ is greater than $n$. (i.e) Take $Id(Q_n) = n + 1$.

Case(i) Let $S$ be maximum independent queen set of $Q_n$ with cardinality $n$. By the theorem 2.3, $S$ is not unique. Therefore, another maximum independent queen set say $S'$ with cardinality $n$. Now assume remaining $n - 1$, independent queen set of cardinality is $n - 1$. Clearly which gives the contradiction [For, by the assumption the total number vertices of $Q_n$ is equal to $(n - 1)(n - 1) + 2n = n^2 + 1$ but the total number of vertices in $n \times n$ chess board is $n^2$].

Case(ii) Suppose no maximum independent queen set is $n$. By our assumption all the $n + 1$ independent sets of cardinality is $n - 1$. Also which gives the contradiction [For, by the assumption the total number vertices of $Q_n$ is equal to $(n - 1)(n + 1) = n^2 + 1$ but the total number of vertices in $n \times n$ chess board is $n^2$]. Both the cases gives the independent domatic number of $Q_n$ is less than or equal to $n$. 

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Figure 2. $\beta(Q_6) = 5$.

Figure 3. $Id(Q_5) = 2$. 

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**Theorem 2.7:** The minimum independent domatic number of queen graph $Q_n$ is less than or equal to 8. (i.e) $id(Q_n) \leq 8$ if the minimum independent queen number of $Q_{n-1}$ is equal to the minimum independent queen number of $Q_n$.

**Proof:** Theorem is proof by induction on $n$. This theorem is obviously true for $n = 1$. Now assume that this theorem is true for all less than $n$ and to proof $n$.

Suppose that minimum independent queen number of $Q_n$ is same as the minimum independent queen number of $Q_{n-1}$ say $m$. By our induction the independent domatic number of queen graph $Q_{n-1}$ is less than or equal to 8. But $(n-1)^2 < n^2$. Therefore, we cannot find minimum independent domiating sets more than 8 [Since $i(Q_n) = i(Q_{n-1}) = m$].

**References:**

1. Buckley F and Harary F, Distance in Graph, Addition - Wesley, reading M. A. 1990.
2. Chockayne E J, Chess board domination number, Discrete Mathematics 86 (1990) 13-20.
3. Cockayne E J and Hedetniemi S T, Towards a theory of domination in graphs, Networks 7(1977), 247-261.
4. Richard Douglas Chatham, Gerd Fricke, and Maureen Doyale, Independence and domination separation on chessboard graphs, Journal of Combinatorial Mathematics and Combinatorial Computing(2008).