NONLINEAR VERSIONS OF KOROVKIN’S ABSTRACT THEOREMS

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Abstract. In this paper we prove Korovkin type theorems for sequences of sublinear, monotone and weak additive operators acting on function spaces $C(X)$, where $X$ is a compact or a locally compact metric space. Our results are illustrated by a series of examples.

1. Introduction

The celebrated theorem of Korovkin [17], [18] gives us conditions for uniform approximation of continuous functions on a compact interval via sequences of positive linear operators. Precisely, if $(T_n)_n$ is a sequence of positive linear operators that map $C([0,1])$ into itself such that the sequence $(T_n(f))_n$ converges to $f$ uniformly on $[0,1]$ for the three special functions $e_k : x \rightarrow x^k$, where $k = 0, 1, 2$, then this sequence also converges to $f$ uniformly on $[0,1]$ for every $f \in C([0,1])$. This statement remains true by replacing $C([0,1])$ by the space $C_{2\pi}(\mathbb{R})$ of continuous and $2\pi$-periodic functions defined on $\mathbb{R}$ and considering as test functions the triplet 1, cos and sin.

Over the years, many generalizations of Korovkin theorem appeared, all in the framework of linear functional analysis. A nice account on the present state of art is offered by the authoritative monograph of F. Altomare and M. Campiti [2] and the excellent survey of F. Altomare [1]. See also [4].

Inspired by the Choquet theory of integrability with respect to a nonadditive measure, we proved in [13] an extension of Korovkin’s theory to a class of sublinear operators, called by us Choquet type operators. The aim of the present paper is to show that our results still work in a context considerably more general.

We shall need some rudiments of ordered Banach theory that can be covered from the textbook of Meyer-Nieberg [19], or from the papers [15] and [22].

Given a metric space $X$, we will denote by $\mathcal{F}(X)$ the vector lattice of all real-valued functions defined on $X$, endowed with the pointwise ordering. Some important vector sublattices of it are

$$C(X) = \{f \in \mathcal{F}(X) : f \text{ continuous}\},$$
$$C_b(X) = \{f \in \mathcal{F}(X) : f \text{ continuous and bounded}\}$$
and $UC_b(X) = \{f \in \mathcal{F}(X) : f \text{ uniformly continuous and bounded}\}$. Notice that the spaces $C(X), C_b(X)$ and $UC_b(X)$ coincide when $X$ is a compact metric space.
If \( d \) denotes the metric on \( X \), then an important family of Lipschitz continuous functions in \( C(X) \) is

\[
d_x : X \to \mathbb{R}, \quad d_x(y) = d(x, y) \quad (x \in X).
\]

It is also worth noticing that the spaces \( C_b(X) \) and \( UC_b(X) \) are Banach lattices with respect to the pointwise ordering and the sup norm,

\[
\|f\|_\infty = \sup \{|f(x)| : x \in X\}.
\]

See [6]. As is well known, all norms on the \( N \)-dimensional real vector space \( \mathbb{R}^N \) are equivalent. When endowed with the sup norm and the coordinate wise ordering, \( \mathbb{R}^N \) can be identified (algebraically, isometrically and in order) with the Banach lattice \( C(\{1, \ldots, N\}) \), where \( \{1, \ldots, N\} \) carries the discrete topology.

Suppose that \( X \) and \( Y \) are two metric spaces and \( E \) and \( F \) are respectively ordered vector subspaces (or the positive cones) of \( F(X) \) and \( F(Y) \) that contain the unity. An operator \( T : E \to F \) is said to be a weakly nonlinear operator (respectively a weakly nonlinear functional when \( F = \mathbb{R} \)) if it satisfies the following three conditions:

1. **(SL) Sublinearity** \( T \) is subadditive and positively homogeneous, that is,

\[
T(f + g) \leq T(f) + T(g) \quad \text{and} \quad T(af) = aT(f)
\]

for all \( f, g \) in \( E \) and \( a \geq 0 \);

2. **(M) Monotonicity** \( f \leq g \) in \( E \) implies \( T(f) \leq T(g) \).

3. **(TR) Translatability** \( T(f + \alpha \cdot 1) = T(f) + \alpha T(1) \) for all functions \( f \in E \) and all numbers \( \alpha \geq 0 \).

A stronger condition than translatability is that of *comonotonic additivity*,

\[
T(f + g) = T(f) + T(g)
\]

whenever the functions \( f, g \in E \) are comonotone in the sense that

\[
(f(s) - f(t)) \cdot (g(s) - g(t)) \geq 0 \quad \text{for all} \ s, t \in X.
\]

This condition occurs naturally in the context of Choquet’s integral (and thus in the case of Choquet type operators). See [14] and [15] and the references therein. For the convenience of the reader, the basic facts on Choquet’s integral are summarized in the Appendix at the end of this paper.

Of a special interest are the *unital* operators, that is, the operators preserving the unity. A simple example of unital weakly nonlinear operator is

\[
T : \ell^\infty \to \ell^\infty, \quad T((x_n)_n) = \left( \limsup_{n \to \infty} x_n \right) \cdot 1;
\]

here \( \ell^\infty \) is the Banach lattice of all bounded real sequences and \( 1 \) denotes the sequence with all components equal to unity. As is well known, \( \ell^\infty \) can be identified with the space \( C_0(\mathbb{N}) \) (where \( \mathbb{N} \) is endowed with the discrete topology) or with the space \( C(\beta\mathbb{N}) \) (where \( \beta\mathbb{N} \) is the Stone-Cech compactification of \( \mathbb{N} \)). See [7]. The operator \( T \) is not a Choquet integral (associated to a lower continuous capacity). Indeed, according to Remark 5(e) in the Appendix, \( T \) would play the property

\[
\lim_{n \to \infty} T(\xi_n) = T(\xi),
\]

whenever \( (\xi_n)_n \) is a nondecreasing sequence of elements of \( \ell^\infty \) that converges coordinatewise to \( \xi \), but this is clearly false.
The permanence properties of weakly nonlinear operators as well as more examples of such operators are presented in Section 2.

In this paper we extend Korovkin’s theorem to the case when the operators $T_n$ are weakly nonlinear operators acting on a function space. This can be $C(X)$, where $X$ is a compact metric space, or $C_b(X)$, where $X$ is a locally compact metric space. The families of test functions are constructed via the separating functions (the functions $\gamma : X \times X \to \mathbb{R}$ which are continuous and nonnegative and have the property that $\gamma(x, y) \neq 0$ if $x \neq y$). The details are presented in Section 3. This section also includes the extension of Korovkin’s theorem to the case of weakly nonlinear operators acting on spaces $C(X)$, where $X$ is a compact space. See Theorem 2. An important consequence of it the following result that extends our Korovkin type theorem from [13] in the particular case of compact metric spaces.

**Theorem 1.** (The nonlinear extension of Korovkin’s theorem for several variables) Suppose that $X$ is a compact subset of the Euclidean space $\mathbb{R}^N$ and let $(T_n)_n$ be a sequence of sublinear and monotone operators from $C(X)$ into itself such that

$$T_n(f)(x) \to f(x) \quad \text{uniformly on } X$$

for each of the test functions $1$, $\pm pr_1, \ldots, \pm pr_N$ and $\sum_{k=1}^N pr_k^2$. Then

$$\lim_{n \to \infty} T_n(f) = f \quad \text{uniformly on } X$$

for all nonnegative functions $f \in C(X)$. The conclusion (1.2) occurs for all functions $f \in C(X)$ when the operators $T_n$ are also translatable.

The family of test functions used here is built via the canonical projections on the Euclidean $N$-dimensional space:

$$pr_k : (x_1, \ldots, x_N) \to x_k, \quad k = 1, \ldots, N.$$

Section 4 is devoted to an extension of Altomare’s Korovkin type theorem (see [1], Theorem 3.5, p. 100) to the framework of weakly nonlinear operators acting on a space $C_b(X)$, where $X$ is a locally compact metric space.

Applications of our new results are presented in Section 5.

### 2. Preliminaries on weakly nonlinear operators

Suppose that $X$ and $Y$ are two locally compact compact spaces and $E$ and $F$ are closed vector sublattices respectively of the Banach lattices $C_b(X)$ and $C_b(Y)$.

Every monotone and subadditive operator $T : E \to F$ verifies the inequality

$$|T(f) - T(g)| \leq T(|f - g|) \quad \text{for all } f, g.$$

Indeed, $f \leq g + |f - g|$ yields $T(f) \leq T(g) + T(|f - g|)$, that is, $T(f) - T(g) \leq T(|f - g|)$, and interchanging the role of $f$ and $g$ we infer that $- (T(f) - T(g)) \leq T(|f - g|)$.

If $T$ is linear, then the property of monotonicity is equivalent to that of positivity, that is, to the fact that

$$T(f) \geq 0 \quad \text{for all } f \geq 0.$$

If the operator $T$ is monotone and positively homogeneous, then necessarily

$$T(0) = 0.$$

Every sublinear operator is convex and a convex function $\Phi : E \to F$ is sublinear if and only if it is positively homogeneous.
The continuity of a sublinear operator $T : E \to F$ is equivalent to its continuity at the origin, which in turn is equivalent to existence of a constant $\lambda \geq 0$ such that
\[ \|T(x)\| \leq \lambda \|x\| \quad \text{for all } x \in E. \]
The smallest constant $\lambda = \|T\|$ with this property will be called the norm of $T$.

**Remark 1.** If $T : C_b(X) \to C_b(X)$ is a sublinear and monotone operator, then $T$ is continuous and $\|T\| = \|T(1)\|$.

Indeed, $|f| \leq \|f\|_{\infty} \cdot 1$, so that, according to (2.1), we infer that $\|T(f)\| \leq \|f\|_{\infty} \|T(1)\|$. This shows that $T$ is continuous and $\|T\| \leq \|T(1)\|$; the other inequality is trivial and thus $\|T\| = \|T(1)\|$. An immediate consequence is that $\|T\| = 1$ when $T$ is in addition unital.

The following variant of Hölder’s inequality is a particular case of Theorem 3 in our paper [14].

**Lemma 1.** (Hölder’s inequality for $p \in (1, \infty)$ and $1/p + 1/q = 1$) Suppose that $X$ is a compact metric space and $T : C(X) \to C(X)$ is a unital, sublinear and monotone operator. Then
\[ T(|fg|) \leq [T(|f|^p)]^{1/p} \cdot [T(|g|^q)]^{1/q}. \]
for all $f, g \in C(X)$.

Concrete examples of sublinear and monotone operators are presented in [22] and references therein. They are ubiquitous in many fields like functional analysis, convex analysis and partial differential equations. As we prove in Section 3, even the sequences of sublinear and monotone operators $T_n : C(X) \to C(X)$ having the property that $T_n(1) = 1$ for every $n \in \mathbb{N}$ offer a natural framework for approximating the nonnegative continuous functions by suitable special classes of functions.

The set $\mathcal{WN}(C_b(X), C_b(X))$ of all weakly nonlinear operators $T : C_b(X) \to C_b(X)$ is a convex cone in the Banach space $\text{Lip}_0(C_b(X), C_b(X))$, of all Lipschitz maps from $C_b(X)$ into itself that vanish at the origin. In turn, it includes the cone $\mathcal{L}_+(C_b(X), C_b(X))$, of all linear, continuous and monotone operators from $C_b(X)$ into itself.

A general procedure to generate new weakly nonlinear operators form old ones is as follows:

**Lemma 2.** (a) If $S, T \in \mathcal{WN}(C_b(X), C_b(X))$ and $S(1) = T(1)$, then the operator $S \vee T$ defined by the formula
\[ (S \vee T)(f) = \sup \{S(f), T(f)\} \quad \text{for } f \in C(X), \]
also belongs to $\mathcal{WN}(C(X), C(X))$.

(b) If $S, T \in \mathcal{WN}(C_b(X), C_b(X))$ and $T$ is unital, then $ST \in \mathcal{WN}(C_b(X), C_b(X))$. 
Proof. Indeed, the fact that the pointwise sup of two sublinear and monotone operators is also sublinear and monotone is obvious. In addition,
\[
(S \vee T)(f + \alpha 1) = \sup \{S(f + \alpha 1), T(f + \alpha 1)\}
\]
\[
= \sup \{S(f) + \alpha S(1), T(f) + \alpha T(1)\}
\]
\[
= \sup \{S(f), T(f)\} + \sup \{\alpha S(1), \alpha T(1)\}
\]
\[
= \sup \{S(f), T(f)\} + \alpha (S \vee T)(1)
\]
for all \(f \in C(X)\) and \(\alpha \geq 0\). The proof is done. \(\square\)

Example 1. The Banach lattice \(c\) of all convergent sequences of real numbers (endowed with the sup norm and the coordinatewise ordering) can be identified with \(C(\hat{\mathbb{N}})\), where \(\hat{\mathbb{N}} = \mathbb{N} \cup \{\infty\}\) is the one point compactification of the discrete space \(\mathbb{N}\). See [17]. According to Lemma 2 (a), the following operators, from \(c\) into itself, are unital and weakly nonlinear:
\[
T_1((x_n)_n) = \left( \sup \{x_n, \lim_{k \to \infty} x_k\} \right)_n
\]
\[
T_2((x_n)_n) = \left( \sup_{n} \left\{ \frac{x_1 + \cdots + x_n}{n}, \lim_{k \to \infty} x_k \right\} \right)_n
\]
\[
T_3((x_n)_n) = \left( \sup \{x_n, \frac{x_1 + 2x_2 + \cdots + 2^n x_n}{2^{n+1} - 1}\} \right)_n.
\]

3. The case of compact metric spaces

The basic ingredient in our approach of extending Korovkin’s theory is a technical estimate for uniformly continuous functions, originating in his paper [17] from 1953, and put here in a slightly more generality.

Lemma 3. If \(X = (X, d)\) is a compact metric space, and \(\gamma : X \times X \to \mathbb{R}\) is a separating function, that is, a nonnegative continuous function such that
\[
\gamma(x, y) = 0 \quad \text{implies} \quad x = y,
\]
then every real-valued continuous function \(f\) defined on \(X\) verifies an estimate of the form
\[
|f(x) - f(y)| \leq \varepsilon + \delta(\varepsilon)\gamma(x, y) \quad \text{for all} \ x, y \in X \quad \text{and} \ \varepsilon > 0.
\]

Proof. We borrow the quick argument from [20], [21]. If the estimate above doesn’t work, then for a suitable \(\varepsilon_0 > 0\) one can find two sequences \((x_n)_n\) and \((y_n)_n\) of elements of \(X\) such that
\[
|f(x_n) - f(y_n)| \geq \varepsilon_0 + 2^n \gamma(x_n, y_n)
\]
for all \(n\). Without loss of generality we may assume (by passing to subsequences) that both sequences \((x_n)_n\) and \((y_n)_n\) are convergent, respectively to \(x\) and \(y\). Since \(f\) is bounded, the inequality (3.1) forces \(x = y\). Indeed,
\[
\frac{|f(x_n) - f(y_n)|}{2^n} \to 0 \quad \text{and} \quad \frac{|f(x_n) - f(y_n)|}{2^n} \geq \gamma(x_n, y_n) \to \gamma(x, y) \geq 0,
\]
which implies that \(\gamma(x, y) = 0\). On the other hand, from (3.1) one can infer that \(|f(x) - f(y)| \geq \varepsilon_0\) and thus \(x \neq y\). This contradiction shows that the assumption made at the beginning of the proof is wrong and the assertion of Lemma 3 is true. \(\square\)
Remark 2. The argument of Lemma 3 also shows that every separating function \( \gamma : X \times X \to \mathbb{R} \) is related to the metric \( d \) on \( X \) via an estimate of the form
\[
d(x, y) \leq \varepsilon + \delta(\varepsilon)\gamma(x, y) \quad \text{for all} \ (x, y) \in X \times X \text{ and } \varepsilon > 0.
\]

If \( X = (X, d) \) is an arbitrary compact metric space and \( \varphi : [0, \infty) \to [0, \infty) \) is a continuous function such that \( \varphi(t) > 0 \) for \( t > 0 \), then \( \gamma(x, y) = \varphi(d(x, y)) \) is an example of separating function.

Every separating function \( \gamma : X \times X \to \mathbb{R} \) generates a family of nonnegative functions on \( X \), precisely,
\[
\gamma_x : X \to \mathbb{R}, \quad \gamma_x(y) = \gamma(x, y) \quad \text{for} \ x, y \in X.
\]

As shows Theorem 2 below, this family can be used as a family of test functions in the same manner as the functions \( f_1, x \) and \( x^2 \) were used in Korovkin's theorem. Therefore we are primarily interested in separating functions producing a minimal number of test functions. A classical example is offered by the case of compact subsets \( X \) of \( \mathbb{R}^N \). Choosing \( f_1, \ldots, f_m \in C(X) \) a family of functions which separates the points of \( X \) and
\[
\gamma(x, y) = \sum_{k=1}^{m} (f_k(x) - f_k(y))^2
\]
is a separating function; when this family consists of the coordinate functions \( \text{pr}_1, \ldots, \text{pr}_N \), then
\[
\gamma(x, y) = \|x - y\|^2.
\]
The following result represents a nonlinear generalization of Korovkin’s theorem and of many other related results existing in the literature.

Theorem 2. Let \( X \) be a compact metric space (endowed with the metric \( d \)) and let \( (T_n)_n \) be a sequence of sublinear and monotone operators from \( C(X) \) into itself such that
\[
T_n(1)(x) \to 1 \quad \text{uniformly on } X.
\]

Suppose that \( \gamma : X \times X \to \mathbb{R} \) is a separating function such that
\[
T_n(\gamma_x)(x) \to 0 \quad \text{uniformly on } X.
\]

Then for all nonnegative functions \( f \in C(X) \),
\[
T_n(f) \to f \quad \text{uniformly on } X.
\]

This convergence occurs for all \( f \in C(X) \) if the operators \( T_n \) are also translatable (that is, when they are weakly nonlinear).

Proof. Let \( f \in C(X) \) be a nonnegative function. Then, according to Lemma 3, for every \( \varepsilon > 0 \) there is \( \delta(\varepsilon) > 0 \) such that
\[
|f - f(x)| \leq \varepsilon + \delta(\varepsilon)\gamma_x
\]
for all \( x \in X \). Since the operators \( T_n \) are subadditive, positively homogeneous and monotone, one can use the inequality (2.1) to show that
\[
|T_n(f) - f(x)T_n(1)| = |T_n(f) - T_n(f(x) \cdot 1)| \leq T_n(|f - f(x)|) \leq \varepsilon T_n(1) + \delta(\varepsilon)T_n(\gamma_x).
\]

According to our hypotheses (3.3) and (3.4), this leads to the conclusion that \( T_n(f) \to f \) uniformly on \( X \).
Suppose now that each operator $T_n$ is also weakly additive. Every function $f \in C(X)$ verifies the inequality $f(x) + \|f\|_\infty \geq 0$, whenever $x \in X$, so that by taking into account the above considerations, we infer that

$$T_n(f + \|f\|_\infty )(x) \to f(x) + \|f\|_\infty ,$$

uniformly on $X$. Taking into account the hypothesis \((3.3)\) and the fact that the operators $T_n$ were assumed to be weakly additive, we have

$$T_n(f + \|f\|_\infty )(x) = T_n(f)(x) + \|f\|_\infty \cdot T_n(1)(x) \to T_n(f)(x) + \|f\|_\infty ,$$

which yields that $T_n(f) \to f$, uniformly on $X$, for any function $f \in C(X)$. The proof is done. \(\square\)

According to Remark 2, if the sequence of operators $T_n$ verifies the condition \((3.4)\) in Theorem 2, it also verifies the condition

$$T_n(d_x)(x) \to 0 \quad \text{uniformly on } X.$$  

This outlines the prominent role played by the distance function among the separating functions.

**Theorem 3.** Under the hypotheses of Theorem 2 if $\gamma = d$ and $f \in C(X)$ is a Lipschitz continuous function with the Lipschitz constant $K$, then the following estimate holds:

$$|T_n(f)(x) - f(x)| \leq K \cdot \sup \left\{|T_n(d_x^2)(x)|^{1/2} : x \in X\right\} \quad \text{for all } x \in X \text{ and } n \in \mathbb{N}.$$

**Proof.** The argument is similar to that of Theorem 2 replacing the starting estimate \((3.0)\) by the condition of Lipschitzianity,

$$f(x) \cdot 1 - K \cdot d_x \leq f \leq f(x) \cdot 1 + K \cdot d_x \quad \text{for all } x \in X.$$

Suppose for a moment that $f \geq 0$. Since the operators $T_n$ are subadditive, monotonic and positively homogeneous one can apply them to the left-hand side inequality (rewritten as $f(x) \cdot 1 \leq f + Kd_x$), resulting that

$$f(x) \leq T_n(f)(x) + KT_n(d_x)(x).$$

Applying these operators to the right hand side inequality one obtains

$$T_n(f)(x) \leq f(x) + KT_n(d_x)(x).$$

Therefore, taking into account Lemma 4, we conclude that

$$\left|T_n(f)(x) - f(x)\right| \leq KT_n(d_x)(x) \leq K \left(T_n(d_x^2)(x)\right)^{1/2} \quad \text{for all } x \in X \text{ and } n \in \mathbb{N}.$$

The case of Lipschitz functions not necessarily nonnegative can be settled as in the proof of Theorem 2. \(\square\)

**Proof of Theorem 1.** When $X$ is a compact subset of $\mathbb{R}^N$ and $\gamma(x,y) = \|x - y\|^2$, Theorem 2 can be restated in a more convenient way by replacing the two conditions \((3.3)\) & \((3.4)\) with a set of $2N + 1$ tests of convergence:

$$T_n(f)(x) \to 1 \quad \text{uniformly on } X,$$

for each of the test functions $1, \pm pr_1, ..., \pm pr_N$ and $\sum_{k=1}^N pr_k^2$. Here we can replace $\sum_{k=1}^N pr_k^2$ by the string of test functions $pr_1^2, ..., pr_N^2$.

Indeed, by denoting

$$M = \sup_{x \in X} \{pr_1(x), ..., pr_N(x), 0\},$$

...
we have
\[
0 \leq T_n(\| \cdot - x \|^2)(x) \leq T_n(\| x \|^2)(x) + 2T_n(\langle \cdot, x \rangle)(x) + \| x \|^2 T_n(1)(x)
\]
\[
= T_n \left( \sum_{k=1}^{N} pr_k^2 \right)(x) + 2T_n \left[ \sum_{k=1}^{N} (- pr_k(x)) \cdot pr_k(\cdot) \right](x) + \| x \|^2 T_n(1)(x)
\]
\[
= T_n \left( \sum_{k=1}^{N} pr_k^2 \right)(x) + 2T_n \left[ \sum_{k=1}^{N} (M - pr_k(x)) \cdot (pr_k(\cdot)) + M \sum_{k=1}^{N} (- pr_k(\cdot)) \right](x)
\]
\[
+ \| x \|^2 T_n(1)(x)
\]
\leq T_n \left( \sum_{k=1}^{N} pr_k^2 \right)(x) + 2 \sum_{k=1}^{N} (M - pr_k(x)) \cdot T_n (pr_k(\cdot))(x) + 2M \sum_{k=1}^{N} T_n (- pr_k(\cdot))
\]
\[
+ \| x \|^2 T_n(1)(x)
\]
and assuming that \( \lim_{n \to \infty} T_n(f)(x) \to f \) uniformly on \( X \) for each of the test functions \( 1, \pm pr_1, ..., \pm pr_N \) and \( \sum_{k=1}^{N} pr_k^2 \) we can easily check that the right-hand side of the precedent string of inequalities converges uniformly to 0 on \( X \). Consequently \( T_n(\| \cdot - x \|^2)(x) \to 0 \) uniformly on \( X \) and Theorem 2 applies. The proof is done.

\[ \Box \]

**Remark 3.** Working with a finite family \( f_1, ..., f_p \) of continuous functions that separates the points of \( X \) and the separating function \( \gamma(x,y) = \sum_{k=1}^{p} |f_k(x) - f_k(y)|^2 \), one can arrive at the conclusion of Theorem 1 by verifying the convergence \( (3.3) \) for the \( 2p + 2 \) test functions, \( \pm f_1, ..., \pm f_p \) and \( \sum_{k=1}^{2} f_k^2 \).

**Example 2.** Consider now the particular case of the unit circle
\[
S^1 = \{ (\cos \varphi, \sin \varphi) : \varphi \in \mathbb{R} \}.
\]

With respect to the metric induced by \( \mathbb{R}^2 \),
\[
d((\cos \varphi, \sin \varphi), (\cos \psi, \sin \psi)) = \sqrt{(\cos \varphi - \cos \psi)^2 + (\sin \varphi - \sin \psi)^2}
\]
\[
= 2 \left| \sin \frac{\varphi - \psi}{2} \right|.
\]
\( S^1 \) is a compact (that is, bounded and close) subset of \( \mathbb{R}^2 \). Choosing as a separating function the square distance,
\[
\gamma((\cos \varphi, \sin \varphi), (\cos \psi, \sin \psi)) = \sin^2 \frac{\varphi - \psi}{2}
\]
\[
= 1 - \cos \varphi \cos \psi - \sin \varphi \sin \psi
\]
one can easily check that the conditions \( (3.3) \) in Theorem 2 can be replaced in this case by the fulfillment of the following 5 tests of convergence:
\[
T_n(f)(x) \to 1 \quad \text{uniformly on } X
\]
for each of the functions \( 1, \pm pr_1 \) and \( \pm pr_2 \). It is well known that the Banach space \( C(S^1) \), can be identified with the space \( C_{2\pi}(\mathbb{R}) \), of all continuous and \( 2\pi \)-periodic functions \( f : \mathbb{R} \to \mathbb{R} \). Modulo this identification, we infer from Theorem 1 that a
sufficient condition for a sequence of weakly nonlinear operators \( T_n : C_{2\pi}(\mathbb{R}) \to C_{2\pi}(\mathbb{R}) \) to verify the condition

\[
\lim_{n \to \infty} T_n(f)(\varphi) \to 0, \quad \text{uniformly on } \mathbb{R}
\]

is to verify this conditions for the test functions 1, \( \pm \cos \varphi \) and \( \pm \sin \varphi \). This was first noticed by Korovkin \[17\], \[18\] in the particular case of linear operators.

**Example 3.** Suppose that \( X = (X, d) \) is a compact metric space. Then the product space \( X \times S^1 \) is also a compact metric space and the space \( C(X \times S^1) \) can be identified with the Banach space \( C_{2\pi}(K \times \mathbb{R}) \), of all continuous functions \( f : K \times \mathbb{R} \to \mathbb{R} \), \( 2\pi \)-periodic in the second variable, endowed with the sup norm. This space is genuine for many results in dynamical systems theory. By considering the separating function

\[
\gamma((x, \varphi), (y, \psi)) = d(x, y)^2 + \sin^2 \varphi - \psi^2,
\]

Popa \[23\] has recently proved the variant of Theorem 1 for the linear and positive operators \( T : C(X \times S^1) \to C(X \times S^1) \). The reader can easily check that actually his results extend to the case of weakly nonlinear operators. In particular, when \( X \) is compact subset of \( \mathbb{R}^N \) then the convergence

\[
T_n(f) \to f \quad \text{uniformly on } X \times S^1,
\]

for all \( f \in C(X \times S^1) \) reduces to its verification for the product functions \( f(x) = u(x)v(\varphi) \), where

\[
u \in \left\{ 1, \pm pr_1, ..., \pm pr_N \text{ and } \sum_{k=1}^{N} pr_k^2 \right\} \text{ and } v \in \{1, \pm \cos \varphi, \pm \sin \varphi\}.
\]

We left to the reader the easy exercise to detail Theorem 1 in some other cases of interest such as the torus \( S^1 \times S^1 \) and the 2-dimensional sphere \( S^2 \).

**Remark 4.** In the absence of the condition of translatability the conclusion of Theorem 2 may fail for functions with variable sign. An example working for \( X = [0, 1] \) is given by the Bernstein like operators

\[
T_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} \sup \{ f(k/n), 0 \},
\]

which are sublinear and monotone (but not translatable). Clearly, \( T_n(f) \to f \) uniformly on \( [0, 1] \) for each of the functions 1, \( x \) and \( x^2 \). According to Theorem 2 this convergence occurs for all nonnegative functions \( f \in C([0, 1]) \). Clearly, it fails for the nonpositive functions. Remarkably, the case of nonnegative functions is strong enough to provide valuable information for all functions in \( C([0, 1]) \), for example, the possibility to approximate them by polynomials. Indeed,

\[
T_n(f + \|f\|_{\infty}) - \|f\|_{\infty} \to f \quad \text{uniformly on } [0, 1].
\]

4. The extension of a result due to Altomare

The next theorem represents a nonlinear analogue of a result due to Altomare (see \[1\], Theorem 3.5, p. 100).
Theorem 4. Let $X$ be a locally compact metric space (endowed with the metric $d$) and consider a vector sublattice $E$ of $\mathcal{F}(X)$ containing the constant functions and all the functions $d^p_x$ for $x \in X$ and some exponent $p \geq 1$. Let $(T_n)_n$ be a sequence of sublinear and monotone operators from $E$ into $\mathcal{F}(X)$ which verifies the following two conditions:

(a) $\lim_{n \to \infty} T_n(1) = 1$, uniformly on compact subsets of $X$;
(b) $\lim_{n \to \infty} T_n(d^p_x)(x) = 0$, uniformly on compact subsets of $X$;

Then, for all nonnegative $f$ in $E \cap C_b(X)$, we have

$$\lim_{n \to \infty} T_n(f) = f, \quad \text{uniformly on compact subsets of } X.$$ 

The convergence occurs for all functions in $E \cap C_b(X)$ when the operators $T_n$ are also translatable.

The proof of Theorem 4, needs the following lemma due to Altomare. See [1], Lemma 3.4, p. 99 for details.

Lemma 4. Let $X$ be a locally compact metric space endowed with the metric $d$. Then for every compact subset $K$ of $X$ and for every $\varepsilon > 0$, there exist $0 < \varepsilon' < \varepsilon$ and a compact subset $K_{\varepsilon'}$ of $X$ such that the open ball $B_{\varepsilon'}(x)$ is included in $K_{\varepsilon'}$ for every $x \in K$.

Proof of Theorem 4. Let $f \in E \cap C_b(X)$ and $\varepsilon > 0$ be arbitrarily fixed. Then for every compact subset $K$ of $X$, choose $\varepsilon' \in (0, \varepsilon)$ and $K_{\varepsilon'}$ as in Lemma [4]. Since $f$ is uniformly continuous on $K_{\varepsilon'}$, there exists $\delta \in (0, \varepsilon')$ such that

$$|f(x) - f(y)| \leq \varepsilon \quad \text{for every } x, y \in K_{\varepsilon'} \text{ with } d(x, y) \leq \delta.$$ 

Suppose that $x \in K$ and $y \in X$. If $d(x, y) \leq \delta$, then $y \in B'(x, \varepsilon) \subset K_{\varepsilon'}$ and therefore, $|f(x) - f(y)| \leq \varepsilon$. If $d(x, y) \geq \delta$, then

$$|f(x) - f(y)| \leq \frac{2\|f\|_{\infty}}{\delta^p} \cdot d^p_x(x, y).$$

Therefore

$$|f - f(x)| \leq \frac{2\|f\|_{\infty}}{\delta^p} \cdot d^p_x + \varepsilon \cdot 1 \text{ for all } x \in K,$$

so that, taking into account the inequality (2.1), we infer in the case of nonnegative functions $f$ that

(4.1) $|T_n(f)(x) - f(x)T_n(1)(x)| = T_n(|f - f(x)|)(x) \leq \frac{2\|f\|_{\infty}}{\delta^p} \cdot d^p_x(x) + \varepsilon T_n(1)(x)$

for all $x \in K$, whence

(4.2) $\lim_{n \to \infty} T_n(f)(x) = f(x), \quad \text{uniformly with respect to } x \in K.$

Assume now that all operators $T_n$ are translatable. Replacing $f$ by $f + \|f\|_{\infty}$, the equality in the left hand side (4.1) still works, so that the inequality in the right hand side occurs for all functions $f \in E \cap C_b(X)$. The same is true concerning the formula (4.2) and the proof is done. \(\square\)

An example illustrating Theorem 4 is exhibited at the end of the next section.
5. Applications

In this section we illustrate the results in the previous sections by several concrete examples. We adopt the convention $0^0 = 1$.

The Bernstein-Kantorovich-Choquet polynomial operators. We proved in [13] that the Bernstein-Kantorovich-Choquet polynomial operators for functions of one real variable,

$$K_{n,\mu}^{(1)} : C([0,1]) \to C([0,1]),$$

defined by the formula

$$K_{n,\mu}^{(1)}(f)(x) = \sum_{k=0}^{n} p_n,k(x) \cdot \frac{(C) \int_{[k/(n+1), (k+1)/(n+1)]}^{[k/(n+1), (k+1)/(n+1)]} f(t) d\mu(t)}{\mu([k/(n+1), (k+1)/(n+1)])},$$

verifies the conditions $K_{n,\mu}^{(1)}(x^k) \to x^k$ uniformly on $[0,1]$ for $k \in \{0, 1, 2\}$, which implies that $K_{n,\mu}^{(1)}(f) \to f$ uniformly on $[0,1]$ for all functions $f \in C([0,1])$.

The Bernstein-Kantorovich-Choquet polynomial operators for functions of two real variables,

$$K_{n,\mu}^{(2)} : C([0,1]^2) \to C([0,1]^2),$$

are defined by the formula

$$K_{n,\mu}^{(2)}(f)(x_1, x_2) = \sum_{k_1=0}^{n} \sum_{k_2=0}^{n} p_{n,k_1}(x_1)p_{n,k_2}(x_2) \cdot \frac{(C) \int_{[k_1/(n+1), (k_1+1)/(n+1)]}^{[k_1/(n+1), (k_1+1)/(n+1)]} f(t_1, t_2) d\mu(t_1)}{\mu([k_1/(n+1), (k_1+1)/(n+1)])\mu([k_2/(n+1), (k_2+1)/(n+1)])},$$

where

$$p_{n,k}(t) = \binom{n}{k} t^k (1-t)^{n-k}, \text{ for } t \in [0,1] \text{ and } n \in \mathbb{N},$$

$\mu = \sqrt{\mathcal{L}}$ is the monotone and submodular (and therefore subadditive) set function associated to the Lebesgue measure $\mathcal{L}$ on the specific interval of integration and $f \in C([0,1]^2)$.

Due to the properties of the Choquet integral mentioned in the Appendix, it follows that each operator $K_{n,\mu}^{(2)}$ is a weekly nonlinear and unital operator from $C([0,1]^2)$ into itself. However, this operator is not comonotonically additive (and thus escapes the theory developed in [13]).

We will show that

$$K_{n,\mu}^{(2)}(f)(x_1, x_2) \to f(x_1, x_2) \text{ uniformly on } [0,1]^2$$

for all test functions $1, \pm pr_1, \pm pr_2, pr_1^2 + pr_2^2$ (which will imply, via Theorem 1, that this convergence occurs for all functions $f \in C([0,1]^2)$.

The case of the unity is clear, while the case of the functions $\pm pr_1$ and $\pm pr_2$ is settled by the aforementioned properties of the operators $K_{n,\mu}^{(1)}$. As concerns
the case of the function \( pr_2^2 + pr_2^2 \), notice that

\[
K_{n,t}^{(2)}(|pr_1^2(t) + pr_2^2(t) - pr_1^2(x) + pr_2^2(x)|) \\
\leq 2 \sum_{i=1}^2 K_{n,\mu}^{(2)}(|pr_i(t) + pr_i(x) - pr_i(t) - pr_i(x)|) \\
\leq 2 \sum_{i=1}^2 K_{n,\mu}^{(2)}(|pr_i(t) - pr_i(x)|) \\
\leq 2 \sum_{i=1}^2 \sqrt{K_{n,\mu}^{(2)}(|pr_i(t) - pr_i(x)|^2)} \\
= 2 \sum_{i=1}^2 \sqrt{K_{n,\mu}^{(2)}(t_i^2(x) + 2x_iK_{n,\mu}^{(2)}(-t_i)(x) + x_i^2)}
\]

according to Lemma 1. Now, by using the calculations for the Bernstein-Kantorovich-Choiquet operators in one variable in [10], [13], it is immediate that \( K_{n,\mu}^{(2)}(t_i^2)(x) \to x_i^2 \) and \( K_{n,\mu}^{(2)}(-t_i)(x) \to -x_i \) as \( n \to \infty \), uniformly with respect to \( x = (x_1, x_2) \in [0,1]^2 \), for \( i \in \{1,2\} \). Therefore, it follows that

\[
K_{n,\mu}^{(2)}(t_i^2)(x) + 2x_iK_{n,\mu}^{(2)}(-t_i)(x) + x_i^2 \to 0,
\]

uniformly with respect to \( x \), for \( i \in \{1,2\} \). Thus the convergence \( (5.1) \) also occurs for the function \( pr_1^2 + pr_2^2 \) (and thus for all functions \( f \in C([0,1]^2] \).

The reader can now easily extend this example to the case of Bernstein-Kantorovich-Choiquet polynomial operators for functions of \( N \) real variables.

Notice that while \( K_{n}^{(2)} \) is only translatable, the one variable corresponding operator \( K_{n}^{(1)} \) is comonotonic additive, see [13]. Also, in the one variable case, the error estimate in approximation of \( f \) by \( K_{n}^{(1)}(f) \) in terms of the modulus of continuity was obtained in [10].

The bivariate possibilistic Bernstein-Durrmeyer and Kantorovich polynomial operators. In the case of one variable, the so-called possibilistic Bernstein-Durrmeyer polynomials operators and possibilistic Kantorovich polynomial operators were considered in [11] by replacing in the expressions of the classical integral operators of Bernstein-Durrmeyer and of Kantorovich, the Lebesgue integral by the so-called possibilistic integral.

The correspondents of these operators in the bivariate case can be defined on \( C([0,1]^2] \) by the formulas

\[
P_n(f)(x_1, x_2) = \sum_{k_1=0}^n \sum_{k_2=0}^n p_{n,k_1}(x_1)p_{n,k_2}(x_2) \\
\cdot \sup \{ f(t_1, t_2)k_1^{k_1}(1-t_1)^{n-k_1}k_2^{k_2}(1-t_2)^{n-k_2} : t_1, t_2 \in [0,1] \}
\]
so that, according to (5.2), we infer that $P_{n,k_1}(x_1)p_{n,k_2}(x_2)$

\[
Q_n(f)(x_1, x_2) = \sum_{k_1=0}^{n} \sum_{k_2=0}^{n} p_{n,k_1}(x_1)p_{n,k_2}(x_2)
\]

\[
\cdot \sup \left\{ f(t_1, t_2) : t_1 \in \left[ \frac{k_1}{n+1}, \frac{k_1+1}{n+1} \right], t_2 \in \left[ \frac{k_2}{n+1}, \frac{k_2+1}{n+1} \right] \right\},
\]

respectively.

It is easy to show that both $P_n$ and $Q_n$ are monotone, unital and sublinear operators. Notice that $Q_n$ is translatable, while $P_n$ is not.

Using the estimate included in the proof of Corollary 3.5 in [11] (for functions of one variable), one can easily show that

\[
P_n(|pr_i(t) - pr_i(x)|)(x) \leq \frac{(1 + \sqrt{2}) \sqrt{pr_i(x)(1 - pr_i(x))} + \sqrt{2} \sqrt{pr_i(x)} + \frac{1}{n}},
\]

for $i \in \{1, 2\}$, $n \in \mathbb{N}$, and all points $x = (x_1, x_2)$ and $t = (t_1, t_2)$ in $[0, 1]^2$.

As a separating function on $[0, 1]^2$ we choose the square distance,

\[
\gamma(x, y) = ||x - y||^2.
\]

We will show that

\[
P_n(f)(x_1, x_2) \to f(x_1, x_2) \quad \text{uniformly on } [0, 1]^2
\]

for all test functions $1, \pm pr_1, \pm pr_2, pr_t^2 + pr_2^2$ (which will imply, via Theorem 1, that this convergence occurs for all nonnegative functions $f \in C([0, 1]^2)$). As in the case of Bernstein-Kantorovich-Choquet polynomial operators, only the status of the test function $pr_t^2 + pr_2^2$ needs attention. Or,

\[
P_n(|pr_t^2(t) + pr_2^2(t) - pr_t^2(x) + pr_2^2(x)|)
\]

\[
\leq 2 \sum_{i=1}^{2} P_n(|pr_i(t) + pr_i(x)| \cdot |pr_i(t) - pr_i(x)|)
\]

\[
\leq 2 \sum_{i=1}^{2} P_n(|pr_i(t) - pr_i(x)|)(x),
\]

so that, according to (5.2), we infer that $P_n(pr_t^2 + pr_2^2) \to pr_t^2 + pr_2^2$, uniformly on $[0, 1]^2$.

The case of the operators $Q_n$ is similar. The fact that they verify the hypotheses of Theorem 1 (for the same family of test functions) can be done as above, by using instead the estimate included in the proof of Theorem 3.7 in [11] (for functions of one variable):

\[
Q_n(|pr_i(t) - pr_i(x)|)(x) \leq \frac{\sqrt{pr_i(x)(1 - pr_i(x))}}{\sqrt{n}} + \frac{2}{n+1},
\]

Since $Q_n$ are translatable, the convergence of $Q_n(f)$ to $f$ holds for all $f \in C([0, 1]^2)$.

**The max-product operators.** An important class of monotone, unital and sublinear operators are the so-called max-product operators, whose theory made the subject of the monograph [3]. Denote $\sqrt{m_{ji}} = \max_{j=0,\ldots,m}$ and $\Delta = \{(x_1, x_2); 0 \leq$
The max-product Bernstein operators $T_n : C(\Delta) \to C(\Delta)$ are defined by the formula

$$T_n(f)(x_1, x_2) = \frac{\sum_{i=0}^{n} \sum_{j=0}^{n} \binom{n}{i} \binom{n}{j} x_1^i x_2^j (1 - x_1 - x_2)^{n-i-j} f(i/n, j/n)}{\sum_{i=0}^{n} \sum_{j=0}^{n} \binom{n}{i} \binom{n}{j} x_1^i x_2^j (1 - x_1 - x_2)^{n-i-j}}.$$ 

As was shown in [3], pp. 139-140, these operators satisfy the estimate

$$T_n(|\text{pr}_i(t) - \text{pr}_i(x)|)(x) \leq \frac{6}{\sqrt{n+1}}, n \in \mathbb{N}, (x_1, x_2) \in \Delta, i = 1, 2.$$ 

for $i \in \{1, 2\}$, $n \in \mathbb{N}$, and all points $x = (x_1, x_2)$ and $t = (t_1, t_2)$ in $\Delta$.

The operators $T_n$ are sublinear, unital, and monotone (but not translatable). Reasoning as in the previous example, one can infer from Theorem 1 that $T_n(f)(x_1, x_2) \to f(x_1, x_2)$ uniformly on $\Delta$ for all nonnegative functions $f \in C(\Delta)$.

As a consequence, for an arbitrary function $f \in C(\Delta)$ we have

$$T_n(f + \|f\|_{\infty}) - \|f\|_{\infty} \to f$$

uniformly on $\Delta$.

**The Gauss-Weierstrass-Choquet operators of two variables.** The bivariate Gauss-Weierstrass-Choquet operators $W_{n,\mu} : C_b(\mathbb{R}^2) \to C_b(\mathbb{R}^2)$ are defined by the formula

$$W_{n,\mu}(f)(x_1, x_2) = \frac{(C) \int_{\mathbb{R}}(C) \int_{\mathbb{R}} f(s_1, s_2) e^{-n^2(x_1-s_1)^2} e^{-n^2(x_2-s_2)^2} d\mu(s_1) d\mu(s_2)}{c(n, x_1, \mu)c(n, x_2, \mu)},$$

where $\mu = \sqrt{E}$ and (according to the calculation in the proof of Theorem 4.1 in [12]) $c(n, x_1, \mu) = (C) \int_{\mathbb{R}} e^{-n^2(x_1-s_1)^2} d\mu(s_1) = \sqrt{2/n} \cdot \Gamma(5/4)$, for $i \in \{1, 2\}$.

The fact that $W_{n,\mu}$ maps $C_b(\mathbb{R}^2)$ into itself follows from [23], Theorem 11.13, p. 239.

Clearly, the operators $W_{n,\mu}$ are sublinear, monotone, unital but not translatable.

Now, by using the estimate included in the proof of Theorem 4.1 in [12] (for functions of one variable), we infer that

$$W_{n,\mu}(|\text{pr}_i(t) - \text{pr}_i(x)|)(x) \leq \frac{4}{n}$$

for $i \in \{1, 2\}$, whenever $x = (x_1, x_2)$ and $t = (t_1, t_2)$ in $\mathbb{R}^2$. The Euclidean space $\mathbb{R}^2$ is locally compact space and this also works for all equivalent metrics on it, in particular to

$$d(x, y) = |x_1 - y_1| + |x_2 - y_2|,$$

for all $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$. Taking into account Theorem 4 (for $p = 1$), one can easily show that $W_{n,\mu}(f) \to f$, uniformly on the compact subsets of $\mathbb{R}^2$, for every nonnegative $f \in C_b(\mathbb{R}^2)$.

### 6. Appendix: Generalities on Choquet’s integral

Very interesting and the integral associated to it. Full details are to be found in the books of D. Denneberg [3], M. Grabisch [10] and Z. Wang and G. J. Klir [22].

Let $(X, \mathcal{A})$ be an arbitrarily fixed measurable space, consisting of a nonempty abstract set $X$ and a $\sigma$-algebra $\mathcal{A}$ of subsets of $X$. 
**Definition 1.** A set function $\mu : A \to [0,1]$ is called a capacity if it verifies the following two conditions:

(a) $\mu(\emptyset) = 0$ and $\mu(X) = 1$;

(b) $\mu(A) \leq \mu(B)$ for all $A, B \in A$, with $A \subset B$ (monotonicity).

An important class of capacities is that of probability measures (that is, the capacities playing the property of $\sigma$-additivity). Probability distortions represents a major source of nonadditive capacities. Technically, one start with a probability measure $P : A \to [0,1]$ and applies to it a distortion $u : [0,1] \to [0,1]$, that is, a nondecreasing and continuous function such that $u(0) = 0$ and $u(1) = 1$; for example, one may chose $u(t) = t^\alpha$ with $\alpha > 0$. The distorted probability $\mu = u(P)$ is a capacity with the remarkable property of being continuous by descending sequences, that is,

$$\lim_{n \to \infty} \mu(A_n) = \mu\left(\bigcap_{n=1}^{\infty} A_n\right)$$

for every nonincreasing sequence $(A_n)_n$ of sets in $A$. Upper continuity of a capacity is a generalization of countable additivity of an additive measure. Indeed, if $\mu$ is an additive capacity, then upper continuity is the same with countable additivity. When the distortion $u$ is concave (for example, when $u(t) = t^\alpha$ with $0 < \alpha < 1$), then $\mu$ is also submodular in the sense that

$$\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B)$$

for all $A, B \in A$.

The next concept of integrability with respect to a capacity refers to the whole class of random variables, that is, to all functions $f : X \to \mathbb{R}$ such that $f^{-1}(A) \in A$ for every Borel subset $A$ of $\mathbb{R}$.

**Definition 2.** The Choquet integral of a random variable $f$ with respect to the capacity $\mu$ is defined as the sum of two Riemann improper integrals,

$$(C) \int_X f \, d\mu = \int_0^{+\infty} \mu \left(\{x \in X : f(x) \geq t\}\right) \, dt + \int_{-\infty}^0 \left[\mu \left(\{x \in X : f(x) \geq t\}\right) - 1\right] \, dt,$$

Accordingly, $f$ is said to be Choquet integrable if both integrals above are finite.

If $f \geq 0$, then the last integral in the formula appearing in Definition 2 is 0.

The inequality sign $\geq$ in the above two integrands can be replaced by $>$; see [24], Theorem 11.1, p. 226.

Every bounded random variable is Choquet integrable. The Choquet integral coincides with the Lebesgue integral when the underlying set function $\mu$ is a $\sigma$-additive measure.

As usually, a function $f$ is said to be Choquet integrable on a set $A \in A$ if $f\chi_A$ is integrable in the sense of Definition 2. We denote

$$(C) \int_A f \, d\mu = (C) \int_X f\chi_A \, d\mu.$$

We next summarize some basic properties of the Choquet integral.
Remark 5. (a) If $\mu : A \rightarrow [0, 1]$ is a capacity, then the associated Choquet integral is a functional on the space of all bounded random variables such that:

- $f \geq 0$ implies $(C) \int_A f d\mu \geq 0$ (positivity)
- $f \leq g$ implies $(C) \int_A f d\mu \leq (C) \int_A g d\mu$ (monotonicity)
- $(C) \int_A a f d\mu = a \cdot \left( (C) \int_A f d\mu \right)$ for $a \geq 0$ (positive homogeneity)
- $(C) \int_A 1 \cdot d\mu(t) = \mu(A)$ (calibration);

see [9], Proposition 5.1 (ii), p. 64, for a proof of the property of positive homogeneity.

(b) In general, the Choquet integral is not additive but, if the bounded random variables $f$ and $g$ are comonotonic, then

\[(C) \int_A (f + g) d\mu = (C) \int_A f d\mu + (C) \int_A g d\mu.\]

This is usually referred to as the property of comonotonic additivity and was first noticed by Delacherie [8]. An immediate consequence is the property of translation invariance,

\[(C) \int_A (f + c) d\mu = (C) \int_A f d\mu + c \cdot \mu(A)\]

for all $c \in \mathbb{R}$ and all bounded random variables $f$. For details, see [9], Proposition 5.1, (vi), p. 65.

(c) If $\mu$ is a lower continuous capacity, then the Choquet integral is lower continuous in the sense that

\[\lim_{n \to \infty} \left( (C) \int_A f_n d\mu \right) = (C) \int_A f d\mu\]

whenever $(f_n)_n$ is a nondecreasing sequence of bounded random variables that converges pointwise to the bounded variable $f$. See [9], Theorem 8.1, p. 94.

(d) Suppose that $\mu$ is a submodular capacity. Then the associated Choquet integral is a subadditive functional, that is,

\[(C) \int_A (f + g) d\mu \leq (C) \int_A f d\mu + (C) \int_A g d\mu\]

for all bounded random variables $f$ and $g$. See [9], Corollary 6.4, p. 78. and Corollary 13.4, p. 161. It is also a submodular functional in the sense that

\[(C) \int_A \sup \{f, g\} d\mu + (C) \int_A \inf \{f, g\} d\mu \leq (C) \int_A f d\mu + (C) \int_A g d\mu\]

for all bounded random variables $f$ and $g$. See [5], Theorem 13, (c).

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