One-dimensional Force-free Numerical Simulations of Alfvén Waves around a Spinning Black String

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Abstract

We performed one-dimensional force-free magnetodynamic numerical simulations of the propagation of Alfvén waves along magnetic field lines around a spinning black hole-like object, the Banados–Teitelboim–Zanelli black string, to investigate the dynamic process of wave propagation and energy transport with Alfvén waves. We considered an axisymmetric and stationary magnetosphere and perturbed the background magnetosphere to obtain the linear wave equation for the Alfvén wave mode. The numerical results show that the energy of Alfvén waves monotonically increases as the waves propagate outwardly along the rotating curved magnetic field line around the ergosphere, where energy seems not to be conserved, in the case of energy extraction from the black string by the Blandford–Znajek mechanism. The apparent breakdown of energy conservation suggests the existence of a wave induced by the Alfvén wave. Considering the additional fast magnetosonic wave induced by the Alfvén wave, the energy conservation is confirmed. Similar relativistic phenomena, such as the amplification of Alfvén waves and induction of fast magnetosonic waves, are expected around a spinning black hole.

Unified Astronomy Thesaurus concepts: Black hole physics (159); General relativity (641); Plasma astrophysics (1261); Alfvén waves (23); Magnetic fields (994); Active galactic nuclei (16); Computational methods (1965)

1. Introduction

High-energy phenomena, such as relativistic jets from active galactic nuclei and in gamma-ray bursts, are considered to be powered by the rotational energy of a spinning black hole as suggested by general relativistic magnetohydrodynamic (GRMHD) numerical simulations (Gammie et al. 2003; Koide 2003; Mizuno et al. 2004; Komissarov 2005; Koide et al. 2006; McKinney 2006; McKinney & Blandford 2009; Nagataki 2009; Paschalidis et al. 2015; Ruiz et al. 2016; Event Horizon Telescope Collaboration 2019; Katheringmaraju et al. 2019; Porth et al. 2019). The GRMHD simulations have shown that the rotational energy of the black hole is extracted through a global magnetic field, and the energy is transported outwardly as the propagating twist of the magnetic field lines. Mizuno et al. (2004) and Koide et al. (2006) reported the propagation of the field line twist as torsional Alfvén waves. Torsional Alfvén waves have also been investigated in solar coronal heating through analytical and numerical methods (Musielak et al. 2007; Antolin et al. 2008). In the studies on the solar corona, the magnetic field lines of torsional Alfvén waves are helical, transverse, and incompressive. Therefore, the wave is a shear Alfvén wave or an Alfvén wave. On the other hand, torsional Alfvén waves with spiral magnetic field lines have not been investigated in solar physics because such spiral magnetic field lines are not related to coronal heating, solar flare, and other related phenomena in solar physics. Spiral torsional Alfvén waves were shown in the GRMHD numerical simulations for relativistic outflow formation with an initially radial magnetic field around a rapidly spinning black hole (Koide 2004; Komissarov 2004). Such spiral torsional Alfvén waves are compressive, and they are identified by fast magnetosonic waves (fast waves). When we consider the perturbational displacement of magnetic fields perpendicular to axisymmetric magnetic surfaces, the wave is incompressive, i.e., an Alfvén wave. Under the force-free condition, Noda et al. (2020) derived a linear equation of an Alfvén wave propagating along a magnetic field line around a Banados–Teitelboim–Zanelli (BTZ) black string. The BTZ spacetime is analogous to the spacetime at the equatorial plane of a spinning black hole, but its translation symmetry is along the rotating axis, and the horizon is cylindrical as shown in Figure 1. Considering the perturbation of an electromagnetic field, a stationary solution is used as a background equilibrium around the central object. To obtain an equilibrium solution of a magnetosphere around a black hole, it is necessary to solve the general relativistic Grad–Shafranov (GS) equation. For a field around a spinning black hole, there is no analytic exact solution of the GS equation. Fortunately, for the BTZ black string, there is a global analytic solution of the equilibrium state; thus, we can discuss the essence of phenomena in spacetime around a spinning black hole. Using the stationary scattering analytical method of the linear force-free equation, Noda et al. (2020) showed that Alfvén waves are reflected around static limit surfaces with a reflection rate greater than unity, indicating superradiance. The dynamic causal process of the waves cannot be clearly observed using stationary scattering methods. In this paper, we perform numerical simulations of the dynamic processes of Alfvén waves to investigate the details of energy transport by the waves.

To investigate the dynamic behaviors of the Alfvén wave and their energy transport by the wave, we performed force-free magnetodynamic (FFMD) simulations of Alfvén waves
We consider the axisymmetric pulse of the Alfvén wave indicated by the gray ring. We discuss the results in Section 6. In this paper, we use the natural unit system, where the light speed and gravitation constant are unity (i.e., \(c = 1\) and \(G = 1\)).

2. Background Magnetosphere and Alfvén Wave

We investigate the propagation of the Alfvén wave along a background force-free magnetic field around a spinning black string, using special-relativistic force-free electromagnetic dynamics, derived by Noda et al. (2020). The numerical results show that the energy of Alfvén waves increases monotonically as the propagation along the background magnetic field lines as well as the associated emergy are different. In these cases, the Alfvén wave also causes the fast wave. The conversion of the Alfvén wave into fast mode is also investigated by Yuan et al. (2021) using special-relativistic force-free electromagnetic dynamics to explain the possible energy-loss mechanism in the pulsar magnetosphere. Our treatment corresponds to its general relativistic extension, while the background magnetic configuration and the polarity of the Alfvén wave are different.

In Section 2, we explain the theoretical background for Alfvén waves along stationary background magnetic field lines around a black string. In Section 3, we investigate the conservation law of energy and angular momentum due to the Alfvén wave and show an apparent breakdown of the force-free conditions with respect to the azimuthal component of the Lorentz force in the second-order perturbation and the energy conservation law of Alfvén waves. To recover the force-free condition, we observed that the second-order variable \(\chi\) should be introduced, which expresses the fast wave. We further show the conservation law of energy in the original coordinates of the BTZ spacetime with \(\chi\). In Section 4, we summarize the numerical methods. The numerical results are shown in Section 5. We summarize and
The 4-velocity of the normal observer frame $\vec{x}$ is given by
\[
N^\lambda = \left( \frac{1}{\alpha}, -\frac{\beta^j}{\alpha} \right), \quad N_\lambda = (-\alpha, 0, 0, 0),
\]
where $\beta_j = g_{0i}$ and $\beta^j = g^{ij}\beta_i$. The electromagnetic field is defined by
\[
E^\lambda = F^{\lambda\nu}N_\nu = -\alpha F^\nu_{\nu0} = -\alpha(\partial^\lambda \phi_1 \partial^0 \phi_2 - \partial^0 \phi_1 \partial^\lambda \phi_2),
\]
\[
B^\lambda = *F^{\lambda\nu}N_\nu = -\alpha *F^\nu_{\nu0} = \frac{1}{\sqrt{\gamma}} F^{\nu0 \lambda \beta} \partial^\nu \phi_1 \partial^\beta \phi_2.
\]
According to Equation (4), the steady-state solution of the force-free field around the black string is given by Jacobson and Rodriguez (2019) as
\[
\vec{\phi}_1 = -\int g_{0\nu} dx \nu = -z, \quad \vec{\phi}_2 = \frac{1}{2\pi} \int \frac{dr}{\rho^2} + \varphi - \Omega_F t,
\]
where $I$ is a constant of the total current and $\Omega_F$ is a constant of the angular velocity of the magnetic field. Note that $\vec{\phi}_1$ corresponds to the stream function and the contour of $\vec{\phi}_1$ expresses steady magnetic surfaces. In this case, $z = \text{const.}$ presents the steady magnetic surface. We have
\[
\vec{E} = -\frac{1}{\alpha} (\Omega - \Omega_F), \quad \vec{B} = \frac{1}{\sqrt{\gamma}} \vec{\alpha}
\]
\[
\vec{B} = \frac{1}{2\pi r^2 \alpha} = -\frac{1}{\sqrt{\gamma}} \frac{I}{2\pi \alpha^2}.
\]
\[
\vec{E} = \vec{E} = 0, \quad \vec{B} = 0.
\]
The continuity of the scalar of the field $F^\lambda_{\nu}F^\nu_{\lambda}$ at the horizon yields the condition with respect to the total current,
\[
I = 2\pi r^2(\Omega_H - \Omega_F).
\]
The corotating vector of the field line, $\xi^\nu_{(p)} \equiv \xi^\nu_{(o)} + \Omega_F \xi^\nu_{(c)}$, is parallel to the 4-velocity of the corotating observer with the magnetic field line. The norm of $\xi^\nu_{(p)}$ is
\[
\Gamma = g_{\mu\nu} \xi^\nu_{(p)} \xi^\nu_{(p)} = -\alpha^2 + r^2(\Omega - \Omega_F)^2 = -\gamma_F(r^2 - r_{LS}^2),
\]
with $\gamma_F = 1 - \Omega_F^2$. The real root of $\Gamma = 0$ gives the location of the light surface uniquely, which is the causal boundary for Alfvén waves, and we denote its location by $r = \eta_S = \sqrt{(1 - a\Omega_F)/(1 - \Omega_F^2)}$.

To reduce the calculations to a simple form, following Noda et al. (2020), we utilize the coordinates $(T, X, \rho, z)$ defined as
\[
T = t + \int I_0 dx, \quad t = T - \int I_0 dx, \quad X = r, \quad \rho = \frac{T}{\Omega}, \quad z = z
\]
where $I_0 = I(X^0 - \Omega t)$, $I' = I/\Omega$, and $I_2 = \Omega_2 I_0 = -I'/\Omega^2$. In this coordinate system, $X$ is the coordinate along the magnetic field line, then we call this coordinate system $(T, X, \rho, z)$ the “corotating magnetically natural frame” or simply the “corotating natural frame”. In this section, hereafter, we use the corotating natural coordinates. The metric tensor of the corotating natural frame is as follows:
\[
g_{\mu\nu} = \begin{pmatrix}
\Gamma & 0 & -X^2 W & 0 \\
0 & Y & J & 0 \\
-X^2 W & J & X^2 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
\[
g_{\mu\nu} = \begin{pmatrix}
\gamma_N & 0 & JW - YW & 0 \\
JW & \alpha^2 & 0 & 0 \\
-YW & \alpha^2 & \gamma^N & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]
where $W = \Omega - \Omega_F$, $Y = \frac{1}{\alpha^2} (\Gamma - I'^2)$, $J = \frac{I'}{\gamma}$, and $\gamma_N \equiv \det(g_{\mu\nu}) = X^2 (\Gamma^2 - I'^2 X^2 W^2)$. We also have $\sqrt{-g} = X$ and the lapse function, $\alpha^2 = \frac{X}{\gamma_N}$, in the coordinates. In the corotating natural coordinates, the steady-state force-free solution given by Equation (9) is described simply as
\[
\vec{\phi}_1 = -z, \quad \vec{\phi}_2 = \rho.
\]

2.2. Wave Equation for Alfvén Wave
In the background magnetosphere given by Equation (19), we consider the propagation of the Alfvén waves. The infinitesimally small perturbation to the Euler potential $\phi_1 \rightarrow \phi_1 + \delta \phi_1$ is given by the displacement vector $\xi^\lambda$ as $\delta \phi_1 = \xi^\lambda \partial_\lambda \phi_1$. We focus on the wave propagating on the magnetic surface given by $z = 0$ and oscillating perpendicularly to the magnetic surface: $\xi^\lambda = (0, 0, 0, \zeta^\lambda)$. Then, the perturbed Euler potentials are
\[
\phi_1 = \vec{\phi}_1 + \delta \phi_1 = -z + \delta \phi_1, \quad \phi_2 = \vec{\phi}_2 = \rho,
\]
because $\delta \phi_2 = \xi^\lambda \partial_\lambda \phi_2 = \zeta^\lambda \partial_\lambda \phi_2 = 0$. We assume the perturbation is translationary invariant for the $z$ direction,
\[
\delta \phi_1 = \psi(T, X, \rho).
\]
In the corotating coordinates, the linearization of Equation (5) with $i = 2$ yields

$$
\partial_{\lambda} \left( \sqrt{-g} W^{\alpha \beta \mu \nu} \partial_{\mu} \psi \partial_{\nu} \tilde{f} \partial_{\lambda} \tilde{f} \right) \partial_{\lambda} \tilde{f} = 0,
$$

and can be simplified as

$$
\left( 1 - \frac{\nu^2}{\Gamma} \right) \partial_{T}^{2} \psi + \alpha^{2} X \partial_{X} \left( \frac{\Gamma}{X} \partial_{X} \psi \right) = 0.
$$

Equation (23) is identical to the time-evolution version of Equation (16) in Noda et al. (2020). Note that Equation (23) does not have a derivative term with respect to $\rho$. This means the perturbation $\delta \phi_{1}$ propagates independently on a two-dimensional sheet spanned by $T$ and $X$, called a field sheet, which represents the time evolution of a magnetic field line. Therefore, we can identify $\psi$ as the perturbation of an Alfvén wave. Furthermore, we can assume the perturbation of the field is axisymmetric, without loss of generality, $\psi(T, X, \rho) = \psi(T, X)$, i.e., $\psi$ is axisymmetric. Introducing the dimensionless
Alfvén and induced fast waves. The balance of the Alfvén wave: negative in the tortoise coordinate with the nonuniform line density \( \lambda \). Notably, the light surface is located at \( x = r_L \) and \( x = r_b \) like the Blandford–Znajek mechanism when \( 0 < \Omega_b < \Omega_H \). The profile of \( \lambda(x) \) indicates \( \omega(x) \) changes at the border point (\( x = x_b = 0 \)) with the characteristic length \( \ell_c \). Then, the linear outwardly propagating Alfvén wave, whose wavelength is much longer than \( \ell_c \), is reflected at the point \( x_b \), which is like a free boundary, as remarked by Noda et al. (2020). On the other hand, the outwardly propagating Alfvén wave, whose wavelength is smaller than or equal to \( \ell_c \), is not reflected and passes through the border point \( x_b \). Only the Alfvén wave with a length longer than \( \ell_c \) outwardly propagating along the magnetic field line can be reflected at the border point \( x_b \) like the reflection along a string with a free boundary condition. The inwardly propagating Alfvén wave is not reflected to propagate smoothly inwardly. Because \( \lambda \) approaches zero exponentially at \( x \to -\infty \) and \( \lambda \) approaches a constant, 

\[
\lambda_{LS} = \frac{r'}{\gamma_F \alpha (r_L s) r_L} \left[ \frac{1}{M} \frac{2 \Omega_b (\Omega_b - \Omega_H)}{\alpha^2 (\Omega_b + 1) - 2 \Omega_b} \right],
\]

at \( x \to -\infty \) (\( r \to r_L \)).

Figure 4. Time evolution of the energy balance of the Alfvén and induced fast waves. \( \delta E(t) \): total energy. \( \delta E_{Al}(t) \): energy of the Alfvén wave only. \( \delta^2 F_1(t) \): energy flux at the left edge. \( \delta F_2(t) \): energy flux at the right edge. The balance: \( \delta^2 E(t) - \delta^2 E_{Al}(t) - \delta^2 F_1(t) + \delta^2 F_2(t) \). The energy balance constancy shows the energy conservation of the Alfvén and induced fast waves. The balance of the Alfvén wave: \( \delta^2 E(t) - \delta^2 E_{Al}(t) - \delta^2 F_1(t) + \delta^2 F_2(t) \).

\[
\frac{dx}{dX} = \frac{X}{X^2 - X_{LS}^2} \frac{2}{\Gamma}, \quad (-\infty < x < \infty),
\]

(24)

Equation (23) reads

\[
\left[ 1 - \frac{T^2}{\Gamma} \right] \frac{\partial^2 \psi}{\partial T^2} + \frac{\alpha^2 \gamma_F}{X^2 - X_{LS}^2} \frac{\partial^2 \psi}{\partial X^2} = 0,
\]

(25)

where \( X_{LS} = r_L s \). Notably, the light surface is located at negative infinity in the tortoise coordinate (\( x \to -\infty \)) because \( x = \frac{1}{\alpha^2} \log(X^2 - X_{LS}^2) \). If we use the variable

\[
\lambda(x) = X^2 - X_{LS}^2 + \frac{T^2}{\gamma_F} \frac{\alpha^2}{X^2} \frac{\partial^2 \psi}{\partial X^2} = \frac{T^2 - \Gamma}{\gamma_F \alpha^2 X^2} = \frac{T^2 - \alpha^2 X^2 (\Omega - \Omega_F)^2}{\gamma_F \alpha^2 X^2},
\]

(26)

Equation (25) is written as

\[
\lambda \frac{\partial^2 \psi}{\partial T^2} = \frac{\partial^2 \psi}{\partial X^2}.
\]

(27)

This equation is identified by the equation of the displacement of the string with the nonuniform line density \( \lambda(x) \) and uniform (unit) tension \( T_{st} = 1 \). Then, the speed of a wave governed by Equation (27) with a small wavelength is \( v_{ph} = 1/\sqrt{\lambda} \). The profiles of \( \lambda \) in the cases of \( \Omega_b = 0.5 \) and \( \Omega_b = 0.7 \) with \( a = 0.9 \) (\( \Omega_H = 0.6268 \)) are shown in Figure 2. It is noted that the rotational energy of the black string is extracted through the Blandford–Znajek mechanism when \( 0 < \Omega_b < \Omega_H \).
3. Energy and Angular Momentum Conservation and Force-free Condition of the Alfvén Wave

In this section, we consider the transport of the energy and angular momentum with respect to the Alfvén wave. In the first two subsections (Sections 3.1 and 3.2), we use the corotating natural coordinates \((T, X, \rho, z)\). In the last subsection (Section 3.3), we show the energy transport in the coordinates \((t, r, j, z)\) of the BTZ black string.

3.1. Apparent Breakdown of Energy and Angular Momentum Conservation

In this subsection, we consider only the first-order perturbation of the Alfvén wave, \(\psi\), and ignore the perturbation of \(f_2\), where \(\psi\) is given by Equation (27). The Maxwell equations yield the energy–momentum conservation law,

\[
\nabla \cdot T^{\mu\nu} = -F_{\mu L}^{\nu},
\]

where \(T^{\mu\nu} = F_{\mu\sigma}F^{\nu\sigma} - \frac{1}{4}g^{\mu\nu}F_{\lambda\delta}F_{\lambda\delta}\) is the energy–momentum tensor of the electromagnetic field and \(F_{\mu L}^{\nu}\) is the 4-Lorentz force density. Here, \(F_{\mu L}^{\nu} = 0\) is called the “force-free condition”.

To investigate the energy and momentum transfer due to the Alfvén wave, we consider the perturbation of \(T^{\mu\nu}\) at the first order or higher order. Hereafter, we denote the stationary background (zeroth-order) term, first-order perturbation, second-order perturbation, \(\cdots\) of a variable \(A\) by \(\bar{A}, \delta A, \delta^2 A, \cdots\), i.e., \(A = \bar{A} + \delta A + \delta^2 A + \cdots\). When we assume

\[
\phi_1 = \bar{\phi}_1 + \delta \phi_1 = -z + \psi, \quad \phi_2 = \bar{\phi}_2,
\]

the first order of \(T^{\mu\nu}\) is calculated as

\[
\delta T^{\mu\nu} = \delta F^{\mu\lambda}F_{\nu\lambda} + F^{\mu\lambda}\delta F_{\nu\lambda} - \frac{1}{2}g^{\mu\nu}F_{\lambda\delta}\delta F_{\lambda\delta},
\]

where the detail derivation is shown in Appendix A. The second and third terms of the right-hand side of Equation (31) vanish because \(\partial_\psi \psi = 0\). The first term of the right-hand side of Equation (31) is

\[
\delta F^{\mu\lambda}F_{\lambda\delta} = \delta F^{\mu\nu}F_{\nu\delta} = -\delta F^{\mu\nu} = -W^{\mu\nu},
\]

which does not always vanish, while if both \(\mu\) and \(\nu\) are not \(z\), it vanishes. Then, when both \(\mu\) and \(\nu\) are not \(z\), we have

\[
\delta T^{\mu\nu}_z = 0.
\]

Equation (33) suggests that as far as we consider the energy and angular momentum transfer of the Alfvén wave along the equatorial plane with respect to the system with the translational symmetry toward the \(z\) direction, \(\delta T^{\mu\nu}\) vanishes. This
shows the second-order perturbation should be considered when investigating the energy transport of the linear Alfvén wave. Then, the force-free condition should be satisfied up to the second order of the perturbation. If the force-free condition of the second-order perturbation is broken, the energy and angular momentum of the Alfvén wave are not conserved. To check the conservation of the energy and the angular momentum, we evaluate the force-free condition up to the second order of the perturbation of the Alfvén wave. For this purpose, we calculate the Lorentz force,

\[ \mathbf{f}_\nu = J^\rho F^\mu_{\nu \rho} = \nabla_{\nu} F^\rho_{\mu \nu} = \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} F^\rho_{\mu \nu}). \]  

(34)

For the background, we can confirm the Lorentz force of the equilibrium vanishes because the 4-current density of the equilibrium vanishes, \( f^\rho_{\mu} = 0 \). For the first-order perturbation, we confirm the Lorentz force with respect to the first-order perturbation of the Alfvén wave vanishes, \( \delta f^L_{\rho \mu} = 0 \). Note that we have \( \delta J^\rho = \delta J^L = 0 \).

For the second-order quantities, we have \( \delta^2 J^\rho = \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} \delta^2 F^\rho_{\mu \nu}) = 0 \) because we ignore the perturbation with respect to \( \phi_2 \). Hence, we find three components of the Lorentz force, except the \( \rho \) component, vanish, \( \delta^2 f^L_{\rho \mu} = 0 \), because \( \delta J^\rho = 0 \). The \( \rho \) component of the Lorenz force is calculated as

\[
\delta^2 J^L_\rho = \frac{1}{\sqrt{-g}} \left[ -\partial_X \left( \frac{\varphi'}{\Gamma} \partial_T \psi \right) \right. \\
\left. + \partial_T \left( \partial_X \psi \left( \frac{\varphi'}{\Gamma} \partial_T \psi + XW \partial_X \psi \right) \right] \right].
\]  

(35)

\( \delta^2 J^L_\rho \) vanishes only if \( \varphi' = 0 \) and \( W = \Omega - \Omega_F = 0 \). When \( \varphi' \) and \( W \) are finite, \( \delta^2 J^L_\rho \) is finite and the force-free condition is broken. Thus, the conservation law of energy and angular momentum does not hold if we consider the Alfvén wave only. This breakdown of the conservation law is apparent and is recovered by an additional wave as shown in the next subsection.

### 3.2. Recovery of the Force-free Condition

To recover the force-free condition, we should take into account the additional first- and second-order perturbations \( \delta^2 \phi_1, \delta \phi_2, \) and \( \delta^2 \phi_2 \) as

\[
\phi_1 = \tilde{\phi}_1 + \delta \phi_1 + \delta^2 \phi_1 = -z + \psi + \delta^2 \phi_1, \\
\phi_2 = \tilde{\phi}_2 + \delta \phi_2 + \delta^2 \phi_2 = \rho + \delta \phi_2 + \delta^2 \phi_2.
\]  

(36)

(37)

We notice that the force-free condition is broken only in one equation with respect to \( \delta^2 J^\rho \). Then, the breakdown of the
force-free condition should be recovered by the addition of one free variable. The above intuitive reason for the Lorentz force acting on the Alfvén wave suggests the variable $\delta f^2$ is the appropriate additional perturbation as the additional freedom. To recover the force-free condition, we consider the following perturbation setting of the force-free Alfvén wave along the magnetic field line around the spinning black string:

$$f(t) = + \pm \chi \left( x, \phi, \psi \right)$$

where $\chi \equiv \delta f^2$.

The time-evolution equation of $\chi$ is given as

$$\Box \chi = \nabla_a \nabla^a \chi = \frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} g^{ab} \partial_b \chi)$$

$$= \frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} W^{\rho \sigma \mu \nu} \partial_{\sigma} \partial_{\mu} \psi) \equiv s,$$  

(38)  

(39)  

where $s$ is recognized as the source term in the wave equation with respect to $\chi$ (see the derivation in Appendix B), and we have

$$\frac{1}{\sqrt{-g}} \left[ \sqrt{-g} g^{\mu \nu} \partial_{\mu} \chi \right] + 2 f \partial_a (\sqrt{-g} g^{\mu \nu} \partial \chi)$$

$$- \partial_a (\sqrt{-g}) \partial \chi + f \partial_a (\sqrt{-g} g^{\varphi \psi} \partial \chi) = s.$$  

(38)  

(39)  

Introducing $\chi$, we confirm that the force-free condition holds. The linear homogeneous equation (Equation (40)) shows that the wave described by $\chi$ propagates isotropically in space, and it is recognized as the fast wave (Bellman 2006), while the complex formalism of the homogeneous linear equation of $\psi$ (Equation (22)) suggests the strong anisotropy of the wave described by $\psi$, which is a characteristic property of the Alfvén wave.

The equilibrium and first order of these values are the same as in the calculation without introducing $\chi$. When $\xi^\mu$ is the Killing vector, we have the conservation law,

$$\frac{\partial}{\partial x^a} (\xi_a T^{0 a}) + \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^a} (\sqrt{-g} \xi_a T^{a b})$$

$$= \frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} T^{a b}) = \nabla_a (\xi_a T^{a b}) = -\xi^c T^c = \xi^b T^b.$$  

(38)  

(39)  

(40)  

(41)  

(42)
When $\xi^\nu T^\mu$ vanishes, $\xi^\nu J^\mu$ represents the density of the conservation value and $\xi^\nu T^i\psi$ represents the density of the conservation quantity flux.

For the time-like Killing vector $x^\mu = (1, 0, 0, 0)$ and the axial Killing vector $x^\mu = (0, 0, 1, 0)$ in the corotating natural frame, we have the conservation laws of the energy and angular momentum:

$$\frac{\partial}{\partial x^\sigma} S^0 + \frac{1}{\sqrt{-g}} \frac{\partial}{\partial X^1} (\sqrt{-g} S^X) = f_0^L,$$

$$\frac{\partial}{\partial x^\sigma} M^0 + \frac{1}{\sqrt{-g}} \frac{\partial}{\partial X^1} (\sqrt{-g} M^X) = -f_\rho^L,$$

where we assume axisymmetry and $z$-direction translation symmetry, and $S^0 = -\xi^\mu_0 T^\mu$ and $M^0 = \xi^\mu_0 T^\mu$ are the 4-energy flux and 4-angular momentum flux, respectively.

Because the energy density and flux $S^0$ and $S^X$ are calculated as $S^0 = -\frac{1}{2} F_\mu^\nu F^\mu_\nu + \frac{1}{2} F_\mu^i F^\mu_i$, $S^X = F_\mu^i F^\mu_i$, respectively, we have the energy density and flux of the second-order perturbation of the linear Alfvén wave and the induced fast wave as follows:

$$\delta^2 S^0 = \frac{1}{2} \left[ \frac{\Gamma^2 - \Gamma}{-\Gamma \alpha^2 X^2} (\partial_T \psi)^2 + \frac{\Gamma^2}{X^2} (\partial_X \psi)^2 \right] + \frac{\Gamma}{X} \partial_X \chi,$$

$$\delta^2 S^X = \frac{\Gamma}{X^2} \partial_T \psi \partial_X \psi - \frac{\Gamma}{X} \partial_T \chi.$$

Because the angular momentum density and flux are $M^0 = F^0_\mu F^\mu_\nu$ and $M^X = F^X_\mu F^\mu_\nu$, we have the angular momentum density and flux of the second perturbation as follows:

$$\delta^2 M^0 = -\partial_X \psi \left[ (\Omega - \Omega_T) \partial_T \chi + \frac{\Gamma}{X^2} \partial_T \psi \right] - \gamma N \partial_T \chi,$$

$$\delta^2 M^X = \partial_T \psi \left[ (\Omega - \Omega_T) \partial_X \chi + \frac{\Gamma}{X^2} \partial_T \psi \right]$$

$$+ \frac{\Gamma}{\Gamma} (\Omega - \Omega_T) \partial_T \chi + \alpha^2 \partial_X \chi.$$

### 3.3. Conservation Laws of Energy in the BTZ Spacetime

Now, we derive the energy density and energy flux for the energy conservation law around the BTZ black string. In this subsection, we use both the original coordinates of the BTZ spacetime $(t, r, \phi, z)$ and the corotating natural coordinates $(T, X, \rho, z)$. To distinguish the two coordinates, we denote the corotating natural coordinates by $x^\mu = (T, X, \rho, z)$ with the underline, while the original coordinates of the BTZ spacetime just by $x^\mu = (t, r, \phi, z)$. First, we formulate the energy conservation in the BTZ coordinates $(t, r, \phi, z)$. The Killing vector for the energy conservation law in the BTZ coordinates

$$\delta^2 E^0 = \frac{\Gamma}{X^2} \partial_T \psi \partial_X \psi - \frac{\Gamma}{X} \partial_T \chi.$$ 


is \( \xi_0^\mu = (1, 0, 0, 0) \). Then, the 4-energy flux density is

\[
S^\mu = -\xi_0^\nu T_\nu^\mu = -T_t^t = -T_r^r.
\]

The energy conservation law is expressed as

\[
\nabla_\nu S^\nu = \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} S^\nu) = \xi_0^\nu f^L_\nu = f^L_t = \frac{\partial \chi^L}{\partial t} f^L_t = 0,
\]

after we introduce \( \chi \). The energy density in the BTZ coordinates is related to the components of density and flux in the corotating natural coordinates as

\[
S^t = S^T + \Omega_T M^T - L_0 (S^X + \Omega_T M^X),
\]

where the detailed derivation is shown in Appendix C. Then, we have the energy density and energy flux density of the equilibrium, the first- and second-order perturbations with respect to the linear Alfvén wave, and the second-order fast wave,

\[
S^t = \frac{1}{2} g^{\rho\rho} + \Omega_T (g^{T\rho} - L_0 g^{X\rho}),
\]

\[
\delta S^t = 0,
\]

\[
\delta^2 S^t = \delta^2 S^t + \delta^2 \nabla S^t,
\]

where

\[
\delta^2 S^t = \frac{1}{2\Gamma^2} \left[ \frac{1}{X^2} \left( \Gamma^2 + I^2 \left( \alpha^2 - X^2 (\Omega^2 - \Omega_T^2) \right) \right) \left( \partial_T \psi \right)^2 + \frac{\Gamma^2}{X^2} \left( \alpha^2 - X^2 (\Omega^2 - \Omega_T^2) \right) \left( \partial_X \psi \right)^2 - \frac{2}{\alpha^2 X^2} \left( \Gamma (\Omega - \Omega_T) + \Omega_T (\alpha^2 + X^2 (\Omega - \Omega_T)^2) \right) \partial_T \psi \partial_X \psi \right].
\]
\[
\delta^{2+0}\psi' = g^{X'} \partial_X \chi + \Omega_F \Gamma'^2 \partial_Y \chi \\
+ \delta(g^{X'} \partial_T \chi - \Omega_F g^{X'} \partial_Y \chi) \\
= \frac{l'}{X} \partial_X \chi - \frac{1}{\alpha^2 \Gamma} (\Omega_F \Gamma - l'^2 \psi) \partial_T \chi. \quad (56)
\]

In the numerical calculations, we use the above formulation (Equations (54)(56)) with the tortoise coordinates.

The energy flux density in the BTZ coordinates is related to the flux components in the corotating natural coordinates as

\[
S' = -\Gamma T' + \Omega_T \Gamma' Y = S^X + \Omega_F M^Y. \quad (57)
\]

Then, we have the energy flux density of the equilibrium, and the first- and second-order perturbations of the linear Alfvén wave,

\[
\delta S' = \frac{\Gamma^2 l'}{X}, \quad (58)
\]

\[
\delta^2 S' = 0, \quad (59)
\]

\[
\delta^2 R' = 0, \quad (60)
\]

where

\[
\delta^1 S' = -\frac{1}{X^2} \partial_T \psi \left[ \left( X^2 - M + \frac{Ma}{2} \Omega_F \right) \partial_X \psi \\
- \frac{l' \Omega_F}{\Gamma} \partial_T \psi \right], \quad (61)
\]

\[
\delta^2 S' = \frac{l'}{X \Gamma} \left( -\Gamma + X^2 \Omega_F \right) \partial_T \chi + \Omega_F \Omega_F \partial_X \chi. \quad (62)
\]

In the numerical calculations, we use the formulation (Equations (60)(62)) with the tortoise coordinates.

### 4. Numerical Method

To perform 1D numerical simulations of the force-free field with Equation (25), we use the multidimensional two-step Lax–Wendroff scheme of \( u = (\psi, v, w, \chi, u, G) \), the details of which are shown in Appendix E. Here, \( v, w, \chi, u, \) and \( G \) are the new variables introduced for the numerical calculation. This scheme sometimes causes numerically artificial structures called the Gibbs phenomena. To avoid such numerical phenomena, we employ a smooth profile of the initial variables as follows. The initial condition of an outwardly/inwardly propagating single pulse of field is given as

\[
\psi(x) = \begin{cases} 
1 - \left( \frac{x - x_0}{w/2} \right)^{2m} & (x_0 - \frac{w}{2} \leq x \leq x_0 + \frac{w}{2}) \\
0 & \text{ (other)}
\end{cases}
\]

\[
v(x) = \pm \psi(x), \quad w(x) = 0, \quad \chi(x) = 0, \quad u(x) = 0, \quad G(x) = 0,
\]

where the plus and minus signs in the equation with respect to \( v(x) \) are taken for the outwardly and inwardly propagating waves, respectively, \( x_0 \) and \( w \) give the center and width of the pulse of the Alfvén wave, and \( m \) and \( n \) are constants of the pulse shape. In this paper, we set \( n = 32 \) and \( m = \log(1 - 2^{-1/3})/\log(1 - 2^{-1/4}) \).

As shown in the next section, we have very smooth numerical results without numerical artificial structures. In this paper, we set the width and the position of the outwardly and inwardly propagating pulses as \( w = 0.5, x_0 = 1.6 \) (outward pulse) and \( w = 3, x_0 = 2 \) (inward pulse), respectively.

At the inner and outer boundaries, \( x = x_{\text{min}} \) and \( x = x_{\text{max}} \), the free boundary conditions \( u_0 - u_i = 0 \) and \( u_f - u_{i-1} = 0 \) are used to mimic the radial boundary condition, where \( u_0 \) and \( u_f \) are the variables at the boundary and \( u_i \) and \( u_{i-1} \) are the values at the neighborhood.

### 5. Numerical Results

We performed the 1D numerical simulations of Alfvén waves along a stationary magnetic field line with \( \Omega_F = 0.5 \) \( (0 < \delta < \Omega_B) \), \( \Omega_F = 0.2 \), and \( \Omega_F = 0.7 \) \( (\delta_F > \Omega_B) \) at \( \ell = 0 \) around a black string with spin parameter \( a = 0.9 \) using Equations (25) and (41) \( (\text{Table 1}) \). In the case, we have the horizon radius \( r_h = 1 + \frac{1}{a^2} / \sqrt{1 - \frac{\ell}{3}} \) \( / \sqrt{1 - \frac{\ell}{3}} \) \( = 0.8473 \), the radius of the light surface \( r_{LS} = 0.8563 \), the static limit surface \( \left( \Omega = 1 \right) \) \( x_{\text{ergo}} = \frac{1}{2} \log(\Omega_F - a - \Omega_F) \) \( (1 - \Omega_F^2)^{1/2} \) \( = 0.6608 \), in the case of \( \Omega_F = 0.5 \), \( r_{LS} = 0.8473 \), \( x_{\text{ergo}} = 0.6328 \) in the case of \( \Omega_F = 0.7 \), \( r_{LS} = 0.852 \), \( x_{\text{ergo}} = 0.6464 \) in the case of \( \Omega_F = 0.7 \), and \( r_{LS} = 0.7 \). The ergosphere is the region inside of the static limit surface, \( r < r_{\text{ergo}} \).

The initial condition of \( \psi \) is given by \( w = 0.5 \) and \( x_0 = 1.5 \) at \( T = 0 \) in the case of the outwardly propagating pulse both in the background magnetic field with \( \Omega_F = 0.5, \Omega_F = 0.6268 \), and \( \Omega_F = 0.7 \). In the case of the inwardly propagating pulse, we set \( w = 3, x_0 = 4 \), and \( \Omega_F = 0.5 \) for the initial condition of \( \psi \) at \( T = 0 \). The initial condition of \( \psi \) is given by \( \chi = 0 \) in all cases at \( T = 0 \).

Figure 3 shows the Alfvén wave propagates outwardly and the fast induced wave is caused by the Alfvén wave and propagates toward both outward and inward sides in the case of \( \Omega_F = 0.5, a = 0.9, \) \( w = 0.5 \), and \( x_0 = 1.5 \). The lines in each panel show the time slot at \( t = 0.062 \) (black dashed line), \( t = 1.0 \) (red dotted line), \( t = 2.0 \) (blue, thick solid line), and \( t = 3.0 \) (violet, thick dashed line). The pulse of the Alfvén wave with the thin width passes through the border point of \( \lambda, x_0 \) without reflection. This is because the length of the pulse is smaller than \( \xi = 3 \). In the outer region of the border point \( (x > x_0) \), the pulse propagates much faster than the propagation speed in the inner region of the border point \( (x < x_0) \) because of the wave speed \( 1/\sqrt{\lambda} \). The energy density of the Alfvén wave \( \delta^{1+1}S' \) is initially negative, but the outside of the ergosphere \( (X > 1) \), the energy density becomes positive, where the total energy density of the Alfvén wave \( \sqrt{-g} \delta^{1+1}S' \) is calculated from only \( \delta^{1+1}S' \) of \( \psi \) without \( \chi \) (see details in Appendix D). The total energy density of the Alfvén wave and the induced wave, \( \delta^{1+1}S' \), remains dominantly negative. The integrated energy over the calculation region \( (E(t) \) in Equation (D4)) is shown in Figure 4. The total energy balance of the Alfvén wave and the induced wave remains constant. This shows that the total energy of the Alfvén wave and the induced wave is conserved. Notably, the negative energy of the fast wave indicates the decrease in the electromagnetic energy of the field with the azimuthally curved field lines. On the other hand, the integrated energy of the Alfvén wave over the
calculation region increases monotonically from negative values to positive as shown in Figure 4. This shows the Alfvén wave obtains energy from the background magnetic field. This phenomenon is explained by the relativistic mechanism, where the Alfvén wave has angular momentum. In the framework of relativity, the Alfvén wave propagating along the magnetic field line has angular momentum. Therefore, if the magnetic field line is curved azimuthally \( (I = 0) \), a torque onto the Alfvén wave is required to trace the magnetic field line. This means the force-free condition is broken and the azimuthal component of the Lorenz force becomes finite. That is, the angular momentum of the Alfvén wave changes. Then, the Alfvén wave should receive external torque to trace the rotating magnetic field line. Now, we confirmed that \( \chi \) recovers the energy conservation law in the BTZ coordinates as expected.

We also performed a calculation of the Alfvén wave along the magnetic field line of \( \Omega_F = \Omega_H = 0.6268 \), where the rotational energy of the black string is not marginally extracted by the Blandford–Znajek mechanism. As shown in Figure 5, the Alfvén wave propagates outwardly and the fast wave is induced. The induced wave propagates toward both sides as in the case of \( \Omega_F = 0.5 \) (the Blandford–Znajek mechanism extracts the black string rotational energy). However, the amplitude of the induced wave is very small compared to the case of \( \Omega_F = 0.5 \) that the energy contribution of the induced wave is negligible and \( \frac{\delta T}{\delta S} \) is almost equal to \( \frac{\delta S}{\delta T} \). In fact, the energy of the Alfvén wave is almost the same as the total energy (Figure 8). The panel with respect to the balance of the Alfvén wave in Figure 8 shows that the energy of the Alfvén wave decreases monotonically and remains positive from the initial stage.

Figure 9 shows the numerical result of the inwardly propagating pulse of the Alfvén wave along the magnetic field line of \( \Omega_F = 0.5 \) around the spinning black string with \( a = 0.9 \). The Alfvén wave pulse propagates inwardly and induces the fast wave. The induced wave propagates mainly inwardly with the Alfvén wave pulse. The energy contribution of the induced wave is not so large that \( \frac{\delta T}{\delta S} \) is almost equal to \( \frac{\delta S}{\delta T} \). In fact, the time evolution of the total energy is almost identical to that of the Alfvén wave (Figure 10). Then, in the case of the inwardly propagating wave from the outer
region, the induced fast wave is negligible for the energy conservation law. In conclusion, the fast wave induced by the Alfvén wave has a significant role in the case of an outwardly propagating Alfvén wave along the rotating curved magnetic field line with $0 < \Omega_F < \Omega_H$. However, strictly speaking, we have to consider the energy contribution of the fast wave induced by the Alfvén wave to guarantee the energy conservation and the force-free condition in the case of the Alfvén wave propagating along the rotating curved magnetic field line in the framework of relativity. In a nonrelativistic framework, the angular momentum (momentum) of the Alfvén wave vanishes and such a fast wave is never induced.

Notably, Noda et al. (2020) suggested the reflection of the inwardly propagating Alfvén wave near the horizon in the black string spacetime with a scalar function $f(z) = \cos \mu z$, which is an extension of the BTZ black hole string spacetime as

$$ds^2 = -\alpha^2 dt^2 + \frac{1}{\alpha^2} dr^2 + r^2 (\varphi - \Omega dt)^2 + f(z)^2 dz^2. \quad (64)$$

They showed that an incident wave can be reflected back outwards only for the case of $\mu^2 < 0$ ($f(z) = \cosh \mu z$) by investigating the structure of the effective potential for Alfvén waves, where the metric is no longer uniform in the $z$ direction. When $\mu$ vanishes, they find no reflection of the wave as we have shown. For the Kerr black hole case, there is no degree of freedom $\mu$ corresponding to the extension in the $z$ direction, and it is possible to consider the reflection of Alfvén waves depending on the structure of the magnetic field lines. Indeed, Noda et al. (2022) recently discussed the reflection of an inwardly propagating Alfvén wave along the magnetic field lines on the equatorial plane of the Kerr spacetime and demonstrated the superradiant scattering of the Alfvén wave is possible.

### 6. Concluding Remarks

We performed FFMD numerical simulations of Alfvén waves along a magnetic field line around a black string using Equation (23) of $\psi$ and Equation (41) of $\chi$, which should satisfy the force-free condition and energy conservation law in the BTZ coordinates. In the case of outwardly propagating Alfvén waves along the rotating curved magnetic field line with $0 < \Omega_F < \Omega_H$, the energy of the Alfvén wave is initially negative in the ergosphere and increases to a positive value out of the ergosphere. The Alfvén wave induces a second-order fast wave along the magnetic field line. The total energy of the Alfvén and induced fast waves is conserved, and $\chi$, which expresses the induced fast wave, plays a significant role in energy transport in the BTZ coordinates. In the other cases, such as with the inwardly propagating Alfvén wave or the background magnetosphere around the BTZ black string with $\Omega_F < 0$ or $\Omega_H < \Omega_F$, the energy of the Alfvén wave is always positive and decreases as the wave propagates along the magnetic field line, while the Alfvén wave also causes the fast wave.

The fast wave induced by the Alfvén wave is attributed to the angular momentum of the Alfvén wave (Figure 11).\(^7\) When the Alfvén wave propagates along a curved magnetic field line, the angular momentum changes due to the torque from the background magnetic field. The reaction of the torque induces the fast wave described by $\chi$. The above mechanism is expected to work in the relativistic magnetohydrodynamic (MHD) framework. Notably, in the nonrelativistic MHD framework, the angular momentum of the Alfvén wave vanishes, and such a fast wave is not induced. The second-order fast wave would be induced by the Alfvén wave in relativistic MHD. This mechanism explains the conversion of the Alfvén wave into a fast magnetosonic wave in the pulsar magnetosphere shown by Yuan et al. (2021). They showed the Alfvén wave loses energy due to the induction of the fast mode. The same energy conversion of the Alfvén wave is found in the case with $\Omega_F = 0.7 > \Omega_h$ (Figure 8). On the other hand, in the case with $\Omega_F = 0.5 < \Omega_h$, the energy of the Alfvén wave pulse increases as it propagates outwardly from the ergosphere (Figure 4). This amplification of the Alfvén wave is caused by the general relativistic effect with the negative energy of the Alfvén wave in the ergosphere.

As shown in Figure 3, with the energy extraction from the black string by the Blandford–Znajek mechanism ($0 < \Omega_F < \Omega_h$), the energy of the Alfvén wave inside the ergosphere is negative, and without energy extraction from the black string by the Blandford–Znajek mechanism ($\Omega_F > \Omega_h$), the energy is positive all the time (Figure 7). The difference between the two cases is explained by the following inequality derived from Equations (46) and (55) with $\chi = 0$,

$$\delta^{1+1}_{\chi I} = \frac{\Gamma^2 - I^2 \Omega^2}{2\alpha^2 X^2 \Delta^2} (\partial_{\tau} \psi)^2 + \frac{-I^2}{2X^2} (\partial_{X} \psi)^2 - \frac{I' \chi}{\alpha^2 \Delta^2} (2\alpha^2 \Omega_F + \Gamma \Omega) \delta^{1+1}_{\chi I},$$

where $\Gamma' = -\alpha^2 + X^2 (\Omega^2 - \Omega_F^2) = \Gamma + 2\Omega_F W = (1 + \Omega_F^2) X^2 - 1$. Here, we consider the condition of the negative energy density ($\delta^{1+1}_{\chi I} < 0$) of the Alfvén wave propagating outwardly ($\delta^{1+1}_{\chi I} > 0$). With respect to the zero point of $\Gamma'$, $X_c = 1/(1 + \Omega_F^2)$, we have

$$X_c - X_{LS} = -\frac{a}{1 - \Omega_F^2} (\Omega_H^{-} - \Omega_F) \Omega_F (\Omega_F - \Omega_H), \quad (65)$$

\(^7\) In the special-relativistic framework, the momentum density of the transverse wave is given by $S/c$, where $S$ is the Poynting flux and $c$ is the speed of light. In the nonrelativistic limit ($c \rightarrow \infty$), the momentum density vanishes.
where \( X_{LS} = r_{LS} \) and \( \Omega^+ = (1 + \sqrt{1 - a^2})/a > 1 > \Omega_f \). Then, if \( \Omega_f(\Omega_f - \Omega_H) > 0 \), we find \( \chi < X_{LS} \), and we have \( \chi' < 0 \). If \( \chi' < 0 \), the first and second terms of the left-hand side of Equation (65) are positive. When \( \Omega_f < 0 \), we have \( \Omega_f(\Omega_f - \Omega_H) > 0 \), so \( \chi = \eta_1(\Omega_H - \Omega_f) > 0 \) and \( 2\alpha^2\Omega_f + \Gamma\Omega < 0 \). Then, Equation (65) shows that if \( \Omega_f < 0 \), \( \delta^+\delta^X > 0 \). When \( \Omega_f > \Omega_H \), we also have \( \Omega_f(\Omega_f - \Omega_H) > 0 \), so \( \chi = \eta_1(\Omega_H - \Omega_f) < 0 \) and \( 2\alpha^2\Omega_f + \Gamma\Omega = \Gamma W + \Omega_f(\alpha^2 + \chi^2) > 0 \) because of \( W = \Omega - \Omega_f < \Omega_H - \Omega_f < 0 \). Then, Equation (65) shows that if \( \Omega_f > \Omega_H \), \( \delta^+\delta^X > 0 \). Considering the outwardly propagating Alfvén wave \( \delta^+\delta^X > 0 \), to realize \( \delta^+\delta^Y < 0 \), we require \( 0 < \Omega_f < \Omega_H \). This condition is the same as that of the black string rotational energy extraction due to the Blandford–Znakje mechanism. Notably, \( 0 < \Omega_f < \Omega_H \) is the necessary condition for the negative energy of the Alfvén wave, which is the necessary and sufficient condition for energy extraction from the black string by the Blandford–Znakje mechanism.

The outwardly propagating Alfvén wave along the magnetic field line with \( 0 < \Omega_f < \Omega_H \) has negative energy in the ergosphere. Then, the process of formation of the outwardly propagating Alfvén wave in the ergosphere generates the residual energy. In an MHD magnetosphere, we can imagine that the residual energy would generate a plasma blob with kinetic energy. Furthermore, the energy of the Alfvén wave increases as it propagates outwardly from the ergosphere because the energy of the Alfvén wave should be positive outside of the ergosphere. The total energy of the Alfvén wave and the induced fast wave is conserved, therefore the negative energy should be responsible for the induced fast wave, where the negative energy of the fast wave corresponds to a decrease in the electromagnetic energy of the azimuthally curved magnetic field. Figure 3 shows the induced fast wave spreads inwardly and outwardly. Thus, part of the fast wave propagates beyond the horizon, and the black string energy decreases. In summary, the rotational energy of the black string is extracted as the plasma blob and the Alfvén wave, where the energy of the black string decreases due to the fast wave with negative energy induced by the Alfvén wave. Such energy extraction of the black string is caused by the outwardly propagating Alfvén wave from the ergosphere around the black string. The Kerr black hole also has the ergosphere around it. Hence, a similar process is expected around the Kerr black hole, which is recognized as an astrophysical object. In the magnetosphere around a supermassive black hole in the active galactic nuclei, the above elementary process of black hole energy extraction by the Alfvén wave would be important to understand the observed high-energy phenomena.

To calculate the force-free field of the Alfvén and induced fast waves, we use Equation (23) for \( \psi \) and Equation (41) for \( \chi \) on the natural coordinates \( x' = (T, X, \rho, \chi) \). There is an apparent singularity point on the natural coordinates, such as \( g^{TT} = -\frac{\rho}{X} = 0 \). The position of the apparent singularity of the natural coordinates is shown in Figure 12. Equation (23) does not contain \( g^{TT} \), and then, we calculate the Alfvén wave over the region \( X > X_{LS} \). However, the time-evolution equation of \( \chi \) contains \( g^{TT} \) explicitly. Thus, \( \chi \) is not to be calculated on the singular point. The natural coordinates ease the calculations of the equations drastically, but it has a restriction in the numerical calculation region with respect to the singularity. We will solve the problem of apparent singularity with the natural coordinates in our next paper.

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## Appendix A

### Energy–Momentum Tensor and Force-free Condition for Alfvén Wave

We show the detailed calculation with respect to the energy–momentum tensor and the force-free condition in the case of the Alfvén wave only, where we assume

\[
\phi_1 = \tilde{\phi}_1 + \delta \tilde{\phi}_1 = -z + \psi, \quad \phi_2 = \tilde{\phi}_2 = \rho. \quad (A1)
\]

First, we show nonzero components of the field tensor of the equilibrium and the first- and second-order perturbations as follows. Nonvanishing components of the field tensor of the equilibrium are only

\[
\tilde{F}_{\mu \nu} = -\tilde{F}_{\nu \mu} = 1, \quad \tilde{F}^{\mu \nu} = -\tilde{F}^{\nu \mu} = g^{\mu \nu} \quad (\mu \neq \nu). \quad (A2)
\]

Nonzero components of the field tensor of the first-order perturbations are only

\[
\delta F_{\lambda \nu} = -\delta F_{\nu \lambda} = \partial_\lambda \psi \quad (\lambda \neq \rho), \quad \delta F^{\mu \nu} = W^{\mu \lambda \nu} \partial_\lambda \psi. \quad (A3)
\]

All components of the field tensor of the second-order perturbations vanish.

Then, we calculate

\[
F^{\mu \lambda} \delta F_{\mu \lambda} = F^{\mu \nu} \delta F_{\nu \mu} = g^{\mu \nu} \delta F_{\nu \mu} = 0, \quad (A4)
\]

\[
F^{\lambda \nu} \delta F_{\lambda \nu} = F_{\mu \nu} \delta F^{\mu \nu} = 2F_{\mu \rho} \delta F^{\mu \nu} = 2W^{\rho \lambda \nu} \partial_\lambda \psi = 2g^{\rho \lambda} g^{\mu \nu} \partial_\lambda \psi = 0. \quad (A5)
\]
We find
\[
\delta F^{\mu\nu} F_{\lambda} = \delta F^{\mu\nu} F_{\mu} + \delta F^{\mu\nu} F_{\nu} = \delta F^{\mu\nu} F_{\mu} - \delta F^{\mu\nu} F_{\nu} = W^{\mu\nu\rho\delta}(\partial_\delta F_{\mu}) - W^{\nu\mu\rho\delta}(\partial_\delta F_{\nu}) = 0.
\]

Because \( \nu = \rho \), we have \( \delta F_{\mu} = - \partial_\mu \psi = 0 \), and otherwise, we have \( \delta F_{\nu} = 0 \). Then, we conclude that when \( \mu \) and \( \nu \) are not \( z \), we have
\[
\delta T^\mu_\mu = 0. \tag{A7}
\]

To check the force-free condition, we calculate the Lorentz force,
\[
J^\mu = J^\mu F^\mu_\nu, \quad J^\mu = \nabla_\nu F^{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_\nu \left( \sqrt{-g} F^{\mu\nu} \right).
\]

First, we can confirm the Lorentz force of the equilibrium vanishes because the 4-current density of the equilibrium vanishes:
\[
J^\mu = \frac{1}{\sqrt{-g}} \partial_\nu \left( \sqrt{-g} F^{\mu\nu} \right) = \frac{1}{\sqrt{-g}} \partial_\nu \left( \sqrt{-g} F^{\mu\nu} \right) = 0,
\]
\[
\tilde{J}^\mu = J^\mu F^\mu_\nu = 0. \tag{A8}
\]

Second, we confirm the Lorentz force with respect to the first-order perturbation of Alfvén wave vanishes:
\[
\delta J^\mu = \delta J^\mu F^\mu_\nu = 0, \tag{A10}
\]
\[
\delta J^\mu = \delta J^\mu F^\mu_\nu = \delta J^\mu F^\mu_\nu = \delta J^\mu F^\mu_\nu = - \delta J^\mu = - \frac{1}{\sqrt{-g}} \partial_\nu \left( \sqrt{-g} F^{\mu\nu} \right) = 0,
\]
\[
\delta J^\mu = \delta J^\mu F^\mu_\nu = \delta J^\mu F^\mu_\nu = - \delta J^\mu = - \frac{1}{\sqrt{-g}} \partial_\nu \left( \sqrt{-g} F^{\mu\nu} \right) = 0,
\]
where \( i \) is \( T \) or \( X \). In the last equation of Equation (A12), we use Equation (23). Then, we confirmed the first order of the Lorentz force vanishes.

It is noted that we have \( \delta J^\mu = \delta J^\mu = 0 \). Using
\[
\delta J^\mu = \frac{1}{\sqrt{-g}} \partial_\nu \left( \sqrt{-g} \delta F^{\mu\nu} \right) = \frac{1}{\sqrt{-g}} \partial_\nu \left( \sqrt{-g} W^{\mu\nu\rho\delta}(\partial_\delta \psi) \right),
\]
we have
\[
\delta J^T = \frac{1}{\sqrt{-g}} \sum_{\lambda = T,X} \partial_\lambda \left( \sqrt{-g} W^{T\lambda\nu\rho}(\partial_\lambda \psi) \right) = \frac{1}{\sqrt{-g}} \sum_{\lambda = T,X} \partial_\lambda \left( \sqrt{-g} W^{T\lambda\nu\rho}(\partial_\lambda \psi) \right), \tag{A14}
\]
\[
\delta J^X = \frac{1}{\sqrt{-g}} \sum_{\lambda = T,X} \partial_\lambda \left( \sqrt{-g} W^{X\lambda\nu\rho}(\partial_\lambda \psi) \right) = \frac{1}{\sqrt{-g}} \sum_{\lambda = T,X} \partial_\lambda \left( \sqrt{-g} W^{X\lambda\nu\rho}(\partial_\lambda \psi) \right), \tag{A15}
\]
\[
\delta J^\rho = \delta J^X = 0, \tag{A16}
\]
where \( W^{TX\rho} = \frac{1}{\lambda_\lambda} \) and \( W^{TX\rho} = \Omega \). We also have
\[
\delta J^\mu = \frac{1}{\sqrt{-g}} \partial_\nu \left( \sqrt{-g} \delta F^{\mu\nu} \right) = 0. \tag{A17}
\]

We find three components of the Lorentz force vanish as
\[
\delta J^T = \delta J^X = \delta J^\rho = 0, \tag{A18}
\]
\[
\delta J^T = \delta J^X = \delta J^\rho = 0, \tag{A19}
\]
\[
\delta J^T = \delta J^X = \delta J^\rho = 0, \tag{A20}
\]

Appendix B

Calculation of the Energy Density and Energy Flux Density Corrected by \( \chi \)

We show the details of the calculation of the energy density and energy flux of the Alfvén wave and the induced fast wave described by \( \chi \). The Alfvén wave and the fast wave are given by the following perturbation, respectively,
\[
\phi_1 = \tilde{\phi}_1 + \delta \phi_1 = - \zeta + \psi(T, X), \tag{B1}
\]
\[
\phi_2 = \tilde{\phi}_2 + \delta \phi_2 = \rho + \chi(T, X). \tag{B2}
\]
To derive the time-evolution equation of \( \chi \), we use Equation (5). When we take \( i = 2 \), we find the trivial equation:
\[
\partial_t (\sqrt{-g} W^{\lambda \mu \nu \rho} \partial_\lambda \partial_\nu \partial_\rho) = 0.
\]
(B3)

When we take \( i = 1 \), we obtain the equation for \( \chi \):
\[
\partial_t (\sqrt{-g} W^{\lambda \mu \nu \rho} \partial_\nu \partial_\rho) = \partial_t (\sqrt{-g} W^{\lambda \mu \nu \rho} \partial_\nu \partial_\rho) = 0.
\]
(B4)

Then, we obtain the time-evolution equation of \( \chi \),
\[
\square \chi = \nabla_\mu \nabla_\nu \chi = \frac{1}{\sqrt{-g}} \partial_\lambda (\sqrt{-g} W^{\lambda \mu \nu \rho} \partial_\nu \partial_\rho) = 0.
\]
(B5)

When we consider \( \chi \), we found the force-free condition recovers as follows. When we take \( \nu = \rho \), we have
\[
\delta^2 f_\mu = \delta J^\mu \delta F_{\nu \rho} + \delta^2 J^\mu \partial_\rho = \delta J^\mu \partial_\rho = 0.
\]
(B6)

Otherwise, we have
\[
\delta^2 f_\mu = \delta J^\mu \delta F_{\nu \rho} + \delta^2 J^\mu \partial_\rho
= - \frac{1}{\sqrt{-g}} \partial_\lambda (\sqrt{-g} W^{\lambda \mu \nu \rho} \partial_\nu \partial_\rho) + \delta^2 J^\mu
= - \frac{1}{\sqrt{-g}} \partial_\lambda (\sqrt{-g} W^{\lambda \mu \nu \rho} \partial_\nu \partial_\rho)
+ \frac{1}{\sqrt{-g}} \partial_\lambda (\sqrt{-g} W^{\lambda \mu \nu \rho} \partial_\nu \partial_\rho) = 0.
\]
(B7)

Eventually, introducing \( \chi \), we recovered and confirmed the force-free condition.

We showed the detailed derivation of the equilibrium and first-order, and second-order of the values with respect to the conservation law. With respect to the energy density, we have
\[
S^0 = -\frac{1}{2} \tilde{F}^{0 \mu} \tilde{F}_{0 \mu} + \frac{1}{4} \tilde{F}^{\mu \nu} \tilde{F}_{\mu \nu} = \frac{1}{2} g^{\mu \nu} = \frac{I^2}{2X^2 \alpha^2},
\]
(B8)

\[
\delta S^0 = -\frac{1}{2} \delta F^{0 \mu} \delta F_{0 \mu} + \frac{1}{2} \delta F^{\mu \nu} \delta F_{\mu \nu} + \frac{1}{4} \delta F^{\mu \nu} \delta F_{\mu \nu} = -g^{\mu \nu} \partial_\sigma = 0,
\]
(B9)

\[
\delta^2 S^0 = -\frac{1}{2} \delta F^{0 \mu} \delta F_{0 \mu} + \frac{1}{2} \delta F^{\mu \nu} \delta F_{\mu \nu} - \frac{1}{2} F^{0 \mu} \delta^2 F_{0 \mu}
+ \frac{1}{4} \delta F^{\mu \nu} \delta F_{\mu \nu} = -\frac{1}{2} \frac{I^2}{\alpha^2 X^2} \partial_\sigma + \frac{1}{2} \frac{I^2}{\alpha^2 X^2} \partial_\sigma
= -\frac{\gamma F}{2 \Gamma} \left[ \lambda (\partial_\sigma) + (\partial_\sigma)^2 \right] + \frac{I^2}{\Gamma} \partial_\sigma,
\]
(B10)

With respect to the energy flux, we have
\[
\delta^X = \delta F^{X \mu} \tilde{F}_{0 \mu} + \delta^X \delta F_{0 \mu} = 0,
\]
(B11)

\[
\delta^2 S^X = \delta F^{X \mu} \delta F_{0 \mu} + \delta F^{X \mu} \delta F_{0 \mu} = \delta F^{X \mu} \delta F_{0 \mu} + \delta F^{X \mu} \delta F_{0 \mu} = \frac{G}{\chi^2} \partial_\sigma \psi - g^{\mu \nu} \partial_\sigma
\]
(B12)

\[
= -\frac{1}{X^2} \partial_\sigma \psi - \psi \partial_\sigma.
\]
(B13)

With respect to the angular momentum, we introduced \( \chi \) to recover the force-free condition of the Alfvén wave up to the second order of the perturbation so that we become able to consider the conservation law of angular momentum up to second order. The axial Killing vector \( \xi^\mu = (0, 0, 1, 0) \) yields the 4-angular momentum flux density \( M^\mu = \xi^\mu T^\nu_\nu \), and we obtain the angular momentum conservation law in the corotating natural coordinates,
\[
\nabla_\mu M^\mu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} M^\mu) = \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} M^\mu \right) = \xi^\mu T^\nu_\nu = f^L_
u = 0.
\]
(B14)

The 4-angular momentum density \( M^\mu \) in the corotating natural coordinates is calculated as
\[
M^\mu = \xi^\nu T^\mu_\nu = T^\mu_\nu = F^{\mu \nu} F_{\nu \rho}.
\]
(B15)

Then, we have the 4-angular momentum density of the equilibrium, and first- and second-order perturbations of the linear Alfvén wave,
\[
\tilde{M}^\mu = \tilde{T}^\mu_\nu = \tilde{F}^{\mu \nu} \tilde{F}_{\nu \rho} = g^{\mu \rho},
\]
(B16)

\[
\delta M^\mu = \delta F^{\mu \nu} \tilde{F}_{\nu \rho} + \delta F^{\mu \nu} \delta F_{\nu \rho} = \delta F^{\mu \nu} \tilde{F}_{\nu \rho} + \delta F^{\mu \nu} \delta F_{\nu \rho} = \delta F^{\mu \nu} - g^{\mu \rho} \partial_\sigma \psi = \mu^{\mu \nu \rho \sigma} \partial_\sigma \psi = 0.
\]
(B17)

\[
\delta^2 M^\mu = \delta^2 F^{\mu \nu} \tilde{F}_{\nu \rho} + \delta^2 F^{\mu \nu} \delta F_{\nu \rho} + \delta F^{\mu \nu} \delta^2 F_{\nu \rho} = \delta^2 F^{\mu \nu} \tilde{F}_{\nu \rho} + \delta^2 F^{\mu \nu} \delta F_{\nu \rho} = -W^{\mu \nu \rho \sigma} \partial_\sigma \psi - \delta F^{\mu \nu} \delta F_{\nu \rho} = -W^{\mu \nu \rho \sigma} \partial_\sigma \psi - \delta F^{\mu \nu} \delta F_{\nu \rho} = -W^{\mu \nu \rho \sigma} \partial_\sigma \psi - \delta F^{\mu \nu} \delta F_{\nu \rho}.
\]
(B18)

where we assume \( \mu = z \). Then, we confirm the conservation of the angular momentum for the second order of the perturbation as
\[
\nabla_\mu \delta^2 M^\mu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} M^\mu) = \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} M^\mu \right) = 0,
\]
(B19)

where we used Equation (23). We have the energy density and energy flux density of the equilibrium, and first- and second-order perturbations of the linear Alfvén wave distinctively,
\[
\tilde{M}^0 = g^{\mu \rho} = -\frac{1}{\alpha^2} \frac{\gamma}{\alpha^2} (\Omega - \Omega_E), \quad \tilde{M}^X = g^{\mu \rho} = \frac{I^2}{\chi^2}.
\]
(B20)
Then, we have the energy density and energy flux of the equilibrium, and first- and second-order perturbations of the linear Alfvén wave, $S' = S^T + \Omega_F M^T - I_0(S^X + \Omega_F M^X)$, $\delta S' = \delta S^T + \Omega_F \delta M^T - I_0(\delta S^X + \Omega_F \delta M^X) = 0$, $\delta^2 S' = \delta^2 S^T + \Omega_F \delta^2 M^T - I_0(\delta^2 S^X + \Omega_F \delta^2 M^X) = \delta^{i+1} S' + \delta^{i+0} S'$, where

$$\delta^{i+1} S' = \frac{1}{2 \alpha^2} \left[ \frac{1}{X^2} (\Gamma^2 + \Pi^2 (\alpha^2 - X^2 (\Omega^2 - \Omega_F^2))) (\partial_T \psi)^2 
+ \frac{1}{\alpha^2 X^2} (\alpha^2 - X^2 (\Omega^2 - \Omega_F^2)) (\partial_X \psi)^2 
- 2 \frac{1}{\alpha^2 X^3} (\Gamma (\Omega - \Omega_F) + \Omega_F (\alpha^2 + X^2 (\Omega - \Omega_F)^2)) \partial_T \psi \partial_X \psi \right].$$

$$\delta^{i+0} S' = g^{3m} \partial_X \chi + \Omega_F g^{3T} \partial_X \chi + I_0(g^{3m} \partial_T \chi + \Omega_F g^{3T} \partial_X \chi), = \frac{I'}{X} \partial_X \chi - \frac{1}{\alpha^2 \Gamma} (\Omega_F \Gamma - \Pi^2 W) \partial_T \chi. \quad (C5)$$

When we use the tortoise coordinate $x$, we have the following expressions,

$$\delta^{i+1} S'_x = \frac{1}{2 \alpha^2} \left[ \frac{1}{X^2} (\Gamma^2 + \Pi^2 (\alpha^2 - X^2 (\Omega^2 - \Omega_F^2))) (\partial_T \psi)^2 
+ \gamma_F^2 (\alpha^2 - X^2 (\Omega^2 - \Omega_F^2)) (\partial_X \psi)^2 
+ 2 \frac{1}{\alpha^2} \gamma_F (\Gamma (\Omega - \Omega_F) + \Omega_F (\alpha^2 + X^2 (\Omega - \Omega_F)^2)) \partial_T \psi \partial_X \psi \right].$$

$$\delta^{i+0} S'_x = \frac{1}{\sqrt{-g}} \partial_x (\sqrt{-g} S') = \frac{1}{\sqrt{-g}} \partial_x (\sqrt{-g} S') + \frac{1}{\sqrt{-g}} \partial_t (\sqrt{-g} S') = 0. \quad (D1)$$

\[\text{Appendix D} \]

Energy Transport in the BTZ Spacetime

We recovered the force-free condition in the BTZ coordinates with the additional variable $\chi$, so we have the energy and momentum conservation, $\nabla_x T_{\mu\nu} = 0$, up to the second order of the perturbations. When we use the time-like Killing vector $\xi_\mu^0 = (1, 0, 0, 0)$, we have the energy conservation law in the BTZ coordinates,

$$\nabla_x S' = \frac{1}{\sqrt{-g}} \partial_x (\sqrt{-g} S') = \frac{1}{\sqrt{-g}} \partial_t (\sqrt{-g} S') + \frac{1}{\sqrt{-g}} \partial_x (\sqrt{-g} S') = 0. \quad (D1)$$

where $S' = -\xi_\mu^0 T_{\mu\nu}$ is the 4-energy flux density in the BTZ coordinates. In the case of axisymmetry and translation

$$\delta^{i+0} S' = \frac{\gamma_F}{\alpha^2} \partial_x (\sqrt{-g} S') + \frac{1}{\sqrt{-g}} \partial_t (\sqrt{-g} S') = 0. \quad (D1)$$
symmetry with respect to the $z$ direction, we have
\[
\frac{\partial}{\partial t} \sqrt{-g} S^t + \frac{\partial}{\partial r} \sqrt{-g} S^r = 0. \tag{D2}
\]

It reads
\[
\int_{r_1}^{r_2} dt \int_{r_1}^{r_2} dr \frac{\partial}{\partial t} \sqrt{-g} S^t + \int_{r_1}^{r_2} dt \int_{r_1}^{r_2} dr \frac{\partial}{\partial r} \sqrt{-g} S^r
\]
\[
= \int_{r_1}^{r_2} dr \sqrt{-g} S^t(r, t) - \int_{r_1}^{r_2} dr \sqrt{-g} S^r(r, t)
\]
\[
+ \int_{r_1}^{r_2} dt \int_{r_1}^{r_2} dr \sqrt{-g} S^t(r, t) - \int_{r_1}^{r_2} dt \int_{r_1}^{r_2} dr \sqrt{-g} S^r(r, t) = 0. \tag{D3}
\]

When we define the total energy between $r = r_1$ and $r = r_2$ and the energy flux at $r = r_b$ ($b = 1, 2$) by
\[
E(t) = \int_{r_1}^{r_2} \sqrt{-g} S^t(r, t) dr,
\]
\[
F_b(t) = \int_{r_1}^{r_2} \sqrt{-g} S^r(r_b, t) dt,
\]
respectively, we obtain the conservation quantity as
\[
E(t) - F_1(t) + F_2(t) = E(t_b). \tag{D6}
\]

When we consider the second order of the perturbations, we have the quantity with respect to the energy conservation of the wave,
\[
\delta^2 E(t) - \delta^2 F_1(t) + \delta^2 F_2(t) = \delta^2 E(t_b), \tag{D7}
\]
where $\delta^2 E(t) = \int_{r_1}^{r_2} \sqrt{-g} \delta^2 S^t(r, t) dr$ and $\delta^2 F_b(t) = \int_{r_1}^{r_2} \sqrt{-g} \delta^2 S^r(r_b, t) dt$ ($b = 1, 2$).

**Appendix E**

**Numerical Method of the 1D Wave Equation**

Equations (27) with the additional term $-\kappa(x)\psi$ in its right-hand side and Equation (41) are written as multidimensional time-development equations:

\[
\frac{\partial \psi}{\partial T} = -\frac{1}{\lambda(x)} \frac{\partial v}{\partial x} - \kappa(x)w,
\]
\[
\frac{\partial v}{\partial T} = -\frac{\partial \psi}{\partial x},
\]
\[
\frac{\partial w}{\partial T} = \psi,
\]
\[
\frac{\partial \chi}{\partial T} = -h(x) \frac{\partial}{\partial x} (b(x)\chi + c(x)u) + k(x)\chi + G,
\]
\[
\frac{\partial u}{\partial T} = \frac{\partial \chi}{\partial x},
\]
\[
\frac{\partial G}{\partial T} = g(x). \tag{E1}
\]

where $u(x, T), v(x, T), w(x, T)$, and $G(x, T)$ are new variables, and $h(x) = -\frac{\gamma}{\sqrt{-g} g^{TT}}, b(x) = -2\sqrt{-g} g^{TX}, k(x) = \frac{1}{\sqrt{-g} g^{TX}}, c(x) = \sqrt{-g} g^{XX}f$, and $g(x) = \frac{1}{g}$. We use the two-step Lax–Wendroff scheme for the multi-dimension time-development equation:
\[
\frac{\partial u}{\partial t} = -h \odot \frac{\partial w}{\partial t} + f. \tag{E2}
\]

where $u$ is the array of the conserved quantity density, $w$ is the array of the flux density of the conserved quantity, and $f$ is the array of the source density of the conserved variable:
\[
u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x} h_j^n \odot (w_{j+1}^n - w_j^n) + \Delta g_j^n, \tag{E3}
\]
\[
u_j^n = \frac{1}{2} \left[ u_j^n + u_j^{n+1} - \frac{\Delta t}{\Delta x} h_j^n \odot (w_{j+1}^{n+1} - w_{j+1}^n) + \Delta g_j^{n+1} \right]. \tag{E4}
\]

Here, we used $\odot$ to express the product of two vectors $a = (a_1, a_2, \cdots)^T$ and $b = (b_1, b_2, \cdots)^T$,
\[
a \odot b = \left( a_1 b_1, a_2 b_2, \cdots \right). \tag{E5}
\]

Equations (E1) are given by
\[
u = \begin{pmatrix} \psi \\ w \\ v \\ \chi \\ a \end{pmatrix}, \quad h = \begin{pmatrix} \frac{1}{\lambda(x)} \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad w = \begin{pmatrix} v \\ \psi \\ b\chi + cu \\ \chi \end{pmatrix}, \quad f = \begin{pmatrix} -\kappa(x)w \\ 0 \\ \psi \\ k\chi + G \end{pmatrix}. \tag{E6}
\]

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