Hook’s law as Lie group

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Abstract. In this work classification of anisotropic linear elastic materials with residual deformations assuming that the material constants depend on the residual deformations is offered. Each generalised Hook’s law is assigned some Lie point group, and this group in its turn is assigned Lie algebra.

1. Introduction

One of the basic tasks of mathematics is classification. An example of such a task very well solved by means of theory of groups is classification of crystals.

A series of authors noticed analogies between the theory of elasticity and groups of continuous transformations (Lie groups). First to be mentioned is A. Love [1]. He showed that linear homogeneous deformations of an elastic body generate a group of continuous transformations. And also one the authors of the monography shared similar views. Some thoughts on this matter are included in Works [2, 3].

The very idea of classification of elastic materials on the basis of Hook’s law is not new. It goes back to classification of crystals. It is believed that the elastic material inherits symmetry properties of its component crystals. That is why elastic materials are classified according to syngonies. Let us remind that Nikola Stenton (1669) devided all the crystals into seven syngonies (similar angled): triclinic, monoclinic, rhombic, trigonal, hexagonal, tetragonal and cubic. Then, using the notion of crystal lattice Auguste Brave (1848) divided crystals into 14 classes. In 1867 Aksel Vilgelmoivch Gadolin introduced 32 crystal classes and in 1890-1894 Evgraf Stepanovich Fyodorov completed the classification by introducing 230 crystal groups.

It is necessary to highlight that all these ingenious findings were made long before experimental discovery of the crystal lattice with the help of X-rays. This classification is still up-to-date and is widely used in various spheres of science.

Elastic materials that are described by linear Hook’s law can also be successfully classified according to the number of constants included in Hook’s law in accordance with the crystal symmetry (syngony). This number can be 81, 21, 13, 9, 5 and 2.

In this work classification of anisotropic linear elastic materials with residual deformations assuming that the material constants depend on the residual deformations is offered. Each generalised Hook’s law is assigned some Lie point group, and this group in its turn is assigned Lie algebra. Each two-dimensional subalgebra of this algebra is matched with some experiment on combined loading which allows us to determine dependence of the material constants on residual deformations. This allows us to carry out classification of such materials and divide them into a considerably large
number of classes.

The plan of the article is as follows: 1) using the definition of Lie group and Lie algebra of continuous transformations to write Hook’s law for materials with residual deformations and to verify that these laws comply with all the properties of the continuous transformations group, 2) as per these groups to build Lie algebras for different variants of anisotropic elasticity theory, 3) to describe possible experiments that allow us to classify elastic materials.

2. Necessary information from the theory of continuous groups of transformations

Let us remind necessary information from the theory of Lie groups of continuous transformations that is required for understanding of further presentation. Here we will adhere to the original source that is the recently translated into Russian Book of Sophus Lie [4].

Assume \( x, x' \in \mathbb{R}^n, a \in \mathbb{R}' \). Let us consider the transformations in the form of

\[
x' = f(x; a)
\]  

(1)

where \( f \) – some smooth function, \( a \) – parameters that as a rule change in some neighbourhood of zero.

Assume \( x'' = f(x'; b) \), then this equation takes place

\[
x'' = f(x'; b) = f(f(x; a); b) = f(x; \varphi(a, b))
\]  

(2)

Here Function \( \varphi(a, b) \) is the law of multiplication of parameters which is assumed by the smooth function.

Let us demand that the continuous transformations (1) create a group. In this case Function \( \varphi(a, b) \) must comply with the following properties

\[
\begin{align*}
\varphi(0, a) &= \varphi(a, 0) = a \\
\varphi(a^{-1}, a) &= \varphi(a, a^{-1}) = 0 \\
\varphi(a, \varphi(b, c)) &= \varphi(\varphi(a, b), c).
\end{align*}
\]  

(3)

If properties (3) are fulfilled in the neighbourhood of zero of Space \( \mathbb{R}' \), then the group built is local and is called local Lie group. If \( a \in \mathbb{R}' \), then such a group of transformations is called one-parameter, and if \( a \in \mathbb{R}^1 \), then correspondingly – \( r \) – parameter. It may be shown that the \( r \) – parameter Lie group denoted as \( G_r \) is fully determined by its one-parameter subgroups \( G_i \).

Each local one-dimensional subgroup \( G_i \) is assigned a tangent vector according to the following rule

\[
x'_i = f_i(x; a) \rightarrow \xi_i = \frac{df_i}{da} \bigg|_{a=0},
\]  

(4)

and each tangent vector is assigned the operator in the form of

\[
X_a = \xi_i \frac{\partial}{\partial x_i}.
\]  

(5)

Operators (4) for Group \( G_r \) create Lie algebra \( L_r \), that can be classified as per their structure determined by the commutators in the form of

\[
[X, Y] = [\xi_i \frac{\partial}{\partial x_i}, \eta_j \frac{\partial}{\partial x_j}] = (\xi_j \frac{\partial \eta_i}{\partial x_j} - \eta_j \frac{\partial \xi_i}{\partial x_j}) \frac{\partial}{\partial x_i}.
\]
Let us give a simple example that will allow us to carry out building of the groups and Lie algebras for the theory of elasticity with initial deformations.

Example [4].

Let us consider the three-parameter group of transformations of the one-dimensional space

\[ x' = \frac{x + a_1}{a_2x + a_3} \]  

Then \[ x'' = \frac{x' + b_1}{b_2x' + b_3} \]

In this case Function \( \varphi(a, b) \) is written as

\[ \phi_1 = \frac{a_1 + b_1a_3}{1 + b_1a_2}, \phi_2 = \frac{b_2 + b_1a_2}{1 + b_1a_2}, \phi_3 = \frac{b_2a_3 + b_3a_3}{1 + b_1a_2}. \]

It is not difficult to see that to the identity transformation there answer the values of \( a_i = 0, a_2 = 0, a_3 = 1 \). To the backward transformations there correspond the parameters of \( -a_1/a_3, -a_2/a_3, 1/a_3 \).

All the properties (3) are fulfilled that is why (6) is Group \( G_3 \). Let us build Lie algebra \( L_3 \) corresponding to this group. We have

\[ \xi_1 = \left. \frac{dx'}{da_1} \right|_{(0,0,1)} = 1, \xi_2 = \left. \frac{dx'}{da_2} \right|_{(0,0,1)} = -x^2, \xi_3 = \left. \frac{dx'}{da_3} \right|_{(0,0,1)} = x. \]

Therefore Lie algebra \( L_3 \) is created by the operators

\[ X_1 = \frac{\partial}{\partial x}, X_2 = x^2 \frac{\partial}{\partial x}, X_3 = x \frac{\partial}{\partial x}. \]

At that \([X_1, X_2] = X_3, [X_1, X_3] = X_1, [X_2, X_3] = X_2 - 2X_3\).

This example shows there are two ways to show that the transformations (6) create a group of continuous transformations: to check properties (3) or to build the corresponding Lie algebra and check its closeness relative to the operation of commutation.

Later on we will use both of the ways when we will be showing that the generalised Hook’s laws built by us create groups of continuous transformations.

3. Classification of elastic bodies

The basis of the linear theory of elasticity of smaller deformations is Hook’s law which for anisotropic materials is written as [4, 5]:

\[ \sigma_{ij} = c_{ijkl} \varepsilon_{kl} \quad (i,j,k,l=1,2,3) \]  

or

\[ \varepsilon_{ij} = a_{ijkl} \sigma_{kl} \quad (i,j,k,l=1,2,3) \]

where \( \sigma_{ij} \) – components of symmetrical stress tensor, \( \varepsilon_{ij} \) components of symmetrical deformation tensor, \( c_{ijkl}, a_{ijkl} \) – mechanical material constants, for the repeated indices summarisation is carried out. Values \( c_{ijkl}, a_{ijkl} \) create tensor that has 81 components. By virtue of the symmetry of Tensors \( \sigma_{ij}, \varepsilon_{ij} \) there will be no more than 21 of independent components. This number of constants correspond to
triclinic crystal system.

Let us introduce the structure of Lie group into the theories of elasticity. Hook’s law (7) in an even more general form can be written in two different ways [2]

\[ \sigma_{ij} = f_{ij}(\epsilon_{kl}) \quad (i,j,k,l=1,2,3) \]  
\[ \epsilon_{ij} = g_{ij}(\sigma_{kl}) \quad (i,j,k,l=1,2,3) \]  

This writing reminds (1), but there is a significant difference: in the left-hand part there is no \( \sigma_{ij} \) or \( \epsilon_{ij} \). Can they be introduced into there? They can be if we assume that in the elastic medium there are residual stresses \( \sigma_{ij}^0 \) (residual deformations \( \epsilon_{ij}^0 \)). Now if we remember that \( \epsilon_{kl} \in \mathbb{R}^6 (\sigma_{kl} \in \mathbb{R}^6) \), then the relations (9)-(10) will be written as

\[ \sigma_{ij}^* = g_{ij}(\sigma_{ij}^0, \epsilon_{kl}) \]  
\[ \epsilon_{ij}^* = f_{ij}(\epsilon_{ij}^0, \sigma_{kl}) \]  

The question arises: which of these formulas is better to be used to build the theory of elasticity, which one of them better corresponds to the group of continuous transformations (1.64)? In the author’s opinion it is better to use Formulas (12), the reasons for that will be explained later. Let us highlight that elastic materials with initial stress condition were started to be looked at as early as by Cauchy, see [4] and the references available there, in recent time the properties of such materials were studied in [6]. The presence of Laws (11) or (12) mean that the constants included in Hook’s law (7) or (8) depend on the initial stresses (deformations). This question if it is so can only be answered after performing experiments for the certain given material. Some of such experiments will be discussed below.

Now it is remaining to demand fulfilling of the properties (3). From (12) we will obtain the group determined by six parameters, in other words Group \( G_6 \).

\[ \epsilon_{ij}^* = f_{ij}(\epsilon_{ij}^0, \sigma_{kl}), \epsilon_{ij}^0 = f_{ij}(\epsilon_{ij}^*, \sigma_{kl}^{-1}), \epsilon_{ij}^0 = f_{ij}(\epsilon_{ij}^0, 0), \]
\[ \epsilon_{ij}^* = f_{ij}(\epsilon_{ij}^0, \sigma_{kl}), \epsilon_{ij}^0 = f_{ij}(\epsilon_{ij}^0, \sigma_{kl}), \epsilon_{ij}^0 = f_{ij}(\epsilon_{ij}^0, \phi(\sigma_{kl}^0, \sigma_{kl}^0)) \].

These conditions are fulfilled by virtue of linearity of the equations of elasticity and the smallness of the deformation tensor components.

Of course we can check all the conditions of the continuous group as it is carried out in the example of S.Lie, but we can do it in another way: to calculate operators of Lie algebra and check if they create Lie algebra of dimension 6. If it is so then Transformations (12) create the group of continuous transformations.

Since \( \epsilon_{ij}, \epsilon_{ij}^0 \) are small, see [5, 7], then we expand \( f_{ij}(\epsilon_{ij}^0, \sigma_{kl}) \) according to Taylor formula in the neighbourhood of zero. We have

\[ f_{ij}(\epsilon_{ij}^0, \sigma_{kl}) = a_{ijkl}(1 + \alpha_{mn} \epsilon_{mn}^0) \sigma_{kl} = a_{ijkl} \alpha_{mn} \epsilon_{mn}^0 \sigma_{kl} + \epsilon_{ij}^0 \]  

Let us highlight that if \( \epsilon_{ij}^0 = 0 \), then Relation (14) coincides with (8). Let us calculate the operator corresponding to Transformation (14), assuming \( \sigma_{kl} \) to be the parameters of the group of transformations. We have

\[ X_{ij} = a_{ijkl} \alpha_{mn} \epsilon_{mn}^0 \frac{\partial}{\partial \epsilon_{kl}}. \]
It is not difficult to see that the commutator of any two operators in the form of (15) is a linear combination of the operators in the form of (15). This depicts that the generalised Hook’s law (14) in the case of small elastic deformations is a group of continuous transformations of deformation tensor components.

4. In this section we will consider some possible variants of the generalised Hook’s law

As the first example of classification we will consider different variants of Hook’s law that are used to describe elastic torsion. In this case only two components of stress tensor \( \tau_{13}, \tau_{23} \) will differ from zero.

4.1. Isotropic case

We have

\[
\varepsilon_{13} = \varepsilon_{13}^0 + \mu \tau_{13}, \quad \varepsilon_{23} = \varepsilon_{23}^0 + \mu \tau_{23},
\]

where \( \varepsilon_{13}^0, \varepsilon_{23}^0 \) - initial stresses, \( \mu \) - is function \( \varepsilon_{13}, \varepsilon_{23} \).

We obtain

\[
X_{13} = \mu \frac{\partial}{\partial \varepsilon_{13}}, \quad X_{23} = \mu \frac{\partial}{\partial \varepsilon_{23}}.
\]

The commutator of operators (17) equals to

\[
[X_{13}, X_{23}] = \mu \frac{\partial \mu}{\partial \varepsilon_{13}} \frac{\partial}{\partial \varepsilon_{23}} - \mu \frac{\partial \mu}{\partial \varepsilon_{23}} \frac{\partial}{\partial \varepsilon_{13}}.
\]

Let us equate the commutator to the linear combination of operators (17), we will see that in this case \( \mu \) is a linear function \( \mu = \alpha \varepsilon_{13}^0 + \beta \varepsilon_{23}^0 + \gamma, \quad \alpha, \beta, \gamma - const \)

4.2. Orthotropic symmetry

We have

\[
\varepsilon_{13} = \varepsilon_{13}^0 + a_{44} \tau_{13}, \quad \varepsilon_{23} = \varepsilon_{23}^0 + a_{55} \tau_{23},
\]

where \( \varepsilon_{13}^0, \varepsilon_{23}^0 \) - initial stresses, \( a_{44}, a_{55} \) - is function \( \varepsilon_{13}, \varepsilon_{23} \).

We obtain

\[
X_{13} = a_{44} \frac{\partial}{\partial \varepsilon_{13}}, \quad X_{23} = a_{55} \frac{\partial}{\partial \varepsilon_{23}}.
\]

The commutator of operators (19) equals to

\[
[X_{13}, X_{23}] = a_{44} \frac{\partial a_{55}}{\partial \varepsilon_{13}} \frac{\partial}{\partial \varepsilon_{23}} - a_{55} \frac{\partial a_{44}}{\partial \varepsilon_{23}} \frac{\partial}{\partial \varepsilon_{13}}.
\]

Let us equate it to the linear combination of operators (19) and we will see that in this case the coefficients satisfy the system of equations

\[
a_{44} \frac{\partial a_{55}}{\partial \varepsilon_{13}} = A a_{55}, a_{55} \frac{\partial a_{44}}{\partial \varepsilon_{23}} = B a_{44},
\]

here \( A, B \) – are arbitrary constants.
4.3. Case of monoclinic crystal symmetry

We have

\[ e_{13} = e_{13}^0 + a_{44} \tau_{13} + a_{45} \tau_{23}, e_{23} = e_{23}^0 + a_{45} \tau_{13} + a_{55} \tau_{23}, \]

where \( \tau_{13}^0, \tau_{23}^0 \) – initial stresses, \( a_{44}, a_{55}, a_{45} \) – is function \( e_{13}^0, e_{23}^0 \).

We obtain

\[ X_{13} = a_{55} \frac{\partial}{\partial e_{13}} + a_{45} \frac{\partial}{\partial e_{23}} X_{23} = a_{55} \frac{\partial}{\partial e_{23}} + a_{45} \frac{\partial}{\partial e_{13}} \]

The commutator of operators (21) equals to

\[ [X_{13}, X_{23}] = \left( a_{44} \frac{\partial}{\partial e_{13}} - a_{45} \frac{\partial}{\partial e_{23}} - a_{45} \frac{\partial}{\partial e_{23}} - a_{44} \frac{\partial}{\partial e_{13}} \right) \frac{\partial}{\partial e_{13}} + \]

\[ + \left( a_{44} \frac{\partial}{\partial e_{13}} + a_{45} \frac{\partial}{\partial e_{23}} - a_{45} \frac{\partial}{\partial e_{23}} - a_{44} \frac{\partial}{\partial e_{13}} \right) \frac{\partial}{\partial e_{23}} \]

Let us equate it to the linear combination of operators \( X_{13}, X_{23} \) and we will see that in this case

\( a_{44}, a_{55}, a_{45} \) satisfy the two differential equations:

\[ \left( a_{44} \frac{\partial}{\partial e_{13}} + a_{45} \frac{\partial}{\partial e_{23}} - a_{45} \frac{\partial}{\partial e_{23}} - a_{44} \frac{\partial}{\partial e_{13}} \right) C_1 a_{44} + C_2 a_{45} = 0, \]

\[ \left( a_{44} \frac{\partial}{\partial e_{13}} + a_{45} \frac{\partial}{\partial e_{23}} - a_{45} \frac{\partial}{\partial e_{23}} - a_{44} \frac{\partial}{\partial e_{13}} \right) C_1 a_{45} + C_2 a_{55} = 0. \]

Here \( C_i \) – arbitrary constants.

4.4. In conclusion of this section we will consider a special case of orthotropic medium

In conclusion of this section we will consider a special case of orthotropic medium [7].

\[ e_{11} = a_{11} \sigma_{11} + a_{12} \sigma_{22} + a_{13} \sigma_{33}, e_{22} = a_{12} \sigma_{11} + a_{11} \sigma_{22} + a_{23} \sigma_{33}, \]

\[ e_{33} = a_{13} \sigma_{11} + a_{33} \sigma_{22} + a_{11} \sigma_{33}, \]

After first loading the body obtains residual deformations

\[ e_{11}^0 = a_{11} \sigma_{11} + a_{12} \sigma_{22} + a_{13} \sigma_{33}, e_{22}^0 = a_{22} \sigma_{11} + a_{11} \sigma_{22} + a_{23} \sigma_{33}, \]

\[ e_{33}^0 = a_{13} \sigma_{11} + a_{33} \sigma_{22} + a_{11} \sigma_{33}, \]

Assume expanding of coefficients according to Taylor formula is written as

\[ a_{11} = a_{11}(1 + \alpha \epsilon_{11}^0 + \beta \epsilon_{22}^0 + \gamma \epsilon_{33}^0), a_{44} = a_{44}(1 + \alpha \epsilon_{11}^0), a_{66} = a_{66}(1 + \alpha \epsilon_{11}^0), a_{55} = a_{55}(1 + \alpha \epsilon_{11}^0). \]

For simplicity we consider other coefficients constant.

Then taking into account (23) we will get the following formulas

\[ e_{11} = a_{11}(\alpha \epsilon_{11}^0 + \beta \epsilon_{22}^0 + \gamma \epsilon_{33}^0) \sigma_{11} + \epsilon_{11}^0, e_{22} = a_{11}(\alpha \epsilon_{11}^0 + \beta \epsilon_{22}^0 + \gamma \epsilon_{33}^0) \sigma_{22} + \epsilon_{22}^0, \]

\[ e_{33} = a_{11}(\alpha \epsilon_{11}^0 + \beta \epsilon_{22}^0 + \gamma \epsilon_{33}^0) \sigma_{33} + \epsilon_{33}^0, e_{12} = a_{44} \epsilon_{11}^0, e_{13} = a_{44} \epsilon_{44}^0, e_{23} = a_{55} \epsilon_{55}^0, e_{13} = a_{44} \epsilon_{11}^0 \]

We build Lie algebra according to Group (25) we have
\[ X_{11} = a_{11}(\alpha e_{11}^0 + \beta e_{22}^0 + \gamma e_{33}^0) \frac{\partial}{\partial e_{11}}, X_{22} = a_{11}(\alpha e_{11}^0 + \beta e_{22}^0 + \gamma e_{33}^0) \frac{\partial}{\partial e_{22}}, \]
\[ X_{33} = a_{11}(\alpha e_{11}^0 + \beta e_{22}^0 + \gamma e_{33}^0) \frac{\partial}{\partial e_{33}}, X_{12} = a_{12,1} \frac{\partial}{\partial e_{12}}, X_{23} = a_{23,2} \frac{\partial}{\partial e_{23}}, X_{13} = a_{13,2} \frac{\partial}{\partial e_{13}}. \]

We calculate the commutators of Operators (25). We have
\[ [X_{11}, X_{22}] = \alpha X_2 - \beta X_1, [X_{11}, X_{33}] = \alpha X_3 - \gamma X_1, [X_{22}, X_{33}] = \beta X_3 - \gamma X_2. \]

Other commutators are equal to zero.

Since Operators (26) create Lie algebra of dimension 6, then Hook’s law in the form of (25) is a six-parameter group of continuous transformations.

5. Experimentation on the classification of materials

For simplicity we will consider mental experiments on combined loading that will allow us to determine the type of Hook’s law.

We have
\[ \varepsilon_{13} = (\alpha e_{13}^0 + \beta e_{23}^0 + \mu) \tau_{13} + e_{13}^0, \varepsilon_{23} = (\alpha e_{13}^0 + \beta e_{23}^0 + \mu) \tau_{23} + e_{23}^0. \]

5.1. Assume in the formulas (28) \( \alpha = \beta = 0 \). In this case (28) is written as
\[ \varepsilon_{13} = \mu \tau_{13} + e_{13}^0, \varepsilon_{23} = \mu \tau_{23} + e_{23}^0. \]

The experiment will consist of four steps. Assume the initial deformations \( \varepsilon_{13} = \varepsilon_{23} = 1 \). This corresponds to Point A in figure 1.

![Figure 1](image-url)

**Figure 1.** Experiment on combined loading with \( \alpha = \beta = 0 \)

Step 1. Assumes \( \tau_{13} = 1, \tau_{23} = 0 \). We have \( \varepsilon'_{13} = 1 + \mu, \varepsilon'_{23} = e_{23}^0 = 1 \). This is Point B in figure 1.
Step 2. Assumes $\tau_{13} = 1, \tau_{23} = 1$. We have $\varepsilon''_{13} = 1 + \mu, \varepsilon''_{23} = 1 + \mu$. We get point C in figure 1.

Step 3. Assumes $\tau_{13} = -1, \tau_{23} = 1$. We have $\varepsilon''_{13} = 1, \varepsilon''_{23} = 1 + \mu$. We get Point D in figure 1.

Step 4. Assumes $\tau_{13} = -1, \tau_{23} = -1$. We have $\varepsilon''_{13} = 1, \varepsilon''_{23} = 1$. As a result we return to Point A in figure 1. We get a commutative experiment.

5.2. Assume in the formulas (28) $\alpha \neq 0, \beta = 0, \mu \neq 0$. In this case (28) is written as

$$e_{13} = (\alpha e_{13} + \mu) \tau_{13} + e_{13}^{0} + (\alpha e_{13} + \mu) \tau_{23} + e_{23}^{0}.$$  

(30)

Let us repeat the same steps as in the previous experiment

Assume the initial deformations $e_{13}^{0} = e_{23}^{0} = 1$. This corresponds to Point A in figure 2.

![Figure 2. Experiment on combined loading with $\alpha \neq 0, \beta = 0, \mu \neq 0$](image)

Step 1. Assumes $\tau_{13} = 1, \tau_{23} = 0$. We have $\varepsilon'_{13} = 1 + \alpha + \mu, e'_{23} = e_{23}^{0} = 1$.

Step 2. Assumes $\tau_{13} = 1, \tau_{23} = 1$. We have $\varepsilon''_{13} = 1 + (\alpha + \mu), e''_{23} = 1 + \alpha(1 + \alpha + \mu) + \mu$.

Step 3. Assumes $\tau_{13} = -1, \tau_{23} = 1$. We have $e'''_{13} = 1 + (\alpha + \mu) - (\alpha(1 + \alpha + \mu) + \mu), e'''_{23} = 1 + \alpha(1 + \alpha + \mu) + \mu$.

Step 4. Assumes $\tau_{13} = -1, \tau_{23} = -1$. We have $e^{iv}_{13} = e''_{13}, e^{iv}_{23} = e''_{23} - (\alpha e''_{13} + \mu)$.

For clarity we will assume $\alpha = 1, \mu = 1$. We get the following points in figure 2: A(1,1), B(3,1), C(3,5), D(-1,5), E(-1,5). We see that contrary to figure 1 we have not got the closed curve. Let us highlight that this result does not depend on the certain values of the parameters $\alpha, \mu$. Here all is determined by the commutator generated by Hook’s law. In the first case it is equal to zero and in the second case it differs from zero (27).

It is known that Lie subalgebras of dimension two can only be of two types: either commutative or not. That is why elastic materials can only be divided into two classes with the use of one two-dimension experiment: «commutative» material or not.

If we take Hook’s law for planar deformations then there we will have three parameters $\sigma_{11}, \sigma_{22}, \tau_{12}$. It is feasible to perform a three-dimension experiment but in practice it is very difficult to
realise. It is simpler to perform a series of two-dimension experiments similar to the ones described above with each pair of the parameters: \((\sigma_{11}, \sigma_{22}), (\sigma_{11}, \tau_{12}), (\sigma_{22}, \tau_{12})\). With the use of such experiments it will be feasible to study most of the three-dimension subalgebras. And because their number is countless then this gives us vast opportunities for the classification of materials. In the same way it will be feasible to classify materials determined by 4, 5 or 6 components of stress tensor. This will allow us to classify most of the materials because all Lie algebras dimension 2-6 over the field of real numbers are described, see [2] and the articles referenced there.

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