Chiral Properties of QCD Vacuum in Magnetars- A Nambu-Jona-Lasinio Model with Semi-Classical Approximation

Sutapa Ghosh, Soma Mandal and Somenath Chakrabarty *
Department of Physics, University of Kalyani, Kalyani 741 235, India
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The breaking of chiral symmetry of light quarks at zero temperature in presence of strong quantizing magnetic field is studied using Nambu-Jona-Lasinio (NJL) model with Thomas-Fermi type semi-classical formalism. It is found that the dynamically generated light quark mass can never become zero if the Landau levels are populated and the mass of light quarks increases with the increase of magnetic field strength.

1. INTRODUCTION

The theoretical investigation of properties of compact stellar objects in presence of strong quantizing magnetic field have gotten a new life after the recent discovery of a few magnetars [1–4]. These stellar objects are believed to be the strongly magnetized young neutron stars. The surface magnetic fields are observed to be \( \geq 10^{15} \text{G} \). Then it is quite possible that the fields at the core region may go up to \( 10^{18} \text{G} \). The exact source of this strong magnetic field is of course yet to be known. These objects are also supposed to be the possible sources of anomalous X-ray and soft gamma emissions (AXP and SGR). If the magnetic field is really so strong, in particular at the core region, they must affect most of the important physical properties of such stellar objects and also some of the physical processes, e.g., the rates / cross-sections of elementary processes, in particular the weak and the electromagnetic decays / reactions taking place at the core region.

The strong magnetic field affects the equation of state of dense neutron star matter. As a consequence the gross-properties of neutron stars [5–8], e.g., mass-radius relation, moment of inertia, rotational frequency etc. should change significantly. In the case of compact neutron stars, the phase transition from neutron matter to quark matter which may occur at the core region is also affected by strong quantizing magnetic field. It has been shown that a first order phase transition initiated by the nucleation of quark matter droplets is absolutely forbidden if the magnetic field strength \( \sim 10^{15} \text{G} \) at the core region [9,10]. However, a second order phase transition is allowed, provided the magnetic field strength \( < 10^{20} \text{G} \). This is of course too high to achieve at the core region. The study of time evolution of nascent quark matter, produced at the core region through some higher order phase transition, shows that in presence of strong magnetic field it is absolutely impossible to achieve chemical equilibrium (\( \beta \)-equilibrium) configuration among the constituents of the quark phase if the magnetic field strength is as low as \( B_m \sim 10^{14} \text{G} \).

The elementary processes, in particular, the weak and the electromagnetic decays/reactions taking place at the core region of a neutron star are strongly affected by such ultra-strong magnetic fields [11,12]. Since the cooling of neutron stars are mainly controlled by neutrino/anti-neutrino emission, the presence of strong quantizing magnetic field should affect the thermal history of strongly magnetized neutron stars. Further, the electrical conductivity of neutron star matter which directly controls the evolution of neutron star magnetic field will also change significantly [12].

Similar to the study of quark-hadron deconfinement transition inside neutron star core in presence of strong quantizing magnetic field, a thorough investigations have also been done on the effect of ultra-strong magnetic field on chiral symmetry breaking. In those studies, quantum field theoretic formalisms were mainly used [13–19]. In reference [20] Inagaki et al have studied the chiral symmetry violation with NJL model using quantum field theoretic approach in presence of strong quantizing magnetic field. In many of these papers, the effect of curvature with or without external magnetic field on chiral symmetry violation have been investigated. In the studies by Gusynin et al in [13,21], have thoroughly investigated the chiral symmetry breaking in presence of strong external quantizing magnetic field. They have used NJL model in \( 2+1 \) and also in \( 3+1 \) dimensions. It has been shown that the external magnetic field acts as a catalyst to generate fermion mass dynamically. In the first paper [13] they have studied it in \( 2+1 \) dimension and showed how the external magnetic field generated dynamical mass of fermion and broke the dynamical flavor symmetry. They have further shown by using NJL model that chiral symmetry breaks dynamically even if the attractive interaction between the fermions is extremely weak. In the second paper [21] they have extended the calculation to

*E-Mail: somenath@klyuniv.ernet.in
In the very nice piece of work by [17] Lee et al have studied the breaking of chiral symmetry for fermions in presence of external magnetic field. It has also been shown in this work that the symmetry is broken dynamically and further the effect of finite density and the temperature of the system on the chiral properties of the fermions have been investigated thoroughly in this paper in presence of strong magnetic field. It has been reported in this paper that there exists a critical density (or chemical potential) above which the chiral symmetry is again restored (which actually indicates the restoration of chiral symmetry at high enough density) and if it is treated as a chiral phase transition, the order will be of first order in nature. On the other hand the chiral symmetry is again restored at high temperature above some critical value. In this case the transition is of second order in nature. In an extensive review work [22], Kleevasky has reported the dynamical chiral symmetry breaking in presence of strong external quantizing magnetic field using NJL model in SU(2) and SU(3) flavor space. In this paper the effect of density (i.e., finite chemical potential) and temperature of the system on chiral symmetry restoration have also been reviewed.

In the present chapter we shall study the effect of strong quantizing magnetic field on the chiral properties of QCD vacuum with the help of NJL model following a semi-classical Thomas-Fermi type mean field approach in presence of strong quantizing external QED magnetic field. Now in NJL model, there is no in-built mechanism of color confinement, however, it can produce two chirally distinct phases - appropriate for confined quark matter within the bag and the matter outside the bag. These phases are also known as the Wigner phase and spontaneously broken chiral phase respectively. Therefore, if one re-formulates the NJL model in presence of strong quantizing magnetic field, it is quite possible to obtain the effect of quantizing magnetic field on these two chirally distinct phases and hence obtain the effect of magnetic field on chiral symmetry breaking. Further, it is also possible to obtain bag pressure from the difference of vacuum energy densities of these two phases and hence its variation with strong magnetic field. Assuming that the confinement and spontaneously broken chiral symmetry are synonymous, Bhaduri et. al. obtained some estimate of bag constant from the difference of energy densities [23] for the conventional case. In the present chapter we shall modify these original calculations of Bhaduri et. al. [23] and Providência et. al. [24] to study the breaking of chiral symmetry of light quarks in presence of strong magnetic fields and show that the chiral symmetry always remains broken in presence of strong quantizing magnetic field if the Landau levels for quarks are populated. Our motivation in this work was to study the effect of strong quantizing magnetic field on two chirally distinct phases and then obtain the vacuum pressure as a function of strong external magnetic field. Unfortunately, we have noticed that Wigner phase does not exist if the Landau levels of quarks are populated and in this formalism, there is no way, either by controlling chemical potential i.e., the density of matter (which is meaningless in our investigation since we have considered QCD vacuum state) or temperature of the system to restore chiral symmetry. Our study is now basically an application of the formalism developed recently to study the equation of state of dense fermionic matter of astrophysical interest in presence of strong quantizing magnetic field [25].

2. BASIC FORMALISM

We start with the density matrix $\rho(x,x')$, defined by

$$\rho(x,x') = \sum_{\text{spin},p} \psi(x)\psi^\dagger(x')\theta(\Lambda - |p_z|)$$

(1)

where $\psi$ and $\psi^\dagger$ are respectively the negative energy Dirac spinor and the corresponding adjoint, satisfy the equation

$$h\psi = E_-\psi$$

(2)

(and similarly for $\psi^\dagger$) with the single particle Hamiltonian

$$h = \gamma_5 \vec{\Sigma}.(\vec{p} - q_f\vec{A}) + \beta m$$

(3)

with

$$\Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$$

(4)

$\gamma_5$ and $\beta$ are the usual Dirac matrices, $\Lambda$ is the ultra-violet cut off in the momentum integral over $p_z$ (since we are considering vacuum, unlike a many body fermionic statistical system we have to put the cut off by hand) and $\vec{A}$ is the electromagnetic field three vector corresponding to the external constant magnetic field of strength $B_m$ along $z$-axis.
Here the light quark mass $m$ is assumed to be generated dynamically. Now in presence of strong quantizing magnetic field along $z$-direction, the up and down spin negative energy spinor solutions are therefore given by

$$\psi(x) = \frac{1}{(L_y L_z)^{1/2}} \exp[i(E_\nu t - p_y y - p_z z)]v_\nu^{(1,+)}, \quad (5)$$

where

$$v_\nu^{(+)\dagger} = \frac{1}{[2E_-(E_--m)]^{1/2}} \begin{pmatrix} \frac{p_z I_\nu}{E_--m} \\
-i(2\nu q f B_m)^{1/2} I_{\nu-1} \\
0 \end{pmatrix} \quad (6)$$

and

$$v_\nu^{(-)} = \frac{1}{[2E_-(E_--m)]^{1/2}} \begin{pmatrix} i(2\nu q f B_m)^{1/2} I_\nu \\
-p_z I_{\nu-1} \\
0 \end{pmatrix} \quad (7)$$

where $E_- = -(p_z^2 + m^2 + 2\nu q f B_m)^{1/2} = -E_\nu$, is the single particle energy eigen value, $\nu = 0, 1, 2, \ldots$, are the Landau quantum numbers, $q_f$ is the magnitude of the charge carried by $f$th flavor and

$$I_\nu = \left(\frac{q_f B_m}{\pi}\right)^{1/4} \left(\frac{1}{\nu!}\right)^{1/2} 2^{-\nu/2} \exp\left[-\frac{1}{2} q_f B_m \left(x - \frac{p_y}{q_f B_m}\right)^2\right] H_\nu \left[(q_f B_m)^{1/2} \left(x - \frac{p_y}{q_f B_m}\right)\right] \quad (8)$$

with $H_\nu$ is the well known Hermite polynomial of order $\nu$, and $L_y$, $L_z$ are respectively length scales along $Y$ and $Z$ directions. Now it can very easily be shown that $\nu = 0$ state is singly degenerate, whereas all other states are doubly degenerate. We now express the density matrix, as the modified version of Wigner transform in presence of strong quantizing magnetic field, in the following form:

$$\rho(x, x') = \sum \rho(x, x', p_y, p_z, \nu) \exp[i\{(t - t')E_- - (y - y')p_y - (z - z')p_z\}] \quad (9)$$

where the sum is over the momentum components $p_y, p_z$ and the Landau quantum number $\nu$. Since the momentum variables are continuous, the sum over momentum components will be replaced by the corresponding integrals. Now we have from eqn.(9)

$$\rho(x, x', p_y, p_z, \nu) = \sum_{\text{spin} = -1/2}^{1/2} v(x, p_y, p_z, \nu)v^{\dagger}(x', p_y, p_z, \nu) \quad (10)$$

Then substituting the negative energy up and down spinors solutions, we have

$$\rho(x, x', p_y, p_z, \nu) = \frac{1}{2E_-}[E_- A - p_z \gamma_z \gamma_0 A + m\gamma_0 A - p_\perp \gamma_0 B] \theta(A - |p_z|) \quad (11)$$

(see Appendix for detail derivation) where the matrices $A$ and $B$ are given by

$$A = \begin{pmatrix} I_\nu I_\nu^* & 0 & 0 & 0 \\
0 & I_{\nu-1} I_{\nu-1}^* & 0 & 0 \\
0 & 0 & I_\nu I_\nu^* & 0 \\
0 & 0 & 0 & I_{\nu-1} I_{\nu-1}^* \end{pmatrix} \quad (12)$$

$$B = \begin{pmatrix} I_{\nu-1} I_{\nu-1}^* & 0 & 0 & 0 \\
0 & I_{\nu} I_{\nu}^* & 0 & 0 \\
0 & 0 & I_{\nu-1} I_{\nu-1}^* & 0 \\
0 & 0 & 0 & I_{\nu} I_{\nu}^* \end{pmatrix} \quad (13)$$

where the primes indicate the functions of $x'$. Now in the evaluation of vacuum energy, we have noticed that it would be more convenient to define a quantity $\mu_f$, similar to the chemical potential for the $f$th flavor in a multi-quark
statistical system in presence of strong quantizing magnetic field (strictly speaking we are not considering a multi-quark statistical system and $\mu_f$ is therefore not the quark chemical potential. However, its minimum value should be $m$ and not zero, i.e., in this simplified model, just like dynamical mass $m$, this quantity is also treated as a parameter and we evaluate numerically $\mu_f$ and $m$ and then obtain the upper limit of Landau quantum number $[\nu_{max}]$ and the cut of $\Lambda$). Then it is very easy to write

$$\Lambda = (\mu_f^2 - m^2 - 2\nu q_f B_m)^{1/2}$$  \hspace{1cm} (14)$$

Since $\Lambda > 0$, it is also possible to express the upper limit of $\nu$, which is the maximum value of Landau quantum number of the levels occupied by $f$th flavor, and is given by

$$\nu^{(f)}_{\max} = \left[ \frac{\mu_f^2 - m^2}{2q_f B_m} \right]$$  \hspace{1cm} (15)$$

where $[ ]$ indicates the nearest integer but less than the actual number. Now to obtain the energy density of the vacuum, we consider the NJL (chiral) Hamiltonian, given by

$$H = \sum_{i=1}^{N} \gamma(i) \cdot \Sigma \cdot (\vec{p}_i - q_f \vec{A}) - \frac{1}{2} \sum_{i \neq j} \delta(\vec{x}_i - \vec{x}_j) \left[ \beta(i) \beta(j) - \beta(i) \gamma_5(i) \beta(j) \gamma_5(j) \right]$$  \hspace{1cm} (16)$$

where $\Sigma$, $\gamma_5$ and $\beta$ are usual $4 \times 4$ matrices and $2g$ is the effective coupling. Here we have used the formulation of da Providencia et al [24] of the mean field density matrix to describe the Dirac vacuum, thereby employing the Thomas-Fermi semi-classical method instead of formal field theory. As we have noticed, the Physics of condensation energy is more transparent in this method than the formal field theoretic technique. Assuming the magnetic field $B_m$ along $z$-direction and is constant, we can choose the gauge $A^\mu \equiv (0, 0, x B_m, 0)$. The energy of the vacuum is then given by

$$\epsilon_v = \sum_{p_{1z} < \nu} \int dx \int_{\nu}^{\nu_{\max}} \int_{0}^{\Lambda} dp_z E$$  \hspace{1cm} (18)$$

where $\rho_{p_1}$ is given by eqn.\,(11), the first term is the kinetic energy part and $\epsilon_v^{(f)}$ indicates the interaction term, including the exchange interaction. To evaluate the vacuum energy, we first calculate the kinetic energy term in eqn.\,(18). This quantity is proportional to the trace defined as $\text{Tr}(ph)$, can easily be evaluated by using $\rho$ from eqn.\,(11) and the single particle Hamiltonian $h$ from eqn.\,(17). Now using the orthonormality relations for the Hermite polynomials at the time of evaluation of integral over $dx$ and also using the anti-commutation relations for $\gamma$-matrices, we have the first term at zero temperature (see Appendix)

$$\epsilon_v^{(0)} = 2N_c \sum_{\nu=0}^{\nu_{\max}} \int_{0}^{\Lambda} dp_z E$$  \hspace{1cm} (19)$$

where $p_1^2 + 2q_f B_m$. $N_c = 3$, the number of colors, and $E_- = -E_\nu$.

In the evaluation of all the traces in this paper we have used the following important relation:

$$\text{Tr}(\gamma^\mu \gamma^\nu A_1 A_2 \ldots B_1 B_2 \ldots) = \text{Tr}(A_1 A_2 \ldots B_1 B_2 \ldots) g^{\mu \nu},$$  \hspace{1cm} (20)$$

$$\text{Tr}(\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma A_1 A_2 \ldots B_1 B_2 \ldots) = \text{Tr}(A_1 A_2 \ldots B_1 B_2 \ldots) (g^{\mu \nu} g^{\sigma \lambda} - g^{\mu \lambda} g^{\nu \sigma} + g^{\mu \sigma} g^{\nu \lambda}),$$  \hspace{1cm} (21)$$

$\text{Tr}$ (product of odd $\gamma$s with $A$ and/or $B$) = 0 etc. The other interesting aspects of $A$ and $B$ matrices are:

i) $k_{1\nu} k_{2\mu} \text{Tr}(A_1 A_2) = (E_1 E_2 - k_{1z} k_{2z}) \text{Tr}(A_1 A_2)$

ii) $k_{1\mu} k_{2\nu} \text{Tr}(B_1 B_2) = k_{1z} k_{2z} \text{Tr}(B_1 B_2)$

iii) $k_{1\mu} k_{2\nu} \text{Tr}(A_1 B_2) = k_{1z} k_{2z} \text{Tr}(B_1 A_2) = 0$

iv) $p_{1\mu} k_{2\nu} k_{2\nu} \text{Tr}(A_1 B_2) \neq 0 = (E_{c_1} E_{c_2} - p_{1z} k_{1z}) \vec{p}_{2z} \times \vec{k}_{2z} \text{Tr}(A_1 B_2)$
These set of relations are very recently obtained by us [25]. Since \( \gamma \) matrices are traceless and both \( A \) and \( B \) matrices are diagonal with identical blocks, it is very easy to evaluate the above traces of the product of \( \gamma \)-matrices multiplied with any number of \( A \) and/or \( B \), from any side with any order.

To evaluate the interaction term, we first consider the direct part which is proportional to \( Tr(\beta \rho_{\eta}) Tr(\beta \rho_{\eta}) \) and it is very easy to show that \( Tr(\beta \gamma_{\eta}) = 0 \) (see Appendix). Then using the orthonormality relations for Hermite polynomials and the anti-commutation relations for the \( \gamma \)-matrices, we have the direct term

\[
V_{\text{dir}} = -4g m^2 [V(\Lambda, m)]^2 \tag{22}
\]

(The four fermion coupling is included in \( V \), it is \( \sim V(1, 2)\rho_1 \rho_2 \)) where

\[
V(\Lambda, m) = \frac{N_c}{2\pi^2} \sum_{f,u} e_f B_m \sum_{\nu=0}^{[\nu_{\max}]} (2 - \delta_{\nu,0}) \int_{\Lambda}^\infty \frac{dp_2}{(p_2^2 + m_p + f)^2} \tag{23}
\]

where \( m_{\nu,f} = (m^2 + 2\nu q_f B_m)^{1/2} \) (see Appendix for derivation).

To evaluate the exchange term, we first calculate \( Tr((\beta \rho_p_1)(\beta \rho_{p_2})) \). Now

\[
\beta \rho = \frac{1}{2E^-}[E^- A + p_\bot A \gamma_z + mA - p_\bot B \gamma_y] \tag{24}
\]

Then to obtain the trace of the product of \( \beta \rho_1 \) and \( \beta \rho_2 \) it is very easy to show that only the direct product terms are non-zero whereas cross product terms do not contribute. Therefore

\[
\beta \rho_1 \beta \rho_2 = \frac{1}{4E_1 E_2}[E_1 \beta A + p_{1z} A \gamma_z + mA - p_{1\bot} B \gamma_y] \nonumber
\]

\[
+ (2 \beta \rho_1 \beta \rho_2) \tag{25}
\]

Then using the orthonormality relations for Hermite polynomials at the time of integration over \( dx_1 \) and \( dx_2 \), the above trace reduces to

\[
\int_{-\infty}^{+\infty} (\beta \rho_1 \beta \rho_2) dx = \frac{1}{4E_1 E_2} [4E_1 E_2 + 4p_{1z} p_{2z} + 4m^2] = \left[ 1 + \frac{p_{1z} p_{2z}}{E_1 E_2} + \frac{m^2}{E_1 E_2} \right] \tag{26}
\]

where both \( E_1 \) and \( E_2 \) are negative. Then in the energy contribution, after integrating over \( p_{1z} \) and \( p_{2z} \), the first term gives

\[
\left( \frac{N_c}{2\pi^2} \sum_{f,u} q_f B_m \sum_{\nu=0}^{[\nu_{\max}]} (2 - \delta_{\nu,0}) \Lambda \right)^2 \tag{27}
\]

Similarly the contribution from second term is given by

\[
\left( \frac{N_c}{2\pi^2} \sum_{f,u} q_f B_m \sum_{\nu=0}^{[\nu_{\max}]} (2 - \delta_{\nu,0}) (\Lambda^2 + m_{\nu,f}^2)^{1/2} \right)^2 \tag{28}
\]

and finally, the third term is given by

\[
m^2 \left( \frac{N_c}{2\pi^2} \sum_{f,u} q_f B_m \sum_{\nu=0}^{[\nu_{\max}]} (2 - \delta_{\nu,0}) \ln \left[ \frac{\Lambda + (\Lambda^2 + m_{\nu,f}^2)^{1/2}}{m_{\nu,f}} \right] \right)^2 \tag{29}
\]

(see Appendix for derivation of these expressions)

To obtain the next term in the exchange part, we evaluate the trace \( Tr((\beta \gamma_5 \rho_1)(\beta \gamma_5 \rho_{p_2})) \), which unlike the direct case, gives non-zero contribution. Using the anti-commutation relations of \( \gamma \)-matrices and as usual with the help of orthonormality relations for Hermite polynomials, we finally arrive to the following result

\[
- \left[ 1 + \frac{p_{1z} p_{2z}}{E_1 E_2} + \frac{m^2}{E_1 E_2} + m \left( \frac{1}{E_1} + \frac{1}{E_2} \right) \right] \tag{30}
\]
(see Appendix for derivation) The contribution to the interaction energy will again be obtained if we integrate over $p_{1z}$ and $p_{2z}$ (done in similar manner as have been done for direct case). Then the first term is given by

$$\left( \frac{N_c}{2\pi^2} \sum_{f=u}^{d} q_f B_m \sum_{\nu=0}^{[\nu(f)_{\max}]} (2 - \delta_{\nu 0}) \Lambda \right)^2$$

(31)

The second term is given by

$$\left( \frac{N_c}{2\pi^2} \sum_{f=u}^{d} q_f B_m \sum_{\nu=0}^{[\nu(f)_{\max}]} (2 - \delta_{\nu 0}) (\Lambda^2 + m_{\nu, f}^2)^{1/2} \right)^2$$

(32)

The third term is given by

$$m^2 \left( \frac{N_c}{2\pi^2} \sum_{f=u}^{d} q_f B_m \sum_{\nu=0}^{[\nu(f)_{\max}]} (2 - \delta_{\nu 0}) \ln \left[ \frac{\Lambda + (\Lambda^2 + m_{\nu, f}^2)^{1/2}}{m_{\nu, f}} \right] \right)^2$$

(33)

and finally the fourth and fifth terms, which are identical, given by

$$m \left( \frac{N_c}{2\pi^2} \sum_{f=u}^{d} q_f B_m \sum_{\nu=0}^{[\nu(f)_{\max}]} (2 - \delta_{\nu 0}) \ln \left[ \frac{\Lambda + (\Lambda^2 + m_{\nu, f}^2)^{1/2}}{m_{\nu, f}} \right] \right) \left( \frac{N_c}{2\pi^2} \sum_{f=u}^{d} q_f B_m \sum_{\nu=0}^{[\nu(f)_{\max}]} (2 - \delta_{\nu 0}) \Lambda \right)$$

(34)

Then combining all these terms we finally obtain the vacuum energy density. Since the mass $m$, which is assumed to be same for both $u$ and $d$ quarks, is generated dynamically, we obtain this quantity by minimizing the total vacuum energy density with respect to $m$, i.e., by putting $\frac{de_v}{dm} = 0$. Simplifying this non-linear equation, we finally get

$$\frac{de_v}{dm} = -P + 2gQ = 0$$

(35)

where

$$P = \frac{N_c}{2\pi^2} \sum_{f=u}^{d} q_f B_m \sum_{\nu=0}^{[\nu(f)_{\max}]} (2 - \delta_{\nu 0}) \left[ \frac{2m^3 \Lambda}{m_{\nu, f}^2 (\Lambda^2 + m_{\nu, f}^2)^{1/2}} - 2mX \right]$$

(36)

$$Q = \frac{N_c}{2\pi^2} \sum_{f=u}^{d} q_f B_m \sum_{\nu=0}^{[\nu(f)_{\max}]} (2 - \delta_{\nu 0}) \left[ X - \frac{m^2 \Lambda}{m_{\nu, f}^2 (\Lambda^2 + m_{\nu, f}^2)^{1/2}} \right]$$

(37)

$$R = \frac{N_c}{2\pi^2} \sum_{f=u}^{d} q_f B_m \sum_{\nu=0}^{[\nu(f)_{\max}]} (2 - \delta_{\nu 0}) \left[ \Lambda - 4mX \right]$$

(38)

with

$$X = \ln \left[ \frac{\Lambda + (\Lambda^2 + m_{\nu, f}^2)^{1/2}}{m_{\nu, f}} \right]$$

(39)

It is therefore obvious from eqn.(35) that the trivial solution $m = 0$ is not possible in this particular situation, or in other wards, the gap equation given by

$$m = 4g\mathcal{V}m$$

(40)

can not exist. On the other hand in a non-magnetic case, or for the magnetic field strength less than the quantum critical value, eqn.(35) reduces to the gap equation as written above (eqn.(40)). Here $\mathcal{V}$ is the overall contribution of interaction terms. Hence it is obvious that $m = 0$, the trivial solution exists in this non-magnetic or the conventional scenario, investigated by Bhaduri et. al. [23]. The phase with $m = 0$ is the Wigner phase and $m \neq 0$ is the so called Goldstone phase. Now eqn.(40) further gives

$$4g\mathcal{V} = 1$$

(41)

which is nothing but the well known gap equation used in BCS theory. The gap equation therefore does not exist in presence of strong quantizing magnetic field if the Landau levels for $u$ and $d$ quarks are populated.
3. CONCLUSION

The non-existence of trivial solution \((m = 0)\) indicates the spontaneously broken chiral symmetry in presence of strong quantizing magnetic field. Therefore as soon as the Landau levels are populated for light quarks in presence of strong external magnetic field, the chiral symmetry gets broken, the quarks become massive and the mass \(m\) (assumed to be same for both \(u\) and \(d\) quarks) is generated dynamically.

Therefore we may conclude here that the Wigner phase does not exist in the case of relativistic Landau diamagnetic system. Further, if the deconfinement transition and restoration of chiral symmetry occur simultaneously, or in other wards, if the chiral symmetry remains restored within the bag, as it is generally assumed, then it puts a big question mark whether the idea of bag model is applicable at all in presence of strong quantizing magnetic field. Questions may also arise, that if Wigner phase still exists inside the bag, then whether the external quantizing magnetic field can penetrate the bag boundary, if not, what is the underlying physics which prevents the external magnetic field from entering the periphery of the bag.

Now to illustrate the variation of dynamical quark mass with magnetic field, we consider the relation

\[
m_{\pi}^2 = -\frac{m_0}{f_\pi^2} < \bar{\psi} \psi >
\]

where \(m_{\pi}\) is the pion mass, \(m_0\) is the quark current mass and \(f_\pi\) is the pion decay constant. Using the spinor solutions given by eqns.(6) and (7) we get

\[
m_{\pi}^2 = \frac{2m_0m}{f_\pi^2} N_c \frac{d}{\pi^2} \sum_{f=\nu} \sum_{\nu=0}^{\nu_{\pi}} (2 - \delta_{\nu,0}) \ln \left[ \frac{\Lambda}{m_{\nu,f}} \right]
\]

We have now solved eqns.(35) and (43) numerically to obtain \(\Lambda\) and \(m\) for various values of magnetic field strength. In the actual numerical work we have solved self-consistently for the dynamical mass \(m\) and the parameter \(\mu_f\) using eqn.(15) for \(\nu_{\max}^f\) and then from eqn.(14) we get the infra red momentum cut off \(\Lambda\). In our calculation we have always used \(\mu_f\) instead of \(\Lambda\) which allows us to obtain \(\nu_{\max}^f\). So we can not compare our result with those obtained with zero chemical potential, since in our calculation it is just a parameter, it has no resemblances with chemical potential in a finite density quark \((u, d)\) matter system. We were forced to make this trick to obtain \(\nu_{\max}^f\) and also \(\nu_{\max}^f\) self-consistently from the numerical solutions of eqns.(35) and (43). In doing numerical calculations, we have considered the following sets of numerical values for the parameters. The current quark mass \(m_0 = 10\text{MeV}\), pion mass \(m_{\pi} = 140\text{MeV}\), pion decay constant \(f_\pi = 93\text{MeV}\), coupling constant \(g = 10\text{GeV}^{-2}\) and electron mass \(m_e = 0.5\text{MeV}\). In fig.(1) we shown the variation of dynamically generated quark mass with the strength of magnetic field. As it is evident that the dynamical quark mass never goes to zero and diverges beyond \(B_m \approx 10^{17}\text{G}\).
FIG. 1. The variation of dynamically generated quark mass with the strength of magnetic field (expressed in terms of $B_{m}^{2}(MRI) = 4.4 \times 10^{13} G$).

APPENDIX A

Evaluation Of Density Matrix:
To obtain the density matrix (eqn.(11)), we use eqns.(6) and (7), then

$$v^\dagger v = \frac{1}{2E(E - m)} \begin{pmatrix} p_z I_{\nu} & i(2\nu q_f B_m)^{1/2} I_{\nu} & (E - m) I_{\nu} & 0 \\ -i(2\nu q_f B_m)^{1/2} I_{\nu} & (E - m) I_{\nu} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} (A1)$$

and

$$v^\dagger v = \frac{1}{2E(E - m)} \begin{pmatrix} i(2\nu q_f B_m)^{1/2} I_{\nu} & -p_z I_{\nu} & 0 & (E - m) I_{\nu} \\ -p_z I_{\nu} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (E - m) I_{\nu} & 0 & 0 & 0 \end{pmatrix} (A2)$$

Adding

$$v^\dagger v + v^\dagger v = \frac{1}{2E} \begin{pmatrix} (E + m) I_{\nu} & 0 & 0 & 0 \\ 0 & (E + m) I_{\nu} & 0 & 0 \\ 0 & 0 & p_z I_{\nu} & i(2\nu q_f B_m)^{1/2} I_{\nu} \\ 0 & 0 & -i(2\nu q_f B_m)^{1/2} I_{\nu} & (E - m) I_{\nu} \end{pmatrix} (A5)$$

which may easily be simplified after a little algebra and reduces to

$$\rho(x, x', p_y, p_z, \nu) = \frac{1}{2E} [E - A - p_z \gamma_z \gamma_0 A + m \gamma_0 A - p_\perp \gamma_y \gamma_0 B] \delta(A - | p_z |) (A6)$$

where the matrices $A$ and $B$ are given by eqns.(12) and (13).

Evaluation of Kinetic Term:
Free Hamiltonian

$$h = \gamma^5 \Sigma \vec{p} + \beta m \quad (A7)$$

substituting $\Sigma$ from eqn.(4) and using the properties of $\gamma$-matrices we get
\[ h = \bar{\alpha} \cdot \bar{p} + \beta m \]  
the usual one. With the gauge choice \( A^\mu = (0, 0, xB_m, 0) \), we have
\[
h = \gamma_5 \hat{S}_i (\bar{p} - q_f \bar{A}) + \beta m
\]  
(A8)
by the same technique as above, we have
\[
h = \alpha (\bar{p} - q_f \bar{A}) + \beta m
\]  
(A9)
Now to evaluate \( \text{Tr}(h\rho) \), where
\[
h = (\gamma_i (\bar{p} - q_f \bar{A}) + m) \gamma_0
\]  
(A10)
we take \( A_y = q_f xB_m \), whereas all other components are zero. Then we can express the above Hamiltonian in the following form
\[
h = (\gamma_x p_x + \gamma_y (p_y - q_f xB_m) + \gamma_z p_z + m) \gamma_0
\]  
(A12)
substituting
\[
(q_f B_m)^{1/2} \left( \frac{p_y}{q_f B_m} - x \right) = -\zeta
\]  
(A13)
we have
\[
h = (\gamma_x p_x - (q_f B_m)^{1/2} \gamma_y \zeta + \gamma_z p_z + m) \gamma_0
\]  
(A14)
In evaluating the trace \( \text{Tr}(h\rho) \) we integrate over x-coordinate, use orthonormality relations for \( H_\nu(\zeta) \) and finally using the anti-commutation relations for \( \gamma \)-matrices, we have the first term of the product \( 0 \) (from the conclusion drawn just after eqn.(21)), second term \( \propto 4p_f^2 \), third term \( \propto 4m^2 \) and the fourth term \( \propto 4p^2 \). Finally adding, we have \( \text{Tr}(\rho h) \propto 2E \). Since \( E < 0 \), this is actually \( -2E \), where \( E = (p^2 + p^2_x + m^2)^{1/2} \) Then the kinetic energy density is given by
\[
\epsilon^{(0)}_\nu = 2N_c \frac{d}{df} \frac{q_f B_m}{2\pi^2} \sum_{\nu=0}^{\nu_{\text{max}}} (2 - \delta_{\nu0}) \int_0^\Lambda dp_z E
\]
\[
= \frac{N_c}{4\pi^2} \sum_{f=0}^{d} q_f B_m \sum_{\nu=0}^{\nu_{\text{max}}} (2 - \delta_{\nu0}) \left[ \Lambda (\Lambda^2 + m^2_{\nu,f})^{1/2} \right. \\
+ \left. m^2_{\nu,f} \ln \left( \Lambda + (\Lambda^2 + m^2_{\nu,f})^{1/2} \right) \right]
\]  
(A15)
where \( m^2_{\nu,f} = m^2 + 2\nu q_f B_m \) and \( e_u = 2/3e, e_d = 1/3e \) and \( N_c = 3 \)

**Interaction Energy (Direct):**

It is very easy to show that \( \text{Tr}(\beta \gamma_5 \rho) = 0 \):
\[
\beta \gamma_5 \rho = \gamma_x \gamma_y \gamma_z \frac{1}{2E} [EA + p_z A \gamma_z + mA - p_\perp B \gamma_y]
\]  
(A16)
Then it is very easy to check that the above result follows from here.

To calculate the direct term of interaction we have to evaluate the \( \text{Tr}(\beta \rho) \) integrated over x-coordinate, i.e.,
\[
\int_{-\infty}^{+\infty} \text{Tr}(\beta \rho) dx = \frac{1}{2E} \int_{-\infty}^{+\infty} [E \beta A + p_z A \gamma_z + mA - p_\perp B \gamma_y] dx
\]  
(A17)
since first, second and fourth terms contain odd (single) number \( \gamma \) with \( A \) the contribution of those terms are zero. Only third term contributes to the integral and is given by
\[
\frac{m}{2E} \int_{-\infty}^{+\infty} \text{Tr}(A) dx = \frac{m}{2E} \int_{-\infty}^{+\infty} 2[I_0 u^2(x) + I_{-1}^2(x)] dx
\]  
(A18)
using orthonormality relation for Hermite polynomials the above integral reduces to \( 2m/E \). Since there is an identical expression with integral over \( x' \), finally we get
\[
V_{\text{dir}} = -4gm^2 [V(\Lambda, m)]^2
\]  
(A19)
Momentum Integral:

First Term:

\[
\left( \frac{N_c}{2\pi^2} \sum_{f=u}^{d} q_f B_m \sum_{\nu=0}^{[\nu(f)]_{\max}} (2 - \delta_{\nu 0}) \int_{0}^{\Lambda} dp_{1z} \right) \times \left( \frac{N_c}{2\pi^2} \sum_{f=u}^{d} q_f B_m \sum_{\nu=0}^{[\nu(f)]_{\max}} (2 - \delta_{\nu 0}) \int_{0}^{\Lambda} dp_{2z} \right) = \left( \frac{N_c}{2\pi^2} \sum_{f=u}^{d} q_f B_m \sum_{\nu=0}^{[\nu(f)]_{\max}} (2 - \delta_{\nu 0}) \Lambda \right)^2
\]

(A20)

Second Term:

\[
\left( \frac{N_c}{2\pi^2} \sum_{f=u}^{d} q_f B_m \sum_{\nu=0}^{[\nu(f)]_{\max}} (2 - \delta_{\nu 0}) \int_{0}^{\Lambda} \frac{p_{1z} dp_{1z}}{E_1} \right) \times \left( \frac{N_c}{2\pi^2} \sum_{f=u}^{d} q_f B_m \sum_{\nu=0}^{[\nu(f)]_{\max}} (2 - \delta_{\nu 0}) \int_{0}^{\Lambda} \frac{p_{2z} dp_{2z}}{E_2} \right) = \left( \frac{N_c}{2\pi^2} \sum_{f=u}^{d} q_f B_m \sum_{\nu=0}^{[\nu(f)]_{\max}} (2 - \delta_{\nu 0}) (\Lambda^2 + m_{\nu,f}^2)^{1/2} m_{\nu,f} \right)^2
\]

(A21)

Third Term:

\[
m^2 \left( \frac{N_c}{2\pi^2} \sum_{f=u}^{d} q_f B_m \sum_{\nu=0}^{[\nu(f)]_{\max}} (2 - \delta_{\nu 0}) \int_{0}^{\Lambda} \frac{dp_{1z}}{E_1} \right) \times \left( \frac{N_c}{2\pi^2} \sum_{f=u}^{d} q_f B_m \sum_{\nu=0}^{[\nu(f)]_{\max}} (2 - \delta_{\nu 0}) \int_{0}^{\Lambda} \frac{dp_{2z}}{E_2} \right) = m^2 \left( \frac{N_c}{2\pi^2} \sum_{f=u}^{d} q_f B_m \sum_{\nu=0}^{[\nu(f)]_{\max}} (2 - \delta_{\nu 0}) \ln \left[ \frac{\Lambda + (\Lambda^2 + m_{\nu,f}^2)^{1/2}}{m_{\nu,f}} \right] \right)^2
\]

(A22)

Interaction Energy (Exchange Terms):

The product

\[
(\beta\gamma_5\rho_1)(\beta\gamma_5\rho_2) = \gamma_1\gamma_2\gamma_3 \frac{1}{2E_1} [E_1 A_1 + p_{1z} A_1 \gamma_3 + m A_1 - p_{1\perp} B_1 \gamma_2] \times \gamma_1\gamma_2\gamma_3 \frac{1}{2E_2} [E_2 A_2 + p_{2z} A_2 \gamma_3 + m A_2 - p_{2\perp} B_2 \gamma_2]
\]

(A23)

Now

\[
\gamma_1\gamma_2\gamma_3 \frac{1}{2E_1} [E_1 A_1 + p_{1z} A_1 \gamma_3 + m A_1 - p_{1\perp} B_1 \gamma_2] = \frac{1}{2E_1} [E_1 \gamma_1\gamma_2\gamma_3 A_1 + p_{1z} \gamma_1\gamma_2 A_1 + \gamma_1\gamma_2\gamma_3 m A_1 - p_{1\perp} \gamma_1\gamma_2 B_1 \gamma_2]
\]

(A24)

Hence

\[
(\beta\gamma_5\rho_1)(\beta\gamma_5\rho_2) = \frac{1}{4E_1 E_2} [E_1 \gamma_1\gamma_2\gamma_3 A_1 + p_{1z} \gamma_1\gamma_2 A_1 + \gamma_1\gamma_2\gamma_3 m A_1 - p_{1\perp} \gamma_1\gamma_2 B_1 \gamma_2] \times [E_2 \gamma_1\gamma_2\gamma_3 A_2 + p_{2z} \gamma_1\gamma_2 A_2 + \gamma_1\gamma_2\gamma_3 m A_2 - p_{2\perp} \gamma_1\gamma_2 B_2 \gamma_2]
\]

(A25)
Now we use the results
\[ \gamma_1 \gamma_2 \gamma_3 = - \gamma_1 \gamma_2 \gamma_1 \gamma_3 \gamma_2 = \gamma_1 \gamma_2 \gamma_1 = -1, \quad \gamma_1 \gamma_2 \gamma_2 = -1, \] (A26)
to obtain the Tr\([\langle \beta \gamma_5 \rho_1 \rangle \langle \beta \gamma_5 \rho_2 \rangle]\), which is given by
\[
\text{Tr}\[\langle \beta \gamma_5 \rho_1 \rangle \langle \beta \gamma_5 \rho_2 \rangle\] = \frac{1}{4E_1 E_2} \left[ -4E_1 E_2 - 4p_{1z} p_{2z} - 4m^2 - 4mE_2 - 4mE_1 \right]
= - \left[ 1 + \frac{p_{1z} p_{2z}}{E_1 E_2} + \frac{m^2}{E_1 E_2} + m \left( \frac{1}{E_1} + \frac{1}{E_2} \right) \right] \quad (A27)
\]
The integration over \(p_{1z}\) and \(p_{2z}\) are done in the same manner as are did for Tr\([\langle \beta \rho_1 \rangle \langle \beta \rho_2 \rangle]\).

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