Stochastic differential equations with coefficients in Sobolev spaces

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Abstract

We consider Itô SDE \( dX_t = \sum_{j=1}^{m} A_j(X_t) \, dw^j_t + A_0(X_t) \, dt \) on \( \mathbb{R}^d \). The diffusion coefficients \( A_1, \ldots, A_m \) are supposed to be in the Sobolev space \( W^{1,p}_{\text{loc}}(\mathbb{R}^d) \) with \( p > d \), and to have linear growth; for the drift coefficient \( A_0 \), we consider two cases: (i) \( A_0 \) is continuous whose distributional divergence \( \delta(A_0) \) w.r.t. the Gaussian measure \( \gamma_d \) exists, (ii) \( A_0 \) has the Sobolev regularity \( W^{1,p'}_{\text{loc}} \) for some \( p' > 1 \). Assume

\[
\int_{\mathbb{R}^d} \exp \left[ \lambda_0 (|\delta(A_0)| + \sum_{j=1}^{m} (|\delta(A_j)|^2 + |\nabla A_j|^2)) \right] \, d\gamma_d < +\infty \quad \text{for some } \lambda_0 > 0,
\]

in the case (i), if the pathwise uniqueness of solutions holds, then the push-forward \( (X_t)_{\#} \gamma_d \) admits a density with respect to \( \gamma_d \). In particular, if the coefficients are bounded Lipschitz continuous, then \( X_t \) leaves the Lebesgue measure \( \text{Leb}_d \) quasi-invariant. In the case (ii), we develop a method used by G. Crippa and C. De Lellis for ODE and implemented by X. Zhang for SDE, to establish the existence and uniqueness of stochastic flow of maps.

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1 Introduction

Let \( A_0, A_1, \ldots, A_m : \mathbb{R}^d \to \mathbb{R}^d \) be continuous vector fields on \( \mathbb{R}^d \). We consider the following Itô stochastic differential equation on \( \mathbb{R}^d \) (abbreviated as SDE)

\[
dX_t = \sum_{j=1}^{m} A_j(X_t) \, dw^j_t + A_0(X_t) \, dt, \quad X_0 = x, \tag{1.1}
\]

where \( w_t = (w^1_t, \ldots, w^m_t) \) is the standard Brownian motion on \( \mathbb{R}^m \). It is a classical fact in the theory of SDE (see [16, 17, 21, 30]) that, if the coefficients \( A_j \) are globally Lipschitz continuous, then SDE (1.1) has a unique strong solution which defines a stochastic flow of homeomorphisms on \( \mathbb{R}^d \); however contrary to ordinary differential equations (abbreviated as ODE), the regularity of the homeomorphisms is only Hölder continuity of order \( 0 < \alpha < 1 \). Thus it is not clear whether the Lebesgue measure \( \text{Leb}_d \) on \( \mathbb{R}^d \) admits a density under the flow \( X_t \). In the case where the vector fields \( A_j, j = 0, 1, \ldots, m \), are in \( C^\infty_b(\mathbb{R}^d, \mathbb{R}^d) \), the SDE (1.1) defines a flow of diffeomorphisms, and Kunita [21] showed that the measures on \( \mathbb{R}^d \) which have a strictly positive

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smooth density with respect to Leb$_d$ are quasi-invariant under the flow. This result was recently generalized in [27] to the case where the drift $A_0$ is allowed to be only log-Lipschitz continuous. Studies on SDE beyond the Lipschitz setting attracted great interest during the last years, see for instance [10, 11, 13, 19, 20, 23, 24, 29, 34, 35].

In the context of ODE, existence of a flow of quasi-invariant measurable maps associated to a vector field $A_0$ belonging to Sobolev spaces appeared first in [6]. In the seminar paper [7], Di Perna and Lions developed transport equations to solve ODE without involving exponential integrability of $|\nabla A_0|$. On the other hand, L. Ambrosio [1] took advantage of using continuity equations which allowed him to construct quasi-invariant flows associated to vector fields $A_0$ with only BV regularity. In the framework for Gaussian measures, the Di Perna-Lions method was developed in [4], also in [2, 12] on the Wiener space.

The situation for SDE is quite different: even for the vector fields $A_0, A_1, \ldots, A_m$ in $C^\infty$ with linear growth, if no conditions were imposed on the growth of the derivatives, the SDE (1.1) could not define a flow of diffeomorphisms (see [25, 26]). More precisely, let $\tau_x$ be the life time of the solution to (1.1) starting from $x$. The SDE (1.1) is said to be complete if for each $x \in \mathbb{R}^d$, $\mathbb{P}(\tau_x = +\infty) = 1$; it is said to be strongly complete if $\mathbb{P}(\tau_x = +\infty, x \in \mathbb{R}^d) = 1$. The goal in [26] is to construct examples for which the coefficients are smooth, but the SDE (1.1) is not strongly complete (see [11, 25] for positive examples). Now consider

$$\Sigma = \{(w, x) \in \Omega \times \mathbb{R}^d; \tau_x(w) = +\infty\}.$$ 

Suppose that the SDE (1.1) is complete, then for any probability measure $\mu$ on $\mathbb{R}^d$,

$$\int_{\mathbb{R}^d} \left( \int_{\Omega} 1_{\Sigma}(w, x) \, d\mathbb{P}(w) \right) \, d\mu(x) = 1.$$

By Fubini’s theorem, $\int_{\Omega} \left( \int_{\mathbb{R}^d} 1_{\Sigma}(w, x) \, d\mu(x) \right) \, d\mathbb{P}(w) = 1$. It follows that there exists a full measure subset $\Omega_0 \subset \Omega$ such that for all $w \in \Omega_0$, $\tau_x(w) = +\infty$ holds for $\mu$-almost every $x \in \mathbb{R}^d$. Now under the existence of a complete unique strong solution to SDE (1.1), we have a flow of measurable maps $x \rightarrow X_t(w, x)$.

Recently, inspired by a previous work due to Ambrosio, Lecumberry and Maniglia [3], Crippa and De Lellis [5] obtained some new type of estimates of perturbation for ODE whose coefficients have Sobolev regularity. More precisely, the absence of Lipschitz condition was filled by the following inequality: for $f \in W_{loc}^{1,1}(\mathbb{R}^d)$,

$$|f(x) - f(y)| \leq C_d |x - y| (M_R|\nabla f|(x) + M_R|\nabla f|(y))$$

holds for $x, y \in N^c$ and $|x - y| \leq R$, where $N$ is a negligible set of $\mathbb{R}^d$ and $M_R g$ is the maximal function defined by

$$M_R g(x) = \sup_{0 < r \leq R} \frac{1}{\text{Leb}_d(B(x, r))} \int_{B(x, r)} |g(y)| \, dy,$$

here $B(x, r) = \{y \in \mathbb{R}^d; |y - x| \leq r\}$; the classical moment estimate was replaced by estimating the quantity

$$\int_{B(0, r)} \log \left( \frac{|X_t(x) - \bar{X}_t(x)|}{\sigma} + 1 \right) \, dx,$$

where $\sigma > 0$ is a small parameter. This method has recently been successfully implemented to SDE by X. Zhang in [36].

The aim in this paper is two-fold: first we shall study absolute continuity of the push-forward measure $(X_t)_{t \geq 0}$ with respect to Leb$_d$, once the SDE (1.1) has a unique strong solution;
secondly we shall construct strong solutions (for almost all initial values) using the approach mentioned above for SDE with coefficients in Sobolev space. The key point is to obtain a priori $L^p$ estimate for the density. To this end, we shall work with the standard Gaussian measure $\gamma_d$; this will be done in Section 2. The main result in Section 3 is the following

**Theorem 1.1.** Let $A_0, A_1, \ldots, A_m$ be continuous vector fields on $\mathbb{R}^d$ of linear growth. Assume that the diffusion coefficients $A_1, \ldots, A_m$ are in the Sobolev space $\cap_{q>1} W^1_q(\gamma_d)$ and that $\delta(A_0)$ exists; furthermore there exists a constant $\lambda_0 > 0$ such that

$$\int_{\mathbb{R}^d} \exp \left[ \lambda_0 \left( |\delta(A_0)| + \sum_{j=1}^m (|\delta(A_j)|^2 + |\nabla A_j|^2) \right) \right] d\gamma_d < +\infty. \quad (1.2)$$

Suppose that pathwise uniqueness holds for SDE (1.1). Then $(X_t)_\# \gamma_d$ is absolutely continuous with respect to $\gamma_d$ and the density is in the space $L^1 \log L^1$.

A consequence of this theorem concerns the following classical situation.

**Theorem 1.2.** Let $A_0, A_1, \ldots, A_m$ be globally Lipschitz continuous. Suppose that there exists a constant $C > 0$ such that

$$\sum_{j=1}^m (x, A_j(x))^2 \leq C (1 + |x|^2) \quad \text{for all } x \in \mathbb{R}^d. \quad (1.3)$$

Then the stochastic flow of homeomorphisms $X_t$ generated by SDE (1.1) leaves the Lebesgue measure $\text{Leb}_d$ quasi-invariant.

Remark that the condition (1.3) not only includes the case of bounded Lipschitz diffusion coefficients, but also, maybe more significant, indicates the role of dispersion: the vector fields $A_1, \ldots, A_m$ should not go radically into infinity. The purpose of Section 4 is to find conditions that guarantee strict positivity of the density, in the case where the existence of the inverse flow is not known, see Theorem 4.4.

The main result in Section 5 is

**Theorem 1.3.** Assume that the diffusion coefficients $A_1, \ldots, A_m$ belong to the Sobolev space $\cap_{q>1} W^1_q(\gamma_d)$ and the drift $A_0 \in W^1_q(\gamma_d)$ for some $q > 1$. Assume (1.2) and that the coefficients $A_0, A_1, \ldots, A_m$ are of linear growth, then there is a unique stochastic flow of measurable maps $X : [0, T] \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, which solves (1.1) for almost all initial $x \in \mathbb{R}^d$ and the push-forward $(X_t(w, \cdot))_\# \gamma_d$ admits a density with respect to $\gamma_d$, which is in $L^1 \log L^1$.

When the diffusion coefficients satisfy the uniform ellipticity, a classical result due to Stroock and Varadhan [32] says that if the diffusion coefficients $A_1, \ldots, A_m$ are bounded continuous and the drift $A_0$ is bounded Borel measurable, then the weak uniqueness holds, that is the uniqueness in law of the diffusion. This result was strengthened by Veretennikov [33], saying that in fact the pathwise uniqueness holds. When $A_0$ is not bounded, some conditions on diffusion coefficients were needed. In the case where the diffusion matrix $a = (a_{ij})$ is the identity, the drift $A_0$ in (1.1) can be quite singular: $A_0 \in L^p_{\text{loc}}(\mathbb{R}^d)$ with $p > d + 2$ implies that the SDE (1.1) has the pathwise uniqueness (see Krylov-Röckner [20] for a more complete study); if the diffusion coefficients $A_1, \ldots, A_m$ are bounded continuous, under a Sobolev condition, namely, $A_j \in W^{1,2(d+1)}_{\text{loc}}(\mathbb{R}^d)$, X. Zhang proved in [34] that the SDE (1.1) admits a unique strong solution. Note that even in this uniformly non-degenerated case, if the diffusion coefficients lose the continuity, there are counterexamples for which the weak uniqueness does not hold, see [19, 31].
Finally we would like to mention that under weaker Sobolev type conditions, the connection between weak solutions and Fokker-Planck equations was investigated in [14, 22], some notions of “generalized solutions”, as well as the phenomena of coalescence and splitting, were investigated in [23, 24]. Stochastic transport equations were studied in [15, 36].

2 \( L^p \) estimate of the density

The purpose of this section is to derive a priori estimates for the density; we assume that the coefficients \( A_0, A_1, \ldots, A_m \) of SDE (1.1) are smooth with compact support in \( \mathbb{R}^d \). Then the solution \( X_t \), i.e., \( x \mapsto X_t(x) \), is a stochastic flow of diffeomorphisms on \( \mathbb{R}^d \). Moreover SDE (1.1) is equivalent to the following Stratonovich SDE

\[
dX_t = \sum_{j=1}^{m} A_j(X_t) \circ dw_t^j + A_0(X_t) \, dt, \quad X_0 = x,
\]

where \( \tilde{A}_0 = A_0 - \frac{1}{2} \sum_{j=1}^{m} \mathcal{L}_{A_j} A_j \) and \( \mathcal{L}_A \) denotes the Lie derivative with respect to \( A \).

Let \( \gamma_d \) be the standard Gaussian measure on \( \mathbb{R}^d \), and \( \gamma_t = (X_t)^\# \gamma_d \), \( \tilde{\gamma}_t = (X_t^{-1})^\# \gamma_d \) the push-forwards of \( \gamma_d \) respectively by the flow \( X_t \) and its inverse \( X_t^{-1} \). To fix ideas, we denote by \( (\Omega, \mathcal{F}, \mathbb{P}) \) the probability space on which the Brownian motion \( \omega_t \) is defined. Let \( K_t = \frac{d\gamma_t}{d\gamma_d} \) and \( \tilde{K}_t = \frac{d\tilde{\gamma}_t}{d\gamma_d} \) be the densities with respect to \( \gamma_d \). By Lemma 4.3.1 in [21], the Radon-Nikodym derivative \( K_t \) has the following explicit expression

\[
\tilde{K}_t(x) = \exp \left( -\sum_{j=1}^{m} \int_0^t \delta(A_j)(X_s(x)) \circ dw_s^j - \int_0^t \delta(\tilde{A}_0)(X_s(x)) \, ds \right),
\]

where \( \delta(A_j) \) denotes the divergence of \( A_j \) with respect to the Gaussian measure \( \gamma_d \):

\[
\int_{\mathbb{R}^d} \langle \nabla \varphi, A_j \rangle \, d\gamma_d = \int_{\mathbb{R}^d} \varphi \delta(A_j) \, d\gamma_d, \quad \varphi \in C^1_c(\mathbb{R}^d).
\]

It is easy to see that \( K_t \) and \( \tilde{K}_t \) are related to each other by the equality below:

\[
K_t(x) = \left[ \tilde{K}_t(X_t^{-1}(x)) \right]^{-1}.
\]

In fact, for any \( \psi \in C^\infty_c(\mathbb{R}^d) \), we have

\[
\int_{\mathbb{R}^d} \psi(x) \, d\gamma_d(x) = \int_{\mathbb{R}^d} \psi[X_t(X_t^{-1}(x))] \, d\gamma_d(x)
\]

\[
= \int_{\mathbb{R}^d} \psi[X_t(y)] \tilde{K}_t(y) \, d\gamma_d(y) = \int_{\mathbb{R}^d} \psi(x) \tilde{K}_t(X_t^{-1}(x)) K_t(x) \, d\gamma_d(x),
\]

which leads to (2.3) due to the arbitrariness of \( \psi \in C^\infty_c(\mathbb{R}^d) \). In the following we shall estimate the \( L^p(\mathbb{P} \times \gamma_d) \) norm of \( K_t \).

We rewrite the density (2.2) with the Itô integral:

\[
\tilde{K}_t(x) = \exp \left( -\sum_{j=1}^{m} \int_0^t \delta(A_j)(X_s(x)) \, dw_s^j - \int_0^t \left[ \frac{1}{2} \sum_{j=1}^{m} \mathcal{L}_{A_j} \delta(A_j) + \delta(\tilde{A}_0) \right](X_s(x)) \, ds \right).
\]
Lemma 2.1. We have
\[ \frac{1}{2} \sum_{j=1}^{m} \mathcal{L}_{A_j} \delta(A_j) + \delta(\bar{A}_0) = \delta(A_0) + \frac{1}{2} \sum_{j=1}^{m} |A_j|^2 + \frac{1}{2} \sum_{j=1}^{m} \langle \nabla A_j, (\nabla A_j)^* \rangle, \] (2.5)
where \( \langle \cdot, \cdot \rangle \) denotes the inner product of \( \mathbb{R}^d \otimes \mathbb{R}^d \) and \( (\nabla A_j)^* \) the transpose of \( \nabla A_j \).

**Proof.** Let \( A \) be a \( C^2 \) vector field on \( \mathbb{R}^d \). From the expression
\[ \delta(A) = \sum_{k=1}^{d} \left( x_k A^k - \frac{\partial A^k}{\partial x_k} \right), \]
we get
\[ \mathcal{L}_{A} \delta(A) = \sum_{\ell,k=1}^{d} \left( A^\ell A^k \delta_{k\ell} + A^\ell x_k \frac{\partial A^k}{\partial x_\ell} - A^\ell \frac{\partial^2 A^k}{\partial x_\ell \partial x_k} \right). \] (2.6)
Note that \( \frac{\partial}{\partial x_k} \left( A^\ell \frac{\partial A^k}{\partial x_\ell} \right) = \frac{\partial A^k}{\partial x_\ell} \frac{\partial A^\ell}{\partial x_k} + A^\ell \frac{\partial^2 A^k}{\partial x_k \partial x_\ell} \). Thus, by means of (2.6), we obtain
\[ \mathcal{L}_{A} \delta(A) = |A|^2 + \delta(\mathcal{L}_{A} A) + \langle \nabla A, (\nabla A)^* \rangle. \] (2.7)
Recall that \( \delta(\bar{A}_0) = \delta(A_0) - \frac{1}{2} \sum_{j=1}^{m} \delta(\mathcal{L}_{A_j} A_j) \). Hence, replacing \( A \) by \( A_j \) in (2.7) and summing over \( j \), gives formula (2.5). \( \square \)

We can now prove the following key estimate.

**Theorem 2.2.** For \( p > 1 \),
\[ \| K_t \|_{L^p(\mathbb{R}^d)} \leq \left[ \int_{\mathbb{R}^d} \exp \left( pt \left[ 2|\delta(A_0)| + \sum_{j=1}^{m} \left( |A_j|^2 + |\nabla A_j|^2 + 2(p-1)|\delta(A_j)|^2 \right) \right] \right) d\gamma_d \right]^{\frac{1}{p-1}}. \] (2.8)

**Proof.** Using relation (2.3), we have
\[ \int_{\mathbb{R}^d} \mathbb{E}[K_t^p(x)] d\gamma_d(x) = \mathbb{E} \int_{\mathbb{R}^d} \left[ \tilde{K}_t(X_t^{-1}(x)) \right]^{1-p} d\gamma_d(x) \]
\[ = \mathbb{E} \int_{\mathbb{R}^d} \left[ \tilde{K}_t(y) \right]^{1-p} \tilde{K}_t(y) d\gamma_d(y) \]
\[ = \int_{\mathbb{R}^d} \mathbb{E} \left[ \left( \tilde{K}_t(x) \right)^{1-p+1} \right] d\gamma_d(x). \] (2.9)
To simplify the notation, denote the right hand side of (2.5) by \( \Phi \). Then \( \tilde{K}_t(x) \) rewrites as
\[ \tilde{K}_t(x) = \exp \left( -\sum_{j=1}^{m} \int_0^t \delta(A_j)(X_s(x)) \, dw_s^j - \int_0^t \Phi(X_s(x)) \, ds \right). \]
Fixing an arbitrary \( r > 0 \), we get
\[
(\bar{K}_t(x))^{-r} = \exp \left( r \sum_{j=1}^{m} \int_0^t \delta(A_j)(X_s(x)) \, dw_s^j + r \int_0^t \Phi(X_s(x)) \, ds \right)
\]
\[
= \exp \left( r \sum_{j=1}^{m} \int_0^t \delta(A_j)(X_s(x)) \, dw_s^j - r^2 \sum_{j=1}^{m} \int_0^t |\delta(A_j)(X_s(x))|^2 \, ds \right)
\]
\[
\times \exp \left( \int_0^t \left( r^2 \sum_{j=1}^{m} |\delta(A_j)|^2 + r\Phi \right)(X_s(x)) \, ds \right).
\]
By Cauchy-Schwarz’s inequality,
\[
\mathbb{E}[\{(\bar{K}_t(x))^{-r}\}] \leq \left[ \mathbb{E} \exp \left( 2r \sum_{j=1}^{m} \int_0^t \delta(A_j)(X_s(x)) \, dw_s^j - 2r^2 \sum_{j=1}^{m} \int_0^t |\delta(A_j)(X_s(x))|^2 \, ds \right) \right]^{1/2}
\]
\[
\times \left[ \mathbb{E} \exp \left( \int_0^t \left( 2r^2 \sum_{j=1}^{m} |\delta(A_j)|^2 + 2r\Phi \right)(X_s(x)) \, ds \right) \right]^{1/2}
\]
\[
= \left[ \mathbb{E} \exp \left( \int_0^t \left( 2r^2 \sum_{j=1}^{m} |\delta(A_j)|^2 + 2r\Phi \right)(X_s(x)) \, ds \right) \right]^{1/2}, \quad (2.10)
\]
since the first term on the right hand side of the inequality in (2.10) is the expectation of a martingale. Let
\[
\tilde{\Phi}_r = 2r|\delta(A_0)| + r \sum_{j=1}^{m} (|A_j|^2 + |\nabla A_j|^2 + 2r|\delta(A_j)|^2).
\]
Then by (2.10), along with the definition of \( \Phi \) and Cauchy-Schwarz’s inequality, we obtain
\[
\int_{\mathbb{R}^d} \mathbb{E}[\{(\bar{K}_t(x))^{-r}\}] \, d\gamma_d \leq \left[ \int_{\mathbb{R}^d} \mathbb{E} \exp \left( \int_0^t \tilde{\Phi}_r(X_s(x)) \, ds \right) \, d\gamma_d \right]^{1/2}. \quad (2.11)
\]
Following the idea of A.B. Cruzeiro ([6] Corollary 2.2, see also Theorem 7.3 in [8]) and by Jensen’s inequality,
\[
\exp \left( \int_0^t \tilde{\Phi}_r(X_s(x)) \, ds \right) = \exp \left( \int_0^t \tilde{\Phi}_r(X_s(x)) \, \frac{ds}{t} \right) \leq \frac{1}{t} \int_0^t e^{t\tilde{\Phi}_r(X_s(x))} \, ds.
\]
Define \( I(t) = \sup_{0 \leq s \leq t} \mathbb{E}[K^p_s(x)] \, d\gamma_d \). Integrating on both sides of the above inequality and by Hölder’s inequality,
\[
\int_{\mathbb{R}^d} \mathbb{E} \exp \left( \int_0^t \tilde{\Phi}_r(X_s(x)) \, ds \right) \, d\gamma_d(x) \leq \frac{1}{t} \int_0^t \mathbb{E} \int_{\mathbb{R}^d} e^{t\tilde{\Phi}_r(X_s(x))} \, d\gamma_d(x) \, ds
\]
\[
= \frac{1}{t} \int_0^t \mathbb{E} \int_{\mathbb{R}^d} e^{t\tilde{\Phi}_r(y)} \, K_s(y) \, d\gamma_d(y) \, ds
\]
\[
\leq \frac{1}{t} \int_0^t \|e^{t\tilde{\Phi}_r}\|_{L^p(d\gamma_d)} \|K_s\|_{L^p(d\gamma_d \times d\gamma_d)} \, ds
\]
\[
= \|e^{t\tilde{\Phi}_r}\|_{L^q(d\gamma_d)} I(t)^{1/p},
\]
Thus we have
\[ \int_{\mathbb{R}^d} \mathbb{E}[(\tilde{K}_t(x))^{-r}] \, d\gamma_d(x) \leq \|e^{t\tilde{\Phi}_r}\|_{L^q(\gamma_d)}^{1/2} I(t)^{1/2p}. \quad (2.12) \]

Taking \( r = p - 1 \) in the above estimate and by (2.9), we obtain
\[ \int_{\mathbb{R}^d} \mathbb{E}[K_t^p(x)] \, d\gamma_d(x) \leq \|e^{t\tilde{\Phi}_{p-1}}\|_{L^q(\gamma_d)}^{1/2} I(t)^{1/2p}. \]

Thus we have \( I(t) \leq \|e^{t\tilde{\Phi}_{p-1}}\|_{L^q(\gamma_d)}^{1/2} I(t)^{1/2p} \). Solving this inequality for \( I(t) \) gives
\[ \int_{\mathbb{R}^d} \mathbb{E}[K_t^p(x)] \, d\gamma_d(x) \leq I(t) \leq \left[ \int_{\mathbb{R}^d} \exp \left( \frac{pt}{p - 1} \tilde{\Phi}_{p-1}(x) \right) \, d\gamma_d(x) \right]^{\frac{1}{p-1}}. \]

Now the desired estimate follows from the definition of \( \tilde{\Phi}_{p-1} \).

**Corollary 2.3.** For any \( p > 1 \),
\[ \|\tilde{K}_t\|_{L^p(\mathbb{P} \times \gamma_d)} \leq \left[ \int_{\mathbb{R}^d} \exp \left( (p + 1)t \left[ 2|\delta(A_0)| + \sum_{j=1}^{m} \left( |A_j|^2 + |\nabla A_j|^2 + 2p|\delta(A_j)|^2 \right) \right] \right) \, d\gamma_d \right]^{\frac{1}{p+1}}. \quad (2.13) \]

**Proof.** Similar to (2.12), we have for \( r > 0 \),
\[ \int_{\mathbb{R}^d} \mathbb{E}[(\tilde{K}_t(x))^r] \, d\gamma_d(x) \leq \|e^{t\tilde{\Phi}_r}\|_{L^q(\gamma_d)}^{1/2} I(t)^{1/2p}, \quad (2.14) \]
where \( \tilde{\Phi}_r \) and \( I(t) \) are defined as above. Since \( I(t) \leq \|e^{t\tilde{\Phi}_{p-1}}\|_{L^q(\gamma_d)}^{p/(2p-1)} \), by taking \( r = p - 1 \), we get
\[ \int_{\mathbb{R}^d} \mathbb{E}[(\tilde{K}_t(x))^{p-1}] \, d\gamma_d(x) \leq \|e^{t\tilde{\Phi}_{p-1}}\|_{L^q(\gamma_d)}^{p/(2p-1)} \]
\[ = \left[ \int_{\mathbb{R}^d} \exp \left( pt \left[ 2|\delta(A_0)| + \sum_{j=1}^{m} \left( |A_j|^2 + |\nabla A_j|^2 + 2(p-1)|\delta(A_j)|^2 \right) \right] \right) \, d\gamma_d \right]^{\frac{p-1}{2p-1}}. \]

Replacing \( p \) by \( p + 1 \) in the last inequality gives the claimed estimate. \( \square \)

### 3 Absolute continuity under flows generated by SDEs

Now assume that the coefficients \( A_j \) in SDE (1.1) are continuous and of linear growth. Then it is well known that SDE (1.1) has a weak solution of infinite life time. In order to apply the results of the preceding section, we shall regularize the vector fields using the Ornstein-Uhlenbeck semigroup \( \{P_\varepsilon\}_{\varepsilon>0} \) on \( \mathbb{R}^d \):
\[ P_\varepsilon A(x) = \int_{\mathbb{R}^d} A(e^{-\varepsilon} x + \sqrt{1-e^{-2\varepsilon}} y) \, d\gamma_d(y). \]

We have the following simple properties.

**Lemma 3.1.** Assume that \( A \) is continuous and \( |A(x)| \leq C(1+|x|^q) \) for some \( q \geq 0 \). Then
(i) there is $C_q > 0$ independent of $\varepsilon$, such that

$$|P_\varepsilon A(x)| \leq C_q (1 + |x|^q), \quad \text{for all } x \in \mathbb{R}^d;$$

(ii) $P_\varepsilon A$ converges uniformly to $A$ on any compact subset as $\varepsilon \to 0$.

Proof. (i) Note that $|e^{-\varepsilon} x + \sqrt{1 - e^{-2\varepsilon} y}| \leq |x| + |y|$ and that there exists a constant $C > 0$ such that $(|x| + |y|)^q \leq C (|x|^q + |y|^q)$. Using the growth condition on $A$, we have for some constant $C > 0$ (depending on $q$),

$$|P_\varepsilon A(x)| \leq \int_{\mathbb{R}^d} |A(e^{-\varepsilon} x + \sqrt{1 - e^{-2\varepsilon} y})| \, d\gamma_d(y)$$

$$\leq C \int_{\mathbb{R}^d} (1 + |x|^q + |y|^q) \, d\gamma_d(y) \leq C \left(1 + |x|^q + M_q\right)$$

where $M_q = \int_{\mathbb{R}^d} |y|^q \, d\gamma_d(y)$. Changing the constant yields (i).

(ii) Fix $R > 0$ and $x$ in the closed ball $B(R)$ of radius $R$, centered at $0$. Let $R_1 > R$ be arbitrary. We have

$$|P_\varepsilon A(x) - A(x)| \leq \int_{\mathbb{R}^d} |A(e^{-\varepsilon} x + \sqrt{1 - e^{-2\varepsilon} y}) - A(x)| \, d\gamma_d(y)$$

$$= \left(\int_{B(R_1)} + \int_{B(R_1)^c}\right) |A(e^{-\varepsilon} x + \sqrt{1 - e^{-2\varepsilon} y}) - A(x)| \, d\gamma_d(y)$$

$$=: I_1 + I_2.$$

By the growth condition on $A$, for some constant $C_q > 0$, independent of $\varepsilon$, we have

$$I_2 \leq \int_{B(R_1)^c} \left(|A(e^{-\varepsilon} x + \sqrt{1 - e^{-2\varepsilon} y})| + |A(x)|\right) \, d\gamma_d(y)$$

$$\leq C_q \int_{B(R_1)^c} (1 + R^q + |y|^q) \, d\gamma_d(y),$$

where the last term tends to $0$ as $R_1 \to +\infty$. For given $\eta > 0$, we may take $R_1$ large enough such that $I_2 < \eta$. Then there exists $\varepsilon_{R_1} > 0$ such that for $\varepsilon < \varepsilon_{R_1}$ and $|y| \leq R_1$,

$$|e^{-\varepsilon} x + \sqrt{1 - e^{-2\varepsilon} y} - x| \leq \varepsilon R + \sqrt{1 - e^{-2\varepsilon}} R_1 \leq R_1.$$

Note that

$$|e^{-\varepsilon} x + \sqrt{1 - e^{-2\varepsilon} y} - x| \leq \varepsilon R + \sqrt{2\varepsilon} R_1, \quad \text{for } |x| \leq R, \ |y| \leq R_1.$$

Since $A$ is uniformly continuous on $B(R_1)$, there exists $\varepsilon_0 \leq \varepsilon_{R_1}$ such that

$$|A(e^{-\varepsilon} x + \sqrt{1 - e^{-2\varepsilon} y}) - A(x)| \leq \eta \quad \text{for all } y \in B(R_1), \ \varepsilon \leq \varepsilon_0.$$

As a result, the term $I_1 \leq \eta$. Therefore by (3.1), for any $\varepsilon \leq \varepsilon_0$,

$$\sup_{|x| \leq R} |P_\varepsilon A(x) - A(x)| \leq 2\eta.$$

The result follows from the arbitrariness of $\eta > 0$. \□

The vector field $P_\varepsilon A$ is smooth on $\mathbb{R}^d$ but does not have compact support. We introduce cut-off functions $\varphi_\varepsilon \in C^\infty_c(\mathbb{R}^d, [0, 1])$ satisfying

$$\varphi_\varepsilon(x) = 1 \quad \text{if } |x| \leq \frac{1}{\varepsilon}, \quad \varphi_\varepsilon(x) = 0 \quad \text{if } |x| \geq \frac{1}{\varepsilon} + 2 \quad \text{and } \|\nabla \varphi_\varepsilon\|_\infty \leq 1.$$
Set
\[ A_j^\varepsilon = \varphi_\varepsilon P_\varepsilon A_j, \ j = 0, 1, \ldots, m. \]

Now consider the Itô SDE (1.1) with \( A_j \) being replaced by \( A_j^\varepsilon \) \((j = 0, 1, \ldots, m)\), and denote the corresponding terms by adding the superscript \( \varepsilon \), e.g., \( X^\varepsilon_t \), \( K_t^\varepsilon \), etc.

In the sequel, we shall give a uniform estimate to \( K_t^\varepsilon \). To this end, we need some preparations in the spirit of Malliavin calculus \([28]\). For a vector field \( A \) corresponding terms by adding the superscript \( \varepsilon \), we say that \( \varepsilon \) is bounded if \( \varepsilon \) is bounded, and denote by \( \varepsilon \) the corresponding terms by adding the superscript \( \varepsilon \), e.g., \( X^\varepsilon_t \), \( K_t^\varepsilon \), etc.

For such \( A \in D^p(\gamma_d) \) if \( A \in L^p(\gamma_d) \) and if there exists \( \nabla A \colon \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d \) in \( L^p(\gamma_d) \) such that for any \( v \in \mathbb{R}^d \),
\[ \nabla A(x)(v) = \partial_v A := \lim_{\eta \to 0} \frac{A(x + \eta v) - A(x)}{\eta} \text{ holds in } L^{p'}(\gamma_d) \text{ for any } p' < p. \]

For such \( A \in D^p(\gamma_d) \), the divergence \( \delta(A) \in L^p(\gamma_d) \) exists and the following relations hold:
\[ \nabla P_\varepsilon A = e^{-\varepsilon} P_\varepsilon (\nabla A), \quad \delta(P_\varepsilon A) = e^\varepsilon P_\varepsilon (\delta(A)). \tag{3.2} \]

If \( A \in L^p(\gamma_d) \), then \( P_\varepsilon A \in D^p(\gamma_d) \) and \( \lim_{\varepsilon \to 0} \| P_\varepsilon A - A \|_{L^p} = 0. \)

**Lemma 3.2.** Assume the vector field \( A \in D^p(\gamma_d) \) with \( p > 1 \), and denote by \( A^\varepsilon = \varphi_\varepsilon P_\varepsilon A \). Then for \( \varepsilon \in [0, 1] \),
\[ |\delta(A^\varepsilon)| \leq P_\varepsilon (|A| + e|\delta(A)|), \]
\[ |A^\varepsilon|^2 \leq P_\varepsilon (|A|^2), \]
\[ |\nabla A^\varepsilon|^2 \leq P_\varepsilon (2(|A|^2 + |\nabla A|^2)), \]
\[ |\delta(A^\varepsilon)|^2 \leq P_\varepsilon (2(|A|^2 + e^2|\delta(A)|^2)). \]

**Proof.** Note that according to (3.2), \( \delta(A^\varepsilon) = \delta(\varphi_\varepsilon P_\varepsilon A) = \varphi_\varepsilon e^\varepsilon P_\varepsilon \delta(A) - (\nabla \varphi_\varepsilon, P_\varepsilon A) \), from where the first inequality follows. In the same way, the other results are obtained. \( \square \)

Applying Theorem 2.2 to \( K_t^\varepsilon \) with \( p = 2 \), we have
\[ \|K_t^\varepsilon\|_{L^2(\mathbb{R} \times \gamma_d)} \leq \left[ \int_{\mathbb{R}^d} \exp \left( 2t \left[ 2|\delta(A_0^\varepsilon)| + \sum_{j=1}^m (|A_j^\varepsilon|^2 + |\nabla A_j^\varepsilon|^2 + 2|\delta(A_j^\varepsilon)|^2) \right] \right) d\gamma_d \right]^{1/6}. \tag{3.3} \]

By Lemma 3.2,
\[ 2|\delta(A_0^\varepsilon)| + \sum_{j=1}^m (|A_j^\varepsilon|^2 + |\nabla A_j^\varepsilon|^2 + 2|\delta(A_j^\varepsilon)|^2) \]
\[ \leq P_\varepsilon \left[ 2|A_0| + 2e|\delta(A_0)| + \sum_{j=1}^m (7|A_j|^2 + 2|\nabla A_j|^2 + 4e^2|\delta(A_j)|^2) \right]. \]

We deduce from Jensen’s inequality and the invariance of \( \gamma_d \) under the action of the semigroup \( P_\varepsilon \) that
\[ \|K_t^\varepsilon\|_{L^2(\mathbb{R} \times \gamma_d)} \leq \left[ \int_{\mathbb{R}^d} \exp \left( 4t \left[ |A_0| + e|\delta(A_0)| + \sum_{j=1}^m (4|A_j|^2 + |\nabla A_j|^2 + 2e^2|\delta(A_j)|^2) \right] \right) d\gamma_d \right]^{1/6}. \tag{3.4} \]

for any \( \varepsilon \leq 1 \). According to (3.4), we consider the following conditions.

**Assumptions (H):**
(A1) For \( j = 1, \ldots, m \), \( A_j \in \cap_{q \geq 1} D^{q}_{\lambda} (\gamma_d) \), \( A_0 \) is continuous and \( \delta(A_0) \) exists.

(A2) The vector fields \( A_0, A_1, \ldots, A_m \) have linear growth.

(A3) There exists \( \lambda_0 > 0 \) such that
\[
\int_{\mathbb{R}^d} \exp \left[ \lambda_0 \left( |\delta(A_0)| + \sum_{j=1}^{m} |\delta(A_j)|^2 \right) \right] d\gamma_d < +\infty.
\]

(A4) There exists \( \lambda_0 > 0 \) such that
\[
\int_{\mathbb{R}^d} \exp \left( \lambda_0 \sum_{j=1}^{m} |\nabla A_j|^2 \right) d\gamma_d < +\infty.
\]

Note that by Sobolev’s embedding theorem, the diffusion coefficients \( A_1, \ldots, A_m \) admit Hölder continuous versions. In what follows, we consider these continuous versions. It is clear that under the conditions (A2)–(A4), there exists \( T > 0 \) small enough, such that
\[
\Lambda_{T_0} := \left[ \int_{\mathbb{R}^d} \exp \left( 4T_0 [ |A_0| + \|e\delta(A_0)| + \sum_{j=1}^{m} (4|A_j|^2 + |\nabla A_j|^2 + 2e^2|\delta(A_j)|^2) \right) \right] d\gamma_d \right]^{1/6} < \infty. \tag{3.5}
\]

In this case, for \( t \in [0, T_0] \),
\[
\sup_{0 < \varepsilon \leq 1} \| K_t^\varepsilon \|_{L^2(\mathbb{R} \times \gamma_d)} \leq \Lambda_{T_0}. \tag{3.6}
\]

**Theorem 3.3.** Let \( T > 0 \) be given. Under (A1)–(A4) in Assumptions (H), there are two positive constants \( C_1 \) and \( C_2 \), independent of \( \varepsilon \), such that
\[
\sup_{0 < \varepsilon \leq 1} \mathbb{E} \int_{\mathbb{R}^d} \log K_t^\varepsilon \, d\gamma_d \leq 2 (C_1 T)^{1/2} \Lambda_{T_0} + C_2 TA_{T_0}^2, \quad \text{for all } t \in [0, T].
\]

**Proof.** We follow the arguments of Proposition 4.4 in [12]. By (2.3) and (2.4), we have
\[
K_t^\varepsilon (X_t^\varepsilon (x)) = [K_t^\varepsilon (x)]^{-1} = \exp \left( \sum_{j=1}^{m} \int_{0}^{t} \delta(A_j^\varepsilon)(X_s^\varepsilon(x)) \, dw_s^j + \int_{0}^{t} \Phi_\varepsilon(X_s^\varepsilon(x)) \, ds \right),
\]
where
\[
\Phi_\varepsilon = \delta(A_0^\varepsilon) + \frac{1}{2} \sum_{j=1}^{m} |A_j^\varepsilon|^2 + \frac{1}{2} \sum_{j=1}^{m} \langle \nabla A_j^\varepsilon, (\nabla A_j^\varepsilon)^* \rangle.
\]
Thus
\[
\mathbb{E} \int_{\mathbb{R}^d} K_t^\varepsilon \log K_t^\varepsilon \, d\gamma_d = \mathbb{E} \int_{\mathbb{R}^d} |\log K_t^\varepsilon (X_t^\varepsilon (x))| \, d\gamma_d(x)
\]
\[
\quad \leq \mathbb{E} \int_{\mathbb{R}^d} \left\| \sum_{j=1}^{m} \int_{0}^{t} \delta(A_j^\varepsilon)(X_s^\varepsilon(x)) \, dw_s^j \right\| \, d\gamma_d(x) + \mathbb{E} \int_{\mathbb{R}^d} \left\| \int_{0}^{t} \Phi_\varepsilon(X_s^\varepsilon(x)) \, ds \right\| \, d\gamma_d(x)
\]
\[
\quad =: I_1 + I_2. \tag{3.7}
\]
Using Burkholder’s inequality, we get
\[
\mathbb{E} \left\| \sum_{j=1}^{m} \int_{0}^{t} \delta(A_j^\varepsilon)(X_s^\varepsilon(x)) \, dw_s^j \right\| \leq 2 \mathbb{E} \left[ \left( \sum_{j=1}^{m} \int_{0}^{t} |\delta(A_j^\varepsilon)(X_s^\varepsilon(x))|^2 \, ds \right)^{1/2} \right].
\]
For the sake of simplifying the notations, write $\Psi_\epsilon = \sum_{j=1}^m |\delta(A_j)|^2$. By Cauchy’s inequality,

$$I_1 \leq 2 \left[ \int_0^t \mathbb{E} \int_{\mathbb{R}^d} \left| \Psi_\epsilon(X^\epsilon_s(x)) \right| d\gamma_d(x) ds \right]^{1/2}.$$  \hspace{1cm} (3.8)

Now we are going to estimate $\mathbb{E} \int_{\mathbb{R}^d} \left| \Psi_\epsilon(X^\epsilon_s(x)) \right|^2 d\gamma_d(x)$ for $\alpha \in \mathbb{Z}_+$ which will be done inductively. First if $s \in [0, T_0]$, then by (3.4) and (3.6), along with Cauchy’s inequality,

$$\mathbb{E} \int_{\mathbb{R}^d} \left| \Psi_\epsilon(X^\epsilon_s(x)) \right|^2 d\gamma_d(x) = \mathbb{E} \int_{\mathbb{R}^d} \left| \Psi_\epsilon(y) \right|^2 K^\epsilon_s(y) d\gamma_d(y)$$

$$\leq \left\| \Psi_\epsilon \right\|_{L^{2\alpha+1}(\gamma_d)}^2 \left\| K^\epsilon_s \right\|_{L^2(\mathbb{R} \times \gamma_d)}$$

$$\leq \Lambda T_0 \left\| \Psi_\epsilon \right\|_{L^{2\alpha+1}(\gamma_d)}^2.$$  \hspace{1cm} (3.9)

Now for $s \in [T_0, 2T_0]$, we shall use the flow property of $X^\epsilon_t$: let $(\theta_{T_0} w)_t := w_{T_0+t} - w_{T_0}$ and $X^\epsilon_{T_0}$ be the solution of the Itô SDE driven by the new Brownian motion $(\theta_{T_0} w)_t$, then

$$X^\epsilon_{T_0+t}(x, w) = X^\epsilon_{T_0}(X^\epsilon_{T_0}(x, w), \theta_{T_0} w),$$

for all $t \geq 0$, and $X^\epsilon_{T_0}$ enjoys the same properties as $X^\epsilon_t$. Therefore,

$$\mathbb{E} \int_{\mathbb{R}^d} \left| \Psi_\epsilon(X^\epsilon_s(x)) \right|^2 d\gamma_d(x) = \mathbb{E} \int_{\mathbb{R}^d} \left| \Psi_\epsilon(X^\epsilon_{s-T_0}(x_{T_0}(x))) \right|^2 d\gamma_d(x)$$

$$= \mathbb{E} \int_{\mathbb{R}^d} \left| \Psi_\epsilon(X^\epsilon_{s-T_0}(y)) \right|^2 K^\epsilon_{T_0}(y) d\gamma_d(y)$$

which is dominated, using Cauchy-Schwarz inequality

$$\left( \mathbb{E} \int_{\mathbb{R}^d} \left| \Psi_\epsilon(X^\epsilon_{s-T_0}(y)) \right|^{2\alpha+1} d\gamma_d(y) \right)^{1/2} \left\| K^\epsilon_{T_0} \right\|_{L^2(\mathbb{R} \times \gamma_d)}$$

$$\leq \left( \Lambda T_0 \left\| \Psi_\epsilon \right\|_{L^{2\alpha+2}(\gamma_d)} \right)^{1/2} \Lambda_{T_0} = \Lambda^{1+2^{-1}} T_0 \left\| \Psi_\epsilon \right\|_{L^{2\alpha+2}(\gamma_d)}.$$  \hspace{1cm} (3.10)

Repeating this procedure, we finally obtain, for all $s \in [0, T]$,

$$\mathbb{E} \int_{\mathbb{R}^d} \left| \Psi_\epsilon(X^\epsilon_s(x)) \right|^2 d\gamma_d(x) \leq \Lambda_{T_0}^{1+2^{-1}+\ldots+2^{-N+1}} \left\| \Psi_\epsilon \right\|_{L^{2\alpha+N}(\gamma_d)}^2,$$

where $N \in \mathbb{Z}_+$ is the unique integer such that $(N - 1)T_0 < T \leq NT_0$. In particular, taking $\alpha = 0$ gives

$$\mathbb{E} \int_{\mathbb{R}^d} \left| \Psi_\epsilon(X^\epsilon_s(x)) \right| d\gamma_d(x) \leq \Lambda_{T_0}^{2} \left\| \Psi_\epsilon \right\|_{L^{2N}(\gamma_d)}.$$  \hspace{1cm} (3.10)

By Lemma 3.2,

$$\sup_{0 < \epsilon \leq 1} \left\| \Psi_\epsilon \right\|_{L^{2N}(\gamma_d)} \leq \left\| \sum_{j=1}^m (|A_j|^2 + e^2 |\delta(A_j)|^2) \right\|_{L^{2N}(\gamma_d)} =: C_1$$

whose right hand side is finite under the assumptions (A2)–(A4). This along with (3.8) and (3.10) leads to

$$I_1 \leq 2 \left( C_1 T \right)^{1/2} \Lambda_{T_0}.$$  \hspace{1cm} (3.11)

The same manipulation works for the term $I_2$ and we get

$$I_2 \leq C_2 T \Lambda_{T_0}^2,$$  \hspace{1cm} (3.12)
Then for any $R > 0$, $u_t \geq 0$, 

\[
\int_{\mathbb{R}^d} E(u_t \log u_t) \, d\gamma_d \leq 2 (C_1 T)^{1/2} \Lambda T_0 + C_2 T \Lambda T_0^2.
\]  

(3.13)

**Theorem 3.4.** Assume conditions (A1)–(A4) and that pathwise uniqueness holds for SDE (1.1). Then for each $t > 0$, there is a full subset $\Omega_t \subset \Omega$ such that for all $w \in \Omega_t$, the density $K_t$ of $(X_t)_{\#} \gamma_d$ with respect to $\gamma_d$ exists and $K_t \in L^1 \log L_1$.

**Proof.** Under these assumptions, we can use Theorem A in [18]. For the convenience of the reader, we include the statement:

**Theorem 3.5 ([18]).** Let $\sigma_n(x)$ and $b_n(x)$ be continuous, taking values respectively in the space of $(d \times m)$-matrices and $\mathbb{R}^d$. Suppose that

\[
\sup_n (\|\sigma_n(x)\| + |b_n(x)|) \leq C (1 + |x|),
\]

and for any $R > 0$,

\[
\lim_{n \to +\infty} \sup_{|x| \leq R} (\|\sigma_n(x) - \sigma(x)\| + |b_n(x) - b(x)|) = 0.
\]

Suppose further that for the same Brownian motion $B(t)$, $X_n(x,t)$ solves the SDE

\[
dX_n(t) = \sigma_n(X_n(t)) \, dB(t) + b_n(X_n(t)) \, dt, \quad X_n(0) = x.
\]

If pathwise uniqueness holds for

\[
dX(t) = \sigma(X(t)) \, dB(t) + b(X(t)) \, dt, \quad X(0) = x,
\]

then for any $R > 0$, $T > 0$, 

\[
\lim_{n \to +\infty} \sup_{|x| \leq R} \mathbb{E} \left( \sup_{0 \leq t \leq T} |X_n(t,x) - X(t,x)|^2 \right) = 0.
\]  

(3.14)

We continue the proof of Theorem 3.4. By means of Lemma 3.1 and Theorem 3.5, for any $T, R > 0$, we get

\[
\lim_{\varepsilon \to 0} \sup_{|x| \leq R} \mathbb{E} \left( \sup_{0 \leq t \leq T} |X_n^\varepsilon(x) - X_t(x)|^2 \right) = 0.
\]  

(3.15)
Now fixing arbitrary \( \xi \in L^\infty(\Omega) \) and \( \psi \in C_c^\infty(\mathbb{R}^d) \), we have

\[
\mathbb{E} \int_{\mathbb{R}^d} |\xi(\cdot)| |\psi(X_t^\varepsilon(x)) - \psi(X_t(x))| \, d\gamma_d(x) \\
\leq \|\xi\|_\infty \left( \int_{B(R)} + \int_{B(R)^c} \right) \mathbb{E} |\psi(X_t^\varepsilon(x)) - \psi(X_t(x))| \, d\gamma_d(x) \\
=: J_1 + J_2. \tag{3.16}
\]

By (3.15),

\[
J_1 \leq \|\xi\|_\infty \|\nabla \psi\|_\infty \int_{B(R)} \mathbb{E} |X_t^\varepsilon(x) - X_t(x)| \, d\gamma_d(x) \\
\leq \|\xi\|_\infty \|\nabla \psi\|_\infty \left[ \sup_{|x| \leq R} \mathbb{E} \left( \sup_{0 \leq t \leq T} |X_t^\varepsilon(x) - X_t(x)|^2 \right) \right]^{1/2} \to 0, \tag{3.17}
\]

as \( \varepsilon \) tends to 0. It is obvious that

\[
J_2 \leq 2 \|\xi\|_\infty \|\psi\|_\infty \gamma_d(B(R)^c). \tag{3.18}
\]

Combining (3.16), (3.17) and (3.18), we obtain

\[
\limsup_{\varepsilon \to 0} \mathbb{E} \int_{\mathbb{R}^d} |\xi| |\psi(X_t^\varepsilon(x)) - \psi(X_t(x))| \, d\gamma_d(x) \leq 2 \|\xi\|_\infty \|\psi\|_\infty \gamma_d(B(R)^c) \to 0
\]
as \( R \uparrow \infty \). Therefore

\[
\lim_{\varepsilon \to 0} \mathbb{E} \int_{\mathbb{R}^d} \xi \psi(X_t^\varepsilon(x)) \, d\gamma_d(x) = \mathbb{E} \int_{\mathbb{R}^d} \xi \psi(X_t(x)) \, d\gamma_d. \tag{3.19}
\]

On the other hand, by Theorem 3.3, for each fixed \( t \in [0, T] \), up to a subsequence, \( K_t^\varepsilon \) converges weakly in \( L^1(\Omega \times \mathbb{R}^d) \) to some \( \hat{K}_t \), hence

\[
\mathbb{E} \int_{\mathbb{R}^d} \xi \psi(X_t^\varepsilon(x)) \, d\gamma_d(x) = \mathbb{E} \int_{\mathbb{R}^d} \xi \psi(y) K_t^\varepsilon(y) \, d\gamma_d(y) \\
\to \mathbb{E} \int_{\mathbb{R}^d} \xi \psi(y) \hat{K}_t(y) \, d\gamma_d(y). \tag{3.20}
\]

This together with (3.19) leads to

\[
\mathbb{E} \int_{\mathbb{R}^d} \xi \psi(X_t(x)) \, d\gamma_d(x) = \mathbb{E} \int_{\mathbb{R}^d} \xi \psi(y) \hat{K}_t(y) \, d\gamma_d(y).
\]

By the arbitrariness of \( \xi \in L^\infty(\Omega) \), there exists a full measure subset \( \Omega_\psi \) of \( \Omega \) such that

\[
\int_{\mathbb{R}^d} \psi(X_t(x)) \, d\gamma_d(x) = \int_{\mathbb{R}^d} \psi(y) \hat{K}_t(y) \, d\gamma_d(y), \quad \text{for any } \omega \in \Omega_\psi.
\]

Now by the separability of \( C_c^\infty(\mathbb{R}^d) \), there exists a full subset \( \Omega_t \) such that the above equality holds for any \( \psi \in C_c^\infty(\mathbb{R}^d) \). Hence \( (X_t)_\# \gamma_d = \hat{K}_t \gamma_d \).

**Remark 3.6.** The \( K_t(w, x) \) appearing in (3.13) is defined almost everywhere. It is easy to see that \( K_t(w, x) \) is a measurable modification of \( \{ \hat{K}_t(w, x); \ t \in [0, T] \} \).
Remark 3.7. Beyond the Lipschitz condition, several sufficient conditions guaranteeing pathwise uniqueness for SDE (1.1) can be found in the literature. For example in [13], the authors give the condition
\[ \sum_{j=1}^{m} |A_j(x) - A_j(y)|^2 \leq C |x - y|^2 r(|x - y|^2), \]
for \( |x - y| \leq c_0 \) small enough, where \( r: [0, c_0] \to [0, +\infty[ \) is \( C^1 \) satisfying
\begin{enumerate}
  \item \( \lim_{s \to 0} r(s) = +\infty, \)
  \item \( \lim_{s \to 0} \frac{sr'(s)}{r(s)} = 0, \) and
  \item \( \int_{0}^{c_0} \frac{ds}{sr(s)} = +\infty. \)
\end{enumerate}

Corollary 3.8. Suppose that the vector fields \( A_0, A_1, \ldots, A_m \) are globally Lipschitz continuous and there exists a constant \( C > 0 \), such that
\[ \sum_{j=1}^{m} (x, A_j(x))^2 \leq C (1 + |x|^2) \quad \text{for all } x \in \mathbb{R}^d. \] (3.21)
Then \( (X_t)_{#} \text{Leb}_d \ll \text{Leb}_d \) for any \( t \in [0, T] \).

Proof. It is obvious that hypotheses (A1), (A2) and (A4) are satisfied, and that for some constant \( C > 0, \)
\[ |\delta(A_0)|(x) \leq C(1 + |x|^2). \]
Hence there exists \( \lambda_0 > 0 \) such that \( \int_{\mathbb{R}^d} \exp(\lambda_0 |\delta(A_0)|) \, d\gamma_d < +\infty. \) Finally we have
\[ \sum_{j=1}^{m} |\delta(A_j)|^2(x) \leq 2 \sum_{j=1}^{m} (x, A_j(x))^2 + 2 \sum_{j=1}^{m} \text{Lip}(A_j)^2. \]
Therefore, under condition (3.21), there exists \( \lambda_0 > 0 \) such that
\[ \int_{\mathbb{R}^d} \exp\left(\lambda_0 \sum_{j=1}^{m} |\delta(A_j)|^2 \right) d\gamma_d < +\infty. \]
Hence, hypothesis (A3) is satisfied as well. By Theorem 3.4, we have \( (X_t)_{#} \gamma_d = \hat{K}_t \gamma_d. \) Let \( A \) be a Borel subset of \( \mathbb{R}^d \) such that \( \text{Leb}_d(A) = 0 \), then \( \gamma_d(A) = 0; \) therefore \( \int_{\mathbb{R}^d} 1_{\{X_t(x) \in A\}} \, d\gamma_d(x) = 0. \)
It follows that \( 1_{\{X_t(x) \in A\}} = 0 \) for \( \text{Leb}_d \) almost every \( x \), which implies \( \text{Leb}_d(X_t \in A) = 0; \) this means that \( (X_t)_{#} \text{Leb}_d \) is absolutely continuous with respect to \( \text{Leb}_d. \)

In the next section, we shall prove that under the conditions of Corollary 3.8, the density of \( (X_t)_{#} \text{Leb}_d \) with respect to \( \text{Leb}_d \) is strictly positive, in other words, \( \text{Leb}_d \) is quasi-invariant under \( X_t. \)

Corollary 3.9. Assume that conditions (A1)–(A4) hold. Let \( \sigma = (A_j^2) \) and suppose that for some \( C > 0, \)
\[ \sigma(x)\sigma(x)^* \geq C \text{Id}, \quad \text{for all } x \in \mathbb{R}^d. \]
Then \( (X_t)_{#} \gamma_d \) is absolutely continuous with respect to \( \gamma_d. \)

Proof. The conditions (A1)–(A4) are stronger than those in Theorem 1.1 of [34] given by X. Zhang, so the pathwise uniqueness holds. Hence Theorem 3.4 applies to this case. \( \square \)
4 Quasi-invariance under stochastic flow

In the sequel, by quasi-invariance we mean that the Radon-Nikodym derivative of the corresponding push-forward measure is strictly positive. First we prove that in the situation of Corollary 3.8, the Lebesgue measure is in fact quasi-invariant under the stochastic flow of homeomorphisms. To this end, we need some preparations. In what follows, \( T_0 > 0 \) is chosen small enough such that (3.5) holds.

**Proposition 4.1.** Let \( q \geq 2 \). Then

\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \mathbb{E} \left( \left| \sup_{0 \leq t \leq T_0} \sum_{j=1}^m \int_0^t [\delta(A_j^\varepsilon)(X_s^\varepsilon) - \delta(A_j)(X_s)] \, dw_s \right|^q \right) \, d\gamma_d = 0. \tag{4.1}
\]

**Proof.** By Burkholder’s inequality,

\[
\mathbb{E} \left( \sup_{0 \leq t \leq T_0} \left| \sum_{j=1}^m \int_0^t [\delta(A_j^\varepsilon)(X_s^\varepsilon) - \delta(A_j)(X_s)] \, dw_s \right|^q \right)
\leq C \mathbb{E} \left( \left( \int_0^{T_0} \sum_{j=1}^m |\delta(A_j^\varepsilon)(X_s^\varepsilon) - \delta(A_j)(X_s)|^2 \, ds \right)^{q/2} \right)
\leq C \int_0^{T_0} \mathbb{E} \left( |\delta(A_j)(X_s^\varepsilon) - \delta(A_j)(X_s)|^q \right) \, ds.
\]

Again by the inequality \((a + b)^q \leq C_q (a^q + b^q)\), there exists a constant \( C_{q,T_0} > 0 \) such that the above quantity is dominated by

\[
C_{q,T_0} \sum_{j=1}^m \int_0^{T_0} \mathbb{E} \left( |\delta(A_j^\varepsilon)(X_s^\varepsilon) - \delta(A_j)(X_s)|^q \right) \, ds + \int_0^{T_0} \mathbb{E} \left( |\delta(A_j)(X_s^\varepsilon) - \delta(A_j)(X_s)|^q \right) \, ds. \tag{4.2}
\]

Let \( I_1^\varepsilon \) and \( I_2^\varepsilon \) be the two terms in the squared bracket of (4.2). Note that

\[
\int_{\mathbb{R}^d} \mathbb{E} \left( |\delta(A_j^\varepsilon)(X_s^\varepsilon) - \delta(A_j)(X_s)|^q \right) \, d\gamma_d
= \mathbb{E} \int_{\mathbb{R}^d} |\delta(A_j^\varepsilon) - \delta(A_j)|^q K_s \, d\gamma_d
\leq \|\delta(A_j^\varepsilon) - \delta(A_j)\|_{L^{2q}(\gamma_d)}^q \|K_s\|_{L^2(\mathbb{P} \times \gamma_d)}. \tag{4.3}
\]

According to (3.5), for \( s \leq T_0 \), we have \( \|K_s\|_{L^2(\mathbb{P} \times \gamma_d)} \leq \Lambda_{T_0} \). Remark that

\[
\delta(A_j^\varepsilon) = \delta(\varphi_\varepsilon P_\varepsilon A_j) = \varphi_\varepsilon e^\varepsilon P_\varepsilon \delta(A_j) - \langle \nabla \varphi_\varepsilon, P_\varepsilon A_j \rangle,
\]

which converges to \( \delta(A_j) \) in \( L^{2q}(\gamma_d) \). By (4.3),

\[
\int_{\mathbb{R}^d} I_1^\varepsilon \, d\gamma_d = \int_0^{T_0} \left[ \int_{\mathbb{R}^d} \mathbb{E} \left( |\delta(A_j^\varepsilon)(X_s^\varepsilon) - \delta(A_j)(X_s)|^q \right) \, d\gamma_d \right] \, ds
\leq T_0 \Lambda_{T_0} \|\delta(A_j^\varepsilon) - \delta(A_j)\|_{L^{2q}(\gamma_d)}^q
\]

which tends to 0 as \( \varepsilon \to 0 \).

For the estimate of \( I_2^\varepsilon \), we remark that \( \int_{\mathbb{R}^d} |\delta(A_j)|^{2q} \, d\gamma_d < +\infty \). Let \( \eta > 0 \) be given. There exists \( \psi \in C_c(\mathbb{R}^d) \) such that

\[
\int_{\mathbb{R}^d} |\delta(A_j) - \psi|^{2q} \, d\gamma_d \leq \eta^2.
\]

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We have, for some constant $C_q > 0$,
\[
\int_{\mathbb{R}^d} \mathbb{E}(|\delta(A_j)(X^\varepsilon_s) - \delta(A_j)(X_s)|^q) \, d\gamma_d \\
\leq C_q \left[ \int_{\mathbb{R}^d} \mathbb{E}(|\delta(A_j)(X^\varepsilon_s) - \psi(X^\varepsilon_s)|^q) \, d\gamma_d + \int_{\mathbb{R}^d} \mathbb{E}(|\psi(X^\varepsilon_s) - \psi(X_s)|^q) \, d\gamma_d \\
+ \int_{\mathbb{R}^d} \mathbb{E}(|\psi(X_s) - \delta(A_j)(X_s)|^q) \, d\gamma_d \right].
\]

(4.4)

Again by (3.6), we find
\[
\mathbb{E} \left[ \int_{\mathbb{R}^d} |\delta(A_j)(X^\varepsilon_s) - \psi(X^\varepsilon_s)|^q \, d\gamma_d \right] = \mathbb{E} \left[ \int_{\mathbb{R}^d} |\delta(A_j) - \psi|^q K^\varepsilon_s \, d\gamma_d \right] \\
\leq \|\delta(A_j) - \psi\|_{L^{2q}(\gamma)}^q A_T \leq \Lambda_{T_0} \eta.
\]

In the same way,
\[
\mathbb{E} \left[ \int_{\mathbb{R}^d} |\delta(A_j)(X_s) - \psi(X_s)|^q \, d\gamma_d \right] \leq \Lambda_{T_0} \eta.
\]

To estimate the second term on the right hand side of (4.4), we use Theorem 3.5: from (3.14), we see that up to a subsequence, $X^\varepsilon_s(w, x)$ converges to $X_s(w, x)$, for each $s \leq T_0$ and almost all $(w, x) \in \Omega \times \mathbb{R}^d$. By Lebesgue’s dominated convergence theorem,
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \mathbb{E}(|\psi(X^\varepsilon_s) - \psi(X_s)|^q) \, d\gamma_d = 0.
\]

In conclusion, $\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} I_s^\varepsilon \, d\gamma_d = 0$. According to (4.2), the proof of (4.1) is complete. \qed

**Proposition 4.2.** Let $\Phi$ be defined by
\[
\Phi = \delta(A_0) + \frac{1}{2} \sum_{j=1}^m |A_j|^2 + \frac{1}{2} \sum_{j=1}^m \langle \nabla A_j, (\nabla A_j)^* \rangle,
\]
and analogously $\Phi^\varepsilon$ where $A_j$ is replaced by $A^\varepsilon_j$. Then
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \int_0^{T_0} \mathbb{E}(|\Phi^\varepsilon(X^\varepsilon_s) - \Phi(X_s)|^q) \, ds \, d\gamma_d = 0.
\]

(4.6)

**Proof.** Along the lines of the proof of Proposition 4.1, it is sufficient to remark that
\[
\lim_{\varepsilon \to 0} \|\Phi^\varepsilon - \Phi\|_{L^{2q}(\gamma)} = 0.
\]

(4.7)

To see this, let us check convergence for the last term in the definition of $\Phi^\varepsilon$. We have
\[
|\langle \nabla A^\varepsilon_j, (\nabla A^\varepsilon_j)^* \rangle - \langle \nabla A_j, (\nabla A_j)^* \rangle| \\
\leq \|\nabla A^\varepsilon_j - \nabla A_j\| \|\nabla A^\varepsilon_j\| + \|\nabla A_j\| \|\nabla A^\varepsilon_j - \nabla A_j\|.
\]

Note that $A^\varepsilon_j = \varphi_\varepsilon P_\varepsilon A_j$. Thus
\[
\nabla A^\varepsilon_j = \nabla \varphi_\varepsilon \otimes P_\varepsilon A_j + e^{-\varepsilon} \varphi_\varepsilon P_\varepsilon (\nabla A_j),
\]
which converges to $\nabla A_j$ in $L^{2q}(\gamma_d)$ as $\varepsilon \to 0$. \qed

Now we can prove
Proposition 4.3. Under the conditions of Corollary 3.8, the Lebesgue measure \( \text{Leb}_d \) is quasi-invariant under the stochastic flow.

Proof. Let \( k_t \) be the density of \( (X_t)_\# \text{Leb}_d \) with respect to \( \text{Leb}_d \). We shall prove that \( k_t \) is strictly positive. Set

\[
K_t^\varepsilon(x) = \exp \left( - \sum_{j=1}^{m} \int_{0}^{t} \delta(A_j^\varepsilon)(X_s^\varepsilon(x)) \, dw_s^j - \int_{0}^{t} \Phi_\varepsilon(X_s^\varepsilon(x)) \, ds \right),
\]

where \( \Phi_\varepsilon \) is defined in Proposition 4.2. By (2.3) we have

\[
\int_{\mathbb{R}^d} \psi(X_t^\varepsilon) K_t^\varepsilon \, d\gamma_d = \int_{\mathbb{R}^d} \psi \, d\gamma_d, \quad \psi \in C_c^1(\mathbb{R}^d).
\]

Applying Propositions 4.1 and 4.2, up to a subsequence, for each \( t \leq T_0 \) and almost every \((w, x)\), the term \( K_t^\varepsilon(w, x) \) defined in (4.8) converges to

\[
K_t(x) = \exp \left( - \sum_{j=1}^{m} \int_{0}^{t} \delta(A_j)(X_s(x)) \, dw_s^j - \int_{0}^{t} \Phi(X_s(x)) \, ds \right).
\]

By Corollary 2.3 and Lemma 3.2, we may assume that \( T_0 \) is small enough so that for any \( t \leq T_0 \), the family \( \{K_t^\varepsilon : \varepsilon \leq 1\} \) is also bounded in \( L^2(\mathbb{P} \times \gamma_d) \). Therefore, by the uniform integrability, letting \( \varepsilon \to 0 \) in (4.9), we get \( \mathbb{P} \)-almost surely,

\[
\int_{\mathbb{R}^d} \psi(X_t) K_t \, d\gamma_d = \int_{\mathbb{R}^d} \psi \, d\gamma_d, \quad \psi \in C_c^1(\mathbb{R}^d).
\]

Now taking a Borel version of \( x \to K_t(w, x) \). Under the assumptions, the solution \( X_t \) is a stochastic flow of homeomorphisms, hence the inverse flow \( X_t^{-1} \) exists. Consequently, if \( t \leq T_0 \), we deduce from (4.11) that the density \( K_t(w, x) \) of \( (X_t)_\# \gamma_d \) with respect to \( \gamma_d \) admits the expression \( K_t(w, x) = [K_t(w, X_t^{-1}(w, x))]^{-1} \) which is strictly positive. For \( X_{t+T_0} \) with \( t \leq T_0 \), we use the flow property: \( X_{t+T_0}(w, x) = X_t(\theta_{T_0}w, X_{T_0}(w, x)) \). Thus, for any \( \psi \in C_c^1(\mathbb{R}^d) \),

\[
\int_{\mathbb{R}^d} \psi(X_{t+T_0}) \, d\gamma_d = \int_{\mathbb{R}^d} \psi(X_t(X_{T_0})) \, d\gamma_d = \int_{\mathbb{R}^d} \psi(X_t) K_{T_0} \, d\gamma_d = \int_{\mathbb{R}^d} \psi(X_t) K_t \, d\gamma_d.
\]

That is to say, the density \( K_{t+T_0} = K_{T_0}(X_t^{-1}) K_t \) is strictly positive. Continuing in this way, we obtain that \( K_t \) is strictly positive for any \( t \geq 0 \).

Now if \( \rho(x) \) denotes the density of \( \gamma_d \) with respect to \( \text{Leb}_d \), then

\[
k_t(w, x) = \rho(X_t^{-1}(w, x))^{-1} K_t(w, x) \rho(x) > 0
\]

which concludes the proof. \( \square \)

In what follows, we will give examples for which existence of the inverse flow is not known.

Theorem 4.4. Let \( A_1, \ldots, A_m \) be bounded \( C^1 \) vector fields on \( \mathbb{R}^d \) such that their derivatives are of linear growth; furthermore let \( A_0 \) be continuous of linear growth such that \( \delta(A_0) \) exists. Define

\[
\hat{A}_0 = A_0 - \sum_{j=1}^{m} \mathcal{L}_{A_j} A_j.
\]
Suppose that \( \delta(\hat{A}_0) \) exists and that
\[
\int_{\mathbb{R}^d} \exp \left( \lambda_0 (|\delta(A_0)| + |\delta(\hat{A}_0)|) \right) \, d\gamma_d < +\infty, \quad \text{for some } \lambda_0 > 0.
\] (4.13)

If pathwise uniqueness holds both for SDE (1.1) and for
\[
dY_t = \sum_{j=1}^{m} A_j(Y_t) \, dw^j_t - \hat{A}_0(Y_t) \, dt,
\] (4.14)
then the solution \( X_t \) to SDE (1.1) leaves the Gaussian measure \( \gamma_d \) quasi-invariant.

**Proof.** Obviously the conditions in Theorem 3.4 are satisfied; hence \((X_t)_{#} \gamma_d = K_t \gamma_d\). Let \( t > 0 \) be given, we consider the dual SDE to (1.1):
\[
dY^t_s = \sum_{j=1}^{m} A_j(Y^t_s) \, dw^j_s - \hat{A}_0(Y^t_s) \, ds
\]
for which pathwise uniqueness holds; here \( w^j_s = w_{t-s} - w_t \) with \( s \in [0,t] \). Let \( A^\varepsilon_j, j = 0,1,\ldots,m \), be the vector fields defined as above. Consider
\[
dY^{t,\varepsilon}_s = \sum_{j=1}^{m} A^\varepsilon_j(Y^{t,\varepsilon}_s) \, dw^j_s - \hat{A}^\varepsilon_0(Y^{t,\varepsilon}_s) \, ds,
\]
where \( \hat{A}^\varepsilon_0 = A_0^\varepsilon - \sum_{j=1}^{m} \mathcal{L}_{A_j^\varepsilon} A_j^\varepsilon \). Then it is known that \((X^\varepsilon_t)^{-1} = Y^{t,\varepsilon}_t \). It is easy to check that for some constant \( C > 0 \) independent of \( \varepsilon \),
\[
|\hat{A}^\varepsilon_0(x)| \leq C (1 + |x|).
\] (4.15)
Moreover,
\[
\mathcal{L}_{A_j^\varepsilon} A_j^\varepsilon = \sum_{k=1}^{d} (A_j^\varepsilon)^k \left[ \frac{\partial \varphi^\varepsilon}{\partial x_k} P^\varepsilon A_j + \varphi^\varepsilon e^{-\varepsilon P^\varepsilon} \left( \frac{\partial A_j}{\partial x_k} \right) \right]
\]
which converges locally uniformly to \( \mathcal{L}_{A_j} A_j \). Therefore \( \hat{A}^\varepsilon_0 \) converges uniformly over any compact subset to \( \hat{A}_0 \). By Theorem 3.5,
\[
\lim_{\varepsilon \to 0} \sup_{|x| \leq R} \mathbb{E} \left( \sup_{0 \leq s \leq t} |Y^{t,\varepsilon}_s - Y^{t}_s|^2 \right) = 0.
\]
It follows that, along a sequence, \( Y^{t,\varepsilon} \) converges to \( Y^t \) for almost every \((w,x)\). Now let \( \psi_1, \psi_2 \in \mathcal{C}_b(\mathbb{R}^d) \), we have for \( t \leq T_0 \),
\[
\int_{\mathbb{R}^d} \psi_1(Y^t_s) \, d\gamma_d = \int_{\mathbb{R}^d} \psi_1(Y^{t,\varepsilon}_s) \cdot \psi_2 \, d\gamma_d.
\]
Letting \( \varepsilon \to 0 \) leads to
\[
\int_{\mathbb{R}^d} \psi_1(Y^t_s) \, d\gamma_d = \int_{\mathbb{R}^d} \psi_1(Y^t_s) \cdot \psi_2 \, d\gamma_d.
\] (4.16)
Taking \( \psi_1 \) and \( \psi_2 \) positive in (4.16) and using a monotone class argument, we see that equation (4.16) holds for any positive Borel functions \( \psi_1 \) and \( \psi_2 \). Hence taking a Borel version of \( K_t \) and setting \( \psi_1 = 1/K_t \) in (4.16), we get
\[
\int_{\mathbb{R}^d} \psi_2(Y^t_s) \, d\gamma_d = \int_{\mathbb{R}^d} \left[ K_t(Y^t_s) \right]^{-1} \psi_2 \, d\gamma_d.
\] (4.17)
It follows that $K_t = [\tilde{K}_t(Y^t_0)]^{-1} > 0$ for $t \leq T_0$. For $X_{t+T_0}$ with $t \leq T_0$, we shall use repeatedly (4.16). By the flow property, $X_{t+T_0}(w, x) = X_t(\theta_{T_0} w, X_{T_0}(w, x))$ where $\theta_{T_0} w = w_{t+T_0} - w_{T_0}$. Letting $t = T_0$ and replacing $\psi$ by $\psi_2(X_t)$ we get

$$\int_{\mathbb{R}^d} \psi_1 \cdot \psi_2(X_{t+T_0}) \tilde{K}_{T_0} \, d\gamma_d = \int_{\mathbb{R}^d} \psi_1(Y^t_{T_0}) \psi_2(X_t) \, d\gamma_d.$$ 

Taking $\psi_1 = 1/\tilde{K}_{T_0}$ in the above equality, we get

$$\int_{\mathbb{R}^d} \psi_2(X_{t+T_0}) \, d\gamma_d = \int_{\mathbb{R}^d} [\tilde{K}_{T_0}(Y^t_{T_0})]^{-1} \psi_2(X_t) \, d\gamma_d = \int_{\mathbb{R}^d} [\tilde{K}_{T_0}(Y^t_{T_0})]^{-1} \psi_2(X_t) \tilde{K}_t^{-1} \, d\gamma_d,$$

where in the last equality we have used (4.16) with $\psi_1 = [\tilde{K}_{T_0}(Y^t_{T_0})]^{-1} \tilde{K}_t^{-1}$. It follows that the density $K_{t+T_0}$ of $(X_{t+T_0})$ with respect to $\gamma_d$ is strictly positive, and so on.

**Corollary 4.5.** Let $A_1, \ldots, A_m$ be bounded $C^2$ vector fields such that their derivatives up to order 2 grow at most linearly, and let $A_0$ be a continuous vector field of linear growth. Suppose that

$$|A_0(x) - A_0(y)| \leq C_R |x - y| \log k \left| \frac{1}{|x - y|} \right| \quad \text{for } |x| \leq R, \ |y| \leq R, \ |x - y| \leq c_0 \text{ small enough},$$

(4.18)

where $\log_k s = (\log s)(\log \log s) \cdots (\log \cdots \log s)$. Suppose further that

$$\text{div}(A_0) = \sum_{j=1}^d \frac{\partial A^j_0}{\partial x_j}$$

exists and is bounded. Then the stochastic flow $X_t$ defined by SDE (1.1) leaves the Lebesgue measure quasi-invariant.

**Proof.** It is obvious that $\hat{A}_0$ defined in (4.12) satisfies condition (4.18); therefore by [13], pathwise uniqueness holds for SDE (1.1) and (4.14). Note that $\delta(A_0) = (x, A_0) - \text{div}(A_0)$. Then condition (4.13) is satisfied; thus Theorem 4.4 yields the result.

**5 The case $A_0$ in Sobolev spaces**

From now on, $A_0$ is not supposed to be continuous, but in some Sobolev space, that is, we replace the condition (A1) in (H) by

(A1') For $i = 1, \ldots, m$, $A_i \in \cap_{q \geq 1} D^q_1(\gamma_d)$, $A_0 \in D^q_1(\gamma_d)$ for some $q > 1$.

First we establish the following *a priori* estimate on perturbations, using the method developed in [36]. Let $\{A_0, A_1, \cdots, A_m\}$ be a family of measurable vector fields on $\mathbb{R}^d$. We shall give a precise definition of solution to the following SDE

$$dX_t = \sum_{i=1}^m A_i(X_t) \, dw^i_t + A_0(X_t) \, dt, \quad X_0 = x.$$  

(5.1)
Definition 5.1. We say that a measurable map \( X : \Omega \times \mathbb{R}^d \to C([0,T], \mathbb{R}^d) \) is a solution to Itô SDE (5.1) if

(i) for each \( t \in [0,T] \) and almost all \( x \in \mathbb{R}^d \), \( w \to X_t(w,x) \) is measurable with respect to \( \mathcal{F}_t \), i.e. the natural filtration generated by the Brownian motion \( \{ w_s; s \leq t \} \);

(ii) for each \( t \in [0,T] \), there exists \( K_t \in L^1(\mathbb{P} \times \mathbb{R}^d) \) such that \((X_t(w,\cdot))\# \gamma_d \) admits \( K_t \) as the density with respect to \( \gamma_d \);

(iii) almost surely

\[
\sum_{i=1}^{m} \int_0^T |A_i(X_s(w,x))|^2 \, ds + \int_0^T |A_0(X_s(w,x))| \, ds < +\infty;
\]

(iv) for almost all \( x \in \mathbb{R}^d \),

\[
X_t(w,x) = x + \sum_{i=1}^{m} \int_0^t A_i(X_s(w,x)) \, dw_i^s + \int_0^t A_0(X_s(w,x)) \, ds;
\]

(v) the flow property holds

\[
X_{t+s}(w,x) = X_t(\theta_s w, X_s(w,x)).
\]

Now consider another family of measurable vector fields \( \{ \hat{A}_0, \hat{A}_1, \cdots, \hat{A}_m \} \) on \( \mathbb{R}^d \), and denote by \( \hat{X}_t \) the solution to the SDE

\[
d\hat{X}_t = \sum_{i=1}^{m} \hat{A}_i(\hat{X}_t) \, dw_i^t + \hat{A}_0(\hat{X}_t) \, dt, \quad \hat{X}_0 = x. \tag{5.2}
\]

Let \( \hat{K}_t \) be the density of \( (\hat{X}_t)\# \gamma_d \) and define

\[
A_{p,T} = \sup_{0 \leq t \leq T} \left( \| \hat{K}_t \|_{L^p(\gamma_d)} \vee \| \hat{K}_t \|_{L^p(\mathbb{P} \times \gamma_d)} \right). \tag{5.3}
\]

Theorem 5.2. Let \( q > 1 \). Suppose that \( A_1, \cdots, A_m \) as well as \( \hat{A}_1, \cdots, \hat{A}_m \) are in \( D_{1,2}^q(\gamma_d) \) and \( A_0, \hat{A}_0 \in D_{1,2}^2(\gamma_d) \). Then for any \( T > 0 \) and \( R > 0 \), there exist constants \( C_{d,q,R} > 0 \) and \( C_T > 0 \) such that for any \( \sigma > 0 \),

\[
\mathbb{E} \left[ \int_{G_R} \log \left( \frac{\sup_{0 \leq t \leq T} |X_t - \hat{X}_t|^2}{\sigma^2} + 1 \right) \, d\gamma_d \right] \leq C_T A_{p,T} \left\{ C_{d,q,R} \left[ \| \nabla A_0 \|_{L^q} + \left( \sum_{i=1}^{m} \| \nabla A_i \|_{L^{2q}}^2 \right)^{\frac{1}{2}} + \sum_{i=1}^{m} \| \nabla A_i \|_{L^{2q}}^2 \right] \right. \\
\left. + \frac{1}{\sigma^2} \sum_{i=1}^{m} \| A_i - \hat{A}_i \|_{L^{2q}}^2 + \frac{1}{\sigma} \left[ \| A_0 - \hat{A}_0 \|_{L^q} + \left( \sum_{i=1}^{m} \| A_i - \hat{A}_i \|_{L^{2q}}^2 \right)^{\frac{1}{2}} \right] \right\},
\]

where \( p \) is the conjugate number of \( q \): \( 1/p + 1/q = 1 \) and

\[
G_R(w) = \left\{ x \in \mathbb{R}^d : \sup_{0 \leq t \leq T} |X_t(w,x)| \vee |\hat{X}_t(w,x)| \leq R \right\}. \tag{5.4}
\]
Thus by H"older’s inequality and according to (5.3), we have
\[d|\xi_t|^2 = 2 \sum_{i=1}^{m} \langle \xi_t, A_i(X_t) - \hat{A}_i(\hat{X}_t) \rangle \, dw^i_t + 2 \langle \xi_t, A_0(X_t) - \hat{A}_0(\hat{X}_t) \rangle \, dt \]
\[+ \sum_{i=1}^{m} |A_i(X_t) - \hat{A}_i(\hat{X}_t)|^2 dt. \]
\[(5.5)\]

For \(\sigma > 0\), \(\log (\frac{|\xi_t|^2}{\sigma} + 1) = \log (|\xi_t|^2 + \sigma^2) - \log \sigma^2\). Again by the Itô formula,
\[d\log(|\xi_t|^2 + \sigma^2) = \frac{d|\xi_t|^2}{|\xi_t|^2 + \sigma^2} - \frac{1}{2} \cdot 4 \sum_{i=1}^{m} \langle \xi_t, A_i(X_t) - \hat{A}_i(\hat{X}_t) \rangle^2 \frac{1}{(|\xi_t|^2 + \sigma^2)^2} dt,
\]
using (5.5), we obtain
\[d\log(|\xi_t|^2 + \sigma^2) = 2 \sum_{i=1}^{m} \frac{\langle \xi_t, A_i(X_t) - \hat{A}_i(\hat{X}_t) \rangle}{|\xi_t|^2 + \sigma^2} \, dw^i_t + 2 \frac{\langle \xi_t, A_0(X_t) - \hat{A}_0(\hat{X}_t) \rangle}{|\xi_t|^2 + \sigma^2} \, dt \]
\[+ \sum_{i=1}^{m} \frac{|A_i(X_t) - \hat{A}_i(\hat{X}_t)|^2}{|\xi_t|^2 + \sigma^2} dt - 2 \sum_{i=1}^{m} \frac{\langle \xi_t, A_i(X_t) - \hat{A}_i(\hat{X}_t) \rangle^2}{(|\xi_t|^2 + \sigma^2)^2} dt \]
\[= dI_1(t) + dI_2(t) + dI_3(t) + dI_4(t). \]
\[(5.6)\]

Let \(\tau_R(x) = \inf\{t \geq 0 : |X_t(x)| \vee |\hat{X}_t(x)| > R\}\). Remark that almost surely, \(G_R \subset \{x : \tau_R(x) > T\}\) and for any \(t \geq 0\), \(\{\tau_R > t\} \subset B(R)\). Therefore
\[\mathbb{E} \left[ \int_{B(R)} \sup_{0 \leq s \leq T} |I_1(t)| \, d\gamma_d \right] \leq \mathbb{E} \left[ \int_{B(R)} \sup_{0 \leq s \leq T \wedge \tau_R} |I_1(t)| \, d\gamma_d \right].
\]
By Burkholder’s inequality,
\[\mathbb{E} \left( \sup_{0 \leq s \leq T \wedge \tau_R} |I_1(t)|^2 \right) \leq 4 \mathbb{E} \left( \int_0^{T \wedge \tau_R} \sum_{i=1}^{m} \frac{\langle \xi_t, A_i(X_t) - \hat{A}_i(\hat{X}_t) \rangle^2}{(|\xi_t|^2 + \sigma^2)^2} dt \right),
\]
which is obviously less than
\[4 \mathbb{E} \left( \int_0^{T \wedge \tau_R} \sum_{i=1}^{m} \frac{|A_i(X_t) - \hat{A}_i(\hat{X}_t)|^2}{|\xi_t|^2 + \sigma^2} dt \right).
\]
Hence
\[\mathbb{E} \left( \int_{B(R) \cap \{\tau_R > t\}} \sup_{0 \leq s \leq T \wedge \tau_R} |I_1(t)| \, d\gamma_d \right) \leq 4 \left[ \int_0^T \mathbb{E} \left( \int_{\{\tau_R > t\}} \sum_{i=1}^{m} \frac{|A_i(X_t) - \hat{A}_i(\hat{X}_t)|^2}{|\xi_t|^2 + \sigma^2} \, d\gamma_d \right) dt \right]^{\frac{1}{2}}. \]
\[(5.7)\]
We have \(A_i(X_t) - \hat{A}_i(\hat{X}_t) = A_i(X_t) - A_i(\hat{X}_t) + A_i(\hat{X}_t) - \hat{A}_i(\hat{X}_t)\). Using the density \(K_t\), it is clear that
\[\mathbb{E} \int_{\{\tau_R > t\}} \frac{|A_i(\hat{X}_t) - \hat{A}_i(\hat{X}_t)|^2}{|\xi_t|^2 + \sigma^2} \, d\gamma_d \leq \frac{1}{\sigma^2} \mathbb{E} \int_{\mathbb{R}^d} |A_i(\hat{X}_t) - \hat{A}_i(\hat{X}_t)|^2 \, d\gamma_d \]
\[= \frac{1}{\sigma^2} \mathbb{E} \int_{\mathbb{R}^d} |A_i - \hat{A}_i|^2 \hat{K}_t \, d\gamma_d.
\]
Thus by Hölder’s inequality and according to (5.3), we have
\[\mathbb{E} \int_{\{\tau_R > t\}} \frac{|A_i(\hat{X}_t) - \hat{A}_i(\hat{X}_t)|^2}{|\xi_t|^2 + \sigma^2} \, d\gamma_d \leq \frac{A_{\text{p,T}}}{\sigma^2} \|A_i - \hat{A}_i\|^2_{L^2_q}, \]
\[(5.8)\]
Now we shall use Theorem 6.1 in the Appendix to estimate another term. Note that on the set \( \{ \tau_R > t \} \), \( X_t, \hat{X}_t \in B(R) \), then \( |X_t - \hat{X}_t| \leq 2R \). Since \( (X_t)_{\#} \gamma_d \ll \gamma_d \) and \( (\hat{X}_t)_{\#} \gamma_d \ll \gamma_d \), we can apply (6.2) so that
\[
|A_i(X_t) - A_i(\hat{X}_t)| \leq C_d |X_t - \hat{X}_t| (M_{2R}|\nabla A_i|(X_t) + M_{2R}|\nabla A_i|(\hat{X}_t)).
\]
Then
\[
\mathbb{E} \left[ \int_{\{\tau_R > t\}} \frac{|A_i(X_t) - A_i(\hat{X}_t)|^2}{|\xi_t|^2 + \sigma^2} \, d\gamma_d \right] \leq C_d^2 \mathbb{E} \int_{\{\tau_R > t\}} (M_{2R}|\nabla A_i|(X_t) + M_{2R}|\nabla A_i|(\hat{X}_t))^2 \, d\gamma_d.
\]
Notice again that on \( \{ \tau_R(x) > t \} \), \( X_t(x) \) and \( \hat{X}_t(x) \) are in \( B(R) \), therefore
\[
\mathbb{E} \left[ \int_{\{\tau_R > t\}} \frac{|A_i(X_t) - A_i(\hat{X}_t)|^2}{|\xi_t|^2 + \sigma^2} \, d\gamma_d \right] \leq 2C_d^2 \mathbb{E} \int_{B(R)} (M_{2R}|\nabla A_i|)^2 (K_t + \hat{K}_t) \, d\gamma_d
\]
\[
\leq 4C_d^2 \Lambda_{p,T} \left( \int_{B(R)} (M_{2R}|\nabla A_i|)^{2q} \, d\gamma_d \right)^{\frac{1}{q}}. \quad (5.9)
\]
Remark that the maximal function inequality does not hold for the Gaussian measure \( \gamma_d \) on the whole space \( \mathbb{R}^d \). However, on each ball \( B(R) \),
\[
\gamma_d|_{B(R)} \leq \frac{1}{(2\pi)^{d/2}} \text{Leb}_d|_{B(R)} \leq e^{R^2/2} \gamma_d|_{B(R)}.
\]
Thus, according to (6.3),
\[
\int_{B(R)} (M_{2R}|\nabla A_i|)^{2q} \, d\gamma_d \leq \frac{1}{(2\pi)^{d/2}} \int_{B(R)} (M_{2R}|\nabla A_i|)^{2q} \, dx \leq C_{d,q} \left( \frac{2\pi}{2} \right)^{d/2} \int_{B(3R)} |\nabla A_i|^{2q} \, dx
\]
\[
\leq C_{d,q} e^{9R^2/2} \int_{B(3R)} |\nabla A_i|^{2q} \, d\gamma_d \leq C_{d,q} e^{9R^2/2} \|\nabla A_i\|_{L^{2q}}.
\]
Therefore by (5.9), there exists a constant \( C_{d,q,R} > 0 \) such that
\[
\mathbb{E} \left[ \int_{\{\tau_R > t\}} \frac{|A_i(X_t) - A_i(\hat{X}_t)|^2}{|\xi_t|^2 + \sigma^2} \, d\gamma_d \right] \leq C_{d,q,R} \Lambda_{p,T} \|\nabla A_i\|_{L^{2q}}.
\]
Combining this estimate with (5.7) and (5.8), we get
\[
\mathbb{E} \left[ \int_{G_R} \sup_{0 \leq t \leq T} |I_1(t)| \, d\gamma_d \right] \leq CT^{\frac{1}{2}} \Lambda_{p,T} \left( C_{d,q,R} \sum_{i=1}^{m} \|\nabla A_i\|_{L^{2q}}^2 + \frac{1}{\sigma^2} \sum_{i=1}^{m} \|A_i - \hat{A}_i\|_{L^{2q}}^2 \right)^{\frac{1}{2}}. \quad (5.10)
\]
Now we turn to deal with \( I_2(t) \) in (5.6). We have
\[
\mathbb{E} \left[ \int_{G_R} \sup_{0 \leq t \leq T} |I_2(t)| \, d\gamma_d \right] \leq 2 \int_0^T \mathbb{E} \left[ \int_{G_R} \frac{|A_0(X_t) - \hat{A}_0(\hat{X}_t)|}{(|\xi_t|^2 + \sigma^2)^{\frac{1}{2}}} \, d\gamma_d \right] \, dt.
\]
Note that for \( x \in G_R, \hat{X}_t(x) \in B(R) \) for each \( t \in [0,T] \), thus
\[
\mathbb{E} \left[ \int_{G_R} \frac{|A_0(\hat{X}_t) - \hat{A}_0(\hat{X}_t)|}{(|\xi_t|^2 + \sigma^2)^{\frac{1}{2}}} \, d\gamma_d \right] \leq \frac{1}{\sigma} \mathbb{E} \int_{B(R)} |A_0 - \hat{A}_0| \hat{K}_t \, d\gamma_d \leq \frac{\Lambda_{p,T}}{\sigma} \|A_0 - \hat{A}_0\|_{L^q}.
\]
Again using (6.2),
\[
E \left[ \int_{G_R} \frac{|A_0(X_t) - A_0(\hat{X}_t)|}{(|\xi|^2 + \sigma^2)^{1/2}} d\gamma_d \right] \leq C_d \int_{G_R} (M_{2R} |\nabla A_0| (X_t) + M_{2R} |\nabla A_0| (\hat{X}_t)) d\gamma_d,
\]
which is dominated by
\[
C_d E \left[ \int_{B(R)} (M_{2R} |\nabla A_0|) \cdot (K_t + \hat{K}_t) d\gamma_d \right] \leq C_{d,q,R} \|\nabla A_0\|_{L^q} \Lambda_{p,T}.
\]
Therefore we get the following estimate for \( I_2 \):
\[
E \left[ \int_{G_R} \sup_{0 \leq t \leq T} |I_2(t)| d\gamma_d \right] \leq 2T \Lambda_{p,T} \left( C_{d,q,R} \|\nabla A_0\|_{L^q} + \frac{1}{\sigma^2} \|A_0 - \hat{A}_0\|_{L^q} \right).
\]
(5.11)

In the same way we have
\[
E \left[ \int_{G_R} \sup_{0 \leq t \leq T} |I_3(t)| d\gamma_d \right] \leq C T \Lambda_{p,T} \left( C_{d,q,R} \sum_{i=1}^m \|\nabla A_i\|_{L^{2q}} + \frac{1}{\sigma^2} \sum_{i=1}^m \|A_i - \hat{A}_i\|_{L^{2q}} \right).
\]
(5.12)
The term \( I_4(t) \) is negative and hence we omit it. Combining (5.6) and (5.10)–(5.12), we complete the proof.

Now we shall construct a solution to SDE (5.1). To this end, we take \( \varepsilon = 1/n \) and we write \( A^n_j \) instead of \( A^{1/n}_j \) introduced in Section 3. Then by assumption (A2) and Lemma 3.1, there is \( C > 0 \) independent of \( n \) and \( i \), such that
\[
|A^n_i(x)| \leq C(1 + |x|).
\]
(5.13)
Let \( X^n_t \) be the solution to Itô SDE (5.1) with the coefficients \( A^n_i \) \((i = 0, 1, \ldots, m)\). Then for any \( \alpha \geq 1 \) and \( T > 0 \), there exists \( C_{\alpha,T} > 0 \) independent of \( n \) such that
\[
E \left( \sup_{0 \leq t \leq T} |X^n_t|^{\alpha} \right) \leq C_{\alpha,T} (1 + |x|^{\alpha}), \quad \text{for all } x \in \mathbb{R}^d.
\]
(5.14)
Let \( K^n_t \) be the density of \( (X^n_t)_{\#} \gamma_d \) with respect to \( \gamma_d \). Under the hypotheses (A2)–(A4), there is \( T_0 > 0 \) such that (recall that \( p \) is the conjugate number of \( q > 1 \)):
\[
\Lambda_{p,T_0} := \int_{\mathbb{R}^d} \exp \left( 2p T_0 \left[ |A_0| + \varepsilon |\delta(A_0)| \right. \right.
\]
\[
+ \sum_{j=1}^m \left( 2p |A_j|^2 + |\nabla A_j|^2 + 2(p - 1)\sigma^2 |\delta(A_j)|^2 \right) \left. \right) d\gamma_d \right)^{\frac{p-1}{p(2p-1)}} < \infty.
\]
(5.15)
Similar to (3.6), we have
\[
\sup_{t \in [0,T_0]} \sup_{n \geq 1} \|K^n_t\|_{L^p(\gamma_d \times \mathbb{R}^d)} \leq \Lambda_{p,T_0} < +\infty.
\]
(5.16)
Now we shall prove that the family \( \{X^n : n \geq 1\} \) is convergent to some stochastic field.

**Theorem 5.3.** Let \( T_0 \) be given in (5.15). Then under the assumptions (A1’ and (A2)–(A4), there exists \( X : \Omega \times \mathbb{R}^d \to C([0,T_0], \mathbb{R}^d) \) such that for any \( \alpha \geq 1 \),
\[
\lim_{n \to \infty} E \left[ \int_{\mathbb{R}^d} \left( \sup_{0 \leq t \leq T_0} |X^n_t - X_t|^{\alpha} \right) d\gamma_d \right] = 0.
\]
(5.17)
Proof. We shall prove that \( \{X^n; n \geq 1\} \) is a Cauchy sequence in \( L^a(\Omega \times \mathbb{R}^d; C([0, T_0], \mathbb{R}^d)) \). Denote by \( \| \cdot \|_{\infty, T_0} \) the uniform norm on \( C([0, T_0], \mathbb{R}^d) \), so what we have to prove is

\[
\lim_{n,k \to +\infty} \mathbb{E} \left( \int_{\mathbb{R}^d} \|X^n - X^k\|_{\infty, T_0}^a \, d\gamma_d \right) = 0. \tag{5.18}
\]

First by (5.14), the quantity

\[
J_{a,T_0} := \sup_{n \geq 1} \mathbb{E} \left( \int_{\mathbb{R}^d} \|X^n\|_{\infty, T_0}^{2a} \, d\gamma_d \right) \leq C_{a,T_0} \int_{\mathbb{R}^d} (1 + |x|^{2a}) \, d\gamma_d \tag{5.19}
\]

is obviously finite. Let \( R > 0 \) and set

\[ G_{n,R}(w) = \{ x \in \mathbb{R}^d; \|X^n(w,x)\|_{\infty, T_0} \leq R \} . \]

Using (5.19), for any \( a \geq 1 \) and \( R > 0 \), we have

\[
\sup_{n \geq 1} \mathbb{E} \left( \gamma_d(G_{n,R}) \right) \leq \frac{J_{a,T_0}}{R^{2a}} .
\]

Now by Cauchy-Schwarz inequality

\[
\mathbb{E} \left( \int_{G_{n,R}^c \cup G_{k,R}^c} \|X^n - X^k\|_{\infty, T_0}^a \, d\gamma_d \right) \leq \left( \mathbb{E} \left[ \gamma_d(G_{n,R}^c \cup G_{k,R}^c) \right] \right)^{1/2} \left( \mathbb{E} \left( \int_{\mathbb{R}^d} \|X^n - X^k\|_{\infty, T_0}^{2a} \, d\gamma_d \right) \right)^{1/2} \leq \left( \frac{2J_{a,T_0}}{R^{2a}} \right)^{1/2} \cdot (2^{2a} J_{a,T_0})^{1/2} .
\]

Let \( \varepsilon > 0 \) be given; choose \( R > 1 \) big enough such that the last quantity in the above inequality is less than \( \varepsilon \). Then we have for any \( n,k \geq 1 \),

\[
\mathbb{E} \left( \int_{G_{n,R}^c \cup G_{k,R}^c} \|X^n - X^k\|_{\infty, T_0}^a \, d\gamma_d \right) \leq \varepsilon . \tag{5.20}
\]

Let

\[ \sigma_{n,k} = \|A_0^n - A_k^n\|_{L^q} + \left( \sum_{i=1}^m \|A_i^n - A_i^k\|_{L^{2q}}^2 \right)^{1/2} , \]

which tends to 0 as \( n,k \to +\infty \) since \( A_0^n \) converges to \( A_0 \) in \( L^q(\gamma_d) \) and \( A_i^n \) converges to \( A_i \) in \( L^{2q}(\gamma_d) \) for \( i = 1, \ldots , m \). Now applying Theorem 5.2 with \( A_i \) and \( A_i \) being replaced respectively by \( A_i^n \) and \( A_i^k \), we get

\[
I_{n,k} := \mathbb{E} \left[ \int_{G_{n,R}^c \cup G_{k,R}^c} \log \left( \frac{\|X^n - X^k\|_{\infty, T_0}^2}{\sigma_{n,k}^2} + 1 \right) \, d\gamma_d \right] \leq C_{T_0,A_p,T_0} \left\{ C_{d,q,U} \left( \|\nabla A_0^n\|_{L^q} + \left( \sum_{i=1}^m \|\nabla A_i^n\|_{L^{2q}}^2 \right)^{1/2} + \sum_{i=1}^n \|\nabla A_i^n\|_{L^{2q}}^2 \right) + 2 \right\} .
\]

Recall that \( A_i^n = \varphi_{1/n} P_{1/n} A_i \), then \( \nabla A_i^n = \nabla \varphi_{1/n} \otimes P_{1/n} A_i + \varphi_{1/n} e^{-1/n} P_{1/n} \nabla A_i \), therefore

\[ \|\nabla A_i^n\| \leq P_{1/n}(|A_i| + |\nabla A_i|) . \]
We get the following uniform estimates
\[ \| \nabla A_n^0 \|_{L^q} \leq \| A_0 \|_{\mathcal{D}^q}, \quad \| \nabla A_i^n \|_{L^2} \leq \| A_i \|_{\mathcal{D}^2}. \]
So the quantity \( I_{n,k} \) is uniformly bounded with respect to \( n, k \). Let \( \hat{\Pi} \) be the measure on \( \Omega \times \mathbb{R}^d \) defined by
\[ \int_{\Omega \times \mathbb{R}^d} \psi(w, x) \, d\hat{\Pi}(w, x) = \mathbb{E} \left[ \int_{G_n, R \cap G_k, R} \psi(w, x) \, d\gamma_d(x) \right]. \]
We have \( \hat{\Pi}(\Omega \times \mathbb{R}^d) \leq 1 \). Let \( \eta > 0 \), consider
\[ \Sigma_{n,k} = \{(w, x); \|X_n(w, x) - X_k(w, x)\|_{\gamma_d} \geq \eta\}, \]
which is equal to
\[ \left\{ (w, x); \log \left( \frac{\|X_n - X_k\|_{\gamma_d}^2}{\sigma_{n,k}^2} + 1 \right) \geq \log \left( \frac{\eta^2}{\sigma_{n,k}^2} + 1 \right) \right\}. \]
It follows that as \( n, k \to +\infty \),
\[ \hat{\Pi}(\Sigma_{n,k}) \leq \frac{I_{n,k}}{\log \left( \frac{\eta^2}{\sigma_{n,k}^2} + 1 \right)} \to 0, \tag{5.21} \]
since \( \sigma_{n,k} \to 0 \) and the family \( \{I_{n,k}; n, k \geq 1\} \) is bounded. Now
\[ \mathbb{E} \left( \int_{G_n, R \cap G_k, R} \|X_n - X_k\|_{\gamma_d}^{\alpha, T_0} \, d\gamma_d \right) = \int_{\Omega \times \mathbb{R}^d} \|X_n - X_k\|_{\gamma_d}^{\alpha, T_0} \, d\hat{\Pi} \]
\[ = \int_{\Sigma_{n,k}} \|X_n - X_k\|_{\gamma_d}^{\alpha, T_0} \, d\hat{\Pi} + \int_{\Sigma_{n,k}} \|X_n - X_k\|_{\gamma_d}^{\alpha, T_0} \, d\hat{\Pi}. \tag{5.22} \]
The first term on the right side of (5.22) is less than \( \eta^\alpha \), while the second one, due to (5.19) and (5.21), is dominated by
\[ \sqrt{\hat{\Pi}(\Sigma_{n,k})} \cdot \sqrt{\mathbb{E} \int_{\mathbb{R}^d} \|X_n - X_k\|_{\gamma_d}^{2\alpha, T_0} \, d\gamma_d} \leq 2^\alpha \sqrt{J_{\alpha, T_0}} \hat{\Pi}(\Sigma_{n,k}) \to 0 \quad \text{as} \ n, k \to +\infty. \]
Now taking \( \eta = \varepsilon^{1/\alpha} \) and combining (5.20) and (5.22), we prove that
\[ \limsup_{n, k \to +\infty} \mathbb{E} \left[ \int_{\mathbb{R}^d} \|X_n - X_k\|_{\gamma_d}^{\alpha, T_0} \, d\gamma_d \right] \leq 2\varepsilon, \]
which implies (5.18).

Let \( X \in L^\alpha(\Omega \times \mathbb{R}^d; C([0, T_0], \mathbb{R}^d)) \) be the limit of \( X^n \) in this space. We see that for each \( t \in [0, T] \) and almost all \( x \in \mathbb{R}^d, w \to X_t(w, x) \) is in \( \mathcal{F}_t \). \( \square \)

**Proposition 5.4.** There exists a family \( \{\hat{K}_t; t \in [0, T_0]\} \) of density functions on \( \mathbb{R}^d \) such that \( X_t \# \gamma_d = \hat{K}_t \gamma_d \) for each \( t \in [0, T_0] \). Moreover, \( \sup_{0 \leq t \leq T_0} \|\hat{K}_t\|_{L^p(\mathbb{R}^d \times \gamma_d)} \leq \Lambda_{p, T_0} \), where \( \Lambda_{p, T_0} \) is given in (5.16).

**Proof.** It is the same as the proof of Theorem 3.4. \( \square \)

The same arguments in the proof of Proposition 4.1 and 4.2 yield the following
Proposition 5.5. For any \( \alpha \geq 2 \), up to a subsequence,

\[
\lim_{n \to \infty} \int_{\mathbb{R}^d} \mathbb{E} \left( \sup_{0 \leq t \leq T_0} \left| \sum_{i=1}^{m} \int_{0}^{t} \left[ A_i^n(x_s^n) - A_i(x_s) \right] dw_i^s \right|^\alpha \right) d\gamma_d = 0,
\]

and

\[
\lim_{n \to \infty} \int_{\mathbb{R}^d} \left[ \mathbb{E} \int_{0}^{T_0} \left| A_0^n(x_s^n) - A_0(x_s) \right|^\alpha ds \right] d\gamma_d = 0.
\]

Now for regularized vector fields \( A_i^n, i = 0, 1, \ldots, m \), we have

\[
X_t^n(x) = x + \sum_{i=1}^{m} \int_{0}^{t} A_i^n(X_s^n) \, dw_i^s + \int_{0}^{t} A_0^n(X_s^n) \, ds. \tag{5.23}
\]

When \( n \to +\infty \), by Theorem 5.3 and Proposition 5.5, the two sides of (5.23) converge respectively to \( X \) and

\[
x + \sum_{i=1}^{m} \int_{0}^{t} A_i(X_s) \, dw_i^s + \int_{0}^{t} A_0(X_s) \, ds
\]

in the space \( L^\alpha(\Omega \times \mathbb{R}^d; C([0, T_0], \mathbb{R}^d)) \). Therefore for almost all \( x \in \mathbb{R}^d \), the following equality holds \( \mathbb{P} \)-almost surely:

\[
X_t(x) = x + \sum_{i=1}^{m} \int_{0}^{t} A_i(X_s) \, dw_i^s + \int_{0}^{t} A_0(X_s) \, ds, \quad \text{for all } t \in [0, T_0].
\]

That is to say, \( X_t \) solves SDE (5.1) over \([0, T_0]\).

The following result proves the pathwise uniqueness to SDE (5.1) for a.e. initial value \( x \in \mathbb{R}^d \).

Proposition 5.6. Under the conditions (A1') and (A2)–(A4), the SDE (5.1) has a unique solution on the interval \([0, T_0]\).

Proof. Let \( (Y_t)_{t \in [0, T_0]} \) be another solution. Set, for \( R > 0 \),

\[
G_R = \left\{ (w, x) \in \Omega \times \mathbb{R}^d; \sup_{0 \leq t \leq T_0} |X_t(w, x) - Y_t(w, x)| \leq R \right\}.
\]

Remark that in Theorem 5.2, the terms involving \( 1/\sigma \) and \( 1/\sigma^2 \) are equal to zero. Therefore the term

\[
I := \mathbb{E} \int_{G_R} \log \left( \frac{\sup_{0 \leq t \leq T_0} |X_t - Y_t|^2}{\sigma^2} + 1 \right) d\gamma_d
\]

\[
\leq C_{T_0} A_{p,T_0} C_{d,q,R} \left[ \|A_0\|_{\mathbb{D}^q_1} + \left( \sum_{i=1}^{m} \|A_i\|_{\mathbb{D}^{2q}_1} \right)^{\frac{1}{2}} + \sum_{i=1}^{m} \|A_i\|_{\mathbb{D}^{2q}_1}^2 \right]
\]

is bounded for any \( \sigma > 0 \). Consider for \( \eta > 0 \),

\[
\Sigma_\eta = \left\{ (w, x) \sup_{0 \leq t \leq T_0} |X_t(w, x) - Y_t(w, x)| \geq \eta \right\}.
\]

Similar to (5.21), we have

\[
\mathbb{E} \left( \int_{G_R} 1_{\Sigma_\eta} d\gamma_d \right) \leq \frac{I}{\log(\frac{\eta^2}{\sigma^2} + 1)} \to 0
\]
as $\sigma \to 0$. So we obtain

$$1_{G_R} \cdot \sup_{0 \leq t \leq T_0} |X_t - Y_t| = 0, \quad (\mathbb{P} \times \gamma_d)\text{-a.s.}$$

Letting $R \to \infty$, we obtain that $(\mathbb{P} \times \gamma_d)$ almost surely, $X_t = Y_t$ for all $t \in [0, T_0]$.

Now we extend the solution to any time interval $[0, T]$. Let $\theta_{T_0} w$ be the time-shift of the Brownian motion $w$ and denote by $X^{T_0}_t$ the corresponding solution to SDE driven by $\theta_{T_0} w$. By Proposition 5.6, $\{X^{T_0}_t(\theta_{T_0} w, x) : 0 \leq t \leq T_0\}$ is the unique solution to the SDE over $[0, T_0]$:

$$X^{T_0}_t(x) = x + \sum_{i=1}^m \int_0^t A_i(X^{T_0}_s(x)) \, d(\theta_{T_0} w)_s^i + \int_0^t A_0(X^{T_0}_s(x)) \, ds.$$

For $t \in [0, T_0]$, define $X_{t+T_0}(w, x) = X^{T_0}_t(\theta_{T_0} w, X^{T_0}_0(w, x))$. Note that $X_t$ is well-defined on the interval $[0, 2T_0]$ up to a $(\mathbb{P} \times \gamma_d)$-negligible subset of $\Omega \times \mathbb{R}$. Replacing $x$ by $X^{T_0}_0(x)$ in the above equation, we get easily

$$X_{t+T_0}(x) = x + \sum_{i=1}^m \int_0^{t+T_0} A_i(X_s(x)) \, dw_s^i + \int_0^{t+T_0} A_0(X_s(x)) \, ds.$$

Therefore $X_t$ defined as above is a solution to SDE on the interval $[0, 2T_0]$. Continuing in this way, we obtain the solution of SDE (5.1) on $[0, T]$.

**Theorem 5.7.** The $\{X_t; t \in [0, T]\}$ constructed above is the unique solution to SDE (5.1) in the sense of Definition 5.1. Moreover for each $t \in [0, T]$, the density $K_t$ of $(X_t)_{\#} \gamma_d$ with respect to $\gamma_d$ is in the space $L^1 \log L^1$.

**Proof.** Let $Y_t$, $t \in [0, T]$ be another solution in the sense of Definition 5.1. First by Proposition 5.6, we have $(\mathbb{P} \times \gamma_d)$-almost surely, $Y_t = X_t$ for all $t \in [0, T_0]$. In particular, $Y^{T_0}_t = X^{T_0}_t$. Next by the flow property, $Y_{t+T_0}$ satisfies the following equation:

$$Y_{t+T_0}(x) = Y^{T_0}_t(x) + \sum_{i=1}^m \int_0^t A_i(Y_{s+T_0}(x)) \, d(\theta_{T_0} w)_s^i + \int_0^t A_0(Y_{s+T_0}(x)) \, ds,$$

that is, $Y_{t+T_0}$ is a solution with initial value $Y^{T_0}_t$. But by the above discussion, $X_{t+T_0}$ is also a solution with the same initial value $X^{T_0}_t = Y^{T_0}_t$. Again by Proposition 5.6, we have $(\mathbb{P} \times \gamma_d)$-almost surely, $X_{t+T_0} = Y_{t+T_0}$ for all $t \leq T_0$. Hence we have proved $X_{|[0,2T_0]} = Y_{|[0,2T_0]}$. Repeating this procedure, we obtain the uniqueness over $[0, T]$. The existence of density $K_t$ of $(X_t)_{\#} \gamma_d$ with respect to $\gamma_d$ beyond $T_0$ is deduced from the flow property. However, to insure that $K_t \in L^1 \log L^1$, we have to use Theorem 3.3 and the following

$$\lim_{n \to \infty} \int_{\mathbb{R}^d} \mathbb{E} \left( \sup_{0 \leq t \leq T} |X^n_t - X_t|^\alpha \right) \, d\gamma_d = 0,$$

which can be checked using the same arguments as in the proof of Propositions 4.1 and 4.2. $\square$

### 6 Appendix

For any locally integrable function $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $R > 0$, the local maximal function $M_R f$ is defined by

$$M_R f(x) = \sup_{0 < r \leq R} \frac{1}{\text{Leb}_d(B(x, r))} \int_{B(x, r)} |f(y)| \, dy,$$

where $B(x, r) = \{y \in \mathbb{R}^d; |y - x| \leq r\}$. The following result is the starting point for the approach concerning Sobolev coefficients, used in [5] and [36].
Theorem 6.1. Let \( f \in L^1_{\text{loc}}(\mathbb{R}^d) \) be such that \( \nabla f \in L^1_{\text{loc}}(\mathbb{R}^d) \). Then there is a constant \( C_d > 0 \) (independent of \( f \)) and a negligible subset \( N \), such that for \( x, y \in N^c \) with \(|x - y| \leq R\),

\[
|f(x) - f(y)| \leq C_d |x - y| \left( (M_R |\nabla f|)(x) + (M_R |\nabla f|)(y) \right); \tag{6.2}
\]

moreover for \( p > 1 \) and \( f \in L^p_{\text{loc}}(\mathbb{R}^d) \), there is a constant \( C_{d,p} > 0 \) such that

\[
\int_{B(r)} (M_R f)^p \, dx \leq C_{d,p} \int_{B(r+R)} |f|^p \, dx. \tag{6.3}
\]

Since the inequality (6.2) played a key role in the proof of Theorem 5.2, we give here its proof for the sake of the reader’s convenience.

We follow the idea of the proof of Claim #2 on p.253 in [9]. For any bounded measurable subset \( U \) in \( \mathbb{R}^d \) such that its Lebesgue measure \( \text{Leb}_d(U) > 0 \), define the average of \( f \in L^1_{\text{loc}}(\mathbb{R}^d) \) on \( U \) by

\[
(f)_U = \frac{1}{\text{Leb}_d(U)} \int_U f(y) \, dy.
\]

Write \((f)_{x,r}\) instead of \((f)_{B(x,r)}\) for simplicity. Then \( M_R f(x) = \sup_{0 < r \leq R} (|f|)_{x,r} \). We will need the following simple inequality: for any \( C \in \mathbb{R} \),

\[
|(f)_U - C| \leq \int_U |f(y) - C| \, dy. \tag{6.4}
\]

First, for any \( x \in \mathbb{R}^d \) and \( r \in (0, R) \), by Poincaré’s inequality with \( p = 1 \) and \( p^* = d/(d-1) \) (see [9] p.141), there is \( C_d > 0 \) such that

\[
\int_{B(x,r)} |f - (f)_{x,r}| \, dy \leq \left( \int_{B(x,r)} |f - (f)_{x,r}|^{d/(d-1)} \, dy \right)^{(d-1)/d} \leq C_d r \int_{B(x,r)} |\nabla f| \, dy \leq C_d M_R |\nabla f|(x) r. \tag{6.5}
\]

In particular, for all \( k \geq 0 \), by (6.4) and (6.5),

\[
|(f)_{x,r/2^{k+1}} - (f)_{x,r/2^k}| \leq \int_{B(x,r/2^{k+1})} |f - (f)_{x,r/2^k}| \, dy \\
\leq 2^d \int_{B(x,r/2^k)} |f - (f)_{x,r/2^k}| \, dy \\
\leq 2^d C_d M_R |\nabla f|(x) r/2^k.
\]

Since \( f \in L^1_{\text{loc}}(\mathbb{R}^d) \), there is a negligible subset \( N \subset \mathbb{R}^d \), such that for all \( x \in N^c \), \( f(x) = \lim_{r \to 0} (f)_{x,r} \). Thus for any \( x \in N^c \), we have by summing up the above inequality that

\[
|f(x) - (f)_{x,r}| \leq \sum_{k=0}^{\infty} |(f)_{x,r/2^{k+1}} - (f)_{x,r/2^k}| \leq 2^{1+d} C_d M_R |\nabla f|(x) r. \tag{6.6}
\]

Next for all \( x, y \in N^c, x \neq y \) and \(|x - y| \leq R\), let \( r = |x - y| \). Then by the triangular inequality, (6.4) and (6.5),

\[
|(f)_{x,r} - (f)_{y,r}| \leq \int_{B(x,r) \cap B(y,r)} \left( |(f)_{x,r} - f(z)| + |f(z) - (f)_{y,r}| \right) \, dz \\
\leq C_d \left[ \int_{B(x,r)} |(f)_{x,r} - f(z)| \, dz + \int_{B(y,r)} |f(z) - (f)_{y,r}| \, dz \right].
\]
\[
\leq \tilde{C}_d C_d (M_R |\nabla f(x)| + M_R |\nabla f(y)|) r. \quad (6.7)
\]

Now (6.2) follows from the triangular inequality and (6.6), (6.7):

\[
|f(x) - f(y)| \leq |f(x) - (f)_{x,r}| + |(f)_{x,r} - (f)_{y,r}| + |(f)_{y,r} - f(y)| \\
\leq 2^{1+d} C_d M_R |\nabla f|(x) r + \tilde{C}_d C_d (M_R |\nabla f|(x) + M_R |\nabla f|(y)) r \\
+ 2^{1+d} C_d M_R |\nabla f|(y) r \\
= C_d (2^{1+d} + \tilde{C}_d) |x - y| (M_R |\nabla f|(x) + M_R |\nabla f|(y)).
\]

We obtain (6.2). \hfill \Box

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