INTERPOLATION BETWEEN PARA-BOSE AND PARA-FERMI STATISTICS

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Abstract

Using deformed Green’s oscillators and Green’s Ansatz, we construct a multi-parameter interpolation between para-Bose and para-Fermi statistics of a given order. When the interpolating parameters $q_{ij}$ satisfy $|q_{ij}| < 1$ ($|q_{ij}| = 1$), the interpolation statistics is ”infinite quon”-like (anyon-like). The proposed interpolation does not contain states of negative norms.
1. Introduction

The properties of presently known particles are usually described in terms of an effective field theory in which the field operators at space-like points commute or anti-commute. The choice of commutation relations for bosons and anti-commutation relations for fermions are uniquely determined by the spin-statistics theorem (Pauli principle). During the past years much effort has been made to construct consistent generalizations of Bose and Fermi statistics. One motivation comes from theoretical and experimental search for possible violation of the Pauli exclusion principle in 3+1 dimensions [1]. The other motivation comes from the study of some phenomena in condensed matter whose dynamics is essentially two-dimensional [2].

Theoretically, there are two types of generalized statistics which follow from the relations between creation and annihilation operators.

(i) One type of interpolation between Bose and Fermi statistics is described by a continuous parameter. The characteristic example is infinite (quon) statistics [3,4] and the other example is anyonic (braid) statistics in 2 + 1 dimensions [5].

The interpolating infinite (quon) statistics [3] is described by the following commutation relations

\[ a_i a_j^\dagger - qa_j^\dagger a_i = \delta_{ij}, \]

\[ \forall i, j \in I, q \in \mathbb{R}, \quad -1 \leq q \leq +1, \]

and with the vacuum condition \( a_i |0\rangle = 0, \forall i \in I \). It has been shown that the corresponding Fock space is positive definite for \(-1 \leq q \leq +1\) [4] and that the different permutations of a given multiparticle state are linearly independent, i.e. all representations of the symmetric group can occur.
In the lower number of dimensions, interpolating statistics are described by fractional statistics of anyons. The anyonic-type exchange algebra of annihilation and creation anyonic operators \( a_i, a_i^\dagger, (i \in I) \) is characterized by a continuous statistical parameter \( \lambda, \lambda \in [0, 1] \) [5]:

\[
a_i a_j^\dagger - e^{i\lambda \pi \text{sgn}(i-j)} a_j^\dagger a_i = 0 \quad i \neq j, \quad i, j \in I,
\]

\[
a_i a_i^\dagger - \cos(\lambda \pi) a_i^\dagger a_i = 1,
\]

\[
a_i a_j - e^{-i\lambda \pi \text{sgn}(i-j)} a_j a_i = 0,
\]

The vacuum condition is \( a_i |0\rangle = 0, \forall i \in I \). The set \( \{I\} \) can be position or momentum space, discrete or continuous. The anyonic algebra can be obtained from the Bose algebra by mapping defined in [6]. It is now widely accepted that particles with fractional statistics are likely to play a role in fractional Hall effects and some other phenomena in condensed matter [2].

Both interpolations, anyonic and quonic, can be unified in a generalized quon algebra [7]

\[
a_i a_j^\dagger - q_{ij} a_j^\dagger a_i = \delta_{ij}, \quad q_{ij}^* = q_{ji}, \quad \forall i, j \in I
\]

with the vacuum condition \( a_i |0\rangle = 0, \forall i \in I \).

(ii) Another type of generalized statistics is parastatistics (para-Bose and para-Fermi statistics), proposed by Green [8]. These were the first consistent generalizations of Bose and Fermi statistics. Parastatistics are characterized by a discrete pa-
rameter $p \in \mathbb{N}$ (order of parastatistics). They are described by trilinear relations:

$$[a_k, [a_l^\dagger, a_m]_\pm] = (\frac{2}{p})\delta_{kl}a_m,$$

$$[a_k, [a_l^\dagger, a_m^\dagger]_\pm] = (\frac{2}{p})(\delta_{kl}a_m^\dagger \pm 2\delta_{km}a_l^\dagger),$$

$$[a_k, [a_l, a_m]_\pm] = 0,$$

with the vacuum conditions $a_k|0\rangle = 0$, $a_k a_l^\dagger |0\rangle = \delta_{kl}|0\rangle$, $k, l \in I$ and $p \in \mathbb{N}$.

The sign $+$ ($-$) corresponds to the para-Bose (para-Fermi) algebra. Note that the last two relations are not independent but follow from the first one. The corresponding Fock space has no states with negative norms. For $p = 1$ the para-Bose (para-Fermi) algebra becomes the Bose (Fermi) algebra and for $p = \infty$ the para-Bose (para-Fermi) algebra becomes the Fermi (Bose) algebra. The possibility of describing some physical models in solids in terms of quasi-particles which obey para-Fermi statistics has recently been discussed by Safonov [9].

There were a few attempts to perform a continuous interpolation between different parastatistics [10]. The proposed interpolating trilinear commutation relations are

$$[a_i a_j^\dagger + q a_j^\dagger a_i, a_k] = \rho \delta_{jk}a_i,$$

with the vacuum conditions $a_k|0\rangle = 0$, $a_k a_l^\dagger |0\rangle = \delta_{kl}|0\rangle$, $k, l \in I$. For $\rho = -(\frac{2}{p})q$, $p \in \mathbb{N}$ and $q = +1 (-1)$, one recovers the para-Bose (para-Fermi) algebra. However, it was shown that for the generic $q$ the corresponding Fock-like space contained states with negative norms [11]. Hence, no small violation of parastatistics, described by Eq.(5), is allowed. Recently, Speicher [12] and two of us [7] indicated the possibility of continuous interpolation between parastatistics without states with negative norms, but no explicit construction was presented.

In this Letter we describe a general construction of a continuous interpolation
between the para-Bose and the para-Fermi algebra of a given order \( p \). The construction is performed in such a way that the Fock space does not contain states with negative norms. Furthermore, for \( p = 1 \) and \( p = (\infty) \) the construction reduces to the quonic interpolation (Eq.(1)) or to the anyonic interpolation (Eq.(2)), depending on interpolating parameters \( q_{ij} \).

2. \( q \)-deformed Green’s Ansatz

We start with the generalized quon algebra, Eq.(3), and use it to deform the Green’s Ansatz for the parastatistics [8] in the following way (the algebraic structure of the Green’s Ansatz and its \( q \)-deformed analogue is described in [13]):

\[
b_i^\alpha b_j^{\beta \dagger} - q_{\alpha \beta} b_j^{\beta \dagger} b_i^\alpha = \delta_{\alpha \beta} \delta_{ij}, \quad i, j \in I, \ \alpha, \beta = 1, 2, \ldots p
\]

\[
q_{\alpha \beta} = q \Delta_{\alpha \beta} \equiv q (2\delta_{\alpha \beta} - 1), \quad q \in \mathbb{R}, \quad -1 \leq q \leq +1,
\]

with the vacuum conditions \( b_i^\alpha |0\rangle = 0, \ b_i^\alpha b_j^{\beta \dagger} |0\rangle = \delta_{\alpha \beta} \delta_{ij} |0\rangle \). Note that for \( |q| < 1 \) there are no commutation relations between \( b_i^\alpha, b_j^{\beta \dagger} \).

If \( q = +1 (-1) \), the oscillators \( b_i^\alpha \) and \( b_j^{\beta \dagger} \) reduce to the ordinary Green’s oscillators, leading to the para-Bose (para-Fermi) algebra, Eq.(4). The main point of our construction is that the Fock space corresponding to the commutation relation (6) does not contain states with negative norms if \( |q| \leq 1 \) (see [14,15]).

Let us define the operators \( A_i, A_i^{\dagger}, i \in I \), in the same way as in the ordinary Green’s Ansatz [8], namely:

\[
A_i = \frac{1}{\sqrt{p}} \sum_{\alpha=1}^{p} b_i^\alpha, \quad A_i^{\dagger} = \frac{1}{\sqrt{p}} \sum_{\alpha=1}^{p} b_i^{\alpha \dagger}.
\]

The Fock space \( \mathcal{F}(A_i) \) created by the \( A_i^{\dagger} \) operators can be built. It is important to note that \( \mathcal{F}(A_i) \) is a subspace of the initial Fock space \( \mathcal{F}(b_i^\alpha) \), and therefore the
space $\mathcal{F}(A_i)$ automatically does not contain states with negative norms since $\mathcal{F}(b_i^\dagger)$ does not contain states with negative norms.

Let us define the matrix $\mathcal{A}^{(n)}$ of inner products with the matrix elements:

$$
\mathcal{A}_{i_1\cdots i_n; j_1\cdots j_n}^{(n)} = \langle 0 | A_{i_1} \cdots A_{i_n} A_{j_1}^\dagger \cdots A_{j_n}^\dagger | 0 \rangle
$$

where $\{i_1 \cdots i_n; j_1 \cdots j_n\} \in I$. If indices $i_1 \cdots i_n$ are mutually different, the matrix $\mathcal{A}^{(n)}$ is an $(n! \times n!)$ matrix whose diagonal elements are equal to 1.

We find that an arbitrary matrix element is

$$
\mathcal{A}_{\pi(i_1\cdots i_n); \sigma(i_1\cdots i_n)}^{(n)} = \frac{1}{p^n} \sum_{a_1,\cdots,a_n} \prod_{a,b} q_{a,a_b}.
$$

Here, $\pi$ and $\sigma$ are elements (permutations) of the permutation group $S_n$. The product is taken over those pairs $a, b = 1, \cdots, n$, which satisfy $a < b$ and $(\sigma^{-1} \pi)(a) > (\sigma^{-1} \pi)(b)$. From the general theorem on positivity [14,15] it follows that the general matrix $\mathcal{A}^{(n)}, n \in \mathbb{N}$, is positive definite for $|q| < 1$, and therefore the states $\pi(A_{i_1}^\dagger \cdots A_{i_n}^\dagger) |0\rangle$, where $\pi \in S_n$ and $(i_1 \cdots i_n)$ are mutually different, are linearly independent as for quons [3,4].

However, the operators $A_i, A_i^\dagger, i \in I$, do not close the commutation relations between themselves, except for $q = \pm 1$, corresponding to the ordinary parastatistics, Eq.(4).

Namely,

$$
A_i A_j^\dagger = \delta_{ij} - q A_j^\dagger A_i + q K_{ij}
$$

$$
K_{ij} = \left(\frac{2}{p}\right) \sum_{\alpha=1}^p b_j^{\alpha\dagger} b_i^\alpha
$$

and $K_{ij}$ cannot be expanded in terms of the $A_i, A_i^\dagger$ operators i.e. cannot be eliminated from the commutation relations except for $q = \pm 1$. For example, the action of the $A_i$ operator on the state $A_j^\dagger A_k^\dagger A_m^\dagger |0\rangle$ is not contained in the space $\mathcal{F}(A)$ if $|q| < 1$. If $q = \pm 1$ it is contained in $\mathcal{F}(A)$. 

5
3. Construction of interpolation between paras-statistics

The operators $A_i, A^\dagger_i$ do not close and hence do not achieve desired interpolation between the para-Bose and para-Fermi algebra. Nevertheless, the matrices $A^{(n)}$, Eq.(9), represent the desired interpolation between the corresponding matrices for the para-Bose and para-Fermi algebras. To perform our construction of interpolating commutation relations between the para-Bose and para-Fermi algebra, Eq.(4), we look for the operators $a_i, a_i^\dagger, i \in I$ with closed commutation relations of the type

$$a_i a_j^\dagger = \Gamma_{ij}(a^\dagger, a) \quad i, j \in I, \quad (11)$$

with the vacuum conditions $a_i |0\rangle = 0, a_i a_j^\dagger |0\rangle = \delta_{ij} |0\rangle, i, j \in I$ and where $\Gamma_{ij}$ denotes a sum of all possible independent normally ordered terms. Then we build a Fock-like space $F(a)$. The main requirement is that the matrix of inner products in $F(a), A^{(n)}_{i_1 \cdots i_n; j_1 \cdots j_n} = \langle 0 | a_{i_1} \cdots a_{i_n} a_{j_1}^\dagger \cdots a_{j_n}^\dagger | 0 \rangle$, should be identical to the corresponding matrix elements, Eq.(9). This requirement ensures that the interpolation between para-Bose and para-Fermi statistics, based on Eq.(9), is continuous and that the corresponding Fock space $F(a)$ does not contain states of negative norms (since all matrices $A^{(n)}$ are positive definite [14,15]). This procedure is well defined [16]. Namely, Eq.(11) can be expanded as

$$\Gamma_{ij} \equiv a_i a_j^\dagger = \delta_{ij} + C^{(ij)} a_j^\dagger a_i + \sum_k \sum_{\pi,\sigma \in S_2} C^{(ij;k)}_{\pi,\sigma} \pi(a_j^\dagger a_k^\dagger)\sigma(a_k a_i) + \cdots$$

$$\cdots + \sum_{k_1, \cdots, k_n} \sum_{\pi,\sigma \in S_{n+1}} C^{(ij;k_1 \cdots k_n)}_{\pi,\sigma} \pi(a_j^\dagger a_k^\dagger \cdots a_k^\dagger)\sigma(a_{k_1} \cdots a_{k_n} a_i) \quad (12)$$

The summation is performed over those permutations $\pi,\sigma \in S_{n+1}$ for which the states $\pi(a_{k_1} \cdots a_{k_n} a_i)$ are independent. Note that the expansion (12) could lead to
Fock space with states with negative norms. However, the negative norm states do not appear, owing to our requirement. The unknown coefficients \( C_{\pi,\sigma} \) can be uniquely determined \([16]\). Moreover, the commutation relations (11) and (12) imply that \( a_i F(a) \subset F(a) \) and vice versa, i.e.

\[
a_i a_j^\dagger |0\rangle = \delta_{ij} |0\rangle,
\]

\[
a_i a_{i_1}^\dagger a_{i_2}^\dagger |0\rangle = \delta_{ii_1} a_{i_2}^\dagger |0\rangle + \Phi_{i_1i_2;i_4}^i \delta_{ii_2} a_{i_1}^\dagger |0\rangle,
\]

\[
a_i a_{i_1}^\dagger \cdots a_{i_n}^\dagger |0\rangle = \sum_{k=1}^{n} \sum_{\pi \in S_{n-1}} \Phi_{i_1 \cdots i_n;\pi(i_1 \cdots i_k \cdots i_n)}^i \pi(a_{i_1}^\dagger \cdots a_{i_k}^\dagger) |0\rangle,
\]

where the summation on the RHS is performed over the linearly independent states. The slash denotes the omission of the corresponding operator \( a_{i_k}^\dagger \). One easily finds the coefficients \( \{ \Phi \} \) as

\[
\{ \Phi \} = \sum_{\pi \in S_{n-1}} [A_{\pi,\sigma}^{(n-1)}]^{-1} A_{(k,\sigma);id}^{(n)}.
\]

Here, \( (k, \sigma) \) denotes the indices \( \{k, \sigma(1), \cdots \sigma(k-1), \sigma(k+1), \cdots \sigma(n)\} \). We point out that Eq.(12) represents recurrent relations for \( C_{\pi,\sigma} \), which always have a unique solution since the determinant of this linear system is always regular.

The solution of recurrent relations, Eq.(12), to the second order in the operators \( a \) and \( a^\dagger \) is:

\[
a_i a_j^\dagger = \delta_{ij} + q \left( \frac{2}{p} - 1 \right) a_j^\dagger a_i + \frac{8p(p-1)q^3}{[p^2 - (p-2)^2q^2]^2} \sum_{k \in I} [Y_{jk}] [Y_{ik}] + \cdots
\]

with \( Y_{ik} = a_i a_k - q \left( \frac{2}{p} - 1 \right) a_k a_i \), \(-1 \leq q \leq 1\) and \( p \in \mathbb{N} \). The limiting cases, \( p = 1 \) and \( p = \infty \) correspond to the quon interpolation between the Bose and Fermi algebra [3,4].

If \( q = \varepsilon = \pm 1 \) and for \( p \in \mathbb{N} \), we find

\[
a_i a_j^\dagger = \delta_{ij} + \varepsilon \left( \frac{2}{p} - 1 \right) a_j^\dagger a_i + \frac{\varepsilon p}{2(p-1)} \sum_{k \in I} [Y_{jk}] [Y_{ik}] + \cdots
\]
with \( Y_{ik} = a_i a_k - \varepsilon \left( \frac{2}{p} - 1 \right) a_k a_i \). For \( \varepsilon = +1(-1) \) this corresponds to the ordinary para-Bose (para-Fermi) commutation relations [4]. It is easy to show that for \( p = 1 \) the above equation reduces to the Bose (Fermi) algebra for \( \varepsilon = +1(-1) \), respectively, since \( Y_{ik} = a_i a_k - \varepsilon a_k a_i \equiv 0 \) and these terms do not appear in the expansion (16). The similar argument holds for \( p = \infty \).

Hence, we have constructed an infinite (quon) statistics interpolating between the para-Bose and para-Fermi algebra of the \( p^{th} \) order. The norms of all Fock states are positive definite for \( |q| < 1 \). Note that, even for the ordinary parastatistics, Eq. (12) contains an infinite set of terms on the RHS [16].

4. Multiparametric deformation of Green’s Ansatz

We point out that our construction can be extended to an arbitrary multiparametric interpolation between Green’s oscillators, Eq. (6), i.e. between the para-Bose and para-Fermi algebras. In this case we write Eq. (6) as

\[
\begin{align*}
 b^\alpha_i b^\beta_j - q_{i\alpha,j\beta} b^\beta_j b^\alpha_i &= \delta_{\alpha\beta}\delta_{ij}, \\
 q_{i\alpha,j\beta} &= q_{ij} \Delta_{\alpha\beta},
\end{align*}
\]

where \( q^\ast_{ij} = q_{ji} \) and \( |q_{ij}| \leq 1 \).

The generic matrices \( \mathcal{A}^{(n)} \), Eq. (8), can be easily calculated (see Eq. (9)):

\[
\begin{align*}
\mathcal{A}^{(n)}_{\pi(i_1 \cdots i_n),\sigma(i_1 \cdots i_n)} &= \frac{1}{p^n} \sum_{\alpha_1 \cdots \alpha_n} (\prod_{a,b} q_{i_a,j_b}) (\prod_{a,b} \Delta_{\alpha_a\alpha_b}), \\
\Delta_{\alpha_a\alpha_b} &= 2\delta_{\alpha_a\alpha_b} - 1,
\end{align*}
\]

where the products are taken over those pairs \( a, b = 1, \ldots n \), which satisfy \( a < b \) and \((\sigma^{-1}\pi)(a) > (\sigma^{-1}\pi)(b)\).
For example, $A^{(2)}$ is given by

$$A^{(2)} = \begin{pmatrix} 1 & q_{ij} \left( \frac{2}{p} - 1 \right) \\ q_{ij}^* \left( \frac{2}{p} - 1 \right) & 1 \end{pmatrix}.$$ 

The multiparametric interpolation between para-Bose and para-Fermi oscillators to the second order in $a_i, a_j^\dagger$ is

$$\Gamma_{ij} \equiv a_i a_j^\dagger = \delta_{ij} + q_{ij} \left( \frac{2}{p} - 1 \right) a_j^\dagger a_i + 8p(p-1) \sum_{k \in I} Q_{ji; k} [Y_{jk}]^\dagger [Y_{ik}] + \cdots,$$ (19)

$$Q_{ji; k} = \left. \frac{q_{ji} q_{jk} q_{ik}}{[p^2 - (p-2)q_{kj}]^2 \left| p^2 - (p-2)q_{ki} \right|^2} \right|,$n

$$Y_{ik} = a_i a_k - q_{ki} \left( \frac{2}{p} - 1 \right) a_k a_i.$$ (20)

Remark

Speicher [12] suggested $q$-para-Bose ($q$-para-Fermi) fields defined through the $q$-deformed Green’s oscillators

$$b_{i}^\alpha b_{j}^\beta \to b_{j}^\alpha b_{i}^\beta = \delta_{ij},$$

$$b_{i}^\alpha b_{j}^\beta = \epsilon q b_{j}^\beta b_{i}^\alpha, \quad \alpha \neq \beta.$$ (20)

For $q = +1$ and $\epsilon = +1(-1)$, this corresponds to Bose (Fermi) oscillators. For $q = -1$, one recovers para-Bose (para-Fermi) oscillators when $\epsilon = +1(-1)$, respectively. This is also a special case of Eq.(17) with

$$q_{i\alpha,j\beta} = \epsilon [(1-q)\delta_{\alpha\beta} + q],$$

and it interpolates between the $p$ Bose (Fermi) oscillators and $p$ para-Bose (para-Fermi) oscillators, respectively.

5. Anyonic deformation of Green’s Ansatz

In the same way we can construct an anyonic-like interpolation between Green’s oscillators, i.e. between the para-Bose and para-Fermi algebras of a given order
\( p \in \mathbb{N}. \) In this case, in Eq. (6) we replace \( q_{\alpha\beta} \) with
\[
q_{\alpha\beta} \rightarrow q_{i\alpha,j\beta} = q_{ij} \Delta_{\alpha\beta},
\]
where \( \Delta_{\alpha\beta} \) is given by Eq. (21).

\[
q_{ij} = \begin{cases} 
\cos(\lambda \pi) & \text{if } i = j \\
e^{\varphi_{ij}} & \text{if } i \neq j
\end{cases}
\]

The generic matrix \( \mathcal{A}^{(n)} \) can be obtained from Eq. (18). The interpolating commutation relations (to the lowest order in \( a_i, a_j^\dagger \)) are given by Eq. (19) with \( q_{ij} \) given in Eq. (21). For \( p = 1 \) and \( p = \infty \), they reduce to the anyonic algebra, Eq. (2).

For \( \varphi_{ij} = 0, \pi \), the relations (19,21) reduce to the ordinary para-Bose (para-Fermi) relations.

Contrary to the quonic interpolation of the preceding section, in this case there exists a well-defined mapping of the Bose algebra (of the Jordan-Wigner type, see [6]) leading to the algebra (17) (with \( q_{\alpha\beta} \) given by Eq. (21)), that is
\[
b_i^\alpha = e^{i \sum_j c_{ij} N_j + \mu \pi \sum_{\beta} \theta_{\alpha\beta} N_{\beta}} B_i^\alpha \sqrt{\frac{[N_{i\alpha}]_\omega}{N_{i\alpha}}}
\]

\[
[N_{i\alpha}]_\omega = \frac{\omega^{N_{i\alpha}} - 1}{\omega - 1}, \quad \omega = - \cos \lambda \pi \cos \mu \pi,
\]

\[
\varphi_{ij} = c_{ij} - c_{ji},
\]

where \( \theta_{AB} \) is a step function (\( \theta_{AB} = 1 \) (0) if \( A > B \) \((A \leq B)\)), \( N_j = \sum_{\alpha} N_{j\alpha} \), \( N_\alpha = \sum_j N_{j\alpha} \) are number operators and \( B_i^\alpha \) are bosonic operators, which satisfy
\[
[N_{i\alpha}, B_j^\beta] = -\delta_{\alpha\beta} \delta_{ij} B_i^\alpha. \]

The resulting algebra is
\[
b_i^\alpha b_j^{\dagger \beta} - e^{\varphi_{ij}} e^{\mu \pi \text{sgn}(\alpha - \beta)} b_j^{\dagger \beta} b_i^\alpha = 0, \quad (i, \alpha) \neq (j, \beta),
\]
\[
b_i^\alpha b_i^{\dagger \alpha} - \omega b_i^{\dagger \alpha} b_i^\alpha = 1,
\]
\[
b_i^\alpha b_j^\beta = e^{-\varphi_{ij}} e^{-\mu \pi \text{sgn}(\alpha - \beta)} b_i^\beta b_i^\alpha,
\]

(23)
The algebra (17)(with $q_{\alpha\beta}$ given by Eq.(21)) is reproduced with $\mu = 1$. However, additional commutation relations between $b^i_\alpha$ and $b^j_\beta$ emerge in the anyonic case ($|q_{ij}| = 1$), in contrast to the case $|q_{ij}| < 1$. Note also that there is another anyonic interpolation between Green’s oscillators characterized by mapping, Eq.(22), with the choice of parameters $c_{ij} = 0$ and $\mu \in [0, 1]$. 

6. Conclusion

We have considered multiparametric interpolations between the para-Bose and para-Fermi oscillator algebras of a given order $p \in \mathbb{N}$, Eqs.(4). The construction has been performed using deformed Green’s oscillators and requiring that the corresponding Fock space does not contain negative norm states. When the interpolating parameters $q_{ij}$ in Eq.(17) satisfy $|q_{ij}| < 1$, the interpolation between parastatistics goes through the generalized infinite (quon) statistics, and if $|q_{ij}| = 1, \forall i, j \in I$, the interpolation is of the anyonic type (i.e. the corresponding $A^{(n)}$ matrices are singular and characteristic anyonic exchange factors appear). In the anyonic-type interpolation, oscillators can be obtained from para-Bose oscillators by a mapping. Let us remark that statistical properties are connected with the rank of the matrices $A^{(n)}$ and this is not a continuous function of the deformation parameters $q_{ij}$ [16]. It would be interesting to explore whether such interpolating statistics are compatible with cosmological arguments presented in [17]. It is also an open question whether the anyonic-like parastatistics described in this Letter might have applications in models of condensed matter physics and field theory.
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