Motivic decomposition of a compactification of a Merkurjev-Suslin variety

N. Semenov

Abstract

We provide a motivic decomposition of a twisted form of a smooth hyperplane section of Gr(3, 6). This variety is a norm variety corresponding to a symbol in $K^M_3/3$. As an application we construct a torsion element in the Chow group of this variety.

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1 Introduction

In the present paper we study certain twisted forms of a smooth hyperplane section of Gr(3, 6). These twisted forms are smooth SL$_1(A)$-equivariant compactifications of a Merkurjev-Suslin variety corresponding to a central simple algebra $A$ of degree 3. On the other hand, these twisted forms are norm varieties corresponding to symbols in $K^M_3/3$ given by the Serre-Rost invariant $g_3$. In the present paper we provide a complete decomposition of the Chow motives of these varieties.

The history of this problem goes back to Rost and Voevodsky. Namely, Rost obtained the celebrated decomposition of a norm quadric (see [18]) and later Voevodsky found some direct summand, called a generalized Rost motive, in the motive of any norm variety (see [19]). Note that the $F_4$-varieties from [16] can be considered as a mod-3 analog of a Pfister quadric (more precisely, of a maximal Pfister neighbour). In its turn, our variety can be considered as a mod-3 analog of a norm quadric.

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The paper is organized as follows. In section 2 we provide a background to the category of Chow motives. In section 3 we define a smooth compactification of a Merkurjev-Suslin variety $MS(A, c)$ with $A$ a central simple algebra of degree 3, describe its geometrical properties, and decompose its Chow motive. Section 4 is devoted to an application of the obtained motivic decomposition. Namely, using the ideas of Karpenko and Merkurjev we construct a 3-torsion element in the Chow group of our variety.

The main ingredients of our proofs are results of Białynicki-Birula [2], Lefschetz hyperplane theorem, and Segre embedding.

2 Notation

2.1. The matrix notation of the present paper follows [12].

We use Galois descent language, i.e., identify a (quasi-projective) variety $X$ over a field $k$ with the variety $X_s = X \times_{\text{Spec} k} \text{Spec } k_s$ over the separable closure $k_s$ equipped with an action of the absolute Galois group $\Gamma = \text{Gal}(k_s/k)$. The set of $k$-rational points of $X$ is precisely the set of $k_s$-rational points of $X_s$ stable under the action of $\Gamma$.

We consider the Chow group $\text{CH}^i(X)$ (resp. $\text{CH}_i(X)$) of classes of algebraic cycles of codimension $i$ (resp. of dimension $i$) on an irreducible algebraic variety $X$ modulo rational equivalence (see [9]).

A Poincaré polynomial or generating function for a variety $X$ is, by definition, the polynomial $\sum a_i t^i \in \mathbb{Z}[t]$ with $a_i = \text{rk } \text{CH}^i(X)$.

The structure of the Chow ring of a Grassmann variety is of our particular interest. We do a lot of computations using formulae from Schubert calculus (see [9] 14.7).

Next we introduce the category of Chow motives over a field $k$ following [13] and [7]. We remind the notion of a rational cycle and state the Rost Nilpotence Theorem following [6].

2.2. Let $k$ be a field and $\text{Var}_k$ be a category of smooth projective varieties over $k$. Let $S$ denote any commutative ring. For any variety $X$ we set $\text{Ch}(X) := \text{CH}(X) \otimes_{\mathbb{Z}} S$. First, we define the category of correspondences with $S$-coefficients (over $k$) denoted by $\text{Cor}_k(S)$. Its objects are smooth projective varieties over $k$. For morphisms, called correspondences, we set $\text{Mor}(X, Y) := \text{CH}_{\dim X}(X \times Y) \otimes_{\mathbb{Z}} S$. For any two correspondences $\alpha \in \text{Ch}(X \times Y)$ and
$\beta \in \text{Ch}(Y \times Z)$ we define their composition $\beta \circ \alpha \in \text{Ch}(X \times Z)$ as
\begin{equation}
\beta \circ \alpha = \text{pr}_{13,4}(\text{pr}_{12}^*(\alpha) \cdot \text{pr}_{23}^*(\beta)),
\end{equation}
where $\text{pr}_{ij}$ denotes the projection on the $i$-th and $j$-th factors of $X \times Y \times Z$ respectively and $\text{pr}_{ij,*}$, $\text{pr}_{ij}^*$ denote the induced push-forwards and pull-backs for Chow groups.

The pseudo-abelian completion of $\text{Cor}_k(S)$ is called the category of Chow motives with $S$-coefficients and is denoted by $\mathcal{M}_k(S)$. The objects of $\mathcal{M}_k(S)$ are pairs $(X, p)$, where $X$ is a smooth projective variety and $p \in \text{Mor}(X, X)$ is an idempotent, that is, $p \circ p = p$. The morphisms between two objects $(X, p)$ and $(Y, q)$ are the compositions $q \circ \text{Mor}(X, Y) \circ p$.

2.3. By construction, $\mathcal{M}_k(S)$ is a tensor additive category, where the tensor product is given by the usual product $(X, p) \otimes (Y, q) = (X \times Y, p \times q)$. For any cycle $\alpha$ we denote by $\alpha^t$ the corresponding transposed cycle.

2.4. Observe that the composition product $\circ$ induces the ring structure on the abelian group $\text{Ch}_{\dim X}(X \times X)$. The unit element of this ring is the class of the diagonal map $\Delta_X$, which is defined by $\Delta_X \circ \alpha = \alpha \circ \Delta_X = \alpha$ for all $\alpha \in \text{Ch}_{\dim X}(X \times X)$. The motive $(X, \Delta_X)$ will be denoted by $\mathcal{M}(X)$.

2.5. Consider the morphism $(e, \text{id}): \{pt\} \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$. Its image by means of the induced push-forward $(e, \text{id})_*$ does not depend on the choice of the point $e: \{pt\} \rightarrow \mathbb{P}^1$ and defines the projector in $\text{CH}_1(\mathbb{P}^1 \times \mathbb{P}^1)$ denoted by $p_1$. The motive $\mathbb{Z}(1) = (\mathbb{P}^1, p_1)$ is called Lefschetz motive. For a motive $M$ and a nonnegative integer $i$ we denote its twist by $M(i) = M \otimes \mathbb{Z}(1)^{\otimes i}$.

2.6. An isomorphism between the twisted motives $(X, p)(m)$ and $(Y, q)(l)$ is given by correspondences $j_1 \in q \circ \text{Ch}_{\dim X+1-m}(X \times Y) \circ p$ and $j_2 \in p \circ \text{Ch}_{\dim Y+1-l}(Y \times X) \circ q$ such that $j_1 \circ j_2 = q$ and $j_2 \circ j_1 = p$.

2.7. Let $X$ be a smooth projective cellular variety. The abelian group structure of $\text{CH}(X)$ is well-known. Namely, $X$ has a cellular filtration and the generators of Chow groups of the bases of this filtration correspond to the free additive generators of $\text{CH}(X)$. Note that the product of two cellular varieties $X \times Y$ has a cellular filtration as well, and $\text{CH}^*(X \times Y) \cong \text{CH}^*(X) \otimes \text{CH}^*(Y)$ as graded rings. The correspondence product of two cycles $\alpha = f_{\beta} \times g_{\alpha} \in \text{Ch}(X \times Y)$ and $\beta = f_{\beta} \times g_{\beta} \in \text{Ch}(X \times Y)$ is given by (see [3] Lemma 5)
\begin{equation}
(f_{\beta} \times g_{\beta}) \circ (f_{\alpha} \times g_{\alpha}) = \deg(g_{\alpha} \cdot f_{\beta})(f_{\alpha} \times g_{\beta}),
\end{equation}
where $\deg: \text{Ch}(Y) \rightarrow \text{Ch}(\{pt\}) = S$ is the degree map.
2.8. Let $X$ be a projective variety of dimension $n$ over a field $k$. Let $k_s$ be the separable closure of $k$ and $X_s = X \times_{\text{Spec } k} \text{Spec } k_s$. We say a cycle $J \in \text{Ch}(X_s)$ is \textit{rational} if it lies in the image of the natural homomorphism $\text{Ch}(X) \to \text{Ch}(X_s)$. For instance, there is an obvious rational cycle $\Delta_{X_s}$ in $\text{Ch}_n(X_s \times X_s)$ that is given by the diagonal class. Clearly, all linear combinations, intersections and correspondence products of rational cycles are rational.

2.9 (Rost Nilpotence). Finally, we shall also use the following fact (see [6] Theorem 8.2) called Rost Nilpotence theorem. Let $X$ be a projective homogeneous variety over $k$. Then for any field extension $l/k$ the kernel of the natural ring homomorphism $\text{End}(\mathcal{M}(X)) \to \text{End}(\mathcal{M}(X_l))$ consists of nilpotent elements.

3 Motivic decomposition

From now on we assume the characteristic of the base field $k$ is 0.

It is well-known (see [10] Ch. 1, § 5, p. 193) that the Grassmann variety $\text{Gr}(l, n)$ can be represented as the variety of $l \times n$ matrices of rank $l$ modulo an obvious action of the group $\text{GL}_l$. Having this in mind we give the following definition.

3.1 Definition. Let $A$ be a central simple algebra of degree 3 over a field $k$, $c \in k^*$. Fix an isomorphism $(A \oplus A)_s \simeq M_{3,6}(k_s)$. Consider the variety $D = D(A, c)$ obtained by Galois descent from the variety

$$\{\alpha \oplus \beta \in (A \oplus A)_s \mid \text{rk}(\alpha \oplus \beta) = 3, \text{Nrd}(\alpha) = c \text{Nrd}(\beta)\}/\text{GL}_1(A_s),$$

where $\text{GL}_1(A_s)$ acts on $A_s \oplus A_s$ by the left multiplication.

This variety was first considered by M. Rost.

Consider the Plücker embedding of $\text{Gr}(3, 6)$ into projective space (see [10] Ch. 1, § 5, p. 209). It is obvious that under this embedding for all $c$ the variety $D(M_3(k), c)$ is a hyperplane section of $\text{Gr}(3, 6)$.

3.2 Lemma. The variety $D$ is smooth.

Proof. (M. Florence) We can assume $k$ is separably closed. Consider first the variety

$$V = \{\alpha \oplus \beta \in M_3(k) \oplus M_3(k) = M_{3,6}(k) \mid \text{rk}(\alpha \oplus \beta) = 3, \det(\alpha) = c \det(\beta)\}.$$
An easy computation of differentials shows that $V$ is smooth. The variety $V$ is a $GL_3$-torsor over $D$ and, since $GL_3$ is smooth, this torsor is locally trivial for étale topology. Therefore to prove its smoothness we can assume that this torsor is split.

Since $D \times_k GL_3$ is smooth, $D \times_k M_3$ is also smooth. Therefore it suffices to prove that if $D \times_k \mathbb{A}^1$ is smooth, then $D$ is smooth. But this is true for any variety. Indeed, for any point $x$ on $D$ we have $T_{(x,0)}(D \times_k \mathbb{A}^1) = T_x D \oplus T_0 \mathbb{A}^1 = T_x D \oplus k$ and $\dim T_x D = \dim T_{(x,0)}(D \times_k \mathbb{A}^1) - 1 = \dim(D \times_k \mathbb{A}^1) - 1 = \dim D$.

3.3 Remark. One can associate to the variety $D$ a Serre-Rost invariant $g_3(D) = [A] \cup [c] \in H^3(k, \mathbb{Z}/3)$ (see [12] § 40). This invariant is trivial if and only if $D$ is isotropic.

It is easy to see that $D^0 := MS(A, c) := \{a \in A \mid \text{Nrd}(a) = c\}$ is an open orbit under the natural right $SL_1(A)$- or $SL_1(A) \times SL_1(A)$-action on $D$. Namely, the open orbit consists of all $\alpha \oplus \beta$ with $\text{rk}(\alpha) = 3$. $D^0$ is called a Merkurjev-Suslin variety. In other words, the variety $D(A, c)$ is a smooth $SL_1(A)$-equivariant compactification of the Merkurjev-Suslin variety $MS(A, c)$.

Denote as $\imath: D \to SB_3(M_2(A))$ the corresponding closed embedding.

3.4 Lemma. For the variety $D_s$ the following properties hold.

1. There exists a $\mathbb{G}_m$-action on $D_s$ with 18 fixed points. In particular, $D_s$ is a cellular variety.

2. The generating function for $CH(D_s)$ is equal to $g = t^8 + t^7 + 2t^6 + 3t^5 + 4t^4 + 3t^3 + 2t^2 + t + 1$.

3. Picard group $\text{Pic}(D_s)$ is rational.

Proof. 1. We can assume $c = 1$. The right action of $\mathbb{G}_m$ on $D_s$ is induced by the following action:

$$(M_3(k_s) \oplus M_3(k_s)) \times \mathbb{G}_m \to M_3(k_s) \oplus M_3(k_s)$$

$$(\alpha \oplus \beta, \lambda) \mapsto \alpha \text{diag}(\lambda, \lambda^5, \lambda^6) \oplus \beta \text{diag}(\lambda^2, \lambda^4, \lambda^7)$$

Note that this action is compatible with the left action of $GL_3(k_s)$.

The 18 fixed points of $D$ are the $\binom{6}{3} = 20$ 3-dimensional standard subspaces of $\text{Gr}(3, 6)$ minus 2 subspaces, generated by the first and by the last 3 basis vectors.
2. By the Lefschetz hyperplane theorem (see [10]) the pull-back \( i^* \) is an isomorphism in codimensions \( i < \frac{\dim(\text{Gr}(3,6)) - 1}{2} \). Therefore \( \text{rk} \, \text{CH}^i(D_s) = \text{rk} \, \text{CH}^i(\text{Gr}(3,6)) \) for such \( i \)’s. Since Poincaré duality holds, we have \( \text{rk} \, \text{CH}^i(D_s) = 2 \text{rk} \, \text{CH}^i(\text{Gr}(3,6)) - 2 = 4 \).

It remains to determine only the rank in the middle codimension. To do this observe that \( \text{rk} \, \text{CH}^4(D_s) = 18 \) (see [2]). Therefore \( \text{rk} \, \text{CH}^4(D_s) = 2 \text{rk} \, \text{CH}^4(\text{Gr}(3,6)) - 2 = 4 \).

3. Consider the following commutative diagram:

\[
\begin{array}{ccc}
\text{Pic}(\text{SB}_3(M_2(A))) & \xrightarrow{i^*} & \text{Pic}(D) \\
\downarrow & & \downarrow \text{res}^* \\
\text{Pic}(\text{Gr}(3,6)) & \xrightarrow{i^*} & \text{Pic}(D_s)
\end{array}
\]

where the vertical arrows are the morphisms of scalar extension. By the Lefschetz hyperplane theorem the map \( i^* \) restricted to \( \text{Pic}(\text{Gr}(3,6)) \) is an isomorphism. Since \( \text{Pic}(\text{SB}_3(M_2(A))) \) is rational (see [15] and [16] Lemma 4.3), i.e., the left vertical arrow is an isomorphism, the restriction map \( \text{res}^* \) is surjective. On the other hand, it follows from a Hochschild-Serre spectral sequence (see [11] § 2) that \( \text{Pic}(D) \) can be identified with a subgroup of \( \mathbb{Z} \). We are done.

3.5 Remark. It immediately follows from this Lemma that the variety \( D \) is not a twisted flag variety. Indeed, the generating functions of all twisted flag varieties over a separable closed field are well-known and all of them are different from the generating function of \( D_s \).

3.6. We must determine a (partial) multiplicative structure of \( \text{CH}(D_s) \). By Lefschetz hyperplane theorem the generators in codimensions 0, 1, 2, and 3 are pull-backs of the canonical generators \( \Delta_{(0,0,0)}, \Delta_{(1,0,0)}, \Delta_{(1,1,0)}, \Delta_{(2,0,0)}, \Delta_{(1,1,1)}, \Delta_{(2,1,0)}, \Delta_{(3,0,0)} \) of \( \text{Gr}(3,6) \) (see [9] 14.7). We denote these pull-backs as \( 1, h_1, h_2^{(1)}, h_2^{(2)}, h_3^{(1)}, h_3^{(2)}, \text{ and } h_3^{(3)} \) respectively. In the codimension 4 the pull-back is injective and the pull-backs \( h_4^{(1)} := i^*_s(\Delta_{(2,1,1)}), h_4^{(2)} := i^*_s(\Delta_{(2,2,0)}), h_4^{(3)} := i^*_s(\Delta_{(3,1,0)}) \), where \( i \) is as above, form a subbasis of \( \text{CH}^4(D_s) \).
Consider the following diagram:

Since pull-backs are ring homomorphisms, it immediately follows that

\[ h_1 \cdot u = \sum_{u \to v} v, \]

where \( u \) is a vertex on the diagram, which corresponds to a generator of codimension less than 4, and the sum runs through all the edges going from \( u \) one step to the right.

Next we compute some products in the middle codimension.

Since \( \Delta_{(3,1,0)} \Delta_{(2,1,1)} = \Delta_{(2,2,0)}^2 = 0 \) and \( \Delta_{(2,2,0)} \Delta_{(2,1,1)} = \Delta_{(3,1,0)} \Delta_{(3,3,2)} \) (see [9] 14.7), we have \( h_4^{(1)} h_4^{(3)} = (h_4^{(2)})^2 = 0 \) and \( (h_4^{(1)})^2 = (h_4^{(3)})^2 = h_4^{(2)} h_4^{(3)} = h_4^{(1)} h_4^{(2)} = i^*_s(\Delta_{(3,3,2)}) = pt \), where \( pt \) denotes the class of a rational point on \( D_s \).

The next theorem shows that the Chow motive of \( D \) with \( \mathbb{Z}/3 \)-coefficients is decomposable. Note that for any cycle \( h \) in \( CH(D_s) \) or in \( CH(D_s \times D_s) \) the cycle \( 3h \) is rational.

**3.7 Theorem.** Let \( A \) denote a central simple algebra of degree 3 over a field \( k \), \( c \in k^* \), and \( D = D(A,c) \). Then

\[ M(D) \simeq R \oplus (\oplus_{i=1}^5 R'(i)), \]

where \( R \) is a motive such that over a separably closed field it becomes isomorphic to \( \mathbb{Z} \oplus \mathbb{Z}(4) \oplus \mathbb{Z}(8) \) and \( R' \simeq M(SB(A)) \).

**Proof.** Consider the following commutative diagram (see [9] 5.5):

(4)

\[
\begin{array}{ccc}
D_s \times \mathbb{P}^2 & \xrightarrow{\iota_s \times \text{id}} & \text{Gr}(3,6) \times \mathbb{P}^2 \\
\downarrow & & \downarrow \text{Seg} \\
D \times SB(A^{\text{op}}) & \xrightarrow{1 \times \text{id}} & SB_3(M_2(A)) \times SB(A^{\text{op}}) \\
\downarrow & & \downarrow \text{Seg} 
\end{array}
\]

\[ \text{Gr}(3,18) \]

\[ SB_3(M_2(A)) \otimes_k A^{\text{op}} \]
where the right horizontal arrows are Segre embeddings given by the tensor product of ideals (resp. linear subspaces) and the vertical arrows are canonical maps induced by the scalar extension $k_s/k$.

This diagram induces the commutative diagram of rings

\[
\begin{array}{c}
\text{Ch}^*(D_s \times \mathbb{P}^2) \xrightarrow{(t_s \times \text{id}_s)^*} \text{Ch}^*(\text{Gr}(3, 6) \times \mathbb{P}^2) \\
\xrightarrow{\text{Seg}^*} \text{Ch}^*(\text{Gr}(3, 18))
\end{array}
\]

Observe that the right vertical arrow is an isomorphism since $M_2(A) \otimes_k A^{op}$ splits.

Let $\tau_3$ and $\tau_1$ be tautological vector bundles on $\text{Gr}(3, 6)$ and $\mathbb{P}^2$ respectively and let $e$ denote the Euler class (the top Chern class). By [5] Lemma 5.7 the cycle $(t_s \times \text{id}_s)^*(e(pr_1^*\tau_3 \otimes pr_2^*\tau_1)) \in \text{Ch}(D_s \times \mathbb{P}^2)$ is rational. A straightforward computation (cf. [5] 5.10 and 5.11) shows that $r := -(t_s \times \text{id}_s)^*(e(pr_1^*\tau_3 \otimes pr_2^*\tau_1)) = h_3^{(1)} \times 1 + h_2^{(1)} \times H + h_1 \times H^2 \in \text{Ch}^3(D_s \times \mathbb{P}^2)$, where $H$ is the class of a smooth hyperplane section of $\mathbb{P}^2$.

Define following rational cycles $\rho_i = r(h_i^1 \times 1) \in \text{Ch}^{3+i}(D_s \times \mathbb{P}^2)$ for $i = 1, \ldots, 4$, $\rho_0 = r + h_3^{(1)} \times 1 \in \text{Ch}^3(D_s \times \mathbb{P}^2)$ and $\rho_1' = r(h_1 \times 1) + h_1^1 \times 1$. A straightforward computation using multiplication rules shows that $(-\rho_1') \circ \rho_3'$ as well as $(-\rho_4-i) \circ \rho_1' \in \text{Ch}_2(\mathbb{P}^2 \times \mathbb{P}^2)$ is the diagonal $\Delta_{2,2}$. Moreover, the opposite compositions $(-\rho_0)^t \circ \rho_4$, $(-\rho_1)^t \circ \rho_3$, $(-\rho_2)^t \circ \rho_2$, $(-\rho_3)^t \circ \rho_1$, and $(-\rho_4)^t \circ \rho_0$ give rational pairwise orthogonal idempotents in $\text{Ch}_6(D_s \times D_s)$.

To finish the proof of the theorem it remains by 2.6 to lift all these rational cycles $\rho_i$, $\rho_1'$ to $\text{Ch}(D \times \text{SB}(A^{op}))$ and to $\text{Ch}(\text{SB}(A^{op}) \times D)$ respectively in such a way that the corresponding compositions of their preimages would give the diagonal $\Delta_{\text{SB}(A^{op})}$.

Fix an $i = 0, \ldots, 4$. Consider first any preimage $\alpha \in \text{Ch}(D \times \text{SB}(A^{op}))$ of $-\rho_{4-i}$ and any preimage $\beta \in \text{Ch}(\text{SB}(A^{op}) \times D)$ of $\rho_i'$. The image of the composition $\alpha \circ \beta$ under the restriction map is the diagonal $\Delta_{2,2}$. Therefore by Rost Nilpotence theorem for Severi-Brauer varieties (see 2.9) $\alpha \circ \beta = \Delta_{\text{SB}(A^{op})} + n$, where $n$ is a nilpotent element in $\text{End}(\mathcal{M}(\text{SB}(A^{op})))$. Since $n$ is nilpotent $\alpha \circ \beta$ is invertible and $((\Delta_{\text{SB}(A^{op})} + n)^{-1} \circ \alpha) \circ \beta = \Delta_{\text{SB}(A^{op})}$. In other words, we can take $(\Delta_{\text{SB}(A^{op})} + n)^{-1} \circ \alpha$ as a preimage of $-\rho_{4-i}$ and $\beta$ as a preimage of $\rho_i'$. Note that $n$ is always a torsion element and since $\text{End}(\mathcal{M}(\text{SB}(A^{op}))) \simeq \text{Mor}(\mathcal{M}(\text{SB}(A)), \mathcal{M}(\text{SB}(A^{op})))$ and $\text{Ch}(\text{SB}(A))$ has no torsion, projective bundle theorem implies that in fact $n = 0$. 

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Denote as $R$ the remaining direct summand of the motive of $D$. Comparing the left and the right hand sides of the decomposition over $k_s$ it is easy to see that $R_s \cong \mathbb{Z} \oplus \mathbb{Z}(4) \oplus \mathbb{Z}(8)$.

3.8 Remark. Using a bit more messy computations one can show that the same proof works for the motive of $D$ with integral coefficients.

4 Torsion

In this section we use Steenrod operations modulo 3 (see [4], [11] § 3, and [14]). We denote the total Steenrod operation by $S^\bullet = S^0 + S^1 + \ldots$.

Let $X$ be a smooth projective variety over $k$. For any cycle $p \in CH(X \times X)$ we define its realization $p_* : CH(X) \to CH(X)$ as $p_*(\alpha) = pr_2_*(pr_1^*(\alpha)p)$, $\alpha \in CH(X)$, where $pr_1, pr_2 : X \times X \to X$ denote the first and the second projections. As $\deg : CH_0(X) \to \mathbb{Z}$ we denote the usual degree map.

The goal of the present section is to prove the following theorem.

4.1 Theorem. Assume that the variety $D$ is anisotropic. Then $CH_2(D)$ contains $3$-torsion.

4.2. The proof of this Theorem consists of several parts. First we define an important element $d$ as follows. The kernel of the push-forward map

$$(\iota_*)_s : CH_4(D_s) \to CH_4(Gr(3, 6))$$

has rank 1, since by Lefschetz hyperplane theorem the push-forward $((\iota_*)_s \otimes \mathbb{Q})$ is surjective. Denote as $d \in CH_4(D_s)$ a generator of this kernel. Projection formula immediately implies that $(\iota_*)_s(\alpha d) = 0$ for any $\alpha \in \text{Im } \iota_*$ and therefore by Lefschetz hyperplane theorem $\alpha d = 0$.

From now on we work with Chow groups modulo 3.

4.3 Lemma. We have

1. $d^2 \neq 0$ mod 3,

2. the total Chern class of the tangent bundle

$$c(-T_{D_s}) = 1 + h_1 + h_1^2 - h_1^3 - h_1^4 - h_1^5$$

and
3. $S^\bullet(d) = d$.

Proof. The first equality is just a routine computation, which uses Poincaré duality on $\text{CH}(D_s)$.

Next we compute the total Chern class of the tangent bundle $T_{D_s}$. Since $D_s$ is a hyperplane section of $\text{Gr}(3,6)$ we have the following exact sequence:

$$0 \rightarrow T_{D_s} \rightarrow i_*^s(T_{\text{Gr}(3,6)}) \rightarrow i_*^s(\mathcal{O}_{\text{Gr}(3,6)}(1)) \rightarrow 0$$

Therefore $c(T_{D_s})i_*^s(c(\mathcal{O}_{\text{Gr}(3,6)}(1))) = i_*^s(c(T_{\text{Gr}(3,6)}))$. Since $i_*^s(c(\mathcal{O}_{\text{Gr}(3,6)}(1))) = 1 + h_1$ and $i_*^s(c(T_{\text{Gr}(3,6)})) = 1 - h_1^2 - h_1^3 + h_1^4$, we have $c(T_{D_s}) = 1 - h_1 - h_1^3 + h_1^4$ and $c(-T_{D_s}) = 1 + h_1 + h_1^2 - h_1^3 - h_1^4 - h_1^5$.

To prove the last assertion note that $\Delta_{D_s} = \Delta' \pm d \times d$, where $\Delta'$ is a part of the diagonal $\Delta_{D_s}$, which does not involve $d$, i.e., which comes from $\text{Gr}(3,6)$. Let $\delta: D_s \rightarrow D_s \times D_s$ denote the diagonal morphism.

Now

$$S^\bullet(\Delta_{D_s} - \Delta') = S^\bullet(\delta_*^s(1) - \Delta') = S^\bullet(\delta_*^s(1)) - S^\bullet(\Delta').$$

To prove that $S^\bullet(d) = d$ we must show that the right hand side does not contain summands of the form $d \times \alpha$, $\alpha \in \text{Ch}(D_s)$, different from $\pm d \times d$. Therefore the summand $S^\bullet(\Delta')$ is not interesting for us.

We have

$$S^\bullet(\delta_*^s(1)) = c(T_{D_s \times D_s})\delta_*^s(S^\bullet_{D_s}(1)c(-T_{D_s})) = c(T_{D_s \times D_s})(c(-T_{D_s}) \times 1)\delta_*^s(1) = c(T_{D_s \times D_s})(c(-T_{D_s}) \times 1)\Delta_{D_s},$$

where the second equality follows from projection formula. But by item 2, the Chern classes $c_i(T_{D_s})$ don’t involve $d$, i.e., lie in the image of $i_*^s$. The lemma is proved. 

In the notation of Theorem 3.7 denote as $p \in \text{Ch}_8(D \times D)$ the projector corresponding to the motive $R$, i.e., $R = (D, p)$. From the proof of Theorem 3.7 it is easy to see that $p_s = 1 \times pt \pm d \times d + pt \times 1$.

Since the natural map $\text{Pic}(D) \rightarrow \text{Pic}(D_s)$ is an isomorphism (see Lemma 3.4(3)), we denote as $h_1$ the canonical generator of $\text{Pic}(D_s)$ as well as the corresponding generator of $\text{Pic}(D)$.
4.4 Lemma. The following properties of \( D \) hold:

1. The natural group homomorphism \( \text{CH}_0(D) \to \text{CH}_0(D_\times) \) is injective. Its image is generated by zero cycles of degree divisible by 3.

2. \( S^1(p_*(h_1^8)) = h_1^8 \).

Proof. 1. By [8] Theorem 6.5 it suffices to show that the class \( A(D) \) of all field extensions \( E/k \) such that \( D(E) \neq \emptyset \) is connected and for any \( L \in A(D) \) the group \( \text{CH}_0(D_L) = \mathbb{Z} \). The first assertion follows from [8] Theorem 11.3, since any field extension \( E/k \) such that \( D = D(A, c) \) has an \( E \)-point splits the Jordan algebra \( J(A, c) \) obtained by the first Tits construction, and vice versa.

To prove that \( \text{CH}_0(D_L) = \mathbb{Z} \) for any \( L \in A(D) \) it suffices to check that for any field extension \( E/L \) any two rational points of \( D_E \) are rationally equivalent (see [8] Lemma 5.2). If the algebra \( A_E \) is not split, then all rational points of \( D_E \) are contained in \( \text{MS}(A_E, 1) \cong \text{SL}_1(A_E) \). Since \( \text{SL}_1(A) \) is rational and homogeneous, this implies that \( D_E \) is \( R \)-trivial, and, hence, \( \text{CH}_0(D_E) = \mathbb{Z} \). If the algebra \( A_E \) splits, then obviously \( \text{CH}_0(D_E) = \mathbb{Z} \).

2. The proof of this item is similar to the proof of Corollary 4.9 [11]. By [11] Lemma 3.1

\[
S^\bullet(p_*(h_1^6)) = S^\bullet(p)_*(h_1^6(1 + h_1^2)^6c(-T_D)).
\]

Therefore \( S^1(p_*(h_1^6)) \) equals the 0-dimensional component of the right hand side. Assume that

\[
S^1(p)_*(h_1^6) = 0.
\]

Then an easy computation using item 1. shows that the right hand side of (6) equals \( p_*(h_1^8) = h_1^8 \).

To prove (7) it suffices to show that \( \text{deg} \, S^1(p)_*(h_1^6) \) is divisible by 9 (cf. [11] Proof of Corollary 4.5). Without loss of generality we can compute this degree over \( k_s \). It follows that \( \text{deg} \, S^1(p)_*(h_1^6) = \text{deg} \, h_1^6 \text{pr}_1_*(S^1(p)) \) (see [11] Proof of Corollary 4.5). But \( \text{pr}_1_*(S^1(p)) \) is divisible by 3 (see Lemma 4.3(3)) and for any \( \alpha \in \text{Ch}_2(D_\times) \) the product \( h_1^6\alpha \) is divisible by 3. We are done. \( \square \)

Now we are ready to prove Theorem 4.1. Consider the cycle \( S^1(p_*(h_1^6)) \). Since \( \text{deg} \, h_1^8 = 42 \) and \( D \) is anisotropic, by lemma 4.4 this cycle is non-zero.

Therefore \( p_*(h_1^6) \in \text{Ch}_2(D) \) is non-zero. On the other hand, \( (p_*)_*(h_1^8) = 0 \). In other words, \( p_*(h_1^6) \) is a non-trivial torsion element in \( \text{Ch}_2(D) \).
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N. Semenov
Fakultät für Mathematik
Universität Bielefeld
Deutschland