Rotating black string in dRGT massive gravity

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Abstract One of the solutions of Einstein field equation with cylindrical symmetry is known as black string solution. In this work, the rotating black string solution in dRGT massive gravity is obtained and then called rotating-dRGT black string solution. This solution is a kind of generalized version of rotating AdS/dS black string solution containing an additional two more terms characterizing the structure of graviton mass. The horizon structures of the black string are explored. The thermodynamical properties of the black string are investigated. We found that it is possible to obtain the Hawking-Page phase transition depending on the additional structure of the graviton mass, while it is not possible for rotating-AdS/dS black string. By analyzing the free energy, we also found that the stable rotating black string is bigger than the non-rotating one.

1 Introduction

Massive gravity theory is a theory which extends Einstein’s general relativity (GR) by adding consistent interaction terms interpreted as a graviton mass. Such a theory can provide the solution for describing our universe, which is currently expanding with acceleration without introducing a cosmological constant. Massive gravity modifies the gravity by weakening it at the large scale compared to GR, which allows the universe to accelerate whereas the predictions at small scale are kept to be the same as those in GR. The cost of introducing the mass to the graviton is that it breaks the diffeomorphism invariance which normally resides in GR. The first attempt was done in 1939 by Fierz and Pauli [1], they added the interaction terms in the linearized level of GR and later on it was found that the theory made by Fierz and Pauli suffered from the discontinuity in predictions which were pointed out by van Dam, Veltman, and Zakharov, the so-called van Dam-Veltman-Zakharov (vDVZ) discontinuity [2,3,4]. This discontinuity problem invoked further studies on the nonlinear generalization of Fierz-Pauli massive gravity. Boulware and Deser found that such nonlinear generalization can only generate an equation of motion which has higher derivative term yielding a ghost instability in the theory, later called Boulware-Deser (BD) ghost [5]. In the same time, Vainshtein found that the origin of the vDVZ discontinuity is that the prediction made by the linearized theory cannot be trusted inside some characteristic “Vainshtein” radius and he also proposed the mechanism which can be used to recover the prediction made by GR for the nonlinear massive gravity [6].

Recently, these main problems of massive gravity could be solved by de Rham, Gabadadze, and Tolley [7,8]. They introduced the massive gravity action which contains a nonlinear interaction term which is free from BD ghost and also admits the Vainshtein mechanism. The de Rham-Gabadadze-
Tolley (dRGT) massive gravity is well constructed so that the equations of motion contains no higher derivative term to avoid BD ghost. As a consequence, such construction gives rise to a certain energy scale to the dRGT massive gravity formally known as $\Lambda_3$ scale. This scale can be parametrically expressed as $\Lambda_3 = (M_{Pl} m_g^{3/2})^{1/3}$ which marks the cutoff, or, in other words, strong coupling scale, of this dRGT theory when viewed as an effective theory. The reviews on these topics are in Ref. 9 10.

Beside the cosmological solutions, there have been various investigations of spherically symmetric solutions 11 12 13 14 15. These kind of solutions allows us to investigate the properties of the local astronomical objects such as the white dwarfs 16, neutron star 17 and black hole 18 19 20 21 22 23 24. Thermodynamical properties of the black hole are also intensively investigated 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39. Other properties, such as the superradiant effect 40 and greybody factor 41, of the black holes in dRGT massive gravity and the mass-radius ratio bounds for compact objects 42 are investigated. The modification of the gravity due to the graviton mass as a dark matter is also determined in terms of the rotation curves of galaxies 43. Furthermore, the motion of a particle around the spherical object is shown to be affected by the modification of the graviton mass 44.

It is well-known that the usual observed astronomical objects do not respect the static and spherical symmetry, they are commonly known as rotating prolate spheroids. Therefore, this allows investigating the astronomical objects satisfying the cylindrical symmetry. Theoretically, the study of the cylindrical solutions provides the better understanding of the hoop conjecture 45, which states that horizons form when and only when a mass gets compacted into a region whose circumference is less than, $4\pi GM$ in all directions. By using this conjecture, it is expected that the cylindrical matter will not form a black hole. However, it is shown that the hoop conjecture may be violated when the cosmological constant is included since the cylindrical black holes are shown to exist in the GR with the existence of the cosmological constant 46 47 48. The black hole with cylindrical symmetry is called black string. The charged and rotating black string solutions were consequently found 49. The quasinormal modes 50 and the greybody factor of the black string have been investigated 51.

For the dRGT massive gravity theory, it is found that the spherically symmetric solution can provide a more general solution than the Schwarzschild-dS/AdS. Therefore, it is possible to obtain the cylindrical solution or black string in the dRGT massive gravity theory 52. From this investigation, it is found that the Hawking-Page phase transition 53 54, a transition from the non-black hole or hot flat space state to a black hole, can be obtained while it is not possible for AdS/dS black string in GR. The quasinormal mode 55 and greybody factor 56 for the dRGT black string solution have been investigated as well. In this work, we investigate the rotating black string solution in dRGT massive gravity theory. The horizon structures in both charged case and the uncharged case of the black string are explored. For asymptotically dS black string, it is possible to obtain two horizons while it is not for the usual dS black string. For asymptotically AdS black string, the maximum number of the horizon is 3 while it is 1 for the usual AdS black string in GR. These modifications come from the existence of the structure of the graviton mass in dRGT massive gravity theory. We discuss this issue in Sec. 3. We then investigate the thermodynamical properties of the black string in Sec. 4. The quantities such as entropy, temperature, mass, heat capacity and free energy are obtained. The stability of the black string, as well as the possibility to obtain the Hawking-Page phase transition, is investigated in this section. The results are summarized in Sec. 5.

2 dRGT massive gravity

We begin by reviewing dRGT massive gravity, which is a well known nonlinear generalization of a massive gravity and is free of the BD ghost by introducing suitable interaction terms into the Lagrangian. The dRGT massive gravity can be represented as Einstein gravity interacting with the non-dynamical field (fiducial or reference metric), and hence its action is the well-known Einstein-Hilbert action plus suitable nonlinear interaction terms as given by 8

$$S = \int d^4x \sqrt{-g} \frac{1}{2} [R + m_g^2 U(g,f)],$$

where $R$ is the Ricci scalar and $U$ is a potential for the graviton which modifies the gravitational sector with the parameter $m_g$ interpreted as graviton mass. It is important to note that the form of the fiducial...
metric $f_{\mu\nu}$ can provide a significant form of the physical metric $g_{\mu\nu}$ \cite{57,58,59}. Note that for the following calculations, we adopt the natural unit by which the Newtonian gravitational constant is unity, i.e. $G = 1$. The effective potential $U$ in four-dimensional spacetime is given by

$$U(g, f) = U_2 + \alpha_3 U_3 + \alpha_4 U_4,$$  
(2)

in which $\alpha_3$ and $\alpha_4$ are dimensionless free parameters of the theory. The dependencies of the terms $U_2$, $U_3$ and $U_4$ on the metric $g$ and scalar fields $\phi^a$ are defined as

$$U_2 \equiv [K]^2 - [K']^2,$$  
(3)

$$U_3 \equiv [K]^3 - 3[K][K']^2 + 2[K^3],$$  
(4)

$$U_4 \equiv [K]^4 - 6[K]^2[K']^2 + 8[K][K^3] + 3[K^2]^2 - 6[K^4]$$  
(5)

where

$$K_{\mu}^\nu = \delta_{\mu}^\nu - \left(\sqrt{g^{-1} f}\right)_{\mu}^\nu,$$  
(6)

and the rectangular brackets denote the traces, namely $[K] = K_{\mu}^\nu$ and $[K'] = (K'^\mu)^{\nu}$. The metric $f_{\mu\nu} = f_{\alpha\beta} \partial_{\alpha} \phi^a \partial_{\nu} \phi^b$ can be defined in terms of the reference metric $f_{\alpha\beta}$ and four scalar fields $\phi^a$ called the St"uckelberg scalars which are introduced to restore general covariance of the theory. One may recognize the interaction terms as symmetric polynomials of $K$; for a particular order, each of the coefficients of possible combinations is chosen so that these terms do not excite higher derivative terms in the equations of motion of a scalar degree of freedom known as BD ghost.

To proceed further, we choose the unitary gauge $\phi^a = x^a \delta^a \phi$. In this gauge, the tensor $g_{\mu\nu}$ is the observable metric describing the five degrees of freedom of the massive graviton. Note that since the St"uckelberg scalars transform according to the coordinate transformation, once the scalars are fixed, for example, due to choosing the unitary gauge, applying a coordinate transformation will break the gauge condition and then introduce additional changes in the St"uckelberg scalars. Also, we redefine the two parameters $\alpha_3$ and $\alpha_4$ of the graviton potential in Eq. (2) by introducing two new parameters $\alpha$ and $\beta$, as follows

$$\alpha_3 = \frac{\alpha - 1}{3}, \quad \alpha_4 = \frac{\beta}{4} + 1 - \frac{\alpha}{12}.$$  
(7)

By varying the action with respect to metric $g_{\mu\nu}$, we obtain the modified Einstein field equations as

$$G_{\mu\nu} + m_g^2 X_{\mu\nu} = 0,$$  
(8)

where $X_{\mu\nu}$ is the effective energy-momentum tensor obtained by varying the potential term with respect to $g_{\mu\nu}$.

$$X_{\mu\nu} = K_{\mu\nu} - K g_{\mu\nu} - \alpha \left(K^2_{\mu\nu} - K K_{\mu\nu} + \frac{U_2}{2} g_{\mu\nu}\right) + 3\beta \left(K^3_{\mu\nu} - K K_{\mu\nu}^2 + \frac{U_2}{2} K_{\mu\nu} - \frac{U_4}{6} g_{\mu\nu}\right).$$  
(9)

In addition to the modified Einstein equations, one can obtain a constraint by using the Bianchi identities as follows

$$\nabla^\mu X_{\mu\nu} = 0,$$  
(10)

where $\nabla^\mu$ denotes the covariant derivative which is compatible with $g_{\mu\nu}$. Henceforth, we shall use $\alpha$ and $\beta$, instead of the parameters $\alpha_3$ and $\alpha_4$.

### 3 Rotating solutions

Most of the real astronomical objects are rotating, therefore, it is worthwhile to investigate the rotating solution of the black string in dRGT massive gravity. However, for the black string solution, it is not difficult to obtain the rotating solution from the non-rotating one since both of them are still respecting the same symmetry. The general line element for static and cylindrically symmetric spacetime in 4-dimension reads as

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2,$$  
(11)

where $d\Omega^2 = d\varphi^2 + \alpha_3^2 dz^2$ is a metric on the 2D surface and compatible with black string solution \cite{46,48}. We choose the cylindrical coordinates system such that black string has symmetry along $z-$direction, $-\infty < t < +\infty$, $0 \leq r < +\infty$, $-\infty < z < +\infty$, and $0 \leq \varphi < 2\pi$. The solution of Einstein-Hilbert action in dRGT massive gravity theory Eq. (8) for ansatz in Eq. (11) has been found in Ref. \cite{52}, it yield two possible branches of solution

$$f_1(r) = -\frac{b}{\alpha r} - \frac{m_g^2 r^2}{3(\alpha + 3\beta)},$$  
(12)

$$f_2(r) = -\frac{b}{\alpha r} + m_g^2 \left(r^2(1 + \alpha + \beta) - h_0 r(1 + 2\alpha + 3\beta) + h_0^2 (\alpha + 3\beta)\right),$$  
(13)

with $b$ as an integration constant expressed as $b = 4M$, and $M$ is ADM mass per unit length along $z$ direction. Out of these two solutions only $f_2(r)$ is
non-trivial because \( f_1(r) \) mimics the Lemos’s \cite{46} black string solution in AdS background in GR with suitable cosmological constant \( \Lambda \) \cite{52},
\[
\Lambda \equiv -3\alpha_g^2 = \frac{m_s^2(1 + \alpha + \alpha^2 - 3\beta)}{(\alpha + 3\beta)},
\]
(14)
which has been already studied in literature including their rotating and charged counterparts, and thermodynamical properties \cite{18}. We must notice that this AdS behavior comes as a natural consequence of non-zero graviton mass in dRGT theory. However, second solution in Eq. (13) is the generalization of Lemos’s black string solution, and contains the correction terms due to the rotating counterpart of the dRGT massive gravity \cite{52}. Therefore, we will be focusing on the rotating counterpart of the \( f_2(r) \) solution only. For mathematical simplicity, to rewrite \( f_2(r) \) in a more compact form we can redefine the variables and parameters as follow
\[
f_2(r) = -\frac{b}{\alpha_g r} + \alpha_m^2(\nu^2 - c_1 r + c_0),
\]
(15)
where
\[
\alpha_m^2 \equiv m_s^2(1 + \alpha + \beta), \quad c_1 \equiv \frac{b_0(1 + 2\alpha + 3\beta)}{1 + \alpha + \beta}, \quad c_0 \equiv \frac{b_0(\alpha + 3\beta)}{1 + \alpha + \beta}.
\]
(16)
Note that \( b \) is an integration constant and supposed to be \( 4M \) where \( M \) is the black string mass for non-rotating spacetime. The black string mass will be modified due to the rotating spacetime depending on the angular frequency as we will see later. Furthermore, the solution of modified Einstein-Maxwell equations in dRGT massive gravity theory for static and cylindrically symmetric spacetime has been studied \cite{52}, which reads as
\[
f_2^q(r) = -\frac{b}{\alpha_g r} + \frac{\gamma^2}{\alpha_g^2 r^2} + \alpha_m^2(\nu^2 - c_1 r + c_0),
\]
(17)
with vector potential \( A_\mu = a(r)\delta_\mu^r \), where \( a(r) \) is arbitrary function of radial coordinate \( r \). In order to have the consistent solution with Maxwell equations, \( a(r) \) can be interpreted as
\[
a(r) = -\frac{\gamma}{\alpha_g r}.
\]
(18)
\( \gamma \) being an integration constant can be fixed as \( \gamma^2 = 4q^2 \), where \( q \) is identified as the linear charge density in \( z \)-direction.

Lemos \cite{46}, showed that the non-rotating black string solution can be extended to the rotating one by using the following simple coordinate transformation,
\[
t = \lambda t - \frac{\omega}{\alpha_g} \tilde{\varphi}, \quad \varphi = \lambda \tilde{\varphi} - \omega t.
\]
(19)
where \( \lambda \) and \( \omega \) are constant parameters. On using the coordinate transformation \cite{19}, the metric of the physical rotating black string spacetime in dRGT massive gravity is given by
\[
ds^2 = (r^2\omega^2 - \lambda^2 f(r)) dt^2 + \frac{dr^2}{f(r)} + \frac{2\lambda \omega}{\alpha_g} (f(r) - \alpha_g^2 r^2)
\]
\[
d\tilde{\varphi}^2 + \frac{r^2 \lambda^2 - \omega^2 f(r)}{\alpha_g^2} d\tilde{\varphi}^2 + r^2 \alpha_g^2 dz^2,
\]
(20)
where \( f(r) \) is given by Eqs. (15) and (17), respectively, for the charged and uncharged case. In the no-rotation (\( \omega = 0 \)), the above line element reverts back to the static and cylindrically symmetric black string spacetime \cite{52}. On the otherhand, the Lemos \cite{46} rotating black string solution in GR can be obtained as a special case of (20) for \( c_1 = c_0 = 0 \), and \( \alpha_g = \alpha_m \). Likewise, to non-rotating black string, the rotating black string also approach the AdS spacetime for asymptotically large \( r \), though for large \( z \) with fixed \( r \) it do not approaches the AdS spacetime.

The line element Eq. (20) describes a black string rotating only along \( \varphi \) direction. For a closed black string with compacted \( z \) coordinate \( 0 \leq \alpha_g z < 2\pi \) \((S^1 \times S^1 \) torus topology), we can have rotation along \( z \) direction too. This will lead to a black toroid rotating in two orthogonal directions \( \varphi \) and \( z \). Though, it will merely lead to any interesting phenomenon, as we can always make a coordinate transformation to eventually get the black string rotating only along \( \varphi \) direction described by Eq. \cite{20} \cite{46}. The electromagnetic field also get transformed under the coordinate transformation in Eq. (19) as
\[
A_\mu = (a(r)\lambda, 0, -a(r)\omega/\alpha_g^2, 0).
\]
(21)
Since the form of the physical metric depends on the form of the fiducial (reference) metric, the fiducial metric is supposed to get modified in the same way as the physical metric to preserve the equation of motion. According to this coordinate transformation, the fiducial metric can be written as
\[
f_{\mu\nu} = \begin{pmatrix}
h^2 \omega^2 & 0 & -h^2 \lambda \omega & 0 \\
0 & 0 & 0 & 0 \\
-h^2 \lambda \omega & 0 & h^2 \lambda^2 & 0 \\
0 & 0 & 0 & \alpha_g^2 h^2
\end{pmatrix}
\]
(22)
One can check that this form of the fiducial metric still provides the solution in Eq. (20) of the modified Einstein Eq. in [8]. The rotating black string metric has coordinate singularity at $g^\sigma\sigma = 0 \Rightarrow f(r) = 0$, whose solutions determined the radial coordinates of horizons viz., $f_2(r) = 0$ for uncharged rotating black string and $f_3^2(r) = 0$ for charged one. Clearly, the number of horizons and their positions have an explicit dependency upon the parameters $b, \alpha_g, \alpha_m, q, c_1, c_0$, and crucially on the sign of parameter $\alpha_m^2$. For $\alpha_m^2 < 0$, the spacetime is asymptotically dS and then the number of horizons is up to two for the uncharged case and up to three for the charged case as shown in Fig. [1]. Whereas for the asymptotically AdS case, $\alpha_m^2 > 0$, the number of horizons is up to three for the uncharged case and up to four for charged case (cf. Fig. [2]). In figure [1] and [2] we plotted $f = g^\sigma\sigma$ vs $r$ for uncharged and charged black string ($q = 0.3$) in left and right panel, respectively. It is also shown that there exist critical values where two horizons merge together corresponding to the extremal case.

Let us point the important issue for the rotating black string in dRGT massive gravity. As we have mentioned, the crucial difference in this solution to the usual black string is the existence of $c_0$ and $c_1$ terms. Without these two terms, it is not possible to have the horizons in uncharged and asymptotically-dS case, though, the structure of the graviton mass allows the existence of the horizon. However, for the asymptotically-AdS case, one horizon can always be found even without these two terms. This will significantly affect the thermodynamics behavior as we will see later. Furthermore, for the charged case, it provides more horizons and then the thermodynamics is significantly changed.

This is evident from Eq. (20), that rotating black string also has another physical relevant surface called “static limit surface” (SLS) governed by the solution of $g_{tt} = 0 \Rightarrow \lambda^2 f(r) - r^2 \omega^2 = 0$, which coincides with the event horizon in the non-rotating limit ($\omega = 0$). As a result, we have two more parameters in order to find the behavior of the SLS. However, the structure of the surface does not significantly change since the additional terms are proportional to $r^2$ which already exist in $f(r)$. Therefore, the number of SLS is the same as those for horizons and we have shown it explicitly. Specifically, we plot how the horizon changes when the term $r^2 \omega^2$ is added as shown in Fig [3]. Note that the other structures, for example, the existence of the extremal case, will not be significantly changed. For asymptotically AdS case, the value of $\omega$ significantly change the structure of the SLS surface since the $\omega^2 r^2$ term in $\lambda f - \omega^2 r^2$ will cancel the contribution in graviton mass in contrast to the dS case, which it will support. As a result, we use $\omega = 0.2$ instead of $\omega = 0.5$ for dS case as shown in Fig. [4].

The linear charge and mass densities come naturally as the integration constants in terms of $\gamma$ and $b$, respectively. Though the integration constants in both Eq. (13) and Eq. (17) may not necessarily the same as in the stationary solution. In order to find the proper constants, we can find the relation of the constants to the physical mass, angular momentum and charge of the black string. The metric described in Eq. (20) has an infinite extension along $z$ direction, therefore it looks obvious that for a far distant observer ($r \to \infty$), the total mass and total charge would be infinite. The physical quantities, in this case, are mass and charge linear densities, which are finite. In order to estimate these quantities we can use the Hamiltonian formalism as suggested by Brown and York [61]. We can redefine the metric into the canonical form as follows

\[
\begin{align*}
&ds^2 = -N_0^2 dt^2 + R^2(N_\varphi d\varphi + d\tilde{r})^2 + \tilde{f}^{-2}dR^2 + r^2 \alpha_2^2 dz^2, \\
&N_0^2 = \frac{r^2}{R^2} \Delta^4 f(r), \quad \Delta^2 = \lambda^2 - \frac{\omega^2}{\alpha_g^2}, \quad \tilde{f}^2 = \left(\frac{dR}{dr}\right)^2 f(r), \\
&R^2 = \lambda^2 r^2 - \frac{\omega^2}{\alpha_g^2} f(r), \quad N_\varphi = \frac{\lambda \omega}{\alpha_g^2 R^2} \left(f(r) - \alpha_m^2 r^2\right), \\
&\Delta_m^2 = \lambda^2 - \frac{\omega^2 \alpha_m^2}{\alpha_g^4} \left(1 - \frac{c_1}{r} + \frac{c_0}{r^2}\right).
\end{align*}
\]  (23)

Here, $N_0$ and $N_\varphi$ are respectively called the lapse and shift functions. In order to estimate the physical quantities, we consider a $t = \text{constant}$ hypersurface $\Sigma$, which foliate the four-dimensional manifold $\mathcal{M}$, and described by the metric $h_{ij}$ and future pointing unit normal $n^\mu$. An element of three-boundary of $\mathcal{M}$, $^3B$ is a timelike three-surface generated by metric $g_{ij}$, and outward unit normal vector $n^\mu$. Let $\sigma$ is the determinant of the metric $\sigma_{ij}$ evaluated on two-boundary $^2B$ of hypersurface $\Sigma$ with constraints $d\Sigma = 0$ and $d\tilde{r} = 0$. With such construction, we can determine the proper surface energy density by the projection of stress tensor defined on three-boundary $^3B$ to the normal of two-boundary $^2B$ [61]. Furthermore, the conserved charges can be defined in terms
Fig. 1 Plot of $g^{rr}$ vs $r$ for asymptotically-dS rotating black string in dRGT massive gravity for particular values of parameters $b = 4, \alpha_g = 1, \alpha_m^2 = -1, c_1 = -6$. The left panel is the uncharged case and the right is the charged case with $q = 0.3$.

Fig. 2 Plot of $g^{rr}$ vs $r$ for asymptotically-AdS rotating black string in dRGT massive gravity for particular values of parameters $b = 4, \alpha_g = 1, \alpha_m^2 = 1, c_1 = 6$. The left panel is the uncharged case and the right is the charged case with $q = 0.3$.

of the Killing vectors on the boundary of the surface and surface stress tensor, which are equal to the value of Hamiltonian required to generate the diffeomorphism along the Killing vectors [61]. In the black string spacetime, $^2\mathcal{B}$ resemble the surface of an infinite cylinder. In order to make the region finite, we consider that $^2\mathcal{B}$ is bounded also between $z = z_1$ and $z = z_2$. This is expected that the two obvious Killing vectors $\xi^\mu_{(t)}$ and $\xi^\mu_{(\phi)}$ of metric (20) corresponding to the time translation and rotation invariance, respectively, will entail the existence of two conserved quantities, which can be identified as en-
Fig. 3 Plot of $g_{tt}$ and $g^{rr}$ vs $r$ for asymptotically-dS rotating black string in dRGT massive gravity for particular values of parameters $b = 4, \alpha_g = 1, \lambda = 1, \alpha_m^2 = -1, c_1 = -6, \omega = 0.5$. The solid lines represent $g^{rr} = f$ corresponding to horizons with $f = 0$ while dashed lines represents $g_{tt}$ corresponding to SLS with $g_{tt} = 0$. The left panel is the uncharged case and the right is the charged case with $q = 0.3$.

Fig. 4 Plot of $g_{tt}$ and $g^{rr}$ vs $r$ for asymptotically-dS rotating black string in dRGT massive gravity for particular values of parameters $b = 4, \alpha_g = 1, \lambda = 1, \alpha_m^2 = 1, c_1 = 6, \omega = 0.2$. The solid lines represent $g^{rr} = f$ corresponding to horizons with $f = 0$ while dashed lines represents $g_{tt}$ corresponding to SLS with $g_{tt} = 0$. The left panel is the uncharged case and the right is the charged case with $q = 0.3$.

Energy (or mass) and angular momentum. The global conserved charge $Q_\xi$ associated with metric (20) can be written as [61]

$$Q_\xi = \int d^2x \sqrt{\sigma} (\epsilon u^\mu + j^\mu) \xi_\mu,$$

(25)

where $\epsilon$ and $j^\mu = (0, j^i)$ are respectively the energy and momentum surface densities on the two-surface.
which are defined as

\[
\epsilon = \frac{k}{8\pi} = \nabla_i n^i,
\]

\[
j^i = \frac{h^i_{jk}K_hj^k}{\sqrt{h}16\pi} = \frac{h^i_{jk}(K_hj^k - K^j_k)}{8\pi},
\]

where \(K_{\mu\nu}\) and \(k_{ab}\) are extrinsic curvature of hypersurface \(\Sigma\) embedded in \(\mathcal{M}\) and of \(\mathcal{B}\) embedded in \(\Sigma\), respectively, whereas \(K\) and \(k\) are their respective traces. \(n^i\) is a spacelike normal vector to the three-space \(\Sigma\). In Eq. (25), \(\epsilon\) and \(j^i\) are defined such that the conserved charge \(Q_\xi\) do not have any contribution from the background spacetime [43]. Following [43], the mass and angular momentum linear densities of black string spacetime at radial infinity \((R \to \infty)\) can be written in terms of the constant parameters as follows

\[
M = \frac{b}{4} \left( \lambda^2 + \frac{\omega^2\alpha_m^2}{2\alpha_y^2} \right) \frac{\Delta^2_0}{\Delta^2},
\]

\[
J = \frac{3\alpha_0\alpha_y}{8\alpha_y^2},
\]

where \(\Delta^2_0 \equiv \Delta^2 (r \to \infty) = \lambda^2 - \omega^2\alpha_m^2/\alpha_y^2\). Again, the results can be reduced to those for the black string solution in GR by setting \(\alpha_m = \alpha_y\) and \(\Delta_0 = \Delta\). In the similar fashion, the electric charge of segment \(\Delta z\) contributed from the vector potential \(A_\mu\) can be written as

\[
Q = \frac{1}{4\pi} \int d^2x \sqrt{\sigma} n^i \varepsilon^i.
\]

\[
\varepsilon^R = \frac{\alpha_0 R}{f(r)N_0} (N_\varphi \partial_R A_\varphi - \partial_R A_\varphi).
\]

Using the metric form in Eq. (23), the electric charge density of black string is computed and expressed in terms of integration constant \(\gamma\)

\[
q = \frac{Q}{\Delta z} = \frac{\lambda_2}{2} \left( 1 + \frac{2\omega^2(\alpha_m^2 - \alpha_y^2)}{\Delta^2_0 \Delta^2} \right).
\]

From Eq. (31), one can see that the electric charge reduces to that for non-rotating black string as \(\omega = 0\) or even \(\alpha_m = \alpha_y\).

4 Thermodynamics of rotating black string

Now we are ready to extract some thermodynamical properties of the rotating black string in dRGT massive gravity. First, let us find the black string mass density obtained by solving \(f(r) = 0\) and using Eq. (28). As a result, the black string mass can be expressed as

\[
M = \frac{\Delta^2 \alpha_y r_+}{4\Delta^2_0} \left( \lambda^2 + \frac{\omega^2\alpha_m^2}{2\alpha_y^2} \right)
\]

\[
\left( \frac{\gamma^2}{\alpha_y^2 r_+^2} + \alpha_m^2 (r_+^2 - c_1 r_+ + c_0) \right).
\]

For asymptotically-dS black string, \(\alpha_m^2 < 0\), we have to impose the conditions \(2\alpha_y^2/\alpha_m^2 < \omega^2/(\alpha_y^2 r_+^2) < 1\) and \(c_0 > \gamma^2/4\) to obtain the positive definite of the black string mass. However, these conditions do not allow the existence of the horizons. Therefore, we will not consider the thermodynamics properties of the asymptotic dS black string. For asymptotic AdS black string, the conditions to obtain the positive definite of the black string mass can be expressed as \(\omega^2/(\alpha_y^2 r_+^2) < 1\) and \(c_0 > \frac{\gamma^2}{4}\). These conditions satisfy the existence of the horizons and then allow us to properly investigate the thermodynamical properties of the black string.

Note that it is sufficient to restrict our consideration to the case where the angular frequency is sufficiently small so that the conditions are safely satisfied.

One of the important thermodynamics quantity is temperature, which can be defined in terms of the surface gravity \(\kappa\)

\[
\xi^\mu = -\frac{1}{2} \nabla^\mu (\xi^\nu \varepsilon^\nu),
\]

where Killing vector \(\xi^\mu\) is the generator of event horizon. Since the black string is axialsymmetric, the Killing vector is given by

\[
\xi^\mu = (1, 0, \Omega_H, 0), \quad \Omega_H = -\frac{g_{r\varphi}}{g_{\varphi\varphi}} |_{r = r_+} = \frac{\omega}{\lambda}.
\]

As a result, the surface gravity can be expressed as follows \(\kappa = \frac{1}{2} \sqrt{g_{\varphi\varphi}} \partial_r \Phi^2 = \frac{1}{2} \Delta^2 1\sqrt{g_{\varphi\varphi}} f'(r_+)\) where \(\Phi^2 = -\xi^\mu \xi^\mu\). Then using \(f(r)\), we can calculate the corresponding Hawking temperature from surface gravity

\[
T_+ = \frac{\Delta^2}{4\pi r_+ \lambda} \left( \frac{\alpha_m^2 (3r_+^2 - 2c_1 r_+ + c_0) - \frac{\gamma^2}{\alpha_y^2 r_+^2}}{ \alpha_m^2 (r_+^2 - c_1 r_+ + c_0)} \right).
\]

Note that in the special setting \(c_1 = c_0 = 0\), \(\alpha_y = \alpha_m\), this temperature coincide with that for Lemos’ black string in GR. The existence of \(c_1\) and \(c_0\) in our expression serves as corrections from the graviton mass to the black string solution and moreover, the
structure of horizon will be different. For charged rotating black string the minimum of temperature occurs at \( r_+ = r_{\text{min}} \equiv \left( \sqrt{c_0 + \frac{\sqrt{c_0^2 - \left( \frac{6\gamma}{\alpha_g \alpha_m} \right)^2}}{6}} \right) / \sqrt{6} \), while for uncharged rotating black string it appears at \( r_+ = r_{\text{min}} \equiv \sqrt{c_0/3} \). The minimum temperature of charged rotating black string reads as

\[
T_{\text{min}} = \frac{-72\gamma^2 + \alpha_g^2 \alpha_m^2 \Xi}{2\sqrt{6\pi\alpha_g^2 \lambda^3} \Xi} \Delta^2,
\]

where \( \Xi = c_0 + \sqrt{c_0^2 - (6\gamma/\alpha_g \alpha_m)^2} \). For uncharged rotating black string it takes relatively simpler form as follow

\[
T_{\text{min}} = \frac{\alpha_g^2 (\sqrt{3c_0} - c_1) \Delta^2}{2\pi \lambda},
\]

which, in the limit \( \omega = 0, \lambda = 1 \), reads as following

\[
T_{\text{min}} = \frac{\alpha_g^2 (\sqrt{3c_0} - c_1)}{2\pi},
\]

and matches with the calculated minimum temperature for static black string [52].

Horizon temperature of rotating black string is shown in Fig. 3. Charged black string witnesses a zero temperature phase where it is cooled down to \( T_r = 0 \) after reaching a finite maximum value, unlike uncharged one whose temperature attains a minimum value and then increases monotonically and eventually attain the unboundedly large value as its horizon shrinks further. This is clearly inferring from Fig. 3 that graviton mass (in terms of \( \alpha_m \)) effectively alter the evaporation profile of black string in dRGT gravity. The maximum value of charged black string temperature increases with increasing \( \alpha_m \). The rotating charged black string temperature study reveals that in the late stage of evaporation it will end up to a finite size zero temperature remnant, whose size depends upon various parameters.

According to the Kerr-AdS black holes [60], the first law of thermodynamics and the area law are satisfied by considering the angular velocity measured by the observer relative to a non-rotating frame at infinity, \( \Omega_{\text{BS}} \), instead of one measured by the observer relative to a rotating frame at infinity, \( \Omega_H \). Since dRGT black string solution is asymptotically AdS/dS rather than asymptotically flat, the thermodynamics of the rotating black string should respect actual angular velocity \( \Omega_{\text{BS}} \). In particular, \( \Omega_{\text{BS}} \) is defined in a way that the angular velocity of the background spacetime does not contribute. From the black string metric in Eq. (20) the angular velocity \( \Omega \) at \( r \to \infty \) can be found as

\[
\Omega_{\infty} = \lim_{r \to \infty} \left( -\frac{g_{\phi\phi}}{g_{\varphi\varphi}} \right) = \frac{\lambda \omega \left( 1 - \frac{\alpha_g^2}{\alpha_m^2} \right)}{\lambda^2 - \frac{\omega \alpha_m^2}{\alpha_g^2}}.
\]

Note that \( \Omega_{\infty} = 0 \) when \( \alpha_m = \alpha_g \). Thus, we can find \( \Omega_{\text{BS}} \) as

\[
\Omega_{\text{BS}} = \Omega_H - \Omega_{\infty} = \frac{\omega \alpha_m^2 \left( \lambda^2 - \frac{\alpha_g^2}{\alpha_m^2} \right)}{\lambda^2 - \frac{\omega \alpha_m^2}{\alpha_g^2}}.
\]

It is worth mentioning that even if there is no rotating black string present in the spacetime, the background can still possess a nonzero angular momentum due to the difference between \( \alpha_m \) and \( \alpha_g \) which can be seen in Eq. (39), viz., \( \Omega_{\text{BS}} = \Omega_H \) when \( \alpha_m = \alpha_g \). From the one-form of four potential, the electrostatic potential of the black string is \( \Phi_q = \frac{q}{2q \lambda / (\alpha_g \varphi)} \). As a result, with these thermodynamical quantities and the differential form of the first law of thermodynamics

\[
TdS = dM - \Omega_{\text{BS}} dJ - \Phi_q dq,
\]

we can calculate the entropy per unit length of black string, which yield

\[
S_+ = \frac{1}{2} \pi \lambda g r_+^2 = \frac{A}{4},
\]

where \( A = 2\pi \lambda g r_+^2 \) is the black string horizon area per unit length.

In order to determine the black hole local thermodynamical stability, we need to study the specific heat behaviour. Local thermodynamical stability signifies that how the system responds to the small fluctuations in its thermodynamical variables. The heat capacity for charged rotating black string

\[
C = \frac{dE}{dT},
\]

where \( E \) is the energy of the black string. This depends on the temperature and the angular velocity. For charged rotating black string the heat capacity is given by

\[
C = \frac{dE}{dT} = -\frac{\Omega_{\text{BS}}}{T} + \frac{\omega \alpha_m^2 \left( \lambda^2 - \frac{\alpha_g^2}{\alpha_m^2} \right)}{\lambda^2 - \frac{\omega \alpha_m^2}{\alpha_g^2}}.
\]

The specific heat at constant angular velocity is given by

\[
C_{\Omega} = \frac{dE}{dT} = -\frac{\Omega_{\text{BS}}}{T} - \frac{\omega \alpha_m^2 \left( \lambda^2 - \frac{\alpha_g^2}{\alpha_m^2} \right)}{\lambda^2 - \frac{\omega \alpha_m^2}{\alpha_g^2}}.
\]

For uncharged rotating black string the heat capacity is given by

\[
C = -\frac{\Omega_{\text{BS}}}{T} + \frac{\omega \alpha_g^2 \left( \lambda^2 - \frac{\alpha_g^2}{\alpha_g^2} \right)}{\lambda^2 - \frac{\omega \alpha_g^2}{\alpha_g^2}}.
\]

The specific heat at constant angular velocity is given by

\[
C_{\Omega} = \frac{dE}{dT} = -\frac{\Omega_{\text{BS}}}{T} - \frac{\omega \alpha_g^2 \left( \lambda^2 - \frac{\alpha_g^2}{\alpha_g^2} \right)}{\lambda^2 - \frac{\omega \alpha_g^2}{\alpha_g^2}}.
\]

The heat capacity for uncharged rotating black string

\[
C = -\frac{\Omega_{\text{BS}}}{T} + \frac{\omega \alpha_g^2 \left( \lambda^2 - \frac{\alpha_g^2}{\alpha_g^2} \right)}{\lambda^2 - \frac{\omega \alpha_g^2}{\alpha_g^2}}.
\]

The specific heat at constant angular velocity is given by

\[
C_{\Omega} = \frac{dE}{dT} = -\frac{\Omega_{\text{BS}}}{T} - \frac{\omega \alpha_g^2 \left( \lambda^2 - \frac{\alpha_g^2}{\alpha_g^2} \right)}{\lambda^2 - \frac{\omega \alpha_g^2}{\alpha_g^2}}.
\]
Fig. 5 The Hawking temperature ($T_\times$) vs horizon radius ($r_\times$) for $c_1 = 3, b = 4, \alpha_g = 1$. The dashed lines represent the uncharged case while the solid lines represent charged case with $\gamma = 0.3$. (Left) for varying $c_0$ and (Right) for varying $\alpha_m$.

Fig. 6 The specific heat ($C_\times$) behaviour with horizon radius ($r_\times$) of uncharged rotating black string for parametric values of $\alpha_g = 1, c_1 = 3, \lambda = 1$, and $\omega = 0.1$. (Right) for $\alpha_m = 1$ and (left) for $c_0 = 4.5$.

in dRGT massive gravity takes the following form

$$C_\times = \left( T \frac{dS}{dT} \right)_{r=r_\times}$$

$$= \frac{\pi r_\times^2 \lambda \alpha_g \left[ \gamma^2 - r_\times^2 (3\gamma^2 - 2r_\times c_1 + c_0) \alpha_m \omega_m \right]}{\left( r_\times^2 (c_0 - 3\gamma^2) \alpha_m \omega_m - 3\gamma^2 \right) \Delta^2}.$$  \hspace{1cm} (43)

which in the limit $\gamma = 0$, gives the value for uncharged rotating black string, reads as

$$C_\times = \frac{\pi \alpha_g r_\times^2 \lambda \left( 3r_\times^2 - 2c_1 r_\times + c_0 \right)}{\left( 3r_\times^2 - c_0 \right) \Delta^2}.$$  \hspace{1cm} (44)

Furthermore, specific heat for uncharged and static black string can be obtained in the special setting of
\[ \lambda = 1, \omega = 0, \gamma = 0, \]
\[ C_+ = \frac{\pi \alpha \gamma r_+^2 (3r_+^2 - 2c_1 r_+ + c_0)}{(3r_+^2 - c_0)}, \quad (45) \]

which matches with the one calculated in Ref. 52.

With the specific heat we can analyse the thermodynamical stability as well as the phase transition during Hawking evaporation process. Indeed the local thermodynamical stability of system depends upon the sign of specific heat, if \( C_+ > 0 \) then it is stable otherwise it is unstable.

Because, during the evaporation rotating black string exhibit an extremum temperature i.e., minima for both uncharged and charged while local maxima only for charged black string, accordingly specific heat \( C_+ \) will diverge at these extremum points \( r_{\text{min}} \) and \( r_{\text{max}} \). As pointed out by Davies 62, discontinuity in specific heat capacity (abruptly changing its sign) implies the second order phase transition in evaporation process. In particular, it is very clear from Eq. 44 that for uncharged black string heat capacity is discontinuous at \( r_+ = r_{\text{min}} \equiv \sqrt{c_0/3} \) and hence it is thermodynamically stable \( (C_+ > 0) \) for \( r_+ > r_{\text{min}} \) while unstable \( (C_+ < 0) \) for \( r_+ < r_{\text{min}} \). It is interesting to find that the phase transition depends upon the structure of graviton mass. In Fig. 6 we have plotted the specific heat \( C_+ \) behaviour of uncharged rotating black string with horizon radius \( r_+ \) for varying \( \alpha_m \) and \( c_0 \). As Fig. 3 ascertain that the location of minimum temperature remains intact with varying \( \alpha_m \) only the temperature value at that point changes. The same situation can be deduced from Fig. 6, the position of second-order phase transition is independent of \( \alpha_m \) but significantly vary with changing \( c_0 \). Black string temperature increases with increasing horizon radius for \( r_+ > r_{\text{min}} \), hence this is thermodynamically stable in this region, whereas it increases as horizon radius decrease for \( r_+ < r_{\text{min}} \) consequently specific heat is negative and it is unstable in this region.

The specific heat of charged black string as a function of horizon radius with varying \( \alpha_m \) and \( c_0 \) is shown in Fig. 7. Since, charged string has a local minima as well as local maxima (cf. Fig. 5) in temperature profile, the specific heat undergoes phase transition at two points \( r_{\text{min}} \) and \( r_{\text{max}} \), unlike uncharged string which has only one local minimum in temperature profile and hence exhibits phase transition at only one point. Phase transition at large value corresponds to the \( r_{\text{min}} \) whereas that at smaller value is associated with \( r_{\text{max}} \). The position of phase transition (corresponding to the minimum of temperature) changes significantly with varying \( c_0 \) while it does not change much with \( \alpha_m \).

The charged rotating black string is thermodynamically stable in the region \( r_+ > r_{\text{min}} \), whereas thermodynamically unstable in \( r_{\text{max}} < r_+ < r_{\text{min}} \). Another stable region lies in the interval of zero temperature to the maximum temperature region (cf. Fig. 5 and Fig. 7). It is important to note that without the structure of the graviton mass, \( c_0 = c_1 = 0 \), the heat capacity of the rotating dRGT black string does not diverge for all \( r_+ \) so that it is not possible to obtain the phase transition. As a result, one found that the structure of the graviton mass significantly provides the Hawking-Page phase transition. This suggests that the graviton mass may play an important role in high-energy physics of black string while the quantum effects become significant and taken into account. In the presence of multiple horizons associated with spacetime, it is interesting to study its global thermodynamical stability, which is concerned with the phase of a system corresponding to the global maximum of the total entropy 53.

One can calculate the Helmholtz free energy to discuss the global thermodynamical stability of black string. If we consider that system is in thermodynamical equilibrium with reservoir such that it exchange only mass \( \Delta M \neq 0 \) while \( \Delta J = \Delta q = 0 \), then in the preferred phase Helmholtz free energy will be minimum

\[ F = M - TS = \frac{r_+ \alpha \gamma \Delta^2}{8} \left[ \frac{\alpha_m^2}{\Delta^2_0} \left( (2 - 3\Delta_0^2)\Delta_+^2 + 2c_1(\Delta_0^2 - 1)r_+ - (\Delta_0^2 - 2)c_0 \right) + \frac{c^2(\Delta_0^2 + 2)}{r_+^2 \alpha_0^2 \Delta_0^3} \right]. \quad (46) \]

In order to find the global stability of the black string, one has to find the condition where the free energy is negative. Therefore, one can solve \( F = 0 \) for \( r_+ \). It is convenient to firstly consider the uncharged case. As a result, the critical horizon can be expressed as

\[ r_c = \frac{c_1(\Delta_0^2 - 1)}{3\Delta_0^2 - 2} \left[ 1 - \sqrt{1 - \frac{3\Delta_0^2 - 2}{\Delta_0^2 - 2}(\Delta_0^2 - 2)} \frac{\Delta_0^2 - 1}{c_0} \right]. \quad (47) \]
Fig. 7 Rotating charged black string specific heat ($C_+$) vs horizon radius ($r_+$) for parametric values of $\alpha_g = 1, c_1 = 3, \lambda = 1$, and $\omega = 0.1$. (Right) for $\alpha_m = 1$ and (left) for $c_0 = 6$.

Fig. 8 The free energy of uncharged black string with various choice of parameters $\alpha_m$ and $\omega$. The other parameters are fixed as follows $b = 4, \alpha_g = 1, \lambda = 1, c_0 = 6, c_1 = 3$.

our attention to the case of small angular frequency, $\omega \ll \lambda \alpha_g^2 / \alpha_m$. Then one obtains the approximation as

$$r_c \approx \sqrt{c_0} + (2\sqrt{c_0} - c_1) \frac{\omega^2 \alpha_m^2}{\alpha_g^4}.$$  \hspace{1cm} (48)

Note that the other value is always negative and we do not consider that here. From this critical horizon, one can see that it reduces to the non-rotating case, $r_c = \sqrt{c_0}$ for $\omega = 0$ as found in [52]. Moreover, if we impose the existence of the minimum potential, $c_0 > c_1^2 / 3$, it is found that the critical horizon is always larger than that from the non-rotating case. As a result, the rotating black string is thermodynamically stable with the horizon larger than its non-rotating counterparts. These results can be confirmed by using numerical plot as shown in Fig.
Since the value of critical horizon is characterized by two parameters, graviton mass parameter $\alpha_0$ and angular frequency $\omega$, the figures show that the more value of the parameters, the larger of the critical horizon as indicated from Eq. 18. This result is also true for the charged case but the equation is very lengthy and is not provided here. The effect of parameter $c_0$ on the free energy is also shown explicitly in Fig. 4 for both uncharged and charged cases. As infer from Eq. 18, the more value of $c_0$ leads to the bigger stable black string. It is important to note that there is a tinniest range of parameters to provides the free energy to be positive again for the larger horizon. From Eq. 18, this can occur when the angular frequency $\omega$ is large enough, for example, $\Delta_0^2 < 2/3$ and $(3\Delta_0^2 - 2)(\Delta_0^2 - 2)c_0 < c_1^2(\Delta_0^2 - 1)^2$. This case will not be considered here since it may encounter the fine tuning in parameters and we aim to find how the thermodynamics properties of the black string change when the black string gets small rotating. Note also that the larger value of $\omega$ will violate the positive definite condition of the thermodynamics mass as we have mentioned earlier.

5 Concluding remarks

In this paper, we obtained both uncharged and charged rotating black string solutions in dRGT massive gravity. All the limiting cases are discussed. The obtained solution is naturally AdS/dS depending upon the suitable choice of the parameters, which has a direct relation from the non-zero graviton mass in the massive gravity theory. Since the black string has an infinite extension along the $z$ axes, therefore it is interesting to determine the actual physical quantities associated with the black string. We use the Hamiltonian formalism to extract the mass density, angular momentum density, and charge density. In the thermodynamical analysis, we found that the horizon temperature has an explicit dependency upon graviton mass. The first law of black hole mechanics still holds valid for rotating black string. We have also analyzed the thermodynamical stability of both charged and uncharged rotating black string. It is found that rotating black string undergoes a second order phase transition during the Hawking evaporation process. For global thermodynamical stability, we studied the behavior of free energy and it is found that the stable rotating black string is bigger than the non-rotating one.

Acknowledgement

This project is supported by the ICTP through grant No. OEA-NT-01. S.G.G. would like to thank SERB-DST Research Project Grant No. SB/S2/HEP-008/2014. R.K. thanks UGC, Govt. of India for financial support through SRF scheme. PW is supported by the Thailand Research Foundation (TRF) through grant no. MGR6180003. L.T. is supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIP) (No.2016R1C1B1010107).

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Fig. 9 Rotating uncharged (Left) and charged (Right) black string free energy ($F$) behaviour with horizon radius ($r_+$).

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