Truncated Overlap Fermions

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In this talk I propose a new computational scheme with overlap fermions and a fast algorithm to invert the corresponding Dirac operator.

1. INTRODUCTION

After many years of research in lattice QCD, it was possible to formulate QCD with chiral fermions on the lattice [1,2].

The basic idea is an expanded flavor space which may be seen as an extra dimension with left and right handed fermions defined in the two opposite boundaries or walls.

Let \( N \) be the size of the extra dimension, \( D_W \) the Wilson-Dirac operator, and \( m \) the bare fermion mass. Then, the theory with \textit{Domain Wall Fermions} is defined by the action [3,4]:

\[
S_{DW} := \bar{\psi} M \psi = \sum_{i=1}^{N} \bar{\psi}_i [(D^\parallel - 1) \psi_i + P_+ \psi_{i+1} + P_- \psi_{i-1}],
\]

\[
P_+ \psi_{n+1} = (D^\parallel + 1) P_+ \psi_{n+1},
\]

\[
P_- \psi_{n-1} = (D^\parallel + 1) P_- \psi_{n-1}
\]

where \( M \) is the five-dimensional fermion matrix of the regularized theory and \( D^\parallel = M - D_W \) with \( M \in (0,2) \) being a mass parameter.

In this talk I define a theory with \textit{Truncated Overlap Fermions} in complete analogy with the Domain Wall Fermions by substituting

\[
D^\parallel \rightarrow \gamma_5 D^{-1} + \gamma_5 = 2a \gamma_5 R,
\]

where \( a \) is the lattice spacing and \( R \) is a local operator trivial in the Dirac space (see below for \( R \)-locality tests). From now on I set \( a = 1 \).

I defined Truncated Overlap Fermions such that in the large \( N \) limit one obtains Overlap Fermions \[5\] with the Dirac operator given by \[6\]:

\[
D_{OV} = \frac{1 + m}{2} - \frac{1 - m}{2} \gamma_5 \text{sgn}(H)
\]

where \( H = \gamma_5 D^\parallel \).

Until now computations with chiral fermions and standard algorithms have been very expensive. The extra fermion flavors introduce a large overhead. One multiplication with the fermion matrix costs \( \mathcal{O}(n) \) \( D_W \)-multiplications with \( n \sim N \) for Domain Wall Fermions and much larger for the overlap operator \[5\].

In this talk I propose a fast algorithm which makes these simulations an order of magnitude faster. The key observation is the lack of gauge connections along the fifth dimension.

2. TRUNCATED OVERLAP FERMIONS

I recall the action of the Truncated Overlap Fermions:

\[
S_{TOV} := \bar{\psi}_1 [(D^\parallel - 1) \psi_1 + (D^\parallel + 1) P_+ \psi_2 - m(D^\parallel + 1) P_- \psi_N] + \sum_{i=2}^{N-1} \bar{\psi}_i [(D^\parallel - 1) \psi_i + (D^\parallel + 1) P_+ \psi_{i+1} + (D^\parallel + 1) P_- \psi_{i-1}] + \bar{\psi}_N [(D^\parallel - 1) \psi_N - m(D^\parallel + 1) P_+ \psi_1 + (D^\parallel + 1) P_- \psi_{N-1}]
\]

Let \( P^T \) be the matrix representing the unitary transformation:

\[
\chi_1 = P_+ \psi_1 + P_- \psi_N,
\]

\[
\chi_i = P_+ \psi_i + P_- \psi_{i-1}, \quad i = 2, \ldots, N
\]

and \( S \) the matrix representing the diagonal transformation: \( \bar{\chi}_i = \psi_i \gamma_5 (H - 1), \quad i = 1, \ldots, N \). Let
Figure 1. Norm of $D$ kernel in spin and color space with the distance $r$ from the origin for $N = 4$ (circles) and $N = 32$ (crosses).

Figure 2. Norm of $R$ kernel in spin and color space with the distance $r$ from the origin for $N = 4$ (circles), $N = 32$ (crosses) and $N = 64$ (pluses).

3. LOCALITY AND GINSPARG-WILSON RELATION TESTS

I test the Ginsparg-Wilson relation for the Truncated Overlap Fermions, so that they can be used like Domain Wall Fermions.

I computed the norm of $D$ and $R$ kernels in spin and color space with the distance $r = \sqrt{x_\mu x_\mu}$ from the origin on $4^4$ lattices at $\beta = 6.0$. In Figs. 1 and 2 I show the maximum values at on-axis distances. Note that $x_\mu, \mu = 1, 2, 3, 4$ is measured modulo lattice size in that direction.

Fig. 1 suggests exponential fall-off of $||D(r)||_2$, whereas Fig. 2 shows that $R$ approaches a Kronecker-Delta function as $N$ grows.

4. A FAST INVERSION ALGORITHM

I use Truncated Overlap fermions to define the following

**ALGORITHM 1 (Generic) for solving the system $D_{OV}x = b$:**

Given $N$, $x_0$, $r_0 (= b)$, $tol$, $tol_1$, set $tol_0 = 1$ and iterate:

\[
\text{for } i = 1, \ldots
\]

\[
tol_0 = tol_0 tol_1
\]

\[
\text{Solve } Dy = r_{i-1} \text{ within } tol_0
\]

\[
x_i = x_{i-1} + y
\]

\[
r_i = b - D_{OV} x_i
\]

\[
if ||r_i||_2 < tol, \text{ end for}
\]

where by $a$ is denoted a vector with zero entries and $tol_1, tol$ are tolerances. $tol_0$ is typically orders of magnitude larger than $tol$ such that the work
per $D_{OV}$ inversion is minimized.

**Remark 1.** Bold face equations represent the smaller system solution and the correction of the right-hand side. The straightforward application of the \textit{ALGORITHM 1} gives a two-level algorithm. By calling it again in solving the smaller system and iterating, one gets a multi-level algorithm.

In Fig. 3 I compare the norm of the residual $r_i = b - D_{OV}x_i$ of the Conjugate Residual (CR) algorithm (which is optimal since $D_{OV}$ is normal [10]) and \textit{ALGORITHM 1}. I gain about an order of magnitude (in average) on 30 $4^4$-configurations at $\beta = 6.0$ and $m = 0.1$. For the coarse lattice I used $N = 6$ with the Truncated Overlap Fermions and the Lanczos method to compute $D_{OV}$ [7].

**Remark 2.** Dynamical fermions can be implemented similarly. The corresponding Hybrid Monte Carlo (HMC) algorithm can be obtained by working with an approximate Hamiltonian in the coarse lattice and by a global correction on the fine lattice.

One may use also as a starting point Truncated Overlap Fermions: $\det D = \det M / \det M_1$. This way, all known simulation algorithms for dynamical fermions apply.

5. **CONCLUSIONS**

I showed that Truncated Overlap Fermions may be used in two ways:

a) to implement Overlap Fermions in the same fashion as Domain Wall Fermions;

b) to construct a multi-level inversion algorithm for Overlap Fermions which saves an order of magnitude of computer time compared to the state of the art methods.

Further tests are needed to verify these results on larger lattices.

Recently, the possibility of a Multigrid algorithm along all dimensions is raised [11]. In this case a gauge fixing is needed.

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