Algebras and universal quantum computations with higher dimensional systems

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Abstract

Here is discussed application of the Weyl pair to construction of universal set of quantum gates for high-dimensional quantum systems. An application of Lie algebras (Hamiltonians) for construction of universal gates is revisited first. It is shown next, how for quantum computation with qubits can be used two-dimensional analog of this Cayley-Weyl matrix algebras, i.e. Clifford algebras, and discussed well known applications to product operator formalism in NMR, Jordan-Wigner construction in fermionic quantum computations. It is introduced universal set of quantum gates for higher dimensional system (“qudit”), as some generalization of these models. Finally it is briefly mentioned possible application of such algebraic methods to design of quantum processors (programmable gates arrays) and discussed generalization to quantum computation with continuous variables.

1 INTRODUCTION

The models discussed here only recently were introduced in theory of quantum computation [1, 2, 3] for description of \( d \)-dimensional quantum systems \( (d > 2) \), but they have rather long history. Initially some special matrices were introduced by Arthur Cayley more than 100 years ago [4], and 30 years later similar construction was used by Hermann Weyl for building of fundament of quantum theory [5]. It is so-called Weyl pair of matrices (or elements of infinite dimensional operator algebra, last case is not considered here in details for simplicity and only briefly discussed due to applications to quantum computations with continuous variables). The Weyl approach seems very important in theory of quantum computation, because it is only way to write an equivalent of Heisenberg relations for finite-dimensional systems used there. In addition algebra has important technical applications. It is widely used in quantum computation community after reintroducing few years ago with new name “generalized Pauli algebra.”
2 Universal Quantum Gates and Lie Algebras

“Quantum gate” is a special name for arbitrary unitary transformation. “Quantum networks” may be constructed using many quantum gates and corresponds to some unitary evolution, “quantum computation.” It is reasonable to consider some sets of elements that may be used for construction (or approximation) of any unitary transformation. It is similar with using universal set of logical gates (Boolean functions) like AND, NOT in classical networks and theory of computations.

In many models of quantum computations an elementary system is described by finite-dimensional Hilbert space, but more difficult continuous case is also may be considered. Quantum gates act on some composite system with Hilbert space described as tensor product (see Eq. (2) and Eq. (3) below). For schemes of quantum networks each “wire” corresponds to such system and it corresponds to exponential growth of dimension of space of states with number of such “quantum wires.”

In simplest case the Hilbert space of the elementary quantum system, qubit, is two-dimensional. More general case with higher-dimensional systems is also used sometimes and considered in present paper. For n qubits Hilbert space has dimension $2^n$ and most general quantum evolution may be described by $2^n \times 2^n$ unitary matrix.

The problem of universality is existence of some set or family of unitary matrices $U_\kappa$ with possibility of decomposition of any unitary matrix $U \in U(2^n)$

$$U = U_{\kappa_1} U_{\kappa_2} \cdots U_{\kappa_p}$$

or approximation $U \approx U_{k_1} U_{k_2} \cdots U_{k_p}$ with necessary or arbitrary precision, if the set is finite $k = 1, \ldots, K$.

It should be mentioned, that formally the same gate acting on different “wires” corresponds to different matrices, but the additional indexes are omitted in Eq. (1) in sake of simplicity. Situation may be more difficult in real applications, then system may not be simply presented as tensor product but it is not considered in present work.

For quantum computation with n qubits the Hilbert space $\mathcal{H}$ has special structure due to representation as tensor power of n two-dimensional spaces $\mathcal{H}_2$

$$\mathcal{H} = \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_2. \quad (2)$$

Due to such structure any unitary transformation of some subsystem with $m < n$ qubits, $U_{\text{sub}} \in U(2^m)$ can be expressed as transformation that acts only on $m$ elements (indexes) in tensor product with $n$ terms Eq. (2) used for representation of the Hilbert space for $n$ qubits. Such transformations are called $m$-qubit gates.

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1For example in theory of decoherence free subspaces, it is more known, but rather specific example.
The constructions may be simply extended to higher-dimensional quantum systems. Let $d > 2$ is dimension of Hilbert space $\mathcal{H}_d$ for one “qudit”, then $n$ systems may be described by $d^n$-dimensional space

$$\mathcal{H} = \mathcal{H}_d \otimes \cdots \otimes \mathcal{H}_d,$$

and unitary group $U(d^n)$. Non-binary quantum $m$-gates correspond to elements of $U(d^m)$, $m \leq n$.

An important result of quantum theory of computation is existence of universal set with two-qubit quantum gates [7, 8, 9].

There is so-called infinitesimal approach to construction of universal set of quantum gates. It was discussed in original papers [7, 9, 10] and described here only briefly. From physical point of view it can be considered as using Hamiltonian approach with Hermitian operators $H_k$ (Hamiltonian), instead of $S$-matrix-like approach with unitary gates (see also Eq. (4) below).

The infinitesimal approach uses Lie algebra $\mathfrak{u}(N)$ of Lie group $U(N)$ of unitary matrices together with finite set of elements $A_k$ of this algebra represented as some anti-Hermitian matrices $A_k = -A_k^\dagger$. It is possible also to write $iA_k \equiv H_k$ for some Hermitian matrices $H_k = H_k^\dagger$. Quantum gate may be expressed as

$$U_k^\tau = \exp(-iH_k\tau)$$

with some real parameter $\tau$. Now instead of universal set of elements of Lie group $U(N)$ used in Eq. (1) it is possible to work with elements $A_k = -iH_k$ of Lie algebra $\mathfrak{u}(N)$.

**Lemma** If the elements $A_k$ generate full Lie algebra $\mathfrak{u}(N)$ by commutators, then it is possible to use set of gates described by Eq. (4) as universal set of gates.

Subtleties of transition from Lie algebra to Lie group of gates based on Eq. (4) is not discussed with more detail and may be found in original works [7, 9, 10, 22, 3]. Description of specific mathematical problems may be found also in Ref. [11].

In physical applications $H_k = iA_k$ correspond to Hamiltonian and real parameter $\tau$ may be infinitesimal [7], irrational [9] or it is possible to consider Eq. (4) as one-parametric family, where $\tau$ is time [10].

### 3 Clifford Algebras

If there is some associative algebra $A$, then it is possible to introduce structure of Lie algebra by using commutators $[A, B] = AB - BA$. Due to relation with universal set of gates discussed above, it is reasonable to look for algebras with “simple commutation law” with purpose to have appropriate tools for construction and description of universal quantum gates.

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2Here $N = 2^n$ for quantum computation with $n$ qubits and $N = d^n$ for higher-dimensional case.
As an example of such a commutation law it is possible to mention
\[ AB = \zeta BA \quad (A, B \in \mathcal{A}, \ \zeta \in \mathbb{C}) \] (5)

It is commutative algebra for \( \zeta = 1 \) and anticommutative one for \( \zeta = -1 \). It is useful also to consider more general case, then \( \zeta \) is arbitrary complex number and not all elements of algebra satisfy equation Eq. (5), but only different generators of the algebra.

Well known examples for \( \zeta = -1 \) are algebra of all \( 2 \times 2 \) complex matrices with generators are Pauli matrices
\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\] (6)

and algebra of all \( 4 \times 4 \) complex matrices with generators are four Dirac matrices
\[
\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & -\sigma \\ \sigma & 0 \end{pmatrix}
\] (7)

Both algebras are Clifford algebras [12, 13] defined by (anti)commutation law
\[
e_k e_j + e_j e_k = 2\delta_{jk},
\] (8)

where \( e_k \) are generators of the algebra.

Main property of Clifford algebras used in mathematical and physical application is algebraic decomposition of square root
\[
\left( \sum_k c_k e_k \right)^2 = \sum_k c_k^2, \quad c_k \in \mathbb{C}.
\] (9)

Really Eq. (8) often used for definition of Clifford algebra instead of Eq. (8). For example, the property is used for representation of Dirac equation as square root of Klein-Gordon equation. On the other hand, Eq. (8) seems not related directly with applications of Clifford algebras in quantum computations discussed in present paper.

The Clifford algebras with even number of generators \( 2n \) are especially convenient for description of quantum computation with \( n \) qubits due to so-called product operator formalism [17, 18, 19] and some other applications [20, 22, 23]. It is enough to look on standard presentation for generators of such algebra [12, 15]
\[
e_{2k} = i 1 \otimes \cdots \otimes 1 \otimes \sigma_x \otimes \sigma_z \otimes \cdots \otimes \sigma_z,
\]
\[
e_{2k+1} = i 1 \otimes \cdots \otimes 1 \otimes \sigma_y \otimes \sigma_z \otimes \cdots \otimes \sigma_z,
\] (10)

Formally only two Pauli matrices are enough. Here is some subtlety, because 3D real Clifford algebra uses all three Pauli matrices as generators and it is not some pure mathematical trick, because may be used in description of real NMR experiments [18, 19].
where \( k = 0, \ldots, n - 1 \) and \( 1 \) is unit matrix.

There are \( 2^{2^n} \) different products of \( 2n \) generators \( e_k \) Eq. (10) and it corresponds to presentation of the Clifford algebra via \( 2^n \times 2^n \) complex matrices. Such representation is useful both for quantum \( n \)-gates as elements of Lie group \( U(2^n) \) and Hamiltonians as elements of Lie algebra \( u(2^n) \) of that group.

Jordan-Wigner representation of fermionic creation and annihilation operators via

\[
a^\dagger_k = \frac{1}{2}(e_{2k} + ie_{2k+1}), \quad a_k = \frac{1}{2}(e_{2k} - ie_{2k+1})
\]

with standard properties

\[
\{a_k, a_j\} = \{a^\dagger_k, a^\dagger_j\} = 0, \quad \{a^\dagger_k, a_j\} = \delta_{kj}
\]

provides useful links with fermionic quantum computation [5, 20, 21].

It should be mentioned, that an approach with creation and annihilation operators was used for description of universal quantum simulators and computers already in first works of Feynman [15, 16], but the question again has some subtleties, because for description of quantum computation it is also possible to use other operators produced via Eq. (11) from slightly changed \( e_k \) with \( 1 \) instead of any \( \sigma_z \) in tensor product Eq. (10). Sometime they are called now parafermionic operators, but the special term may be a bit misjudged, because such operators even simpler and more natural in formal theory of quantum computation, than “usual” fermionic operators. Formally, both kind of operators may be used equally in most constructions with creation and annihilation operators used in Refs. [15, 16].

Let us return to theory of universal quantum gates. It can be shown, that the commutators of elements Eq. (10) generate only \((2n^2 + n)\)-dimensional space and so set of gates based on these elements via Eq. (4) produces only \((2n^2 + n)D\) subgroup in \( 2^n2^D \) group of all unitary gates [22]. It is interesting counter-example of widespread belief about “omnipresent universality.” Anyway, appending of only one element of third order like \( ie_0e_1e_2 \) produces universal set of gates [22]. For fermionic quantum computation sometime is used representation with even number of operators Eq. (11) and so the result is in agreement with necessity of elements of second and fourth order for universality in such a case [21].

The representation Eq. (11) is good for general algebraic approach or fermions, but in usual theory of quantum computation more useful elements have minimal amount of non-unit terms in tensor product. Formally Eq. (10) produces universal set of \( k \)-gates with \( k = 1, \ldots, n \), but it is very simple to construct universal
set of two-gates. It is enough to consider elements of second order \( e_k e_{k+1} \), because any such element is one- or two-gate. So universal set of \( 2n + 1 \) two-gates may be based on following elements of Lie algebra \( \mathfrak{u}(2^n) \) (Hamiltonians) \[12\]

\[
e_0, \quad e_{k,k+1} \equiv i e_k e_{k+1} \quad (k = 0, \ldots, n - 2), \quad e_{012} \equiv i e_0 e_1 e_2. \tag{13}
\]

It may be checked, the last element of third order corresponds to one-gate.

### 4 Generalization with Weyl Pair

The construction with Clifford algebra described above may not be applied directly for non-binary quantum circuits, when each system described by \( d \)-dimensional Hilbert space, \( d > 2 \). Hilbert space of composite system with \( n \) qudits is \( d^n \)-dimensional and gates belong to \( U(d^n) \). But even in such a case the general principles are very close \[3\].

First, let us consider instead of Pauli matrices Weyl pair \[3, 14\] described by Eq. (5) with \( \zeta = \exp(2\pi i/d) \):

\[
UV = \exp(2\pi i/d) VU, \quad VV^\dagger = UU^\dagger = 1, \tag{14}
\]

where \( U \) and \( V \) are \( d \times d \) complex matrices \[3\]

\[
U = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
1 & 0 & 0 & \cdots & 0
\end{pmatrix}, \quad V = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & \zeta & 0 & \cdots & 0 \\
0 & 0 & \zeta^2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \zeta^{d-1}
\end{pmatrix}. \tag{15}
\]

Lately it is called sometime “generalized Pauli group” \[1\].

It is possible to consider three elements

\[
\tau_x = U, \quad \tau_y = \zeta (d-1)/2 UV, \quad \tau_z = V, \tag{16}
\]

with properties \[3\]

\[
\tau_x \tau_y = \zeta \tau_y \tau_x, \quad \tau_y \tau_z = \zeta \tau_z \tau_y, \quad \tau_x \tau_z = \zeta \tau_z \tau_x, \quad \tau_d = 1. \tag{17}
\]

Let us introduce set of complex \( d^n \times d^n \) matrices similar with construction Eq. (10) above:

\[
t_{2k} = \begin{pmatrix}
1 & \cdots & 1 \\
k - n & - 1 & n - k - 1
\end{pmatrix} \otimes \tau_x \otimes \tau_z \otimes \cdots \otimes \tau_z,
\]

\[
t_{2k+1} = \begin{pmatrix}
1 & \cdots & 1 \\
k - n & - 1 & n - k - 1
\end{pmatrix} \otimes \tau_y \otimes \tau_z \otimes \cdots \otimes \tau_z, \tag{18}
\]

where \( k = 0, \ldots, n - 1 \).
The $2n$ elements $t_k$ itself meet to analogue of commutation law Eq. (3):
\[ t_j t_k = \zeta t_k t_j \quad (j < k), \quad (t_k)^d = 1. \] (19)

As it was suggested earlier, the rule makes test of universality simpler and the elements are really useful for construction of universal quantum gates for higher dimensional case ($d > 2$) [3].

It is interesting, an analogue of Eq. (6) is also valid [24]
\[ \left( \sum_k c_k t_k \right)^d = \sum_k c_k^d, \quad c_k \in \mathbb{C}. \] (20)

So the elements may be used for expression of algebraic $d$-th root, but it is not relevant to present work.

The elements Eq. (18) are not Hermitian, but for construction of universal gates via Eq. (4) it is possible to use [3]
\[ t_k^+ = i(t_k + t_k^\dagger), \quad t_k^- = (t_k - t_k^\dagger). \] (21)

Commutators of the elements generate whole Lie algebra and so exponents Eq. (4) of elements Eq. (21) may be used as universal set of quantum gates [3].

Formally initial expressions Eq. (18) represent $k$-gates ($k = 1, \ldots, n$), but it is again possible to use pairwise products of the elements
\[ t_0, \quad t_{k,k+1} \equiv t_k t_{k+1} \quad (k = 0, \ldots, n - 2) \] (22)
to construct universal set of two-gates. It is also necessary to use generators Eq. (4) with Hermitian operators expressed as sums
\[ t_{k,k+1}^+ = i(t_{k,k+1} + t_{k,k+1}^\dagger), \quad t_{k,k+1}^- = (t_{k,k+1} - t_{k,k+1}^\dagger). \] (23)

It should be emphasized, that for higher-dimensional quantum systems with $d > 2$ it is enough only $2n$ elements Eq. (18) and $4n$ elements Eq. (21) of first order and it is not necessary to add some generator of fourth or third order for $d = 2$ (qubits and fermionic quantum computation). Such point may be essential, for example elements of high order have difficulties of implementation for fermionic quantum computation [21], and even in more general case such elements may have specific properties [22].

If to recall that without such additional element quantum gates generate only state of space with quadratic dimension (“complexity”) instead of exponential [21, 24], then it is possible to suggest, that for $d = 2$ the question about additional element of third order may be essential from point of view of universality, correspondence between classical and quantum complexity, etc.. So absence of such subtleties with one additional element for higher-dimensional system $d > 2$ maybe not so formal, as it could be suggested at first sight.

It was mentioned earlier relations of different standard techniques with theory of quantum computation: unitary gates ($S$-matrix), Hamiltonian, occupation numbers (annihilation and creation operators). The approach discussed in
present section has relation with yet another standard model, Weyl quantization. The Weyl pair $U, V$ Eq. (13) has many other applications in theory of quantum computations [1, 2], but specific new terminology used in some papers sometimes hides it.

The Eq. (14) is Weyl representation of Heisenberg commutation relation. The Weyl approach may be even more significant in area or research under consideration, because Heisenberg relation itself may not be satisfied in finite-dimensional Hilbert spaces used in many applications of quantum computations.

5 Universal Quantum Processors

Formally, the quantum network approach used above (and in many other papers) resembles classical situation, then some chosen set of logical gates (“chips”) are fastening on some board to build networks with necessary functions, on the other hand, usual electronic computers have fixed structure and different functioning is ensured by programmability.

An analogue of such classical computer is programmable quantum gate arrays [25, 26] or, simply, quantum processors [27, 28, 29, 30]. Here are exists two different designs “stochastic” [25, 26] and “deterministic” [27, 28, 29, 30]. Formally “deterministic” and “stochastic” processors initially have very different design and only recently was found possibility of some confluence in continuous limit [30].

Anyway only first kind of processors will be discussed in present paper, so word “deterministic” is omitted further for simplicity. Such quantum processor uses some version of “controlled” or “conditional” quantum dynamics [31]. In such approach some quantum “wire(s)” are used for encoding of operations on second set of “wires.” Quantum processor is special network with quantum system represented as tensor product of two subsystem: “program bus” and “data bus,” but there is some specific difficulties, because different states of “program bus” should be orthogonal [23] and so in some meaning all valid programs are “pseudo-classical,” i.e., a superposition may not be used and would produce “malfunction” of quantum processor (entanglement of data and program, more details may be found in original works [25, 26, 27, 28, 29, 30]).

Let us consider a particular design of quantum processors [25, 27, 29]. There are three buses: quantum data bus, there any superposition are possible, intermediate bus used for choice of particular quantum gate applied to data bus on current state. Intermediate bus is also some quantum system, but only fixed number of orthogonal basic states may be used to produce proper work of quantum processor. And, finally, there is (pseudo-)classical bus necessary to programming of changes in intermediate bus for performing necessary sequence of quantum gates on quantum data bus.

In simplest case the pseudo-classical bus is simply “cyclic ROM,” but sequences of universal gates used for implementation of some quantum algorithms may be too long for realistic implementations using the ROM based approach. So it is better, if sequence of gates may be described by some simple law and
processor may use some program of minimal size, i.e. it should be set of
instructions like

\[
\text{repeat } \{ \text{repeat } U_5 \text{ 10 times}; \text{repeat } U_7 \text{ 20 times } \} \text{ 3 times},
\]

but these issues related with application of \textit{reversible} classical programming are
out of scope of present work.

It is clear, that theory of (deterministic) universal quantum processors re-
sembles theory of universal quantum networks, but sequence of quantum gates
is not implemented statically, and each next operation is dynamically chosen on
each step of evolution, accordingly with program.

At present time a statement, that some particular set of gates is universal,
has limited significance, because property of universality is already proved for
very general sets of quantum gates. Discussed algebraic approach is important
due to constructivity, i.e., there are described concrete and not very difficult
algorithms [22, 3] for decomposition or approximation of arbitrary unitary op-
erators.

It may be not very actual for some standard quantum algorithms, when con-
struction of quantum networks is already developed and optimized during many
years in works of different researchers, but \textit{universality} of quantum processors
suggests existence of algorithm for decomposition of arbitrary unitary matrix
with given precision. Maybe realistic applications are related with mixed ap-
proach, when the universal algorithm of decomposition is used for \( k \)-gates with
not very big \( k \) (i.e., \( d^k \) has appropriate order) and the gates are used as building
blocks for more difficult quantum algorithms.

\section{Continuous Case}

The Eq. (14) is \textit{Weyl representation of Heisenberg commutation relations} and
valid both in finite and continuous (infinite) case \[.\] For \( d \to \infty \) two families
of operators with property Eq. (\[]) may be expressed as

\[
U_a; \psi(q) \mapsto \psi(q + a), \quad V_b; \psi(q) \mapsto e^{ibq}\psi(q),
\]

where \( a, b \) are two real parameters and \( \zeta = \exp(iab) \) in Eq. (\[]). It is possible
also to write the operators in \textit{exponential form}

\[
U_a = e^{ia\mathbf{p}}, \quad V_b = e^{ib\mathbf{q}},
\]

where \( \mathbf{p}, \mathbf{q} \) are operators of coordinate and momentum \[.\]

All necessary construction used for finite \( d \) above may be generalized on
continuous case \[.\], but such universal gates for continuous case have \textit{“bi-
exponential”} structure, like

\[
\exp[(U_a - U_a^\dagger)\tau] = \exp([\exp(iap) - \exp(-iap)]\tau),
\]

\[
\exp[i(U_a + U_a^\dagger)\tau] = \exp(i[\exp(iap) + \exp(-iap)]\tau).
\]
Such approach not only produces universal set of gates for one quantum continuous variable like in some other works [32], but for arbitrary amount of such variables. It is only necessary instead of $t_k$ Eq. (14) in all expressions uses their continuous analogues described in Ref. [24]. All constructions are based on straightforward change of $U, V$ Eq. (19) to $U_a, V_b$ Eq. (24). Here infinite-dimensional Hilbert space is represented as space of complex function with $n$ real variables.

Most difficult part of proof used in Ref. [3] related with consideration of different nonstandard cases with $\zeta^k = 1$, but it becomes trivial in infinite-dimensional case, because it is possible to choose $\zeta^k \neq 1$ for any integer $k$. Here is again possible to construct set of universal two-gates for arbitrary finite number of continuous quantum variables.

Physical meaning may depend on choice of particular conjugated variables $p, q$. For example, if $q$ is usual coordinate, expressions

$$e^{(V_b - V_b^*) \tau} = \exp(-i \sin(bx) \tau), \quad e^{(V_b + V_b^*) \tau} = \exp(i \cos(bx) \tau),$$

look more realistic as Hamiltonian for particle in periodic potentials, but more difficult analogues of operators Eq. (23) maybe look more strange. On the other hand, the continuous generalizations of $t_k$ are based on some non-canonical form of Weyl commutation relations [24] and so may be relevant to description of real systems via Weyl quantization (and analogues of Wigner functions).

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