Avalanche Merging and Continuous Flow in a Sandpile Model

Álvaro Corral and Maya Paczuski

† The Niels Bohr Institute, University of Copenhagen, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark
‡ Department of Physics, University of Houston, Houston, TX 77204-5506, USA

A dynamical transition separating intermittent and continuous flow is observed in a sandpile model, with scaling functions relating the transport behaviors between both regimes. The width of the active zone diverges with system size in the avalanche regime but becomes very narrow for continuous flow. The change of the mean slope, \( \Delta z \), on increasing the driving rate, \( r \), obeys \( \Delta z \sim r^{1/\theta} \). It has nontrivial scaling behavior in the continuous flow phase with an exponent \( \theta \) given, paradoxically, only in terms of exponents characterizing the avalanches \( \theta = (1 + z - D)/(3 - D) \).

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Granular systems can exhibit continuous flow or intermittent avalanches depending on the driving rate. Experiments on rice piles have demonstrated that when grains are slowly added, transport of grains through the pile takes place in terms of avalanches of all sizes [3]; these rice piles exhibit self-organized criticality (SOC) [3]. Additional experiments have established that the transport process is dispersive, in the sense that the transit times of grains through the pile are broadly distributed [3]. A simple one dimensional “Oslo” model was proposed to mimic these experiments, and numerical simulations showed that the model exhibits SOC with dispersive transport [3]. This model was subsequently shown to represent a large universality class of avalanche phenomena including interface depinning, a slip-stick model for earthquakes [1], and maybe other sandpile models [3].

Although it is known that SOC can be reached only if the driving rate is very low, very little is known about the transition out of SOC as the driving rate is increased. In particular, for a given system size, there will be some driving rate at which the motion in the system never stops and the avalanches become infinite, signaling a new type of behavior which may or may not be related to the avalanche regime. Previously, Tang and Bak [6] measured the increase in average height in the BTW sandpile model [3] as the driving rate was increased. They found a power law behavior (at high rates) for the average height vs. driving rate, but did not study the system at low rates (see in addition Ref. [7]). Here we show, using the Oslo model, that there is an abrupt change in transport behaviors distinguishing two regimes, an avalanche regime and a continuous flow phase, with a different critical exponent \( \theta \) for each one. This change is associated with a pronounced contraction in the width of the active zone of transport. We utilize the active zone behavior to relate the scaling coefficients in the continuous flow phase to the exponents characterizing the avalanches in the SOC state.

The Oslo model is defined as follows: In a one dimensional system of size \( L \), an integer variable \( h(x) \) gives the height of the pile at position \( x \), and \( z(x) = h(x) - h(x+1) \) is the local slope. Grains are dropped at \( x = 1 \) with the opposite boundary open, i.e., \( h(L+1) \equiv 0 \). At each time step, all sites are tested for stability. Each unstable site with \( z(x) > z^c(x) \) topples in parallel. In a toppling event at site \( x \), \( h(x) \rightarrow h(x) - 1 \) and \( h(x+1) \rightarrow h(x+1) + 1 \). The key ingredient making this model different from previous critical slope models [3] is that the critical slopes \( z^c(x) \) are dynamical variables chosen randomly to be 1 or 2 every time a site topples. This randomness describes in a simple way the changes in the local slopes observed in the rice pile experiments [3].

We drive the model by adding grains at \( x = 1 \) at a uniform rate \( r = 1/\Delta T \), where \( \Delta T \) is the number of lattice updates separating grain additions [3]. In the limit of zero driving, corresponding to SOC, no grains are added to the pile until the avalanche resulting from adding a sand grain ends and the system reaches a stable state with \( z(x) \leq z^c(x) \) for all \( x \). Clearly, for a finite system size \( L \), well defined avalanches composed of activity separating intervals where the system returns to a metastable state are observed when the driving rate is nonvanishing but small. To be precise, for finite \( r \) an avalanche is defined as an interval separating two metastable configurations, where no topplings occur. The transition to continuous flow can be viewed as a depinning transition away from the set of metastable states, where the avalanches become infinite. In general, if the time between additions \( \Delta T \) is much larger than the mean avalanche lifetime in the SOC limit, \( \langle t(r \rightarrow 0, L) \rangle \) the system will display intermittent avalanches; on the other hand, if \( \Delta T \ll \langle t(r \rightarrow 0, L) \rangle \) the flow will never stop. Therefore, the relevant parameter for the avalanche to continuous flow transition is the ratio between these two quantities,

\[
R \propto \frac{\langle t(r \rightarrow 0, L) \rangle}{\Delta T} \sim r L^x,
\]

where \( \langle t(r \rightarrow 0, L) \rangle \sim L^x \) with \( x = 1 + z - D \approx 0.19 \) for the boundary driven model. The dynamical exponent \( z \approx 1.42 \pm 0.03 \) determines the cutoff in the lifetimes of
avalanches $t_{co} \sim L^z$ and the exponent $D \simeq 2.23 \pm 0.03$ determines the cutoff in the total number of topplings of avalanches $s_{co} \sim L^D$ in the SOC limit.

The key geometrical feature distinguishing the avalanche regime and continuous flow is the active zone, whose width is denoted as $\lambda_L(r)$ and is computed as $\lambda_L(r) \equiv \langle (h - \langle h \rangle)^2 \rangle^{1/2}$. Here $h$ refers to the height of the pile. In the SOC state the active zone diverges with system size according to a power law $\lambda_L(r) \sim L^x$ giving a rough surface for the pile, with $x = D - 2$; see Ref. [4]. One can argue that for small rates and fixed $L$ the scaling of $\lambda_L$ remains unchanged. In fact, we observe that the scaling of $\lambda_L$ remains unchanged over the entire avalanche regime. In the case of continuous flow, however, the surface is smooth and the active zone is narrow with a width that is independent of system size. Thus at high driving, $\lambda_L(r) \to \lambda(r)$. Based on this fact, we propose the following finite size scaling ansatz,

$$\lambda_L(r) \sim L^x f(r L^x),$$

where the scaling function is constant for the avalanche regime and a decreasing power law for the continuous-flow phase,

$$f(R) \propto \begin{cases} \text{constant} & \text{for } R < R_c, \\ 1/R^{\chi/x} & \text{for } R > R_c. \end{cases}$$

The exponent $\chi/x$ is obtained by imposing the independence of the active zone width on system size for fixed $r > r_c(L)$. Figure 2 supports all of these results, giving a good estimation for the roughness exponent, $\chi = 0.24 \pm 0.01$ and also $\chi/x \simeq 1.2 \pm 0.1$, in concordance with the values of $x$ and $\chi$. Notice then that the properties of the continuous-flow regime depend on the exponents characterizing the SOC limit, without apparently introducing any new critical coefficients.

These results suggest that the existence of an active zone that increases with system size is a unique feature of avalanche dynamics. In addition, the constant value of the roughness exponent $\chi$ in the avalanche regime explains the previously obtained constant value for the exponent $D$ observed in Fig. 1 and shows the validity of the scaling relation $D = \chi + 2$ for the entire avalanche regime.

We also measured the dispersive transport properties of this system, by recording the transit time of each grain, $T(r, L)$, which corresponds to the time a grain needs to travel through the entire pile. The distribution of transit times, $P(T, L, r)$, is measured to be a (decreasing) power law for long times, with an exponent $\alpha \simeq 2.2 \pm 0.1$ that does not change with the driving rate. Therefore, the scaling relation $\alpha = D$ found in Ref. [1] seems to hold also for finite rate. A finite size scaling of this distribution is obtained scaling both axis with $L^z$. 

![FIG. 1. Divergence of the mean size of the external avalanches with the rate, scaled with system size using $x = 0.20$ and $D = 2.24$. The fit parameters in Eq. (1) are $a = 1.0$, $R_c = 1.1$, $e = 1.2$, and $b = 1.64$. The inset displays the same data but versus $R_c/R$, with a double logarithmic vertical axis.]

In order to characterize the transition from intermittent avalanches to continuous flow we look at the mean size of the external avalanches $\langle s(r, L) \rangle_{out}$, which are the avalanches that drop grains outside the pile. Thus in determining the average, we only count avalanches that lead to an outflow. In the SOC limit the average external avalanche size resulting from a single grain addition is $\langle s(r \to 0, L) \rangle_{out} \sim L^D$. As we increase the driving rate we find an extra, extremely fast divergence for these large avalanches,

$$\langle s(r, L) \rangle_{out} \sim \exp \left( \frac{A}{(r_c - r)^e} \right),$$

when $r$ approaches a transition rate $r_c$. A collapse of the data is possible using the rescaled variable $r L^z$ and the scaling of $\langle s \rangle_{out}$ in the SOC limit, which gives,

$$\langle s(r, L) \rangle_{out} \sim L^D f(r L^z).$$

The consistency of this scaling ansatz is checked in Fig. 1, allowing an accurate determination of the scaling exponents, $x = 0.205 \pm 0.015$ and $D = 2.24 \pm 0.02$. Also, we show a power-law fit of the form

$$\log \frac{\langle s(r, L) \rangle_{out}}{L^D} \simeq a \left( 1 - \frac{r L^z}{R_c} \right)^{-e} - b$$

which locates the transition point at

$$r_c(L) \simeq \frac{R_c}{L^z} \text{ with } R_c \simeq 1.1 \pm 0.1,$$
At small driving rate the average transit time of a grain scales as \( \langle T \rangle \sim L^{D-1/r} \), in agreement with Ref. 3. Consistent with this limit we can consider a general crossover function for both regimes,

\[
\langle T(r, L) \rangle \sim L^z f(rL^x) ,
\]

using \( z = D - 1 + x \) with \( f(R) \propto 1/R \) for \( R < R_c \).

As a consequence of the behavior of \( \lambda_\alpha(r) \) above \( r_c(L) \), the mean transit time scales linearly with \( L \) when \( r \) is kept fixed. Considering this limit in the crossover function the scaling function must also be a decreasing power law in the continuous flow regime, i.e., \( f(R) \propto 1/R^{\alpha'} \) for \( R > R_c \), with an exponent

\[
\alpha' = \frac{z - 1}{x} .
\]

This behavior is demonstrated in Fig. 3 with \( \alpha' = 2.2 \pm 0.1 \), in agreement with the previous scaling relation.

Clearly, as the system is driven harder the mean slope tends to increase. A power law behavior \( \Delta z \sim r^{1/\theta} \) may be observed, where \( \Delta z \) is the increment of the mean slope with respect to the SOC limit, i.e., \( \Delta z(r, L) \equiv \bar{z}(r, L) - \bar{z}(r \rightarrow 0, L) \). We find that the observed power law varies depending on which regime the system operates in. Since in the SOC limit, avalanches happen instantaneously, they do not contribute to the mean slope and then \( \bar{z}(r \rightarrow 0, L) \) is the mean slope corresponding to the angle of repose of the pile. Therefore, for very small rates the mean slope will be given by \( \bar{z}(r \rightarrow 0, L) \) plus the contribution of the avalanches. As the vertical scale of the avalanches is set by \( \lambda_L \sim L^\chi \), the mean discharge in slope during an avalanche will be proportional to \( \lambda_L/L \sim 1/L^{1-\chi} \). For each addition \( \Delta T \) this contribution will have to be taken into account for a typical time \( \langle t(r \rightarrow 0, L) \rangle \) and then

\[
\Delta z(r, L) \sim \frac{\lambda_L}{L} \frac{\langle t(r \rightarrow 0, L) \rangle}{\Delta T} \sim \frac{r}{L^{1-\chi-x}} ,
\]

which means that \( \theta = 1 \). We propose a scaling ansatz

\[
\Delta z(r, L) \sim \frac{1}{L^{1-\chi}} f(rL^x) ,
\]

with \( f(R) \propto R \) for small \( R \). At \( r_c \) the profile of the pile overcomes the maximum of the active zone fluctuations of the stable pile and then avalanches never stop.

As the average transit time is linear with \( L \) for fixed \( r \) above the transition point, the grains see the same profile, independent of \( L \), no matter how large the system size. Since \( \bar{z}(r \rightarrow 0, L) \) is also independent of \( L \), for large \( L \), then \( \Delta z \) is \( L \)-independent as well. Introducing this in the scaling ansatz we get our final result

\[
f(R) \propto R^{1/\theta} \text{ for } R > R_c ,
\]

with

\[
\theta = \frac{x}{1 - \chi} = \frac{1 + z - D}{3 - D} .
\]
These results are displayed in Fig. 4 which determines $\theta^{-1} = 3.7 \pm 0.15$, that is, $\theta = 0.27 \pm 0.01$, in agreement again with the obtained scaling relation. The fact of having $\theta \neq 1$ in the continuous flow regime can be used to argue that the infinite avalanche is not a superposition of finite avalanches, in contrast with the other regime.

It is worth mentioning that we have found a similar change in the value of $\theta$ in the BTW sandpile model and in the Manna model between low and high driving rates. However, this transition is difficult to observe since in these cases the value of $\theta$ in the continuous flow region is close to 1. This reconciles the results of Tang and Bak, valid for high rates, and a claim by Grassberger and Manna in favor of $\theta = 1$. The transition we are reporting could also account for the different values of exponents reported in the literature of interface depinning. At present we are trying to extend our arguments to other models and compare the results with other approaches.

In conclusion, we have analyzed the transition from intermittent avalanches to continuous flow that appears in a sandpile model when the driving rate is finite. For a finite but arbitrarily large system, the self-organized critical state extends to finite driving rates with no change in the scaling properties associated with the roughness of the pile. Above a critical driving rate, the flow becomes continuous. We argue that it is possible to relate the scaling properties of both regimes to each other with scaling arguments that are supported by results from numerical simulations. Remarkably, the nonlinear scaling properties of the continuous flow regime, where the pile appears physically to be completely different (having a narrow active zone) depend only on critical exponents associated with avalanches in the self-organized critical state. Since the Oslo model is a good approximate description for transport and fluctuations of the profile of a real rice pile, it would be very interesting to test our theoretical predictions with experiments.

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