The CPR black hole with acceleration

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Abstract Different deformations and modifications have been proposed in the Kerr black hole solution. In the so-called non-Kerr metric a deformation function was proposed. This approach has been generalized to include two different deformation functions to obtain the CPR black hole (Cardoso et al. in Phys Rev D 89:064007, 2014). In this letter we develop the accelerating version of this spacetime and study its thermodynamics.

The theory of general relativity (GR) has been a subject of continuous tests and verifications since its discovery. Black holes are considered an ideal laboratory to perform different experiments in the strong gravitational field region. Kerr solution is expected to represent the astrophysical black holes. It is also useful to test GR in the regime of weak gravitational field. This framework is obtained by introducing some deviation parameters which help to measure deviations from the Kerr solution. These kinds of metrics must be able to describe the gravitational field that exists in the surrounding of compact objects, in any theory of gravitation, by making suitable choices of these parameters.

There exists a large number of spacetimes in literature that present vacuum solutions of some unknown field equations (which are different from Einstein’s field equations) and are axisymmetric and asymptotically flat. The non-Kerr spacetime is regular for the single deformation parameter and it does not contain any unphysical properties above the event horizon and the maximum range of spin parameters can be considered to represent a black hole. It is also useful to test the no-hair theorem near the black holes that do not explicitly depend on the field equations. This metric is further extended to include more deviation functions and electric charge [6–11].

The non-Kerr metric introduced by Johannsen and Psaltis has gained a particular importance in the gravity community. Taking first a deformed Schwarzschild black hole (multiplying the components $gtt$ and $g_{rr}$ of the Schwarzschild black hole by a function $(1 + h)$, a non-Kerr black hole which has infinite number of deviation parameters was obtained by applying Newman–Janis algorithm. This approach was extended by Cardoso et al. [7] to construct a metric having two different deformation functions in the components $g_{tt}$ and $g_{rr}$ of the Schwarzschild black hole and is named as the CPR black hole. The line element of this black hole is given by

$$
\begin{align*}
\text{ds}^2 &= -(1 + h^t) \left(1 - \frac{2Mr}{\rho^2} + \frac{q^2}{\rho^2}\right)dt^2 \\
&\quad - 2a \sin^2 \theta \left[H - (1 + h^t) \left(1 - \frac{2Mr}{\rho^2} + \frac{q^2}{\rho^2}\right)\right]dt d\phi \\
&\quad + \frac{\rho^2(1 + h^r)}{\Delta + a^2 \sin^2 \theta r^2} dr^2 + \sin^2 \theta \left[\rho^2 + a^2 \sin^2 \theta \right] d\phi^2 + \rho^2 d\theta^2,
\end{align*}
$$

(1)

where $H = \sqrt{(1 + h^t)(1 + h^r)}$, $\rho^2 = r^2 + a^2 \cos^2 \theta$, $\Delta = r^2 + a^2 - 2Mr + q^2$ and $h^t(r, \theta)$ has the general expression...
\[ h^i(r, \theta) = \sum_{k=0}^{\infty} \left( e_{2k}^i + e_{2k+1}^i \frac{M r}{\rho^2} \right) \left( \frac{M^2}{\rho^2} \right)^k, \quad i = t, r. \]  

(2)

\( M \) and \( q \) represent the mass and charge of the black hole and \( a = J/M \) is the rotation parameter, \( J \) being the angular momentum. Taking \( e_k^i = e_k^j \) for any value of \( k \) yields the charged non-Kerr metric \[7\]. From these components we observe that \( g_{tt} \) is only depending on \( h^t \) and \( g_{rr} \) contains the terms of \( h^r \) only. But \( g_{\phi\phi} \) and \( g_{\theta\theta} \) are depending on both \( h^t \) and \( h^r \) which gives a complex expression. This CPR metric reduces to the non-Kerr when the deformations are set to equal values. Kerr black hole is obtained when we set \( e_2 \) ways, \( e_2 \) and \( e_3 \) being the only nonzero parameters. For further study, they will be referred to as only \( e^t \) and \( e^r \) for convenience.

In this letter we construct an accelerating version of the CPR metric \[7\] and propose its two forms, one of them is electrically charged and the other is uncharged.

Plebaniński and Demiański introduced a spacetime in 1976 which covers a large family of electro-vacuum solutions in GR. The solution of accelerating and rotating black holes plays a vital role in this family of spacetimes. Here the parameter \( \alpha \) measures the acceleration of the black hole. This metric represents the uniformly accelerating Kerr-type black holes. If the cosmological constant is taken to be zero, the charged accelerating black hole solution is \[12–17\]

\[
d s^2 = \frac{1}{\Omega^2} \left\{ -\left( \frac{Q}{\rho^2} - \frac{a^2 P \sin^2 \theta}{\rho^2} \right) dt^2 + \frac{\rho^2}{Q} d r^2 + \frac{\rho^2}{P} d \theta^2 + \sin^2 \theta \left\{ \frac{P (r^2 + a^2)}{\rho^2} - \frac{Q a^2 \sin^2 \theta}{\rho^2} \right\} d \phi^2 \right\}
  - \frac{2 a \sin^2 \theta}{\rho^2 \Omega^2} \left[ P (a^2 + r^2) - Q \right] d t d \phi, \tag{3}
\]

where

\[
\Omega = 1 - a r \cos \theta, \tag{4}
\]

\[
\rho^2 = r^2 + a^2 \cos^2 \theta, \tag{5}
\]

\[
P = 1 - 2 a M \cos \theta + a^2 (a^2 + q^2) \cos^2 \theta, \tag{6}
\]

\[
Q = (a^2 + q^2 - 2 M r + r^2)(1 - a^2 r^2). \tag{7}
\]

Putting \( \alpha = 0 \) gives the Kerr-Newman metric. Setting \( \alpha = 0 = q \) yields the Kerr metric.

We propose the following CPR black hole spacetime which is accelerating as well:

\[
d s^2 = \frac{1}{\Omega^2} \left\{ -\left( \frac{Q}{\rho^2} - \frac{a^2 P \sin^2 \theta}{\rho^2} \right) (1 + h^t) d t^2 + \rho^2 (1 + h^r) \frac{d r^2}{Q + a^2 h^r \sin^2 \theta} + \frac{Q}{P} d \theta^2 \right\}
  + \left\{ \left( \frac{P (r^2 + a^2)}{\rho^2} - \frac{Q a^2 \sin^2 \theta (1 + h^t)}{\rho^2} \right) \sin^2 \theta \right\} d \phi^2 \right\}
  + a^2 \sin^4 \theta \left\{ 2 (H - 1) + \frac{a^2 h^r \sin^2 \theta}{\rho^2} \right\} d \phi^2 \right\}
  - \frac{2 a \sin^2 \theta}{\rho^2 \Omega^2} \left[ H - (1 + h^t) \left\{ 1 - \frac{P (a^2 + r^2) - Q}{\rho^2} \right\} \right] d t d \phi, \tag{8}
\]

where

\[
\Omega = 1 - a r \cos \theta, \tag{9}
\]

\[
\rho^2 = r^2 + a^2 \cos^2 \theta, \tag{10}
\]

\[
P = 1 - 2 a M \cos \theta + a^2 (a^2 + q^2) \cos^2 \theta, \tag{11}
\]

\[
Q = (a^2 + q^2 - 2 M r + r^2)(1 - a^2 r^2). \tag{12}
\]

When deformations vanish we recover the original spacetimes as expected. If we put \( \alpha = 0 \) and \( Q = 0 \), we obtain, respectively, the charged CPR and simple CPR black holes \[6,7\]. On the other hand, we get accelerating non-Kerr spacetime if we set \( e^t = e^r \) and simple non-Kerr by putting \( \alpha = 0 \) as well \[2,9\]. The vector potential \( A_\mu \) of the accelerating CPR is defined as

\[
A_\mu = \left( -\frac{q r}{\rho^2}, \frac{\sqrt{1 + h^t}}{\sqrt{1 + h^r}}, \frac{q r}{Q + a^2 h^r \sin^2 \theta}, 0, \frac{a q r \sin^2 \theta}{\rho^2} \right). \tag{13}
\]

The event horizon of the metric (8) is located at the solution of the equation \( g^{rr} = 0 \) giving \( Q + a^2 h^r \sin^2 \theta = 0 \) which is depending only on the deformation function \( h^r \). We denote it by \( r_+ \) and it clearly depends on the mass, charge and spin of the black hole. The redshift surface of accelerating CPR black holes is obtained from \( g^{tt} = 0 \) which has dependence only on \( h^t \) and has no new contributions. The horizon and the redshift functions have similarity with the corresponding formulas for the accelerating non-Kerr black holes \[9\].

We study horizon thermodynamics of these objects here. The surface gravity \( \kappa \) of a black hole is given as

\[
\kappa = \frac{1}{\sqrt{-h}} \frac{\partial}{\partial x^a} \left( \sqrt{-h} h^{ab} \frac{\partial \gamma}{\partial x^b} \right), \tag{14}
\]
where $h^{ab}$ denotes the inverse of the metric deduced from the $t - r$ sector of the spacetime, $h = \det h_{ab}$, $\partial/\partial x^a$ denotes the partial derivative and the indices $a, b$ take the values $0, 1$. Putting the values of $h^{11}$ and $\sqrt{-h}$ from Eq. (8), the surface gravity $\kappa_+$ at the event horizon $r_+$, takes the form

$$
\kappa_+ = \frac{\Omega^2}{2(1 + h')}\left\{ (1 - a^2 r_+^2) (2r_+ - 2M) - 2a^2 r_+ (a^2 + q^2 + r_+^2 - 2Mr_+) + a^2 e^r M^3 \sin^2 \theta \left[ \frac{1}{(r_+^2 + a^2 \cos^2 \theta)^2} - \frac{4r_+^2}{(r_+^2 + a^2 \cos^2 \theta)^3} \right] \right\}.
$$

(15)

The Hawking temperature, $T = \kappa_+/2\pi$, at the event horizon can be obtained by taking the value of $\kappa_+$ from the above equation as

$$
T = \frac{\Omega^2}{4\pi (1 + h') r_+^2} \left\{ (1 - a^2 r_+^2) (2r_+ - 2M) - 2a^2 r_+ (a^2 + q^2 + r_+^2 - 2Mr_+) + a^2 e^r M^3 \sin^2 \theta \left[ \frac{1}{(r_+^2 + a^2 \cos^2 \theta)^2} - \frac{4r_+^2}{(r_+^2 + a^2 \cos^2 \theta)^3} \right] \right\}.
$$

(16)

To discuss entropy of the black holes, the first thing to be determined is the angular velocity $\omega$, which leads to the horizon area. We have

$$
\omega = \frac{d\phi}{dt} = -\frac{8\phi}{\Omega^2}.
$$

(17)

Substituting the required metric components, the angular velocity at the event horizon becomes

$$
\omega_+ = \frac{2a [H \rho^2 - (1 + h') \{ \rho^2 - P (a^2 + r_+^2) - Q \}]}{P (a^2 + r_+^2)^2 + a^2 \sin^2 \theta \left[ 2\rho^2 (H - 1) - Q (1 + h') + a^2 h' \sin^2 \theta \right]}.
$$

(18)

The horizon area of a rotating black hole is defined [18] by $A = 4\pi a/\omega_+$. Using Eq. (18) we obtain the entropy, $S = A/4$, of the black hole as

$$
S = \frac{\pi}{2} \left( \frac{P (a^2 + r_+^2)^2 + a^2 \sin^2 \theta \left[ 2\rho^2 (H - 1) - Q (1 + h') + a^2 h' \sin^2 \theta \right]}{H \rho^2 - (1 + h') \{ \rho^2 - P (a^2 + r_+^2) - Q \}} \right).
$$

(19)

We note that $S$ increases with $M$ in accordance with the second law of black hole thermodynamics. Now, the first law of thermodynamics can be written in the form of the law of conservation of mass as [19,20]

$$
dM = \frac{\kappa_+}{8\pi} dA + \omega_+ dJ + \varphi_+ dq,
$$

(20)

where $\varphi_+$ denotes the electrostatic potential of the black hole

$$
\varphi_+ = \frac{4\pi qr_+}{A}.
$$

(21)

Substituting the area expression in the above equation, we get the electrostatic potential for the accelerating CPR black hole

$$
\varphi_+ = \frac{2qr_+ [H \rho^2 - (1 + h') \{ \rho^2 - P (a^2 + r_+^2) - Q \}]}{a \sin^2 \theta \left[ 2\rho^2 (H - 1) - Q \right] + P (a^2 + r_+^2)^2 + h' a^2 \sin^2 \theta \left[ a^2 \sin^2 \theta - Q \right]}.
$$

(22)

By substituting the values of $\kappa_+$, $\omega_+$ and $\varphi_+$ from Eqs. (15), (18) and (22) in (20), the first law takes the form

$$
dM = \frac{1}{8\pi (1 + h') r_+^2} \left\{ (1 - a^2 r_+^2) (2r_+ - 2M) - 2ar_+ (a^2 + q^2 + r_+^2 - 2Mr_+) + a^2 e^r M^3 \sin^2 \theta \left[ \frac{1}{(r_+^2 + a^2 \cos^2 \theta)^2} - \frac{4r_+^2}{(r_+^2 + a^2 \cos^2 \theta)^3} \right] \right\} dA
$$

$$
+ \frac{2a [H \rho^2 - (1 + h') \{ \rho^2 - P (a^2 + r_+^2) - Q \}]}{P (a^2 + r_+^2)^2 + a^2 \sin^2 \theta \left[ 2\rho^2 (H - 1) - Q (1 + h') + a^2 h' \sin^2 \theta \right]} dJ
$$

$$
+ \frac{2qr_+ [H \rho^2 - (1 + h') \{ \rho^2 - P (a^2 + r_+^2) - Q \}]}{a \sin^2 \theta \left[ 2\rho^2 (H - 1) - Q \right] + P (a^2 + r_+^2)^2 + h' a^2 \sin^2 \theta \left[ a^2 \sin^2 \theta - Q \right]} dq.
$$

(23)
If we put \( q = 0 \) in (8) we obtain the spacetime for the uncharged accelerating CPR black hole. The values of \( P \) and \( Q \) in this case become

\[
P = 1 - 2\alpha M \cos \theta + \alpha^2 a^2 \cos^2 \theta, \quad (24)
\]

\[
Q = (a^2 - 2Mr + r^2)(1 - \alpha^2 r^2). \quad (25)
\]

The surface gravity and the Hawking temperature at \( r_+ \) will reduce to

\[
\kappa_+ = \frac{\Omega^2}{2(1+h')r^2} \left\{ (1 - a^2 r_+^2)(2r_+ - 2M) - 2\alpha r_+(a^2 + e^2 + r_+^2 - 2Mr_+) \right\}
\]

\[

\begin{align*}
dM &= \frac{1}{8\pi(1+h')r^2} \left\{ (1 - a^2 r_+^2)(2r_+ - 2M) - 2\alpha r_+(a^2 + e^2 + r_+^2 - 2Mr_+) \right\} dA \\
&+ a^2 e' M \sin^2 \theta \left\{ \frac{1}{(r_+^2 + a^2 \cos^2 \theta)^2} - \frac{4r_+^2}{(r_+^2 + a^2 \cos^2 \theta)^3} \right\} dA \\
&+ \frac{2a}{P(a^2 + r_+^2)^2 + a^2 \sin^2 \theta} \left[ 2\rho^2 (H - 1) - Q(1 + h') + a^2 h' \sin^2 \theta \right] dJ.
\end{align*}
\]

\[
\Omega^2 = \frac{\Omega^2}{4\pi(1+h')r^2} \left\{ (1 - a^2 r_+^2)(2r_+ - 2M) - 2\alpha r_+(a^2 + e^2 + r_+^2 - 2Mr_+) + a^2 e' M^3 \sin^2 \theta \right\}
\]

\[
\times \left\{ \frac{1}{(r_+^2 + a^2 \cos^2 \theta)^2} - \frac{4r_+^2}{(r_+^2 + a^2 \cos^2 \theta)^3} \right\}.
\]

\[
T = \frac{\Omega^2}{4\pi(1+h')r^2} \left\{ (1 - a^2 r_+^2)(2r_+ - 2M) - 2\alpha r_+(a^2 + e^2 + r_+^2 - 2Mr_+) + a^2 e' M^3 \sin^2 \theta \right\}
\]

\[
\times \left\{ \frac{1}{(r_+^2 + a^2 \cos^2 \theta)^2} - \frac{4r_+^2}{(r_+^2 + a^2 \cos^2 \theta)^3} \right\}.
\]

\[
S = \frac{\pi}{2} \left( \frac{P(a^2 + r_+^2) + a^2 \sin^2 \theta \left[ 2\rho^2 (H - 1) - Q(1 + h') + a^2 h' \sin^2 \theta \right]}{H\rho^2 - (1+h') \left[ \rho^2 - P \left( a^2 + r_+^2 \right) - Q \right] \sin^2 \theta} \right).
\]

Table 1 Entropy for the accelerating CPR black hole, for the charged and uncharged cases, for some values of mass \( M \)

| \( M \)  | \( S \) (charged case) | \( S \) (uncharged case) |
|-------|-------------------------|--------------------------|
| 50    | 14,810.78637            | 14,811.17421             |
| 55    | 17,898.67605            | 17,899.47966             |
| 60    | 21,301.05788            | 21,301.86151             |
| 65    | 24,999.33684            | 25,000.14047             |
| 70    | 28,993.48780            | 28,994.29143             |

To conclude, we have formulated two deformed CPR black hole spacetimes with acceleration and studied the effects of the deviation parameters \( e', e' \) along with acceleration parameter \( \alpha \), on the structure and thermodynamics of these holes. The well-known accelerating black hole solution and its thermodynamical properties are the limiting case (when the deformation parameters vanish) of these new metrics. Further, taking \( \alpha = 0 \), all the results of Kerr black hole can be obtained.

The study of thermodynamics reveals that the black hole cannot be represented for the large values of acceleration parameter because the Hawking temperature becomes negative which violates the third law of thermodynamics. Only a small range of acceleration parameter provides the physically significant results [21,22].

The entropy behaviour is further elaborated in Table 1. It is clear from here that the presence of charge has led to a decrease in the entropy.
ics. This study can be extended by considering some other parameters like NUT parameters and cosmological constant along with the acceleration parameter $\alpha$ to analyze the effect of these deformations similar to the one presented in this study. It would be interesting to investigate particle dynamics and energy processes in the accelerating CPR spacetime presented here.

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