FIELD THEORY ASPECTS OF COSMOLOGY
AND
BLACK HOLES

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Certificate from the supervisor

This is to certify that the thesis entitled “Field theory aspects of cosmology and black holes” submitted by Sri. Kulkarni Shailesh Gajanan, who got his name registered on October 8, 2007 for the award of Ph.D. (Science) degree of Jadavpur University, is absolutely based upon his own work under the supervision of Professor Rabin Banerjee at S.N. Bose National Centre for Basic Sciences, Kolkata, India, and that neither this thesis nor any part of it has been submitted for any degree/diploma or any other academic award anywhere before.

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TO

MY MOTHER
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   This thesis is based on the papers numbered by [2,3,4,5,6,7] whose reprints are attached at the end of the thesis.
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Chapter 1

Introduction

1.1 Overview

Recently, there has been significant development in the field of cosmology. Detail study of cosmology enable us to understand the origin and ultimate fate of our universe. Research in cosmology has become astonishingly lively in the early 1980s. An idea of the cosmic inflation [1, 2, 3] (for review see [4, 5]) offered a way to understand some outstanding cosmological puzzles and provided a mechanism for the origin of large-scale structure. Results predicted by the theory of inflation could be tested by observations of anisotropies in the cosmic microwave background. In the late 1990, observations of Type Ia supernovae led to the discovery that the expansion of the universe is accelerating [6]. Further, CMB data [7] and cluster mass distribution [8] seem to favor models in which the energy density contributed by the negative pressure component should be roughly twice as much as the energy density of the matter, thus leading to the flat universe i.e the fraction of total density $\Omega_{tot} = 1$ with $\Omega_M \sim 0.4$ and $\Omega_\Lambda \sim 0.6$. Therefore the universe should be presently dominated by a smooth component with effective negative pressure; this is infact the most general requirement in order to explain the observed accelerated expansion of the universe.

Recently, it has been suggested that the change of behavior of the missing energy density, responsible for the accelerated expansion of the Universe, might be governed by a change in the equation of state of the background fluid instead of the form of the
potential, thereby avoiding the fine-tuning problems present in the above approaches. This is achieved via introduction of an exotic background fluid, the Chaplygin gas [9], described by the equation of state

\[ P = \frac{-B}{\rho^\alpha}. \]  

(1.1)

Where \( P \) and \( \rho \) are the pressure and density of the fluid in comoving frame, respectively with \( \rho > 0 \) and \( B \) is some constant. The exponent \( \alpha \) is bounded by \( 0 < \alpha \leq 1 \). When \( \alpha = 1 \), the model (1.1) is referred as standard Chaplygin gas.

S. Chaplygin introduced this equation of state [10] as a convenient soluble model to study the lifting force on a plane wing in aerodynamics. Later on, the same equations were rediscovered in [11], again in an aerodynamical context.

Apart from its applications in cosmology, the Chaplygin gas model has drawn considerable interest because of its many remarkable and intriguingly unique features. In the action formulation, the standard \( \alpha = 1 \) Chaplygin gas has very deep connection with string theory [12, 13, 14]. Indeed, it was shown in [14] that, in the light cone parameterization there is a one to one correspondence between the reparameterization invariant Nambu-Goto action for \( d \)-brane in \((d + 1, 1)\) dimensions and the Galileo invariant (nonrelativistic) action for \((d, 1)\) dimensional Chaplygin gas. While, in the Cartesian parameterization, the reparameterization invariant Nambu-Goto action for \( d \)-brane in \((d + 1, 1)\) dimensions is dual to the Poincare invariant (relativistic) action for Born-Infeld model in \((d, 1)\) dimensional spacetime [15]. In the nonrelativistic limit, \((d, 1)\) dimensional Born-Infeld action reduces to action for \((d, 1)\) dimensional Chaplygin gas. In addition to this, in the nonrelativistic decent from the Born-Infeld theory to the Chaplygin gas, there exists a mapping of one system to another, and between solutions of one system to another, because both, with the certain choice of parameterizations, reduces to the Nambu-Goto action [14]. Also, the Chaplygin gas is the only fluid which, up to now, admits a supersymmetric generalization [16, 17]. All this analysis was done for \( \alpha = 1 \) Chaplygin gas model. Thus, it is clear that the field theoretical techniques play a vital role in the understanding of some issues in the modern cosmology.

Now we focus our attention on the applications of field theory to black hole physics. Black holes are among the most fascinating predictions of Einstein’s theory of gravitation. One of the basic results of general relativity is that matter affects the spacetime geometry.
The gravitational field produced by the matter could become so strong as to substantially modify the causal structure of spacetime and eventually produce a region from which nothing can escape. A boundary of such region of spacetime is called an event horizon. Black hole is characterized by certain parameters like mass, charge, angular momentum. Classical black hole mechanics can be summarized by the following three basic laws,

1. Zeroth law : The surface gravity $\kappa$ of a black hole is constant on the horizon.

2. First law : The variations in the black hole parameters, i.e mass $M$, area $A$, angular momentum $L$, and charge $Q$, obey

$$\delta M = \frac{\kappa}{8\pi} \delta A + \Omega \delta L - V \delta Q \quad (1.2)$$

where $\Omega$ and $V$ are the angular velocity and the electrostatic potential, respectively.

3. Second law : The area of a black hole horizon $A$ is nondecreasing in time [18],

$$\delta A \geq 0 \quad (1.3)$$

These laws have a close resemblance to the corresponding laws of thermodynamics. The zeroth law of thermodynamics says that the temperature $T$ is constant throughout a system in thermal equilibrium. The first law states that in small variations between equilibrium configurations of a system, the changes in the energy $M$ and entropy $S$ of the system obey equation (1.2), if the surface gravity $\kappa$ is replaced by a term proportional to $T$ (other terms on the right hand side are interpreted as work terms). The second law of thermodynamics states that, for a closed system, entropy always increases in any (irreversible or reversible) process, i.e $\delta S \geq 0$. Jacob Bekenstein in 1973 [19] suggested that a physical identification does hold between the laws of thermodynamics and the laws of black hole mechanics. The surface gravity $\kappa$ and horizon area $A$ are identified with multiple of temperature $T$ and entropy $S$, respectively. However, he was unable to elevate this analogy to a more formal level. For example, if the correspondence among the laws of black hole mechanics and thermodynamics were true then black holes must radiate. However black holes do not radiate.

This discrepancy was successfully removed by Stephen Hawking. In 1975 he published his famous paper “Particle Creation by Black Holes” [20] where he explicitly showed that
black holes do radiate if one takes into the account the quantum mechanical nature of
matter fields in the spacetime. The key idea behind quantum particle production in
curved spacetime is that the definition of a particle is observer dependent. It depends on
the choice of reference frame. Since the theory is generally covariant, any time coordinate,
possibly defined only locally within a patch, is a legitimate choice with which to define
positive and negative frequency modes. Hawking considered a massless quantum scalar
field moving in the background of a collapsing star. If the quantum field was initially
in the vacuum state (no particle state) defined in the asymptotic past, then at late
times it will appear as if particles are present in that state. Hawking showed [20], by
explicit computation of the Bogoliubov coefficients (see also [21, 22] for detail calculation
of Bogoliubov coefficients) between the two sets of vacuum states defined at asymptotic
past and future respectively, that the spectrum of the emitted particles is identical to
that of black body with the temperature

\[ T_H = \frac{\hbar \kappa}{2\pi}, \quad (1.4) \]

known as the Hawking temperature [20]. This astonishing result is obtained using the
approximation that the matter field behaves quantum mechanically but the gravitational
field (metric) satisfy the classical Einstein equation. This semiclassical approximation
holds good for energies below the Planck scale [20]. Although it is a semiclassical result,
Hawking’s computation is considered an important clue in the search for a theory of
quantum gravity. Any theory of quantum gravity that is proposed must predict black
hole evaporation.

Apart from Hawking’s original calculation, this effect has been studied by different
methods. S. Hawking and G. Gibbons, in 1977 [23] developed an approach based on the
Euclidean quantum gravity. In this approach they computed an action for gravitational
field, including the boundary term, on the complexified spacetime. The purely imaginary
values of this action gives a contribution of the metrics to the partition function for
a grand canonical ensemble at Hawking temperature \( T_H \). Using this, they were able
to show that the entropy associated with these metrics is always equal to \( \frac{A}{4} \), where
\( A \) is an area of the event horizon. In the same year Christensen and Fulling [26], by
exploiting the structure of trace anomaly, were able to obtain the expectation value
for each component of the stress tensor \( \langle T_{\mu\nu} \rangle \), which eventually lead to the Hawking
flux. This approach is exact in $(1 + 1)$ dimensions, however in $3 + 1$ dimensions, the requirements of spherical symmetry, time independence and covariant conservation are not sufficient to fix completely the flux of Hawking radiation in terms of the trace anomaly [21, 26]. There is an additional arbitrariness in the expectation values of the angular components of the stress tensor. Another intuitive way to understand the Hawking effect was proposed independently by T. Padmanabhan, K. Srinivasan [24] and F. Wilczek, M. Parikh [25]. This approach is based on the quantum tunneling. The essential idea is that a particle-antiparticle pair forms close to the event horizon which is similar to pair formation in an external electric field. The ingoing mode is trapped inside the horizon while the outgoing mode can quantum mechanically tunnel through the event horizon. It is observed at infinity as a Hawking flux. Within the tunneling mechanism the expressions for temperature and entropy for a black hole in presence of gravitational back reaction were also computed by R. Banerjee and B. Majhi [27].

Recently, S. Robinson and F. Wilczek [28] gave a new approach to compute the Hawking flux from a black hole. This approach is based on gravitational or diffeomorphism anomaly. Basic and essential fact used in their analysis is that the theory of matter fields (scalar or fermionic) in the $3 + 1$ dimensional static black hole background can effectively be represented, in the vicinity of event horizon, by an infinite collection of free massless $1 + 1$ dimensional fields, each propagating in the background of an effective metric given by the $r - t$ sector of full $3 + 1$ dimensional metric $^{1}$. By definition the horizon is null surface and hence the region inside it is causally disconnected from the exterior. Thus, in the region near to the horizon the modes which are going into the black hole do not affect the physics outside the horizon. In other words, the theory near the event horizon acquires a definite chirality. Any two dimensional chiral theory in general curved background possesses gravitational anomaly [31]. This anomaly is manifested in the nonconservation of the stress tensor. The theory far away from the event horizon is $3 + 1$ dimensional and anomaly free and the stress tensor in this region satisfies the usual conservation law. Consequently, the total energy-momentum tensor, which is a sum of two contribution from the two different regions, is also anomalous. However, it becomes anomaly free once

$^{1}$Such a dimensional reduction of matter fields has been already used in the analysis of [29, 30] to compute the entropy of $2 + 1$ dimensional BTZ black hole.
Chapter 1. Introduction

we take into account the contribution from classically irrelevant ingoing modes. This imposes restrictions on the structure of the energy-momentum tensor and is ultimately responsible for the Hawking radiation [28]. The expression for energy-momentum flux obtained by this anomaly cancellation approach is in exact agreement with the flux from the perfectly black body kept at Hawking temperature [28]. Soon this analysis was extended to compute Hawking fluxes from the Reissner-Nordstrom (charged) black hole [32] and Kerr (rotating) black hole [33].

It is worth to note that there are certain similarities between the trace [26] and the gravitational [28] anomaly method. Both the approaches uses two inputs: the usual conservation law, and the trace [26] or gravitational [28] anomaly. Further, since the structure of trace as well as gravitational anomaly, apart from the overall multiplicative factor, is identical for different field species (e.g scalar, fermionic etc.) the methodology of two approaches would not alter for different field species. However, the analysis of [26] is restricted to 1 + 1 dimensional conformal fields. In this sense the anomaly cancellation method [28] is more appealing compared to the trace anomaly approach [26].

1.2 Outline of the thesis

This thesis, based on the work [36, 37, 38, 39, 40, 41], is focussed towards the applications of field theory, classical as well as quantum, in the context of cosmology and black holes. On the cosmology side, we study some theoretical aspects of generalized Chaplygin gas, a strong candidate for explaining the origin of accelerated expansion of the Universe. In the remaining part of the thesis, we discuss thoroughly, the relationship between the quantum gauge and gravitational anomalies and Hawking effect. We propose two different approaches, based on the covariant anomalies [37] and chiral effective actions [38], to compute the fluxes of Hawking radiation. Further, we provide a way to understand the covariant boundary condition used in the analysis of [32, 37, 38]. A connection of this boundary condition with the various vacuum states defined in the black hole spacetime is also elucidated.

\(^2\)Comparison among these two approaches and their connection with the \(W-\text{infinity}\) algebra has been discussed in detail by L. Bonora and collaborators [34, 35].
Chapter wise summary of the thesis is given below.

In chapter-2, we focus our attention on generalized Chaplygin gas model, which is considered as an alternative model for explaining the accelerated expansion of the Universe. In the nonrelativistic regime, we give the most general action for the generalized Chaplygin gas. This construction has been done in two versions. In one case the action involves the density and the velocity potential. Elimination of density is possible leading to the second version involving the velocity potential only. The form for density independent action is similar to the Born-Infeld type action in the nonrelativistic limit. In the case of relativistic generalized Chaplygin gas a Born-Infeld action involving only velocity potential is proposed which has the correct nonrelativistic limit. We also provide a form for the action of a relativistic generalized Chaplygin gas involving both density and velocity potential which also has a proper nonrelativistic limit. Our whole analysis of the generalized Chaplygin gas is consistent in the $\alpha = 1$ limit which corresponds to the standard Chaplygin gas model.

In chapter-3 we provide a derivation of Hawking radiation using covariant gauge and gravitational anomalies. Our derivation is essentially linked with the approach given in [28, 32], but with important distinctions. A crucial ingredient in the analysis of [28] is that quantum field theory in the region near the event horizon becomes two dimensional and chiral. A two dimensional chiral theory is anomalous with respect to gauge and general coordinate transformation. Such theories admit two types of anomalous currents and energy-momentum tensors - the consistent and the covariant. The covariant divergence of these currents and energy-momentum tensors yields either the consistent or the covariant form of the gauge and gravitational anomaly, respectively [42, 43]. The consistent current and anomaly satisfy the Wess-Zumino condition but do not transform covariantly under the gauge transformation. Expressions for covariant current and anomaly, on the other hand, transform covariantly under the gauge transformation but do not satisfy the Wess-Zumino condition. The covariant and consistent structures are connected by a local counterterm [42, 44]. In fact this difference between the covariant and consistent currents is the germ of the anomaly. For usual (anomaly free) theory the covariant and consistent expressions are identical. Similar conclusions also hold for the gravitational case. In [28, 32] the fluxes of Hawking radiation were obtained by cancellation of consistent gauge and gravitational anomalies. However, the analysis of [28, 32] raises several issues, both
technically and conceptually. The Hawking flux is obtained from the consistent expression for gauge and gravitational anomaly but the boundary condition, necessary to fix the form of current and energy-momentum tensor, involves the covariant form. Note that the Hawking flux is measured at infinity where there is no anomaly, so that covariant and consistent structures are identical. Hence, one can also obtain the flux from the covariant expressions. In our derivation we completely reformulate the analysis of [28, 32] totally in terms of covariant expressions leading to a simple and conceptually clean way to understand the Hawking effect.

We begin this chapter by giving a brief discussion on some aspects of gauge and gravitational anomalies highlighting the peculiarities of two dimensional spacetime. Then we compute Hawking charge and energy-momentum flux by using the covariant gauge and gravitational anomalies. Since the boundary condition involves the vanishing of covariant current and energy-momentum tensor at event horizon, all calculations involve only covariant expressions. We discuss essential differences among the consistent and covariant anomaly based methods, emphasizing the utility of our approach. Also, we show that the analysis of [28, 32] is resilient and the results are unaffected by taking more general expressions for the consistent gauge and gravitational anomalies, which occur due to peculiarities of two dimensional spacetime. We then implement our covariant anomaly approach to compute the Hawking radiation from non-trivial black hole geometries arising in the string theory. Finally, we provide an appendix discussing the dimensional reduction of real and complex scalar fields.

In chapter-4 we present a new formalism, based on the chiral effective action, to compute the Hawking fluxes from generic spherically symmetric static black hole. The expressions for current and energy-momentum tensor are obtain from the chiral effective action, suitably modified by a local counterterm. The covariant divergence of this current and energy-momentum tensor satisfy covariant gauge and gravitational anomaly, respectively. The role of chirality in imposing constraints on the structure of current and energy-momentum tensor is elucidated. The arbitrary constants appearing in the current and energy-momentum tensor are fixed by imposing the covariant boundary condition. Since the covariant gauge and gravitational anomaly vanish in the asymptotic infinity limit, we can obtain the Hawking charge and energy-momentum flux by appropriately taking the asymptotic limit of the covariant anomalous current and energy-momentum
tensor. Since this approach uses only the near horizon structure of effective action and covariant boundary condition, splitting of spacetime into two different regions and consequently the use of discontinuous step functions, as required in the earlier approaches [28, 32, 37], are not mandatory in the computation of Hawking flux. As an application of this chiral effective action approach, we compute the correction to the Hawking flux due to the effect of one loop back reaction.

One of the most important and crucial step in the anomaly [28, 32, 37] or in the chiral effective action [38] approach of computing the Hawking flux was the implementation of covariant boundary condition; namely, the vanishing of the covariant current and energy-momentum tensor at event horizon. Apart from the fact that it is covariant under the gauge or general coordinate transformation, there was no other justification or physical interpretation in favor of this boundary condition. In chapter-5 we address this issue by giving a detailed explanation for the covariant boundary condition.

We begin this exercise by giving a derivation of Hawking charge and energy-momentum flux from generic spherically symmetric static black hole. Here we adopt the technique developed in [45]. This method [45], like the chiral effective action approach, uses only the near horizon structures for the covariant currents and energy-momentum tensors and the covariant boundary condition.

Next, we use the structures of covariant current/energy-momentum tensor derived earlier from the chiral effective action, appropriately modified by the local counterterm. In order to make the chiral nature of the theory more transparent we transform the various components of current and energy-momentum tensor to null coordinates. Only one component of the covariant chiral current and energy-momentum tensor is independent, while other components get fix by chirality and trace anomaly. The independent components of current and energy-momentum tensor involve arbitrary constants, which are then fixed by imposing the condition that, a freely falling observer must see a finite amount of flux across the event horizon. This is the regularity condition and it implies that the current and energy-momentum tensor in Kruskal coordinates must be regular at future event horizon. For the chiral theory however, this is the same condition on outgoing modes in either the Unruh vacuum [46] or Hartle-Hawking vacuum [47]. The structures for currents and energy-momentum tensors obtained from this analysis are seen to be in exact
agreement with that found by solving the anomaly equations subjected to the covariant boundary condition. The fact that the Unruh and Hartle-Hawking vacua gives identical results is a consequence of chirality. This provides a clear justification for the covariant boundary condition used in determining the Hawking flux form the gauge/gravitational anomalies. Further, we compare our findings with the results derived from the conventional analysis of the various vacua states [48]. For $1 + 1$ dimensional chiral theory it is possible to give a connection between the trace and gravitational anomaly. This is explained in the appendix-5.A.

Finally, in chapter-6 we present our conclusion and outlook.
Chapter 2

Generalized Chaplygin gas

The recent observation of accelerated expansion of the universe, concluded [6] from the study of luminosity of type Ia distant supernova, has put Cosmology at the center stage. Our inability to explain the origin of this expansion has led to the naming of this phenomenon as "Dark Energy" effect. The coinage obviously matches the other fuzzy area in Cosmology, i.e. the existence of "Dark Matter". There exist several plausible models at hand that attempt to explain the astronomical data [6]. The traditional one - vacuum energy or non-zero cosmological constant - fits well with the observational data. Unfortunately it is plagued with serious conceptual difficulties: smallness of the value of the cosmological constant in comparison with Planck mass scale and the coincidence problem, (that questions the reason for the near equality between energy densities of Dark Energy and dust-like matter in the present epoch), to name a few. The latter is circumvented by introducing scalar field (or Quintessence) models [49] inducing a dynamical vacuum energy, but only at the expense of fine tuning the scalar potential parameters.

An alternative dynamical model [9] for Dark Energy, featuring Chaplygin Gas [10] or its generalization [50] - the Generalized Chaplygin Gas (GCG) - has created some interest in recent times. Conventional analysis of the model [51] allows a smooth interpolation between a dust dominated era (at early times) to the Cosmological constant dominated era (at present times). A further generalization [52] to inhomogeneous GCG model allows one to address the issue of Dark Matter as well. The GCG model has passed several experimental tests of various nature, such as high precision Cosmic Microwave Background Radiation data [53], supernova data [54] and gravitational lensing [55]. Naive
analysis [56] seemed to suggest a disturbing phenomenon in the GCG model: possible existence of unphysical oscillations or even an exponential blow-up in the matter power spectrum at present. However, this problem has been solved in [57] by taking into account the interaction between Dark Matter, Dark Energy and phantom-type Dark Energy [58]. (For a detailed exposition of these issues, see [59].)

The above mentioned ideal fluid system was introduced long ago by Chaplygin [10] as an effective model in computing the lifting force on a wing of an airplane. It obeys an exotic equation of state,

\[ P = -\frac{B}{\rho} \tag{2.1} \]

where \( P(x) \) and \( \rho(x) \) denote pressure and density respectively and \( B \) is a constant parameter. However, the interest in Chaplygin gas model actually goes beyond Cosmology (see [12, 13, 14] for a review, oriented towards the High Energy Physics community). It has a deep connection with the \( D \)-branes in a higher dimensional Nambu-Goto formulation in light-cone parameterization [15]. It is also unique in admitting a supersymmetric generalization [12] for a fluid. The dynamical role of Chaplygin gas in cosmology has been shown in [60]. The above discussion clearly underlines the relevance of GCG models in Cosmology and High Energy Physics.

In the present work, we follow the same theme and focus our attention on the GCG models, where the generalization amounts to postulating the Chaplygin equation of state as,

\[ P = -\frac{B}{\rho^\alpha}; \quad B > 0; \quad 0 < \alpha < 1 \tag{2.2} \]

where the standard Chaplygin pressure equation (2.1) is recovered for \( \alpha = 1 \). In the nonrelativistic regime, we have constructed the most general action for GCG consistent with (2.2). This construction has been done in two versions. In one case, the action involves the density \( \rho \) and the velocity potential \( \theta \). Elimination of \( \rho \) is possible leading to the second version involving only \( \theta \). This may be interpreted as a Born-Infeld type action in the nonrelativistic limit [14].

Next, the construction of relativistic action for GCG is discussed. Here a Born-Infeld action involving only \( \theta \) is proposed which has the correct nonrelativistic limit. Also for
2.1 Normal $(\alpha = 1)$ Chaplygin gas: a brief review

$\alpha = 1$, it reproduces the standard Born-Infeld action for the Chaplygin gas. We may mention that our action is different from the one given in the literature [50]. Since the density plays an important role it becomes worthwhile to write the relativistic action for GCG involving both $\rho$ and $\theta$ analogous to the usual $\alpha = 1$ case [14]. However here we are faced with certain problems. Our suggested form for GCG action has the correct nonrelativistic and $\alpha = 1$ limits. But for $\alpha \neq 1$ it is relativistic only for large $\rho$. This is found by an explicit check of the Poincare algebra.

The chapter is organized as follows: In Section-2.1 we provide a brief review of the non-relativistic and relativistic action formulations of the normal Chaplygin gas $(\alpha = 1)$. This helps us to fix the notation and charts the course of our subsequent analysis. Section-2.2 comprises an analogous study for the GCG $(\alpha \neq 1)$ and introduces new expressions for the nonrelativistic GCG action. Section-2.3 is devoted to the construction and subsequent analysis of the relativistic GCG. In Section-2.4 we provide our conclusion and propose avenues for future study.

2.1 Normal $(\alpha = 1)$ Chaplygin gas: a brief review

Before concentrating on the Chaplygin gas, let us discuss some basic notions of fluid dynamics, in the Eulerian formulation [14].

We start with the non-relativistic scenario. The equations of motion, governing an ideal fluid in arbitrary space dimensions, are given by

$$\partial_t \rho(t, x) + \nabla \cdot (\rho(t, x)v(t, x)) = 0,$$  \hspace{1cm} (2.3)

$$\partial_t v(t, x) + v(t, x) \cdot \nabla v(t, x) = f(t, x)$$ \hspace{1cm} (2.4)

where $\rho(t, x)$ and $v(t, x)$ are the matter density and velocity fields respectively. The first identity reflects the matter conservation and the second is the Euler equation of motion. We consider the motion of fluid to be isentropic. Hence the force $f$ is derived from a $\rho$-dependent potential $V(\rho)$,

$$f = -\frac{1}{\rho} \nabla P.$$ \hspace{1cm} (2.5)
Here \( P \) is the pressure. For isentropic motion \( P \) is a function of \( \rho \) only. Hence we can write (2.5) as
\[
f = -\nabla V'(\rho).
\]
(2.6)
prime denotes derivative with respect to \( \rho \). Note that \( V'(\rho) \) is the enthalpy, given by,
\[
P(\rho) = \rho V'(\rho) - V(\rho).
\]
(2.7)
In the case of irrotational fluid further simplification occurs. For this case, the vorticity vanishes, which implies
\[
v = \nabla \theta.
\]
(2.8)
where \( \theta(x, t) \) is some scalar field. The (non-relativistic) Hamiltonian for irrotational motion is just the sum of kinetic and potential energy,
\[
H = \int dr (\frac{1}{2} \rho (\partial_i \theta)^2 + V(\rho)).
\]
(2.9)
Now, the first order form of the Lagrangian \( L \), corresponding to (2.9) is given by,
\[
L = \int dr (\theta \dot{\rho} - \frac{1}{2} \rho (\partial_i \theta)^2 - V(\rho)).
\]
(2.10)
From the symplectic structure it is clear that \( \rho \) and \( \theta \) are conjugate variables, satisfying the canonical Poisson bracket,
\[
\{\theta(x), \rho(y)\} = \delta(x - y).
\]
(2.11)
The nature of the potential function \( V(\rho) \) will specify the particular fluid model under study. For Chaplygin gas the potential profile is given by,
\[
V(\rho) = \frac{\lambda}{\rho}
\]
(2.12)
where \( \lambda \) is the interaction strength. Using (2.12) the Lagrangian for the Chaplygin gas model is given by
\[
L = \int dr (\theta \dot{\rho} - \frac{1}{2} \rho (\partial_i \theta)^2 - \frac{\lambda}{\rho}).
\]
(2.13)
Varying \( L \) with respect to \( \rho \), yields
\[
\dot{\rho} + \frac{1}{2} (\partial_i \theta)^2 = \frac{\lambda}{\rho^2}.
\]
(2.14)
This the Bernoulli equation.

It is possible to eliminate $\rho$ from the Lagrangian to obtain a non-relativistic Born-Infeld like structure in $\theta$,

$$L(\theta) = -2\sqrt{\lambda} \int dr \sqrt{\left(\dot{\theta} + \frac{1}{2}(\partial_i \theta)^2\right)}$$

with the equation of motion,

$$\partial_t \left( \dot{\theta} + \frac{1}{2}(\partial_i \theta)^2 \right)^{-\frac{1}{2}} + \partial_i \left[ \partial_i \theta \left( \dot{\theta} + \frac{1}{2}(\partial_i \theta)^2 \right)^{-\frac{1}{2}} \right] = 0.$$  \hspace{1cm} (2.16)

Now we come to the relativistic generalization of Chaplygin gas\cite{14}. A Lagrangian has been suggested for the normal ($\alpha = 1$) Chaplygin gas in \cite{14},

$$L = \int dr (\partial_\mu \dot{\theta} - \sqrt{\rho^2 c^2 + a^2}) \sqrt{c^2 + (\partial_\mu \theta)^2},$$

where $a$ is a interaction strength. Although (2.17) does not have a manifestly relativistic form, its Poincare invariance has been demonstrated explicitly in \cite{60} in a Hamiltonian framework (see section-2.3 of the present chapter, also). On the other hand, one can eliminate $\rho$ once again from (2.17) to obtain the Born-Infeld form,

$$L = -a \int dr \sqrt{c^2 - \partial_\mu \theta \partial^\mu \theta}$$

which is manifestly relativistic.

To get the correct non-relativistic limit, one has to consider the map \cite{14},

$$\theta \rightarrow \theta - tc^2.$$  \hspace{1cm} (2.19)

Under this transformation, the relativistic model (2.17) will reduce to the non-relativistic one in (2.13) with the identification $\lambda \equiv \frac{a^2}{2}$. This concludes our review \cite{14} of action formulation of Chaplygin gas.

\section{2.2 Nonrelativistic generalized ($\alpha \neq 1$) Chaplygin gas}

In GCG the equation of state (2.1) is replaced by a more flexible one, given in (2.2). In order to incorporate this generalization in the action formulation, our starting point is
to find a suitable potential \(V(\rho)\) compatible with (2.2). To find the general solution for \(V(\rho)\) we start from an ansatz,

\[
V(\rho) = \left(\frac{B}{\alpha + 1}\right) \frac{1}{\rho^{\alpha}} + u(\rho) \tag{2.20}
\]

where \(u(\rho)\) is such that \(V(\rho)\) satisfies

\[
-\frac{B}{\rho^{\alpha}} = \rho \frac{dV(\rho)}{d\rho} - V(\rho). \tag{2.21}
\]

This follows from (2.2) and the enthalpy relation (2.7). This implies \(u(\rho)\) must satisfy

\[
\rho \frac{du}{d\rho} - u = 0. \tag{2.22}
\]

The solution for above equation is

\[
u(\rho) = I\rho. \tag{2.23}\]

where \(I\) is an integration constant.

Hence the most general form of the potential \(V(\rho)\) is

\[
V(\rho) = \left(\frac{B}{\alpha + 1}\right) \frac{1}{\rho^{\alpha}} + I\rho. \tag{2.24}
\]

For the irrotational fluid we can write the Hamiltonian (2.9), with \(V(\rho)\) as given in (2.24)

\[
H = \int dr \left(\frac{1}{2} \rho (\partial_i \theta)^2 + \frac{B}{(\alpha + 1)\rho^{\alpha}} + I\rho\right). \tag{2.25}
\]

Now the first order form of the Lagrangian follows from (2.25),

\[
L^\alpha = \int dr \left[\dot{\theta} \rho - \frac{1}{2} \rho (\partial_i \theta)^2 - \frac{B}{(\alpha + 1)\rho^{\alpha}} - I\rho\right] \tag{2.26}
\]

where the superscript \(\alpha\) on \(L\) reveals the fact that we are dealing with GCG.

Variation of \(\rho\) yields the Bernoulli equation,

\[
\dot{\theta} + \frac{1}{2} (\partial_i \theta)^2 = \frac{B\alpha}{(\alpha + 1)\rho^{\alpha+1}} - I. \tag{2.27}
\]

To obtain the \(\rho\) independent Lagrangian for GCG, one can use the Bernoulli equation to reexpress \(\rho\) in terms of \(\theta\),

\[
\rho = \left[\frac{\alpha B}{\alpha + 1} \left(\dot{\theta} + \frac{1}{2} (\partial_i \theta)^2 + I\right)^{-1}\right]^\frac{1}{\alpha+1}. \tag{2.28}
\]
It is very convenient to rewrite the Lagrangian given in (2.26) in the following form

\[ L^\alpha = - \int dr \left( \dot{\theta} \rho + \frac{\rho}{2} (\partial_i \theta)^2 + \frac{B}{(\alpha + 1)\rho^\alpha} + I\rho \right). \]  

(2.29)

In the above equation we have omitted total derivative terms. Substituting \( \rho \) from (2.28) in (2.29) we find,

\[ L^\alpha(\theta) = - \left( \frac{\alpha}{\alpha + 1} \right) \frac{\alpha^\alpha}{\alpha + 1} B^{\frac{1}{\alpha + 1}} \int dr \sqrt{\left( \frac{\dot{\theta}}{2} (\partial_i \theta)^2 + I \right)^{\frac{2\alpha}{\alpha + 1}}}. \]  

(2.30)

This is the most general form of GCG Lagrangian and is a central result of our analysis. It is the Born-Infeld version of nonrelativistic GCG.

The equation of motion for \( \theta \) turns out to be,

\[ \partial_t \left( \frac{\dot{\theta}}{2} (\partial_i \theta)^2 \right) + \partial_i \left[ \partial_i \theta \left( \frac{\dot{\theta}}{2} (\partial_i \theta)^2 + I \right)^{\frac{1}{\alpha + 1}} \right] = 0. \]  

(2.31)

A definite simplification occurs by setting \( I = 0 \). Then the Lagrangians (2.26) and (2.30) reduce to

\[ L^\alpha = \int dr \left( \dot{\theta} \rho - \frac{1}{2} \rho (\partial_i \theta)^2 - \frac{B}{(\alpha + 1)\rho^\alpha} \right) \]  

(2.32)

and

\[ L^\alpha(\theta) = - \left( \frac{\alpha}{\alpha + 1} \right)^{\frac{\alpha^\alpha}{\alpha + 1}} B^{\frac{1}{\alpha + 1}} \int dr \sqrt{\left( \frac{\dot{\theta}}{2} (\partial_i \theta)^2 \right)^{\frac{2\alpha}{\alpha + 1}}}. \]  

(2.33)

Putting \( \alpha = 1 \) in (2.32) and (2.33) reproduces the expressions for the usual Chaplygin gas [14].

2.3 Relativistic generalized Chaplygin gas

Now we turn to the relativistic form of GCG. Any relativistic version of GCG must satisfy two conditions: it should have the correct nonrelativistic limit (2.26) or (2.30), secondly, for \( \alpha = 1 \) it should reduce to (2.13) or (2.15).

To begin with we suggest a manifestly Poincare invariant model for GCG, given by

\[ L^\alpha = -(a')^{\frac{1}{\alpha + 1}} \int dr \sqrt{(c^2 - \partial_{\mu} \theta \partial^\mu \theta)^{\frac{2\alpha}{\alpha + 1}}}. \]  

(2.34)
Chapter 2. Generalized Chaplygin gas

In the nonrelativistic limit it agrees with (2.33). To show this we exploit (2.19) and use the fact that \( \partial_\mu \theta \partial^\mu \theta = \frac{\dot{\theta}^2}{c^2} - \partial_\mu \theta^2 \), to simplify the above Lagrangian,

\[
L^\alpha = -(a')^\frac{1}{\alpha+1} \int dr \left[ -\frac{\dot{\theta}}{c^2} + 2\dot{\theta} + (\partial_\mu \theta)^2 \right]^\frac{\alpha}{\alpha+1}.
\]  

(2.35)

Now taking the large \( c \) limit we get

\[
\lim_{c \to \infty} L^\alpha = -(2)^\frac{\alpha}{\alpha+1} (a')^\frac{1}{\alpha+1} \int dr \sqrt{\left[ \dot{\theta} + \left( \frac{\partial_\mu \theta}{2} \right)^2 \right]^\frac{2\alpha}{\alpha+1}}.
\]  

(2.36)

After making the identification

\[
a' \equiv \left( \frac{\alpha}{2(\alpha+1)} \right)^\alpha B
\]  

(2.37)

we see that (2.36) agrees with (2.33). Also, in the \( \alpha = 1 \) limit our Lagrangian (2.34) reduces to that of usual relativistic Chaplygin gas (2.18). This shows that it is possible to interpret (2.34) as a viable form for the relativistic GCG Lagrangian.

At this point we should mention that there exists in the literature a Poincare invariant form for GCG [50]

\[
L_b = -A^\frac{1}{1+\alpha} \int dr \sqrt{\left[ c^2 - \left( \partial_\mu \theta \partial^\mu \theta \right)^{\frac{1+\alpha}{2\alpha}} \right]^\frac{\alpha}{\alpha+1}}.
\]  

(2.38)

Note that, for \( \alpha \neq 1 \) the above Lagrangian is different from (2.34). However for \( \alpha = 1 \) it agrees with normal Chaplygin gas Lagrangian [14]. GCG of similar nature [50] coupled to gravity has been considered in [61].

Now consider the nonrelativistic limit of (2.38). Following the same procedure as discussed above we get in this limit

\[
L_b = -A^\frac{1}{1+\alpha} (2Y)^\frac{\alpha}{\alpha+1} \sqrt{\left[ \frac{Z}{2Y} + \left( \dot{\theta} + \frac{1}{2} \nabla \theta^2 \right)^{\frac{2\alpha}{1+\alpha}} \right]}.
\]  

(2.39)

where

\[
Y = \frac{1 + \alpha}{2\alpha} c^{\frac{1+\alpha}{\alpha}},
\]

(2.40)

\[
Z = c^2 - c^{\frac{1+\alpha}{\alpha}}.
\]

(2.41)

This Lagrangian is same as that of (2.30) provided we identify \( I \) with \( \frac{Z}{2Y} \). Thus both (2.34) and (2.38) are valid forms for the relativistic GCG whose nonrelativistic limits
2.3. Relativistic generalized Chaplygin gas

correspond to different parameterizations of the general form for nonrelativistic GCG given in (2.30).

Let us next attempt to construct the relativistic GCG model by including the density field \( \rho \). Also, since the density field plays an important role in the observational analysis of GCG it is worthwhile to have a relativistic version for GCG involving \( \rho \) and the velocity potential \( \theta \).

To this end, we consider the following Lagrangian for relativistic GCG:

\[
L^\alpha = \int dr \left( \theta \dot{\rho} - \sqrt{\left( \rho^2 c^2 + \frac{a^2}{\rho^{\alpha - 1}} \right) c^2 + (\partial_i \theta)^2} \right)
\]

(2.42)

where \( a \) is a constant parameter. To ensure the correct nonrelativistic limit we use the same map as (2.19), and explicitly check that in the \( c \to \infty \) limit, the nonrelativistic GCG model (2.32) is reproduced, provided we identify,

\[
a = \sqrt{\frac{2B}{\alpha + 1}}.
\]

(2.43)

We put \( c = 1 \) and obtain the equations of motion,

\[
\dot{\rho} + \partial_i \left( \frac{\sqrt{\left( \rho^2 c^2 + \frac{a^2}{\rho^{\alpha - 1}} \right)}}{\sqrt{1 + (\partial_i \theta)^2}} \partial_i \theta \right) = 0,
\]

(2.44)

\[
\dot{\theta} = -\frac{\sqrt{1 + (\partial_i \theta)^2}}{\sqrt{\left( \rho^2 c^2 + \frac{a^2}{\rho^{\alpha - 1}} \right)}} \left[ \rho c^2 - \left( \frac{\alpha - 1}{2} \right) \frac{a^2}{\rho^\alpha} \right].
\]

(2.45)

They also have the correct \( \alpha = 1 \) limit [14].

As we have pointed out before, the Lagrangian (2.42) has been posited by us in analogy with the relativistic Lagrangian given in (2.17) [14]. Also we show that (2.42) has the correct \( \alpha = 1 \) limit. Since the model (2.42) is not manifestly Lorentz invariant, it becomes imperative to check the Poincare algebra. To this end, we follow the method discussed in [60] and compute the canonical energy-momentum tensor \( T_{\mu\nu} \) (in the Noether prescription),

\[
T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial (\partial^\mu \psi_i)} \partial_\nu \psi^i - g_{\mu\nu} \mathcal{L}.
\]

(2.46)
Using the above definition, the explicit form of the components of $T_{\mu\nu}$ are given by,

\[ T_{00} = \sqrt{(\rho^2 + \frac{a^2}{\rho^{\alpha-1}})} \sqrt{1 + (\partial_i \theta)^2}, \]  
\[ T_{0i} = \theta \partial_i \rho, \]  
\[ T_{i0} = -\frac{\sqrt{\rho^2 + \frac{a^2}{\rho^{\alpha-1}}}}{\sqrt{1 + (\partial_i \theta)^2}}(\partial_i \theta)\dot{\theta} = \left( \rho(x) + \left( \frac{1 - \alpha}{2} \right) \frac{a^2}{\rho^\alpha(x)} \right) \partial_i \theta, \]  
\[ T_{ij} = -\frac{\sqrt{\rho^2 + \frac{a^2}{\rho^{\alpha-1}}}}{\sqrt{1 + (\partial_i \theta)^2}}(\partial_i \theta)(\partial_j \theta) - g_{ij} \mathcal{L}^\alpha. \]  

Notice that $T_{0i} \neq T_{i0}$. Using the equations of motion (2.44), (2.45) one can explicitly verify the conservation law,

\[ \partial^\mu T_{\mu\nu} = 0. \]  

Hence $T_{\mu\nu}$ is a conserved but non-symmetric energy-momentum tensor.

Once we have the forms of $T_{00}$ and $T_{0i}$ we can easily obtain the expression for the momenta $P_\mu$ and the angular momenta $M_{\mu\nu}$. They are related to the components of the energy-momentum tensor as

\[ P_\mu = \int d^3x \ T_{0\mu}, \]  
\[ M_{\mu\nu} = \int d^3x \ (T_{0\mu} x_\nu - T_{0\nu} x_\mu). \]

By using (2.47) and (2.48) we get,

\[ P_0 = \int d^3x \ \sqrt{(\rho^2 + \frac{a^2}{\rho^{\alpha-1}})} \sqrt{1 + (\partial_i \theta)^2}, \]  
\[ P_i = \int d^3x \ \theta \partial_i \rho, \]  
\[ M_{0i} = \int d^3x \ \sqrt{(\rho^2 + \frac{a^2}{\rho^{\alpha-1}})} \sqrt{1 + (\partial_i \theta)^2} \ x_i - \theta \partial_i \rho \ x_0, \]  
\[ M_{ij} = \int d^3x \ (\theta \partial_i \rho \ x_j - \theta \partial_j \rho \ x_i). \]

Using the Poisson bracket (2.11) we are able to compute the following algebra,

\[ \{M_{ij}, M_{kl}\} = (g_{jk}M_{il} - g_{ik}M_{jl} - g_{il}M_{kj} + g_{jl}M_{ki}), \]  
\[ \{M_{ci}, M_{kl}\} = (g_{ik}M_{cl} - g_{il}M_{ck}), \]  
\[ \{M_{0i}, M_{0j}\} = -g_{00} \int d^3x \ (\theta \partial_i \rho \ x_j - \theta \partial_j \rho \ x_i) \left( 1 - \alpha \left( \frac{1}{2} \right) \frac{a^2}{\rho^{\alpha+1}} \right). \]
Similarly the algebra between $P_{\mu}M_{\mu\nu}$ is given by

\begin{align}
\{M_{0i}, P_j\} &= P_0 g_{ij}, \quad (2.61) \\
\{M_{ij}, P_k\} &= g_{jk} P_i - g_{ik} P_j, \quad (2.62) \\
\{M_{0i}, P_0\} &= -g_{00} \int d^3 x \theta \partial_i \rho \left( 1 - \alpha \left( \frac{1 - \alpha}{2} \right) \frac{a^2}{\rho^{\alpha+1}} \right). \quad (2.63)
\end{align}

Finally, the algebra between $P_{\mu}P_{\nu}$ is found out to be,

\[ \{P_{\mu}, P_{\nu}\} = 0. \quad (2.64) \]

Concentrate on the two Poisson brackets (2.60) and (2.63). We find that for $\alpha = 1$ the complete Poincare algebra is satisfied. This corresponds to the usual Chaplygin Poincare algebra [60]. However for $\alpha \neq 1$ (which corresponds to the GCG model) the Poincare algebra closes only in the large density limit ($\rho >> 1$). It is, however, reassuring to note that the Schwinger condition,

\[ \{T_{00}(x), T_{00}(y)\} = (T_{i0}(x) + T_{i0}(y)) \partial_i \delta(x - y), \]

is satisfied for any $\alpha$ and $\rho$.

### 2.4 Discussions

To conclude, we have studied various aspects of the Generalized Chaplygin Gas (GCG) models. In the nonrelativistic regime, we have constructed a general form of the Lagrangian for GCG, that obeys the generalized equation of state. Different parameterizations of this master Lagrangian yield different *inequivalent* models for GCG, such as the one studied here and the one in [50]. In this sense the construction of nonrelativistic GCG is not unique. Naturally, the same conclusion extends for a relativistic formulation of GCG.

For the relativistic scenario, we have proposed a Born-Infeld like model for GCG, which in the nonrelativistic limit, reduces to the conventional GCG. However, unlike the usual $\alpha = 1$ Chaplygin gas case, the construction of a relativistic GCG model, including both density field and velocity potential is nontrivial. In this context, our model reduces to the usual one, quoted in literature [14] for $\alpha = 1$ and also has the correct nonrelativistic limit. However the Poincare algebra closes only in the limit of large density.
Chapter 3

Hawking fluxes from covariant gauge and gravitational anomalies

Hawking effect provides an important step towards understanding the quantum aspects of black holes. Specifically, it arises in a background spacetime with event horizons. Hawking studied quantum effects of matter in the black hole background formed during collapse and concluded that the black hole emits thermal radiation as if it was a black body at a temperature proportional to the surface gravity of the black hole [20]. The radiation emitted from the black holes has a spectrum with Planck distribution. Hawking’s original derivation [20] was based on the computation of Bogoliubov coefficients between ‘in’ and ‘out’ states. Apart from this derivation there are several ways to compute the flux of thermal radiation emitted by the black hole [23, 24, 25, 26], each having its own merits and demerits. This has led to open problems leading to alternative approaches with fresh insights. In this chapter we would discuss a different approach to derive the Hawking flux from a black hole. This approach is based on the covariant gauge and gravitational anomalies.

A relationship between gravitational anomalies and Hawking radiation was first noted by S. Robinson and F. Wilczek [28]. They considered quantum scalar fields propagating on the (3 + 1) dimensional Schwarzschild black hole background. Their analysis rests on the fact that quantum field theory in the region near to the event horizon can effectively be described by a (1 + 1) dimensional [29, 30] chiral [28] theory. Any 2-dimensional chiral theory on a curved background possesses gravitational anomaly [31].
In the region away from the horizon, the theory is still $3 + 1$ dimensional and also usual (anomaly free). The energy flux of the Hawking radiation, which is necessary to cancel the anomaly present near the horizon, was then computed by solving the anomalous as well as usual conservation laws in the respective regions together with implementation of certain boundary conditions. This method is expected to hold in any dimensions. In this sense it is distinct from the approach given by Christiansen and Fulling [26], where the form for energy-momentum tensor of massless quantum field in a $(1 + 1)$ dimensional black hole background was obtained by exploiting the structure of trace anomaly. The flux obtained via trace anomaly is in quantitative agreement with Hawking’s original result. However, the masslessness of the quantum fields and also limitation to $(1 + 1)$ dimensional black holes are quite essential ingredients in this analysis. The approach of [28] was also applied to compute the Hawking charge and energy flux coming from the Reissner-Nordstrom black hole [32]. Further advances and application of this anomaly cancellation approach may be found in a host of papers ([62]-[72]).

However, an unpleasant feature of [28, 32] was that whereas the expressions for chiral anomalies were taken to be consistent, the boundary conditions required to fix the arbitrary constants were covariant. In this chapter we present a derivation that is solely based on covariant expressions. The expressions for the covariant anomalous currents and energy-momentum tensors, in the region near to the event horizon, are obtained by solving the covariant anomalous gauge and gravitational Ward identities, respectively. The arbitrary constants appearing in these expressions are fixed by imposing the covariant boundary condition: namely, the vanishing of covariant current and energy-momentum tensor at the event horizon. On the other hand the corresponding expressions for the current and energy-momentum in the region far away from the horizon are derived by solving the usual conservation laws. The charge/energy-momentum flux is then obtained by demanding that the total current/energy-momentum tensor of the theory must be anomaly free. The charge and energy fluxes obtained by our anomaly approach matches with the standard expression of Hawking flux [20, 21]. Further, as a side calculation, we show that the analysis of [28, 32] is resilient and the results are unaffected by taking more general expressions for the consistent anomaly which occur due to peculiarities of two dimensional spacetime.

This chapter is organized in the following manner. Section-3.1 discusses the general
3.1 General discussion on covariant and consistent anomalies

Symmetries play an important role in physics in general and in quantum field theory in particular. A continuous symmetry of the classical action is a transformation of the fields that leaves the action invariant. Corresponding to each such symmetry operation there exist a conserved charge. This is the Noether’s theorem. Standard examples are Lorentz, or more generally Poincare transformations, and gauge transformations in gauge theories. In the functional integral formulation of quantum field theory, symmetries of the classical action are easily seen to translate into the Ward identities for the correlation functions computed from the quantum effective action. Naturally, it becomes important to know whether a certain classical symmetry is still valid in the quantum theory.

An anomaly in quantum field theory is a breakdown of some classical symmetry due to the process of quantization. This surprising feature of quantum theory plays a fundamental role in physics (for reviews, see [42, 44, 73, 74, 75, 76]). Specifically, for instance,
a gauge anomaly is an anomaly in gauge symmetry, taking the form of nonconservation of the gauge current. Such anomalies characterize a theoretical inconsistency, leading to problems with the probabilistic interpretation of quantum mechanics. The cancellation of gauge anomalies gives strong constraints on model building. Likewise, a gravitational anomaly [31, 43] is an anomaly in general coordinate invariance, taking the form of nonconservation of the energy-momentum tensor. There are other types of anomalies but here we shall be concerned with only gauge and gravitational anomalies. The simplest case for these anomalies, which is also relevant for the present analysis, occurs for 1 + 1 dimensional chiral fields.

In this section we would discuss some important aspects of gauge and gravitational anomalies in (1 + 1) dimensions. In general, the anomalous theories admit two types of currents and energy-momentum tensors; the consistent and the covariant [42, 43, 44, 78]. The covariant divergence of these currents and energy-momentum tensors yields either consistent or covariant gauge and gravitational anomalies, respectively [31, 42, 43, 44, 77, 78, 79]. The consistent current and anomaly satisfy the Wess-Zumino integrability condition but do not transform covariantly under a gauge transformation. Expressions for covariant current and anomaly, on contrary, transform covariantly under gauge the transformation but do not satisfy the Wess-Zumino integrability condition. Similar conclusions also hold for the gravitational case, except that currents are now replaced by energy-momentum tensors and gauge transformations by general coordinate transformations. The consistent and covariant of currents (energy-momentum tensors) are interrelated by a local counterterm.

For simplicity, we mainly focus our attention on the gauge anomaly. For the gravitational case we just quote some basic results which shall be important in discussing the Hawking effect.

Consider the chiral (Weyl) fermions moving in the presence of external abelian gauge field \( A_\mu \) on a flat 1 + 1 dimensional spacetime. The action for this chiral theory is

\[
S = - \int d^2 x \bar{\Psi} \gamma^\mu \left( \partial_\mu - i A_\mu \frac{1 + \gamma_5}{2} \right) \Psi \tag{3.1}
\]

where \(+(-)\) corresponds to the left and right moving fermions, respectively. In the Minkowski space the Dirac matrices satisfy

\[
(i\gamma^0)^\dagger = i\gamma^0 ; \quad (\gamma^1)^\dagger = \gamma^1 ; \quad \gamma_5^\dagger = \gamma_5 . \tag{3.2}
\]
The chiral gauge current, derived from (3.1) is

$$ J_\mu = i \bar{\Psi} \gamma_\mu \frac{1 \pm \gamma_5}{2} \Psi . $$

(3.3)

On using the equations of motion for $\Psi$ and $\bar{\Psi}$ we can easily show that the chiral current given above is conserved

$$ \partial_\mu J^\mu = 0 . $$

(3.4)

Also, it transform covariantly under the chiral gauge transformations,

$$ \Psi(x) \rightarrow \exp \left( i \alpha(x) \frac{1 \pm \gamma_5}{2} \right) \Psi(x) $$

(3.5)

$$ \bar{\Psi}(x) \rightarrow \bar{\Psi}(x) \exp \left( -i \alpha(x) \frac{1 \pm \gamma_5}{2} \right) $$

(3.6)

$$ A_\mu(x) \rightarrow \exp(i \alpha(x))[A_\mu - i \partial_\mu] \exp(-i \alpha(x)) , $$

(3.7)

However, when we quantize the theory described by (3.1), the regularized current $\langle J_\mu(x) \rangle$ does not conserve. In fact, it satisfies

$$ \partial_\mu \langle J^\mu(x) \rangle = \pm G $$

(3.8)

where $G$ is the chiral abelian gauge anomaly. The explicit form for $G$ depends upon how we regularize $\langle J_\mu \rangle$. Let $W$ be the quantum effective action for the chiral theory, defined as

$$ e^{iW[A]} = \int [D\Psi D\bar{\Psi}] e^{iS[\Psi, \bar{\Psi}, A_\mu]} $$

(3.9)

where $S$ is the classical action (3.1). The current $\langle \tilde{J}_\mu(x) \rangle$ obtained by taking the functional derivative of the quantum effective action, i.e

$$ \langle \tilde{J}^\mu(x) \rangle = \frac{\delta}{\delta A_\mu(x)} W $$

(3.10)

does not transform covariantly under the chiral gauge transformations (3.5, 3.6, 3.7). Rather, it satisfies the Wess-Zumino integrability condition [78, 79].

$$ \frac{\delta \langle \tilde{J}_\mu(x) \rangle}{\delta A^\nu(x')} = \frac{\delta \langle \tilde{J}_\nu(x') \rangle}{\delta A^\mu(x)} . $$

(3.11)
This current is called the consistent current \([42, 78]\). Divergence of the consistent current yields the consistent anomaly

\[
\partial_\mu \langle \tilde{J}^\mu \rangle = \pm \frac{e^2}{4\pi} \epsilon^{\mu\nu} \partial_\mu A_\nu .
\]  

(3.12)

where \(\epsilon^{\mu\nu}\) is the numerical antisymmetric tensor with

\[
\epsilon^{01} = -\epsilon^{10} = 1 .
\]

(3.13)

The relation (3.12) can be easily extended to a general curved background defined by the metric \(g_{\mu\nu}\) by replacing partial derivative with the covariant derivative compatible with the metric \(g_{\mu\nu}\) and \(\epsilon^{\mu\nu}\) by \(\bar{\epsilon}^{\mu\nu}\)

\[
\nabla_\mu \langle \tilde{J}^\mu \rangle = \pm \frac{e^2}{4\pi} \bar{\epsilon}^{\mu\nu} \partial_\mu A_\nu .
\]

(3.14)

where

\[
\bar{\epsilon}^{\mu\nu} = \frac{\epsilon^{\mu\nu}}{\sqrt{-g}} .
\]

(3.15)

The structure appearing in (3.14) is the minimal form, since only odd parity terms occur. However it is possible that normal parity terms appear in (3.14). Indeed, as we now argue, such a term is a natural consequence of two dimensional properties. To emphasize this point we note that in \(1 + 1\) dimensions, \(\gamma_\mu\) satisfy

\[
\gamma_5 \gamma^\mu = -\frac{\epsilon^{\mu\nu}}{\sqrt{-g}} \gamma_\nu .
\]

(3.16)

Using this it is found that the gauge field \(A_\mu\) couples as a chiral combination \((g^{\mu\nu} \pm \epsilon^{\mu\nu})A_\nu\). Hence the expression for the consistent anomaly in (3.14) generalizes to

\[
\nabla_\mu \langle \tilde{J}^{\mu} \rangle = \pm \frac{e^2}{4\pi} \nabla_\alpha \left[ (\epsilon^{\alpha\beta} \pm g^{\alpha\beta}) A_\beta \right] .
\]

(3.17)

This is the nonminimal form for the consistent gauge anomaly dictated by the symmetry of the Lagrangian, and has already appeared earlier in the literature \([43]\). The current \(\langle \tilde{J}^{\mu} \rangle\) is related to \(\langle \tilde{J}^\mu \rangle\) as

\[
\langle \tilde{J}^{\mu} \rangle = \langle \tilde{J}^\mu \rangle + \frac{e^2}{4\pi} A_\mu .
\]

(3.18)

so that the covariant divergence of \(\langle \tilde{J}^{\mu} \rangle\) yields the nonminimal form of the consistent gauge anomaly (3.17). Also, note that the \(\langle \tilde{J}^{\mu} \rangle\) is the consistent current since the extra piece satisfies the Wess-Zumino integrability condition (3.11).
As mentioned earlier, value of the anomaly (3.8) depends upon the regularization prescription. If the current \( \langle J^\mu(x) \rangle \) is defined by a gauge invariant regularization then it transform covariantly under the chiral gauge transformations (3.5, 3.6, 3.7). Let us denote this current by \( \langle \hat{J}^\mu \rangle \),  then

\[
\langle \hat{J}^\mu(x) \rangle \rightarrow e^{-i\alpha} \langle \hat{J}^\mu(x) \rangle e^{i\alpha} .
\]  

This defined in such a way is called the covariant current. Divergence of the covariant current yields the covariant chiral gauge anomaly [42, 78, 79]

\[
\partial_\mu \langle \hat{J}^\mu \rangle = \pm \frac{e^2}{4\pi} \epsilon^{\alpha\beta} F_{\alpha\beta} .
\]  

As before, the curved space generalization of this is given by

\[
\nabla_\mu \langle \hat{J}^\mu \rangle = \pm \frac{e^2}{4\pi} \bar{\epsilon}^{\alpha\beta} F_{\alpha\beta}.
\]  

It is possible to modify the consistent current (3.10), by adding a local counterterm, so that it becomes covariant,

\[
\langle \hat{J}^\mu \rangle = \langle \tilde{J}^\mu \rangle \pm \frac{e^2}{4\pi} A_\alpha \epsilon^{\alpha\mu} .
\]  

Note that the covariant current (3.22) does not satisfy the Wess-Zumino consistency condition since the counterterm violates the integrability condition (3.11). Moreover the gauge covariant anomaly (3.20) or its curved space generalization (3.21) has a unique form dictated by the gauge transformation properties. This is contrary to the consistent anomaly which may have a minimal (3.14) or non-minimal (3.17) structure.

Now we will concentrate our attention on the gravity sector. It was shown by Alvarez-Gaume and E. Witten [31] that in \( 4k + 2 \) (\( k = 0, 1, 2, \ldots \)) dimensions, Einstein’s general coordinate transformation can contain an anomaly in the chiral sector (see for a review [42, 44, 77]). The breakdown of general coordinate invariance is manifested in the non-conservation of the energy-momentum tensor. As in the case of \( U(1) \) gauge current, the energy-momentum tensor for the chiral theory on the general curved background is

\footnote{In this section all the covariantly regularized objects are indicated by the hatted variables. From the next section onwards, we would denote them by usual (unhatted) variables.}
either consistent or covariant depending on the choice of regularization adopted to quantize the theory. We denote the consistent energy-momentum tensor by $\tilde{T}_{\mu\nu}$. The covariant divergence of the consistent energy-momentum tensor yields the consistent anomaly \[31, 43, 77\]. In $1+1$ dimensions the form of consistent gravitational anomaly, for right moving fermions, is given by

$$\nabla_{\mu}\langle \tilde{T}^{\mu}_{\nu}\rangle = \frac{1}{96\pi} \bar{\epsilon}^{\beta\delta} \partial_\delta \partial_\alpha \Gamma_{\nu\beta}^\alpha.$$ (3.23)

It is worthwhile to point out that consistent gravitational anomaly and the consistent gauge anomaly are analogous satisfying similar consistency conditions (3.11). This is easily observed here by comparing (3.23) with (3.14) where the affine connection plays the role of the gauge potential. We therefore omit the details and write the generalized gravitational consistent anomaly by an inspection of (3.17) on how to include the normal parity term. The result is

$$\nabla_{\mu}\langle \tilde{T}^{\mu}_{\nu}\rangle = \frac{1}{96\pi} \bar{\epsilon}^{\nu\mu} \nabla_\mu R = \mathcal{A}_{\nu}.$$ (3.24)

The covariant energy-momentum tensor $\hat{T}_{\mu\nu}$, on the other hand, has the divergence anomaly,

$$\nabla_{\mu}\langle \hat{T}^{\mu}_{\nu}\rangle = \frac{1}{96\pi} \bar{\epsilon}_{\nu\mu} \nabla^{\mu} R = \mathcal{A}_{\nu}$$ (3.25)

where $R$ is the Ricci scalar corresponding to the metric $g_{\mu\nu}$. This is the covariant form of the gravitational anomaly. Note that the right hand side of (3.25) is manifestly covariant, since it contains terms proportional to the derivative of the Ricci scalar. However this is not true for the consistent anomaly, (3.23). The consistent and covariant energy-momentum tensors are related via local counterterm

$$\langle \hat{T}_{\mu\nu}\rangle = \langle \tilde{T}_{\mu\nu}\rangle + \mathcal{P}_{\mu\nu}$$ (3.26)

where $\mathcal{P}_{\mu\nu}$ satisfies \[43\]

$$\nabla^{\mu}\mathcal{P}_{\mu\nu} = \frac{1}{96\pi} [\bar{\epsilon}_{\nu\sigma} \nabla^{\sigma} R - \bar{\epsilon}^{\beta\delta} \partial_\delta \partial_\alpha \Gamma_{\nu\beta}^\alpha].$$ (3.27)

### 3.2 Covariant anomalies and Hawking fluxes

Now we discuss the relationship between Hawking radiation and the gauge and gravitational anomalies. In [28], it was proposed that the flux of Hawking radiation can be
obtain by a knowledge of the gravitational anomaly at the horizon. An essential observation in [28] is that quantum fields near the horizon of \((d + 1)\) dimensional black hole behave as an infinite collection of two dimensional massless fields propagating on \(r - t\) sector of the full \((d + 1)\) dimensional black hole metric. Then, by transforming into the null coordinate and using the equations of motion for fields under consideration, we can easily decompose the field into two parts, propagating either ‘in’ to the horizon or ‘out’ from the horizon. We interpret ingoing modes as left moving and outgoing modes as right moving. Once the left moving modes fall into the black hole, they never come out classically and cannot affect the physics outside the black hole. Classically, modes inside the event horizon are causally disconnected from the outer region. Consequently, the effective field theory near the horizon is two dimensional and chiral. If we then integrate over the relevant right moving modes to obtain the quantum effective action in the exterior region, it becomes anomalous with respect to gauge or general coordinate symmetry. However, the original theory is of course gauge and diffeomorphism invariant. Therefore the anomalies, which are present near the horizon, must be cancelled by the quantum effects of classically irrelevant left moving modes. This fixes the flux of the outgoing modes which is interpreted as the Hawking flux as measured by an observer at the asymptotic infinity. Also, since the source of anomaly is located in an arbitrary small region near the event horizon, the fluxes of radiation are universally determined by the properties of black holes at the horizon.

To put the above considerations in a proper perspective, it is important to realize that \((1 + 1)\) dimensional chiral theories admit two types of anomalous currents and energy-momentum tensors- the consistent and the covariant (section-3.1). The analysis of [28, 32] uses the consistent form for the gauge and gravitational anomalies to obtain the form for the current and energy-momentum tensor in the vicinity of the event horizon. However, the boundary conditions, required to fix the arbitrary constants appearing in the expressions for current and energy-momentum tensor, were covariant. This raises several issues, both technically and conceptually. Note that the flux is measured at infinity. Since, in the region far away from the horizon, the theory is anomaly free, covariant and consistent structures are identical. Note that the mismatch between the covariant and consistent currents (energy-momentum tensors) is the germ of the anomaly [78, 79]. Hence if the anomaly approach is viable the fluxes of Hawking radiation should equally
Chapter 3. Hawking fluxes from covariant gauge and gravitational anomalies

well be obtainable from the covariant expressions. Also, as we shall demonstrate below, the use of covariant expressions entails considerable technical simplification. For example the shift between covariant and consistent expressions through local counterterms, as is mandatory in [28, 32], is not necessary if we use the covariant expressions for the gauge and gravitational anomalies. We now elaborate step by step the covariant anomaly method to compute the flux of Hawking radiation. We do this analysis for the Reissner-Nordstrom black hole.

Consider the Einstein-Maxwell theory represented by the action

\[ S_{EM} = \int d^4x \sqrt{-\gamma} \left[ R^{(4)} - F_{ab}F^{ab} \right] \]  

(3.28)

where \( \gamma_{ab} \) is the metric on \((3 + 1)\) dimensional spacetime\(^2\) while \( \gamma = det\gamma_{ab} \). \( R^{(4)} \) is the curvature scalar associated with \( \gamma_{ab} \). The electromagnetic field strength tensor \( F_{ab} \) is defined in terms of gauge potential \( A_a \) as

\[ F_{ab} = \nabla_a A_b - \nabla_b A_a. \]  

(3.29)

Variation of (3.28) with respect to the metric and gauge potential gives the coupled Einstein and Maxwell equations, respectively. The solutions for Einstein equations are given by

\[ ds^2 = \left( 1 - \frac{2M}{r} + \frac{Q^2 + P^2}{r^2} \right) dt^2 - \left( 1 - \frac{2M}{r} + \frac{Q^2 + P^2}{r^2} \right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \]  

(3.30)

The metric (3.30) is static and it is known as the Reissner-Nordstrom metric. The solutions for the Maxwell’s equations are given by

\[ E_r = F_{rt} = \frac{Q}{r^2}; B_r = \frac{F_{\theta \phi}}{r^2 \sin \theta} = \frac{P}{r^2}. \]  

(3.31)

Here \( M \) is the mass of the black hole and \( Q, P \) are electric and magnetic charges respectively. For our purpose we consider only electrically charged black hole (i.e \( P = 0 \)). Also, we shall work in the gauge \( A_r = 0 \). Then the metric (3.30) becomes

\[ ds^2 = \gamma_{ab} dx^a dx^b = f(r) dt^2 - \frac{1}{f(r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \]  

(3.32)

\(^2\)Latin indices \( a, b \), unless otherwise stated, represent \((3 + 1)\) dimensional spacetime while the Greek indices \( \mu, \nu \) are reserved for \((1 + 1)\) dimensions.
with
\[ f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} = \frac{(r - r_+)(r - r_-)}{r^2}, \tag{3.33} \]
while the gauge field \( A_a \) is given by
\[ A_t(r) = -\frac{Q}{r}; \quad A_r = A_\theta = A_\phi = 0. \tag{3.34} \]
The location of inner \((r_-)\) and outer \((r_+)\) horizons are given by
\[ r_\pm = M \pm \sqrt{M^2 - Q^2}. \tag{3.35} \]
Now we consider charged (complex) scalar fields moving on the background represented by the metric given in (3.32). The metric \( \gamma_{ab} \) and the gauge field \( A_a \) serves as external fields.

In the region near the outer event horizon, upon transforming to \( r^* \) (tortoise) coordinate and performing the partial wave decomposition, we can show that the effective radial potential corresponding to each partial wave mode is proportional to the metric function \( f(r(r_*)) \) which decay exponentially fast near the event horizon. The same reasoning holds for the mass terms (in the matter Lagrangian). Hence, the matter action in the region near the event horizon can be described by an infinite collection of \((1+1)\) dimensional free massless partial wave modes, each propagating in a spacetime with the effective metric given by \( r - t \) sector of the full spacetime metric (3.32), i.e
\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu = f(r)dt^2 - \frac{1}{f(r)}dr^2 \tag{3.36} \]
with \( \mu, \nu = t, r \). This is a kind of dimensional reduction, of the field theory under consideration, from \((3 + 1)\) to \((1 + 1)\) dimensions\(^3\). Now we further split each partial wave into left moving and right moving parts. This splitting is always possible. Consider for example the free massless complex scalar field \( \Phi(t,r) \) satisfying the Klein-Gordon equation
\[ \nabla_\mu \nabla^\mu \Phi(t,r) = 0 \tag{3.37} \]
and a similar equation with \( \Phi \) replaced by its complex conjugate \( \Phi^* \). Upon transforming to null coordinates, defined as
\[ u = t - r_*; \quad v = t + r_* \]
\[ \frac{dr}{dr_*} = f(r) \tag{3.38} \]
\(^3\)See appendix of this chapter for a detailed discussion of dimensional reduction.
the equation (3.37) becomes
\[ \partial_u \partial_v \Phi(u, v) = 0 . \] (3.39)
The general solution for (3.39) can be taken as
\[ \Phi(u, v) = \Phi^R(u) + \Phi^L(v) \] (3.40)
where \( \Phi^R(u) \) and \( \Phi^L(v) \) are right moving and left moving modes, satisfying
\[ \partial_v \Phi^R = 0 ; \quad \partial_u \Phi^R \neq 0 \]
\[ \partial_u \Phi^L = 0 ; \quad \partial_v \Phi^L \neq 0 . \] (3.41)
Similar analysis holds for \( \Phi^* \).

Since the horizon is a null hypersurface, all the left moving \( (\Phi^L) \) modes at the horizon cannot classically affect the theory outside the horizon. In other words, in equation (3.40) we have \( \Phi^L(v) = 0 \) and hence the field \( \Phi(u, v) \) possesses definite handedness (in our case it is right handed).

**Hawking fluxes for the Reissner-Nordstrom black hole:**

We now compute the Hawking charge and energy fluxes coming from the Reissner-Nordstrom black hole by using the covariant expressions for the gauge and gravitational anomalies. First, let us consider the charge flux. We denote the expectation value of the covariant current very near the outer event horizon by \( \langle J^\mu_{(H)} \rangle \). This covariant current satisfies the \((1+1)\) dimensional chiral covariant gauge anomaly [42, 44]. For the right-handed fields it is given by (3.21)
\[ \nabla_\mu \langle J^\mu_{(H)} \rangle = -\frac{e^2}{4\pi} \varepsilon^{\mu\nu} F_{\mu\nu} \] (3.42)
where \( \varepsilon^{\mu\nu} \) is an antisymmetric tensor defined in (3.15). For the effective \((1+1)\) dimensional Reissner-Nordstrom metric (3.36) the left hand side of (3.42) becomes
\[ \nabla_\mu \langle J^\mu_{(H)} \rangle = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \langle J^\mu_{(H)} \rangle) = \partial_r \langle J^r_{(H)} \rangle \] (3.43)
while the right hand side of (3.42) is
\[ -\frac{e^2}{4\pi} \varepsilon^{\mu\nu} F_{\mu\nu} = \frac{e^2}{2\pi} \partial_r A_t . \] (3.44)
Here we have used the fact that \( \sqrt{-g} = 1 \) for the metric (3.36)\(^4\). Hence the equation

\(^4\)An example where \( \sqrt{-g} \neq 1 \) is discussed in the section-3.5.
\[ \partial_r \langle J^r_H \rangle = \frac{e^2}{2\pi} \partial_r A_t. \] (3.45)

The solution for the above equation is given by
\[ \langle J^r_H(r) \rangle = c_H + \frac{e^2}{2\pi} [A_t(r) - A_t(r_+)] \] (3.46)

where \( c_H \) is an integration constant. This is the expression for the chiral covariant current. By construction, (3.46) is valid only in the region near to the event horizon.

Next, we consider the theory away from the event horizon. Theory away from the horizon is still \((3 + 1)\) dimensional (since the dimensional reduction procedure is valid only in the vicinity of the horizon). Also, in this region, both the modes, left and right handed, are present which makes the theory anomaly free. Consequently, the \((3 + 1)\) dimensional current, denoted by \( \langle J^a_{(4)} \rangle \) satisfy the usual conservation law
\[ \nabla_a \langle J^a_{(4)} \rangle = 0 \] (3.47)

where \( \gamma \) is determinant of the full Reissner-Nordstrom metric \( \gamma_{ab} \) given in (3.32) and \( a \in t, r, \theta, \phi \). Now since the the current \( \langle J^a_{(4)} \rangle \) only depends upon the radial coordinate (since the metric is static and spherically symmetric), equation (3.47) becomes
\[ \partial_r (r^2 \sin \theta \langle J^a_{(4)}(r) \rangle) + [\langle J^\theta_{(4)}(r) \rangle + \langle J^\phi_{(4)}(r) \rangle] r^2 \cos \theta = 0. \] (3.48)

We now define the effective \((1 + 1)\) dimensional current \( \langle J^a_{(o)} \rangle \) corresponding to \(3 + 1\) dimensional one \([32, 33]\) as
\[ \langle J^a_{(o)} \rangle = \int d\theta d\phi \sin \theta \ r^2 \langle J^a_{(4)} \rangle. \] (3.49)

Then by integrating (3.48) over the angular degrees of freedom and using (3.49), we arrive at
\[ \partial_r \langle J^r_{(o)} \rangle = 0. \] (3.50)

The solution of (3.50) is given by
\[ \langle J^r_{(o)} \rangle = c_o. \] (3.51)
where $c_o$ is an integration constant. We would like to point out that by construction $\langle J_{(o)}^\mu \rangle$ is an integrated current and hence it gives the amount of current passing through the spatial hypersurface defined by $\theta$ and $\phi$. As we shall see below the Hawking charge flux is related to the radial component $\langle J_r^\mu (o) \rangle$.

Now we write the total current $\langle J^\mu \rangle$ as a sum of two contributions from the two regions - the region near to the horizon ranging from $[r_+, r_++\epsilon]$ and the other region ranging from $[r_++\epsilon, \infty]$. Then we have

$$\langle J^\mu \rangle = \langle J_{(o)}^\mu \rangle \Theta (r - r_+ - \epsilon) + \langle J_{(H)}^\mu \rangle H(r)$$

(3.52)

where $\Theta (r - r_+ - \epsilon) = 1$ for $r > r_+ + \epsilon$ and otherwise zero while, $H(r)$ is the top hat function given by $H(r) = 1 - \Theta (r - r_+ - \epsilon)$. Taking the covariant divergence of $\langle J^\mu \rangle$ the Ward identity becomes

$$\nabla_\mu \langle J^\mu \rangle = \partial_r \langle J_r \rangle$$

$$= \partial_r \langle J_{(o)}^r \rangle \Theta (r - r_+ - \epsilon) + \partial_r \langle J_{(H)}^r \rangle H(r)$$

$$+ [(\langle J_{(o)}^r \rangle - \langle J_{(H)}^r \rangle) \delta (r - r_+ - \epsilon)]$$

(3.53)

By using the conservation relations for $\langle J_{(H)}^r \rangle$ and $\langle J_{(o)}^r \rangle$ given in (3.45) and (3.50) respectively, we get

$$\partial_r \langle J^r \rangle = \partial_r \left( \frac{e^2}{2\pi} A_t H \right) + [(\langle J_{(o)}^r \rangle - \langle J_{(H)}^r \rangle) + \frac{e^2}{2\pi} A_t \delta (r - r_+ - \epsilon)]$$

(3.54)

To make the the current anomaly free the first term must be cancelled by quantum effects of the classically irrelevant left moving modes. This is the Wess-Zumino term induced by these modes near the horizon. Effectively it implies a redefinition of the current as

$$\langle J'^r \rangle = \langle J^r \rangle - \frac{e^2}{2\pi} A_t H(r)$$

(3.55)

Since the cancellation of the anomaly occurs in the arbitrary small region near the horizon, it is expected that the redefinition of $\langle J'^r \rangle$ (3.55) should not affect the current conservation (3.50) valid far away from the horizon. Indeed, the explicit appearance of the top hat function $H(r)$ in (3.55) assures that in the region far away from the horizon $\langle J'^r (r) \rangle = \langle J^r (r) \rangle$. Once we take into account quantum effects of the left moving modes
3.2. Covariant anomalies and Hawking fluxes

the current (3.55) becomes anomaly free provided the coefficient of the delta function in (3.54) vanishes, leading to the relation

\[ \langle J^r_o \rangle = \langle J^r_H \rangle - \frac{e^2}{2\pi} A_t(r) . \]  

(3.56)

Substituting (3.46) and (3.51) in (3.56) gives a relation among the integration constants \( c_o \) and \( c_H \)

\[ c_o = c_H - \frac{e^2}{2\pi} A_t(r_+) . \]  

(3.57)

The coefficient \( c_H \) is fixed by imposing a boundary condition requiring the vanishing of the covariant current (3.46) at the horizon i.e

\[ \langle J^r_H(r = r_+) \rangle = 0 . \]  

(3.58)

Using (3.58) in (3.46) we get \( c_H = 0 \). The other constant \( c_o \) is now obtained from (3.57)

\[ c_o = -\frac{e^2}{2\pi} A_t(r_+) . \]  

(3.59)

Thus, the integrated flux \( \langle J^r_o \rangle \) given in (3.51) now reads

\[ \langle J^r_o \rangle = c_o = -\frac{e^2}{2\pi} A_t(r_+) = \frac{e^2 Q}{2\pi r_+} . \]  

(3.60)

Now we focus our attention on the gravity sector. In the region near to the horizon the theory is \((1 + 1)\) dimensional and chiral. Such a \((1 + 1)\) dimensional chiral theory is anomalous with respect to the general coordinate invariance. Consequently, in the region near the horizon, the covariant divergence of the energy-momentum tensor will satisfy either the consistent or the covariant gravitational anomaly. We denote the covariant chiral energy-momentum tensor for the charged scalar field by \( \langle T^\mu_{\nu[H]} \rangle \). This energy-momentum tensor satisfies the covariant gravitational anomaly [31, 77] and for the right moving fields it is given by (3.25)

\[ \nabla_\mu \langle T^\mu_{\nu[H]} \rangle = \frac{1}{96\pi} \varepsilon_{\mu\nu} \nabla^\nu R = A_\nu . \]  

(3.61)

Here \( R \) is the Ricci scalar corresponding to the \((1 + 1)\) dimensional metric (3.36). For the Reissner-Nordstrom black hole background, however, we have to take into account

\[ ^5 \text{In chapter-5 we will elaborate on the meaning and interpretation of this boundary condition.} \]
the effect of $U(1)$ gauge field leading to the modification in (3.61). The corresponding anomalous Ward identity for the covariantly regularized energy-momentum tensor is then given by

$$\nabla_\mu \langle T^\mu_\nu (H) \rangle = \mathcal{A}_\nu + F_{\mu\nu} \langle J_\nu^{\mu} (H) \rangle$$  \hspace{1cm} (3.62)

where $\langle J_\nu^{\mu} (H) \rangle$ is given by (3.46). The first term in the above expression represents the covariant gravitational anomaly and it is purely a quantum effect while the second one is the classical Lorentz force term which arises due to the effect of gauge field on the charged matter. Here we would like to point out that since the current $\langle J_\mu (H) \rangle$ itself is anomalous one might envisage the possibility of an additional term in (3.62) proportional to the gauge anomaly. Indeed this happens in the Ward identity for consistently regularized objects [32]. Such a term is ruled out here because there is no such covariant piece with the correct dimensions, having single a free index [37]. Further, we observe that for the metric (3.36) radial component of $\mathcal{A}_\nu$ vanishes. Thus the covariant gravitational anomaly (3.61) is purely timelike. Substituting $\nu = t$ in (3.61) we obtain

$$\mathcal{A}_t = \frac{1}{96\pi} f \partial_r R.$$  \hspace{1cm} (3.63)

For the metric (3.36) the explicit form for the Ricci scalar is

$$R = \partial^2 f = f''.$$  \hspace{1cm} (3.64)

Substituting (3.64) in (3.63), we get

$$\mathcal{A}_t = \partial_r N^r_t$$  \hspace{1cm} (3.65)

where $N^r_t$ is given by

$$N^r_t = \frac{1}{192\pi} \left( 2 f f'' - f'^2 \right).$$  \hspace{1cm} (3.66)

Taking $\nu = t$ component of (3.62) and then using (3.46, 3.65), yields

$$\nabla_\mu \langle T^\mu_t (H) \rangle = \partial_r \langle T^r_t (H) \rangle = \partial_r \left[ \frac{e^2}{4\pi} \left( A^2_t (r) - 2 A_t (r) A_t (r_+) \right) + N^r_t (r) \right].$$  \hspace{1cm} (3.67)
Solving the above equation we get the form for energy-momentum tensor in the vicinity of the horizon
\[
\langle T^r_{t(H)} \rangle = a_H + \left[ \frac{e^2}{4\pi} (A_t^2(r) - 2A_t(r)A_t(r_+)) + N_t^r(r) \right] \bigg|_{r_+}^r
\]
(3.68)
where \(a_H\) is an integration constant.

Now let us consider the theory away from the event horizon. In this region of spacetime, the theory is 3 + 1 dimensional and anomaly free. Consequently, the covariant divergence of 3 + 1 dimensional energy-momentum tensor, denoted as \(\langle T^{ab}_{(4)} \rangle\) satisfies the usual Lorentz force law
\[
\nabla_a \langle T^{a(b)}_{(4)} \rangle = F_{ab} \langle J^r_{(4)} \rangle .
\]
(3.69)
For the static spherically symmetric metric (3.32), \(b = t\) component of the above equation is given by
\[
\partial_r (r^2 \sin \theta \langle T^r_{t(4)} \rangle) + r^2 \cos \theta \langle T^\theta_{t(4)} \rangle = F_{rt} \langle J^r_{(4)} \rangle .
\]
(3.70)
Further using (3.49) and its tensorial analog
\[
\langle T^a_{b(o)} \rangle = \int d\phi d\sigma \ r^2 \sin \theta \ \langle T^a_{b(4)} \rangle
\]
(3.71)
the equation (3.70), after performing angular integrations, reduces to
\[
\partial_r \langle T^r_{t(o)} \rangle = (\partial_r A_t) \langle J^r_{(o)} \rangle .
\]
(3.72)
By substituting the known expression for \(\langle J^r_{(o)} \rangle\), given by (3.60), in the above equation and then performing the integral, yields the solution
\[
\langle T^r_{t(o)}(r) \rangle = a_o - \frac{e^2}{2\pi} A_t(r_+)A_t(r) .
\]
(3.73)
As before, writing the energy-momentum tensor as a sum of two combinations
\[
\langle T^r_t \rangle = \langle T^r_{t(o)} \rangle \Theta(r - r_+ - \epsilon) + \langle T^r_{t(H)} \rangle H(r)
\]
(3.74)
we find
\[
\nabla_\mu \langle T^\mu_t \rangle = \partial_r \langle T^r_t \rangle = -\frac{e^2}{2\pi} A_t(r_+) \partial_r A_t(r) + \partial_r \left[ \left( \frac{e^2}{4\pi} A_t^2 + N_t^r \right) H \right]
\]
\[
+ \left( \langle T^r_{t(o)} \rangle - \langle T^r_{t(H)} \rangle + \frac{e^2}{4\pi} A_t^2 + N_t^r \right) \delta(r - r_+ - \epsilon) .
\]
(3.75)
The first term is a classical effect coming from the Lorentz force. The second term has to be cancelled by the quantum effect of the left moving modes. As before, it implies the existence of a Wess-Zumino term modifying the energy-momentum tensor as

\[
⟨T^{\mu \nu}_t⟩ = ⟨T^{\mu \nu}_t⟩ - \left[ \frac{e^2}{4\pi} A_t^2 + N_r^r \right] H
\]

which is anomaly free provided the coefficient of the delta function vanishes. This gives a relation among the integration constants \(a_o\) and \(a_H\)

\[
a_o = a_H + \frac{e^2}{4\pi} A_t^2(r_+) - N_r^r(r_+) \tag{3.77}
\]

where \(a_H\) is now fixed by boundary condition\(^6\) requiring that the covariant energy-momentum tensor vanishes at the horizon, i.e

\[
⟨T^r_t(r_H)⟩ = 0 \tag{3.78}
\]

Using (3.78) in (3.68) gives \(a_H = 0\). Then from (3.77), we have

\[
a_o = \frac{e^2}{4\pi} A_t^2(r_+) - N_r^r(r_+) \tag{3.79}
\]

The Hawking flux is given by the asymptotic \((r \to \infty)\) limit of anomaly free energy momentum tensor. Substituting \(a_o\) (3.77) in \(⟨T^r_t(o)⟩\) (3.73) and then taking its asymptotic limit, we get the expression for energy-momentum flux of the charge particles emitted from the horizon

\[
⟨T^r_t(o)⟩(r \to \infty) = a_o = \frac{e^2}{4\pi} A_t^2(r_+) - N_r^r(r_+) \tag{3.80}
\]

Since \(f(r_+) = 0\) we find from (3.66) that

\[
N_r^r(r_+) = -\frac{f^2(r_+)}{192\pi} \tag{3.81}
\]

Further, by using the known expressions

\[
κ = \frac{2\pi}{β} = \frac{f'(r_+)}{2} \tag{3.82}
\]

for the surface gravity \(κ\) and \(β\) the inverse of Hawking temperature \(T_H\), we write (3.80) into a more familiar form

\[
⟨T^r_t(o)⟩(r \to \infty) = a_o = \frac{e^2 Q^2}{4πr_+^2} + \frac{π}{12β^2} \tag{3.83}
\]

\(^6\)See earlier footnote.
This is the expression for the energy-momentum flux obtained from the covariant anomaly method.

Further, since the basic structure of the covariant anomalous gauge and gravitational Ward identities (3.42, 3.62), apart from the coupling constant, is identical both for complex scalar field and fermionic field, the results given in (3.60) and (3.83) would remain unchanged if one uses the fermionic field instead of the complex scalar field.

Now we compare our findings (3.60) and (3.83), with the fluxes of Hawking radiation coming from the Reissner-Nordstrom black hole. Hawking radiation spectrum is given by the Bose distribution

\[ N_b^\pm(\omega) = \frac{1}{e^{\beta(\omega \pm \mu)} - 1} \] (3.84)

in the case of bosons, and the Fermi-Dirac distribution

\[ N_f^\pm(\omega) = \frac{1}{e^{\beta(\omega \pm \mu)} + 1} \] (3.85)

for fermions. Here \( \mu = \frac{eQ}{r_+} \) is the chemical potential [20]. \( N_b^\pm \) and \( N_f^\pm \) correspond to the distribution of particles with charge \( \pm e \). In the following we calculate the flux in the case of fermions in order to avoid the problem of superradiance present in the bosonic case [80].

The charge flux of fermionic particles is given by

\[ \langle J^r \rangle = e \int_0^{\infty} \frac{d\omega}{2\pi} \left[ N_f^-(\omega) - N_f^+(\omega) \right] . \] (3.86)

After substituting (3.85) in the above and performing the integral, we get

\[ \langle J^r \rangle = \frac{e}{2\pi \beta} \ln \left| \frac{1 + e^{\beta \mu}}{1 + e^{-\beta \mu}} \right| . \] (3.87)

Expanding the right hand side of (3.87) about \( \beta = 0 \), finally yields

\[ \langle J^r \rangle = \frac{e^2 Q}{2\pi r_+} . \] (3.88)

Similarly, the flux of energy-momentum tensor is given by

\[ \langle T^r_t \rangle = \int_0^{\infty} \frac{d\omega}{2\pi} \omega \left[ N_f^-(\omega) + N_f^+(\omega) \right] . \] (3.89)
Using (3.85) in (3.89) and following similar steps as before, yields
\[ \langle T^r_r \rangle = \frac{e^2 Q^2}{4\pi r^4} + \frac{\pi}{12 \beta^2} . \] (3.90)

The results (3.60, 3.83), derived from the covariant anomaly cancellation mechanism coincide with (3.88, 3.90). Anomalies in the covariant current and energy-momentum tensor, present in the region near to the horizon, are compensated by the charge and energy-momentum flux emitted from the Reissner-Nordstrom black hole. Note that the flux computed either from the cancellation of covariant anomaly or from the consistent anomaly [28, 32] matches with the pure thermal flux of the blackbody radiation. The actual Hawking spectrum is obtained by propagating the emission, originated from the horizon, through the centrifugal barrier. Effectively, the particles emitted from the horizon back scatter before reaching to spatial infinity. This leads to the corrections in the standard Hawking flux (3.88, 3.90). The resulting radiation observed at infinity is that of a \((3 + 1)\) dimensional gray body at the Hawking temperature [81]. These gray body factors are not accounted in our derivation of Hawking flux.

### 3.3 Comparison between consistent and covariant anomaly approach

In the last section we saw that conditions imposed by the vanishing of covariant gauge and gravitational anomalies are capable of giving the expressions for Hawking charge and energy-momentum flux. Similar results, based on the use of consistent anomalies, were already derived in [32]. Therefore it is important to compare the efficiency of both, the consistent [32] and the covariant anomaly approaches.

We would emphasize this point by considering the gauge sector of the total Hawking radiation. In the analysis of [32], the explicit form for the universal component of the consistent current \( \langle \tilde{J}^r_r \rangle \), in the region near to the horizon, was obtained by solving the consistent gauge anomaly [42, 44] for the right handed fields (3.14)
\[ \nabla_\mu \langle \tilde{J}_\mu^r \rangle = \partial_r \langle \tilde{J}^r \rangle = \frac{e^2}{2\pi} \partial_r A_t . \] (3.91)
Solution of (3.91) gives us the form for consistent gauge current

$$\langle \tilde{J}_r^{(H)} \rangle = \tilde{c}_H + \frac{e^2}{4\pi}[A_t(r) - A_r(r_+)] .$$

(3.92)

Like before, the integration constant $\tilde{c}_H$ is fixed by demanding the vanishing of the covariant current $\langle J_\mu^{(H)} \rangle$ at the horizon. However, since the current (3.92) is the consistent current, knowledge of the local counterterm which connects the consistent and covariant current becomes necessary. The explicit relation among these two anomalous currents is given by (3.22)

$$\langle J_r^{(H)} \rangle = \langle \tilde{J}_r^{(H)} \rangle + \frac{e^2}{4\pi}A_t \bar{\epsilon}_{tr} .$$

(3.93)

Now by implementing the covariant boundary condition ($\langle J_r^{(H)}(r = r_+) \rangle = 0$) in the above expression and then using (3.92), yields

$$\tilde{c}_H = -\frac{e^2}{4\pi}A_t(r_+) .$$

(3.94)

This fixes the form for the consistent current (3.92) completely [32]. On the other hand, in the region away from the horizon there is no anomaly in the gauge current. Hence the issue of covariant and consistent current would not arise. Consequently, the Hawking charge flux, which is measured at asymptotic infinity, computed either from the covariant or consistent anomaly, agrees. It is therefore clear from the above discussion that unlike in the covariant anomaly based approach, where only the boundary condition on the covariant current and the expression for covariant gauge anomaly were essential inputs, the consistent anomaly method, apart from the boundary condition and the consistent gauge anomaly, also requires the knowledge of local counterterms relating the different currents. The computation of this local counterterm, although quite straightforward for the case of gauge current, becomes cumbersome in the case of higher rank tensors. For example, the universal component of the covariant energy-momentum tensor $\langle T^r_{\mu(H)} \rangle$ is related to its consistent counterpart $\langle \tilde{T}^r_{\mu(H)} \rangle$ by the Bardeen polynomial (3.26) [43, 77]. For the metric (3.36) the relation among the two types of energy-momentum tensor is given by

$$\langle T^r_{\mu(H)} \rangle = \langle \tilde{T}^r_{\mu(H)} \rangle + \frac{1}{192\pi}[ff'^2 - 2f^2] .$$

(3.95)

All these computations become essential in dealing with the consistent anomaly cancellation approach to compute the fluxes of Hawking radiation [32]. In this sense, therefore,
the covariant anomaly cancellation approach, presented in the last section, is more efficient and lucid compared to the one given in [32].

3.4 Generalized consistent anomaly and flux

Here we show that the conclusions of [28, 32] remain unaffected by taking the general form of the consistent anomaly (3.17) and (3.24), rather than the minimal expressions (3.12, 3.23) considered in [28, 32]. Instead of repeating the analysis of [28, 32] we just point out the reasons for this robustness.

For static configuration and for the specific choice of the gauge potential \((A_r = 0)\), it is clear that the normal parity terms in (3.17) vanishes. Likewise the normal parity term in the counterterm relating the generalized consistent current \((\tilde{J}^\mu)\) (3.17) and the covariant current \((J^\mu)\) would also vanish since only the \(\mu = r\) component in \((\tilde{J}^\mu)\) is relevant. Hence, effectively the same structures of the consistent gauge anomaly and the counterterm relating the consistent and covariant currents, as used in [32], are valid. Since these were the two basic inputs, the results concerning the charge flux associated with Hawking radiation remain intact.

Identical conclusions also hold for the gravity sector. Although not immediately obvious, a little algebra shows that the normal parity term in \(\tilde{A}_r^\prime\) (3.24) vanishes. Hence the energy-momentum flux obtained by solving the generalized consistent anomaly (3.24) agrees with the one given in [32].

3.5 Application to stringy black holes

In section 3.2 we gave a derivation, based on the treatment of covariant gauge and gravitational anomalies, of Hawking radiation from the Reissner-Nordstrom black hole. As discussed earlier, such an approach can be applicable to a variety of black hole spacetimes. In this section we will adopt our covariant anomaly mechanism to discuss Hawking radiation from the black hole backgrounds that arise in string theory.

Many interesting properties and physics of black holes can be acquired by studying other types of black hole solutions that may appear in theories aiming to generalize
Einstein’s theory of gravity. Of particularly interest are considering those black hole configurations that emerge as classical solutions of the low energy limit of superstring theory \cite{82}. These black hole solutions were discovered when the dilaton scalar field was included in the Einstein-Maxwell theory \cite{83, 84}. This dilaton field couples in a nontrivial fashion to the metric and the gauge field. When the electromagnetic (gauge) field vanishes, the only static and spherically symmetric black hole solution is the Schwarzschild black hole with a constant dilaton field. However, the dilaton field cannot remain constant in the presence of a gauge field (i.e. in the case of charged black holes). This fact separates the stringy black holes from the Reissner-Nordstrom black holes. Nevertheless, these black holes also satisfy the usual laws of black hole thermodynamics.

Another example, motivated by string theory, is that of the charged non-extremal five-dimensional black hole in string theory. This black hole solution is obtained from a specific $D$-brane configuration and often called the non-extremal $D1 - D5$ black hole \cite{85} (for review see \cite{86}). This background is particularly interesting since it is related to various black hole solutions by taking different limits on parameters appearing in the background of five-dimensional Reissner-Nordstrom and Schwarzschild solutions, six-dimensional black string solution \cite{87}, black five-brane solution \cite{88}, dyonic black string solution \cite{89}. All these black hole solutions possess similar thermodynamical properties. Naturally, it would be an interesting exercise to implement the covariant anomaly cancellation approach to these stringy black holes. First we will study the Hawking radiation from the Garfinkle-Horowitz-Strominger (GHS) \cite{84} black hole. In the extremal limit of this black hole solution, we compute the flux of Hawking radiation by using our covariant anomaly approach. Next, we calculate the Hawking flux for D1-D5 nonextremal black holes.

### 3.5.1 Hawking Fluxes from GHS black hole

The Garfinkle-Horowitz-Strominger (GHS) black hole is a member of a family of solutions to low-energy string theory described by the $3 + 1$ dimensional action \cite{84} (in the string frame)

$$S_{GHS} = \int d^4 x \sqrt{-\gamma} e^{-2\phi} \left[ -R - 4(\nabla \phi)^2 + F^2 \right]$$

(3.96)
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where \( \phi \) is the dilaton field and \( F_{\mu\nu} \) is the Maxwell field associated with a \( U(1) \) subgroup of \( E_8 \times E_8 \) or \( Spin(32)/Z_2 \). Its charged black hole solution is given by

\[
ds^2_{\text{string}} = \gamma_{ab} dx^a dx^b = f(r) dt^2 - \frac{1}{h(r)} dr^2 - r^2 d\Omega
\]

where,

\[
f(r) = \left( 1 - \frac{2M e^{\phi_0}}{r} \right) \left( 1 - \frac{Q^2 e^{2\phi_0}}{M r} \right)^{-1}
\]

\[
h(r) = \left( 1 - \frac{2M e^{\phi_0}}{r} \right) \left( 1 - \frac{Q^2 e^{2\phi_0}}{M r} \right)
\]

with \( \phi_0 \) being the asymptotic constant value of the dilaton field, and \( Q \) the magnetic charge. We consider the case when \( Q^2 < 2e^{-2\phi_0} M^2 \) for which the above metric describes a black hole with an event horizon situated at \[84\]

\[
r_H = 2M e^{\phi_0} .
\]

Now consider scalar fields propagating on the background \(3.97\). In the near horizon region, with the aid of dimensional reduction procedure, we can effectively describe scalar fields with a metric given by the by the “\( r - t \)” sector of the full spacetime metric \(3.97\), i.e

\[
ds^2 = f(r) dt^2 - \frac{1}{h(r)} dr^2
\]

where the metric functions \( f(r) \) and \( h(r) \) are given in \(3.98\). It is important to realize that, unlike the Reissner-Nordstrom black hole, the \(1 + 1\) dimensional effective metric for GHS black hole \(3.100\) has nontrivial determinant, i.e \( \sqrt{-g} = \sqrt{-g_{tt} g_{rr}} \neq 1 \). In fact \(3.97\) represents the most general spherically symmetric metric. We shall see below that the anomaly method works without any difficulty for the GHS black hole also.

As mentioned earlier, the theory near the horizon is \(1 + 1\) dimensional and chiral and the energy-momentum tensor \( \langle T^{\mu\nu}_{(H)} \rangle \) in this region satisfies the covariant gravitational anomaly \(3.61\). For the metric \(3.100\) the radial component of the covariant anomaly \( A_\nu \) vanishes. Consequently, the covariant anomaly \(3.61\) is timelike. This feature of the covariant anomaly is common for all stationary black holes. The point is that for stationary black holes, the Ricci scalar \( R \) corresponding to the \( r - t \) sector of the full \(3+1\)}
dimensional metric is time independent. On the other hand, the radial component of the covariant anomaly $A_\nu$ (3.61), due to the presence of $\bar{\epsilon}_{\nu\mu}$, would always be proportional to the time derivative of $R$. Hence for the stationary black holes we have $A_r = 0$. Our task now is to compute $A_t$. For the metric (3.100), expression for the Ricci scalar $R$ is given by

$$R = \frac{f'' h}{f} + \frac{f' h'}{2f} - \frac{f'^2 h}{2f^2}.$$  

(3.101)

Then taking the $\nu = t$ component of (3.61), we find

$$\nabla_\mu \langle T^{\mu}_{\nu (H)} \rangle = \frac{1}{\sqrt{-g}} \partial^t (\sqrt{-g} \langle T^r_{t (H)} \rangle) = \frac{1}{\sqrt{-g}} \partial_r N^t_r$$

(3.102)

with

$$N^t_r = \frac{1}{96\pi} \left( h f'' + \frac{f' h'}{2} - \frac{f'^2 h}{f} \right).$$

(3.103)

Solution of (3.102) yields,

$$\langle T^r_{t (H)} \rangle = \frac{1}{\sqrt{-g}} [a_H + N^t_r (r) - N^t_r (r_H)]$$

(3.104)

where, $a_H$ is an integration constant.

In the region far away from the horizon the theory is $3 + 1$ dimensional and anomaly free. Hence the energy-momentum tensor $\langle T^{a b (4)} \rangle$ is conserved, i.e

$$\nabla_a \langle T^{a b (4)} \rangle = 0.$$  

(3.105)

The effective $1 + 1$ dimensional anomaly free energy-momentum tensor, defined in (3.71), then satisfies,

$$\partial_r (\sqrt{-g} \langle T^r_{t (o)} \rangle) = 0,$$

(3.106)

which, after integrating, yields

$$\langle T^r_{t (o)} \rangle = \frac{a_o}{\sqrt{-g}}$$

(3.107)

where $a_o$ is an integration constant.

Writing the total energy-momentum tensor $\langle T^r_\nu \rangle$ as a sum of two combinations and following the same reasoning given in section-3.2, we arrive at

$$\langle T^r_{t (o)} \rangle - \langle T^r_{t (H)} \rangle + \frac{N^r_r (r)}{\sqrt{-g}} = 0.$$  

(3.108)
Substituting (3.104) and (3.107) in the above equation, yields

\[ a_o = a_H - N_t^r(r_H) \]  (3.109)

The integration constant \( a_H \) can be fixed by imposing the covariant boundary condition (3.78). This gives \( a_H = 0 \). Hence the total flux of the energy-momentum tensor is given by

\[ \langle T^r_t(r \to \infty) \rangle = a_o = -N_t^r(r_H) = \frac{1}{192\pi} f'(r_H) h'(r_H) = \frac{\pi}{12} T_H^2 \]  (3.110)

where \( T_H \) is the Hawking temperature given by \([84]\)

\[ T_H = \frac{1}{8\pi M e^{\phi_0}} \]  (3.111)

Equation (3.110) represents the energy-momentum flux of Hawking radiation coming from the GHS black hole \([84]\). Moreover, the flux (3.110) which is obtained by the present approach is compatible with that calculated by using the consistent anomaly \([91]\).

**Extremal limit:**

At the extremality, i.e. when \( Q^2 = 2e^{-2\phi_0} M^2 \), the GHS black hole solution (3.97, 3.98) becomes

\[ ds^2 = dt^2 - \left( 1 - \frac{2M e^{\phi_0}}{r} \right)^{-2} dr^2 - r^2 d\Omega^2 . \]  (3.112)

It is easy to check that in this case the Hawking temperature vanishes. Indeed, by substituting

\[ f(r) = 1 \]
\[ h(r) = \left( 1 - \frac{2M e^{\phi_0}}{r} \right)^{-2} \]  (3.113)

in (3.110), we see that the energy flux vanishes.

### 3.5.2 Hawking radiation from D1-D5 non-extremal black hole

As another example of the covariant anomaly approach, we consider a non-extremal five dimensional black hole which originates as a brane configuration in Type IIB superstring
3.5. Application to stringy black holes

theory compactified on $S^1 \times T^4$[85]. The configuration relevant to the present case is composed of D1-branes wrapping $S^1$, D5-branes wrapping $S^1 \times T^4$ and momentum modes along $S^1$. The solution of the Type IIB supergravity corresponding to this configuration is a supersymmetric background known as the extremal five-dimensional D1-D5 black hole having zero Hawking temperature. Hence, in order to consider Hawking radiation we study the non-extremal D1-D5 black hole.

The ten-dimensional supergravity background corresponding to the non-extremal D1-D5 black hole has the following form in the string frame [85]:

$$ds_{10}^2 = f^{-1/2} f_5^{-1/2}(-h f_n^{-1} dt^2 + f_n(dx_5 + (1 - \tilde{f}_1^{-1})dt)^2) + f_1^{1/2} f_5^{-1/2}(dx_6^2 + \cdots + dx_9^2) + f_1^{1/2} f_5^{1/2}(h^{-1} dr^2 + r^2 d\Omega_3^2)$$

$$e^{-2\phi} = f_1^{-1} f_5 , \quad C_{05} = \tilde{f}_1^{-1} - 1$$

$$F_{ijk} = \frac{1}{2} \epsilon_{ijkl} \partial_l \tilde{f}_5 , \quad i, j, k, l = 1, 2, 3, 4$$

$x_5$ is the periodic coordinate along $S^1$ with period $2\pi R_5$ while $x_6, \cdots, x_9$ are periodic coordinates on $T^4$. Each of $x_6, \cdots, x_9$ is periodically identified with $2\pi V_4^1$, where $V$ is the volume of $T^4$. $F$ is the three-form field strength of the RR 2-form gauge potential $C$, $F = dC$. Also various functions appearing in the above background are functions of coordinates $x_1, \ldots, x_4$ given by

$$h = 1 - \frac{r_0^2}{r^2} , \quad f_{1,5,n} = 1 + \frac{r_{1,5,n}^2}{r^2}$$

$$\tilde{f}_1^{-1} = 1 - \frac{r_0^2 \sinh \alpha_{1,n} \cosh \alpha_{1,n}}{r^2} f_{1,n}^{-1}$$

$$r_{1,5,n}^2 = r_0^2 \sinh^2 \alpha_{1,5,n} , \quad r^2 = x_1^2 + \cdots + x_4^2 .$$

This black hole solution is parameterized by six independent quantities $\alpha_{1,5,n}, r_0, R_5$ and $V$. Functions $h, f_{1,5,n}$, are harmonic functions representing the non-extremality and the presence of D1, D5, and momentum modes respectively.

Dimensional reduction of (3.114) along $S^1 \times T^4$ following the procedure of [90] yields the Einstein metric of the non-extremal five-dimensional black hole as

$$ds_{5}^2 = -\lambda^{-2/3} h \ dt^2 + \lambda^{1/3}(h^{-1} dr^2 + r^2 d\Omega_3^2) \quad (3.116)$$

---

7This is the dimensional reduction of a background metric and it is different from the the dimensional reduction of the scalar field action in the near horizon limit, outlined in the appendix of this chapter.
where $\lambda$ is defined by

$$\lambda = f_1 f_5 f_n .$$

(3.117)

The event horizon $r_H$ of this black hole geometry is located at

$$r_H = r_0 .$$

(3.118)

Apart from the metric, the dimensional reduction gives us three kinds of gauge fields. The first one is the Kaluza-Klein gauge field $A^{(K)}_\mu$ coming from the metric and the second one, say $A^{(1)}_\mu$, basically stems from $C_{\mu 5}$ (here $\mu = 0, 1, 2, 3, 4$). From the background (3.114), the first two gauge fields are obtained as

$$A^{(K)} = -(\tilde{f}_n^{-1} - 1) dt , \quad A^{(1)} = (\tilde{f}_1^{-1} - 1) dt .$$

(3.119)

Unlike these gauge fields which are one-form in nature, the last one is the two-form gauge field $A_{\mu \nu}$, originating from $C_{\mu \nu}$ whose field strength is given by the expression of $F$ in (3.114).

Now if we consider a free complex scalar field in the black hole background (3.116) and (3.119) and perform a partial wave decomposition of $\Phi$ in terms of the spherical harmonics, then it can be shown that the action near the horizon becomes (see the Appendix-3.A)

$$S[\Phi] = - \sum_i \int dt dr \; r^3 \lambda^{1/2} \Phi_i^*(t, r) \left( -\frac{1}{f}(\partial_t - i A_t)^2 + \partial_r f \partial_r \right) \Phi_i(t, r)$$

(3.120)

where, $a$ is the collection of angular quantum numbers of the spherical harmonics, $A_t = e_1 A_t^{(1)} + e_K A_t^{(K)}$ and

$$f(r) = \frac{\lambda^{1/2}}{h} .$$

(3.121)

This action describes an infinite set of massless two-dimensional complex scalar fields in the following effective background :

$$ds^2 = -f(r) dt^2 + \frac{1}{f(r)} dr^2 , \quad \phi = r^3 \lambda^{1/2}$$

(3.122)

$$A_t(r) = -\frac{e_1 r_o^2 \sinh \alpha_1 \cosh \alpha_1}{r^2 + r_1^2} + \frac{e_k r_o^2 \sinh \alpha_n \cosh \alpha_n}{r^2 + r_n^2}$$

(3.123)
where $\phi$ is the two-dimensional dilaton field.

Once we mapped the theory from $3 + 1$ to $1 + 1$ dimensions, the computation of the charge and energy-momentum flux will follow exactly in a similar way illustrated in section-3.2. To calculate the charge flux we first note that since there are two kinds of $U(1)$ gauge symmetries, we have two $U(1)$ gauge currents $J^{(1)}_\mu$ and $J^{(K)}_\mu$ corresponding to $A^{(1)}_\mu$ and $A^{(K)}_\mu$ respectively. The form for covariant gauge anomaly for these two currents are identical in nature and given by (3.42) (but with two different coupling constants $e_1$ and $e_K$ corresponding to $A^{(1)}_\mu$ and $A^{(K)}_\mu$, respectively). Solving the covariant anomalous gauge Ward identities (3.42) for two different gauge currents $\langle J^{(1)}_\mu (o) \rangle$ and $\langle J^{(K)}_\mu (o) \rangle$, yields

\[
\langle J^{(1)}_\mu \rangle = c^{(1)}_H + \frac{e_1}{2\pi} [A_t(r) - A_t(r_H)] \quad (3.124)
\]
\[
\langle J^{(K)}_\mu \rangle = c^{(K)}_H + \frac{e_K}{2\pi} [A_t(r) - A_t(r_H)] . \quad (3.125)
\]

While, in the region away from the horizon, solving the usual conservation equations for $\langle J^{(1)}_\mu (H) \rangle$ and $\langle J^{(K)}_\mu (H) \rangle$, we get

\[
\langle J^{(1)}_\mu (o) \rangle = c^{(1)}_o \quad (3.126)
\]
\[
\langle J^{(K)}_\mu (o) \rangle = c^{(K)}_o \quad (3.127)
\]

The relations among the integration constants $c^{(1)}_o$, $c^{(1)}_H$, and $c^{(K)}_o$, $c^{(K)}_H$ are obtained by splitting the total currents $\langle J^{(1)}_\mu \rangle$ and $\langle J^{(K)}_\mu \rangle$ and demanding them to be anomaly free. After doing this, we get

\[
c^{(1)}_o = c^{(1)}_H - \frac{e_1}{2\pi} A_t(r_H) ; \quad c^{(K)}_o = c^{(K)}_H - \frac{e_K}{2\pi} A_t(r_H) . \quad (3.128)
\]

As before, the integration constants $c^{(1)}_H$ and $c^{(1)}_H$ are fixed by imposing the covariant boundary condition, leading to $c^{(1)}_H = c^{(K)}_H = 0$. Finally, the Hawking charge flux corresponding to $\langle J^{(1)}_\mu \rangle$ and $\langle J^{(K)}_\mu \rangle$ is given by

\[
c^{(1)}_o = -\frac{e_1}{2\pi} A_t(r_H) = \frac{e_1}{2\pi} (e_1 \tanh \alpha_1 - e_K \tanh \alpha_n) \quad (3.129)
\]
\[
c^{(K)}_o = -\frac{e_K}{2\pi} A_t(r_H) = \frac{e_K}{2\pi} (e_1 \tanh \alpha_1 - e_K \tanh \alpha_n) . \quad (3.130)
\]

Hence total charge flux is given by

\[
c_o = c^{(1)} + c^{(K)}_o = -\frac{e}{2\pi} A_t(r_H) = \frac{e}{2\pi} (e_1 \tanh \alpha_1 - e_K \tanh \alpha_n) \quad (3.131)
\]
where, $e = e_1 + e_K$.

Now consider the energy-momentum flux. Since we have an external gauge field, the energy-momentum tensor will not satisfy the conservation law even at classical level. Rather it gives rise to the Lorentz force law, $\nabla_\mu \langle T^{\mu \nu} \rangle = F_{\mu \nu} \langle J^{\nu} \rangle$ \(^8\). Hence the corresponding expression for the anomalous Ward identity for covariantly regularized quantities is given by (3.62)

$$\nabla_\mu \langle T^{\mu \nu(1)} \rangle = F_{\mu \nu} \langle J^{\nu} \rangle + A_\nu$$ (3.132)

where, $A_\nu$ is the two-dimensional gravitational covariant anomaly (3.61). In the region outside the horizon, there is no anomaly and hence the Ward identity reads

$$\nabla_\mu \langle T^{\mu \nu(o)} \rangle = \partial_\nu \langle T^{\nu \xi} \rangle = F_{\nu \xi} \langle J^{\xi} \rangle$$ (3.133)

Using (3.126, 3.127), the above equation can be solved as

$$\langle T^{\nu \xi} \rangle = a_o + c_\nu A_t$$ (3.134)

where, $a_o$ is an integration constant.

In the region near to the horizon, the Ward identity (3.132) reads

$$\partial_\nu \langle T^{\nu \xi(H)} \rangle = F_{\nu \xi} \langle J^{\xi(H)} \rangle + \partial_\xi N_\xi$$ (3.135)

where, $N_\xi$ is given by (3.66). Now substituting $\langle J^{\xi(H)} \rangle = \langle J^{(1)}_{\xi(H)} \rangle + \langle J^{(K)}_{\xi(H)} \rangle$ and using (3.124, 3.125), we get

$$\langle T^{\nu \xi(H)} \rangle = a_H + \int_{r_H}^r dr \partial_r \left[ c_\nu A_t + \frac{e}{4\pi} A_t^2 + N_\xi \right].$$ (3.136)

Following the same procedure as given in the gauge part, we arrive at the relation

$$a_o = a_H + \frac{e}{4\pi} A_t^2(r_H) - N_\xi(r_H).$$ (3.137)

Implementing (as before) the boundary condition that covariant energy momentum tensor vanishes at the horizon, fixes $a_H$ to be zero. Therefore, $a_o$ is given by

$$a_o = \frac{e}{4\pi} A_t^2(r_H) - N_\xi(r_H).$$ (3.138)

\(^8\)Note that here $\langle J^{\nu} \rangle = \langle J^{(1)}^{\nu} \rangle + \langle J^{(K)}^{\nu} \rangle$. 

Finally, substituting \( a_o \) (3.138) in (3.134) and then taking its asymptotic infinity limit, we get the expression for the energy-momentum flux

\[
\langle T^r_t(r \to \infty) \rangle = \frac{e}{4\pi} A^2_t(r_H) - N^r_t(r_+)
\]

\[
= \frac{e}{4\pi}(e_1 \tanh \alpha_1 - e_K \tanh \alpha_n)^2 + \frac{\pi}{12} T_H^2
\]

(3.139)

where,

\[
T_H = \frac{1}{2\pi r_0 \cosh \alpha_1 \cosh \alpha_n \cosh \alpha_n}
\]

(3.140)

is the Hawking temperature. Expression (3.139) agrees with the energy-momentum flux from black body radiation with two chemical potentials \( \mu_1 = e_1 \tan \alpha_1 \) and \( \mu_2 = e_2 \tan \alpha_2 \) [92].

3.6 Discussions

Using only the expressions of covariant gauge and gravitational anomalies we have given a derivation of Hawking radiation from the Reissner-Nordstrom black hole. The quantum field theory in the region near the event horizon is anomalous. In this region the expressions for covariant anomalous current and energy-momentum tensor were obtained by solving the covariant anomalous gauge and gravitational Ward identities. On the other hand, far away from the horizon the theory is anomaly free. The corresponding expressions for current and energy-momentum tensor were then computed by solving the usual conservation laws. Both, the anomalous as well as the anomaly free expressions for currents/energy-momentum tensors contains the arbitrary constants of integration. A relation among these constants were obtained by demanding that the total current/energy-momentum tensor, which is a sum of two combinations coming from the regions near to and away from the event horizon, must be anomaly free. This condition fixes one of the two integration constants. The remaining integration constant were then fixed by imposing the covariant boundary condition; namely, the vanishing of the covariant anomalous current and energy-momentum tensor at event horizon. This condition together with the fact that the total current/energy-momentum tensor is anomaly free, fixes the form for
current/energy-momentum tensor away from the event horizon i.e. \( \langle J^r_{(o)} \rangle / \langle T^r_{t(o)} \rangle \), completely. Finally, by extrapolating this anomaly free current/energy-momentum tensor gives the charge/energy-momentum fluxes of Hawking radiation.

Our approach of deriving the Hawking fluxes, uses only covariant expressions. Neither the consistent anomaly nor the counterterm relating the different currents, which were essential inputs in [32], were required. Consequently our analysis was economical and, we feel, also conceptually clean. It should be pointed out that the charge (energy) flux is identified with \( \langle J^r_{(o)} \rangle \) \( \langle (T^r_{t(o)}) \rangle \) which are the expressions for the currents (energy-momentum tensors) exterior to the horizon. Here these currents are anomaly free, implying that there is no difference between the covariant and consistent expressions. Actually the germ of the anomaly lies in this difference [78, 79]. Hence it becomes essential, and not just desirable, to obtain the same flux in terms of the covariant expressions. In other words, the Hawking flux must yield identical results whether one uses the consistent or the covariant anomalies. But the boundary condition must be covariant. This is consistent with the universality of the Hawking radiation and gives further credibility to the anomaly based approach.

It was shown [28, 32], performing a partial wave decomposition, that physics near the horizon is described by an infinite collection of massless \((1 + 1)\) dimensional fields, each partial wave propagating in spacetime with a metric given by the \( 'r - t' \) sector of the complete spacetime metric (3.32). This simplification, which effects a dimensional reduction from \( d \)-dimensions to \( d = 2 \) is also exploited here. It is however noted that greybody factors have not been included. In that case dimensional reduction will not yield the real Hawking radiation for \( d > 2 \). For instance, it is known [81] that in case of \( d = 4 \) reduction to \( d = 2 \) and keeping only the \( s \)-wave (\( i.e. l = 0 \)) reduces the Hawking flux with respect to its \( 2 - D \) value.

There are distinct advantages of the covariant anomaly approach. First, all the expressions are manifestly covariant. Also, the functional forms for the covariant anomalies are unique, being governed solely by the gauge (diffeomorphism) transformation properties. This is not so for consistent anomalies. They can and do have normal parity terms, apart from the odd parity ones. In fact, the special property (3.16) of two dimensions yields a natural form for this anomaly which has normal parity terms. Our observation,
that the results of [28, 32] are still valid, lend further credibility to this scheme of deriving Hawking radiation.

Further, we apply the covariant anomaly approach to compute the fluxes of Hawking radiation from stringy black holes. In particular, we discuss the Hawking radiation from GHS and five dimensional non-extremal D1-D5 blackhole. For GHS blackhole, the energy-momentum flux was obtained when \( Q^2 < 2M e^{-2\phi_0} \). At extremality there is no energy flux and hence Hawking temperature is zero. In the case of \( D1 - D5 \) blackhole, fluxes of electric charge flow and energy-momentum tensor were obtained. The resulting fluxes are the same as that of the two dimensional black body radiation at the Hawking temperature. The present approach based on covariant anomalies has also been applied to various other black hole geometries ([93]-[96]).
Appendix

3.A Dimensional reduction

Consider matter fields moving on the $3 + 1$ dimensional static spherically symmetric black hole background. In general, equations of motion governing the matter fields are complicated to solve. However, if we consider the theory near the event horizon, the action and hence the equations of motion, get simplified. The point is that in the near horizon region, matter field can be decomposed into an infinite collection of free, massless fields propagating on the $r - t$ sector of the original $3 + 1$ dimensional metric. Consequently, in the near horizon limit the matter field action becomes conformally invariant. In this appendix, we explicitly show the dimensional reduction procedure for the neutral and charged (complex) scalar fields.

Let us first consider the scalar field $\Phi$ moving on the $3+1$ dimensional static Schwarzschild black hole background represented by the metric

$$ds^2 = \gamma_{ab} dx^a dx^b = f(r) dt^2 - \frac{1}{f(r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$  \hspace{1cm} (3A.1)

where $f(r)$ is the metric function and, for the Schwarzschild black hole, is given in terms of mass $M$ of the black hole as

$$f(r) = 1 - \frac{2M}{r}.$$  \hspace{1cm} (3A.2)

The event horizon $r_h$ is defined by $f(r = r_h) = 0$. The action for scalar field moving on the background (3A.1) is given by

$$S = - \int d^4x \sqrt{-\gamma} \left[ \Phi \Box \Phi + m^2 \Phi^2 \right]$$  \hspace{1cm} (3A.3)
where $m$ is the mass of scalar field $\Phi$ while $\sqrt{-\gamma} = \sqrt{-\det \gamma_{ab}}$. In order to study the near horizon behavior of the theory, it is convenient to use the tortoise coordinate $r_*$, defined as

$$ \frac{dr}{dr_*} = f(r). $$

Then, for the Schwarzschild metric (3A.1), we have

$$ r_* = \int dr \left( \frac{1}{1 - 2M/r} \right) = r + 2M \ln \left| \frac{r}{2M} - 1 \right|. $$

Thus, in the tortoise coordinate event horizon is located at $r_* = -\infty$. The metric (3A.1), in $(t, r_*, \theta, \phi)$ coordinates now reads

$$ ds^2 = f(r(r_*))(dt^2 - dr_*^2) - r^2(r_*)(d\theta^2 + \sin^2 \theta d\phi^2). $$

and for the metric (3A.6) we have $\sqrt{-\gamma} = f(r(r_*))r^2(r_*)\sin \theta$. Then by writing the scalar field action (3A.3) in the tortoise coordinate, we obtain,

$$ S = -\int dt dr_* d\theta d\phi \sin \theta \Phi \left[ r^2(r_*)(\partial_t^2 - \partial_{r_*}^2) - 2r(r_*)f(r(r_*))\partial_r \right] \Phi $$

$$ + \int dt dr_* d\theta d\phi f(r(r_*)) \sin \theta \Phi \left[ \frac{1}{\sin^2 \theta} \partial_\phi^2 + \cot \theta \partial_\theta + \partial_\theta^2 + r^2(r_*)m^2 \right] \Phi. $$

Since the black hole metric is spherically symmetric, we decompose $\Phi(t, r_*, \theta, \phi)$ in terms of spherical harmonics

$$ \Phi(t, r_*, \theta, \phi) = \sum_{l,n} R_l(t, r_*) Y_{l,n}(\theta, \phi) $$

Substituting this ansatz in (3A.7) and integrating over the angular variables, we get

$$ S = -\sum_l \int dt dr_* r^2(r_*) R_l \left[ \partial_t^2 - \partial_{r_*}^2 - \frac{1}{r(r_*)} \partial_r f(r(r_*)) \right] R_l $$

$$ - \sum_l \int dt dr_* r^2(r_*) R_l f(r(r_*)) \left( -\frac{l(l+1)}{r^2(r_*)} + m^2 \right) R_l. $$

Now we expand $f(r)$ about the horizon ($r = r_h$) as,

$$ f(r) = f(r_h) + f'(r_h)(r - r_h) + \cdots. $$
Keeping only the leading order term in the above expansion, the near horizon expression for the metric coefficient \( f(r) \) reads

\[
f(r) \approx f'(r_h)(r - r_h) = 2\kappa(r - r_h)
\]

(3A.11)

where \( \kappa = \frac{f'(r_h)}{2} \) is the surface gravity. Substituting (3A.11) in (3A.4) and performing the integral, we arrive at

\[
r \approx Ae^{2\kappa r_*} + \delta; \quad A = constant .
\]

(3A.12)

Finally, the equations (3A.11) and (3A.12) give

\[
f(r(r_*)) \approx 2\kappa Ae^{2\kappa r_*} .
\]

(3A.13)

Hence, near the horizon \( f(r(r_*)) \) decay exponentially fast (since \( r_* \to -\infty \)). By similar reasoning, the term proportional to \( \partial_{r_*}f(r(r_*)) \) also vanishes exponentially. Therefore, in the near horizon limit the action for the scalar field (3A.9) becomes

\[
S \approx \sum_l \int dt dr_* r_h^2 \frac{R_l}{f} \left( \frac{1}{f} \partial_t^2 - \partial_r \left( f \partial_r \right) \right) R_l .
\]

(3A.14)

Transforming back to the Schwarzschild coordinate \((t, r)\), yields

\[
S \approx \sum_l \int dt dr r_h^2 \frac{R_l}{f} \left( \frac{1}{f} \partial_t^2 - \partial_r \left( f \partial_r \right) \right) R_l .
\]

(3A.15)

The factor of \( r_h^2 \) in the action can be interpreted as a dilaton background coupled to the scalar field. Thus, physics near the horizon can be effectively described by an infinite collection (for each value of \( l \)) of \((1+1)\) dimensional free massless scalar fields, each propagating in a \((1+1)\) dimensional spacetime given by the \( t - r \) sector of the \( 3 + 1 \) dimensional metric, that is

\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu = f(r) dt^2 - \frac{1}{f(r)} dr^2
\]

(3A.16)

with \((\mu, \nu = t, r)\). In deriving the Hawking flux, by using the anomaly approach, we consider only one component among the infinite collection of scalar fields. Analysis for all other component is essentially the same. However, how to extract four dimensional information by summing over all \( l \) values is still an open issue [34].
Next, we consider the charged (complex) scalar field moving on the 3 + 1 dimensional Reissner-Nordstrom black hole described by the metric (3.32) and the gauge field $A_b$ (3.34). The action for the charged scalar field is given by

$$S_{cs} = \int d^4x \sqrt{-\gamma \gamma^{ab}(\nabla_a - ieA_a)\Phi(\nabla_b + ieA_b)\Phi^*}$$

(3A.17)

By writing (3A.17) in the tortoise coordinate $r_*$ appropriate for the Reissner-Nordstrom black hole, we get

$$S_{cs} = -\int dtdr_*$ d$ \theta d\phi \sin \theta \Phi^* \left[ r_*^2(\partial_t^2 - \partial_{r_*}^2) - 2r_*(r_*)f(r_*)\partial_{r_*}\right] \Phi$$

$$+ \int dtdr_* d\theta d\phi \left[ \frac{1}{\sin^2 \theta} \partial^2_{\phi} + \cot \theta \partial_{\theta} + \partial^2_{\theta} + r^2(r_*)m^2 \right] \Phi$$

$$+ \int dtdr_* d\theta d\phi \sin \theta \left[ ieA_t[\Phi^* \partial_t \Phi - (\partial_t \Phi^*)\Phi] + e^2 (A_t^2 \Phi^* \Phi) \right]$$

(3A.18)

As before, by using the ansatz (3A.8) (together with its complex conjugate) and then integrating over the angular variable, (3A.18) reduces to

$$S_{cs} = \sum_l \int dtdr_* r_*^2 [((\partial_t - ieA_t)R_l)^2 - |\partial_{r_*} R_l|^2]$$

$$- \sum_l \int dtdr_* r_*^2 (r_*) f(r_*) \left( m^2 - \frac{l(l+1)}{r_*^2(r_*)} \right) |R_l|^2$$

(3A.19)

In the near horizon limit, by dropping the terms proportional to $f(r_*)$, the action for the complex scalar field becomes

$$S_{cs} \approx \sum_l \int dtdr_* r_*^2_h [((\partial_t - ieA_t)R_l)^2 - |\partial_{r_*} R_l|^2]$$

(3A.20)

As before, we interpret $r_*^2_h$ as the dilaton background coupled to the charged scalar field. Consequently, the (3 + 1) dimensional charged scalar field can be considered as an infinite set of $d = 2$ conformal fields near the horizon in $(t, r_*)$ coordinates. Any other field interaction terms can also be shown to be proportional to the damping factor $f(r_*)$ and hence they can be neglected in the near horizon limit. Similar analysis will also hold for the fermionic fields. Further, we would like to point out that the above discussion can be easily extended to the higher dimensional as well as non-spherical black holes [97, 98].
Chapter 4

Hawking Fluxes and Effective Actions

In chapter 3 we studied the relationship between the Hawking flux and the covariant anomalies. An important aspect of our analysis was that only covariant expressions are used throughout and consistent expressions were completely bypassed. The point is that since covariant boundary conditions, namely; the vanishing of covariant anomalous current (energy-momentum tensor) at the event horizon, are mandatory, it is rather conceptually clean and compact to discuss everything from the covariant point of view. The local counterterms relating the consistent and covariant expressions, which were essential in the approach based on the consistent anomalies [32], were not at all required in our approach. Consequently, the calculation of Hawking flux was simplified considerably. In both the approaches however, a splitting of space into two different regions (near to and away from the horizon) using discontinuous step functions like $\Theta(r-r_+ - \epsilon)$ and $H(r)$ was essential to obtain the Hawking flux. This split, which enforces the use of both the normal and anomalous Ward identities, also poses certain conceptual issues. Particularly, the definition of path integral in this context is not clear.

In this chapter we provide an algorithm to compute the Hawking flux from a generic spherically symmetric black hole spacetime, based on the chiral effective actions defined near to the event horizon, which only require the boundary conditions at the event horizon. This approach completely bypasses the use of discontinuous step functions and also solely depend on the properties of the theory near the event horizon. We adopt the ar-
guments given in [28, 32] and chapter-3 which imply that effective field theories are two
dimensional and chiral near the event horizon. Then, exploiting the known structures of
two dimensional chiral effective action appropriately modified by the local counterterms
[99], the relevant expressions for the covariant currents and energy-momentum tensors are
derived. These currents and energy- momentum tensors are anomalous. Again, as before
the arbitrary constants appearing in the expressions of currents and energy-momentum
tensors are fixed by imposing the covariant boundary conditions at the event horizon.
Further, we note that in the asymptotic limit, anomalies in the current and energy-
momentum tensor vanish. Then the Hawking charge and energy-momentum flux, which
are measured at the asymptotic infinity, can be computed by taking appropriately the
asymptotic infinity limit of these chiral currents and energy-momentum tensor, respect-
ively. The results obtain by this chiral effective action approach are in exact agreement
with the that of obtained in previous chapter and also in [32].

The chapter is arranged in the following manner. In section-4.1 we introduce the
chiral effective action. The currents and energy-momentum tensors following from the
chiral effective action, suitably modified by a local counterterm, are obtained. These cur-
rents and energy-momentum tensors satisfy covariant gauge and gravitational anomalies,
respectively [31, 42, 43, 44]. Using these expressions for the covariant current and energy-
momentum tensor and implementing the covariant boundary condition at the horizon,
Hawking fluxes from the generic spherically symmetric charged black hole are derived in
section-4.2. Our results match exactly with the one obtained by solving the covariant
anomalous Ward identities (see section-3.2). In section-4.3 we implement the chiral effec-
tive action approach to study the Hawking flux for the Reissner-Nordstrom black hole in
the presence of gravitational back reaction. The corrections to the Hawking charge and
energy-momentum flux are also obtained. Finally, we conclude this chapter in section-4.4

4.1 General setting and chiral effective action

Consider a general form of static spherically symmetric charged black hole represented
by the metric,

\[ ds^2 = \gamma_{ab}dx^adx^b = f(r)dt^2 - \frac{1}{h(r)}dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2) \]  (4.1)
where \( f(r) \) and \( h(r) \) are the metric coefficients. The gauge field is given by
\[
A = A_t(r)dt .
\]
(4.2)

Since we are discussing static, spherically symmetric black hole solutions, the above choice of gauge field is always possible. The outer horizon is given by
\[
f(r_h) = h(r_h) = 0 .
\]
(4.3)

Here we consider the asymptotically Minkowski flat black hole, i.e
\[
f(r \to \infty) = h(r \to \infty) = 1 \text{ and } \nonumber \n f''(r \to \infty) = f'''(r \to \infty) = h''(r \to \infty) = h'''(r \to \infty) = 0 .
\]
(4.4)

Now consider fermionic (or complex scalar) fields propagating on this background. It was shown in appendix-3.A that, by using a dimensional reduction technique, the effective field theory near the event horizon becomes two dimensional with the metric given by the \( r-t \) section of (4.1)
\[
ds^2 = f(r)dt^2 - \frac{1}{h(r)}dr^2 .
\]
(4.5)

Note that \( \sqrt{-g} = \sqrt{-detg_{\mu\nu}} = \sqrt{\frac{f}{h}} \neq 1 \) (unless \( f(r) = h(r) \)). On this two dimensional background, the modes which are going in to the black hole (left moving modes) are lost and the effective theory become chiral.

We now summarize, step by step, our methodology. For a two dimensional theory the expressions for the effective actions, whether anomalous (chiral) [99] or normal [100, 101, 102, 103], are known in the literature. For deriving the Hawking flux, only the form of the anomalous (chiral) effective action [99], which describes the theory near the horizon, is required. The currents and energy momentum tensors are computed by taking appropriate functional derivatives of this effective action. Next, the parameters appearing in these solutions are fixed by imposing the vanishing of covariant currents (energy momentum tensors) at the horizon. Once we have the complete form for current and energy-momentum tensor, the Hawking fluxes are obtained from the asymptotic \((r \to \infty)\) limits of the currents and energy momentum tensors.
For the right handed Weyl fermion propagating in the presence of external gravitational and $U(1)$ gauge field, the classical action is given by

$$S[Ψ, \bar{Ψ}, A, g] = \int d^2x \sqrt{-g} \bar{Ψ} \gamma^\mu \left( \partial_\mu - i\Gamma_\mu - iA_\mu \frac{1 - \gamma_5}{2} \right)Ψ$$ \hspace{1cm} (4.6)

where $\Gamma_\mu$ is the spin connection given by \[99\]

$$\Gamma_\mu = -\frac{1}{2} \epsilon^{(a)}_{\alpha\beta} \partial_\alpha e_{(a)\beta} \hspace{1cm} (4.7)$$

and $\epsilon^{\alpha\beta}$ is given by (3.15). The Zweibein vectors $e_{(a)\mu}$ and its inverse, defined by a relation $e_{(b)\mu} e_{(a)\mu} = \delta_{(a)(b)}$, are connected to the metric \footnote{Here indices in the parenthesis are defined with respect to flat spacetime.} 

$$e_{(a)\mu} e_{\mu(b)} = \eta_{(a)(b)}$$

$$e_{(a)\mu} e_{\nu(a)} = g^{\mu\nu} \hspace{1cm} (4.8)$$

Now we consider the quantization of $Ψ$ and $\bar{Ψ}$ in the presence of external gravitational and gauge fields. The corresponding quantum effective action $Γ_{(H)}$ is given by \[99\]

$$Γ_{(H)} = -\frac{1}{3} z(ω) + z(A) \hspace{1cm} (4.9)$$

where

$$z(ν) = \frac{1}{4\pi} \int d^2x d^2y e^{\mu\nu} \partial_\mu v_\nu(x) \Delta^{-1}(x, y) \partial_\rho [(e^{\rho\sigma} + \sqrt{-g}g^{\rho\sigma})v_\sigma(y)]$$ \hspace{1cm} (4.10)

and $Δ^{-1}$ is the inverse of Laplace-Beltrami operator $\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)$,

$$\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu) \Delta^{-1}(x, y) = δ(x - y) \hspace{1cm} (4.11)$$

Note that the effective action given in (4.10) contain $Δ^{-1}(x, y)$ and hence it is non-local. The local form is obtain by introducing auxiliary fields. From a variation of this effective action the energy momentum tensor and the gauge current are computed. These are shown in the literature \[31, 42, 43, 44, 77, 78, 79\] as consistent forms. To get their covariant forms in which we are interested, however, appropriate local polynomials have
4.1. General setting and chiral effective action

to be added. This is possible since energy momentum tensors and currents are only defined modulo local polynomials. Hence we have,

$$\delta \Gamma_{(H)} = \int d^2x \sqrt{-g} \left( \frac{1}{2} \delta g_{\mu\nu} T^{\mu\nu} + \delta A_\mu J^\mu \right) + l \tag{4.12}$$

where the local polynomial is given by [99],

$$l = \frac{1}{4\pi} \int d^2x \epsilon^{\mu\nu}(A_\mu \delta A_\nu - \frac{1}{3} w_\mu \delta w_\nu - \frac{1}{24} R \epsilon^{(a)}_\mu \delta e_{(a)\nu}) \tag{4.13}$$

The covariant gauge current $\langle J^\mu \rangle$ and the covariant energy momentum tensor $\langle T^{\mu\nu} \rangle$ are read-off from the above relations as [99],

$$\langle J^\mu \rangle = \delta \Gamma_{(H)} \delta A_\mu = -\frac{e^2}{2\pi} D^\mu B \tag{4.14}$$

$$\langle T^{\mu\nu} \rangle = \delta \Gamma_{(H)} \delta g^{\mu\nu}$$

$$= \frac{e^2}{4\pi} (D_\mu B D_\nu B) + \frac{1}{4\pi} \left( \frac{1}{48} D_\mu G D_\nu G - \frac{1}{24} D_\mu D_\nu G + \frac{1}{24} g_{\mu\nu} R \right). \tag{4.15}$$

Note the presence of the chiral covariant derivative $D_\mu$ expressed in terms of the usual covariant derivative $\nabla_\mu$,

$$D_\mu = \nabla_\mu - \bar{\epsilon}_{\mu\nu} \nabla^\nu = -\bar{\epsilon}_{\mu\nu} D^\nu, \tag{4.16}$$

The auxiliary fields $B(x)$ and $G(x)$, necessary to make the effective action (4.9) local, are defined as

$$B(x) = \int d^2y \sqrt{-g} \Delta^{-1}(x,y) \epsilon^{\mu\nu} \partial_\mu A_\nu(y) \tag{4.17}$$

$$G(x) = \int d^2y \sqrt{-g} \Delta^{-1}(x,y) R(y) \tag{4.18}$$

By acting $\nabla_\mu \nabla^\mu$ on both sides of (4.17) we get the differential equation for $B(x)$

$$\nabla_\mu \nabla^\mu B(x) = \frac{1}{\sqrt{-g(x)}} \int d^2y \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu) \Delta^{-1}(x,y) \epsilon^{\alpha\beta} \partial_\alpha A_\beta(y)$$

$$= \frac{\epsilon^{\alpha\beta}}{\sqrt{-g(x)}} \int d^2y \delta(x-y) \partial_\alpha A_\beta(y)$$

$$= \epsilon^{\alpha\beta}(x) \partial_\alpha A_\beta(x) \tag{4.19}$$

2In this chapter, we have suppressed the suffix on $\langle J^\mu \rangle$ and $\langle T^{\mu\nu} \rangle$ since most of the time we will be using the near horizon expressions for current and energy-momentum tensor.
where we have used (4.11). Similarly, by operating $\nabla_\mu \nabla^\mu$ on both sides of (4.18), we obtain
\[
\nabla_\mu \nabla^\mu G(x) = R(x)
\] (4.20)
where $R$ is the Ricci scalar, which for the metric (4.5), is given by
\[
R = \frac{f'' h}{f} + \frac{f' h'}{2f} - \frac{f'^2 h}{2f^2}.
\] (4.21)

Determination of the near horizon structures for the covariant current (4.14) and energy-momentum tensor (4.15) hence eventually reduces to finding the solutions of the differential equations (4.19, 4.20).

Before proceeding further we provide some consistency checks and properties of the covariant current and energy-momentum tensor given in (4.14, 4.15). The covariant divergence of current $\langle J^\mu \rangle$ and energy-momentum tensor $\langle T^{\mu\nu} \rangle$ satisfy the covariant Ward identities (3.42, 3.62), respectively. For example, using (4.14) and (4.17) in (4.19), we find
\[
\nabla_\mu \langle J^\mu \rangle = -\frac{e^2}{2\pi} \nabla_\mu \nabla^\mu B = -\frac{e^2}{4\pi} \epsilon^{\alpha\beta} F_{\alpha\beta}.
\] (4.22)
This is precisely the expression of covariant gauge anomaly (3.42). Similarly, by using (4.15) and (4.18) in (4.20) reproduces the covariant anomalous gravitational Ward identity (in the presence of gauge field),
\[
\nabla_\mu \langle T^{\mu\nu} \rangle = \langle J_\mu \rangle F^{\mu\nu} + \frac{1}{96\pi} \epsilon_{\nu\mu} \nabla^\mu R.
\] (4.23)

The trace of the covariant energy-momentum tensor is obtained by contracting $\langle T^{\mu\nu} \rangle$ (4.15) with respect to the metric, and yields
\[
\langle T^\alpha_{\ldots\alpha} \rangle = \frac{e^2}{4\pi} (D^\mu B D_\mu B) + \frac{1}{4\pi} \left( \frac{1}{48} D^\mu G D_\mu G - \frac{1}{24} D^\mu D_\mu G + \frac{1}{24} \delta_\mu^\mu R \right).
\] (4.24)
Now by using (4.16) we can easily see that terms like $D_\mu B D^\mu B$, $D_\mu G D^\mu G$ and $D^\mu D_\mu G$ drop out, leading to the chiral trace anomaly
\[
\langle T^\alpha_{\ldots\alpha} \rangle = \frac{R}{48\pi}.
\] (4.25)
Note that the chiral theory (4.9) has both a diffeomorphism anomaly (3.62) and a trace anomaly (4.25). This is distinct from the usual vector theory, represented by the Polyakov type action [100], where there is only trace anomaly \( \langle T_{\mu\nu} \rangle = \frac{R}{24\pi} \). No diffeomorphism anomaly exists. However, it is important to note that for a vector theory, it is possible to shift the trace anomaly to diffeomorphism anomaly by adopting a certain regularization prescription [104].

The covariant current (4.14) and covariant energy-momentum tensor (4.15) are chiral. Consequently, not all components of \( \langle J^\mu \rangle \) and \( \langle T_{\mu\nu} \rangle \) are independent. Let us first consider the expression for current (4.14). Contraction on both sides of (4.14) with respect to the antisymmetric tensor \( \bar{\epsilon}^\rho_{\mu} \) gives

\[
\bar{\epsilon}^\rho_{\mu} \langle J^\mu \rangle = -\frac{e^2}{2\pi} \epsilon^\rho_{\mu\nu} D^\mu B = \frac{e^2}{2\pi} D^\mu B = -\langle J_\rho \rangle . \tag{4.26}
\]

This is the chirality relation for the covariant current (4.14). In order to get a chirality relation for the covariant energy-momentum tensor, we first consider the contraction of \( \langle T^\rho_{\nu} \rangle \) with \( \bar{\epsilon}^\nu_{\rho} \)

\[
\epsilon^\nu_{\rho} \langle T^\rho_{\nu} \rangle = -\left[ \frac{e^2}{4\pi} D^\nu B D^\rho B + \frac{1}{4\pi} \left( \frac{1}{48} D^\rho G D^\nu G - \frac{1}{24} D^\rho D^\nu G - \frac{1}{24} \epsilon^\rho_{\nu\mu} R \right) \right]. \tag{4.27}
\]

where we have used (4.16). Interchanging \( \mu \) and \( \nu \) in the above equation, we get a similar relation

\[
\bar{\epsilon}^\nu_{\rho} \langle T^\rho_{\mu} \rangle = -\left[ \frac{e^2}{4\pi} D^\rho B D^\nu B + \frac{1}{4\pi} \left( \frac{1}{48} D^\rho G D^\nu G - \frac{1}{24} D^\rho D^\nu G - \frac{1}{24} \epsilon^\rho_{\nu\mu} R \right) \right]. \tag{4.28}
\]

Adding (4.27) with (4.28) and using the fact that

\[
[D^\mu, D^\nu] G = 0 ; \quad \text{for some scalar function } G, \tag{4.29}
\]

we arrive at

\[
\epsilon^\nu_{\rho} \langle T^\rho_{\nu} \rangle + \epsilon^\rho_{\nu} \langle T^\rho_{\mu} \rangle = -2 \langle T_{\mu\nu} \rangle + \frac{1}{48\pi} g_{\mu\nu} R , \tag{4.30}
\]

which can be further simplified by using (4.25),

\[
\langle T_{\mu\nu} \rangle = -\frac{1}{2} [\epsilon^\nu_{\rho} \langle T^\rho_{\nu} \rangle + \epsilon^\rho_{\nu} \langle T^\rho_{\mu} \rangle] + \frac{1}{2} g_{\mu\nu} \langle T^\alpha_{\alpha} \rangle . \tag{4.31}
\]
These chirality properties (4.26, 4.31) constrain the structure of the covariant current \( \langle J^\mu \rangle \) and energy-momentum tensor \( \langle T^{\mu\nu} \rangle \).

**Solutions for \( B \) and \( G \):**

Now we solve the differential equations for the auxiliary fields \( B(r, t) \) and \( G(r, t) \). First we solve (4.19) for \( B(r, t) \). For the metric (4.5), equation (4.19) becomes

\[
\nabla_\mu \nabla^\mu B(r, t) = \frac{1}{\sqrt{fh}} \partial_t^2 B(r, t) - \partial_r [\sqrt{fh} \partial_r B(r, t)] = - \partial_r A_t
\]

where we have used the fact that metric (4.5) is static and \( A_t(t, r) \equiv A_t(r) \). The general solution for (4.32) is

\[
B(r, t) = B_o(r) - at + b, \quad \text{with} \quad \partial_r B_o(r) = \frac{1}{\sqrt{fh}} [A_t + c].
\]

Here \( a, b \) and \( c \) are integration constants. Similarly, the differential equation (4.20) for \( G(r, t) \) on using (4.21), can be written as

\[
\frac{1}{\sqrt{fh}} \partial_t^2 G(r, t) - \partial_r [\sqrt{fh} \partial_r G(r, t)] = -gR = \partial_r \left[ \sqrt{\frac{h}{f'}} \right],
\]

which after solving, yields

\[
G(r, t) = G_o - 4pt + q, \quad \text{with} \quad \partial_r G_o(r) = -\frac{1}{\sqrt{fh}} \left[ \frac{f'}{\sqrt{-g}} + z \right]
\]

where \( p, q \) and \( z \) are integration constants.

### 4.2 Charge and energy flux

Now we are in a position to calculate the charge and energy-momentum flux from the black hole background (4.1). We will see that the results are the same as that obtained either from the consistent [32] or from the covariant (section-3.2) anomaly based approach. First we derive the charge flux. The covariant current (4.14) can be written in terms of ordinary partial derivative

\[
\langle J^\mu \rangle = -\frac{e^2}{2\pi} D^\mu B(r, t) = \frac{e^2}{2\pi} [\epsilon^{\mu\nu} \partial_\nu B(r, t) - g^{\mu\sigma} \partial_\sigma B(r, t)].
\]
Taking the $\mu = r$ component of above equation and then using $B(r,t)$ (4.33), yields

$$\langle J^r(r) \rangle = \frac{e^2}{2\pi \sqrt{-g}} [\bar{A}_t(r)]$$

(4.37)

where $\bar{A}_t(r)$ is defined as

$$\bar{A}_t(r) = A_t(r) + c + a$$

(4.38)

The other component $\langle J^t(r) \rangle$ is fixed by the chirality constraint (4.26)

$$\langle J^t(r) \rangle = -\bar{\epsilon}^r_t \langle J_r(r) \rangle = -\bar{\epsilon}^r_g r_{rr} \langle J^r(r) \rangle = \sqrt{-g} \frac{f}{f'} \langle J^r(r) \rangle$$

(4.39)

Our task now is to determine the integration constants $c$ and $a$. For that, we impose the covariant boundary condition (3.58), i.e vanishing of the covariant current (4.37) at the horizon. This leads to a relation among $c$ and $a$

$$c + a = -A_t(r_h)$$

(4.40)

Hence the expression for $\langle J^r(r) \rangle$ takes the form

$$\langle J^r(r) \rangle = \frac{e^2}{2\pi \sqrt{-g}} [A_t(r) - A_t(r_h)]$$

(4.41)

Now the charge flux is given by the asymptotic ($r \to \infty$) limit of the anomaly free current [28, 32]. We had observed that for the gauge fields which vanish at asymptotic infinity, the covariant gauge anomaly (4.22) vanishes, and hence we directly compute the flux from (4.41) by taking the ($r \to \infty$) limit. This yields,

$$\langle J^r(r \to \infty) \rangle = -\frac{e^2}{2\pi} (A_t(r_h))$$

(4.42)

This is the desired Hawking charge flux and agrees with our previous result given in section-3.2 (see also [28, 32]).

We next consider the energy momentum flux by adopting the same technique. After using the solutions for $B(x)$ (4.33) and $G(x)$ (4.35) in the general expression for covariant energy-momentum tensor (4.15), we get

$$\langle T^r_r \rangle = \frac{e^2}{4\pi \sqrt{-g}} \bar{A}_t^2(r) + \frac{1}{12\pi \sqrt{-g}} \bar{P}^2(r) + \frac{1}{24\pi \sqrt{-g}} [\frac{f'}{\sqrt{-g}} \bar{P}(r) + \bar{Q}(r)]$$

(4.43)

$$\langle T^r_t \rangle = \frac{R}{96\pi} - \frac{\sqrt{-g}}{f'} \langle T^r_r \rangle$$

(4.44)

$$\langle T^t_t \rangle = -\langle T^r_r \rangle + \frac{R}{48\pi}$$

(4.45)
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where \( \bar{A}(r) \) is given in (4.38) and

\[
\bar{P}(r) = p - \frac{1}{4} \left( \frac{f'}{\sqrt{-g}} + z \right) \quad (4.46)
\]

\[
\bar{Q}(r) = \frac{1}{4} hf'' - \frac{f'}{8} \left( \frac{hf'}{f} - h' \right) \quad (4.47)
\]

Relation (4.45) is a consequence of the trace anomaly (4.25) while (4.44) follows from the chirality criterion (4.31). Now we implement the boundary condition; namely the vanishing of the universal component of the covariant energy momentum tensor at the horizon, i.e \( \langle T^\tau_{\tau}(r_h) \rangle = 0 \). Then from (4.40) and (4.43), we have

\[
\langle T^\tau_{\tau}(r = r_h) \rangle = \bar{P}^2(r_h) + 2 \left[ \frac{f'}{\sqrt{-g}} \bar{P}(r_h) + \bar{Q}(r_h) \right] = 0 \quad (4.48)
\]

\[
\Rightarrow p = \frac{1}{4} (z \pm \sqrt{f'(r_h)h'(r_h)}) \quad (4.49)
\]

Substituting either of the above relations (among \( p \) and \( z \)) in (4.43) we get

\[
\langle T^\tau_{\tau}(r) \rangle = \frac{e^2}{4\pi \sqrt{-g}} (A_t(r) - A_t(r_h))^2
\]

\[
+ \frac{1}{192\pi \sqrt{-g}} \left[ f'(r_h)h'(r_h) - \frac{2f(r)h'(r)}{h(r)} + 2f''(r)h(r) + f'(r)h'(r) \right] .
\]

Further, by noting that for metric (4.5)

\[
f'(r_h)h'(r_h) - \frac{2f(r)h'(r)}{h(r)} + 2f''(r)h(r) + f'(r)h'(r) = N^r_t(r) - N^r_t(r_h) \quad (4.51)
\]

(see equation (3.103), we can write (4.50) as

\[
\langle T^\tau_{\tau}(r) \rangle = \frac{e^2}{4\pi \sqrt{-g}} (A_t(r) - A_t(r_h))^2 + \frac{1}{192\pi \sqrt{-g}} \left[ N^r_t(r) - N^r_t(r_h) \right] .
\]

This expression is in agreement with the one obtained by solving the covariant anomalous Ward identities in the region near to the event horizon (3.68).

To obtain the energy flux, we recall that it is given by the asymptotic expression for the anomaly free energy momentum tensor. As for the charge case, here too it is found from (4.23) that the gravitational Ward identity vanishes in this limit. Hence the energy flux is abstracted by taking the asymptotic infinity limit of (4.52). This yields,

\[
\langle T^\tau_{\tau}(r \to \infty) \rangle = \frac{e^2}{4\pi} A^2_t(r_+) + \frac{1}{192\pi} f'(r_h)h'(r_h) ,
\]

(4.53)
which correctly reproduces the Hawking flux from the generic spherically symmetric charged black hole.

For the Reissner-Nordstrom black hole (3.32), the charge and energy-momentum flux can be obtained by substituting the expressions for metric function and the gauge potential in (4.42) and (4.53)

\[
\langle J^r(r \to \infty) \rangle = \frac{e^2 Q}{2 \pi r_+} \tag{4.54}
\]

\[
\langle T^r_t(r \to \infty) \rangle = \frac{e^2 Q^2}{4 \pi r_+^2} + \frac{\pi T_H^2}{12} . \tag{4.55}
\]

These expressions for the charge and energy-momentum flux are in agreement with our previous results (3.60, 3.83) obtained by using the covariant anomaly approach. Hence, from the above analysis, we observe that only the structure of chiral effective action (4.9) and the covariant boundary conditions at the event horizon are sufficient to determine the charge and energy-momentum flux completely.

### 4.3 Back reaction effect and chiral effective action

In this section we shall implement the chiral effective action approach to study the Hawking flux in the presence of gravitational back reaction. The modified expressions for the charge and energy-momentum flux, due to the effect of one loop back reaction are obtained.

Back reaction, it might be recalled, is an effect of non-zero expectation value of the energy-momentum tensor on the spacetime geometry, which acts as a source of curvature. It is possible to include the effect of gravitational back reaction in the derivation of Hawking radiation. Indeed, using the conformal anomaly method the effect on the spacetime geometry by one loop back reaction was computed in [105, 106]. Based on this approach, corrections to the Hawking temperature were obtained in [106, 107]. Correction to the Hawking temperature using the back reaction equation for linearised quantum fluctuation was derived in [108]. Recently, more useful and intuitive way to understand the effect of back reaction through the quantum tunneling formalism [25] was developed in [27].
We are interested in discussing the Hawking effect from the Reissner-Nordstrom black hole \( (3.32) \) in the presence of back reaction. However, we cannot use the standard Reissner-Nordstrom metric directly to compute the Hawking fluxes since it gets modified when we take into account the effect of one loop back reaction. Instead, we shall use the modified Reissner-Nordstrom metric given in \([106]\)

\[
ds^2 = f(r)dt^2 - \frac{1}{h(r)}dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2) \tag{4.56}
\]

where

\[
f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{A(\alpha)}{r} \tag{4.57}
\]

\[
h(r) = \frac{1}{f(r)}B(\alpha) \tag{4.58}
\]

\(A(\alpha)\) and \(B(\alpha)\) depend on the parameters \(\alpha\). The event horizon for the modified metric \((4.56)\) is now defined by \(f(r = r_M) = h(r = r_M)\) where \(r_M\) is the modified horizon radius given by \([106]\)

\[
r_M = r_+ \left(1 + \frac{\alpha}{M^2}\right)^{-\frac{1}{2}}. \tag{4.59}
\]

Here \(r_+\) is the radius of the outer event horizon of the original (in the absence of back reaction) Reissner-Nordstrom black hole \((3.32)\).

Such a form is also dictated by simple scaling arguments. As is well known, a loop expansion is equivalent to an expansion in powers of the Planck constant \(h\). Since, in natural units, \(\sqrt{h} = M_p\) (the Planck mass), the one loop correction has a form given by \(\frac{\alpha}{M^2}\). Where parameter \(\alpha\) is the related to the trace anomaly coefficient, taking into account the degrees of freedom of the fields, and its explicit form is given by \([105, 106, 107]\).

\[
\alpha = \frac{1}{360\pi}(-N_o - \frac{7}{4}N_\frac{1}{2} + 13N_1 + \frac{233}{4}N_\frac{3}{2} - 212N_2). \tag{4.60}
\]

where \(N_s\) denotes the number of fields with spin \(s\) entering into the theory \([21, 109]\). In our case, only the complex scalar field \((s = 0)\) exits. Therefore, we have

\[
\alpha = -\frac{1}{360\pi}N_o. \tag{4.61}
\]
Further, we note that the generic form for the metric (4.56) in the presence of the gravitational backreaction was obtained [105, 106] by solving the semiclassical Einstein equations

$$ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \langle T^{(4)}_{\mu\nu}(g_{\mu\nu}) \rangle $$

or in the more convenient form

$$ R_{\mu\nu} = \langle T^{(4)}_{\mu\nu}(g_{\mu\nu}) \rangle - \frac{1}{2} \langle T^{(4)}_{\rho\rho} \rangle g_{\mu\nu}, $$

with the aid of confomral (trace) anomaly in 4D by keeping the spherical symmetry intact. Here $\langle T^{(4)}_{\mu\nu}(g_{\mu\nu}) \rangle$ is the renormalized energy-momentum tensor in 3 + 1 dimensions. In our formalism, we consider the generic form for the 4D metric (4.56), in the presence of the back reaction, as a starting point. As mentioned before, by using a dimensional reduction technique the effective field theory near the horizon becomes two dimensional. The metric of this two dimensional theory is identical to the $r - t$ component of the full metric (4.56). On this effective 1 + 1 dimensional background, if we omit the classically ingoing (left moving) modes, then the theory becomes chiral.

Now we compute the modified charge and energy-momentum flux by using the chiral effective action approach discussed earlier. Instead of repeating the whole analysis we just use the structures for the covariant current $\langle J^r \rangle$ and the covariant energy-momentum tensor $\langle T^r_{\rho} \rangle$ given in (4.41) and (4.52), respectively. Then the Hawking charge and energy-momentum flux, in the presence of gravitational back reaction can be obtain by appropriately taking the asymptotic infinity limit of the covariant current and energy-momentum tensor.

First, let us consider the expression for the chiral covariant current as given in (4.41)

$$ \langle J^r(r) \rangle = \frac{e^2}{2\pi \sqrt{-g}} [A_t(r) - A_t(r_M)] $$

where the gauge potential is given by

$$ A_t(r) = -\frac{Q}{r}. $$

The charge flux is determined by the asymptotic infinity limit of the anomaly free current. As is evident from the expression (4.22), the covariant gauge anomaly vanishes in this limit.
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and therefore we can obtain the charge flux directly from (4.64) by taking its asymptotic limit. This gives

\[ \langle J^r (r \to \infty) \rangle = -\frac{e^2}{2\pi} A_t (r_M) \ . \]

(4.66)

Finally, by using (4.59) and (4.65) in (4.66), we obtain

\[ \langle J^r (r \to \infty) \rangle = \frac{e^2 Q}{2\pi r_+} \left( 1 + \frac{\alpha}{M^2} \right)^{\frac{3}{2}} \ . \]

(4.67)

This is the expression for the Hawking charge flux from the Reissner-Nordstrom black hole in the presence of gravitational back reaction.

Further, by expanding \( \left( 1 + \frac{\alpha}{M^2} \right)^{\frac{3}{2}} \) and keeping only leading order terms in \( \alpha \), we find

\[ \langle J^r (r \to \infty) \rangle \approx \frac{e^2 Q}{2\pi r_+} + \frac{e^2 Q \alpha}{4\pi r_+ M^2} \ . \]

(4.68)

The first term in the above expression is the usual charge flux for the Reissner-Nordstrom black hole while the next term represents correction to the standard value of charge flux due to the effect of one loop back reaction. Note that, since the trace anomaly coefficient \( \alpha \) is negative (4.61), there is a net decrease in the Hawking charge flux compare to its standard value.

Next, we consider the expression for the chiral covariant energy-momentum tensor (4.52)

\[ \langle T^{r \ell} (r) \rangle = \frac{e^2}{4\pi \sqrt{-g}} (A_t (r) - A_t (r_h))^2 + \frac{1}{192\pi \sqrt{-g}} [N^{r \ell}_t (r) - N^{r \ell}_t (r_M)] \ . \]

(4.69)

The Hawking energy-momentum flux is given by the asymptotic limit of the anomaly free energy-momentum tensor. In this limit, we observe that the covariant gravitational anomaly (4.23) vanishes. Hence the energy-momentum flux can be easily obtained by taking the asymptotic infinity limit of (4.69)

\[ \langle T^{r \ell} (r \to \infty) \rangle = \frac{e^2 A_t^2 (r_+)}{4\pi} + \frac{1}{192\pi} f'(r_h) h'(r_M) \ . \]

(4.70)

We can also write the above expression in terms of the modified surface gravity \( \kappa_M = \frac{1}{2} \sqrt{f'(r_M) h'(r_M)} \) as

\[ \langle T^{r \ell} (r \to \infty) \rangle = \frac{e^2 Q^2}{4\pi r_+^2} \left( 1 + \frac{\alpha}{M^2} \right) + \frac{1}{48\pi} \kappa_M^2 \ . \]

(4.71)
4.4. Discussions

\( \kappa_M \) is the modification in the surface gravity due to the effect of one loop back reaction. Following similar arguments given below equation (4.59) we can relate \( \kappa_M \) to the usual surface gravity for the Reissner-Nordstrom black hole \( \kappa \) [106, 107]

\[
\kappa_M = \kappa \left( 1 + \frac{\alpha}{M^2} \right), \tag{4.72}
\]

Substituting (4.72) in (4.71), yields

\[
\langle T^r_t (r \to \infty) \rangle = \frac{e^2 Q^2}{4 \pi r_H^2} \left( 1 + \frac{\alpha}{M^2} \right) + \frac{1}{48 \pi} \kappa^2 \left( 1 + \frac{\alpha}{M^2} \right)^2 \]

\[
= \frac{e^2 Q^2}{4 \pi r_H^2} \left( 1 + \frac{\alpha}{M^2} \right) + \frac{\pi T_H^2}{12} (1 + \frac{\alpha}{M^2})^2, \tag{4.73}
\]

where \( T_H = \frac{\kappa}{2 \pi \alpha} \) is the usual Hawking temperature of the Reissner-Nordstrom black hole. Further, by expanding \( (1 + \frac{\alpha}{M^2})^2 \) and keeping terms up to leading order in \( \alpha \), we arrive at

\[
\langle T^r_t (r \to \infty) \rangle \approx \frac{e^2 Q^2}{4 \pi r_H^2} + \frac{\pi T_H^2}{12} + \frac{e^2 Q^2 \alpha}{2 \pi r_H^2 M^2} + \frac{\pi T_H^2 \alpha}{6 M^2}. \tag{4.74}
\]

The first two terms in the above expression represent energy flux from the usual charged black hole, while the last two terms are corrections due to the effect of one loop back reaction. Since the trace anomaly coefficient \( \alpha \) is negative, the overall effect of gravitational back reaction, at one loop level, is to reduce the Hawking flux from its usual value. This feature is also valid for the fermions (\( s = \frac{1}{2} \)) and gravitons (\( s = 2 \)). On contrary, for the fields with \( s = 1, \frac{3}{2}, \cdots \), the situation is exactly opposite. In these cases the trace anomaly coefficient \( \alpha \) becomes positive, leading to the increase in the Hawking charge and energy-momentum flux [107].

4.4 Discussions

We have given a derivation of the Hawking flux from charged black holes, based on the effective action approach, which only employs the boundary conditions at the event horizon. It might be mentioned that generally such (effective action based) approaches require, apart from conditions at the horizon, some other boundary condition, as for example, the vanishing of ingoing modes at infinity [26, 46, 110]. The latter obviously
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going against the universality of the Hawking effect which should be determined from conditions at the horizon only. In this we have succeeded. Also, the specific structure of the effective action, which gives the Hawking radiation, is valid only at the event horizon. This is the anomalous (chiral) effective action. Other effective action based techniques do not categorically specify the structure of the effective action at the horizon. Rather, they use the usual (anomaly free) form for the effective action and are restricted to two dimensions only [102, 103, 110].

An important factor concerning this analysis is to realize that effective field theories become two dimensional and chiral near the event horizon [28]. Yet another ingredient was the implementation of a specific boundary condition: namely the vanishing of the covariant form of the current and energy-momentum tensor. As emphasized in chapter-3, the anomaly based approach was simplified considerably if, instead of consistent anomalies used in [28, 32], covariant anomalies were taken as the starting point. Indeed, in the present computations, we have taken that form of the effective action which yields anomalous Ward identities having covariant gauge and gravitational anomalies. This is distinct from the standard (Polyakov type) effective action [32, 101]. The connection these two as well as their correspondence with the Unruh vacuum is the topic of next chapter.

The arbitrary constants in the covariant energy momentum tensor and the covariant current derived from the anomalous effective action were fixed by a boundary condition at the event horizon. The Hawking fluxes, which are measured at infinity, are then obtained by taking the asymptotic infinity limit of the covariant current and energy-momentum tensor. This may be compared with the anomaly based approach given in chapter-3 and [32] where the Hawking radiation is derived by solving both the anomalous Ward identities as well as the usual conservation laws together with the imposition of covariant boundary condition at the event horizon. Apart from this the use of discontinuous step functions, which are essential in the anomaly based approach, is avoided. Consequently, we have shown that aspects like covariant anomalies and covariant boundary conditions are not merely confined to discussing the Hawking effect in the anomaly based approach. Rather they have a wider applicability since our effective action based approach is different from (although connected with) the anomaly based approach.
Finally, we implement the chiral effective action approach to compute the Hawking charge and energy-momentum flux for the Reissner-Nordstrom black hole taking into the account the effect of one loop back reaction. The point is that the $r-t$ part of usual charged black hole ($\sqrt{-g} = 1$) gets modified to a more general ($\sqrt{-g} \neq 1$) due to the effect of back reaction without disturbing the spherical symmetry. For this general metric, the expressions for the covariant current and energy momentum tensor were obtained. This indicates the generality of the chiral effective action approach. The corrections to charge and energy flux due to (one loop) back reaction effect were then obtained by appropriately taking asymptotic limit of the current and energy momentum tensor.

Apart from this example, the chiral effective action approach discussed here has been implemented in the computation of Hawking radiation from other several black hole geometries [111, 112, 113, 114].
Chapter 5

Covariant boundary conditions and connection with vacuum states

The motivation of this chapter is to provide a clear understanding of the covariant boundary condition used in the analysis of chapter-3, chapter-4 and also in [32], of deriving the Hawking flux using chiral gauge and gravitational anomalies. Besides this we also reveal certain new features in chiral currents and energy-momentum tensors which are useful in exhibiting their connection with the standard nonchiral expressions.

In chapter-3 we gave a method, based on the covariant gauge and gravitational anomalies, to compute the fluxes of Hawking radiation. Hawking fluxes were obtained by solving the covariant anomalous gauge/gravitational Ward identities (valid near the horizon) as well as the usual conservation laws (valid away from the horizon) together with the implementation of covariant boundary condition; namely, the vanishing of covariant anomalous current and energy-momentum tensor at the horizon. It is important to note that the analysis of [32] also uses the same boundary condition, however, the expressions for gauge and gravitational anomalies were taken to be consistent. Consequently, the knowledge of local counterterms, connecting consistent and covariant expressions, becomes essential in the consistent anomaly based approach [32].

In another new development (see chapter-4), we obtained the Hawking charge and energy-momentum flux by exploiting the structure of the chiral effective action [99], appropriately modified by local counterterms. This chiral effective action is defined in the
neighborhood of the event horizon. The currents and energy-momentum tensors obtained from this effective action satisfy the covariant gauge and gravitational Ward identities. Again, the arbitrary constants appearing in the expressions for covariant current and energy-momentum tensor were fixed by imposing the covariant boundary condition at the event horizon. Finally, the Hawking charge and energy-momentum fluxes were obtained by taking appropriately the asymptotic infinity limit of these covariant currents and energy-momentum tensors, respectively.

Apart from these approaches, there is an alternative procedure to compute the Hawking fluxes [45]. Like the chiral effective action approach, this method uses only the near horizon structures for the covariant current and energy-momentum tensor obtained by solving the covariant gauge and gravitational Ward identities. As before, the arbitrary constants appearing in the expressions for current and energy-momentum tensor were fixed by imposing the covariant boundary condition. Since the gauge and gravitational anomalies vanish in the asymptotic limit, it is expected that the asymptotic behavior of the chiral covariant current and energy-momentum tensor would be identical as that of anomaly free current and energy-momentum tensor. This expectation is also confirmed by actual computation of Hawking fluxes, which were obtained by taking the asymptotic infinity limit of the covariant current and energy-momentum tensor.

It is thus clear from the above discussion that the covariant boundary condition plays an important role in the computation of Hawking fluxes. Also, the imposition of this boundary condition at the event horizon helps to make the whole anomaly approach consistent with the universality of the Hawking effect.

Here we give a detailed analysis for this particular boundary condition, clarifying its role in the computation of the Hawking flux. First, we compute the Hawking flux by adopting the approach of [45]. It turns out that, with this choice of boundary condition, the components for covariant current/energy-momentum tensor \( \langle J^r \rangle, \langle T^{r \tau} \rangle \) obtained from solving the anomaly equation match exactly with the expectation values of the current/energy-momentum tensor, obtained from the chiral effective action, taken by imposing the regularity condition on the outgoing modes at the future horizon. Furthermore, we discuss the connection of our results with those found by a standard use of boundary conditions on nonchiral (anomaly free) currents and energy-momentum ten-
5.1 Charge and energy flux from covariant anomaly: A direct approach

Consider a generic spherically symmetric charged black hole background represented by the metric

$$ds^2 = \gamma_{ab} dx^a dx^b = f(r) dt^2 - \frac{1}{h(r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

(5.1)

where $f(r)$ and $h(r)$ are the metric coefficients\(^1\). The event horizon for this black hole is defined by

$$f(r_h) = h(r_h) = 0 .$$

(5.2)

Now consider charged scalar fields propagating on this background. As discussed earlier, the effective field theory near the event horizon becomes two dimensional with the metric.

---

\(^1\)This metric is same as the one given in section-4.1. Consequently, the vector potential $A$ and the metric coefficients $f(r)$ and $h(r)$ satisfy all the properties given in (4.2) and (4.4), respectively.
given by the $r-t$ section of (5.1)

$$ds^2 = f(r)dt^2 - \frac{1}{h(r)}dr^2.$$  \hspace{1cm} (5.3)

On this two dimensional background, the modes which are going in to the black hole (for example left moving modes) are lost and the effective theory become chiral. Two dimensional chiral theory possesses gravitational anomaly and, if gauge fields are present, also gauge anomaly [31, 42, 43, 44, 77, 78, 79]. These anomalies are further classified in two groups - the consistent and the covariant [42, 43, 78, 79]. A derivation of Hawking flux by using the consistent gauge and gravitational anomalies was given in [32]. However, the boundary condition used to fix the arbitrary constants was covariant. A complete reformulation of this approach using only covariant structures was given in chapter-3, while the corresponding effective action based approach was developed in chapter-4.

An efficient and quite simple way to obtain the Hawking flux was discussed in [45] where the computation involved only the expressions for anomalous covariant Ward identities and the covariant boundary condition. An important advantage of this approach was that the splitting of space into two different regions (see [32] and chapter-3) is avoided. In this section we would first generalize this approach for the generic black hole background (5.1). This would also help in setting up the conventions and introduce certain expressions that are essential for subsequent analysis.

\section*{Charge flux:}

We now compute the Hawking charge flux by using only the covariant gauge anomaly and the covariant boundary condition. The expression for covariant gauge anomaly [42, 44] for the right moving modes (3.42) is

$$\nabla\mu\langle J^\mu \rangle = -\frac{e^2}{4\pi\sqrt{-g}}e^{\alpha\beta}F_{\alpha\beta}.$$  \hspace{1cm} (5.4)

For a static background, the above equation becomes,

$$\partial_r(\sqrt{-g}\langle J^r \rangle) = \frac{e^2}{2\pi}\partial_rA_t.$$  \hspace{1cm} (5.5)

Solving this equation we get

$$\sqrt{-g}\langle J^r \rangle = c_H + \frac{e^2}{2\pi}[A_t(r) - A_t(r_h)].$$  \hspace{1cm} (5.6)
Here $c_H$ is an integration constant which can be fixed by imposing the covariant boundary condition i.e covariant current $\langle J^r \rangle$ must vanish at the event horizon,

$$\langle J^r (r = r_h) \rangle = 0.$$  \hspace{1cm} (5.7)

Hence we get $c_H = 0$ and the expression for the current becomes,

$$\langle J^r \rangle = \frac{e^2}{2\pi \sqrt{-g}} [A_t(r) - A_t(r_h)].$$  \hspace{1cm} (5.8)

Note that the Hawking flux is measured at infinity where there is no anomaly. This necessitated a split of space into two distinct regions - one near the horizon and one away from it - and the use of two Ward identities (see chapter-3 and [32]) This is redundant if we observe that the anomaly (5.4) vanishes at asymptotic infinity. Consequently, in this approach, the flux is directly obtained from the asymptotic infinity limit of (5.8):

$$\text{Charge flux} = \langle J^r (r \to \infty) \rangle = -\frac{e^2 A_t(r_h)}{2\pi}.$$  \hspace{1cm} (5.9)

This reproduces the familiar expression for the charge flux obtained earlier in chapter-3.

**Energy-momentum flux:**

Next, we consider the expression for the two dimensional covariant gravitational Ward identity (3.62)

$$\nabla_\mu \langle T^{\mu\nu} \rangle = \langle J_\mu \rangle F^{\mu\nu} + \frac{\epsilon^{\nu\mu}}{96\pi \sqrt{-g}} \nabla_\mu R$$  \hspace{1cm} (5.10)

where the first term is the classical contribution (Lorentz force) and the second is the covariant gravitational anomaly [31, 77, 115]. Here $R$ is the Ricci scalar and for the metric (5.3) it is given by

$$R = \frac{f''}f + \frac{f'h'}{2f} - \frac{f'^2h}{2f^2}.$$  \hspace{1cm} (5.11)

By simplifying (5.10) we get, in the static background,

$$\partial_r (\sqrt{-g} \langle T_{rr} \rangle) = \partial_r N_f^r (r) - \frac{e^2 A_t(r_h)}{2\pi} \partial_r A_t(r) + \partial_r \left( \frac{e^2 A_t^2(r)}{4\pi} \right)$$  \hspace{1cm} (5.12)

where

$$N_f^r = \frac{1}{96\pi} \left( h f'' + \frac{f'h'}{2} - \frac{f'^2h}{f} \right).$$  \hspace{1cm} (5.13)
The solution for (5.12) is given by
\[ \sqrt{-g} \langle T^r_t \rangle = b_H + [N^r_t(r) - N^r_t(r_h)] + \frac{e^2}{4\pi} [A_t(r) - A_t(r_h)]^2. \] (5.14)

Here \( b_H \) is an integration constant. Implementing the covariant boundary condition, namely, the vanishing of covariant EM tensor at the event horizon,
\[ \langle T^r_t(r = r_h) \rangle = 0 \] (5.15)
yields \( b_H = 0 \). Hence (5.14) reads
\[ \sqrt{-g} \langle T^r_t(r) \rangle = [N^r_t(r) - N^r_t(r_h)] + \frac{e^2}{4\pi} [A_t(r) - A_t(r_h)]^2. \] (5.16)

Since the covariant gravitational anomaly vanishes asymptotically, we can compute the energy flux as before by taking the asymptotic limit of (5.16)
\[ \text{energy flux} = \langle T^r_t(r \rightarrow \infty) \rangle = -N^r_t(r_h) + \frac{e^2 A_t^2(r_h)}{4\pi}. \] (5.17)

This reproduces the expression for the Hawking flux found by earlier using the anomaly based approach (chapter-3, [32]) or by using the chiral effective action (chapter-4).

### 5.2 Covariant boundary condition and vacuum states

It is now clear that the covariant boundary conditions play a crucial role in the computation of Hawking fluxes using chiral gauge and gravitational anomalies, either in the approaches discussed in [32, 33], chapter-3, chapter-4 or in the more direct method [45] reviewed here. Therefore it is worthwhile to study it in some detail. We adopt the following strategy. We consider the expressions for the expectation values of the covariant current and energy-momentum tensor deduced from the chiral effective action [99], suitably modified by a local counterterm. These are already given in section-4.1. We then transform the components of current and energy-momentum tensor into null coordinates. The arbitrary constants appearing in the expressions for current and energy-momentum tensor are now fixed by imposing regularity conditions on the outgoing modes at the future event horizon. The final results are found to match exactly with the corresponding expressions for the covariant current (5.8) and EM tensor (5.16), which were derived
by using the covariant boundary conditions (5.7, 5.15). Subsequently we show that our results are consistent with the imposition of the Unruh vacuum on usual (nonchiral) expressions.

We begin our analysis by considering the theory near the event horizon. An expression for the chiral covariant current obtained from the chiral effective action (4.9) is given in (4.14). Substituting the solution for the auxiliary field $B(r, t)$ (4.33) in (4.14) and then taking $\mu = r$ component of the covariant current $\langle J^\mu \rangle$, we get

$$\langle J^r(r) \rangle = \frac{e^2}{2\pi \sqrt{-g}} [\tilde{A}_t(r)]$$  \hspace{1cm} (5.18)

where $\tilde{A}_t(r)$ is defined by (4.38). The $\mu = t$ component of the covariant current is obtained by exploiting the chirality condition (4.26)

$$\langle J^t(r) \rangle = -\tilde{\epsilon}^{tr}\langle J^r(r) \rangle = -\tilde{\epsilon}^{tr} g_{tr} \langle J^r(r) \rangle = \frac{\sqrt{-g}}{f} \langle J^r(r) \rangle .$$  \hspace{1cm} (5.19)

Next, we consider the chiral covariant energy-momentum tensor given in (4.15). Using the solutions for the auxiliary fields $B(r, t)$ (4.33) and $G(r, t)$ (4.35) in (4.15), the expressions for the various components of $\langle T^\mu_\nu \rangle$ follow from (4.43-4.45)

$$\langle T^t_t \rangle = \frac{e^2}{4\pi \sqrt{-g}} \tilde{A}_t^2(r) + \frac{1}{12\pi \sqrt{-g}} \tilde{P}^2(r) + \frac{1}{24\pi \sqrt{-g}} \left[ -\frac{f'}{\sqrt{-g}} \tilde{P}(r) + \tilde{Q}(r) \right]$$  \hspace{1cm} (5.20)

$$\langle T^r_r \rangle = \frac{R}{96\pi} - \frac{\sqrt{-g}}{f} \langle T^r_t \rangle$$  \hspace{1cm} (5.21)

$$\langle T^t_t \rangle = -\langle T^r_r \rangle + \frac{R}{48\pi}$$  \hspace{1cm} (5.22)

where $\tilde{A}(r), \tilde{P}(r)$ and $\tilde{Q}(r)$ are defined by the relations (4.38, 4.46) and (4.47), respectively. As illustrated in section-4.1, the chirality conditions (4.26, 4.31) imposes certain restrictions on the components of current (5.18, 5.19) and the energy-momentum tensor (5.20-5.22). For example the relation (5.21) is obtained by using (4.31) and (5.20), while the remaining component $\langle T^t_t \rangle$ can be fixed by using the expression for the chiral trace anomaly (4.25) \(^2\)

$$\langle T^\mu_\mu \rangle = \frac{R}{48\pi} .$$  \hspace{1cm} (5.23)

\(^2\)For a (1 + 1) dimensional chiral theory, it is possible to derive a relation among the trace and gravitational anomalies. See appendix of this chapter for more details.
To further illuminates the chiral nature of the theory near the event horizon, we transform the various components of the covariant current and energy-momentum tensor to null coordinates

\[ v = t + r^* \]
\[ u = t - r^* \] (5.24)

where \( r^* \) is the tortoise coordinate defined by the relation

\[ \frac{dr}{dr^*} = \sqrt{fh} . \] (5.25)

The metric (5.3) in these coordinates looks like

\[ ds^2 = \bar{g}_{\alpha\beta}dx^\alpha dx^\beta = f(r)^2 (dudv + dvdu) ; \, \alpha, \beta = u, v . \] (5.26)

The metric coefficients \( \bar{g}_{\alpha\beta} \) are:

\[ \bar{g}_{uu} = \bar{g}_{vv} = 0 ; \, \bar{g}_{uv} = \bar{g}_{vu} = \frac{f(r)}{2} \] (5.27)

Now the components of the covariant current in \((u,v)\) and \((r,t)\) coordinates are related as

\[ \langle J_u \rangle = \frac{\partial t}{\partial u} \langle J_t \rangle + \frac{\partial r}{\partial u} \langle J_r \rangle \] (5.28)
\[ \langle J_v \rangle = \frac{\partial t}{\partial v} \langle J_t \rangle + \frac{\partial r}{\partial v} \langle J_r \rangle . \] (5.29)

After using (5.18, 5.19, 5.24, 5.25) in (5.28, 5.29) we arrive at the expressions for the components of chiral covariant current in \(u,v\) coordinates

\[ \langle J_u(r) \rangle = \frac{1}{2}[\langle J_t \rangle - \sqrt{fh} \langle J_r \rangle] = \frac{e^2}{2\pi} \tilde{A}_t(r) \] (5.30)
\[ \langle J_v(r) \rangle = \frac{1}{2}[\langle J_t \rangle + \sqrt{fh} \langle J_r \rangle] = 0 . \] (5.31)

Following similar steps, the components of the chiral covariant energy-momentum tensor in \(u,v\) coordinates are given by

\[ \langle T_{uu}(r) \rangle = \frac{1}{4}[f\langle T^t_t \rangle - f\langle T^r_r \rangle + 2\sqrt{-\bar{g}}\langle T^r_t \rangle] \]
\[ = \frac{e^2}{4\pi} \tilde{A}_t^2(r) + \frac{1}{12\pi} \bar{P}^2(r) + \frac{1}{24\pi} \left[ \frac{f'}{\sqrt{-g}} \bar{P}(r) + \bar{Q}(r) \right] \] (5.32)
\[ \langle T_{uv}(r) \rangle = \frac{f}{4}[\langle T^t_t \rangle + \langle T^r_r \rangle] = \frac{1}{192\pi} fR \] (5.33)
\[ \langle T_{vv}(r) \rangle = \frac{1}{4}[f\langle T^t_t \rangle - f\langle T^r_r \rangle - 2\sqrt{-\bar{g}}\langle T^r_t \rangle] = 0 . \] (5.34)
We now observe that, due to the chiral property, the $\langle J_v \rangle$ and $\langle T_{vv} \rangle$ components vanish everywhere. These correspond to the ingoing modes and are compatible with the fact, stated earlier, that the near horizon theory is a two dimensional chiral theory where the ingoing modes are lost. Further, by rewriting (5.33) as

$$\langle T_{uv} \rangle = f(r)R_{192 \pi} = f(r)\frac{4}{\langle T_{\alpha \alpha} \rangle}$$  

(5.35)

we observed that the structure of $\langle T_{uv} \rangle$ is fixed by the trace anomaly (5.23). Only the $\langle J_u \rangle$ and $\langle T_{uu} \rangle$ components involve the undetermined constants. These will now be determined by considering various vacuum states.

5.2.1 Vacuum states

In a generic spacetime three different vacua [48] are defined by appropriately choosing ‘in’ and ‘out’ modes.

1) The Unruh vacuum:

In this state the ‘in’ modes are chosen as to be positive frequency with respect to the Schwarzschild time ‘$t'$’. With this choice, in the asymptotic past the Unruh vacuum $|U\rangle$ coincides with the usual Minkowski vacuum. On the other hand, out modes are taken to be positive frequency with respect to the Kruskal coordinate $U = -\kappa e^{-\kappa u}$; $\kappa$ is the surface gravity (5.36)

The Kruskal coordinate $U$ acts as the affine parameter along the past horizon. This mimics the late time behavior of modes coming out of a collapsing star as its surface approaches the horizon [46]. By this choice $\langle U|T_{\mu\nu}|U\rangle$ is regular on the future event horizon $H^+$ i.e a freely falling observer must see a finite amount of flux at the future event horizon $H^+$. In the asymptotic future $\langle U|T_{\mu\nu}|U\rangle$ has the form of a flux of radiation at the Hawking temperature $T_H$ [101, 102]. This state is the most appropriate to discuss evaporation of black holes formed by gravitational collapse of matter.

2) Hartle-Hawking vacuum:

The Hartle-Hawking state $|H\rangle$ [47] is obtained by choosing in modes to be positive frequency with respect to the Kruskal coordinate

$$V = \kappa e^{\kappa u}$$  

(5.37)
the affine parameter on the future horizon, whereas outgoing modes are defined in the
same way as for Unruh vacuum. By construction this state is regular on both the future
and past event horizons. Consequently \( \langle H|T_{\mu\nu}|H \rangle \) is also regular on the future and past
horizons. Hartle-Hawking vacuum is appropriate to describe a black hole in thermal
equilibrium with quantum field under consideration.

3) Boulware vacuum:
Boulware vacuum \( |B \rangle \) \[116\] is obtained by choosing both ‘in’ and ‘out’ modes to be
positive frequency with respect to the Schwarzschild time coordinate \( t \). This state most
closely reproduces the familiar notion of Minkowski vacuum asymptotically. However,
since the Schwarzschild coordinates are not well define near the horizon, the expectation
value of the energy-momentum in the Boulware vacuum \( \langle B|T_{\mu\nu}|B \rangle \) blows up at the event
horizon.

This general picture is modified when dealing with a chiral theory since, as shown
before, the ‘in’ modes always vanish. Consequently this leads to a simplification and
conditions are imposed only on the ‘out’ modes. Moreover, these conditions have to be
imposed on the horizon since the chiral theory is valid only there. The natural condition,
leading to the occurrence of Hawking flux, is that a freely falling observer must see
a finite amount of flux at the horizon. This implies that the current (EM tensor) in
Kruskal coordinates must be regular at the future horizon. Effectively, this is the same
condition on the ‘out’ modes in either the Unruh vacuum \[46\] or the Hartle-Hawking
vacuum \[47\]. As far as our analysis is concerned this is sufficient to completely determine
the form of \( \langle J_{\mu} \rangle \) or \( \langle T_{\mu\nu} \rangle \). We show that their structures are identical to those obtained
in the previous section using the covariant boundary condition.

A more direct comparison with the conventional results obtained from Unruh or
Hartle-Hawking states is possible. In that case one has to consider the nonchiral ex-
pressions \[33, 101\] containing both ‘in’ and ‘out’ modes. We show that, at asymptotic
infinity where the flux is measured, our expressions agree with that calculated from Unruh
vacuum only. We discuss this in some detail.
Regularity conditions, Unruh and Hartle-Hawking vacua:

We now fix the arbitrary constants appearing in the covariant current (5.30) and energy-momentum tensor (5.32) by imposing the regularity conditions, appropriate for the Unruh [46] and Hartle-Hawking [47] vacua, on outgoing modes at future event horizon.

First, consider the Unruh vacuum. As mentioned earlier, in this vacuum, ‘out’ modes are defined with respect to Kruskal coordinate $U$ (5.36). Therefore, we first transform $\langle J_u \rangle$ to $\langle J_U \rangle$ defined in Kruskal coordinate. $\langle J_u \rangle$ and $\langle J_U \rangle$ are related by

$$\langle J_U \rangle = -\frac{\langle J_u \rangle}{\kappa U} \tag{5.38}$$

Now the regularity condition tells us that a freely falling observer must see a finite amount of charge flux at the future event horizon. However, since near the future event horizon $U \to \sqrt{r - r_h}$ ($r \to r_h$), it implies that $\langle J_u \rangle$ must vanish at $r \to r_h$. Hence from (4.38) and (5.30) we find

$$c + a = -A_t(r_h) \tag{5.39}$$

Similarly, imposing the condition that $\langle T_{UU} \rangle = \left(\frac{1}{\kappa U}\right)^2 \langle T_{uu} \rangle$ must be finite at future horizon leads to $\langle T_{uu}(r \to r_h) \rangle = 0$. This yields, from (4.38, 4.46, 4.47) and (5.32),

$$p = \frac{1}{4}(z \pm \sqrt{f'(r_h)h'(r_h)}) \tag{5.40}$$

Substituting (5.39) and (5.40) in (5.30) and (5.32) we get

$$\langle J_u(r) \rangle = \frac{e^2}{2\pi}[A_t(r) - A_t(r_h)] \tag{5.41}$$

$$\langle T_{uu}(r) \rangle = \frac{e^2}{4\pi}[A_t(r) - A_t(r_h)]^2 + [N_t'(r) - N_t'(r_h)] \tag{5.42}$$

where $N_t'(r)$ is given by (5.13). Finally, by transforming back to $r - t$ coordinates, we obtain the expressions

$$\langle J^r(r) \rangle = \frac{e^2}{2\pi\sqrt{-g}}[A_t(r) - A_t(r_h)] \tag{5.43}$$

$$\langle J^t(r) \rangle = \frac{\sqrt{-g}}{f} \langle J^r(r) \rangle \tag{5.44}$$
for the covariant current. While the energy-momentum tensor is given by,
\[
\sqrt{-g} \langle T^r_i \rangle = \frac{e^2}{4\pi} [A_t(r) - A_t(r_h)]^2 + [N^r_t(r) - N^r_t(r_h)].
\] (5.45)
Likewise, \(\langle T^r_r \rangle\) and \(\langle T^t_t \rangle\) follow from (5.21, 5.22).

The expressions for \(\langle J^r \rangle (5.43)\) and \(\langle T^r_t \rangle (5.45)\) agree with the corresponding ones given in (5.8) and (5.16). This shows that the structures for the universal components \(\langle J^r \rangle, \langle T^r_t \rangle\) obtained by solving the anomalous Ward identities (5.4, 5.10) subjected to the covariant boundary conditions (5.7, 5.15) exactly coincide with the results computed by demanding regularity at the future event horizon.

Next, consider the Hartle-Hawking vacuum. In this case, both \(\langle T^U_U \rangle\) and \(\langle T^V_V \rangle\) are regular at the future and past event horizons, respectively. In the null coordinates \((u, v)\), the above regularity condition is translated into the vanishing of \(\langle T^u_u \rangle\) and \(\langle T^v_v \rangle\), at the future and past event horizons. In our case however, only the outgoing modes are present in the region near the horizon. Consequently, the regularity condition on ingoing modes i.e., \(\langle T^v_v(r \to r_h) \rangle = 0\) is trivially satisfied (see equation (5.34)). While, the condition on outgoing modes i.e., \(\langle T^u_u(r \to r_h) \rangle = 0\) is same as for the Unruh vacuum. Naturally, the result (5.45) remain unchanged for Hartle-Hawking vacuum also. This is different from the results derived for the usual (nonchiral) theory [48]. As we shall see below, in the conventional analysis, Hartle-Hawking vacuum corresponds to the no flux state and is suitable for describing the black hole in a thermal equilibrium with surrounding quantum fields.

It is possible to compare our findings with conventional (nonchiral) computations where the Hawking flux is obtained in the Unruh vacuum. We begin by considering the conservation equations for a nonchiral theory that is valid away from the horizon.

Such equations were earlier used in [28, 32, 33] (see also chapter-3). Conservation of the gauge current yields \(^3\),
\[
\nabla_\mu \langle \tilde{J}^\mu \rangle = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \langle \tilde{J}^\mu \rangle) = 0 \tag{5.46}
\]
which, in a static background, leads to,
\[
\langle \tilde{J}^r \rangle = \frac{C_1}{\sqrt{-g}} \tag{5.47}
\]

\(^3\)We use a tilde \(\langle \tilde{J}^\mu \rangle\) to distinguish nonchiral expressions from chiral ones.
where $C_1$ is some constant.

As is well know there is no regularisation that simultaneously preserves the vector as well as axial vector gauge invariance. Indeed, for a vector gauge invariant regularisation resulting in (5.46), the following anomaly is found in the axial current,

$$\nabla_\mu \langle \tilde{J}^{5\mu} \rangle = \frac{e^2}{2\pi \sqrt{-g}} \epsilon^{\mu
u} F_{\mu\nu} ; \langle \tilde{J}^{5\mu} \rangle = \frac{1}{\sqrt{-g}} \epsilon^{\mu\nu} \langle \tilde{J}_\nu \rangle .$$

(5.48)

The solution of this Ward identity is given by,

$$\langle \tilde{J}^t \rangle = -\frac{1}{f} [C_2 - \frac{e^2}{\pi} A_t(r)]$$

(5.49)

where $C_2$ is another constant.

In the null coordinates introduced in (5.24) the various components of the current are defined as,

$$\langle \tilde{J}_u \rangle = \frac{1}{2} [C_1 - C_2 + \frac{e^2}{\pi} A_t(r)] ,$$

(5.50)

$$\langle \tilde{J}_v \rangle = -\frac{1}{2} [C_1 + C_2 - \frac{e^2}{\pi} A_t(r)] .$$

(5.51)

The constants $C_1, C_2$ are now determined by using appropriate boundary conditions corresponding to first, the Unruh state, and then, the Hartle-Hawking state. For the Unruh state $\langle \tilde{J}_u (r \to r_h) \rangle = 0$ and $\langle \tilde{J}_v (r \to \infty) \rangle = 0$ yield,

$$C_1 = -C_2 = -\frac{e^2}{2\pi} A_t(r_h) ,$$

(5.52)

so that, reverting back to $(r,t)$ coordinates, we obtain,

$$\langle \tilde{J}^r \rangle = -\frac{e^2}{2\pi \sqrt{-g}} A_t(r_h) ,$$

(5.53)

$$\langle \tilde{J}^t \rangle = \frac{e^2}{\pi f} [A_t(r) - \frac{1}{2} A_t(r_h)] .$$

(5.54)

The Hawking charge flux, identified with $\langle \tilde{J}^r (r \to \infty) \rangle$, reproduces the desired result (5.9). Expectedly, (5.53, 5.54) differ from our relations (5.43, 5.44) which are valid only near the horizon. However, at asymptotic infinity where the Hawking flux is measured, both expressions match, i.e

$$\langle \tilde{J}^r (r \to \infty) \rangle = \langle J^r (r \to \infty) \rangle ,$$

(5.55)

$$\langle \tilde{J}^t (r \to \infty) \rangle = \langle J^t (r \to \infty) \rangle .$$

(5.56)
implying the important consequence,
\[
\langle \tilde{J}^\mu (r \to \infty) \rangle = \langle J^\mu (r \to \infty) \rangle .
\] (5.57)

All the above considerations follow identically for the stress tensor. Now the relevant conservation law in the presence of an external gauge field, is
\[
\nabla_\mu \langle \tilde{T}^\mu_\nu \rangle = \langle \tilde{J}^\mu \rangle F_{\mu\nu}
\] (5.58)

Also, there is a trace anomaly given by,
\[
\langle \tilde{T}^\mu_\mu \rangle = \frac{R}{24\pi}
\] (5.59)

Taking \( \nu = t \) component of (5.58) and using (5.47) we get
\[
\partial_r (\sqrt{-g} \langle \tilde{T}^r_t \rangle) = \langle \tilde{J}^r \rangle \partial_r A_t (r) = C_1 \partial_r A_t .
\] (5.60)

which on static background, leads to
\[
\langle \tilde{T}^r_t \rangle = \frac{1}{\sqrt{-g}} \left[ D_1 + C_1 A_t (r) \right]
\] (5.61)

where \( D_1 \) is an integration constant. Similarly, by taking the \( \nu = r \) component of (5.58) and using (5.49), we obtain,
\[
\frac{1}{\sqrt{-g}} \partial_r (\sqrt{-g} \langle \tilde{T}^r_r \rangle) = \frac{f'}{2f} \langle \tilde{T}^t_t \rangle - \frac{h'}{2h} \langle \tilde{T}^r_r \rangle + \frac{1}{f} \partial_r \left( C_2 A_t - \frac{e^2}{2\pi} A_t^2 \right) .
\] (5.62)

Now by using the expression for trace anomaly (5.59) we can eliminate \( \langle \tilde{T}^t_t \rangle \) from the above equation. Then (5.62) becomes,
\[
\frac{1}{\sqrt{-g}} \partial_r (\sqrt{-g} \langle \tilde{T}^r_r \rangle) = \frac{f'}{2f} \left[ \frac{R}{24\pi} - \langle T^r_r \rangle \right] - \frac{h'}{2h} \langle \tilde{T}^r_r \rangle + \frac{1}{f} \partial_r \left( C_2 A_t - \frac{e^2}{2\pi} A_t^2 \right) .
\] (5.63)

Here the Ricci scalar \( R \) is given in (5.11). The solution for the above equation is given by,
\[
\langle \tilde{T}^r_r \rangle = \frac{f'^2 h}{96\pi f^2} + \frac{1}{f} \left[ C_2 A_t - \frac{e^2}{2\pi} A_t^2 \right] + \frac{D_2}{f} .
\] (5.64)

Where \( D_2 \) is a constant of integration. The remaining component \( \langle \tilde{T}^t_t \rangle \) is determined by the trace anomaly (5.59).
After transforming the components of the energy-momentum tensor (5.61, 5.64) into null coordinates (5.24), we get
\[
\langle \tilde{T}_{uu} \rangle = \frac{1}{4} \frac{hf''}{24\pi} + \frac{f'h'}{48\pi} - \frac{f'^2h}{24\pi f} - 2 \left[ (C_2 - C_1)A_t - \frac{e^2}{2\pi} A_t^3 \right] + 2(D_1 - D_2) \tag{5.65}
\]
\[
\langle \tilde{T}_{vv} \rangle = \frac{1}{4} \frac{hf''}{24\pi} + \frac{f'h'}{48\pi} - \frac{f'^2h}{24\pi f} - 2 \left[ (C_2 + C_1)A_t - \frac{e^2}{2\pi} A_t^3 \right] - 2(D_2 + D_1) \tag{5.66}
\]

The arbitrary constants \(D_1\) and \(D_2\) are now fixed by imposing the boundary conditions appropriate for the Unruh and Hartle-Hawking vacua. Let us first consider the Unruh vacuum. In this state, the energy-momentum tensor in the Kruskal coordinate is regular across the horizon, leading to the condition \(\langle \tilde{T}_{uu}(r \to r_h) \rangle = 0\). Using this in (5.65), and noting the expressions of \(C_1\) and \(C_2\) (for the Unruh vacuum) given in (5.52), we get a relation among the integration constants
\[
D_2 - D_1 = \frac{f'(r_h)h'(r_h)}{96\pi} + \frac{e^2A_t^2(r_h)}{2\pi} \tag{5.67}
\]

Also, in the Unruh vacuum, there is no incoming flux at past null infinity i.e \(\langle \tilde{T}_{vv}(r \to \infty) \rangle = 0\). This condition gives another relation
\[
D_2 = -D_1. \tag{5.68}
\]

Combining (5.67) and (5.68) we have
\[
D_1 = -D_2 = \frac{1}{192\pi} f'(r_h)h'(r_h) + \frac{e^2A_t^2(r_h)}{2\pi}. \tag{5.69}
\]

Substituting (5.69) in (5.61) we get
\[
\langle \tilde{T}^r_t \rangle = \frac{1}{192\pi\sqrt{-g}} f'(r_h)h'(r_h) + \frac{e^2}{4\pi\sqrt{-g}} A_t^2(r_h) - 2A_t(r_h)A_t(r_h) \tag{5.70}
\]
\[
\langle \tilde{T}^r_r \rangle = \frac{1}{96\pi f} \left[ \frac{f'^2h}{f} - \frac{1}{2} f'^2(r_h)h'(r_h) \right] - \frac{e^2}{2\pi f} \left( A_t^2(r) - A_t(r_h)A_t + \frac{1}{2} A_t^2(r_h) \right) \tag{5.71}
\]

while, \(\langle \tilde{T}^t_t \rangle\) can be fixed from the trace anomaly (5.59). The Hawking energy-momentum flux, identified with \(\langle \tilde{T}^r_t(r \to \infty) \rangle\), gives the desired result (5.17). Once again \(\langle \tilde{T}^\mu_\nu \rangle\) will not agree with our \(\langle T^\mu_\nu \rangle\) (5.45). However, at asymptotic infinity, all components agree:
\[
\langle \tilde{T}^\mu_\nu(r \to \infty) \rangle = \langle T^\mu_\nu(r \to \infty) \rangle, \tag{5.72}
\]
leading to the identification of the Hawking flux with \( \langle \tilde{T}_{rr}(r \to \infty) \rangle \).

The equivalences (5.57, 5.72) reveal the internal consistency of our approach. They are based on two issues. First, in the asymptotic limit the covariant chiral gauge (5.4) and gravitational (5.10) anomalies vanish and, secondly, the boundary conditions (5.7, 5.15) get identified with the Unruh state that is appropriate for discussing Hawking effect. It is important to note that, asymptotically, all the components, and not just the universal component that yields the flux, agree.

In the Hartle-Hawking state, the conditions \( \langle \tilde{J}_u(r \to r_h) \rangle = 0 \) and \( \langle \tilde{J}_t(r \to r_h) \rangle = 0 \) yield,

\[
C_1 = 0 ; \quad C_2 = \frac{e^2}{\pi} A_t(r_h) \tag{5.73}
\]

so that,

\[
\langle \tilde{J}^r(r) \rangle = 0, \tag{5.74}
\]

\[
\langle \tilde{J}^t(r) \rangle = \frac{e^2}{\pi f} (A_t(r) - A_t(r_h)) , \tag{5.75}
\]

Expectedly, there is no Hawking (charge) flux now. The above expressions, even at asymptotic infinity, do not agree with our expressions (5.43, 5.44).

Now consider the stress tensor. In the Hartle-Hawking vacuum both \( \langle T_{uu} \rangle \) (5.65) and \( \langle T_{vv} \rangle \) (5.66) are regular on the future and past event horizon respectively i.e

\[
\langle \tilde{T}_{uu}(r \to r_h) \rangle = \langle \tilde{T}_{vv}(r \to r_h) \rangle = 0 . \tag{5.76}
\]

By evaluating (5.65, 5.66) at \( r = r_h \) and then equating to zero, we get the relations among \( D_1 \) and \( D_2 \)

\[
D_1 - D_2 = \frac{f'(r_h)h'(r_h)}{96\pi} + \frac{e^2 A_t^2(r_h)}{2\pi} \tag{5.77}
\]

\[
D_1 + D_2 = -(D_1 - D_2) . \tag{5.78}
\]

Hence, for the Hartle-Hawking state, we have

\[
D_1 = 0 ; \quad D_2 = -\frac{f'(r_h)h'(r_h)}{96\pi} - \frac{e^2 A_t^2(r_h)}{2\pi} . \tag{5.79}
\]
Substituting (5.73, 5.79) in (5.61) and (5.64), yields the expression for various components of the energy-momentum tensor:

\[
\langle \tilde{T}^r_r \rangle = 0 \tag{5.80}
\]

\[
\langle \tilde{T}^r_t \rangle = \frac{1}{96\pi f} \left[ \frac{f'^2 h}{f} - f'(r_h) h'(r_h) \right] - \frac{e^2}{2\pi f} [A_t - A_t(r_h)]^2. \tag{5.81}
\]

There is no energy-momentum flux in the Hartle-Hawking vacuum. Also, the relations (5.80, 5.81), even at the asymptotic limit, do not agree with our expressions (5.45).

**Boulware vacuum:**

Apart from the Unruh and Hartle-Hawking vacua there is another vacuum named after Boulware [116] which closely resembles the Minkowski vacuum asymptotically. In this vacuum, there is no radiation in the asymptotic future. In other words this implies \( \langle J^r \rangle \) and \( \langle T^r_t \rangle \) given in (5.18) and (5.20) must vanish at \( r \to \infty \) limit. Therefore, for the Boulware vacuum, we get

\[
c + a = 0 \tag{5.82}
\]

\[
p = \frac{1}{4} \tag{5.83}
\]

By substituting (5.82) in (5.18) and (5.19) we have

\[
\langle J^r(r) \rangle = \frac{e^2}{2\pi \sqrt{-g}} A_t(r) \tag{5.84}
\]

\[
\langle J^t(r) \rangle = \frac{e^2}{2\pi f} A_t(r). \tag{5.85}
\]

Similarly, by substituting (5.82) and (5.83) in equations (5.20 -5.22), we get

\[
\langle T^r_t \rangle = \frac{e^2 A_t^2(r)}{4\pi \sqrt{-g}} + \frac{1}{\sqrt{-g}} N_t^r(r) \tag{5.86}
\]

\[
\langle T^r_r \rangle = -\frac{e^2 A_t^2(r)}{4\pi f} - \frac{1}{f} N^r_t(r) + \frac{R}{96\pi} \tag{5.87}
\]

\[
\langle T^t_t \rangle = \frac{e^2 A_t^2(r)}{4\pi f} + \frac{1}{f} N^t_t(r) + \frac{R}{96\pi} \tag{5.88}
\]

Observe that there is no radiation in the asymptotic region in the Boulware vacuum. Also, the trace anomaly (5.23) is reproduced since this is independent of the choice of
quantum state. Further, we note that, in the Kruskal coordinates, \( \langle J_U \rangle \) and \( \langle T_{VV} \rangle \) components of current and energy-momentum tensors diverge at the horizon. This can be seen by substituting equations (5.84-5.85) in (5.30). Then the expression for \( \langle J_u \rangle \) in Boulware vacuum becomes,

\[
\langle J_u \rangle = \frac{e^2}{2\pi} A_i(r) \tag{5.89}
\]

while, by putting (5.86-5.88) in (5.32), we obtain, for \( \langle T_{uu} \rangle \)

\[
\langle T_{uu} \rangle = \frac{e^2 A_i^2(r)}{4\pi} + N^r_i(r) \tag{5.90}
\]

Note that in the limit \( (r \to r_h) \) \( \langle J_u \rangle \) and \( \langle T_{uu} \rangle \) do not vanish. Hence, in the Kruskal coordinates, the current and EM tensor diverge. This is expected since the Boulware vacuum is not regular near the horizon.

### 5.3 Discussions

We have analyzed in details a method, briefly introduced in [45], of computing the Hawking flux using covariant gauge and gravitational anomalies. Contrary to earlier approaches discussed in chapter-3 and [28, 32], a split of space into distinct regions (near to and away from horizon) using step functions was avoided. This method is different from the one given in chapter-3 and [28, 32] where the fluxes of Hawking radiation were obtained by demanding that the complete theory composed from contributions from inside the horizon, near the horizon and away from the horizon must be anomaly free. However, the present approach uses identical (covariant) boundary conditions. It reinforces the crucial role of these boundary conditions, the study of which has been the principal objective of this paper.

In order to get a clean understanding of these boundary conditions we first computed the explicit structures of the covariant current \( \langle J_\mu \rangle \) and the covariant energy-momentum tensor \( \langle T_{\mu\nu} \rangle \) from the chiral (anomalous) effective action, appropriately modified by adding a local counterterm [99]. The chiral nature of these structures became
more transparent by passing to the null coordinates. In these coordinates the contribution from the ingoing (left moving) modes was manifestly seen to vanish. The outgoing (right moving) modes involved arbitrary parameters which were fixed by imposing regularity conditions at the future horizon. No condition on the ingoing (left moving) modes was required as these were absent as a result of chirality. These findings by themselves are new. They are also different from the corresponding expressions for $\langle J^\mu \rangle$, $\langle T_{\mu\nu} \rangle$, obtained from the standard nonanomalous (Polyakov type) action [100], satisfying $\nabla_\mu \langle J^\mu \rangle = 0$, $\nabla_\mu \langle T^{\mu\nu} \rangle = \langle J_\mu \rangle F^{\mu\nu}$ and $\langle T^\mu_{\mu} \rangle = \frac{R}{24\pi}$, implying the absence of any gauge or gravitational (diffeomorphism) anomaly. Only the trace anomaly is present. Details of the latter computation may be found in [33, 102, 103].

We have then established a direct connection of these results (obtained from the chiral currents) with the choice of the covariant boundary condition used in determining the Hawking flux from chiral consistent [32, 33] or covariant (see chapter-3) gauge and gravitational anomalies and also from the near horizon chiral effective action given in chapter-4. The relevant universal component $\langle J^r \rangle$ or $\langle T_{rt} \rangle$ obtained by solving the anomaly equation subject to the covariant boundary condition (5.7, 5.15) agrees exactly with the result derived from imposing regularity condition on the outgoing modes at the future horizon: namely, a free falling observer sees a finite amount of flux at outer horizon indicating the possibility of Hawking radiation. Our findings, therefore, provide a clear justification of the covariant boundary condition.

Finally, we put our computations in a proper perspective by comparing our findings with the standard implementation of the various vacua states on nonchiral expressions. Specifically, we show that our results are compatible with the choice of Unruh vacuum for a nonchiral theory which eventually yields the Hawking flux. Further, we showed that, in the Unruh vacuum the asymptotic forms for the components of the covariant current and energy-momentum tensor obtained from the chiral effective action matches exactly with corresponding components of the current and energy-momentum tensor computed from the usual (nonchiral) theory. However, for the Hartle-Hawking vacuum there exist no such equivalence between the chiral and usual theories, even at the asymptotic infinity.
Appendix

5.A Relation between chiral trace and gravitational anomalies

Unlike the case of vector theory, where the diffeomorphism invariance is kept intact inspite of the presence of trace anomaly, the chiral theory has both a diffeomorphism anomaly (gravitational anomaly) and a trace anomaly. In 1 + 1 dimensions it is possible to obtain a relation between the coefficients of the diffeomorphism anomaly and the trace anomaly by exploiting the chirality criterion.

To see this let us write the general structure of the covariant gravitational Ward identity in the presence of an external gauge field,

\[ \nabla_\mu \langle T^\mu_\nu \rangle = \langle J^\mu_\nu \rangle F^{\mu_\nu} + N_a \bar{\epsilon}^{\nu\mu} \nabla^\mu R \] (5A.1)

where \( N_a \) is an undetermined normalisation. The functional form of the anomaly follows on grounds of dimensionality, covariance and parity. Likewise, the structure of the covariant trace anomaly is written as,

\[ \langle T^{\mu_\mu} \rangle = N_t R \] (5A.2)

with \( N_t \) being the normalisation. In the null coordinates (5.24, 5.25) and (5.26) for \( \nu = v \), the left hand side of (5A.1) becomes

\[ \nabla_\mu \langle T^\mu_v \rangle = \nabla_u \langle T^v_v \rangle + \nabla_v \langle T^v_v \rangle \]

\[ = \nabla_u (g^{uu} \langle T^u_v \rangle) + \nabla_v (g^{uv} \langle T^u_v \rangle) = \nabla_v (g^{uv} \langle T^u_v \rangle) \] (5A.3)
where we have used the fact that for a chiral theory \( \langle T_{vv} \rangle = 0 \) (see equation 5.34). Also, in null coordinates, we have,
\[
\langle T_{uv} \rangle = \frac{1}{2}(g_{uv} \langle T^{v}_{\ v} \rangle + g_{uv} \langle T^{u}_{\ u} \rangle) = \frac{g_{uv}}{2} \langle T^{\mu}_{\mu} \rangle = \frac{f}{4} \langle T^{\mu}_{\mu} \rangle .
\]
By using (5A.2), (5A.4) and (5A.3) we obtain,
\[
\nabla_{\mu} \langle T^{\mu}_{\nu} \rangle = \frac{N_t}{2} \nabla_{\nu} R.
\]
where we used \( g^{uv} = \frac{2}{f} \) (5.27).

The right hand side of (5A.1) for \( \nu = v \), with the use of the chirality constraint \( \langle J_v \rangle = 0 \) (5.31), yields
\[
\langle J_{\mu} \rangle F^{\mu}_{\ \nu} + N_a \epsilon_{\nu\mu} \nabla^{\mu} R = N_a \nabla_{\nu} R .
\]
Hence, by equating (5A.5) and (5A.6) we find a relationship between \( N_a \) and \( N_t \)
\[
N_a = \frac{N_t}{2}
\]
which is compatible with (5.10) and (5.23) with \( N_a = \frac{N_t}{2} = \frac{1}{36\pi} \). It is clear that chirality enforces both the conformal and diffeomorphism anomalies. The trivial (anomaly free) case \( N_a = N_t = 0 \) is ruled out because, using general arguments based on the unidirectional property of chirality, it is possible to prove the existence of the diffeomorphism anomaly in 1 + 1 dimensions [117].
Chapter 6

Conclusions

The motivation of this thesis was to study certain field theory aspects of cosmology and black holes. We now summarize the results obtained in last four chapters and briefly comment on future prospects.

In the second chapter we studied a generalized Chaplygin gas (GCG) model containing a parameter $\alpha$, which is a strong contender for explaining the accelerated expansion of the Universe. In particular, we gave an action formulation of GCG model, both in nonrelativistic as well as relativistic regimes. In the nonrelativistic case, we constructed a general form of the Lagrangian for GCG. This Lagrangian contained both the density and velocity fields. By using Bernoulli’s equation we expressed this master Lagrangian into a nonrelativistic Born-Infeld form. Further, $\alpha = 1$ limit of our model was shown to be consistent with the corresponding normal Chaplygin gas model. In the relativistic domain, we proposed a Born-Infeld like Lagrangian for GCG. This model was manifestly Poincare invariant and in the nonrelativistic limit, reduced to the conventional GCG. We also suggested a Lagrangian for GCG, which included both density and velocity fields. In order to check its Poincare invariance, we computed the algebra among the generators of Poincare group. We observed that the Poincare algebra closed only in the large density limit.

The relativistic Lagrangian formulation for GCG, initiated here, opens up a host of avenues for future study. One possibility is to extend the analysis [60] of the Chaplygin matter in FRW spacetime, to the case of a GCG, to observe its cosmological implications.
Also, study of symmetry properties of GCG (first elucidated in [14] for usual Chaplygin gas), as well as its connection to $d$-branes, offer further scope.

In the third chapter, we provided a new mechanism, based on covariant gauge and gravitational anomalies, to compute the fluxes of Hawking radiation. In contrast to the earlier approaches [28, 32], where the expressions for gauge/gravitational anomalies were taken to be consistent whereas the boundary conditions were covariant, the analysis presented here used only covariant expressions. The point was that since the covariant boundary condition was mandatory in deriving the Hawking flux, it was conceptually clean to discuss everything from the covariant point of view. There are two important reasons in favor of the covariant anomaly approach that was adopted here:

- No counterterms connecting the consistent and covariant expressions for currents and energy-momentum tensors were required.
- Manifest covariance was preserved at all stages of the computations.

We also discussed applications of our covariant anomaly technique to the case of stringy black holes. In particular, we computed the Hawking energy-momentum flux from Garfinkle-Horowitz-Strominger (GHS) and D1-D5 nonextremal black holes.

In the fourth chapter we discussed yet another way to derive the fluxes of Hawking radiation. This approach used only the structure of chiral effective action, which was defined in the vicinity of event horizon. The current and energy-momentum tensors derived from the chiral effective action, suitably modified by local counterterms, yielded the covariant gauge and gravitational anomalies, respectively. The arbitrary constants appearing in the expressions for the chiral covariant current and energy-momentum tensor were then fixed by imposing the covariant boundary condition at event horizon. Once we knew the forms for current and energy-momentum tensor, in the region near the event horizon, the Hawking fluxes were easily obtained by taking the asymptotic infinity limit of the current/energy-momentum tensor. Novelty of this chiral effective action approach was that, unlike the previous approaches based on the consistent [28, 32] or covariant (see chapter-3) anomaly cancellation method, it used only the properties of the theory near the event horizon. The structure of the chiral effective action and the covariant boundary condition were the only necessary inputs and we showed that they were sufficient to
determine the Hawking fluxes. Also, our method did not require the introduction of any discontinuous step functions. This was consistent with the universality of Hawking effect. Next, we used this approach to obtain the expressions for charge and energy-flux from the Reissner-Nordstrom black hole in the presence of gravitational back reaction.

The last chapter was devoted to discussion of the covariant boundary condition used in the analysis of chapters-3,4 and also in [32]. We used the structures of covariant current/energy-momentum tensor, derived from the chiral effective action, suitably modified by a local counterterm. The arbitrary constants appearing in the expressions for current and energy-momentum tensor were fixed by imposing the regularity condition on the outgoing modes. Because our theory was chiral, in the vicinity of horizon, no further condition on the ingoing modes was necessary. The regularity condition states that; a freely falling observer must see a finite amount of flux at the future horizon. This condition was sufficient to determine completely the forms for current and energy-momentum tensor. The expressions for the universal components of current and energy-momentum tensor were in exact agreement with the corresponding ones obtained by solving the covariant gauge/gravitational anomaly and imposing the covariant boundary condition (see chapter-3). This provided a clear physical interpretation for the covariant boundary condition.

Next, we compared our results with the standard implementation of the various vacua [48] on nonchiral expressions. In the conventional analysis, the expressions for the expectation values of current and energy-momentum tensor were derived by solving simultaneously the conservation law and the trace anomaly [26]. Similar results can also be derived by using the structure of Polyakov type effective action (see [33, 101] for details). The arbitrary constants were fixed by imposing the conditions appropriate for Unruh, Hartle-Hawking and Boulware vacuum.

The Unruh vacuum, by construction, is appropriate for discussing the Hawking flux [46]. Unruh vacuum is characterized by two properties:

1. A finite amount of flux at the future horizon. This implies that the outgoing component of the current and energy-momentum tensor in Kruskal coordinates (i.e. $\langle J_U \rangle$ and $\langle T_{UU} \rangle$) must be regular at future event horizon. However, since the outgoing components of the currents/energy-momentum tensors in the Kruskal...
Chapter 6. Conclusions

coordinates \((U, V)\) are related to the null coordinates \((u, v)\), as \(\langle J_U \rangle = -\langle J_u \rangle \kappa_U\) and \(\langle T_{UU} \rangle = \left(\frac{1}{\kappa_U}\right)^2 \langle T_{uu} \rangle\), the regularity condition stated above translates into vanishing of current \(\langle J_u \rangle\) and energy-momentum tensor \(\langle T_{uu} \rangle\) in the null coordinates at future horizon.

2. No ingoing flux at past null infinity, i.e \(\langle T_{VV} (r \to \infty) \rangle = 0\).

In the conventional analysis, based on the trace anomaly [26], both the above conditions were essential to fix the structures of current and energy-momentum tensor. We showed that, in the asymptotic infinity limit our results were compatible with the choice of Unruh vacuum for conventional (nonchiral) theory. For the Hartle-Hawking state no such equivalence, between the chiral and nonchiral expressions, was possible. Thus, we conclude that: the imposition of covariant boundary condition on the chiral expressions is equivalent to implementing the conditions for the Unruh vacuum on the nonchiral expressions.

There are certain issues which are worthwhile for future study. For example, the inclusion of grey body effect within the anomaly approach would be an interesting excersise. The approaches given in chapter-3,4 and [28, 32] did not include the grey body effect. Consequently, the flux obtained from these approaches were compared with the fluxes associated with the perfect black body. Another important issue is the computation of black hole entropy by using the anomaly approach. There are strong reasons to believe that the black hole entropy, like Hawking flux can be related to the diffeomorphism anomaly. For example, in the analysis of [118, 119] the counting of microstates was done by imposing the “horizon constraints”. The algebra among these “horizon constraints” commutes only after modifying the generators for diffeomorphism symmetry. This modification in the generators give rise to desired central charge, which ultimately leads to Bekenstein-Hawking entropy. This is roughly similar to the diffeomorphism anomaly mechanism, illustrated in this thesis. Thus, it is clear that the covariant anomaly mechanism and the effective action approach, provided in this thesis, could illuminate the subject of black hole entropy.
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