The Fuzzy Natural Transformations, the Algebra $\mathcal{P}(\omega)/\text{fin}$ and Generalized Encoding Theory

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ABSTRACT A category theory constitutes a convenient conceptual apparatus to organize the worlds of mathematical entities. The concept of fuzzy natural transformation as an abstract mapping on functors is one of the essential concepts of this theory. If we admit a piece of non-commutativity in its definition diagrams, then the ‘upward’ and the ‘downward’ diagram parts generate different result sets. In this way, we can introduce the concept of fuzzy natural transformation. We deal with the multi-fuzzy natural transformation if such a transformation is based on a multi-diagram. This paper aims to describe the multi-fuzzy natural transformation situation when the symmetric difference of the result is finite. It allows us to organize the whole spectrum of the result sets in a unique quotient algebra $\mathcal{P}^{\text{comp}}(\omega)/\text{fin}$. Different algebraic properties of this structure will be explored, and a piece of classical encoding theory will be reconstructed in the environment determined by this algebra. In particular, the concepts of the $k$-multi similarity, the abstract $k$-multi similarity, the $k$-similarity balls, and the abstract $k$-similarity balls are introduced as some generalization of the idea of Hamming’s distances and Hamming’s balls. Finally, it is shown how to automate some verification processes in this context using an R-based programming environment.

INDEX TERMS The fuzzy natural transformation, $\mathcal{P}(\omega)/\text{fin}$, generalized encoding theory, Hamming distances, R programming.

I. INTRODUCTION
A category theory forms a general algebra-based conceptual apparatus for grasping the dynamic nature of different mathematical structures and relations between these structures. The founding idea of category theory stems from S. Eilenberg and S. McLane’s groundbreaking idea from [1]–[3] to comprehend the processes of preserving mathematical structure. The same dynamic way of thinking found its reflection in the majority of younger positions devoted to category theory, such as [3]–[6]. Consequently, this theory may pretend to play a role of a new paradigm in the foundations of formal sciences. This postulate is comprehensive in light of the increasing tendency to grasp many set-theoretic operations on sets, collections, structures in a more dynamic and mapping-based depiction, such as embeddings or morphisms.

From another and no less sophisticated perspective – category theory might be viewed as an art of clever manipulations of mathematical entities in terms of arrows and commutative diagrams. Meanwhile, the idea of diagram commutativity plays a fundamental role in different areas of formal reasoning. Indeed, the commutative diagrams constitute a reasoning device that clarifies the relationship between certain processes, problems, or mappings. Often, the diagram commutativity means that we can express the same result alternatively in two (or more) ways. One can indicate several examples of such commutative diagrams to illustrate the thesis.

- Having defined an initial problem instance (i.e., a differential equation), we can adopt a Fourier transform to it to obtain its modified instance. If the transformed problem is easier to be solved, we can return from its transformed solution to the initial instance solution via inverse Fourier transform.
- Having defined a relational semantics (given by a Kripke frame-based model and its ultrafilter extension) for a given modal logic system $Th$, we can exploit the idea of diagram commutativity to elaborate its corresponding algebraic semantics (in terms of modal algebra $A$ and its dual frame $A^+$) as depicted in Fig. 1. – due to the original Stone’s ideas from [7].
Relational structures: Algebraic structures:

\[ F \xrightarrow{ueF} F^+ = A \]

\[ (F^+)^+ = A^+ \]

FIGURE 1. The mutual relationships between Kripke frame \( F \), its dual frame \( F_\omega = ueF \), its corresponding modal algebra \( A \) – denoted as \( F^+ \) – and the dual algebra for the modal algebra \( A_\omega \). The semantic sense and properties of the diagram may be found in [8]. We omit explaining the diagram details as redundant from the perspective of further analysis.

As already mentioned – the idea of diagram commutativity manifests itself in many ways in category theory. One can venture to state that preserving diagram commutativity plays a role of a fundamental methodological principle of this theory and a necessary condition of existence of various categorical entities, such as products, co-products, etc. The same idea of diagram commutativity finds its transparent manifestation in the main reasoning line in category theory. It runs from categories by different types of functors (the covariants or the contravariants ones) towards the so-called natural transformations. The last ones form the most sophisticated and abstract mappings operating between functors. Its general form may be predicted employing the so-called Yoneda’s lemma - initially elaborated in [9], [10].\(^1\) Meanwhile, the existence of a natural transformation between two given functors exactly means that the appropriate diagram for them commutes. This property is usually expressed in terms of diagram commutativity. It may doubt how the situation changes if we admit a fuzzy commutative diagram.

Independently of the category theory-oscillated research, an increasing development of algebraic and set-theoretic research on the so-called \( P(\omega)/\text{fin} \), its founding role in the advanced theory of Boolean algebras may be observed. This unique quotient algebraic structure enables of introducing a quite serious portion of set-theoretic considerations around the so-called Haussdorf gaps or Čech-Stone’s compactification – as described, e.g., in [16], [17] – makes this algebraic structure hardly elusive from the operational perspective and a dynamic paradigm of category theory.

Admittedly, research on \( P(\omega)/\text{fin} \) shows a high affinity with combinatorics, but it refers to infinity combinatorics. As a result, a majority of classical results of finite combinatorics (such as partitioning properties of finite sets, Ramsey’s theory) loses its natural operational, practical dimension and classical combinatorial sense\(^2\) – even if some results are extrapolable for infinite sets. It may arise a question how one can restore or extract some connections between the theory of \( P(\omega)/\text{fin} \)-algebras and finite combinatorics.

A. PAPER MOTIVATION, ITS OBJECTIVES, AND ORGANIZATION

1) MOTIVATION

Although these open questions might potentially be a sufficient research motivation, it remains a methodologically fundamental problem why to combine fuzzy category theory with the theory of \( P(\omega)/\text{fin} \)-algebras at all?

Indeed, the author of the paper decides to do it expecting that

- the appropriate \( P(\omega)/\text{fin} \)-algebras – already discussed in set-theoretic works of Hausdorff [18], [19], of Parovičenko [20] and of Shelah [12] – help to organize all sets created from fuzzy natural transformation diagrams - (not only the pairs of sets ordered by inclusion relation), and

- the algebras may be immersed in such an environment, which shows a deep affinity to the classical combinatorial encoding theory and allows us to extrapolate the combinatorial idea of a Hamming distance – as depicted in [21] – and Hamming balls for arbitrary sets. It seems that there is a couple of premises for such optimism. At first, \( P(\omega)/\text{fin} \) shows some inherent relationships with the class of finite sets – independently of a leading research tendency to immerse this structure in cardinal arithmetic and infinite set theory.\(^3\) Secondly, the same class of finite sets – connected to \( P(\omega)/\text{fin} \) – is a key to mutually connecting the sets, which have (at most) finite intersection.

Meanwhile, the idea of a reasonable organization of relative sets – created at each stage of the fuzzy natural transformation diagrams – was developed in [22], [23] in a relative idealistic way. Indeed, it was idealistically assumed in [22], [23] that either the result of the ‘upward’ multi-composition of mappings, say \( A \), is included in the result of the ‘downward’ multi-composition of mappings, say \( B \) or – conversely. Meanwhile, the compositions’ results may differ in a general

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\(^1\) Yoneda’s lemma introduces another type of algebraic duality called Yoneda’s duality. In the author’s opinion – it constitutes a convenient bridgehead to comprehend more sophisticated Grothendieck’s duality, which lives in the worlds of functors between ringed spaces and modules. Their characterization and their deep mutual relation exceed the thematic scope of the paper.

\(^2\) This sense manifests itself by some elusiveness or intuitiveness of combinatorial operations and objects for finite sets.

\(^3\) It will be explained in detail later. It is enough to state here that the class of finite sets is crucial for the proper definition of the equivalence relation, which determines \( P(\omega)/\text{fin} \).
case or even have an empty intersection. Last but not least – there is another, more general reason to develop fuzzy category theory as considered in the paper. Namely, even in the most recent papers in fuzzy category theory – such as [24]–[26] – it dominates more a tendency to elaborate the category theory-based approaches to fuzzy sets than to consider fuzzy sets inside categorial entities.

All these lacks correspond well with the next premise to combine the theory of $\mathcal{P}(\omega)/\text{fin}$ with fuzzy theory category: the following particular questions demand a solving (and they seem to be solvable thanks to the machinery of the $\mathcal{P}(\omega)/\text{fin}$ algebra theory):

1) How to define the multi-fuzzy natural transformation if the ‘upward’ composition mapping is essentially different from the ‘downward’ composition mapping than relative sets are finite?
2) What can we state about the class $\mathcal{C}$ of all pairs $(A, B)$ such that their symmetric difference is finite (i.e., $A \Delta B \in \text{fin}$)?
3) Can it be ordered in some way?
4) Which combinatorial properties does the class $\mathcal{C}$ have? Finally, $\mathcal{P}(\omega)/\text{fin}$ forms a convenient conceptual frame to explore combinatorial properties of the class of relative sets – obtained at each construction stage of the multi-natural transformation diagram. Indeed, it seems that $\mathcal{P}(\omega)/\text{fin}$ might be a convenient bridgehead to construct some unique similarities balls and to reconstruct in this way a piece of classical encoding theory in this new algebraic environment.

Last but not least – one thing requires a piece of justification. Namely, we decided to explore the R language’s programming-wise machinery as operational support in some operationally workable fragments of our analysis instead of Haskell - ordinary exploited in category theory contexts. This solution has been mainly dictated by further data analysis-sensitive automating the processes and different verification procedures discussed in the paper. $R$ is a natural language for such attempts.

2) THE PAPER OBJECTIVES
Due to these motivation factors – the general objectives of the paper are:

- to propose a realistic (less idealistic) approach to the multi-fuzzy natural transformation itself and to the organization of the spectrum of relative sets by the appropriate equivalence relation between these sets,
- to organize the relative sets into the so-called composition algebra $\mathcal{P}^{\text{Comp}}(\omega)$ for relative sets – as a quotient algebra with formal properties of $\mathcal{P}(\omega)/\text{fin}$,
- to exploit formal properties of $\mathcal{P}^{\text{Comp}}(\omega)$ to reconstruct a piece of classical encoding theory in this concept-

tual environment in terms of $k$-multi similarity, $k$-multi abstract similarity, $k$-multi similar balls, and $k$-multi abstract balls,

- to extract some combinatorial properties of $\mathcal{P}^{\text{Comp}}(\omega)$ with an algorithmic anti-chain partitioning of this structure as a special focus area,
- to propose a method to automate some procedures in this area (such as verification whether a given equivalent class belongs to a given $k$-multi abstract ball).

These general paper objectives will be materialized by performing some sub-objectives, such as:

- establishing the type of orders (partial, linear), which is created by $\mathcal{P}(\omega)/\text{fin}$ or
- establishing the maximal chain cardinality in this algebra.

Nevertheless, the paper’s objectives might be measured differently, considering that the paper analysis combines two mutually different paradigms in reasoning about the foundation of formal and technical sciences: the set-theoretic and the categorical one. In this perspective, the paper is focused on creating a methodological sound, synergy approach to fuzzy natural transformation in its operational depiction. From this point of view, the paper has one extended goal only.

Last but not least, there also exists a more practical perspective. From this perspective – the paper analysis may be viewed as elaborating a theoretic tissue to solve an initial computational problem being a specification of the above questions 2), 3) and 4). The exact formulation of the problem will be given in Section II in terms of the algebraic apparatus presented in this section.

3) THE PAPER ORGANIZATION
The rest of the paper is organized as follows. The conceptual framework of the paper analysis is given in Section II. Section III describes the idea of the multi-fuzzy natural transformation in a more and less idealistic perspective. In Section IV, it is shown how the class of relative sets may be algebraically organized employing the formal apparatus of the $\mathcal{P}(\omega)/\text{fin}$ theory. In particular, the so-called composition algebra $\mathcal{P}^{\text{Comp}}(\omega)/\text{fin}$ is introduced here. Finally, a piece of combinatorics is introduced in terms of $k$-multi-similarity, $k$-multi abstract similarity, $k$-multi-similar balls, and $k$-multi-similar abstract balls. Section V contains solving the leading problems. Section VI describes state of the art, but Section VII contains conclusions of the previous analysis and closing remarks.

II. THE CONCEPTUAL FRAMEWORK OF FURTHER ANALYSIS AND THE LEADING PROBLEM FORMULATION
In this part, the conceptual framework of the analysis is put forward, and the leading problem is formulated. The definitions of the lattice, distributive lattice, Boolean algebra,
and its extensions may be found in each handbook of lattice theory and universal algebra. (See: [13].)

\section*{A. CONCEPTUAL FRAMEWORK OF FURTHER ANALYSIS}

The notion of the category stems from the algebraic concept of the group of transformations. The term ‘transformation’ itself is exploited here as synonymous with the concept of mapping.

\textit{Definition 1:} A category \( K \) is a triple consisting of:

1) a class \( O_{K} \) of objects;
2) a class \( \text{Morp}_{K} \) of morphisms (also called maps or arrows) between objects from \( O_{K} \);
3) a binary operation \( \cdot : (\text{Morp}_{K})^{2} \rightarrow \text{Morp}_{K} \) called composition of morphisms, which satisfy the following conditions:

- \( \text{If } \alpha, \beta \in \text{Morp}_{K}, \text{ then } \beta \cdot \alpha \in \text{Morp}_{K} \).
- \( \text{Identity: There is a unique neutral element in } \text{Morp}_{K} \) (the so-called identity morphism \( \text{id} \)), i.e.,, for each \( \alpha \in \text{Morp}_{K} \), it holds: \( \text{id} \cdot \alpha = \alpha = \alpha \cdot \text{id} \).
- \( \text{Associativity: For each } \alpha, \beta, \gamma \in \text{Morp}_{K}, \text{ it holds } \alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma \).

We sometimes consider categories as the pairs of the form:

\[ K = (O_{K}, \text{Morp}_{K}) \]

where \( O_{K} \) is a class of objects (usually denoted by \( X, Y, \ldots, A, B, \ldots \) etc.) and \( \text{Morp}_{K} \) is a class of arrows between these objects. We usually decide for simplification provided that an arrow composition operation is defined to satisfy the conditions from point 3) of Definition 1.

\textit{Example 1:} The following table collects some typical examples of categories.

| Categories | Category objects | Category morphisms |
|------------|------------------|--------------------|
| RelA       | sets             | binary relations   |
| Pos        | posets           | monotone functions |
| Lat        | lattices         | lattice homomorphisms |
| Bool       | Boolean algebras | Boolean algebras   |
| Grp/Gr     | groups           | homomorphisms      |
| Vect       | vector spaces    | morphisms          |
| Metr       | metric spaces    | linear mappings    |

\textit{Definition 2:} Let us assume that \( K = (O_{K}, \text{Morp}_{K}) \) and \( L = (O_{L}, \text{Morp}_{L}) \) are two categories. A mapping \( F : K \rightarrow L \) forms a covariant functor if and only if it

\( a) \) associates an element \( F(X) \in O_{L} \) to each element \( X \in O_{K} \) and

\( b) \) associates to each morphism \( f : X \rightarrow Y \) from \( \text{Morp}_{K} \) a new morphism \( F(f) : F(X) \rightarrow F(Y) \) from \( \text{Morp}_{L} \) such that the following holds:

- \( F(\text{id}_{X}) = \text{id}_{F(X)}, \) for each object \( X \) of \( K \),
- \( F(g \cdot f) = F(g) \cdot F(f), \) for every morphisms: \( f : X \rightarrow Y, g : Y \rightarrow Z \).

\textit{Example 2:} An example of a covariant functor is a Haskell \( \text{fmap} \) function defined by Haskell’s clause:

\[ \text{fmap} :: \text{Functor } f \Rightarrow (a \rightarrow b) \rightarrow f \ a \rightarrow f \ b. \]

It works as follows. Given a function \( a \rightarrow b \) and an \( f \) \( a \), you can produce \( f \ b \).

\textit{Definition 3:} A functor \( F : K \rightarrow L \) is a contravariant functor, if it reverses the direction of morphisms (arrows) in category \( L \) (with respect to \( K \)).

\textit{Example 3:} An example of a contravariant functor is a Haskell \( \text{contramap} \) function defined by the Haskell’s clause:

\[ \text{contramap} :: \text{Covariant } f \Rightarrow (a \rightarrow b) \rightarrow f \ a \rightarrow f \ b. \]

\textit{Definition 4:} Let \( C \) and \( D \) be two categories, and functors \( F, G \) from \( K \rightarrow L \) are given (see: Fig.2). Independently of their nature, one could establish a new and more abstract mapping from \( F \) into \( G \) to be so-called a \textit{natural transformation} between \( F \) and \( G \). Its rigorous definition is as follows.

\textit{Definition 5:} A partial order is any binary relation, which is reflexive, anti-symmetric, and transitive.

\textit{Example 4:} In Fig. 2., the natural transformation between functors \( F \) and \( G \) is represented by the pair of \( \eta \)-components, i.e.,, by \((\eta_{0}, \eta_{1})\).

Let us repeat the following algebraic definitions at the end of this subsection.

\textit{Definition 5:} A partial order is any binary relation, which is reflexive, anti-symmetric, and transitive.
Definition 6: Each partial order \( P \), in which there exists supremum and infimum for each pair of elements of \( P \), is said to be a lattice.

Definition 7: Each lattice as a structure \((A, \lor, \land)\) satisfying the distributivity laws:

\[
\begin{align*}
    a \lor (b \land c) &= (a \lor b) \land (a \lor c) \quad (1) \\
    a \land (b \lor c) &= (a \land b) \lor (a \land c) \quad (2)
\end{align*}
\]

is said to be a distributive lattice.

Definition 8: Let a lattice \( A \) has the minimal element \((0)\) and the maximal element \((1)\). For any element \( a \in A \), if there exists an element such that \( a \lor x_a = 1 \) and \( a \land x_a = 0 \) (a complement of \( a \)) \( x_a \), then the lattice is a complemented lattice.

Definition 9: Each distributive and complemented lattice is said to be a Boolean algebra.

The power set \( 2^X \) of a given set \( X \), consisting of all subsets of \( X \) with inclusion relation, forms a standard example of Boolean algebra. \( X \) may be here any set: empty, finite, infinite, or even uncountable.

We now define a unique and important Boolean algebra \( \mathcal{P}(\omega)/\text{fin} \) defined as a quotient structure. Let assume that \( \text{fin}(X) \) denotes a family of all finite subsets of \( X \). We will write \( \text{fin} \) for \( X = \omega \) (a linearly ordered set of natural numbers).

Definition 10 (\( \mathcal{P}(\omega)/\text{fin} \)): The algebra \( \mathcal{P}(\omega)/\text{fin} \) is a quotient set consisting of all equivalence classes \([A]\), for \( A \subseteq \omega \), determined by the following equivalence relation \( \sim_{\text{fin}} \):

\[
A \sim_{\text{fin}} B \iff A \Delta B \in \text{fin}.
\]

B. THE LEADING PROBLEM FORMULATION

Let us consider a fuzzy natural transformation multi-diagram (as in Fig. 4), i.e., where the commutativity condition is violated, and different relative sets arise. Assuming that we intend to organize the class of these sets, say \( C \), into disjoint equivalence classes using the appropriate equivalence relation, let us decide the following problems.

1) Which equivalence relation \( \sim \) may be defined for \( C \) to enable of comparing even these sets, which have (at most) finite common intersection?

2) Assuming that \([A], [B]\) are two equivalence relations determined by \( \sim \), for two sets \( A, B \) such that \( A \Delta B \leq 20 \) – decide if \([B]\) may belong the neighborhood of the class \([A]\) interpreted as a ball in the center \([A]\) of a radius of the length 19 (symbolically: \( B_{19}([A]) \)).

3) Assuming that all sets from \( C \) organized by \( \sim \) create a quotient algebra being a partial order of exactly 13 elements – establish the maximal possible cardinality of anti-chains if there are four chains in this structure. How does its cardinality increases if this quotient algebra has 100 elements and 11 chains?\(^8\)

4) Can we automate the verification procedures for belonging to the equivalent class of finite sets or a ball of a given radius? Which conditions must be satisfied for the task feasibility?

III. THE multi-FUZZY NATURAL TRANSFORMATIONS WITH FINITE RELATIVE SETS – IN A MORE AND LESS IDEALISTIC SCENARIO

In a case of a non-fuzzy natural transformation, the diagram commutativity condition (see: point b) of Definition 4) warranties that no relative set arises between the composition \( \eta_Y \circ F \) and \( G(f) \circ \eta_X \). The ‘upward’ mapping composition of \( \eta_Y \circ F \circ f \) gives the same result as the ‘download’ one, i.e., \( G(f) \circ \eta_X \).

A radically different situation holds for fuzzy natural transformation, where different non-empty relative sets may arise as the difference between the appropriate ‘upward’ and ‘downward’ composition mappings. The non-empty relative sets may arise as either the difference set \( \eta_Y \circ F - G(f) \circ \eta_X \) or \( G(f) \circ \eta_X - \eta_Y \circ F \). A quite similar situation holds for the multi-fuzzy natural transformations. (See: [23].)

Unfortunately, the approach to the multi-fuzzy natural transformation from [23] seems to be too idealistic. Indeed, it only refers to the following two situations: the ‘upward’ mapping composition contains the ‘download’ mapping composition, or the inverse inclusion holds.

Against these circumstances, this chapter will elaborate on a new, less idealistic approach to the multi-fuzzy natural transformations. We intend to do it by relaxing the requirement of strict inclusions between (set-theoretic results of) the ‘upward’ and the ‘downward’ mapping compositions. It means that we admit the situation where the results differ arbitrarily. However, the finite relative sets will constitute our particular focus area.

A. THE multi-FUZZY NATURAL TRANSFORMATIONS WITH FINITE RELATIVE SETS – THE MORE IDEALISTIC APPROACH

The typology of the multi-fuzzy natural transformations – as proposed in [22], [23] – contains two main types of them: the ‘upward’ and the ‘downward’ multi-fuzzy natural transformations. Informally speaking, we deal with the upward ones when the ‘upward’ diagram composition of mappings constitutes a ‘greater’ set than the set obtained via the ‘downward’ diagram composition. If the ‘downward’ composition generates a set with greater cardinality than the ‘upward’ composition – we deal with the ‘downward’ multi-fuzzy natural transformations. If the relative set (as a Formally, they are introduced as follows.

Definition 11 (The Upward Multi-Fuzzy Natural Transformation\(^9\)): Let us assume that a finite sequence of categories \( C_i \) with two corresponding sequences \( \{F_i\}_{i=1}^{k-1} \) and \( \{G_i\}_{i=1}^{k-1} \) of functors operating between the categories are given, for a finite \( k \) and \( i = 1, 2, \ldots, k \). Then the upward

\(^8\) Obviously, the numbers of chains and anti-chains play an exemplary role only.

\(^9\) As in [22], the name – as previously – was motivated by the observation that the composition \( G(f) \circ \eta_X \) by the upward diagram part gives the set of values with cardinality greater than \( \eta_Y \circ F(f) \) – obtained by the downward diagram part.
multi-fuzzy natural transformation $\eta$ from $\{F_i\}_{i=1}^{k-1}$ to $\{G_i\}_{i=1}^{k-1}$ forms a family of mapping such that the following requirements are satisfied:

- to each object $X \in C_i$ a mapping $\eta^X_i : F_i(X) \rightarrow G_i(X)$ in $C_{i+1}$ is associated (it forms an $i$-component of $\eta$ at $X$), for $i = 1, 2, \ldots, k$.
- it holds the upward fuzzy commutativity:

$$\eta^X_i \bullet (F_k \bullet \ldots \bullet F_1(f)) \subset (G_k \bullet \ldots \bullet G_1(f)) \bullet \eta^X_1 \quad (3)$$

(the result set of the ‘upward’ functor composition contains the result set of the ‘downward’ composition of the functors.)

If a relative set, say $A$, determined by inclusion (3) has a finite cardinality (i.e., $\text{card}(A) < \infty$), then we deal with the finite-valued upward multi-fuzzy natural transformation. The same ideas constitute a construction basis not only for further variants of the upward multi-fuzzy natural transformation but also for a variety of their downward counterparts; thus, we can omit details of their constructions.\textsuperscript{10}

![Figure 3](image-url) An example of the multi-fuzzy natural transformation diagram.

multi-fuzzy natural transformation with the (almost) commutativity condition $\eta$ from $\{F_i\}_{i=1}^{k-1}$ to $\{G_i\}_{i=1}^{k-1}$ forms a family of mapping such that the following requirements are satisfied:

- to each object $X \in C_i$ a mapping $\eta^X_i : F_i(X) \rightarrow G_i(X)$ in $C_{i+1}$ is associated (it forms an $i$-component of $\eta$ at $X$), for $i = 1, 2, \ldots, k$.
- it holds the fuzzy (almost) commutativity:

$$\eta^X_k \bullet (F_k \bullet \ldots \bullet F_1(f)) \triangle (G_k \bullet \ldots \bullet G_1(f)) \bullet \eta^X_1 \in \text{fin}. \quad (5)$$

where ‘fin’ denotes a class of finite sets (of a given space, universe, domain, etc.).

We will denote this type of relative sets as in (5) in terms of $\equiv_{\text{fin}}$ – read ‘almost equal’ (Fig. 4):

$$\eta^X_k \bullet (F_k \bullet \ldots \bullet F_1(f)) \equiv_{\text{fin}} (G_k \bullet \ldots \bullet G_1(f)) \bullet \eta^X_1 \quad (6)$$

Let us assume that the condition of finite symmetry difference (the $\equiv_{\text{-}}$-type relation between different composition sets) enables of introducing some unification and avoiding the dichotomy between the ‘upward’ and ‘downward’ variants of the multi-fuzzy natural transformation from [22], [23]. Simultaneously, it also delivers a way to modify the concept of a multi-similarity from [23] as follows.

**Definition 13 (Multi-Similarity up to the Difference Set $A$):** Let $K, M$ be two arbitrary sets. It will be said that $K$ is multi-similar to $M$ (or $M$ is multi-similar to $K$) up to d.s. $A$ if and only if $K \triangle M = A \in \text{fin}$, where $A = \bigtriangleup_1 A_i$ for some $A_i \in \text{fin}$, for a fixed $k$. We will write $K \sim M (M \sim K \text{ resp.})$ up to d.s. $A$.

These three properties of the unification-based approach to the multi-fuzzy natural transformation from Definition 15 will be developed in chapter IV. In its framework, we intend to introduce a piece of algebraic machinery to organize the entire spectrum of the relative sets created at

\textsuperscript{10} They may be easily found in [23].
each stage of the diagram construction for the multi-fuzzy natural transformations. The pairs of them, which has finite symmetry difference, will constitute a particular focus area of our interests.

### IV. THE MULTI-FUZZY NATURAL TRANSFORMATION WITH RELATIVE SETS IN \( \mathcal{P}(\omega)/\text{fin} \)

Let us repeat that the entire spectrum of the relative sets is the effect of the diagram construction for the multi-fuzzy natural transformation. We intend to organize a spectrum of these sets. The algebraic structure – exploited in it – will be a unique \( \mathcal{P}(\omega)/\text{fin} \), the so-called composition algebra \( \mathcal{P}^\text{Comp}(\omega)/\text{fin} \).

In order to introduce this structure in the categorial contexts of the multi-fuzzy natural transformations, let us assume that the following ‘upward’ and ‘downward’ mapping compositions \( \eta^X_k \bullet (F_k \bullet \ldots \bullet F_1(f)) \) and \( (G_k \bullet \ldots \bullet G_1(f)) \bullet \eta_1^X \) are given. (See: Fig. 3.)

Let us now introduce the following \( \sim_{\text{fin}} \) relation between these mapping compositions.

\[
\eta^X_k \bullet (F_k \bullet \ldots \bullet F_1(f)) \sim_{\text{fin}} (G_k \bullet \ldots \bullet G_1(f)) \bullet \eta_1^X
\]

\[
\Leftrightarrow \eta^X_k \bullet (F_k \bullet \ldots \bullet F_1(f)) \triangle (G_k \bullet \ldots \bullet G_1(f)) \in \text{fin}
\]

We will prove that \( \sim_{\text{fin}} \) constitutes an equivalence relation. For simplicity of further explanation, let us divide the class of all mapping compositions into the ‘upwards’ and the ‘downward’ group and denote them by \( \text{Comp}^U \) and \( \text{Comp}^D \) (resp.).

The elements of \( \text{Comp}^U \) and \( \text{Comp}^D \), and the methods of their creation are collected in the following table created for the situation presented in Fig. 4. (For editorial reasons, we will use the abbreviation \( H_i \) for \( \text{Hom}_i \) in the table, \( i = 1, 2, \ldots, k - 1 \).)

| \( \text{Comp}^U \) | \( \text{Comp}^D \) |
|-------------------|-------------------|
| \( F_1 \bullet \eta_0 \) | \( \eta_1 \bullet H_1(c, -) \) |
| \( F_2 \bullet F_1 \bullet \eta_0 \) | \( \eta_2 \bullet H_2(c, -) \bullet H_1(c, -) \) |
| \( \ldots \ldots \) | \( \ldots \ldots \) |
| \( F_k-1 \bullet \ldots \bullet F_2 \bullet F_1 \bullet \eta_0 \) | \( \eta_{k-1} \bullet \ldots \bullet H_{k-1}(c, -) \bullet \ldots \bullet H_1(c, -) \) |
| \( F_2 \bullet \eta_1 \) | \( \eta_2 \bullet H_2(c, -) \) |
| \( F_3 \bullet F_2 \bullet \eta_1 \) | \( \eta_3 \bullet H_3(c, -) \bullet H_2(c, -) \) |
| \( F_k-1 \bullet \ldots \bullet F_2 \bullet \eta_1 \) | \( \eta_{k-1} \bullet \ldots \bullet H_{k-1}(c, -) \bullet \ldots \bullet H_1(c, -) \) |
| \( \ldots \ldots \) | \( \ldots \ldots \) |
| \( F_k-1 \bullet \eta_{k-2} \) | \( \eta_{k-1} \bullet H_{k-1}(c, -) \) |

It is noteworthy to make a methodological remark in this place. Although we define the classes \( \text{Comp}^U \) and \( \text{Comp}^D \) as the collections of composition mappings, we will also have a tendency to treat them in a more set-theoretic way, i.e., as a collection of the composition mapping values. The interpretation method will stem from the context of analysis.

It is not difficult to show that \( \sim_{\text{fin}} \in \text{Comp}^U \times \text{Comp}^D \) (defined as above by the formula taken from condition (5)) forms an equivalence relation and divide the class \( \text{Comp}^U \times \text{Comp}^D \) of all composition mappings over a given multi-fuzzy diagram into disjoint abstract classes. We propose to prove this fact in the form of the following lemma.

**Lemma 1:** The relation \( \sim_{\text{fin}} \in \text{Comp}^U \times \text{Comp}^D \) forms an equivalent relation and divide a class \( \text{Comp}^U \times \text{Comp}^D \) of all composition mappings over a given multi-fuzzy diagram into disjoint equivalent classes.

**Proof:** Let us establish \( \text{Comp}^* = \text{Comp}^U \cup \text{Comp}^D \).

1. Obviously, since \( \forall x \in \text{Comp}^* \), \( x \Delta x = \emptyset \in \text{fin} \), thus \( \forall x \in \text{Comp}^U \cup \text{Comp}^D(x \sim_{\text{fin}} x) \). Hence, \( \sim_{\text{fin}} \) is reflexive.

2. Obviously, \( \forall x, y \in \text{Comp}^* \) \( x \sim_{\text{fin}} y \iff y \Delta x \in \text{fin} \). \( \sim_{\text{fin}} \) is symmetric.\(^{12}\)

3. Transitivity follows from the fact that for each sets \( A, B, C \):

\[
A \Delta C \subset A \Delta B \cup B \Delta C.
\]

Since \( A \Delta B \in \text{fin} \) and \( B \Delta C \in \text{fin} \), it implies that also \( A \Delta C \in \text{fin} \). In particular, \( \forall x, y, z \in \text{Comp}^*(x \sim_{\text{fin}} y \land y \sim_{\text{fin}} z) \Rightarrow x \sim_{\text{fin}} z \).

Instead of \( \text{Comp}^* \), we will sometimes write \( \mathcal{P}^\text{Comp}(\omega) \) to underline the set-theoretic aspects of mapping compositions, more as mappings themselves, but as the sets of their values. The set-theoretic perspective in defining of \( \mathcal{P}^\text{Comp}(\omega) \)

\(^{11}\)Although both \( \text{Comp}^U \), \( \text{Comp}^D \) are considered as finite classes, at each \( k \), it is admissible to consider them as potentially infinite.

\(^{12}\)One can also infer this property immediately from commutativity of symmetric difference.
finds its reflection in the concept of the composition quotient algebra $\mathcal{P}^{Comp}(\omega)/\text{fin}$, which stems from the concept of $\mathcal{P}^{Comp}(\omega)$.

**A. $\mathcal{P}^{Comp}(\omega)/\text{fin}$ FOR RELATIVE SETS AND ITS ALGEBRAIC FOUNDATION**

The nature of $\sim_{\text{fin}}$ as an equivalent relation allows us to introduce a new quotient algebra to be called a composition quotient algebra $\mathcal{P}^{Comp}(\omega)/\text{fin}$ for relative sets. This quotient algebra constitutes a unique subalgebra of the so-called algebra $\mathcal{P}(\omega)/\text{fin}$\textsuperscript{13}. Formally, it is defined as follows.

**Definition 14 (A Family of Relative Sets):** Let us assume that $\text{Comp}^*$ is given as associated to a multi-fuzzy natural transformation diagram. Each set $A \in \text{fin}$ such that

$$A = B \Delta C,$$

for some $B, C \in \text{Comp}^*$ is said to be a relative set for the pair $(B, C)$. A class of all such relative sets is said to be a family of relative sets and denoted by $\text{Comp}_R^*$.

**Example 5:** Let us establish that

$$\text{Comp}^* = \{F_1 \circ \eta_0 \circ H_1(c, -)\},$$

for some functors $F_1, F_2, H_1(c, -)$, and the natural transformation components $\eta_0, \eta_1$, and let us assume that $F_1 \circ \eta_0 \circ H_1(c, -) = A \in \text{fin}$. Due to Definition 14 – $A$ is a relative set for the pair $(F_1 \circ \eta_0 \circ H_1(c, -))$ and $\text{Comp}^* \ominus \{A\}$, i.e., it forms the singleton.

**Definition 15 A Composition $\mathcal{P}^{Comp}(\omega)/\text{fin}$ Algebra for Relative Sets:** Let us assume that $\text{Comp}^*_R$ forms a non-empty family of relative sets (as defined in Definition 14) associated with a family of composition morphisms $\text{Comp}^*$, let $\sim_{\text{fin}}$ – as previously – be a class\textsuperscript{14} of all finite subsets of $\omega$. Let us finally assume that $\sim_{\text{fin}} \in \text{Comp}^U \times \text{Comp}^D$ is an equivalent relation defined by the condition:

$$\forall A, B \in \text{Comp}^* \quad A \sim_{\text{fin}} B \iff A \Delta B \in \text{fin}.$$

Then the quotient algebra

$$\mathcal{P}^{Comp}(\omega)/\text{fin} = \left\{ [A]_{\sim_{\text{fin}}} : A \subseteq \text{Comp}^*, \text{and} \right\}$$

is said to be a composition algebra of relative sets, and denoted by $\mathcal{P}^{Comp}(\omega)/\text{fin}$.

Obviously, for each $A \subseteq \text{Comp}^*$, we define the equivalent class of this algebra as follows.

$$[A]_{\sim_{\text{fin}}} = \{ B \in \text{Comp}^* : A = \sim_{\text{fin}} B \}.$$  \text{(7)}

In other words,

- $B$ belongs to the equivalent class determined by $A$ and $\sim_{\text{fin}}$ if $A = \sim_{\text{fin}} B$, or the symmetric difference $A \Delta B$ is finite, and

\textsuperscript{13}We omit a detailed explanation of the fact. The concept of the quotient algebra $\mathcal{P}(\omega)/\text{fin}$ itself and its properties – the more and less advanced may be found, for example, in [29]. A nice and compact introduction to this concept may be found in the initial pages of [30].

\textsuperscript{14}One can show that it forms an ideal in the class of all subsets of $\omega$.

$\mathcal{P}^{Comp}(\omega)/\text{fin}$ is created by all equivalent classes determined by sets from $\text{Comp}^*$ (as the class representatives) and the equivalent relation $\sim_{\text{fin}}$ between them.

In order to introduce a piece of dynamism to $\mathcal{P}^{Comp}(\omega)/\text{fin}$, and to grasp its algebraic nature, we associate two following operations $+ \text{ and } \cdot$ into $\mathcal{P}^{Comp}(\omega)/\text{fin}$.

To perform the task – let us assume that $\text{Comp}^*$ constitutes a $\rho$-algebra, i.e., it forms a collection $\Sigma$ of subsets of $\omega$, which is closed under sums $\cup$ and complements $\setminus$.\textsuperscript{15}

It allows us to define the required operations in the quotient $\mathcal{P}^{Comp}(\omega)/\text{fin}$ as follows.

$$[A]_{\sim_{\text{fin}}} + [B]_{\sim_{\text{fin}}} = [A \cup B]_{\sim_{\text{fin}}},$$

$$[A]_{\sim_{\text{fin}}} \cdot [B]_{\sim_{\text{fin}}} = [A \cap B]_{\sim_{\text{fin}}},$$

and

$$-[A]_{\sim_{\text{fin}}} = [\text{Comp}^*/A]_{\sim_{\text{fin}}},$$

for $A, B \in \text{Comp}^*$.

**Example 6:**

1) Due to (7) and Definition 15 – we can state that $[\emptyset]_{\sim_{\text{fin}}} = \{ B \in \text{Comp}^* : \emptyset = \sim_{\text{fin}} B \} = \{ B \in \text{Comp}^* : \emptyset \Delta B \in \text{fin} \}$. Since the class of finite sets as the only one satisfies the condition $\emptyset \Delta B$, thus $[\emptyset]_{\sim_{\text{fin}}}$, where $\sim_{\text{fin}} = \sim_{\text{fin}}$ – is as previously – an ideal of finite subsets of $\mathcal{P}^{Comp}(\omega)$.

2) Let us assume now that $\text{Comp}^*$ contains infinite (denumerable) sets only. Due to (10) – the inverse element for the equivalent class $[\emptyset]_{\sim_{\text{fin}}}$, i.e., $-[\emptyset]_{\sim_{\text{fin}}}$, is identical with the equivalent class $[\text{Comp}^*]_{\sim_{\text{fin}}}$.

Meanwhile – due to (7) – the equivalent class $[\text{Comp}^*]_{\sim_{\text{fin}}} = \{ B \in \text{Comp}^* : B = \sim_{\text{fin}} \text{Comp}^* \} = \{ B \in \text{Comp}^* : B \Delta \text{Comp}^* \in \text{fin} \}$. Obviously, the only subclass of $\text{Comp}^*$, which satisfies the condition, is the class of its infinite (denumerable) subsets, i.e., $[\text{Comp}^*] = \omega/\text{fin}$.

In order to extract a couple of more advanced algebraic features of $\mathcal{P}^{Comp}(\omega)/\text{fin}$ let us enlarge the list of possible $\star$-relations on relative sets $A, B \in \text{Comp}^*$ by introducing:

$$A \leq \star \ B \iff A/B \in \text{fin}$$

$$A \not\leq \star \ B \iff A/B \notin \text{fin}, \text{ but } B/A \notin \text{fin},$$

$$A \cap B = \emptyset \iff A \cap B \in \text{fin}.$$  \text{(13)}

These $\star$-relations between relative sets find their reflection in the corresponding relations $\leq, \prec, \cdot$ between abstract classes for the sets $A, B$ from $\text{Comp}^*$. More precisely, for all $A, B \in \text{Comp}$ the following connections between these two classes of relations hold:

$$A \leq \star \ B \equiv [A] = [B] \quad \text{(in } \mathcal{P}^{Comp}(\omega)/\text{fin}),$$

$$A \not\leq \star \ B \equiv [A] \leq [B],$$

$$A \not\geq \star \ B \equiv [A] < [B],$$

$$A \cap B = \emptyset \equiv [A] \cdot [B] = 0,$$

\text{(17)}

\textsuperscript{15}It means that 1) if $A, B \in \Sigma$, then $A \cup B \in \Sigma$, and 2) if $A \in \Sigma$, then also $\omega/A \in \Sigma$.\textsuperscript{15}
where 0 is a zero of \( \mathcal{P}^{\text{Comp}}/\text{fin} \), and the order \( \leq \) in \( \mathcal{P}^{\text{Comp}}(\omega)/\text{fin} \) is determined as follows

\[
[A] \leq [B] \equiv [A] + [B] = [B] \equiv (A \cup B) = \star B \quad (18)
\]

\[
\equiv (A \cup B) \Delta B \in \text{fin}. \quad (19)
\]

Since \( (A \cup B) \Delta B = A/B \), we can alternatively write

\[
[A] \leq [B] \equiv A/B \in \text{fin}. \quad (20)
\]

**Example 7:** Let us consider a relative set (singleton) \( A = \{1\} \), obtained at some stage of the multi-diagram construction for a multi-fuzzy natural transformation. It is easy to see that \( [\emptyset] \sim \text{fin} \leq \{1\} \sim \text{fin} \). In fact, \( \emptyset/\{1\} = \emptyset \in \text{fin} \). Simultaneously, \( \{1\} \sim \text{fin} \leq [\emptyset] \) because \( \{1\}/\emptyset = \{1\} \in \text{fin} \). Thus, \( [\emptyset] \sim \text{fin} = \{1\} \sim \text{fin} \), and \( \emptyset \) and \( 1 \) are some different representatives of the same equivalent class only. The same equality may be achieved in another way taking into account the fact that \( \{1\} \sim \text{fin} = [B] \in \text{Comp}^\ast \). \( B \Delta \{1\} = \text{fin} = [\emptyset] \sim \text{fin} \). The same reasoning way may be incorporated for other finite relative sets.

It is noteworthy to observe that the relation \( \leq \) between the equivalent classes from \( \mathcal{P}^{\text{Comp}}(\omega)/\text{fin} \) – expressible in terms of \( \leq \star \) – enables of defining our quotient algebra as an ordered structure \((\mathcal{P}^{\text{Comp}}(\omega)/\text{fin}, \leq)\). It has some interesting algebraic properties. One of them is communicated by the following fact.

**Proposition 1:** \( \mathcal{P}^{\text{Comp}}(\omega)/\text{fin} \) has atoms.\(^{17} \)**

**Proof:** Obviously, \((\mathcal{P}(\omega)/\text{fin}, \leq \star)\) as a partially ordered structure is finite. In fact, \(\text{card}(\text{Comp}^\ast) < \infty\) and only infinite subsets from \(\text{Comp}^\ast\) are capable of creating new equivalent classes – different than the class \(\text{fin}\). Obviously, the number of these is no greater than \(\text{card}(\text{Comp}^\ast)\), thus \(\text{card}(\mathcal{P}(\omega)/\text{fin}) < \infty\), too. The thesis follows now from the fact that each finite and partially ordered structure is atomless. \(\square\)

Obviously, \(\mathcal{P}(\omega)^{\text{Comp}}/\text{fin}\) just defined is embeddable in \(\mathcal{P}(\omega)/\text{fin}\) because we consider \(\text{Comp}^\ast\) as finite (even if arbitrarily large). It is noteworthy to state that (even if) we admit \(\text{Comp}^\ast\) to be infinite (denumerable), \(\mathcal{P}(\omega)^{\text{Comp}}/\text{fin}\) will constitute some substructure of \(\mathcal{P}(\omega)/\text{fin}\) because only a selection of subsets of \(\mathcal{P}(\omega)\) will be contained in it.

Although the assumption on infiniteness of \(\text{Comp}^\ast\) seems to be a convenient bridgehead to incorporate some results of cardinal arithmetic for \(\mathcal{P}(\omega)/\text{fin}\), such as a serious of theorems concerning the existence of the so-called Hausdorff gaps and limits in it, we will omit their considering for a cost of exposing some combinatorial features of \(\mathcal{P}^{\text{Comp}}(\omega)/\text{fin}\) and for a cost of extending the previous analysis thanks to a further specification of the concept of the multi-similarity. This issue is the subject of the following subsection.

**B. \(\mathcal{P}^{\text{Comp}}(\omega)/\text{fin}\) AND THE RECAPITULATED IDEA OF \(k\)-MULTI-SIMILARITY BETWEEN SETS**

The analysis of the previous subsection delivers only a general method (or a frame) to organize the internal structure of \(\mathcal{P}^{\text{Comp}}(\omega)/\text{fin}\) and the mutual relations between relative sets – obtained at each construction stage of a given multi-fuzzy natural transformation diagram.

This general approach is not sensitive to the differences between finite sets (all of them fall into the same equivalent class \(\text{fin}\)); thus, it is hardly exploitable in computational contexts. Therefore, arises a need to specify the relations slightly more. It will be executed by further specification the concept of multi-similarity towards the concept of a \(k\)-multi-similarity for a fixed natural \(k\). It delivers a piece of more substantial knowledge about similarities between two given sets.

In this chapter, \(k\)-multi-similarity is firstly introduced for sets from \(\mathcal{P}^{\text{Comp}}(\omega)\). Secondly, we propose to extrapolate it for equivalence classes from \(\mathcal{P}^{\text{Comp}}(\omega)/\text{fin}\). It allows us to define the so-called \(k\)-similarity balls as a notion conceptually close-related to the concept of Hamming’s balls. Being equipped with these two concepts, we can computationally explore the internal structure of \(\mathcal{P}^{\text{Comp}}(\omega)\). It will be a matter of the first subsection of this section. Similarly – the analogy notions of abstract \(k\)-similarity and the abstract \(k\)-similar balls allow us to computationally explore the internal structure of \(\mathcal{P}^{\text{Comp}}(\omega)/\text{fin}\). This task will be the subject of 2. subsection of this section.

1) THE IDEA OF \(k\)-MULTI-SIMILARITIES OF SETS IN \(\mathcal{P}^{\text{Comp}}(\omega)\)

**Definition 16 (The \(k\)-Multi-Similarity up to the Difference Set A):** Let \(K, M\) be two arbitrary sets. It will be said that \(K\) is \(k\)-multi-similar to \(M\) (or \(M\) is \(k\)-multi-similar to \(M\)) up to d.s. \(A\) if and only if

- \(\text{card}(K \Delta M = A) = k,\) and \(k\) is finite.

We will write \(K \sim M (M \sim^k K\) resp.) up to d.s. \(A\).

This idea may be incorporated for defining the so-called finite sum \(k\)-multi-similarities and for the finite intersection \(k\)-multi-similarities.

**Definition 17 (The Finite Sum \(k\)-Multi-Similarity up to the Difference Set A):** Let \(K\) be an arbitrary set and \(M = \bigcup_{i=1}^n M_i\), for some \(M_i\) and \(i = 1, \ldots, n\). It will be said that \(K\) is finite sum \(k\)-multi-similar to \(\bigcup_{i=1}^n M_i\) (or \(\bigcup_{i=1}^n M_i\) is finite sum \(k\)-multi-similar to \(K\)) up to d.s. \(A\) if and only if

- \(\text{card}(K \bigcup_{i=1}^n M_i = A) = k,\) for some \(M_i\), and \(i = 1, \ldots, n\).

**Definition 18 (The Finite Intersection \(k\)-Multi-Similarity up to the Difference Set A):** Let \(K\) be an arbitrary set and \(M = \bigcap_{i=1}^n M_i\), for some \(M_i\) and \(i = 1, \ldots, n\). It will be said that \(K\) is finite intersection \(k\)-multi-similar to \(\bigcap_{i=1}^n M_i\) (or \(\bigcap_{i=1}^n M_i\) is finite intersection \(k\)-multi-similar to \(K\)) up to d.s. \(A\) if and only if

- \(\text{card}(K \bigcap_{i=1}^n M_i = A) = k,\) and \(\text{card}(A) = k,\) for some \(M_i\), and \(i = 1, \ldots, n\).
These definitions may be completed by the concept of k-similarity ball.

**Definition 19 (k-Similarity Ball):** Let \( A \subset \mathcal{P}^{\text{Comp}}(\omega) \) be an arbitrary set, and \( k \) be a non-negative integer. The set
\[
\mathcal{B}_k(A) := \{ B \in \mathcal{P}^{\text{Comp}}(\omega) : |A \Delta B| \leq k \}
\]
is said to be a \( k \)-similarity ball of a center \( A \) and radius \( k \).

It allows us to formulate the following theorems. It states that no \( k-1 \) similar ball is \( k \)-multi-similar. Informally speaking, we can always find such a finite sum (of some subsets from the \( \mathcal{P}^{\text{Comp}}(\omega) \) universe), which are too far from the ball center \( A \). In other words, they differ from the ball center \( A \) more than is admissible by the ball radius.

**Theorem 1:** \( \mathcal{B}_{k-1}(A) \) is not finite sum \( k \)-multi-similar (for each \( A \subset \mathcal{P}^{\text{Comp}}(\omega) \)).

**Proof:** The goal of the proof is to show that we can always find such a finite sum of subsets of \( \mathcal{P}^{\text{Comp}}(\omega) \), which is further from a fixed ball center \( A \) than admissible value \( k-1 \).

For that reason, let \( Y \subset \mathcal{P}^{\text{Comp}}(\omega) \), and \( |Y| = k \). Let \( S = \bigcup_{i=1}^{n} S_i \) (i.e., \( S \) be an arbitrary finite sum of some \( S_i \subset \mathcal{P}^{\text{Comp}}(\omega) \) such that \( \bigcup_{i=1}^{n} S_i \cap Y = Y - A \)), i.e., \( \bigcup_{i=1}^{n} S_i \cap Y \) does not contain any elements from \( A \). Establish \( \bigcup_{i=1}^{n} S_i \cap Y = Y - A = B \).

1. \( B \subset (\bigcup_{i=1}^{n} S_i - A) \), since \( Y - A \subset (\bigcup_{i=1}^{n} S_i - A) \), and
2. \( (A - \bigcup_{i=1}^{n} S_i) \cap Y \subset A - \bigcup_{i=1}^{n} S_i \).

To prove the thesis, it is enough to compute the symmetric difference (‘distance’) \( \bigcup_{i=1}^{n} S_i \Delta A \) now. From 1 and 2 – we obtain the inequality:

\[
\bigcup_{i=1}^{n} S_i \Delta A = (\bigcup_{i=1}^{n} S_i - A) \cup (A - \bigcup_{i=1}^{n} S_i) \supset B \cup (A - \bigcup_{i=1}^{n} S_i) \cap Y.
\]

Simultaneously – because of \( \bigcup_{i=1}^{n} S_i \cap Y = B \) and \( B = Y - A \):

\[
B \cup (A - \bigcup_{i=1}^{n} S_i) \cap Y = B \cup (A \cap Y - \bigcup_{i=1}^{n} S_i \cap Y) = B \cup (A \cap Y - B) = (Y - A) \cup (A \cap Y) = Y.
\]

Therefore, \( Y \subset \bigcup_{i=1}^{n} S_i \). It exactly means that \( |\bigcup_{i=1}^{n} S_i \Delta A| \geq k \), i.e., \( \bigcup_{i=1}^{n} S_i \notin \mathcal{B}_{k-1}(A) \). □

**Theorem 2:** \( \mathcal{B}_{k-1}(A) \) is not finite intersection \( k \)-multi-similar (for each \( A \subset \mathcal{P}^{\text{Comp}}(\omega) \)).

**Proof:** The proof runs as previously. It is only enough to consider the finite intersections instead of finite sums in the whole reasoning.

One needs to underline that we implicitly assume that the finite sums and intersections create non-empty sets. Otherwise, the situation from the theses of theorems cannot hold (empty set belongs to each ball).

18|\( A \)| denotes a cardinality of \( A \).

20It easy to see that we can consider \( f^A \) as a multi-argumental automorphism, and – for a generality of the depiction – no further conditions are imposed on it. We omit a detailed definition of the notion of the algebraic structure of a given signature as slightly redundant from this point of view. It may be found in each handbook of general algebra, such as [13] and many others.
\{(\mathbb{Z}, +, - , \cdot)\} is given, where \((\mathbb{Z})\) is a set of integers, and
+ , - , \cdot are standard arithmetic operations on pairs of elements
from \(Z\). Let us finally establish that the relation \(\equiv_n\)
holds between two \(a, b \in (\mathbb{Z})\) if and only if
\[a \equiv_n b \iff n \mid a - b.\]

It is not difficult to see that not only \(\equiv_n\) but also it forms a congruence on \(A\). Considering
(modules of) \(a, b\)'s as cardinalities of some subsets from \(Z\),
one can also associate the relation, but also it forms a congruence on \(\equiv_n\) as elements of \(P(\mathbb{Z})\).

It allows us to note the following obvious fact.

Fact 1: Let us assume that \(P^{\text{Comp}}(\omega)\)/\(\text{fin} – \text{as determined by \(P\)}\)
by Definition 15 – is given. Let also 'fin' denote the ideal
of all finite subsets of \(\omega\), i.e., \(\text{fin} = [\emptyset]_{n}\). Let also assume
that \(\equiv \subset A \times B, \text{ for } A, B \in \text{fin} \text{ is a congruence. Then the}
equivalence class 'fin' forms a quotient structure consisting of
a pairwise disjoint equivalence class.

Proof: It follows from the definition of each congruence
as a unique equivalence class and from the abstraction
principle.

Being equipped with the definitions of \(\leq_k\)-type relations,
the definition of congruence and its properties, we are in
a position to introduce the corresponding relations \(\equiv_k\),
\(\leq_k\), \(\leq_k\) between the equivalence classes – newly
created by \(\equiv\) congruence on \([\emptyset]_{\text{fin}}\), e.g., the 'fin' class of \(P^{\text{Comp}}(\omega)\)/\(\text{fin} \). In general, the following situations should be
distinguished:

1) when the sets \(A, B \in [\emptyset]_{\text{fin}}\) (i.e., both are finite), and
2) when \(A, B\) are such that \([A]_{\text{fin}} \neq [B]_{\text{fin}}\) \(21\)

In general, we intend to express the new relations (i.e., \(\equiv_k\), \(\leq_k\), \(\leq_k\)) between equivalence classes determined by a
congruence, say \(\equiv\), in terms of the old \(\leq_k\)-type relations.
Since the existence of such relations between two equivalence classes
should be independent of a choice of the relation representatives, it forces a need to use general quantifiers in
explanans of the appropriate definition of these relations.

This postulate is reflected in the following definition.

Definition 22: Let \([\emptyset]_{\text{fin}} \in P^{\text{Comp}}(\omega)\)/\(\text{fin} \) determined by
\(\equiv\) equivalence relation, and \(\equiv\) be a congruence on \([\emptyset]_{\text{fin}}\).
Let us also establish a natural \(k \geq 1\). Then:

1) If \(A, B \in [\emptyset]_{\text{fin}}\), then
\[a) [A]_{\equiv_k} = [B]_{\equiv_k}\]
\[\iff \forall C \in [A]_{\equiv_k} \forall D \in [B]_{\equiv_k} \left( C \leq_k D \land D \leq_k C \right).\]

\[b) [A]_{\equiv_k} \leq_k [B]_{\equiv_k}\]
\[\iff \forall C \in [A]_{\equiv_k} \forall D \in [B]_{\equiv_k} \left( C \leq_k D \right).\]

\[c) [A]_{\equiv_k} < k [B]_{\equiv_k}\]
\[\iff \forall C \in [A]_{\equiv_k} \forall D \in [B]_{\equiv_k} \left( C \leq_k D \right).\]

\[21\text{In particular, } A \text{ may belong to } [\emptyset]_{\text{fin}}, \text{ while } B \text{ may not.}\]

2) If \([A]_{\text{fin}} \neq [B]_{\text{fin}}\) and \(\equiv\) are congruences \(22\)
from \([A]_{\text{fin}}\) and \([B]_{\text{fin}}\) (resp.), then
\[a) [A]_{\equiv_k} = [B]_{\equiv_k}\]
\[\iff \forall C \in [A]_{\equiv_k} \forall D \in [B]_{\equiv_k} \left( C \leq_k D \land D \leq_k C \right).\]

\[b) [A]_{\equiv_k} \leq_k [B]_{\equiv_k}\]
\[\iff \forall C \in [A]_{\equiv_k} \forall D \in [B]_{\equiv_k} \left( C \leq_k D \right).\]

\[c) [A]_{\equiv_k} < k [B]_{\equiv_k}\]
\[\iff \forall C \in [A]_{\equiv_k} \forall D \in [B]_{\equiv_k} \left( C \leq_k D \right).\]

Before exemplifying the definition, it is reasonable to make
a couple of explanatory remarks.

1) The common sense of clause b) of 2) may be explained
as follows. If we take two sets, \(A\) and \(B\) from two
different \(\equiv\)-equivalence classes, and we intend to
check whether they remain in \(\leq_k\)-relation, then
we should only check whether the corresponding \(\leq_k\)
holds between all sets-representatives of the classes \([A]\) and
\([B]\). Similar reasoning leads to formulating the appro-
priate conditions for other clauses of 2).

2) Since the condition \(C \leq_k D \sim \leq_k D\) and the similar ones
from points a) and c) – is expressible by \(\text{card}(A/B) \leq k\),
the essence consists in computing the cardinality of the
difference set \(A/B\), for all such \(A\)'s and \(B\)'s. Universal
feasibility of such a computation determines the feasi-
bility of the operations (existence of the relations) on the left side of 1) and 2).

3) It is also possible to consider a non-negative value, say
\(r > 0\), instead of the natural \(k\)'s.

Example 9: 1) Let us establish \(A = \{a_1, a_2\}, B = \{a_1, a_2, a_3, a_4, \ldots, a_n\}\), and \([A]_{\text{fin}} = [A, \{a_1\}, \{a_2\}],
\([B]_{\text{fin}} = [B].\) Obviously, \(\text{card}(A/B) = k \leq k, \text{ and}\)
\(\text{card}(A/B) = k = k > k.\) However, \(\text{card}(B/A) = k + 2 > k.\)\n
Similarly, \(\text{card}(B/A_2) = k + 2 > k, \) so not for all
representatives of \([A]_{\text{fin}}\) and \([B]_{\text{fin}}\), \(A \neq B\).

2) Let us take \(A = (2n : n \in N)\), \(B = N\), and \([A]_{\text{fin}} = \{A = (2n : n \in N)\), \(A^* = (2n : n = 1, 2, \ldots, k)\}.

In this case, \([A]_{\text{fin}} \neq [B]_{\text{fin}}\) as \(\Delta B = \{2n + 1 : n \in N\} \neq \emptyset.\) In addition, \(\text{card}(B/A) = (2n + 1 : n \in N) \geq k, \text{ and}\)
\(\text{card}(B/A) = k > k, \) not for each natural
\(k = n.\) Thus, for all sets \(C\) from \([A]_{\text{fin}}\), and for the set
\(D = N\) from \([B]_{\text{fin}}\), it holds \(C \leq_k D\), thus the left-side
condition of 2c) is satisfied. Hence, \([A] < k [B] – \text{ due to}\)
point c) in Definition 22.\(23\)

Having already defined the new relations – we can venture
to extrapolate the idea of \(k\)-multi-similarity for sets of \(P^{\text{Comp}}\)

\(22\text{Let us observe that in a general case we need two congruences in this case as }\{A\}_{\text{fin}} \neq [B]_{\text{fin}}\).\n
\(23\text{Note that we are sometimes interested in equivalence classes }\{A\}_{\text{fin}},\)
\([B]_{\text{fin}}, \text{ i.e., with respect to } \equiv\text{ relation, sometimes in }\{A\}_{\text{fin}}, [B]_{\text{fin}}, \text{ i.e.,}
\text{with respect to } \equiv\text{ relations.}\)
for equivalent classes in $\mathcal{P}^{\text{Comp}}/\text{fin}$. In this way, we introduce a new type of $k$-multi-similarity to be called abstract $k$-multi-similarity, and the abstract $k$-similar balls. Finally, it allows us to formulate and prove the fact that $(k-1)$-balls are not abstract $k$-multi-similar. Whereas, the $k$-multi-similarity and the abstract $k$-similarity ball may be proposed for the equivalence classes of $\mathcal{P}^{\text{Comp}}/\text{fin}$, i.e., with respect to $\sim_{\text{fin}}$ relation, we will decide to introduce them for the equivalence classes inside the equivalence class ‘fin,’ i.e., with respect to a congruence inside $[\emptyset]_{\sim_{\text{fin}}}$. It reflects the idea of the distance measurement for finite sets only.

Definition 23 (The Abstract $k$-Multi-Similarity): Let $[\emptyset]_{\text{fin}} \in \mathcal{P}^{\text{Comp}(\omega)}/\text{fin}$ determined by $\sim_{\text{fin}}$ equivalence relation, and $\simeq$ be a congruence on $[\emptyset]_{\sim_{\text{fin}}}$. Let us also establish a natural $k \geq 1$. We say that the classes $[A]_{\sim}$ and $[B]_{\sim}$ are abstract $k$-multi-similar if and only if

$$[A]_{\sim} \simeq^k [B]_{\sim}, \quad \text{i.e., } \forall C \in [A]_{\sim}, \forall D \in [B]_{\sim}, \left(\text{card}(C/D) \leq k\right),$$

(24)

for a fixed $k$.

It is easy to see that the abstract $k$-multi-similarity is based on the mutual relationship between $[A]_{\sim}$ and $[B]_{\sim}$ as in Definition 22 (2b)). It only extracts it in another way.

It is also clear that the situation when $[A] \neq [B]$ is just expected for the proper definition of the abstract $k$-multi-similar balls. In fact, it forms a necessary condition to introduce non-trivial balls with a non-zero radius. They are defined as follows.

Definition 24 (The Abstract $k$-Similarity Ball): Let $[\emptyset]_{\text{fin}} \in \mathcal{P}^{\text{Comp}(\omega)}/\text{fin}$ determined by $\sim_{\text{fin}}$ equivalence relation, and $\simeq$ be a congruence on $[\emptyset]_{\sim_{\text{fin}}}$. Let us also establish a non-negative $k$. The abstract $k$-similarity ball of a center $[\emptyset]_{\sim}$ and a non-negative radius $k$ is the set $\mathcal{B}(\emptyset, \sim, k)$, defined as follows:

$$\mathcal{B}_k([\emptyset]_{\sim}) \quad = \quad \left\{ [B]_{\sim} \in [\emptyset]_{\sim_{\text{fin}}} : [A]_{\sim} \simeq^k [B]_{\sim} \right\} \quad = \quad \left\{ [B]_{\sim} \in [\emptyset]_{\sim_{\text{fin}}} : \forall C \in [A]_{\sim}, \forall D \in [B]_{\sim}, \text{card}(C/D) \leq k \right\}.$$

The informal sense of this definition is as follows. The abstract $k$-similar ball $\mathcal{B}_k([\emptyset]_{\sim})$ collects (and only these) equivalence classes from $[\emptyset]_{\sim_{\text{fin}}}$, which contain only sets $D$’s close-related to all sets $C$’s from $[A]_{\sim}$, i.e., card$(C/D) \leq k$, for all such $D$’s and $C$’s.

Theorem 3: For all $[A]_{\sim} \in [\emptyset]_{\sim_{\text{fin}}}$, the ball $\mathcal{B}_{k-1}([A]_{\sim})$ is not abstract $k$-multi-similar.

Proof: Reasoning as previously for $k$-multi-similarity, it is easy to find such an equivalent class $[B]_{\sim} \in [\emptyset]_{\sim_{\text{fin}}}$, for some set $B \in \mathcal{P}^{\text{Comp}}$, that $[A]_{\sim}$ and $[B]_{\sim}$ are abstract $k$-multi-similar, but $[B]_{\sim} \notin \mathcal{B}_{k-1}([A]_{\sim})$. Due to Definition 23 it is enough to indicate (at least one) such a set $D \in [B]_{\sim}$ that $\text{card}(C/D) > k - 1$, for all $C$’s from the ball center $[A]_{\sim}$.

Let us assume now that $[A]_{\sim}$ and $[B]_{\sim}$ are abstract $k$-multi-similar, and let us establish $D$ such that card$(C/D) = k$, for all $C$’s in$[A]_{\sim}$. Since card$(C/D) > k - 1$, for all $C$’s and for this set $D \in [B]_{\sim}$, thus $[B]_{\sim} \notin \mathcal{B}_{k-1}([A]_{\sim})$—due to Definition 24.

It is noteworthy to observe that this forms a far and general echo of the well-known result from the error encoding theory, which may be expressed as follows. If a word $b$ does not belong to the (Hamming) ball of a radius $r$, and a center in a given word $a$, then the transmission of $a$ with at most $d$ errors does not enable of achieving the word $b$. Obviously, a difference between these two situations manifests itself in the difference between the entities or objects that participate in the procedure. (See: [21], pp. 376-377.) In the encoding theory-determined situation, we deal with words (as finite sequences). In the conceptual ‘entourage of the paper analysis – we deal with equivalence classes determined by sets and the equivalence relation $\sim_{\text{fin}}$ defined in terms of asymmetry difference as previously.

Remark 1: It seems that the thesis of Theorem may be generalized for equivalence classes of the entire $\mathcal{P}(\omega)/\text{fin}$ determined by $\sim$ equivalence relation, i.e., for all $[A] \in \mathcal{P}(\omega)/\text{fin}$, the ball $\mathcal{B}_{k-1}([A]_{\sim})$ is not abstract $k$-multi-similar.

Reasoning as previously for $k$-multi-similarity, we could find such an equivalent class $[D]$, for some set $D \in \mathcal{P}^{\text{Comp}(\omega)}/\text{fin}$, that $\Delta D \geq k$, i.e., $D$ does not belong to the ball $\mathcal{B}_{k-1}(A)$. Thus, it is not true that $[A] \sim_k [D]$. It remain a question whether $[A] \sim_k [D]$, i.e., card$(A/D) \leq k$. It is easy to see that this condition must not be satisfied. Indeed, let us take $D = \mathbb{N}$ and $A = \{2n : n \in \mathbb{N}\}$. Obviously, for all $k \in \mathbb{N}$, $\Delta A \geq k$ and even $A \not\subset \mathbb{N}$. $D$ does not hold as $\{2n : n \in \mathbb{N}\}/\mathbb{N} = \emptyset \in \text{fin}$, but $\mathbb{N}/\{2n : n \in \mathbb{N}\} \notin \text{fin}$. Thus $[D]$ cannot belong to $\mathcal{B}_{k-1}([A]_{\sim})$.

C. $\mathcal{P}^{\text{Comp}(\omega)}/\text{fin}$ FOR RELATIVE SETS AND ITS CHOSEN COMBINATORIAL PROPERTIES

In order to grasp different combinatorial properties of $\mathcal{P}^{\text{Comp}(\omega)}/\text{fin}$ algebra, we will consider it as the ordered structure $(\mathcal{P}^{\text{Comp}(\omega)}/\text{fin}, \leq_{\sim})$. It has been already established that $(\mathcal{P}^{\text{Comp}(\omega)}/\text{fin}, \leq_{\sim})$ is atomless. One could extend the proof argumentation line in order to extract the fact that $(\mathcal{P}^{\text{Comp}(\omega)}/\text{fin}, \leq_{\sim})$ – as partially ordered structure – cannot constitute any linearly ordered structure. Indeed, if it is possible, all anti-chains in $(\mathcal{P}^{\text{Comp}(\omega)}/\text{fin}, \leq_{\sim})$ should form (at most) singletons. (It exactly means that each element of this structure may be compared with any other in the sense of $\leq_{\sim}$.)

Meanwhile, anti-chains in $(\mathcal{P}^{\text{Comp}(\omega)}/\text{fin}, \leq_{\sim})$ may contain more elements – due to the condition defining $\leq_{\sim}$-order. In fact, if $A, B \in \{\mathcal{P}^{\text{Comp}}\}$ are infinite, then each of them creates the same abstract class with all these infinite subsets.

24In practice and for some convenience – a natural value as a radius is considered. Thus, we use $k$ instead of $r$ to denote the ball radius.
$C_j, D_k$ of $\{\text{Comp}\}^\ast$, $j \in J, k \in K$, which create only finite relative sets with it, i.e., it holds $A/C_j \in \text{fin}$, $B/D_k \in \text{fin}$ and for each $j \in J, k \in K$. Simultaneously, if $A, B$ do not satisfy the same condition, i.e., $A/B \not\in \text{fin}$, then also $[A] \not\leq \star [B]$. Thus, the algebra cannot be linearly ordered structure, but it forms a proper partially ordered structure.

This fact determines a non-triviality of different combinatorial features of $(\text{P}^{\text{Comp}}(\omega)/\text{fin}, \leq \star)$. In order to illustrate this conjecture, we propose to prove the theorem about cardinalities of chains and anti-chains of $nm + 1$-elemental $(\text{P}^{\text{Comp}}(\omega)/\text{fin}, \leq \star)$. It requires a unique version of dual Dilworth’s Theorem.\footnote{We omit its proof as it runs in a standard version of the theorem. The proof of Dilworth’s Theorem and its dual may be found in each handbook of combinatorics. See, for example: [21].}

**Theorem 4 (A Dual of Dilworth’s Theorem for $(\text{P}^{\text{Comp}}(\omega)/\text{fin}, \leq \star)$):** The size of the largest chain in $(\text{P}^{\text{Comp}}(\omega)/\text{fin}, \leq \star)$ equals the smallest number of anti-chains into which the $(\text{P}^{\text{Comp}}(\omega)/\text{fin}, \leq \star)$ may be partitioned.

**Theorem 5:** Let us assume that $(\text{P}^{\text{Comp}}(\omega)/\text{fin}, \leq \star)$ has exactly $nm + 1$ elements, for some fixed $n, m \in \mathbb{N}$. Then $(\text{P}^{\text{Comp}}(\omega)/\text{fin}, \leq \star)$ contains a chain of the length $n + 1$ or an anti-chain of the length $m + 1$.

**Proof:** Let us assume that no chain of the length $n + 1$ exists in $(\text{P}^{\text{Comp}}(\omega)/\text{fin}, \leq \star)$. We will show that there exists an anti-chain of the length $m + 1$. Meanwhile, the dual Dilworth’s theorem allows us to write $(\text{P}^{\text{Comp}}(\omega)/\text{fin}, \leq \star) = \bigcup_{i=1}^{n} A_i$, i.e., represent the algebra as a sum of $n$ anti-chains. Since $nm + 1 = \text{card}((\text{P}^{\text{Comp}}(\omega)/\text{fin}, \leq \star)) = \text{card}(\bigcup_{i=1}^{n} A_i) \leq |A_1| + \ldots + |A_n|$, thus:

\[
\text{so it must be } |A_m| \geq m + 1, \text{ for some natural } m. \text{ Otherwise, if } \forall i \in 1, \ldots, n, |A_i| \text{ is at most } m, \text{ then the inequality (25) cannot be satisfied.} \tag{25}
\]

The proof of dual Dilworth’s Theorem delivers a general framework of the operational method of achieving the anti-chain partition of $\text{P}^{\text{Comp}}(\omega)/\text{fin}$. Indeed, the main idea is to take the minimal elements of the newly created subsets of $\text{P}^{\text{Comp}}(\omega)/\text{fin}$. The method of their construction is simple: at first we take the minimal elements of $P = \text{dom}(\text{P}^{\text{Comp}}(\omega)/\text{fin})$, and take them as the first anti-chain, say $A_1$. Secondly, we take the minimal elements of the difference set $P - A_1$ as the second anti-chain, say $A_2$. Next, the third anti-chain $A_3$, is (are) the minimal element(s) of $P - (A_1 \cup A_2)$, etc. In this way, we can achieve an anti-chain partition of $P$, i.e., $P = A_1 \cup \ldots \cup A_h$. The exact order of the steps is presented by Algorithm 1.

**Algorithm 1 Anti-Chain Partition of $P = \text{dom}(\text{P}^{\text{Comp}}(\omega)/\text{fin})$**

1. for all $j := 0$ to $h$ and $P$ and do
2. if $j := 1$; set := $P$ then
3. $A_1 = \text{min}(P)$;
4. end if
5. while $j > 1$ and set $\subseteq P$ do
6. if $j := 2$ then
7. set := $P - A_1$;
8. $A_2 = \text{min}(P - A_1)$;
9. end if
10. if $j := 3$ then
11. set := $P - (A_1 \cup A_2)$;
12. $A_3 = \text{min}(P - (A_1 \cup A_2))$;
13. end if
14. if $j := 4$ then
15. set := $P - (A_1 \cup A_2 \cup A_3)$;
16. $A_4 = \text{min}(P - (A_1 \cup A_2 \cup A_3))$;
17. end if
18. if $j := h$ then
19. set := $P - (A_1 \cup A_2 \ldots \cup A_h)$;
20. $A_h = \text{min}(P - (A_1 \cup A_2 \ldots \cup A_h))$;
21. end if
22. end while
23. $A_1 = \text{min}(P - (\bigcup_{i=1}^{j-1} A_i))$;
24. $P = A_1 \cup \ldots \cup A_h$;
25. end for

2) Assuming that $[A], [B]$ are two equivalence relations determined by $\sim$, for two sets $A, B$ such that $A \Delta B \leq 20$ – decide if $[B]$ may belong the neighborhood of the class $[A]$ interpreted as a ball in the center $[A]$ of a radius of the length 19 (symbolically: $B_{19}([A])$).

3) Assuming that all sets from $\mathcal{C}$ organized by $\sim$ create a quotient algebra being a partial order of exactly 13 elements – establish the maximal possible cardinality of anti-chains if there are four chains in this structure. How does their cardinality increase if this quotient algebra has 100 elements and 11 chains?\footnote{Obviously, the numbers of chains and anti-chains play an exemplary role only.}

4) Can we automate the verification procedures for belonging to the equivalent class of finite sets or to a ball of a given radius? Which conditions must be satisfied for the task feasibility?

Being equipped with the results, just elaborated, we can venture to formulate the answers to these questions.

**Ad. 1:** Due to arrangements from Section IV – the class $\mathcal{C}$ of all pairs $(A, B)$ constitute a unique quotient algebra $\text{P}^{\text{Comp}}(\omega)/\text{fin}$ determined by the following equivalence binary relation $\sim_{\text{fin}}$ between two composition sets (results of V. SOLVING THE LEADING PROBLEMS

In Section I, the following questions have been formulated.

1) Which equivalence relation $\sim$ may be defined for $\mathcal{C}$ to enable of comparing even these sets, which have (at most) finite common intersection?
the ‘upward’ and ‘downward’ composition mappings)

\[ \forall A, B \in \mathcal{P} \text{fin} \quad A \sim \text{fin} B \iff A \Delta B \in \text{fin}. \]

It was also shown that \( \mathcal{P}^{\text{comp}}(\omega) \)/fin may be partially ordered by relation \( \subseteq \) defined by the condition:

\[ A \subseteq \star B \iff A/B \in \text{fin}, \]

which introduces an order \( \leq \) among entities of \( \mathcal{P}^{\text{comp}}(\omega) \)/fin due to the relation:

\[ A \subseteq \star B \iff [A] \leq [B]. \]

**Ad. 2:** It has been shown in Section IV (part B) that for all \( [A] \in \mathcal{P}^{\text{comp}} \)/fin, the ball \( B_{k-1}([A]) \) is not (abstract) \( k \)-multi-similar. In particular, the ball \( B_{19}([A]) \) is not (abstract) 20-multi-similar. It exactly means that if sets \( A, B \) are such that \( A \cap B \geq 20 \) (as expected in the example formulation), the equivalent class \( [B] \) is not contained in the \( B_{19} \)-neighborhood of the equivalence class \( [A] \).

**Ad. 3:** In Section IV (part C) – by means of dual Dilworth-Theorem – it was also proven that if \( (\mathcal{P}^{\text{comp}}(\omega) \)/fin, \( \leq \star \) has exactly \( nm + 1 \) elements, for some fixed \( n, m \in \mathbb{N} \), then \( (\mathcal{P}^{\text{comp}}(\omega) \)/fin, \( \leq \star \) contains a chain of the length \( n + 1 \) or an anti-chain of the length \( m + 1 \). Hence, if the quotient algebra has 13 elements and 4 chains, then the maximal number of anti-chains might be equal to \( (13 - 1)/4 = 3 \). Similarly, if the quotient algebra has 100 elements and 11 chains, then the maximal number of anti-chains might be equal to \( (100 - 1)/11 = 9 \).

**Ad. 4:** Whereas the most general and theoretic constructions of the approach (such as constructions of \( \mathcal{P}^{\text{comp}}(\omega) \)/fin, its equivalent classes, the class of finite sets, say FIN, as a quotient set, etc.) often resist the effort of their programming-based depiction, one can venture to automate some verification procedures. We demonstrate two exemplary verification procedures in terms of R-language:

1. for belonging to a FIN equivalent class, and
2. for belonging to a given ball of a radius \( p \).

In both cases, the appropriate control functions may be defined: ‘FIN-cont’ and ‘BALL-cont3’ as depicted in Fig. 5 and Fig. 6 (resp.) for the task performing.

In the first case, we consider the FIN-class as restricted to a given natural \( p \) (i.e., as a class of all finite sets with cardinalities up to \( p \)).29 The FIN-cont function is defined by the if-else instruction: if cardinality of a symmetry difference of two sets - denoted by variables \( x \) and \( y \) - is no greater than \( p \) (the blue marked line), then it is confirmed that the sets belong to FINp. Otherwise – they do not. The if-else instruction also defines the ‘BALL-cont3’ function. The core line (blue marked) of the definition body for the function introduces a sum of cardinalities \( \Delta \) of the differences sets in different combinations \( (x[i] - y[j]), x[i] - z[k], y[j] - z[k], \) for \( i, j, k = 1, 2, 3 \). If the entire sum is no greater than \( 3 * 3 * p \), then classes \( x, y, z \) belong to pBall.31 Otherwise they do not.

Fig. 8 illustrates a simplified version of the situation from Fig. 6. – restricted to two equivalence classes (two variables \( x \) and \( y \)). Fig. 7 delivers an exemplary R-code for FIN-cont and the situation of two 1-set equivalence classes \( A, B \) for verification procedure whether these sets belong to FIN3-class. Fig.9 illustrates an exact implementation of R-commend as in Fig. 6 for some sets \( A, B, C \) previously defined.

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29 The main idea is to consider \( p \) as an arbitrary natural number.
30 In practice, we identify cardinalities of finite sets with lengths of the vectors representing the sets.
31 This multiplication follows from the fact that we should compute all the ‘cardinality distances’ between all sets in each equivalence class to multiply them by the accepted distance \( p \) for each pair of such sets. Assuming that we have three equivalence classes \( x, y, z \) and three elements in each class, we get \( 3 * 3 * p \) as the supremum of the accepted values of these cardinal distances.
VI. STATE OF THE ART

As already mentioned, category theory aspires to be a new, functional paradigm in the foundations of the entire area of formal sciences. However, it stems from homology algebra, and its general frame was elaborated in McLane’s and Eilenberg’s works from the 40s. The idea of functors and the natural transformation itself was introduced in [1], [2], whereas the concept of duality was introduced in [3]. Independently of the relative early provenance of category theory, a renaissance of research on practice-motivated aspects of it in the frame of computer science coincides both with the development of such programming-wise tools as Haskell and its enhancements (see: [31]), and different attempts of category theory popularization outside mathematics and theoretical computer science as in [4], [5], [27].

By contrast – the proper birth of the algebraic theory of $\mathcal{P}^{\text{Comp}}(\omega)/\text{fin}$ is – due to the author’s best knowledge – difficult to indicate. Indeed, this idea stems from the classical Haussdorf’s papers [18], [19] on the so-called Haussdorf’s gaps and limits, and from research on Stone’s representation as in [7]. In later decades, a development of the theory of $\mathcal{P}^{\text{Comp}}(\omega)/\text{fin}$ algebras was - on the one hand – supported by research on the so-called Parovicenko’s algebras, as in [20]. On the other hand, a new catalyzation of this development was the idea of forcing – broadly exploited in research on the algebras in such sophisticated works as [12], [17] and in many others. Although some metamathematical monographs on Boolean algebras, such as: [14], [15] treated the issue perfunctorily (if any), $\mathcal{P}^{\text{Comp}}(\omega)/\text{fin}$ returned to scientific court in [13] and in many other works, such as [17], [32]. It would be challenging to discuss the entire spectrum of directions and contexts in which these algebras are immersed and described in scientific literature. It is not coherent with the main task of the paper. One only needs to underline that all the directions have been developed independently of the category theory in a purely set-theoretic paradigm. Indeed, the theory of $\mathcal{P}^{\text{Comp}}(\omega)/\text{fin}$ algebras – as an integral part of the theory of gaps and limits – is immersed in the arithmetic of high cardinals. As such one – it has almost nothing to do with classical finite combinatorics – as described in [21].

Against these tendencies – another approach to $\mathcal{P}^{\text{Comp}}(\omega)/\text{fin}$ was elaborated in this paper. Its first goal is not to elaborate some new metamathematical properties of these algebras in terms of the arithmetic of high cardinals but to organize the spectrum of relative sets in the context of the multi-fuzzy natural transformations. As previously indicated, this approach was incorporated to build some new connections with classical encoding theory. Finally, this paper research breaks the common tendency to exploit the Haskell language in a role of programming-wise support for category theory – although Haskell’s functionalities and the entire philosophy of programming in Haskell is strongly motivated by an algebraic tissue of category theory. The reason for the solution has already been explained in ‘Introduction.’

This paper may be viewed as a development of considerations from [22], [23], [28]. However, the condition of the diagram commutativity is violated in the paper approach in another way. Here, we consider pairs of almost equal relative sets, while the appropriate inclusions between the sets from the pairs were required in the previous papers.

VII. CONCLUSION AND CLOSING REMARKS

By introducing a piece of machinery of the Boolean algebra theory – we showed how to organize the spectrum of the so-called relative sets due to fuzzifying the commutativity condition for the natural transformation diagrams. We have initially introduced an equivalence relation (‘identifying’ two sets if they have a finite common intersection). Secondly, we collected the equivalent classes into a quotient structure called the composition algebra for relative sets. Next, we equipped this algebra with the appropriate order relation to organize it and extract a piece of algebraic and combinatorial properties of this structure. In this way, we have also illustrated how one could reconcile the dynamic, categorial paradigm with the purely set-theoretic one – the seemingly irreconcilable paradigms. It seems that this project could be performed thanks to our decision to treat the class Comp$^*$ more set-theoretically – as values of the mapping compositions than the mappings compositions themselves. It allowed us to break the difficulty with the different nature of these two approaches.

From a broader perspective, we can specify the paper’s analysis as a ‘reflection on differences’ (between the ‘upward’ and the ‘downward’ mappings compositions), which introduces a piece of fuzziness to the multi-natural transformations. It is noteworthy to underline that the formal algebraic apparatus enabled considering pairs of the sets with only a finite common intersection. Simultaneously, one
needs to underline that only several conceptual ‘bridgeheads’ have been constructed in the area of a typical interaction of these two paradigms for the use of the analysis of the paper. Fortunately, their construction delivers a piece of optimism regarding the possibility to reconstruct a more significant portion of encoding theory in the conceptual framework of the analysis. In particular, it seems that the mutual relationships between the idea of distance measurements in terms of $k$-multi-similarity and $k$-multi-balls and the same idea of distance measurement in terms of Hamming’s distances may be deeper emphasized in the future. It seems to be a promising subject of further analysis.

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**REFERENCES**

[1] S. Eilenberg and S. MacLane, “General theory of natural equivalences,” Trans. Amer. Math. Soc., vol. 28, no. 2, pp. 247–294, 1945.

[2] S. Eilenberg and S. MacLane, “Natural isomorphisms in group theory,” Proc. Nat. Acad. Sci. USA, vol. 28, no. 12, pp. 537–543, Dec. 1942.

[3] S. MacLane, “Duality for groups,” Bull. Amer. Math. Soc., vol. 56, no. 6, pp. 485–516, 1950.

[4] F. Lawvere and R. Rosebrugh, Sets for Mathematics. Oxford, U.K.: Oxford Univ. Press, 2003.

[5] F. Lawvere and S. Schnauel, Conceptual Mathematics: A 1st Introduction to Categories. Cambridge, U.K.: Cambridge Univ. Press, 2009.

[6] S. Awodey, Category Theory. Oxford, U.K.: Oxford Univ. Press, 2010.

[7] M. Stone, “The theory of representation for Boolean algebras,” Trans. Amer. Math. Soc., vol. 40, no. 1, pp. 37–111, 1936.

[8] P. Blackburn and M. de Rijke, “Algebras and general frames,” in Modal Logic. Cambridge, U.K.: Cambridge Univ. Press, 2014, pp. 261–331.

[9] N. Yoneda, “On the homology theory of modules,” J. Fac. Sci., Univ. Tokyo, vol. 7, pp. 193–227, May 1954.

[10] N. Yoneda, “On Ext and exact sequences,” J. Fac. Sci., Univ. Tokyo, vol. 8, pp. 507–576, Apr. 1960.

[11] K. Kunen. Set Theory: An Introduction to Independence Proofs. Berlin, Germany: North Holland, 1980.

[12] S. Shelah, Proper forcing (Lecture Notes in Mathematics), vol. 940. Berlin, Germany: Springer-Verlag, 1982.

[13] S. Koppelberg, I. D. Monk, and R. Bonnet, Handbook Boolean Algebras, vol. 1. Amsterdam, The Netherlands: North Holland, 1989.

[14] R. Sikorski, Boolean Algebras. Berlin, Germany: Springer-Verlag, 1969.

[15] R. Sikorski, “On an analogy between measures and homomorphisms,” Ann. Soc. Pol. Math., vol. 23, pp. 1–20, Jul. 1950.

[16] P. Borodulin-Nadzieja and D. Chodounský, “Hausdorff gaps and towers in $P(\omega)/\text{fin}$,” 2013, arXiv:1302.4550.

[17] D. Chodounský, V. Fischer, and J. Grebik, “Free sequences in $P(\omega)/\text{fin}$,” Arch. Math. Log., vol. 58, no. 7, pp. 1035–1051, 2019.

[18] F. Hausdorff, “Summen von aleph-zero mengen,” Fundam. Math., vol. 56, pp. 485–516, Jun. 1950.

[19] F. Hausdorff. “Ueber zwei satze von g. fichtenholz und l. kantorovitch,” Studia Math., vol. 6, pp. 18–19, Apr. 1936.

[20] J. Parovicenko, “A universal bicomplete weight of aleph,” Sov. Math. Dokl., vol. 4, pp. 592–595, Jul. 1963.

[21] W. Lipski and W. Marek, Combinatorial Analysis. PWN: Warsaw, Poland, 1986.

[22] K. A. Jobczyk and P. Gaździński, “The natural transformations with fuzzified commutativity,” in Proc. IEEE Int. Conf. Fuzzy Syst. (FUZZ-IEEE), New Orleans, LA, USA, Jun. 2019, pp. 1–6.

[23] K. A. Jobczyk and A. Ligeza, “The natural transformations with the multi-fuzzified commutativity condition,” in Proc. IEEE Int. Conf. Fuzzy Syst. (FUZZ-IEEE), Jul. 2020, pp. 1–9.

[24] J. Mockor, “Relational variants of categories of fuzzy sets defined by monads,” in Proc. Conf. Int. Fuzzy Syst. Assoc. Eur. Soc. Fuzzy Log. Technol. (EUSFLAT), 2019, pp. 44–51.

[25] J. Mockor, “Fuzzy type relations and transformation operators defined by monads,” Int. J. Comput. Intell. Syst., vol. 13, no. 1, pp. 1530–1538, 2020.

[26] J. Mockor, “Fuzzy transforms for hesitant, soft or intuitionistic fuzzy sets,” Int. J. Comput. Intell. Syst., vol. 14, no. 1, p. 164, Dec. 2021.

[27] D. Spivak, Category Theory for the Sciences. Cambridge, U.K.: Cambridge Univ. Press, 2014.

[28] K. Jobczyk, “The fuzzified natural transformation between categorical functors and its selected categorial aspects,” Symmetry, vol. 12, no. 9, p. 1578, Sep. 2020.

[29] P. Borodulin-Nadzieja and D. Chodounský, “Hausdorff gaps and towers in $P(\omega)/\text{fin}$,” Fundam. Math., vol. 229, no. 3, pp. 197–229, 2015.

[30] D. Talayco, “Applications of cohomology to set theory I: Hausdorff gaps,” Ann. Pure Appl. Log., vol. 71, no. 1, pp. 69–106, 1995.

[31] P. Hudak, J. Hughes, S. Jones, and P. Wadler, A History of Haskell: Being Lazy With Class. 2007.

[32] W. Brian, “Universal flows and automorphisms of $P(\omega)/\text{fin}$,” Isr. J. Math., vol. 233, no. 1, pp. 453–500, 2019.

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