New considerations on Einstein equations in anisotropic spaces

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November 26, 2009

Abstract

We find the generalization of Einstein equations to Finsler spaces by variational means and, based on the invariance of the Finslerian Hilbert action to infinitesimal transformations, we find the analogous of the energy-momentum conservation law in these spaces.

1 Introduction

As already shown by many researchers, Finsler geometry is an important alternative to be taken into account for a theory of gravitational field which could solve some important problems of modern astrophysics, such as, for instance, [7]: the rotation curves of spiral galaxies, the 3D-problem for spiral galaxies (usual gravity theory doesn’t work in the plane of the galaxy but works in the orthogonal direction), or the location of globular clusters (which is close to the center of the galaxy and not on its periphery, as expected) etc.

In this context, finding a generalization of Einstein equations to Finsler spaces is a necessary and natural step. Several different variants have already been proposed.

Thus, R. Miron and M. Anastasiei ([3], 1984) proposed a generalization of Einstein equations for \( h \)-metrics on the tangent bundle \( TM \):

\[
\begin{align*}
G &= g_{ij}(x, y)dx^i \otimes dx^j + v_{ab}(x, y)\delta y^a \otimes \delta y^b. \\
R_{ij} - \frac{1}{2}(R + S)g_{ij} &= \mathcal{X}T_{ij}, \\
\mathcal{P}_{ij} &= \mathcal{X}T_{ij}, \\
\mathcal{P}_{jib} &= \mathcal{X}T_{jib}, \\
S_{bc} - \frac{1}{2}(R + S)v_{bc} &= \mathcal{X}T_{ab}.
\end{align*}
\]

where \( R_{\alpha\beta} = (R_{ij}, P_{ia}, P_{ai}, S_{ab}) \) are the local components of the curvature tensor of the canonical metrical linear connection on \( TM \) (whose torsion tensor does
not vanish) and $\mathcal{T}_{\alpha\beta}$ are the components of the generalized energy-momentum tensor. The number of equations in this theory is $4 \dim M$.

Later, S. Rutz proposed in 1993, starting from the geodesic deviation equations in (pseudo-)Finsler spaces,

$$\frac{Dw^i}{ds^2} = (R_{j\;kl}^i y^j y^l)w^k,$$

as generalization of Einstein equations in vacuum in these spaces:

$$H^i_k := R^i_{j\;kl}y^j y^l = 0.$$

This is an intuitive approach, pointing out the importance of the deviation operator $H^i_k$. [5]

More recently, in a series of three papers, (2007, 2008, 2009), Xin Li, Zhe Chang, proposed another generalization for a special class of Finsler spaces, namely, for Berwald spaces. In the cited papers, they started from Akbar-Zadeh’s definition of Ricci tensor:

$$Ric_{ik} = \frac{1}{2} \frac{\partial^2}{\partial y^i \partial y^k} (R^h_{j\;kl} y^j y^l),$$

and from Bianchi identities for the Chern connection), thus getting:

$$\{Ric_{jl} - \frac{1}{2} g_{jl} S\} + \{\frac{1}{2} B^h_{j\;kl} + B^h_{j\;lh}\} = 8\pi GT_{jl},$$

where

$$S = g^{ij} R_{ij}, \quad B_{ijkl} = -\frac{1}{2} g_{ij\;u} R^u_{kl}.$$

All the above approaches provide different variants, with different ranges of applicability. Moreover, none of them is based on variational methods.

A first variational approach for a gravitational field equation in Finslerian spaces is proposed by G. I. Garas’ko (2009), by taking a Lagrangian which is proportional to the inverse of the volume of the indicatrix of a pseudo-Finslerian space. Still, the obtained field equations do not provide Einstein equations in the particular case of pseudo-Riemannian spaces.

In this paper, we propose a new generalization of Einstein equations to Finsler space, based entirely on variational approaches and also a generalization of the energy conservation law, starting from the invariance of our generalized Hilbert type action to infinitesimal diffeomorfisms.

## 2 Pseudo-Finslerian spaces

Let $M$ be a 4-dimensional differentiable manifold of class $C^\infty$, $(TM, \pi, M)$ its tangent bundle and $(x^i, y^i)_{i=1}^4$ the coordinates in a local chart on $TM$. By "smooth" we shall always mean $C^\infty$-differentiable.

A pseudo-Finslerian function on $M$, is a function $\mathcal{F} : TM \rightarrow \mathbb{R}$ with the properties:
1. $\mathcal{F} = \mathcal{F}(x, y)$ is smooth for $y \neq 0$;

2. $\mathcal{F}$ is positive homogeneous of degree 1, i.e., $\mathcal{F}(x, \lambda y) = \lambda \mathcal{F}(x, y)$ for all $\lambda > 0$;

3. The pseudo-Finslerian metric tensor:
   \begin{equation}
   g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 \mathcal{F}^2}{\partial y^i \partial y^j} \tag{1}
   \end{equation}

   is nondegenerate: $\det(g_{ij}(x, y)) \neq 0$, $\forall x \in M$, $y \in T_x M \setminus \{0\}$.

In the literature, pseudo-Finslerian spaces are also sometimes called simply Finslerian.

The equations of geodesics $s \mapsto (x^i(s))$ of a Finsler space $(M, \mathcal{F})$,
\[ \frac{dy^i}{ds} + 2G^i(x, y) = 0, \quad y^i = \dot{x}^i, \]
give rise to an Ehresmann (nonlinear) connection on $TM$, called the Cartan nonlinear connection, of local coefficients $N^i_j = \frac{\partial G^i}{\partial y^j}$. Let
\[ \delta_i = \frac{\partial}{\partial x^i} - \tilde{N}_a^i \frac{\partial}{\partial y^a}, \quad \hat{\delta}_a = \frac{\partial}{\partial y^a} \]
be the corresponding adapted basis and
\[ (dx^i, \delta y^a = dy^a + G^a_i dx^i), \]
its dual basis. Then, the equations of geodesics of $(M, \mathcal{F})$ become $\frac{dy^i}{ds} = 0$, $y^i = \dot{x}^i$, $i = 1, \ldots, 4$.

We can also obtain other nonlinear connections $N$ on $TM$ by adding to $(\tilde{N}_a^i)$ the components of a $(1,1)$-type tensor field:
\[ N^a_i = \tilde{N}^a_i + X^a_i. \tag{2} \]

Any vector field $V$ on $TM$ can be written as
\[ V = V^i \delta_i + \tilde{V}^a \hat{\delta}_a; \]
the component $hV = V^i \delta_i$ is a vector field, called the horizontal component of $V$, while $vV = \tilde{V}^a \hat{\delta}_a$ is its vertical component. Similarly, a 1-form $\omega$ on $TM$ can be decomposed as $\omega = \omega_i dx^i + \tilde{\omega}_a \delta y^a$, with $h\omega = \omega_i dx^i$ called the horizontal component, and $v\omega = \tilde{\omega}_a \delta y^a$ the vertical one.
We shall usually denote indices corresponding to horizontal geometrical objects by $i, j, k, ...$, indices corresponding to vertical ones by $a, b, c...$ and $\alpha, \beta, \gamma, ... \in \{i, j, k, ..., a, b, c...\}$ will denote those which can correspond to either of the distributions.

The Chern-type linear connection $\nabla^\alpha = (F^i_{jk}, 0)$ on $TM$ has the local coefficients:

$$F^i_{jk} = \frac{1}{2} g^{ih} (\delta_k g_{hj} + \delta_j g_{hk} - \delta_h g_{jk}).$$

(3)

We denote by $\iota_i$ and $\cdot_i$ the corresponding covariant derivations

$$W^\alpha_{\iota_i} = \delta_i W^\alpha + F^j_{ki} W^\alpha,$$
$$W^\alpha_{\cdot_i} = \frac{\partial W^\alpha}{\partial y^i},$$

(4)

(where $W^\alpha$ are local coordinates of a vector field $W$ on $TM$).

There holds

$$g_{ij|k} = 0, \quad \forall i, j, k = 1...4.$$

The curvature tensor $R(X, Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z$ has the following essential local components:

$$R(\delta_l, \delta_k)\delta_j = R^i_{jkl} \delta_i,$$
$$R(\dot{\partial}_l, \delta_k)\delta_j = P^i_{jkl} \delta_i,$$

where

$$R^i_{jkl} = \delta_l F^i_{jk} - \delta_k F^i_{jl} + F^h_{jl} F^i_{kh} - F^h_{jk} F^i_{hl},$$
$$P^i_{jkl} = F^i_{jkl}.$$

Geodesics $s \mapsto x^i(s)$ on space-time $M$ are characterized by

$$\frac{Dy^i}{ds} = \frac{dy^i}{ds} + F^i_{jk} y^j y^k = 0,$$

(4)

and their deviations, by

$$\frac{D^2w^i}{ds} = R^i_{jkl} y^j y^l.$$

(5)

**Remark 1** The above relations (4) and (5) do not depend on the nonlinear connection we use. Namely, if we choose some other nonlinear connection of coefficients $N^i_{jk}$ and $F^i_{jk}$ are of Chern type (3), then geodesics and their deviations are still described by these relations. Hence, it appears as convenient to consider momentarily an arbitrary nonlinear connection $N = (N^i_{jk})$.  


Also, in our further considerations, the following quantities:

\[ C_{ijk} = \frac{1}{2} g_{ij} \]

(components of the Cartan tensor) will be important.

3 h-v metric structures on \( TM \). Divergence of vector fields on \( TM \)

A tensor on \( TM \) is called distinguished (or Finslerian), \( [3] \), if, with respect to coordinate changes on \( TM \) induced by coordinate changes \( (x^i) \rightarrow (x'^i) \) on \( M \), its local components transform by the same rule as those of a tensor on \( M \):

\[
T_{i_1'...i_k'}^{i_1...i_k}(x, y) = \frac{\partial x^{i_1'}}{\partial x^{i_1}} \frac{\partial x^{i_1}}{\partial x^{i_1}} ... \frac{\partial x^{i_k}}{\partial x^{i_k}} \frac{\partial x^{i_k}}{\partial x^{i_k}} T_{i_1'...i_k'}^{i_1...i_k}(x, y).
\]

For instance, Finslerian metric tensors \( [1] \) are distinguished tensors on \( TM \). Also, the elements \( \delta_i, \partial_a \) are distinguished vector fields, while \( dx^i, \delta y^a \) are distinguished covector fields.

**Definition 2** An h-v metric on \( TM \) is a metric structure of the form

\[
G_{\alpha\beta}(x, y) = g_{ij}(x, y)dx^i \otimes dx^j + v_{ab}(x, y)\delta y^a \otimes \delta y^b
\]

where \( g_{ij} \) and \( v_{ab} \) denote (0,2)-type distinguished tensors with \( \det(g_{ij}) \neq 0 \), \( \det(v_{ab}) \neq 0 \).

**Remark 3** An h-v metric structure is actually a pseudo-Riemannian metric on the manifold \( TM \).

Let us denote

\[
\begin{align*}
g &= \det(g_{ij}), v = \det(v_{ab}), \\
G &= \det(G_{\alpha\beta}) = gv.
\end{align*}
\]

The (Riemannian) volume element on \( TM \) is

\[
dV = * (1)
\]

(where \( * \) denotes the Hodge star operator for differential forms). That is,

\[
dV = \sqrt{|G|} dx^i \wedge dx^2 \wedge ... \wedge \delta y^4.
\]

For simplicity, we shall denote in the following, \( dx^i \wedge dx^2 \wedge ... \wedge \delta y^4 =: d\Omega \), hence

\[
dV = \sqrt{|G|} d\Omega.
\]
The divergence of a vector field \( X = X^i \delta_i + \tilde{X}^a \partial_a \in \mathcal{X}(TM) \) is defined as
\[
\text{div}(X) = d(*X^b),
\]
where \(^b\) denotes the musical isomorphism (lowering indices).

In local coordinates,
\[
\text{div}(X) = \text{div}(X^H) + \text{div}(X^V),
\]
where
\[
\begin{align*}
\text{div}(X^H) &= X^i |_i - P_i X^i, \quad P_i = N^a_{\ i - a} - \delta_i (\ln \sqrt{g}), \\
\text{div}X^V &= \tilde{X}^a_a + \tilde{P}_a \tilde{X}^a, \quad \tilde{P}_a = \partial_a (\sqrt{G})
\end{align*}
\]
(8)

(9)

(and \( |_i \) denotes covariant derivative with respect to the Chern-type linear connection).

### 4 Hilbert action and Einstein equations

Our aim in the following is to define a Hilbert action for Finslerian spaces, which should:
- be as simple as possible and yield as simple equations as possible;
- in the particular case of Riemannian spaces, provide the regular Einstein equations;

Hilbert proposed as a "simplest scalar" (in the pseudo-Riemannian case):
\[
R = g^{ij} R_{ij}.
\]

This "simplest scalar" characterizes geodesic deviation, in the following sense:
\[
\text{geodesic deviation : } \frac{Dw^i}{ds^2} = H^i_k y^k, \quad H^i_k = R_{j\ kl} y^j y^l, \quad y^i = \frac{dx^i}{ds}.
\]
(10)

Then
\[
R_{ij} y^j y^l = \text{trace}(H^i)
\]

In particular, Einstein equations in vacuum can be written as:
\[
H^i_k = 0.
\]
(11)

(which justifies Rutz’s intuitive approach).

In the Finslerian case, we can think of the following possibilities:

1. by means of Ricci tensor \( \text{Ric}_{ij} = \frac{1}{2} \frac{\partial^2 H^i_k}{\partial y^i \partial y^j} \) as defined by Akbar-Zadeh;
2. by means of Chern, Cartan or Berwald connection curvature?
3. eventually add the contracted vertical curvature \( S = g^{ij} S_{ij} \) (for Cartan connection) and get \( R + S \) as "simplest scalar" (as suggested by Miron and Anastasiei’s approach).
We notice that, among the above variants, $R_{jk} = R^i_{jk;i}$ built from the curvature of the Chern connection provides the simplest computations, while the intuitive interpretation (10) and (11) is satisfied.

There is still a technical problem, represented by the non-holonomy of the frame $(\delta_i, \dot{\delta}_a)$, which yields extra terms when performing variation - expectedly leading to complicated equations. In order to solve this problem, we propose the use of another nonlinear connection than Cartan’s. Thus, the intuitive interpretation (10) and (11) will be still sufficed, and the obtained equations will acquire a simple form.

Briefly, we shall perform the following steps:

1. Define an analogue of Hilbert action, for anisotropic spaces;
2. Find a most convenient nonlinear connection (appropriate frame on $TM$).
3. Variate the Hilbert action w.r.t. the metric tensor $g_{ij}$ and get the Einstein equations;
4. Find the analogue of the conservation law for the energy-momentum tensor (some identity verified by the divergence of energy–momentum tensor).

4.1 Step 1: define Hilbert action

Let

$$G_{\alpha \beta}(x, y) = g_{ij}(x, y)dx^i \otimes dx^j + v_{ab}(x, y)\delta y^a \otimes \delta y^b,$$

be an h-v metric on $TM$, such that:
- $g_{ij}$ defines a pseudo-Finslerian metric on $M$, of signature $(+, -, -, -)$.
- $v_{ab}$ denotes an arbitrary and fixed $(0,2)$ distinguished tensor on $TM$, of constant signature. For convenience, let us suppose $v = \det(v_{ab}) < 0$, which entails $G > 0$.

Let

$$R_{jk} = R^i_{jk;i}, \quad R = R^{jk}g_{jk}$$

where $R^i_{jk;l} = F^i_{jk;l} - F^i_{j;k} + F^h_{jk}F^i_{hl} - F^h_{jl}F^i_{hk}$ is the curvature of Chern connection.

**Interpretation:** $R_{jk}y^jy^k$ is the trace (with a minus sign) of the deviation operator (cf. Rutz): $H^i_l = R^i_{jk}y^jy^k$.

We define the Finslerian Hilbert action as

$$S = \int R\sqrt{G}d\Omega,$$

where integration is made upon some fixed domain $D \subset TM$ in the tangent bundle. We have: $S = S(g_{ij}, N^i_j, g_{ij;k}, g_{ij;k;l})$. 
4.2 Step 2: find a convenient nonlinear connection

It is convenient for further purposes to choose, if possible, a nonlinear connection for which \( P^i_a = 0 \), that is, \( N^a_{i,a} - \delta_i (\ln \sqrt{G}) = 0 \). According to (8), in this case, the divergence of a horizontal vector field \( X^H \) on \( TM \) will be written simply as:

\[
\text{div}(X^H) = X^i_{\mid i}.
\]

(15)

Hence, let us consider:

\[
N^a_i = G^a_i + y^a V_i,
\]

where \( V_j = V_j(x, y) \) are the components of a 0-homogeneous in \( y \) distinguished vector field to be determined.

By imposing the condition \( N^a_{i,a} - \delta_i (\ln \sqrt{G}) = 0 \), we get

\[
\delta_i (\ln \sqrt{G}) = \tilde{N}^a_i + y^a V_i + y^a V_{i,a}.
\]

By 0-homogeneity of \( V_i \), it follows that \( y^a V_{i,a} = 0 \), which leads to

\[
V_i = \frac{1}{4} (\delta_i (\ln \sqrt{G}) - \tilde{N}^a_{i,a})
\]

and

\[
N^a_i = G^a_i + \frac{1}{4} (\delta_i (\ln \sqrt{G}) - \tilde{N}^a_{i,a})
\]

(16)

By direct computation, it can be proven that \( N^a_i \) obey the rule of transformation of the coefficients of a nonlinear connection ([3]), with respect to coordinate changes on \( TM \).

Moreover, for this nonlinear connection, the Chern-type connection coefficients are

\[
L^i_{jk} = \frac{1}{2} g^{ih} (g_{hj} \mid k + g_{hk} \mid j - g_{jk} \mid h) = F^i_{jk} - \frac{1}{2} g^{ih} (y^h V_k g_{hj,l} + y^j V_j g_{hk,l} - y^l V_h g_{jk,l}).
\]

By the 0-homogeneity of \( g_{ij} \), the terms in the last bracket vanish, which means that \( L^i_{jk} = F^i_{jk} \). Thus, there holds

Proposition 4 1. The functions (16) are the local coefficients of a nonlinear connection on \( TM \).

2. The coefficients \( L^i_{jk} = \frac{1}{2} g^{ih} (g_{hj} \mid k + g_{hk} \mid j - g_{jk} \mid h) \) of the corresponding Chern type connection coincide with the usual Chern connection coefficients \( F^i_{jk} \).

That is, we can formally use nonlinear connection (16) instead of the usual Cartan one, without changing either the Ricci scalar or the Ricci tensor, but having (15).
4.3 Step 3: variation w.r.t. the metric $g_{ij}$, Einstein equations

Now, having the nonlinear connection (16) and Chern covariant derivation (3), let us perform the variation with respect to the metric $g_{ij}$ of the Finslerian Hilbert action (for vacuum case). That is, we shall variate the horizontal part of the hv-metric structure $G_{\alpha \beta}$ and keep its vertical part $v_{ab}$ fixed.

We get

$$\delta g S = \int \delta (g^{ij} R_{ij} \sqrt{G}) d\Omega =$$

$$= \int R_{ij} \delta g^{ij} \sqrt{G} d\Omega + \int g^{ij} \delta (R_{ij}) \sqrt{G} d\Omega + \int g^{ij} R_{ij} \delta (\sqrt{G}) d\Omega.$$

By means of relation (15), we get that the divergence of a horizontal vector field $V = V^i \delta_i$ on $TM$ can be written simply as $div(V) = V^i \sqrt{g}$, and the second integral is

$$\int g^{ij} \delta (R_{ij}) \sqrt{g} d\Omega = \int \{g^{jk} (\delta F^i_{jk}) - g^{ij} (\delta F^k_{ij})\} \sqrt{g} d\Omega = \int (V^k \sqrt{g})_{,k} d\Omega = 0$$

(we suppose that we can make the variations $\delta F^i_{jk}$ vanish on the boundary of the domain of integration).

The third integral is

$$\int g^{ij} R_{ij} \delta (\sqrt{G}) d\Omega = \int g^{ij} R_{ij} \delta (\sqrt{g}) \sqrt{G} d\Omega = -\frac{1}{2} \int R g_{ij} \sqrt{G} d\Omega.$$

We get:

$$\delta g S = \int (R_{ij} - \frac{1}{2} R g_{ij}) \delta g^{ij} \sqrt{-g} d\Omega,$$

which yields the Einstein equations in vacuum for the Finsler space $(M, F)$:

$$R_{ij} - \frac{1}{2} R g_{ij} = 0. \quad (17)$$

By adding to the action some term $L_{\text{matter}}$, the corresponding stress-energy tensor is

$$\frac{\delta L_{\text{matter}}}{\delta g_{ij}} = \lambda T_{ij},$$

(where $\lambda$ is a constant), and we get

**Theorem 5** The Einstein field equations in the Finsler space $(M, F)$ are

$$R_{ij} - \frac{1}{2} R g_{ij} = \lambda T_{ij}. \quad (18)$$
The quantity

\[ G_{ij} = R_{ij} - \frac{1}{2} R g_{ij} = \delta L_{\text{Hilbert}} \]

is called the Einstein tensor for the Finsler space \((M, F)\). With this notation, Einstein equations read as

\[ G_{ij} = \kappa T_{ij}. \]

### 4.4 Step 4: diffeomorphism invariance of Finslerian Hilbert action and energy-momentum conservation

The correct way of posing the problem of energy momentum conservation, is deducing it from the invariance to (infinitesimal) diffeomorphisms of Hilbert action, [6].

An infinitesimal transformation on \(M\) induces an infinitesimal transformation on \(TM\) as

\[ \tilde{x}^i = x^i + \epsilon \xi^i(x), \]
\[ \tilde{y}^i = y^i + \epsilon \xi^i_j(x)y^j, \]

where \(\xi = \xi(x)\) is an arbitrary vector field on the base manifold \(M\).

We can regard variations \(\delta g_{ij}\), generated by infinitesimal transformations. Then, the variation of the metric tensor is given by

\[ \delta g_{ij} \equiv L_{\xi} g_{ij} = \xi_{ij} + \xi_{jl}^i + 2 \xi^k |_l C_{ijkl}. \]

It can be easily checked (similarly to [6]), that:

**Remark 6** Any scalar action on \(TM\) is invariant to infinitesimal diffeomorphisms.

That is, with respect to such transformations, we shall have for the Finslerian Hilbert action

\[ \delta S = \int G_{ij} \delta g_{ij} \sqrt{G} d\Omega = 0. \]

(the above is independent on the fact that the metric extremizes the action or not!) It is more convenient to write it as

\[ \int G^{ij} \delta g_{ij} \sqrt{G} d\Omega = 0 \]

Replacing \(\delta g_{ij}\) and integrating by parts, we get

\[ 0 = - \int \xi^k \{ G_{lj}^i + (y^j G^{ij} C_{ijk})_l \} \sqrt{G} d\Omega. \]

Since the vector field \(\xi\) is arbitrary and \(G_{ij} = \kappa T_{ij}\) for solutions of Einstein equations we have proved
Theorem 7 The energy-momentum tensor in Finsler spaces satisfies the identities
\[ T^j_{\ k|j} + (y^l T^{ij} C_{ijk})|l = 0, \quad k = 1, ..., 4. \] (19)

The above is the way that the "energy conservation" relation \( T^j_{\ k|j} = 0 \) (which is not even in the Riemannian case a true conservation law, since \( T^j_{\ k|j} = 0 \) does not entail \( \text{div} T = 0 \)) translates to Finsler spaces.

5 Examples

1) For weak metrics
\[ g_{ij}(x, y) = \eta_{ij} + \varepsilon_{ij}(x, y), \]
where \( \eta_{ij} = \text{diag}(1, -1, -1, -1) \) is the Minkowski metric and \( \varepsilon_{ij} \) is a small deformation \( ((\varepsilon_{ij})^2 \approx 0) \), Chern-type connection coefficients reduce to regular Christoffel symbols (with respect to \( x \))
\[ F^i_{\ jk} \simeq \Gamma^i_{\ jk}, \]
hence Einstein equations formally look as in the Riemannian case. In the Lorentz gauge (\( \Box \)), they reduce to
\[ \Box \varepsilon_{ij} \equiv \eta^{kl} \varepsilon_{ij,kl} = 0. \]

Relations (19) reduce to
\[ T^j_{\ k,j} = 0, \]
(regular divergence of the energy-momentum tensor vanishes), hence energy-momentum tensor is conserved.

2) For weak conformal deformations
\[ g_{ijkl} = \varepsilon(x) \gamma_{ijkl}, \]
of the Berwald-Moor metric, the Einstein equations become again
\[ \Box \varepsilon = 0. \]

Acknowledgment: The work was supported by the grant No. 4 / 03.06.2009, between the Romanian Academy and Politehnica University of Bucharest.

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