HRT CONJECTURE AND LINEAR INDEPENDENCE OF TRANSLATES ON THE HEISENBERG GROUP

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Abstract. We prove that the HRT (Heil, Ramanathan, and Topiwala) conjecture is equivalent to the conjecture that finite translates of square-integrable functions on the Heisenberg group are linearly independent.

1. Preliminaries and overview of the paper

Given $x, y \in \mathbb{R}$, define unitary operators $T_x$ and $M_y$ by

$$T_x f(t) = f(t - x), \quad M_y f(t) = e^{2\pi i y t} f(t).$$

The following conjecture known as the HRT conjecture [10, 11, 12, 2, 17, 9] is an open problem deeply rooted in time-frequency analysis. It was posed about twenty years ago by Chris Heil, Jay Ramanathan, and Pankaj Topiwala in [11] as follows

**Conjecture 1.** (The HRT Conjecture) Let $\phi \in L^2(\mathbb{R}), \phi \neq 0$, and let $F$ be a finite subset of $\mathbb{R}^2$. Then the set

$$\{ M_y T_x \phi : (x, y) \in F \}$$

is linearly independent in $L^2(\mathbb{R})$.

Although the HRT conjecture is still unresolved, there are quite a few results that might be regarded as evidence for an affirmative answer. One substantial contribution in the literature is due to Linnell. In [14], Linnell proves that for nonzero $\phi \in L^2(\mathbb{R})$, $\{ M_y T_x \phi : (x, y) \in F \}$ is linearly independent when $F$ is a subset of a full-rank lattice of the time-frequency plane. For a full account of partial results available in the literature, we refer the interested reader to [12].

As is well-known, this conjecture can be recast in terms of the Heisenberg group. First, observe that

$$T_x M_y = e^{-2\pi i xy} M_y T_x$$

holds for all $x, y \in \mathbb{R}$. Second, the joint action of the operators $T_x$ and $M_y$ is irreducible, in the following sense.

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Lemma 2. Let $\mathcal{H} \subset L^2(\mathbb{R})$ be a closed and non-trivial subspace which is stable under all the operators $T_x$ and $M_y, x, y \in \mathbb{R}$. Then $\mathcal{H} = L^2(\mathbb{R})$.

Proof. Fix a nonzero vector $\phi \in \mathcal{H}$ and suppose that $f \in L^2(\mathbb{R})$ is orthogonal to the set
\[ \{ M_yT_x\phi : x, y \in \mathbb{R} \}. \]
We aim to show that $f$ is the zero element in $L^2(\mathbb{R})$. Now
\[ \int_{\mathbb{R}} e^{-2\pi ity}\phi(t - x)f(t)dt = 0 \]
for all $x, y \in \mathbb{R}$, so for each $x \in \mathbb{R}$, the Fourier transform of the function $t \mapsto \phi(t - x)f(t)$ is identically zero, and hence
\[ 0 = \int_{\mathbb{R}} (|\phi(t - x)|^2 \cdot |f(t)|^2) dt \text{ for all } x \in \mathbb{R}. \]
By Fubini’s theorem,
\[ 0 = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |\phi(t - x)|^2 \cdot |f(t)|^2 dt \right) dx \]
\[ = \int_{\mathbb{R}} |f(t)|^2 \cdot \left( \int_{\mathbb{R}} |\phi(t - x)|^2 dx \right) dt \]
\[ x \mapsto t - x = \left( \int_{\mathbb{R}} |f(t)|^2 dt \right) \cdot \left( \int_{\mathbb{R}} |\phi(x)|^2 dx \right). \]
Since $\phi$ is nonzero, we have $\|f\| = 0$, as desired. \qed

That the relation (1.1) is canonical among jointly irreducible two-parameter families of operators is the content of the Stone-von Neumann Theorem, proved independently by Stone and von Neumann in the late 1920’s.

Theorem 3. (Stone-von Neumann) Let $x \mapsto A_x$ and $y \mapsto B_y$ be unitary representations of the additive group $\mathbb{R}$ acting in a Hilbert space $\mathcal{H}$ such that for each $x, y \in \mathbb{R}$,
\[ (1.2) \quad A_x B_y = e^{-2\pi ixy} B_y A_x. \]
Suppose further that $\mathcal{H}$ admits no non-trivial, proper, closed subspace that is invariant under all operators $A_x, B_y, x, y \in \mathbb{R}$. Then there is a unitary map $U : \mathcal{H} \to L^2(\mathbb{R})$ such that for all $x, y \in \mathbb{R},$
\[ UA_x U^{-1} = T_x, \quad UB_y U^{-1} = M_y \]

The three-dimensional Heisenberg group $\mathbb{H}$ can be defined as a subgroup of unitary operators on $L^2(\mathbb{R})$:
\[ \mathbb{H} = \{ zM_yT_x : y, x \in \mathbb{R}, z \in \mathbb{T} \}. \]
When $\mathbb{H}$ is identified with $\mathbb{T} \times \mathbb{R} \times \mathbb{R}$ in the obvious way, the group operation is given by
\[ (z_1, y_1, x_1) (z_2, y_2, x_2) = \left( z_1 z_2 e^{-2\pi i(x_1 y_2)}, y_1 + y_2, x_1 + x_2 \right), \]
where \((z_1, y_1, x_1), (z_2, y_2, x_2) \in \mathbb{T} \times \mathbb{R} \times \mathbb{R}\). With the usual topology on \(\mathbb{T} \times \mathbb{R} \times \mathbb{R}\), \(\mathbb{H}\) is a connected topological group with center

\[ Z = \{ (z, 0, 0) : z \in \mathbb{T} \}. \]

Moreover, \(\mathbb{H}\) is a unimodular group and Lebesgue measure on \(\mathbb{T} \times \mathbb{R} \times \mathbb{R}\) is a left-invariant measure on the group. We remark that \(\mathbb{H}\) is sometimes called the reduced Heisenberg group so as to distinguish it from the simply connected Heisenberg group \(\tilde{\mathbb{H}} = \mathbb{R}^3\), whose group operation is such that the canonical covering map \((u, y, x) \mapsto (e^{2\pi i u}, y, x)\) is a homomorphism.

Next we recall a few facts about unitary representations of \(\mathbb{H}\). A strongly continuous unitary representation \(\pi : \mathbb{H} \to U(\mathcal{H})\), denoted by \((\pi, \mathcal{H})\), is said to be irreducible if \(\mathcal{H}\) admits no non-trivial, proper, closed subspace that is invariant under all operators \(\pi(z, y, x)\). As an example, let \(k \in \mathbb{Z} \setminus \{0\}\) and for each \((z, y, x) \in \mathbb{H}\), put

\[ \pi_k(z, y, x) = z^k M_{ky} T_x. \]

The relation (1.1) shows that \((\pi_k, L^2(\mathbb{R}))\) is a homomorphism of \(\mathbb{H}\) into the unitary group \(U(L^2(\mathbb{R}))\), and it is easy to check that \(\pi_k\) is strongly continuous. Lemma 2 shows that \(\pi_k\) is irreducible.

Unitary representations \((\pi, \mathcal{H})\) and \((\rho, \mathcal{K})\) are equivalent if there is a unitary operator \(U : \mathcal{H} \to \mathcal{K}\) such that \(U \pi(z, y, x) = \rho(z, y, x) U\) holds for all \((z, y, x) \in \mathbb{H}\). Formally, \(\mathbb{H}\) is the space of all equivalence classes of unitary irreducible representations of \(\mathbb{H}\). The following is almost immediate.

**Corollary 4.** Let \(\mathcal{H}\) be a Hilbert space and \((\pi, \mathcal{H})\) an irreducible unitary representation of \(\mathbb{H}\) such that \(\pi|_Z\) is non-trivial. Then there is \(k \in \mathbb{Z} \setminus \{0\}\) such that \((\pi, \mathcal{H})\) is equivalent with \((\pi_k, L^2(\mathbb{R}))\).

**Proof.** As a consequence of Schur’s Lemma, the restriction of \(\pi\) to \(Z\) consists of unitary scalar operators \(\pi(z, 0, 0) = \varphi(z) \text{Id}|_\mathcal{H}\). Since \(z \mapsto \varphi(z)\) is a non-trivial homomorphism of \(\mathbb{T}\), we have \(k \in \mathbb{Z} \setminus \{0\}\) such that \(\varphi(z) = z^k, z \in \mathbb{T}\). Now let \(A_x = \pi(0, 0, x)\) and \(B_y = \pi(0, y/k, 0)\). The group operation in \(\mathbb{H}\) shows that (1.2) holds for each \(x, y\), and hence by Theorem 3 there is \(U : \mathcal{H} \to L^2(\mathbb{R})\) with \(T_x U = U A_x\) and \(M_y U = U B_y\). Since \(B_y^k = \pi(y, 0, 0)\) and \(M_y^k = M_{ky}\), we get \(U \pi(z, y, x) = \pi_k(z, y, x) U\) as desired. \(\square\)

Now suppose that \((\pi, \mathcal{H})\) is an irreducible unitary representation of \(\mathbb{H}\) that vanishes on \(Z\) and let \(p : \mathbb{H} \to \mathbb{H}/Z\) be the canonical quotient map. Then \(\pi\) defines a unitary representation \(\bar{\pi}\) of \(\mathbb{H}/Z\) so that \(\pi = \bar{\pi} \circ p\). Since \(\mathbb{H}/Z\) is just the additive group \(\mathbb{R}^2\), then (again by Schur’s Lemma [6, Proposition 3.5]) we have \(\mathcal{H} = \mathbb{C}\) and there is \(\omega \in \mathbb{R}^2\) such that

\[ \pi(z, y, x) = \chi_\omega(z, y, x) = e^{2\pi i \omega \cdot (x, y)}. \]

Define

\[ \Sigma = \{ \chi_{0, \omega} : \omega \in \mathbb{R}^2 \} \cup \{ \pi_k : k \in \mathbb{Z} \setminus \{0\} \}. \]

**Corollary 5.** Each irreducible representation of \(\mathbb{H}\) is equivalent with exactly one element of \(\Sigma\).
Proof. We have just shown that each unitary irreducible representation is equivalent with some element of $\Sigma$. It remains to observe that for $k_1, k_2 \in \mathbb{Z} \setminus \{0\}$, $k_1 \neq k_2$ implies that $\pi_{k_1}$ and $\pi_{k_2}$ are not equivalent. Similarly, $\omega_1 \neq \omega_2$ implies $\chi_{\omega_1}$ and $\chi_{\omega_2}$ are inequivalent. □

It is now clear that Conjecture 1 is equivalent with the following.

**Conjecture 6.** (Restatement of HRT) Let $k$ be any nonzero integer. Let $\phi \in L^2(\mathbb{R})$, $\phi \neq 0$, and let $\mathcal{F}$ be a finite subset of $\mathbb{H}$ such that the cosets $h\mathbb{Z}, h \in \mathcal{F}$ are distinct. Then the set
\[
\{ \pi_k(h)\phi : h \in \mathcal{F} \}
\]
is linearly independent in $L^2(\mathbb{R})$.

The purpose of this note is to show that the Conjectures 1 and 6 are equivalent with the conjecture that translates in the Heisenberg group are independent. For $h, k \in \mathbb{H}$ and $F$ in $C_c(\mathbb{H})$, put
\[
L_k F(h) = F\left(k^{-1}h\right).
\]
Then for each $k \in \mathbb{H}$, $L_k$ extends to a unitary operator on $L^2(\mathbb{H})$.

**Conjecture 7.** (The Heisenberg-Translate Conjecture) Let $F$ in $L^2(\mathbb{H})$, $F \neq 0$, and let $\mathcal{F}$ be a finite subset of $\mathbb{H}$, such that the cosets $h\mathbb{Z}, h \in \mathcal{F}$ are distinct. Then the collection of vectors $\{L_h F : h \in \mathcal{F}\}$ is linearly independent in $L^2(\mathbb{H})$.

The following remark due to Rosenblatt [19] shows the necessity of the assumption that the cosets $h\mathbb{Z}, h \in \mathcal{F}$ are distinct.

**Remark 8.** Choose a point $z \in \mathbb{T}$ of $H$ such that $z$ has a finite order $n$, and let $K$ be a compact subset of $\mathbb{H}$. Put
\[
F = \sum_{\ell = 1}^n L_{z^\ell}1_K.
\]
Then for a fixed natural number $m$, the following is clearly true
\[
L^{zm}F = \sum_{\ell = 1}^n L_{z^\ell}1_K = F.
\]

The primary objective of this note is to prove the following.

**Theorem 9.** The HRT conjecture fails if and only if the Heisenberg-Translate conjecture fails.

Let $C_c(\mathbb{H}/\mathbb{Z}) = \{F \in C_c(\mathbb{H}) : L_z F = F, z \in \mathbb{Z}\}$; note that $C_c(\mathbb{H})$ projects onto $C_c(\mathbb{H}/\mathbb{Z})$ by
\[
P : F \mapsto \int_{\mathbb{T}} F(\cdot, (z, 0, 0))dz.
\]
It is easily seen that for $p > 1$, $\|PF\|_p \leq \|F\|_p$, so $P$ extends to a continuous map with image $L^p(\mathbb{H}/\mathbb{Z})$, the closure of $C_c(\mathbb{H}/\mathbb{Z})$ in $L^p(\mathbb{H})$. Of course $L^p(\mathbb{H}/\mathbb{Z})$ is canonically isomorphic with $L^p(\mathbb{R}^2)$. It is worth noting that when $p$ is greater than 4, the analog of Conjecture 7 fails.
Proposition 10. [10] Theorem 9.18]

(1) Let \( F \) be a finite subset of \( \mathbb{H} \), such that the elements \( hZ, h \in F \) are distinct elements of \( \mathbb{H}/Z \). If \( F \in L^p(\mathbb{H}/Z) \) is non-zero and \( 1 \leq p \leq 4 \), then the collection of vectors \( \{L_hF : h \in F\} \) is linearly independent.

(2) If \( 4 < p \leq \infty \) then there exist \( F \in L^p(\mathbb{H}/Z) \) and a finite set \( F \) of \( \mathbb{H} \), such that the cosets \( hZ, h \in F \) are distinct, and \( \{L_hF : h \in F\} \) is linearly dependent.

Proof. Given \( F \in L^p(\mathbb{H}/Z) \) and \((z, y, x), (z_j, y_j, x_j) \in \mathbb{H},
\[
F\left( (z_j, y_j, x_j)^{-1} (z, y, x) \right) = F \left( ze^{2\pi i x_j y}, y - y_j, x - x_j \right)
= F \left( 1, y - y_j, x - x_j \right).
\]
Thus, for complex numbers \( c_1, \ldots, c_n \),
\[
\sum_{j=1}^{n} c_j F \left( (z_j, y_j, x_j)^{-1} (z, y, x) \right) = 0
\]
if and only if
\[
\sum_{j=1}^{n} c_j F \left( 1, y - y_j, x - x_j \right) = 0.
\]
The results of this proposition follow from a straightforward application of [10] Theorem 9.18] which is due to the work of Rosenblatt and Edgar [5, 18]. In fact, a function satisfying the claim of the second part of the proposition can be constructed as follows. Define
\[
F(z, y, x) = \int_{1/3}^{2/3} e \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} \arccos(t) \\ \arccos(1-t) \end{bmatrix} \right\} dt.
\]
It is shown in [5] that
\[
2F(1, y, x) = F(1, y, x + 1) + F(1, y, x - 1) + F(1, y + 1, x) + F(1, y - 1, x)
\]
and \( F \) is a continuous function in \( L^p(\mathbb{R}^2) = L^p(\mathbb{H}/Z) \). 

2. Proof of Theorem 9

We begin with a proof of a standard result; see also [4, 7, 16].

Lemma 11. Fix \( k \in \mathbb{Z} \setminus \{0\} \) and let \( f, g \in L^2(\mathbb{R}) \). Then the function \( h \mapsto \langle g, \pi_k(h)f \rangle \) is continuous and square-integrable on \( \mathbb{H} \).
Proof. The fact that $F : h \mapsto \langle g, \pi_k(h) f \rangle$ is continuous is a consequence of the strong continuity of the representation $\pi_k$. The square-integrability of $F$ is due to the following straightforward calculations:

$$
\int_0^1 \int_\mathbb{R} \int_\mathbb{R} \left| \langle g, \pi_k(e^{2\pi i \theta}, y, x) f \rangle \right|^2 \, dx \, dy \, d\theta = \int_0^1 \int_\mathbb{R} \int_\mathbb{R} \left| e^{-2\pi ik \theta} \langle g, \pi_k(1, y, x) f \rangle \right|^2 \, dx \, dy \, d\theta
$$

$$
= \left( \int_0^1 d\theta \right) \int_\mathbb{R} \int_\mathbb{R} \left| \langle g, \pi_k(1, y, x) f \rangle \right|^2 \, dx \, dy
$$

$$
= \int_\mathbb{R} \int_\mathbb{R} \left| \langle g, M_{ky} T_x f \rangle \right|^2 \, dx \, dy.
$$

Now

$$
\int_\mathbb{R} \int_\mathbb{R} \left| \langle g, M_{ky} T_x f \rangle \right|^2 \, dx \, dy = \int_\mathbb{R} \int_\mathbb{R} \left| ([M_{ky} g] * f^*) (x) \right|^2 \, dx \, dy.
$$

In the last equality above, $*$ stands for the usual convolution and $f^* (x) = \overline{f(-x)}$. For each $y \in \mathbb{R}$, the function $x \mapsto ([M_{ky} g] * f^*) (x)$ belongs to $C_0(\mathbb{R})$, and is $L^2$ if and only if

$$
\widehat{M_{-ky} g} \hat{f}^* : \xi \mapsto \hat{g}(\xi + ky) \overline{\hat{f}(\xi)}
$$

belongs to $L^2(\mathbb{R})$. We conclude that

$$
\int_0^1 \int_\mathbb{R} \int_\mathbb{R} \left| \langle g, \pi_k(e^{2\pi i \theta}, y, x) f \rangle \right|^2 \, dx \, dy \, d\theta = \int_\mathbb{R} \int_\mathbb{R} \left| \hat{g}(\xi + ky) \overline{\hat{f}(\xi)} \right|^2 \, d\xi \, dy
$$

$$
= \int_\mathbb{R} \left( \int_\mathbb{R} \left| \hat{g}(\xi + ky) \right|^2 \, dy \right) \left| \hat{f}(\xi) \right|^2 \, d\xi
$$

$$
= k^{-1} \| f \|^2 \| g \|^2 < \infty.
$$

□

The following result now has a short proof.

**Lemma 12.** If Conjecture 1 fails then Conjecture 7 fails as well.

Proof. Suppose that Conjecture 1 fails; then Conjecture 6 fails as well, so we have a nonzero function $\phi \in L^2(\mathbb{R})$, elements $h_1, \ldots, h_n \in \mathbb{H}$, and nonzero complex numbers $c_1, \ldots, c_n$, such that the cosets $h_1 Z, \ldots, h_n Z$ are distinct, and

$$
\sum_{j=1}^n c_j \pi_k(h_j) \phi = 0.
$$
Put $F(h) = \langle \phi, \pi_k(h) \phi \rangle$. Since $\phi$ is a non-zero, according to Lemma 11, $F$ is a non-zero element of $L^2(\mathbb{H})$. Since $\pi_k$ is unitary, we have
\[
0 = \left\langle \sum_{t=1}^{n} c_t \pi(h_t) \phi, \pi_k(h) \phi \right\rangle = \sum_{j=1}^{n} c_j \langle \phi, \pi_k(h^{-1}_j h) \phi \rangle = \sum_{j} c_j L_{h_j} F(h).
\]
Thus Conjecture 7 fails.

It is worth noting that Lemma 12 was also proved in [13, Proposition 1.1].

The proof of the converse of Lemma 12 requires a bit more work. Note that by Proposition 10, for the proof of the converse of Lemma 12, it is enough to consider functions in the closed subspace $\mathcal{K} = \ker P = \{F \in L^2(\mathbb{H}) : PF = 0\}$.

Let $F \in C_c(\mathbb{H})$; for each $k \in \mathbb{Z} \setminus \{0\}$, define a sesquilinear form $s_k$ on $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ by
\[
s_k : (f, g) \mapsto \int_{\mathbb{H}} F(h) \langle \pi_k(h) f, g \rangle dh.
\]
Since $F$ is integrable on $\mathbb{H}$ then $s_k$ is bounded, and hence defines a bounded linear operator $\pi_k(F)$ on $L^2(\mathbb{R})$:
\[
s_k(f, g) = \langle \pi_k(F) f, g \rangle
\]
Straightforward computations show that
(a) for each $h \in \mathbb{H}$, $\pi_k(L_h F) = \pi_k(h) \pi_k(F)$, and
(b) $\pi_k(F)$ is an integral operator with kernel $K^F_k(t, x) = \mathcal{F}_1 \mathcal{F}_2 F(k, -kt, t - x)$.

where $\mathcal{F}_1 \mathcal{F}_2 F$ is the partial Fourier transform of $F(z, y, x)$ with respect to the variables $z \in \mathbb{T}$ and $y \in \mathbb{R}$. Since $K^F_k(t, x) \in L^2(\mathbb{R}^2)$, then $\pi_k(F)$ is a Hilbert-Schmidt operator.

Observe that $L^2(\mathbb{H}) = \mathcal{K} \oplus L^2(\mathbb{H}/\mathbb{Z})$ and $\mathcal{D} = (I - P) C_c(\mathbb{H})$ is dense in $\mathcal{K}$, since $I - P$ is a projection.

**Proposition 13.** The map $F \mapsto (|k|^{1/2} \pi_k(F))_{k \in \mathbb{Z} \setminus \{0\}}$ extends to a linear isometry $\mathcal{K} \to \bigoplus_{k \in \mathbb{Z} \setminus \{0\}} \text{HS}(L^2(\mathbb{R}))$.

**Proof.** Let $F \in \mathcal{D}$; then $0 = PF = \mathcal{F}_1 F(0, \cdot, \cdot)$ so
\[
\|F\|_{L^2(\mathbb{H})}^2 = \sum_{k \in \mathbb{Z} \setminus \{0\}} \int_{\mathbb{R}} \int_{\mathbb{R}} |\mathcal{F}_1 F(k, y, x)|^2 dy dx.
\]
We claim that for each $k$, $\|\pi_k(F)\|_{HS}^2 = |k|^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} |\mathcal{F}_1 F(k, y, x)|^2 dy dx$. Recall that $\pi_k(F)$ is given by

$$(\pi_k(F)\phi)(t) = \int_{\mathbb{R}} K_k^F(t, x) \phi(x) dx, \quad \phi \in L^2(\mathbb{R})$$

where $K_k^F(t, x)$ is defined as above, so

$$\|\pi_k(F)\|_{HS}^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} |K_k^F(t, x)|^2 dt dx = \int_{\mathbb{R}} \int_{\mathbb{R}} |\mathcal{F}_1 \mathcal{F}_2 F(k, -kt, t - x)|^2 dt dx.$$ 

Changing variables gives

$$\|\pi_k(F)\|_{HS}^2 = \frac{1}{|k|} \int_{\mathbb{R}} \int_{\mathbb{R}} |\mathcal{F}_1 \mathcal{F}_2 F(k, t, x)|^2 dt dx = \frac{1}{|k|} \int_{\mathbb{R}} \int_{\mathbb{R}} |\mathcal{F}_1 F(k, t, x)|^2 dt dx$$

as claimed. Thus for all $F \in \mathcal{D}$,

$$\|F\|_{L^2(\mathbb{H})}^2 = \sum_{k \in \mathbb{Z} \setminus \{0\}} \int_{\mathbb{R}} \int_{\mathbb{R}} |\mathcal{F}_1 F(k, y, x)|^2 dy dx = \sum_{k \in \mathbb{Z} \setminus \{0\}} |k| \|\pi_k(F)\|_{HS}^2.$$ 

\[\square\]

**Lemma 14.** If Conjecture 7 fails then Conjecture 1 fails as well.

**Proof.** Suppose that Conjecture 7 fails: there exists a non-zero function $F$ in $L^2(\mathbb{H})$, elements $h_1, \ldots, h_n \in \mathbb{H}$, and non-zero complex numbers $c_1, \ldots, c_n$, such that the cosets $h_1Z, \ldots, h_nZ$ are distinct, and

$$\sum_{j=1}^n c_j L_{h_j} F = 0.$$ 

Recall that we may assume that $F \in \mathcal{K}$.

By Lemma 13, we have $k \in \mathbb{Z} \setminus \{0\}$ such that

$$\|\pi_k(F)\|_{HS}^2 \neq 0$$

so choose $\phi \in L^2(\mathbb{R})$ such that $\psi = \pi_k(F)\phi \neq 0$. But

$$\sum_{j=1}^n c_j \pi_k(h_j) \psi = \sum_{j=1}^n c_j \pi_k(L_{h_j} F) \phi = \pi_k \left( \sum_{j=1}^n c_j L_{h_j} F \right) \phi = 0,$$

showing that Conjecture 6 fails, and hence Conjecture 1 fails. \[\square\]

**Remark 15.** The proof of Theorem 7 is a direct application of Lemma 12 and its converse: Lemma 14.
3. Additional observations on Conjecture 7

Let $\mathcal{B}(L^2(\mathbb{H}))$ be the algebra of bounded linear operators acting on $L^2(\mathbb{H})$. Next, let $\mathcal{C}(L)$ be the linear space of all bounded operators on $L^2(\mathbb{H})$ commuting with $L_h$, $h \in \mathbb{H}$. It is closed under weak limits and taking adjoints, and as such it is a von Neumann algebra.

Define the right regular representation $R$ of $\mathbb{H}$ as follows. For $h \in \mathbb{H}$, we define a unitary operator acting by right translation on $L^2(\mathbb{H})$ as $R_h F(x) = F(xh)$. According to a well-known result of Takesaki, $\mathcal{C}(L)$ is the von Neumann algebra generated by the right regular representation [20].

Proposition 16. The right regular representation of $\mathbb{H}$ admits a cyclic vector. In other words, there exists a vector $F \in L^2(\mathbb{H})$ such that the linear span of $R_h F, h \in \mathbb{H}$ is a dense subspace of $L^2(\mathbb{H})$.

For a proof Proposition 16, we refer the interested reader to a paper of Losert and Rindler [15] which gives a construction of a cyclic vector for the regular representation of any first countable locally compact group. A non-constructive proof of Proposition 16 can also be found in [8].

Proposition 17. If $F$ is a cyclic vector for the right regular representation of the Heisenberg group then Conjecture 7 holds for $F$.

Proof. Suppose by ways of contradiction that $\sum_{j=1}^n c_j L_{h_j} F = 0$ for some nonzero scalars $c_1, \ldots, c_n$ and distinct cosets $h_1 Z, \ldots, h_n Z$. Then the linear span of the vectors $R_{h_j} F, h \in \mathbb{H}$ is a dense subset of $L^2(\mathbb{H})$ contained the kernel of the bounded operator $J = \sum_{j=1}^n c_j L_{h_j}$. The continuity of $J$ implies that $J$ is the zero operator in $\mathcal{B}(L^2(\mathbb{H}))$. This gives a contradiction since it is easy to construct a function $F_1 \in L^2(\mathbb{H})$ such that $JF_1 \neq 0$ (see Proposition 18 and Corollary 19 for example.)

Proposition 18. Conjecture 7 holds for non-trivial functions which are Schwartz in the $(y,x)$-variable and supported on a half-line in the $x$-variable.

Proof. Let $F$ be a non-zero function on the Heisenberg group, Schwartz in the $(y,x)$-variable and supported on a half-line in the $x$-variable. Suppose that $\sum_{j=1}^n c_j L_{h_j} F = 0$ for some nonzero scalars $c_1, \ldots, c_n$ and distinct cosets $h_1 Z, \ldots, h_n Z$. Without loss of generality, we may assume that $F \in \mathcal{K}$. Since the set of compactly supported and continuous functions is dense in $L^2(\mathbb{R})$, there exist $\phi \in C_c(\mathbb{R})$ and a nonzero integer $k$ such that $\pi_k (F) \phi$ is nonzero in $L^2(\mathbb{R})$. By assumption,

$$0 = \pi_k \left( \sum_{j=1}^n c_j L_{h_j} F \right) \phi = \sum_{j=1}^n c_j \pi_k (h_j) \pi_k (F) \phi.$$
On the other hand, it is not hard to verify that \( \pi_k (F) \phi \) is necessarily supported on a half-line in \( L^2 (\mathbb{R}) \). However, it is known that the time-frequency shifts of such a function must be linearly independent [11, Proposition 3]. This gives a contradiction. \( \square \)

A straightforward application of Proposition 18 gives the following.

**Corollary 19.** Conjecture \([7]\) holds for all non-trivial functions which are in \( C_c^\infty (\mathbb{R}) \).

**Proposition 20.** Let \( A \) be an invertible operator in \( \mathcal{C}(L) \). Then Conjecture \([7]\) holds for non-trivial functions of the type \( AF \) where \( F \) is Schwartz in the \((y, x)\)-variable and is supported on a half-line in the \( x \)-variable.

**Proof.** Suppose that \( \sum_{j=1}^{n} c_j L_{h_j} F = 0 \) for some nonzero scalars \( c_1, \ldots, c_n \) and distinct cosets \( h_1 Z, \ldots, h_n Z \). Since \( A \) commutes with the operators \( L_{h_j} \), the vector \( \sum_{j=1}^{n} c_j L_{h_j} F \) must be in the kernel of \( A \). The fact that \( A \) is invertible implies that \( \sum_{j=1}^{n} c_j L_{h_j} F = 0 \). However, \( F \) is Schwartz in the \((y, x)\)-variable and supported on a half-line in the \( x \)-variable. This contradicts Proposition 18. \( \square \)

We conclude our work by giving an example describing a large class of functions for which Conjecture \([7]\) holds.

**Example 21.** Given complex numbers \( c_1, \ldots, c_n \), it is easy to verify that

\[
A = e^{\sum_{j=1}^{n} c_j R_{h_j}} = \sum_{k=0}^{\infty} \left( \sum_{j=1}^{n} c_j R_{h_j} \right)^k \frac{1}{k!}
\]

is an invertible operator in \( \mathcal{C}(L) \). In light of Proposition 20, the following is immediate. The Heisenberg-Translate Conjecture holds for any non-zero function of the type \( AF \) where \( F \) is Schwartz in the \((y, x)\)-variable and supported on a half-line in the \( x \)-variable.

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