Complex dynamics of a heterogeneous network of Hindmarsh-Rose neurons

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Keywords: heterogeneous Hindmarsh-Rose neurons, electrical synapse, ring-star network, chimera state, cluster state, analog electronic circuit

Abstract
This contribution is devoted to the study of the collective behavior of two HR neurons followed by a network of HR neurons. The collective behavior of the two coupled neuron was obtained from the connection between the traditional 3D HR and a memristive 2D HR neuron via a gap junction. The dynamical properties of this first topology revealed that it is dissipative therefore can support complex phenomena. From numerical simulations, it is found that the coupled neurons display a variety of behaviors just by varying the control parameter. Amongst these behaviors found, we have periodic bursting or spiking, quasi-periodic bursting or spiking, and chaotic bursting or spiking. Non-synchronized motion is observed when the electrical coupling strength is weak. However, synchronized cluster states are observed when the coupling strength is increased. Also varied of cross ring networks made of combination of \( N = 100 \) these different HR neurons in the network are also investigated. It is discovered that the spatiotemporal patterns are affected by the network topology. The cluster states are represented in the non-homogenous network’s ring and star structures. The ring and ring-star structures contain single and double-well chimera states. Finally, in the PSIM simulation environment, a comparable electronic circuit for the two coupled heterogeneous neurons is designed and investigated. The results obtained from the designed analog circuit and the mathematical model of the two coupled neurons match perfectly.

1. Introduction

Communications between many neurons and between neurons and cells are realized in a specialized contact region, called a synapse. The number of synapses in the human nervous system is estimated to be around \( 10^{14} \) (100 trillion), but given newly discovered synaptic diversity, it could be substantially greater [1]. Several research studies have been carried out to study the synapse based on artificial mathematical models. Among others, we can quote the chemical synapse [2], electrical synapse [3, 4], hybrid synapse [5, 6], Josephson junction synapse [7], and memristive synapse [8, 9]. These varieties of artificial synapses are used to couple artificial models of neurons. Among the artificial models of neurons proposed in the literature, the Hopfield neural network model [10–13], the Hodgkin-Huxley neuron [14], the Chay model [15], the Izhikevich neuron [16], the FitzHugh-Nagumo (FN) model [17], the Morris-Lecar neuron [18], the 2-D Hindmarsh-Rose (HR), the 3-D-HR model [19, 20], the Rulkov model [21] and the Wilson neuron [22] are widely used. For example, the dynamical behavior of a single non-autonomous Hopfield neuron with a memristive self-synaptic connection was studied in [23]. They demonstrated that the proposed model might produce homogenous extreme multistability.
proposed a non-invasive control mechanism that allows them to choose any attractor among the coexisting ones, which is very interesting to observe. Telksnys et al. [24] investigated the phenomena of broken symmetry in the single solutions of the Hodgkin-Huxley neuron model. They also showed the necessary and sufficient conditions for bright and dark solitary solutions in that model. Xu et al. [22] predicted the dynamics of an improved Wilson neuron. They have considered a memristor model to imitate the effect of electromagnetic induction on it. As a result, rich electrical activities, including the asymmetric coexisting electrical activities and the anti-monotonicity phenomenon, were recorded in their model. The global dynamics of a Chay neuron was explored in [25]. The authors demonstrated the occurrence of some neuron characteristics, such as bursting and spiking, using several basic nonlinear analytic methods. More importantly, their proposed neuron model was implemented on an FPGA to validate their theoretical investigations. The authors explored a Fitzhugh-Nagumo neuron model with memristive autapse in [26]. According to their findings, the proposed model can display extreme multistability. Finally, the circuit implementation of the examined model supported their findings. The multistable dynamics of an autonomous Morris-Lecar neuron have been addressed in [27]. During their study, diverse neuronal behaviors such as chaotic bursting, chaotic tonic-spiking, and periodic bursting behaviors were found. Also, the results were further validated based on a microcontroller development board. A model of a 4-D memristive Hindmarsh-Rose has been investigated in [28].

Apart from the dynamical behavior, several researchers have studied the collective behavior of the network of neurons. In [29], the authors considered a discretized version of the Izhikevich neuron model and found that the electromagnetic flux can act as an order parameter in the sense that it can tune different firing patterns under the variation of electromagnetic flux. Hussain et al. [30] investigated the dynamics of a network of multi-weighted Fitzhugh-Nagumo neurons, taking into account the effect of electrical, chemical, and ephaptic couplings. They analyzed the impact of those coupling on the chimera states and the complete synchronization exhibited by the networks. When the temperature coefficient was varied, chimera states were able to occur in a network of thermosensitive Fitzhugh-Nagumo neurons [31]. In [32], the authors investigated a Hindmarsh-Rose neuronal network. They found the interconnected neurons were able to exhibit chimera states. They also investigated the effect of $\alpha$ stable noise. They found that when the control parameter of the noise is increased, the chimera states are progressively suppressed. In [33], considering the impacts of electrical synapses and autapses, the authors investigated the behavior of a network of Hindmarsh-Rose neurons. They demonstrated the presence of chimera states in the studied network using numerical simulations. More interestingly, Simo et al. [34] revealed traveling chimera patterns in a 2-D HR neural network. Simo et al. [35] studied a set of 3-D HR neurons in the presence of an electric field. As a result, traveling chimera states and multicluster oscillating breathers were discovered in the absence of the field. In the presence of the field, chimera states, multi-chimera states, alternating chimera states, and multiclustel traveling chimeras were discovered. In a ring network of bistable systems, like FitzHugh-Nagumo neurons [36], Chua circuits [37], double-well chimera states were revealed. It was conjectured that the mechanism behind the formation of double-well chimera states was due to double-scroll chaotic attractors or even attractors that span both positive and negative values of the state variable. Interestingly, double-well chimera states were found in the current heterogeneous network under study.

In all of the preceding works, great emphasis has been given to investigating single neuron and the networks of neurons. Concerning networks of neurons, most studies generally considered only networks with the same types of neurons. Such networks are well known to be homogeneous. In a nutshell, the homogeneous coupled neurons or networks showed the complete synchronization, phase synchronization, chimera states, multi-chimera states, alternating chimera states, and multiclustel traveling chimeras. In contrast, the dominant behaviors observed in our heterogeneous coupled neurons network were robust chaos, clusters of synchronization, and single and double-well chimera states, cluster states. It motivates us to consider the heterogeneous coupling between HR neuron models, which will be addressed with the following objectives:

- Investigate the global behavior of a 2-D mHR neuron coupled with a 3-D HR via a memristive synapse.
- Investigate the network of up to 100 neurons using the various topological configuration of the 2D mHR and the 3D HR neurons.
- Propose an electronic circuit of the coupled two neurons to validate our numerical results.

This article has been structured as follows. In section 2, we have described the considered model, which shows the heterogeneous interactions, and have studied their properties. Section 3 shows the dynamical behaviors of the supposed system, as a transition from periodicitics to co-existence of different attractors, etc. In section 5, we have analyzed the networks of oscillators. Section 4 depicts the experimental validations of the numerical predictions of the two coupled neuron models. Section 6 is the conclusion of our work.
2. Model description and its properties

2.1. Model description

A brain-like complex structure is made of the interconnection of many neurons. Those neurons have a variety of functionality and thus perform many tasks. Most studies in the literature are generally focused on the networks of neurons that are homogeneous, considering either the effect or not of different types of fields [29–33]. As a result, because they focused on homogeneous networks of neurons to explain complex brain behavior, these works are excellent examples of ideal cases. To consider a more realistic system and for more robust applications, we have considered two variants of the HR neuron as the nodes of our network under different configurations. In addition, it is good to emphasize that most of the studied neurons in the literature derive from the well-known Hodgkin-Huxley neuron [14]. As can be seen in equation (1), the first neuron used comes from the traditional 2D HR neuron [19] by introducing the effect of a memristive autapse.

\[
\begin{align*}
\dot{x}_1 &= y_1 - a_1 x_1^3 + b_1 x_2^2 + i_1 + \alpha \cos(z_1) x_1 \\
\dot{y}_1 &= c_1 - d_1 x_1^2 - y_1 \\
\dot{z}_1 &= \sin(z_1) + e x_1 
\end{align*}
\]

(1)

The second variant of the neuron used in equation (2) is the well-known traditional 3D HR neuron system, which has been extensively studied in the literature [38].

\[
\begin{align*}
\dot{x}_2 &= y_2 - a_2 x_2^3 + b_2 x_2^2 - z_2 + i_2 \\
\dot{y}_2 &= c_2 - d_2 x_2^2 - y_2 \\
\dot{z}_2 &= r(s(x_3 - \bar{x}_2) - z_2)
\end{align*}
\]

(2)

From equations (1) and (2), \(x_i\) are the membrane potentials of each HR neuron, also called fast variables, \(y_i\) denote the retrieval variables associated to a fast current of either Na\(^+\) or K\(^+\), which are known as recovery variables. The state variable \(z_i\) of the 2-D memristive HR neuron represents the inner variable of the memristive autapse. The term \(\sin(z_i)\) stands for the sum of the magnetic flux leakage, and the membrane potential enables the variation of the magnetic flux, respectively. The term \(\alpha \cos(z_i) x_i\) is the memductance of the memristive synapse. It depicts the modulation of a time-varying field on the membrane’s gap junction. The state variable, \(z_1\), is the slow adaptation current of the traditional 3-D HR neuron. \(i_1\) represent external forcing currents and \(\alpha\) is the memristive autapse strength. Before the investigation of the network, a simple model of the coupled heterogeneous neurons of equation (3) will be studied. The latter model is obtained from the 2D HR neuron with memristive autapse coupled with the traditional 3D HR neuron via a gap junction.

\[
\begin{align*}
\dot{x}_1 &= y_1 - a_1 x_1^3 + b_1 x_2^2 + i_1 + \alpha \cos(z_1) x_1 + \sigma(x_2 - x_1) \\
\dot{y}_1 &= c_1 - d_1 x_1^2 - y_1 \\
\dot{z}_1 &= \sin(z_1) + e x_1 \\
\dot{x}_2 &= y_2 - a_2 x_2^3 + b_2 x_2^2 - z_2 + i_2 - \sigma(x_2 - x_1) \\
\dot{y}_2 &= c_2 - d_2 x_2^2 - y_2 \\
\dot{z}_2 &= r(s(x_3 - \bar{x}_2) - z_2)
\end{align*}
\]

(3)

In equation (3), the two neurons are coupled through an electrical synapse having a coupling strength, \(\sigma\). An electrical synapse, also known as a gap junction, is a mechanical contact between two neurons that permits electricity to pass through. The channels in electrical synapses allow charges (or ions) to flow from one cell to another. In this way, the neurons can exchange information’s and communicate with millions of others via the complex chain reactions within interconnected neurons. The dynamics of the coupled model are investigated using standard nonlinear analysis methods in the parameter space, such as one-parameter bifurcation diagrams, maximal Lyapunov exponent spectrums, and two-parameter bifurcation diagrams. Also, a heterogeneous ring-star network of up to 100 neurons will be used to explore the collective dynamics of the coupled neurons. The investigation of the collective behavior of the proposed coupled neurons and the network will be based on the following values of the parameters: \(a_1 = a_2 = 1, b_1 = b_2 = 3, c_1 = c_2 = 1, d_1 = d_2 = 5, r = 0.008, \bar{x}_3 = -1.6, s = 4, e = 0.5, f = 0.5, i_1 = m \sin(2\pi ft)\). The parameters \(\alpha\) and \(i_2\) are tuneable parameters for different values of \(\sigma, m\).

2.2. Dissipation properties

The volume contraction rate of the heterogeneous coupled neurons considered in the first part of this work is given in equation (4). The contraction rate enables us to estimate an ideal nature of firing patterns generated by the proposed model. In such a way, three types of patterns can be identified among which, the dissipative one with \(\nabla. V < 0\), the conservative one with \(\nabla. V = 0\), and the repelled one with \(\nabla. V > 0\).
From the equation (5), it is evident that the volume contraction rate of the coupled neurons depends on the average values of the membrane potentials of both the neurons as well as the inner variable of the memristive autapse of the 2-D HR neuron. Therefore, the coupled model will be dissipative if and only if

$$\nabla.V = -(3\alpha(x_1^2 + x_2^2) + 2\sigma + 2 + r) > 2b_1(x_1 + x_2) + \cos(z_1)(\alpha + 1)$$

Based on the condition given in equation (6), in the analysis, we have always used parameters such that the system is dissipative, and hence can support attractors in the system.

3. Dynamical behavior of the coupled neurons

This section is devoted to investigating the dynamical behavior of two coupled heterogeneous HR neuron models via an electrical synapse. The two-parameters maximal Lyapunov exponents, bifurcation diagrams, characterize the global behavior of the coupled neurons. In addition, numerical simulations have been performed using parameters and variables in extended precision mode with a fixed time step of $5 \times 10^{-3}$. When the two parameters of the coupled neurons are simultaneously varied and the maximum Lyapunov exponent is recorded at each iteration, the color-coded figures (as shown in figure 1) are obtained. The system’s dynamics (such as periodic attractors, quasi periodicities, and chaos) are recorded from these two-parameters Lyapunov exponent diagrams. Periodic behaviors with the regular attractors are characterized by $\lambda_{\text{max}} < 0$, the quasiperiodic behaviors are supported by $\lambda_{\text{max}} = 0$, while the chaotic behaviors with bounded random patterns are supported by $\lambda_{\text{max}} > 0$.

It can be seen in the two-parameter diagrams that there are several windows of different attractors, like the periodic, quasiperiodic, and chaotic attractors. The switching between the attractors has been captured in the varying parameter space. In order to further support the dynamics of the coupled system, the bifurcation diagrams have been computed, which are shown in figures 2(a), (c), and (e). The corresponding maximal Lyapunov exponents in the parameter space are shown in the figures 2(b), (d), and (f), respectively. In the figure 2(a), the bifurcation parameter is $\sigma$. We can see a transition from the quasiperiodic orbit to the chaotic
orbit in the bifurcation diagram. The corresponding maximal Lyapunov exponent confirm this type of transition. The bifurcation diagram in figure 2(c) is obtained when the external forcing current, \( i_2 \), of the second neuron, is varied. From the figure, we can see an interplay between different periodic orbits and the chaotic attractors in the bifurcation diagram. The periodic windows with different attractors exist between the chaotic attractors in the parameter space. The corresponding maximal Lyapunov exponent is shown in figure 2(d).

Additionally, for some fixed values of the control parameters, the effects of the memristive autapse strength of the first neuron on the collective behavior of the coupled neurons are also evaluated. In the bifurcation diagram shown in figure 2(e), when the memristive autapse \( \alpha \) is null, the coupled model exhibits chaotic dynamics. When \( \alpha \) is increased in the positive direction, the chaotic dynamic is metamorphosed to give birth to a quasiperiodic attractor. While expanding the control parameter \( \alpha \) further, the quasiperiodic attractor disappears, and a period-1 limit cycle exists. The corresponding maximal Lyapunov exponent, as shown in figure 2(f), confirms the transition between the attractors with the variation of \( \alpha \).

The coupled neuron model, for example, can exhibit robust chaos when the electrical coupling strength \( \sigma \) is varied within a specific range, as shown in figure 2(a). The persistence of chaotic dynamics in the coupled neurons model when the electrical coupling between the neurons is smoothly varied over a wide range characterizes this phenomenon [39–41].

For a deep analysis of the behaviors that can occur in the coupled neurons under the consideration of varying the electrical coupling strength, we have computed the time-series waveforms of \( x_1 \) and \( x_2 \), shown in figure 3. These waveforms depict the time evolution of the membrane potential of each coupled neuron for a particular value of the electrical coupling strength, chosen from figure 2(a). Figures 3(a) and (b) are the time-series waveforms of \( x_1 \) and \( x_2 \), when the parameter \( \sigma = 0 \), i.e., when there is no couple between the two neuron models. In this case, the membrane potential of the first neuron exhibits chaotic spiking while the second neuron exhibits chaotic bursting. The irregular periodicities in the time series waveforms confirm the chaotic behaviors. When the parameter \( \sigma = 1 \), both the neurons exhibit chaotic bursting with different shapes, as shown in figures 3(c) and (d) respectively. The absence of synchronization between the behavior of the coupled neurons is because of
the feeble electrical coupling strength $\sigma$. For large enough coupling strength, i.e., when $\sigma = 100$, chaotic burstings of figures 3(e) and (f) are obtained. The identical shape of the time series supports the partial/cluster of synchronization of the coupled model. Indeed, for $\sigma = 100$, the membrane potential and the recovery potentials of the first and the second neurons are synchronized. In contrast, the inner variable of the memristive autapse of the first neuron and the slow adaptation current of the second neuron is not synchronized.

Figure 4 shows the time-series waveforms of $z_1$ and $z_2$ of the coupled neurons. These state variables signify the inner variable of the first neuron’s memristive autapse and the slow adaptation current of the second neuron, respectively. As from the figure, we can say that for $\sigma = 100$, these state variables are not in synchronism, which supports the lack of synchronization statements of the heterogeneous HR neuron model.
When $\sigma$ has a lower value close to zero, the two coupled neurons are completely out of synchronization. We now explore the behaviors in the coupled neurons under the absence of synchronization conditions, i.e., for a small constant parameter value of $\sigma$. For that, we have computed the time series of the model using some constant parameter values of the external forcing current, $i_2$. When $i_2 = 0$, both the neurons exhibit periodic firing activities, as shown in figures 5(a) and (b). Chaotic bursting and spiking are exhibited by both neurons of the coupled model as shown in figures 5(c)–(f) for the parameter values $i_2 = 2$ and $i_2 = 7$, respectively. The absence of synchronization between these firing activities is related to the weak electrical coupling strength $\sigma$.

In the next section, we perform PSIM simulations to validate the theoretical and numerical results obtained.

4. PSIM simulations

In order to validate the numerically predicted results, an equivalent circuit diagram of the considered system (as shown in equation (3)) is developed. The electronic circuit diagram is shown in figure 6. To achieve the circuit, we have rewritten the equation (3) as below:

$$C R \dot{x}_1 = - \left[ \left( \frac{R}{R_{39}} \right) \cdot (-x_1) + \left( \frac{R}{R_{42}} \right) \cdot (-x_2) \cdot (x_1) + \left( \frac{R}{R_{46}} \right) \cdot (-x_2^2) \cdot (-x_3) + \left( \frac{R}{R_{47}} \right) \cdot (-x_1) \cdot (\cos(x_3)) + \left( \frac{R}{R_{52}} \right) \cdot (-m \cdot \sin(2\pi ft)) + \left( \frac{R}{R_{56}} \right) \cdot (x_1 - x_2) \right]$$

\[(7)\]
where, $R_{42} = b_1$, $R_{71} = b_2$, $R_{46} = a_1$, $R_{72} = a_2$, $R_{82} = \alpha_1$, $R_{83} = \alpha_2$, $R_{52} = \sigma_1$, $R_{62} = d_1$, $R_{63} = d_2$, $R_{80} = e$, $R_{81} = r_s$, $R_{86} = r$, and $A = rsA_1$.

From figure 6, the outputs of the Op-Amps OA2, OA9, and OA6 are the state variables $x_1, y_1$, and $z_1$ of the first oscillator, respectively. Similarly, the outputs of the Op-Amps OA11, OA14, and OA16 are the state variables $x_2, y_2$, and $z_2$ of the second oscillator, respectively. The state variables from the circuit diagram have a unit, which is in volt. We have provided the initial conditions as initial voltages of the six capacitor voltages in $V_c (−0.54, −5.01, 0.0, 0.688)$. The

\begin{align*}
CR\dot{y}_1 &= -\left(\frac{R}{R_{57}}\right) (-C_1) + \left(\frac{R}{R_{58}}\right) (-x_1) (-x_1) \\
&\quad + \left(\frac{R}{R_{59}}\right) (y_1) \\
CR\dot{z}_1 &= -\left(\frac{R}{R_{62}}\right) (-x_1) + \left(\frac{R}{R_{63}}\right) (-\sin(z_1)) \\
CR\dot{x}_1 &= -\left(\frac{R}{R_{68}}\right) (-y_1) + \left(\frac{R}{R_{71}}\right) (-x_1) (-x_1) + \left(\frac{R}{R_{72}}\right) \\
&\quad \cdot (-x_1^2) (-x_3) + \left(\frac{R}{R_{73}}\right) (z_2) + \left(\frac{R}{R_{80}}\right) (-x_1 + x_2) \\
&\quad \quad + \left(\frac{R}{R_{82}}\right) (-i) \\
CR\dot{y}_2 &= -\left(\frac{R}{R_{86}}\right) (-x_1) + \left(\frac{R}{R_{87}}\right) (-x_2) \\
&\quad + \left(\frac{R}{R_{83}}\right) (y_2) \\
CR\dot{z}_2 &= -\left(\frac{R}{R_{86}}\right) (-x_1) + \left(\frac{R}{R_{87}}\right) (z_2) + \left(\frac{R}{R_{91}}\right) (-A) \\
\end{align*}
ac sine wave in the expression 3 is expressed here in terms of $V$. In case of non-dimensional equation (3), the value of frequency $f$ is 0.5. But the circuit diagram equations are the dimensional one. So, to maintain the equivalence between them, the frequency of $V$ is chosen 500 Hz. To simulate the system in the PSIM platform, we have chosen the time-step as 5 $\mu$s, total time as 11 sec. We have eliminated the first 10 sec as the transient-time and have plotted the last second. We have used unity gain inverting amplifiers to achieve the state variables’ negative values.

In order to obtain the experimental validations of the numerical predictions, we have varied $i$ and $\sigma$, keeping the remaining parameters fixed.

Figure 7 is the time-series waveforms of the two oscillators. Figure 7(a) is the time-series waveform of the state variable $x_1$ of the first oscillator. Figure 7(b) is the time-series waveform of the state variable $x_2$ of the second oscillator. The two figures are well-agreed with the numerical results as shown in figures 3(c) and (d) in the parameter values as mentioned in the caption.

Figure 8 is the time-series waveforms of the two oscillators. Figure 8(a) is the time-series waveform of the state variable $x_1$ of the first oscillator. Figure 8(b) is the time-series waveform of the state variable $x_2$ of the second oscillator. The two figures are well-agreed with the numerical results in the parameter values as mentioned in the caption.

Looking at the two waveforms carefully, we find that the waveforms are identical, making the two oscillators synchronized in the parameter ranges mentioned in the caption. If we plot the phase-space, $x_1$-$x_2$, we confirm that the first state variable of the two oscillators, i.e., $x_1$ and $x_2$, are synchronized.

Figure 9 is the time-series waveforms of the two oscillators. Figure 9(a) is the time-series waveform of the state variable $x_1$ of the first oscillator. Figure 9(b) is the time-series waveform of the state variable $x_2$ of the second oscillator. The two figures are well-agreed with the numerical results in the parameter values as mentioned in the caption.
Since two coupled neurons are insufficient to study their collective behaviors, the next section of our work, we shall analyze heterogeneous ring-star networks made up of the interconnection of up to 100 coupled neurons under various coupling configurations.

5. Network analysis

After exploring a single coupled heterogeneous neuron model in the section 3, we have considered in this section a heterogeneous ring-star system of both the traditional Hindmarsh neuron and memristive Hindmarsh neuron models.
The dynamical equations of the ring-star network are given by

$$\dot{x}_i = f_x + \mu(x_i - x_0) + \frac{\sigma}{2P} \sum_{n=i-P}^{n=i+P} (x_i - x_n),$$

$$\dot{y}_i = f_y,$$

$$\dot{z}_i = f_z.$$  \hspace{1cm} (13)

For $i = 1$ (central node) the dynamical equations are:

$$\dot{x}_i = f_x + \sum_{j=1}^{N} \mu(x_j - x_i),$$

$$\dot{y}_i = f_y,$$

$$\dot{z}_i = f_z.$$  \hspace{1cm} (14)

where depending upon one of the three network configurations, the functional form of $f$ can be either traditional HR neuron or memristive HR neuron, with periodic boundary conditions:

$$x_{i+N}(t) = x_i(t),$$

$$y_{i+N}(t) = y_i(t),$$

$$z_{i+N}(t) = z_i(t),$$

for $i = 2, 3, \ldots, N$.

The network size is considered to be of 100 nodes with $P$ nearest neighbors connected with each other having periodic boundary conditions. The network parameters, such as the ring coupling strength $\sigma$, star coupling strength $\mu$, and the coupling range $P$, will be varied to explore different synchronization patterns arising in the ring-star network memristive Hindmarsh–Rose neuron system. It is good to emphasize that, in the connection
topology of the figures 10(a)–(c), the red nodes are associated with the 3D traditional HR neuron; and the blue nodes are associated with the memristive 2D HR neuron.

In this study, we show three different heterogeneous networks. The motivation is to introduce heterogeneity in the network gradually. Network A is composed of a traditional HR neuron as the central node, and the other end nodes are comprised of memristive HR neurons. The schematic representation is shown in figure 10(a). Network B is constructed with the odd number of nodes made of traditional HR neurons and the even number of nodes made of the memristive HR neuron models, as shown in figure 10(b). Network C has been set up for N = 100 nodes, the right hemisphere is connected to memristive HR neurons, and the left hemisphere comprises traditional HR neurons. The schematic representation of the Network C is shown in figure 10(c).

Since the heterogeneity is due to the presence of both traditional HR neurons and memristive HR neurons, the functional form of \( f \) for the \( i \)-th node in the equations (13) and (14) changes according to the configuration of the network A, B, C. The functional form of \( f \) for \( i \)-th node can be expressed as either in the equation (1) or the equation (2). The working parameter set for this section of the paper are fixed as follows: for the traditional HR neuron in equation (1), the parameters are \( b_2 = 3, d_2 = 5, c_2 = 1, a_2 = 1, \gamma = 0.008, \bar{x} = -1.6, i_2 = 5.5, s = 4 \). For the memristive HR neuron, the parameters are \( a_1 = 1, c_1 = 5, d_1 = 5, e = 0.5, m = 0.02, f = 0.5, \alpha = 0.01 \).

5.1. Quantitative metrics
In order to quantify chimera states, coherent and incoherent states, we use two metrics: (a) root mean-square deviation and (b) cross correlation coefficient. In [36], authors have characterized chimera states (both single-well and double-well), coherent states, and incoherent states using the above two characterizations. We next define the two characterizations used:

5.2. Root mean square deviation
The root mean-square deviation (denoted by \( \Delta_i \)) of element states is given by:

\[
\Delta_i = \langle (2x_i(t) - x_{i+1}(t) - x_{i-1}(t))^2 \rangle, \tag{15}
\]

for \( i = 1, \ldots, N \). It measures the difference between the adjacent nodes and then considers an average of them. Thus, this can enable us to detect chimera states, coherent states. The local maxima observed in \( \Delta_i \) are a signature of incoherent clusters.

5.3. Cross-correlation coefficient
The cross-correlation coefficient denoted by \( \Gamma_{i,m} \) (cross-correlation of the \( i \)th node with the \( m \)th node) of a ring-star network is given by

\[
\Gamma_{i,m} = \frac{\langle x_i(n) \bar{x}_m(n) \rangle}{\sqrt{\langle (x_i(n))^2 \rangle \langle (\bar{x}_m(n))^2 \rangle}}, \tag{16}
\]
The averaged cross-correlation coefficient over all the units of the network is given by,

$$\Gamma = \left( \frac{1}{N} \right)_m \sum_{m=1}^{N} \Gamma_{i,m},$$

where $\Gamma$ denotes the average cross-correlation coefficient over time with the transients removed. We have $\bar{x}(n) = x(n) - \langle x(n) \rangle$, which quantifies the variation of $x$ from its mean and $\langle \cdot \rangle$ denotes the average taken over time. If the value of the cross-correlation coefficient is $\pm 1$, it denotes the regions without oscillations (coherent regions), and the sign of the cross-correlation coefficient actually tells which well the element lies on. A value of cross-correlation coefficient in the neighborhood of zero signifies an incoherent state.

5.4. Network A

We consider a ring-star network of the system as shown in figure 10(a). We reveal different spatiotemporal patterns upon the variation of the star and ring coupling strengths $\mu$ and $\sigma$ respectively.

In figure 11(a), we have shown a double-well chimera state with the coexistence of coherence and incoherent nodes. The nodes oscillate in both the negative and positive values of the state variables $x$ in the ring-star network when the ring coupling strength $\sigma$ and star coupling strength $\mu$ are 14 and $-1$, respectively. It is confirmed via spatiotemporal plot on the left and recurrence plot on the right comprising blue (coherent) and red (incoherent) colors confirming a chimera. Since the structure traverses both the positive and negative values (both wells), it is called a double-well chimera state. This can also be seen by the root mean square deviation (R.M.S.D), denoted by $\Delta_i$. There are nodes with a zero value of $\Delta_i$ signifying coherence, and also there are many local peaks around the nodes behaving incoherently in another well. The average cross-correlation coefficient also gives a value of $\Gamma = -0.4$, implying that the elements are oscillating in two wells (both positive and negative).

When the ring coupling strength is increased, the chimera state gets destroyed and settles down to a double well two-cluster state, as shown in figure 11(b). It is also confirmed via the recurrence plot on the right due to many small square structures. This can be observed better with the R.M.S.D plot on the right. We can see that it takes two values, signifying the presence of clusters. The cross-correlation coefficient is $\Gamma = 0.7$, which is close to one and indicates coherence.

In figure 12(a), a star network of figure 10(a) is considered by setting the ring coupling strength $\sigma$ at 0. When $\mu = -3$, the nodes oscillate incoherently as evident from the nodes plot and the spatiotemporal pattern plots in figure 12(a). The R.M.S.D plot shows incoherence as there are almost no nodes with zero value of $\Delta_i$. The cross-correlation coefficient is also close to zero ($\Gamma = 0.04$), indicating incoherence. When the memristive strength $m$ is decreased to 1, the oscillators start to gather around in a two-cluster state as shown in figure 12(b). It can also be confirmed via the spatiotemporal plot and square-shaped repetitive structures in the recurrence plot. The R.M.S.D plot also shows a similar trend and treats it as incoherence, with the cross-correlation coefficient $\Gamma = 0.3$.

5.5. Network B

We further add more heterogeneity to the network A. One of the ways to proceed is to consider even-numbered nodes to be of memristive HR neurons and the odd-numbered nodes to be of standard HR neurons. The schematic representation of this kind of model is shown in figure 10(b). It can be observed that the number of
memristive HR neuron nodes has decreased in this case as compared to network A. We then explore different spatiotemporal patterns in this case.

For the ring network type B, we have observed a double-well-like state with clusters as shown in figure 13. Although the system steers the double-well-shaped state, it is qualitatively different from a double-well chimera state (as compared with figure 11). The recurrence plots on the right confirm the previous statement. This can be observed in the R.M.S.D plot, where there is a group of oscillators with zero $\Delta_i$. While there is oscillatory behavior in the $\Delta_i$ plot in the incoherent region, indicating that the oscillators are toggling between two wells. The cross-correlation coefficient in this case, $\Gamma = -0.3$.

We can verify the cluster property by forming the small square structures using the recurrence plot of network type B. In the case of the ring network, such double-well chimera state is shown in figure 13(a) for the parameter value $\sigma = 10$ and in the case of ring-star network, the double-well chimera state is shown in figure 13(b). Observe that the R.M.S.D plot just shifted to the right and has a group of nodes with zero $\Delta$, and another group with the presence of local maxima and minima signifying double-well chimera.

We next consider the star configuration of network B by setting $\sigma = 0$, and $\mu 
eq 0$, as shown in figure 14. When the star coupling strength, $\mu$, is set to 0.4, we observe a double cluster state in figure 14(a). It is evident from the space-time plot on the left and the recurrence plot on the right via the formation of tiny squares of figure 14(a). R.M.S.D plot, where the oscillators have many local peaks, and it treats this state as incoherent. The average cross-correlation coefficient turns out to be $\Gamma = 0.7$. The cluster state configuration is robust with the coupling strength $\mu$ variation. When the coupling strength is decreased to $\mu = 0.05$, we observe a three clustered state, which is shown in figure 14(b). The recurrence plot also confirms this three-cluster state by forming many.
tiny squared structures. The $\Delta_i$ plot has a lot of peaks and can be treated as incoherent behavior, but the values of $\Delta_i$ are nearby for adjacent nodes.

5.6. Network C
In this network, we consider a bipartite network, where the ring-star network is divided into two hemispheres. Memristive HR neurons are connected on the right hemisphere (in blue), and on the left, traditional HR neurons are interconnected (in red color). The schematic representation of this type of network is shown in figure 10(c). We next explore variety of spatiotemporal patterns exhibited by this network.

In the case of the ring network ($\mu = 0$), we observe that a double-well cluster state exist, that is evident from the state variable plots and also from the recurrence plot, as shown in figure 15(a). The R.M.S.D plot shows two local maxima peaks, one indicating the nodes to be incoherent and the other having zero values of $\Delta_i$ indicating coherence. Thus, a two cluster state can be verified. The average cross-correlation coefficient is also 0.8112, indicating closer to synchronization. This deviation is due to the formation of clusters. If it is fully synchronized, we would have noticed the value of $\Gamma$ to be 1. A similar two cluster state is observed in a ring-star configuration when $\sigma = 10, \mu = 3.5$ in figure 15(b). When $\sigma = 1.1, \mu = 1.1$, the prevalence of chimera state is recorded and can be verified by observing the state variable plot of the nodes of the network and also from the recurrence plot on the right in figure 15(c).

A time series plot for figure 15(c) supports the existence of a chimera state, where both synchronous and asynchronous regimes having four nodes were identified from figure 15(c). The evolution of the central node is marked in cyan, the node number less than 50 in sync is shown in blue, node number 55 kept in red, node number 95 marked in black. The time series is shown in figure 16, where we can see the coexistence of both synchronous and asynchronous nodes confirming a chimera state.

In the case of a star network, when $\mu = 1.5$, we observe a three-cluster state, as shown in figure 17(a) making a transition in both positive and negative values of $X$ (it toggles between positive and negative well). As can be seen from the $\Delta_i$ plot on the right, a group of nodes is in sync or with zero mean square deviation, and then the other half of the network toggles between the two wells, and the $\Delta_i$ plot has oscillatory behavior. The value of the cross-correlation coefficient is 0.752, which is in close agreement with the synchrony. It is evident from both the state variable plot and the spatiotemporal plot. When $\mu$ is decreased to 0.5, we observe a close two cluster states as in figure 17(b). Even though in this state the transition towards two clusters can be observed, the RMSD plot on the right shows a higher number of peaks with many elements with a higher value of deviation and hence treats it as incoherence. The cross-correlation coefficient is also low, around 0.3.

5.7. Cross-correlation coefficient with number of nodes
We next consider the variation of the cross-correlation coefficient $\Gamma$ with the number of nodes $N$ for network A as it can be seen in figure 18. We have observed that in network A, with an increase in the coupling strength $\sigma$, the cross-correlation coefficient follows an increased trend and reaches close to one or to a synchronized state.

For the same range of coupling strength $\sigma$, we also considered the cross-correlation coefficient $\Gamma$ for different sizes of the network $N$. As we can see, the trend is not much affected, but there is some variations in the value of the cross-correlation coefficient. We also expect a similar scaling trend for the other two networks B and C.
detailed study on the proper scaling of the number of nodes on the characteristics considered can be considered in the future.

6. Conclusion

This paper explores the collective behavior of two coupled and ring-star networks of heterogeneous neurons consisting of memristive 2-D HR and traditional 3-D HR neurons. The dynamical behavior of two heterogeneous coupled neurons via an electrical synapse has been investigated using the nonlinear analysis techniques based on the two-parameters maximal Lyapunov exponent graphs, bifurcation diagrams, and time-series waveforms. It has been found that the coupled model was able to exhibit a rich kind of dynamics, such as periodic, quasi-periodic, and chaotic dynamics involving bursting and spiking oscillations when parameters

Figure 15. A ring-star configuration of network C is considered. In (a), (b) a double-well cluster state is shown. observe that the oscillators are in both positive and negative x values. In (c), a chimera state is shown evident from the recurrence and spatiotemporal plot.

Figure 16. Time series of chimera state for the ring-star network configuration C. Four nodes $i = 1$ (central node) marked in blue, $i = 25$ node marked in blue, $i = 55$ node marked in red, $i = 95$ node marked in black. It clearly shows the coexistence of synchronous and asynchronous states or a chimera.
were varied smoothly in one direction. The absence of synchronization was observed for weak electrical coupling strength, while a synchronization cluster was observed between those neurons for higher values of the coupling strength. We have considered three different topological heterogeneous configurations concerning the collective behavior of up to 100 coupled neurons in a ring-star network. We also investigated various spatiotemporal patterns exhibited by them. Root mean square deviation and average cross-correlation coefficient are considered to characterize various spatiotemporal patterns obtained in the study. We found that they were effective in detecting chimera, double-well chimera, coherent state, but they were not able to distinguish between double-well cluster and double-well chimera state. A future direction would be to systematically analyse the parameter space and initial conditions to claim the prevalence of one spatiotemporal pattern over the other as heterogeneity is added. A detailed study on the variation, scaling of the root mean square deviation, cross-correlation coefficient can be carried out in the future. To further support the fact that our obtained result was not related to an artifact, the electronic circuit of the electronic heterogeneous coupled neurons was built and run in a PSIM environment to support our numerically obtained results further. Since the network of heterogeneous Hindmarsh-Rose neuron model showcase double-well chimera states, this motivates us to further consider coexistence of spatiotemporal patterns along similar lines of [42–44]. Various other network topologies like lattice network [45–48], multilayer networks [49] in which there can be spiral waves, antiphase synchronization phenomenon. A discrete version of this system might reveal new dynamical behaviors [29, 50] as the bifurcation scenarios for both the continuous and discrete systems are different.
Acknowledgments

This work has been supported by the Polish National Science Centre, Poland under the grant OPUS 18 No. 2019/35/B/ST8/00980.

Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

Conflict of interest

The authors declare that they have no conflict of interest.

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