A Note on Probe Cographs

Ton Kloks

Department of Computer Science
National Tsing Hua University
Taiwan

Abstract. Let G be a graph and let \( N_1, \ldots, N_k \) be \( k \) independent sets in \( G \). The graph \( G \) is a \( k \)-probe cograph if \( G \) can be embedded into a cograph by adding edges between vertices that are contained in the same independent set. We show that there exists an \( O(k \cdot n^5) \) algorithm to check if a graph \( G \) is a \( k \)-probe cograph.

1 Introduction

Definition 1. A decomposition tree of a graph \( G \) is a pair \((T, f)\) where \( T \) is a ternary tree and where \( f \) is a bijection from the leaves of \( T \) to the vertices of \( G \).

Definition 2. A graph is a cograph if it has no induced \( P_4 \).

The class of cographs is exactly the class of graphs which contains the one-vertex graph and which is closed under complementation and disjoint union. That is, a graph \( G \) is a cograph if and only if every induced subgraph \( H \) of \( G \) with at least two vertices is either disconnected or its complement \( \overline{H} \) is disconnected.

There are various other characterizations. For example, a graph \( G = (V, E) \) is a cograph if and only if every induced subgraph \( G' = (V', E') \) with at least two vertices has a twin, that is, it has two vertices \( x \) and \( y \) that have the same neighbors in \( V' \setminus \{x, y\} \).

Cographs have a decomposition tree which is called a cotree. This is a rooted binary tree in which each internal node is labeled with \( \otimes \) or \( \oplus \). The \( \otimes \)-operator connects every vertex in the left subtree with every vertex in the right subtree. The \( \oplus \)-operator unions the two subgraphs induced by the left- and right subtree.

Every edge in the cotree induces a partition of the vertices in two parts, say \( A \) and \( B \), which are the sets of vertices that are mapped to the leaves of the two subtrees that are separated by the edge. The submatrix of the adjacency matrix with rows indexed by the vertices of \( A \) and the columns indexed by the vertices of \( B \) has the form

\[
\begin{pmatrix}
J & 0 \\
0 & 0
\end{pmatrix}
\]

or the transpose of this. Here \( J \) is the all-ones matrix. This follows from the fact that the vertices in every rooted subtree form a module. Cographs can be recognized in linear time. The recognition algorithm builds a cotree \( \Pi \).
Let $G = (V, E)$ be a cograph and let $N_1, \ldots, N_k$ be $k$ subsets of vertices, not necessarily disjoint. Remove all edges $\{x, y\} \in E$ for which there is a subset $N_i$ that contains both $x$ and $y$. We call the graphs that are obtained in this manner $k$-probe cographs. For the recognition problem of $k$-probe cographs we refer to the labeled case when sets $N_i$ are a part of the input. In that case each vertex has a label which is a 0/1-vector of length $k$ with a 1 in position $i$ if the vertex is in $N_i$. By Kruskal’s theorem \cite{3} $k$-probe cographs are characterized by a finite collection of forbidden induced subgraphs (either labeled or unlabeled). (It follows that in $k$-probe cographs all induced paths have a length which is bounded by a function of $k$.)

By Equation \eqref{1} $k$-probe cographs have rankwidth $k$, since the adjacencies of every vertex across a line in the cotree is characterized by its label, which is a vector of length $k$. The recognition of (labeled or unlabeled) $k$-probe cographs can be expressed in monadic second-order logic and it follows that recognizing $k$-probe cographs is fixed-parameter tractable (see \cite{2}).

In this note we show that there is an efficient recognition algorithm to recognize labeled $k$-probe cographs.

\section{Recognition of $k$-probe cographs}

\textbf{Theorem 1.} There exists an $O(k \cdot n^5)$ algorithm for the recognition of labeled $k$-probe cographs.

\textbf{Proof.} Let $G = (V, E)$ be a labeled graph, that is, every vertex $x$ has a label which is a 0/1-vector of length $k$. For $i \in \{1, \ldots, k\}$, let $N_i$ be the set of vertices that have a 1 in position $i$ of their label. Then the graph induced by $N_i$ is an independent set in $G$. The algorithm that we describe below builds a cotree for an embedding of $G$ or it concludes that $G$ is not a labeled $k$-probe cograph.

Let $X \subseteq V$ be a subset of vertices. Call the set $X$ a module if every vertex $z \in V \setminus X$ is either

(1) not adjacent to any vertex of $X$, or
(2) $z$ is adjacent to all vertices $x \in X$ of which the label is orthogonal to the label of $z$.

Let $X$ and $Y$ be two disjoint modules. Call $X$ and $Y$ twins if

(a) either no vertex of $X$ is adjacent to any vertex of $Y$ or every vertex $x \in X$ is adjacent to every vertex $y \in Y$ which has a label that is orthogonal to the label of $x$, and
(b) $X \cup Y$ is a module.

\footnote{Here, we say that two vectors are orthogonal if they don’t have a 1 in any common entry. In particular, the 0-vector is orthogonal to every other vector.}
Let $X$ and $Y$ be twins. Notice that there is a cotree embedding with $X \cup Y$ as a rooted subtree if and only if there is a cotree embedding with one of $X$ and $Y$ as a subtree.

The algorithm builds a cotree as follows. Starting with subtrees $X$ which consist of one vertex, it grows the subtrees by looking for twins. If $\{X, Y\}$ is a twin, the subtrees of $X$ and $Y$ are replaced by the subtree for $X \cup Y$. If no vertex of $X$ is adjacent to any vertex of $Y$ then the root of the subtree for $X \cup Y$ is labeled by $\oplus$, and otherwise it is labeled by $\otimes$.

At each stage there is a collection of $O(n)$ feasible subtrees. To look for a twin, the algorithm tries all pairs. To check if two subtrees $X$ and $Y$ form a twin the algorithm checks if either no vertex of $X$ is adjacent to any vertex of $Y$, or if every vertex $x \in X$ is adjacent to those vertices $y \in Y$ of which the label is orthogonal to the label of $x$. Furthermore, the algorithm checks if $X \cup Y$ is a module. Adjacencies are checked in constant time by using the adjacency matrix of $G$. To check if the labels of two vertices are orthogonal takes $O(k)$ time. Thus it can be checked in $O(k \cdot n^2)$ time if two modules $X$ and $Y$ are twins. It follows that within $O(k \cdot n^3)$ time either a twin is found or the conclusion is drawn that $G$ is not a $k$-probe cograph. Since the cotree has $O(n)$ nodes, it follows that a cotree embedding is built in $O(k \cdot n^5)$ time, if it exists. $\square$

Remark 1. For the unlabeled case, the recognition of $k$-probe cographs is NP-complete. As noted above, the recognition problem is fixed-parameter tractable.

References

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