Abstract. The aim of this article is to define and study a notion of unstable algebras over an operad that generalises the classical notion of unstable algebras over the Steenrod algebra. For this study we focus on the case of the characteristic 2. We define $\star$-unstable $\mathcal{P}$-algebras, where $\mathcal{P}$ is an operad and $\star$ is a commutative binary operation in $\mathcal{P}$. We then build a functor that takes an unstable module $M$ to the free $\star$-unstable $\mathcal{P}$-algebra generated by $M$. Under some hypotheses on $\star$ and on $M$, we identify this unstable algebra to a free $\mathcal{P}$-algebra. Finally, we give some examples of this result, and we show how to use our main theorem to obtain a new construction of the unstable modules studied by Carlsson, Brown-Gitler, and Campbell-Selick, that takes into account their inner product.

1. Introduction

In this paper we study unstable modules over the mod 2 Steenrod algebra from an algebraic point of view, taking as a starting point the operations that appear naturally on these objects.

Apart from cohomology modules, which are endowed with the associative and commutative cup-product, other unstable modules are classically described with an inner product, like Brown-Gitler modules, Carlsson modules, and Campbell-Selick modules. These unstable modules appear notably when studying the injective objects in the category of unstable modules. In [5], Davis shows that the Carlsson module of weight 1, with the multiplication studied by Carlsson, which is commutative but not associative and satisfies an interchange law, is in fact isomorphic to the free ‘depth-invariant’ algebra generated by an element of degree 1. His definition of a ‘depth-invariant’ algebra corresponds to the definition of level algebras of Chataur and Livernet [4].
Here, we use the formalism of (symmetric) algebraic operads to define and study algebraic operations on unstable modules. Following the classical definitions, we define a notion of \(\ast\)-unstable \(\mathcal{P}\)-algebras, relatively to an operad \(\mathcal{P}\) endowed with a commutative operation \(\ast\). Given such an operad \(\mathcal{P}\) endowed with an operation \(\ast \in \mathcal{P}(2)^{S_2}\), a \(\mathcal{P}\)-algebra \(M\) is said to be unstable if, for all \(x \in M\), we have \(Sq^{|x|}x = \ast(x, x)\). We construct a functor \(K^P_{\mathcal{P}}\) that takes an unstable module \(M\) to the free \(\ast\)-unstable \(\mathcal{P}\)-algebra generated by \(M\).

Given a \(\ast\)-unstable \(\mathcal{P}\)-algebra, we notice that the operation \(\ast\) is compatible, in a certain sense, with the other operations of \(\mathcal{P}\). For instance, for \(\mu \in \mathcal{P}(n)\), \(x_1, \ldots, x_n \in M\), one has (see Proposition 5.2):

\[
\ast(\mu(x_1, \ldots, x_n), \mu(x_1, \ldots, x_n)) = Sq^{|x_1|+\cdots+|x_n|} \mu(x_1, \ldots, x_n) = \mu(\ast(x_1, x_1), \ldots, \ast(x_n, x_n)).
\]

An operation of \(\mathcal{P}\) satisfying such a compatibility relation is here called \(\mathcal{P}\)-central. Such an operation is, in particular, a level operation. The \(\mathcal{P}\)-centrality condition is a strong hypothesis: an operad does not necessarily contain a non-trivial \(\mathcal{P}\)-central operation. For instance, the operad \(As\) of associative algebras does not contain any such non-trivial operation. Nevertheless, there are some interesting examples of operads endowed with such an operation (see Lemma 9.10).

In the case where \(\ast\) is \(\mathcal{P}\)-central and where \(M\) is a connected reduced unstable module (see Definition 6.1), we show that the free \(\ast\)-unstable \(\mathcal{P}\)-algebra \(K^P_{\mathcal{P}}(M)\) is in fact free as a \(\mathcal{P}\)-algebra:

**Theorem A** (6.11). Let \(\mathcal{P}\) be an operad in \(F_{\text{vect}}\), \(\ast \in \mathcal{P}(2)^{S_2}\) a \(\mathcal{P}\)-central operation, and \(M\) a connected reduced unstable module. There exists a graded \(\mathcal{P}\)-algebra isomorphism between \(K^P_{\mathcal{P}}(M)\) and the free \(\mathcal{P}\)-algebra generated by \(\Sigma \Omega M\), where \(\Sigma\) is the classical suspension functor and \(\Omega\) is left adjoint to \(\Sigma\).

When setting \(\mathcal{P} = u\text{Com}\), the operad of commutative associative unital algebras, this result is already known notably for free unstable modules \(M\) (see for example [1]). Moreover, when \(M\) is freely generated by one element, the result of this theorem corresponds to the computations that were conducted by Serre on the mod 2 cohomology of the Eilenberg-MacLane spaces of \(\mathbb{Z}/2\mathbb{Z}\) [12].

A direct application of this result shows (see Remark 9.11) that the Carlsson module of weight 1 with its inner product is the free unstable level algebra. This assertion gives more precision to the result of Davis.

**Plan of the paper:**

In Section 2 we recall the symmetric monoidal structure on the category of unstable modules. We study algebras in the category of unstable modules over an operad concentrated in degree 0. We notice that a natural compatibility condition between the actions of the Steenrod algebra and of the operad translates to a generalization of the Cartan formula: \(Sq^i(x \cdot y) = \sum_{m+n=i} (Sq^m x) \cdot (Sq^n y)\).

In Section 3 we recall the notion of ideal of an algebra over an operad, and we adapt this definition in the setting of unstable modules.

In Section 4 we define the notion of \(\ast\)-unstable \(\mathcal{P}\)-algebras over an operad \(\mathcal{P}\) relatively to a commutative operation \(\ast\) of the operad.

In Section 5 we define the condition of \(\mathcal{P}\)-centrality for an element of an operad \(\mathcal{P}\).

In Section 6 we build the functor \(K^P_{\mathcal{P}}\), for a fixed operad \(\mathcal{P}\) endowed with a commutative binary operation \(\ast\), and we state Theorem 6.11.

Section 7 is dedicated to the proof of Theorem 6.11.

In Section 8 we clarify the action of the Steenrod algebra on the \(\mathcal{P}\)-algebra \(S(\mathcal{P}, \Sigma \Omega M)\) obtained by transfer from \(K^P_{\mathcal{P}}(M)\), we give the first application of our result, as well as one counter-example in the case where the unstable module \(M\) is not reduced.
In Section 9 we recall the definition of the classical unstable modules due to Brown-Gitler, Carlsson and Campbell-Selick. Then we introduce a list of operads that allow us to give a new presentation of those classical modules and their operations in an operadic point of view.

Notation.

- The base field is $F := \mathbb{F}_2$ for the whole article.
- $F_{\text{vect}}$ is the category of $F$-vector spaces.
- $A$ is the mod 2 Steenrod algebra. $A_\text{mod}$ is the category of left $A$-modules.
- $U$ is the category of unstable modules over the Steenrod algebra.

Recollections about operads:
We assume that the reader has a basic knowledge of operad theory in the algebraic setting. Our reference on the subject is the book [10] of Loday and Vallette. Let us recall the basic definition, in order to set our notation:

Definition.

- A symmetric sequence $M$ is a sequence of vector spaces $(M(n))_{n \in \mathbb{N}}$ such that, for all $n \in \mathbb{N}$, $\mathfrak{S}_n$ acts on $M(n)$.
- Symmetric sequences form a category. This category is endowed with a tensor product such that:
  $$(M \otimes N)(n) = \bigoplus_{i+j=n} \text{Ind}_{\mathfrak{S}_i \times \mathfrak{S}_j}^{\mathfrak{S}_n} M(i) \otimes N(j),$$
  where $\text{Ind}_{\mathfrak{S}_i \times \mathfrak{S}_j}^{\mathfrak{S}_n}$ denotes the induced representation from the Young subgroup $\mathfrak{S}_i \times \mathfrak{S}_j$ of the group $\mathfrak{S}_n$.
- The category of symmetric sequences is endowed with another tensor product $\circ$ such that:
  $$(M \circ N)(n) = \bigoplus_{k \geq 0} M(k) \otimes (N \otimes k(n)),$$
  and the unit of this tensor product is the sequence $F$ with $F(i) = \begin{cases} F, & \text{if } i = 1, \\ 0, & \text{otherwise.} \end{cases}$
- Operads are monoids in the monoidal category of symmetric sequence with the product $\circ$. For an operad $P$, we denote by $\mu_P : P \circ P \to P$ its composition morphism, and $1_P \in P(1)$ its unit element. When $P$ and $Q$ are two operads, $\nu \in P(n)$, and $\xi_1, \ldots, \xi_n \in Q$, we denote by $(\nu; \xi_1, \ldots, \xi_n)$ the element $[\nu \otimes \xi_1 \otimes \cdots \otimes \xi_n]_{\mathfrak{S}_n} \in P \circ Q$. When $Q = P$, we denote by $\nu(\xi_1, \ldots, \xi_n)$ the element $\mu_P(\nu; \xi_1, \ldots, \xi_n) \in P$.
- For any operad $P$, $P_{\text{alg}}$ is the category of $P$-algebras. We denote by $S(P, -)$ the Schur functor $F_{\text{vect}} \to F_{\text{vect}}$ associated to $P$, and the ‘free $P$-algebra’ functor $F_{\text{vect}} \to P_{\text{alg}}$, depending on the context. We recall that in $F_{\text{vect}}$, this functor is a monad and is defined by:
  $$S(P, V) = \bigoplus_{n \geq 0} P(n) \otimes_{\mathfrak{S}_n} V^\otimes n.$$ 
- For an operad $P$, a $P$-algebra is an algebra over the monad $S(P, -)$. In other terms, it is a couple $(V, \theta)$ where $V$ is a vector space and $\theta : S(P, V) \to V$ is compatible with the composition and unit of $P$. 

• For \( \mathcal{P} \) an operad, \( V \) a vector space, \( \nu \in \mathcal{P}(n) \), and \( v_1, \ldots, v_n \in V \), we denote by \( (\nu; v_1, \ldots, v_n) \) the element \([\nu \otimes v_1 \otimes \cdots \otimes v_n]_{v_\nu} \in S(\mathcal{P}, V)\). For \((V, \theta)\) a \( \mathcal{P} \)-algebra, with \( \theta : S(\mathcal{P}, V) \to V \), we denote by \( \nu(v_1, \ldots, v_n) \) the element \( \theta(\nu(v_1, \ldots, v_n)) \in V \).

• \text{Com} (resp. \text{uCom}) is the operad of commutative, associative algebra. (resp. of commutative, associative, unital algebras).

• \text{Lev} is the operad of level algebras, that is (see [4]), vector spaces endowed with a commutative bilinear operation \( \star \) satisfying, for all \( a, b, c, d \in V \),

\[(a \star b) \star (c \star d) = (a \star c) \star (b \star d).
\]

2. Algebras over an operad in \( \mathcal{U} \)

In this section, we recall the symmetric monoidal structure on the category of unstable modules over the Steenrod algebra, which gives a notion of operad in this category. We focus on operads that come from operads in \( \mathbb{F}_{\text{vect}} \) concentrated in degree 0. We explain why algebras over such an operad \( \mathcal{P} \) are graded \( \mathcal{P} \)-algebras satisfying a generalised Cartan formula.

**Notation 2.1.** Recall that the mod 2 Steenrod algebra \( \mathcal{A} \) is the \( \mathbb{F}_2 \) associative, non-commutative, graded algebra generated by the elements denoted by \( Sq^i \) of degree \( i \) for \( i > 0 \), satisfying the Adem relations:

\[ Sq^i Sq^j - \sum_{k=0}^{[i/2]} \binom{i-k-1}{i-2k} Sq^{i+k} Sq^j = 0, \]

for all \( i, j > 0 \) such that \( i < 2j \), where \([\cdot]\) is the floor function, and where we denote by \( Sq^0 \) the unit of \( \mathcal{A} \). We refer to [12] for notation and classical results about the Steenrod algebra.

Recall that the category \( \mathcal{U} \) is the subcategory of \( \mathcal{A} \)-modules satisfying the condition: \( Sq^j x = 0 \) for all \( j > |x| \). These modules are called unstable modules.

The category of \( \mathcal{A} \)-modules is endowed with a symmetric monoidal tensor product: if \( M, N \) are two \( \mathcal{A} \)-modules, one can endow the graded tensor product \( M \otimes N \) of the graded \( \mathbb{F} \)-vector spaces \( M \) and \( N \) with the \( \mathcal{A} \)-module structure given by:

\[ Sq^i (x \otimes y) := \sum_{k+i=i} Sq^k x \otimes Sq^i y, \]

for all \( x \in M, y \in N, i \in \mathbb{N} \). This monoidal structure actually comes from a Hopf algebra structure on \( \mathcal{A} \) with cocommutative coproduct (see [12], [11]).

If \( M \) and \( N \) are unstable modules, then \( M \otimes N \) is still unstable.

We define a functor \( \mathbb{F}_{\text{vect}} \to \mathcal{U} \subset \mathcal{A}_{\text{mod}} \) that maps an \( \mathbb{F} \)-vector space \( V \) to the unstable module \( M \) concentrated in degree 0 with \( M^0 = V \). For all \( v \in V \) and \( i > 0 \), one has \( Sq^i v = 0 \). This functor is fully faithful and strictly symmetric monoidal. For any operad \( \mathcal{P} \) in \( \mathbb{F}_{\text{vect}} \), we deduce a notion of \( \mathcal{P} \)-algebra in \( \mathcal{A} \), and in \( \mathcal{U} \), regarding \( \mathcal{P} \) as concentrated in degree 0. We denote by \( \mathcal{P}_{\text{alg}} \) and \( \mathcal{P}_{\text{alg}}^{\mathcal{U}} \) the categories of \( \mathcal{P} \)-algebras in \( \mathcal{A} \)-modules and in unstable modules. A morphism between \( \mathcal{P} \)-algebras in \( \mathcal{A} \)-modules or in \( \mathcal{U} \) is a \( \mathcal{P} \)-algebra morphism that is compatible with the action of \( \mathcal{A} \).

**Lemma 2.2.** All Com-algebras in \( \mathcal{U} \) satisfy the Cartan formula.

**Proof.** Given an unstable module \( M \), and \( \theta : S(\text{Com}, M) \to M \) a morphism in \( \mathcal{U} \) endowing \( M \) with the structure of a Com-algebra in \( \mathcal{U} \). Then \( \theta \) is compatible to the action of the Steenrod algebra. Denote by \( X_2 \) the generator of \( \text{Com}(2) \). Recall that the (associative and commutative) multiplication of \( M \) is then
defined by \((x, y) \mapsto \theta(X_2; x, y) = X_2(x, y)\). For all \(x, y \in M\), \(i \in \mathbb{N}\), one has:
\[
Sq^i(X_2(x, y)) = \sum_{j+k+l=i} (Sq^j X_2)(Sq^k x, Sq^l y) = \sum_{k+l=i} X_2(Sq^k x, Sq^l y).
\]

The preceding lemma can be readily generalised:

**Proposition 2.3.** A \(P\)-algebra in \(U\) is a graded \(P\)-algebra \(M\) endowed with an action of the Steenrod algebra that satisfy the (generalised) Cartan formula, that is, for all \(\mu \in P(n)\), \((x_i)_{1 \leq i \leq n} \in M^n\),
\[
Sq^i(\mu(x_1, \ldots, x_n)) = \sum_{i_1 + \cdots + i_n = i} \mu(Sq^{i_1} x_1, \ldots, Sq^{i_n} x_n).
\]

**Proof.** A \(P\)-algebra in \(U\) is a graded \(P\)-algebra \(M\) endowed with an action of the Steenrod algebra such that the structural morphism \(S(P, M) \to M\) is compatible to the action of \(A\). This compatibility condition corresponds exactly to the Cartan formula. The proof of the preceding lemma gives an example of this computation. \(\square\)

**Lemma 2.4** (see \(\cite{[6]}\), section 1.3.2). The forgetful functor \(P_{alg}^U \to U\) admits, as a left adjoint functor, the functor \(S(P, -) : U \to P_{alg}^U\), where, if \(M\) is an unstable \(A\)-module, \(S(P, M)\) is an unstable \(A\)-module for the action induced by \(M\) and the Cartan formula.

**Lemma 2.5.** Let \(M\) be an unstable module, \(P\) an operad in \(F_{vect}\). For all \(\mu \in P(k)\), \(x_1 \in M^{n_1}, \ldots, x_k \in M^{n_k}\), one has the following equality in \(S(P, M)\).
\[
Sq^{n_1 + \cdots + n_k}(\mu; x_1, \ldots, x_k) = (\mu; Sq^{n_1} x_1, \ldots, Sq^{n_k} x_k).
\]

**Proof.** Let \(M\) be an unstable module and \(P\) be an operad in \(F_{vect}\). For all \(\mu \in P(k)\), \(x_1 \in M^{n_1}, \ldots, x_k \in M^{n_k}\), Proposition 2.3 gives:
\[
Sq^{n_1 + \cdots + n_k}(\mu; x_1, \ldots, x_k) = \sum_{i_1 + \cdots + i_k = n_1 + \cdots + n_k} (\mu; Sq^{i_1} x_1, \ldots, Sq^{i_k} x_k).
\]

Because \(M\) is an unstable module, one has \(Sq^i x_j = 0\) as soon as \(i > n_j\). So the only terms of this sum that are not zero are the ones with \(i_j \leq n_j\) for all \(j \in \{1, \ldots, k\}\). The condition \(i_1 + \cdots + i_k = n_1 + \cdots + n_k\) implies that:
\[
Sq^{n_1 + \cdots + n_k} = (\mu; x_1, \ldots, x_k) = (\mu; Sq^{n_1} x_1, \ldots, Sq^{n_k} x_k).
\]

3. \(P\)-ideals in \(U\)

In this section, we recall the definition of a \(P\)-ideal, where \(P\) is an operad, as well as the definition of the \(P\)-ideal generated by a subset of a \(P\)-algebra. We then extend these definitions to those of a \(P\)-ideal in \(U\) and of the \(P\)-ideal in \(U\) generated by a subset of a \(P\)-algebra in \(U\). These objects have the desired universal properties in the corresponding category.

**Definition 3.1** (\(\cite{[7]}\)).
- Let \(A\) be a \(P\)-algebra. A \(P\)-ideal of \(A\) is a linear subspace \(I\) of \(A\) such that, for all \(\mu \in P(n)\), \(a_1, \ldots, a_n \in A\), \(a_n \in I \Rightarrow \mu(a_1, \ldots, a_n) \in I\).

The structure of \(P\)-algebra of \(A\) induces a structure of \(P\)-algebra on the vector space \(A/I\).
• Let $X \subseteq A$ be a subset of the $\mathcal{P}$-algebra $A$. The $\mathcal{P}$-ideal generated by $X$, denoted by $(X)_p$, is the smallest $\mathcal{P}$-ideal of $A$ that contains $X$. It satisfies the following universal property: For all $\mathcal{P}$-algebras $B$ and all $\mathcal{P}$-algebra morphisms $\varphi : A \to B$, $\varphi(X) = 0$ if and only if $\varphi$ factors in a unique way into a $\mathcal{P}$-algebra morphism $\tilde{\varphi} : A/(X)_p \to B$.

**Remark 3.2.** One has:  

$$(X)_p = \left\{ \sum_{i \in E} \mu_i(a_{i,1}, \ldots, a_{i,n}) : E \text{ is a finite set, } \mu_i \in \mathcal{P}(n), a_{i,1}, \ldots, a_{i,n-1} \in A \text{ and } a_{i,n} \in X \right\}.$$  

**Definition 3.3.**

- Let $M$ be a $\mathcal{P}$-algebra in $\mathcal{U}$. A $\mathcal{P}$-ideal in $\mathcal{U}$ of $M$ is an unstable submodule $I$ of $M$ such that, for all $\mu \in \mathcal{P}(n)$, $a_1, \ldots, a_n \in M$,  

$$a_n \in I \Rightarrow \mu(a_1, \ldots, a_n) \in I.$$  

Just as before, $M/I$ is a $\mathcal{P}$-algebra in $\mathcal{U}$.

- Let $X \subseteq M$ be a subset of the $\mathcal{P}$-algebra $M$ in $\mathcal{U}$. The $\mathcal{P}$-ideal in $\mathcal{U}$ generated by $X$, denoted by $(X)_{\mathcal{P},\mathcal{U}}$, is the smallest $\mathcal{P}$-ideal in $\mathcal{U}$ of $M$ that contains $X$.

**Proposition 3.4.** Let $M$ be a $\mathcal{P}$-algebra in $\mathcal{U}$, $X$ a subset of $M$. One has $(X)_{\mathcal{P},\mathcal{U}} = (A \cdot X)_p$.

**Proof.** The subspace $(A \cdot X)_p$ of $M$ is stable under the action of $A$. Indeed, every element is a sum of monomials of the type $t = \mu(a_1, \ldots, \rho a_n)$, with $\mu \in \mathcal{P}(n)$, $a_1, \ldots, a_{n-1} \in M$, $\rho \in A$ and $a_n \in X$. For all $i \in \mathbb{N}$, one has:  

$$Sq^i t = \sum_{i_1 + \cdots + i_n = i} \mu(Sq^{i_1} a_1, \ldots, Sq^{i_n} a_{n-1}, Sq^{i_n} \rho a_n).$$  

Yet, for all $i_n \in \mathbb{N}$, $Sq^{i_n} \rho \in A$, so $Sq^{i_n} \rho a_n \in A \cdot X$, and hence $Sq^i t \in (A \cdot X)_p$.

Let $J \subseteq M$ be a $\mathcal{P}$-ideal in $\mathcal{U}$ containing $X$. Let $\mu \in \mathcal{P}(n)$, $a_1, \ldots, a_{n-1} \in M$, $\rho \in A$ and $a_n \in X$. Since $X \subseteq J$, $a_n \in J$. Since $J$ is stable under the action of $A$, $\rho a_n \in J$. Finally, since $J$ is a $\mathcal{P}$-ideal of $M$, $\mu(a_1, \ldots, a_{n-1}, \rho a_n) \in J$. So $(A \cdot X)_p \subseteq J$. \qed

## 4. $\ast$-unstable $\mathcal{P}$-algebra over the Steenrod algebra

In this section, we define a notion of unstable algebra over the Steenrod algebra with respect to the data of an operad endowed with a commutative operation. This definition generalises the classical notion of unstable algebra over the Steenrod algebra. With this aim in mind, we recall the definition of the endofunctor $\Phi$ of the category of unstable modules, and we study the natural transformations linking $\Phi$ to $S(\mathcal{P}, \cdot)$.

**Definition 4.1 (see [9], [12]).** There is a functor $\Phi : \mathcal{U} \to \mathcal{U}$ such that:  

$$(\Phi(M))^n = \begin{cases} M^2, & \text{if } n \equiv 0 \ [2], \\ 0, & \text{otherwise.} \end{cases}$$  

For all $x \in M^n$, $\Phi x$ denotes the corresponding element in $(\Phi M)^{2n}$. For all $i \in \mathbb{N}$, one has:  

$$Sq^i \Phi x = \begin{cases} \Phi(Sq^2 x), & \text{if } i \equiv 0 \ [2], \\ 0, & \text{otherwise.} \end{cases}$$  

There is a natural transformation $\lambda : \Phi \to id_\mathcal{U}$ such that, for all $x \in M$, $\lambda_M(\Phi x) = Sq^{[x]} x$. We also set  

$$S_0 x = Sq^{[x]} x.$$
Remark 4.2 (see [12]). In order to check if $\lambda$ is a natural transformation, one has to show that, for all $M$, $\lambda_M : \Phi M \to M$ is compatible to the action of $\mathcal{A}$, which boils down to showing for all $x \in M^i$, $j \in \mathbb{N}$, the equality:

$$Sq^iSq_{0}x = \begin{cases} \hspace{1em} Sq_{0}Sq^{i/2}x, & \text{if } j = 0 \ [2], \\ 0, & \text{otherwise}, \end{cases}$$

and this is a direct application of the Adem relations.

**Lemma 4.3.** Let $\mathcal{P}$ be an operad, $* \in \mathcal{P}(2)^{S_2}$, $M$ an $\mathcal{A}$-module. One has, in $S(\mathcal{P}, M)$,

$$Sq^{i}(*)x, x = \begin{cases} (*) Sq^{*}x, Sq^{*}x, & \text{if } i = 0 \ [2], \\ 0, & \text{otherwise}. \end{cases}$$

**Proof.** One checks that

$$Sq^{i}(*)x, x = \sum_{j+k=i} (*) Sq^{j}x, Sq^{k}x, \quad \sum_{j+k=i, j<k} (*) Sq^{j}x, Sq^{k}x, + (*) Sq^{k}x, Sq^{j}x, + Y,$$

where $Y := \begin{cases} (*) Sq^{*}x, Sq^{*}x, & \text{if } i = 0 \ [2], \\ 0, & \text{otherwise}. \end{cases}$. Yet, $*$ being stable under the action of $S_2$, one has $(*) Sq^{i}x, Sq^{i}x) = (*) Sq^{k}x, Sq^{j}x)$. Hence the result. \hfill \Box

**Proposition 4.4** (Proposition/Definition). Let $\mathcal{P}$ be an operad, $* \in \mathcal{P}(2)^{S_2}$ a commutative operation. There is a natural transformation $\alpha^* : \Phi \to S(\mathcal{P}, *)$ such that, for all $x \in M$,

$$\alpha^*_M(\Phi x) = (*x, x).$$

**Proof.** The map $\alpha^*_M$ is linear in characteristic 2 because $*$ is stable under the action of $S_2$, and is clearly natural in $M$. It remains to check that $\alpha^*_M$ is compatible with the action of the Steenrod algebra. Let $i \in \mathbb{N}$. One has:

$$\alpha^*_M(Sq^i\Phi x) = \begin{cases} \alpha^*_M(\Phi Sq^i(x)), & \text{if } i = 0 \ [2], \\ 0, & \text{otherwise}. \end{cases}$$

Following Lemma 4.3, we then have $Sq^i\alpha^*_M(\Phi x) = \alpha^*_M(Sq^i\Phi x)$. \hfill \Box

For instance, there is a natural transformation $\alpha^{X_2} : \Phi \to S(u\text{Com}, *)$, and one has:

**Lemma 4.5.** An unstable algebra over the Steenrod algebra is an $u\text{Com}$-algebra $(M, \theta)$ in $\mathcal{U}$ such that $\theta \circ \alpha^{X_2}_M = \lambda_M$.

**Proof.** By definition, an unstable algebra over the Steenrod algebra is an $u\text{Com}$-algebra $(M, \theta)$ such that $X_2(x, x) = Sq_0x$ for all $x \in M$. But $X_2(x, x) = \theta \circ \alpha^{X_2}_M(\Phi x)$ and $Sq_0x = \lambda_M(\Phi x)$. \hfill \Box

**Definition 4.6.** Let $\mathcal{P}$ be an operad, and $* \in \mathcal{P}(2)^{S_2}$. A $*$-unstable $\mathcal{P}$-algebra over the Steenrod algebra is a $\mathcal{P}$-algebra $(M, \theta)$ in $\mathcal{U}$ such that $\theta \circ \alpha^*_M = \lambda_M$, that is, such that for all $x \in M$,

$$Sq_0x = *x, x.$$
5. $\mathcal{P}$-Central Operations

In this section, we define the condition of $\mathcal{P}$-centrality for an operation $\star \in \mathcal{P}(2)^{S_2}$. Such an operation is said $\mathcal{P}$-central if it satisfies the interchange relation with respect to all other operations in $\mathcal{P}$. The condition of $\mathcal{P}$-centrality for an operation $\star \in \mathcal{P}(2)^{S_2}$ is necessary for the proof of Theorem 6.11 in order to identify certain free $\star$-unstable $\mathcal{P}$-algebras.

**Notation.** In this section, the operad $\mathcal{P}$ in $\mathbb{F}_{\text{vect}}$ is fixed.

**Definition 5.1.**

- Let $\star \in \mathcal{P}(2)^{S_2}$. The operation $\star$ is said to be $\mathcal{P}$-central if it satisfies, for all $\mu \in \mathcal{P}(n)$, the interchange law:

$$(E) \quad \star (\mu, \mu) = \left( \mu(\underbrace{\star, \ldots, \star}_n) \right) \cdot \sigma_{2n},$$

where $\sigma_{2n} \in S_{2n}$ maps $2i$ to $n+i$ and $2i-1$ to $i$ for all $i \in [n]$.

- Let $\star \in \mathcal{P}(2)^{S_2}$. A $\mathcal{P}$-algebra $A$ is said to be $\star$-compatible if, for all $\mu \in \mathcal{P}(n)$, $a_1, \ldots, a_n \in A$, one has:

$$\star (\mu(a_1, \ldots, a_n), \mu(a_1, \ldots, a_n)) = \mu(\star(a_1, a_1), \ldots, \star(a_n, a_n)).$$

**Remark.** If $\star$ is $\mathcal{P}$-central, then all $\mathcal{P}$-algebras are $\star$-compatible.

**Proposition 5.2.** All $\star$-unstable $\mathcal{P}$-algebras are $\star$-compatible.

**Proof.** Let $M$ be a $\star$-unstable $\mathcal{P}$-algebra, $\mu \in \mathcal{P}(n)$, $x_1 \in M^{k_1}, \ldots, x_n \in M^{k_n}$, and $K := \sum_{j=1}^n k_i$. One has

$$\star (\mu(x_1, \ldots, x_n), \mu(x_1, \ldots, x_n)) = Sq^K \mu(x_1, \ldots, x_n),$$

$$= \sum_{j_1 + \ldots + j_n = K} \mu(Sq^{j_1}x_1, \ldots, Sq^{j_n}x_n),$$

$$= \mu(Sq^{j_1}x_1, \ldots, Sq^{j_n}x_n),$$

$$= \mu(\star(x_1, x_1), \ldots, \star(x_n, x_n)),$$

where the third equality is a consequence of Lemma 2.5. \qed

**Proposition 5.3.** Let $\star \in \mathcal{P}(2)^{S_2}$. Let $F$ be a $S$-subset of $\mathcal{P}$. Suppose that $F$ generates the operad $\mathcal{P}$.

Then $\star$ is $\mathcal{P}$-central if and only if it satisfies relation $(E)$ of Definition 5.1 for all $\mu \in F$.

**Proof.** It suffices to check that if $\mu, \nu \in F$ satisfy $(E)$, then $\mu \circ \nu$ satisfies $(E)$, and, in this setting, if $\mu$ and $\nu$ have same arity, $\mu + \nu$ satisfies $(E)$. Let $\mu, \nu \in F$. Denote by $m$ and $n$ the respective arities of $\mu$
and of $\nu$. If $\mu$ and $\nu$ satisfy (E), then:

$$
\star(\mu \circ_i \nu, \mu \circ_i \nu) = ((\star(\mu, \mu)) \circ_{n+i} \nu) \circ_i \nu,
$$

$$
= ((\mu(\star, \ldots, \star) \cdot \sigma_{2m}) \circ_{n+i} \nu) \circ_i \nu,
$$

$$
= ((\mu(\star, \ldots, \star) \circ_{i+1} \nu) \circ_i \nu) \cdot \sigma',
$$

$$
= \left( \mu(\star, \ldots, \star, \nu(\mu, \nu), \star, \ldots, \star) \right) \cdot \sigma',
$$

$$
= \left( \mu(\star, \ldots, \star, (\nu(\star, \ldots, \star) \cdot \sigma_{2m}, \star, \ldots, \star)) \right) \cdot \sigma',
$$

$$
= (\mu \circ_i \nu)(\star, \ldots, \star) \cdot \sigma_{2(m+n-1)}.
$$

Here, $\sigma' \in S_{2m+2n-2}$ is the block permutation obtained by applying $\sigma$ to $2m$ blocks of size 1, except for the $i$-th and the $(i + 1)$-th, of size $n$.

Moreover, if $\mu$ and $\nu$ are of arity $n$, then since the characteristic of the base field is 2, and since $\star$ is commutative, one has:

$$
\star(\mu + \nu, \mu + \nu) = \star(\mu, \mu) + \star(\nu, \nu),
$$

$$
= \mu(\star, \ldots, \star) + \nu(\star, \ldots, \star),
$$

$$
= (\mu + \nu)(\star, \ldots, \star).
$$

\[\square\]

Remark 5.4. A $P$-central operation is a level operation. Indeed, one has $\star(\star, \star) = \star(\star, \star) \cdot \sigma_4$, and $\sigma_4$ is the transposition of 2 and 3.

We will give in Lemma 9.10 a list of examples of operads endowed with a $P$-central operation.

6. $\star$-unstable $P$-algebras generated by an unstable module

In this section, we build, for any operad $P$ endowed with a commutative operation $\star \in P(2)^{S_2}$, a functor that assigns to an unstable module $M$ the free $\star$-unstable $P$-algebra generated by $M$.

In the beginning of this section, we recall some basic notions for the study of unstable modules. We refer to [12] for these notation.

The main result of this section is Theorem 6.11, which gives a concise description of the free $\star$-unstable $P$-algebra generated by a connected reduced unstable module, when the operation $\star$ satisfies the $P$-centrality condition defined in Section 5. Under these conditions, we identify this $\star$-unstable $P$-algebra, which is a quotient of a $P$-algebra, to a free $P$-algebra.

When working with the operad $uCom$, endowed with its multiplication $X_2 \in uCom(2)^{S_2}$, and when we consider free unstable algebras generated by a free, monogeneous unstable module, the result of Theorem 6.11 corresponds to the computations that were conducted by Serre on the mod 2 cohomology of the Eilenberg-MacLane spaces of $\mathbb{Z}/2\mathbb{Z}$ [13].

Notation. Throughout this section, we fix an operad $P$ and a commutative operation $\star \in P(2)^{S_2}$.

Definition 6.1 (voir [12], [12], [9]).
Lemma 6.2 (voir [12], [14]). The unstable module $F(n)$ is isomorphic to:

$$
\Sigma^n A/(Sq^I : I \text{ admissible, and } e(I) > n).
$$

Therefore, $F(n)$ admits as a graded vector space basis the set of $Sq^I \iota_n$ such that $\iota_n$ is the generator of degree $n$ and $I$ satisfies $e(I) \leq n$.

Remark 6.3.

- The unstable module $F(1)$ has a basis of the form $\{j_k\}_{k \in \mathbb{N}}$, where $j_k := Sq^{2^{k-1}} \cdots Sq^1 \iota_1 \in F(1)^{2^k}$ ($j_0 = \iota_1$).
- For all $n > 0$, $\Omega F(n)$ is isomorphic as an unstable module to $F(n-1)$. Indeed, for $M$ an unstable module, there is a one-to-one correspondence (natural in $M$):

$$
\text{Hom}_U(\Omega F(n), M) \cong \text{Hom}_U(F(n), \Sigma M) \cong (\Sigma M)^n \cong M^{n-1} \cong \text{Hom}_U(F(n-1), M).
$$
- The counit of the adjunction $(\Omega, \Sigma)$ is a natural isomorphism $\Omega \Sigma M \cong M$.

Definition 6.4. The category of $*$-unstable $\mathcal{P}$-algebras is denoted by $K^*_\mathcal{P}$. It is a full subcategory of the category $\mathcal{P}^{dl}_{\text{alg}}$.

Notation 6.5.

- For all $k \in \mathbb{N}$, the element $*_k \in \mathcal{P}(2^k)$ is inductively defined by:

$$
*_0 = 1_\mathcal{P}, \quad _1 = *, \quad \text{and, } \forall k > 1, \quad _k = (*_{k-1}, _{k-1}).
$$

Let us remark that, when $*$ is $\mathcal{P}$-central, $*_k$ belongs to $\mathcal{P}(2^k)^{\mathcal{P}_{\text{alg}}}$.  

- For all $\mu \in \mathcal{P}(n)$, $x \in M$, where $M$ is an unstable module, the element $\mu(x, \ldots, x) \in S(\mathcal{P}; M)$ is denoted by $(\mu; x^n)$. If $M$ is a $\mathcal{P}$-algebra, the element $\mu(x, \ldots, x) \in M$ is denoted by $\mu(x^n)$.

Proposition 6.6. Let $M$ be an unstable module. We set:

$$
K^*_\mathcal{P}(M) := S(\mathcal{P}; M)/(\{Sq_0 + * (t, t) : t \in S(\mathcal{P}; M)\})_{\mathcal{U}, \mathcal{U}}.
$$

Then $K^*_\mathcal{P}(M)$ is a $*$-unstable $\mathcal{P}$-algebra. Moreover, $K^*_\mathcal{P} : \mathcal{U} \to K^*_\mathcal{P}$ gives a left adjoint functor for the forgetful functor $U : K^*_\mathcal{P} \to \mathcal{U}$. 

- The suspension functor $\Sigma : \mathcal{U} \to \mathcal{U}$ takes an unstable module $M$ to the unstable module $\Sigma M$ defined by $(\Sigma M)^d = M^{d-1}$, and $Sq^I(\sigma x) = \sigma(Sq^I x)$, where $\sigma x \in M^{d+1}$ corresponds to the element $x \in M^d$.
- The functor $\Sigma$ admits a left adjoint denoted by $\Omega$.
- Let $I = (i_1, \ldots, i_k)$ be a (finite) sequence of integers. Then $I$ is called admissible if for all $h \in \{1, 2, \ldots, k-1\}$, $i_h \geq 2i_{h+1}$. The excess of an admissible sequence is the integer $e(I) := i_1 - i_2 - \cdots - i_k$.
- Let $I = (i_1, \ldots, i_k)$ be a (finite) sequence of integers. We denote by $Sq^I$ the product $Sq^{i_1} \cdots Sq^{i_k}$ in $\mathcal{A}$.
- The free unstable module generated by an element $\iota_n$ of degree $n$ is denoted by $F(n)$.
- The linear map sending $x \in M$ to $Sq^ix$ is denoted by $S\phi_0 : M \to M$. Recall (see Section 4) that for all $x \in M$, one has $S\phi_0 x = \lambda_M(\Phi x)$.
- An unstable module $M$ is said to be reduced if the application $S\phi_0 : M \to M$ is injective.
- An unstable module $M$ is said to be connected if $M^0 = 0$.
Proof. Let \( t \in S(\mathcal{P}, M) \). One has \( Sq_0[t] = [Sq_0t] = \ast(t, t) = \ast([t], [t]) \) in \( K^P_*(M) \), so \( K^P_*(M) \) is \( \ast \)-unstable.

Let \( N \) be a \( \ast \)-unstable \( \mathcal{P} \)-algebra, and \( g : M \to N \) a morphism of \( \mathcal{A} \)-modules. Since \( S(\mathcal{P}, M) \) is the free \( \mathcal{P} \)-algebra in \( \mathcal{U} \) generated by \( M \), there exists a unique morphism \( g' : S(\mathcal{P}, M) \to N \) of \( \mathcal{P} \)-algebras in \( \mathcal{U} \) that extends \( g \).

Now, as \( K^P_*(M) \) is \( \ast \)-unstable, \( g'(\{ [Sq_0t + \ast(t, t) : t \in S(\mathcal{P}, M) \})_{\mathcal{P}, \mathcal{U}} = 0 \). So there is a unique factorisation morphism \( g'' : K^P_*(M) \to N \). Thus, since \( K^P_*(M) \) is \( \ast \)-unstable, it is the free \( \ast \)-unstable \( \mathcal{P} \)-algebra generated by \( M \). \( \square \)

Remark 6.7. The functor \( K^P_* : \mathcal{U} \to K^P_* \) can be defined as the coequaliser of the following diagram:

\[
\begin{array}{ccc}
\Phi \circ S(\mathcal{P}, -) & \xrightarrow{\alpha^* \circ id_{S(\mathcal{P}, -)}} & S(\mathcal{P}, S(\mathcal{P}, -))
\end{array}
\]

between the natural transformation \( \lambda \circ id_{S(\mathcal{P}, -)} \), and the composite:

\[
\begin{array}{ccc}
\Phi \circ S(\mathcal{P}, -) & \xrightarrow{\alpha^* \circ id_{S(\mathcal{P}, -)}} & S(\mathcal{P}, S(\mathcal{P}, -)) & \xrightarrow{\mu_{\mathcal{P}}} & S(\mathcal{P}, -)
\end{array}
\]

where \( \alpha^* : \Phi \to S(\mathcal{P}) \) is defined in \([4,3]\).

Lemma 6.8 (see \([12]\)). Let \( M \) be an unstable module. The unstable modules \( \Sigma \Omega M \) and \( \text{Coker} \lambda_M \) are isomorphic.

The following diagram is a short exact sequence when \( M \) is reduced:

\[
\begin{array}{ccc}
\Phi M & \xrightarrow{\lambda_M} & M & \xrightarrow{pr} & \Sigma \Omega M
\end{array}
\]

Moreover, the morphism \( pr : M \to \Sigma \Omega M \) is the unit of the adjunction \((\Omega, \Sigma)\).

Definition 6.9. Let \( M \) be a reduced unstable module.

- A graded section, denoted by \( s : \Sigma \Omega M \to M \), is the data, for all \( d \in \mathbb{N} \), of a linear section \( s : (\Sigma \Omega M)^d \to M^d \) of the map \( pr : M^d \to (\Sigma \Omega M)^d \) (that is, such that \( pr \circ s = id_{\Sigma \Omega M} \)). We draw the reader’s attention on the fact that a graded section is not, in general, compatible with the action of \( \mathcal{A} \), but is only a graded linear map.
- A graded basis of \( \Sigma \Omega M \) is a graded set \( B = (B_i)_{i \in \mathbb{N}} \) such that, for all \( i \in \mathbb{N} \), \( B_i \) is a basis of \((\Sigma \Omega M)^i\).

Remark 6.10. Suppose that \( M \) is reduced. When \( d \) is even, \( (\Phi M)^d = M^{d/2} \). According to the classical isomorphism theorems, since \( Sq_0 \) is injective, the choice of a section \( s : (\Sigma \Omega M)^d \to M^d \) is equivalent to the choice of a linear retraction \( r : M^d \to M^{d/2} \) of \( Sq_0 : M^{d/2} \to M^d \) (that is, such that \( r \circ Sq_0 = id_{M^{d/2}} \)), and this is also equivalent to the choice of a linear bijection \( M^d \cong (\Sigma \Omega M)^d \oplus (\text{Im} Sq_0)^d \). This furnishes a decomposition of all \( x \in M^d \): \( x = Sq_0 \circ r(x) + s \circ pr(x) \). When \( d \) is odd, \( (\Phi M)^d = 0 \), so \( pr \) is an isomorphism, and we can set \( s = pr^{-1} \) and \( r = 0 \).

Theorem 6.11. Let \( \mathcal{P} \) be an operad in \( \mathcal{F}_{\text{vect}} \), \( \ast \in \mathcal{P}(2)^{\mathbb{Z}_2} \) be a \( \mathcal{P} \)-central operation. For all connected reduced unstable module \( M \), there exists an isomorphism of graded \( \mathcal{P} \)-algebras between the \( \ast \)-unstable \( \mathcal{P} \)-algebra \( K^P_*(M) \) and the free \( \mathcal{P} \)-algebra generated by \( \Sigma \Omega M \).

This isomorphism is not natural in \( M \), and from any choice of a graded section \( s : \Sigma \Omega M \to M \) and of a graded basis \( B \) of \( \Sigma \Omega M \) (see Definition \([6,9]\)), one can deduce a graded \( \mathcal{P} \)-algebra isomorphism:

\[
\hat{\phi}_{s, B} : K^P_*(M) \to S(\mathcal{P}, \Sigma \Omega M).
\]

Proof. We refer to Section \([7]\) \( \square \)
7. Proof of Theorem 6.11

The aim of this section is to present the proof of Theorem 6.11.

Throughout this section, we fix an operad $P$ in $F_{\text{vect}}$, endowed with a $P$-central operation $\star \in P(2)^{\otimes 2}$, and $M$ a connected unstable module endowed with a graded section $s: \Sigma \Omega M \to M$ (see Definition 6.9), and with a graded basis $B = (B_i)_{i \in \mathbb{N}}$ of $\Sigma \Omega M$. For the proof of Lemma 7.2 and 7.3 we do not assume that $M$ is reduced.

The $P$-algebra $K_P^P(M)$ is described as a quotient of the free $P$-algebra $S(P, M)$ by an ideal. In Lemmas 7.2, 7.3 and 7.6, we simplify this ideal, by replacing it by a $P$-ideal generated by a set built from $B$.

In Lemma 7.7, we give a construction for the desired $P$-algebra isomorphism between $K_P^P(M)$ and the free $P$-algebra $S(P, \Sigma \Omega M)$.

Notation 7.1. Let $\text{Inst} = \{Sq_0x + (\star; x, x) : x \in M\} \subset S(P, M)$, and $X = \{Sq_0t + (\star; t, t) : t \in S(P, M)\} \subset S(P, M)$.

Lemma 7.2. In $S(P, M)$, one has $(X)_{P, \mathcal{U}} = (\text{Inst})_{P, \mathcal{U}}$, where $X$ and $\text{Inst}$ are defined in 7.1.

In particular, one has:

$$K_P^P(M) = S(P, M)/(\text{Inst})_{P, \mathcal{U}}.$$  

Proof. Let us show that $\text{Inst} \subset X$ and $X \subset (\text{Inst})_P$. The first inclusion is clear, let us prove the second one. Let $t := (\mu; x_1, \ldots, x_n) \in S(P, M)$ be a $P$ monomial, with $x_1, \ldots, x_n \in M$. Following Lemma 2.5, one has:

$$Sq_0t = (\mu; Sq_0x_1, \ldots, Sq_0x_n),$$  

So the following element is in $(\text{Inst})_P$:

$$Sq_0t + \mu((\star; x_1, x_1), \ldots, (\star; x_n, x_n)) = Sq_0t + (\mu(\star, \ldots, \star; x_1, x_1, \ldots, x_n, x_n),$$

$$= Sq_0t + (\mu(\star, \ldots, \star) \cdot \sigma_2^n; x_1, x_1, \ldots, x_n, x_1, \ldots, x_n).$$

Since $\star$ is $P$-central, this element is equal to:

$$Sq_0t + (\star(\mu; x_1, \ldots, x_n, x_1, \ldots, x_n) = Sq_0t + (\star; t, t).$$

Hence, $X \subset (\text{Inst})_P$.

We proved that $(X)_{P} = (\text{Inst})_{P}$. The result then ensues from $(X)_{P, \mathcal{U}} = A \cdot (X)_{P} = A \cdot (\text{Inst})_{P} = (\text{Inst})_{P, \mathcal{U}}$. 

Lemma 7.3. In $S(P, M)$, one has $(\text{Inst})_{P, \mathcal{U}} = (\text{Inst})_P$, where $\text{Inst}$ is defined in 7.1.

In particular, one deduces from Lemma 7.2 that:

$$K_P^P(M) = S(P, M)/(\text{Inst})_P.$$  

Proof. Recall, from Proposition 6.2 that $(\text{Inst})_{P, \mathcal{U}} = (A \cdot \text{Inst})_P$. The set $A \cdot \text{Inst}$ is, by definition, the set of sums of elements of the form:

$$\rho(Sq_0x + (\star; x, x)),$$

where $\rho \in A$, and $x \in M$. Let us show that, for all $x \in M$ and $j \in \mathbb{N}$, $Sq^j(Sq_0x + (\star; x, x))$ is in $(\text{Inst})_P$.

On the one hand, according to Remark 1.2, one has

$$Sq^j Sq_0x = \begin{cases} 
Sq_0 Sq^{j/2} x, & \text{if } j \equiv 0 \mod 2, \\
0, & \text{otherwise}.
\end{cases}$$

But, following Lemma 1.3, one has

$$Sq^j (\star; x, x) = \begin{cases} 
(\star; Sq^{j/2} x, Sq^{j/2} x), & \text{if } j \equiv 0 \mod 2, \\
0, & \text{otherwise}.
\end{cases}$$
Thus, one has:

\[ S^j \left( S q_0 x + (\ast; x, x) \right) = \begin{cases} S q_0 S q^j x + (\ast; S q^j x, S q^j x) & \text{if } j \equiv 0 \mod{2} \in \text{Inst}, \\ 0 & \text{otherwise.} \end{cases} \]

Hence the result.

\[ \square \]

**Definition 7.4.**
- Let \( C \) be the graded set \((C_i)_{i \in \mathbb{N}}, \) where \( C_i := \{ S q_0^k s(b) : k \in \mathbb{N}, b \in B_{i/2^k} \}, \) where \( B_{i/2^k} = \emptyset \) if \( \frac{i}{2^k} \notin \mathbb{N}. \)
- We set:

\[ E := \left\{ S q_0^k s(b) + \left( \ast_k; (s(b))^{x_{2k}} \right) : S q_0^k s(b) \in C_i, i \in \mathbb{N} \right\} \subset S(\mathcal{P}, M), \]

**Lemma 7.5.** Assume that \( M \) is reduced. The set \( C_i, \) defined in \( \text{[7.4]} \), is a vector space basis for \( M^i. \)

**Proof.** Since \( B_i \) is a basis of \((\Sigma \Omega M)^i, \) and since \( s : (\Sigma \Omega M)^i \to M^i \) is injective, it suffices to show that \( D_i := \{ S q_0^k s(b) : k > 0, b \in B_{i/2^k} \} \) is a basis of \((\text{Im } S q_0)^i, \) where we set \( B_0 = \emptyset \) if \( q \notin \mathbb{N}. \) Since \( M \) is connected, there is nothing to prove for \( i = 0. \)

Let us prove the result by induction on the 2-adic valuation of \( i > 0. \)

If \( i \) is odd, then \((\text{Im } S q_0)^i = 0, \) but \( D_i = \emptyset, \) hence the result.

Let us assume that \( D_j \) is a basis of \((\text{Im } S q_0)^j \) for all \( j \) of 2-adic valuation \( v, \) and that the 2-adic valuation of \( i \) is \( v + 1. \) By induction hypothesis, since the 2-adic valuation of \( l := i/2 \) is equal to \( v, \) the set \( D_l \) is a basis for \((\text{Im } S q_0)^l. \) So, \( D_l \cup \text{Im} (B_l) \) is a basis of \( M^l. \) Since \( S q_0 \) induces a linear bijection \( M^l \to (\text{Im } S q_0)^l, \) this implies that \( S q_0 (D_l \cup \text{Im} (B_l)) \) is a basis for \((\text{Im } S q_0)^l. \) Yet, one checks that \( D_i = S q_0 (D_l \cup \text{Im} (B_l)). \)

Hence the result.

\[ \square \]

**Lemma 7.6.** Suppose that \( M \) is reduced. One has \((E)_p = (\text{Inst})_p, \) where \( E \) is defined in \( \text{[7.4]} \) and \( \text{Inst} \) is defined in \( \text{[7.3]} \) that:

\[ \mathcal{K}_p^f (M) = S(\mathcal{P}, M) / (E)_p. \]

**Proof.** Let us show that \( E \subset (\text{Inst})_p \) and \( \text{Inst} \subset (E)_p. \)

**Proof of the inclusion \text{Inst} \subset (E)_p.**

Let \( x \in \text{Inst}. \) According to Lemma \( \text{[7.3]} \) \( x \) can be written as a sum of elements of the form \( y := S q_0^k s(b) \in M \) where \( k \in \mathbb{N}, b \in \Sigma \Omega M. \)

On the one hand, \( S q_0 y = S q_0^{k+1} s(b), \) so the following element \( \alpha \) is in \((E)_p:\)

\[ \alpha := S q_0 y + \left( \ast_{k+1}; (s(b))^{x_{2k+1}} \right) \]

On the other hand, \((\ast; y, y) = (\ast; S q_0^k s(b), S q_0^k s(b)). \) So the following element is in \((E)_p:\)

\[ \beta := (\ast; y, y) + \left( \left( \ast_k; (s(b))^{x_k} \right), \left( \ast_k; (s(b))^{x_k} \right) \right) = (\ast; y, y) + \left( \ast_{k+1}; (s(b))^{x_{2k+1}} \right). \]

Thus, the elements of the form \( \alpha + \beta = S q_0 y + (\ast; y, y) \) are in \((E)_p, \) so \( \text{Inst} \subset (E)_p. \)

**Proof of the inclusion \( E \subset (\text{Inst})_p. \)**

Let \( k \in \mathbb{N}, b \in B. \) Let us show, by induction on \( k \in \mathbb{N}, \) that \( S q_0^k s(b) + \left( \ast_k; (s(b))^{x_k} \right) \in (\text{Inst})_p. \) On the one hand, if \( k = 0, \) one has \( S q_0^k s(b) + \left( \ast_k; (s(b))^{x_k} \right) = 0 \in (\text{Inst})_p. \)
Let us assume that, for all \( l \in \mathbb{N} \), one has \( Sq_0^k s(b) + \left( *; (s(b))^{x^2k} \right) \in (\text{Inst})_P \). Let \( k = l + 1 \). Note that the following element is in \( \text{Inst} \):

\[
\alpha := Sq_0^k s(b) + \left( *; Sq_0^l s(b), Sq_0^l s(b) \right).
\]

By induction hypothesis, the following element is in \( (\text{Inst})_P \):

\[
Sq_0^l s(b) + \left( *; (s(b))^{x^2l} \right).
\]

Thus, the following element is in \( (\text{Inst})_P \):

\[
\beta := * \left( Sq_0^l s(b) + \left( *; (s(b))^{x^2l} \right), Sq_0^l s(b) + \left( *; (s(b))^{x^2l} \right) \right) = * \left( Sq_0^l s(b), Sq_0^l s(b) + \left( *; (s(b))^{x^2l} \right) \right).
\]

Finally, the following element is in \( (\text{Inst})_P \):

\[
\alpha + \beta = Sq_0^k s(b) + \left( *; (s(b))^{x^2k} \right). \tag*{\Box}
\]

**Lemma 7.7.** Suppose that \( M \) is reduced. There exists a graded \( \mathcal{P} \)-algebra isomorphism \( \varphi_{s,B} : K_*^\mathcal{P}(M) \cong S(\mathcal{P}, \Sigma \Omega M) \), which depends on the choice of the graded section \( s : \Sigma \Omega M \to M \) and on the graded basis \( B \) of \( \Sigma \Omega M \).

**Proof.** Recall that Lemma 7.3 furnishes, for all \( i \in \mathbb{N} \), a basis \( C_i \) of \( M^i \).

Let \( \varphi_{s,B} : M \to S(\mathcal{P}, \Sigma \Omega M) \) be the graded linear map sending \( Sq_0^k s(b) \in C_i \) to \( \left( \nu; b^{x^k} \right) \in (S(\mathcal{P}, \Sigma \Omega M))^i \). This induces a \( \mathcal{P} \)-algebra morphism \( \varphi_{s,B} : S(\mathcal{P}, M) \to S(\mathcal{P}, \Sigma \Omega M) \).

Let us show that \( \varphi_{s,B} \) factorises into a \( \mathcal{P} \)-algebra morphism \( \varphi_{s,B} : K_*^\mathcal{P}(M) \to S(\mathcal{P}, \Sigma \Omega M) \). Following Lemma 7.2, Lemma 7.3 and Lemma 7.6, one just has to show that all elements of \( E \) are sent to 0 by \( \varphi_{s,B} \), where \( E \) is defined in 7.6. Let \( Sq_0^k s(b) \in C_i \). Since \( \varphi_{s,B} \) is compatible to the action of \( \mathcal{P} \), and since \( \varphi_{s,B}(s(b)) = b \) for all \( b \in \Sigma \Omega M \), one has

\[
\varphi_{s,B}(Sq_0^k s(b)) = \left( \nu; (\varphi_{s,B}(s(b)))^{x^2k} \right) = \varphi_{s,B} \left( \nu; (s(b))^{x^2k} \right).
\]

We have proved the existence of the factorisation morphism \( \varphi_{s,B} : K_*^\mathcal{P}(M) \to S(\mathcal{P}, \Sigma \Omega M) \). Let us show that this morphism \( \varphi_{s,B} \) is a bijection.

The vector space \( (S(\mathcal{P}, \Sigma \Omega M))^i \) is spanned by elements of the form:

\[
s := (\nu; b_1, \ldots, b_m),
\]

where \( \nu \in \mathcal{P}(n) \) and \( b_1, \ldots, b_m \in \bigsqcup_{j > 0} B_j \) satisfy \( |b_1| + \cdots + |b_m| = i \). But, one has:

\[
\varphi_{s,B}(\nu; b_1, \ldots, b_m) = (\nu; b_1, \ldots, b_m).
\]

So \( \varphi_{s,B} \) is onto.

Let us show that \( \varphi_{s,B} \) is injective. For this purpose, note that, by definition, \( \varphi_{s,B} \circ S(\mathcal{P}, s) = id_{S(\mathcal{P}, \Sigma \Omega M)} \). Let \( \pi : S(\mathcal{P}, M) \to S(\mathcal{P}, M)/(E)_P \) denote the canonical projection. One has \( \varphi_{s,B} \circ \pi \circ S(\mathcal{P}, s) = id_{\Sigma \Omega M} \). In particular, \( \pi \circ S(\mathcal{P}, s) \) is injective. Let \( X \in S(\mathcal{P}, M) \). Since \( S(\mathcal{P}, M) \) is generated by the elements of the form \( \nu; Sq_0^k s(b_1), \ldots, Sq_0^k s(b_m) \), with \( \nu \in \mathcal{P}(m) \), and \( S\nu \cdot s(b_1), \ldots, S\nu \cdot s(b_m) \in \bigsqcup_{d \in \mathbb{N}} C_d \). X can be decomposed:

\[
X = \sum_{i \in I} (\nu_i; Sq_0^{k_1} s(b_{i,1}), \ldots, Sq_0^{k_{m_i}} s(b_{i,m_i})),
\]
where $I$ is a finite set and for all $i \in I$, $\nu_i \in P(m_i)$, and $Sq^{k_i,1} s(b_{i,1}), \ldots, Sq^{k_i,m_i} s(b_{i,m_i}) \in \bigcup_{d \in \mathbb{N}} C_d$. This equality can be rephrased:

$$
X = \sum_{i \in I} \left( \nu_i; Sq^{k_i,1} s(b_{i,1}), \ldots, Sq^{k_i,m_i} s(b_{i,m_i}) \right) + \left( \nu_i(*_{k_i,1}, \ldots, *_{k_i,m_i}); (s(b_{i,1}) \times 2^{k_i,1}, \ldots, (s(b_{i,m_i}) \times 2^{k_i,m_i}) \right)
\in (E)_P

\left. \begin{array}{l}
\in \operatorname{im}(\Sigma \Omega) \\
\in \operatorname{im}(S(P,s))
\end{array} \right\}
$$

Let $Y$ be the following element of $\Sigma \Omega M$:

$$
Y = \sum_{i \in I} \left( \nu_i(*_{k_i,1}, \ldots, *_{k_i,m_i}); b^1_{i,1}, \ldots, b^{2^{k_i,m_i}}_{i,1} \right).
$$

Note that the class of $X$ modulo $(E)_P$ is $\pi \circ S(P,s)(Y)$. Now, if $\varphi_s, B(X) = 0$, since $\varphi_s, B(X) = \varphi_{s,B} \circ \pi \circ S(P, s)(Y) = Y$, this implies that $X$ is equal to 0 modulo $(E)_P$. Hence, $\varphi_{s,B}$ is injective. □

8. First examples, and applications

In this section, we study some applications of Theorem 6.11 when we take $M$ to be the free unstable module $F(n)$. Theorem 6.11 gives, under some assumptions on the unstable module $M$ and the operation $\ast \in P(\mathcal{E}^2)$, an isomorphism of $P$-algebras $K^P_*(M) \cong S(P, \Sigma \Omega M)$. This isomorphism is highly non-natural with respect to $M$, and depends on a graded section $s$ of $M$ and a graded basis $B$ of $\Sigma \Omega M$ (see Definition 6.9). In the case where $M = F(1)$, we will see that there is a unique choice for $s$ and for $B$. When $M = F(n)$, we will give a somewhat natural choice. In these cases, we will study the action of $\mathcal{A}$ obtained on $S(P, \Sigma \Omega M)$ by transfer from the action of $\mathcal{A}$ on $K^P_*(M)$. We will then expose a counter-example to Theorem 6.11 in the case where $M$ is not reduced.

Remark 8.1. Let $P$ be an operad in $\mathbb{F}_{\text{vect}}$, and let $M$ be a connected reduced unstable module endowed with a graded section $s$ and a graded basis $B$ of $\Sigma \Omega M$. Following the constructions of the proof of Lemma 7.7, the inverse of $\varphi_{s,B}: K^P_*(M) \rightarrow S(P, \Sigma \Omega M)$ is the $P$-algebra morphism sending $x \in \Sigma \Omega M$ to $[s(x)]_E \in K^P_*(M)$.

Definition 8.2. Let $P$ be an operad in $\mathbb{F}_{\text{vect}}$, $M$ be a connected reduced unstable module, endowed with a graded section $s$ and a graded basis $B$ of $\Sigma \Omega M$. One defines an action $\mathcal{A}$ on the graded $P$-algebra $S(P, \Sigma \Omega M)$ by setting:

$$
Sq^i \circ t := \varphi^{-1}(Sq^i \varphi(t)).
$$

Definition 8.3.

- Let $n > 0$. Recall (see Remark 6.3) that $\Omega F(n) \cong F(n-1)$. When $M = F(n)$, the classical choice for the graded basis $B$ of $\Sigma \Omega M$ is:

$$
B_i = \{ \sigma(Sq^{i_{n-1}}) \in \Sigma F(n-1) : I = (i_1, \ldots, i_k) \text{ is admissible, } e(I) < n, \text{ and } i_1 + \cdots + i_k = i - n \}.
$$

- The associated classical graded section $s: \Sigma F(n-1) \rightarrow F(n)$ sends $\sigma(Sq^{i_{n-1}}) \in B_i$ to $Sq^{i_{n}}$.

- In the case $n = 1$, since $\Sigma F(0)$ only contains one non-zero element $\sigma_{i_0}$ of degree 1. The graded basis $B = (B_i)_{i \in \mathbb{N}}$ with

$$
B_i = \begin{cases} 
\{ \sigma_{i_0} \}, & \text{if } i = 1, \\
\emptyset, & \text{otherwise}
\end{cases}
$$

is the only graded basis of $\Sigma \Omega F(1)$. Since $F(1)^1$ only contains one non-zero element $\iota_1$, the only graded section $\Sigma F(0) \rightarrow F(1)$ sends $\sigma_{i_0}$ to $\iota_1$. 
**Proposition 8.4.** \( \mathcal{K}^P(F(1)) \) is the free \( \mathcal{P} \)-algebra generated by one element \( \iota_1 \) of degree 1 endowed with the unstable action \( \mathcal{A} \) defined by:

\[
\text{Sq}^j \iota_1 := \begin{cases} 
\iota_1, & \text{if } j = 0, \\
\star(\iota_1, \iota_1), & \text{if } j = 1, \\
0, & \text{otherwise},
\end{cases}
\]

and satisfying the Cartan formula.

**Proof.** Let us describe the action \( \mathcal{A} \) obtained on the graded \( \mathcal{P} \)-algebra \( S(\mathcal{P}, \Sigma F(0)) \) from the action of \( \mathcal{A} \) on \( \mathcal{K}^P(F(1)) \) through the isomorphism \( \hat{\varphi}_{s,B} \) deduced from \( s \) and \( B \) as defined above. Since \( \hat{\varphi}_{s,B} \) is an isomorphism of \( \mathcal{P} \)-algebras, and since \( \mathcal{K}^P(F(1)) \) satisfies the Cartan formula, it suffices to describe the action of \( \mathcal{A} \) on the generator \( \iota_1 \). Since \( \text{Sq}^0 \iota_1 = \iota_1 \) and since \( \text{Sq}^i \iota_1 = 0 \) for all \( i \neq 0, 1 \), it suffices to compute \( \text{Sq}^1 \iota_1 \). But, because \( \mathcal{K}^P(F(1)) \) is \( \ast \)-unstable, one necessarily gets \( \text{Sq}^1 \iota_1 = (\ast; \iota_1, \iota_1) \). \( \square \)

**Example 8.5.** This example shows that for \( x \in \Omega M \), even if \( i \leq |x| \), the equality \( \text{Sq}^i \iota (\sigma x) = \sigma(\text{Sq}^i x) \) does not hold in \( S(\mathcal{P}, \Sigma \Omega M) \).

Set \( x := \sigma(\text{Sq}^4 \text{Sq}^2 \text{Sq}^1 \iota_1) \in \Sigma F(1) \subset S(\mathcal{P}, \Sigma F(1)) \). Note that the Adem relations give \( \text{Sq}^1 \text{Sq}^4 = \text{Sq}^5 \).

So, the action of \( \mathcal{A} \) on \( \Sigma F(1) \) gives \( \text{Sq}^1 x = \sigma(\text{Sq}^5 \text{Sq}^2 \text{Sq}^1 \iota_1) \), which is 0 because of the unstability condition. However, the action of \( \mathcal{K}^P(F(2)) \) transferred to \( S(\mathcal{P}, \Sigma F(1)) \) through the isomorphism \( \hat{\varphi}_{s,B} \) with \( s \) and \( B \) as defined above yields:

\[
\text{Sq}^1 \iota x = \hat{\varphi}(\text{Sq}^1 \iota) (x) = \hat{\varphi}(\text{Sq}^3 \text{Sq}^2 \text{Sq}^1 \iota_2) = \hat{\varphi}(\text{Sq}^0 \text{Sq}^2 \text{Sq}^1 \iota_2) = \hat{\varphi}(\ast; \text{Sq}^2 \text{Sq}^1 \iota_2, \text{Sq}^2 \text{Sq}^1 \iota_2)
\]

\[
= \hat{\varphi}(\ast; \sigma \text{Sq}^3 \text{Sq}^1 \iota_1, \sigma \text{Sq}^3 \text{Sq}^1 \iota_1, \sigma \text{Sq}^3 \text{Sq}^1 \iota_1).
\]

**Lemma 8.6.** Let \( \mathcal{P} \) be an operad in \( \mathbb{F}_{\text{vect}} \), \( \ast \in \mathcal{P}(2)^{G_2} \) be a \( \mathcal{P} \)-central operation. Let \( M \) be a connected reduced unstable module. For all graded sections \( s : \Sigma \Omega M \rightarrow M \), and all graded bases \( B \) of \( \Sigma \Omega M \), the action of \( \mathcal{K}^P(M) \) transferred on \( S(\mathcal{P}, \Sigma \Omega M) \) through the \( \mathcal{P} \)-algebra isomorphism \( \hat{\varphi}_{s,B} \) always yields:

\[
\text{Sq}^0 \circ (\mu; x_1, \ldots, x_n) = \ast ((\mu; x_1, \ldots, x_n), (\mu; x_1, \ldots, x_n)),
\]

where \( \mu \in \mathcal{P}(n) \) in \( x_1, \ldots, x_n \in \Sigma \Omega M \).

**Proof.** It is a consequence of the \( \ast \)-unstability of \( \mathcal{K}^P(M) \). More precisely, one has, with the notation of the proof of Lemma 7.7:

\[
\text{Sq}^0 \circ (\mu; x_1, \ldots, x_n) = \hat{\varphi}(\text{Sq}^0 \mu(s(x_1), \ldots, s(x_n)))
\]

\[
= \hat{\varphi}(\ast(\mu(s(x_1), \ldots, s(x_n)), \mu(s(x_1), \ldots, s(x_n))).
\]

Since \( \hat{\varphi} \) is a \( \mathcal{P} \)-algebra morphism, one deduces that:

\[
\text{Sq}^0 \circ (\mu; x_1, \ldots, x_n) = \hat{\varphi}(\text{Sq}^0 \mu(s(x_1), \ldots, s(x_n)))
\]

\[
= \ast (\hat{\varphi}(\mu(s(x_1), \ldots, s(x_n))), \hat{\varphi}(\mu(s(x_1), \ldots, s(x_n))).
\]

The result then ensues from Remark 8.31. \( \square \)

Let us now expose a counter-example to Theorem 6.11 in the case where the generating unstable module is not reduced.

**Example.** Set \( M = \Sigma F(0), \mathcal{P} = \text{Lev} \) and \( \ast \) be the generator of \( \text{Lev} \). Recall that \( \Sigma F(0) \) is a dimension 1 vector space concentrated in degree 1 with generator \( \sigma_0 \) satisfying \( \text{Sq}^0(\sigma_0) = \text{Sq}^1, \sigma_0 = 0 \). Hence, it is not reduced. In the unstable level algebra \( \mathcal{K}_{\text{Lev}}(\Sigma F(0)) \), one has \( \ast(\sigma_0, \sigma_0) = \text{Sq}^1, \sigma_0 = 0 \). So \( \mathcal{K}_{\text{Lev}}(\Sigma F(0)) \) is isomorphic to \( \Sigma F(0) \) endowed with a trivial multiplication. On the other hand, recall (see Remark 6.33) that \( \Omega \Sigma F(0) \cong F(0) \), so \( S(\text{Lev}, \Sigma \Omega (\Sigma F(0))) \cong S(\text{Lev}, \Sigma F(0)) \) is the free level algebra
generated by a degree one element. It is clear that, as level algebras, \( K^*_{\text{Lev}}(\Sigma F(0)) \) is not isomorphic to \( S(\text{Lev}, \Sigma \Omega \Sigma F(0)) \).

9. Further Applications

In this section, we recall the definition and structures of several classical unstable modules, such as Brown-Gitler modules, Carlsson modules, and Campbell-Selick modules. These modules come equipped with an inner product satisfying different properties. We give the definition for a list of operads such that these classical modules, with their operations, are free unstable algebras over these operads.

We refer to [9] and [12] for the definition and results on Brown-Gitler modules, the Brown-Gitler algebra, Carlsson modules and the Carlsson algebra, and we refer to [2] for the definition and results on Campbell-Selick modules.

Recollections about classical unstable modules.

**Definition 9.1** ([9]). Let \( n \in \mathbb{N} \). The \( n \)-th Brown-Gitler module \( J(n) \) is the representation of the functor \( H^n : \mathcal{U} \to \text{Set} \), mapping \( M \) to \( \text{Hom}_F(M^n, F) \).

**Lemma 9.2** ([9]). For all \( n, m > 0 \), there is a linear correspondence \( J(n)^m \cong (F(m)^n)^2 \), where \( V^2 \) is the dual vector space of \( V \).

**Definition 9.3** ([9]).
- The unstable modules \( J(n) \) are endowed with an external product \( \mu_{m,n} : J(n) \otimes J(m) \to J(n+m) \).
  The map \( \mu_{m,n} \in \text{Hom}_F(J(n) \otimes J(m), J(n + m)) \) corresponds to the only non-zero element of \( \text{Hom}_F((J(n) \otimes J(m))^{n+m}, F) \).
- The unstable module \( J(n) \) is endowed with an inner product obtained as a composite:
  \[
  J(n) \otimes J(n) \xrightarrow{\mu_{n,n}} J(2n) \xrightarrow{Sq^n} J(n),
  \]
  where the second map \( \cdot Sq^n \in \text{Hom}_F(J(2n), J(n)) \) corresponds, in the set \( \text{Hom}_F(J(2n)^n, J(n)) \cong (J(2n)^n)^2 \equiv F(n)^{2n} \), to the element \( Sq^n \).
- The direct sum \( J := \bigoplus_{n \in \mathbb{N}} J(n) \), with the multiplication given by the outer products \( \mu_{m,n} \), is the Brown-Gitler algebra, also called the Miller algebra. An element of \( J(n) \), seen in \( J \), is said to have weight \( n \), and this weight is additive with respect to the multiplication of \( J \).

**Theorem 9.4** (Miller [11]). The Brown-Gitler algebra \( J \) is isomorphic to the polynomial algebra \( \mathbb{F}[x_i, i \in \mathbb{N}] \), with \( |x_i| = 1 \), and \( x_i \) has weight \( 2^i \), endowed with the unstable action of \( A \) induced by

\[
Sq^j x_i := \begin{cases} 
x_i, & \text{if } j = 0, 
x_i^2, & \text{if } j = 1, 
0, & \text{otherwise,}
\end{cases}
\]

where we set \( x_{-1} = 0 \), and satisfying the Cartan formula.

**Remark.** The Brown-Gitler algebra is not, in the classical sense, an unstable algebra.

**Definition 9.5** ([3], [9]).
- The \( n \)-th Carlsson module \( K(n) \) is the limit of the following diagram in \( \mathcal{U} \):
  \[
  J(n) \xrightarrow{Sq^n} J(2n) \quad \cdots \quad J(2^n) \xrightarrow{Sq^2} J(2^{n+1}) \quad \cdots .
  \]

The outer products \( \mu_{n,m} \) and the inner products on \( J(n) \) pass to the limits, yielding an outer product \( \mu_{n,m} : K(n) \otimes K(m) \to K(n + m) \) and an inner product \( K(n) \otimes K(n) \to K(n) \).
The direct sum $K := \bigoplus_{n \in \mathbb{N}} K(n)$, with the multiplication given by the outer products $\mu_{m,n}$, is the Carlsson algebra.

**Theorem 9.6** (Carlsson [3], [4]). The Carlsson algebra $K$ is isomorphic to the polynomial algebra $\mathbb{F}[x_i, i \in \mathbb{Z}]$, with $|x_i| = 1$, endowed with the unstable action of $A$ induced by

$$Sq^j x_i := \begin{cases} x_i, & \text{if } j = 0, \\ x_i^{j-1}, & \text{if } j = 1, \\ 0, & \text{otherwise,} \end{cases}$$

and satisfying the Cartan formula.

**Definition 9.7.** The $s$-th Campbell-Selick module $N_s$ is the polynomial algebra $\mathbb{F}[x_i, i \in \mathbb{Z}/s\mathbb{Z}]$, with $|x_i| = 1$, endowed with the unstable action of $A$ induced by

$$Sq^j x_i := \begin{cases} x_i, & \text{if } j = 0, \\ x_i^{j-1}, & \text{if } j = 1, \\ 0, & \text{otherwise,} \end{cases}$$

**Remark.** The Carlsson algebra and the Campbell-Selick modules are not unstable algebras in the classical acceptation of the term.

**Operadic constructions.**

**Definition 9.8.**

- Set $D = \mathbb{F}[d]$, the polynomial algebra in one indeterminate $d$, seen as an operad concentrated in arity 1, with unit 1 $\in \mathbb{F}$. A $D$-algebra is a vector space endowed with an endomorphism $d$.
- For all $s > 0$, set $Q_s D = \mathbb{F}[d]/(d^s - 1)$, seen as an operad concentrated in arity 1. A $Q_s D$-algebra is a vector space endowed with an endomorphism $d$ such that $d^s$ is the identity morphism.
- For all $q > 0$, set $T_q D = \mathbb{F}[d]/(d^{q+1})$, seen as an operad concentrated in arity 1. A $T_q D$-algebra is a vector space endowed with a nilpotent endomorphism $d$ of order $\leq q + 1$.
- Set $D^\pm = \mathbb{F}[d, d^{-1}]$, the Laurent polynomial in one indeterminate $d$, seen as an operad concentrated in arity 1. A $D^\pm$-algebra is a vector space endowed with an automorphism $d$.
- Let $n \geq 1$, and recall (see [3]) that the operad $\mathcal{L}$ is spanned by the set operad $\mathcal{L}'$, where $\mathcal{L}'(n)$ is the set of maps $h : \{1, \ldots, n\} \to \mathbb{N}$ satisfying $\sum_{i=1}^{n} \frac{1}{2^h(i)} = 1$. Set $T_q \mathcal{L}$ for the $q$-truncation of the operad $\mathcal{L}$. Alternatively, $T_q \mathcal{L}$ is the quotient of the operad $\mathcal{L}$ by the operadic ideal generated by the functions $h : \{1, \ldots, n\} \to \mathbb{N}$ such that $\max(h) \geq q$.

**Lemma 9.9.** Let $P$ be an operad. The following morphism of $G$-modules is a distributive law (see [10]):

$$D \circ P \to P \circ D$$

$$(d^m; \mu) \mapsto (\mu; d^{m_1}, \ldots, d^{m_n})$$

and it induces a distributive law on $P \circ Q_s D$ and on $P \circ D^\pm$.

**Proof.** It is a straightforward verification.

**Lemma 9.10.** The generating operation $\ast \in \mathcal{L}((2)^{\mathbb{Z}})$ is a $\mathcal{L}$-central operation and a $T_q \mathcal{L}$ Levr central operation. The generating operation $X_2 \in u\text{Com}(2)$ is a $u\text{Com}$-central operation. The operation $(X_2; d, d)$ is an $u\text{Com} \circ D$-central operation, an $u\text{Com} \circ D^\pm$-central operation, and an $u\text{Com} \circ Q_s D$-central operation. More generally, if $\ast \in P(2)$ is $P$-central, then $(\ast; d^i, d^j)$ is $P \circ D$, $P \circ D^\pm$, and $P \circ Q_s D$-central, for all $i \in \mathbb{N}$.
Proof. All these assertions are proved by use of Proposition 5.3. □

Remark 9.11. With this definition, one can check that the Brown-Gitler algebra $J$ is a $(X_2; d, d)$-unstable uCom ◦ D-algebra, that the Carlsson algebra $K$ is a $(X_2; d, d)$-unstable uCom ◦ D-algebra, that $J(n)$ and $K(n)$ with their inner products are unstable Lev-algebras, and that $N_s$ is a $(X_2; d, d)$-unstable $Q_s$ D-algebra. Some of these modules are free as algebra over their respective operads, and can be identified to free unstable algebras. The following table shows, for several triplets $(\mathcal{P}, \star, M)$, where $\mathcal{P}$ is an operad, $\star \in \mathcal{P}(2)^{S_2}$ is $\mathcal{P}$-central, and where $M$ is a connected reduced unstable module, how to identify the associated $\star$-unstable $\mathcal{P}$-algebra $K^\star_{\mathcal{P}}(M)$:

| $(\mathcal{P}, \star, M)$ | $K^\star_{\mathcal{P}}(M)$ |
|---------------------------|---------------------------|
| (uCom, $X_2$, $F(1)$)    | $H^*(\mathbb{R}P^\infty, \mathbb{F}_2) \cong S^*(\Sigma F_2) = S(\text{Com}, \Sigma \Omega F(1))$ |
| (uCom, $X_2$, $F(n)$)    | $H^*(K(\mathbb{Z}/2\mathbb{Z}, n), \mathbb{F}_2) \cong S^*(\Sigma F(n - 1)) = S(\mathcal{P}, \Sigma \Omega F(n))$ |
| (uCom ◦ $D$, $(X_2; d, d), F(1)$) | Brown-Gitler module $J(2^n)$ |
| (uCom ◦ $Q_s$ $D$, $(X_2; d, d), F(1)$) | Carlsson module $K(1) \cong S(\text{Lev}, \Sigma F_2) = S(\text{Lev}, \Sigma \Omega F(1))$ |
| (uCom ◦ $T_q$ $D$, $(X_2; d, d), F(1)$) | Campbell-Selick module $N_s$ |
| (uCom ◦ $T_q$ $D$, $(X_2; d, d), F(1)$) | $\bigoplus_{i=1}^{2^n} J(i)$ |

Remark 9.12. On the one hand, the unstable module $K(1)$ is defined as a limit of the $J(2^n)$’s under the morphism $\cdot Sq^{2^n} : J(2^{n+1}) \rightarrow J(2^n)$. On the other hand, the operad Lev is the limit of the operads $T_q$ $\text{Lev}$ under the quotient morphisms. One checks that $K(1) = K^*_\text{Lev}(F(1))$ is in fact the limit of the $J(2^n) = K^*_{T_q \text{Lev}}(F(1))$ under the morphisms $d := \cdot Sq^{2^n}$ in the category of unstable Lev-algebras.
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