Induced cosmological constant in braneworlds with warped internal spaces

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December 19, 2021

Abstract

We investigate the vacuum energy density induced by quantum fluctuations of a bulk scalar field with general curvature coupling parameter on two codimension one parallel branes in a \((D+1)\)-dimensional background spacetime \(\text{AdS}_{D+1} \times \Sigma\) with a warped internal space \(\Sigma\). It is assumed that on the branes the field obeys Robin boundary conditions. Using the generalized zeta function technique in combination with contour integral representations, the surface energies on the branes are presented in the form of the sums of single brane and second brane induced parts. For the geometry of a single brane both regions, on the left (L-region) and on the right (R-region), of the brane are considered. The surface densities for separate L- and R-regions contain pole and finite contributions. For an infinitely thin brane taking these regions together, in odd spatial dimensions the pole parts cancel and the total surface energy is finite. The parts in the surface densities generated by the presence of the second brane are finite for all nonzero values of the interbrane separation. The contribution of the Kaluza-Klein modes along \(\Sigma\) is investigated in various limiting cases. It is shown that for large distances between the branes the induced surface densities give rise to an exponentially suppressed cosmological constant on the brane. In the higher dimensional generalization of the Randall-Sundrum braneworld model, for the interbrane distances solving the hierarchy problem, the cosmological constant generated on the visible brane is of the right order of magnitude with the value suggested by the cosmological observations.

PACS 04.62.+v, 03.70.+k, 11.10.Kk

1 Introduction

Motivated by the problems of the radion stabilization and the generation of cosmological constant, the role of quantum effects in braneworlds has attracted great deal of attention \[1-47\]. A class of higher dimensional models with the topology \(\text{AdS}_{D+1} \times \Sigma\), where \(\Sigma\) is a one-parameter compact manifold, and with two branes of codimension one located at the orbifold fixed points, is considered in Refs. \[23,24\]. In both cases of the warped and unwarped internal manifold, the quantum effective potential induced by bulk scalar fields is evaluated and it has been shown that this potential

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can stabilize the hierarchy between the Planck and electroweak scales without fine tuning. In addition to the effective potential, the investigation of local physical characteristics in these models is of considerable interest. Local quantities contain more information on the vacuum fluctuations than the global ones and play an important role in modelling a self-consistent dynamics involving the gravitational field. In the previous papers [40, 41] we have studied the Wightman function, the vacuum expectation values of the field square and the energy-momentum tensor for a scalar field with an arbitrary curvature coupling parameter obeying Robin boundary conditions on two codimension one parallel branes embedded in the background spacetime AdS$_{D_1+1} \times \Sigma$ with a warped internal space $\Sigma$. For an arbitrary internal space $\Sigma$, the application of the generalized Abel-Plana formula [48] allowed us to extract from the vacuum expectation values the part due to the bulk without branes and to present the brane induced parts in terms of exponentially convergent integrals for the points away from the branes.

The braneworld corresponds to a manifold with boundaries and the physical quantities, in general, receive both volume and surface contributions. In particular, the contributions located on the visible brane are of special interest as they are direct observables in the theory. In Ref. [32] the vacuum expectation value of the surface energy-momentum tensor is evaluated for a massive scalar field subject to Robin boundary conditions on two parallel branes in $(D + 1)$-dimensional AdS bulk. It has been shown that for large distances between the branes the induced surface densities give rise to an exponentially suppressed cosmological constant on the brane. In the Randall-Sundrum braneworld model, for the interbrane distances solving the hierarchy problem between the gravitational and electroweak mass scales, the cosmological constant generated on the visible brane is of the right order of magnitude with the value suggested by the cosmological observations. In the present talk based on [47], we describe similar issues within the framework of higher dimensional braneworld models with warped internal spaces.

2 Surface energy-momentum tensor and the zeta function

Consider a scalar field $\varphi(x)$ with curvature coupling parameter $\zeta$ satisfying the equation of motion

$$ (\nabla^M \nabla_M + m^2 + \zeta R) \varphi(x) = 0, \quad (1) $$

where $\nabla_M$ is the covariant derivative operator and $R$ is the scalar curvature for a $(D+1)$-dimensional background spacetime. We will assume that the bulk has the topology AdS$_{D_1+1} \times \Sigma$ and is described by the line element

$$ ds^2 = g_{MN} dx^M dx^N = e^{-2k_D y} \left( \eta_{\mu\sigma} dx^\mu dx^\sigma - \gamma_{ik} dX^i dX^k \right) - dy^2, \quad (2) $$

where $\eta_{\mu\sigma}$ is the metric tensor for $D_1$-dimensional Minkowski spacetime, $k_D$ is the inverse AdS radius, and the coordinates $X^i$, $i = 1, \ldots, D_2$, cover the internal manifold $\Sigma$, $D = D_1 + D_2$. The Ricci scalar corresponding to line element (2) is given by formula $R = -D(D+1)k_D^2 e^{-2k_D y} R(\gamma)$, with $R(\gamma)$ being the scalar curvature for the metric tensor $\gamma_{ik}$. In the discussion below, in addition to the radial coordinate $y$ we will also use the coordinate $z = e^{k_D y}/k_D$, in terms of which the line element is written in the form conformally related to the metric in the direct product spacetime $R^{(D_1,1)} \times \Sigma$ by the conformal factor $(k_D z)^{-2}$.

We are interested in one-loop vacuum effects induced by quantum fluctuations of the bulk field $\varphi(x)$ on two parallel branes of codimension one, located at $y = a$ and $y = b$, $a < b$. We assume that on the branes the field obeys Robin boundary conditions

$$ (\tilde{A}_y + \tilde{B}_y \partial_y) \varphi(x) = 0, \quad y = a, b, \quad (3) $$
with constant coefficients $\tilde{A}_y, \tilde{B}_y$. In a higher dimensional generalization of the Randall-Sundrum braneworld model the coordinate $y$ is compactified on an orbifold $S^1/Z_2$, of length $l$, $-l \leq y \leq l$, and the orbifold fixed points $y = 0$ and $y = l$ are the locations of two branes. The corresponding line-element has the form \([2]\) with the warp factor $e^{-2k_D|y|}$. In these models the region between the branes is employed only. For an untwisted bulk scalar with brane mass terms $c_a$ and $c_b$, the corresponding ratio of the coefficients in the boundary condition \([3]\) is determined by the expression (see, e.g., Refs. \[50\] \[13\] \[35\] for the case of the bulk $\text{AdS}_{D+1}$ and Refs. \[23\] \[40\] for the geometry under consideration)

$$\tilde{A}_y/\tilde{B}_y = -\frac{n^{(j)} c_j + 4D\zeta k_D}{2}, \quad n^{(a)} = 1, \quad n^{(b)} = -1.$$  

(4)

In the supersymmetric version of the model \([50]\) one has $c_b = -c_a$ and the boundary conditions are the same for both branes. For a twisted scalar, Dirichlet boundary conditions are obtained on both branes.

For the geometry of two parallel branes in $\text{AdS}_{D+1} \times \Sigma$ with boundary conditions \([3]\), the Wightman function and the vacuum expectation values (VEVs) of the field square and the bulk energy-momentum tensor are investigated in Refs. \[40\] \[41\]. On manifolds with boundaries the energy-momentum tensor is investigated in Refs. \[40\] \[41\]. On manifolds with boundaries the energy-momentum tensor in addition to the bulk part contains a contribution located on the boundary. For an arbitrary smooth boundary $\partial M$ with the inward-pointing unit normal vector $n^L$, the surface part of the energy-momentum tensor for a scalar field \([51]\) is given by the formula $T^{(s)}_{MN} = \delta(x; \partial M)\tau_{MN}$, where the ”one-sided” delta-function $\delta(x; \partial M)$ locates this tensor on $\partial M$ and

$$\tau_{MN} = \zeta \varphi^2 K_{MN} - (2\zeta - 1/2)h_{MN}\varphi n^L \nabla_L \varphi.$$  

(5)

In Eq. \([5]\), $h_{MN} = g_{MN} + n_M n_N$ is the induced metric on the boundary and $K_{MN} = h_{LM} h^{KP} \nabla_L n_P$ is the corresponding extrinsic curvature tensor.

By expanding the field operator over a complete set of eigenfunctions $\{ \varphi_\alpha(x), \varphi^*_\alpha(x) \}$ obeying the boundary conditions and using the standard commutation rules, for the VEV of the operator $\tau_{MN}$ one finds $\langle 0|\tau_{MN}|0 \rangle = \sum_\alpha \tau_{MN} \langle \varphi_\alpha(x), \varphi^*_\alpha(x) \rangle$, where the bilinear form $\tau_{MN} \{ \varphi, \psi \}$ is determined by the classical energy-momentum tensor \([5]\). Below in this section, we will consider the region between the branes. The corresponding quantities for the regions $y \leq a$ and $y \geq b$ are obtained as limiting cases. As we have mentioned before, in the orbifolded version of the model, the only bulk is the one between the branes. In this region the inward-pointing normal to the brane at $y = j$, $j = a, b$, and the corresponding extrinsic curvature tensor are given by the relations $n^{(j)M} = n^{(j)N} \delta^M_D$ and $K^{(j)}_{MN} = -n^{(j)}_{KM} h_{MN}$, where $n^{(j)}$ is defined in formula \([4]\). By using these relations and the boundary conditions, the VEV of the surface energy-momentum tensor on the brane at $y = j$ is presented in the form

$$\langle 0|\hat{T}^{(j)}_{MN}|0 \rangle = -h_{MN}n^{(j)}_{KD} C_j \langle 0|\varphi^2|0 \rangle_{y = j}, \quad C_j \equiv \zeta - (2\zeta - 1/2)\tilde{A}_j/(k_D \tilde{B}_j).$$  

(6)

From the point of view of physics on the brane, Eq. \([6]\) corresponds to the gravitational source of the cosmological constant type with the surface energy density $\varepsilon^{(s)}_j = \langle 0|\tau_0^{(j)0}|0 \rangle$ (surface energy per unit physical volume on the brane at $y = j$ or brane tension), stress $p_j^{(s)} = -\langle 0|\tau_1^{(j)1}|0 \rangle$, and the equation of state $\varepsilon^{(s)}_j = -p_j^{(s)}$. It is remarkable that this relation takes place for both subspaces on the brane. It can be seen that this result is valid also for the general metric $g_{\mu\sigma}$ instead of $\eta_{\mu\sigma}$ in line element \([2]\). For an untwisted bulk scalar in the higher dimensional generalization of the Randall-Sundrum braneworld based on the bulk $\text{AdS}_{D+1} \times \Sigma$ the coefficient in Eq. \([6]\) is given by the formula

$$C_j = \zeta(4D\zeta - D + 1) + (\zeta - 1/4)n^{(j)}c_j/k_D,$$  

(7)
In particular, the corresponding surface energy vanishes for minimally and conformally coupled scalar fields with zero brane mass terms.

In order to evaluate the expectation value of the surface energy-momentum tensor we need the corresponding eigenfunctions. These functions can be taken in the decomposed form

\[ \varphi_{\alpha}(x^M) = \frac{f_n(y)e^{-in\omega k^ix^x}}{\sqrt{2\omega_\beta n(2\pi)^{D-1}}} \psi_\beta(X), \quad k^\mu = (\omega_\beta n, k), \quad \omega_\beta n = \sqrt{k^2 + m_n^2 + \lambda_\beta^2}, \quad k = |k|, \quad (8) \]

where the mass spectrum \(m_n\) is determined by the boundary conditions. The modes \(\psi_\beta(X)\) are eigenfunctions for the internal subspace:

\[ [\Delta(\gamma) + \zeta R(\gamma)] \psi_\beta(X) = -\lambda_\beta^2 \psi_\beta(X), \quad (9) \]

with the eigenvalues \(\lambda_\beta^2\) and the orthonormalization condition \(\int d^Dx \sqrt{\gamma} \psi_\beta(X) \psi_\gamma^*(X) = \delta_{\beta\gamma}\). In Eq. (9), \(\Delta(\gamma)\) is the Laplace-Beltrami operator for the metric \(\gamma_{ij}\). From the field equation (1), one obtains the equation for the function \(f_n(y)\) with the solution

\[ f_n(y) = C_n e^{DkDy/2} g_\nu^{(a)}(m_n z_a, m_n z_b), \quad g_\nu^{(a)}(u, v) \equiv J_\nu(v)Y_\nu^{(a)}(u) - Y_\nu(v)J_\nu^{(a)}(u), \quad j = a, b, \quad (10) \]

where the normalization coefficient \(C_n\) is determined below. Here \(J_\nu(x), Y_\nu(x)\) are the Bessel and Neumann functions of the order

\[ \nu = \sqrt{D^2/4 - D(D+1)}(\zeta + m^2/k_D^2), \quad (11) \]

and the \(z\)-coordinates of the branes are denoted by \(z_j = e^{kDj}/k_D, \quad j = a, b\). In formula (10) and in what follows for a given function \(F(x)\) we use the notation

\[ \bar{F}^{(j)}(x) = A_j F(x) + B_j x F'(x), \quad A_j = \bar{A}_j + B_j k_D D/2, \quad B_j = \bar{B}_j k_D, \quad j = a, b. \quad (12) \]

Note that for conformally and minimally coupled massless scalar fields one has \(\nu = 1/2\) and \(\nu = D/2\), respectively.

The function (10) satisfies the boundary condition on the brane at \(y = b\). Imposing the boundary condition on the brane \(y = b\) we find that the eigenvalues \(m_n\) are solutions to the equation

\[ g_\nu^{(a)}(m_n z_a, m_n z_b) \equiv \bar{J}_\nu^{(a)}(m_n z_a) \bar{Y}_\nu^{(b)}(m_n z_b) - \bar{Y}_\nu^{(a)}(m_n z_a) \bar{J}_\nu^{(b)}(m_n z_b) = 0. \quad (13) \]

Equation (13) determines the Kaluza-Klein (KK) spectrum along the transverse dimension. We denote by \(u = \gamma_{\nu, n}, \quad n = 1, 2, \ldots\), the positive zeros of the function \(g_\nu^{(ab)}(u, uz_b/za)\), arranged in the ascending order, \(\gamma_{\nu, n} < \gamma_{\nu, n+1}\). The eigenvalues for \(m_n\) are related to these zeros as \(m_n = \gamma_{\nu, n}/z_a\). From the orthonormality condition of the radial functions, for the coefficient \(C_n\) in Eq. (10) one finds

\[ C_n^2 = \frac{\pi m_n \sqrt{\bar{Y}_\nu^{(b)}(m_n z_b)/\bar{Y}_\nu^{(a)}(m_n z_a)}}{k_D \frac{\partial}{\partial u} g_\nu^{(ab)}(uz_a, uz_b)|_{u=m_n}}. \quad (14) \]

Note that, as we consider the quantization in the region between the branes, \(a \leq y \leq b\), the modes defined by (10) are normalizable for all real values of \(\nu\) from Eq. (11).

Substituting the eigenfunctions (8) into the corresponding mode sum and integrating over the angular part of the vector \(k\), for the expectation value of the energy density on the brane at \(y = j\) we obtain

\[ \varepsilon^{(s)}_j = -2n^{(j)} C_j k_D z_j B_j \beta_{D-1} \int_0^\infty dk \, k^{D-1} \sum_{\beta} \sum_{n=1}^\infty |\psi_\beta(X)|^2 m_n g_\nu^{(b)}(m_n z_t, m_n z_j) \frac{\partial}{\partial u} g_\nu^{(ab)}(uz_a, uz_b)|_{u=m_n}, \quad (15) \]
where \( j, l = a, b, \) and \( l \neq j, \) and
\[
\beta_{D_1} = \frac{1}{(4\pi)^{D_1/2} \Gamma(D_1/2)}.
\] (16)

To regularize the divergent expression on the right of this formula we define the function
\[
\Phi_j(s) = 2z_j^D B_j \beta_{D_1-1} \sum_\beta |\psi_\beta(X)|^2 \int_0^\infty dk k^{D_1-2} \sum_{n=1}^{\infty} \omega_{\beta, n} \frac{m_n g_{\beta}^{(l)}(m_n z_l, m_n z_j)}{\partial \alpha \nu (u z_a, u z_b) |_{u = m_n}},
\] (17)
where an arbitrary mass scale \( \mu \) is introduced to keep the dimension of the expression. After the evaluation of the integral over \( k, \) this expression can be presented in the form
\[
\Phi_j(s) = \frac{z_j^D B_j}{(4\pi)^{(D_1-1)/2}} \sum_\beta |\psi_\beta(X)|^2 \zeta_j \beta(s),
\] (18)
where the generalized partial zeta function
\[
\zeta_j \beta(s) = \frac{\Gamma(-\alpha_s)}{\Gamma(-s/2) \mu^{s+1}} \sum_{n=1}^{\infty} (m_n^2 + \lambda_{\beta}^2)^{\alpha_s} \frac{m_n g_{\beta}^{(l)}(m_n z_l, m_n z_j)}{\partial \alpha \nu (u z_a, u z_b) |_{u = m_n}}, \quad \alpha_s = \frac{D_1 + s - 1}{2}.
\] (19)

The computation of the VEV of the surface energy-momentum tensor requires the analytic continuation of the function \( \Phi_j(s) \) to the value \( s = -1. \) In order to obtain this analytic continuation we will follow the procedure multiply used for the evaluation of the Casimir energy (see, for instance, [52]). On the base of this procedure it can be seen that in the strip \(-D_1 - 1 < \Re s < -D_1 \) of the complex plane \( s \) we have the following integral representation
\[
\zeta_j \beta(s) = \zeta_j^{(j)}(s) - \frac{\mu^{-s-1} B_j}{\Gamma(-s/2) \Gamma(\alpha_s + 1)} \int_0^\infty du (u^2 - \lambda_{\beta}^2)^{\alpha_s} \Omega_j \nu (u z_a, u z_b),
\] (20)
where
\[
\zeta_j^{(j)}(s) = -\frac{n_j \mu^{-s-1}}{\Gamma(-s/2) \Gamma(\alpha_s + 1)} \int_0^\infty du (u^2 - \lambda_{\beta}^2)^{\alpha_s} \frac{F_\nu (u z_j)}{F_\nu (u z_j)}. \quad (21)
\]
In (20) we have defined
\[
\Omega_j \nu (u, v) = \frac{\bar{K}_\nu^{(b)}(v)}{K_\nu^{(a)}(u)} G_\nu^{(a)}(u, v), \quad \Omega_j \nu (u, v) = \frac{\bar{I}_\nu^{(b)}(u)}{I_\nu^{(a)}(u)} G_\nu^{(a)}(u, v),
\] (22)
with the modified Bessel functions \( I_\nu(u) \) and \( K_\nu(u) \). In formula (21) we use the notation \( F = K \) for \( j = a \) and \( F = I \) for \( j = b \). The contribution of the second term on the right of Eq. (20) is finite at \( s = -1 \) and vanishes in the limits \( z_a \to 0 \) or \( z_b \to \infty \). The first term corresponds to the contribution of a single brane at \( z = z_j \) when the second brane is absent. The surface energy density corresponding to this term is located on the surface \( y = a + 0 \) for the brane at \( y = a \) and on the surface \( y = b - 0 \) for the brane at \( y = b \). To distinguish on which side of the brane is located the corresponding term we use the superscript \( J = L \) for the left side and \( J = R \) for the right side. As we consider the region between the branes, in formula (21) \( J = R \) for \( j = a \) and \( J = L \) for \( j = b \). The further analytic continuation is needed for the function \( \zeta_j^{(j)}(s) \) only and this is done in the next section. The integral representation for the partial zeta function given by Eqs. (20), (21) is valid for the case of the presence of the zero mode as well.
3 Surface energy density for a single brane

In this section we consider the geometry of a single brane placed at \( y = j \). The orbifolded version of this model corresponds to the higher dimensional generalization of the Randall-Sundrum 1-brane model with the brane location at the orbifold fixed point \( y = 0 \). The corresponding partial zeta function is given by Eq. (21). Now in this formula \( J = L, R \) for the left and right sides of the brane, respectively, and \( F = K \) for \( J = R \) and \( F = I \) for \( J = L \). In addition, the replacement \( n^{(j)} \rightarrow n^{(j)} \) should be done with \( n^{(R)} = 1 \) and \( n^{(L)} = -1 \). The integral representation (21) for a single brane partial zeta function is valid in the strip \( -D_1 - 1 < \text{Re} s < -D_1 \) and under the assumption that the function \( \bar{F}^{(j)}(u) \) has no real zeros. For the analytic continuation to \( s = -1 \) we employ the asymptotic expansions of the modified Bessel functions for large values of the argument [53]. For \( B_j \neq 0 \) from these expansions one has

\[
\frac{F_\nu(u)}{\bar{F}^{(j)}_\nu(u)} \sim \frac{1}{B_j} \sum_{l=1}^{\infty} v_l^{(F,j)} \lambda^l, \tag{23}
\]

where the coefficients \( v_l^{(F,j)}(\nu) \) are combinations of the corresponding coefficients in the expansions for the functions \( F_\nu(u) \) and \( F_\nu'(u) \). Note that one has the relation \( v_l^{(K,j)} = (-1)^l v_l^{(I,j)} \), assuming that the coefficients in the boundary conditions are the same for both sides of the brane. For the nonzero modes along the internal space \( \Sigma \) we subtract and add to the integrand in (21) the leading terms of the corresponding asymptotic expansion and exactly integrate the asymptotic part. For the zero mode we first separate the integral over the interval \((0,1)\) and apply the described procedure to the integral over \((1,\infty)\). As a result, the corresponding function

\[
\Phi_j^{(j)}(s) = \frac{z_j^D B_j}{(4\pi)^{(D_1-1)/2}} \sum_\beta \left| \psi_\beta(X) \right|^2 v_{\nu\beta}^{(j)}(s), \tag{24}
\]

is written in the form

\[
\Phi_j^{(j)}(s) = -\frac{(4\pi)^{(1-D_1)/2} n^{(j)} z_j^{D_2}}{\Gamma(-\frac{D_1}{2}) \Gamma(\alpha_s + 1)(\mu z_j)^{s+1}} \left\{ \sum_\beta \left| \psi_\beta(X) \right|^2 \left[ \delta_{0\lambda_\beta} B_j \int_0^1 du u^{D_1+\lambda_s} \frac{F_\nu(u)}{\bar{F}^{(j)}_\nu(u)} \right] \right. \\
+ \int_{u_\beta}^{\infty} du u (u^2 - \lambda_\beta^2 z_j^2) \alpha_s \left( B_j \frac{F_\nu(u)}{\bar{F}^{(j)}_\nu(u)} - \sum_{l=1}^N v_l^{(F,j)} \lambda^l \right) - \sum_{l=1}^N \frac{\left| \psi_0(X) \right|^2 v_l^{(F,j)}}{D_1 + s - l + 1} \\
\left. + \frac{1}{2} \Gamma(\alpha_s + 1) \sum_{l=1}^N v_l^{(F,j)} z_j^{D_1+s-l+1} \Gamma\left(\frac{l}{2} - \alpha_s - 1\right) \zeta_{\Sigma} \left(\frac{l}{2} - \alpha_s - 1, X\right) \right\}, \tag{25}
\]

where \( u_\beta = \lambda_\beta z_j + \delta_{0\lambda_\beta} \). In Eq. (25) we have introduced the local spectral zeta function associated with the massless laplacian defined on the internal subspace \( \Sigma \):

\[
\zeta_{\Sigma}(z, X) = \sum_\beta \left| \psi_\beta(X) \right|^2 \lambda_\beta^{-2z}, \tag{26}
\]

where the prime on the summation sign means that the zero mode should be omitted. Both integrals in Eq. (25) are finite at \( s = -1 \) for \( N \geq D_1 - 1 \). For large values \( \lambda_\beta \) the second integral behaves as \( \lambda_\beta^{D_1+s-N} \) and the series over \( \beta \) in Eq. (25) is convergent at \( s = -1 \) for \( N > D_1 - 1 \). For these values \( N \) the poles at \( s = -1 \) are contained only in the last two terms on the right.
are related to the corresponding coefficients $C_{\frac{D_z}{2-p}}(X)$ for the corresponding non-minimal laplacian, and the dots denote the terms vanishing at $z = p$. In the way similar to that used in Ref. [24], it can be seen that the coefficients $C_p(X)$ are related to the corresponding coefficients $C_p(X, m)$ for the massive zeta function

$$\zeta(s, X; m) = \sum_{\beta} \frac{|\psi_\beta(X)|^2}{(\lambda_\beta^2 + m^2)^s},$$

by the formula

$$C_p(X) = C_p(X, 0) - |\psi_0(X)|^2 \delta_{p, D_z/2}.$$

The VEV of the energy density on a single brane is derived from

$$\bar{\epsilon}^{(J)}_{j} = -n^{(J)}k^D_{D_z} \Phi^j_{\bar{J}}(s)|_{s=-1}.$$

By using relation (27), the energy density is written as a sum of pole and finite parts: $\bar{\epsilon}^{(J)}_{j} = \bar{\epsilon}^{(J)}_{j,p} + \bar{\epsilon}^{(J)}_{j,f}$. Laurent-expanding the expression on the right of Eq. (25) near $s = -1$, one finds

$$\bar{\epsilon}^{(J)}_{j,p} = -\frac{2k^D_{D_z} C_j}{(4\pi)^{D_1/2}(s + 1)} \sum_{i=1}^{D} \frac{v_i^{(F,j)} z_j^{D_1-l}}{\Gamma(l/2)} [C_{(D_1-l)/2}(X) + |\psi_0(X)|^2 \delta_{D_1}]$$

for the pole part, and

$$\bar{\epsilon}^{(J)}_{j,f} = 2k^D_{D_z} C_j \beta D_1 \sum_{\beta} |\psi_\beta(X)|^2 \left[ \delta_0 \lambda_\beta B_j \int_0^1 du \frac{u^{D_1-1} F_{\nu}(u)}{F_{\nu}(u)} \right]$$

$$+ \int_{u_\beta}^{\infty} du \, \frac{u^2 - \lambda_\beta^2 z_j^2}{\nu^{D_1/2}} \left[ B_j \int_0^1 du \frac{F_{\nu}(u)}{F_{\nu}(u)} - \sum_{l=1}^{N} \frac{v_l^{(F,j)}}{\nu^l} \right]$$

$$- k^D_{D_z} C_j \beta D_1 |\psi_0(X)|^2 \left\{ \sum_{l=1}^{N} \frac{2v_l^{(F,j)}}{D_1-l} - v_l^{(F,j)} \left[ 2\ln(\mu z_j) + \psi \left( \frac{D_1}{2} \right) - \psi \left( \frac{1}{2} \right) \right] \right\}$$

$$+ \frac{k^D_{D_z} C_j}{(4\pi)^{D_1/2}} \sum_{l=1}^{N} \frac{v_l^{(F,j)}}{\Gamma(l/2)} z_j^{D_1-l} \left\{ C_{(D_1-l)/2}(X) \left[ 2\ln(\mu z_j) + \psi \left( \frac{1}{2} \right) \right] + \Omega(l-D_1)/2(X) \right\}$$

for the finite part, with $\psi(x)$ being the di-gamma function. In this formula the prime on the summation sign means that the term with $l = D_1$ should be omitted and it is understood that $C_p(X) = 0$ for $p < 0$. In the pole part the second term in the square brackets comes from the zero mode along $\Sigma$ and this term is cancelled by the delta term on the right of Eq. (29).

The renormalization of the surface energy density can be done modifying the procedure used previously for the renormalization of the Casimir energy in the Randall-Sundrum model [13, 6, 9] and in its higher-dimensional generalizations with compact internal spaces [24, 23]. The form of the counterterms needed for the renormalization is determined by the pole part of the surface energy.
density given by Eq. (31). For an internal manifold with no boundaries, this part has the structure
\[ \sum_{l=0}^{[(D-1)/2]} a_l^{(s)} (z_j/L)^{2l} . \]
By taking into account that the intrinsic scalar curvature \( R_j \) for the brane at \( y = j \) contains the factor \( (z_j/L)^2 \), we see that the pole part can be absorbed by adding to the brane action counterterms of the form
\[ \int d^D x \sqrt{|h|} \sum_{l=0}^{[(D-1)/2]} b_l^{(s)} R_l^j, \]
where the square brackets in the upper limit of summation mean the integer part of the enclosed expression. By taking into account that there is the freedom to perform finite renormalizations, we see that the renormalized surface energy density on a single brane is given by the formula
\[ \varepsilon^{(1, \text{ren})}_j = \varepsilon^{(f)}_j + \sum_{l=0}^{[(D-1)/2]} c_l^{(s)} (z_j/L)^{2l} . \]
(34)
The coefficients \( c_l^{(s)} \) in the finite renormalization terms are not computable within the framework of the model under consideration and their values should be fixed by additional renormalization conditions.

The total surface energy density for a single brane at \( y = j \) is obtained by summing the contributions from the left and right sides:
\[ \varepsilon^{(LR)}_j = \varepsilon^{(L)}_j + \varepsilon^{(R)}_j . \]
In formulas (31), (32) we should take \( F = I \) for \( J = L \) and \( F = K \) for \( J = R \). Now we see that, assuming the same boundary conditions on both sides of the brane, the coefficients \( C_p(X,0) \) enter into the sum of pole terms in the form
\[ 2 \sum_{l=1}^{[D/2]} \frac{v_l^{(I,j)}}{\Gamma(l)} z_j^{D-2l} C_{D/2-l}(X,0) . \]
(35)
If the internal manifold contains no boundaries and \( D \) is an odd number, one has \( C_{D/2-l}(X,0) = 0 \) and, hence, the pole parts coming from the left and right sides cancel out. For a one parameter internal space of size \( L \) the surface energy density on the brane at \( y = j \) is a function of the ratio \( L/z_j \) only. Note that in the case of the AdS bulk the corresponding quantity does not depend on the brane position. To discuss the physics from the point of view of an observer residing on the brane, it is convenient to introduce rescaled coordinates
\[ x'^M = e^{-kDj} x^M, \quad M = 0, 1, \ldots, D - 1. \]
(36)
With this coordinates the warp factor in the metric is equal to 1 on the brane and they are physical coordinates for an observer on the brane. For this observer the physical size of the subspace \( \Sigma \) is \( L_j = L e^{-kDj} \) and the corresponding KK masses are rescaled by the warp factor: \( \lambda^{(j)}_\beta = \lambda_\beta e^{kDj} \). Now we see that the surface energy density is a function on the ratio \( L_j/(1/kD) \) of the physical size for the internal space (for an observer residing on the brane) to the AdS curvature radius.

As an application of the general results presented above, we can consider a simple example with \( \Sigma = S^1 \). In this case the bulk corresponds to the AdS\(_D+1 \) spacetime with one compactified dimension \( X \). The corresponding normalized eigenfunctions are \( \psi_\beta(X) = e^{2\pi i \beta X/L}/\sqrt{L} \) with \( \beta = 0, \pm 1, \pm 2, \ldots \), where \( L \) is the length of the compactified dimension. The surface energy density induced on the brane is obtained from general formulas by the replacements
\[ \sum_\beta |\psi_\beta(X)|^2 \to \frac{2}{L} \sum_{\beta=0}^{\infty'} \lambda_\beta \to \frac{2\pi}{L} |\beta|, \quad D_2 = 1, \]
(37)
where the prime means that the summand \( \beta = 0 \) should be taken with the weight \( 1/2 \). For the local zeta function from Eq. (26) one has \( \zeta_X(s, \lambda) = 2L^{2s-1} \zeta_R(2s)/(2\pi)^{2s} \), where \( \zeta_R(z) \) is the Riemann zeta function. Now the only poles of the function \( \Gamma(z)\zeta_X(z, X) \) are the points \( z = 0, 1/2 \). By using the standard formulas for the gamma function and the Riemann zeta function (see, for instance, [53]), it can be seen that one has \( C_{1/2}(X) = -1/L \), \( C_0(X) = 1/2\sqrt{\pi} \), for the residues appearing in (27)

\[
\Omega_0(X) = \frac{\gamma - 2\ln L}{L}, \quad \Omega_\frac{1}{2}(X) = \frac{\gamma + 2\ln(L/4\pi)}{2\sqrt{\pi}}, \quad \Omega_p(X) = \frac{2L^{2p-1}}{(2\pi)^{2p}} \Gamma(p)\zeta_R(2p), \quad p \neq 0, \frac{1}{2},
\]

for the finite parts, with \( \gamma \) being the Euler constant.

4 Two-brane geometry and induced cosmological constant

As it has been shown in Section 2, the partial zeta function related to the surface energy density on the brane at \( y = j \) is presented in the form (20), where the second term on the right is finite at the physical point \( s = -1 \). For two-brane geometry the VEV of the surface energy density on the brane at \( y = j \) is presented as the sum

\[
\epsilon_j^{(s)} = \epsilon_j^{(j)} + \Delta\epsilon_j^{(s)}.
\]

The first term on the right is the energy density induced on a single brane when the second brane is absent. The second term is induced by the presence of the second brane and is given by the formula

\[
\Delta\epsilon_j^{(s)} = 2C_j n^{(j)}(k_D z_j)^D B_j^2 \beta_D \sum_\beta |\psi_\beta(X)|^2 \int_{\lambda_\beta}^{\infty} du u(u^2 - \lambda_\beta^2)^{D_1/2-1} \Omega_{j\nu}(uz_a, uz_b).
\]

As we consider the region \( a \leq y \leq b \), the energy density (40) is located on the surface \( y = a + 0 \) for the left brane and on the surface \( y = b - 0 \) for the right brane. Consequently, in formula (39) we take \( J = R \) for \( j = a \) and \( J = L \) for \( j = b \). The energy densities on the surfaces \( y = a - 0 \) and \( y = b + 0 \) are the same as for the corresponding single brane geometry. The expression on the right of Eq. (40) is finite for all nonzero distances between the branes and is not touched by the renormalization procedure. For a given value of the AdS energy scale \( k_D \) and one parameter manifold \( \Sigma \) with the length scale \( L \), it is a function on the ratios \( z_b/z_a \) and \( L/z_a \). The first ratio is related to the proper distance between the branes, \( z_b/z_a = \exp[k_D(b - a)] \), and the second one is the ratio of the size of the internal space, measured by an observer residing on the brane at \( y = a \), to the AdS curvature radius \( k_D^{-1} \).

For the comparison with the case of the bulk spacetime AdS\(_{D+1} \) when the internal space is absent, it is useful in addition to the VEV (40) to consider the corresponding quantity integrated over the subspace \( \Sigma \):

\[
\Delta\epsilon_{D_1j}^{(s)} = \int_{\Sigma} dD_2 X \sqrt{\gamma} \Delta\epsilon_j^{(s)} e^{-D_2 k_D j} = e^{-D_2 k_D j} \sum_\beta \Delta\epsilon_j^{(s)}_{j\beta},
\]

where \( \Delta\epsilon_j^{(s)}_{j\beta} \) is defined by the relation

\[
\Delta\epsilon_j^{(s)}_{j\beta} = \sum_\beta |\psi_\beta(X)|^2 \Delta\epsilon_j^{(s)}_{j\beta}.
\]

Comparing this integrated VEV with the corresponding formula from Ref. [32], we see that the contribution of the zero KK mode (\( \lambda_\beta = 0 \)) in Eq. (41) differs from the VEV of the energy density.
in the bulk AdS\(_{D+1}\) by the order of the modified Bessel functions: for the latter case \(\nu \to \nu_1\) with \(\nu_1\) defined by Eq. (11) with the replacement \(D \to D_1\).

Now we turn to the investigation of the part (10) in the surface energy density in asymptotic regions of the parameters. For large values of AdS radius compared with the interbrane distance, \(k_D(b-a) \ll 1\), the main contribution to the integral on the right of Eq. (10) comes from large values of \(u z_a \sim |k_D(b-a)|^{-1}\). Assuming that \(\bar B_a/(b-a)\) and \(m(b-a)\) are fixed, we see that the order of the modified Bessel functions is large. Replacing these functions by their uniform asymptotic expansions for large values of the order \([53]\), it can be seen that to the leading order the corresponding surface energy on the branes in the bulk geometry \(R^{(D_1-1,1)} \times \Sigma\) is obtained.

For large KK masses along \(\Sigma\), \(z_a \lambda_\beta \gg 1\), \(\lambda_\beta \gg 1\), we can replace the modified Bessel functions by the corresponding asymptotic expansions for large values of the argument. For the contribution of a given KK mode to the leading order this gives

\[
\Delta \varepsilon_{j\beta}^{(s)} \approx 4n^{(j)}k_D^Dz^{D+1}B^D_2C_j\beta_{D_1}\int_{\lambda_\beta}^{\infty} du \frac{u^2(A_j^2 - B_j^2u^2z_{\lambda_\beta}^2)^{-1}(u^2 - \lambda_\beta^2)^D_{\beta} - 1}{c_a(u z_a)c_b(u z_b)e^{2au(z_b - z_a)} - 1},
\]

where

\[
c_j(u) = \frac{A_j - n^{(j)}B_j u}{A_j + n^{(j)}B_j u}, \quad j = a, b.
\]

If in addition one has the condition \(\lambda_\beta(z_b - z_a) \gg 1\), the dominant contribution into the \(u\)-integral comes from the lower limit and we have the formula

\[
\Delta \varepsilon_{j\beta}^{(s)} \approx \frac{2n^{(j)}B^2_jC_j}{A_j^2 - (\lambda_\beta z_b B_j)^2} \frac{k_D^Dz^{D+1}}{(4\pi)^{D_1/2}} \frac{\lambda_\beta^{D_1/2+1}e^{-2\lambda_\beta(z_b - z_a)}}{c_a(\lambda_\beta z_a)c_b(\lambda_\beta z_b)(z_b - z_a)^{D_1/2}}.
\]

In particular, for sufficiently small length scale of the internal space this formula is valid for all nonzero KK masses and the main contribution to the surface densities comes from the zero KK mode. In the opposite limit of large internal space, to the leading order we obtain the corresponding result for parallel branes in AdS\(_{D+1}\) bulk \([32]\). For small interbrane distances, \(k_D(b-a) \ll 1\), which is equivalent to \(z_b/z_a - 1 \ll 1\), the main contribution into the integral in Eq. (10) comes from large values \(u\) and to the leading order we obtain formula (13). If in addition one has \(\lambda_\beta(z_b - z_a) \ll 1\) or equivalently \(\lambda_\beta^{(s)}(b-a) \ll 1\), we can put in this formula \(\lambda_\beta = 0\). Assuming \((b-a) \ll |\bar B_j/A_j|\), for \(\bar B_j \neq 0\) to the leading order one finds

\[
\Delta \varepsilon_{j\beta}^{(s)} \approx -4k_D\sigma_jn^{(j)}C_j\frac{(D_{\beta} - 1)}{(4\pi)^{D_1/2}(b-a)^{D_{\beta} - 1}}C_{j\beta}^{(s)}\kappa_{\nu\nu}^{(D_1-1)D_{\beta}},
\]

where \(\sigma_j = 1\) for \(|B_a/A_a|, |B_b/A_b| \gg k_D(b-a)\), and \(\sigma_j = 2^{D_1-D} - 1\) for \(|B_j/A_j| \gg k_D(b-a)\) and \(B_l/A_l = 0\), with \(l = b\) for \(j = a\) and \(l = a\) for \(j = b\). We see that for small interbrane distances the sign of the induced surface energy density is determined by the coefficient \(C_j\) and this sign is different for two cases of \(\sigma_j\).

Now we consider the limit \(\lambda_\beta z_b \gg 1\) assuming that \(\lambda_\beta z_a \ll 1\). Using the asymptotic formulas for the Bessel modified functions containing in the argument \(z_b\), for the contribution of a given nonzero KK mode we find the following results

\[
\Delta \varepsilon_{a\beta}^{(s)} \approx \frac{k_D^Dz^{D+1}B^D_2C_a(\lambda_\beta z_b)^{D_1/2}e^{-2\lambda_\beta z_b}}{2^{D_{\beta} - 1}} \frac{\lambda_\beta^{D_1/2+1}e^{-2\lambda_\beta z_b}}{c_b(\lambda_\beta z_b)K_{\nu\nu}^{(a)(D_1-1)D_{\beta}}(\lambda_\beta z_a)},
\]

\[
\Delta \varepsilon_{b\beta}^{(s)} \approx -\frac{k_D^Dz^{D+1}B^D_2C_b(\lambda_\beta z_b)^{D_1/2}e^{-2\lambda_\beta z_b}}{2^{D_{\beta} - 1}} \frac{\lambda_\beta^{D_1/2+1}e^{-2\lambda_\beta z_b}}{c_b(\lambda_\beta z_b)K_{\nu\nu}^{(a)(D_1-1)D_{\beta}}(\lambda_\beta z_a)}.
\]
This limit corresponds to the interbrane distances much larger compared with the AdS curvature radius and with the inverse KK masses measured by an observer on the left brane: \((b - a) \gg 1/k_{D}, 1/\lambda^2_{(a)}\). For a single parameter manifold \(\Sigma\) with length scale \(L\) and \((b - a) \gg L_{a}\) these conditions are satisfied for all nonzero KK modes.

In the limit \(z_{a}\lambda_{\beta} \ll 1\) for fixed \(z_{b}\lambda_{\beta}\), by using the asymptotic formulas for the modified Bessel functions for small values of the argument and assuming \(|A_{a}| \neq |B_{a}|\nu\), one finds

\[
\Delta \varepsilon^{(s)}_{a,\beta} \approx \frac{k_{D}^{2}z_{a}D_{2}^{2} + 2\nu B_{a} C_{a} \lambda_{\beta} D_{1}}{2^{2\nu - 3}T^{2}(\nu)(A_{a} - \nu B_{a})^{2}} \int_{\lambda_{\beta}}^{\infty} du \frac{u^{2\nu + 1}(u^{2} - \lambda^{2}_{\beta})^{D_{1}/2 - 1} \bar{K}_{\nu}(u z_{a})}{I_{\nu}(u z_{b})},
\]

(49)

\[
\Delta \varepsilon^{(s)}_{b,\beta} \approx - \frac{k_{D}^{2}}{e_{a}(\nu)} \left( \frac{z_{a}}{z_{b}} \right)^{2\nu} k_{D}^{2} z_{b}^{2} f_{\nu}^{(b)}(b),
\]

(50)

where we have introduced the notation

\[
f_{\nu}^{(b)} = \frac{4D_{b}^{2} C_{b} \lambda_{\beta} D_{1}}{2^{2\nu + 1} \Gamma^{2}(\nu)} \int_{\lambda_{\beta} z_{b}}^{\infty} du \frac{u^{2\nu + 1}}{I_{\nu}^{(b)2}(u)}(u^{2} - \lambda^{2}_{\beta} z_{b}^{2})^{D_{1}/2 - 1}.
\]

(51)

For \(|A_{a}| = |B_{a}|\nu\) we should take into account the next terms in the corresponding expansions of the modified Bessel functions. The integral in Eq. (49) is negative for small values of the ratio \(A_{b}/B_{b}\) and is positive for large values of this ratio. As it follows from Eq. (50), for large interbrane separations the sign of the quantity \(\Delta \varepsilon^{(s)}_{b}\) is determined by the combination \((B_{a}^{2} - A_{a}^{2})C_{b}\) of the coefficients in the boundary conditions. In the limit under consideration the KK masses measured by an observer on the brane at \(y = a\) are much less than the AdS energy scale, \(\lambda_{\beta}^{(a)} \ll k_{D}\), and the interbrane distance is much larger than the AdS curvature radius. In particular, substituting \(\lambda_{\beta} = 0\), from these formulas we obtain the asymptotic behavior for the contribution of the zero mode to the surface energy density induced by the second brane in the limit \(z_{a}/z_{b} \ll 1\). Now combining the corresponding result with formulas \((47), (48)\), we see that under the conditions \((b - a) \gg 1/k_{D}, L_{a}\), and \(L_{a}k_{D} \gg 1\), the contribution of the nonzero KK modes along \(\Sigma\) is suppressed with respect to the contribution of the zero mode by the factor \((z_{b}/z_{a})^{2\nu + D_{1}/2 + \beta}\) \(\exp(- 2\lambda_{\beta} z_{a})\).

From the analysis given above it follows that in the limit when the right brane tends to the AdS horizon, \(z_{b} \to \infty\), the energy density \(\Delta \varepsilon^{(s)}_{a,\beta}\) vanishes as \(e^{-2\lambda_{\beta} z_{b}/z_{b}^{D_{1}}/2}\) for the nonzero KK mode along \(\Sigma\) and as \(z_{b}^{-D_{1} - 2\nu}\) for the zero mode. The energy density on the right brane, \(\Delta \varepsilon^{(s)}_{b,\beta}\), vanishes as \(z_{b}^{D_{2} + D_{1}/2 + 1} e^{-2\lambda_{\beta} z_{b}}\) for the nonzero KK mode and behaves like \(z_{b}^{-2\nu - 2\nu}\) for the zero mode. In the limit when the left brane tends to the AdS boundary, \(z_{a} \to 0\), the contribution of a given KK mode vanishes as \(z_{a}^{D_{2} + 2\nu}\) for \(\Delta \varepsilon^{(s)}_{a,\beta}\) and as \(z_{a}^{2\nu}\) for \(\Delta \varepsilon^{(s)}_{b,\beta}\). For small values of the AdS curvature radius corresponding to strong gravitational fields, assuming \(\lambda_{\beta} z_{a} \gg 1\) and \(\lambda_{\beta}(z_{b} - z_{a}) \gg 1\), we can estimate the contribution of the nonzero KK modes to the induced energy densities by formula \((45)\). In particular, for the case of a single parameter internal space with the length scale \(L\), under the assumed conditions the length scale of the internal space measured by an observer on the brane at \(y = a\) is much smaller compared to the AdS curvature radius, \(L_{a} \ll k_{D}^{-1}\). If \(L_{a} \gg k_{D}^{-1}\) one has \(\lambda_{\beta} z_{a} \lesssim 1\) and to estimate the contribution of the induced surface densities we can use formulas \((47)\) and \((48)\), and the suppression is stronger compared with the previous case. For the zero KK mode, under the condition \(k_{D}(b - a) \gg 1\) we have \(z_{a}/z_{b} \ll 1\) and to the leading order the corresponding energy densities are described by relations \((49)\) and \((50)\). From these formulae it follows that the induced energy densities integrated over the internal space behave as \(k_{D}^{D_{1} + 1} \exp((D_{1}\delta_{j} + 2\nu)k_{D}(a - b))\) for the brane at \(y = j\) and are exponentially suppressed.

Introducing the rescaled coordinates defined by Eq. \((46)\), after the Kaluza-Klein reduction of the higher dimensional Hilbert action, by the way similar to that in the Randall-Sundrum braneworld,
it can be seen that effective $D_1$-dimensional Newton’s constant $G_{D_1j}$ measured by an observer on the brane at $y = j$ and the fundamental $(D + 1)$-dimensional Newton’s constant $G_{D+1}$ are related by the formula

$$G_{D_1j} = \frac{(D - 2)kDG_{D+1}}{V_{\Sigma j}} \left[ e^{(D-2)kD(b-a)} - 1 \right] e^{(D-2)kD(b-j)}, \quad V_{\Sigma j} = e^{-DkD} \int d^{D_1}X \sqrt{\gamma}, \quad (52)$$

where $V_{\Sigma j}$ is the volume of the internal space measured by the same observer. In the orbifolded version of the model an additional factor 2 appears in the denominator of the expression on the right. Formula (52) explicitly shows two possibilities for the hierarchy generation by the gravitational fluctuations (for the discussion of the cosmological constant problem within the framework of brane. For an observer living on the brane at $y = j$ the corresponding effective $D_1$-dimensional cosmological constant is determined by the relation

$$\Lambda_{D_1j} = 8\pi G_{D_1j} \Delta \varepsilon_{D_1j}^{(s)} = 8\pi M_{D_1j}^{2-D} \Delta \varepsilon_{D_1j}^{(s)}, \quad (53)$$

where $M_{D_1j}$ is the $D_1$-dimensional effective Planck mass scale for the same observer and $\Delta \varepsilon_{D_1j}^{(s)}$ is defined by Eq. (41). Denoting by $M_{D+1}$ the fundamental $(D + 1)$-dimensional Planck mass, $G_{D+1} = M_{D+1}^{1-D}$, from Eq. (52) one has the following relation

$$\left( \frac{M_{D_1j}}{M_{D+1}} \right)^{D_1-2} = \frac{(z_b/z_a)^{D-2} - 1}{(D-2)(z_b/z_j)^{D-2}} \frac{V_{\Sigma j}}{kDG_{D+1}} M_{D+1}, \quad (54)$$

for the ratio of the effective and fundamental Planck scales. By using the asymptotic relations given above, for large interbrane distances one obtains the following estimate for the ratio of the induced cosmological constant (53) to the corresponding Planck scale quantity in the brane universe:

$$h_j \equiv \frac{\Lambda_{D_1j}}{8\pi G_{D_1j} M_{D_1j}} \sim \left( \frac{k_{Dj}^{D-1}}{V_{\Sigma j} M_{D+1}} \right)^{D_1} \frac{1}{a^{1-2}} \exp \left[ k_D(a - b) \left( 2\nu + \frac{D_1}{D_1 - 2} \delta_b^j \right) \right]. \quad (55)$$

For the model without an internal space this ratio is of the same order of magnitude for both branes.

In the higher dimensional version of the Randall-Sundrum braneworld the brane at $z = z_b$ corresponds to the visible brane. For large interbrane distances, by taking into account Eq. (50), for the ratio of the induced cosmological constant (53) to the Planck scale quantity in the corresponding brane universe one obtains

$$h_b \approx -\frac{1}{c_a(\nu)} \left( \frac{k_D}{M_{D1b}} \right)^{D_1} \left( \frac{z_a}{z_b} \right)^{2\nu} \sum_{\beta} f_{\nu\beta}^{(b)}, \quad (56)$$

where the function $f_{\nu\beta}^{(b)}$ is defined by Eq. (51).
Using relation (54) with \( j = b \), we can express the corresponding interbrane distance in terms of the ratio of the Planck scales

\[
\frac{z_a}{z_b} \approx \left[ M_{D+1}^D \frac{V_{2b}}{k_D} \left( \frac{M_{D+1}}{M_D} \right)^{-\nu(b-1)/2} \right]^{1/2}.
\]

Substituting this into Eq. (56), for the ratio of the cosmological constant on the brane at \( j = b \) to the corresponding Planck scale quantity one finds

\[
h_b \approx -\frac{1}{c_{\nu}(\nu)} \left( \frac{k_D}{M_{D+1}} \right)^{D_1-\tilde{\nu}} \left( V_{2b} M_{D+1}^D \right)^{\tilde{\nu}} \left( \frac{M_{D+1}}{M_{D,b}} \right)^{D_1+\tilde{\nu}(D_1-2)} \sum_{\beta} f_{\nu\beta},
\]

with \( \tilde{\nu} = 2\nu/(D - 2) \). The higher dimensional Planck mass \( M_{D+1} \) and AdS inverse radius \( k_D \) are two fundamental energy scales in the theory which in the Randall-Sundrum model are usually assumed to be of the same order, \( k_D \sim M_{D+1} \). In this case one obtains the induced cosmological constant which is exponentially suppressed compared with the corresponding Planck scale quantity on the visible brane. In the model with \( D_1 = 4 \), \( k_D \sim M_{D+1} \sim 1 \) TeV, \( M_{D,b} = M_{Pl} \sim 10^{16} \) TeV, assuming that the compactification scale on the visible brane is close to the fundamental Planck scale, \( V_{2b} M_{D+1}^2 \sim 1 \), for the ratio of the induced cosmological constant to the Planck scale quantity on the visible brane we find the estimate \( h_b \sim 10^{-32(2+\tilde{\nu})} \). From (57) one has \( k_D(b-a) \approx 74/(D_2 + 2) \) and the corresponding interbrane distances generating the required hierarchy between the electroweak and Planck scales are smaller than those for the model without an internal space. In the model proposed in Ref. [23], a separation between the fundamental Planck scale and curvature scale is assumed: \( k_D \sim M_{D+1} z_a/z_b \sim 1 \) TeV. Under the assumption \( V_{2b} M_{D+1}^2 \sim 1 \), in this model we have \( h_b \sim 10^{-64[1+\nu/(D+1)]} \) and \( k_D(b-a) \approx 74/(D_2 + 3) \).

5 Conclusion

We have investigated the expectation value of the surface energy-momentum tensor induced by the vacuum fluctuations of a bulk scalar field with an arbitrary curvature coupling parameter satisfying Robin boundary conditions on two parallel branes in background spacetime \( AdS_{D+1} \times \Sigma \) with a warped internal space \( \Sigma \). Vacuum stresses on the brane are the same for both subspaces and the energy-momentum tensor on the brane corresponds to the source of the cosmological constant type in the brane universe. It is remarkable that the latter property is valid also for the more general model with the metric \( g_{\mu\nu} \) instead of \( \eta_{\mu\nu} \) in line element (2). As a regularization procedure for the surface energy density we employ the zeta function technique. The corresponding zeta function is presented as the sum of single brane and second brane induced parts. The latter is finite at the physical point and the further analytical continuation is necessary for the first term only. As the first step we subtract and add to the integrand the leading terms of the corresponding asymptotic expansion for large values of the argument and explicitly integrate the asymptotic part. Further, for the regularization of the sum over the modes along the internal space we use the local zeta function related to these modes. By making use of the formula for the pole structure of this function, we have presented the energy density on a single brane as the sum of the pole and finite parts. The pole parts in the surface energy density are absorbed by adding to the brane action the counterterms having the structure given by Eq. (23). The renormalized energy density on the corresponding surface of a single brane is determined by formula (31), where the second term on the right presents the finite renormalization part. The coefficients in this part cannot be determined within the model under consideration and their values should be fixed by additional renormalization conditions which relate them to observables.
Unlike to the single brane part, the surface energy density induced by the presence of the second brane contains no renormalization ambiguities and is investigated in Section 4. This part is given by formula (40). We have investigated the induced energy density in various asymptotic regions for the parameters of the model. In the model under discussion the hierarchy between the fundamental Planck scale and the effective Planck scale in the brane universe is generated by the combination of redshift and large volume effects. The corresponding effective Newton’s constant on the brane at \( y = j \) is related to the higher-dimensional fundamental Newton’s constant by formula (52) and for large interbrane separations is exponentially small on the brane \( y = b \). We show that this mechanism also allows obtaining a naturally small cosmological constant generated by the vacuum quantum fluctuations of a bulk scalar. For large interbrane distances the ratio of the induced cosmological constant to the the corresponding Planck scale quantity in the brane universe is estimated by formula (55) and is exponentially small. For the visible brane in the higher dimensional generalization of the Randall-Sundrum two-brane model, this ratio is given in terms of the effective and fundamental Planck masses by Eq. (58). We have considered two classes of models with the compactification scale on the visible brane close to the fundamental Planck scale. For the first one the higher dimensional Planck mass and the AdS inverse radius are of the same order and in the second one a separation between these scales is assumed. In both cases the corresponding interbrane distances generating the hierarchy between the electroweak and Planck scales are smaller than those for the model without an internal space and the required suppression of the cosmological constant is obtained without fine tuning.

Acknowledgments

The work was supported by the Armenian Ministry of Education and Science Grant No. 0124 and by PVE/CAPES Program.

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