INSTABILITY OF NON-UNIFORM TOROIDAL MAGNETIC FIELDS IN ACCRETION DISKS

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ABSTRACT
We present a new type of instability that is expected to drive magnetohydrodynamic (MHD) turbulence from a purely toroidal magnetic field in an accretion disk. It is already known that in a differentially rotating system, the uniform toroidal magnetic field is unstable due to magnetorotational instability (MRI) under a non-axisymmetric and vertical perturbation, while it is stable under a purely vertical perturbation. Contrary to the previous study, this paper proposes an unstable mode completely confined to the equatorial plane, driven by the expansive nature of the magnetic pressure gradient force under a non-uniform toroidal field. The basic nature of this growing eigenmode, which we name “magneto-gradient driven instability,” is studied using linear analysis, and the corresponding nonlinear evolution is then investigated using two-dimensional ideal MHD simulations. Although a single localized magnetic field channel alone cannot provide sufficient Maxwell stress to contribute significantly to the angular momentum transport, we find that the mode coupling between neighboring toroidal fields under multiple localized magnetic field channels drastically generates a highly turbulent state and leads to the enhanced transport of angular momentum, which is comparable to the efficiency seen in previous studies on MRIs. This horizontally confined mode may play an important role in the saturation of an MRI through complementary growth with the toroidal MRIs and coupling with magnetic reconnection.

Key words: accretion, accretion disks – instabilities – magnetohydrodynamics (MHD) – methods: numerical – turbulence

1. INTRODUCTION

Accretion disks are some of the most ubiquitous astrophysical objects, including dynamics such as astrophysical jets, disk winds, and particle acceleration. It is widely believed that these dynamical phenomena are driven by the anomalous transport of angular momentum and the subsequent release of gravitational binding energy. Several mechanisms have been proposed in the attempt to explain the origin of this angular momentum transport. Examples include magnetic braking by external, large-scale magnetic fields (e.g., Blandford & Payne 1982; Stone & Norman 1994), non-axisymmetric wave excitation (e.g., Fragile & Blaes 2008), and hydrodynamic/hydromagnetic turbulence (e.g., Papaloizou & Pringle 1984; Balbus & Hawley 1998). In this paper, we consider a mechanism related to the third example, which is the sole candidate that possesses a high correlation with conventional, α-disk models (Shakura & Sunyaev 1973). In α-disk models, the efficiency of angular momentum transport, which is determined by the \( R\phi \) component of the stress tensor, is determined as the product of the pressure and a given parameter \( \alpha \). The value of \( \alpha \) depends significantly on viscosity physics, but the simple molecular viscosity in an accretion disk cannot provide the high efficiency of angular momentum transport suggested by observations (Cannizzo et al. 1988). Since the astrophysical importance of magnetorotational instability (MRI) as the origin of the required turbulence was pointed out (Balbus & Hawley 1991, 1998), a number of authors have investigated the nature of MRIs and the resultant turbulence in accretion disks over a wide range of plasma parameters (e.g., Stone et al. 1996; Sano & Stone 2002; Simon & Hawley 2009; Simon et al. 2012; Kunz & Lesur 2013; Bai 2015; Hoshino 2015; Zhu et al. 2015).

In order to study the basic behavior behind the nonlinear time evolution of MRIs, most numerical studies of the local properties of MRI-induced turbulence have adopted the shearing box model (Hawley et al. 1995; Sano & Inutsuka 2001; Sharma et al. 2006), which can capture the wave vector toward an arbitrary direction in a differentially rotating plasma. Since MRI with a vertical wave vector has the maximum growth rate for an axisymmetric perturbation when the background magnetic field is purely poloidal, fully three-dimensional simulations, or at least two-dimensional ones including a vertical axis, are necessary. (Note that the final states in the two- and three-dimensional cases are rather different from each other, and that the three-dimensional simulations are required to investigate the saturation stage.)

The situation is similar when the unperturbed magnetic field is purely toroidal. For example, Balbus & Hawley (1992) investigated the linear stability of an accretion disk threaded by a uniform toroidal magnetic field assuming three-dimensional wavevectors in the cylindrical coordinates, whose x component varies with time because of the background shear velocity. They showed that the perturbation satisfying \( k \cdot \mathbf{V}_s \leq \Omega \) can become unstable, in the sense that the amplitude of oscillation increases with time. Moreover, a finite vertical wavenumber, \( k_z \), is required for the instability to occur, and a larger \( k_z \) leads to faster amplification. The nonlinear evolution of this oscillatory instability was also examined by Hawley et al. (1995) using three-dimensional ideal magnetohydrodynamic (MHD) simulations, and its contribution to turbulence generation was confirmed.

Other examples include linear eigenvalue analyses and the corresponding MHD simulations by Matsumoto & Tajima (1995). They revealed that purely growing eigenmodes can exist in a shearing plasma, in contrast to the above oscillatory unstable modes. For a Keplerian disk, only non-axisymmetric perturbations with \( k_z^2/k_\phi^2 < 0.015 \) become purely growing modes, where \( k_\phi \) and \( k_z \) are the azimuthal and vertical wavenumbers, respectively. The vertical waves, therefore,
again contribute most significantly to the unstable modes, although a finite azimuthal wavenumber is required.

Such a situation, wherein the toroidal magnetic field is dominant, is thought to appear easily during the nonlinear stage of an MRI, even when starting from a poloidal field. To understand the dynamics and the nature of turbulence in well-developed disks, it is therefore important to investigate plasma stability under a purely toroidal field. In this paper, we provide the results of linear and nonlinear analyses on this issue, particularly focusing on an initially non-uniform toroidal field, and we suggest another possible path leading to the generation of turbulence. The unstable modes that we propose here, which we will call magneto-gradient driven instability (MGDI; reflecting its driving source as shown in the succeeding sections), are confined within the equatorial plane, i.e., \( k_z = 0 \), in contrast to the previous studies that required \( k_z^2 > k_r^2 \). An instability bound to the equatorial plane may play a crucial role in plasma transport, as it could potentially couple with magnetic reconnection occurring in the plane and contribute to the saturation mechanism of MRIs.

The outline of this paper is as follows. In Section 2, we briefly introduce the setup of our theoretical study and show the existence of unstable eigenmodes by linear analysis. Section 3 discusses the results of the fully nonlinear two-dimensional numerical simulations. We also present nonlinear calculations corresponding to the linear study and those which can lead to more turbulent states. Finally, Section 4 is devoted to the summary and conclusion of our results.

2. LINEAR ANALYSIS

In this section, we investigate the linear stability of a non-uniform toroidal magnetic field in a differentially rotating system. In particular, we consider a simplified situation with a localized toroidal magnetic field channel to extract the physical essence of possible unstable modes.

2.1. Equilibrium State and Linearized Equations

The ideal MHD equations incorporated with the standard shearing box model are employed as the basic equations throughout this paper (Stone & Gardiner 2010):

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = -\rho \nabla \cdot \mathbf{v},
\]

\[
\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla \left( \rho + \frac{\mathbf{B}^2}{2} \right) + \mathbf{B} \cdot \nabla \mathbf{B} - 2\rho \Omega \times \mathbf{v} - 2\rho \Omega v_y \partial_z \mathbf{e}_x,
\]

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}),
\]

\[
\frac{\partial p}{\partial t} + \nabla \cdot \rho \mathbf{v} = -\gamma p \nabla \cdot \mathbf{v}.
\]

The radial and azimuthal directions are then interpreted as the \( x \) and \( y \) axes in the local Cartesian coordinate system, where the differential rotation is described by a linearly changing background velocity defined as \( v_\Omega(x) = -q \Omega x \), using an angular velocity at the center of the computational domain, \( \Omega \), and a positive constant, \( q \). The specific heat ratio, \( \gamma \), is set to be \( 5/3 \), and the other notations are standard.

When a purely toroidal magnetic field is imposed, the background shearing plasma is kept in an equilibrium state as long as the total pressure is spatially constant. Therefore, we can choose an arbitrary magnetic structure with a finite gradient. Here, we focus on the idealized case with a simple localized toroidal field:

\[
B_{\psi 0}(x) = B_0 \cosh^{-2} (x/d),
\]

where \( B_0 \) is the field strength at \( x = 0 \) and \( d \) is the typical width of the localized field. The gas pressure is determined so as to satisfy the total pressure balance, and the background density is distributed so as to keep the temperature uniform.

Next, we linearize the MHD equations around this equilibrium state, assuming the functional form of a small perturbation as

\[
f_i(x, y, t) = f_i(x) \exp(-i\omega t + ik_y y).
\]

The vertical dependence is ignored so as to pick up horizontally confined modes. Using the vector potential instead of the magnetic field, the linearization of the basic equations leads to the following eigenvalue problem:

\[
\omega \mathbf{U}_i = M \mathbf{U}_i,
\]

where \( \mathbf{U}_i = (v_{i x} , v_{i y} , A_{i x} , p_i) \) is a first-order perturbation vector and \( M \) is a coefficient matrix whose components are given as follows:

\[
M = k_y v_{i 0} \mathbf{I} + D,
\]

\[
D = \begin{pmatrix} 0 & 2\Omega & D_{13} & - (i/p_0) \partial_x \\ D_{21} & 0 & -k_y B_{\psi 0}/\rho_0 & k_y/\rho_0 \\ i B_{\psi 0} & 0 & 0 & 0 \\ D_{41} & \gamma k_y \rho_0 & 0 & 0 \end{pmatrix},
\]

\[
D_{13} = (i/p_0)[B_{\psi 0}(\partial_x^2 - k_y^2) + B_{\psi 0} \partial_x],
\]

\[
D_{21} = -iv_{y 0} - 2i\Omega,
\]

\[
D_{41} = -ip_0 - iv_{0} \partial_x.
\]

Here, \( \mathbf{I} \) is the identity matrix, and \( \partial_x \) and \( \partial_x' \) indicate differentiation operators by \( x \). Note that the equations for \( v_{i x} \) and \( B_{\psi 1} \) are decoupled as usual shear Alfvén waves. Although the density perturbation, \( \rho_1 \), appears to have disappeared from the basic equations, compressional modes remain in the system and \( \rho_1 \) can be obtained from \( \nabla \cdot \mathbf{v}_1 \).

Finally, Equation (7) is discretized in the computational domain \( |x|/L_c \leq 1 \) with 400 grid points using a fourth-order central difference. The width of the localized field is set to be \( d = 0.05L_c \). As a boundary condition, a conducting wall is assumed at \( |x|/L_c \leq 1 \). The eigenvalues, \( \omega \), and the eigenvectors, \( \mathbf{U}_i \), are then computed numerically.

2.2. Growth Rates

From our calculations of the eigenvalue problem described above, at most we obtained one growing mode for the MGDI for each particular wavenumber value. The results are summarized in Figure 1 where the color contour shows the imaginary parts of the eigenvalues as a function of the wavenumber, normalized by the width of the localized field, and the plasma beta, \( \beta = 2\rho_0/B_0^2 \), measured at \( x = 0 \). Note that the real parts are zero at machine precision.
Figure 1 indicates that the purely growing mode appears if $\beta$ is lower than approximately 100, and that the growth rate increases as the initial magnetic field strength increases. When $\beta$ is equal to unity, the maximum growth rate reaches $0.765\Omega$, which is comparable to that of the axisymmetric MRI, i.e., $0.75\Omega$, as far as the linear approximation is appropriate. We should emphasize here that, for theoretical simplicity, the magnitude of the velocity shear, $q$, is assumed to be unity in Figure 1, which is smaller than in the case of Keplerian rotation, where $q = 1.5$. In the Keplerian rotation case, we expect more unstable eigenmodes due to the stronger shear motion.

A physical picture of the MGDI can be explained as follows. Let us consider an outward travelling perturbation in $v_{x1}$ away from $x = 0$, i.e., positive for $x > 0$ and negative for $x < 0$. Since a linearized equation for the azimuthal magnetic field can be written as:

$$\frac{dB_{1y}}{dt} = -B_{0y} \frac{\partial v_{x1}}{\partial x} - B_{x1} v_{x1} - q\Omega B_{x1}, \quad (8)$$

where $d/dt = \partial/\partial t + v_0 \cdot \nabla$ is a Lagrangian derivative, such outward $v_{x1}$ directly induces the increment in $B_{x1}$ through the second term in the right-hand side. This term comes from the linearized advection effect, which represents the fact that a fluid element brings the frozen-in magnetic field line from the original position with the stronger magnetic field. On the other hand, a linearized version of the equation of motion for the radial velocity is as follows:

$$\rho_0 \frac{dv_{x1}}{dt} = -\frac{\partial p_0}{\partial x} - \frac{\partial}{\partial x}(B_{0y}B_{x1})$$

$$+ B_{y0} \frac{\partial B_{x1}}{\partial y} + 2\rho_0 \Omega v_{x1}. \quad (9)$$

The second and third terms in the right-hand side represent the magnetic pressure gradient and the magnetic tension force, respectively. The magnetic pressure can be further decomposed into two contribution from $-B_{x0}B_{y1}$ and $-B_{y0}\partial_x B_{y1}$. The increase in $B_{x1}$ then leads to further expansion force via the first component of the magnetic pressure, which implies positive feedback. This feedback process will continue to work as long as the finite gradient in the background magnetic field is available.

In addition to the growth rates, the range of unstable wavenumbers also tends to broaden as the plasma beta decreases, especially toward the long-wavelength side. The smallest scale, on the other hand, always seems to be limited roughly by $k_yd < 0.5$, which corresponds to the wavelength one order of magnitude larger than $d$. This boundary could be understood qualitatively by the competition between the magnetic pressure gradient force, which is a driver here, and the magnetic tension force working as a restoring force. For the feedback mechanism described above to occur, it is clear that the expansive nature of the magnetic pressure needs to be greater than the tension effect. These promoting and restoring effects can be rearranged into the form of the Lorentz force, $J_0 \times B_0$ and $J_1 \times B_1$. The schematic view of the situation is illustrated in Figure 2. Then, the condition that the expansive term outpaces the restoring force is roughly estimated as:

$$\left| \frac{\partial B_{y0}}{\partial x} B_{y1} \right| > \left| \frac{\partial B_{x1}}{\partial y} B_{y0} \right|.$$

Replacing the derivatives by typical scales such as $1/d$ and $k_y$, we obtain an estimate, $1/d > k_y$, which is consistent with the unstable range in Figure 1.

The destabilization mechanism described above does not seem to be related to the nature of the differential rotation. For the purpose of comparison, the growth rates in a rigid-rotating plasma are shown in Figure 3 with the same format as in Figure 1. This panel indicates the existence of unstable modes over a range which is similar to that in the differentially rotating case. The qualitative dependence on $k_y$ and $\beta$ also resembles that in Figure 1, but the magnitude of the growth rate becomes smaller by a factor of 2 for $\beta = 1$, and much more for larger $\beta$. Therefore, it can be concluded that this instability arises originally from the gradient of the magnetic field, and
can attain a large growth rate comparable to that of the standard MRI only when combined with the shearing motion.

2.3. Eigenfunctions

Let us discuss the structure of the eigenfunctions. Figure 4 shows two-dimensional representations of the Fourier decomposed eigenfunction, $U_i$, superposed on the background equilibrium state, $U_{0i}$, where $U_{0i}$ is normalized to satisfy $|U_{0i}| = 1$. Based on the normalized case in panel (b), the states before and after twice the $\epsilon$-folding time are shown in panels (a) and (c), respectively. The plasma beta and the wavenumber are chosen to be $\beta = 100$ and $k_d = 0.223$, respectively, and the corresponding point on the $k_d$-$\beta$ diagram is plotted in Figure 1 by an outlined circle. In each panel, the color contour, the solid lines, and the vector field show the gas pressure distribution, the lines of magnetic force, and the bulk velocity, respectively.

Figure 4 shows that the bending of the field line broadens with time, and eventually spreads out beyond the typical width of the equilibrium field, $d = 0.05L_e$. This broadening of the field line can be explained by the destabilization process described in the previous subsection, i.e., the outward magnetic pressure gradient exceeding the inward magnetic tension force may further explosively expand the magnetic field. In addition to the expansion in the $x$ direction, the field lines are also stretched in the $y$ direction by the so-called $\Omega$ effect due to the background shear motion, and thus the magnetic field lines become inclined downward to the right. In other words, $B_v$ and $B_t$ tend to have a negative correlation. Therefore, it can be expected that the MGDI has the potential to contribute to powerful angular momentum transport, leading to the averaged Maxwell stress, $\langle -B_x B_y\rangle$, once it has developed to nonlinear turbulence.

Other important features include vortex structures around the nodes of the magnetic field lines in panel (c). In particular, the clockwise vortices at every other node, which align with the differential rotation, are selectively enhanced. Since the Coriolis force works rightward in relation to the direction of motion, the inside of the clockwise vortex is compressed while the counterclockwise vortex is expanded. On the other hand, the selective enhancement causes the magnetic field lines to loosen around the clockwise vortex, like a tightly stretched rope being reeled up. This leads to the negative correlation between the gas pressure and the magnetic pressure, which implies that the present unstable mode essentially arises from slow-magnetosonic waves.

Looking at Figure 4, one may associate the MGDI with current-driven instabilities (CDIs), which are thought to contribute to the fast dissipation of magnetic energy in various astrophysical contexts (e.g., Mignone et al. 2010; O’Neill et al. 2012; Mizuno et al. 2014). Although the characteristic of both unstable modes being driven by magnetic non-uniformity is common, we consider the MGDI to be a fundamentally different mode from the CDI. To make the difference clear, for example, note that the wave vector in the CDI is essentially parallel to the background electric current which drives the instability. In the case of the MGDI, on the other hand, the background current has only an out-of-plane component which is obviously perpendicular to wave vector of the perturbation.

Figures 5 and 6 show the eigenfunctions for the cases with $\beta = 10$ and 1, respectively, in the same format as in Figure 4. The wavenumber is again assumed to be $k_d = 0.223$. Since the magnetic tension force, working as a restoring force, becomes stronger with an increase in the magnetic field strength, it becomes more and more difficult to bend the magnetic field lines significantly, and the unstable mode seems to be localized around the initial localized channel. It should be noted that the localization of the eigenfunction and its growth rate is a different matter. The growth rate actually becomes greater as the background magnetic field strength increases, as shown in Figure 1, due to the large gradient of the field.

This localization implies a small $B_{y,1}$ compared with the initial $B_x$, but how it affects the average value of the Maxwell stress is not trivial. The estimate of the efficiency of the angular momentum transport is tightly connected to its nonlinear behavior and the problem of the saturation mechanism. In the next section, we will discuss the significant contribution of the present instability to the stress tensor using nonlinear simulations.

3. NONLINEAR SIMULATIONS

This section provides the results of fully nonlinear MHD simulations designed to validate the presence of MGDI suggested in the previous section and to investigate the nonlinear time evolution. Specifically, we focus on the efficiency of angular momentum transport.

3.1. Basic Equations and Simulation Codes

We solve the same equations as used in the previous section, i.e., from Equations (1)–(4), but rewritten in a semi-conservative form:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,$$

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot \left[ \rho \mathbf{v} \mathbf{v} + \left( p + \frac{B^2}{2} \right) \mathbf{I} - \mathbf{BB} \right] = -2\rho \Omega \times \mathbf{v} - 2\rho \Omega \nabla \times \mathbf{B},$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}),$$

Figure 3. Color contour of growth rates in a rigid-rotating plasma, $q = 0$, with the same format as in Figure 1.
where $e = \rho v^2/2 + p/(\gamma - 1) + B^2/2$ is the total energy density, and the other notation is the same as in the linear analysis. The specific heat ratio is again set to be $\gamma = 5/3$. All of the quantities are spatially discretized by using the finite difference approach in a computational domain, $-1 \leq x/L_x \leq 1$ and $0 \leq y/L_y \leq 2\pi$, which is resolved with $200 \times 600$ grid points.

We calculate the flux with the help of the HLL approximate Riemann solver (Harten et al. 1983) at the face center where the primitive variables, i.e., $\rho, \nu, B,$ and $p$, are evaluated as point values by combining a fifth-order weighted essentially non-oscillatory (WENO) interpolation (Liu et al. 1994; Jiang & Shu 1996) and the monotonicity preserving limiter (Suresh & Huynh 1997). The point-value flux is then converted to the appropriate numerical flux with a sixth-order formula (Shu & Osher 1988). The cell-centered conservative variables are finally updated using the third-order TVD Runge–Kutta method (Shu & Osher 1988). To avoid a spurious magnetic monopole, the HLL-upwind constrained transport (UCT; Londrillo & Del Zanna 2004) treatment is employed to update the face-centered magnetic field, in which the edge-centered electric field is evaluated using WENO interpolation and the HLL average. Note that the results to be shown in this section largely will not change even if the HLLD Riemann solver, which is more accurate than the HLL Riemann solver (especially in high-$\beta$ plasmas; Mignone et al. 2007, 2009), is employed instead. The quantitative behavior of the statistics,
however, slightly differs due to the higher resolution of each wave mode. In particular, less diffusivity is preferable to larger stress related to the turbulent motion on small scales.

3.2. A Single Localized $B$-field

Our initial condition is set to be a superposition of exactly the same equilibrium state considered in our linear analysis and a random perturbation of the in-plane velocity, the amplitude of which is fixed to 1% of the sound speed measured in an unmagnetized region. Without the random perturbation, the system would remain in the initial equilibrium state. To calculate the long time evolution of the system, we implement the standard shearing periodic boundary condition. Even if one adopts a conducting wall boundary, the results do not change in the early stage before the distorted magnetic fields approach the radial boundary.

From left to right in Figure 7, we show snapshots of simulations with $\beta = 100$, 10, and 1 taken at time $\tilde{t}/2\pi = 20$. The format of each panel is the same as in Figures 4–6, except for the range of the $y$ coordinates. All of these cases show the negative correlation between $B_x$ and $B_y$ reflected as the downward-sloping magnetic field lines, which contributes to the angular momentum transport. While the linear theory discussed in the previous section predicts broadband growth for a stronger initial field, the typical scale

![Figure 6. Unstable eigenfunction with $\beta = 1$ and $k_y a = 0.223$. The format is the same as in Figure 4. The growth rate is 0.675$\Omega$.](image)

![Figure 7. Snapshots of the MHD simulations at 20 times the orbital period. The color contours, solid lines, and arrows represent the gas pressure, the lines of magnetic force, and the in-plane velocity, respectively.](image)
of the bending of the field line is clearly larger for smaller $\beta$, which suggests that the magnetic tension force works more efficiently in nonlinear evolution, and thus the growth of short waves is suppressed. In larger $\beta$ cases, on the other hand, the bent mean structure and other small-scale magnetic structures appear. Such structures first grow along both sides of the initial field where the large gradient $|\partial B_\beta/\partial \chi|$ exists, and then they are torn off from the mean field by magnetic reconnection. (Note that magnetic reconnection occurs via numerical resistivity, but nonlinear evolution does not change by assuming finite resistivity.) In any case, the nonlinear growth up to the torsion of the localized magnetic field is ascertained. Note that the stage during which the linear theory is applicable finishes instantly, since the growing perturbed field quickly breaks the background structure of the initial magnetic field.

Figure 8 summarizes the box-averaged stress as a function of time. The $xy$ components of the Reynolds stress and the Maxwell stress normalized by the averaged initial gas pressure, that is, the $\alpha$ parameters.

![Figure 8. Time histories of the box-averaged stress. The left and right panels show the $xy$ components of the Reynolds stress and the Maxwell stress normalized by the averaged initial gas pressure, that is, the $\alpha$ parameters.](image)

motions, respectively. Specifically, the energy increase through the background velocity shear, $B_y B_y v_0$, is referred to as the $\Omega$-dynamo. In a radially periodic system like the shearing box, if a fluid element moves largely across radial boundaries, then the total velocity shear which the element feels can become much larger than the shear just inside one simulation domain, $q\Omega L_x$. Larger radial fluctuation therefore leads to stronger azimuthal magnetic energy. In the next subsection, we suggest a possible path leading to a more amplified magnetic field and a resultant large Maxwell stress.

### 3.3. Multiple Localized $B$-fields

As a phenomenon expected to occur in accretion disks, let us consider the situation where a toroidal magnetic field has multiple structures rather than a single localized field. The motivation of this idealized setup comes from, for example, the existence of parasitic instabilities occurring on a current sheet, which induce periodic variation of plasma parameters along an equatorial plane (Goodman & Xu 1994; Pessah & Goodman 2009; Rembiasz et al. 2016). The physical mechanism driving instability in the linear stage does not change even in this case, but a more turbulent state could be expected in the nonlinear stage as a result of coupling between neighboring fields. This section shows a possible path through which the MGDIs may contribute to turbulence generation and anomalous angular momentum transport.

The simulation setup is the same as described in the previous subsection, except for the initial profile of the magnetic field. Here, we assume the functional form of the toroidal magnetic field as follows:

$$B_{y0}(x) = B_c \cos^4 \left(\frac{3\pi x}{L_x}\right).$$

where $B_c$ is the amplitude of the field, $\Omega$ is the angular velocity of the disk, and $L_x$ is the characteristic length scale of the turbulence. The pressure and density profiles are also modified to retain dynamical and thermal
equilibrium. The plasma beta is defined using the peak value of the magnetic field, $B_c$, and the gas pressure at the same site.

Figures 9–11 show snapshots at characteristic stages in cases with $\beta = 100$, 10, and 1, respectively. The three panels in each figure are taken at times $\Omega t/2\pi = 5$, 10, and 20 from left to right, and the format of each panel is the same as in Figure 7. In the leftmost panel in Figure 9, we can see that the discrete magnetic field lines are distorted individually in the early stage by nonlinear growth, as demonstrated in the simulations with a single localized field. The bending of the field lines grows as time goes on, and before 10 orbital periods they drastically overlap and merge with the neighboring magnetic fields. The mixing of the magnetic field is completed by 20 orbital periods, and the simulation domain is filled with many magnetic islands as a result of the repetitive reconnection process. Recall that in a single channel case, the reconnected field is simply torn off of the background field and shows no further turbulent development. At this stage, the energy contained in the magnetic field increases to about 10% of the kinetic energy of the background differential rotation. Even if the difference of the initial magnetic energy is taken into account, this ratio is rather large compared with the case of a single localized field where the magnetic energy at the saturated stage is smaller by three orders of magnitude.

The detailed time history of the box-averaged energy is summarized in Figure 12(a), which shows that the dynamo process works efficiently on both $B_x$ and $B_y$ during the early stage before 20 orbital periods, and roughly speaking, an equipartition is eventually attained between the kinetic energy in the $x$ direction and the magnetic energy in the $x$ and $y$ directions. The production of the strong azimuthal field is understood to be a natural consequence from Equation (14), which relates the effect of the $\Omega$-dynamo. A similar relation holds with regard to the radial field energy as

$$\frac{d}{dt}\left(\frac{B_r^2}{2}\right) = B_x B_y \frac{\partial v_x}{\partial y} - B_x^2 \frac{\partial^2 v_x}{\partial y^2},$$

which shows that shear motion in the radial velocity newly generates the radial magnetic field. It is clear that in the MGDI, this radial velocity is driven by the magnetic pressure gradient force, while the gravity-related terms play the same role in the case of MRI. In this sense, the dynamo process working here looks quite similar to that in the MRI, not only in the azimuthal field, but also in the radial field. At the saturated stage, the gas pressure gradually increases through magnetic reconnection, creating many magnetic islands.

Even when the initial magnetic energy is 10 times as large, that is, $\beta = 10$, the same instability grows as shown in Figure 10. Compared with the case of $\beta = 100$, it can be seen that the typical scale of the growing mode becomes larger, as do the sizes of the magnetic islands in the final stage. The gas pressure at 20 orbital periods increases about twofold, which implies that the total amount of energy input into the system across the boundaries is enhanced and continuously converted to the internal energy of the plasma through magnetic reconnection. In other words, the total stress integrated along the radial boundary becomes larger, since the time variation of the total energy, $e$, can be described as follows:

$$\frac{\partial}{\partial t} \int e \, dV = 2 \int \nu_0(L_x) \int_{x=L_x} W_{xy} dy - \int 2\rho \Omega \nu_0(L_x) v, dV,$$

where $W_{xy} = \rho v_0 \delta v_y - B_x B_y$ is the total stress. The second term on the right-hand side of Equation (17), which represents the change in the total gravitational potential in the simulation

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**Figure 9.** Snapshots of the simulation started from a wavy toroidal field with $\beta = 100$, taken at 50, 100, and 200 orbits. The format of each panel is the same as in Figure 7.
domain, becomes nearly zero on average. The detailed time history of each energy component is plotted in Figure 12(b). As in the case of $\beta = 100$ shown in panel (a), the growing mode is saturated before 20 orbital periods, after which the kinetic energy related to $v_x$ and the magnetic energy reach a level comparable to the background differential rotation, but slightly larger than those in the case with a weaker initial field.

However, this situation changes significantly for the strong magnetic field with $\beta = 1$ shown in Figure 11. The initial equilibrium state holds and no growing mode can be observed. It is interesting to note that nonlinear evolution is accomplished in the single field case discussed in the previous subsection. The energy history in Figure 12(c) also shows a quite calm variation. The suppression of the growing mode is indeed the result of nonlinear magnetic tension force, since we confirmed
the presence of unstable modes in linear analyses under the initial magnetic profile described here.

We summarize the time histories of the Reynolds and Maxwell stresses in Figures 13(a) and (b), respectively. The result of $\beta = 1$, denoted by the red line, shows no stress for either the Reynolds or Maxwell components because the initial equilibrium state is almost conserved. In the cases with $\beta = 10$ and 100, denoted by the green and blue lines, the Reynolds and Maxwell stresses averaged after 20 orbital periods are $-0.00109$ and 0.0235 for $\beta = 100$, and $-0.0164$ and 0.126 for $\beta = 10$, respectively. The Maxwell stress, therefore, is larger by about two orders of magnitude than the results in the single field case, which indicates a qualitative change in the nonlinear behavior, rather than simple quantitative superposition due to an increase in the initial total magnetic flux. In addition, the result for a stronger shear motion with $q = 1.5$, which corresponds to the Kepler rotation, is also plotted as a cyan line. The basic mechanism driving the instability is the same as in the case with $q = 1.0$, but thanks to the more powerful $\Omega$-dynamo effect, the MGDI can grow more quickly nonlinearly and a larger Maxwell stress can be attained at the saturated stage.

The $\beta$-dependence of the Maxwell stress averaged during the saturated stage in the range $30 \leq \Omega t/2\pi \leq 50$ is summarized in Figure 14. It can clearly be seen that the results are well fit by a power law of $\beta^{-1/2}$ as long as $\beta > 2$, which indicates proportionality to the initial magnetic flux rather than magnetic energy density, while the cases starting with $\beta < 2$ result in almost no stress.

The suppression of instabilities for a small $\beta$ is highly relevant to the large characteristic spatial scale under the strong magnetic tension force, which works less efficiently for longer waves. Although this mode has a broad, unstable range in wavelength with respect to the linear theory around a single channel, the nonlinear growth is actually quenched in the multiple channel case. Once the simulation box is extended by twice its length in the azimuthal direction, however, the drastic growth in the Maxwell stress via the channel merging process

Figure 12. Temporal development of the box-averaged energy divided into the contribution from the $x$ and $y$ directions, all of which are normalized by the gas pressure measured in a magnetized region. Each panel shows the results for (a) $\beta = 100$, (b) $\beta = 10$, and (c) $\beta = 1$, respectively.
is activated even for $\beta = 1$. This certainly happens because the extended box allows for the growth of fluctuations on a larger scale larger than for the original domain size. Figure 15 shows the time histories of the Reynolds and Maxwell stress for $\beta = 1, 10, \text{and } 100$, respectively, normalized by the volume averages of the instantaneous thermal pressure, $\langle p \rangle$. From the right panel, we can clearly observe the enhancement of the Maxwell stress by non-linear growth after 50 orbits for $\beta = 1$. Except for the shifted time in which this drastic merging is switched on, the statistical behavior is quite similar in all of the cases, which implies that MGDI works as the common underlying mechanism driving the turbulence.

4. SUMMARY AND CONCLUSION

In the present paper, we propose a new plasma instability, MGDI, which can generate a highly turbulent state in an accretion disk and contribute to the enhanced transport of angular momentum. Driven by the spatial non-uniformity of a toroidal magnetic field, an unstable mode completely confined within the equatorial plane can be realized, in contrast to the previous studies on the toroidal MRI, in which a vertical wavenumber always dominates over a finite azimuthal component.

The growth rates and eigenfunctions of this instability are calculated by linear eigenvalue analysis, and the corresponding nonlinear evolution is then demonstrated using two-dimensional MHD simulations. While the simulations beginning with a single localized toroidal field reveal the nonlinear growth of the MGDI, it is shown that the angular momentum transport does not work as efficiently in the saturated stage. If the instabilities occur in the neighboring field lines under multiple localized magnetic fields rather than in an isolated situation, however, they drastically overlap with each other and a well-developed turbulent state can be realized. In such a case, dynamo action due to differential rotation begins to work efficiently on magnetic field lines crossing the radial boundaries, which contributes to a large Maxwell stress. Furthermore, we have shown that a toroidal field with a larger magnetic flux is favorable for the Maxwell stress to reach a large value, but this drastic transition does not occur for a magnetic field that is too strong with $\beta < 2$ as long as we employ a box size of $(L_x, L_y, L_z)$. Once the simulation domain is elongated, the transition is reactivated, but it takes much longer to switch on the drastic merging. It is worth noting that the situation with $\beta \sim 1$, which is favorable to the growth of the MGDI, often appears during a nonlinear phase in a local simulation of the MRI for a relatively small initial beta, $\beta \sim O(10^2)$. In the cases with larger initial beta values, such as $\beta \sim O(10^{3-6})$, the final states still seem to be in the unstable regime (e.g., Hawley et al. 1995; Sharma et al. 2006; Minoshima et al. 2015).

Although the profile of the toroidal magnetic field discussed here is one of the most idealized situations, the physical essence driving the instability does not change even under a different structure, as long as sufficient radial gradient of the magnetic field is available. There is, therefore, the possibility that the present unstable mode will arise around various kinds of fluctuations in the toroidal field, such as via parasitic modes.
including the Kelvin–Helmholtz instability and the tearing instability.

It should be emphasized again that the efficiency of the angular momentum transport obtained here is comparable to that obtained in evaluation in three-dimensional simulations of MRIs assuming an initial toroidal field, despite the low dimensionality, and therefore the MGDI alone are capable of driving strong turbulence. In addition to the sole contribution of the MGDI, one can also expect coupling with magnetic reconnection occurring parallel to the equatorial plane during the nonlinear phase of MRI, and with the toroidal MRIs as considered in previous studies if vertical variation is also taken into account. It is still not obvious how large a contribution the instabilities have in fully three-dimensional shearing boxes. Nevertheless, since they provide new paths toward turbulence without any finite \( k_z \), in contrast to the conventional toroidal MRIs, complementary growth between toroidally and vertically propagating waves, rather than competitive growth, should be expected. The present instability could potentially play a wide variety of crucial roles in the mechanism of turbulence generation in differentially rotating systems.

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Figure 15. Time histories of the Reynolds and Maxwell stress normalized box averages of the instantaneous gas pressure in the elongated simulation box, \((x, y) \in (2L_x, 4\pi L_x)\), for different initial \( \beta \).