SYMPLECTIC RIGIDITY OF REAL BIDISC

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Abstract. Let $D$ be the unit disc in $C$, then $D^n(r)$ is the complex or symplectic $n$-discs of radius $r$. Let $z_j = x_j + iy_j \in C, j = 1, 2$ and $D^2_2 = \{(z_1, z_2) : |x_1|^2 + |x_2|^2 < 1, |y_1|^2 + |y_2|^2 < 1\}$ be the real bidisc. In this paper we will prove the following two theorems:

1) If $T \in O(4)$ is an orthogonal transformation on $\mathbb{R}^4$, then $T(D^2)$ is symplectomorphic to $D^2$ w.r.t. the standard symplectic form on $\mathbb{R}^4$ if and only if $T$ is unitary or conjugate to unitary.

2) For $r \geq 1$ and $n \geq 2$, $D_2^n \times D^{n-2}(r)$ and $D^2 \times D^{n-2}(r)$ are not symplectomorphic w.r.t. the standard symplectic form on $\mathbb{C}^n$.

1. Introduction

Let $x_1, y_1, \ldots, x_n, y_n$ be the standard coordinates on the $2n$-dimensional Euclidean space $\mathbb{R}^{2n} \cong \mathbb{C}^n$, the standard symplectic form on the space is given by $dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$. All symplectic embeddings considered in this paper will be with respect to the standard symplectic form, unless otherwise specified. Define the standard disc in $\mathbb{C}$ of radius $R$ by $D(R) = \{z \in \mathbb{C} : |z| < R\}$, also define $D^2_2(r) = \{(z_1, z_2) \in \mathbb{C}^2 : |x_1|^2 + |x_2|^2 < r, |y_1|^2 + |y_2|^2 < r\}$ be the real bidisc of radius $r$. We denote $D(1) \times D(1)$ by $D^2(1)$ by $D^2_2$. We denote by $B^{2n}(a)$ the $2n$-dimensional Euclidean ball of radius $a$ in $\mathbb{R}^{2n}$.

It is proved by Sukhov and Tumanov [7] that the real bi-disc $D^2_2$ cannot be symplectically embedded into the complex cylinder $D \times C$. If we consider the real bidisc as obtained from a non-holomorphic change of coordinates

$$T_0 : (x_1, y_1, x_2, y_2) \mapsto (x_1, x_2, y_1, y_2)$$

of $D^2$, then the result of Sukhov and Tumanov shows that $T_0(D^2)$ is not symplectomorphic to $D^2$ itself. The first main result of this paper generalizes this observation: if $T \in O(4)$ is any orthogonal transformation on $\mathbb{R}^4 = \mathbb{C}^2$, then $T(D^2)$ is symplectomorphic to $D^2$ if and only if $T$ is unitary or conjugate to unitary. We will give a more precise statement in Section 3.

The second result of this paper considers a high dimensional analogy of the previous result. We will show that for $r \geq 1$ and $n \geq 2$, $D^2_2 \times D^{n-2}(r)$ is not symplectomorphic to $D^2 \times D^{n-2}(r)$.

The first striking result on symplectic rigidity was obtained by Gromov [3], which states that one can symplectically embed a sphere into a cylinder only if the radius of the sphere is less than or equal to the radius of the cylinder. Following Gromov’s

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work, many results on symplectic rigidity were obtained for various domains. For example, McDuff [5] studied when a 4-dimensional ellipsoid can be symplectically embedded in a ball; Guth [4] gave an asymptotic result on when a polydisc $D(r_1) \times \cdots \times D(r_n)$ can be symplectically embedded into another. Our results have the same spirit, but we deal with essentially different domains: real bidisc and its modifications.

There are a lot of open problems concerning symplectic rigidity, for instance it is not known that whether $D^2 \times D(r)$ is symplectomorphic to $D^2 \times D(r)$ when $r < 1$. The results in this paper only show that such symplectomorphism does not exist when $r \geq 1$. Another interesting open problem is to characterize when two given polydiscs are symplectomorphic.

2. J-holomorphic discs and symplectic manifolds

In this section we will recall some basic properties of J-holomorphic discs and symplectic manifolds.

Definition 2.1. A smooth map $\phi : (M, J) \to (M', J')$ from one almost complex manifold to another is said to be $(J, J')$-holomorphic if its derivative $d\phi$ is complex linear, that is

\[ d\phi \circ J = J' \circ d\phi. \]  

Denote by $J_{st}$ the standard complex structure of $\mathbb{C}^n$. A $J$-holomorphic disc or pseudo-holomorphic disc is a $(J_{st}, J)$-holomorphic map

$u : D \to M$

from $D$ to an almost complex manifold $(M, J)$.

In local coordinates $z \in \mathbb{C}^n$, an almost complex structure $J$ is represented by a $\mathbb{R}$-linear operator $J(z) : \mathbb{C}^n \to \mathbb{C}^n$ such that $J(z)^2 = -I$, where $I$ is the identity map. Now the Cauchy-Riemann equations (2.1) for a $J$-holomorphic disc $z : D \to \mathbb{C}^n$ can be written in the form

\[ z_\xi = J(z)z_\xi, \, \zeta = \xi + i\eta \in D. \]

We represent $J$ by a complex $n \times n$ matrix function $A = A(z)$ and obtain the equivalent equations

\[ z_\xi = A(z)^{-1}z_\xi \in D. \]  

We recall the relation between $J$ and $A$ for fixed $z$. Let $J : \mathbb{C}^n \to \mathbb{C}^n$ be a $\mathbb{R}$-linear map so that $\det(J_{st} + J) \neq 0$, where $J_{st}v = iv$. Set

\[ Q = (J_{st} + J)^{-1}(J_{st} - J). \]

Lemma 2.2. (1) $J^2 = -I$ if and only if $Q J_{st} + J_{st} Q = 0$.

Notice that $Q J_{st} + J_{st} Q = 0$ is equivalent to $Q$ being a complex anti-linear operator. Therefore Lemma 2.2 implies that there is a unique matrix $A \in \text{Mat}(n, \mathbb{C})$ such that

\[ Av = Q^2 v, \, v \in \mathbb{C}^n. \]

Let $M$ be a smooth manifold of real dimension $2n$. A closed non-degenerate exterior 2-form $\omega$ on $M$ is called a symplectic form on $M$. A couple $(M, \omega)$ is
called a symplectic manifold. A basic example is $M = \mathbb{C}^n$ with the coordinates $z_j = x_j + iy_j, j = 1, \ldots, n$. The form $\omega_\text{st} = \sum_{j=1}^{n} dx_j \wedge dy_j = \sum_{j=1}^{n} dz_j \wedge d\overline{z}_j$ is called the standard symplectic form on $\mathbb{C}^n$.

A symplectic form $\omega$ tames an almost complex structure $J$ on $M$ if $\omega(u, Ju) > 0$, for all $u \neq 0$. A basic example is $(M, \omega, J) = (\mathbb{C}^n, \omega_\text{st}, J_\text{st})$.

**Lemma 2.3.** Let $J$ be an almost complex structure on $\mathbb{C}^n$, then $J$ is tamed by $\omega_\text{st}$ if and only if the complex matrix $A$ of $J$ satisfies the condition

$$\|A(z)\| < 1, \text{ for all } z \in \mathbb{C}^n.$$  

Here the matrix norm is induced by the Euclidean inner product, that is, $\|A\| = \max_{0 \neq v \in \mathbb{R}^{2n}} |Av|_{\mathbb{R}^{2n}} / |v|_{\mathbb{R}^{2n}}$.

For a map $u : D \to \mathbb{C}^n$, the (symplectic) area of $u$ is given by

$$\text{Area}(u) = \int_{\overline{D}} u^* \omega_\text{st}.$$  

If $J$ is $\omega_\text{st}$ tamed, we can consider the canonical Riemannian metric $g_J(X,Y) = \frac{1}{2}(\omega_\text{st}(X, JY) + \omega_\text{st}(Y, JX))$ determined by $J$ and $\omega_\text{st}$. Suppose $u$ is a $J$-holomorphic disc, then the symplectic area of $u$ coincides with the area induced by $g_J$; in particular, it coincides with the Euclidean area if $J = J_\text{st}$ (see [1] for more details).

## 3. Orthogonal Transformation of Complex Bidisc

Let $T \in O(4)$ be an orthogonal transformation on $\mathbb{R}^4 \cong \mathbb{C}^2$, let $\mathbb{D}^2 = \{|z_j| < 1: j = 1, 2\}$ be the complex bidisc. In this section we will give a necessary and sufficient condition for $T(\mathbb{D}^2)$ to be symplectomorphic to $\mathbb{D}^2$ with respect to the standard symplectic form on $\mathbb{C}^2$.

First of all, we define the notion of holomorphic radius and state a theorem proved by A. Sukhov and A. Tumanov [7] which provides a necessary condition on holomorphic radius for the existence of symplectic embedding.

**Definition 3.1.** Let $\Omega$ be a complex manifold. A closed set $A \subset \Omega$ is called a (complex) analytic set if it is, in a neighborhood of each of its points, the set of common zeros of a certain finite family of holomorphic functions. In this paper we only consider closed analytic sets.

**Definition 3.2.** A point $p$ of an analytic set $A$ in a complex manifold $\Omega$ is called regular if there is a neighborhood $U$ in $\Omega$ containing $p$ such that $A \cap U$ is a complex submanifold of $U$. The complex dimension of this submanifold is said to be the dimension of $A$ at its regular point $p$, and is denoted by $\dim_p A$. The set of all regular points of $A$ is denoted by $\text{reg} A$.

It is a fundamental result of complex analytic sets that the set of all regular points of an analytic set $A$ is dense in $A$ (see, for example, [2]).

**Definition 3.3.** A purely $m$-dimensional analytic set $A$ is an analytic set such that for every $p \in \text{reg} A$, we have $\dim_p A = m$.

**Definition 3.4.** Let $G$ be a domain in $\mathbb{C}^n$ containing the origin. Denote by $O^1_0(G)$ the set of closed complex purely one-dimensional analytic sets in $G$ passing through
the origin. Denote by $E(X)$ the Euclidean area of $X \in \mathcal{O}_0^1(G)$. The holomorphic radius $\text{rh}(G)$ of $G$ is defined as

$$\text{rh}(G) = \inf\{\lambda > 0 : \exists X \in \mathcal{O}_0^1(G), E(X) = \pi \lambda^2\}.$$  

**Example 3.5.** Let $B^4(r)$ be the Euclidean ball of $\mathbb{C}^2$ with radius $r$, then $\text{rh}(B^4(r)) = r$. In fact the area $E(X)$ of $X \in \mathcal{O}_0^1(B^4(r))$ is bounded from below by the area $\pi r^2$ of a section of the ball by a complex line through the origin (Lelong, 1950; see [2]).

The following theorem is known as Bishop’s convergence theorem (see, for example, [2]), it will be used in the rest of the paper:

**Theorem 3.6.** Let $\{A_j\}$ be a sequence of purely $p$-dimensional analytic subsets in a complex manifold $\Omega$ with locally uniformly bounded volumes:

$$\text{Vol}_p(A_j \cap K) \leq M_K < \infty$$

for any compact set $K \subset \Omega$. Here $M_K$ is a constant depending only on $K$. Then we can extract a subsequence from $\{A_j\}$ converging on compact subsets in $\Omega$ (in Hausdorff sense) to a purely $p$-dimensional analytic subset or to the empty set.

The following result is due to A. Sukhov and A. Tumanov [7], it provides a necessary condition on holomorphic radius for the existence of symplectic embedding. This result will be used in the proof of Theorem 3.9.

**Theorem 3.7.** (7) Let $G_1$ be a domain in $\mathbb{C}^2$ containing the origin and let $G_2$ be a domain in $\mathbb{D}(R) \times \mathbb{C}$ for some $R > 0$. Assume there exists a symplectomorphism $\phi : G_1 \to G_2$, then $\text{rh}(G_1) \leq R$.

For $v = (v_1, \ldots, v_4), w = (w_1, \ldots, w_4) \in \mathbb{R}^4$, we denote the real inner product by $\langle v, w \rangle_{\mathbb{R}^4} = \sum_{j=1}^4 v_j w_j$. Similarly for $v = (v_1, v_2), w = (w_1, w_2) \in \mathbb{C}^2$, we denote the complex inner product by $\langle v, w \rangle_{\mathbb{C}^2} = \sum_{j=1}^2 v_j w_j$. Notice that $\langle v, w \rangle_{\mathbb{R}^4} = \text{Re} \langle v, w \rangle_{\mathbb{C}^2}$.

By using the properties of inner product, the following lemma can be proved easily.

**Lemma 3.8.**

1. Let $L \in \mathbb{C}^2$ be a real two dimensional plane. Denote by $L_{-x^2}$ the orthogonal complement of $L$ with respect to $\langle \cdot, \cdot \rangle_{\mathbb{R}^4}$ and by $L_{-c^2}$ the orthogonal complement of $L$ with respect to $\langle \cdot, \cdot \rangle_{\mathbb{C}^2}$. If $L$ is a complex line, that is $v \in L$ if and only if $iv \in L$ for all $v \in \mathbb{C}^2$, then $L_{-x^2} = L_{-c^2}$.

2. If $L \in \mathbb{C}^2$ is a complex line, then $L_{-c^2}$ is also a complex line.

We denote by $\mathcal{J}$ the set consisting of four diagonal matrices:

$$J = \left\{ \begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix} : a = \pm 1, b = \pm 1 \right\}$$

The following is the main theorem of this section. We used the canonical identification between complex matrices on $\mathbb{C}^2$ and real matrices on $\mathbb{R}^4$.

**Theorem 3.9.** Let $T \in O(4)$ be an orthogonal transformation. $T \mathbb{D}^2$ is symplectomorphic to $\mathbb{D}^2$ with respect to the standard symplectic form on $\mathbb{R}^4$ if and only if there exists $U \in U(2)$ such that $UT \in \mathcal{J}$. 

Proof: \( \iff \) Suppose there exists an \( U \in U(2) \) such that \( UT \in \mathfrak{I} \), then we know that \( UT \mathbb{D}^2 = \mathbb{D}^2 \) as a set. Furthermore \( U \in U(2) \) is a linear symplectomorphism on \( \mathbb{C}^2 \). Hence \( T \mathbb{D}^2 \) is symplectomorphic to \( \mathbb{D}^2 \).

\( \Rightarrow \) Let \( (z_1, z_2) \) be the coordinate on \( \mathbb{C}^2 \). First of all, let \( \partial[\mathbb{D}^2 \cap \partial \mathbb{B}^4(1)] = S_1 \cup S_2 \) where \( S_1 = \{ |z_1| = 1, z_2 = 0 \} \) and \( S_2 = \{ z_1 = 0, |z_2| = 1 \} \). Therefore \( S_1 \) and \( S_2 \) are contained in the complex line \( H_1 = \{ z_2 = 0 \} \) and \( H_2 = \{ z_1 = 0 \} \) respectively. For \( i = 1, 2 \), let \( u_i, v_i \in \mathbb{C}^2 \) be orthonormal basis of \( TH_i \) under the real inner product \( \langle \cdot, \cdot \rangle_{\mathbb{R}} \) on \( \mathbb{R}^4 \). Note that \( TS_i \) can be parameterized by

\[
\frac{1}{2} \left( t + \frac{1}{t} \right) u_i + \frac{1}{2t} \left( t - \frac{1}{t} \right) v_i
\]

for \( |t| = 1 \) in \( \mathbb{C} \). The complexification of \( TS_i \), denoted by \( \overline{TS}_i \), is given by the same parametrization but allowing \( t \in \mathbb{C} \). Here \( \mathbb{C}^n \) is the complex projective space of complex dimension \( n \). Hence \( \overline{TS}_i \) is a complex algebraic curve in \( \mathbb{C}^2 \) parameterized by \( t \in \mathbb{C}^n \).

Notice that for \( i = 1, 2 \), \( \overline{TS}_i \) passes through the origin in \( \mathbb{C}^2 \) if and only if \( u_i \) and \( v_i \) are \( \mathbb{C} \)-dependent.

Suppose \( T \mathbb{D}^2 \) is symplectomorphic to \( \mathbb{D}^2 \), then Theorem 4.1 implies that \( \text{rh}(T \mathbb{D}^2) \leq 1 \). By Theorem 4.1, there exists \( X \in \mathcal{O}_0(\mathbb{D}^2) \) such that \( E(X) = \pi(\text{rh}(T \mathbb{D}^2))^2 \). Suppose there exist \( p \in \partial X \cap \partial \mathbb{B}^4(1) \) such that \( p \in \text{Int}(T \mathbb{D}^2) \); then \( X \) is not entirely contained in \( \mathbb{B}^4(1) \). Hence \( E(X) > E(X \cap \mathbb{B}^4(1)) \geq \pi \) (see Example 4.2), which implies \( \text{rh}(T \mathbb{D}^2) > 1 \), a contradiction. Therefore \( \partial X \subset \partial \mathbb{B}^4(1) \cap \partial T \mathbb{D}^2 = TS_1 \cup TS_2 \)
and \( X \) is a complex one dimensional analytic subset in \( \mathbb{C}^2 \setminus (TS_1 \cup TS_2) \). Since \( TS_1 \cup TS_2 \) is a real one dimensional curve, it is totally real. Hence, by the reflection principle for analytic sets (see, for example, Section 20.5 of [2]), \( X \) extends as a complex one dimensional analytic set to a neighborhood of \( TS_1 \cup TS_2 \). By the uniqueness theorem \( X \) is contained in the complex algebraic curve \( \overline{TS}_1 \cup \overline{TS}_2 \).

Since \( X \) contains the origin in \( \mathbb{C}^2 \), without loss of generality we can assume \( \overline{TS}_i \) contains the origin. By the discussion above, we know that \( u_1 \) and \( v_1 \) are \( \mathbb{C} \)-dependent. Hence \( TH_1 = \text{span}_{\mathbb{R}} \{ u_1, v_1 \} = \text{span}_{\mathbb{C}} \{ u_1, iv_1 \} = \text{span}_{\mathbb{C}} \{ u_1 \} = \overline{TS}_1 \).
This shows that \( TH_1 \) is a complex line.

By Lemma 3.8, \( H_2 = H_1^{1-c^2} = H_1^{1+s^2} \). Since \( T \) is an orthogonal matrix, we have \( TH_2 = (TH_1)^{1+s^2} = (TH_1)^{1-c^2} \). where the last equality follows from Lemma 3.8 and the fact that \( TH_1 = \overline{TS}_1 = \text{span}_{\mathbb{C}} \{ u_1 \} \) is a complex line. Therefore Lemma 3.8 implies that \( TH_2 \) is a complex line.

We’ve shown that if \( T \) is orthogonal and \( T \mathbb{D}^2 \) is symplectomorphic to \( \mathbb{D}^2 \), then \( T \) maps the complex lines \( H_1 = \{ z_2 = 0 \}, H_2 = \{ z_1 = 0 \} \) to complex lines \( TH_1, TH_2 \). Therefore there exists a unitary matrix \( U \in U(2) \) such that \( UT \in \mathfrak{I} \). \( \square \)

4. SYMPLECTIC RIGIDITY IN HIGH DIMENSIONAL CASE

Let \( \mathbb{D}^m(r) = \{ (z_1, \ldots, z_m) \in \mathbb{C}^m : |z_j| < r \text{ for } j = 1, \ldots, m \} \) be the \( m \)-th product of discs of radius \( r \). The following is the main theorem in this section:

**Theorem 4.1.** For \( r \geq 1 \) and \( n \geq 2 \), the domains \( \mathbb{D}_R^2 \times \mathbb{D}^{n-2}(r) \) and \( \mathbb{D}^2 \times \mathbb{D}^{n-2}(r) \) in \( \mathbb{C}^n \) equipped with the standard symplectic form are not symplectomorphic.

We will first give the proof for the case \( r > 1 \) by adapting the idea in the proof of Theorem 2.2 in [3]. We will then develop a new method to prove Theorem 4.1 for the case \( r = 1 \).
4.1. The case $r > 1$. In the case $r > 1$, theorem 4.1 follows from a more general result:

**Theorem 4.2.** Given $r > 1$, for any real number $R > 0$, if there exists a symplectic embedding $\phi : \mathbb{D}_R^2 \times \mathbb{D}^{n-2} \to \mathbb{D}(R) \times \mathbb{C}^{n-1}$, then $R > 1$.

**Proof.** For $R > 0$, suppose there exists a symplectic embedding from $\mathbb{D}_R^2 \times \mathbb{D}^{n-2}(r)$ into $\mathbb{D}(R) \times \mathbb{C}^{n-1}$. It is proved in [3] that for every $1 \leq r_1 < \frac{R}{\sqrt{n}}$, there is a symplectic embedding from $\mathbb{B}^4(r_1)$ into $\mathbb{D}_R^2$. Take $1 < r_1 < \frac{R}{\sqrt{n}}$, then by combining these two embeddings, we obtain an embedding from $\mathbb{B}^{2n}(a)$ into $\mathbb{D}(R) \times \mathbb{C}^{n-1}$ where $a = \min(r, r_1) > 1$. Therefore, by Gromov’s non-squeezing theorem [3], we have $R > 1$. \qed

4.2. The case $r = 1$. In order to prove the case $r = 1$, we need the following theorem regarding the existence of $J$-holomorphic discs, which is due to A. Sukhov and A. Tumanov [6]. The original statement was about the triangular cylinder $\Delta \times \mathbb{C}^{n-1}$ where $\Delta = \{ z \in \mathbb{C} : 0 < \text{Im} z < 1 - |\text{Re} z| \}$ instead of the circular cylinder $\mathbb{D} \times \mathbb{C}^{n-1}$. However, one can see the result still holds for the circular cylinder by applying an area preserving map of the triangle to the disc.

**Theorem 4.3.** (A. Sukhov and A. Tumanov [6]) Let $A$ be a continuous $n \times n$ matrix function on $\mathbb{C}^n$ with compact support in $\mathbb{D} \times \mathbb{C}^{n-1}$. Suppose there is a constant $0 < a < 1$ such that

\[
\|A(z)\| \leq a, \forall z \in \mathbb{D} \times \mathbb{C}^{n-1}.
\]

Then there exists $p > 2$ such that for every point $x \in \mathbb{D} \times \mathbb{C}^{n-1}$ there is a solution $Z \in W^{1,p}(\mathbb{D})$ (Sobolev space) of equation (4.2)

\[
Z_\gamma = A(Z)Z_\gamma
\]

such that $Z(\mathbb{D}) \subset \overline{\mathbb{D} \times \mathbb{C}^{n-1}}$, $x \in Z(\mathbb{D})$, $\text{Area}(Z) = \pi$ and $Z(\partial \mathbb{D}) \subset \partial(\mathbb{D} \times \mathbb{C}^{n-1}) = (\partial \mathbb{D}) \times \mathbb{C}^{n-1}$.

Furthermore, if we denote the components of $Z$ by $Z = (f_1, \ldots, f_n)$, then we have the following area property

\[
\text{Area}(f_1) = \pi, \text{Area}(f_j) = 0, \text{ for } j = 2, \ldots, n.
\]

For $1 \leq j \leq n$, let $M_j$ be the holomorphic disc $M_j = (m_1, \ldots, m_n) : \mathbb{D} \to \mathbb{C}^n$ where $m_k(z) = 0$ if $k \neq j$ and $m_j(z) = z$. Notice that the minimal area of an analytic set passing through the origin in $\mathbb{D}_R^2 \times \mathbb{D}^{n-2}$ is $\pi$, this is because $\mathbb{B}^{2n} \subset \mathbb{D}_R^2 \times \mathbb{D}^{n-2}$ and the minimal area of analytic set of $\mathbb{B}^{2n}$ passing through the origin is $\pi$ (Lelong 1950; see [2]).

**Lemma 4.4.** The minimal analytic set of $\mathbb{D}_R^2 \times \mathbb{D}^{n-2}$ through the origin is given by one of the $n - 2$ distinct holomorphic discs $M_3, \ldots, M_n$.

**Proof.** Let $S_1 = \{ x_1^2 + x_2^2 = 1, y_1 = y_2 = 0, z_3 = \cdots = z_n = 0 \}$, $S_2 = \{ y_1^2 + y_2^2 = 1, x_1 = x_2 = 0, z_3 = \cdots = z_n = 0 \}$, $S_j = \{|z_j| = 1, z_k = 0 \text{ for } k \neq j \}$ for $3 \leq j \leq n$. By using Lelong’s result (see [2]) and the argument in proof of Theorem 3.9, we conclude that the boundary of the analytic set $E$ of minimal area in $\mathbb{D}_R^2 \times \mathbb{D}^{n-2}$ through the origin must lie in the intersection of the boundary of $\mathbb{B}^{2n}$ and the boundary of $\mathbb{D}_R^2 \times \mathbb{D}^{n-2}$, notice that this intersection consists of $n$ circles $S_1, \ldots, S_n$. 
Suppose a boundary point of $E$ lies in $S_1 \cup S_2$, then $E$ must have a component lying in the complexification of $S_1 \cup S_2$, which is given by \( \{ z_1^2 + z_2^2 = 1, z_3 = \cdots = z_n = 0 \} \), in fact all of $E$ lies in this set since $E$ is of minimal area. However \( \{ z_1^2 + z_2^2 = 1, z_3 = \cdots = z_n = 0 \} \) does not pass through the origin, so the boundary of $E$ is contained in the circles $S_3 \cup \cdots \cup S_n$. Hence $E$ is one of the discs $M_3, \ldots, M_n$. \( \square \)

The following lemma is a consequence of Lemma 4.1 and Theorem 3.6.

**Lemma 4.5.** Let $E_j$ be a convergent sequence of analytic sets in $\mathbb{D}^2_\mathbb{R} \times \mathbb{D}^{n-2}$ passing through the origin so that

$$\lim_{j \to \infty} \text{Area}(E_j) = \pi.$$

Then the limiting analytic set $E_\infty$ is one of the $n-2$ distinct holomorphic discs $M_3, \ldots, M_n$.

Our proof of Theorem 4.1 in the case $r = 1$ is based on the fact that the domains $\mathbb{D}^2_\mathbb{R} \times \mathbb{D}^{n-2}$ and $\mathbb{D}^n$ have different number of analytic sets of minimum area through the origin. We are now ready to prove the main theorem in this section.

**Theorem 4.6.** The domains $\mathbb{D}^2_\mathbb{R} \times \mathbb{D}^{n-2}$ and $\mathbb{D}^n$ equipped with the standard symplectic form on $\mathbb{C}^n$ are not symplectomorphic.

**Proof.** Suppose on the contrary that $\psi : \mathbb{D}^2_\mathbb{R} \times \mathbb{D}^{n-2} \to \mathbb{D}^n$ is a symplectomorphism. By composing a symplectomorphism of $\mathbb{D}^n$, we can assume that $\psi(0) = 0$.

Consider the standard almost complex structure $J_{st}$ on $\mathbb{D}^2_\mathbb{R} \times \mathbb{D}^{n-2}$ and let $J = \psi^*J_{st}$ be the complex structure on $\mathbb{D}^n$ given by the push-forward of $J_{st}$ by $\psi$. Since $\psi^*\omega_{st} = \omega_{st}$, the almost complex structure $J$ is tamed by $\omega_{st}$. Then the complex matrix $A$ of $J$ satisfies $\|A(z)\| < 1$ for $z \in \mathbb{D}^n$.

Let $\{K_l\}_{l=1}^\infty$ be a compact exhaustion of $\mathbb{D}^n$ so that each $K_l$ is a closed polydisc with radius less than 1, that is, $K_l \subset K_{l+1}$, $K_1$ is a compact subset of $\mathbb{D}^n$ for all $l$ and $\cup_{l=1}^\infty K_l = \mathbb{D}^n$. For each $l$, let $\chi_l$ be a smooth cut-off function on $\mathbb{C}^n$ with support in $\mathbb{D}^n$ and equal to 1 on $K_l$. Define $A_l = \chi_l A$ to be a $n \times n$ matrix function on $\mathbb{C}^n$ such that $A_l = 0$ outside $\mathbb{D}^n$. Since $\|A\| < 1$ on $\mathbb{D}^n$, there is a constant $0 < a < 1$ such that (4.1) holds for $A_l$. Let $J_l$ be the almost complex structure on $\mathbb{C}^n$ corresponding to the complex matrix $A_l$.

By considering $\mathbb{D}^n$ as a subset of $\mathbb{D} \times \mathbb{C}^{n-1}$, we can apply Theorem 4.3 so that for each $l$, there exists a $J_l$-holomorphic disc $f_l : \mathbb{D} \to \mathbb{D} \times \mathbb{C}^{n-1}$ such that the image of $f_l$ passes through the origin. Also if we write $f_l = (f_{l,1}, \ldots, f_{l,n})$, then we have $\text{Area}(f_{l,j}) = \delta_{j1} \pi$ for all $l$, here $\delta_{j1}$ is the Kronecker delta.

Fix an integer $N$, for each $l \geq N$, $\psi^{-1}(f_l(\mathbb{D}) \cap K_N)$ is an analytic set in $\psi^{-1}(K_N) \subset \mathbb{D}^2_\mathbb{R} \times \mathbb{D}^{n-2}$ passing through the origin. Since $\psi$ is a symplectomorphism, we have

$$\text{Area}(\psi^{-1}(f_l(\mathbb{D}) \cap K_N)) \leq \text{Area}(f_l(\mathbb{D}) \cap K_N) \leq \pi.$$ 

Therefore by Theorem 3.6 after passing to a subsequence, $F_N = \lim_{l \to \infty} \psi^{-1}(f_l(\mathbb{D}) \cap K_N)$ exists and $\text{Area}(F_N) \leq \pi$. Notice that $F_N$ is not an empty set for $N$ sufficiently large, this is because $0 \in \psi^{-1}(f_l(\mathbb{D}) \cap K_N)$ for all $l \geq N$.

The above argument holds for all $N$, so we can apply Theorem 3.6 again to the sequence of analytic set $F_N$ as $N \to \infty$. After passing to a subsequence, denote the
limit of $F_N$ by $F$. Now $F$ is an analytic set in $\mathbb{D}_R^2 \times \mathbb{D}^{n-2}$ passing through the origin with $\text{Area}(F) \leq \pi$ and $\partial F \subset \partial (\mathbb{D}_R^2 \times \mathbb{D}^{n-2})$. Since the minimal area of analytic set in $\mathbb{D}_R^2 \times \mathbb{D}^{n-2}$ through the origin is $\pi$, so we must have $\text{Area}(F) = \pi$. Therefore $F$ is one of the holomorphic discs $M_j$ for $3 \leq j \leq n$ by Lemma 4.5.

Let $E = \psi(F)$. We now know that $\text{Area}_{f_l} = \pi$ for all $l$ and $f_l(\mathbb{D}) \cap \mathbb{D}^n \to E$ as $l \to \infty$, also we have $\text{Area}(E) = \pi$. We want to show that $f_l(\mathbb{D}) \to E$ as $l \to \infty$. Let $X_l = f_l(\mathbb{D}) \setminus \mathbb{D}^n$, that is the image of $f_l$ which is not in $\mathbb{D}^n$. By the construction of $A_l$ and $J_l$, we know that $J_l = J_{st}$ outside $\mathbb{D}^n$, hence $X_l$ is an usual analytic set in $(\mathbb{D} \times \mathbb{C}^{n-1}) \setminus \mathbb{D}^n$. Since $\text{Area}_{X_l} \leq \text{Area}_{f_l} = \pi$ for all $l$, we can apply Theorem 3.6 to conclude that, after passing to a subsequence, $X_l$ converges to an analytic set $X$. However $f_l(\mathbb{D}) \cap \mathbb{D}^n \to E$ as $l \to \infty$ and $\text{Area}(E) = \pi$ implies that

$$\lim_{l \to \infty} \text{Area}(f_l(\mathbb{D}) \cap \mathbb{D}^n) = \pi,$$

and by construction $\text{Area}(f_l) = \pi$ for all $l$, hence we have $\text{Area}(X_l) \to 0$ as $l \to \infty$. Therefore $X$ is an empty set and we can conclude that

$$\lim_{l \to \infty} f_l(\mathbb{D}) \subset \mathbb{D}^n,$$

and hence

$$\lim_{l \to \infty} f_l(\mathbb{D}) = E.$$

Since $\text{Area}(f_{l,j}) = \delta_{j,l} \pi$, if we write $\omega_{st} = \omega_1 + \cdots + \omega_n$ where $\omega_j = dx_j \wedge dy_j$ for $j = 1, \ldots, n$, then we have

$$\int_E \omega_j = \delta_{j,l} \pi.$$

Now for $1 \leq k \leq n$, by considering $\mathbb{D}^n$ as a subset of the cylinder $\mathbb{C}^{k-1} \times \mathbb{D} \times \mathbb{C}^{n-k} \cong \mathbb{D} \times \mathbb{C}^{n-1}$, we can apply the above argument to obtain a real 2-dimensional set $E_k$ in $\mathbb{D}^n$ passing through the origin, satisfying the following conditions:

1. $\int_{E_k} \omega_j = \delta_{j,k} \pi$ for $j = 1, \ldots, n$, hence all $E_k$ are distinct for $1 \leq k \leq n$.
2. The preimage $F_k = \psi^{-1}(E_k)$ is an analytic set in $\mathbb{D}_R^2 \times \mathbb{D}^{n-2}$ passing through the origin.
3. $F_k$ are distinct analytic sets for $1 \leq k \leq n$ since $E_k$’s are distinct and $\psi$ is a bijection.
4. $\text{Area}(F_k) = \pi$ for $1 \leq k \leq n$.

Hence for each $1 \leq k \leq n$, $F_k$ must be one of the holomorphic discs $M_j$ for $3 \leq j \leq n$ according to Lemma 4.5 but this is impossible since all $F_k$’s are distinct, so we arrived at a contradiction. Therefore $\mathbb{D}_R^2 \times \mathbb{D}^{n-2}$ and $\mathbb{D}^n$ equipped with the standard symplectic form on $\mathbb{C}^n$ are not symplectomorphic.

□

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