On properties of the space of quantum states and their application to construction of entanglement monotones

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Abstract. We consider two properties of the set of quantum states as a convex topological space and some their implications concerning the notions of a convex hull and of a convex roof of a function defined on a subset of quantum states.

By using these results we analyze two infinite-dimensional versions (discrete and continuous) of the convex roof construction of entanglement monotones, which is widely used in finite dimensions. It is shown that the discrete version may be 'false' in the sense that the resulting functions may not possess the main property of entanglement monotones while the continuous version can be considered as a 'true' generalized convex roof construction. We give several examples of entanglement monotones produced by this construction. In particular, we consider an infinite-dimensional generalization of the notion of Entanglement of Formation and study its properties.

Keywords: convex hull and convex roof of a function, quantum state, entanglement monotone, entanglement of formation.

Introduction

In study of finite dimensional quantum systems and channels such notions of the convex analysis as the convex hull and the convex closure (called also the convex envelope) of a function defined on the set of quantum states as well as the convex roof of a function defined on the set of pure quantum states (introduced in [1] as a special convex extension of this function to the set of all quantum states) are widely used. The last notion plays the basic role in construction of entanglement monotones – functions on the set of states of a composite quantum system characterizing entanglement of these states [2], [3].

The main problem of using these functional constructions in the infinite dimensional case consists in necessity to apply them to functions with singular properties such as discontinuity and unboundedness (including possibility of the infinite values). For instance, the von Neumann entropy – one of the main characteristics of quantum states – is a continuous and bounded function in finite dimensions, but

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it takes the value $+\infty$ on a dense subset of the set of states of infinite dimensional quantum system. The other problems are noncompactness of the set of quantum states and nonexistence of inner points of this set (considered as a subset of the Banach space of trace class operators). All these features lead to very "unnatural" behavior of the above functional constructions and to breaking validity of several "elementary" results (for example, the well known Jensen's inequality may not hold for a measurable convex function). So, a special analysis is required to overcome these problems. The main tools of this analysis are the following two properties of the set of quantum states as a convex topological space:

1) the weak compactness of the set of measures, whose barycenters form a compact set,

2) the openness of the barycenter map (in the weak topology),

proved in [4] and [5] respectively and described in detail in §1. These properties reflect the special relations between the topology and the convex structure of the set of quantum states.

In §2 the infinite dimensional versions of the notions of the convex hull of a function defined on the set $\mathcal{S}(\mathcal{H})$ and of the convex roof of a function defined on the set $\mathcal{E}(\mathcal{H})$ are considered. Their continuity properties are explored. Continuity of the operation of convex closure with respect to monotone pointwise convergence on the class of lower semicontinuous lower bounded functions on $\mathcal{S}(\mathcal{H})$ is proved.

In §3 sufficient conditions for continuity and for coincidence of restrictions of different convex hulls of a given function to the set of states with bounded mean generalized energy (nonnegative lower semicontinuous affine function) are obtained. This result implies several useful properties of the output Renyi entropy (in particular, of the output von Neumann entropy) of a quantum channel.

In §4 applications of the obtained results to the theory of entanglement in composite quantum system are considered [6]. The two infinite dimensional versions (discrete and continuous) of the convex roof construction of entanglement monotones widely used in finite dimensions are considered. It is shown that the discrete version may be "false" in the sense that the functions constructed by using this method may not possess the main property of entanglement monotones (even if the generating function is bounded and lower semicontinuous), while the continuous version produces "true" entanglement monotones under weak requirements on the generating functions. So, the last method is considered as a generalized convex roof construction. It can be applied to obtain infinite dimensional generalization of the Entanglement of Formation (EoF) – one the basic entanglement measures in finite dimensional composite quantum systems [7]. Comparison of this approach to generalization of EoF with the approach proposed in [8] is considered in §5.

§1. Preliminaries

Let $\mathcal{H}$ be a separable Hilbert space, $\mathcal{B}(\mathcal{H})$ – the algebra of all linear bounded operators in $\mathcal{H}$, $\mathcal{B}_{h}(\mathcal{H})$ – the Banach space of bounded Hermitian operators in $\mathcal{H}$ containing the cone $\mathcal{B}_{+}(\mathcal{H})$ of positive operators, $\mathcal{Z}(\mathcal{H})$ and $\mathcal{Z}_{h}(\mathcal{H})$ – the separable Banach spaces of all trace class operators in $\mathcal{H}$ and of all trace class Hermitian operators with the trace norm $\| \cdot \|_{1} = \text{Tr} | \cdot |$ (cf. [9]).
The closed subsets
\[ \mathcal{I}_1(\mathcal{H}) = \{ A \in \mathcal{I}(\mathcal{H}) \mid A \geq 0, \text{Tr} A \leq 1 \}, \quad \mathcal{G}(\mathcal{H}) = \{ A \in \mathcal{I}_1(\mathcal{H}) \mid \text{Tr} A = 1 \} \]
of \( \mathcal{I}(\mathcal{H}) \) are complete separable metric spaces with the metric defined by the trace norm. An operator \( \rho \) in \( \mathcal{G}(\mathcal{H}) \) determines the linear functional \( A \mapsto \text{Tr} A \rho \) on the algebra \( \mathcal{B}(\mathcal{H}) \) called state in the theory of operator algebras. So, in what follows, we will use the term state for operators in \( \mathcal{G}(\mathcal{H}) \). The rank of a positive operator (state) is the dimension of the orthogonal complement of its kernel.

We will denote by \( \text{co} \ A \) (correspondingly, \( \text{co} A \)) the convex hull (correspondingly, closure) of a set \( A \) \[10\]. We will denote by \( \text{extr} A \) the set of all extreme points of a convex set \( A \).

We will denote by \( \mathcal{P}(A) \) the set of all Borel probability measures on a complete separable metric space \( A \) endowed with the topology of weak convergence. This set can be considered as a complete separable metric space as well \[11\], Ch. II, § 6. The subset of \( \mathcal{P}(A) \) consisting of measures with finite support will be denoted \( \mathcal{P}^f(A) \). In what follows we will also use the abbreviations \( \mathcal{P} = \mathcal{P}(\mathcal{G}(\mathcal{H})) \), \( \hat{\mathcal{P}} = \mathcal{P}(\text{extr} \mathcal{G}(\mathcal{H})) \).

The barycenter of the measure \( \mu \in \mathcal{P} \) is the state defined by the Bochner integral
\[ \bar{\rho}(\mu) = \int_{\mathcal{G}(\mathcal{H})} \sigma \mu(d\sigma). \]

For an arbitrary subset \( A \subset \mathcal{G}(\mathcal{H}) \) denote by \( \mathcal{P}A \) (correspondingly, by \( \hat{\mathcal{P}}A \)) the subset of \( \mathcal{P} \) (correspondingly, of \( \hat{\mathcal{P}} \)), consisting of all measures with the barycenter in \( A \).

A finite or countable collection of states \( \{ \rho_i \} \) with corresponding probability distribution \( \{ \pi_i \} \) is conventionally called ensemble and is denoted \( \{ \pi_i, \rho_i \} \). In this paper we will consider ensemble of states as a particular case of probability measure on the set of quantum states.

The von Neumann entropy of a state \( \rho \) and the relative entropy of states \( \rho \) and \( \sigma \) are defined respectively by the expressions
\[ H(\rho) = -\sum_i \langle i | \rho \log \rho | i \rangle, \quad H(\rho \| \sigma) = \sum_i \langle i | (\rho \log \rho - \rho \log \sigma) | i \rangle, \]
where \( \{ | i \rangle \} \) is a basis of eigenvectors of \( \rho \), and it is assumed that \( H(\rho \| \sigma) = +\infty \) if the support of \( \rho \) (the orthogonal complement of the kernel of the operator \( \rho \)) is not contained within the support of the state \( \sigma \). The entropy and the relative entropy are lower semicontinuous functions of their arguments taking values in \([0, +\infty]\).

The first of them is concave while the second one is jointly convex \[12\].

An arbitrary positive unbounded operator \( H \) in a space \( \mathcal{H} \) with discrete spectrum of finite multiplicity will be called \( \mathcal{H} \)-operator.

The set of quantum states \( \mathcal{G}(\mathcal{H}) \) has the following two properties:
A) for an arbitrary compact subset \( A \subset \mathcal{G}(\mathcal{H}) \) the set \( \mathcal{P}A(\mathcal{G}(\mathcal{H})) \) is compact (see \[14\]);
B) the barycenter map \( \mathcal{P}(\mathcal{G}(\mathcal{H})) \ni \mu \mapsto \bar{\rho}(\mu) \in \mathcal{G}(\mathcal{H}) \) is an open surjection (see \[3\], \[13\]).

Property A) provides generalization to the case of \( \mathcal{G}(\mathcal{H}) \) of some well known results concerning compact convex sets (see \[14\], Lemma 1, or the below Propositions...
and hence it may be considered as a kind of "weak" compactness. In fact, this property is not purely topological (in contrast to compactness), but it reflects the special relation between the topology and the convex structure of the set $\mathcal{S}(\mathcal{H})$. Following [13], [15], we will call it the \textit{\(\mu\)-compactness property}.

Note that the \(\mu\)-compactness of the positive part of the unit ball is a specific feature of the Banach space of trace class operators (the Shatten class of order \(p = 1\)) within the family of Shatten classes of order \(p \geq 1\).

Moreover, it can be shown that the set $\mathcal{S}_1(\mathcal{H})$ loses the \(\mu\)-compactness property being endowed with the $\| \cdot \|_p$-norm topology with $p > 1$ and that in the Shatten class of order $p = 2$ (the Hilbert space of Hilbert-Schmidt operators) there exists no \(\mu\)-compact set which is not compact. These and other results concerning the \(\mu\)-compactness property as well as examples of \(\mu\)-compact sets are considered in [15].

Property B) reflects another relation between the topology and the convex structure of the set $\mathcal{S}(\mathcal{H})$. The characterization of the analog of this property for arbitrary \(\mu\)-compact convex set is obtained in [13], Theorem 1. By this theorem property B) is equivalent to continuity of the convex hull of any continuous bounded function on the set $\mathcal{S}(\mathcal{H})$ and to openness of the map

$$\mathcal{S}(\mathcal{H}) \times \mathcal{S}(\mathcal{H}) \times [0, 1] \ni (\rho, \sigma, \lambda) \mapsto \lambda \rho + (1 - \lambda)\sigma \in \mathcal{S}(\mathcal{H}).$$

The analog of the last property for any convex set seems to be the simplest for verification and (its equivalent but formally stronger form) is called the \textit{stability} property (see [17], [18] and references therein).

\section{The convex hulls and the convex roofs}

In this section we consider several notions and constructions for functions defined on the set $\mathcal{S}(\mathcal{H})$. Note that the all definitions are universal, they can be formulated in terms of functions defined on a convex closed bounded subset $A$ of a locally convex space (instead of $\mathcal{S}(\mathcal{H})$). So, the main results obtained in this section can be proved in this extended context under the particular conditions imposed on $A$ (which are valid for $\mathcal{S}(\mathcal{H})$). Possibilities of such generalizations are discussed in the Appendix.

\subsection{Several notions of convexity of a function}

In what follows we will consider functions on the set $\mathcal{S}(\mathcal{H})$ taking values in $[-\infty, +\infty]$, which are \textit{semibounded} (lower or upper bounded) on this set.

We will use the following two strengthened versions of the well known notion of a convex function.

A semibounded function $f$ on the set $\mathcal{S}(\mathcal{H})$ is called \(\sigma\)-\textit{convex} if

$$f\left(\sum_i \pi_i \rho_i\right) \leq \sum_i \pi_i f(\rho_i)$$

for any \textit{countable} ensemble $\{\pi_i, \rho_i\}$ of states in $\mathcal{S}(\mathcal{H})$.\footnote{This theorem is a partial noncompact generalization of the results in [16], concerning compact convex sets. The complete generalization of these results to the class of \(\mu\)-compact convex sets is obtained in [15].}
A semibounded universally measurable\footnote{This means that the function f is measurable with respect to any measure in $P(\mathcal{S}(\mathcal{H}))$.} function $f$ on the set $\mathcal{S}(\mathcal{H})$ is called $\mu$-convex if

$$f\left(\int_\mathcal{S}(\mathcal{H}) \rho \mu(d\rho)\right) \leq \int_\mathcal{S}(\mathcal{H}) f(\rho) \mu(d\rho)$$

for any measure $\mu$ in $P(\mathcal{S}(\mathcal{H}))$.

The simplest example of a convex Borel function on the set $\mathcal{S}(\mathcal{H})$, which is not $\sigma$-convex and $\mu$-convex, is the function taking the value 0 on the convex set of finite rank states and the value $+\infty$ on set of infinite rank states. Difference between the above convexity properties can be also illustrated by functions in the below examples $[1, 2]$ (the first of them is convex but not $\sigma$-convex while the second one is $\sigma$-convex but not $\mu$-convex).

Convexity implies $\sigma$-convexity for all upper bounded functions on $\mathcal{S}(\mathcal{H})$ (Proposition A-1 in the Appendix).

By the integral Jensen’s inequality (Proposition A-2 in the Appendix) all these convexity properties are equivalent for the classes of lower semicontinuous functions and of upper bounded upper semicontinuous functions on the set $\mathcal{S}(\mathcal{H})$.

\subsection*{2.2. The convex hulls and the convex closure.}

The convex hull $\text{co} f$ of a semibounded function $f$ on the set $\mathcal{S}(\mathcal{H})$ is defined as the greatest convex function majorized by $f$, which means that

$$\text{co} f(\rho) = \inf_{\{\pi_i, \rho_i\} \in \mathcal{P}(\rho)} \sum_i \pi_i f(\rho_i), \quad \rho \in \mathcal{S}(\mathcal{H})$$  \hspace{1cm} (1)

(the infimum is over all finite ensembles $\{\pi_i, \rho_i\}$ of states with the average state $\rho$).

The $\sigma$-convex hull $\sigma$-co $f$ of a semibounded function $f$ on the set $\mathcal{S}(\mathcal{H})$ is defined as follows

$$\sigma$-co $f(\rho) = \inf_{\{\pi_i, \rho_i\} \in \mathcal{P}(\rho)} \sum_i \pi_i f(\rho_i), \quad \rho \in \mathcal{S}(\mathcal{H})$$  \hspace{1cm} (2)

(the infimum is over all countable ensembles $\{\pi_i, \rho_i\}$ of states with the average state $\rho$). The function $\sigma$-co $f$ is $\sigma$-convex, since for any countable ensemble $\{\lambda_i, \sigma_i\}$ with the average state $\sigma$ and any family $\{\{\pi_{i,j}, \rho_{i,j}\}\}_{i,j}$ of countable ensembles such that $\sigma_i = \sum_j \pi_{i,j} \rho_{i,j}$ for all $i$ the countable ensemble $\{\lambda_i \pi_{i,j}, \rho_{i,j}\}_{i,j}$ has the average state $\sigma$. Thus $\sigma$-co $f$ is the greatest $\sigma$-convex function majorized by $f$.

The $\mu$-convex hull $\mu$-co $f$ of a Borel semibounded function $f$ on the set $\mathcal{S}(\mathcal{H})$ is defined as follows

$$\mu$-co $f(\rho) = \inf_{\mu \in \mathcal{P}(\rho)} \int_\mathcal{S}(\mathcal{H}) f(\sigma) \mu(d\sigma), \quad \rho \in \mathcal{S}(\mathcal{H})$$  \hspace{1cm} (3)

(the infimum is over all probability measures $\mu$ with the barycenter $\rho$). If the function $\mu$-co $f$ is universally measurable\footnote{By using the results in [19] universal measurability of the function $\mu$-co $f$ can be proved for any bounded Borel function $f$.} and $\mu$-convex then it is the greatest $\mu$-convex function majorized by $f$. By Propositions $[1, 2]$ below (used with evident convexity of the function $\mu$-co $f$ and Proposition A-2 in the Appendix) this...
holds if the function \( f \) is either lower bounded and lower semicontinuous or upper bounded and upper semicontinuous.

The \textit{convex closure} \( \overline{\text{co}} f \) of a lower bounded function \( f \) on the set \( \mathcal{G}(\mathcal{H}) \) is defined as the greatest convex lower semicontinuous (closed) function majorized by \( f \). By Fenchel’s theorem (see [10], [20], [21]) the function \( \overline{\text{co}} f \) coincides with the double Fenchel transformation of the function \( f \), which means that

\[
\overline{\text{co}} f(\rho) = f^{**}(\rho) = \sup_{A \in \mathfrak{B}_+(\mathcal{H})} [\text{Tr} A \rho - f^*(A)], \quad \rho \in \mathcal{G}(\mathcal{H}),
\]

where

\[
f^*(A) = \sup_{\rho \in \mathcal{G}(\mathcal{H})} [\text{Tr} A \rho - f(\rho)], \quad A \in \mathfrak{B}_+(\mathcal{H}).
\]

It follows from the definitions and Proposition A-2 in the Appendix that

\[
\overline{\text{co}} f(\rho) \leq \mu\text{-co } f(\rho) \leq \sigma\text{-co } f(\rho) \leq \text{co } f(\rho), \quad \rho \in \mathcal{G}(\mathcal{H}),
\]

for any Borel lower bounded function \( f \) on the set \( \mathcal{G}(\mathcal{H}) \). It is possible to prove (see Corollary 1 below) that the equalities hold in the above inequalities for any continuous bounded function \( f \) on the set \( \mathcal{G}(\mathcal{H}) \). The following examples show that the last assertion is not true in general.

\textbf{Example 1.} Let \( H \) be the von Neumann entropy (see §1) and \( \rho_0 \) be a state such that \( H(\rho_0) = +\infty \). Since the set of quantum states with finite entropy is convex, \( \text{co } H(\rho_0) = +\infty \) while the spectral theorem implies \( \sigma\text{-co } H(\rho_0) = 0 \).

\textbf{Example 2.} Let \( f \) be the indicator function of the complement of the closed set \( \mathcal{A}_s \) of pure product states in \( \mathcal{G}(\mathcal{H} \otimes \mathcal{H}) \) and \( \omega_0 \) be the separable state in \( \overline{\text{co}} \mathcal{A}_s \) constructed in [14] such that any measure in \( \mathcal{P}_{\{\omega_0\}}(\mathcal{G}(\mathcal{H} \otimes \mathcal{H})) \) have no atoms in \( \mathcal{A}_s \). It is easy to show that \( \sigma\text{-co } f(\omega_0) = 1 \). By Lemma 1 in [14] there exists a measure \( \mu_0 \) in \( \mathcal{P}_{\{\omega_0\}}(\mathcal{G}(\mathcal{H} \otimes \mathcal{H})) \) supported by the set \( \mathcal{A}_s \). Hence \( \mu\text{-co } f(\omega_0) = 0 \). Note that \( \sigma\text{-co } f \) is a \( \mu_0 \)-integrable \( \sigma\text{-convex} \) bounded function on the set \( \mathcal{G}(\mathcal{H} \otimes \mathcal{H}) \), for which Jensen’s inequality does not hold:

\[
1 = \sigma\text{-co } f(\omega_0) > \int_{\mathcal{G}(\mathcal{H} \otimes \mathcal{H})} \sigma\text{-co } f(\omega)\mu_0(d\omega) = 0
\]

(since the functions \( \sigma\text{-co } f \) and \( f \) coincide on the support of the measure \( \mu_0 \)).

\textbf{Example 3.} Let \( f \) be the indicator function of a set consisting of one pure state. Then \( \mu\text{-co } f = f \) while \( \overline{\text{co}} f \equiv 0 \).

Since the set \( \mathcal{G}(\mathcal{H}) \) is \( \mu \)-compact, Proposition 3 in [13] implies the following assertion.

\textbf{Proposition 1.} Let \( f \) be a lower semicontinuous lower bounded function \( f \) on the set \( \mathcal{G}(\mathcal{H}) \). Then the \( \mu \)-convex hull of this function is lower semicontinuous, which means that

\[
\overline{\text{co}} f(\rho) = \mu\text{-co } f(\rho) = \inf_{\mu \in \mathcal{P}(\rho)} \int_{\mathcal{G}(\mathcal{H})} f(\sigma)\mu(d\sigma), \quad \rho \in \mathcal{G}(\mathcal{H}).
\]

\textsuperscript{4}To obtain the below expression from the Fenchel theorem it is necessary to consider the extension \( \hat{f} \) of the function \( f \) to the real Banach space \( \mathfrak{T}_h(\mathcal{H}) \) by setting \( \hat{f} = +\infty \) on \( \mathfrak{T}_h(\mathcal{H}) \setminus \mathcal{G}(\mathcal{H}) \) and to use coincidence of the space \( \mathfrak{B}_h(\mathcal{H}) \) with the dual space to \( \mathfrak{T}_h(\mathcal{H}) \).
The infimum in (5) is achieved at some measure in $P_{\{\rho\}}$.

The $\mu$-compactness of the set $\mathcal{S}(\mathcal{H})$ is an essential condition of validity of representation (5) for the convex closure [15], Proposition 7. Representation (5) implies, in particular, that the convex closure of an arbitrary lower semicontinuous lower bounded function on the set $\mathcal{S}(\mathcal{H})$ coincides with this function on the set $\text{extr} \mathcal{S}(\mathcal{H})$ of pure states.

Note also that the condition of lower boundedness in Proposition 1 is essential, since Lemma 2 below shows that if a convex lower semicontinuous function is not lower bounded on the set $\mathcal{S}(\mathcal{H})$ then it is equal to $-\infty$ everywhere.

Stability of the set $\mathcal{S}(\mathcal{H})$ implies the following result.

Proposition 2. Let $f$ be an upper semicontinuous function on the set $\mathcal{S}(\mathcal{H})$. Then the convex hull $\text{co} f$ of this function is upper semicontinuous. If, in addition, the function $f$ is upper bounded then the convex hull, the $\sigma$-convex hull and the $\mu$-convex hull of this function coincide: $\text{co} f = \sigma\text{-co} f = \mu\text{-co} f$.

Proof. Upper semicontinuity of the function $\text{co} f$ can be proved by using the more general assertion of Lemma 4 below, since for an arbitrary sequence $\{\rho_n\}$ of states in $\mathcal{S}(\mathcal{H})$, converging to a state $\rho_0$, Lemma 3 in [14] implies existence of such $H$-operator $H$ in the space $H$ that $\sup_{n \geq 0} \text{Tr} H \rho_n < +\infty$.

Coincidence of the functions $\text{co} f$ and $\mu\text{-co} f$ under the condition of upper boundedness of the function $f$ is easily proved by using upper semicontinuity of the functional $\mu \mapsto \int_{\mathcal{S}(\mathcal{H})} f(\rho) \mu(d\rho)$ on the set $\mathcal{P}(\mathcal{S}(\mathcal{H}))$ and density of measures with finite support in the set of all measures with given barycenter [4], Lemma 1.

Example 3 shows that the condition of Proposition 2 does not imply coincidence of the function $\text{co} f$ with the function $\mu\text{-co} f = \sigma\text{-co} f = \text{co} f$.

Propositions 1 and 2 have the following obvious corollary.

Corollary 1. Let $f$ be a continuous lower bounded function on the set $\mathcal{S}(\mathcal{H})$. Then the convex hull $\text{co} f$ is continuous on any subset of $\mathcal{S}(\mathcal{H})$, where it coincides with the $\mu$-convex hull $\mu\text{-co} f$.

If, in addition, the function $f$ is bounded then its convex hull, $\sigma$-convex hull, $\mu$-convex hull, convex closure coincide: $\text{co} f = \sigma\text{-co} f = \mu\text{-co} f = \text{co} f$ and this function is continuous.

By using Proposition 1 it is easy to show that a necessary and sufficient condition of coincidence of the functions $\text{co} f$ and $\mu\text{-co} f$ at a state $\rho_0 \in \mathcal{S}(\mathcal{H})$ consists in validity of the Jensen inequality $\text{co} f(\rho_0) \leq \int \text{co} f(\rho) \mu(d\rho)$ for any measure $\mu$ in $\mathcal{P}_{\{\rho_0\}}$ (the convex function $\text{co} f$ is Borel by Proposition 2). The particular sufficient condition of this coincidence is considered in Corollary 6 below.

The second assertion of Corollary 1 shows that

$$\text{co} f(\rho) = \text{co} f(\rho) = \inf_{\{\pi_i, \rho_i\} \in \mathcal{P}_{\{\rho\}}} \sum_i \pi_i f(\rho_i), \quad \rho \in \mathcal{S}(\mathcal{H}),$$

(6)

for any continuous bounded function $f$ on the set $\mathcal{S}(\mathcal{H})$. This representation for a convex closure is a noncompact generalization of Corollary I.3.6 in [22].

We will use the following approximation result.
Lemma 1. Let \( f \) be a Borel lower bounded function on the set \( \mathcal{G}(\mathcal{H}) \). For an arbitrary state \( \rho_0 \) in \( \mathcal{G}(\mathcal{H}) \) there exists a sequence \( \{\rho_n\} \), converging to the state \( \rho_0 \), such that
\[
\limsup_{n \to +\infty} \sigma\text{-co} f(\rho_n) \leq \limsup_{n \to +\infty} \text{co} f(\rho_n) \leq \mu\text{-co} f(\rho_0).
\]
If, in addition, the function \( f \) is lower semicontinuous then
\[
\lim_{n \to +\infty} \sigma\text{-co} f(\rho_n) = \lim_{n \to +\infty} \text{co} f(\rho_n) = \mu\text{-co} f(\rho_0).
\]

Proof. It is sufficient to consider the case of nonnegative function \( f \). For given natural \( n \) let \( \mu_n \) be a measure in \( \mathcal{P}(\rho_0) \) such that
\[
\mu\text{-co} f(\rho_0) \geq \int_{\mathcal{G}(\mathcal{H})} f(\rho) \mu_n(d\rho) - \frac{1}{n}.
\]
Since the set \( \mathcal{G}(\mathcal{H}) \) is separable there exists a sequence \( \{A_{i}^n\} \) of Borel subsets of \( \mathcal{G}(\mathcal{H}) \) with diameter \( \leq 1/n \) such that \( \mathcal{G}(\mathcal{H}) = \bigcup_i A_i^n \) and \( A_i^n \cap A_j^n = \emptyset \) if \( j \neq i \). Let \( m = m(n) \) be such number that \( \sum_{i=m+1}^{+\infty} \mu_n(A_i^n) < 1/n \). Without loss of generality we may assume that \( \mu_n(A_i^n) > 0 \) for \( i = 1, m \). For each \( i \) the set \( A_i^n \) contains a state \( \rho_i^n \) such that \( f(\rho_i^n) \leq (\mu_n(A_i^n))^{-1} \int_{A_i^n} f(\rho)d\rho_n(d\rho) \).

Let \( B_n = \bigcup_{i=1}^m A_i^n \). Consider the state \( \rho_n = (\mu_n(B_n))^{-1} \sum_{i=1}^m \mu_n(A_i^n) \rho_i^n \). We want to show that
\[
\lim_{n \to +\infty} \rho_n = \rho_0. \tag{7}
\]

For each \( i \) the state \( \hat{\rho}_i^n = (\mu_n(A_i^n))^{-1} \int_{A_i^n} \rho \mu_n(d\rho) \) lies in the set \( \overline{\sigma}(A_i^n) \) with diameter \( \leq 1/n \). It follows that \( \|\rho_i^n - \hat{\rho}_i^n\|_1 \leq 1/n \) for \( i = 1, m \). By noting that \( \mu_n(B_n) = \sum_{i=1}^m \mu_n(A_i^n) \), we have
\[
\|\rho_n - \rho_0\|_1 = \left\| (\mu_n(B_n))^{-1} \sum_{i=1}^m \mu_n(A_i^n) \rho_i^n - \sum_{i=1}^m \int_{A_i^n} \rho \mu_n(d\rho) - \int_{\mathcal{G}(\mathcal{H}) \setminus B_n} \rho \mu_n(d\rho) \right\|_1
\leq \sum_{i=1}^m \mu_n(A_i^n) \|\rho_i^n - \rho_0\|_1 + \left\| \int_{\mathcal{G}(\mathcal{H}) \setminus B_n} \rho \mu_n(d\rho) \right\|_1
\leq (1 - \mu_n(B_n)) + \sum_{i=1}^m \mu_n(A_i^n) \|\rho_i^n - \hat{\rho}_i^n\|_1 + \mu_n(\mathcal{G}(\mathcal{H}) \setminus B_n) < \frac{3}{n},
\]
which implies (7).

By the choice of the states \( \rho_i^n \) we have
\[
\text{co} f(\rho_n) \leq (\mu_n(B_n))^{-1} \sum_{i=1}^m \mu_n(A_i^n) f(\rho_i^n) \leq (\mu_n(B_n))^{-1} \sum_{i=1}^m \int_{A_i^n} f(\rho) \mu_n(d\rho)
\leq (\mu_n(B_n))^{-1} \int_{\mathcal{G}(\mathcal{H})} f(\rho) \mu_n(d\rho) \leq \left(1 - \frac{1}{n}\right)^{-1} \left(\mu\text{-co} f(\rho_0) + \frac{1}{n}\right).
\]
This implies the first assertion of the lemma. By Proposition the second assertion follows from the first one (since \( \sigma\text{-co} f \geq \mu\text{-co} f = \overline{\sigma} f \)). The lemma is proved.
We will also use the following corollary of boundedness of the set $\mathcal{S}(\mathcal{H})$ as a subset of $\mathfrak{F}(\mathcal{H})$.

**Lemma 2.** Let $f$ be a concave upper semicontinuous function on a convex subset $A \subseteq \mathcal{S}(\mathcal{H})$. If the function $f$ is finite at a particular state in $A$ then this function is upper bounded on the set $A$.

*Proof.* Let $\rho_0$ be such state in $A$ that $f(\rho_0) = c_0 \neq \pm \infty$. Without loss of generality we can consider that $c_0 = 0$. If there exists a sequence $\{\rho_n\} \subseteq A$ such that $\lim_{n \to +\infty} f(\rho_n) = +\infty$ then the sequence of states $\sigma_n = (1 - \lambda_n)\rho_0 + \lambda_n\rho_n$ in $A$, where $\lambda_n = (f(\rho_n))^{-1}$, converges to the state $\rho_0$ by boundedness of the set $A$ and $f(\sigma_n) \geq \lambda_n f(\rho_n) = 1$ by concavity of the function $f$, contradicting to upper semicontinuity of this function.

2.3. The convex roofs. In the case $\dim \mathcal{H} < +\infty$ any state in $\mathcal{S}(\mathcal{H})$ can be represented as the average state of some finite ensemble of pure states. This provides correctness of the following convex extension to the set $\mathcal{S}(\mathcal{H})$ of an arbitrary function $f$ defined on the set $\text{extr} \mathcal{S}(\mathcal{H})$ of pure states

$$f_*(\rho) = \inf_{\{\pi_i, \rho_i\} \in \mathcal{P}_\rho} \sum_i \pi_i f(\rho_i), \quad \rho \in \mathcal{S}(\mathcal{H}) \quad (8)$$

(the infimum is over all finite ensembles $\{\pi_i, \rho_i\}$ of pure states with the average state $\rho$).

Following [1] we will call this extension the convex roof of the function $f$. The notion of a convex roof plays essential role in quantum information theory, where it is used, in particular, for construction of entanglement monotones (see §4).

In the case $\dim \mathcal{H} = +\infty$ one can consider the following two generalizations of the above construction.

Let $f$ be a semibounded function $f$ on the set $\text{extr} \mathcal{S}(\mathcal{H})$ of pure states. The $\sigma$-convex roof $f_\sigma^*$ of the function $f$ is defined as follows

$$f_\sigma^*(\rho) = \inf_{\{\pi_i, \rho_i\} \in \mathcal{P}_\rho} \sum_i \pi_i f(\rho_i), \quad \rho \in \mathcal{S}(\mathcal{H}) \quad (9)$$

(the infimum is over all countable ensembles $\{\pi_i, \rho_i\}$ of pure states with the average state $\rho$). Similar to the case of function $\sigma$-co $f$ it is easy to show $\sigma$-convexity of the function $f_\sigma^*$.

Let $f$ be a semibounded Borel function $f$ on the set $\text{extr} \mathcal{S}(\mathcal{H})$ of pure states. The $\mu$-convex roof $f_\mu^*$ of the function $f$ is defined as follows

$$f_\mu^*(\rho) = \inf_{\mu \in \mathcal{P}_\rho} \int_{\text{extr} \mathcal{S}(\mathcal{H})} f(\sigma) \mu(d\sigma), \quad \rho \in \mathcal{S}(\mathcal{H}) \quad (10)$$

(the infimum is over all probability measures $\mu$ supported by pure states with the barycenter $\rho$). If the function $f_\mu^*$ is universally measurable and $\mu$-convex then it is the greatest $\mu$-convex extension of the function $f$ to the set $\mathcal{S}(\mathcal{H})$. By propositions

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5By using the results in [19] universal measurability of the function $f_\mu^*$ can be proved for any bounded Borel function $f$. 
and below (used with evident convexity of the function $f^*_\mu$ and Proposition A-2 in the Appendix) this holds if the function $f$ is either lower bounded and lower semicontinuous or upper bounded and upper semicontinuous.

Note that the notions of a $\sigma$-convex roof and of a $\mu$-convex roof can be reduced respectively to the notions of a $\sigma$-convex hull and of a $\mu$-convex hull introduced in Section 2.2. Indeed, it is easy to see that $f^*_\sigma = \sigma\text{-co} \hat{f}$ and $f^*_\mu = \mu\text{-co} \hat{f}$ for any function $f$ on the set $\text{extr} \mathcal{S}(\mathcal{H})$, where

$$
\hat{f}(\rho) = \begin{cases} f(\rho), & \rho \in \text{extr} \mathcal{S}(\mathcal{H}), \\ +\infty, & \rho \in \mathcal{S}(\mathcal{H}) \setminus \text{extr} \mathcal{S}(\mathcal{H}). 
\end{cases}
$$

Since lower semicontinuity of the function $f$ on the set $\text{extr} \mathcal{S}(\mathcal{H})$ implies lower semicontinuity of the function $\hat{f}$ on the set $\mathcal{S}(\mathcal{H})$, Proposition 1 implies the following result (also derived from assertion A of Theorem 2 in [13] by means of $\mu$-compactness of the set $\text{extr} \mathcal{S}(\mathcal{H})$).

**Proposition 3.** Let $f$ be a lower semicontinuous lower bounded function on the set $\text{extr} \mathcal{S}(\mathcal{H})$. Then

- the function $f^*_\mu$ is the greatest lower semicontinuous convex extension of the function $f$ to the set $\mathcal{S}(\mathcal{H})$;
- for arbitrary state $\rho$ in $\mathcal{S}(\mathcal{H})$ the infimum in the definition of the value $f^*_\mu(\rho)$ in (10) is achieved at some measure in $\mathcal{P}(\rho)$.

Importance of the $\mu$-compactness property of the set $\mathcal{S}(\mathcal{H})$ in the proof of this proposition is illustrated by the examples in [15].

By Theorem 1 in [13] stability of the set $\mathcal{S}(\mathcal{H})$ implies openness of the map $\mathcal{P}(\text{extr} \mathcal{S}(\mathcal{H})) \ni \mu \mapsto \bar{\rho}(\mu) \in \mathcal{S}(\mathcal{H})$. Hence assertion B of Theorem 2 in [13] implies the following result.

**Proposition 4.** Let $f$ be an upper semicontinuous upper bounded function on the set $\text{extr} \mathcal{S}(\mathcal{H})$. Then

- the $\sigma$-convex roof and the $\mu$-convex roof of the function $f$ coincide: $f^*_\sigma = f^*_\mu$;
- the function $f^*_\sigma = f^*_\mu$ is upper semicontinuous on the set $\mathcal{S}(\mathcal{H})$ and coincides with the greatest upper bounded convex extension of the function $f$ to this set.

Propositions 3 and 4 have the following obvious corollary.

**Corollary 2.** Let $f$ be a continuous bounded function on the set $\text{extr} \mathcal{S}(\mathcal{H})$. Then its $\sigma$-convex roof and its $\mu$-convex roof coincide and the function $f^*_\sigma = f^*_\mu$ is continuous on the set $\mathcal{S}(\mathcal{H})$.

By this corollary an arbitrary continuous bounded function on the set of pure states has at least one continuous bounded convex extension to the set of all states.

### 2.4. The convex hulls of concave functions.

In the case $\dim \mathcal{H} < +\infty$ it is easy to show that the convex hull of arbitrary concave function $f$ defined on the set $\mathcal{S}(\mathcal{H})$ coincides with the convex roof of the restriction $f|_{\text{extr} \mathcal{S}(\mathcal{H})}$ of this function to
the set \( \text{extr} \mathcal{S}(\mathcal{H}) \). By stability of the set \( \mathcal{S}(\mathcal{H}) \) continuity of the function \( f \) implies continuity of the function \( \text{co} f = (f|_{\text{extr} \mathcal{S}(\mathcal{H})})_\ast \).

In the case \( \dim \mathcal{H} = +\infty \) the analog of this observation is established in the following proposition.

**Proposition 5.** Let \( f \) be a concave function on the set \( \mathcal{S}(\mathcal{H}) \).

If the function \( f \) is lower bounded then \( \sigma\text{-co} f = (f|_{\text{extr} \mathcal{S}(\mathcal{H})})^\sigma_\ast \). If, in addition, the function \( f \) is lower semicontinuous then \( \mu\text{-co} f = (f|_{\text{extr} \mathcal{S}(\mathcal{H})})^\mu_\ast \) and this function is lower semicontinuous.

If the function \( f \) is upper semicontinuous (correspondingly, continuous and lower bounded) then

\[
\text{co} f = \sigma\text{-co} f = \mu\text{-co} f = (f|_{\text{extr} \mathcal{S}(\mathcal{H})})^\sigma_\ast = (f|_{\text{extr} \mathcal{S}(\mathcal{H})})^\mu_\ast
\]

and this function is upper semicontinuous (correspondingly, continuous).

**Proof.** To show coincidence of the functions \( \sigma\text{-co} f \) and \( (f|_{\text{extr} \mathcal{S}(\mathcal{H})})^\sigma_\ast \) (correspondingly, of the functions \( \mu\text{-co} f \) and \( (f|_{\text{extr} \mathcal{S}(\mathcal{H})})^\mu_\ast \)) it is sufficient to prove the inequality \( \sigma\text{-co} f \geq (f|_{\text{extr} \mathcal{S}(\mathcal{H})})^\sigma_\ast \) (correspondingly, the inequality \( \mu\text{-co} f \geq (f|_{\text{extr} \mathcal{S}(\mathcal{H})})^\mu_\ast \)).

The first inequality for the concave lower bounded function \( f \) directly follows from the discrete Jensen’s inequality (Proposition A-1 in the Appendix).

Let \( f \) be a lower bounded lower semicontinuous concave function and \( \rho_0 \) be an arbitrary state. By Lemma 1 there exists a sequence \( \{\rho_n\} \), converging to the state \( \rho_0 \), such that \( \lim_{n \to +\infty} \sigma\text{-co} f(\rho_n) = \mu\text{-co} f(\rho_0) \). By the proved part of the proposition we have

\[
\sigma\text{-co} f(\rho_n) = (f|_{\text{extr} \mathcal{S}(\mathcal{H})})^\sigma_\ast(\rho_n) \geq (f|_{\text{extr} \mathcal{S}(\mathcal{H})})^\mu_\ast(\rho_n) \quad \forall n.
\]

By Proposition 3 passing to the limit \( n \to +\infty \) in this inequality leads to the inequality \( \mu\text{-co} f(\rho_0) \geq (f|_{\text{extr} \mathcal{S}(\mathcal{H})})^\mu_\ast(\rho_0) \).

Let \( f \) be an upper semicontinuous concave function taking finite value at least at one state. By Lemma 2 this function is upper bounded. Propositions 2 and 4 imply respectively \( \text{co} f = \sigma\text{-co} f = \mu\text{-co} f \) and \( (f|_{\text{extr} \mathcal{S}(\mathcal{H})})^\sigma_\ast = (f|_{\text{extr} \mathcal{S}(\mathcal{H})})^\mu_\ast \) as well as upper semicontinuity of these functions. Since \( \text{co} f \geq (f|_{\text{extr} \mathcal{S}(\mathcal{H})})^\sigma_\ast \) by Proposition A-2 in the Appendix and \( \mu\text{-co} f \leq (f|_{\text{extr} \mathcal{S}(\mathcal{H})})^\mu_\ast \) by the definitions, we obtain the main part of the second assertion of the proposition.

The assertion concerning concave continuous lower bounded function \( f \) follows from the previous ones.

### 2.5. One result concerning the convex closure.

It is well known\(^7\) that for an arbitrary increasing sequence \( \{f_n\} \) of continuous functions on a convex compact set \( A \), pointwise converging to a continuous function \( f_0 \), the corresponding sequence \( \{\text{co} f_n\} \) converges to the function \( \text{co} f_0 \). It turns out that the \( \mu \)-compactness of the set \( \mathcal{S}(\mathcal{H}) \) implies the analogous observation.

\(^7\)It follows from Dini’s lemma. The importance of the compactness condition can be shown by the sequence of the functions \( f_n(x) = \exp(-x^2/n) \) on \( \mathbb{R} \), converging to the function \( f_0(x) \equiv 1 \), such that \( \text{co} f_n(x) \equiv 0 \) for all \( n \).
Proposition 6. For arbitrary increasing sequence \( \{f_n\} \) of lower semicontinuous lower bounded functions on the set \( \mathcal{G}(\mathcal{H}) \) and arbitrary converging sequence \( \{\rho_n\} \) of states in \( \mathcal{G}(\mathcal{H}) \) the following inequality holds

\[
\liminf_{n \to +\infty} \overline{\sigma} f_n(\rho_n) \geq \overline{\sigma} f_0(\rho_0),
\]

where \( f_0 = \sup_n f_n \) and \( \rho_0 = \lim_{n \to +\infty} \rho_n \). In particular,

\[
\lim_{n \to +\infty} \overline{\sigma} f_n(\rho) = \overline{\sigma} f_0(\rho) \quad \forall \rho \in \mathcal{G}(\mathcal{H}).
\]

In fact, \( \mu \)-compactness of the set \( \mathcal{G}(\mathcal{H}) \) is equivalent to validity of the last assertion of Proposition 6 (see [23]).

Proof. For an arbitrary Borel function \( g \) on the set \( \mathcal{G}(\mathcal{H}) \) and any measure \( \mu \in \mathcal{P} \) we will use the following notation:

\[
\mu(g) = \int_{\mathcal{G}(\mathcal{H})} g(\sigma) \mu(d\sigma).
\]

Without loss of generality we may assume that the sequence \( \{f_n\} \) consists of nonnegative functions. Suppose there exists a sequence \( \{\rho_n\} \), converging to a state \( \rho_0 \), such that

\[
\overline{\sigma} f_n(\rho_n) + \Delta \leq \overline{\sigma} f_0(\rho_0), \quad \Delta > 0, \quad \forall n.
\]

We will assume that \( \overline{\sigma} f_0(\rho_0) < +\infty \). The case \( \overline{\sigma} f_0(\rho_0) = +\infty \) is considered similarly.

By representation (4) there exists a continuous affine function \( \alpha \) on the set \( \mathcal{G}(\mathcal{H}) \) such that

\[
\alpha(\rho) \leq f_0(\rho) \quad \forall \rho \in \mathcal{G}(\mathcal{H}), \quad \overline{\sigma} f_0(\rho_0) \leq \alpha(\rho_0) + \frac{1}{4} \Delta.
\]

(11)

Let \( N \) be such number that \( |\alpha(\rho_n) - \alpha(\rho_0)| < \frac{1}{4} \Delta \) for all \( n \geq N \).

By Proposition 1 for each \( n \) there exists a measure \( \mu_n \in \mathcal{P}_{\{\rho_n\}} \) such that \( \overline{\sigma} f_n(\rho_n) = \mu_n(\rho_n) \). Since the function \( \alpha \) is affine we have

\[
\mu_n(\alpha) - \mu_n(f_n) = \alpha(\rho_n) - \overline{\sigma} f_n(\rho_n)
\]

\[
= [\alpha(\rho_n) - \alpha(\rho_0)] + [\alpha(\rho_0) - \overline{\sigma} f_0(\rho_0)] + [\overline{\sigma} f_0(\rho_0) - \overline{\sigma} f_n(\rho_n)]
\]

\[
\geq -\frac{1}{4} \Delta - \frac{1}{4} \Delta + \Delta = \frac{1}{2} \Delta \quad \forall n \geq N.
\]

(12)

The \( \mu \)-compactness of the set \( \mathcal{G}(\mathcal{H}) \) implies relative compactness of the sequence \( \{\mu_n\} \). By Prokhorov’s theorem (see [24], § 6) this sequence is tight, which means existence of such compact subset \( \mathcal{K}_\varepsilon \subset \mathcal{G}(\mathcal{H}) \) for each \( \varepsilon > 0 \) that \( \mu_n(\mathcal{G}(\mathcal{H}) \setminus \mathcal{K}_\varepsilon) < \varepsilon \) for all \( n \).

Let \( M = \sup_{\rho \in \mathcal{G}(\mathcal{H})} |\alpha(\rho)| \) and \( \varepsilon_0 = \frac{\Delta}{4M} \). By (12) for all \( n \geq N \) we have

\[
\int_{\mathcal{K}_\varepsilon} (\alpha(\rho) - f_n(\rho)) \mu_n(d\rho) \geq \frac{1}{2} \Delta - \int_{\mathcal{G}(\mathcal{H}) \setminus \mathcal{K}_\varepsilon} (\alpha(\rho) - f_n(\rho)) \mu_n(d\rho) \geq \frac{1}{4} \Delta.
\]
Hence, the set \( C_n = \{ \rho \in \mathcal{K}_{\varepsilon_0} \mid \alpha(\rho) \geq f_n(\rho) + \frac{1}{2}\Delta \} \) is nonempty for all \( n \geq N \).

Since the sequence \( \{f_n\} \) is increasing, the sequence \( \{C_n\} \) of \emph{closed} subsets of the compact set \( \mathcal{K}_{\varepsilon_0} \) is monotone: \( C_{n+1} \subseteq C_n \ \forall \ n \). Hence there exists \( \rho_* \in \bigcap_n C_n \). This means that \( \alpha(\rho_*) \geq f_n(\rho_*) + \frac{1}{2}\Delta \) for all \( n \), and hence \( \alpha(\rho_*) \geq f_0(\rho_*) \), contradicting to (11).

**Corollary 3.** For arbitrary increasing sequence \( \{f_n\} \) of lower semicontinuous lower bounded functions on the set \( \text{extr} \mathcal{S}(\mathcal{H}) \) and arbitrary converging sequence \( \{\rho_n\} \) of states in \( \mathcal{S}(\mathcal{H}) \) the following inequality holds

\[
\liminf_{n \to +\infty} (f_n)_*^\mu(\rho_n) \geq (f_0)_*^\mu(\rho_0),
\]

where \( f_0 = \sup_n f_n \) and \( \rho_0 = \lim_{n \to +\infty} \rho_n \). In particular,

\[
\lim_{n \to +\infty} (f_n)_*^\mu(\rho) = (f_0)_*^\mu(\rho) \quad \forall \rho \in \mathcal{S}(\mathcal{H}).
\]

**Proof.** By Theorems 1 and 2 in [13] for an arbitrary lower semicontinuous lower bounded function \( f \) on the set \( \text{extr} \mathcal{S}(\mathcal{H}) \) the function

\[
f^*(\rho) \triangleq \sup_{\mu \in \mathcal{P}(\rho)} \int_{\text{extr} \mathcal{S}(\mathcal{H})} f(\sigma)\mu(d\sigma) = \sup_{\{\pi_i, \rho_i\} \in \mathcal{P}(\rho)} \sum_i \pi_i f(\rho_i), \quad \rho \in \mathcal{S}(\mathcal{H}),
\]

is a lower semicontinuous lower bounded concave extension of the function \( f \) to the set \( \mathcal{S}(\mathcal{H}) \). It is clear that for an arbitrary increasing sequence \( \{f_n\} \) of lower semicontinuous lower bounded functions on the set \( \text{extr} \mathcal{S}(\mathcal{H}) \), converging pointwise to the function \( f_0 \), the corresponding increasing sequence \( \{f_n^*\} \) converges pointwise to the function \( f_0^* \) on the set \( \mathcal{S}(\mathcal{H}) \). Thus the assertion of the corollary can be derived from Proposition [10] by using Propositions [1] and [5].

**Remark 1.** The \( \mu \)-convex roof can not be replaced by the \( \sigma \)-convex roof in Corollary [3]. Indeed, let \( f \) be the indicator function of the set \( \text{extr} \mathcal{S}(\mathcal{H} \otimes \mathcal{H}) \setminus \mathcal{A}_s \) and \( \omega_0 \) be the separable state considered in Example [2]. This function \( f \) can be represented as a limit of an increasing sequence \( \{f_n\} \) of continuous bounded functions on the set \( \text{extr} \mathcal{S}(\mathcal{H} \otimes \mathcal{H}) \). Since by Corollary [2] we have \( (f_n)^*_\sigma = (f_n)^*_\mu \) for all \( n \), Corollary [3] and the property of the state \( \omega_0 \) imply \( \lim_{n \to +\infty} (f_n)^*_\sigma(\omega_0) = (f_0)^*_\sigma(\omega_0) = 0 \) and \( (f_0)^*_\sigma(\omega_0) = 1 \).

**Remark 2.** The monotone convergence theorem implies the following results dual to the second assertions of Proposition [10] and of Corollary [3].

1) For an arbitrary decreasing sequence \( \{f_n\} \) of Borel upper bounded functions on the set \( \mathcal{S}(\mathcal{H}) \) the following relation holds

\[
\lim_{n \to +\infty} \mu\text{-co} f_n(\rho) = \mu\text{-co} f_0(\rho), \quad \forall \ \rho \in \mathcal{S}(\mathcal{H}), \quad \text{where} \quad f_0 = \inf_n f_n;
\]

2) For an arbitrary decreasing sequence \( \{f_n\} \) of Borel upper bounded functions on the set \( \text{extr} \mathcal{S}(\mathcal{H}) \) the following relation holds

\[
\lim_{n \to +\infty} (f_n)_*^{\mu}(\rho) = (f_0)_*^{\mu}(\rho), \quad \forall \rho \in \mathcal{S}(\mathcal{H}), \quad \text{where} \quad f_0 = \inf_n f_n.
\]
By using Corollary 1, Proposition 6, the first assertion of Remark 2 and Dini’s lemma the following result can be easily proved.

**Corollary 4.** Let \( \{f_t\}_{t \in T \subseteq \mathbb{R}} \) be a family of continuous bounded functions on the set \( \mathcal{G}(\mathcal{H}) \) such that:

1) \( f_{t_1}(\rho) \leq f_{t_2}(\rho) \) for all \( \rho \in \mathcal{G}(\mathcal{H}) \) and all \( t_1, t_2 \in T \) such that \( t_1 < t_2 \);

2) the function \( T \ni t \mapsto f_t(\rho) \) is continuous for all \( \rho \in \mathcal{G}(\mathcal{H}) \).

Then the function \( \mathcal{G}(\mathcal{H}) \times T \ni (\rho, t) \mapsto \text{co} f_t(\rho) \) is continuous.

By using Corollary 2, Corollary 3, the second assertion of Remark 2 and Dini’s lemma the analogous result for the \( \mu \)-convex roof of a family of continuous bounded functions on the set \( \text{extr} \mathcal{G}(\mathcal{H}) \) can be proved.

### § 3. The main theorem

Let \( \alpha \) be a lower semicontinuous affine function on the set \( \mathcal{G}(\mathcal{H}) \) taking values in \([0, +\infty]\). Consider the family of closed subsets

\[
A_c = \{ \rho \in \mathcal{G}(\mathcal{H}) \mid \alpha(\rho) \leq c \}, \quad c \in \mathbb{R}_+, \tag{13}
\]

of the set \( \mathcal{G}(\mathcal{H}) \). In the following theorem the properties of restrictions of convex hulls of a given function to the subsets of this family are considered.

**Theorem 1.** Let \( f \) be a Borel lower bounded function on the set \( \mathcal{G}(\mathcal{H}) \) and \( \alpha \) be the above affine function. If the function \( f \) has upper semicontinuous bounded restriction to the set \( A_c \) for each \( c > 0 \) and

\[
\limsup_{c \to +\infty} c^{-1} \sup_{\rho \in A_c} f(\rho) < +\infty, \tag{14}
\]

then

\[
\text{co} f(\rho) = \sigma\text{-co} f(\rho) = \mu\text{-co} f(\rho)
\]

for all \( \rho \in \bigcup_{c>0} A_c \) and the common restriction of these functions to the set \( A_c \) is upper semicontinuous for each \( c > 0 \).

If, in addition, the function \( f \) is lower semicontinuous on the set \( \mathcal{G}(\mathcal{H}) \) then

\[
\text{co} f(\rho) = \sigma\text{-co} f(\rho) = \mu\text{-co} f(\rho) = \overline{\mathcal{W}} f(\rho)
\]

for all \( \rho \in \bigcup_{c>0} A_c \) and the common restriction of these functions to the set \( A_c \) is continuous for each \( c > 0 \).

**Proof.** Without loss of generality we can assume that \( f \) is a nonnegative function.

Let \( \rho_0 \) be a state such that \( \alpha(\rho_0) = c_0 < +\infty \). By the condition \( \mu\text{-co} f(\rho_0) \leq f(\rho_0) < +\infty \). For arbitrary \( \varepsilon > 0 \) let \( \mu_0 \) be a measure in \( \mathcal{P}_{\{\rho_0\}} \) such that

\[
\int_{\mathcal{G}(\mathcal{H})} f(\rho) \mu_0(d\rho) < \mu\text{-co} f(\rho_0) + \varepsilon.
\]

Condition (14) implies existence of such positive numbers \( c_* \) and \( M \) that \( f(\rho) \leq M\alpha(\rho) \) for all \( \rho \in \mathcal{G}(\mathcal{H}) \setminus A_{c_*} \).
Note that \( \lim_{c \to +\infty} \mu_0(A_c) = 1 \). Indeed, it follows from the inequality

\[
c_0 \leq \int_{A_c} \alpha(\rho) \mu_0(d\rho) + \int_{\mathcal{S}(\mathcal{H}) \setminus A_c} \alpha(\rho) \mu_0(d\rho) = \alpha(\rho_0) = c_0,
\]

obtained by using Corollary A-1 in the Appendix that

\[
\mu_0(\mathcal{S}(\mathcal{H}) \setminus A_c) \leq \frac{c_0}{c}.
\]

Thus the monotone convergence theorem implies

\[
\lim_{c \to +\infty} \int_{\mathcal{S}(\mathcal{H}) \setminus A_c} \alpha(\rho) \mu_0(d\rho) = \lim_{c \to +\infty} \left( \alpha(\rho_0) - \int_{A_c} \alpha(\rho) \mu_0(d\rho) \right) = 0.
\]

Let \( c^* > c_\ast \) be such that \( \int_{\mathcal{S}(\mathcal{H}) \setminus A_{c^\ast}} \alpha(\rho) \mu_0(d\rho) < \varepsilon \). By Lemma 3 below there exists a sequence \( \{\mu_n\} \) of measures in \( \mathcal{P}_{\{\rho_0\}}^f \) weakly converging to the measure \( \mu_0 \) such that \( \mu_n(A_{c^\ast}) = \mu_0(A_{c^\ast}) \) and \( \int_{\mathcal{S}(\mathcal{H}) \setminus A_{c^\ast}} \alpha(\rho) \mu_n(d\rho) < \varepsilon \) for all \( n \). Since the function \( f \) is upper semicontinuous and bounded on the set \( A_{c^\ast} \) we have (see [24], § 2)

\[
\limsup_{n \to +\infty} \int_{A_{c^\ast}} f(\rho) \mu_n(d\rho) \leq \int_{A_{c^\ast}} f(\rho) \mu_0(d\rho).
\]

Hence by noting that

\[
\int_{\mathcal{S}(\mathcal{H}) \setminus A_{c^\ast}} f(\rho) \mu_n(d\rho) \leq M \int_{\mathcal{S}(\mathcal{H}) \setminus A_{c^\ast}} \alpha(\rho) \mu_n(d\rho) < M \varepsilon, \quad n = 0, 1, 2, \ldots,
\]

we obtain

\[
\co f(\rho_0) \leq \liminf_{n \to +\infty} \int_{\mathcal{S}(\mathcal{H})} f(\rho) \mu_n(d\rho) \leq \limsup_{n \to +\infty} \int_{A_{c^\ast}} f(\rho) \mu_n(d\rho) + M \varepsilon
\]

\[
\leq \int_{\mathcal{S}(\mathcal{H})} f(\rho) \mu_0(d\rho) + M \varepsilon \leq \co f(\rho_0) + \varepsilon (M + 1).
\]

Since \( \varepsilon \) is arbitrary this implies \( \co f(\rho_0) = \co f(\rho_0) \).

The proof of the first assertion of the theorem is completed by applying Lemma 4 below.

By Proposition 1 the second assertion of the theorem follows from the first one.

**Lemma 3.** Let \( \alpha \) be a lower semicontinuous affine function on the set \( \mathcal{S}(\mathcal{H}) \) taking values in \( [0, +\infty] \) and \( \mu_0 \) be a measure in \( \mathcal{P} \) such that \( \alpha(\rho(\mu_0)) < +\infty \). For given arbitrary \( c > 0 \) there exists a sequence \( \{\mu_n\} \) of measures in \( \mathcal{P}_{\{\rho(\mu_0)\}}^f \) converging to the measure \( \mu_0 \) such that

\[
\mu_n(A_c) = \mu_0(A_c), \quad \int_{\mathcal{S}(\mathcal{H}) \setminus A_c} \alpha(\rho) \mu_n(d\rho) = \int_{\mathcal{S}(\mathcal{H}) \setminus A_c} \alpha(\rho) \mu_0(d\rho)
\]

for all \( n \), where \( A_c \) is the subset of \( \mathcal{S}(\mathcal{H}) \) defined by [13].
**Proof.** This lemma can be proved by the simple modification of the proof of Lemma 1 in [4], consisting in finding for given $n$ of such decomposition of the set $\mathcal{S}(H)$ into collection $\{A_i^n\}_{i=1}^{m+2}$ of $m+2$ ($m = m(n)$) disjoint Borel subsets that

1. the set $A_i^n$ has diameter $< 1/n$ for $i = \overline{1, m}$;
2. $\mu_0(A_i^{m+1}) < 1/n$ and $\mu_0(A_i^{m+2}) < 1/n$;
3. the set $A_i^n$ is contained either in $A_c$ or in $\mathcal{S}(H) \setminus A_c$ for $i = \overline{1, m+2}$.

The essential points in this construction are the following implication

\[ A \subseteq B \Rightarrow (\mu_0(A))^{-1} \int_A \rho \mu_0(d\rho) \in B, \quad B = A_c, \quad \mathcal{S}(H) \setminus A_c, \]

and the equality

\[ \int_A \alpha(\rho) \mu_0(d\rho) = \mu_0(A) \alpha \left( \frac{1}{\mu_0(A)} \int_A \rho \mu_0(d\rho) \right), \quad A \subseteq \mathcal{S}(H), \quad \mu_0(A) \neq 0, \]

easily obtained by using Corollary A-1 in the Appendix. The lemma is proved.

Stability of the set $\mathcal{S}(H)$ is used in the proof of the above Theorem [4] via the following lemma.

**Lemma 4.** Let $\alpha$ be a lower semicontinuous affine function on the set $\mathcal{S}(H)$ taking values in $[0, +\infty]$ and $f$ be a function on the set $\mathcal{S}(H)$ having upper semicontinuous restriction to the set $A_c$ defined by (13) for each $c > 0$. Then the function $\text{co} f$ has upper semicontinuous restriction to the set $A_c$ for each $c > 0$.

**Proof.** Let $\rho_0 \in A_{c_0}$ and let $\{\rho_n\} \subset A_{c_0}$ be an arbitrary sequence converging to the state $\rho_0$. Suppose there exists

\[ \lim_{n \to +\infty} \text{co} f(\rho_n) > \text{co} f(\rho_0). \quad (15) \]

For given arbitrary $\varepsilon > 0$ let $\{\pi^0_i, \rho^0_i\}_{i=1}^m$ be an ensemble in $P_{\{\rho_0\}}$ such that

\[ \sum_{i=1}^m \pi^0_i f(\rho^0_i) < \text{co} f(\rho_0) + \varepsilon. \]

By stability of the set $\mathcal{S}(H)$ (see [5]) there exists a sequence $\{\{\pi^n_i, \rho^n_i\}_{i=1}^m\}_n$ of ensembles such that $\sum_{i=1}^m \pi^n_i \rho^n_i = \rho_n$ for each $n$, $\lim_{n \to +\infty} \pi^n_i = \pi^0_i$ and $\lim_{n \to +\infty} \rho^n_i = \rho^0_i$ for all $i = \overline{1, m}$. Let $\pi_* = \min_{1 \leq i \leq k} \pi^0_i$. Then there exists such $N$ that $\pi^0_i \geq \pi_*/2$ for all $n \geq N$ and $i = \overline{1, m}$. It follows from the inequality $\sum_{i=1}^m \pi^n_i \alpha(\rho^n_i) = \alpha(\rho_n) \leq c_0$ that $\rho^n_i \in A_{2c_0/\pi_*}$ for all $n \geq N$ and $i = \overline{1, m}$. By upper semicontinuity of the function $f$ on the set $A_{2c_0/\pi_*}$ we have

\[ \limsup_{n \to +\infty} \text{co} f(\rho_n) \leq \limsup_{n \to +\infty} \sum_{i=1}^m \pi^n_i f(\rho^n_i) \leq \sum_{i=1}^m \pi^0_i f(\rho^0_i) < \text{co} f(\rho_0) + \varepsilon, \]

which contradicts to (15) since $\varepsilon$ is arbitrary.

**Remark 3.** If $f$ is a concave function then condition [14] follows from boundedness of the restriction of this function to the set $A_c$ for each $c$. Indeed, for arbitrary affine function $\alpha$ concavity of the function $f$ on the set $\mathcal{S}(H)$ implies concavity of the function $c \mapsto \sup_{\rho \in A_c} f(\rho)$ on the set $\mathbb{R}^+$, hence finiteness of the last function guarantees validity of condition (14).
By Remark 3, Theorem 1, Lemma 2 and Proposition 5 imply the following result.

**Corollary 5.** Let $f$ be a concave lower semicontinuous lower bounded function and $\alpha$ be a lower semicontinuous affine function on the set $\mathcal{S}(\mathcal{H})$ taking values in $[0, +\infty]$. If the function $f$ has continuous restriction to the set $\mathcal{A}_c$ defined by (13) for each $c > 0$ then

$$
\text{co } f(\rho) = \sigma\text{-co } f(\rho) = \mu\text{-co } f(\rho) = \overline{\text{co}} f(\rho) = (f|_{\text{extr } \mathcal{S}(\mathcal{H})})_c^\mu(\rho) = (f|_{\text{extr } \mathcal{S}(\mathcal{H})})_c^\sigma(\rho)
$$

for all $\rho \in \bigcup_{c > 0} \mathcal{A}_c$ and the common restriction of these functions to the set $\mathcal{A}_c$ is continuous for each $c > 0$.

Theorem 1 implies the following sufficient conditions of coincidence and continuity of convex hulls.

**Corollary 6.** Let $f$ be a Borel lower bounded function on the set $\mathcal{S}(\mathcal{H})$ and $\rho_0$ be an arbitrary state in $\mathcal{S}(\mathcal{H})$. If there exists an affine lower semicontinuous function $\alpha$ on the set $\mathcal{S}(\mathcal{H})$ taking values in $[0, +\infty]$ such that $\alpha(\rho_0) < +\infty$, the function $f$ has upper semicontinuous bounded restriction to the set $\mathcal{A}_c$ defined by (13) for each $c > 0$ and condition (14) holds, then

$$
\text{co } f(\rho_0) = \sigma\text{-co } f(\rho_0) = \mu\text{-co } f(\rho_0).
$$

**Corollary 7.** Let $f$ be a lower semicontinuous lower bounded function on the set $\mathcal{S}(\mathcal{H})$ and $\{\rho_n\}$ be an arbitrary sequence of states in $\mathcal{S}(\mathcal{H})$ converging to a state $\rho_0$. If there exists an affine lower semicontinuous function $\alpha$ on the set $\mathcal{S}(\mathcal{H})$ taking values in $[0, +\infty]$ such that $\sup_n \alpha(\rho_n) < +\infty$, the function $f$ has continuous bounded restriction to the set $\mathcal{A}_c$ defined by (13) for each $c > 0$ and condition (14) holds, then

$$
\text{co } f(\rho_n) = \sigma\text{-co } f(\rho_n) = \mu\text{-co } f(\rho_n) = \overline{\text{co}} f(\rho_n), \quad n = 0, 1, 2, \ldots, \quad (16)
$$

$$
\lim_{n \to +\infty} \text{co } f(\rho_n) = \text{co } f(\rho_0). \quad (17)
$$

**Remark 4.** If $f$ is a concave function then condition (14) in Corollaries 6 and 7 can be omitted by Remark 3.

**Example 4.** In study of informational properties of a quantum channel the output Renyi entropy, in particular, the output von Neumann entropy and their convex closures play important role [25].

Let $\Phi : \mathcal{S}(\mathcal{H}) \mapsto \mathcal{S}(\mathcal{H}')$ be a quantum channel – a linear completely positive trace preserving map (see [9], § 3.1) and $\mathcal{S}(\mathcal{H}) \ni \rho \mapsto (R_p \circ \Phi)(\rho) = \frac{\log \text{Tr } \Phi(\rho)^p}{1-p}$ be the output Renyi entropy of this channel of order $p \in (0, +\infty)$ (the case $p = 1$ corresponds to the output von Neumann entropy $-\text{Tr } \Phi(\rho) \log \Phi(\rho)$, the case $p = +\infty$ corresponds to the function $-\log \lambda_{\text{max}}(\Phi(\rho))$, where $\lambda_{\text{max}}(\Phi(\rho))$ is the maximal eigenvalue of the state $\Phi(\rho)$). For $p \in (0, 1]$ the function $R_p \circ \Phi$ is lower semicontinuous concave and takes values in $[0, +\infty]$, while for $p \in (1, +\infty]$ it is continuous and finite but not concave. The output von Neumann entropy $H \circ \Phi = R_1 \circ \Phi$ is the supremum (pointwise limit as $p \to 1 + 0$) of the monotonic family $\{R_p \circ \Phi\}_{p>1}$ of continuous functions. By Proposition 3 the convex closure $\overline{\text{co}}(H \circ \Phi)$
of the output von Neumann entropy coincides with the supremum (pointwise limit as \( p \to 1 + 0 \)) of the monotonic family of functions \( \{ \mathcal{C}(R_p \circ \Phi) \}_{p > 1} \).

Corollary 6 makes it possible to show that

\[
\mathcal{C}(R_p \circ \Phi)(\rho_0) = \sigma \mathcal{C}(R_p \circ \Phi)(\rho_0) = \mu \mathcal{C}(R_p \circ \Phi)(\rho_0) = \mathcal{C}(R_p \circ \Phi)(\rho_0)
\]

for any state \( \rho_0 \) such that \( (H \circ \Phi)(\rho_0) < +\infty \). Indeed, the condition \( H(\Phi(\rho_0)) < +\infty \) implies existence of such \( H \)-operator \( H' \) in the space \( \mathcal{H}' \) that

\[
g(H') = \inf \{ \lambda > 0 \mid \text{Tr} \exp(-\lambda H') < +\infty \} < +\infty
\]

and \( \text{Tr} H'\Phi(\rho_0) < +\infty \). By Proposition 1 in [26] the conditions of Corollary 6 are fulfilled for the function \( f(\rho) = (R_p \circ \Phi)(\rho) \leq (H \circ \Phi)(\rho) \) with \( p \in [1, +\infty) \) provided \( \alpha(\rho) = \text{Tr} H'\Phi(\rho) \). Note that if \( (H \circ \Phi)(\rho_0) = +\infty \) then (18) may not be valid (see [26], Proposition 7).

By Corollary 1 the above coincidence of the convex hulls and continuity of the Renyi entropy for \( p > 1 \) imply continuity of the function \( \mathcal{C}(R_p \circ \Phi) \) for \( p > 1 \) on the convex subset \( \{ \rho \in \mathcal{S}(\mathcal{H}) \mid (H \circ \Phi)(\rho) < +\infty \} \).

If the output von Neumann entropy \( H \circ \Phi \) is continuous on a particular set \( \mathcal{A} \subseteq \mathcal{S}(\mathcal{H}) \) then by Theorem 1 in [25] its convex closure \( \mathcal{C}(H \circ \Phi) \) is also continuous and coincides with the convex hull \( \mathcal{C}(H \circ \Phi) \) on this set. If the set \( \mathcal{A} \) is compact then the above assertion on continuity of the function \( \mathcal{C}(R_p \circ \Phi) \) and Dini’s lemma imply uniform convergence of the continuous function \( \mathcal{C}(R_p \circ \Phi)|_{\mathcal{A}} = \mathcal{C}(R_p \circ \Phi)|_{\mathcal{A}} \) to the continuous function \( \mathcal{C}(H \circ \Phi)|_{\mathcal{A}} = \mathcal{C}(H \circ \Phi)|_{\mathcal{A}} \) as \( p \to 1 + 0 \). This shows, in particular, that the Holevo capacity\(^8\) of the \( \mathcal{A} \)-constrained channel \( \Phi \) (see [4]) can be determined by the expression

\[
\mathcal{C}(\Phi, \mathcal{A}) = \lim_{p \to 1+0} \sup_{\rho \in \mathcal{A}} \left( (R_p \circ \Phi)(\rho) - \mathcal{C}(R_p \circ \Phi)(\rho) \right).
\]

This expression can be used for approximation of the Holevo capacity (since the Renyi entropy for \( p > 1 \) is more "computable" than the von Neumann entropy) and in analysis of continuity of the Holevo capacity as a function of a channel (since the Renyi entropy is continuous for \( p > 1 \)).

\section*{4. Entanglement monotones}

4.1. The basic properties. Entanglement is an essential feature of quantum systems, which can be considered as a special quantum correlation having no classical analogue. It is this property that provides a base for construction of different quantum algorithms and cryptographic protocols (see [6]). One of the basic tasks of the theory of entanglement consists in finding appropriate quantitative characteristics of entanglement of a state in composite system and in studying their properties (see [3], [27] and references therein). Entanglement monotones form an important class of such characteristics [2]. In this section we consider infinite dimensional generalization of the "convex roof construction" of entanglement monotones and

\(^8\)This value is closely related to the classical capacity of a quantum channel (see [6]).
investigate its properties. This generalization is based on the results presented in the previous sections.

Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces. A state $\omega \in \mathcal{S}(\mathcal{H} \otimes \mathcal{K})$ is called **separable** or **nonentangled** if it belongs to the convex closure of the set of all product pure states in $\mathcal{S}(\mathcal{H} \otimes \mathcal{K})$, otherwise it is called **entangled**.

A key role in the entanglement theory is played by the notion of **LOCC-operation** in a composite quantum system defined as a composition of Local Operations on each of the subsystems and Classical Communications between these subsystems [3], [27]. Action of a **selective** LOCC-operation on any state of a composite system results in a particular **ensemble** – collection of states of this system with the corresponding probability distribution (in general – probability measure on the set of states of this system). A typical example of a selective LOCC-operation is a quantum measurement on one of the subsystems, which "transforms" an arbitrary a priori state to the set of posterior states, corresponding to the outcomes of the measurement, and the probability distribution of these outcomes [9], Ch. 2. Averaging of the output ensemble of a selective LOCC-operation gives the corresponding **nonselective** LOCC-operation. Thus action of a nonselective LOCC-operation on any state of a composite system results in a particular state of this system. In the above example this averaging corresponds to a quantum measurement in which the result of the measurement is ignored (but a measured state may be changed).

An **entanglement monotone** is an arbitrary nonnegative function $E$ on the set $\mathcal{S}(\mathcal{H} \otimes \mathcal{K})$ having the following two properties (see [2], [3]).

**EM-1)** $\{E(\omega) = 0\} \iff \{\text{the state } \omega \text{ is separable}\}$.

**EM-2a)** **Monotonicity of the function $E$ under nonselective LOCC-operations.** This means that

$$E(\omega) \geq E\left(\sum_i \pi_i \omega_i\right)$$

(19)

for any state $\omega \in \mathcal{S}(\mathcal{H} \otimes \mathcal{K})$ and any LOCC-operation mapping the state $\omega$ to the finite or countable ensemble $\{\pi_i, \omega_i\}$.

This requirement is often strengthened by the following one.

**EM-2b)** **Monotonicity of the function $E$ under selective LOCC-operations.** This means that

$$E(\omega) \geq \sum_i \pi_i E(\omega_i)$$

(20)

for any state $\omega \in \mathcal{S}(\mathcal{H} \otimes \mathcal{K})$ and any LOCC-operation mapping the state $\omega$ to the finite or countable ensemble $\{\pi_i, \omega_i\}$.

In infinite dimensions the last requirement is naturally generalized to the following one.

**EM-2c)** **Monotonicity of the function $E$ under generalized selective LOCC-operations.** This means that for arbitrary state $\omega \in \mathcal{S}(\mathcal{H} \otimes \mathcal{K})$ and local instrument $^{9}$

---

$^9$An instrument in the set of states $\mathcal{S}(\mathcal{H})$ with a measurable space of outcomes $\mathcal{X}$ is a set-function $\mathcal{M}$ defined on the $\sigma$-algebra $\mathcal{B}(\mathcal{X})$ satisfying the following conditions (see [4], Ch. 4): $\mathcal{M}(B)$ is a linear completely positive trace-nonincreasing transformation of the space $\mathcal{T}(\mathcal{H})$ for any $B \in \mathcal{B}(\mathcal{X})$; $\mathcal{M}(\mathcal{X})$ is a trace-preserving transformation; if $\{B_j\} \subset \mathcal{B}(\mathcal{X})$ is a finite or countable disjoint decomposition of the set $B \in \mathcal{B}(\mathcal{X})$ then $\mathcal{M}(B)[T] = \sum_j \mathcal{M}(B_j)[T]$, $T \in \mathcal{T}(\mathcal{H})$, where the series converges in the norm of the space $\mathcal{T}(\mathcal{H})$. 


\[ M \] with the set of outcomes \( \mathcal{X} \) the function \( x \mapsto E(\sigma(x|\omega)) \) is \( \mu_\omega \)-measurable on the set \( \mathcal{X} \) and
\[
E(\omega) \geq \int_{\mathcal{X}} E(\sigma(x|\omega)) \mu_\omega(dx), \tag{21}
\]
where \( \mu_\omega(\cdot) = \text{Tr} \ M(\cdot)[\omega] \) and \( \{\sigma(x|\omega)\}_{x \in \mathcal{X}} \) are respectively the probability measure on the set \( \mathcal{X} \) describing the results of the measurement and the family of posteriori states corresponding to the a priori state \( \omega \) [9], [28].

**Remark 5.** By definition the function \( x \mapsto \sigma(x|\omega) \) is \( \mu_\omega \)-measurable with respect to the minimal \( \sigma \)-algebra on \( \mathcal{X}(H \otimes K) \) for which the all linear functionals \( \omega \mapsto \text{Tr} \ A \omega, \ A \in \mathcal{B}(H \otimes K) \), are measurable. By Corollary 1 in [29] this \( \sigma \)-algebra coincides with the Borel \( \sigma \)-algebra on \( \mathcal{X}(H \otimes K) \). Thus the function \( x \mapsto E(\sigma(x|\omega)) \) is \( \mu_\omega \)-measurable for any Borel function \( \omega \mapsto E(\omega) \).

According to [3] an entanglement monotone \( E \) is called entanglement measure if \( E(\omega) = H(\text{Tr} \ K \ \omega) \) for any pure state \( \omega \) in \( \mathcal{G}(H \otimes K) \), where \( H \) is the von Neumann entropy.

Sometimes the following requirement is included in the definition of entanglement monotone (cf. [27]).

**EM-3a)** *Convexity of the function* \( E \) *on the set* \( \mathcal{G}(H \otimes K) \), *which means that*
\[
E\left( \sum_i \pi_i \omega_i \right) \leq \sum_i \pi_i E(\omega_i)
\]
for any finite ensemble \( \{\pi_i, \omega_i\} \) of states in \( \mathcal{G}(H \otimes K) \).

This requirement is due to the observation that entanglement can not be increased by taking convex mixtures (describing classical noise in preparing of a quantum state).

The following two stronger forms of the convexity requirement are motivated by necessity to consider countable and continuous ensembles of states dealing with infinite dimensional quantum systems (cf. [4]).

**EM-3b)** *\( \sigma \)-convexity of the function* \( E \) *on the set* \( \mathcal{G}(H \otimes K) \), *which means that*
\[
E\left( \sum_i \pi_i \omega_i \right) \leq \sum_i \pi_i E(\omega_i)
\]
for any countable ensemble \( \{\pi_i, \omega_i\} \) of states in \( \mathcal{G}(H \otimes K) \).

If this requirement holds then EM-2b) \( \Rightarrow \) EM-2a).

**EM-3c)** *\( \mu \)-convexity of the function* \( E \) *on the set* \( \mathcal{G}(H \otimes K) \), *which means that*
\[
E\left( \int_{\mathcal{G}(H \otimes K)} \omega \mu(d\omega) \right) \leq \int_{\mathcal{G}(H \otimes K)} E(\omega) \mu(d\omega)
\]
for any Borel probability measure \( \mu \) on the set \( \mathcal{G}(H \otimes K) \), which can be considered as a generalized (continuous) ensemble of states in \( \mathcal{G}(H \otimes K) \).

In §2 it is shown that these convexity properties are not equivalent in general.
EM-4) **Subadditivity of the function** $E$, which means that

$$E(\omega_1 \otimes \omega_2) \leq E(\omega_1) + E(\omega_2)$$  \hspace{1cm} (22)

for any states $\omega_1 \in \mathcal{S}(\mathcal{H}_1 \otimes \mathcal{K}_1)$ and $\omega_2 \in \mathcal{S}(\mathcal{H}_2 \otimes \mathcal{K}_2)$.

This property guarantees existence of the regularization

$$E^*(\omega) = \lim_{n \to +\infty} \frac{E(\omega^{\otimes n})}{n}, \quad \omega \in \mathcal{S}(\mathcal{H} \otimes \mathcal{K}).$$

In the finite dimensional case it is natural to require continuity of an entanglement monotone $E$ on the set $\mathcal{S}(\mathcal{H} \otimes \mathcal{K})$. In infinite dimensions this requirement is very restrictive. Moreover, discontinuity of the von Neumann entropy implies discontinuity of any entanglement measure on the set $\mathcal{S}(\mathcal{H} \otimes \mathcal{K})$ in this case. Nevertheless some weaker continuity requirements may be considered.

**EM-5a)** **Lower semicontinuity of the function** $E$ **on the set** $\mathcal{S}(\mathcal{H} \otimes \mathcal{K})$. **This means that**

$$\lim\inf_{n \to +\infty} E(\omega_n) \geq E(\omega_0)$$

for any sequence $\{\omega_n\}$ of states in $\mathcal{S}(\mathcal{H} \otimes \mathcal{K})$ converging to a state $\omega_0$ or, equivalently, that the set of states defined by the inequality $E(\omega) \leq c$ is closed for any $c > 0$. This requirement is motivated by the natural physical observation that entanglement can not be increased by an approximation procedure. It is essential that lower semicontinuity of the function $E$ guarantees that this function is Borel and that requirements EM-3a – EM-3c are equivalent for this function (by Proposition A-2 in the Appendix).

From the physical point of view it is natural to require that entanglement monotones must be continuous on the set of states produced in a physical experiment. This leads to the following requirement.

**EM-5b)** **Continuity of the function** $E$ **on subsets of** $\mathcal{S}(\mathcal{H} \otimes \mathcal{K})$ **with bounded mean energy**. **Let** $H_\mathcal{H}$ **and** $H_\mathcal{K}$ **be the Hamiltonians of the quantum systems associated with the spaces** $\mathcal{H}$ **and** $\mathcal{K}$ **correspondingly** [9], § 1.2. **Then the Hamiltonian of the composite system has the form** $H_\mathcal{H} \otimes I_\mathcal{K} + I_\mathcal{H} \otimes H_\mathcal{K}$ **and hence the set of states of the composite system with the mean energy not exceeding** $\hbar$ **is defined by the inequality**

$$\text{Tr}(H_\mathcal{H} \otimes I_\mathcal{K} + I_\mathcal{H} \otimes H_\mathcal{K}) \omega \leq \hbar.$$  

Requirement EM-5b) means continuity of the restrictions of the function $E$ to the subsets of $\mathcal{S}(\mathcal{H} \otimes \mathcal{K})$ defined by the above inequality for all $h > 0$.

The strongest continuity requirement is the following one.

**EM-5c)** **Continuity of the function** $E$ **on the set** $\mathcal{S}(\mathcal{H} \otimes \mathcal{K})$.

Despite infinite dimensionality there exists a nontrivial class of entanglement monotones for which this requirement holds (see Example 5 in the next subsection.)

**4.2. The generalized convex roof constructions.** In the finite dimensional case a general method of producing of entanglement monotones is the "convex roof construction" [3], [27], [30]. By this construction for a given concave continuous nonnegative function $f$ on the set $\mathcal{S}(\mathcal{H})$ such that

$$f^{-1}(0) = \text{extr} \mathcal{S}(\mathcal{H}), \quad f(\rho) = f(U \rho U^*)$$  \hspace{1cm} (23)
for any state $\rho$ in $\mathcal{G}(\mathcal{H})$ and any unitary $U$ in $\mathcal{H}$, the corresponding entanglement monotone $E^f$ is defined as the convex roof $(f \circ \Theta|_{\text{extr } \mathcal{G}(\mathcal{H} \otimes \mathcal{K})})_*$ of the restriction of the function $f \circ \Theta$ to the set $\text{extr } \mathcal{G}(\mathcal{H} \otimes \mathcal{K})$, where $\Theta: \omega \mapsto \text{Tr}_K \omega$ is a partial trace. By using the von Neumann entropy in the role of function $f$ in the above construction we obtain the Entanglement of Formation $E_F$ – one of the most important entanglement measures \[7\].

In the infinite dimensional case there exist two possible generalizations of the above construction: the $\sigma$-convex roof $(f \circ \Theta|_{\text{extr } \mathcal{G}(\mathcal{H} \otimes \mathcal{K})})^{\sigma}_*$ and the $\mu$-convex roof $(f \circ \Theta|_{\text{extr } \mathcal{G}(\mathcal{H} \otimes \mathcal{K})})^{\mu}_*$ of the function $f \circ \Theta|_{\text{extr } \mathcal{G}(\mathcal{H} \otimes \mathcal{K})}$. To simplify notations in what follows we will omit the symbol of restriction and will denote the above functions $(f \circ \Theta)^{\sigma}_*$ and $(f \circ \Theta)^{\mu}_*$ correspondingly.

The results of the previous sections make it possible to prove the following assertions concerning the main properties of these generalized convex roof constructions.

**Theorem 2.** Let $f$ be a nonnegative concave function on the set $\mathcal{G}(\mathcal{H})$ satisfying condition \([23]\).

A-1) If the function $f$ is upper semicontinuous then

$$(f \circ \Theta)^{\sigma}_* = (f \circ \Theta)^{\mu}_* = \mu\text{-co}(f \circ \Theta) = \sigma\text{-co}(f \circ \Theta) = \text{co}(f \circ \Theta),$$

the function $(f \circ \Theta)^{\sigma}_* = (f \circ \Theta)^{\mu}_*$ is upper semicontinuous and satisfies requirements EM-1), EM-2c) and EM-3c).

A-2) If the function $f$ is lower semicontinuous then the function $(f \circ \Theta)^{\sigma}_*$ satisfies requirements\([14]\), EM-2b) and EM-3b), while the function $(f \circ \Theta)^{\mu}_*$ coincides with the function $\overline{\text{co}}(f \circ \Theta)$ and satisfies requirements EM-1), EM-2c), EM-3c) and EM-5a).

B) If the function $f$ is subadditive\([14]\), then the functions $(f \circ \Theta)^{\sigma}_*$ and $(f \circ \Theta)^{\mu}_*$ satisfy requirement EM-4).

C) Let $H_\mathcal{H}$ be a positive operator in the space $\mathcal{H}$. If the function $f$ is lower semicontinuous and for each $h > 0$ it has finite continuous restriction to the subset $K_{H_\mathcal{H},h} = \{\rho \in \mathcal{G}(\mathcal{H}) \mid \text{Tr } H_\mathcal{H} \rho \leq h\}$ then

$$(f \circ \Theta)^{\mu}_*(\omega) = (f \circ \Theta)^{\sigma}_*(\omega) = \overline{\text{co}}(f \circ \Theta)(\omega) = \text{co}(f \circ \Theta)(\omega) \quad \forall \omega \in \bigcup_{h>0} K_{H_\mathcal{H} \otimes I_\mathcal{K},h},$$

where $K_{H_\mathcal{H} \otimes I_\mathcal{K},h} = \{\omega \in \mathcal{G}(\mathcal{H} \otimes \mathcal{K}) \mid \text{Tr}(H_\mathcal{H} \otimes I_\mathcal{K}) \omega \leq h\}$, and the common restriction of these functions to the set $K_{H_\mathcal{H} \otimes I_\mathcal{K},h}$ is continuous for each $h > 0$. In particular, if $H_\mathcal{H}$ is the Hamiltonian of the quantum system associated with the space $\mathcal{H}$ then the functions $(f \circ \Theta)^{\mu}_*$ and $(f \circ \Theta)^{\sigma}_*$ satisfy requirement EM-5b).

D) If the function $f$ is continuous on the set $\mathcal{G}(\mathcal{H})$ then

$$(f \circ \Theta)^{\mu}_* = (f \circ \Theta)^{\sigma}_* = \overline{\text{co}}(f \circ \Theta) = \mu\text{-co}(f \circ \Theta) = \sigma\text{-co}(f \circ \Theta) = \text{co}(f \circ \Theta)$$

and the function $(f \circ \Theta)^{\sigma}_* = (f \circ \Theta)^{\mu}_*$ satisfies requirement EM-5c).

---

\[10\] The example in Remark \([\text{below}]\) shows that the function $(f \circ \Theta)^{\sigma}_*$ may not satisfy requirements EM-1), EM-3c) and EM-5a) even for bounded lower semicontinuous function $f$.

\[11\] This means that $f(\rho_1 \otimes \rho_2) \leq f(\rho_1) + f(\rho_2)$ for any states $\rho_1 \in \mathcal{G}(\mathcal{H}_1)$ and $\rho_2 \in \mathcal{G}(\mathcal{H}_2)$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are separable Hilbert spaces (we implicitly use the isomorphism of all such spaces).
Proof. A) By Lemma 2 upper semicontinuity and concavity of the function $f$ guarantees its boundedness while Proposition 5 implies 

$$(f \circ \Theta)^{\mu}_\sigma = \mu - \text{co}(f \circ \Theta) = \sigma - \text{co}(f \circ \Theta) = \text{co}(f \circ \Theta)$$

and upper semicontinuity of this function. Proposition A-2 in the Appendix provides validity of requirement EM-3c) for the function $(f \circ \Theta)^{\mu}_\sigma$ in this case.

By Proposition 3 lower semicontinuity of the function $f$ implies lower semicontinuity of the function $(f \circ \Theta)^{\mu}_\sigma$ (validity of requirement EM-5a). Hence Proposition A-2 in the Appendix provides validity of requirement EM-3c) for the function $(f \circ \Theta)^{\mu}_\sigma$ in this case.

Validity of requirement EM-3b) for the function $(f \circ \Theta)^{\sigma}_\sigma$ follows from its definition.

By repeating the arguments used in the proof of LOCC-monotonicity of the convex roof of the function $f \circ \Theta$ in the finite dimensional case (see [3, 7]) and by using discrete Jensen’s inequality (Proposition A-1 in the Appendix) validity of requirement EM-2b) for the function $(f \circ \Theta)^{\sigma}_\sigma$ can be proved.

Consider requirement EM-2c). Let $\mathcal{M}$ be an arbitrary instrument acting in the subsystem associated with the space $\mathcal{K}$. If the function $f$ is lower (correspondingly, upper) semicontinuous then the function $(f \circ \Theta)^{\mu}_\sigma$ is lower (correspondingly, upper) semicontinuous and hence it is Borel. By Remark 5 this guarantees $\mu_\sigma$-measurability of the function $x \mapsto E(\sigma(x|\omega))$ for any state $\omega \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$.

Let $\omega$ be a pure state. By locality of the instrument $\mathcal{M}$ we have

$$\Theta(\omega) = \int_{\mathcal{X}} \Theta(\sigma(x|\omega))\mu_\omega(dx).$$

Since the function $f$ is nonnegative concave and either lower or upper semicontinuous, Proposition A-2 in the Appendix implies

$$f \circ \Theta(\omega) \geq \int_{\mathcal{X}} f \circ \Theta(\sigma(x|\omega))\mu_\omega(dx) \geq \int_{\mathcal{X}} (f \circ \Theta)^{\mu}_\sigma(\sigma(x|\omega))\mu_\omega(dx),$$

where the last inequality follows from Proposition 5.

Let $\omega$ be a mixed state. Prove first that

$$(f \circ \Theta)^{\sigma}_\sigma(\omega) \geq \sum_i \pi_i (f \circ \Theta)^{\mu}_\sigma(\sigma(x|\omega_i))\mu_{\omega_i}(dx).$$

For given $\varepsilon > 0$ let $\{\pi_i, \omega_i\}$ be such ensemble in $\widehat{\mathcal{P}}_{\{\omega\}}(\mathfrak{S}(\mathcal{H} \otimes \mathcal{K}))$ that

$$(f \circ \Theta)^{\sigma}_\sigma(\omega) > \sum_i \pi_i f \circ \Theta(\omega_i) - \varepsilon.$$

By the above observation concerning a pure state $\omega$ we have

$$(f \circ \Theta)^{\sigma}_\sigma(\omega) > \sum_i \pi_i \int_{\mathcal{X}} (f \circ \Theta)^{\mu}_\sigma(\sigma(x|\omega_i))\mu_{\omega_i}(dx) - \varepsilon. \quad (25)$$
By the Radon-Nicodym theorem the decomposition
\[ \mu_\omega(\cdot) = \text{Tr } \mathcal{M}(\cdot)[\omega] = \sum_i \pi_i \text{Tr } \mathcal{M}(\cdot)[\omega_i] = \sum_i \pi_i \mu_{\omega_i}(\cdot) \]
implies existence of a family \( \{p_i\} \) of \( \mu_\omega \)-measurable functions on \( \mathcal{X} \) such that
\[ \pi_i \mu_{\omega_i}(\mathcal{X}_0) = \int_{\mathcal{X}_0} p_i(x) \mu_\omega(dx) \]
for any \( \mu_\omega \)-measurable subset \( \mathcal{X}_0 \subseteq \mathcal{X} \) and \( \sum_i p_i(x) = 1 \) for \( \mu_\omega \)-almost all \( x \) in \( \mathcal{X} \).

Since
\[ \int_{\mathcal{X}_0} \sigma(x|\omega) \mu_\omega(dx) = \sum_i \int_{\mathcal{X}_0} \sigma(x|\omega_i) \mu_{\omega_i}(dx) = \sum_i \int_{\mathcal{X}_0} \sigma(x|\omega_i)p_i(x) \mu_\omega(dx) \]
for any \( \mu_\omega \)-measurable subset \( \mathcal{X}_0 \subseteq \mathcal{X} \) we have
\[ \sum_i p_i(x) \sigma(x|\omega_i) = \sigma(x|\omega) \]
for \( \mu_\omega \)-almost all \( x \) in \( \mathcal{X} \).

Note that the function \( (f \circ \Theta)^\mu_* \) is \( \sigma \)-convex in the both cases. Indeed, if \( f \) is an upper semicontinuous function this follows from its coincidence with the function \( (f \circ \Theta)^\sigma_* \), if \( f \) is a lower semicontinuous function then the convex function \( (f \circ \Theta)^\mu_* \) is lower semicontinuous and hence \( \mu \)-convex (by Proposition \( \text{A-2} \) in the Appendix).

By using (25) and \( \sigma \)-convexity of the function \( (f \circ \Theta)^\mu_* \) we obtain
\[ (f \circ \Theta)^\sigma_*(\omega) > \int_{\mathcal{X}} \sum_i p_i(x)(f \circ \Theta)^\mu_*(\sigma(x|\omega_i)) \mu_\omega(dx) - \varepsilon \]
\[ \geq \int_{\mathcal{X}} (f \circ \Theta)^\mu_*(\sigma(x|\omega)) \mu_\omega(dx) - \varepsilon, \]
which implies (24) since \( \varepsilon \) is arbitrary.

If \( f \) is an upper semicontinuous function then \( (f \circ \Theta)^\sigma_* = (f \circ \Theta)^\mu_* \) and (24) means (21) for the function \( E = (f \circ \Theta)^\sigma_* = (f \circ \Theta)^\mu_* \).

If \( f \) is a lower semicontinuous function then for an arbitrary state \( \omega \in \mathcal{S}(\mathcal{H} \otimes \mathcal{K}) \) Lemma 1 and Proposition 5 imply existence of a sequence \( \{\omega_n\} \subset \mathcal{S}(\mathcal{H} \otimes \mathcal{K}) \) converging to the state \( \omega \) such that
\[ \lim_{n \to +\infty} (f \circ \Theta)^\sigma_*(\omega_n) = (f \circ \Theta)^\mu_*(\omega). \]

Inequality (21) for the function \( E = (f \circ \Theta)^\mu_* \) can be proved by applying inequality (24) for each state in the sequence \( \{\omega_n\} \) and passing to the limit \( n \to +\infty \) by means of Lemma \( \text{A-1} \) in the Appendix and due to lower semicontinuity of the function \( (f \circ \Theta)^\mu_* \).

Consider requirement EM-1). Note that a state \( \omega \) is separable if and only if there exists a measure \( \mu \) in \( \hat{P}(\omega) (\mathcal{S}(\mathcal{H} \otimes \mathcal{K})) \) supported by pure product states [14].

Let \( f \) be a lower semicontinuous function. By Proposition 3 for an arbitrary state \( \omega \) in \( \mathcal{S}(\mathcal{H} \otimes \mathcal{K}) \) there exists a measure \( \mu_\omega \) in \( \hat{P}(\omega) (\mathcal{S}(\mathcal{H} \otimes \mathcal{K})) \) such that
Let $f$ be an upper semicontinuous function. Then the function $(f \circ \Theta)^\mu_\omega = (f \circ \Theta)^\mu_\omega$ equals to zero on the set of separable states by the above characterization of this set.

Suppose this function equals to zero at some entangled state $\omega_0$. Then there exists a local operation $\Lambda$ such that the state $\Lambda(\omega_0)$ is entangled and has reduced states of finite rank. By LOCC-monotonicity of the function $(f \circ \Theta)^\mu_\omega = (f \circ \Theta)^\mu_\omega$ proved before this function equals to zero at the entangled state $\Lambda(\omega_0)$.

Let $H_0$ be the finite dimensional support of the state $\text{Tr}_K \Lambda(\omega_0)$. Then the upper semicontinuous concave function $f$ satisfying condition (23) has continuous restriction to the set $\mathcal{G}(H_0)$. Indeed, continuity of this restriction at any pure state in $\mathcal{G}(H_0)$ follows from upper semicontinuity of the nonnegative function $f$ and condition (23), while continuity of this restriction at any mixed state in $\mathcal{G}(H_0)$ can be easily derived from the well known fact that any concave bounded function is continuous at any internal point of a convex subset of a Banach space [21], Proposition 3.2.3. Since

$$(f \circ \Theta|_{\mathcal{G}(H_0 \otimes K)})^\mu_\omega = (f \circ \Theta)^\mu_\omega|_{\mathcal{G}(H_0 \otimes K)},$$

we can apply the previous observation concerning lower semicontinuous function $f$ to show that equality $(f \circ \Theta)^\mu_\omega(\Lambda(\omega_0)) = 0$ implies separability of the state $\Lambda(\omega_0)$, contradicting to the above assumption.

B) If the function $f$ is subadditive then the function $f \circ \Theta$ is subadditive as well. Let $\mu_i \in \hat{P}_{\{\omega_i\}}(\mathcal{G}(L_i))$, where $L_i = H_i \otimes K_i$, $i = 1, 2$, be arbitrary measures. The set of product states in $\text{extr} \mathcal{G}(L_1 \otimes L_2)$ can be considered as the Cartesian product of the sets $\text{extr} \mathcal{G}(L_1)$ and $\text{extr} \mathcal{G}(L_2)$. Hence on this set one can define the Cartesian product of the measures $\mu_1$ and $\mu_2$, denoted by $\mu_1 \otimes \mu_2$, which can be considered as a measure in $\hat{P}_{\{\omega_1 \otimes \omega_2\}}(\mathcal{G}(L_1 \otimes L_2))$ supported by the set of product states. By using this construction it is easy to prove subadditivity of the function $(f \circ \Theta)^\mu_\omega$. By the same argumentation with atomic measures $\mu_1$ and $\mu_2$ one can prove subadditivity of the function $(f \circ \Theta)^\mu_\omega$.

C) If the function $f$ is lower semicontinuous and satisfies the additional conditions in assertion C of the theorem, then the function $f \circ \Theta$ satisfies the conditions of Corollary 5 with the affine function $\alpha(\omega) = \text{Tr}(H_\mathcal{H} \otimes I_K)\omega$.

D) Assertion D follows from Proposition 5.

Remark 6. The function $(f \circ \Theta)^\mu_\omega$ may not satisfy the basic requirement EM-1) even for bounded lower semicontinuous function $f$ (see assertion A-2) of Theorem 2). Indeed, let $f$ be the indicator function of the set of all mixed states in $\mathcal{G}(H)$ and $\omega_0$ be a separable state such that any measure in $\hat{P}_{\{\omega_0\}}(\mathcal{G}(H \otimes K))$ has no atoms within the set of separable states [14]. Then it is easy to see that $(f \circ \Theta)^\mu_\omega(\omega_0) = 1$ (while $(f \circ \Theta)^\mu_\omega(\omega_0) = 0$).

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12In this case the measure $\mu_1 \otimes \mu_2$ corresponds to the tensor product of countable ensembles of pure states corresponding to the measures $\mu_1$ and $\mu_2$. 
The function \((f \circ \Theta)^*\) in Remark 6 does not also satisfy requirements EM-3c) and EM-5a). This is a general feature of any \(\sigma\)-convex roof not coinciding with the corresponding \(\mu\)-convex roof.

Remark 6 and Theorem 2 show that the function \((f \circ \Theta)^*\) either coincides with the function \((f \circ \Theta)^\mu^*\) (if \(f\) is upper semicontinuous) or may not satisfy the basic requirement EM-1 of entanglement monotones (if \(f\) is lower semicontinuous). Thus the \(\mu\)-convex roof construction seems to be more preferable candidate on the role of infinite dimensional generalization of the convex roof construction of entanglement monotones. Thus we will use the following notation:

\[
E^f = (f \circ \Theta)^\mu^*
\]

for any function \(f\) satisfying the conditions of Theorem 2.

**Example 5.** Generalizing to the infinite dimensional case the observation in [30] consider the family of functions

\[
f_\alpha(\rho) = 2(1 - \text{Tr} \rho^\alpha), \quad \alpha > 1,
\]
on the set \(\mathcal{G}(\mathcal{H})\) with \(\dim \mathcal{H} = +\infty\). The functions of this family are nonnegative concave continuous and satisfy conditions [23]. By Theorem 2 \(E^{f_\alpha}\) is an entanglement monotone, satisfying requirements EM-1), EM-2c), EM-3c) and EM-5c).

In the case \(\alpha = 2\) the entanglement monotone \(E^{f_2}\) can be considered as the infinite dimensional generalization of the I-tangle [31]. By Corollary 4 the function \((\omega, \alpha) \mapsto E^{f_\alpha}(\omega)\) is continuous on the set \(\mathcal{G}(\mathcal{H} \otimes \mathcal{K}) \times [1, +\infty)\). By Corollary 3 the least upper bound of the monotonic family \(\{E^{f_\alpha}\}_{\alpha > 1}\) of continuous entanglement monotones coincides with the indicator function of the set of entangled states.

**Example 6.** Let \(R_p(\rho) = \frac{\log \text{Tr} \rho^p}{1-p}\) be the Renyi entropy of the state \(\rho \in \mathcal{G}(\mathcal{H})\) of order \(p \in [0, 1]\) (the case \(p = 0\) corresponds to the function \(\log \text{rank}(\rho)\), the case \(p = 1\) corresponds to the von Neumann entropy), \(R_p\) is a concave lower semicontinuous subadditive function on the set \(\mathcal{G}(\mathcal{H})\) with the range \([0, +\infty]\), satisfying condition [23]. By Theorem 2 the function \(E^{R_p}\) is an entanglement monotone, satisfying requirements EM-1), EM-2c), EM-3c), EM-4) and EM-5a). In the case \(p = 0\) the entanglement monotone \(E^{R_0}\) is an infinite dimensional generalization of the Schmidt measure [27]. In the case \(p = 1\) the entanglement monotone \(E^{R_1} = E^H\) is an entanglement measure, which can be considered as an infinite dimensional generalization of the Entanglement of Formation [7] (see the next section). If \(g(H_\mathcal{H}) = \inf\{\lambda > 0 \mid \text{Tr} \exp(-\lambda H_\mathcal{H}) < +\infty\} = 0\) then Theorem 2.C) implies that the entanglement measure \(E^{R_1} = E^H\) satisfies requirement EM-5b), since the von Neumann entropy \(H = R_1\) is continuous on the set \(\mathcal{K}_{H, H}\) (see [12] or [26], Proposition 1).

**4.3. Approximation of entanglement monotones.** In general entanglement monotones produced by the \(\mu\)-convex roof construction are unbounded and discontinuous (only lower or upper semicontinuous), which may lead to analytical problems in dealing with these functions. Some of these problems can be solved by using the following approximation result.
Proposition 7. Let $f$ be a concave nonnegative lower semicontinuous (correspondingly, upper semicontinuous) function on the set $\mathcal{S}(\mathcal{H})$ satisfying condition (23), which is represented as a pointwise limit of some increasing (correspondingly, decreasing) sequence $\{f_n\}$ of concave continuous nonnegative functions on the set $\mathcal{S}(\mathcal{H})$ satisfying condition (23). Then the entanglement monotone $E^f$ is a pointwise limit of the increasing (correspondingly, decreasing) sequence $\{E^{f_n}\}$ of continuous entanglement monotones.

If, in addition, the function $f$ satisfies condition C in Theorem 2, then the sequence $\{E^{f_n}\}$ converges to the entanglement monotone $E^f$ uniformly on compact subsets of the set $\mathcal{K}_{\mathcal{H} \otimes \mathcal{I}_K}$ for each $h > 0$.

Proof. The first assertion of this proposition follows from Theorem 2, Corollary 3 and Remark 2. The second assertion follows from the first one and Dini’s lemma.

§ 5. Entanglement of Formation

5.1. The two definitions. The Entanglement of Formation (EoF) of a state $\omega$ of a finite dimensional composite system is defined in [7] as the minimal possible average entanglement over all pure state discrete finite decompositions of $\omega$ (entanglement of a pure state is defined as the von Neumann entropy of its reduced state). In our notations this means that

$$E^F = (H \circ \Theta)_* = \overline{\text{co}}(H \circ \Theta) = \text{co}(H \circ \Theta).$$

The possible generalization of this notion is considered in [8], where the Entanglement of Formation of a state $\omega$ of an infinite dimensional composite system is defined as the minimal possible average entanglement over all pure state discrete countable decompositions of $\omega$, which means $E^d_F = (H \circ \Theta)_c^\sigma$.

The generalized convex roof construction considered in Section 4.2 with the von Neumann entropy $H$ in the role of function $f$ leads to the proposed in [25] definition of the EoF: $E^c_F = E^H = (H \circ \Theta)_c^\mu = \overline{\text{co}}(H \circ \Theta)$, by which the Entanglement of Formation of a state $\omega$ of an infinite dimensional composite system is defined as the minimal possible average entanglement over all pure state continuous decompositions of $\omega$.

An interesting open question is a relation between $E^d_F$ and $E^c_F$. It follows from the definitions that

$$E^d_F(\omega) \geq E^c_F(\omega) \quad \forall \omega \in \mathcal{S}(\mathcal{H} \otimes \mathcal{K}).$$

In [25] it is shown that

$$E^d_F(\omega) = E^c_F(\omega)$$

for any state $\omega$ such that either $H(\text{Tr}_\mathcal{H} \omega) < +\infty$ or $H(\text{Tr}_\mathcal{K} \omega) < +\infty$. Equality (26) obviously holds for all pure states and for all nonentangled states, but its validity for arbitrary state $\omega$ is not proved (as far as I know). The example in Remark 3 shows that this question can not be solved by using only such analytical properties of the von Neumann entropy as concavity and lower semicontinuity. Note that the question of coincidence of the functions $E^d_F$ and $E^c_F$ is equivalent to the question of lower semicontinuity of the function $E^d_F$, since $E^c_F$ is the greatest lower
semicontinuous convex function coinciding with the von Neumann entropy on the set of pure states. Despite the fact that the definition of the function $E^d_F$ seems more reasonable from the physical point of view (since it involves optimization over ensembles of quantum states rather than measures) the assumption of existence of a state $\omega_0$ such that $E^d_F(\omega_0) \neq E^c_F(\omega_0)$ leads to the following "nonphysical" property of the function $E^d_F$. For each natural $n$ consider the local measurement $\{M^n_k\}_{k \in \mathbb{N}}$, where

$$M_1 = \left( \sum_{i=1}^{n} |i\rangle \langle i| \right) \otimes I_K, \quad M_k = |n+k-1\rangle \langle n+k-1| \otimes I_K, \quad k > 1.$$  

It is clear that the sequence $\{\Phi_n\}_n$, where $\Phi_n = \{M^n_k\}_{k \in \mathbb{N}}$, of nonselective local operations tends to the trivial operation – the identity transformation (in the strong operator topology). Since the functions $E^d_F$ and $E^c_F$ satisfy requirement EM-2b) and EM-3b), for each $n$ we have

$$E^d_F(\omega_0) \geq \sum_{k=1}^{+\infty} \pi^n_k E^d_F(\omega^n_k) \geq E^d_F\left( \sum_{k=1}^{+\infty} \pi^n_k \omega^n_k \right) = E^d_F(\Phi_n(\omega_0)),$$

$$E^c_F(\omega_0) \geq \sum_{k=1}^{+\infty} \pi^n_k E^c_F(\omega^n_k),$$

where $\pi^n_k = \text{Tr} M^n_k \omega_0 M^n_k$ is the probability of $k$-th outcome and $\omega^n_k = (\pi^n_k)^{-1} M^n_k \omega_0 M^n_k$ is the posteriori state corresponding to this outcome [9], Ch. 4.

Since for each $n$ and $k$ the state $\text{Tr}_K \omega^n_k$ has finite rank, the above-mentioned result in [25] implies $E^d_F(\omega^n_k) = E^c_F(\omega^n_k)$. Thus the above two inequalities show that

$$E^d_F(\Phi_n(\omega_0)) = E^d_F\left( \sum_{k=1}^{+\infty} \pi^n_k \omega^n_k \right) \leq E^c_F(\omega_0)$$

for all $n$ and hence,

$$\limsup_{n \to +\infty} E^d_F(\Phi_n(\omega_0)) \leq E^d_F(\omega_0) - \Delta, \quad \Delta = E^d_F(\omega_0) - E^c_F(\omega_0) > 0,$$

despite the fact that the sequence $\{\Phi_n\}_n$ of nonselective local operations tends to the identity transformation. In contrast to this lower semicontinuity and LOCC-monotonicity of the function $E^c_F$ implies

$$\lim_{n \to +\infty} E^c_F(\Phi_n(\omega_0)) = E^c_F(\omega_0)$$

for any state $\omega_0$ and any sequence $\{\Phi_n\}_n$ of nonselective LOCC-operations tending to the identity transformation.

The another advantage of the function $E^c_F$ consists in its generalized LOCC-monotonicity (validity of requirements EM-2c)) following from Theorem [2] while the assumption $E^d_F \neq E^c_F$ means that the function $E^d_F$ is not lower semicontinuous, which is a real obstacle to prove the analogous property for this function.
5.2. The approximation of EoF. For given natural $n > 1$ consider the function $H_n$ on the set $\mathcal{S}(\mathcal{H})$ defined as follows
\[
H_n(\rho) = \sup_i \pi_i H(\rho_i),
\]
where the supremum is over all countable ensembles $\{\pi_i, \rho_i\}$ of states of rank $\leq n$ such that $\sum_i \pi_i \rho_i = \rho$. It is easy to see that the function $H_n$ is concave, satisfies condition (23), has the range $[0, \log n]$ and coincides with the von Neumann entropy on the subset of $\mathcal{S}(\mathcal{H})$ consisting of states of rank $\leq n$. By using the strengthened version of the stability property of the set $\mathcal{S}(\mathcal{H})$ in (32) it is shown that the function $H_n$ is continuous on the set $\mathcal{S}(\mathcal{H})$ and that the increasing sequence $\{H_n\}$ pointwise converges to the von Neumann entropy on this set.

By Theorem 2 the function $E_F^n = (H_n \circ \Theta)_{\mathcal{H}}^\mathcal{K}$ is an entanglement monotone satisfying requirements EM-1, EM-2c), EM-3c), EM-4) and EM-5c). It is easy to see that the function $E_F^n$ has the range $[0, \log n]$ and coincides with the function $E_F^n$ on the set
\[
\{\omega \in \mathcal{S}(\mathcal{H} \otimes \mathcal{K}) \mid \min\{\text{rank Tr}_\mathcal{K} \omega, \text{rank Tr}_\mathcal{H} \omega\} \leq n\}.
\]

By Proposition 7 the sequence $\{E_F^n\}$ provides approximation of the function $E_F^n$ on the set $\mathcal{S}(\mathcal{H} \otimes \mathcal{K})$, which is uniform on each compact set of continuity of the function $E_F^n$, in particular, on compact subsets of the set $\mathcal{K}_{\mathcal{H} \otimes \mathcal{I}_\mathcal{K}, h}$ for all $h > 0$, where $H_\mathcal{H}$ is a $\mathcal{H}$-operator in the space $\mathcal{H}$ such that $\text{Tr} e^{-\lambda H_\mathcal{H}} < +\infty$ for any $\lambda > 0$. Conditions of continuity of the function $E_F^n$ are considered in the next subsection.

5.3. Continuity conditions for EoF. Theorem 1 in [25] implies the following continuity condition for the function $E_F^n$, which can be also formulated as a continuity condition for the function $E_F^n$, since this condition implies coincidence of these functions.

**Proposition 8.** The function $E_F^n$ has continuous restriction to a set $A \subset \mathcal{S}(\mathcal{H} \otimes \mathcal{K})$ if either the function $\omega \mapsto H(\text{Tr}_\mathcal{H} \omega)$ or the function $\omega \mapsto H(\text{Tr}_\mathcal{K} \omega)$ has continuous restriction to the set $A$.

This condition implies the result mentioned in Example 6 (validity of requirement EM-5b) as well as the following observation.

**Corollary 8.** Let $\rho$ be a state in $\mathcal{S}(\mathcal{H})$. The function $E_F^n$ has continuous restriction to the set $\{\omega \mid \text{Tr}_\mathcal{K} \omega = \rho\}$ if and only if $H(\rho) < +\infty$.

**Proof.** It is sufficient to note that if $H(\rho) = +\infty$ then there exists a pure state $\omega \in \mathcal{S}(\mathcal{H} \otimes \mathcal{K})$ such that $\text{Tr}_\mathcal{K} \omega = \rho$.

By Corollary 8 for arbitrary continuous family $\{\Psi_t\}_t$ of local operations on the quantum system associated with the space $\mathcal{K}$ and arbitrary state $\omega \in \mathcal{S}(\mathcal{H} \otimes \mathcal{K})$ such that $\text{Tr}_\mathcal{K} \omega < +\infty$ the function $t \mapsto E_F^n(\Psi_t(\omega))$ is continuous.

For an arbitrary state $\sigma$ let $d(\sigma) = \inf\{\lambda \in \mathbb{R} \mid \text{Tr} \sigma^\lambda < +\infty\}$ be the characteristic of the spectrum of this state. It is clear that $d(\sigma) \in [0, 1]$. Proposition 8, Proposition 2 in [26] and the monotonicity of the relative entropy imply the following condition of continuity of the function $E_F^n$ with respect to the convergence
defined by the relative entropy (which is stronger than the convergence defined by the trace norm).

**Corollary 9.** Let \( \omega_0 \) be a state in \( \mathcal{S}(\mathcal{H} \otimes \mathcal{K}) \) such that either \( d(\text{Tr}_\mathcal{H}\omega) < 1 \) or \( d(\text{Tr}_\mathcal{K}\omega) < 1 \). If \( \{\omega_n\} \) is a sequence such that \( \lim_{n \to +\infty} H(\omega_n\|\omega_0) = 0 \) then \( \lim_{n \to +\infty} E_{\mathcal{F}}\left(\omega_n\right) = E_{\mathcal{F}}\left(\omega_0\right) \).

### § 6. Possible generalizations

The definitions of \( \sigma \)-convexity and \( \mu \)-convexity are naturally generalized to functions defined on an arbitrary convex closed subset of a locally convex space if any probability measure on this set has a well defined barycenter. The definitions of \( \sigma \)-convex and \( \mu \)-convex roofs also admit such generalizations but it is necessary to impose conditions providing correctness of these constructions.

There exists a class of convex subsets of locally convex spaces including all metrizable compact sets as well as several noncompact sets (in particular, the set \( \mathcal{S}(\mathcal{H}) \) of quantum states), to which the main results obtained in §2, 3 can be extended. This class of subsets called \( \mu \)-compact in [13] is studied in detail in [15], where possibility to extend several results well known for convex compact sets (in particular, the Choquet theorem on barycenter decomposition and the Versterstrom-O’Brien theorem) to \( \mu \)-compact sets is shown [16]. The last theorem states equivalence of the stability property of a convex \( \mu \)-compact set (which means openness of the convex mixture map) and several other properties, in particular, openness of the barycenter map and openness of the restriction of this map to the set of measures supported by extreme points.

By using results in [13], [15] it is easy to show that the all assertions in §2, 3 are valid for arbitrary convex stable \( \mu \)-compact set \( \mathcal{A} \) (instead of \( \mathcal{S}(\mathcal{H}) \)) such that \( \mathcal{A} = \sigma-\text{co}(\text{extr} \mathcal{A}) \). Stability of \( \mathcal{A} \) is used only in the proofs of Propositions 2, 4, Corollaries 1, 2, 4 the second part of Proposition 5, Theorem 1 and its corollaries while in the proofs of Proposition 3 and Corollary 3 it can be replaced by the weaker requirement of closedness of the set \( \text{extr} \mathcal{A} \), which is necessary for definition of the \( \mu \)-convex roof. The condition \( \mathcal{A} = \sigma-\text{co}(\text{extr} \mathcal{A}) \) is necessary for definition of the \( \sigma \)-convex roof and is used in the proofs of all assertions related with this construction.

### Appendix

**A1. Jensen’s inequalities for functions on Banach spaces.** Here sufficient conditions for validity of Jensen’s inequality (in discrete and integral forms) for convex functions on Banach spaces taking values in \([-\infty, +\infty] \) are presented. As a simple example showing importance of the conditions in the below propositions one can consider the affine Borel function on the simplex of all probability distributions with countable number of outcomes taking the value 0 on finite rank distributions and the value \(+\infty\) on infinite rank distributions. Other examples are considered in §2

By using Jensen’s inequality for finite convex combinations and a simple approximation it is easy to prove the following assertion.
Proposition A-1 (discrete Jensen’s inequality). Let \( f \) be a convex upper bounded function on a closed convex bounded subset \( A \) of a Banach space. Then for arbitrary countable set \( \{x_i\} \subset A \) with the corresponding probability distribution \( \{\pi_i\} \) the following inequality holds

\[
f \left( \sum_{i=1}^{+\infty} \pi_i x_i \right) \leq \sum_{i=1}^{+\infty} \pi_i f(x_i).
\]

Proposition A-2 (integral Jensen’s inequality). Let \( f \) be a convex function on a closed bounded convex subset \( A \) of a separable Banach space which is either lower semicontinuous or upper bounded and upper semicontinuous. Then for arbitrary Borel probability measure \( \mu \) on the set \( A \) the following inequality holds

\[
f \left( \int_A x \mu(dx) \right) \leq \int_A f(x) \mu(dx). \tag{27}
\]

(If \( A \) is a subset in \( \mathbb{R}^n \) then inequality (27) holds for any Borel function \( f \) taking values in \([-\infty, +\infty]\) and any Borel measure \( \mu \).)

Proof. Let \( \mu_0 \) be an arbitrary probability measure on the set \( A \). Let \( f \) be an upper bounded upper semicontinuous function. Then the functional \( \mu \mapsto \int_A f(x) \mu(dx) \) is upper semicontinuous on the set \( \mathcal{P}(A) \) of Borel probability measures on \( A \) endowed with the weak convergence topology \( [24], \S \, 2 \). Let \( \{\mu_n\} \) be a sequence of measures with finite support and the same barycenter as the measure \( \mu_0 \) weakly converging to the measure \( \mu_0 \). By convexity of the function \( f \) inequality (27) holds with \( \mu = \mu_n \) for each \( n \). By upper semicontinuity of the functional \( \mu \mapsto \int_A f(x) \mu(dx) \) passing to the limit \( n \to +\infty \) in this inequality implies inequality (27) with \( \mu = \mu_0 \).

Let \( f \) be a lower semicontinuous function. By using the arguments from the proof of Lemma [2] one can show that the function \( f \) is either lower bounded or does not take finite values. It is sufficient to consider the first case. Suppose that \( \int_A f(x) \mu(dx) < +\infty \). By applying the construction used in the proof of Lemma [1] it is possible to obtain a sequence \( \{\mu_n\} \) of measures on the set \( A \) with finite support such that

\[
\limsup_{n \to +\infty} \int_A f(x) \mu_n(dx) \leq \int_A f(x) \mu_0(dx), \quad \lim_{n \to +\infty} \int_A x \mu_n(dx) = \int_A x \mu_0(dx).
\]

By convexity of the function \( f \) inequality (27) holds with \( \mu = \mu_n \) for each \( n \). By lower semicontinuity of the function \( f \) passing to the limit \( n \to +\infty \) implies inequality (27) with \( \mu = \mu_0 \).

Corollary A-1. Let \( f \) be an affine lower semicontinuous function on a closed bounded convex subset \( A \) of a separable Banach space. Then for arbitrary Borel probability measure \( \mu \) on the set \( A \) the following equality holds

\[
f \left( \int_A x \mu(dx) \right) = \int_A f(x) \mu(dx). \tag{28}
\]
A2. One property of posteriori states. Let $\mathcal{M}$ be an arbitrary instrument on the set $\mathcal{G}(\mathcal{H})$ with the set of outcomes $\mathcal{X}$ [9], Ch. 4. For a given arbitrary state $\rho \in \mathcal{G}(\mathcal{H})$ let $\mu_\rho(\cdot) = \text{Tr} \mathcal{M}(\cdot)\rho$ be the posteriori measure on the set $\mathcal{X}$ and $\{\sigma(x|\rho)\}_{x \in \mathcal{X}}$ be the family of posteriori states corresponding to the a priori state $\rho$ [9, 28].

Lemma A-1. For arbitrary convex lower semicontinuous function $f$ on the set $\mathcal{G}(\mathcal{H})$ and arbitrary sequence $\{\rho_n\} \subset \mathcal{G}(\mathcal{H})$ converging to a state $\rho_0$ the following relation holds

$$\lim_{n \to +\infty} \inf \int_{\mathcal{X}} f(\sigma(x|\rho_n)) \mu_{\rho_n}(dx) \geq \int_{\mathcal{X}} f(\sigma(x|\rho_0)) \mu_{\rho_0}(dx).$$

Proof. It is sufficient to show that the assumption

$$\lim_{n \to +\infty} \int_{\mathcal{X}} f(\sigma(x|\rho_n)) \mu_{\rho_n}(dx) \leq \int_{\mathcal{X}} f(\sigma(x|\rho_0)) \mu_{\rho_0}(dx) - \Delta, \quad \Delta > 0, \quad (29)$$

leads to a contradiction.

Let $\nu_0 = \mu_{\rho_0} \circ \sigma^{-1}(\cdot|\rho_0)$ be the image of the measure $\mu_{\rho_0}$ under the map $x \mapsto \sigma(x|\rho_0)$. It is clear that $\nu_0 \in \mathcal{P}$ (see Remark 5) and that

$$\int_{\mathcal{X}} f(\sigma(x|\rho_0)) \mu_{\rho_0}(dx) = \int_{\mathcal{G}(\mathcal{H})} f(\rho) \nu_0(d\rho).$$

By separability of the set $\mathcal{G}(\mathcal{H})$ for given $m$ one can find a family $\{B_i^m\}_i$ of Borel subsets of $\mathcal{G}(\mathcal{H})$ such that $\nu_0(B_i^m) > 0$ for all $i$ and the sequence of measures

$$\nu_m = \left\{ \nu_0(B_i^m), \frac{1}{\nu_0(B_i^m)} \int_{B_i^m} \rho \nu_0(d\rho) \right\}_i$$

weakly converges to the measure $\nu_0$ (see the proof of Lemma 1 in [4]). Lower semicontinuity of the functional $\mu \mapsto \int_{\mathcal{G}(\mathcal{H})} f(\rho) \mu(d\rho)$ implies existence of such $m_0$ that

$$\sum_i \nu_0(B_i^{m_0}) f\left( \frac{1}{\nu_0(B_i^{m_0})} \int_{B_i^{m_0}} \rho \nu_0(d\rho) \right)$$

$$= \int_{\mathcal{G}(\mathcal{H})} f(\rho) \nu_{m_0}(d\rho) \geq \int_{\mathcal{G}(\mathcal{H})} f(\rho) \nu_0(d\rho) - \frac{1}{3} \Delta. \quad (30)$$

By using the finite family $\{\mathcal{X}_i\}$, $\mathcal{X}_i = \sigma^{-1}(B_i^{m_0}|\rho_0)$, of $\mu_{\rho_0}$-measurable subsets of $\mathcal{X}$ we can construct the family $\{\mathcal{X}_i'\}$ consisting of the same number of Borel subsets of $\mathcal{X}$ such that $\mu_{\rho_0}((\mathcal{X}_i' \setminus \mathcal{X}_i) \cup (\mathcal{X}_i \setminus \mathcal{X}_i')) = 0$ and $\bigcup_i \mathcal{X}_i' = \mathcal{X}$. For each $i$ the state

$$\sigma_0^i = \frac{1}{\nu_0(B_i^{m_0})} \int_{B_i^{m_0}} \rho \nu_0(d\rho) = \frac{1}{\mu_{\rho_0}(\mathcal{X}_i')} \int_{\mathcal{X}_i'} \sigma(x|\rho_0) \mu_{\rho_0}(dx) = \frac{\mathcal{M}(\mathcal{X}_i')|\rho_0}{\text{Tr} \mathcal{M}(\mathcal{X}_i')|\rho_0}$$

is the posteriori state, corresponding to the set of outcomes $\mathcal{X}_i'$ and the a priori state $\rho_0$. 

For each $i$ let $\sigma^i_n = \frac{\mathcal{M}(\mathcal{X}'_i) | \rho_n}{\text{Tr} \mathcal{M}(\mathcal{X}'_i) | \rho_n}$ be the posteriori state, corresponding to the set of outcomes $\mathcal{X}'_i$ and the a priori state $\rho_n$. By lower semicontinuity of the function $f$ and since $\lim_{n \to +\infty} \mathcal{M}(\mathcal{X}'_i) | \rho_n = \mathcal{M}(\mathcal{X}'_i) | \rho_0$ we have

$$\sum_i \mu_{\rho_n}(\mathcal{X}'_i) f(\sigma^i_n) \geq \sum_i \mu_{\rho_0}(\mathcal{X}'_i) f(\sigma^i_0) - \frac{1}{3} \Delta$$

(31)

for all sufficiently large $n$.

By Jensen’s inequality (Proposition A-2) convexity and lower semicontinuity of the function $f$ implies

$$\mu_{\rho_n}(\mathcal{X}'_i) f(\sigma^i_n) \leq \int_{\mathcal{X}'_i} f(\sigma(x) | \rho_n)) \mu_{\rho_n}(dx) \quad \forall i, n.$$  

(32)

By using (30)–(32) we obtain

$$\int_{\mathcal{X}'} f(\sigma(x) | \rho_n)) \mu_{\rho_n}(dx) = \sum_i \int_{\mathcal{X}'_i} f(\sigma(x) | \rho_n)) \mu_{\rho_n}(dx) \geq \sum_i \mu_{\rho_n}(\mathcal{X}'_i) f(\sigma^i_n)$$

$$\geq \sum_i \mu_{\rho_0}(\mathcal{X}'_i) f(\sigma^i_0) - \frac{1}{3} \Delta \geq \int_{\Theta(\mathcal{H})} f(\rho) \nu_0(\rho) - \frac{2}{3} \Delta$$

for all sufficiently large $n$, which contradicts to (29).

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13Since $\text{Tr} \mathcal{M}(\mathcal{X}'_i) | \rho_0 = \mu_{\rho_0}(\mathcal{X}'_i) > 0$ the state $\sigma^i_n$ is correctly defined for all sufficiently large $n$. 


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