Naked Singularity, Black-hole and the Mass Loss in a Spherically Symmetric Gravitational Collapse

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Abstract

A distinguishable physical property between a naked singularity and a black-hole, formed during a gravitational collapse has important implications for both experimental and theoretical relativity. We examine the energy radiated during the spherically symmetric Gravitational collapse in this context within the framework of general relativity. It is shown that total energy radiated (Mass Loss) during the collapse ending in a naked singularity scenario cannot be more than the case when a collapse scenario ends in a black-hole. In cases of interest (for example stars having same mass, size, and internal composition) considered here the total energy released in the collapse ending in a black hole can be considerably more than the case otherwise.
1 Introduction

The universe is full of super high energy events. For example quasars (and other active galactic nuclei) radiating $10^3$ times more than the luminosity of a galaxy, supernova explosions emitting huge amounts of energy within a few days to a year and gamma ray bursts which radiate energy of the order of $10^{51}-10^{54}$ ergs within a duration of few milliseconds to few seconds. Despite numerous attempts, an ideal explanation is yet to be found. It is, however, widely believed that probably gravitational collapse is responsible for such events and that during a dynamic collapse huge amount of energy would be released, possibly due to the conversion of the rest mass of the star into energy. A gravitational collapse occurs when a star with sufficient mass has exhausted its nuclear fuel. As this stellar collapse progresses further, gravitational forces become very strong resulting in the development of closed trapped surfaces. The development of this surface signals the formation of a region of space time from where no causal particles could escape. According to singularity theorems a space time singularity must develop generically during a gravitational collapse specially once a trapped surface develops. These theorems, however do not say any thing about the nature of such singularities and its physical properties.

With the advent of cosmic censorship conjecture (CCH) and subsequent development of Black-hole physics, efforts have been devoted over the last two decades on the study of the dynamics of a gravitational collapse of physically reasonable matter in the context of CCH. Conclusions from these are that singularities both naked and covered would form generically
during the collapse (see [1, 2, 3] and the references therein). The implication of these studies is that even if the collapse produces unbounded curvature and density, trapped surface may not develop early to prevent the exposure of the singular regions to an outside observer.

Considerable work has appeared in the literature which attribute these high energy events (for example gamma ray bursts) in the cosmos to the formation and existence of a naked singularity during the gravitational collapse (See [4, 5] and further references therein). The suggestion is that contrary to the black hole scenario, in case of naked singularity, the high curvature region of the space time (where the naked singularity is formed) is open to the outside and therefore energy of this region somehow (either due to quantum particle production [4, 5] or formation of a compact fireball [4]) would lead to the high energy emission from the vicinity of the naked singularity and onward to the outside world. This is quite natural and one would expect that since an extra high curvature region of the space time with extreme conditions is available to radiate energy in the case of a naked singularity, the energy released should be more than the case of a black hole. Thus this could also possibly be a observable signature of naked singularity.

Based on CCH Black-hole physics has developed significantly. Similarly could the formation of a naked singularity also signal a physical phenomena responsible for huge amounts of energy release. An important aspect is whether the occurrence of a singularity (naked or otherwise) is just a mathematical problem due to the mathematical structure of the field equations or do they have a deeper physical meaning?. Investigation, therefore, of the two scenarios namely naked singularity and black-hole from the perspective of the total mass
loss (energy radiated) during the gravitational collapse is of theoretical interest in its own right in general relativity. Moreover is there a distinguishable difference between the amount of energy released during a collapse process which is ending either as a black hole or a naked singularity under similar conditions. The implication of such a study within the framework of general relativity could not be overemphasized.

We make such an attempt in this paper and consider spherically symmetric gravitational collapse from the point of view of examining the energy radiated (Mass Loss) during the collapse ending in a singularity within the framework of general relativity. We calculate the remnant mass of the collapsing star which is shown to be related to the initial mass and the initial pressure at the boundary of the star. It is then shown that the total energy radiated in the model of a collapsing star considered here can be same for both end states (naked singularity or black hole) i.e. One can have, for equal amount of energy radiated during the collapse, different sets of parameters at the onset of collapse leading the star either to a naked singularity or a black hole. Thus a naked singularity scenario in general would not radiate more energy than a black hole scenario. An equally important result is that for two stars having the same mass, size, and the internal structure, the star ending in a black hole would suffer a greater mass loss then its counterpart that is naked singularity.

Here is the summary of what we intend to do and how we organize rest of our paper. We would of course consider the collapse scenario within the framework of classical general relativity. A typical scenario involves a massive star on a continuous gravitational collapse starting at some time \( t = t_i \). In the next section we therefore consider such a spherically
symmetric matter cloud representing type I matter field [7]. The radiating Vaidya space
time describes the space-time outside the moving boundary of such a collapsing star. The
conditions on the initial density etc. are then examined for the collapse to end either as a
black-hole or a globally naked singularity. Next in the section on Mass Loss we discuss the
mass function in Vaidya space-time and it’s relation to the interior parameters. The remnant
mass of the star is then calculated in terms of mass, density and pressure of the star at the
onset of collapse. In appendix A we discuss the interior matter field in the context of field
equations and energy conditions. Appendix B is devoted to the star model and the boundary
conditions.
2 The Collapsing Star

Despite the numerous exact solutions of the field equations, very few describe a physically reasonable collapsing matter cloud. In fact in nearly all the studies of spherically symmetric collapse, Tolman-Bondi-Lemitre metric (TBL) \cite{8} has been used extensively and very well may be the only reasonable exact solution available. While dust is a possible equation of state describing late stages of collapse \cite{11}, it is an idealized form of matter with vanishing pressure. Total mass enclosed in the dust cloud is time independent (Exterior space-time is Schwarzschild) and no energy is radiated during such a collapse. We therefore consider a general matter field with non vanishing pressures. The metric describing a spherically symmetric space-time is given by

\[ ds^2 = -dt^2 + \frac{R'^2}{1 + f}dr^2 + R^2 d\Omega^2, \]  

(1)

where \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \), \( R = R(t, r) \) and \( f = f(t, r) \) are arbitrary functions of \( t \) and \( r \).

The metric in (1) has to satisfy field equations which can be put in a better form if we put

\[ \dot{R}^2 = f + \frac{F}{R} \Rightarrow \dot{R} = -\sqrt{f + \frac{F}{R}} \]  

(2)

\( F = F(t, r) \), \( R = R(t, r) \) and \( f(t, r) > -1 \) are \( C^2 \) functions of \( t \) and \( r \) throughout the cloud. Notation (’) and (\cdot) are used to denote partial differentiation with respect to \( r \) and \( t \). \( F = F(t, r) \) is interpreted as the mass function and for physical reasons \( F(t, r) \geq 0 \), \( F'(t, r) \geq 0 \). \( f(t, r) \) may be interpreted as total energy and classifies the space time as
bound, marginally bound and unbounded as per $f < 0$, $f = 0$ and $f > 0$. For collapse $\dot{R} < 0$. $R(t, r)$ is called the area radius in the sense that $4\pi R^2(t, r)$ gives the proper area of the mass shells for a given value of coordinate $r$ at time $t$ and the area of such a shell vanishes when $R(t, r) = 0$. In this sense curve $t = t(r)$ such that $R(t(r), r) = 0$ describe the singularity in the space-time where mass shells collapse to zero volume and where curvatures diverge. To avoid shell crossing we would require $R' > 0$. The specification of arbitrary functions $F$ and $f$ gives a particular solution of the field equations, for example $F = F(r)$, $f = f(r)$ gives the TBL models. (For the metric (1) satisfying field equations in general see Appendix A.)

Our intentions and aim in this paper are certainly not to find another exact solution of the field equations prescribing a particular matter field satisfying a particular equation of state, but to investigate within the framework of general relativity mass loss when physically allowed reasonable evolution develop into a singularity naked or covered. The answer to the problem would have been simple if we had at our disposal both an exact closed equation of state describing the state of a collapsing matter and an exact solution of the field equation. However both of these are little understood in highly dense regions in relativity. Dust solutions being an exact solution has been widely used for investigating the occurrence of naked singularities and black holes (see [1] and references therein), however as pointed out earlier it is of limited use in the context of our work here. We therefore consider a general physically reasonable matter field (i.e. type I matter fields) satisfying energy conditions. Let
us consider matter cloud given by

\[ f(t, r) = -f_0(r) + f_1(t, r), \quad F(t, r) = F_0(r) - f_1(t, r)R \] (3)

Where \( F_0(r) \geq 0, f_0(r) \geq 0 \) and \( f_1(t, r) \geq 0 \) are functions which are \( C^2 \) throughout the matter cloud. For type I fields satisfying energy conditions it follows (For details of the solution for such a consideration in equation (3), including energy conditions etc. see Appendix A.)

\[ F'_0 \geq f'_1 R, 1 > f_0 - f_1 \geq 0, \quad \dot{F} \leq 0, \quad F'_0 \geq 0 \] (4)

Because of equation (3) equation (2) now reads

\[ \dot{R} = \sqrt{-f_0 + \frac{F_0}{R}} \] (5)

We now elaborate on equation (3) and such a particular choice. First note that in case \( f_1(t, r) = 0 \) the space-time given in metric (1) reduces to TBL dust models. The structure of differential equation (5) determining \( R(t, r) \) remains as in dust case implying same functional dependence of \( R(t, r) \) as a function of \( t \) and \( r \) as in dust. The metric in (1) is therefore similar to TBL model except for the extra term \( f_1 \) in \( f = f_0(r) - f_1(t, r) \) appearing in the denominator of the \( g_{rr} \) term in the metric. The requirement that the space-time be singularity free, in the sense that Krichmann scalar, principle density and pressures etc. do not diverge prior to the formation of shell focusing singularity, implies \( f_1 = r^n h(t, r), n \geq 2 \) with \( h(t, r) \) being at least \( C^2 \) function. Therefore the analysis regarding the formation of
a naked singularity or a black-hole in the gravitational collapse for such a model remains similar to dust case. The advantage of such a consideration is that while the analysis and conditions for occurrence of singularity either naked or covered would be same as dust, we now have in the space time a type I matter field satisfying energy conditions with non vanishing pressures and which could be matched to Vaidya space-time outside the star.

The dust models have been extensively investigated for occurrence of singularities naked and otherwise. Because of the relevance of dust models to our work we give, before a general consideration, a very brief account of dust solutions and use the needed results while referring the reader to articles in literature for details [1]. By suitably scaling coordinate $r$ at time $t = 0$ (when the collapse starts) such that $R(0, r) = r$, we get from equation (5)

$$t - t_0(r) = -\frac{R^{3/2}}{\sqrt{F_0}} G\left(\frac{R}{r}\right),$$

(6)

$$G(y) = \frac{\arcsin\sqrt{y}}{y^{3/2}} - \frac{\sqrt{1 - y}}{y}, G(0) = \frac{2}{3}, \quad t_0(r) = \frac{\pi r^{3/2}}{2\sqrt{F_0}}$$

(7)

Where we have used the fact that for the mass shells to start from rest at $t = 0$ $\dot{R}(0, r) = 0 \Rightarrow F_0(r) = r f_0(r)$. For physical reasons we have restricted ourselves to bound cases only that is $0 \geq f > -1$ throughout. In order to avoid shell crossing we require $t'_0(r) \geq 0$. The function $f_0(r) = f_r r^2 g(r), g(0) > 0$ is such that $g(r)$ is at least $C^2$ function of $r$ throughout. This ensures that there are no singularities in the space-time prior to the formation of first singularity at the center $r = 0$ which occurs at time $t = t_s = t_0(0)$ such that $R(t_s, 0) = 0$. Only the first central singularity could be naked while all other for $r > 0$ are necessarily
covered. The central shell focusing singularity is globally naked if

\[ \frac{3\pi f_c}{4f_{3/2}^3} > 13 + \frac{15\sqrt{3}}{2} \]  

(8)

Where \( f_c > 0 \) and \( f_{co} \geq 0 \) are constants given by

\[ f_0(r) = f_cr^2g(r) = f_cr^2(1 - f_{co}r^3 + r^4f_{01}(r)), \]  

(9)

\( f_{01}(r) \) being \( C^2 \) differentiable function throughout the dust cloud. In general \( f_0(r)/r^2 \) would have terms of the order of \( r \) and \( r^2 \) as well, however the resulting naked singularity is gravitationally weak. For occurrence of strong curvature singularity \( f_0(r) \) must have the form in equation (9). If the condition in equation (8) is not satisfied than the singularity is covered (i.e. black hole).

Let us now consider a general scenario. If the spherically symmetric star considered above is to radiate energy the most general external metric is Vaidya space time

\[ ds^2 = -(1 - \frac{2m(u)}{r})du^2 - 2dudr + r^2d\Omega^2, \]  

(10)

where the arbitrary function \( m(u) \) of the retarded time \( u \) represent the mass of the system outside the star. The Matching conditions are given in equations (29) to (33)(see appendix B for details). The motion of the boundary in Vaidya space-time is given by \( r = R(t(u), r_b) \)

Equation (32) relates retarded time \( u \) to time \( t \). At \( t = 0, u = u_i \) and as \( t \rightarrow t_{ah}, u \rightarrow \infty \),

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$t_{ab}$ is the time when the outermost shell at $r = r_b$ meets the apparent horizon $R(t_{ab}, r_b) = F(t_{ab}, r_b)$. Using equation (3) and (29) we have

\[
F_0(r_b) - f_1(t, r_b)R(t, r_b) = 2m(u), \tag{11}
\]

The matching of the two space-time across the boundary $r = r_b$ restricts the function $f_1(t, r)$ to the form given in equation (33) and we have for $f(t, r_b)$

\[
f_1(t, r_b) = \frac{4c(r_b)(1 - f_0(r_b)) \exp(2Y_0(r_b) \arccos(\sqrt{\frac{R(t, r_b)}{r_b}})))}{(1 - c(r_b) \exp(2Y_0(r_b) \arccos(\sqrt{\frac{R(t, r_b)}{r_b}}))))^2} \tag{12}
\]

where functions $c(r), Y_0(r)$ are related to initial pressure $p_r(r)$ and $p_\theta(r)$ (see appendix A) via equations (27) and (28). $f_0(r)$ is an important quantity in our calculations and we get from equations (27) and (11)

\[
f_0 = r^2 p_r + \frac{2M(r)}{r}, \quad 2M(r) = k \int \rho r^2 dr \tag{13}
\]

\[
f_0(r_b) = \frac{2m_0}{r_b} + kr_b^2 p_r(r_b), \quad m_0 = M(r_b) = m(0) \tag{14}
\]

where the initial density $\rho \geq 0$ and pressure $p_r \geq 0$ and $p_\theta \geq 0$ are $C^2$ functions throughout the cloud. For physical reasonableness we require that $\rho' \leq 0, p'_r \leq 0, p'_\theta \leq 0$. Thus we now have a matter cloud with non vanishing pressures and where the space outside the collapsing star is described by the Vaidya space time. The mass function $m(u)$ is related to initial density and pressures of interior matter cloud making it possible to examine mass loss and that is precisely what we look for in the next section.
3 Mass Loss

Prior to the onset of collapse the star might appear to be a normal radiating star to an outside observer receiving radiation at some rate. These radiation could have been coming for a long time in the past as they do in most cases. Our interest here, however, is only to estimate the energy radiated (mass loss) once the collapse starts.

The exterior metric outside the moving boundary \( r = R(t(u), r_b) \) is given by Vaidya metric. Thus the star with radius \( r = r_b \) at the onset of collapse at \( t = 0 \) continues radiating till the outermost shell given by \( r = r_b \) meets the surface \( R = F \). Actually the surface \( R = F \) is characterized as apparent horizon and represent the boundary beyond which all causal particles are ingoing. From equation (32) it follows that as \( R(t, r_b) \to F(t, r_b) \) \( u \to \infty \) thus while the time \( t \) is finite for an inside observer, an outside observer would take infinite amount of time for the surface of the star to reach the apparent horizon. Once the boundary of the star meets the apparent horizon there would not be any outgoing radiation. This is the stage at which the remaining mass of the collapsing cloud disappears into the black hole and becomes the mass of the black hole. The mass of the star at the onset of collapse is given by equation (14). we have

\[
x_0 = \frac{2m_0}{r_b} = E_0 - T_0, \quad E_0 = f_0(r_b), \quad T_0 = k r_b^2 p_r(r_b)
\]

\( x_0 \) actually is the ratio of the Schwarzschild radius \( 2m_0 \) of the star with mass \( m_0 \) at the onset of collapse, to the radius of the star \( r_b \). At the time \( t = 0 \) the boundary of the star \( r = r_b \)
starts collapsing. The star continues to radiate till its boundary \( r = r_b \) meets the apparent horizon \( R = F \) at time \( t = t_{ah} \) which is given by

\[
R(t_{ah}, r_b) = F(t_{ah}, r_b) \tag{16}
\]

Using equations (3)

\[
1 + f_1(t_{ah}, r_b) = \frac{f_0(r_b)}{R(t_{ah}, r_b)} \tag{17}
\]

Using equations (3), (11) to (16) and eliminating \( c(r_b) \) by using equation (27) we get

\[
\frac{\sqrt{1-x_0} - \sqrt{1-E_0}}{\sqrt{1-x_0} + \sqrt{1-E_0}} = \frac{\sqrt{E_0(1-x_s)} - \sqrt{x_s(1-E_0)}}{\sqrt{E_0(1-x_s)} + \sqrt{x_s(1-E_0)}} \exp(-2y_0 \arccos(\sqrt{x_s})) \tag{18}
\]

where we have put

\[
x_s = \frac{2m_r}{r_b} = \frac{2m(\infty)}{r_b}, \quad y_0 = \sqrt{\frac{1-E_0}{E_0}} \quad \frac{x_s}{x_o} = \frac{m_r}{m_o} \tag{19}
\]

\( 2m_r = F(t_{ah}, r_b) = 2m(\infty) \) is the remnant mass of the star. The above equation relates final mass of the collapsing cloud to the initial mass of the star at the onset of collapse. The value of \( E_0 \) is the parameter which determines the exact form of this relation via the above equation. Thus for a given initial or remnant mass and \( E_0 \) at the onset of collapse the remnant mass or the initial mass as the case may be, can be calculated from equation (18).

It should be noted at this point from equation (8) that condition for a strong curvature naked singularity or a black hole depend on the ratio \( f_c/f_{c0}^{3/2} \), whereas in equation (18)
\[ E_0 = f_0(r_b) \] is the value of the \( f_0(r) \) (equation (9)) at the boundary of the star \( r = r_b \). Thus for a given value of \( E_0 \) one still has the freedom to choose the different set of parameters \((f_c, f_{co})\) so as to satisfy the condition for either naked singularity or a black hole. Therefore for any set of initial and remnant mass in the case of naked singularity resulting in a certain mass loss one can have a black hole also with a different set of structure in the form of density and pressure distribution functions at the onset of collapse. Hence we can conclude that naked singularity scenario can not radiate more energy than a black hole scenario. The same is true even if the naked singularity is weak.

We next consider the case when the total mass, size and the internal structure of the star is same. Let us consider two stars with same mass \( m_o \), radius \( r = r_b \), density distribution \( \rho = \rho_c \rho_o(r) \) and pressure distribution \( p_r = p_c p_o(r), p_\theta = p_{\theta c} p_{\theta o}(r) \) at the initial epoch of collapse. Here only central density \( \rho_c \) and central pressure \( p_c \) could be different while the internal structure is kept same i.e. \( \rho_o(r), p_o(r) \) are same. This imply that for both cases \( f_0(r) = f_c r^2 g(r) \) where \( g(r) \) is kept the same but different values of \( f_c \) are allowed. In fact \( f_c = 3 \rho_c + p_c \). It follows that for \( f_c > f_{crit} \) the collapse would terminate in a black hole while for \( f_c \leq f_{crit} \) would end in a naked singularity, \( f_{crit} \) being the critical value determined by equation (8). Physically what it means that higher central density and pressure at the onset of collapse lead to black-hole while lesser ones lead the collapse to naked singularity. Therefore black hole scenarios occur for all values of \( E_0 \) greater than certain critical value while less than the critical value would end in a naked singularity. Equation (18) implies that for a constant value of \( x_0 \), \( x_s \) decreases for increasing value of \( E_0 \) for the typical behavior.
see Fig 1). Therefore scenarios which end up in black hole would have a less remnant mass than the naked singularity scenario having the same initial mass. Hence the collapsing stars which terminate in a black hole would suffer a higher mass loss and radiate more energy than the stars which end up in naked singularity having the same mass, size and same internal structure at the onset of collapse.

Another interesting scenario is where the collapse scenario is determined completely by a single parameter. Let us consider at the onset of collapse, matter cloud satisfying adiabatic equation of state, i.e. \( p_r(r) = p_\theta(r) = a \rho(r) \), \( a \) being a constant. Here we consider the two scenarios of naked singularity and black-hole which occur due to the different values of \( a \) and where the total mass, size, and the density composition of the star are same. We have

\[
E_0 = x_0 + a \rho_b
\]  

(20)

\( \rho_b = \rho(r_b) \) is the initial surface density of the star. Using equations (9), (27) and (28) the condition for strong curvature naked singularity given by equation (8) becomes.

\[
\frac{3\pi(1 + 6a)}{4(1 + 3a)^{3/2}} \rho_1 \geq 13 + \frac{15}{2} \sqrt{3}, \rho_1 = \frac{\rho'''(0)}{\rho(0)^2}
\]  

(21)

Thus for a given \( \rho_1 \) naked singularity occurs for \( a \leq a_{\text{crit}} \) and a black hole for \( a > a_{\text{crit}} \).

The behavior of the ratio of remnant mass \( m_r \) to the initial mass \( m_o \) remains the same as in previous consideration. That is for increasing \( a \) this ratio decreases implying that black hole scenario would radiate more. A typical behavior is shown in Fig 2.
Within the framework of classical general relativity the overall consideration of the collapsing star (model incorporating an interior and a exterior space-time) as considered here does not seem to point towards the significance of the formation of a naked singularity in respect of energy radiated. Instead black hole scenario radiate more energy. Thus the occurrence of a naked singularity may not be physically more relevant than a black-hole and even if naked singular scenario does free up a region of space-time for radiation, the overall energy radiated does not seem to increase. We have considered here a typical model (though the space-time considered represent quite a wide class). However in general for spherically symmetric collapse satisfying physically reasonable conditions as considered here (for example type I field, energy conditions etc.) the results should be the same qualitatively.

4 Appendix A.

In general relativity the metric describing the space time must satisfy the field equations $G_{ab} = kT_{ab}$, where $k$, $G_{ab}$ and $T_{ab}$ are the coupling constant, Einstein and stress energy tensors respectively. For a spherically symmetric space-time described by the metric in equation (1) we have

$$-k_0 T_{t}^t = \frac{F'}{R^2 R^e} - \frac{\dot{f} \dot{R}}{R(1 + f)} = \frac{(F_o - f_1 R)'}{R^2 R^e} - \frac{\dot{f}_1 \dot{R}}{R(1 - f_0 + f_1)}$$

(22)

$$k T_{v}^v = \left( \frac{\dot{F}}{R^2 R^e} + \frac{\dot{f}}{R R^e} \right) = \frac{f_1}{R^2}, \quad k T_{t}^t = \frac{-\dot{f} \dot{R}}{R(1 + f)} = \frac{-f_1 \dot{R}}{R(1 - f_0 + f_1)}.$$  

(23)

$$2T_{\theta}^\theta = \left( f - R \ddot{R} + \dot{R}^2 \right)' \frac{1}{2 R R^e} + \frac{\dot{f}}{1 + f} \left( \frac{\dot{R}}{R} + \dot{\dot{f}} + \frac{\dot{R}^e}{R^e} - 3 \frac{\dot{f} \dot{R}}{2(1 + f)} \right) = \frac{f_1}{R R^e}$$
\[ + \frac{\dot{f}_1}{1 - f_0 + f_1} \left( \frac{\ddot{f}_1}{2f_1} - \frac{3\dot{f}_1}{2(1 - f_0 + f_1)} + \frac{F_0'}{RRR'} - \frac{f_0'}{RR'} - \frac{F_0}{R^2R} \right) \]  

(24)

Second part of the above give the non vanishing components of the stress energy tensor for the matter cloud given by equation (3). We are interested only in type I matter field satisfying energy conditions [10, 9]. It follows that throughout the cloud

\[ F_0' \geq f_1'R, \dot{F} \leq 0 \]
\[ \frac{F_0'}{R^2R'} - \frac{f_1'}{RR'} - \frac{2f_1}{R^2} - \frac{\dot{f}_1\dot{R}}{2R(1 - f_0 + f_1)} \geq 0 \]  

(25)

Using the notation \( \sigma = -T_t^t, P = T_r^r \) and \( q = T_t^r \).

\[ \epsilon = \frac{\sigma - P + \sqrt{(\sigma + P)^2 - 4q^2}}{2}, \quad P_r = \frac{-\sigma + P + \sqrt{(\sigma + P)^2 - 4q^2}}{2}, P_\theta, P_\phi \]  

(26)

where the four real eigenvalues \( \epsilon = \epsilon(t, r), P_r = P_r(t, r) \) and \( P_\theta = P_\phi = T_\theta^\theta \) represent the principle density, principle radial pressure and principle tangential pressure respectively. At \( t = 0 \) \( T_{tr} = q = 0 \) and therefore initial density \( \rho(r) = \sigma(0, r) = \epsilon(0, r) \) and initial radial pressure \( p_r(r) = P(0, r) = P_r(0, r) \). Using equations (3), (12), and (22) to (25) we get after some simplification

\[ kr^2\rho(r) = (rf_0 - kr^3p_r)', kr^2p_r = f_1(0, r) \rightarrow \frac{4c}{(1 - c)^2} = \frac{kr^2p_r}{1 - f_0} \]  

(27)

\[ p_\theta = P_\theta(0, r) = \frac{(r^2p_r)'}{2r} + \frac{p_rY_0^2f_0}{2(1 - f_0 + r^2P_r)} \]  

(28)
5 Appendix B.

Nearly all distant objects (stars) in our universe radiate energy in the form of light like particles. Thus the exterior space time outside a spherically symmetric star is rightfully described by the Vaidya space time. The energy for these radiation comes from within the star. The spherically symmetric matter cloud prior to the onset of collapse may be radiating energy in the form of light like particles and in the very distant past may even not be radiating (i.e. in the very distant past the space time outside the star may be Schwarzschild). We therefore discuss the related boundary conditions relevant to our work in this appendix.

We first start with the interior space time metric in equation (1) for which describe the collapsing spherically symmetric matter cloud

\[ ds^2 = -dt^2 + \frac{R'^2}{1 + f}dr^2 + R^2 d\Omega^2, \]

At the onset of collapse at \( t = 0 \) \( R(0, r) = r \). The exterior Vaidya space time is given by equation (11)

\[ ds^2 = -(1 - \frac{2m(u)}{r})du^2 - 2dudr + r^2 d\Omega^2 \]

The boundary \( \Sigma \) in the internal metric is \( r = r_b = constant \). The Darmois conditions for matching the space-time require that the first and second fundamental forms of the interior and exterior metric are continuous across \( \Sigma \) \( \square \). This is straightforward and we get the
following after using equation (3) and some simplification

\[ r = R(t(u), r_b), \quad F(t, r_b) = 2m(u) \]  

(29)

\[ \frac{dt}{du} = \sqrt{1 + f(t, r_b)} - \sqrt{f(t, r_b) + \frac{F(t, r_b)}{R(t, r_b)}} \]  

(30)

\[ \hat{F}|_{r=r_b} = - \left( fR(1 - \sqrt{f + \frac{f}{R}})^{r=r_b} \right) \]  

(31)

Using equation (3)

\[ u - u_i = \int_0^t \frac{dt}{\sqrt{1 - f_0(r_b) + f_1(t, r_b) - \sqrt{f_0(r_b)\sqrt{r_b} - 1}}} \]  

(32)

At the onset of collapse at \( t = 0, u = u_i \). The apparent horizon is given by \( R = F \) and \( t = t_{ah} \) is the time when the boundary of the star \( r = r_b \) meets the apparent horizon, it therefore follows from the above equation (32) \( t \to t_{ah}, u \to \infty \) and thus \( \infty > u \geq u_i \). Equation (31) imply that function \( f_1(t, r) \) must of the form

\[ f_1(t, r) = \frac{4c(r)(1 - f_0(r)) \exp\left(2Y_0(r) \arccos\left(\frac{\sqrt{R}}{r}\right)\right)}{(1 - c(r) \exp\left(2Y_0(r) \arccos\left(\frac{\sqrt{R}}{r}\right)\right)^2 + r^2 f_2(t, r), Y_0(r)} = \sqrt{\frac{1 - f_0(r_b)}{f_0(r_b)}} \]  

(33)

where \( f_2(t, r) \) is a arbitrary \( C^2 \) function through out the cloud such that \( f_2(0, r) = f_2(t, r_b) = 0 \). Though not essential from the point of view of our purpose however we should discuss briefly on the nature of the space time prior to gravitational collapse not only from the theoretical interest but also to complete the discussion. This intermediate state prior to
the collapse should preferably be a compact star in equilibrium with outside space time preferably a radiating metric like Vaidya space times (or some generalization of it) matched at some fix boundary \( r = r_b \). The exact form of interior metric for \( t < 0 \) would depend actually on the specific form of the arbitrary function \( f_1(t, r) \) and its behavior near \( t = 0 \). However for the sake of completeness we consider the same. Let us consider the interior metric of the star for \( t < 0 \) which may be described by

\[
ds^2 = -A dT^2 + B dr^2 + r^2 d\Omega^2,
\]

where \( A(t, r) > 0 \) \( B(t, r) > 0 \) are at least \( C^2 \) functions through out the cloud \( r_b \geq r \geq 0 \).

Taking into account the fact that at \( t = 0 \) \( R(0, r) = r, \dot{R}(t, r) = 0 \) for the metric in equation (1) Darmois matching conditions for these interior metric at the surface \( t = 0 \) are

\[
B(0, r) = f_0(r) - f_1(0, r), \quad \frac{\dot{B}(0, r)}{\sqrt{A(0, r)}} = \frac{-\dot{f}_1(0, r)}{1 - f_0 + r^2 p_r}.
\]

Prior to collapse the exterior metric outside the static boundary \( r = r_b = constant \) of the star could be Vaidya space-time.

\[
ds^2 = (1 - \frac{2M_p(u)}{r})du^2 - 2udu dr + r^2 d\Omega^2,
\]

Matching conditions at the boundary \( r = r_b \) require

\[
\frac{dT}{du} = \sqrt{\frac{1 - \frac{2M_p(u)}{r_b}}{A(T, r_b)}}.
\]
\[
\frac{1}{B(T, r_b)} = 1 - \frac{2M(u)}{r_b}
\]

\[
A(T, r_b)A'(T, r_b) = \left(1 - \frac{2M_p(u)}{r_b}\right) \frac{2M_p(u)}{r_b} - 2 \frac{dM_p(u)}{du}
\]

(39)

Thus a set of functions \((A, B)\) satisfying the above conditions at the onset of collapse would describe the interior and exterior of the star prior the onset of collapse. Because of the generality of the two arbitrary functions \(A(t, r)\) and \(B(t, r)\) the suitable solution satisfying boundary conditions would exist for a given \(f_1(t, r)\). Though the exact expressions for \(A(T, r)\) and \(B(T, r)\) would depend on the exact form of \(f_1\) we can consider a simple example. Consider For the function \(f_1\) given in equation (33) such that \(f_2 = 0\) The space-time prior to onset of collapse therefore is given by

\[
A = (1 + a_1(r)M_p(1 - \frac{2M_p}{r_b} - r_b \frac{dM_p}{du})^{1/2})
\]

(40)

\[
B = (1 - B_0(1 - \frac{B_1}{b_1}) - 2B_1 \frac{dM_p}{b_1 r_b \frac{du}{du}})^{-1}
\]

(41)

where \(a_1(r)\) and \(M_p(u)\) are arbitrary functions of \(r\) and \(u\) respectively such that

\[
a_1'(r_b) = 2r_b, \quad M_p(u_i) = M_0, \quad \left(\frac{dM_p}{du}\right)_{u=u_i} = -(E_0 - x_0)\sqrt{1 - x_0}
\]

(42)

\[
B_0 = B_0(r) = f_0(r) - r^2p_r(r), \quad b_0 = B_0(r_b)
\]

(43)

\[
B_1 = B_1(r) = -r^2p_r \sqrt{1 - f_0 + r^2p_r}, \quad b_1 = B_1(r_b)
\]

(44)
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[7] In fact all known and observed physical matter field are of type I except the directed radiations (Vaidya space-time) which are type II. A type I matter field is characterized by the existence of four real eigenvalues $\lambda_{(i)}$ of the energy momentum tensor i.e. $T^{ab} = \sum_{i=0}^{3} \lambda_{(i)} E^{a}_{(i)} E^{b}_{(i)}$, $E^{a}_{(i)}$ being an orthonormal basis. $\lambda_{0} = \rho$ associated with timelike basis vector $E^{(0)}$ represent the principle density while $P_{1} = \lambda_{1}, P_{2} = \lambda_{2}, P_{3} = \lambda_{3}$ are principal presseurs. The energy conditions require $\rho \geq 0, \rho + P_{1} \geq 0, \rho + P_{2} \geq 0$ and $\rho + P_{3} \geq 0$. For more details see [10].
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Figure 1: Plot showing the ratio of the remnant mass to the initial mass (i.e. \( \frac{m_r}{m_0} \)) vs \( E_0 \). The critical value being \( E_0 = 0.202 \) for a star of radius \( r_0 = 10m_o \rightarrow x_0 = 0.2 \) with initial mass \( m_o \). Thus region \( E_0 > 0.202 \) correspond to black-hole region while \( E_0, \leq 0.202 \) is the naked singular region.
Figure 2: Graph showing the mass ratio $m_r/m_o$ vs $a$. Here $a \approx 0.001$ is the critical value of $a$. Thus for a star of radius $r_b \approx 45m_0 \to x0 \approx 0.04408$ and with surface density $\rho(r_b) \approx 0.001$. Region $a > 0.001$ is black-hole region while the other is naked singular region.