Qubit regularization of asymptotic freedom

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We provide strong evidence that the asymptotically free $(1 + 1)$-dimensional non-linear $O(3)$ sigma model can be regularized using a quantum lattice Hamiltonian, referred to as the “Heisenberg-comb”, that acts on a Hilbert space with only two qubits per spatial lattice site. The Heisenberg-comb consists of a spin-half anti-ferromagnetic Heisenberg-chain coupled anti-ferromagnetically to a second local spin-half particle at every lattice site. Using a world-line Monte Carlo method we show that the model reproduces the universal step-scaling function of the traditional model up to correlation lengths of 200,000 in lattice units and argue how the continuum limit could emerge. We provide a quantum circuit description of time-evolution of the model and argue that near-term quantum computers may suffice to demonstrate asymptotic freedom.

Formulating quantum field theories (QFTs) so that they can be implemented on a quantum computer has become an active area of research recently [1–10]. One of the first steps in this process is to construct a suitable lattice quantum Hamiltonian that acts on a Hilbert space realized by $n$-qubits at each lattice site where $n$ is small. For bosonic quantum field theories, including gauge theories, an exact realization of the canonical commutation relation

$$[\phi_x, \pi_y] = i \delta_{x,y}$$

forces $n$ to be infinite. For this reason all traditional formulation of lattice QFTs with bosonic degrees of freedom need reformulation to be solvable on a quantum computer. A simple way to proceed is to truncate the infinite dimensional Hilbert space to an $n$-qubit subspace while preserving the long distance physics. This procedure of truncation and constructing a $n$-qubit lattice Hamiltonian can be viewed as an extra regularization necessary for quantum computation and we refer to it as “qubit regularization” of the original QFT [11–13].

As with any form of regularization, a procedure to define the continuum limit of the $n$-qubit model is necessary. How this limit emerges is not obvious in the $n$-qubit model, at least when $n$ remains finite. A common approach followed by many groups is to approach the continuum limit using the naïve procedure of making $n$ large, that takes us back to the traditional theory [6, 14–17]. An interesting unanswered theoretical question is whether all QFTs can in principle be obtained using suitable continuum limits of $n$-qubit models where $n$ remains finite; and if so, what is the minimal value of $n$ necessary for each QFT?

In this work, we show that the asymptotically-free $(1 + 1)$-dimensional $O(3)$ non-linear sigma model, described in Euclidean time $\tau$ by the continuum action

$$S[n] = \frac{1}{2g^2} \int d\tau dx \partial_\tau n \cdot \partial_x n,$$

with $n(x, \tau) \in O(3)$, can be regularized successfully using the 2-qubit-per-site Hamiltonian

$$H = \sum_i J_p H_{(i,1),(i,2)} + J H_{(i,1),(i+1,1)},$$

which we illustrate pictorially in Fig. 1 and refer to as the Heisenberg-comb. The continuum limit emerges when $J/J_p \to \infty$. Note that $n(x, \tau)$ in Eq. (2) is a classical 3-vector field of unit magnitude, while $H_{(i,a),(j,b)} = S_{i,a} \cdot S_{j,b}$ is the standard Heisenberg interaction between spin-half operators $S_{i,a}$ and $S_{j,b}$, where $i, j$ label one-dimensional spatial lattice sites and $a, b = 1, 2$ label the two qubit spaces.

The idea of qubit regularization of field theories is not new and was introduced many years ago in the $D$-theory formulation of field theories [18, 19]. The challenge is to search the space of all $n$-qubit lattice Hamiltonians in $d$ spatial dimensions to discover the correct quantum critical point where the original continuum quantum field theory in $d+1$ space-time dimensions is recovered. The fine tuning to the quantum critical point would then naturally define the procedure to obtain the continuum limit of the $n$-qubit model. Qubit regularization of conformal field theories that emerge naturally at second order quantum critical points between two different phases are easy to construct with finite values of $n$. In this case one has to just preserve the important symmetries of the original QFT and tune a single relevant parameter to the correct quantum critical point. While this is well known in the literature, this approach has found new applications recently [20].

In contrast, qubit regularization of asymptotically-free theories like QCD and the non-linear sigma model given in Eq. (2) is much more challenging, especially when $n$ is finite and fixed. This is the central topic of our paper. In this case, in addition

FIG. 1. A pictorial representation of the Heisenberg-comb whose Hamiltonian is given in Eq. (3). The asymptotically free QFT described by Eq. (2) is reproduced in the limit $J/J_p \to \infty.$
to preserving the symmetries of the QFT, one has to discover the critical point with the correct marginally relevant coupling that preserves the physics at all length scales from the infrared (IR) to the ultraviolet (UV). The IR is usually characterized by a physical correlation length $\xi$, while the UV is characterized by a minimum lattice size $L_{\text{min}} \ll \xi$ at which universal physics of the QFT can be observed using the lattice theory, where even $L_{\text{min}}$ is much larger than the lattice spacing. In the $D$-theory formulation this marginal operator is obtained through the size of an extra spatial dimension [21]. It has been shown that asymptotic freedom in the two dimensional $CP(N - 1)$ models can be reproduced when the size of this extra dimension grows [22, 23]. Since the size of the extra dimension naturally increases the number of qubits per spatial lattice site, one can say that asymptotic freedom in the traditional $D$-theory approach can be obtained if $n$ is allowed to grow.

In this work, we explore whether asymptotic freedom may be achieved even with a fixed value of $n$ by discovering the correct marginally relevant coupling not related to an extra dimension. We show this explicitly in the case of the asymptotically free two dimensional non-linear $O(3)$ QFT described by Eq. (2). Traditionally, this theory is regularized using the lattice Euclidean action

$$S = -\kappa \sum_{\langle i,j \rangle, a} \phi_a^i \cdot \phi_a^j,$$

where $i, j$ now label space-time lattice sites on a square lattice and $\phi_a^i$ $(a = 1, 2, 3)$ are the three components of a unit vector associated with the lattice site $i$. The continuum limit is obtained when $\kappa \rightarrow \infty$. Here we will argue that the same continuum physics can also be obtained using the Heisenberg-comb Hamiltonian discussed above.

The question of whether 2-qubit models can reproduce the physics of the traditional model has been partially explored previously. The first exploration was performed in traditional lattice field theory using a Nienhuis-type action, that can be viewed as a space-time lattice formulation of a two-qubit Hamiltonian [24]. Evidence was provided that a step scaling function similar to that for the traditional model is reproduced in the physics, we can narrow the search to the space of $SO(3) \subset O(3)$ symmetry plays an important role in the physics, we can narrow the search to the space of 2-qubit models invariant under this symmetry. The phase diagram of these models is quite rich with several phases and quantum critical points separating them. In particular there are at least five distinct phases: phase-A where the two qubits form local spin-singlets and spin-triplet excitations are massive, phase-B where the spin-triplets dominate and form a ferromagnet, phase-C where the spin-triplets on neighboring sites form singlets and break translation invariance spontaneously, phase-D where spin-triplets form a massive topological phase also referred to as the Haldane phase [26–29] and phase-E where the long distance physics is a critical gapless phase described by level-one $SU(3)$ Wess-Zumino-Witten conformal field theory [30]. Such a rich phase structure already suggests that previous studies could have missed the asymptotically free fixed point.

One way to parameterize the space of 2-qubit models is to begin with spin-half ladders whose Hamiltonian takes the form

$$H = \sum_x J_p H_{(x,1),(x,2)} + J_1 H_{(x,1),(x+1,1)} + J_2 H_{(x,2),(x+1,2)},$$

with three tunable couplings. When $J_p < 0$ and large compared to $J_1$ and $J_2$, we can access the physics of spin-1 chains. In these chains, when $J_1, J_2 < 0$ we obtain the ferromagnetic phase $B$. On the other hand when $J_1, J_2 > 0$ we can access the physics of anti-ferromagnetic spin-1 chains which is the starting point to accessing phases $C$, $D$ and $E$. When $J_p = 0$, it is well known that each of the two decoupled spin-half chains...
describe the long distance physics of

\[ S[n] = \int d\tau dx \left\{ \frac{1}{2g^2} \partial_\mu n \cdot \partial_\mu n + \frac{i \theta}{4\pi} n \cdot (\partial_\mu n \times \partial_\mu n) \right\} \quad (6) \]

at \( \theta = \pi \) [32]. This theory is known to be critical and described by the \( k = 1 \) \( SU(2) \) WZW theory [33–35]. When \( J_p > 0 \), \( \theta \) is constrained to be zero and the IR physics of Eq. (2) is reproduced [36]. So it is likely that there exists a critical point in the three parameter space of \((J_1, J_2, J_p)\) with \( J_p > 0 \) that reproduces both the UV and IR physics of Eq. (2) correctly. In particular three critical points seem interesting candidates to explore: (1) the symmetric ladder where \( J_1 = J_2 = 1 \) and \( J_p \to 0^+ \), (2) the asymmetric ladder \( J_1 > J_2 \) and \( J_p \to 0^+ \), and (3) the Heisenberg comb \( J_2 = 0 \), \( J_p = 1 \) and \( J_1 = J \to \infty \). We will show below that the Heisenberg comb is the correct 2-qubit model.

In order to study whether the traditional model and the 2-qubit model quantitatively reproduce the same physics in the continuum limit, we need to match the two theories at all physical scales from the IR to the UV. Asymptotically free theories are massive, and the correlation length \( \xi \) defined by the mass gap sets the natural IR length scale. In order to probe the UV physics we put the system in a small box of physical size \( L \ll \xi \). In fact, one can use a suitably defined finite size correlation length \( \xi(L) \) even in the UV such that \( \xi(L \to \infty) = \xi \). One definition of such a length scale is the second moment definition [37], \( \xi(L) = (\sqrt{\langle G_0/G_1 \rangle - 1})/(2\sin(\pi/L)) \) where

\[ G_k = \sum_{(x,t)} \langle \phi^3_{(x,t)} \phi^3_{(0,0)} \rangle e^{i2\pi k x / L}. \quad (7) \]

A quantitative way to probe all physical scales from the UV to the IR using \( \xi(L) \) is the universal step scaling function \( F(z) = \xi(2L)/\xi(L) \) as a function of \( z = \xi(L)/L \). The dark line in both the plots is this function reproduced from [31], where it was calculated using the traditional model defined by Eq. (4). The dashed line is the function \( F(z) = 2(1 - 0.0276/z^2 - 0.00258/z^4) \), computed in Ref. [31] using perturbation theory near the asymptotically free fixed point. The left plot shows results for (1) symmetric ladder \( J_1 = J_2 = 1, J_p = 0.10, 0.20 \) and (2) asymmetric ladder \( J_1 = 1, J_2 = 0.5; J_p = 0.05, 0.10 \). The right plot shows results for the Heisenberg comb at \( J_1 = J \to 3, 5, 10; J_2 = 0; J_p = 1 \). Each data point is constructed using two calculations, one at lattice size \( L \) and another at 2\( L \), with \( 12 \leq L \leq 256 \).

\[ \int d\tau dx \left\{ \frac{1}{2g^2} \partial_\mu n \cdot \partial_\mu n + \frac{i \theta}{4\pi} n \cdot (\partial_\mu n \times \partial_\mu n) \right\} \]

FIG. 3. Universal step scaling function \( F(z) = \xi(2L)/\xi(L) \) as a function of \( z = \xi(L)/L \). The dark line in both the plots is this function reproduced from [31], where it was calculated using the traditional model defined by Eq. (4). The dashed line is the function \( F(z) = 2(1 - 0.0276/z^2 - 0.00258/z^4) \), computed in Ref. [31] using perturbation theory near the asymptotically free fixed point. The left plot shows results for (1) symmetric ladder \( J_1 = J_2 = 1, J_p = 0.10, 0.20 \) and (2) asymmetric ladder \( J_1 = 1, J_2 = 0.5; J_p = 0.05, 0.10 \). The right plot shows results for the Heisenberg comb at \( J_1 = J \to 3, 5, 10; J_2 = 0; J_p = 1 \). Each data point is constructed using two calculations, one at lattice size \( L \) and another at 2\( L \), with \( 12 \leq L \leq 256 \).
We study lattice sizes in the range $12 \leq L \leq 512$. In a Hamiltonian formulation, spatial correlation functions, Eq. (8), will, in general, be different from temporal correlation functions

$$\tilde{G}_k = \frac{1}{Z} \int d\tau \sum_x \text{Tr} \left( \mathcal{O}(x, \tau) \mathcal{O}(0,0) e^{-\beta H} \right) e^{i 2\pi k \tau / \beta},$$

(9)

where, as in the traditional model, Eq. (4), these two are identical on square lattices due to space-time rotational symmetry. In our calculations we tune $\beta$ as a function of $L$ to make $\tilde{G}_k \approx \tilde{G}_k$. These fine tuned values of $\beta$ are plotted as a function of $L$ in Fig. 2, which shows that for each of our study $\beta / L$ becomes a constant for large $L$, as expected in a relativistic theory.

Our results for $F(z)$ in the qubit models are shown in two plots in Fig. 3, along with results from the traditional model recreated from [31]. In the left plot we show our results for the symmetric ladder and the asymmetric ladder. These results show that neither of these models reproduce the traditional model in the UV although they are $SO(3)$ symmetric and massive in the IR. Thus, the spin ladders may be described by Eq. (2) in the IR [36], but in the UV they are most likely described by two decoupled $k = 1$ SU(2) WZW CFTs as one might expect. In fact for small values of $J_p$ the mass gap in the symmetric case is known to increase linearly as $0.41(1)J_p$ implying that $J_p$ is not the marginally relevant coupling we are looking for.

In contrast to spin-ladders, when $F(z)$ is computed in the Heisenberg comb, it matches the traditional model well for all values of $L > L_{\text{min}}$. When $J = 3, 5, 10$ we find that, in lattice units, $L_{\text{min}} \approx 30, 100, 400$ and $\xi \approx 25, 600, 200\,000$ respectively. We observe that the UV scale, $L_{\text{min}}$, increases with $J$, and in the limit $J \to \infty$ the asymptotically free critical point is recovered. From Wilson’s renormalization group perspective, after blocking to a scale of $L_{\text{min}}$, the 2-qubit model turns into an $n$-qubit model where $n = 2L_{\text{min}}$. The traditional infinite Hilbert space of the continuum asymptotically free fixed point is recovered in the $J \to \infty$ limit.

Note that we define the physics of the Heisenberg-comb by setting $J_p = 1$, $J_1 = J$ and study the limit $J \to \infty$. We could have instead set $J_1 = 1$ and $J_p = 1/J$, and obtained the same physics. However, this will force us to study asymmetric lattices since now the values of $\beta$ in Fig. 2 will need to be multiplied by $J$. One way to understand our choice is to recognize that the value of $J_p$ sets a UV energy scale for our problem. The physics of Eq. (2) only emerges at temperatures much smaller than $J_p$. So it is natural in the qubit regularization to set $J_p = 1$ and study the physics at large values of $J_1 = J$.

An important motivation for discovering a qubit regularization with a finite $n$ is the ability to study the real time evolution of the QFT on a quantum computer. The simplicity of the Heisenberg-comb Hamiltonian allows us to implement the Trotterized time evolution operator of the theory using a short-depth quantum circuit. To construct this we write the Hamiltonian as a sum of three commuting terms $H = J_p H_1 + J(H_2 + H_3)$ where $H_1 = \sum_{i \in x} H_{(x,1),(x,2)}$, $H_2 = \sum_{i \in y} H_{(x,1),(x,2)}$, and $H_3 = \sum_{i \in z} H_{(x,1),(x,2)}$, with $x$, $y$, and $z$ being even sites and $x, y, z$ being odd sites.

For terms in $H_1$ we can view the Hamiltonian $H$ as the sum of $z$ terms

$$H_i = \sum_j e^{-i J H_j} e^{-i J H_j} e^{-i J H_j} + O(t^2).$$

In the computational basis, that is the $|\pm\rangle$ basis for each spin, we can view the Hamiltonian $H_{(x,a),(x,b)}$ as a two-qubit operator, whose matrix elements are given by a $4 \times 4$ matrix. The corresponding time-evolution operator $e^{-i H_{(x,a),(x,b)} + 1/4}t$ (with an additional global phase that can be easily undone) takes the form

$$e^{-i H t} = e^{-i J H_1 t} e^{-i J H_2 t} e^{-i J H_3 t} + O(t^2).$$

This unitary transformation can be implemented using two CNOT gates, one controlled-unitary gate which implements the $X$-rotation $R_X(\phi)$ and one single-qubit phase rotation $P(\phi)$, given by

$$R_X(\phi) = \begin{pmatrix} \cos \phi & -i \sin \phi \\ -i \sin \phi & \cos \phi \end{pmatrix}, \quad P(\phi) = \begin{pmatrix} e^{-i \phi} & 0 \\ 0 & 1 \end{pmatrix}.$$
demonstrates that asymptotic freedom does not necessarily require an infinite dimensional local Hilbert space in a lattice model, although intuition might lead us to believe this is necessary. By suitably adjusting the lattice size $L$ as $J$ is increased to approach the continuum limit, one can stay on the universal scaling curve and thus deduce the properties of the continuum theory. The question remains—which approach is more efficient to implement on a quantum computer? Here we have shown that the quantum circuit for the Heisenberg comb is simple but whether the growth of complexity with lattice size eventually makes the other approach more efficient remains to be investigated.

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