On spanning tree congestion of Hamming graphs

Kyohei Kozawa\textsuperscript{*} Yota Otachi\textsuperscript{†}

October 18, 2011

Abstract

We present a tight lower bound for the spanning tree congestion of Hamming graphs.

1 Preliminaries

The spanning tree congestion of graphs has been studied intensively [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. In this note, we study the spanning tree congestion of Hamming graphs. We present a lower bound for the spanning tree congestion of Hamming graphs. That is, in our terminology, we show that \( \text{stc}(K^d_n) \geq \frac{1}{d}(n^d - 1) \log_n d \). It is known that \( \text{stc}(K^d_n) = O\left(\frac{1}{d}n^d \log_n d\right) \) [10]. Thus our lower bound is asymptotically tight.

For a graph \( G \), we denote its vertex set and edge set by \( V(G) \) and \( E(G) \), respectively. For \( S \subseteq V(G) \), let \( G[S] \) denote the subgraph induced by \( S \). For an edge \( e \in E(G) \), we denote by \( G - e \) the graph obtained from \( G \) by deleting \( e \). Let \( N_G(v) \) denote the neighborhood of \( v \in V(G) \) in \( G \); that is, \( N_G(v) = \{ u \mid \{ u, v \} \in E(G) \} \). We denote the degree of a vertex \( v \in V(G) \) by \( \deg_G(v) \), and the maximum degree of \( G \) by \( \Delta(G) \); that is, \( \deg_G(v) = |N_G(v)| \) and \( \Delta(G) = \max_{v \in V(G)} \deg_G(v) \). A graph \( G \) is \( r \)-regular if \( \deg_G(v) = r \) for every \( v \in V(G) \).

For \( S \subseteq V(G) \), we denote the edge set of \( G[S] \) by \( E_G(S) \), and the boundary edge set by \( \theta_G(S) \); that is, \( E_G(S) = \{ \{ u, v \} \mid u, v \in S \} \) and \( \theta_G(S) = \{ \{ u, v \} \in E(G) \mid \text{exactly one of } u, v \text{ is in } S \} \). We define the function \( \iota \) and \( \theta \) also for a positive integer \( s \leq |V(G)| \) as \( \iota_G(s) = \max_{S \subseteq V(G), |S|=s} |E_G(S)| \) and \( \theta_G(s) = \min_{S \subseteq V(G), |S|=s} |\theta_G(S)| \). Let \( T \) be a spanning tree of a connected graph \( G \). The congestion of \( e \in E(T) \) is \( \text{cng}_G(e) = |\theta_G(L_e)| \), where \( L_e \) is the vertex set of one of the two components of \( T - e \). The congestion of \( T \) in \( G \), denoted by \( \text{cng}_G(T) \), is the maximum congestion over all edges in \( T \). We define the spanning tree congestion of \( G \), denoted by \( \text{stc}(G) \), as the minimum congestion over all spanning trees of \( G \).

The \( d \)-dimensional Hamming graph \( K^d_n \) is the graph with vertex set \( \{0, \ldots, n-1\}^d \) in which two vertices are adjacent if and only if their corresponding \( d \)-dimensional vectors differ in exactly one place. It is evident that \( K^d_n \) is \( d(n-1) \)-regular. The exact value of \( \text{stc}(K^d_n) \) is known [6]. Also, \( \text{stc}(K^d_n) \) is determined asymptotically [3].

\textsuperscript{*}Electric Power Development Co., Ltd., 6-15-1, Ginza, Chuo-ku, Tokyo, 104-8165 Japan.
\textsuperscript{†}Graduate School of Information Sciences, Tohoku University, Sendai 980-8579, Japan. E-mail address: otachi@dais.is.tohoku.ac.jp
2 The lower bound

Here, we present the lower bound. We need the following three lemmas.

Lemma 2.1 ([11]). If $G$ is $r$-regular and $S \subseteq V(G)$, then $|\theta_G(S)| = r|S| - 2|\kappa_G(S)|$.

Lemma 2.2 ([17]). Let $G$ be a subgraph of $K_n^d$. If $G$ has $s$ vertices and $t$ edges, then $2t \leq (n-1)s \log n s$.

Lemma 2.3 ([4, 7]). For any connected graph $G$, $\text{stc}(G) \geq \min_{s=\lceil |V(G)|/2 \rceil} \{ 2\min \{ f(\lceil \frac{n^d-1}{d(n-1)} \rceil), f(\lceil \frac{n^d}{2} \rceil) \} \}$.

Theorem 2.4. $\text{stc}(K_n^d) \geq (n^d-1) \log_n d$ for $n, d \geq 3$.

Proof. Since $K_n^d$ is $d(n-1)$-regular, Lemmas 2.1 and 2.2 imply that $\theta_{K_n^d}(s) \geq (n-1)s(d - \log n s)$. Let $f(s) = (n-1)s(d - \log n s)$ and $f'(s)$ be the derived function of $f(s)$. Then $f'(s) = (n-1)(d - 1/\ln n - \log n s)$, and thus, $f(s)$ is increasing for $(n^d - 1)/(d(n-1)) \leq s \leq n^d/1\ln n$ and decreasing for $n^d/1\ln n \leq s \leq n^d/2$. Therefore,

$$\min_{s=\lceil (n^d-d)/d(n-1) \rceil} f(s) = \min \left\{ f\left( \frac{n^d-1}{d(n-1)} \right), f\left( \frac{n^d}{2} \right) \right\} \geq \min \left\{ \frac{n^d-1}{d} \left( d - \log_n \frac{n^d-1}{d(n-1)} \right), \frac{(n-1)n^d}{2} \left( d - \log_n \frac{n^d}{2} \right) \right\} \geq \min \left\{ \frac{n^d-1}{d} \log_n d, \frac{(n-1)n^d}{2} \log_n 2 \right\}.$$

Thus, by Lemma 2.3, it holds that

$$\text{stc}(K_n^d) \geq \min \left\{ \frac{n^d-1}{d} \log_n d, \frac{(n-1)n^d}{2} \log_n 2 \right\}.$$

By a simple calculation, we can see that $\frac{n^d-1}{d} \log_n d \leq \frac{(n-1)n^d}{2} \log_n 2$ for $d = 2, 3$. Since $n^d - 1 \leq (n-1)n^d$ and $(\log_n d)/d \leq (\log_n 2)/2$ for $d \geq 4$, the theorem holds. \qed

References

[1] S. L. Bezrukov, Edge isoperimetric problems on graphs, in: L. Lovász, A. Gyárfás, G. O. H. Katona, A. Recski, L. Székely (eds.), Graph Theory and Combinatorial Biology, vol. 7 of Bolyai Soc. Math. Stud., János Bolyai Math. Soc., Budapest, 1999, pp. 157–197.

[2] H. L. Bodlaender, F. V. Fomin, P. A. Golovach, Y. Otachi, E. J. van Leeuwen, Parameterized complexity of the spanning tree congestion problem, Algorithmica.
[3] H. L. Bodlaender, K. Kozawa, T. Matsushima, Y. Otachi, Spanning tree congestion of $k$-outerplanar graphs, Discrete Math. 311 (2011) 1040–1045.

[4] A. Castejón, M. I. Ostrovskii, Minimum congestion spanning trees of grids and discrete toruses, Discuss. Math. Graph Theory 29 (2009) 511–519.

[5] S. W. Hruska, On tree congestion of graphs, Discrete Math. 308 (2008) 1801–1809.

[6] K. Kozawa, Y. Otachi, Spanning tree congestion of rook’s graphs, Discuss. Math. Graph Theory 31 (2011) 753–761.

[7] K. Kozawa, Y. Otachi, K. Yamazaki, On spanning tree congestion of graphs, Discrete Math. 309 (2009) 4215–4224.

[8] H.-F. Law, Spanning tree congestion of the hypercube, Discrete Math. 309 (2009) 6644–6648.

[9] H.-F. Law, M. I. Ostrovskii, Spanning tree congestion: duality and isoperimetry; with an application to multipartite graphs, Graph Theory Notes N. Y. 58 (2010) 18–26.

[10] H.-F. Law, M. I. Ostrovskii, Spanning tree congestion of some product graphs, Indian J. Math. 52 (2011) 103–111.

[11] C. Löwenstein, D. Rautenbach, F. Regen, On spanning tree congestion, Discrete Math. 309 (2009) 4653–4655.

[12] Y. Okamoto, Y. Otachi, R. Uehara, T. Uno, Hardness results and an exact exponential algorithm for the spanning tree congestion problem, in: Proceedings of the 8th Annual Conference on Theory and Applications of Models of Computation (TAMC 2011), vol. 6648 of Lecture Notes in Comput. Sci., pp. 452–462, Springer-Verlag, 2011.

[13] M. I. Ostrovskii, Minimal congestion trees, Discrete Math. 285 (2004) 219–226.

[14] M. I. Ostrovskii, Minimum congestion spanning trees in planar graphs, Discrete Math. 310 (2010) 1204–1209.

[15] M. I. Ostrovskii, Minimum congestion spanning trees in bipartite and random graphs, Acta-Math. Sci. 31B (2011) 634–640.

[16] S. Simonson, A variation on the min cut linear arrangement problem, Math. Syst. Theory 20 (1987) 235–252.

[17] R. Squier, B. Torrence, A. Vogt, The number of edges in a subgraph of a Hamming graph, Appl. Math. Lett. 14 (2001) 701–705.