Diagonalization of the strongly coupled lattice QCD Hamiltonian

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Abstract

We construct a solution to the equation of motion of Hamiltonian lattice QCD in the strong coupling limit using Wilson fermions which exactly diagonalizes the Hamiltonian to second order in the field operators. This solution obeys the free lattice Dirac equation with a dynamical mass which is identified with the gap. The equation determining this gap is derived and it is found that the dynamical quark mass is a constant to lowest order in $N_c$ but becomes momentum dependent once $O(1/N_c)$ corrections are taken into account. We interpret our solution within the framework of the N–quantum approach to quantum field theory and discuss how our formalism may be systematically extended to study bound states at finite temperature and chemical potential.

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I. INTRODUCTION

The strong coupling approximation has played an important role in the development of QCD lattice gauge theory from its very inception. In the renowned paper by Wilson [1] this approximation was invoked to demonstrate quark confinement on the Euclidean space–time lattice. Soon thereafter Kogut and Susskind [2] formulated the Hamiltonian lattice gauge theory and concluded that in the strong coupling limit the quark dynamics is best described by a collection of non–Abelian electric flux tubes with quarks attached at their ends. This was followed by the work of Baluni and Willemse n who used a variant of the Kogut–Susskind formalism to demonstrate quantitatively that dynamical chiral symmetry breaking indeed takes places in lattice QCD at strong coupling [3]. Finally, calculations by Kogut, Pearson and Shigemitsu [4] and by Creutz [5] suggesting the absence of a phase transition between the strong and weak coupling regimes of QCD motivated numerous studies in this subject which continues to date.

The Hamiltonian formulation of strong coupling QCD using Wilson fermions was first studied in detail by Smit [6] who derived an effective Hamiltonian using the $1/N_c$ approximation and applied the spin wave theory of magnetism to determine the vacuum energy density and the excitation spectrum. In the absence of the current quark mass and the Wilson term the theory possesses a $U(4N_f)$ symmetry. Smit has shown how this symmetry is spontaneously broken down to $U(2N_f) \otimes U(2N_f)$ accompanied by the appearance of $8N_f^2$ Goldstone bosons. A finite current quark mass also breaks the original $U(4N_f)$ symmetry explicitly down to $U(2N_f) \otimes U(2N_f)$. Introduction of the Wilson term explicitly breaks the latter symmetry further down to $U(N_f)$ solving the fermion doubling problem. In addition Smit has proposed how quantum corrections appearing at $O(1/N_c)$ may be used to resolve the $U_A(1)$ problem. Le Yaouanc, Oliver, Pène and Raynal [7] later reexamined Smit’s work by applying the Bogoliubov–Valatin variational method to the same effective Hamiltonian and found that the vacuum is not chirally degenerate and concluded that the massless pseudoscalar boson found by Smit is not a Goldstone boson.

In this letter we report on a new approach to study strongly coupled Hamiltonian lattice QCD using Wilson fermions. Our approach differs from those adopted in [6] and [7] in that we explicitly construct a solution to the field equations of motion derived from the Hamiltonian. This solution exactly diagonalizes the Hamiltonian to second order in field operators and obeys the free lattice Dirac equation where the mass plays the role of the gap. Using the equation of motion we derive the gap equation for the dynamical quark mass to $O(1/N_c)$. Our results for the dynamical quark mass is qualitatively different from the one found in [7].

With our solution the interpretation of the chiral condensate being the order parameter becomes transparent as well as the observation that the broken symmetry phase is the energetically favored phase. In addition, our approach admits to a systematic extension to the study of bound states both in free space and at finite temperature and chemical potential. This is accomplished by interpreting our solution within the context of the N–quantum approach to quantum field theory [8,9] which we shall discuss in the concluding section.
II. NON–INTERACTING WILSON FERMIONS

We begin with a brief review of the free lattice Dirac Hamiltonian with the Wilson term. Although elementary, we shall see that the results presented here will play crucial roles in the construction of a solution of the strongly coupled QCD. We adopt the notation of Smit [6] with unit lattice spacing \( a = 1 \) and temporarily suppress color and flavor indices. Then the free lattice Dirac Hamiltonian is

\[
H^0 = \frac{1}{2i} \sum_{\vec{x},l} \left[ \Psi^\dagger(\vec{x}) \gamma_0 \gamma_l \Psi(\vec{x} + \hat{n}_l) - \Psi^\dagger(\vec{x} + \hat{n}_l) \gamma_0 \gamma_l \Psi(\vec{x}) \right] + M \sum_{\vec{x}} \Psi^\dagger(\vec{x}) \gamma_0 \Psi(\vec{x})
\]

\[
- \frac{r}{2} \sum_{\vec{x},l} \left[ \Psi^\dagger(\vec{x}) \gamma_0 \Psi(\vec{x} + \hat{n}_l) + \Psi^\dagger(\vec{x} + \hat{n}_l) \gamma_0 \Psi(\vec{x}) \right]
\]

(2.1)

where the third term is the Wilson term with \( 0 \leq r \leq 1 \). For \( r = 0 \) there is an eightfold fermion multiplicity which is removed when \( r \neq 0 \).

At each lattice site the free Dirac field in configuration space is given by

\[
\Psi_\mu(t, \vec{x}) = \sum_{\vec{p}} \left[ b(\vec{p}) \xi_\mu(\vec{p}) e^{-i(\omega(\vec{p})t - \vec{p} \cdot \vec{x})} + d^\dagger(\vec{p}) \eta_\mu(\vec{p}) e^{i(\omega(\vec{p})t - \vec{p} \cdot \vec{x})} \right]
\]

(2.2)

with \( \mu \) denoting the Dirac index. The excitation energy \( \omega(\vec{p}) \) will be determined shortly.

The annihilation operators \( b \) and \( d \) annihilate the non–interacting vacuum state \( |0\rangle \). For our purpose it is not necessary to know the structure of the spinors \( \xi \) and \( \eta \). The only assumption that we shall make is that the creation and annihilation operators obey the free fermion anti–commutation relations

\[
\left[ b(\vec{p}), b(\vec{q}) \right]_+ = \left[ d^\dagger(\vec{p}), d(\vec{q}) \right]_+ = \delta_{\vec{p},\vec{q}}
\]

(2.3)

Using this assumption we can recover the anti–commutation relations for the field operators

\[
\left[ \Psi_\mu(t, \vec{x}), \Psi^\dagger_\nu(t, \vec{y}) \right]_+ = \delta_{\vec{x},\vec{y}} \delta_{\mu\nu}
\]

(2.4)

provided that \( \xi \) and \( \eta \) satisfy the relation

\[
\xi_\mu(\vec{p}) \xi^\dagger_\mu(\vec{p}) + \eta_\mu(\vec{p}) \eta^\dagger_\mu(\vec{p}) = \delta_{\mu\nu}
\]

(2.5)

We normalize the spinors by demanding that the number density is given by

\[
\mathcal{N} = \sum_{\vec{x}} : \Psi^\dagger(t, \vec{x}) \Psi(t, \vec{x}) : = 2 \sum_{\vec{p}} \left( b^\dagger(\vec{p}) b(\vec{p}) - d^\dagger(\vec{p}) d(\vec{p}) \right)
\]

(2.6)

where the symbol \( : : \) denotes normal ordering and the factor of 2 accounts for the spin degrees of freedom. Eq. (2.6) fixes the normalizations of \( \xi \) and \( \eta \) to be

\[
\xi_\mu(\vec{p}) \xi^\dagger_\mu(\vec{p}) = \eta_\mu(\vec{p}) \eta^\dagger_\mu(\vec{p}) = 2
\]

(2.7)

\[
\xi^\dagger_\mu(\vec{p}) \eta_\mu(\vec{p}) = \eta^\dagger_\mu(\vec{p}) \xi_\mu(\vec{p}) = 0
\]

(2.8)

which are consistent with Eq. (2.5).
We now go over into momentum space where we perform all our calculations. In momentum space the charge conjugation symmetric form of $H^0$ is

$$H^0 = \frac{1}{2} \sum_{\vec{p}} \left( - \sum_l \sin(\vec{p} \cdot \hat{n}_l) \gamma_0 \gamma_l + M(\vec{p}) \gamma_0 \right) \mu_\nu \left[ \Psi^\dagger_\mu(t, \vec{p}), \Psi_\nu(t, -\vec{p}) \right]_\mu \gamma_\nu$$

(2.9)

where the momentum dependent mass term is given by

$$M(\vec{p}) \equiv M - r \sum_l \cos(\vec{p} \cdot \hat{n}_l)$$

(2.10)

The free Dirac field becomes

$$\Psi_\mu(t, \vec{p}) = b(\vec{p}) \xi_\mu(\vec{p}) e^{-i\omega(\vec{p})t} + d^\dagger(-\vec{p}) \eta_\mu(-\vec{p}) e^{+i\omega(\vec{p})t}$$

(2.11)

which is used to derive the equation of motion corresponding to Eq. (2.9)

$$i \dot{\Psi}(t, \vec{p}) = :\left[ \Psi(t, \vec{p}), H^0 \right]_\mu : \quad \left( \sum_l \sin(\vec{p} \cdot \hat{n}_l) \gamma_0 \gamma_l + M(\vec{p}) \gamma_0 \right) \Psi(t, \vec{p})$$

(2.12)

(2.13)

From Eq. (2.13) one obtains the excitation energy

$$\omega(\vec{p}) = \left( \sum_l \sin^2(\vec{p} \cdot \hat{n}_l) + M^2(\vec{p}) \right)^{1/2}$$

(2.14)

and the equations of motion for the $\xi$ and $\eta$ spinors

$$\omega(\vec{p}) \xi(\vec{p}) = \left( \sum_l \sin(\vec{p} \cdot \hat{n}_l) \gamma_0 \gamma_l + M(\vec{p}) \gamma_0 \right) \xi(\vec{p})$$

(2.15)

$$\omega(\vec{p}) \eta(-\vec{p}) = - \left( \sum_l \sin(\vec{p} \cdot \hat{n}_l) \gamma_0 \gamma_l + M(\vec{p}) \gamma_0 \right) \eta(-\vec{p})$$

(2.16)

When $r = 0$ these equations of motion are relativistic near the eight corners of the Brillouin zone denoted by $\vec{p}_0 = (0, 0, 0)$, $\vec{p}_x = (\pi, 0, 0)$, $\vec{p}_y = (0, \pi, 0)$, $\vec{p}_z = (0, 0, \pi)$, $\vec{p}_{xy} = (\pi, \pi, 0)$, $\vec{p}_{xz} = (\pi, 0, \pi)$, $\vec{p}_{yz} = (0, \pi, \pi)$ and $\vec{p}_{xyz} = (\pi, \pi, \pi)$. The excitation energies at these values of momenta are equal which corresponds to the eightfold multiplicity mentioned above. This degeneracy is lifted when $r \neq 0$ due to the momentum dependent mass term Eq. (2.10). Using the equations of motion for $\xi$ and $\eta$ it is a simple exercise to show that the off-diagonal Hamiltonian vanishes and that the vacuum energy is given by $\langle 0|H^0|0 \rangle = -2V \sum_{\vec{p}} \omega(\vec{p})$. Finally, we construct positive and negative energy projection operators $\Lambda^+(\vec{p})$ and $\Lambda^-(\vec{p})$ as follows

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1Our convention for the Fourier transform from configuration to momentum space is $\Psi(\vec{x}) = \sum_{\vec{p}} \Psi(\vec{p}) e^{i\vec{p} \cdot \vec{x}}$ which implies that the volume $V$ is given by $V = \sum_{\vec{x}} = \delta_{\vec{p},\vec{p}}$. 

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4
\[ \Lambda^+(\vec{p}) \equiv \xi(\vec{p}) \otimes \xi^\dagger(\vec{p}) \]
\[ = \frac{1}{2} \left[ 1 + \frac{1}{\omega(\vec{p})} \sum_l \sin(\vec{p} \cdot \hat{n}_l)\gamma_0\gamma_l + \frac{M(\vec{p})}{\omega(\vec{p})}\gamma_0 \right] \quad (2.17) \]
\[ \Lambda^-(\vec{p}) \equiv \eta(-\vec{p}) \otimes \eta^\dagger(-\vec{p}) \]
\[ = \frac{1}{2} \left[ 1 - \frac{1}{\omega(\vec{p})} \sum_l \sin(\vec{p} \cdot \hat{n}_l)\gamma_0\gamma_l - \frac{M(\vec{p})}{\omega(\vec{p})}\gamma_0 \right] \quad (2.18) \]

These projection operators will be used extensively below. Note that they obey the condition
\[ \left[ \Lambda^+(\vec{p}) + \Lambda^-(\vec{p}) \right]_{\alpha\beta} = \delta_{\alpha\beta} \quad (2.19) \]

as is required by Eq. (2.5).

### III. EFFECTIVE HAMILTONIAN AND THE ANSATZ

For the sake of comparison we use the same effective Hamiltonian obtained by Smit in Eq. (3.13) of [6] which is also used in [7]. The charge conjugation symmetric form of Smit’s Hamiltonian in momentum space is
\[ H_{\text{eff}} = \frac{1}{2} M_0 \sum_{\vec{p}_1, \vec{p}_2} \delta_{\vec{p}_1+\vec{p}_2,0} \left( \gamma_0 \right)_{\mu\nu} \left[ \left( \Psi_{a\alpha}^\dagger \right)_\mu (\vec{p}_1), \left( \Psi_{a\alpha} \right)_\nu (\vec{p}_2) \right] \]
\[ - \frac{K}{8N_c} \sum_{\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4} \sum_l \delta_{\vec{p}_1+\vec{p}_2+\vec{p}_3+\vec{p}_4,0} \left[ e^{i((\vec{p}_1+\vec{p}_2)\cdot\hat{n}_l)} + e^{i((\vec{p}_3+\vec{p}_4)\cdot\hat{n}_l)} \right] \]
\[ \otimes \left[ \left( \Sigma_l \right)_{\mu\nu} (\vec{p}_1), (\vec{p}_2) (\vec{p}_3), (\vec{p}_4) \right] \]
\[ \otimes \left[ \left( \Sigma_l \right)_{\gamma\delta} (\vec{p}_1), (\vec{p}_2) (\vec{p}_3), (\vec{p}_4) \right] \quad (3.1) \]

where \( \Sigma_l \equiv -i(\gamma_0\gamma_l - ir\gamma_l) \). We denote color, flavor and Dirac indices by \((ab), (\alpha\beta)\) and \((\mu\nu\gamma\delta)\), respectively. Summation convention for repeated indices is implied. The three parameters in this theory are the Wilson parameter \( r \), the current quark mass \( M_0 \) and the effective coupling constant \( K \) which behaves as \( 1/g^2 \) with \( g \) being the QCD coupling constant.

Our ansatz which diagonalizes Eq. (3.1) to second order in the field operators is deceptively simple. It is the solution of the free lattice Dirac equation Eq. (2.13) with \( r = 0 \) and is given by Eq. (2.14) appropriately modified to include color and flavor degrees of freedom. However, the mass term \( M(\vec{p}) \) in Eq. (2.13) is now the dynamical and not the current quark mass. We shall show below that this dynamical mass is the solution of the gap equation. In addition, the annihilation operators \( b \) and \( d \) in our ansatz now annihilate an interacting vacuum state \(|G\rangle\) and not \(|0\rangle\). Since we shall be working only in the space of quantum

\[ ^2 \text{This modification will of course affect the anti–commutation relation Eq. (2.3).} \]
field operators there is no need to specify the structure of $|\mathcal{G}\rangle$. Aside from these non-trivial modifications our ansatz obeys all the properties of the free Dirac field described above. For example, with this ansatz the chiral condensate is given by

$$\langle \mathcal{G} | \bar{\Psi} \Psi | \mathcal{G} \rangle = \frac{N_c N_f}{2} \sum_{\vec{p}} \left[ \text{Tr}(\gamma_0 \Lambda^- (\vec{p})) - \text{Tr}(\gamma_0 \Lambda^+ (\vec{p})) \right]$$

$$= -2 N_c N_f \sum_{\vec{p}} \frac{M(\vec{p})}{\omega(\vec{p})}$$

(3.2)

Hence the chiral condensate is determined by the dynamical quark mass and clearly plays the role of the order parameter. The basic idea of our approach is to use the equation of motion for $H_{\text{eff}}$ to derive and solve the gap equation for the dynamical mass.

Using our ansatz we proceed to derive the vacuum energy density and the second order off–diagonal Hamiltonian by renormal ordering $H_{\text{eff}}$ with respect to $|\mathcal{G}\rangle$. The former is given by terms involving two contractions while the latter is derived by retaining terms involving a single contraction. Our results for the vacuum energy density is

$$\frac{1}{V} \langle \mathcal{G} | H_{\text{eff}} | \mathcal{G} \rangle = -2 M_0 N_c \sum_{\vec{p}} \text{Tr} \left[ \Lambda^+ (\vec{p}) \gamma_0 \right]$$

$$- K \sum_{\vec{p}, \vec{q}} \sum_{l} \Lambda^+_{\mu\gamma}(\vec{p}) \Lambda^+_{\delta\gamma}(\vec{q})$$

$$\otimes \left\{ \cos (\vec{p} - \vec{q}) \cdot \hat{n}_l \left[ (\Sigma_l)_{\mu\nu} (\Sigma_l)_{\gamma\delta}^{\dagger} + (\Sigma_l)_{\mu\nu} (\Sigma_l)_{\gamma\delta}^{\dagger} \right] + \cos (\vec{p} + \vec{q}) \cdot \hat{n}_l \left[ (\Sigma_l)_{\mu\nu} (\Sigma_l)_{\gamma\delta}^{\dagger} + (\Sigma_l)_{\mu\nu} (\Sigma_l)_{\gamma\delta}^{\dagger} \right] \right\}$$

$$- \frac{1}{2} K \sum_{\vec{p}, \vec{q}} \sum_{l} \left[ \Lambda^+_{\mu\gamma}(\vec{q}) - \delta_{\nu\gamma} \right] \Lambda^+_{\delta\mu}(\vec{p})$$

$$\otimes \left\{ N_c \left[ (\Sigma_l)_{\mu\nu} (\Sigma_l)_{\gamma\delta}^{\dagger} + (\Sigma_l)_{\mu\nu} (\Sigma_l)_{\gamma\delta}^{\dagger} \right] + \cos (\vec{p} + \vec{q}) \cdot \hat{n}_l \left[ (\Sigma_l)_{\mu\nu} (\Sigma_l)_{\gamma\delta}^{\dagger} + (\Sigma_l)_{\mu\nu} (\Sigma_l)_{\gamma\delta}^{\dagger} \right] \right\}$$

(3.3)

while the second order off–diagonal Hamiltonian is found to be

$$H_{\text{off}} |\mathcal{G}\rangle = \left\{ M_0 \sum_{\vec{p}} (\gamma_0)_{\mu\nu} \xi_{\mu\gamma}(\vec{q}) \eta_{\nu}(\vec{q}) \right\}$$

$$+ \frac{1}{N_c} K \sum_{\vec{p}, \vec{q}} \sum_{l} \Lambda^+_{\mu\gamma}(\vec{q}) \xi_{\gamma}(\vec{q}) \eta_{\delta}(\vec{q})$$

$$\otimes \left\{ \cos (\vec{p} - \vec{q}) \cdot \hat{n}_l \left[ (\Sigma_l)_{\mu\nu} (\Sigma_l)_{\gamma\delta}^{\dagger} + (\Sigma_l)_{\mu\nu} (\Sigma_l)_{\gamma\delta}^{\dagger} \right] + \cos (\vec{p} + \vec{q}) \cdot \hat{n}_l \left[ (\Sigma_l)_{\mu\nu} (\Sigma_l)_{\gamma\delta}^{\dagger} + (\Sigma_l)_{\mu\nu} (\Sigma_l)_{\gamma\delta}^{\dagger} \right] \right\}$$

$$- \frac{1}{N_c} K \sum_{\vec{p}, \vec{q}} \sum_{l} \left[ 2 \Lambda^+_{\nu\gamma}(\vec{p}) - \delta_{\nu\gamma} \right] \xi_{\mu}(\vec{q}) \eta_{\delta}(\vec{q})$$

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\begin{align*}
\otimes & \left[ N_c \left( (\Sigma_l)_{\mu\nu}(\Sigma_l)_{\gamma\delta}^\dagger + (\Sigma_l)_{\mu\nu}^\dagger(\Sigma_l)_{\gamma\delta} \right) \\
& \quad + \cos (\vec{p} + \vec{q}) \cdot \hat{n}_l \left( (\Sigma_l)_{\gamma\nu}^\dagger(\Sigma_l)_{\mu\delta} + (\Sigma_l)_{\mu\nu}^\dagger(\Sigma_l)_{\gamma\delta} \right) \right] e^{i2\omega(\vec{q})t} \\
& \otimes b_{\alpha,a}^\dagger(\vec{q}) d_{\alpha,a}^\dagger(-\vec{q}) \langle G \rangle 
\end{align*}

(3.4)

We see from Eq. (3.4) that the excitation spectrum of the effective Hamiltonian involves color singlet quark–anti–quark mesonic excitations coupled to zero total three momentum.

### IV. EQUATION OF MOTION AND DIAGONALIZATION OF \( H_{\text{EFF}} \)

We shall now demonstrate that \( H_{\text{eff}} \) vanishes exactly by exploiting the equation of motion for \( H_{\text{eff}} \). For this purpose it suffices to consider the equation of motion for the up quark field \( u(t, \vec{q}) \) given by

\begin{align*}
i(\dot{u}_a)_{\mu}(t, \vec{q}) &= : \left( (u_a)_{\mu}(t, \vec{q}), H_{\text{eff}} \right) : \\
&= \left\{ M_0 (\gamma_0)_{\mu\delta} \\
& \quad + \frac{1}{N_c} K \sum_{\vec{p}} \sum_{l} \lambda^+_l(\vec{p}) \\
& \quad \otimes \left[ \cos (\vec{p} - \vec{q}) \cdot \hat{n}_l \left( (\Sigma_l)_{\gamma\nu}(\Sigma_l)_{\mu\delta}^\dagger + (\Sigma_l)_{\mu\nu}^\dagger(\Sigma_l)_{\gamma\delta} \right) \\
& \quad \quad + \cos (\vec{p} + \vec{q}) \cdot \hat{n}_l \left( (\Sigma_l)_{\gamma\nu}^\dagger(\Sigma_l)_{\mu\delta} + (\Sigma_l)_{\mu\nu}(\Sigma_l)_{\gamma\delta} \right) \right] \\
& \quad - \frac{1}{N_c} K \sum_{\vec{p}, \vec{q}} \sum_{l} \left[ 2\lambda^+_{\nu\gamma}(\vec{p}) - \delta_{\nu\gamma} \right] \\
& \quad \otimes \left[ N_c \left( (\Sigma_l)_{\mu\nu}(\Sigma_l)_{\gamma\delta} + (\Sigma_l)_{\mu\nu}^\dagger(\Sigma_l)_{\gamma\delta} \right) \\
& \quad \quad + \cos (\vec{p} + \vec{q}) \cdot \hat{n}_l \left( (\Sigma_l)_{\gamma\nu}^\dagger(\Sigma_l)_{\mu\delta}^\dagger + (\Sigma_l)_{\mu\nu}^\dagger(\Sigma_l)_{\gamma\delta} \right) \right] \right\} (u_a)_{\delta}(t, \vec{q}) 
\end{align*}

(4.1)

\begin{align*}
\left[u_a, H_{\text{eff}} \right] &= \left[u_a, H_{0}^{r=0} \right] 
\end{align*}

(4.4)

The crucial step in diagonalizing \( H_{\text{eff}} \) is to recall that \( u_a \) is also the solution of the free lattice Dirac equation with \( r = 0 \) and therefore one can identify Eq. (1.11) with Eq. (2.13).

\begin{align*}
H_{\text{off}} \langle G \rangle &= \xi^\dagger(\vec{q}) \left( \sum_l \sin(\vec{q} \cdot \hat{n}_l) \gamma_0 \gamma_l + M(\vec{q}) \gamma_0 \right) \eta(-\vec{q}) e^{i2\omega(\vec{q})t} b_{\alpha,a}^\dagger(\vec{q}) d_{\alpha,a}^\dagger(-\vec{q}) \langle G \rangle \\
&= 0 
\end{align*}

(4.5)
We can also use Eq. (4.4) to simplify the vacuum energy density to the following form.

\[ \frac{1}{V} \langle G| H_{\text{eff}} |G \rangle = -N_c \sum_{\vec{q}} \left\{ 3K(1 + r^2) + 2\omega(\vec{q}) + \frac{M(\vec{q})}{\omega(\vec{q})}M_0 \right\} \]  

(4.6)

Therefore, the difference in the vacuum energy densities between the phase with \(M(\vec{q}) = 0\) and \(M(\vec{q}) \neq 0\) is given by

\[ \Delta E = \frac{1}{V} \langle G| H_{\text{eff}} |G \rangle |_{M(\vec{q})=0} - \frac{1}{V} \langle G| H_{\text{eff}} |G \rangle |_{M(\vec{q}) \neq 0} \]

\[ = N_c \sum_{\vec{q}} \left[ \frac{M(\vec{q})}{\omega(\vec{q})}M_0 + 2 \left( \sum_l \sin^2(\vec{q} \cdot \hat{n}_l) + M^2(\vec{q}) \right)^{1/2} - 2 \left( \sum_l \sin^2(\vec{q} \cdot \hat{n}_l) \right)^{1/2} \right] \]

\[ > 0 \]  

(4.7)

Hence the phase with a finite gap is the energetically favored phase. It remains to derive the gap equation and to solve it for the dynamical quark mass.

**V. THE GAP EQUATION AND THE DYNAMICAL QUARK MASS**

The gap equation is obtained by using Eq. (4.4) and equating the coefficients of the \(\gamma_0\) operator. Using Eq. (2.17) we find

\[ M(\vec{q}) = M_0 + \frac{3}{2}K(1 - r^2) \sum_{\vec{\rho}} \frac{M(\vec{\rho})}{\omega(\vec{\rho})} \]

\[ + \frac{1}{N_c} K \sum_{\vec{\rho},l} \frac{M(\vec{\rho})}{\omega(\vec{\rho})} \left\{ 8r^2 \cos(\vec{\rho} \cdot \hat{n}_l) \cos(\vec{q} \cdot \hat{n}_l) - \frac{1}{2}(1 + r^2) \cos(\vec{\rho} + \vec{q}) \cdot \hat{n}_l \right\} \]  

(5.1)

Thus the dynamical quark mass is determined by a three dimensional non–linear integral equation and is momentum dependent. This dependence comes from the \(O(1/N_c)\) correction to the gap equation.

The solution to the gap equation to lowest order in \(N_c\) is shown in Figure 1a for the case of \(M_0 = 0\) and \(r = 0\). We see that the dynamical symmetry breaking takes place only above a critical coupling constant of about \(K_C \approx 0.73\). From Eq. (5.1) we see that \(K_C\) must increase as \(r\) is increased in order for the gap equation to have a solution and this situation is shown in Figure 1b. In fact, the critical coupling constant approaches infinity as \(r \to 1\) and the assumption of strong coupling (small \(K\)) breaks down in this limit.

However, when \(r = 1\) there is no \(O(N_c^0)\) contribution to Eq. (5.1) and the dynamical quark mass may be written as

\[ M(\vec{q}) = M_0 + B(K) \sum_l \cos(\vec{q} \cdot \hat{n}_l) \]  

(5.2)

assuming that \(M(-\vec{q}) = M(\vec{q})\). In this case the solution to the gap equation is shown in Figure 2 again for the \(M_0 = 0\) case. At this value of the Wilson parameter we find solutions to the gap equation only above a value of \(K_C \approx 0.84\). However, the dynamical quark mass is one order of magnitude less than that obtained with only the \(O(N_c^0)\) contribution. This
result suggests that the momentum dependence of the dynamical quark mass may be treated as a correction to the dominant constant term.

Our results for the dynamical mass is qualitatively different to the one obtained by Le Yaouanc, Oliver, Pène and Raynal in Eq. (6.6) of [7]. Not only their dynamical mass is momentum independent, but the chiral symmetry breaking takes place for all values of the coupling constant $K > 0$. In addition, from Eq. (3.2) we see that in the $M_0 \to 0$ limit the chiral condensate is finite only above a certain value of the critical coupling constant $K_C$ for the examples presented here. Above $K_C$, the chiral condensate is not a constant but a function of $K$. This situation is in contrast to the behaviour of the chiral condensate obtained in [7] where it is always a finite constant in the broken symmetry mode.

VI. CONCLUSION AND OUTLOOK

In this work we have constructed a simple solution to Smit’s effective Hamiltonian for the strongly coupled lattice QCD. Our solution obeys the free lattice Dirac equation with a dynamical quark mass. This mass is the solution of the gap equation which is determined from the equation of motion. The creation and annihilation operators for quarks and anti–quarks obey the well–defined free fermion anti–commutation relations and annihilate an interacting vacuum state. Since quantities of interest are solely determined by the anti–commutation relations there is no need to specify the structure of this state.

The solution presented here exactly diagonalizes the Hamiltonian to second order in the quark field operators. We find that the elementary excitations of the theory consist of quark–anti–quark color singlet states coupled to zero total three momentum and that the phase with broken chiral symmetry is the energetically favored phase. Moreover in this phase the gap of the theory is found to depend both on momentum and on the effective coupling constant.

However, our solution is by no means complete. It may be systematically improved to calculate the masses, widths and coupling constants of all the bound states allowed by the Hamiltonian if our approach is to be interpreted within the context of the N–quantum approach (NQA) to quantum field theory [8,9] as follows. NQA is a method to solve the field equations of motion by expanding the Heisenberg fields in terms of asymptotic fields obeying the free field equations of motion. This expansion, which varies from theory to theory, is known as the Haag expansion [10].

The first term in the Haag expansion is simply a single free asymptotic field. Previous applications of the NQA to various field theories have shown that the use of this first order term alone can reproduce the known mean field results not only in free space but also at finite temperature ($T$) and chemical potential ($\mu$) [11]. These additional variables are introduced by subjecting each of the field operators in the Haag expansion to thermal Bogoliubov transformations just as in thermal field dynamics. The solution presented in this work can be interpreted as the first order Haag expansion of the Heisenberg quark fields in the effective strong coupling QCD Hamiltonian. Since the time coordinate is not discreticized in the Hamiltonian (Kogut–Susskind) formulation of lattice gauge theory one can naturally introduce and work with the concept of an asymptotic field.

We can improve our solution by taking the second order terms in the Haag expansion into account. In our case each of these terms would be a product of a fermionic (quark) and
a bosonic field, the latter corresponding to the various bound states allowed by the Hamiltonian. They are the elementary excitations identified in this work. The coefficients of the second order terms are interpreted as amplitudes for formations of these bound states. Self-consistency equations for these bound state amplitudes are derived by using the equation of motion. These bound state amplitudes have the same number of kinematical variables as in non-relativistic theories and are independent of Bethe–Salpter amplitudes. The idea of the NQA is to solve for these amplitudes and thus construct a solution to the field theory beyond the mean field approximation.

The program described above have recently been carried out for the effective instanton induced ‘t Hooft interaction at finite $T$ and $\mu$ [12]. In constructing a solution to the ‘t Hooft model beyond the mean field theory the masses, widths and coupling constants of the $\sigma$ and diquark states have been determined. It would be interesting to apply the second order NQA to the Hamiltonian considered in this work and to calculate the properties of bound states of the theory and to construct an improved solution to the strong coupling QCD Hamiltonian at any $T$ and $\mu$. This solution can then be used to obtain a qualitative picture of the QCD equation of state.

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FIGURE CAPTIONS

FIGURE 1. The solution to the gap equation Eq. (5.1) to lowest order in $N_c$ for $M_0 = 0$. (a) Dynamical quark mass $m$ for $r = 0$ as a function of the effective coupling constant $K$. The critical coupling constant $K_C$ is approximately 0.73. (b) $K_C$ as a function of the Wilson parameter $r$.

FIGURE 2. The solution to the gap equation Eq. (5.1) for $r = 1$ and $M_0 = 0$. The coefficient of the momentum dependent dynamical quark mass $B$ from Eq. (5.2) is shown as a function of the effective coupling constant $K$. The critical coupling constant $K_C$ is approximately 0.84.
Figure 1

(a)

(m vs r)

(b)

(K vs r)

(Kc vs r)
