Beta Process Non-negative Matrix Factorization with Stochastic Structured Mean-Field Variational Inference

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Abstract

Beta process is the standard nonparametric Bayesian prior for latent factor model. In this paper, we derive a structured mean-field variational inference algorithm for a beta process non-negative matrix factorization (NMF) model with Poisson likelihood. Unlike the linear Gaussian model, which is well-studied in the nonparametric Bayesian literature, NMF model with beta process prior does not enjoy the conjugacy. We leverage the recently developed stochastic structured mean-field variational inference to relax the conjugacy constraint and restore the dependencies among the latent variables in the approximating variational distribution. Preliminary results on both synthetic and real examples demonstrate that the proposed inference algorithm can reasonably recover the hidden structure of the data.

1 Introduction

Non-negative matrix factorization (NMF) model, which approximately decomposes a non-negative matrix into the product of two non-negative matrices (usually referred as the latent component and the activation), is widely used in many application domains, such as music signal analysis [Smaragdis and Brown, 2003] and recommender systems [Gopalan et al., 2013]. One hyperparameter in the NMF model is the number of latent components, which is usually set via model selection (e.g. cross validation). Nonparametric Bayesian latent factor models, on the other hand, offer an alternative solution by putting an infinite-dimensional prior on the latent component and activation matrices, and allow the data to “speak for itself” via posterior inference.

Most of the literature on nonparametric Bayesian latent factor models focuses on conjugate linear Gaussian models, for example, beta process factor analysis [Paisley and Carin, 2009]. However, such models are not appropriate for problems where non-negativity should be imposed. To address this limitation, Liang et al. [2013] proposed beta process NMF model by introducing a binary mask, the same as in Paisley and Carin [2009], and adopted Laplace approximation variational inference [Wang and Blei, 2013] for this non-conjugate model.

However, Gaussian likelihood model was chosen for mathematical convenience; in order to perform inference, numerical optimization is required, which is computationally intensive. Besides the computational burden, naively applying mean-field variational inference to beta process NMF model breaks the strong dependencies among the binary mask, the latent components, and the activations, and it introduces additional local optima [Wainwright and Jordan, 2008]. The stochastic structured mean-field (SSMF) variational inference [Hoffman, 2014] was recently developed as an attempt to restore the dependencies among latent variables in the approximating distribution. In this paper, we utilize SSMF to address two problems: First, we develop an inference algorithm for beta process NMF models that are inherently non-negative, which, to our knowledge, has not been
and take the following forms:

\[ q(W_{fk}) = \text{Gamma}(\nu_k^W, \rho_k^W); \quad q(H_{kt}) = \text{Gamma}(\nu_k^H, \rho_k^H); \quad q(\pi_k) = \text{Beta}(\alpha_k^\pi, \beta_k^\pi) \]

Comparing with the regular mean-field where the variational distributions are completely factorized among \( W, H, \) and \( S, \) here we allow the approximated joint posterior of binary mask \( s_t \) and auxiliary variables \( Z_t \) to depend on \( W \) and \( H \) for each \( t \in \{1, \ldots, T\}. \) The evidence lower bound (ELBO):

\[ \mathcal{L} \equiv \mathbb{E}_q[\log p(W, H, \pi)] + \sum_t \mathbb{E}_q[\log p(x_t, Z_t, s_t | W, H, \pi)] \leq \log p(X) \]

As noted in Hoffman [2014], the second term corresponds to the “local ELBO”:

\[ \mathcal{L}_t \equiv \mathbb{E}_q[\log p(x_t, Z_t, s_t | W, H, \pi)] - \mathbb{E}_q[\log q(Z_t, s_t | W, H, \pi)] \leq \log p(x_t | W, H, \pi) \]

The basic idea behind SSMF is that we can first sample global parameters from the variational distribution and then optimize the local ELBO (with respect to the local parameters) using these sampled global parameters, followed by taking a (natural) gradient step on the global parameters. This local ELBO will reach the optimum if \( q(Z_t, s_t | W, H, \pi) \) equals the exact conditional \( p(Z_t, s_t | x_t, W, H, \pi), \) which is intractable to compute. Fortunately, SSMF only requires that we get a sample from it to construct a noisy gradient. We will resort to Collapsed Gibbs sampling to sample \( s_t \) by marginalizing out \( Z_t. \)
3.1 Collapsed Gibbs sampler for $s_{it}$

The construction of the auxiliary variables $Z_t$ makes them straightforward to marginalize. We can then derive the proportion $s_{ik}$ being active or not by computing the following two quantities (Define $X_{ft} = \sum_{l \neq k} W_{ft}H_{jt}S_{lt}$).

$\mathbb{P}(S_{kt} = 1|S_{-,k,t}, x_t, W, H, \pi) \propto \pi_k \cdot p(x_t|W, h_t, S_{-,k,t}, S_{kt} = 1)$

$\mathbb{P}(S_{kt} = 0|S_{-,k,t}, x_t, W, H, \pi) \propto (1 - \pi_k) \cdot p(x_t|W, h_t, S_{-,k,t}, S_{kt} = 0)$

Finally, we can sample $S_{kt} \sim \text{Bernoulli}(\frac{\pi_k}{\pi_k + \pi_{l}})$ after the burn-in period. To recover the per-component contribution $Z_t$, note that by the property of the Poisson distribution, the conditional is multinomial-distributed: $z_{ft}\sim \text{Multi}(X_{ft}, \chi_{ft})$ where $\chi_{ft} \sim W_{ft}H_{kt}S_{kt}$. Thus, we can use the conditional expectation $\mathbb{E}[z_{ft}|X_{ft}, W_{ft}, H_{kt}, S_{kt}] = X_{ft}\phi_{ftk}$ as a proxy.

3.2 Update global parameters $W, H, \pi$

By introducing the auxiliary variables $Z$, the model in Equation 1 enjoys the conditional conjugacy when conditioning on the binary mask $S$. Therefore, the full global posterior can be factorized into conjugate pairs with respect to $W, H, \pi$ separately. Applying SSMF\textsuperscript{3} on the corresponding variational parameters, we can obtain the full SSMF variational inference algorithm as described in Algorithm 1.

4 Experimental results

We evaluated the proposed SSMF variational inference algorithm on synthetic examples for sanity check, as well as on real data on the task of blind source separation (BSS). In the BSS task, for comparison, we also derived a Gibbs sampler, which is slower but asymptotically exact, as an upper bound on how well the inference can potentially be. The Gibbs sampler is briefly described below.

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\textsuperscript{3}For simplicity, we actually applied an approximated version of SSMF (referred as “SSMF-A” in Hoffman [2014]).
and one clarinet note are played simultaneously at different pitches. The corresponding activation parameters whose activations are greater than 0 are included. As we can see, the learned latent components have clear harmonic structure and capture the notes which are activated at different time. This is also implicitly reflected in Figure 1. Here we show the posterior mean (using variational distribution as a proxy) for the latent activations \( \{ h_t \} \). Only 20 out of 100 \( \pi_t \)'s are significantly greater than 0.

\[ \pi_t \sim \text{Beta}(\frac{\alpha}{K} + \sum_t S_{kt}, \frac{\beta_t(K-1)}{K} + T - \sum_t S_{kt}). \]

**Algorithm 2** Gibbs sampler for beta process NMF

Randomly initialize \( W, H, S, \) and \( \pi. \)

for \( i = 1, 2, \ldots \) do

Sample \( S_{kt} \) using Gibbs sampler in Section 3.1 and compute \( \phi_{ftk} = \frac{W_{ftk}H_{kt}S_{kt}}{\sum_t W_{ftk}H_{kt}S_{kt}}. \)

Sample \( W_{ftk} \sim \text{Gamma}(a + \sum_t X_{ftk}\phi_{ftk}, b + \sum_t H_{kt}S_{kt}). \)

Sample \( H_{kt} \sim \text{Gamma}(c + \sum_t X_{ftk}\phi_{ftk}, d + S_{kt}\sum_t W_{ftk}). \)

Sample \( \pi_k \sim \text{Beta}(\frac{\alpha_k}{K} + \sum_t S_{kt}, \frac{\beta_t(K-1)}{K} + T - \sum_t S_{kt}). \)

end for

**Gibbs sampling** We sampled \( s_t \) and estimated \( Z_t \) the same way we did for SSMF, as in Section 3.1. The same complete conditionals can be adopted for the global parameters, which leads to Algorithm 2. Notice the similarity between Gibbs sampler and SSMF for this particular model: if we set the step-size \( \eta^{(t)} \) in SSMF to 1, then SSMF effectively transitions into Gibbs sampler, yet we also lose the convergence guarantee on the stochastic optimization procedure.

### 4.1 Synthetic data

We randomly sampled synthetic data following the generative process: We first sampled the hyperparameters: \( A_{lf}, B_{lf} \sim \text{Gamma}(1, 1), C_{lt}, D_{lt} \sim \text{Gamma}(3, 5), \pi_l \sim \text{Beta}(0.05, 0.95), \) for \( f \in \{1, \ldots, 75\}, t \in \{1, \ldots, 1000\}, \) and \( l \in \{1, \ldots, 100\}. \) Then we sampled the latent variables: \( W_{lf} \sim \text{Poisson}(\sum_t W_{ft}H_{lt}S_{lt}), \) and \( S_{lt} \sim \text{Bernoulli}(\pi_l). \) Finally the data was sampled \( X_{lf} \sim \text{Poisson}(\sum_t W_{ft}H_{lt}S_{lt}). \) Only 20 out of 100 \( \pi_t \)'s are significantly greater than 0.

We fit the model with the hyperparameters \( a = b = 0.5, c = d = 5, a_0 = b_0 = 1, \) and truncation level \( K = 500. \) After the algorithm converged, roughly 20 out of 500 \( \pi_k \)'s had values significantly greater than 0, and the synthetic data was clearly recovered from the posterior mean.

We synthesized a short clip of audio with 5 distinct piano notes and 5 distinct clarinet notes using ChucK\(^2\) which is based on physical models of the instruments. At any given time, one piano note and one clarinet note are played simultaneously at different pitches.\(^3\)

The audio clip was resampled to 22.05 kHz and we computed Fast Fourier Transform (FFT) of 512 points (23.2ms) with 50% overlap, which yielded a matrix of 257 by 238. We fit the model using the same hyperparameter setting as used above. The NMF decomposition results are illustrated in Figure 1. Here we show the posterior mean (using variational distribution as a proxy) for the latent components \( W \) (left) and the activations \( H \oplus S \) (right). Only the components with \( \pi_k \) significantly greater than 0 are included. As we can see, the learned latent components have clear harmonic structure and capture the notes which are activated at different time. This is also implicitly reflected by the clear patterns from the activations \( H \oplus S \) on the right.

### 4.2 Blind source separation

We compared the performance of SSMF and Gibbs sampler on the task of audio blind source separation. We used MIREX \( F_0 \) estimation data, a woodwind quintet recording, which consists of bassoon, clarinet, flute, horn, and oboe. The goal is to separate individual instruments (sources) from the mixture audio signals. We resampled the audio to 22.05 kHz and computed FFT of 1024 samples with 50% overlap. We fit the model for both SSMF and Gibbs sampler with the hyperparameters \( a = 0.5, c = d = 5, a_0 = b_0 = 1, \) and truncation level \( K = 500. \) For the Gibbs sampler, we ran 200 iterations as burn-in. Given the fairly random behavior of the binary mask, we did not take multiple samples and average, instead we only took one sample for the binary mask \( S. \)

There is no direct information to determine how the latent components and instruments correspond, thus we used the heuristic in Liang et al. [2013]: for each instrument, we picked the single component whose corresponding activation \( \{ h_t \oplus s_t \} \) had the largest correlation with the power

\[ \text{http://chuck.stanford.edu/} \]

\[ \text{http://www.ee.columbia.edu/~dliang/files/demo.mp3} \]
Table 1: Instrument-level bss_eval results with standard error in the parenthesis. The last column represents the number of components whose corresponding $\pi_k$’s are significantly greater than 0.

|       | SDR       | SIR       | SAR       | K   |
|-------|-----------|-----------|-----------|-----|
| SSMF  | 1.84 (0.92) | 6.95 (1.80) | 4.82 (0.32) | 37  |
| Gibbs | 3.58 (1.22) | 13.46 (3.69) | 5.47 (1.26) | 60  |

The bss_eval [Vincent et al., 2006] was used to quantitatively evaluate the separation performance. Table 1 lists the average SDR (Source to Distortion Ratio), SIR (Source to Interferences Ratio), and SAR (Sources to Artifacts Ratio) across instruments for SSMF and Gibbs sampler (higher ratios are better). As we can see, Gibbs sampler yielded better separation performance (even better than the ones reported in Liang et al. [2013]) with running time about twice as long\(^4\). The higher SIR by Gibbs sampler may be partially due to that more components were discovered (the last column), introducing less interference between components.

On the other hand, comparably better results were obtained for SSMF than Liang et al. [2013], while admittedly the model assumptions are slightly different. Ideally it would be more convincing if we can also compare against the regular mean-field method. However, the binary mask $S$ breaks the conditional conjugacy and we cannot directly apply the mean-field variational inference. A workaround is to use a degenerate delta function as the variational distribution for $S$, as used in Gopalan et al. [2014]. This will effectively estimate $S$ via maximum a posteriori (MAP), possibly with numerical optimization involved. This will be part of the future work.

The relative time difference may also come from the implementation details, but the similarity between Gibbs sampler and SSMF\(^5\) ensures this should not be a contributing factor.

### 5 Conclusion and discussion

We present a stochastic structured mean-field variational inference algorithm for beta process KL-NMF model, which is infamously vulnerable to local optima. On synthetic examples, the model can reasonably recover the hidden structure. On a blind source separation task, SSMF performs on par with the asymptotically exact Gibbs sampler.

There is one caveat regarding the hyperparameters. The model $a$ priori has two independent channels to impose sparsity on the activation: the binary mask $S$ and the unmasked activation $H$. Partic-

\(^4\)The time for Gibbs sampler only includes 200 iterations as burn-in.
ularly, when the prior on $H$ is sparse (i.e. with $c < 1$), which produces sparse $H$, $S$ can always turn on a few factors (with the corresponding $\pi_k$ being close to 1) and leave the remaining factors almost completely off. This is not a desirable scenario, since we hope the sparsity pattern is captured by the binary mask. Therefore, in the experiments, we set the prior for $H$ to be relatively dense with $c = d = 5$ to encourage the binary mask be sparse to “mask out” the dense $H$. As an alternative, negative-binomial process Poisson factor model [Zhou et al., 2012] can be adopted where $H \odot S$ is modeled together as a random drawn from a negative-binomial distribution and efficient variational inference can thus be derived.

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