Quasi multipartite entanglement measure based on quadratic functions

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We develop a new entanglement measure by extending Jaeger’s Minkowskian norm entanglement measure. This measure can be applied to a much wider class of multipartite mixed states, although still ”quasi” in the sense that it is still incapable of dividing precisely the sets of all separable and entangled states. As a quadratic scalar function of the system density matrix, the quasi measure can be easily expressed in terms of the so-called coherence vector of the system density matrix, by which we show the basic properties of the quasi measure including (1) zero-entanglement for all separable states, (2) invariance under local unitary operations, and (3) non-increasing under local POVM (positive operator-valued measure) measurements. These results open up perspectives in further studies of dynamical problems in open systems, especially the dynamic evolution of entanglement, and the entanglement preservation against the environment-induced decoherence effects.

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I. INTRODUCTION

In recent years, research on quantum information science [1] has made a tremendous leap inspired by its potential impacts on the existing information technologies. Among the rich theoretical studies from various fields, quantum entanglement [2, 3, 4, 5, 6, 7, 8, 9, 10], as a pure quantum phenomenon, has been recognized to be the key of high computation ability and communication security in quantum information implementations. It is practically very important to quantify the amount of entanglement in quantum states that embodies the capacity of quantum information process. Considerable efforts have been addressed in the literature. For two-qubit pure states, this problem has been completely solved in various ways such as the partial entropy entanglement measure [11], the distillable entanglement [11, 13, 14], and the entanglement of formation [15, 16, 17, 18, 19]. The third class is related to the eigenvalues of some matrix, e.g. concurrence [15] and the entanglement measures based on the PPT (positive partial transpose) condition [21, 22, 23].

The above approaches are physically intuitive, but not convenient in calculations. Following the idea of Wootters [15], Jaeger [24, 25, 26] proposed a novel entanglement measure for multi-qubit states. This measure can be explicitly expressed as a quadratic function of the coherence vector of quantum states under consideration. Since this measure is not yet perfect for some entangled states, even if it is better than Jaeger’s measure, we prefer to call it quasi entanglement measure.

The paper is organized as follows: in section II the concept of quasi entanglement measure is introduced. In section III we present the (expanded) coherence vector representation for multipartite density matrices, by which the quasi entanglement measure introduced in section II is re-expressed. In this picture, properties of this measure are discussed in section IV followed by several typical examples in section V. Finally, summary and perspectives for future studies are given in section VI.
II. QUASI ENTANGLEMENT MEASURE

Basically, in quantum physics, separable states refer to quantum states that can be prepared by classical means such as local unitary operations and local measurements. In the mathematical language, a $n$-partite separable state $\rho$ can be written as

$$\rho = \sum p_i |\psi_i^k\rangle \langle \psi_i^k| \otimes \cdots \otimes |\psi_i^n\rangle \langle \psi_i^n|,$$

where $p_i \geq 0$ and $|\psi_i^k\rangle$ is a pure state of the $k^{th}$ subsystem. The quantum states that are not separable are called entangled states. For a general entangled state $\rho$, a perfect entanglement measure $E(\rho)$ should satisfy the following conditions:

- $E(\rho) \geq 0$; $E(\rho) = 0$ if and only if $\rho$ is separable;
- Local unitary operations leave $E(\rho)$ invariant, i.e. $E(\rho) = E(U\rho U^\dagger)$ for arbitrary $U = U_1 \otimes \cdots \otimes U_n$,
- where $U_i$ is a unitary transformation acting on the $i^{th}$ subsystem;
- $E(\rho)$ is non-increasing under LOCC (local operation and classical communication) operations. Note that the LOCC operation can be mathematically expressed as $\Theta(\rho) = \sum_r M_r \rho M_r^\dagger$, where $M_r = L_{r,1} \otimes \cdots \otimes L_{r,k}$ and $\sum_r M_r^\dagger M_r = I$.

Among the enormous efforts to seek entanglement measures that fulfill the above criteria, Jaeger [24] proposed a scheme to measure entanglement in multi-qubit states borrowing ideas from Wootters [32]:

$$E(\rho) = \text{tr}\rho F(\rho),$$

where $F(\rho) = \sigma_y \otimes_n \rho^* \sigma_y \otimes_n$ is the flip operation on $\rho$ in which $\sigma_y$ is the $y$-axis Pauli matrix and $\rho^*$ denotes the complex-conjugate of the system density matrix $\rho$.

Jaeger’s measure is a good measure for pure multi-qubit states, and in particular, coincides with the so-called concurrence squared [33] for pure two-qubit states. The remarkable advantage is that the measure can be expressed as the Minkowskian norm of the Stokes parameters or, equivalently, the coherence vector of $\rho$, which are easy to be computed in practice.

However, Jaeger’s measure fails to precisely quantify entanglement in general mixed states, because it might be non-zero for separable mixed states. For example, one can verify that his measure is equal to $\frac{1}{n^2} I_2 \otimes_n I_2$ for the separable and completely mixed state $\frac{1}{n^2} I_2 \otimes_n I_2$, where $I_2$ is the two-dimensional identity matrix.

Actually, having an insight into the definition, we may find that the flip operation $F(\rho)$ for multi-qubit pure states flips a separable pure state

$$|\psi\rangle \langle \psi| = \otimes_{k=1}^n |\psi^k\rangle \langle \psi^k|,$$

to the separable pure state

$$|\tilde{\psi}\rangle \langle \tilde{\psi}| = \otimes_{k=1}^n |\tilde{\psi}^k\rangle \langle \tilde{\psi}^k|,$$

where $|\psi^k\rangle$ and $|\tilde{\psi}^k\rangle = \sigma_y(|\psi^k\rangle)^*$ satisfy:

$$\langle \psi^k|\tilde{\psi}^k\rangle = 0, \quad |\psi^k\rangle \langle \psi^k| + |\tilde{\psi}^k\rangle \langle \tilde{\psi}^k| = I.$$

Hence, for pure separable states, we have:

$$E(|\psi\rangle \langle \psi|) = |\langle \psi| \tilde{\psi}^k\rangle|^2 = \prod_{k=1}^n |\langle \psi^k|\tilde{\psi}^k\rangle|^2 = 0.$$

However, the measure becomes much more complicated for mixed separable states. In fact, for mixed separable states in the form of [10], one can find that:

$$E(\rho) = \sum_{i,j} p_i p_j \prod_{k} |\langle \psi^k_i|\tilde{\psi}^k_j\rangle|^2,$$

in which $\langle \psi^k_i|\tilde{\psi}^k_j\rangle$ is generally non-zero for $i \neq j$. Hence the measure $E(\rho)$ is improper because it usually gives positive values for mixed separable states.

The main reason for the above imperfection is that there are more than one term in the decomposition [10]. This reflects the mixedness of the state that is somewhat related to the classical correlation information [10] and should not be taken into account for the measure of quantum entanglement. In this regard, we will remove this amount of information from the original Jaeger’s measure in order to obtain a better one for more general multipartite mixed states.

Firstly, we generalize the aforementioned "flip" operation to multipartite systems:

**Definition 1** The "flip" operation $F(\rho)$ and the "unflip" operation $\bar{F}(\rho)$ on the multipartite density matrix $\rho$ are expressed as follows:

$$F(\rho) = \sum_{1 \leq i_1 < j_1 \leq N_1} \cdots \sum_{1 \leq i_n < j_n \leq N_n} \left( \bigotimes_{k=1}^n \sigma_{i_kj_k}^{(k)} \right) \rho \left( \bigotimes_{k=1}^n \sigma_{i_kj_k}^{(k)} \right)^*, \quad (2)$$

$$\bar{F}(\rho) = \sum_{1 \leq i_1 < j_1 \leq N_1} \cdots \sum_{1 \leq i_n < j_n \leq N_n} \left( \bigotimes_{k=1}^n \sigma_{i_kj_k}^{(k)} \right) \rho \left( \bigotimes_{k=1}^n \sigma_{i_kj_k}^{(k)} \right), \quad (3)$$

where $\otimes_{k=1}^n A_k = A_1 \otimes \cdots A_n$. $\rho^*$ is the complex-conjugate of $\rho$. The $N_k$ dimensional matrices $\sigma_{i_kj_k}^{(k)}$ and $\sigma_{i_kj_k}^{(k)*}$ act on the $k^{th}$ subsystem with the entries defined as:

$$(\sigma_{i,j}^{(k)})_{rs} = \frac{i}{\sqrt{N_k-1}} (\delta_{ir}\delta_{js} - \delta_{jr}\delta_{is}), \quad (\bar{\sigma}_{i,j}^{(k)})_{rs} = \frac{1}{\sqrt{N_k-1}} (\delta_{ir}\delta_{js} + \delta_{jr}\delta_{is}).$$

As a generalization of Jaeger’s measure, we use the quadratic function $\text{tr}[\rho F(\rho)]$ defined by the "flip" operation to quantify the amount of entanglement in multipartite pure states. One can verify that, owing to the factor
we introduce, this measure vanishes for arbitrary pure separable states (see Theorem 2) of general multipartite quantum systems, and, as a special case, is reduced to Jaeger’s Minkowskian norm entanglement measure for multi-qubit states. However, as analyzed before, this measure will become non-zero for mixed separable states that contain classical correlation information among the subsystems, which is related to the mixedness of the quantum states. We introduce the following function:

$$M(\rho) = \left( \frac{2^n}{\prod_{k=1}^{N} N_k} - tr[\rho \bar{F}(\rho)] \right),$$

to partially reflect the mixedness of the states. Note that, except for the multi-qubit case, $M(\rho)$ is different from the traditional mixedness expression $M(\rho) = 1 - tr\rho^2$, because the measure is required to vanish for pure separable states (see Theorem 2) and fulfill some other conditions that we are going to prove later (see Theorem 3).

Finally, by subtracting $M(\rho)$ from the gross entanglement measure $trpF(\rho)$, we draw a quadratic quasi entanglement measure

$$E_q(\rho) = \max \{f(\rho), 0\}, \quad (4)$$

with the function $f(\rho)$ defined as follows:

$$f(\rho) = tr[\rho F(\rho)] - \left( \frac{2^n}{\prod_{k=1}^{N} N_k} - tr[\rho \bar{F}(\rho)] \right), \quad (5)$$
or, in a more compact form:

$$f(\rho) = trp[F(\rho) + \bar{F}(\rho)] - \frac{2^n}{\prod_{k=1}^{N} N_k}. \quad (6)$$

In the following parts of this paper, we will show that the function $E_q(\rho)$ satisfies most of the conditions to be a perfect entanglement measure, except that the function may be zero for some entangled states. In this regard, we call the function $E_q(\cdot)$ a quasi entanglement measure. For example, the two-qubit Werner state $\tilde{w}$

$$w = \frac{1}{4} I \otimes I - \frac{1}{4} \cdot \frac{2\Phi + 1}{3} \sum_{k=1}^{3} \sigma_k \otimes \sigma_k$$

has been shown to be separable if and only if $-1 \leq \Phi \leq 0$, but our measure gives a wider range $-\frac{\sqrt{3}}{4} \leq \Phi \leq -\frac{\sqrt{3}}{2}$, which is 0.112 for $E_q(w) = 0$. Nevertheless, it can be proven that $E_q(\rho) = 0$, i.e. $f(\rho) \leq 0$, for all separable states (see the proof of Theorem 4). Therefore, the set of separable states is a proper subset of the set of zero-entanglement states. The relationship between these two sets is shown in Figure 1.

**Remark 1** For states of which $f(\rho)$ are negative, the negative values reflect the mixedness of the states due to the subtraction of $M(\rho)$. For example, for separable states, $f(\rho)$ is zero for all pure separable states (see Theorem 2), and goes below zero for mixed separable states in which the minimum $\frac{2^n}{\prod_{k=1}^{N} N_k} (1 - 2^{n-1})$ is reached at the completely mixed (separable) state $\frac{1}{\prod_{k=1}^{N} N_k} I \otimes \cdots \otimes I$.

To obtain a non-negative measure, we artificially cut off the negative part of $f(\rho)$ in (4) so that all separable states have zero entanglement. Such a definition amounts to a not so mathematically elegant non-smooth function, which also appears in Wooters’s concurrence entanglement measure [11]. Nevertheless, once we are able to find a perfect $f(\rho)$ for which $\rho$ satisfies $f(\rho) \leq 0$ if and only if $\rho$ is separable, $E(\rho) = \max\{f(\rho), 0\}$ becomes a perfect entanglement measure. In this case, negative $f(\rho)$ does not contain any entanglement information, because this only happens to separable states. On the other hand, the non-smoothness of such entanglement measures would help to explain why entanglement may be lost in finite time [12]. Roughly speaking, consider a Markovian open system that exponentially decays to an equilibrium separable distribution $\rho_\infty$ such that $f(\rho_\infty) < 0$. Although the equilibrium can be reached only in infinite time, the function $f(\rho(t))$ will touch and go below zero within a finite time interval because of the continuity of the function $f$. This is to say, in this case, the entanglement in $\rho(t)$ will disappear completely in finite time.

**III. EXPANDED COHERENCE VECTOR**

In this section, we will represent the entanglement measure $E_q(\rho)$ in the simpler coherence vector picture [12, 24, 25, 26, 27, 28, 29], which has been widely applied to describe the evolution of open quantum systems. For N-level systems, the coherence vector of a density matrix $\rho$ is derived as follows. Firstly, we choose an orthonormal basis $\{\Omega_0, \Omega_1, \cdots, \Omega_{N^2-1}\}$ of $N \times N$ complex matrices with respect to the inner product $(X,Y) = tr(X^\dagger Y)$, where $\Omega_0 = \frac{1}{\sqrt{N}} I$ is the normalized $N \times N$ identity matrix and $\Omega_1, \cdots, \Omega_{N^2-1}$ are $N \times N$ normalized traceless Hermitian matrices. A natural choice of the basis for N-level
systems is the generalization of Pauli matrices in two-level systems: \( \{ \Omega_{ij}^x, \Omega_{ij}^y, \Omega_{ij}^z; 1 \leq i < j \leq N, 2 \leq p \leq N \} \), where the entries of these matrices are:

\[
\begin{align*}
(\Omega_{ij}^x)_{rs} &= \frac{1}{\sqrt{2}} (\delta_{ir} \delta_{js} + \delta_{jr} \delta_{is}), \\
(\Omega_{ij}^y)_{rs} &= \frac{i}{\sqrt{2}} (\delta_{ir} \delta_{js} - \delta_{jr} \delta_{is}), \\
(\Omega_{ij}^z)_{rs} &= \begin{cases} \\
\sqrt{p(1-p)} & r < p \\
-\sqrt{p-1} & r = p \\
0 & r > p
\end{cases},
\end{align*}
\]

and \( 1 \leq r, s \leq N \). Under this basis, any \( N \times N \) complex matrix \( A \) can be expanded as \( A = \tilde{a} \cdot \tilde{\Omega} := \sum_{i=0}^{N-1} a_i \Omega_i \) where \( \tilde{a} = (a_0, \cdots, a_{N^2-1})^T \in \mathbb{C}^{N^2} \) (if \( A \) is Hermitian) and \( \tilde{\Omega} = (\Omega_0, \cdots, \Omega_{N^2-1})^T \). Denote the Hermitian density matrix of \( \rho \) as \( \rho = \tilde{m} \cdot \tilde{\Omega} \), where \( m_0 = \text{tr}(\Omega_0 \rho) \equiv \frac{1}{\sqrt{N}} \). The \( N^2 - 1 \) dimensional vector \( m = (m_0, \cdots, m_{N^2-2})^T \) is called the coherence vector of \( \rho \), while \( \tilde{m} = (m_0, m_1, \cdots, m_{N^2-2})^T \) is called the expanded coherence vector. For convenience of the following computations involving tensor product quantum states, we will frequently use \( \tilde{m} \) instead of \( m \). Obviously, since \( \{ \Omega_i \}_{i=0, \cdots, N^2-1} \) is an orthonormal basis, we have \( m^T \tilde{m} = \text{tr} \rho^2 \leq 1 \), which implies that the expanded coherence vector resides in the solid unit ball of \( \mathbb{R}^{N^2} \).

Generally, for a \( n \)-partite system of which the \( k \)-th subsystem is \( N_k \) dimensional, the orthonormal matrix basis can be naturally chosen as the tensor product of basis matrices of the subsystems:

\[ \{ \Omega_{i_1}^{(1)} \otimes \cdots \otimes \Omega_{i_n}^{(n)}; i_k = 0, \cdots, N_k^2 - 1; k = 1, \cdots, n \}, \]

where \( \{ \Omega_{i_k}^{(k)} \}_{i_k=0, \cdots, N_k^2-1} \) is the matrix basis of the \( k \)-th subsystem. Under this basis, a \( n \)-partite system density matrix can be written as:

\[
\rho = \sum_{0 \leq i_1 \leq N_1^2-1} \cdots \sum_{0 \leq i_n \leq N_n^2-1} m_{i_1 \cdots i_n} \Omega_{i_1}^{(1)} \otimes \cdots \otimes \Omega_{i_n}^{(n)},
\]

where the coefficients are:

\[
m_{i_1 \cdots i_n} = \text{tr} \left[ \rho \Omega_{i_1}^{(1)} \otimes \cdots \otimes \Omega_{i_n}^{(n)} \right].
\]

A novel property of the basis \( \{ \Omega_{i_1}^{(1)} \otimes \cdots \otimes \Omega_{i_n}^{(n)} \} \) is that the "flip" ("unflip") operations acting on the basis matrix \( \otimes_{k=1}^{n} \Omega_{r_k}^{(k)} \) can be decomposed into local "flip" ("unflip") operations acting on the local basis matrix \( \Omega_{r_k}^{(k)} \), i.e.

\[
F \left( \bigotimes_{k=1}^{n} \Omega_{r_k}^{(k)} \right) = \sum_{1 \leq i_1 < j_1 \leq N_1} \cdots \sum_{1 \leq i_n < j_n \leq N_n} \bigotimes_{k=1}^{n} \left( \sigma_{i_k j_k}^{(k)} \Omega_{r_k}^{(k)} \sigma_{i_k j_k}^{(k)} \right)
\]

\[
= \bigotimes_{k=1}^{n} \left( \sum_{1 \leq i_k < j_k \leq N_k} \left( \sigma_{i_k j_k}^{(k)} \right) \Omega_{r_k} \left( \sigma_{i_k j_k}^{(k)} \right) \right)
\]

\[
= \bigotimes_{k=1}^{n} F_k \left( \Omega_{r_k}^{(k)} \right),
\]

where \( 0 \leq r_k \leq N_k^2 - 1 \). For each local flip operation \( F_k \), one can examine by routine calculations that it keeps the basis matrix \( \Omega_{r_k}^{(k)} \) invariant, while flipping the other traceless matrices \( \Omega_{i_k}^{(k)} \) to \(-\frac{1}{N_k-1} \Omega_{i_k}^{(k)} \). In the coherence vector picture, the local operation keeps \( m_{0(k)} \) invariant and reverses the direction of the coherence vector \( m_{i(k)} \) with the norm shrinking to its \( \frac{1}{N_k-1} \) multiple. Similarly, the local "unflip" operation \( F_k \) keeps both \( m_{0(k)} \) and the direction of the coherence vector \( m_{i(k)} \) invariant, while shortening the coherence vector to its \( \frac{1}{N_k-1} \) multiple.

Therefore, the quadratic function \( f \) can be expressed in the coherence vector picture as follows:

\[
f \left( \tilde{m} \right) = \tilde{m}^T (S + \tilde{S}) \tilde{m} - \frac{2^n}{\prod_{k=1}^{n} N_k} \cdot \frac{1}{N_k-1} \cdot \frac{1}{N_k-1},
\]

where \( S = S^{(1)} \otimes \cdots \otimes S^{(n)} \) and \( \tilde{S} = \tilde{S}^{(1)} \otimes \cdots \otimes \tilde{S}^{(n)} \) with the \( N_k^2 \) dimensional matrices:

\[
S^{(k)} = \left( \frac{1}{N_k^2-1} \right), \quad \tilde{S}^{(k)} = \left( \frac{1}{N_k^2-1} \right)
\]

\[
\sum_{k=1}^{n} \frac{1}{N_k-1} \cdot \frac{1}{N_k-1},
\]

\[
S = \text{diag} \left\{ 1, \bigoplus_{k=i_1, \cdots, i_k} (-1)^k \frac{I_{N_k^2-1}}{N_k-1} \otimes \cdots \otimes \frac{I_{N_k^2-1}}{N_k-1} \right\},
\]

\[
\tilde{S} = \text{diag} \left\{ 1, \bigoplus_{k=i_1, \cdots, i_k} \frac{I_{N_k^2-1}}{N_k-1} \otimes \cdots \otimes \frac{I_{N_k^2-1}}{N_k-1} \right\},
\]

based on which we get the decomposition of \( G = S + \tilde{S} \):

\[
G = 2 \cdot \text{diag} \left\{ 1, \bigoplus_{2k=i_1, \cdots, i_k} \frac{0_{N_k^2-1}}{N_k-1} \otimes \cdots \otimes \frac{0_{N_k^2-1}}{N_k-1} \right\}.
\]

\[
\bigoplus_{2k=i_1, \cdots, i_k} \frac{I_{N_k^2-1}}{N_k-1} \otimes \cdots \otimes \frac{I_{N_k^2-1}}{N_k-1} \right\}.
\]
Within the established coherence vector picture, we may obtain the following equivalent expressions for future applications (see Appendix A for proof):

**Lemma 1** The fundamental concepts and operations in multipartite systems can be rephrased as follows:

1. **Separability:** $\bar{m}$ corresponds to a separable state $\rho$ if and only if it can be written as:
   \[ \bar{m} = \sum_{i} p_{i} \bar{m}_{i}^{(1)} \otimes \cdots \otimes \bar{m}_{i}^{(n)}, \]
   where $\bar{m}_{i}^{(k)}$ is the $N_{k}^{2}$ dimensional expanded coherence vector of a system density matrix $\rho_{i}^{(k)}$ of the $k^{th}$ subsystem;

2. **Local unitary operation** can be represented by a tensor product matrix $O^{(1)} \otimes \cdots \otimes O^{(n)}$ acting on the expanded coherence vector $\bar{m}$, where $O^{(k)} = \text{diag}(1, O^{(k)})$ and $O^{(k)}$ is a $N_{k}^{2} - 1$ dimensional orthonormal matrix;

3. **Local measurements** can be expressed as the tensor product matrix $D = D^{(1)} \otimes \cdots \otimes D^{(n)}$ acting on $\bar{m}$, where the matrix $D^{(k)}$ is $N_{k}^{2}$ dimensional. If the local measurements are POVM measurements, $D^{(k)} = \text{diag}(1, D^{(k)})$ and the $N_{k}^{2} - 1$ dimensional matrix $D^{(k)}$ is contractive, i.e. $D^{(k)T}D^{(k)} \leq I$.

**IV. PROPERTIES OF THE QUASI ENTANGLEMENT MEASURE**

Firstly, we show that our quasi entanglement measure vanishes for pure separable states:

**Theorem 2** For arbitrary pure separable states $\rho$, we have:

\[ \text{tr}[\rho F(\rho)] = 0, \quad \frac{2^n}{\prod_{k=1}^{n} N_k} - \text{tr}[\rho \bar{F}(\rho)] = 0, \]

which means $E_q(\rho) = f(\rho) = 0$.

**Proof:** From the first item in lemma 1, the coherence vector of a pure separable state $\rho$ can be written as:

\[ \bar{m} = \bar{m}_{i}^{(1)} \otimes \cdots \otimes \bar{m}_{i}^{(n)}, \]

where $\bar{m}_{i}^{(k)}$ is the $N_{k}^{2}$ dimensional expanded coherence vector of a pure state $\rho_{i}^{(k)}$ of the $k^{th}$ subsystem whose norm is 1, i.e. $\bar{m}_{i}^{(k)T}\bar{m}_{i}^{(k)} = \text{tr}(\rho^{(k)}) = 1$.

Let $\bar{m}^{(k)} = (m_{0}^{(k)}, m_{1}^{(k)})^{T} = (\frac{1}{\sqrt{N_{k}}}, m_{1}^{(k)})^{T}$, we have

\[ m_{i}^{(k)T}m_{i}^{(k)} = 1 - \frac{1}{N_{k}}, \]

by which it can be deduced that:

\[ \text{tr}[\rho F(\rho)] = \bar{m}^{T}S\bar{m} = \prod_{k=1}^{n} \bar{m}_{i}^{(k)T}S^{(k)}\bar{m}_{i}^{(k)} = 0, \]

\[ \text{tr}[\rho \bar{F}(\rho)] = \bar{m}^{T}\bar{S}\bar{m} = \prod_{k=1}^{n} \bar{m}_{i}^{(k)T}\bar{S}^{(k)}\bar{m}_{i}^{(k)} = \frac{2^n}{\prod_{k=1}^{n} N_k}. \]

**Theorem 3** The quadratic function $E_{q}(\rho)$ is a quasi entanglement measure that possesses the following properties:

1. $E_{q}(\rho) \geq 0$; $E_{q}(\rho) = 0$ if $\rho$ is separable;

2. Invariant under local unitary operations;

3. Non-increasing under local POVM measurements.

**Proof:** From $E_{q}(\rho) = \max\{ f(\rho), 0 \}$ and lemma 1, it is sufficient to prove in the coherence vector picture:

- $f(\bar{m}) \leq 0$ if $\bar{m} = \sum_{i} p_{i} \bar{m}_{i}^{(1)} \otimes \cdots \otimes \bar{m}_{i}^{(n)}$;
- $f(\bar{O}\bar{m}) = f(\bar{m})$ for arbitrary local unitary operation $\bar{O} = \bar{O}^{(1)} \otimes \cdots \otimes \bar{O}^{(n)}$, where $\bar{O}^{(k)} = \text{diag}(1, O^{(k)})$ and $O^{(k)}$ is an orthonormal matrix;
- $f(\bar{D}\bar{m}) \leq f(\bar{m})$ for the operation $\bar{D} = \bar{D}^{(1)} \otimes \cdots \otimes \bar{D}^{(n)}$, where $\bar{D}^{(k)} = \text{diag}(1, D_{k})$ and the matrix $D_{k}$ is contractive, i.e. $D_{k}^{(k)T}D_{k}^{(k)} \leq I$.

For the first property, we can directly compute that

\[ f(\sum_{i} p_{i} \bar{m}_{i}^{(1)} \otimes \cdots \otimes \bar{m}_{i}^{(n)}) = \sum_{i,j} p_{i}p_{j} \left( \prod_{k=1}^{n} \bar{m}_{i}^{(k)T}S^{(k)}\bar{m}_{j}^{(k)} + \prod_{k=1}^{n} \bar{m}_{i}^{(k)T}\bar{S}^{(k)}\bar{m}_{j}^{(k)} \right) - \frac{2^n}{\prod_{k=1}^{n} N_{k}}. \]

From the fact that

\[ \bar{m}_{i}^{(k)T}\bar{m}_{i}^{(k)} = \frac{1}{N_{k}} + m_{i}^{(k)T}m_{i}^{(k)} \leq 1 \Rightarrow m_{i}^{(k)} \leq \left[ 1 - \frac{1}{N_{k}} \right]^{\frac{1}{2}}, \]

it can be shown that $m_{i}^{(k)T}S^{(k)}\bar{m}_{i}^{(k)}$ and $m_{i}^{(k)T}\bar{S}^{(k)}\bar{m}_{i}^{(k)}$ in the above equation are both non-negative. In fact, the first term is non-negative because

\[ m_{i}^{(k)T}S^{(k)}\bar{m}_{i}^{(k)} = \frac{1}{N_{k}} - m_{i}^{(k)T}m_{i}^{(k)} \geq \frac{1}{N_{k}} - \frac{1}{N_{k} - 1} = 0, \]

\[ m_{i}^{(k)T}\bar{S}^{(k)}\bar{m}_{i}^{(k)} = \frac{1}{N_{k}} - m_{i}^{(k)T}m_{i}^{(k)} \geq \frac{1}{N_{k}} - \frac{1}{N_{k} - 1} = 0, \]
and the non-negativity of the latter term is because
\[ \tilde{m}_i^{(k)T} S^{(k)} \tilde{m}_j^{(k)} = \frac{1}{N_k} + \frac{m_i^{(k)T} m_j^{(k)}}{N_k - 1} \geq \frac{1}{N_k} - \frac{\|m_i^{(k)}\| \cdot \|m_j^{(k)}\|}{N_k - 1} \geq 0. \]

Employing the inequality
\[ \prod_{k=1}^n a_k + \prod_{k=1}^n b_k \leq \prod_{k=1}^n (a_k + b_k); \quad a_k, b_k \geq 0, \]
we arrive at the first property as follows:
\[ f(\sum_i p_i \tilde{m}_i^{(1)} \otimes \cdots \otimes \tilde{m}_i^{(n)}) \leq \sum_{i,j} p_i p_j \prod_{k=1}^n \tilde{m}_i^{(k)T} S^{(k)} \tilde{m}_j^{(k)} - \frac{2^n}{N_k} = \sum_{i,j} p_i p_j \prod_{k=1}^n \frac{2}{N_k} - \frac{2^n}{N_k} = 0. \]

As to the second property, it is sufficient to prove:
\[ f(\tilde{m}) - f(\tilde{O} \tilde{m}) = \tilde{m}^T (G - \tilde{O}^T \tilde{O}) \tilde{m} = 0, \]
where \( G = S + \tilde{S} \). It can be deduced from \( G - \tilde{O}^T \tilde{O} = 0 \) that can be easily verified from the commutativity relationship:
\[ [\tilde{O}^{(1)} \otimes \cdots \otimes \tilde{O}^{(n)}, G] = 0. \]

The third property requires that
\[ f(\tilde{m}) - f(\tilde{D} \tilde{m}) = \tilde{m}^T (G - \tilde{D}^T \tilde{D}) \tilde{m} \geq 0, \]
for which it is sufficient to prove that \( G - \tilde{D}^T \tilde{D} \) is a non-negative matrix.

From the expression of \( G \) in \( \{\mathcal{D}\} \) and
\[ \tilde{D} = \tilde{D}_1 \otimes \cdots \otimes \tilde{D}_n \]
\[ = \text{diag} \left\{ 1, \bigoplus_{k: i_1, \ldots, i_k} D^{(i_k)} \otimes \cdots \otimes D^{(i_k)} \right\}, \]
one can easily show that \( G - \tilde{D}^T \tilde{D} \) is block-diagonal with the non-zero diagonal blocks:
\[ \bigotimes_{i=1}^k \left[ \frac{1}{N_{i_1} - 1} \left(I_{N_{i_1}^2 - 1} - D^{(i_1)T} D^{(i_1)} \right) \right], \]
where \( 2k \). Thus \( G - \tilde{D}^T \tilde{D} \geq 0 \) is obvious from the contractive property of \( D^{(i_k)} \). The end of proof. \( \Box \)

Moreover, we can give an estimation of the bounds of the quadratic quasi entanglement measure:

**Theorem 4** For arbitrary quantum states \( \rho \), the quasi entanglement measure satisfies \( 0 \leq E_q(\rho) \leq 1 \).

**Proof:** The lower bound is obvious. Let \( \tilde{m} \) be the corresponding expanded coherence vector of \( \rho \). Suppose the dimensions of the first \( p \) subsystems \( N_1, \ldots, N_p \) \( \geq 3 \) and \( N_{p+1} = \cdots = N_n = 2 \) for the remaining subsystems. Writing \( \tilde{m}^T (S + \tilde{S}) \tilde{m} \) as the quadratic sum of the entries of \( \tilde{m} \), i.e. \( m_{i_1, \ldots, i_n} \) given in \( \{\mathcal{D}\} \), we have:
\[ \tilde{m}^T (S + \tilde{S}) \tilde{m} = \frac{2}{N_k} + \sum_{1 \leq i_1 \leq N_{ik} \leq N_k^2 - 1} \sum_{1 \leq i_{k+1} \leq N_k^2 - 1} \frac{m_{i_1, \ldots, i_n}^2}{N_k} = 0. \]

Divide the terms in the summation into three groups: the first group only depends on the first \( p \) subsystems whose dimensions are no less than 3, i.e. the non-zero indices \( i_1 \ldots i_n \) in subscripts only come from the first \( p \) subsystems; the second group is related to both the first \( p \) and the latter two-dimensional subsystems, in which non-zero indices distribute in both the two groups of subsystems with \( i_1 \ldots i_k \) from the first and \( i_{k+1} \ldots i_s \) from the second; the third group depends only on the latter \( n - p \) subsystems from which all non-zero indices \( i_1 \ldots i_s \) come. Noting that \( N_{ik} - 1 \) will automatically disappear from the denominator for \( i_k > p \) because \( N_{ik} = 2 \), we have:
\[ \tilde{m}^T (S + \tilde{S}) \tilde{m} = \frac{2}{N_k} + \sum_{1 \leq i_1 \leq N_{ik} \leq N_k^2 - 1} \sum_{1 \leq i_{k+1} \leq N_k^2 - 1} \frac{m_{i_1, \ldots, i_n}^2}{N_k} = 0. \]

Because, for the first two groups,
\[ \frac{2}{N_k} \leq 1, \quad \frac{2}{N_k} \leq 1, \]
we can derive that
\[ \tilde{m}^T (S + \tilde{S}) \tilde{m} \leq \frac{2}{N_k} + \sum_{1 \leq i_1 \leq N_k^2 - 1} \sum_{1 \leq i_{k+1} \leq N_k^2 - 1} \frac{m_{i_1, \ldots, i_n}^2}{N_k} = 0. \]
which means it can be further calculated that

\[
\bar{m}^T \rho^2 = \frac{1}{\prod_{k=1}^{n} N_k} + \sum_{1 \leq i, k \leq N_k^2 - 1} m_{i, i}^2 + \sum_{1 \leq i, k \leq N_k^2 - 1} m_{i, i}^2(0, 0)(0) = \frac{1}{\prod_{k=1}^{n} N_k} + \sum_{1 \leq i, k \leq N_k^2 - 1} m_{i, i}^2(0, 0)(0) = \frac{1}{\prod_{k=1}^{n} N_k} + \sum_{1 \leq i, k \leq N_k^2 - 1} m_{i, i}^2(0, 0)(0)
\]

From the equation

\[
tr \rho^2 = \bar{m}^T \bar{m} = \frac{1}{\prod_{k=1}^{n} N_k} + \sum_{1 \leq i, k \leq N_k^2 - 1} m_{i, i}^2(0, 0)(0) \]

it can be further calculated that

\[
\bar{m}^T (S + \tilde{S}) \bar{m} \leq tr \rho^2 + \frac{1}{\prod_{k=1}^{n} N_k} + \sum_{1 \leq i, k \leq N_k^2 - 1} m_{i, i}^2(0, 0)(0) \leq 1 + \frac{1}{\prod_{k=1}^{n} N_k} + \sum_{1 \leq i, k \leq N_k^2 - 1} m_{i, i}^2(0, 0)(0) \leq 1 + \frac{1}{\prod_{k=1}^{n} N_k} + \sum_{1 \leq i, k \leq N_k^2 - 1} m_{i, i}^2(0, 0)(i_{p+1} \ldots i_{n})
\]

Denote \(\rho_{p+1 \ldots n}\) the reduced density matrix for the last \(n-p\) subsystems. From (5), one can show that:

\[
\rho_{p+1 \ldots n} = tr_{1 \ldots p} \rho = \frac{1}{2^{n-p}} I + \prod_{k=1}^{n} N_k \sum_{i_{p+1} \ldots i_n} m_{(0, 0)}(i_{p+1} \ldots i_n) \Omega_i \otimes \ldots \otimes \Omega_i \otimes \Omega_i, \]

so we have

\[
tr \rho_{p+1 \ldots n}^2 = \frac{1}{2^{n-p}} + \left( \prod_{k=1}^{n} N_k \right) \sum_{1 \leq i_k \leq N_k^2 - 1} m_{(0, 0)}(i_{p+1} \ldots i_n) \leq 1,
\]

which means

\[
\sum_{1 \leq i_k \leq N_k^2 - 1} m_{(0, 0)}(i_{p+1} \ldots i_n) \leq \frac{1}{\prod_{k=1}^{n} N_k} - \frac{1}{\prod_{k=1}^{n} N_k}
\]

This from inequality, one obtains that

\[
\bar{m}^T (S + \tilde{S}) \bar{m} \leq 1 + \frac{1}{\prod_{k=1}^{n} N_k} + \sum_{1 \leq i_k \leq N_k^2 - 1} m_{(0, 0)}(i_{p+1} \ldots i_n) \leq 1 + \frac{1}{\prod_{k=1}^{n} N_k} \leq 1 + \frac{2^n}{\prod_{k=1}^{n} N_k}
\]

Therefore,

\[
0 \leq E_q(\rho) = \max\left\{ f(\bar{m}), 0 \right\} \leq 1 + \frac{2^n}{\prod_{k=1}^{n} N_k} - \frac{2^n}{\prod_{k=1}^{n} N_k} = 1.
\]

The end of proof.

Note that the upper bound can be reached for some multi-qubit states. For example, the entanglement value of the well-known GHZ state \(\frac{1}{\sqrt{2}} (|0 \cdots 0 \rangle + |1 \cdots 1 \rangle)\) with even \(n\) is equal to 1.

\[\text{V. EXAMPLES}\]

\textbf{Example 1} Consider \(n\)-qubit quantum states that are widely used in the quantum information theory. All the \(n\) subsystems are 2-dimensional. It is not difficult to verify from (6) that the corresponding quadratic quasi entanglement measure can be expressed as:

\[
E_q(\rho) = \max\{ tr(\rho F(\rho)) - [1 - tr(\rho^2)], 0 \},
\]

where \(F(\rho) = \sigma_y \otimes \sigma_y \rho \sigma_y \otimes \sigma_y \) and \(\rho^*\) denotes the complex-conjugate of \(\rho\). \(\sigma_y\) is the \(y\)-component of the well-known Pauli matrices.

For \(n\)-qubit pure states for which \(tr \rho^2 = 1\), we have \(f(\rho) = tr(\rho F(\rho))\). As shown by Jaeger, \(tr(\rho F(\rho)) \geq 0\) for physically meaningful states. Thus in this case the quasi entanglement measure \(E_q(\rho) = \max\{ f(\rho), 0 \} = f(\rho)\) is reduced to Jaeger’s Minkowskian norm entanglement measure \(E(\rho) = tr(\rho F(\rho))\).

\textbf{Example 2} Consider the entanglement measure of two-partite systems with dimensions \(N_1\) and \(N_2\) respectively. The two-partite "flip" operation \(F(\rho)\) is equivalent to the so-called universal state inverter \(F(\rho) = tr(\rho F(\rho))\) as follows:

\[
F(\rho) = \frac{tr(\rho I) - \rho_1 \otimes I - I \otimes \rho_2 + \rho}{(N_1 - 1)(N_2 - 1)},
\]

where \(\rho_1\) and \(\rho_2\) denote the reduced density matrices of the two subsystems.

In fact, in the coherence vector picture, we have:

\[
\rho = \frac{1}{N_1 N_2} I + \sum_{1 \leq i_1 \leq N_1^2 - 1} m_{i_1, i_1} \Omega_{i_1} \otimes \frac{1}{\sqrt{N_2}} I + \sum_{1 \leq i_2 \leq N_2^2 - 1} m_{i_2, i_2} \frac{1}{\sqrt{N_1}} I \otimes \Omega_{i_2} + \sum_{1 \leq i_2 \leq N_2^2 - 1} m_{i_1 i_2} \Omega_{i_1} \otimes \Omega_{i_2},
\]
\[ F(\rho) = \frac{1}{N_1 N_2} I - \frac{1}{N_1 - 1} \sum_{i_1} m_{i_1,0} \Omega_{i_1} \otimes \frac{1}{\sqrt{N_2}} I \]
\[ - \frac{1}{N_2 - 1} \sum_{i_2} m_{0, i_2} \frac{1}{\sqrt{N_1}} I \otimes \Omega_{i_2} \]
\[ + \frac{1}{(N_1 - 1)(N_2 - 1)} \sum_{i_1, i_2} m_{i_1 i_2} \Omega_{i_1} \otimes \Omega_{i_2}, \]

and
\[ \rho_1 = \frac{1}{N_1} I + \sqrt{N_2} \sum_{1 \leq i_1 \leq N_1^2 - 1} m_{i_1,0} \Omega_{i_1}, \]
\[ \rho_2 = \frac{1}{N_2} I + \sqrt{N_1} \sum_{1 \leq i_2 \leq N_2^2 - 1} m_{0, i_2} \Omega_{i_2}. \]

\[ E_q(\rho) = \max\{2(N_1 M(\rho_1) + N_2 M(\rho_2) - N_1 N_2 M(\rho)) / (N_1 - 1)(N_2 - 1)N_1 N_2, 0\}, \]

where \( M(\mu) = 1 - tr \mu^2 \) is the mixedness function of some quantum state \( \mu \).

Suppose the two-partite state is a pure state, i.e. \( M(\rho) = 0 \), we find that the more entangled the global state is, the more mixed the local states are. This shows that entanglement will increase the uncertainties in local measurements.

VI. CONCLUSION

In summary, we have developed a quadratic quasi entanglement measure for the general multiparticle quantum states. This measure is a generalization of several well-known measures that have been studied in the literature. The advantage of our measure is that it can be expressed as a simple quadratic function of the coherence vector that can be explicitly calculated. However, this measure is still not perfect for most general quantum states, for which we call it quasi entanglement measure, because it is not necessarily non-zero for all entangled states and we are still not able to prove the non-increasing property under more general LOCC transformations except for local POVM measurements. Nevertheless, the improvements of this measure comparing to the existing ones open up many perspectives such as analysis of the mechanism of the entanglement loss, and more importantly, the control of preserving entanglement against environment-induced decoherence effects, which used to be studied mainly from numerical or experimental perspectives. These remain to be studied in future work.

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APPENDIX A: PROOF OF THE LEMMA

(1) Separable states in the expanded coherence vector picture.

By the definition of separable states, we have:
\[ \rho = \sum_i p_i |\psi_i^1\rangle \langle \psi_i^1| \otimes \cdots \otimes |\psi_i^n\rangle \langle \psi_i^n| \]
\[ = \sum_i p_i \tilde{m}_i^{(1)} \cdot \tilde{\Omega}_i^{(1)} \otimes \cdots \tilde{m}_i^{(n)} \cdot \tilde{\Omega}_i^{(n)} \]
\[ = (\sum_i p_i \tilde{m}_i^{(1)} \otimes \cdots \tilde{m}_i^{(n)}) \cdot \tilde{\Omega}_i^{(1)} \otimes \cdots \otimes \tilde{\Omega}_i^{(n)} \]

Therefore, the expanded coherence vector of the separable states must be in the form of
\[ \tilde{m} = \sum_i p_i \tilde{m}_i^{(1)} \otimes \cdots \tilde{m}_i^{(n)}. \]

(2) Local unitary operation in the expanded coherence vector picture.

Firstly, the unitary operation of the \( N \)-level systems can be expressed by the expanded coherence vector:

\[ U \rho U^\dagger = U \left( \frac{1}{N} I + \sum_{i=1}^{N^2-1} m_i \Omega_i \right) U^\dagger \]
\[ = \frac{1}{N} I + \sum_{i=1}^{N^2-1} m_i U \Omega_i U^\dagger \]
\[ = \frac{1}{N} I + \sum_{i=1}^{N^2-1} \tilde{m}_i \tilde{\Omega}_i. \]

Obviously, \( \tilde{m}_0 = m_0 \). Denote \( m = (m_1, \cdots, m_{N^2-1})^T \), \( \tilde{m} = (\tilde{m}_1, \cdots, \tilde{m}_{N^2-1})^T \), we have
\[ tr \rho^2 = \frac{1}{N} + m^T m, \quad tr(U \rho U^\dagger)^2 = \frac{1}{N} + \tilde{m}^T \tilde{m}, \]

which imply that \( \tilde{m}^T \tilde{m} = m^T m \). Therefore, there exists an orthonormal matrix \( O \in so(N^2 - 1) \), such that \( \tilde{m} = O m \). Correspondingly, the expanded coherence vectors satisfy that \( \tilde{m} = O m \) with \( O = \text{diag}(1, O) \).

Furthermore, it is easy to show that local unitary operations \( U_1 \otimes \cdots U_n \) acting on the system density matrices can be expressed as the tensor product operations \( \tilde{O}^{(1)} \otimes \cdots \tilde{O}^{(n)} \) on the corresponding expanded coherence vectors.
(3) Local measurements in the expanded coherence vector picture.

For single partite case, measurements of $N$-level systems can be expressed as linear trace-preserving Kraus maps:

$$\epsilon(\rho) = \sum_j L_j \rho L_j^\dagger$$

with $\sum_j L_j^\dagger L_j = I$. One can always express in the coherence vector picture the measurement by a linear operation on the expanded coherence vector, i.e. the state after a measurement can be written as $\bar{m} = D\tilde{m}$, where $D$ is a constant matrix with proper dimensions.

If the measurement is further restricted to be a POVM measurement, i.e. $[L_j, L_k^\dagger] = 0$, it can be calculated that

$$\sum_j L_j \rho L_j^\dagger = \frac{1}{N} I + \sum_{i=1}^{N^2-1} m_i L_i \Omega_i L_i^\dagger.$$

The first term is equal to $\frac{1}{N} I$ because $L_j L_j^\dagger = L_j^\dagger L_j$ and $\sum_j L_j^\dagger L_j = I$. The second term can be expanded as a linear combination of $\Omega_i, i = 1, \cdots, N^2 - 1$, because it is traceless. Thus, for a POVM measurement, we have

$$\sum_j L_j \rho L_j^\dagger = \frac{1}{N} I + \sum_{i=1}^{N^2-1} \tilde{m}_i \Omega_i.$$

Writing $D$ as

$$D = \begin{pmatrix} a & h^T \\ g & D \end{pmatrix},$$

one can show that

$$D\tilde{m} = (am_0 + h^T m_0 g + Dm)^T = (m_0, m^T)^T.$$

The trace-preserving property requires that $m_0 = \frac{1}{\sqrt{N}}$, which implies $a = 1$ and $h^T = 0$. Further, let $m = 0$, i.e. $\rho = \frac{1}{N} I$, it can be verified that $\epsilon(\rho) = \frac{1}{N} I$. This is to say:

$$0 = \tilde{m} = Dm + m_0 g = m_0 g = \frac{1}{\sqrt{N}} g,$$

so we have $g = 0$. Therefore, $D$ can be written in a block-diagonal form $D = diag(1, D)$. The fact that $D$ is contractive for the POVM measurement $\epsilon(\rho)$ can be proved by showing that

$$tr\epsilon(\rho)^2 \leq tr\rho^2.$$

To prove this fact, note that the inequality $tr(A - B)(A^\dagger - B^\dagger) \geq 0$ gives:

$$trBA^\dagger + trAB^\dagger \leq trAA^\dagger + trBB^\dagger.$$

Let $A = \rho L_j^\dagger L_k$ and $B = L_j L_k \rho$. Applying the above inequality together with the properties $[L_i, L_j^\dagger] = 0$ and $\sum_j L_j^\dagger L_j = I$, we have

$$tr\epsilon(\rho)^2 = \sum_{j,k} trL_j \rho L_k^\dagger L_k \rho L_k^\dagger = \frac{1}{2} \sum_{j,k} (tr(\rho L_j^\dagger L_k \rho L_k^\dagger L_j) + tr(L_j^\dagger L_k \rho L_k^\dagger L_j \rho)) \leq \frac{1}{2} \sum_{j,k} (tr(\rho L_j^\dagger L_k \rho L_k^\dagger L_j) + tr(L_j^\dagger L_k \rho L_k^\dagger L_j \rho)) = \sum_{j,k} tr\rho^2(L_j^\dagger L_k \rho L_k^\dagger L_j) = tr\rho^2.$$

Thus it can be calculated that

$$\frac{1}{N} + m^T D^T Dm = \tilde{m}^T D^T D\tilde{m} = tr\epsilon(\rho)^2 \leq tr\rho^2 = \tilde{m}^T m = \frac{1}{N} + m^T m,$$

so we have $m^T D^T Dm \leq m^T m$ for any $N^2 - 1$ dimensional vector $m$ which means $D^T D \leq I$.

For $n$-partite case, a local POVM measurement can be written as:

$$\epsilon(\rho) = \sum_{i_1, \cdots, i_n} L_{i_1}^{(1)} \otimes \cdots \otimes L_{i_n}^{(n)} \rho L_{i_1}^{(1)^\dagger} \otimes \cdots \otimes L_{i_n}^{(n)^\dagger},$$

where $\sum_{i_k} L_{i_k}^{(k)^\dagger} L_{i_k}^{(k)} = I$ and $[L_{i_k}^{(k)}, L_{i_k}^{(k)^\dagger}] = 0$. One can decompose $\epsilon(\rho)$ into the product of the local operations,

$$\epsilon_k(\rho) = \sum_{i_k} M_{i_k} \rho M_{i_k}^\dagger,$$

where $M_{i_k} = I \otimes \cdots \otimes L_{i_k}^{(k)^\dagger} \otimes \cdots I$. It is easy to verify that the resulting expanded coherence vector of $\epsilon(\rho)$ can be written as $D\tilde{m} = (D^{(1)} \otimes \cdots \otimes D^{(n)})\tilde{m}$, where $D^{(k)} = diag(1, D^{(k)})$, and $D^{(k)^T} D^{(k)} \leq I$.

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