The $\infty$-Categorical Reflection Theorem and Applications

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Abstract

In this paper we prove an $\infty$-categorical version of the reflection theorem of [AR89]. Namely, that a full subcategory of a presentable $\infty$-category which is closed under limits and $\kappa$-filtered colimits is a presentable $\infty$-category. We then use this theorem in order to classify subcategories of a symmetric monoidal $\infty$-category which are equivalent to a category of modules over an idempotent algebra.

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1 Introduction

The goal of this paper is to prove an $\infty$-categorical version of the reflection theorem. Namely:

**Theorem 1.1.** (Theorem 6.2) Let $\mathcal{C}$ be a presentable category and let $\mathcal{D} \subset \mathcal{C}$ be a full subcategory which is closed under limits and $\kappa$-filtered colimits for some regular cardinal $\kappa$. Then, $\mathcal{D}$ is presentable.

The term presentable $\infty$-category refers to an $\infty$-category equivalent to an accessible localization of an $\infty$-category of presheaves. One can think about this definition as follows. Since the $\infty$-category $\text{Fun}(\mathcal{E}, \mathcal{S})$ is the large $\infty$-category generated freely by colimits from $\mathcal{E}$, it can be viewed as the free large $\infty$-category on $\mathcal{E}$. Taking an accessible localization of $\text{Fun}(\mathcal{E}, \mathcal{S})$ can be seen as adding a small space of relations. Thus, we can think of a presentable $\infty$-category as an $\infty$-category generated from a small space of objects and small spaces of relations.

It would be interesting to try retelling this story in reverse. Instead of starting with an $\infty$-category of presheaves and then localizing, we could start with a subcategory of presheaves and ask if it is obtained from an accessible localization. A necessary condition is that the inclusion commutes with limits and $\kappa$-filtered colimits for $\kappa$ large enough. Theorem 1.1 (the reflection theorem) then states that this condition is also sufficient.

Our work on the $\infty$-categorical case of the reflection theorem proves a conjecture posed by [Yan22, 2.15]. Our approach parallels work by [AR89] which proves the Theorem 1.1 for the case of ordinary 1-categories.

1.1 Outline of the Proof of the Reflection Theorem

For the convenience of the reader, we sketch an outline of the proof.

**Special class of morphisms:** In section 2 we define a special class of morphisms that we call $\kappa$-pure morphisms (see Definition 2.2), depending on a cardinal $\kappa$. We then show that every $\kappa$-pure morphism is a $\kappa$-filtered colimit of split morphisms with a fixed domain, where split morphisms are morphisms that have a left inverse (see Proposition 2.5). Therefore, we get that a $\kappa$-pure morphism $f : A \to B$ gives rise to a functor $F : I^\leq \to \mathcal{C}$ where $I$ is $\kappa$-filtered and all the maps $F(-\infty \to i)$ admit a retract (here $-\infty$ is the cone point). We call such functors quasi-split cones on $I$.

**Cone with retracts:** Let $F : I^\leq \to \mathcal{C}$ be a quasi-split cone. By choosing a left inverse for each map $F(-\infty \to i)$, $F$ gives rise to a functor from the pushout (see Construction 3.4)

$$L_I := I^\leq \coprod_{\text{Ob}(I) \times \Delta^1} (\text{Ob}(I) \times \text{Ret}) \to \mathcal{C}$$

where $\text{Ret}$ is the universal category with a retraction (see Definition 3.2). We call the $\infty$-category $L_I$ the cone with retracts on $I$. In section 3 we show that if $I$ is equivalent to a 1-category then
$L_I$ is equivalent to a 1-category. Furthermore, when $I$ is a poset we give an explicit description of $L_I$ (see Corollary 3.26). The idea is that gluing a copy of Ret along the morphism $-\infty \to i$ is equivalent to gluing a free arrow from $i$, i.e. taking the pushout

$$
\begin{array}{ccc}
\Delta^0 & \longrightarrow & I \\
\downarrow & & \downarrow \\
\Delta^1 & \longrightarrow & C_I
\end{array}
$$

and then inverting the unique arrow from $-\infty$ to the “new” object (i.e. the object in $C_I$ and not in $C$) in $C_I^{\triangleright}$. Therefore, it suffices to show that an $\infty$-category obtained by gluing a free arrow to a 1-category is equivalent to a 1-category (see Proposition 3.15), and that the $\infty$-category obtained by inverting the arrow from $-\infty$ to the new object is equivalent to a 1-category (see Proposition 3.20).

**Closedness under pure morphisms:** Let $C$ be a presentable $\infty$-category and $D \subset C$ be a full subcategory. We say that $D$ is closed under $\kappa$-pure morphisms if for any $B \in D$ and a $\kappa$-pure morphism $f : A \to B$ the object $A$ is in $D$ as well. In section 4 we show that a full subcategory closed under limits and $\kappa$-filtered colimits in a category of presheaves is closed under $\mu$-pure morphisms for $\mu$ large enough (see Proposition 4.2). This is the heart of the argument. The general approach is to show that if $f : A \to B$ is $\kappa$-pure then $A$ can be obtained from $B$ by limits and $\kappa$-filtered colimits. We use the explicit description of $L_I$ in ?? in order to build the desired diagram.

**Factoring through pure morphism:** Let $C$ be a category of presheaves. In section 5 we show that for any large enough cardinal $\kappa$, there exists a cardinal $\gamma \gg \kappa$ such that for any morphism $f : A \to B$ in $C$ with $A$ $\gamma$-compact there exists a factorization:

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
A' & \xrightarrow{f'} &
\end{array}
$$

where $f'$ is $\kappa$-pure and $A'$ is $\gamma$-compact.

**Finishing the proof:** Let $C$ be a presentable $\infty$-category and $D \subset C$ a full subcategory closed under limits and $\kappa$-filtered colimits. Since $D$ is closed under $\kappa$-filtered colimits it is closed under $\mu$-filtered colimits for all $\mu \geq \kappa$. Therefore, from the result of section 4, by enlarging $\kappa$, we may assume that $D$ is closed under $\kappa$-pure morphisms. Fix $B \in D$. Let $\gamma \gg \kappa$ as above and denote $\text{Pure}_\kappa(C^{\triangleright})/B$ the subcategory of $C^{\triangleright}_B$ spanned by the pure morphisms to $B$. Using the results of section 5, we get that the inclusion

$$
\text{Pure}_\kappa(C^{\triangleright})/B \hookrightarrow (C^{\triangleright})/B
$$

is cofinal. This means that every object of $D$ is a filtered colimit of objects in $C^{\triangleright} \cap D$, thus $D$ is an accessible category. From our assumption on $D$ we get that the inclusion $i : D \to C$ is an accessible functor between accessible categories. The adjoint functor theorem [NRS20, 3.2.5] then says that $i$ admits an accessible left adjoint. Hence $D$ is presentable (see Theorem 6.2).
One should note that the general structure of the proof is analogous to the structure of the proof of the 1-categorical case as presented in [AR94]. The main difference is the proof of Proposition 4.2 which is much more straight-forward in the 1-categorical case.

1.2 Applications for Recognizing Smashing Localizations

Given a symmetric monoidal ∞-category $C$ and an object $A \in C$ we say that a map $u: \mathbb{I}_C \to A$ exhibits $A$ as an idempotent algebra if the map

$$A \cong A \otimes \mathbb{I} \xrightarrow{1 \otimes u} A \otimes A$$

is an equivalence. By [Lur11, 4.8.2.9] in this case $A$ admits a unique commutative algebra structure for which $u$ is the unit. The fundamental feature of idempotent algebras is that the forgetful functor $\text{Mod}_A(C) \to C$ is fully faithful. Thus, it is a property for an object in $C$ to have the structure of an $A$-module.

So, given a property of objects of $C$, we can ask whether this property is classified by an idempotent algebra. That is, when the full subcategory of $C$ on such objects is of the form $\text{Mod}_A(C)$ for some idempotent algebra $A$. In the last section of the paper we use Theorem 1.1 in order to answer this question in two cases of interest.

First, we treat the case where $C \in \text{CAlg}(\text{Pr}^L)$\textsuperscript{1}. In this case the adjoint functor theorem provides us an “internal Hom functor”

$$\text{Hom}^C(-,-): C^{\text{op}} \times C \to C.$$

Using $\text{Hom}^C(-,-)$, we give a necessary and sufficient condition for a subcategory of $C$ to be classified by an idempotent algebra:

**Theorem 1.2.** *(Theorem 7.6)* For $C \in \text{CAlg}(\text{Pr}^L)$ and $D \hookrightarrow C$ a full-subcategory the following are equivalent:

1. $D$ is the category of modules over an idempotent algebra.

2. $D$ is closed under limits and colimits in $C$, and if $d \in D$ and $c \in D$, then $d \otimes c, \text{Hom}^C(c,d) \in D$.

Second, we deal with the case $C = \text{Pr}^L$. In [CSY21] idempotent algebras in $\text{Pr}^L$ are studied and are called “Modes”. The result we obtain is similar to Theorem 1.2, however since $\text{Pr}^L$ is not itself presentable there are some additional set theoretic assumptions to consider:

**Theorem 1.3.** *(Theorem 7.15)* For $P \hookrightarrow \text{Pr}^L$ a full-subcategory the following are equivalent:

1. $P$ is equivalent to the category of modules over an idempotent algebra.

2. (a) $P$ is closed under colimits in $\text{Pr}^L$.

(b) If $D \in P$, then $D^\Delta \coloneqq \text{Fun}(\Delta^1, D) \in P$.

\textsuperscript{1}Here $\text{Pr}^L$ is the ∞-category of presentable ∞-categories and left adjoints
There exists a regular cardinal \( \kappa \) such that for all \( \kappa \leq \pi \) if \( p : I \to \mathcal{P}_\pi \), then for all \( \kappa \leq \mu \leq \pi \) \( \text{Ind}^\mu(\lim p(i)) \in \mathcal{P} \).

1.3 Conventions

We shall generally follow [Lur09] in notation and terminology regarding \( \infty \)-categories. Having said that, a category with no more adjectives will always mean for us an \( \infty \)-category, unlike [Lur09]. However, since the 1-categories in this project usually play the role of "indexing" categories, we will distinguish between 1-categories and \( \infty \)-categories notationally, as indicated below.

**Notation 1.4.**

- \( \infty \)-categories will be denoted by calligraphic font e.g. \( \mathcal{C}, \mathcal{D}, \mathcal{E} \).
- 1-categories will be denoted by capital letters e.g. \( C, D, E \).
- For a 1-category \( C \), we denote the discrete category on the objects of \( C \) by \( C_\delta \). \(^2\) We will use the same notion for the set of vertices of a simplicial set \( K \).
- For a category \( \mathcal{C} \) we let \( \mathcal{C}^\kappa \) be the full subcategory on \( \kappa \)-compact objects.
- For a symmetric monoidal category \( \mathcal{C} \) we let \( \text{CAlg}(\mathcal{C}) \) be the category of commutative algebra in \( \mathcal{C} \).
- For a symmetric monoidal category \( \mathcal{C} \) and \( R \in \text{CAlg}(\mathcal{C}) \) we let \( \text{Mod}_R(\mathcal{C}) \) be the category of \( R \) modules in \( \mathcal{C} \).
- \( \text{Set} \) will denote the category of small sets.
- \( \text{Cat}_1 \) will denote the category of small 1-categories.
- \( \text{Cat}_\infty \) will denote the category of small \( \infty \)-categories.
- \( \mathcal{S} \) will denote the category of small spaces.
- \( \Delta \) will denote the category of finite non-empty sets and monotone increasing maps.
- \( \Delta^+ \) will denote the category of finite non-empty sets and strictly monotone increasing maps.
- \( \text{Set}^{\Delta^\text{op}} \) will denote the category of simplicial sets i.e. the category \( \text{Fun}(\Delta^\text{op}, \text{Set}) \), and \( \Delta^n, \partial\Delta^n \in \text{Set}^{\Delta^\text{op}} \) are defined as usual.
- \( \text{Pr}^L \) will denote the category of presentable categories and left adjoints.
- \( \text{Pr}^R \) will denote the category of presentable categories and accessible right adjoints.

1.4 Acknowledgments

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\(^2\) One should note that the operation \( C \mapsto C_\delta \) is not invariant under equivalence of categories. However, we will only use it when \( C \) is a poset and then it is an invariant notion.
2 From Pure Morphisms to Quasi-Split Cones

In this section we define a class of morphisms, dependent on a cardinal \( \kappa \), called \( \kappa \)-pure morphisms (see Definition 2.2), and show that every \( \kappa \)-pure morphism is a \( \kappa \)-filtered colimit of split morphisms, (see Proposition 2.5).

We will need the following lemma:

**Lemma 2.1.** Let \( \kappa \) be a regular cardinal, \( C \) be a \( \kappa \)-presentable category and \( K \) be a \( \kappa \)-small simplicial set. Then there exists a regular cardinal \( \pi \geq \kappa \) such that for every cardinal \( \mu \geq \pi \), and any \( f \in \text{Fun}(K, C) \), \( f \) is a \( \mu \)-compact object (in the category \( \text{Fun}(K, C) \)) if and only if for each vertex \( x \in K_0 \), \( f(x) \) is a \( \mu \)-compact object in \( C \).

**Proof.** First, the assumptions on \( K \) and \( C \) implies that \( \text{Fun}(K, C) \) is presentable. Since colimits in a functor category are computed point-wise, the evaluation functor \( \text{ev}_x : \text{Fun}(K, C) \to C \) is a colimit preserving functor between presentable categories for any \( x \in K_0 \). It follows that there exists a cardinal \( \mu_x \) such that for every \( \mu \geq \mu_x \), the functor, \( \text{ev}_x \) sends \( \mu \)-compact objects to \( \mu \)-compact objects. Hence, by taking the supremum \( \pi := \sup_{x \in K_0} \mu_x \) we have that if \( f \) is \( \mu \)-compact for \( \mu \geq \pi \) then \( f(x) \in \text{Ob}(C) \) is also \( \mu \)-compact. For the other implication see [Lur09, 5.3.4.13].

**Definition 2.2.** Let \( C \) be a presentable category and let \( A, B \in \text{Ob}(C) \). We will say that a morphism \( f : A \to B \) in \( C \) is \( \kappa \)-pure if for any two \( \kappa \)-compact objects \( A', B' \in \text{Ob}(C) \) and a commutative diagram:

\[
\begin{array}{ccc}
A' & \xrightarrow{f'} & B' \\
\downarrow{u} & & \downarrow{v} \\
A & \xrightarrow{f} & B
\end{array}
\]

in \( C \), there exists a 2 simplex filling the following diagram:

\[
\begin{array}{ccc}
A' & \xrightarrow{f'} & B' \\
\downarrow{u} & \nearrow{\bar{u}} & \\
A & \xrightarrow{f} & B
\end{array}
\]

for some \( \bar{u} : B' \to A \). In other words, there exists a factorization \( u \simeq \bar{u} \circ f' \).

**Definition 2.3.** Let \( C \) be a category, following [Lur09, 4.4.5.2] we will say that a commutative diagram of the form:

\[
\begin{array}{ccc}
X & \xrightarrow{i} & Y \\
\downarrow{r} & & \downarrow{1_Y} \\
Y & \xrightarrow{1_Y} & Y
\end{array}
\]

is a weak retraction. For any such weak retraction we will call \( i \) a split morphism and \( r \) a retraction.

**Example 2.4.** Note that a split morphism is \( \kappa \)-pure for all \( \kappa \).

**Proposition 2.5.** Let \( C \) be a \( \kappa \)-presentable category. We claim that there exists a regular cardinal \( \pi \geq \kappa \), such that for all \( \mu \geq \pi \), every \( \mu \)-pure morphism \( f \in \text{Fun}(\Delta^1, C) \) is a \( \mu \)-filtered colimit of split morphisms.
Proof. By Lemma 2.1 there exists $\pi \geq \kappa$ such that for all $\mu \geq \pi$, $\text{Fun}(\Delta^1, C)$ is $\mu$-presentable and $A \to B \in \text{Fun}(\Delta^1, C)$ is $\mu$-compact if and only if $A$ and $B$ are $\mu$-compact in $C$. Let $f : A \to B$ be a $\mu$-pure morphism, since $\text{Fun}(\Delta^1, C)$ is $\mu$-presentable, by definition there exists a functor

$$F : J \to \text{Fun}(\Delta^1, C)^\mu \simeq \text{Fun}(\Delta^1, C^\mu)$$

where $J$ is $\mu$-filtered and $\text{colim} F = f$. For each $i \in \text{Ob}(J)$ we denote $f_i := F(i) : A_i \to B_i$ and by $u_i$ the canonical map

$$u_i : f_i \to \text{colim}_J f_j = f.$$

Let $\mathcal{B}_i$ be the pushout:

$$
\begin{array}{ccc}
A_i & \xrightarrow{f_i} & B_i \\
\downarrow^{ev_0(u_i)} & & \downarrow \\
A & \xrightarrow{f} & \mathcal{B}_i
\end{array}
$$

By the functionality of pushouts we get a $\mu$-filtered diagram $F' : J \to \text{Fun}(\Delta^1, C)$ with $F'(i) = f'_i$. We will now show that $f'_i$ are split and that their colimit is $f$ i.e. that $\text{colim} F' = f$.

We begin by showing that $\text{colim} F' = f$: Let $\mathcal{D} := \text{Fun}(\Delta^1, C) \times_C C/A$ be the category of morphisms over $A$. Informally $\mathcal{D}$ is spanned by the objects :

$$
\begin{array}{ccc}
X & \xrightarrow{f'} & Y \\
\downarrow{g} & & \downarrow \\
A & & 
\end{array}
$$

with morphisms:

$$
\begin{array}{ccc}
X' & \xrightarrow{f'_{ij}} & Y' \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
A & & 
\end{array}
$$

In other words it is the category of arrows in $C$ together with a morphism from the source to $A$. Since $\mathcal{D}$ is a pullback, the functors $F' : J \to \text{Fun}(\Delta^1, C)$ and $ev_0(u_i) : J \to C/A$ induce a functor $F'' : J \to \mathcal{D}$, adding the fact that the forgetful $\mathcal{D} \to \text{Fun}(\Delta^1, C)$ commutes with colimits we have that $\text{colim} F'' = f$ (here, we abuse notation by identifying $f$ and the object “$(f, \text{Id})$” $\in \mathcal{D}$). As the pushout functor $\mathcal{D} \to C/A$ also commutes with colimits, we get that $\text{colim} F' = f$.

We now turn to show that $f'_i$ are split: By construction we have a commutative diagram:

$$
\begin{array}{ccc}
A_i & \xrightarrow{f_i} & B_i \\
\downarrow^{ev_0(u_i)} & & \downarrow^{ev_1(u_i)} \\
A & \xrightarrow{f} & B
\end{array}
$$

7
where \( A_i \) and \( B_i \) are \( \mu \)-compact. Therefore, since \( f \) is \( \mu \)-pure, we have a 2-simplex of the form

\[
\begin{array}{ccc}
A_i & \xrightarrow{f} & B_i \\
\downarrow{ev_0(u_i)} & & \downarrow{g_i} \\
A & \; & A
\end{array}
\]

Thus, by the universal property of pushouts we get

\[
\begin{array}{ccc}
A_i & \xrightarrow{f_i'} & B_i \\
\downarrow{ev_0(u_i)} & & \downarrow{g_i} \\
A & \xrightarrow{\text{Id}} & A
\end{array}
\]

and the claim follows. \( \square \)

3 Quasi-Split Cones

We have just seen that a pure morphism gives rise to a cone on a filtered category where all maps from the initial object have retracts. We will call such cones quasi-split (see Definition 3.1).

This section is devoted to the study of quasi-split cones and the categories derived from them.

**Definition 3.1.** Let \( K \in \text{Set}^\Delta \) and \( F : K^{<\cdot} \to \mathcal{C} \) a cone on \( K \). We say that \( F \) is a quasi-split cone if \( F(\rightarrow \to k) \) is split \( \forall k \in K \).

**Definition 3.2.** \([\text{Lur09}, 4.4.5.2]\) Let \( \text{Ret} \) be the 1-category defined as follows

- The set of objects is given by \( \{X, Y\} \).
- The sets of morphisms are given by

\[
\begin{array}{c}
\text{Map}(X, X) = \{\text{Id}, e\}, \quad \text{Map}(X, Y) = \{r\}, \quad \text{Map}(Y, X) = \{i\}, \quad \text{Map}(Y, Y) = \{\text{Id}\}.
\end{array}
\]

The composition law is determined by \( r \circ i = \text{Id} \) and \( i \circ r = e \).

**Definition 3.3.** In the category \( \text{Ret} \) there exists a commutative diagram

\[
\begin{array}{ccc}
X \\
\downarrow{i} & \uparrow{r} \\
Y & \text{Id} & Y
\end{array}
\]

Which is determined by a map of simplicial sets \( \sigma : \Delta^2 \to \text{Ret} \). We let \( \text{wRet} \) denote the image of \( \sigma \) in \( \text{Ret} \).

**Construction 3.4.** Given a quasi-split cone \( F' : K^{<\cdot} \to \mathcal{C} \), by choosing a retract \( g_k : F'(k) \to \)
$F'(-\infty)$ for each $h_k := F'(-\infty \to k)$ we get a functor from the pushout in $\text{Set}^{\Delta^{op}}$

$$F : K \sqcup \coprod_{K \times \Delta^1} (K \times \text{Ret}) =: \tilde{L}_K \to C.$$ 

We denote by $L_K$ a fibrant replacement of $\tilde{L}_K$ in the Joyal model structure i.e. a $L_K$ is a quasi-category equivalent to $\tilde{L}_K$. We shall call $L_K$ the cone with retracts on $K$.

The main goal of this section is to prove the following:

**Proposition 3.5.** If $C$ is equivalent to the nerve of a 1-category $N(C)$ then, $L_C$, the cone with retracts on $C$ is equivalent to a nerve of a 1-category. Furthermore, if $C = I$ is a poset then $L_I$ admits an explicit description as in Corollary 3.26.

The outline of the proof of Proposition 3.5 is as follows:

- In subsection 3.1 we will show that the category obtained by gluing a “free arrow” to a 1-category is still a 1-category.

More precisely we show that the pushout in $\text{Cat}_{\infty}$

$$C \coprod_{\Delta^0} \Delta^1$$

is equivalent to a 1-category - Proposition 3.15.

- Denote by $D_C$ the category obtained by iterating the previous construction for each object in $C$.

In subsection 3.2 we will show that inverting the arrows from the initial object in $D_C$ to the objects “which are not in $C$”, gives rise to a 1-category.

More precisely the category $D_C$ is defined by the pushout:

$$D_C := C \coprod_{C \times \Delta^0} (C \times \Delta^1)$$

and we will show that inverting all morphisms in $D_C$ of the form $-\infty \to x$ where $x$ is not an object of $C$ (i.e. not in the image of the natural inclusion $C \to D_C$) is still a 1-category - Proposition 3.20.

- In subsection 3.3 we will show that the previous construction gives the same result as gluing Ret along the morphism from the initial object to the source of the free arrow - Lemma 3.25.

### 3.1 Gluing a Free Arrow

This subsection is devoted to showing that the pushout

$$\begin{array}{ccc}
\Delta^0 & \longrightarrow & C \\
\downarrow & & \downarrow \\
\Delta^1 & \longrightarrow & P
\end{array}$$

in $\text{Cat}_{\infty}$ is a 1-category, when $C$ is a 1-category.
Our first step toward proving the above claim will be to prove the following (rather technical) lemmas:

**Lemma 3.6.** Let \( X \in \text{Set}^{\Delta^\text{op}} \) and let \( \sigma_1 \in X_{k_1} \) and \( \sigma_2 \in X_{k_2} \) be two non-degenerate simplices. Assume that for some \( n \geq 0 \) there are two surjective order preserving maps

\[
\phi_l : [n] \to [k_l], \quad l = 1, 2
\]

such that \( \phi_1^*(\sigma_1) = \phi_2^*(\sigma_2) \), then \( k_1 = k_2 \), \( \sigma_1 = \sigma_2 \) and \( \phi_1 = \phi_2 \).

**Proof.** Without loss of generality assume that \( k_1 \leq k_2 \). Let \( f : \Delta^{k_1} \to X \) be the map classified by \( \sigma_1 \). Denote the unique non-degenerate simplex in \( \Delta^{k_1} \) by \( \tau \). Let \( \psi : [k_2] \to [n] \) be a right inverse to \( \phi_2 \) we have that

\[
* \quad \sigma_2 = \psi^*(\phi_2^*(\sigma_2)) = \psi^*(\phi_1^*(\sigma_1)) = \psi^*(\phi_1^*(\sigma_1)) = f(\psi^*(\phi_1^*(\sigma_1))).
\]

In particular the image of \( f \) contains \( \sigma_2 \). Now, since \( \Delta^{k_1} \) does not contain any non-degenerate simplices of dimension larger than \( k_1 \) we get that \( k_1 = k_2 \). The other two implications are immediate, as the image of \( f \) can contain at most one non-degenerate simplex of dimension \( k_1 \), and it contains both \( \sigma_1 \) and \( \sigma_2 \) so \( \sigma_1 = \sigma_2 \).

We now show that \( \phi_1 = \phi_2 \). By the above \( \sigma_1 = \sigma_2 \), and thus by the chain of equivalences \( * \) we have that \( \phi_1 \circ \psi = \text{Id} \). We conclude that every right inverse of \( \phi_2 \) is a right inverse of \( \phi_1 \) which implies that \( \phi_1 = \phi_2 \). \( \square \)

**Notation 3.7.** Let \( f : X \to Y \) be a map of simplicial sets, we denote by

\[
f_k^{\text{nd}} : X_k^{\text{nd}} \to Y_k
\]

the induced map we get by restricting \( f_k \) to the non degenerate simplices in \( X_k \).

**Lemma 3.8.** Let \( A \) be a set and \( 0 \leq i \leq n \) be natural numbers. Assume we have a commutative diagram in \( \text{Set}^{\Delta^\text{op}} \) of the form:

\[
\begin{array}{ccc}
A \times \Lambda^n_i & \longrightarrow & X \\
\downarrow & & \downarrow f \\
A \times \Delta^n & \underset{j}{\longrightarrow} & Y
\end{array}
\]

and let

\[
g : A \times \Delta^{n-1} \xrightarrow{\text{Id} \times \delta_i} A \times \Delta^n \to Y
\]

where \( \delta_i \) embeds \( \Delta^{n-1} \) as the face in front of the \( i \)-th vertex.

Assume further that:

1. The map \( f : X \hookrightarrow Y \) is a level-wise injection.
2. \( g \) induces an injection

\[
g_{n-1}^{\text{nd}} : (A \times \Delta^{n-1})_{n-1}^{\text{nd}} \to Y_{n-1}.
\]
3. $\text{Im}(g_{n-1}^{\text{nd}}) \cap \text{Im}(f_{n-1}) = \text{Im}(j_n^{\text{nd}}) \cap \text{Im}(f_n) = \emptyset$.

4. Both the maps $j_n$ and $g_{n-1}$ send non degenerate simplices to non degenerate simplices.

Then the induced natural map from the pushout in $\text{Set}^{\Delta^{op}}$

$$h : P := (A \times \Delta^n) \coprod_{A \times \Lambda^n_l} X \to Y$$

is a level-wise injection.

**Proof.** Let $\sigma_1, \sigma_2 \in P_k$, such that $h_k(\sigma_1) = h_k(\sigma_2)$. We shall show that $\sigma_1 = \sigma_2$ by analyzing each possible case individually.

- Assume that $\sigma_1$ and $\sigma_2$ are in the image of the natural map $X \to P$.
  In this case the claim follows from (1).

- Assume that $\sigma_1$ and $\sigma_2$ are in the image of the natural map $A \times \Delta^n \to P$ and not in the image of the map $X \to P$.
  Let $\sigma'_l \in (A \times \Delta^n)_{k_l}$ for $l = 1, 2$ be two non degenerate simplices that degenerate to the $\sigma_l$-s, i.e. there exists two order preserving and surjective maps $\phi_l : [k] \to [k_l]$ such that $\phi_l(\sigma'_l) = \sigma_l$. If one of the $\sigma'_l$ is in the image of the natural map $A \times \Lambda^n_l \to P$ then it is also in the image of the natural map $X \to P$ which gives us that $\sigma_1$ is in the image of the natural map $X \to P$, contradicting our assumption.
  By the above, we conclude that both of the $\sigma'_l$-s are not in the image of the natural map $A \times \Lambda^n_l \to P$, so either $k_l = n - 1$ and $h_{n-1}(\sigma'_l) = g_{n-1}(\sigma'_l)$ or $k_l = n$ and $h_n(\sigma'_l) = j_n(\sigma'_l)$.
  Since both $j_n$ and $g_{n-1}$ send non degenerate simplices to non degenerate simplices, we get by Lemma 3.6 that $h_k(\sigma'_1) = h_k(\sigma'_2)$ and $\phi_1 = \phi_2$. All in all, since $g_{n-1}^{\text{nd}}$ and $j_n^{\text{nd}}$ are one-to-one, we also have that $\sigma'_1 = \sigma'_2$ and thus $\sigma_1 = \sigma_2$.

- Assume that $\sigma_1$ is in the image of the natural map $X \to P$ and that $\sigma_2$ is in the image of the natural map $A \times \Delta^n \to P$.
  Let $\sigma'_2 \in (A \times \Delta^n)_{m}$ be a non degenerate simplex that degenerates to $\sigma_2$ i.e. $\phi^*(\sigma'_2) = \sigma_2$.
  If $\sigma'_2$ is in the image of the natural map $A \times \Lambda^n \to P$, we are back to the first case.
  If $\sigma'_2$ is not in the image of the natural map $A \times \Lambda^n \to P$, then either $m = n$ and $h_n(\sigma'_2) = j_n(\sigma'_2)$ or $m = n - 1$ and $h_{n-1}(\sigma'_2) = g_{n-1}(\sigma'_2)$. Assume that $m = n$ and choose a right inverse $\psi$ for $\phi$. Since maps of simplicial sets are neutral transformations we get

$$f_k(\sigma_1) = \phi^*(j_n(\sigma'_2)) \implies f_n(\psi^*(\sigma_1)) = j_n(\sigma'_2).$$

Noting that $j_n$ sends non degenerate simplices to non degenerate simplices, we get a contradiction to our assumption that $\text{Im}(j_n^{\text{nd}}) \cap \text{Im}(f_n) = \emptyset$. Similar argument shows that if $m = n - 1$, then we get a contradiction to our assumption that $\text{Im}(g_{n-1}^{\text{nd}}) \cap \text{Im}(f_{n-1}) = \emptyset$.

Recall the main objective of this subsection: We want to show that for any 1-category $C$ the
pushout
\[
\begin{array}{ccc}
\Delta^0 & \longrightarrow & C \\
\downarrow & & \downarrow \\
\Delta^1 & \longrightarrow & P
\end{array}
\]

in \text{Cat}_\infty is a 1-category. We shall do so by explicitly constructing its fibrant replacement in the Joyal model structure on \text{Set}^{\Delta^\text{op}}.

Let us fix for the remainder of this section a quasi-category \(C \in \text{Set}^{\Delta^\text{op}}\) and an object \(x \in C_0\). For each \(n \geq 0\) denote by \(C^x_n \subset C_n\) the subset of simplices with "last vertex \(x\)". We consider \(C^x_n\) as a pointed set pointed by the degenerate simplex on \(x\).

We now define our candidate for the fibrant replacement of \(P\).

**Definition 3.9.** Let \(D^\infty \in \text{Set}^{\Delta^\text{op}}\) be the following simplicial set:

- the simplices are given by:
  \[D^\infty_n = C_n \sqcup \prod_{i=0}^{n} C^x_i.\]

- The face maps are defined as follows:
  If \(\sigma \in C_{n+1} \subset D^\infty_{n+1}\), then
  \[\partial_D^k(\sigma) = \partial_C^k(\sigma) \in C_n \subset D_n^\infty.\]
  If \(\sigma \in C^x_{n+1} \subset D^\infty_{n+1}\), then
  \[\partial_D^k(\sigma) = \begin{cases} 
  \partial_C^k(\sigma) & \text{if } k \leq n, \\
  \partial_C^k(\sigma) & \text{if } k = n+1.
  \end{cases}\]

  If \(\sigma \in C^x_l \subset D^\infty_{n+1}\) and \(l \neq n+1\), then
  \[\partial_D^k(\sigma) = \begin{cases} 
  \partial_C^k(\sigma) & \text{if } k \leq l-1, \\
  \sigma & \text{if } k \geq l.
  \end{cases}\]

- The degeneracies are defined as follows:
  If \(\sigma \in C_n \subset D^\infty_n\), then
  \[s_D^k(\sigma) = s_C^k(\sigma) \in C_{n+1} \subset D^\infty_{n+1}.\]
  If \(\sigma \in C^x_l \subset D^\infty_{n+1}\), then
  \[s_D^k(\sigma) = \begin{cases} 
  s_C^k(\sigma) & \text{if } k \leq l-1, \\
  \sigma & \text{if } k \geq l.
  \end{cases}\]

One can check directly that this defines a simplicial set.
We proceed to show that $D^\infty$ is a fibrant replacement for the pushout

\[
\begin{array}{ccc}
\Delta^0 & \xrightarrow{x} & \mathcal{C} \\
\downarrow & & \downarrow \\
\Delta^1 & \xrightarrow{s} & P
\end{array}
\]

in Set$^\Delta^{op}$, when $\mathcal{C}$ is the nerve of a 1-category.

**Remark 3.10.** We believe that whenever $\mathcal{C}$ is a quasi-category, $D^\infty$ is the fibrant replacement of the pushout. Since we are only interested in the case where $\mathcal{C}$ is the nerve of the 1-category, we shall not prove this more general case.

**Lemma 3.11.** Assume that $\mathcal{C}$ is the nerve of a 1-category $C$ and $x \in \text{Ob}(C)$. Let $D$ be the 1-category described below:

- The objects are given by:
  \[ \text{Ob}(D) = \text{Ob}(C) \sqcup \{x'\} \]

- The morphisms sets are defined as follows:
  for all $a, b \in \text{Ob}(D)$
  \[
  \text{hom}_D(a, b) = \begin{cases}
  \text{hom}_C(a, b), & a, b \in \text{Ob}(C) \\
  \text{hom}_C(a, x'), & a \in \text{Ob}(C), b = x' \\
  \{\ast\}, & a = b = x' \\
  \emptyset, & a = x', b \in \text{Ob}(C)
  \end{cases}
  \]

Then $D^\infty \simeq N(D)$.

**Proof.** Note that we have a canonical bijection between $\text{hom}_D(a, x')$ and $\text{hom}_D(a, x)$ for $a \neq x'$, which we will denote by $f \mapsto \bar{f}$. We adopt the following notation - given a simplex, $\sigma \in N(D)_n$, denote by $i_\sigma$ the number of times that $x'$ appears as a vertex of $\sigma$. Note that $x'$ can only appear as the last $i_\sigma$ vertices of $\sigma$ and that the “maps” from $x'$ to itself can only be identities. Furthermore, we note that if $i_\sigma = 1$ then, by the above bijection, there exists a canonical simplex $\sigma_x \in N(D)_n$ in which we change the last vertex from $x'$, to $x$, i.e.

if $\sigma = [y_0 \xrightarrow{f_0} \ldots \xrightarrow{f_n} y_n = x']$ then $\sigma_x = [y_0 \xrightarrow{f_0} \ldots \xrightarrow{f_n} x]$.

We proceed to define an isomorphism $f : N(D) \to D^\infty$ as follows:

\[
f_n : N(D)_n \to D^\infty_n, \quad f(\sigma) = \begin{cases}
\sigma, & i_\sigma = 0 \\
(\sigma_{[0, \ldots, n-i_\sigma+1]})_x \in C_{n-i_\sigma+1}, & i_\sigma \neq 0.
\end{cases}
\]

The above functions are obviously bijective, and one can also check that the different $f_n$ assemble to a map $f : N(D) \to D^\infty$. We conclude that $D^\infty \simeq N(D)$. \qed
We will now define a sequence of simplicial sets “converging” to $D^\infty$ i.e.

$$\text{colim}_n(D^0 \subset D^1 \subset \cdots) = D^\infty$$

where the colimit is taken in $\text{Set}^{\Delta^{op}}$. Let $(C^x)^k$ be the union of the images of the maps $C_k \to C_n$ intersected with $C^x_n$. For example, for $k \geq n$ we have that $(C^x)^k = C^x_n$ and for $k = n - 1$ we have that $(C^x)^{n-1}$ is the set of degenerate simplices in $C^x_n$ (where we think of $C^x_n \subset C_n$). We define a sub-simplicial set $D^m \subset D^\infty$:

$$D^m_n = C_n \sqcup \prod_{k=0}^{n}(C^x_k)^m = C_n \sqcup \prod_{k=m+1}^{n}(C^x_k)^m \sqcup \prod_{i=0}^{\min\{m,n\}} C^x_i.$$

Informally the difference between $D^m$ and $D^{m+1}$ is the $m+1$ composable arrows missing in $D^m$ (a formal version of this statement is Lemma 3.14).

We also denote by $C^x_k \subset C^x_n$ the subset of simplices whose last two coordinates are $x$ and the “map” from $x$ to $x$ is the “identity”. Note that if $k \leq n - 2$, then all the simplices in $(C^x_k)^k$ are degenerate, when viewed as simplices of $D^m$. And if $k = n - 1$, then the non degenerate simplices in $(C^x_n)^{n-1}$ are contained in $C^x_n$.

**Lemma 3.12.** There exists a pushout diagram in $\text{Set}^{\Delta^{op}}$ of the form

$$\begin{CD}
\Delta^0 @>x>> C \\
\downarrow s \downarrow \downarrow \downarrow \\
\Delta^1 @>>> D^0
\end{CD}$$

where $x$ is the morphism that chooses $x$ and $s$ is the morphism that chooses 0.

**Proof.** Define $C \to D^0$ as the obvious map

$$C_n \hookrightarrow C_n \sqcup \prod_{k=0}^{n}(C^x_k)^0 = D^0_n$$

and $\Delta^1 \to D^0$ as the map that chooses the unique simplex in $(C^x_0)^0$. These two maps together form a map from the pushout of the diagram in the lemma $C \coprod_{\Delta^0} \Delta^1 \to D^0$. Using the fact that $D^0_n = C_n \prod_{k=0}^{n}*$, one can verify that this map is level-wise bijective. \hfill $\square$

Note that for each $m \geq 0$ we can define $m+1$ maps $r_k : C^x_m \to D^m_{m-1}$ for $0 \leq k \leq m$:

For $k < m$ the maps are given by

$$r_k : C^x_m \xrightarrow{\partial^k} C^x_{m-1} \xrightarrow{r_m-1} C^x_m \to C^x_m \subset D^m_{m-1}, \quad k \leq m$$

and $r_m$ is the forgetful map $C^x_m \to C_m \subset D^m_{m-1}$ which forgets that $x$ is “marked”. We observe that given a $m$ simplex $\sigma \in C^x_m$ we can get a $m+1$ simplex $\sigma' \in D^\infty_{m+1}$ via the map $C^x_m \xrightarrow{s_m} C^x_{m+1} \subset D^\infty_{m+1}$ and that the $r_k(\sigma)$-s are all the faces of $\sigma'$ different then the face in front of the $m$-th vertex. It
follows that the above maps assemble into a map of simplicial sets:
$$
\Lambda^{m+1}_m \times C^x_m \to D^{m-1}.
$$

**Remark 3.13.** One can think about the map
$$
C^x_{m-1} \xrightarrow{s^x_m} C^x_{m,x} \subset D^{m-1}
$$
as taking elements of $C^x_{m-1}$ thought of as $m-1$ composable arrows in $C$ ending in $x$, and composing them with the identity morphism on $x$ e.g.

$$
\begin{array}{c}
y \\ z \end{array} \xleftarrow{x} \begin{array}{c}
y \\ z \end{array} \xrightarrow{x} \begin{array}{c}
y \\ z \end{array} \xleftarrow{x} \begin{array}{c}
y \\ z \end{array}.
\end{array}
$$

**Lemma 3.14.** Let $(C^x_m)^{nd} \subset C^x_m$ be the subset of non-degenerate simplices. Here, when we say non-degenerate we think of the elements of $C^x_m$ as simplices in $C_m$. We claim that the map described above defines a pushout diagram in $Set^{\Delta^{op}}$

$$
\begin{array}{ccc}
\Lambda^{m+1}_m \times (C^x_m)^{nd} & \rightarrow & D^{m-1} \\
\downarrow & & \downarrow \\
\Delta^{m+1} \times (C^x_m)^{nd} & \rightarrow & D^m
\end{array}
$$

In particular the inclusion $D^m \to D^{m+1}$ is a categorical equivalence.

**Proof.** Denote the pushout by $P$. Let $i : D^{m-1} \hookrightarrow D^m$ be the natural inclusion. Define

$$
f : \Delta^{m+1} \times (C^x_m)^{nd} \to D^m
$$

by sending $\sigma \in (C^x_m)^{nd}$ to the simplex in $C^x_{m+1}$ we get by composing with $\text{Id} : x \to x$ see **Remark 3.13** e.g.

$$
\begin{array}{c}
y \\ z \end{array} \xleftarrow{x} \begin{array}{c}
y \\ z \end{array} \xrightarrow{x} \begin{array}{c}
y \\ z \end{array} \xleftarrow{x} \begin{array}{c}
y \\ z \end{array}.
\end{array}
$$

By the universal property of pushout we get a map $g : P \to D_m$. We will show that $g$ is an isomorphism by verifying that $g_n : P_n \to D^m_n$ is bijective.

- $g_n$ is surjective:

Note that since $g$ is a natural transformation it suffices to show surjectivity on the non-degenerate simplices of $D^m$. Let $\sigma \in D^m_n$ be non-degenerate. If

$$
\sigma \in C_n \sqcup \prod_{k=0}^n (C^x_k)^{m-1} = D^m_{n-1}
$$

then $\sigma$ is obviously in the image $i$. Therefore, we may assume $n \in \{m, m+1\}$. If $n = m$, then by our analysis of the non degenerate simplices in $(C^x_k)^m$ we may also assume that $\sigma$ is
contained in one of the following two subsets of $C^x_m$ -

$$\sigma \in (C^x_m)^{nd} \text{ or } \sigma \in C^x_m.$$ 

We already handled the case of $\sigma \in C^x_m \subset D^{m-1}_m$, so we assume that $\sigma \in (C^x_m)^{nd}$. By assumption we have we have that

$$\partial^m f_{m+1}(\sigma) = \sigma \in D^m_m.$$ 

Finally, if $n = m + 1$, then we may assume that $\sigma \in C^x_{m+1}$ and we have

$$f_{m+1}(\partial^n(\sigma)) = \sigma \in D^m_{m+1}.$$ 

We conclude that $g_n$ is surjective.

- $g_n$ is injective: We shall check that the diagram in the lemma satisfies the conditions of Lemma 3.8. Indeed, by assumption $i$ is level-wise injective. Note that the $g_m$ from Lemma 3.8 that corresponds to $f$ is represented by the obvious map

$$(C^x_m)^{nd} \hookrightarrow C^x_m \subset D^m_m.$$ 

Hence, $g_m$ sends non degenerate simplices to non degenerate simplices and by the definition of $D^m$, $\text{Im}(g^{nd}_m) \cap \text{Im}(i_m) = \emptyset$. From the definition of $D^m$ and $D^{m-1}$ one can see that $\text{Im}(f^{nd}_{m+1}) \cap \text{Im}(i_{m+1}) = \emptyset$ and that $f_{m+1}$ sends non degenerate simplices to non degenerate simplices.

Based on all of the above we get:

**Proposition 3.15.** There is a pushout diagram in $\text{Cat}_\infty$

$$\begin{array}{ccc}
\Delta^0 & \rightarrow & C \\
\downarrow & & \downarrow \\
\Delta^1 & \rightarrow & D^\infty
\end{array}$$

where $C$ is the nerve of a 1-category. Furthermore, $D^\infty \simeq N(D)$ where $D$ is as in Lemma 3.11.\(^3\)

**Proof.** By Lemma 3.12 $D^0$ is the pushout in $\text{Set}^{\Delta^{op}}$. Due to the fact that a filtered colimit of weak equivalences remains a weak equivalence (in the Joyal model structure), it follows from Lemma 3.14 that $D^0 \rightarrow D^\infty$ is a categorical equivalence, proving the claim. \(\square\)

### 3.2 Inverting the Morphism

Let $C$ be a 1-category and let $D^1_C$ be the 1-category one gets by gluing arrows to all the objects in $C$, see Construction 3.18, and then taking a cone. In this subsection we show that inverting the

\(^3\)This proposition is true for a general quasi-category $C$ (i.e. not necessarily the nerve of a 1-category), but as we will only need the lemma for the case of a 1-category the general proof is omitted from this paper.
arrows from the initial object in $D^<_C$ to the objects “which are not in $C$”, gives rise to a 1-category. As a first step, we will present a criterion for when an object is initial in terms of “mapping to” property.

**Lemma 3.16.** Let $C \in \text{Cat}_\infty$. Then, $x \in C$ is initial if and only if for every $T \in \text{Cat}_\infty$ and $t \in T$, the forgetful map $T_{t/-} \to T$ induces an equivalence

$$
\text{Fun}^{x \to (\text{Id}: t \to t)}(C, T_{t/-}) \to \text{Fun}^{x \to t}(C, T)
$$

where $\text{Fun}^{x \to \bullet}$ is the full subcategory of functors which send $x$ to $\bullet$.

**Proof.** Let $T \in \text{Cat}_\infty$ and $t \in T$. We denote by $p_T : T_{t/-} \to T$ the forgetful functor and by $i : C \to C_{\subseteq}$ the natural inclusion. We have a commutative diagram

$$
\begin{array}{c}
\ast & \xrightarrow{\sim} & \text{Fun}^{x \to (\text{Id}: t \to t)}(C, T_{t/-}) \\
& & \downarrow{\sim} \\
& & \text{Fun}^{x \to t}(C, T)
\end{array}
$$

where the upper arrow in the diagram

$$
\text{Fun}^{x \to (\text{Id}: t \to t)}(C, T_{t/-}) \to \text{Fun}^{(- \infty \to x) \to (\text{Id}: t \to t)}(C_{\subseteq}, T)
$$

is the equivalence arising from the universal property of the under category.

Assume that $x$ is initial. In this case, any $\bar{F} \in \text{Fun}^{(- \infty \to x) \to (\text{Id}: t \to t)}(C_{\subseteq}, T)$ is the right Kan extension of its restriction to $C$. We conclude that composition with $i$ induces an equivalence:

$$
\text{Fun}^{(- \infty \to x) \to (\text{Id}: t \to t)}(C_{\subseteq}, T) \to \text{Fun}^{x \to t}(C, T).
$$

By the diagram $\ast$ we have that $p_T$ induces an equivalence:

$$
\text{Fun}^{x \to (\text{Id}: t \to t)}(C, T_{t/-}) \to \text{Fun}^{x \to t}(C, T).
$$

For the other implication: Assume that for all $T \in \text{Cat}_\infty$ and $t \in T$, composition with $p_T$ induces an equivalence

$$
\text{Fun}^{x \to (\text{Id}: t \to t)}(C, T_{t/-}) \to \text{Fun}^{x \to t}(C, T).
$$

Choosing $T = C$ and $t = x$ we have that $p_C$ has a right inverse $q$. Since $q \in \text{Fun}^{x \to (\text{Id}: x \to t)}(C, C_{\subseteq})$, by definition $q(x) = (x, \text{Id})$. Furthermore, for all $y \in C$ the composition

$$
\text{Map}_C(x, y) \xrightarrow{q(-)} \text{Map}((x, \text{Id}), q(y)) \xrightarrow{p(-)} \text{Map}(x, y)
$$

is an equivalence. We conclude that $\text{Map}_C(x, y)$ is a retract of a contractible space and thus contractible. $\square$
Corollary 3.17. Let $C$ be a 1-category and assume that it has an initial object $\emptyset$. Let $W$ be a set of morphisms. Then, the image of $\emptyset$ in $C | W^{-1}$ is an initial object.

Proof. Let $\mathcal{T} \in \text{Cat}_\infty$ and $t \in \mathcal{T}$. By the universal property of localization and Lemma 3.16 we have the following chain of equivalences:

$$
\text{Fun}^{\emptyset \rightarrow t}(C | W^{-1}, \mathcal{T}) \simeq \text{Fun}^{W \rightarrow \text{Eq}} \emptyset \rightarrow t(C, \mathcal{T}) \simeq \text{Fun}^{W \rightarrow \text{Eq}} (\text{Id} : t \rightarrow t)(C | W^{-1}, \mathcal{T}_t).
$$

Therefore, the claim follows from Lemma 3.16. \qed

Construction 3.18. Let $C$ be a 1-category. Choose a good order on $C$ i.e. an isomorphism $C \cong \mu$ for some ordinal $\mu$. We shall define categories $D_j \in \text{Cat}_1$ for $j \leq \mu$ by induction.

Let $D_0 = C$.

For $j$ successor, we define $D_j$ to be the following pushout in $\text{Cat}_\infty$:

$$
\Delta^0 \overset{x_{j-1}}{\rightarrow} D_{j-1} \ar[d]^s \ar[r] & D_j \\
\Delta^1 \ar[r] & D_j
$$

where $s$ chooses 0 and $x_{j-1}$ chooses $x_{j-1} \in C$.

For $j$ a limit ordinal we let $D_j$ be the colimit in $\text{Cat}_\infty$:

$$
D_j := \text{colim}_{k<j} D_k.
$$

According to the previous section, $D_\mu := D_\mu$ is also a 1-category. We say that $D_\mu$ is the category obtained from $C$ by gluing arrows to all the objects. We denote the objects of $D_\mu$ which are not in $C$ by $\{x'_x\}_{x \in C}$.

Recall that our goal is to show that the category obtained by inverting the morphisms $- \infty \rightarrow x'_x$ in $D_\mu$ is a 1-category. We will do so by using the hammock localization construction. For the convenience of the reader we shall repeat the construction here:

Construction 3.19. [DK80, 2.1] Let $C$ be a 1-category and $W \subset C$ be a wide-subcategory i.e. one which contains all the objects. The hammock localization of $C$ with respect to $W$ is a simplicial category $L^H(C, W)$ defined as follows: for every two objects $x, y \in C$ the $k$-simplices of $\text{Hom}_{L^H(C, W)}(x, y)$ will be the “reduced hammocks of depth $k$ and any length” between $x$ and $y$ i.e. commutative diagram of the form:

in which:
1. \( n \) is an integer larger than 0.

2. All the vertical maps are in \( W \).

3. In each column, all maps point in the same direction; if they point to the left, then they are in \( W \).

4. The maps in adjacent columns point in different directions.

5. No column contains only the identity map.

Faces and degeneracies are defined by omitting or repeating rows. If the resulting hammock is not reduced, i.e. does not satisfy condition 4 or 5, then we make it reduced, by composing adjacent columns whenever their maps point in the same direction and omitting columns which contain only the identity map. This gives a model for the localization of \( C \) with respect to \( W \) by [Ste15, 17 section 5].

**Proposition 3.20.** Let \( C \) be a 1-category and denote by \( D := D_C \) the category obtained from \( C \) by gluing arrows to all the objects. Let \( W' \subset D \) be the full subcategory on \( \{-\infty\} \cup \{x'\}_{x \in C} \) where \( -\infty \) is the cone point, and denote by \( W = W' \cup D_{\delta} \) the wide-subcategory generated by \( W' \). Then, the hammock localization \( E := D^\square[W^{-1}] = L^H(D^\square, W) \) is equivalent to a 1-category.

**Proof.** We will show that \( \text{Map}_E(x, y) \) is equivalent to a discrete space for any \( x, y \in E \). We will do so by case by case analyses.

- Assume that \( x = -\infty \).
  Since \( -\infty \) is initial in \( D^\square \), by Corollary 3.17, we have that \( \text{Map}_E(-\infty, y) \) is contractible and in particular equivalent to a discrete space for all \( y \in E \).

- Assume that \( x \) and \( y \) are in the image of \( C \) in \( E \).
  We will show that under the above assumption, one can build an explicit isomorphism of simplicial sets between \( \text{Map}_E(x, y) \) and a discrete simplicial set.
  Assume that we have a reduced hammock of length bigger than 0

  \[
  \begin{array}{c}
  x \\
  \downarrow \\
  z_{k,1} \\
  \downarrow \\
  \vdots \\
  \downarrow \\
  z_{k,n-1} \\
  y \\
  \end{array}
  \begin{array}{c}
  z_{0,1} \\
  \downarrow \\
  z_{0,n-1} \\
  \end{array}
  \begin{array}{c}
  z_{1,1} \\
  \downarrow \\
  z_{1,n-1} \\
  \end{array}
  \begin{array}{c}
  z_{0,2} \\
  \downarrow \\
  \vdots \\
  \downarrow \\
  \vdots \\
  \end{array}
  \begin{array}{c}
  z_{0,1} \\
  \end{array}
  \begin{array}{c}
  z_{2,2} \\
  \downarrow \\
  \vdots \\
  \downarrow \\
  \vdots \\
  \end{array}
  \begin{array}{c}
  z_{k,2} \\
  \downarrow \\
  \vdots \\
  \downarrow \\
  \vdots \\
  \end{array}
  \begin{array}{c}
  z_{k,n-1} \\
  \end{array}
  \]

  which represent a simplex in \( \text{Map}_E(x, y) \). One may observe that for all \( 0 \leq m \leq k \), \( z_{m,1} = z' \) for some \( z' \in D \) (which by definition, is not in \( C \)) and \( z_{m,n-1} = -\infty \). Furthermore, all the other \( z_{i,j} \)-s are in \( \{-\infty\} \cup \{z'\}_{z \in C} \). Let

  \[
  |(C_{\delta})^\square| = L^H((C_{\delta})^\square, (C_{\delta})^\square)
  \]
be the hammock localization of \((Cδ)^3\) in which we invert all morphisms. Note that \(|(Cδ)^3|\) is equivalent to the terminal category.

We now construct a map of simplicial sets

\[
\bigsqcup_{g \in \text{Hom}_C(x,y)} \Delta^0 \sqcup \bigsqcup_{z \in C \sqcap \text{Hom}_D(z,z')} \text{Hom}_{LH}((Cδ)^3,(Cδ)^3)(z, -\infty) \rightarrow \text{Hom}_E(x, y)
\]

as follows:

Given

\[
\Delta^0 \in \bigsqcup_{g \in \text{Hom}_C(x,y)} \Delta^0
\]

indexed on some \(g : x \rightarrow y\), we will send it to the length zero hammock that corresponds to that map.

Given

\[
\sigma_{z,f} \in \left( \bigsqcup_{z \in C} \bigsqcup_{f \in \text{Hom}_D(x,z)} \text{Hom}_{LH}((Cδ)^3,(Cδ)^3)(z, -\infty) \right)_k
\]

we define its image by expanding its source and target with identities, i.e.

\[
\sigma_{z,f} = \left( \begin{array}{c} z \\ \vdots \\ \vdots \end{array} \right) \quad \text{and} \quad \left( \begin{array}{c} x \\ \vdots \\ \vdots \end{array} \right)
\]

By our analysis above this defines an isomorphism of simplicial sets, concluding that \(\text{Map}_E(x, y)\) is equivalent to a discrete space.

- Assume that \(y = -\infty\).

We will use a similar argument to the previous case.

As before one can check that the reduced hammocks all begin with \(z' \in E\) (again, those \(z'\) are not in \(C\)) and all other vertices are in \((-\infty) \cup \{z'\}_{z \in C}\).

We now contract a map of simplicial sets

\[
\bigsqcup_{z \in C} \bigsqcup_{f \in \text{Hom}_D(x,z')} \text{Hom}_{LH}((Cδ)^3,(Cδ)^3)(z, -\infty) \rightarrow \text{Hom}_E(x, y)
\]

by expanding the source of a given \(\sigma_{z,f} \in \left( \bigsqcup_{z \in C} \bigsqcup_{f \in \text{Hom}_D(x,z')} \text{Hom}_{LH}((Cδ)^3,(Cδ)^3)(z, -\infty) \right)_k\) with identities i.e.

\[
\sigma_{z,f} = \left( \begin{array}{c} z \\ \vdots \\ \vdots \end{array} \right) \quad \text{and} \quad \left( \begin{array}{c} x \\ \vdots \\ \vdots \end{array} \right)
\]

As before, our analysis shows that this is an isomorphism of simplicial sets, concluding that
Map\(_E(x, -\infty)\) is equivalent to a discrete space.

All in all we have that \(E\) is equivalent to a 1-category. \(\square\)

**Definition 3.21.** Under the assumptions of **Proposition 3.20** we will say that \(E\) is the cone with left inverses on \(C\) and we will denote it by \(E_C\).

**Corollary 3.22.** Let \(I\) be a poset and denote by \(E_I\) the cone with left inverses on \(I\). Denote the distinct map (if it exists) from \(i\) to \(j\) in \(I\) by \(b_{i,j}\). Then \(E_I\) is equivalent to the 1-category described below:

- The objects are given by:
  \[
  \text{Ob}(E_I) = \text{Ob}(I) \cup \{-\infty\}.
  \]
- The morphisms sets are defined as follows:
  \[
  \text{hom}_{E_I}(a, b) = \begin{cases} 
  \{q_k\}_{k \geq i}, & a, b \in I, \ a = i, \ b = j, \ i \neq j \\
  \{q_k\}_{k \geq i} \cup \{b_{ij}\}, & a, b \in I, \ a = i, \ b = j, \ i < j \\
  \{g_k\}_{k \geq i}, & a = i \in I, \ b = -\infty \\
  \{h_i\}, & b = i \in I, \ a = -\infty \\
  \{\text{Id}\}, & a = b = -\infty 
  \end{cases}.
  \]

where the composition law is determined by:
\[
q_p q_k = q_k, \quad g_p q_k = g_k, \quad q_k b_{ij} = q_k, \quad b_{ij} q_k = q_k, \quad h_i g_k = q_k.
\]

### 3.3 Showing That \(L_I \in \text{Cat}_1\)

Given a poset \(I\), we have defined the cone with retracts, \(L_I\), on \(I\) see **Construction 3.4**. In this subsection we will show that \(L_I\) is equivalent to a 1-category.

**Lemma 3.23.** For any quasi-category \(C\), the restriction map:
\[
\text{Fun}(\text{Ret}, C) \to \text{Fun}(\text{wRet}, C)
\]
is a trivial fibration of simplicial sets (here Ret and wRet are defined as in **Definition 3.2**).

*Proof.* This is \([Lur09, 4.4.5.7]\) \(\square\)

**Lemma 3.24.** Let \(C \in \text{Cat}_\infty\).
We denote by
\[
\text{Fun}^{(0 \to 2) \to \text{Eq}}(\Delta^2, C) \subset \text{Fun}(\Delta^2, C)
\]
the full subcategory on functors which send \((0 \to 2)\) to an equivalence and by
\[
\text{Fun}^{(0 \to 2) \to \text{Id}}(\Delta^2, C) \subset \text{Fun}(\Delta^2, C)
\]
the full subcategory on functors which send \((0 \to 2)\) to the identity. Following the above notions, the natural inclusion
\[
\text{Fun}^{(0 \to 2) \to \text{Id}}(\Delta^2, C) \subset \text{Fun}^{(0 \to 2) \to \text{Eq}}(\Delta^2, C)
\]
is an equivalence of categories.

Proof. The inclusion above is clearly fully-faithful. It remains to show that any $F \in \text{Fun}(0 \to 2)^{-\to \text{Eq}}(\Delta^2, C)$ is equivalent to a functor which sends $0 \to 2$ to the identity. Indeed, let $\sigma$ be the 2-simplex in $C$

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{h} & & \downarrow{g} \\
Z & \xleftarrow{h^{-1}} & 
\end{array}
\]

which represents $F$. Choose an inverse, $h^{-1}$, of $h$, and let $\alpha$ be a 3-simplex we get by the horn filling property:

\[
\begin{array}{ccc}
X & \xrightarrow{\text{Id}} & X \\
\downarrow{h} & & \downarrow{g} \\
Y & \xleftarrow{h^{-1}g} & Z \\
\end{array}
\]

in $C$. Define $F' \in \text{Fun}(0 \to 2)^{-\to \text{Id}}(\Delta^2, C)$ as the composition

\[\Delta^2 \xrightarrow{i} \Delta^3 \xrightarrow{\alpha} C\]

where $i$ is the embedding of $\Delta^2$ as the face in front of 2. One can verify that $\alpha$ induces a natural equivalence between $F$ and $F'$ via the map $\Delta^2 \times \Delta^1 \to \Delta^3$ that collapses $\{0\} \times \Delta^1$ and $\{1\} \times \Delta^1$. \qed

**Lemma 3.25.** Let $C$ be a 1-category. Denote by $L_C$ the cone with retracts on $C$ and by $E_C$ the cone with left inverses on $C$. We claim that we have an equivalence of $\infty$-category

$L_C \simeq E_C$.

In particular $L_C$ is a 1-category

Proof. Let $D_C$ be the category obtained from $C$ by gluing arrows to all the objects. By the universal property of localization, for every $T \in \text{Cat}_\infty$, composing with the natural map $D_C \to E_C$ induces a fully-faithful functor

$$\text{Fun}(E_C, T) \to \text{Fun}(D_C, T)$$

whose essential image is the full subcategory on functors that sends the maps $-\infty \to x'$ to equivalences. By the definition of $D_C$ we can write:

$$\text{Fun}(-\infty \to x')^{-\to \text{Eq}}(D_C, T) \simeq \text{Fun}(-\infty \to x')^{-\to \text{Eq}}(\bigtimes_{\Delta^\circ \times C_\delta} (\Delta^1 \times C_\delta), T).$$

Note that the pushout $\bigtimes_{\Delta^\circ \times C_\delta} (\Delta^1 \times C_\delta)$ is equivalent to the pushout $\bigtimes_{\Delta^1 \times C_\delta} (\Delta^2 \times C_\delta)$ where the latter is defined via the map

$$\Delta^1 \times C_\delta \to C^\triangleleft$$

that chooses all the maps from the initial object and the map

$$\Delta^1 \times C_\delta \to \Delta^2 \times C_\delta$$
that chooses the maps from 0 to 1.

From the above together with Lemma 3.24 we may write

\[
\text{Fun}((-\infty \to x') \to \text{Eq}(C^{<} \coprod \coprod_{\Delta^2 \times C_{\delta}} \Delta^1 \times C_{\delta}, T)) \simeq \text{Fun}(0 \to 2) \to \text{Eq}(C^{<} \coprod \coprod_{\Delta^2 \times C_{\delta}} \Delta^1 \times C_{\delta} \times C_{\delta} \times C_{\delta} \times C_{\delta}, T)) \simeq
\]

\[
\text{Fun}(C^{<}, T) \times \text{Fun}(0 \to 2 \to \text{Eq}(C^{<} \coprod \coprod_{\Delta^2 \times C_{\delta}} \Delta^1 \times C_{\delta} \times C_{\delta} \times C_{\delta} \times C_{\delta}, T)) \simeq \text{Fun}(C^{<}, T) \times \text{Fun}(0 \to 2 \to \text{Id}(\Delta^2, T)).
\]

Applying Lemma 3.25 we get

\[
\text{Fun}(0 \to 2 \to \text{Id}(\Delta^2, T) = \text{Fun}(\text{wRet}, T) \simeq \text{Fun}(\text{Ret}, T),
\]

which gives

\[
\text{Fun}(C^{<}, T) \times \text{Fun}(\text{Id}(\Delta^2, T) \simeq \text{Fun}(C^{<}, T) \times \text{Fun}(\text{Id}(\Delta^2, T) \simeq \text{Fun}(L_I, T)
\]

as desired. \[\square\]

From the above and Corollary 3.22 we get an explicit description of \(L_I\) when \(I\) is a poset.

**Corollary 3.26.** Let \(I\) be a poset and denote by \(L_I\) the cone with retracts on \(I\). Denote the distinct map (if it exists) from \(i\) to \(j\) in \(I\) by \(b_{i,j}\). Then \(L_I\) is equivalent to the 1-category described below:

- The objects are given by:
  \[\text{Ob}(L_I) = \text{Ob}(I) \cup \{-\infty\}.
  \]

- The morphisms sets are defined as follows:

\[
\text{hom}_{L_I}(a, b) = \begin{cases}
\{q_k\}_{k \geq i}, & a, b \in I, \ a = i, \ b = j, \ i \neq j \\
\{q_k\}_{k \geq i} \cup \{b_{i,j}\}, & a, b \in I, \ a = i, \ b = j, \ i < j \\
\{g_k\}_{k \geq i}, & a = i \in I, \ b = -\infty \\
\{h_i\}, & b = i \in I, \ a = -\infty \\
\{\text{Id}\}, & a = b = -\infty
\end{cases}
\]

where the composition law is determined by:

\[
q_pq_k = q_k, \ g_pq_k = g_k, \ q_kb_{i,j} = q_k, \ b_{i,j}q_k = q_k, \ h_ig_k = q_k.
\]

### 4 Subcategories That Are Closed Under Pure Morphisms

In this section we prove that a subcategory of a category of presheaves closed under limits and \(\kappa\)-filtered colimits is closed under \(\kappa\)-pure morphisms provided that \(\kappa\) is sufficiently large. In more formal terms, we define:
Definition 4.1. Let $\mathcal{C} \in \text{Cat}_\infty$ and $\mathcal{D} \subset \mathcal{C}$ a full subcategory. We say that $\mathcal{D}$ is closed under $\kappa$-pure morphisms in $\mathcal{C}$ if for every $B \in \mathcal{D}$ and every $A \rightarrow B$ a $\kappa$-pure morphism in $\mathcal{C}$, $A$ is also in $\mathcal{D}$.

The main objective of section 4 is then to prove:

Proposition 4.2. Let $\mathcal{E}$ be a small $\infty$-category and $\mathcal{D} \subset \text{Fun}(\mathcal{E}, \mathcal{S}) := \mathcal{C}$ be a full subcategory closed under limits and $\kappa$-filtered colimits. Then, there exists $\mu \geq \kappa$ such that the $\infty$-category $\mathcal{D}$ is closed under $\mu$-pure morphisms in $\mathcal{C}$.

The above proposition is the key to proving Theorem 6.2 (the reflection theorem). We will begin by proving some preliminary lemmas for the proof of Proposition 4.2, which will be presented in the next subsection.

Lemma 4.3. Let $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \text{Cat}_\infty$ where $\mathcal{E}$ is complete, $F : \mathcal{C} \rightarrow \mathcal{E}$ a functor and $p : \mathcal{C} \rightarrow \mathcal{D}$ a right fibration. Assume that we have a pullback square

$$
\begin{array}{ccc}
\mathcal{C}' & \xrightarrow{u} & \mathcal{C} \\
q \downarrow & & \downarrow p \\
\mathcal{D}' & \xrightarrow{v} & \mathcal{D}
\end{array}
$$

Then

$$v^* p_*(F) \simeq q_* u^*(F)$$

where $(-)^*$ is post-composition and $(-)_*$ is the right Kan extension.

Proof. By [Cis19, 4.4.11], a left fibration is proper. Therefore, the claim is just proper base change [Cis19, 6.4.13] for

$$
\begin{array}{ccc}
\mathcal{C}'^{\text{op}} & \xrightarrow{u^{\text{op}}} & \mathcal{C}^{\text{op}} \\
q^{\text{op}} \downarrow & & \downarrow p^{\text{op}} \\
\mathcal{D}'^{\text{op}} & \xrightarrow{v^{\text{op}}} & \mathcal{D}^{\text{op}}
\end{array}
$$

\hfill \square

Lemma 4.4. Let $A$ be a set and let $F_a : T_a \rightarrow \mathcal{S}$ be a collection of functors from filtered posets indexed on $A$. Then, there is a canonical equivalence:

$$
\prod_{a \in A} \text{colim} F_a \simeq \text{colim}_{(c_a) \in \prod_{a \in A} T_a} \prod_{a \in A} F_a(c_a).
$$

Proof. This is [BSS20, 3.10]. \hfill \square

Lemma 4.5. Let $\mathcal{S}$ be a simplicial set. Assume that for every functor $F : K \rightarrow \mathcal{S}$ with $K$ finite simplicial set, there exists a natural transformation in $\text{Fun}(K, \mathcal{S})$ to another functor $F' \rightarrow F'$ where $F'$ admits a cone i.e. extents to $K^{op}$. Then, $\mathcal{S}$ is contractible.

Proof. We will use Kan’s, $\text{Ex}^\infty(-)$ construction for the fibrant replacement in the Kan model structure of simplicial sets.

Recall that $\text{Ex}^\infty(X)$ is defined as the following colimit in $\text{Set}^{\Delta^{op}}$

$$
\text{Ex}^\infty(X) = \text{colim} \{ X \subset \text{Ex}(X) \subset \text{Ex}^2(X) \subset \cdots \}
$$
where each $\text{Ex}^k : \text{Set}^{\Delta^{op}} \to \text{Set}^{\Delta^{op}}$ is the right adjoint of the $k$-th subdivision functor.

By definition of fibrant replacement, any homotopy class in $\pi_n([S])$ can be represented by a map:

$$\partial \Delta^n \to \text{Ex}^\infty(S).$$

Since $\partial \Delta^n$ has only finitely many non-degenerate simplices, this map must factor through some $\text{Ex}^k(S)$, which, by adjunction, means that the homotopy class we started with can be represented by a map of simplicial sets

$$\text{Sb}^k(\partial \Delta^n) \to S.$$

This map factors through the contractible simplicial set

$$(\text{Sb}^k(\partial \Delta^n) \times \Delta^1) \coprod_{\text{Sb}^k(\partial \Delta^n) \times \{1\}} \text{Sb}^k(\partial \Delta^n)^\nabla$$

by our assumption on $S$, which implies that $S$ is contractible. 

\[ \square \]

4.1 Proof of Proposition 4.2:

Proof. In this subsection we prove Proposition 4.2. Hence, we will assume that we are in the setting of Proposition 4.2, i.e. we have $C := \text{Fun}(\mathcal{E}, S)$ where $\mathcal{E}$ is a small $\infty$-category, and $D \subset C$ a full subcategory closed under limits and $\kappa$-filtered colimits. Fixing some more notation, let $\pi$ be the cardinal from Lemma 2.1 that corresponds to $C$ and choose $\mu \geq \max\{\pi, \kappa, \omega^+\}$. We also fix a $\mu$-pure morphism $f : A \to B$ with $B \in D$, our goal is to show that we can write $A$ in terms of limits and $\mu$-filtered colimits from $B$.

First, we show that $f$ gives rise to a functor from the cone with retracts on a filtered poset to $C_{A/}$.

Construction 4.6. From Proposition 2.5 there exists a $\mu$-filtered category, $I$ together with a quasi-split cone $Z' : I^{\leq} \to C$ such that $Z'(-\infty) = A$ and colim $Z'_i(i) = \text{colim } B_i = B$ where $B_i = Z'(i)$. By [Lur09, 5.3.1.18] we may, and will, assume that $I$ is a $\mu$-filtered poset. We choose retracts $g'_i$ for all $h'_i := Z'(-\infty \to i)$ and denote the resulting functor from the cone with retracts on $I$ (see Construction 3.4) by

$$Z'' : L \to C.$$

By Corollary 3.26 $L$ is a 1-category and we will denote its elements (throughout this section) as we did in Corollary 3.26. Note that $-\infty$ is the initial object of $L$, and $Z''$ sends it to $A$, thus, by Lemma 3.16, $Z''$ extends to a functor

$$Z : L \to C_{A/}.$$

We will use the following notation:

Notation 4.7. Denote by $B$ the smallest full subcategory of $C$ that is closed under limits and $\mu$-filtered colimits and contains $B$.

Our goal is to show that $A \in B$. We will do so in three steps:
1. First, we will build a $\Delta^+$ indexed diagram in $\mathcal{C}_A$ i.e. we construct a functor

$$X : \Delta^+ \to \mathcal{C}_A.$$  

2. Second, we will show that $X([n]) \in \mathcal{B}_{A^\prime}$. 

3. Third, we will show that $(\text{Id} : A \to A) \in \mathcal{C}_{A^\prime}$ is a retract of $\lim_{\Delta^+} X$ in $\mathcal{C}_{A^\prime}$.

These three steps together conclude the proof.

### 4.1.1 First Step - Constructing a Diagram

The main goal of this subsection is to build a diagram $X : \Delta^+ \to \mathcal{C}_A$. We will do so by constructing a functor $\phi : \text{Fun}(L, \mathcal{C}_A^\prime) \to \text{Fun}(\Delta^+, \mathcal{C}_A^\prime)$ and applying it to $Z$ of Construction 4.6. We will construct $\phi$ by first constructing a functor $\phi' : \text{Fun}(L, \mathcal{C}_A^\prime) \to \text{Fun}(J \times \Delta^+, \mathcal{C}_A^\prime)$ for a category $J$ that we will now define, and then taking a colimit over $J$.

**Construction 4.8.** Let $\prod_{m \in \mathbb{N}} I^m_{\delta}$ be the $\mu$-filtered poset whose objects are series of functions of sets

$$(f_0, f_1, ...)$$

where $f_m : I^m_{\delta} \to I_{\delta}$

with the order $(f_0, ...)$ if $f_m(i_0, ..., i_{m-1}) \leq f'_m(i_0, ..., i_{m-1})$ for all $i_0, ..., i_{m-1} \in I$, $m \in \mathbb{N}$. We will denote by $J \subset \prod_{m \in \mathbb{N}} I^m_{\delta}$ the full subcategory on $(f_0, f_1, ...)$ satisfying

$$f_m(i_0, ...i_{k-1}, i_{k+1}, ..., i_m) \leq f_{m+1}(i_0, ...i_{k-1}, i_k, i_{k+1}, ..., i_m)$$

for all $m \in \mathbb{N}$, $0 \leq k \leq m$ and for all $i_0, ..., i_m \in I$.

We will also denote:

**Notation 4.9.** For $(f_0, ..., f_n) \in \prod_{m \leq n} I^m_{\delta}$ we let

$$I_{f_0, ..., f_n} \subseteq I^{n+1}_{\delta}$$

be the subset

$$I_{f_0, ..., f_n} := \{(i_0, ..., i_n) \in I^{n+1}_{\delta} \mid i_k \geq f_k(i_0, ..., i_{k-1}) \quad \forall 0 \leq k \leq n\}.$$  

**Example 4.10.** $I_0$ is a set which contains a single point, and $I_{f_0}$ is $I_{\geq f_0}$.

We will need the following lemma for the second step.

**Lemma 4.11.** The following are true:

1. Let $J$ be as in Construction 4.8. Then $J$ is $\mu$-filtered and cofinal in $\prod_{m \in \mathbb{N}} I^m_{\delta}$.

2. Let $i \in I$ and $J^i \subset \prod_{m \in \mathbb{N}} I^m_{(i)^{\delta}}$ be the full subcategory on $(f_0, f_1, ...)$ satisfying

$$f_m(i_0, ...i_{k-1}, i_{k+1}, ..., i_m) \leq f_{m+1}(i_0, ...i_{k-1}, i_k, i_{k+1}, ..., i_m)$$

for all $m \in \mathbb{N}$, $0 \leq k \leq m$ and for all $i_0, ..., i_m \in I$. 

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for all \( m \in \mathbb{N}, 0 \leq k \leq m \) and for all \( i_0, \ldots, i_m \in I_i \) (this is Construction 4.8 when we replace \( I \) by \( I_i \)). Then there exists a cofinal map \( \xi : J_i \to J \) such that \( \xi(f_k)_{\vert (i_k)_k'} = f_k \) for all \( k \in \mathbb{N} \).

**Proof.**

1. We first prove that \( J \) is cofinal in \( \coprod_{m \in \mathbb{N}} I_m^\circ \). Let \((f_0, \ldots) \in \coprod_{m \in \mathbb{N}} I_m^\circ \). Our goal is to define \((f_0, \ldots) \in J \) such that \((f_0, \ldots) \leq (f_0, \ldots)\) and we will do so by induction on \( n \). Let \( f_0 = f_0 \) and assume \( f_n \) is defined. For every \((i_0, \ldots, i_n) \in I_n^\circ \) we pick

\[
p^k_{i_0, \ldots, i_n} \geq f_{n+1}(i_0, \ldots, i_n), \quad f_n(i_0, \ldots, i_n, \hat{i}_k, \ldots, i_n)
\]

for \( 0 \leq k \leq n \) and we choose \( p_{i_0, \ldots, i_n} \geq p^0_{i_0, \ldots, i_n}, \ldots, p^n_{i_0, \ldots, i_n} \). We define \( f_{n+1}(i_0, \ldots, i_n) = p_{i_0, \ldots, i_n} \) and by construction we have that \( f_n(i_0, \ldots, i_n, \hat{i}_k, \ldots, i_n) \leq f_{n+1}(i_0, \ldots, i_n) \) and that \((f_0, \ldots) \leq (f_0, \ldots)\).

We now show that \( J \) is \( \mu \)-filtered. Let \( F : K \to J \) be a functor with \( K \) a \( \mu \)-small simplicial set. Since \( \coprod_{m \in \mathbb{N}} I_m^\circ \) is \( \mu \)-filtered, after composing with the inclusion \( J \subset \coprod_{m \in \mathbb{N}} I_m^\circ \), \( F \) admits a cocone, which we denote by \((f_0, \ldots) \in \coprod_{m \in \mathbb{N}} I_m^\circ \). Thus \((f_0, \ldots) \in J \) form the previous paragraph is a cocone on \( F \).

2. We will construct \( \xi \) explicitly. For every \( j \in I \) such that \( i \not\leq j \) we pick \( p_j \geq i, j \). Given \( f : (I^\circ)_j \to (I^\circ)_i \) we define

\[
\tilde{f} : I^\circ_j \to I^\circ_i, \quad (i_0, \ldots, i_{n-1}) \mapsto f(i_0', \ldots, i_{n-1}') \quad \text{where} \quad \tilde{i}_k = \begin{cases} i_k, & i_k \geq i \\ p_i, & i_k \not\geq i \end{cases}.
\]

We let \( \xi : J_i \to J \) be defined by \( \xi(f_0, \ldots) = (\tilde{f}_0, \ldots) \). One can verify that this map is cofinal.

Recall, that our first goal is to define a functor \( \phi' : \text{Fun}(L, C_{A_f}) \to \text{Fun}(J \times \Delta^+, C_{A_f}) \). We will do so by defining a category \( M \), functors \( F : M \to J \times \Delta^+, H : M \to L \) and taking a right Kan extension:

\[
\begin{array}{ccc}
M & \xrightarrow{H} & L \\
\downarrow F & & \downarrow (-)_H \\
J \times \Delta^+ & \xrightarrow{F, (-)_H} & C_{A_f}
\end{array}
\]

We begin by defining \( F : M \to J \times \Delta^+ \) as a right fibration:

**Construction 4.12.** Let \( G : (J \times \Delta^+)^{op} \to \text{Set} \) be defined as follows:

For an object \(((f_0, \ldots), [n]) \in J \times \Delta^+ \) let

\[
G(((f_0, \ldots), [n]) = I_{f_0 \ldots f_{n-1}}.
\]

For a morphism \((j, h)\) where \( j : (f_0, \ldots) \to (f'_0, \ldots) \in J \) and \( h : [k] \to [n] \) is a strictly increasing function let

\[
G(j, h) : I_{f_0 \ldots f_{n-1}} \to I_{f'_0 \ldots f_{k-1}}, \quad (i_0, \ldots, i_{n-1}) \mapsto (i_{h(0)}, \ldots, i_{h(k-1)}).
\]
One can easily check that this defines a functor. We let \( F : M \to J \times \Delta^+ \) be the right fibration over \( J \times \Delta^+ \) classified by \( G \).

**Remark 4.13.** Note that the objects of \( M \) are 3-tuples \(((f_0, \ldots), [n], (i_0, \ldots, i_{n-1}))\) with \((f_0, \ldots) \in J\), \( n \in \mathbb{N} \) and \((i_0, \ldots, i_{n-1}) \in I_{f_0, \ldots, f_{n-1}}, \) and a morphism:

\[
\varphi : ((f_0, \ldots), [k], (i_0, \ldots, i_{k-1})) \to ((f'_0, \ldots), [n], (i'_0, \ldots, i'_{n-1}))
\]

over \( h : [k] \to [n] \) implies that \( i'_{h(m)} = i_m \) if \( m \neq k \).

We shall now define a functor \( H : M \to L \), as follows (see Corollary 3.26 for the explicit description of \( L \)):

For \(((f_0, \ldots), [n], (i_0, \ldots, i_{n-1})) \in M \) let:

\[
H((f_0, \ldots), [n], (i_0, \ldots, i_{n-1})) = f_n(i_0, \ldots, i_{n-1}).
\]

For \( \varphi_{j,h} : ((f_0, \ldots), [k], (i_0, \ldots, i_{k-1})) \to ((f'_0, \ldots), [n], (i'_0, \ldots, i'_{n-1})) \) over \((j : (f_0, \ldots) \to (f'_0, \ldots), h : [k] \to [n]) \) let:

\[
H(\varphi_{j,h}) = \begin{cases} 
\frac{b_i}{f_k(i_0, \ldots, i_{k-1}), f'_n(i'_0, \ldots, i'_{n-1})}, & n = h(k) \\
q_{h(k)}, & n \neq h(k)
\end{cases}
\]

By the definition of \( J \) this is well defined as a function of Hom sets.

**Lemma 4.14.** \( H : M \to L \) defined above is a functor.

**Proof.** We shall check that this is a functor using the explicit description of \( L \) as in Corollary 3.26. Let \([k] \xrightarrow{\delta_k} [m] \xrightarrow{\delta_h} [n] \) be 2 composable arrows in \( \Delta^+ \). There are 2 possibilities:

- If \( n = g \circ h(k) \), then
  \[
  H(\varphi_{j,g}) \circ H(\varphi_{j,h}) = \frac{b_i}{f_k(i_0, \ldots, i_{k-1}), f'_n(i'_0, \ldots, i'_{n-1})} \cdot b_{f_l(i_0, \ldots, i_{m-1}), f'_n(i'_0, \ldots, i'_{n-1})} =
  \frac{b_i}{f_k(i_0, \ldots, i_{k-1}), f'_n(i'_0, \ldots, i'_{n-1})} = H(\varphi_{j,g \circ h}).
  \]

- If \( n \neq g \circ h(k) \). Then, either \( m \neq h(k) \) which gives:
  \[
  H(\varphi_{j,g}) \circ H(\varphi_{j,h}) = s \circ q_{h(k)} = q_{h(k)} = q_{g(h(k))} = H(\varphi_{j,g \circ h})
  \]
  where \( s \) is some morphism. Or \( m = h(k) \) and \( n \neq g(m) \) which gives
  \[
  H(\varphi_{j,g}) \circ H(\varphi_{j,h}) = q_{g(m)} \cdot b_{f_k(i_0, \ldots, i_{m-1}), f'_n(i'_0, \ldots, i'_{n-1})} = q_{g(m)} = q_{g(h(k))} = H(\varphi_{j,g \circ h}).
  \]

We can now define \( \phi \) and \( \phi' \):

**Definition 4.15.** Let \( F \) be as in Construction 4.12 and \( H \) be as in Lemma 4.14. Denote

\[
\phi' := F_* H^* : \text{Fun}(L, \mathcal{C}_{A_f}) \to \text{Fun}(J \times \Delta^+, \mathcal{C}_{A_f}).
\]
By adjunction, we can view $\phi'$ as a functor $\text{Fun}(L, C_A) \to \text{Fun}(J, \text{Fun}(\Delta^+, C_A))$, so we can define

$$\phi := \colim_J \phi' : \text{Fun}(L, C_A) \to \text{Fun}(\Delta^+, C_A).$$

We now apply our functors to $Z$ of Construction 4.6:

**Notation 4.16.** Let $Z$ be as in Construction 4.6 and $\phi, \phi'$ be as in Definition 4.15. We denote $Y := \phi'(Z)$ i.e. $Y$ is the right Kan extension:

$$M \xrightarrow{H} L \xrightarrow{Z} C_A$$

$$J \times \Delta^+$$

and we let $X := \phi(Z)$ i.e.

$$X = \colim_Y Y : \Delta^+ \to C_A.$$

It will be useful for the second step to describe $Y$ more explicitly:

**Remark 4.17.** Let $Y$ be as in Notation 4.16. By Lemma 4.3, we can “describe” $Y$ more explicitly:

For an object $((f_0, \ldots), [n]) \in J \times \Delta^+$ we have:

$$Y((f_0, \ldots), [n]) \simeq \prod_{(i_0, \ldots, i_{n-1}) \in I_{f_0\ldots f_{n-1}}} Z(f_n(i_0, \ldots, i_{n-1})) = \prod_{(i_0, \ldots, i_{n-1}) \in I_{f_0\ldots f_{n-1}}} (g'_{f_n(i_0, \ldots, i_{n-1})} : A \to B_{f_n(i_0, \ldots, i_{n-1})})$$

and for a morphism $(j, h) \in \text{Mor}(J \times \Delta^+)$ we have that $Y(j, h)$ is the product of the maps over it.

This concludes the first step.

### 4.1.2 Second Step - Finding a Diagram We Can Write in Terms of Limits and $\mu$-Filtered Colimits From $B$

In this subsection our goal is to show that $X([n]) \in B_{C_A}$. We begin by defining a functor

**Definition 4.18.** Let $I$ be the $\mu$-filtered poset in Construction 4.6. Define

$$(-)^* : C \to C \quad C \mapsto \colim_{i \in I} \prod_{(I_i)_i} C$$

where the maps $\prod_{(I_i)_i} C \to \prod_{(I_j)_j} C$ for $i \leq j$ are projections.

**Remark 4.19.** Informally, for $C \in C_{A^*}$, $C^*$ is the space of $I$-indexed series in $C$ when we identify two series that “agree at some point”.

**Definition 4.20.** For an object $C \in C$ we let $C^{(0)} = C$ and define recursively $C^{(n)} = (C^{(n-1)})^*$.

**Lemma 4.21.** Let $X : \Delta^+ \to C_{A^*}$ be as in Notation 4.16 i.e. $X = \phi(Z)$ where $Z : L \to C_A$ is a functor that preserves the initial object and $\phi$ is as in Definition 4.15. Recall that $\colim Z|_I = f : A \to B$. Then,

$$X([n]) \simeq (h : A \to B^{(n)}) \in C_{A^*},$$

where $h : A \to B^{(n)}$ is some morphism.
Proof. Before proving the lemma we fix some notation. Let \( p_C : C_{A/} \to C \) be the forgetful functor and denote \( X' = p_C \circ X, Y' = p_C \circ Y \). In our revised notation we need to show that \( X'([n]) = B^{(n)} \).

It should be noted, that because \( \mu \)-filtered colimits in \( C_{A/} \) are computed in \( C \), we have that \( X' = \text{colim}_J Y' \).

We shall now prove the lemma by induction on \( n \):

For \( n = 0 \): By Remark 4.17, \( Y'((f_0, ...), [0]) = B_{f_0} \) and

\[
Y'(j, [0]) = Z'(b_{f_0, f_0}) = b'_{f_0, f_0} : B_{f_0} \to B_{f_0}
\]

for a morphism \( j : (f_0, ...) \to (f'_0, ...) \) in \( J \). Therefore, as the map

\[
J \to I, \quad (f_0, ...) \mapsto f_0
\]

is cofinal, we have that \( X'([0]) \simeq B \).

We now prove for \( n > 0 \): Let \( i \in I \) and \( J^i \subset \prod_{m \in \mathbb{N}} I^m \) be the full subcategory on \( (f_0, f_1, ...) \) satisfying

\[
f_m(i_0, ...i_{k-1}, i_k, i_{k+1}, ..., i_m) \leq f_{m+1}(i_0, ...i_{k-1}, i_k, i_{k+1}, ..., i_m)
\]

for all \( m \in \mathbb{N}, 0 \leq k \leq m \) and for all \( i_0, ..., i_m \in I_i \) (this is Construction 4.8 when we replace \( I \) by \( I_i \)). Note, that by Lemma 4.11 we can choose a cofinal map \( \xi : J^i \to J \) such that \( \xi(f_k)_{|[I_i]^k} = f_k \) for all \( k \in \mathbb{N} \) and denote by \( J^i_n \), the image of \( J^i \) under the projection

\[
\prod_{m \in \mathbb{N}} I^m \to \prod_{m \leq n} I^m.
\]

Based on the above and the fact that \( Y'(-, [n]) \) only depends on the first \( n \) coordinates of \( J \), we can write:

\[
B^{(n+1)} = \text{colim}_{f_0 \in I} \prod_{(I_0)_{i_0}} B^{(n)} \simeq \text{colim}_{f_0 \in I} \prod_{(I_0)_{i_0}} X'([n]) = \text{colim}_{f_0 \in I} \prod_{(I_0)_{i_0}} \text{colim}_J Y'(-, [n]) \simeq \text{colim}_{f_0 \in I} \prod_{(I_0)_{i_0}} \text{colim}_J \prod_{I_{f_0, ..., i_{n-1}}} B_{f_0}(i_0, ..., i_{n-1})
\]

where all the equalities are by definition, the first equivalence follows from our induction hypothesis, the second equivalence follows from Remark 4.17, and the last equivalence follows from our last remark (cofinality of \( J^i_n \) in \( J_n \) and the fact that \( Y'(-, [n]) \) only depends on the \( n \) first coordinates).

Furthermore, as limits and colimits of functors are computed point-wise, by Lemma 4.4 we can
write:

\[
\colim_{f_0 \in I} \prod_{(I_{f_0})_k} \prod_{i_0 \in (I_{f_0})_{i_k}} \prod_{i_1 \in (I_{f_0}^f_{i_0})_{i_k}} \ldots \prod_{i_{n-1} \in (I_{f_0}^f_{i_{n-1}})_{i_k}} B_{f_{n+1}(i_0, \ldots, i_n)} \simeq \\
\colim_{f_0 \in I} \prod_{(J^f_{l_{f_0}})_k} \prod_{i_0 \in (I_{f_0})_{i_k}} \prod_{i_1 \in (I_{f_0}^f_{i_0})_{i_k}} \ldots \prod_{i_{n-1} \in (I_{f_0}^f_{i_{n-1}})_{i_k}} B_{f_{n+1}(i_0, \ldots, i_n)}
\]

where we think of \((J^f_{l_{f_0}})_{l_{f_0}}\) as a subset of \(\prod_{m=1}^{n+1} J_{f_0}^m\).

Finally, let \(J^f_{l_{f_0}}\) be the image of the projection \(J^f_{l_{f_0}} \to \prod_{m=1}^{n+1} J_{f_0}^m\) and note that, by a similar argument to the one in the proof of 1 in Lemma 4.11, it is cofinal in \((J^f_{l_{f_0}})_{l_{f_0}}\). Therefore, we can write:

\[
\colim_{f_0 \in I} \prod_{(J^f_{l_{f_0}})_{l_{f_0}}} \prod_{i_0 \in (I_{f_0})_{i_k}} \prod_{i_1 \in (I_{f_0}^f_{i_0})_{i_k}} \ldots \prod_{i_{n-1} \in (I_{f_0}^f_{i_{n-1}})_{i_k}} B_{f_{n+1}(i_0, \ldots, i_n)} \simeq \\
\colim_{f_0 \in I} \prod_{J^f_{l_{f_0}}} \prod_{i_0 \in (I_{f_0})_{i_k}} \prod_{i_1 \in (I_{f_0}^f_{i_0})_{i_k}} \ldots \prod_{i_{n-1} \in (I_{f_0}^f_{i_{n-1}})_{i_k}} B_{f_{n+1}(i_0, \ldots, i_n)} = \\
\colim_{J_{f_0}^{l_{f_0}}} \prod_{f_{n+1}(i_0, \ldots, i_n)} = X'([n + 1])
\]

where the second to last equality follows from the definition of \(J_{f_0}^{l_{f_0}}\).

This concludes the second step. \(\square\)

4.1.3 Third Step - Showing That \((\text{Id}: A \to A)\) Is a Retract of \(\lim_{\Delta^+} X\)

This subsection concludes the proof of Proposition 4.2. We begin by explaining why showing that \((\text{Id}: A \to A)\) is a retract of \(\lim_{\Delta^+} X\) proves the proposition.

**Lemma 4.22.** Let \(X: \Delta^+ \to C_{A/}\) be as in Notation 4.16 i.e. \(X = \phi(Z)\) where \(Z: L \to C_{A/}\) is as in Construction 4.6 and \(\phi\) is as in Definition 4.15. Then, if \((\text{Id}: A \to A)\) is a retract of \(\lim_{\Delta^+} X\), then Proposition 4.2 follows.

**Proof.** Assume that \((\text{Id}: A \to A)\) is a retract of \(\lim X\). As limits in \(C_{A/}\) are computed in \(C\), it follows that \(A\) is a retract of \(\lim_{\Delta^+} pc \circ X\) where \(pc : C_{A/} \to C\) is the forgetful. By Lemma 4.21 we get that \(A\) is a retract of an object in \(B\) where \(B\) is as in Notation 4.7. Thus, since \(B\) is complete and therefore by [Lur09, 4.4.5.14] it is idempotent complete, it follows that \(A \in B\). \(\square\)

We will need the following definitions

**Definition 4.23.** Let \(Z: L \to C_{A/}\) be a functor that preserves the initial object. We define \(\psi(Z) \in C_{A/}\) to be

\[\psi(Z) := (F_1)_*(F_*)H^*(Z)\]
For the functors on the diagram:

\[
\begin{array}{ccc}
M & \xrightarrow{H} & L \\
\downarrow F & & \downarrow Z \\
J \times \Delta^+ & \xrightarrow{F_1} & J
\end{array}
\]

where \(H\) is as in Lemma 4.14, \(F\) is as in Construction 4.12 and \(F_1\) is the projection.

**Notation 4.24.** Let \(i : * \to L\) be the functor that chooses \(-\infty\) and \(Z : L \to \mathcal{C}_{\mathcal{A}/}\) be as in Construction 4.6. We denote \(i^* Z =: Z_{\prod \mathcal{A}}\).

Our proof that ((\Id : \mathcal{A} \to \mathcal{A})) is a retract of \(\lim \Delta^+ X\) will be divided into four parts: First, we will write \(\psi(Z_{\prod \mathcal{A}}(j))\) as a constant limit on ((\Id : \mathcal{A} \to \mathcal{A})) in \(\mathcal{C}_{\mathcal{A}/}\) indexed on some diagram which we denote by \(M_{j,-\infty}\). Second, we will prove that \(M_{j,-\infty}\) is contractible and thus \(\colim_j \psi(Z_{\prod \mathcal{A}}) = (\Id : \mathcal{A} \to \mathcal{A})\). Third, we will show \(\colim_j \psi(Z_{\prod \mathcal{A}}) = \lim \psi(Z)\) where \(\phi\) is as in Definition 4.15. Lastly, we will observe that there exists a map \(\phi(Z) \to \phi(Z_{\prod \mathcal{A}})\) and that ((\Id : \mathcal{A} \to \mathcal{A})) is the initial object in \(\mathcal{C}_{\mathcal{A}/}\), which together with the previous parts proves the claim.

We first want to show that for a fixed \(j \in J\), \(\psi(Z_{\prod \mathcal{A}}(j))\) is equivalent to the limit of the constant diagram on ((\Id : \mathcal{A} \to \mathcal{A})) indexed on some category defined in Lemma 4.25.

**Lemma 4.25.** Let \(j \in J\) and denote the pullback

\[
\begin{array}{ccc}
M_j & \xrightarrow{\bar{p}} & M \\
\downarrow \bar{F} & & \downarrow F \\
\{j\} \times \Delta^+ & \xrightarrow{p} & J \times \Delta^+
\end{array}
\]

by \(M_j\). Denote the comma category over \(-\infty \in L\), defined by \(H : M \to L\) (see Lemma 4.14 for the definition of \(H\)) by \(M_{j,-\infty}\). Let \(Z_{\prod} : L \to \mathcal{C}_{\mathcal{A}/}\) be as in Notation 4.24. Then \(\psi(Z_{\prod}(j))\) is equivalent to the limit of the constant \(M_{j,-\infty}\) indexed diagram on ((\Id : \mathcal{A} \to \mathcal{A})) where \(\psi\) is as in Definition 4.23.

**Proof.** Applying Lemma 4.3 to the following pullback diagram

\[
\begin{array}{ccc}
M_j & \xrightarrow{\bar{p}} & M \\
\downarrow \bar{F} & & \downarrow F \\
\{j\} \times \Delta^+ & \xrightarrow{p} & J \times \Delta^+
\end{array}
\]

yields an equivalence \(p^* F_* \simeq \bar{F}_* \bar{p}^*\). And similarly, denoting \(q = H \circ \bar{p}\), and applying [Lur09, 4.3.2.13] to the following square

\[
\begin{array}{ccc}
M_{j,-\infty} & \xrightarrow{\bar{q}} & * \\
\downarrow \bar{i} & & \downarrow i \\
M_j & \xrightarrow{q} & L
\end{array}
\]

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yields an equivalence \( q^* i_* \simeq \tilde{i}_* \tilde{q}^* \).

We can arrange both of the above squares in the following commutative diagram

\[
\begin{array}{ccc}
M^j_{/-\infty} & \xrightarrow{\tilde{q}} & *_{\mathbf{A} \rightarrow \mathbf{A}}
\\
i \downarrow & & \downarrow
\\
M^j & \xrightarrow{p'} & M
\\
F \downarrow & & \downarrow F
\\
j \times \Delta^+ & \xrightarrow{p} & J \times \Delta^+
\end{array}
\]

where \( c_{(\text{Id}: A \rightarrow A)} \) is the functor \(* \rightarrow \mathbf{A} /\) that chooses \((\text{Id}: A \rightarrow A)\). Recall that \( \phi'(Z_\Pi) = F_* H^* i_*(c_{(\text{Id}: A \rightarrow A)}) \). Hence, by the previous paragraph:

\[
\phi'(Z_\Pi)(j, -) = p^* F_* H^* i_*(c_{(\text{Id}: A \rightarrow A)}) \simeq \tilde{F}_* \tilde{p}^* H^* i_*(c_{(\text{Id}: A \rightarrow A)}) \simeq \tilde{F}_* \tilde{q}^* (c_{(\text{Id}: A \rightarrow A)}).
\]

From the above, and the fact that a limit is the right Kan extension to a point, we conclude that \( \psi_\Pi(j) = \lim \phi'(Z_\Pi)(j, -) \) is equivalent to the limit of the constant \( M^j_{/-\infty} \) indexed diagram on \((\text{Id}: A \rightarrow A)\).

According to our program we now want to show that \( M^j_{/-\infty} \) is contractible. In order to do so we will first need to describe \( M^j_{/-\infty} \) more explicitly.

**Remark 4.26.** Let us write an explicit description of \( M^j_{/-\infty} \). First, as \( j \in J \), by definition \( j = (f_0, ...) \), i.e. \( j \) is a tuple of functions \( f_m : I^n \rightarrow I_\delta \) satisfying

\[
f_m(i_0, ..., i_{k-1}, i_{k+1}, ..., i_m) \leq f_{m+1}(i_0, ..., i_{k-1}, i_k, i_{k+1}, ..., i_m)
\]

where \( m \in \mathbb{N}, 0 \leq k \leq m \) and \( i_0, ..., i_m \in I \) (see Construction 4.8). Recall that \( M^j \) is the fiber over \( j \in J \) of the map \( M \rightarrow J \times \Delta^+ \), thus, from Remark 4.13, we can describe \( M^j \) as follows:

An object in \( M^j \) is a 2-tuples \([n], (i_0, ..., i_{n-1})\) with \( n \in \mathbb{N} \) and \((i_0, ..., i_{n-1}) \in I_{f_0, ..., f_{n-1}} \) (i.e. \( i_k \geq f_k(i_0, ..., i_{k-1}) \)).

A morphism \( \varphi : ([k], (i_0, ..., i_{k-1})) \rightarrow ([n], (i'_0, ..., i'_{n-1})) \) in \( M^j \) over \( h : [k] \rightarrow [n] \) exists (and is the unique morphism over \( h \), with the same source and target as \( \varphi \)) if and only if \( i'_{h(m)} = i_m \) for all \( m \neq k \).

Lastly, recall how the functor which defines the comma category, \( H : M^j \rightarrow L \) \(^4\), is defined (see Corollary 3.26 for the explicit description of \( L \)) -

For objects:

\[
H([n], (i_0, ..., i_{n-1})) = f_n(i_0, ..., i_{n-1})
\]

and for morphisms:

\[
H(\varphi) = \begin{cases}
q_{f_k(i_0, ..., i_{k-1}), f_n(i'_0, ..., i'_{n-1})}, & n = h(k) \\
q_{i'_{h(m)}}, & n \neq h(k)
\end{cases}
\]

\(^4\)Here, and in what follows, we abuse notation by identifying \( H \) with its restriction to \( M^j \).
where \( \varphi \) is over \( h : [k] \to [n] \) as before (this is a functor by Lemma 4.14).

Adding everything together, we have the following description of \( M_{j/-\infty}^j \):

The objects of \( M_{j/-\infty}^j \) consists of 3-tuples \( ([n], \, (i_0, \ldots, i_{n-1}), \, g_n : f_n(i_0, \ldots, i_{n-1}) \to -\infty) \) with \( ([n], \, (i_0, \ldots, i_{n-1})) \in M^j \) and \( g_n : f_n(i_0, \ldots, i_{n-1}) \to -\infty \) a morphism in \( L \) as in Corollary 3.26.

Since a morphism \( g_n : f_n(i_0, \ldots, i_{n-1}) \to -\infty \) is determined by an element \( i_n \geq f_n(i_0, \ldots, i_{n-1}) \) we get that we can view objects of \( M_{j/-\infty}^j \) as 2-tuples \( ([n], \, (i_0, \ldots, i_n)) \) with \( [n] \in \Delta^+ \) and \( (i_0, \ldots, i_n) \in I_{f_0 \ldots f_n} \).

The morphisms of \( M_{j/-\infty}^j \) are defined as morphism in the comma category i.e. a morphism:

\[
\varphi : ([k], \, (i_0, \ldots, i_k),) \to ([n], \, (i_0', \ldots, i_n'))
\]

is a morphism \( \tilde{\varphi} : ([k], \, (i_0, \ldots, i_{k-1})) \to ([n], \, (i_0', \ldots, i_{n-1}')) \) in \( M^j \) such that the following diagram commutes:

\[
\begin{array}{ccc}
f_k(i_0, \ldots, i_{k-1}) & \xrightarrow{H(\tilde{\varphi})} & f_n(i_0', \ldots, i_{n-1}') \\
g_k & \downarrow & \downarrow g_n \\
-\infty & & -\infty
\end{array}
\]

In other words a morphism \( \varphi : ([k], \, (i_0, \ldots, i_k)) \to ([n], \, (i_0', \ldots, i_n')) \) over \( h : [k] \to [n] \) exists (and is the unique morphism over \( h \), with the same source and target as \( \varphi \)) if and only if \( i_m = i_{h(m)}' \) for all \( 0 \leq m \leq k \).

Using the above description we will show:

**Lemma 4.27.** For every \( j \in J \) the category \( M_{j/-\infty}^j \) is contractible, where \( M_{j/-\infty}^j \) is as in Lemma 4.25.

**Proof.** We will show that \( M_{j/-\infty}^j \) satisfies the conditions of Lemma 4.5. Indeed, let \( E : K \to M_{j/-\infty}^j \) be a finite diagram. Because \( M_{j/-\infty}^j \) is a 1-category we may assume that \( K \) is also a 1-category.

For any \( k \in K \) denote

\[
E(k) = ([nk], \, (i_0^k, \ldots, i_n^k)).
\]

Since \( I \) is \( \mu \)-filtered, we can pick \( p \in I \) such that:

\[
p \geq f_{nk+1}(i_0^k, \ldots, i_n^k) \quad \forall \ k \in K.
\]

We proceed to define \( E' : K \to M_{j/-\infty}^j \) (which will play the role of \( F' \) in Lemma 4.5) as follows:

For objects:

\[
E'(k) = ([nk+1], \, (i_0^k, \ldots, i_n^k, p))
\]

For morphisms: Given, \( s : k \to k' \) in \( \text{Mor}(K) \) such that \( E(s) \) is over \( h : [nk] \to [nk'] \) in \( \text{Mor}(\Delta^+) \), define

\[
h' : [nk+1] \to [nk'+1], \quad m \mapsto \begin{cases} h(m), & m \neq nk+1 \\ nk'+1, & m = nk+1 \end{cases}
\]

and let

\[
E'(s) : ([nk+1], \, (i_0^k, \ldots, i_n^k, p)) \to ([nk'+1], \, (i_0^{k'}, \ldots, i_n^{k'}, p))
\]

be the unique morphism over \( h' \), with the above source and target. Note that by Remark 4.26
such a morphism exists (uniqueness is evident from the discussion in Remark 4.26) in \( M^j_{/-\infty} \) if and only if \( i^k_{h(m)} = i^k_m \) for all \( 0 \leq m \leq n_k \) and \( p = p \) i.e. if and only if \( i^k_{h(m)} = i^k_m \) for all \( 0 \leq m \leq n_k \) which follows since \( E(s) \) is a morphism in \( M^j_{/-\infty} \). Furthermore, since the maps in \( M^j_{/-\infty} \) are completely determined by their source, target and image in \( \Delta^+ \) it is obvious that the definition of \( E' \) assembles to a functor.

Now, it remains to show that we have a natural transformation \( \theta : E \to E' \) and that \( E' \) admits a cone.

We turn to define \( \theta \). We define

\[
\theta_k : ([n], (i^k_0, \ldots, i^k_{n_k})) \to ([n+1], (i^k_0, \ldots, i^k_{n_k}, p))
\]

as the unique morphism in \( M^j_{/-\infty} \) over \( d^n : [n] \to [n+1] \) (the map in \( \Delta^+ \) which skips \( n_k + 1 \)). Note that such a morphism exists in \( M^j_{/-\infty} \) because \( i^k_m = i^k_{d^n(m)} \) for \( 0 \leq m \leq n_k \). Furthermore, by the commutativity of the following diagram

\[
\begin{array}{ccc}
[n+1] & \xrightarrow{h'} & [n'+1] \\
\downarrow{d^n} & & \downarrow{d^n} \\
[n] & \xrightarrow{h} & [n']
\end{array}
\]

the different \( \theta_k \) assembles to a natural transformation.

Finally, we need to provide a natural transformation from a constant functor to \( E' \). We define

\[
\psi_k : ([0], p) \to ([n_k+1], (i^k_0, \ldots, i^k_{n_k}, p))
\]

as the unique map in \( M^j_{/-\infty} \) over the map \( \{0\} \to \{n+1\} \to [n+1] \) in \( \Delta^+ \). The different \( \psi_k \) obviously assembles to a natural transformation and thus we conclude that \( M^j_{/-\infty} \) satisfies the conditions of Lemma 4.5 and therefore contractible.

We deduce:

**Corollary 4.28.** Let \( Z_\prod : L \to \mathcal{C}_A/ \) be as in Notation 4.24. Then \( \lim_J \psi(Z_\prod) = (\text{Id} : A \to A) \), where \( \psi \) is as in Definition 4.23.

**Proof.** By Lemma 4.25 and Lemma 4.27 \( \psi(j) = (\text{Id} : A \to A) \). Since \( J \) is contractible and \( (\text{Id} : A \to A) \) is the initial object of \( \mathcal{C}_A/ \) the corollary follows.

\[
\begin{align*}
\text{Lemma 4.29.} & \quad \text{Let } Z_\prod : L \to \mathcal{C}_A/ \text{ be as in Notation 4.24. Then } \\
& \quad \text{colim}_J \psi(Z_\prod) = \lim_{\Delta^+} \phi(Z_\prod), \text{ where } \psi \text{ is as in Definition 4.23 and } \phi \text{ is as in Definition 4.15.}
\end{align*}
\]

**Proof.** Since \( \mu \)-filtered colimits commute with \( \mu \)-small limits in \( \mathcal{S} \) and since limits and colimits in a category of presheaves are computed point-wise we get that \( \mu \)-filtered colimits commute with \( \mu \)-small limits in \( \mathcal{C} \). Furthermore, as limits and \( \mu \)-filtered colimits in \( \mathcal{C}_A/ \) are computed in \( \mathcal{C} \), we get
that \( \mu \)-filtered colimits commute with \( \mu \)-small limits in \( C_A/ \). Now, since \( J \) is \( \mu \)-filtered and \( \mu > \omega \), we have

\[
\lim_{\Delta^+} \phi(Z_{\Pi}) = \lim_{\Delta^+} \colim_J F_* H^*(Z_{\Pi}) \simeq \colim_{\Delta^+} \lim_J F_* H^*(Z_{\Pi}) = \colim_{\Delta^+} \psi(Z_{\Pi}).
\]

We can finally prove the claim.

**Lemma 4.30.** Let \( X : \Delta^+ \to C_A/ \) be as in Notation 4.16 i.e. \( X = \phi(Z) \) where \( Z : L \to C_A/ \) is a functor that preserves the initial object and \( \phi \) is as in Definition 4.15. Then \( (\text{Id}: A \to A) \) is a retract of \( \lim_{\Delta^+} X \).

**Proof.** Let \( i : \ast \to L \) be the map that chooses \(-\infty\) and

\[
\eta : Z \to i_* i^* Z = Z_{\Pi A}.
\]

be the unit map. Applying \( \phi \) from Definition 4.15 to the above map, yields a morphism

\[
\phi(\eta) : X \to \phi(Z_{\Pi})
\]

in \( \text{Fun}(\Delta^+, C_A/) \). By taking limits, and using Lemma 4.29 and Corollary 4.28 we get a map

\[
\lim_{\Delta^+} X \to (\text{Id}: A \to A).
\]

Since \( (\text{Id}: A \to A) \) is the initial object in \( C_A/ \) it follows that it is a retract of \( \lim_{\Delta^+} X \).

The proposition now follows from Lemma 4.22 and Lemma 4.30.

---

5 Factoring Through Pure Morphisms

In this section we show that every map in a category of presheaves can be factored through a \( \mu \)-pure morphism for \( \mu \) large enough, and that we can bound the size \(^5\) of the source of the factorization, in terms of the size of the source of the original map (for precise formulation, see Proposition 5.6).

We will first need some preliminary lemmas.

**Lemma 5.1.** Let \( K \) be a \( \kappa \)-small simplicial set. Then, there exists a \( \kappa^\omega \)-small quasi-category \( C \) and a categorical equivalence \( K \to C \).

**Proof.** We shall build a fibrant replacement for \( K \) using the small object argument.

Let \( S \) be the set of inner horn inclusions:

\[
S := \{ \Lambda_i^n \hookrightarrow \Delta^n \mid 0 < i < n, n \in \mathbb{N} \}
\]

Define \( E^0(K) := K \), and let \( \theta^0_k \) be the set of maps from inner horns to \( E^0(K) \). Define \( E^1(K) \) as

[^5]: In a presentable category, we say that an object \( C \in C \) is of size \( \kappa \), where \( \kappa \) is a cardinal, if \( \kappa \) is the minimal cardinal such that \( C \) is \( \kappa \)-compact.
the pushout in $\text{Set}^{\Delta^{op}}$

\[
\begin{array}{ccc}
\coprod_{\emptyset^0_n} \Lambda^n \longrightarrow E^0(K) \\
\downarrow \\
\coprod_{\emptyset^0_n} \Delta^n \longrightarrow E^1(K)
\end{array}
\]

and repeat the process to get a filtered diagram:

\[
K \longrightarrow E^1(K) \longrightarrow E^2(K) \longrightarrow \cdots .
\]

We denote the colimit of the above diagram in $\text{Set}^{\Delta^{op}}$ by $C$. By the small object argument, $C$ is a quasi-category and the natural map $K \to C$ is a categorical equivalence.

It remains to analyse the cardinality of each object in the above (filtered) diagram. First, note that there are $\omega$ horns and for a specific horn, $\Lambda^n_i$, the number of maps $\Lambda^n_i \to K$ is less than $\kappa$.$\omega$. So, the size of $\emptyset^0_n$ is strictly less than $\kappa$.$\omega$. Now, as each map in $\emptyset^0_n$ adds less than $\omega$ non degenerate simplices, $E^1(K)$ is $\kappa + \omega \kappa$.$\omega$-small, and because $(\kappa$.$\omega$)$^2 = \kappa$.$\omega$, we conclude that $E^1(K)$ is $\kappa$.$\omega$-small. All in all we conclude that $C$ is $\kappa$.$\omega$-small.

**Lemma 5.2.** Let $\kappa > \omega$ be a cardinal and $C$ be $\kappa$-compact object of $\text{Cat}_{\infty}$. Let $F : C \to S$ be a functor that factors through $S^\kappa$. Then, $\text{Un}(F)$ is also a $\kappa$-compact object of $\text{Cat}_{\infty}$ where $\text{Un}(F)$ is the unstraightening of $F$.

**Proof.** By [GHN15, 1.1] we have the following chain of equivalences:

\[
\text{Un}(F) \simeq \text{op}(F) \simeq \text{colim}_{\text{Tw}(C)} F(\cdot) \times C_{/\cdot}.
\]

We claim that the colimit above is a $\kappa$-small colimit of $\kappa$-compact objects. Indeed from the definition of the over category the $n$ simplices of $C_{/c}$ are $n + 1$ simplices in $C$ that begins in $c$ and thus by [Lur09, 5.4.1.2] $C_{/c}$ is $\kappa$-compact. Invoking [Lur09, 5.4.1.2] again, we get that $F(c) \times C_{/c}$ is $\kappa$-compact for all $c$. Hence, recalling that

\[
\text{Tw}(C) = \text{Hom}_{\text{Set}(\Delta^{+})^{op}}((\Delta^n)^{op} \ast (\Delta^n), C)
\]

where $\text{Hom}_{\text{Set}(\Delta^{+})^{op}}$ is the Hom set and invoking [Lur09, 5.4.1.2] again, we can write $\text{Un}(F)$ as a $\kappa$-small colimit of $\kappa$-compact objects. Since $\kappa$-compact objects are closed under $\kappa$-small colimits, it follows that $\text{Un}(F)$ is $\kappa$-compact. \hfill \Box

**Remark 5.4.** Note that $(\kappa$.$\mu)^{<\mu} = \kappa^{<\mu}$ for $\kappa \geq \mu$.

**Lemma 5.5.** Let $I$ be $\mu$-small simplicial set and let $F : I \to S$ be a functor that factors through $S^\kappa$. Assume further that $\mu, \kappa > \omega$. Then, $\lim F$ is $\gamma$-compact, where $\gamma := \max\{\mu^{<\mu}, \kappa^{<\mu}\}$.

**Proof.** From [Lur09, 3.3.3.4] $\lim F$ is the fiber over $\text{Id}$ of the map $\text{Map}(I, \text{Un}(F)) \to \text{Map}(I, I)$ where again, $\text{Un}(F)$ denotes the unstraightening of $F$. Thus, by [Lur09, 5.4.1.5] and the long exact
sequence in homotopy groups, it suffices to show that \( \text{Map}(I, \text{Un}(F)) \) and \( \text{Map}(I, I) \) are \( \gamma \)-compact.

We will show that \( \text{Map}(I, \text{Un}(F)) \) and \( \text{Map}(I, I) \) are \( \gamma \)-compact by finding simplicial models with less than \( \gamma \) simplices in each simplicial degree. First, by Lemma 5.2, \( \text{Un}(F) \) is \( \max\{\kappa, \mu\} \)-compact object of \( \text{Cat}_{\infty} \). Therefore, by [Lur09, 5.4.1.2] there exists a simplicial model \( \text{Un}(F)' \in \text{Set}^{\Delta^\text{op}} \) for \( \text{Un}(F) \), with less than \( \max\{\kappa, \mu\} \) simplices in each simplicial degree. Invoking Lemma 5.1, we get quasi-category models \( I', \text{Un}(F)' \in \text{Set}^{\Delta^\text{op}} \), for \( I \) and \( \text{Un}(F) \), that have less than \( \mu^\omega, \max\{\kappa^\omega, \mu^\omega\} \) simplices in each simplicial degree respectively. Second, by [Lur09, 1.2.7.3] \( \text{Hom}_{\text{Set}^{\Delta^\text{op}}}(I, I') \) and \( \text{Hom}_{\text{Set}^{\Delta^\text{op}}}(I, \text{Un}(F)') \) are simplicial models for \( \text{Fun}(I, I) \) and \( \text{Fun}(I, \text{Un}(F)) \) respectively (here, \( \text{Hom}_{\text{Set}^{\Delta^\text{op}}}(\cdot, \cdot) \) denotes the inner-Hom in \( \text{Set}^{\Delta^\text{op}} \)). Lastly, as the \( n \)-simplices of the inner-Hom in simplicial sets are given by \( \text{Hom}_{\text{Set}^{\Delta^\text{op}}}(\Delta^n \times (-), (-)) \) (where \( \text{Hom}_{\text{Set}}(\cdot, \cdot) \) is the Hom set) and we have the following identity

\[
\max\{\mu^{<\mu}, \kappa^{<\mu}\} = \max\{(\mu^\omega)^{<\mu}, (\kappa^\omega)^{<\mu}\},
\]

we have found models for \( \text{Fun}(I, \text{Un}(F)) \) and \( \text{Fun}(I, I) \) with less than \( \gamma \) simplices in each simplicial degree. We conclude that both \( \text{Map}(I, \text{Un}(F)) \) and \( \text{Map}(I, I) \) are \( \gamma \)-compact.

**Proposition 5.6.** Let \( E \) be a small category and denote \( C := \text{Fun}(E, S) \). Let \( \kappa \) be a cardinal for which \( C \) is \( \kappa \)-presentable and \( E \) \( \kappa \)-small. Let \( \pi \) be the cardinal from Lemma 2.1. Then, for every \( \mu \geq \pi \) there exists \( \gamma \geq \mu \) such that for every map \( f : A \to B \), where \( A \) is \( \gamma \)-compact in \( C \), there exists a 2-simplex in \( C \)

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow f' \\
A' & \xrightarrow{f'} & B'
\end{array}
\]

such that \( f' \) is \( \mu \)-pure and \( A' \) is \( \gamma \)-compact in \( C \).

**Proof.** (The proof is similar to the proof of [AR94, 2.33]) Let \( \gamma' \) be such that \( C^\mu \) is \( \gamma' \)-small, denote \( \gamma := (\max\{\gamma'^{<\gamma'}, \mu^{<\mu}\})^{+} \), and note that \( \gamma^{<\mu} = \gamma \).

Let \( f : A \to B \) be a map where \( A \) is \( \gamma \)-compact, we shall construct the desired factorization of \( f \) by transfinite induction on \( \mu \).

**Base case:** Let \( f_0 := f \).

**Successive case:** Given a map \( f_i : A_i \to B \) where \( A_i \) is \( \gamma \)-compact, we shall define \( f_{i+1} : A_{i+1} \to B \), where \( A_{i+1} \) is \( \gamma \)-compact, factoring \( f_i \). Let \( D_i \) be the set of representatives of spans

\[
\begin{array}{ccc}
A_i & \xrightarrow{u} & A' \\
\downarrow & & \downarrow g' \\
A_i & \xrightarrow{f_i} & B'
\end{array}
\]
where $A'$ and $B'$ are $\mu$-compact and there exists a factorization:

$$
\begin{array}{c}
A' \\
\downarrow u \\
A_i \\
\downarrow f_i \\
B \\
\end{array}
\quad \begin{array}{c}
A_i \\
\downarrow v \\
B' \\
\end{array}

Here, when we say a representative, we mean a representative of a span in the homotopy category of $\mathcal{C}$. We denote elements of $D_i$ by

$$
\begin{array}{c}
\quad \\
\downarrow \\
A_i \\
\end{array}
= 
\quad 
\begin{array}{c}
A^j \\
B^j \\
\end{array}
\quad \begin{array}{c}
\quad \\
\downarrow \\
A_i \\
\end{array}
$$

Given an element $j \in D_i$ we define $A_{i,j} := \bigsqcup_{A^j} A_i \bigsqcup B^j$ where we view each $A_{i,j}$ as an object of $(\mathcal{C}_{A_i})_{/B}$ (i.e. $A_{i,j} \in (\mathcal{C}_{A_i})_{/B}$). Finally, let

$$
A_{i+1} = \bigsqcup_{j \in D_i} A_{i,j}
$$

where the coproduct is taken in $(\mathcal{C}_{A_i})_{/B}$. By construction, we get a map $f_{i+1} : A_{i+1} \to B$ that factors $f_i$.

We now show that $A_{i+1}$ is $\gamma$-compact. Let us bound the size of $D_i$ i.e. the number of spans above. First, we claim that for a fixed $A' \in \mathcal{C}^\mu$ the space $\text{Map}_{\mathcal{C}}(A', A_i)$ is $\gamma$-compact. Indeed, write $A'$ as a $\mu$-small colimit of representables, then by the Yoneda lemma

$$
\text{Map}_{\mathcal{C}}(A', A_i) \simeq \text{Map}_{\mathcal{C}}(\text{colim Map}_{\mathcal{C}}(e, -), A_i) \simeq \text{lim Map}_{\mathcal{C}}(\text{Map}_{\mathcal{C}}(e, -), A_i) \simeq \text{lim} A_i(e)
$$

is a $\mu$-small limit of $\gamma$-compact object and therefore $\gamma$-compact by Lemma 5.5 and Lemma 2.1. Hence, for a fixed $A' \to B'$ the number of spans:

$$
\begin{array}{c}
A' \\
\downarrow u \\
A_i \\
\downarrow g_i \\
B' \\
\end{array}
$$

is less than $\gamma$. Second, since $\mathcal{C}^\mu$ is $\gamma$-small, the cardinality of the set of maps $A' \to B'$ is less than $\gamma$. We conclude that the size of $D_i$ is bounded by $\gamma$, and therefore, we can write $A_{i+1}$ as a $\gamma$-small colimit of $\gamma$-compact objects. All in all we have that $A_{i+1}$ is $\gamma$-compact.

**Limit step:** For a limit ordinal $i \leq \mu$, Let $f_i := \text{colim}_{m < i} f_m$ where the colimit is taken in $\mathcal{C}_{/B}$ and note that $\text{colim}_{m < i} A_m$ is $\gamma$-compact.
By construction we get a factorization

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \ f_\mu \\
A_\mu & \xrightarrow{f_\mu} & B \\
\end{array}
\]

where \( A_\mu \) is \( \gamma \)-compact. Hence, it remains to show that \( f_\mu : A_\mu \to B \) is \( \mu \)-pure. Indeed, assume that we are given a commutative diagram

\[
\begin{array}{ccc}
A' & \xrightarrow{u} & B' \\
\downarrow & & \downarrow \\
A_\mu & \xrightarrow{u_\mu} & B \\
\end{array}
\]

where \( A', B' \) are \( \mu \)-compact. Because \( A_\mu \) is a \( \mu \)-filtered colimit there exists an \( i < \mu \) such that

\[
\begin{array}{ccc}
A' & \xrightarrow{u} & B' \\
\downarrow & & \downarrow \\
A_i & \xrightarrow{u_i} & B' \\
\end{array} \in D_i
\]

and a 2-simplex:

\[
\begin{array}{ccc}
A' & \xrightarrow{u} & A_\mu \\
\downarrow & & \downarrow \\
A_{i+1} & \xrightarrow{u_{i+1}} & A_\mu \\
\end{array}
\]

but, by construction we also have a 2-simplex:

\[
\begin{array}{ccc}
A' & \xrightarrow{u'} & A_{i+1} \\
\downarrow & & \downarrow \\
B' & \xrightarrow{u_{i+1}} & A_{i+1} \\
\end{array}
\]

Therefore, we get a horn \( A^3_2 \) in \( \mathcal{C} \):

\[
\begin{array}{ccc}
A_\mu & \xleftarrow{u'} & B' \\
\downarrow & & \downarrow \\
A' & \xrightarrow{u} & A_{i+1} \\
\end{array}
\]

which yields a 2-simplex

\[
\begin{array}{ccc}
A' & \xrightarrow{u} & B' \\
\downarrow & & \downarrow \\
A_\mu & \xrightarrow{u_\mu} & A_{i+1} \\
\end{array}
\]

in the quasi-category \( \mathcal{C} \), as desired.

\[\square\]

6 Proving the Reflection Theorem

In this section we will finally prove our main theorem, i.e. that a category closed under limits and sufficiently filtered colimits in a presentable category is presentable. Before doing so, we will need
the following lemma:

**Lemma 6.1.** Let $\mathcal{C}$ be a $\kappa$-filtered category and $\mathcal{D} \subset \mathcal{C}$ a full subcategory such that every $c \in \mathcal{C}$ admits a map to an object of $\mathcal{D}$. Then, the inclusion $i : \mathcal{D} \hookrightarrow \mathcal{C}$ is cofinal.

**Proof.** By Quillen theorem A it suffices to show that for every $c \in \mathcal{C}$, the comma category $\mathcal{D}_{c/}$ is contractible. We shall do so by showing that $\mathcal{D}_{c/}$ is $\kappa$-filtered. Let $F : K \rightarrow \mathcal{D}_{c/}$ be a diagram, where $K$ is $\kappa$-small. Since $\mathcal{C}_{c/}$ is $\kappa$-filtered, after composing with the inclusion $\mathcal{D}_{c/} \hookrightarrow \mathcal{C}_{c/}$, $F$ admits a cocone, which we denote by $c'$. Composing with a map $c' \rightarrow d$ for $d \in \mathcal{D}$ we get a cocone on $F$, which implies that $\mathcal{D}_{c/}$ is $\kappa$-filtered and therefore contractible.

**Theorem 6.2.** Let $\mathcal{C}$ be a presentable category and let $\mathcal{D} \subset \mathcal{C}$ be a full subcategory which is closed under limits and $\kappa$-filtered colimits for some regular cardinal $\kappa$. Then, $\mathcal{D}$ is presentable.

**Proof.** We start by reducing to the case in which $\mathcal{C} = \text{Fun}(A, S)$ where $A$ is a small category. Recall that every presentable category is a reflective accessible localization of $\text{Fun}(A, S)$ where $A$ is some small category [Lur09, 5.5.1.1], and as such, $\mathcal{C}$ is equivalent to a full subcategory of $\text{Fun}(A, S)$ which is closed under limits and $\mu$-filtered colimits (for some regular cardinal $\mu$). As a consequence, any full subcategory of $\mathcal{C}$ which is closed under limits and $\kappa$-filtered colimits in $\mathcal{C}$, is equivalent to a full subcategory of $\text{Fun}(A, S)$ closed under limits and $\mu + \kappa$-filtered colimits. Therefore, we may assume that $\mathcal{C} = \text{Fun}(A, S)$.

We proceed to prove the claim, under said assumption. Let $\mu$ be larger then the cardinal from **Lemma 2.1** (i.e. the one which corresponds to $\mathcal{C}$) by enlarging $\mu$ we may also assume that:

- $\mathcal{D}$ is closed under $\mu$-pure morphisms (this follows from **Proposition 4.2**).
- $\mathcal{C}$ is $\mu$-presentable (this follows from the definition of presentable category).
- $\mathcal{D}$ is closed under $\mu$-filtered colimits in $\mathcal{C}$ (this follows from our assumption on $\mathcal{D}$).

We shall denote by Pure$_\mu(\mathcal{C})$ the wide-subcategory of $\mathcal{C}$ spanned by all $\mu$-pure morphisms.

Let $\gamma$ be a regular cardinal bigger then the cardinal that corresponds to $\mu$ in the setting of **Proposition 5.6**, i.e. for $f : A \rightarrow B$ a morphism in $\mathcal{C}$ where $A$ is $\gamma$-compact there exists a factorization

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
A' & \xrightarrow{f'} & \\
\end{array}
$$

with $f'$ $\mu$-pure and $A'$ $\gamma$-compact.

The rest of the proof of the theorem will be divided into two parts: First, we will show that for $B \in \mathcal{D}$, the category $\mathcal{P} := \text{Pure}_\mu(\mathcal{C})/B \cap (\mathcal{C}\gamma)/B$ is a full subcategory of $(\mathcal{C}\gamma)/B$. Then, we will conclude that $\mathcal{D}$ is presentable.
**First part -** \( \mathcal{P} \) is a full subcategory of \((\mathcal{C}^\gamma)_B\): Observe, that if we have a commutative diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{b} & A' \\
\downarrow{f'} & & \downarrow{f} \\
B & & B
\end{array}
\]

with \( f \) and \( f' \) \( \gamma \)-pure then \( h \) is also \( \gamma \)-pure (in fact one only needs \( f' \) to be \( \gamma \)-pure), this implies that \( \text{Pure}_\mu(\mathcal{C})_B \) is the pullback of the following diagram:

\[
\begin{array}{ccc}
\text{Fun}^\text{Pure}(\Delta^1, \mathcal{C}) & \xrightarrow{\text{target}} & \mathcal{C} \\
\downarrow & & \downarrow \text{target} \\
\Delta^0 & \xrightarrow{B} & \mathcal{C}
\end{array}
\]

where \( \text{Fun}^\text{Pure}(\Delta^1, \mathcal{C}) \subset \text{Fun}(\Delta^1, \mathcal{C}) \) is the full subcategory on pure morphisms. Since there is a natural map between the following two diagrams:

\[
\begin{array}{ccc}
\text{Fun}(\Delta^1, \mathcal{C}) & \xrightarrow{\text{target}} & \mathcal{C} \\
\downarrow & & \downarrow \text{target} \\
\Delta^0 & \xrightarrow{B} & \mathcal{C}
\end{array}
\]

and the induced map on limits is the natural map \( \text{Pure}_\mu(\mathcal{C})_B \to (\mathcal{C})_B \), it follows that \( \mathcal{P} \) is a full subcategory of \((\mathcal{C}^\gamma)_B\).

**Second Part -** \( \mathcal{D} \) is presentable: First, note that by definition \((\mathcal{C}^\gamma)_B\) admits \( \gamma \)-small colimits and thus it is \( \gamma \)-filtered. Therefore, invoking Proposition 5.6 and Lemma 6.1 we have that the inclusion \( \mathcal{P} \hookrightarrow (\mathcal{C}^\gamma)_B \) is cofinal. Second, as \( \mathcal{D} \) is closed under \( \mu \)-pure morphisms we have that \( \mathcal{P} \subset (\mathcal{D}^\gamma)_B \) (note that since \( \mathcal{D} \) is closed under \( \gamma \)-filtered colimits, a \( \gamma \)-compact object in \( \mathcal{C} \) that lies in \( \mathcal{D} \) is also \( \gamma \)-compact in \( \mathcal{D} \) ), which implies, by the above, that \( B \) is \( \gamma \)-filtered colimit of objects in \( \mathcal{D}^\gamma \). Since \( B \) was arbitrary we get that every object of \( \mathcal{D} \) is a \( \gamma \)-filtered colimit of \( \gamma \)-compact objects of \( \mathcal{D} \) i.e. \( \mathcal{D} \) is accessible. Finally, using our assumptions on \( \mathcal{D} \), it follows that the functor \( i : \mathcal{D} \hookrightarrow \mathcal{C} \) is accessible and thus satisfies the solution set condition. The adjoint functor theorem \([\text{NRS20}, 3.2.5]\) then asserts that \( i \) has a left adjoint, which implies that \( \mathcal{D} \) is also co-complete.

**Corollary 6.3.** For a category \( \mathcal{C} \) the following are equivalent:

1. \( \mathcal{C} \) is presentable.

2. \( \mathcal{C} \) is complete and there exist a set of \( \kappa \)-compact objects, \( S \), for some cardinal \( \kappa \), such that
restricted Yoneda to the full subcategory on the elements of $S$ is fully-faithful.\textsuperscript{6}

Proof. 1 $\implies$ 2: By definition $\mathcal{C}$ is complete. Choose a cardinal $\kappa$ such that $\mathcal{C}$ is $\kappa$-presentable. Using the model for $\text{Ind}^\kappa$ as the full subcategory of $\text{Fun}(\mathcal{C}^\kappa, S)$ on the functors that commutes $\kappa$-small limits given in [Lur09, 5.3.5.4] we see that $\mathcal{C} \simeq \text{Ind}^\kappa(\mathcal{C}^\kappa) \hookrightarrow \text{Fun}(\mathcal{C}^\kappa, S)$ can be identified with the restricted Yoneda to $\mathcal{C}^\kappa$.

2 $\implies$ 1: Choose a set $S$ as in 2 and denote the full subcategory that on $S$ by $\mathcal{C}_0$. By assumption the restricted Yoneda $\mathcal{C} \hookrightarrow \text{Fun}(\mathcal{C}_0, S) := \mathcal{D}$ is fully-faithful. Thus, by Theorem 6.2, it suffices to show that $\mathcal{C}$ is closed under limits and sufficiently filtered colimits in $\mathcal{D}$. As the Yoneda embedding commutes with limits, $\mathcal{C}$ is closed under limits in $\mathcal{D}$, and since all the objects of $S$ are $\kappa$-compact, $\mathcal{C}$ is also close under $\kappa$-filtered colimits in $\mathcal{D}$. \hfill $\square$

7 Applications - Recognizing Smashing Localization

In this section we discuss the use of the Theorem 6.2 in recognizing subcategories of the ambient symmetric monoidal category $\mathcal{C}$ for which inclusion admits a smashing left adjoint, as defined in Definition 7.1. We first deal with the case where $\mathcal{C} \in \text{CAlg}(\text{Pr}^L)$ (see Theorem 7.6) and then with the case $\mathcal{C} = \text{Pr}^L$ (see Theorem 7.15).

7.1 Recognition Result for Smashing Localizations of $\mathcal{C} \in \text{CAlg}(\text{Pr}^L)$

In this subsection, we prove a necessary and sufficient condition for when a full-subcategory of $\mathcal{C} \in \text{CAlg}(\text{Pr}^L)$ is classified by an idempotent algebra. Recall that we say that a morphism $u : \mathbb{1} \to X$ in $\mathcal{C}$ exhibits $X$ as an idempotent object of $\mathcal{C}$, if

$$X \simeq X \otimes \mathbb{1} \xrightarrow{1 \otimes u} X \otimes X$$

is an equivalence. By [Lur11, 4.8.2.9], an idempotent object $u : \mathbb{1} \to X$ admits a unique commutative algebra structure for which $u$ is the unit. Conversely, the unit $u : \mathbb{1} \to R$ of a commutative algebra $R$ exhibits it as an idempotent object if and only if the multiplication map $R \otimes R \to R$ is an isomorphism. We call such $R$-s idempotent algebras. More precisely, the functor $\text{CAlg}(\mathcal{C}) \to \mathcal{C}_1/\mathbb{1}$ which forgets the algebra structure and remembers only the unit map, induces an equivalence of categories from the full-subcategory of idempotent algebras $\text{CAlg}^{\text{idem}}(\mathcal{C})$ to the full-subcategory of idempotent objects [Lur11, 4.8.2.9].

The fundamental feature of an idempotent algebra $R$ is that the forgetful functor $\text{Mod}_R(\mathcal{C}) \to \mathcal{C}$ is fully faithful. Thus, it is a property of an object in $\mathcal{C}$ to have the structure of an $R$-module. We shall say that $R$ classifies the property of being an $R$-module. So, if we are interested in a property of objects of $\mathcal{C}$, we can ask whether this property is classified by an idempotent algebra. When this is the case, we get a universal object in $\mathcal{C}$ that satisfies this property. Therefore, one can consider this subsection’s results as classification results of properties classified by idempotent algebras.

Let us recall some definitions and known results.

\textsuperscript{6}Some sources call such set a ‘dense’ set of $\mathcal{C}$.
Definition 7.1. Let $C \in \text{CAlg}(\text{Cat}_{\infty})$ and $\iota : D \hookrightarrow C$ a reflective subcategory. Denote the left adjoint of $\iota$ by $L$. We say that $L$ is a *smashing localization* if there exists an object $X$ and an equivalence of functors $L \simeq L_X$ where $$L_X(Y) = X \otimes Y.$$ 

Definition 7.2. ([Lur11, 2.2.1.6]) Let $C$ be a symmetric monoidal category. Let $\iota : D \hookrightarrow C$ be a reflective full-subcategory and denote the left adjoint of $\iota$ by $L$. We say that $L$ is compatible with the symmetric monoidal structure if the tensor product of an $L$ equivalence is an $L$ equivalence.

There is a bijective correspondence between smashing localizations and idempotent algebras.

Proposition 7.3. For $R \in \text{CAlg}(C)$ the following are equivalent:

1. $R \in \text{CAlg}_{\text{idem}}(C)$.
2. The functor $L_R$ is a smashing localization.

Proof. This is [Lur11, 4.8.2.4].

We will need the following lemma:

Lemma 7.4. Let $C$ be a closed symmetric monoidal category. Let $\iota : D \hookrightarrow C$ be a reflective full-subcategory. Then the following are equivalent:

1. The left adjoint of the inclusion, $L$, is compatible with the symmetric monoidal structure.
2. For all $d \in D$ and $c \in C$, $\text{Hom}^C(c, d) \in D$ where $\text{Hom}^C(\_ , \_)$ is the inner-Hom.

Proof. $1 \implies 2$: Assume that $L$ is compatible with the symmetric monoidal structure. We will show that the natural map $\text{Hom}^C(c, d) \to L\text{Hom}^C(c, d)$ is an equivalence. By [Lur11, 2.2.1.9] $D$ is symmetric monoidal and the natural map $L(c \otimes c') \to L(L(c) \otimes L(c'))$ is an equivalence, where we can view the latter as a “formula” for the tensor product in $D$. Therefore, for all $c, c' \in C$ and $d \in D$ the unit map induces an equivalence:

$$\text{Map}_C(Lc, \text{Hom}^C(c', d)) \to \text{Map}_C(c, \text{Hom}^C(c', d)).$$

Indeed, by adjunction we have

$$\text{Map}_C(c, \text{Hom}^C(c', d)) \simeq \text{Map}_C(c \otimes c', d) \simeq \text{Map}_C(L(c \otimes c'), d) \simeq \text{Map}_C(Lc \otimes c', d) \simeq \text{Map}_C(Lc, \text{Hom}^C(c', d)).$$

Choosing $c = \text{Hom}^C(c', d)$ we get that the unit of the adjunction

$$\eta_{\text{Hom}^C(c', d)} : \text{Hom}^C(c', d) \to L\text{Hom}^C(c', d)$$

has a right inverse, which will be denoted by $v$. Since both $\eta_{\text{Hom}^C(c', d)} \circ v$ and $\text{Id}_{L\text{Hom}^C(c', d)}$ fit in the dotted line

$$\text{Hom}^C(c', d) \xrightarrow{\eta_{\text{Hom}^C(c', d)}} L\text{Hom}^C(c', d)$$

$$\eta_{\text{Hom}^C(c', d)}$$

$$L\text{Hom}^C(c', d)$$
we get that $v$ is also a left inverse by the universal property of $\eta$.

2 $\implies$ 1: Assume that for all $d \in \mathcal{D}$ and $c \in \mathcal{C}$, $\text{Hom}^\mathcal{C}(c, d) \in \mathcal{D}$. We will show that $L(c \otimes c') \simeq L(Lc \otimes Lc')$. Fix $c, c' \in \mathcal{C}$ and $d \in \mathcal{D}$ and note that we have the following chain of equivalences

$$\text{Map}(L(c \otimes c'), d) \simeq \text{Map}(c, \text{Hom}^\mathcal{C}(c', d)) \simeq \text{Map}(Lc, \text{Hom}^\mathcal{C}(c', d)) \simeq \text{Map}(L(Lc \otimes c'), d).$$

So, by the Yoneda lemma, $L(c \otimes c') \simeq L(Lc \otimes Lc')$ and by repeating the argument we get that $L(c \otimes c') \simeq L(Lc \otimes Lc')$.

**Corollary 7.5.** Let $\mathcal{C}$ be a closed symmetric monoidal category and let $\iota : \mathcal{D} \hookrightarrow \mathcal{C}$ be a reflective full-subcategory. Assume that the left adjoint of the inclusion, $L$, is compatible with the symmetric monoidal structure. Then the unit map $c \mapsto Lc$ induces an equivalence

$$\text{Hom}^\mathcal{C}(Lc, d) \rightarrow \text{Hom}^\mathcal{C}(c, d) \quad \forall d \in \mathcal{D}$$

where $\text{Hom}^\mathcal{C}$ is the inner-Hom.

**Proof.** By **Lemma 7.4** $\text{Hom}^\mathcal{C}(L(c), d)$, $\text{Hom}^\mathcal{C}(c, d) \in \mathcal{D}$. Hence, by the Yoneda lemma, it suffices to show that the unit map induces an equivalence:

$$\text{Map}(d', \text{Hom}^\mathcal{C}(Lc, d)) \simeq \text{Map}(d', \text{Hom}^\mathcal{C}(c, d))$$

But by adjunction we have

$$\text{Map}(d', \text{Hom}^\mathcal{C}(c, d)) \simeq \text{Map}(d' \otimes c, d) \simeq \text{Map}(L(Ld' \otimes Lc), d) \simeq \text{Map}(d' \otimes Lc, d) \simeq \text{Map}(d', \text{Hom}^\mathcal{C}(Lc, d))$$

We now prove the main theorem of this subsection.

**Theorem 7.6.** For $\mathcal{C} \in \text{CAlg}^{\text{idem}}(\text{Pr}^L)$ and $\iota : \mathcal{D} \hookrightarrow \mathcal{C}$ a full-subcategory the following are equivalent:

1. $\mathcal{D} = \text{Mod}_R(\mathcal{C})$ for some $R \in \text{CAlg}^{\text{idem}}(\mathcal{C})$.

2. $\iota$ admits a left adjoint $L$ which is a smashing localization.

3. $\mathcal{D}$ is closed under limits and colimits in $\mathcal{C}$, and if $d \in \mathcal{D}$ and $c \in \mathcal{C}$, then $d \otimes c, \text{Hom}^\mathcal{C}(c, d) \in \mathcal{D}$.

**Proof.** The equivalence between 1 and 2 is **Proposition 7.3** (which is just [Lur11, 4.8.2.4]).

2 $\implies$ 3: Assume that $\iota$ admits a left adjoint, $L$, and that $L$ is smashing. First, it is immediate that $\mathcal{D}$ is closed under limits and $c \otimes d \in \mathcal{D}$ for all $c \in \mathcal{C}$ and $d \in \mathcal{D}$. Second, since the tensor product in $\mathcal{C}$ commutes with colimits, $\mathcal{D}$ is also closed under colimits. Finally, since the localization is compatible with the symmetric monoidal structure, $\text{Hom}^\mathcal{C}(c, d) \in \mathcal{D}$ for all $c \in \mathcal{C}$ and $d \in \mathcal{D}$ by **Lemma 7.4**.
3 \implies 2: Assume that the conditions of 2 are satisfied. Theorem 6.2 then implies that \(i\) admits a left adjoint, \(L\). We need to show that \(L\) is smashing i.e. that there exists an object \(X\) such that \(L\) is given by \(c \mapsto X \otimes c\). We will show that \(X = L(\mathbb{1})\) satisfies this condition. By assumption \(L(\mathbb{1}) \otimes c \in D\) and \(\text{Hom}_C^c(c, d) \in D\) for all \(c \in C\) and \(d \in D\). Therefore, it follows from Corollary 7.5 that

\[
\text{Map}(Lc, d) \simeq \text{Map}(c, d) \simeq \text{Map}(c, \text{Hom}_C(L\mathbb{1}, d)) \simeq \text{Map}(c \otimes L\mathbb{1}, d).
\]

Hence, from the Yoneda lemma \(Lc \simeq L\mathbb{1} \otimes c\) as desired. \(\square\)

The following situation is often of interest to us. Given a map \(f : X \to Y\) between objects in a symmetric monoidal presentable category \(C\), can we characterize the category spanned by the objects \(Z\) such that \(Z \otimes X \xrightarrow{\text{Id} \otimes f} Z \otimes Y\) is an equivalence? By [Lur18, 4.3.17] if \(X\) and \(Y\) are invertible with respect to the tensor product then this category is equivalent to a category of modules over an idempotent algebra. It follows from Theorem 7.6 that under relatively mild assumptions the same is true when \(X\) and \(Y\) are only dualizable.

Definition 7.7. Let \(C \in \text{CAlg}(\text{Pr}^L)\). We say that \(C\) is generated under colimits from dualizable objects if the smallest full subcategory that contains all the dualizable objects and is closed under colimits in \(C\) is all of \(C\).

Corollary 7.8. Let \(C \in \text{CAlg}(\text{Pr}^L)\) and assume that \(C\) is generated under colimits from dualizable objects. Let \(f : D_1 \to D_2\) be a map between dualizable objects, and let \(D\) be the full subcategory on objects, \(X\), such that \(X \otimes D_1 \xrightarrow{\text{Id} \otimes f} X \otimes D_2\) is an equivalence. Then, \(D\) is equivalent to a category of modules over an idempotent algebra in \(C\).

Proof. We will check that \(D\) satisfies the equivalent conditions of Theorem 7.6. Since the tensor product in \(C\) commutes with colimits in each variable it is immediate that \(D\) is closed under colimits. Since \(D_1, D_2\) are dualizable it is also immediate that \(D\) is closed under limits in \(C\). It is also evident from the definition of \(D\) that if \(X \in D\) and \(Y \in C\) then \(X \otimes Y \in D\). It remains to show that if \(X \in D\) and \(Y \in C\) then \(\text{Hom}_C^c(Y, X) \in D\). If \(Y\) is dualizable then \(\text{Hom}_C^c(Y, X) \simeq X \otimes Y^\vee \in D\). Now, as the functor \(\text{Hom}_C^c(-, X) : C^{\text{op}} \to C\) commutes with limits, \(D\) is closed under limits and the dualizable objects generate \(C^{\text{op}}\) under limits the claim follows. \(\square\)

7.2 Recognition Result for Smashing Localizations of \(\text{Pr}^L\)

In this subsection, we prove a necessary and sufficient condition for when a full-subcategory of \(\text{Pr}^L\) is classified by an idempotent algebra. In [CSY21] idempotent algebras in \(\text{Pr}^L\) are studied and are called “Modes”. The result we obtain is similar to Theorem 7.6, however since \(\text{Pr}^L\) is not itself presentable there are some additional set theoretic assumptions to consider - see Theorem 7.15.

We first recall some definitions and known results:

Proposition 7.9. \(\text{Pr}^L\) has a closed symmetric monoidal structure where the inner-Hom between two categories, \(C\) and \(D\), is the category of colimits preserving functors denoted by \(\text{Fun}^R(C, D)\). Furthermore, in this symmetric monoidal structure we have the formula:

\[
C \otimes D \simeq \text{Fun}^R(C^{\text{op}}, D)
\]
where $\text{Fun}^R(\mathcal{C}, \mathcal{D})$ is the category of limits preserving accessible functors.

**Proof.** This is [Lur11, 4.8.1.17].

**Definition 7.10.** [Lur09, 5.5.7.7] Let $\kappa$ be a regular cardinal. We denote by $\text{Cat}_{\kappa}^{\text{Rex}}$ the category whose objects are categories with $\kappa$-small colimits and morphisms are functors which preserve $\kappa$-small colimits.

**Definition 7.11.** [Lur09, 5.5.7.7] Let $\kappa$ be a regular cardinal. We denote by $\text{Pr}^L_{\kappa}$ the category whose objects are $\kappa$-presentable categories and morphisms are functors that preserve colimits and send $\kappa$-compact objects to $\kappa$-compact objects. For $\mathcal{P} \subset \text{Pr}^L$ we denote $\mathcal{P}_\kappa := \mathcal{P} \cap \text{Pr}^L_{\kappa}$.

Though $\text{Pr}^L$ is not presentable, $\text{Pr}^L_{\kappa}$ is presentable for all $\kappa$ and

$$\text{colim}_\kappa \text{Pr}^L_{\kappa} = \text{Pr}^L$$

where the colimit is taken in the category of huge categories. Furthermore, by [Lur11, 5.3.2.9] the natural maps

$$\text{Pr}^L_{\kappa} \to \text{Pr}^L$$

commutes with colimits for all $\kappa$.

Since $\kappa$-compact objects in a presentable category are closed under $\kappa$-small colimits we have a functor $(-)^{\kappa} : \text{Pr}^L_{\kappa} \to \text{Cat}_{\infty}^{\text{Rex}(\kappa)}$ which sends a $\kappa$-accessible presentable category to its $\kappa$-compact objects. For $\kappa > \omega$ this functor is an equivalence and its left adjoint is given by freely adding $\kappa$-filtered colimits [Lur09, 5.5.7.10].

We can summarize the above discussion as follows. There exists a commutative diagram

$$
\begin{array}{ccc}
\text{Pr}^L_{\pi} & \xrightarrow{(-)^{\pi}} & \text{Cat}_{\infty}^{\text{Rex}(\pi)} \\
\downarrow \text{id} & & \uparrow \text{Ind}^\kappa_{\pi} \\
\text{Pr}^L_{\kappa} & \xrightarrow{(-)^{\kappa}} & \text{Cat}_{\infty}^{\text{Rex}(\kappa)}
\end{array}
$$

where the horizontal maps are equivalences and the vertical maps are left adjoints. Note that the right adjoint of $\text{id}$ is $\text{Ind}^{\kappa}(\text{Ind}^{\kappa})$ and that the right adjoint of $\text{Ind}^\kappa_{\pi}$ is the forgetful. In particular we get:

**Lemma 7.12.** Let $I$ be a small category and $p : I \to \text{Pr}^L_{\kappa}$.

Then

$$\text{lim} \ p = \text{Ind}^{\kappa}(\text{lim} p(i)^{\kappa})$$

where the limit on the left hand side is taken in $\text{Pr}^L_{\kappa}$, and the limit on the right hand side is taken in $\text{Cat}_{\infty}$.

Before proving our main theorem we need to understand how compact objects behave with respect to the tensor product in $\text{Pr}^L$.
Lemma 7.13. For $\mathcal{C}, \mathcal{D} \in \Pr^L_\kappa$ we have

$$(\mathcal{C} \otimes \mathcal{D})^\kappa \simeq \mathcal{C}^\kappa \otimes^\kappa \mathcal{D}^\kappa$$

where $\otimes^\kappa$ is the tensor product of the symmetric monoidal structure on $\text{Cat}^{\text{Rex}(\kappa)}_\infty$ from [Lur11, 4.8.1.4].

Proof. By our assumption on $\mathcal{C}$ and $\mathcal{D}$ we may assume that $\mathcal{D} = \text{Ind}^\kappa(\mathcal{D}^\kappa)$, $\mathcal{C} = \text{Ind}^\kappa(\mathcal{C}^\kappa)$. From the universal property of the tensor product in $\Pr^L$, we have a natural equivalence

$$\text{Fun}^L(\text{Ind}^\kappa(\mathcal{D}^\kappa) \otimes \text{Ind}^\kappa(\mathcal{C}^\kappa), T) \simeq \text{Fun}^{L,L}(\text{Ind}^\kappa(\mathcal{D}^\kappa) \times \text{Ind}^\kappa(\mathcal{C}^\kappa), T)$$

for all $T \in \Pr^L$, where $\text{Fun}^{L,L}((-,),(-))$ is the subcategory of $\text{Fun}((-,),(-))$ spanned by functors that commute with colimits in each variable. Using the universal property of $\text{Ind}^\kappa$ twice we get that

$$\text{Fun}^{L,L}(\text{Ind}^\kappa(\mathcal{D}^\kappa) \times \text{Ind}^\kappa(\mathcal{C}^\kappa), T) \simeq \text{Fun}^\kappa(\mathcal{D}^\kappa \times \mathcal{C}^\kappa, T)$$

where $\text{Fun}^\kappa((-),(-))$ is the subcategory of $\text{Fun}((-),(-))$ spanned by functors that commute with $\kappa$-small colimits in each variable. But, from the universal property of the tensor product in $\text{Cat}^{\text{Rex}(\kappa)}_\infty$ we get that

$$\text{Fun}^\kappa(\mathcal{D}^\kappa \times \mathcal{C}^\kappa, T) \simeq \text{Fun}^\kappa(\mathcal{D}^\kappa \otimes^\kappa \mathcal{C}^\kappa, T).$$

Finally using the universal property of $\text{Ind}^\kappa$ again we get

$$\text{Fun}^\kappa(\mathcal{D}^\kappa \otimes^\kappa \mathcal{C}^\kappa, T) \simeq \text{Fun}^L(\text{Ind}^\kappa(\mathcal{D}^\kappa \otimes^\kappa \mathcal{C}^\kappa), T)$$

as desired. \qed

We will also need the following claim.

Lemma 7.14. $\Pr^L$ is generated under colimits by $S^{\Delta^1}$.

Proof. Since $S$ is a retract of $S^{\Delta^1}$ its suffice to show that $\Pr^L$ is generated under colimits by $S^{\Delta^1}$ and $S$. Let $C \in \Pr^L$. Then there exists a cardinal $\kappa$ such that $C \in \Pr^L_\kappa$. One easily sees that

$$\text{Map}_{\text{Cat}^{\text{Rex}(\kappa)}_\infty}(S^\kappa, -) \text{ and } \text{Map}_{\text{Cat}^{\text{Rex}(\kappa)}_\infty}((S^{\Delta^1})^\kappa, -)$$

are jointly conservative. Therefore by [Yan22, 2.5] we can write $C$ as an iterative colimit of $S$ and $S^{\Delta^1}$ in $\Pr^L_\kappa$. Since the natural map $\Pr^L_\kappa \to \Pr^L$ commutes with colimits, the lemma follows. \qed

We now prove the main theorem of this subsection

Theorem 7.15. For $\iota : \mathcal{P} \hookrightarrow \Pr^L$ a full-subcategory the following are equivalent:

1. $\iota$ admits a left adjoint $L$ which is a smashing localization.
2. (a) $\mathcal{P}$ is closed under colimits in $\Pr^L$. 

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(b) If $D \in \mathcal{P}$, then $D^{\Delta^1} := \text{Fun}(\Delta^1, D) \in \mathcal{P}$.

(c) There exists a regular cardinal $\kappa$ such that for all $\kappa \leq \pi$ if $p : I \to \mathcal{P}_\pi$, then for all $\kappa \leq \mu \leq \pi$ $\text{Ind}^\mu(\lim p(i)^\pi) \in \mathcal{P}$.

**Proof.** $(2) \implies (1)$: Assume that $\mathcal{P}$ satisfies the conditions in 2. We first construct a left adjoint for $\iota$. Let $\mu \geq \kappa$. By condition (c) and Lemma 7.12 the embedding $\mathcal{P}_\mu \hookrightarrow \text{Pr}_\mu^L$ commutes with limits. Furthermore, since $\text{Pr}_\mu^L$ is closed under colimits in $\text{Pr}_\mu^L$, $\iota_\mu$ also commutes with colimits. Thus, since $\text{Pr}_\mu^L$ is presentable, by Theorem 6.2 $\iota_\mu$ has a left adjoint $L_\mu$.

We will show that the different $L_\mu$ assemble to a natural transformation between the functors $\text{Pr}_\pi^L \times \text{Crd}_{\geq \kappa} \to \text{Pr}_\pi^R$, $\text{Pr}_\pi^L(\pi \geq \mu) : \text{Pr}_\pi^L \to \text{Ind}^{\pi(\text{Fun}(\Delta^1, \text{C})^\text{op})}_\pi \to \mathcal{P}_\pi^L$, $\mathcal{P}_\pi(\pi \geq \mu) : \mathcal{P}_\pi^\text{Ind}(\text{Fun}(\Delta^1, \text{C})^\text{op})_\pi \to \mathcal{P}_\pi$ where $\text{Crd}$ is the poset of small cardinals and $\text{Pr}_\pi^R$ is the category of presentable categories and accessible right adjoints. Moving to left adjoints we conclude that the $L_\mu$-s assemble to a natural transformation between the functors $\text{Pr}_\pi^L \times \text{Crd}_{\geq \kappa} \to \text{Pr}_\pi^L$ where the maps $\text{Pr}_\mu^L \to \text{Pr}_\pi^L$ and $\mathcal{P}_\mu \to \mathcal{P}_\pi$ are $\overline{i}_\mu^\pi$ and $i_\mu^\pi$ respectively. Taking a colimit and using the description of mapping spaces in a filtered colimit of categories given by [Roz12, 0.2.1], we get that the embedding $\mathcal{P} \to \text{Pr}_\pi^L$ has a left adjoint $L := \text{colim} L_\mu$.

We now show that $\mathcal{P}$ is an ideal in $\text{Pr}_\mu^L$ i.e. if $D \in \mathcal{P}$ and $\mathcal{C} \in \text{Pr}_\mu^L$ and let $A$ be the full-subcategory of $\text{Pr}_\mu^L$ spanned by the categories $\mathcal{C}$ for which $\mathcal{C} \otimes D \in \mathcal{P}$. Recall that the tensor product in $\text{Pr}_\mu^L$ commutes with colimits in each variable, and so $A$ is closed under colimits. Furthermore, by (b):

$$\mathcal{S}^{\Delta^1} \otimes \mathcal{C} \simeq \text{Fun}^R((\mathcal{S}^{\Delta^1})^\text{op}, \mathcal{C}) \simeq \text{Fun}^L((\mathcal{S}^{\Delta^1})^{\text{op}})^\text{op} \simeq \text{Fun}(\mathcal{S}^{\Delta^1}, \mathcal{C}) = \mathcal{C}^{\Delta^1} \in \mathcal{P} \implies \mathcal{S}^{\Delta^1} \in A.$$  

Hence, the claim follows from lemma Lemma 7.14.

We finally show that $L$ is smashing using the above. By the previous paragraph it suffices to show that for all $\mathcal{T} \in \mathcal{P}$ the map, coming from pre-composing with the unit of the adjunction,

$$\text{Fun}^L(X \otimes \mathcal{S}, \mathcal{T}) \to \text{Fun}^L(X, \mathcal{T})$$

is an equivalence. Note that $\text{Fun}^L(X \otimes \mathcal{S}, \mathcal{T}) \simeq \text{Fun}^L(X, \text{Fun}^L(\mathcal{L}_\mathcal{S}, \mathcal{T}))$ and so it suffice to prove the above for the case $X = \mathcal{S}$ i.e. to prove that the map $\text{Fun}^L(\mathcal{L}_\mathcal{S}, \mathcal{T}) \to \text{Fun}^L(\mathcal{S}, \mathcal{T})$ is an equiva-
lence. Since $L$ is a left adjoint this map is an equivalence on the space of objects and since $D^{\Delta^1} \in \mathcal{P}$ it is also an equivalence on the space of arrows. It follows that it is an equivalence.

(1) $\implies$ (2): Assume the inclusion $\mathcal{P} \hookrightarrow \text{Pr}^L$ admits a left adjoint, $L$, which is a smashing localization. We immediately get that $\mathcal{P}$ is closed under colimits in $\text{Pr}^L$. Furthermore $L$ is obviously compatible with the symmetric monoidal structure, hence by Lemma 7.4 if $D \in \mathcal{P}$ then

$$\text{Fun}^L(\text{Fun}((\Delta^1)^{op}, S), D) \simeq \text{Fun}(\Delta^1, D) \in \mathcal{P}.$$ 

We now show that there exists a regular cardinal $\kappa > \omega$ such that for all $\kappa \leq \mu \leq \pi$ and $C \in \mathcal{P}_\pi$ we have that $\text{Ind}^\mu(C^\pi) \in \mathcal{P}$. Let $\kappa$ be a regular cardinal such that $LS$ is $\kappa$-compactly generated and the unit of the adjunction $\eta_S : S \to LS$ sends $\kappa$-compact objects to $\kappa$-compact objects. By lemma 7.13 we get that $LS^\mu$ is an idempotent algebra in $\text{Cat}^{\text{Rex}(\mu)}$ for all $\mu \geq \kappa$. Therefore, the inclusion $\iota_\mu : \mathcal{P}_\mu \hookrightarrow \text{Pr}^L_\mu$ has a left adjoint, $L_\mu$, given by tensoring with $LS$. We conclude that we have a commutative diagram

$$\begin{array}{ccc}
\text{Pr}^L_\mu & \xrightarrow{p} & \text{Pr}^L_\pi \\
L_\mu \downarrow & & \downarrow L_\pi \\
\mathcal{P}_\mu & \xrightarrow{i_\mu} & \mathcal{P}_\pi
\end{array}$$

where all the maps are left adjoints. By passing to right adjoints we get that the right adjoint of $i_\mu^\pi$ is given by $\text{Ind}^\mu((\cdot)^\pi)$ and the claim follows.

Let $\pi \geq \mu \geq \kappa$ be cardinals and $p : I \to \mathcal{P}_\pi$ be a functor, we need to show that $\text{Ind}^\mu(\lim p(i)^\pi) \in \mathcal{P}$. By the previous paragraph the induced diagram

$$p' : I \to \text{Pr}^L_\mu, \quad p'(i) = \text{Ind}^\mu(p(i)^\pi)$$

lands in $\mathcal{P}_\mu$. Furthermore, in the previous paragraph we also saw that the inclusion $\mathcal{P}_\mu \to \text{Pr}^L_\mu$ admits a left adjoint. We conclude that $\mathcal{P}_\mu$ is closed under limits in $\text{Pr}^L_\mu$. Thus, by Lemma 7.12 we get

$$\lim p' \simeq \text{Ind}^\mu(\lim p'(i)^\mu) = \text{Ind}^\mu(\lim(\text{Ind}^\mu(p(i)^\pi))^\mu) \simeq \text{Ind}^\mu(\lim p(i)^\pi) \in \mathcal{P}$$

as desired.

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