Non-linear resonance in relativistic preheating

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Abstract. Inflation in the early Universe can be followed by a brief period of preheating, resulting in rapid and non-equilibrium particle production through the dynamics of parametric resonance. However, the parametric resonance effect is very sensitive to the linearity of the reheating sector. Additional self-interactions in the reheating sector, such as non-canonical kinetic terms like the DBI Lagrangian, may enhance or frustrate the parametric resonance effect of preheating. In the case of a DBI reheating sector, preheating is described by parametric resonance of a damped relativistic harmonic oscillator. In this paper, we illustrate how the non-linear terms in the relativistic oscillator shut down the parametric resonance effect. This limits the effectiveness of preheating when there are non-linear self-interactions.

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1 Introduction

The energy that drives inflation in the very early Universe is trapped in the homogeneous inflaton scalar field $\phi$. As inflation ends and the inflaton field settles down into its minimum, this energy is transferred to observable matter and radiation through an epoch of reheating. When the inflaton couples to another scalar field $\chi$, called the “reheaton” field, the dynamics often contain a period of explosive particle production preceding the usual perturbative reheating called preheating (see [1–3] for an introduction to preheating).

For example, the simple Lagrangian

$$L = \frac{1}{2}(\partial \phi)^2 - V(\phi) + \frac{1}{2}(\partial \chi)^2 - \frac{1}{2}g^2 \phi^2 \chi^2 - \frac{1}{2}m_\chi^2 \chi^2,$$

(1.1)

describes the interaction between the inflaton and reheating fields. In an expanding Universe the equation of motion for large scale modes of $\chi$ becomes that of a damped harmonic oscillator with a time-dependent frequency:

$$\ddot{\chi}(t) + 3H \dot{\chi}(t) + \left( m_\chi^2 + g^2 \phi_0(t)^2 \right) \chi(t) = 0,$$

(1.2)

where $\phi_0(t)$ is the time-dependent background solution for the inflaton field. At the end of inflation, the inflaton oscillates about its potential minimum, resulting in an oscillating frequency for the reheaton field in (1.2). A harmonic oscillator with a time-dependent frequency as in (1.2) can exhibit explosive growth in the amplitude of $\chi$ through the non-linear phenomenon of parametric resonance (see [4, 5] for an introduction to parametric resonance).

The interaction described in (1.1)–(1.2) is a simple toy model for the physics of reheating and preheating — in practice, the inflaton and reheaton sectors could be considerably more complicated. In particular, additional interactions between and within the inflaton and reheaton sectors in (1.1)–(1.2) may enhance, or frustrate, the parametric resonance effect of preheating.

One type of self-interactions, known as non-canonical kinetic terms, arise naturally in low-energy effective field theories [6–9]:

$$L = p(X_\phi, \phi) + \tilde{p}(X_\chi, \chi) - \frac{1}{2}g^2 \phi^2 \chi^2,$$

(1.3)
where $X_\phi \equiv -\frac{1}{2} (\partial \phi)^2$, $X_\chi \equiv -\frac{1}{2} (\partial \chi)^2$. Note that we have left the interaction between the inflaton and reheaton sectors unchanged; some previous work which has explored modified couplings between these sectors can be found in [10].

There has been considerable interest in the use of non-canonical kinetic terms such as those in (1.3) to describe alternative models of inflation [6, 11, 12]. Non-canonical kinetic terms are interesting because perturbations about a cosmological background can have a sound speed $c_s^2$ different than 1, and can have interesting observational signatures such as violation of the single-field consistency relation and enhanced non-gaussianity [11–14]. Recent cosmological data suggests that the sound speed of the inflaton during inflation cannot be too small, $c_s^2 \geq 0.02$ [15–19]. In this paper, we are interested in investigating the role that non-canonical kinetic terms in the reheating sector alone can play in the dynamics of preheating, so we will consider the inflaton-reheaton Lagrangian to consist of a canonical inflaton field coupled to a non-canonical reheaton field:

$$L = \frac{1}{2} (\partial \phi)^2 - V(\phi) + \tilde{p}(X_\chi, \chi) - \frac{1}{2} g^2 \phi^2 \chi^2. \quad (1.4)$$

We have chosen to couple the inflaton and reheaton sectors with a simple quartic cross-coupling, which has the advantage that it does not lead to tachyonic effective masses for $\chi$. In principle, other cross-couplings between the sectors such as $\lambda \phi \chi$ could be considered, as well as higher-order powers of $\phi$ and $\chi$ such as $L_{\text{int}} \sim -g_{(n,m)} \phi^n \chi^m / \Lambda^{4-n-m}$. A general analysis of all such possible terms involves many non-linear terms of different types, and as such is beyond the scope of this initial analysis.

As discussed in [7, 8], non-canonical kinetic terms of the form $\tilde{p}(X_\chi, \chi)$ can arise as an effective field theory below some scale of new physics $\Lambda$. Integrating out the physics above $\Lambda$ leads to an effective field theory in terms of some non-renormalizable operators

$$\tilde{p}(X_\chi, \chi) = \frac{1}{2} (\partial \chi)^2 - \frac{1}{2} m_\chi^2 \chi^2 + \sum_{n>4} c_n \frac{\mathcal{O}_n}{\Lambda^{n-4}}. \quad (1.5)$$

To be more precise, we will consider $\tilde{p}(X_\chi, \chi)$ to be of the Dirac-Born-Infeld (DBI) form [11, 12]:

$$\tilde{p}(X_\chi, \chi) = -\Lambda^4 \left[ \sqrt{1 - 2X_\chi / \Lambda^4} - 1 \right] - \frac{1}{2} m_\chi^2 \chi^2, \quad (1.6)$$

where we will take the “warp factor” $\Lambda^4$ to be a constant for simplicity. This Lagrangian can arise as a low-energy effective field theory for the motion of a D-brane in warped extra dimensions in string theory.

The equation of motion for long-wavelength modes of $\chi$ in an expanding background arising from (1.4), (1.6) is then:

$$\frac{\ddot{\chi}}{(1 - \frac{\dot{\chi}^2}{\Lambda^4})^{3/2}} + 3H \dot{\chi} + (m_\chi^2 + g^2 \phi(t)^2) \chi = 0. \quad (1.7)$$

Now the reason for choosing the specific DBI Lagrangian is clear — this equation of motion has the form of a damped relativistic harmonic oscillator with a time-dependent effective frequency, so we can apply much of our intuition from that simple model to understand preheating with non-canonical kinetic terms. This is no accident — the DBI action (1.6)
is a generalization of the relativistic point particle action to an spacetime filling dynamical membrane, where the field $\chi$ denotes the embedding of the membrane in an extra dimension. We will take (1.7) as our starting point to understand how non-linearities introduced by non-canonical kinetic terms affect the dynamics of preheating. As will be discussed later, a more general Lagrangian with non-linear terms of the form $\mathcal{L}_{\text{non-linear}} \sim c_8 \mathcal{O}(\chi^4)/\Lambda^4$ will exhibit similar behavior in preheating, so our conclusions are not just restricted to the DBI/relativistic oscillator case.

The role of the DBI Lagrangian (1.6) in preheating has been considered before. In [20–22], the leading order effects of an inflaton field with a DBI kinetic term for preheating were studied at low speeds. Ref [9], in contrast, considered the effects of a DBI inflaton on preheating at high speeds. More recently, [23] studied numerical lattice simulations of preheating for an inflaton with a DBI kinetic term. All of these previous works consider the non-canonical kinetic term to be in the inflaton sector; however, as discussed above, it is also possible for the reheaton sector to have non-canonical kinetic terms.\footnote{See also [24], which studied the effects on preheating of non-linear terms in the reheaton sector arising from a Generalized Uncertainty Principle.}

Preheating is a complex phenomenon, involving not just parametric resonance but also rescattering and backreaction effects, and inhomogeneous evolution. We do not intend to include all of these effects here; that approach is best suited to complex numerical simulations, such as [23, 25, 26]. Instead, our goal is to attempt to gain some analytic and conceptual understanding of how the non-linearities inherent in (1.7) affect the onset of parametric resonance for preheating. As we will see below, analytic techniques will help us to generate some insight into solutions of (1.7), and shed light on the effect of non-linear terms on parametric resonance.

The rest of the paper is organized as follows. In section 2, we describe the equation of motion for long-wavelength modes of a DBI reheaton field, and cast it as a relativistic oscillator with a time-dependent frequency. In section 2.1, we review some of the properties of the relativistic oscillator, including its dependence of oscillation frequency with amplitude, and its behavior when subject to a resonant driving force. In section 2.2, we analyze parametric resonance of the relativistic oscillator, making use of the technique of multiple scales. We find there that in the traditional “resonance band” of the non-relativistic oscillator where solutions grow exponentially, solutions to the relativistic oscillator do not exhibit parametric resonance and their amplitudes are bounded. In section 2.3, we perform a similar analysis for the damped relativistic oscillator, finding similar behavior. In section 3 we discuss the implications of our analysis for preheating in the early Universe. We conclude in section 4 with some final remarks.

2 The relativistic oscillator

Our starting point is the Lagrangian (1.4), (1.6) in a flat FRW spacetime

$$ds^2 = -dt^2 + a(t)^2 dx^2,$$

which gives rise to the long-wavelength equations of motion:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3M_p^2} \left[\frac{1}{2}\dot{\phi}_0^2 + V(\phi_0) + \frac{1}{2}g^2\phi_0^2\chi^2 + \Lambda^4\left(\frac{1}{\sqrt{1 - \chi^2/\Lambda^4}} - 1\right) + \frac{1}{2}m_\chi^2\chi^2\right];$$

(2.2)
\begin{align}
\ddot{\phi}_0 + 3H\dot{\phi}_0 + V'(\phi_0) &= 0; \\
\frac{\ddot{\chi}}{(1 - \chi^2/\Lambda^4)^{3/2}} + 3H\dot{\chi} + (m_\chi^2 + g^2\phi_0(t)^2) \chi &= 0.
\end{align}

For simplicity, we will take the inflaton to have the potential \( V(\phi) = \frac{1}{2}m_\phi^2\phi^2 \), arising for example from a quadratic expansion of the inflaton potential about the inflaton’s minimum. For this choice of potential, at the end of inflation the inflaton oscillates about its minimum as a function of time as \[ \phi_0(t) = \Phi(t) \sin(m_\phi t) = \Phi_0 \sin(m_\phi t). \] Inserting (2.5) into (2.4), we obtain the equation of motion for the long-wavelength reheaton field during preheating:

\begin{equation}
\frac{\ddot{\chi}}{(1 - \chi^2/\Lambda^4)^{3/2}} + 3H\dot{\chi} + (m_\chi^2 + g^2\Phi(t) \sin^2(m_\phi t)) \chi = 0.
\end{equation}

By making the redefinition \( \tau \equiv m_\phi t \), (2.6) can be written as:

\begin{equation}
\frac{\chi''(\tau)}{(1 - \chi^2/\Lambda^4)^{3/2}} + \mu \chi'(\tau) + (A - 2q \cos(2\tau))\chi(\tau) = 0,
\end{equation}

where \( \tau' = \frac{d\tau}{d\chi} \), \( v \equiv \Lambda^2/m_\phi \) is the “speed limit” of \( \chi(\tau) \), \( \mu \equiv 3H/m_\phi \) is a damping coefficient, and \( A, q \) are defined as

\begin{equation}
A \equiv m_\chi^2 + \frac{1}{2}g^2\Phi^2, \quad q \equiv \frac{g^2\Phi^2}{4m_\phi^2}.
\end{equation}

Since the inflaton amplitude \( \Phi(t) \) is decaying with time, in principle the parameters \( A, q, \mu \) are also time-dependent. However, for our analysis in section 2 we will assume that these parameters are approximately constant so that their timescale is much longer than the timescale for preheating. In section 3 we will allow \( A, q, \mu \) to be time-dependent, and find that this does not affect our conclusions.

Equation (2.7) is the relativistic generalization of the damped Mathieu equation. At small speeds \( |\chi'| \ll v \), (2.7) reduces to the usual Mathieu equation. It is well known that the non-relativistic Mathieu equation exhibits a phenomenon known as parametric resonance — for certain values of \( q \), \( A \) the amplitude of \( \chi(\tau) \) grows exponentially with time [4, 5]. However, at high speeds the non-linearity inherent in (2.7) can be important. In the rest of this section, we will focus on how the non-linearity of (2.7) affects the explosive growth of parametric resonance. In the next section, we will apply these results to preheating.

### 2.1 Properties of the relativistic oscillator

To begin, let us review the behavior of the relativistic harmonic oscillator in general, taking \( \mu = 0 = q \):

\begin{equation}
\frac{\chi''(\tau)}{(1 - \chi^2/v^2)^{3/2}} + A\chi(\tau) = 0.
\end{equation}

It is possible to solve (2.9) in quadrature to obtain \( \tau = \tau(\chi) \) in terms of elliptic integrals [27, 28] (see also [29–31] for more studies of the relativistic oscillator). The details of these
Figure 1. The period of the relativistic harmonic oscillator (2.9) depends on the amplitude, in contrast to its non-relativistic counterpart. This amplitude dependence arises because of the speed limit of the relativistic oscillator — any increase in amplitude must be accompanied by a decrease in frequency. This is shown in plots of $\chi$ versus $\tau$ (left) and $\chi'$ versus $\tau$ (right), where we have set the relativistic speed $v = 1$.

solutions will not be important to us here. However, there is one important feature to solutions of (2.9) — unlike the non-relativistic harmonic oscillator, the frequency of oscillation depends on the amplitude of the oscillation. The reason for this is quite simple — as the amplitude increases, so does the speed $\chi'$ of the oscillator. However, because of the speed limit imposed by (2.9) $|\chi'| < v$, when the oscillator reaches relativistic speeds any further increase in amplitude must be accompanied by a corresponding decrease in the frequency of oscillation so that the speed $|\chi'|$ does not exceed the speed limit. See figure 1 for a graphical illustration of this effect. As we will discuss in more detail below, this has important consequences for the existence of resonance of the relativistic oscillator.

Including damping, (2.7) becomes the damped relativistic harmonic oscillator,

$$\frac{\chi''(\tau)}{(1 - \chi'^2/v^2)^{3/2}} + \mu \chi' + A\chi(\tau) = 0.$$  (2.10)

Solutions to (2.10) behave much as in the non-relativistic case — for small values of $\mu$, $\chi$ oscillates with a decaying amplitude (underdamped), while for large values of $\mu$, $\chi$ decays asymptotically to zero (overdamped). The primary difference between solutions of the relativistic and non-relativistic damped oscillators is that the rate of decay is slower for the former, as shown in figure 2. The physical reason is straightforward to see — since $\chi(\tau)$ is speed limited $|\chi'| < v$, at high speeds the relativistic oscillator loses less energy to the velocity-dependent dissipation than its non-relativistic counterpart.

Before studying the relativistic Mathieu equation (2.7) in more detail, let us consider one more similar system, the forced relativistic harmonic oscillator:

$$\frac{\chi''(\tau)}{(1 - \chi'^2/v^2)^{3/2}} + A\chi(\tau) = F_0 \sin(\Omega t).$$  (2.11)

As a reminder, in the forced non-relativistic harmonic oscillator, when the driving frequency is equal to the natural frequency of the oscillator ($\Omega = A^{1/2}$), the system resonates and the amplitude of $\chi(\tau)$ grows linearly with time. The behavior of (2.11) is much more complex, however. For small initial amplitudes and speeds, where (2.11) behaves effectively like a
Figure 2. The relativistic damped oscillator (2.10) (blue, darker) \((v = 1, \mu = .1, A = 1)\) loses energy less rapidly at large amplitudes than its non-relativistic counterpart (red, lighter) because the non-relativistic oscillator is able to reach higher speeds, and thus have higher damping energy losses.

2.2 Relativistic parametric resonance

Finally, we are ready to analyze the relativistic Mathieu equation (2.7). For simplicity, let us ignore damping for now and set \(\mu = 0\) — we will return to the effects of damping in the next subsection. In general, solutions to (2.7) for arbitrary values of the parameters can only be obtained numerically, and due to the non-linear nature of (2.7) their behavior can be difficult to understand. Before we study these solutions, then, let us comment on some expected qualitative features of solutions of (2.7).

The non-relativistic Mathieu equation has solutions which exhibit resonance (and thus unrestricted growth of the amplitude of \(\chi\)) for small \(q\) when the average natural frequency \(A\) of the oscillator is approximately an integer \(A \approx 1, 2, \ldots n\). From our knowledge of the driven relativistic oscillator discussed above, we know that while resonance may initially occur for (2.7) when the initial amplitude is small, it will soon stop when the relativistic corrections to the frequency cause the oscillations to be out of phase with the oscillations of the effective frequency \(\omega_{\text{eff}}^2 = A - 2q \cos 2\tau\).

Another perspective can be obtained by rearranging (2.7) in the absence of damping as

\[
\chi''(\tau) + (A - 2q \cos 2\tau) \left(1 - \frac{\chi^2}{v^2}\right)^{3/2} \chi(\tau) = 0 .
\]
Figure 3. The forced relativistic oscillator (blue, solid) \((v = 948, F_0 = 200 \, A)\) and forced non-relativistic oscillator (red, dashed) both initially grow in amplitude when their natural frequency is equal to the driving frequency. However, once the amplitude is sufficiently large, the natural frequency of the relativistic oscillator begins to decrease as in figure 1, causing it to fall out of resonance. After the amplitude of the relativistic oscillator has decreased sufficiently, the oscillator’s natural frequency again matches the driving frequency, causing growth again. This process repeats, resulting in a “beating” like pattern of oscillation.

As discussed before, as the amplitude of \(\chi\) increases, so does the speed \(|\chi’|\) until the speed limit is reached. However, as the speed limit is approached the coefficient of the new effective frequency in (2.12), namely \((A - 2q \cos 2\tau) \left(1 - \frac{\chi’^2}{v^2}\right)^{3/2}\), vanishes so that at large speeds the oscillator becomes a nearly free particle without resonance. Thus, on basic principles we expect that while the non-relativistic Mathieu equation exhibits exponential growth and resonance for certain values of \(A,q\), the relativistic Mathieu equation (2.7) or (2.12) may contain some initial growth, but should eventually saturate at some maximum amplitude.

Let us now check this expectation against solutions of (2.12). As mentioned before, it is difficult to interpret generic numerical solutions of (2.12) for arbitrary values of the parameters. However, some analytic progress can be made perturbatively when we set \(q = \epsilon \hat{q}, \, \nu^2 = \hat{\nu}^2 \epsilon^{-1}\), where \(\hat{q}, \hat{\nu} \sim O(1)\) and \(\epsilon \ll 1\) is a small parameter, so that \(q \ll 1\) and \(\nu^2 \gg 1\). It is then possible to expand (2.12) to leading order in \(\epsilon\) as

\[
\chi'' + (A - 2\hat{q}\epsilon \cos 2\tau)\chi = \frac{3}{2}\epsilon A \frac{Y^2}{\nu^2} \chi + O(\epsilon^2).
\]  

While (2.13) seems like a significant simplification of (2.12), as we will discuss solutions of (2.13) contain all of the same qualitative features as numerical solutions to (2.12).

Solutions to (2.13) can be found perturbatively\(^2\) using the method of multiple scales (see [5]). In particular, to leading order the solution to (2.13) is

\[
\chi(\tau) = a(\epsilon \tau) \cos \left(A^{1/2}\tau + \beta(\epsilon \tau) \right) + O(\epsilon),
\]  

\(^2\) See also [24], which studied parametric resonance of a similar equation using different techniques.
Figure 4. Solutions to the relativistic Mathieu equation (2.12) through (Left) the approximation of the method of multiple scales (2.15), and (Right) the exact numerical solution, with \( \hat{q} = 1.2, \hat{v} = 4, \hat{\sigma} = 0.3, \) and \( \epsilon = 0.03. \) Notice that the exact and approximate solutions have the same qualitative beating pattern; however, the amplitude and frequency of the beats are slightly different.

where the amplitude and phase \( a, \beta \) are functions of the “slow time” \( T_1 = \epsilon \tau. \) In the vicinity of the first resonance band \( A \sim 1 \) of the non-relativistic Mathieu equation, the amplitude and phase are determined by the equations:

\[
\begin{align*}
\frac{da}{dT_1} &= \frac{\hat{q}}{2A^{1/2}} a \sin \psi; \\
\frac{d\psi}{dT_1} &= 2\sigma + \frac{\hat{q}}{A^{1/2}} \cos \psi - \frac{3}{2} \frac{Aa^2}{\hat{v}^2},
\end{align*}
\]

where we made the redefinition \( \psi = 2\sigma \epsilon t - 2\beta \) and we introduced a “de-tuning parameter” \( \sigma \) defined as \( A^2 = (1 - \epsilon \sigma)^2 \) with \( \sigma \sim O(1). \) Solutions to (2.13) through the method of multiple scales (2.15) give good qualitative agreement with solutions to the exact relativistic Mathieu equation (2.12), as can be seen in figure 4. We see there a characteristic beating pattern, similar to that found in the forced relativistic oscillator; however, the amplitude and frequency of the beating is slightly different between the analytic method of multiple scales and the numerical solution. For our purposes in this paper, this qualitative agreement is sufficient to illustrate the absence of unbounded resonance when the non-linear terms of (2.12) are included. Solutions to (2.13) and (2.12) do agree at early times, as seen in figure 5; for smaller values of \( \epsilon, \) the solutions can be made to agree for longer periods of time.

Since we are ultimately interested in the growth of particles by parametric resonance, we should also examine the behavior of the number density of particles \( n_\chi. \) The number density is defined to be the energy density of a mode \( \rho_\chi \) divided by the energy of that mode \( \omega; \) for our relativistic oscillator, the number density is thus proportional to

\[
\begin{align*}
n_\chi &= \frac{v^2}{\omega(t)} \left( \frac{1}{\sqrt{1 - \chi^2/v^2}} - 1 \right) + \frac{1}{2} \omega(t) \chi^2
\end{align*}
\]

where \( \omega(t)^2 \equiv A - 2q \cos 2\tau \) is the energy of a mode. The number density for the exact numerical solution to (2.12) shown in figure 4 is shown in figure 6. We see clearly that the number density tracks the amplitude of \( \chi, \) as expected. Note that in some models of preheating the amplitude of the reheaton field does not necessarily indicate growth of particle
Figure 5. Solutions to the relativistic Mathieu equation (2.12) through (blue, thick) the approximation of the method of multiple scales (2.15), and (red, thin) the exact numerical solution, with the same parameters as in the previous figure. The two solutions agree up to some time, after which they begin to diverge. Nevertheless, as we see in the previous figure, the two solutions have the same qualitative behavior. The time over which they agree can be made longer by making $\epsilon$ smaller.

Figure 6. The number density $n_k$ for the relativistic oscillator for the exact solution shown in figure 4 shows that the number density grows and falls with the amplitude $\chi$ of the reheaton field. However, this failure only occurs when the amplitude of the time-dependent frequency $\omega(t)$ is changing in time, overwhelming the time-dependence of the amplitude $\chi(t)$. For example, for the model shown in [3], after the end of preheating the amplitude $\chi(t)$ increases slightly while the amplitude of the frequency $\omega(t)$ also decreases slightly; the two effects balance each other out, resulting in no particle production. This is clearly not the case here — the amplitude of $\omega(t)$ in this section is fixed, as we are considering the quantities $A, q$ to be fixed in time. As we will show in the next section, even when we allow time-dependence for the background values of the inflaton (so that the effective $A, q$ become time-dependent), the parametric growth of the energy density of the reheaton field is still suppressed when the non-linear terms become important, so this particular caveat does not concern us here.

As a basis for comparison, let’s investigate the nature of solutions in the absence of the non-linear term from the relativistic correction, $\hat{v} \rightarrow \infty$. The equation of motion (2.13) then
becomes the well-known non-relativistic Mathieu equation:

$$\chi'' + (A - 2\hat{q}\epsilon \cos 2\tau)\chi = 0. \quad (2.17)$$

The equations governing the amplitude and phase of (2.14) are then a special case of (2.15):

$$\frac{da}{dT_1} = \frac{\hat{q}}{2A^{1/2}} a \sin \psi; \quad (2.18)$$

$$\frac{d\psi}{dT_1} = 2\sigma + \frac{\hat{q}}{A^{1/2}} \cos \psi.$$

In this limit, there are two types of solutions: a runaway solution with $\psi \rightarrow |\cos^{-1} \frac{2\sigma}{\hat{q}}|$, $a \rightarrow \infty$, and a decaying solution $\psi \rightarrow |\cos^{-1} \frac{2\sigma}{\hat{q}}|$, $a \rightarrow 0$, representing the two independent growing and decaying solutions of (2.17). A phase space plot of non-relativistic solutions of (2.15) shown in figure 7 demonstrates this behavior. Note that growing mode solutions are only possible for $|\sigma| < \hat{q}/2$, demonstrating the finite width of the resonance band, while for $|\sigma| > \hat{q}/2$ the amplitude $a(T_1)$ is bounded.

Let us now examine the effect of the relativistic correction to (2.15). It is straightforward to see that there are 3 fixed points of (2.15):

(i) Stable fixed point $a_i = \frac{\hat{v}}{A^{1/2}} \sqrt{\frac{2}{3} \left(2\sigma + \frac{\hat{q}}{A^{1/2}}\right)}$, $\psi_i = 2n\pi, n \in \mathbb{N}$.

(ii) Saddle point $a_{ii} = \frac{\hat{v}}{A^{1/2}} \sqrt{\frac{2}{3} \left(2\sigma - \frac{\hat{q}}{A^{1/2}}\right)}$, $\psi_{ii} = (2n + 1)\pi, n \in \mathbb{N}$. 

Figure 7. Phase space plot of the equations (2.18), describing the non-relativistic Mathieu equation (2.17), for oscillation frequencies (Left) $|\sigma| < \hat{q}/2$ inside the resonance band, and (Right) $|\sigma| > \hat{q}/2$ outside the resonance band. Notice the presence of growing mode solutions in the resonance band (left-hand plot), indicating the presence of parametric resonance, while outside the resonance band solutions are bounded. The bold (blue) trajectories are some sample trajectories to illustrate the typical behavior of solutions in phase space.
(iii) Saddle point \( a_{iii} = 0, \psi_{iii} = \cos^{-1} \left( \frac{2\sigma A^{1/2}}{q} \right) \).

Depending on the values of the parameters \( \sigma, \hat{q} \), some or none of these fixed points will be present. The behavior of solutions depends on the existence and types of fixed points present, so we will now discuss this in some detail. When \( 2|\sigma|/\hat{q} < 1 \), the non-relativistic Mathieu equation (2.17) would have exponentially growing solutions. The relativistic Mathieu equation, however, has fixed points of type (i) and (iii) above. Instead of growing indefinitely, small initial amplitudes orbit the stable fixed point (i), as shown in the left-hand plot of figure 8. These trajectories have a similar behavior as with the forced relativistic oscillator — after an initial period of growth, the frequency of oscillation shifts so that \( \chi(\tau) \) and the external force are no longer in phase, limiting the amount of growth. This is the origin of the “beating” pattern seen in the exact and multiple scales solutions shown in figure 4, and has the same physical origin as the beating as seen in the forced relativistic oscillator in figure 3. This qualitative understanding of the origin of the beating is justification for considering the multiple scales approximation to solutions to the full relativistic Mathieu equation (2.12): even though the numerical solutions to (2.15) and (2.12) are not exactly the same, they have the same qualitative behavior, so we expect that solutions to the full relativistic Mathieu equation (2.12) are governed by phase space plots very similar to those in figure 8. We also see from figure 8 that if the amplitude is large enough, solutions change in nature from closed to open trajectories in the \((a, \psi)\) phase plane. Regardless of initial conditions, however, we see that the amplitude is bounded and does not grow indefinitely as in the non-relativistic case.

Continuing to other values of the parameters, when \( \sigma < 0 \) and \( 2|\sigma|/\hat{q} > 1 \) the behavior is similar to the non-relativistic Mathieu equation — there are no fixed points, and solutions are oscillatory with bounded amplitude; see the center plot of figure 8. Finally, for \( \sigma > 0 \) and \( 2\sigma/\hat{q} > 1 \), the non-relativistic Mathieu equation does not have resonance; however, solutions of (2.15) in this parameter range have the same behavior as solutions in the relativistic “resonance band” \( 2|\sigma|/\hat{q} < 1 \), with fixed points of type (i) and (ii). Solutions for \( a, \psi \) form orbits in the neighborhood of the stable fixed point (i), while for large amplitudes trajectories become open with bounded amplitude, as seen in the right-hand plot of figure 8.

To conclude this subsection, let us summarize our findings. Based on qualitative arguments, we expect that solutions of the relativistic Mathieu equation do not exhibit any resonance or unbounded growth, even when the corresponding Mathieu equation does have resonant growth. We have shown this explicitly by reducing the full relativistic Mathieu equation (2.12) to the equations (2.15) for the amplitude and phase of the solution (2.14). The phase diagram of (2.15) for the values of the parameters that would normally give rise to resonance instead contains a stable fixed point at finite amplitude, which nearby solutions orbit, as shown in figure 8. At large amplitudes, the phase diagram of (2.15) resembles that of the non-resonant Mathieu equation — solutions are oscillatory with bounded amplitude.

In the next section, we will discuss how the inclusion of non-zero damping in (2.12) modifies these observations.

### 2.3 Damped relativistic Mathieu equation

Now let us include non-zero damping in the relativistic Mathieu equation,

\[
\chi''(\tau) + \mu \chi' \left( 1 - \frac{\chi'^2}{v^2} \right)^{3/2} + (A - 2q \cos 2\tau) \left( 1 - \frac{\chi'^2}{v^2} \right)^{3/2} \chi(\tau) = 0. \quad (2.19)
\]

We expect similar behavior for the damped Mathieu equation above as from the previous section — as the amplitude and speed increase, the damping and effective frequency vanish...
so that (2.19) becomes the equation for a free particle without resonance. Further, as the amplitude of $\chi$ increases, so does the frequency of oscillation of $\chi$, so that if $\chi$ does experience resonance initially, it soon becomes out of phase with the time-dependent effective frequency and resonance quickly ceases.

We can construct semi-analytic solutions to (2.19) as in the previous subsection: we let $q = \epsilon \hat{q}$, $v^2 = \hat{v}^2 \epsilon^{-1}$, and $\mu = \epsilon \hat{\mu}$, where $\epsilon \ll 1$ and $\hat{q}, \hat{v}, \hat{\mu} \sim O(1)$. Expanding (2.19) to leading order, we have,

$$\chi'' + \epsilon \hat{\mu} \chi' + (A - 2\epsilon \hat{q} \cos 2\tau)\chi = \frac{3}{2} \epsilon A \frac{\chi'^2}{\hat{v}^2} \chi. \quad (2.20)$$

Solutions to (2.20) have the same form as in the undamped case:

$$\chi(\tau) = a(\epsilon \tau) \cos \left( A^{1/2} \tau + \beta(\epsilon \tau) \right) + O(\epsilon). \quad (2.21)$$

In the vicinity of the first resonance band, the amplitude and phase of (2.21) obey the equations:

$$\frac{da}{d\tilde{T}_1} = \hat{q} a \frac{A^{1/2}}{2} \sin \psi - \frac{\hat{\mu}}{2} a;$$

$$\frac{d\psi}{d\tilde{T}_1} = 2\sigma + \hat{q} A^{1/2} \cos \psi - \frac{3}{2} \frac{Aa^2}{\hat{v}^2} \sigma, \quad (2.22)$$

where $\psi, \sigma$ are defined as in the previous subsection.
Figure 9. Stability chart for the damped Mathieu equation. See text below (2.22) for explanation.

The fixed points of (2.22) are now shifted to:

(i) Fixed point

\[ a_i = \frac{\hat{v}}{A^{1/2}} \sqrt{\frac{2}{3} \left( 2\sigma + \frac{\hat{q}}{A^{1/2}} \sqrt{1 - \hat{\mu}^2 A/\hat{q}} \right)} \]

\[ \psi_i = \left| \cos^{-1} \left( \sqrt{1 - \hat{\mu}^2 A/\hat{q}} \right) \right| \]

(ii) Fixed point

\[ a_i = \frac{\hat{v}}{A^{1/2}} \sqrt{\frac{2}{3} \left( 2\sigma - \frac{\hat{q}}{A^{1/2}} \sqrt{1 - \hat{\mu}^2 A/\hat{q}} \right)} \]

\[ \psi_i = -\left| \cos^{-1} \left( \sqrt{1 - \hat{\mu}^2 A/\hat{q}} \right) \right| \]

(iii) Fixed point \( a_{iii} = 0, \psi_{iii} = \cos^{-1} \left( \frac{2\pi A^{1/2}}{q} \right) \).

The pattern of fixed points is much more complicated than in the undamped case, as seen in figure 9. The values of the parameters \( \hat{\mu}, \sigma \), affect the existence and stability of the fixed points. In Region A, no stable fixed points exist; trajectories are bounded and open, as in the middle plot of figure 8. In Region B, the only fixed points are those of type (iii), which are stable. Notice that this means that in Region B, solutions decay to zero amplitude — there are no fixed points with non-zero amplitude. In Region C, fixed points of type (i) and (iii) both exist, but only the fixed points of type (i) are stable in this region — here there are fixed points with non-zero amplitudes. In Region D, fixed points of type (i), (ii) and (iii) all exist simultaneously, and the fixed points of type (i) and (iii) are both stable in this region. Finally, Region E, fixed points of type (i) and (ii) exist, but only those of type (i) are stable here.
Despite the more complicated pattern of fixed points, many of the same features persist as in the undamped case. First, there are no resonant $a \to \infty$ solutions as there are in the non-relativistic Mathieu equation. Even with damping, the non-relativistic Mathieu equation allows for resonant solutions. However, the presence of the relativistic correction term in (2.19), (2.20) affects the phase of the oscillation at large amplitude as we can see in (2.22), effectively shutting down the resonance. The presence of damping does affect the nature of the solutions in one qualitative way — in the presence of damping, stable fixed points become attractors. At late times, all solutions eventually end up at the attractor solutions, as demonstrated in figure 10.

Thus, again we see that the non-linearities of the relativistic corrections to (2.19) prevent parametric resonance from occurring.

3 Implications for preheating

Now let us apply our results from the previous section to preheating with a non-canonical kinetic term. Specifically, we will numerically solve for the dynamics of the equations of motion (2.2)–(2.4), explicitly allowing for time-dependence in the coefficients of the $\chi$ equation of motion (2.4). As we found in the previous section, including the non-linearities from the reheaton’s non-canonical kinetic terms affects the existence of parametric resonance.

Let us numerically investigate a specific example, with the parameter values $g = 2 \times 10^{-3}, m_\phi = 10^{-5} M_p, \Phi_0 = 0.01 M_p, m_\chi = 10^{-7} M_p$ and $\chi(0) = \frac{H}{\pi} \sim 6 \times 10^{-9} M_p$. Since perturbations of massive scalar fields should be damped for long-wavelengths, the latter initial condition is chosen to be a concrete (and not arbitrary) initial condition, as would be expected.
for large wavelength modes of a light scalar field in a nearly de-Sitter background [33, 34]. As discussed in the introduction, the parameter $\Lambda$ corresponds to some UV degrees of freedom that have been integrated out. For consistency of the effective theory, then, $\Lambda$ cannot be too small; in particular, it should at least be larger than the mass of the inflation $\Lambda > m_\phi$.

Numerical solutions of (2.2)–(2.4) for the energy density $\rho_\chi$ are shown in figure 11 for several different values of the parameter $\Lambda$ as well as the energy density for a reheaton with a canonical kinetic term.

A few things are worth noting here. First, the non-canonical reheaton field behaves much like a canonical reheaton field for the initial stages of preheating. However, as the amplitude of $\chi$ increases the non-canonical reheaton field eventually falls out of resonance with the oscillating effective frequency, and preheating terminates at an earlier time and lower energy density. Thus, a smaller number density of $\chi$ particles is produced by preheating, and we thus expect similar reductions in possible violations of adiabaticity conditions. Second, as $\Lambda$ increases, the final energy density in the reheaton field approaches that of a canonical reheaton field, as expected. Note that the parameters chosen above imply $q_0 = 1$, $A_0 = 2.0$, and $\mu = 1200$ in terms of the parameters of (2.7), (2.8). Thus, while the analysis of the previous sections is strictly only valid for small $q$, the lessons learned are valid for a much wider range of parameters.

Finally, we note that while our analysis has so far been restricted to the DBI Lagrangian (1.6), many different types of non-linear self-interactions may have similar behavior during preheating. In particular, the quartic self-interaction:

$$\tilde{p}(X_\chi, \chi) = \frac{1}{2}(\partial \chi)^2 - \frac{1}{2} m_\chi^2 \chi^2 - \frac{1}{4} \lambda \chi^4,$$

(3.1)
gives rise to a cubic-order term in the equation of motion for $\chi$ similar to the term on the right hand side of (2.13). The equation of motion then takes the form of the parametrically excited Duffing equation, which exhibits similar behavior as the DBI Lagrangian studied here [5].
Similarly, keeping one of the leading non-renormalizable operators in the expansion (1.3):

\[ \tilde{p}(X, \chi) = \frac{1}{2} (\partial \chi)^2 - \frac{1}{2} m_\chi^2 \chi^2 + c_8 \frac{\Lambda^4}{(\partial \chi)^4}, \]  

will also lead to an equation of motion for \( \chi \) similar to (2.13). Thus, while our analysis has been restricted to the DBI Lagrangian, the lessons learned are more general.

4 Conclusion

We have investigated preheating when the reheaton field has non-canonical kinetic terms of the DBI type. The equation of motion for long-wavelength modes is the same as for a damped relativistic oscillator with time-dependent effective frequency — the relativistic Mathieu equation. In contrast to the usual Mathieu equation, the relativistic Mathieu equation does not exhibit unbounded parametric resonance, so that growth in the reheaton field cannot be arbitrarily large. The primary reason for the absence of unbounded resonance is that the relativistic oscillator is anharmonic — the frequency of oscillation depends on the amplitude. When the amplitude is initially small, the relativistic Mathieu equation behaves like the non-relativistic Mathieu equation and solutions grow due to resonance in the usual resonance bands. However, as the amplitude grows the frequency changes, and the reheaton is soon out of phase with the oscillating effective frequency. This effect limits the growth of the reheaton field with non-canonical kinetic terms. We studied this in some detail by finding perturbative solutions to the full relativistic Mathieu equation, and verified that this effect limits the growth of the reheaton field during preheating.

We made a number of simplifications in order to make progress on this question. Most significant, we considered only long-wavelength modes of the reheaton field. In non-linear systems such as the one studied in this paper, different Fourier modes with finite wavelength do not decouple can interact in complex and interesting ways — a full analysis of the \( k \neq 0 \) case requires a complex lattice simulation. Further, we did not include backreaction of the reheaton field on the expanding background and inflaton field, or rescattering of fluctuations, all of which are important effects [3]. Instead, our focus was only on the non-linear terms introduced by non-canonical kinetic terms in the long-wavelength approximation. A firmer understanding of the effects of these terms on parametric resonance in this simple case will help our intuition for when we eventually can relax these simplifications.

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References

[1] B.A. Bassett, S. Tsujikawa and D. Wands, Inflation dynamics and reheating, Rev. Mod. Phys. 78 (2006) 537 [astro-ph/0507632] [inSPIRE].

[2] R. Allahverdi, R. Brandenberger, F.-Y. Cyr-Racine and A. Mazumdar, Reheating in Inflationary Cosmology: Theory and Applications, Ann. Rev. Nucl. Part. Sci. 60 (2010) 27 [arXiv:1001.2600] [inSPIRE].
[3] L. Kofman, A.D. Linde and A.A. Starobinsky, *Towards the theory of reheating after inflation*, Phys. Rev. D 56 (1997) 3258 [hep-ph/9704452] [inSPIRE].

[4] L. Landau and E. Lifshitz, *Mechanics. Vol. 1*, third edition, Elsevier Ltd. (1976).

[5] A.H. Nayfeh and D.T. Mook, *Nonlinear Oscillations*, John Wiley & Sons, Inc. (1979).

[6] C. Armendariz-Picon, T. Damour and V.F. Mukhanov, *k-inflation*, Phys. Lett. B 458 (1999) 209 [hep-th/9904075] [inSPIRE].

[7] A.J. Tolley and M. Wyman, *The Gelaton Scenario: Equilateral non-Gaussianity from multi-field dynamics*, Phys. Rev. D 81 (2010) 043502 [arXiv:0910.1853] [inSPIRE].

[8] P. Franche, R. Gwyn, B. Underwood and A. Wissanji, *Attractive Lagrangians for Non-Canonical Inflation*, Phys. Rev. D 81 (2010) 123526 [arXiv:0912.1857] [inSPIRE].

[9] J. Karouby, B. Underwood and A.C. Vincent, *Preheating with the Brakes On: The Effects of a Speed Limit*, Phys. Rev. D 84 (2011) 043528 [arXiv:1105.3982] [inSPIRE].

[10] J. Lachapelle and R.H. Brandenberger, *Preheating with Non-Standard Kinetic Term*, JCAP 04 (2009) 020 [arXiv:0808.0936] [inSPIRE].

[11] E. Silverstein and D. Tong, *Scalar speed limits and cosmology: Acceleration from D-cceleration*, Phys. Rev. D 70 (2004) 103505 [hep-th/0310221] [inSPIRE].

[12] M. Alishahiha, E. Silverstein and D. Tong, *DBI in the sky*, Phys. Rev. D 70 (2004) 123505 [hep-th/0404084] [inSPIRE].

[13] J. Garriga and V.F. Mukhanov, *Perturbations in k-inflation*, Phys. Lett. B 458 (1999) 219 [hep-th/9904176] [inSPIRE].

[14] X. Chen, M.-x. Huang, S. Kachru and G. Shiu, *Observational signatures and non-Gaussianities of general single field inflation*, JCAP 01 (2007) 002 [hep-th/0605045] [inSPIRE].

[15] PLANCK collaboration, P.A.R. Ade et al., *Planck 2013 results. I. Overview of products and scientific results*, arXiv:1303.5062 [inSPIRE].

[16] PLANCK collaboration, P.A.R. Ade et al., *Planck 2013 results. XVI. Cosmological parameters*, arXiv:1303.5076 [inSPIRE].

[17] PLANCK collaboration, P.A.R. Ade et al., *Planck 2013 results. XXII. Constraints on inflation*, arXiv:1303.5082 [inSPIRE].

[18] PLANCK collaboration, P.A.R. Ade et al., *Planck 2013 results. XXIII. Isotropy and statistics of the CMB*, arXiv:1303.5083 [inSPIRE].

[19] PLANCK collaboration, P.A.R. Ade et al., *Planck 2013 Results. XXIV. Constraints on primordial non-Gaussianity*, arXiv:1303.5084 [inSPIRE].

[20] A.C. Davis and R.H. Ribeiro, *Enhanced (p)reheating in DBI Inflation*, arXiv:0908.4217 [inSPIRE].

[21] N. Bouatta, A.-C. Davis, R.H. Ribeiro and D. Seery, *Preheating in Dirac-Born-Infeld inflation*, JCAP 09 (2010) 011 [arXiv:1005.2425] [inSPIRE].

[22] J. Zhang, Y. Cai and Y.-S. Piao, *Preheating in a DBI Inflation Model*, arXiv:1307.6529 [inSPIRE].

[23] H.L. Child, J.T. Giblin Jr., R.H. Ribeiro and D. Seery, *Preheating with Non-Minimal Kinetic Terms*, Phys. Rev. Lett. 111 (2013) 051301 [arXiv:1305.0561] [inSPIRE].

[24] W. Chemissany, S. Das, A.F. Ali and E.C. Vagenas, *Effect of the Generalized Uncertainty Principle on Post-Inflation Preheating*, JCAP 12 (2011) 017 [arXiv:1111.7288] [inSPIRE].
[25] G.N. Felder and I. Tkachev, *LATTICEASY: A Program for lattice simulations of scalar fields in an expanding universe*, Comput. Phys. Commun. 178 (2008) 929 [hep-ph/0011159] [inSPIRE].

[26] A.V. Frolov, *DEFROST: A New Code for Simulating Preheating after Inflation*, JCAP 11 (2008) 009 [arXiv:0809.4904] [inSPIRE].

[27] L. MacColl, *Theory of the Relativistic Oscillator*, Am. J. Phys. 25 (1957) 535.

[28] H. Goldstein, *Classical Mechanics*, third edition, Pearson Education Inc. (2002).

[29] T.P. Mitchell and D.L. Pope, *On the Relativistic Damped Oscillator*, J. Soc. Ind. Appl. Math. 10 (1962) 49.

[30] R. Struble and T. Harris, *Motion of a Relativistic Damped Oscillator*, J. Math. Phys. 5 (1964) 138.

[31] W. Moreau, R. Easther and R. Neutze, *Relativistic (an)harmonic oscillator*, Am. J. Phys. 62 (1994) 531.

[32] J.-H. Kim and H.-W. Lee, *Relativistic chaos in the driven harmonic oscillator*, Phys. Rev. E 51 (1995) 1579.

[33] A.D. Linde, *Scalar Field Fluctuations in Expanding Universe and the New Inflationary Universe Scenario*, Phys. Lett. B 116 (1982) 335 [inSPIRE].

[34] D. Wands, *Multiple field inflation*, Lect. Notes Phys. 738 (2008) 275 [astro-ph/0702187] [inSPIRE].