On finitely Lipschitz space mappings.

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Abstract

It is established that a ring $Q$-homeomorphism with respect to $p$-modulus in $\mathbb{R}^n$, $n \geq 2$, is finitely Lipschitz if $n-1 < p < n$ and if the mean integral value of $Q(x)$ over infinitesimal balls $B(x_0, \varepsilon)$ is finite everywhere.

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1 Introduction

Recall that, given a family of paths $\Gamma$ in $\mathbb{R}^n$, a Borel function $\varrho : \mathbb{R}^n \to [0, \infty]$ is called admissible for $\Gamma$, abbr. $\varrho \in \text{adm}_\Gamma$, if

$$\int_{\gamma} \varrho \, ds \geq 1 \quad \text{for all } \gamma \in \Gamma.$$  

The $p$-modulus of $\Gamma$ is the quantity

$$M_p(\Gamma) = \inf_{\varrho \in \text{adm}_\Gamma} \int_{\mathbb{R}^n} \varrho^p(x) \, dm(x).$$

Here the notation $m$ refers to the Lebesgue measure in $\mathbb{R}^n$.

Let $G$ and $G'$ be domains in $\mathbb{R}^n$, $n \geq 2$, and let $Q : G \to [0, \infty]$ be a measurable function. A homeomorphism $f : G \to G'$ is called a $Q$–homeomorphism with respect to the $p$-modulus if

$$M_p(f\Gamma) \leq \int_{G} Q(x) \cdot \varrho^p(x) \, dm(x)$$

for every family $\Gamma$ of paths in $G$ and every admissible function $\varrho$ for $\Gamma$.

This conception is a natural generalization of the geometric definition of a quasiconformal mapping: if $Q(x) \leq K < \infty$ a.e., then $f$ is quasiconformal under $p = n$, see 13.1 and 34.6 in [Va], and quasisiometric under $1 < p \neq n$, see [Ge].

This class of $Q$-homeomorphisms with respect to the $n$-modulus was first considered in the papers [MRSY1]-[MRSY3], see also the monograph [MRSY]. The...
main goal of the theory of $Q$-homeomorphisms is to clear up various interconnections between properties of the majorant $Q(x)$ and the corresponding properties of the mappings themselves. In particular, the problem of the local and boundary behavior of $Q$-homeomorphisms has been studied in $\mathbb{R}^n$ first in the case $Q \in BMO$ (bounded mean oscillation) in the papers [MRSY_1]-[MRSY_3] and then in the case of $Q \in FMO$ (finite mean oscillation) and other cases in the papers [IR_1], [IR_2], [RS_1].

Note that the estimate of the type (1.3) was first established in the classical quasiconformal theory. Namely, it was obtained in [LV], p. 221, for quasiconformal mappings in the complex plane that

\begin{equation}
M(f) \leq \int_{\Gamma} K(z) \cdot \rho^2(z) \, dx dy
\end{equation}

where

\begin{equation}
K(z) = \frac{|f_z| + |f_z^*|}{|f_z| - |f_z^*|}
\end{equation}

is a (local) maximal dilatation of the mapping $f$ at a point $z$. Next, it was obtained in [BGMV], Lemma 2.1, for quasiconformal mappings in space, $n \geq 2$, that

\begin{equation}
M(f) \leq \int_{D} K_I(x,f) \rho^n(x) \, dm(x)
\end{equation}

where $K_I(x,f)$ stands for the inner dilatation of $f$ at $x$, see (1.8) below.

Given a mapping $f : G \to \mathbb{R}^n$ with partial derivatives a.e., $f'(x)$ denotes the Jacobian matrix of $f$ at $x \in D$ if it exists, $J(x) = J(x,f) = \det f'(x)$ the Jacobian of $f$ at $x$, and $|f'(x)|$ the operator norm of $f'(x)$, i.e., $|f'(x)| = \max\{|f'(x)h| : h \in \mathbb{R}^n, |h| = 1\}$. We also let $l(f'(x)) = \min\{|f'(x)h| : h \in \mathbb{R}^n, |h| = 1\}$. The outer dilatation of $f$ at $x$ is defined by

\begin{equation}
K_O(x,f) = \begin{cases} 
\frac{|f'(x)|^n}{|J(x,f)|}, & \text{if } J(x,f) \neq 0 \\
1, & \text{if } f'(x) = 0 \\
\infty, & \text{otherwise},
\end{cases}
\end{equation}

the inner dilatation of $f$ at $x$ by

\begin{equation}
K_I(x,f) = \begin{cases} 
\frac{|J(x,f)|}{|f'(x)|^n}, & \text{if } J(x,f) \neq 0 \\
1, & \text{if } f'(x) = 0 \\
\infty, & \text{otherwise},
\end{cases}
\end{equation}

The following notion generalizes and localizes the above notion of $Q$-homeomorphism. It is motivated by the ring definition of Gehring for quasiconformal mappings, see, e.g., [Ge_2], introduced first by V. Ryazanov, U. Srebro, and E. Yakubov in the plane and later on extended by V. Ryazanov and E. Sevostyanov to the space case, see, e.g., [RS_2], [RSY] and Chapters 7 and 11 in [MRSY].

Let $E, F \subset \mathbb{R}^n$ be arbitrary sets. Denote by $\Delta(E, F, G)$ a family of all curves $\gamma : [a, b] \to \mathbb{R}^n$ joining $E$ and $F$ in $G$, i.e., $\gamma(a) \in E, \gamma(b) \in F$ and $\gamma(t) \in G$.
for $t \in (a, b)$. Given a domain $G$ in $\mathbb{R}^n$, $n \geq 2$, a (Lebesgue) measurable function $Q : G \to [0, \infty]$, $x_0 \in G$, a homeomorphism $f : G \to \mathbb{R}^n$ is said to be a ring $Q$–homeomorphism at the point $x_0$ if

\begin{equation}
M_p(f(\Delta(S_1, S_2, A))) \leq \int_A Q(x) \cdot \eta^p(|x - x_0|) \, dm(x)
\end{equation}

for every ring $A = A(r_1, r_2, x_0) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}$ and the spheres $S_i = S(x_0, r_i) = \{x \in \mathbb{R}^n : |x - x_0| = r_i\}$, where $0 < r_1 < r_2 < r_0 := \text{dist}(x_0, \partial D)$, and every measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

\begin{equation}
\int_{r_1}^{r_2} \eta(r) \, dr \geq 1.
\end{equation}

$f$ is called a ring $Q$–homeomorphisms with respect to the $p$-modulus in the domain $G$ if $f$ is a ring $Q$–homeomorphism at every point $x_0 \in G$.

Let $f : G \to \mathbb{R}^n$, $n \geq 2$ be a quasiconformal mapping. The angular dilatation of the mapping $f : G \to \mathbb{R}^n$ at the point $x \in G$ with respect to $x_0 \in G, x_0 \neq x$ is defined by

\begin{equation}
D_f(x, x_0) = \frac{J(x, f)}{l_f(x, x_0)},
\end{equation}

where

\begin{equation}
l_f(x, x_0) = \min_{|h| = 1} \frac{|\partial_h f(x)|}{|\langle h, \frac{x - x_0}{|x - x_0|}\rangle|}.
\end{equation}

Here $\partial_h f(x)$ denotes the derivative of $f$ at $x$ in the direction $h$ and the minimum is taken over all unit vectors $h \in \mathbb{R}^n$, see [GG].

We recall that the estimate of the type (1.9) was first established in the classical quasiconformal theory in complex plane, see [GS]. Next, it was obtained in [GG], for quasiconformal mappings in space, $n \geq 2$, that

\begin{equation}
M(f(\Delta(S_1, S_2, A))) \leq \int_A D_f(x, x_0) \cdot \eta^p(|x - x_0|) \, dm(x)
\end{equation}

for every ring $A = A(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}$ and the spheres $S_i = S(x_0, r_i) = \{x \in \mathbb{R}^n : |x - x_0| = r_i\}$, where $0 < r_1 < r_2 < r_0 := \text{dist}(x_0, \partial G)$, and every measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that (1.10) holds.

Note that, in particular, homeomorphisms $f : G \to \mathbb{R}^n$ in the class $W^{1,n}_{\text{loc}}$ with $K_f(x, f) \in L^1_{\text{loc}}$ are ring $Q$–homeomorphisms as well as $Q$–homeomorphisms with $Q(x) = K_f(x, f)$, see, e.g., Theorem 6.10 and Corollary 4.9 in [MRSY], or Theorem 4.1 in [MRSY].

\section{Preliminaries}

Here a condenser is a pair $E = (A, C)$ where $A \subset \mathbb{R}^n$ is open and $C$ is a non–empty compact set contained in $A$. $E$ is a ringlike condenser if $B = A \setminus C$ is a
ring, i.e., if $B$ is a domain whose complement $\mathbb{R}^n \setminus B$ has exactly two components where $\mathbb{R}^n = \mathbb{R}^n \cup \{\infty\}$ is the one point compactification of $\mathbb{R}^n$. $E$ is a bounded condenser if $A$ is bounded. A condenser $E = (A, C)$ is said to be in a domain $G$ if $A \subset G$.

The following proposition is immediate.

2.1. Proposition. If $f : G \to \mathbb{R}^n$ is open and $E = (A, C)$ is a condenser in $G$, then $(fA, fC)$ is a condenser in $fG$.

In the above situation we denote $fE = (fA, fC)$.

Let $E = (A, C)$ be a condenser. Then $W_0(E) = W_0(A, C)$ denotes the family of non-negative functions $u : A \to R^1$ such that (1) $u \in C_0(A)$, (2) $u(x) \geq 1$ for $x \in C$, and (3) $u$ is ACL. We set

\begin{equation}
\begin{aligned}
cap_p E &= \inf_{u \in W_0(E)} \int_A |\nabla u|^p \, dm \\
&= \inf_{u \in W_0(E)} \int_A |\nabla u|^p \, dm
\end{aligned}
\end{equation}

where

$$|\nabla u| = \left( \sum_{i=1}^n (\partial_i u)^2 \right)^{1/2}$$

and call the quantity (2.2) the $p$-capacity of the condenser $E$.

For the next statement, see, e.g., [Ge], [He] and [Sh].

2.3. Proposition. Suppose $E = (A, C)$ is a condenser such that $C$ is connected. Then

$$\cap_p E = M_p(\Delta(\partial A, \partial C; A \setminus C)).$$

We give here also the following two useful statements, see Proposition 5 and 6 in [Kr].

2.4. Proposition. Let $E = (A, C)$ be a condenser such that $C$ is connected. Then

$$\cap_p E \geq \left( \inf m_{n-1} \sigma \right)^p \frac{m(A \setminus C)}{[m(A \setminus C)]^{p-1}}$$

where $m_{n-1} \sigma$ denotes the $(n-1)$-dimensional area of the $C^\infty$-manifold $\sigma$ that is the boundary $\sigma = \partial U$ of open set $U$ containing $C$ and contained along with its closure $\overline{U}$ in $A$ and the infimum is taken over all such $\sigma$.

2.5. Proposition. Let $E = (A, C)$ be a condenser such that $C$ is connected. Then for $n-1 < p \leq n$

$$\left( \cap_p E \right)^{n-1} \geq \gamma \frac{d(C)^p}{m(A)^{1-n+p}}$$

where $\gamma$ is a positive constant that depends only on $n$ and $p$, $d(A)$ is a diameter and $m(A)$ is the Lebesgue measure of $A$ in $\mathbb{R}^n$. 
3 Characterization of ring \( Q \)-homeomorphisms with respect to the \( p \)-modulus

The theorems of this section extend the corresponding results in [RS₂], see also Section 7.3 in the monograph [MRSY], from the case of \( p = n \) to the case of \( p \in (1, n] \). Below we use the standard conventions: \( a/\infty = 0 \) for \( a \neq \infty \) and \( a/0 = \infty \) if \( a > 0 \) and \( 0 \cdot \infty = 0 \), see e.g. [Sa], p. 6.

3.1. Lemma. Let \( G \) be a domain in \( \mathbb{R}^n, n \geq 2, 1 < p \leq n \), \( Q : G \to [0, \infty] \) a measurable function and \( q_{x_0}(r) \) the mean of \( Q(x) \) over the sphere \( |x - x_0| = r \).

Set

\[
I = I(x_0, r_1, r_2) = \int_{r_1}^{r_2} \frac{dr}{r^{n-1} q_{x_0}^{1/p} (r)}
\]

and \( S_j = \{ x \in \mathbb{R}^n : |x - x_0| = r_j \}, j = 1, 2 \), where \( x_0 \in G \) and \( 0 < r_1 < r_2 < r_0 = \text{dist} \, (x_0, \partial G) \). Then whenever \( f : G \to \mathbb{R}^n \) is a ring \( Q \)-homeomorphism with respect to the \( p \)-modulus at a point \( x_0 \)

\[
M_p \left( \Delta (f S_1, f S_2, f G) \right) \leq \frac{\omega_{n-1}}{I^p}
\]

where \( \omega_{n-1} \) is the area of the unit sphere in \( \mathbb{R}^n \).

Proof. With no loss of generality, we may assume that \( I \neq 0 \) because otherwise (3.3) is trivial, and that \( I \neq \infty \) because otherwise we can replace \( Q(x) \) by \( Q(x) + \delta \) with arbitrarily small \( \delta > 0 \) and then take the limit as \( \delta \to 0 \) in (3.3). The condition \( I \neq \infty \) implies, in particular, that \( q_{x_0}(r) \neq 0 \) a.e. in \((r_1, r_2)\).

Set

\[
\psi(t) = \begin{cases} 
1/\left[t^{n-1} q_{x_0}^{1/p} (t)\right] , & t \in (r_1, r_2) , \\
0 , & t \notin (r_1, r_2).
\end{cases}
\]

Then

\[
\int_A Q(x) \cdot \psi^p (|x - x_0|) \, dm(x) = \omega_{n-1} I
\]

where \( A = A(x_0, r_1, r_2) \).

Let \( \Gamma \) be a family of all paths joining the spheres \( S_1 \) and \( S_2 \) in \( A \). Let also \( \psi^* \) be a Borel function such that \( \psi^*(t) = \psi(t) \) for a.e. \( t \in [0, \infty] \). Such a function \( \psi^* \) exists by the Lusin theorem, see, e.g., 2.3.5 in [Fu] and [Sa], p. 69. Then the function

\[
\rho(x) = \psi^* (|x - x_0|)/I
\]

is admissible for \( \Gamma \) and since \( f \) is a ring \( Q \)-homeomorphisms with respect to the \( p \)-modulus we get by (3.5) that

\[
M_p(f \Gamma) \leq \int_A Q(x) \cdot \rho^p (x) \, dm(x) = \frac{\omega_{n-1}}{I^p}.
\]

and the proof is complete.
The following lemma shows that the estimate (3.3) cannot be improved for ring $Q$–homeomorphisms with respect to the $p$-modulus.

3.7. Lemma. Let $G$ be a domain in $\mathbb{R}^n$, $n \geq 2$, $1 < p \leq n$, $x_0 \in G$, $0 < r_1 < r_2 < r_0 = \text{dist}(x_0, \partial G)$, $A = A(x_0, r_1, r_2) = \{ x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2 \}$, $Q : \rightarrow [0, \infty]$ be a measurable function. Set

\[ \eta_0(r) = \frac{1}{Ir^{p-1} q_{x_0}^0(r)} \]

(3.8)

where $q_{x_0}(r)$ is the mean of $Q(x)$ over the sphere $|x - x_0| = r$ and $I$ is given by (3.2). Then

\[ \frac{\omega_{n-1}^p}{I^{p-1}} = \int_A Q(x) \cdot \eta_0^p(|x - x_0|) \, dm(x) \leq \int_A Q(x) \cdot \eta_0^p(|x - x_0|) \, dm(x) \]

(3.9)

whenever $\eta : (r_1, r_2) \rightarrow [0, \infty]$ is measurable and

\[ \int_{r_1}^{r_2} \eta(r) \, dr = 1. \]

(3.10)

Proof. If $I = \infty$, then the left hand side in (3.9) is equal to zero and the inequality is obvious. If $I = 0$, then $q_{x_0}(r) = \infty$ for a.e. $r \in (r_1, r_2)$ and the both sides in (3.9) are equal to $\infty$. Hence we may assume below that $0 < I < \infty$. Now, by (3.8) and (3.10) $q_{x_0}(r) \neq 0$ and $\eta(r) \neq \infty$ a.e. in $(r_1, r_2)$. Set $\lambda(r) = r^{p-1} q_{x_0}^0(r) \eta(r)$ and $w(r) = 1/r^{p-1} q_{x_0}^{-1}(r)$. Then by the standard conventions $\eta(r) = \lambda(r)w(r)$ a.e. in $(r_1, r_2)$ and

\[ C := \int_A Q(x) \cdot \eta^p(|x - x_0|) \, dm(x) = \omega_{n-1} \int_{r_1}^{r_2} \lambda^p(r) \cdot w(r) \, dr. \]

(3.11)

By Jensen’s inequality with weights, see, e.g., Theorem 2.6.2 in [Ra], applied to the convex function $\varphi(t) = t^p$ in the interval $\Omega = (r_1, r_2)$ with the probability measure

\[ \nu(E) = \frac{1}{I} \int_E w(r) \, dr \]

(3.12)

we obtain that

\[ \left( \int \lambda^p(r)w(r) \, dr \right)^{1/p} \geq \int \lambda(r)w(r) \, dr = \frac{1}{I} \]

(3.13)

where we also applied that $\eta(r) = \lambda(r)w(r)$ satisfies (3.10).

Thus

\[ C \geq \frac{\omega_{n-1}}{I^{p-1}} \]

(3.14)

and the proof of (3.9) is complete.
Finally, combining Lemmas 3.1 and 3.7, we obtain the following statement.

3.15. Theorem. Let $G$ be a domain in $\mathbb{R}^n$, $n \geq 2$, and $Q : G \to [0, \infty]$ a measurable function. A homeomorphism $f : G \to \mathbb{R}^n$ is ring $Q$–homeomorphism with respect to the $p$-modulus at a point $x_0 \in G$ if and only if for every $0 < r_1 < r_2 < d_0 = \operatorname{dist}(x_0, \partial G)$

\[
M_p(\Delta (fS_1, fS_2 fG)) \leq \frac{\omega_{n-1}}{I^{p-1}}
\]

where $S_1$ and $S_2, S_1 = \{ x \in \mathbb{R}^n : |x-x_0| = r_1 \}$ and $S_2 = \{ x \in \mathbb{R}^n : |x-x_0| = r_2 \}$. $
\omega_{n-1}$ is the area of the unit sphere in $\mathbb{R}^n$, $I = I(x_0, r_1, r_2) = \frac{r_2}{r_1} \int \frac{dr}{r^{n-1} q_{x_0}(r)^{p-1}}$, $q_{x_0}(r)$ is the mean value of $Q$ over the sphere $|x-x_0| = r$.

Note that the infimum from the right hand side in (1.9) holds for the function

\[
\eta_0(r) = \frac{1}{Ir^{n-1} q_{x_0}(r)}.
\]

Theorem 3.15 will have many applications in the theory of ring $Q$–homeomorphisms with respect to the $p$-modulus, see, e.g., the next section.

4 On finite by Lipschitz mappings

Given a mapping $\varphi : E \to \mathbb{R}^n$ and a point $x \in E \subseteq \mathbb{R}^n$, set

\[
L(x, \varphi) = \limsup_{y \to x \ y \in E} \frac{|\varphi(y) - \varphi(x)|}{|y-x|}
\]

and

\[
l(x, \varphi) = \liminf_{y \to x \ y \in E} \frac{|\varphi(y) - \varphi(x)|}{|y-x|}
\]

Given a set $A \subseteq \mathbb{R}^n$, $n \geq 1$, we say that a mapping $f : A \to \mathbb{R}^n$ is called Lipschitz if there is number $L > 0$ such that the inequality

\[
|f(x) - f(y)| \leq L |x-y|
\]

holds for all $x$ and $y$ in $A$. Given an open set $\Omega \subseteq \mathbb{R}^n$, we say that a mapping $f : \Omega \to \mathbb{R}^n$ is finitely Lipschitz if $L(x, f) < \infty$ for all $x \in \Omega$.

4.4. Lemma. Let $G$ and $G'$ be bounded domains in $\mathbb{R}^n$, $n \geq 2$, $Q : G \to [0, \infty]$ be a measurable function and let $f : G \to G'$ be a ring $Q$–homeomorphism with respect to $p$-modulus at a point $x_0 \in G$, $1 < p < n$. Then

\[
m(fB(x_0, r_1)) \leq \frac{c_{n,p}}{I^{n(p-1)}(x_0, r_1, r_2)}
\]
for every $0 < r_1 < r_2 < d_0 = \text{dist} (x_0, \partial G)$ where $I(x_0, r_1, r_2)$ is defined by (3.2) and $c_{n,p}$ is a positive constant that depends only on $n$ and $p$.

**Proof.** Let us consider the condenser $(A_{t+\Delta t}, C_t)$, where $C_t = \overline{B(x_0, t)}$, $A_{t+\Delta t} = B(x_0, t+\Delta t)$. Note that $(A_{t+\Delta t}, fC_t)$ is a ringlike condenser in $\mathbb{R}^n$ and according to Proposition 2.3, we have

$$\text{cap}_p (fA_{t+\Delta t}, fC_t) = M_p (\triangle (\partial fA_{t+\Delta t}, \partial fC_t; fR_t)).$$

(4.6)

In view of Proposition 2.4, we obtain

$$\text{cap}_p (fA_{t+\Delta t}, fC_t) \geq (\inf m_{n-1} \sigma)^p / m (fA_{t+\Delta t} \setminus fC_t)^{p-1};$$

(4.7)

where $m_{n-1} \sigma$ denotes the $(n-1)$-dimensional area of a $C^\infty$-manifold $\sigma$ that is the boundary of an open set $U$ containing $fC_t$ with its closure $\overline{U}$ in $fA_{t+\Delta t}$ and the infimum is taken over all such $\sigma$.

On the other hand, by Lemma 3.1, we have

$$M_p (\triangle (\partial fA_{t+\Delta t}, \partial fC_t; fR_t)) \leq \omega_{n-1} / \left( \frac{t+\Delta t}{t} \frac{ds}{s^{n-1} q_{t_0}^{p-1} (s)} \right)^{p-1}.$$  

(4.8)

Combining (4.6)-(4.8), we obtain

$$\frac{(\inf m_{n-1} \sigma)^p}{m (fA_{t+\Delta t} \setminus fC_t)^{p-1}} \leq \frac{\omega_{n-1}}{\left( \frac{t+\Delta t}{t} \frac{ds}{s^{n-1} q_{t_0}^{p-1} (s)} \right)^{p-1}}.$$  

Applying the isoperimetric inequality to the numerator of the fraction on the left-hand side we came to the inequality

$$n \cdot \Omega_n^{1/p} (m(fC_t))^{\frac{n-1}{p}} \leq \frac{1}{\omega_{n-1} p} \left( \frac{m(fA_{t+\Delta t} \setminus fC_t)}{\left( \frac{t+\Delta t}{t} \frac{ds}{s^{n-1} q_{t_0}^{p-1} (s)} \right)} \right)^{\frac{n-1}{p}}$$

(4.9)

where $\Omega_n$ is the volume of the unit ball in $\mathbb{R}^n$.

Now, setting $\Phi(t) := m(fB_t)$, we see from (4.9) that

$$n \cdot \Omega_n^{1/p} (\Phi(t))^{\frac{n-1}{p}} \leq \frac{1}{\omega_{n-1} p} \left( \frac{\Phi(t+\Delta t) - \Phi(t)}{\Delta t} \frac{1}{\left( \frac{t+\Delta t}{t} \frac{ds}{s^{n-1} q_{t_0}^{p-1} (s)} \right)} \right)^{\frac{n-1}{p}}.$$  

(4.10)
Since the function \( \Phi(t) \) is nondecreasing, has finite derivative \( \Phi'(t) \) for a.e. \( t \).
Letting \( \Delta t \to 0 \) in (4.10) and taking into account that \( \omega_{n-1} = n\Omega_n \), we obtain

\[
\frac{n\Omega_n^{\frac{p-n}{p-1}}}{t^{\frac{n-p}{p-1} q_{x_0}^{-1}}(t)} \leq \frac{\Phi'(t)}{\Phi_n^{\frac{p-n}{p-1}}(t)}.
\]

Integrating (4.11) under \( 1 < p < n \) with respect to \( t \in [r_1, r_2] \), since

\[
\int_{r_1}^{r_2} \frac{\Phi'(t)}{\Phi_n^{\frac{p-n}{p-1}}(t)} \, dt \leq \frac{n(p-1)}{p-n} \left( \Phi_n^{\frac{p-n}{p-1}}(r_2) - \Phi_n^{\frac{p-n}{p-1}}(r_1) \right),
\]

see, e.g., Theorem IV. 7.4 in [Sa], we observe that

\[
\Omega_n^{\frac{p-n}{p-1}} \int_{r_1}^{r_2} \frac{dt}{t^{\frac{n-p}{p-1} q_{x_0}^{-1}}(t)} \leq \frac{p-1}{p-n} \left( \Phi_n^{\frac{p-n}{p-1}}(r_2) - \Phi_n^{\frac{p-n}{p-1}}(r_1) \right).
\]

From (4.12) we conclude that

\[
\Phi(r_1) \leq \left( \Phi_n^{\frac{p-n}{p-1}}(r_2) + \Omega_n^{\frac{p-n}{p-1}} \frac{n-p}{p-1} \int_{r_1}^{r_2} \frac{dt}{t^{\frac{n-1}{p-1} q_{x_0}^{-1}}(t)} \right)^{\frac{n(p-1)}{p-n}}
\]

and hence

\[
\Phi(r_1) \leq \Omega_n \left( \frac{p-1}{n-p} \right)^{\frac{n(p-1)}{n-p}} \left( \int_{r_1}^{r_2} \frac{dt}{t^{\frac{n-1}{p-1} q_{x_0}^{-1}}(t)} \right)^{-\frac{n(p-1)}{n-p}}.
\]

Combining Lemmas 3.7 and 4.4, we have the following statement.

**4.13. Lemma.** Let \( G \) and \( G' \) be bounded domains in \( \mathbb{R}^n, n \geq 2 \), \( Q : G \to [0, \infty) \) be a measurable function and let \( f : G \to G' \) be a ring \( Q \)-homeomorphism with respect to the \( p \)-modulus. Then for \( 1 < p < n \)

\[
m(fB(x_0, r_1)) \leq c'_{n,p} \left[ \int_{A(x_0, r_1, r_2)} Q(x) \eta^p(|x - x_0|) \, dm(x) \right]^{\frac{n}{n-p}}.
\]

for every ring \( A = A(x_0, r_1, r_2), \ 0 < r_1 < r_2 < d_0 = \text{dist} (x_0, \partial G) \) and for every measurable function \( \eta : (r_1, r_2) \to [0, \infty] \), such that

\[
\int_{r_1}^{r_2} \eta(r) \, dr \geq 1.
\]

where \( c'_{n,p} \) is a positive constant that depends only on \( n \) and \( p \).

**4.16. Theorem.** Let \( G \) and \( G' \) be domains in \( \mathbb{R}^n, n \geq 2 \), and \( Q : G \to [0, \infty) \) be a measurable function such that
\[ Q_0 = \lim_{r \to 0} \frac{1}{\Omega_n r^n} \int_{B(x_0, \varepsilon)} Q(x) \, dm(x) < \infty. \]

Then for every ring \( Q \)-homeomorphism \( f : G \to G' \) with respect to the \( p \)-modulus, \( n - 1 < p < n \),

\[ L(x_0, f) = \limsup_{x \to x_0} \frac{|f(x) - f(x_0)|}{|x - x_0|} \leq \lambda_{n,p} Q_0^{\frac{1}{n-p}} \]

where \( \lambda_{n,p} \) is a positive constant that depends only on \( n \) and \( p \).

**Proof.** Let us consider the spherical ring \( A(x_0, \varepsilon, 2\varepsilon) = \{ x : \varepsilon < |x - x_0| < 2\varepsilon \} \), \( x \in G, \varepsilon > 0 \) such that \( A(x_0, \varepsilon, 2\varepsilon) \subset G \). Since \((fB(x_0, 2\varepsilon), fB(x_0, \varepsilon)) = (fB(x_0, 2\varepsilon), fB(x_0, \varepsilon))\) are ringlike condensers in \( G' \) and, according to Proposition 2.3, we obtain

\[ \text{cap}_p (fB(x_0, 2\varepsilon), fB(x_0, \varepsilon)) = M_p(\Delta(\partial fB(x_0, 2\varepsilon), \partial fB(x_0, \varepsilon); fA(x_0, \varepsilon, 2\varepsilon))). \]

Note that, in view of the homeomorphism of \( f \),

\[ \Delta(\partial fB(x_0, 2\varepsilon), \partial fB(x_0, \varepsilon); fA(x_0, \varepsilon, 2\varepsilon)) = f(\Delta(\partial B(x_0, 2\varepsilon), \partial B(x_0, \varepsilon); A(x_0, \varepsilon, 2\varepsilon))). \]

By Proposition 2.5

\[ \text{cap}_p (fB(x_0, 2\varepsilon), fB(x_0, \varepsilon)) \geq \left( \frac{\gamma d^p(fB(x_0, \varepsilon))}{m^{1-n+p}(fB(x_0, 2\varepsilon))} \right)^{\frac{1}{n-1}} \]

where \( \gamma \) is a positive constant that depends only on \( n \) and \( p \), \( d(A) \) is the diameter and \( m(A) \) is the Lebesgue measure of \( A \) in \( \mathbb{R}^n \).

By the definition of ring \( Q \)-homeomorphisms with respect to the \( p \)-modulus

\[ \text{cap}_p (fB(x_0, 2\varepsilon), fB(x_0, \varepsilon)) \leq \frac{1}{\varepsilon^p} \int_{A(x_0, \varepsilon, 2\varepsilon)} Q(x) \, dm(x) \]

because the function

\[ \eta(t) = \begin{cases} \frac{1}{\varepsilon}, & \text{if } t \in (\varepsilon, 2\varepsilon), \\ 0, & \text{if } t \in \mathbb{R} \setminus (\varepsilon, 2\varepsilon) \end{cases} \]

satisfies (1.10) for \( r_1 = \varepsilon \) and \( r_2 = 2\varepsilon \).

Next, the function

\[ \tilde{\eta}(t) = \begin{cases} \frac{1}{2\varepsilon}, & \text{if } t \in (2\varepsilon, 4\varepsilon) \\ 0, & \text{if } t \in \mathbb{R} \setminus (2\varepsilon, 4\varepsilon) \end{cases} \]

satisfies (1.10) for \( r_1 = 2\varepsilon \) and \( r_2 = 4\varepsilon \) and hence by Lemma 4.13 we have the following estimates:
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(4.21) \[ m(fB(x_0,2\varepsilon)) \leq c''_{n,p} \varepsilon^n \left[ \frac{1}{m(B(x,4\varepsilon))} \int_{B(x_0,4\varepsilon)} Q(x) \, dm(x) \right]^{\frac{n}{n-p}}, \]

where \( c''_{n,p} \) is a positive constant that depends only on \( n \) and \( p \).

Combining (4.21), (4.20) and (4.19), we obtain

\[ d(fB(x_0,\varepsilon)) \leq \lambda_{n,p} \left( \frac{1}{m(B(x_0,4\varepsilon))} \int_{B(x_0,4\varepsilon)} Q(y) \, dy \right)^{\frac{n(1-n+p)}{p(n-p)}} \left[ \frac{1}{m(B(x,2\varepsilon))} \int_{B(x_0,2\varepsilon)} Q(x) \, dm(x) \right]^{\frac{n-1}{p}}, \]

and hence

\[ L(x_0,f) = \limsup_{x \to x_0} \frac{|f(x) - f(x_0)|}{|x - x_0|} \leq \limsup_{\varepsilon \to 0} \frac{d(fB(x_0,\varepsilon))}{\varepsilon} \leq \lambda_{n,p} Q_0^{\frac{1}{n-p}}, \]

where \( \lambda_{n,p} \) is a positive constant that depends only on \( n \) and \( p \).

4.22. Corollary. Let \( G \) and \( G' \) be domains in \( \mathbb{R}^n \), \( n \geq 2 \), \( f : G \to G' \) be a ring \( Q \)-homeomorphism with respect to the \( p \)-modulus, \( n-1 < p < n \), such that

(4.23) \[ \lim_{r \to 0} \frac{1}{\Omega_n \varepsilon^n} \int_{B(x_0,\varepsilon)} Q(x) \, dm(x) < \infty \quad \forall \ x_0 \in G. \]

Then \( f \) is finitely Lipschitz.

Note that the theory of ring \( Q \)-homeomorphisms with respect to \( p \)-modulus can be applied to mappings in the Orlich-Sobolev classes \( W^{1,p}_{loc} \) with a Calderon type condition on \( \varphi \) and, in particular, to the Sobolev classes \( W^{1,p}_{loc} \) for \( p > n-1 \), cf. [KRSS].

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