Phases of
\[ \mathcal{N} = 1 \ SO(N_c) \] Gauge Theories with Flavors

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Abstract

We studied the phase structures of \[ \mathcal{N} = 1 \] supersymmetric \[ SO(N_c) \] gauge theory with \( N_f \) flavors in the vector representation as we deformed the \[ \mathcal{N} = 2 \] supersymmetric QCD by adding the superpotential of arbitrary polynomial for the adjoint chiral scalar field. Using weak and strong coupling analyses, we determined the most general factorization forms for various breaking patterns. We observed all kinds of smooth transitions for quartic superpotential.
1 Introduction and summary

The $\mathcal{N} = 1$ supersymmetric gauge theories in four dimensions have rich structures and one obtains nonperturbative aspects by studying the holomorphic effective superpotential which determines the quantum moduli space. A large class of interesting gauge theories can be obtained from the choice of geometries in which D5-branes wrap partially over the nontrivial cycles [1, 2, 3, 4]. As far as the effective superpotential is concerned, the geometry in which the four dimensional gauge theories are realized on the worldvolume of D5-branes wrapping around $S^2$ is replaced by a dual geometry in which D5-branes are replaced by RR fluxes and the $S^2$ by $S^3$. These RR fluxes provide the effective superpotential which corresponds to one of four dimensional gauge theories on the D5-branes [5, 6, 7]. This equivalence has been tested for several different models in different contexts [8, 9, 10, 11, 12, 13].

A new recipe for the computation of the effective superpotential was proposed by Dijkgraaf and Vafa [14, 15, 16] through the free energies in a certain matrix model. This matrix model analysis could be interpreted within purely field theoretical point of view without a string theory [17]. Based on the anomalous Ward identity of a generalized Konishi anomaly, a powerful machinery for the nonperturbative aspects was obtained. Moreover, a new kind of duality where one can transit several vacua with different broken gauge groups continuously and holomorphically by changing the parameters of the superpotential was found in [18]. The extension of [18] to the $\mathcal{N} = 1$ supersymmetric gauge theories with the gauge group $SO/USp$ was found in [19] in which there were no flavors and the phase structures of these theories, a matrix model curve and a generalized Konishi anomaly equation, were obtained. More recently [20], by adding the flavors in the fundamental representation to the theory of [18], the vacuum structures in classically and quantum mechanically were described and an addition map as well as multiplication map were developed.

In this paper, we study the phase structures of $\mathcal{N} = 1$ supersymmetric $SO(N_c)$ gauge theory with $N_f$ flavors in the vector representation by deforming the $\mathcal{N} = 2$ supersymmetric QCD with the superpotential of arbitrary polynomial for adjoint chiral scalar field, by applying the methods in [18, 19, 20]. These kinds of study were initiated by Cachazo, Seiberg and Witten in [18], where a kind of new duality was found. This paper is a generalization of [19] to the case with flavors. We found that with flavors the phase structure is richer and that more interesting dualities show up. We list some partial papers [21]-[63] on the recent works, along the line of [14, 15, 16].

In section 2, we describe the classical moduli space of $\mathcal{N} = 2$ SQCD deformed to $\mathcal{N} = 1$ theory by adding the superpotential $W(\Phi)$ (2.2). The gauge group $SO(N_c)$ will break to $SO(N_0) \times \prod_{j=1}^{n} U(N_j)$ with $N_0 + \sum_{j=1}^{n} 2N_j = N_c$ by choosing the adjoint chiral field $\Phi$ to be the root of $W'(x)$ or $\pm m_i$, the mass parameters. To have pure Coulomb branch where no factor
$U(N_j)$ is higgsed, we restrict ourselves to the case $W'(\pm m_i) = 0$. For each factor with some effective massless flavors, there exists a rich structure of Higgs branches, characterized by an integer $r_i$, that meets the Coulomb branch along the submanifold called the root of the $r$-th Higgs branch.

In section 3.1, we discuss the quantum moduli space of $SO(N)$ by both the weak and strong coupling analyses. When the difference between the roots of $W'(x)$ is much larger than $N = 2$ dynamical scale $\Lambda$ (in other words, in the weak coupling region), the adjoint scalar field $\Phi$ can be integrated out and it gives a low energy effective $\mathcal{N} = 1$ superpotential. Under this condition, one can suppress higher order terms except the quadratic piece in the superpotential (2.2). Then the effective superpotential consists of the classical part plus nonperturbative part. There exist two groups of solutions, i.e., Chebyshev vacua and Special vacua, according to the unbroken flavor symmetry, meson matrix $M$ (3.1) and various phases of vacua.

At the scales below the $\mathcal{N} = 1$ scale $\Lambda_1$ (when the roots of $W'(x)$ are almost the same), the strong coupling analysis is relevant. We need to look for the special points where some number of magnetic monopoles (mutually local or non local) become massless, on the submanifold of the Coulomb branch of $\mathcal{N} = 2$ $SO(N)$ which is not lifted by the $\mathcal{N} = 1$ deformation. The conditions for these special points are translated into a particular factorization form of the corresponding Seiberg-Witten curve. We discuss the characters of these curves at the Chebyshev branch or the Special branch, especially, the power of factor $t = x^2$ and the number of single roots.

In section 3.2, combining the quantum moduli space of $SO(N)$ group with the quantum moduli space of $U(N)$ group studied in [20], we give the most general factorization forms of the curves with the proper number of single roots and double roots for various symmetry breaking patterns, which generalize the results in [9, 10, 11, 12]. From the point of view of the geometry, these various breaking patterns correspond to the various distributions of wrapping D5-branes among the roots of $W'(x)$, i.e., some of the roots do not have wrapping D5-branes. We also summarize the counting of vacua for various phases which will be used heavily in the examples in sections 4 and 5. For $U(N_i)$ without flavors, the number of vacua is given by the Witten index $w(N_i) = N_i$. For $U(N_i)$ with flavors, it is given by (3.11) (for more details, see [20]). For $SO(N_i)$ without flavors, it is also given by the Witten index and finally for $SO(N_i)$ with $M_i$ flavors, the number of vacua is given by $(N_i - M_i - 2)$ for the Chebyshev vacua and one for Special vacua.

In section 4, we analyze the simplest nontrivial examples for quartic tree level superpotential with *massive* flavors for $SO(4)$, $SO(5)$, and $SO(6)$ gauge groups. For these cases, there are two values which can be chosen by $\Phi$ and we have the following breaking patterns: $SO(N_c) \rightarrow$

\footnote{Although for the pure case [19], the description for $SO(7)$ and $SO(8)$ gauge theories was given, in this paper we will skip it because on the one hand, it will give rise to the complicated solutions and it is hard to analyze and on the other hand, we expect one cannot see any new interesting phenomena.}
$SO(N_0) \times U(\tilde{N}_1)$, $SO(N_c) \rightarrow SO(N_c)$, and $SO(N_c) \rightarrow U([\tilde{N}_c/2])$. Depending on both the $SO(N)$ factor at the Chebyshev branch or the Special branch and the $U(\tilde{N})$ factor at the various $r$ baryonic or non-baryonic branches, the curve takes a different factorization form. By solving the factorization of the curve, we find various phases predicted by weak and strong coupling analyses that lead to the matches of the counting of vacua. We find also three interesting smooth transitions among these three breaking patterns: $SO(N_c) \leftrightarrow SO(M_0) \times U(\tilde{M}_1)$, $SO(N_0) \times U(\tilde{N}_1) \leftrightarrow SO(M_0) \times U(\tilde{M}_1)$, and $U([\tilde{N}_c/2]) \leftrightarrow SO(M_0) \times U(\tilde{M}_1)$. Especially the smooth transitions of $SO(N_c) \leftrightarrow SO(M_0) \times U(\tilde{M}_1)$ and $U([\tilde{N}_c/2]) \leftrightarrow SO(M_0) \times U(\tilde{M}_1)$ cannot be found if we do not consider the breaking patterns $SO(N_c) \rightarrow U([\tilde{N}_c/2])$ and $SO(N_c) \rightarrow SO(N_c)$. We also discuss carefully how the smooth transition arises when we tune the parameters of the deformed superpotential. The phase structures for various gauge groups have been summarized in the Tables which can be found in sections 4 and 5.

In section 5, we move to the massless flavors with quartic deformed superpotential for $SO(N_c)$ where $N_c = 4, 5, 6$ and 7. In this case, contrary to section 5, at the IR limit, the $SO(\tilde{N}_0)$ factor has massless flavors instead of a $U(N_i)$ factor having effective massless flavors as in section 5. Because of this difference, new features arise. For example, there is Special branch for general $SO(N_c)$ with massless flavors while for massive case, only $SO(2)$, $SO(3)$, and $SO(4)$ gauge groups have the Special branch. Also with massless flavors, there are no smooth transitions as in $SO(\tilde{N}_0) \leftrightarrow SO(\tilde{N}_0) \times U(N_1)$ and $SO(\tilde{N}_0) \leftrightarrow U([\tilde{N}_c/2])$. Furthermore, for the smooth transition $SO(\tilde{N}_0) \times U(N_1) \leftrightarrow SO(\tilde{M}_0) \times U(M_1)$ in the Special branch, we have $M_0 = (2N_f - N_0 + 4)$, which is the relationship between $SO(\tilde{N}_0)$ and $SO(\tilde{M}_0)$ to be the Seiberg dual pair. In fact, the smooth transition in this case may be rooted in the Seiberg duality. In this section, we also give the general discussion for the possible smooth transition in the case of massless flavors.

In Appendix A, by using the $N = 2$ curve together with monopole constraints we are interested in and applying the contour integral formula, we derive the matrix model curve (A.19) for deformed superpotential with an arbitrary degree and the relationship (A.6) (or the most general expression (A.8)) between the matrix model curve (i.e., the single-root part of the factorized Seiberg-Witten curve) and the deformed superpotential $W'(x)$. The formula (A.8) will be used to determine the number of vacua for fixed tree level superpotential. Using these results, we have also checked the generalized Konishi anomaly equation for our gauge theory with flavors (A.24).

In Appendix B, in order to understand the vacua of different theories, we discuss both the addition map and multiplication map. The addition map with massive flavors relates the vacua of $SO(N_c)$ gauge theory with $N_f$ flavors in the $r$-th branch to those of $SO(N'_c)$ gauge theory

\footnote{In fact, with a little modification, the result can also be applied to the massive flavors.}
with \(N'_f\) flavors in the \(r'-\)branch where both \(N_c\) and \(N'_c\) are even or odd. This phenomenon is the same exactly as the one in the \(U(N_c)\) gauge theory with flavors. However, the addition map with massless flavors can relate \(SO(2N_c)\) gauge theory to \(SO(2N_c + 1)\) gauge theory. For the multiplication map, we present the most general form. Through this multiplication map, one can obtain the unknown factorization of the gauge group with higher rank from the known factorization of the gauge group with lower rank. In this derivation, the properties of Chebyshev polynomials are used. In several specific examples, we demonstrate the general results. Some interesting points of these general multiplication maps are: (1) We can map the results of \(SO(2M)\) to \(SO(2N + 1)\) and vice versa; (2) We can map the results without massless flavors to the case with some massless flavors. This is a somewhat unusual result.

One can also study the phase structures of \(\mathcal{N} = 1\) supersymmetric \(USp(2N_c)\) gauge theory with \(N_f\) flavors in the fundamental representation with equal footing described in this paper. However, due to the length of the paper, we will publish our results in a separate paper.

Finally let us mention a few interesting directions to be done in the future:

- In this paper, we restrict ourselves to the case \(W'(\pm m) = 0\). It would be interesting to study the general case without this constraint \(W'(\pm m) = 0\) along the line of [50]. For the general case, the parameter space is bigger and we expect to have more smooth transitions in this bigger space.

- We have discussed some indices which distinguish between different phases. It will be useful to have more concrete discussion of phases with the complete list of indices, along the line of [18].

- In this paper, we have discussed the phase structure with adjoint chiral field \(\Phi\). It would be interesting to discuss the phase structure with other representations, especially the second rank tensor representation. The descriptions with these representations have attracted some attention recently [47, 54, 55, 60].

### 2 The classical moduli space of \(SO(N_c)\) supersymmetric QCD

In this section, we will discuss the classical moduli space of \(SO(N_c)\) with flavors and the \(\mathcal{N} = 1\) deformation (2.2) so that the total superpotential will be (2.1). Although the classical picture will be modified by quantum corrections, it does give some useful information about the quantum moduli, especially in the weak coupling region.

Let us consider an \(\mathcal{N} = 1\) supersymmetric \(SO(N_c)\) gauge theory with \(N_f\) flavors of quark \(Q_a^i (i = 1, 2, \cdots, 2N_f, a = 1, 2, \cdots, N_c)\) in the vector representation. The tree level superpotential of the theory is obtained from \(\mathcal{N} = 2\) SQCD by adding the arbitrary polynomial of the
adjoint scalar $\Phi_{ab}$ belonging to the $\mathcal{N} = 2$ vector multiplet:

$$W_{\text{tree}}(\Phi, Q) = \sqrt{2} Q_a^i \Phi_{ab}^i Q_b^j J_{ij} + \sqrt{2} m_{ij} Q_a^i Q_b^j + \sum_{s=1}^{k+1} \frac{g_{2s}}{2s} \text{Tr} \Phi^{2s} \quad (2.1)$$

where the first two terms come from the $\mathcal{N} = 2$ theory and the third term, $W(\Phi)$, can be described as a small perturbation of $\mathcal{N} = 2$ $SO(N_c)$ gauge theory \cite{64, 65, 66, 67, 68, 69, 70, 71, 72, 8, 11}

$$W(\Phi) = \sum_{s=1}^{k+1} \frac{g_{2s}}{2s} \text{Tr} \Phi^{2s} \equiv \sum_{s=1}^{k+1} g_{2s} u_{2s}, \quad u_{2s} \equiv \frac{1}{2s} \text{Tr} \Phi^{2s} \quad (2.2)$$

where $\Phi_{ab}$ is an adjoint scalar chiral superfield that plays the role of a deformation breaking $\mathcal{N} = 2$ supersymmetry to $\mathcal{N} = 1$ supersymmetry. Note that in \cite{64} the coefficient for the mass term of quark in the tree superpotential is different from $\sqrt{2}$ used here, but it can be absorbed in the mass matrix and the only quadratic mass deformation for $\Phi$ with $g_2 = \mu$ (other parameters are vanishing) was considered in \cite{64}. The $J_{ij}$ is the symplectic metric used to raise and lower the $SO(N_c)$ flavor indices while $m_{ij}$ is a quark mass matrix, and they are defined as

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes I_{N_f \times N_f}, \quad m = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \text{diag}(m_1, \ldots, m_{N_f}) \quad (2.3)$$

where $I_{N_f \times N_f}$ is the $N_f \times N_f$ identity matrix. The $\mathcal{N} = 2$ theory without $W(\Phi)$ is asymptotically free for $N_f < N_c - 2$ (and generates $\mathcal{N} = 2$ strong-coupling scale $\Lambda$), conformal for $N_f = N_c - 2$ (scale invariant) and infrared (IR) free for $N_f > N_c - 2$.

The classical vacuum structure, the zeroes of the scalar potential, can be obtained by solving D-terms and F-terms. We summarize the following results from the mechanism of adjoint vevs as follows:

1) The eigenvalues of $\Phi$, $\pm i \phi_i$, can only be the roots of $W'(x)$ or $\pm m_i$; thus, the gauge group $SO(N_c)$ with $N_f$ flavors is broken to the product of blocks with or without the effective massless flavors. Among these blocks at most one block is $SO(\mu_{n+1})$ and others, $U(\mu_i)$ where $i = 1, 2, \ldots, n$.

2) If $\phi_i = \pm m_i$ but $W'(\pm m_i) \neq 0$ ($W'(x)$ has some roots which are not equal to $\pm m_i$), the corresponding gauge symmetry of that block will be completely higgsed; Because of this, we will restrict the following discussions to the case of $W'(\pm m) = 0$ for which there are richer structures.

3) For each block, if it has “effective” massless flavors, there are various Higgs branches classified by some integers $r$. 

5
3 The quantum moduli space of $SO(N_c)$ supersymmetric QCD

From the above section we see that the gauge group $SO(N_c)$ is broken to the product of various $U(N_i)$ with at most one $SO(N)$. To get the quantum moduli space we need first to understand the quantum moduli for various factors. The quantum theory of $U(N_i)$ with effective massless flavors has been discussed in [73, 74, 20] and we will not repeat it here. Here we will focus on the quantum moduli of $SO(N)$ \(^3\) with massless flavors discussed in [68, 64].

3.1 The quantum theory of $SO(N)$ supersymmetric QCD

The quantum theory of $SO(N)$ with mass deformation $\frac{1}{2} \mu \text{Tr} \Phi^2$ has been presented in [68, 64] for both weak and strong coupling analyses. Notice that when $N \leq 4$, it is not a simple Lie group, so we will restrict our discussion to the case $N \geq 5$ \(^4\).

3.1.1 The weak coupling analysis

Special corners in parameter space will place $\mathcal{N} = 1$ vacua in regions where the gauge symmetry breaking scale is much larger than the $\mathcal{N} = 2$ dynamical scale, $\Lambda$. Then the gauge coupling is small. When the mass $\mu$ for adjoint scalar $\Phi$ is larger compared with the dynamical scale $\Lambda$, the adjoint scalar $\Phi$ can be integrated out to lead to a low energy effective $\mathcal{N} = 1$ superpotential from which one can recover the original theory as in [75]. The weak coupling analysis is valid when the difference between the roots of the polynomial $W'(x)$ are much bigger than $\Lambda$. Under this condition we can only consider the quadratic part of the effective superpotential since the higher powers of it will be suppressed by $\mu$. Then the relevant superpotential is

$$W_{\text{tree}}(\Phi, Q) = \sqrt{2} Q^a_\alpha \Phi_{ab} Q^b_\beta J_{ij} + \frac{\mu}{2} \text{Tr} \Phi^2.$$ 

Integrating out $\Phi$ will give

$$W_{\text{tree}}(\Phi, Q) = -\frac{1}{2\mu} \text{Tr} (M J M J), \quad M^{ij} = Q^i \cdot Q^j. \quad (3.1)$$

Then the effective superpotential will consist of the above classical part (3.1) plus nonperturbative effects. To find $\mathcal{N} = 1$ vacua, the effective superpotential should be minimized. We can

\(^3\)The number of colors $N$ is less than or equal to $N_c$ and it can be $2, 4, \cdots, N_c$ for $N_c$ even and $3, 5, \cdots, N_c$ for $N_c$ odd.

\(^4\)For $N \geq 5$ the Witten index $w(N)$ of $SO(N)$ gauge group is $(N - 2)$ while for $N \leq 4$ the Witten indices are $1, 1, 1, 2, 4$ for $N = 0, 1, 2, 3, 4$ respectively. We will give some explanations why the Witten index is different for $N \leq 4$. 

study the vacuum structure, the number of vacua, and global symmetry breakings according to the range of the number of flavors $N_f$.

We summarize two groups of solutions as follows:

1) The first group has non-degenerated meson matrix $M$. They are all in a confining phase with unbroken flavor symmetry $U(N_f)$ and the number of vacua is given by $(N - N_f - 2)$. As we will show, they are given by *Chebyshev point* in the corresponding Seiberg-Witten(SW)-curve. Its position in the $\mathcal{N} = 2$ moduli space is located by the roots of the Chebyshev polynomial; the light degrees of freedom are mutually nonlocal and the theory flows to an interacting $\mathcal{N} = 2$ superconformal theory. Since the superconformal theory is nontrivial, one does not have a local effective Lagrangian description for those theories. The symmetry breaking pattern is obtained by the analysis at large $\mu >\gg \Lambda$ where there exists an effective description of the theory by integrating out the adjoint scalar $\Phi$. The dynamical condensation of mesons breaks the $USp(2N_f)$ flavor symmetry to $U(N_f)$.

2) The second group exists only when $2N_f \geq N - 4$ with unbroken flavor symmetry $USp(2N_f)$. Unlike the first group, the vacua can be in any of Higgs, Coulomb, or confining phases. They are given by the Special point in the SW-curve. In the *Special point* observed in [68], the gauge symmetry is enhanced to IR free $SO(\tilde{N}) = SO(2N_f - N + 4)$ which is the Seiberg dual gauge group [75]. The full $USp(2N_f)$ global symmetry remains unchanged since there are no meson condensates and no dynamical symmetry breaking occurs. These vacua are in the non-Abelian free magnetic phase.

### 3.1.2 The strong coupling analysis

When $W(\Phi)$ is very small compared with dynamical scale $\Lambda$, the $\mathcal{N} = 1$ quantum theory can be considered as a small perturbation of a strongly coupled $\mathcal{N} = 2$ gauge theory without $W(\Phi)$. In this region of parameters, we can use the Seiberg-Witten curve in which all the nontrivial dynamicses are already encoded. The $\mathcal{N} = 2$ Seiberg-Witten curve derivation in the context of the matrix model was found in [38, 40]. Turning on the perturbed superpotential lifts most points on the Coulomb branch except some points (the Higgs roots) where a certain number of mutually local or nonlocal monopoles becomes massless.

The strong coupling analysis has been done in [68, 64]. Let us recall that the curve of

\[\text{From this condition we see that when } N_f = 0, \text{ it is satisfied only when } N \leq 4. \text{ This explains why the Witten index of } SO(N) \text{ without flavors is different for } N \leq 4 \text{ because there are extra contributions from this second group. These special cases are also related to the fact that there is a factor } x^4 \text{ (or } x^2) \text{ in the Seiberg-Witten curve as we will see in the later examples.}\]
SO(N) is given by

\[ y^2 = \prod_{j=1}^{[N/2]} (t - \phi_j^2)^2 - 4\Lambda^2(N-2-N_f) t^{1+\epsilon} \prod_{k=1}^{N_f} (t - m_k^2) \]

where \( t = x^2 \) and \( \epsilon = 0 \) for \( N \) odd and \( \epsilon = 1 \) for \( N \) even \([76, 77]\). For our massless case, it is reduced to

\[ y^2 = \prod_{j=1}^{N/2} (t - \phi_j^2)^2 - 4\Lambda^2(N-2-N_f) t^{N_f+2}, \quad \text{for } N \text{ even,} \]

\[ y^2 = \prod_{j=1}^{(N-1)/2} (t - \phi_j^2)^2 - 4\Lambda^2(N-2-N_f) t^{N_f+1}, \quad \text{for } N \text{ odd.} \]

By setting \( r \phi_j \)'s to vanish, the curve is

\[ y^2 = t^{2r} \left[ \prod_{j=1}^{([N/2]-r)} (t - \phi_j^2)^2 - 4\Lambda^2(N-2-N_f) t^{1+\epsilon+N_f-2r} \right] \]

which has a factor \( t \) with some power. The results in \([68, 64]\) are that there are unlifted vacua under the mass deformation \( \frac{1}{2} \mu \text{Tr} \Phi^2 \) only when the power of factor \( t \) in the curve \( y^2 \) is some particular numbers, i.e.,

\[ \tilde{N}_c = (2N_f - N + 4)/(2N_f - N + 3) \]

for \( N \) is even/odd or

\[ (N_f + 3)/(N_f + 1)/(N_f + 2)/(N_f + 2) \]

for \( (N, N_f) = (e, e)/(e, o)/(o, e)/(o, o) \) where we denote \( e \) by an even number and \( o \) by an odd number. The first case \([68, 64]\) is the character of the Special branch which in some sense corresponds to the baryonic branch of \( U(N) \) theory. The second case indicates the Chebyshev branch where the low energy effective \( \mathcal{N} = 2 \) theory is a nontrivial conformal theory.

The reason for the above conclusion is the following. For general \( r \), it corresponds to the trivial superconformal theory classified as class 1 in \([78]\). To have the vacua, we must have enough massless monopoles which can happen if and only if \( r = \tilde{N}_c = 2N_f - N + 4 \) when \( N \) is even. It is easy to show that there are \( (N - N_f - 2) \) mutual local monopoles by noticing that from (3.2) (here we use the example where \( N \) is even)

\[ y^2 = t^{2N_f-N+4} \left( P_{N-N_f-2}^2(t) - 4\Lambda^2(N-2-N_f) t^{N_f-N-2} \right) \]

\[ = t^{2N_f-N+4} \left( t^{N_f-N-2} - \Lambda^2(N-2-N_f) \right)^2 \]

\(^6\text{Notice that there is a factor } (t - m_k^2) \text{ instead of } (t + m_k^2) \text{. The reason is as follows. The } SU(N) \text{ case has a factor } (x + m) \text{ for } \det(x + m) \text{ in the second term for diagonalized mass matrix } m \text{. For } USp(2N) \text{ group, the masses are given by } i\sigma_2 \otimes \text{diag}(m_1, m_2, \ldots, m_{N_f}) \text{ so } \det(x + m) \text{ contains a factor } (x^2 + m_2^2) \text{. For } SO(N) \text{ case, the masses are given by } (2.3) \text{ so } \det(x + m) \text{ has a factor } (x^2 - m_k^2).\)
where to have a perfect square form in the bracket implying the maximal degeneracy of the Riemann surface, we have chosen $P_{N-N_f-2}(t) = t^{N-2-N_f} + \Lambda^{2(N-2-N_f)}$ [68]. This particular case is called the “Special branch” where the curve $y^2$ is a square form, which is a typical character of this branch. The Special branch, in fact, gives the second group of the solution analyzed in the weak coupling region.

Besides the trivial superconformal fixed point, we can obtain a non-trivial superconformal fixed point in IR by having a proper power of $t$ in the curve, where mutually non-local monopoles are massless, and call it the “Chebyshev branch”. To see these nontrivial fixed points, let us assume that both $N_f$ and $N$ are even. The curve is given by $y^2 = t^{N_f+2} \left( P_{(N-N_f-2)/2}(t) - 4\Lambda^{2(N-N_f-2)} \right)$. If we take the characteristic function as

$$P_{(N-N_f-2)/2}(t) = 2(\eta\Lambda)^{(N-N_f-2)} \mathcal{T}_{(N-N_f-2)} \left( \frac{\sqrt{t}}{2\eta\Lambda} \right)$$

with $\eta^{2(N-N_f-2)} = 1$, we have the curve \footnote{Here we have used a useful relation between the Chebyshev polynomials $\mathcal{T}_K(x) - 1 = (x^2 - 1)\mathcal{U}_{K-1}(x)$. They are defined as $\mathcal{T}_K(x) = \cos(Kx)$ and $\mathcal{U}_{K-1}(x) = \frac{1}{2} \frac{\partial \mathcal{T}_K(x)}{\partial x}$.}

$$y^2 = t^{N_f+2} 4(\eta\Lambda)^{2(N-N_f-2)} \left[ \mathcal{T}_{(N-N_f-2)} \left( \frac{\sqrt{t}}{2\eta\Lambda} \right) - 1 \right]$$

$$= t^{N_f+2} 4(\eta\Lambda)^{2(N-N_f-2)} \left[ \left( \frac{\sqrt{t}}{2\eta\Lambda} \right)^2 - 1 \right] \mathcal{U}_{(N-N_f-3)}^2 \left( \frac{\sqrt{t}}{2\eta\Lambda} \right).$$

Notice that although $\eta^{2(N-N_f-2)} = 1$, the characteristic function $P_{(N-N_f-2)/2}(t)$ is a function of $t$ and $\eta^2$, so we will have only the $(N-N_f-2)$ solution predicted from the weak coupling analysis. Notice also that except a factor $t^{N_f+2}$, there are single roots at $t = 0$ and $t = 4(\eta\Lambda)^2$ from the factor $\left[ \left( \frac{\sqrt{t}}{2\eta\Lambda} \right)^2 - 1 \right] \mathcal{U}_{(N-N_f-3)}^2 \left( \frac{\sqrt{t}}{2\eta\Lambda} \right)$, so finally we have a factor $t^{N_f+3}$ for this case \footnote{The second kind of Chebyshev polynomial $\mathcal{U}_K(x)$ contains a factor $x$ for odd degree $K$.}.

Notice that $(N_f+3)$ is an odd number which will be the property of the Chebyshev branch in the following discussions. Similar arguments can be made for other three cases. The Chebyshev branch discussed here will give the first group of the solutions in the weak coupling region.

### 3.2 The factorized form of a hyperelliptic curve

Now combining the proper factorization form of the $SO(N)$ curve with the massless flavors described in the previous subsection with the proper factorization form of the $U(N_i)$ curve with the flavors in [20], we can describe the general curve form. The basic idea is that first we need to have the proper prefactor (like $t^p$ for $SO(N)$ part and $(t-m^2)^{2r}$ for the $U(N_i)$ part at the $r$-th branch. More details about the power $p$ will be presented in a later section.). After
factorizing out these prefactors, we require the remaining curve to have the proper number of double roots and single roots, which will fix the form of factorization eventually.

The key part of the above procedure is the number of single roots. The basic rule is the following. For every $U(N_i)$ factor in the non-baryonic branch, we have two single roots while there is no single root in the baryonic branch. For the possible $SO(N)$ factor, if it is at the Chebyshev branch, we have two single roots where one of them is at the origin; i.e., we have a factor $t$ for that single root. However, if it is at the Special branch, there is no single root for the block $SO(N)$. Adding the single roots together for all blocks, we obtain the final number of the single roots. If there are $2n$ single roots, we will have a factor $F_{2n}(t)$ in the curve. Furthermore, as we will discuss soon, the polynomial $F_{2n}(t)$ has some relationship with the deformed superpotential $W'(x)$. One key point of all the above discussions is that the Special branch (or the Baryonic branch) will have one more double root than the Chebyshev branch (or the Non-baryonic branch) in the factorization form. This fact will be used repeatedly in the following study.

To demonstrate our idea, we consider the following examples in which $SO(2N_c)$ group is broken into the following three cases:

$$SO(2N_c) \rightarrow SO(2N_0) \times U(N_1), \quad SO(2N_c) \rightarrow U(N_c), \quad SO(2N_c) \rightarrow SO(2N_c),$$

where the generalization to multiple blocks will be trivial and straightforward. Similar analysis corresponding to $SO(2N_c + 1)$ can be done with no difficulty.

For the broken pattern $SO(2N_c) \rightarrow SO(2N_0) \times U(N_1)$, by counting the number of the single roots, we derive the following four possible curves

$$y^2 = tF_3(t)H_{N_c-2}^2(t), \quad (3.3)$$
$$y^2 = F_2(t)H_{N_c-1}^2(t), \quad (3.4)$$
$$y^2 = tF_1(t)H_{N_c-1}^2(t), \quad (3.5)$$
$$y^2 = H_{N_c}^2(t). \quad (3.6)$$

Curve (3.3) is for $SO(2N_0)$ at the Chebyshev branch and $U(N_1)$ at the non-baryonic branch. The reason is that first we need have four single roots. Secondly, because $SO(2N_0)$ is at the Chebyshev branch, one of the four single roots must be at the origin $t = 0$ and finally $F_1(t) = tF_3(t)$. Curve (3.4) is for $SO(2N_0)$ at the Special branch and $U(N_1)$ at the non-baryonic branch. Factor $F_2(t)$ with two single roots will record the information of $U(1) \subset U(N_1)$. Curve (3.5) is for $SO(2N_0)$ at the Chebyshev branch and $U(N_1)$ at the baryonic branch. Since $SO(2N_0)$ is at the Chebyshev branch we should have $F_2(t) = tF_1(t)$. Finally curve (3.6) is

\[\text{For the breaking pattern } SO(2N_c) \rightarrow U(N_c), \text{ the eigenvalues of } \Phi \text{ are the same and are nonzero while for } SO(2N_c) \rightarrow SO(2N_c), \text{ all of those are vanishing.}\]
for $SO(2N_0)$ at the Special branch and $U(N_1)$ at the baryonic branch, where no single root is required. Notice that the function $H_p(t)$ will have a proper number of $(t - m^2)$ or $t$ to count the prefactor for various branches.

For the broken pattern $SO(2N_c) \to U(N_c)$, we have

\begin{align}
y^2 & = F_2(t)H_{N_c-1}^2(t), \quad (3.7) \\
y^2 & = H_{N_c}^2(t), \quad (3.8)
\end{align}

where curve (3.7) is for $U(N_c)$ at the non-baryonic branch and curve (3.8), for $U(N_c)$ at the baryonic branch. Again, function $H_p(t)$ has a proper number of $(t - m^2)$ to count the prefactor for various $r$-th branches. It is also interesting to see that the curve form (3.7) is identical to the (3.4), which as we will show later, provides the possibility for smooth transition between these two breaking patterns.

For the broken pattern $SO(2N_c) \to SO(2N_c)$, we have

\begin{align}
y^2 & = tF_1(t)H_{N_c-1}^2(t), \quad (3.9) \\
y^2 & = H_{N_c}^2(t), \quad (3.10)
\end{align}

where curve (3.9) is for $SO(2N_c)$ at the Chebyshev branch and curve (3.10), for $SO(2N_c)$ at the Special branch. The function $H_p(t)$ will have a proper number of factor $t$ to count the prefactor required by Special or Chebyshev branch. It is noteworthy to notice that although both curves (3.5) and (3.9) appear to be identical, they can be distinguished by a factor $H_p(t)$ where different powers of $t$ and $(t - m^2)$ can arise. For example, there may be a factor $(t - m^2)$ in $H_p(t)$ of the curve (3.5), but it does not exist in $H_p(t)$ of the curve (3.9).

Before ending this subsection, let us give the rules for the counting of vacua. The total number of vacua is the product of the number of vacua of various blocks. For each block, there are four cases: $U(N_i)$ without flavors, $U(N_i)$ with flavors, $SO(N)$ without flavors, and $SO(N)$ with flavors. For $U(N_i)$ without flavors, the counting is given by the Witten index $w(N_i) = N_i$. For $U(N_i)$ with $M_i$ flavors, the result has been given in [20] and is summarized as follows:

\[
\text{The number of vacua} = \begin{cases} 
2N_i - M_i & r < M_i/2, \\
N_i - M_i/2 & r = M_i/2, \\
2N_i - M_i & r \geq M_i - N_i, \\
N_i - r & r < M_i - N_i, \\
1 & r = N_i - 1(\text{nonbaryonic}), \\
1 & r = M_i - N_i, N_i(\text{baryonic}).
\end{cases} \quad (3.11)
\]

For $SO(N)$ without flavors, the counting is given by the Witten index $w(N) = (N - 2)$ for $N \geq 5$ and 4, 2, 1, 1, 1 for $N = 4, 3, 2, 1, 0$ (notice that in previous subsection we have given an explanation of the Witten index for $N \leq 4$). For $SO(N)$ with $M$ flavors, there are $(N - M - 2)$ vacua from the Chebyshev branch and one vacuum from the Special branch (if it exists).
4 Quartic superpotential with massive flavors

In this section we will deal with and analyze the explicit examples with quartic tree level superpotential of degree 4 (that is, $k = 1$ from [2.2]) satisfying $W'(\pm m) = 0$,

$$W'(x) = x(x^2 - m^2)$$ (4.1)

where we do not have a constant term in $x$ because the superpotential $W(x)$ is a function of $x^2$ with a constant term; therefore, after differentiating this with respect to $x$, superpotential $W(x)$ will give rise to the above expression (4.1). In these examples the gauge group $SO(N_c)$ can break into three cases: two blocks with $SO(N_c) \rightarrow SO(N_0) \times U(N_1)$, $N_c = N_0 + 2N_1$ under the semiclassical limit $\Lambda \rightarrow 0$ which we will call the non-degenerated case and one block with

$$SO(N_c) \rightarrow SO(N_c) \quad \text{or} \quad SO(N_c) \rightarrow U([N_c/2])$$

which we will call the degenerated case. Since for the sake of simplicity, we consider the case with equal masses of flavors, we can write the Seiberg-Witten curves for these gauge theories as follows:

$$y^2 = P_{2N_c}^2(x) - 4x^4A^{4N_c-2N_f-4}(x^2 - m^2)^{N_f} \quad \text{for } SO(2N_c),$$

$$y^2 = P_{2N_c}^2(x) - 4x^2A^{4N_c-2N_f-2}(x^2 - m^2)^{N_f} \quad \text{for } SO(2N_c + 1),$$

where $m$ in these curves is the same as the one in (4.1) \footnote{As in [20], we use the notation for hat in $U(N_1)$ to denote a gauge theory with flavors charged under the $U(N_1)$ group.}. On the $r$-th branch these Seiberg-Witten curves are factorized as follows, based on the discussion of previous section 3 and [73, 20],

$$(x^2 - m^2)^{2r}\left[P_{2(N_c-r)}^2(x) - 4x^4A^{4N_c-2N_f-4}(x^2 - m^2)^{N_f-2r}\right] \quad \text{for } SO(2N_c), \quad (4.2)$$

$$(x^2 - m^2)^{2r}\left[P_{2(N_c-r)}^2(x) - 4x^2A^{4N_c-2N_f-2}(x^2 - m^2)^{N_f-2r}\right] \quad \text{for } SO(2N_c + 1), \quad (4.3)$$

where the $(x^2 - m^2)^{2r}$ is the prefactor for that particular branch. For the expressions in the bracket we need to count the proper number of single roots and double roots as given in previous \footnote{For gauge theories with massless flavors we will study in next section in which the tree level superpotential (5.25) is different from (4.1) and all the flavors must be charged under $SO(N_0)$, not $U(N_1)$. For degenerated case, $SO(N_c) \rightarrow SO(N_c)$, the flavors are charged under the factor $SO(N_c)$.}
section 3. Furthermore, there are two special $r$’s which give the baryonic branch of $U(N_1)$ factor

$$r = N_1, \quad \text{or} \quad r = N_f - N_1. \quad (4.4)$$

If the $r$ does not satisfy this condition, it is called a nonbaryonic branch.

For convenience we list the proper factorization form of the curve in various situations. Note that the roots for $W'(x)$ are $x = 0$ and $x = \pm m$ from (4.1). For the non-baryonic branch of $U(N_1)$ factor with some D5-branes wrapping around the origin $x = 0$, the curve corresponding to (3.3) is

$$P^2_{2(N_c-r)} - 4x^2\Lambda^{4N_c-2N_f-4}(x^2 - m^2)^{N_f-2r} = x^2H_{2N_c-2r-4}^2(x)F_6(x) \quad \text{for } SO(2N_c), \quad (4.5)$$

$$P^2_{2(N_c-r)} - 4x^2\Lambda^{4N_c-2N_f-2}(x^2 - m^2)^{N_f-2r} = x^2H_{2N_c-2r-4}^2(x)F_6(x) \quad \text{for } SO(2N_c+1). \quad (4.6)$$

For the baryonic branch of $U(N_1)$ factor with some D5-branes wrapping around the origin $x = 0$ (or for the degenerated case where all D5-branes are wrapping around the origin $x = 0$, the curve has the same form: (3.9)), the curve corresponding to (3.5) is

$$P^2_{2(N_c-r)} - 4x^2\Lambda^{4N_c-2N_f-4}(x^2 - m^2)^{N_f-2r} = x^2H_{2N_c-2r-2}^2(x)F_2(x) \quad \text{for } SO(2N_c), \quad (4.7)$$

$$P^2_{2(N_c-r)} - 4x^2\Lambda^{4N_c-2N_f-2}(x^2 - m^2)^{N_f-2r} = x^2H_{2N_c-2r-2}^2(x)F_2(x) \quad \text{for } SO(2N_c+1). \quad (4.8)$$

For the non-baryonic branch of $U(N_1)$ factor (or for degenerated case with all D5-branes wrapping around the $x = \pm m$, the curve is the same: (3.7)) the curve corresponding to (3.4) is

$$P^2_{2(N_c-r)} - 4x^2\Lambda^{4N_c-2N_f-4}(x^2 - m^2)^{N_f-2r} = H_{2N_c-2r-2}^2(x)F_4(x) \quad \text{for } SO(2N_c), \quad (4.9)$$

$$P^2_{2(N_c-r)} - 4x^2\Lambda^{4N_c-2N_f-2}(x^2 - m^2)^{N_f-2r} = H_{2N_c-2r-2}^2(x)F_4(x) \quad \text{for } SO(2N_c+1). \quad (4.10)$$

Finally for the baryonic branch of $U(N_1)$ factor (or for degenerated case with all D5-branes wrapping around the $x = \pm m$, the curve takes the same form: (3.8) or (3.10)) the curve corresponding to (3.6) is

$$P^2_{2(N_c-r)} - 4x^2\Lambda^{4N_c-2N_f-4}(x^2 - m^2)^{N_f-2r} = H_{2N_c-2r}^2(x) \quad \text{for } SO(2N_c), \quad (4.11)$$

$$P^2_{2(N_c-r)} - 4x^2\Lambda^{4N_c-2N_f-2}(x^2 - m^2)^{N_f-2r} = H_{2N_c-2r}^2(x) \quad \text{for } SO(2N_c+1). \quad (4.12)$$

In fact, these curves have been given in section 3.2 where we used the fact that $t = x^2$ and $r = 0$.

Next we want to clarify the number of flavors $N_f$ that we will study below. For the exponent of $\Lambda$ in (4.2) and (4.3) to be positive, we will concentrate on $SO(2N_c)$ gauge theories with the

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12 This holds also for the degenerated case: $SO(N_c) \to U([N_c/2])$. In this case, $N_1$ becomes $N_c/2$.

13 Moreover, the curves corresponding to (3.7), (3.8), (3.9) and (3.10) can be written similarly and we will describe them in the following discussions when we need to explain each case.
condition $N_f < 2N_c - 2$ and $SO(2N_c + 1)$ gauge theories with the condition $N_f < 2N_c - 1$, which are the conditions for the theories to be asymptotically free. Thus taking into account of $r = \min \left( N_c, \frac{N_f}{2} \right)$, we will study the examples with the condition

$$r \leq \frac{N_f}{2}. \quad (4.13)$$

Finally let us emphasize the comments on the relation between function $F_6(x)$ appearing in (4.5) and (4.6) and $W'(x)$ from (4.1). According to relation (A.6), we have to take care of the second term of (A.6) on the left hand side when the number of flavors $N_f$ is greater than $(2N_c - 4)$ for $SO(2N_c)$ gauge theories and $(2N_c - 3)$ for $SO(2N_c + 1)$ gauge theories. This is a kind of new phenomenon that did not appear in the pure case [19]. For example, on the non-baryonic branch of $U(N_1)$ factor for $SO(4)$ with $N_f = 1$, $SO(5)$ with $N_f = 2$, $SO(6)$ with $N_f = 3$ and $SO(7)$ with $N_f = 4$ cases, we must be attentive to these modifications due to the presence of flavors.

Now we are ready to deal with the explicit examples for $SO(N_c)$ gauge theory with massive flavors where $N_c = 4, 5, 6, 7$. The number of flavors $N_f$ is restricted to $N_f < N_c - 2$ and the index $r$ satisfies (4.13). Therefore, for given number of colors $N_c$, the quantities $N_f$ and $r$ are fixed.

### 4.1 $SO(4)$ case

For this gauge theory we will discuss the number of flavors $N_f = 0, 1$ cases. There are three breaking patterns $SO(4) \to SO(2) \times \widehat{U}(1)$, $SO(4) \to SO(4)$, and $SO(4) \to \widehat{U}(2)$.

#### 4.1.1 $N_f = 0$

1. Non-degenerated case

In [19], the case with $N_f = 0$ was discussed. Since the factorization problem was trivial, the characteristic function $P_4(x)$ could be represented as

$$P_4(x) = x^2(x^2 - v^2).$$

From this $P_4(x)$, we could obtain a tree level superpotential and a deformed function $f_2(x)$,

$$W'(x) = x(x^2 - v^2), \quad f_2(x) = -4x^2\Lambda^4.$$ 

There is only one vacuum for a given $W'(x)$. To satisfy the condition $W'(\pm m) = 0$ discussed above, we should have $v^2 = m^2$. Under the semiclassical limit $\Lambda \to 0$, the gauge group $SO(4)$ breaks into $SO(2) \times U(1)$. Notice that the Witten index $w(2)$ of $SO(2)$ is one and therefore
the total number of vacua becomes one from the weak coupling analysis. We realize that the number of vacua from the strong coupling analysis coincides with the one from the weak coupling analysis.

In fact, by factoring out $x^2$ common to $W'(x)^2$ and $f_2(x)$, the full curve can be written as

$$y^2 = x^2 \left(W'(x)^2 + f_2(x)\right) = [(x^2 - v^2)^2 - 4\Lambda^4](x^2)^2 \equiv F_4(x)H_2^2(x),$$

which indicates that the $SO(2)$ is at the Special branch where the power of $x^2$ is two (even) in the above curve and that $U(1)$ at the nonbaryonic $r = 0$ branch because the condition (4.4) is not satisfied. See also (4.9).

2. Degenerated case

The above calculations for nondegenerated case assume that every root of $W'(x)$ has at least one D5-brane wrapping around it. However, there are the situations where some roots do not have wrapping D5-branes around them. For our example, there are two cases; one is that there is no D5-brane around zero $(x = 0)$ and the other is that there is no D5-brane around $x = \pm m$. For both cases, the curve corresponding to (3.7) should be factorized with factor $F_4(x)$ as

$$y^2 = P_4^2(x) - 4\Lambda^4x^4 = F_4(x)H_2^2(x),$$

where $F_4(x)$ can have a factor $x^2$ further depending on whether the $SO(N)$ factor is at the Chebyshev branch or not. See also (3.9). Writing $P_4(x) = (x^4 - s_1x^2 + s_2)$, $H_2(x) = (x^2 + a)$ and $F_4(x) = (x^4 + bx^2 + c)$, we have the following solutions

$$b = 2(a \mp 2\Lambda^2), \quad c = a^2, \quad s_1 = -2a \pm 2\Lambda^2, \quad s_2 = a^2.$$  

If $\Lambda \to 0$, but $a \neq 0$, we get a symmetry breaking $SO(4) \to U(2)$ where $U(2)$ is at the nonbaryonic $r = 0$ branch. If the classical limit goes to $\Lambda, a \to 0$ where $F_4(x)$ contains the $x^2$ factor, we get $SO(4) \to SO(4)$. It is obvious that there is no smooth interpolation between these vacua because there exists a $U(1)$ at the IR for the former symmetry breaking while there exists no $U(1)$ for the latter symmetry breaking.

To count the vacua, we need to determine the value of $a$. If there are D5-branes wrapping around the $x = \pm m$, we get $b = -2m^2$. Putting it back to the expression for $b$, we found two values for $a$ (each sign gives one solution for $a$); thus, there exist two vacua for $SO(4) \to U(2)$. If all D5-branes wrap around the origin $x = 0$, there are two cases to be distinguished. The first case is that the $SO(4)$ factor is at Chebyshev vacua, so we need a factor $x^2$ in the curve which determines the value of $c = 0$ (so $a = 0$ too). There are two vacua corresponding to the $\pm$ sign. It is noteworthy that the curve is $y^2 = (x^2)^3(x^2 + b)$ with the power of $x^2$ being three. This is
the character of $SO(4)$ without flavors at the Chebyshev branch. More detailed discussion of the power of $t = x^2$ can be found in the next section. The second case is that the $SO(4)$ factor is at the Special vacua, so $F_4(x)$ should be a complete square form. This determines the value of $a$ as $a = \Lambda^2$ if $b = 2(a - 2\Lambda^2)$ or $a = -\Lambda^2$ if $b = 2(a + 2\Lambda^2)$. Although it seems that we have two solutions, in fact, the curve is the same for both cases and presumably we should count them as one.

However, it is known that for $SO(4) = SU(2)_L \times SU(2)_R$, we should have $2 \times 2 = 4$ vacua. Among these four vacua, two of them have the relations $S_L = S_R$ and $S_L + S_R \neq 0$, so they correspond to the above two solutions at the Chebyshev vacua. The other two have the relation $S_L + S_R = 0$, but we found only one solution for the curve at the Special vacua. Our understanding is that since $S_L + S_R = 0$ for these two vacua, they correspond to the same point in the $\mathcal{N} = 2$ curve. Later we will meet same problem again and again, where from the curve we find only one solution for $SO(4)$ at the Special branch, but it should be counted as two vacua.

4.1.2 $N_f = 1$

In this case, from the condition $r \leq \frac{N_f}{2}$ we have only the $r = 0$ branch. For the breaking pattern $SO(4) \rightarrow SO(2) \times \widehat{U}(1)$, the $r = 0$ branch can be a non-baryonic or baryonic branch. On the other hand, for the breaking pattern $SO(4) \rightarrow \widehat{U}(2)$, the $r = 0$ is non-baryonic since $r \neq N_f - N_1$ and $r \neq N_1$.

- **Baryonic:** $r = 0$ branch

Since we are considering a baryonic branch, we have an extra massless monopole (that is, one more double root) and from the relation (4.7) the following factorization problem arises,

$$y^2 = P_4^2(x) - 4x^4\Lambda^2(x^2 - m^2) = x^2H_2^2(x)F_2(x).$$

To be consistent on both sides, the characteristic function $P_4(x)$ on the left hand side must have an $x^2$ factor, i.e., $P_4(x) = x^2(x^2 - a^2)$ due to the presence of an $x^2$ factor in the right hand side. In addition, we assume $H_2(x) = (x^2 - b^2)$. Assuming that $b \neq 0$ (the $b = 0$ case will correspond to $SO(4) \rightarrow SO(4)$ and will be discussed later), we must have $F_2(x) = x^2$, $a^2 = (m^2 - \Lambda^2)$, and $b^2 = (m^2 + \Lambda^2)$; thus, the characteristic function and $F_2(x)$ behave as

$$P_4(x) = x^2\left(x^2 + \Lambda^2 - m^2\right), \quad F_2(x) = x^2.$$

Under the semiclassical limit $\Lambda \rightarrow 0$, the gauge group $SO(4)$ breaks into $SO(2) \times \widehat{U}(1)$. In this case, we have only one vacuum. This number matches the one obtained from the weak coupling analysis, because both $SO(2)$ and $\widehat{U}(1)$ with $N_f = 1$ theories have only one vacuum. Notice also that for this special case, the curve $y^2$ has a total square form with a factor $x^4$ implying that $SO(2)$ is at Special branch.
• Non-baryonic $r = 0$ branch

1. Non-degenerated case

In this case the factorization problem (4.5) becomes as

$$P_4^2(x) - 4x^4\Lambda^2(x^2 - m^2) = x^2F_6(x).$$

This equation is easily solved if we assume that $P_4(x) = x^2P_2(x) \equiv x^2(x^2 - A)$ by recognizing that there is a factor $x^2$ on the right hand side and finally one obtains

$$F_6(x) = x^2\left[(x^2 - A)^2 - 4\Lambda^2(x^2 - m^2)\right].$$

Taking into account of (A.6) we have $m^2 = A$. Under the semiclassical limit $\Lambda \rightarrow 0$, the characteristic function goes to $P_4(x) = x^2(x^2 - m^2)$, which means that the gauge group $SO(4)$ breaks into $SO(2) \times \hat{U}(1)$. We have only one vacuum, which matches the counting obtained by general analysis given in previous section (weak coupling analysis). The curve $y^2$ has a total square form with a factor $x^4$ implying that $SO(2)$ is at Special branch.

2. Degenerated case

The curve should be factorized as, together with a term $F_4(x)$ on the right hand side as before,

$$y^2 = P_4^2(x) - 4x^4\Lambda^2(x^2 - m^2) = F_4(x)H_2^2(x).$$

Parameterized by $P_4(x) = (x^4 - s_1x^2 + s_2)$, $H_2(x) = (x^2 + a)$ and $F_4(x) = (x^4 + bx^2 + c)$ we have the following relations

$$s_1 = \frac{-2a - b - 4\Lambda^2}{2},$$
$$s_2 = \frac{4c + 4a(b - 4\Lambda^2) - (b + 4\Lambda^2)^2 - 16\Lambda^2m^2}{8},$$
$$c = -\left(\frac{32a^2\Lambda^2 + (b + 4\Lambda^2)^3 + 16\Lambda^2(b + 4\Lambda^2)m^2 + a(-2b^2 + 16b\Lambda^2 + 32\Lambda^2(3\Lambda^2 + m^2))}{8a - 4(b + 4\Lambda^2)}\right),$$
$$0 = a\left(4a^2 - 4a(b - 5\Lambda^2) + (b + 4\Lambda^2)^2\right)$$
$$+ \left((-2a + b)^2 + 8(4a + b)\Lambda^2 + 16\Lambda^4\right)m^2 + 16\Lambda^2m^4.$$  

To see the limit where there is D5-brane wrapping around $x = \pm m$, we put $b = -2m^2 - 4\Lambda^4$.

\(^{14}\)To discuss the smooth transition, we take $a$ to be free parameter while the mass $m$ is determined by $a$, i.e., $m$ will change as the $a$ does. This is the philosophy used in [18, 20] (see, for example, the equation (6.30) in [20]) which is very convenient for this purpose. We can also use the direct method which will be demonstrated later by one simple example. To count the vacua, we really need to fix $m$ and then solve $a$.  

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by the relationship of $F_4(x)$ and $(x^2 - m^2)^{15}$ and get

$$0 = a^2 \left( a + 9\Lambda^2 \right) + 3a \left( a + 4\Lambda^2 \right) m^2 + \left( 3a + 4\Lambda^2 \right) m^4 + m^6.$$ 

There are two limits we can take: (1) $\Lambda \to 0$, but $a \to$ constant. In this limit, all three solutions provide $m^2 \to -a$ and it is $SO(4) \to \widehat{U}(2)$; (2) $\Lambda \to 0$, $a \to 0$ which gives $SO(4) \to SO(4)$. However, from the above equation by keeping the last term $m^6$, we see that all of the solutions give $m^2 \to 0$ in this limit. Since we require $m^2$ to be nonzero (that is, we are considering massive flavors), this limit is inconsistent with our assumption and should not be taken. In fact, as we have seen, the non-baryonic $r = 0$ branch of $SO(2) \times \widehat{U}(1)$ theory will have a factor $x^4$ while for $\widehat{U}(2)$ theory, the curve does not have any factor $x^2$, so we should not expect any smooth transition between them. To count the number of vacua, we fix $m$ and then solve $a$. There exist three solutions. They match the counting, $(2N - N_f) = (2 \times 2 - 1) = 3$, of $\widehat{U}(2)$ with one flavor.

If all D5-branes wrap around the origin $x = 0$, there are two cases to be considered. The first one is when the extra $x^2$ factor occurs in $F_4(x)$: $F_4(x) = x^2(x^2 + d)$ or $c = 0$. In fact, it is the same curve of the baryonic $r = 0$ branch, but with $H_2(x) = x^2$. There are two solutions with the expression $P_4(x) = x^2(x^2 \pm 2\Lambda^2)$ by $\pm$ sign providing a breaking $SO(4) \to SO(4)$ under the semiclassical limit $\Lambda \to 0$. These two vacua are the Chebyshev vacua of $SO(4)$ gauge theory (notice that the counting becomes $N - N_f - 2 = 4 - 0 - 2 = 2$) with an overall factor $x^6$ in the curve. The second one is that function $F_4(x)$ should be a complete square form. There are two solutions. One of them is $P_4(x) = x^2(x^2 - [m^2 - \Lambda^2])$ which gives a breaking $SO(4) \to SO(2) \times \widehat{U}(1)$ under the semiclassical limit $\Lambda \to 0$ (it is the baryonic $r = 0$ branch found above) and other, $P_4(x) = x^4 + \Lambda^2 x^2 - m^2 \Lambda^2$ which provides a breaking $SO(4) \to SO(4)$ under the semiclassical limit $\Lambda \to 0$ and $SO(4)$ is at the Special branch. In other words, among the four vacua of $SO(4)$, two of them are found in the Chebyshev vacua and two of them correspond to the same point in the Special vacua.

Now we summarize what we have obtained in Table 1 by specifying the flavors $N_f$, symmetry breaking patterns, various branches, the exponent of $t = x^2$ in the curve, $U(1)$ at the IR, the number of vacua, and the possibility of smooth transition. It turned out that the number of vacua is exactly the same as from the weak coupling analysis.

### 4.2 $SO(5)$ case

For this gauge theory we will discuss the number of flavors $N_f = 0, 1, 2$ cases. There are three breaking patterns $SO(5) \to SO(3) \times \widehat{U}(1)$, $SO(5) \to SO(5)$ and $SO(5) \to \widehat{U}(2)$.

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15If we use $t = x^2$, the curve is effectively the one for $U(2)$ with three flavors where two of them are massless and one is massive. Then using the relation $F_4(x) + 4\Lambda^2 x^2 \sim (x^2 - m^2)^2 + O(x^0)$, we can read off $F_4(x) = x^4 - (2m^2 + 4\Lambda^2)x^2 + m^4 + d$. 

---
| $N_f$ | Group | Branch | Power of $t(=x^2)$ | $U(1)$ | Number of vacua | Connection |
|------|-------|--------|-------------------|--------|----------------|------------|
| 0    | $SO(2) \times U(1)$ | $(S,0_{NB})$ | $t^2$ | 1 | 1 |           |
|      | $SO(4)$ | (C) | $t^4$ | 0 | 2 |           |
|      |        | (S) | $t^0$ | 0 | 2 |           |
|      | $U(2)$ | (0_{NB}) | $t^0$ | 1 | 2 |           |
| 1    | $SO(2) \times U(1)$ | $(S,0_{NB})$ | $t^2$ | 1 | 1 |           |
|      |        | (S,0_{B}) | $t^0$ | 0 | 1 |           |
|      | $SO(4)$ | (C) | $t^4$ | 0 | 2 |           |
|      |        | (S) | $t^0$ | 0 | 2 |           |
|      | $U(2)$ | (0_{NB}) | $t^0$ | 1 | 3 |           |

Table 1: The summary of the phase structure of the $SO(4)$ gauge group. Here we use $C/S$ for Chebyshev or Special branch for $SO(N_i)$ factor and $r_{NB}/r_B$ for the $r$-th non-baryonic or $r$-th baryonic branch. In this table, we list the power of $t$ and the $U(1)$ which is present in the nonbaryonic branch, at the IR. They are the indices to see whether two phases could have smooth transition. In the case of massive flavors, the power of $t$ is 3 for $SO(N_i)$ for $N_i$ even and 1 for $SO(N_i)$ for $N_i$ odd at the Chebyshev branch. For massive flavors, only $SO(N_i)$ where $N_i = 2, 3, 4$ can be at the Special branch and the power is 2, 0, 0 for $N_i = 2, 3, 4$. For this table, we see that for both $N_f = 0$ and $N_f = 1$, there are no two phases having the same factor $t^p$ where $p$ is some number and the number of $U(1)$, so there exists no smooth transition between them.

4.2.1 $N_f = 0$

1. Non-degenerated case

The theory with $N_f = 0$ was discussed in [19]. Since the factorization problem was trivial, the characteristic function was given by,

\[ P_4(x) = x^2(x^2 - l^2). \]

To satisfy the condition for $W'(\pm m) = 0$, we must have the relation, $l^2 = m^2$. Under the semiclassical limit $\Lambda \to 0$, the gauge group $SO(5)$ breaks into $SO(3) \times U(1)$ where $SO(3)$ is at the Chebyshev branch with a factor $x^2$ in the curve while $U(1)$ is at the nonbaryonic branch. However, since the Witten index $w(3) = 2$, we expect to find two solutions instead of one. In fact, another solution will be found in the degenerated case $^{16}$.

2. Degenerated case

$^{16}$For the two vacua of $SO(3)$, one is at the Chebyshev branch having factor $t = x^2$ and another, at the Special branch with factor $t^0$. However, since the Special branch has extra double roots just like the one in the degenerated case, they have the same factorization form of the curve and for simplicity we use the degenerated case to include all of them.
For D5-branes wrapping around only one root of $W'(x)$ (either the origin or $x = \pm m$), the curve is

$$y^2 = P_2^2(t) - 4\Lambda^6 t = F_2(t)H_1^2(t), \quad t \equiv x^2.$$  

First let us discuss a case in which there are D5-branes wrapping around $(\pm \sqrt{3}m - \Lambda^6/3)$ curve is

Now we can discuss the smooth interpolation. First using the relationship of

Let us consider the following expression in the above

$$\gamma \equiv \frac{2}{3},$$

so the breaking pattern is $SO(5) \rightarrow U(2)$. If $\Lambda \rightarrow 0$, but $\Lambda^6/b \equiv \gamma \neq 0, a = m^2 \rightarrow -4\gamma$ and $P_2(t) \rightarrow t(t + 4\gamma)$, the breaking pattern is $SO(5) \rightarrow SO(3) \times U(1)$ where $SO(3)$ is at the Special branch. From this we see that $U(2)$ is smoothly connected to $SO(3) \times U(1)$. It is noteworthy that at the Special point of $SO(3)$ without flavor, the power of $t$ is given by

$$(2N_f - N + 3) = (0 - 3 + 3) = 0$$

According to the argument in section 3.1.2.  

Because of this, both $SO(3) \times U(1)$ and $U(2)$ can have the same form of curve.

Now let us count the number of vacua. To do so, we need to fix the mass $m^2$ and solve $b$. Using the relation $a = m^2$, we find three solutions for $b$. It is obvious that two of them correspond to the limit $b^2/\Lambda^6 \equiv \beta \neq 0$ which gives the two vacua of $U(2)$ and the third one, to the limit $\Lambda^6/b \equiv \gamma \neq 0$ which gives the vacua of $SO(3) \times U(1)$. The third one is the missing one for $SO(3)$ mentioned in the non-degenerate case.

The above discussion shows the smooth transition clearly. However, to deepen our understanding, we use another method by solving the $b$ in terms of the $m^2$. One of these three solutions will turn out

$$b = \frac{-4\left(\frac{2}{3}\right)^4 \Lambda^6 m^2}{\left(-9\Lambda^{12} + \sqrt{3} \Lambda^{18} (27\Lambda^6 - 4m^6)\right)^{\frac{3}{4}}} - 2 \left(\frac{2}{3}\right)^4 \left(-9\Lambda^{12} + \sqrt{3} \Lambda^{18} (27\Lambda^6 - 4m^6)\right)^{\frac{1}{3}}.$$  

Let us consider the following expression in the above

$$\left(-9\Lambda^{12} + \sqrt{3} \Lambda^{18} (27\Lambda^6 - 4m^6)\right)^{\frac{1}{3}} = \Lambda^3 \left(-9\Lambda^3 + \sqrt{3} \Lambda^{18} (27\Lambda^6 - 4m^6)\right)^{\frac{1}{3}}$$

$$= 12^{\frac{1}{3}} \Lambda^3 m\omega_6, \quad \omega_6^6 = -1, \quad \Lambda \rightarrow 0$$

\[17\text{The proper power of } t \text{ at the Chebyshev branch and Special branch will be summarized in the next section, where more details can be found.}\]
where we have taken the limit $\Lambda \to 0$ in the last line, but carefully kept the phase factor $\omega_6$. The phase factor comes from the expression
\[
(27\Lambda^6 - 4m^6)^\frac{1}{6} = \left(4(m^2 - \alpha_1)(m^2 - \alpha_2)(m^2 - \alpha_3)\right)^\frac{1}{6}
\]
where three roots $\alpha_i$ are the order $\Lambda^2$. If we start from very large $m^2$, surround only one root once and go back to large $m^2$, there will be a phase factor $e^{\frac{2\pi i}{6}} = \eta$ where $\eta^6 = 1$. Thus depending on the path, in general we have
\[
b \sim -\frac{4}{\sqrt{3}}\Lambda^3 m[\omega_6 \eta^k + \omega_6^{-1} \eta^{-k}]\]
For general $k$, we find $[\omega_6 \eta^k + \omega_6^{-1} \eta^{-k}] \neq 0$ and $b \sim \Lambda^3$. This case gives the breaking pattern $SO(5) \to U(2)$ in the limit $\Lambda \to 0$. However, when $k = 1$, we have $(\omega_6 \eta)^2 = \eta^3 = -1$ (where we have used $\omega^2_6 = \eta$) and $[\omega_6 \eta^k + \omega_6^{-1} \eta^{-k}] = 0$. This case will give rise to $b \sim \Lambda^6$ and the breaking pattern $SO(5) \to SO(3) \times U(1)$. From this analysis we see that starting from $k = 0$ and surrounding one root once, we make a smooth transition from one phase to another phase.

The above analysis demonstrates explicitly the origin of the smooth transition among the phases in the parameter space of deformed superpotential. However, in most cases it is hard to make such a detailed analysis. It is more convenient to use $b$ as a free parameter while $m^2$ is determined by $b$ when our focus is just the smooth transition. Later we mainly use this method. To count the number of vacua, we really need to fix $m^2$ and find the number of solutions of $b$. Although we cannot solve this exactly most of the time, the counting can be done easily by observing the degree of the equation and various limits we take.

Finally we consider a case in which all D5-branes wrap around the origin $x = 0$. In this case, we need to have a factor $t$ in $F_2(t)$, or $a^2 + b = 0$ using the above results. There are three solutions $b = -4\Lambda^4, 2(1 \pm i\sqrt{3})\Lambda^4$ which give three vacua with the breaking pattern $SO(5) \to SO(5)$ where $SO(5)$ is at the Chebyshev branch. The counting goes like $(N - 2) = 3$ which is consistent with the one from both the strong and weak coupling analyses.

### 4.2.2 $N_f = 1$

In this case, there is only one nondegenerated breaking pattern $SO(5) \to SO(3) \times \hat{U}(1)$ where we can have both non-baryonic and baryonic $r = 0$ branch due to the fact $r = N_f - N_1$ and $r \neq N_1$. Besides that, we have two degenerated breaking patterns $SO(5) \to SO(5)$ and $SO(5) \to \hat{U}(2)$ where only the non-baryonic $r = 0$ branch exists because the relations $r \neq N_f - N_1$ and $r \neq N_1$ hold.

---

18Notice that for this case, $a, b$ are fixed by this equation $a^2 + b = 0$, so there is no free parameter in the curve to be adjusted. This explains why the above smooth transition, where $b$ is a free parameter, does not include this phase.
Baryonic $r = 0$ branch

The curve should be factorized together with a function $F_2(x)$ as

$$y^2 = P_4^2(x) - 4x^2\Lambda^4(x^2 - m^2) = x^2H_2^2(x)F_2(x).$$

Setting $H_2(x) = (x^2 - b)$, $F_2(x) = x^2 - a$ and $P_4(x) = x^2(x^2 - s_1)$, we get

$$s_1 = b - \frac{2\Lambda^4m^2}{b^2}, \quad a = -\frac{4\Lambda^4m^2}{b^2}, \quad -b^4 + b^3m^2 + \Lambda^4m^4 = 0. \quad (4.14)$$

From (4.14) we see two limits: (1) If $\Lambda \to 0$ and $b \to$ constant, by keeping the first and second terms we find $m^2 \to b$ and the gauge group is broken to $SO(5) \to SO(3) \times \hat{U}(1)$ where $SO(3)$ is at the Chebyshev branch and $U(1)$, the baryonic $r = 0$ branch; (2) If $\Lambda \to 0$, but $b \sim \Lambda^4/3$, by keeping the second and the third terms we find $m^2 \to b^3/\Lambda^4 \sim$ finite and the gauge group is broken to $SO(5) \to SO(5)$. Thus, we see a smooth transition between these two phases. This is also consistent with the indices. Both phases have the same $t^1$ and zero $U(1)$ factor at IR.

Now we can count the vacua by solving $b$ for fixed $m^2$. Clearly there are four solutions. Keeping the first and second terms we get one solution $b \sim -m^2$ which has the classical limit $SO(3) \times \hat{U}(1)$ where $SO(3)$ is at the Chebyshev branch. Keeping the second and third terms we get three solutions $b \sim \Lambda^4/3$ which give the three vacua of $SO(5)$ and are consistent with the counting $N - 2 = 3$ from the weak coupling analysis.

To show clearly the rightness of the above result, we use the $m^2$ to solve $b$ directly. One of these four solutions will be

$$\Delta = -9\Lambda^4m^8 - \sqrt{3}\Lambda^4m^6\sqrt{256\Lambda^4 + 27m^4},$$

$$\Sigma = \frac{\Delta^+}{2\sqrt{3}^+} + \frac{m^4}{4} - \frac{4\Delta^+}{3\Delta^+} \Lambda^4m^4,$$

$$b = \frac{m^2}{4} - \frac{1}{2}\sqrt{\Sigma} - \frac{1}{2}\sqrt{-\Sigma + \frac{3m^4}{4} - \frac{m^6}{4\sqrt{\Sigma}}}.$$ 

Let us consider the classical limit $\Lambda \to 0$. First we have $\Delta \leq \Lambda^4$. Using this factor and considering the phase factor as we did for the $SO(5)$ with $N_f = 0$, we have $\sqrt{\Sigma} \sim \eta \frac{m^2}{2}$ with $\eta^2 = 1$. Using this, we finally have

$$b \sim \frac{m^4}{4} \left(1 - \eta - \omega \sqrt{2(1 - \eta)}\right), \quad \omega^2 = 1$$

with two phase factors $\eta, \omega$ depending on the path of $m^2$. For $\eta = 1$ we have $b \sim 0$ at the classical limit. For $\eta = -1$, $\omega = 1$ again $b \sim 0$. However, for $\eta = -1$, $\omega = -1$ we have $b \sim m^2$. The limit of $b \sim 0$ gives the breaking $SO(5) \to SO(5)$ while the limit $b \sim m^2$ gives the breaking $SO(5) \to SO(3) \times U(1)$. These calculations confirm our conclusions derived by the simpler method.
For another vacuum of $SO(3) \times \widetilde{U(1)}$ where $SO(3)$ is at the Special branch, we just need to choose $P_4(x) = x^2(x^2 - m^2) + \Lambda^4$ so that $y^2 = (x^2(x^2 - m^2) + \Lambda^4)^2$ is a total square form.

- **Non-baryonic $r = 0$ branch**

1. **Non-degenerated case**

The curve should be

$$y^2 = P_2^2(x) - 4x^2\Lambda^4(x^2 - m^2) = x^2F_6(x).$$

It is easy to solve and it turns out $P_4(x) = x^2(x^2 - s)$ and $F_6(x) = x^2[(x^2 - s)^2 - 4\Lambda^4(x^2 - m^2)]$. To determine the parameter $s$, by noticing that for $N_f = 1$, $F_6(x) = W'(x)^2 + a x^2 + b$, we get $s = m^2$. Here we found only one solution for $SO(5) \to SO(3) \times \widetilde{U(1)}$ where $SO(3)$ is at the Chebyshev branch with a factor $x^2$ in the curve. Another one will be given from the Special vacua of $SO(3)$ in the degenerated case.

2. **Degenerated case**

For the degenerated case where D5-branes wrap around only one root, the curve should be the form with a factor $F_4(x)$

$$y^2 = P_2^2(x) - 4x^2\Lambda^4(x^2 - m^2) = F_4(x)H_2^2(x).$$

Parameterizing as $P_4(x) = (x^4 - s_1x^2 + s_2)$, $H_2(x) = (x^2 + a)$, and $F_4(x) = (x^2 - b)^2 + c$, we found the following solutions

$$s_1 = -a + b, \quad s_2 = \frac{-2ab + c + 4\Lambda^4}{2}, \quad c = \frac{4\Lambda^4(a - b + m^4)}{a + b}.$$

Using the relationship between $F_4(x)$ and $(x^2 - m^2)$, we get $b = m^2$. Putting it into the last equation above we get

$$0 = (a + m^2)^3 a - 4\Lambda^4 a(a + m^2) - \Lambda^4 m^4. \quad (4.15)$$

This equation (4.15) has three solutions of $m^2$. If $\Lambda \to 0$, but $a \to$ constant, keeping the first and third terms we have all three $m^2$ going to $-a$ and $(a + m)^2 \sim \Lambda^{4/3}$. This limit gives $SO(5) \to \widetilde{U(2)}$. If $\Lambda \to 0$ and $a \to 0$ but $a \sim \Lambda^4$, keeping the first and third terms we get one $m^2 \to \Lambda^4/a \neq 0$ (another two solutions of $m^2$ go to 0). This limit gives $SO(5) \to SO(3) \times \widetilde{U(1)}$ where $SO(3)$ is at the Special branch. From this discussion we see that at least one solution of $m^2$ gives a smooth transition between these two phases.

To count the vacua, notice that for fixed $m^2$ we have four solutions $a$ from (4.15). Keeping the first and third terms we see that there are three solutions $a \to -m^2$ and one solution $a \to 0$; i.e., there are three vacua for $SO(5) \to \widetilde{U(2)}$ by counting $2N - N_f = 4 - 1 = 3$ and one vacuum for $SO(5) \to SO(3) \times \widetilde{U(1)}$ where $SO(3)$ is at the Special branch.
4.2.3 \( N_f = 2 \)

In this case we have the following situations. For \( SO(5) \to SO(3) \times \hat{U}(1) \), we can have the \( r = 0 \) non-baryonic branch with two vacua and the \( r = 1 \) baryonic branch with two vacua. For \( SO(5) \to SO(5) \), there are three vacua, while for \( SO(5) \to \hat{U}(2) \) we can have the \( r = 0 \) non-baryonic branch with two vacua, the \( r = 0 \) baryonic branch with one vacuum, and the \( r = 1 \) non-baryonic branch with one vacuum. We will study these vacuum structures in detail.

- **Baryonic \( r = 1 \) branch**
  As previously discussed, on \( r = 1 \) branch, the Seiberg-Witten curve is factorized as follows:
  \[
y^2 = \left( x^2 - m^2 \right)^2 \left[ P_2^2(x) - 4x^2\Lambda^2 \right].
  \]
  In addition to this factor, we have the following factorization problems,
  \[
P_2^2(x) - 4x^2\Lambda^2 = x^2F_2(x).
  \]
  The solution for this equation is easily obtained. \( P_2(x) = x^2 \) because this characteristic function should have an \( x^2 \) factor in order for both sides to be consistent with each other. Thus, we have \( P_4(x) = x^2(x^2 - m^2) \), which means that the gauge group \( SO(5) \) breaks into \( SO(3) \times \hat{U}(1) \). For fixed \( m \), there is only one vacuum which comes from the Chebyshev point with a factor \( x^2 \) in the curve. To get another vacuum, we must choose \( P_2(x) = (x^2 + \Lambda^2) \), so that
  \[
P_2^2(x) - 4x^2\Lambda^2 = (x^2 - \Lambda^2)^2.
  \]
  This vacuum comes from the Special point. Thus, we get all of two vacua for \( r = 1 \) baryonic branch.

- **Non-baryonic \( r = 1 \) branch**
  In this case, we need use the relationship of \( F_4(x) = P_2(x)^2 - 4x^2\Lambda^2 \) and \( (x^2 - m^2)^2 \) so that \( P_2(x) = (x^2 - m^2) \). There is only one solution which gives the one vacuum of \( SO(5) \to \hat{U}(2) \) at the \( r = 1 \) non-baryonic branch.

- **Non-baryonic \( r = 0 \) branch**
  1. Non-degenerated case
     In this case, from (4.6) we have the following factorization problem,
     \[
P_4^2(x) - 4x^2\Lambda^2(x^2 - m^2)^2 = x^2F_6(x).
     \]
     To satisfy this equation we must have \( x^2 \) factor in \( P_4(x) \). Thus, assuming that \( P_4(x) = x^2(x^2 - a^2) \), we can obtain \( F_6(x) \) as follows:
     \[
     F_6(x) = x^2(x^2 - a^2)^2 - 4\Lambda^2(x^2 - m^2)^2.
     \]
At this point we want to determine $W'(x)$ from $F_6(x)$. As already discussed above, we should take into account the second term of (A.6) \(^{19}\).

$$F_6(x) + 4\Lambda^2 x^2 \left( x^2 - m^2 \right) = W'(x)^2 + O(x^2).$$

From this equation, we can read off $W'(x) = x^2(x^2 - a^2)$. Thus, according to the condition $W'(\pm m) = 0$, we identify $a^2$ with $m^2$. Finally the characteristic function becomes

$$P_4(x) = x^2 \left( x^2 - m^2 \right),$$

which means that the gauge group $SO(5)$ breaks into $SO(3) \times \hat{U}(1)$ where $SO(3)$ is at the Chebyshev branch with a factor $x^2$ in the curve. For fixed $m$, there is only one vacuum. We will find another vacuum coming from the degenerated case.

2. **Degenerated case**

Now let us consider the degenerated case with the curve (note the factor $F_4(x)$),

$$y^2 = P_4^2(x) - 4x^2\Lambda^2(x^2 - m^2)^2 = F_4(x)H_2^2(x).$$

Writing $P_4(x) = (x^4 - s_1x^2 + s_2)$, $H_2(x) = (x^2 + a)$ and $F_4(x) = x^4 + bx^2 + c$ we found

\[
\begin{align*}
  s_1 &= \frac{-2a - b - 4\Lambda^2}{2}, \\
  s_2 &= \frac{4c + 4a(b - 4\Lambda^2) - (b + 4\Lambda^2)^2 - 32\Lambda^2m^2}{8}, \\
  c &= -\left(\frac{32a^2\Lambda^2 - 2a(b - 12\Lambda^2)(b + 4\Lambda^2) + (b + 4\Lambda^2)^3}{8} + 4\Lambda^2(2a + b + 4\Lambda^2)m^2 + 4\Lambda^2m^4\right) a - \frac{b}{2} - 2\Lambda^2, \\
  0 &= a \left(4a^2 - 4a(b - 5\Lambda^2) + (b + 4\Lambda^2)^2\right) + 24a\Lambda^2m^2 + 4\Lambda^2m^4.
\end{align*}
\]

Putting $b = -2m^2 - 4\Lambda^2$ we can solve \(^{20}\)

$$0 = 4a^2 \left(a + 9\Lambda^2\right) + 8a \left(a + 3\Lambda^2\right) m^2 + 4 \left(a + \Lambda^2\right) m^4 \Rightarrow m^2 = \frac{-a^{3/2} \pm 3ia\Lambda}{\sqrt{a} \mp i\Lambda}.$$  

At the limit $\Lambda \to 0$ and $a \to$ constant, we get both $m^2 \to -a$ and $SO(5) \to \hat{U}(2)$. At the limit $\Lambda \to 0$, but $\sqrt{a} \to i\Lambda + \alpha\Lambda^3$, we have $m^2 \neq 0$ and $SO(5) \to SO(3) \times \hat{U}(1)$ where $SO(3)$ is at the Special branch. Thus, there is smooth transition between these two phases. To count the vacua, notice that for fixed $m^2$, we have three solutions for $a$. Setting $\Lambda = 0$, the equation becomes $4a(a + m^2)^2 = 0$. Thus we have two vacua for $\hat{U}(2)$ and one vacuum for $SO(3) \times \hat{U}(1)$.

\(^{19}\)In this case, we are considering the tree level superpotential (4.1), thus $g_{2n+2}$ is 1. In addition, since the equation (A.6) is the result for $SO(2N_c)$ theory, we have to extend the result to $SO(2N_c + 1)$ theories by replacing $x^2\Lambda^{4N_c-2N_f-4}$ in the second term to $\Lambda^{4N_c-2N_f-2}$.

\(^{20}\)In this case, $c = -4a\Lambda^2 + 4\Lambda^2m^2 + m^4$ is very simple compared with the above messy expression.
Next we discuss the case in which all D5-branes wrap around the origin \( x = 0 \); thus, \( F_4(x) \) should have a factor \( x^2 \). Writing \( P_2(t) = t(t - s_1) \), \( H_1(t) = t + a \) and \( F_2(t) = t(t + b) \) (where we have used \( t = x^2 \) for simplicity), we get \( s_1 = -a - 2\Lambda^2 + \frac{2\Lambda^2 m^4}{a^2} \), \( b = -\frac{4\Lambda^2 m^4}{a^2} \) and

\[
0 = a^2 (a + \Lambda^2) - 2a\Lambda^2 m^2 + \Lambda^2 m^4, \quad \text{or } m^2 = \pm \frac{a^{3/2}}{\Lambda} + a.
\]

There is only one limit \( a \sim \Lambda^{2/3} \) to get finite \( m^2 \). For fixed \( m^2 \), there are three solutions of \( a \) which give the three wanted vacua of \( SO(5) \to SO(5) \) where \( SO(5) \) is at the Chebyshev branch with a factor \( x^2 = t \) in the curve.

Finally we want to get the \( r = 0 \) baryonic branch of \( \widehat{U}(2) \). To achieve this, we require \( y^2 \) to be square. See also (4.12). There are two solutions. One of them is \( P_2(t) = (t - m^2)(t + \Lambda^2) \) (for simplicity we have used \( t = x^2 \)) which gives the \( r = 1 \) baryonic branch of \( SO(3) \times \overline{U}(1) \). Another is \( P_2(t) = (t - m^2)^2 + \Lambda^2 t \) which is the one we want. It is easy to see that the latter solution gives \( y^2 = [(t - m^2)^2 - \Lambda^2 t^2] \neq H_1(t)^2 F_1(t)^2 \), which explains why we could not get it in the previous paragraph. In other words, for the baryonic branch, the relationship \( F_4(x) + 4\Lambda^2 x^2 = (\frac{W'(x)}{x})^2 + O(x^0) \) fails.

Now we summarize the results in Table 2 by specifying the flavors \( N_f \), symmetry breaking patterns, various branches, the exponent of \( t = x^2 \) in the curve, \( U(1) \) at the IR, the number of vacua, and the possibility of smooth transition. It turned out that the number of vacua is exactly the same as from the weak coupling analysis.

### 4.3 SO(6) case

The next example is \( SO(6) \) gauge theory, which is more interesting. For this gauge theory we will discuss the number of flavors \( N_f = 0, 1, 2, 3 \) cases. There are three breaking patterns \( SO(6) \to SO(4) \times \overline{U}(1) \), \( SO(6) \to SO(2) \times \overline{U}(2) \), \( SO(6) \to SO(6) \), and \( SO(6) \to \overline{U}(3) \).

#### 4.3.1 \( N_f = 0 \)

1. **Non-degenerated case**

   In [19] the theory with \( N_f = 0 \) was discussed intensively. For the breaking pattern \( SO(6) \to SO(2) \times U(2) \) where \( SO(2) \) is at the Special branch, there are two vacua which are confining vacua. These confining vacua are constructed from the Coulomb branch with the breaking pattern \( SO(4) \to SO(2) \times U(1) \) by the multiplication map. In addition to these vacua, there are two vacua with the breaking pattern \( SO(6) \to SO(4) \times U(1) \). These two vacua come from the Chebyshev branch of \( SO(4) \) factor, and we will find another two vacua from the Special branch of \( SO(4) \) in the degenerated case.

2. **Degenerated case**
Table 2: The summary of the phase structure of $SO(5)$ gauge group. There are three pairs of smoothly connected phases indicated by the letters A, B, C. That is, the two branches denoted by A, B, or C are smoothly connected with each other. For $N_f = 2$, since we can have $r = 0, 1$, the $r$ is also an index for phases and should be preserved under the smooth transition. One application is that although for $N_f = 2$, $SO(3) \times \hat{U}(1)$ at $(S, 1_B)$ and $\hat{U}(2)$ at $(0_B)$ have the same $t^0$ and zero $U(1)$, they can not be smoothly connected because the index $r$ is different.
To see all the vacua including the one with the breaking pattern $SO(6) \to U(3)$, it is necessary to consider the degenerated case. The factorization problem becomes

$$P_6^2(x) - 4x^4\Lambda^8 = H_4^2(x)F_4(x).$$

After solving this factorization problem, we obtain the following solutions,

$$P_6(x) = x^6 - \epsilon \frac{3G^4 + \Lambda^8}{GA^4}x^4 + \left(G^2 - \frac{3G^6}{\Lambda^8}\right)x^2 + \epsilon \frac{G(G^4 - \Lambda^8)^2}{\Lambda^{12}},$$

$$F_4(x) = \left(x^2 - \epsilon \frac{G^4 + \Lambda^8}{GA^4}\right)^2 + 4G^2,$$

where $\epsilon^2 \equiv -1$. Note that this use of $\epsilon$ is slightly different from previous one. If we remember the relation $F_4(x) \equiv \frac{W'(x)^2}{x^2} + d$, we can see the relation,

$$\Delta = \epsilon \frac{G^4 + \Lambda^8}{GA^4}, \quad \text{where} \quad W'(x) \equiv x \left(x^2 - \Delta\right).$$

In this case we can take two semiclassical limits with $\Lambda \to 0$:

1. $G \to 0$ with fixed $\frac{G^2}{\Lambda^4} \equiv v$: Under these limits, since the characteristic function goes to $P_6(x) \to (x^2 - v)^3$, the gauge group $SO(6)$ breaks into $U(3)$. From the condition (4.17), $\Delta = \epsilon \frac{G^2}{\Lambda^4}$, we obtain three values for $\epsilon G = (-\Delta \Lambda^4)^{\frac{1}{4}}$. Thus substituting (4.16), we can see three vacua, which agree with the number obtained by weak coupling analysis, because pure $U(3)$ theory has three vacua.

2. $G \to 0$ with fixed $\frac{\Lambda^4}{G} \equiv w$: Under this limit since the characteristic function behaves as $P_6(x) \to x^4(x^2 - w)$, the gauge group $SO(6)$ breaks into $SO(4) \times U(1)$. From condition (4.17), $\Delta = \epsilon \frac{\Lambda^4}{G}$, we obtain two values for $G$. These two vacua of $SO(4) \times U(1)$ are the missing two vacua mentioned in the non-degenerated case. Adding all the contributions from the degenerated case and nondegenerated case, the result agrees with the one from the weak coupling analysis, because the Witten index for pure $SO(4)$ gauge theory has four vacua.

Notice that from the above analysis we get a smooth transition between $U(3)$ and $SO(4) \times U(1)$ at the Special branch. The reason is that the power of $t = x^2$ of $SO(4)$ at the Special branch is $(2N_f - N + 4) = 0$, which is the same as the case of $U(3)$. Similar phenomena have been observed in the previous subsection of the $SO(5)$ gauge group.

Finally, to get the breaking $SO(6) \to SO(6)$ (see also (3.9)), we require a $x^2$ factor in $F_4(x)$ which can be obtained when $G^4 - \Lambda^8 = 0$. There are four solutions for $G$. Combining the $\epsilon$, it seems we will get eight solutions. However, careful study shows that there are, in fact, only four solutions which match the counting $(N - 2) = 4$ of the Witten index for $SO(6)$ gauge theory where $SO(6)$ is at the Chebyshev branch.
4.3.2 \( N_f = 1 \)

In this example, there is only the \( r = 0 \) branch. For the breaking pattern \( SO(6) \to SO(4) \times U(1) \), this \( r = 0 \) branch can be non-baryonic or baryonic due to the fact \( r = N_1 \) while for the breaking pattern \( SO(6) \to SO(2) \times U(2) \), this branch is non-baryonic because both relations \( r \neq N_f - N_1 \) and \( r \neq N_1 \) hold in this case. In addition, for the breaking pattern \( SO(6) \to U(3) \) it is a non-baryonic branch because \( r \neq N_f - N_1 \) and \( r \neq N_1 \). For \( SO(6) \to SO(6) \), only the Chebyshev branch exists with four vacua.

• Non-baryonic \( r = 0 \) branch
  1. Non-degenerated case

Since this branch is non-baryonic we have the following factorization problem,

\[
P_6^2(x) - 4x^4\Lambda^6(x^2 - m^2) = x^2H_2^2(x)F_6(x).
\]

To satisfy this equation, the characteristic function must have an \( x^2 \) factor, \( P_6(x) = x^2P_4(x) \).

Then one of \( H_2(x) \) and \( F_6(x) \) on the right hand side must have a factor \( x^2 \). If the factor \( x^2 \) is contained in the function \( F_6(x) \), it belongs, in fact, to the degenerated case, so we consider the case \( H_2(x) = x^2 \). Parameterizing \( P_6(x) = x^2(x^4 - s_1x^2 + s_2) \) and \( F_6(x) = x^6 + ax^4 + bx^2 + c \), we found the solutions:

\[
s_1 = -\frac{a}{2}, \quad s_2 = \mp 2i\Lambda^3m, \quad b = \frac{a^2}{4} \mp 4i\Lambda^3m, \quad c = -4\Lambda^6 \mp 2ia\Lambda^3m.
\]

Obviously the classical limit is \( SO(6) \to SO(4) \times \widehat{U}(1) \). By the relationship of \( W'(x) \) and \( F_6(x) \) we find \( a = -2m^2 \), so there are two vacua of \( SO(4) \times \widehat{U}(1) \) which come from the Chebyshev branch of the \( SO(4) \) factor where there is a factor \( x^6 \) in the curve.

2. Degenerated case

Next we go to the degenerated case. The factorization problem of this case is given by

\[
P_6^2(x) - 4x^4\Lambda^6(x^2 - m^2) = H_2^2(x)F_4(x).
\]

Parameterizing \( P_3(t) = t^3 - s_1t^2 + s_2t + s_3, \) \( H_2(t) = t^2 - at + b, \) \( F_2(t) = t^2 - 2m^2t + m^4 + c \) where we have used the relationship between \( F_2(t) \) and \( (t - m^2)^2 \) to simplify the calculation \(^{21}\).

The solution is

\[
b = \frac{c}{4} - \frac{4a\Lambda^6}{c} + am^2 - m^4,
\]

\(^{21}\)The logic of the calculation is following. First we factorize the curve with some parameters. Then we establish the relationship between these parameters and \( m^2 \). This is the method used in \( SO(4) \) and \( SO(5) \). However, we can use the relationship of \( m^2 \) and \( F_4(x) \) to solve some parameters firstly, then to solve the factorization form. The latter method is more convenient for our purpose.
There are three solutions: $SO(6)$ gives always one. We have met this situation many times already.

From the last equation in (4.18), we see that at $\Lambda \to 0$ we must have $c \to 0$. Therefore the solution of $m^2$ is roughly

$$m^8 \sim \frac{(c^3 - 64 \Lambda^{12})^2}{1024 c \Lambda^{12}}.$$ 

The first limit is that $c \sim \Lambda^{2.4}$, so that all four solutions provide $m^2 \neq 0$ and $a \sim 2m^2$ and $b \sim m^4$. This limit gives $SO(6) \to \hat{U}(3)$. The second limit is that $c \sim \alpha \Lambda^{12}$, so that all four solutions give $m^2 \neq 0$, $a \sim \frac{c}{\alpha} \Lambda^6 \sim 0$, $b \sim 0$. This limit gives $SO(6) \to SO(4) \times \hat{U}(1)$ where $SO(4)$ is at the Special branch. Thus, we have a smooth transition between these two phases.

To count the vacua, for fixed $m^2$ there are six solutions of $c$. Keeping the first and third terms of (4.18) we have five $c \sim \Lambda^{2.4}$, which is consistent with the counting $(2N - N_f) = 6 - 1 = 5$ of $\hat{U}(3)$. Keeping the second and third terms of (4.18) we have one solution of $c \sim \Lambda^{12}$, which gives the one vacuum of $SO(4) \times \hat{U}(1)$ where $SO(4)$ is at the Special branch.

Now we consider other special cases. The first one is that $b = 0$ so that $H_2(t) = t(t - a)$. There are three solutions: $a = -\omega_3 \Lambda^2 + m^2$, $c = -4 \omega_3^{-1} \Lambda^4$ with $\omega_3^{-1} = -1$. They are the three vacua of $SO(6) \to SO(2) \times \hat{U}(2)$ by counting number $(2N - N_f) = 4 - 1 = 3$.

The next one is that $F_2(t) = t(t + a)$ and $H_2(t) = t(t + b)$. The solutions are given by

$$b = \frac{a}{2} + \frac{16 \Lambda^6}{a^2}, \quad a^6 + 4096 \Lambda^{12} + a^2 \Lambda^6 (128a + 256m^2) = 0.$$ 

From the (4.19) we see that $a \to 0$ in the limit $\Lambda \to 0$. There are two limits we can take. If $a \sim 4 \Lambda^{3/2}$, we have $m^2 \sim -1$, $b \sim 0$ which gives $SO(6) \to SO(6)$. If $a \sim \Lambda^3$, we have $m^2 \sim -16$, $b \sim 16$ which gives $SO(6) \to SO(4) \times \hat{U}(1)$ where $SO(4)$ is at Chebyshev branch and $\hat{U}(1)$ is at the $r = 0$ baryonic branch.

From these considerations we see that there is a smooth transition between $SO(6) \to SO(6)$ and $SO(6) \to SO(4) \times \hat{U}(1)$. Indices for these two phases are the same: $t^3$ and non IR $\hat{U}(1)$.

It is interesting to compare them with the baryonic $r = 0$ branch of $SO(5)$ with $N_f = 1$, where we do not find a smooth transition between $SO(5) \to SO(5)$ and $SO(5) \to SO(3) \times \hat{U}(1)$. For $SO(5)$, these two phases come from different solutions of $m^2$ while for $SO(6)$ there is only one solution for $m^2$. This different behavior of $m^2$ as a function of parameters explains the different phase structure, but the physics behinds these calculations is still unclear.

To count the vacua at fixed $m^2$, keeping the first and the third terms of the second equation (4.19) we find four solutions $a \sim m^{1/2} \Lambda^{3/2}$ which give the limit $SO(6) \to SO(6)$. Keeping

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\[\text{Notice that although the Special branch of } SO(4) \text{ should have two vacua, but the counting from the curve gives always one. We have met this situation many times already.}\]
the second and the third terms we find two solutions $a \sim \pm 4i\Lambda^3/m$ and $b \sim -m^2$. These two solutions give the breaking pattern $SO(6) \to SO(4) \times \hat{U}(1)$ where $SO(4)$ is at Chebyshev branch and $\hat{U}(1)$ is at the $r = 0$ baryonic branch.

- **Baryonic $r = 0$ branch**

Since this branch is baryonic, from the relation (4.7) we have the following factorization problem,

$$P_3^2(t) - 4t^2\Lambda^6(t - m^2) = tH_2^2(t)F_1(t).$$

In fact, for the $SO(4)$ at the Chebyshev branch, we need to have $H_2(t) = t(t + b)$. This is exactly the same problem discussed above.

To get the missing vacua, we just need to choose $P_3(t) = t^3 - m^2t^2 + \Lambda^6$, so that $y^2 = (t^3 - m^2t^2 - \Lambda^6)^2$. This gives the $r = 0$ baryonic branch of $SO(6) \to SO(4) \times \hat{U}(1)$ where $SO(4)$ factor is at the Special branch.

### 4.3.3 $N_f = 2$

In this case, we have two branches $r = 0, 1$. For the breaking pattern $SO(6) \to SO(4) \times \hat{U}(1)$, the $r = 1$ branch is baryonic because $r = N_f - N_1 = N_1$ while the $r = 0$ branch is non-baryonic. For the breaking pattern $SO(6) \to SO(2) \times \hat{U}(2)$ the $r = 1$ branch is non-baryonic while the $r = 0$ branch can be non-baryonic or baryonic because $r = N_f - N_1$. Moreover, for the breaking pattern $SO(6) \to \hat{U}(3)$ both branches of $r = 0, 1$ are non-baryonic. For the breaking pattern, $SO(6) \to SO(6)$, there is only Chebyshev branch.

- **Baryonic $r = 1$ branch**

The Seiberg-Witten curve is factorized as,

$$y^2 = \left(x^2 - m^2\right)^2 \left[P_4^2(x) - 4x^4\Lambda^4\right]. \quad (4.20)$$

In addition to this factor, we have the following factorization problem with a factor $F_2(x)$,

$$y^2 = P_4^2(x) - 4x^4\Lambda^4 = x^2H_2^2(x)F_2(x).$$

To satisfy this constraint, the characteristic function $P_4(x)$ must have $x^2$ factor, i.e. $P_4(x) = x^2(x^2 - a^2)$. In addition, we assume that $H_2(x) = (x^2 - b^2)$. After solving the factorization problem we obtain the following results,

$$P_4(x) = x^2(x^2 - 2\eta\Lambda^2), \quad F_2(x) = (x^2 - 4\eta\Lambda^2),$$

where $\eta$ is 2-nd roots of unity. Thus taking into account the $(x^2 - m^2)$ factor, we obtain

$$P_6(x) = x^2(x^2 - m^2)(x^2 - 2\eta\Lambda^2).$$
Under the semiclassical limit, $\Lambda \to 0$, the characteristic function becomes $P_6(x) \to x^4(x^2 - m^2)$, which means that the gauge group $SO(6)$ breaks into $SO(4) \times \tilde{U}(1)$. Since there is $\eta$ we have two vacua for fixed $m$ which come from the Chebyshev branch of $SO(4)$ factor.

In addition to this solution, we have other solutions from the following case,

$$P_4^2(x) - 4x^4\Lambda^4 = H_2^2(x).$$

If we parameterize $H_4(x) = (x^2 - a)(x^2 - b)$, we have two solutions $a \to -\Lambda^2, b \to \Lambda^2$ or $a \to \Lambda^2, b \to -\Lambda^2$. Both cases lead to the conclusion that $P_4(x)$ becomes $P_4(x) = x^4 + \Lambda^4$.

As above, these solutions also exhibit the breaking pattern $SO(6) \to SO(4) \times \tilde{U}(1)$. However, these come from the Special branch of $SO(4)$. Therefore, we have four solutions from the strong coupling analysis. This number agrees with the one from the previous analysis of the weak coupling. Although $SO(4)$ gauge theory has four vacua, $\tilde{U}(1)$ with $N_f = 2$ gauge theory has only one vacuum.

- Non-baryonic $r = 1$ branch
  1. Non-degenerated case

As in the previous case, we can factorize the Seiberg-Witten curve as (4.20). In this case, since the branch is non-baryonic, we have different factorization problem,

$$y^2 = P_4^2(x) - 4x^4\Lambda^4 = x^2P_6(x)$$

in which the factor $(x^2 - m^2)^2$ has been factorized out. To solve this equation, we assume that $P_4(x) = x^2(x^2 - a^2)$. After inserting this relation, we find that $P_6(x)$ must have an $x^2$ factor. However, this implies the degenerated case. Therefore, we have no new solution for this non-degenerated case.

2. Degenerated case
Next we will consider the degenerated case. The factorization problem is given by

$$P_4^2(x) - 4x^4\Lambda^4 = H_2^2(x)F_4(x).$$

Solving this factorization problem, we obtain two kinds of solutions. One is given by

$$P_4(x) = (x^2 - a)^2 + 2\eta\Lambda^2 x^2,$$

$$F_4(x) = (x^2 - a + 2\eta\Lambda^2)^2 + 4a\eta\Lambda^2 - 4\Lambda^4,$$

where $\eta$ is the 2-nd root of unity. Taking into account that $W'(\pm m) = 0$, we have one constraint,

$$m^2 = a + 2\eta\Lambda^2.$$
In this case, we can take only one semiclassical limit, \( \Lambda \to 0 \). Under the limit, the characteristic function goes to \( P_6(x) \to (x^2 - m^2)^3 \), which means that the gauge group \( SO(6) \) breaks into \( \widetilde{U}(3) \). Thus, we have two vacua from the strong coupling approach. This number of vacua agrees with the weak coupling analysis given in (3.11). The \( \widetilde{U}(3) \) theory with \( N_f = 2 \) and \( r = 1 \) has two vacua.

The other solutions are derived from \( H_2(x) = x^2 \). Inserting \( P_4(x) = x^2(x^2 - a^2) \) into the factorization problem, we obtain

\[
F_4(x) = (x^2 - a^2)^2 - 4\Lambda^4 \equiv \left( \frac{W'(x)}{x} \right)^2 + d.
\]

The condition \( W'(\pm m) = 0 \) leads to the solution \( a^2 = m^2 \). After all, we obtain the trivial solution,

\[
P_6(x) = x^2(x^2 - m^2)^2.
\]

The gauge group \( SO(6) \) breaks into \( SO(2) \times \widetilde{U}(2) \). Thus, we obtained only one vacuum from the strong coupling analysis which comes from the Special branch of \( SO(2) \). This number matches the one from the weak coupling analysis. The \( \widetilde{U}(2) \) with \( N_f = 2 \) gauge theory has only one vacuum from (3.11). Since \( SO(2) \) theory has only one vacuum, the total number of vacua is only one, which agrees with the results of the strong coupling analysis.

- **Non-baryonic \( r = 0 \) branch**

1. **Non-degenerated case**

Next we consider a non-baryonic \( r = 0 \) branch. From (4.5) we have the following factorization problem,

\[
P_6^2(x) - 4x^4\Lambda^4(x^2 - m^2)^2 = x^2H_2^2(x)F_6(x).
\]  

Assuming that \( P_6(x) = x^2P_4(x) \), we can rewrite (4.21) as follows:

\[
x^2 \left( P_4(x) - 2\Lambda^2(x^2 - m^2) \right) \left( P_4(x) + 2\Lambda^2(x^2 - m^2) \right) = H_2^2(x)F_6(x).
\]

If we include the additional factor \( x^2 \) in a function \( F_6(x) \) it goes to the degenerated case. Here we want to discuss the non-degenerated case so we consider the \( H_2(x) = x^2 \) case. The solution is given by

\[
P_6(x) = x^2 \left[ x^4 + Ax^2 - 2\eta\Lambda^2m^2 \right],
\]

\[
F_6(x) = x^2 \left( x^2 + A \right)^2 - 4 \left( \Lambda^4 + \eta\Lambda^2m^2 \right) \left( x^2 + \frac{A - 2\eta\Lambda^2m^2}{m^2 + \eta\Lambda^2m^2} \right),
\]

where \( \eta \) is the 2-nd root of unity. Taking into account that \( W'(\pm m) = 0 \) we have the condition \( m^2 = -A \). Under the semiclassical limit, \( \Lambda \to 0 \), the characteristic function goes to \( P_6(x) \to \)
$x^4(x^2 - m^2)$, which means that the gauge group $SO(6)$ breaks into $SO(4) \times \hat{U}(1)$. These vacua come from the Chebyshev branch of the $SO(4)$ factor.

**2. Degenerated case**

Now we consider the degenerated $r = 0$ branch. Using $t = x^2$, the curve should be factorized as

$$P_3^2(t) - 4\Lambda^4t^2(t - m^2)^2 = H_2^2(t)F_2(t).$$

We have two cases to be discussed for this $r = 0$ branch. The first case is $H_2(t) = tH_1(t)$ which constrains to $P_3(t) = tP_2(t)$. Then we have the following solution

$$P_2(t) - 2\eta\Lambda^2(t - m^2) = H_2^2(t) = (t - a)^2,$$

$$P_2(t) + 2\eta\Lambda^2(t - m^2) = (t - a)^2 + 4\eta\Lambda^2(t - m^2),$$

with $\eta^2 = 1$. There are two situations we need to consider for this kind of solution. The first is that there are D5-branes wrapping around the $x = \pm m$; then we have

$$a = m - 2\eta\Lambda^2,$$

by the relationship of $F_2(t)$ and $(t - m^2)^2$. It gives $SO(6) \rightarrow SO(2) \times \hat{U}(2)$ at the classical limit where $SO(2)$ is at the Special branch. There are two vacua corresponding to $\eta = \pm 1$, which is consistent with the counting that $U(2)$ with two flavors at $r = 0$ non-baryonic branch has two vacua. The second situation is that $F_2(t)$ has a further factor $t$ which leads to

$$a^2 - 4\eta\Lambda^2m^2 = 0.$$  

It gives $SO(6) \rightarrow SO(6)$ at the classical limits. There are four solutions, which is consistent with the counting that $SO(6)$ at the Chebyshev branch has a Witten index $(N - 2) = 4$. Notice that because of the second situation, $F_2(t)$ has a factor $t$, and the first situation is not smoothly connected to the second situation.

Having considered the case that $H_2(t) = tH_1(t)$, we move to the case that there is no factor $t$ in the curve $y^2$. Then $P_3(t) - 2\eta\Lambda^2t(t - m^2)$ has only one double root for every $\eta$, or

$$P_3(t) + 2\Lambda^2t(t - m^2) = (t - a - b)^2(t + c),$$

$$P_3(t) - 2\Lambda^2t(t - m^2) = (t - a + b)^2(t + d),$$

where

$$c = 2b + 2\Lambda^2 - \frac{a(b + 2\Lambda^2)}{b} + \frac{\Lambda^2m^2}{b},$$

$$d = -2b - 2\Lambda^2 - \frac{a(b + 2\Lambda^2)}{b} + \frac{\Lambda^2m^2}{b},$$

$$0 = b^2(b + \Lambda^2) + a\Lambda^2(m^2 - a).$$
Using the relationship of $F_2(t) = (t + c)(t + d)$ and $(t - m^2)$, we get

$$\frac{2ab(b + 2\Lambda^2) - 2b\Lambda^2m^2}{b^2} = 2m^2,$$

from which we can solve

$$a = \frac{(b + \Lambda^2)m^2}{b + 2\Lambda^2}, \quad c = 2(b + \Lambda^2) - m^2, \quad d = -2(b + \Lambda^2) - m^2$$

and

$$(b + \Lambda^2) \left(b^2 + \frac{\Lambda^4m^4}{(b + 2\Lambda^2)^2}\right) = 0. \quad (4.22)$$

Forgetting about the factor $(b + \Lambda^2)$ we found that there is only one limit $b \sim \Lambda$ to get finite $m^2$. In this limit $m^4 \sim -1$, $a \sim m^2$, $c, d \sim -m^2$, so it gives $SO(6) \to \hat{U}(3)$. For fixed $m^2$, there are four solutions of $b$ which give the four vacua of $\hat{U}(3)$ by counting $(2N - N_f) = 6 - 2 = 4$.

However, where is the breaking $SO(6) \to SO(4) \times \hat{U}(1)$? It comes from $(b + \Lambda^2) = 0$, so that $a = 0$ and $c = d = -m^2$. In fact, the curve has the same form as the curve of the $r = 1$ branch. Similar phenomena have been observed in [20]; for example, the $U(2)$ with $N_f = 2$.

Since this non-baryonic $r = 0$ branch is coincident with the baryonic $r = 1$ branch in this particular example, we do not have a smooth transition from $\hat{U}(3)$ to $SO(4) \times \hat{U}(1)$. It is very interesting to recall that there is smooth transition for $N_f = 0, 1$.

- **Baryonic** $r = 0$ branch

  For the baryonic branch, the curve should be

  $$P_3^2(t) - 4\Lambda^4t^2(t - m^2)^2 = H_3^2(t).$$

  Parameterizing $H_3(t) = t^3 - at^2 + bt + c$, there are two solutions. The first one is $a = 2m^2$, $b = m^4 - \Lambda^4$, and $c = 0$ which gives $r = 0$ baryonic branch of $SO(2) \times \hat{U}(2)$. The $SO(2)$ is at the Special branch with a factor $t^2$ in the curve. The second solution is $a = m^2$, $b = -\Lambda^4$, and $c = -\Lambda^4m^2$ which gives the $r = 1$ baryonic branch of $SO(4) \times \hat{U}(1)$ discussed before.

4.3.4 $N_f = 3$

In this case, some of the branches can be described by the addition map we have discussed in previous section.

- **Non-baryonic** $r = 1$ branch

  In this case, the curve is simplified as

  $$y^2 = (x^2 - m^2)^2 \left[P_2^2(x) - 4x^4\Lambda^2(x^2 - m^2)\right],$$

  where the term in the bracket is, in fact, the curve of $SO(4)$ with $N_f = 1$. This is just the familiar application of addition maps. Using the corresponding result of $SO(4)$, we find three
vacua for $SO(6) \to U(3)$ and one vacua for $SO(6) \to SO(2) \times U(2)$. There is no smooth transition in this case.

- **Baryonic $r = 1$ branch**

Again, we can use the addition map to reduce the problem to the $r = 0$ baryonic branch of $SO(4)$ with $N_f = 1$. From there, we get one vacuum for $SO(6) \to SO(2) \times U(2)$, two vacua for $SO(6) \to SO(4) \times U(1)$ where $SO(4)$ is at the Chebyshev branch, and two vacua for $SO(6) \to SO(4) \times U(1)$ where $SO(4)$ is at the Special branch.

- **Non-baryonic $r = 0$ branch**

1. **Non-degenerated case**

In this case, we have the following factorization problem,

$$P_6^2(x) - 4x^4\Lambda^2(x^2 - m^2)^3 = x^2H_2^2(x)F_6(x).$$

To satisfy this equation, we must have a $x^2$ factor in $P_6(x) = x^2P_4(x)$. Let us assume that

$$P_4(x) = x^4 + Ax^2 + B, \quad H_2(x) = x^2 - C,$$

$$F_6(x) = \left[x(x^2 - D)\right]^2 + G\left(x^2 + F\right).$$

After solving the factorization problem, we have three kinds of solutions. However, two of them have an $x^2$ factor in $F_6(x)$, which means that this becomes the degenerated case. Therefore, in the non-degenerated case, we have only the solution below.

The third solution is given by

$$P_6(x) = x^2\left[x^4 + Ax^2 - 2i\eta\Lambda m^3\right], \quad G = 4\left(4\Lambda^2 - \Lambda^4 + 3\Lambda m^2 - i\eta\Lambda m^3\right),$$

$$F = \frac{m^3(A - 3i\eta\Lambda m)}{i\Lambda\eta\Lambda - i\eta\Lambda^3 + 3i\eta\Lambda m^2 + m^3}, \quad D = -A + 2\Lambda^2.$$

Thus taking into account the relation $W'(x)$ and $F_6(x)$, we have $m^2 = -A$. Under the semi-classical limit, the characteristic function behaves as $P_6(x) = x^4(x^2 - m^2)$, which means that the classical group $SO(6)$ breaks into $SO(4) \times U(1)$. Thus, we have obtained two vacua from the analysis of strong coupling, which comes from the Chebyshev branch of the $SO(4)$ factor.

2. **Degenerated case**

For this case, the curve should be factorized as

$$P_3^2(t) - 4\Lambda^2 t^2(t - m^2)^3 = F_2(t)H_t^2(t).$$

There are two cases we should discuss. The first case is that $H_2(t) = tH_1(t)$. After dividing out the $t^2$, at two sides we parameterize $P_2(t) = t^2 - s_1 t + s_2$, $H_1(t) = (t - a)$, and $F_2(t) = t^2 + bt + c$. Solving $s_1, s_2, c$ in terms of $a$ and $b$ we get the following constraint

$$\Lambda^2(a - m^2)^3\left(4a^2 + 4a(b - 5\Lambda^2) + (b + 4\Lambda^2)^2 + 36\Lambda^2 m^2\right) = 0.$$
At the classical limit, we have \( b = -2a^2 \). If there are D5-branes wrapping around the \( x = \pm m \), we will have \( b \sim -2m^2 \), thus giving \( SO(6) \rightarrow SO(2) \times \tilde{U}(2) \). For given \( b \), there are two solutions for \( a \) which are consistent with the counting of \( U(2) \) with \( N_f = 3 \) at the \( r = 0 \) non-baryonic branch. If there are no D5-branes wrapping around the \( x = \pm m \), we need to have an extra \( t \) factor in \( F_2(t) \); i.e., \( c = 0 \). There are four solutions having \( a, b \rightarrow 0 \) which give the four vacua of \( SO(6) \rightarrow SO(6) \).

Now we move to the second case in which there is no factor \( t \) in \( H_2(t) \). This case will give the \( r = 0 \) non-baryonic branch of \( SO(6) \rightarrow \tilde{U}(3) \) and the \( r = 0 \) non-baryonic branch of \( SO(6) \rightarrow SO(4) \times \tilde{U}(1) \) where \( SO(4) \) is at the Special branch. Although the indices for these two phases are the same, it is not clear whether they are smoothly connected or not. To demonstrate the smooth transition, we must use the analytic expression, which is too complex to be solved. To count the number of vacua we can use a numerical method. Since we are unable to solve it, we will not discuss if any further.

- Baryonic \( r = 0 \) branch

This can happen only for \( SO(6) \rightarrow \tilde{U}(3) \) and the corresponding curve should be factorized as

\[
P_3^2(t) - 4\Lambda^2t^2(t - m^2)^3 = H_3^2(t).
\]

There are three solutions for \( H_3(t) = t^3 - at^2 + bt - c \). The first one is \( a = 2m^2 + \Lambda^2 \), \( b = m^2(m^2 + \Lambda^2) \), and \( c = 0 \) which gives the \( r = 1 \) baryonic branch of \( SO(2) \times \tilde{U}(2) \). The second one is \( a = m^2 + \Lambda^2 \), \( b = 2\Lambda^2m^2 \), and \( c = \Lambda^2m^4 \) which gives the \( r = 1 \) baryonic branch of \( SO(4) \times \tilde{U}(1) \). The third one is \( a = 3m^2 + \Lambda^2 \), \( b = 3m^4 \), and \( c = m^6 \) which gives the baryonic \( r = 0 \) branch of \( \tilde{U}(3) \) we are looking for.

The results are summarized in Table 3 by specifying the flavors \( N_f \), symmetry breaking patterns, various branches, the exponent of \( t = x^2 \) in the curve, \( U(1) \) at the IR, the number of vacua, and the possibility of a smooth transition. It turns out that the number of vacua is exactly the same as from the weak coupling analysis.

For the \( SO(7) \) case, we refer to version one in the hep-th archive for detail. In this paper we only summarize the results in Table 4 and Table 5 by specifying the flavors \( N_f \), symmetry breaking patterns, various branches, the exponent of \( t = x^2 \) in the curve, \( U(1) \) at the IR, the number of vacua, and the possibility of a smooth transition. It turned out that the number of vacua is exactly the same as from the weak coupling analysis.

5 Quartic superpotential with massless flavors

Thus far we have discussed the phase structures of massive flavors. In this section we will focus on massless flavors. Classically, under the breaking pattern \( SO(N_c) \rightarrow SO(N_0) \times
Table 3: The summary of phase structure of \(SO(6)\) gauge group. Here again we use capital letters to indicate the three phases having smooth transitions. The * means that it is not clear whether they are smoothly connected or not.
| $N_f$ | Group | branch | Power of $t(=x^2)$ | $U(1)$ | Number of vacua | Connection |
|-------|--------|--------|-------------------|-------|----------------|------------|
| 0     | $SO(3) \times U(2)$ | $(S,0_{NB})$ | $t^0$ | 1 | 2 | A |
|       |        | $(C,0_{NB})$ | $t^1$ | 1 | 2 | B |
|       | $SO(5) \times U(1)$ | $(C,0_{NB})$ | $t^1$ | 1 | 3 | B |
|       | $SO(7)$ | $(C)$ | $t^1$ | 0 | 5 |   |
|       | $U(3)$ | $(0_{NB})$ | $t^0$ | 1 | 3 | A |
| 1     | $SO(3) \times U(2)$ | $(C,0_{NB})$ | $t^1$ | 1 | 3 | C |
|       |        | $(S,0_{NB})$ | $t^0$ | 1 | 3 | D |
|       | $SO(5) \times U(1)$ | $(C,0_{NB})$ | $t^1$ | 1 | 3 | C |
|       |        | $(C,0_{B})$ | $t^1$ | 0 | 3 | N |
|       | $SO(7)$ | $(C)$ | $t^1$ | 0 | 5 | N |
|       | $U(3)$ | $(0_{NB})$ | $t^0$ | 1 | 5 | D |
| 2     | $SO(3) \times U(2)$ | $(S,0_{NB})$ | $t^1$ | 1 | 3 | G |
|       |        | $(C,0_{NB})$ | $t^0$ | 1 | 2 | F |
|       | $SO(5) \times U(1)$ | $(C,0_{NB})$ | $t^1$ | 1 | 3 | O |
|       | $SO(7)$ | $(C)$ | $t^1$ | 0 | 5 | E |
|       | $U(3)$ | $(1_{NB})$ | $t^0$ | 1 | 2 | G |
|       |        | $(0_{NB})$ | $t^0$ | 1 | 4 | F |
| 3     | $SO(3) \times U(2)$ | $(S,0_{NB})$ | $t^0$ | 1 | 3 | I |
|       |        | $(C,0_{NB})$ | $t^1$ | 1 | 1 | J |
|       | $SO(5) \times U(1)$ | $(C,0_{NB})$ | $t^1$ | 0 | 1 |   |
|       | $SO(7)$ | $(C)$ | $t^1$ | 0 | 5 |   |
|       | $U(3)$ | $(1_{NB})$ | $t^0$ | 1 | 3 |   |
|       |        | $(0_{B})$ | $t^0$ | 0 | 1 |   |

Table 4: Summary of the phase structures of $SO(7)$ gauge group. Here again we use capital letters in the last column to indicate the various phases having a smooth transition.
Table 5: Continued. Summary of the phase structures of $SO(7)$ gauge group.

\(\prod_{j=1}^{n} U(N_j)\), these massless flavors will be charged under $SO(N_0)$ since $W'(x)|_{x=\pm m=0} = 0$ is always true in our considerations ²³.

To demonstrate the new features coming from the massless flavors and compare them with the weak and strong coupling analyses, let us start with the following quadratic superpotential first

\[
W(\Phi) = \frac{1}{2} \mu \text{Tr} \Phi^2 .
\]

The curve is given by

\[
y^2 = P_{2[N_c/2]}^2(x) - 4 \Lambda^{2(N_c-N_f-2)} x^{2(1+\epsilon+N_f)}
\]

where \(\epsilon = 0\) for \(N_c\) odd and \(1\) for \(N_c\) even and \(N_f < N_c - 2\) for asymptotic free theory. For the non-baryonic branch, the curve should be factorized as

\[
y^2 = F_4(x) H_{2[N_c/2]-2}^2(x) \quad \text{(5.23)}
\]

with \(F_4(x) = x^2 F_2(x)\) where the appearance of \(x^2\) in this case is characteristic of the non-baryonic branch. See for example, (4.7) and (4.8). Furthermore, we have the following relationship

\[
F_2(x) = W'(x)^2 + d = x^2 + d .
\]

²³For degenerated cases, we have $SO(N_c) \rightarrow SO(N_c)$ and $SO(N_c) \rightarrow U([N_c/2])$ for quartic superpotential.
Because of the special form of $F_{4}(x)$, we do not have the $r$-th branch in the $U(N_{c})$ case.

To show it more clearly, let us consider the example of $SO(2N_{c})$ gauge theory with $2M$ flavors. If we use $t = x^{2}$, the curve is given by

$$y^{2} = \prod_{j=1}^{N_{c}}(t - \phi_{j}^{2})^{2} - 4\Lambda^{2(2N_{c} - 2 - 2M)}t^{2+2M}.$$ 

The non-baryonic vacua should be factorized as

$$y^{2} = F_{2}(t)H_{N_{c}-1}^{2}(t), \quad F_{2}(t) = t(t + d).$$

Due to the factor $t$ of $F_{2}(t)$, we should have, for example, $\phi_{N_{c}} = 0$ and the curve becomes

$$y^{2} = t^{2}\left[\prod_{j=1}^{N_{c}-1}(t - \phi_{j}^{2})^{2} - 4\Lambda^{2(2N_{c} - 2 - 2M)}t^{2+2(M-1)}\right] = t(t + d)H_{N_{c}-1}^{2}(t)$$

which means that the function $H_{N_{c}-1}(t)$ must have a factor $t$. Cancelling out factor $t^{2}$ at both sides, we get

$$y_{1}^{2} = \prod_{j=1}^{N_{c}-1}(t - \phi_{j}^{2})^{2} - 4\Lambda^{2(2N_{c} - 2 - 2M)}t^{2+2(M-1)} = t(t + d)H_{N_{c}-2}^{2}(t)$$

which is the non-baryonic curve of $SO(2(N_{c} - 1))$ with $2(M - 1)$ flavors. Since we require that the number of flavors satisfies $2M < 2N_{c} - 2$ for asymptotic free theory, the above operation can be iterated at most $(M + 1)$ steps and the factorization problem is reduced to

$$y^{2} = t^{2(M+1)}[F_{N_{c}-(M+1)}^{2}(t) - 4\Lambda^{2(2N_{c} - 2 - 2M)}] = t(t + d)t^{2(M+1)}H_{N_{c}-(M+1)-1}^{2}(t).$$

The presence of factor $t^{2(M+1)}$ indicates that no matter what factor $t^{2r}$ with $r < M + 1$ we start with, we always end up with $r = M + 1$ branch. This particular branch is nothing but the Chebyshev point emphasized in the weak and strong coupling analyses given in section 3.1.2. It is easy to check that for other $N_{c}$ and $N_{f}$, the factorized curves are also at the Chebyshev point. For the Special branch, we have already written down the form of curves in the strong coupling analysis where the basic feature of this branch is that the curve is a complete square.

Having obtained the quadratic superpotential, we can progress to the general superpotential with degree $2(n + 1)$ where $k = n$. Again, for example, the $SO(2N_{c})$ with $2M$ flavors at the general non-baryonic branch should have the factorization, by generalizing (5.23)

$$y^{2} = \prod_{j=1}^{N_{c}}(t - \phi_{j}^{2})^{2} - 4\Lambda^{2(2N_{c} - 2 - 2M)}t^{2+2M} = tF_{2n+1}(t)H_{N_{c}-(n+1)}^{2}(t).$$

Similar to the above derivation, factor $t$ tells us that at least one of $\phi_{i}$'s is vanishing, so that there is a factor $t^{2}$ on the left hand side. Then either $H_{N_{c}-(n+1)}(t)$ or $F_{2n+1}(t)$ must be divided by factor $t$. 

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If \( t \) is present in the \( F_{2n+1}(t) \), we should have \( F_{2n+1}(t = x^2) = x^2 \left[ (W'(x))^2 + f_{n-1}(x^2) \right] \) and the curve becomes
\[
y^2 = F_{2n}(t)[tH_{N-(n+1)}(t)]^2
\]
with, (for example see (A.9)),
\[
F_{2n}(t = x^2) = \left( \frac{W'_{2n+1}(x)}{x} \right)^2 + f_{n-1}(x^2).
\]
The factorization form (5.24) is, in fact, the expression for the Special branch because there is no overall factor \( t \) in the curve. Geometrically, the function \( F_{2n+1}(t) \) divided by \( t \) means that the cycle around the origin \( t = 0 \) on the reduced Riemann surface is degenerated and the remaining non-trivial cycles are around \( U(N_i) \) factors. In other words, the non-baryonic solution found in this case comes from the Special root discussed in the weak and strong coupling analyses.

Instead, if \( t \) is present in the function \( H_{N-(n+1)}(t) \), then we can cancel the \( t^2 \) factor on both sides of the curve and reduce the problem to \( SO(2(N_c - 1)) \) with \( 2(M - 1) \) massless flavors. Iterating this procedure, we can reach the Chebyshev point as in the quadratic superpotential if it does not stop at the above Special point.

In summary, the above analysis shows that the factorization of curves will stop only at either the Chebyshev point or the Special point, which is consistent with the results from the weak and strong coupling analyses.

So considered, we can now focus on the following quartic superpotential \( W(\Phi) = \frac{1}{4} \text{Tr} \Phi^4 - \frac{\alpha^2}{2} \text{Tr} \Phi^2 \), so that
\[
W'(x) = x(x^2 - \alpha^2).
\]
We also consider only the case \( N_f < N_c - 2 \) for asymptotic free theory. For convenience of the following discussions, we would like to summarize the essential points of the form of Seiberg-Witten curves:

a). At the Chebyshev point, the power of \( t = x^2 \) in the curve is \( (N_f + 3) \) for \( (N_f, N) = (\text{even, even}) \), \( (N_f + 1) \) for \( (N_f, N) = (\text{even, odd}) \), \( (N_f + 2) \) for \( (N_f, N) = (\text{odd, even}) \), and \( (N_f + 2) \) for \( (N_f, N) = (\text{odd, odd}) \). Notice that for all four cases, the power is always an odd number which is consistent with the factorization form \( tH^2(t) \); i.e., except one factor of \( t \), the other power of \( t \) can be considered as double roots. The number is given by \( (N - N_f - 2) \).

b). At the Special point which exists only when \( \tilde{N}_c = 2N_f - N + 4 \geq 0 \), the characters of the curve are: (1) When \( N_f < N - 2 \), the power of \( t \) is \( (2N_f - N + 4) \) for \( N \) even and \( (2N_f - N + 3) \) for \( N \) odd. When \( N_f \geq N - 2 \), the power of \( t \) is \( N \) for \( N \) even or \( (N - 1) \) for \( N \) odd. Notice that the power is always even, so it can be considered to belong into the double root in the factorization form of curve; (2) the number of the double root increases by one compared with that at the Chebyshev point.
c). From (a) and (b) the simplest way to distinguish between the Chebyshev branch and the Special branch is to see whether the power of $t$ is odd or even.

d). Because of point (b), the Special branch has a similar form of factorization as one of degenerated cases and will be discussed together in the following discussion.

5.1 $SO(4)$ case

Because of $N_f < N_c - 2$ for asymptotic free theory, we only have two choices: $N_f = 0$ and $N_f = 1$. The $N_f = 0$ case has been discussed in section 4 under massive flavors where we only need to replace $m$ by $\alpha$. However, the $N_f = 1$ will give new results. There are three classical limits $SO(4) \to \hat{SO}(2) \times U(1)$, $SO(4) \to \hat{SO}(4)$, and $SO(4) \to U(2)$. For $SO(4)$ with one flavor, there exists only the Special branch which can also be seen by taking into consideration the counting of the Chebyshev branch, $N - N_f - 2 = 2 - 1 - 2 < 0$, which does not exist. For $\hat{SO}(4)$ with one flavor, we can have both the Chebyshev branch and the Special branch because it is possible that $2N_f - N + 4 = 2 > 0$ and $N - N_f - 2 = 1 > 0$.

1 Non-degenerated case

The curve should be

$$y^2 = P_4^2(x) - 4x^6\Lambda^2 = x^2F_6(x)$$

by taking the $m \to 0$ limit into (3.3). Using $P_4(x) = x^2(x^2 - A)$, we get $F_6(x) = x^2[(x^2 - A)^2 - 4x^2\Lambda^2]$. From the relationship between $F_6(x)$ and $W'(x)$ we find that $A = \alpha^2$. In the classical limit $\Lambda \to 0$, we get $SO(4) \to \hat{SO}(2) \times U(1)$. Notice that there is a $x^4 = t^2$ factor in the $y^2$ indicating that the solution found comes from the Special root instead of the Chebyshev point.

2. Degenerated case and Special branch

As previously noted, since the Special branch has one extra double root and the degenerated case has one root of $W'(x)$ without wrapping D5-branes, the starting factorization forms of the curve are the same and we should consider them simultaneously. The curve should be

$$y^2 = P_4^2(x) - 4x^6\Lambda^2 = F_4(x)H_2^2(x)$$

or

$$y^2 = (t^2 - s_1t + s_2)^2 - 4t^3\Lambda^2 = (t^2 + at + b)(t + c)^2, \quad t = x^2.$$ (5.26)

Given the $a$ there are three solutions for $(s_1, s_2, b, c)$. The first is $(-a-4\Lambda^2, 0, (-a-4\Lambda^2)^2, 0)$, where $c = 0$ gives a factor $t^2$ in the curve. Notice that there is no smooth transition found in this solution.

To determine the classical limit, we need to determine the behavior of $a$. There are two cases to be analyzed. The first case is that there are some D5-branes wrapping around $x = \pm \alpha$. 43
Table 6: The phase structure of $SO(4)$ with one massless flavor. There are no smooth transitions between these vacua because there is no vacua which has the same number of the power of $t$ and the number of $U(1)$.

Then we get $a = -(2\alpha^2 + 4\Lambda^2)$, which produces the classical limit $SO(4) \rightarrow SO(2) \times U(1)$.

In fact, this solution is identical to the one given in the previous paragraph where we claimed that it belongs to the Special branch. There is only one solution which is consistent with the counting. The second case is that all D5-branes wrap around the origin $x = 0$, so curve (5.26) describes the $SO(4) \rightarrow SO(4)$ classically. With one flavor, we can have the Special branch with a factor $t^2$ by $(2N_f - N + 4) = 2$ or the Chebyshev branch with $t^3$ where $N_f + 2 = 3$. To get the Special branch we must set $a = -2\Lambda^2$ so that $(t^2 + at + b) = (t - \Lambda^2)^2$. There is only one solution. To get the Chebyshev branch, we need $b = 0$ so that $a = -4\Lambda^2$. Notice the counting of vacua matches the formula $(N - N_f - 2 = 4 - 1 - 2 = 1)$.

The second and third cases have a somewhat complex expression as a function of $a$ and neither $b$ or $c$ is zero for general $a$

\[
\begin{align*}
    s_1 &= \frac{-2a + \Lambda^2 \mp 3\Lambda\sqrt{-2a + \Lambda^2}}{2}, \\
    s_2 &= \frac{a^2 - 19a\Lambda^2 + 16\Lambda^4 \pm 5a\Lambda\sqrt{-2a + \Lambda^2} \mp 16\Lambda^3\sqrt{-2a + \Lambda^2}}{4}, \\
    b &= \frac{a^2}{4} + 2\Lambda^4 \pm 2\Lambda^3\sqrt{-2a + \Lambda^2} + a(-3\Lambda^2 \pm \Lambda\sqrt{-2a + \Lambda^2}), \\
    c &= \frac{a - 5\Lambda^2 \pm 3\Lambda\sqrt{-2a + \Lambda^2}}{2}.
\end{align*}
\]

Under the limit $\Lambda \rightarrow 0$, $(s_1, s_2, b, c) \rightarrow (-a, \frac{a^2}{4}, \frac{a^2}{4}, \frac{a}{2})$ which gives $SO(4) \rightarrow U(2)$ with two vacua coming from the $U(2)$. Again by the relationship between $F_4(x)$ and $W'(x)/x$ we get $a = -(2\alpha^2 + 4\Lambda^2)$. Notice that we cannot take the limit $\Lambda \rightarrow 0, a \rightarrow 0$ because the curve form (5.26) has no factor $t$ before the limit is reached, whereas if there are D5-branes wrapping around origin $x = 0$, the starting curve must have a factor $t$. Because of this inconsistency, we can not take that limit. Similar observations have been presented in [20].

The phase structure has been summarized in Table 6.


5.2 \textit{SO}(5) case

For the \textit{SO}(5) gauge group, \(N_f\) can be zero, one, or two. There are three breaking patterns \(\text{SO}(5) \to \widehat{\text{SO}}(3) \times U(1), \text{SO}(5) \to U(2),\) and \(\text{SO}(5) \to \widehat{\text{SO}}(5).\) The \(N_f = 0\) case has been discussed in section 4 under massive flavors. For the \(N_f = 2\) case, notice that the curve is

\[y^2 = P_4^2(x) - 4\Lambda^2 x^6\]

which is exactly the same form as the curve of \(\text{SO}(4)\) with \(N_f = 1.\) Thus, the results are the same as \(\text{SO}(4)\) with one flavor. In fact, it is an example of the addition map previously discussed in Appendix B. For \(N_f = 1,\) the curve

\[y^2 = P_4^2(x) - 4\Lambda^4 x^4\]

is the same as that of \(\text{SO}(4)\) without any flavors, so the results can be applied here. However, now we need to distinguish between Special vacua and Chebyshev vacua for the case of massless flavors. Because of this, we will repeat the \(\text{SO}(5)\) example, but not discuss \(\text{SO}(7)\) at all because everything in \(\text{SO}(7)\) can be reduced to the results of \(\text{SO}(6).\) Additionally, we will use a somewhat different method to redo the calculations. It will be interesting to compare these two methods.

1. Non-degenerated case

The curve should be

\[y^2 = P_4^2(x) - 4x^4\Lambda^4 = x^2F_6(x).\]

Using \(P_4(x) = x^2(x^2 - u)\) and \(F_6(x) = W'(x)^2 + ax^2 + b,\) we found that \(b = 0, u = \alpha^2,\) and \(a = - (\alpha^4 - 4\Lambda^4),\) which gives one vacuum of \(\text{SO}(5) \to \widehat{\text{SO}}(3) \times U(1).\) Again, the fact that \(y^2 = x^4F_4(x)\) indicates that it is the Special vacuum. The reason that the Chebyshev branch of \(\widehat{\text{SO}}(3) \times U(1)\) does not exist can be seen from the counting \(N - N_f - 2 = 3 - 1 - 2 = 0\) for the Chebyshev branch.

2. Degenerated case and Special branch

The curve should be

\[y^2 = P_4^2(x) - 4x^4\Lambda^4 = F_4(x)H_2^2(x)\]

or

\[y^2 = (t^2 - s_1t + s_2)^2 - 4t^2\Lambda^4 = (t^2 + at + b)(t + c)^2, \quad t = x^2.\]

There are three solutions for \((s_1, s_2, b, c).\) The first is \((-\alpha^2, 0, \alpha^2 - 4\Lambda^4, 0).\) To determine the classical limit we need to determine \(a.\) Similar to the discussion of \(\text{SO}(4)\) with one flavor we
need to consider various situations. The first is that there are D5-branes wrapping around the 
\( x = \pm \alpha \), so \( a = -2\alpha^2 \), which gives the Special vacuum of \( SO(5) \to \tilde{SO}(3) \times U(1) \) presented 
in the previous paragraph. The second is that all D5-branes are wrapped around the origin 
\( x = 0 \) so we have \( SO(5) \to \tilde{SO}(5) \). Then we need a factor \( t^3 \) for the Chebyshev vacua where 
\( N_f + 2 = 3 \) and \( t^0 \) for Special vacua where \( 2N_f - N + 3 = 0 \). Since \( c = 0 \) gives at least \( t^2 \), we 
we can only get two Chebyshev vacua by setting \( b = 0 \) or \( a = \pm 4\Lambda^2 \), which match the counting 
\( (N - N_f - 2) = 5 - 1 - 2 = 2 \). The missing Special vacuum will be discussed immediately.

The second and third solutions are \((-a \pm 2\Lambda^2, (\frac{a+4\Lambda^2}{2})^2, (\frac{a+\Lambda^2}{2})^2, \frac{a+4\Lambda^2}{2})\). For nonzero D5-
branes wrapping around the \( x = \pm \alpha \), again we have \( a = -2\alpha^2 \), so there are two vacua for 
\( SO(5) \to U(2) \). Besides that, if we set \( a = 2\Lambda^2 \) for the second solution or \( a = -2\Lambda^2 \) for the 
third solution, both will give the same curve having a complete square form, which is identical 
with the Special vacuum of \( SO(5) \to \tilde{SO}(5) \).

We want to emphasize again that the calculations here are exactly the same calculations 
of \( SO(4) \) with \( N_f = 0 \). The only difference is how to explain and describe these results in the 
phase structure of \( SO(5) \) gauge group.

### 5.3 \( SO(6) \) case

For the \( SO(6) \) gauge group, the number of flavors \( N_f \) can be 0, 1, 2, or 3. There exist the 
following four breaking patterns: \( SO(6) \to \tilde{SO}(2) \times U(2) \), \( SO(6) \to \tilde{SO}(4) \times U(1) \), \( SO(6) \to 
\tilde{SO}(6) \), and \( SO(6) \to U(3) \). For the \( SO(2) \) factor, only the Special branch exists because for 
the Chebyshev branch there is a negative vacuum number \( N - N_f - 2 < 0 \), which does not 
exist when for \( \tilde{SO}(4) \) and \( \tilde{SO}(6) \) factors, both the Special branch and Chebyshev branch exist.
The \( N_f = 0 \) case has been discussed in section 4 under massive flavors, so we are left only with 
\( N_f = 1, 2, 3 \).

- \( N_f = 1 \)

#### 1. Non-degenerated case

The curve should be

\[
y^2 = P_6^2(x) - 4x^6\Lambda^6 = x^2F_6(x)H_2^2(x).
\]

Using \( F_6(x) = W'(x)^2 + bx^2 + d \), \( H_2(x) = x^2 + a \), and \( P_6(x) = x^2(x^2 - t_1)(x^2 - t_2) \), we found 
four solutions for \((t = t_1t_2, u = t_1 + t_2, a, b, d)\). The first is \((0, \alpha^2, 0, 0, -4\Lambda^6)\), which gives 
\( y^2 = x^6(W'(x)^2 - 4\Lambda^6) \) and \( F_6(x) = x^4(x^2 - \alpha^2)^2 \). Classically it gives the Chebyshev branch of 
\( SO(6) \to \tilde{SO}(4) \times U(1) \) with the counting \( N - N_f - 2 = 4 - 1 - 2 = 1 \). The factor \((x^2)^3\) is exactly 
the one we need for \( SO(4) \) with \( N_f = 1 \) at the Chebyshev point by noticing \((N_f + 2) = 3\).

The second solution has \( d = 0 \), so the curve is \( y^2 = \left[ \left( \frac{W'(x)}{x} \right)^2 + \hat{b} \right] [x^2(x^2 + a)]^2 \). At the 
classical limit it becomes \((0, \alpha^2, 0, 0, 0)\) which gives \( SO(6) \to \tilde{SO}(4) \times U(1) \). Because of the

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factor \((x^2)^2\), it is the Special branch by noticing \((2N_f - N + 4) = 2\). The counting for the number of vacua is also consistent.

The third and fourth solutions have \(d = 0\) and the classical limit becomes \((\alpha^4, 2\alpha^2, -\alpha^2, 0, 0)\), which gives \(SO(6) \to \hat{SO}(2) \times U(2)\). Since \(d = 0\), they are in fact the Special vacua with the counting two coming from the \(U(2)\) factor.

To discuss the smooth transition, we can use the above solutions directly by taking various limits. However, since the solutions are so complex, it is not easy to see the results. However, it is easy to see that the first solution cannot be smoothly interpolated to the other three solutions. We will discuss the relationship of the other three solutions immediately by another method.

2. Degenerated case and Special branch

The curve should be factorized as

\[
y^2 = P_6^2(x) - 4x^6 \Lambda^6 = F_4(x)H_4^2(x),
\]

or

\[
y^2 = P_3^2(t) - 4t^3 \Lambda^6 = F_2(t)H_2^2(t), \quad t = x^2.
\]

To solve the problem, we parameterize as follows:

\[
F_2(t) = (t - a)^2 + b, \quad H_2(t) = t^2 + ct + d, \quad P_3(t) = t^3 - s_1 t^2 + s_2 t - s_3.
\]

There are four solutions. The first is given by

\[
s_1 = \frac{-3b}{4\Lambda^2}, \quad s_2 = \frac{3b^2}{16\Lambda^4} + \frac{3b}{4}, \quad s_3 = -\frac{(b + 4\Lambda^4)^3}{64\Lambda^6},
\]

\[
a = \frac{-b}{4\Lambda^2} + \Lambda^2, \quad c = \frac{b}{2\Lambda^2} + \Lambda^2, \quad d = \frac{(b + 4\Lambda^4)^2}{16\Lambda^4}.
\]

There is only one sensible limit \(b/\Lambda^2 = \beta \neq 0\) where \(P_3(t) \to (t + \beta^3)^3\), so it gives \(SO(6) \to U(3)\). The second and third solutions have

\[
s_1 = 3(1 \pm i\sqrt{3}) \frac{b}{8\Lambda^2},
\]

which has also only one sensible limit \(b/\Lambda^2 = \beta \neq 0\) and gives \(SO(6) \to U(3)\). In fact, for all of these solutions we should consider two cases: one is \(b = 0\) and the other, \(b \neq 0\). For \(b = 0\), all three solutions give the same curve \((5.27)\) as a square form with \(P_3(t) = t^3 + \Lambda^3\). This vacuum is the Special vacuum of \(SO(6) \to \hat{SO}(6)\). For \(b \neq 0\), these three solutions give the three vacua of \(SO(6) \to U(3)\). Since \(SO(6) \to \hat{SO}(6)\) has \(b = 0\) and a factor \(t\) at the curve, there is no smooth connection between \(SO(6)\) and \(U(3)\).
The fourth solution has
\[
\begin{align*}
    s_1 &= \frac{b^2}{8\Lambda^6} - \frac{4\Lambda^6}{b}, \\
    s_2 &= \frac{b}{256} \left( 64 + \frac{b^3}{\Lambda^{12}} \right), \\
    s_3 &= 0,
\end{align*}
\]
\[
a &= \frac{b^2}{16\Lambda^6} - \frac{4\Lambda^6}{b}, \\
e &= \frac{-b^2}{16\Lambda^6}, \\
d &= 0.
\]

In fact, since \(d = 0\) and \(s_3 = 0\), we can factorize out the \(t^2\) factor and the curve (5.27) is reduced to the degenerated case of \(SO(5)\) without flavor. Thus, we immediately get the following results:

1. There is a smooth interpolation between \(SO(2) \times U(2)\) and \(SO(4) \times U(1)\);
2. There are two Special vacua for \(SO(2) \times U(2)\), one Special vacuum for \(SO(4) \times U(1)\), and three Chebyshev vacua \((N - N_f - 2) = (6 - 1 - 2) = 3\) for \(SO(6) \to SO(6)\).

   - \(N_f = 2\)

1. **Non-degenerated case**
   For simplicity we will use \(t = x^2\). The curve is given by
   \[
y^2 = P^2_3(t) - 4\Lambda^4 t^4 = tF_3(t)H^2_1(t).
   \]
   Due to the factor \(t\) on the right hand side, we write \(P_3(t) = t(t^2 - s_1 t + s_2)\), \(H_1(t) = t + a\), and \(F_3(t) = t^3 + bt^2 + ct + d\). There are three solutions for \((s_1, s_2, a, c, d)\) as a function of \(b\). The first is \((-b/2, 0, 0, b^2/4 - 4\Lambda^4, 0)\) which gives \(SO(6) \to SO(4) \times U(1)\). Since \(F_3(t) = t(t^2 + bt + (b^2/4 - 4\Lambda^4))\), it is, in fact, the solution of Special or degenerated case. The other two solutions are \((-b \pm 2\Lambda^2, \frac{1}{8}(b \mp 4\Lambda^2)^2, \frac{1}{2}(b \mp 4\Lambda^2), \frac{1}{4}(b \mp 4\Lambda^2)^2, 0)\), which give \(SO(6) \to SO(2) \times U(2)\).
   Again, since \(d = 0\) they are the solutions of the Special or degenerated case. The counting number two comes from the \(U(2)\) factor. Finally, using the relationship of \(W'(x)\) and \(F_3(t)\) we get \(b = -2\alpha^2\). There is no smooth transition between these three solutions. It is noteworthy that there is no Chebyshev vacua of \(SO(6) \to SO(4) \times U(1)\) because the counting leads to \((N - N_f - 2) = (4 - 2 - 2) = 0\).

2. **Degenerated case and Special branch**
   The curve should be
   \[
y^2 = P^2_3(t) - 4\Lambda^4 t^4 = F_2(t)H^2_2(t).
   \]
   Let us consider the solutions case by case. First, if there is a factor \(t^4\), we have \(P_3(t) = t^2(t - s)\), \(H_2(x) = t^2\), and \(F_2(t) = (t - s)^2 - 4\Lambda^4\). There are two situations to be considered. If there are D5-branes wrapping around the \(x = \pm \alpha\), we need to have \(F_2(t) = (t - \alpha)^2 + d\), so \(s = \alpha^2\), \(d = -4\Lambda^4\) and \(SO(6) \to SO(4) \times U(1)\). It is the same solution presented in the previous paragraph. The power 4 = \((2N_f - N + 4)\) of \(t\) is the expected character of \(SO(4)\) at the Special vacua. If there are no D5-branes wrapping around the \(x = \pm \alpha\), we need a factor \(t\) from \(F_2(t)\) which is equal to set \(s^2 = 4\Lambda^4\). The curve has a factor \(t^2\) which gives the Chebyshev vacua of \(SO(6) \to SO(6)\). The counting two is also consistent with the fact that \((N - N_f - 2) = 2\).
If there is only one \( t^2 \) factor in (5.28), we write \( P_3(t) = tP_2(t) \) and \( H_2(t) = t(t + a) \). Canceling out the factor \( t^2 \), we get \( P_2(t) = 4\Lambda^4 t^2 = F_2(t)(t + a)^2 \). It can be solved if we require \( P_2(t) = 2\eta\Lambda^2 t = (t + a)^2 \), so \( F_2(t) = (t + a)^2 + 4\eta\Lambda^2 t \). If there are D5-branes wrapping around \( x = \pm \alpha \), we get \( a = -\alpha^2 \) and two Special vacua for \( SO(6) \to \hat{SO}(2) \times U(2) \), as found in the previous paragraph. If there is no D5-brane wrapping around \( x = \pm \alpha \), there are two choices: either \( F_2(t) \) has a factor \( t \) so that \( a = 0 \) or \( F_2(t) \) is a square. The former where \( a = 0 \) gives a factor \( t^5 \), which is the one discussed before. The latter is given by \( P_2(t) = t^2 + \Lambda^4 \), which gives the one vacuum of \( SO(6) \to \hat{SO}(6) \) at the Special branch.

Finally we consider the case in which there is no \( t^2 \) factor at all, so all D5-branes must wrap around \( x = \pm \alpha \). To achieve this, we must have

\[
P_3(t) - 2\Lambda^2 t^2 = (t + a)^2(t + b), \quad P_3(t) + 2\Lambda^2 t^2 = (t + c)^2(t + d).
\]

This problem is the same problem of \( U(3) \) with \( N_f = 4 \) which has been solved in [20]

\[
a = s_1 + s_2, \quad c = s_1 - s_2, \quad b = s_1 - 2\Lambda^2 - 2s_2 + \frac{2s_1\Lambda^2}{s_2}, \quad d = s_1 + 2\Lambda^2 + 2s_2 + \frac{2s_1\Lambda^2}{s_2},
\]

with \( s_2^2(s_2 + \Lambda^2) = s_1^2\Lambda^2 \). Furthermore, using \( (t + b)(t + d) = (t - \alpha^2)^2 + e \) we get

\[
bd = -2\alpha^2 \to s_1 \left( 1 + \frac{2\Lambda^2}{s_2} \right) = -\alpha^2.
\]

For fixed \( \alpha \), there are three solutions which have the classical limits \( (s_1, s_2) \to (\pm \alpha^2, 0) \). These three vacua give \( SO(6) \to U(3) \).

It is worth remarking about the different behaviors. We observe a smooth interpolation between \( \hat{SO}(4) \times U(1) \) and \( \hat{SO}(2) \times U(2) \) for \( N_f = 1 \) at Special branch, but not for \( N_f = 2 \). The reason for that comes from the power of \( t \). The Special branch of \( SO(2) \) always has a factor \( t^2 \) in the curve. For \( SO(4) \), it is \( (2N_f - N + 4) = 2N_f \), so only for the case with \( N_f = 1 \) can we have a factor \( t^2 \). This also explains why there is no smooth interpolation of \( SO(4) \times U(1) \) or \( \hat{SO}(2) \times U(2) \) to \( U(3) \) because there is no \( t \) factor for the curve of \( U(3) \).

- \( N_f = 3 \)

1. Non-degenerated case

Setting \( t = x^2 \), the curve of non-degenerated case should be

\[
y^2 = P_3^2(t) - 4\Lambda^2 t^5 = tF_3(t)H_1^2(t).
\]

Writing \( P_3(t) = t(t^2 - s_1 t + s_2) \), \( H_1(t) = t + a \), and \( F_3(t) = t^3 + bt^2 + ct + d \), we found three solutions. The first one has \( (s_1, s_2, a, c, d) = (-\frac{b - 4\Lambda^2}{2}, 0, 0, \frac{(b + 4\Lambda^2)^2}{4}) \). It gives one vacuum of \( SO(6) \to \hat{SO}(4) \times U(1) \). However, since \( F_3(t) = t(t^2 + bt + c) \), it is in fact at the Special branch. It is also noteworthy that the power of \( t \) is 4, which is not equal to \( (2N_f - N + 4) = 6 \). The
reason is that $SO(4)$ with three flavors is not asymptotically free (recalling that $N_f < N - 2$); thus, the power stops at $N$. In fact, we have a similar result that there is a factor $t^2$ for $SO(2)$ with flavors. The other two solutions have $d = 0$ and the classical limits $(-b, b^2/4, b/2, b^2/4, 0)$, so give $SO(6) \rightarrow \hat{SO}(2) \times U(2)$. Again, since $d = 0$, these two vacua are in fact at the Special branch. The counting number two comes from the $U(2)$ factor. Finally, we can fix as follows: $b = \frac{-2\alpha^2 - 4\Lambda^2}{\alpha^2}$. The reason why we did not find the Chebyshev branch in the non-degenerated case is that the number of counting becomes $(N - N_f - 2) < 0$ for both the $SO(2)$ and $SO(4)$ gauge groups.

2. Degenerated case and Special branch

Let us now consider the Special branch or the degenerated case with the following curve

$$y^2 = P_3^2(t) - 4\Lambda^2 t^5 = F_2(t)H_2^2(t).$$

We will discuss them case by case. First, if there is a $t^4$ factor in the curve, we must take $P_3(t) = t^2(t - s)$, $H_2(t) = t^2$ and $F_2(t) = (t - s)^2 - 4\Lambda^2 t$. There are three possible values for $s$. If $s = -\Lambda^2$, the curve is a complete square and gives the Special vacuum of $SO(6) \rightarrow \hat{SO}(6)$. Another case is $s = \alpha^2$ which gives one Special vacuum of $SO(6) \rightarrow \hat{SO}(4) \times U(1)$. As we have remarked before, since $N_f = 3 > (N - 2) = 1$, the power of $t$ is $N = 4$. Finally, if $F_2(t)$ has just one factor $t$, or if $s = 0$, it gives the one vacuum of the Chebyshev branch $SO(6) \rightarrow \hat{SO}(6)$.

Second, if there is only $t^2$ factor in the curve, we have $P_3(t) = t(t^2 - s_1t + s_2)$, $H_2(t) = t(x + c)$. Cancelling the $t^2$ at both sides, we get the reduced curve

$$P_2^2(t) - 4\Lambda^2 t^3 = F_2(t)H_1^2(t).$$

This is exactly the same curve (5.26) and we get two Special vacua of $SO(6) \rightarrow \hat{SO}(2) \times U(2)$ by keeping only those solutions with a factor $t^2$.

The last case is when there is no factor $t$ in the curve and all D5-branes wrap around the root $x = \pm\alpha$, so the gauge group is broken to $SO(6) \rightarrow U(3)$. Using this fact, we know immediately $F_2(t) = (t - \alpha^2)^2 + d - 4\Lambda^2 t$. There are three solutions $^{24}$.

The phase structure of $SO(6)$ with massless flavors has been summarized in Table 7.

5.4 $SO(7)$ case

The curve is

$$y^2 = P_3^2(t) - 4\Lambda^{10 - 2N_f}t^{1+N_f}.$$  

$^{24}$It is very complicated to solve the factorization. However, notice that at the limit $\Lambda \rightarrow 0$, there are $a \rightarrow \alpha^2, c \rightarrow -2\alpha^2, d \rightarrow \alpha^4$ and $b \rightarrow 0$. If we set $\alpha = 1$ and $\Lambda = 0.001$, numerically it is easy to see only three solutions that satisfy this limit.
### Table 7: Summary of the phase structures of $SO(6)$ gauge group with massless flavors. It is worth comparing with the phase structures of $SO(6)$ with massive flavors. Here again we use capital letters in the last column to indicate the phases having smooth transition.

| $N_f$ | Group              | Branch      | Power of $t(= x^2)$ | $U(1)$ | Number of vacua | Connection |
|-------|--------------------|-------------|---------------------|--------|-----------------|------------|
| 1     | $SO(2) \times U(2)$ | $(S, 0_{NB})$ | $t^2$               | $1$    | $2$             | A          |
|       | $SO(4) \times U(1)$ | $(C, 0_{NB})$ | $t^3$               | $1$    | $1$             |            |
|       |                    | $(S, 0_{NB})$ | $t^4$               | $1$    | $1$             | A          |
|       | $SO(6)$            | $(C)$       | $t^3$               | $0$    | $3$             |            |
|       |                    | $(S)$       | $t^0$               | $0$    | $1$             |            |
| 2     | $SO(2) \times U(2)$ | $(S, 0_{NB})$ | $t^2$               | $1$    | $2$             |            |
|       | $SO(4) \times U(1)$ | $(S, 0_{NB})$ | $t^4$               | $1$    | $1$             |            |
|       | $SO(6)$            | $(C)$       | $t^5$               | $0$    | $2$             |            |
|       |                    | $(S)$       | $t^4$               | $0$    | $1$             |            |
| 3     | $SO(2) \times U(2)$ | $(S, 0_{NB})$ | $t^2$               | $1$    | $2$             |            |
|       | $SO(4) \times U(1)$ | $(S, 0_{NB})$ | $t^4$               | $1$    | $1$             |            |
|       | $SO(6)$            | $(C)$       | $t^5$               | $0$    | $1$             |            |
|       |                    | $(S)$       | $t^4$               | $0$    | $1$             |            |
|       | $U(3)$             | $(0_{NB})$  | $t^0$               | $1$    | $3$             |            |
For $N_f = 0$, it has been discussed in the section 5 of massive flavors. For $N_f \geq 1$, the curve can be written as

$$y^2 = P_3^2(t) - 4\Lambda^{10-2N_f}t^{2N_f-1}$$

which is the same curve of $SO(6)$ with $(N_f - 1)$ flavors. Therefore, all the results of $SO(6)$ can be applied here.

## 5.5 The smooth interpolation

Now let us discuss in this subsection the general picture of smooth interpolations. The classical limit has three types: $SO(N_c) \to SO(N_0) \times U(N_1)$, $SO(N_c) \to U([N_c/2])$, and $SO(N_c) \to SO(N_c)$. The first two have $U(1)$ left at IR while for the last one, there is no $U(1)$ left at all. Because of this fact, only the first two types can have smooth interpolations by realizing the presence of the number of $U(1)$ left.

Now for the type $SO(N_c) \to U([N_c/2])$, the curve has no factor $t$ in the factorization form, while for the type $SO(N_c) \to SO(N_0) \times U(N_1)$, depending on the Chebyshev branch or Special branch, it has a different power of $t$.

For the Chebyshev branch, the power of $t = x^2$ is always an odd number and mainly the function of $N_f$. Thus, the type $SO(N_c) \to SO(N_0) \times U(N_1)$ cannot be smoothly connected to the type $SO(N_c) \to U([N_c/2])$, but can have smooth interpolations inside itself. For example, if $SO(N_c) \to SO(N_0) \times U(N_1)$ and $SO(N_c) \to SO(M_0) \times U(M_1)$ with $N_f < N_0 - 2$ and $N_f < M_0 - 2$ (so the Chebyshev branch exists for both $SO(N_0)$ and $SO(M_0)$), the curves will have the same power of $t$ and can give the above two different classical limits. To demonstrate the analysis, let us consider the $SO(8)$ with one flavor. The curve is

$$y^2 = P_8^2(x) - 4x^6\Lambda^{10} = x^2F_6(x)H_4^2(x).$$

The required $x^6$ factor forces us that $P_8(x) = x^4(x^4 - s_1x^2 + s_2)$, $H_4(x) = x^2(x^2 + a)$. Writing $F_6(x) = x^6 + bx^4 + cx^2 + d$, we have the following solutions

$$b = 2a \mp \frac{i\Lambda^5}{a^{3/2}}, \quad c = a^2 \mp 6\frac{i\Lambda^5}{a^{1/2}} - \frac{\Lambda^{10}}{a^3}, \quad d = -4\frac{\Lambda^{10}}{a^2},$$

$$s_1 = -2a \pm \frac{i\Lambda^5}{a^{3/2}}, \quad s_2 = a^2 \mp 3\frac{i\Lambda^5}{a^{1/2}}.$$

We can take the following limits: (1) $\Lambda \to 0$, $a$ fixed, we get $SO(8) \to SO(4) \times U(2)$; (2) $\Lambda \to 0$, $a \to 0$, but $\frac{i\Lambda^5}{a^{3/2}} \to \beta \neq 0$, we get $SO(8) \to SO(6) \times U(1)$. We have not shown that it is always true that $SO(N_0) \times U(N_1)$ is connected to $SO(M_0) \times U(M_1)$ under the above conditions, but we expect it to be true.
Now let us consider the Special branch in which the power of $t = x^2$ is always an even number. 1) First, if the power is zero, it is possible to connect $SO(N_c) \rightarrow \hat{SO}(N_0) \times U(N_1)$ to $SO(N_c) \rightarrow U([N_c/2])$. Recalling that the power is given by $(2N_f - N + 4)/(2N_f - N + 3)$ for $N$ is even/odd and $N_f < N - 2$ and $N/(N-1)$ for $N_f \geq N - 2$, we need $(2N_f - N + 4)/(2N_f - N + 3) = 0$ or $N_f = (N - 4)/2$ for $N$ even and $N_f = (N - 3)/2$ for $N$ odd. This has been observed for $SO(5)$ with $N_f = 0$ where there is, in fact, a smooth interpolation in $\hat{SO}(3) \times U(1) \leftrightarrow U(2)$.

2) Second, if the power is not zero, we can only expect a smooth interpolation inside the first type. Since the power is a function of both $N_f$ and $N$, the symmetry breaking pattern

$$SO(N_0) \times U(N_1)$$

is connected to the following symmetry breaking pattern

$$SO(M_0) \times U(M_1)$$

(assuming $M_0 < N_0$ and both are even numbers) only if $N_f, M_f < N_0 - 2$ and $(2N_f - N_0 + 4) = M_0$. One such example is $SO(6)$ with one flavor where $\hat{SO}(2) \times U(2)$ is smoothly connected to $\hat{SO}(4) \times U(1)$. In fact, case 1) where the power of $t$ is zero and case 2) where the power of $t$ is not zero are related to each other by the addition map.

It is noteworthy that for the smooth transition in the Special branch, the condition $(2N_f - N_0 + 4) = M_0$ is exactly the condition of the Seiberg dual pair between $SO(N_0)$ and $SO(M_0)$. Therefore, the smooth transition can be connected through the Seiberg duality. However, for a smooth transition in the Chebyshev branch, no such relationship exists and a smooth transition cannot be understood from the Seiberg duality.

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Appendix A Strong gauge coupling approach: superpotential and a generalized Konishi anomaly equation for $SO(2N_c)$ case

In [19] the strong gauge coupling approach for $SO/USp$ pure gauge theories were studied generally. That analysis was an extension of [8] to allow a more general superpotential in
which the degree can be arbitrary without any restrictions. We extend the analysis discussed in [19] to $SO(2N_c)$ gauge theory with $N_f$ flavors by using the method of [72, 79, 80]. Let us consider superpotential regarded as a small perturbation of an $\mathcal{N} = 2$ $SO(2N_c)$ gauge theory [65, 66, 67, 68, 69, 70, 71, 72, 8, 11] 

$$W(\Phi) = \sum_{s=1}^{k+1} \frac{g_{2s}}{2s} \Tr \Phi^{2s} \equiv \sum_{s=1}^{k+1} g_{2s} u_{2s}, \quad u_{2s} \equiv \frac{1}{2s} \Tr \Phi^{2s}$$  \hspace{1cm} (A.1)

where $\Phi$ is an adjoint scalar chiral superfield and we denote its eigenvalues by $\pm \phi_I (I = 1, 2, \cdots, N_c)$. The degree of the superpotential $W(\Phi)$ is $2(k+1)$. Since $\Phi$ is an antisymmetric matrix we can transform to the following simple form,

$$\Phi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \text{diag}(i\phi_1, i\phi_2, \cdots, i\phi_{N_c-r}, 0, 0, \cdots, 0).$$  \hspace{1cm} (A.2)

When we replace $\Tr \Phi^{2j}$ with $\langle \Tr \Phi^{2j} \rangle$, the superpotential becomes an effective superpotential. We introduce a classical $2N_c \times 2N_c$ matrix $\Phi_{cl}$ such that $\langle \Tr \Phi^{2j} \rangle = \Tr \Phi_{cl}^{2j}$ for $j = 1, 2, \cdots, N_c$. Additionally, $u_{2j} \equiv \frac{1}{2j} \Tr \Phi_{cl}^{2j}$ are independent. However, for $2j > 2N_c$, both $\Tr \Phi^{2j}$ and $\langle \Tr \Phi^{2j} \rangle$ can be written as the $u_{2j}$ of $2j \leq 2N_c$. Classical vacua can be obtained by putting all the eigenvalues of $\Phi$ and $\Phi_{cl}$ equal to the roots of $W'(z) = \sum_{s=1}^{k+1} g_{2s} z^{2s-1}$. We will take the degree of superpotential to be $2(k+1) \leq 2N_c$ first in which the $u_{2j}$ are independent and $\langle \Tr \Phi^{2j} \rangle = \Tr \Phi_{cl}^{2j}$. Then we will take the degree of superpotential to be arbitrary. Until now we have reviewed the discussion given in [19]. Next we will study some restriction that is an important idea in the discussion below.

Let us consider the special $(N_c - r)$ dimensional submanifold of the Coulomb branch where some of the branching points of the moduli space collide. The index $r$ runs from 0 to $\min(N_c, N_f/2)$ [68, 80] and classifies the branch. On the $r$-th branch, the effective theory becomes $SO(2r) \times U(1)^{N_c-r}$ with $N_f$ massless flavors. Thus, after turning on the tree level superpotential since there exist $U(1)^n$ gauge groups with $2n \leq 2k$, the remaining $(N_c - r - n)$ $U(1)$ factors are confined, which lead to $(N_c - r - n)$ massless monopoles or dyons. This occurs only at points where at $W' = 0$ the monopoles are massless on some particular submanifold $< u_{2r} >$. This can be done by including the $(N_c - r - n)$ monopole hypermultiplets in the superpotential. Then the exact effective superpotential by adding (A.1), near a point with $(N_c - r - n)$ massless monopoles, is given by

$$W_{eff} = \sqrt{2} \sum_{l=1}^{N_c-n-r} M_l(u_{2s}) q_l \bar{q}_l + \sum_{s=1}^{k+1} g_{2s} u_{2s}.$$ 

By varying this with respect to $u_{2s}$, we get an equation of motion similar to pure Yang-Mills theory except that the extra terms on the left hand side since the $u_{2s}$ with $2s > 2(N_c - r)$ are dependent on $u_{2s}$ with $2s \leq 2(N_c - r)$. There exist $(N_c - n - r)$ equations for the $(k+1)$
parameters \(g_{2s}\). Here \(q_l\) and \(\tilde{q}_l\) are the monopole fields and \(M_l(u_{2s})\) is the mass of \(l\)-th monopole as a function of the \(u_{2s}\). The variation of \(W_{\text{eff}}\) with respect to \(q_l\) and \(\tilde{q}_l\) vanishes. However, variation of \(W_{\text{eff}}\) with respect to \(u_{2s}\) does not lead to the vanishing of \(q_l\tilde{q}_l\) and the mass of monopoles should vanish for \(l = 1, 2, \ldots, (N_c - r - n)\) in a supersymmetric vacuum. Therefore, the superpotential in this supersymmetric vacuum becomes \(W_{\text{exact}} = \sum_{r=1}^{k+1} g_{2s} < u_{2s}>\). The masses \(M_l\) are equal to the periods of some meromorphic one-form over some cycles of the \(\mathcal{N} = 2\) hyperelliptic curve.

It is useful and convenient to consider a singular point in the moduli space where \((N_c - r - n)\) monopoles are massless. Then the \(\mathcal{N} = 2\) curve of genus \((2N_c - 2r - 1)\) degenerates to a curve of genus \(2n\) \([8, 11]\) and it is given by

\[
y^2 = P_{2(N_c - r)}^2(x) - 4A^{4(N_c - r) - 2(N_j - 2r + 2)}x^{2(N_j - 2r + 2)} = x^2H_{2N_c - 2r - 2n - 2}^2(x)F_{2(2n + 1)}(x),
\]

(A.3)

where

\[
H_{2N_c - 2r - 2n - 2}(x) = \prod_{i=1}^{N_c - n - r - 1} (x^2 - p_i^2), \quad F_{2(2n + 1)}(x) = \prod_{i=1}^{2n + 1} (x^2 - q_i^2).
\]

Here \(H_{2N_c - 2n - 2r - 2}(x)\) is a polynomial in \(x\) of degree \((2N_c - 2r - 2n - 2)\) that gives \((2N_c - 2r - 2n - 2)\) double roots and \(F_{2(2n + 1)}(x)\) is a polynomial in \(x\) of degree \((4n + 2)\) that is related to the deformed superpotential. Both functions are even functions in \(x\). That is, a function of \(x^2\) which is peculiar to the gauge group \(SO(2N_c)\). Since we concentrate on the \((N_c - r)\) dimensional subspace of Coulomb phase the characteristic function \(P_{2(N_c - r)}(x)\) is defined by \(2(N_c - r) \times 2(N_c - r)\) matrix \(\Phi_{\text{cl}}\) as follows:

\[
P_{2(N_c - r)}(x) = \det(x - \Phi_{\text{cl}}) = \prod_{l=1}^{N_c - r} (x^2 - \phi_l^2).
\]

(A.4)

The degeneracy of the above curve can be checked by computing both \(y^2\) and \(\frac{\partial y^2}{\partial x^2}\) at the point \(x = \pm p_i\) and \(x = 0\) obtaining a zero. The factorization condition (A.3) can be described and encoded by Lagrange multipliers [18]. When the degree \((2k + 1)\) of \(W'(x)\) is equal to \((2n + 1)\), we obtain the result (A.6).

- **Field theory analysis for superpotential**

In [20] it was noted that when the degree of \(W'(x)\), \(n\), was greater than or equal to \((2N_c - N_f)\) the structure of the matrix model curve of \(U(N_c)\) gauge theories was changed by the effect of the flavors. Thus, we first extended the discussion in [20] to \(SO(2N_c)\) gauge theory and then

\[25\]As we classified in previous section, there exist various curves (3.4), (3.5) and (3.6) depending on the number of single and double roots. It is straightforward to proceed to other cases so we restrict ourselves to here the particular case (A.3).
later to the more general cases. The massless monopole constraint for \( SO(2N_c) \) gauge theory with \( N_f \) flavors is described as follows:

\[
y^2 = P_{2N_c}^2(x) - 4A^{4N_c-4-2N_f}A(x) = x^2H_{2N_c-2n-2}^2(x)F_{2(2n+1)}(x),
\]

\[
y = x^2 \prod_{i=1}^{l-1} \left(x^2 - p_i^2\right)^2 F_{2(2n+1)}(x), \quad A(x) \equiv x^4 \prod_{j=1}^{N_f} (x^2 - m_j^2)
\]

(A.5)

where we used \( l \) as the number of massless monopoles. From the equation (A.5) we can find double roots at \( x = 0, \pm p_i \), \( i = 1, \ldots, l - 1 \). We do not need take into account all points. Since \( P_{2N_c}(x) \) and \( A(x) \equiv x^4 \det_{N_f}(x + m) \) are an even function of \( x \), we have only to consider \( x = 0, x = +p_i \). For simplicity of equations we introduce \( p_i = 0 \); thus, index \( i \) runs from 1 to \( l \). With these conventions we have an effective superpotential with \( l \) massless monopole constraints (A.5),

\[
W_{low} = \sum_{i=1}^{k+1} g_{2t} u_{2t} + \sum_{i=1}^{l} \left[ L_i \left( P_{2N_c}(p_i) - 2\epsilon_i \Lambda^{2N_c-2-N_f} \sqrt{A(p_i)} \right) + Q_i \frac{\partial}{\partial p_i} \left( P_{2N_c}(p_i) - 2\epsilon_i \Lambda^{2N_c-2-N_f} \sqrt{A(p_i)} \right) \right]
\]

where \( L_i \) and \( Q_i \) are Lagrange multipliers and \( \epsilon_i = \pm 1 \). From the equation of motion for \( p_i \) and \( Q_i \) we obtain the following relations,

\[
Q_i = 0, \quad \frac{\partial}{\partial p_i} \left( P_{2N_c}(p_i) - 2\epsilon_i \Lambda^{2N_c-2-N_f} \sqrt{A(p_i)} \right) = 0.
\]

The variation of \( W_{low} \) with respect to \( u_{2t} \) leads to

\[
g_{2t} + \sum_{i=1}^{l} L_i \frac{\partial}{\partial u_{2t}} \left( P_{2N_c}(p_i) - 2\epsilon_i \Lambda^{2N_c-2-N_f} \sqrt{A(p_i)} \right) = 0.
\]

Since \( A(p_i) \) is independent of \( u_{2t} \) the third term vanishes. By using \( P_{2N_c}(p_i) = \sum_{j=0}^{N_c} s_{2j} p_i^{2N_c-2j} \)

\[26\]

we obtain

\[
g_{2t} = \sum_{i=1}^{l} L_i \frac{\partial}{\partial u_{2t}} P_{2N_c}(p_i) = \sum_{i=1}^{l} \sum_{j=0}^{N_c} L_i p_i^{2N_c-2j} s_{2j-2t}.
\]

With this relation as in [12] we can obtain the following relation for \( W'(x) \).

\[
W'(x) = \sum_{i=1}^{l} x P_{2N_c}(x) L_i - x^{-1} \sum_{i=1}^{l} 2\epsilon_i L_i \Lambda^{2N_c-2-N_f} \sqrt{A(p_i)} + O(x^{-3}).
\]

\[26\text{The polynomial } P_{2N_c}(p_i) \text{ is given by } \sum_{j=0}^{N_c} s_{2j} x^{2N_c-2j} = \prod_{j=1}^{N_c} (x^2 - \delta_j^2) \text{ where } s_{2j} \text{ and } u_{2j} \text{ are related each other by so-called Newton's formula } s_{2j} + \sum_{i=1}^{j} s_{2j-2i} u_{2i} = 0 \text{ where } j = 1, 2, \cdots, N_c \text{ with } s_0 = 1. \text{ From this recurrence relation we obtain } \frac{\partial s_{2j}}{\partial u_{2t}} = -s_{2j-2t} \text{ for } j \geq t.\]
Defining a new polynomial \( B_{2(l-1)} \) of order \( 2(l-1) \) as in [2],

\[
\sum_{i=1}^{l} \frac{L_i}{x^2 - p_i^2} = \frac{B_{2(l-1)}(x)}{x^2 H_{2(l-1)}(x)},
\]

the \( W'(x) \) is rewritten as

\[
W'(x) = \frac{xp_2N_c(x)B_{2(l-1)}(x)}{x^2 H_{2(l-1)}(x)} + \mathcal{O}(x^{-1}).
\]

Now we can compare them with both sides, in particular, for the power behavior of \( x \). It is easy to see that the left hand side behaves like \( (2n+1) \) while the right hand side behaves like \( 2N_c - 1 - 2(l-1) \) except the factor \( B_{2(l-1)}(x) \) and therefore the condition \( l = (N_c - n) \) will give rise to the consistency and the polynomial \( B_{2(l-1)}(x) \) becomes constant. By using this relation and substituting the characteristic polynomial \( P_{2N_c}(x) \) from the monopole constraints (A.5), we can obtain the following relation together with the replacement of the polynomial \( H_{2(l-1)}(x) \),

\[
F_{4n+2}(x) + \frac{4x^2\Lambda^{4N_c-2N_f-4} \prod_{i=1}^{N_f} (x^2 - m_i^2)}{\prod_{j=1}^{N_c-n-1} (x^2 - p_j^2)^2} \cdot \frac{1}{g_{2n+2}^2} \left( (W'_{2n+1}(x))^2 + \mathcal{O}(x^{2n}) \right).
\]  

(A.6)

Thus, if \( n > -N_f + 2N_c - 3 \), the effect of flavor changes the geometry. This is a new feature compared with pure gauge theory with no flavors. This is due to the fact that the flavor-dependent part, \( A(x) \) provides a new contribution.

The formula (A.6) makes the assumption that every root of \( W'(x) \) has some D5-branes wrapping around it. If the assumption is not true, for example, for the breaking pattern \( SO(N_c) \to \prod_{j=1}^{n} U(N_j) \), instead of \( SO(N_c) \to SO(N_0) \times \prod_{j=1}^{n} U(N_j) \), the formula (A.6) should be modified. The general form can be obtained as follows. Assuming a curve is factorized as

\[
y^2 = P_M^2(x) - A(x) = f_{2s}(x)H_M^2(x)
\]

(A.7)

where \( A(x) \) counts the contribution of flavors (massive or massless) and \( H_M(x) \) includes all the double roots (for example, the \( x^2 \) in [A.3] should take into account \( H_M(x) \)), we will have

\[
f_{2s}(x) + \frac{A(x)}{H_M^2(x)} = W'_s(x)^2 + \mathcal{O}(x^{s-1})
\]

(A.8)

where \( W'_s(x) = \prod_{j=1}^{s} (x - \lambda_j) \) and the \( \lambda_j \)'s are the roots of \( W'(x) \) with wrapping D5-branes. For example, for the case considered in (A.6), \( \lambda_j = 0, \pm \alpha_j \) so \( W'_s(x) = W'_{2n+1}(x) \). If among these roots there is only the origin without wrapping D5-branes, \( W'_s(x) = \frac{W'_{2n+1}(x)}{x} \), so

\[
F_{4n}(x) + \frac{4x^2\Lambda^{4N_c-2N_f-4} \prod_{i=1}^{N_f} (x^2 - m_i^2)}{\prod_{j=1}^{N_c-n} (x^2 - p_j^2)^2} \cdot \frac{1}{g_{2n+2}^2} \left[ \left( \frac{W'_{2n+1}(x)}{x} \right)^2 + \mathcal{O}(x^{2n-2}) \right].
\]  

(A.9)
As soon as \( n > -N_f + 2N_c - 3 \), the effect of flavor also changes the geometry.

- **Superpotential of degree \( 2(k + 1) \) less than \( 2N \)**

Now we generalize (A.6) to \( 2n < 2k \) by introducing the constraints (A.3). We follow the basic idea of [18, 19] and repeat the derivations of (A.6) and generalize to the arbitrary degree of superpotential later. At first, in the range \( 2n + 2 \leq 2k + 2 \leq 2N \), let us consider the superpotential for \( SO(2N_c) \) theory with massless \( N_f \) flavors in the \( r \)-th branch for simplicity (the massive flavors can be obtained by including the mass dependent parts appropriately. For massless case, as we remarked before the \( r \) is fixed to be some number for Chebyshev or Special branches.) under these constraints (A.3) by starting from the curve in pure case and adding the flavor dependent parts,

\[
W_{\text{eff}} = \sum_{s=1}^{k+1} g_{2s} u_{2s} + \sum_{i=0}^{2N_c-2r-2n-2} L_i \oint_{\Lambda} \frac{P_{2(N_c-r)}(x) - 2\epsilon_i x^{N_f-2r+2} \Lambda^{2N_c-N_f-2}}{(x-p_i)^2} dx \\
+ B_i \oint_{\Lambda} \frac{P_{2(N_c-r)}(x) - 2\epsilon_i x^{N_f-2r+2} \Lambda^{2N_c-N_f-2}}{(x-p_i)^2} dx
\]  

(A.10)

where \( L_i \) and \( B_i \) are Lagrange multipliers imposing the constraints and \( \epsilon_i = \pm 1 \). What we have done here is to include the \( r \)-dependence and \( N_f \) dependence coming from the generalization of pure gauge theory [19]. For an equal massive case, we simply replace \( x^{N_f-2r+2} \) in the numerator with \( (x^2 - m^2)^{N_f/2-r}x^{2r} \). The contour integration encloses all \( p_i \)'s and the factor \( 1/2\pi i \) is absorbed in the symbol of \( \oint \) for simplicity. The \( p_i \)'s where \( i = 0, 1, 2, \ldots, (2N_c-2r-2n-2) \) are the locations of the double roots of the curve \( y^2 = P_{2(N_c-r)}^2(x) - 4\Lambda^{4N_c-2N_f-4}x^{2N_f-4r+4} \) according to the constraints (A.3) where the function \( H_{2N_c-2r-2n-2}(x) \) has a factor \( (x^2 - p_i^2) \).

The function \( P_{2(N_c-r)}(x) \) depends on \( u_{2s} \). Note that the massless monopole points appear in pair \((p_i, -p_i)\) where \( i = 1, 2, \ldots, (N_c - r - n - 1) \). Therefore, we denote half of the \( p_i \)'s by \( p_{N_c-r-n+1+i} = -p_i, i = 1, 2, \ldots, (N_c - r - n - 1) \). Moreover, we define \( p_0 = 0 \) and note that the summation index \( i \) in the above starts from \( i = 0 \). Since \( P_{2(N_c-r)}(x) \) is an even function in \( x \) and the property of \( \epsilon_i \), if the following constraints are satisfied at \( x = p_i \) where \( i = 1, 2, \ldots, (N_c - r - n - 1) \),

\[
\left. \left( P_{2(N_c-r)}(x) - 2x^{N_f-2r+2} \epsilon_i \Lambda^{2N_c-N_f-2} \right) \right|_{x=p_i} = 0,
\]

\[
\frac{\partial}{\partial x} \left( P_{2(N_c-r)}(x) - 2x^{N_f-2r+2} \epsilon_i \Lambda^{2N_c-N_f-2} \right) \bigg|_{x=p_i} = 0,
\]

(A.11)

then they also automatically are satisfied at \( x = -p_i \). Then the numbers of constraints that we should consider are \( (N_c - r - n - 1) \). Thus, we denote the half of the Lagrange multipliers by \( L_{N_c-r-n+1+i} = L_i \) and \( B_{N_c-r-n+1+i} = B_i \) where \( i = 1, 2, \ldots, (N_c - r - n - 1) \). Due to the fact that the second derivative of \( y \) with respect to \( x \) at \( x = p_i \) does not vanish, there are no higher order terms such as \( (x-p_i)^{−a} \), \( a = 3, 4, 5, \cdots \) in the effective superpotential (A.10). In other
words, for given constraints (A.3), there exist only a Lagrange multiplier $L_i$ with $(x - p_i)^{-1}$ and a Lagrange multiplier $B_i$ with $(x - p_i)^{-2}$.

The variation of $W_{\text{eff}}$ with respect to $B_i$ leads to an integral expression which is the coefficient function of $B_i$ in (A.10) and by the formula of contour integral, it is evaluated at $x = p_i$ and we have changed the derivative of $P_{2(N_c - r)}(x)$ with respect to $x$ into $P_{2(N_c - r)}(x)$ and the trace of some quantity

$$0 = \int \frac{P_{2(N_c - r)}(x) - 2x^{N_f - 2r + 2} \epsilon_i \Lambda^{2N_c - N_f - 2}}{(x - p_i)^2} dx$$

$$= \left( P_{2(N_c - r)}(x) - 2x^{N_f - 2r + 2} \epsilon_i \Lambda^{2N_c - N_f - 2} \right)' |_{x = p_i}$$

$$= \left( P'_{2(N_c - r)}(x) - \frac{N_f - 2r + 2}{x} \epsilon_i x^{N_f - 2r + 2} \Lambda^{2N_c - N_f - 2} \right) |_{x = p_i}$$

$$= \left( P_{2(N_c - r)}(x) \sum_{j=1}^{N_c - r} \frac{2x}{x^2 - \Phi_j^2} - \frac{N_f - 2r + 2}{x} P_{2(N_c - r)}(x) \right) |_{x = p_i}$$

$$= P_{2(N_c - r)}(x) \left( \frac{1}{x - \Phi_{cl}} - \frac{N_f + 2}{x} \right) |_{x = p_i}$$

where we used the equation of motion for $L_i$ when we replace $2x^{N_f - 2r + 2} \epsilon_i \Lambda^{2N_c - N_f - 2}$ with $P_{2(N_c - r)}(x)$ at $x = p_i$. 27 The equation of motion for $B_i$ can be summarized as 28

$$\left( \frac{\text{Tr}}{x - \Phi_{cl}} - \frac{N_f + 2}{x} \right) |_{x = p_i} = 0, \quad P_{2(N_c - r)}(x = p_0 = 0) = 0, \quad P_{2(N_c - r)}(x = p_i) \neq 0$$

where the characteristic function by solving the first order differential equation is given by

$$P_{2(N_c - r)}(x) = \left[ x^{2(N_c - r)} \exp \left( - \sum_{s=1}^{2s} \frac{u_{2s}}{x^{2s}} \right) \right]_+$$

(A.12)

and the polynomial part of a Laurent series inside the bracket is denoted here by $+$. By putting the negative power of $x$ to zero, the $u_{2s}$ with $2s > 2(N_c - r)$ can be obtained the $u_{2s}$ with $2s \leq 2(N_c - r)$.

Next we consider the variation of $W_{\text{eff}}$ with respect to $p_i$,

$$0 = 2B_i \int \frac{P'_{2(N_c - r)}(x) - 2x^{N_f - 2r + 2} \epsilon_i \Lambda^{2N_c - N_f - 2}}{(x - p_j)^2} dx$$

27 The last equality comes from the following relation, together with (A.2), $\text{Tr} \frac{1}{x - \Phi_{cl}} = \sum_{k=0}^{\infty} x^{-k-1} \text{Tr} \Phi_{cl}^k = \sum_{i=0}^{\infty} x^{-(2i+1)} \text{Tr} \Phi_{cl}^{2i} = \sum_{i=0}^{\infty} x^{-i} \left( 2x^{2} \right)^{i-1} \sum_{j=1}^{N_c} \left( \Phi_j^2 \right)^{i-1} = \left( \sum_{j=1}^{N_c} \frac{2x}{x^2 - \Phi_j^2} \right) + \frac{2r}{x}$ where $\Phi_{cl}$ is antisymmetric matrix, and the odd power terms are vanishing.

28 Through the definition (A.4), the derivative $P_{2(N_c - r)}(x)$ with respect to $x$ is given by $P'_{2(N_c - r)}(x) = \left( \prod_{j=1}^{N_c - r} \left( x^2 - \Phi_j^2 \right) \right)' = 2x \sum_{j=1}^{N_c - r} \prod_{j \neq j}^{N_c - r} \left( x^2 - \Phi_j^2 \right) = P_{2(N_c - r)}(x) \sum_{j=1}^{N_c - r} \frac{2x}{x^2 - \Phi_j^2}$. Taking into account the result of trace given in previous footnote 27, we can rewrite this result as $\frac{P'_{2(N_c - r)}(x)}{P_{2(N_c - r)}(x)} = \text{Tr} \frac{1}{x - \Phi_{cl}} - \frac{2r}{x}$. Therefore, we have $\frac{P'_{2(N_c - r)}(x)}{P_{2(N_c - r)}(x)} = \sum_{i=0}^{\infty} x^{-(2i+1)} \text{Tr} \Phi_{cl}^{2i} = \sum_{i=0}^{\infty} x^{-(2i+1)} 2i u_{2i}$. It is easy to check that for equal massive flavors, the corresponding relation leads to $\left( \text{Tr} \frac{1}{x - \Phi_{cl}} - \frac{2N_f}{x^2 - \Phi_j^2} - \frac{2}{x} \right) |_{x = p_i} = 0$, $P_{2(N_c - r)}(x = p_0 = 0) = 0$, $P_{2(N_c - r)}(x = p_i) \neq 0$. 28
where there is no \( L_i \) term because we have used the equation of motion for \( B_i \). In general since this integral does not vanish, we should have \( B_j = 0 \) because the curve does not have more than cubic roots due to the fact that \( y^2 \) contains a polynomial \( H_{2N_c-2r-2n-2}^2(x) \) as we have discussed before.

Let us consider variation of \( W_{eff} \) with respect to \( u_{2s} \), by using the relation \( \frac{\partial P_{2(N_c-r)}(x)}{\partial u_{2s}} = -\left[ \frac{P_{2(N_c-r)}(x)}{x^{2s}} \right]_{+} \), which can be checked from the definition (A.12) and remembering the fact that the function \( P_{2(N_c-r)}(x) \) depends on \( u_{2s} \),

\[
0 = g_{2s} - \sum_{i=0}^{k+1} \int \left[ \frac{P_{2(N_c-r)}(x)}{x^{2s}} \right]_{+} \frac{L_i}{x - p_i} dx,
\]

where we used \( B_i = 0 \) at the level of equation of motion. Multiplying this with \( z^{2s-1} \) and summing over \( s \) where \( z \) is inside the contour of integration, we can obtain the first derivative \( W'(z) \),

\[
W'(z) = \sum_{s=1}^{2N_c-2r-2n-2} g_{2s} z^{2s-1} = \sum_{i=0}^{k+1} \int \sum_{s=1}^{k+1} z^{2s-1} \frac{P_{2(N_c-r)}(x)}{x^{2s}} \frac{L_i}{x - p_i} dx. \tag{A.13}
\]

Let us introduce a new polynomial \( Q(x) \) defined as

\[
\sum_{i=0}^{2N_c-2r-2n-2} \frac{x L_i}{(x - p_i)} = L_0 + \sum_{i=1}^{N_c-r-n-1} \frac{2x^2 L_i}{x^2 - p_i^2} = \frac{Q(x)}{H_{2N_c-2r-2n-2}(x)} \tag{A.14}
\]

where we used the fact that \( L_{N_c-r-n-1+i} = L_i \) and \( p_{N_c-r-n-1+i} = -p_i \) for \( i = 1, 2, \ldots, (N_c - r - n - 1) \). By using this new function we can rewrite (A.13) as

\[
W'(z) = \int \sum_{s=1}^{k+1} \frac{z^{2s-1}}{x^{2s}} \frac{Q(x) P_{2(N_c-r)}(x)}{x H_{2N_c-2r-2n-2}(x)} dx. \tag{A.15}
\]

Since \( W'(z) \) is a polynomial of degree \((2k+1)\), we found the order of \( Q(x) \) as \((2k-2n)\), so we denote it by \( Q_{2k-2n}(x) \). Thus, we have found the order of polynomial \( Q(x) \) and therefore the order of integrand in (A.13) is \( \mathcal{O}(x^{2k-2s+1}) \). Thus, if \( s \geq k + 1 \) it does not contribute to the integral because the power of \( x \) in this region implies that the Laurent expansion around the origin vanishes. We can replace the upper value of summation, \( k + 1 \), with the infinity and by computing the infinite sum over \( s \) we get

\[
W'(z) = \int \sum_{s=1}^{\infty} \frac{z^{2s-1}}{x^{2s}} \frac{Q_{2k-2n}(x) P_{2(N_c-r)}(x)}{x H_{2N_c-2r-2n-2}(x)} dx = \int z \frac{Q_{2k-2n}(x) P_{2(N_c-r)}(x)}{x(x^2 - z^2) H_{2N_c-2r-2n-2}(x)} dx.
\]

From (A.3) we have the relation,

\[
P_{2(N_c-r)}(x) = x \sqrt{F_{2(n+1)}(x) H_{2N_c-2r-2n-2}(x) + \mathcal{O}(x^{-2N_c+2N_f-2r+4})}. \tag{A.16}
\]

Let us emphasize the presence of the second term which will play the role and cannot be ignored because the power of \( x \) can be greater than or equal to \(-1\) depending on the number of flavor
$N_f$ which can vary in various regions. For pure cases, there were no contributions from the second terms. Therefore, we have

$$W'(z) = \int z \frac{y_m(x)}{x^2 - z^2} dx + \int O\left(x^{-4N_c+2N_f+2k+3}\right) dx,$$  \hspace{1cm} (A.17)

where we have defined $y_m$ as follows,

$$y_m^2(x) = F_{2(2n+1)}(x)Q_{2k-2n}^2(x)$$  \hspace{1cm} (A.18)

corresponding to the matrix model curve. In (A.17), the second term does contribute if the condition $-4N_c + 2N_f + 2k + 3 \geq -1$ holds. Then we get an expected and generalized result:

$$y_m^2(x) = F_{2(2n+1)}(x)Q_{2k-2n}^2(x) = \begin{cases} W_{2k+1}'^2(x) + O\left(x^{2k}\right) & k \geq 2N_c - N_f - 2 \\ W_{2k+1}^2(x) + O\left(x^{4N_c-2N_f-4}\right) & k < 2N_c - N_f - 2 \end{cases}$$

$$\equiv W_{2k+1}'^2(x) + f_{2M}(x), \hspace{1cm} 2M = \max(2k, 4N_c - 2N_f - 4)$$  \hspace{1cm} (A.19)

where both $F_{2(2n+1)}(x)$ and $Q_{2k-2n}(x)$ are functions of $x^2$, then $f_{2M}(x)$ is also a function of $x^2$. We put the subscript $m$ in the $y_m$ in order to emphasize the fact that this corresponds to the matrix model curve. When $2k = 2n$, we reproduce (A.6) with $Q_0 = g_{2n+2}$. The second term on the left hand side of (A.6) behaves like $x^{2+2N_f-4(N_c-n-1)} = x^{2N_f-4N_c+4n+6}$. Depending on whether the power of this is greater than or equal to $2n = 2k$, the role of flavor is effective or not. When $n = k > -N_f + 2N_c - 3$, the flavor dependent part will contribute to the $W'(x)$. The above relation determines a polynomial $F_{2(2n+1)}(x)$ in terms of $(2n + 1)$ unknown parameters by assuming the leading coefficient of $W(x)$ to be normalized by 1. These parameters can be obtained from both $P_{2(N_c-r)}(x)$ and $H_{2N_c-2r-2n-2}(x)$ through the factorization condition (A.3)

When $k$ is arbitrary large, we refer to the version one in the hep-th archive for details.

- A generalized Konishi anomaly

Now we are ready to study the derivation of the generalized Konishi anomaly equation based on the results of previous section. As in [18], we will restrict ourselves to the case with

\[\text{One can proceed the case where there are no D5-branes wrapping around the origin similarly. Our corresponding monopole constraint equation (A.16) becomes } F_{2(N_c-r)}(x) = \sqrt{F_{4n}(x)}H_{2N_c-2r-2n}(x) + O(x^{-2N_c+2N_f-2r+4}). \text{ Using this relation, it is straightforward to arrive the following matrix model curve in this case:} \]

$$y_m^2(x) = F_{4n}(x)Q_{2k-2n}(x) = \begin{cases} \left(\frac{W_{2k+1}'}{x}\right)^2 + O\left(x^{2k-2}\right) & k \geq 2N_c - N_f - 2 \\ \left(\frac{W_{2k+1}}{x}\right)^2 + O\left(x^{4N_c-2N_f-6}\right) & k < 2N_c - N_f - 2 \end{cases}$$

$$\equiv \left(\frac{W_{2k+1}'}{x}\right)^2 + f_{2M}(x), \hspace{1cm} 2M = \max(2k - 2, 4N_c - 2N_f - 6).$$
\[ \langle \text{Tr} W'(\Phi) \rangle = \text{Tr} W'(\Phi_{cl}) \] and assume that the degree of superpotential \((2k+2)\) is less than \(2N_c\). By substituting (A.14) into (A.15) we can write the derivative of superpotential \(W'(\phi_I)\)

\[
W'(\phi_I) = \sum_{i=0}^{2N_c-2r-2n-2} \phi_i P_{2(N_c-r)}(x) \frac{L_i}{(x^2 - \phi_i^2)} dx
\]

where we varied \(W'(\phi_I)\) with respect to \(\phi_I\) rather than \(u_{2k}\) and used the result of \(B_i = 0\). Note that the characteristic function is given by \(P_{2(N_c-r)}(x) = \prod_{i=1}^{N_c-r} (x^2 - \phi_i^2)\). Using the above expression, one obtains the following relation \(^{30}\),

\[
\frac{\text{Tr} W'(\Phi_{cl})}{z - \Phi_{cl}} = 2 \sum_{i=1}^{N_c-r} \phi_i W'(\phi_I) \frac{1}{(z^2 - \phi_i^2)}
\]

\[
= \sum_{i=1}^{N_c-r} \frac{2\phi_i^2}{(z^2 - \phi_i^2)} \sum_{i=0}^{2N_c-2r-2n-2} \phi_i P_{2(N_c-r)}(x) \frac{L_i}{(x^2 - \phi_i^2)} (x - p_i) dx
\]

\[
= \int \sum_{i=0}^{2N_c-2r-2n-2} P_{2(N_c-r)}(x) L_i \left( z \text{Tr} \frac{1}{z - \Phi_{cl}} - x \text{Tr} \frac{1}{x - \Phi_{cl}} \right) dx. \quad (A.20)
\]

As in the case of [18], we can rewrite outside contour integral in terms of two parts as follows:

\[
\int_{z_{out}} = \int_{z_{in}} - \int_{C_z+C_{-z}} \quad (A.21)
\]

where \(C_z\) and \(C_{-z}\) are the small contour around \(z\) and \(-z\) respectively. Thus, the first term in (A.20) (corresponding to the second term in [B.3] of [18]) can be written as, by exploiting the relation (A.14) to write in terms of \(Q_{2k-2n}(x)\) and \(H_{2N_c-2r-2n-2}(x)\)

\[
\left( \text{Tr} \frac{1}{z - \Phi_{cl}} \right) \int_{z_{out}} \frac{z Q_{2k-2n}(x) P_{2(N_c-r)}(x)}{x H_{2N_c-2r-2n-2}(x)(x^2 - z^2)} dx.
\]

Let us emphasize that in this case also we cannot drop the terms of order \(O(x^{-2N_c-2N_f-2r+4})\) in the characteristic function \(P_{2(N_c-r)}(x)\). In order to take into account of this, we have to use the above change of integration, then the first term of (A.20) is given by

\[
\left( \text{Tr} \frac{1}{z - \Phi_{cl}} \right) \left( \int_{z_{in}} \frac{z Q_{2k-2n}(x) P_{2(N_c-r)}(x)}{x H_{2N_c-2r-2n-2}(x)(x^2 - z^2)} dx - \int_{C_z+C_{-z}} \frac{z Q_{2k-2n}(x) P_{2(N_c-r)}(x)}{x H_{2N_c-2r-2n-2}(x)(x^2 - z^2)} dx \right)
\]

\[
= \left( \text{Tr} \frac{1}{z - \Phi_{cl}} \right) \left( W'(z) - \frac{y_m(z) P_{2(N_c-r)}(z)}{\sqrt{P_{2(N_c-r)}(z) - 4z^{2N_f-4r+4}} A^{4N_c-2N_f-4}} \right), \quad (A.22)
\]

\(^{30}\)When we change the summation index from \(k\) to \(i\), the only odd terms appear because effectively the product of \(\Phi_{cl}\) and \(W'(\Phi_{cl})\) does contribute only under that condition \(\text{Tr} \frac{W'(\Phi_{cl})}{z - \Phi_{cl}} = \text{Tr} \sum_{k=0}^{\infty} z^{-k-1} \Phi_k W'(\Phi_{cl}) = \sum_{i=0}^{\infty} z^{-(2+i+1)-1} A^{2+i+1} W'(\phi_I) = 2 \sum_{i=1}^{N_c-r} \phi_i W'(\phi_{cl}) \frac{1}{(x^2 - z^2)} \). The even terms do not contribute. Here \(z\) is outside the contour of integration. We recognize that the following factor can be written as, by simple manipulation between the property of the trace we have seen before, \(2\delta_i^2 \left( z - \phi_i^2 \right) \left( x^2 - \phi_i^2 \right) = \frac{1}{(x^2 - z^2)} \left( z \text{Tr} \frac{1}{z - \Phi_{cl}} - x \text{Tr} \frac{1}{x - \Phi_{cl}} \right) \).
where the first term was obtained by the method done previously and the second term was calculated at the poles and we used
\[ H_{2N_c-2r-2n-2}(z) = \sqrt{\frac{P_{2(N_c-r)}^2(z)}{2} - 4\Lambda^4N_c-2N_f-4z^2N_f-4r+4} \cdot z^2F_{2(2n+1)}(z), \]
\[ y_m^2(z) = F_{2(2n+1)}(z)Q_{2k-2n}^2(z). \]

The crucial difference between \( U(N_c) \) case and \( SO(2N_c) \) case comes from the second term of (A.20) (corresponding to the first term in (B.3) of [18]), which vanishes in \( U(N_c) \) case. Now we use the result of the equation of motion for \( B_i \) (the equation just above (A.12)) in order to change the trace part and arrive at the final contribution of the second term (A.20) as follows:

\[
- \sum_{i=0}^{2N_c-2r-2n-2} (N_f + 2) \frac{L_iP_{2(N_c-r)}(x = p_i)}{(p_i^2 - z^2)} \]
\[
= - \sum_{i=0}^{2N_c-2r-2n-2} \int \frac{(N_f + 2)P_{2(N_c-r)}(x)}{(x^2 - z^2)} \frac{L_i}{(x - p_i)} dx
\]
\[
= -(N_f + 2) \frac{W'(z)}{z} + \frac{(N_f + 2)}{z} \frac{y_m(z)P_{2(N_c-r)}(z)}{\sqrt{P_{2(2N_c-r)}^2(z) - 4z^2N_f-4r+4\Lambda^4N_c-2N_f-4}}. \quad \text{(A.23)}
\]

We used here again the property of contour integration (A.21). Therefore, we obtain the (A.20) by combining the two contributions (A.22) and (A.23)

\[
\frac{\text{Tr} W'(\Phi_{cl})}{z - \Phi_{cl}} = \left( \text{Tr} \frac{1}{z - \Phi_{cl}} \right) \left( W'(z) - \frac{y_m(z)P_{2(N_c-r)}(z)}{\sqrt{P_{2(2N_c-r)}^2(z) - 4z^2N_f-4r+4\Lambda^4N_c-2N_f-4}} \right)
\]

\[
- (N_f + 2) \frac{W'(z)}{z} + \frac{(N_f + 2)}{z} \frac{y_m(z)P_{2(2N_c-r)}(z)}{\sqrt{P_{2(2N_c-r)}^2(z) - 4z^2N_f-4r+4\Lambda^4N_c-2N_f-4}}.
\]

Then the second term of above expression can be written as 31

\[
- \left( \text{Tr} \frac{1}{z - \Phi_{cl}} \right) \frac{y_m(z)P_{2(N_c-r)}(z)}{\sqrt{P_{2(2N_c-r)}^2(z) - 4z^2N_f-4r+4\Lambda^4N_c-2N_f-4}}
\]

31Let us write it as, after an integration over \( x \) and using the (A.11) to change the trace part into \( 1/p_i \) term
\[- \int \frac{L_iP_{2(N_c-r)}(x = p_i)}{(x - p_i)(x - z^2)} \frac{1}{\Phi_{cl}(x)} \frac{dx}{z} = - \frac{L_iP_{2(N_c-r)}(x = p_i)}{(p_i^2 - z^2)} \frac{1}{\Phi_{cl}(x)} \frac{1}{z} = - (N_f + 2) \frac{L_iP_{2(N_c-r)}(x = p_i)}{(p_i^2 - z^2)}. \]

For equal massive flavors, this becomes \(- \left( \frac{N_f}{1 - \frac{1}{r^2}} + 2 \right) \frac{L_iP_{2(N_c-r)}(x = p_i)}{(p_i^2 - z^2)} \). Due to the mass dependent term containing \( p_i \) dependent term, we can not simplify the second term (A.20) (as we go to the contour integral around \( x \) again, when we consider massive flavors) further like as (A.23) for the massless case.

32One can multiply \( z^{2r} \) both in the numerator and denominator and then the orders of the characteristic polynomial are changed into \( 2N_c \). In other words, \( z^{2r}P_{2(N_c-r)}(z) = P_{2N_c}(z) \). Remember that \( \text{Tr} \frac{1}{x - \Phi_{cl}} = \frac{(z - N_f + 2P_{2N_c}(z))}{z - N_f + 2z^2P_{2N_c}(z)} + \frac{(N_f + 2)}{z} \) and the quantum mechanical expression \( \left< \text{Tr} \frac{1}{x - \Phi_{cl}} \right> = \frac{d}{dx} \log \left( P_{2(N_c-r)}(x) + \sqrt{P_{2(2N_c-r)}^2(x) - 4x^2N_f-4r+4\Lambda^4N_c-2N_f-4} \right) \).
Therefore, we obtain (A.20) as
\[
\text{Tr} \left( W'(\Phi_{cl}) - W'(z) \right) = -(N_f + 2) \frac{(W'(z) - y_m(z))}{z} - \left\langle \frac{1}{z - \Phi} \right\rangle y_m(z).
\]
Taking into account the relation
\[
\text{Tr} \frac{W'(\Phi_{cl}) - W'(z)}{z - \Phi_{cl}} = \left\langle \frac{W'(\Phi) - W'(z)}{z - \Phi} \right\rangle,
\]
we can write the quantum mechanical expression as follows:
\[
\left\langle \frac{W'(\Phi)}{z - \Phi} \right\rangle = \left\langle \frac{1}{z - \Phi} \right\rangle W'(z) + \text{Tr} \frac{W'(\Phi_{cl}) - W'(z)}{z - \Phi_{cl}} = \left( \left\langle \frac{1}{z - \Phi} \right\rangle - \frac{(N_f + 2)}{z} \right) (W'(z) - y_m(z))
\]
which is the generalized Konishi anomaly equation for the $SO(2N_c)$ case. The resolvent of the matrix model $R(z)$ is related to $W'(z) - y_m(z)$. When $N_f = 0$, this result reproduces the result in [19] exactly. It does not depend on the $r$ with which we start from. For massive flavors, one can generalize the above description similarly, but as we mentioned in the footnote 31, we would get rather involved expressions.

For $SO(2N + 1)$ case, we refer to the version one in the hep-th archive for details.

**Appendix B  Addition and multiplication maps**

In [18] it was noted that all the confining vacua of higher rank gauge groups can be constructed from the Coulomb vacua with lower rank gauge groups by using the Chebyshev polynomial through the multiplication map. Under this map, which was called multiplication map, the vacua with the classical gauge group $\prod_{i=1}^n U(N_i)$ for a given superpotential are mapped to the vacua with the gauge group $\prod_{i=1}^n U(KN_i)$ for the same superpotential where $K$ is a multiplication index. This multiplication map was extended to the $SO/USp$ gauge theories in [19] and $U(N_c)$ gauge theory with flavors in [20]. In addition to this multiplication map,
another map (called the addition map) was introduced in [20]. The addition map reduces many analyses to simpler cases. Thus, we also introduce these two maps, the addition map and the multiplication map for the $SO(N_c)$ gauge theory with flavors.

- **Addition map**

Unlike the unitary gauge group, for $SO(N_c)$ with flavors, there are two kinds of addition maps: one is for massive flavors and the other is for massless flavors. They should be treated separately. As we will show shortly, the addition map with massive flavors can only connect the $SO(N_c)$ and $SO(M_c)$ for both $N_c$ and $M_c$ are an even numbers (or they are odd at the same time), but the addition map with massless flavors can connect the $SO(2N_c)$ to $SO(2N_c + 1)$.

Let us start with *massive* flavors first by using the $SO(2N_c)$ gauge group as an example. On assumption that we have two theories, $SO(2N_c)$ with $N_f$ flavors in the $r$-th branch and $SO(2N'_c)$ with $N'_f$ flavors in the $r'$-th branch. On these branches, the SW curves are described by

\[
\begin{align*}
    y_1^2 &= (x^2 - m^2)^{2r} \left[ P_{2(N_c-r)}^2(x) - 4x^4\Lambda^{4N_c-2N_f-4}(x^2 - m^2)^{N_f-2r} \right], \\
    y_2^2 &= (x^2 - m^2)^{2r'} \left[ P_{2(N'_c-r')}^2(x) - 4x^4\Lambda^{4N'_c-2N'_f-4}(x^2 - m^2)^{N'_f-2r'} \right].
\end{align*}
\]

For the two curves, if we have the following relations, we find that the polynomials in the bracket are identical.

\[
2(N_c - r) = 2(N'_c - r'), \quad 4N_c - 2N_f = 4N'_c - 2N'_f, \quad N_f - 2r = N'_f - 2r'. \quad (B.1)
\]

These relations are exactly the same as the ones for the $U(N_c)$ case. We expect that certain vacua of $SO(2N_c)$ gauge theory with $N_f$ flavors in the $r$-th branch are related to the one of $SO(2N'_c)$ gauge theory with $N'_f$ flavors in the $r'$-th branch. For the $SO(2N_c + 1)$ case we have the same relation, because the only difference comes from the factor of $x^4\Lambda^{-4}$ which does not matter for the relation. They are a common factor in both SW curves. Namely, we have only to replace $x^4\Lambda^{4N_c-2N_f-4}$ with $x^2\Lambda^{4N_c-2N_f}$. It is also noteworthy that starting with $SO(N_c)$ for $N_c$ even (or odd), we can only get $SO(M_c)$ for $M_c$ even (or odd).

Thus, if on the $r$-th branch, the $SO(N_c)$ theory with $N_f$ massive flavors has a classical limit $SO(N_0) \times U(N_1)$ with the *effective* $N_f$ massless flavors charged under the $U(N_1)$ factor in the quartic superpotential, then a $SO(N_c + 2d)$ theory with $(N_f + 2d)$ massive flavors will have a classical limit $SO(N_0) \times U(N_1 + d)$ with the *effective* $(N_f + 2d)$ massless flavors charged under $U(N_1 + d)$ on the $(r + d)$-th branch, according to (B.1). Notice that for massive flavors, the counting of vacua of the $U(N_1)$ factor is $2N_1 - N_f = 2(N_1 + d) - (N_f + 2d)$, which is invariant under the above addition map, so it is consistent. In fact, for massive flavors, it is the factor $U(N_i)$ with the *effective* massless flavors so we get the same results as in [20].
Now let us discuss the addition map of *massless* flavors. For \( SO(2N_c + 1) \) with \( N_f \) massless flavors, the curve is characterized by

\[
y^2 = P_{2N_c}^2(x) - 4\Lambda^{4N_c - 2N_f - 2}x^2(x^2)^{N_f}
\]

which can be rewritten as

\[
y^2 = P_{2N_c}^2(x) - 4\Lambda^{4N_c - 2(N_f - 1) - 4}x^4(x^2)^{N_f-1}.
\]  \hspace{1cm} (B.2)

However, the form of (B.2) can be interpreted as the curve of \( SO(2N_c) \) with \((N_f - 1)\) massless flavors. In other words, \( SO(2N_c + 1) \) gauge group with \( N_f \) massless flavors is equivalent to the \( SO(2N_c) \) gauge group with \((N_f - 1)\) massless flavors. It is obvious that the above interpretation can only be done for massless flavors where \( SO(2N_c) \) is related to \( SO(2N_c + 1) \). We call it the “special addition maps”. In fact, this special addition map comes from a well known result in the field theory: giving one flavor in \( SO(2N + 1) \) non-zero vacuum expectation value, it can be higgsed down to the \( SO(2N_c) \) gauge group.

It seems that except the reinterpretation of \( SO(2N_c + 1) \) curve to \( SO(2N_c) \) curve, we should have a similar addition map of massive flavors (we will call it the “general addition map”) by just setting \( m = 0 \). However, we need to be careful in applying the general addition map. As we did in the weak and strong coupling analyses, for the \( SO(N_c) \) gauge group with massless flavors, there is no concept of \( r \)-th branch and what we have is the Chebyshev branch or the Special branch where the power of \( x \) is fixed for a given \( SO(N_c) \) factor. The general addition map is used to relate the Chebyshev branches (or the Special branches) of two different gauge groups. We will give one explicit example to demonstrate the general idea. For \( SO(2N_c) \) with \( 2M \) massless flavors, the Chebyshev branch requires a factorization of the curve to be

\[
y^2 = (x^2)^{2M+2}\left[P_{2N_c-(2M+2)}^2(x) - 4\Lambda^{4N_c - 2(2M-2)}\right].
\]

Similarly, for \( SO(2N'_c) \) with \( 2M' \) massless flavors, the curve at the Chebyshev branch is

\[
y^2 = (x^2)^{2M'+2}\left[P_{2N'_c-(2M'+2)}^2(x) - 4\Lambda^{4N'_c - 2(2M'-2)}\right].
\]

Therefore, if

\[
N_c - (M + 1) = N'_c - (M' + 1),
\]

we can reduce the Chebyshev branch of \( SO(2N'_c) \) with \( 2M' \) massless flavors to the Chebyshev branch of \( SO(2N_c) \) with \( 2M \) massless flavors. Notice that there is no index \( r \) in the above relationship.

- **Multiplication map**
Now we will discuss the multiplication map. The method we used is from the bottom to the top, i.e., starting from the known factorization of gauge group with lower rank to the unknown factorization of gauge group with higher rank. First let us assume that the factorization of curve is given by

\[ y^2 = P_{2N_c}^2(x) - 4x^{2s}A_0^{4N_c-2s-2N_f}A(x) = f_2p(x)H_{2N_c-p}^2(x) \]

where \( s = 2 \) for the \( SO(2N_c) \) group and \( s = 1 \) for \( SO(2N_c+1) \) group and \( A(x) \) the contribution of flavors (it can be very general with different masses). Next let us divide out the common factor between \( P_{2N_c}^2(x) \) and \( x^{2s}A(x) \). For example, if one of the eigenvalues of \( \Phi \) in the \( SO(2N_c) \) gauge theory is zero, we can divide out the \( x^4 \) factor at the two sides of the curve. Or if \( A(x) \) has factor \( (x^2 - m^2)^2 \) and we are at the \( r = 1 \) branch, the factor \( (x^2 - m^2)^2 \) can also be divided out. After dividing out all the common factors, the remaining curve is given by

\[ P_M^2(x) - 4A_0^{4N_c-2s-2N_f}\tilde{A}(x) = \tilde{f}_2p(x)H_{M-p}^2(x) \]

where \( \tilde{p} \) is some number. Now let us define

\[ \tilde{x} = \frac{P_M(x)}{2\eta A_0^{2N_c-s-N_f}\sqrt{A(x)}} \quad (B.3) \]

and

\[ P_{KM}(x) = 2\left(\eta A_0^{2N_c-s-N_f}\sqrt{A(x)}\right)^K T_K(\tilde{x}) \quad (B.4) \]

with \( \eta^{2K} = 1 \). It is worth to notice that although \( \tilde{x} \) has \( \sqrt{A(x)} \) in the denominator, the \( P_{KM}(x) \) is a perfect polynomial of \( x \). The reason is very simple: \( T_K(t) \) has only an even (odd) power of \( t \) in the polynomial if \( K \) is an even (odd) number, so \( \sqrt{A(x)}^{K-q} = \tilde{A}(x)^{\frac{K-q}{2}} \) is a polynomial of \( x \) because \( K-q \) is always an even number \(^{33} \). Using (B.3) and (B.4) it is easy to see that one obtains

\[ P_{KM}^2(x) - 4A^{(4N_c-2s-2N_f)K}\tilde{A}(x)^K \]

\[ = 4A^{(4N_c-2s-2N_f)K}\tilde{A}(x)^K \left[ \left( \frac{P_{KM}(x)}{2\left(\eta A_0^{2N_c-s-N_f}\sqrt{A(x)}\right)^K} \right)^2 - 1 \right] \]

\[ = 4A^{(4N_c-2s-2N_f)K}\tilde{A}(x)^K \left[ T_K^2(\tilde{x}) - 1 \right] \]

\[ = \left[ 2\left(\eta A_0^{2N_c-s-N_f}\sqrt{A(x)}\right)^K U_{K-1}(\tilde{x}) \right]^2 (\tilde{x}^2 - 1) \]

\(^{33}\)We want to thank Freddy Cachazo and Oleg Lunin for discussing this general situation of the multiplication map.

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\[ = \left[ \eta \Lambda_0^{2N_c-s-N_f} \sqrt{\Lambda(x)} \right]^{K-1} U_{K-1}(\bar{x})^2 \left[ P_{KM}^2(x) - 4\Lambda_0^{4N_c-2s-2N_f} \tilde{A}(x) \right] \]
\[ = f_{2p}(x) \left[ H_{M-\bar{p}}(x) \left( \eta \Lambda_0^{2N_c-s-N_f} \sqrt{\Lambda(x)} \right)^{K-1} U_{K-1}(\bar{x}) \right]^2. \]

Notice that a similar reason guarantees that \( \sqrt{\Lambda(x)}^{K-1} U_{K-1}(\bar{x}) \) is a polynomial of \( x \). The above calculation shows that we have the solution of factorization of \( P_{KM}(x) \) with matter dependent part \( \tilde{A}(x)^K \). Finally to recover to the \( SO(M_c) \) gauge group, we need to multiply back some factor of \( x^p \) to make sure that \( (KM + p/2) \) is even and \( x^p \tilde{A}(x)^K \) has \( x^2 \) or \( x^4 \) factor.

To demonstrate the above general results, let us consider several concrete examples:

1). The first example is that we consider the breaking pattern \( SO(2N_c) \to \prod_{j=1}^n U(N_j) \) at the \( r = 0 \) branch, so the \( P_{2N_c}(x) \) does not have a common factor with \( x^4 \tilde{A}(x) = x^4(x^2 - m^2)^{N_f} \).

2). For the breaking \( SO(2N_c) \to SO(2N_0) \times \prod_{j=1}^n U(N_j) \), we can divide out at least the factor \( x^4 \) from the SW curve because the characteristic function \( P_{2N_c}(x) \) and \( 4x^4 \Lambda_0^{4N_c-4-2N_f} A(x) \) possess a factor \( x^2 \), so \( M = (2N_c - 2) \) and \( \tilde{A}(x) = (x^2 - m^2)^{N_f} \). The multiplication map will give the factorization of \( SO(K(2N_c - 2) + 2) \) with \( KN_f \) flavors. It is significant that the +2 of the rank \( K(2N_c - 2) + 2 \) comes from putting back the \( x^4 \) factor to \( P_{KM}^2(x) - 4\Lambda_0^{4N_c-2s-2N_f} \tilde{A}(x)^K \). Since we can not multiply a factor \( x^2 \) back \( ((KM + p/2) \) is not even), it is impossible to obtain the map from \( SO(2N_c) \) to \( SO(2N_c + 1) \) which reveals the same phenomena observed in [19] for the pure \( SO(2N_c) \) gauge theory. However, if there are massless flavors, one gets from \( SO(2N_c) \) to \( SO(2M_c + 1) \) through the special addition map. The multiplication map has another application. We have seen that the relation between \( f_{2p}(x) \) and \( W'(x) \) is changed in certain situation under the presence of the flavors. For \( SO(2N_c) \) gauge theory with \( N_f \) flavors at the \( n \)-th branch, when \( n > -N_f + 2N_c - 3 \), the relation between \( f_{2p}(x) \) and \( W'(x) \) is modified.

\(^{34}\)For simplicity, we have assumed that all \( N_f \) flavors have the same mass. Generalization to different masses is obvious.
different for $SO(2N_c)$ gauge and $SO(2KN_c - 2K + 2)$ gauge theories. If $K$ is large enough, the presence of flavors will not modify the relationship between $f_{2p}(x)$ and $W'(x)$ and the geometric proofs [10, 11, 12] can go through [20]. In other words, the geometric picture is really for a large $N_c$ limit.

3). To break $SO(2N_c+1) \rightarrow SO(2N_0+1) \times \prod_{j=1}^{n} U(N_j)$, we can divide out at most the factor $x^2$ if all masses are not zero, so $M = (2N_c-1)$ and $\tilde{A}(x) = (x^2 - m^2)^N$. The multiplication map will give us $KM = K(2N_c-1)$. Notice that when we multiply back $x^p$ we require $(KM + p/2)$ is even. If $K$ is even, we can only multiply back $x^4$ factor so it is $SO(K(2N_c - 1) + 2)$ with $KN_f$ flavors. If $K$ is odd, we can only multiply back $x^2$ factor so it is $SO(K(2N_c - 1) + 1)$ with $KN_f$ flavors. Notice that this conclusion is made under the assumption that there are no massless flavors at the beginning. If there are massless flavors, by the “special addition maps”, we can jump from $SO(2N_c)$ gauge theory to $SO(2N_c + 1)$ gauge theory.

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