THE TWO-WEIGHT HARDY INEQUALITY: A NEW ELEMENTARY AND UNIVERSAL PROOF

AMIRAN GOGATISHVILI AND LUBOŠ PICK

Abstract. We give a new proof of the known criteria for the inequality
\[
\left( \int_0^\infty \left( \int_0^t f \right)^q w(t) \, dt \right)^{\frac{1}{q}} \leq C \left( \int_0^\infty f^p v \right)^{\frac{1}{p}}.
\]
The innovation is in the elementary nature of the proof and its versatility.

1. Introduction

Consider the two-weight Hardy inequality
\[
\left( \int_0^\infty \left( \int_0^t f \right)^q w(t) \, dt \right)^{\frac{1}{q}} \leq C \left( \int_0^\infty f^p v \right)^{\frac{1}{p}},
\]
in which \( C \) is a positive constant independent of a nonnegative measurable function \( f \) on \((0, \infty)\), \( v \) and \( w \) are fixed nonnegative measurable functions on \((0, \infty)\) (weights), \( p \in [1, \infty) \), and \( q \in (0, \infty) \). The requirement \( p \in [1, \infty) \) is reasonable since for \( p \in (0, 1) \) there are functions in weighted \( L^p \) which are not locally integrable.

The problem of characterizing pairs of weights for which (1.1) is true has a long and rich history and it would be impossible to mention here every contribution. For \( p = q > 1 \), \( v = 1 \), \( w(t) = t^{-q} \) and \( C = p' \), it is just the boundedness of the integral averaging operator on \( L^p(0, \infty) \), a result almost one century old, which appears in classical Hardy’s papers in 1920’s, see [5]. The beginning of investigation of a general weighted case goes back to 1950’s, and it starts with the paper by Kac and Krein [6] in which a characterization for \( p = q = 2 \) and \( v = 1 \) can be found. In 1950’s and 1960’s, plenty of partial results were obtained by Beesack, see e.g. [1]. In late 1960’s and in 1970’s, a boom in the so-called convex case \((p \leq q, \text{ named after the convexity of } t \mapsto t^{\frac{q}{p}})\) was seen. For \( p = q \), a characterization was obtained by Tomaselli [15], Talenti [14] and Muckenhoupt [9]. It was extended to \( p \leq q \) by Bradley [4], the same result is also stated without proof in [7]. Many authors referred further to an untitled and unpublished manuscript by Artola, and in [10], a paper by D.W. Boyd and J.A. Erdős was quoted, which most likely was never published. In any case, (1.1) holds if and only if
\[
\sup_{t \in (0, \infty)} \left( \int_t^\infty w \right)^{\frac{1}{q}} \left( \int_0^t \left( v^{1-p'} \right)^{\frac{1}{p'}} \right)^{\frac{1}{p'}} < \infty \quad \text{for } 1 < p \leq q
\]
and
\[ \sup_{t \in (0, \infty)} \left( \int_t^\infty w \right)^{\frac{1}{q}} \operatorname{ess sup}_{s \in (0,t)} \frac{1}{v(s)} < \infty \quad \text{for } 1 = p \leq q. \]

Here and throughout, if \( p \in (0, \infty] \), then \( p' \) denotes the conjugate exponent defined by \( \frac{1}{p} + \frac{1}{p'} = 1 \). Observe that 1 and \( \infty \) are conjugate exponents and that \( p' \) is negative when \( p \in (0,1) \).

The non-convex case \( (p > q) \) turned out to be more difficult to handle, and it had to wait till 1980’s and 1990’s for appropriate treatment. The first characterization, for \( 1 \leq q < p < \infty \), was obtained by Maz’ya and Rozin, see [8], who proved that a necessary and sufficient condition is
\[ \int_0^\infty \left( \int_t^\infty w \right)^{\frac{1}{q}} \left( \int_0^t v^{1-p'} \right)^{\frac{1}{q'}} v(t)^{1-p'} dt < \infty, \]
where \( r = \frac{pq}{p-q} \). A universal characterization, sheltering both the convex and the non-convex cases and involving more general norms was obtained by Sawyer [11], but the condition in the non-convex case is expressed in terms of a discretized condition. While discretization techniques proved later to be of colossal theoretical importance, conditions expressed in terms of discretizing sequences are difficult to verify. Later, Sinnamon [12] characterized the inequality for \( 0 < q < 1 < p < \infty \). The criterion turns out to be the same as that of Maz’ya and Rozin but the proof, based on Halperin’s level function, is very different. The case \( 0 < q < p = 1 \) was treated by Sinnamon and Stepanov [13], who moreover observed that, unless \( p = 1 \), Sinnamon’s and Mazya-Rozin’s results can be proved in a unified manner. The case \( p = 1 \), however, still required separate treatment. In [3], restriction of (1.1) to the cone of non-increasing functions is studied, together with its discrete version. Some ideas developed there are useful also for the unrestricted case.

In this note we present a short, uniform and elementary proof, in which
\begin{itemize}
  \item all cases are covered,
  \item \( p > 1 \) is not separated from \( p = 1 \),
  \item only Fubini’s theorem, Hölder’s inequality, Minkowski’s integral inequality and Hardy’s lemma are used.
\end{itemize}

2. THE THEOREM AND ITS PROOF

**Theorem 2.1.** Let \( v, w \) be weights on \((0, \infty)\), \( p \in [1, \infty) \) and \( q \in (0, \infty) \). For \( t \in (0, \infty) \), denote
\[ V(t) = \begin{cases} 
\left( \int_0^t v^{1-p'} \right)^{\frac{1}{p'}} & \text{if } p \in (1, \infty), \\
\operatorname{ess sup}_{s \in (0,t)} \frac{1}{v(s)} & \text{if } p = 1,
\end{cases} \]
and
\[ W(t) = \int_t^\infty w. \]

Then there exists a positive constant \( C \) such that (1.1) holds for every nonnegative measurable function \( f \) on \((0, \infty)\) if and only if \( A < \infty \), where
\[ A = \begin{cases} 
\sup_{t \in (0, \infty)} V(t)W(t)^{\frac{1}{q}} & \text{if } p \leq q, \\
\int_0^\infty W^{\frac{p}{p-q}} dV^{\frac{pq}{p-q}} & \text{if } p > q,
\end{cases} \]
in which the latter integral should be understood in the Lebesgue–Stieltjes sense with respect to the (monotone) function $V_{\frac{1}{p-q}}$.

**Proof.** Sufficiency. Fix $\varepsilon \in (0, 1)$. We claim that, for every nonnegative measurable function $f$ on $(0, \infty)$, one has

\begin{equation}
\int_0^t f \lesssim \left( \int_0^t f^{pV_{\varepsilon p}}v \right)^{\frac{1}{p}} V(t)^{1-\varepsilon} \quad \text{for } t > 0.
\end{equation}

(We write $\lesssim$ when the expression to the left of it is majorized by a constant times that on the right.) To show (2.1), fix $t \in (0, \infty)$. If $p \in (1, \infty)$, then, by Hölder’s inequality,

\[ \int_0^t f = \int_0^t fV_{\varepsilon p}V^{-\varepsilon v^{-\frac{1}{p}}} \leq \left( \int_0^t f^{pV_{\varepsilon p}}v \right)^{\frac{1}{p}} \left( \int_0^t V^{-\varepsilon v^{-1}} \right)^{\frac{1}{p}}. \]

By a change of variables, we obtain

\[ \int_0^t V^{-\varepsilon v^{-1}} = \int_0^t \left( \int_0^s v^{-1} ds \right)^{-\varepsilon} v^{-1} ds = \frac{1}{1-\varepsilon} \left( \int_0^s v^{-1} ds \right)^{1-\varepsilon} = \frac{1}{1-\varepsilon} V(t)^{(1-\varepsilon)q'}, \]

hence

\[ \int_0^t f \lesssim \left( \int_0^t f^{pV_{\varepsilon p}}v \right)^{\frac{1}{p}} V(t)^{1-\varepsilon}, \]

and (2.1) follows. If $p = 1$, then we get (2.1) from

\[ \int_0^t f = \int_0^t f v^{-\varepsilon} v^{-1-\varepsilon} \leq \left( \int_0^t f^{\varepsilon v}v \right) V(t)^{1-\varepsilon}. \]

Let $p \leq q$. Then $A < \infty$ implies $V \lesssim W^{-\frac{1}{q}}$. Using this and (2.1), we get

\[ \int_0^t f \lesssim \left( \int_0^t f^{pW^{-\frac{\varepsilon p}{q}} v} \right)^{\frac{1}{p}} W(t)^{\frac{1-\varepsilon}{q}} \quad \text{for } t > 0. \]

Raising to $q$ and integrating with respect to $w(t) \, dt$, we obtain

\[ \int_0^\infty \left( \int_0^t f \right)^q w(t) \, dt \lesssim \int_0^\infty \left( \int_0^t f(s)^pW(s)^{-\frac{\varepsilon p}{q}} v(s) \, ds \right)^{\frac{1}{p}} W(t)^{\frac{1}{q}} w(t) \, dt. \]

Next we apply Minkowski’s integral inequality (note that $\frac{q}{p} \geq 1$ and all the expressions in the play are nonnegative) in the form

\[ \int_0^\infty \left( \int_0^\infty F(s, t) \, d\mu_1(s) \right)^{\frac{q}{p}} d\mu_2(t) \leq \left( \int_0^\infty \left( \int_0^\infty F(s, t)^{\frac{q}{p}} \, d\mu_2(t) \right)^{\frac{q}{q}} \, d\mu_1(s) \right)^{\frac{q}{p}}, \]
in which \( F(s,t) = \chi_{(0,t)}(s)f(s)^p \), \( \chi \) denotes the characteristic function, \( d\mu_1(s) = W(s)^{-\frac{\alpha p}{q}} v(s)ds \) and \( d\mu_2(t) = W(t)^{\frac{\alpha}{q} - 1} w(t) dt \). We thus obtain

\[
\int_0^\infty \left( \int_0^t f(s)^p W(s)^{-\frac{\alpha p}{q}} v(s) ds \right) \frac{q}{p} W(t)^{\frac{\alpha}{q} - 1} w(t) dt \\
\leq \left( \int_0^\infty f(s)^p W(s)^{-\frac{\alpha p}{q}} v(s) \left( \int_s^\infty W(t)^{\frac{\alpha}{q} - 1} w(t) dt \right)^{\frac{q}{p}} ds \right)^{\frac{p}{q}} \\
\approx \left( \int_0^\infty f^p v \right)^{\frac{q}{p}}.
\]

(We write \( \approx \) when both \( \lesssim \) and \( \gtrsim \) apply.) Altogether, we arrive at

\[
\int_0^\infty \left( \int_0^t f \right)^q w(t) dt \lesssim \left( \int_0^\infty f^p v \right)^{\frac{q}{p}},
\]

and (1.1) follows.

Let \( p > q \). Fix \( \alpha \in (0, \infty) \). We shall use the symbol \( V(\infty) \) for \( \lim_{t \to \infty} V(t) \) (this limit always exists, either finite or infinite, owing to the monotonicity of \( V \)). By (2.1),

\[
\int_0^\infty \left( \int_0^t f \right)^q w(t) dt \lesssim \int_0^\infty \left( \int_0^t f^p V^{\frac{\alpha q}{p}} v \right)^{\frac{q}{p}} V(t)^{-\alpha q} V(t)^{(1-\varepsilon+\alpha)q} w(t) dt \\
\lesssim \int_0^\infty \left( \int_0^t f^p V^{\frac{\alpha q}{p}} v \right)^{\frac{q}{p}} (V(t)^{-\alpha p} - V(\infty)^{-\alpha p})^{\frac{q}{p}} V(t)^{(1-\varepsilon+\alpha)q} w(t) dt \\
+ \int_0^\infty \left( \int_0^t f^p V^{\frac{\alpha q}{p}} v \right)^{\frac{q}{p}} V(t)^{(1-\varepsilon+\alpha)q} w(t) dt \cdot V(\infty)^{-\alpha q} = I + II.
\]

If \( V(\infty) = \infty \), one has \( II = 0 \). Since

\[
V(t)^{(1-\varepsilon+\alpha)q} \approx \int_0^t V(1-\varepsilon+\alpha)^q - m V^{\frac{\alpha q}{p-q}} d(V^{\frac{\alpha q}{p-q}}) \text{ for } t > 0
\]

and

\[
V(t)^{-\alpha p} - V(\infty)^{-\alpha p} = \int_t^\infty d(-V^{-\alpha p}) \text{ for } t > 0,
\]

monotonicity and Fubini’s theorem yield

\[
I \lesssim \int_0^\infty \left( \int_t^\infty \left( \int_0^s f^p V^{\frac{\alpha q}{p}} v \right) d(-V^{-\alpha p})(s) \right)^{\frac{q}{p}} \left( \int_0^t V(1-\varepsilon+\alpha)^q - m V^{\frac{\alpha q}{p-q}} d(V^{\frac{\alpha q}{p-q}}) \right) w(t) dt \\
\lesssim \int_0^\infty \left( \int_0^t \left( \int_0^\infty f^p V^{\frac{\alpha q}{p}} v \right) d(-V^{-\alpha p})(\tau) \right)^{\frac{q}{p}} V(s)^{(1-\varepsilon+\alpha)q - \frac{\alpha q}{p-q}} dV^{\frac{\alpha q}{p-q}}(s) w(t) dt \\
= \int_0^\infty \left( \int_0^\infty f^p V^{\frac{\alpha q}{p}} v \right) d(-V^{-\alpha p})(\tau) \right)^{\frac{q}{p}} V(s)^{(1-\varepsilon+\alpha)q - \frac{\alpha q}{p-q}} W(s) dV^{\frac{\alpha q}{p-q}}(s).
\]
Thus, owing to \( A < \infty \), Hölder’s inequality, and Fubini’s theorem,
\[
I \lesssim \left( \int_0^\infty W_\nu^p \frac{dV_{\nu^p}}{V_{\nu^p}} \right)^\frac{1}{p} \left( \int_0^\infty \left( \int_0^T f^{pV_{\nu^p}} \right) d(-V^{-\alpha p})(\tau) \right)^{\frac{p}{q}} \left( \int_0^\infty V_\nu^{(1-\alpha)p} \frac{dV_{\nu^p}}{V_{\nu^p}} \right)^{\frac{p}{q}}
\]
\[
\lesssim \left( \int_0^\infty \left( \int_0^T V_{\nu^p} \right) \left( \int_0^\infty V_\nu^{(1-\alpha)p} \frac{dV_{\nu^p}}{V_{\nu^p}} \right) d(-V^{-\alpha p})(\tau) \right)^{\frac{p}{q}}
\]
\[
\approx \left( \int_0^\infty \left( \int_0^T f^{pV_{\nu^p}} \right) V(\tau)^{(\alpha-\epsilon)p} d(-V^{-\alpha p})(\tau) \right)^{\frac{p}{q}}
\]
\[
= \left( \int_0^\infty f(y)^p V(y)^{\epsilon p} v(y) \int_y^\infty V(\alpha-\epsilon)p d(-V^{-\alpha p}) dy \right)^{\frac{1}{p}} \approx \left( \int_0^\infty f^{pV} \right)^{\frac{1}{p}}
\]
If \( V(\infty) < \infty \), we have
\[
II \lesssim \left( \int_0^\infty \left( \int_0^T f^{pV} \right)^\frac{1}{p} V(t)^{(1+\alpha)q} w(t) dt \cdot V(\infty)^{-\alpha q} \leq \left( \int_0^\infty f^{pV} \right)^{\frac{p}{q}} \left( \int_0^\infty V^{(1+\alpha)q} w \right) V(\infty)^{-\alpha q}.
\]
Owing to \( A < \infty \), Fubini’s theorem, and Hölder’s inequality, we get
\[
\int_0^\infty V^{(1+\alpha)q} w \approx \int_0^\infty \left( \int_0^T V^{\alpha q + \alpha q - p' v^{1-p'}} \right) w(t) dt = \int_0^\infty V^{\alpha q + q - p' v^{1-p'}} W
\]
\[
\lesssim \left( \int_0^\infty V^{\alpha q - p' v^{1-p'}} \right)^{\frac{1}{q}} \left( \int_0^\infty W(t)^{\frac{p}{p-q}} \frac{dV_{\nu^p}}{V_{\nu^p}} \right)^{\frac{1}{p-q}} \lesssim V(\infty)^{\alpha q},
\]
establishing II \( \lesssim \left( \int_0^\infty f^{pV} \right)^{\frac{1}{p}} \). This shows sufficiency.

Necessity. Let \( p \leq q \) and assume that (1.1) holds. Fix \( t \in (0, \infty) \). Then
\[
\int_0^\infty \left( \int_0^s f \right)^q w(s) ds \geq \int_t^\infty \left( \int_0^s f \right)^q w(s) ds \geq W(t) \left( \int_0^t f \right)^q.
\]
Therefore, (1.1) yields
\[
(2.2) \quad C \geq W(t)^{\frac{1}{q}} \sup_{f \geq 0} \frac{\int_0^t f}{\left( \int_0^\infty f^{pV} \right)^{\frac{1}{p}}}
\]
We claim that
\[
(2.3) \quad \sup_{f \geq 0} \frac{\int_0^t f}{\left( \int_0^\infty f^{pV} \right)^{\frac{1}{p}}} = V(t).
\]
Indeed, if \( p > 1 \), then we have, by Hölder’s inequality,
\[
\int_0^t f = \int_0^t f^{\frac{1}{p}} v^{\frac{1}{q}} v^{\frac{1}{p}} \leq \left( \int_0^t f^{pV} \right)^{\frac{1}{p}} \left( \int_0^t v^{\frac{1}{p}} \right)^{\frac{1}{p}} \leq \left( \int_0^\infty f^{pV} \right)^{\frac{1}{p}} V(t)
\]
for every measurable \( f \geq 0 \). On the other hand, this inequality is saturated by the choice \( f = v^{1-p'} \chi_{(0,t)} \), since \( f^{pV} = f \), and, consequently,
\[
\int_0^t f = \left( \int_0^t f^{pV} \right)^{\frac{1}{p}} \left( \int_0^t f^{pV} \right)^{\frac{1}{p}} = \left( \int_0^\infty f^{pV} \right)^{\frac{1}{p}} V(t).
\]
If \( p = 1 \), then we, once again, obtain
\[
\int_0^t f = \int_0^t f v v^{-1} \leq V(t) \int_0^t f v \leq V(t) \int_0^\infty f v
\]
for every measurable \( f \geq 0 \). In order to saturate this inequality, fix any \( \lambda < V(t) \). Then there exists a set \( E \subset (0, t) \) of positive measure such that \( \frac{1}{v} \geq \lambda \) on \( E \). Set \( f = \frac{\chi_E}{v} \). Then
\[
\int_0^t f = \int_E \frac{1}{v} \geq \lambda |E| = \lambda \int_0^\infty f v.
\]
On letting \( \lambda \to V(t)_- \), we get
\[
\int_0^t f \geq V(t) \int_0^\infty f v.
\]
In any case, (2.3) follows. Since \( p > q \), denote \( r = \frac{pq}{p-q} \) and \( B = \int_0^\infty V^r W^{q/r} \). Let \( \theta \in (\frac{r}{q}, \infty) \) and set
\[
f(t) = \left( \int_t^\infty W^\frac{r}{q} v W^{\theta - \theta p'} \right)^{\frac{q}{p}} V(t)^{(\theta-1)(p'-1)} v(t)^{1-p'} \quad \text{for } t > 0.
\]
By Fubini’s theorem,
\[
\begin{aligned}
\int_0^\infty f^r v &= \int_0^\infty \left( \int_t^\infty W^\frac{r}{q} v W^{\theta - \theta p'} \right) V(t)^{(\theta-1)p'} v(t)^{1-p'} dt \\
&= \int_0^\infty W(s)^{\frac{r}{q}} v(s) V(s)^{r-\theta p'} \left( \int_0^s V(s)^{(\theta-1)p'} v^{1-p'} \right) ds \approx B.
\end{aligned}
\]
On the other hand, by monotonicity,
\[
\begin{aligned}
\int_0^\infty \left( \int_0^t f \right)^q w(t) dt &\geq \int_0^\infty \left( \int_0^t V^{(\theta-1)(p'-1)} v^{1-p'} \right)^q \left( \int_t^\infty W^\frac{r}{q} v W^{\theta - \theta p'} \right)^{\frac{q}{p}} w(t) dt \\
&\geq \int_0^\infty \left( \int_0^t V^{(\theta-1)(p'-1)+\frac{r}{q} - \frac{\theta p'}{p} v^{1-p'}} \right)^q \left( \int_t^\infty W^\frac{r}{q} \right)^{\frac{q}{p}} w(t) dt \approx B.
\end{aligned}
\]
Altogether, (1.1) implies \( B^{\frac{1}{q}} \lesssim B^{\frac{1}{p}} \). Using a standard approximation argument, we obtain \( B^{\frac{1}{q}} < \infty \), hence \( B < \infty \). Since \( A \approx B \) owing to integration by parts, we get \( A < \infty \).

Finally, let \( p = 1 \) and \( p > q \). Fix some \( \sigma > 1 \) and define
\[
E_k = \{ t \in (0, \infty) : \sigma^k < V(t) \leq \sigma^{k+1} \} \quad \text{for } k \in \mathbb{Z}.
\]
Set \( \mathbb{A} = \{ k \in \mathbb{Z} : E_k \neq \emptyset \} \). Then \( (0, \infty) = \bigcup_{k \in \mathbb{A}} E_k \), in which the union is disjoint and each \( E_k \) is a nondegenerate interval (which could be either open or closed at each end) with endpoints \( a_k \) and \( b_k \), \( a_k < b_k \). For every \( k \in \mathbb{A} \), we find \( \delta_k > 0 \) so that \( a_k + \delta_k < b_k \) and
\[
\int_{a_k}^{b_k} W^{\frac{a}{1-q}} \leq \sigma \int_{a_k+\delta_k}^{b_k} W^{\frac{a}{1-q}}.
\]
which is clearly possible, and then we define the set

\[ G_k = \left\{ t \in (a_k, a_k + \delta_k) : \frac{1}{v(t)} > \sigma^k \right\}. \]

Since \( V \) is non-decreasing and left-continuous, \(|G_k| > 0\) for every \( k \in \mathbb{A} \). Set \( h = \sum_{k \in \mathbb{A}} \frac{x_{G_k}}{|G_k|} \). Then, for every \( k \in \mathbb{A} \), one has

\[
\int_0^{a_k + \delta_k} h v^{-1} V^{\frac{q}{1-q}} \geq \int_{a_k}^{a_k + \delta_k} h v^{-1} V^{\frac{q}{1-q}} = \frac{1}{|G_k|} \int_{G_k} V^{\frac{q}{1-q}} v^{-1} \geq \sigma^k \int_{G_k} V^{\frac{q}{1-q}} v^{-1}.
\]

Fix \( t \in (0, \infty) \). Then there is a uniquely defined \( k \in \mathbb{A} \) such that \( t \in (a_k, b_k) \). Consequently,

\[
\int_0^t h V^{\frac{q}{1-q}} \leq \sum_{j \in \mathbb{K}, j \leq k} \frac{1}{|G_j|} \int_{G_j} V^{\frac{q}{1-q}} \leq \sum_{j = -\infty}^k \sigma^{q(j+1)} = \frac{\sigma^{q(k+2)}}{\sigma^{q-1} - 1}.
\]

On the other hand,

\[
\int_0^t dV^{\frac{q}{1-q}} \geq \int_0^{a_k} dV^{\frac{q}{1-q}} = V(a_k)^{\frac{q}{1-q}} \geq \sigma^k.
\]

The last two estimates yield

\[
\int_0^t h V^{\frac{q}{1-q}} \lesssim \int_0^t dV^{\frac{q}{1-q}} \quad \text{for } t > 0.
\]

Since \( W^{\frac{1}{1-q}} \) is non-increasing, we can apply Hardy’s lemma (whose version for Lebesgue integrals can be found in [2, Chapter 2, Proposition 3.6]) - note that the proof presented there works verbatim for Lebesgue–Stieltjes integrals) to (2.6) and get

\[
\int_0^\infty h V^{\frac{q}{1-q}} W^{\frac{1}{1-q}} \lesssim \int_0^\infty W^{\frac{1}{1-q}} dV^{\frac{q}{1-q}}.
\]

Finally, using subsequently integration by parts, decomposition of \((0, \infty)\) into \( \bigcup_{k \in \mathbb{A}} E_k \), the definition of \( E_k \), the fact that each \( E_k \) is an interval with endpoints \( a_k, b_k \), (2.4), (2.5), monotonicity of functions given by integrals, (1.1) applied to \( p = 1 \) and \( f = h v^{-1} V^{\frac{q}{1-q}} W^{\frac{1}{1-q}} \), and (2.7), we get

\[
\int_0^\infty W^{\frac{1}{1-q}} dV^{\frac{q}{1-q}} \leq 2 \int_0^\infty V^{\frac{q}{1-q}} W^{\frac{1}{1-q}} w = 2 \sum_{k \in \mathbb{A}} \int_{E_k} V^{\frac{q}{1-q}} W^{\frac{q}{1-q}} w \lesssim \sum_{k \in \mathbb{A}} \int_{E_k} W^{\frac{q}{1-q}} w
\]

\[
\lesssim \sum_{k \in \mathbb{A}} \int_{a_k + \delta_k}^{b_k} W^{\frac{1}{1-q}} w \lesssim \sum_{k \in \mathbb{A}} \left( \int_{0}^{a_k + \delta_k} h v^{-1} V^{\frac{q}{1-q}} \right)^q \int_{a_k + \delta_k}^{b_k} W^{\frac{q}{1-q}} w
\]

\[
\lesssim \sum_{k \in \mathbb{A}} \int_{a_k + \delta_k}^{b_k} \left( \int_{0}^{t} h v^{-1} V^{\frac{q}{1-q}} \right)^q W(t)^\frac{q}{1-q} w(t) dt \lesssim \int_0^\infty \left( \int_{0}^{t} h v^{-1} V^{\frac{q}{1-q}} W^{\frac{1}{1-q}} \right)^q w(t) dt
\]

\[
\lesssim \left( \int_{0}^{\infty} h V^{\frac{q}{1-q}} W^{\frac{1}{1-q}} \right)^q \lesssim \left( \int_{0}^{\infty} W^{\frac{1}{1-q}} dV^{\frac{q}{1-q}} \right)^q,
\]

in which the multiplicative constants depend only on \( C \) and \( q \). This establishes, via a standard approximation argument, that \( A^{1-q} < \infty \), which in turn yields \( A < \infty \). The proof is complete. \( \square \)
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Email address, A. Gogatishvili: gogatish@math.cas.cz
ORCID: 0000-0003-3459-0355

Email address, L. Pick: pick@karlin.mff.cuni.cz
ORCID: 0000-0002-3584-1454

Amiran Gogatishvili, Institute of Mathematics of the Czech Academy of Sciences, Žitná 25, 115 67 Praha 1, Czech Republic
L. N. Gumilyov Eurasian National University, 5 Munaytpasov St., 010008 Nur-Sultan, Kazakhstan

Luboš Pick, Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic