A homological estimate for the Thurston norm

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Abstract We establish a new homological lower bound for the Thurston norm on 1-cohomology of 3-manifolds. This generalizes previous results of C. McMullen, S. Harvey, and the author. We also establish an analogous lower bound for 1-cohomology of 2-dimensional CW-complexes.

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Introduction

We establish a new homological lower bound for the Thurston norm on 1-cohomology of 3-manifolds. This bound generalizes a row of previously known results and most notably the Seifert inequality saying that the genus of a knot in the 3-sphere is greater than or equal to half the span of the Alexander polynomial.

Recall first the definition of the Thurston norm, see [Th]. For a finite CW-space $Y$, set $\chi_-(Y) = \sum_i \chi_-(Y_i)$ where the sum runs over all connected components $Y_i$ of $Y$ and $\chi_-(Y_i) = \max(-\chi(Y_i), 0)$. For a compact connected orientable 3-manifold $M$, the Thurston norm of $\psi \in H^1(M) = H^1(M; \mathbb{Z})$ is defined by $\|\psi\|_T = \min_Y \chi_-(Y)$ where $Y$ runs over proper oriented em-
bedded (not necessarily connected) surfaces in $M$ dual to $\psi$ with respect to some orientation of $M$.

The recent interest to finding lower estimates for the Thurston norm on $H^1(M)$ was inspired by the famous (and much deeper) adjunction inequality in dimension 4, see [KM1] and [MST]. Suppose that $\partial M = \emptyset$. The adjunction inequality applied to the 4-manifold $M \times S^1$ and to surfaces lying in the slice $M \times pt$ provides a finite set $B \subset H_1(M)$ such that

$$||\psi||_T \geq \max_{h \in B} |\psi(h)|.$$  \hspace{1cm} (0.1)

The set $B$ is defined in terms of spin$^c$-structures on $M$ and their Seiberg-Witten invariants. Namely, the elements of $B$ are dual to the Chern classes of those spin$^c$-structures on $M$ whose Seiberg-Witten invariant is non-zero. For more on this, see [Ak], [Au], [Kr], [KM2], [Vi]. The Seiberg-Witten theory and the related Ozsvath-Szabo theory yield also stronger estimates formulated in terms of Floer-type homology (see [OS]).

A homological approach to lower estimates for the Thurston norm was developed by C. McMullen [McM] who showed that the Thurston norm is bounded from below by a norm derived from the Alexander polynomial of $\pi_1(M)$. (Here we do not need to suppose that $\partial M = \emptyset$). This result was generalized in two directions. The author showed that the same is true for the twisted Alexander polynomials of $M$ associated with complex characters of the finite abelian group $\text{Tors}H_1(M)$, see [Tu1], [Tu2]. For closed $M$, the resulting set of inequalities is equivalent to the adjuction inequality (0.1). In an independent important development, S. Harvey [Ha] derived lower bounds for the Thurston norm from higher-order Alexander modules introduced in [COT] (for the exteriors of knots in $S^3$ the Harvey bounds were also obtained by T. Cochran [Co]).

We now state our main result. Fix a skew field $K$, i.e., a possibly non-commutative field with $1 \neq 0$. Recall that any right (or left) $K$-module $H$ is free (see for instance [Coh1, Theorem 4.4.8]). The cardinality of a basis of $H$ is called the rank of $H$ and denoted $\text{rk}_KH$. It does not depend on the
choice of the basis (see [Coh1, Theorems 4.4.6-7]). If $H$ is finitely generated, then $\text{rk}_K H$ is a non-negative integer.

Fix a field automorphism $\alpha : K \to K$. One defines the skew Laurent polynomial ring $\Lambda = K[t^{\pm 1}; \alpha]$ in one variable $t$ with coefficients in $K$ as follows. The ring $\Lambda$ consists of expressions $t^{-m}a_{-m} + \ldots + t^{-1}a_{-1} + a_0 + ta_1 + \ldots + t^k a_k$ where $m,k \geq 0$ and $a_i \in K$ for all $i$. These expressions add as ordinary Laurent polynomials. Multiplication is defined on monomials by $t^i a t^j b = t^{i+j} \alpha^j(a) b$, where $i,j \in \mathbb{Z}$ and $a,b \in K$, and then extended to arbitrary polynomials by additivity. An easy computation shows that $\Lambda$ is an associative ring with unit. Any $\Lambda$-module will be regarded as a $K$-module via the inclusion $K \subset \Lambda$.

For a pointed connected CW-space $X$, a ring homomorphism $\varphi : \mathbb{Z}[\pi_1(X)] \to \Lambda$ allows us to regard $\Lambda$ as a local system of coefficients on $X$ and to consider the corresponding right $\Lambda$-module of twisted homology $H_*(X; \Lambda)$. (We shall discuss twisted homology in detail in Sect. 2.1). We say that $\varphi$ is compatible with a cohomology class $\psi \in H^1(X) = H^1(X; \mathbb{Z})$ if for any $\gamma \in \pi_1(X)$ we have $\varphi(\gamma) \in t^{\psi(\gamma)} K$ where $\psi(\gamma) \in \mathbb{Z}$ is the evaluation of $\psi$ on $\gamma$.

**Theorem 1.** Let $K, \alpha, \Lambda = K[t^{\pm 1}; \alpha]$ be as above. Let $M$ be a compact connected orientable 3-manifold. Set $\varepsilon_M = 2$ if $\partial M$ is void or consists of 2-spheres and set $\varepsilon_M = 1$ otherwise. Let $\varphi : \mathbb{Z}[\pi_1(M)] \to \Lambda$ be a ring homomorphism compatible with a non-zero cohomology class $\psi \in H^1(M)$ and such that the right $\Lambda$-module $H_1(M; \Lambda)$ is finitely generated over $K$. If the multiplicative group $\varphi(\pi_1(M)) \subset \Lambda \setminus \{0\}$ is not cyclic, then

$$||\psi||_T \geq \text{rk}_K H_1(M; \Lambda). \quad (0.2)$$

If the group $\varphi(\pi_1(M)) \subset \Lambda \setminus \{0\}$ is cyclic, then

$$||\psi||_T \geq \max(\text{rk}_K H_1(M; \Lambda) - \varepsilon_M |\psi|, 0) \quad (0.3)$$

where $|\psi|$ is the maximal positive integer dividing $\psi$ in the lattice $H^1(M)$.

Note that the number $\text{rk}_K H_1(M; \Lambda)$ depends only on $\pi_1(M)$ and $\varphi$. 

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Theorem 1 generalizes the results of Seifert, McMullen, Harvey, Cochran, and the author mentioned above. In particular, Theorem 1 implies the adjunction inequality in dimension 3.

The next theorem is a rather straightforward supplement to Theorem 1.

**Theorem 2.** If under conditions of Theorem 1, \( \psi \) is dual to the fiber of a fibration \( M \to S^1 \), then \( H_1(M; \Lambda) \) is finitely generated over \( K \) and (0.2), (0.3) become equalities.

The paper is organized as follows. In Sect. 1–3 we collected several preliminary definitions and lemmas used in Sect. 4 to prove Theorem 1. Theorem 2 is proven in Sect. 5. In Sect. 6 we discuss special cases and relations with the previously known results. In Sect. 7 we consider an analogue of Theorem 1 for 2-dimensional CW-complexes.

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1. **Codimension 1 subspaces of CW-spaces**

1.1. **Codimension 1 subspaces.** Let \( X \) be a CW-space. A codimension 1 subspace of \( X \) is a CW-subspace \( Y \subset X \) having a (closed) neighborhood homeomorphic to \( Y \times [-1,1] \) so that \( Y = Y \times 0 \). It is understood that the product CW-structure on \( Y \times [-1,1] \) should extend to a CW-structure of \( X \) compatible with the given one (in the sense that they have a common
A typical example of a codimension 1 subspace is a proper two-sided $n$-dimensional PL-submanifold of an $(n+1)$-dimensional PL-manifold. (A submanifold is proper if it is compact and meets the boundary of the ambient manifold exactly along its own boundary). The language of codimension 1 subspaces is convenient in our context. However only the case of proper surfaces in 3-manifolds will be actually needed for the proof of Theorem 1. The reader unwilling to consider the general setting of codimension 1 subspaces may safely restrict him/herself to this case.

Each component $Y_i$ of a codimension 1 subspace $Y \subset X$ splits its cylinder neighborhood into 2 components called the sides of $Y_i$. A choice of one of these sides (considered as “positive”) determines a coorientation of $Y_i$. We say that $Y$ is cooriented if all its components are cooriented.

A cooriented codimension 1 subspace $Y \subset X$ determines a 1-dimensional cohomology class $\psi_Y \in H^1(X)$. Its value on a loop in $X$ is the algebraic intersection number of this loop with $Y$. Here is a more formal definition of $\psi_Y$. Choose a cylinder neighborhood $Y \times [-1,1] \subset X$ of $Y = Y \times 0$ such that $Y \times [0,1]$ lies on the positive side of $Y$. Define a map $g : X \to S^1 = \{ z \in \mathbb{C} | |z| = 1 \}$ which sends the complement of this neighborhood to $-1$ and sends any point $(y,t) \in Y \times [-1,1]$ to $\exp(\pi i t)$. Then $\psi_Y = g^*(s)$ where $s$ is the generator of $H^1(S^1) = \mathbb{Z}$ determined by the counterclockwise orientation of $S^1$.

We shall need a notion of a weighted codimension 1 subspace of $X$ which formalizes subspaces with parallel components. A codimension 1 subspace $Y$ of $X$ is weighted if each its component $Y_i$ is endowed with an integer $w_i \geq 1$ called the weight of $Y_i$. We write $Y = \cup_i (Y_i, w_i)$ where $i$ runs over $\pi_0(Y)$. Such $Y$ gives rise to an (unweighted) codimension 1 subspace $Y^\# \subset X$ obtained by replacing every $Y_i$ by $w_i$ parallel copies $Y_i \times (1/w_i), Y_i \times (2/w_i), ..., Y_i \times 1$ in a cylinder neighborhood of $Y_i$. A coorientation of $Y$ induces a coorientation of $Y^\#$ in the obvious way. For a cooriented weighted
codimension 1 subspace \( Y = \bigcup_i(Y_i, w_i) \) of \( X \), set
\[
\psi_Y = \psi_Y^\# = \sum_i w_i \psi_{Y_i} \in H^1(X).
\]
If \( Y \) is finite (as a CW-complex), then its components are finite CW-complexes and their number is finite. Any \( \mathbb{Z} \)-valued topological invariant \( \kappa \) of finite connected CW-complexes extends to such \( Y \) by linearity:
\[
\hat{\kappa}(Y) = \sum_i w_i \kappa(Y_i).
\]

1.2. Lemma. Let \( \kappa \) be a \( \mathbb{Z} \)-valued topological invariant of finite connected CW-complexes. Let \( X \) be a connected CW-space and \( Z \subset X \) be a weighted cooriented codimension 1 finite CW-subspace. Then there is a weighted cooriented codimension 1 finite CW-subspace \( Y \subset X \) such that \( \psi_Y = \psi_Z \), \( \hat{\kappa}(Y) \leq \hat{\kappa}(Z) \), and \( X \setminus Y \) is connected.

Proof. We define several transformations on a weighted cooriented codimension 1 subspace \( Y = \bigcup_i(Y_i, w_i) \) of \( X \). By “decreasing the weight of \( Y_i \) by 1” we mean the transformation which reduces \( w_i \) by 1 and keeps the other weights. If \( w_i = 1 \), then this transformation removes \( Y_i \) from \( Y \).

Assume that \( X \setminus Y = X \setminus \bigcup_i Y_i \) is not connected. For a component \( N \) of \( X \setminus Y \), we define a reduction of \( Y \) along \( N \). Let \( \alpha_+ \) (resp. \( \alpha_- \)) be the set of all \( i \in \pi_0(Y) \) such that \( N \) is adjacent to \( Y_i \) only on the positive (resp. negative) side. Since \( N \neq X \setminus Y \) and \( X \) is connected, at least one of the sets \( \alpha_+, \alpha_- \) is non-void. Counting how many times a loop in \( X \) goes into or out of \( N \), we obtain that \( \sum_{i \in \alpha_+} \psi_{Y_i} = \sum_{i \in \alpha_-} \psi_{Y_i} \). We modify \( Y \) as follows. If \( \alpha_+ \neq \emptyset \) and \( \sum_{i \in \alpha_+} \kappa(Y_i) \geq \sum_{i \in \alpha_-} \kappa(Y_i) \), then we decrease by 1 the weights of all \( \{Y_i\}_{i \in \alpha_+} \) and increase by 1 the weights of all \( \{Y_i\}_{i \in \alpha_-} \). If \( \alpha_+ = \emptyset \) or \( \sum_{i \in \alpha_+} \kappa(Y_i) < \sum_{i \in \alpha_-} \kappa(Y_i) \) then we increase by 1 the weights of all \( \{Y_i\}_{i \in \alpha_+} \) and decrease by 1 the weights of all \( \{Y_i\}_{i \in \alpha_-} \). This yields another weighted cooriented codimension 1 subspace \( Y' \subset X \) such that \( \psi_{Y'} = \psi_Y \) and \( \hat{\kappa}(Y') \leq \hat{\kappa}(Y) \). Iterating this transformation, we eventually remove from \( Y \) at least one component incident to \( N \) on one side. We call
this iteration the reduction of $Y$ along $N$. The reduction does not increase $\hat{\kappa}$, preserves $\psi_Y$ and strictly decreases the number of components of $X \setminus Y$. If $N$ is adjacent to only one component of $Y$ then the reduction removes this component from $Y$.

We can now prove the lemma. If $X \setminus Z$ is connected then $Z$ satisfies the requirements of the lemma. If $X \setminus Z$ is not connected then iteratively applying to $Z$ reductions along the components of $X \setminus Z$ we eventually obtain a weighted codimension 1 subspace $Y \subset X$ such that $X \setminus Y$ is connected. Clearly, $\psi_Y = \psi_Z$ and $\hat{\kappa}(Y) \leq \hat{\kappa}(Z)$.

2. Twisted homology and the ring $\Lambda$

2.1. Twisted homology. Let $X$ be a connected CW-space with base point $x$. A ring homomorphism $\varphi$ from $\mathbb{Z}[\pi_1(X, x)]$ to an associative ring with unit $R$ gives rise to a right $R$-module of twisted homology $H_*(X; R)$ as follows. Consider the universal covering $p : \tilde{X} \to X$ and denote by $\pi$ the group of covering transformations of $p$. To stay in line with [COT], [Co], [Ha], we consider $\tilde{X}$ as a right $\pi$-set. The right action of $\gamma \in \pi$ is just the usual left action of $\gamma^{-1} \in \pi$. The cellular chain complex $C_*(\tilde{X})$ of $\tilde{X}$ is then a right free chain complex over $\mathbb{Z}[\pi]$. To specify a basis of $C_*(\tilde{X})$, it is enough to orient all cells of $X$ and choose their lifts to $\tilde{X}$. We can identify $\pi = \pi_1(X, x)$ as usual and provide $R$ with the structure of a $\mathbb{Z}[\pi]$-bimodule via $arb = \varphi(a)r\varphi(b)$ for any $a, b \in \mathbb{Z}[\pi]$ and $r \in R$. Then

$$H_*(X; R) = H_*(C_*(\tilde{X}) \otimes_{\mathbb{Z}[\pi]} R).$$

Note that the $n$-th boundary homomorphism $C_n(\tilde{X}) \to C_{n-1}(\tilde{X})$ is presented by a matrix over $\mathbb{Z}[\pi]$. Applying $\varphi$ to its entries we obtain the matrix of the $n$-th boundary homomorphism of the chain complex $C_*(\tilde{X}) \otimes_{\mathbb{Z}[\pi]} R$.

The identification $\pi = \pi_1(X, x)$ used above depends on a lift of the base point $x \in X$ to $\tilde{X}$. For a different lift, this identification is composed with the conjugation by an element of $\pi$. However, the isomorphism class of the
right $R$-module $H_*(X; R)$ is easily seen to be well defined. Similarly, the isomorphism class of $H_*(X; R)$ does not change when $\varphi$ is composed with conjugation by an invertible element of $R$.

The twisted homology extend to CW-subspaces $A \subset X$ by

$$H_*(A; R) = H_*(C_*(p^{-1}(A)) \otimes \mathbb{Z}[\pi] R).$$

Clearly, $H_*(A; R) = \bigoplus_i H_*(A_i; R)$ where $A_i$ runs over components of $A$.

The twisted homology are invariant under cellular subdivisions and form usual exact homology sequences such as the Mayer-Vietoris sequence. Using a CW-decomposition of $X$ with one 0-cell, one can check that $H_0(X; R) = R/\varphi(I)R$ where $I$ is the augmentation ideal of $\mathbb{Z}[\pi_1(X, x)]$. If $X$ is a closed orientable $n$-dimensional PL-manifold, then using a CW-decomposition of $X$ with one $n$-cell one easily computes that $H_n(X; R) = \{ r \in R | \varphi(I)r = 0 \}$.

A CW-subspace $A \subset X$ is said to be bad (with respect to $\varphi$) if all loops in $A$ are mapped by $\varphi$ to $1 \in R$. This condition does not depend on the way in which the origins of the loops are connected to the base point $x \in X$.

2.2. The ring $\Lambda$. We shall need a few algebraic properties of the skew polynomial ring $\Lambda = K[t^{\pm 1}; \alpha]$ defined in the introduction. Considering the highest degree terms of polynomials one immediately obtains that $\Lambda$ has no zero-divisors. A slightly more elaborated argument shows that all right (and left) ideals of $\Lambda$ are principal (see [Coh2, Prop. 2.1.1] or [Co, Prop. 4.5]). This implies that $\Lambda$ satisfies the right Ore condition which says that $a\Lambda \cap b\Lambda \neq 0$ for all non-zero $a, b \in \Lambda$ (see [Coh2, Sect. 1.3]). Therefore the formal expressions $ab^{-1}$ with $a, b \in \Lambda, b \neq 0$ form a skew field under natural addition and multiplication rules. This field, called the classical right field of quotients of $\Lambda$, will be denoted $Q$. The ring $\Lambda$ embeds into $Q$ via $a \mapsto a1^{-1}$ so that $Q$ becomes a $\Lambda$-bimodule. It is known that $Q$ is a flat left $\Lambda$-module, i.e., the functor $\otimes_\Lambda Q$ is exact (see [St, Prop. II.3.5]).

For a right $\Lambda$-module $H$, the torsion submodule $\text{Tors}_\Lambda H$ of $H$ consists of $h \in H$ such that $ha = 0$ for a non-zero $a \in \Lambda$. Observe that if $H$ is
finitely generated over $K \subset \Lambda$, then $\text{Tors}_\Lambda H = H$. Indeed, for any $h \in H$ the elements $h, ht, ht^2, \ldots$ can not be linearly independent over $K$. Therefore $x$ is annihilated by a non-zero element of $\Lambda$. This implies that for such $H$, we have $H \otimes_\Lambda Q = 0$.

In Sect. 3 we shall use an estimate, established by Harvey [Ha], on the $K$-rank of $\Lambda$-modules presented by certain matrices. Let $H$ be a right $\Lambda$-module with presentation matrix of the form $A + tB$ where $A$ and $B$ are $(l \times m)$-matrices over $K$ with $l, m = 1, 2, \ldots$. Then $\text{Tors}_\Lambda H$ is finitely generated over $K$ and $\text{rk}_K \text{Tors}_\Lambda H \leq \min(l, m)$, see [Ha], Prop. 9.1. (This is well known when $K$ is a commutative field).

2.3. Lemma. Let $X$ be a connected CW-space with base point $x$. Let $\varphi : \mathbb{Z}[\pi_1(X, x)] \to \Lambda$ be a ring homomorphism compatible with the cohomology class $\psi_Y \in H^1(X)$ determined by a weighted cooriented codimension 1 subspace $Y \subset X$ with finite number of components. Let $c$ be the number of bad components of $Y$ and $d$ be the number of bad components of $X \setminus Y$. Then

$$c \leq d + \text{rk}_Q (H_1(X; \Lambda) \otimes_\Lambda Q) \quad (2.1)$$

where $Q$ is the right field of quotients of $\Lambda$. In particular, if $H_1(X; \Lambda)$ has finite rank over $K$, then $c \leq d$.

Proof. Note first that both $Y$ and $X \setminus Y$ have a finite number of components so that the numbers $c, d$ are well defined.

We begin by computing the twisted 0-homology $H_0(A; \Lambda)$ for a connected CW-subspace $A \subset X$ such that the restriction of $\psi$ to $A$ is zero. If $A$ is bad then $H_*(A; \Lambda)$ is the usual untwisted homology of $A$ with coefficients in $\Lambda$ and in particular $H_0(A; \Lambda) = \Lambda$. Suppose that $A$ is not bad. Then $\varphi(\gamma) \neq 1$ for a certain $\gamma \in \pi_1(X, x)$ whose conjugacy class is represented by a loop in $A$. Since $\psi|_A = 0$, we have $\varphi(\gamma) \in K$. Then $\varphi(\gamma) - 1$ is invertible in $K \subset \Lambda$. This implies that $H_0(A; \Lambda) = 0$. 9
Let $U = Y \times [-1, 1]$ be a closed cylinder neighborhood of $Y = Y \times 0$ in $X$ and $N = X \setminus U$. The Mayer-Vietoris sequence of the triple $(X = N \cup U, N, U)$ gives an exact sequence

$$H_1(X; \Lambda) \to H_0(N \cap U; \Lambda) \to H_0(N; \Lambda) \oplus H_0(U; \Lambda). \quad (2.2)$$

Observe that the restrictions of $\psi = \psi_Y$ to all components of $Y$ and $N$ are zero. The computation above shows that $H_0(N; \Lambda) = \Lambda^d$ and $H_0(U; \Lambda) = \Lambda^c$. Similarly, $H_0(N \cap U; \Lambda) = \Lambda^{2c}$. Tensoring the exact sequence (2.2) with $Q$ we obtain an exact sequence

$$H_1(X; \Lambda) \otimes_{\Lambda} Q \to Q^{2c} \to Q^{c+d}. \quad (2.3)$$

Note that the $Q$-rank is additive with respect to short exact sequences of $Q$-modules (all such sequences split). Therefore taking $Q$-ranks in (2.3) we obtain that $2c \leq c + d + \text{rk}_Q(H_1(X; \Lambda) \otimes_{\Lambda} Q)$. This implies (2.1).

### 3. More on twisted homology

#### 3.1. Homology with coefficients in $K$.

Consider a connected CW-space $X$ with base point $x$ and a ring homomorphism $\varphi : \mathbb{Z}[\pi_1(X, x)] \to \Lambda$ compatible with a cohomology class $\psi \in H^1(X)$. Let $A \subset X$ be a connected CW-subspace of $X$ such that the restriction $\psi|_A \in H^1(A)$ of $\psi$ to $A$ is zero. Pick a base point $a \in A$ and a path (or a homotopy class of paths) $\nu$ in $X$ leading from $x$ to $a$. Pushing the origin of loops from $a$ to $x$ along $\nu^{-1}$ we obtain an inclusion homomorphism $\pi_1(A, a) \to \pi_1(X, x)$. Extending it by linearity to the group rings and composing with $\varphi$ we obtain a ring homomorphism $\mathbb{Z}[\pi_1(A, a)] \to K \subset \Lambda$. This allows us to consider the corresponding twisted homology $H_*^\nu(A; K)$. To simplify notation, we shall sometimes omit $\nu$.

We define a homomorphism $g_A^\nu : H_*^\nu(A; K) \to H_*^\nu(A; \Lambda)$ linear over the ring inclusion $K \subset \Lambda$. Let $p_A : \tilde{A} \to A, p : \tilde{X} \to X$ be universal coverings with base points $\tilde{a} \in \tilde{A}, \tilde{x} \in \tilde{X}$ lying over $a$ and $x$, respectively. Let $q = q_A :$
\( \tilde{A} \to \tilde{X} \) be the unique map such that \( pq = p_\tilde{A} \) and \( \nu \) lifts to a path in \( \tilde{X} \) leading from \( \tilde{x} \) to \( q(\tilde{a}) \). The map \( q \) and the inclusion \( K \subset \Lambda \) induce a chain homomorphism

\[
C_\ast(\tilde{A}) \otimes_{\mathbb{Z}[\pi_1(A,a)]} K \to C_\ast(p^{-1}(A)) \otimes_{\mathbb{Z}[\pi_1(X,x)]} \Lambda.
\]  

(3.1)

The induced homomorphism in homology is \( g_\nu^\ast : H_\ast^\nu(A; K) \to H_\ast(A; \Lambda) \).

Observe that the chain \( \Lambda \)-complex on the right hand side of (3.1) is obtained from the chain \( K \)-complex on the left hand side via \( \otimes_K \Lambda \) where \( \Lambda \) is regarded as a left \( K \)-module via the inclusion \( K \subset \Lambda \). Clearly, \( \Lambda \) is a free \( K \)-module and therefore \( \Lambda \) is flat over \( K \). Therefore \( g_\nu^\ast \) induces an isomorphism

\[
H_\ast^\nu(A; K) \otimes_K \Lambda = H_\ast(A; \Lambda).
\]  

(3.2)

The \( K \)-rank of \( H_\ast^\nu(A; K) \) does not depend on the choice of \( \nu \). However, the image of \( g_\nu^\ast \) does depend on \( \nu \). An easy computation shows that for any \( \gamma \in \pi_1(X,x) \), we have \( \text{Im}(g_\nu^\ast \varphi(\gamma)) = \text{Im}(g_\nu^\ast \varphi(\gamma)) \).

The homology \( H_\ast(A; K) \) is natural with respect to certain inclusions \( A \subset A' \) where \( A \subset A' \) are pointed connected CW-subspaces of \( X \) such that \( \psi|_{A'} = 0 \). Namely, let \( a \in A, a' \in A' \) be the base points of \( A, A' \) and let \( \nu \) (resp. \( \nu' \)) be the distinguished path leading from \( x \) to \( a \) (resp. to \( a' \)). Suppose that the path \( \nu^{-1} \nu' \) is homotopic (modulo the endpoints) to a path in \( A' \). Then \( q_\Lambda(\tilde{A}) \subset q_{A'}(\tilde{A'}) \). This inclusion induces a \( K \)-linear homomorphism \( H_\ast^\nu(A; K) \to H_\ast^{\nu'}(A'; K) \) in the usual way.

The homology \( H_\ast(A; K) \) can be extended to non-connected CW-subspaces \( A \subset X \) with \( \psi|_{A} = 0 \) by \( H_\ast(A; K) = \bigoplus_i H_\ast(A_i; k) \) where \( A_i \) runs over components of \( A \). Here we need to assume that all components of \( A \) are pointed and \( x \) is connected to their base points by distinguished paths.

If \( A \) is finite as a CW-space then \( H_\ast(A; K) \) is finitely generated over \( K \). Moreover, \( \text{rk}_K H_n(A; K) \) does not exceed the minimal number of \( n \)-cells in a CW-decomposition of \( A \). This follows from definitions and the additivity of the rank with respect to short exact sequences.
In particular, if \( \psi = \psi_Y \) is determined by a weighted cooriented codimension 1 subspace \( Y \) of \( X \), then \( \psi|_Y = 0 \) and we can apply the constructions above to \( A = Y \). By definition, \( H_s(Y; K) = \oplus_i H_s(Y_i; k) \).

In the sequel, we call a class \( \psi \in H^1(X) \) primitive if \( \psi \) is not divisible by integers \( \geq 2 \) in the lattice \( H^1(X) \).

3.2. Lemma. Let \( X \) be a finite connected CW-space with base point \( x \) and let \( \varphi : \mathbb{Z}[\pi_1(X, x)] \to \Lambda \) be a ring homomorphism compatible with the primitive cohomology class \( \psi_Y \in H^1(X) \) determined by a weighted cooriented codimension 1 finite CW-subspace \( Y = \cup_i (Y_i, w_i) \) of \( X \). Then for any \( n \geq 0 \),

\[
\text{rk}_K \text{Tors}_\Lambda H_n(X; \Lambda) \leq \sum_i w_i \text{rk}_K H_n(Y_i; K). \tag{3.3}
\]

In particular, if \( H_n(X; \Lambda) \) has finite rank over \( K \) then

\[
\text{rk}_K H_n(X; \Lambda) \leq \sum_i w_i \text{rk}_K H_n(Y_i; K). \tag{3.4}
\]

Proof. The right hand side of (3.3) does not change if \( Y \) is replaced by \( Y^\# \). Therefore we can assume that the weights of all components of \( Y \) are equal to 1. Then the right hand side of (3.3) is just \( \text{rk}_K H_n(Y; K) \).

Let \( U = Y \times [-1, 1] \) be a closed cylinder neighborhood of \( Y = Y \times 0 \) in \( X \) such that \( Y \times [0, 1] \) lies on the positive side of \( Y \). Set \( Y^\pm_i = Y_i \times (\pm 1) \) and \( Y^\pm = \cup_i Y^\pm_i = Y \times (\pm 1) \). Set \( N = \overline{X \setminus U} \). Clearly, \( N \cap U = Y^+ \cup Y^- \).

We provide \( X \) with a CW-decomposition (compatible with the given one) so that \( N, U, Y^+, Y^- \) are subcomplexes.

The Mayer-Vietoris sequence of the triple \( (X = N \cup U, N, U) \) gives an exact sequence

\[
H_n(N; \Lambda) \oplus H_n(U; \Lambda) \xrightarrow{j} H_n(X; \Lambda) \xrightarrow{\partial} H_{n-1}(N \cap U; \Lambda).
\]

Set \( J = \text{Im} \ j = \text{Ker} \ \partial \subset H_n(X; \Lambda) \). Note that \( \text{Tors}_\Lambda J = \text{Tors}_\Lambda H_n(X; \Lambda) \). Indeed \( H_s(N \cap U; \Lambda) = H_s(Y^-; \Lambda) \oplus H_s(Y^+; \Lambda) \) and the right \( \Lambda \)-module
\[ H_*(Y^\pm; \Lambda) = H_*(Y^\pm; K) \otimes_K \Lambda \] is free since \( H_*(Y^\pm; K) \) is a free \( K \)-module. Therefore \( \text{Tors}_\Lambda H_*(X; \Lambda) \subset \text{Ker} \partial = J \) so that \( \text{Tors}_\Lambda H_*(X; \Lambda) = \text{Tors}_\Lambda J \).

Writing down the next term of the Mayer-Vietoris sequence, we obtain an exact sequence

\[ H_n(N \cap U; \Lambda) \to H_n(N; \Lambda) \oplus H_n(U; \Lambda) \to J \to 0. \tag{3.5} \]

Clearly, the inclusion homomorphisms \( H_n(Y^\pm; \Lambda) \to H_n(U; \Lambda) \) are isomorphisms. Therefore the exact sequence (3.5) yields an exact sequence

\[ H_n(Y^+; \Lambda) \xrightarrow{\eta} H_n(N; \Lambda) \to J \to 0 \]

where \( \eta \) can be computed as follows. We identify \( H_n(Y^+; \Lambda) = H_n(Y^-; \Lambda) \) using the homeomorphism \( y \times 1 \mapsto y \times (-1) : Y^+ \to Y^- \) where \( y \) runs over \( Y \). Denote by \( \eta^\pm \) the inclusion homomorphism \( H_n(Y^\pm; \Lambda) \to H_n(N; \Lambda) \). Then \( \eta = \eta^+ - \eta^- \).

Observe that the restrictions of \( \psi \) to all components of \( Y \) and \( N \) are zero. Since \( \psi \) is primitive, there is \( \gamma \in \pi_1(X, x) \) such that \( \psi(\gamma) = 1 \). Let us fix base points in all components of \( N \) and choose paths in \( X \) connecting them to \( x \). Composing if necessary these paths with loops representing powers of \( \gamma \), we can assume that the algebraic number of intersections of each of these paths with \( Y \) is 0. Pick a point \( y_i \) on every component \( Y_i \) of \( Y \) and fix the base points \( y_i \times (\pm 1) \) on \( Y^\pm_i \). Let us connect the latter base points to the base points of the adjacent components of \( N \) (by paths in \( N \)) and then further to \( x \) along the paths chosen above. Now we can consider the twisted homology \( H_*(Y^\pm; K) = \oplus_i H_*(Y^\pm_i; K) \) and \( H_*(N; K) = \oplus_j H_*(N_j; K) \) where \( j \) numerates the components of \( N \).

A basis of each \( K \)-module \( H_*(Y^\pm_i; K) \) determines a basis of the \( \Lambda \)-module \( H_*(Y^\pm_i; \Lambda) \) using (3.2). Similarly, for each component \( N_j \) of \( N \) a basis of \( H_*(N_j; K) \) determines a basis of \( H_*(N_j; \Lambda) \). With respect to these bases the homomorphisms \( \eta^\pm \) are presented by matrices with entries in \( K \). The bases of \( H_*(Y^+_i; \Lambda) = H_*(Y^-_i; \Lambda) \) obtained in this way from bases in \( H_*(Y^+_i; K), H_*(Y^-_i; K) \) are related by a matrix over \( \Lambda \). We claim that all
its entries lie in $tK$. Indeed, by the computations of Sect. 3.1, these entries lie in $\varphi(\gamma_i)K$ where $\gamma_i$ is the loop formed by the distinguished path leading from $x$ to $y_i \times (-1)$, the interval $y_i \times [-1,1]$, and the distinguished path leading from $y_i \times 1$ to $x$. The algebraic number of intersections of $\gamma_i$ with $Y$ is 1. Therefore $\psi(\gamma_i) = 1$ and $\varphi(\gamma_i) \in tK$. This proves that the homomorphism $\eta = \eta^+ - \eta^-$ can be presented by a matrix of the form $A + tB$ where $A,B$ are matrices over $K$. The Harvey inequality stated in Sect. 2.2 implies that $\text{rk}_K \text{Tors}_A H_n(X; \Lambda) = \text{rk}_K \text{Tors}_A J \leq \text{rk}_K H_n(Y; K)$. 

Finally, if $H_n(X; \Lambda)$ has finite rank over $K$, then $\text{Tors}_A H_n(X; \Lambda) = H_n(X; \Lambda)$ so that (3.3) implies (3.4).

### 3.3. Remarks

1. The assumption that $\psi$ is primitive can be removed using for instance the argument in [Th, p. 103], but we shall not need this.

2. The right hand side of (3.3) can be often easily estimated from above which may give useful estimates for $\text{rk}_K \text{Tors}_A H_n(X; \Lambda)$. Assume, for concreteness, that in Lemma 3.2, the space $X$ is a compact orientable 3-manifold and $Y \subset X$ is a proper cooriented embedded surface with weights of all components equal to 1. Then $\text{rk}_K H_1(Y; K) \leq b_1(Y)$ where $b_1$ is the first Betti number. This follows from the fact that each component $Y_i$ either has a CW-decomposition with $b_1(Y_i)$ one-cells (if $\partial Y_i = \emptyset$) or collapses onto a graph with $b_1(Y_i)$ one-cells (if $\partial Y_i \neq \emptyset$). Another useful estimate:

$$\text{rk}_K H_1(Y; K) = -\chi(Y) + \text{rk}_K H_0(Y; K) + \text{rk}_K H_2(Y; K)$$

$$= -\chi(Y) + c + c' \leq ||\psi_Y|| + c + c'$$

where $c$ is the number of bad components of $Y$ and $c'$ is the number of closed bad components of $Y$. For example, let $X$ be the exterior of an oriented $m$-component link in $S^3$ and let $\psi \in H^1(X)$ be the cohomology class taking values $k_1, ..., k_m \in \mathbb{Z}$ on the meridians of the link. One can easily show (cf. [Ha, Cor. 10.4]) that there is a Thurston norm minimizing proper surface in $X$ dual to $\psi$, having $\sum_j |k_j|$ boundary circles and no closed components.
Such a surface has at most $\sum_j |k_j|$ components and therefore for all $\varphi$ as in Lemma 3.2, $\text{rk}_K \text{Tors}_\Lambda H_1(X; \Lambda) \leq ||\psi||_T + \sum_j |k_j|$.

4. Proof of Theorem 1

4.1. We first reduce the theorem to the case of primitive $\psi$. Any $\psi \in H^1(M)$ can be presented in the form $\psi = n\psi'$ where $n = |\psi| \geq 1$ and $\psi' \in H^1(M)$ is primitive. Consider the skew polynomial ring $\Lambda' = K[u^{\pm 1}; a^n]$ in the variable $u$. There is a ring embedding $j : \Lambda' \to \Lambda$ which maps any monomial $u^{mn}a$ to $t^{mn}a$ where $a \in K, m \in \mathbb{Z}$. If a ring homomorphism $\varphi : \mathbb{Z}[\pi_1(M)] \to \Lambda$ is compatible with $\psi$ then it takes values in $j(\Lambda')$ and splits therefore as a composition of $j$ with a ring homomorphism $\varphi' : \mathbb{Z}[\pi_1(M)] \to \Lambda'$ compatible with $\psi'$. Observe that the right $\Lambda$-module $H_*(M; \Lambda)$ viewed as a $\Lambda'$ module splits as a direct sum of $n$ copies of $H_*(M; \Lambda')$ (this is true already on the level of chain complexes). Hence $\text{rk}_K H_1(M; \Lambda) = n \text{rk}_K H_1(M; \Lambda')$. This and the homogeneity of the Thurston norm $||\psi||_T = n||\psi'||_T$ imply that if the claim of Theorem 1 holds for $\psi'$ then it holds also for $\psi$. We assume from now on that $\psi$ is primitive.

4.2. Fixing an orientation on $M$ we can identify orientations of surfaces in $M$ with their coorientations. We can represent $\psi \cap [M] \in H_2(M, \partial M)$ by a proper oriented surface $Z \subset M$ such that $\chi_-(Z) = ||\psi||_T$. In the notation of Sect. 1.1, we have $\psi_Z = \psi$. We can regard $Z$ as a weighted codimension 1 subspace of $M$ with weights of all components equal to 1. We apply Lemma 1.2 to $\kappa = \chi_-$ and $X = M$. This lemma yields a weighted proper oriented surface $Y = \cup_i (Y_i, w_i)$ embedded in $M$ such that $M \setminus Y$ is connected, $\psi_Y = \psi_Z = \psi$ and $\hat{\chi}_-(Y) \leq \hat{\chi}_-(Z) = \chi_-(Z)$. The inequalities

$$||\psi||_T \leq \chi_-(Y^\#) = \hat{\chi}_-(Y) \leq \chi_-(Z) = ||\psi||_T$$

imply that $||\psi||_T = \hat{\chi}_-(Y) = \sum_i w_i \chi_-(Y_i)$. Note that all components $Y_i$ of $Y$ are proper embedded oriented surfaces in $M$. 

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We now separate 2 cases depending on whether $M \setminus Y$ is bad or not.

4.3. Suppose that $M \setminus Y$ is not bad with respect to $\varphi$. Since $H_1(M; \Lambda)$ is finitely generated over $K$, Lemma 2.3 implies that $Y$ has no bad components. For any component $Y_i$ of $Y$ we have $H_0(Y_i; K) = 0$, cf. Sect. 2.1 or the proof of Lemma 2.3. If $\partial Y_i \neq \emptyset$, then $Y_i$ is contractible onto a 1-dimensional complex and therefore $H_2(Y_i; K) = 0$. If $\partial Y_i = \emptyset$ then using a CW-decomposition of $Y_i$ with one 2-cell, one can easily check that $H_2(Y_i; K) = 0$. If $\partial Y_i \neq \emptyset$ then using a CW-decomposition of $Y_i$ with one 2-cell, one can easily check that $H_2(Y_i; K) = 0$. Then $\chi(Y_i) = \sum_{j \geq 0} (-1)^j \text{rk}_K H_j(Y_i; K) = -\text{rk}_K H_1(Y_i; K)$ and $\chi_-(Y_i) = \text{rk}_K H_1(Y_i; K)$. By Lemma 3.2,

$$\text{rk}_K H_1(M; \Lambda) \leq \sum_i w_i \text{rk}_K H_1(Y_i; K) = \sum_i w_i \chi_-(Y_i) = ||\psi||_T.$$  

This implies the claim of the theorem in this case.

4.4. Suppose that $M \setminus Y$ is bad with respect to $\varphi$. All components of $Y$ are parallel to surfaces lying in $M \setminus Y$ and therefore they all are bad. Since $H_1(\Lambda)$ is finitely generated over $K$, Lemma 2.3 implies that $Y$ is connected. Since the dual cohomology class $\psi = \psi_Y$ is primitive, the weight of the only component of $Y$ is equal to 1.

It is clear that $\varphi(\pi_1(M)) \subset \Lambda$ is a cyclic group generated by the value of $\varphi$ on the homotopy class of a loop in $M$ intersecting $Y$ once transversely. We must therefore prove the inequality (0.3). Since $||\psi||_T \geq 0$ and $|\psi| = 1$, we need to prove that $||\psi||_T \geq \text{rk}_K H_1(M; \Lambda) - \varepsilon_M$.

Set $b = b_1(Y)$. Assume first that $\partial M$ is void or consists of 2-spheres so that $\varepsilon_M = 2$. Gluing 3-balls to the components of $\partial M$, we obtain a closed 3-manifolds with the same 1-homology (twisted or untwisted) and the same Thurston norm. Therefore without loss of generality we can assume that $\partial M = \emptyset$. Then $Y$ is a closed surface and $||\psi||_T = \chi_-(Y) = \max(b - 2, 0)$. Since $Y$ is bad the twisted homology $H_1(Y; K)$ coincide with the untwisted homology, i.e., $H_1(Y; K) = K^b$. By Lemma 3.2,

$$||\psi||_T \geq b - 2 = \text{rk}_K H_1(Y; K) - \varepsilon_M \geq \text{rk}_K H_1(M; \Lambda) - \varepsilon_M.$$
Assume from now on that $\partial M$ has a component of non-zero genus so that $\varepsilon_M = 1$. We shall prove below that in this case $\partial Y \neq \emptyset$. Then $||\psi||_T = \chi_-(Y) = \max(b - 1, 0)$. As in the previous paragraph, $\text{rk}_KH_1(Y; K) = b$. Hence, by Lemma 3.2, $b \geq \text{rk}_KH_1(M; \Lambda)$. Therefore $||\psi||_T \geq b - 1 \geq \text{rk}_KH_1(M; \Lambda) - \varepsilon_M$.

It remains to prove that $\partial Y \neq \emptyset$. We begin with algebraic preliminaries. If the unit $1 \in K$ is annihilated by a positive integer then the minimal such integer is prime and the additive subgroup of $K$ generated by $1$ is a finite commutative field. It is denoted $F$. If $1 \in K$ is not annihilated by positive integers then $\mathbb{Q}$ embeds in $K$ and we set $F = \mathbb{Q} \subset K$. It is easy to check that $F$ lies in the center of $K$ and any automorphism of $K$ acts on $F$ as the identity. Let $\Lambda_0 = F[t^{\pm 1}]$ be the usual commutative Laurent polynomial ring in variable $t$ with coefficients in $F$. The inclusion $F \subset K$ extends to a ring inclusion $\Lambda_0 \hookrightarrow \Lambda$. It is clear that $K$ is a free left $F$-module and therefore $\Lambda$ is a free left $\Lambda_0$-module. Hence $\Lambda$ is a flat left $\Lambda_0$-module.

Consider the infinite cyclic covering $\hat{M} \to M$ determined by $\psi$. Since $\psi$ is primitive, the manifold $\hat{M}$ is connected. The surface $Y$ lifts to a homeomorphic surface $\hat{Y} \subset \hat{M}$ splitting $\hat{M}$ into two connected pieces, $M_-$ and $M_+$ such that $M_- \cap M_+ = \hat{Y}$. Let $t$ be the generating deck transformation of the covering $\hat{M} \to M$ such that $tM_+ \subset M_+$. The action of $t$ on the usual (untwisted) homology $H_s(\hat{M}; F)$ makes it a module over $\Lambda_0 = F[t^{\pm 1}]$. It follows from definitions and the flatness of $\Lambda$ as a left $\Lambda_0$-module that

$$H_s(M; \Lambda) = H_s(\hat{M}; F) \otimes_{\Lambda_0} \Lambda = H_s(\hat{M}; F) \otimes_F K.$$ 

The assumption that $H_1(M; \Lambda)$ has finite rank over $K$ implies that $H_1(\hat{M}; F)$ must be a finite dimensional vector space over $F$.

Suppose that $\partial Y = \emptyset$ so that $Y \cap \partial M = \emptyset$. Let $Z$ be a component of $\partial M$ of genus $g \geq 1$. Cutting $M$ open along $Y$ we obtain a compact connected 3-manifold homeomorphic to $N = M_+ \setminus t(\text{Int}M_+)$. Therefore $\partial N$ consists of $\hat{Y} \cup t(\hat{Y})$, a copy of $Z$, and possibly other closed surfaces. For any $r \geq 1$, the boundary of the 3-manifold $N_r = N \cup tN \cup ... \cup t^{r-1}N$ includes $\hat{Y} \cup t^r(\hat{Y})$ and...
copies of \( Z \). By a well known corollary of the Poincaré duality, the image of the inclusion homomorphism \( H_1(\partial N_r; F) \to H_1(N_r; F) \) has dimension \((1/2) \dim_F H_1(\partial N_r; F) \geq gr\). Therefore \( \dim_F H_1(N_r; F) \geq gr \). On the other hand, the kernel of the inclusion homomorphism \( H_1(N_r; F) \to H_1(\hat{M}; F) \) has dimension \( \leq 2 \dim_F H_1(\hat{Y}; F) \). This shows that \( H_1(\hat{M}; F) \) can not be finite dimensional. This contradiction implies that \( \partial Y \neq \emptyset \) and completes the proof of the theorem.

5. Proof of Theorem 2

Suppose that \( \psi \) is dual to the fiber, \( Y \), of a fibration \( M \to S^1 \). Since \( M \) is connected, all components of \( Y \) are homeomorphic to each other and represent the same class \( \psi' = \psi/|\psi| \in H^1(M) \). The inequalities (0.2) and (0.3) for \( \psi \) are obtained from the corresponding inequalities for \( \psi \) via multiplication by \( |\psi| \), cf. Sect. 4.1. Therefore, replacing \( \psi \) by \( \psi' \) we can assume that \( \psi \) is primitive and \( Y \) is connected.

If \( Y \) is a 2-sphere or a 2-disc, then \( \pi_1(M) = \mathbb{Z}, ||\psi||_T = 0 \), and (0.3) becomes an equality \( 0 = 0 \). Assume from now on that \( Y \) is neither a 2-sphere nor a 2-disc. Then \( \chi(Y) \leq 0 \) and \( \chi_-(Y) = -\chi(Y) \).

The universal covering \( \tilde{M} \to M \) factors through the infinite cyclic covering \( Y \times \mathbb{R} \to M \) determined by \( \psi \). Therefore \( \tilde{M} = \hat{Y} \times \mathbb{R} \) where \( \hat{Y} \to Y \) is a regular covering of \( Y \). This easily implies that \( H_*(M; \Lambda) = H_*(Y; K) \) as right \( K \)-modules. In particular \( H_1(M; \Lambda) \) is finitely generated over \( K \).

If the fiber \( Y \subset M \) is not bad with respect to \( \varphi \), then \( \varphi(\pi_1(M)) \) is not a cyclic group and we must prove that (0.2) becomes an equality. It suffices to prove the inequality opposite to (0.2). By the computations of Sect. 4.3, \( ||\psi||_T \leq -\chi(Y) = -\chi(Y) = \text{rk}_K H_1(Y; K) = \text{rk}_K H_1(M; \Lambda) \).

Assume that \( Y \subset M \) is bad with respect to \( \varphi \). Then \( \varphi(\pi_1(M)) \) is a cyclic group and we must prove that (0.3) becomes an equality. It suffices to prove
the inequality opposite to (0.3). If \( \partial Y \neq \emptyset \), then

\[
||\psi||_T \leq \chi_-(Y) = -\chi(Y) = \text{rk}_K H_1(Y; K) - \text{rk}_K H_0(Y; K)
\]

\[
= \text{rk}_K H_1(Y; K) - 1 = \text{rk}_K H_1(M; \Lambda) - \varepsilon_M |\psi|
\]

since |\psi| = 1 and \( \varepsilon_M = 1 \). If \( \partial Y = \emptyset \), then similarly

\[
||\psi||_T \leq -\chi(Y) = \text{rk}_K H_1(Y; K) - \text{rk}_K H_0(Y; K) - \text{rk}_K H_2(Y; K)
\]

\[
= \text{rk}_K H_1(Y; K) - 2 = \text{rk}_K H_1(M; \Lambda) - \varepsilon_M |\psi|
\]

since |\psi| = 1 and \( \varepsilon_M = 2 \).

6. Special cases

Throughout this section \( M \) is a compact connected orientable 3-manifold with fundamental group \( \pi \).

6.1. Seifert’s inequality. Let \( \Lambda = F[t^{\pm 1}] \) be the usual commutative ring of Laurent polynomials on the variable \( t \) with coefficients in a commutative field \( F \). Pick \( \psi \in H^1(M) \) and let \( \varphi \) be the ring homomorphism \( \mathbb{Z}[\pi] \to \Lambda \) defined by \( \varphi(\gamma) = t^{\psi(\gamma)} \) for \( \gamma \in \pi \). It follows from definitions that \( \varphi \) is compatible with \( \psi \) and \( H_1(M; \Lambda) = H_1(\hat{M}; F) \) where \( \hat{M} \to M \) is the infinite cyclic covering determined by \( \psi \). Clearly, \( \varphi(\pi) \approx \mathbb{Z} \). The inequality (0.3) says that if \( H_1(\hat{M}; F) \) is finite dimensional over \( F \) then

\[
||\psi||_T \geq \dim_F H_1(\hat{M}; F) - \varepsilon_M |\psi|.
\] (6.1)

This was known to H. Seifert at least in the case where \( F = \mathbb{Q} \), \( M \) is the exterior of a knot \( L \) in \( S^3 \) and \( \psi \) is the generator of \( H^1(M) = \mathbb{Z} \). In this case the condition \( \dim_F H_1(\hat{M}; F) < \infty \) is always satisfied, the number \( \dim_F H_1(\hat{M}; F) \) is equal to the span of the Alexander polynomial \( \Delta_L \) of \( L \), and \( ||\psi||_T = \max(2g(L) - 1, 0) \) where \( g(L) \) is the genus of \( L \). Clearly, \( \varepsilon_M = |\psi| = 1 \). Therefore (6.1) yields \( 2g(L) \geq \text{span}\Delta_L \).
One can modify a little the definition of \( \varphi \). Namely, any group homomorphism \( \sigma : H_1(M) \to F^* = F\setminus\{0\} \) gives rise to a ring homomorphism \( \mathbb{Z}[\pi] \to \Lambda \) sending any \( \gamma \in \pi \) to \( \sigma([\gamma])t^{\psi(\gamma)} \) where \( [\gamma] \in H_1(M) \) is the homology class of \( \gamma \). Applying Theorem 1 to this ring homomorphism we obtain a “twisted” version of (6.1).

### 6.2. McMullen’s inequality.

McMullen [McM] defined for any finitely generated group \( \Gamma \) a (semi-)norm \( ||...||_A \) on \( H^1(\Gamma) \) called the Alexander norm. It is derived from the Alexander polynomial \( \Delta_\Gamma \) as follows. Consider the free abelian group \( G = H_1(\Gamma)/\text{Tors}H_1(\Gamma) \). Recall that \( \Delta_\Gamma \) is an element of \( \mathbb{Z}[G] \) defined up to multiplication by \( \pm 1 \) and elements of \( G \). Pick a representative of \( \Delta_\Gamma \) and expand it as a finite sum \( \sum_{g \in G} c_g g \) where \( c_g \in \mathbb{Z} \). For any \( \psi \in H^1(\Gamma) \),

\[
||\psi||_A = \max_{g,g' \in G, c_g c_g' \neq 0} |\psi(g) - \psi(g')|
\]

where \( \psi(g) \in \mathbb{Z} \) is the evaluation of \( \psi \) on \( g \). This norm does not dependent on the choice of the representative of \( \Delta_\Gamma \). It is understood that if \( \Delta_\Gamma = 0 \), then \( ||\psi||_A = 0 \) for all \( \psi \).

Applying these constructions to \( \pi = \pi_1(M) \) we obtain the Alexander norm on \( H^1(M) = H^1(\pi) \). McMullen [McM] proved that for all \( \psi \in H^1(M) \), the Alexander and Thurston norms satisfy \( ||\psi||_T \geq ||\psi||_A \) if \( b_1(M) \geq 2 \) and \( ||\psi||_T \geq ||\psi||_A - \varepsilon_M ||\psi||_A \) if \( b_1(M) = 1 \). It is clear from Harvey’s argument [Ha, Prop. 5.12] that McMullen’s inequality is a special case of Theorem 1. We shall prove this in a more general context in the next subsection.

### 6.3. Alexander-Fox norms.

In generalization of McMullen’s Alexander norm, the author [Tu1] defined for any finitely generated group \( \Gamma \) a set of (semi-)norms on \( H^1(\Gamma) \) numerated by \( \sigma \in \text{Hom}(\text{Tors}H_1(\Gamma), \mathbb{C}^*) \). They are called Alexander-Fox norms or (generalized) Alexander norms. Recall their definition. Set \( H = H_1(\Gamma), G = H/\text{Tors}H \), and pick \( \sigma \in \text{Hom}(\text{Tors}H, \mathbb{C}^*) \). Fix a splitting \( H = \text{Tors}H \times G \) (we use multiplicative notation for the group operation in \( H \)). Consider the first elementary ideal \( E(\Gamma) = E_1(\Gamma) \subset \mathbb{Z}[H] \)
of $\Gamma$ (see [Fox]). Consider the ring homomorphism $\tilde{\sigma}: \mathbb{Z}[H] \to \mathbb{C}[G]$ sending $hg$ with $h \in \text{Tors}H, g \in G$ to $\sigma(h)g$ where $\sigma(h) \in \mathbb{C}^* \subset \mathbb{C}$. The ring $\mathbb{C}[G]$ is a unique factorization domain and we set $\Delta^\sigma(\Gamma) = \gcd(\tilde{\sigma}(E(\Gamma)))$. This gcd is an element of $\mathbb{C}[G]$ defined up to multiplication by elements of $G$ and nonzero complex numbers. Pick a representative $\sum_{g \in G} c_g g \in \mathbb{C}[G]$ of $\Delta^\sigma(\Gamma)$. For $\psi \in H^1(\Gamma)$, set

$$||\psi||^\sigma = \max_{g,g' \in G, c_g \neq 0} |\psi(g) - \psi(g')|.$$  

This (semi-)norm does not depend on the choice of the representative of $\Delta^\sigma(\Gamma)$ and does not depend on the choice of the splitting $H = \text{Tors}H \times G$. If $\Delta^\sigma(\Gamma) = 0$, then the norm $||\cdots||^\sigma$ is identically zero. For $\sigma = 1$, the norm $||\cdots||^1$ coincides with McMullen’s Alexander norm.

The author proved in [Tu2] that for all $\psi \in H^1(M) = H^1(\pi)$ and all $\sigma \in \text{Hom}(\text{Tors}H_1(M), \mathbb{C}^*)$,

$$||\psi||^\sigma_T \geq \begin{cases} ||\psi||^\sigma, & \text{if } b_1(M) \geq 2, \\ ||\psi||^\sigma - \delta^1_\sigma \varepsilon_M|\psi|, & \text{if } b_1(M) = 1. \end{cases} \tag{6.2}$$  

Here $\delta^1_\sigma = 1$ if $\sigma = 1$ and $\delta^1_\sigma = 0$ otherwise. It is also explained in [Tu2] that varying $\sigma$ in (6.2) we obtain a set of inequalities equivalent to the adjunction inequality (0.1).

We now deduce (6.2) from Theorem 1. Since all the norms involved in (6.2) are homogeneous, it suffices to prove (6.2) for primitive $\psi$. Set $H = H_1(M)$ and denote by $U$ the kernel of the epimorphism $H/\text{Tors}H \to \mathbb{Z}$ induced by $\psi$. Clearly, $U$ is a free abelian group of rank $b_1(M) - 1$. The group ring $\mathbb{C}[U]$ is a commutative domain so that we can consider its field of quotients denoted $K$. Let $\Lambda = K[t^\pm 1]$ be the commutative ring of Laurent polynomials on the variable $t$ with coefficients in $K$.

Pick $\tau \in H$ such that $\psi(\tau) = 1$ and fix a splitting $H = \text{Tors}H \times U \times (\tau)$ where $(\tau)$ is the infinite cyclic subgroup of $H$ generated by $\tau$. Consider the ring homomorphism $\mathbb{Z}[H] \to \Lambda$ sending $h u^m \tau \in H$ to $\sigma(h)u t^m$ for any $h \in \text{Tors}H, u \in U, m \in \mathbb{Z}$. Composing with the obvious ring projection
\[ \mathbb{Z}[\pi] \to \mathbb{Z}[H] \] we obtain a ring homomorphism, \( \varphi : \mathbb{Z}[\pi] \to \Lambda \), compatible with \( \psi \).

**Claim.** If \( H_1(M; \Lambda) \) is finitely generated over \( K \), then \( \operatorname{rk}_K H_1(M; \Lambda) = ||\psi||^\sigma \). If \( H_1(M; \Lambda) \) is not finitely generated over \( K \), then \( ||\psi||^\sigma = 0 \).

This Claim and Theorem 1 imply (6.2). Indeed, if \( H_1(M; \Lambda) \) is not finitely generated over \( K \) then the right hand side of (6.2) is \( \leq 0 \) and (6.2) is obvious. Assume that \( H_1(M; \Lambda) \) is finitely generated over \( K \) and \( b_1(M) = 1 \), then \( \varphi(\pi) \) is a cyclic group and (0.3) yields

\[
||\psi||_T \geq \operatorname{rk}_K H_1(M; \Lambda) - \varepsilon_M |\psi| = ||\psi||^\sigma - \delta_1 \varepsilon_M |\psi|.
\]

If \( \sigma \neq 1 \) or \( b_1(M) \geq 2 \), then \( \varphi(\pi) \) is not a cyclic group and (0.2) similarly implies (6.2).

Let us prove the Claim. For a non-zero Laurent polynomial \( \lambda \in \Lambda \), we define the span \( \text{span}(\lambda) \) as the maximal difference \( m - n \) where both \( t^m \) and \( t^n \) appear in \( \lambda \) with non-zero coefficients. Clearly, \( \mathbb{C}[H/TorsH] = \mathbb{C}[U][\tau^{\pm 1}] \) is the ring of Laurent polynomials on \( \tau \) with coefficients in \( \mathbb{C}[U] \). Therefore the inclusion \( \mathbb{C}[U] \subset K \) extends to a ring embedding \( \nu : \mathbb{C}[H/TorsH] \to \Lambda \) sending \( \tau \) to \( t \). It follows from definitions that \( ||\psi||^\sigma = \text{span}(\nu(\Delta)) \) for any representative \( \Delta \) of \( \Delta^\sigma(\pi) \). Let \( E \subset \Lambda \) be the ideal generated by the set \( \nu(\tilde{\sigma}(E(\pi))) \subset \Lambda \). We claim that \( \nu(\Delta) = \gcd E \). Recall that elements of a commutative ring are mutually prime if all their common divisors are invertible. To prove the equality \( \nu(\Delta) = \gcd E \) it suffices to show that for any mutually prime elements of \( \mathbb{C}[H/TorsH] \), their images in \( \Lambda \) also are mutually prime. This easily follows from the fact that \( \mathbb{C}[H/TorsH] \) is a unique factorization domain and the following property: if a non-zero element of \( \mathbb{C}[U] \) is divisible by \( a \in \mathbb{C}[H/TorsH] \) in \( \mathbb{C}[H/TorsH] = \mathbb{C}[U][\tau^{\pm 1}] \), then \( a \in \tau^m \mathbb{C}[U] \) for some \( m \in \mathbb{Z} \). This property follows from the additivity of the span with respect to multiplication.

The ideal \( E \subset \Lambda \) is nothing but the first elementary ideal of the \( \Lambda \)-
module $H_1(M, x; \Lambda)$ where $x$ is the base point of $M$. This means that for any presentation matrix of $H_1(M, x; \Lambda)$ with $s$ rows and $\geq s$ columns, $E$ is generated by the $(s-1) \times (s-1)$-minors of this matrix.

Consider the exact homological sequence

$$0 \to H_1(M; \Lambda) \to H_1(M, x; \Lambda) \to H_0(x; \Lambda) \to H_0(M; \Lambda).$$

Clearly, $H_0(x; \Lambda) = \Lambda$ and $H_0(M; \Lambda) = \Lambda/\varphi(I)\Lambda$ where $I$ is the augmentation ideal of $\mathbb{Z}[\pi]$. If $\sigma = 1$ then $H_0(M; \Lambda) = K$ and the inclusion homomorphism $H_0(x; \Lambda) \to H_0(M; \Lambda)$ is the augmentation (summation of coefficients) $\Lambda = K[t^\pm 1] \to K$. If $\sigma \neq 1$ then $H_0(M; \Lambda) = 0$. In both cases the kernel of the inclusion homomorphism $H_0(x; \Lambda) \to H_0(M; \Lambda)$ is a free $\Lambda$-module of rank one. Hence $H_1(M, x; \Lambda) = H_1(M; \Lambda) \oplus \Lambda$. Therefore for any presentation matrix of the $\Lambda$-module $H_1(M; \Lambda)$ with $s$ rows and $\geq s$ columns, the ideal $E$ is generated by the $s \times s$-minors of this matrix. It remains to observe that $\Lambda$ is a principal ideal domain and therefore $H_1(M; \Lambda)$ can be presented by a diagonal matrix $\text{diag}(p_1, \ldots, p_s)$ where $p_1, \ldots, p_s \in \Lambda$.

Then $\nu(\Delta) = \gcd E = \prod_i p_i$ up to multiplication by non-zero elements of $K$. If $p_i = 0$ for some $i$ then $H_1(M; \Lambda)$ is not finitely generated over $K$ and $||\psi||^\sigma = \text{span}(\nu(\Delta)) = 0$. If $p_i \neq 0$ for all $i$ then $H_1(M; \Lambda)$ is finitely generated over $K$ and has $K$-rank

$$\sum_i \text{span}(p_i) = \text{span}\left(\prod_i p_i\right) = \text{span}(\nu(\Delta)) = ||\psi||^\sigma.$$ 

**6.4. PTFA groups and Harvey’s inequalities.** A source of skew fields naturally appearing in the context of knot groups and 3-manifold groups, is the theory of PTFA groups, cf. [COT], [Co], [Ha]. We call a group $\Gamma$ poly-torsion-free-abelian (PTFA) if it admits a normal series $\{1\} = \Gamma_0 \subset \Gamma_1 \subset \ldots \subset \Gamma_n = \Gamma$ such that each factor $\Gamma_{i+1}/\Gamma_i$ is torsion-free abelian. Obviously, any PTFA group is torsion-free and solvable (the converse is not true). Any subgroup $H$ of a PTFA group $\Gamma$ is itself PTFA: for a normal series $\{1\} = \Gamma_0 \subset \Gamma_1 \subset \ldots \subset \Gamma_n = \Gamma$ as above, the groups $H_i = \Gamma_i \cap H$ form a
normal series \( \{1\} = H_0 \subset H_1 \subset \ldots \subset H_n = H \) such that each factor \( H_{i+1}/H_i \) is a subgroup of \( \Gamma_{i+1}/\Gamma_i \) and hence is torsion-free abelian. Any group \( \Gamma \) gives rise to a sequence of PTFA groups as follows. Consider the normal filtration 
\[ \Gamma = \Gamma^{(0)} \supset \Gamma^{(1)} \supset \ldots \] defined inductively by the condition that \( \Gamma^{(i)}/\Gamma^{(i+1)} \) is the maximal torsion-free abelian quotient of \( \Gamma^{(i)} \) for \( i = 0, 1, \ldots \). It is clear that \( \Gamma/\Gamma^{(i)} \) is PTFA for all \( i \).

For any PTFA group \( \Gamma \) and any commutative field \( F \), the group ring \( F[\Gamma] \) is a right (and left) Ore domain, i.e., it has no zero divisors and satisfies the Ore condition mentioned in Sect. 2.2 (see [Pa, pp. 588–592 and 611]). Therefore \( F[\Gamma] \) embeds into its classical right ring of quotients, \( F(G) \), which is a skew field. We shall use this construction to embed \( F[\Gamma] \) into a skew Laurent polynomial ring with coefficients in a skew field \( K_\psi \) determined by a primitive cohomology class \( \psi \in H^1(\Gamma; \mathbb{Z}) \). Namely, let \( \Gamma_\psi \) be the kernel of the epimorphism \( \Gamma \to \mathbb{Z} \) induced by \( \psi \). Pick any \( t \in \Gamma \) such that \( \psi(t) = 1 \). Every element of \( \Gamma \) can be uniquely expanded in the form \( t^mg \) with \( m \in \mathbb{Z} \) and \( g \in \Gamma_\psi \). This allows us to identify \( F[\Gamma] \) with the skew Laurent polynomial ring \( F[\Gamma_\psi][t^{\pm 1}] \) (the ring automorphism of \( F[\Gamma_\psi] \) needed in the definition of the skew Laurent polynomial ring is induced by the conjugation by \( t \in \Gamma \)). Therefore \( F[\Gamma] \subset K_\psi[t^{\pm 1}] \) where \( K_\psi = F(\Gamma_\psi) \). Note for the record that \( K_\psi[t^{\pm 1}] \) naturally embeds in \( F(\Gamma) \).

Coming back to our geometric setting, we can use PTFA groups to construct ring homomorphisms \( \varphi \) as in Theorem 1. Consider a PTFA group \( \Gamma \) and a commutative field \( F \). Suppose that we have group homomorphisms 
\[ \varphi_0 : \pi = \pi_1(M) \to \Gamma \] and \( \sigma : H_1(M) \to F^* = F\setminus\{0\} \). Pick a primitive cohomology class \( \psi \in H^1(\Gamma) \) and recall the skew field \( K_\psi \) and the embedding \( F[\Gamma] \subset K_\psi[t^{\pm 1}] \) constructed in the previous paragraph. Consider the ring homomorphism \( \varphi : \mathbb{Z}[\pi] \to K_\psi[t^{\pm 1}] \) which maps any \( \gamma \in \pi \) to \( \sigma([\gamma])\varphi_0(\gamma) \in F[\Gamma] \subset K_\psi[t^{\pm 1}] \) where \( [\gamma] \in H_1(M) \) is the homology class of \( \gamma \). It is clear that \( \varphi \) is compatible with the cohomology class \( \varphi_0^*(\psi) \in H^1(\pi) = H^1(M) \). Thus we can apply Theorem 1 to estimate 
\[ ||\varphi_0^*(\psi)||_T \] from \( \text{rk}_{K_\psi} H_1(M; K_\psi[t^{\pm 1}]) \).
Set now $F = \mathbb{C}$, $\Gamma = \pi/\pi^{(i)}$ with $i \geq 1$, and let $\varphi_0$ be the projection $\pi \to \Gamma$. For any $\sigma \in \text{Hom}(H_1(M), \mathbb{C}^*)$ and any primitive $\psi \in H^1(M)$, we obtain an estimate of $||\psi||_T$ (here we identify $H^1(\Gamma) = H^1(\pi)$ via $\varphi_0^*$). This estimate extends by linearity to arbitrary $\psi \in H^1(M)$. In the case $\sigma = 1$, the resulting estimates were first obtained by S. Harvey [Ha]. (She uses $F = \mathbb{Q}$ rather than $\mathbb{C}$ but this does not change the number $\text{rk}_{K_1} H_1(M; K_\psi[\pm 1])$).

For $i = 1$ we recover the estimates discussed in Sect. 6.3. Fixing $i \geq 2$ and varying $\sigma$ in $\text{Hom}(H_1(M), \mathbb{C}^*)$ we obtain a set of estimates which can be collectively considered as the adjunction inequality of level $i$. This set of estimates is actually finite: one can check that the number $\text{rk}_{K_1} H_1(M; K_\psi[\pm 1])$ does not change when $\sigma$ is multiplied by a homomorphism $H_1(M) \to \mathbb{C}^*$ sending $\text{Tors} H_1(M)$ to 1.

Note finally that in this construction $\varphi(\pi) \approx \mathbb{Z}$ if and only if $b_1(M) = 1$, $\sigma(\text{Tors} H_1(M)) = 1$, and either $i = 1$ or $i \geq 2$ and $\pi^{(1)} = \pi^{(2)} = \pi^{(3)} = \ldots$. In the case where $i \geq 2$ and $\pi^{(1)} = \pi^{(2)} = \pi^{(3)} = \ldots$ the inequality (0.3) is of no interest since then $H_1(M; K_\psi[\pm 1]) = 0$ (cf. [Ha]).

7. A homological estimate for 2-complexes

7.1. A norm on 1-cohomology. The author [Tu1] defined a homogeneous (semi-)norm on 1-cohomology of finite 2-complexes analogous to the Thurston norm on 1-cohomology of 3-manifolds. We recall this definition in a slightly generalized form. By a finite 2-complex we mean the underlying topological space of a finite 2-dimensional CW-complex such that each its point has a neighborhood homeomorphic to the cone over a finite graph. The latter condition is aimed at eliminating all kinds of local wilderness.

Let $X$ be a finite 2-complex. Note that a codimension 1 subspace of $X$ (in terminology of Sect. 1) is just a finite graph embedded in $X$ and having a cylinder neighborhood in $X$. We call codimension 1 subspaces of $X$ regular graphs on $X$. Any $\psi \in H^1(X)$ can be represented by a cooriented regular
graph on $X$: it can be obtained as the preimage of a regular value of a map $X \to S^1$ inducing $\psi$. Set $|| \psi ||_X = \min_Y \chi_-(Y)$ where $Y$ runs over cooriented regular graph on $X$ such that $\psi_Y = \psi$. The function $\psi \mapsto || \psi ||_X : H^1(X) \to \mathbb{Z}$ is non-negative and homogeneous in the sense that $||n \psi||_X = |n| ||\psi||_X$ for any $n \in \mathbb{Z}, \psi \in H^1(X)$. Here the inequality $||n \psi||_X \leq |n| ||\psi||_X$ is obvious (use $|n|$ parallel copies of any cooriented regular graph representing $\psi$). The opposite inequality can be obtained by the argument of [Th, p. 103] or deduced from Lemma 1.2. We verify in Sect. 7.4 that $||\psi||_X$ is a semi-norm. It does not depend on the choice of a CW-decomposition of $X$.

7.2. Theorem. Let $K, \alpha, \Lambda = K[t^{\pm 1}; \alpha]$ be as in the introduction. Let $X$ be a connected finite 2-complex and $\varphi : \mathbb{Z}[\pi_1(X)] \to \Lambda$ be a ring homomorphism compatible with a non-zero cohomology class $\psi \in H^1(X)$ and such that the right $\Lambda$-module $H_1(X; \Lambda)$ is finitely generated over $K$. Then

$$||\psi||_X \geq \max(\text{rk}_K H_1(X; \Lambda) - \delta_\varphi |\psi|, 0)$$

(7.1)

where $|\psi|$ is the maximal positive integer dividing $\psi$ in $H^1(X)$ and $\delta_\varphi = 1$ if the multiplicative group $\varphi(\pi_1(X)) \subset \Lambda \setminus \{0\}$ is cyclic and $\delta_\varphi = 0$ otherwise. If $\psi$ is induced by a fibration $X \to S^1$, then $H_1(X; \Lambda)$ is finitely generated over $K$ and (7.1) becomes an equality.

Proof. The proof of (7.1) goes along the same lines as the proof of Theorem 1. First one reduces the theorem to the case of primitive $\psi$. Then using Lemma 1.2 one obtains a weighted cooriented regular graph $Y = \cup_i (Y_i, w_i)$ on $X$ such that $X \setminus Y$ is connected, $\psi_Y = \psi$, and $||\psi||_X = \sum_i w_i \chi_-(Y_i)$. If $X \setminus Y$ is not bad with respect to $\varphi$, then by Lemma 2.3, $Y$ has no bad components so that $\chi_-(Y_i) = \text{rk}_K H_1(Y_i; K)$ for all $i$. This and Lemma 3.2 imply that $||\psi||_X \geq \text{rk}_K H_1(X; \Lambda)$.

Suppose that $X \setminus Y$ is bad with respect to $\varphi$. Then $Y$ is connected, its weight is 1, and $\delta_\varphi = 1$. We must prove that $\chi_-(Y) \geq \text{rk}_K H_1(X; \Lambda) - 1$. Let $F \subset K$ be a commutative subfield of $K$ defined in Sect. 4.4. As there,
$H_s(X; \Lambda) = H_s(\hat{X}; F) \otimes_F K$ where $\hat{X}$ is the infinite cyclic cover of $X$ determined by $\psi$. The assumptions of the theorem imply that the vector space $H_1(\hat{X}; F)$ is finite dimensional. Observe that $Y$ lifts to a homeomorphic graph $\hat{Y} \subset \hat{X}$ splitting $\hat{X}$ into two connected pieces, $\hat{X}_-$ and $\hat{X}_+$. Let $t$ be the generating deck transformation of the covering $\hat{X} \to X$ such that $t\hat{X}_+ \subset \hat{X}_+$. Since $H_1(\hat{X}; F)$ is finite dimensional, its basis can be represented by 1-cycles lying in $t^{-m}\hat{X}_+$ for sufficiently big $m$. Applying $t^m$ we obtain a basis of $H_1(\hat{X}; F)$ represented by 1-cycles in $\hat{X}_+$. Therefore the inclusion homomorphism $H_1(\hat{Y}; F) \to H_1(\hat{X}; F)$ is surjective. Similarly, the inclusion homomorphism $H_1(\hat{X}_-; F) \to H_1(\hat{X}, F)$ is also surjective. The Mayer-Vietoris sequence for $\hat{X} = \hat{X}_+ \cup \hat{X}_-$ implies the surjectivity of the inclusion homomorphism $H_1(\hat{Y}; F) \to H_1(\hat{X}; F)$. Hence

$$\chi_-(Y) \geq -\chi(Y) = \dim_F H_1(Y; F) - 1 \geq \dim_F H_1(\hat{X}; F) - 1 = \text{rk}_K H_1(X; \Lambda) - 1.$$}

The proof of the last claim of the theorem follows the proof of Theorem 2 with obvious changes.

### 7.3. Special cases

1. Let $\Lambda = F[t^{\pm 1}]$ be the usual commutative ring of Laurent polynomials on the variable $t$ with coefficients in a commutative field $F$. Pick $\psi \in H^1(X)$ and let $\varphi$ be the ring homomorphism $\mathbb{Z}[\pi_1(X)] \to \Lambda$ defined by $\varphi(\gamma) = t^{\psi(\gamma)}$ for $\gamma \in \pi_1(X)$. Then $H_1(X; \Lambda) = H_1(\hat{X}; F)$ where $\hat{X} \to X$ is the infinite cyclic covering determined by $\psi$. The inequality (7.1) says that if $H_1(\hat{X}; F)$ is finite dimensional over $F$ then $||\psi||_X \geq \dim_F H_1(\hat{X}; F) - |\psi|$. Using group homomorphisms $H_1(X) \to F^*$, we can obtain “twisted” versions of this inequality, cf. Sect. 6.1.

2. For any $\psi \in H^1(X)$ and any $\sigma \in \text{Hom}(\text{Tors}H_1(X), \mathbb{C}^*)$, it is proven in [Tu1] that

$$||\psi||_X \geq \begin{cases} ||\psi||^\sigma, & \text{if } b_1(X) \geq 2, \\ ||\psi||^\sigma - \delta_\sigma^1|\psi|, & \text{if } b_1(X) = 1 \end{cases}$$
where \(|\ldots||^\sigma|\) is the Alexander-Fox norm on \(H^1(X)\) determined by \(\sigma\) and \(\delta_\sigma^1 = 1\) if \(\sigma = 1\) and \(\delta_\sigma^1 = 0\) otherwise. This inequality can be deduced from Theorem 7.2 following the lines of Sect. 6.3.

3. We can use PTFA groups in this setting exactly as in Sect. 6.4. Consider a PTFA group \(\Gamma\) and a commutative field \(F\). Suppose that we have group homomorphisms \(\phi_0: \pi = \pi_1(X) \to \Gamma\) and \(\sigma: H_1(X) \to F^*\).

Pick a primitive cohomology class \(\psi \in H^1(\Gamma)\) and recall the skew field \(K_\psi\) and the embedding \(F[\Gamma] \subset K_\psi[t^{\pm 1}]\) constructed in Sect. 6.4. Consider the ring homomorphism \(\phi: \mathbb{Z}[\pi] \to K_\psi[t^{\pm 1}]\) which maps any \(\gamma \in \pi\) to \(\sigma([\gamma])\phi_0(\gamma) \in F[\Gamma] \subset K_\psi[t^{\pm 1}]\) where \([\gamma] \in H_1(X)\) is the homology class of \(\gamma\). It is clear that \(\phi\) is compatible with \(\phi_0^*(\psi) \in H^1(X)\). Thus we can apply Theorem 7.2 to estimate \(||\phi_0(\psi)||_X|\) from below.

In particular, set \(F = \mathbb{C}\), \(\Gamma = \pi/\pi^{(i)}\) with \(i \geq 1\), and let \(\phi_0\) be the projection \(\pi \to \Gamma\). Let us identify \(H^1(\Gamma) = H^1(\pi)\) via \(\phi_0^*\). For any \(\psi, \psi' \in H^1(X)\), we obtain the following: if \(H_1(M; K_\psi[t^{\pm 1}])\) has finite rank over \(K_\psi\), then

\[
||\psi||_T \geq \text{rk}_{K_\psi}H_1(M; K_\psi[t^{\pm 1}]) - \delta_\phi.
\] (7.2)

One can check that the number \(\text{rk}_{K_\psi}H_1(M; K_\psi[t^{\pm 1}])\) depends only on the restriction of \(\sigma\) to \(\text{Tors}H_1(M)\). The inequality (7.2) extends by linearity to arbitrary \(\psi \in H^1(X)\). For \(i = 1\) we recover the estimates discussed above.

Note that \(\delta_\phi = 1\) if and only if \(b_1(X) = 1\), \(\sigma(\text{Tors}H_1(X)) = 1\) and either \(i = 1\) or \(i \geq 2\) and \(\pi^{(1)} = \pi^{(2)} = \pi^{(3)} = \ldots\). In all other cases \(\delta_\phi = 0\). If \(i \geq 2\) and \(\pi^{(1)} = \pi^{(2)} = \pi^{(3)} = \ldots\) then \(H_1(M; K_\psi[t^{\pm 1}]) = 0\) and (7.2) is of no interest.

7.4. **Theorem.** For any finite 2-complex \(X\), the function \(\psi \mapsto ||\psi||_X: H^1(X) \to \mathbb{Z}\) is a semi-norm.

**Proof.** We need to show that the function \(\psi \mapsto ||\psi||_X\) is convex, i.e., \(||\psi + \psi'||_X \leq ||\psi||_X + ||\psi'||_X\) for any \(\psi, \psi' \in H^1(X)\). In the case where each
1-cell of $X$ is adjacent to at least two 2-cells of $X$ (counting with multiplicity). This is verified in [Tu1, Lemma 1.4]. The key point is that in this case every vertex of a regular graph $Y \subset X$ is incident to at least 2 edges of $Y$ (counting with multiplicity) and therefore $\chi(Y) \leq 0$ and $\chi_-(Y) = -\chi(Y)$. A regular graph representing $\psi + \psi'$ can be obtained from the union of regular graphs representing $\psi, \psi'$ by smoothing at all crossing points. This preserves the Euler characteristic and hence $||\psi + \psi'||_X \leq ||\psi||_X + ||\psi'||_X$.

Consider now the general case. We choose a CW-decomposition of $X$ such that all 2-cells are glued to the 1-skeleton along loops formed by sequences of oriented 1-cells (possibly with repetitions). Suppose that $X$ has a 1-cell $e$ which has no adjacent 2-cells. All cells of $X$ besides $e$ form a subcomplex $X' \subset X$. If the endpoints of $e$ belong to different components of $X'$, then $H^1(X) = H^1(X')$. It follows from definitions and the fact that $\chi_-(point) = 0$ that the functions $||...||_X, ||...||_{X'}$ coincide. If the endpoints of $e$ lie on one component of $X'$, then similarly the function $||...||_X$ is the composition of the restriction homomorphism $H^1(X) \to H^1(X')$ with $||...||_{X'}$. In both cases the convexity of $||...||_X$ would follow from the convexity of $||...||_{X'}$.

Suppose that $X$ has a 1-cell $e$ adjacent to exactly one 2-cell, $f$, with multiplicity 1. All cells of $X$ besides $e, f$ form a subcomplex $X' \subset X$. It is clear that $X'$ is a deformation retract of $X$ so that $H^1(X) = H^1(X')$. We claim that $||\psi||_X = ||\psi||_{X'}$ for any $\psi \in H^1(X) = H^1(X')$. To see this, let $e, e_1, ..., e_n$ be the 1-cells of $X$ (possibly with repetitions) forming the image of the gluing map of $f$. Let us call a regular graph on $X$ normal if it meets $f$ along a finite set of disjoint intervals connecting points on $e$ to points on $\cup_i e_i$. Observe that for any cooriented regular graph $Z \subset X$ the intersection $Z \cap f$ consists of a finite number of disjoint intervals connecting points on $e \cup \cup_i e_i$. If some of them have both endpoints on $\cup_i e_i$, then we can push the interiors of these intervals towards $e$ and eventually replace each of them with two intervals connecting points on $\cup_i e_i$ to points on $e$. The intervals in $Z \cap f$ with both endpoints on $e$ are simply eliminated. These transformations produce a
normal cooriented regular graph $Y \subset X$ representing the same cohomology class as $Z$. It is easy to compute from the definition of $\chi_-$, that $\chi_-(Y) \leq \chi_-(Z)$. Therefore for any $\psi \in H^1(X)$, we have $||\psi||_X = \min_Y \chi_-(Y)$ where $Y$ runs over normal cooriented regular graphs on $X$ representing $\psi$. Each such $Y$ gives rise to a cooriented regular graph $Y' = Y \cap X'$ on $X'$. Clearly $\psi_{Y'} = \psi_Y$ and $\chi_-(Y') = \chi_-(Y)$. Conversely, any cooriented regular graph $Y'$ on $X'$ arises in this way from a (unique up to ambient isotopy) normal cooriented regular graph on $X$. This implies that $||\psi||_X = ||\psi||_{X'}$ for any $\psi \in H^1(X) = H^1(X')$. Thus, the convexity of $||...||_X$ would follow from the convexity of $||...||_{X'}$.

Iterating the transformations $X \mapsto X'$ as above we eventually arrive to a 2-complex $X_0$ whose all 1-cells are adjacent to at least two 2-cells. As we know, the function $||...||_{X_0}$ is convex. Therefore the function $||...||_X$ is also convex.

7.5. Remark. It follows from general properties of homogeneous semi-norms on lattices (cf. [Th]), that the function $\psi \mapsto ||\psi||_X : H^1(X) \rightarrow \mathbb{Z}$ uniquely extends to a continuous homogeneous semi-norm $H^1(X; \mathbb{R}) \rightarrow \mathbb{R}$.

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