High-Energy Smoothing Estimates for Selfadjoint Operators

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Abstract. We prove the limiting absorption principle on the non-compact interval $I$, on which the uniformly positive Mourre estimate holds. We reveal that such a result yields so-called smoothing estimates.

Keywords: Mourre theory; Kato’s smoothing; Limiting absorption principle; scattering theory.

1 Introduction

In this paper, we show the limiting absorption principle (LAP) for generalized selfadjoint operators $H$ by using Mourre’s theory. In the paper by Mourre [9], the energy localized LAP was deduced for $H$. Hence in this paper, we deduce the global in the high-energy LAP and a smoothing-type LAP by modifying Mourre’s approach. The key to doing so is the so-called positive commutator $A$ (called the conjugate operator), such that $f(H)i[H,A]f(H) \geq cf(H)^2$ with $c > 0$ and energy cut-off $f \in C_0^\infty(R)$. In the background of Mourre theory, a pair of Schrödinger operators $H_S = p^2$ are considered, along with the generator of dilation group $A_S = x \cdot p + p \cdot x$, where $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and $p = -i\nabla_x$.

Formally, it then holds that $f(H_S)i[H_S,A_S]f(H_S) = f(H_S)(4p^2 - 2\nabla V)f(H_S)$ on $L^2(\mathbb{R}^n)$. By selecting $f$ such that $\{0\} \cup \{\sigma_{pp}(H)\} \notin \text{supp}(f)$, we have $f(H_S)4p^2f(H_S) \geq 2c_0f(H_S)^2$, $c_0 > 0$, and by selecting a very narrow $f$, $f(H_S)|2\nabla V|f(H_S) \leq c_0f(H_S)^2$ holds if $V$ and $f$ satisfy the suitable conditions. Consequently, we can deduce that $f(H_S)i[H_S,A_S]f(H_S)$ is positive with a suitable energy cut-off. The condition for the support of $f$ is mainly used to deduce the smallness of $Vf(H_S)$ and to deduce the positiveness and boundedness of the commutator $f(H_S)i[H_S,A_S]f(H_S)$. On the other hand, the Yokoyama-type conjugate operator $A_Y := x \cdot p(1 + p^2)^{-1} + (1 + p^2)^{-1}p \cdot x$ (see Yokoyama [13]) gives $i[H_S,A_Y] = 4p^2\langle p \rangle^{-2} + \langle p \rangle^{-2} \times (\text{bounded operators})$ if $V$ is smooth and satisfies a suitable decaying condition, where $\langle \cdot \rangle = (1 + \cdot^2)^{1/2}$. For the high-energy cut-off $f(H_S)$ (i.e., $p^2 \gg 1$ holds on $f(H_S)$), it holds that $f(H_S)i[H_S,A_Y]f(H_S) \geq 3f(H_S)^2$ even if the support of $f$ is not narrow. Moreover $i[H_S,A_Y]$ can be extended to a bounded operator without the energy cut-off. Additionally, in many physical situations the Hamiltonian $H$ has a conjugate operator $A$, which gives the positive and bounded commutator for the high-energy cut-off. Judging from these, we expect that we can deduce the limiting absorption principle from Mourres theory with a pair of $H$, $A$ and $\varphi$ such that $i[H,A]$ is positive and bounded with a high-energy cut-off $\varphi(H)$, with $\varphi \in C^\infty((-\infty,-R] \cup [R,\infty))$ (not $C_0^\infty(R)$), where $R$ is a given large constant.

Consider the Hamiltonian $H$, where $H$ is a selfadjoint operator acting on Hilbert space $\mathcal{H}$. The norm on $\mathcal{H}$ and the operator norm on $\mathcal{H}$ are denoted by the same notation, $\| \cdot \|$, and $\langle \cdot, \cdot \rangle$ denotes the inner product of $\mathcal{H}$. Let $A$ be a selfadjoint operator and suppose $D = \mathcal{D}(H) \cap \mathcal{D}(A) \subset \mathcal{H}$ is dense. We define the form $g_{H,A}(\cdot,\cdot)$ on $D$ as
Theorem 1.3. For any fixed 0 < ε < 1, under Assumption 1.1, we have the following nonexistence-of-eigenvalues property and limiting absorption principle for large R.

**Assumption 1.1.** Let H and A be selfadjoint operators satisfying the following statements: (a), (b), (c), (d), (e), and (f) (all conditions stated in Mourre [9] are fulfilled, regarding which see also [2]):

(a). $\mathcal{H}|_{\mathcal{D}(H)\cap \mathcal{D}(A)} = H$.

(b). $e^{itA}\mathcal{D}(H) \subset \mathcal{D}(H)$, $|t| < 1$ and for $u \in \mathcal{D}(H)$, $\sup_{|t| < 1} \|He^{itA}u\| < \infty$.

(c). The commutator $i[H,A]^0$ can be defined in the form and can be extended to a bounded operator from $\mathcal{H}$ to $\mathcal{H}$.

(d). There exists a constant $c_2 > 0$ such that for all $u, v \in \mathcal{D}(A)$,

$$\|(Au,i[H,A]^0v) - (i[H,A]^0u,Av)\| \leq c_2 \|u\| \|v\|.$$

(e). Let $R > 0$ be a given constant and $\varphi = \varphi_R \in C^\infty(\mathbb{R})$ with $0 \leq \varphi \leq 1$ is smooth cut-off function such that $\varphi(s) = 0$ if $|s| \leq R$ and $\varphi(s) = 1$ if $|s| \geq 2R$. Then there exists $0 < c_0 \leq c_1$, such that

$$c_0 \varphi(H)^2 \leq \varphi(H)i[H,A]^0 \varphi(H) \leq c_1 \varphi(H)^2 \leq c_1.$$

Moreover, $\varphi'(s) > 0$ for all $s$ in $R < |s| < 2R$.

(f). $i[H,i[H,A]^0]H^0$ can be extended to the bounded operators.

Under this assumption, we have the Mourre estimates for the high-energy case, which provide the following nonexistence-of-eigenvalues property and limiting absorption principle for H:

**Theorem 1.2.** Under Assumption 1.1, H has no eigenvalues on $(-\infty, -3R) \cup [3R, \infty)$.

**Theorem 1.3.** For any fixed $0 < \tilde{\varepsilon} < 1$, we select $\tilde{R}$ arbitrarily in $(0, R^{1-\tilde{\varepsilon}}]$. Under Assumption 1.1 for large $R$, there exists a constant $\delta = \delta_R > 1$ with $\delta_R \to 1$ (as $R \to \infty$) such that the limiting absorption principle

$$\sup_{|\lambda| \geq 3R, \mu > 0} \|\langle A \rangle^{-s}(H - \lambda \mp i\mu)^{-1}\langle A \rangle^{-s}\phi\|$$

$$\leq c_0^{-1} \left( \frac{\delta}{\tilde{R}} \right)^{1/2} + \delta(\tilde{R})^{s-1/2}(1-s)^{1-s}(2-s)^2(2s-1) \right)^2 e^{\delta \tilde{R}/\mu} \|\phi\|$$

holds for all $1/2 < s \leq 1$ and $\phi \in \mathcal{H}$, where $\langle \cdot \rangle = (1 + \cdot^2)^{1/2}$. Moreover, for all $s \geq 1$,

$$\sup_{|\lambda| \geq 3R, \mu > 0} \|\langle A \rangle^{-s}(H - \lambda \mp i\mu)^{-1}\langle A \rangle^{-s}\phi\|$$

$$\leq c_0^{-1} \left( \frac{\delta}{\tilde{R}} \right)^{1/2} + 2\delta s^{-s}(s-1)^{s-1}(\tilde{R})^{1/2} \right)^2 e^{\delta \tilde{R}/\mu} \|\phi\|$$
holds. In particular, if we can select $s > 0$ such that it is sufficiently large and $c_2 = c_2(R)$ satisfies $c_2\tilde{R} \to 0$ as $\tilde{R} \to \infty$, then for some $0 < \varepsilon_{R,s} \ll 1$,

$$\sup_{|\lambda| \geq 3R, \mu > 0} \| (A)^{-s} (H - \lambda) - i\mu)^{-1} (A)^{-s} \phi \| \leq \varepsilon_{R,s} \| \phi \|$$

holds.

Here, we define $\hat{\varphi} \in C^\infty(\mathbb{R})$, which satisfies $\hat{\varphi}(s) = 1$ if $|s| \geq 5R$ and $= 0$, if $|s| \leq 4R$. Then, it follows that

$$\sup_{|\lambda| \leq 3R, \mu > 0} \| \varphi(H)(H - \lambda) - i\mu)^{-1} \hat{\varphi}(H) (A)^{-s} \phi \| \leq \sup_{|\lambda| \leq 3R, \mu > 0} \| \varphi(H)(H - \lambda) - i\mu)^{-1} \hat{\varphi}(H) \| \| (A)^{-s} \phi \|$$

where $1/2 < s' \leq s$ with $s' \leq 1$. By noting commutator expansion (see, §2), we get

$$\sup_{|\lambda| \leq 3R, \mu > 0} \| \varphi(H)(H - \lambda) - i\mu)^{-1} \hat{\varphi}(H) \| \leq CR^{-1},$$

which yields the LAP:

$$\sup_{|\lambda| \leq 3R, \mu > 0} \| (A)^{-s} \hat{\varphi}(H)(H - \lambda) - i\mu)^{-1} \hat{\varphi}(H) (A)^{-s} \phi \| \leq c_0^{-1}\delta \left( \delta \tilde{R}^{1/2} + \delta(\tilde{R})^{1/2} \frac{(1 - s)^{1-s}(2 - s)}{(2s - 1)} \right)^2 e^{\delta c_2 \tilde{R}/c_0} \| \phi \|$$

for all $1/2 < s \leq 1$. As the direct consequence of Kato [7] (see also D’Ancona [3]), we obtain the following theorem:

**Theorem 1.4.** Under the same assumptions in Theorem [1.3],

$$\frac{1}{2\pi} \int_{\mathbb{R}} \| (A)^{-s} \hat{\varphi}(H)e^{-itH} \|^2 dt \leq c_0^{-1}\delta \left( \delta \tilde{R}^{1/2} + \delta(\tilde{R})^{s-1/2} \frac{(1 - s)^{1-s}(2 - s)}{(2s - 1)} \right)^2 e^{\delta c_2 \tilde{R}/c_0} \| \phi \|^2$$

holds for all $1/2 < s \leq 1$ and $\phi \in \mathcal{H}$, where $\hat{\varphi} \in C^\infty(\mathbb{R})$, which satisfies $\hat{\varphi}(s) = 1$ if $|s| \geq 5R$ and $= 0$ if $|s| \leq 4R$.

Theorems [1.2] and [1.3] yield the following corollary:

**Corollary 1.5.** Let $I := \sigma(H) \cap ((-\infty, -5R) \cup [5R, \infty))$. Then $I \subset \sigma_{ac}(H)$, where $\sigma_{ac}$ denotes a set of all absolutely continuous spectra of $H$.  

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Remark 1.6. As usual, $c_0 > 0$ is not a large number. For example, consider $H = p^2$, $p = -id/dx$, $\mathcal{H} = L^2(\mathbb{R})$ and the Yokoyama-type conjugate operator $A = (xp(1 + p^2)^{-1} + (1 + p^2)^{-1}px)/2, x \in \mathbb{R}$. Formally, it then follows that
\[
i[H, A]^0 = 2p^2(1 + p^2)^{-1} = 2 + O(R^{-1}), \quad i[[H, A]^0, A]^0 = O(R^{-2}).
\]
Hence, deducing the smallness exclusively with (1) may be difficult. On the other hand, by taking $R = 4$ and $s = 4$, (2) deduces $\sup_{|\lambda| \geq 3R} \| (A)^{-s} (H - \lambda \mp i\mu)^{-1} (A)^{-s} \phi \| < 1$, such that the estimate can be applied to an operator written as a form (selfadjoint operator + complex perturbation). See, e.g., Wang \cite{12} and the references therein.

Remark 1.7. Under some situations, it follows that
\[
\sup_{|\lambda| \geq 3R} \| (A)^{-s} (H - \lambda \mp i\mu)^{-1} (A)^{-s} \phi \| \to 0, \quad \text{as } R \to \infty,
\]
and it may be expected that the same estimate holds for the general Hamiltonian $H$ and its conjugate operator $A$. However, there are some counterexamples, e.g., by taking $H = p$ on $L^2(\mathbb{R})$. Then, $i[H, x]^0 = 1 \geq 0$, such that for any fixed $\lambda_0$ there exists $\nu > 0$, where
\[
\lim_{\mu \to 0^+} \| (x)^{-s} (H - \lambda_0 - i\mu)^{-1} (x)^{-s} \| = \nu \quad \text{(3)}
\]
holds. On the other hand, suppose that for all $\varepsilon > 0$, there exists $R = R_\varepsilon \gg 1$ such that
\[
\lim_{\mu \to 0^+} \| (x)^{-s} (H - R - i\mu)^{-1} (x)^{-s} \| \leq \varepsilon \quad \text{(4)}
\]
holds. Then, by the unitary transform $e^{i(R - \lambda_0)x}$, we notice that (3) is equivalent to
\[
\lim_{\mu \to 0^+} \| (x)^{-s} (H - \lambda_0 - i\mu)^{-1} (x)^{-s} \| \leq \varepsilon,
\]
which contradicts (4) by taking a sufficiently small $\varepsilon > 0$ compared to $\nu$. Hence, some additional assumptions are needed with regard to $H$, $A$, and $\varphi$ in order to prove the high-energy decay property for the resolvent. (The decaying estimate with a micro local parameter has been proven by Royer \cite{10}.)

With a similar approach, we can prove a smoothing-type limiting absorption principle.

In this paper, we additionally assume the following statements:

Assumption 1.8. Suppose Assumption \cite{11} and let $B := i[H, A]^0$. Then, commutators
\[
ad_A^2(H) := [B, A]^0, \quad \ad_A^3(H) := [B, A]^0, A]^0, \quad \ad_A^4(H) := i[\ad_A^3(H), A]^0
\]
are well-defined and can be extended to bounded operators. Moreover, for some $c_0 \leq \tilde{c} \leq c_1$, the commutator $i[H, A]^0$ can be written as
\[
i[H, A]^0 = \tilde{c}I + J + K, \quad \text{(5)}
\]
where $I$ is the identity operator on $\mathcal{H}$, and $J$ and $K$ are bounded operators. Moreover, they satisfy the following four conditions:

(g). For large $R$, there exists a sufficiently small constant $\delta > 0$, such that
\[
\| K \varphi(H) \| \leq \delta.
\]

(h). $J$ commutes with $H$.

(i). For the same $0 \leq \beta \leq 1/2$, $[\langle H \rangle^\beta, K]^0$ is a bounded operator.

(k). For the same $0 \leq \beta \leq 1/2$, the operator $\langle H \rangle^\beta i[H, A]^0, A]^0 \langle H \rangle^\beta$ is a bounded operator.
**Theorem 1.9.** For the same $\beta$ in Assumption 1.8 let us define

$$P := \langle H \rangle^{2\beta} A + A \langle H \rangle^{2\beta}.$$  

Then, under Assumptions 1.1 and 1.8 for all $s > 1/2$, the following limiting absorption principle holds:

$$\sup_{|\lambda| \geq 3R, \mu > 0} \| \langle P \rangle^{-s} \langle H \rangle^{\beta} (H - \lambda \mp i\mu)^{-1} \langle H \rangle^{\beta} \langle P \rangle^{-s} \phi \| \leq C \| \phi \|. \quad (6)$$

Moreover, for all $1/2 < s \leq 1$, the high-energy smoothing estimates are

$$\int_{\mathbb{R}} \left\| \langle P \rangle^{-s} \langle H \rangle^{\beta} \tilde{\varphi}(H)e^{itH}\phi \right\|^2 dt \leq C \| \tilde{\varphi}(H)\phi \|^2,$$

where $\tilde{\varphi} \in C^\infty(\mathbb{R})$, which satisfies $\tilde{\varphi}(s) = 1$, if $|s| \geq 5R$, and $= 0$, if $|s| \leq 4R$.

**2 Preliminaries**

In this section, we introduce the most important Lemma in this paper:

**Lemma 2.1.** Under Assumption 1.1 the following statements hold:

(a1) $\mathcal{D}(A)$ reduces $\varphi(H)$, i.e.,

$$\varphi(H)\mathcal{D}(A) \subset \mathcal{D}(A).$$

(a2) $\| [\varphi(H), A] \| \leq CR^{-1}$ holds, (denoted by $[\varphi(H), A] = O(R^{-1})$).

(a3) Additionally, if we suppose Assumption 1.8 then for all $0 \leq \beta \leq 1/2$, $[\langle H \rangle^{2\beta}, i[H, A]^{0}]^{0}$ is a bounded operator. Moreover, $[\langle H \rangle^{\beta}, A]^{0} \langle H \rangle^{\beta}$ is also a bounded operator.

**Proof.** We employ so-called commutator expansion: Let $A_0$ and $B_0$ be selfadjoint operators with

$$\| i[A_0, B_0]^{0} \| < \infty, \quad \| \text{ad}^{j}_{A_0}(B_0) \| < 0,$$

for some integer $2 \leq j$. For $0 \leq \rho \leq 1$, suppose $f \in C^\infty(\mathbb{R})$ satisfies $|\hat{\phi}_s^k f(s)| \leq C_k \langle s \rangle^{\rho-k}$, $k \geq 0$. Then,

$$i[f(A_0), B_0] = \sum_{k=1}^{j-1} \frac{1}{k!} f^{(k)}(A_0) \text{ad}^{k}_{A_0}(B_0) + R_j(f, A_0, B_0)$$

where $R_j(f, A_0, B_0)$ satisfies

$$\| (A_0 + i)^{j-1} R_j(f, A_0, B_0) \| \leq C(f^{(j)}) \| \text{ad}^{j}_{A_0}(B_0) \|.$$

The proof of this lemma is given in Sigal–Soffer [11] and as Lemma C.3.1 in Dereziński and Gérard [4]. Commutator expansion with $j = 2$ is such that

$$A_0 \varphi(H) \langle A \rangle^{-1} = [A, \varphi(H)] \langle A \rangle^{-1} + \varphi(H) A \langle A \rangle^{-1}.$$
will be a bounded operator, since \( \varphi' \) and \( \varphi'' \) are bounded functions. Moreover, by the construction of \( \varphi \), we have \(|\varphi'|, |\varphi''| \leq O(R^{-1})\), and hence (a1) and (a2) are proven. Now we prove (a3): by \( 2\beta \leq 1 \) and Assumption \( \|B, H\|^0 \leq C \) with \( B = i[H, A]^0 \), we can use commutator expansion to a pair of operators, \( B \) and \( \langle H \rangle^{2\beta} \). Hence, \( \|\langle H \rangle^{2\beta}, B\|^0 \leq C \).

Moreover, formally we have

\[
i[H, A]^0 \langle H \rangle^\beta = i \langle H \rangle^\beta A \langle H \rangle^\beta - i A \langle H \rangle^{2\beta} = \langle H \rangle^\beta i[A, \langle H \rangle^\beta]^0 + i\langle H \rangle^{2\beta}, A\|^0.
\]

By commutator expansion with \( j = 2 \), we have \( \langle H \rangle^\beta i[A, \langle H \rangle^\beta]^0 \) as a bounded operator, since \( |\langle s \rangle^\beta (\langle s \rangle^\beta)'| \leq C \langle s \rangle^{2\beta-1} \leq C. \)

3 Proof of Theorems

We now prove Theorems 1.2 and 1.3. The proof for Theorem 1.3 involves imitates the approach of [9] (see Isozaki [6]).

3.1 Proof of Theorem 1.2

Let \( \lambda_0 \in (-\infty, -3R] \cup [3R, \infty) \) be an eigenvalue of \( H \), and let \( \psi \in \mathcal{H} \) be an eigenfunction: i.e., \( H \psi = \lambda_0 \psi \). Then, using Virial's theorem, Assumption (e), and \( \varphi(\lambda_0) = 1 \), we have

\[
0 = (i[\lambda_0, A]\varphi(H)\psi, \varphi(H)\psi) = (i[H, A]^0 \varphi(H)\psi, \varphi(H)\psi) \geq c_0 \|\varphi(H)\psi\|^2 = c_0 \|\psi\|^2
\]

which implies \( \psi \equiv 0 \).

3.2 Proof of Theorem 1.3

For a small parameter \( \varepsilon > 0 \), \( \mu > 0 \) and a large parameter \( \lambda \in \mathbb{R} \) with \( |\lambda| \geq 3R \), we define

\[
G(\varepsilon) := (H - \lambda - i\mu - i\varepsilon M^* M)^{-1},
\]

\[
T(\varepsilon) := H - \lambda - i\mu - i\varepsilon M^* M
\]

with

\[
M = (\varphi(H)i[H, A]^0 \varphi(H))^{1/2}.
\]

The following lemma immediately holds:

**Lemma 3.1.** Under Assumption 1.1, the following statements hold:

(b1). \( T(\varepsilon) \) is a closed operator, and for all \( z \in \mathbb{C}_+ \), \( 0 \in \rho(T(\varepsilon)) \), and \( G(\varepsilon) \) is analytic on \( C_+ \).

(b2). Let \( M_1 \) satisfy \( M_1^* M_1 \leq M^* M \). Then, for all bounded operators \( \mathcal{B} \),

\[
\|M_1 G(\varepsilon) \mathcal{B}\| \leq \varepsilon^{-1/2} \|B G(\varepsilon) \|^{1/2}
\]

holds.

(b3). For some \( 0 < \varepsilon < 1 \), suppose that \( \varepsilon < R^{1-\delta} \). Then there exists \( \delta = \delta(R) \), with \( \delta \to 1 \) as \( R \to \infty \) such that \( \|G(\varepsilon)\| \leq \delta(c_0 \varepsilon)^{-1} \).

(b4). \( ||(\varphi(H) - 1)G(\varepsilon)|| \leq CR^{-1} \).
Proof. We only need to prove (b3) and (b4). Proofs for (b1) and (b2) can be found in [9] and [2] (or see the proof for Lemma 4.2 with $\beta = 0$). We divide $G(\varepsilon)$ into $\varphi(H)G(\varepsilon) + (1 - \varphi(H))G(\varepsilon)$. By (b2) with $(\varphi(H)\sqrt{c_0})^2 \leq M^*M$ and $B = 1$, we have $\|\varphi(H)G(\varepsilon)\| \leq (c_0\varepsilon)^{-1/2}\|G(\varepsilon)\|^{1/2}$. On the other hand, by

$$G(\varepsilon) - (H - \lambda - i\mu)^{-1} = i\varepsilon(H - \lambda - i\mu)^{-1}M^*MG(\varepsilon)$$

and for $|\lambda| \geq 3R$,

$$\|\varphi(H) - 1)(H - \lambda - i\mu)^{-1}\| \leq R^{-1},$$

holds. Thus,

$$\|\varphi(H) - 1)G(\varepsilon)\| \leq R^{-1} + \varepsilon c_1 R^{-1}\|G(\varepsilon)\|.$$  (7)

Hence, we derive inequality

$$\|G(\varepsilon)\| \leq (c_0\varepsilon)^{-1/2}\|G(\varepsilon)\|^{1/2} + R^{-1} + \varepsilon c_1 R^{-1}\|G(\varepsilon)\|.$$  

By assuming $\varepsilon c_1 \leq R^{1-\delta}$ for some $0 < \delta < 1$, we get

$$\|G(\varepsilon)\| \leq \varepsilon^{-1}c_0^{-1}(1 - \frac{\varepsilon c_1}{R})^{-1} \leq \varepsilon^{-1}c_0^{-1}(1 - R^{-\delta})^{-1} = \varepsilon^{-1}c_0^{-1}\delta.$$ 

By using (b3) for (7), we immediately obtain (b4).

Define

$$W(\varepsilon) := (|A| + 1)^{-s}(\varepsilon|A| + 1)^{s-1}$$

and

$$F(\varepsilon) := W(\varepsilon)G(\varepsilon)W(\varepsilon).$$

For simplicity, we denote $W(\varepsilon) = W$, $F(\varepsilon) = F$, $G(\varepsilon) = G$, $i[H, A]^0 = B$ and $d/d\varepsilon = \cdot$. Then,

$$-iF' = -i(W'GW + WG') + WGM^2GW =: \sum_{j=1}^{4} L_j$$

holds, where

$$L_1 = -i(W'GW + WG')$$

$$L_2 = W\varphi(\varphi - 1)B(\varphi - 1)GW;$$

$$L_3 = W\varphi(\varphi - 1)BGW + WGB(\varphi - 1)GW.$$  

$$L_4 = WGBGW.$$  

First, we prove that

$$\|W'\| \leq (1 - s)c(s)\varepsilon^{s-1},$$  (8)

$$\|AW\| \leq \varepsilon^{s-1},$$  (9)

\[7\]
where \( c(s) := (1 - s)^{1-s}(2 - s)^{s-2} \leq 1 \), because
\[
\| A(\| A \| + 1)^{-s} \| = \sup_{\lambda \geq 0} \lambda (\lambda + 1)^{-s}(\varepsilon \lambda + 1)^{s-2} = \varepsilon^{s-1} \sup_{t \geq 0} t^{1-s}(t + 1)^{s-2}
\]
provides (5). Second, we estimate \( L_j \), \( j = 1, 2, 3, \) and Lemma 3.1(b2) provides
\[
\| \varphi(H)GW \| \leq c_0^{-1/2} \varepsilon^{-1/2} \| WG \|^{1/2} = (c_0 \varepsilon)^{-1/2} \| F \|^{1/2}.
\]
By Lemma 3.1(b4),
\[
\| (1 - \varphi(H))GW \| \leq CR^{-1}
\]
holds, and thus we have
\[
\| GW \| \leq (c_0 \varepsilon)^{-1/2} \| F \|^{1/2} + O(R^{-1}).
\]
By (8) and (11), we have
\[
\| L_1 \| \leq 2(1 - s)c(s) \varepsilon^{s-1}(O(R^{-1}) + \| F/(\varepsilon c_0) \|^{1/2}).
\]
By (10) and \( \| B \| \leq C \), we have
\[
\| L_2 \| \leq O(R^{-2}).
\]
By (10), (11), and \( \| B \| \leq C \), we have
\[
\| L_3 \| \leq O(R^{-1})(O(R^{-1}) + \| F/(\varepsilon c_0) \|^{1/2}) = O(R^{-2}) + O(R^{-1})\| F/(\varepsilon c_0) \|^{1/2}
\]
We divide \( -iL_4 \) into \( L_5 + L_6 \) with
\[
L_5 := WAGW - WGAW; \quad L_6 := i\varepsilon W[G^2, A]GW.
\]
By (9) and (11),
\[
\| L_5 \| \leq 2c(s) \varepsilon^{s-1}(O(R^{-1}) + \| F/(\varepsilon c_0) \|^{1/2}).
\]
By Lemma 2.1(a2) and
\[
[M^2, A] = [\varphi(H), A][H, A][\varphi(H) + \varphi(H)i[H, A], A] \varphi(H) + \varphi(H)i[H, A][\varphi(H), A],
\]
we have
\[
\| L_6 \| \leq \varepsilon (O(R^{-1}) + \| F/(\varepsilon c_0) \|^{1/2})^2 (O(R^{-1}) + c_2) \leq ((c_2/c_0) + O(R^{-1}))\| F \| + \varepsilon^{1/2}O(R^{-1})\| F \|^{1/2} + \varepsilon O(R^{-2}).
\]
Then, for some $\delta = \delta_R > 1$ with $\delta \to 1$ as $R \to \infty$,
\[
\| F' \| \leq \sum_{j=1}^{6} \| L_j \| \leq 2\delta(2-s)c(s)c_{0}^{-1/2} \varepsilon s^{-3/2}\| F \|^{1/2} + (\delta c_{2}/c_{0})\| F \| + \mathcal{O}(R^{-1})
\]
holds. By following the argument in [3], for all $0 < \tilde{R} \leq R^{1-\varepsilon}$ and for some $\delta$ it holds that
\[
F(\varepsilon) \leq \left( (\delta F(\tilde{R}))^{1/2} + \frac{1}{2} \int_{\varepsilon}^{\tilde{R}} \left( 2\delta(2-s)c(s)c_{0}^{-1/2} t^{s-3/2} \right) e^{-\delta c_{2}(\tilde{R}-t)/(2c_{0})} dt \right)^{2} e^{\delta c_{2}/c_{0}},
\]
which proves (1). Here, we remark that
\[
(2-s)c(s)(2s-1)^{-1} = (1-s)^{-1-s}(2-s)(2s-1)^{-1} \geq 1.
\]
Next we prove (2). Except for replacing $W(\varepsilon)$ to $W := (1+|A|)^{-s}$, the approach is the same as in the proof for (1). The important estimate is
\[
\| W' \| = 0, \quad \text{and} \quad \| WA \| \leq s^{-s}(s-1)^{-1} = \left( 1 - \frac{1}{s} \right)^{s} (s-1)^{-1} =: C(s).
\]
Then, it follows that
\[
\| L_{1} \| \leq 0, \quad \text{and} \quad \| L_{5} \| \leq 2C(s)(\mathcal{O}(R)^{-1} + \| F/(\varepsilon c_{0}) \|^{1/2})
\]
Hence, we have
\[
\| F' \| \leq 2\delta C(s)c_{0}^{-1/2} \varepsilon^{-1/2}\| F \|^{1/2} + (\delta c_{2}/c_{0})\| F \| + \mathcal{O}(R^{-1}).
\]
Let $\tilde{R}$ be a positive constant with $\tilde{R} < R$ and then
\[
F(\varepsilon) \leq \left( (\delta F(\tilde{R}))^{1/2} + \frac{1}{2} \int_{\varepsilon}^{\tilde{R}} \left( 2\delta C(s)c_{0}^{-1/2} t^{-1/2} \right) e^{-\delta c_{2}(\tilde{R}-t)/(2c_{0})} dt \right)^{2} e^{\delta c_{2}(\tilde{R}-\varepsilon)/c_{0}},
\]
holds, which gives (2).

4 **Smoothing estimate with high-energy cut-off**

In this section, we prove Theorem 1.9. In the case where $\varphi \in C_{0}^{\infty}(\mathbb{R})$, a more generalized estimate was obtained by Moller-Skibsted [3]. For $0 \leq \beta \leq 1/2$, we define
\[
G_{\beta}(\varepsilon) := (\mathcal{H} - \lambda - i\mu - i\varepsilon \varphi(H) (\mathcal{H})^{\beta} i[H, A]^{0} (\mathcal{H})^{\beta} \varphi(H))^{-1},
\]
\[
T_{\beta}(\varepsilon) := \mathcal{H} - \lambda - i\mu - i\varepsilon \varphi(H) (\mathcal{H})^{\beta} i[H, A]^{0} (\mathcal{H})^{\beta} \varphi(H),
\]
\[
M_{\beta} := (\varphi(H) (\mathcal{H})^{\beta} i[H, A]^{0} (\mathcal{H})^{\beta} \varphi(H))^{1/2},
\]
where we always assume that $\mu > 0$ and $\varepsilon > 0$ are sufficiently small. For simplicity, we may denote
\[
\langle \mathcal{H} \rangle^{\beta} \varphi(H) = g_{\beta}(H) = g(H), \quad M_{\beta} = M.
\]
Lemma 4.1. For any fixed $\varepsilon$ and $\mu$, we define $H(\varepsilon) := \langle H \rangle^\beta G(\varepsilon) \langle H \rangle^\beta$. Then, $H(\varepsilon)$ satisfies

$$\|H(\varepsilon)\| \leq C(\mu^2/\lambda^4 + \varepsilon^2)^{-1/2}.$$  

Proof. Here, let us consider the operator

$$\hat{T} := \langle H \rangle^{-\beta} (H - \lambda - i\mu - i\varepsilon M^* M) \langle H \rangle^{-\beta}.$$  

Clearly, $\hat{T}$ can be defined on $D(\langle H \rangle^{1-2\beta})$. Then, for all $u \in D(\langle H \rangle^{1-2\beta})$ with $\|u\| = 1$,

$$\left\| \hat{T} u \right\|^2 = \left\| \langle H \rangle^{-\beta} (H - \lambda) \langle H \rangle^{-\beta} u \right\|^2 + \left\| \langle H \rangle^{-\beta} (\mu + \varepsilon M^* M) \langle H \rangle^{-\beta} u \right\|^2 \geq I_1 + I_2 + I_3,$$

where

$$I_1 := \left\| \langle H \rangle^{-\beta} (H - \lambda) \langle H \rangle^{-\beta} u \right\|^2,$$

$$I_2 := \mu^2 \left\| \langle H \rangle^{-2\beta} u \right\|^2 + \mu \varepsilon c_0 \|\varphi(H) u\|^2 + \varepsilon^2 c_0^2 \|\varphi(H)^2 u\|^2,$$

$$I_3 := -\varepsilon \left( i \langle H \rangle^{-\beta} (H - \lambda) \langle H \rangle^{-\beta}, \varphi(H) i[H, A]^0 \varphi(H) u, u \right).$$

By (5) and the boundedness of $\left\| \langle H \rangle^{-\beta} (H - \lambda) \langle H \rangle^{-\beta} u \right\|$, we get

$$I_3 \leq -\varepsilon \left( i \langle H \rangle^{-\beta} (H - \lambda) \langle H \rangle^{-\beta}, \tilde{c} \varphi(H)^2 + J \varphi(H)^2 \right) \langle H - \lambda \rangle \langle H \rangle^{-\beta} \varphi(H) u.$$  

By the assumption of $J$ and $K$, we have

$$\|K \varphi(H)\| \leq \delta, \quad \text{and} \quad i \langle H \rangle^{-\beta} (H - \lambda) \langle H \rangle^{-\beta}, \tilde{c} \varphi(H)^2 + J \varphi(H)^2 \right) = 0,$$

and we also have

$$|I_3| \leq 4 \varepsilon \delta \| \langle H \rangle^{-\beta} (H - \lambda) \langle H \rangle^{-\beta} (1 - \varphi(H)) \varphi(H) u\| \quad + 4 \varepsilon \delta \| \langle H \rangle^{-\beta} (H - \lambda) \langle H \rangle^{-\beta} \varphi(H)^2 u\|.$$

Here, we define

$$I_4 := \left\| \langle H \rangle^{-\beta} (H - \lambda) \langle H \rangle^{-\beta} (1 - \varphi(H)^2) u \right\|^2,$$

$$I_5 := \left\| \langle H \rangle^{-\beta} (H - \lambda) \langle H \rangle^{-\beta} (1 - \varphi(H)) u \right\|^2,$$

$$I_6 := \left\| \langle H \rangle^{-\beta} (H - \lambda) \langle H \rangle^{-\beta} \chi(|H| \leq 2\lambda) \varphi(H)^2 u \right\|^2,$$

$$I_7 := \left\| \langle H \rangle^{-\beta} (H - \lambda) \langle H \rangle^{-\beta} \varphi(H)(1 - \varphi(H)) u \right\|^2.$$
where we use $\varphi(H)\chi(|H| \geq 2\lambda) = \chi(|H| \geq 2\lambda)$. Then, we have $I_1 \geq I_4 + I_5 + I_6$, $I_3 \leq 4\varepsilon\delta(\sqrt{I_6} + \sqrt{I_6} + \sqrt{I_7})$ and

$$\begin{align*}
I_5/2 - 4\varepsilon\delta\sqrt{I_5} &\geq -16\varepsilon^2\delta^2 = -16\varepsilon^2\delta^2\|u\|^2 \\
&\geq -64\varepsilon^2\delta^2 (\|(1 - \varphi(H))u\|^2 + \|\varphi(H)\chi(|H| \leq 2\lambda)u\|^2 + \|\chi(|H| \geq 2\lambda)u\|^2).
\end{align*}$$

Hence, we have

$$\begin{align*}
I_1 - |I_3| \geq I_4 + I_5/2 + I_6 - 4\varepsilon\delta \left(\sqrt{I_6} + \sqrt{I_7}\right) - 64\varepsilon^2\delta^2 (\|(1 - \varphi(H))u\|^2 + \|\varphi(H)\chi(|H| \leq 2\lambda)u\|^2 + \|\chi(|H| \geq 2\lambda)u\|^2).
\end{align*}$$

We thus divide into

$$I_1 + I_2 + I_3 \geq J_1 + J_2 + J_3,$$

where

$$\begin{align*}
J_1 &:= I_6 - 4\varepsilon\delta\sqrt{I_6} - 64\varepsilon^2\delta^2 \|\chi(|H| \geq 2\lambda)u\|^2, \\
J_2 &:= I_4/2 + I_5/2 - 4\varepsilon\delta\sqrt{I_7} - 64\varepsilon^2\delta^2 (1 - \varphi(H))u\|^2, \\
J_3 &:= I_4/2 + I_2 - 64\varepsilon^2\delta^2 \|\varphi(H)\chi(|H| \leq 2\lambda)u\|^2.
\end{align*}$$

First, we estimate $J_1$. It is clear that for large $|y| \geq 2\lambda$

$$\left| \langle y - \lambda \rangle \langle y \rangle^{-2\beta} \right| \geq \frac{1}{2},$$

and hence by taking $R > 0$ to be sufficiently large, we have

$$\|\langle H \rangle^{-\beta} (H - \lambda) \langle H \rangle^{-\beta} \chi(|H| \geq 2\lambda)u\| \geq \|\chi(|H| \geq 2\lambda)u\| / 4 \tag{12}$$

which gives

$$I_6 - 64\varepsilon^2\delta^2 \|\chi(|H| \geq 2\lambda)u\|^2 - 4\varepsilon\delta\sqrt{I_6} \geq (1 - 1028\varepsilon^2\delta^2)I_6 - 4\varepsilon\delta\sqrt{I_6} \tag{13}$$

$$= v_1 \left(\sqrt{I_6} - \frac{2\varepsilon\delta}{v_1}\right)^2 - \frac{4\varepsilon^2\delta^2}{v_1},$$

where we assume $v_1 := (1 - 1028\varepsilon^2\delta^2) \geq 1/2$. If $\chi(|H| \geq 2\lambda)u \equiv 0$, then $\langle H \rangle = 0 = \|\chi(|H| \geq 2\lambda)u\|^2$, and if $\chi(|H| \geq 2\lambda)u \neq 0$, $\langle H \rangle$ implies $\sqrt{I_6} \geq 1/4$ then for some certain $\varepsilon$-independent positive constant $v_2 > 0$, $\langle H \rangle \geq v_2 = v_2 \|\chi(|H| \geq 2\lambda)u\|^2$

holds, which gives

$$J_1 \geq v_2 \|\chi(|H| \geq 2\lambda)u\|^2.$$

Next we estimate $J_2$. By the definition of $\varphi$, $\sqrt{I_7}$ can be divided into

$$\sqrt{I_7} = \|\langle H \rangle^{-\beta} (H - \lambda) \langle H \rangle^{-\beta} \varphi(H)(1 - \varphi(H))\chi(3R/2 \leq |H| \leq 2\lambda)u\|$$

$$+ \|\langle H \rangle^{-\beta} (H - \lambda) \langle H \rangle^{-\beta} \varphi(H)(1 - \varphi(H))\chi(|H| \leq 3R/2)u\|$$

$$=: I_8 + I_9.$$
On the support of \((1 - \varphi(H))\),
\[
\left\| (H)^{-\beta} (H - \lambda) (H - \beta) (1 - \varphi(H)) u \right\| \geq R(1 + 4R^2)^{-\beta} \| (1 - \varphi(H)) u \| \\
\geq 5^{-\beta} R^{1-2\beta} \| (1 - \varphi(H)) u \|
\]
holds. Hence, there is a positive constant \(v_3 \geq 1/4\) such that
\[
I_4/2 - 64 \varepsilon^2 \delta^2 \| (1 - \varphi(H)) u \|^2 - 4 \varepsilon \delta I_9 \geq v_3 I_4 - 4 \varepsilon \delta I_9
\]
(14)
\[
\geq v_3 \left\| (H)^{-\beta} (H - \lambda) \langle H \rangle^{-\beta} (1 - \varphi(H)) \chi(|H| \leq 3R/2) u \right\|^2 \\
- 4 \varepsilon \delta \left\| (H)^{-\beta} (H - \lambda) \langle H \rangle^{-\beta} (1 - \varphi(H)) \chi(|H| \leq 3R/2) u \right\|^2
\]
holds. On the support of \(\chi(|H| \leq 3R/2)\), \((1 - \varphi(H))\) is not always 0, since \(\varphi'(s) > 0\) on \(R < |s| < 2R\), which gives (14) \(\geq 0\), by selecting \(\varepsilon \delta > 0\) enough small. In the same manner, we get \(I_5/2 - 4 \varepsilon \delta I_8 \geq 0\). Thus, we have \(J_2 \geq 0\). These yield
\[
I_1 + I_2 + I_3 \geq v_2 \right\| \chi(|H| \geq 2\lambda) u \right\|^2 + J_3
\]
Since \(J_3\) is greater than
\[
\frac{1}{4} \left\| (H)^{-\beta} (H - \lambda) \langle H \rangle^{-\beta} (1 - \varphi(H)^2) u \right\|^2 \\
+ \left( \varepsilon^2 c_0^2 \right) \| \chi(3R/2 \leq |H| \leq 2\lambda) \varphi(H)^2 u \|^2 - 64 \varepsilon^2 \delta^2 \| \chi(3R/2 \leq |H| \leq 2\lambda) \varphi(H) u \|^2 \\
+ \left( \frac{1}{4} \right) \left\| (H)^{-\beta} (H - \lambda) \langle H \rangle^{-\beta} (1 - \varphi(H)^2) \chi(|H| \leq 3R/2) u \right\|^2 \\
- 64 \varepsilon^2 \delta^2 \| \chi(|H| \leq 3R/2) \varphi(H) u \|^2
\]
\[
\geq v_4 \| (1 - \varphi(H)^2) u \|^2 + \mu^2 \langle \lambda \rangle^{-4\beta} \| \varphi(H)^2 \chi(|H| \leq 2\lambda) u \|^2 \\
+ v_5 \varepsilon^2 c_0^2 \| \chi(3R/2 \leq |H| \leq 2\lambda) \varphi(H)^2 u \|^2 + v_6 \| \chi(|H| \leq 3R/2) u \|^2 \\
\geq v_4 \| (1 - \varphi(H)^2) u \|^2 + \varepsilon^2 + (\mu / |\lambda|^{2\beta}) \| \varphi(H)^2 \chi(|H| \leq 2\lambda) u \|^2
\]
for some \((\varepsilon, \mu, \lambda)\)-independent positive constants \(v_4, v_5, v_6, \) and \(v_7\). Finally, we obtain
\[
I_1 + I_2 + I_3 \geq v_4 \| (1 - \varphi(H)^2) u \|^2 + \varepsilon^2 + (\mu / |\lambda|^{2\beta}) \| \varphi(H)^2 \chi(|H| \leq 2\lambda) u \|^2 \\
+ \varepsilon^2 + (\mu / |\lambda|^{2\beta}) \| u \|^2 \\
\geq C \varepsilon^2 + (\mu / |\lambda|^{2\beta}) \| u \|^2
\]
(15)
where we take \(0 < \delta^2 \ll c_0^2\) and we use \(\varphi^2 \leq \varphi\). Inequality (15) entails that there exists a strict inverse operator of \(T\):
\[
\hat{T}^{-1} = (H)^{\beta} (H - \lambda - i\mu - i\varepsilon M^* M)^{-1} \langle H \rangle^{\beta}
\]
satisfying
\[
\| \hat{T} \| \leq C((\mu / |\lambda|^{2\beta})^2 + \varepsilon^2)^{-1/2}.
\]
\]
Lemma 4.2. The following statements hold:

(c1). Operator $H(\varepsilon) := \langle H \rangle^{\beta} G_\beta(\varepsilon) \langle H \rangle^{\beta}$ is differentiable in $\varepsilon$ and satisfies

$$H'(\varepsilon) = i \langle H \rangle^{\beta} G_\beta(\varepsilon) \Gamma^{\beta} G_\beta(\varepsilon) \langle H \rangle^{\beta}.$$ 

(c2). Let $Q_1$ be an operator satisfying $\mathcal{D}(\langle H \rangle^{\beta}) \subset \mathcal{D}(Q_1)$ and $Q_1^* Q_1 \leq M_2^{\beta} M_\beta$. Then, for all bounded selfadjoint operators $\mathcal{B}$,

$$\|Q_1 G_\beta(\varepsilon) \langle H \rangle^{\beta} \mathcal{B}\| \leq \varepsilon^{-1/2} \|\mathcal{B}\| \|\|G_\beta(\varepsilon) \langle H \rangle^{\beta}\|\|H(\varepsilon)\|^{1/2}$$

holds.

(c3). $\|G_\beta(\varepsilon)\| \leq \|\langle H \rangle^{\beta} G_\beta(\varepsilon) \langle H \rangle^{\beta}\| \leq C e^{-1}$.

(c4). $\|\langle H \rangle^{\beta} (1 - \varphi(H)) G_\beta(\varepsilon) \langle H \rangle^{\beta}\| \leq C$.

Proof. The resolvent formula

$$\langle H \rangle^{\beta} G_\beta(\varepsilon) \langle H \rangle^{\beta} - \langle H \rangle^{\beta} G_\beta(\varepsilon') \langle H \rangle^{\beta} = i(\varepsilon - \varepsilon') \langle H \rangle^{\beta} G_\beta(\varepsilon) \Gamma^{\beta} G_\beta(\varepsilon') \langle H \rangle^{\beta}$$

gives (c1). By a similar calculation and $\mu > 0$, we have

$$(i/2\varepsilon) \mathcal{B} \langle H \rangle^{\beta} (G_\beta(\varepsilon)^* - G_\beta(\varepsilon)) \langle H \rangle^{\beta} \mathcal{B} \geq \mathcal{B} \langle H \rangle^{\beta} G_\beta(\varepsilon)^* M_2^{\beta} M_\beta G_\beta(\varepsilon) \langle H \rangle^{\beta} \mathcal{B}$$

$$\geq \mathcal{B} \langle H \rangle^{\beta} G_\beta(\varepsilon)^* Q_1^* Q_1 G_\beta(\varepsilon) \langle H \rangle^{\beta} \mathcal{B},$$

which proves (c2), where we remark that

$$\|Q_1 G_\beta(\varepsilon) \langle H \rangle^{\beta} \mathcal{B}\| \leq \|Q_1 \langle H \rangle^{\beta} \mathcal{B}\| \|\|G_\beta(\varepsilon) \langle H \rangle^{\beta}\|\|H(\varepsilon)\|$$

is well defined. By \[\[\],

$$\|g(H) G_\beta(\varepsilon) \langle H \rangle^{\beta}\| \leq Ce^{-1/2} \|\langle H \rangle^{\beta} G_\beta(\varepsilon) \langle H \rangle^{\beta}\|^{1/2}$$

holds. Since $(1 - \varphi) \in C_0^\infty(\mathbb{R})$, $|(1 - \varphi(a))(a - \lambda - i\mu)^{-1} (a)^{2\beta}| \leq C$ and

$$G_\beta(\varepsilon) - (H - \lambda - i\mu)^{-1} = (H - \lambda - i\mu)^{-1} (i\varepsilon M_2^{\beta}) G_\beta(\varepsilon)$$

we also have

$$\|(1 - \varphi(H)) \langle H \rangle^{\beta} G_\beta(\varepsilon) \langle H \rangle^{\beta}\| \leq C(1 + \varepsilon \|H(\varepsilon)\|).$$

Consequently we get $\|H(\varepsilon)\| \leq C(1 + \varepsilon^{-1/2} \|H(\varepsilon)\|^{1/2} + \varepsilon \|H(\varepsilon)\|)$, which proves (c3) and (c4).

\[\]

4.1 Proof of Theorem \[\[\]

Now, we prove \[\]. The fundamental approach is exactly the same as in the proof of \[\].

We define

$$P = P_\beta := \langle H \rangle^{2\beta} A + A \langle H \rangle^{2\beta},$$

$$W = W_\beta(\varepsilon) := (1 + |P|)^{-1} (1 + |P|)^{\varepsilon} - 1,$$

$$G = G_\beta(\varepsilon) := (H - \lambda - i\mu - i\varepsilon \varphi(H) \langle H \rangle^{\beta} i[H, A]^0 \varphi(H) \langle H \rangle^{\beta})^{-1},$$

$$F = F_\beta(\varepsilon) := W_\beta(\varepsilon) \langle H \rangle^{\beta} G_\beta(\varepsilon) \langle H \rangle^{\beta} W_\beta(\varepsilon),$$

$$T = T_\beta(\varepsilon) := H - \lambda - i\mu - i\varepsilon \varphi(H) \langle H \rangle^{\beta} i[H, A]^0 \varphi(H) \langle H \rangle^{\beta} \varphi(H)$$

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For simplicity, we use notations $F_\beta(\epsilon) = F$, $G_\beta(\epsilon) = G$, $T_\beta(\epsilon) = T$, $\langle H \rangle^\beta = Z$, $B = i[H, A]^0$, $g = g(H) = Z\varphi(H)$, $M = (\varphi(H)ZBZ\varphi(H))^{1/2}$ and $d/d\epsilon = \epsilon'$. A straightforward calculation shows

$$-iF' = -i(W'ZGZW + WZGZW') + WZGM^2GZW =: \sum_{j=1}^4 L_j,$$

where

$L_1 = -i(W'ZGZW + WZGZW')$, 
$L_2 = WZG(\varphi - 1)ZBZ(\varphi - 1)GZW$, 
$L_3 = WZG(\varphi - 1)ZBZGZW + WZGZBZ(\varphi - 1)GZW$, 
$L_4 = WZGZBZGZW =: L_5 + L_6$ 
$L_5 = i(WZGZA\varphi ZGW - WZGZATZGZW)$, 
$L_6 = i\varphi WZGZgBg, A][ZGZW.$

By (8), Lemma 4.2(c4) and 

$$\|\varphi(H)ZGZW\| \leq C\epsilon^{-1/2} \|F\|^{1/2}$$

we have

$$\|L_1\| \leq C\epsilon^{-1}(1 + \epsilon^{-1/2} \|F\|^{1/2}).$$

By Lemma 4.2(c4),

$$\|L_2\| \leq C$$

holds. By Lemma 4.2(c4), we have

$$\|WZG(1 - \varphi)ZB\varphi GZW\| \leq C\|\varphi(H)ZGZW\|,$$

which yields

$$\|L_3\| \leq C(1 + \epsilon^{-1/2} \|F\|^{1/2}).$$

Now we estimate $\|L_4\|$. By (5), we have

$$T = H - \lambda - i\mu - i\varphi gBg$$

$$= H - \lambda - i\mu - i\varphi g^2 - i\varphi g J^2 - i\varphi g Kg$$

Hence,

$$ZT = TZ - i\varphi g[Z, K]g$$

holds and this yields

$$WZGZA\varphi ZGW = WZ^2A \cdot ZGZW - i\varphi WZGZ \cdot Z^{-1}g[Z, K]gA \cdot ZGZW.$$
By Assumption 1.8 condition (i), we have
\[ \| Z^{-1} g[Z, K] g A \| \leq \| Z, K \| Z A \| + C \leq C. \]

By \( 2Z^2 A = AZ^2 + Z^2 A + [Z^2, A] = P + \) (bounded operator), \( WZ^2 A = WP/2 + \) (bounded operator) is bounded. Consequently, we get
\[ \| L_5 \| \leq C(1 + \| F \|) \leq C. \]

Lemma 2.1 (a3) yields
\[ \| \langle H \rangle^\beta [\langle H \rangle^\beta, A] \| \leq C. \]

Moreover, by Assumption 1.8 condition (k),
\[ \| g[B, A] g \| \leq C \]
holds. Hence, we get
\[ \| L_5 \| \leq C(1 + \| F \|). \]

Consequently, we get (6).

5 Application to Schrödinger-type operators

We now apply the main theorem to dissipative operators. Unfortunately, complex and long calculations are needed in order to check all of the conditions stated in Assumption 1.8. Thus, we here only provide a sketch of the calculation. In what follows, let \(| \lambda | \gg 1\), \( \mathcal{H} = L^2(\mathbb{R}^n) \), \( p = -i \nabla \) on \( \mathbb{R}^n \), and where \( V \) is smooth and satisfies the condition that for all multi-index \( \alpha \) and for some \( \rho > 0 \), there exists a constant \( C_{\alpha} > 0 \) such that
\[ \left| \left\langle x \right\rangle^{\rho + |\alpha|} \partial_x^\alpha V(x) \right| \leq C_{\alpha} \]

Let us consider the Schrödinger-type operators
\[ H = a(p^2 + b^2)^\gamma + V, \]
where \( a > 0\), \( b \in \mathbb{R}\), and \( \gamma \geq 1/2\), and its conjugate operator
\[ A = (1 + (p^2 + b^2))^{-\gamma} p \cdot x + x \cdot p(1 + (p^2 + b^2))^{-\gamma}. \]

By simple calculation, formally, we have
\[ i[H, A] = 4\gamma a p^2 (p^2 + b^2)^{\gamma - 1} (1 + (p^2 + b^2))^{-\gamma} - x \cdot \nabla V \times (1 + p^2 + b^2)^{-\gamma} + \text{(similar terms),} \]
\[ i[i[H, A], A] = 16\gamma a p^2 (p^2 + b^2)^{\gamma - 1} (1 + p^2 + b^2)^{-2\gamma} + \text{(similar terms)} + x^2 (\Delta V)(1 + p^2 + b^2)^{-2\gamma} + \text{(similar terms) } \]
and
\[ i[H, i[H, A]] = -8\gamma a \nabla V \cdot p(p^2 + b^2)^{-\gamma-1}(1 + (p^2 + b^2))^{-\gamma} + \text{(similar terms)} \\
+ 2a\gamma p \cdot \nabla V \times (1 + p^2 + b^2)^{-\gamma} + \text{(similar terms)}. \]

Here, let \( R \) be sufficiently larger than \( C_\alpha \) with \( a, b, \) and \( \gamma \) as constants. Then, on the support of \( \varphi(H) \), it follows that \( H_0 = H - V \) is sufficiently large. Now we prove (5) and the four statements in Assumption 1.8. A straightforward calculation gives
\[ \phi \text{ support of } \gamma \text{ and } \beta \text{ and prove for } \beta \geq \beta^* \]
\[ \| P \langle \gamma \rangle^{-4\beta\gamma-1+2\gamma} \langle x \rangle^{-1} \| \leq C \]
(22)

Proof. Let us define
\[ \beta^* := \frac{2\gamma - 1}{4\gamma} \]
and prove for \( \beta \geq \beta^* \]
and for $\beta < \beta^*$
\[
\| P \langle x \rangle^{-1} \| \leq C. \tag{23}
\]

Here, we only consider the case of $\beta \geq \beta^*$. A simple calculation shows
\[
\langle H \rangle^{2\beta} x \cdot p (1 + p^2 + b^2) - \gamma = \langle H \rangle^{2\beta} (1 + p^2 + b^2) - \gamma ((\text{bounded operator}) + p \cdot x) = (\text{bounded operator}) + \langle H \rangle^{2\beta} (1 + p^2 + b^2) - \gamma p \cdot x.
\]

Moreover, by $[p \cdot x, \langle p \rangle^{-4\beta \gamma - 1 + 2\gamma}] = (\text{bounded operator})$, we have
\[
\left\| P \langle p \rangle^{-4\beta \gamma - 1 + 2\gamma} \langle x \rangle^{-1} \right\| = C + \left\| \langle H \rangle^{2\beta} (1 + p^2 + b^2) - \gamma \langle p \rangle^{-4\beta \gamma - 1 + 2\gamma + 1} \right\| \leq C.
\]

In the same way, for $1/2 < s \leq 1$,
\[
\left\| \langle p \rangle^s \langle \langle p \rangle^{-4\beta \gamma - s + 2\gamma} \langle x \rangle^{-s} \right\| \leq C.
\]

Now we prove that for $\theta = -4\beta \gamma + 2\beta - s - 2\gamma s \leq -4\beta s + 2\beta + s \leq 1$,
\[
\left\| \langle x \rangle^{-s} \langle p \rangle^\theta \langle H \rangle^{-\beta} \langle x \rangle^s \langle p \rangle^{4\beta \gamma + s + 2\gamma s} \right\| \leq C \tag{24}
\]
holds if $\beta \geq \beta^*$ and for $\theta = 2\beta \gamma$,
\[
\left\| \langle x \rangle^{-s} \langle p \rangle^\theta \langle H \rangle^{-\beta} \langle x \rangle^s \right\| \leq C \tag{25}
\]
holds if $\beta \leq \beta^*$. Again, we only consider the case of $\beta \geq \beta^*$. By Helffer–Sjöstrand’s formula (see [3]), we get
\[
\langle p \rangle^\theta \langle H \rangle^{-\beta} \langle x \rangle^s \langle p \rangle^{4\beta \gamma + s + 2\gamma s} = \langle p \rangle^\theta \langle x \rangle^s \langle H \rangle^{-\beta} \langle p \rangle^{4\beta \gamma + s + 2\gamma s} + \langle p \rangle^\theta \mathcal{L} \langle p \rangle^{4\beta \gamma + s + 2\gamma s}
\]
with
\[
\left\| \langle p \rangle^\theta \mathcal{L} \langle p \rangle^{4\beta \gamma + s + 2\gamma s} \right\| \leq C \left\| \langle p \rangle^\theta \langle x \rangle^{-s} \right\| \left\| \langle p \rangle^{4\beta \gamma + s + 2\gamma s - 2\gamma - 1} \right\| \leq C,
\]
where we use $\beta \geq \beta^*$, $\theta - 2\gamma \leq 2\beta - 2\gamma \leq 0$ and $4\beta \gamma + s - 2\gamma s - 1 \leq s - 1 \leq 0$.

Moreover, a symbol calculation of the pseudo-differential operator gives
\[
\left\| \langle x \rangle^{-s} \langle p \rangle^\theta \langle x \rangle^s \langle p \rangle^{-s} \right\| \leq C,
\]
which proves (24). By (22) and (24) or (23) and (25), we get
\[
\left\| \langle x \rangle^{-s} \langle p \rangle^\theta \tilde{\varphi} \langle H \rangle e^{-itH} \phi \right\| \leq \left\| \langle x \rangle^{-s} \langle p \rangle^\theta \langle H \rangle^{-\beta} \langle P \rangle^s \right\| \left\| \langle P \rangle^{\gamma - 1/2} \tilde{\varphi} \langle H \rangle e^{-itH} \phi \right\|
\leq C \left\| \langle P \rangle^{\gamma - 1/2} \tilde{\varphi} \langle H \rangle e^{-itH} \phi \right\|
\]
and, together with (21), we obtain (21).

By $\max_{0 \leq \beta \leq 1/2} \theta = \gamma - 1/2$, we finally obtain the following smoothing estimate:

**Theorem 5.2.** For the operator $H = a(p^2 + b^2) + V$ in (13),
\[
\sup_{|\lambda| \geq 3R} \left\| \langle x \rangle^{-s} \langle p \rangle^{\gamma - 1/2} (H - \lambda \mp 0)\langle p \rangle^{\gamma - 1/2} \langle x \rangle^{-s} \phi \right\| \leq C \| \phi \|,
\]
\[
\int_{\mathbb{R}} \left\| \langle x \rangle^{-s} \langle p \rangle^{\gamma - 1/2} \tilde{\varphi} \langle H \rangle e^{-itH} \phi \right\|^2 dt \leq C \| \phi \|^2.
\]
holds.

If $\gamma = 1$, this estimate corresponds to the smoothing estimate for the Schrödinger operator with a high-energy cut-off (see, e.g., [1]).
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