**Abstract.** We evaluate certain multidimensional integrals in terms of the Lerch transcendent function \( \Phi \), generalizing Guillera-Sondow’s formulas. As an application, we get new representations of classical constants like Euler’s constant \( \gamma \) and \( \ln(4/\pi) \).

The *Lerch transcendent* \( \Phi \) is defined as the analytic continuation of the series

\[
\Phi(z, s, u) = \frac{1}{u^s} + \frac{z}{(u + 1)^s} + \frac{z^2}{(u + 2)^s} + \cdots,
\]

which converges for any complex number \( u \) with \( \Re u > 0 \) if \( z \) and \( s \) are any complex numbers with either \( |z| < 1 \), or \( |z| = 1 \) and \( \Re s > 1 \) (we suppose \( \zeta^s = \exp(s \log \zeta) \), where \( \log \zeta \) is the principal branch of the logarithm). The function \( \Phi \) is holomorphic in \( z \) and \( s \), for \( z \in \mathbb{C} \setminus [1, \infty] \) and all complex \( s \) (see [1, section 1.11] or [2, section 2]). From the definition it follows that

\[
\Phi(z, s, u + 1) = \frac{1}{z} \left( \Phi(z, s, u) - \frac{1}{u^s} \right),
\]

(1)

\[
\Phi(z, s + 1, u) = -\frac{1}{s} \frac{\partial \Phi}{\partial u}(z, s, u).
\]

(2)

In the paper [2] J. Sondow and J. Guillera proved the following two theorems.

**Theorem 1** Suppose \( u > 0, v > 0, u \neq v \), either \( z \in \mathbb{C} \setminus [1, \infty) \) and \( \Re s > -2 \), or \( z = 1 \) and \( \Re s > -1 \). Then

\[
\int_{[0,1]^2} \frac{x_1^{u-1} x_2^{v-1}}{1 - z x_1 x_2} (-\ln x_1 x_2)^s dx_1 dx_2 = \Gamma(s + 1) \frac{\Phi(z, s + 1, v) - \Phi(z, s + 1, u)}{u - v},
\]

\[
\int_{[0,1]^2} \frac{x_1^{u-1} x_2^{v-1}}{1 - z x_1 x_2} (-\ln x_1 x_2)^s dx_1 dx_2 = \Gamma(s + 2) \Phi(z, s + 2, u).
\]

**Theorem 2** Suppose \( u > 0 \), either \( z \in \mathbb{C} \setminus [1, \infty) \) and \( \Re s > -3 \), or \( z = 1 \) and \( \Re s > -2 \). Then

\[
\int_{[0,1]^2} \frac{1 - x_1}{1 - z x_1 x_2} (x_1 x_2)^{u-1} (-\ln x_1 x_2)^s dx_1 dx_2
\]

\[
= \Gamma(s + 2) \left[ \Phi(z, s + 2, u) + \frac{(1 - z) \Phi(z, s + 1, u) - u^{-s-1}}{z(s + 1)} \right].
\]

The purpose of this paper is to prove the following \( m \)-dimensional analogs of Theorems 1 and 2 (in what follows \( d\mathbf{x} \) means \( dx_1 dx_2 \cdots dx_m \), where \( m \) is the dimension of an integral).

**Theorem 3** Suppose \( m \) is a positive integer, \( \Re u > 0, \Re v > 0, u \neq v \), either \( z \in \mathbb{C} \setminus [1, \infty) \) and \( \Re s > -m \), or \( z = 1 \) and \( \Re s > 1 - m \). For the case \( m > 1 \) define the function

\[
F_{m,u,v}(x_1, x_2, \ldots, x_m) = (x_1 x_2 \cdots x_m)^{u-1} (x_1^{u-v} + (x_1 x_2)^{u-v} + \cdots + (x_1 x_2 \cdots x_{m-1})^{u-v}).
\]

Then

\[
\int_{[0,1]^m} \frac{F_{m,u,v}(x_1, x_2, \ldots, x_m)}{1 - z x_1 x_2 \cdots x_m} (-\ln x_1 x_2 \cdots x_m)^s d\mathbf{x}
\]

\[
= \frac{\Gamma(s + m - 1)}{(m - 2)!} \cdot \frac{\Phi(z, s + m - 1, v) - \Phi(z, s + m - 1, u)}{u - v} \quad \text{for } m > 1,
\]

(3)

\[
\int_{[0,1]^m} \frac{(x_1 x_2 \cdots x_m)^{u-1}}{1 - z x_1 x_2 \cdots x_m} (-\ln x_1 x_2 \cdots x_m)^s d\mathbf{x} = \frac{\Gamma(s + m)}{(m - 1)!} \Phi(z, s + m, u) \quad \text{for } m \geq 1.
\]

(4)
Theorem 4 Suppose $m$ is an integer $> 1$, $Re\ u > 0$, either $z \in \mathbb{C}\setminus[1, \infty)$ and $Re\ s > -m-1$, or $z = 1$ and $Re\ s > -m$. Then

$$\int_{[0,1]^m} \frac{m - 1 - x_1 - x_1 x_2 - \cdots - x_1 x_2 \cdots x_m}{1 - z x_1 x_2 \cdots x_m} (x_1 x_2 \cdots x_m)^{u-1} (-\ln x_1 x_2 \cdots x_m)^s d\tilde{x}$$

$$= \frac{\Gamma(s+m)}{(m-2)!} \left[ \Phi(z, s+m, u) + \frac{(1-z)\Phi(z, s+m-1, u) - u^{-s-m+1}}{z(s+m-1)} \right]. \quad (5)$$

In the case $m = 2$ Theorems 3 and 4 give Theorems 1 and 2. As an example, we give also the case $m = 3$.

Example 1 a) If $Re\ u > 0$, $Re\ v > 0$, $u \neq v$, either $z \in \mathbb{C}\setminus[1, \infty)$ and $Re\ s > -3$, or $z = 1$ and $Re\ s > -2$, then

$$\int_{[0,1]^3} \frac{(1-x_1 x_2 x_3) u-1}{1 - z x_1 x_2 x_3} (-\ln x_1 x_2 x_3)^s d\tilde{x} = \frac{\Gamma(s+3)}{2} \Phi(z, s+3, u).$$

b) If $Re\ u > 0$, either $z \in \mathbb{C}\setminus[1, \infty)$ and $Re\ s > -3$, or $z = 1$ and $Re\ s > -2$, then

$$\int_{[0,1]^3} \frac{(1-x_1 x_2 x_3)^{u-1}}{1 - z x_1 x_2 x_3} (-\ln x_1 x_2 x_3)^s d\tilde{x} = \Gamma(s+3) \left[ \Phi(z, s+3, u) + \frac{1-z)\Phi(z, s+2, u) - u^{-s-2}}{z(s+2)} \right].$$

c) If $Re\ u > 0$, either $z \in \mathbb{C}\setminus[1, \infty)$ and $Re\ s > -4$, or $z = 1$ and $Re\ s > -3$, then

$$\int_{[0,1]^3} \frac{2 - x_1 - x_1 x_2}{1 - z x_1 x_2 x_3} (x_1 x_2 x_3)^{u-1} (-\ln x_1 x_2 x_3)^s d\tilde{x}$$

$$= \Gamma(s+3) \left[ \Phi(z, s+3, u) + \frac{1-z)\Phi(z, s+2, u) - u^{-s-2}}{z(s+2)} \right].$$

In [2] many interesting applications of Theorems 1 and 2 are given. All of them can be generalized by Theorems 3 and 4; indeed, by these four theorems we have

$$\int_{[0,1]^2} \frac{x_1^{u-1} x_2^{v-1}}{1 - z x_1 x_2} (-\ln x_1 x_2)^s d\tilde{x}$$

$$= (m-2)! \int_{[0,1]^m} F_{m,u,v}(x_1, x_2, \ldots, x_m) \frac{(x_1 x_2 \cdots x_m)^{u-1}}{1 - z x_1 x_2 \cdots x_m} (-\ln x_1 x_2 \cdots x_m)^{s-m+2} d\tilde{x} \quad \text{for } m > 1,$$

$$\int_{[0,1]^2} \frac{(x_1 x_2)^{u-1}}{1 - z x_1 x_2} (-\ln x_1 x_2)^s d\tilde{x}$$

$$= (m-1)! \int_{[0,1]^m} \frac{(x_1 x_2 \cdots x_m)^{u-1}}{1 - z x_1 x_2 \cdots x_m} (-\ln x_1 x_2 \cdots x_m)^{s-m+2} d\tilde{x} \quad \text{for } m \geq 1,$$

$$\int_{[0,1]^2} \frac{1 - x_1}{1 - z x_1 x_2} (x_1 x_2)^{u-1} (-\ln x_1 x_2)^s d\tilde{x}$$

$$= (m-2)! \int_{[0,1]^m} \frac{m - 1 - x_1 - x_1 x_2 - \cdots - x_1 x_2 \cdots x_m}{1 - z x_1 x_2 \cdots x_m} (x_1 x_2 \cdots x_m)^{u-1} \times (-\ln x_1 x_2 \cdots x_m)^{s-m+2} d\tilde{x} \quad \text{for } m > 1.$$
Example 2 Let \( m \) be an integer \( > 1 \), and \( \gamma = \lim_{n \to \infty} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n \right) \) be Euler’s constant. Then

\[
\gamma = (m-2)! \int_{[0,1]^m} \frac{m-1 - x_1 - x_1 x_2 - \cdots - x_1 x_2 \cdots x_{m-1}}{(1 - x_1 x_2 \cdots x_m)(- \ln x_1 x_2 \cdots x_m)^{m-1}} \, dx.
\]

Example 3 For an integer \( m > 1 \) the following identity holds:

\[
\ln \frac{4}{\pi} = (m-2)! \int_{[0,1]^m} \frac{m-1 - x_1 - x_1 x_2 - \cdots - x_1 x_2 \cdots x_{m-1}}{(1 + x_1 x_2 \cdots x_m)(- \ln x_1 x_2 \cdots x_m)^{m-1}} \, dx.
\]

We omit details of proofs of these examples and refer to the case \( m = 2 \), which was considered by J. Sondow [3].

To prove Theorem 3, we will require two lemmas. The first is the identity (4) for \( m = 1 \), and is classical.

Lemma 1 Suppose \( \Re u > 0 \), either \( z \in \mathbb{C} \setminus [1, \infty) \) and \( \Re s > -1 \), or \( z = 1 \) and \( \Re s > 0 \). Then

\[
\int_0^1 \frac{x^{u-1}}{1 - zx} (-\ln x)^s \, dx = \Gamma(s+1) \Phi(z, s+1, u).
\]

Proof. The integral, call it \( I \), defines a holomorphic function of \( z \) and \( s \) under the conditions stated. We can prove the statement for \( |z| < 1 \) and \( \Re s > 0 \) and then use analytic continuation. Expand \( 1/(1 - zx) \) into a geometric series and then integrate:

\[
I = \sum_{n=0}^{\infty} z^n \int_0^1 x^{u+n-1} (-\ln x)^s \, dx.
\]

Making the substitution \( x = e^{-y} \), we obtain

\[
I = \sum_{n=0}^{\infty} z^n \int_0^\infty e^{-(u+n)y} y^s \, dy = \sum_{n=0}^{\infty} \frac{\Gamma(s+1)z^n}{(u+n)^{s+1}} = \Gamma(s+1) \Phi(z, s+1, u),
\]

and the lemma follows.

Lemma 2 Let \( \alpha \neq 0 \) and \( x \in (0, 1] \). Then the following identities hold for \( k \geq 1 \):

a)

\[
\int_{1 \geq t_1 \geq t_2 \geq \cdots \geq t_k \geq x} \frac{1}{t_1 t_2 \cdots t_k} \, dt_1 dt_2 \cdots dt_k = \frac{(-\ln x)^k}{k!},
\]

b)

\[
\int_{1 \geq t_1 \geq t_2 \geq \cdots \geq t_k \geq x} \frac{t_1^\alpha + t_2^\alpha + \cdots + t_k^\alpha}{t_1 t_2 \cdots t_k} \, dt_1 dt_2 \cdots dt_k = \frac{(-\ln x)^{k-1}}{(k-1)!} \cdot \frac{1 - x^\alpha}{\alpha}.
\]

Proof. The identity (6) is easily proved using induction and the equality

\[
\int_{1 \geq t_1 \geq t_2 \geq \cdots \geq t_k \geq x} \frac{dt_1 dt_2 \cdots dt_k}{t_1 t_2 \cdots t_k} = \int_x^1 \left( \int_{1 \geq t_1 \geq t_2 \geq \cdots \geq t_{k-1} \geq t_k} \frac{dt_1 dt_2 \cdots dt_{k-1}}{t_1 t_2 \cdots t_{k-1}} \right) \cdot \frac{dt_k}{t_k}
\]

Denote the integral in (7) by \( I_k(x) \). We prove by induction; the case \( k = 1 \) is true. Suppose \( k > 1 \) and the statement is true for \( k - 1 \). We have

\[
I_k(x) = \int_x^1 I_{k-1}(t_k) \frac{dt_k}{t_k} + \int_x^1 \left( \int_{1 \geq t_1 \geq t_2 \geq \cdots \geq t_{k-1} \geq t_k} \frac{dt_1 dt_2 \cdots dt_{k-1}}{t_1 t_2 \cdots t_{k-1}} \right) t_k^\alpha \, dt_k.
\]
Apply (6) to the integral in parentheses:

\[ I_k(x) = \int_x^1 I_{k-1}(t_k) \frac{dt_k}{t_k} + \int_x^1 \frac{(-\ln t_k)^{k-1}}{(k-1)!} t_k^{\alpha - 1} dt_k. \]

Using the induction hypothesis, we obtain

\[ I_k(x) = \int_x^1 \left( \frac{(-\ln t_k)^{k-2}}{(k-2)!} \cdot \frac{1 - t_k^{\alpha}}{\alpha t_k} + \frac{(-\ln t_k)^{k-1}}{(k-1)!} t_k^{\alpha - 1} \right) dt_k \]

\[ = \frac{(-\ln t_k)^{k-1} - t_k^{\alpha} - 1}{(k-1)!} \bigg|_x = \frac{(-\ln x)^{k-1}}{(k-1)!} - \frac{1 - x^{\alpha}}{\alpha}. \]

Now the lemma is completely proved.

**Proof of Theorem 3.** The integrals \( J_1 \) and \( J_2 \) in (3) and (4) define holomorphic functions of \( s \) under the conditions stated. We can prove the statement for \( \text{Re} s > 0 \) and then use analytic continuation.

First we prove (4). Make the substitution

\[ x_1 = t_1, \quad x_2 = t_2/t_1, \quad x_3 = t_3/t_2, \ldots, \quad x_m = t_m/t_{m-1} \]

in \( J_2 \). We obtain

\[ J_2 = \int_{1 \geq t_1 \geq t_2 \geq \cdots \geq t_m \geq 0} \frac{t_m^{\nu-1}}{1 - z t_m} \frac{1}{t_1 t_2 \cdots t_{m-1}} (-\ln t_m)^s dt_1 dt_2 \cdots dt_m \]

\[ = \int_0^1 \frac{t_m^{\nu-1}}{1 - z t_m} (-\ln t_m)^s \left( \int_{1 \geq t_1 \geq t_2 \geq \cdots \geq t_{m-1} - t_m} \frac{1}{t_1 t_2 \cdots t_{m-1}} dt_1 dt_2 \cdots dt_{m-1} \right) dt_m. \]

Applying (6) with \( x = t_m \) and \( k = m - 1 \), we get

\[ J_2 = \frac{1}{(m - 1)!} \int_0^1 \frac{t_m^{\nu-1}}{1 - z t_m} (-\ln t_m)^{s + m - 1} dt_m. \]

It remains to apply Lemma 1.

Now we prove (3). Denote \( \alpha = u - v \), then

\[ J_1 = \int_{[0,1]^m} \frac{(x_1 x_2 \cdots x_m)^{\nu-1}}{1 - z x_1 x_2 \cdots x_m} (x_1^{\alpha} + (x_1 x_2)^{\alpha} + \cdots + (x_1 x_2 \cdots x_{m-1})^{\alpha}) (-\ln x_1 x_2 \cdots x_m)^s dx. \]

Make the substitution (8)

\[ J_1 = \int_0^1 \frac{t_m^{\nu-1}}{1 - z t_m} (-\ln t_m)^s \left( \int_{1 \geq t_1 \geq t_2 \geq \cdots \geq t_{m-1} - t_m} \frac{t_1^{\alpha} + t_2^{\alpha} + \cdots + t_{m-1}^{\alpha}}{t_1 t_2 \cdots t_{m-1}} dt_1 dt_2 \cdots dt_{m-1} \right) dt_m \]

and apply (6)

\[ J_1 = \frac{1}{(m - 2)!} \alpha \left( \int_0^1 \frac{t_m^{\nu-1}}{1 - z t_m} (-\ln t_m)^{s + m - 2} dt_m - \int_0^1 \frac{t_m^{\nu + \alpha - 1}}{1 - z t_m} (-\ln t_m)^{s + m - 2} dt_m \right). \]

It remains to apply Lemma 1 to both integrals and get back to \( u \) from \( \alpha \). The theorem is proved.

**Remark.** The formula (4) can be also obtained by letting \( v \to u \) in (3) and using the identity (2).
Proof of Theorem 4. The integral $J$ in (5) defines a function which is holomorphic in $s$, when $\Re s > -m - 1$ if $z \in \mathbb{C} \backslash [1, \infty]$, and when $\Re s > -m$ if $z = 1$. We prove the statement for $\Re s > 0$ and then use analytic continuation. We have

$$J = (m - 1) \int_{[0,1]^m} \frac{(x_1 x_2 \cdots x_m)^{u-1}}{1 - zx_1 x_2 \cdots x_m} (-\ln x_1 x_2 \cdots x_m)^s d\vec{x}$$

$$- \int_{[0,1]^m} \frac{F_{m,u+1,u}(x_1, x_2, \ldots, x_m)}{1 - zx_1 x_2 \cdots x_m} (-\ln x_1 x_2 \cdots x_m)^s d\vec{x}.$$

Apply Theorem 3 to both integrals:

$$J = (m - 1) \frac{\Gamma(s + m)}{(m - 1)!} \Phi(z, s + m, u)$$

$$- \frac{\Gamma(s + m - 1)}{(m - 2)!} (\Phi(z, s + m - 1, u) - \Phi(z, s + m - 1, u + 1))$$

$$= \frac{\Gamma(s + m)}{(m - 2)!} \left[ \Phi(z, s + m, u) + \frac{\Phi(z, s + m - 1, u) - \Phi(z, s + m - 1, u + 1)}{(s + m - 1)} \right].$$

Use (1) and the theorem follows.

The way which we prove Theorem 3 can be applied to any integral

$$\int_{[0,1]^m} \frac{x_1^{u_1} x_2^{u_2} \cdots x_m^{u_m}}{1 - zx_1 x_2 \cdots x_m} (-\ln x_1 x_2 \cdots x_m)^s d\vec{x}.$$

We give the formula for the case when all $u_i$ are different.

Theorem 5 Suppose $m \geq 1$ and $\Re u_1 > 0$, $\Re u_2 > 0$, $\ldots$, $\Re u_m > 0$, $u_i \neq u_j$ whenever $i \neq j$, and either $z \in \mathbb{C} \backslash [1, \infty)$ and $\Re s > -1$, or $z = 1$ and $\Re s > 0$. Then the following identity holds:

$$\int_{[0,1]^m} \frac{x_1^{u_1-1} x_2^{u_2-1} \cdots x_m^{u_m-1}}{1 - zx_1 x_2 \cdots x_m} (-\ln x_1 x_2 \cdots x_m)^s d\vec{x} = \frac{\Gamma(s + 1)}{\prod_{j=1, j \neq i} (u_j - u_i)} \sum_{i=1}^m \Phi(z, s + 1, u_i).$$

To prove Theorem 5 we require the following

Lemma 3 Let $k \geq 1$ and $u_1, u_2, \ldots, u_{k+1}$ be arbitrary numbers with $u_i \neq u_j$ whenever $i \neq j$, and $x \in (0, 1]$. Then the following identity hold:

$$\int_{t_1 \geq t_2 \geq \cdots \geq t_k \geq x} t_1^{u_1-u_2-1} t_2^{u_2-u_3-1} \cdots t_k^{u_k-u_{k+1}-1} \, dt_1 dt_2 \cdots dt_k = \sum_{i=1}^{k+1} \frac{x^{u_i-u_{k+1}}}{\prod_{j=1, j \neq i} (u_j - u_i)}.$$
Thus the statement of the lemma is equivalent to the identity
\[
\sum_{i=1}^{k} \frac{1}{(u_i - u_{k+1})} \prod_{j=1, j \neq i}^{k} (u_j - u_i) = \frac{1}{\prod_{j=1}^{k} (u_j - u_{k+1})}.
\] (11)

To prove it, consider the polynomial
\[
P(x) = \sum_{i=1}^{k} \frac{\prod_{j=1, j \neq i}^{k} (u_j - x)}{\prod_{j=1, j \neq i}^{k} (u_j - u_i)},
\]
of degree \(k - 1\). We have \(P(u_i) = 1\) for any \(i \in \{1, 2, \ldots, k\}\). Hence \(P(x) \equiv 1\). The equality \(P(u_{k+1}) = 1\) yields (11) and the lemma follows.

**Proof of Theorem 5.** In the case \(m = 1\) the theorem is equivalent to Lemma 1. Now let \(m > 1\). Make the substitution (8) in the integral \(J\) in (9)
\[
J = \int_{t_1 \geq t_2 \geq \cdots \geq t_m \geq 0} \frac{t_{u_1}^{u_1-1}t_{u_2}^{u_2-1} \cdots t_{u_m}^{u_m-1}}{1 - zt_m} (-\ln t_m)^s dt_1 dt_2 \cdots dt_m.
\]

Applying (10) for \(k = m - 1\) and \(x = t_m\), we obtain
\[
J = \sum_{i=1}^{m} \frac{1}{\prod_{j=1, j \neq i}^{m} (u_j - u_i)} \int_{0}^{1} \frac{t_{u_i}^{u_i-1}}{1 - zt_m} (-\ln t_m)^s dt_m.
\]

Use Lemma 1 and the theorem follows.

**Remark.** Theorem 5 is another generalization of the first equality in Theorem 1.

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