Discrete Sparse Signals: Compressed Sensing by Combining OMP and the Sphere Decoder

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Abstract—We study the reconstruction of discrete-valued sparse signals from underdetermined systems of linear equations. On the one hand, classical compressed sensing (CS) is designed to deal with real-valued sparse signals. On the other hand, algorithms known from MIMO communications, especially the sphere decoder (SD), are capable to reconstruct discrete-valued non-sparse signals from well- or overdefined system of linear equations. Hence, a combination of both approaches is required. We discuss strategies to include the knowledge of the discrete nature of the signal in the reconstruction process. For brevity, the exposition is done for combining the orthogonal matching pursuit (OMP) with the SD; design guidelines are derived. It is shown that by suitably combining OMP and SD an efficient low-complexity scheme for the detection of discrete sparse signals is obtained.

I. INTRODUCTION

In some applications like multiple-access schemes with a very small number of active users (e.g., sensor networks) or peak-to-average power ratio reduction in orthogonal frequency-division multiplexing (OFDM) a discrete-valued sparse signal has to be estimated based on an under-determined system of linear equations. Even in source coding, the direct estimation of the quantized transform-domain coefficients may be beneficial.

Usually, in compressed sensing (CS) a real-valued s-sparse (column) vector \( \mathbf{x} \in \mathbb{R}^L \) has to be reconstructed from an under-determined system of linear equations \( \mathbf{y} = \mathbf{A} \mathbf{x} + \mathbf{n} \). Specifically, if \( \mathbf{A} \in \mathbb{R}^{K \times L} \) is the measurement matrix and \( \mathbf{y} = \mathbf{A} \mathbf{x} + \mathbf{n} \in \mathbb{R}^K \), \( K \ll L \), is the noisy observation (AWGN with variance \( \sigma_n^2 \) per component), the following problem has to be solved:

\[
\hat{x} = \arg\min_{\mathbf{z} \in \mathbb{R}^L} \| \mathbf{z} \|_0 \quad \text{with} \quad \| \mathbf{A} \hat{x} - \mathbf{y} \|_2 \leq \epsilon . \tag{1}
\]

This can be (approximately) done by using one of the standard CS algorithms; in view of the computational complexity greedy approaches like the orthogonal matching pursuit (OMP) or the compressive sampling matching pursuit (CoSaMP) are of special interest.

If the non-zero elements of the sparse vector \( \mathbf{x} \) are chosen from a finite set \( \mathcal{C} \) and \( \mathbb{C}_0^L = \mathcal{C} \cup \{0\} \), we have to solve (1) with trial vector \( \hat{x} \in \mathbb{C}_0^L \). Please note that in contrast to “one-bit CS”, e.g., [1], here still \( \mathbf{y} \in \mathbb{R}^K, i.e., \) the sparse vector \( \mathbf{x} \), not the measurement vector \( \mathbf{y} \), is discrete.

Stating from (1), the obvious strategy is to run conventional CS to obtain a real-valued estimate followed by quantizing the elements of the vector to the set \( \mathbb{C}_0 \). The main drawback is that the knowledge about the discrete nature of the sparse signal is not used in the reconstruction step, which—whenever side information is ignored—causes a loss.

However, the CS problem (1) with trial vector \( \hat{x} \in \mathbb{C}_0^L \) can be rewritten in the form

\[
\hat{x} = \arg\min_{\mathbf{z} \in \mathbb{C}_0^L} \| A \hat{x} - y \|_2 \quad \text{with} \quad \| \hat{x} \|_0 \leq s . \tag{2}
\]

Looking at (2), since a number of discrete signals are interfering with each other, the field of multiple-input/multiple-output (MIMO) schemes has to be considered. Lattice decoding algorithms, in particular the so-called sphere decoder (SD) [1], solve a well-defined or over-determined system \( \mathbf{y} = \mathbf{H} \mathbf{x} + \mathbf{n} \), where \( \mathbf{H} \in \mathbb{R}^{K \times E}, K \geq E \), w.r.t. minimum Euclidean distance. In CS with discrete-valued signals, CS recovery and lattice decoding/MIMO equalization meet each other. Hence, either a sparsity constraint is introduced in the SD, or the discrete nature of the signal is incorporated into the CS recovery algorithm, cf. Fig. 1.

![Fig. 1. The connection between CS, discrete CS, and MIMO detection.](image)

In this paper, we combine both worlds in order to benefit from the different features they have. After briefly reviewing known approaches for adapting the SD to sparse signals, we show how OMP (as prominent representative of greedy recovering algorithms) can be used in connection with a SD, operating on a much lower-dimensional problem as the known approaches. Moreover, we show that the decoding metric in the SD has to be properly adjusted to the given sparsity. In each case we assume that the sparsity is known, and we restrict ourselves to binary signals, i.e., \( \mathbb{C} = \{-1, +1\} \).

The paper is organized as follows. In Sec. II different methods of incorporating quantization/SD into compressed sensing recovery algorithms are introduced. The performance of the schemes is assessed via numerical simulations and design guidelines are derived in Sec. III. Brief conclusions are drawn in Sec. IV.

II. CS USING THE SPHERE DECODER

In this section, we first review how to directly use the SD for sparse signals and then show how OMP can be combined with the SD to obtain a low-complexity scheme for the detection of discrete sparse signals.
A. SD with Sparsity Constraint

In order to use the SD for sparse under-determined problems (approaching the problem “from the right” in Fig. 1), essentially two modifications to the standard SD are needed. On the one hand, it has to be adapted to the sparsity constraint. Several attempts have already been made in the literature, e.g., \cite{13, 11, 8, 15}. On the other hand, the standard SD does not work for under-determined systems of linear equations. Basically, two different approaches to solve this problem exist. First, the set of equations may artificially be enlarged to the full dimensionality \( L \), see, e.g., \cite{13}. This augmentation has to be carefully done to avoid close-to-singular matrices, which, in addition to the huge dimensionality, dramatically increases the search complexity.

Second, the SD may only be applied to a \( K \)-dimensional part of \( x \), while a brute-force search over the remaining \( L - K \) components is carried out, e.g., \cite{4, 16}. Unfortunately, for \( L - K \gg 1 \), this method also has a tremendous computational complexity (cf. Section 1.1.3). A possible solution to overcome this problem is presented subsequently.

B. CS with Discrete-Value Constraint

The straightforward approach to use CS for discrete-valued signals (approaching the problem “from the left” in Fig. 1) is to run a conventional CS algorithm and to quantize the output to the given alphabet \( C \) in a final step (denoted as \( Q_{C}(\cdot) \)). The procedure is illustrated for OMP\cite{3} in pseudocode representation in Alg. 1.

\begin{algorithm}
\caption{OMP (\( y, A, E \))}
\begin{algorithmic}
\STATE \( \hat{x} = 0 \), \( r = y \), \( S = \{ \} \), \( i = 0 \) // init
\WHILE {\( i < E \)}
\STATE \( i = i + 1 \)
\STATE \( \hat{x} = A^T r \), \( \varsigma_{\text{best}} = \arg\max_{\varsigma \in \bar{S}} |\hat{x}_\varsigma| \)
\STATE \( S = S \cup \{ \varsigma_{\text{best}} \} \) // extend support
\STATE \( \tilde{x}_S = (A_S)^+ y \) // calculate signal at \( S \)
\STATE \( r = y - A \hat{x} \) // calculate residual
\STATE \( } \)
\STATE \( \tilde{x}_S = Q_{C_0}(\tilde{x}_S) \) // quantize signal
\end{algorithmic}
\end{algorithm}

In OMP, in each iteration one new support position is added to the support set \( S \) in a greedy fashion. Specifically, the element with the largest correlation with the residual\footnote{To a large extent, the respective steps are also valid for CoSaMP and other greedy approaches.} is selected. Usually, knowing the sparsity, \( E = s \) iterations are carried out. One disadvantage of OMP is that, once a support element has been chosen, it can never be removed again which leads to a degradation of the performance. To avoid this fact, in \cite{17, 12} it has been proposed to run some additional iterations, i.e., \( E > s \), in order to be able to find all support elements even if some wrong elements have been chosen. Quantization has then to be done w.r.t. \( C_0 \).

\footnote{This choice is justified from a signal representation perspective. When discrete symbols have to be detected, a reliability measure should be used. Since for binary transmission log-likelihood ratios are proportional to the observation, the selection criterion is reasonable in the present setting.} C. Adaptation of the Branch Metric in SD

Applying SD in the above settings, the branch metric should be adapted. Specifically, the fact that the a-priori probabilities of the elements of \( C_0 \) are non-uniformly distributed should be included. Applying the maximum-a-posteriori criterion gives, cf. \cite{7},

\[
\hat{x}_S = \arg\max_{\hat{x}_S | y} \Pr\{ \tilde{x}_S | y \} = \arg\max_{\hat{x}_S | y} \Pr\{ y | \tilde{x}_S \} \Pr\{ \tilde{x}_S \}
\]

\[
= \arg\min_{\hat{x}_S \in C_0} \left\{ \| y - A \hat{x}_S \|_2^2 - 2\sigma_x^2 s \sum_{i=1}^{\| S \|} \ln(\Pr\{ x_{S(i)} \}) \right\}.
\]

In contrast to the approach given in \cite{7}, where the a-priori probability is not updated, we take already available decisions into account. If the decoder has reached depth \( \ell \) of the decoding tree, i.e., still \( j = \| S \| - \ell + 1 \) decisions are missing and already \( m \) non-zero elements have been detected, the a-priori probability of the symbols \( x_{S(i)} \) is given as

\[
\Pr\{ x_{S(i)} \} = \begin{cases} \frac{s-m}{2}, & x_{S(i)} \neq 0 \\ \frac{s-2m}{2}, & x_{S(i)} = 0 \end{cases},
\]

Please note that this approach guarantees \( \hat{x} \) to have the desired (known) sparsity \( s \).

1) Obvious Concatenation: Using this obvious concatenation of OMP and subsequent quantization (we denote this by “OMP/Q”), the real-valued signal estimate is the basis for the final decisions. Please note, for each realization of the vector \( \tilde{x}_S \) the threshold of the quantizer is optimized, such that exactly \( s \) non-zero samples are obtained (fixed sparsity).

However, the estimate can be discarded and only the support set estimate \( \hat{S} \) may be utilized; CS just serves for finding the support. But, knowing \( \hat{S} \), a MIMO detection problem with “channel” matrix \( A_S \in \mathbb{R}^{K \times |S|} \) results; since \( |S| < K \), an over-determined problem is present. Consequently, Line 9 may be replaced by any advanced MIMO detection scheme (e.g., decision-feedback equalization, lattice-reduction-aided techniques) to improve the estimate. The SD (strategy denoted by “OMP/SD”) is again of particular interest.

2) Embedding the Detection: An alternative strategy to cascading OMP and a MIMO detection scheme is to embed the detection into the algorithms. Thereby, the knowledge about the finite alphabet is directly utilized in the reconstruction. An obvious procedure is an element-wise quantization of the current signal estimate within the algorithm, i.e., to replace Line 6 of Alg. 1 by \( \tilde{x}_S = Q_{C_0}(A_S)^{+} y \) and delete Line 9 (strategy denoted by “Q-OMP”).

In terms of communications, Line 6 is nothing else than zero-forcing (ZF) linear equalization applied to an over-determined MIMO detection problem. Consequently, in this step any MIMO detection strategy can be utilized. Since channel noise is present, the minimum mean-squared error (MMSE) approaches may be preferred over the ZF one. Advanced MIMO detection schemes are again of special interest; in particular the SD can be employed at this step. We denote this strategy of OMP with embedded SD by “SD-OMP”.

D. Complexity Analysis

For comparison, a brief overview of the complexity of the discussed algorithms is given in Table I. OMP/SD requires an OMP with $E$ iterations and one run of the SD with dimension (depth of the decoding tree) $E$. SD-OMP requires to run the SD $E$ times, with dimensionality $s$ in the $i$th iteration. Note, the complexity of these two algorithms depends only on the sparsity but not on the dimension $L$ of the sparse vector.

In contrast, both pure-SD-based approaches depend on the dimensionality of the sparse vector $x$ and are hence computationally infeasible for high-dimensional problems. The SD with split matrix, proposed in [4], [16], needs to solve an $K$-dimensional problem up to $|C|^{L-K}$ times. The SD with enlarged matrix [15] results in an $L$-dimensional problem, which, moreover, tends to be ill conditioned.

III. NUMERICAL RESULTS

In this section, the performance of the proposed algorithms is evaluated in terms of the symbol error rate $\text{SER} = E\{\hat{x}_i \neq x_i\}$ by numerical simulations. The measurement matrix $A$ is obtained by randomly selecting $K$ rows from a $L \times L$ unitary matrix and then normalizing the columns to unit norm.

1) Number of Iterations in OMP: Fig. 2 shows the SER over the number of iterations, $E$, in the OMP for $1/\sigma_n^2 \geq 18$ dB. The sparsity $s = 20$ is marked by a dashed black line. In each case, additional iterations are rewarding. If OMP/Q (red, symbol-wise quantization after OMP guaranteeing sparsity $s$) is used, $E$ has to be selected carefully—choosing $E$ too large, the signal estimation via the pseudoinverse fails and causes a degradation. Q-OMP (green) does only reach similar performance but is much more tolerant to the choice of $E$, which may be an advantage if the sparsity $s$ is not known exactly.

Increasing $E$ the probability that the set $\mathcal{S}$ contains the correct support increases (and tends to one as $E \rightarrow L$). In turn, the SD (OMP/SD, blue) is able to most likely recover it. However, the gain comes at the cost of higher computational complexity (larger dimensionality of SD). OMP with embedded SD (SD-OMP, purple) shows a slightly better performance than OMP/SD for $E \approx s$, but for $E > s$, OMP/SD outperforms SD-OMP et even lower computational complexity. The problem in Q-OMP and SD-OMP is that the residual $r$ (Line 7 of Alg. 1) is no longer orthogonal to $\hat{x}$, which affects the selection of the next support elements.

For reference, the SER when OMP finds the (enlarged) support set but perfect (error-free) decisions of these symbols would be obtained at the final quantization/sphere decoding step is shown (OMP, genie-aided values). Via this curve it can be concluded that the main problem is to find a set $\mathcal{S}$ which indeed contains the correct support. Using the SD the MIMO detection problem is solved almost perfectly.

2) Variants of OMP: The above discussed variants of OMP are compared in Figs. 3 and 4 where the SER is plotted over $1/\sigma_n^2$ in dB. Fig. 3 shows the results for the common approach $E = s = 20$, while in Fig. 4 the number $E$ of iterations is randomized. In view of Fig. 2 we choose $E = 24$ for OMP/Q and (to limit complexity) $E = 30$ for all other approaches.

For $E = s$, a quantization embedded in the OMP gives slight gains at negligible computational effort. Using the SD instead of scalar quantization does not enable further gains but would only waste complexity. Once again, as can be seen from the genie-aided reference curve, the problem is that the OMP does not provide the correct support set.
Choosing $E > s$ the situation changes. Quantization embedded in the OMP does not gain in performance compared to OMP with subsequent quantization. Here, the additional iterations provide some tolerance of the algorithm to wrong selections of support elements. If the sparsity is not known exactly, one can benefit from the robustness of Q-OMP against additional iterations.

Using the SD for detection clearly outperforms symbol-wise quantization as long as adapted a-priori probabilities are taken into account. From this observation one can clearly conclude that by constraining the final detection step, which is based on the enlarged set $\mathcal{S}$, to the correct sparsity $s$ (which is assumed to be known) significant gains are possible. Neither OMP/Q, Q-OMP nor OMP/SD, SD-OMP with fixed priors guarantee the correct sparsity.

In summary, allowing the OMP to carry out some additional iterations, the embedding of quantization or even the SD is not rewarding. The combination of i) selecting an enlarged set of candidate positions for the support via OMP and ii) detecting the discrete-valued symbols at these positions via SD is a powerful and efficient approach. Noteworthy, the known sparsity should be utilized and the SD has to be adapted to guarantee this fact. Again, comparing the performance of proposed approaches with the genie-aided curves clearly indicate the source of losses. If perfect decisions were available for the positions, actually provided by OMP, only approximately 0.5 dB could be gained (solid black curve). Conversely, if the correct support set (plus random extra positions) was guaranteed to be included in $\mathcal{S}$ and the SD worked on this set, much better performance would be possible (dashed black).

### IV. Conclusions

In this paper, we have proposed and assessed approaches for the recovery of discrete-valued sparse signals. Combining CS algorithms, which essentially serve to find a candidate set $\mathcal{S}$ which contains the correct support, and subsequent MIMO detection schemes, in particular the sphere decoder, which recover the discrete symbols, efficient low-complexity approaches are enabled. Choosing the number of iterations of OMP large enough, an embedding of the MIMO detection into the algorithm is not required. The adaptation of the decoding metric in the SD, guaranteeing the desired/known sparsity, is crucial.

However, as the genie-aided reference curves show, the probability that the correct support set is found should be increased. One way is to use the CoSaMP instead of the OMP; almost everything shown for the OMP is equivalently valid for the CoSaMP. Its known performance gains over OMP can be transferred to the present situation of discrete sparse signals. An even more reliable support set recovery, taking into account the finite nature of the symbols—in particular via reliabilities in the selection of the support positions—, is still a field of current research.

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### TABLE I

| Algorithm      | Detection of $\mathcal{S}$ | #iter in OMP | Dim. of SD |
|----------------|-----------------------------|--------------|------------|
| OMP/SD         | $x \subset \mathcal{S}$ after OMP | $E$          | $E$        |
| SD-OMP         | $x \subset \mathcal{S}$ within OMP | $E$          | $1, \ldots, E$ |
| SD, split matrix [13], [15] | $x$ | — | $K$ (up to $|\mathcal{S}|^{L-K}$ times) |
| SD, enlarged matrix [13] | $x$ | — | $L$ |