Mass Formula for a Stationary Axisymmetric Configuration and
the Physical Realization of the Kerr Metric

R. M. Avakian $^a$ and G. Oganessyan $^{a, b, *}$

$^a$ Dept. of Theoretical Physics, Yerevan State University, Yerevan 375049, Armenia

$^b$ Theoretical Astrophysics Group, Tata Institute of Fundamental Research, Bombay-400 005,
India $^\dagger$

Abstract

We analyse the expression for the mass of a stationary axisymmetric configuration in general relativity obtained in our previous work [1]. From the generality of our formula and its incompatibility with the corresponding expression in Kerr space-time we argue that a stationary equilibrium distribution of a real matter cannot be a source of the Kerr metric.

There exists an expression relating the mass of a stationary axisymmetric configuration to the pressure distribution inside it [1]. The line element for the metric created by such a configuration is written in the coordinates which in the spherically symmetric limit, for instance when the angular velocity $\Omega$ becomes zero, go over into the isotropic form:

\[ ds^2 = (e^\nu - \omega^2 r^2 \sin^2 \theta e^\mu) dt^2 - e^\lambda (dr^2 + r^2 d\theta^2) - r^2 \sin^2 \theta e^\mu d\phi^2 \]

\[ - 2\omega r^2 \sin^2 \theta e^\mu d\phi dt, \]

where $\nu, \mu, \lambda$ and $\omega$ are functions of $r, \theta$ and $\Omega$.

*Corresponding author

$^\dagger$Address for correspondence. E-mail: gurgen@tifrvax.tifr.res.in
Since the components of the metric tensor do not depend on $x^0 = ct$ and $x^3 = \phi$, the components $R^0_0$ and $R^3_3$ of the Ricci tensor can be written as
\begin{align*}
R^0_0 &= \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} (\sqrt{-g} g^{0i} \Gamma^i_{0\alpha}), \\
R^3_3 &= \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} (\sqrt{-g} g^{3i} \Gamma^i_{3\alpha}),
\end{align*}
where $g = \text{det} g_{ik} = -r^4 \sin^2 \theta e^{\nu+2\lambda+\mu}$, $i = 0, 1, 2, 3$, $\alpha = 1, 2, 3$. This covariant divergence form of the components of the Ricci tensor allows us to apply Gauss’s theorem and use only the asymptotic form of the metric functions while integrating expressions containing $R^0_0$ and $R^3_3$ over the entire space.

From Einstein equations we then obtain the following mass formula [1]:
\[ M^2 = -32 \int_0^\pi \int_0^{r_s(\theta)} (T^1_1 + T^2_2) r^3 \sin^2 \theta e^{\nu+2\lambda+\mu/2} dr d\theta, \]
where $T^1_1$ and $T^2_2$ are the corresponding components of the energy-momentum tensor, and $r_s(\theta)$ is the boundary of the configuration, $T^1_1[r_s(\theta)] = T^2_2[r_s(\theta)] = 0$.

In the case of rotating ideal fluid (4) reduces to
\[ M^2 = 64 \int_0^\pi \int_0^{r_s(\theta)} P r^3 \sin^2 \theta e^{\nu+2\lambda+\mu/2} dr d\theta. \]

The derivation of (4) is very general, the only assumption being that the configuration is stationary (equations (2) and (3) hold) and equilibrium ($T^1_1$ and $T^2_2$ are nonzero). One could try to proceed in the same manner and derive a similar formula for the Kerr metric, which also can be written in isotropic coordinates [5]:
\[ ds^2 = -X^2 dt^2 + r^2 \sin^2 \theta \frac{B^2}{X^2} (d\phi - \omega dt)^2 + \frac{\psi^2}{X^2} (dr^2 + r^2 d\theta^2) \]

The metric functions in (5) are given by
\begin{align*}
X^2 &= \frac{\Sigma \Delta}{A} \quad (7) \\
\omega &= 2a \frac{R}{A} M \quad (8)
\end{align*}
\[ \psi^2 = \frac{\Sigma^2 \Delta}{r^2 A} \]  
\[ B^2 = \frac{\Delta}{r^2} \]  

where

\[ R = \frac{(2r + M)^2 - a^2}{4r} \]  
\[ \Sigma = R^2 + a^2 \cos^2 \theta \]  
\[ \Delta = R^2 - 2RM + a^2 \]  
\[ A = (R^2 + a^2)^2 - a^2 \Delta \sin^2 \theta, \]

\( M \) is the mass and \( a = L/M \) is the specific angular momentum of the black hole.

Now, following the procedure by which (4) was derived, we obtain the following mass formula:

\[ M^2 - a^2 = -32 \int_0^\pi \int_0^{r_s(\theta)} (T_1 + T_2^2) r^3 \sin^2 \theta e^{-\frac{\mu + 2\lambda + \nu}{2}} dr d\theta, \]  

Formula (15) as compared to (4) has an extra term \(-a^2\). This difference is essential and cannot be neglected, except for the trivial case of \( \Omega = 0 \), when both the line elements (1) and (6) reduce to the Schwarzschild form. This suggests that the conditions used in the derivation of (4), namely that the configuration of a real matter must be equilibrium and stationary, do not hold in the case of the Kerr metric.

In a less general form this result has been known from the work of one of the authors [2], where it is shown that, in the case of a real gas of baryons, no coordinate transformation can bring the exterior solution for a stationary axisymmetric stellar configuration in \( \Omega^2 \) approximation, known as Hartle-Thorne-Sedrakian-Chubarian solution [3, 4], to the Kerr line element written in the same approximation.
The generality of the mass formula (4) and its crucial disagreement with (15) allow us to make a generic statement: stationary rotating equilibrium stellar configurations cannot be a source of the Kerr metric.

A similar result has been recently obtained by S. Sengupta [6], who suggests an interpretation of the parameter $a$ associated with the Kerr metric as the specific angular momentum corresponding to the unphysical situation of a hollow rotating object.

Acknowledgement

It is a pleasure to thank E. Gourgoulhon for initiating an important discussion, as well as S. M. Chitre, E. V. Chubarian, P. Ghosh and S. Sengupta for helpful comments.
REFERENCES

[1] R. M. Avakian, G. Oganessyan, *Mod. Phys. Lett. A*, **10** (1995) 841.

[2] R. M. Avakian, in *Contemporary Theoretical and Experimental Problems of Relativity Theory and Gravitation* (Yerevan, 1988).

[3] D. M. Sedrakian, E. V. Chubarian, *Astrofizika*, **4** (1968) 239; **4** (1968) 481.

[4] J. B. Hartle, K. S. Thorne, *Ap. J.*, **153** (1968) 807.

[5] A. Lanza, *Class. Quantum Grav.* **9** (1992) 677.

[6] S. Sengupta, preprint [gr-qc/9501037](http://arxiv.org/abs/gr-qc/9501037)