Abstract – In this work, starting by suitable superpositions of equal-frequency Bessel beams, we develop a theoretical and experimental methodology to obtain localized \textit{stationary} wave fields, with high transverse localization, \textit{whose longitudinal intensity pattern can approximately assume any desired shape} within a chosen interval $0 \leq z \leq L$ of the propagation axis $z$. Their intensity envelope remains static, i.e. with velocity $v = 0$; so that we have named “Frozen Waves” (FW) these new solutions to the wave equations (and, in particular, to the Maxwell equations). Inside the envelope of a FW only the carrier wave does propagate: And the longitudinal shape, within the interval $0 \leq z \leq L$, can be chosen in such a way that no nonnegligible field exists outside the pre-determined region (consisting, e.g., in one or more high intensity peaks). Our solutions are noticeable also for the different and interesting applications they can have, especially in electromagnetism and acoustics, such as optical tweezers, atom guides, optical or acoustic bistouries,
various important medical apparata, etc.

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1 Introduction

Over many years the theory of localized waves (LW), or nondiffracting waves, and in particular of the so-called X-shaped waves, has been developed, generalized, and experimentally verified in many fields, such as optics, microwaves and acoustics[1]. These new solutions to the wave equations (and, in particular, to the Maxwell equations) have the noticeable characteristic of resisting the diffraction effects for long distances, i.e., of possessing a large depth of field.

Such waves can be divided into two classes: the localized beams and the localized pulses. With regard to the beams, the most popular is the Bessel beam.

Much work was made about the properties and applications of a single Bessel beam, while some work has been done in connection with Bessel beam superpositions performed by summing or integrating over their frequency (by producing, e.g., the well-known “X-shaped pulses” and/or their velocity (for example, it has been studied the space-time focusing of different-speed X-shaped pulses). By contrast, only a few papers have been addressed to the properties and applications of superpositions of Bessel beams with the same frequency, but with different longitudinal wave numbers. The few works existing on this subject have shown some surprising possibilities associated with this particular type of superpositions, mainly the possibility of controlling the transverse shape of the resulting beam[2, 3].

The other important point, i.e., that of controlling the longitudinal shape, has been even more rarely addressed, and the relevant papers have been so far confined to numerical optimization processes[4, 5], aimed at finding out an appropriate computer-generated hologram.
In this work we develop a very simple method**, having recourse to superpositions of forward propagating and equal-frequency Bessel beams only, that allows controlling the beam-intensity longitudinal shape within a chosen interval $0 \leq z \leq L$, where $z$ is the propagation axis and $L$ can be much greater than the wavelength $\lambda$ of the monochromatic light (or sound) which is being used. Inside such a space interval, indeed, we succeed in constructing a stationary envelope whose longitudinal intensity pattern can approximately assume any desired shape, including, for instance, one or more high-intensity peaks (with distances between them much larger than $\lambda$); and which results—in addition—to be naturally endowed also with a good transverse localization.***

This intensity envelope remains static, i.e., has velocity $v = 0$; and because of this in a previous paper[6] we have called “Frozen Waves” (FW) these new solutions to the wave equations (and, in particular, to the Maxwell equations). Inside the envelope of a FW only the carrier wave does propagate: And the longitudinal shape, within the interval $0 \leq z \leq L$, can be chosen in such a way that no nonnegligible field exists outside the pre-determined high-intensity region.

We also suggest a simple apparatus capable of generating the mentioned stationary fields.

Static wave solutions like these are noticeable also for the different and interesting applications they can have, especially in electromagnetism and acoustics, such as optical tweezers, atom guides, optical or acoustic bistouries, optical micro-lithography, electromagnetic or ultrasound high-intensity fields for various important medical purposes, and so on.**

** Patent pending.

*** When we get a complete control on the longitudinal shape, we cannot have—however—a total control also on the transverse localization, since our stationary fields are of course constrained to obey the wave equation.
2 The mathematical methodology:** Stationary wavefields with arbitrary longitudinal shape, obtained by superposing equal-frequency Bessel beams

Let us start from the well-known axis-symmetric zeroth order Bessel beam solution to the wave equation:

$$\psi(\rho,z,t) = J_0(k_\rho \rho)e^{i\beta z}e^{-i\omega t}$$  \hspace{1cm} (1)$$

with

$$k_\rho^2 = \frac{\omega^2}{c^2} - \beta^2,$$  \hspace{1cm} (2)$$

where $\omega$, $k_\rho$ and $\beta$ are the angular frequency, the transverse and the longitudinal wave numbers, respectively. We also impose the conditions

$$\frac{\omega}{\beta} > 0 \text{ and } k_\rho^2 \geq 0$$  \hspace{1cm} (3)$$

(which imply $\omega/\beta \geq c$) to ensure forward propagation only (with no evanescent waves), as well as a physical behavior of the Bessel function $J_0$.

Now, let us make a superposition of $2N + 1$ Bessel beams with the same frequency $\omega_0$, but with different (and still unknown) longitudinal wave numbers $\beta_n$:

$$\Psi(\rho,z,t) = e^{-i\omega_0 t} \sum_{n=-N}^{N} A_n J_0(k_{\rho n} \rho)e^{i\beta_n z},$$  \hspace{1cm} (4)$$

where $n$ are integer numbers and $A_n$ are constant coefficients. For each $n$, the parameters $\omega_0$, $k_{\rho n}$ and $\beta_n$ must satisfy Eq.(2), and, because of conditions (3), when considering $\omega_0 > 0$, we must have
\[ 0 \leq \beta_n \leq \frac{\omega_0}{c}. \] (5)

Let us now suppose that we wish \(|\Psi(\rho, z, t)|^2\) of Eq. 4 to assume on the axis \(\rho = 0\) the pattern represented by a function \(|F(z)|^2\), inside the chosen interval \(0 \leq z \leq L\). In this case, the function \(F(z)\) can be expanded, as usual, in a Fourier* series:

\[ F(z) = \sum_{m=-\infty}^{\infty} B_m e^{i \frac{2\pi}{L} m z}, \]

where

\[ B_m = \frac{1}{L} \int_0^L F(z) e^{-i \frac{2\pi}{L} m z} \, dz. \]

More precisely, our goal is finding out, now, the values of the longitudinal wave numbers \(\beta_n\) and of the coefficients \(A_n\), of Eq. 4, in order to reproduce approximately, within the said interval \(0 \leq z \leq L\) (for \(\rho = 0\)), the predetermined longitudinal intensity-pattern \(|F(z)|^2\). Namely, we want to have

\[ \left| \sum_{n=-N}^{N} A_n e^{i \beta_n z} \right|^2 \approx |F(z)|^2 \quad \text{with} \quad 0 \leq z \leq L. \] (6)

Looking at Eq. (6), one might be tempted to take \(\beta_n = 2\pi n/L\), thus obtaining a truncated Fourier series, expected to represent approximately the desired pattern \(F(z)\).

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*Such a choice of the longitudinal intensity pattern does imply an interesting freedom, since we can consider more in general any expansion \(\sum_{m=-\infty}^{\infty} B_m \exp i \frac{2\pi}{L} m z = F(z) \exp i \phi(z)\), quantity \(\phi(z)\) being an arbitrary function of the coordinate \(z\).
Superpositions of Bessel beams with $\beta_n = 2\pi n / L$ have been actually used in some works to obtain a large set of transverse amplitude profiles\cite{2,3}. However, for our purposes, this choice is not appropriate, due to two principal reasons: 1) It yields negative values for $\beta_n$ (when $n < 0$), which implies backwards propagating components (since $\omega_0 > 0$); 2) In the cases when $L >> \lambda_0$, which are of our interest here, the main terms of the series would correspond to very small values of $\beta_n$, which results in a very short field-depth of the corresponding Bessel beams (when generated by finite apertures), preventing the creation of the desired envelopes far from the source.

Therefore, we need to make a better choice for the values of $\beta_n$, which allows forward propagation components only, and a good depth of field. This problem can be solved by putting

$$\beta_n = Q + \frac{2\pi}{L} n ,$$

where $Q > 0$ is a value to be chosen (as we shall see) according to the given experimental situation, and the desired degree of transverse field localization. Due to Eq.(5), we get

$$0 \leq Q \pm \frac{2\pi}{L} N \leq \frac{\omega_0}{c} .$$

Inequality (8), can be used to determine the maximum value of $n$, that we call $N_{\text{max}}$, once $Q$, $L$ and $\omega_0$ have been chosen.

As a consequence, for getting a longitudinal intensity pattern approximately equal to the desired one, $|F(z)|^2$, in the interval $0 \leq z \leq L$, Eq.(4) should be rewritten as

$$\Psi(\rho = 0, z, t) = e^{-i\omega_0 t} e^{i Q z} \sum_{n=-N}^{N} A_n e^{i \frac{2\pi}{L} n z} ,$$

with

$$A_n = \frac{1}{L} \int_0^L F(z) e^{-i \frac{2\pi}{L} n z} dz .$$
Obviously, one obtains only an approximation to the desired longitudinal pattern, because the trigonometric series (9) is necessarily truncated ($N \leq N_{\text{max}}$). Its total number of terms, let us repeat, will be fixed once the values of $Q$, $L$ and $\omega_0$ are chosen.

When $\rho \neq 0$, the wavefield $\Psi(\rho, z, t)$ becomes

$$\Psi(\rho, z, t) = e^{-i\omega_0 t} e^{iQz} \sum_{n=-N}^{N} A_n J_0(k_{\rho n} \rho) e^{i\frac{2\pi n L}{L} z},$$

with

$$k_{\rho n}^2 = \omega_0^2 - \left(Q + \frac{2\pi n}{L}\right)^2.$$  \hspace{1cm} (12)

The coefficients $A_n$ will yield the amplitudes and the relative phases of each Bessel beam in the superposition.

Because we are adding together zero-order Bessel functions, we can expect a high field concentration around $\rho = 0$. Moreover, due to the known non-diffractive behavior of the Bessel beams, we expect that the resulting wavefield will preserve its transverse pattern in the entire interval $0 \leq z \leq L$.

The methodology developed here deals with the longitudinal intensity pattern control. Obviously, we cannot get a total 3D control, due the fact that the field must obey the wave equation. However, we can use two ways to have some control over the transverse behavior too. The first is through the parameter $Q$ of Eq.(7). Actually, we have some freedom in the choice of this parameter, and FWs representing the same longitudinal intensity pattern can possess different values of $Q$. The important point is that, in superposition (11), using a smaller value of $Q$ makes the Bessel beams possess a higher transverse concentration (because, on decreasing the value of $Q$, one increases the value of the Bessel beams transverse numbers), and this will reflect in the resulting field, which will present a narrower central transverse spot. We will exemplify this fact in the next Section. The second way to control the transverse intensity pattern is using higher order Bessel beams,
but we shall show this in Section 5.

3 Some examples

In this Section we shall present a few examples of our methodology.

First example:
Let us suppose that we want an optical wavefield with $\lambda_0 = 0.632 \ \mu m$, that is, with $\omega_0 = 2.98 \times 10^{15} \ Hz$, whose longitudinal pattern (along its $z$-axis) in the range $0 \leq z \leq L$ is given by the function

$$F(z) = \begin{cases} 
-4 \frac{(z-l_1)(z-l_2)}{(l_2-l_1)^2} & \text{for } l_1 \leq z \leq l_2 \\
1 & \text{for } l_3 \leq z \leq l_4 \\
-4 \frac{(z-l_5)(z-l_6)}{(l_6-l_5)^2} & \text{for } l_5 \leq z \leq l_6 \\
0 & \text{elsewhere} 
\end{cases}$$

(13)

where $l_1 = L/5 - \Delta z_{12}$ and $l_2 = L/5 + \Delta z_{12}$ with $\Delta z_{12} = L/50$; while $l_3 = L/2 - \Delta z_{34}$ and $l_4 = L/2 + \Delta z_{34}$ with $\Delta z_{34} = L/10$; and, at last, $l_5 = 4L/5 - \Delta z_{56}$ and $l_6 = 4L/5 + \Delta z_{56}$ with $\Delta z_{56} = L/50$. In other words, the desired longitudinal shape, in the range $0 \leq z \leq L$, is a parabolic function for $l_1 \leq z \leq l_2$, a unitary step function for $l_3 \leq z \leq l_4$, and again a parabola in the interval $l_5 \leq z \leq l_6$, it being zero elsewhere (within the interval $0 \leq z \leq L$, as we said). In this example, let us put $L = 0.2 \ m$.

We can then easily calculate the coefficients $A_n$, which appear in the superposition (11), by inserting Eq. (13) into Eq. (10). Let us choose, for instance, $Q = 0.999 \omega_0/c$: This choice allows the maximum value $N_{\max} = 316$ for $n$, as one can infer from Eq. (8). Let
us emphasize that one is not compelled to use just \( N = 316 \), but can adopt for \( N \) any values smaller than it; more generally, any value smaller than that calculated via Eq. (8). Of course, on using the maximum value allowed for \( N \), one gets a better result.

In the present case, let us adopt the value \( N = 30 \). In Fig.1(a) we compare the intensity of the desired longitudinal function \( F(z) \) with that of the Frozen Wave, \( \Psi(\rho = 0, z, t) \), obtained from Eq.(9) by adopting the mentioned value \( N = 30 \).

![Figure 1](image1.png)

**Figure 1:** (a) Comparison between the intensity of the desired longitudinal function \( F(z) \) and that of our Frozen Wave (FW), \( \Psi(\rho = 0, z, t) \), obtained from Eq.(9). The solid line represents the function \( F(z) \), and the dotted one our FW. (b) 3D-plot of the field-intensity of the FW chosen in this case by us.

One can verify that a good agreement between the desired longitudinal behavior and our approximate FW is already got with \( N = 30 \). The use of higher values for \( N \) can only improve the approximation. Figure 1(b) shows the 3D-intensity of our FW, given by Eq.(11). One can observe that this field possesses the desired longitudinal pattern, while being endowed with a good transverse localization.

*Second example:*
Let us now suppose we want an optical wavefield with $\lambda = 0.632 \, \mu m \ (\omega_0 = 2.98 \times 10^{15} \, \text{Hz})$, whose longitudinal pattern (on its axis) in the range $0 \leq z \leq L$ consists in a pair of parabolas, for $l_1 \leq z \leq l_2$ and $l_3 \leq z \leq l_4$, the intensity of the second parabola being twice as much as that of the first one. Outside the intervals $l_1 \leq z \leq l_2 \cup l_3 \leq z \leq l_4$, the desired field has a null intensity. Summarizing, we want:

$$F(z) = \begin{cases} 
-4 \frac{(z - l_1)(z - l_2)}{(l_2 - l_1)^2} & \text{for } l_1 \leq z \leq l_2 \\
-4\sqrt{2} \frac{(z - l_3)(z - l_4)}{(l_4 - l_3)^2} & \text{for } l_3 \leq z \leq l_4 \\
0 & \text{elsewhere ,}
\end{cases} \quad (14)$$

where $l_1 = 3L/10 - \Delta z_{12}$ and $l_2 = 3L/10 + \Delta z_{12}$ with $\Delta z_{12} = L/70$; while $l_3 = 7L/10 - \Delta z_{34}$ and $l_4 = 7L/10 + \Delta z_{34}$ with $\Delta z_{34} = L/70$. In this example we choose $L = 0.02 \, \text{m}$.

Again, we can calculate the coefficients $A_n$ by inserting Eq.(14) into Eq.(10), and use them in our superposition (9). In this case, we chose $Q = 0.995 \omega_0/c$: This choice allows a maximum value of $n$ given by $N_{\text{max}} = 158$ (one can see this by exploiting Eq.(8)). But for simplicity we adopt once more $N = 35$, hoping that Eq.(11) will yield a good enough approximation of the desired function.

We compare in Fig.2(a) the intensity of the desired longitudinal function $F(z)$ with that of our FW, $\Psi(\rho = 0, z, t)$, obtained from Eq.(9) by using $N = 35$: We can verify a good agreement between the desired longitudinal behaviour and our FW. Obviously we can improve the approximation by using larger values of $N$.

In Fig.2(b) we show the 3D field intensity of our FW, forwarded by Eq.(11). We can see that this field has a good transverse localization and possesses the desired longitudinal pattern.
Figure 2: (a) Comparison between the intensity of the desired longitudinal function \( F(z) \), given by Eq. (14), and that of our FW, \( \Psi(\rho = 0, z, t) \), obtained from Eq. (9). The solid line represents the function \( F(z) \), and the dotted one our FW. (b) 3D plot of the field intensity of the FW chosen by us in this new case.

Third example (controlling the transverse shape too):

We want to take advantage of this new example for addressing an important question: We can expect that, for a desired longitudinal pattern of the field intensity, by choosing smaller values of the parameter \( Q \) one will get FWs with narrower transverse width [for the same number of terms in the series entering Eq. (11)], because of the fact that the Bessel beams in Eq. (11) will possess larger transverse wave numbers, and, consequently, higher transverse concentrations. We can verify this expectation by considering, for instance, inside the usual range \( 0 \leq z \leq L \), the longitudinal pattern represented by the function

\[
F(z) = \begin{cases} 
-4 \frac{(z - l_1)(z - l_2)}{(l_2 - l_1)^2} & \text{for } l_1 \leq z \leq l_2 \\
0 & \text{elsewhere}
\end{cases},
\]  

(15)

with \( l_1 = L/2 - \Delta z \) and \( l_2 = L/2 + \Delta z \). Such a function has a parabolic shape, with its
peak centered at $L/2$ and with longitudinal width $2\Delta z/\sqrt{2}$. By adopting $\lambda_0 = 0.632 \, \mu m$ (that is, $\omega_0 = 2.98 \times 10^{15} \, Hz$), let us use the superposition (11) with two different values of $Q$: We shall obtain two different FWs that, in spite of having the same longitudinal intensity pattern, will possess different transverse localizations. Namely, let us consider $L = 0.06 \, m$ and $\Delta z = L/100$, and the two values $Q = 0.999 \, \omega_0/c$ and $Q = 0.995 \, \omega_0/c$. In both cases the coefficients $A_n$ will be the same, calculated from Eq.(10), on using this time the value $N = 45$ in superposition (11). The results are shown in Figs.3(a) and 3(b). Both FWs have the same longitudinal intensity pattern, but the one with the smaller $Q$ is endowed with a narrower transverse width.

![Figure 3: (a) The Frozen Wave with $Q = 0.999 \, \omega_0/c$ and $N = 45$, approximately reproducing the chosen longitudinal pattern represented by Eq.(15). (b) A different Frozen wave, now with $Q = 0.995 \, \omega_0/c$ (but still with $N = 45$) forwarding the same longitudinal pattern. We can observe that in this case (with a lower value for $Q$) a higher transverse localization is obtained.](image)

In Section 5 we shall show that a better control of the transverse shape can be obtained by using higher order Bessel beams in superposition (11).
4 Spatial resolution and Residual intensity

In connection with a FW of a given frequency, it is of practical (and theoretical) interest to investigate its Spatial resolution, its Residual intensity, the Size of the source necessary to generate it, as well as the minimum distance from the source needed to get such a FW.

Let us first comment that, in lossless media, the theory of FWs can furnish results similar to the free-space ones. This happens because FWs are suitable superpositions of Bessel beams with the same frequency, so that there is no problem with the material dispersion.

Here, we deal with lossless media only.

Let us address the question of the longitudinal and transverse spatial resolution for the FWs.

In connection with the longitudinal case, once we choose a desired longitudinal intensity field configuration, \(|F(z)|^2\), given, for example, by a single peak (or a few peaks) with a certain longitudinal width \(\Delta z\), we wish to investigate whether it is possible to obtain such a spatial resolution, and what are the relevant parameters for getting good results.

As one can expect, this question is directly related to the number \(2N + 1\) of terms in superposition (11): More specifically, in superposition (9).

Once the values of the frequency \(\omega_0\), and the parameters \(L\) and \(Q\) are chosen, the best approximation that we can get for a given longitudinal intensity-field configuration, \(|F(z)|^2\), is obtained by using Eq. (9) with the maximum number of terms \(2N_{\text{max}} + 1\), where \(N_{\text{max}}\) is calculated from is calculated through inequality (8).

As we have seen in the previous Sections, it is not always necessary to use \(N = N_{\text{max}}\), and frequently a smaller value of \(N\) can provide us with good results. But even in this
cases, when a value \( N < N_{\text{max}} \) is quite sufficient to furnish the desired spatial resolution, nevertheless it can be desirable to increase the value of \( N \) for lowering the longitudinal residual intensities, as we are going to see.

In the cases in which not even the value \( N = N_{\text{max}} \) yields a good result, we have to adopt a smaller value for the parameter \( Q \) so to increase, in this way, the value of \( N_{\text{max}} \) itself. For quantifying mathematically the precision of our approximation, one may have recourse to the mean square deviation, \( D \),

\[
D = \int_0^L |F(z)|^2 \, dz - L \sum_{-N}^N |A_n|^2 ,
\]

where \( A_n \), the coefficients of superposition (9), are given by Eq.(10).

The case of the transverse spatial resolution cannot be tackled in such a detail, since, as we know, one cannot have a complete three-dimensional control of the field. In the previous Section we have seen that we can get, however, some control on the transverse spot size through the parameter \( Q \). Actually, Eq.(11), that defines our FW, is a superposition of zero-order Bessel beams, and, due to this fact, the resulting field is expected to possess a transverse localization around \( \rho = 0 \). Each Bessel beam in superposition (11) is associated with a central spot with transverse size, or width, \( \Delta \rho_n \approx 2.4/k_{\rho n} \). On the basis of the expected expected convergence of series (11), we can estimate the width of the transverse spot of the resulting beam as being

\[
\Delta \rho \approx \frac{2.4}{k_{\rho n=0}} = \frac{2.4}{\sqrt{\omega_0^2/c^2 - Q^2}} ,
\]

which is the same value as that for the transverse spot of the Bessel beam with \( n = 0 \) in superposition (11). Relation (16) can be useful: Once we have chosen the desired longitudinal intensity pattern, we can choose even the size of the transverse spot, and use relation (16) for evaluating the needed, corresponding value of parameter \( Q \).
In spite of the fact that the transverse spot size happens to be approximately equal to that of a Bessel beam with \( k_\rho = \sqrt{\omega_0^2/c^2 - Q^2} \), it may happen that the decay of the field transverse intensity for \( \rho > \Delta \rho \) is much faster than that of an ordinary Bessel beam! This happens when the desired field intensity presents a longitudinal width \( \Delta z \) much smaller than \( L \), i.e., \( \Delta z \ll L \), as we will see below.

An illustrative example:
Let us consider the situation in which, within the interval \( 0 \leq z \leq L \), the desired longitudinal intensity pattern is given by a well-concentrated peak, represented by expression (15), with \( \lambda_0 = 0.632 \mu m \) (that is, \( \omega_0 = 2.98 \times 10^{15} \) Hz), \( Q = 0.98 \omega_0/c \), \( L = 0.01 \) m, and \( \Delta z = L/500 \).

Figures 4(a), 4(b) and 4(c) show the resulting FWs obtained by using, in superposition (9), \( N = 100 \), \( N = 250 \) and \( N = 300 \), respectively. We can see in the first case that \( N = 100 \) is not enough for yielding a good result. On the other hand, the second and third cases, with \( N = 250 \) and \( N = 300 \), seem to reproduce the desired pattern very well, with no apparent difference between the two cases. However, Fig.5 shows that the residual intensity for the third case is smaller than for the second one, confirming the previous conclusions.

Figure 6(a) shows the transverse intensity pattern of the peak (in the plane \( z = L/2 \)) for the case with \( N = 250 \). We can see that the value of the transverse spot width agrees very well with our estimate (16), which furnishes, in this case, the value \( \Delta \rho \approx 1.22 \mu m \). From Fig.6(b) one can visually evaluate the residual intensity of the transverse pattern: One can observe that the transverse decay is strong and much faster than that presented by Bessel beams. This figure too confirms our previous conclusions.
Figure 4: Comparison of the desired longitudinal intensity pattern (solid line) with those of the resulting FWs (dotted line), when using: (a) $N = 100$; (b) $N = 250$; (c) $N = 300$.

Figure 5: (a) Longitudinal residual intensity of the considered FW with $N = 250$. (b) The same with $N = 300$.

Figure 6: (a) Transverse intensity pattern for the peak of the considered FW with $N = 250$. (b) The transverse residual intensity for this case.
5 Increasing the control on the transverse shape by using higher-order Bessel beams

We have shown in the previous Section how to get a very strong control over the longitudinal intensity pattern of a beam using suitable superposition of zero-order Bessel beams.

As already mentioned, due to the fact that the resulting beam must obey the wave equation, we cannot get a total three-dimensional control of the wave pattern; but we have shown that we can have some control on the transverse behavior: More specifically, we can control the transverse spot size through the parameter $Q$, which defines the values of the transverse wave numbers of the Bessel beams entering superposition (11).

In this Section we are going to argue that it is possible to increase even more our control of the transverse shape by using higher-order Bessel beams in our fundamental superposition (11). Despite the method presented in this Section is not yet demonstrated in a rigorous mathematical way, it can be understood and accepted on the basis of simple and intuitive arguments. The basic idea is obtaining the desired longitudinal intensity pattern, not along the axis $\rho = 0$, but on a cylindrical surface corresponding to $\rho = \rho' > 0$. This allows one to get interesting stationary field distributions, as static annular structures (tori), or cylindrical surfaces, of stationary light (or electromagnetic or acoustic field), and so on, with many possible applications.** To realize this, let us initially start with the same procedure in the previous Section; i.e., let us choose some desired longitudinal intensity pattern, within the interval $0 \leq z \leq L$, and calculate the coefficients $A_n$ by using Eq.(10). Afterwards, let us replace the zero-order Bessel beams $J_0(k_{\rho n}\rho)$, in superposition (11), with higher-order Bessel beams, $J_\mu(k_{\rho n}\rho)$, to get

$$
\Psi(\rho, z, t) = e^{-i\omega t} e^{i Q z} \sum_{n=-N}^{N} A_n J_\mu(k_{\rho n}\rho) e^{i \frac{2\pi}{L} n z},
$$

(17)
with \( A_n = (1/L) \int_0^L F(z) \exp(-i2\pi nz/L) \, dz \), and \( k_{\rho n} = \sqrt{\omega_0^2 - (Q + 2\pi n/L)^2} \).

In superposition (17), the Bessel functions \( J_\mu(k_{\rho n}\rho) \), with different values of \( n \), reach their maximum values at \( \rho = \rho'_n \), where \( \rho'_n \) is the first positive root of the equation \( (d J_\mu(k_{\rho n}\rho)/d\rho)|_{\rho'_n} = 0 \). The values of \( \rho'_n \) are located around the central value \( \rho'_{n=0} \), at which the Bessel function \( J_\mu(k_{\rho n=0}\rho) \) assumes its maximum value. We can intuitively expect that the desired longitudinal intensity pattern, initially constructed for \( \rho = 0 \), will approximately shift to \( \rho = \rho'_{n=0} \). We have found such a conjecture to hold in all situations explicitly considered by us. By such a procedure, one can obtain very interesting stationary configurations of field intensity, as the mentioned “donuts” and cylindrical surfaces, and much more.

In the following example, we show how to obtain, e.g., a cylindrical surface of stationary light. To get it, within the interval \( 0 \leq z \leq L \), let us first select the longitudinal intensity pattern given by Eq.(15), with \( l_1 = L/2 - \Delta z \) and \( l_2 = L/2 + \Delta z \), and with \( \Delta z = L/300 \). Moreover, let us choose \( L = 0.05 \) m, \( Q = 0.998 \) \( \omega_0/c \), and use \( N = 150 \).

Then, after calculating the coefficients \( A_n \) as before,

\[
A_n = \frac{1}{L} \int_0^L F(z) e^{-i\frac{2\pi n}{L} z} \, dz ,
\]

we have recourse to superposition (17). In this case, we choose \( \mu = 4 \). According to the previous discussion, one can expect the desired longitudinal intensity pattern to appear shifted to \( \rho' \approx 5.318/k_{\rho n=0} \rho = 8.47 \) \( \mu m \), where 5.318 is the value of \( k_{\rho n=0} \rho \) for which the Bessel function \( J_4(k_{\rho n=0}\rho) \) assumes its maximum value, with \( k_{\rho n=0} = \sqrt{\omega_0^2 - Q^2} \). The figures below show the resulting intensity field.

Figure 7(a) depicts the transverse intensity pattern for \( z = L/2 \). The transverse peak intensity is located at \( \rho = 7.75 \) \( \mu m \), with a 8.5% difference w.r.t. the predicted value of 8.47 \( \mu m \). In Fig.7(b) the transverse section of the resulting beam for \( z = L/2 \) is shown.

Figure 8 depicts the three-dimensional pattern of such a higher-order FW. In Fig.8(a)
the orthogonal projection of its 3D pattern is shown, which corresponds to nothing but a cylindrical surface of stationary light (or other fields). In Fig.8(b) the same field is shown, but from a different point of view.

Figure 7: (a) Transverse intensity pattern at $z = L/2$ of the considered, higher-order FW. (b) Transverse section of the resulting stationary field for $z = L/2$.

Figure 8: (a) Orthogonal projection of the three-dimensional intensity pattern of the higher-order FW depicted in Figs.7. (b) The same field but under a different perspective.

We can see that the desired longitudinal intensity pattern has been approximately obtained, but, as wished, shifted from $\rho = 0$ to $\rho = 7.75 \mu m$: and the resulting field resembles
a cylindrical surface of stationary light with radius 7.75 µm and length 238 µm. Donut-like configurations of light (or sound) are also possible.

6 Generation of Frozen Waves

In the previous Sections, we have shown how suitable superpositions of Bessel beams of the same frequency can provide impressive results: Namely, can produce stationary wavefields with high transverse localization, and with an arbitrary longitudinal shape, within a chosen space interval $0 \leq z \leq L$; that is, Frozen Waves with a static envelope. As we already mentioned, such waves are rather interesting, not only from the theoretical point of view, but also because of their great variety of possible applications, ranging from ultrasonics to laser surgery, and from tumor destruction to optical tweezers.

But how to produce our FWs? Regarding the generation of FWs, one has to recall that superpositions (11), which define them, consist of sums of Bessel beams. Let us also recall that a Bessel beam, when generated by a finite aperture (as it must be, in any real situations), maintains its nondiffracting properties till a certain distance only (called its field-depth), given by

$$Z = \frac{R}{\tan \theta},$$  \hspace{1cm} (18)

where $R$ is the aperture radius and $\theta$ is the so-called axicon angle, related to the longitudinal wave number by the known expression $\cos \theta = c\beta/\omega$.

So, given an apparatus whatsoever capable of generating a single (truncated) Bessel beam, we can use an array of such apparatus to generate a sum of them, with the appropriate longitudinal wave numbers and amplitudes/phases [as required by Eq.(11)], thus producing the desired FW. Here, it is worthwhile to notice preliminarily that we shall be
able to generate the desired FW in the the range $0 \leq z \leq L$ if all Bessel beams entering the superposition \[11\] are able to reach the distance $L$ resisting the diffraction effects. We can guarantee this, for instance, if $L \leq Z_{\text{min}}$, where $Z_{\text{min}}$ is the field-depth of the Bessel beam with the smallest longitudinal wave number $\beta_{n=-N} = Q - 2\pi N / L$, that is, with the shortest depth of field. In such a way, once we have the values of $L$, $\omega_0$, $Q$, $N$, from Eq.\[18\] and from the above considerations it results that the radius $R$ of the finite aperture has to be

$$R \geq L \sqrt{\frac{\omega_0^2}{c^2 \beta_{n=-N}^2}} - 1$$ \hspace{1cm} (19)

The simplest apparatus capable of generating a Bessel beam is that adopted by Durnin et al.\[7\], which consists in an annular slit located at the focus of a convergent lens and illuminated by a cw laser. Then, an array of such annular rings with the appropriate radii and transfer functions, able to yield both the correct longitudinal wave numbers\[†\] and the coefficients $A_n$ of the fundamental superposition \[11\], can generate the desired FW.

In the next Section we shall just consider such a simple apparatus, even if, of course, other powerful tools, like the computer generated holograms, may be used to produce the FWs.

### 6.1 A very simple apparatus for producing FWs

Let us work out an example, by having recourse to an array of the very simple Durnin et al.’s experimental apparatus.

Since 1987, let us repeat, it has been used a simple experimental mean for creating a Bessel beam, consisting in an annular slit located at the focus of a convergent lens and illuminated by a cw laser. Let us call $\delta a$ the width of the annular slit, $\lambda$ the wavelength of

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\[†\]Once a value for $Q$ has been chosen.
the laser, and $f$ and $R$ the focal length and the aperture radius of the lens, respectively. On illuminating the annular slit with a cw laser with frequency $\omega_0$, and provided that condition $\delta a \ll \lambda f / R$ is satisfied, the Durnin et al.’s apparatus creates, after the lens, a wavefield closely similar to a Bessel beam along a certain depth of field. Within such field-depth, $z < R / \tan \theta$, and to $\rho << R$, the generated Bessel beam can be approximately written

$$\psi(\rho, z, t) \approx \Lambda J_0(k_\rho \rho) e^{i\beta z} e^{i\omega_0 t}$$

(20)

with $\Lambda$ a constant depending on the values of $a$, $f$, $\omega_0$ and $\delta a$,

$$k_\rho = \frac{\omega_0 a}{c f} ,$$

(21)

and

$$\beta^2 = \frac{\omega_0^2}{c^2} - k_\rho^2 .$$

(22)

Thus, as Durnin et al. suggested, we can see that the transverse and longitudinal wave numbers are determined by radius and focus of slit and lens, respectively. Once more, let us recall also that the wavefield has approximately a Bessel beam behavior (when $\rho << R$), in the range $0 \leq z \leq Z \approx R f / a$ that we have called the field-depth of the Bessel beam. Our FWs are to be obtained by suitable superpositions of Bessel beams. So we can experimentally produce the FWs by using several concentric annular slits (Fig.9), where each radius is chosen in order yield the correct longitudinal wave number, and where
the transfer function of each annular slit is chosen in order to furnish the coefficients $A_n$ of Eq. (9) which are needed for the desired longitudinal pattern to be obtained.

Figure 9: A set of suitable, concentric annular slits, as a simple means for generating a Frozen Wave.

Let us examine all this in more details. Suppose we have $2N + 1$ concentric annular slits with their radii given by $a_n$, with $-N \leq n \leq N$. Along a certain range, after the lens, one will have a wavefield given by the sum of the Bessel beams produced by each slit, namely

$$\Psi(\rho, z, t) = e^{-i\omega_0 t} \sum_{n=-N}^{N} \Lambda_n T_n J_0(k_{n\rho} \rho) e^{i\beta_n z},$$

(23)

$T_n$ being the transfer function of the $n$-th annular slit (which regulates amplitude and phase of the emitted Bessel beam, and is a constant function for each slit); while the $\Lambda_n$

$^\dagger$The same apparatus could also be used to generate higher order FWs, when the zero-order Bessel beams in superposition (23) are replaced with higher order Bessel functions. Experimentally, it can be performed by angular modulation of the slits.
are constants depending on the characteristics of the apparatus: Namely depending, in
general, on the values of $a$, $f$, $\omega_0$ and $\delta a$. It is possible to obtain a simple expression for the
$\Lambda_n$ by making some simplifying, rough considerations[8]. The transverse and longitudinal
wave numbers are given by

$$k_{\rho n} = \frac{\omega_0 a_n}{c} f$$  \hspace{1cm} (24)$$
and

$$\beta_n^2 = \frac{\omega_0^2}{c^2} - k_{\rho n}^2.$$  \hspace{1cm} (25)$$

On the other hand, we know from the present theory that for constructing the FWs
quantity $\beta$ is to be given by Eq.(7):

$$\beta_n = Q + \frac{2\pi}{L} n.$$  \hspace{1cm} (7)$$

On combining Eqs.(7,24),(25), one gets

$$\left(Q + \frac{2\pi}{L} n\right)^2 = \frac{\omega_0^2}{c^2} - \left(\frac{\omega_0 a_n}{c f}\right)^2$$  \hspace{1cm} (26)$$
and, solving with respect to $a_n$,

$$a_n = f \sqrt{1 - \frac{c^2}{\omega_0^2} \left(Q + \frac{2\pi}{L} n\right)^2}.$$  \hspace{1cm} (27)$$
Equation (27) yields the radii of all the annular slits that provide the correct longitudinal wave numbers, needed for the generation of the FWs. We may notice that the radii of the annular slits do not depend on the specific desired longitudinal intensity-pattern, and that many different sets of values for the radii are possible on making different choices for the parameter $Q$.

Notice that the procedure is not yet finished. Indeed, once the desired longitudinal pattern $F(z)$ has been chosen, one has necessarily to meet in Eq. (9) the coefficients $A_n$ given by Eq. (10); and such coefficients have to be the coefficients of Eq. (23). For obtaining them, it is necessary that each annular slit be endowed by the appropriate transfer function, which regulates amplitude and phase of the Bessel beam emitted by that slit. By using Eqs. (10), (11), (23), we get the transfer function $T_n$ of the $n$-th annular slit to be

$$T_n = \frac{A_n}{\Lambda_n} = \frac{1}{L \Lambda_n} \int_0^L F(z) e^{-i \frac{2\pi}{\Lambda} n z} \, d z,$$

(28)

Finally, with the radius of each annular slit given by Eqs. (27) and the transfer functions of each slit given by Eqs. (28), we do obtain a FW endowed with the desired longitudinal behaviour, inside the interval $0 \leq z \leq L$. Of course, one has to guarantee also that the distance $L$ is smaller than the smallest field-depth of the Bessel beams entering superposition (23). In other words, one must have also

$$L \leq Z_{\text{min}} \approx \frac{R f}{a_{\text{max}}},$$

(29)
where $a_{\text{max}}$ is the largest radius of the concentric annular slits.

7 Conclusions

In this work we have expounded the theory of Frozen Waves, and depicted some possible experimental apparatus to generate them. The present results can find applications in many fields: Just to make an example, in optical tweezers modelling, since we can construct stationary optical (but also acoustic, etc.) fields with a great variety of shapes; capable, e.g., of trapping particles or tiny objects at different locations.** These topics will be reported elsewhere.

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References

[1] For a review, see: E.Recami, M.Zamboni-Rached, K.Z.Nóbrega, C.A.Dartora, and H.E.Hernández-Figueroa: “On the localized superluminal solutions to the Maxwell equations,” IEEE Journal of Selected Topics in Quantum Electronics 9, 59-73 (2003); and references therein.

[2] Z.Bouchal and J.Wagner: “Self-reconstruction effect in free propagation wavefield,” Optics Communications 176, 299-307 (2000).

[3] Z. Bouchal, “Controlled spatial shaping of nondiffracting patterns and arrays,” Optics Letters 27, 1376-1378 (2002).

[4] J.Rosen and A.Yariv: “Synthesis of an arbitrary axial field profile by computer-generated holograms,” Optics Letters 19, 843-845 (1994).

[5] R.Piestun, B.Spektor and J.Shamir: “Unconventional light distributions in three-dimensional domains,” Journal of Modern Optics 43, 1495-1507 (1996).

[6] M.Zamboni-Rached: “Stationary optical wavefields with arbitrary longitudinal shape, by superposing equal frequency Bessel beams: Frozen Waves”, Optics Express 12, 4001-4006 (2004).

[7] J.Durnin, J.J.Miceli and J.H.Eberly, “Diffraction-free beams,” Physical Review Letters 58, 1499-1501 (1987). Cf. also A.O.Barut, G.D.Maccarrone and E.Recami: Nuovo Cimento A71, 509-533 (1982).

[8] C.A.Dartora, M.Zamboni-Rached, K.Z.Nóbrega, E.Recami and H.E.Hernández-Figueroa, “General formulation for the analysis of scalar diffraction-free beams using angular modulation: Mathieu and Bessel beams”, Optics Communications 222, 75-80 (2003).