Superradiant modes in Fibonacci quantum wells under resonant conditions

C H Chang, C W Tsao and W J Hsueh
Photonics Group, Department of Engineering Science and Ocean Engineering, National Taiwan University, 1, Sec. 4, Roosevelt Road, Taipei, 10660, Taiwan
E-mail: hsuehwj@ntu.edu.tw

Received 29 July 2014, revised 9 October 2014
Accepted for publication 20 October 2014
Published 26 November 2014

Abstract
It is first presented that superradiant modes exist in Fibonacci quantum wells within the exact regions that are obtained using the gap map diagram, rather than the traditional resonant Bragg condition. The results show that three limited regions are derived from the diagram, which correspond to bandgaps with widths that differ from each other. The regions in which the superradiant modes do not occur are also defined clearly. Moreover, the proposed method can be used to determine whether superradiant modes occur in multiple quantum wells that have non-periodical arrangements, including quasiperiodic sequences and correlated disorder sequences.

Keywords: superradiant modes, quasiperiodic structure, exciton polaritons, quantum wells

1. Introduction
Since the concept of superradiance was first proposed by Dicke, in the system of two-level atoms, the phenomenon of collective effects has been extensively investigated in various theoretical and experimental studies of semiconductor laser structures and optical nanoantennas [1–4]. During the last decade, it has been confirmed that the propagation of exciton polaritons in a one-dimensional resonant photonic crystal with quantum wells that are arranged periodically, namely a periodic quantum well (PQW), also exhibits a phenomenon that is similar to the Dicke
superradiance [5–11]. For PQWs, the superradiant mode arises under the Bragg condition \( d = j\lambda (\omega_0)/2 \) for \( j = 1, 2, \ldots \), where \( d \) is the period and \( \lambda (\omega_0) \) is the wavelength at the exciton resonance frequency, \( \omega_0 \) [12]. The reflection spectrum for the superradiant mode can be divided into two stages, the superradiant and the photonic crystal regimes, in accordance with the increase in the number of QWs, \( N \) [6, 10]. PQWs under the Bragg condition have been used for studies of many optical properties [13–15].

Recently, the concept of quasicrystals has attracted much interest in numerous fields of research [16–21]. The propagation of exciton polaritons in multiple quantum well structures that have different quasiperiodic arrangements has been extensively studied [22–25]. For quasiperiodic systems, the Fibonacci sequence is one of the most common structures [26, 27]. Previous studies of the structure factor for a one-dimensional quasicrystal in the limit, \( N \to \infty \), have shown that there is also an analogous condition in Fibonacci quantum wells (FQWs) that is similar to the Bragg condition in PQWs. This condition is called the resonant Bragg condition, which is described by the average period of the structure and two integers, \( h \) and \( h' \) [12, 28]. The reflection spectra in the FQWs under the resonant Bragg condition also exhibit similar superradiant and photonic crystal behaviors. In contrast to the PQWs, the significant structural dips occur near the exciton resonance frequency in the reflection spectra of the FQWs [12].

Most previous studies have focused on the superradiant modes in the canonical FQWs under the resonant Bragg condition (CBFQWs), wherein the ratio of the long to the short intervals is equal to the golden ratio, \( \tau \equiv (\sqrt{5} + 1)/2 \). There have also been studies of noncanonical FQWs under the resonant Bragg condition, where the ratio is not unity or \( \tau \) [28]. This paper investigates the region in which superradiant modes occur in FQWs, where the ratio of the long to the short intervals is not limited to a specific value. Although the region of the superradiant modes in the FQWs can be obtained from the resonant Bragg condition, it is also found that the superradiant modes only occur within a part of this region. However, these regions where the superradiant modes arise are not clearly defined by the resonant Bragg condition. In addition to the traditional resonant Bragg condition that is determined by two integers, \( h \) and \( h' \), it is determined whether the regions in which the superradiant modes occur in the FQWs can be defined precisely using the proposed method.

### 2. Model and formulation

A FQW that is composed of two intervals, \( A \) and \( B \), and which obeys the iteration rule: \( A \to AB \) and \( B \to A \), is considered [22, 27]. The structures of the different generation orders, \( v \), are given by \( A, AB \) and \( ABA \) for \( v = 1, 2 \) and 3. Since each of the intervals contains a QW, the number of QWs in a \( v \)th-order FQW is calculated by \( N_v = N_{v-1} + N_{v-2} \), where \( N_1 = 1 \) and \( N_2 = 2 \). The FQWs are placed between semi-infinite barriers. It is assumed that there is no dielectric contrast between the QWs and the barriers. The barrier thickness is also assumed to be sufficient to prevent interaction between excitons in the QWs, so the excitons only couple with the electromagnetic field.

According to the classical Maxwell’s equations, the electric field for normal incidence in a single layer, where the left and right boundaries are at \( z_- \) and \( z_+ \) for a QW located at \( z=0 \), is described as
ωε + ω = −z qE z,

where the coordinate, \( z \), is the growth direction for the structure, \( q = \omega n_b / c \) is the light wave vector, \( \varepsilon_0 \) is the permittivity of a vacuum, \( c \) is the velocity of light in a vacuum and \( n_b \) is the background refractive index of the QWs and the barriers [9, 10, 15]. The contribution of the exciton to the polarization of a single QW is written as

\[
\varepsilon_{\text{exc}} = \int \Phi''(z') E(z') dz'.
\]

For the 1s excitons, the exciton susceptibility, \( \chi_{\text{exc}}(\omega) \), has a single-pole form and the wave function of the exciton, \( \Phi_{1s}(z) \), is an even function. By solving this equation, the relationship between the amplitudes of the electric fields at the left and right surfaces of layer \( j \) is

\[
E_+(z_j+) = M_j(E_+, E_-)(z_j+),
\]

where the transfer matrix, \( M_j \), passes through layer \( j \). The total transfer matrix through the \( v \)th-order FQW is written as

\[
M_T = M_{N_v}...M_1.
\]

The bandgaps are determined by the band structure of the exciton polaritons that propagate in the material. The dispersion equation for an infinite periodic structure can be written as

\[
\cos(KL) = (M_{T11} + M_{T22})/2,
\]

where \( K \) is the Bloch wave vector and \( L \) is the entire thickness of the \( v \)th-order FQW [11, 16]. In the absence of exciton nonradiative decay, the allowed band and the forbidden gap are respectively determined by the conditions, \( |\cos(KL)| \leq 1 \) and \( |\cos(KL)| > 1 \). This study uses the gap map diagram for infinite periodic FQWs and the effect of the thickness filling factor, defined by \( F = d_A/D \) where \( D = d_A + d_B \), on the bandgaps is determined by varying \( d_A = FD \) and \( d_B = (1 - F)D \) simultaneously for a fixed value of \( D \) [7, 16, 20]. The reflection spectra in the FQWs are also calculated using the total transfer matrix.

### 3. Results and discussion

Initially, the gap map diagram in a FQW where \( v = 12 \) and \( D = 1.02\lambda(\omega_0) \) is considered for various filling factors, as shown in figure 1. Only the major gaps in the gap map diagram are shown. The number of forbidden gaps increases as the generation order increases. The width of some forbidden gaps within some ranges for the filling factor does not decrease as the generation order increases. Therefore, it is clearly seen that there are two pairs of major gaps in...
the gap. This study defines the filling factors that correspond to the maximum gapwidths of the major gaps as $F_C$. The smaller and larger values of $F_C$ respectively equal the values obtained from the resonant Bragg condition for $(h, h') = (1, 0)$ and $(0, 2)$. The crosses denote the CBFQWs. The green, blue and red lines pertain to SMFQW-1, SMFQW-2 and SMFQW-3, except for the positions $D/\lambda(\omega_0) = 1$ and 1.5.

To allow a clear comparison, the values of $F_C$ in the FQW for $v = 12$ are plotted for different values of $D$ in figure 2(a). The analytical description of the resonant Bragg condition, $\bar{D} = (h + h'/\tau)\lambda(\omega_0)/2$, where $\bar{D} = d_B + (d_A - d_B)/\tau$, in terms of the parameters, $F$ and $D$, is presented in [12, 28]

$$F = \frac{1}{2\tau - 3} \left[ \frac{h + h'/\tau}{2D/\lambda(\omega_0)} + (\tau - 2) \right].$$ (2)

According to equation (2), it is more convenient to discuss the relationship between the values of $F_C$ that are obtained using the proposed method and the resonant Bragg condition with two arbitrary integers, $h$ and $h'$, for different values of $D$. At $D/\lambda(\omega_0) = 1$, three values of $F_C = 0$, 0.5 and 1 correspond to the Bragg condition for the PQW. The resonant Bragg condition occurs for $D/\lambda(\omega_0) \neq 1$, which can be seen as an extension of the Bragg condition for $j = 1$ and 2. The blue, green and red lines around $D/\lambda(\omega_0) = 1$ and 1.5 indicate the FQWs within the regions.
where the superradiant modes that are derived from the diagram occur, which are referred to as the SMFQWs in this study. These regions are different from those obtained for the traditional resonant Bragg condition. This is clearly shown by comparing the green, blue and red lines and the wide black lines that are obtained from the resonant Bragg condition for two given integers, \( h \) and \( h' \), in figure 2(a). It is seen that the black line for the resonant Bragg condition for \((h, h') = (1, 0)\) ranges from \( F_C = 0 \) to 1 and \( D\lambda(\omega_0) \sim 0.8 \) to 1.3, which does not precisely define the region in which the superradiant modes occur [12, 28]. However, the regions in which the superradiant modes occur are precisely derived from the regions where the maximum gapwidths exist, which are shown by the blue, red and green lines, except for the positions, \( D\lambda(\omega_0) = 1 \) and 1.5. Specifically, for values of \( D\lambda(\omega_0) \) around unity, the regions where the superradiant modes for the SMFQW-1, the SMFQW-2 and the SMFQW-3 occur are respectively within \( F_C \sim 0.789 \)–0.273 and \( D\lambda(\omega_0) \sim 0.88 \)–1.12, \( F_C \sim 0.221 \)–0 and \( D\lambda(\omega_0) \sim 0.88 \)–1, and \( F_C \sim 1 \)–0.719 and \( D\lambda(\omega_0) \sim 0 \)–1.12. In other words, the regions where the superradiant modes do not arise are defined by the regions where the maximum gapwidths do not exist. For example, there are no superradiant modes in the FQWs that have values of \( D\lambda(\omega_0) \sim 1.12 \)–1.38, as seen in figure 2(a). The CBFQWs are denoted by crosses.

Figure 2(b) shows the gap positions for the maximum gapwidths of the major gaps in SMFQW-1, SMFQW-2, SMFQW-3 and the CBFQW for different values of \( D\lambda(\omega_0) \) near unity. It is seen that the region where the maximum gapwidths exist for SMFQW-1 ranges from \( D\lambda(\omega_0) \sim 0.88 \) to 1.12, which corresponds to the formation of almost vertically symmetrical fan-shaped patterns. However, the maximum gapwidths for SMFQW-2 and SMFQW-3 only respectively exist in the left and right halves of the region in which they exist in the SMFQW-1. For \( D = \lambda(\omega_0) \), the maximum gapwidths for the SMFQW-1 are smaller and larger than those for the SMFQW-2 and the SMFQW-3, respectively. This is explained by noting that a structure \( D = \lambda(\omega_0) \) and \( F_C = 0.5 \) for the SMFQW-1 is the same as the PQW under the Bragg condition for \( j = 1 \), in which the width of the bandgap is equal to \( 2\sqrt{2}T_0\omega_0/\pi \), since it is composed of QWs with an exciton radiative damping rate that equals \( T_0 \) [6, 7]. In the limit of \( F_C \rightarrow 0 \) and 1 for \( D = \lambda(\omega_0) \), i.e. \( d_A \rightarrow 0 \) and \( \lambda(\omega_0) \), the SMFQW-2 and the SMFQW-3 both approximate to the PQW under the Bragg condition for \( j = 2 \), where the gapwidth is also \( 2\sqrt{2}T_0\omega_0/\pi \), because the QWs have twice the exciton radiative damping rate, \( 2T_0 \) [7, 12]. In contrast, for the limit of \( F_C \rightarrow 0 \) and 1, the SMFQW-2 and the SMFQW-3 have respectively broader and narrower gapwidths than the PQW under the Bragg condition for \( j = 2 \) since they include QWs with exciton radiative damping rates that are respectively larger and smaller than \( 2T_0 \), because of their Fibonacci arrangement. Therefore, the maximum gapwidths for SMFQW-2 and SMFQW-3 with values of \( D\lambda(\omega_0) \) around unity are respectively wider and narrower than those for SMFQW-1. The distances between the upper and the lower gap positions for SMFQW-2 and SMFQW-3 are also respectively larger and smaller than those for SMFQW-1. This is clearly seen from a comparison of the maximum gapwidths for SMFQW-2 and those for the CBFQW with the same value of \( D = 0.9472\lambda(\omega_0) \). This value is determined by the resonant Bragg condition for \((h, h') = (1, 0)\) and \( d_A/d_B = \tau \) [28]. The gap positions in the CBFQW are nearly consistent with the band gaps in the Bragg Fibonacci structure in [25].

The effect of the generation order on \( F_C \) and the gap positions in SMFQW-1, SMFQW-2 and SMFQW-3 is shown for fixed values of \( D \) in figures 3(a) and (b). SMFQW-1 has the same respective values of \( D \) as SMFQW-2 and SMFQW-3. It is seen that the values of \( F_C \) and the gap positions oscillate, when the generation order is small. When the generation order exceeds 6, these values approach fixed values. Therefore, the values of \( F_C \) and the gap positions in the
FQWs are almost the same for higher generation orders. In the following, these values are used to study the reflection spectra, in order to explain the phenomenon of superradiance. Figure 3(b) also shows that the broadest and narrowest maximum gapwidths respectively occur in SMFQW-2 and SMFQW-3 in comparison to those in SMFQW-1.

In order to verify the phenomenon of superradiance, the reflection spectra in SMFQW-2 and SMFQW-3 for increasing generation orders are shown in figures 4(a) and (b), respectively. The values of $D$ and $F_C$ for SMFQW-3 correspond to the case for a larger value of $F_C$ in figure 1. It is seen that the magnitude and the linewidths of the reflection spectra both increase as the generation order increases. It is worthy of note that the significant structural dips in the middle of the reflection spectra of SMFQW-2 and SMFQW-3 are an important feature, which differs from the spectra of the PQWs under the Bragg condition \cite{22, 23}. When the generation order is small, namely small $N$, the reflection profiles for SMFQW-2 and SMFQW-3 both have a Lorentzian form, where there is a linear increase in the half width as $N$ increases, if the dips are neglected. This directly shows the superradiant regime, which is similar to that in PQWs under the Bragg condition. For sufficiently large $N$, the half widths of the reflection profiles begin to saturate. The frequency ranges for high reflectance are consistent with the gap positions in SMFQW-2 and SMFQW-3, respectively, which shows a transition from the superradiant to the photonic crystal regime \cite{12, 23}. A comparison of figures 4(a) and (b) shows that the frequency ranges and the depths of the dips in SMFQW-2 are respectively broader and deeper than those in SMFQW-3. In contrast, the frequency ranges for high reflectance in SMFQW-2

---

**Figure 3.** (a) $F_C$ and (b) the gap positions in SMFQW-1 for $D=0.98\lambda(\omega_0)$ and $1.02\lambda(\omega_0)$, SMFQW-2 for $D=0.98\lambda(\omega_0)$ and SMFQW-3 for $D=1.02\lambda(\omega_0)$ for $\nu=3$ to 12.
are smaller than that in SMFQW-3. This is expected from the maximum gapwidths of the major gaps for SMFQW-2 and SMFQW-3, as shown in Figure 2(b).

Figures 5(a) and (b) show a comparison of the reflection spectra in FQWs with a higher generation order. Figure 5(a) shows that the frequency ranges for high reflectance and the maximum value in the reflection spectrum for SMFQW-3 are respectively smaller than and almost identical to those for SMFQW-1, although the values of FC are both obtained from the same gap map diagram. SMFQW-3 and SMFQW-1 are also detuned from their respective values of FC by ±6% of FC = 0.4587 for SMFQW-1. This results in a steep decrease in the linewidths of the reflection spectra. Therefore, SMFQW is very sensitive to slight variations in the filling factor, which is similar to the respective sensitivity of the FQWs and the PQWs to the resonant Bragg condition and the Bragg condition [12, 28]. A similar situation is also seen in figure 5(b). In contrast, although the maximum reflectance value for SMFQW-2 is smaller than that for the CBFQW, the frequency range for high reflectance in the former is larger.

Figure 4. The reflection spectra in (a) SMFQW-2 for D = 0.9472λ(ω0) and FC = 0.09 and in (b) SMFQW-3 for D = 1.02λ(ω0) and FC = 0.9489 for different generation orders. The gray areas in (a) and (b) denote the gap positions in SMFQW-2 and SMFQW-3 for v = 12. The other parameters are the same as those for figure 1, except that \( \hbar \Gamma = 100 \mu \text{eV} \).
4. Conclusions

The region where superradiant modes exist in FQWs is first accurately proposed using the gap map diagram. In contrast to the resonant Bragg condition, in which there are only wide regions for two given integers, there are three limited regions for the SMFQW. The maximum gapwidths of the major gaps for these regions are different from each other. The results also show that the exact regions in which the superradiant modes do not occur are defined in terms of the regions in which the maximum gapwidths do not exist. The values of $F_C$ and the gap positions in SMFQW-1, SMFQW-2 and SMFQW-3 for specific values of $D$ approach fixed values as the generation order increases. It is confirmed that superradiant modes occur in SMFQWs if these fixed values are used for the reflection spectra for increasing generation order. The sensitivity of SMFQWs to slight variations in the filling factor is also demonstrated. Therefore, it is possible to find the phenomenon of superradiance not only in FQWs, but also in multiple QWs that are arranged in other quasiperiodic sequences or correlated disorder sequences, using the gap map diagram.

**Figure 5.** (a) The reflection spectra in SMFQW-3 for $D=1.02\lambda(\omega_0)$ and $F_C=0.9489$, in SMFQW-1 for $D=1.02\lambda(\omega_0)$ and $F_C=0.4587$ and in these structures where the values of $F_C$ are detuned by ±6% of $F_C=0.4587$. (b) The reflection spectra in SMFQW-2 for $D=0.9472\lambda(\omega_0)$ and $F_C=0.09$, in the CBFQW for $D=0.9472\lambda(\omega_0)$ and $F_C=0.618$ and in these structures where the values of $F_C$ are detuned by ±5% of $F_C=0.618$. The numerical calculations are performed for $v=12$ and $N=233$. The pale red and gray areas in (a) respectively correspond to the gap positions in SMFQW-3 and SMFQW-1. The pale blue and pale green areas in (b) respectively denote the gap positions in SMFQW-2 and the CBFQW.
Acknowledgments

The authors acknowledge support in part by the National Science Council of Taiwan under grant numbers NSC 102-2221-E-002-105 and MOST 103-2221-E-002-118.

References

[1] Dicke R H 1954 Phys. Rev. 93 99–110
[2] Scully M O and Svidzinsky A A 2009 Science 325 1510–1
[3] Boiko D L and Vasil’ev P P 2012 Opt. Express 20 9501–15
[4] Teperik T V and Degiron A 2012 Phys. Rev. Lett. 108 147401
[5] Hüblner M, Prineas J P, Ell C, Brick P, Lee E S, Khitrova G, Gibbs H M and Koch S W 1999 Phys. Rev. Lett. 83 2841–4
[6] Ikawa T and Cho K 2002 Phys. Rev. B 66 085338
[7] Ivchenko E L, Voronov M M, Erementchouk M V, Deych L I and Lisyansky A A 2004 Phys. Rev. B 70 195106
[8] Gibbs H M, Khitrova G and Koch S W 2011 Nat. Photonics 5 275–82
[9] Andreani L C, Panzarini G, Kavokin A V and Vladimirova M R 1998 Phys. Rev. B 57 4670–80
[10] Pilozzi L, D’Andrea A and Cho K 2004 Phys. Rev. B 69 205311
[11] Erementchouk M V, Deych L I and Lisyansky A A 2006 Phys. Rev. B 73 115321
[12] Poddubny A N, Pilozzi L, Voronov M M and Ivchenko E L 2008 Phys. Rev. B 77 113306
[13] Askitopoulos A, Mouchliadis L, Iorsh I, Christmann G, Baumberg J J, Kaliteevski M A, Hatzopoulos Z and Savvidis P G 2011 Phys. Rev. Lett. 106 076401
[14] Prineas J P, Johnston W J, Yildirim M, Zhao J and Smirl A L 2006 Appl. Phys. Lett. 89 241106
[15] Sadeghi S M, Li W, Li X and Huang W-P 2006 Phys. Rev. B 74 161304
[16] Hsueh W J, Chen C T and Chen C H 2008 Phys. Rev. A 78 013836
[17] Rechtsman M C, Jeong H-C, Chaikin P M, Torquato S and Steinhardt P J 2008 Phys. Rev. Lett. 101 073902
[18] Ghulinyan M, Oton C J, Negro L D, Pavesi L, Sapienza R, Colocci M and Wiersma D S 2005 Phys. Rev. B 71 094204
[19] Vardeny Z V, Nahata A and Agrawal A 2013 Nat. Photonics 7 177–87
[20] Hsueh W J, Chang C H, Cheng Y H and Wun S J 2012 Opt. Express 20 26618–23
[21] Kruck S S et al 2013 Phys. Rev. B 88 201404(R)
[22] Chang C H, Chen C H and Hsueh W J 2013 Opt. Express 21 14656–61
[23] Poshakinskiy A V, Poddubny A N and Tarasenko S A 2012 Phys. Rev. B 86 205304
[24] Hsueh W J, Chang C H and Lin C T 2014 Opt. Lett. 39 489–92
[25] Poddubny A N, Pilozzi L, Voronov M M and Ivchenko E L 2009 Phys. Rev. B 80 115314
[26] Shechtman D, Blech I, Gratias D and Cahn J W 1987 Phys. Rev. Lett. 53 1951–3
[27] Albuquerque E L and Cottam M G 2003 Phys. Rep. 376 225–337
[28] Werchner M et al 2009 Opt. Express 17 6813–28