ON CERTAIN PROPERTIES OF THE CLASS $\mathcal{U}(\lambda)$

NAJLA M. ALARIFI, MILUTIN. OBRADOVIĆ, AND NIKOLA. TUNESKI

Abstract. Let $\mathcal{A}$ be the class of functions analytic in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and normalized such that $f(0) = f'(0) - 1 = 0$, i.e., have expansion $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. In this paper we study the class $\mathcal{U}(\lambda)$, $0 < \lambda \leq 1$, consisting of functions $f$ from $\mathcal{A}$ satisfying

$$\left| \left( \frac{z}{f(z)} \right)^2 f'(z) - 1 \right| < \lambda (z \in \mathbb{D}).$$

and give results regarding the Zalcman Conjecture, the generalised Zalcman conjecture, the Krushkal inequality and the second and third order Hankel determinant.

1. Introduction and Preliminaries

Let $\mathcal{A}$ be the class of functions that are analytic in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$ and normalised such that $f(0) = f'(0) - 1 = 0$, i.e., have expansion $f(z) = z + a_2z^2 + a_3z^3 + \cdots$.

Further, let $\mathcal{U}(\lambda) = \{f \in \mathcal{A} : |U_f(z) - 1| < \lambda, z \in \mathbb{D}\}$, where $0 < \lambda \leq 1$ and

$$U_f(z) := \left( \frac{z}{f(z)} \right)^2 f'(z).$$

The functions from $\mathcal{U}(\lambda)$ are univalent, its special case when $\lambda = 1$ first studied in [3] and more details on them can be found in [4, 5, 10]).

An intriguing fact about $\mathcal{U}(\lambda)$ is that in spite the class $\mathcal{S}^*$ of starlike functions is very large and contains most classes of univalent functions, it doesn’t contain the class $\mathcal{U} \equiv \mathcal{U}(1)$, i.e., $\mathcal{U}$ is not in $\mathcal{S}^*$, nor vice versa. Namely, the function $-\ln(1-z)$ is convex, thus starlike, but not in $\mathcal{U}$ because $U_f(0.99) = 3.621\ldots > 1$, while the function $f$ defined by $\frac{z}{f(z)} = 1 - \frac{2}{9}z + \frac{2}{9}z^3 = (1-z)^2 (1 + \frac{2}{9})$ is in $\mathcal{U}$ and such that $\frac{zf'(z)}{f(z)} = -\frac{2(z^2+z+1)}{z^2+z-2} = -\frac{1}{9} + \frac{14i}{9}$ for $z = i$. This rear property is the main reason why the class $\mathcal{U}$ (and $\mathcal{U}(\lambda)$) attracts huge attention in the past decades.

In this paper we study class $\mathcal{U}(\lambda)$ regarding the Zalcman Conjecture, the generalised Zalcman conjecture, the Krushkal inequality and the second and third order Hankel determinant which will be defined in corresponding sections further in the paper.

For the study we will need the following result proven in [6] as a part of the proof of Theorem 1.

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Lemma 1.1. For each function \( f \) in \( U(\lambda), \ 0 < \lambda \leq 1 \), there exists function \( \omega_1 \), analytic in \( D \), such that \( |\omega_1(z)| \leq |z| < 1 \), and \( |\omega'_1(z)| \leq 1 \), for all \( z \in D \), with

\[
\frac{z}{f(z)} = 1 - a_2z - \lambda z \omega_1(z).
\]

Additionally, for \( \omega_1(z) = c_1z + c_2z^2 + \cdots \),

\[
|c_1| \leq 1, \quad |c_2| \leq \frac{1}{2}(1 - |c_1|^2) \quad \text{and} \quad |c_3| \leq \frac{1}{3} \left[ 1 - |c_1|^2 - \frac{4|c_2|^2}{1 + |c_1|} \right].
\]

(1.1)

Using (1.1), we have

\[
z = [1 - a_2z - \lambda z \omega_1(z)]f(z),
\]

and after equating the coefficients,

\[
a_3 = \lambda c_1 + a_2^2,
\]

(1.3)

\[
a_4 = \lambda c_2 + 2\lambda a_2c_1 + a_2^3,
\]

\[
a_5 = \lambda c_3 + 2\lambda a_2c_2 + \lambda^2 c_1^2 + 3\lambda a_2^2c_1 + a_2^4,
\]

that we will use later on.

We will also use the next result (8).

Lemma 1.2. For each function \( f \) in \( U(\lambda), \ 0 < \lambda \leq 1 \), the next estimates are valid

\[
|a_2| \leq 1 + \lambda, \quad |a_3| \leq 1 + \lambda + \lambda^2.
\]

(1.4)

The results are sharp as the function

\[
f_{\lambda}(z) = \frac{1}{(1 - z)(1 - \lambda z)} = \sum_{n=1}^{\infty} \frac{1 - \lambda^n}{1 - \lambda} z^n = z + (1 + \lambda)z^2 + (1 + \lambda + \lambda^2)z^3 + \cdots.
\]

(1.5)

shows. Here \( \frac{1 - \lambda^n}{1 - \lambda} \big|_{\lambda=1} = n \) for all \( n = 1, 2, 3, \ldots \).

Let note that from (1.3) and (1.4),

\[
|a_3| = |\lambda c_1 + a_2^2| \leq 1 + \lambda + \lambda^2,
\]

i.e.,

\[
|\lambda c_1 + a_2^2| \leq 1 + \lambda + \lambda^2,
\]

(1.6)

which we will use further in the proofs.

2. ZALCMAN AND GENERALISED ZALCMAN CONJECTURE FOR THE CLASS \( U(\lambda) \)

In 1960 Zalcman posed the conjecture:

\[
|a_n^2 - a_2a_{n-1}| \leq (n - 1)^2 \quad (n \in \mathbb{N}, n \geq 2),
\]

proven in 2014 by Krushkal (1) for the whole class \( S \) by using complex geometry of the universal Teichmüller space. In 1999, Ma (2) proposed a generalized Zalcman conjecture,

\[
|a_m a_n - a_{m+n-1}| \leq (m - 1)(n - 1) \quad (m, n \in \mathbb{N}, m \geq 2, n \geq 2),
\]

which is still an open problem, closed by Ma for the class of starlike functions and for the class of univalent functions with real coefficients. Ravichandran and Verma
in [9] closed it for the classes of starlike and convex functions of given order and for the class of functions with bounded turning.

In [7], the authors treated some particulars cases of those problems and obtained sharp results. Since \( \mathcal{U}(\lambda) \subseteq \mathcal{U} \) for \( 0 < \lambda \leq 1 \), the same results are valid for the class \( \mathcal{U}(\lambda) \), but those results are not sharp for the class \( \mathcal{U}(\lambda) \). In this part of the paper we find better and sharp results for \( \mathcal{U}(\lambda) \).

**Theorem 2.1.** Let \( f \) in \( \mathcal{U}(\lambda) \) be of the form \( f(z) = az^2 + \alpha_3z^3 + \cdots \). Then

(i) \( |a_2^2 - a_3| \leq \lambda, 0 < \lambda \leq 1 \);

(ii) \( |a_3^2 - a_5| \leq \lambda(1 + \lambda^2), \frac{1}{2} \left( \frac{\sqrt{17}}{4} - 1 \right) \leq \lambda \leq 1 \).

Both results are sharp as the function \( f_\lambda \) given by (1.5) shows.

**Proof.**

(i) From \( a_3 = \lambda c_1 + a_2 \), we have \( |a_3^2 - a_3| = | - \lambda c_1| \leq \lambda \).

(ii) Using the relations (1.2), (1.4) and (1.6), we have

\[
|a_2^2 - a_3| = |\lambda c_3 + 2\lambda a_2 c_2 + \lambda a_2 c_1|
= \lambda|c_3 + 2a_2 c_2 - \lambda c_1^2 + c_1(\lambda c_1 + a_2^2)|
\leq \lambda(|c_3| + 2|a_2||c_2| + |\lambda c_1| + |c_1|)|c_1 + a_2^2|
\leq \lambda \left[ \frac{1}{3} \left( 1 - |c_1|^2 - \frac{4|c_2|^2}{1 + |c_1|} \right) + 2(1 + \lambda)|c_2| + \lambda|c_1|^2 + (1 + \lambda + \lambda^2)|c_1| \right]
\leq \lambda g_1(|c_1|, |c_2|),
\]

where

\[
g_1(x, y) = \frac{1}{3} \left( 1 - x^2 - \frac{4y^2}{1 + x} \right) + 2(1 + \lambda)y + \lambda x^2 + (1 + \lambda + \lambda^2)x,
\]

\( 0 \leq x = |c_1| \leq 1, 0 \leq y = |c_2| \leq \frac{1}{2}(1 - x^2) \).

Since \( \frac{\partial g_1(x,y)}{\partial y} > 0 \) for all \( (x, y) \in E := \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq \frac{1}{2}(1 - x^2)\} \), then \( g_1 \) has no singular points in the interior of \( E \) and \( g_1 \) attains its maximum on the boundary of \( E \).

For \( x = 0 \), we have \( 0 \leq y \leq \frac{1}{2} \) and

\[
g_1(0, y) = \frac{1}{3}(1 - 4y^2) + 2(1 + \lambda)y \]
\[
= \frac{1}{3} [-4y^2 + 6(1 + \lambda)y + 1]
\leq 1 + \lambda.
\]

Similarly, for \( y = 0 \), we have \( 0 \leq x \leq 1 \) and

\[
g_1(x, 0) = \frac{1}{3}(1 - x^2) + \lambda x^2 + (1 + \lambda + \lambda^2)x
= \left( \lambda - \frac{1}{3} \right) x^2 + (1 + \lambda + \lambda^2)x + \frac{1}{3}
\leq (1 + \lambda)^2.
\]
Finally, for $0 \leq x \leq 1$ and $y = \frac{1}{2}(1 - x^2)$, we have
\[
g_1\left(x, \frac{1}{2}(1 - x^2)\right) = 1 + \lambda + \left(\frac{4}{3} + \lambda + \lambda^2\right)x - x^2 - \frac{1}{3}x^3 = \varphi_1(x).
\]

Since $\varphi'_1(x) = \frac{4}{3} + \lambda + \lambda^2 - 2x - x^2 \geq \lambda^2 + \lambda - \frac{5}{3} \geq 0$ for $\frac{1}{2} \leq \lambda \leq 1$, then $\varphi_1$ is an increasing function for $0 \leq x \leq 1$, and $\varphi_1(x) \leq (1 + \lambda)^2$.

Using (2.1) and all these facts, we have the statement of Theorem 3.1 (ii). \qed

**Remark 2.1.** For $0 < \lambda < \frac{1}{2} \left(\sqrt{\frac{23}{3}} - 1\right) = 0.8844\ldots$ we have that the function $\varphi_1$ attains its maximum for $x_0 = \sqrt{\lambda^2 + \lambda + \frac{7}{3}} - 1$ and $\varphi_1(x_0) = \frac{2}{3} (\lambda^2 + \lambda + \frac{7}{3})^2 - 1 - \lambda^2$, and so $|a_3^2 - a_5| \leq \lambda \left(\frac{2}{3} (\lambda^2 + \lambda + \frac{7}{3})^2 - 1 - \lambda^2\right)$.

Next we consider the Generalized Zalcman conjecture.

**Theorem 2.2.** Let $f$ in $\mathcal{U}(\lambda)$ be of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. Then

(i) $|a_2a_3 - a_4| \leq \lambda(\lambda + 1)$, $0 < \lambda \leq 1$;

(ii) $|a_2a_4 - a_5| \leq \lambda(1 + \lambda + \lambda^2)$, $\sqrt{\frac{3}{2}} \leq \lambda \leq 1$.

Both results are sharp as the function $f_\lambda$ given by (1.5) shows.

**Proof.**

(i) Using (1.2), (1.3) and (1.4) we obtain
\[
|a_2a_3 - a_4| = |\lambda c_2 + \lambda a_2c_1|
\leq \lambda(|c_2| + |a_2||c_1|)
\leq \lambda \left(\frac{1}{2}(1 - |c_1|^2) + (1 + \lambda)|c_1|\right)
\leq \lambda(1 + \lambda).
\]

(ii) Similarly,
\[
|a_2a_4 - a_5| = \lambda|c_3 + a_2c_2 + a_3^2c_1 + \lambda c_2^2|
\leq \lambda(|c_3| + |a_2||c_2| + |c_1||a_2^2 + \lambda c_1|)
\leq \lambda \left[\frac{1}{3} \left(1 - |c_1|^2 - \frac{4|c_2|^2}{1 + |c_1|}\right) + (1 + \lambda)|c_2| + (1 + \lambda + \lambda^2)|c_1|\right]
= \lambda g_2(|c_1|, |c_2|),
\]
where
\[
g_2(x, y) = \frac{1}{3} \left(1 - x^2 - \frac{4y^2}{1 + x}\right) + (1 + \lambda)y + (1 + \lambda + \lambda^2)x,
\]
$0 \leq x = |c_1| \leq 1$, $0 \leq y = |c_2| \leq \frac{1}{2}(1 - x^2)$.

Since $\frac{\partial g_2(x, y)}{\partial x} > 0$ for all $(x, y) \in E$, then the function $g_2$ attains its maximum on the boundary of $E$. In that sense, let consider all cases
For \( x = 0 \), we have \( 0 \leq y \leq \frac{1}{2} \) and
\[
g_2(0, y) = \frac{1}{3}(1 - 4y^2) + (1 + \lambda)y
= \frac{1}{3}[-4y^2 + 3(1 + \lambda)y + 1]
\leq \frac{1}{2}(1 + \lambda),
\]
since \( \lambda \geq \sqrt{\frac{2}{3}} > \frac{1}{3} \).

For \( 0 \leq x \leq 1 \) and \( y = 0 \) we have
\[
g_2(x, 0) = \frac{1}{3}(1 - x^2) + (1 + \lambda + \lambda^2)x
= \frac{1}{3}[-x^2 + 3(1 + \lambda + \lambda^2)x + 1]
\leq 1 + \lambda + \lambda^2.
\]

Finally, for \( 0 \leq x \leq 1 \), \( y = \frac{1}{2}(1 - x^2) \) we have
\[
g_2 \left( x, \frac{1}{2}(1 - x^2) \right) = \frac{1}{2}(1 + \lambda) + \left( \lambda^2 + \lambda + \frac{4}{3} \right) x - \frac{1}{2}(1 + \lambda)x^2 - \frac{1}{3}x^3 \equiv \varphi_2(x).
\]
Since for \( \lambda^2 \geq \frac{2}{3} \),
\[
\varphi_2'(x) = \lambda^2 + \lambda + \frac{4}{3} - (1 + \lambda)x - x^2 \geq \lambda^2 - \frac{2}{3} \geq 0,
\]
we realize that \( \varphi_2 \) is an increasing function and
\[
\varphi_2(x) \leq \varphi_2(1) = 1 + \lambda + \lambda^2.
\]

From all the previous facts, we have the statement of the theorem. \( \square \)

3. **Krushkal inequality for the class \( \mathcal{U}(\lambda) \)**

The inequality
\[
|a_n^p - a_2^p(n-1)| \leq 2^p(n-1) - n^p
\]
was introduced by Krushkal who proved its sharpness for the whole class of univalent functions in \( \mathcal{H} \). In this section we give direct proof over the class \( \mathcal{U}(\lambda) \) for the cases \( n = 4, p = 1 \) and \( n = 5, p = 1 \).

**Theorem 3.1.** Let \( f \) in \( \mathcal{U}(\lambda) \) be of the form \( f(z) = z + a_2z^2 + a_3z^3 + \cdots \). Then
\[(i) \ |a_4 - a_2^3| \leq 2\lambda(\lambda + 1);
(ii) \ |a_5 - a_2^4| \leq \lambda(3 + 5\lambda + 3\lambda^2).
\]
Both results are sharp as the function \( f_\lambda \) given by (1.3) shows.
Proof.

(i) Similarly as in the proofs of two previous theorem, we have

\[ \begin{align*}
|a_4 - a_3^2| &= \lambda|c_2 + 2a_2c_1| \\
&\leq \lambda(|c_2| + 2|a_2||c_1|) \\
&\leq \lambda \left[ \frac{1}{2}(1 - |c_1|^2) + 2(1 + \lambda)|c_1| \right] \\
&\leq 2\lambda(1 + \lambda).
\end{align*} \]

(ii) We can easily verify that

\[ \begin{align*}
|a_5 - a_4^2| &= \lambda|c_3 + 2a_2c_2 + \lambda c_1^2 + 3a_2^2c_1| \\
&= \lambda|c_3 + 2a_2c_2 - 2\lambda c_1^2 + 3c_1(\lambda c_1 + a_2^2)| \\
&\leq \lambda(|c_3| + 2|a_2||c_2| + 2\lambda|c_1|^2 + 3|c_1||\lambda c_1 + a_2^2|) \\
&\leq \lambda \left[ \frac{1}{3} \left( 1 - |c_1|^2 - \frac{4|c_2|^2}{1 + |c_1|} \right) + 2(1 + \lambda)|c_2| + 2\lambda|c_1|^2 + 3(1 + \lambda + \lambda^2)|c_1| \right] \\
&= \lambda g_3(|c_1|, |c_2|),
\end{align*} \]

where

\[ g_3(x, y) = \frac{1}{3} \left( 1 - x^2 - \frac{4y^2}{1 + x} \right) + 2(1 + \lambda)y + 2\lambda x^2 + 3(1 + \lambda + \lambda^2)x, \]

\[ 0 \leq x = |c_1| \leq 1, \quad 0 \leq y = |c_2| \leq \frac{1}{2}(1 - x^2). \]

Since \( \frac{\partial g_3(x, y)}{\partial x} > 0 \), \((x, y) \in E\), then we have that \( g_3 \) attains its maximum on the boundary of \( E \).

For \( x = 0 \), \( 0 \leq y \leq \frac{1}{2} \) we have

\[ g_3(0, y) = \frac{1}{3}(1 - 4y^2) + 2(1 + \lambda)y \leq (1 + \lambda). \]

For \( 0 \leq x \leq 1 \), \( y = 0 \) we have

\[ g_3(x, 0) = \left( 2\lambda - \frac{1}{3} \right) x^2 + 3(1 + \lambda + \lambda^2)x + \frac{1}{3} \leq 2\lambda + 3(1 + \lambda + \lambda^2) = 3 + 3\lambda + 3\lambda^2. \]

Finally, for \( 0 \leq x \leq 1 \), \( y = \frac{1}{2}(1 - x^2) \),

\[ g_3 \left( x, \frac{1}{2}(1 - x^2) \right) = 1 + \lambda + \left( \frac{1}{3} + 3(1 + \lambda + \lambda^2) \right) x - (1 - \lambda)x^2 - \frac{1}{3} x^3 \equiv \varphi_3(x). \]

Since

\[ \varphi_3'(x) = \frac{1}{3} + 3(1 + \lambda + \lambda^2) - 2(1 - \lambda)x - x^2 \geq \frac{1}{3} + 5\lambda + 3\lambda^2 > 0, \]

\( \varphi_3 \) is an increasing function and

\[ \varphi_3(x) \leq \varphi_3(1) = 3 + 5\lambda + 3\lambda^2. \]
All those facts imply the conclusion of theorem.

4. Hankel determinant of second and third order

Let \( f \in A \). Then the \( q \)th Hankel determinant of \( f \) is defined for \( q \geq 1 \), and \( n \geq 1 \) by

\[
H_q(n) = \begin{vmatrix}
  a_n & a_{n+1} & \cdots & a_{n+q-1} \\
  a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2}
\end{vmatrix}.
\]

Thus, the second and the third Hankel determinants are, respectively,

\[
H_2(2) = a_2a_4 - a_3^2,
\]
\[
H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2).
\]

In [6] the authors showed that for the class \( U \), the following estimates are sharp:

\[
|H_2(2)| \leq 1 \quad \text{and} \quad |H_3(1)| \leq \frac{1}{4}.
\]

Here we generalize this result for the classes \( U(\lambda) \).

**Theorem 4.1.** Let \( f \in U(\lambda) \), \( 0 < \lambda \leq 1 \), and \( f(z) = z + a_2z^2 + a_3z^3 + \cdots \). Then

\[
|H_2(2)| \leq \frac{\lambda(\lambda + 1)}{2} \quad \text{and} \quad |H_3(1)| \leq \frac{\lambda^2}{4}.
\]

The second result is sharp due to the function \( f(z) = \frac{z}{1-\lambda z^2} = z + \frac{\lambda z^4}{2} + \frac{\lambda^2 z^7}{4} + \cdots \).

**Proof.** Using the relation (1.3), after some calculations we have:

\[
|H_2(2)| = |a_2a_4 - a_3^2| = \lambda |a_2c_2 - \lambda c_1^2|
\]
\[
\leq \lambda (|a_2||c_2| + \lambda |c_1|^2)
\]
\[
\leq \lambda (\lambda + 1) \left( \frac{1 - |c_1|^2}{2} + \lambda |c_1|^2 \right)
\]
\[
\leq \lambda \left( \frac{\lambda + 1}{2} + \left( \frac{\lambda - 1}{2} \right) |c_1|^2 \right)
\]
\[
\leq \frac{\lambda(\lambda + 1)}{2},
\]

when \( 0 \leq |c_1|^2 \leq 1 \), because \( \lambda - 1 \leq 0 \).
In a similar way, using (1.2) and (1.3), after some calculations, for the third order Hankel determinant we have

\[ |H_3(1)| = \lambda^2|c_1c_3 - c_2^2| \leq \lambda^2(|c_1||c_3| + |c_2|^2) \]
\[ \leq \lambda^2 \left[ \frac{|c_1|}{3} \left( 1 - |c_1|^2 - \frac{4|c_2|^2}{1+|c_1|} \right) + |c_2|^2 \right] \]
\[ = \frac{\lambda^2}{3} \left[ |c_1| - |c_1|^3 + \frac{3 - |c_1|}{1+|c_1|} \cdot |c_2|^2 \right] \]
\[ \leq \frac{\lambda^2}{3} \left[ |c_1| - |c_1|^3 + \frac{3 - |c_1|}{1+|c_1|} \cdot \frac{(1 - |c_1|^2)^2}{4} \right] \]
\[ = \frac{\lambda^2}{12} (3 - 2|c_1|^2 - |c_1|^4) \leq \frac{3\lambda^2}{12} = \frac{\lambda^2}{4}. \]

Equality is attained for the function \( f(z) = \frac{z}{1 - (\lambda/2)z} = z + \frac{\lambda^2}{2} + \frac{\lambda^2 z^2}{4} + \cdots. \) \( \square \)

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