Many-body correlations in nuclear superfluidity

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The two-fermion two-point correlation function in the pairing channel is discussed in the equation of motion framework. Starting from the bare two-fermion interaction, we derive the equation of motion for the two-fermion pair propagator in a strongly-correlated medium. The resulting equation is of the Dyson type with the kernel having a static and a one frequency dependent components and, thus, can be regarded as Dyson Bethe-Salpeter equation (Dyson-BSE). The many-body hierarchy generated by the dynamical interaction kernel is truncated on the level of two-body correlation functions, thus neglecting the explicit three-body and higher-rank correlations. The truncation is performed via a cluster expansion of the intermediate three-particle-one-hole correlation function irreducible in the particle-particle channel, that leads to the coupling between single fermions and emergent collective modes of excitation. The latter couplings are, thus, derived in terms of the exact mapping of the in-medium two-fermion correlation functions onto the domain of bosonic quasibound states (phonons) without introducing new parameters. The approach is applied to calculations of the pairing gaps in medium-mass nuclear systems, that include calcium, nickel and tin isotopic chains.

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I. INTRODUCTION

Theoretical description of strongly-interacting many-body systems remains one of the most difficult areas of physical sciences and, despite the many years of effort, still requires more elaborate modeling. An accurate treatment of many-body correlations is the key to unraveling the mechanisms of emergent phenomena in the strongly-coupled systems at various scales of physics, however, it is very difficult in the non-perturbative regimes.

Atomic nuclei are among the systems, where not only the many-body correlations are extremely difficult to treat in an accurate and systematic way, but even the underlying nucleon-nucleon forces are not known precisely. Being rooted in quantum chromodynamics (QCD) on the fundamental level, the nucleon-nucleon interaction still can not be consistently derived from QCD in the form of potentials. The latter can be, instead, modeled by the meson-nucleon dynamics and parametrized by scattering data \cite{1,2}. However, the use of such potentials in the standard many-body frameworks does not yet lead to an accurate description of nuclear phenomena. Thus, both the nucleon-nucleon interactions and the strongly-coupled many-body models require further refinement.

One of the most interesting problems is the understanding and predictive description of phenomena related to nuclear superfluidity. It was noticed shortly after the appearance of the Bardeen-Cooper-Schrieffer (BCS) theory of superconductivity \cite{9} that atomic nuclei behave in some respects similarly to superconducting metals \cite{4}. Indeed, the reduction of nuclear moments of inertia, compared to the case of rigid rotation, the odd-even mass differences, low-lying vibrational states, nuclear shapes and level densities can only be reproduced under the assumption of the presence of an interaction acting between particles with equal and opposite momenta (pairing). Over the decades the BCS and the more general Bogoliubov’s theory are widely used for the description of open-shell nuclei \cite{5,6}.

It has become clear quite early that the underlying mechanism of nuclear pairing can be more complex than it is implied in the BCS and Bogoliubov’s approaches. In particular, the coupling between single-particle and emergent collective degrees of freedom (phonons), which plays a significant role in the formation of the nuclear ground and excited states \cite{8,13}, may also affect nuclear pairing. This idea was investigated in various phenomenological frameworks \cite{14,15} that concluded, in particular, that coupling between nucleons and collective surface vibrations (particle-vibration coupling, or PVC) can be responsible for a large fraction of the nuclear pairing. The PVC effects are widely known to be of prime importance in electronic condensed matter systems, where they can even reverse the sign of the repulsive Coulomb interaction to give rise to superconductivity \cite{19,20}.

The common deficiency of the state-of-the-art approaches to nuclear structure, which may also affect the current understanding of the nuclear superfluidity, is that they tend to combining different techniques for the static and dynamical parts of the in-medium interactions. This, however, may lead to uncontrollable inconsistencies, double counting and missing effects, especially those of collective character. Indeed, the exact equations of motion (EOM) for the many-body fermionic correlation functions, which are known across the many areas of quantum physics from condensed matter to quantum chemistry \cite{21,24}, show that the static and dynamical kernels of these equations are mutually dependent. A consistent treatment of both kernels in a uni-
fied framework is, therefore, necessary for reproducing the collective emergent phenomena from first principles. This has been justified to be a feasible while yet a highly accurate approach, if the infinite EOM hierarchies of the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) type are truncated by cluster expansions of the dynamical kernels in terms of the many-body correlation functions corresponding to the relevant degrees of freedom \[24,25\]. For instance, in the low and intermediate-energy regimes of nuclear physics truncations of the BBGKY chains on the level of two-body or three-body correlation functions should be sufficient for a highly accurate theory applicable to a wide range of nuclear phenomena.

The model-independent EOM method \[25,29\] was shown to produce a hierarchy of approximations to the dynamical kernels of the equations for one-fermion and two-time two-fermion propagators. As we discussed in our recent study of Ref. \[30\], the non-perturbative versions of those kernels, approximated by cluster expansions in terms of the single-particle, particle-hole and particle-particle correlation functions, can be mapped to the kernels of the phenomenological nuclear field theories (NFT) \[8,12,51,62\] and quasiparticle-phonon models (QPM) \[13,33,35\]. This mapping provides an understanding of the emergent collective phenomena and a microscopic foundation for the effective Hamiltonians used in these models, connecting the bare nucleon-nucleon interaction and its modification in the strongly-coupled medium. Moreover, such insights allow for considerable extensions of this type of theories to more complex correlations, which are necessary for achieving spectroscopic accuracy in the description of atomic nuclei in a wide energy range.

In Ref. \[30\] such an extension was presented and implemented numerically on the base of the covariant density functional theory (DFT), thus advancing the previously developed relativistic version of the NFT \[36,41\]. Although the latter demonstrated a considerable progress implementing the PVC models in the covariant self-consistent framework and described satisfactory some low-energy nuclear phenomena \[12,46\], it was still lacking the spectroscopic accuracy because of the absence of more complex correlations than in the conventional NFT. In Ref. \[30\] we have shown, in particular, that the higher-order correlations beyond the two-quasiparticle-plus-phonon (2q\@phonon) ones, for instance, 2q\@2phonon configurations can introduce some noticeable improvements in the description of the nuclear spectra at both low and high energies. Other types of correlations, which are rarely addressed in the literature on the NFT and other PVC models, such as the PVC-induced ground state correlations \[31,47,49\] and the coupling to charge-exchange phonons \[50,51\] were also shown to be important for spectroscopically-accurate theories. Recent finite-temperature extensions can be found in Refs. \[52–55\].

In this article we continue to elaborate on the EOM method for fermionic correlation functions and its connections to the phenomenological NFT’s. While Ref. \[30\] was focused on the two-point single-particle and particle-hole fermionic propagators, here we discuss the two-point two-fermion propagator and the associated pairing gap equation. Some of the closely related ideas on the theory of correlated fermion pairs and ab-initio particle-vibration coupling approach were discussed recently in Ref. \[56\]. The formalism starts along the lines of Refs. \[29,50\] and then advances to non-perturbative approximations for the dynamical interaction kernel. In the theoretical sections we discuss fermionic Hamiltonians with unspecified interactions, while the equations of motion are confined by the two-body interactions. Thus, the theory can be naturally extended to multiparticle forces and bosonic degrees of freedom.

## II. FERMIONIC PROPAGATORS IN A CORRELATED MEDIUM

The formalism of correlation functions, such as the Green functions, or propagators, is one of the most convenient and powerful ones in the description of phenomena that occur in strongly-coupled media. The propagators are directly related to observed excitation spectra and ground state properties of the many-body systems.

The single-fermion propagator is commonly defined as the amplitude:

$$G(1, 1') = G_{11'}(t - t') = -i\langle T\psi(1)\psi^\dagger(1')\rangle,$$  \hspace{1cm} (1)

where \(T\) is the operator of the chronological ordering and \(\psi(1), \psi^\dagger(1)\) are the one-fermion fields in the Heisenberg picture:

$$\psi(1) = e^{iHt_1}\psi_1 e^{-iHt_1}, \quad \psi^\dagger(1) = e^{iHt_1}\psi_1^\dagger e^{-iHt_1},$$  \hspace{1cm} (2)

while the subscript ‘1’ stands for the full set of the single-particle quantum numbers in a given representation. In the present work the fermionic degrees of freedom are associated with nucleons which compose a many-nucleon system. The averaging in Eq. (1) and in the following is performed over the formally exact correlated ground state while the time evolution is determined by the many-body Hamiltonian

$$H = H^{(1)} + V^{(2)} + W^{(3)} + \ldots,$$  \hspace{1cm} (3)

Where the operator \(H^{(1)}\) is the one-body contribution to the Hamiltonian:

$$H^{(1)} = \sum_{12} t_{12} \psi_1^\dagger \psi_2 + \sum_{12} v_{12}^{(MF)} \psi_1^\dagger \psi_2 \equiv \sum_{12} h_{12} \psi_1^\dagger \psi_2,$$  \hspace{1cm} (4)

with the matrix elements \(h_{12}\) which, in general, combine the kinetic energy \(t\) and the mean-field \(v^{(MF)}\) part of the interaction. The operator \(V^{(2)}\) describes the two-body sector associated with the two-fermion interaction

$$V^{(2)} = \frac{1}{4} \sum_{1234} v_{1234} \psi_1^\dagger \psi_2^\dagger \psi_3 \psi_4,$$  \hspace{1cm} (5)

and the operator \(V^{(3)}\) generates the three-body forces

$$W^{(3)} = \frac{1}{36} \sum_{123456} w_{123456} \psi_1^\dagger \psi_2^\dagger \psi_3 \psi_4 \psi_5 \psi_6.$$  \hspace{1cm} (6)
with the antisymmetrized matrix elements $\bar{v}_{1234}$ and $\bar{v}_{123456}$, respectively. The ellipsis in Eq. (8) stands for further multiparticle forces which can be, in principle, included in the theory. We will make an explicit derivation of the equations of motion assuming that the Hamiltonian is confined by the two-body interaction, however, the theory can be naturally extended to multiparticle forces.

We will work in the basis, which diagonalizes the one-body (also named single-particle or mean-field) part of the Hamiltonian $\tilde{h}_{12} = \delta_{12} \tilde{c}_1$. We will see, however, that on the way to the final equations of motion this basis should be redefined as soon as the one-body part of the Hamiltonian absorbs additional contributions from the two-body sector.

The fermionic field operators satisfy the anticommutation relations:

$$[\psi_1, \psi^\dagger_1],_+ \equiv \psi_1 \psi^\dagger_1 + \psi^\dagger_1 \psi_1 = \delta_{11},$$

$$[\psi_1, \psi^\dagger_1],_+ = [\psi^\dagger_1, \psi_1],_+ = 0.$$  (7)

The Fourier transform of the single-particle propagator $G^{(1)}$, which depends explicitly on the time difference $\tau = t - t'$, is known as the spectral (Lehmann) representation:

$$G^{(1)}(\varepsilon) = \sum_n \frac{\eta_n^{\mu \nu}}{\varepsilon - \varepsilon_n + i\delta} + \sum_m \frac{\chi_m^{\mu \nu}}{\varepsilon + \varepsilon_m - i\delta}.$$  (8)

with the poles $\varepsilon^+_n = \varepsilon_n + (E^{(N+1)}_n - E^{(N)}_0)$ and $\varepsilon^-_m = -(E^{(N-1)}_m - E^{(N)}_0)$ at the energies of the states in $(N+1)$- and $(N-1)$-particle systems related to the ground state of the initial $N$-particle system. The residues of Eq. (8) are, in turn, composed of matrix elements of the field operators between the ground state $|0^{(N)}\rangle$ of the $N$-particle system and the states $|\rho^{(N+1)}\rangle$ and $|\rho^{(N-1)}\rangle$ of the $(N+1)$- and $(N-1)$-particle systems, respectively:

$$\eta^{\mu}_n = \langle 0^{(N)} | \psi_1 | \rho^{(N+1)} \rangle, \quad \chi^{\mu}_m = \langle \rho^{(N-1)} | \psi_1 | 0^{(N)} \rangle.$$  (9)

These matrix elements represent the weights of the given single-particle (single-hole) configuration on top of the ground state $|0^{(N)}\rangle$ in the formally exact $n$-th $(m$-th$)$ state of the systems with $(N+1)$ and $(N-1)$ particles system. The residues correspond to the observed occupation probabilities of the corresponding states and related to the spectroscopic factors.

In analogy to Eq. (1), the two-fermion, three-fermion and, in general, $n$-fermion propagators are defined as follows:

$$G(12, 1'2') = -(i)^2 \langle T \psi(1) \psi(2) \psi^\dagger(2') \psi^\dagger(1') \rangle,$$  (10)

$$G(123, 1'2'3') = -(i)^3 \langle T \psi(1) \psi(2) \psi(3) \psi^\dagger(3') \psi^\dagger(2') \psi^\dagger(1') \rangle,$$

$$G(12...n, 1'2'...n') = -(i)^n \langle T \psi(1) \psi(2) ... \psi(n) \psi^\dagger(n') ... \psi^\dagger(2') \psi^\dagger(1') \rangle.$$  (11)

In this work we will focus on the two-time-two-fermion Green function (10) with $t_1 = t_2 = t, t_{1'} = t_{2'} = t'$, which depends on the single time difference $t - t'$. In this case, with the help of Eqs. (7), Eq. (10) can be transformed to the energy (frequency) domain as:

$$iG_{12, 1'2'}(\omega) = \sum_{\mu} \frac{\alpha^{\mu \mu}_1 \alpha^{\mu \mu}_{2'} + i\delta}{\omega - \omega^{(+)}_\mu + i\delta} - \sum_{\kappa} \frac{\beta^{\kappa \mu}_2 \beta^{\mu \kappa}_{2'}}{\omega + \omega^{(-)}_\kappa - i\delta}.$$  (12)

where the poles $\omega^{(++)}_\mu = E^{(N+2)}_{\mu} - E^{(N)}_0$ and $\omega^{(--)}_\kappa = E^{(N-2)}_{\kappa} - E^{(N)}_0$ are the formally exact states of the systems with $(N+2)$ and $(N-2)$ particles, respectively, and the residues are the products of the matrix elements:

$$\alpha^{\mu \mu}_{12} = \langle 0^{(N)} | \psi_2 \psi_1 | \mu^{(N+2)} \rangle, \quad \beta^{\kappa \mu}_{12} = \langle 0^{(N)} | \psi_2^\dagger \psi_1^\dagger | \mu^{(N-2)} \rangle.$$  (13)

As they connect the states of the $(N+2)$- and $(N-2)$-particle systems to the ground state of the initial $N$-particle system, the two-body propagator of Eq. (12) describes the response to the probes with pair transfer (addition and removal of two fermions, respectively). For further analysis it is convenient to include the phase factor "$i$" into the two-body propagator, so that here we start to use the modified definition:

$$G(12, 1'2') = -i \langle T \psi(1) \psi(2) \psi^\dagger(2') \psi^\dagger(1') \rangle,$$  (14)

i.e. replace $iG(12, 1'2') \rightarrow G(12, 1'2')$.

Earlier in Ref. [30] we have considered the two-time-two-fermion propagator in the particle-hole channel. It was shown, in particular, that an accurate description of this response function requires the EOM with the dynamical kernel where the particle-hole (ph) and particle-particle (pp) channels are coupled. Similarly, we will see below that the EOM for the pp-channel which describes the propagator of Eq. (12) will require the knowledge about the particle-hole response function:

$$R(12, 1'2') = -i \langle T \psi^\dagger(1) \psi(2) \psi^\dagger(2') \psi^\dagger(1') \rangle,$$  (15)

which also depends on two times as $t_1 = t_2 = t, t_{1'} = t_{2'} = t'$ and whose spectral image, or Fourier transform, reads:

$$R_{12, 1'2'}(\omega) = \sum_{\nu > 0} \frac{\rho^{\nu \nu}_{21} \rho^{\nu \nu}_{2'} \rho^{\nu \nu}_{12} \rho^{\nu \nu}_{1'} - \rho^{\nu \nu}_{21} \rho^{\nu \nu}_{2'} \rho^{\nu \nu}_{12} \rho^{\nu \nu}_{1'}}{\omega - \omega_{\nu} + i\delta - \omega - \omega_{\nu} - i\delta}.$$  (16)

Similarly to the ones for the one-fermion and two-fermion propagators (8,12), it satisfies the general quantum field theory requirements of locality and unitarity with the residues composed of the properly normalized matrix elements of the transition densities:

$$\rho^{\nu \nu}_{12} = \langle 0 | \psi_2 \psi_1 | \nu \rangle.$$  (17)

They describe the weights of the pure particle-hole configurations on top of the ground state $|0\rangle$ in the model (ideally, exact) excited states $|\nu\rangle$ of the (even-even) $N$-particle system. The corresponding poles are the excitation energies of this system $\omega_{\nu} = E_{\nu} - E_0$.

Obviously, the spectral representation of the propagators given by Eqs. (8,12,16) are model independent: they are valid regardless how the many-body states $|\nu\rangle, |\nu^\prime\rangle, |\nu^{\prime\prime}\rangle, |\mu\rangle$ and $|\kappa\rangle$ are modeled. The sums in Eqs. (8,12,16) run over both the discrete and continual sectors of the excitation spectra, i.e. formally complete.

Summarising, because of their simple relations to various observables, fermion propagators are important characteristics of strongly-coupled many-body systems, in particular,
of atomic nuclei. In Ref. [30] we have investigated the propagators of Eqs. (8) and (15) by generating the equations of motion for them. In the present work we will focus on the particle-particle Green function of Eqs. (10, 12) by considering its time evolution and investigating its potential of describing nuclear pairing vibrations.

III. EQUATION OF MOTION FOR THE TWO-POINT PAIR CORRELATION FUNCTION

A. The model-independent EOM

The time evolution of the correlation function of a fermionic pair can be investigated by the differentiation with respect to the time variable applied to Eq. (10):

$$\partial_t G_{12,1'2'}(t-t') = -i\delta(t-t')[\psi_1 \psi_2, \psi_1^{\dagger} \psi_2^{\dagger}].$$

(18)

where we defined

$$[H, A](t) = e^{iHt}[H, A]e^{-iHt}$$

(19)

for an arbitrary operator $A$ and adopted the notation $G_{12,1'2'}(t-t') = G(12,1'2')$ for the two-time particle-particle propagator with $t_1 = t_2 = t, t_1 = t_2' = t'$. The commutators can be computed straightforwardly:

$$[\psi_1 \psi_2, \psi_1^{\dagger} \psi_2^{\dagger}] = \delta_{22'} \psi_1 \psi_1^{\dagger} - \delta_{11'} \psi_2 \psi_2^{\dagger} - \delta_{1'2} \psi_2 \psi_1^{\dagger} + \delta_{12'} \psi_1 \psi_1^{\dagger},$$

(20)

$$[H, \psi_1 \psi_2] = -\epsilon_1 \psi_1 + \epsilon_2 \psi_2 + [V, \psi_1 \psi_2],$$

(21)

so that the first EOM takes the form:

$$(i\partial_t - \epsilon_1 - \epsilon_2)G_{12,1'2'}(t-t') = \delta(t-t')N_{121'2'} + i\langle T(V, \psi_1 \psi_2)(t)(\psi_1^{\dagger} \psi_2^{\dagger}),(t')\rangle,$$

(22)

where we introduced the norm matrix in the pp-channel as the ground state average of the commutator of Eq. (20):

$$N_{121'2'} = \langle [\psi_1 \psi_2, \psi_1^{\dagger} \psi_2^{\dagger}] \rangle.$$
by introducing the new kernel $K(\omega)$ which can be obtained from $T(\omega)$ by retaining only the terms irreducible with respect to the uncorrelated pp-propagator \cite{42}:

$$T_{121'2'}(\omega) = K_{121'2'}(\omega) +$$

$$+ \frac{1}{4} \sum_{343'4'} K_{1234}(\omega)G_{34,3'4'}^{(0)}(\omega)T_{3'4'1'2'}(\omega)$$

(34)

or $K(\omega) = T^{(irr)}(\omega)$. Obviously, the removal of the reducible contributions affects only the dynamical part of $T$. Remarkably, as in the case of the particle-hole response, Eq. \cite{33} has the form of the Dyson equation. Its interaction kernel contains a static and a one frequency dependent parts, in full analogy to the Dyson equation for the one-body propagator. In other words, the Bethe-Salpeter equation for the two-point two-body Green functions can be regarded as Dyson Bethe-Salpeter equation (Dyson-BSE) \cite{33}.

In the next two subsections we consider the static and dynamical parts of the kernel $T$ \cite{32} in the explicit form.

\section*{B. The static kernel}

The static part requires evaluating the commutator:

$$[V, \psi_1 \psi_2]_{\omega} = \frac{1}{4} \sum_{ijkl} \bar{v}_{ijkl}[\psi_1^\dagger \psi_1^\dagger \psi_2^\dagger \psi_2] =$$

$$= \frac{1}{2} \sum_{ijkl} \bar{v}_{ijkl}(\delta_{ij} \psi_i^\dagger \psi_2 + \delta_{2j} \psi_1^\dagger \psi_k)\psi_1 \psi_k =$$

$$= \frac{1}{2} \sum_{ijkl} \bar{v}_{ijkl}(\delta_{ij} \psi_i^\dagger \psi_2 + \delta_{2j} \delta_{i1} - \delta_{ij} \psi_1^\dagger \psi_1)\psi_1 \psi_k,$$

so that

$$T_{121'2'}^{(0)} = \left< \left< [V, \psi_1 \psi_2], \psi_1^\dagger \psi_2^\dagger \psi_1^\dagger \right> \right> = \frac{1}{2} \sum_{kl} \bar{v}_{kl} N_{1k,1'2'} +$$

$$+ \left< \left< \sum_{ikl} \bar{v}_{ikl}(\delta_{2k} \psi_k^\dagger \psi_2^\dagger \psi_1^\dagger \psi_1, \delta_{1k} \psi_1^\dagger \psi_1^\dagger \psi_2^\dagger \psi_k +$$

$$+ \frac{1}{2} \delta_{2k} \psi_k^\dagger \psi_1^\dagger \psi_1^\dagger \psi_1, \psi_1^\dagger \psi_1^\dagger \psi_1^\dagger \psi_1) - (1 \leftrightarrow 2) \right>.$$  

(36)

By reorganizing the fermionic field operators into the two-body densities

$$\rho_{ijkl} = \left< \psi_i^\dagger \psi_j^\dagger \psi_j \psi_i \right> = \rho_{ij}, \rho_{jk} - \rho_{il} \rho_{jk} + \sigma_{ijkl}^{(2)},$$

and introducing the mean-field single-particle energies

$$\tilde{\Sigma}_{1'2'} = \sum_l \bar{v}_{l1'2'} n_l, \quad \tilde{\Sigma}_{11'} = \delta_{11'} \tilde{\Sigma},$$

Eq. (36) can be written as follows:

$$T_{121'2'}^{(0)} = -\delta_{121'2'} n_{12} (\tilde{\Sigma}_1 + \tilde{\Sigma}_2) + K_{121'2'}^{(0)},$$

(39)

FIG. 1. Diagrammatic form of the static part $K^{(0)}$ of the interaction kernel given by Eq. (40). Lines with arrows represent one-fermion propagators. Rectangular blocks $\psi$ and $\bar{\psi}$ stand for the non-antisymmetrized and antisymmetrized bare two-fermion interaction and those marked with $\sigma^{(2)}$, together with the attached long fermionic lines, for the fully correlated part of the two-body density. The abbreviation "AS" denotes the full antisymmetrization. The cross stands for the Kronecker delta symbol $\delta_{22'}$. 

$$K_{121'2'}^{(0)} = -\bar{v}_{121'2'} n_{12} n_{1'2'} +$$

$$+ \left[ \sum_{i} \bar{v}_{i12'} \sigma_{i21'}^{(2)} + \frac{\delta_{22'} \sigma_{ijkl}^{(2)}}{2} \right] - \left[ 1 \leftrightarrow 2 \right],$$

(40)

which are consistent with the obvious antisymmetry properties of the static kernel: $T^{(0)}_{121'2'} = -T^{(0)}_{121'2'} = T^{(0)}_{211'2'} = T^{(0)}_{212'1'}$. The first term on the right hand side of Eq. (39) contains the mean-field single-particle energies which can be absorbed in the uncorrelated propagator, so that

$$G_{121'2'}(\omega) = G_{121'2'}^{(0)}(\omega) +$$

$$+ \frac{1}{4} \sum_{343'4'} G_{12,34}^{(0)}(\omega) K_{343'4'}(\omega) G_{3'4',1'2'}(\omega),$$

(41)

where

$$G_{121'2'}^{(0)}(\omega) = \frac{N_{121'2'}}{\omega - \tilde{\epsilon}_1 - \tilde{\epsilon}_2}, \quad \tilde{\epsilon}_1 = \tilde{\epsilon}_1 + \tilde{\Sigma}_1,$$

(42)

and the kernel $K$ does not contain the mean-field term in its static part while the dynamical part remains unchanged: $K = N^{-1}(K^{(0)} + K^{(c)})N^{-1}$. The obtained static part of the interaction kernel $K_{121'2'}^{(0)}$ is shown in Fig. 2 in the diagrammatic representation. The form of Eq. (40) for the static kernel is consistent with Ref. [29], where it was derived in a similar way.

\section*{C. The dynamical kernel}

The dynamical part of the interaction kernel can be calculated with the aid of the commutator:

$$[V, \psi_2^\dagger \psi_2^\dagger \psi_1]_{\omega} = \frac{1}{4} \sum_{ijkl} \bar{v}_{ijkl}[\psi_i^\dagger \psi_j^\dagger \psi_j \psi_i, \psi_2^\dagger \psi_2^\dagger \psi_1] =$$

$$= \frac{1}{2} \sum_{ijkl} \bar{v}_{ijkl}(\delta_{2k} \psi_k^\dagger \psi_2^\dagger \psi_1^\dagger \psi_1, \delta_{2k} \psi_2^\dagger \psi_1^\dagger \psi_1) \bar{v}_{ijkl} =$$
\[
K^{(r;11)}_{121'2'} = \sum_{ijkl} \psi_i^+ \psi_j^+ (\delta_{k2}' \delta_{l1}' - \delta_{k2} \delta_{l1} \psi_i \psi_j + \delta_{k1} \delta_{l2} \psi_i \psi_j) \bar{v}_{ijkl},
\]
(43)

so that
\[
T^{(r)}_{121'2'}(t - t') = \frac{i}{4} \sum_{ijkl} \sum_{mnq} \bar{v}_{ijkl} \langle \delta_{i1} \psi_i \psi_2 + \delta_{i2} \psi_i \psi_1 \rangle \times \psi_i \psi_j (t) \psi_m \psi_n (\delta_{p1} \delta_{q1} \psi_i \psi_j + \delta_{p2} \delta_{q2} \psi_i \psi_j) (t') \bar{v}_{ijkl},
\]
(44)
or
\[
T^{(r)}_{121'2'}(t - t') = \frac{i}{4} \times \sum_{ijkl} \sum_{mnq} \bar{v}_{ijkl} \langle T(\psi_i^+ \psi_j^+ \psi_i \psi_j) (t) (\psi_m^+ \psi_n^+ \psi_i^+ \psi_j^+) (t') \bar{v}_{mn1'q} + \bar{v}_{i1kl} \langle T(\psi_i^+ \psi_j^+ \psi_i \psi_j) (t) (\psi_m^+ \psi_n^+ \psi_i^+ \psi_j^+) (t') \bar{v}_{mn2'q} + \bar{v}_{i2kl} \langle T(\psi_i^+ \psi_j^+ \psi_i \psi_j) (t) (\psi_m^+ \psi_n^+ \psi_i^+ \psi_j^+) (t') \bar{v}_{mn1'q} + \bar{v}_{i2kl} \langle T(\psi_i^+ \psi_j^+ \psi_i \psi_j) (t) (\psi_m^+ \psi_n^+ \psi_i^+ \psi_j^+) (t') \bar{v}_{mn2'q} \rangle.
\]
(45)

Thus, in complete analogy to the case of the particle-hole response \[19\], the dynamical kernel of the EOM for the particle-particle propagator is determined by the irreducible four-fermion correlation functions. The nature of these correlation functions is, however, different. Namely, each term of Eq. \[45\] contains a propagator of three particles and one hole (3p – 1h). Therefore, an approximate cluster expansion truncated on the two-body level, should contain all possible products of the particle-particle and particle-hole correlation functions. For instance, the internal propagator in the first term of Eq. \[45\] can be factorized as follows:
\[
G^{(11)irr}_{2klq,nm1'}(t - t') = (T(\psi_i^+ \psi_j^+ \psi_i \psi_j) (t) (\psi_m^+ \psi_n^+ \psi_i^+ \psi_j^+) (t')) \bar{v}_{mn1'q} + (T(\psi_i^+ \psi_j^+ \psi_i \psi_j) (t) (\psi_m^+ \psi_n^+ \psi_i^+ \psi_j^+) (t')) \bar{v}_{mn2'q} + (T(\psi_i^+ \psi_j^+ \psi_i \psi_j) (t) (\psi_m^+ \psi_n^+ \psi_i^+ \psi_j^+) (t')) \bar{v}_{mn1'q} + (T(\psi_i^+ \psi_j^+ \psi_i \psi_j) (t) (\psi_m^+ \psi_n^+ \psi_i^+ \psi_j^+) (t')) \bar{v}_{mn2'q}
\]
(46)
if the fully correlated three-body and four-body terms are neglected. Using the definitions \[14\]\[15\], the 3p – 1h propagator of Eq. \[46\] can be rewritten as
\[
G^{(11)irr}_{2klq,nm1'}(t - t') = -R_{i2,q1'}(t - t') G_{k1,nm1'}(t - t') - R_{i1,qn}(t - t') G^{(22)}_{klq,nm1'}(t - t') - R_{i1,qn}(t - t') G^{(22)}_{klq,nm1'}(t - t') - AS,
\]
(47)

The other three 3p – 1h correlation functions can be factorized similarly, so that
\[
G^{(22)}_{2klq,nm1'}(t - t') = -R_{i1,q1'}(t - t') G^{(22)}_{klq,nm1'}(t - t') - R_{i1,qn}(t - t') G^{(22)}_{1k,nm1'}(t - t') - AS,
\]
(49)

and the irreducible kernel takes the form:
\[
K^{(r)}_{121'2'}(t - t') = -\frac{i}{4} \times \sum_{ijkl} \sum_{mnq} \bar{v}_{ijkl} G^{(11)irr}_{121'2'}(t - t') \bar{v}_{mn1'q} + \bar{v}_{i1kl} G^{(12)irr}_{121'2'}(t - t') \bar{v}_{mn2'q} + \bar{v}_{i2kl} G^{(21)irr}_{121'2'}(t - t') \bar{v}_{mn1'q} + \bar{v}_{i2kl} G^{(21)irr}_{121'2'}(t - t') \bar{v}_{mn2'q}
\]
(50)

The cluster expansion of Eq. \[46\], thereby, shows how the many-body problem can be truncated on the level of two-body correlations in the pairing, or particle-particle, channel. The presence of the particle-hole propagators in the dynamical kernel of the pairing propagator expresses explicitly the coupling of the particle-hole and particle-particle channels in the many-body systems. This points out to the formulation of the truncated EOM's for fermionic correlation functions in a closed form \[24\]\[30\]\[37\].
IV. TRUNCATION SCHEMES FOR THE TWO-POINT CORRELATION FUNCTIONS

Alternatively to the symmetric form of the dynamical kernel $K^{(r)}$, which is obvious from Eq. (51) and Fig. 1 it can be expressed via the three-fermion propagator, if only the first EOM [22] is generated. More generally, the dynamical kernel of the EOM for the $n$-fermion Green function contains all up to $(n + 1)$-fermion Green functions in the non-symmetric form and all up to $(n + 2)$-fermion ones in the symmetric form, so that at least the one-fermion, two-fermion, ..., $(n + 1)$-fermion Green functions are needed to define the dynamics of the $n$-fermion Green function. Thus, the general structure of the many-body theory in terms of the two-point Green functions can be expressed as a series of EOM's of the Dyson's type:

\[
\hat{G}^{(1)} = \hat{G}^{(1)0} + \hat{G}^{(1)0} K^{(1)} [\hat{G}^{(1)}, \hat{G}^{(2)}] \hat{G}^{(1)} \\
\hat{G}^{(2)} = \hat{G}^{(2)0} + \hat{G}^{(2)0} K^{(2)} [\hat{G}^{(1)}, \hat{G}^{(2)}, \hat{G}^{(3)}] \hat{G}^{(2)} \\
\hat{G}^{(3)} = \hat{G}^{(3)0} + \hat{G}^{(3)0} K^{(3)} [\hat{G}^{(1)}, \hat{G}^{(2)}, \hat{G}^{(3)}, \hat{G}^{(4)}] \hat{G}^{(3)} \\
\cdots \\
\hat{G}^{(n)} = \hat{G}^{(n)0} + \hat{G}^{(n)0} K^{(n)} [\hat{G}^{(1)}, \hat{G}^{(2)}, \hat{G}^{(3)}, \ldots, \hat{G}^{(n+1)}] \hat{G}^{(n)} \\
\cdots,
\]

where $\hat{G}^{(n)}$ stand for the $n$-particle Green functions, or propagators, in the medium, and the positive integer index $n$ is, in principle, limited by the total particle number. The notion of the propagator $\hat{G}^{(n)}$ absorbs all possible channels ($q$-particle-$p$-hole, $q + p = n$) as well as the respective $n$-body densities, which can be obtained from $\hat{G}^{(n)}$ in the static limit, thus the generalized kernels $K^{(n)} [\hat{G}^{(1)}, \hat{G}^{(2)}, \hat{G}^{(3)}, \ldots, \hat{G}^{(n+1)}]$ include both static and dynamical parts. The index '0' marks the free propagators.

The idea of a truncation of this complex hierarchy on the $n$-body level suggests that the kernel of the $n$-body propagator does not depend explicitly on the higher-rank ones:

\[
\hat{G}^{(1)} = \hat{G}^{(1)0} + \hat{G}^{(1)0} K^{(1)} [\hat{G}^{(1)}, \hat{G}^{(2)}] \hat{G}^{(1)} \\
\hat{G}^{(2)} = \hat{G}^{(2)0} + \hat{G}^{(2)0} K^{(2)} [\hat{G}^{(1)}, \hat{G}^{(2)}, \hat{G}^{(3)}] \hat{G}^{(2)} \\
\hat{G}^{(3)} = \hat{G}^{(3)0} + \hat{G}^{(3)0} K^{(3)} [\hat{G}^{(1)}, \hat{G}^{(2)}, \hat{G}^{(3)}, \hat{G}^{(4)}] \hat{G}^{(3)} \\
\cdots \\
\hat{G}^{(n)} = \hat{G}^{(n)0} + \hat{G}^{(n)0} K^{(n)} [\hat{G}^{(1)}, \hat{G}^{(2)}, \hat{G}^{(3)}, \ldots, \hat{G}^{(n+1)}] \hat{G}^{(n)} \\
\cdots, \\
\]

where $m > n$. However, the higher-rank propagators are not neglected, but enter the dynamical kernels in the form of their cluster expansions in terms of the low-rank propagators. The essential feature of this truncation is that the EOM’s up to the rank $n$ form the closed system and can, in principle, be solved exactly. In particular, the truncation on one-body level

\[
\hat{G}^{(1)} = \hat{G}^{(1)0} + \hat{G}^{(1)0} K^{(1)} [\hat{G}^{(1)}] \hat{G}^{(1)} \\
\hat{G}^{(2)} = \hat{G}^{(2)0} + \hat{G}^{(2)0} K^{(2)} [\hat{G}^{(1)}, \hat{G}^{(2)}] \hat{G}^{(2)} \\
\hat{G}^{(3)} = \hat{G}^{(3)0} + \hat{G}^{(3)0} K^{(3)} [\hat{G}^{(1)}, \hat{G}^{(2)}] \hat{G}^{(3)} \\
\cdots \\
\hat{G}^{(n)} = \hat{G}^{(n)0} + \hat{G}^{(n)0} K^{(n)} [\hat{G}^{(1)}, \hat{G}^{(2)}, \hat{G}^{(3)}, \ldots, \hat{G}^{(n)}] \hat{G}^{(n)} \\
\cdots,
\]

implies that all the dynamical kernels are composed of one-body propagators, which can be found, in principle, as a self-consistent solution of the first equation. A static version of this approach is the traditional density functional theory, where the one-body density is the only variable. The approach considered in Ref. [30] and in the present work implies that

\[
\hat{G}^{(1)} = \hat{G}^{(1)0} + \hat{G}^{(1)0} K^{(1)} [\hat{G}^{(1)}] \hat{G}^{(1)} \\
\hat{G}^{(2)} = \hat{G}^{(2)0} + \hat{G}^{(2)0} K^{(2)} [\hat{G}^{(1)}, \hat{G}^{(2)}] \hat{G}^{(2)} \\
\hat{G}^{(3)} = \hat{G}^{(3)0} + \hat{G}^{(3)0} K^{(3)} [\hat{G}^{(1)}, \hat{G}^{(2)}] \hat{G}^{(3)} \\
\cdots \\
\hat{G}^{(n)} = \hat{G}^{(n)0} + \hat{G}^{(n)0} K^{(n)} [\hat{G}^{(1)}, \hat{G}^{(2)}] \hat{G}^{(n)} \\
\cdots,
\]

i.e that the many-body problem is truncated on the two-body level. This means that all the complex propagators, including those which are more complex than $G^{(2)}$, are calculated by making use the one- and two-body propagators $G^{(1)}$ and $G^{(2)}$ as building blocks.

As far as the pairing correlations are concerned, one may notice that the theory truncated on the two-body level does not involve the anomalous one-fermion Green functions of the Gorkov’s type [53]. Instead, the pairing correlations influence the one-fermion propagator via the two-fermion Green functions of the particle-particle type, which enter the dynamical kernel of the one-body equation of motion [30]. This fact has been noted already in Ref. [59] and briefly discussed there in a different context. On one hand, the approach with the two-fermion particle-particle correlation functions may look more complicated because it requires a solution of the coupled one-fermion and two-fermion EOM’s. But, on the other hand, it avoids working in the doubled quasiparticle space, that can be technically quite demanding in the models with explicit dynamical kernels.

V. EMERGENT PHONONS AND MAPPING TO THE PARTICLE-PHONON COUPLING

The EOM for the fermionic pair propagator [53] in the energy (frequency) domain contains the Fourier transform of the dynamical kernel [51]. As all the terms of its propagators’ expansion [47-50] consist of non-contracted products of one particle-particle and one particle-hole propagators, they can be treated with the aid of the following
Then, each product should be contracted with two matrix elements of the two-fermion interaction $\bar{v}$, as given by Eq. 51, so that the components of the dynamical kernel take the form:

$$K^{(r;11)}_{121'2'}(\omega) = -\frac{i}{4} \sum_{klm} \langle \Gamma \rangle_{\nu \mu} \times \left( [R_{12',q2'}G_{ik,mn}](\omega) + [R_{ik,qn}G_{12',m}](\omega) + [R_{il,qn}G_{k2',n}](\omega) - \text{AS} \right) \bar{v}_{mnq},$$

(57)

Similarly to Ref. 30, one can introduce the vertices $g^{(\pm)}_{13}$ and propagators $D^{(\sigma)}(\omega)$ of the normal phonons:

$$g^{(\pm)}_{13} = \delta_{\sigma,+}g^{(\pm)}_{13} + \delta_{\sigma,-}g^{(\pm)*}_{13}, \quad g^{(\pm)}_{13} = \sum_{34} \bar{v}_{134} \rho_{124}^{(\pm)};$$

$$D^{(\sigma)}(\omega) = \frac{\sigma}{\omega - \sigma(\omega + \omega_{\pm} - i\delta)}, \quad \omega_{\pm} = E_{\pm} - E_0,$$

(61)

as well as the particle-phonon coupling amplitude

$$\Gamma^{(ph)}_{13',1'3} = \sum_{242'd'} \bar{v}_{134} R^{(ph)}_{242'd'}(\omega) \bar{v}_{d'2'}(\omega) = \sum_{\nu,\sigma=\pm} \bar{v}_{134} D^{(\sigma)}(\omega) g^{(\sigma)*}_{134},$$

(62)

Analogously, the vertices $\gamma^{(\pm)}_{12}$ and propagators $\Delta^{(\sigma)}_{\nu}(\omega)$ of the pairing, or superfluid, phonons are

$$\gamma^{(\pm)}_{12} = \sum_{34} \bar{v}_{134} \rho_{124}^{(\pm)}; \quad \bar{v}_{134} = \sum_{34} \rho_{123}^{(\pm)};$$

$$\Delta^{(\sigma)}_{\nu}(\omega) = \frac{\sigma}{\omega - \sigma(\omega_{\pm} - i\delta)},$$

(63)

with the corresponding coupling amplitude $\Gamma^{(pp)}_{121'2'}(\omega)$:

$$\Gamma^{(pp)}_{121'2'}(\omega) = \sum_{343'd'} v_{1234} G^{(pp)}_{34,34'}(\omega) v_{d'2'}^* = \sum_{\mu,\sigma=\pm} \bar{g}^{(\sigma)}_{12} D^{(\sigma)}(\omega) \bar{g}^{(\sigma)*}_{134},$$

(64)

The mapping to emergent particle-hole and particle-particle (pairing) phonons is depicted in Fig. 3 in the diagrammatic form.

With the notions of these emergent phonons, the first component of the dynamical kernel associated with the self-energy graph (the upper line of Fig. 2) takes the following form:

$$K^{(r;11)}_{121'2'}(\omega) = \left[ \sum_{kn;\nu\mu} \frac{g^{(\pm)}_{13,\nu\mu\nu\mu}}{\omega - \omega_{\pm} - \omega_{\nu\mu} - \omega_{\nu\mu} - i\delta} \right] - \left[ \sum_{nmq} \frac{\bar{g}^{(\pm)}_{12,\nu\mu\nu\mu}}{\omega - \omega_{\nu\mu} - \omega_{\nu\mu} - \omega_{\nu\mu} - i\delta} \right]$$

(65)

while the second component represented by the "twisted" graph in the second line of Fig. 2 reads:

$$K^{(r;12)}_{121'2'}(\omega) = \left[ \sum_{kn;\nu\mu} \frac{g^{(\pm)}_{13,\nu\mu\nu\mu}}{\omega - \omega_{\nu\mu} - \omega_{\nu\mu} - \omega_{\nu\mu} - i\delta} \right] - \left[ \sum_{nmq} \frac{\bar{g}^{(\pm)}_{12,\nu\mu\nu\mu}}{\omega - \omega_{\nu\mu} - \omega_{\nu\mu} - \omega_{\nu\mu} - i\delta} \right]$$

(66)

The two remaining components $K^{(r;21)}_{121'2'}(\omega)$ and $K^{(r;22)}_{121'2'}(\omega)$ can be found from Eqs. 65 and 66 with the help of the symmetry relations of Eqs. 59 and 60.

The diagrammatic representation of the components $K^{(r;11)}_{121'2'}(\omega)$ and $K^{(r;12)}_{121'2'}(\omega)$ of the dynamical kernel are shown.
in Figs. 4 and 5 respectively. In the case of $K^{(r;11)}(\omega)$, the mapping to the PVC leads to two topologically similar terms of the self-energy type with the pairing and normal phonons. The "twisted" component $K^{(r;12)}(\omega)$ contains a typical phonon-exchange term (the third term in Eq. (66) and in Fig. 5) with the normal phonon, however, its counterpart with the single pairing phonon would violate the particle number conservation and, therefore, is absent in this component. Instead, mixed contributions of the normal and pairing phonons appear, as it is clear from the graphical form of the first two terms of Eq. (66) in Fig. 5

VI. THE STATIC LIMITS: PAIRING GAP AND TWO-BODY DENSITY MATRIX

From Eq. (33) it is easy to obtain the equations for the pairing transition densities $\alpha^\mu$ and $\beta^\nu$. Indeed, considering the frequency argument of $G_{121'2'}(\omega)$ in the vicinity of a pole in the $(N + 2)$-particle system $\omega = \omega_s$, leads to the following equation for $\alpha^\mu$

$$\alpha^\mu_{21} = \frac{1 - n_1 - n_2}{\omega_s - \tilde{\epsilon}_1 - \tilde{\epsilon}_2} \sum_{343'4'} \delta_{1234} K_{343'4'}(\omega_s) \alpha^\mu_{4'3'}$$ (67)

and a similar equation for $\beta^\nu$. Furthermore, if the ground state of the reference nucleus is approximated by the BCS-like approach, where

$$n_1 = v_1^2 = \frac{E_1 - (\tilde{\epsilon}_1 - \lambda)}{2E_1}, \quad E_1 = \sqrt{(\tilde{\epsilon}_1 - \lambda)^2 + \Delta^2}$$ (68)

and $\lambda$ being the chemical potential, the pairing gap $\Delta_1$ can be related to the pairing transition density as

$$\Delta_1 = 2E_1 \alpha^\mu_{11},$$ (69)

and at $\omega_s = 2\lambda$ Eq. (67) takes the form of the well-known pairing gap equation:

$$\Delta_1 = -\sum_2 \nu_{122'} \frac{\Delta_2}{2E'_2},$$ (70)

where the bar denotes the time-reversed state and the interaction matrix elements read:

$$\nu_{121'2'} = \frac{1}{4} \sum_{34} \delta_{1234} K_{341'2'}(2\lambda) = \frac{1}{2} \left( K^{(r)}_{121'2'} + K^{(r)}_{121'2'}(2\lambda) \right)$$. (71)

The integral part of the gap equation (70), thus, contains all the microscopic effects of the kernel $K$ "on shell", regardless the approximations made for its static $K^{(0)}$ dynamical $K^{(r)}$ parts.

Another ingredient for the evaluation of the interaction kernel is the two-body density which enters its static part (40). The two-body density can be related to the static limit of the two-body propagators:

$$\rho_{212'1'} = \lim_{t' \to t+0} G_{12,1'2'}(t-t')$$ (72)

$$\rho_{21'2'1} = \delta_{11'} \rho_{22'} - \lim_{t' \to t+0} R_{12,1'2'}(t-t'),$$ (73)

as follows from their definitions given in Eqs. (14,15). In turn, the static limits of $G_{12,1'2'}(t-t')$ and $R_{12,1'2'}(t-t')$...
can be expressed via their expansions over the transition densities:

\[
\lim_{t' \to t+0} G_{12,1/2'}(t - t') = -i \sum_{12} \beta_{12}^{\nu} \beta_{12}^{\nu'}
\]

(74)

\[
\lim_{t' \to t+0} R_{12,1/2'}(t - t') = -i \sum_{12} \tilde{\rho}_{12}^{\nu} \tilde{\rho}_{12}^{\nu'}.
\]

(75)

VII. CALCULATIONS: DETAILS, RESULTS AND DISCUSSION

In order to test the developed approach in realistic conditions, we have performed some illustrative calculations for finite nuclei. As in Ref. [30], in these calculations we were focused mainly on the dynamical kernel, but now for the propagator of the fermion pair in the static limit, that is determined by the nuclear pairing gaps. Thus, we solved Eq. (70) with the kernel of Eq. (71) that includes both the static and dynamical parts, together with the usual BCS constraint on the average particle number [20]. In this first study we approximated the static part of the interaction by the phenomenological 'monopole force' which is detailed, for instance, in Ref. [37]. Consistently, we used the basis of the relativistic mean field (RMF) with NL3 parametrization [61] to approximate the one-body part of the Hamiltonian.

In this first study of the PVC effects on the pairing gaps in a relativistic framework, the dynamical kernel \( K^{(r)}(\omega) \) was computed in the leading approximation that (i) omits the exchange of the pairing phonons, thus, keeping only the last terms shown in Figs. 4, 5 and (ii) neglects the particle-particle correlations in the \( G \) parts of those terms. This form of the dynamical kernel corresponds to the leading 'two-quasiparticle plus phonon' \((2q \oplus \text{phonon})\) approximation, which is commonly employed in the nuclear field theories. Although more sophisticated approaches are already available for the particle-hole response from Refs. [30, 49, 64], here we investigated only the major PVC effect on the nuclear pairing gaps. The latter are known to be linked to the observed odd-even mass differences as discussed, for instance, in Ref. [62], and can thus be extracted from the nuclear mass tables [64].

Since we do not address an ab-initio calculation of the pairing gaps, but rather keep the static part of the interaction kernel purely phenomenological, the calculations have, besides the six RMF parameters, one free parameter which is adjusted to reproduce the average pairing gap in a \( \approx 20 \) MeV energy window around the Fermi energy, or chemical potential. In this way, it is possible to determine the relative contributions of the static and dynamical kernels. Based on the RMF for the given nucleus, the phonon vertices \( g_{\nu}^{r} \) and their frequencies \( \omega_{\nu} \), were extracted from the relativistic random phase approximation (RRPA) calculations [55]. The latter were performed for the angular momenta and parities \( J^{\pi}_{\nu} = 2^{+}, 3^{-}, 4^{+}, 5^{-}, 6^{+} \) forming the phonon model space commonly used in the PVC applications. This model space was slightly truncated to select the modes which provide the most important contributions. In particular, in this work we used the same truncation criteria as in Ref. [30]. After that, Eq. (70) with the kernel of Eq. (71) and the BCS particle number constraint were solved in a self-consistent cycle, where the value of the state-dependent pairing gaps and the chemical potential were determined with a \( 10^{-3} \) MeV accuracy.

Fig. 6 illustrates the results of calculations of the neutron pairing gaps in the two open-shell nickel isotopes \(^{66}\text{Ni}\) and \(^{68}\text{Ni}\) calculated in the purely static constant-gap approximation with a phenomenological kernel (blue symbols) and taking into account the dynamical PVC effects (red symbols). The vertical dashed lines mark the Fermi energy.
First we performed the calculation with both the static phenomenological and the PVC kernels, where the strength parameter in front of the static kernel is tuned in a way that the resulting averaged pairing gaps reproduce the empirical value extracted from Ref. [64] with the help of the three-point formula [63]. The results of these calculations are shown by the red symbols and marked as ‘Static+PVC’ on the legends. After that, the PVC, or dynamical, part of the interaction kernel was turned off and the calculations were repeated with the same static kernel. These results are shown by the blue symbols and marked as ‘Static’. Thus, we isolated the PVC effects of the dynamical kernel $K^{(r)}$ that can be assessed quantitatively by the difference between the two results.

The first observation from Fig. 6 is that the calculations with the PVC contribution produce larger average pairing gaps. The latter contribution is a result of an interplay between the self-energy terms of the type shown in Fig. 4, with PVC contributions produce larger average pairing gaps of the purely static character. Another observation is that the PVC mechanism brings a sizable state dependence of the pairing gap as compared to the calculations with nearly constant static kernel. In all the cases investigated in the present work, which include nickel, calcium and tin isotopic chains, we found a remarkable state dependence of the neutron pairing gaps obtained with the PVC kernel. In particular, an enhancement of the gaps is found for the single-particle states at or near the Fermi surface. Our calculations show 153% ($1g_{9/2}$) and 200% ($2p_{1/2}$) enhancement of the peak values of the neutron pairing gap in these nickel isotopes, as compared to their ‘static’ values while the average pairing gap increases by factors 1.77 and 1.69, respectively.

Fig. 7 shows the results of similar calculations for two calcium isotopes, $^{44}$Ca and $^{46}$Ca. In these two open-shell nuclei as well as in $^{42}$Ca which we also investigated, the enhancement of the average pairing gap due to the PVC effects is less pronounced than in nickel isotopes adding up to the factors of 1.06 and 1.25, respectively. This may indicate a stronger cancellation between the self-energy and exchange PVC terms in these nuclei. However, one can still notice a remarkable state dependence of $\Delta_k$. In both $^{44}$Ca and $^{46}$Ca it peaks at the $2s_{1/2}$ and $1d_{3/2}$ hole states located next to the Fermi surface. The peak values of the neutron pairing gaps exhibit about 20% and 60% enhancement with respect to their ‘static’ values in the two calcium isotopes, respectively.

In heavier systems the trends are similar. The left panel of Fig. 8 illustrates the behavior of the pairing gaps in $^{120}$Sn. Again, we find that the PVC contribution to the interaction kernel brings a noticeable state dependence, especially at the Fermi surface. The average value, however, remains nearly unchanged. In the right panel of Fig. 8 we plot the average pairing gaps in some stable tin isotopes obtained with and without the PVC effects. One can see that the PVC enhancement of the average value varies and can reach 50% as it occurs, for instance, in $^{112}$Sn. The peak

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**FIG. 7.** Same as in Fig. 6 but for $^{44}$Ca and $^{46}$Ca.
values of the pairing gaps at the Fermi surface obtained in the 'Static+PVC' calculation are also plotted and show a soft minimum in the middle of the shell.

The behavior of the pairing gaps around the Fermi surface is in a qualitative agreement with Refs. [16, 59] and may occur due to the specific structure of the PVC contributions to the interaction kernel. Nevertheless, our attempts to perform calculations with only the dynamical PVC kernel did not result in the realistic pairing gaps even for the states around the Fermi energy, in contrast to the latter two studies. The calculations of Refs. [16, 20] employed the nuclear field theory with the effective particle-phonon Hamiltonian and the Bloch-Horowitz second-order perturbation theory for the pairing interaction induced by the particle-phonon coupling, while in Refs. [14, 59] a generalized Dyson equation was solved for the single-quasiparticle propagator in the doubled quasiparticle space, which treats pairing and PVC effects on the equal footing. The latter method is still based on the Gorkov factorization and on the use of the anomalous one-fermion Green functions. As a consequence, these approaches commonly contain the dynamical kernel with only coherent PVC contributions, that may explain the overall stronger PVC effects.

The present study is considered as only the initial step towards a microscopic theory of nuclear superfluidity. After quantifying the contribution of the dynamical kernel $K^{(r)}$ to the observed pairing gaps, the next natural move would be considering an accurate calculation of the static kernel $K^{(0)}$. Instead of employing the simple monopole-force ansatz, the static kernel should be computed based on more realistic effective or bare interactions, that can use the insights from both the relativistic [20, 66–70] and non-relativistic [20, 71–73] studies. Adding the knowledge about the treatment of two-fermion propagators with particles in the continuum, as outlined in Refs. [73, 75], would be also very instructive for applications to loosely bound and light nuclei.

VIII. SUMMARY AND OUTLOOK

We introduce a many-body approach to the pairing correlation function in fermionic systems. The equation of motion method is formulated for the two-time two-fermion propagator in the particle-particle channel in a strongly-coupled medium. Both the static and dynamical interaction kernels of the EOM for this propagator, which appears to be Dyson Bethe-Salpeter equation, are derived from the underlying bare two-fermion interaction. The exact symmetric dynamical kernel, which contains a four-fermion propagator, is approximated by a cluster decomposition into the two-fermion propagators of both particle-particle and particle-hole type. In this way, the nuclear many-body problem is truncated at the level of two-body correlation functions whose EOM’s, together with those for the one-fermion particle-hole correlation function discussed in Ref. [30] form a closed system of integral equations.

Although a complete solution of such a system is not yet available for finite nuclei, some aspects of the formulated approach can be studied for these systems. For instance, the resulting particle-particle correlation function appears to be related to the observables associated with the nuclear superfluidity. The equation for the pairing gap, which is directly related to a residue of the two-time particle-particle propagator, is therefore formulated as a static limit of the...
EOM for this propagator. Assuming the ground state wave function of the BCS type, a BCS-like equation for the pairing gap is obtained. The interaction kernel of this equation, as the one of the corresponding EOM, has the purely static part as well as the dynamical part taken in the static limit. The latter contribution thus represents an extension of the BCS approximation to the inclusion of higher complexity correlations.

We investigated the effects of this additional term on pairing gaps in medium-light and medium-heavy nuclei. Namely, the neutron pairing gaps were calculated for calcium, nickel and tin isotopes. The developed method was implemented numerically on the base of quantum hadrodynamics and relativistic mean field. The beyond-mean-field effects on the pairing gaps are found quite pronounced. They lead to a sizable state dependence of the pairing gaps with the tendency to an enhancement around the Fermi surface, in a qualitative agreement with existing NFT calculations. We found, however, that the static part of the interaction gives a relatively large contribution to the pairing gap values. Thus, we conclude that this part should be also accurately determined from the underlying microscopic interaction. This is recognized as the most natural further advancement that will be addressed by future effort.

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