The Manneville map: topological, metric and algorithmic entropy

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Abstract

We study the Manneville map $f(x) = x + x^z \pmod{1}$, with $z > 1$, from a computational point of view, studying the behaviour of the Algorithmic Information Content. In particular, we consider a family of piecewise linear maps that gives examples of algorithmic behaviour ranging from the fully to the mildly chaotic, and show that the Manneville map is a member of this family.

1 Introduction

The Manneville map was introduced by Manneville in [17], as an example of a discrete dissipative dynamical system with intermittency, an alternation between long regular phases, called laminar, and short irregular phases, called turbulent. This behaviour had been observed in fluid dynamics experiments and in chemical reactions. Manneville introduced his map, defined on the interval $I = [0, 1]$ by

$$f(x) = x + x^z \pmod{1} \quad z > 1,$$

(1)
to have a simple model displaying this complicated behaviour. In Figure 1, it is plotted the Manneville map for $z = 2$. His work has attracted much attention, and the dynamics of the Manneville map has been found in many
other systems. We can find applications of the Manneville map in dynamical approaches to DNA sequences ([1],[2]) and ion channels ([20]), and in non-extensive thermodynamical problems ([7]).

![The Manneville map for $z = 2$](image)

**Figure 1: The Manneville map $f$ for $z = 2$**

The Manneville map has also been studied by Gaspard and Wang ([13]), using the notion of *Algorithmic Information Content* of a string, briefly explained below. Given a dynamical system, any orbit of this system can be translated into a string $\sigma$ of symbols by a partition of the phase space of the system (symbolic dynamics). For any finite string $\sigma^n$ of length $n$, it has been introduced by Chaitin ([8]) and Kolmogorov ([15]), the notion of *Algorithmic Information Content* (AIC) or *Kolmogorov complexity*) of the string, that we denote by $I_{AIC}(\sigma^n)$, defined as the binary length of the shortest program $p$ that outputs the string on a universal machine $C$,

$$I_{AIC}(\sigma^n) = \{|p| \mid C(p) = \sigma^n\}. \quad (2)$$

It is then possible, using the symbolic dynamics, to define the notion of Algorithmic Information Content for a finite orbit of a dynamical system. This extension requires some attention on the choice of the partition. The
first results have been obtained by Brudno ([6]) using open covers of the phase space. Another possible approach, using computable partitions, is introduced in [4].

To generalize the notion of AIC to infinite strings, it is natural to consider the mean of the AIC. We call complexity of an infinite string $\sigma$ the maximum limit of the AIC of the first $n$ symbols of the string divided by $n$. Then, if we denote the complexity of an infinite string by $K(\sigma)$, we have

$$K(\sigma) = \lim_{n \to +\infty} \sup I_{AIC}(\sigma^n)/n,$$

(3)

where $\sigma^n$ is the string given by the first $n$ digits of the infinite string $\sigma$. Symbolic dynamics is again the tool to define the complexity of an infinite orbit of a dynamical system.

Moreover, we can ask whether it is possible to define a notion of information content for the dynamical system, without to consider any particular orbit. To do this, we have to introduce a probability measure $\mu$ on the phase space $X$ of the system and we can define the algorithmic entropy $h_\mu$ of a dynamical system by

$$h_\mu = \int_X K(x) \, d\mu,$$

(4)

where $K(x)$ denotes the complexity of the orbit of the system with initial condition $x \in X$.

There exist some results connecting the information content of a string generated by a dynamical system and the Kolmogorov-Sinai entropy $h^{KS}$ of the system.

First of all it is proved that for a compact phase space $X$ and for an invariant measure $\mu$, we have $h_\mu = h^{KS}_\mu$. Then in particular, in a dynamical system with an ergodic invariant measure $\mu$ with positive K-S entropy $h^{KS}_\mu$, the AIC of a string $n$ symbols long behaves like $I_{AIC}(\sigma^n) \sim h^{KS}_\mu n$ for almost any initial condition with respect to the measure $\mu$ ([8]).

Instead, in a periodic dynamical system, we expect to find $I_{AIC}(\sigma^n) = O(\log(n))$. Indeed, the shortest program that outputs the string $\sigma^n$ would contain only information on the period of the string and on its length.

It is possible to have also intermediate cases, in which the K-S entropy is null for all the invariant measures that are physically relevant and the system is not periodic. These systems, whose behaviour has been defined weak chaos,
are an important challenge for research on dynamical systems. Indeed no information are given by the classical properties, such as K-S entropy or Lyapunov exponents, and in the last years some generalized definitions of entropy of a system have been introduced to characterize the behaviour of such systems (for example see [21]). We believe that an approach to weakly chaotic systems using the infinite order of their AIC could be a powerful way to classify these systems (no information are obtained by the complexity and the algorithmic entropy defined as above).

The Manneville map with parameter $z > 2$ is a non periodic map with null K-S entropy for all the physically relevant invariant measures, then the analysis of the AIC of the strings generated by the map is interesting. Gaspard and Wang ([13]) showed that the Manneville map exhibits a behaviour that they called sporadicity. Namely, the Algorithmic Information Content, $I_{AIC}(\sigma^n)$, of a string $n$ symbols long, behaves in mean like $n^\alpha(\log(n))^{\beta}$, with either $0 < \alpha < 1$ or $\alpha = 1$ and $\beta < 0$.

In this paper, we give a formal proof of the results obtained by Gaspard and Wang ([13]) for the Manneville map, giving more precise estimates for the AIC of a string generated by the map. But the most important generalization is that we find our results for the Manneville map as a particular case of a general theorem concerning a large family of maps $L$, defined in equation (6). This family of maps exhibits an extremely wide range of behaviours (for the AIC of the generated strings), and sporadicity is only one possible case. Then we find a family of maps that can be classified with respect to the order of the AIC of a “typical” (in the sense of Lebesgue measure) initial condition. Moreover we study some topological and metric properties of the maps $L$, useful to obtain a prediction of the behaviour of the AIC of the related symbolic dynamics (see Section 5).

In Section 2, we introduce the family of maps $L$, and show how the Manneville map $f$ can be thought of as one member of the family.

In Section 3, we study the maps $L$ from the topological point of view. In Section 4, we show that the maps in the class $L$ are equivalent to a Markov chain in a suitable sense. This equivalence is extensively used in Subsection 5.1, where we present our results relative to the behaviour of the AIC of the strings obtained from the maps $L$.

Finally, in Subsection 5.2, we study the computational aspect of the maps $L$ from a practical point of view. So, we restrict ourselves to consider the Lebesgue measure $l$ on the interval $I$. 
2 The family of piecewise linear maps \( L \)

In this section we present what we shall use as our formulation of the Manneville map \( f \). In the following, we study a family of piecewise linear maps \( L \) on the interval \( I = [0, 1] \), which are topologically equivalent to the Manneville map \( f \). Using the maps \( L \), all the theorems have an easier interpretation and computations can be done exactly. Moreover all our results are extendible through metric isomorphism, hence we shall define on the interval \( I \), two different measures that make the topological equivalence between \( L \) and \( f \) a metric isomorphism. Then we can extend all the results that we find for the maps \( L \) to the Manneville map \( f \).

Let’s start defining the piecewise linear maps \( L \) that we consider. We use here the same approach as in \([12]\). A natural way to get a partition of the interval \( I = [0, 1] \) from the Manneville map \( f \) is the following: let’s call \( x_0 \) the point of \( I \) such that \( f(x_0) = 1 \) with \( x_0 \neq 0, 1 \), and \( x_1 \) the preimage of \( x_0 \) in the interval \([0, x_0]\); then we define recursively \( x_n = \{f^{-1}(x_{n-1})\} \cap [0, x_{n-1}] \). Then the sub-intervals \( B_k = (x_k, x_{k-1}] \), for \( k \geq 1 \), and \( B_0 = (x_0, 1] \) are a partition of \( I \).

Define \( \{\epsilon_k\}_{k \in \mathbb{N}} \) a sequence of positive real numbers, that is strictly monotonically decreasing and converging towards zero, with the property that

\[
\frac{\epsilon_{k-1} - \epsilon_k}{\epsilon_{k-2} - \epsilon_{k-1}} < 1 \quad \forall \, k \in \mathbb{N}.
\]

The piecewise linear maps that we consider are defined by

\[
L(x) = \begin{cases} 
\frac{\epsilon_{k-2} - \epsilon_{k-1}}{\epsilon_{k-1} - \epsilon_k} (x - \epsilon_k) + \epsilon_{k-1} & \epsilon_k < x \leq \epsilon_{k-1}, \quad k \geq 1 \\
\frac{x - \epsilon_0}{1 - \epsilon_0} & \epsilon_0 < x \leq 1 \\
0 & x = 0 
\end{cases}
\]

where we define \( \epsilon_{-1} = 1 \). These piecewise linear maps \( L \) clearly depend on the definition of the sequence \( \{\epsilon_k\}_{k \in \mathbb{N}} \), but a particular choice for this sequence is important only for Section \([3]\). For the moment we consider any possible sequence, with the properties specified above. Let’s define sub-intervals \( A_i = (\epsilon_i, \epsilon_{i-1}] \), and \( A_0 = (\epsilon_0, 1] \). These interval form a partition of the interval \( I \). We prove the following

**Theorem 2.1.** Any piecewise linear map \( L \) defined as in equation \([3]\) is topologically equivalent to the Manneville map \( f \).
Proof. We have to find a homeomorphism \( h : I \to I \) such that \( h(f(x)) = L(h(x)) \) for each \( x \in I \). To find such a homeomorphism we use the partitions \((B_j)\) and \((A_j)\). We define \( h(x_n) = \epsilon_n \) for each \( n \in \mathbb{N} \) and \( h(0) = 0 \), \( h(1) = 1 \), and such that \( h(B_j) = A_j \), for all \( j \geq 0 \). To define the homeomorphism \( h \) we use a dense set of \( I \), define \( h \) on this set and, then, simply extends the definition of \( h \) to the whole interval \( I \) by continuity.

Let’s consider a sub-interval \( B_k \). By definition of the Manneville map \( f \), we have that \( f(B_k) = B_{k-1} \), for \( k \geq 1 \), and \( f(B_0) = I \). Then it follows that \( f^k(B_k) = B_0 \) and \( f^{k+1}(B_k) = I \). So, within each \( B_k \) we can find sub-intervals \( B_{kj} \), with \( k, j \in \mathbb{N} \), defined by \( f^{k+1}(B_{kj}) = B_j \). These sub-intervals form a partition of each \( B_k \). We can continue this partition of the intervals \( B_k \), defining, by the same rule, sub-intervals \( B_{kji} \) that form a partition of \( B_{kj} \). We write then any sub-interval of the form \( B_{k_1k_2...k_n} \), with \( k_i \in \mathbb{N} \) for each \( i = 1, \ldots, n \), as \( B_{k_1k_2...k_n} = (x_{k_1k_2...k_n}, x_{k_1k_2...((k_n-1)})] \).

The set \( \{x_{k_1k_2...k_n}, \ n \in \mathbb{N}, \ k_i \in \mathbb{N}\} \) is a countable dense set of the interval \( I \). Analogously, we can define a set of points \( \{\epsilon_{k_1k_2...k_n}, \ n \in \mathbb{N}, \ k_i \in \mathbb{N}\} \), for the map \( L \), with the same property as \( x_{k_1k_2...k_n} \). We define then \( h(x_{k_1k_2...k_n}) = \epsilon_{k_1k_2...k_n} \) for each \( n \in \mathbb{N} \), and extend the function \( h \) continuously to the whole interval \( I \). We have thus obtained a continuous function \( h \) such that \( h(B_{k_1k_2...k_n}) = A_{k_1k_2...k_n} \), for each \( n \in \mathbb{N} \), where the sub-intervals \( A_{k_1k_2...k_n} \) are defined as \( A_{k_1k_2...k_n} = (\epsilon_{k_1k_2...k_n}, \epsilon_{k_1k_2...((k_n-1)})] \).

The injectivity of \( h \) follows by contradiction. Let’s suppose to have two points \( x < y \in I \) such that \( h(x) = h(y) = z \). By the density of the set \( \{x_{k_1k_2...k_n}, \ n \in \mathbb{N}, \ k_i \in \mathbb{N}\} \), we can find a point \( \tilde{x}_{k_1k_2...k_n} \in (x, y) \). This implies that there exists a \( \bar{n} \) such that \( x \in B_{k_1k_2...((k_n-1)} \) and \( y \in B_{k_1k_2...k_n} \). Then we have \( h(x) \neq h(y) \). The subjectivity of \( h \) follows immediately by the definition.

Then the inverse function \( h^{-1} \) exists and \( h \) is a homeomorphism because it is a continuous invertible function from a compact to a Hausdorff space. \( \square \)

The topological equivalence between \( L \) and \( f \) can be used in particular to obtain a metric isomorphism. If we have a measure \( \mu \) on the interval \((I, \mathcal{B}, L)\), where \( \mathcal{B} \) is the Borel \( \sigma \)-algebra, then the homeomorphism \( h \) carries \( \mu \) into another measure \( \nu = h^* \mu \) on \((I, \mathcal{B}, f)\), and with respect to these measures \( h \) is a metric isomorphism.

**Theorem 2.2 (Radon-Nikodym).** Given two measures \( \mu \) and \( \nu \) on \((I, \mathcal{B})\), such that \((I, \mathcal{B}, \nu)\) is \( \sigma \)-finite, \( \mu \ll \nu \) if and only if there exists a real function \( f \) on \( I \), integrable with respect to \( \nu \) on all sets \( B \in \mathcal{B} \) such that \( \nu(B) < +\infty \),
satisfying the following condition for every $B \in \mathcal{B}$:

$$\mu(B) = \int_B f \ d\nu$$

**Theorem 2.3.** If the measure $\mu$ on $(I, \mathcal{B}, L)$ is absolutely continuous with respect to the Lebesgue measure $l$ ($\mu << l$), then also the measure $\nu = h^* \mu$ on $(I, \mathcal{B}, f)$ is absolutely continuous with respect to the Lebesgue measure $l$, and vice-versa.

**Proof.** We can apply Theorem 2.2 to the measures $\mu$ and $l$. Then we obtain a real function $f_\mu$ on $I$, integrable with respect to $l$ and such that for all $B \in \mathcal{B}$

$$\mu(B) = \int_B f_\mu \ dl.$$

By definition of the measure $\nu$, we have that $\nu(B) = \mu(h(B))$ for all $B \in \mathcal{B}$, then

$$\nu(B) = \int_{h(B)} f_\mu \ dl = \int_B (f_\mu \circ h)(dh) \ dl,$$

where $dh$ is defined almost everywhere with respect to $l$, being $h$ a monotone continuous function. Then we have found a function $f_\nu = (f_\mu \circ h)(dh)$, which satisfies the hypotheses of the Radon-Nikodym Theorem. Then $\nu << l$.

The vice-versa is proved in the same way. \qed

At this point, thanks to Theorem 2.3, we can use our linear map in all our applications of topological and metric methods.

### 3 The topological approach

In this section we start a procedure of equivalences of the maps $L$ with well-known maps, that can be used to establish the results of Section 3. The first step is to study the relationship between the maps $L$ and a sub-shift of finite type.
3.1 Symbolic dynamics

We briefly recall the definition of a sub-shift of finite type. Let’s consider a finite set of symbols \( S = \{0, 1, 2, \ldots, N\} \), with \( N \geq 1 \), and build a set \( \Sigma^N = S^N \) as the product of countable factors \( S \). On \( \Sigma^N \) it is defined a map \( T \), called the shift map, that acts on the elements of \( \Sigma^N \), by shifting forward the indexes. Namely, if \( \sigma = (\sigma_0\sigma_1\ldots\sigma_n\ldots) \in \Sigma^N \), with \( \sigma_i \in \{0,\ldots,N\} \) for all \( i \in \mathbb{N} \), then \( T(\sigma) = (\sigma_1\ldots\sigma_n\ldots) \). The set \( \Sigma^N \) is endowed with a metric \( d \) defined by

\[
    d(\sigma, \sigma') = \sum_{n=0}^{\infty} \frac{\delta_{\sigma_n, \sigma'_n}}{2^n},
\]

where \( \delta_{ij} \) is the Kronecker symbol, that makes it a compact space. A sub-shift of finite type is obtained from the set \( (\Sigma^N, T) \), by means of a \( (N+1) \times (N+1) \) matrix \( M = (m_{ij}) \), called the transition matrix, such that \( m_{ij} \in \{0, 1\} \) for all \( i, j = 0, \ldots, N \). We define a subset \( \Sigma^N_M \) of \( \Sigma^N \), by

\[
    \Sigma^N_M = \{ \sigma \in \Sigma^N \mid m_{\sigma_i, \sigma_{i+1}} = 1 \ \forall \ i \in \mathbb{N} \},
\]

then a sub-shift of finite type is simply the compact, metric space \( (\Sigma^N_M, T_M) \) (see [14]).

We have

**Theorem 3.1.** For any \( N \geq 1 \), there exists a particular transition matrix \( M \) such that any map \( L \) of the family of piecewise linear maps (6) is a factor of the sub-shift of finite type \( (\Sigma^N_M, T_M) \). This means that there exists a subjective continuous map \( \pi : \Sigma^N_M \to I \) such that \( \pi \circ T_M = L \circ \pi \).

**Proof.** Let a \( (N+1) \times (N+1) \) matrix \( M \) be defined by

\[
    M = \begin{pmatrix}
        1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
        1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
        0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
        \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
        0 & 0 & 0 & \cdots & 0 & 1 & 1
    \end{pmatrix},
\]

we show that this is the matrix we need.

Let’s consider a partition of the interval \( I \) defined by using the partition \( A_i = (\epsilon_i, \epsilon_{i-1}] \), introduced in Section [2]. From this partition we obtain a
partition of the interval $I$, by $B_i = A_i$ for $i = 0, \ldots, N - 1$, and $B_N = \{0\} \cup \left( \bigcup_{j=N}^{\infty} A_j \right) = [0, \epsilon_{N-1}]$. From this partition and by using the properties of any of the map $L$, we obtain a $(N+1)$-nary representation of the interval $I$.

The $(N+1)$-nary representation of a point $x \in I$ is given by a string $\sigma$ such that $L^n(x) \in B_{\sigma_n}$ with $\sigma_n = 0, \ldots, N$, for any $n \in \mathbb{N}$. This representation is nothing else that a map $\pi : \Sigma^N \to I$. Hence we just need to show that this map $\pi$ is continuous and subjective, and verifies the commutation rule with $L$ and $T$.

First of all we notice that our map $\pi$ is not defined on the whole space $\Sigma^N$, because of the restrictions given by the particular form of the map $L$. If we want to reduce the space $\Sigma^N$, we have to consider a transition matrix $M$, and we use the matrix $M$ defined in equation (7). It is easy to verify that for any $\sigma \in \Sigma^N_M$ there is a point $x \in I$ such that $\pi(\sigma) = x$. Then we show that $\pi : \Sigma^N_M \to I$ is a subjective continuous map such that $\pi \circ T_M = L \circ \pi$.

We have then a semi-conjugacy between our piecewise linear maps $L$ and symbolic dynamics. We have not a conjugacy because of the lack of injectivity of the map $\pi$. Indeed, as in any $n$-ary representation of the real numbers in the interval $I = [0,1]$, there is a countable set $X$ of points that are images of two sequences. Moreover, in our case, these points can be characterized by the property that for any $x$ in this set there exists a $N \in \mathbb{N}$ such that $L^N(x) = 1$.

Commutation. The commutation rule $\pi \circ T_M = L \circ \pi$ is an immediate consequence of the definition of $\pi$.

Subjectivity. It follows immediately from the definition of the map $\pi$.

Continuity. We have to prove that given any $\epsilon > 0$ there exists a $\delta > 0$ such that if $d(\sigma_1, \sigma_2) < \delta$ then $|\pi(\sigma^1) - \pi(\sigma^2)| < \epsilon$. But from the definition of the metric $d$ on the space $\Sigma^N_M$, we have that $d(\sigma^1, \sigma^2) < \delta$ is equivalent to: there exists a $K > 0$ such that $\sigma^1_j = \sigma^2_j$ for all $j = 0, \ldots, K$. So given any $\epsilon > 0$ we have to find a $K > 0$ such that $\sigma^1_j = \sigma^2_j$ for all $j = 0, \ldots, K$ implies $|\pi(\sigma^1) - \pi(\sigma^2)| < \epsilon$. From the definition of the map $\pi$ it is clear that if $\sigma^1_j = \sigma^2_j$ for all $j = 0, \ldots, K$ for any $K > 0$, then $L^j(\pi(\sigma^1))$ and $L^j(\pi(\sigma^2))$ belong to the same subset $B_{\sigma^1_j}$ of the partition $(B_i)$ for all $j = 0, \ldots, K$.

Then, if we consider a partition of the subset $B_{\sigma^1_j}$, given by $(B_{\sigma^1_j})_{j_1, j_2, \ldots, j_n}$, with $j_i = 0, \ldots, N$ for all $i$, where $L^r((B_{\sigma^1_j})_{j_1, j_2, \ldots, j_n}) = B_{j_r}$ for all $r = 1, \ldots, n$, we have that $\text{diam}((B_{\sigma^1_j})_{j_1, j_2, \ldots, j_n}) \to 0$ as $n \to +\infty$, thanks to the particular form of the map $L$. This argument gives the continuity of the map $\pi$. 

\[ \square \]
We can extend Theorem 3.1 to the case of $N = \infty$. The space $\Sigma = \Sigma^\infty$ is defined in the same way, but we cannot extend the metric $d$, defined as before, and we can only define a topology on $\Sigma$, where the open balls are the same as before. In this case $\Sigma$ is not anymore a compact space. The transition matrix $M$ is $\infty \times \infty$ dimensional and it is defined by

$$M = \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots \\
1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}. \quad (8)$$

On the space $\Sigma_M$, we define then a map $T_M$, given by the forward shift. Then we can prove

**Theorem 3.2.** Any of our maps $L$ on $I$ is a topologically conjugate to the dynamical system given by $(\Sigma_M, T_M)$ built on countable symbols and transition matrix $M$ given by equation (8).

**Proof.** The proof is the same as in Theorem 3.1. The difference is that now we obtain a conjugacy with the set $\Sigma_M$, thanks to the fact that the matrix $M$, being infinite, exactly simulate the dynamics of the piecewise linear maps $L$ from a topological point of view.

3.2 Topological and Kolmogorov-Sinai entropy

The semi-conjugacy of the maps $L$ with the symbolic dynamical system on finite symbols is useful to compute some topological and metric quantities of our map. Indeed the lack of injectivity of the map $\pi$ is on a set that doesn’t change the dynamical richness of the systems. In this subsection we compute the topological entropy and the Kolmogorov-Sinai entropy for some measures (see [14]).

From the theory of dynamical systems, we know that the following theorems hold (see [14]):

**Theorem 3.3.** The topological entropy is invariant for topological equivalence.

**Theorem 3.4.** The topological entropy $h_{top}$ of a sub-shift of finite type is $\log \lambda_{\text{max}}$, where $\lambda_{\text{max}}$ is the largest eigenvalue of the transition matrix.
Then we have just to compute the eigenvalues of the transition matrix $M$ on finite symbols defined in equation (7), and then, thanks to the previous theorems, the topological entropy $h_{\text{top}}(L)$ of our linear map $L$ is given by $\log \lambda_{\text{max}}$. For any $N$, we find that $\lambda_{\text{max}} = 2$, then $h_{\text{top}}(L) = \log 2$, for any possible sequence $(\epsilon_k)$ defined as before.

At this point we start to consider measures on the space $\Sigma_M^N$. We have to introduce first a $\sigma$-algebra $\mathcal{C}$. We take as a basis of $\mathcal{C}$ the sets of the form

$$C^n_r = \{ \sigma \in \Sigma_M^N \mid \sigma_i = r_i \forall i = 0, \ldots, n \}, \quad (9)$$

for any $n \in \mathbb{N}$ and $r \in S^{n+1}$. These sets are called cylinders. At this point we use a classical result of dynamical systems (see [16]):

**Theorem 3.5 (Variational Principle).** Given a continuous map $f : X \to X$ of a compact metric space $X$, the topological entropy $h_{\text{top}}(f)$ is the maximum of the K-S entropies $h_\nu(f)$ on the set of all the $f$-invariant probability measures $\nu$ on $X$.

Then we look for the probability measures $\nu$ on $\Sigma_M^N$ with K-S entropy $h^K_\nu$ equal to $\log 2$. For sub-shift of finite type, a particular class of $T_M$-invariant measures are defined by a stochastic matrix associated to the sub-shift. These measures are called Markov measures, and among them there is a measure that maximize the K-S entropy. This measure is called Parry measure, and is denoted by $\nu_{\Pi}$ (see [14]). This measure is defined by a particular choice of the stochastic matrix $\Pi$. In words, the Parry measure represent the asymptotic distribution of the periodic orbits, that is if $C$ is a cylinder in $\Sigma_M^N$, we have that

$$\nu_{\Pi}(C) = \lim_{n \to \infty} \frac{\text{periodic orbits of period } n \text{ contained in } C}{\text{all the periodic orbits of period } n}.$$

We compute the Parry measures for some $N$, then we define on $I$ the induced measure $\mu_{\Pi} = \pi^* \nu_{\Pi}$, to obtain an $L$-invariant probability measure on $I$ of K-S entropy $h^K_{\mu_{\Pi}}(L) = \log 2$.

For $N = 1$, we have that $\nu_{\Pi}(C_0^0) = \nu_{\Pi}(C_1^0) = \frac{1}{2}$. So $\mu_{\Pi}(B_0) = \mu_{\Pi}(B_1) = \frac{1}{2}$. For any $N$, we obtain

$$\mu_{\Pi}(B_i) = \frac{1}{2^{i+1}} \quad i = 0, \ldots, N - 1$$

$$\mu_{\Pi}(B_N) = \frac{1}{2^{N-1}}$$

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So for the limit \( N \to \infty \) (where we directly have topological equivalence between \((I, L)\) and \((\Sigma_M, T_M)\)), we obtain \( \mu_\Pi(B_i) = \mu_\Pi(A_i) = \frac{1}{2^{i+1}} \). We have thus found a countable family of \( L \)-invariant measures on \( I \) with K-S entropy \( \log 2 \).

### 4 The metric approach

We start now to consider what happens if we start from our interval \( I \), endowed with a probability measure \( \mu \) on the Borel \( \sigma \)-algebra \( \mathcal{B} \), and with the dynamics induced by the maps \( L \). In particular we show that we obtain an equivalence with a Markov chain that will be useful for the computational approach (Section 5).

From our space \((I, \mathcal{B}, L, \mu)\), where we remark that we haven’t supposed \( \mu \) to be \( L \)-invariant, we have a metric isomorphism with the space \((\Sigma_M, \mathcal{C}, T_m)\) on countable symbols, with the probability measure \( P = \pi_* \mu \) induced by the homeomorphism \( \pi \) found in Theorem 3.2.

At this point we use some notions and results introduced by Parry [18].

**Definition 4.1.** A non-atomic stochastic process is \((X, \mathcal{A}, T, m)\) where \( X = \{x = x_0, x_1, \ldots \mid x_i \in \mathbb{N} \ \forall i \in \mathbb{N}\} \), \( \mathcal{A} \) is the \( \sigma \)-algebra generated by the cylinders \( \mathcal{C}_n \), \( m \) is a non-atomic probability measure on \( \mathcal{A} \), and \( T \) is the forward shift on \( X \). In the theory of stochastic processes the transition matrix \( M \) on \( \Sigma_M \) is called a structure matrix.

**Definition 4.2.** A stochastic process is called transitive of order \( k \) if for all \((x_1, \ldots, x_k)\) and \((y_1, \ldots, y_k)\) with \( m(x_1, \ldots, x_k) > 0 \) and \( m(y_1, \ldots, y_k) > 0 \), there exists a finite \((z_1, \ldots, z_n)\) such that

\[
m(x_1, \ldots, x_k; z_1, \ldots, z_n; y_1, \ldots, y_k) > 0.
\]

When \( k = 1 \), a stochastic process is simply called transitive.

**Definition 4.3.** A stochastic process is said to be intrinsically Markovian of order \( k \) if \( m(x_1, \ldots, x_n) > 0 \) and \( m(x_{n-k+1}, \ldots, x_{n+1}) > 0 \) imply

\[
m(x_1, \ldots, x_n, x_{n+1}) > 0.
\]

When \( k = 1 \) it is simply called intrinsically Markovian.
We can easily prove

**Proposition 4.4.** Our space \((\Sigma_M, C, T_M, P)\) is a non-atomic stochastic process, which is transitive and intrinsically Markovian.

**Definition 4.5.** Given a stochastic process \((X, A, T, m)\), a measure \(p\) makes the process \((X, A, T, p)\) *compatible* with the original when \(p(C^m_r) > 0\) if and only if \(m(C^m_r) > 0\), for any cylinder \(C^m_r\).

At this point we use the notion of non-atomic stochastic processes to obtain a compatibility between the maps \(L\) and a Markov chain, through the symbolic dynamical system on countable symbols. Before giving the theorems in this direction, we briefly recall the theory of Markov chains (see [9]).

**Definition 4.6.** Given a probability space \((\Lambda, F, P)\) and a countable space \(Y\) with the discrete \(\sigma\)-algebra, a *Markov chain* is a sequence \((Z_n)_{n \in \mathbb{N}}\) of random variables \(Z_n : \Lambda \to Y\) such that

i) If, given \(y_0, \ldots, y_{n+1} \in Y\), we have \(P[Z_n = y_n, Z_{n-1} = y_{n-1}, \ldots, Z_0 = y_0] > 0\), then

\[
P[Z_{n+1} = y_{n+1} \mid Z_n = y_n, \ldots, Z_0 = y_0] = P[Z_{n+1} = y_{n+1} \mid Z_n = y_n],
\]

ii) If \(x, y \in Y\) and \(m, n \in \mathbb{N}\) are such that \(P[Z_m = x] > 0\) and \(P[Z_n = x] > 0\), then

\[
P[Z_{m+1} = y \mid Z_m = x] = P[Z_{n+1} = y \mid Z_n = x].
\]

In particular the numbers \(p(x, y) = P[Z_{n+1} = y \mid Z_n = x]\) form a matrix \(\Pi = (p(x, y))_{x, y \in X}\), called the *transition matrix*. Moreover the probability measure on \(Y\) defined by \(\nu(y) = P[Z_0 = y]\) is called the *initial distribution*.

The transition matrix \(\Pi\) is a *stochastic matrix*, that is \(p(x, y) \geq 0\) and \(\sum_{y \in X} p(x, y) = 1\) for all \(x \in Y\).

**Theorem 4.7.** Given any countable space \(Y\), a transition matrix \(\Pi\) and an initial distribution \(\nu\), it is possible to construct a probability space \((\Lambda, F, P)\) and a sequence of random variables \(Z_n : \Lambda \to Y\), such that the constructed Markov chain has \(\Pi\) as transition matrix and \(\nu\) as initial distribution.
Proof. For the proof of the theorem see Chung [4]. We simply say what is the constructed probability space \((\Lambda, \mathcal{F}, P)\). The space \(\Lambda\) is \(Y^\mathbb{N}\) and is called the realizations space, the \(\sigma\)-algebra \(\mathcal{F}\) is given by the cylinders defined as in equation \((3)\), and the probability \(P\) is defined on the cylinders by
\[
P(C_r^n) = \nu(r_0)p(r_0, r_1)p(r_1, r_2)\ldots p(r_{n-1}, r_n).
\]
The random variables are defined as the projections of \(\Lambda\) on \(Y\).

Theorem 4.8. Given any intrinsically Markovian, transitive stochastic process \((X, \mathcal{A}, T, m)\) and a stochastic matrix \(\Pi\) such that \(p(i, j) > 0\) if and only if \(m_{ij} = 1\), for the structure matrix \(M\) of the process, there is a probability \(p\) on \(X\) such that \((X, \mathcal{A}, T, p)\) is compatible with \((X, \mathcal{A}, T, m)\), and it is a Markov chain with \(\Pi\) as transition matrix.

Proof. We have just to apply Theorem 4.7 to the matrix \(\Pi\) and to an initial distribution \(\nu\), being \(X\) already in the form of the realizations space. The probability \(p\) is then the probability \(P\) defined as above.

Corollary 4.9. Our space \((\Sigma_M, \mathcal{C}, T_M, P)\) is compatible with a Markov chain.

Proof. The corollary is proved thanks to Theorem 4.8 and Proposition 4.3. We have just to choose a stochastic matrix that satisfies the hypothesis, that is \(p(i, j) > 0\) if and only if \(m_{ij} = 1\) for the structure matrix defined as in equation \((8)\). The measure \(P\) can be used as initial distribution.

We have thus completed our equivalence, in the sense of Corollary 4.9, between the maps \(L\) defined on the space \((I, \mathcal{B}, \mu)\), for any probability measure \(\mu\), and a Markov chain, that is denoted simply by a stochastic matrix \(\Pi\) and an initial distribution \(\nu\).

5 The algorithmic entropy

The results of Section 4 are useful for the computational approach to the Manneville map \(f\) defined by equation \((1)\). As remarked before, also in this section we shall restrict ourselves to the AIC for the maps \(L\), which are equivalent to the Manneville map \(f\) in the sense described above. Using this restriction it is possible to perform explicit computations which, by Theorems
2.1 and 2.3, can be extended to the Manneville map \( f \). The investigation on
the maps \( L \) that we present in this section is meant to be a generalization of
the work of Gaspard and Wang on the Manneville map (see [13]).

Given our dynamical system \((I, \mathcal{B}, L, \mu)\), where \( \mu \) is any probability mea-
sure on the Borel \( \sigma \)-algebra of the interval \( I \), we translate the orbit of a point \( x \in I \) into a string \( \sigma = \pi(x) \in \Sigma_M \), with the transition matrix \( M \) given in
equation (8), and study the AIC of the string.

5.1 General results

In Section 4, we proved that for any map \( L \) our dynamical system \((I, \mathcal{B}, L, \mu)\)
is equivalent, in a sense specified above, with a Markov chain with a stochastic
matrix \( \Pi \) defined by means of the transition matrix \( M \) of equation (8), and
a given initial distribution that can be considered to be the measure \( \mu \) itself
(see Corollary 4.9). We remark that it is not necessary to choose the measure \( \mu \) on
\( I \) to be \( L \)-invariant. Now, we want to relate the dynamics of the Markov
chain with our dynamical system. A natural choice for the stochastic matrix
\( \Pi \) is

\[
\Pi = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\
1 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\
0 & 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\
0 & 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\
0 & 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\
0 & 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\
0 & 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\
0 & 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix},
\] (10)

where \( p_i \) are the probabilities of transition from the sub-interval \( A_0 \) to the
sub-interval \( A_i \), and in terms of the measure \( \mu \) are given by

\[
p_i = \frac{\mu(A_0 \cap L^{-1}(A_i))}{\mu(A_0)}.
\] (11)

It is evident in the definition of \( \Pi \) the dependence on the particular se-
quence \((\epsilon_k)\) that we consider to define the map \( L \) and on the probability
measure \( \mu \).

It has been shown by Gaspard and Wang ([13]), that a way to estimate
the AIC of a string obtained from our dynamical system is the theory of
recurrent events applied to Markov chains ([11],[10]). From the theory of
Markov chains, we have that our stochastic matrix \( \Pi \) is irreducible, and that
the state \( A_0 \) is persistent, in the sense that

\[
p[Z_m = 0 \text{ for some } m > n | Z_n = 0] = 1 \quad \forall \ n,
\]
where \( p \) is the probability measure on \((\Sigma \cup \mathcal{C}, T_M)\) that makes it a Markov chain with \((Z_n)\) as random variables from \(\Sigma\) to \(\mathbb{N}\) (see Theorems 4.7, 4.8).

If we consider as recurrent event \(E\) the passage from the sub-interval \(A_0\), we have that \(E\) is certain and that the mean recurrence time \(m_0\) is given by

\[
m_0 = \sum_{k=1}^{+\infty} k p_{k-1}.
\] (12)

If \(m_0 = +\infty\) the state \(A_0\) is called null, otherwise it is ergodic. Thanks to the irreducibility of the stochastic matrix \(\Pi\), we have that all the states \(A_i\) are of the same kind of \(A_0\), and then either \(m_i = +\infty\) for all \(i \in \mathbb{N}\) or \(m_i < +\infty\) for all \(i \in \mathbb{N}\).

For the recurrent event \(E\) two random variables can be introduced: \(X_k : \Sigma \to \mathbb{N}\) given by 1 plus the number of trials between the \((k-1)\)-th and \(k\)-th occurrence of \(E\); \(N_k : \Sigma \to \mathbb{N}\) given by the number of realizations of \(E\) in \(k\) trials. The random variables \(X_k\) have all the same probability distribution given by

\[
p[X_k = r] = p_{r-1},
\]
and their mean \(E_p[X_k] = m_0\), where the subscript \(p\) specifies the measure we use to find the mean. For our problem it will be very important also the form of the distribution function \(F(x) = \sum_{r=0}^{\lfloor x \rfloor} p_r\) of the \(X_k\). Finally, we consider also the probabilities \(u_n\) that \(E\) occurs at the \(n\)-th trial. It holds that

\[
\lim_{n \to +\infty} u_n = \frac{1}{m_0}.
\] (13)

Let’s now explain the link between the AIC of a string generated by the map \(L\) and the theory of recurrent events ([3]). Given an initial point \(x \in I\) we obtain a string \(\sigma \in \Sigma^*\) such that \(L^k(x) \in A_{\sigma_k}\) for all \(k \in \mathbb{N}\). The string \(\sigma\) is, for example, of the form

\[
\sigma = (7654321054321002103210\ldots).
\] (14)

One possible way to give an estimate for the AIC of the string is to consider a compression of the string, and study the binary length of the compressed string. One possible compression of the string \(\sigma\) is given by

\[
S = (75023\ldots),
\] (15)

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that is the sequence of recurrent times for $\mathcal{E}$. So to the finite string $\sigma^n$, obtained by the first $n$ symbols of $\sigma$, we associate the string

$$S^{N_n} = (\sigma_{q_1}\sigma_{q_2} \ldots \sigma_{q_{N_n}})$$

with $\sigma_{q_i-1} = 0$ for all $i$, where $N_n$ is the number of realizations of $\mathcal{E}$. Then to have an idea of the behaviour of the AIC of a string we have to estimate the behaviour of the random variables $N_n$.

In [10], some possible behaviours for $E_p[N_n]$ have been studied, for particular forms of the distribution function $F(x)$. In particular

**Theorem 5.1 (Feller).** If the recurrence time of $\mathcal{E}$ has finite mean $m_0$ and variance $V$, then

$$E_p[N_n] \sim \frac{n}{m_0} + \frac{V - m_0 + m_0^2}{2m_0^2}.$$  

If, instead $V = +\infty$, and the distribution function $F(x)$ satisfies

$$F(x) \sim 1 - Ax^{-\alpha}$$

with a constant $A$ and $0 < \alpha < 2$, then:

i) If $1 < \alpha < 2$,

$$E_p[N_n] \sim \frac{n}{m_0} + \frac{A}{(\alpha - 1)(2 - \alpha)m_0^2} n^{(2-\alpha)};$$

ii) If $0 < \alpha < 1$,

$$E_p[N_n] \sim \frac{\sin \alpha \pi}{A \alpha \pi} n^\alpha.$$  

If the variance $V$ of the recurrence time is infinite but the distribution function has a form different from that studied in Theorem 5.1, then we can show that

**Theorem 5.2.** If the state $A_0$ is ergodic then $E_p[N_n] \sim kn$, if instead $A_0$ is a null state then $E_p[N_n]$ is an infinite of order less than $n$. 

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Proof. The proof is based on a characterization of the mean $E_p[N_n]$. In [10], it is shown that $E_p[N_n] = U_n - 1$, where

$$U_n = \sum_{i=0}^{n} u_i.$$ 

Then it follows that

$$\lim_{n \to +\infty} \frac{E_p[N_n]}{n} = \lim_{n \to +\infty} \frac{U_n}{n} = \lim_{n \to +\infty} u_n = \frac{1}{m_0}.$$ 

Then, if $m_0 < +\infty$, that is $A_0$ is ergodic, then $E_p[N_n]$ is linear on $n$. Whereas if $m_0 = +\infty$, that is $A_0$ is a null state, then $E_p[N_n]$ is an infinite of order less than $n$.

The AIC calculated with the compression we have chosen can be linked to the random variables $N_n$ by the following theorem

**Theorem 5.3.** For any string $\sigma \in \Sigma_M$ it holds

$$(N_n - 1) + \log_2(n - N_n + 2) \leq I_{AIC}(\sigma^n) \leq N_n \log_2 \left( \frac{n + N_n}{N_n} \right)$$  \hspace{1cm} (16)

**Proof.** Given $n \in \mathbb{N}$, we have that $\Sigma_M = \cup C_r^n$, where the union is made on all the possible cylinders $C_r^n$, with $r \in \mathbb{N}^n$. Moreover we write

$$I_{AIC}(\sigma^n) = \sum_{i=1}^{N_n} \log_2(\sigma_{q_i} + 2)$$

for the AIC, where $\sigma_{q_i} + 2$ is used instead of $\sigma_{q_i}$, to have $\log_2(\sigma_{q_i} + 2) \geq 1$ for all $i$.

First of all, we consider only the cylinders $C_r^n$ with $r_n = 0$. This is done because we want to study the strings whose compression changes when we increase our given $n$. Indeed, if the compression wouldn’t change, we wouldn’t have any hint on the behaviour of the AIC with respect to the length of the string.

We start with some special cases. Let’s consider first the case $N_n = n$. The only possible cylinder is then $C_r^n$ with $r = (0, \ldots, 0)$. Then our
compression doesn’t change any string in this cylinder, and \( I_{AIC}(\sigma^n) = n \) for any string.

In the case \( N_n = (n-1) \), the possible cylinders are given by \( r \in \mathbb{N}^n \) with only one symbol 1 and all the others 0. For all the strings in these cylinders, the compression is \((n-1)\) symbols long, and \( I_{AIC}(\sigma^n) = (n-2) + \log_2 3 \).

In the case \( N_n = 1 \), the only possible cylinder is given by \( r_i = (n-i) \), and for the strings in this cylinder \( I_{AIC}(\sigma^n) = \log_2(n+1) \).

Let now be in general \( N_n = n-h \), for some \( h < n \). The compression of such strings is then \( N_n\)-symbols long. Moreover the compression is such that \( \sum_{i=1}^{N_n} \sigma_{q_i} = h \). We now want to find the maximum and the minimum of the function

\[
\sum_{i=1}^{N_n} \log_2 (\sigma_{q_i} + 2)
\]

with the condition \( \sum_{i=1}^{N_n} \sigma_{q_i} = k \). The maximum is attained for equal \( \sigma_{q_i} \neq 0 \), and the minimum for all the \( \sigma_{q_i} = 0 \) but one which is equal to \( h \). Then the maximum is given by \( \sigma_{q_i} = \frac{h}{n-h} \) for all \( i \), and the AIC for the strings in such a cylinder is given by

\[
\sum_{i=1}^{n-h} \log_2 \left( \frac{h}{n-h} + 2 \right) = N_n \log_2 \left( \frac{n+N_n}{N_n} \right),
\]

and the minimum is given by

\[
(n-h-1) + \log_2(h+2) = (N_n - 1) + \log_2(n-N_n + 2).
\]

Looking back at the special cases we studied before, we see that for \( N_n = n \) and \( N_n = 1 \), the maximum and the minimum are the same, and give exactly the value of the AIC we found building the sequences. For the case \( N_n = (n-1) \), the only possible value for the AIC is equal to the minimum we found. This shows that the maximum is not attained always, but there are cases in which it is attained. For example let \( n = 8 \), \( N_n = 4 \), and consider the string \( \sigma = (10101010) \). Its compression is then given by \( S = (1111) \), and

\[
I_{AIC}(\sigma^n) = 4 \log_2 3 = N_n \log_2 \left( \frac{n+N_n}{N_n} \right).
\]

Finally we remark that we have tacitly assumed that our strings do not begin with the symbol 0. If it happened, the estimates wouldn’t change significantly.

We have thus proved that for a subset of \( \Sigma_M \) of full measure with respect to all the cylinders with 0 as last symbol, the AIC of a string can be estimated using the value of the random variables \( N_n \).

Then our plan is the following: given a probability measure \( \mu \) on \((I, B, L)\), we find the stochastic matrix \( \Pi \), equation (10), the distribution function

\[
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\]
$F(x)$, and, the mean $E_p[N_n]$. Then we can link the mean $E_\mu[I_{AIC}(\sigma^n)]$, where $\mu$ is the probability measure on $(I, \mathcal{B})$, with the mean $E_p[N_n]$, by Theorem 5.3.

Another important aspect is the existence of an invariant measure for the Markov chain associated to our dynamical system. Given a stochastic matrix $\Pi$ of the form of equation (10) we have

**Theorem 5.4.** There is a measure $\bar{p}$ on the space $(\Sigma_M, \mathcal{C}, T_M)$, invariant for the stochastic matrix $\Pi$, defined by

$$\bar{p}(k) = \sum_{n=0}^{+\infty} p_{k+n}.$$  

This measure is a probability measure if and only if the mean recurrence time $m_0$ is finite.

This theorem allows us to induce on $(I, \mathcal{B}, L)$ an $L$-invariant measure $\mu$, which is a probability measure if and only if $A_0$ is ergodic.

### 5.2 Restriction to the Lebesgue measure

In the previous subsection we have shown that given a probability measure $\mu$ on the space $(I, \mathcal{B}, L)$, we have that the mean of the AIC behaves differently according to the mean recurrence time of the passage for the sub-interval $A_0$. These results clearly depend on the choice of the measure $\mu$ and of the sequence $(\epsilon_k)$ used in the definition of the map $L$. In this subsection we want to study these two problems from a practical point of view.

When we apply the notion of AIC to a string obtained from a dynamical system, the choice of this string depends on the choice of the initial point $x$ which we use to generate the orbit of the dynamical system. This choice can be made randomly, and the most natural way to introduce a probability distribution on the choice of the initial point is by using the Lebesgue measure $l$ on the space. Hence we apply the results of Subsection 5.1 to the system $(I, \mathcal{B}, L, l)$. Using the Lebesgue measure $l$, thanks to the piecewise linearity of the map $L$, the probabilities of transition $p_i$ given in equation (11) assume a particular simple form. Indeed we find that $p_i = l(A_i)$ for all $i \in \mathbb{N}$.

With respect to the Lebesgue measure $l$, it is also possible to prove that the compression we have introduced for strings given by any map $L$ is the best possible. We have
**Theorem 5.5.** Given any piecewise linear map \(L\) of the form (6), the best compression for the strings generated by the dynamical system \((I, \mathcal{B}, L, l)\), where \(l\) is the Lebesgue measure, is the compression given in equations (14) and (15), for \(l\)-almost any initial condition.

**Proof.** The compression we introduced gives a bijective relation between our space \((\Sigma_M, \mathcal{C}, T_M, p)\) and the space of all possible infinite sequences built on countable symbols, without any restriction given by a transition matrix. We denote this space as \(\Sigma\). We introduce on \(\Sigma\) the \(\sigma\)-algebra of the cylinders \(C'_n r\), given as in equation (9), and a probability measure \(p'\) inherited in some way from \(p\). We define \(p'\) by

\[
p'(C'^n_R) = p(C^N_R),
\]

where the cylinder \(C'^n_R\) is built in such a way that compression of strings that belong to it gives strings belonging to the cylinder \(C'_n\). At this point we ask if it is possible to compress any more strings belonging to the space \((\Sigma, C', p')\). But, if on this space we consider the usual shift map, we find a \(p'\)-invariant map with positive K-S entropy \(h_{KS}\). This is given by a direct computation, using the piecewise linearity of the map \(L\) that gives a simple form for the measure \(p'\). Then it is well known that the AIC for this dynamical system behaves like \(h_{KS}n\), for \(p'\)-almost any string \(\sigma\), then for \(l\)-almost any initial condition \(x \in I\). This clearly implies that for \(l\)-almost any initial condition \(x \in I\), ours is the best possible compression. \(\square\)

**Remark 5.6.** According, for example, to Brudno’s approach ([6]) to obtain a definition of AIC for finite orbits of a dynamical system, we should evaluate the supremum of \(I_{AIC}(\sigma^n)\) varying the open covers of the interval \(I = [0, 1]\).

Theorem 3.2 can be established as well if we consider an open cover of the form \(A_i = (\epsilon_i, \epsilon_{i-1} + \eta_i)\) for \(0 < \eta_i << 1\). Theorem 5.3 suggests that the AIC of any sequence generated by the system \((I, \mathcal{B}, L)\) with a non-trivial open cover has to be related to the random variables \(N_n\) as in the case we are considering. Then we can prove that the AIC of the particular strings we are considering is the AIC of the dynamical system \((I, L)\). This can also be proved in a more general contest ([12]).

The second point is the choice of the sequence \((\epsilon_k)\). We know that for any sequence, we can find an isomorphism of the map \(L\) with the Manneville map \(f\) of equation (1). At this point we present some cases for the choice of \((\epsilon_k)\), and then study a particular choice.
Example 5.7. Let the sequence be given by $\epsilon_k = \frac{1}{k^\alpha}$ with $\alpha > 0$. This sequence has all the properties we need for the definition of the map $L$. Then

$$p_i = l(A_i) = \frac{1}{(i - 1)^\alpha} - \frac{1}{i^\alpha} \sim \frac{1}{i^{\alpha+1}}.$$ 

The mean recurrence time $m_0$ is given by

$$m_0 = \sum_{r=1}^{+\infty} r \ p_r$$

and it is finite if and only if $\alpha > 1$. Then we can find an invariant measure $\bar{\mu}$ for the system $(I, B, L)$, such that $\bar{\mu}(A_i) \sim \frac{1}{i^{\alpha}}$. This measure $\bar{\mu}$ is a probability measure if and only if $\alpha > 1$.

The variance $V$ of the recurrence time is given by

$$V = \sum_{r=1}^{+\infty} r^2 \ p_r$$

and then $V < +\infty$ if and only if $\alpha > 2$. If we compute the distribution function $F(x)$, we have that

$$F(x) \sim 1 - Ax^{-\alpha},$$

then we can apply Theorems 5.1 and 5.3, and obtain that:

i) if $\alpha > 1$ then $E_l[I_{AIC}(\sigma^n)] \sim E_p[N_n] \sim n$;

ii) if $\alpha < 1$ then $E_p[N_n] \sim n^\alpha$ and $n^\alpha \leq E_l[I_{AIC}(\sigma^n)] \leq n^\alpha \log_2 n$ (see Theorem 5.3).

Example 5.8. Let now the sequence $(\epsilon_k)$ be given by $\epsilon_k = \frac{1}{a^k}$, where $a \in \mathbb{N}$ and $a > 1$. In this case it is easy to verify that $p_i \sim \frac{1}{a^i}$. Then the mean recurrence time $m_0$ and the variance $V$ of the recurrence time are always finite, and we can deduce that $E_l[I_{AIC}(\sigma^n)] \sim E_p[N_n] \sim n$. The invariant probability measure $\bar{\mu}$ exists and is given by $\bar{\mu}(A_i) \sim \frac{1}{a^i}$. 

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Example 5.9. Finally let’s consider the case in which the sequence \((\epsilon_k)\) is given by \(\epsilon_k = \frac{1}{k^\alpha (\log k)^\beta}\) with either \(\alpha = 1\) and \(\beta > 1\) or \(\alpha > 1\). If we compute the variance \(V\) of the recurrence time, we obtain that it is finite if and only if either \(\alpha > 3\) or \(\alpha = 3\) and \(\beta > 1\). In this case we obtain from Theorem 5.1 that \(E_l[I_{AIC}(\sigma^n)] \sim n\). But if \(V = +\infty\), we cannot find an explicit form for the distribution function \(F(x)\) similar to that of Theorem 5.2, but we can use Theorem 5.3. Indeed we have that the state \(A_0\) is ergodic if and only if either \(3 > \alpha > 2\) or \(\alpha = 2\) and \(\beta > 1\), in which case \(E_l[I_{AIC}(\sigma^n)] \sim n\). For other values of \(\alpha\) and \(\beta\), we know that \(E_p[N_n]\) is an infinite of order less than \(n\), and for \(E_l[I_{AIC}(\sigma^n)]\) we can apply Theorem 5.3 to obtain an estimate for its order of infinite.

Example 5.10. In a particular case, we can say something more about the order of infinite of \(E_p[N_n]\) using the theory of recurrent events ([10]) and of power series ([19]). Indeed, choosing the sequence \(\epsilon_k \sim \frac{1}{\log k}\), we have that the distribution function \(F(x) \sim \left(1 - \frac{1}{\log x}\right)\). Then from the characterization of \(E_p[N_n]\) in terms of the generating function of the random variables \(X_k\), we obtain that asymptotically \(E_p[N_n] < n^\alpha\) for all \(\alpha > 0\). Hence, from Theorem 5.3, \(E_l[I_{AIC}(\sigma^n)]\) is an infinite of order smaller than any power law.

We have thus found that changing the sequence \((\epsilon_k)\) it is possible to obtain all a set of behaviours for the mean of the AIC with respect to the Lebesgue measure. It is clear that the different behaviours are induced by the rate with which the derivative of the map \(L\) increases, rate that depends on the order with which \((\epsilon_k)\) tends to 0. It is then evident that this must be also the criterion to distinguish between different behaviours of the information function of the Manneville map \(f\) for different values of the parameter \(z\) in equation (1). We have then to find a way to associate to a particular value of \(z\) a given sequence \((\epsilon_k)\). Since we have to maintain a given rate of increasing of the derivative, given a value for \(z\), we look for the sequence \((\epsilon_k)\) such that \(\epsilon_k \sim x_k\), where \(x_k\) is the sequence of preimages of the point \(x_0\), as defined in Section 2.

We have that \(x_k \sim \frac{1}{k^\alpha}\) with \(\alpha = \frac{1}{z-1}\). We are then in the first case we studied, and \(\alpha > 0\) being \(z > 1\). We can apply all the results we found, in particular:

- if \(z < 2\) then \(E_l[I_{AIC}(\sigma^n)] \sim n\) and there is an \(L\)-invariant probability measure \(\mu\) such that \(\mu(A_i) \sim \frac{1}{n}\).
• if \( z > 2 \) then \( n^\alpha \leq E[I_{\text{AIC}}(\sigma^n)] \leq n^\alpha \log_2 n \) and the invariant measure \( \bar{\mu} \) is not a probability measure.

We have then found as a particular case the same results of [13]. But, we would like to have a behaviour of the AIC valid for almost any orbit with respect to the Lebesgue measure \( l \). We have

**Theorem 5.11.** For almost any point \( x \in I \) with respect to the Lebesgue measure \( l \) and for all \( \delta > 0 \), we have that \( E_{\mu_\delta}[I_{\text{AIC}}(\sigma^n)] \) is asymptotically equivalent to \( E[I_{\text{AIC}}(\sigma^n)] \), where with \( \mu_\delta \) we denote the measure given by the Lebesgue measure \( l \) concentrated on \( U_\delta = (x - \delta, x + \delta) \).

**Proof.** The proof is based on a simple application of the method we used before. Indeed, given \( x \in I \), let’s consider \( U_\delta = (x - \delta, x + \delta) \) for some \( \delta \). If we now want to estimate the value of \( E_\delta[N_n] \), that is the mean made with respect to the measure on \( \Sigma_M \) induced by \( \mu_\delta \), we notice that, from the properties of the map \( L \), it is clear that there exists a \( R(\delta) \in \mathbb{N} \) such that \( L^{R(\delta)}(U_\delta) = [0, 1] \). We can deduce that \( E_\delta[N_n] \) is the same as \( E_\mu[N_n] \), where \( p \) is the measure on \( \Sigma_M \) induced by \( l \), for \( n > R(\delta) \). Then it follows that \( E_{\mu_\delta}[I_{\text{AIC}}(\sigma^n)] \sim E[l][I_{\text{AIC}}(\sigma^n)] \) for \( n > R(\delta) \). \( \square \)

**Corollary 5.12.** The AIC of the Manneville map \( f \) is given by

- \( n^\alpha \leq E_{\mu_\delta}[I_{\text{AIC}}(\sigma^n)] \leq n^\alpha \log_2 n \) with \( \alpha = \frac{1}{z-1} \), for \( z > 2 \);
- \( E_{\mu_\delta}[I_{\text{AIC}}(\sigma^n)] \sim n \) for \( z < 2 \)

for almost any point \( x \in I \) with respect to the Lebesgue measure \( l \).

We remark that there are points \( x \in I \) for which \( I_{\text{AIC}}(x^n) \sim n \) also for the Manneville map \( f \) with \( z > 2 \). Indeed, in Subsection 3.2, we proved the existence of \( f \)-invariant measures \( \mu \) on \((I, \mathcal{B})\), for which the K-S entropy \( h^K_S \) = \( \log 2 \). This is not a contradiction since such measures have support on a set of zero Lebesgue measure.

### 6 Conclusions

In this paper we have proved that the Manneville map with \( z > 2 \) exhibits, from the AIC point of view, a behaviour which is intermediate between the so-called full chaos (positive K-S entropy) and periodicity. Then we obtain that
the complexity (see Section 1, equations (3)) of the Manneville map with \( z > 2 \) is null for almost any initial condition with respect to the Lebesgue measure \( l \) and then the algorithmic entropy \( h_l = 0 \) (see Section 1, equations (4)). In particular we have found a family of piecewise linear maps \( L \) that, for fixed sequences \( \epsilon_k \) (see equation (6)), can be used as a model for the Manneville map. Moreover this family of maps presents a rich set of possible algorithmic behaviours (depending on the choice of the map \( L \) of the family). It is evident that changing the sequence \( \epsilon_k \), the algorithmic behaviour varies from full chaos to mild chaos, which is characterized by \( I_{AIC}(\sigma^n) \) of order smaller than any power law. This behaviour can be achieved from the Manneville map at the limit \( z \to +\infty \) (see Theorem 5.3), and Example 5.10 seems to suggest a way to find a particular sequence generating mild chaos. We remark that it would be impossible using the notions of complexity and algorithmic entropy to distinguish many of the maps of the family \( L \), since we would obtain \( K(x) = h_l = 0 \) for almost every \( x \in (I, \mathcal{B}, l) \). Then the AIC is for those maps a powerful tool for the classification and it can be actually estimated.

Finally we observe that it has been proved that the AIC of a string is not a computable function and, in particular, it cannot be computed by any algorithm (8). Then, apart from particular dynamical systems for which we can estimate the AIC, the classification of dynamical systems using the AIC cannot be obtained explicitly. Nevertheless we believe that it is fundamental to obtain an explicit estimation of the AIC for as many dynamical systems as possible. However the AIC can be approximated by different notions of information content of a string. In particular we can define

\[
I_A(\sigma^n) = |A(\sigma^n)|,
\]

the information function, where \( |A(\sigma^n)| \) is the binary length of the output obtained from a string \( \sigma^n \) by means of a compression algorithm \( A \) (see [4]). A particular compression algorithm, called CASToRe, has been built to analyze dynamical systems which present rich dynamics, but having zero K-S entropy for all invariant measures that are physically relevant (3). The algorithm CASToRe has been tested on the Manneville map, giving as result \( I_{CASToRe}(\sigma^n) \sim n^\alpha \) for \( \alpha < 1 \), confirming our results (3, 4), and on the logistic map at the chaos threshold, giving the presence of mild chaos (5). At the moment it is not clear whether the algorithm CASToRe can approximate the AIC for any dynamical system, but from this paper on the Manneville map and from many experimental results on the logistic map, it
seems that at least in these two cases there is evidence of accordance between the theoretical predictions and the experiments with \textit{CASToRe}.

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