Cocycle Enhancements of Psyquandle Counting Invariants

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Abstract

We bring cocycle enhancement theory to the case of psyquandles. Analogously to our previous work on virtual biquandle cocycle enhancements, we define enhancements of the psyquandle counting invariant via pairs of a biquandle 2-cocycle and a new function satisfying some conditions. As an application we define new single-variable and two-variable polynomial invariants of oriented pseudoknots and singular knots and links. We provide examples to show that the new invariants are proper enhancements of the counting invariant and are not determined by the Jablan polynomial.

Keywords: Psyquandles, cocycle invariants, pseudoknots, singular knots

2020 MSC: 57K12

1 Introduction

In [2], a (co)-homology theory for quandles was introduced and used to enhance the quandle counting invariant of classical (and later virtual) knots and links with 2-cocycles. The idea was extended to biquandles in [1] and to virtual biquandles by the present authors in [3], obtaining a two-variable enhancement in the case of strongly compatible cocycles and a single-variable polynomial enhancement in the case of weakly compatible cocycles.

The term pseudoknot was first used in the natural sciences to describe knotted structures with only partial crossing information, such as DNA strands in images with insufficient resolution to determine the crossing information at some crossings, for example [4]. A rigorous mathematical definition for pseudoknots was established in [6] and has been studied in further work such as [7] [8] [9]. In addition to classical crossings, pseudoknots have precrossings which are understood to be classical crossings about which we lack crossing information. In particular, a pseudoknot can be understood as a kind of probability distribution with the classical knot resolutions obtained by assigning crossing information to precrossings as outcomes.

Singular knots are rigid vertex isotopy classes of 4-valent spatial graphs. They have been studied particularly for their connection with Vassiliev invariants; see [13] for example. In particular, the singular Reidemeister moves are the same as the pseudoknot Reidemeister moves, aside from one move, if we replace precrossings with singular crossings with precrossings.

In [11], an algebraic structure called psyquandle was introduced, algebraically encoding the Reidemeister moves for pseudoknots and singular knots into one unified structure. Psyquandles were used to define a new polynomial invariant of pseudoknots and singular, the Jablan polynomial, which can be understood as a weighted sum of Alexander polynomials of the classical resolutions of the pseudoknot. An integer-valued psyquandle counting invariant was also introduced.

In this paper we enhance the psyquandle counting invariant with cocycles analogously to our work in [3], obtaining a new infinite family of single-variable and two-variable polynomial invariants of pseudoknots and singular knots and links. The paper is organized as follows. In Section 2 we revisit psyquandles and recall their basics. In Section 3 we recall the basics of biquandle (co)homology. In Section 4 we introduce the new invariants and provide examples and computations, including and example to show that the new invariants are not determined by the Jablan polynomial. We conclude in Section 6 with some questions for future research.

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2 Psyquandles

In this section we recall the basics of psyquandles. See [11] for more.

**Definition 1.** Let X be a set. A *psyquandle structure* on X is set of four binary operations $\sqcup, \sqcap, \bullet, \cdot : X \times X \to X$ satisfying the conditions

(i) For all $x \in X$, $x \sqcup y = x \sqcap y$,

(ii) For all $x, y \in X$, the maps $S, S' : X \times X \to X \times X$ defined by

$$S(x, y) = (y \sqcap x, x \sqcup y) \quad \text{and} \quad S'(x, y) = (y \bullet x, x \cdot y)$$

are invertible,

(iii) For all $x, y, z \in X$,

$$\begin{align*}
(x \sqcup y) \sqcup (z \sqcup y) &= (x \sqcup z) \sqcup (y \sqcup z) \\
(x \sqcup y) \sqcap (z \sqcap y) &= (x \sqcap z) \sqcap (y \sqcap z) \\
(x \sqcap y) \sqcap (z \sqcap y) &= (x \sqcap z) \sqcap (y \sqcap z)
\end{align*}$$

(iv) For all $x, y \in X$ we have

$$\begin{align*}
x \cdot ((y \sqcap x) \sqcup^{-1} x) &= [(x \sqcup y) \sqcap^{-1} y] \sqcap [(y \sqcap x) \sqcup^{-1} x] \\
y \bullet ((x \sqcup y) \sqcap^{-1} y) &= [(y \sqcap x) \sqcap^{-1} x] \sqcup [(x \sqcup y) \sqcap^{-1} y],
\end{align*}$$

and

(v) For all $x, y, z \in X$ we have

$$\begin{align*}
(x \sqcap y) \sqcap (z \sqcap y) &= (x \sqcap z) \sqcap (y \sqcap z) \\
(x \sqcup y) \sqcup (z \sqcup y) &= (x \sqcup z) \sqcup (y \sqcup z) \\
(x \sqcap y) \sqcap (z \sqcap y) &= (x \sqcap z) \sqcap (y \sqcap z) \\
(x \sqcup y) \sqcup (z \sqcup y) &= (x \sqcup z) \sqcup (y \sqcup z) \end{align*}$$

A psyquandle which also satisfies $x \bullet x = x \cdot x$ for all $x \in X$ is said to be *pI-adequate*.

**Example 1.** Every biquandle is a psyquandle by setting $x \bullet y = x \sqcap y$ and $x \cdot y = x \sqcup y$.

**Example 2.** A module over $\mathbb{Z}[t^{\pm 1}, s^{\pm 1}, a^{\pm 1}, b^{\pm 1}]/(t + s - a - b)$ is a psyquandle with operations

$$\begin{align*}
x \sqcup y &= tx + (s - t)y \\
x \sqcap y &= sx \\
x \bullet y &= ax + (s - a)y \\
x \cdot y &= bx + (s - b)y
\end{align*}$$

known as an *Alexander psyquandle*. 
**Example 3.** Given a finite set $X = \{1, 2, \ldots, n\}$, we can specify a psyquandle structure on $X$ by explicitly listing the operation tables of the four psyquandle operations. In practice it is convenient to put these together into an $n \times 4n$ block matrix, so the psyquandle structure on $X = \{1, 2, 3\}$ specified by

\[
\begin{array}{cccc|cccc|cccc|cccc}
\triangleright & 1 & 2 & 3 & \triangleright & 1 & 2 & 3 & \cdot & 1 & 2 & 3 & \triangleright & 1 & 2 & 3 \\
1 & 2 & 2 & 2 & 1 & 2 & 2 & 1 & 3 & 3 & 3 & 1 & 3 & 3 & 3 \\
2 & 3 & 3 & 3 & 2 & 3 & 3 & 3 & 2 & 1 & 1 & 1 & 2 & 1 & 1 \\
3 & 1 & 1 & 1 & 3 & 1 & 1 & 3 & 2 & 2 & 2 & 3 & 2 & 2 & 2 \\
\end{array}
\]

is encoded as the block matrix

\[
\begin{pmatrix}
2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 \\
1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2
\end{pmatrix}
\]

See [11] for more examples of psyquandles.

The psyquandle axioms are motivated by the semiarc coloring rules for singular knots and pseudoknots below:

\[x \triangleright y \iff y \triangleright x\]

\[x \triangleright y \iff y \triangleright x\]

\[(x \triangleright y) \triangleright y^{-1} = x\]

\[(y \triangleright x) \triangleright x^{-1} = y\]

\[(y \triangleright x) \triangleright x^{-1} = y\]

\[(x \triangleright y) \triangleright y^{-1} = x\]

\[(y \triangleright x) \triangleright x^{-1} = y\]

\[(x \triangleright y) \triangleright y^{-1} = x\]

\[(y \triangleright x) \triangleright x^{-1} = y\]

\[(x \triangleright y) \triangleright y^{-1} = x\]

\[(y \triangleright x) \triangleright x^{-1} = y\]

**Remark 1.** We have reformulated axiom (iv) from its description in [10] in order to simplify the Reidemeister IV conditions for the Boltzmann weights we will define later. To see how these arise, consider the labeled move below:

**Definition 2.** Let $X$ be a finite psyquandle (respectively, a $pI$-adequate finite psyquandle). An $X$-coloring of an oriented singular knot or link diagram $L$ (respectively, an oriented pseudoknot or pseudolink diagram $L$) is an assignment of an element of $X$ to each semiarc in $L$ such that the coloring rules (1) and (2) are satisfied at every crossing.
In [11] we find the following result:

**Theorem 1.** Let $X$ be a finite psyquandle (respectively, a finite $pI$-adequate psyquandle) and let $L$ be an oriented singular knot or link diagram (respectively, an oriented pseudoknot or pseudolink diagram). Then the cardinality of the set of $X$-colorings of $L$,

$$\Phi_X^\mathbb{Z}(L) = |\text{Hom}(P(L), X)|$$

is an integer-valued invariant of singular knots and links (respectively, pseudoknots and pseudolinks).

In the remainder of this paper we will define Boltzmann weight enhancements of the psyquandle counting invariant.

## 3 Biquandle Cohomology

In this section we recall the basics of biquandle homology and cohomology. See [1, 5] etc. for more on this topic.

Let $X$ be a finite biquandle and $R$ a commutative ring with identity. The set $C_n^\mathbb{Z}(X; R) = R[X^n]$ is the free $R$-module on the set ordered $n$-tuples of element of $X$; its dual, $C_n^\mathbb{Z}(X; R) = \text{Hom}(C_n^\mathbb{Z}(X; R), R)$ is the set of $R$-module homomorphisms from $C_n^\mathbb{Z}(X; R)$ to $R$.

The map $\partial_n : C_n^\mathbb{Z}(X; R) \to C_{n-1}^\mathbb{Z}(X; R)$ defined on generators by

$$\partial(x_1, \ldots, x_n) = \sum_{k=1}^{n} (\sum_{\sigma \in S_n} (-1)^{\sigma(k)} (x_1 \circ x_{k-1} \circ x_{k+1} \circ \cdots \circ x_n) - (x_1 \circ x_{k-1} \circ x_k \circ x_{k+1} \circ \cdots \circ x_n))$$

and extended linearly is a boundary map, with coboundary map $\delta_n : C_n^\mathbb{Z}(X; R) \to C_{n-1}^\mathbb{Z}(X; R)$ given by $\delta_n(f) = f \partial_n$. The resulting homology groups $H_n^\mathbb{Z}(X; R) = \text{Ker}\partial_n/\text{Im}\partial_{n-1}$ and cohomology groups $H^n^\mathbb{Z}(X; R) = \text{Ker}\delta_n/\text{Im}\delta_{n-1}$ are known as the birack homology and birack cohomology groups of $X$ with coefficients in $R$.

The subgroups $D_n(X; R)$ and $D^n(X; R)$ generated by elements $(x_1, \ldots, x_n)$ with $x_k = x_{k+1}$ for some $k \in 1, \ldots, n-1$ form degenerate subcomplexes; modding out by these subcomplex yields biquandle homology and biquandle cohomology.

**Example 4.** A function $\phi : X \times X \to R$ represents a biquandle cohomology class in the biquandle cohomology of $X$ with $R$ coefficients if it satisfies

(i) For all $x \in X$,

$$\phi(x, x) = 0$$

and

(ii) For all $x, y, z \in X$,

$$\phi(x, y, z) - \phi(x \circ z, y \circ z) - \phi(x, z) + \phi(x \circ y, z \circ y) + \phi(y, z) - \phi(y \circ x, z \circ x) = 0.$$
A biquandle 2-cocycle for a finite biquandle structure on $X = \{1, \ldots, n\}$ can be written as a linear combination of characteristic functions

$$\phi = \sum_{(j,k) \in X \times X} \phi_{jk} \chi_{(j,k)}$$

which we can conveniently encode as an $n \times n$ matrix whose $(j, k)$ entry is $\phi_{jk}$.

### 4 Ccocycle Enhancements

We will now generalize biquandle cocycle invariants to the case of psyquandles.

**Definition 3.** Let $X$ be a psyquandle and $R$ a commutative ring with identity. A *Boltzmann weight* for $X$ is a pair of maps $\phi, \psi : X \times X \to R$ satisfying

(i) For all $x \in X$, $\phi(x, x) = 0$

(ii) For all $x, y \in X$,

$$\phi(x, y) + \psi(y, (x \geq y) \bullet^{-1} y) = \phi((y \triangleright x) \bullet^{-1} x, (x \geq y) \bullet^{-1} y) + \psi(x, (y \triangleright x) \bullet^{-1} x).$$

(iii) For all $x, y, z \in X$,

$$\begin{align*}
\phi(x, y) + \phi(y, z) + \phi(x \geq y, z \triangleright y) & = \phi(x \geq z, y \triangleright z) + \phi(x, z) + \phi(y \triangleright x, z \triangleright x) \\
\psi(x, y) + \phi(y, z) + \phi(x \bullet y, z \triangleright y) & = \psi(x \geq z, y \triangleright z) + \phi(x, z) + \phi(y \triangleright x, z \triangleright x) \\
\psi(z, y) - \phi(x, y) - \phi(x \geq y, z \bullet y) & = \psi(z \triangleright x, y \triangleright x) - \phi(x, z) - \phi(x \geq z, y \triangleright z).
\end{align*}$$

If $X$ is pI-adequate, we say that $(\phi, \psi)$ is *pI-adequate* if

(v) For all $x \in X$,

$$\psi(x, x) = 0$$

and we say that $\phi$ and $\psi$ are *strongly compatible* if we also have

(vi) For all $x, y, z \in X$,

$$\psi(x, y) = \psi(x \geq z, y \geq z) \quad \text{and} \quad \psi(z, y) = \psi(z \triangleright x, y \triangleright x).$$

The Boltzmann weight axioms are motivated by the Reidemeister moves for singular knots and pseudo-knots following below, using the contribution rule:
Definition 4. Let $X$ be a psyquandle (respectively, a $pI$-adequate psyquandle), $R$ a commutative ring with identity and $(\phi, \psi)$ a Boltzmann weight (respectively, a $pI$-adequate Boltzmann weight). Let $L$ be an oriented singular knot or link (respectively, any oriented pseudoknot or pseudolink).

(1) For each $X$-coloring $L_c$ of $L$ in the set $C(L, X)$ of $X$-colorings of $L$, we define the Boltzmann weight of
Let \( L_c \), denoted \( BW(L_c) \), to be the sum over all crossings in \( L_c \) of crossing contributions as shown:

\[
\begin{align*}
    x & \quad \quad y \\
    y & \quad \quad x \\
    +\phi(x,y) & \quad -\phi(x,y) \\
    +\psi(x,y)
\end{align*}
\]

(2) We define the single-variable Boltzmann-enhanced psyquandle polynomial to be

\[
\Phi_{\phi,\psi}^X(L) \sum_{L_c \in \mathcal{C}(L,X)} w^{BW(L_c)}
\]

(3) If \( \phi \) and \( \psi \) are strongly compatible, we define the partial Boltzmann weights \( BW_{\phi}(L_c) \) and \( BW_{\psi}(L_c) \) to be the sums of \( \phi \) contributions and \( \psi \) contributions respectively; then we define the two-variable Boltzmann-enhanced psyquandle polynomial to be

\[
\Phi_{\phi,\psi}^X(L) \sum_{L_c \in \mathcal{C}(L,X)} u^{BW_{\phi}(L_c)} v^{BW_{\psi}(L_c)}
\]

By construction, we have

**Proposition 2.** Let \( X \) be a psyquandle, \( R \) a commutative ring with identity and \((\phi, \psi)\) a Boltzmann weight with coefficients in \( R \). Then:

(1) \( \Phi_{\phi,\psi}^X \) is an invariant of singular knots and links,

(2) If \( X \) and \( \psi \) are pI-adequate, then \( \Phi_{\phi,\psi}^X \) is an invariant of pseudoknots and pseudolinks.

### 5 Examples

In this section we collect some computations and examples.

**Example 5.** Let \( X = \mathbb{Z}_5 \) with the following operations,

\[
\begin{align*}
    x \triangledown y &= 3x + 4y \\
    x \triangledown y &= 2x \\
    x \blacklozenge y &= 4x + 3y \\
    x \blacklozenge y &= x + y
\end{align*}
\]

is an Alexander psyquandle which is pI-adequate, for detail see [11]. Consider the Boltzmann weight on \( X \),
defined by $\phi, \psi : X \times X \to \mathbb{Z}_4$ with $\phi(x, y) = 0$ and $\psi(x, y) = 2$ which is $pI$-adequate. The singular knot $K_1$

\[ x_1 \quad \bullet \quad x_2 \quad \bullet \quad x_3 \quad \bullet \quad x_4 \]

has the system of coloring equations given by

\[
4x_1 - 2x_2 = x_1 \bullet x_2 = x_3 \\
x_1 + x_2 = x_2 \circ x_1 = x_4 \\
x_1 + x_2 = x_1 \circ x_2 = x_3 \\
-2x_1 + 4x_2 = x_2 \bullet x_1 = x_4
\]

which we can solve by row-reduction over $\mathbb{Z}_5$:

\[
\begin{bmatrix}
4 & -2 & -1 & 0 \\
1 & 1 & 0 & -1 \\
1 & 1 & -1 & 0 \\
-2 & 4 & 0 & -1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 4 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

Now, consider the following singular knot $K_2$

\[ x_1 \quad \bullet \quad x_2 \quad \circ \quad x_3 \quad \bullet \quad x_4 \]

has the system of coloring equations given by

\[
4x_1 - 2x_2 = x_1 \bullet x_2 = x_3 \\
x_1 + x_2 = x_2 \circ x_1 = x_4 \\
2x_1 = x_1 \circ x_2 = x_3 \\
x_1 + 3x_2 = x_2 \circ x_1 = x_4
\]

which we can solve by row-reduction over $\mathbb{Z}_5$:

\[
\begin{bmatrix}
4 & -2 & -1 & 0 \\
1 & 1 & 0 & -1 \\
2 & 0 & -1 & 0 \\
-1 & 3 & 0 & -1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 4 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]
These two systems have 5 solutions, therefore, both diagrams have 5 coloring by this psyquandle. We will now consider the Boltzmann weight enhanced invariant in order to distinguish these two singular knots. Using the Boltzmann weight above, we obtain enhanced invariant values $\Phi_{X}(K_1) = 5$ and $\Phi_{X}(K_2) = 5v^2$, demonstrating that the enhanced invariant is not determined by the number of colorings and hence is a proper enhancement.

**Example 6.** Using our custom Python code, we computed the Boltzmann weight for certain singular knots known as *two-bouquet graphs of type K* (with choice of orientation) listed in [12] using the psyquandle $X$ given by the operation matrix

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 1 & 1 & 1 & 1
\end{bmatrix}.
$$

The weight function $\phi : X \times X \rightarrow \mathbb{Z}_{14}$ is given by the following matrix

$$
\begin{bmatrix}
0 & 0 \\
7 & 0
\end{bmatrix}
$$

and the weight function $\psi : X \times X \rightarrow \mathbb{Z}_{14}$ is given by the following matrix

$$
\begin{bmatrix}
0 & 0 \\
3 & 0
\end{bmatrix}.
$$

We compute the Boltzmann weight enhancement for $5k^2$ and $5k^3$ from [12]. We obtain the following Boltzmann weight enhanced invariant values: $\Phi_{X}^{\phi,\psi}(5k^2) = 2u^7$ and $\Phi_{X}^{\phi,\psi}(5k^3) = 2$. This example shows that $\Phi_{X}^{\phi,\psi}$ detects additional information beyond counting invariant $\Phi_{X}^{Z}(5k^2) = 2 = \Phi_{X}^{Z}(5k^3)$.

**Example 7.** Using our custom Python code, we compute the Boltzmann weights invariant for the 2-bouquet graphs of type L (with choice of orientation) in [12] using the psyquandle given by the operation matrix

$$
\begin{bmatrix}
2 & 4 & 4 & 6 & 6 & 2 & 6 & 2 & 6 & 2 & 6 & 2 & 6 & 2 & 2 & 2 & 6 & 4 & 6 & 6 & 6 \\
3 & 5 & 5 & 1 & 1 & 3 & 1 & 5 & 1 & 5 & 1 & 5 & 3 & 5 & 5 & 5 & 1 & 5 & 1 & 1 & 1 & 3 \\
4 & 6 & 6 & 2 & 2 & 4 & 6 & 4 & 6 & 4 & 6 & 6 & 6 & 2 & 6 & 4 & 4 & 6 & 4 & 2 & 4 \\
5 & 1 & 1 & 3 & 3 & 5 & 5 & 3 & 5 & 3 & 5 & 3 & 1 & 3 & 3 & 3 & 5 & 1 & 5 & 3 & 5 & 5 \\
6 & 2 & 2 & 4 & 4 & 6 & 4 & 2 & 2 & 4 & 2 & 4 & 4 & 4 & 4 & 6 & 6 & 2 & 2 & 2 & 4 & 2 \\
1 & 3 & 3 & 5 & 5 & 1 & 3 & 1 & 3 & 1 & 3 & 1 & 1 & 1 & 3 & 1 & 5 & 1 & 3 & 3 & 3 & 5 & 3 & 1
\end{bmatrix}
$$

with weight function $\phi : X \times X \rightarrow \mathbb{Z}_2$ given by the matrix

$$
\begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

and weight function $\psi : X \times X \rightarrow \mathbb{Z}_2$ given by the matrix

$$
\begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}.
$$

The results are collected in the table.
\[
\begin{array}{|c|c|c|c|}
\hline
\Phi_X^\phi & \Phi_X^\phi(L) & L \\
\hline
12 & 12w & 5_2, 6_1^1 \\
 & 6w + 6 & 3_1, 4_{11}, 5_3, 6_2, 6_6^1 \\
24 & 24w & 6_3, 6_9, 6_{10}, 6_{11} \\
 & 6w + 18 & 5_1, 6_5, 6_7^1 \\
 & 18w + 6 & 1_1^1 \\
36 & 18w + 18 & 6_4, 6_12. \\
\hline
\end{array}
\]

**Example 8.** Using our custom Python code, we computed the counting invariant \( \Phi_X^\phi \) and Boltzmann weight enhanced invariant \( \Phi_X^{\phi, \psi} \) for a choice of orientation for the pseudoknots in [7] using the psyquandle \( X \) given by the operation matrix

\[
\begin{pmatrix}
1 & 3 & 1 & 1 & 1 & 1 & 3 & 1 & 3 & 3 & 3 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 1 & 3 & 3 & 3 & 3 & 1 & 3 & 1 & 1 & 1
\end{pmatrix}
\]

and the weight function \( \phi : X \times X \to \mathbb{Z}_6 \) given by the matrix

\[
\begin{pmatrix}
0 & 3 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

and weight function \( \psi : X \times X \to \mathbb{Z}_6 \) given by the matrix

\[
\begin{pmatrix}
0 & 5 & 4 \\
2 & 0 & 5 \\
4 & 5 & 0
\end{pmatrix}
\]

We have a pi-adequate psyquandle since the two right blocks have the same diagonal and the pair \((\phi, \psi)\) is pi-adequate since the entries along the diagonal of \( \psi \) are all zero. The results are collected in the table

\[
\begin{array}{|c|c|c|}
\hline
\Phi_X^\phi & \Phi_X^{\phi, \psi}(L) & L \\
\hline
3 & 3 & 3, 1, 3, 1, 3, 1, 4, 1, 4, 1, 3, 4, 1, 3, 4, 5, 5, 1, 5, 1, 5, 1, 5, 1, 5, 1, 5, 2, 1, 5, 2, 3, 5, 2, 6, 5, 2, 6, 5, 2, 8, 5, 2, 10 \\
2w^2 + 1 & 3, 1, 2, 4, 1, 2, 4, 1, 4, 1, 4, 1, 4, 5, 2, 4, 5, 2, 7, 5, 2, 9 \\
2w^4 + 1 & 5, 1, 2, 5, 2, 2. \\
\hline
\end{array}
\]

**Example 9.** Using the psyquandle and weight functions from Example 8 the Boltzmann weight of a pseudoknot can detect additional information for pseudoknots with the same Jablan polynomial. We collect the results in the following two tables

\[
\begin{array}{|c|c|c|}
\hline
\Delta_J(L) & \Phi_X^{\phi, \psi}(L) & L \\
\hline
1 & 3 & 3, 1, 4, 1, 1, 4, 1, 3, 5, 1, 1, 5, 2, 1, 5, 2, 6 \\
2w^2 + 1 & 3, 1, 2, 4, 1, 2, 5, 2, 9 \\
2w^4 + 1 & 5, 1, 2, 5, 2, 2. \\
\hline
\end{array}
\]

and

\[
\begin{array}{|c|c|c|}
\hline
\Delta_J(L) & \Phi_X^{\phi, \psi}(L) & L \\
\hline
s^2 + t^2 & 3 & 3, 1, 3, 5, 2, 3 \\
2w^2 + 1 & 5, 2, 4. \\
\hline
\end{array}
\]

**6 Questions**

We end with a few questions for future work.
In [3], biquandle homology was enhanced for virtual knots and links with a weight function at virtual crossings, and these were found to be cocycles themselves in a cohomology theory we called $S$-cohomology. Is something similar true for psyquandles? How should the singular crossing weight $\psi$ be interpreted in terms of cohomology?

What other enhancements of the psyquandle counting invariant and Boltzmann weight-enhanced invariants can be defined?

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