Some preliminary results on the dynamics of Einstein–Yang–Mills–Higgs systems *

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Abstract. A mathematical and numerical framework for the study of dynamical properties of spherically symmetric Einstein–Yang-Mills–Higgs systems is introduced. The quantitative investigations of the time evolution of some simple magnetic monopole type configurations are presented. Long living breather type states are found to develop.

1. Introduction

The study of the evolution of nonlinear dynamical systems, in particular, that of the properties of radiative tails, is at the center of various analytic and numerical investigations (see, e.g., references Zenginoglu 2008, Pürrer & Aichelburg 2009, Bizoń et al. 2009, Bizoń & Zenginoglu 2009). Although most of these studies are restricted to spherically symmetric configurations—due to the fact that some of the involved fields are radiative—the results beside being of interest on their own rights have also some relevance concerning the plausible fall off properties of gravitational waves. In order to get the gravitational degrees of freedom to become dynamical one needs to use matter fields whenever the geometry and the fields are spherically symmetric. The main motivation for the study of these type of systems is provided by the fact that the linearization of a nonlinear system fails to predict the correct asymptotic behavior of the involved fields—see, e.g., Table 1 below—even in regions far away from the truly dynamical and visible nonlinear regions.

In this paper the dynamics of Einstein–Yang-Mills–Higgs systems is investigated with the assumption that the Yang-Mills field is in the adjoint representation of the $SU(2)$ gauge group. Our eternal aim will be the study of the asymptotic properties of these systems. Nevertheless in this paper considerations will be restricted to the investigation of dynamics in the inner region containing a localized particle type soliton.

The Einstein–Yang-Mills–Higgs systems—beside being of great interest from the point of view of nonlinear dynamics—are known to host various particle like solutions discovered by Bartnik & McKinnon 1988 and later these static systems were investigated by many authors see, e.g., Bizoń 1990, Breitenlohner et al. 1994. The reader interested in these more particle physics oriented motivations may find the review papers Goddard & Di Olive 1978, and Volkov & Gal’tsov 1999 to be useful.

* Dedicated to the memory of our co-author and friend Péter Csizmadia who was missing—along with three other young Hungarian climbers—during an expedition in Himalayas at the end of October 2009.
Table 1. The value of the power $p$ of the $\sim \Lambda \tau^p$ type fall off law of the matter field variable is indicated while approaching future timelike infinity $i^+$ and future null infinity $\mathcal{I}^+$, respectively, where in both cases $\tau$ denotes the coordinate time of the pertinent spacetime. In all of these investigations the evolution of a single Yang-Mills field was considered.

| $i^+$ | $\mathcal{I}^+$ | Spacetime | Technique       |
|-------|----------------|-----------|-----------------|
| -5    | -3            | Schwarzschild | Linear Perturbation |
| -4    | -2            | Schwarzschild and Minkowski | Nonlinear Perturbation |
| -4    | -2            | Schwarzschild  | Numerical Method |
| -4    | -2            | Spherically Symmetric | Numerical Method |

2. The gauge choices

This section is to provide a precise specification of the gravity matter model to which our results apply.

2.1. The spacetime

Regarding the geometry we shall adopt the choices made in Csizmadia & Rácz 2010. Accordingly the metric of the spherically symmetric spacetime is supposed to be given by line element

$$ds^2 = \alpha^2 d\tau^2 - \alpha d\rho^2 - r^2 dS^2$$

where $\alpha$, $\beta$, and $r$ are smooth functions of the coordinates $\tau, \rho$, and $dS^2$ stands for the line element of the two-dimensional unite sphere, i.e., $dS^2 = d\theta^2 + \sin^2 \theta d\phi^2$. In virtue of the particular form of the metric it is straightforward to see that $\beta$ is the coordinate speed of light.

2.2. The matter system

The Yang-Mills part is represented by its vector potential $A_a = A^i_a T_i$, where $T_i$ denotes the $\mathfrak{su}(2)$ generators which are related to the Pauli matrices as $T_i = \frac{1}{2} \sigma_i$. The corresponding field strength reads then as

$$F_{ab} = 2 \nabla_a \mathcal{A}_b + ig [\mathcal{A}_a, \mathcal{A}_b], \quad F_{ab}^l = 2 \nabla_a A^l_b - g \varepsilon_{kjl} A^k_a A^j_b$$

where $F_{ab} = F_{ab}^l T_l$, while the Yang-Mills Lagrangian is given as

$$\mathcal{L}_{YM} = -\frac{1}{2} \text{Tr} \mathcal{F}_{ab} \mathcal{F}^{ab} = -\frac{1}{4} \gamma_{kj} F_{ab}^k F^{j,k}$$

where $\gamma_{kj} = K^{-1} \text{Tr} \langle T_k, T_j \rangle$, with $K^{-1} = 2$, is the positive definite metric on $\mathfrak{su}(2)$ which takes the simple form $\gamma_{kj} = \delta_{kj}$.

The Higgs-field is represented as $\Phi = \Phi^l T_l$, with Lagrangian

$$\mathcal{L}_H = \frac{1}{2} [2 \text{Tr} \mathcal{D}_a \Phi \mathcal{D}^a \Phi - V(\Phi)] = \frac{1}{2} [\gamma_{ij} \mathcal{D}_a \Phi^i \mathcal{D}^a \Phi^j - V(\Phi)]$$

where the gauge covariant derivative of $\Phi$ is given as

$$\mathcal{D}_a \Phi = \nabla_a \Phi + ig [\mathcal{A}_a, \Phi], \quad \mathcal{D}_a \Phi^l = \nabla_a \Phi^l - g \varepsilon_{kjl} A^k_a \Phi^j.$$
and the self-interaction potential $V$ is assumed to possess the standard form

$$V(\Phi) = \frac{\lambda}{4} \left[ \gamma_{ij}(\Phi^i \Phi^j - \Phi_0^i \Phi_0^j) \right].$$  (6)

The coupled Yang-Mills–Higgs system is characterized by the Lagrangian $\mathcal{L}_{YMH} = \mathcal{L}_{YM} + \mathcal{L}_H$ while its stress-energy tensor is given as

$$T_{ab} = -\frac{1}{4\pi} \{ \text{Tr} \mathcal{F}_a \mathcal{F}_b - \text{Tr} \mathcal{D}_a \Phi \mathcal{D}_b \Phi + \mathcal{L}_{YMH} g_{ab} \}. $$ (7)

3. The dynamics

It can then be justified that the above specified coupled Einstein–Yang-Mills–Higgs system is governed by the field equations

$$\nabla^c \mathcal{F}_{ca} + ig \left[ [\mathcal{F}^E, \mathcal{F}_{ca}] + [\Phi, \mathcal{D}_a \Phi] \right] = 0, $$  (8)

$$\nabla^c \mathcal{D}^c \Phi + ig \left[ [\mathcal{F}^E, \mathcal{D}_c \Phi] + \frac{1}{2} \frac{\partial V}{\partial \Phi} \right] = 0, $$  (9)

$$E^a_b = R^a_b - \frac{1}{2} g^a_b R - 8\pi T^a_b = 0. $$  (10)

Restricting our attention—as in Fodor & Rácz 2008, Fodor & Rácz 2004—to the minimal spherically symmetric configurations the Yang-Mills and Higgs fields can be given in the Abelian gauge as

$$A = -\frac{1}{g} \left[ w (T_2 d\theta - \sin\theta T_1 d\varphi) + \cos\theta T_3 d\varphi \right], \quad \Phi = HT_3, $$ (11)

with $\Phi_0 = H_0 T_3$, for some $H_0 \in \mathbb{R}$. By introducing then $h(\tau, \rho)$ via $H(\tau, \rho) = \frac{h(\tau, \rho)}{\rho} + H_\infty$, where $H_\infty = \lim_{\rho \to \infty} H$, as a new Higgs variable the field equations (8) and (9) can be seen to take the form

$$w_{\tau \tau} = \beta^2 w_{\rho \rho} + \beta_\rho w_{\rho} + \frac{\beta_\tau w_{\tau}}{\beta} + \alpha \beta^2 w \left[ \frac{1 - w^2}{r^2} - g^2 \left( \frac{h}{\rho} + H_\infty \right)^2 \right], $$ (12)

$$h_{\tau \tau} = \beta^2 h_{\rho \rho} + \beta_\rho h_{\rho} + \frac{\beta_\tau h_{\tau}}{\beta} - \frac{2\beta^2 h_{\rho}}{\rho} + \frac{2r_h h_{\tau} - 2\beta^2 h_{\rho}}{r} \left( \frac{h}{\rho} - \frac{h}{\rho^2} \right) - \alpha \beta^2 \left( \frac{h}{\rho} + H_\infty \right) \left[ \frac{2r w^2}{r^2} + \frac{\lambda h}{\rho} + 2H_\infty \right] + \frac{2\beta^2 h}{\rho^2}, $$ (13)

where the abbreviating notation $f_\sigma = \partial_\sigma f$ is applied (for $f = \alpha, \beta, r, w$ and $h$).

The combinations $E^\rho_\varphi + E^\varphi_\rho - E^\rho_\rho - E^\varphi_\varphi = 0$, and $E^\rho_\rho + E^\varphi_\varphi = 0$ yield similar wave equations for $\alpha$ and $r$, while $\beta$ remains freely specifiable. However, as in Csizmadia & Rácz 2010 by introducing the Misner-Sharp mass as

$$m = \frac{r}{2} \left( 1 + g^{\mu \nu} \partial_\mu \varphi \partial_\nu \varphi \right) = \frac{r}{2} \left( 1 + \frac{r^2 - \beta^2 r^2}{\alpha \beta^2} \right), $$ (14)

these latter equations become simpler

$$\alpha_{\tau \tau} = \beta^2 \alpha_{\rho \rho} + \beta_\rho \alpha_{\rho} + \frac{\beta_\tau \alpha_{\tau}}{\beta} + \frac{\alpha^2 - \beta^2 \alpha^2}{\alpha} + 2\alpha \beta^2 \alpha_{\rho \rho} + \frac{4\alpha^2 \beta^2 m}{r^3} + \nabla \text{geometrical part}$$

$$8\pi \alpha^2 \beta^2 \left( T_{\varphi \varphi} + T_{\rho \rho} - T_{\rho \rho} - T_{\varphi \varphi} \right), $$ (15)

matter part
\[
r_{\tau\tau} = \beta^2 r_{\rho\rho} + \beta r_{\rho} + \frac{\beta r_{\tau}}{\beta} - \frac{2\alpha\beta^2 m}{r^2} + 4\pi \rho \alpha \beta^2 (T^\rho_\rho + T^\tau_\tau). \quad (16)
\]

and, more importantly, numerically stable at the origin.

The full set of evolution equations can be put into the form

\[
\partial_\tau u = A \partial_\rho u + B \quad (17)
\]

with the 11-dimensional vector variable \( u = (m, \alpha, \alpha_\tau, r, r_\rho, w, w_\tau, w_\rho, h, h_\tau, h_\rho)^T \) and a suitable source term \( B \), the nontrivial components of which can be determined by making use of (8), (9), (15) and (16). What is even more important the pertinent form of (17) can be seen to possess the form of a first order strongly hyperbolic system to which the existence and uniqueness of solutions is guaranteed.

In addition, by making use of the Bianchi identity, \( \nabla_a E_{ab} \equiv 0 \), it can be shown that the 2-dimensional vector variable \( u = (E_{\tau\tau}, E_{\tau\rho})^T \) —built up from the left hand sides of the constraint equations \( E_{\tau\tau} = 0 \) and \( E_{\tau\rho} = 0 \)—do satisfy a first order strongly hyperbolic system of the form (17) with a source term \( B \) the components of which are homogeneous and linear in \( E_{\tau\tau} \) and \( E_{\tau\rho} \). Thereby they possess identically zero solution for vanishing initial data specifications or, in other words, it is satisfactory to solve them only on the initial data surface \( \Sigma_0 \) as the constraints propagate. For more details see Csizmadia & I Rácz 2010 and Csizmadia & Kovács & Rácz 2010.

4. Numerical results

The first order strongly hyperbolic system of evolution equations (17) were solved numerically— as in Csizmadia & Rácz 2010—with our finite difference code called GridRipper AMR, within which the time integration is done by making use of the method of lines based on a fourth order Runge-Kutta scheme. GridRipper does also incorporate the techniques of adaptive mesh refinement Csizmadia 2007.

4.1. The initial data

The methods for the selection of suitable initial data on \( \Sigma_0 \) are exactly the same as in Csizmadia & Rácz 2010, with obvious changes associated with the fact that in the present case we have the Yang-Mills and Higgs fields variables, \( w \) and \( h \), instead of a single Klein-Gordon field investigated in Csizmadia & Rácz 2010.

For simplicity and for demonstrative purposes we shall use only the type (A) initial data specification described in details in Csizmadia & Rácz 2010. In particular, in all of the numerical simulations discussed below the choices \( r = \rho, r_\tau = 0 \),

\[
w = \frac{gC}{\sinh (gC\rho)}, \quad h = C\rho \left[ \cosh (gC\rho) - 1 - \frac{1}{gC\rho} \right], \quad (18)
\]

and \( w_\tau = h_\tau = 0 \) were made on \( \Sigma_0 \), with parameter values \( C = H_\infty = 0.5, g = 0.05 \) and \( \lambda = 0 \).

4.2. Time development

Notice first that in (18) the functional form of \( w \) and \( h \) was chosen to possess the form that of the t’Hoft-Polyakov magnetic monopole which is known to be a static solution to the Yang-Mills–Higgs equations on the flat Minkowski background. However, in the considered case the geometry is also dynamical and as it is indicated by Figs. 1 and 2 the geometry differs from that
Figure 1. The functions $w$ and $\alpha$, along with the rescaled Higgs variable $h_{\text{res}} = h/20$ and the rescaled Misner-Sharp mass $m_{\text{res}} = m/10$, are shown on the initial data surface.

Figure 2. On the left the matter energy density $T_{\tau\tau}$ while on the right the "weighted" energy density of matter $E_{\text{M}} = 4\pi r^2 \sqrt{\alpha} T_{\tau\tau}$ and the weighted combined energy density of gravity and matter $E_{\text{GM}} = 4\pi r^2 (r_{\rho} T_{\tau\tau} - r_{\tau} T_{\tau\rho})$ are indicated on $\Sigma_0$.

Figure 3. Time evolution of the functions $\alpha$ and the rescaled $r_{\text{res}} = r/70$ are shown at $\rho = 69.7388$. Although there is a slight oscillation on the graph of $\alpha$ apparently the breathing state of the system is dominated by that of the matter.

of the Minkowski spacetime. Correspondingly, a dynamical process starts as it is indicated by Figs. 3 and 4. It is also clearly visible that a breather type oscillation develops which, in virtue of Fig. 3 is mainly dominated by the excitations of the matter system.

On Fig. 4 the weighted matter and combined gravity and matter energy densities $E_{\text{M}}, E_{\text{GM}}$
Figure 4. The time evolution of the energy density of matter $\varepsilon_M$, the combined energy density of gravity and matter $\varepsilon_{GM}$ and the binding energy density of gravity $\varepsilon_G$ are indicated at $\rho = 69.7388$.

(and their difference, the binding gravitational energy density, $\varepsilon_G = \varepsilon_{GM} - \varepsilon_M$) are defined such that the energy of matter, $E_M(\rho)$, or gravity and matter, $E_{GM}(\rho)$, within a ball of radius $\rho$ can be given in terms of them given as $E_M(\rho) = 4\pi \int_0^\rho \varepsilon_M \, d\rho$ and $E_{GM}(\rho) = 4\pi \int_0^\rho \varepsilon_{GM} \, d\rho$.

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