SHARP WEYL-TYPE FORMULAS OF THE SPECTRAL FUNCTIONS
FOR BIHARMONIC STEKLOV EIGENVALUES

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Abstract. In this paper, by explicitly calculating the principal symbols of pseudodifferential operators and by applying Hörmander’s spectral function theorem, we obtain the Weyl-type asymptotic formulas with sharp remainder estimates for the counting functions of the two classes of biharmonic Steklov eigenvalues $\lambda_k$ and $\mu_k$ in a smooth bounded domain of a Riemannian manifold. This solves a longstanding challenging problem.

1. Introduction

Let $(M, g)$ be a $C^\infty$ Riemannian manifold of dimension $n$ with a positive definite metric tensor $g$, and let $\Omega \subset M$ be a bounded domain with $C^\infty$ boundary $\partial \Omega$. Assume $g$ is a non-negative bounded function defined on $\partial \Omega$. We consider the following two biharmonic Steklov eigenvalue problems:

$$\begin{cases}
\Delta_g^2 u = 0 \quad &\text{in} \, \Omega, \\
u = 0 \quad &\text{on} \, \partial \Omega, \\
\Delta_g u + \lambda_g \partial u / \partial \nu = 0 \quad &\text{on} \, \partial \Omega
\end{cases}$$

and

$$\begin{cases}
\Delta_g^2 v = 0 \quad &\text{in} \, \Omega, \\
\partial v / \partial \nu = 0 \quad &\text{on} \, \partial \Omega, \\
\partial(\Delta_g v) / \partial \nu - \mu g^3 v = 0 \quad &\text{on} \, \partial \Omega,
\end{cases}$$

where $\nu$ denotes the inward unit normal vector to $\partial \Omega$, and $\Delta_g$ is the Laplace-Beltrami operator defined in local coordinates by the expression,

$$\Delta_g = \frac{1}{\sqrt{|g|}} \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( \sqrt{|g|} g^{ij} \frac{\partial}{\partial x_j} \right).$$

Here $|g| := det(g_{ij})$ is the determinant of the metric tensor, and $g^{ij}$ are the components of the inverse of the metric tensor $g$.

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(1.1) and (1.2) are the biharmonic Steklov eigenvalue problems (see [6], [10], [17], [19], [26], [41] and [43]). In each of the two cases, the spectrum is discrete and we arrange the eigenvalues in non-decreasing order (repeated according to multiplicity)

\[ 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots, \]
\[ 0 = \mu_1 \leq \mu_2 \leq \cdots \leq \mu_k \leq \cdots. \]

The corresponding eigenfunctions on \( \partial \Omega \) are expressed as \( \frac{\partial u_1}{\partial \nu}, \frac{\partial u_2}{\partial \nu}, \ldots, \frac{\partial u_k}{\partial \nu}, \ldots; v_1, v_2, \ldots, v_k, \ldots. \) It is clear that \( \lambda_k \) and \( \mu_k \) can be characterized variationally as

\[ \lambda_k = \frac{\int_{\Omega} |\Delta_g u_k|^2 \, dx}{\int_{\partial \Omega} \vartheta \left( \frac{\partial u_k}{\partial \nu} \right)^2 \, ds}, \]
\[ \mu_k = \frac{\int_{D} |\Delta_g v_k|^2 \, dx}{\int_{\partial D} \vartheta^3 v_k^2 \, ds}, \]

and

\[ \mu_0 = 0, \quad \mu_3^1 = \frac{\int_{D} |\Delta_g v_1|^2 \, dx}{\int_{\partial D} \vartheta^3 v_1^2 \, ds}, \]
\[ \mu_k^3 = \frac{\int_{D} |\Delta_g v_k|^2 \, dx}{\int_{\partial D} \vartheta^3 v_k^2 \, ds}, \]

where \( H^m(\Omega) \) is the Sobolev space, and where \( dx \) and \( ds \) are the Riemannian elements of volume and area on \( \Omega \) and \( \partial \Omega \), respectively.

The boundary value problems (1.1) and (1.2) have very interesting interpretations in theory of elasticity. We refer the reader to [10], [26] and [43] for more details. In view of the important applications, one is interested in finding the asymptotic formulas for \( \lambda_k \) and \( \mu_k \) as \( k \to \infty \). Let us introduce the counting functions \( A(\tau) \) and \( B(\tau) \) defined as the numbers of eigenvalues \( \lambda_k \) and \( \mu_k \) less than or equal to a given \( \tau \), respectively. Then our asymptotic problems for the eigenvalues are reformulated as the study of the asymptotic behavior of \( A(\tau) \) and \( B(\tau) \) as \( \tau \to +\infty \).

The simpler harmonic Steklov problem was first introduced by V. A. Steklov for bounded domains in the plane in [36]. This problem is to find function \( v \) satisfying

\[ \begin{cases} 
\Delta_g v = 0 & \text{in } \Omega, \\
\frac{\partial v}{\partial \nu} + \eta \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial \Omega,
\end{cases} \]

where \( \eta \) is a real number (The function \( v \) represents the steady state temperature on \( \Omega \) such that the flux on the boundary is proportional to the temperature). The harmonic Steklov spectrum of the domain is also called as the spectrum of the Dirichlet-to-Neumann map (see [7], [11]) or [37]. For the harmonic Steklov eigenvalue problem (1.3), in 1955 Sandgren [28] established the asymptotic formula of the counting function \( N(\tau) = \# \{ k | \eta_k \leq \tau \} \):

\[ N(\tau) = \omega_{n-1} \tau^{n-1} \left( \frac{2\pi}{n} \right)^{n-1} \int_{\partial \Omega} \vartheta^{n-1} \, ds + o(\tau^{n-1}) \quad \text{as } \tau \to +\infty, \]

where \( \omega_{n-1} \) is the volume of the unit ball of \( \mathbb{R}^{n-1} \). In the case that Riemannian manifold \( \mathcal{M} \) and the boundary of \( \Omega \) are smooth, the author [21] further gave a sharp remainder
estimate for the counting function of the harmonic Steklov eigenvalues \( \{\eta_k\}_{k=1}^{\infty} \):

\[
N(\tau) = \frac{\omega_{n-1}}{(2\pi)^{n-1}} \int_{\partial\Omega} q^{n-1}(x)dx + O(\tau^{n-2}) \quad \text{as} \quad \tau \to +\infty.
\]

For the biharmonic Steklov eigenvalue problem (1.1) with general domain, in [21] the author also established the leading asymptotic formula with remainder \( o(\tau^{n-1}) \) as \( \tau \to +\infty \).

For the Dirichlet Steklov eigenvalue problem with general domain, in [14], [15] in 1912 proved the following asymptotic formula which answered a question posed in 1908 by the physicist Lorentz:

\[
N(\tau) = (2\pi)^{-n} \omega_n(\text{vol}(\Omega))\tau^{n/2} + o(\tau^{n/2}) \quad \text{as} \quad \tau \to +\infty,
\]

where \( N(\tau) = \#\{ k \in \mathbb{N} | \alpha_k \leq \tau \} \), \( N(\tau) = \#\{ k \in \mathbb{N} | \beta_k \leq \tau \} \), and \( 0 < \alpha_1 < \alpha_2 \leq \cdots \leq \alpha_k \leq \cdots \) and \( 0 = \beta_1 < \beta_2 \leq \cdots \leq \beta_k \leq \cdots \) are all the Dirichlet and Neumann eigenvalues on \( \Omega \), respectively. As far back as in 1912, H. Weyl [47] conjectured (see also, Clark [8]) that the second term of the asymptotic formula for \( N(\tau) \) contain an \((n-1)\)-dimensional measure (‘area’) of the boundary \( \partial\Omega \), i.e.,

\[
N(\tau) = \frac{1}{4}(2\pi)^{-n+1} \omega_{n-1}(\text{vol}(\partial\Omega))\tau^{(n-1)/2} + O(\tau^{(n-1)/2}) \quad \text{as} \quad \tau \to +\infty.
\]

In 1980, Ivrii [20] proved this conjecture for domains having smooth boundary under the following condition regarding the billiard trajectories of \( \Omega \), where the billiard trajectory in \( \Omega \) is taken with the usual reflections at the boundary (see also, p. 100 of [9]). Melrose [23] independently obtained the second term in Weyl’s conjecture for manifolds with concave boundary.

Note that it was already observed by Avakumović [5] that for the Laplacian on the sphere \( S^n \), the high multiplicities of the eigenvalues make it impossible to improve (1.5) to (1.6). Seeley (see, [30] and [31]) in 1980 gave the sharp asymptotic formula of the counting function (see also [4], [9], [27]):

\[
N(\tau) = (2\pi)^{-n} \omega_n(\text{vol}(\Omega))\tau^{n/2} + O(\tau^{(n-1)/2}) \quad \text{as} \quad \tau \to +\infty.
\]

Applying the sharp asymptotic result (1.7), Sogge invented the unit band spectral projection operator (see [33] and [34]) and established the well-known asymptotic estimates of the mapping norm \( \| \chi_{\tau} \|_{L^p \to L^p} \) \( (2 \leq p \leq \infty) \) (cf. [35]). However, it has been a longstanding challenging problem to get the sharp Weyl-type asymptotic formulas for the biharmonic Steklov eigenvalues (see [21]).

In this paper, by explicitly calculating the principal symbols of the corresponding pseudodifferential operators for the problems (1.1) and (1.2) in the boundary of a \( C^\infty \) bounded domain, we obtain the sharp asymptotic formulas for the counting functions \( A(\tau) \) and \( B(\tau) \), respectively. The main results are the following:

**Theorem 1.1.** Let \( (\mathcal{M}, g) \) be an \( n \)-dimensional \( C^\infty \) Riemannian manifold, and let \( \Omega \subset \mathcal{M} \) be a bounded domain with \( C^\infty \) boundary \( \partial\Omega \). Then

\[
A(\tau) = \frac{\omega_{n-1}}{(4\pi)^{n-1}} \int_{\partial\Omega} q^{n-1}ds + O(\tau^{n-2}) \quad \text{as} \quad \tau \to +\infty.
\]

Moreover, the above remainder estimate is sharp.
Theorem 1.2. Let \((M, g)\) be an \(n\)-dimensional \(C^\infty\) Riemannian manifold, and let \(\Omega \subset M\) be a bounded domain with smooth boundary \(\partial \Omega\). Then

\[
B(\tau) = \frac{\omega_{n-1} \tau^{n-1}}{(\sqrt{16\pi})^{n-1}} \int_{\partial D} u^{n-1} ds + O(\tau^{n-2}) \quad \text{as } \tau \to +\infty.
\]

Moreover, the above remainder estimate is also sharp.

The plan of the paper is as follows. In Section 2 we give some definitions and lemmas. In Section 3, by a key technique we calculate the principal symbols of the corresponding “Neumann-to-Laplacian map” and “Dirichlet-to-Laplacian derivative map”. Section 4 is devoted to the proofs of the sharp Weyl-type asymptotic formulas for \(A(\tau)\) and \(B(\tau)\). In Section 5, we give two counterexamples, which show that Theorem 1.1 and 1.2 cannot be improved.

2. Some definitions and lemmas

Definition 2.1. If \(U\) is an open subset of \(\mathbb{R}^n\), we denote by \(S^m = S^m(U, \mathbb{R}^n)\) the set of all \(p \in C^\infty(U, \mathbb{R}^n)\) such that for every compact set \(K \subset U\) we have

\[
|D^\beta_x D^\alpha_\xi p(x, \xi)| \leq C_{K, \alpha, \beta} (1 + |\xi|)^{m-|\alpha|}, \quad x \in K, \ \xi \in \mathbb{R}^n
\]

for all \(\alpha, \beta \in \mathbb{N}_0^n\). The elements of \(S^m\) are called symbols of order \(m\).

It is clear that \(S^m\) is a Fréchet space with semi-norms given by the smallest constants which can be used in (2.1) (i.e.,

\[
\|p\|_{K, \alpha, \beta} = \sup_{x \in K} \left| (D^\beta_x D^\alpha_\xi p(x, \xi)) (1 + |\xi|)^{|\alpha|-|m|} \right|.
\]

Let \(p(x, \xi) \in S^m\). A pseudo-differential operator in an open set \(U \subset \mathbb{R}^n\) is essentially defined by a Fourier integral operator (cf. 13):

\[
P(x, D)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} p(x, \xi) e^{ix \cdot \xi} \hat{u}(\xi) d\xi.
\]

Here \(u \in C^\infty_0(U)\) and \(\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-iy \cdot \xi} u(y) dy\) is the Fourier transform of \(u\).

Definition 2.2. A pseudodifferential operator \(P\) with its symbol \(p\) in \(S^m\) is called classical or polyhomogeneous if there is a sequence of symbols \(p_j \in S^{m-j}\), \(j = 0, 1, 2, \cdots\), such that \(p_j(x, t\xi) = t^{m-j} p_j(x, \xi)\) for \(t > 1, |\xi| > 1\), and

\[
\left| D^\beta_x D^\alpha_\xi \left( p(x, \xi) - \sum_{j=0}^N p_j(x, \xi) \right) \right| \leq C_{\alpha, \beta, N} |\xi|^{m-N-1-|\alpha|}
\]

for all \(\alpha, \beta, |\xi| > 1\) and all integers \(N \geq 0\). In this case the notation \(p \sim \sum_{j=0}^\infty p_j\) is used. The function \(p_0\) is known as the principal symbol of pseudodifferential operator \(P\), and the class of such symbol is denoted by \(S^m_{cl}\).

Given a diffeomorphism \(\iota: U_1 \to U_2\), from one open set \(U_1 \subset \mathbb{R}^n\) onto another open set \(U_2 \subset \mathbb{R}^n\), the induced transformation \(\iota^*: C^\infty_0(U_2) \to C^\infty_0(U_1)\), taking a function \(u\) to the function \(u \circ \iota\), is an isomorphism and transforms \(C^\infty_0(U_2)\) into \(C^\infty_0(U_1)\). Let \(P_1\) be a
pseudodifferential operator on \( U_1 \) and define \( P_2 : C_0^\infty(U_2) \rightarrow C_0^\infty(U_2) \) with the help of the commutative diagram

\[
\begin{array}{ccc}
C_0^\infty(U_1) & \xrightarrow{P_1} & C^\infty(U_1) \\
\kappa'^* & & \kappa'^* \\
C_0^\infty(U_2) & \xrightarrow{P_2} & C^\infty(U_2)
\end{array}
\]

i.e.,

(2.4) \( P_2u = [P_1(u \circ \kappa')] \circ \kappa'^{-1} \).

(2.4) can also be written as

\[
P_2u = (\kappa'^{-1})^* P_1(\kappa'^* u).
\]

It follows from this that \( P_2 \) is also a pseudodifferential operator on \( U_2 \).

Let \( \mathcal{M} \) be a smooth \( n \)-dimensional Riemannian manifold (of class \( C^\infty \)). We will denote by \( C^\infty(\mathcal{M}) \) and \( C_0^\infty(\mathcal{M}) \) the space of all smooth complex-valued functions on \( \mathcal{M} \) and the subspace of all functions with compact support, respectively. Assume that we are given a linear operator

\[
P : C_0^\infty(\mathcal{M}) \rightarrow C^\infty(\mathcal{M}).
\]

If \( G \) is some chart in \( \mathcal{M} \) (not necessarily connected) and \( \kappa : G \rightarrow U \) its diffeomorphism onto an open set \( U \subset \mathbb{R}^n \), then let \( \tilde{P} \) be defined by the diagram

\[
\begin{array}{ccc}
C_0^\infty(G) & \xrightarrow{P} & C^\infty(G) \\
\kappa'^* & & \kappa'^* \\
C_0^\infty(U) & \xrightarrow{P} & C^\infty(U)
\end{array}
\]

(note, in the upper row is the operator \( r_G \circ P \circ i_G \), where \( i_G \) is the natural embedding \( i_G : C_0^\infty(G) \rightarrow C_0^\infty(M) \) and \( r_G \) is the natural restriction \( r_G : C^\infty(M) \rightarrow C^\infty(G) \); for brevity we denote this operator by the same letter \( P \) as the original operator).

**Definition 2.3.** An operator \( P : C_0^\infty(\mathcal{M}) \rightarrow C^\infty(\mathcal{M}) \) is called a pseudodifferential operator on \( \mathcal{M} \) if for any chart diffeomorphism \( \kappa : G \rightarrow U \), the operator \( \tilde{P} \) defined above is a pseudodifferential operator on \( U \).

**Lemma 2.4** (see, for example, Proposition 0.3.C of [10]) If \( A \) and \( B \) are two pseudodifferential operators of order \( m \) and \( m' \), respectively, then the composition \( C = A \circ B \) is a pseudodifferential operator of order \( m + m' \) with the symbol

\[
c(x, \xi) \sim \sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha a(x, \xi) D_x^\alpha b(x, \xi)
\]

(2.5)

where \( a(x, \xi) \) and \( b(x, \xi) \) are the symbols of \( A \) and \( B \), respectively. In particular, the principal symbol of \( A \circ B \) is \( a_0(x, \xi) b_0(x, \xi) \), where \( a_0(x, \xi) \) and \( b_0(x, \xi) \) are the principal symbols of \( A \) and \( B \), respectively.
Lemma 2.5. Let

\[
A = \begin{pmatrix}
  a_{11} & a_{12} & \ldots & a_{1,n-1} & 0 \\
  a_{21} & a_{22} & \ldots & a_{2,n-1} & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  a_{n-1,1} & a_{n-1,2} & \ldots & a_{n-1,n-1} & 0 \\
  0 & 0 & \ldots & 0 & a_{nn}
\end{pmatrix}
\]

be a positive definite, real symmetric constant matrix. Let \( \phi(x') \) and \( h(x') \) be \( C^\infty \) functions of compact support in \((n-1)-\text{space}\). Then the problem

\[
\begin{cases}
\left( \frac{\partial}{\partial x_j} \right)^2 u + a_{nn} \frac{\partial^2}{\partial x_n^2} u = 0 & \text{in } \mathbb{R}_n^+, \\
\frac{\partial u}{\partial x_n} = 0 & \text{on } \partial \mathbb{R}_n^+,
\end{cases}
\]

has a solution

\[
u(x',x_n) = \int_{\mathbb{R}_{n-1}} K_1(x' - y', x_n) \phi(y') dy' + \int_{\mathbb{R}_{n-1}} K_2(x' - y', x_n) h(y') dy',
\]

where \( \mathbb{R}_n^+ = \{x = (x_1, \ldots, x_{n-1}, x_n) \in \mathbb{R}^n | x_n > 0 \} \), and

\[
K_1(x', x_n) = (-1)^{n-1} \frac{(n-2)!}{(2\pi)^{n-1}} \int_{|\eta'|=1} \left( x' \cdot \eta' + ix_n \sum_{j,k=1}^{n-1} \frac{a_{jk} \eta_j \eta_k}{a_{nn}} \right)^{1-n} \times \frac{+ (n-1)ix_n \sum_{j,k=1}^{n-1} \frac{a_{jk} \eta_j \eta_k}{a_{nn}} \left( x' \cdot \eta' + ix_n \sum_{j,k=1}^{n-1} \frac{a_{jk} \eta_j \eta_k}{a_{nn}} \right)^{-n} \times ds_{\eta'},
\]

\[
K_2(x', x_n) = (-1)^{n-1} \frac{(n-2)!}{(2\pi)^{n-1}} \int_{|\eta'|=1} \left[ \frac{x_n}{\sqrt{a_{nn}}} \left( x' \cdot \eta' + ix_n \sum_{j,k=1}^{n-1} \frac{a_{jk} \eta_j \eta_k}{a_{nn}} \right)^{1-n} \times \frac{ds_{\eta'}}{ds_{\eta'}} \right].
\]

Here \( \eta' = (\eta_1, \ldots, \eta_{n-1}) \) and \( ds_{\eta'} \) is the area element on the unit sphere \( |\eta'| = 1 \).

Proof. Writing \( x = (x', x_n) \). Then the bi-Laplace operator \( P \) has characteristic form \( P(\eta', \tau) = (\sum_{j,k=1}^{n-1} a_{jk} \eta_j \eta_k + a_{nn} \tau^2)^2 \). It is easy to see that the roots of \( P(\eta', \tau) \) with positive imaginary parts are \( \tau_1^+ (\eta') = \tau_2^+ (\eta') = i \sqrt{\sum_{j,k=1}^{n-1} a_{jk} \eta_j \eta_k / a_{nn}} \). Thus we have (see, Chapter I, §1 of [1])

\[
M^+(\eta', \tau) = \left( \tau - i \sqrt{\sum_{j,k=1}^{n-1} a_{jk} \eta_j \eta_k / a_{nn}} \right)^2 = \tau^2 - 2i \sum_{j,k=1}^{n-1} a_{jk} \eta_j \eta_k / a_{nn} \
\]

so that

\[
N_1(\eta', \tau) = M_1^+(\eta', \tau) = \tau - 2i \sqrt{\sum_{j,k=1}^{n-1} a_{jk} \eta_j \eta_k / a_{nn}}, \quad N_2(\eta', \tau) = \frac{1}{\sqrt{a_{nn}}} M_0^+(\eta', \tau) = \frac{1}{\sqrt{a_{nn}}}.
\]
It follows from p. 635 of [1] and the well-known residue theorem (see, for example, p. 150 of [3]) that

\[
K_1(x', x_n) = (-1)^{n-1} \frac{(n-2)!}{(2\pi i)^n} \int_{|\eta'|=1} \left[ \int_{\gamma} \left( \frac{N_1(\eta', \tau)}{M^+_{\eta'}(\eta', \tau) (x' \cdot \eta' + x_n \tau)^n} \right) d\tau \right] d\eta',
\]

\[
= (-1)^{n-1} \frac{(n-2)!}{(2\pi i)^n} \int_{|\eta'|=1} \left[ \int_{\gamma} \left( \frac{\tau - 2i \sqrt{\sum_{j,k=1}^{n-1} \frac{a_{jk} \eta_j \eta_k}{\alpha_{jk}}}}{\sum_{j,k=1}^{n-1} \frac{a_{jk} \eta_j \eta_k}{\alpha_{jk}}} (x' \cdot \eta' + x_n \tau)^{-1-n} \right) d\tau \right] d\eta',
\]

\[
= (-1)^{n-1} \frac{(n-2)!}{(2\pi i)^n} \int_{|\eta'|=1} \left[ \left( x' \cdot \eta' + i x_n \sqrt{\sum_{j,k=1}^{n-1} \frac{a_{jk} \eta_j \eta_k}{\alpha_{jk}}} \right)^{1-n} \right] d\eta',
\]

\[
K_2(x', x_n) = (-1)^{n-2} \frac{(n-3)!}{(2\pi i)^n} \int_{|\eta'|=1} d\eta' \left[ \int_{\gamma} \left( \frac{N_2(\eta', \tau)}{M^+_{\eta'}(\eta', \tau) (x' \cdot \eta' + x_n \tau)^{-n}} \right) d\tau \right]
\]

\[
= (-1)^{n-2} \frac{(n-3)!}{(2\pi i)^n} \int_{|\eta'|=1} \left[ \int_{\gamma} \left( x' \cdot \eta' + i x_n \sqrt{\sum_{j,k=1}^{n-1} \frac{a_{jk} \eta_j \eta_k}{\alpha_{jk}}} \right)^{-n} \right] d\eta',
\]

where \( \gamma \) is a Jordan contour in \( \text{Im} \tau > 0 \) enclosing all the points \( i \sqrt{\sum_{i,j=1}^{n-1} \frac{a_{ij} \eta_i \eta_j}{\alpha_{ij}}} \) for all \( |\eta'| = 1 \). Applying Theorem 2.1 of [1], we obtain (2.8). \( \square \)

Let \( \{ E_r \} \) be the spectral resolution of pseudodifferential operator \( P \), and let \( e(x, y, \tau) \) be the kernel of \( E_r \). This is an element of \( C^\infty(\Omega \times \Omega) \) called the spectral function of \( P \).

The following Lemma will be used later.

**Lemma 2.6 (Hömander’s spectral function theorem, see, Theorem 5.1 of [13], [12] or [32])** Let \( P \) be a non-negative pseudodifferential operator, acting on a \( C^\infty \) subdomain \( \Omega \) of an \( n \)-dimensional \( C^\infty \) manifold. Let \( p_0(x, \xi) \) be the principal symbol of \( P \), which is a real homogeneous polynomial of degree \( m \) on the cotangent bundle \( T^*\Omega \). The measure \( dx \) defines a Lebesgue measure \( dx^* \) in each fiber of \( T^*\Omega \); which is a vector space of dimension \( n \). Then

\[
(2.9) \quad \tau^{-n/m} e(x, x, \tau) - (2\pi)^{-n} \int_{B_x} dx^* = O(\tau^{-1/m}) \quad \text{as } \tau \to \infty,
\]

where \( B_x = \{ \xi \in T^*_x(\Omega) | p_0(x, \xi) < 1 \} \).

3. The principal symbols

3.1. Let \( (\mathcal{M}, g) \) be a \( C^\infty \) Riemannian manifold, and let \( \Omega \) be a bounded domain with \( C^\infty \) boundary in \( \mathcal{M} \). The “Neumann-to-Laplacian map” is the map

\[
F : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)
\]
defined by the following problem: Let \( h \in H^{1/2}(\partial \Omega) \) and let \( u \in H^2(\Omega) \) be the solution of

\[
\begin{align*}
\Delta_g^2 u &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega, \\
\frac{\partial u}{\partial \nu} &= h \quad \text{on } \partial \Omega,
\end{align*}
\]

we set \( Fh := (-\Delta_g u)|_{\partial \Omega} \). Multiplying (3.1) by \( u \), integrating the result over \( \Omega \), and using Green’s formula, we derive

\[
0 = \int_{\Omega} u (\Delta_g^2 u) \, dx = \int_{\partial \Omega} u \frac{\partial (\Delta_g u)}{\partial \nu} \, ds + \int_{\partial \Omega} (\Delta_g u) \frac{\partial u}{\partial \nu} \, ds,
\]

so that

\[
\langle Fh, h \rangle = \int_{\partial \Omega} (Fh) h \, ds = \int_{\Omega} |\Delta_g u|^2 \, dx \geq 0, \quad \text{for any } h \in H^{1/2}(\partial \Omega).
\]

This shows that \( F \) is a non-negative, self-adjoint, pseudodifferential operator on \( H^{1/2}(\partial \Omega) \).

We shall calculate the principal symbol of \( F \).

**Lemma 3.1** Let \( A \) be a positive definition, real-valued constant matrix as in (2.6). Assume that \( F_0 : C^\infty_0(\mathbb{R}^{n-1}) \to C^\infty(\mathbb{R}^{n-1}) \) defined by the following problem: Let \( h \in C^\infty_0(\mathbb{R}^{n-1}) \) and let \( u \in C^\infty(\mathbb{R}^n_+) \) be the solution of

\[
\begin{align*}
\left( \sum_{j,k=1}^{n-1} a_{jk} \frac{\partial^2}{\partial x_j \partial x_k} + a_{nn} \frac{\partial^2}{\partial x_n^2} \right)^2 u &= 0 \quad \text{in } \mathbb{R}^n_+, \\
u &= 0 \quad \text{on } \partial \mathbb{R}^n_+, \\
\sqrt{a_{nn}} \frac{\partial u}{\partial x_n} &= h \quad \text{on } \partial \mathbb{R}^n_+.
\end{align*}
\]

we set \( F_0 h := - \left( \sum_{j,k=1}^{n-1} a_{jk} \frac{\partial^2}{\partial x_j \partial x_k} + a_{nn} \frac{\partial^2}{\partial x_n^2} \right)|_{\partial \mathbb{R}^n_+} \). Then the principal symbol of \( F_0 \) is

\[
p_0(x', \eta') = 2 \sqrt{\sum_{j,k=1}^{n-1} a_{jk} \eta_j \eta_k}, \quad \forall (x', \eta') \in \mathbb{R}^{n-1} \times (\mathbb{R}^{n-1} \setminus 0).
\]

**Proof.** Writing \( x = (x', x_n) \), it follows from Lemma 2.5 that

\[
u(x', x_n) = \int_{\mathbb{R}^{n-1}} K_2(x' - y', x_n) h(y') \, dy',
\]

where

\[
K_2(x', x_n) = (-1)^{n-1} \frac{(n-2)!}{(2\pi i)^{n-1}} \int_{|\eta'|=1} \left[ \frac{x_n}{\sqrt{a_{nn}}} \left( x' \cdot \eta' + ix_n \sqrt{\sum_{j,k=1}^{n-1} a_{jk} \eta_j \eta_k} \right) \right]^{1-n} d\eta'.
\]
(3.3) shows that \( u(x) \) is uniquely determined by the data of \( h \) on its support set. (3.2) can be rewritten as

\[
\begin{aligned}
\left\{ \begin{array}{l}
(a^{nn})^2 \frac{\partial^2 u}{\partial x_n^2} + 2 a^{nn} \frac{\partial^2}{\partial x_n^2} \left( \sum_{j,k=1}^{n-1} a^{jk} \frac{\partial^2}{\partial x_j \partial x_k} \right) u \\
+ \left( \sum_{j,k=1}^{n-1} a^{jk} \frac{\partial^2}{\partial x_j \partial x_k} \right)^2 u = 0 \quad \text{in } \mathbb{R}^n,
\end{array} \right.
\]

(4.2)

\[
\begin{aligned}
\frac{\partial u}{\partial x_n} = h
\end{aligned}
\]

Taking the Fourier transform to (3.4) with respect to \( x_1, \ldots, x_{n-1} \), we have

\[
\left\{ \begin{array}{l}
\left( a^{nn} \right)^2 \frac{\partial^2 \hat{u}}{\partial x_n^2} - 2 a^{nn} \left( \sum_{j,k=1}^{n-1} a^{jk} \eta_j \eta_k \right) \frac{\partial^2 \hat{u}}{\partial x_n^2} \\
+ \left( \sum_{j,k=1}^{n-1} a^{jk} \eta_j \eta_k \right)^2 \hat{u} = 0 \quad \text{in } \mathbb{R}^n,
\end{array} \right.
\]

(5.5)

\[
\begin{aligned}
\frac{\partial u}{\partial x_n} = \hat{h}(\eta')
\end{aligned}
\]

We denote \( |\xi'| := \sqrt{\sum_{j,k=1}^{n-1} a^{jk} \eta_j \eta_k} \). Then, the general solution of (3.5) has the form:

\[
\hat{u}(\eta', x_n) = C_1 e^{i|\xi'|x_n} + C_2 e^{-i|\xi'|x_n} + C_3 x_n e^{i|\xi'|x_n} + C_4 x_n e^{-i|\xi'|x_n},
\]

where \( C_1, C_2, C_3, C_4 \) are arbitrary functions in \( \eta' \). From the boundary conditions of (5.5), it follows that

\[
\hat{u}(\eta', x_n) = C_2 \left( -e^{i|\xi'|x_n} + e^{-i|\xi'|x_n} + 2x_n |\xi'| e^{-|\xi'|x_n} \right)
\]

\[
+ C_3 \left( x_n e^{i|\xi'|x_n} - x_n e^{-i|\xi'|x_n} \right) + \frac{\hat{h}(\eta')}{\sqrt{a^{nn}}} x_n e^{-|\xi'|x_n},
\]

Therefore

\[
\hat{u}(\eta', x_n) = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{i(x', \eta')} \left[ C_2 \left( -e^{i|\xi'|x_n} + e^{-i|\xi'|x_n} + 2x_n |\xi'| e^{-|\xi'|x_n} \right)
\]

\[
+ C_3 \left( x_n e^{i|\xi'|x_n} - x_n e^{-i|\xi'|x_n} \right) + \frac{\hat{h}(\eta')}{\sqrt{a^{nn}}} x_n e^{-|\xi'|x_n} \right] \, d\eta',
\]

from which we have

\[
\frac{\partial^2 u}{\partial x_j \partial x_k} = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{i(x', \eta')} (-\eta_j \eta_k) \left[ C_2 \left( -e^{i|\xi'|x_n} + e^{-i|\xi'|x_n} + 2x_n |\xi'| e^{-|\xi'|x_n} \right)
\]

\[
+ C_3 \left( x_n e^{i|\xi'|x_n} - x_n e^{-i|\xi'|x_n} \right) + \frac{\hat{h}(\eta')}{\sqrt{a^{nn}}} x_n e^{-|\xi'|x_n} \right] \, d\eta',
\]

\[
\frac{\partial^2 u}{\partial x_n^2} = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{i(x', \eta')} \left[ C_2 \left( -e^{i|\xi'|x_n} + |\xi'|^2 e^{-|\xi'|x_n} - 4|\xi'|^2 e^{-|\xi'|x_n} + 2x_n |\xi'|^3 e^{-|\xi'|x_n} \right)
\]

\[
+ 2x_n |\xi'|^2 e^{-|\xi'|x_n} \right) + C_3 \left( 2 |\xi'| e^{i|\xi'|x_n} + x_n |\xi'|^2 e^{i|\xi'|x_n} + 2 |\xi'| e^{-|\xi'|x_n} - x_n |\xi'|^2 e^{-|\xi'|x_n} \right)
\]

\[
- \frac{\hat{h}(\eta')}{\sqrt{a^{nn}}} |\xi'| e^{-|\xi'|x_n} + \frac{\hat{h}(\eta')}{\sqrt{a^{nn}}} x_n |\xi'|^2 e^{-|\xi'|x_n} \right] \, d\eta'.
\]
Then
\[
\begin{aligned}
\left( \sum_{j,k=1}^{n-1} a^{jk} \frac{\partial^2 u}{\partial x_j \partial x_k} + a^{nn} \frac{\partial^2 u}{\partial x_n^2} \right) \bigg|_{x_n=0} = \left( a^{nn} \frac{\partial^2 u}{\partial x_n^2} \right) \bigg|_{x_n=0} = \\
\frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} a^{nn} \left[ -4C_2|\xi'|^2 + 4C_3|\xi'| - 2|\xi'| \frac{\hat{h}(\eta)}{\sqrt{a^{nn}}} \right] e^{i(x',\eta')} \, d\eta'.
\end{aligned}
\]

In order to take a bounded solution of the equation (3.5), we may let \( C_2 = C_3 = 0 \). Hence
\[
\begin{aligned}
\left( \sum_{j,k=1}^{n-1} a^{jk} \frac{\partial^2 u}{\partial x_j \partial x_k} + a^{nn} \frac{\partial^2 u}{\partial x_n^2} \right) \bigg|_{x_n=0} = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} a^{nn} \left[ -2|\xi'| \frac{\hat{h}(\eta)}{\sqrt{a^{nn}}} \right] e^{i(x',\eta')} \, d\eta' = \\
\frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} \left( -2 \sum_{j,k=1}^{n-1} a^{jk} \eta_j \eta_k \right) e^{i(x',\eta')} \hat{h}(\eta') d\eta',
\end{aligned}
\]
i.e.,
\[
F_0 h = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} \left( 2 \sum_{j,k=1}^{n-1} a^{jk} \eta_j \eta_k \right) e^{i(x',\eta')} \hat{h}(\eta') \, d\eta'.
\]

This shows that the principal symbol of the pseudodifferential operator \( F_0 \) is
\[
2 \sqrt{\sum_{j,k=1}^{n-1} a^{jk} \eta_j \eta_k}.
\]

\[\square\]

**Theorem 3.2** Let \((M, g)\) be an \(n\)-dimensional \(C^\infty\) Riemannian manifold, and let \(\Omega\) be a bounded domain with \(C^\infty\) boundary. Assume that the pseudodifferential operator \(F\) is defined as before. Then for any coordinate chart \(\kappa : \partial \Omega \supset U \rightarrow U^\kappa \subset \mathbb{R}^{n-1}\) there is a pseudodifferential operator \(\Lambda \in \Psi^1(U^\kappa)\) such that for every \(h \in H^{1/2}(\partial \Omega)\) we have
\[
\kappa_* (Fh) - \Lambda(\kappa_* h) \in C^\infty(U^\kappa),
\]
where \(\kappa_*\) is the linear tangent mapping of \(\kappa\). \(\Lambda\) is elliptic and has a coordinate invariant positively 1-homogeneous principal symbol \(p_0 \in C^\infty(T^*U^\kappa \setminus 0) = C^\infty(U^\kappa \times (\mathbb{R}^{n-1} \setminus 0))\) given by
\[
p_0(x', \eta') = 2 \sqrt{\sum_{j,k=1}^{n-1} g^{jk} \eta_j \eta_k}, \quad \forall (x', \eta') \in T^*U^\kappa \setminus 0.
\]

**Proof.** It is well-known (see, for example, [21]) that there is a \(T > 0\) and a neighborhood \(G \subset M\) of the boundary \(\partial \Omega\) together with a diffeomorphism \(\psi : G \rightarrow \partial \Omega \times [0, T)\) such that
i) \(\psi(q) = (q, 0)\) for every \(q \in \partial \Omega\),
ii) The unique geodesic normal to \(\partial \Omega\) (with the unit-speed \(\sqrt{g^m}\) with respect to \(g\)) starting in \(q \in \partial \Omega\) is given by
\[
[0, T) \rightarrow \Omega, \quad x_n \rightarrow \psi^{-1}(q, x_n).
\]
Moreover, \(\psi\) is unique with i) and ii) and has the following additional properties: Let \(\kappa : \partial \Omega \supset U \rightarrow U^\kappa \subset \mathbb{R}^{n-1}\) be any coordinate chart on \(\partial \Omega\) and \(\tilde{k} : M \supset \tilde{U} \rightarrow U^\kappa \times [0, T)\) be its extension via \(\psi\).
in addition to (3.8) we have

\[ \tilde{\kappa}_s g = \sum_{j,k=1}^{n-1} (g_{jk} \, dx_j \otimes dx_k) + g_{nn} dx_n \otimes dx_n, \]

for any given \( h \). By the regularity of elliptic equations, we get functions \( F \) subordinate to the open cover \( q, \epsilon \subset q, \epsilon \)
\( G \). Hence, we may assume that the support set of \( h \) is contained in some small neighborhood \( G_{q, \epsilon} \cap \partial \Omega \) for some \( q \in \partial \Omega \). (Let us point out that \( \tilde{\kappa}(G_{q, \epsilon}) = U^\epsilon_{q, \epsilon} \times [0, T] \)) such that \( \text{diam}(G_{q, \epsilon}) < \epsilon \). Then there is a partition of unity subordinate to the open cover \( \{ G_{q, \epsilon} \cap \partial \Omega \mid q \in \partial \Omega \} \), i.e., a collection of real-valued \( C^\infty \)
functions \( \phi_i \) on \( \partial \Omega \) satisfying the following conditions:

1. The supports of the \( \phi_i \) are compact and locally finite;
2. The support of \( \phi_i \) is completely contained in \( G_\alpha \) for some \( \alpha \);
3. The \( \phi_i \) sum to one at each point of \( \partial \Omega \):

\[ \sum_i \phi_i(x) = 1. \]

Since \( h = \sum_i h\phi_i \), we may assume that the support set of \( h \) is contained in some small neighborhood \( G_{q, \epsilon} \cap \partial \Omega \) for some \( q \in \partial \Omega \). (Let us point out that \( \tilde{\kappa}(G_{q, \epsilon}) \cap \{ x_n = 0 \} = U^\epsilon_{q, \epsilon} \cap \partial \mathbb{R}_+^n \)). It is clear that we can always choose a fine cover \( \{ G_{q, \epsilon} \cap \partial \Omega \} \) of \( \partial \Omega \), so that in addition to (3.8) we have

\[ |(g^{jk}(x', x_n) - g^{jk}(0)) \eta_j \eta_k| \leq \epsilon \sum_{j=1}^n \eta_j^2, \quad ((x', x_n) \in \tilde{\kappa}(G_{q, \epsilon})) \]

for any given \( \epsilon > 0 \), all real \( \eta_1, \ldots, \eta_n \) and all \( \tilde{\kappa}(G_{q, \epsilon}) \). The finer cover does not influence the fact, which is obviously true in the original cover, that (3.9) holds.

Let \( u \) be the solution of (3.1). For any fixed \( \epsilon_0 > 0 \), in the local coordinates, the \( u \) satisfies

\[ \left\{ \begin{array}{ll}
\sum_{j,k=1}^{n-1} \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_j} \left( \sqrt{|g|} g^{jk} \frac{\partial}{\partial x_k} \right) + g^{nn} \frac{\partial^2}{\partial x_n^2} u = 0 & \text{in } \tilde{\kappa}(G_{q, \epsilon}), \\
u = 0 & \text{on } \tilde{\kappa}(G_{q, \epsilon}^0) \cap \partial \mathbb{R}_+^n, \\
\sqrt{g^{nn}(x)} \frac{\partial u}{\partial x_n} = h & \text{on } \tilde{\kappa}(G_{q, \epsilon}) \cap \partial \mathbb{R}_+^n.
\end{array} \right. \]

By the regularity of elliptic equations, we get \( u \in C^{4, \alpha}(\tilde{\kappa}(G_{q, \epsilon})) \). We define the operator \( F_{q, \epsilon_0} : C_0^\infty ((\tilde{\kappa}(G_{q, \epsilon})) \cap \partial \mathbb{R}_+^n) \rightarrow C_0^\infty ((\tilde{\kappa}(G_{q, \epsilon})) \cap \partial \mathbb{R}_+^n) \) by \( F_{q, \epsilon_0} h := (-\Delta u)|_{\tilde{\kappa}(G_{q, \epsilon}) \cap \partial \mathbb{R}_+^n} \) for any \( h \in C_0^\infty ((\tilde{\kappa}(G_{q, \epsilon})) \cap \partial \mathbb{R}_+^n) \). It is easy to check that \( F_{q, \epsilon_0} \) is a pseudodifferential operator on the open set \( (\tilde{\kappa}(G_{q, \epsilon})) \cap \partial \mathbb{R}_+^n \). We denote its principal symbol as \( p_0^{(\epsilon_0)}(x', \eta') \).

Next, noticing that (see, for example, Theorem 1.4.4 of [15])

\[ \frac{\partial g^{jk}}{\partial x_l}(0) = 0 \quad \text{for all } 1 \leq j, k, l \leq n, \]
we get
\[
\Delta^2 g(0) = \left( \sum_{j,k=1}^{n-1} g^{jk}(0) \frac{\partial^2}{\partial x_j \partial x_k} + g^{nn}(0) \frac{\partial^2}{\partial x_n^2} \right)^2,
\]
where 0 is the boundary normal coordinates of \( q \in \partial \Omega \). Let \( v \) be the solution of the problem
\[
\begin{cases}
\Delta^2 g(0) v - \Delta_g^2 u = \Delta^2 g(0) (v - u) + (\Delta^2 g(0) - \Delta_g^2) u & \text{in } \tilde{\mathcal{K}}(G,q,\varepsilon),

v = 0 & \text{in } \mathbb{R}^n_+,

\sqrt{g^{nn}(0)} \frac{\partial v}{\partial x_n} = h & \text{on } \partial \mathbb{R}^n_+.
\end{cases}
\]
Since
\[
0 = \Delta^2 g(0) v - \Delta_g^2 u = \Delta^2 g(0) (v - u) + (\Delta^2 g(0) - \Delta_g^2) u \text{ in } \tilde{\mathcal{K}}(G,q,\varepsilon),
\]
we get that \( v - u \) satisfies
\[
\begin{cases}
\Delta^2 g(0) (v - u) = (\Delta^2 g - \Delta_g^2) u & \text{in } \tilde{\mathcal{K}}(G,q,\varepsilon),

v - u = 0 & \text{in } (\tilde{\mathcal{K}}(G,q,\varepsilon) \cap \partial \mathbb{R}^n_+),

\sqrt{g^{nn}(0)} \frac{\partial (v-u)}{\partial x_n} = (\sqrt{g^{nn}(x)} - \sqrt{g^{nn}(0)}) \frac{\partial u}{\partial x_n} & \text{on } (\tilde{\mathcal{K}}(G,q,\varepsilon) \cap \partial \mathbb{R}^n_+).
\end{cases}
\]
It follows from \( L^p \)-estimates of elliptic equations (see, for example, Theorem 15.3 of [1]) that
\[
\|v - u\|_{L^p(\tilde{\mathcal{K}}(G,q,\varepsilon))} \leq C_1 \left( \|(\Delta^2 g - \Delta_g^2) u\|_{L^p(\tilde{\mathcal{K}}(G,q,\varepsilon))} \right.
\]
\[
\left. + \left\| \sqrt{g^{nn}(x)} - \sqrt{g^{nn}(0)} \frac{\partial u}{\partial x_n} \right\|_{W^{3-\frac{1}{p},\frac{1}{p}}(\tilde{\mathcal{K}}(G,q,\varepsilon) \cap \partial \mathbb{R}^n_+)} + \|v - u\|_{L^p(\tilde{\mathcal{K}}(G,q,\varepsilon))} \right),
\]
where the constant \( C_1 \) is independent of \( v - u \), \( W^{3,p}(\tilde{\mathcal{K}}(G,q,\varepsilon)) \) is the Sobolev space with \( p > n \), and \( 0 < \varepsilon < \varepsilon_0 \). From
\[
\lim_{\varepsilon \to 0} \|v - u\|_{L^p(\tilde{\mathcal{K}}(G,q,\varepsilon))} = 0,
\]
and
\[
\lim_{\varepsilon \to 0} \left\| \sqrt{g^{nn}(x)} - \sqrt{g^{nn}(0)} \frac{\partial u}{\partial x_n} \right\|_{W^{3-\frac{1}{p},\frac{1}{p}}(\tilde{\mathcal{K}}(G,q,\varepsilon) \cap \partial \mathbb{R}^n_+)} = 0,
\]
we obtain
\[
\lim_{\varepsilon \to 0} \|v - u\|_{W^{4,p}(\tilde{\mathcal{K}}(G,q,\varepsilon/2))} = 0.
\]
Combining this and the Sobolev embedding theorem, we find
\[
\|v - u\|_{C^{3,\alpha}(\tilde{\mathcal{K}}(G,q,\varepsilon/2))} = 0.
\]
Furthermore, applying the Schauder estimates (see, for example, the proof of Theorem 7.2 in [1]), we have that for any \( \varepsilon < \varepsilon_0 \),
\[
\|v - u\|_{C^{3,\alpha}(\tilde{\mathcal{K}}(G,q,\varepsilon/4))} \leq C_2 \left( \|(\Delta^2 g - \Delta_g^2) u\|_{C^{\alpha}(\tilde{\mathcal{K}}(G,q,\varepsilon/2))} \right.
\]
\[
\left. + \left\| \sqrt{g^{nn}(x)} - \sqrt{g^{nn}(0)} \frac{\partial u}{\partial x_n} \right\|_{C^{3,\alpha}(\tilde{\mathcal{K}}(G,q,\varepsilon/2)) \cap \partial \mathbb{R}^n_+} + \|v - u\|_{C^{\alpha}(\tilde{\mathcal{K}}(G,q,\varepsilon/2))} \right),
\]
where the constant \( C_2 \) is independent of \( v - u \). By
\[
\lim_{\varepsilon \to 0} \|(\Delta^2 g - \Delta_g^2) u\|_{C^{\alpha}(\tilde{\mathcal{K}}(G,q,\varepsilon/2))} = 0
\]
and
\[
\lim_{\epsilon \to 0} \left\| \frac{\sqrt{g^{nn}(x)} - \sqrt{g^{nn}(0)}}{g^{nn}(0)} \frac{\partial u}{\partial x_n} \right\|_{C^{3, \alpha}(\tilde{\kappa}(G_{\epsilon,1/4}) \cap \partial \mathbb{R}^n_+)} = 0,
\]
we have
\[
\lim_{\epsilon \to 0} \| v - u \|_{C^{3, \alpha}(\tilde{\kappa}(G_{\epsilon,1/4}))} = 0,
\]
which implies
\[
\lim_{\epsilon \to 0} \| \Delta_g u - \Delta_g(0)v \|_{C^{2, \alpha}(\tilde{\kappa}(G_{\epsilon,1/4}))} = 0.
\]
Applying the trace theorem, we obtain (3.12)

(3.12) \quad (\Delta_g u) \big|_{x' = 0} \big|_{(x', 0) \in \mathbb{R}^{n-1}} = (\Delta_g(0)v) \big|_{x' = 0} \big|_{(x', 0) \in \mathbb{R}^{n-1}}.

It follows from Lemma 3.1 that \( F_0 \) has the principal symbol \( 2 \sqrt{\sum_{j,k=1}^{n-1} g^{jk}(0) \eta_j \eta_k} \), where \( F_0 : C_0^\infty(\tilde{\kappa}(G_{\epsilon,0}) \cap \partial \mathbb{R}^n_+) \to C_0^\infty(\tilde{\kappa}(G_{\epsilon,0}) \cap \partial \mathbb{R}^n_+) \) defined by \( F_0 h := (\Delta v) \big|_{\tilde{\kappa}(G_{\epsilon,0}) \cap \partial \mathbb{R}^n_+} \), and \( v \) is the solution of (3.11). From (3.12), we get that the principal symbol of \( F_{q, \epsilon_0} \) is also \( p^{(\epsilon_0)}(0, \xi') = 2 \sqrt{\sum_{j,k=1}^{n-1} g^{jk}(x') \eta_j \eta_k} \). Since \( q \) is an arbitrary point at \( \partial \Omega \), it follows that the principal symbol of \( F \) is \( 2 \sqrt{\sum_{j,k=1}^{n-1} g^{jk}(x') \eta_j \eta_k} \) in the local boundary normal coordinate system \( (x') \) of \( \partial \Omega \).

3.2. We define another pseudodifferential operator \( \Theta : H^{3/2}(\partial \Omega) \to H^{-3/2}(\partial \Omega) \) as follows: Let \( \phi \in H^{3/2}(\partial \Omega) \), and let \( u \in H^2(\Omega) \) be the solution of

(3.13) \quad \begin{cases}
\Delta_g^2 u = 0 & \text{in } \Omega, \\
u = \phi & \text{on } \partial \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega,
\end{cases}

we set \( \Theta h := \frac{\partial (\Delta_g u)}{\partial \nu} \bigg|_{\partial \Omega} \). By Green’s formula, we have

\[
\langle \Theta h, h \rangle = \int_{\partial \Omega} \langle \Theta h, h \rangle \, ds = \int_{\partial \Omega} u \left( \frac{\partial (\Delta_g u)}{\partial \nu} \right) \, ds = \int_{\Omega} |\Delta_g u|^2 \, dx \geq 0, \quad \text{for any } h \in H^{1/2}(\partial \Omega),
\]
which implies that \( \Theta \) is a non-negative, self-adjoint pseudodifferential operator from \( H^{3/2}(\partial \Omega) \) to \( H^{-3/2}(\partial \Omega) \).

Lemma 3.3 Let \( A \) be a positive definition, real-valued constant matrix as in (2.10). Assume that

\[
\Theta_0 : C_0^\infty(\mathbb{R}^{n-1}) \to C^\infty(\mathbb{R}^{n-1})
\]

defined by the following problem: Let \( \phi \in C_0^\infty(\mathbb{R}^{n-1}) \) and let \( u \in C^\infty(\mathbb{R}^n_+) \) be the solution of

(3.14) \quad \begin{cases}
\sum_{j,k=1}^{n-1} a^{jk} \frac{\partial^2 u}{\partial x_j \partial x_k} + a^{nn} \frac{\partial^2 u}{\partial x_n^2} = 0 & \text{in } \mathbb{R}^n_+, \\
u = \phi & \text{on } \partial \mathbb{R}^n_+, \\
\frac{\partial u}{\partial x_n} = 0 & \text{on } \partial \mathbb{R}^n_+.
\end{cases}
we set \( \Theta_0 \phi := \sqrt{a^{\alpha n}} \frac{\partial (\Delta_{\alpha n} \phi)}{\partial x_n} |_{\partial \mathbb{R}^n} \). Then the principal symbol of \( \Theta_0 \) is

\[
p_0(x', \eta') = 2 \left( \sum_{j,k=1}^{n-1} a^{\alpha k} \eta_j \eta_k \right)^{\frac{3}{2}}, \quad \forall (x', \eta') \in \mathbb{R}^{n-1} \times (\mathbb{R}^{n-1} \setminus 0).
\]

**Proof.** Similar to Lemma 3.2, it follows from Lemma 2.5 that

\[
u(x', x_n) = \int_{\mathbb{R}^{n-1}} K_1(x' - y', x_n) \phi(y') dy',
\]

where

\[
K_1(x', x_n) = (-1)^{n-1} \frac{(n-2)!}{(2\pi i)^{n-1}} \int_{|\eta'| = 1} \left[ \left( x' \cdot \eta' + ix_n \right)^n \left( \sum_{j,k=1}^{n-1} a^{\alpha k} \eta_j \eta_k \right)^{1-n} \right] ds_{\eta'}.
\]

Taking the Fourier transform for (3.14) with respect to \( x_1, \cdots, x_{n-1} \), we have

\[
\left\{
\begin{array}{l}
\frac{\partial}{\partial x_n} \left[ \left( a^{\alpha n} \right)^2 \frac{\partial^2 \hat{u}}{\partial x_n^2} \right] - 2a^{\alpha n} \left( \sum_{j,k=1}^{n-1} a^{\alpha k} \eta_j \eta_k \right) \frac{\partial^2 \hat{u}}{\partial x_n^2} + \left( \sum_{j,k=1}^{n-1} a^{\alpha k} \eta_j \eta_k \right)^2 \hat{u} = 0 \quad \text{in} \ \mathbb{R}^n_+,
\frac{\partial \hat{u}}{\partial x_n} = 0 \quad \text{on} \ \partial \mathbb{R}^n_+, \\
\end{array}
\right.
\]

Then, the general solution of (3.16) has the form:

\[
\hat{u}(\eta', x_n) = C_1 e^{i|\xi'| x_n} + C_2 e^{-|\xi'| x_n} + C_3 x_n e^{i|\xi'| x_n} + C_4 x_n e^{-|\xi'| x_n},
\]

where \( |\xi'| := \sqrt{\sum_{j,k=1}^{n-1} a^{\alpha k} \eta_j \eta_k} \), and \( C_1, C_2, C_3, C_4 \) are arbitrary functions in \( \eta' \). In order to obtain a bounded solution of (3.16), we put \( C_1 = C_3 = 0 \), so that we find by the boundary conditions of (3.16) that \( C_2 = \hat{\phi}(\eta'), C_4 = \hat{\phi}(\eta') \). That is,

\[
\hat{u}(\eta', x_n) = \hat{\phi}(\eta') e^{-|\xi'| x_n} (1 + x_n |\xi'|).
\]

Thus

\[
u(x) = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{i(x', \eta')} \hat{\phi}(\eta') e^{-|\xi'| x_n} (1 + x_n |\xi'|) \ dy'.
\]

Since

\[
\frac{\partial^3 u}{\partial x_j \partial x_k \partial x_n} = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} x_n e^{i(x', \eta') \eta_j \eta_k} \hat{\phi}(\eta') |\xi'|^2 e^{-|\xi'| x_n} dy',
\]

\[
\frac{\partial^3 u}{\partial x_n^3} = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{i(x', \eta')} \hat{\phi}(\eta') e^{-|\xi'| x_n} (2|\xi'|^3 - x_n |\xi'|^4) dy',
\]
it follows that

\[
\Theta_0 \phi = \left. \left[ \sqrt{a^{nn}} \frac{\partial}{\partial x_n} \left( \sum_{j,k=1}^{n-1} a^{jk} \frac{\partial^2 u}{\partial x_j \partial x_k} + a^{nn} \frac{\partial^2 u}{\partial x_n^2} \right) \right] \right|_{x_n=0} \\
= \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} 2 \left( \sqrt{a^{nn}} |\xi'| \right)^3 e^{i(x',\eta')} \phi(\eta') \, d\eta'
\]

This shows that the principal symbol of the pseudodifferential operator \( \Theta_0 \) on \( \partial \mathbb{R}^n_+ \) is

\[
2 \sqrt{\sum_{j,k=1}^{n-1} a^{jk} \eta_j \eta_k}.
\]

\( \square \)

**Theorem 3.4** Let \((\mathcal{M}, g)\) be an \(n\)-dimensional \(C^\infty\) Riemannian manifold, and let \(\Omega\) be a bounded domain with \(C^\infty\) boundary. Assume that the pseudodifferential operator \(\Theta\) is defined as before. Then for any coordinate chart \(\kappa: \partial \Omega \supset U \rightarrow U^\kappa \subset \mathbb{R}^{n-1}\) there is a pseudodifferential operator \(\Upsilon \in \Psi^1(U^\kappa)\) such that for every \(\phi \in H^{3/2}(\partial \Omega)\) we have

\[
\kappa_* (\Theta \phi) - \Upsilon (\kappa_* \phi) \in C^\infty(U^\kappa).
\]

\(\Upsilon\) is elliptic and has a coordinate invariant positively \(3\)-homogeneous principal symbol \(p_0 \in C^\infty(T^*U^\kappa \setminus 0) = C^\infty(U^\kappa \times (\mathbb{R}^{n-1} \setminus 0))\) given by

\[
p_0(x', \eta') = 2 \left( \sum_{j,k=1}^{n-1} g^{jk} \eta_j \eta_k \right)^{3/2}, \quad \forall (x', \eta') \in T^*U^\kappa \setminus 0.
\]

**Proof.** The proof is similar to Theorem 3.2.

4. Proofs of main results

**Proof of Theorem 1.1.** Let \(F: H^{1/2}(\partial \Omega) \rightarrow H^{-1/2}(\partial \Omega)\) is defined as in §3.1. It follows from the discussion in §3.1 that \(F\) is a self-adjoint, elliptic, pseudodifferential operator on \(H^{1/2}(\partial \Omega)\) whose principal symbol is \(2 \sqrt{\sum_{j,k=1}^{n-1} g^{jk}(x') \eta_j \eta_k}\), where \(x'\) is the local boundary normal coordinate on \(\partial \Omega\). We define the operator \(Z_\epsilon\) by

\[
Z_\epsilon f(x') = \left( \frac{1}{\varphi(x') + \epsilon} \right) f(x') \quad \text{for all } f \in H^{1/2}(\partial \Omega) \text{ and } x' \in \partial \Omega,
\]

where \(\epsilon > 0\) is a sufficiently small constant. Applying Lemma 2.4, we obtain that the operator \(Q_\epsilon = Z_\epsilon \circ F : H^{1/2}(\partial \Omega) \rightarrow H^{-1/2}(\partial \Omega)\) defined by \(Q_\epsilon h = \left( \frac{1}{\varphi(x') + \epsilon} \right) (-\Delta u) \bigg|_{\partial \Omega}\)

is a pseudodifferential operator with the principal symbol \(2 \sqrt{\sum_{j,k=1}^{n-1} g^{jk}(x') \eta_j \eta_k} / \varphi(x') + \epsilon\), where \(u\) is the solution of (3.1). It is easily seen that the operator \(Q_\epsilon\) has the same eigenvalues \(\lambda_k(\epsilon)\) and corresponding normalized eigenfunctions \(\frac{\partial u}{\partial \eta_k}\) on \(\partial \Omega\) as the following biharmonic
Steklov eigenvalue problem:

\[
\begin{aligned}
\Delta_g u_k &= 0 &\text{in } \Omega, \\
\frac{\partial u_k}{\partial \nu} &= 0 &\text{on } \partial \Omega, \\
\Delta_g u_k + (\lambda_k(\epsilon))(g(x') + \epsilon) \frac{\partial u_k}{\partial \nu} &= 0 &\text{on } \partial \Omega.
\end{aligned}
\]

Let \( E_{\tau} \) be the spectral resolution of \( Q_{\epsilon} \), and let \( e(x', y', \tau) \) be the kernel of \( E_{\tau} \) (here \( e(x', y', \tau) = \sum_{\lambda \leq \tau} \sqrt{(g(x') + \epsilon)(g(y') + \epsilon)} \frac{\partial u_k}{\partial \nu} \)). It follows from Lemma 2.6 (see, Theorem 1.1 of [13], or [12]) that

\[
(4.1) \quad \tau^{-(n-1)} e(x', x', \tau) - (2\pi)^{-(n-1)} \int_{B_{x'}} d\xi^* = O(\tau^{-1}) \quad \text{as } \tau \to +\infty,
\]

where \( B_{x'} = \{ \eta' \in T_{x'}(\partial \Omega)|p_0(x', \eta') < 1 \} \), \( p_0(x', \eta') \) denotes the principal symbol of \( Q_{\epsilon} \).

By \( A_{\tau}(\epsilon) = \int_{\partial \Omega} e(x', x', \tau) dx' \) and \( p_0(x', \eta') = 2\sqrt{\sum_{j,k=1}^{n-1} g^{ik}(x') \eta_j \eta_k} \), we obtain that

\[
A_{\tau}(\epsilon) = \frac{1}{(2\pi)^{n-1}} \int_{\partial \Omega} dx' \int_{2(\epsilon)_{x'}} d\xi^* \\
\left( \int_{2(\epsilon)_{x'}} \int_{\epsilon} \int_{\epsilon} \frac{d\xi^*}{\det C(x')} |\det C(x')| \sqrt{|g'(x')|} \right) \tau^{n-1} \quad \text{as } \tau \to +\infty.
\]

For any fixed local boundary normal coordinate \( x' \in ^{R}(\partial \Omega) \), since \( (n-1) \times (n-1) \) matrix \( g' = (g^{jk}(x')) \) is positive definite, there exists an \( (n-1) \times (n-1) \) matrix \( C(x') = (c_{jk}(x')) \) such that \( ^{t}C(x')g'(x')C(x') = (\delta_{jk}) \), where \( \delta_{jk} \) is the Kronecker delta. Note that \( d\xi^* = \sqrt{|g'(x')|} d\zeta_1 \cdots d\zeta_{n-1} \), in each fiber of \( T^*(\partial \Omega) \), which is a vector space of dimension \( n-1 \).

With the change of variables \( \eta_j = \sum_{k=1}^{n-1} c_{jk}(x') \zeta_k \), we obtain

\[
\int \left( \int_{2(\epsilon)_{x'}} \int_{\epsilon} \int_{\epsilon} \frac{d\xi^*}{\det C(x')} |\det C(x')| \sqrt{|g'(x')|} \right) \tau^{n-1}
\]

\[
= \int \left( \int_{(\zeta_1, \cdots, \zeta_{n-1}) \in \mathbb{R}^{n-1}} |\sqrt{\sum_{j,k=1}^{n-1} g^{ik}(x') \eta_j \eta_k}| \right) \left( \int_{2(\epsilon)_{x'}} \int_{\epsilon} \int_{\epsilon} \frac{d\xi^*}{\det C(x')} |\det C(x')| \sqrt{|g'(x')|} \right) \\
= \int \left( \int_{(\zeta_1, \cdots, \zeta_{n-1}) \in \mathbb{R}^{n-1}} \left( \frac{d\zeta_1 \cdots d\zeta_{n-1}}{\sqrt{\sum_{j,k=1}^{n-1} g^{ik}(x') \eta_j \eta_k}} \right) \right) \\
= \omega_{n-1} \left( \frac{\phi(x') + \epsilon}{2} \right)^{n-1}.
\]

Here we have used the fact that \( |\det C(x')| \sqrt{|g'(x')|} = 1 \), and where \( \omega_{n-1} \) is the volume of the unit ball of \( \mathbb{R}^{n-1} \). Therefore

\[
A_{\tau}(\epsilon) = \frac{1}{(2\pi)^{n-1}} \omega_{n-1} \tau^{n-1} \int_{\partial \Omega} \left( \frac{\phi(x') + \epsilon}{2} \right)^{n-1} dx' + O(\tau^{n-2}) \quad \text{as } \tau \to +\infty.
\]

Letting \( \epsilon \to 0 \), we obtain

\[
A(\tau) = \frac{1}{(2\pi)^{n-1}} \omega_{n-1} \tau^{n-1} \int_{\partial \Omega} \left( \frac{\phi(x')}{2} \right)^{n-1} dx' + O(\tau^{n-2}) \quad \text{as } \tau \to +\infty,
\]

that is,

\[
A(\tau) = \frac{\omega_{n-1} \tau^{n-1}}{(4\pi)^{n-1}} \int_{\partial \Omega} (\phi(s))^{n-1} ds + O(\tau^{n-2}) \quad \text{as } \tau \to +\infty.
\]

\[\square\]
Proof of Theorem 1.2. Let $R_\epsilon : H^{3/2}(\partial \Omega) \to H^{-3/2}(\partial \Omega)$ be defined as follows: For any $\phi \in H^{3/2}(\partial \Omega)$, we put $R_\epsilon \phi = \left( \frac{1}{(\epsilon + \tau)^3} \frac{\partial (\Delta_a \phi_\tau)}{\partial \nu} \right) |_{\partial \Omega}$, where $\nu$ satisfies
\[
\begin{align*}
\Delta_\nu \phi_\tau = 0 & \quad \text{in } \Omega, \\
u = \phi & \quad \text{on } \partial \Omega, \\
\frac{\partial \nu}{\partial \nu} = 0 & \quad \text{on } \partial \Omega,
\end{align*}
\]
and $\epsilon > 0$ is a sufficiently small constant. Clearly, $R_\epsilon$ is a self-adjoint, elliptic, non-negative pseudodifferential operator of order 3. By Lemmas 2.4 and Theorem 3.4, we get that the principal symbol of $R_\epsilon$ has the form $\frac{2}{(\epsilon + \tau)^3} \left( \sum_{j,k=1}^{n-1} g_{jk}^\eta \eta_k \right)^{3/2}$, where $\eta^p \in \mathbb{R}^{n-1}$. It is easily seen that the operator $R_\epsilon$ has the same eigenvalues $\mu_k^3(\epsilon)$ and corresponding normalized eigenfunctions $\nu_k$ on $\partial \Omega$ as the following biharmonic Steklov eigenvalue problem:
\[
\begin{align*}
\frac{\Delta_\nu \nu_k}{\partial \nu} = 0 & \quad \text{in } \Omega, \\
\nu_k = \phi & \quad \text{on } \partial \Omega, \\
\frac{\partial \nu_k}{\partial \nu} = 0 & \quad \text{on } \partial \Omega.
\end{align*}
\]
It follows from Lemma 2.6 (Theorem 1.1 of [14], also see [12] or [2]) that
\[
\tau^{(n-1)/3} e(x', \tau, \tau) - (2\tau)^{(n-1)} \int_{B_{x'}} d\xi^* = O(\tau^{1/3}) \quad \text{as } \tau \to +\infty,
\]
where $B_{x'} = \{ \eta^p \in T^*_{x'}(\partial \Omega) | p_0(x', \eta^p) < 1 \}$, $p_0(x', \eta^p)$ denotes the principal symbol of $F_\epsilon$. Since $B_\epsilon(\tau) = \int_{\partial \Omega} e(x', x, \tau) dx'$ and $p_0(x', \eta^p) = \frac{2}{\sqrt{(\sum_{j,k=1}^{n-1} g_{jk}^\eta \eta_k)^3}}$, we have
\[
B_\epsilon(\tau) = \frac{1}{(2\pi)^{n-1}} \left( \int_{\partial \Omega} dx' \int_{2g(x') + \epsilon} (\sqrt{(\sum_{j,k=1}^{n-1} g_{jk}^\eta \eta_k)^3})^{3/2} < 1 \right) \tau^{n-1} + O(\tau^{n-2}) \quad \text{as } \tau \to +\infty,
\]
where $d\xi^* = \sqrt{g'(x')} d\zeta_1 \cdots d\zeta_{n-1}$. With the change of variables $\eta_j = \sum_{k=1}^{n-1} c_{jk}(x') \zeta_k$, where $(n-1) \times (n-1)$ matrix $C(x') = (c_{jk}(x'))$ satisfies $C(x') g'(x') C(x') = (\delta_{jk})$, we obtain
\[
\int_{2g(x') + \epsilon} (\sqrt{(\sum_{j,k=1}^{n-1} g_{jk}^\eta \eta_k)^3})^{3/2} < 1 \left( \int_{\zeta_1 \cdots \zeta_{n-1} < \frac{\sqrt{g'(x')}}{\sqrt{2}}} d\zeta_1 \cdots d\zeta_{n-1} \right) \omega_{n-1} \left( \frac{g(x') + \epsilon}{\sqrt{2}} \right)^{n-1},
\]
which implies
\[
B_\epsilon(\tau) = \frac{1}{(2\pi)^{n-1}} \omega_{n-1} \tau^{n-1} \int_{\partial \Omega} \left( \frac{g(x') + \epsilon}{\sqrt{2}} \right)^{n-1} dx' + O(\tau^{n-2}) \quad \text{as } \tau \to +\infty.
\]
Letting $\epsilon \to 0$, we obtain
\[
B(\tau) = \frac{1}{(2\pi)^{n-1}} \omega_{n-1} \tau^{n-1} \int_{\partial \Omega} \left( \frac{g(x')}{\sqrt{2}} \right)^{n-1} dx' + O(\tau^{n-2}) \quad \text{as } \tau \to +\infty,
\]
\[
i.e.,
A(\tau) = \frac{\omega_{n-1} \tau^{n-1}}{(\sqrt{16\pi})^{n-1}} \int_{\partial D} \phi^{n-1}(s) ds + O(\tau^{n-2}) \quad \text{as } \tau \to +\infty.
\]
The asymptotic formulas are sharp

Hörmander (see [14] or [32]) proved that, for a pseudodifferential operator of order $m$ with principal symbol $p_0(x, \xi)$,

$$\left| e(x, x, \tau) - (2\pi)^{-n} \int_{p_0(x, \xi) < \tau} d\xi \right| \leq C(1 + |\tau|)^{-n-1}$$

uniformly in compact subsets of $\mathcal{M}$, where $C$ is independent of $x, \tau$. Applying this result to our cases, we immediately obtain

$$\left| A(\tau) - \frac{\omega_{n-1} \tau^{n-1}}{(4\pi)^{n-1}} \int_{\partial \Omega} g^{n-1} ds \right| \leq C(1 + |\tau|)^{n-2},$$

$$\left| B(\tau) - \frac{\omega_{n-1} \tau^{n-1}}{(\sqrt{16\pi})^{n-1}} \int_{\partial D} g^{n-1} ds \right| \leq C(1 + |\tau|)^{n-2}.$$ 

In this section, we shall show that (5.2) and (5.3) cannot further be improved. More precisely, we shall show by two counterexamples (letting $\Omega$ respectively be the unit ball of $\mathbb{R}^n$ and the unit disk of of $\mathbb{R}^2$, and $\varrho \equiv 1$) that the asymptotic formulas (1.8) and (1.9) are the “best possible”.

First we give some well-known facts concerning spherical harmonics (See e.g. Müller [25]). When $\Omega = B$ we may explicitly determine all the biharmonic Steklov eigenvalues of (1.1). In fact, for each integer $m \geq 0$, let $P_m(\mathbb{R}^n)$ denote the set of homogeneous polynomials of degree $m$ in $n$ variables, i.e., the set of functions $u$ of the form

$$u(x) = \sum_{|\alpha| = m} a_\alpha x^\alpha \quad \text{for } x \in \mathbb{R}^n,$$

with coefficients $a_\alpha \in \mathbb{C}$. A solid spherical harmonic of degree $m$ is an element of the subspace

$$\mathcal{H}_m(\mathbb{R}^n) = \{ u \in P_m(\mathbb{R}^n) | \Delta u = 0 \text{ on } \mathbb{R}^n \}.$$

Let

$$N(n, m) = \dim \mathcal{H}_m(\mathbb{R}^n) \quad \text{for } n \geq 1 \text{ and } m \geq 0.$$ 

Note that $\mathcal{P}_0 = \mathcal{H}_0$ is just the space of constant functions, and $\mathcal{P}_1 = \mathcal{H}_1$ is just the space of homogeneous linear functions, so

$$N(n, 0) = 1 \quad \text{and} \quad N(n, 1) = n \quad \text{for } n \geq 1.$$ 

It follows from p. 251-252 of [22] that

$$N(1, m) = \begin{cases} 1 & \text{if } m = 0 \text{ or } 1, \\ 0 & \text{if } m \geq 2, \end{cases}$$

$$N(2, m) = \begin{cases} 1 & \text{if } m = 0, \\ 2 & \text{if } m \geq 1, \end{cases}$$
and
\[ N(n, m) = \frac{2m + n - 2}{n - 2} \left( \frac{m + n - 3}{n - 3} \right) \quad \text{for } n \geq 3 \text{ and } m \geq 0. \]

The following Lemma was obtained by Ferrero, Gazzola and Weth (see, Theorem 1.3 of [10]):

**Lemma 5.1.** If \( n \geq 2 \) and \( \Omega = B \), then for all \( m = 0, 1, 2, 3, \cdots \):

(i) the eigenvalues of (1.1) are \( \tilde{\lambda}_m = n + 2m \);
(ii) the multiplicity of \( \lambda_m \) equals \( N(n, m) \);
(iii) for all \( \tilde{\psi}_m \in \mathcal{H}_m(\mathbb{R}^n) \), the function \( \tilde{\phi}_m(x) := (1 - |x|^2)\tilde{\psi}_m(x) \) is an eigenfunction corresponding to \( \tilde{\lambda}_m \).

Now, let \( 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \) be all biharmonic Steklov eigenvalues for the ball \( B \). From the above lemma and the formula
\[
\sum_{j=1}^m \binom{a+j-1}{j} = \frac{(a+1)(a+2)\cdots(a+m)}{m!} - 1,
\]
we get that for \( n \geq 2 \),
\[
A(\tilde{\lambda}_m) = \# \{ i | \lambda_i \leq \tilde{\lambda}_m \} = \sum_{k=0}^m N(n, k) = 1 + \sum_{k=1}^m \frac{(n+2k-2)(n+k-3)!}{(n-2)!k!} + \sum_{k=2}^m \frac{(n+k-3)!}{(n-2)!(k-1)!} 
\]
\[
= 1 + \sum_{k=1}^m \binom{n+k-2}{k} + \sum_{k=2}^m \binom{n+k-3}{k-1} 
\]
\[
= 1 + \sum_{k=1}^m \binom{n-1+k-1}{k} + \sum_{k=2}^m \binom{n-1+(k-1)-1}{k-1} 
\]
\[
= 2 + \sum_{k=1}^m \binom{n-1+k-1}{k} + \sum_{j=1}^m \binom{n-1+j-1}{j} - \binom{n-1+m-1}{m} 
\]
\[
= 2 + \frac{n(n+1)\cdots(n-1+m-1)}{m!} - 1 + \frac{n(n+1)\cdots(n-1+m)}{m!} - 1 - \frac{n+m-2}{m} 
\]
\[
= 2 \left( \frac{n(n+1)\cdots(n-1+m)}{m!} - 1 \right) - \left( \frac{n+m-2}{m} \right) \]
\[
= 2 \left( \frac{n+m-1}{m} \right) - \left( \frac{n+m-2}{m} \right) = 2 \left( \frac{n+m-1}{n-1} \right) - \left( \frac{n+m-2}{n-2} \right).
\]

By applying the formula
\[
\binom{p+1}{r} = \binom{p}{r-1} = \binom{p}{r}
\]
and \( m = \frac{\lambda_m}{2} - \frac{n}{2} \), we obtain
\[
A(\lambda_m) = \binom{n + m - 1}{n - 1} + \binom{n + m - 2}{n - 1} \\
= \frac{(2m + n - 1)(m + n - 2)(m + n - 3) \cdots (m + 1)}{(n - 1)!} \\
= \frac{1}{2^{2n-2}(n-1)!}(\lambda_m - 1)(\lambda_m + n - 4)(\lambda_m + n - 6) \cdots (\lambda_m - n + 2).
\]

In view of \( \frac{\sqrt{\pi} \Gamma(2z)}{2^{z-1}} = \Gamma(z)\Gamma(\frac{1}{2} + z) \), we get
\[
\frac{1}{2^{2n-2}(n-1)!} = \frac{2^n \pi^{n-1}}{(4\pi)^{n-1}(n-1)!} = \frac{n \pi^{n-\frac{1}{2}}}{(4\pi)^{n-1}\pi^{(n+1)}} \\
= \frac{n \pi^{n-\frac{1}{2}}}{(4\pi)^{n-1}\Gamma\left(\frac{n}{2} + 1\right)} = \frac{1}{(4\pi)^{n-1}} \cdot \frac{\pi^{n-\frac{1}{2}}}{\Gamma\left(\frac{n}{2} + 1\right)} \\
= \frac{\omega_{n-1}}{(4\pi)^{n-1} \cdot n \omega_n} = \frac{\omega_{n-1}}{(4\pi)^{n-1}(\text{vol}(\partial B))}.
\]

Hence
\[
A(\lambda_m) = \frac{\omega_{n-1}}{(4\pi)^{n-1}(\text{vol}(\partial B))(\lambda_m - 1)(\lambda_m + n - 4)(\lambda_m + n - 6) \cdots (\lambda_m - n + 2)} \\
= \frac{\omega_{n-1}}{(4\pi)^{n-1}(\text{vol}(\partial B))} \left[\lambda_m^{(n-1)} + (1 - n)\lambda_m^{n-2} + \cdots + (n - 4)(n - 6) \cdots (-n + 2)\right].
\]

Since \( 1 - n \neq 0 \), this shows that the formula (1.8) is sharp.

Similar to the above \( A(\lambda_m) \), we can also give a counter-example to show that the remainder term estimate in the asymptotic formula (1.9) is sharp. Let \( B \subset \mathbb{R}^2 \) be the unit disk. If \( m \geq 1 \), then the functions
\[
\psi_{m,1}(r, \theta) = r^m \cos m\theta \quad \text{and} \quad \psi_{m,2}(r, \theta) = r^m \sin m\theta
\]
form an orthogonal basis of harmonic function in the space \( H_m(\mathbb{R}^2) \). Let us consider the following Neumann boundary value problem:

\[
\begin{cases}
\Delta u_{m,j} = \psi_{m,j} & \text{in } B, \\
\frac{\partial u_{m,j}}{\partial \nu} = 0 & \text{on } \partial B,
\end{cases}
\]

where \( m = 1, 2, 3, \cdots ; j = 1, 2 \). We claim that the above solutions \( u_{m,j} \) satisfy
\[
\begin{cases}
\Delta^2 u_{m,j} = 0 & \text{in } B, \\
\frac{\partial u_{m,j}}{\partial \nu} = 0 & \text{on } \partial B,
\end{cases}
\]

where
\[
\frac{1}{\mu_{m,j}^2} = \frac{\int_B |\Delta u_{m,j}|^2 \, dx}{\int_{\partial B} \left(\frac{\partial (\Delta u_{m,j})}{\partial \nu}\right)^2 \, ds} = \frac{\int_B |\psi_{m,j}|^2 \, dx}{\int_{\partial B} \left(\frac{\partial \psi_{m,j}}{\partial \nu}\right)^2 \, ds} = \frac{1}{2m^2(m + 1)}.
\]

In fact, for the Dirichlet problem
\[
\begin{cases}
\Delta u_{m,j} = \psi_{m,j}(r, \theta) & \text{in } B, \\
u_{m,j} = \frac{1}{\mu_{m,j}^2} \psi_{m,j}(1, \theta) & \text{on } \partial B,
\end{cases}
\]

where \( m = 1, 2, 3, \cdots ; j = 1, 2, \ldots \)
from the formula of the solution to the Dirichlet boundary value problem for the Poisson equation in the unit disk, we have
\[
u_{m,j}(\rho, \alpha) = -\frac{1}{4\pi} \int_0^1 \int_0^{2\pi} r dr \int_0^{2\pi} \ln \frac{1 + \rho^2 r^2 - 2 \rho r \cos(\alpha - \theta)}{r^2 + \rho^2 - 2 \rho \cos(\alpha - \theta)} \psi_{m,j}(r, \theta) d\theta - \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - \rho^2}{1 + \rho^2 - 2 \rho \cos(\alpha - \theta)} \left( \psi_{m,j}(1, \theta) \right) d\theta.
\]
Then
\[
\frac{\partial \nu_{m,j}(\rho, \alpha)}{\partial \rho} \bigg|_{\rho=1} = \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} r \frac{(1 - r^2)}{1 + r^2 - 2 \rho \cos(\alpha - \theta)} \psi_{m,j}(r, \theta) dr d\theta - \frac{1}{2\pi} \left[ \frac{\partial}{\partial \rho} \int_0^{2\pi} \frac{1 - \rho^2}{1 + \rho^2 - 2 \rho \cos(\alpha - \theta)} \left( \psi_{m,j}(1, \theta) \right) d\theta \right]_{\rho=1}.
\]
It is well-known that
\[
\frac{1 - r^2}{1 + r^2 - 2 \rho \cos(\alpha - \theta)} = 1 + 2 \sum_{l=1}^{\infty} r^l \cos(l(\alpha - \theta)).
\]
Therefore
\[
\frac{\partial \nu_{m,j}(\rho, \alpha)}{\partial \rho} \bigg|_{\rho=1} = \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} \left( r + 2 \sum_{l=1}^{\infty} r^{l+1} \cos(l(\alpha - \theta)) \right) \psi_{m,j}(r, \theta) dr d\theta - \frac{1}{2\pi} \left[ \frac{\partial}{\partial \rho} \int_0^{2\pi} \left( r + 2 \sum_{l=1}^{\infty} r^{l+1} \cos(l(\alpha - \theta)) \right) \frac{\psi_{m,j}(1, \theta)}{2m(m+1)} d\theta \right]_{\rho=1}
\]
\[
= - \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} \frac{1 - \rho^2}{1 + \rho^2 - 2 \rho \cos(\alpha - \theta)} \psi_{m,j}(r, \alpha - \theta) dr d\theta + \frac{1}{2\pi} \left[ \int_0^{2\pi} \left( 2 \sum_{l=1}^{\infty} r^{l-1} \cos(l(t)) \right) \frac{\psi_{m,j}(1, \alpha - t)}{2m(m+1)} dt \right]_{\rho=1}.
\]
By (5.7) and a simple calculation, we get \(\frac{\partial \nu_{m,j}}{\partial \rho} \bigg|_{\rho=1} = 0\). Combining this and the result
\[
\begin{cases}
\Delta \psi_{m,j} = 0 & \text{in } B, \\
\frac{\partial \psi_{m,j}}{\partial \nu} = \eta_{m,j} \psi_{m,j} & \text{on } \partial B, \\
m = 1, 2, 3, \cdots; j = 1, 2,
\end{cases}
\]
where
\[
\frac{1}{\eta_{m,j}} = \frac{\int_B |\nabla(\psi_{m,j})|^2 dx}{\int_{\partial B} \left( \frac{\partial \psi_{m,j}}{\partial \nu} \right)^2 ds} = \frac{1}{m}, \quad m = 1, 2, 3, \cdots; j = 1, 2,
\]
we show the desired claim. It follows that
\[
\mu_{m,j}^3 = 2m^2(m+1), \quad m = 1, 2, 3, \cdots; j = 1, 2,
\]
Thus, for \(n = 2\),
\[
B(\mu_{m,j}) = \# \{ i | \mu_i \leq \mu_{m,j} \} = \sum_{k=0}^{m} N(2, k) = 2(m + \frac{1}{2}) \]
\[
= \frac{\omega_1(\text{vol}(\partial B))}{2\pi} \left( m + \frac{1}{2} \right).
\]
By
\[ m \sim \frac{\mu_{m,j}}{\sqrt{2}} \quad \text{as} \quad m \to +\infty, \]
we have
\[ B(\mu_{m,j}) \sim \frac{\omega_1(\text{vol}(\partial B))}{\sqrt{16\pi}} \left( \mu_{m,j} + \frac{1}{2} \right) \quad \text{as} \quad m \to +\infty, \quad j = 1, 2. \]

Since \( \frac{1}{2} \neq 0 \), this shows that asymptotic formula (1.9) cannot be improved on unit disk of \( \mathbb{R}^2 \).

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