Modified Massive Arratia Flow and Wasserstein Diffusion

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Abstract

Extending previous work by the first author we present a variant of the Arratia flow, which consists of a collection of coalescing Brownian motions starting from every point of the unit interval. The important new feature of the model is that individual particles carry mass that aggregates upon coalescence and that scales the diffusivity of each particle in an inverse proportional way. In this work we relate the induced measure-valued process to the Wasserstein diffusion of von Renesse and Sturm. First, we present the process as a martingale solution to an SPDE similar to that of von Renesse and Sturm. Second, as our main result we show a Varadhan formula for short times that is governed by the quadratic Wasserstein distance. © 2018 Wiley Periodicals, Inc.

1 Introduction and Statement of Main Results

1.1 Motivation

Since its introduction in [33], Otto’s formal infinite-dimensional Riemannian calculus for optimal transportation has been the inspiration for numerous new results both in pure and applied mathematics; see, e.g., [1, 31, 34, 40, 45]. It can be considered a lift of conventional calculus of points to point ensembles or spatially continuous mass distributions. It is therefore natural to ask whether this lifting procedure from points to mass configurations has a probabilistic counterpart. The fundamental object of such a theory would need to be an analogue of Brownian motion on the space of probability measures adapted to Otto’s Riemannian structure of optimal transportation. In [46] the second author together with Sturm proposed a first candidate of such a measure-valued Brownian motion (with drift), calling it Wasserstein diffusion, and showed among other things that its short-time asymptotics are indeed governed by the geometry of optimal transport in the sense of a Varadhan formula for short times governed by the Wasserstein distance. However, the construction in [46] has several limitations since it is strictly restricted to...
diffusing measures on the real line, it brings about additional seemingly nonphysical correction/renormalization terms, and lastly it is obtained by abstract Dirichlet form methods that, e.g., do not allow for generic starting points of the evolution. Hence, in spite of several ad hoc finite-dimensional approximations [3, 39, 41], the process remained a rather obscure object. Given the strong similarity of the SPDE representation of the Wasserstein diffusion to the Dean-Kawasaki equation in physics [9, 24], it is natural to ask for related measure-valued diffusion processes that share a similar multiplicative noise structure, giving rise to the same large-deviation principles on short time scales.

1.2 Modified Massive Arratia Flow

In this paper we give a different and very explicit construction of another diffusion process in the space of probability measures on the real line that exhibits a similarity to the Wasserstein diffusion as discussed above. The construction is based on a modification of the so-called Arratia flow of coalescing Brownian motions, which was introduced in [4] and later extensively studied by Dorogovtsev and coauthors [14–17, 32] and Le Jan-Raimond [30]. As an important extension the Brownian web [19] has also received significant attention in recent studies.

Our point of departure is another modification of the Arratia flow in [25] by assigning a mass to each particle, which is aggregated when particles coalesce and which controls the diffusivity of each particle in inverse proportion. In [27] it was shown for the first time that such a system can be constructed starting with an infinitesimal mass particle at each point of the unit interval, i.e., such that the empirical measure of the particles almost surely converges in weak topology to the uniform measure on the unit interval as time tends to 0. The resulting model, which we shall call modified massive Arratia flow (MMAF), can best be described in terms of a family of continuous martingales that describe the motion of the particles. Letting $D([0, 1], C[0, T])$ denote the Skorokhod space of càdlàg functions from $[0, 1]$ into the metric space $(C[0, T], d_{\infty})$ of continuous real-valued trajectories over the time interval $[0, T]$ with the uniform distance $d_{\infty}$ and $\lambda$ denote the Lebesgue measure on $[0, 1]$, the main result of [27] is as follows.

**Theorem 1.1.** There is a process $y \in D([0, 1], C[0, T])$ such that

(C1) for all $u \in [0, 1]$ the process $y(u, \cdot)$ is a continuous square-integrable martingale with respect to the filtration $\mathcal{F}_t = \sigma(y(u, s), u \in [0, 1], s \leq t)$, $t \in [0, T]$;

(C2) for all $u \in [0, 1]$, $y(u, 0) = u$;

(C3) for all $u < v$ from $[0, 1]$ and $t \in [0, T]$, $y(u, t) \leq y(v, t)$;

(C4) for all $u, v \in [0, 1]$ the joint quadratic variation of $y(u, \cdot)$ and $y(v, \cdot)$ is

$$[y(u, \cdot), y(v, \cdot)]_t = \int_0^t \frac{\mathbb{I}_{[\tau_u, v] \leq s} ds}{m(u, s)},$$
where $m(u, t) = \lambda\{v : \exists s \leq t, \ y(v, s) = y(u, s)\}$, $\tau_{u,v} = \inf\{t : y(u, t) = y(v, t)\} \wedge T$.

Note that uniqueness in law of $y$ satisfying properties (C1)–(C4) remains an important open problem. However, all subsequent results derived in this paper deal just with some field of martingales satisfying properties (C1)–(C4) above. In particular, uniqueness is not needed for any of our arguments.

We also point out that, in contrast to the classical Arratia flow, the family of maps $\{y(\cdot, s)\}_{s \geq 0}$ does not induce a (stochastic) flow on the real line, i.e., does not satisfy a cocycle property. Our terminology of a “modified massive Arratia flow” refers rather to the corresponding measure-valued process

$$\mu_t := \mathcal{H}y(\cdot, t)\#\lambda, \quad t \in [0, T],$$

which is obtained via the image (push-forward) of the uniform measure $\lambda$ on $[0, 1]$ under the random maps $y(\cdot, t)$. The process $\mu_t$, $t \in [0, T]$, is the central object of our interest. In particular, $\mu_0 = \lambda$ in the present case, but our arguments and constructions below can be modified to the case of a more general starting measure; cf. [28]. For the sake of presentation, in what follows we stick to the $\mu_0 = \lambda$ case.

For illustration and comparison to the standard Arratia flow we include here some numerical simulations. The red trajectory in the picture below is the evolution of the center of mass of the particles, which is a Brownian motion.

### 1.3 Main Results for the Modified Massive Arratia Flow

**New Construction and Stochastic Calculus for the MMAF**

The first result of this paper is a new simplified construction of a modified massive Arratia flow, using spatial discretization and a tightness argument. Second, we analyze the process $y(t) = y(\cdot, t), t \in [0, T]$, as an $L_2(\lambda)$-valued martingale and develop an associated stochastic calculus. We show that for each $g \in L_2(\lambda)$,

$$s \mapsto J(g)(s) := (g, y(s))_{L_2(\lambda)},$$

where $(\cdot, \cdot)_{L_2(\lambda)}$ denotes the inner product in

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1 In fact, $\mu_t, t \in [0, T]$, turns out to be a Markov process, but we will not stress this here.
$L_2(\lambda)$, is a continuous square-integrable martingale with quadratic variation process

$$[J(g)]_t = \int_0^t \|\text{pr}_{\gamma(s)}g\|_{L_2(\lambda)}^2 ds, \quad t \in [0, T].$$

Here $\text{pr}_h g$ is the orthogonal projection in $L_2(\lambda)$ of $g$ onto the subspace of $\sigma(h)$-measurable functions. This shows that the process $\gamma(\cdot)$ is a martingale solution to the infinite-dimensional SDE

$$dy(s) = \text{pr}_{\gamma(s)}dW_s,$$

where $W$ is cylindrical Brownian motion in the Hilbert space $L_2(\lambda)$.

By (C3) the map $[0, 1] \ni u \mapsto y(u, t)$ is monotone (and càdlàg); hence the one-to-one map between probability measures on $\mathbb{R}$ and their quantile functions on $[0, 1]$ yield an equivalent parametrization of $y$ by the induced measure-valued flow

$$y(\cdot, t) = y(u, t)_{\#} \mathcal{P}, \quad t \in [0, T],$$

where $y(\cdot, t)_{\#} \mathcal{P}(A) = \mathbb{P}\{y(u, t) \in A\}, A \in \mathcal{B}(\mathbb{R})$, denotes the image measure of $\lambda$ under the map $y(\cdot, t)$.

The process $\mu_t$, $t \in [0, T]$, and its relation to the Wasserstein diffusion, is our main interest in this paper. Our first observation follows from the Itô formula for $y(t)$, $t \in [0, T]$, obtained in [27].

**Proposition 1.2.** Let $\mu_t := y(\cdot, t)_{\#} \lambda$. Then, for each twice continuously differentiable function $f$ on $\mathbb{R}$ with bounded derivatives up to the second order,

$$M^f_t := \langle f, \mu_t \rangle - \int_0^t \langle f, \Gamma(\mu_s) \rangle ds$$

is a continuous local martingale with quadratic variation process

$$[M^f]_t = \int_0^t \langle (f')^2, \mu_s \rangle ds,$$

where $\Gamma$ is defined as follows:

$$\langle f, \Gamma(v) \rangle = \frac{1}{2} \sum_{x \in \text{supp}(v)} f''(x).$$

We point out that $\Gamma(\mu_t)$ is well-defined since property (P4) of Section 2.4 below implies that $\text{supp}(\mu_t)$ is a finite set for all $t \in (0, T]$ almost surely.

As a consequence of Proposition 1.2, $\mu_t$, $t \in [0, T]$, is a probability-valued martingale solution to the SPDE

$$d\mu_t = \Gamma(\mu_t)dt + \text{div}(\sqrt{\mu_t} dW_t),$$

which follows from a standard application of Itô’s formula in finite dimensions.
The SPDE (1.2) should be compared to the corresponding SPDE for the Wasserstein diffusion [3, 46]:

$$d\mu_t = \beta \Delta \mu_t \, dt + \tilde{\Gamma}(\mu_t) \, dt + \text{div}(\sqrt{\mu_t} \, dW_t),$$

with

$$\langle f, \tilde{\Gamma}(v) \rangle = \sum_{I \in \text{gaps}(v)} \left[ \frac{f''(I_+) + f''(I_-)}{2} - \frac{f'(I_+) - f'(I_-)}{|I|} \right].$$

Thus, besides the apparent similarity of the second-order part in the drift operators $\Gamma$ and $\tilde{\Gamma}$, both models share the same singular multiplicative noise that gives rise to the characteristic density $\langle (f')^2, \mu \rangle$ in the quadratic variation process. Of course, this is the same expression as the one appearing in Otto’s definition [33] of the Riemannian energy of an infinitesimal (tangential) perturbation of a measure or in the Benamou-Brenier formula [5] for optimal transportation.

**Varadhan Formula for the Short-Time Asymptotics of the MMAF**

As the main achievement of the present paper we will make this connection more rigorous by showing that the small-time fluctuations of the process are in fact governed, on an exponential scale, by the Wasserstein metric.

To this aim, recall that the (quadratic) Wasserstein metric is defined as follows. For probability measures $\nu_1, \nu_2$ on the real line with finite second moments, it is defined by

$$d_W(\nu_1, \nu_2) = \left( \inf_{\chi \in \mathcal{X}(\nu_1, \nu_2)} \int_{\mathbb{R}^2} |\xi - \eta|^2 \nu(d\xi, d\eta) \right)^{1/2},$$

where $\mathcal{X}(\nu_1, \nu_2)$ denotes the set of all probability measures on $\mathbb{R}^2$ with marginals $\nu_1, \nu_2$. The main result of the present paper is the following version of the Varadhan formula [42] for the measure-valued diffusion $\mu_t, t \in [0, T]$. The precise conditions for a set $A \subset \mathcal{P}(\mathbb{R})$ to be properly chosen for the statement are specified in Section 1.3.

**THEOREM 1.3.** Let $y$ satisfy (C1)–(C4) and let $\mu_t, t \in [0, T]$, be defined by (1.1). Then, for properly chosen sets $A \subset \mathcal{P}(\mathbb{R})$

$$\lim_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}\{\mu_\varepsilon \in A\} = -\frac{(d_W(\lambda, A))^2}{2},$$

where the uniform distribution $\lambda$ on $[0, 1]$ is considered as an element of $\mathcal{P}(\mathbb{R})$.

It should be noted that in Theorem 1.3 we do not make any assumptions on the system $y$ other than (C1)–(C4). Here, the particular construction leading to a system with these properties does not play any role.

\[^2\text{See also [1] for the connection to the Dean-Kawasaki equation (cf. [9]).}\]
Theorem 1.3 is a large-deviation statement for the family of random measures \( \mu_\varepsilon \) involving the rate function

\[
I(\eta) = \frac{1}{2} \left( \inf_{\nu \in \mathcal{X}(\lambda, \eta)} \iint_{\mathbb{R}^2} |\xi - \eta|^2 \nu(d\xi, d\eta) \right)^2 = \frac{1}{2} d_{\nu}^2(\lambda, \eta).
\]

We obtain it by contraction from a full large-deviation principle for the family of processes \( \{y^\varepsilon(\cdot)\}_{\varepsilon \in (0, 1]} = \{y(\varepsilon \cdot)\}_{\varepsilon \in (0, 1]} \). The latter is the main technical achievement of the present paper and comprises the biggest part of it.

**Large-Deviation Principle for the MMAF**

For a precise statement of the large-deviation principle for the sequence \( \{y^\varepsilon(\cdot)\}_{\varepsilon \in (0, 1]} = \{y(\varepsilon \cdot)\}_{\varepsilon \in (0, 1]} \) we need some notation. Let \( L_2(\rho) = L_2([0, 1], \rho), \rho(du) = \kappa(u)du, \) where \( \kappa : [0, 1] \to [0, 1], \)

\[
(1.4) \quad \kappa(u) = \begin{cases} u^{\beta}, & u \in [0, \frac{1}{2}], \\ (1 - u)^{\beta}, & u \in (\frac{1}{2}, 1], \end{cases}
\]

for some fixed \( \beta > 1 \), and

\[
D^\dagger = \{h \in D([0, 1], \mathbb{R}) : h \text{ is nondecreasing}\}.
\]

Denote

\[
\mathcal{H} = \{\varphi \in C([0, T], L_2(\lambda) \cap D^\dagger) : \varphi(0) = id \}
\]

and \( t \to \varphi(t) \in L_2(\lambda) \) is absolutely continuous. \(^3\)

\[
(1.5) \quad I(\varphi) = \begin{cases} \frac{1}{2} \int_0^T \|\dot{\varphi}(t)\|_{L_2(\lambda)}^2 dt, & \varphi \in \mathcal{H}, \\ +\infty, & \text{otherwise}. \end{cases}
\]

**THEOREM 1.4.** The family of processes \( \{y^\varepsilon\}_{\varepsilon \in (0, 1]} \) satisfies a large-deviation principle in the space \( C([0, T], L_2(\rho)) \) with the good rate function \( I; i.e., for any open set \( G \in C([0, T], L_2(\rho)) \)

\[
\lim_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}\{y^\varepsilon \in G\} \geq -\inf_G I
\]

and for any closed set \( F \)

\[
\lim_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}\{y^\varepsilon \in F\} \leq -\inf_F I.
\]

\(^3\) A function \( f(t), t \in [0, T] \), taking values in a Hilbert space \( H \) is called absolutely continuous if there exists an integrable function \( t \to h(t) \in H \) (in the Bochner sense) such that

\[
f(t) = f(0) + \int_0^t h(s) ds,
\]

and we will denote the function \( h \) by \( \dot{f} \).
Since the processes \( y^\varepsilon(\cdot) \) solve
\[
\frac{dy^\varepsilon}{ds}(s) = \text{pr}_{y^\varepsilon(s)} \sqrt{\varepsilon} dW_s.
\]
Theorem 1.4 appears as an instance of the classical Freidlin-Wentzel LDP for solutions of SDE, but here we have to deal with additional difficulties since the diffusion operator \( g \to \sigma(g) = \text{pr}_g \) is not continuous as an operator-valued map on \( L_2(\lambda) \), and generally little is known about such large-deviation principles for solutions of a SDE with nonsmooth coefficients even in finite dimensions. In our case we can overcome these difficulties with additional arguments, using the fact that \( \sigma \) is continuous on strictly monotone \( y \in L_2(\lambda) \).

**Properly Chosen Subsets** \( A \subset \mathcal{P}(\mathbb{R}) \)

In order to specify the conditions on the set \( A \) for the validity of formula (1.3), let \( \tau_\rho \) denote the image topology on \( \mathcal{P}(\mathbb{R}) \) of the \( L_2(\rho) \)-topology on \( D^\uparrow([0, 1]) \) induced from the bijection
\[
\iota : g \mapsto g#^\lambda.
\]
We call a set \( A \subset \mathcal{P}(\mathbb{R}) \) displacement convex if it is the image of a convex subset of \( D^\uparrow([0, 1]) \) under the map \( \iota \). A set is properly chosen for the validity of Varadhan’s formula as in Theorem 1.3, for instance, if it is displacement convex \( \tau_\rho \)-closed with nonempty \( \tau_\rho \)-interior.

**Remark 1.5.** It is possible to construct a process \( y \) in a similar fashion on a circle \( S \) with a proper notion of martingale on \( S \). In this case the family \( \{y^\varepsilon(\cdot) = y(\varepsilon \cdot)\}_{\varepsilon \in (0, 1]} \) will be exponentially tight in \( C([0, T], L_2(\lambda)) \), since the state space \( L_2^T(\lambda) \) is compact. Consequently, the large-deviation principle can be proved in \( C([0, T], L_2(\lambda)) \), and thus it will imply that the Varadhan formula (1.3) holds for any measurable set \( A \) that belongs to the space \( \mathcal{P}(S) \) of probability measures on \( S \) and satisfies \( \text{int} A = \overline{A} \), for instance.

**Organization of the Paper.** In Section 2 we give a streamlined review of the construction of the modified massive Arratia flow from [27] \(^4\). In Section 3 we introduce some elements of a stochastic calculus relative to \( y \) to the extent needed in what follows. Finally, Section 4 is devoted to the proof of the large-deviation principle Theorem 1.4.

## 2 Construction by a System of Coalescing Heavy Diffusion Particles

### 2.1 A Finite Number of Particles

We consider a finite system of particles that start from the points \( k_n, k = 1, \ldots, n \), with the mass \( \frac{1}{n} \), where \( n \in \mathbb{N} \) is fixed.

\(^4\) Here we construct the process directly on \([0, T]\) as a limit of particle systems, whereas in [27] the construction also included an \( \varepsilon \to 0 \) limit for a sequence of processes on \([\varepsilon, T]\).
Proposition 2.1. For each $n$ there exists a set of processes $\{x^n_k(t), k = 1, \ldots, n, t \in [0, T]\}$ that satisfies the following conditions:

(F1) for each $k$, $x^n_k$ is a continuous square-integrable martingale with respect to the filtration

$$\mathcal{F}^n_t = \sigma(x^n_l(s), s \leq t, l = 1, \ldots, n);$$

(F2) for all $k$, $x^n_k(0) = \frac{k}{n}$;

(F3) for all $k < l$ and $t \in [0, T]$, $x^n_k(t) \leq x^n_l(t)$;

(F4) for all $k$ and $l$,

$$[x^n_k, x^n_l]_t = \int_0^t \frac{I_{\{r^n_{k,l} \leq s\}}ds}{m^n_k(s)},$$

where $m^n_k(t) = \frac{1}{n} \# \{j : \exists s \leq t : x^n_j(s) = x^n_k(s)\}$, $r^n_{k,l} = \inf\{t : x^n_k(t) = x^n_l(t)\} \wedge T$, and $\#A$ denotes the number of points of $A$.

Such a system of processes can be constructed from a family of independent Wiener processes, coalescing their trajectories. Moreover, (F1)–(F4) uniquely determined the distribution of $x^n = (x^n_1, \ldots, x^n_n)$ in $(C[0, T])^n$ (see [25]).

2.2 Tightness of a Finite System in the Space $D([0, 1], C[0, T])$

Let

$$y_n(u, t) = \begin{cases} x^n_{[un]+1}(t), & u \in [0, 1), \\ x^n_{un}(t), & u = 1, \end{cases} \quad t \in [0, T].$$

Proposition 2.2. The sequence $\{y_n(u, t), u \in [0, 1], t \in [0, T]\}$ is tight in $D([0, 1], C[0, T])$.

The statement will follow from theorems 3.8.6 and 3.8.8 and remark 3.8.9 in [18]. Lemmas 2.3, 2.4, and 2.5 can be used to check conditions (8.39) and (8.30) of [18] and (a) of theorem 3.7.2 in [18], respectively.

Lemma 2.3. For all $n \in \mathbb{N}$, $u \in [0, 2]$, $h \in [0, u]$, and $\lambda > 0$,

$$\mathbb{P}\{d_\infty(y_n(u + h, \cdot), y_n(u, \cdot)) > \lambda, \ d_\infty(y_n(u, \cdot), y_n(u - h, \cdot)) > \lambda\} \leq \frac{4h^2}{\lambda^2}.$$

Here $y_n(u, \cdot) = y_n(1, \cdot)$, $u \in [1, 2]$, and $d_\infty$ is the uniform distance on $[0, T]$.

Lemma 2.4. For all $\beta > 1$

$$\lim_{\delta \to 0} \sup_{n \geq 1} \mathbb{E}[d_\infty(y_n(\delta, \cdot), y_n(0, \cdot))^{\beta} \wedge 1] = 0.$$

Lemmas 2.3 and 2.4 were proved in [27] (see lemmas 2.2 and 2.3). The following statement is a new result.

Lemma 2.5. For all $u \in [0, 1]$ the sequence $\{y_n(u, t), t \in [0, T]\}_{n \geq 1}$ is tight in $C[0, T]$. 

Proof. To prove the lemma we use the Aldous tightness criterion (see, e.g., theorem 3.6.5 in [8]); namely, we show that

(A1) for all \( t \in [0, T] \) the sequence \( \{y_n(u, t)\}_{n \geq 1} \) is tight in \( \mathbb{R} \);

(A2) for all \( r > 0 \), each set of stopping times \( \{\sigma_n\}_{n \geq 1} \) taking values in \( [0, T] \) and each sequence \( \delta_n \searrow 0 \), we have

\[
\lim_{n \to \infty} \mathbb{P}\{ |y_n(u, \sigma_n + \delta_n) - y_n(u, \sigma_n) | \geq r \} = 0.
\]

Note that (A1) follows from Chebyshev’s inequality and the estimate

\[
\mathbb{E}|y_n(u, t)| \leq \mathbb{E}|y_n(u, t) - \int_0^1 y_n(q, t) dq| + \mathbb{E}\left| \int_0^1 y_n(q, t) dq \right|
\]

\[
\leq \mathbb{E}(y_n(1, t) - y_n(0, t)) + \mathbb{E}\left| \int_0^1 y_n(q, t) dq \right|
\]

\[
= 1 + \mathbb{E}\left| \int_0^1 y_n(q, t) dq \right|,
\]

where \( \int_0^1 y_n(q, t) dq \) is a Wiener process.

Condition (A2) can be checked as follows. Similarly to the proof of lemma 2.16 in [27], we have that for each \( \alpha \in (0, \frac{3}{2}) \) there exists a constant \( C \) such that for all \( u \in [0, 1] \) and \( n \geq 1 \)

\[
\mathbb{E}\left( \frac{1}{m_n^\alpha(u, t)} \right) \leq \frac{C}{\sqrt{t}},
\]

where

\[
m_n(u, t) = \begin{cases} m_n([u]) + 1 & u \in [0, 1), \\ m_n(t) & u = 1, \\ t \in [0, T]. 
\end{cases}
\]

Thus, one can estimate

\[
\lim_{n \to \infty} \mathbb{P}\{ |y_n(u, \sigma_n + \delta_n) - y_n(u, \sigma_n) | \geq r \}
\]

\[
\leq \frac{1}{r^2} \lim_{n \to \infty} \mathbb{E}(y_n(u, \sigma_n + \delta_n) - y_n(u, \sigma_n))^2
\]

\[
= \frac{1}{r^2} \lim_{n \to \infty} \mathbb{E} \left| \int_{\sigma_n}^{\sigma_n + \delta_n} \frac{1}{m_n(u, s)} ds \right|
\]

\[
= \frac{1}{r^2} \lim_{n \to \infty} \mathbb{E} \int_0^T \mathbb{I}(\sigma_n, \sigma_n + \delta_n) \frac{1}{m_n(u, s)} ds
\]

\[
\leq \frac{1}{r^2} \lim_{n \to \infty} \left( \mathbb{E} \int_0^T \mathbb{I}(\sigma_n, \sigma_n + \delta_n) ds \right)^{\frac{1}{2}} \left( \mathbb{E} \int_0^T \frac{1}{m_n^{4/3}(u, s)} ds \right)^{\frac{3}{4}}
\]

\[
\leq (2C)^{\frac{3}{2}} T^{\frac{3}{2}} \lim_{n \to \infty} \delta_n^{1/4} = 0.
\]
2.3 Martingale Characterization of Limit Points (Proof of Theorem 1.1)

Since the space \( D([0, 1], C[0, T]) \) is Polish, the tightness implies the relative compactness of \( \{y_n(u, t), u \in [0, 1], t \in [0, T]\} \) in \( D([0, 1], C[0, T]) \). In this section we explain how one can prove that every limit point of \( \{y_n\} \) satisfies (C1)–(C4), which proves Theorem 1.1. The idea is the same as in [27].

Let \( \{y_{n'}\} \) converge to \( y \) weakly in the space \( D([0, 1], C[0, T]) \) for some subsequence \( \{y_{n'}\} \). By Skorokhod’s theorem (see [18, theorem 3.1.8]) we may suppose that the \( \{y_{n'}\} \) converge to \( y \) in \( D([0, 1], C[0, T]) \) a.s. For convenience of notation we will suppose that the \( \{y_n\} \) converge to \( y \) in \( D([0, 1], C[0, T]) \) a.s. Next, to prove the theorem, first we show that \( y_n(u, \cdot) \) tends to \( y(u, \cdot) \) in \( C[0, T] \) a.s. Note that, in general, this does not follow from convergence in the space \( D([0, 1], C[0, T]) \).

Lemmas 2.6 and proposition 9.1.17 in [21] immediately imply properties (C1)–(C3). Property (C4) can be proved by the following lemma and the representation of \( m(u, t) \) and \( m_n(u, t) \) via \( \tau_{u,v}, v \in [0, 1] \), and \( \tau_{u,v}, v \in [0, 1] \), i.e.,

\[
m(u, t) = \int_0^1 I_{\{\tau_{u,v} \leq t\}} dv, \quad m_n(u, t) = \int_0^1 I_{\{\tau_{u,v} \leq t\}} dv,
\]

similarly as was done in the proofs of lemmas 2.13 and 2.15 [27].

Corollary 2.7 and proposition 9.1.17 in [21] immediately imply properties (C1)–(C3). Property (C4) can be proved by the following lemma and the representation of \( m(u, t) \) and \( m_n(u, t) \) via \( \tau_{u,v}, v \in [0, 1] \), and \( \tau_{u,v}, v \in [0, 1] \), i.e.,

\[
m(u, t) = \int_0^1 I_{\{\tau_{u,v} \leq t\}} dv, \quad m_n(u, t) = \int_0^1 I_{\{\tau_{u,v} \leq t\}} dv,
\]

similarly as was done in the proofs of lemmas 2.13 and 2.15 [27].

Lemma 2.8. Let \( \{z_n(t), t \in [0, T]\}_{n \geq 1} \), be a sequence of continuous local martingales (not necessary with respect to the same filtration) such that for all \( n \geq 1 \) and \( s, t \in [0, \tau_n], s < t \),

\[
[z_n(\cdot)]_t - [z_n(\cdot)]_s \geq p(t - s),
\]

where \( \tau_n = \inf\{t : z_n(t) = 0\} \wedge T \) and \( p \) is a nonrandom positive constant. Let \( z(t), t \in [0, T] \), be a continuous process such that

\[
z(\cdot \wedge \tau) = \lim_{n \to \infty} z_n(\cdot \wedge \tau_n) \quad \text{in } C([0, T], \mathbb{R}) \text{ a.s.},
\]

where \( \tau = \inf\{t : z(t) = 0\} \wedge T \). Then

\[
\tau = \lim_{n \to \infty} \tau_n \text{ in probability.}
\]

Proof. The proof of this technical lemma can be found in [27] lemma 2.10.
2.4 Some Properties of the Modified Massive Arratia Flow

Let \( y \) satisfy (C1)–(C4). Then the following properties hold:

**P1** For each \( \alpha \in (0, \frac{3}{2}) \) there exists a constant \( C \) such that for all \( u \in [0, 1] \)

\[
\mathbb{E} \frac{1}{m^\alpha(u, t)} \leq \frac{C}{\sqrt{t}}, \quad t \in (0, T].
\]

**P2** There exists a constant \( C \) such that for all \( u \in [0, 1] \),

\[
\mathbb{E} \int_0^t \frac{ds}{m(u, s)} \leq C \sqrt{t}, \quad t \in [0, T].
\]

**P3** There exists a constant \( C \) such that for all \( u \in [0, 1] \)

\[
\mathbb{E}(y(u, t) - u)^2 \leq C \sqrt{t}, \quad t \in [0, T].
\]

**P4** Almost surely for all \( t \in (0, T] \) the function \( y(u, t) \), \( u \in [0, 1] \), is a step function in \( D([0, 1], \mathbb{R}) \) with a finite number of jumps. Moreover,

\[
\mathbb{P}\{\forall u, v \in [0, 1], t \in [0, T], \ y(u, t) = y(v, t)
\]

implies \( y(u, t + \cdot) = y(v, t + \cdot) \} = 1.

Remark 2.9. According to (P4), hereafter we will suppose that for all \( \omega \in \Omega \) and \( t \in [0, T] \), \( y(\cdot, t, \omega) \) is a step function in \( D([0, 1], \mathbb{R}) \) with a finite number of jumps. Also, we assume that for all \( u, v \in [0, 1], \omega \in \Omega, \) and \( t \in [0, T] \),

\( y(u, t, \omega) = y(v, t, \omega) \) implies \( y(u, t + \cdot, \omega) = y(v, t + \cdot, \omega) \).

Here (P1) is the statement of Lemma 2.16 [27]. (P2) immediately follows from (P1). Property (P3) follows from (P2) and (C4).

**Proof of (P4).** We set

\[
\Omega' = \{ \forall u, v \in [0, 1] \cap \mathbb{Q}, t \in [0, T], \ y(u, t) = y(v, t)
\]

implies \( y(u, t + \cdot) = y(v, t + \cdot) \}

\[
\cap \{ \forall n \in \mathbb{N}, \int_0^{1/n} \frac{du}{m(u, t_n)} < \infty \text{ where } t_n = \frac{1}{n} \wedge T \}.
\]

Since the set \([0, 1] \cap \mathbb{Q}\) is countable, proposition 2.3.4 in [35] and (P1) imply \( \mathbb{P}\{\Omega'\} = 1 \).

Next we prove that

\[
\text{for every } \omega \in \Omega', \ u \in [0, 1] \cap \mathbb{Q}, \ v \in [0, u], \text{ and } t \in [0, T),
\]

\[
y(u, t, \omega) = y(v, t, \omega) \text{ implies } y(u, t + \cdot, \omega) = y(v, t + \cdot, \omega).
\]

Indeed, if \( y(u, t, \omega) = y(v, t, \omega) \), then by the monotonicity of \( y(\cdot, t, \omega) \) (see (C3)), \( y(u, t, \omega) = y(\tilde{v}, t, \omega) \) for all \( \tilde{v} \in [v, u) \cap \mathbb{Q} \). Hence \( y(u, t + s, \omega) = y(\tilde{v}, t + s, \omega) \) for all \( s \in [0, T - t] \). By using the right-continuity of \( y(\cdot, t, \omega) \), we have \( y(u, t + s, \omega) = y(v, t + s, \omega) \). This proves (2.4).

Let \( \omega \in \Omega', u \in [0, 1], v \in [0, u], \text{ and } t \in [0, T) \) be fixed, and let \( y(u, t, \omega) = y(v, t, \omega) \). If we show that there exists \( \tilde{u} \in [u, 1] \cap \mathbb{Q} \) satisfying \( y(\tilde{u}, t, \omega) = y(u, t, \omega) \), then...
y(u, t, \omega), then (2.4) will immediately imply (2.3). To check this, we will use the
fact that \int_0^1 d\hat{u}/m(\hat{u}, t_n, \omega) is finite for all \( n \in \mathbb{N} \).

We fix some element \( \tilde{t} \) from \( \{ t_n, \ n \in \mathbb{N} \} \) such that \( \tilde{t} \leq t \) and assume that, for all
\( \tilde{u} \in (u, 1] \cap \mathbb{Q}, y(\tilde{u}, t, \omega) > y(u, t, \omega) \). Then the right-continuity of \( y(\cdot, t, \omega) \) and
its monotonicity imply that there exists a sequence \( \{ u_n \}_{n \geq 1} \) strongly decreasing
to \( u \) such that \( y(u_{n+1}, t, \omega) < y(u_n, t, \omega) \) for all \( n \in \mathbb{N} \). Next, we set
\[
\bar{u}_n = \inf \{ u': y(u', t, \omega) = y(u_n, t, \omega) \}, \quad n \in \mathbb{N}.
\]
Since \( y(\cdot, t, \omega) \) is right-continuous, we have \( y(\bar{u}_n, t, \omega) = y(u_n, t, \omega) \). Moreover,
\( \{ \bar{u}_n \}_{n \geq 1} \) strongly decreases to \( u \) and \( y(\bar{u}_{n+1}, t, \omega) < y(\bar{u}_n, t, \omega) \) for all \( n \in \mathbb{N} \).
Consequently, for all \( u' \in (\bar{u}_{n+1}, \bar{u}_n) \cap \mathbb{Q} \) and \( u'' \in (\bar{u}_n, \bar{u}_{n+1}) \cap \mathbb{Q}, y(u'', t, \omega) < y(u', t, \omega) \)
by the monotonicity of \( y(\cdot, t, \omega) \) and the choice of the
sequence \( \{ \bar{u}_n \}_{n \geq 1} \). Thus, \( y(u'', r, \omega) < y(u', r, \omega) \) also for each \( r \in [0, t] \), since
\( u', u'' \) are rational and \( \omega \) was taken from \( \Omega' \). Now we can estimate for every
\( \hat{u} \in (\bar{u}_{n+1}, \bar{u}_n), n \in \mathbb{N}, \)
\[
m(\hat{u}, t, \omega) = \lambda \{ u': \exists \bar{r} \leq \tilde{t} \ y(\hat{u}', r, \omega) = y(\hat{u}, r, \omega) \} \leq \bar{u}_n - \bar{u}_{n+1}.
\]
So,
\[
\int_0^1 \frac{d\hat{u}}{m(\hat{u}, \tilde{t}, \omega)} \geq \sum_{n=1}^\infty \int_{\bar{u}_{n+1}}^{\bar{u}_n} \frac{d\hat{u}}{m(\hat{u}, \tilde{t}, \omega)} \geq \sum_{n=1}^\infty \frac{\bar{u}_n - \bar{u}_{n+1}}{\bar{u}_n} = +\infty.
\]
But this contradicts the finiteness of the integral \( \int_0^1 d\hat{u}/m(\hat{u}, \tilde{t}, \omega) \). Consequently,
(2.3) holds.

Next, let \( t \in (0, T] \) be fixed. We are going to show that \( y(\cdot, t) \) is a step function
with a finite number of jumps a.s. Let \( N(t) \) be a number of distinct points of
\( B_t = \{ y(u, t), u \in [0, 1] \} \) (that can be equal to \( +\infty \) if \( B_t \) has infinitely many points). Then under (2.3) one can see that
\[
\text{Indeed, let } \pi(u, t) = \{ v : y(v, t) = y(u, t) \}, u \in [0, 1]. Then by (2.4), we have
\( m(\cdot, t) = \lambda(\pi(\cdot, t)) \) a.s. Consequently, (2.5) holds if \( N(t) \) is finite.

Next, we suppose that \( N(t) = +\infty \) and set \( A_t = \{ u : m(u, t) > 0 \} \). Note
that \( \int_0^1 \frac{du}{m(u, t)} = +\infty \) is enough to check only for the case \( \lambda(A_t) = 1 \). So,
assuming that \( \lambda(A_t) = 1 \) and using the fact that the number of distinct points of
\( B_t \) is infinite and \( y(\cdot, t) \) is nondecreasing, it is easily seen that there exists a set of
\( \{ u_k, k \in \mathbb{N} \} \subset [0, 1] \) such that \( y(u_k, t) \neq y(u_l, t) \) for all \( k \neq l \) and \( m(u_k, t) > 0, k \in \mathbb{N} \). Now, we can estimate
\[
\int_0^1 \frac{du}{m(u, t)} = \sum_{k=1}^n \int_{\pi(u_{k})} \frac{du}{m(u, t)} = \sum_{k=1}^n \int_{\pi(u_{k})} \frac{du}{\lambda(\pi(u_{k}))} = n.
\]
Letting \( n \to \infty \), we get (2.5).

Thus, \( N(t) \) must be finite a.s., by (P1).
Also, we would like to note here that (2.3) yields that, almost surely for all 
\( t \in (0, T] \), \( y(\cdot, t) \) is a step function with a finite number of jumps. □

3 Some Elements of Stochastic Analysis for the System
of Heavy Diffusion Particles

In this section \( L_2 \) will denote the space of square-integrable measurable functions on \([0, 1]\) with respect to Lebesgue measure, and \( \| \cdot \|_{L_2} \) the usual norm in \( L_2 \).

3.1 Definition of a Stochastic Integral for Predictable \( L_2 \)-Valued Processes

In this section we give a self-contained construction of the stochastic integral with respect to \( y \) with emphasis on a simpler class of integrands than, for example, in Krylov-Rozovskii [29].

As before, let \( \text{pr}_a \) denote the projection of \( a \) onto the space of \( \sigma (b) \)-measurable functions from \( L_2 \).

**Lemma 3.1.** For each \( a \in L_2 \) the process \( (y(t), a), t \in [0, T] \), is a continuous square-integrable \( (\mathcal{F}_t) \)-martingale with the quadratic variation

\[
[(y(\cdot), a)]_t = \int_0^t \| \text{pr}_{y(s)}a \|^2_{L_2} ds.
\]

**Proof.** First note that \( M(t) := (y(t), a), t \in [0, T], \) is a continuous square-integrable \( (\mathcal{F}_t) \)-martingale, since for each \( u \in [0, 1], y(u, t), t \in [0, T], \) is. Hence, it is enough to check that for all \( 0 \leq s < t \leq T \)

\[
\mathbb{E}[(M(t) - M(s))^2 \mid \mathcal{F}_s] = \mathbb{E} \left[ \int_s^t \| \text{pr}_{y(r)}a \|^2_{L_2} dr \mid \mathcal{F}_s \right].
\]

Since for each \( u, v \in [0, 1] \) the joint quadratic variation of \( y(u, \cdot) \) and \( y(v, \cdot) \) equals \( \int_s^t \frac{1}{m(u, \cdot)} dr \), we have

\[
\mathbb{E}[(M(t) - M(s))^2 \mid \mathcal{F}_s]
\]

\[
= \mathbb{E} \left( \int_0^1 \int_0^1 a(u)a(v)(y(u, t) - y(u, s))(y(v, t) - y(v, s)) du dv \mid \mathcal{F}_s \right)
\]

\[
= \mathbb{E} \left( \int_0^1 \int_0^1 a(u)a(v) \left[ \int_s^t \frac{1}{m(u, r)} dr \right] du dv \mid \mathcal{F}_s \right)
\]

\[
= \mathbb{E} \left( \int_s^t \left[ \int_0^1 \int_0^1 a(u)a(v) \frac{1}{m(u, r)} du dv \right] dr \mid \mathcal{F}_s \right).
\]
By Fubini’s theorem, we obtain
\[
\int_0^1 \int_0^1 a(u)a(v) \frac{\mathbb{1}_{\{\tau_{u,v} \leq r\}}}{m(u,r)} \, du \, dv
\]
(3.1)
\[
= \int_0^1 \frac{a(u)}{m(u,r)} \left( \int_0^1 a(v) \mathbb{1}_{\{\tau_{u,v} \leq r\}} \, dv \right) \, du
\]
\[
= \int_0^1 \frac{a(u)}{m(u,r)} \left( \int \pi(u,r) a(v) \, dv \right) \, du,
\]
where \( \pi(u,t) = \{v : y(v,t) = y(u,t)\} \). Here we have used the equality \( \pi(u,t) = \{v : \tau_{u,v} \leq r\} \), which follows from Remark 2.9. We note that for each \( \omega \) and \( r \) the operator \( \text{pr}_{y(r,\omega)} \) is a usual projection (in \( L^2 \)) onto the subspace of all \( \sigma(y(r,\omega)) \)-measurable functions, and moreover
\[
(\text{pr}_{y(r,\omega)} a)(u) = \frac{1}{m(u,r,\omega)} \int_{\pi(u,r,\omega)} a(v) \, dv
\]
because \( y(r,\omega) \) is a step function according to Remark 2.9. Consequently, the left-hand side of (3.1) equals
\[
\int_0^1 a(u)(\text{pr}_{y(r)} a)(u) \, du = \|\text{pr}_{y(r)} a\|^2_{L^2}.
\]
The lemma is proved. \( \square \)

By the polarization equality, the following corollary holds.

**Corollary 3.2.** For each \( a, b \in L^2 \) we have
\[
[(y(\cdot), a), (y(\cdot), b)]_t = \int_0^t (\text{pr}_{y(s)} a, \text{pr}_{y(s)} b) \, ds.
\]

Let \( \{e_n\}_{n \geq 1} \) be a fixed orthonormal basis in \( L^2 \), and \( f(t), t \in [0,T] \), be a predictable process taking values in \( L^2 \) with
\[
\mathbb{E} \int_0^T \|f(t)\|^2_{L^2} \, dt < \infty.
\]
(3.2)

We define the integral of \( f \) with respect to \( y \) as the series
\[
\int_0^t (f(s), dy(s)) = \int_0^t \int_0^1 f(u,s) dy(u,s) \, du
\]
(3.3)
\[
:= \sum_{n=1}^{\infty} \int_0^t (f(s), e_n) d(y(s), e_n),
\]
which converges in \( M^2 \) according to the following proposition, where \( M^2 \) denotes the space of real-valued continuous square-integrable \( (\mathcal{F}_t) \)-martingales \( M(t), t \in [0,T] \), with the norm
\[
\|M\|_{M^2} = (\mathbb{E} M^2(T))^{\frac{1}{2}}.
\]
PROPOSITION 3.3. The series \((3.3)\) converges in \(\mathcal{M}_2\) and is a continuous square-integrable \((\mathcal{F}_t)\)-martingale with the quadratic variation

\[
\left[ \int_0^t (f(s), dy(s)) \right]_t = \int_0^t \| \text{pr}_y f(s) \|_{L_2}^2 ds. 
\]

PROOF. We set for each \(n \in \mathbb{N}\)

\[
S_n(t) = \sum_{k=1}^{n} \int_0^t (f(s), e_k)d(y(s), e_k). \quad t \in [0, T].
\]

Corollary 3.2 and a simple calculation yield that \(S_n\) belongs to \(\mathcal{M}_2\) and has the quadratic variation

\[
[S_n]_t = \int_0^t \left\| \text{pr}_y \sum_{k=1}^{n} f_k(s)e_k \right\|_{L_2}^2 ds,
\]

where \(f_k(s) := (f(s), e_k)\). Moreover, for each \(1 \leq n < p\),

\[
\|S_p - S_n\|_{\mathcal{M}_2}^2 = \mathbb{E}(S_p(T) - S_n(T))^2 \leq \mathbb{E} \int_0^T \left\| \sum_{k=n+1}^{p} f_k(t)e_k \right\|_{L_2}^2 dt.
\]

By the dominated convergence theorem and assumption (3.2), \(\|S_p - S_n\|_{\mathcal{M}_2} \to 0\) as \(n, p \to \infty\). Thus, the sequence \(\{S_n\}_{n \geq 1}\) converges in \(\mathcal{M}_2\) by the completeness of \(\mathcal{M}_2\). Next, (3.4) follows from lemma B.11 in [7]. The proposition is proved. \(\square\)

Remark 3.4. Let \(f(t), t \in [0, T]\), be a predictable \(L_2\)-valued process such that

\[
\int_0^T \| f(t) \|_{L_2}^2 dt < \infty \quad \text{a.s.}
\]

Then using a localization sequence of stopping times, one can define the stochastic integral \(\int_0^T (f(s), dy(s))\) that is a continuous local square-integrable \((\mathcal{F}_t)\)-martingale with the quadratic variation given by (3.4).

3.2 Girsanov’s Theorem

In this section we construct a system of coalescing diffusion particles with drift that will be needed in Section 4.2 for the proof of the lower bound in LDP. So, we fix a predictable \(L_2\)-valued process \(\varphi\) satisfying (3.5) and consider on \((\Omega, \mathcal{F})\) the new measure

\[
\mathbb{P}^\varphi(A) = \mathbb{E}_A \exp \left\{ \int_0^T (\varphi(s), dy(s)) - \frac{1}{2} \int_0^T \| \text{pr}_y \varphi(t) \|_{L_2}^2 dt \right\}, \quad A \in \mathcal{F}.
\]

If

\[
\mathbb{E} \exp \left\{ \int_0^T (\varphi(s), dy(s)) - \frac{1}{2} \int_0^T \| \text{pr}_y \varphi(t) \|_{L_2}^2 dt \right\} = 1,
\]

then \(\mathbb{P}^\varphi\) is a probability measure.
Theorem 3.5. Let $\varphi$ satisfy (3.6). Then the random element $\{y(u,t), u \in [0,1], \ t \in [0,T]\}$ in $D([0,1], C[0,T])$ satisfies the following properties under $P^\varphi$:

(D1) for all $u \in [0,1]$ the process

$$\eta(u, \cdot) = y(u, \cdot) - \int_0^\cdot (\text{pr}_{y(s)} \varphi(s))(u) \, ds$$

is a continuous local square-integrable $(\mathcal{F}_t)$-martingale;

(D2) for all $u \in [0,1], \ y(u,0) = u$;

(D3) for all $u < v$ from $[0,1]$ and $t \in [0,T]$, $y(u,t) \leq y(v,t)$;

(D4) for all $u, v \in [0,1]$ and $t \in [0,T]$,

$$[\eta(u, \cdot), \eta(v, \cdot)]_t = \int_0^t \frac{\mathbb{1}_{\{\tau_{u,v} \leq s\}} ds}{m(u,s)}.$$

Note that (D2) and (D3) immediately follow from the absolute continuity of $P^\varphi$.

To prove (D1) and (D4) we state an auxiliary lemma.

Lemma 3.6. For each $u \in [0,1]$

$$y(u,t) = u + \int_0^t \int_0^1 \frac{\mathbb{1}_{\pi(u,s-)}(q)}{m(u,s-)} \, dy(q,s) dq.$$

Proof. Setting $f(q,s) = \frac{\mathbb{1}_{\pi(u,s-)}(q)}{m(u,s-)}$ and using (P2), we have

$$E \int_0^T \|f(s)\|_{L^2}^2 ds = \int_0^T \int_0^1 \frac{\mathbb{1}_{\pi(u,s-)}(q)}{m^2(u,s-)} \, dq \, ds \, dq$$

$$= \int_0^T \int_0^1 \frac{1}{m^2(u,s)} \int_\pi(u,s) dq \, ds$$

$$= \int_0^T \frac{1}{m(u,s)} ds < \infty.$$

Next, put

$$\sigma_0 = t,$$

$$\sigma_k = \inf \{s : N(s) \leq k \} \wedge t, \quad k \in \mathbb{N},$$

where $N(t) = \int_0^1 \frac{1}{m(u,t)} \, du$, $t \in [0,T]$, denotes a number of distinct points in $\{y(u,t), u \in [0,1]\}$ and is an $(\mathcal{F}_t)$-adapted càdlàg process, and note that $\sigma_k$ is an $(\mathcal{F}_t)$-stopping time, $\sigma_k \geq \sigma_{k+1}$. So,

$$\int_0^t \int_0^1 f(q,s) \, dy(q,s) dq$$

$$= \sum_{k=0}^\infty \int_{\sigma_k}^{\sigma_{k+1}} \int_0^1 \frac{\mathbb{1}_{\pi(u,s-)}(q)}{m(u,s-)} \, dq \, dy(q,s) dq = \int_0^T \frac{1}{m(u,s)} ds < \infty.$$
\[
\begin{align*}
&= \sum_{k=0}^{\infty} \int_0^1 \frac{\|\pi(u,\sigma_{k+1})(q)\|}{m(u, \sigma_{k+1})} (y(q, \sigma_k) - y(q, \sigma_{k+1})) dq \\
&= \sum_{k=0}^{\infty} \int_0^1 \pi(u,\sigma_{k+1}) dq \frac{1}{m(u, \sigma_{k+1})} (y(u, \sigma_k) - y(u, \sigma_{k+1})) \\
&= \sum_{k=0}^{\infty} (y(u, \sigma_k) - y(u, \sigma_{k+1})) = y(u, t) - u.
\end{align*}
\]

The lemma is proved. \qed

**Corollary 3.7.** For each predictable \( L_2 \)-valued process \( f \) satisfying (3.5) and \( u \in [0, 1] \)

\[
\left[ \int_0^t (f(s), dy(s)), y(u, \cdot) \right]_t = \int_0^t (pr_y(s)f(s))(u) ds, \quad t \in [0, T].
\]

**Proof of Theorem 3.5.** The proof of the assertion follows from Girsanov’s theorem (see theorem 5.4.1 in [20]) and Corollary 3.7. \qed

**Remark 3.8.** For predictable \( L_2 \)-valued functions \( f \) satisfying (3.2) or (3.5) we can construct the stochastic integral with respect to the flow \( \{y(u, t), u \in [0, 1], t \in [0, T]\} \) satisfying conditions (D1)–(D4) in the same way as in the case of conditions (C1)–(C4). Moreover,

\[
\int_0^t (f(s), dy(s)) = \int_0^t (f(s), pr_y(s)\varphi(s)) ds + \int_0^t (f(s), d\eta(s))
\]

and \( \int_0^t (f(s), \varphi(s)) ds \) is a continuous square-integrable (resp., local square-integrable) \((\mathcal{F}_t)\)-martingale with the quadratic variation

\[
\left[ \int_0^t (f(s), d\eta(s)) \right]_t = \int_0^t \|pr_y(s)f(s)\|_{L_2}^2 ds.
\]

**4 Large-Deviation Principle for the Modified Massive Arratia Flow**

**4.1 Exponential Tightness**

In this section we prove exponential tightness of the modified massive Arratia flow. In order to prove this we will use an “exponentially fast” version of Jakubowski’s tightness criterion (see theorem A.1 in [11]). So, let \( \{y(u, t), u \in [0, 1], t \in [0, T]\} \) be a random element in \( D([0, 1], C[0, T]) \) satisfying (C1)–(C4) and \( \rho(du) = \kappa(u) du \), where \( \kappa \) is given by (1.4).

Since \( y(\cdot, t), t \in [0, T], \) is a continuous \( L_2(\rho) \)-valued process, we will establish exponential tightness of \( \{y^\varepsilon\}_{\varepsilon \in (0, 1]} \) in the space \( C([0, T], L_2(\rho)) \), where \( y^\varepsilon(t) = y(\cdot, \varepsilon t), t \in [0, T] \).
By theorem A.1 of [11], \( \{ y^\varepsilon \}_{\varepsilon \in (0,1)} \) is exponentially tight in \( C([0,T], L_2(\rho)) \); i.e., for every \( M > 0 \) there exists a compact \( K_M \subset C([0,T], L_2(\rho)) \) such that
\[
\lim_{\varepsilon \to 0} \varepsilon \ln \mathbb{P} \{ y^\varepsilon \notin K_M \} \leq -M
\]
if and only if

(E1) for every \( M > 0 \) there exists a compact \( K_M \subset L_2(\rho) \) such that
\[
(4.1) \quad \lim_{\varepsilon \to 0} \varepsilon \ln \mathbb{P} \{ \exists t \in [0,T] : y^\varepsilon(t) \notin K_M \} \leq -M;
\]

(E2) for every \( h \in L_2(\rho) \) the sequence \( \{ (h, y^\varepsilon(\cdot))_{L_2(\rho)} \}_{\varepsilon \in (0,1)} \) is exponentially tight in \( C([0,T], \mathbb{R}) \), where \((\cdot, \cdot)_{L_2(\rho)}\) denotes the inner product in \( L_2(\rho) \).

Since \( y(u,t), u \in [0,1], \) is nondecreasing for all \( t \in [0,T], \) to find the compact \( K_M \subset L_2(\rho) \) satisfying (4.1) it suffices to control the behavior of processes \( y(u,t), t \in [0,T], \) for \( u \) close to 0 or 1. Note that the diffusion rate of the process \( y(u,t), t \in [0,T], \) tends to infinity as \( t \to 0. \) But

\[
(4.2) \quad M_*(u,t) \leq y(u,t) \leq M^*(u,t), \quad t \in [0,T],
\]

where
\[
M^*(u,t) = \frac{1}{1-u} \int_u^1 y(v,t) dv, \quad M_*(u,t) = \frac{1}{u} \int_0^u y(v,t) dv,
\]
and
\[
(4.3) \quad \frac{d[M^*(u,\cdot)]_t}{dt} \leq \frac{1}{1-u}, \quad \frac{d[M_*(u,\cdot)]_t}{dt} \leq \frac{1}{u}.
\]
The latter inequalities follow from (3.4) and the simple relations
\[
M^*(u,t) = \frac{1}{1-u} \int_0^t (\mathbb{I}_{[u,1]}, dy(s)), \quad M_*(u,t) = \frac{1}{u} \int_0^t (\mathbb{I}_{[0,u]}, dy(s)),
\]
where these sorts of integrals were defined in Section 3.1. Indeed,
\[
\frac{d[M^*(u,\cdot)]_t}{dt} = \frac{1}{(1-u)^2} \| \mathbb{I}_{[u,1]} \|_{L_2(\lambda)}^2 \leq \frac{1}{(1-u)^2} \| \mathbb{I}_{[u,1]} \|_{L_2(\lambda)}^2 = \frac{1}{1-u}.
\]
The inequality \( \frac{d[M_*(u,\cdot)]_t}{dt} \leq \frac{1}{u} \) can be obtained in the same way.

In Lemma 4.2 we will use inequalities (4.2) in order to find a compact \( K_M \subset L_2(\rho) \) for each \( M > 0 \) such that (4.1) holds.

Since \( y(u,t), u \in [0,1], \) belongs to \( D([0,1], \mathbb{R}) \) and is a nondecreasing function, we will often work with nondecreasing functions
\[
D^\dagger = \{ h \in D([0,1], \mathbb{R}) : h \text{ is nondecreasing} \}.
\]

**Lemma 4.1.** The set
\[
A_M = \{ h \in D^\dagger : h(1/n) \geq -Mn \text{ and } h(1-1/n) \leq Mn, \quad n \in \mathbb{N} \}
\]
is compact in \( L_2(\rho) \) for all positive \( M \).
PROOF. First we prove that \( A_M \subset L_2(\rho) \). Let \( h \in A_M \). Without loss of generality, let \( h \) be positive on \([\frac{1}{2}, 1]\) and negative on \([0, \frac{1}{2}]\). Then
\[
\int_0^1 h^2(u)\rho(du) = \int_0^1 h^2(u)\kappa(u)du \leq C \sum_{n=2}^{\infty} \frac{M^2n^2}{n^\beta} \left( \frac{1}{n-1} - \frac{1}{n} \right) < C_1
\]
and \( C_1 \) is independent of \( h \).

Next, take a sequence \( \{h_k\}_{k \geq 1} \) in \( A_M \). Since \( \{h_k\}_{k \geq 1} \subset D^\dagger \), there exists a subsequence \( \{h_{k'}\} \) that converges to \( h \in D^\dagger \) for all \( u \in [0, 1] \) except possibly countably many points, and hence also \( \rho \)-a.e. Since \( |h_k(u)| \leq f(u), u \in [0, 1] \), where
\[
f(u) = \begin{cases} 
Mn, & u \in [1 - 1/(n - 1), 1 - 1/n), \\
Mn, & u \in [1/n, 1/(n - 1)),
\end{cases}
\]
and \( f \in L_2(\rho) \), \( \|h_{k'}\|_{L_2(\rho)} \to \|h\|_{L_2(\rho)} \), by the dominated convergence theorem. Consequently, this and lemma 1.32 in [23] imply \( h_{k'} \to h \) in \( L_2(\rho) \). The lemma is proved.

**Lemma 4.2.** The family of processes \( \{y^\varepsilon\}_{\varepsilon \in (0, 1]} \) satisfies (E1); i.e., for every \( M > 0 \) there exists a compact \( K_M \subset L_2(\rho) \) such that (4.1) holds.

**Proof.** By Lemma 4.1 we can take \( K_M = A_L \) and show that for some \( L > 0 \) (4.1) holds. So,
\[
P\{\exists t \in [0, T] : y^\varepsilon(t) \notin A_L \} \leq \sum_{n=1}^{\infty} P\{\exists t \in [0, T] : y^\varepsilon(1/n, t) < -Ln \} + \sum_{n=1}^{\infty} P\{\exists t \in [0, T] : y^\varepsilon(1 - 1/n, t) > Ln \}.
\]
Using (4.2) and (4.3), we estimate for fixed \( n \in \mathbb{N} \)
\[
P\{\exists t \in [0, T] : y(1 - 1/n, \varepsilon t) > Ln \}
\]
\[
= P\left\{ \sup_{t \in [0, T]} y(1 - 1/n, \varepsilon t) > Ln \right\}
\]
\[
\leq P\left\{ \sup_{t \in [0, T]} M^\varepsilon (1 - 1/n, \varepsilon t) > Ln \right\}
\]
\[
\leq P\left\{ \sup_{t \in [0, T]} (w_n(n\varepsilon t) + 1) > Ln \right\}
\]
\[
= \frac{2}{\sqrt{2\pi n\varepsilon T}} \int_{Ln-1}^{\infty} e^{-\frac{x^2}{2\varepsilon T}} dx \leq C \exp\left\{-\frac{L^2n}{2\varepsilon T} + \frac{L}{\varepsilon T}\right\},
\]
where \( C \) is independent of \( \varepsilon, L, \) and \( n \).
Similarly,
\[ \mathbb{P}\{\exists t \in [0, T] : y(1/n, \varepsilon t) < -LT\} \leq C \exp\left\{ -\frac{L^2 n}{2 \varepsilon T} + \frac{L}{\varepsilon T} \right\}. \]

Now, for \( M > 0 \) we can estimate
\[
\lim_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}\{ \exists t \in [0, T] : y^\varepsilon(t) \notin A_L \}
\leq \lim_{\varepsilon \to 0} \varepsilon \ln \left( 2C \sum_{n=1}^{\infty} \exp\left\{ -\frac{L^2 n}{2 \varepsilon T} + \frac{L}{\varepsilon T} \right\} \right)
\leq -\frac{L^2}{2T} + \frac{L}{T} < -M,
\]
where \( L \) is taken large enough. The lemma is proved. \( \square \)

**Lemma 4.3.** The sequence of processes \( \{y^\varepsilon\}_{\varepsilon \in (0, 1]} \) satisfies (E2); specifically, for every \( h \in L^2(\rho) \) the sequence \( \{(h, y^\varepsilon(\cdot))_{L^2(\rho)}\}_{\varepsilon \in (0, 1]} \) is exponentially tight in \( C([0, T], \mathbb{R}) \).

**Proof.** To prove the lemma, we will use corollary 7.1 in [37] (see also theorem 3 in [36]). It is enough to show that for each \( h \) there exist positive constants \( \alpha, \gamma, \) and \( k \) such that for all \( s, t \in [0, T], s < t, \)
\[
\mathbb{E} \exp\left\{ \frac{\gamma}{(t-s)^\alpha} |M_h(\varepsilon t) - M_h(\varepsilon s)| \right\} \leq k^{1/\varepsilon} \quad \forall \varepsilon \leq \varepsilon_0,
\]
where \( M_h(t) = (h, y(t))_{L^2(\rho)}, \ t \in [0, T]. \)

Using (3.4), for
\[
M_h(t) = \int_0^1 h(u) y(u, t) \kappa(u) du = \int_0^t (h\kappa, dy(s))_{L^2(\lambda)}
\]
we have
\[
[M_h]_t = \int_0^t \|\text{pr}_y(h\kappa)\|_{L^2(\lambda)}^2 ds \leq \int_0^t \|h\kappa\|_{L^2(\lambda)}^2 ds \leq \|h\|_{L^2(\rho)}^2 t.
\]
The inequality for the quadratic variation of \( M_h \) and Novikov’s theorem imply
\[
\mathbb{E} \exp\left\{ \delta \int_s^t (h\kappa, dy(r))_{L^2(\lambda)} - \frac{\delta^2}{2} \int_s^t \|\text{pr}_y(h\kappa)\|_{L^2(\lambda)}^2 dr \right\} = 1.
\]
So, for \( \delta > 0 \)
\[
\mathbb{E} \exp\{\delta |M_h(\varepsilon t) - M_h(\varepsilon s)|\}
\leq \mathbb{E} \exp\{\delta (M_h(\varepsilon t) - M_h(\varepsilon s))\} + \mathbb{E} \exp\{\delta (M_h(\varepsilon s) - M_h(\varepsilon t))\} =
\]
\[\begin{align*}
&= \mathbb{E} \exp \left\{ \delta \int_{t}^{s} (h \kappa, dy(r))_{L_{2}^{+}(\lambda)} - \frac{\delta^{2}}{2} \int_{t}^{s} \| \text{pr}_{y}(r) (h \kappa) \|_{L_{2}^{+}(\lambda)}^{2} dr \\
&\quad + \frac{\delta^{2}}{2} \int_{t}^{s} \| \text{pr}_{y}(r) (h \kappa) \|_{L_{2}^{+}(\lambda)}^{2} dr \right\} + \mathbb{E} \exp \{ \delta (M_{-h}(\varepsilon t) - M_{-h}(\varepsilon s)) \}
&\leq 2 \mathbb{E} \exp \left\{ \frac{\varepsilon \delta^{2}}{2} \| h \|_{L_{2}^{+}(\rho)}^{2} (t - s) \right\}.
\end{align*}\]

Taking \( \delta = \frac{\sqrt{2}}{\varepsilon(t-s)^{1/2} \| h \|_{L_{2}^{+}(\rho)}} \), we have

\[\mathbb{E} \exp \left\{ \frac{\sqrt{2} | M_{h}(\varepsilon t) - M_{h}(\varepsilon s) |}{\varepsilon \| h \|_{L_{2}^{+}(\rho)}(t - s)^{1/2}} \right\} \leq 2e^{1/\varepsilon} \leq (2e)^{1/\varepsilon}.\]

This finishes the proof of the lemma. \(\square\)

From the two previous lemmas we get the exponential tightness of \( \{ y^{\varepsilon} \}_{\varepsilon \in (0,1]} \).

**Proposition 4.4.** The sequence \( \{ y^{\varepsilon} \}_{\varepsilon \in (0,1]} \) is exponentially tight in \( C([0,T], L_{2}(\rho)) \).

### 4.2 Proof of Theorem 1.4

We set

\[ L_{2}^{+}(\rho) = \{ g \in L_{2}(\rho) : \exists \tilde{g} \in D^{+}, g = \tilde{g}, \rho\text{-a.e.} \} \]

and

\[ C_{id}([0,T], L_{2}^{+}(\rho)) = \{ \varphi \in C([0,T], L_{2}^{+}(\rho)) : \varphi(0) = \text{id} \}. \]

**Remark 4.5.** Since the set \( C_{id}([0,T], L_{2}^{+}(\rho)) \) is closed in \( C([0,T], L_{2}(\rho)) \), it is enough to state LDP for \( \{ y^{\varepsilon} \}_{\varepsilon \in (0,1]} \) in the metric space \( C_{id}([0,T], L_{2}^{+}(\rho)) \).

Due to the exponential tightness, for the upper bound it is enough to consider compact sets. According to [10, theorem 4.1.11], it suffices to show that \( I \) is a lower-semicontinuous function and

- (B1) weak upper bound:
  \[ \lim_{r \to 0} \lim_{\varepsilon \to 0} \varepsilon \ln \mathbb{P} \{ y^{\varepsilon} \in B_{r}(\varphi) \} \leq -I(\varphi), \]

  where \( \varphi \in C_{id}([0,T], L_{2}^{+}(\rho)) \) and \( B_{r}(\varphi) \) is the open ball in \( C_{id}([0,T], L_{2}^{+}(\rho)) \) with center \( \varphi \) and radius \( r \);

- (B2) lower bound: for every open set \( A \subseteq C_{id}([0,T], L_{2}^{+}(\rho)) \)
  \[ \lim_{\varepsilon \to 0} \varepsilon \ln \mathbb{P} \{ y^{\varepsilon} \in A \} \geq - \inf_{\varphi \in A} I(\varphi). \]

To prove the upper and lower bounds we will follow the idea in [12,13] based on exponential change of measure and the Girsanov transformation.
Upper Bound

First we check (B1). We set

\[ H = \{ h \in C([0, T], L_2(\rho^{-1})) : \hat{h} \in L_2([0, T], L_2(\rho^{-1})) \}, \]

where \( \rho^{-1}(du) = \frac{1}{x(u)} du \).

For \( h \in H \) let

\[ (4.5) \quad M^{\varepsilon,h}_t = \exp \left\{ \frac{1}{\varepsilon} \left[ \int_0^t (h(s), dy^\varepsilon(s))_{L_2(\lambda)} - \frac{1}{2} \int_0^t \| \text{pr}_{y^\varepsilon(s)} h(s) \|_{L_2(\lambda)}^2 ds \right] \right\}. \]

By Novikov’s theorem, \( M^{\varepsilon,h}_t, t \in [0, T] \), is a martingale with \( \mathbb{E} M^{\varepsilon,h}_T = 1 \) (see also (4.4)). By an integration by parts (Lemma A.1), we can write

\[ M^{\varepsilon,h}_T = \exp \left\{ \frac{1}{\varepsilon} F(y^\varepsilon, h) \right\}, \]

where

\[ F(\varphi, h) = (h(T), \varphi(T))_{L_2(\lambda)} - (h(0), \text{id})_{L_2(\lambda)} \]

\[ - \int_0^T (\hat{h}(s), \varphi(s))_{L_2(\lambda)} ds \]

\[ - \frac{1}{2} \int_0^T \| \text{pr}_{y^\varepsilon(s)} h(s) \|_{L_2(\lambda)}^2 ds, \quad \varphi \in C_{\text{id}}([0, T], L_2^T(\rho)). \]

For \( \varphi \in C_{\text{id}}([0, T], L_2^T(\rho)) \) we have

\[ \mathbb{P}\{ y^\varepsilon \in B_r(\varphi) \} = \mathbb{E} \left[ \mathbb{I}_{\{ y^\varepsilon \in B_r(\varphi) \}} \frac{M^{\varepsilon,h}_T}{M^{\varepsilon,h}_T} \right] \]

\[ \leq \exp \left\{ - \frac{1}{\varepsilon} \inf_{\varphi \in B_r(\varphi)} F(\varphi, h) \right\} \mathbb{E} M^{\varepsilon,h}_T \]

\[ = \exp \left\{ - \frac{1}{\varepsilon} \inf_{\varphi \in B_r(\varphi)} F(\varphi, h) \right\}. \]

Using the inequality \( \| \text{pr}_{y^\varepsilon(s)} h(s) \|_{L_2(\lambda)}^2 \leq \| h(s) \|^2_{L_2(\lambda)} \), we obtain

\[ \lim_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}\{ y^\varepsilon \in B_r(\varphi) \} \leq - \inf_{\varphi \in B_r(\varphi)} F(\varphi, h) \leq - \inf_{\varphi \in B_r(\varphi)} \Phi(\varphi, h), \]

where

\[ \Phi(\varphi, h) = (h(T), \varphi(T))_{L_2(\lambda)} - (h(0), \text{id})_{L_2(\lambda)} \]

\[ - \int_0^T (\hat{h}(s), \varphi(s))_{L_2(\lambda)} ds \]

\[ - \frac{1}{2} \int_0^T \| h(s) \|_{L_2(\lambda)}^2 ds, \quad \varphi \in C_{\text{id}}([0, T], L_2^T(\rho)). \]
Since the map $\Phi(\varphi, h), \varphi \in C[id([0, T], L^2_2(\rho)))$, is continuous for fixed $h$,
$$\lim_{r \to 0} \lim_{\varepsilon \to 0} \varepsilon \ln P\{y^\varepsilon \in B_r(\varphi)\} \leq -\Phi(\varphi, h).$$
Minimizing in $h \in H$, we obtain
$$\lim_{r \to 0} \lim_{\varepsilon \to 0} \varepsilon \ln P\{y^\varepsilon \in B_r(\varphi)\} \leq -\sup_{h \in H} \Phi(\varphi, h).$$

Now (B1) will follow from the following.

**Proposition 4.6.** For each $\varphi \in C[id([0, T], L^1_2(\rho)))$, 
$$\sup_{h \in H} \Phi(\varphi, h) = I(\varphi).$$

**Proof.** First we prove the assertion of the proposition for $\varphi$ satisfying
$$J(\varphi) := \sup_{h \in H} \Phi(\varphi, h) < \infty.$$
Replacing $h$ by $\theta h$, $\theta \in \mathbb{R}$, and maximizing the expression $\Phi(\varphi, \theta h)$ over $\theta$ for fixed $h$, we get
$$J(\varphi) = \frac{1}{2} \sup_{h \in H} \int_0^T \frac{G^2(\varphi, h)}{||h(s)||^2_{L^2_2(\lambda)}} ds < \infty.$$ 

By (4.6) and Lemma A.2 the linear map
$$G : h \to G(\varphi, h)$$
can be extended to the space $L_2([0, T], L_2(\lambda))$ and consequently,
$$G(\varphi, h) = \int_0^T (k_\varphi(s), h(s)) ds$$
for some function $k_\varphi \in L_2([0, T], L_2(\lambda))$. Thus, by the integration-by-parts formula for Bochner integrals, $\varphi$ is absolutely continuous and $\dot{\varphi} = k_\varphi$. Applying the Cauchy-Schwarz inequality to (4.7) we get
$$G(\varphi, h)^2 \leq 2I(\varphi) \int_0^T ||h(s)||^2_{L^2_2(\lambda)} ds.$$ 
This yields $J(\varphi) \leq I(\varphi)$, and since $H$ is dense in $L_2([0, T], L_2(\lambda))$, we get the equality $J(\varphi) = I(\varphi)$.

If $I(\varphi) < \infty$, then $\varphi$ is absolutely continuous and $k_\varphi = \dot{\phi}$ in (4.7). So, $J(\varphi) \leq I(\varphi) < \infty$, and consequently we have $J(\varphi) = I(\varphi)$. This completes the proof of the proposition.

**Corollary 4.7.** $I$ is lower-semicontinuous as a supremum of continuous functions.
Lower Bound

In order to obtain the lower bound (B2), it is enough to find a subset $\mathcal{R} \subset C_{id}([0, T], L_2^\uparrow(\rho))$ such that for each $\varphi \in \mathcal{R}$

$$\lim_{r \to 0} \lim_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}[y^\varepsilon \in B_r(\varphi)] \geq -I(\varphi),$$

and prove that for each $\varphi$ satisfying $I(\varphi) < \infty$, there exists a sequence $\{\varphi_n\} \subset \mathcal{R}$ such that $\varphi_n \to \varphi$ in $C_{id}([0, T], L_2^\uparrow(\rho))$ and $I(\varphi_n) \to I(\varphi)$.

We denote

$$D \uparrow \uparrow = \{g \in D([0, 1], \mathbb{R}) : \forall u < v \in [0, 1], g(u) < g(v)\}$$

and define $L_2^\uparrow(\rho)$ in the same way as $L_2^{\uparrow}(\rho)$, replacing $D\uparrow$ by $D \uparrow \uparrow$. Set

$$\mathcal{R} = \left\{ \varphi \in C([0, T], L_2^\uparrow(\lambda)) : I(\varphi) < \infty, \varphi \in H_{L_2^\uparrow(\lambda)}, \left. \varphi\right|_{[0, t]} \text{ is continuously differentiable in } u \text{ with bounded (uniformly in } t, u) \right\},$$

where

$$H_{L_2^\uparrow(\lambda)} = \{h \in C([0, T], L_2(\lambda)) : \dot{h} \in L_2([0, T], L_2(\lambda))\}.$$

For $h \in H_{L_2^\uparrow(\lambda)}$ define the new probability measure $\mathbb{P}^{\varepsilon, h}$ with density

$$\frac{d\mathbb{P}^{\varepsilon, h}}{d\mathbb{P}} = M^{\varepsilon, h}_T,$$

where $M^{\varepsilon, h}_T$ is defined by (4.5). By Novikov’s theorem and Theorem 3.5 the random element $y^\varepsilon$ in $D([0, 1], C[0, T])$ satisfies (w.r.t. $\mathbb{P}^{\varepsilon, h}$) the following properties:

\begin{itemize}
  \item [(D\varepsilon 1)] for all $u \in [0, 1]$ the process
    $$\eta^\varepsilon(u, \cdot) = y^\varepsilon(u, \cdot) - \int_0^r (\text{pr}_{y^\varepsilon(s)}h(s))(u)ds$$
    is a continuous local square-integrable $(\mathcal{F}_t)$-martingale;
  \item [(D\varepsilon 2)] for all $u \in [0, 1]$, $y^\varepsilon(u, 0) = u$;
  \item [(D\varepsilon 3)] for all $u < v$ from $[0, 1]$ and $t \in [0, T]$, $y^\varepsilon(u, t) \leq y^\varepsilon(v, t)$;
  \item [(D\varepsilon 4)] for all $u, v \in [0, 1]$ and $t \in [0, T]$,
    $$[y^\varepsilon(u, \cdot), \eta^\varepsilon(v, \cdot)]_t = \varepsilon \int_0^t \frac{\mathbb{1}_{\{v^\varepsilon(u, s) \leq s\}}ds}{m^\varepsilon(u, s)},$$
    where $\tau^\varepsilon$ and $m^\varepsilon$ is defined in the same way as $\tau$ and $m$, replacing $y$ by $y^\varepsilon$.
\end{itemize}

Note that if $\varphi \in \mathcal{R}$, then

$$\lim_{\varepsilon \to 0} \mathbb{P}^{\varepsilon, \varphi}[y^\varepsilon \in B_r(\varphi)] = 1$$

for all $r > 0$, by Proposition B.1.
Let us fix $\varphi \in \mathcal{R}$ and set $h = \psi$. Noting that $y^\varepsilon \in L_2([0, T], L_2(\lambda))$ a.s., we estimate
\[
\mathbb{P}\{y^\varepsilon \in B_\varepsilon(\varphi)\} = \mathbb{E}^{\varepsilon, h} \frac{\mathbb{I}_{\{y^\varepsilon \in B_\varepsilon(\varphi)\}}}{M_{\varepsilon}^{T, h}} \geq \exp\left\{-\frac{1}{\varepsilon} \sup_{\psi \in B_\varepsilon(\varphi) \cap L_2([0, T], L_2(\lambda))} F(\psi, h)\right\} \mathbb{P}^{\varepsilon, h}\{y^\varepsilon \in B_\varepsilon(\varphi)\},
\]
where $\mathbb{E}^{\varepsilon, h}$ denotes the expectation w.r.t. $\mathbb{P}^{\varepsilon, h}$. Thus, by (4.10),
\[
(4.11) \lim_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}\{y^\varepsilon \in B_\varepsilon(\varphi)\} \geq \sup_{\psi \in B_\varepsilon(\varphi) \cap L_2([0, T], L_2(\lambda))} F(\psi, h).
\]

Next we prove the continuity of the map $g \mapsto \text{pr}_g f$ on $L^\uparrow_2(\rho)$ for each $f \in L_2(\lambda)$.

**Lemma 4.8.** Let $g \in L^\uparrow_2(\rho)$ and $f \in L_2(\lambda)$. If a sequence $\{g_n\}_{n \geq 1}$ of elements $L^\uparrow_2(\rho)$ converges to $g$ a.e., then $\{\text{pr}_{g_n} f\}_{n \geq 1}$ converges to $f$ in $L_2(\lambda)$ and
\[
(4.12) \lim_{n \to \infty} \|\text{pr}_{g_n} f\|_{L_2(\lambda)} = \|f\|_{L_2(\lambda)}.
\]

**Proof.** First we note that $\sigma(g)$ is a Borel $\sigma$-algebra on $[0, 1]$. Moreover, since $\{g_n\}_{n \geq 1}$ converges to $g$ a.e., for almost all $a, b \in [0, 1]$ and $a < b$ one can show that there exists a sequence $\{c_n, d_n, n \in \mathbb{N}\}$ such that $\lambda((a, b) \setminus g_n^{-1}(c_n, d_n)) \to 0$ as $n \to \infty$. This immediately implies that for all $A \in \sigma(g)$ there exist $A_n \in \sigma(g_n), n \in \mathbb{N}$, such that $\lambda(A \setminus A_n) \to 0$. Thus, by [2] prop. 1, $\text{pr}_{g_n} f \to \text{pr}_g f = f$ in $L_2$. Consequently, we also have $\|\text{pr}_{g_n} f\|_{L_2(\lambda)} \to \|f\|_{L_2(\lambda)}$. The lemma is proved. \(\square\)

So, by Lemma 4.8 (4.11) yields
\[
\lim_{r \to 0} \lim_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}\{y^\varepsilon \in B_\varepsilon(\varphi)\} \geq -F(\varphi, h) = -I(\varphi).
\]

Here the last equality follows from the form of the map $F$, the choice of $h$, and Lemma A.1.

**Proposition 4.9.** For each $\varphi \in \mathcal{H}$, where $\mathcal{H}$ is defined by (1.5) satisfying $I(\varphi) < \infty$, there exists a sequence $\{\varphi_n\} \subset \mathcal{R}$ such that $\varphi_n \to \varphi$ in $C_\text{id}([0, T], L^\uparrow_2(\rho))$ and $I(\varphi_n) \to I(\varphi)$.

**Proof.** First we note that it is enough to check the statement only for functions $\varphi$ with bounded derivative, i.e.,
\[
\sup_{(u, t) \in [0, 1] \times [0, T]} |\dot{\varphi}(u, t)| < \infty.
\]

Then the proposition can be proved using the approximation of $\dot{\varphi}$ in $L_2([0, T], L_2(\lambda))$ by functions
\[
\psi_{\delta, \alpha}(u, t) = u + \int_0^t \tilde{\varphi} * \varsigma_\delta(u, s) ds + \alpha tu, \quad u \in [0, 1], \ t \in [0, T], \ \alpha, \delta > 0,
\]
where

\[
\tilde{\phi}(u, t) = \begin{cases} 
\dot{y}(u, t), & (u, t) \in [0, 1] \times [0, T], \\
C, & (u, t) \in (1, 2] \times [0, T], \\
-C, & (u, t) \in [-1, 0) \times [0, T], \\
0, & \text{otherwise},
\end{cases}
\]

\[C = \sup_{(u, t) \in [0,1] \times [0, T]} |\dot{\phi}(u, t)|,\] and \(\zeta_\delta(u, t), (u, t) \in \mathbb{R}^2\), is a standard mollifier.

\[\square\]

PROOF OF THEOREM 1.3. First of all we note that the family \(\{y(\epsilon)\}_{\epsilon \in (0, T]}\) satisfies large deviations in \(L_2(\rho)\) with the rate function \(\frac{1}{2} \| \text{id} - \cdot \|_{L_2(\lambda)}^2\) (for simplicity of notation we suppose that \(\| \text{id} - g \|_{L_2(\lambda)} = +\infty\) whenever \(g \notin L_2(\lambda)\)). This immediately follows from Theorem 1.4 and the contraction principle for large deviations.

Next, let us show that for each convex closed set \(C\) in \(L_2(\rho)\) with nonempty interior we have

\[
\lim_{\epsilon \to 0} \epsilon \ln P\{y(\epsilon) \in C\} = -\frac{1}{2} \inf_{g \in C} \| \text{id} - g \|_{L_2(\lambda)}^2.
\]

To prove this, it is enough to show that

\[
\inf_{g \in C^o} \| \text{id} - g \|_{L_2(\lambda)} \leq \inf_{g \in C} \| \text{id} - g \|_{L_2(\lambda)},
\]

where \(C^o\) denotes the interior of \(C\) in \(L_2(\rho)\). Let \(\inf_{g \in C} \| \text{id} - g \|_{L_2(\lambda)} < \infty\) and \(\delta \in (0, 1]\) be fixed. Then there exists \(g_0 \in L_2(\lambda) \cap C\) such that

\[
\| \text{id} - g_0 \|_{L_2(\lambda)} \leq \inf_{g \in C} \| \text{id} - g \|_{L_2(\lambda)} + \delta.
\]

Since \(C^o\) is nonempty, there exists \(g_1 \in C^o\) and \(\delta > 0\) such that \(B(g_1, \delta) := \{g \in L_2(\rho) : \| g - g_1 \|_{L_2(\rho)} < \delta \} \subseteq C\). Moreover, \(g_1\) can be chosen from \(L_2(\lambda)\) because \(L_2(\lambda)\) is dense in \(L_2(\rho)\). Next, let \(g_c = (1 - \delta_0)g_0 + \delta_0g_1\), where

\[
\delta_0 = \frac{\delta}{1 + \| g_0 + g_1 \|_{L_2(\lambda)}}.
\]

Using the convexity of \(C\), we can see that \(B(g_c, \delta_0r) \subseteq C\) and consequently, \(g_c\) belongs to \(C^o\). Now we can estimate

\[
\inf_{g \in C^o} \| \text{id} - g \|_{L_2(\lambda)} \leq \inf_{g \in C^o} \| g_c \|_{L_2(\lambda)} \leq \inf_{g \in C} \| g_0 \|_{L_2(\lambda)} + \| g_0 - g_c \|_{L_2(\lambda)} \\
\leq \inf_{g \in C} \| g_0 \|_{L_2(\lambda)} + \| g_0 - g_1 \|_{L_2(\lambda)} + \| g_1 \|_{L_2(\lambda)} \\
< \inf_{g \in C} \| \text{id} - g \|_{L_2(\lambda)} + 2\delta.
\]

Making \(\delta \to 0\), we obtain (1.14).

Thus, the Varadhan formula (1.3) is obtained from a straightforward combination (4.13), the contraction principle for large deviations applied to the endpoint map \(C([0, 1], L_2(\rho)) \to L_2(\rho), \mu_t \in [0, 1] \to \mu_1\), and the fact that the map

\[
i : D^\uparrow([0, 1]) \ni g \mapsto g\#\lambda \in \mathcal{P}(\mathbb{R})
\]
is an isometry from the $L_2(\lambda)$-metric to the quadratic Wasserstein metric $d_W$ (see, e.g., [6, sec. 2.1]).

**Appendix A Some Properties of Absolutely Continuous Functions**

The following lemma follows from the definition of the stochastic integral, given in Section 3.1, and the integration-by-parts formula for integrals with respect to real-valued continuous martingales.

**Lemma A.1.** For every absolutely continuous function $f(t), t \in [0, T]$, with values in $L_2(\lambda)$,

$$
\int_0^t (f(s), dy(s)) = (f(t), y(t))_{L_2(\lambda)} - (f(0), y(0))_{L_2(\lambda)}
$$

(A.1)

almost surely, where the integral in the left-hand side was defined in Section 3.1.

**Lemma A.2.** The set $H$, which is defined in Section 4.2, i.e.,

$$
H = \{ h \in C([0, T], L_2(\rho^{-1})) : \hat{h} \in L_2([0, T], L_2(\rho^{-1})) \},
$$

where $\rho^{-1}(du) = \frac{1}{k(u)} du$, is dense in $L_2([0, T], L_2(\lambda))$.

**Proof.** The lemma can be proved using the density of the space of continuously differentiable functions $C^1([0, T])$ on $[0, T]$ in $L_2([0, T], \mathbb{R})$ and the approximation of each function $f \in L_2([0, T], L_2(\rho^{-1}))$ by $f_n = \sum_{k=1}^{n} \tilde{f}_k e_k, n \in \mathbb{N}$, where $\{e_n\}_{n \geq 1}$ is an orthonormal basis in $L_2(\rho^{-1})$ and $\tilde{f}_k(t) = (f(t), e_k)_{L_2(\rho^{-1})}$, $t \in [0, T]$.

**Appendix B Convergence of the Flow of Particles with Drift**

In this section we prove that the process $\{z_{\varepsilon}\}_{\varepsilon \in (0,1)}$ satisfying $(D^\varepsilon 1)$–$(D^\varepsilon 4)$ with $h = \phi$ tends to $\phi$. Note that $z_{\varepsilon}$ is a weak martingale solution to the equation

$$
dz_{\varepsilon}(t) = p_{z_{\varepsilon}} \psi(t) dt + \sqrt{\varepsilon} p_{z_{\varepsilon}} dW_t.
$$

If we show that $z_{\varepsilon}$ converges to a process $z$ taking values from $L_2(\varepsilon)(\rho)$, then by Lemma 4.8, $z$ should be a solution of the equation

$$
dz(t) = \psi(t) dt.
$$

It gives $z = \phi$.

Thus, we prove first that the family $\{z_{\varepsilon}\}_{\varepsilon \in (0,1)}$ is tight. Then we show that any limit point $z$ of $\{z_{\varepsilon}\}_{\varepsilon \in (0,1)}$ is a $L_2(\varepsilon)(\rho)$-valued process. As we noted, it immediately gives $z = \phi$. Since $\{z_{\varepsilon}\}_{\varepsilon \in (0,1)}$ has only one nonrandom limit point, we obtain that $\{z_{\varepsilon}\}_{\varepsilon \in (0,1)}$ tends to $\phi$ in probability (not only in distribution).
PROPOSITION B.1. Let \( \varphi \in \mathcal{R} \) and a family of random elements \( \{ z^\varepsilon \}_{\varepsilon \in (0, 1]} \) satisfy properties (D\( \varepsilon \)) with \( h = \varphi \); then \( z^\varepsilon \) tends to \( \varphi \) in the space \( C([0, T], L_2(\rho)) \) in probability.

To prove the proposition, we first establish tightness of \( \{ z^\varepsilon \} \) in \( C([0, T], L_2(\rho)) \) using the boundedness of \( \varphi \) and the same argument as in the proof of exponential tightness of \( \{ y^\varepsilon \} \). Next, testing the convergent subsequence \( \{ z^\varepsilon \} \) by functions \( l \) from \( C([0, 1] \times [0, T], \mathbb{R}) \) and using integration by parts, we will obtain

\[
\int_0^T \int_0^1 l(u, t)(z^\varepsilon(u, t) - u)dt du = \int_0^T \int_0^1 L(u, t)(\text{pr}_{z^\varepsilon(t)} \varphi(t))(u)dt du + \int_0^T \int_0^1 L(u, t)d\eta^\varepsilon(u, t)du,
\]

where \( L(u, t) = \int_0^T l(u, s)ds \) and \( z^\varepsilon \to z \). If \( z(t) \) belongs to \( L_2^{\uparrow}(\rho) \) for all \( t \in [0, T] \), then passing to the limit and using Lemma 4.8, we obtain

\[
\int_0^T \int_0^1 l(u, t)(z(u, t) - u)dt du = \int_0^T \int_0^1 L(u, t)\varphi(u, t)dt du,
\]

which implies \( z = \varphi \).

The fact that \( z(t) \in L_2^{\uparrow}(\rho) \) will follow from the following lemma.

LEMMA B.2. Let \( \varphi \) and \( \{ z^\varepsilon \} \) be as in Proposition B.1. Then for each \( u < v \) there exists \( \delta > 0 \) such that

\[
\lim_{\varepsilon \to 0} \mathbb{P}\{ z^\varepsilon(v, t) - z^\varepsilon(u, t) \leq \delta \} = 0.
\]

Let \( \mathcal{S}(u, v, t) \) be a finite set of intervals contained in \([u, v]\) for all \( t \in (0, T] \) such that

1. if \( \pi_1, \pi_2 \in \mathcal{S}(u, v, t) \) and \( \pi_1 \neq \pi_2 \), then \( \pi_1 \cap \pi_2 = \emptyset \);
2. \( \bigcup \mathcal{S}(u, v, t) = [u, v] \);
3. for all \( s < t \) and \( \pi_1 \in \mathcal{S}(u, v, s) \) there exists \( \pi_2 \in \mathcal{S}(u, v, t) \) containing \( \pi_1 \);
4. there exists a decreasing sequence \( \{ t_n \}_{n \geq 1} \) on \((0, T]\) that tends to 0 and \( \mathcal{S}(u, v, t) = \mathcal{S}(u, v, t_n), \ t \in [t_n, t_{n-1}], \ n \in \mathbb{N}, \ t_0 = T \);
5. for each monotone sequence \( \pi(t) \in \mathcal{S}(u, v, t), \ t > 0 \), \( \bigcap_{t > 0} \pi(t) \) is a one-point set.

LEMMA B.3. Let \( \varphi \in \mathcal{R} \) and \( [\tilde{u}, \tilde{v}] \subset (0, 1) \). Then there exists \( \gamma > 0 \) such that for each interval \( (u, v) \supset [\tilde{u}, \tilde{v}] \) there exist \( u_0 \in (u, \tilde{u}) \) and \( v_0 \in (\tilde{v}, v) \) such that

\[
\inf_{t \in [0, T]} \left[ v_0 - u_0 + \int_0^t (\text{pr}_{\mathcal{S}(s)} \varphi(s))(v_0)ds - \int_0^t (\text{pr}_{\mathcal{S}(s)} \varphi(s))(u_0)ds \right] = \delta > 0
\]

for all \( \mathcal{S}(t) = \mathcal{S}(0 \vee (u - \gamma), (v + \gamma) \wedge 1, t), \ t \in (0, T], \) such that \( u_0 \) and \( v_0 \) belong to separate intervals from \( \mathcal{S}(T) \), and \( \text{pr}_{\mathcal{S}(t)} \) denotes the projection in \( L_2(\lambda) \) onto the space of \( \sigma(\mathcal{S}(t)) \)-measurable functions.
Proof. Let \( u \in [0, 1] \) and \( \mathcal{S}(t) = \mathcal{S}(0, 1, t), t \in [0, T] \). Then we can choose a sequence of intervals \( \{\pi_n\}_{n \geq 1} \) and a decreasing sequence \( \{s_n\}_{n \geq 1} \) from \((0, T]\) converging to 0 such that \( \pi_{n+1} \subseteq \pi_n \subseteq [0, 1], \{u\} = \bigcap_{n=1}^{\infty} \pi_n \), and

\[
\begin{align*}
&u + \int_0^t (pr_{\mathcal{S}(t)}\hat{\psi}(s))(u)ds \\
&= u + \sum_{n=1}^{\infty} \int_{s_{n-1} \wedge t}^{s_n \wedge t} \left( \frac{1}{|\pi_n|} \int_{\pi_n} \hat{\psi}(q, r) dq \right) dr \\
&= u + \sum_{n=1}^{\infty} \frac{1}{|\pi_n|} \int_{\pi_n} (\varphi(q, s_{n-1} \wedge t) - \varphi(q, s_n \wedge t)) dq \\
&= \frac{1}{|\pi_k|} \int_{\pi_k} \varphi(q, t) dq \\
&+ \sum_{n=k}^{\infty} \left[ \frac{1}{|\pi_{n+1}|} \int_{\pi_{n+1}} \varphi(q, s_n) dq - \frac{1}{|\pi_n|} \int_{\pi_n} \varphi(q, s_n) dq \right],
\end{align*}
\]

where \( t \in [s_k, s_{k-1}] \).

We estimate the \( n \)th term of the sum. For convenience of calculations, let \([a, b] \subseteq [c, d]\) and \( f : [0, 1] \to \mathbb{R} \) be a nondecreasing absolutely continuous function with bounded derivative. So,

\[
\begin{align*}
\frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{d-c} \int_c^d f(x) dx &\leq \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{b-c} \int_c^b f(x) dx \\
&= \frac{a-c}{(b-a)(b-c)} \int_a^b f(x) dx - \frac{1}{b-c} \int_c^a f(x) dx \\
&\leq \frac{a-c}{b-c} f(b) - \frac{a-c}{b-c} f(c) = \frac{a-c}{b-c} (f(b) - f(c)) \\
&\leq \sup_{x \in [0, 1]} f'(x)(a - c).
\end{align*}
\]

Taking \( c = a_n, d = b_n, a = a_{n+1}, b = b_{n+1}, \) and \( f = \varphi(\cdot, s_n) \), where \( a_n < b_n \) are the ends of \( \pi_n \), we get

\[
\begin{align*}
&u + \int_0^t (pr_{\mathcal{S}(t)}\hat{\psi}(s))(u)ds \leq \frac{1}{|\pi_k|} \int_{\pi_k} \varphi(q, t) dq + \sum_{n=k}^{\infty} \tilde{C}(a_{n+1} - a_n) \\
&\leq \frac{1}{|\pi_k|} \int_{\pi_k} \varphi(q, t) dq + \tilde{C}(u - a_k),
\end{align*}
\]

where \( \tilde{C} = \sup_{(u, t) \in [0, 1] \times [0, T]} \frac{\partial \varphi}{\partial u}(u, t) \).
Similarly, we can obtain
\[ u + \int_0^t (\text{pr}_{\mathbb{E}(s)} \psi(s))(u)ds \geq \frac{1}{|\pi_k|} \int_{\pi_k} \psi(q, t) dq - \tilde{C}(b_k - u). \]

Next, let \( \bar{u} \) and \( \bar{v} \) be from the statement of the lemma. Since \( \varphi \) is continuous on \([0, 1] \times [0, T]\) and increasing by the first argument, the function
\[ G(a, b, t) = \frac{1}{\bar{v} - b} \int_b^{\bar{v}} \varphi(q, t) dq - \frac{1}{a - \bar{u}} \int_{\bar{u}}^a \varphi(q, t) dq \]
is positive and continuous on \( E = \{(a, b, t) : \bar{u} \leq a \leq b \leq \bar{v}, \ t \in [0, T]\} \). Hence
\[ \delta_1 = \inf_E G > 0. \]

Take \( \gamma = \delta_1/8\tilde{C} \) and for \( u, v \in [0, 1] \) satisfying \( (u, v) \supseteq [\bar{u}, \bar{v}] \) set
\[ u_0 = (u + \gamma) \land \bar{u}, \quad v_0 = (v - \gamma) \lor \bar{v}. \]

Let \( \mathcal{G}(t) = \mathcal{G}(0 \lor (u - \gamma), (v + \gamma) \land 1, t, t \in (0, T), \) such that \( u_0 \) and \( v_0 \) belong to separate intervals from \( \mathcal{G}(T) \) and \( t \in (0, T) \) be fixed. For \( \pi_1, \pi_2 \) belonging to \( \mathcal{G}(t) \) and containing \( u_0, v_0 \), respectively, we obtain
\[ v_0 - u_0 + \int_0^t (\text{pr}_{\mathbb{E}(s)} \psi(s))(v_0)ds - \int_0^t (\text{pr}_{\mathbb{E}(s)} \psi(s))(u_0)ds \geq \frac{1}{|\pi_2|} \int_{\pi_2} \psi(q, t) dq - \frac{1}{|\pi_1|} \int_{\pi_1} \psi(q, t) dq - \tilde{C}(d - v_0) - \tilde{C}(u_0 - a) \]
and
\[ \geq G(b \lor \bar{u}, c \land \bar{v}, t) - \tilde{C}(d - v_0) - \tilde{C}(u_0 - a). \]
where \( c < d \) and \( a < b \) are the ends of \( \pi_1 \) and \( \pi_2 \), respectively, and \( b \leq c \) because \( u_0 \) and \( v_0 \) belong to separate intervals from \( \mathcal{G}(T) \). Since \( u - \gamma \leq a < u_0 \leq u + \gamma \) and \( v - \gamma \leq v_0 < b \leq v + \gamma \), we have
\[ v_0 - u_0 + \int_0^t (\text{pr}_{\mathbb{E}(s)} \psi(s))(v_0)ds - \int_0^t (\text{pr}_{\mathbb{E}(s)} \psi(s))(u_0)ds \geq \delta_1 - \tilde{C}(v + \gamma - v + \gamma) - \tilde{C}(u + \gamma - u + \gamma) \]
\[ = \delta_1 - 4\tilde{C}\gamma = \frac{\delta_1}{2} > 0. \]
This finishes the proof of the lemma. \( \square \)

**Proof of Lemma B.2** Let \( u < v \) be fixed points from \((0, 1)\) and \( \delta, u_0, v_0, \gamma \) be defined in Lemma B.3 for some \([\bar{u}, \bar{v}] \subset (u, v)\). Suppose that
\[ \lim_{\varepsilon \to 0} \mathbb{P} \left\{ \left| z^\varepsilon(v, t) - z^\varepsilon(u, t) \right| \leq \frac{\delta}{2} \right\} > 0. \]
Set

\[ B_1^\varepsilon = \{ z^\varepsilon(u, t) = z^\varepsilon((u - \gamma) \vee 0, t) \} , \]
\[ B_2^\varepsilon = \{ z^\varepsilon(v, t) = z^\varepsilon((v + \gamma) \wedge 1, t) \} , \]
\[ A^\varepsilon = \left\{ z^\varepsilon(v, t) - z^\varepsilon(u, t) \leq \frac{\delta}{2} \right\} . \]

Since the diffusion rate of \( z^\varepsilon(u, \cdot) \) grows to infinity as the time tends to 0, it is convenient to work with the mean of \( z^\varepsilon \) because we can control the growing of the diffusion rate in this case. So, denote

\[ \xi^\varepsilon_u(t) = \frac{1}{u_0 - u} \int_u^{u_0} z^\varepsilon(q, t) \, dq, \quad \xi^\varepsilon_v(t) = \frac{1}{v - v_0} \int_{v_0}^v z^\varepsilon(q, t) \, dq. \]

It is easy to see that

\[ \tilde{A}^\varepsilon := \left\{ \xi^\varepsilon_v(t) - \xi^\varepsilon_u(t) \leq \frac{\delta}{2} \right\} \supseteq A^\varepsilon . \]

Next, using the processes \( \xi^\varepsilon_u \) and \( \xi^\varepsilon_v \), we want to show that

\[ \lim_{\varepsilon \to 0} \mathbb{P} \{ A^\varepsilon \cap (B_1^\varepsilon \cup B_2^\varepsilon)^c \} = 0 . \]

Note that \( \xi^\varepsilon_u \) and \( \xi^\varepsilon_v \) are diffusion processes, namely,

\[ \xi^\varepsilon_u(t) = \frac{u_0 + u}{2} + \int_0^t a^\varepsilon_u(s) \, ds + \chi^\varepsilon_u(t) , \]
\[ \xi^\varepsilon_v(t) = \frac{v_0 + v}{2} + \int_0^t a^\varepsilon_v(s) \, ds + \chi^\varepsilon_v(t) , \]

where

\[ a^\varepsilon_u(t) = \frac{1}{u_0 - u} \int_u^{u_0} (pr_{z^\varepsilon(t)}(\psi(t))(q) \, dq . \]
\[ a^\varepsilon_v(t) = \frac{1}{v - v_0} \int_{v_0}^v (pr_{z^\varepsilon(t)}(\psi(t))(q) \, dq . \]
\[ \chi^\varepsilon_u(t) = \frac{1}{u_0 - u} \int_0^t \int_u^{u_0} d\eta^\varepsilon(q, s) \, dq . \]
\[ \chi^\varepsilon_v(t) = \frac{1}{v - v_0} \int_0^t \int_{v_0}^v d\eta^\varepsilon(q, s) \, dq . \]

By choosing \( u_0, v_0, \) and, we have

\[ L^\varepsilon(t, \omega) = \frac{v_0 + v}{2} - \frac{u_0 + u}{2} + \int_0^t a^\varepsilon_v(s, \omega) \, ds - \int_0^t a^\varepsilon_u(s, \omega) \, ds \geq \delta , \]

\[ t \in [0, T] , \omega \in (B_1^\varepsilon \cup B_2^\varepsilon)^c . \]
Denote the difference \( \xi^\varepsilon_v - \xi^\varepsilon_u \) by \( \xi^\varepsilon \). Note that the quadratic variation of the martingale part \( \chi^\varepsilon \) of \( \xi^\varepsilon \) satisfies
\[
[\chi^\varepsilon]_t \leq \varepsilon C t,
\]
where \( C = \frac{1}{u_0 - u} + \frac{1}{v - v_0} \). So, denoting
\[
\sigma^\varepsilon = \inf \left\{ t : \xi^\varepsilon(t) = \frac{\delta}{2} \right\},
\]
we get
\[
\mathbb{P} \{ A^\varepsilon \cap (B_1^\varepsilon \cup B_2^\varepsilon) \} \leq \mathbb{P} \{ \sigma^\varepsilon \leq t \} \cap (B_1^\varepsilon \cup B_2^\varepsilon),
\]
which implies (B.1). Indeed, by theorem 2.7.2 in [20], there exists a standard Wiener process \( w^\varepsilon(t), t \geq 0 \), such that
\[
\chi^\varepsilon(t) = w^\varepsilon([\chi^\varepsilon]_t).
\]
Define \( \tau^\varepsilon = \inf \{ t : w^\varepsilon(\varepsilon C t) = -\frac{\delta}{2} \} \). Since
\[
\xi^\varepsilon(t) = \delta + (L^\varepsilon(t) - \delta) + \chi^\varepsilon(t)
\]
and the term \( L^\varepsilon - \delta \) is nonnegative on \( (B_1^\varepsilon \cup B_2^\varepsilon) \), the process \( \delta + w^\varepsilon(\varepsilon C \cdot) \) hits at the point \( \frac{\delta}{2} \) sooner than \( \xi^\varepsilon \). Consequently,
\[
\{ \sigma^\varepsilon \leq t \} \cap (B_1^\varepsilon \cup B_2^\varepsilon) \subseteq \{ \tau^\varepsilon \leq t \} \cap (B_1^\varepsilon \cup B_2^\varepsilon).
\]
This yields (B.1).

Next, the relation \( \mathbb{P} \{ A^\varepsilon \cap (B_1^\varepsilon \cup B_2^\varepsilon) \} = \mathbb{P} \{ A^\varepsilon \} - \mathbb{P} \{ A^\varepsilon \cap (B_1^\varepsilon \cup B_2^\varepsilon) \} \), (B.1), and the assumption \( \lim_{\varepsilon \to 0} \mathbb{P} \{ A^\varepsilon \} > 0 \) imply
\[
\lim_{\varepsilon \to 0} \mathbb{P} \{ A^\varepsilon \cap (B_1^\varepsilon \cup B_2^\varepsilon) \} > 0.
\]
Thus, we obtain
\[
\lim_{\varepsilon \to 0} \mathbb{P} \{ A^\varepsilon \cap B_1^\varepsilon \} > 0 \quad \text{or} \quad \lim_{\varepsilon \to 0} \mathbb{P} \{ A^\varepsilon \cap B_2^\varepsilon \} > 0.
\]
It means that we can extend the interval \( [u, v] \) to \( [(u - \gamma) \lor 0, v] \) or \( [u, (v + \gamma) \land 1] \), i.e.,
\[
\lim_{\varepsilon \to 0} \mathbb{P} \left\{ \zeta^\varepsilon(v, t) - \zeta^\varepsilon((u - \gamma) \lor 0, t) \leq \frac{\delta}{2} \right\} > 0
\]
or
\[
\lim_{\varepsilon \to 0} \mathbb{P} \left\{ \zeta^\varepsilon((v + \gamma) \land 1, t) - \zeta^\varepsilon(u, t) \leq \frac{\delta}{2} \right\} > 0.
\]
Noting that \( \gamma \) only depends on \( [\bar{u}, \bar{v}] \) and applying the same argument for new start points of the particles in finitely many steps, we obtain
\[
\lim_{\varepsilon \to 0} \mathbb{P} \left\{ \zeta^\varepsilon(v_1, t) - \zeta^\varepsilon(u_1, t) \leq \frac{\delta}{2} \right\} > 0,
\]
where \((u_1, v_1) \supset [\overline{u}, \overline{v}]\) and \(u_1 = 0\) or \(v_1 = 1\). Next, applying the same argument for new start points of the particles but replacing \(B_1^\varepsilon \cup B_2^\varepsilon\) by \(B_1^\varepsilon\) if \(v_1 = 1\) or by \(B_2^\varepsilon\) if \(u_1 = 0\), in finitely many steps we get

\[
\lim_{\varepsilon \to 0} \mathbb{P} \left\{ \varrho_{\varepsilon} (1, t) - \varrho_{\varepsilon} (0, t) \leq \frac{\delta}{2} \right\} > 0.
\]

But it is not possible because the same argument (without \(B_1^\varepsilon\) and \(B_2^\varepsilon\)) gives

\[
\lim_{\varepsilon \to 0} \mathbb{P} \left\{ \varrho_{\varepsilon} (1, t) - \varrho_{\varepsilon} (0, t) \leq \frac{\delta}{2} \right\} = 0.
\]

The lemma is proved.

**Proof of Proposition B.1.** Using Jakubowski’s tightness criterion (see theorem 3.1 in [22]) and the boundedness of \(\varphi\), as in the proof of exponential tightness of \(\{y^\varepsilon\}\) (see Proposition 4.4), we can prove that \(\{\varrho^\varepsilon\}_{\varepsilon \in (0, 1]}\) is tight in \(C([0, T], L_2(\rho))\).

Let \(\{\varrho^\varepsilon\}\) be a convergent subsequence and \(\varrho\) its limit. By Skorokhod’s theorem (see theorem 3.1.8 in [18]), we can define a probability space and a sequence of random elements \(\{\varrho^\varepsilon\}\) on this space, such that \(\text{Law} (\varrho^\varepsilon) = \text{Law} (\varrho)\). If we show that \(\varrho^\varepsilon \rightarrow \varrho\) in probability, and since \(\varrho\) is nonrandom, \(\varrho^\varepsilon \rightarrow \varrho\) in probability, and thus, it will easily yield that \(\varrho^\varepsilon \rightarrow \varrho\) in probability.

So, for convenience of notation we will assume that \(\varrho^\varepsilon \rightarrow \varrho\) a.s., instead of \(\varrho^\varepsilon \rightarrow \varrho\). First we check that \(\varrho(t) \in L_2^\uparrow(\rho)\) for all \(t \in [0, T]\). Let \(t\) be fixed. One can show that

\[
\varrho^\varepsilon (t) \rightarrow \varrho(t) \quad \text{in measure } \mathbb{P} \otimes \rho.
\]

By lemma 4.2 in [23], there exists subsequence \(\{\varepsilon'\}\) such that

\[
\varrho^\varepsilon (t) \rightarrow \varrho(t) \quad \text{in measure } \mathbb{P} \otimes \rho \text{-a.e.}
\]

Set \(A = \{(\omega, u) : \varrho^\varepsilon (u, t, \omega) \rightarrow \varrho(u, t, \omega)\}\). Since \(\mathbb{P} \otimes \rho (A) = 0\), it is easy to see that there exists a set \(U \subseteq [0, 1]\) such that \(\rho(U^c) = 0\) and \(\mathbb{P} (A_u) = 1\) for all \(u \in U\), where \(A_U = \{\omega : (\omega, u) \in A\}\). Note that this implies that for each \(u \in U\)

\[
\varrho^\varepsilon (u, t) \rightarrow \varrho(u, t) \quad \text{a.s.}
\]

Let \(U_{\text{count}}\) be a countable subset of \(U\) that is dense in \([0, 1]\). From Lemma B.2 it follows that

\[
z(u, t) < z(v, t) \quad \text{a.s.}
\]

for all \(u, v \in U_{\text{count}}, u < v\).

Denote

\[
\Omega^\prime = \bigcap_{u < v, u, v \in U_{\text{count}}} \{z(u, t) < z(v, t)\}.
\]
Since $U_{\text{count}}$ is countable, $\mathbb{P}(\Omega') = 1$. Next, define
\[ \tilde{z}(u, t, \omega) = \inf_{u \leq v, v \in U_{\text{count}}} z(v, t, \omega), \quad u \in [0, 1], \ \omega \in \Omega'. \]
Then for all $\omega \in \Omega'$, $\tilde{z}(\cdot, t, \omega) \in D^{\uparrow\uparrow}$. We show that $\rho\{u : \tilde{z}(u, t) \neq z(u, t)\} = 0$ a.s.

Denote
\[ \tilde{\Omega} = \left( \bigcap_{u \in U_{\text{count}}} A_u \right) \cap \Omega' \cap \{z^\varepsilon(t) \to z(t) \text{ in } L_2(\rho)\}. \]
Then
\[ (B.2) \quad z^\varepsilon(u, t, \omega) \to z(u, t, \omega) = \tilde{z}(u, t, \omega) \]
for all $u \in U_{\text{count}}$ and $\omega \in \tilde{\Omega}$. Fix $\omega \in \tilde{\Omega}$. Since $\tilde{z}(\cdot, t, \omega)$ is nondecreasing, it has a countable set $D\tilde{z}(\cdot, t, \omega)$ of discontinuous points. The countability implies that $\rho(D\tilde{z}(\cdot, t, \omega)) = 0$. Take $u \in D\tilde{z}(\cdot, t, \omega)$; then from the monotonicity of $\tilde{z}(\cdot, t, \omega)$ and $z^\varepsilon(\cdot, t, \omega)$, the density of $U_{\text{count}}$, and (B.2) we can obtain
\[ z^\varepsilon(u, t, \omega) \to \tilde{z}(u, t, \omega). \]
Thus,
\[ z^\varepsilon(\cdot, t, \omega) \to \tilde{z}(\cdot, t, \omega) \quad \rho\text{-a.e.} \]
On the other hand,
\[ z^\varepsilon(\cdot, t, \omega) \to z(\cdot, t, \omega) \quad \text{in } L_2(\rho). \]
Consequently, $z(\cdot, t, \omega) = \tilde{z}(\cdot, t, \omega)$ $\rho$-a.e. for all $\omega \in \tilde{\Omega}$. So, it means that $z(t) \in L^{\uparrow\uparrow}_2(\rho)$ a.s. for all $t \in [0, T]$.

Now we can prove that $z = \varphi$. Take $l \in C([0, 1] \times [0, T], \mathbb{R})$. By the dominated convergence theorem, $\int_0^T \int_0^1 l(u, t)(z^\varepsilon(u, t) - u)dt\,du$ converges to $\int_0^T \int_0^1 l(u, t)(z(u, t) - u)dt\,du$ a.s. Integrating by parts, we get
\[
\int_0^T \int_0^1 l(u, t)(z^\varepsilon(u, t) - u)dt\,du = \int_0^T \int_0^1 L(u, t)(pr_{z^\varepsilon(t)}(\varphi(t))(u)dt\,du \\
+ \int_0^T \int_0^1 L(u, t)d\eta^\varepsilon(u, t)du,
\]
where $L(u, t) = \int_t^T l(u, s)ds$. The first term in the right-hand side of the previous relation converges to $\int_0^T \int_0^1 L(u, t)\varphi(u)dt\,du$ by Lemma 4.8. The second term is the stochastic integral, so we can estimate the expectation of its second moment
\[
\mathbb{E}\left( \int_0^T \int_0^1 L(u, t)d\eta^\varepsilon(u, t)du \right)^2 \leq \varepsilon \mathbb{E}\int_0^T \int_0^1 (pr_{z^\varepsilon(t)}L(t))^2(u)dt\,du \\
\leq \varepsilon \mathbb{E}\int_0^T \int_0^1 L^2(u, t)dt\,du \to 0, \quad \varepsilon \to 0.
\]
Consequently, we obtain
\[ \int_0^T \int_0^1 l(u,t)(z(u,t) - u) \, dt \, du = \int_0^T \int_0^1 L(u,t)\psi(u,t) \, dt \, du, \]
which easily implies \( z = \varphi \). The proposition is proved. \( \square \)

Bibliography

[1] Adams, S.; Dirr, N.; Peletier, M. A.; Zimmer, J. From a large-deviations principle to the Wasserstein gradient flow: a new micro-macro passage. Comm. Math. Phys. 307 (2011), no. 3, 791–815. doi:10.1007/s00220-011-1328-4
[2] Alonso, A.; Brannibia-Paz, F. \( L^p \)-continuity of conditional expectations. J. Math. Anal. Appl. 221 (1998), no. 1, 161–176. doi:10.1006/jmaa.1998.5818
[3] Andres, S.; von Renesse, M.-K. Particle approximation of the Wasserstein diffusion. J. Funct. Anal. 258 (2010), no. 11, 3879–3905. doi:10.1016/j.jfa.2009.10.029
[4] Arratia, R. A. Coalescing Brownian motion on the line. Ph.D. thesis, The University of Wisconsin-Madison, 1979.
[5] Benamou, J.-D.; Brenier, Y. A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem. Numer. Math. 84 (2000), no. 3, 375–393. doi:10.1007/s002110050002
[6] Brenier, Y.; Gangbo, W.; Savaré, G.; Westdickenberg, M. Sticky particle dynamics with interactions. J. Math. Pures Appl. (9) 99 (2013), no. 5, 577–617. doi:10.1016/j.matpur.2012.09.013
[7] Cherny, A. S.; Engelbert, H.-J. Singular stochastic differential equations. Lecture Notes in Mathematics, 1858. Springer, Berlin, 2005. doi:10.1007/6104187
[8] Dawson, D. A. Measure-valued Markov processes. Ecole d’Eté de Probabilités de Saint-Flour XXI—1991, 1–260. Lecture Notes in Mathematics, 1541. Springer, Berlin, 1993. doi:10.1007/BFb0084190
[9] Dean, D. S. Langevin equation for the density of a system of interacting Langevin processes. J. Phys. A 29 (1996), no. 24, L613–L617. URL http://dx.doi.org/10.1088/0305-4470/29/24/001
[10] Dembo, A.; Zeitouni, O. Large deviations techniques and applications. Stochastic Modelling and Applied Probability, 38. Springer, Berlin, 2010. doi:10.1007/978-3-642-03311-7
[11] Djeihiche, B.; Schied, A. Large deviations for hierarchical systems of interacting jump processes. J. Theoret. Probab. 11 (1998), no. 1, 1–24. doi:10.1023/A:1021690707556
[12] Donati-Martin, C. Large deviations for Wishart processes. Probab. Math. Statist. 28 (2008), no. 2, 325–343.
[13] Donati-Martin, C.; Rouault, A.; Yor, M.; Zani, M. Large deviations for squares of Bessel and Ornstein-Uhlenbeck processes. Probab. Theory Related Fields 129 (2004), no. 2, 261–289. doi:10.1007/s00440-004-0338-y
[14] Dorogovtsev, A. A. One Brownian stochastic flow. Theory Stoch. Process. 10 (2004), no. 3-4, 21–25.
[15] Dorogovtsev, A. A. Measure-valued processes and stochastic flows. Proceedings of Institute of Mathematics of NAS of Ukraine. Mathematics and Its Applications, 66. Natsional’nna Akadem’ya Nauk Ukraini, Institut Matematiki, Kiev, 2007.
[16] Dorogovtsev, A. A.; Nishchenko, I. I. An analysis of stochastic flows. Commun. Stoch. Anal. 8 (2014), no. 3, 331–342.
[17] Dorogovtsev, A. A.; Ostapenko, O. V. Large deviations for flows of interacting Brownian motions. Stoch. Dyn. 10 (2010), no. 3, 315–339. doi:10.1142/S0219493710002978
[18] Ethier, S. N.; Kurtz, T. G. *Markov processes. Characterization and convergence*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. Wiley, New York, 1986. doi:10.1002/9780470316658

[19] Fontes, L. R. G.; Isopi, M.; Newman, C. M.; Ravishankar, K. The Brownian web: characterization and convergence. *Ann. Probab.* 32 (2004), no. 4, 2857–2883. doi:10.1214/009117904000000568

[20] Ikeda, N.; Watanabe, S. *Stochastic differential equations and diffusion processes*. Second edition. North-Holland Mathematical Library, 24. North-Holland, Amsterdam; Kodansha, Tokyo, 1989.

[21] Jacod, J.; Shiryaev, A. N. *Limit theorems for stochastic processes*. Second edition. Grundlehren der mathematischen Wissenschaften, 288. Springer, Berlin, 2003. doi:10.1007/978-3-662-05265-5

[22] Jakubowski, A. On the Skorokhod topology. *Ann. Inst. H. Poincaré Probab. Statist.* 22 (1986), no. 3, 263–285.

[23] Kallenberg, O. *Foundations of modern probability*. Second edition. Probability and Its Applications (New York). Springer, New York, 2002. doi:10.1007/978-1-4757-4015-8

[24] Kawasaki, K. Stochastic model of slow dynamics in supercooled liquids and dense colloidal suspensions. *Phys. A* 208 (1994), no. 1, 35–64. doi:10.1016/0378-4371(94)90533-9

[25] Konarovski˘ı, V. V . On an infinite system of diffusing particles with coalescing. *Teor. Veroyatn. Primen.* 55 (2010), no. 1, 157–167; translation in *Theory Probab. Appl.* 55 (2010), no. 1, 134–144. doi:10.1137/S0040585X97984693

[26] Konarovskiy, V. V. Large deviations principle for finite system of heavy diffusion particles. *Theory Stoch. Process.* 19 (2014), no. 1, 37–45.

[27] Konarovskiy, V. A system of coalescing heavy diffusion particles on the real line. *Ann. Probab.* 45 (2017), no. 5, 3293–3335. doi:10.1214/16-AOP1137

[28] Krylov, N. V.; Rozovskii, B. L. Stochastic evolution equations. *J. Soviet Math.* 16 (1981), no. 4, 1233–1277. doi:10.1007/BF01084893

[29] Le Jan, Y.; Raimond, O. Flows, coalescence and noise. *Ann. Probab.* 32 (2004), no. 2, 1247–1315. doi:10.1214/009117904000000207

[30] Lott, J.; Villani, C. Ricci curvature for metric-measure spaces via optimal transport. *Ann. of Math. (2)* 169 (2009), no. 3, 903–991. doi:10.4007/annals.2009.169.903

[31] Malovichko, T. V . Girsanov’s theorem for stochastic flows with interaction. *Ukrain. Mat. Zh.* 61 (2009), no. 3, 365–383; translation in *Ukrainian Math. J.* 61 (2009), no. 3, 435–456. doi:10.1007/s11253-009-0216-y

[32] Otto, F. The geometry of dissipative evolution equations: the porous medium equation. *Comm. Partial Differential Equations* 26 (2001), no. 1-2, 101–174. doi:10.1081/PDE-100002243

[33] Otto, F.; Villani, C. Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality. *J. Funct. Anal.* 173 (2000), no. 2, 361–400. doi:10.1006/jfan.1999.3557

[34] Revuz, D.; Yor, M. *Continuous martingales and Brownian motion*. Third edition. Grundlehren der mathematischen Wissenschaften, 293. Springer, Berlin, 1999. doi:10.1007/978-3-662-06400-9

[35] Schied, A. *Criteria for exponential tightness in path spaces*. Unpublished preprint, 1995.

[36] Schied, A. Moderate deviations and functional LIL for super-Brownian motion. *Stochastic Process. Appl.* 72 (1997), no. 1, 11–25. doi:10.1016/S0304-4149(97)00078-1

[37] Shamov, A. Short-time asymptotics of one-dimensional Harris flows. *Commun. Stoch. Anal.* 5 (2011), no. 3, 527–539.

[38] Stannat, W. Two remarks on the Wasserstein Dirichlet form. *Seminar on Stochastic Analysis, Random Fields and Applications VII*, 235–255. Progress in Probability, 67. Birkhäuser/Springer, Basel, 2013.
[40] Sturm, K.-T. On the geometry of metric measure spaces. I. Acta Math. 196 (2006), no. 1, 65–131. doi:10.1007/s11511-006-0002-8

[41] Sturm, K.-T. A monotone approximation to the Wasserstein diffusion. Singular phenomena and scaling in mathematical models, 25–48. Springer, Cham, 2014. doi:10.1007/978-3-319-00786-1_2

[42] Varadhan, S. R. S. On the behavior of the fundamental solution of the heat equation with variable coefficients. Comm. Pure Appl. Math. 20 (1967), 431–455. doi:10.1002/cpa.3160200210

[43] Villani, C. Optimal transport. Old and new. Grundlehren der mathematischen Wissenschaften, 338. Springer, Berlin, 2009. doi:10.1007/978-3-540-71050-9

[44] von Renesse, M.-K. An optimal transport view of Schrödinger’s equation. Canad. Math. Bull. 55 (2012), no. 4, 858–869. doi:10.4153/CMB-2011-121-9

[45] von Renesse, M.-K.; Sturm, K.-T. Transport inequalities, gradient estimates, entropy, and Ricci curvature. Comm. Pure Appl. Math. 58 (2005), no. 7, 923–940. doi:10.1002/cpa.20060

[46] von Renesse, M.-K.; Sturm, K.-T. Entropic measure and Wasserstein diffusion. Ann. Probab. 37 (2009), no. 3, 1114–1191. doi:10.1214/08-AOP430

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