Type IIA Superstrings, Chiral Symmetry, and $N = 1$ 4D Gauge Theory Dualities

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We study $N = 1$ four dimensional gauge theories as the world volume theory of D4-branes between NS 5-branes. We find constructions for a number of known field theory dualities involving $SU(N_c) \times SU(N'_c)$ groups, coupled by matter fields $F$ in the $(N_c, \bar{N}_c)$ representation, in terms of branes of type IIA string theory. The dual gauge group follows from simply reversing the ordering of the NS 5-branes and the D6-branes while conserving magnetic charge on the world volume of the branes. We interpret many field theory phenomena such as deformation of the superpotential $W = \text{Tr} (F \tilde{F})^{k+1}$ in terms of the position of branes. By looking to D-branes for guidance, we find new $N = 1$ dualities involving arbitrary numbers of gauge groups. We propose a mechanism for enhanced chiral symmetry in the brane construction which, we conjecture, is associated with tensionless threebranes in six dimensions.

4/97
1. Introduction

The world volume field theory on the branes of superstring theory has received much attention recently. Many new field theory results as well as new implications for string theory dynamics have been obtained by studying the low energy physics of branes in space-time. One particular construction was introduced in [1] to study the dynamics of 2+1 dimensional gauge theories using Type IIB physics. This construction has lead to applications in various dimensions and supersymmetries [2, 3, 4, 5, 6, 7, 8, 9]. A common approach in all these papers is to stretch 4 (or 3)-branes of type IIA (or IIB) string theory between two NS 5-branes producing what looks to be at large distances a $U(N_c)$ four (three) dimensional gauge theory. D6(5)-branes, when they intersect the D4(3)-branes, are interpreted as fundamental fields charged under the gauge groups. In [1] a particularly interesting phenomenon was noticed in which a D5-brane can pass through an NS 5-brane and create a D3-brane. This phenomenon has led to a discovery of a new phase transition between two different gauge theories in $N = 4$ supersymmetric gauge theory in three dimensions. Concretely, the two theories are a $U(N_c)$ gauge theory coupled to $N_f$ flavors in the fundamental representation and a $U(N_f - N_c)$ gauge theory coupled to $N_f$ flavors. When the two NS fivebranes adjoining the $N_c$ 3-branes interchange their position, there is a transition from one theory to the other which passes through the point of strong coupling for both theories. In this sense the two theories were called in [1] a continuation of each other past infinite coupling.

Exploiting the methods of [1], Elitzur, Giveon, and Kutasov [3] showed how one could move an NS 5-brane of type IIA string theory through D6-branes to continuously connect a gauge theory of one rank, $SU(N_c)$, to a gauge theory of a different rank, $SU(N_f - N_c)$, manifesting Seiberg’s duality in brane language. In the 3d $N = 4$ case, the theories have different Coulomb branches, and this motion of the NS 5 branes is interpreted as a phase transition in which the massless matter content changes during the transition. In contrast, the 4d $N = 1$ case has no Coulomb branch, and thus the Higgs branch parameterizes all of moduli space. In this sense, this transition may be called a duality. There are however problems with this interpretation which will be discussed in the sequel.

[3] also showed how to add an adjoint chiral superfield, $X$, to the brane picture and how to interpret the deformations of a particular superpotential $W = \text{Tr} \, X^{k+1}$ as the motion of $k$ NS 5-branes. They were able to connect an $SU(N_c)$ gauge theory to an $SU(kN_f - N_c)$ gauge theory by continuously moving $k$ NS 5-branes past $N_f$ D6-branes finding a result that had been known from field theory [10, 11].
In section 2, of this paper, we will review the Elitzur, Giveon, and Kutasov construction and comment on the Higgs branch and the problem of realizing chiral symmetry. In section 3, we will generalize the work of [1] and [3] by suspending two sets of D4-branes between three NS 5-branes, thus constructing product gauge groups $\text{SU}(N_c) \times \text{SU}(N'_c)$ coupled with matter field $F$ in the $(N_c, \bar{N}'_c)$ representation and conjugate field $\bar{F}$. The theory in section 3 will have a superpotential of the form $W = \text{Tr} (F \bar{F})^2$. By moving D4-branes, D6-branes, and NS 5-branes through the ten dimensional space of type IIA string theory and making [1]-type transitions, we will arrive at a new configuration of branes whose world volume theory is the dual described in [12]. After studying the simplest $N = 1$ product duality in detail from the brane point of view, we will move on in section 4 and consider the theory of [12] with a more general superpotential $W = \text{Tr} (F \bar{F})^{k+1}$ and see how that translates into brane language. In section 5, we will add adjoint matter to both gauge groups and arrive at a duality of [13]. Finally, in section 6, we will discuss the field theory and the brane configuration for a new duality involving three gauge groups and show how that naturally generalizes to a duality for $n$ gauge groups.

2. Brane configurations

2.1. Supersymmetry.

The configurations we will study involve three kinds of branes in type IIA string theory: a Neveu-Schwarz (NS) fivebrane, Dirichlet (D) sixbrane and Dirichlet fourbrane. Specifically, the branes are:

(1) NS fivebrane with worldvolume $(x^0, x^1, x^2, x^3, x^4, x^5)$, which lives at a point in the $(x^6, x^7, x^8, x^9)$ directions. The NS fivebrane preserves supercharges of the form$\epsilon_L Q_L + \epsilon_R Q_R$, with

$$\begin{align*}
\epsilon_L &= \Gamma^0 \cdots \Gamma^5 \epsilon_L \\
\epsilon_R &= \Gamma^0 \cdots \Gamma^5 \epsilon_R.
\end{align*}$$

(2.1)

(2) D6-brane with worldvolume $(x^0, x^1, x^2, x^3, x^7, x^8, x^9)$, which lives at a point in the $(x^4, x^5, x^6)$ directions. The D6-brane preserves supercharges satisfying

$$\epsilon_L = \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^7 \Gamma^8 \Gamma^9 \epsilon_R.$$  

(2.2)

$Q_L, Q_R$ are the left and right moving supercharges of type IIA string theory in ten dimensions. They are (anti-) chiral: $\epsilon_R = -\Gamma^0 \cdots \Gamma^9 \epsilon_R$, $\epsilon_L = \Gamma^0 \cdots \Gamma^9 \epsilon_L$. 

2
(3) D4-brane with worldvolume \((x^0, x^1, x^2, x^3, x^6)\) which preserves supercharges satisfying
\[
\epsilon_L = \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^6 \epsilon_R.
\]  

(2.3)

We will also use branes with different orientation in spacetime. One particular brane will be called NS’ brane and is rotated 90 degrees with respect to the NS brane in item 1.

(4) NS’ fivebrane with worldvolume \((x^0, x^1, x^2, x^3, x^8, x^9)\) preserving the supercharges
\[
\epsilon_L = \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^8 \Gamma^9 \epsilon_L
\]
\[
\epsilon_R = \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^8 \Gamma^9 \epsilon_R.
\]  

(2.4)

There are four supercharges satisfying equations (2.1)-(2.4); \(\frac{1}{8}\) of the original supersymmetry of type IIA string theory is preserved. Each relation (2.1)-(2.4) by itself would break \(\frac{1}{2}\) of the supersymmetry. Equations (2.1) and (2.2) are independent and together break to \(\frac{1}{4}\). Equation (2.3) is not independent of (2.1) and (2.2) and so breaks no more of the supersymmetry. If we only had the NS 5-brane, the D6-brane, and the D4-brane we could only make \(N = 2\) four-dimensional supersymmetric gauge theories as were studied in [3]. However, by introducing the NS’-5 brane as was done in [3], we have a new equation (2.4) that is independent of (2.1)-(2.3). Altogether the branes preserve \(\frac{1}{8}\) of the supercharges. In general we can consider rotating the 5-branes to some arbitrary angle in \((x^4, x^5, x^8, x^9)\) [14] (see a related discussion in [15]). As long as all 5-branes don’t live exclusively in \((x^4, x^5)\), we will have \(N = 1\) supersymmetric configurations.

Instead of an NS’ 5-brane, we can rotate the 6-branes to \((x^0, x^1, x^2, x^3, x^4, x^5, x^7)\), which will be called D’ branes. This gives us the relation
\[
\epsilon_L = \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^4 \Gamma^5 \Gamma^7 \epsilon_R.
\]  

(2.5)

The D’6-branes, together with the NS 5-branes and the D4-branes, break \(\frac{1}{8}\) of the supersymmetry. Of course we can have configurations with NS, NS’ D, D’ and D4 which will not break the supersymmetry further.

The presence of these branes breaks space-time Lorentz group \(SO(1, 9)\) to \(SO(1, 3) \times SO(2) \times SO(2)\), where the first group is the Lorentz group of the 3+1 dimensional theory we want to study, the second factor is the rotation group which acts on the \((x^4, x^5)\) coordinates, and the third factor is the rotation group which acts on the \((x^8, x^9)\) coordinates. The spinor representations of \(SO(1, 9)\) decompose as \([([2, 1] + (1, 2)]_{\pm \pm}\). For the 16 representation, out of these spinors, only one is not broken, and we can choose it to be \((2, 1)_{++}\). For the
representation, the unbroken spinor is \((1, 2)_{--}\). and thus both \(SO(2)\) groups act on the spinors. As such, they are both global R-symmetries. Their action on the spinors is identical, and thus we can find a linear combination which does not act on the spinors. This reduces the global symmetries into a \(U(1)_R \times U(1)_g\) symmetry. This symmetry is consistent with the global symmetry for the four dimensional \(N = 1\) supersymmetric theories that we are interested. In fact, they are identified with the symmetries of the models.

2.2. D-brane boundary conditions.

As in [1] the four branes will be finite in the \(x^6\) coordinate. The effective low energy theory on the world volume of the D4-branes will thus be macroscopically 3+1 dimensional. Because 4 supercharges are preserved, we indeed see that the theories have \(N = 1\) supersymmetry in four dimensions. The finite D4-branes can end on either 5-branes or 6-branes, a fact which can be deduced, by a sequence of T- and S-dualities from known configurations. The world volume of the D4 brane contains a \(N = 4\) \(U(1)\) multiplet (with 16 supercharges). When the D4-brane ends on the 5-brane or the 6-brane it is subject to boundary conditions which project out some of the massless states from the theory. This gives few cases in which different multiplets survive. Below, we review, following [1], what states exist for specific boundary conditions on 4-branes.

1) For a 4-brane between two NS 5-branes, there is a \(N = 1\) vector multiplet and chiral multiplet in the adjoint representation living on the world volume theory of the 4-brane (the adjoint chiral multiplet corresponds to motion of the 4-brane in the \((x^4, x^5)\) direction). When \(n\) D4-branes coincide, it seems that we get a \(U(n)\) gauge theory with a chiral adjoint matter. However, as in [3], requiring finite energy solutions on the 5-branes leads to freezing the \(U(1)\) multiplet and instead introducing a mass term for the adjoint field. Thus we get a \(SU(n)\) gauge group with chiral adjoint matter.

2) For a D4-brane between an NS 5-brane and an NS’ 5-brane there is only a vector multiplet. In general, the mass of the adjoint chiral multiplet that appeared in item one is proportional to the angle between the two 5-branes. Since the NS 5-brane and the NS’ 5-brane are at 90 degrees, the mass of the adjoint is infinite. For \(n\) such D4-branes we get \(SU(n)\) gauge theory.

3) For a D4-brane between two D6-branes, there are two chiral multiplets. These chiral multiplets are associated with motion of the D4-brane along the \((x^7, x^8, x^9)\) coordinates together with the \(A_6\) component of the gauge field. The \(A_6\) component is compact with a
radius proportional to the gauge coupling. These four scalars are conveniently written in terms of two chiral multiplets $x_8 + ix_9; x_7 + iA_6$.

4) For a 4-brane between a D6-brane and NS’ 5-brane there is one chiral multiplet $x_8 + ix_9$. In general, for D6 brane and NS brane which are parallel, namely a rotated configuration of the D6 NS’ system, there is one chiral multiplet which corresponds to motion of the D4 brane in between the two branes.

5) For a D4-brane between a D6-brane and an NS 5-brane there are no massless moduli which contribute to the low energy field theory. In general, this is true for any 5-brane that is not parallel to the 6-brane. For cases in which there are more than one brane in between the 5-brane and 6-brane these states are called S-configurations.

In addition to the above massless states, we will need also other types of massless states coming from configurations in which D6-branes meet D4-branes in space. Fundamental type IIA strings stretching between the 4-branes and the 6-branes look like particles to an observer living on the 4-brane. The endpoints of the strings are electrically charged with respect to the gauge group on the D4-brane. These strings are the fundamental fields $Q$ and $\tilde{Q}$, the quarks.

In the configuration in item five, in which a NS brane is not parallel to a D6 brane, the branes meet in space whenever their $x^6$ positions coincide. That is they meet on a line. If they pass through each other a D4 brane is created as in [1]. This is a slight generalization of the configuration discussed in [1] to an arbitrary angle, excluding one case in which the branes are parallel. In the case that the branes are parallel, they meet in space whenever three of the coordinates are tuned. In this case the branes can easily avoid each other when changing the $x^6$ coordinate by passing through the other coordinates. Thus a D4 brane will not be created during such a transition. In this sense the case when the branes are parallel is a special case. We may ask, however, what happens when the branes do meet in space by tuning their three coordinates to coincide. We will argue in section 2.4 that this leads to enhancement of gauge symmetry on the world volume mutual to both the branes which is interpreted as enhanced global chiral symmetry on the world volume of the D4 brane.

This discussion suggest two cases. Either the NS brane and D6 brane are parallel and give rise to enhanced chiral symmetry when they meet in space or the branes are not parallel and when they pass through each other a D4 brane is created.

2.3. Review of Elitzur, Giveon, and Kutasov’s transition

5
First let’s review what was done in [3]. The brane configuration of [3] is shown in fig. 1. In the electric configuration shown on the left, we have $N_c$ D4-branes stretched between an NS 5-branes on the left and an NS’ 5-brane on the right. In addition there are $N_f$ D6-branes with an $x^6$ coordinate in between the two coordinates of the 5-branes. According to our rules above, this gives us $SU(N_c)$ gauge theories with $N_f$ fundamental flavors in four-dimensions.

fig. 1 shows D4-branes which are frozen in space; there are no moduli associated with this configuration. This is consistent with the fact that $N = 1$ supersymmetric gauge theories have no Coulomb branch. The Higgs branch was described in [1] for twice the amount of supersymmetry, and we can repeat the analysis here. Transition to the Higgs branch is done by moving the D6-branes in the $(x^4, x^5)$ coordinates to a point where they touch the D4-branes. The quarks become massless and a transition to the Higgs branch is possible by breaking the D4-branes along the D6-branes and moving in the $(x^8, x^9)$.

fig. 2 shows the maximal Higgsing possible. If one counts the number of chiral multiplets according to the rules above, we see that the dimension of the Higgs branch is $2N_fN_c - N_c^2$ for $N_f > N_c$ and $N_f^2$ for $N_f \leq N_c$ in agreement with field theory.

Now we consider the transition caused by exchanging the two NS branes. We first move the NS 5-brane past the D6-branes where the transition described in [1] occurs. $N_f$ 4-branes are created stretching between the $N_f$ 6-branes on the left and the NS 5-brane on the right. As in [1] in order to avoid magnetic charge transition between the two 5-branes, we move the NS in the $x^7$ direction. This has the effect of turning on a FI term and moving to the Higgs branch. If there are D4-branes stretched between the two NS branes, such a process changes the orientation of the D4-branes in space and thus breaks supersymmetry. In order to avoid such a thing we need to reconnect $N_c$ of the 4-branes,
Fig. 2: The Higgs branch of $N = 1$ supersymmetric QCD versus the Higgs branch of $N = 2$ supersymmetric QCD. The numbers which are assigned to each D4-brane denote the number of chiral superfields which are associated with motion of the D4-brane along the branes at its ends. In the bottom picture, the $N = 2$ case, some of the branes, which carry no moduli, are not broken to avoid S-configurations. In the top picture, such a restriction does not exist for the NS’ brane. Instead there are extra moduli as indicated in the figure.

which are stretched between the 6-branes and the NS brane, to all of the 4-branes which are between the NS brane and the NS’ brane. Such a reconnection corresponds to moving the 4-branes in the $(x^4, x^5)$ direction, and thus to massless quarks. At this point there are no 4-branes between the NS brane and the NS’ 5-brane, and thus it can be moved in the $x^7$ direction without breaking the supersymmetry. Then we can exchange the position of the NS branes in the $x^6$ direction and move back to the origin of the Higgs branch by setting the $x^7$ coordinate of the NS brane to zero. Note that in this process the NS’ brane plays a passive role, namely it stays in place while all other branes move around. As such this transition can be connected to the transition in [1] by a rotation of the NS brane to NS’ brane after T-duality in the 3rd coordinate.

We end up with the configuration on the right in fig. 1. From left to right in $x^6$, $N_f$ 4-branes stretch between the $N_f$ 6-branes and the NS’ 5-brane, and $N_f - N_c$ 4-branes stretch between the NS’ and the NS. Here we have Seiberg’s dual $SU(N_f - N_c)$ gauge group. The
strings stretching between the two sets of 4-branes are the dual quarks $q$ and $\bar{q}$. The 4-branes can break completely along the rightmost 6-brane, and thus generate $N_f$ 4-branes which move freely in the 89 directions. These branes give rise to $N_f^2$ singlet fields $M$ which are the states stretched between the $N_f$ 4-branes in the $(x^8, x^9)$ direction. Only $N_f$ fields, which correspond to the “Cartan generators” of these mesons are visible in the brane picture. Moving D6-branes in the $(x^4, x^5)$ direction corresponds to giving a flavor a mass in the electric theory while this motion corresponds to Higgsing in the magnetic theory. Breaking a D4-brane on a D6-brane and moving the D4-brane in the $(x^8, x^9)$ direction corresponds in the electric theory to Higgsing. Motion of one of the $N_f$ D4-branes in the $(x^8, x^9)$ direction in the magnetic theory corresponds to giving a fundamental $Q$ field a mass. In this way, we can see that there is a superpotential $W = Mqq\bar{q}$.

![Diagram](image)

**Fig. 3:** The Higgs branch of Seiberg’s dual $SU(N_f - N_c)$ gauge theory. The dimension of the dual Higgs branch is the same as the dimension of original Higgs branch $2N_fN_c - N_c^2$.

It is interesting to check that the dual Higgs branch shown in fig. 2 has the same dimension as the original Higgs branch shown in fig. 3. This is a necessary condition if these two theories are to be called dual.

There are a few comments in order. In making the transition from one theory to another, the gauge coupling changes continuously and passes through infinite coupling of both theories. In this sense this process is a phase transition not a duality. One the other hand, the space of vacua of these theories is parameterized by Higgs expectation values which give, as is clear from the brane construction, identical moduli spaces in both theories. In this sense the name duality is justified.

Another problem is with the interpretation of the $x^7$ coordinate as a FI term for the $U(1)$ gauge group. In [6] it was argued that the gauge group which naively looks like $U(N_c)$ gauge theory is really $SU(N_c)$ by requiring finite energy solutions of the 5-branes.
This restriction removes the $U(1)$ from the theory and thus does not allow for a FI term to appear as a parameter in the field theory, according to conventional wisdom. From the point of view of the branes, this is nevertheless a well defined procedure which by analogy to the $d = 3$ $N = 4$ case bares the name FI term.

Another way to calculate the dual gauge group is to use linking numbers. As in [1], we assign a number to each NS 5-brane and D6-brane depending on the types of branes to its left or right. 6-branes or 5-branes to the right of the brane under consideration contribute $-\frac{1}{2}$ linking number while 6-branes or 5-branes to the left contribute $+\frac{1}{2}$ linking number. 4-branes to the right contribute $+1$ linking number while 4-branes to the left contribute $-1$ linking number. These linking numbers were assigned in the $N = 4$ $d = 3$ setup. However by T-duality, we can assign these numbers to the $N = 2$ $d = 4$ setup. Now the motion of branes can be calculated for the 8 supercharges case, namely when the NS branes are parallel. Once the transition is done it is possible to make a 90 degree rotation of the NS brane to become an NS' brane. This process corresponds to adding a mass term to the adjoint and thus lead from the $N = 2$ system to the $N = 1$ system in the spirit of [10] (for a related discussion, see [15]). The rotation of the NS brane, which is done without encountering any other brane in this process preserves the magnetic charge of the brane, or the linking number, and thus allows us to use these linking numbers as conserved quantities of these transitions. In the brane configuration described above, the NS 5-brane started off on the left with linking number $-\frac{N_f}{2} + N_c$. After it moved over to the right, it’s linking number became $+\frac{N_f}{2} - \tilde{N}_c$. Since the linking number for all branes must be conserved, the dual gauge group is $\tilde{N}_c = N_f - N_c$.

In [3], they also discussed a brane configuration for another field theory duality proposed by Kutasov and Schwimmer in [10,11]. This theory is an $SU(N_c)$ gauge theory with $N_f$ fundamentals $Q$ and $\tilde{Q}$, an adjoint field $X$, and a superpotential $W = \text{Tr} \; X^{k+1}$. The brane configuration is the same as above except we have $k$ NS 5-branes instead of a single NS 5-brane. Relative motion of the NS 5-branes in the $(x^8, x^9)$ directions corresponds to turning on lower order operator in the superpotential, giving the adjoint field a vacuum expectation value, and breaking the gauge group to $k$ copies of Seiberg’s duality.

2.4. Realization of Chiral Symmetry

A problem with the brane construction of [3] is that it does not have the same global symmetries as the theory of Seiberg, namely, it lacks chiral symmetry. Because of this, the brane configuration of Elitzur, Giveon, and Kutasov resembles a theory that “remembers”
it’s $N = 2$ origin. In this section, we discuss how this problem with the brane configuration can be corrected.

If we look at the $N = 2$ supersymmetric QCD, we notice that there is a term in the superpotential,

$$W = \lambda Q X \tilde{Q}. \quad (2.6)$$

$\lambda$ is fixed to $\sqrt{2}$ by the supersymmetry. This term requires that when we give a vacuum expectation value to the adjoint field, the hypermultiplets receive a mass. This term also breaks the global chiral symmetry from $SU(N_f)_R \times SU(N_f)_L$ to $SU(N_f)$. Turning off the coefficient $\lambda$ in (2.6), breaks the $N = 2$ supersymmetry to $N = 1$ while restoring the chiral symmetry. Chiral symmetry can be broken explicitly whenever the quarks receive a mass or dynamically when the quarks receive a vacuum expectation value. How can we see this chiral symmetry in the brane picture? $N = 2$ supersymmetric QCD corresponds in brane language to two parallel 5-branes extended in the $(x^4, x^5)$ direction and $N_f$ 6-branes extended in the $(x^8, x^9)$ direction. $N_c$ 4-branes connect the 5-branes. Motion of the 4-branes in the $(x^4, x^5)$ direction corresponds to giving the adjoint field a vacuum expectation value and Higgsing the gauge group to it’s Cartan subalgebra. Moving the 6-branes in the $(x^4, x^5)$ direction corresponds to giving the hypermultiplets a mass, so we naturally see that the adjoint field is coupled to the hypermultiplets as in (2.6). If we consider rotating the 6-branes from extending in the $(x^8, x^9)$ to extending in the $(x^4, x^5)$ direction, we see that now giving a mass to the hypermultiplets corresponds to motion of the 6-branes in the $(x^8, x^9)$ directions. In the brane picture, the adjoint and the fundamentals have decoupled in the superpotential. This rotation of the 6-branes corresponds [17] to turning off the parameter $\lambda$ in equation (2.6). Now we should be in a position to see the chiral symmetry. We propose that chiral symmetry is restored when the 6-branes touch a parallel 5-brane. The only direction that the 6-branes and the 5-brane do not share is $x^7$. Therefore, a 6-brane can have a boundary on the 5-brane in the $x^7$ direction. Indeed, this was proposed in [18], [19].

To support this proposal, we note that the distance between, say a NS’ brane and a D6 brane is a 3-vector in $(x^4, x^5, x^6)$ space. To make the two branes coincide, we need to tune the three coordinates in these directions. When they touch, a massless multiplet comes down in the world volume $(x^0, x^1, x^2, x^3, x^8, x^9)$ which is mutual to both branes. This world volume is six dimensional, and thus we expect to get a massless state in six dimensions when both branes meet in space. We also note that locally, in the absence of
D4 branes, the supersymmetry is broken to 8 supercharges, and thus we are dealing with (0, 1) supersymmetry in six dimensions. There are two possible types of multiplets which may become massless at the point when the branes meet: a hypermultiplet and a vector multiplet. Supersymmetry implies that vector multiplets become massless when three FI parameters are set to zero. These are the three distances mentioned above, and thus this rules out the possibility of having a massless hypermultiplet at the point in question. The distance in the \((x^4, x^5, x^6)\) directions now gets the interpretation of a FI parameter for a \(U(1)\) gauge field that becomes massless when the two branes meet in space.

In the presence of D4 branes the local supersymmetry is broken from 8 supercharges to 4 supercharges, as well as the local Lorentz invariance from \(SO(1, 5)\) to \(SO(1, 3)\). The \(U(1) \times U(1)\) gauge symmetry on the six dimensional world-volume theory becomes now global \(U(1) \times U(1)\) chiral symmetry of the four dimensional theory.

This setup can be generalized in the following way. When \(N_f\) 6-branes touch a 5-brane, they split in half, and the \((x^0, x^1, x^2, x^3, x^8, x^9)\) world volume theory becomes \(SU(N_f)_R \times SU(N_f)_L\). Strings stretching from the \(N_c\) 4-branes to the \(N_f\) 6-branes to the right in \(x^7\), are the quarks \(Q_R\) charged under \((N_c, N_f, 1)\). Strings stretching from the \(N_c\) 4-branes to the \(N_f\) 6-branes to the left in \(x^7\), are the quarks \(Q_L\) charged under \((\bar{N}_c, 1, N_f)\). Moving the 6-branes off the 5-brane in the \((x^4, x^5)\) direction breaks the six dimensional symmetry spontaneously while breaking the four dimensional chiral symmetry explicitly, as it should since motion in the \((x^4, x^5)\) corresponds in the field theory to giving the quarks a mass.

Breaking the 4-branes on the 6-branes is only possible if the 6-branes move off the 5-brane in the \((x^8, x^9)\) directions. This corresponds to Higgsing, and in agreement with field theory, Higgsing breaks chiral symmetry.

The string theory setup also predicts a phenomenon which is hard to understand from a field theory point of view. This corresponds to moving the 6-branes in the \(x^6\) direction. This motion, like the \((x^4, x^5)\) motion, corresponds to breaking the six dimensional symmetry spontaneously while breaking explicitly the four dimensional chiral symmetry. The parameter which governs that is the \(x^6\) distance between the NS’ brane and the D6 brane which is real. Such a parameter is problematic since it has no natural complex partner and thus violates holomorphy if it appears in the superpotential. Thus we expect it to appear only in D-terms. On the other hand such a parameter has no natural interpretation in the field theory. A similar problem was encountered in [1] where “magnetic couplings” were introduced by mirror symmetry which did not have a natural interpretation in the field theory setup. In [1] it was also suggested that the relative distance between NS and D
branes is irrelevant for the field theory. This is also valid when the branes are oriented in any angle which is not the parallel case. In this case we see that, when the NS and D branes are parallel, these parameters become relevant as they control the breaking of the chiral symmetry of the theory.

Finally for this section, one may wonder what are the objects for which quantization leads to the appearance of massless vector multiplets. As is, by now well known, quantization of open strings lead to massless hypermultiplets whenever two D-branes (which break to 1/4 of the supersymmetry) meet in space. It is not known however what are the states which get massless when a D brane meets a NS brane. The above analysis predicts that the states are vector multiplets. Now the only virtual states which end on both a NS brane and a D6 brane are D4 branes. There is no other brane which has this property, on the other hand, we have been using this property for D4 throughout this paper. Thus we can have virtual open D4-branes which have threebrane boundaries which propagate on the worldvolume of the D6 and NS branes. To a D6 observer they look like monopoles while for a NS observer they look like vortices. When these two branes touch, the tension of the threebranes vanishes. The worldvolume of the D4-branes consists of 0123 and a real line in the 456 space which connects the NS and D6 branes. A supersymmetric configuration which is consistent with the supersymmetries in this problem implies that the D6 and NS branes will have identical 45 positions and different $x^6$ positions. This again gives a special case to the point where in the field theory the masses are zero and chiral symmetry is expected. Thus we are led to predict that quantization of tensionless threebranes in six dimensions gives rise to massless vector multiplet in six dimensions! It would be very interesting to provide further support for this scenario.

3. **Duality with two gauge groups:** $SU(N_c) \times SU(N'_c)$

3.1. *The theory with superpotential* $W = \text{Tr} \ (F \tilde{F})^2$. 
Fig. 4: This figure is in \((x^4, x^8, x^6)\) space. The thick lines are the 5-branes which point at arbitrary angles in \((x^4, x^8)\) while the 4-branes marked by their numbers, \(N_c\) and \(N'_c\), point in the \(x^6\) direction.

In this section, unlike in section 2.3, we will find we cannot suffice with only 5-branes oriented at 0 degrees \((x^4, x^5)\) and 90 degrees \((x^8, x^9)\). We will need 5-branes at arbitrary angles in \((x^4, x^5, x^8, x^9)\). Therefore, referring to the 5-branes as NS and NS’ no longer makes sense. In this section we will label the 5-branes with letters \(A\), \(B\), and \(C\). The \(B\) 5-brane is in the middle of the other two 5-branes and is oriented at zero degrees (i.e. it points along the \((x^4, x^5)\) direction). The \(A\) 5-brane is oriented at angle \(\theta_1\) with respect to the \(B\) 5-brane, and the \(C\) 5-brane is oriented at an angle \(\theta_2\) with respect to the middle \(B\) 5-brane. We orient the \(N_f\) 6-branes in the direction parallel to the \(A\) 5-brane, and the \(N'_f\), 6-branes in the direction parallel to the \(C\) 5-brane. In this way we get the brane realization of chiral symmetries which are present in the field theory. The configuration is shown in fig. 4.

Let’s consider fig. 5. We have from left to right, \(N_c\) 4-branes stretched between the \(A\) 5-brane and the \(B\) 5-brane which is connected to the \(C\) 5-brane by \(N'_c\) 4-branes. This gives us an \(SU(N_c) \times SU(N'_c)\) gauge theory. \(N_f\) 6-branes intersect the \(N_c\) 4-branes, and \(N'_f\) 6-branes intersect the \(N'_c\) 4-branes in the \(x^6\) direction. Strings stretching between the \(N_f\) 6-branes and the \(N_c\) 4-branes are chiral multiplets \(Q\) and \(\bar{Q}\) in the fundamental representation of \(SU(N_c)\), while strings stretching between the \(N'_f\) 6-branes and the \(N'_c\) 4-branes are the chiral multiplets \(Q'\) and \(\bar{Q}'\) in the fundamental representation of \(SU(N'_c)\). Strings can also stretch between the \(N_c\) 4-branes and the \(N'_c\) 4-branes. These fields we will call \(F\) and \(\bar{F}\) and are in the \((N_c, N'_c)\) and \((\bar{N}_c, N'_c)\) representation of the gauge group respectively. We will argue below that this theory has a superpotential \(W = \text{Tr} (F \bar{F})^2\). The field theory that this brane configuration corresponds to was described in \([12]\).
Fig. 5: This figure is a different view of fig. 4. The vertical lines point in $(x^4, x^5, x^8, x^9)$. The horizontal lines point in $x^6$. $SU(N_c) \times SU(N'_c)$ gauge fields live on the D4-branes. Strings stretching between the two sets of 4-branes are the fields $F$ in the $(N_c, \tilde{N}_c)$ representation. There are strings stretching from $N_c$ 4-branes to the $N_f$ 6-branes. These are the fundamental fields $Q$ and $Q'$. There is a superpotential $W = Tr (F \tilde{F})^2$. The strings drawn near B are somewhat misleading since these strings are really inside the NS brane. It must be so because of BPS condition and minimal length criterion. 

it was shown that the dual gauge group is $SU(2N'_f + N_f - N'_c) \times SU(2N_f + N'_f - N_c)$ with singlet fields that are a one-to-one map of the mesons of the electric theory. Let’s see how this works in the brane language.

Fig. 6: This theory is the dual of fig. 4. It has $SU(2N'_f + N_f - N'_c) \times SU(2N_f + N'_f - N_c)$ gauge fields living on the D4-branes. There are hypermultiplets $F, \tilde{F}$ in the $(\tilde{N}_c, \tilde{N}'_c)$ representation coming from strings stretched along the B NS brane. There are $N_f, (N'_f)$ strings stretching from a fourbrane to the $\tilde{N}'_c(\tilde{N}_c)$ 4-branes along the A (C) 5-brane. These are the fundamental fields $q', \tilde{q}'(q, \tilde{q})$. The gauge theory also has singlet fields coming from 4-branes stretched between the NS’ 5-branes and the 6-branes.

One way to find the dual theory is to use linking numbers as we did for the Seiberg duality in section 2.3. Although in [1] the linking numbers were only used for NS 5-branes intersecting 6-branes at 90 degrees, we claim that a 4-brane is created whenever a 5-brane crosses a 6-brane as long as the 5-brane and the 6-brane are not parallel. The
$N_f$ D6-branes start out with linking number $\frac{-1}{2}$. When we move them to the right past the $B$ and $C$ 5-branes, their linking number becomes $\frac{+3}{2}$. Since the linking number for a particular brane must be conserved, we must add two 4-branes to the left side of all $N_f$ 6-branes. These are the singlets corresponding to the mesons $Q\bar{Q}$ and $Q\bar{F}F\bar{Q}$. The singlets corresponding to the mesons $Q\bar{F}Q'$ and $\bar{Q}F\bar{Q}'$ presumably become visible in the IR when the $N_f$ and $N'_f$ 6-branes intersect. The $A$ 5-branes on the far left starts out with linking number $L = -\frac{N_f}{2} - \frac{N'_f}{2} + N_c$. After the duality, the $A$ 5-brane ends up on the far right with linking number $L = -\frac{N_f}{2} + \frac{N'_f}{2} - \tilde{N}_c + 2N_f$. In order to conserve linking number the dual gauge group must be $\tilde{N}_c' = 2N_f + N'_f - N_c$. We can consider what happens when we move the $C$ 5-brane from the right to the left. This gives us the other dual gauge group $\tilde{N}_c = 2N_f' + N_f - N_c'$. It is easy to check that the linking numbers for the middle $B$ 5-brane are consistent with this dual configuration. The dual brane configuration is shown in fig. 6.

A point on the relation between chiral symmetry and the singlet mesons is now in order. In item 5 of section 2.2 we stated that there are no moduli for branes which are not parallel. On the other hand in item 4 of that section there is a chiral multiplet whenever the branes are parallel. These states are precisely the states which give the chiral mesons of the dual theory as can be seen by breaking the D4 branes to the right of the A NS brane of figure 6. Had we chosen the $N_f$ D6 branes to be nonparallel to the A NS brane, such a breaking would lead to S-configuration which is assumed to break supersymmetry. On the other hand for the parallel case, we have chiral mesons and enhanced chiral symmetry when the branes meet in space. This is in perfect agreement with the field theory expectations which thus provides further support to our claim for enhanced chiral symmetry.

Another (more tedious) way to see the dual configuration is to dualize each gauge group independently of the other. We can label each 5-brane from left to right $A, B, C$. We then move $B$ past $A$ which constitutes a Seiberg duality on the first gauge group. We have the configuration $BAC$. Then we perform another Seiberg duality on the second gauge group which brings us to $BCA$. We dualize again the first (left most) gauge group and end up with $CBA$. We have effectively switched the two end 5-branes while keeping the middle 5-brane in place. It is not hard to see that in the process, we have switched the D6-branes and put them outside of the 5-branes. It is curious to note that although there are many ways we can arrange the 6-branes and 5-branes, the dual turns out to be the one which is the “mirror” of the original configuration. We will see that this is true for theories with arbitrary numbers of gauge groups.
3.2. Derivation from $N = 2$

The brane configuration of section 3.1 can be derived from an $N = 2$ configuration. In field theory language, we begin with an $SU(N_c) \times SU(N'_c)$ gauge theory with $N_f$ hypermultiplets charged under $SU(N_c)$, $N'_f$ hypermultiplets charged under $SU(N'_c)$, and a hypermultiplet in the $(N_c, N'_c)$ representation. $N = 2$ supersymmetry forces us to have a superpotential of the form

$$W = \lambda_1 Q X_1 \tilde{Q} + \lambda_2 Q' X_2 \tilde{Q}' + F X_1 \tilde{F} + F X_2 \tilde{F}.$$  \hspace{1cm} (3.1)

Breaking the $N = 2$ supersymmetry, we set $\lambda_1 = \lambda_2 = 0$ rather than to $\sqrt{2}$. Giving the adjoint fields $X_1$ and $X_2$ masses, $m_1$ and $m_2$, respectively and integrating them out, we are left with

$$W = -\frac{1}{2} \left( \frac{1}{m_1} + \frac{1}{m_2} \right) \text{Tr} (F \tilde{F})^2.$$ \hspace{1cm} (3.2)

Notice that if $m_1 = -m_2$ then the superpotential vanishes and if we take the limit that the masses are infinite, then the superpotential again vanishes.

![Figure 7](image)

**Fig. 7:** We see the NS 5-brane configuration in the $(x^4, x^8)$ plane. The brane configuration on the right has a superpotential $W = (F \tilde{F})^2$. The configuration on the left is the same theory perturbed by a mass term $W = (F \tilde{F})^2 + \mu F \tilde{F}$. The dotted lines are not branes.

In brane language, the derivation of the superpotential amounts to beginning with the brane configuration described in section 3.1; the only difference being, as explained in section 2.1, for $N = 2$ supersymmetry we must have all 5-branes parallel in the $(x^4, x^5)$ with the 6-branes extending in the $(x^8, x^9)$ directions. We can rotate the 6-branes to 6'-branes such that they extend in the $(x^4, x^5)$ directions. This corresponds in the field
theory to setting the terms $\lambda_1$ and $\lambda_2$ to zero as explained in 2.4. We now rotate the NS 5-branes. The mass of the adjoint field on the 4-brane is proportional to the angle between the adjoining NS 5-branes. We do not want to rotate the 5-branes 90 degrees since this would give the adjoint fields infinite mass and drive the resulting superpotential term (3.2) to zero. We also don’t want to have the angles between the 5-branes equal and opposite since that also would result in $m_1 = -m_2$ and (3.2) equaling zero. We rotate each end 5-brane and all 6'-branes an angle $\theta_1$ and $\theta_2$ with respect to the center 5-brane such that $0 < \theta_1 < \frac{\pi}{2}$ and $0 < \theta_2 < \frac{\pi}{2}$ as well as $\theta_1 \neq -\theta_2$. The final configuration is shown in fig. 4 and fig. 7.

3.3. Giving the superpotential a mass

As was done in [12], we can consider what happens when we give the superpotential a mass

$$W = -\frac{1}{2} \left( \frac{1}{m_1} + \frac{1}{m_2} \right) \text{Tr} (F\tilde{F})^2 + \mu \text{Tr} F\tilde{F}. \quad (3.3)$$

The equations of motion for equation (3.3) are $F\tilde{F}F - \nu F = 0$ and $\tilde{F}F\tilde{F} - \nu \tilde{F} = 0$ where we have defined $\nu = \frac{\mu}{\frac{1}{m_1} + \frac{1}{m_2}}$. A general solution to the supersymmetric vacuum condition is to have $<F\tilde{F}> = \nu$; breaking the gauge group $SU(N_c) \times SU(N'c)$ to $SU(N_c - r)$ with $N_f$ flavors, $SU(N'_c - r)$ with $N'_f$ flavors, and $U(r)$ with $N_f + N'_f$. What happens in the brane configuration? The center $B$ 5-brane moves in the $(x^4, x^5)$ direction such that instead of the three 5-branes intersecting at a point in $(x^4, x^5, x^8, x^9)$ they intersect in three points. $N_c - r$ 4-branes move up the sloping $C$ 5-brane and connect to the middle $B$ 5-brane and intersect $N_f$ 6-branes. $N'_c - r$ 4-branes move down the sloping $A$ 5-brane and intersect $N'_f$ 6-branes. $r$ 4-branes remain in place and connect from the $A$ 5-brane to the $C$ 5-brane intersecting $N_f + N'_f$ 6-branes. This agrees nicely with what happens in field theory. We can even check the trigonometry! The distance from the $r$ 4-branes to the middle $B$ 5-brane is $\nu$. The distance from the $N_c - r$ 4-branes to the $N'_c - r$ 4-branes is $\mu$. Using elementary trigonometry and fig. 7 it is not difficult to show that

$$\mu = \nu \left( \frac{1}{\tan(\theta_1)} + \frac{1}{\tan(\theta_2)} \right). \quad (3.4)$$

From the relation for the adjoint fields mass, $m_1 = \tan(\theta_1)$ and $m_2 = \tan(\theta_2)$, we see that this is precisely what we need for agreement with the field theory.
4. A more general superpotential \( W = \text{Tr} (F \tilde{F})^{k+1} \)

4.1. Field Theory considerations

We now study the theory described in section 3.1 with a more general superpotential. Let's first examine the field theory of the theory we are trying to describe by branes. This theory was discussed in [12]. The gauge group is \( SU(N_c) \times SU(N_c') \) with matter content

| \( Q; \tilde{Q} \) | \( SU(N_c) \) | \( SU(N_c') \) | \( SU(N_f)_L \) | \( SU(N_f')_L \) | \( SU(N_f)_R \) | \( SU(N_f')_R \) | \( U(1)_R \) |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| \( N_c; \overline{N_c} \) | 1; 1 | \( N_f; 1 \) | 1; 1 | 1; \( N_f \) | 1; \( N_f \) | 1 + \( \frac{kN_c'-(k+1)N_c}{N_f'(k+1)} \) |
| \( Q'; \tilde{Q}' \) | 1; 1 | \( N_c'; \overline{N_c} \) | 1; 1 | \( N_f'; 1 \) | 1; 1 | 1; \( N_f' \) | 1 + \( \frac{kN_c-(k+1)N_c'}{N_f'(k+1)} \) |
| \( F; \tilde{F} \) | \( N_c; \overline{N_c} \) | \( N_c'; \overline{N_c} \) | 1; 1 | 1; 1 | 1; 1 | 1; 1 | \( \frac{1}{k+1} \) |

The superpotential

\[
W = \text{Tr} (F \tilde{F})^{k+1}
\]  

(4.1)

truncates the chiral ring; it’s equations of motion equate higher order operators with lower order ones. We studied the case \( k = 1 \) in section 3.1. The chiral mesons are \( M_j = Q(\tilde{F}F)^j \tilde{Q}, M'_j = Q'(\tilde{F}F)\tilde{Q}' \), \( P_r = Q(\tilde{F}F)^{r-1} \tilde{F}Q' \) and \( \tilde{P}_r = \tilde{Q}F(\tilde{F}F)^{r-1} \tilde{Q}' \), where \( j = 0 \ldots k \) and \( r = 0 \ldots k - 1 \).

The dual theory has gauge group \( SU(\tilde{N}_c) \times SU(\tilde{N}_c') \), with \( \tilde{N}_c = (k+1)(N_f + N_f') - N_f - N_c' \) and \( \tilde{N}_c' = (k+1)(N_f' + N_f) - N_f' - N_c \). The matter content is

| \( q; \bar{q} \) | \( SU(\tilde{N}_c) \) | \( SU(\tilde{N}_c') \) | \( SU(N_f)_L \) | \( SU(N'_f)_L \) | \( SU(N_f)_R \) | \( SU(N'_f)_R \) | \( U(1)_R \) |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| \( \tilde{N}_c; \overline{\tilde{N}_c} \) | 1; 1 | \( \overline{N_f}; 1 \) | 1; 1 | 1; \( \overline{N_f} \) | 1; 1 | 1 + \( \frac{kN_c'-(k+1)N_c}{N_f'(k+1)} \) |
| \( q'; \bar{q}' \) | 1; 1 | \( \tilde{N}_c'; \overline{\tilde{N}_c'} \) | \( \overline{N_f}; 1 \) | 1; 1 | 1; \( \overline{N_f} \) | 1; 1 | 1 + \( \frac{kN_c-(k+1)N_c'}{N_f'(k+1)} \) |
| \( f; \bar{f} \) | \( \tilde{N}_c; \overline{\tilde{N}_c} \) | \( \tilde{N}_c'; \overline{\tilde{N}_c'} \) | 1; 1 | 1; 1 | 1; 1 | 1; 1 | \( \frac{1}{k+1} \) |

Notice that in the dual, the fundamental fields have switched global symmetries. There are also singlet fields \( M_j, M_j', P_r, \) and \( \tilde{P}_r \) which are a one-to-one map of the mesons of the electric theory. The dual superpotential is

\[
W = \text{Tr} (f \tilde{f})^{k+1} + \sum_{j=0}^{k} M_{k-j} q' (f \tilde{f})^j q' + M'_{k-j} \tilde{q} (f \tilde{f})^j q
\]  

(4.2)

\[
+ \sum_{r=0}^{k-1} P_{k-r-1} q \tilde{f} (f \tilde{f})^r q' + \tilde{P}_{k-r-1} \tilde{q}' (f \tilde{f})^r \tilde{f} \tilde{q}'
\]
Under perturbation by a mass term, $W = \text{Tr} \left( F\tilde{F} \right)^{k+1} + m \text{Tr} F\tilde{F}$, the gauge group breaks to a product of decoupled defining models

$$SU(N_c - N'_c + p_0) \times SU(p_0) \times U(p_1) \times \ldots \times U(p_k)$$

(for $N_c \geq N'_c$) with $\sum_{\ell=0}^{k} p_\ell = N'_c$. The first (second) factor has $N_f$ ($N'_f$) fundamental flavors, while the others have $N_f + N'_f$ flavors. The magnetic theory flows to the dual of this product.

If $N'_f + N_c = N'_c + 1$ the $SU(N'_c)$ factor can confine \cite{20} leaving a theory of $\mathbb{I}[\mathbb{I}]$, \cite{21} (see sect. 2.3) with $SU(N_c)$, an adjoint tensor $\tilde{X} \sim F\tilde{F}$, $N_f + N'_f + 1$ flavors and $W = \text{Tr} \tilde{X}^{k+1}$. In the magnetic theory $SU(\tilde{N}'_c)$ confines similarly (since $N'_f + \tilde{N}_c = \tilde{N}'_c + 1$) leaving a theory with $SU(\tilde{N}_c) = SU[k(N_f + N'_f) + 1 - N_c]$ and a similar superpotential; this is dual $\mathbb{I}[\mathbb{I}]$ to the confined electric model.

4.2. Brane configuration

We will use the notation of \cite{3} to label the 5-branes. In brane language we claim that the theory of Intriligator, Leigh, and Strassler described in section 4.1 corresponds to, from left to right, $k$ 5-branes in the orientation of $A$ connected to a single $B$ 5-brane by $N_c$ 4-branes. The single $B$ 5-brane is in turn connected to $k$ more $C$ 5-branes on the right by $N'_c$ 4-branes. $N_f$ and $N'_f$ 6-branes intersect the $N_c$ and $N'_c$ 4-branes, respectively.

We obtain the dual configuration by reversing the order of the 5-branes and 6-branes while preserving their linking numbers. From left to right, we have $N'_f$ 6-branes connected to $k$ $C$ 5-branes by $(k + 1)N'_f$ 4-branes. The $k$ $C$ 5-branes are connected to the single $B$ 5-brane by $\tilde{N}_c$ 4-branes. $\tilde{N}'_c$ 4-branes connect the single $B$ 5-brane to $k$ $A$ 5-branes. $(k + 1)N_f$ 4-branes in turn connect the $k$ $A$ 5-branes to $N_f$ 6-branes. The linking numbers show that the number of 4-branes is $\tilde{N}_c = (k + 1)(N_f + N'_f) - N_f - N'_f$ and $\tilde{N}'_c = (k + 1)(N_f + N'_f) - N'_f - N_c$ in agreement with the field theory.

Moving the $k$ 5-branes apart separates the 4-branes and breaks the gauge groups. This corresponds in field theory to giving the superpotential a mass $W = \text{Tr} \left( F\tilde{F} \right)^{k+1} + m \text{Tr} F\tilde{F}$. Similarly to section 3.3, we see that we get the same gauge groups as we found in the field theory under this perturbation to the superpotential.

We can consider what happens if we move the single $B$ 5-brane past $k$ $A$ 5-branes. This constitutes a Seiberg duality since we have moved a single 5-brane through $N_f$ 6-branes and avoided $k$ $A$ 5-branes in $x^7$. If $N'_f + N_c = N'_c + 1$, then there will be one 4-branes between the $B$ and $A$ 5-branes; the gauge group has confined. The brane configuration between the $A$ and $C$ 5-branes is that of \cite{3} reviewed in section 2.3. We have realized in terms of branes the confinement described in the field theory in section 4.1.
5. A product of two gauge groups with adjoint matter.

5.1. Field theory considerations

We can now use what we have learned about branes to understand another theory in terms of branes that was originally formulated in terms of field theory in [13] and [22]. The gauge groups is $SU(N_c) \times SU(N'_f)$ with matter content

|         | $SU(N_c)$     | $SU(N'_c)$   | $SU(N_f)_L$ | $SU(N'_f)_L$ | $SU(N_f)_R$ | $SU(N'_f)_R$ | $U(1)_R$          |
|---------|---------------|--------------|-------------|--------------|-------------|--------------|-------------------|
| $Q; \tilde{Q}$ | $N_c; \overline{N_c}$ | $1; 1$       | $N_f; 1$    | $1; 1$       | $1; N_f$    | $1; 1$       | $1 + \frac{N'_f - 2N_c}{N_f(k+1)}$ |
| $Q'; \tilde{Q}'$ | $1; 1$       | $N'_c; \overline{N_c}$ | $1; 1$       | $1; 1$       | $1; N'_f$ | $1; 1$       | $1 + \frac{N_c - 2N'_f}{N_f(k+1)}$ |
| $F; \tilde{F}$ | $N_c; \overline{N_c}$ | $N'_c; \overline{N'_c}$ | $1; 1$       | $1; 1$       | $1; N'_f$ | $1; 1$       | $\frac{k}{k+1}$ |
| $X_1$ | $N_c^2 - 1$ | $1$           | $1$          | $1$          | $1$ | $1$ | $1 + \frac{2}{k+1}$ |
| $X_2$ | $1$           | $(\overline{N'_c})^2 - 1$ | $1$          | $1$          | $1$ | $1$ | $\frac{2}{k+1}$ |

The superpotential is

$$W = \text{Tr} \, X_1^{k+1} + \text{Tr} \, X_2^{k+1} + \text{Tr} \, X_1 \tilde{F} F - \text{Tr} \, X_2 \tilde{F} F + \rho_1 \text{Tr} \, X_1 + \rho_2 \text{Tr} \, X_2.$$  \hspace{1cm} (5.1)

$\rho_1$ and $\rho_2$ are Lagrange multipliers to enforce the tracelessness condition. Here $k$ can be any positive integer. It follows from the conditions for a supersymmetric vacuum that the chiral ring truncates. The gauge invariant mesons in the theory are $QX_1^j \tilde{Q}$, $Q'X_2^j \tilde{Q}'$, $QX_1^j \tilde{F} F', QF X_2^j \tilde{Q}'$, $Q \tilde{F} F X_1^j \tilde{Q}$, $Q' \tilde{F} F X_2^j \tilde{Q}'$, where $j = 0 \cdots k - 1$.

The dual theory is described by an $SU(2kN'_f + kN_f - N'_c) \times SU(2kN_f + kN'_f - N_c)$ gauge theory with matter content

|         | $SU(\tilde{N}_c)$  | $SU(\tilde{N}'_c)$ | $SU(N_f)_L$ | $SU(N'_f)_L$ | $SU(N_f)_R$ | $SU(N'_f)_R$ | $U(1)_R$          |
|---------|---------------------|--------------------|-------------|--------------|-------------|--------------|-------------------|
| $q; \tilde{q}$ | $\tilde{N}_c; \overline{N_c}$ | $1; 1$           | $1; 1$       | $\overline{N'_f}; 1$ | $1; 1$       | $1; \overline{N'_f}$ | $1 + \frac{N'_f - 2N_c}{N_f(k+1)}$ |
| $q'; \tilde{q}'$ | $1; 1$      | $\tilde{N}'_c; \overline{N'_c}$ | $N_f; 1$    | $1; 1$       | $1; \overline{N_f}$ | $1; 1$       | $1 + \frac{N_c - 2N'_f}{N_f(k+1)}$ |
| $\tilde{F}; \tilde{F}$ | $\tilde{N}_c; \overline{N_c}$ | $\tilde{N}'_c; \overline{N'_c}$ | $1; 1$       | $1; 1$       | $1; \overline{N_f}$ | $1; 1$       | $\frac{k}{k+1}$ |
| $\tilde{X}_1$ | $\tilde{N}_c^2 - 1$ | $1$               | $1$          | $1$          | $1$ | $1$ | $1 + \frac{2}{k+1}$ |
| $\tilde{X}_2$ | $1$           | $(\overline{N'_c})^2 - 1$ | $1$          | $1$          | $1$ | $1$ | $\frac{2}{k+1}$ |

There are also gauge singlet mesons in the magnetic theory which are the images of mesons in the electric theory [13]. The dual superpotential is analogous to (5.1) with the addition
of coupling terms between singlets and dual mesons. For the case $k = 1$, this reduces to the model discussed in section 3.1.

When $kN_f + N'_c = N_c + 1$ the $SU(N_c)$ theory confines. Fields $F\tilde{F}$ becomes one field $\hat{Y}$, and in the superpotential we have three adjoints fields $X_1, X_2,$ and $\hat{Y}$ where only one linear combination of $V = X_1 + X_2$ is massless. The magnetic theory confines similarly (since $kN'_f + \tilde{N}'_c = \tilde{N}_c + 1$) leaving $SU(k(N_f + N'_f) + 1 - N'_c)$ and $W = V^{k+1}$. The theory therefore confines to a Kutasov-type theory with $N_f + N'_f + 1$ flavors [10,11], [21].

5.2. Brane configuration

Here we use $A, B,$ and $C$ to denote 5-branes with orientations as discussed in section 3.1. In brane language we claim that the model discussed in section 5.1 corresponds to, from left to right, $k$ $A$ 5-branes connected to $k$ $B$ 5-branes by $N_c$ D4-branes. The $k$ $B$ 5-branes are in turn connected to $k$ $C$ 5-brane by $N'_c$ 4-branes. The adjoint fields $X_1$ and $X_2$ correspond to the motion of the $k$ 5-branes in $(x^4, x^5, x^8, x^9)$. $N_f$ and $N'_f$ 6-branes intersect the $N_c$ and $N'_c$ 4-branes respectively.

The dual configuration for this theory is $k$ copies of the theory discussed in section 3.1. Linking numbers can be used to show that the dual gauge group is $SU(2kN_f + kN'_f - N_c) \times SU(2kN'_f + kN_f - N'_c)$.

If we switch all of the $k$ $A$ 5-branes with the $k$ $B$ 5-branes, and we satisfy the condition $kN_f + N'_c = N_c + 1$, then we will have one 4-brane between the $B$ and $A$ 5-branes and $N'_c$ 4-branes between the $A$ and $C$ 5-branes. Now, if we reverse the order of the $A$ and $C$ branes, the linking numbers give $kN_f + kN'_f + 1 - N'_c$ 4-branes. The $SU(N_c)$ group has confined, and we are left with the brane configuration of the Kutasov and Schwimmer duality discussed in section 2.3 and in [3] in agreement with the field theory discussed in section 5.1.

6. More than two groups dualities

Given that we have been able to form product dualities by generalizing a brane configuration involving one set of 4-branes suspended between two 5-branes to a configuration involving two sets of 4-branes suspended between three NS 5-branes, it is natural to ask if we can generalize this to configurations with three or more sets of 4-branes suspended between four or more NS 5-branes. We find that the answer is yes; such duals do exist. The field theory corresponding to such brane configurations have not been analyzed prior to this paper.
6.1. $SU(N_c) \times SU(N'_c) \times SU(N''_c)$

Consider $SU(N_c) \times SU(N'_c) \times SU(N''_c)$ with a field $F$ charged under $(N_c, \bar{N}_c')$ and a field $G$ charged under $(N'_c, \bar{N}_c'')$ and their conjugates $\tilde{F}$ and $\tilde{G}$. There are also $N_f$ fields $Q$, $N_f'$ fields $Q'$, $N_f''$ fields $Q''$ and their conjugates, $\tilde{Q}$, $\tilde{Q}'$, and $\tilde{Q}''$, that transform in the fundamental representation of their respective gauge groups (where we have suppressed all indices).

|           | $Q; \tilde{Q}$ | $Q'; \tilde{Q}'$ | $Q''; \tilde{Q}''$ | $F; \tilde{F}$ | $G; \tilde{G}$ |
|-----------|----------------|------------------|---------------------|----------------|----------------|
| $SU(N_c)$ | $N_c; \bar{N}_c$ | 1; 1             | 1; 1                | $N_c; \bar{N}_c$ | 1; 1           |
| $SU(N'_c)$| 1; 1            | $N'_c; \bar{N}'_c$ | 1; 1                | $N'_c; \bar{N}'_c$ | $N'_c; \bar{N}'_c$ |
| $SU(N''_c)$| 1; 1            | 1; 1             | $N''_c; \bar{N}''_c$ | 1; 1 | $N''_c; \bar{N}''_c$ |
| $SU(N_f)_{R}$| 1; $N_f$         | 1; 1             | 1; 1                | 1; 1           | 1; 1           |
| $SU(N_f)_{L}$| $\bar{N}_f; 1$ | 1; 1             | 1; 1                | 1; 1           | 1; 1           |
| $SU(N'_f)_{R}$| 1; 1            | 1; $N'_f$        | 1; 1                | 1; 1           | 1; 1           |
| $SU(N'_f)_{L}$| 1; 1            | $\bar{N}'_f; 1$ | 1; 1                | 1; 1           | 1; 1           |
| $SU(N''_f)_{R}$| 1; 1            | 1; 1             | 1; $N''_f$          | 1; 1           | 1; 1           |
| $SU(N''_f)_{L}$| 1; 1            | 1; 1             | $\bar{N}''_f; 1$   | 1; 1           | 1; 1           |
| $U(1)_R$   | $1 + \frac{N'_c-2N_c}{2N_f}$ | $1 + \frac{N''_c+N_c-2N'_c}{2N'_f}$ | $1 + \frac{N'_c-2N''_c}{2N''_f}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |

To truncate the chiral ring we add the superpotential

$$W = \frac{1}{2} \text{Tr} (F \tilde{F})^2 + \text{Tr} F \tilde{F} G \tilde{G} - \frac{1}{2} \text{Tr} (G \tilde{G})^2.$$ \hfill (6.1)

The equations determining the supersymmetric minima are

$$\tilde{F} F \tilde{F} + \tilde{F} G \tilde{G} = 0$$
$$\tilde{G} F F - \tilde{G} G \tilde{G} = 0$$
$$F \tilde{F} G + G \tilde{G} F = 0$$
$$F \tilde{F} G - G \tilde{G} F = 0$$ \hfill (6.2)

Multiplying the first equation in (6.2) by $\tilde{F} F$ from the left and the right, we obtain

$$\tilde{F} F \tilde{F} F F F F = -\tilde{F} F \tilde{F} G \tilde{G} \tilde{G} = \tilde{F} G \tilde{G} \tilde{G} \tilde{G} \tilde{G}$$
$$\tilde{F} F \tilde{F} F F F F = -\tilde{F} G \tilde{G} F \tilde{F} = -\tilde{F} G \tilde{G} \tilde{G} \tilde{G}$$ \hfill (6.3)
where we have used the first equation in (6.2) in the first equation in (6.3) and the second equation in (6.2) in the second equation in (6.3). Thus, \( \bar{F}F\bar{F}FF = GGG\bar{G}\bar{G} = 0 \) and \( F\bar{F}F\bar{F} = -\bar{G}GG\bar{G} \).

The mesons in the theory are \( M_0^{0,0} = Q\bar{Q}, M_2^{0,0} = \bar{Q}\bar{F}F\bar{Q}, M_4^{0,0} = Q\bar{F}F\bar{FF}\bar{Q}, M_1^{0,1} = Q\bar{F}Q', M_3^{0,1} = Q\bar{FF}\bar{F}\bar{Q}' \), \( \tilde{M}_2^{0,2} = \bar{Q}F\bar{G}\bar{Q}' \), \( M_0^{1,1} = Q'\bar{Q}' \), \( M_2^{1,1} = Q'\bar{FF}\bar{Q}' \), \( M_4^{1,1} = Q'\bar{F}F\bar{FF}\bar{Q}' \), \( M_3^{1,2} = Q'\bar{GG}\bar{G}' \), \( M_4^{2,2} = Q''\bar{Q}' \), \( M_2^{2,2} = Q''\bar{GG}\bar{G}' \), \( M_4^{1,2} = Q'\bar{GG}'' \), \( M_3^{1,2} = Q'\tilde{G}\tilde{G}Q'' \).

The dual gauge group is \( SU(3N_f' + 2N_f + N_f' - N_c') \times SU(4N_f' + 2N_f + 2N_f' - N_c) \times SU(3N_f + 2N_f' + N_f' - N_c) \) with a field \( f \) charged under \( (\bar{N}_c, \bar{N}_c') \) and a field \( \tilde{g} \) charged under \( (\tilde{N}_c', \tilde{N}_c'') \) and their conjugates \( f \) and \( \tilde{g} \). There are also \( N_f' \) fields \( q', N_f \) fields \( q'' \) and their conjugates, \( \tilde{q}', \tilde{q}'' \), and \( \tilde{q}'', \) that transform in the fundamental representation of \( SU(\bar{N}_c) \times SU(\bar{N}_c') \times SU(\bar{N}_c'') \) respectively. We note that as in section 4.1, the fundamentals have reversed their global symmetries in the dual.

|  |  |  |  |  |  |
|---|---|---|---|---|
| \( SU(\bar{N}_c) \) | \( \bar{N}_c; \bar{N}_c \) | \( 1; 1 \) | \( 1; 1 \) | \( \bar{N}_c; \bar{N}_c \) | \( 1; 1 \) |
| \( SU(\bar{N}_c') \) | \( 1; 1 \) | \( \bar{N}_c'; \bar{N}_c' \) | \( 1; 1 \) | \( \bar{N}_c'; \bar{N}_c' \) | \( \bar{N}_c'; \bar{N}_c' \) |
| \( SU(\bar{N}_c'') \) | \( 1; 1 \) | \( \bar{N}_c''; \bar{N}_c'' \) | \( 1; 1 \) | \( \bar{N}_c''; \bar{N}_c'' \) | \( \bar{N}_c''; \bar{N}_c'' \) |
| \( SU(N_f)_R \) | \( 1; 1 \) | \( 1; 1 \) | \( 1; N_f \) | \( 1; 1 \) | \( 1; 1 \) |
| \( SU(N_f)_L \) | \( 1; 1 \) | \( 1; 1 \) | \( \bar{N}_f; 1 \) | \( 1; 1 \) | \( 1; 1 \) |
| \( SU(N_f')_R \) | \( 1; 1 \) | \( 1; N_f' \) | \( 1; 1 \) | \( 1; 1 \) | \( 1; 1 \) |
| \( SU(N_f')_L \) | \( 1; 1 \) | \( \bar{N}_f'; 1 \) | \( 1; 1 \) | \( 1; 1 \) | \( 1; 1 \) |
| \( SU(N'_f)_R \) | \( 1; N_f' \) | \( 1; 1 \) | \( 1; 1 \) | \( 1; 1 \) | \( 1; 1 \) |
| \( SU(N'_f)_L \) | \( \bar{N}_f'; 1 \) | \( 1; 1 \) | \( 1; 1 \) | \( 1; 1 \) | \( 1; 1 \) |
| \( U(1)_R \) | \( 1 + \frac{2N_e - N_c - 4N_f}{2N_f} \) | \( 1 + \frac{2N_e - N_c - N_c' - 4N_f}{2N_f} \) | \( 1 + \frac{2N_e - N_c - 4N_f}{2N_f} \) | \( \frac{1}{2} \) | \( \frac{1}{2} \) |

There is a one-to-one map from the mesons of the original theory to the singlets of the
The dual superpotential is analogous to (6.1), but has coupling to singlets.

\[ W = \text{Tr} (f \tilde{f})^2 + \text{Tr} f \tilde{f} g \tilde{g} - \text{Tr} g \tilde{g} g \tilde{g} + M_0^{0,0} q'' f \tilde{f} \tilde{f} q'' + M_2^{0,0} q'' f \tilde{f} q'' + M_4^{0,0} q'' \\
+ M_1^{0,1} q' f \tilde{f} q'' + M_3^{0,1} q' f \tilde{f} q'' + \tilde{M}_1^{0,1} q' f \tilde{f} q'' + \tilde{M}_3^{0,1} q' f \tilde{f} q'' + M_2^{0,2} q f g q'' + \tilde{M}_2^{0,2} q f g q'' \\
+ M_0^{1,1} q' f \tilde{f} q q' + M_2^{1,1} q' f \tilde{f} q q' + M_2^{1,1} q' g g q' + M_4^{1,1} q' \\
+ M_0^{2,2} q f \tilde{f} q + M_2^{2,2} q f \tilde{f} q + M_4^{2,2} q \\
+ M_1^{1,2} q' f \tilde{f} q q' + M_3^{1,2} q' f \tilde{f} q q' + \tilde{M}_1^{1,2} q' f \tilde{f} q q' + \tilde{M}_3^{1,2} q' f \tilde{f} q q'. \]

(6.4)

The fact that the dual superpotential is consistent with the global symmetries is a non-trivial test of the duality. We have checked that the 't Hooft anomaly matching conditions are satisfied.

On the electric side, it is possible to add a mass term for $F$.

\[ W = \text{Tr} (F \tilde{F})^2 + \text{Tr} F \tilde{F} G \tilde{G} - \text{Tr} (G \tilde{G})^2 + m \text{Tr} F \tilde{F}. \]

(6.5)

The supersymmetric minima allows for a generic solution where $< F \tilde{F} >$ and $< G \tilde{G} >$ are not zero, and the gauge group is broken to $SU(N_c - r_1 - r_2)$ with $N_f$ flavors, $SU(N'_c - r_1 - r_2 - r_3)$ with $N'_{f}$ flavors, $SU(N''_c - r_2 - r_3)$ with $N''_{f}$ flavors, $U(r_1)$ with $N_{f} + N'_{f}$ flavors, $U(r_3)$ with $N_{f} + N'_{f}$ flavors, and $U(r_2)$ with $N_{f} + N'_{f} + N''_{f}$ flavors. The fields $F$ and $G$ are generically massive. In the dual theory, we add the dual operator $m f \tilde{f}$ to the dual superpotential. The dual theory breaks to $SU(N_{f} - N_{c} + r_1 + r_2)$ with $N_{f}$ flavors, $SU(N'_{f} - N'_{c} + r_1 + r_2 + r_3)$ with $N'_{f}$ flavors, $SU(N''_{f} - N''_{c} + r_2 + r_3)$ with $N''_{f}$ flavors, $U(N_{f} + N'_{f} - r_1)$ with $N_{f} + N'_{f}$ flavors, $U(N'_{f} + N''_{f} - r_3)$ with $N'_{f} + N''_{f}$ flavors, and $U(N_{f} + N'_{f} + N''_{f} - r_2)$ with $N_{f} + N'_{f} + N''_{f}$ flavors.

There is a special case where $r_1 = r_2 = r_3 = 0$ and $< F \tilde{F} >= < G \tilde{G} >= 0$ where the fields $G$ and $\tilde{G}$ are massless. The theory then flows to a Seiberg duality and the duality discussed in section 3.1 and in [12]. The fact that we can connect this duality to other known dualities is another important consistency check.

6.2. Brane configurations of theories with three gauge groups

To describe a theory with three gauge groups we must have four 5-branes. We will label these branes $A$, $B$, $C$, and $D$ (we hope that the reader will not find a D NS 5-brane confusing). As before, the four 5-branes are oriented at different angles in $(x^4, x^5, x^8, x^9)$; it is important that no two 5-branes are parallel. Consider the brane configuration described
The theory has $SU(N_c) \times SU(N'_c) \times SU(N''_c)$ gauge fields living on the D4-branes. Strings stretching between the first two sets of 4-branes are the fields $F$ in the $(N_c, \bar{N}'_c)$ representation. Strings stretching between the second two sets of 4-branes are the fields $G$ in the $(N'_c, \bar{N}''_c)$ representation. There are $N_f$ strings stretching from a 4-brane to the $N_f$ 6-branes. These are the fundamental fields $Q, Q',$ and $Q''$. There is a superpotential $W = \text{Tr} \ (F \tilde{F})^2 + \text{Tr} \ F\tilde{F}GG - \text{Tr} \ (GG)^2$.

**Fig. 8:** The theory has $SU(N_c) \times SU(N'_c) \times SU(N''_c)$ gauge fields living on the D4-branes. Strings stretching between the first two sets of 4-branes are the fields $F$ in the $(N_c, \bar{N}'_c)$ representation. Strings stretching between the second two sets of 4-branes are the fields $G$ in the $(N'_c, \bar{N}''_c)$ representation. There are $N_f$ strings stretching from a 4-brane to the $N_f$ 6-branes. These are the fundamental fields $Q, Q'$, and $Q''$. There is a superpotential $W = \text{Tr} \ (F \tilde{F})^2 + \text{Tr} \ F\tilde{F}GG - \text{Tr} \ (GG)^2$.

In the electric theory, the 5-branes are ordered $A, B, C, D$. To find the magnetic theory, we claim we should reverse the ordering to $D, C, B, A$. The 6-branes also have their ordering inverted. In order to have the linking numbers match up with the results of field theory we found it necessary to introduce some semi-infinite 4-branes. The dual brane configuration is shown in fig. 9. From left to right we have $2N'_f$ semi-infinite 4-branes ending on $N'_f$ 6-branes. $2N'_f$ 4-branes stretch between the $N'_f$ 6-branes and the NS 5-brane which has been labeled $D$. $2N''_f$ more 4-branes stretch between the $N'_f$ 6-branes and the $C$ 5-brane. Proceeding to the right, $3N''_f$ 4-branes stretch between the $N''_f$ 6-branes and
Fig. 9: Here we have the dual configuration $SU(\tilde{N}_c) \times SU(\tilde{N}'_c) \times SU(\tilde{N}''_c)$ on the world volume of the 4-branes. Note the appearance of semi-infinite 4-branes.

6.3. Three product dualities with adjoints

It is also possible to form products of three gauge groups with adjoint matter. Consider $SU(N_c) \times SU(N'_c) \times SU(N''_c)$ with fields $X_1$ in the adjoint representation of $SU(N_c)$, $X_2$ in the adjoint representation of $SU(N'_c)$, and $X_3$ in the adjoint representation of $SU(N''_c)$. As before, we also have fields $F$ charged under $(N_c, \bar{N}'_c)$ and fields $G$ charged under $(N'_c, \bar{N}''_c)$ and their conjugates $\tilde{F}$ and $\tilde{G}$. There are also $N_f$ fields $Q$, $N'_f$ fields $Q'$, $N''_f$ fields $Q''$ and their conjugates, $\tilde{Q}$, $\tilde{Q}'$, and $\tilde{Q}''$, that transform in the fundamental representation of their respective gauge groups. To truncate the chiral ring we add the superpotential

$$W = \frac{1}{k+1} \text{Tr} \ X_1^{k+1} + \text{Tr} \ X_1 F \tilde{F} + \frac{1}{k+1} \text{Tr} \ X_2^{k+1} + \text{Tr} \ X_2 G \tilde{G} + \text{Tr} \ X_2 F \tilde{F}$$

$$+ \frac{1}{k+1} \text{Tr} \ X_3^{k+1} - \text{Tr} \ X_3 G \tilde{G} + \rho_1 \text{Tr} \ X_1 + \rho_2 \text{Tr} \ X_2 + \rho_3 \text{Tr} \ X_3.$$  \hspace{1cm} (6.6)
The conditions for supersymmetric minima are

\[ X_1^k + \bar{F}F + \rho_1 = 0 \]
\[ X_2^k + F\bar{F} - \bar{G}G + \rho_2 = 0 \]
\[ X_3^k + G\bar{G} + \rho_3 = 0 \]
\[ X_1F - FX_2 = 0 \]
\[ X_1\bar{F} - \bar{F}X_2 = 0 \]
\[ X_2G - GX_3 = 0 \]
\[ X_2\bar{G} - \bar{G}X_3 = 0 \]

(6.7)

For \( k \) odd it is possible to show that equations (6.7) reduce to equations (6.2), and thus the chiral ring truncates. The mesons in the theory are \( M_{0,j} = QX_1^j\bar{Q}, M_{2,j} = QX_1^j\bar{F}F\bar{Q} \), \( M_{0,j}^{0,0} = QX_1^j\bar{F}F\bar{F}F\bar{Q}, M_{1,j}^{0,1} = QX_1^j\bar{F}F\bar{F}Q', M_{0,j}^{0,2} = QX_1^j\bar{F}G\bar{G}Q'' \), \( M_{1,j}^{0,1} = \bar{Q}FX_1^j\bar{Q}, M_{1,j}^{0,1} = \bar{Q}FF\bar{F}\bar{X}_2^jQ', M_{2,j}^{0,2} = \bar{Q}F\bar{G}X_3^jQ'', M_{0,j}^{1,1} = Q'X_2^j\bar{Q}', M_{2,j}^{1,1} = Q'X_2^j\bar{F}F\bar{Q}', M_{0,j}^{2,2} = Q''X_3^j\bar{Q}'', M_{2,j}^{2,2} = Q''X_3^j\bar{G}\bar{G}Q''', M_{3,j}^{1,2} = Q'X_2^j\bar{G}\bar{G}Q''' \) and \( M_{1,j}^{1,2} = Q'X_2^j\bar{G}\bar{G}Q''' \) where \( j = 0, \ldots, k \).

The dual gauge group is \( SU(3kN'_f + 2kN'_f + kN_f - N''_c) \times SU(4kN'_f + 2kN_f + 2kN'_{f''} - N'_c) \times SU(3kN_f + 2kN'_f + kN''_f - N_c) \). There is similar matter content charged under the dual gauge group, and as usual, a map from mesons of the original theory to singlets of the dual theory. The dual superpotential is

\[ W = \text{Tr} \, \bar{X}_1^{k+1} + \text{Tr} \, \bar{X}_1 \bar{f} \bar{f} + \text{Tr} \, \bar{X}_2 \bar{f} + \text{Tr} \, \bar{X}_2 \bar{g} \bar{g} + \text{Tr} \, \bar{X}_3^{k+1} - \text{Tr} \, \bar{X}_3 \bar{g} \bar{g} \]

\[ + \rho_1 \text{Tr} \, \bar{X}_1 + \rho_2 \text{Tr} \, \bar{X}_2 + \rho_3 \text{Tr} \, \bar{X}_3 \]

\[ + M_{0,j}^{0,0} \bar{q}' \bar{X}_3 f \bar{f} \bar{f} \bar{q}'' + M_{2,j}^{0,0} \bar{q}'' \bar{X}_3^j \bar{f} \bar{f} \bar{q}'' + M_{4,j}^{0,0} \bar{q}'' \bar{X}_3^j \bar{q}'' \]

(6.8)

The fact that the dual superpotential is consistent with the global symmetries is a non-trivial test of the duality. We have checked that the ’t Hooft anomaly matching conditions are satisfied.
6.4. Generalization to $n$ gauge groups and $n + 1$ 5-branes

We consider theories with $n$ gauge groups: $SU(N_{c_0}) \times SU(N_{c_1}) \times \ldots \times SU(N_{c_{n-1}})$. Each gauge group has fundamental fields $N_{f_0}$, $N_{f_1}$, $\ldots$, $N_{f_{n-1}}$. The fields that link the gauge groups together are $F_{01}$ in the $(N_{c_0}, \tilde{N}_{c_1}, 1, 1, \ldots, 1)$, $F_{12}$ in the $(1, N_{c_1}, \tilde{N}_{c_2}, 1, \ldots, 1)$, $F_{23}$ in the $(1, 1, N_{c_2}, \tilde{N}_{c_3}, \ldots, 1)$, $\ldots$ $F_{n-2,n-1}$ in the $(1, 1, 1, \ldots, N_{c_{n-2}}, \tilde{N}_{c_{n-1}})$ and their conjugate fields. The superpotential is

$$W = \left(F_{01} \tilde{F}_{10}\right)^2 + F_{01} \tilde{F}_{10} F_{12} \tilde{F}_{21} + \left(F_{12} \tilde{F}_{21}\right)^2 + F_{12} \tilde{F}_{21} F_{23} \tilde{F}_{32}$$

$$-(F_{23} \tilde{F}_{32})^2 + \ldots + F_{n-3,n-2} \tilde{F}_{n-2,n-3} F_{n-2,n-1} \tilde{F}_{n-1,n-2} - (F_{n-2,n-1} \tilde{F}_{n-1,n-2})^2$$

(6.9)

The dual gauge group is

$$SU(nN_{f_{n-1}} + (n-1)N_{f_{n-2}} + \ldots + N_{f_0} - N_{c_{n-1}})$$

$$\times SU(p_0 - N_{f_{n-1}} + (n-1)N_{f_{n-2}} + (n-2)N_{f_{n-3}} + \ldots + N_{f_0} - N_{c_{n-2}})$$

$$\ldots \times SU(p_{r-1} - N_{f_{n-1}} - 2N_{f_{n-2}} - \ldots - rN_{f_{n-r}} + (n-r)N_{f_{n-r-1}})$$

$$+(n-r-1)N_{f_{n-r-2}} + \ldots + N_{f_0} - N_{c_{n-r-1}})$$

$$\ldots \times SU(nN_{f_0} + (n-1)N_{f_1} + (n-2)N_{f_2} + \ldots + N_{f_{n-1}} - N_{c_0})$$

(6.10)

where $p_r$ is the rank of the $r$-th dual gauge group minus $N_{c_{n-r}} - 1$. It is possible to add adjoint matter fields to the gauge theory described above in a similar manner to that discussed in section 5.3.

7. Conclusions.

D-branes constructions of type IIA string theory have allowed us to explore four dimensional quantum field theories. We see familiar field theory phenomena, such as chiral symmetry and duality, in a new light. We have shown that in a number of cases it is easy to compute the dual gauge group of a theory by reversing the ordering of the 5-branes and 6-branes while preserving the linking numbers. Branes have shown to be a powerful tool in guiding us to interesting, new field theory dualities. In section 6, we saw that in order to have linking numbers and field theory agree, we had to introduce semi-infinite 4-branes. It would be interesting to understand this transition better. In section 2.4, we saw that incorporating chiral symmetry in the brane picture led us to predict that quantization of tensionless threebranes in six dimensions gives rise to massless vector multiplet in six dimensions. It would be very interesting to provide further support for this scenario. We expect much progress in the near future.
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