On Testing Homological Equivalence

Satish Kumar
IIT Kanpur
Department of Mathematics and Statistics
Kanpur 208016, India
email: satsh@iitk.ac.in

Subhra Sankar Dhar
IIT Kanpur
Department of Mathematics and Statistics
Kanpur 208106, India
email: subhra@iitk.ac.in

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Abstract: In this article, we develop a test to check whether the support of the unknown distribution generating the data is homologically equivalent to the support of some specified distribution. Similarly, it is also checked whether the supports of two unknown distributions are homologically equivalent or not. In the course of this study, test statistics based on the Betti numbers are formulated, and the consistency of the tests are established. Moreover, some simulation studies are conducted when the specified population distributions are uniform distribution over circle and 3-D torus, which indicate that the proposed tests are performing well. Furthermore, the practicability of the tests are shown on two well-known real data sets also.

keywords: Topological data analysis, Simplicial complex, Homology, Betti numbers, Persistent homology, Persistent Betti numbers.

1 Introduction

Data visualization is an essential part of data analysis but visualizing data, in more than three dimensions, is not possible in reality without reducing the dimension of the data. In such a scenario, geometric features of the data plays a fundamental role in data visualization. In addition, when the data is sampled from the high dimensional spaces, then these data points are not uniformly distributed in the embedding space but carry some geometric structure. In
such a situation, it is important to extract the geometric structure of the data to understand
the underlying phenomenon. One such way of extracting intrinsic geometric features from the
data is Topological data analysis abbreviated as TDA, and for introduction to TDA, one can
refer to Wasserman (2018), Carlsson (2009), Chazal and Michel (2021), Carlsson (2014) and
Zomorodian (2012). TDA provides topological summaries of the data by means of techniques
developed in a sub branch of topology named as algebraic topology (see text book by Edels-
brunner and Harer (2010), Ghrist (2014), Hatcher (2002) and Munkres (1984)), with the only
difference is that in TDA, we treat data as random points sampled from some unknown proba-
bility distribution, whereas in algebraic topology data are usually viewed as non-stochastic fix
points.

TDA based methods can reveal many stimulating insights from complex data sets, which is
usually not possible from conventional data analysis techniques. For this reason, TDA is widely
used in very diverse range of domains including biological sciences (Karisani u. a. (2022)), fi-
nance (Gidea and Katz (2018)), astronomy (Pranav u. a. (2016)) and many other areas. Gen-
erally speaking, TDA is an emerging field in data analysis, and not much work has been done
in Statistical literature. For the recent development of Statistical literature in TDA, see the
articles by Wasserman (2018), Chazal u. a. (2017), Fasy u. a. (2014), Chazal u. a. (2011), Biscio
u. a. (2020), Krebs and Hirsch (2022) Bubenik and Kim (2007), Niyogi u. a. (2011), Bubenik
(2015), Chazal u. a. (2014b) and Chazal u. a. (2014a). This article attempts to work in this
direction associated with a certain statistical testing of hypothesis problems.

Suppose that we observe a random sample \(X = \{X_1, X_2, \ldots, X_n\}\), where \(X_i (i = 1, \ldots, n)\)
is associated with some probability measure \(P\) supported on a compact set \(\mathcal{X} \subset \mathbb{R}^d, d \geq 1\).
Now, the goal is to infer the topology of \(\mathcal{X}\) from the observed data \(X\). Note that one cannot
consider \(X\) as a topological space to estimate the topology of \(\mathcal{X}\), since topology of \(X\) is trivial,
and hence, \(X\) does not contain any useful topological information. Therefore, one needs to
find a way to transform \(X\) into richer topological spaces that contain useful information about
\(\mathcal{X}\). The notion of simplicial complex, from algebraic topology, plays a fundamental role in this
regard. Simplicial complexes provide a way to convert the discrete set of points into richer
topological objects that contain useful information about the space underlying the data. Thus,
the first step towards calculating the topology of \(\mathcal{X}\) from the data is to convert the set \(X\) into
a simplicial complex. There are various ways of constructing simplicial complexes from the
data but in this article we are mainly concerned with the Čech complex and the Vietoris-Rips
complex that allows us to define and compute the homology efficiently. Precisely speaking, this
article concerns the topology of a topological space that can be approximated by the simplicial
complex, and we are interested in topological properties in terms of homology. Homology of a
simplicial complex is also referred to as simplicial homology (See: Section 2.2).
Homology is a concept from algebraic topology that distinguishes two topological spaces based on the number of connected components and holes in a space. Homology characterizes sets based on the connected components and holes in higher dimensions. In particular, $0^{th}$ order homology corresponds to the number of connected components of the set, which is same as the number of clusters in statistical sense. Homology admits group structure and the rank of homology groups, are important topological invariants in TDA, also referred as Betti numbers (See: Section 2.2). In this article, we are concerned with testing one sample and two sample homological equivalence using the calculated Betti numbers from the data since the Betti numbers characterize the homological equivalence.

From the Statistical point of view, this article develops one sample and two sample homological equivalence. For one-sample problem, we investigate whether the support of the unknown distribution is homologically equivalent to the support of the specified distribution or not, and in the two-sample problem, we want to know whether supports of two unknown distributions are homologically equivalent or not. In the course of this study, we calculate Betti numbers associated with a Čech or Vietoris rips complex constructed from the observed data. We propose test statistics based on the Betti numbers, and the consistency of the tests is also established. This test is expected to be effective in practice as homology of the support of the distribution often has a substantial impact in exploratory data analysis, cluster analysis, reduction of dimension and many other statistical applications. In other words, homological information enables us to extract surprising insights from complex data sets, which can further be used in data analysis for knowledge discovery.

The rest of the article is organized as follows. In Section 2, we have provided necessary background on algebraic topology needed to comprehend this article such as homology, Betti number, persistent homology etc. Section 3 formulates the research problem along with the consistency of the test. In Section 4, a few simulation studies are performed when the data are generated from the uniform distribution over circle and three dimensional torus. Section 5 shows the usefulness of the test on two well-known data sets, and Section 6 consists of a few concluding remarks. Finally, the Appendix (i.e., Section 7) contains technical details.

The codes of all numerical studies are available at https://github.com/satsh636/TDA.git.

2 Preliminaries

Given a random sample $X_1, \ldots, X_n$ from an unknown probability measure $P$, one may want to extract geometric features of the data to understand the inherent structure of the data. In
TDA, one can extract topological features of the data. Strictly speaking, one may want to know
the topology of the continuous object that underlies the data, i.e., topology of the support of
the distribution. We now give the formal definition of topology and related notions.

**Definition 2.1 Topology:** Given a set of points, \( X \), topology of the set \( X \) denoted as \( \tau \), is a
collection of subsets of \( X \) which satisfies the following properties:

- Both the empty set \( \emptyset \) and \( X \) are in \( \tau \).
- \( \tau \) is closed under union.
- \( \tau \) is closed under finite intersection.

**Remark 2.1** If \( \tau = \{ \emptyset, X \} \), then \( \tau \) is called **trivial topology**.

**Definition 2.2 Topological space:** A topological space is a set \( X \) equipped with its topology
\( \tau \), i.e., \((X, \tau)\) is a topological space.

**Remark 2.2** For any arbitrary set \( X \), the topological space is generally denoted by the set \( X \)
itself.

**Example 2.1** Topology of the set of real numbers \( \mathbb{R} \) is the collection of all open sets \( \mathcal{O} \) and
the topological space is \((\mathbb{R}, \mathcal{O})\).

**Definition 2.3 Hausdorff space:** A topological space \( X \) is said to be Hausdorff space if for
any two distinct points \( x \) and \( y \) in \( X \), there exists a neighbourhood \( N_x \) of \( x \) and a neighbourhood
\( N_y \) of \( y \) such that \( N_x \cap N_y = \emptyset \).

**Definition 2.4 Homeomorphism:** Two topological spaces, \( X \) and \( Y \), are said to be topo-
logically same or **homeomorphic** to each other if there exist a continuous bijective map with
continuous inverse, such a map is called homeomorphism between the topological spaces \( X \) and \( Y \).

**Definition 2.5 Manifold:** A manifold of dimension \( d \) or a \( d \)-manifold is a Hausdorff space
in which each point has an open neighborhood homeomorphic to the euclidean space \( \mathbb{R}^d \).
One can distinguish two topological spaces by the property of homeomorphism. A property of a topological space that is invariant under homeomorphism, is called a topological invariant. If two topological spaces are homeomorphic to each other, then topological invariants associated with them coincide but converse is not true.

In general, classifying a topological space up to homeomorphism is not always possible. In TDA, one usually resorts to the homology of a topological space in order to distinguish two topological spaces. In particular, we are interested in simplicial homology which is defined for spaces that can be approximated through points, lines, triangles and its higher dimensional generalizations, and these are called simplicial complexes. In the following two subsections, we shall briefly define the concept of simplicial complex and simplicial homology. For details, one may refer to Munkres (1984).

2.1 Simplicial complex

In this section, we shall define the notion of a simplicial complex which belongs to an important class of topological spaces that we are interested in this article.

Definition 2.6 Complex: A complex is a space that is constructed from a union of simple pieces (see below), if the pieces are topologically easily tractable and their common intersections are lower dimensional pieces of the same kind. (Edelsbrunner und Harer (2010))

In particular, if a complex is made up of simple pieces like, points, line, edges, triangles and their higher dimensional analogues, then it is called a Simplicial Complex. Elements of a simplicial complex are called simplices. Points are 0-simplices, also called vertices; lines are 1-simplices, also called edges; triangles are 2-simplices and so on.

Definition 2.7 Affine independence: A collection of points \( \{x_0, x_1, \ldots, x_n\} \) is said to be affinely independent if the points \( \{x_1 - x_0, x_2 - x_0 \ldots x_n - x_0\} \) are linearly independent.

Definition 2.8 Simplex: Suppose that \( A = \{x_0, x_1, \ldots, x_n\} \) is a set of affinely independent points in \( \mathbb{R}^d, d \geq 1 \). An \( n \)-simplex or \( n \)-dimensional simplex spanned by \( \{x_0, x_1, \ldots, x_n\} \), denoted by \( \sigma \) and defined as the set of convex combinations of \( x_0, x_1, \ldots, x_n \), i.e.,

\[
\sigma = \left\{ z \in \mathbb{R}^d : z = \sum_{i=0}^{n} \alpha_i x_i \text{ such that } \sum_{i=0}^{n} \alpha_i = 1; \alpha_i \geq 0 \text{ for all } i = 0, 1, \ldots n \right\}
\]
Here, the points $x_0, x_1, \ldots, x_n$ are called vertices of $\sigma$. Simplices spanned by the subsets of $A$ are called faces of $\sigma$.

**Definition 2.9** Simplicial complex: A simplicial complex is a finite collection of simplices, denoted by $\mathcal{K}$, in $\mathbb{R}^d$, $d \geq 1$ which satisfies the following condition:

- If $\tau$ is any face of a simplex $\sigma$ in $\mathcal{K}$, then $\tau \in \mathcal{K}$.
- If $\tau$ and $\sigma$ are two simplices in $\mathcal{K}$, then $\tau \cap \sigma = \emptyset$ or $\tau \cap \sigma$ is common face of both $\tau$ and $\sigma$.

The simplicial complexes defined in this way are also referred as geometric simplicial complexes. Note that $\mathcal{K}$ can also be regarded as a topological space through its underlying space which is the union of its simplices. The union of simplices of $\mathcal{K}$ is a subset of $\mathbb{R}^d$ that inherits topology from $\mathbb{R}^d$. The simplices of $\mathcal{K}$ are called the faces of $\mathcal{K}$. The dimension of $\mathcal{K}$ is the largest dimension of its simplices.

**Definition 2.10** Subcomplex: A subset of $\mathcal{K}$, which is itself a simplicial complex, is called a subcomplex of $\mathcal{K}$. In particular, a subcomplex, which contains all the simplices of $\mathcal{K}$ of dimension at most $k$, is called $k$-skeleton of $\mathcal{K}$.

In particular, 1-skeleton of a simplicial complex is same as geometric graph. Thus, simplicial complexes are higher dimensional generalisations of geometric graphs. Therefore connectivity properties of a simplicial complex is same as its 1-skeleton which is a geometric graph.

Now, observe that construction of a simplicial complex involves specifying all the simplices and ensuring that simplices should intersect in a specified manner which is difficult in practice since it involves complicated geometric details. Therefore, in practice, one may work with the simplicial complexes which are fully characterized by only the list of its simplices, this leads to the notion of abstract simplicial complex.

**Definition 2.11** Abstract simplicial complex: Given a finite set $A = \{x_0, x_1, \ldots, x_n\}$, a non-empty and finite collection of subsets of $A$, denoted by $\mathcal{A}$, is called an abstract simplicial complex if the following conditions are satisfied:

- The elements of $A$ belong to $\mathcal{A}$. 


• If $\tau \in \mathcal{A}$ and $\sigma \subseteq \tau$, then $\sigma \in \mathcal{A}$.

Note that given an abstract simplicial complex $\mathcal{A}$, one can always construct a geometric simplicial complex $\mathcal{K}$. Moreover, one can associate a topological space with the abstract simplicial complex $\mathcal{A}$ (denoted as $|\mathcal{A}|$) such that $|\mathcal{A}|$ is homeomorphic to the underlying space of $\mathcal{K}$, such a $\mathcal{K}$ is referred as geometric realization of $\mathcal{A}$. It can also be verified that any $n$-dimensional abstract simplicial complex can be realized as a geometric simplicial complex in $\mathbb{R}^{2n+1}$. Therefore, one can consider abstract simplicial complexes as a topological space from which one can derive the topological properties.

**Example 2.2** Given a set $A = \{a, b, c\}$, $\mathcal{A} = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}\}$ is an abstract simplicial complex representing boundary of the triangle shown below:

![Diagram of a triangle with vertices a, b, c.]

If the 2-simplex $\{a, b, c\}$ is added to the abstract simplicial complex $\mathcal{A}$, i.e.,

$$\mathcal{A} = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}\},$$

then it represent the triangle shown below:

![Diagram of a filled triangle with vertices a, b, c.]
Data to abstract simplicial complex:

Suppose that we have observed a random sample, \( X = \{x_1, x_2, \ldots, x_n\} \), of size \( n \), from the unknown probability measure \( P \) supported on the unknown manifold \( \mathcal{X} \subseteq \mathbb{R}^d, d \geq 1 \), and one may want to estimate the topology of \( \mathcal{X} \) from the observed sample \( X \). Note that the set \( X \) itself does not provide any useful topological information of \( \mathcal{X} \), since the topology of \( X \) is trivial. Since topology is a characteristics of a continuous space, therefore, in order to extract substantial topological information from \( X \), one needs to connect the data points that are close or similar to each other in some sense so that one can have some idea about the continuous space underlying the data.

**Example 2.3** Suppose that we are given 20 sample points, from a unit circle, drawn according to uniform probability measure. Then, the goal is to connect these data points so that one can estimate the topological properties of the underlying continuous space, that is the circle in this example.

Therefore, one needs to define a criterion which specifies the connectivity of points in \( X \). We can define a criterion based on neighbourhood relationship between the data points which can be done by introducing a metric on the set \( X \) or some sort of dissimilarity measure for sampled data points. Thus, TDA takes input not in the form of usual data matrix but in the form of some distance matrix or metric space \( (X, \delta) \), called **point cloud**, where \( \delta \) is any metric such as euclidean metric.
Now, given a point cloud \((X, \delta)\) or a distance matrix, to specify the connectivity of points in \(X\), one has to choose a threshold, \(\epsilon \geq 0\), to decide which points are close or similar to each other. For instance, we consider two points \(x_i\) and \(x_j\) in \(X\) to be connected, or similar, in the sense that \(\delta(x_i, x_j) \leq \epsilon\). Thus, given a point cloud \((X, \delta)\) and a threshold \(\epsilon\), a natural approximation of the underlying topological space is the cover of \(X\), i.e., the union of balls \(\bigcup_{x \in X} B_\epsilon(x) \subset \mathbb{R}^n\), where for any \(x \in X\), \(B_\epsilon(x) = \{ v \in \mathbb{R}^n : \delta(v, x) \leq \epsilon \}\). However, union of balls \(\bigcup_{x \in X} B_\epsilon(x)\) is not useful since it is not algorithmically tractable. Therefore, one can construct an abstract simplicial complex which is homologically equivalent to \(\bigcup_{x \in X} B_\epsilon(x)\). In algebraic topology, a well known theorem, The Nerve theorem (See: Chazal und Michel (2021)), guarantees that homology of \(\bigcup_{x \in X} B_\epsilon(x)\) is same as homology of the \(\check{C}\)ech complex which is defined as follows:

**Definition 2.12 The \(\check{C}\)ech complex:** Suppose that we are given a point cloud \((X, \delta)\), where \(X = \{x_1, x_2, \ldots, x_n\}\), is a set consist of observed sample points and a threshold \(\epsilon \geq 0\), then one can construct an abstract simplicial complex, called \(\check{C}\)ech complex, denoted as \(C(X, \epsilon)\), in the following way:

1. Elements in \(X\) belongs to \(C(X, \epsilon)\), that is, \(\{x_i\} \in C(X, \epsilon)\), for all \(i = 1, 2 \ldots n\).

2. A \(k\)-simplex \([x_{i_0}, x_{i_1} \ldots x_{i_k}] \in C(X, \epsilon)\) if \(\bigcap_{j=0}^{k} B_\epsilon(x_{i_j}) \neq \phi\), where \(B_\epsilon(x)\) is ball of radius \(\epsilon\) around the point \(x\).

The \(\check{C}\)ech complex provides a procedure to construct an abstract simplicial complex from the data but its construction is not computationally tractable. Therefore, one usually constructs the following abstract simplicial complex which is computationally feasible.

**Definition 2.13 The Vietoris-Rips complex:** Suppose that we are given a point cloud \((X, \delta)\), where \(X = \{x_1, x_2, \ldots, x_n\}\), is a set consisting of observed sample points and a threshold \(\epsilon \geq 0\), then one can construct an abstract simplicial complex, called Vietoris-Rips complex or Rips complex, denoted as \(\mathcal{R}(X, \epsilon)\), in the following way:

1. Elements in \(X\) belongs to \(\mathcal{R}(X, \epsilon)\), that is, \(\{x_i\} \in \mathcal{R}(X, \epsilon)\), for all \(i = 1, 2 \ldots n\).

2. A \(k\)-simplex \([x_{i_0}, x_{i_1} \ldots x_{i_k}] \in \mathcal{R}(X, \epsilon)\) if \(B_\epsilon(x_{i_j}) \bigcap B_\epsilon(x_{i_m}) \neq \phi\) for all \(0 \leq j < m \leq k\)
The usefulness of the Rips complex is justified by the following nested relationship with the Čech complex which ensures that the Rips complex is an appropriate approximation of the Čech complex to estimate the homology of $\mathcal{X}$:

$$\mathcal{C}(X, \epsilon) \subseteq \mathcal{R}(X, 2\epsilon) \subseteq \mathcal{C}(X, 2\epsilon).$$

### 2.2 Simplicial homology

Here, we shall briefly define the notion of homology for simplicial complexes. Homology counts the number of holes in a topological space. We will require the following notions to define the holes in a topological space.

**Definition 2.14 Orientation of a simplex:** Given a finite simplicial complex $K$, let $\sigma$ be a $p$-simplex in $K$ with the vertex set $\{x_1, \ldots, x_p\}$, where $p > 0$ be an integer. Define two orderings of its vertex set to be equivalent if they differ from one another by an even permutation, and thus, ordering of vertices falls into two equivalence classes. Each of these equivalence classes is called an orientation of $\sigma$. We denote an oriented $k$-simplex with the vertex set $\{x_1, \ldots, x_p\}$ as $[x_1, \ldots, x_p]$.

**Definition 2.15 Simplicial chains:** Let $K$ be a finite simplicial complex then, for any integer $p \geq 0$, a $p$-chain on $K$ is defined to be formal linear combinations of $p$-simplices in $K$, that is any $p$-chain can be written as $\sum_{i=1}^{m} \alpha_i \sigma_i$, where $\sigma_i$ is a $p$-simplex in $K$, $m$ is the number of $p$-simplices in $K$ and $\alpha_i$'s are integers from some field $\mathbb{F}$. An empty chain is denoted as $0$ and defined as $0 = \sum_{i=1}^{m} 0.\sigma_i$.

In TDA, one usually takes coefficients $\alpha_i$’s from the field $\mathbb{Z}/2\mathbb{Z} := \mathbb{Z}_2$ for computational simplicity. Here $\mathbb{Z}_2$ is a field of elements $\alpha$ modulo 2, where $\alpha \in \mathbb{Z}$, $\mathbb{Z}$ is the set of integers. We denote $C_p(K, \mathbb{F})$ to be the space of $p$-chains over the field $\mathbb{F}$. In this article, we take $\mathbb{F}$ to be $\mathbb{Z}_2$. Thus, if $m$ denotes the number of $p$-simplices in $K$ and $\sigma_i$ denotes a $p$-simplex in $K$, $i = 1, \ldots, m$, then space of $p$-chains is defined as follows:

$$C_p(K, \mathbb{F}) := \left\{ \gamma = \sum_{i=1}^{m} \alpha_i \sigma_i : \alpha_i \in \mathbb{Z}_2 \right\}.$$

It can be verified that $C_p(K, \mathbb{F})$ is a vector space over the field $\mathbb{F}$.
Definition 2.16 **Boundary homomorphism:** Given a simplicial complex $K$, the $p^{th}$ boundary homomorphism is a linear transformation from $C_p(K, F)$ to $C_{p-1}(K, F)$, also referred as boundary operator. We denote the boundary of a $p$-simplex $\sigma$ by $\partial_p(\sigma)$.

Let $\sigma = [x_1, x_2, \ldots, x_p]$ be an oriented $p$-simplex with $p > 0$, we define

$$\partial_p(\sigma) = \partial_p[x_1, x_2, \ldots, x_p] = \sum_{i=0}^{p} (-1)^i [x_1, \ldots, \hat{x}_i, \ldots, x_p],$$

where $[x_1, \ldots, \hat{x}_i, \ldots, x_p]$ is $(p-1)$- face of $\sigma$ obtained by deleting $x_i$ from $[x_1, \ldots, x_p]$. Here the symbol $\hat{x}_i$ means that we obtain $(p-1)$- face of $\sigma$ by deleting the vertex $x_i$ from $[x_1, \ldots, x_p]$.

Note that for $F = \mathbb{Z}_2$, boundary of a $p$-simplex is defined as follows:

$$\partial_p(\sigma) = \partial_p[x_1, x_2, \ldots, x_p] = \sum_{i=0}^{p} [x_1, \ldots, \hat{x}_i, \ldots, x_p].$$

It can be verified that for any $p$-simplex $\sigma$, $p \geq 1$, we have $\partial_{p-1}\partial_p(\sigma) = 0$. This property of boundary operators is also referred as **fundamental lemma of simplicial homology**, which is essential to define homology. Here, it should be noted that by convention, the boundary operator $\partial_0$ maps a 0-simplex to an empty chain 0. Besides, the fundamental lemma of simplicial homology allows us to define an algebraic structure, called chain complex which allows us to compute homology of $K$.

Definition 2.17 **Chain complex:** The chain complex of a finite simplicial complex $K$ of dimension $n$ is the sequence of vector spaces $C_n(K, F)$ connected with the corresponding boundary operators:

$$0 \xrightarrow{\partial_{n+1}} C_n(K, F) \xrightarrow{\partial_n} C_{n-1}(K, F) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1(K, F) \xrightarrow{\partial_1} C_0(K, F) \xrightarrow{\partial_0} 0.$$  

For, $p \in \{0, \ldots, n\}$, we define the set of $p$-cycles as follows:

$$Z_p(K) := \text{kernel}(\partial_p : C_p(K, F) \rightarrow C_{p-1}(K, F)) = \{\gamma \in C_p(K, F) : \partial_p(\gamma) = 0\}.$$  

The set of $p$-boundaries is defined as follows:

$$B_p(K) := \text{image}(\partial_{p+1} : C_{p+1}(K, F) \rightarrow C_p(K, F))$$

$$= \{\gamma \in C_p(K, F) : \gamma = \partial_{p+1}(\gamma^*), \text{ for some } \gamma^* \in C_{p+1}(K, F)\}.$$  

Definition 2.18 **Homology:** Given a finite simplicial complex $K$ and its associated chain complex, it is evident that $B_p(K)$ and $Z_p(K)$ are subspaces of $C_p(K, F)$ and according to fundamental lemma of simplicial homology, we have the following:

$$B_p(K) \subset Z_p(K) \subset C_p(K, F).$$
For any non-negative integer \( p \), we denote the \( p \)th homology of \( K \) as \( H_p(K) \), which is defined to be the following quotient vector space:

\[
H_p(K) := Z_p(K) / B_p(K)
\]

Note that \( H_p(K) \) is a vector space and its elements are called homology classes of \( K \).

**Definition 2.19 Betti numbers:** The \( p \)th Betti number \( \beta_p \) is defined to be the dimension of the vector space \( H_p(K) \), i.e \( \beta_p := \dim (H_p(K)) \).

Note that calculation of Betti numbers involves choosing a value \( \epsilon > 0 \) to construct a simplicial complex \( K \), which makes Betti numbers unstable. Suppose that \( K_\epsilon \) denotes a simplicial complex constructed at a fix value of \( \epsilon \), then note that homology of \( K_\epsilon \) changes as \( \epsilon \) changes. Therefore, instead of choosing a single value of \( \epsilon \), one tracks the evolution of topological features over all the possible values of \( \epsilon \). This idea leads to the notion of persistent homology introduced by Edelsbrunner und Harer (2008).

**Definition 2.20 Filtrations:** A filtration of a simplicial complex \( K \) is a nested family of sub-complexes \( (K_\epsilon)_{\epsilon \in T} \), where \( T \subset \mathbb{R} \), such that for any \( \epsilon, \epsilon' \in T \) we have the following:

(i) \( K = \bigcup_{\epsilon \in T} K_\epsilon \).

(ii) \( K_\epsilon \subseteq K_{\epsilon'} \), for any \( \epsilon \leq \epsilon' \in T \).

**Definition 2.21 Persistent homology:** For a given filtration \( (K_\epsilon)_{\epsilon \in T} \) of a simplicial complex \( K \) and for a non-negative integer \( p \), we obtain a sequence of \( p \)th-homology vector spaces \( H_p(K_\epsilon) \), where the inclusions \( K_\epsilon \subseteq K_{\epsilon'} \), for any \( \epsilon \leq \epsilon' \in T \), induce linear maps \( f_{\epsilon,\epsilon'}^p \) between \( H_p(K_\epsilon) \) and \( H_p(K_{\epsilon'}) \). Such a sequence of vector spaces together with the linear maps connecting them is called the persistence module. Therefore, given a persistence module associated with the simplicial complex \( K \), we define the \( p \)th persistent homology \( H_p^{\epsilon,\epsilon'}(K) \) as follows:

\[
H_p^{\epsilon,\epsilon'}(K) := Z_p(K_\epsilon) / (B_p(K_\epsilon) \cap Z_p(K_{\epsilon'})),
\]

where \( Z_p(K) \) and \( B_p(K) \) denote the set of \( p \)-cycles and the set of \( p \)-boundaries of a simplicial complex \( K \), respectively. Note that here, \( H_p^{\epsilon,\epsilon'}(K) \) contains \( p \)-dimensional holes of \( K_\epsilon \) that are still present in \( K_{\epsilon'} \). Also, note that \( H_p^{\epsilon,\epsilon}(K) = H_p(K_\epsilon) \).
Definition 2.22 **Persistent Betti numbers:** For any $\epsilon \leq \epsilon'$, the $p^{th}$ persistent Betti number $\beta_{p}^{\epsilon, \epsilon'}$ is defined to be the dimension of the vector space $H_{p}^{\epsilon, \epsilon'}(K)$, i.e., $\beta_{p}^{\epsilon, \epsilon'} := \dim (H_{p}^{\epsilon, \epsilon'}(K))$. Note that, for $\epsilon = \epsilon'$, the $p^{th}$ persistent Betti number $\beta_{p}^{\epsilon, \epsilon'}$ coincide with the usual Betti number of $K_{\epsilon}$.

3 Problem Formulation and Main Results

Let $X_{1}, \ldots, X_{n}$ be a random sample of size $n$, sampled according to uniform probability measure supported on an unknown compact set $\mathcal{X} \subset \mathbb{R}^{d}, d \geq 1$. Since we are interested in topological properties of the set $\mathcal{X}$, we assume $\mathcal{X}$ to be a topological space. In particular, we assume the topological space $\mathcal{X}$ to be a manifold (See, Definition 2.5) embedded in $\mathbb{R}^{d}, d \geq 1$, since arbitrary topological spaces are not suitable for computational purposes. Our goal is to perform one sample test for homological equivalence of $\mathcal{X}$ to some hypothesised topological space $\mathcal{X}_{0}$. Moreover, we shall extend this set up to two sample problems. Since Betti numbers of a topological space characterize homology of the topological space, we want to test statistical significance of calculated Betti numbers of $\mathcal{X}$ to the Betti numbers of $\mathcal{X}_{0}$. Since $\mathcal{X}$ is known only up to the random sample $X_{1}, X_{2}, \ldots, X_{n}$, we approximate the topological space $\mathcal{X}$ by a simplicial complex constructed from the random sample $X_{1}, X_{2}, \ldots, X_{n}$. We consider Čech and Vietoris-Rips complex as a simplicial complex constructed from the data to approximate the topological space $\mathcal{X}$. We consider Čech complex for theoretical purposes and Vietoris-Rips complex for computational purposes to calculate the Betti numbers.

Let us now formulate the testing of hypothesis for one sample problem.

$H_{0} : \mathcal{X}$ is homologically equivalent to $\mathcal{X}_{0}$ against $H_{1} : \mathcal{X}$ is not homologically equivalent to $\mathcal{X}_{0}$, where $\mathcal{X}_{0}$ is a specified topological space.

Suppose that $\mathcal{X}$ and $\mathcal{X}_{0}$ are subsets of $d$-dimensional euclidean space, and we denote $\beta_{i}$ and $\beta_{i,0}$ as the $i^{th}$ Betti numbers of $\mathcal{X}$ and $\mathcal{X}_{0}$, respectively, for all $i = 0, 1, \ldots, d - 1$. Let $\beta = (\beta_{0}, \beta_{1}, \ldots, \beta_{d-1})^{T}$ and $\beta_{0} = (\beta_{0,0}, \beta_{1,0}, \ldots, \beta_{d-1,0})^{T}$, where $\beta$ is unknown and $\beta_{0}$ is known. As the Betti numbers characterize homology of a topological space, one can equivalently formulate the testing of hypothesis, in terms of $\beta$ and $\beta_{0}$, as follows:

$$H_{0}^{*} : \beta = \beta_{0} \text{ vs } H_{1}^{*} : \beta \neq \beta_{0},$$

where $\beta$ and $\beta_{0}$ are vectors containing betti numbers of $\mathcal{X}$ and $\mathcal{X}_{0}$, respectively. Here $=$ and $\neq$ denote the component wise equality and inequality, respectively.

Note that in the assertion of testing of hypothesis problem described in $H_{0}^{*}$ against $H_{1}^{*}$,
\(\beta_i\)'s (\(i = 0, \ldots d - 1\)) are unknown, and \(\beta_{i,0}\)'s (\(i = 0, \ldots d - 1\)) are known. Hence, one needs to estimate \(\beta_i\) for \(i = 0, \ldots d - 1\) based on the random sample \(X_1, X_2, \ldots X_n\) to carry out the testing of hypothesis problem \(H_0^*\) against \(H_1^*\). Let \(\hat{\beta}_{0,n}, \hat{\beta}_{1,n}, \ldots, \hat{\beta}_{d-1,n}\) be the estimators of \(\beta_0, \beta_1, \ldots, \beta_{d-1}\), respectively, where \(\hat{\beta}_{i,n}\) (\(i = 0, \ldots, d - 1\)) are precisely the Betti numbers of the Čech or Rips complex constructed by \(X_1, X_2, \ldots X_n\), and note that \(n\) in the subscript \(\hat{\beta}_{i,n}\)'s, emphasise the fact that estimated Betti numbers depend on the sample size \(n\). Now, in order to test \(H_0^*\) against \(H_1^*\), one can formulate the test statistic in the following way: In view of the testing of hypothesis problem \(H_0^*\) against \(H_1^*\), the test statistic should be based on the appropriate differences between \(\hat{\beta}_{i,n}\) and \(\beta_{i,0}\). In this work, we consider the test statistic as the sum of the absolute difference between \(\hat{\beta}_{i,n}\) and \(\beta_{i,0}\) (\(i = 0, \ldots d - 1\)), i.e., the test statistic \(T_n\) is as follows:

\[
T_n = \sum_{i=0}^{d-1} \left| \hat{\beta}_{i,n} - \beta_{i,0} \right|.
\]

In this context, observe that for a given data, the larger value of \(T_n\) indicates the rejection of the null hypothesis \(H_0^*\), i.e., \(H_0^*\) will be rejected when \(T_n > c\), where \(c\) is such that \(P[T_n > c | H_0^*] = \alpha\). Here \(\alpha \in (0, 1)\) denotes the level of significance of the test. It is an appropriate place to mention that one can consider any other distance also to formulate the test statistic.

Suppose that \((X_n, \delta)\) is a point cloud, where \(X_n = \{x_1, x_2, \ldots, x_n\}\) is a collection of observed sample points, and \(\delta\) is any metric such as euclidean metric. The following theorem states the consistency of the test based on \(T_n\) under the following two assumptions:

**Assumptions:**

(A.1) The underlying support of the data is a convex set.

(A.2) For a fix sample size \(n\), the proximity parameter \(\epsilon \in (0, \epsilon_m)\), where \(\epsilon_m\) is some element in an increasing sequence \(\{\epsilon_k : k = 1, 2, \ldots, m, \ldots\}\). Here \(m\) is the largest positive integer for which the Rips complex constructed at \(\epsilon_m\), is not connected and the Rips complex becomes connected at \(\epsilon_{m+1}\).

**Theorem 3.1** Suppose that \(X_1, X_2, \ldots, X_n\) is a random sample of size \(n\) from uniform distribution supported on the manifold \(\mathcal{X} \subseteq \mathbb{R}^d, d \geq 1\). Then, under the conditions (A.1) and (A.2), the test based on \(T_n\) is consistent i.e

\[
P_{H_1^*}[T_n > c] \rightarrow 1 \text{ as } n \rightarrow \infty.
\]

Here \(c\) is such that \(P_{H_0^*}[T_n > c] \rightarrow \alpha \text{ as } n \rightarrow \infty\) and \(\alpha \in (0, 1)\) is the level of significance of the test.
Let us now formulate the set-up for two sample problem. Suppose that we are given two random samples $X_{n_1} = \{X_1, X_2, \ldots, X_{n_1}\}$ and $Y_{n_2} = \{Y_1, Y_2, \ldots, Y_{n_2}\}$ of size $n_1$ and $n_2$, respectively. In addition, $X_{n_1}$ and $Y_{n_2}$ are drawn independently from the uniform probability measure supported on the unknown manifolds $X \subset \mathbb{R}^d$ and $Y \subset \mathbb{R}^d$, $d \geq 1$, respectively. Suppose that $\hat{\beta}_i$ and $\hat{\beta}_{i}^*$ denotes the $i^{th}$ Betti numbers of Čech or Rips complex constructed from $X_{n_1}$ and $Y_{n_2}$ respectively, for $i = 0, 1, \ldots, d - 1$. Also, let $\beta = (\beta_0, \beta_1, \ldots, \beta_{d-1})^T$ and $\beta^* = (\beta_0^*, \beta_1^*, \ldots, \beta_{d-1}^*)^T$ denote the vector containing the unknown Betti numbers of $X$ and $Y$, respectively. We now want to test the homological equivalence of $X$ and $Y$, and we formulate the testing of hypothesis problem as follows:

$$H_0^{**} : \beta = \beta^* \text{ against } H_1^{**} : \beta \neq \beta^*,$$

where $\beta = (\beta_0, \beta_1, \ldots, \beta_{d-1})^T$ and $\beta^* = (\beta_0^*, \beta_1^*, \ldots, \beta_{d-1}^*)^T$ denote the vector containing the unknown Betti numbers of $X$ and $Y$, respectively. In the same spirit of formulation of $T_n$ for one-sample problem, we now propose the following test statistic to test $H_0^{**}$ against $H_1^{**}$:

$$T_{n_1,n_2} = \sum_{i=0}^{d-1} \left| \hat{\beta}_{i,n_1} - \hat{\beta}_{i,n_2}^* \right| .$$

Here, observe that the larger value of $T_{n_1,n_2}$ indicates the rejection of the null hypothesis $H_0^{**}$, i.e., we reject $H_0^{**}$ when $T_{n_1,n_2} > c$, where $c$ is such that $\mathbb{P}[T_{n_1,n_2} > c|H_0^{**}] = \alpha$. Here $\alpha \in (0, 1)$ denotes the level of significance of the test.

Suppose that $(X_{n_1}, \delta)$ and $(Y_{n_2}, \delta)$ are two point clouds, where $X_{n_1}$ and $Y_{n_2}$ are two independent samples of size $n_1$ and $n_2$, respectively. The following theorem states the consistency of the test based on $T_{n_1,n_2}$ under the assumptions (A.1) and (A.2):

**Theorem 3.2** Suppose that we are given two independent samples $X_{n_1} = \{X_1, X_2, \ldots, X_{n_1}\}$ and $Y_{n_2} = \{Y_1, Y_2, \ldots, Y_{n_2}\}$ of size $n_1$ and $n_2$ from the uniform distribution supported on the manifolds $X \subset \mathbb{R}^d$ and $Y \subset \mathbb{R}^d$, $d \geq 1$, respectively. If we assume that $n_1$ and $n_2$ are such that $\frac{n_1}{n_1 + n_2} \to \lambda \in (0, 1)$ as $\min(n_1, n_2) \to \infty$, then under the assumptions (A.1) and (A.2), the test based on $T_{n_1,n_2}$ is consistent i.e.

$$\mathbb{P}_{H_0^{**}}[T_{n_1,n_2} > c] \to 1 \text{ as } \min(n_1, n_2) \to \infty.$$

Here $c$ is such that $\mathbb{P}_{H_0^{**}}[T_{n_1,n_2} > c] \to \alpha$ as $\min(n_1, n_2) \to \infty$, and $\alpha \in (0, 1)$ is the level of significance of the test.
4 Simulated data study

We perform Monte Carlo study based on the random samples, generated according to uniform probability measures supported on a set $X$. We have taken, $r = 100$ replications, denoted as $s_1, \ldots, s_r$, each of which contains $n$ number of data points, where $n \in \{20, 50, 100, \ldots, 500\}$. We denote $X_0$ and $X_1$ as the hypothesised support under $H_0$ and $H_1$, respectively. We take $n = 20$ and fix the choice of $\epsilon$ such that it satisfies the Assumption (A.2) for each of the $r$ replications. We denote $\epsilon_0$ and $\epsilon_1$ as the choice of $\epsilon$ that satisfies the Assumption (A.2) for each of the $r$ replications, under $H_0$ and $H_1$, respectively. We use $\epsilon_0$ and $\epsilon_1$ to construct Rips complex from the data, generated uniformly from the set $X_0$ and $X_1$, respectively. Now, we take $n = 20$ and construct Rips complex at the feature scale $\epsilon_0$, for each of the $r$ replications, $s_1, \ldots, s_r$ from $X_0$, and compute $r$ values of the test statistics, denoted as $T_{1,n}, \ldots, T_{r,n}$. We have fixed the level of significance $\alpha = 0.05$ to estimate the critical value. We obtain 0.95th quantile of $T_{1,n}, \ldots, T_{r,n}$ to estimate the critical value that will be used in computing power of the test.

Next, we follow the same Monte Carlo procedure as above, for the support $X_1$. We generate $r$ number of replications from $X_1$, and construct Rips complex at the feature scale $\epsilon_1$ to compute $r$ values of the test statistics, under $H_1$. We denote $T'_{1,n}, \ldots, T'_{r,n}$ as the values of the test statistics under $H_1$, then we estimate power of the test as follows:

$$\text{Power} = \frac{\# \{ T'_{i,n} > c, i = 1, 2, \ldots, r \}}{r},$$

where $c$ is the estimated critical value at $\alpha = 0.05$. Now, we iterate the Monte Carlo procedure as above with the same fix choice of $\epsilon_0$ and $\epsilon_1$, for $n \in \{50, 100, \ldots, 500\}$ to investigate the performance of the test for finite sample sizes. In the following two subsections, we examine performance of the test when the support is circle and 3-D torus, under various choice of alternative support.

4.1 Circle

Here, we generate 100 replications, each of which consists of a random sample of size 20, from a uniform probability measure supported on the unit circle. We want to test the following hypothesis for the Betti numbers associated with the unit circle:

$$H_0 : \beta = \beta_0 \text{ vs } H_1 : \beta \neq \beta_0,$$

where $\beta_0 = (\beta_0, \beta_1)^T = (1, 1)^T$.

Now, in order to compute the power of the test, we have taken random samples according to uniform measure from two alternative supports, unit disk and 2-D spiral. Followings are
the specifications of $\epsilon$ under $H_0$ and $H_1$, that satisfies the Assumption (A.2) for each of the $r$ replications:

- We take $\epsilon = 0.5$, for 20 samples generated from uniform distribution supported on the unit circle, under $H_0$.
- We take $\epsilon = 0.5$, for 20 samples generated from uniform distribution supported on the unit disk, under $H_1$.
- We take $\epsilon = 2$, for 20 samples generated from uniform distribution supported on the 2-D spiral, under $H_1$.

The plots in Figure 2 suggests that our proposed test perform well:

(a) Power plot when the support is unit disk.  
(b) Power plot when the support is 2-D spiral.

Figure 2: Power plots.

### 4.2 3-D Torus

We generate 100 replications, each of which consists random sample of size 20, from uniform probability measure supported on the 3-D torus, which is defined as follows:

$$\mathcal{X}_0 = \left\{ ((R + r \cos \theta \cos \psi), (R + r \cos \theta \sin \psi), r \sin \theta) : 0 \leq \theta, \psi \leq 2\pi, 0 < r < R < \infty \right\},$$
where $R$ is the distance from the center of the tube to the center of the torus and $r$ is the radius of the tube. We have taken $R = 2$ and $r = 1$. As we have the prior knowledge about the Betti numbers of $\mathcal{X}_0$, we want to test the following hypothesis:

$$H_0 : \beta = \beta_0 \text{ vs } H_1 : \beta \neq \beta_0,$$

where $\beta_0 = (\beta_0, \beta_1, \beta_2)^T = (1, 2, 1)^T$.

Now, in order to compute the power of the test, we have taken random samples according to uniform measure from two alternative supports, 3-D sphere and 3-D trefoil knot. Followings are the specifications of $\epsilon$ under $H_0$ and $H_1$, that satisfies the Assumption (A.2) for each of the $r$ replications:

- We take $\epsilon = 1.5$, for 20 samples generated from uniform distribution supported on the 3-D torus, under $H_0$.
- We take $\epsilon = 2$, for 20 samples generated from uniform distribution supported on the 3-D sphere, under $H_1$.
- We take $\epsilon = 1.5$, for 20 samples generated from uniform distribution supported on the 3-D trefoil knot, under $H_1$.

The plots in Figure 3 suggest that our proposed test perform well:

(a) Power plot when the support is 3-D Sphere.  
(b) Power plot when the support is 3-D trefoil knot.

Figure 3: Power plots.
5 Real data study

Here, we have implemented the proposed methodology on two real data sets, and in the course of this study, the TDA package in R-software has been used. We have taken 100 bootstrap samples to compute p-values of the test. Besides, the value of $\epsilon$ is chosen, as per the assumption (A.2) (see Section 3), from an increasing sequence $\{\epsilon_k; k = 1, 2, \ldots\}$, where $\epsilon_k = \epsilon_{k-1} + 0.5$, $\epsilon_0 = 0$. In this study, the data is standardised using the mean and the standard deviation for computational simplicity. Note that this standardization does not alter the hypothesis problem since the Betti numbers are invariant under the continuous transformation. In the following, both data sets are analysed using the concepts of Betti numbers.

Data set 1: We have taken a cross sectional data (Movie Buzz Data) consisting 62 observations on 13 variables from the book, Econometric analysis, by William Greene. (Data source: https://pages.stern.nyu.edu/~wgreene/Text/Edition7/tablelist8new.htm)

This data consists of observations on different variables, like Movie rating, genres and different social media indicators, affecting the first run U.S. box office collection of a movie. One would like to examine the effect of different variables on the variable Box (first run U.S. box office collection of a movie). One can investigate the shape of the data to form a hypothesis about the model that can explain the data well. Following plots suggest that data lies on the fitted regression lines, depicted as red lines in the plots.

![Box vs Budget plot](image1)

![Box vs StarPowr plot](image2)

(a) Box vs Budget plot  
(b) Box vs StarPowr plot

Figure 4: Plots of two continuous variables, Budget and StarPowr, against the variable Box.
Thus, in terms of Betti numbers, one can say that there is only one connected component in both the Figure 4 and Figure 5 above, and there is no higher dimensional homological features, such as loop, present in the data. As a result, following the initial inspection about the shape, we may hypothesise that the data lies on a hyperplane, since the pairwise relationship, of variables with the variable Box, is linear. Therefore, in terms of Betti numbers, we formulate the following hypothesis:

\[ H_0 : \beta_0 = 1, \beta_i = 0, \text{ for all } i = 1, 2 \ldots \text{ vs } H_1 : \beta_0 \neq 1 \text{ or } \beta_i \neq 0 \text{ for some } i. \]

We have calculated the sample Betti numbers from the Rips complex at \( \epsilon = 3.5 \), which we consider as an appropriate threshold satisfying the assumption (A.2), since R-code run-time for the bootstrap procedure, with 100 replications, is shorter compare to threshold greater than 3.5. Also, calculating Betti numbers up to dimension 12 becomes infeasible as we increase the value of \( \epsilon \). Thus, our choice of epsilon is large enough to capture the higher dimensional topological features and small enough to make the computations feasible.

Figure 6 shows Rips complex at \( \epsilon = 5 \) and \( \epsilon = 5.5 \) which suggests that if we choose \( \epsilon \in \{0, 0.5, 1, \ldots, 5\} \), then it satisfies the assumption (A.2), since at \( \epsilon = 5.5 \), complex becomes connected.

We investigate the performance of the test for this data for 100 bootstrap samples. We
construct Rips complex at $\epsilon = 3.5$ and calculate Betti numbers from the data, which are $\hat{\beta}_0 = 7, \hat{\beta}_i = 0$ for all $i=1,2,\ldots$, which tells that there are 7 connected components in the data and there are no higher dimensional homological features in the data. We obtain p-value 0.99, which suggests that data favours the null hypothesis, which is consistent with the initial inspection of shape of the data (see Figures 4 and 5). These observations from the study, suggests that our test is performing well for this data.

**Data set 2:** We have taken the second data set (Life Cycle Savings, from R-software), which is based on the life-cycle savings hypothesis as developed by Franco Modigliani. As per the life-cycle savings hypothesis, savings ratio, which is defined as aggregate personal savings divided by disposable income, is explained by the following four variables:

- Per-capita disposable income (dpi).
- The percentage rate of change in per-capita disposable income (ddpi).
- The percentage of the population less than 15 years old (pop15).
- The percentage of the population over 75 years old (pop75).

One would like to examine the relationship of these four variables with the savings ratio. Following figures gives the insight into shape of the data:

Thus, from the Figure 7 above, it is clear that most of the data points lie near the fitted regression line, depicted as red lines. Therefore, based on this initial inspection of the shape of
the data, we can hypothesise that the data lies on a hyperplane. Therefore, in terms of Betti numbers, we formulate the following hypothesis:

\[ H_0 : \beta_0 = 1, \beta_i = 0, \text{ for all } i = 1, 2 \ldots \text{ vs } H_1 : \beta_0 \neq 1 \text{ or } \beta_i \neq 0 \text{ for some } i. \]

We have calculated sample Betti numbers from Rips complex at \( \epsilon = 2 \), which satisfies the
assumption (A.2), see figure below:

![Rips complex plots](image)

(a) Rips complex at $\epsilon = 2$
(b) Rips complex at $\epsilon = 2.5$

Figure 8: Plots of Rips complex.

We investigate the performance of the test for this data for 100 bootstrap samples. We construct Rips complex at $\epsilon = 2$ and calculate Betti numbers from the data, which are $\hat{\beta}_0 = 3, \hat{\beta}_i = 0$ for all $i = 1, 2, \ldots$, which tells that there are 3 connected components in the data and there are no higher dimensional homological features in the data. We obtain p-value 0.98, which suggests that data favours the null hypothesis, which is consistent with the initial inspection of shape of the data from Figure 7. These observations from the study, suggests that our test is performing well for this data.

6 Conclusion

In this article, we have proposed a statistical test based on the Betti numbers to investigate the homological equivalence of the support of the data. We perform one sample and two sample tests for the homological equivalence of the support of the data and establish the consistency of the test in both the cases. Moreover, we have also performed the numerical study based on simulated and real data to support our claim in the case of one sample test. Due to constraint on the number of pages, we did not report the similar numerical studies for the two-sample problem, but it has been observed for a few examples that the proposed test for two sample problem also performs well.

We have considered the Betti numbers of Čech complex for theoretical purposes and Betti numbers of the Rips complex for computational purposes. In addition, we have constructed Čech or Rips complex to restrict the choice of proximity parameter $\epsilon$ such that Čech or Rips complex are not connected. Our results are valid for the case when the data is generated from the uniform distribution supported on a submanifold of the euclidean space. For future work, we would like to investigate the validity of these results for the case when the data is generated...
from the non-uniform distribution supported on a sub manifold of the euclidean space.

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## 7 Appendix

In this section, we provide proof of the Theorem 3.1 and Theorem 3.2, before that we will state the two results from Bobrowski und Kahle (2018) in the following two lemmas that will be used in the proof. These Lemmas are true under the supercritical regime that is when for $\epsilon > 0$ we have $n\epsilon^d \to \infty$ as $n \to \infty, d \geq 1$. Note that the supercritical regime is trivial for any fixed positive value of $\epsilon$. Thus, in this article the supercritical regime is satisfied due the Assumption (A.2) (See: Section 3).

**Lemma 7.1** Suppose $\hat{\beta}_{0,n}$ denotes the $0^{th}$ Betti number of Čech or Rips complex, then for the choice of $\epsilon$ satisfying the Assumption (A.2) (See: Section 3), then $\frac{\hat{\beta}_{0,n}}{n}$ is asymptotically almost surely bounded i.e there exists constants $0 < k < l < \infty$ such that:

$$\mathbb{P} \left[ k \leq \frac{\hat{\beta}_{0,n}}{n} \leq l \right] \to 1 \text{ as } n \to \infty.$$

**Remark 7.1** In Section 3.3 of Bobrowski und Kahle (2018), it is stated that under the supercritical regime that is when $n\epsilon^d \to \infty$ as $n \to \infty$, if the radius $\epsilon$ is large enough yet still satisfying $\epsilon \to 0$, the graph becomes connected. This phase is referred to as “connected regime”. As the radius increases, starting at the critical regime where $\beta_0(n) = \Theta(n)$ (In our notation $\beta_0(n)$ is same as $\hat{\beta}_{0,n}$) and ending that connected regime where $\beta_0(n) = \Theta(1)$, the number of connected components in the graph should exhibit some kind of a decay within the supercritical regime. Here, $\beta_0(n) = \Theta(n)$ means that the ratio of Betti numbers of the random Čech or Rips complex to the sample size $n$ is asymptotically almost surely bounded. We have restricted our choice of $\epsilon$ between starting at the critical regime and ending at the connected regime by postulating the Assumption (A.2). Thus, observing the fact that results related to the connected components of random Čech or Rips complex is same for random graph (See: Section 2.1), we have accumulated these facts to propose the Lemma 7.1.
**Lemma 7.2** Suppose that \( \hat{\beta}_{i,n} \) denotes the \( i \)th Betti number of Čech or Rips complex, generated by a uniform distribution on a unit-volume convex body in \( \mathbb{R}^d \). Then under the Assumption (A.2), we have \( E(\hat{\beta}_{i,n}) = o(n) \) for all \( i = 1, \ldots d - 1 \). (See: Section 4.3, Theorem 4.9 of Bobrowski and Kahle (2018))

**Proposition 7.1** Let \( \hat{\beta}_{i,n} \) denotes the \( i \)th Betti number of Čech or Rips complex, generated by a uniform distribution supported on a compact set \( X \subseteq \mathbb{R}^d \), then under the assumption (A.1) and (A.2), for \( i = 0, 1 \ldots d - 1 \), we have the following:

\[
\frac{\hat{\beta}_{0,n}}{n} + \frac{\sum_{i=1}^{d-1} \hat{\beta}_{i,n}}{n} \xrightarrow{D} V \quad \text{as} \quad n \to \infty.
\]

Here \( V \) is a positive random variable defined as follows:

\[
V = \begin{cases} 
  k, & \text{with probability } p \\
  l, & \text{with probability } 1-p
\end{cases}, \quad \text{where } p \in (0, 1) \land 0 < k < l < \infty.
\]

**Proof:** From the Lemma 7.1, we have \( \hat{\beta}_{0,n} \) asymptotically almost surely bounded i.e there exists constants \( 0 < k < l < \infty \) such that:

\[
P\left[ \frac{\hat{\beta}_{0,n}}{n} \geq k \right] \to 1 \quad \text{as} \quad n \to \infty \quad \text{and} \quad P\left[ \frac{\hat{\beta}_{0,n}}{n} \leq l \right] \to 1 \quad \text{as} \quad n \to \infty.
\]

Therefore, for any \( t \in \mathbb{R} \), we have:

\[
P\left[ \frac{\hat{\beta}_{0,n}}{n} \leq t \right] \to \begin{cases} 
  0, & t < k \\
  p, & k \leq t < l \\
  1, & t \geq l
\end{cases}, \quad \text{where } p \in (0, 1).
\]

Therefore, we have:

\[
P\left[ \frac{\hat{\beta}_{0,n}}{n} \leq t \right] \to P[V \leq t] \Rightarrow \frac{\hat{\beta}_{0,n}}{n} \xrightarrow{D} V \quad \text{as} \quad n \to \infty. \quad (7.1)
\]

Also, from the Lemma 7.2 we have:
\( E(\hat{\beta}_{i,n}) = o(n) \) for all \( i = 1, \ldots, d - 1 \).

Since \( \hat{\beta}_{i,n} \)'s are non-negative random variables, using the Lemma 7.2, we have:

\[
\mathbb{E}(\hat{\beta}_{i,n}) = o(n) \Rightarrow \mathbb{E}\left|\frac{\hat{\beta}_{i,n}}{n}\right| \rightarrow 0 \Rightarrow \frac{\hat{\beta}_{i,n}}{n} \overset{L^1}{\rightarrow} 0 \text{ as } n \rightarrow \infty.
\]

Since convergence in mean implies convergence in probability, we have \( \frac{\hat{\beta}_{i,n}}{n} \overset{P}{\rightarrow} 0 \).

Now, since \( \frac{\hat{\beta}_{i,n}}{n} \overset{P}{\rightarrow} 0 \) for all \( i = 1, \ldots d - 1 \) and \( d \) is a fixed integer, we have:

\[
\sum_{i=1}^{d-1} \frac{\hat{\beta}_{i,n}}{n} \overset{P}{\rightarrow} 0 \text{ as } n \rightarrow \infty. \quad (7.2)
\]

Now, using Slutsky’s theorem with (7.1) and (7.2), we have:

\[
\frac{\hat{\beta}_{0,n}}{n} + \sum_{i=1}^{d-1} \frac{\hat{\beta}_{i,n}}{n} \overset{D}{\rightarrow} V \text{ as } n \rightarrow \infty.
\]

\[\square\]

**Corollary 7.1** Suppose \( \hat{\beta}_{i,n_1} \) and \( \hat{\beta}_{i,n_2}^* \) denote the \( i \)th Betti numbers of Čech or Rips complex, generated by a uniform distribution supported on the manifolds \( \mathcal{X} \subseteq \mathbb{R}^d \) and \( \mathcal{Y} \subseteq \mathbb{R}^d, d \geq 1 \), respectively. Then under the assumption (A.1) and (A.2), for \( i = 0, 1 \ldots d - 1 \), we have the following:

(i) \( \frac{\hat{\beta}_{0,n_1}}{n_1} + \sum_{i=1}^{d-1} \frac{\hat{\beta}_{i,n_1}}{n_1} \overset{D}{\rightarrow} V_1 \) as \( n_1 \rightarrow \infty \).

(ii) \( \frac{\hat{\beta}_{0,n_2}^*}{n_2} + \sum_{i=1}^{d-1} \frac{\hat{\beta}_{i,n_2}^*}{n_2} \overset{D}{\rightarrow} V_2 \) as \( n_2 \rightarrow \infty \).

Here \( V_j \)'s, \( j = 1, 2 \), are independent positive random variables defined as follows:

\[
V_j = \begin{cases} k_j, & \text{with probability } p_j \smallskip \\
l_j, & \text{with probability } 1 - p_j \end{cases}, \quad \text{where } p_j \in (0, 1) \land 0 < k_j < l_j < \infty.
\]
Proof (Theorem 3.1): For some positive constant $c$ and let $M = \sum_{i=0}^{d-1} \beta_{i,0}$, where $\beta_{i,0}$’s are hypothesised Betti numbers (See Section 3), for $i = 0, 1, \ldots d - 1$. Then, to establish the consistency of the test based on $T_n$, consider the following probability under $H^*_1$:

\[
P_{H^*_1}[T_n > c] = P_{H^*_1}\left[\sum_{i=0}^{d-1} |\hat{\beta}_{i,n} - \beta_{i,0}| > c \right] \geq P_{H^*_1}\left[\sum_{i=0}^{d-1} (\hat{\beta}_{i,n} - \beta_{i,0}) > c \right]
\]

\[
= P_{H^*_1}\left[\sum_{i=0}^{d-1} \hat{\beta}_{i,n} > c + \sum_{i=0}^{d-1} \beta_{i,0} \right] = P_{H^*_1}\left[\hat{\beta}_{0,n} + \sum_{i=1}^{d-1} \hat{\beta}_{i,n} > c + M \right]
\]

\[
= P_{H^*_1}\left[\frac{\hat{\beta}_{0,n}}{n} + \frac{\sum_{i=1}^{d-1} \hat{\beta}_{i,n}}{n} > \frac{c + M}{n} \right] \to P_{H^*_1}[V > 0] = 1
\]

Here the last line follows from Proposition 7.1 which says that

\[
\frac{\hat{\beta}_{0,n}}{n} + \frac{\sum_{i=1}^{d-1} \hat{\beta}_{i,n}}{n} \Rightarrow V \text{ as } n \to \infty.
\]

Also, since $\frac{c + M}{n} \to 0$, using Slutsky’s theorem we have:

\[
\frac{\hat{\beta}_{0,n}}{n} + \frac{\sum_{i=1}^{d-1} \hat{\beta}_{i,n}}{n} - \frac{c + M}{n} \Rightarrow V \text{ as } n \to \infty.
\]

\[\square\]

Proof (Theorem 3.2): For some positive constant $c$ and assuming that $l_2$ and $k_1$ in the Corollary 7.1 are such that $(1 - \lambda)l_2 < \lambda k_1$. We consider the following probability, under $H^*_{1}^{**}$,
to establish the consistency of the test based on $T_{n_1,n_2}$, we have:

$$P_{H^*_1} [T_{n_1,n_2} > c] = P_{H^*_1} \left[ \sum_{i=0}^{d-1} \beta_{i,n_1} - \beta_{i,n_2}^* > c \right] \geq P_{H^*_1} \left[ \sum_{i=0}^{d-1} (\hat{\beta}_{i,n_1} - \hat{\beta}_{i,n_2}^*) > c \right]$$

$$= P_{H^*_1} \left[ \beta_{0,n_1} + \sum_{i=1}^{d-1} \beta_{i,n_1} - \beta_{0,n_2}^* - \sum_{i=1}^{d-1} \beta_{i,n_2}^* > c \right]$$

$$= P_{H^*_1} \left[ \frac{\beta_{0,n_1} + \sum_{i=1}^{d-1} \beta_{i,n_1}}{n_1 + n_2} - \frac{\beta_{0,n_2}^* + \sum_{i=1}^{d-1} \beta_{i,n_2}^*}{n_1 + n_2} > \frac{c}{n_1 + n_2} \right]$$

$$\Rightarrow P_{H^*_1} \left[ \lambda V_1 - (1 - \lambda) V_2 > 0 \right] \text{ as } \min (n_1, n_2) \to \infty$$

$$= P_{H^*_1} \left[ V_1 > \frac{(1 - \lambda) V_2}{\lambda} \right] + P_{H^*_1} \left[ V_1 > \frac{(1 - \lambda) k_2}{\lambda} \right] (1 - p_2)$$

$$= 1,$$

where the last line follows from the Corollary 7.1 and the fact that we have assumed $(1 - \lambda) l_2 < \lambda k_1$ implies that $\max \left( \frac{(1 - \lambda) k_2}{\lambda}, \frac{(1 - \lambda) l_2}{\lambda} \right) < k_1$. Thus, using the fact that $P[V_1 > t] = 1$ for all $t < k_1$, consistency of the test based on $T_{n_1,n_2}$ is established.

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