Abstract. The “qualitative” extension theorem of Demailly guarantees existence of holomorphic extensions of holomorphic sections on some subvariety under certain positive-curvature assumption, but that comes without any estimate of the extensions, especially when the singular locus of the subvariety is non-empty and the holomorphic section to be extended does not vanish identically there. Residue functions are analytic functions which connect the $L^2$ norms on the subvarieties (or their singular loci) to $L^2$ norms with specific weights on the ambient space. Motivated by the conjectural “dlt extension”, this note discusses the possibility of retrieving the $L^2$ estimates for the extensions in the general situation via the use of the residue functions. It is also shown in this note that the 1-lc-measure defined via the residue function of index 1 is indeed equal to the Ohsawa measure in the Ohsawa–Takegoshi $L^2$ extension theorem.

This note reviews the “qualitative” extension theorems obtained by Demailly in [8] and together with Junyan Cao and Shin-ichi Matsumura in [1], and then discusses the possibility of retrieving the $L^2$ estimates for the extensions. This is the contents of Section 1. The residue functions introduced in [4] by the author are used to facilitate the re-establishment of the possible estimates, which are discussed in Section 2.

All results stated in this note have been proved somewhere else, except for Proposition 2.2.1, which states that the 1-lc-measure introduced in [5] and [3] is indeed equal to the Ohsawa measure in the Ohsawa–Takegoshi $L^2$ extension theorem, and Corollary 2.2.2, which identifies the ad hoc ideal sheaf $\mathcal{I}(m_k-1)$ introduced by Demailly in [8, Def. (2.11)] to be the adjoint ideal sheaf of index 1 introduced in [4].

1. $L^2$ Extension Theorem of Demailly

Based on the techniques developed through the Ohsawa–Takegoshi $L^2$ extension theorems ([21], [19], [7], [22], ...), Demailly proves in [8] an extension theorem for holomorphic sections on possibly non-reduced subvarieties defined by some multiplier ideal sheaves on compact Kähler manifolds. More precisely, let

- $X$ be a compact Kähler manifold, \footnote{The case where $X$ is a weakly pseudconvex Kähler manifold is also discussed in [8].}
- $(L, e^{-\varphi L})$ be a holomorphic line bundle over $X$ equipped with a singular hermitian metric $e^{-\varphi L}$, \footnote{The extension theorem of Demailly is also proved in [8] for the case where $(L, e^{-\varphi L})$ is replaced by $(E, h)$, a holomorphic vector bundle $E$ over $X$ equipped with a smooth hermitian metric $h$.}
- $\psi \leq -1$ be a bounded global function on $X$ with neat analytic singularities.

Here a function $\varphi$ is said to have neat analytic singularities if it is locally the difference $\varphi_1 - \varphi_2$ of quasi-plurisubharmonic (quasi-psh) functions of the form

$$\varphi_i = c_i \log \left( \sum_{j=1}^{N} |g_{ij}|^2 \right) + \alpha_i,$$

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$$\varphi_i = c_i \log \left( \sum_{j=1}^{N} |g_{ij}|^2 \right) + \alpha_i,$$
where \( c_i \in \mathbb{R}_{>0}, \ g_{ij} \in \mathcal{O}_X \) and \( \alpha_i \in \mathcal{C}^\infty_X \) for \( i = 1, 2 \). The function \( \varphi \) is said to have \textit{analytic singularities} if the local functions \( \alpha_i \) are bounded (need not be smooth).

Suppose that \( \varphi_L \) and \( \varphi_L + m\psi \) are quasi-psh for some \( m \geq 0 \) so that the multiplier ideal sheaves \( \mathcal{I}(\varphi_L) \) and \( \mathcal{I}(\varphi_L + m\psi) \) are coherent. The subvarieties considered in [8] (also in [1]) are those defined by the annihilator of \( \mathcal{I}(\varphi_L) \), denoted by \( Y^{(m)} \). (When \( \varphi_L = 0 \) and \( \psi = \log|z_1|^2 \) on a coordinate neighbourhood, \( \text{Ann}\, \mathcal{O}_X \left( \frac{\mathcal{I}(\varphi_L)}{\mathcal{I}(\varphi_L + m\psi)} \right) = \mathcal{I}(m\psi) = \mathcal{I}_{\{z_1=0\}} \), where \( \mathcal{I}_{\{z_1=0\}} \) is the defining ideal sheaf of \( \{z_1 = 0\} \) and \([m]\) is the round-down of \( m \). The subvariety \( Y^{(m)} \) is then non-reduced for \( m \geq 2 \).

1.1. Extension theorem without estimates. The extension result is stated as follows.

**Theorem 1.1.1** ([8, Thm. (2.14b)] and [1, Thm. 1.1]). Given any fixed real number \( m > 0 \), suppose that there is a constant \( \delta > 0 \) such that

\[
i\bar{\partial}((\varphi_L + \beta\psi) \geq 0 \quad \text{for all} \quad \beta \in [m, m+\delta]
\]

in the sense of currents. Then the restriction morphism

\[
H^0(X, K_X \otimes L \otimes \mathcal{I}(\varphi_L)) \to H^0 \left( Y^{(m)}, K_X \otimes L \otimes \frac{\mathcal{I}(\varphi_L)}{\mathcal{I}(\varphi_L + m\psi)} \right)
\]

is surjective.

**Remark 1.1.2.** [1, Thm. 1.1] indeed states that the corresponding statement on holomorphically convex Kähler manifold and for higher cohomology groups also holds true.

This theorem, while being proved via \( L^2 \) method, does not require (explicitly) the convergence of any \( L^2 \) norms of the sections to be extended from the subvariety. This is considered as an advantage since the \textit{Ohsawa measure} on the subvariety \( Y^{(m)} \), the measure appears in the estimate of the Ohsawa–Takegoshi \( L^2 \) extension theorem (see [20] or [8, (2.4)]), diverges in general around the singular locus of \( Y^{(m)} \) (see [17] or [15] for more discussion on the singularities on the Ohsawa measure). The classical theorem, which requires the convergence of sections with respect to the Ohsawa measure, is deemed inapplicable to extend sections non-vanishing on the singular locus. It was hoped that this feature of Theorem 1.1.1 can be exploited in order to solve the conjectural “dlt extension” in [9, Conj. 1.3]. The hurdle is, while the sections which can be extended via Theorem 1.1.1 have to sit inside a quotient of multiplier ideal sheaves (namely, \( \frac{\mathcal{I}(\varphi_L)}{\mathcal{I}(\varphi_L + m\psi)} \)), the conjecture of the “dlt extension” demands the extension of sections which are not confined in any multiplier ideal sheaves. One either has to show that the quotient of the multiplier ideal sheaves of some suitably chosen potentials in the setup of the “dlt extension” is trivial, or has to improve Theorem 1.1.1 in the specific case so that any holomorphic sections on the corresponding \( Y^{(m)} \) can also be extended. At the time of writing, the author does not know any successful attempt in the latter approach. For the former approach, the known strategy involves the use of the \( L^2 \) estimates with universal constant from the Ohsawa–Takegoshi \( L^2 \) extension theorem (cf. proof of the “dlt extension” in [9, Thm. 1.7] or [5, Thm. 1.6.1]), which, roughly speaking, corresponds to the case where the subvariety \( Y^{(m)} \) is smooth). The “universal constant” here means that the multiplicative constant involved in the estimate is independent of the involving sections and metrics.

However, as a trade-off of the non-requirement of the convergence of any \( L^2 \) norm on \( Y^{(m)} \), Theorem 1.1.1 does not provide any estimate for the extensions in general. In [8], it is shown that an \( L^2 \) estimate can be obtained for extensions corresponding to successive jumping numbers, which is explained below.

1.2. Extension theorem with \( L^2 \) estimates. Thanks to the solution to the strong openness conjecture for psh functions by Guan and Zhou ([11], see also [12] and [13]),
under the compactness assumption on $X$, there exists a strictly increasing sequence of jumping numbers

$$0 \leq m_0 < m_1 < \cdots < m_k < \cdots$$

such that, for each $k \in \mathbb{N}$,

$$\mathcal{I}(\varphi_L + m_k \psi) \subsetneq \mathcal{I}(\varphi_L + m \psi) = \mathcal{I}(\varphi_L + m_{k-1} \psi) \quad \text{on } X \text{ for all } m \in [m_{k-1}, m_k).$$

**Remark 1.2.1.** If $\varphi_L$ (as well as $\psi$) has only analytic singularities, then the sequence of jumping numbers $\{m_k\}_{k \in \mathbb{N}}$ has no accumulation point, as can be seen via a log-resolution of the polar ideal sheaves of $\varphi_L$ and $\psi$ (whose existence is guaranteed by Hironaka’s result [14]). The number $m_0$ can be set to 0 and the sequence diverges to $+\infty$ in this case. When $\varphi_L$ has more general singularities, the sequence $\{m_k\}_{k \in \mathbb{N}}$ may converge to $\lim_{k \to +\infty} m_k =: m_0^{(1)} < +\infty$ (see [10] for an example of such $\varphi_L$ (with $\psi := \log |z|^2$ and $m_0^{(1)} = 1$) and [16, §5] for some further discussion). The strong openness property again guarantees the existence of another sequence of jumping numbers

$$0 < m_0^{(1)} < m_1^{(1)} < \cdots < m_k^{(1)} < \cdots$$

satisfying the same property of $\{m_k\}_{k \in \mathbb{N}}$ on the family $\{\mathcal{I}(\varphi_L + m \psi)\}_{m \in \mathbb{R} \geq 0}$ of multiplier ideal sheaves. Since only extensions corresponding to successive jumping numbers $(m_{k-1}$ and $m_k$ or $m_{k-1}^{(1}$ and $m_k^{(1)}$) is under concern in this section, for the sake of generality, the number $m_0$ is not assumed to be 0 as does in [8] and it is assumed that $m_k$ is not an accumulation point of jumping numbers in what follows.

The subvariety $S := S^{(m_k)} (\subset Y^{(m_k)})$ defined by $\text{Ann}_{\mathcal{O}_X} \left( \frac{\mathcal{J}(\varphi_L + m_{k-1} \psi)}{\mathcal{I}(\varphi_L + m_k \psi)} \right)$, which is the scheme-theoretic difference between $Y^{(m_k)}$ and $Y^{(m_{k-1})}$, is reduced (see [8, Lemma (4.2)]). The statement of a “quantitative” extension in [8] is recalled as follows.

**Theorem 1.2.2 ([8, Thm. (2.12a)]).** Given a fixed jumping number $m_k$ (which is not an accumulation point) of the family $\{\mathcal{I}(\varphi_L + m \psi)\}_{m \in \mathbb{R} \geq 0}$, suppose that there is a constant $\delta > 0$ such that one has the curvature assumption

$$i \partial \bar{\partial} (\varphi_L + \beta \psi) \geq 0 \quad \text{for all } \beta \in [m_k, m_k + \delta]$$

in the sense of currents. If $f \in H^0 \left( S, K_X \otimes L \otimes \mathcal{I}(\varphi_L + m_{k-1} \psi) \right)$ has finite $L^2$ norm with respect to the (generalised) Ohsawa measure, i.e.

$$\int_S |J^{m_k} f|^2 d\text{vol}_{\varphi_L} [\psi] := \lim_{t \to -\infty} \int_{\{t < \psi < t + 1\} \subset X} |\tilde{f}'|^2 e^{-\varphi_L - m_k \psi} < +\infty,$$

where $\tilde{f} \in K_X \otimes L \otimes \mathcal{O}_X \cdot \mathcal{I}(\varphi_L + m_{k-1} \psi)(S)$ is some smooth extension of $f$, then there exists $F \in H^0 (X, K_X \otimes L \otimes \mathcal{I}(\varphi_L + m_{k-1} \psi))$ which is a holomorphic extension of $f$, i.e. $F \equiv f \mod \mathcal{I}(\varphi_L + m_k \psi)$ on $X$, such that

$$\int_X \frac{|F|^2 e^{-\varphi_L - m_k \psi}}{|\delta \psi|^2 + 1} \leq \frac{34}{\delta} \int_S |J^{m_k} f|^2 d\text{vol}_{\varphi_L} [\psi].$$

**Remark 1.2.3.** In [5, Thm. 1.4.5 and Thm. 3.4.1], the estimate, at least in the case where $\varphi_L$ has only neat analytic singularities, is improved to

$$\int_X \frac{|F|^2 e^{-\varphi_L - m_k \psi}}{|\delta \psi|(|\log |\delta \psi||)^2} \leq \int_S |f|^2 d\text{lc}_{\varphi_L}^{1, (m_k)} [\psi]$$

(see [3, Remark 1.1.4] for an explanation for the slightly different form from [5, Thm. 1.4.5] on the left-hand-side), where $\ell \geq e$ is a sufficiently large constant, depending only on $\delta$ and $\psi$, and the right-hand-side is the $L^2$ norm of $f$ with respect to the 1-lc-measure introduced in [5], which is also discussed in Section 2. The constant $\ell$ is the multiplicative constant.
in the classical Ohsawa–Takegoshi extension theorem in disguise and is “universal” in this case (it does not depend on \( \varphi_L, m_k \) and the involving sections \( F \) and \( f \)). It is shown in Proposition 2.2.1 below that, indeed, the 1-Lc-measure is equal to the Ohsaw measure.

Let \( \mathcal{I}'(m_{k-1}) \) be the subsheaf of \( \mathcal{I}(\varphi_L + m_{k-1}) \) consisting of all the germs \( F \in \mathcal{I}(\varphi_L + m_{k-1}) \) which are locally \( L^2 \) with respect to the Ohsawa measure, as defined in \([8, \text{Def. (2.11)}]\). It follows that \( \mathcal{I}(\varphi_L + m_\psi) \subset \mathcal{I}'(m_{k-1}) \subset \mathcal{I}(\varphi_L + m_{k-1}) \), and Theorem 1.2.2 indeed states that, under the compactness assumption on \( X \) (so that locally \( L^2 \) implies globally \( L^2 \)), all sections in \( H^0(S, L \otimes \mathcal{O}(\varphi_L + m_\psi)) \) can be extended holomorphically to some \( F \in H^0(X, L \otimes \mathcal{I}(m_{k-1})) \) with \( L^2 \) estimates. Given the fact that Theorem 1.1.1 also guarantees the existence of extensions of sections taking values in the complement \( \mathcal{I}(\varphi_L + m_\psi) \setminus \mathcal{I}'(m_{k-1}) \), the goal of this study is to obtain reasonable estimates for these extensions. The residue functions and the corresponding adjoint ideal sheaves discussed in Section 2 are introduced to facilitate the quest. The sheaf \( \mathcal{I}'(m_{k-1}) \) indeed equals \( \mathcal{I}_1(\varphi_L; m_k \cdot \psi) \), the adjoint ideal sheaf of index 1 defined via the residue functions (see Corollary 2.2.2).

If the desired estimate for some holomorphic extension of every section taking values in \( \mathcal{I}(\varphi_L + m_{k-1}) \) can be obtained for every \( k \geq 1 \), then, in view of the short exact sequence

\[
0 \rightarrow \mathcal{I}(\varphi_L + m_\psi) \rightarrow \mathcal{I}(\varphi_L + m_{k-1}) \rightarrow \mathcal{I}(\varphi_L + m_k) \rightarrow 0
\]

for all integers \( k \geq \ell \geq 1 \), one can also obtain an extension with estimate for any \( f \) taking values in \( \mathcal{I}(\varphi_L + m_{k-1}) \) by treating \( f \) as an \( \mathcal{I}(\varphi_L + m_k) \)-valued section to obtain an \( \mathcal{I}(\varphi_L + m_\psi) \)-valued extension \( F_0 \) on \( X \) with estimate and repeating the process to \( f - F_0 \) mod \( \mathcal{I}(\varphi_L + m_\psi) \), which is now \( \mathcal{I}(\varphi_L + m_k) \)-valued. Iterating this procedure results in \( \mathcal{I}(\varphi_L + m_{j+1}) \)-valued sections \( F_j \) for \( j = 0, \ldots, k-1 \), each having an estimate, and the sum \( \sum_{j=0}^{k-1} F_j \) is an extension of \( f \).

**Remark 1.2.4.** At the moment of writing, the author cannot even make a prediction on whether it is possible to obtain a holomorphic extension, with estimate, for an \( \mathcal{I}(\varphi_L + m_{k-1}^0) \)-valued section, where \( m_{k-1}^0 := \lim_{k \rightarrow +\infty} m_k < +\infty \).

2. Residue functions and adjoint ideal sheaves

The definition of residue functions is based on the following model: given the function \( \psi := \sum_{j=1}^n \log x_j - 1 \) on the cube \([0, 1]^n\) (where \( \sigma \leq n \)) and any compactly supported smooth function \( G \in C^\infty_c([0, 1]^n) \), a direct computation with Fubini’s theorem and integration by parts yields

\[
\mathfrak{R}^G(\varepsilon)_s := \varepsilon \int_{[0,1]^n} G \, dx_1 \cdots dx_n \frac{1}{x_1 \cdots x_\sigma |\psi|^s (\log |\psi|)^{1+s}} \]

\[
= -\varepsilon c_1(s, \sigma, \varepsilon) \mathfrak{R}^G(1+\varepsilon)_s \cdots -\varepsilon c_{\sigma-1}(s, \sigma, \varepsilon) \mathfrak{R}^G(\sigma - 1 + \varepsilon)_s + \frac{(-1)^\sigma}{(\sigma - 1)!} \int_{[0,1]^n} \frac{G}{\partial x_\sigma \cdots \partial x_1} \frac{dx_1 \cdots dx_n}{|\log |\psi||^s} \]

if \( s = \sigma \),

\[
\varepsilon \rightarrow 0^+ \frac{1}{(\sigma - 1)!} \int_{[0,1]^n-x_\sigma} G \, dx_\sigma \cdots dx_n \]

where \( \varepsilon > 0 \) and \( c_j(s, \sigma, \varepsilon) > 0 \) for \( j = 1, \ldots, \sigma - 1 \) are positive coefficients which are polynomials in \( \varepsilon \) (see the proof of [3, Prop. 2.2.1] for the computation in a more general setup). One can also show that the integral \( \mathfrak{R}^G(\varepsilon)_s \) diverges for any \( \varepsilon > 0 \) when \( s < \sigma \).
and $G|_{x_1=\ldots=x_s=0} \neq 0$, and $\lim_{\varepsilon \to 0} \delta^G_{\{\varepsilon\} s} = 0$ when $s > \sigma$. Using the formula obtained after applying successive integration by parts, the function $\varepsilon \mapsto \delta^G_{\{\varepsilon\} s}$ can be continued analytically to the whole complex plane (see [3, Thm. 2.3.1]). This illustrates that such kind of functions connect analytically the norm on an subvariety (at $\varepsilon = 0$) with a norm on the ambient space with a specific weight (at some $\varepsilon > 0$).

2.1. Definitions of residue functions and related notions. The definition of residue functions in the setup as in Section 1 is given as follows.

**Definition 2.1.1** ([3, Def. 1.1.1]). Given the potential $\varphi_L$, the function $\psi \leq -1$ and the jumping numbers $\{m_k\}_{k \in \mathbb{N}}$ described as in Section 1, on any open set $V \subset X$, the residue function $\mathbb{R}_{>0} \ni \varepsilon \mapsto \delta^G_{\{\varepsilon\} V,\sigma}$ of index $\sigma$ for any $L \otimes \mathcal{L}$-valued $(n,n)$-form $G$ with respect to the data $(V,\varphi_L,\psi,m_k)$ is given by

$$
\delta^G_{\{\varepsilon\} V,\sigma} := \delta^G_{\{\varepsilon\} V,\varphi_L,\psi,m_k,\sigma} := \varepsilon \int_V G e^{-\varphi_L - m_k \psi} |\psi|^\sigma (\log |\psi|)^{1+\varepsilon} \quad \text{for } \varepsilon > 0.
$$

When $\varphi_L$ has only neat analytic singularities and $G = |f|^2$ for some $f \in K_X \otimes L \otimes \mathcal{A}(\varphi_L + m_{k-1} \psi)(V)$ such that $\delta^G_{\{\varepsilon\} V,\sigma} < +\infty$ for all $\varepsilon > 0$, the function can then be continued analytically to an entire function (see [3, Thm. 2.3.1] with the log-resolution of $(X,\varphi_L,\psi)$ described as in [4, §2.3] considered). This illustrates that such $\sigma$-lc-measures more properly, the following version of adjoint ideal sheaves is introduced.

**Definition 2.1.2** ([4, Def. 1.2.1]). The (analytic) adjoint ideal sheaf of index $\sigma$ with respect to the data $(X,\varphi_L,\psi,m_k)$, denoted by $\mathcal{I}_{\sigma}(\varphi_L;\psi) := \mathcal{I}_{\sigma}(\varphi_L;m_k;\psi)$, is an ideal sheaf on $X$ such that its stalk at each $x \in X$ is given by

$$
\mathcal{I}_{\sigma}(\varphi_L;m_k;\psi)_x = \left\{ f \in \mathcal{O}_{X,x} \left| \begin{array}{c} \exists \text{ open set } V_x \ni x, \forall \varepsilon > 0, \delta^G_{\{\varepsilon\} V_x,\varphi_L,\psi,m_k,\sigma} < +\infty \end{array} \right. \right\}.
$$

When $\varphi_L$ has only neat analytic singularities, it is shown in [4, Thm. 4.1.2] that there exists an integer $\sigma_{mlc} \in [0,n]$ such that

$$
\mathcal{I}_{\sigma_{mlc}}(\varphi_L;\psi) \subset \cdots \subset \mathcal{I}_{\sigma_{mlc+1}}(\varphi_L;\psi) = \cdots = \mathcal{I}_{\sigma_{mlc+1}}(\varphi_L;\psi) = \mathcal{I}_{\sigma_{mlc+1}}(\varphi_L;\psi) = \cdots = \mathcal{I}_{\sigma_{mlc+1}}(\varphi_L;\psi)
$$

The filtration gives rise to the lc centres.

**Definition 2.1.3** ([4, Def. 1.2.4]). A $\sigma$-lc centre of $(X,\varphi_L,\psi,m_k)$ is an irreducible component $S_\sigma^p$ of the reduced subvariety $\text{lc}_X^\sigma(\varphi_L;\psi) := \bigcup_{p \in I^p_s} S_\sigma^p$ in $X$ defined by the annihilator

$$
\mathcal{I}_{\text{lc}_X^\sigma(\varphi_L;m_k;\psi)} := \text{Ann}_{\mathcal{O}_X} \left( \frac{\mathcal{I}_{\sigma}(\varphi_L;m_k;\psi)}{\mathcal{I}_{\sigma-1}(\varphi_L;m_k;\psi)} \right)
$$

(see [4, Thm. 5.2.1] for a proof on $\text{lc}_X^\sigma(\varphi_L;m_k;\psi)$ being reduced).

Note that one has $S = \text{lc}_X^1(\varphi_L;\psi) \cup \text{lc}_X^2(\varphi_L;\psi) \cup \cdots \cup \text{lc}_X^{\sigma_{mlc}}(\varphi_L;\psi)$ by [4, Prop. 5.2.6]. When $\varphi_L$ has only neat analytic singularities, after passing to a log-resolution of $(X,\varphi_L,\psi)$ as discussed in [4, §2.3] such that $S$ is a reduced snc divisor in particular, each $\sigma$-lc centre $S_\sigma^p$ defined in Definition 2.1.3 is just a component of the intersection of any $\sigma$ distinct pieces of irreducible components of $S$, which coincides with the lc centre of codimension $\sigma$ of the log-smooth and lc pair $(X,S)$ in the study of minimal model program (see [18, Def. 4.15];
see also [4, Example 6.2.1] for an example on which the two concepts of lc centres differ). Moreover, in this case, one has $\mathcal{I}(\varphi_L; m_k \cdot \psi) = \mathcal{I}(\varphi_L + m_{k-1} \psi) \cdot I_{k+1}^{X, \psi_{\varphi_L}}$ (see [4, Thm. 4.1.2]).

A direct computation (see [5, Prop. 2.2.1]) shows that, for any $f \in \mathcal{I}(\varphi_L; \psi)(\overline{V})$ on the closure of any open set $V$ in $X$, the value $\mathcal{S}(f^2)_{0, \varphi_L}$ is finite and is an $L^2$ norm of $f$ on $\text{lc}_X(\varphi_L; \psi)$. In view of this, for any $f \in H^0 \left( S, K_X \otimes L \otimes \mathcal{I}(\varphi_L + m_{k-1} \psi) \right)$, the $\sigma$-lc-measure with respect to $(\varphi_L, \psi, m_k, f)$ is defined to be the measure $|f|^2 \text{d} \text{lc}^{\sigma}(\varphi_L, m_k)[\psi]$ given by the functional

$$
\mathcal{C}^0_\sigma \left( \text{lc}_X(\varphi_L; m_k \cdot \psi) \setminus \text{lc}_X^{\sigma+1}(\varphi_L; m_k \cdot \psi) \right) \ni g \quad \mapsto \quad \mathcal{S}(f^2)_{0, X, \sigma} =: \int_{\text{lc}_X(\varphi_L; m_k \cdot \psi)} g |f|^2 \text{d} \text{lc}^{\sigma}(\varphi_L, m_k)[\psi].
$$

It follows from [4, Thm. 4.1.2] that, if the function $f$ takes values in $\mathcal{I}(\varphi_L; \psi)$, the measure $|f|^2 \text{d} \text{lc}^{\sigma}(\varphi_L, m_k)[\psi]$ is non-trivial and is nowhere divergent if and only if $s = \sigma$.

Another use of such adjoint ideal sheaves can be found in [6].

2.2. Relation with the Ohsawa measure and extension theorem of Demailly.

The following proposition shows that the 1-lc-measure is nothing other than the Ohsawa measure.

**Proposition 2.2.1.** Assume that $\varphi_L$ has only neat analytic singularities. For any $f \in H^0 \left( S, K_X \otimes L \otimes \mathcal{I}(\varphi_L + m_{k-1} \psi) \right)$ and $g \in \mathcal{C}^0_\sigma(S \setminus \text{Sing}(S))$,

$$
\mathcal{S}(f^2)_{0, X, \sigma} = \int_S g |f|^2 \text{d} \text{lc}^{\sigma+1}(\varphi_L, m_k)[\psi] = \int_S g |J^m f|^2 \text{d} \text{vol}_{\varphi_L} \psi .
$$

In other words, $|f|^2 \text{d} \text{lc}^{\sigma+1}(\varphi_L, m_k)[\psi] = |J^m f|^2 \text{d} \text{vol}_{\varphi_L} \psi$ in the sense of measures (which can possibly diverge around $\text{Sing}(S)$).

**Proof.** Passing to a log-resolution of $(X, \varphi_L, \psi)$ (see [4, §2.3]), one can assume that $\psi^{-1}(-\infty)$ (hence $S$), $\varphi^{-1}_L(-\infty)$ and $\psi^{-1}(-\infty) \cup \varphi^{-1}_L(-\infty)$ are all snc divisors (the triple $(X, \varphi_L, \psi)$ is said to be in an snc configuration). Since $g$ has compact support away from $\text{Sing}(S)$, by decomposing $g$ into a sum of functions supported on different components of $S$, one can assume that $g$ is supported only on a single component, say, $D_1 := S_1$, of $S$ (while remaining in $\mathcal{C}^0(S \setminus \text{Sing}(S))$). Via the use of a partition of unity, one can arrange further that $g$ is supported on $V \cap D_1$, where $V$ is an admissible open set in $S$ such that $V \cap S = \{ z_1, \ldots, z_\sigma = 0 \}$ and $V \cap D_1 = \{ z_1 = 0 \}$ (see [4, §4.1] for more precise definition of admissible open sets in this context). Moreover, $\psi|_V$ can be expressed as

$$
\psi|_V = \sum_{j=1}^{\sigma_V} \nu_j \log |z_j|^2 + \sum_{k=\sigma_V+1}^n a_k \log |z_k|^2 + \alpha ,
$$

where $\nu_j > 0$ for $j = 1, \ldots, \sigma_V$ and $a_k \geq 0$ for $k = \sigma_V + 1, \ldots, n$ are constants and $\alpha$ is a smooth function on $V$. The set $V$ can be decomposed into $U \times W$ where $U$ is a 1-disc about the origin in the $z_1$-plane and $W \cong V \cap D_1$.

Following the discussion in [4, §2.3] (in particular, [4, Remark 2.3.8 and Lemma 2.3.9]), there exist an effective snc $(\mathbb{Z})$-divisor $S_0$ with supp $S_0 \subset S$ and a quasi-psh potential $\varphi$ on $L \otimes S_0^{-1} \otimes S^{-1}$ such that

(eq 2.2.1) \[ \varphi_L + m_k \psi = \varphi + \phi_S + \phi_{S_0} , \]

where $\phi_{S_0}$ and $\phi_S$ are potentials defined from some holomorphic canonical sections of $S_0$ and $S$ respectively, and $e^{-\varphi}$ is locally integrable at general points of $S$ in $X$. Indeed,
since $\varphi$ has only neat analytic singularities with snc, this means that $\varphi^{-1}(\infty) \not\subset D_f$ for any irreducible component $D_f$ of $S$. Moreover, given the canonical section $s_0$ of $S_0$ such that $\phi_s = \log|s_0|^2$ and any local lifting $\tilde{f} \in K_X \otimes L \otimes \mathcal{F}(\varphi_L + m_{k-1}\psi)(V)$ of $f$, one has $\tilde{f}_0 := \frac{\tilde{f}}{s_0}$ being holomorphic (see [4, Remark 2.3.8]). Also write $\tilde{f}_0 := \int f''_0 d\zeta_1 \wedge \cdots \wedge d\zeta_n$ and $|\tilde{f}_0|^2 := |\tilde{f}_0|^2 \bigwedge_{j=1}^{n} \pi \partial d\zeta_j \wedge d\overline{\zeta}_j = \int |\tilde{f}_0|^2 \ d\text{vol}_V$ (where $i := \frac{\sqrt{-1}}{2\pi}$).

Write $S_1^{(1)} := S - D_1$ for convenience. Consider the projection $\text{pr} : V \cong U \times W \to W \cong V \cap D_1$ given by $(z_1, z_2, \ldots, z_n) \mapsto (z_2, \ldots, z_n)$. Choose a compactly supported smooth cut-off function $\rho : U \to [0, 1]$ such that $\rho \equiv 1$ on a neighbourhood of the origin. Let $\widetilde{g} := \rho(z_1) \cdot \text{pr}^* g \in \mathcal{C}_c^0(V \setminus S_1^{(1)})$, which is an extension of $g$ (note that $\text{pr}^* g$ is independent of the variable $z_1$), and let $s$ be the canonical section of $S$ such that $\phi_s = \log|s|^2$ and $s = z_1 \cdots z_{n_r}$ on $V$. Write $s =: z_1 s_1^{(1)}$ and $\phi_1 := \log|s_1^{(1)}|^2$ for convenience. Let $(r_1, \theta_1)$ be the polar coordinates of the $z_1$-plane. Recall that, by the definition of admissible open sets (see [4, §4.1]), one has $r_1^2 \frac{\partial}{\partial r_1} \psi > 0$ on $V$ and $(r_1^2 \frac{\partial}{\partial r_1} \psi)_{\{r_1 = 0\}} = \nu_1$. In view of Fubini’s theorem, the norm with respect to the 1-ic-measure can be computed as

$$
\widetilde{f}_0^2|g| \cdot \frac{e^{-\varphi_L - m_k \psi}}{|\varphi|^{1+\varepsilon}} = \varepsilon \int V \frac{d\psi}{2} \cdot \bigg( \frac{1}{|\varphi|} \bigg) \cdot d\text{vol}_W
$$

(Note that $e^{-\varphi_1}$ is smooth on supp $g$.) For the norm with respect to the Ohsawa measure, note that the norm is independent of the choice of the extension $\widetilde{g}$ of $g$ (see, for example, [17, Lemma 3.5] for a detailed proof). Choose $\widetilde{g}$ to be

$$
\widetilde{g} := \rho(z_1) \cdot \text{pr}^* \left( g \left( |\tilde{f}_0|^2 \right)^{-\frac{m_k}{2}} \right)_{D_1} \frac{1}{\nu_1} \cdot \frac{r_1^2 \partial}{\partial r_1} \psi
$$

(one may take a further log-resolution such that $\text{div}(\tilde{f}_0) + S$ is in snc and the poles coming from $\frac{1}{|\tilde{f}_0|^2}$ in the above formula can be cancelled out by $\text{pr}^* \left( |\tilde{f}_0|^2 \right)^{1+\varepsilon}_{D_1}$). Let $t$ be sufficiently negative such that $\rho \equiv 1$ on $\{ \psi < t + 1 \} \cap \text{supp} \ \widetilde{g}$ (recall the $g \in \mathcal{C}_c^0(D_1 \setminus S_1^{(1)})$).

It then follows from Fubini’s theorem and integration by parts that

$$
\int_{\{t < \psi < t + 1\}} \frac{e^{-\varphi_L - m_k \psi}}{|\varphi|^{1+\varepsilon}} = \int_{\{t < \psi < t + 1\}} \frac{\pi \partial d\zeta_j \wedge d\overline{\zeta}_j}{|z_1|^2} \bigwedge_{j=2}^{n} \pi \partial d\zeta_j \wedge d\overline{\zeta}_j
$$

$$
= \int_{\{t < \psi < t + 1\}} \frac{e^{-\varphi_1}}{r_1^2 \partial}{\partial r_1} \psi \cdot d\text{vol}_W
$$

The notation is chosen by mimicking the reduced Planck constant $h = \frac{\hbar}{2\pi}$. It is typeset with the code \{\raisebox{-0.9ex}{$\mathchar'26$}\mkern-6.7mu i\}.\label{1}
and the constant has a holomorphic extension \( L \) to together with \( [5, \text{Thm. 1.4.5}] \) (see Remark 1.2.3) into a statement which says that, un-

\[ \exists \in \mathcal{H} \phi \in \mathcal{H}(\mathcal{F} \mathcal{L} + m_{k-1} \psi) \text{ consisting of all the germs which are locally } L^2 \text{ with respect to the Ohsawa measure, as defined in } [8, \text{Def. (2.11)}]. \]

**Proof.** The results in [4, Thm. 4.1.2] imply that, when \( \mathcal{F} \phi \) has only neat analytic singularities,

\[ \mathcal{J}_\sigma(\mathcal{F} \phi; m_k \cdot \psi)_x = \left\{ f \in \mathcal{H}(\mathcal{F} \phi + m_{k-1} \psi)_x \left| \exists \text{ open } V \ni x \text{ such that } \mathcal{F}(f)^2 < +\infty \right. \right\} \]

for any integer \( \sigma \geq 0 \). The claim then follows directly from Proposition 2.2.1. \( \square \)

Set \( \mathcal{F}(f, \ell)_{X, \sigma} = \mathcal{F}(f, \ell)_{X, \sigma} \) if \( \mathcal{F}(f, \ell)_{X, \sigma} \) is independent of \( \sigma \). The above results indeed translate Theorem 1.2.2 together with [5, Thm. 1.4.5] (see Remark 1.2.3) into a statement which says that, under the given curvature assumption and the assumption that \( \mathcal{F} \phi \) has only neat analytic singularities, there exists a “universal constant” \( \ell \geq e \) such that every \( f \in H^0 \left( S, K_X \otimes L \otimes \mathcal{F}_X(\mathcal{F} \phi, \psi) \right) \) can be extended to a holomorphic section \( F \in H^0 \left( X, K_X \otimes L \otimes \mathcal{J}_X(\mathcal{F} \phi, \psi) \right) \) with an estimate

\[ \mathcal{F}(f, \ell)_{X, 1} \leq \mathcal{F}(f, \ell)_{X, \sigma} \cdot \]

Note that the residue norm \( \mathcal{F}(f, \ell)_{X, \sigma} \) is independent of \( \sigma \geq e \) (see [3, Cor. 2.3.3]) and thus the argument \( \ell \) is omitted.

In order to achieve the goal stated at the end of Section 1.2 (after Remark 1.2.3), one is led to search for a proof of the following conjecture.

**Conjecture 2.2.3 ([3, Conj. 1.1.3]).** Under the curvature assumption given in Theorem 1.2.2, there exists a sufficiently large constant \( \ell \geq e \) (depending only on the function \( \psi \) and the constant \( \delta \) in the curvature assumption) such that, for any integer \( \sigma \geq 1 \), every

\[ f \in H^0 \left( \mathcal{J}_X(\mathcal{F} \phi; m_k \cdot \psi), K_X \otimes L \otimes \mathcal{J}_X(\mathcal{F} \phi; m_k \cdot \psi) \right) \]

has a holomorphic extension \( F \in H^0 \left( X, K_X \otimes L \otimes \mathcal{J}_X(\mathcal{F} \phi; m_k \cdot \psi) \right) \) satisfying the estimate

\[ \mathcal{F}(f, \ell)_{X, \sigma, \psi, m, \sigma} \leq \mathcal{F}(f, \ell)_{X, \sigma, \psi, m, \sigma} \cdot \]

If the conjecture holds true, then, in view of the short exact sequence

\[ 0 \to \mathcal{J}_{\sigma-1}(\mathcal{F} \phi; m_k \cdot \psi) \xrightarrow{\mathcal{J}_0(\mathcal{F} \phi; m_k \cdot \psi)} \mathcal{J}_\sigma(\mathcal{F} \phi; m_k \cdot \psi) \xrightarrow{\mathcal{J}_0(\mathcal{F} \phi; m_k \cdot \psi)} \mathcal{J}_{\sigma-1}(\mathcal{F} \phi; m_k \cdot \psi) \to 0 \]
for any integer \( \sigma \geq 1 \), the procedure similar to the one described at the end of Section 1.2 will result in extensions of any \( f \) taking values in \( \mathcal{F}_{m_k} = \mathcal{F}(\varphi_{L} + m_k \chi, \psi) \) with estimates in terms of the residue norms. The estimates thus obtained are compatible with the estimate in the example of a bidisc obtained by Berndtsson in [2, §A.3].

While the conjecture is still open, [4, Thm. 1.2.3(3) and Cor. 4.3.2] guarantee a local version of the statement.

**Theorem 2.2.4** ([4, Thm. 1.2.3(3) and Cor. 4.3.2]). Suppose that \( \varphi_L \) has only neat analytic singularities and assume that \((X, \varphi_L, \psi)\) is in an snc configuration (as in the proof of Proposition 2.2.1). Let \( \varphi \) be the potential given in \((eq2.2.1)\). Then, on any admissible open set \( V \subset X \) (see [4, §4.1]), given a constant \( C \geq 0 \) such that

\[
(eq2.2.2) \quad \varphi|_{S_p^\sigma} \leq \varphi + C \quad \text{on } V
\]

for every \( \sigma \)-lc centre \( S_p^\sigma \) (where \( \varphi|_{S_p^\sigma} \) is the restriction of \( \varphi \) to \( S_p^\sigma \) and treated as a function on \( V \)), every \( f \in H^0\left(\overline{V}, K_X \otimes L \otimes \mathcal{F}(\varphi_L; \psi)\right) \) has a holomorphic extension \( F \in H^0\left(\overline{V}, K_X \otimes L \otimes \mathcal{F}(\varphi_L; \psi)\right) \) satisfying the estimate

\[
3 |f|^2(\varepsilon)_{V, \sigma} \leq 2eC \mathcal{F}(0)_{V, \sigma} \quad \text{for all } \varepsilon \geq 0 .
\]

The proof of Theorem 2.2.4 does not use the machinery of the \( L^2 \) method as in the proof of the Ohsawa–Takegoshi extension theorem. Instead, it is obtained through a direct computation with the aid from Taylor expansion (see [4, Cor. 4.3.2] for the proof). Note that the constant \( C \geq 0 \) in \((eq2.2.2)\) in the theorem exists for any quasi-psh \( \varphi \) having only neat analytic singularities with snc such that \( \varphi^{-1}(-\infty) \) contains no \( \sigma \)-lc centres of \((V, \varphi_L, \psi)\).

Although the constant in the estimate is not “universal” (as \( C \) depends on \( \varphi \), hence \( \varphi_L \)), it is worth noting that, if \( \varphi \) is psh and toric on \( V \), the mean-value-inequality yields (suppose that \( \varphi \) depends only on \((z_1, \ldots, z_2)\) and consider an \( 1 \)-lc centre \( S_p^1 \) given by \( \{z_1 = 0\} \) for example)

\[
\varphi|_{S_p^1} = \varphi(0, z_2) \leq \frac{1}{2\pi} \int_0^{2\pi} \varphi(r_1 e^{i\theta_1}, z_2) \, d\theta_1 = \varphi(z_1, z_2) ,
\]

that means, the constant \( C \) can be chosen to be 0. Moreover, if the psh potential \( \varphi \) has more general singularities and does satisfy \((eq2.2.2)\) for some \( C \geq 0 \), it can be shown that each member of its Bergman kernel approximation \( \{\varphi^{(k)}\}_{k \in \mathbb{N}} \) also satisfies \((eq2.2.2)\) with the same constant \( C \geq 0 \). This fact may be useful to compensate for the loss of the universal constant in some applications.

Using a partition of unity, Theorem 2.2.4 guarantees a smooth (global) extension with estimate for every \( f \in H^0\left(\mathbb{C}^n(\varphi_L; \psi), K_X \otimes L \otimes \mathcal{F}(\varphi_L; \psi)\right) \). It is hoped that this would be useful in constructing the desired global holomorphic extension with estimate.

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