Solving the Kerzman’s problem on the sup-norm estimate for $\overline{\partial}$ on product domains

Song-Ying Li

Revised by July 10, 2023

Abstract. In this paper, the author solves the long term open problem of Kerzman on sup-norm estimate for Cauchy-Riemann equation on polydisc in $n$-dimensional complex space. The problem has been open since 1971. He also extends and solves the problem on a bounded product domain $\Omega^n$, where $\Omega$ is any bounded domain in $\mathbb{C}$ with $C^{1,\alpha}$ boundary for some $\alpha > 0$.

1 Introduction

Let $\Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^n$. Let $f \in L^2_{(0,1)}(\Omega)$ be any $\overline{\partial}$-closed $(0,1)$-form with coefficients $f_j \in L^2(\Omega)$. By Hörmander’s theorem [25], there is a unique $u \in L^2(\Omega)$ with $u \perp \text{Ker}(\overline{\partial})$ such that $\overline{\partial}u = f$. The regularity theory for Cauchy-Riemann equations has been a very important research area in several complex variables for many decades. In particular, sup-norm estimate for $\overline{\partial}$ is the most difficult one. When $\Omega$ is a bounded smooth strictly pseudoconvex domain in $\mathbb{C}^n$, in 1970, Henkin [22], Grauert and Lieb [20] constructed a solution formula for $\overline{\partial}u = f$ satisfying $\|u\|_{L^\infty} \leq C_\Omega \|f\|_{L^\infty_{(0,1)}}$. In 1971, Kerzman [27] improved the above result in [22] and [19], he proved that $\|u\|_{C^{\alpha,\Omega}} \leq C_{\alpha,\Omega} \|f\|_{L^\infty_{(0,1)}}$ for any $0 < \alpha < 1/2$. In 1971, Henkin and Romanov [23] proved the sharp estimate: $\|u\|_{C^{1/2,\Omega}} \leq C_{\Omega} \|f\|_{L^\infty_{(0,1)}}$. Recently, X. Gong [18] generalized Henkin and Romanov’s results. He reduced the assumption $\partial \Omega \in C^\infty$ to $\partial \Omega \in C^2$ and proved that $\|u\|_{C^{\gamma+1/2,\Omega}} \leq \|f\|_{C^{\gamma,\Omega}_{(0,1)}}$ for any $\gamma$ with that $\gamma + 1/2$ is not an integer. In [27], when $\Omega = D^n$ is the unit polydisc
in \( \mathbb{C}^n \), Kerzman asked the following question: Does \( \overline{\partial} u = f \) have a solution satisfying \( \| u \|_{C^\alpha} \leq C_{\alpha} \| f \|_{L^\infty(0,1)} \) for some \( \alpha > 0 \)? Let \( f_j(\lambda) \in L^\infty(D) \) be holomorphic in \( D \) such that \( u_0 = \overline{\partial}_1 f_1(z_2) + \overline{\partial}_2 f_2(z_1) \not\in C(\overline{D}^2) \). Let \( f(z) = f_1(z_2)dz_1 + f_2(z_1)dz_2 \). Then \( \overline{\partial} f = 0 \) and \( u_0 \in L^\infty(D^2) \setminus C(\overline{D}^2) \) with \( u_0 \perp \text{Ker}(\overline{\partial}) \) solves \( \overline{\partial} u = f \). Then the Kerzman’s question can be refined by: Does \( \overline{\partial} u = f \) have a solution \( u \) satisfying \( \| u \|_{L^\infty} \leq C \| f \|_{L^\infty(0,1)} \)? The problem was studied by Henkin [24], he proved that if \( f \in C^1(0,1)(\overline{D}^2) \) is \( \overline{\partial} \)-closed, then \( \overline{\partial} u = f \) has a solution \( u \) satisfying estimate \( \| u \|_{L^\infty} \leq C \| f \|_{L^\infty(0,1)} \), where \( C \) is a scalar constant. Notice that a \( \overline{\partial} \)-closed form \( f \in L^\infty(0,1)(\overline{D}^n) \) cannot be approximated by \( \overline{\partial} \)-closed forms in \( C^1(0,1)(\overline{D}^n) \) in \( L^\infty(\overline{D}^n) \)-norm. Henkin’s result only partially answered Kerzman’s question and left the Kerzman’s question remaining open.

In [33], Landucci was able to improve the solution \( u \) of \( \overline{\partial} u = f \) in [24] to the canonical solution which is the solution \( u_0 \perp \text{Ker}(\overline{\partial}) \). Recently, Chen and McNeal [4] introduced a new space \( \mathcal{B}^p(0,1)(\overline{D}^n) \) of \((0,1)\)-forms over \( D^n \) which is smaller than \( L^p(0,1)(\overline{D}^n) \) and proved \( L^p \)-norm estimates for \( f \in \mathcal{B}^p(0,1)(\overline{D}^n) \) for \( 1 < p \leq \infty \). Their result generalized Henkin’s result. For a simple example, they reduced Henkin’s assumption: \( f = f_1 dz_1 + f_2 dz_2 \in C^1(0,1)(\overline{D}^2) \) to \( f \in L^\infty(0,1)(\overline{D}^2) \) satisfying \( \frac{\partial f_1}{\partial \overline{z}_2} \in L^\infty(\overline{D}^2) \). Dong, Pan and Zhang [10] proved a very clean and pretty theorem: If \( \Omega \) is any bounded domain in \( \mathbb{C} \) with \( C^2 \) boundary and \( f \in C(\overline{\Omega}^n) \) is \( \overline{\partial} \)-closed, then the canonical solution \( u_0 \) of \( \overline{\partial} u = f \) satisfies \( \| u_0 \|_{L^\infty} \leq C \| f \|_{L^\infty(0,1)} \). However, \( C(\overline{\Omega}^n) \) is strictly smaller than \( L^\infty(0,1)(\overline{\Omega}^n) \), the Kerzman’s question remains open (see [35]).

Main purpose of the current paper is to give a complete solution of the Kerzman’s long open problem on the unit polydisc in \( \mathbb{C}^n \). More general, we will prove that the canonical solution \( u \) satisfying estimate \( \| u \|_{\infty} \leq C \| f \|_{\infty} \) on the product domains \( \Omega^n \), where \( \Omega \) is a bounded domains \( \Omega \subset \mathbb{C} \) with \( C^{1,\alpha} \) boundary for some \( \alpha > 0 \). The main theorem is stated as follows.

**Theorem 1.1** Let \( \Omega \) be a bounded domain in \( \mathbb{C} \) with \( C^{1,\alpha} \) boundary for some \( \alpha > 0 \). Let \( f \in L^\infty(0,1)(\Omega^n) \) be \( \overline{\partial} \)-closed. Then the canonical solution \( u_0 \) of \( \overline{\partial} u = f \) is constructed and satisfies

\[
\| u_0 \|_{L^\infty(\Omega^n)} \leq C \| f \|_{L^\infty(0,1)(\Omega^n)}.
\]
More information for $\bar{\partial}$-estimates, one may find from the following references as well as the references therein. For examples, Chen and Shaw [6], Fornaess and Sibony [15], Krantz [29, 32], Range [40], Range and Siu [41, 42], Shaw [43] and Siu [46]. For product domains, one may also see [6], [9], [31] and other related articles in the reference.

The paper is organized as follows. In section 2, we first provide an estimate for the Green’s function and its derivatives on a bounded domain in $\mathbb{C}$ with $C^{1,\alpha}$ boundary for some $\alpha > 0$. In Section 3, we provide a formula solution for canonical solution of $\bar{\partial}u = f$ on the product domains. In Section 4, technically, we translate the formula in Section 3 to one, from which we can get a uniform $L^p$ estimates. In Section 5, we will prove Theorem 1.1. In fact, we have proved that Theorem 1.1 remains true for all $1 < p \leq \infty$.

Finally, in Section 6, based on $\bar{\partial}$-estimate on the disc $D \subset \mathbb{C}$, we give a more sharp theorem (Theorem 6.1) which is better than Theorem 1.1.

**Acknowledgment.** The author would like to thank R-Y. Chen who read through the first draft of manuscript and Sun-sig Byun for providing some useful reference on Green’s function (Theorem 2.1). The author deeply appreciate Xianghong Gong who went through the draft of the revision very carefully and gave many invaluable comments and suggestions. The author also appreciates the referees for their invaluable comments and suggestions which are very helpful for the revision.

## 2 Green’s function and Bergman kernel

### 2.1 Green’s functions

Let $\Omega$ be a bounded domain in $\mathbb{C}$ and let $G(\lambda, \xi)$ be the Green’s function for the Laplace operator $\Delta' = \frac{\partial^2}{\partial \lambda \partial \xi} = \frac{1}{4} \Delta$ on $\Omega$. Then the Green’s operator $G$ is defined by

\[(2.1) \quad G[f](z) = \int_{\Omega} G(z, w)f(w)dA(w)\]

and $G[f]$ satisfies

\[(2.2) \quad \frac{\partial^2 G[f]}{\partial \lambda \partial \xi}(\lambda) = f(\lambda), \lambda \in \Omega \quad \text{and} \quad G[f] = 0 \quad \text{on} \quad \partial \Omega.\]
Notice that

\[(2.3) \quad G(z, w) = \frac{1}{\pi} \log |z - w|^2 - U(z, w), \]

where \(U(z, w)\) is harmonic in \(z\) and in \(w\) separately which is the solution of

\[(2.4) \quad \begin{cases} \Delta_w U(z, w) = 0, & w \in \Omega, \\ U(z, w) = \frac{1}{\pi} \log |z - w|^2, & w \in \partial \Omega. \end{cases} \]

Then

\[(2.5) \quad \frac{\partial^2 G(z, w)}{\partial z \partial w} = \delta_{w=z} - \frac{\partial^2 U(z, y)}{\partial z \partial w}, \quad \text{and} \quad \frac{\partial U(z, w)}{\partial z} = \frac{1}{\pi(z - w)}, \quad w \in \partial \Omega. \]

where \(\delta_{w=z} = \delta_z(w)\) is the Dirac mass concentrated at \(z\) as measure of \(w\). Moreover, \(-\frac{\partial^2 U(z, w)}{\partial z \partial w}\) is holomorphic in \(z\) and anti-holomorphic in \(w\). In fact, one can show that it is the Bergman kernel function for \(\Omega\) (see the details in [16]).

Let \(A^2(\Omega)\) be the Bergman space over \(\Omega\) which is the holomorphic subspace of \(L^2(\Omega)\). Let \(\mathcal{P} : L^2(\Omega) \to A^2(\Omega)\) be the Bergman projection. Then

\[(I - \mathcal{P})f(z) = -\int_{\Omega} \frac{\partial^2 G(z, w)}{\partial z \partial w} f(w) dA(w), \]

where the Bergman kernel function of \(\Omega\) is given by

\[(2.6) \quad K(z, w) = \frac{\partial^2 G(z, w)}{\partial z \partial w} = -\frac{\partial^2 U(z, w)}{\partial z \partial w}, \quad z \neq w. \]

By Theorem 0.5 in Jerison and Kenig [26], if \(\partial \Omega\) is Lipschitz, there is a \(p_1 > 4\) such that the Green’s operator \(G : W^{-1,p}(\Omega) \to W^{1,p}(\Omega)\) is bounded for \(p_1' < p < p_1\). (2.6) and the expression of \(I - \mathcal{P}\) imply that if \(\partial \Omega\) is Lipschitz, then \(\mathcal{P} : L^p(\Omega) \to A^p(\Omega)\) is bounded for \(p_1' < p < p_1\). One may find further information on regularity of Bergman projections in [36].

We need some properties of the Green’s function and estimations on the Green’s function and its derivatives based on the regularity of \(\partial \Omega\). We recall a definition. We say that a bounded domain \(\Omega \subset \mathbb{R}^n\) satisfies a uniform exterior ball condition if there is a positive number \(r\) such that for any \(z_0 \in \partial \Omega\), there is \(z_0 \in \mathbb{R}^n \setminus \overline{\Omega}\) such that \(B(z_0(r), r) \cap \overline{\Omega} = \{z_0\}\), where \(B(x, r)\) is
ball in \( \mathbb{R}^n \) centered at \( x \) with radius \( r \). It is easy to see that if \( \partial \Omega \) is \( C^2 \), then \( \Omega \) satisfies a uniform exterior (and interior) ball condition.

The following theorem on the Green’s function was proved by Grüter and Widman [21] (Theorem 3.3 for \( n > 2 \)).

**THEOREM 2.1** If \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) which satisfies a uniform exterior ball condition, then its associated Green function satisfies the following five properties for all \( x, y \in \Omega \):

(i) \( |G(x, y)| \leq Cd_\Omega(x)|x - y|^{1-n} \);
(ii) \( |G(x, y)| \leq Cd_\Omega(x)d_\Omega(y)|x - y|^{-n} \);
(iii) \( |
abla_x G(x, y)| \leq C|x - y|^{1-n} \);
(iv) \( |
abla_x G(x, y)| \leq Cd_\Omega(y)|x - y|^{-n} \);
(v) \( |
abla_x \nabla_y G(x, y)| \leq C|x - y|^{-n} \).

Here \( C \) is a constant depending only on \( \Omega \) and \( d_\Omega(x) \) is distance from \( x \) to \( \partial \Omega \).

Notice that \( \Omega \) having \( C^{1,\alpha} \) boundary with \( \alpha \in (0, 1) \) may not satisfy a uniform exterior ball condition. We will give a formula for the Green’s function on a bounded domain in \( \mathbb{C} \) with \( C^{1,\alpha} \) boundary. First, we give the formula for simply connected domain.

Applying the argument by Kerzman [28] and regularity theorem (Theorem 8.34 in [17]), one can prove the following result.

**Proposition 2.2** Let \( \Omega \) be a bounded domains in \( \mathbb{C} \) with \( C^{1,\alpha} \) boundary for some \( 0 < \alpha < 1 \). Let \( D(0, 1) \) be the unit disc centered at \( 0 \).

(i) If \( \psi : \Omega \to D(0, 1) \) is a proper holomorphic map, then \( \psi \in C^{1,\alpha}(\overline{\Omega}) \);
(ii) If \( \phi : \Omega \to D(0, 1) \) is biholomorphic, then the Green’s function \( G_\Omega \) for \( \frac{\partial^2}{\partial \overline{\zeta} \partial \zeta} \) in \( \Omega \) is given by

\[
G_\Omega(z, w) = \frac{1}{\pi} \log \left| \frac{\phi(z) - \phi(w)}{1 - \phi(z)\overline{\phi(w)}} \right|^2
\]

which satisfies (i)–(v) in Theorem 2.1.

**Proof.** By Theorem 8.34 in [17], if \( g \in L^\infty(D) \) with \( D = D(0, 1) \), then

\[
\Delta' u = g \text{ in } D, \quad u = 0 \text{ on } \partial D
\]
has a unique solution \( u \in C^{1,\alpha}(\overline{D}) \). Let \( g \in C_0^\infty(D) \) be a non-negative function on \( D \) such that \( \{ z \in D : g(z) > 0 \} \) is a non-empty, relatively compact subset in \( D \). Let \( v(z) = u(\psi(z)) \) be a function on \( \Omega \) which solves the Dirichlet boundary problem:

\[
\begin{cases}
\Delta' v(z) = g(\psi(z))|\psi'(z)|^2, & z \in \Omega, \\
v(z) = 0, & z \in \partial \Omega.
\end{cases}
\]

By the elliptic theory (Theorem 3.34 in [17]), one has \( v \in C^{1,\alpha}(\overline{\Omega}) \). Then

\[
\frac{\partial v}{\partial z}(z) = \frac{\partial u}{\partial w}(\psi(z))\psi'(z).
\]

Since \( D \) satisfies an interior ball condition, by Hopf’s lemma, one has \( \frac{\partial u}{\partial w}(w) \neq 0 \) on \( \partial D \). Since \( u \in C^{1,\alpha}(\overline{D}) \), one has \( \frac{\partial u}{\partial w}(w) \neq 0 \) on the closed annulus \( A(0, 1-\epsilon, 1] = \{ w \in D : 1-\epsilon \leq |w| \leq 1 \} \) for some small \( \epsilon > 0 \). This implies

(2.8) \[
\psi'(z) = \frac{\partial v(z)}{\partial z} / \frac{\partial u}{\partial w}(\psi(z)) \quad \text{on} \psi^{-1}(A(0, 1-\epsilon, 1]).
\]

This implies that \( \psi \in C^{1}(\overline{\Omega}) \) since \( \psi \) is holomorphic in \( \Omega \). Applying (2.8) again, one can see that \( \psi'(z) \in C^\alpha(\overline{\Omega}) \). Therefore, \( \psi \in C^{1,\alpha}(\overline{\Omega}) \).

It is well known that the Green’s function for \( \frac{\partial^2}{\partial z \partial \overline{z}} \) in the unit disc \( D \) is:

(2.9) \[
G(z, w) = \frac{1}{\pi} \log \left| \frac{w - z}{1 - \overline{w} z} \right|^2, \quad z, w \in D.
\]

If \( \phi : \Omega \to D \) is a bilomorphic map, then it is easy to check that the Green’s function for \( \Omega \) is given by (2.7). Moreover, one can check that \( G_\Omega \) satisfies Properties (i)-(v) in Theorem 2.1 when \( n = 2 \).

When \( \Omega \) is finite multiple connected, the Green’s function of \( \Omega \) was studied by S. Bell in [1] by using Ahlfors map. We will use the partition of unity and (2.7) to study the Green’s function.

Let \( \tilde{P}_\Omega[f] \) denote the unique solution of the Dirichlet boundary problem:

(2.10) \[
\begin{cases}
\Delta'_u u(w) = 0, & w \in \Omega, \\
u(w) = f(w), & w \in \partial \Omega.
\end{cases}
\]

Let \( D = D(0, 1) \) and let

(2.11) \[
r(w) = 1 - |w|^2, \quad r(z, w) = \begin{cases} 1 - \langle z, w \rangle, & z \in \overline{D}, \\
z - w, & \text{if} \ z \not\in \overline{D}, \quad w \in D.
\end{cases}
\]

6
Then for \( f \in C^1(D) \)

\[
(2.12) \quad \tilde{P}_D[f](w) = \frac{1}{2\pi} \int_{\partial D} \frac{r(w)}{|r(w, \xi)|^2} f(\xi) d\sigma(\xi) \\
= \frac{r(w)}{\pi} \int_D \frac{\partial}{\partial \xi} \left( \frac{f(\xi)\overline{\xi}}{|r(w, \xi)|^2} \right) A(\xi) \\
= \frac{r(w)}{\pi} \int_D \frac{\partial}{\partial \xi} \left( \frac{f(\xi)\overline{\xi}}{|r(w, \xi)|^2} \right) A(\xi), \quad w \in D.
\]

**Lemma 2.3** For \( \psi \in C^{1,\alpha}(\overline{D}) \) for some \( \alpha > 0 \) with \( D = D(0,1) \), we let

\[
(2.13) \quad H(z, w) = \tilde{P}_D[\psi(\cdot) \frac{\log|z - \cdot|^2}{\pi}](w) - \psi(w) \tilde{P}_D[\frac{\log|z - \cdot|^2}{\pi}](w).
\]

Then for any \( 0 < \epsilon \leq 1 \), there is constant \( C_\epsilon \) depending only on \( \epsilon, R \) and \( \|\psi\|_{C^{1,\alpha}(\overline{D})} \) such that for any \( (z, w) \in D(0,R) \times D \) for any \( R \geq 1 \), one has

\[
(2.14) \quad |H(z, w)| \leq C_\epsilon \frac{r(w)}{|r(z, w)|} \log \left( \frac{1 + r(w)}{r(w)} \right),
\]

\[
(2.15) \quad |\nabla_z H(z, w)| \leq \frac{C_\epsilon r(w)}{|r(z, w)|^{1+\epsilon}}
\]
and

\[
(2.16) \quad |\nabla_w \frac{\partial}{\partial z} H(z, w)| \leq \frac{C_\epsilon}{|r(z, w)|^{1+\epsilon}}.
\]

**Proof.** Define

\[
(2.17) \quad X(w, \xi) = \frac{\partial}{\partial \xi} \left( \frac{\psi(\xi) - \psi(w)}{|r(w, \xi)|^2} \right) \quad \text{and} \quad Y(w, \xi) = \frac{\psi(\xi) - \psi(w)}{|r(w, \xi)|^2} \overline{\xi}.
\]

By (2.12), (2.13) and (2.17), one has

\[
(2.18) \quad H(z, w) = -\frac{ir(w)}{2\pi^2} \int_D \left[ X(w, \xi) \log|z - \xi|^2 + Y(w, \xi) \frac{1}{\xi - z} \right] d\xi \wedge d\xi \\
= \frac{r(w)}{\pi^2} \left( \int_D X(w, \xi) \log|z - \xi|^2 dA(\xi) + \int_D Y(w, \xi) \frac{1}{\xi - z} dA(\xi) \right).
\]
Coupling this with
\[ \frac{\partial}{\partial z} \left( \frac{1}{\pi} \int_D \frac{g(w)}{z-w} dA(w) \right) = g(z), \]
on one has
\[ \frac{\partial H(z, w)}{\partial z} = \frac{r(w)}{\pi^2} \left( \int_D X(w, \xi) \frac{1}{z-\xi} dA(\xi) + \pi Y(w, z) \right). \]

Similarly, if one let
\[ \tilde{X}(w, \xi) = \frac{\partial}{\partial \xi} \left[ \frac{\psi(\xi) - \psi(w)}{|r(w, \xi)|^2} \right] \quad \text{and} \quad \tilde{Y}(w, \xi) = \frac{\psi_j(\xi) - \psi_j(w)}{|r(w, \xi)|^2} \xi. \]

Then
\[ \frac{\partial H(z, w)}{\partial z} = \frac{r(w)}{\pi^2} \left( \int_D \tilde{X}(w, \xi) \frac{1}{z-\xi} dA(\xi) + \pi \tilde{Y}(w, z) \right). \]

Since \( \psi \in C^1(\overline{D}) \), one has
\[ |X(w, \xi)| \leq \frac{C}{|r(w, \xi)|^2} \quad \text{and} \quad |Y(w, \xi)| \leq \frac{C}{|r(w, \xi)|}. \]

where \( C = C_0 \| \psi \|_{C^1(\overline{D})} \), and
\[ |\nabla_w X(w, \xi)| \leq \frac{C}{|r(w, \xi)|^2} \quad \text{and} \quad |\nabla_w Y(w, \xi)| \leq \frac{C}{|r(w, \xi)|^2}. \]

For \( 0 < \epsilon \leq 1 \), notice that
\[ 1 = \frac{r(z, w)}{r(z, w)} = \frac{r(\xi, w) + \overline{w}(\xi - z)}{r(z, w)}, \quad 1 \leq \frac{|r(w, \xi)|^\epsilon + |\xi - z|^\epsilon}{|r(z, w)|^\epsilon} \]
and
\[ \int_D \frac{1}{|1 - w\xi|^{\beta}} dA(\xi) \leq \begin{cases} C \log \left( \frac{1+r(w)}{r(w)} \right), & \beta = 2; \\ \frac{C}{2-\beta}, & \beta < 2; \\ \frac{C}{\beta-2} r(w)^{2-\beta}, & \beta > 2. \end{cases} \]

8
For $\epsilon > 0$, let

$$C_\epsilon = \sup\{ |t^\epsilon \log t| : 0 < t \leq R \} + \| \log |z| - |t|^2 \|_{L^{2/\epsilon}(D)} < \infty.$$  

Then $C_\epsilon \geq \frac{1}{\epsilon^\epsilon}$. By (2.23), (2.24) and the Hölder’s inequality, one has

$$\begin{align*}
\int_D |X(w, \xi)||\log |z - \xi||dA(\xi) &\leq \frac{1}{|r(z, w)|^{\epsilon}} \int_D |X(w, \xi)||(|r(w, \xi)|^\epsilon + |z - \xi|^\epsilon)||\log |z - \xi||dA(\xi) \\
&\leq \frac{C}{|r(z, w)|^{\epsilon}} \int_D |r(w, \xi)|^{-2+\epsilon}||\log |z - \xi||dA(\xi) \\
&\quad + \frac{C}{|r(z, w)|^{\epsilon}} \int_D |r(w, \xi)|^{-2}|z - \xi|^\epsilon||\log |z - \xi||dA(\xi) \\
&\leq \frac{C}{|r(z, w)|^{\epsilon}} \left( \frac{1}{|r(w, \cdot)|^{2-\epsilon}} \| L^{2-\epsilon} \right) \frac{1}{|r(w, \cdot)|^{2-\epsilon}} \left( \int_D |z - \xi|^{-2+\epsilon}dA(\xi) \right)^{\frac{\epsilon}{2-\epsilon}} \\
&\quad + \frac{C}{|r(z, w)|^{\epsilon}} \left( \int_D |r(w, \xi)|^{-2+\epsilon}dA(\xi) \right)^{\frac{\epsilon}{2-\epsilon}} \left( \int_D |z - \xi|^{-2+\epsilon}dA(\xi) \right)^{\frac{2-\epsilon}{\epsilon}} \\
&\leq \frac{C_\epsilon}{|r(z, w)|^{\epsilon}}.
\end{align*}$$

By taking $p' = \frac{2}{2-\epsilon}$ and $p = 2/\epsilon$. Since

$$(2.25) \int_D |r(w, \xi)|^\epsilon + |z - \xi|^\epsilon dA(\xi)$$

$$\begin{align*}
&\leq \frac{C}{|r(z, w)|^{\epsilon}} \left( \frac{1}{|r(w, \cdot)|^{1-\epsilon}} \| L^{1-\epsilon} \right) \left( \frac{1}{|r(w, \cdot)|^{1-\epsilon}} \| L^{1-\epsilon} \right) \left( \frac{1}{|r(w, \cdot)|^{1-\epsilon}} \| L^{1-\epsilon} \right) \\
&\leq \frac{C}{|r(z, w)|^{\epsilon}} \left( \int_D |r(w, \xi)|^{-2+\epsilon}dA(\xi) \right)^{\frac{\epsilon}{2-\epsilon}} \left( \int_D |z - \xi|^{-2+\epsilon}dA(\xi) \right)^{\frac{2-\epsilon}{\epsilon}} \\
&\quad + \frac{C}{|r(z, w)|^{\epsilon}} \left( \int_D |r(w, \xi)|^{-2+\epsilon}dA(\xi) \right)^{\frac{\epsilon}{2-\epsilon}} \left( \int_D |z - \xi|^{-2+\epsilon}dA(\xi) \right)^{\frac{2-\epsilon}{\epsilon}} \\
&\leq \frac{C_\epsilon}{|r(z, w)|^{\epsilon}}.
\end{align*}$$

By (2.23) and (2.25), one has

$$\begin{align*}
\int_D |Y(w, \xi)| \frac{1}{|z - \xi|} dA(\xi) &\leq \int_D \frac{C}{|r(w, \xi)|^{\epsilon}} \frac{1}{|z - \xi|} dA(\xi) \\
&\leq \frac{C}{|r(z, w)|^{\epsilon}} \int_D \frac{|r(w, \xi)|^\epsilon + |z - \xi|^\epsilon}{|r(w, \xi)||z - \xi|} dA(\xi)
\end{align*}$$
\[ \leq \frac{C_{\varepsilon}}{|r(z, w)|^\varepsilon}. \]

Combining the above two estimates and (2.18) (the expression of \( H \)), one has proved (2.14).

By (2.23), one has

\[ \int_{\partial D} \frac{1}{r(w, \xi)} X(w, \xi) dA(\xi) = \frac{1}{r(z, w)} \int_{D} \frac{r(w, \xi)}{z - \xi} X(w, \xi) - \overline{w} X(w, \xi) dA(\xi). \]

Since \((\xi - w)\) and \(\overline{\xi} - \overline{w}\) is harmonic, \(\overline{\xi} d\xi = i d\theta\) if \(\xi = e^{i\theta}\), by Poisson integral formula, one has

\[ \int_{\partial D} \overline{\xi} (\xi - w) \frac{1}{|r(w, \xi)|^2} d\xi = 0, \quad \int_{\partial D} \overline{\xi} (\overline{\xi} - \overline{w}) \frac{1}{|r(w, \xi)|^2} d\xi = 0. \]

Therefore,

\[ |\int_{\partial D} X(w, \xi) dA(\xi)| = \left| \frac{1}{2i} \int_{\partial D} \frac{\overline{\psi}(\xi) - \psi(w)}{|r(w, \xi)|^2} d\xi \right| \]

\[ \leq \frac{1}{2} \int_{\partial D} \frac{\psi(\xi) - \psi(w)}{|r(w, \xi)|^2} \left| \frac{\partial \psi(w)}{\partial w} (\xi - w) - \frac{\partial \psi(w)}{\partial \overline{w}} (\overline{\xi} - \overline{w}) \right| \left| \overline{\xi} d\xi \right| \]

\[ + \frac{1}{2} \left| \frac{\partial \psi(w)}{\partial w} \right| \left| \int_{\partial D} \overline{\xi}(\xi - w) \frac{1}{|r(w, \xi)|^2} d\xi \right| + \frac{1}{2} \left| \frac{\partial \psi(w)}{\partial \overline{w}} \right| \left| \int_{\partial D} \overline{\xi}(\overline{\xi} - \overline{w}) \frac{1}{|r(w, \xi)|^2} d\xi \right| \]

\[ \leq C \|\psi\|_{C^{1, \alpha}}. \]

Applying (2.23) with \(0 < \varepsilon < 1\) and by (2.25), one has

\[ \int_{\partial D} |X(w, \xi)| \frac{|r(w, \xi)|}{|z - \xi|} dA(\xi) \]

\[ \leq \frac{C}{|r(z, w)|^\varepsilon} \left( \int_{\partial D} \left( \frac{|r(w, \xi)|^\varepsilon}{|z - \xi|^\varepsilon} + \frac{1}{|z - \xi|^{1-\varepsilon}} \right) |r(w, \xi)| X(w, \xi) dA(\xi) \right) \]

\[ \leq \frac{C}{|r(z, w)|^\varepsilon} \int_{\partial D} \frac{|r(w, \xi)|^\varepsilon + |z - \xi|^\varepsilon}{|r(w, \xi)||z - \xi|} dA(\xi) \]

\[ \leq \frac{C_{\varepsilon}}{|r(z, w)|^\varepsilon}. \]
By (2.21), one has that
\[ |Y(w, z)| \leq \frac{C}{|r(z, w)|}. \]

Combing the above two estimates and (2.18), one has
\[ \left| \frac{\partial H(z, w)}{\partial z} \right| \leq \frac{C_r(w)}{|r(z, w)|^{1+\varepsilon}}. \]

Using \( \tilde{X} \) and \( \tilde{Y} \) and (2.20), one has the same estimates for \( \left| \frac{\partial H(z, w)}{\partial z} \right| \). Therefore, (2.15) is proved.

By (2.25), one has
\[ \int_D \frac{1}{|r(w, \xi)|^{2-\varepsilon}} dA(\xi) \leq \frac{C}{r(w)} \int_D \frac{|r(w, \xi)|^\varepsilon}{|z - \xi|} dA(w) \leq \frac{C}{\varepsilon r(w)}. \]

Thus, applying the inequality (2.23) with \( \varepsilon = 1 \) first then with \( 0 < \varepsilon < 1 \), and the estimate for \( \nabla_w X(w, \xi) \) in (2.22), (2.24) and (2.25), one has
\[
\int_D \frac{1}{|z - \xi|} |\nabla_w X(w, \xi)||dA(\xi)
\leq \frac{C}{|r(z, w)|} \int_D \frac{|r(w, \xi)|}{|z - \xi|} |\nabla_w X(w, \xi)||dA(\xi) + \frac{C}{|r(z, w)|} \int_D |\nabla_w X(w, \xi)||dA(\xi)
\leq \frac{C}{|r(z, w)|^{1+\varepsilon}} \int_D \frac{|r(w, \xi)|^{1+\varepsilon}}{|z - \xi|} + \frac{|r(w, \xi)||z - \xi|^\varepsilon}{|z - \xi|} \frac{1}{|r(w, \xi)|^\varepsilon} dA(\xi) + \frac{C}{|r(z, w)| r(w)}
\leq \frac{C}{|r(z, w)|^{1+\varepsilon}} \int_D \frac{|r(w, \xi)|}{|z - \xi||r(w, \xi)|} dA(\xi)
\leq \frac{C}{|r(z, w)|^{1+\varepsilon}}. \]

By (2.24):
\[ |\nabla_w Y(w, z)| \leq \frac{C}{|r(w, z)|^2} \leq \frac{C}{|r(z, w)|r(w)}, \]
one has
\[
\left| \frac{\partial}{\partial z} \nabla_w H(z, w) \right| \leq C r(w) \left( \int_D \frac{1}{|z - \xi|} |\nabla_w X(w, \xi)||dA(\xi) + |\nabla_w Y(w, z)| \right)
+ C \left| \int_D \frac{1}{z - \xi} X(w, \xi) dA(\xi) \right| + |Y(w, z)|
\leq \frac{C}{|r(z, w)|^{1+\varepsilon}}. \]
Therefore, (2.16) is proved, and so is the lemma.

We come to study the Green’s function for a bounded domain \( \Omega \) with \( C^{1,\alpha} \) boundary. Choose \( z_1, \ldots, z_m \in \partial \Omega \) and \( \delta > 0 \) such that \( \Omega \cap D(z_j, 4\delta) \) is simply connected and

\[
\partial \Omega \subset \bigcup_{j=1}^{n} D(z_j, \frac{\delta}{2}).
\]

By the partition of the unity, we choose \( \psi_j \in C_0^\infty(D(z_j, \delta)) \) such that \( 0 \leq \psi_j(z) \leq 1 \),

\[
\psi_0(z) + \sum_{j=1}^{m} \psi_j(z) = 1, \quad z \in \overline{\Omega}
\]

and

\[
\text{supp}(\psi_0) \subset \{z \in \Omega : \text{dist}(z, \partial \Omega) > \delta/10\}.
\]

Choose \( D(z_j, 2\delta) \subset \overline{\Omega}_j \subset D(z_j, 3\delta) \) with \( C^\infty \) boundary such that \( \Omega_j = \Omega \cap \overline{\Omega}_j \) has \( C^{1,\alpha} \) boundary and

\[
(2.26) \quad \Omega \cap D(z_j, 2\delta) \subset \Omega_j \subset \Omega \cap D(z_j, 3\delta), \quad 1 \leq j \leq m.
\]

Let \( \phi_j : \overline{\Omega}_j \rightarrow D(0, 1) \) be a fixed biholomorphic map. By Proposition 2.2, one has \( \phi_j \in C^{1,\alpha}(\overline{\Omega}_j) \). For \( \varepsilon_0 > 0 \), we let \( \Omega_j(2\varepsilon_0) \) be the \( 2\varepsilon_0 \)-neighborhood of \( \overline{\Omega}_j \). We first extend \( \phi_j \in C^{1,\alpha}(\overline{\Omega}_j(2\varepsilon_0) \cap \overline{\Omega}) \) from \( \overline{\Omega}_j \). Choose \( \eta_j \in C_0^\infty(\Omega_j(2\varepsilon_0)) \) and \( \eta_j(z) = 1 \) on \( \Omega_j \). Let

\[
(2.27) \quad \phi_j(w) = \begin{cases} 
\phi_j(w)\eta_j(w), & \text{if } w \in \Omega_j(2\varepsilon_0) \cap \Omega, \\
0, & \text{if } w \in \Omega \setminus \Omega_j(2\varepsilon_0).
\end{cases}
\]

Then \( \phi_j \in C^{1,\alpha}(\overline{\Omega}) \) which is an extension from \( \phi_j \) on \( \overline{\Omega}_j \).

Using the holomorphic changes of variable by \( \phi_j(w) \) and the Poisson kernel for \( D(0, 1) \), one has

\[
(2.28) \quad \tilde{P}_{\Omega_j}[f](w) = \frac{1}{2\pi} \int_{\partial \Omega_j} \frac{1 - |\phi_j(w)|^2}{\overline{1 - \phi_j(w)\overline{\phi_j(\xi)}}} |f(\xi)||\phi_j'(\xi)||d\xi|, \quad w \in \Omega_j.
\]

Let

\[
(2.29) \quad H_j(z, w) = \tilde{P}_{\Omega_j}\left[ \frac{\psi_j(\cdot) - \psi_j(w)}{\pi} \log |z - \cdot|^2 \right](w), \quad (z, w) \in \Omega \times \Omega_j.
\]
Let
\[(2.30)\quad d_j(w) = d_{\Omega_j}(w), \quad d_j(z, w) = \begin{cases} 1 - \phi_j(z)\overline{\phi_j(w)}, & z \in \Omega_j, w \in \Omega_j, \\ z - w, & z \notin \Omega_j, w \in \Omega_j. \end{cases}\]

Applying Lemma 2.3 and holomorphic map \(\phi_j : \Omega_j \to D(0, 1)\), one has

**Corollary 2.4** For \(z \in \Omega \times \Omega_j\), \(H_j(z, w)\) satisfies
\[(2.31)\quad |H_j(z, w)| \leq C_\epsilon \frac{d_j(w)}{|d_j(z, w)|^\epsilon} \log \left(\frac{1 + d_j(w)}{d_j(w)}\right),\]
\[(2.32)\quad |\nabla_z H_j(z, w)| \leq \frac{C_\epsilon d_j(w)}{|d_j(z, w)|^{1+\epsilon}}\]
and
\[(2.33)\quad |\nabla_w \frac{\partial}{\partial z} H_j(z, w)| \leq \frac{C_\epsilon}{|d_j(z, w)|^{1+\epsilon}}.\]

For \(z \in \Omega\) and \(w \in \Omega_j\), we let
\[(2.34)\quad u_j(z, w) = \begin{cases} \int_{\partial \Omega_j} \frac{\psi_j(\xi)}{|1 - \phi_j(w)\phi_j(\xi)|} |\nabla \psi_j(\xi)| \log |z - \xi|^2 \pi d\xi, & \xi \in D(z_j, 3\delta) \cap \Omega, \\ 0, & w \in \Omega_j \setminus D(z_j, 4\delta/3). \end{cases}\]

Since \(D(z_j, 2\delta) \cap \Omega \subset \Omega_j \subset D(z_j, 3\delta) \cap \Omega\), one can see that
\(|1 - \phi_j(w)\phi_j(\xi)| \geq \frac{1}{C(\delta)}, \quad \xi \in D(z_j, \delta) \cap \Omega\) and \(w \in \{\lambda \in \Omega : |\lambda - z_j| > 3\delta/2\}\).

Since \(\phi_j \in C^{1,\alpha}(\Omega)\) which is an extension of \(\phi_j\) from \(\overline{\Omega_j}\), then there is an \(0 < \epsilon_1 < \epsilon_0\) such that
\(|1 - \phi_j(w)\phi_j(\xi)| \geq \frac{1}{2C(\delta)}\)
for all \(\xi \in D(z_j, \delta) \cap \Omega, w \in \{\lambda \in \Omega \cap \Omega_j(2\epsilon_1) \setminus D(z_j, 4\delta/3)\}\). Choose \(\zeta_j \in C_0^\infty(\Omega_j(2\epsilon_1))\) with \(\zeta_j = 1\) on \(\Omega_j(\epsilon_1)\). We define
\[(2.35)\quad u_j(z, w) = \begin{cases} \zeta_j(w)u_j(z, w), & \text{if } w \in \Omega_j(2\epsilon_1) \cap \Omega, \\ 0, & \text{if } w \in \Omega_j(2\epsilon_1)^c \cap \Omega. \end{cases}\]
Then
\[
\|u_j(z, \cdot)\|_{C^{1,\alpha}(\mathbb{M}\setminus D(z_j, 3\delta/2))} \leq C(\delta), \quad z \in \Omega.
\]
and
\[
\|\nabla z u_j(z, \cdot)\|_{C^{1,\alpha}(\mathbb{M}\setminus D(z_j, 3\delta/2))} \leq C(\delta), \quad z \in \Omega.
\]

Choose \(\chi \in C^\infty(\mathbb{C})\) such that \(\chi(w) = 0\) if \(w \in D(z_j, 3\delta/2)\) and \(\chi(w) = 1\) if \(w \not\in D(z_j, 5\delta/3)\). Now since
\[
\Delta'_u u_j(z, w) = \Delta'_u (\chi(w) u_j + (1 - \chi(w)) u_j)
\]
\[
= \frac{\partial}{\partial w} \frac{\partial (u_j(z, w)\chi)}{\partial w} - \Delta'_u \chi u_j(z, w) - 2\mathrm{Re} \frac{\partial \chi}{\partial w} \frac{\partial u_j(z, w)}{\partial w}
\]
\[
= \frac{\partial}{\partial w} F^1_j(z, w) + F^2_j(z, w)
\]
and
\[
\|F^1(z, \cdot)\|_{C^\alpha(\mathbb{M})} \leq C(\delta) \quad \text{and} \quad \|F^2(z, \cdot)\|_{L^\infty(\Omega)} \leq C(\delta)
\]
uniformly for \(z \in \Omega\). Thus,
\[
\begin{cases}
\Delta_u (U(z, w) - \sum_{j=1}^m u_j(z, w)) = -\frac{\partial}{\partial w} \frac{\sum_{j=1}^m F^1(z, w)}{\partial w} - \sum_{j=1}^m F_j(z, w), \quad w \in \Omega, \\
(U(z, w) - \sum_{j=1}^m u_j(z, w)) = 0, \quad w \in \partial \Omega.
\end{cases}
\]

By Theorem 8.34 and (8.90) in [17], there is a constant \(C_\Omega\) depending only on the diameter of \(\Omega\) and \(\delta\) such that
\[
\|(U(z, w) - \sum_{j=1}^m u_j(z, w))\|_{C^{1,\alpha}(\mathbb{M})} \leq C_\Omega, \quad z \in \Omega
\]
and
\[
\left\| \frac{\partial}{\partial z} (U(z, \cdot) - \sum_{j=1}^m u_j(z, \cdot)) \right\|_{C^{1,\alpha}(\mathbb{M})} \leq C_\Omega, \quad z \in \Omega.
\]

Let
\[
v(z, w) = -U(z, w) + \sum_{j=1}^m u_j(z, w).
\]

(2.38)
Then \(v(z, w), \nabla_z v(z, w) \in C^{1,\alpha}(\Omega)\) and
\[
(2.39) \quad U(z, w) = -v(z, w) + \sum_{j=1}^{m} u_j(z, w).
\]

Let \(G_j\) be the Green’s function of \(\Omega_j\). Then we extend the definition of \(G_j\) to be defined on \(\Omega \times \Omega_j\) and on \(\Omega_j \times \Omega\). For \((z, w) \in \Omega \times \Omega_j\), since
\[
u_j(z, w) - \psi_j(w) \left[ \log \left| z - w \right|^2 \right] / \pi - U(z, w)
= \tilde{P}_{\Omega_j} \left[ (\psi_j(\cdot) - \psi_j(w)) \left( \log |z - \cdot|^2 \right) \right](w) - \psi_j(w)G_j(z, w)
= H_j(z, w) - \psi_j(w)G_j(z, w),
\]
one has
\[
G(z, w) = \frac{\log |z - w|^2}{\pi} - U(z, w)
= v(z, w) + \psi_0(w) \left[ \log |z - w|^2 \right] / \pi + \sum_{j=1}^{m} \psi_j(w) \left[ \log |z - w|^2 \right] / \pi - u_j(z, w))
= v(z, w) + \psi_0(w) \left[ \log |z - w|^2 \right] / \pi + \sum_{j=1}^{m} \psi_j(w)G_j(z, w) - \sum_{j=1}^{m} H_j(z, w).
\]

Note. Let
\[
(2.40) \quad \mathcal{H}(z, w) =: \sum_{j=1}^{m} H_j(z, w).
\]

Our \(H_j(z, w)\) was defined on \(\Omega \times \Omega_j\), now we have extended it to be defined on \(\Omega \times \Omega\) through the extension of \(\phi_j\) from \(\Omega_j\) to \(\Omega\) given by (2.27) and the extension of \(u_j(z, w)\) from \(\Omega_j\) to \(\Omega\) given by (2.35). Combining these extensions and Corollary 2.4, we have the following estimates.
\[
(2.41) \quad |\mathcal{H}(z, w)| \leq C \frac{d_{\Omega}(w)}{d(z, w)^{\epsilon}} |\log d_{\Omega}(w)|,
\]
\[
(2.42) \quad |\nabla_z \mathcal{H}(z, w)| \leq C \frac{d_{\Omega}(w)}{d(z, w)^{\epsilon}}
\]
and
\begin{equation}
|\nabla_w \frac{\partial H}{\partial z}| \leq \frac{C_\epsilon}{d(z, w)^{1+\epsilon}},
\end{equation}
where
\begin{equation}
d(z, w) = d_\Omega(w) + |z - w|.
\end{equation}
Therefore, we have proved the following theorem.

**Theorem 2.5** Let \( \Omega \) be a bounded domain in \( \mathbb{C} \) with \( C^{1,\alpha} \) boundary for some \( \alpha > 0 \). Let \( G(z, w) \) be the Green's function for \( \Delta \) in \( \Omega \). Then
\begin{equation}
G(z, w) = \psi_0(w) \frac{\log |z - w|^2}{\pi} + \sum_{j=1}^m \psi_j(w) G_j(z, w) - \mathcal{H}(z, w) + v(z, w),
\end{equation}
where \( v(z, w), \nabla_z v(z, w) \in C^{1,\alpha}(\overline{\Omega}) \) as functions of \( w \) and \( H \) is given by (4.40) satisfy the estimates (2.41)–(2.44).

**Corollary 2.6** Let \( \Omega \) be a bounded domain in \( \mathbb{C} \) with \( C^{1,\alpha} \) boundary for some \( \alpha > 0 \). Then the Bergman projection \( P \) is bounded on \( L^p(\Omega) \) for \( 1 < p < \infty \).

**Proof.** Since
\begin{equation}
K(z, w) = \frac{\partial^2 G(z, w)}{\partial z \partial \overline{w}}, \quad z \neq w,
\end{equation}
Let
\begin{equation}
V(z, w) = \frac{1}{\pi(z - w)} \frac{\partial \psi_0(w)}{\partial \overline{w}} + \frac{\partial^2 v(z, w)}{\partial z \partial \overline{w}}.
\end{equation}
By Theorem 2.5, we have
\begin{equation}
K(z, w)
= V(z, w) + \sum_{j=1}^m \frac{\partial^2 \psi_j(w) G_j(z, w)}{\partial z \partial \overline{w}} - \frac{\partial^2 \mathcal{H}(z, w)}{\partial z \partial \overline{w}}
\end{equation}
We know that

\[(2.49) \quad |V(z, w) + \sum_{j=1}^{m} \frac{\partial \psi_j(w)}{\partial w} \frac{\partial G_j(z, w)}{\partial z}| \leq \frac{C}{|w - z|}.
\]

By (2.33) with \( \epsilon = 1/2 \) and Green’s function \( G_j \), one has

\[(2.50) \quad |\sum_{j=1}^{m} \psi_j(w) \frac{\partial^2 G_j(z, w)}{\partial z \partial w} - \frac{\partial^2 \mathcal{H}(z, w)}{\partial z \partial w}| \leq \frac{C}{|d(z, w)|^2},
\]

where \( d(z, w) \) are given by (2.44). For any \( 1 < p < \infty \), choose \( \epsilon > 0 \) such that \( p\epsilon, p'\epsilon \leq 1/2 \). One can easily see that

\[
\int_{\Omega} |K(z, w)| r(w)^{-p\epsilon} dA(w) \leq \frac{C}{p\epsilon} d_{\Omega}(z)^{-p\epsilon}
\]

and

\[
\int_{\Omega} |K(z, w)| r(w)^{-p'\epsilon} dA(w) \leq \frac{C}{p'\epsilon} d_{\Omega}(z)^{-p'\epsilon}
\]

By the Schur’s lemma, we have the Bergman projection \( P[f] = \int_{\Omega} K(z, w)f(w)dA(w) \) is bounded on the both \( L^p \) and \( L^{p'} \) with norm less than or equal to \( \frac{C}{\epsilon} \). Therefore, the corollary is proved.  

\[\ldots\]

### 3 Formula solution to \( \bar{\partial} \)-equations

Let \( \rho \in C^{1,\alpha}(\mathbb{C}) \) and \( \Omega = \{ z \in \mathbb{C} : \rho(z) < 0 \} \). There is a scalar constant \( C_0 > 1 \) such that \( \frac{1}{C_0} \leq |\nabla \rho(z)| \leq C_0 \) on \( \partial \Omega \). Let \( C_\Omega \) denote a constant depending only on \( C_0 \) and \( \|\rho\|_{C^{1,\alpha}(\overline{\Omega})} \). It may not be the same at each appearance.

Let \( G = G_\Omega \) be the Green’s function for \( \Delta' = \frac{\partial^2}{\partial z \partial \bar{w}} \) on \( \Omega \). Define

\[(3.1) \quad k(z, w) = \frac{\partial G_\Omega(z, w)}{\partial z} \]

and

\[(3.2) \quad T[f](z) = \int_{\Omega} k(z, w)f(w)dA(w).\]
Proposition 3.1 Let $\Omega$ be a bounded domain in $\mathbb{C}$ with $C^{1,\alpha}$ boundary for some $\alpha > 0$ and $1 \leq p \leq \infty$. Then

(i) $T[f]$ is the canonical solution of $\overline{\partial}u = fd\bar{z}$;

(ii) $T : L^p(\Omega) \to L^{2\frac{p}{p-1}}(\Omega)$ if $1 \leq p < 2$;

(iii) $T : L^p(\Omega) \to L^\infty(\Omega)$ is bounded if $2 < p \leq \infty$;

(iv) $T : L^p(\Omega) \to C^{1-2/p}(K)$ for any compact set $K \subset \Omega$.

Proof. By (2.2), the definition of $T[f]$ and the definition of the Green’s function, one can easily see that

$$\frac{\partial T[f]}{\partial \lambda}(\lambda) = \frac{\partial^2 G[f]}{\partial \lambda \partial \overline{\lambda}} = f(\lambda), \quad \lambda \in \Omega.$$

For any $z_0 \in \Omega$, since $G(z, w) = 0$ on $\partial \Omega \times \Omega$, by Proposition 2.2, Theorem 2.5, and integration by parts, one has $K(z_0, \cdot) \in C^\alpha(\Omega)$ and

$$\int_{\partial \Omega} G(\lambda, w)K(z_0, \lambda)d\lambda = 0, \quad w \in \Omega.$$

Thus, by (3.1) and $K(z_0, \cdot) \in C^\alpha(\Omega)$, for $w \in \Omega$, one has

$$\int_{\Omega} k(\lambda, w)K(z_0, \lambda)dA(\lambda) = \int_{\partial \Omega} G(\lambda, w)K(z_0, \lambda)\frac{id\overline{\lambda}}{2} - \int_{\Omega} G(\lambda, w)\frac{\partial K(z_0, \lambda)}{\partial \lambda}dA(\lambda) = 0.$$

By the above identity, one has

$$\int_{\Omega} T[f](\lambda)K(z_0, \lambda)dA(\lambda) = \int_{\Omega} \int_{\Omega} k(\lambda, w)K(z_0, \lambda)dA(\lambda)f(w)dA(w)$$

$$= \int_{\Omega} 0 \cdot f(w)dA(w)$$

$$= 0.$$

Therefore, $T[f] \in A^2(\Omega)^\perp$ since $K$ is the Bergman kernel of $\Omega$. So, $T[f]$ is the canonical solution of $\overline{\partial}u = fd\bar{z}$ in $\Omega$. Part (i) is proved.

Let

$$v(z) = \frac{1}{\pi} \int_{\Omega} \frac{f(w)}{z-w}dA(w).$$

Then $\frac{\partial v}{\partial \bar{z}} = f$. If $f \in L^p(\Omega)$ for $1 \leq p < 2$, we have $v \in W^{1,p}(\Omega)$. By the Sobolev embedding theorem, one has $v \in L^{2\frac{p}{2-p}}(\Omega)$. Since the Bergman
projection $P$ is bounded on $L^q(\Omega)$ for any $1 < q < \infty$ by Corollary 2.6. This implies that $T[f] = v - P[v] \in L^{\frac{2p}{p-2}}(\Omega)$. Therefore, Part (ii) is proved.

By Proposition 2.2 and Theorem 2.5, one has

$$|k(z, w)| = \left| \frac{\partial G(z, w)}{\partial z} \right| \leq \frac{C_{\Omega}}{|z - w|}. \tag{3.4}$$

This implies that if $2 < p \leq \infty$, then

$$|T[f](z)| \leq C_{\Omega} \int_{\Omega} \frac{|f(w)|}{|w - z|} dA(w) \leq \frac{C_{\Omega}}{2 - p'} \|f\|_{L^p} \leq C_{\Omega} \frac{p - 1}{p - 2} \|f\|_{L^p}.$$  

This means $\|T[f]\|_{L^\infty} \leq C_{\Omega} \frac{p - 1}{p - 2} \|f\|_{L^p}$ if $p > 2$. Part (iii) is proved.

By Sobolev embedding theorem, one has that $v \in W^{1,p}(\Omega) \subset C^{1-2/p'}(\Omega)$ for $2 < p < \infty$. Thus,

$$T[f] = v - P[v] \in C^{1-2/p'}(K), \quad \text{for any compact set } K \subset \Omega.$$  

So, the proof of Part (iv) is completed. Therefore, the proof of the proposition is complete. 

**Remark.** For (iv), one expects that $T : L^p(\Omega) \to C^{\alpha_0}(\overline{\Omega})$, where $\alpha_0 = \min\{\alpha, 1 - 2/p\}$. This can be done easily when $\Omega$ is simply connected. For a general case, one needs to do some more works from estimations of Green’s function given by Theorem 2.5. Since we don’t use this property, we omit here.

For any $1 \leq j \leq n$ and $z \in \mathfrak{C}^n$, write

$$z^{(j)} = (z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n), \quad z = (z_j; z^{(j)}). \tag{3.5}$$

Let $f \in L^2(\Omega^n)$, we define the Bergman projection $P_j : L^2(\Omega) \to A^2(\Omega)$ by

$$P_j f(z) = \mathcal{P}[f(\cdot, z^{(j)})](z_j) = \int_{\Omega} K(z_j, w_j)f(w_j; z^{(j)})dA(w_j), \tag{3.6}$$

for almost every $z^{(j)} \in \Omega^{n-1}$. We also use the notations $P_0 = P_{n+1} = I$. Similarly, we also use the following notation:

$$T_j f(z) = T[f(\cdot; z^{(j)})](z_j), \quad 1 \leq j \leq n. \tag{3.7}$$

The following theorem is a very important formulation for the canonical solution of $\overline{\partial}u = f$. 

19
**THEOREM 3.2** Let $\Omega$ be a bounded domain in $\mathbb{C}$ with $C^{1,\alpha}$ boundary for some $\alpha > 0$. For $1 \leq p \leq \infty$ and any $\overline{\partial}$-closed $(0,1)$-form $f = \sum_{j=1}^{n} f_j dz_j \in L^p_{(0,1)}(\Omega^n)$, the canonical solution $u = S[f] \in L^p(\Omega^n)$ to $\overline{\partial} u = f$ satisfies

\begin{equation}
S[f](z) = \sum_{j=1}^{n} T_j P_{j-1} \cdots P_0 f_j = \sum_{j=1}^{n} T_j P_{j+1} \cdots P_{n+1} f_j.
\end{equation}

**Note.** $S[f] \in L^p(\Omega^n)$ is the canonical solution here means $P_{\Omega^n} [S[f]] = 0$.

**Proof.** For each $1 \leq j \leq n$, since $\frac{\partial u(z_j; z^{(j)})}{\partial z_j} = f_j(z_j; z^{(j)}) \in L^p(\Omega)$, by the estimates on the Green’s function given by Proposition 2.2 and Theorem 2.5, one has that

\begin{equation}
u(z_j; z^{(j)}) - P_j [u(\cdot; z^{(j)})](z_j) = T_j [f_j(\cdot; z^{(j)})](z_j),
\end{equation}

for almost every $z^{(j)} \in \Omega^{n-1}$.

Since $u - P_1[u]$ is the canonical solution of $\overline{\partial} u = f_1$, one has

$$P_0 u - P_1 P_0 u = u - P_1[u] = T_1 f_1 = T_1 P_0 f_1.$$ 

Similarly,

$$P_1 P_0[u] - P_2 P_1 P_0[u] = P_1 [(I - P_2) u] = P_1 T_2 \left[ \frac{\partial u}{\partial z_2} \right] = P_1 T_2[f_2] = T_2 P_1[f_2].$$

Keeping the same process, one has

$$P_{j-1} \cdots P_1 P_0 u - P_j P_{j-1} \cdots P_1 P_0 u = T_j P_{j-1} \cdots P_1 P_0 f_j, \quad 1 \leq j \leq n.$$ 

Since $P_1 \cdots P_n u = P_{\Omega^n} u = 0$ and $P_0 = I$, one has

$$S[f] = u = \sum_{j=1}^{n} (P_{j-1} \cdots P_0 u - P_j P_{j-1} \cdots P_0 u) = \sum_{j=1}^{n} T_j P_{j-1} \cdots P_1 P_0 f_j.$$ 

On the other hands, let $P_{n+1} = I$, then

$$u - P_n u = T_n f_n.$$ 

With the same process, one has

$$P_n \cdots P_j u - P_n \cdots P_j P_{j-1} u = T_{j-1} P_j \cdots P_n f_{j-1}.$$
Since \( u \) is the canonical solution of \( \bar{\partial}u = f \), one has \( P_{n+1}P_n \cdots P_1u = 0 \) and 
\[
\sum_{j=1}^{n} T_j P_{j+1} \cdots P_{n+1} f_j = \sum_{j=1}^{n} (P_{n+1}P_n \cdots P_{j+1}u - P_{n+1}P_n \cdots P_ju) = u.
\]
These prove (3.8), so the proof of Theorem 3.2 is complete. \( \square \)

**THEOREM 3.3** Let \( 1 < p < \infty \) and let \( \Omega \) be a bounded domain in \( \mathbb{C} \) with \( C^{1,\alpha} \) boundary for some \( \alpha > 0 \). If \( f_m, f \in L^p_{(0,1)}(\Omega^n) \) with \( f_m \to f \) in \( L^p_{(0,1)}(\Omega^n) \), then

\[
\lim_{m \to \infty} \left\| S[f_m] - S[f] \right\|_{L^p(\Omega^n)} = 0.
\]

**Proof.** Notice that \( T_j \) and \( P_j \) are bounded on \( L^p(\Omega) \) with finite norms \( \|T_j\|_p \) and \( \|P_j\|_p \). Write
\[
f_m = \sum_{k=1}^{n} (f_m)_k d\zeta_k, \quad f = \sum_{k=1}^{n} f_k d\zeta_k.
\]
By (3.8), one has
\[
\left\| S[f_m] - S[f] \right\|_{L^p(\Omega^n)} \leq \sum_{j=1}^{n} \|T_j P_{j-1} \cdots P_0 [(f_m)_j - f_j]\|_{L^p(\Omega^n)}
\leq \sum_{j=1}^{n} \|T_j\|_p \|P_{j-1}\|_p \cdots \|P_1\|_p \| (f_m)_j - f_j\|_{L^p(\Omega^n)}
\leq \sum_{j=1}^{n} \|T_j\|_p \|P_{j-1}\|_p \cdots \|P_1\|_p \| f_m - f\|_{L^p_{(0,1)}(\Omega^n)}
\]
which converges to 0 as \( m \to \infty \). The proof is complete. \( \square \)

### 4 Regularity and a new formula solution

To prove a uniform \( L^p \)-estimate for \( S[f] \) when \( p \geq 1 \). By viewing
\[
T_j P_{j-1}[f] = \frac{1}{\pi} \int_{\Omega} \int_{\Omega} k(z_j, w_j) K(z_{j-1}, w_{j-1}) f(w) dA(w),
\]
one see that the singularity of $k(z_j, w_j)$ is about $\frac{1}{|w_j - z_j|}$ and $|K(z_{j-1}, w_{j-1})|$ is about $\frac{1}{d_{\Omega}(w_{j-1})^2 + |z_{j-1} - w_{j-1}|^\sigma}$. In this section, through the integration by part, we try to move the higher singularity partially to the lower singularity. In the end, we can average them to get kernel on $\Omega \times \Omega$ with the singularity like $\frac{1}{|z_j - w_j|^{\sigma/2}}|z_{j-1} - w_{j-1}|^\sigma$. Which becomes integrable on $\Omega^2$. For this purpose, we need to introduce some notation and to do preparations. First, we state the following lemma.

**Lemma 4.1** Let $(X, d\mu)$ be a measurable space with positive Borel measure $d\mu$. Let $K$ be measurable function on $X \times X$ such that

$$
(4.1) \quad \int_X |K(z, w)| d\mu(w) + \int_X |K(z, w)| d\mu(z) \leq C, \quad \text{for all } z, w \in X.
$$

Then the integral operator

$$
(4.2) \quad T[f](z) = \int_X K(z, w) f(w) d\mu(w)
$$

is bounded on $L^p(X)$ and $\|T\| \leq C$ for all $1 \leq p \leq \infty$.

**Proof.** The proof can be followed from Schur’s lemma with test function 1. One can also prove it by using the interpolation theorem. Since $T$ is bounded on $L^1$ with the norm $\leq C$ and it is also bounded on $L^\infty$ with norm $\leq C$. The interpolation theorem implies that $T$ is bounded on $L^p$ with the norm $\leq C$. \( \square \)

For any $1 \leq i \neq j \leq n$, define

$$
(4.3) \quad \tau_{i,j} = |z_i - w_i|^2 + |z_j - w_j|^2 \quad \text{and} \quad \tau_{i,j}^k = \frac{|w_k - z_k|^2}{|w_i - z_i|^2 + |w_j - z_j|^2}.
$$

Define $\partial_j = \frac{\partial}{\partial w_j}$ and

$$
(4.4) \quad b^{ij}(z, w) \quad := \quad \partial_j \left( k(z_j, w_j) \tau_{i,j}^j \right)
\quad = \quad k(z_j, w_j) \partial_j \tau_{i,j}^j + \tau_{i,j}^j \partial_j k(z_j, w_j)
\quad = \quad k(z_j, w_j) \left( \frac{|w_j - z_j| |w_i - z_i|^2}{(\tau_{i,j})^2} + \tau_{i,j}^j \partial_j k(z_j, w_j) \right)
$$

22
\begin{align*}
&= k(z_j, w_j) \frac{(w_j - z_j)}{\tau_{i,j}} - k(z_j, w_j) \frac{(w_j - z_j)}{\tau_{i,j}} \frac{1}{\tau_{i,j}} k(z_j, w_j) + \frac{|w_j - z_j|^2}{\tau_{i,j}} \partial_j k(z_j, w_j) \\
&= \left( h(z_j, w_j) + (\overrightarrow{w_j - z_j}) (\partial_j h(z_j, (w_j)) \frac{1}{\tau_{i,j}} - h(z_j, w_j) \frac{1}{\tau_{i,j}} \tau_{i,j}^j, \right) \\
\end{align*}

where

(4.5) \quad h(z_j, w_j) = (w_j - z_j)k(z_j, w_j).

By Proposition 2.2 and Theorem 2.5, one has

(4.6) \quad |k(z_j, w_j)| \leq \frac{C_{\Omega}}{|z_j - w_j|}

and

(4.7) \quad |k(z_j, w_j)| \leq C_{\Omega} \frac{d_{\Omega}(w_j)}{|z_j - w_j|^2}.

These imply that

(4.8) \quad |h(z_j, w_j) + (\overrightarrow{w_j - z_j}) \frac{\partial h(z_j, w_j)}{\partial w_j}| \leq C_{\Omega} \quad \text{and} \quad |h(z_j, w_j)| \leq \frac{C_{\Omega} d_{\Omega}(w_j)}{|z_j - w_j|}.

Therefore

(4.9) \quad |\hat{b}_{i,j}(z, w)| \leq \frac{C_{\Omega}}{\tau_{i,j}(z, w)}.

Thus,

(4.10) \quad |k(z_j, w_j)||\hat{b}_{i,j}(z, w)| + |k(z_i, w_i)||\hat{b}_{i,j}(z, w)|

\leq \frac{C_{\Omega}}{|z_j - w_j|\tau_{i,j}} + \frac{C_{\Omega}}{|w_i - z_i|\tau_{i,j}}

\leq \frac{C_{\Omega}}{|w_i - z_i|^{3/2}|w_j - z_j|^{3/2}}.

By (4.7) and integration by parts, one has

\begin{align*}
T_j P_i[f_j] &= T_j[f_j] - T_j T_i \left[ \frac{\partial f_j}{\partial \hat{w}_i} \right] \\
&= T_j[f_j] + \int_{\Omega^2} k_j \frac{\partial (k_i \tau_{i,j}^i)}{\partial \hat{w}_i} f_j + \int_{\Omega^2} k_i \frac{\partial (k_j \tau_{i,j}^j)}{\partial \hat{w}_j} f_i \\
&= T_j[f_j] + \int_{\Omega^2} k_j \hat{b}_{i,j} f_j + \int_{\Omega^2} k_i \hat{b}_{i,j} f_i.
\end{align*}
By Lemma 4.1, estimations (4.6), (4.10) and the above identity, we have proved the following proposition.

**Proposition 4.2** Let $\Omega$ be a bounded domain with $C^{1,\alpha}$ boundary. Let $f \in C^{1,\alpha}_0(\overline{\Omega}^\circ)$ be $\overline{\partial}$-closed. Then

\begin{equation}
\|T_j P_t f_j\|_{L^p(\Omega^\circ)} \leq C_{\Omega} (\|f_j\|_{L^p(\Omega^\circ)} + (\|f_i\|_{L^p(\Omega^\circ)}),
\end{equation}

for all $1 \leq p < \infty$, where $C_{\Omega}$ is a constant defined in the beginning of Section 3.

Let $k_j = k(z_j, w_j)$. Then

\[
T_j P_t P_i f_j = T_j P_t [f_j] + P_t \left[ \int_{\Omega^2} k_j b^{i,j} f_j + \int_{\Omega^2} k_i b^{j,i} f_j \right] \\
= T_j P_t [f_j] + \int_{\Omega^2} k_j b^{i,j} f_j + \int_{\Omega^2} k_i b^{j,i} f_i \\
- T_t [\int_{\Omega^2} k_j b^{i,j} \frac{\partial f_j}{\partial \mu_t}] - T_t [\int_{\Omega^2} k_i b^{j,i} \frac{\partial f_i}{\partial \mu_t}] \\
= T_j P_t [f_j] + T_j P_i [f_j] - T_j [f_j] \\
- T_t [\int_{\Omega^2} k_j b^{i,j} \frac{\partial f_j}{\partial \mu_t}] - T_t [\int_{\Omega^2} k_i b^{j,i} \frac{\partial f_i}{\partial \mu_t}].
\]

The estimate (4.11) holds for the first and the second in the above identity. Their term is similar to the fourth term. We need only to consider $T_t [\int_{\Omega^2} k_j b^{i,j} \frac{\partial f_j}{\partial \mu_t}]$. Since $\ell \neq i, j$, one has

\begin{equation}
\partial_\tau \left( k_{\ell} \tau_{j,\ell} k_j b^{i,j} \right) = k_j b^{i,\ell} b^{j,i},
\end{equation}

and

\begin{equation}
\partial_\tau \left( \tau_{j,\ell} k_j b^{i,j} \right) = k_i b^{\ell,j} b^{i,j} - k_\ell \frac{1}{\tau_{t,j}} a^{i,j},
\end{equation}

where

\begin{equation}
a^{i,j} = |z_j - w_j|^2 k_j \partial_\tau (b^{i,j}).
\end{equation}

Since

\begin{equation}
\partial_\tau b^{i,j} = \left( -b^{i,j} + \frac{h(z_w_j)}{\tau_{i,j}} \tau_{i,j} \right) \partial_\tau \log \tau_{i,j} = \left( -b^{i,j} + \frac{h(z_w_j)}{\tau_{i,j}} \tau_{i,j} \right) \left( w_i - z_i \right).
\end{equation}
By (4.6), one has

\[(4.16) \quad \left| \frac{\partial b^{i,j}}{\partial w_i} \right| \leq \frac{|z_i - w_i|}{(\tau_{i,j})^2}.\]

Thus,

\[(4.17) \quad \left| a^{i,j} \right| = |k_j(z_j - w_j)^2 \partial_x b^{i,j}| \leq |k_j|z_j - w_j|^2 \frac{|z_j - w_j|}{(\tau_{i,j})^2} \leq \frac{C_\Omega}{\tau_{i,j}}.\]

(4.18) $T_\ell \left[ \int_{\Omega^3} k_j b^{i,j} \frac{\partial f_j}{\partial w_\ell} \right] = \int_{\Omega^3} k_\ell k_j b^{i,j} \frac{\partial f_j}{\partial w_\ell}$

\[= -\int_{\Omega^3} \partial_\ell \left(k_\ell \tau_{j,\ell} k_j b^{i,j} \right) f_j - \int_{\Omega^3} \partial_j \left(\tau_{j,\ell} k_\ell k_j b^{i,j} \right) f_\ell \]

\[= -\int_{\Omega^3} (k_j b^{i,j} \partial_{\ell} b^{i,j} f_j - k_\ell b^{\ell,j} b^{i,j} f_\ell) - \int_{\Omega^3} \frac{k_\ell}{\tau_{j,\ell}} a^{i,j} f_\ell.\]

Let

(4.19) $E_{j,i}^1 = b^{i,j}$, $E_{j,i}^2 = b^{i,j}$ or $a^{i,j}$

Then

(4.20) $|E_{j,i}^2| \leq \frac{C_\Omega}{|z_j - w_j|^{1+\epsilon}|w_i - z_i|^{2-\epsilon}}$.

Using the inequality

(4.21) $a^{2/p} b^{2/q} \leq \frac{1}{p}a^2 + \frac{1}{p'}b^2 \leq (a^2 + b^2),$

with $n \epsilon = 1$, one has

(4.22) $|k_j b^{i,\ell} E_{j,i}^1| \leq \frac{C_\Omega}{|z_j - w_j|} \frac{1}{|z_j - w_j|^{2\epsilon}} \frac{1}{|z_i - w_i|^{2-\epsilon}} |w_\ell - z_\ell|^{2-\epsilon}$

and

(4.23) $|k_\ell \frac{1}{\tau_{j,\ell}} E_{j,i}^2| \leq \frac{C_\Omega}{|w_\ell - z_\ell|} \frac{1}{|w_\ell - z_\ell|^{2\epsilon}} \frac{1}{|z_j - w_j|^{2-\epsilon}} |z_j - w_j|^{2-\epsilon} |w_i - z_i|^{2-\epsilon}$.\]
By Lemma 4.1 and the above estimates, one has proved

\[(4.24) \quad \|T_jP_\ell P_i[f_j]\|_{L^p(\Omega^n)} \leq C_\Omega\|f\|_{L^p_{(0,1)}(\Omega^n)}.\]

Let

\[(4.25) \quad c^{i,\ell} = k_\ell\left|z_\ell - w_\ell\right|^2 \frac{\partial}{\partial w_\ell} \left(\frac{1}{c_{\tau_{\ell,j}}}\right) = k_\ell\left|w_\ell - z_\ell\right|^2 \frac{z_\ell - w_\ell}{(c_{\tau_{\ell,j}})^2}.\]

Then

\[(4.26) \quad |c^{i,\ell}| \leq \frac{C_\Omega}{\tau_{j,\ell}}.\]

For \(I = \{i_1, \cdot \cdot \cdot, i_k\}\) consisting of \(k\) different numbers in \(\{1, 2, \cdot \cdot \cdot, n\}\), we define \(E_{j,I}\) to the \(k\)-pairs products from elements in \(\{E^1_{i,j}, E^2_{i,j}, c^{i,j}, \frac{1}{\tau_{j,i}}\}\) such that and

\[(4.27) \quad |E_{j,I}| \leq \frac{C_\Omega}{|z_j - w_j|^{k\epsilon} \prod_{\ell=1}^{k} \left|z_\ell - w_\ell\right|^{2-\epsilon}}.\]

Then

\[
P_q\left[\int_{\Omega^{k+1}} k_j b^{j,\ell} E_{\ell,I} f_j \right] = \int_{\Omega^{k+2}} k_j b^{j,\ell} E_{\ell,I}^2 f_j - T_q\left[\int_{\Omega^{k+1}} k_j b^{j,\ell} E_{\ell,I} \partial_{j} f_j \right]
\]

The function \(F\) defined by the first term has been proved in \(L^p\) with \(\|F\|_{L^p} \leq C_\Omega\|f\|_{L^p_{(0,1)}}\). The second term

\[
-T_q\left[\int_{\Omega^{k+2}} k_j b^{j,\ell} E_{\ell,I} \partial_{j} f_j \right]
\]

\[
= \int_{\Omega^{k+2}} k_j b^{j,\ell} E_{\ell,I} f_j + \int_{\Omega^{k+1}} k_q b^{j,q} b^{i,\ell} E_{j,I} f_j + \int_{\Omega^{k+1}} k_q \left|\frac{1}{\tau_{j,q}} a^{i,\ell} E_{j,I}^2 f_j \right|
\]

\[
= \int_{\Omega^{k+2}} k_j b^{j,\ell} E_{j,I} f_j + \int_{\Omega^{k+1}} k_q b^{j,q} E_{j,I} f_j + \int_{\Omega^{k+1}} k_q \left|\frac{1}{\tau_{j,q}} E_{j,I} f_q \right|
\]

where \(J = \{j, I\}\) and \(|J| = k + 1\). It is easy to see that \(E_{j,J}\) satisfies \((4.27)\) by replacing \(k\) by \(k + 1\). The function \(F\) defined by the above three integrals are in \(L^p(\Omega^n)\) with \(\|F\|_{L^p(\Omega^n)} \leq C_\Omega\|f\|_{L^p_{(0,1)}(\Omega^n)}\) by applying Lemma 4.1 and
Moreover,
\[-T_q \left[ \int_{\Omega^k} k_j \frac{1}{\tau_{j,\ell}} E_{\ell,I} \partial_{\tau_{j,\ell}} f_j \right] = \int_{\Omega^{k+1}} k_j b^{q,j} \frac{1}{\tau_{j,\ell}} E_{\ell,I} f_j + \int_{\Omega^{k+1}} k_k b^{q,j} \frac{1}{\tau_{j,\ell}} E_{\ell,I} f_k + \int_{\Omega^{k+1}} k_{k_k} b^{q,j} \frac{1}{\tau_{j,\ell}} E_{\ell,I} f_{q_k},\]

where \(E_{j,J}\) satisfies (4.27) by replacing \(k\) by \(k + 1\). Applying Lemma 4.1 and (4.21), one can show that the function \(F\) defined by the above three integrals satisfy the estimate \(\|F\|_{L^p(\Omega^n)} \leq C_{\Omega} \|f\|_{L^p(0,1)}\).

As a summary, we have proved the following theorem.

**THEOREM 4.3** Let \(\Omega\) be a bounded domain in \(\mathbb{C}\) with \(C^{1,\alpha}\) boundary for some \(\alpha > 0\). Then there is a constant \(C = C_{\Omega}\) defined in the beginning of Section 3 such that for any \(\overline{\partial}\)-closed \((0,1)\)-form \(f \in C^1_{(0,1)}(\Omega^n)\), one has

\[(4.28) \quad \|S[f](z)\|_{L^p(\Omega^n)} \leq C_{\Omega} \|f\|_{L^p_{(0,1)}(\Omega^n)}, \quad \text{for all } 1 \leq p \leq \infty.\]

## 5 Proof of Theorem 1.1

### 5.1 Approximation

**THEOREM 5.1** Let \(\Omega\) be a bounded simply connected domain in \(\mathbb{C}\) with \(C^{1,\alpha}\) boundary for some \(\alpha > 0\). For any \(1 \leq p < \infty\) and \(f \in L^p_{(0,1)}(\Omega^n)\) be \(\overline{\partial}\)-closed, then there is a \(\overline{\partial}\)-closed sequence \(\{f_m\}_{m=1}^\infty \subset C^1_{(0,1)}(\Omega^n)\) such that

\[(5.1) \quad \lim_{m \to \infty} \|f_m - f\|_{L^p_{(0,1)}} = 0.\]

**Proof.** When \(\Omega\) is the unit disk \(D\), let \(\chi^j \in C^\infty_0(D)\) be nonnegative and \(\int_D \chi^j dA = 1\). Let \(\chi^j_\varepsilon = \chi^j(z/\varepsilon)\varepsilon^{-2}\) and \(\chi_\varepsilon(z) = \chi^1_\varepsilon \cdots \chi^n_\varepsilon\) on \(D^n\). The proof for this case is very simple. For any \(0 < r < 1\) and \(\varepsilon = (1 - r)/2\), since \(f_r(z) = f(rz)\) is \(\overline{\partial}\)-closed in \(D(0,1/r)\) and then

\[(5.2) \quad F_r(z) = f_r * \chi_\varepsilon \in C^\infty_{(0,1)}(\overline{D^n}),\]
is $\partial$-closed in $D^n$ and
\begin{equation}
\|F_r - f\|_{L^p_r(D^n)} \to 0
\end{equation}
as $r \to 1^-$ and any $p \in [1, \infty)$. This argument remains true when $\Omega$ is a simply connected domain in $\mathbb{C}$ with $C^{1,\alpha}$ boundary for any $0 < \alpha < 1$. Let $\phi : \Omega \to D$ be a biholomorphic mapping. Then $\phi \in C^{1,\alpha}(\Omega)$, and $\Omega = \phi^{-1}(D)$, with slightly modification of the unit disc case, one can similarly prove the theorem.

Now we are ready to prove Theorem 1.1 when $\Omega$ is bounded simply connected.

### 5.2 Proof of Theorem 1.1 if $\Omega$ is simply connected

**Proof.** For any $1 < p < \infty$, by Theorem 4.1, there is a sequence $\{f_m\}_{m=1}^{\infty} \subset C^{1,\alpha}(\Omega)$ which are $\partial$-closed such that
\begin{equation}
\lim_{m \to \infty} \|f_m - f\|_{L^p_r(\Omega^n)} = 0.
\end{equation}
By estimations obtained in Section 4, one has that
\begin{equation}
\partial S[f_m] = f_m
\end{equation}
and $S[f_m]$ is a canonical solution. Moreover,
\begin{equation}
\lim_{m \to \infty} \|S[f_m] - S[f]\|_{L^p(\Omega^n)} = 0.
\end{equation}

For $1 < p < \infty$, by Corollary 2.6, one has
\begin{align*}
\|S[f]\|_{L^p(\Omega^n)} &\leq \|S[f_m]\|_{L^p(\Omega^n)} + \|S[f_m] - S[f]\|_{L^p(\Omega^n)} \\
&\leq C_{\Omega} \|f_m\|_{L^p_{(0,1)}(\Omega^n)} + \|S[f_m] - S[f]\|_{L^p(\Omega^n)} \\
&\leq C_{\Omega} \|f\|_{L^p_{(0,1)}(\Omega^n)} + C_{\Omega} \|f_m - f\|_{L^p_{(0,1)}(\Omega^n)} + \|S[f_m] - S[f]\|_{L^p(\Omega^n)},
\end{align*}
where $C_{\Omega}$ is a constant depends neither on $m$ nor $p$. Let $m \to \infty$, one has
\begin{equation}
\|S[f]\|_{L^p_{(0,1)}(\Omega^n)} \leq C_{\Omega} \|f\|_{L^p_{(0,1)}(\Omega^n)}, \quad 1 < p < \infty.
\end{equation}
Letting $p \to +\infty$, one has
\begin{equation}
\|S[f]\|_{L^\infty_{(0,1)}(\Omega^n)} \leq C_{\Omega} \|f\|_{L^\infty_{(0,1)}(\Omega^n)}.
\end{equation}
The proof of Theorem 1.1 is complete if $\Omega$ is simply connected with $C^{1,\alpha}$ boundary.

28
5.3 Proof of Theorem 1.1

Since \( \Omega \) is a bounded domain in \( \mathbb{C} \) with \( C^{1,\alpha} \) boundary for some \( \alpha > 0 \), there is \( \rho \in C^{1,\alpha}(\mathbb{C}) \) with \( \Omega = \{ z \in \mathbb{C} : \rho(z) < 0 \} \) and there is a scalar constant \( C_0 \) such that

\[
\frac{1}{C_0} \leq |\nabla \rho| \leq C_0, \quad \text{if} \quad \rho(z) = 0.
\]

The continuity of \( |\nabla \rho| \) implies that there is an \( \epsilon_0 > 0 \) such that

\[
\frac{1}{2C_0} \leq |\nabla \rho(z)| \leq 2C_0, \quad \text{if} \quad -\epsilon_0 \leq \rho(z) \leq 0.
\]

For \( \ell \in \mathbb{N} \) such that \( \frac{1}{\ell} < \epsilon_0 \), define

\[
(5.9) \quad \Omega_\ell = \{ z \in \mathbb{C} : \rho(z) < -\ell^{-1} \}.
\]

Then \( \partial \Omega_\ell \) is uniformly \( C^{1,\alpha} \) boundary for all \( \ell \geq 2/\epsilon_0 \) and

\[
\Omega_\ell \subset \overline{\Omega_\ell} \subset \Omega_{\ell+1} \subset \overline{\Omega_{\ell+1}} \subset \Omega \quad \text{and} \quad \lim_{\ell \to \infty} \Omega_\ell = \Omega.
\]

Moreover, \( C_{\Omega_\ell} \) is a constant depending only on \( 2C_0 \) and \( \|\rho\|_{C^{1,\alpha}(\overline{\Omega})} \) uniformly on \( \ell > 2/\epsilon_0 \). Therefore, \( C_{\Omega_\ell} = C_\Omega \) for all \( \ell > 2/\epsilon_0 \).

Notice that

\[
(5.10) \quad f * \chi_\epsilon \in C^{\infty}_{(0,1)}(\Omega_\ell^n)
\]

is \( \mathcal{D} \)-closed in \( \Omega_\ell \) if \( \epsilon < \text{dist}(\partial \Omega_\ell, \partial \Omega)/n \). By Theorem 4.3, we have

\[
(5.11) \quad \|S_\ell[f]\|_{L^p(\Omega_\ell^n)} \leq C_\Omega \|f\|_{L^p(\Omega_\ell^n)} \quad \text{for} \quad 1 \leq p \leq \infty,
\]

where \( C_\Omega = C(C_0 \|\rho\|_{C^{1,\alpha}(\overline{\Omega})}^n \) is a constant independent of \( p \) and \( \ell \). For any \( 1 < p < \infty \), since the unit ball is weakly compact in \( L^p(\Omega_\ell) \), there is a subsequence \( \{S_{\ell_j}[f]\}_{j=1}^\infty \) converges to a function in \( L^p(\Omega) \), denoted by \( \tilde{S}[f] \) weakly on \( L^p(\Omega_\ell) \) for any \( \ell \geq 2/\epsilon_0 \). Thus,

\[
(5.12) \quad \|\tilde{S}[f]\|_{L^p(\Omega^n)} \leq C_\Omega \|f\|_{L^p_{(0,1)}(\Omega_\ell^n)} \leq C_\Omega \|f\|_{L^p_{(0,1)}(\Omega^n)}, \quad \ell \geq 2/\epsilon_0.
\]

This implies that \( \tilde{S}[f] \in L^p(\Omega^n) \) and

\[
(5.13) \quad \|\tilde{S}[f]\|_{L^p(\Omega^n)} \leq C_\Omega \|f\|_{L^p_{(0,1)}(\Omega^n)}.
\]
By the uniqueness of weak limit for each $L^p(\Omega^n)$, one has $S[f] = \tilde{S}[f]$ for all $p \in (1, \infty)$. Since $C_\Omega$ in (5.13) does not depend on $p$, letting $p \to \infty$, one has

$$\|\tilde{S}[f]\|_{L^\infty(\Omega^n)} \leq C_\Omega \|f\|_{L^\infty_{(0,1)}(\Omega^n)}.$$  (5.14)

Since $S_\ell[f]$ is the canonical solution for $\partial u = f$ in $\Omega_\ell$, it is easy to check $\partial \tilde{S}[f] = f$ in $\Omega$ in the sense of distribution. Moreover, for any $z \in \Omega$, one has

$$\int_{\Omega^n} \tilde{S}[f](w)K_{\Omega^n}(z, w)dv(w) = \lim_{\ell \to \infty} \int_{\Omega^n_\ell} S_\ell[f](w)K_{\Omega^n}(z, w)dv(w) = 0.$$  (5.15)

Therefore, $\tilde{S}[f]$ is the canonical solution of $\partial u = f$ in $\Omega$. So, $S[f] = \tilde{S}[f]$. Moreover, $\|S[f]\|_{L^\infty(\Omega^n)} \leq C_\Omega \|f\|_{L^\infty_{(0,1)}(\Omega^n)}$. Therefore, the proof of Theorem 1.1 is complete. □

**REMARK 1** In fact, for any $\bar{\partial}$-closed $(0,1)$-form $f$ in $L^p_{(0,1)}(\Omega^n)$, we have

$$\|S[f]\|_{L^p(\Omega^n)} \leq C_\Omega \|f\|_{L^p_{(0,1)}(\Omega^n)}, \quad \text{for all } 1 < p \leq \infty.$$  (5.16)

6 Remarks

For any $\gamma \in [0,1)$, we choose $\epsilon$ such that $n \epsilon = 1 - \gamma$. Thus, for example, $n = 3$,

$$d_\Omega(w)^{-\gamma}|k(z_j, w_j)| \leq C d_\Omega(w)^{-\gamma} \frac{1}{|z_j - w_j|^2 + |w_k - z_k|^2} \frac{1}{|w_k - z_k|^2 + |w_\ell - z_\ell|^2} \leq C d_\Omega(w)^{-\gamma} \frac{1}{|z_j - w_j|^{1+(n-1)\epsilon}} \frac{1}{|w_k - z_k|^{2-\epsilon}} \frac{1}{|w_\ell - z_\ell|^{2-\epsilon}}$$

and

$$\int_{\Omega^n} \left( \frac{1}{d_\Omega(w)^\gamma} \frac{1}{|z_j - w_j|^{1+(n-1)\epsilon}} \frac{1}{|w_k - z_k|^{2-\epsilon}} \frac{1}{|w_\ell - z_\ell|^{2-\epsilon}} \right)^{p'} dv(w) \leq C_\epsilon.$$

for any $1 < p' < \frac{4-\epsilon}{4-2\epsilon}$.

By the arguments given in Sections 4 and 5, we have proved the following theorem.

30
THEOREM 6.1 Let $\Omega$ be a bounded domain in $\mathbb{C}$ with $C^{1,\alpha}$ boundary. Let $f = \sum_{j=1}^{n} f_j d\sigma_j \in L^\infty_{(0,1)}(\Omega^n)$ be $\overline{\partial}$-closed. Then there is a positive constant $C_\Omega$ depending only on $C^{1,\alpha}$ regularity of $\partial \Omega$ such that

$$
\|S[f]\|_{L^\infty(\Omega^n)} \leq \frac{C_\Omega}{(1-\gamma)^n} \sum_{k=1}^{n} \|d\Omega(z_k)\gamma f_k(z)\|_{L^\infty(\Omega^n)},
$$

for any $0 < \gamma < 1$.

References

[1] S. Bell, Green’s function and Ahlfors map, Indiana Univ. Math. Jour., 57 (2008), 3049–3063.

[2] B. Berndtsson, Uniform estimates with weights for the $\overline{\partial}$-equation, J. Geom. Anal., 7 (1997), 195–215.

[3] D. Chakrabarti and M.-C. Shaw, The Cauchy-Riemann equations on product domains, Math. Ann., 349 (2011), 977–998.

[4] L. Chen and J. D. McNeal, Product domains, Multi-Cauchy transforms, and the $\overline{\partial}$-equation, Advanced in Math, arXiv:1904.09401.

[5] R.-Y. Chen and S.-Y. Li, Graham type theorem on classical bounded symmetric domains, Calc. Var., 58:81 (2019).

[6] S.-C. Chen and M.-C. Shaw, Partial differential equations in several complex variables, AMS/IP Stud. Adv. Math. 19, Amer. Math. Soc., Providence, RI; Int. Press, Boston, MA, 2001.

[7] M. Christ and S-Y. Li, On the real analytic hypoellipticity for $\overline{\partial}$ and $\overline{\partial}_b$, Math. Z., 32(1997), 373-399.

[8] J.-P. Demailly, Estimations $L^2$ pour l’opérateur $\overline{\partial}$ d’un fibré vectoriel holomorphe semi-positif au-dessus d’une variété kählérienne complète (French), Ann. Sci. École Norm. Sup., 15 (1982), 457–511.

[9] X. Dong, S.-Y. Li and J. N. Treuer, Sharp pointwise and uniform estimates for $\overline{\partial}$, Anal. PDE, To appear.
[10] X. Dong, Y. Pan and Y. Zhang, Uniform estimates for the canonical solution to the $\bar{\partial}$-equation on product domains, Preprint.

[11] M. Fassina and Y. Pan, Supnorm estimates for $\bar{\partial}$ on product domains in $\mathbb{C}^n$, arXiv:1903.10475.

[12] C. Fefferman, The Bergman kernel and biholomorphic mappings of pseudoconvex domains, Invent. Math., 26 (1974), 1–65.

[13] J. E. Fornaess, Sup-norm estimates for $\bar{\partial}$ in $\mathbb{C}^2$, Ann. of Math., 123 (1986), 335–345.

[14] J. E. Fornæss, L. Lee and Y. Zhang, On supnorm estimates for $\bar{\partial}$ on infinite type convex domains in $\mathbb{C}^2$, J. Geom. Anal., 21 (2011), 495–512.

[15] J. E. Fornæss and N. Sibony, Smooth pseudoconvex domains in $\mathbb{C}^2$ for which the corona theorem and $L^p$ estimates for $\bar{\partial}$ fail, Complex analysis and geometry, 209–222, Univ. Ser. Math., Plenum, New York, 1993.

[16] P. R. Garabedian, A Partial Differential Equation Arising in Conformal Mapping, Pacific J. of Math. 1 (1951), 485–524.

[17] D. Gilbarg and N. S. Trudinger, Elliptic partial Differential Equations of Second Order, 2nd Edition, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983.

[18] X. Gong, Hölder estimates for homotopy operators on strictly pseudoconvex domains with $C^2$ boundary, Math. Ann., 374 (2019), 841–880.

[19] X. Gong and L. Lanzani, Regularity of a $\bar{\partial}$-solution operator for strongly $C$-linearly convex domains with minimal smoothness, J. Geom. Anal., 31 (2021), 6796–6818.

[20] H. Grauert and I. Lieb, Das Ramirezsche Integral Und Die Lösung Der Gleichung $\bar{\partial}f = \alpha$ Im Bereich Der Beschränkten Formen (German), Rice Univ. Stud. 56 (1970), 29–50.

[21] M. Grüter and K. O. Widman, The Green function for uniformly elliptic equations, Manuscripta Math. 37 (1982), no. 3, 303342.
[22] G. M. Henkin, \textit{Integral representation of functions in strictly pseudo-convex domains and applications to the }\overline{\partial}\textit{-problem}, Math. USSR Sb. 11 (1970), 273–281.

[23] G. M. Henkin and A. Romanov, \textit{Exact Hölder estimates of solution of the }\overline{\partial}\textit{-equations}, Izvestija Akad. SSSR, Ser. Mat. 35 (1971), 1171-1183, Math. U.S.S.R. Sb. 5 (1971), 1180–1192.

[24] G. M. Henkin, \textit{Uniform estimates for solutions to the }\overline{\partial}\textit{-problem in Weil domains} (Russian), \textit{Uspehi Mat. Nauk}, 26 (1971), 211–212.

[25] L. Hörmander, \textit{L}^2 \textit{estimates and existence theorems for the }\overline{\partial}\textit{ operator}, \textit{Acta Math.} 113 (1965), 89–152.

[26] D. Jerison and C. Kenig, \textit{The Inhomogeneous Dirichlet Problem in Lipschitz Domains}, \textit{J. Funct. Anal.}, 130(1995), 161–219.

[27] N. Kerzman, \textit{Hölder and }L^p \textit{estimates for solutions of }\overline{\partial}u = f \textit{ in strongly pseudoconvex domains}, Comm. Pure. Appl. Math., 24 (1971) 301–379.

[28] N. Kerzman, \textit{A Monge-Ampère equation in complex analysis}, Proc. of Symposia in Pure Math., 30 (1977), 161–167.

[29] S. G. Krantz, \textit{Function Theory of Several Complex Variables}, 2nd Edition, Wadsworth & Brooks/Cole, Math Series, 1992.

[30] S. G. Krantz and S-Y. Li, \textit{On the Existence of Smooth Plurisubharmonic Solutions for Certain Degenerate Monge-Ampère Equations}, Complex Variables, 41(2000), 207–219.

[31] S. G. Krantz and S-Y. Li, \textit{The explicit solution for the Lipschitz Corona problem in the polydisc}, Pacific J. of Math., vol iii (1996), 287–302.

[32] S. G. Krantz, \textit{Optimal Lipschitz and }L^p \textit{ estimates for the equation }\overline{\partial}u = f \textit{ on strongly pseudoconvex domains}, Math. Ann., 219 (1976), 233–260.

[33] M. Landucci, \textit{On the projection of }L^2(D) \textit{ into }H(D), \textit{Duke Math. J.}, 42 (1975), 231–237.
[34] C. Laurent-Thiébaut and J. Leiterer, *Uniform estimates for the Cauchy-Riemann equation on q-convex wedges*, Ann. Inst. Fourier (Grenoble), 43 (1993), 383–436.

[35] S-Y. Li, *Some Analysis Theorems and Problems on Classical Bounded Symmetric Domains*, Proceeding of ICCM, 2018, to appear.

[36] L. Lanzani, *The Bergman projection in $L^p$ for domains with monimal smoothness*, Illinois J. of Math., 56(2012), 127–154.

[37] L. Ma and J. Michel, *Local regularity for the tangential Cauchy-Riemann complex*, J. Reine Angew. Math., 422 (1993), 63–90.

[38] J. Michel and M.-C. Shaw, *The $\overline{\partial}$-problem on domains with piecewise smooth boundaries with applications*, Trans. Amer. Math. Soc., 351 (1999), 4365–4380.

[39] K. Peters, *Solution operators for the $\overline{\partial}$-equation on nontransversal intersections of strictly pseudoconvex domains*, Math. Ann., 291 (1991), 617–641.

[40] R. M. Range, *Holomorphic functions and integral representations in several complex variables*, Graduate Texts in Mathematics, 108, Springer-Verlag, New York, 1986. 2nd corrected printing 1998.

[41] R. M. Range and Y.-T. Siu, *Uniform estimates for the $\overline{\partial}$—equation on intersections of strictly pseudoconvex domains*, Bull. Amer. Math. Soc., 78 (1972), 721–723.

[42] R. M. Range and Y.-T. Siu, *Uniform estimates for the $\overline{\partial}$—equation on domains with piecewise smooth strictly pseudoconvex boundaries*, Math. Ann., 206 (1973), 325–354.

[43] M.-C. Shaw, *Optimal Hölder and $L^p$ estimates for $\overline{\partial}_b$ on the boundaries of real ellipsoids in $\mathbb{C}^n$*, Trans. Amer. Math. Soc., 324 (1991), 213–234.

[44] Z. Shi, *Weighted Sobolev $L^p$ estimates for homotopy operators on strictly pseudoconvex domains with $C^2$ boundary*, T eprint arXiv:1907.00264
[45] N. Sibony, *Un exemple de domaine pseudoconvexe regulier où l’équation $\overline{\partial}u = f$ n’admet pas de solution bornée pour $f$ bornée* (French), Invent. Math., 62 (1980), 235–242.

[46] Y.-T. Siu, *The $\overline{\partial}$ problem with uniform bounds on derivatives*, Math. Ann., 207 (1974), 163–176.

Department of Mathematics, University of California, Irvine, CA 92697-3875, USA

Email addresses: sli@math.uci.edu