Symmetric bilinear forms and local epsilon factors of isolated singularities in positive characteristic

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Abstract

Let $f : X \to \mathbb{A}^1_k$ be a morphism from a smooth variety to an affine line with an isolated singular point. For such a singularity, we have two invariants. One is a non-degenerate symmetric bilinear form (de Rham), and the other is the vanishing cycles complex (étale).

In this article, we give a formula which expresses the local epsilon factor of the vanishing cycles complex in terms of the bilinear form. In particular, the sign of the local epsilon factor is determined by the discriminant of the bilinear form. This formula can be thought as a refinement of the Milnor formula, which compares the total dimension of the vanishing cycles and the rank of the bilinear form.

In characteristic 2, we find a generalization of the Arf invariant, which can be regarded as an invariant for non-degenerate quadratic singularities, to general isolated singularities.

1 Introduction

Let $k$ be a perfect field of characteristic $p > 0$ and $X$ be a smooth $k$-scheme. Let $f : X \to C$ be a morphism of $k$-schemes to a smooth $k$-curve with an isolated singular point. To such a singularity, one can attach an arithmetic invariant, namely the vanishing cycles complex.

Fix a prime number $\ell$ different from $p$. Let $x \in X$ be an isolated singular point with respect to $f$. Then, the vanishing cycles complex $R\Phi_f(\mathbb{Q}_\ell)_x$ supported at $x$ is a bounded complex of finite dimensional $\mathbb{Q}_\ell$-representations of the absolute Galois group of the local field of $C$ at $f(x) \in C$. It is expected and observed in many cases that the Galois action on $R\Phi_f(\mathbb{Q}_\ell)_x$ respects some complexity of the singularity; as the simplest example, it is acyclic if $x$ is a smooth point. Aside from such special cases, it is usually difficult to describe $R\Phi_f(\mathbb{Q}_\ell)_x$ explicitly as Galois representations. On the other hand, since singular points are of geometric nature, it is natural to ask whether there are relations between $R\Phi_f(\mathbb{Q}_\ell)_x$ and some invariants of singularities which arise geometrically.

In this direction, Deligne considers an analogue of the Milnor formula in the complex geometry to an arithmetic setting in SGA 7 [9]. There he considers the total dimension $\dim_{\text{tot}} R\Phi_f(\mathbb{Q}_\ell)_x$ and proves in positive characteristic cases that it coincides with the Milnor number

$$(-1)^{n+1} \dim_{\text{tot}} R\Phi_f(\mathbb{Q}_\ell)_x = \mu(f, x).$$

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Here $n$ is the dimension of $X$ and $\mu(f, x)$ is the Milnor number, which is defined purely algebro-geometrically from the isolated singular point.

In this article, we give a formula which expresses the local epsilon factors, instead of the total dimensions, of the vanishing cycles complexes in terms of coherent sheaves (more precisely, symmetric bilinear forms as described below) arising from singularities. The local epsilon factor is defined by Langlands and Deligne to be a candidate of Galois-theoretic counterparts of the constant terms in the functional equations of automorphic local $L$-functions, in view of the local Langlands correspondence. As is well-known, and is what follows from our main results, the total dimensions of the vanishing cycles complexes has as much information as the (any) complex absolute values of the local epsilon factors. In this sense, our formula for local epsilon factors can be regarded as a refinement of the Milnor formula (1.1).

To motivate us to give such a refinement, let us consider the Hasse-Weil zeta function $Z(X/\mathbb{F}_q; t)$ of an $n$-dimensional projective smooth variety $X$ over a finite field $\mathbb{F}_q$. It is defined to be the infinite product

$$Z(X/\mathbb{F}_q; t) = \prod_{x \in |X|} \frac{1}{1 - t^{\deg(x/\mathbb{F}_q)}}$$

indexed by the closed points of $X$. Similarly as the usual zeta function and $L$-functions, it admits a functional equation of the following type

$$Z(X/\mathbb{F}_q; t) = \varepsilon(X, \mathbb{Q}_\ell) t^{-\chi(X)} Z(X/\mathbb{F}_q; \frac{1}{q^n} t).$$

Here $\chi(X)$ is the Euler characteristic, say, of the $\ell$-adic cohomology groups $H^i(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$ of $X$ and $\varepsilon(X, \mathbb{Q}_\ell) = \prod_i \det(-\text{Frob}_q, H^i(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell))^{(-1)^{n+1}}$ is the global epsilon factor, which is roughly the alternating product of the determinants of $H^i(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$. It is known that these correcting terms of the functional equation admit Euler product expansions, as those of the usual zeta function have.

To explain this precisely, take a map $f : X \to C$ to a projective smooth curve. According to the well-known philosophy that curves are geometric analogues of the rings of integers of number fields, a pair $(X, f)$ of a variety $X$ with a map to a curve in geometry should correspond to a variety over the ring of integers in arithmetic, and taking such a map should give decompositions of $\chi(X)$, $\varepsilon(X, \mathbb{Q}_\ell)$ into local contributions. Indeed, these decompositions are realized by the Grothendieck–Ogg–Shafarevich formula (for $\chi(X)$) and Laumon’s product formula (for $\varepsilon(X, \mathbb{Q}_\ell)$) applied to the pushforward $Rf_*\mathbb{Q}_\ell$ to the curve. From this point of view, the total dimensions of the vanishing cycles are nothing but the local terms appearing in the decompositions of $\chi(X)$, and on the other hand the local epsilon factors are the one appearing in those of the global epsilon factor $\varepsilon(X, \mathbb{Q}_\ell)$. Our results in this paper are thus to determine such local factors of the global epsilon factors in terms of geometry of singularities, with the assumption that the singularities are isolated.

To formulate the refinement for local epsilon factors, we will use linear-algebraic enhancements of the Milnor numbers, non-degenerate symmetric bilinear forms. Here is our main result in the odd characteristic cases.

**Theorem 1.1.** (Theorem 5.4.1, $p$ is odd) Let $k$ be a finite field of odd characteristic $p$. Let $X$ be a smooth $k$-scheme of dimension $n$ and let $f : X \to \mathbb{A}_k^1$ be a $k$-morphism with an
isolated singular point \( x \in X \). The singular point \( x \) is assumed \( k \)-rational for simplicity in the introduction. Let \( \varepsilon_0(A_k^1(f,x)), R\Phi_f(\mathbb{Q}_\ell)_x, dt \) be the local epsilon factor and \((\varphi_f, B_{f,-dt})\) be the non-degenerate symmetric bilinear form over \( k \) associated with \((f,x)\) (Definition [3.12]). Then we have

\[
(-1)^{\dim_{\text{tot}} R\Phi_f(\mathbb{Q}_\ell)_x} \varepsilon_0(A_k^1(f,x)), R\Phi_f(\mathbb{Q}_\ell)_x, dt) = (\frac{\text{disc}B_{f,-dt}}{k}) \cdot \tau_\psi^{-1} \varepsilon_0(T, V, \omega) \]

where \((\tau)\) denotes the Legendre symbol and \( \tau_\psi \) is the quadratic Gauss sum \( \tau_\psi = -\sum_{a \in k} \psi(a^2) \). Here \( \psi: k \rightarrow \mathbb{Q}_\ell^\times \) is a non-trivial additive character which is used in defining local epsilon factors (cf. [20], (3.1.5.4)).

In this article, we treat local epsilon factors multiplied with the sign as in the left-hand side of the theorem, rather than the local epsilon factors themselves. This is because the natural generalization of local epsilon factor to a general perfect field \( k \) is obtained with this sign, as is observed in [15] and [32]. It is a character of the absolute Galois group of \( k \), and if \( k \) is finite, the value at the geometric Frobenius is the local epsilon factor with the sign. For this reason, we will use the symbol \( \varepsilon_{0,k} \), where \( \bar{k} \) stands for an algebraic closure of \( k \) used to define the Galois group, for local epsilon factors as characters in [15], [32] and keep in mind the relation

\[
\varepsilon_{0,k}(T, V, \omega)(\text{Frob}_k) = (-1)^{\dim_{\text{tot}} V} \varepsilon_0(T, V, \omega)
\]

with the classical ones, in order to avoid any confusion on signs, which are main subjects of this paper.

Attaching bilinear forms to isolated singularities is considered for example in [19] and in [21], Section 12], in the spirit of refining classical formulae of Euler characteristics, which are valued in the integers, to the Grothendieck–Witt ring, i.e. the ring of non-degenerate symmetric bilinear forms over \( k \). In our situation, taking the ranks of the bilinear forms gives the classical Milnor formula for the total dimensions. From this point of view, it can be said that what is done in this paper is to give étale counterparts of the discriminants of them.

The most surprising results at least to the author in this paper appear in characteristic 2, which we now explain. The construction of the bilinear forms from isolated singularities always works over any base scheme, even in characteristic 2. However, the discriminants live in the group \( k^\times/(k^\times)^2 \), which is trivial if \( k \) is of characteristic 2 (and perfect). Of course, in order to construct a character of order 2 of the absolute Galois group of \( k \) (which will be a sign in the finite field case, by evaluating it at the geometric Frobenius), we need to find an element in \( k/\varphi(k) \), rather than the multiplicative group, according to Artin–Schreier theory. Here \( \varphi: k \rightarrow k \) is the map sending \( x \mapsto x^2 - x \).

To get elements in \( k/\varphi(k) \), we consider liftings to the \( p \)-adic base. Precisely speaking, we consider a formal scheme formally smooth and formally of finite type over the Witt ring \( W(k) \) with a morphism to the formal \( p \)-adic completion of \( \mathbb{A}_W^1(k) \) whose reduction mod \( p \) is isomorphic to the initial one. First, take such one lift and fix it. Then, as is in the scheme case, we obtain a non-degenerate symmetric bilinear form over \( W(k) \), whose discriminant lives in \( W(k)^\times/(W(k)^\times)^2 \). As is similar as in the case of Arf invariant (cf. [8], 1.4]), we can prove that the discriminant does not depend on the choice of lifts and it lives in \( \pm(1 + 4W(k)) \). Therefore, choosing the sign appropriately, which is done in 4.3
in this paper, we obtain an element of the form $1 + 4[a]$, hence an element $a \in k$, which is determined up to modulo $\varphi(k)$. Here $[-]$ denotes the Teichmüller lift. The constant $a \in k/\varphi(k)$ so obtained we call the Arf invariant, and we denote by $\text{Arf}(f, x)$. For the detail, see Section 4, especially Theorems 4.7, 4.12.

The terminology of Arf invariant is justified by the following example: Let $Q \in k[T_0, \ldots, T_n]$ be a non-degenerate quadratic form with $k$-coefficient. Then the map $Q: \mathbb{A}_{k}^{n+1} \to \mathbb{A}_{k}^{1}$ has an isolated singular point at the origin and our Arf invariant $\text{Arf}(Q, 0)$ coincides with the original one.

Actually, for our purpose, it is enough to consider liftings to $W_3$, the ring of Witt vectors of length 3, instead of full liftings to $W$. This is because the reduction map $W \to W_3$ induces an isomorphism of the classifying spaces (i.e., $H^1$) of $\mu_2$-torsors over $W$ and $W_3$ (Lemma 2.13), where the discriminants of bilinear forms live. For $\mu_2$-torsors whose induced line bundles under the inclusion $\mu_2 \to \mathbb{G}_m$ are trivial, this is equivalent to saying that $(W^\times)^2$ contains $1 + 8W$. For this reason, we state and prove theorems in characteristic 2 in terms of $W_3$-liftings in Section 4, which allows notations to be relatively simple.

Using this Arf invariant, we can state the main result in characteristic 2 as follows.

**Theorem 1.2.** *(Theorem 5.4.2, $p = 2$)* Let $k$ be a finite field with $q$ elements which is a power of 2. Let $X$ be a smooth $k$-scheme of dimension $n$ and let $f: X \to \mathbb{A}_{k}^{1}$ be a morphism with a $k$-rational isolated singular point $x \in X$. Then $\frac{n_{\mu}(f, x)}{2}$ is an integer and the ratio

$$(-1)^{\dim_{\text{tot}} R\Phi_{f}(\mathbb{Q}_{\ell}) \times \xi_0(\mathbb{A}_{k}^{1}, f(x)), R\Phi_{f}(\mathbb{Q}_{\ell}) \times dt) / q^{\frac{(q-1)n_{\mu}(f, x)}{2}}$$

is $\pm 1$. This sign is determined by the Arf invariant $a = \text{Arf}(f, x)$. Namely, the sign is 1 if and only if $a$ belongs to $\varphi(k)$.

It is very interesting to seek a theory which deals with the vanishing cycles of arbitrary $\ell$-adic sheaves and contains our results as the special case of constant sheaves. For the total dimensions, this is achieved by the theory of characteristic cycles given by T. Saito [27]. For the local epsilon factors, epsilon cycles are defined in [30] as refinements of characteristic cycles. They treat the local epsilon factors of the vanishing cycles of $\ell$-adic sheaves in a geometric way using cotangent bundles, but require that we ignore roots of unity, namely we treat the local epsilon factors in $\overline{\mathbb{Q}_{\ell}}^\times \otimes \mathbb{Q}$, a quotient of the multiplicative group $\overline{\mathbb{Q}_{\ell}}^\times$. This paper tells us that, even for constant sheaves, we need such complicated enhancements as symmetric bilinear forms to treat the local epsilon factors without taking modulo roots of unity. The author hopes that this paper can shed some new light on the theory of epsilon cycles, so that we eventually handle the local epsilon factors themselves without taking modulo roots of unity. In [16], Guignard gives another method for computing global epsilon factors in higher dimension, in a different way from epsilon cycles. It will also be an interesting work to clarify a relation between his results and ours.

We briefly explain the strategy of the proof in the cases of odd characteristic. The proof for the case of characteristic 2 is quite similar, although it is more involved.

We explain the proof of Theorem 1.1. In contrast to Deligne’s global approach in [9], our approach is local. Instead of referring to global results, such as the product formula of epsilon factors by Laumon, we use the continuity of local epsilon factors [29]. Consider a family of morphisms from smooth $k$-schemes to $\mathbb{A}_{k}^{1}$ parametrized by a smooth base scheme...
S. Namely we consider a commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\tilde{f}} & \mathbb{A}^1_S \\
\downarrow{\pi} & & \downarrow{\pi} \\
S & & 
\end{array}
\]

of \(k\)-schemes of finite type where \(\pi\) is smooth and \(S\) is a smooth \(k\)-scheme. If a \(k\)-rational point \(s \in S(k)\) is specified, we call such a family a deformation of \(\tilde{f}_s\), where \(\tilde{f}_s\) is the fiber of \(\tilde{f}\) over \(s\). The continuity says that, if the singular locus of \(\tilde{f}\) is finite over the base \(S\), the local epsilon factors vary continuously over \(S\). More precisely, they satisfy the reciprocity law and give a character of the fundamental group of \(S\). See Theorem 5.1 for the precise statement, or [29, Theorem 4.8] in a more general form of it. A geometric counterpart of this continuity is the flatness of the bilinear forms over \(S\), which is rather trivial from the definition.

Using this continuity, the proof goes as follows. Let \(f : X \to \mathbb{A}^1_k\) be a morphism with an isolated singularity \(x\). We argue by induction on the Milnor number \(\mu(f, x)\). We construct a deformation of \(f\) as in (1.2) so that its generic fibers contain at least one ordinary quadratic point with Milnor number 1 or 2 (Lemma 3.23). By the continuity of local epsilon factors, the flatness of the bilinear forms, and the induction hypothesis on the Milnor number, we reduce the proof to the case where the singularity is ordinary quadratic with the Milnor number 1 or 2, in which we manage to compute explicitly the both sides of the equality in the theorem (Proposition 3.20, Lemma 3.22).

Let us explain the construction of this paper. We collect some preliminaries on Witt rings in Section 2. We prove that a finite étale algebra \(A\) over a Witt ring is integrally closed in the fraction ring \(A[\frac{1}{p}]\) (Lemma 2.8). We also prove that the property for a finite étale covering over the generic fiber of a Witt ring to extend to a finite étale covering of the Witt ring is Zariski-local on the special fiber, in a certain sense (Lemma 2.11). Using these results, we prove results on \(\mu_2\)-torsors and \(\mathbb{Z}/2\)-coverings over a 2-typical Witt ring in Lemma 2.13 which will be needed in Section 4. Section 3 develops our main tools. After recalling basic results on residue symbol, we construct bilinear forms from isolated singularities and study some of their basic properties. We also include calculations on quadratic singularities which are the first pieces of key results for our main theorems. The latter part is devoted to develop a machinery by which we reduce theorems to the case of quadratic singularities. In Section 4, we define a finite étale \(\mathbb{Z}/2\)-covering from an isolated singularities, especially in characteristic 2. Finally, in Section 5, we state and prove the main theorem of this paper. After proving the main theorem, as an example of our results, we record a computation on the discriminants of the bilinear forms associated to homogeneous functions in the last subsection 5.3. In particular, in characteristic 2, the Arf invariants of homogeneous functions relate with the determinants of the étale cohomologies of middle degree of hypersurfaces (cf. [26]).

We collect terminologies we use throughout the paper.

1. For a scheme \(X\), we write \(k(x)\) for the residue field at a point \(x \in X\).

2. Let \(S\) be a scheme. For a point \(s \in S\), we write \(S(s)\) for the henselization at \(s\). On the other hand, for a geometric point \(\bar{s}\) of \(S\), we write \(S(\bar{s})\) for the strict henselization in \(\bar{s}\).
3. Let $S$ be a scheme and $s \to S$ be a morphism from the spectrum of a field. Subscripts $(-)_s$ indicates the base change to $s$. For example, for a morphism $f: X \to Y$ of $S$-schemes, we write $f_s: X_s \to Y_s$ for the base change to $s$. When $s = \text{Spec}(k)$, we also write $(-)_k$ for $(-)_s$.

4. For a henselian local ring $(R, \mathfrak{m}_R)$, we write $R\{t_1, \ldots, t_n\}$ for the henselization of $R[t_1, \ldots, t_n]$ at $(\mathfrak{m}_R, t_1, \ldots, t_n)$.

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2 On finite étale algebras over Witt rings

Let $p$ be a prime number, which we fix in the subsection 2.2. Let $A$ be a perfect $\mathbb{F}_p$-algebra, i.e. the Frobenius map $A \xrightarrow{\phi} A$ is an isomorphism. Let $W(A)$ be the ($p$-typical) Witt ring with coefficients in $A$. In this preliminary section, we give a criterion for a $W(A)$-algebra of a certain type to be finite étale (Lemma 2.11). Using this lemma, we give some results on $\mathbb{Z}/2$-coverings and $\mu_2$-torsors when $p = 2$ (Lemma 2.13).

2.1 Finite modules on adic rings

We start with recalling some basics on finite modules over adic rings. Let $R$ be a ring with a non-zero divisor $\varpi$ such that $R$ is $\varpi$-adically complete and separated. We write $R_n$ for $R/\varpi^{n+1}$.
Definition 2.1. We define \( \text{fMod}_\bullet(R) \) to be the category of projective systems \((M_n)_{n \geq 0}\) indexed by the integers \( n \geq 0 \) with the following properties:

1. For each \( n \geq 0 \), \( M_n \) is an \( R_n \)-module and the transition map \( M_{n+1} \to M_n \) is \( R_{n+1} \)-linear. \( M_0 \) is a finite \( R_0 \)-module.

2. The transition maps induce quasi-isomorphisms \( M_{n+1} \otimes_{R_{n+1}}^L R_n \to M_n \) of complexes of \( R_n \)-modules.

For a \( \varpi \)-torsion free \( R \)-module \( M \), the system \((M/\varpi^{n+1}M)_n\) is an object of \( \text{fMod}_\bullet(R) \) provided that \( M/\varpi M \) is finite.

Remark 2.2. 1. By the condition 2, we especially have \( M_n \otimes_{R_n} R_0 \cong M_0 \). Thus, \( M_n \) is finite since \( M_0 \) is finite and the kernel of \( R_n \to R_0 \) is nilpotent.

2. Let \( n \geq i \geq 0 \) be integers. Composing the quasi-isomorphisms \( M_{n+1} \otimes_{R_{n+1}}^L R_i \to M_i \) repeatedly, we know that the map \( M_{n+1} \otimes_{R_{n+1}}^L R_i \to M_i \) is a quasi-isomorphism. Since \( \varpi \) is not a zero-divisor in \( R \),

\[
\cdots \to R_{n+1} \xrightarrow{\varpi^{n+1-i}} R_{n+1} \xrightarrow{\varpi^{i+1}} R_{n+1} \xrightarrow{\varpi^{n+1-i}} R_{n+1} \xrightarrow{\varpi^{i+1}} R_{n+1}
\]

gives an \( R_{n+1} \)-free resolution of \( R_i \). Therefore, the condition that \( M_{n+1} \otimes_{R_{n+1}}^L R_i \to M_i \) is a quasi-isomorphism for any \( n \geq i \geq 0 \) is equivalent to the following two conditions:

(a) The map \( M_{n+1} \to M_i \) induces an isomorphism \( M_{n+1}/\varpi^{i+1}M_{n+1} \to M_i \).

(b) \( M_{n+1} \xrightarrow{\varpi^{n+1-i}} M_{n+1} \xrightarrow{\varpi^{i+1}} M_{n+1} \) is exact.

Lemma 2.3. Let \((M_n)_n\) be an object in \( \text{fMod}_\bullet(R) \) (Definition 2.1). Let \( M := \lim_{\leftarrow n} M_n \) be the projective limit.

1. The limit \( M \) is a finite \( \varpi \)-torsion free \( R \)-module. The projection \( M \to M_n \) induces an isomorphism \( M/\varpi^{n+1}M \to M_n \).

2. If \( M_0 \) is finitely presented as an \( R_0 \)-module, \( M \) is finitely presented as an \( R \)-module.

3. If \( M_0 \) is projective as an \( R_0 \)-module, \( M \) is projective as an \( R \)-module.

Proof. 1. First we show that \( M/\varpi^{n+1}M \cong M_n \). By Remark 2.2 we have a short exact sequence

\[
0 \to M_s \xrightarrow{\varpi^{s+1}} M_{n+s+1} \to M_n \to 0.
\]

Taking the limit with respect to \( s \), we have

\[
0 \to M \xrightarrow{\varpi^{n+1}} M \to M_n \to 0.
\]

Here the exactness on the right comes from that \((M_s)_s \) satisfies the Mittag-Leffler condition. Therefore we have \( M/\varpi^{n+1}M \cong M_n \).

Next we show that \( M \) is finite. Since \( M_0 \) is finite, we can choose finitely many generators \( x_{0,0}, \ldots, x_{0,m} \in M_0 \). Inductively on \( n \), we choose elements \( x_{n,j} \in M_n \) such that the images of \( x_{n+1,j} \) in \( M_n \) is \( x_{n,j} \). Since the kernel of \( R_n \to R_0 \) is nilpotent, \( x_{n,0}, \ldots, x_{n,m} \) generate
Let $x_j := (x_{n,j})_n \in M$ generate $M$. Indeed, take an element $x = (x_n) \in M$. First choose $a_j \in R$ so that the images $\bar{a}_j \in R_0$ of $a_j$ satisfy the relation $x_0 = \sum_j a_j x_{0,j}$. Then $x' := x - \sum_a a_j x_j$ has 0 in its 0-th component. By the isomorphism $M/\varpi M \cong M_0$, $x'$ can be written in the form $\varpi y$ for some $y \in M$. Applying the same procedure to $y$, we find $b_j \in R$ such that $y - \sum_j b_j x_j$ has 0 in its 0-th component. Construct sequences $a_j, b_j, \ldots$ for each $j$ inductively and we define $a_j \in R$ to be the limits $a_j + \varpi b_j + \ldots$. Then we have $x = \sum_j a_j x_j$, which shows the finiteness.

We prove that $M$ is $\varpi$-torsion free. Consider an exact sequence

$$0 \to K_n \to M_n \xrightarrow{\cong} M_n.$$

By the left exactness of $\varprojlim_n$, the kernel of $\varpi: M \to M$ is isomorphic to $\varprojlim_n K_n$. Since $K_{n+1}$ maps to 0 via $M_{n+1} \to M_n$ (Remark 2.2), this limit vanishes, which implies the claim.

2. Let $R^m \to M$ be a surjection from a free module and let $N$ be its kernel. Taking modulo $\varpi^{n+1}$, we have a short exact sequence

$$(2.1) \quad 0 \to N/\varpi^{n+1} N \to R^m \to M_n \to 0.$$ 

Since $M_0$ is a finitely presented $R_0$-module, $N/\varpi N$ is finitely generated. Thus $N/\varpi^{n+1} N$ is finitely generated. Note that $(N/\varpi^{n+1} N)_n$ satisfies the condition that $N/\varpi^{n+2} N \otimes_{R_n} R_n \to N/\varpi^{n+1} N$ is a quasi-isomorphism, as $N$ is $\varpi$-torsion free. Applying the five lemma to the limit of (2.1), $N$ is isomorphic to $\varprojlim_n N/\varpi^{n+1} N$, which is finitely generated by 1.

3. We know that $M_n$ is finitely presented as an $R_n$-module by 2 and $R_n$-flat by [22, 22.3], hence projective.

Take a surjection $R^m \to M$ and let $f_n: R^m \to M_n$ be its reduction mod $\varpi^{n+1}$. We construct a splitting $\varphi_n: M_n \to R^m$ for $f_n$ by induction on $n$. For $n = 0$, we may take any splitting as $\varphi_0$. Suppose that we choose $\varphi_n$. By the projectivity of $M_{n+1}$, we find a map $\varphi': M_{n+1} \to R^m_{n+1}$ which makes the diagram

$$\begin{array}{ccc}
M_{n+1} & \longrightarrow & M_n \\
\downarrow \varphi' & & \downarrow \varphi_n \\
R^m_{n+1} & \longrightarrow & R^m_n
\end{array}$$

where the horizontal arrows are the transition maps commutative. Since the composition $f_{n+1} \circ \varphi'$ becomes the identity after taking modulo $\varpi^{n+1}$, it is an isomorphism. Then we can take $\varphi_{n+1} := \varphi' \circ (f_{n+1} \circ \varphi')^{-1}$. $\square$

**Remark 2.4.** Let $\mathsf{fMod}(R)$ be the category of finitely generated $R$-modules which are $\varpi$-torsion free. Lemma 2.3.1 implies that the assignment $(M_n)_n \mapsto \varprojlim_n M_n$ defines a functor $\mathsf{fMod}_e(R) \to \mathsf{fMod}(R)$. This is fully faithful and its essential image consists of the elements $M \in \mathsf{fMod}(R)$ which are $\varpi$-adically complete and separated. By Lemma 2.3.1 and Nakayama’s lemma, an element $M \in \mathsf{fMod}(R)$ is always $\varpi$-adically complete, i.e. $M \to \varprojlim_n M/\varpi^{n+1} M$ is surjective, but not separated in general. However, the following holds.

The following lemma is not used in the sequel.

**Lemma 2.5.** Assume that $M \in \mathsf{fMod}(R)$ is finitely presented. Then it is $\varpi$-adically complete and separated.
Proof. Let us denote by $\hat{L}$ the $\varpi$-adic completion of an $R$-module $L$.

The surjectivity of $M \to \hat{M}$ follows from Lemma 2.3.1 and Nakayama’s lemma, for which we merely use the property that $M$ is finitely generated. Take a short exact sequence

\begin{equation}
0 \to N \to R^m \to M \to 0
\end{equation}

where $N$ is a finitely generated $R$-module. As $M$ is $\varpi$-torsion free, the $\varpi^{n+1}$-mod reductions of (2.2) remain exact. As $(N/\varpi^{n+1}N)_n$ satisfies the Mittag-Leffler condition, the $\varpi$-adic completion of (2.2) is exact. We have a morphism of short exact sequences

\[
\begin{array}{c}
0 \\ \downarrow \\ 0
\end{array}
\begin{array}{ccc}
N & \longrightarrow & R^m \\
\cong & & \cong \\
\hat{N} & \longrightarrow & \hat{R}^m
\end{array}
\begin{array}{c}
M \\ \downarrow \\ \hat{M}
\end{array}
\rightarrow 0.
\]

The injectivity of $M \to \hat{M}$ follows from the surjectivity of $N \to \hat{N}$. \hfill $\square$

Proposition 2.6. 1. Let $(R'_n)_n$ be an object in $\text{fMod}_R(R)$. Assume that $R'_n$ are equipped with structures of $R_n$-algebras such that the transition maps $R'_{n+1} \to R'_n$ are morphisms of $R$-algebras. Further assume that $R'_0$ is a finite étale $R_0$-algebra. Then the limit $R' := \varprojlim_n R'_n$ is a finite étale $R$-algebra.

2. The modulo-$\varpi$ reduction gives an equivalence of categories from finite étale algebras over $R$ to those over $R_0$.

Proof. 1. By Lemma 2.3.3, we know that $R'$ is finite projective as an $R$-module. Thus it is enough to show that $\Omega^{1}_{R'/R}$ vanishes, which follows from the vanishing of $\Omega^{1}_{R'_0/R_0}$ and Nakayama’s lemma.

2. We construct a quasi-inverse functor. Let $R'_0$ be a finite étale $R_0$-algebra. For $n \geq 0$, the closed immersion $\text{Spec}(R_0) \to \text{Spec}(R_n)$ is defined by a nilpotent ideal. Hence $R'_n$ uniquely lifts to a finite étale $R_n$-algebra $R'_n$. Then the limit $R' := \varprojlim_n R'_n$ is a finite étale $R$-algebra by 1, which is quasi-inverse to the modulo-$\varpi$ reduction. \hfill $\square$

We recall that the Picard groups of adic rings are isomorphic to those of the reductions.

Lemma 2.7. Let $R$ be a ring which is $\varpi$-adically complete and separated. Then the reduction map $R \to R_0 = R/\varpi$ induces an isomorphism of groups $H^1(\text{Spec}(R), \mathbb{G}_m) \to H^1(\text{Spec}(R_0), \mathbb{G}_m)$.

Proof. The injectivity follows from Nakayama’s lemma. We show the surjectivity. Take an invertible $R_0$-module $M_0$. We construct an invertible $R_n$-module $M_n$ for $n > 0$ such that $M_n \otimes_{R_0} R_{n-1} \cong M_{n-1}$. Then the limit $M := \varprojlim_n M_n$ is an invertible $R$-module with $M \otimes_R R_0 \cong M_0$ by Lemmas 2.3.1,3.

The construction goes by induction on $n$. Suppose that we are given an invertible $R_n$-module $M_n$. To show that $M_n$ lifts to an invertible $R_{n+1}$-module, we show that the map $H^1(S, \mathbb{G}_m) \to H^1(S_n, \mathbb{G}_m)$ is surjective, where we set $S_m = \text{Spec}(R_m)$ for integers $m \geq 0$. We have a short exact sequence

$0 \to \varpi^{n+1}O_{S_{n+1}} \xrightarrow{x \mapsto 1+x} O_{S_{n+1}}^\times \to O_{S_n}^\times \to 0.$

Then the surjectivity follows from $H^2(S_{n+1}, \varpi^{n+1}O_{S_{n+1}}) = 0$, as $S_{n+1}$ is affine. \hfill $\square$
2.2 On Witt rings

From now on, we suppose that $R = W(A)$ is the Witt ring of a perfect $\mathbb{F}_p$-algebra $A$. This is $p$-adically complete separated, and $p$ is not a zero-divisor in it. Therefore, we can apply the previous results to $(R, \varpi) = (W(A), p)$.

First we show that finite étale algebras over the Witt rings are “normal”.

Lemma 2.8. Let $R'$ be a finite étale $W(A)$-algebra. Then $R'$ is the integral closure of $W(A)$ in $R'[\frac{1}{p}]$.

Proof. We show that $R'$ is integrally closed in $R'[\frac{1}{p}]$. Take $a \in R'[\frac{1}{p}]$ which is integral over $R'$. It can be written in the form $a = b/p^n$ for some $b \in R'$ and a non-negative integer $n \geq 0$. We choose a presentation so that $n$ is minimal. Since $a$ is integral over $R'$, there exist $a_1, \ldots, a_m \in R'$ such that

$$a^m + a_1a^{m-1} + \cdots + a_m = 0.$$  

Multiplying $p^m$, we get

$$b^m + a_1p^nb^{m-1} + \cdots + a_mp^m = 0.$$  

Therefore the class $\bar{b}$ of $b$ in $R'/pR'$ is nilpotent if $n > 0$. But $R'/pR'$ is reduced since it is étale over $W(A)/p = A$, which is reduced. This implies that $\bar{b}$ is zero, i.e. $b$ is divisible by $p$ in $R'$, a contradiction to the minimality of $n$. Thus we have $n = 0$, hence $a \in R'$.

Next we give a criterion for a normalization to be étale in Lemma 2.11. We start with the following lemmas.

Lemma 2.9. Let $\{\eta_i\}_{i \in I}$ be the set of generic points of $A$. Then the map $A \to \prod_i k(\eta_i)$ is injective.

Proof. It follows since $A$ is reduced.

Lemma 2.10. Let $R'$ be a finite $W(A)$-algebra which is projective as a $W(A)$-module. Assume that $R'[\frac{1}{p}]$ is finite étale over $W(A)[\frac{1}{p}]$. Let $R''$ be the integral closure of $R'$ in $R'[\frac{1}{p}]$.

1. There exists a finite projective $W(A)$-submodule $R''$ of $R'[\frac{1}{p}]$ which contains $R''$.

2. $R''$ is $p$-adically complete and separated, i.e. the canonical map $R'' \to \varinjlim_n R''/p^{n+1}R''$ is an isomorphism.

Proof. 1. The trace on $R'$ induces a pairing

$$R' \times R' \to W(A), \ (x, y) \mapsto \text{Tr}_{R'/W(A)}(xy).$$  

This induces a $W(A)$-linear map $f: R' \to R''$ where we set $R'' = \text{Hom}_{W(A)}(R', W(A))$. As $R'[\frac{1}{p}]$ is finite étale over $W(A)[\frac{1}{p}]$, the map $f$ becomes an isomorphism after inverting $p$. Thus $R''$ can be seen as a $W(A)$-submodule of $R'[\frac{1}{p}]$.

To show $R'' \subset R''$, it suffices to show that, for $x \in R'[\frac{1}{p}]$ integral over $W(A)$, we have $\text{Tr}_{R'/W(A)}(x) \in W(A)$. By Lemma 2.9 it is enough to show that the image of $\text{Tr}_{R'/W(A)}(x)$ in $W(k(\eta))[\frac{1}{p}]$ for any generic point $\eta$ of $A$ is contained in $W(k(\eta))$. Hence we reduce it
to the case where $A$ is a perfect field. In this case, it is well-known as $W(A)$ is a normal domain.

2. We show that the $p$-adic topology on $R''$ induces the $p$-adic topology on $R'$ and that $R'$ is open in $R''$ in that topology. Indeed, for any integer $n \geq 0$, there exists $N \geq 0$ such that $p^N R'' \subset p^n R'$ as $R''$ is finitely generated and $R''/R'$ is $p$-torsion. Hence we have $p^N R'' \subset p^n R'$ for such an $N$.

We find that $R''$ contains an open submodule $R'$ which is complete and separated in the $p$-adic topology of $R''$. Thus $R''$ itself is $p$-adically complete and separated.

We denote the Teichmüller lift $A \to W(A)$ by $a \mapsto [a]$.

**Lemma 2.11.** Let $R'$ be a finite $W(A)$-algebra which is projective as a $W(A)$-module. Assume that $R'[\frac{1}{p}]$ is finite étale over $W(A)[\frac{1}{p}]$. Let $R''$ be the integral closure of $W(A)$ in $R'[\frac{1}{p}]$.

1. Let $g \in A$ be an element. The $p$-adic completion of $R'' \otimes_{W(A)} W(A)[\frac{1}{g}]$ is isomorphic to the $p$-adic completion of $R'' \otimes_{W(A)} W(A)_g$. We denote it by $R'' \hat{\otimes}_{W(A)} W(A)_g$.

2. The $p$-adic completion $R'' \hat{\otimes}_{W(A)} W(A)_g$, defined in 1, is the integral closure of $W(A)_g$ in $R'[\frac{1}{p}] \otimes_{W(A)} W(A)_g$.

3. Let $g_1, \ldots, g_n \in A$ be a sequence of elements which generates the unit ideal of $A$. If $R'' \hat{\otimes}_{W(A)} W(A)_g$ is a finite étale $W(A)_g$-algebra for each $i$, $R''$ is a finite étale $W(A)$-algebra.

**Proof.** By Lemma 2.10.1, there exists a finite projective $W(A)$-submodule $R''$ of $R'[\frac{1}{p}]$ which contains $R''$, i.e. $R' \subset R'' \subset R''$. It remains to show that $R'' \hat{\otimes}_{W(A)} W(A)_g$ is integral over $W(A)_g$. Take any element $x \in R'' \hat{\otimes}_{W(A)} W(A)_g$. Then there exists $y \in R'' \otimes_{W(A)} W(A)[\frac{1}{g}]$ such that $x - y \in R' \otimes_{W(A)} W(A)_g$ as $R' \otimes_{W(A)} W(A)_g$ is an open subring. Thus $x$ is integral.

3. Assume that $R'' \hat{\otimes}_{W(A)} W(A)_g$ is finite étale over $W(A)_g$. By the Zariski descent, $R''/pR''$ is finite étale over $A$. Thus, by Proposition 2.10.1, $\lim R''_p/p^{n+1} R''_p$ is a finite étale $W(A)$-algebra. On the other hand, we have $R'' \cong \lim R''_p/p^{n+1} R''_p$ by Lemma 2.10.2, hence the assertion.

We recall a well-known result on $W(A)\times/(W(A)\times)^2$ in the case of $p = 2$.

**Lemma 2.12.** Let $p = 2$ and let $A$ be a perfect $\mathbb{F}_2$-algebra.

1. We have the inclusion $1 + 8W(A) \subset (W(A)\times)^2$. 

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2. There exists a canonical short exact sequence

\[
0 \to A/\varphi(A) \to W(A)^\times/(W(A)^\times)^2 \to A \to 0
\]

where \( \varphi: A \to A \) denotes the map \( x \mapsto x^2 - x \). The first arrow sends \( a \in A \) to the class of \( 1 + 4[a] \in W(A)^\times \) and the second arrow sends \( 1 + 2[a] \) to \( a \).

3. Let \( \alpha \) be an element of \( W(A)^\times \). The following are equivalent.

(a) The \( W(A)[\frac{1}{2}] \)-algebra \( W(A)[\frac{1}{2}][u]/(u^2 - \alpha) \) extends to a (necessarily unique) finite étale \( W(A) \)-algebra.

(b) For any generic point \( \eta \) of \( A \), the extension \( F_\eta[u]/(u^2 - \bar{\alpha}) \) where \( F_\eta \) denotes the fraction field of \( W(k(\eta)) \) and \( \bar{\alpha} \) is the image of \( \alpha \) under the map \( W(A) \to W(k(\eta)) \) is unramified.

(c) The class of \( \alpha \) in \( W(A)^\times/(W(A)^\times)^2 \) comes from \( A/\varphi(A) \) under the injection \( A/\varphi(A) \to W(A)^\times/(W(A)^\times)^2 \) in (2.3).

Suppose that these conditions are satisfied and set \( \alpha \equiv 1 + 4[b] \) in \( W(A)^\times/(W(A)^\times)^2 \). Let \( R' \) be a finite étale \( W(A) \)-algebra such that \( R'[\frac{1}{2}] \cong W(A)[\frac{1}{2}][u]/(u^2 - \alpha) \). Then the quadratic character in \( H^1(\text{Spec}(A), \mathbb{Z}/2) \) given by the quadratic extension \( R' \otimes_{W(A)} A \) of \( A \) corresponds to \( b \) via the Artin–Schreier isomorphism \( A/\varphi(A) \cong H^1(\text{Spec}(A), \mathbb{Z}/2) \).

**Proof.** The exponential \( \exp(x) = 1 + x + \frac{x^2}{2!} + \cdots \) converges for \( x \in 4W(A) \) and the logarithm \( \log(x) = (x - 1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots \) converges for \( x \in 1 + 4W(A) \). They define group isomorphisms between \( 1 + 4W(A) \) and \( 4W(A) \), each of which is the inverse to the other. They also respect the filtrations \( 1 + 2^iW(A) \subset 1 + 4W(A) \) and \( 2^iW(A) \subset 4W(A) \) for \( i \geq 2 \). In particular, we have \( (1 + 4W(A))^2 = 1 + 8W(A) \), hence the assertion 1. On the other hand, since \( A \) is perfect of characteristic 2, we have \( ([A^\times])^2 = [A^\times] \). Thus, letting \( M := (1 + 2W(A))/(1 + 8W(A)) \), whose group law we write additively, we have \( W(A)^\times/(W(A)^\times)^2 \cong M/2M \).

Consider a commutative diagram

\[
\begin{array}{ccc}
0 & \to & A & \to & M & \to & A & \to & 0 \\
& & \downarrow 2 & & \downarrow 2 & & \downarrow 2 & & \\
0 & \to & A & \to & M & \to & A & \to & 0
\end{array}
\]

where the left horizontal arrows \( A \to M \) are \( a \mapsto 1 + 4[a] \) and the right horizontal arrows \( M \to A \) are \( 1 + 2[a] + 4[b] \mapsto a \). By the snake lemma, we have

\[
A \to A \to M/2M \to A \to 0.
\]

It is straightforward to check that the connecting homomorphism \( A \to A \) is \( \varphi \). The assertion 2 follows.

3. The uniqueness in (a) follows from Lemma 2.8. The implication (a) \( \Rightarrow \) (b) is clear.

We show (c) \( \Rightarrow \) (a). Write \( \alpha = \beta^2(1 + 4[b]) \) where \( \beta \in W(A)^\times, b \in A \). Putting \( \beta(1 + 2v) = u, W(A)[\frac{1}{2}][u]/(u^2 - \alpha) \) is isomorphic to \( W(A)[\frac{1}{2}][v]/(v^2 + v - [b]) \). This extends to a finite étale \( W(A) \)-algebra \( W(A)[v]/(v^2 + v - [b]) \). The last assertion on the quadratic character follows from this presentation.
We show \((b) \Rightarrow (c)\). Let \(\alpha \in W(A)^\times\) be an element such that \(F_\eta[u]/(u^2 - \tilde{\alpha})\) is unramified for any \(\eta\). Write \(\alpha = \beta^2(1 + 2[a] + 4[b])\) where \(\beta \in W(A)^\times\), \(a, b \in A\). We need to show that \(a = 0\). Suppose otherwise. By Lemma \(2.9\) there exists a generic point \(\eta\) such that the image \(\tilde{\alpha}\) of \(a\) in \(W(k(\eta))\) is non-zero. This means that \(2[a] + 4[b]\) maps to a uniformizer \(\pi\) in \(W(k(\eta))\). Putting \(u = \beta(w + 1)\), we have \(F_\eta[u]/(u^2 - \alpha) \cong F_\eta[w]/(w^2 + 2w - \pi)\), which is a non-trivial totally ramified extension, a contradiction. □

Using the above results, we prove some properties on \(\mu_2\)-torsors and \(\mathbb{Z}/2\)-coverings over Witt rings which will be needed in Section 4. Let \(A\) be a perfect \(\mathbb{F}_2\)-algebra and \(W_3(A)\) be the ring of Witt vectors of length 3. Write \(S_2 := \text{Spec}(W_3(A))\) (we use the subscript 2 in the left since we write \(S_0\) for \(\text{Spec}(A)\) in Section 4). Write \(\mu_2\) for the fppf sheaf on schemes sending a scheme \(T\) to the group

\[ \{a \in \Gamma(T, \mathcal{O}_T)|a^2 = 1\}. \]

This fits into the short exact sequence of fppf sheaves

\[ (2.4) \quad 0 \to \mu_2 \to \mathbb{G}_m \xrightarrow{a \mapsto a^2} \mathbb{G}_m \to 0. \]

Lemma 2.13. Let \(p = 2\). Consider a map \(\mathbb{Z}/2 \to \mu_2\) of sheaves sending \(\mathbb{Z}/2 \ni 1 \mapsto -1 \in \mu_2\).

1. Write \(S_\infty := \text{Spec}(W(A))\). Then the maps \(H^1(S_\infty, \mathbb{Z}/2) \to H^1(S_2, \mathbb{Z}/2)\) and 
\(H^1(S_\infty, \mu_2) \to H^1(S_2, \mu_2)\) are isomorphisms.

2. Write \(S_\infty[\frac{1}{2}] = \text{Spec}(W(A)[\frac{1}{2}]) = S_\infty \times_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}]\). The map \(\mathbb{Z}/2 \to \mu_2\) induces a commutative diagram of cohomology groups

\[ (2.5) \quad \begin{array}{ccc}
H^1(S_2, \mathbb{Z}/2) & \to & H^1(S_2, \mu_2) \\
\downarrow & & \downarrow \\
H^1(S_\infty[\frac{1}{2}], \mathbb{Z}/2) & \to & H^1(S_\infty[\frac{1}{2}], \mu_2).
\end{array} \]

The slant arrow is injective. Consequently, the horizontal map \(H^1(S_2, \mathbb{Z}/2) \to H^1(S_2, \mu_2)\) is injective.

3. Let \(\{\eta_i\}_{i \in I}\) be the set of generic points of \(A\). Set \(S_{2,i} := \text{Spec}(W_3(k(\eta_i)))\). Then the diagram

\[ \begin{array}{ccc}
H^1(S_2, \mathbb{Z}/2) & \to & H^1(S_2, \mu_2) \\
\downarrow & & \downarrow \\
\prod_i H^1(S_{2,i}, \mathbb{Z}/2) & \to & \prod_i H^1(S_{2,i}, \mu_2)
\end{array} \]

is cartesian.

4. Further assume that \(A\) is normal, i.e. \(A\) is the finite product of normal domains. Then the vertical arrow \(H^1(S_2, \mathbb{Z}/2) \to \prod_i H^1(S_{2,i}, \mathbb{Z}/2)\) in 3 is injective.

Proof. 1. The first map \(H^1(S_\infty, \mathbb{Z}/2) \to H^1(S_2, \mathbb{Z}/2)\) is an isomorphism by Proposition \(2.6.2\).
We show that the second one is also an isomorphism. The Kummer exact sequence (2.4) gives a commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & W(A)^x/(W(A)^x)^2 & \rightarrow & H^1(S_\infty, \mu_2) & \rightarrow & H^1(S_\infty, \mathbb{G}_m)[2] & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & W_3(A)^x/(W_3(A)^x)^2 & \rightarrow & H^1(S_2, \mu_2) & \rightarrow & H^1(S_2, \mathbb{G}_m)[2] & \rightarrow & 0 
\end{array}
\]

where the horizontal lines are exact and \(G[2]\) for an abelian group \(G\) denotes the 2-torsion part. By Lemma 2.12, the left vertical arrow is an isomorphism. Applying Lemma 2.7 to \((R, \varpi) = (W(A), p^2)\), we know that the right vertical arrow is an isomorphism. Therefore, the assertion follows from the five lemma.

2. Since the map \(\mathbb{Z}/2 \rightarrow \mu_2\) is an isomorphism on \(S_\infty[\frac{1}{2}]\), the map \(H^1(S_\infty, \mathbb{Z}/2) \rightarrow H^1(S_\infty[\frac{1}{2}], \mathbb{Z}/2)\) factors through \(H^1(S_\infty, \mu_2)\). Then the diagram is obtained from this under the identifications given in 1. The injectivity of the horizontal arrow follows from that of the slant one, which is a consequence of Lemma 2.8.

3. We may replace \(S_2, S_{2,i}\) in the diagram by \(S_\infty\) and \(S_{\infty,i} = \text{Spec}(W(k(\eta_i)))\) by 1. We already know that the horizontal arrows are injective by 2. Hence, it is enough to show that a \(\mu_2\)-torsor \(S'\) on \(S_\infty\) comes from a (unique) \(\mathbb{Z}/2\)-covering on \(S_\infty\) provided that its restrictions to \(S_{\infty,i}\) come from \(\mathbb{Z}/2\)-coverings.

Let \(R' := \Gamma(S', \mathcal{O}_S)\), which is finite projective as a \(W(A)\)-module. We show that the normalization \(R''\) of \(R'\) in \(R'[\frac{1}{2}]\) is finite étale over \(W(A)\) (cf. Lemma 2.8). Then the action of \(\mathbb{Z}/2\) on \(R'[\frac{1}{2}]\) uniquely lifts to an action on \(R''\), which gives an element in \(H^1(S_\infty, \mathbb{Z}/2)\) mapping to \(S'\).

The exact sequence (2.4) gives an exact sequence

\[
\begin{array}{cccccc}
0 & \rightarrow & W(A)^x/(W(A)^x)^2 & \rightarrow & H^1(S_\infty, \mu_2) & \rightarrow & H^1(S_\infty, \mathbb{G}_m) 
\end{array}
\]

Let \(M\) be an invertible \(W(A)\)-module which is the image of \(S'\) via the right arrow in (2.7). Take elements \(g_1, \ldots, g_m \in A\) such that \(M \otimes_{W(A)} A_{g_i}\) are free and \(g_1, \ldots, g_m\) generate the unit ideal of \(A\). Then Nakayama’s lemma implies that \(M \otimes_{W(A)} W(A_{g_i})\) are free. By Lemma 2.11, we may replace \(A\) by each \(A_{g_i}\) to show that \(R''\) is finite étale over \(W(A)\). Hence we may assume that \(M\) itself is free. In this case, the exact sequence (2.7) tells us that there exists an element \(\alpha \in W(A)^x\) such that \(R'\) is isomorphic to \(W(A)[u]/(u^2 - \alpha)\). Then the assertion follows from the equivalence \((a) \iff (b)\) in Lemma 2.12.3.

4. We may assume that \(A\) is a normal domain. Then the assertion follows from the surjectivity of the map \(\pi_1^{ab}(\eta) \rightarrow \pi_1^{ab}(\text{Spec}(A))\) where \(\eta\) is the generic point of \(A\).

\[\square\]

3 Isolated singularities and symmetric bilinear forms

3.1 Reminder on residue symbols

In this and next subsections, we recall the notion of residue symbol and construct symmetric bilinear forms from isolated singular points.

Residue symbol is introduced in [17, III 9] at least for (locally) noetherian schemes. In the sequel, we will need the same construction for non-noetherian cases, in which the arguments in [17] also work. However, we will need explicit descriptions of some
isomorphisms appearing in this context, in order to complete the proofs of the main results of this paper. For this reason, we record the arguments in loc. cit. in a manner which is suitable for our purposes, but content ourselves with explaining only what is necessary in this paper. Affine cases are treated in [18] in detail.

For a ringed space \((X, \mathcal{A})\), \(\text{Qcoh}(\mathcal{A})\) denotes the category of quasi-coherent \(\mathcal{A}\)-modules. We also write \(\text{Qcoh}(X)\) for it if no confusions occur.

**Definition 3.1.** (cf. [17] III 6) Let \(Z, S\) be schemes. For a finite morphism \(g: Z \to S\), define a functor

\[ g^1: \text{Qcoh}(S) \to \text{Qcoh}(Z) \]

by

\[ \mathcal{F} \mapsto \mathcal{O}_Z \otimes_{g^{-1}\mathcal{O}_S} g^{-1}\text{Hom}_{\mathcal{O}_S}(g_*\mathcal{O}_Z, \mathcal{F}), \]

where \(g^{-1}\text{Hom}_{\mathcal{O}_S}(g_*\mathcal{O}_Z, \mathcal{F})\) is regarded as a \(g^{-1}g_*\mathcal{O}_Z\)-module via the left term of \(\text{Hom}\).

When \(S = \text{Spec}(A)\), \(Z = \text{Spec}(B)\) are affine and \(\mathcal{F}\) is associated with an \(A\)-module \(M\), \(g^1\mathcal{F}\) is the quasi-coherent sheaf associated with the \(B\)-module \(\text{Hom}_A(B, M)\).

**Lemma 3.2.** Let \(g: Z \to S\) be a finite morphism.

1. The functor \(g_*g^1(-)\) is canonically isomorphic to \(\text{Hom}_{\mathcal{O}_S}(g_*\mathcal{O}_Z, -)\).

2. \(g^1\) is a right adjoint to \(g_*\). When \(S = \text{Spec}(A)\), \(Z = \text{Spec}(B)\) are affine and \(\mathcal{F}\) is the associated quasi-coherent sheaf with an \(A\)-module \(M\), the counit corresponds to the evaluation map \(\text{Hom}_A(B, M) \to M\) at \(1 \in B\).

3. Let \(\mathcal{M}\) be a quasi-coherent subsheaf of \(g^1\mathcal{O}_S\). Suppose that the composition \(g_*\mathcal{M} \to g_*g^1\mathcal{O}_S \to \mathcal{O}_S\) is zero where the latter map is the counit of the adjunction. Then \(\mathcal{M}\) is zero as a quasi-coherent sheaf.

**Proof.** 1. The map \(\text{Hom}_{\mathcal{O}_S}(g_*\mathcal{O}_Z, \mathcal{F}) \to g_*g^1\mathcal{F}\) is the composition of

\[ \text{Hom}_{\mathcal{O}_S}(g_*\mathcal{O}_Z, \mathcal{F}) \to g_*g^{-1}\text{Hom}_{\mathcal{O}_S}(g_*\mathcal{O}_Z, \mathcal{F}) \to g_*\left(\mathcal{O}_Z \otimes g^{-1}\mathcal{O}_Z g^{-1}\text{Hom}_{\mathcal{O}_S}(g_*\mathcal{O}_Z, \mathcal{F})\right). \]

To check that this is an isomorphism, we may assume that \(S\) (hence also \(Z\)) is affine, in which case it follows from the description of \(g^1\) for affine cases.

2. Under the identification \(g_*: \text{Qcoh}(Z) \xrightarrow{\sim} \text{Qcoh}(g_*\mathcal{O}_Z)\), \(g^1\) corresponds to \(\text{Hom}_{\mathcal{O}_S}(g_*\mathcal{O}_Z, -)\) by 1 and \(g_*\) corresponds to the forgetful functor. Then the adjunction is well-known. The last assertion follows from these descriptions of \(g_*\), \(g^1\).

3. Let \(\mathcal{M}\) be a quasi-coherent subsheaf of \(g^1\mathcal{O}_S\) such that \(g_*\mathcal{M} \to \mathcal{O}_S\) is zero. Via the adjunction in 2, the map \(g_*\mathcal{M} \to \mathcal{O}_S\) corresponds to the inclusion map \(\mathcal{M} \to g^1\mathcal{O}_S\). This is zero if and only if \(\mathcal{M}\) itself is zero. \(\square\)

For the computation of \(g^1\) in the sequel, it is convenient to record [17 III 7.2] in the following manner.

Let \(X\) be a scheme and \(\mathcal{E}\) be a locally free sheaf of constant rank \(n\) on \(X\). Let \(s \in \Gamma(X, \mathcal{E})\) be a global section. Let \(Z \subset X\) be the vanishing locus of \(s\), i.e. the fibered product of the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{s} & Z \\
\downarrow & & \downarrow \\
X & \xrightarrow{0} & \mathbb{V}(\mathcal{E})
\end{array}
\]
where \( V(\mathcal{E}) \) is the spectrum of the symmetric \( \mathcal{O}_X \)-algebra \( \text{Sym}^* \mathcal{E}^\vee \) and \( 0: X \to V(\mathcal{E}) \)

The resolution (3.1) defines a quasi-isomorphism \( R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}, \mathcal{O}_X, \mathcal{E}^\vee) \) since it is a bounded complex of locally finite free \( \mathcal{O}_X \)-modules.

Lemma 3.3. 1. The surjection \( \text{Sym}^+ \mathcal{E}^\vee \to \mathcal{I}_Z \) induces an isomorphism \( i^* \mathcal{E}^\vee \to \mathcal{I}_Z/\mathcal{I}_Z^2 \) of locally free \( \mathcal{O}_Z \)-modules.

2. Let \( (\mathcal{E}_1, s_1) \) be another pair as \( (\mathcal{E}, s) \) such that the vanishing locus of \( s_1 \) is \( Z \). Suppose that an isomorphism \( \alpha: \mathcal{E} \to \mathcal{E}_1 \) which sends \( s \) to \( s_1 \) is given. Then the diagram

is commutative.

Proof. 1. As \( s^*: \text{Sym}^* \mathcal{E}^\vee \to \mathcal{O}_X \) sends \( \text{Sym}^+ \mathcal{E}^\vee \) onto \( \mathcal{I}_Z \), we have a surjection

\[ \mathcal{E}^\vee \cong \text{Sym}^+ \mathcal{E}^\vee / (\text{Sym}^+ \mathcal{E}^\vee)^2 \to \mathcal{I}_Z/\mathcal{I}_Z^2. \]

Hence we get a surjection \( i^* \mathcal{E}^\vee \to \mathcal{I}_Z/\mathcal{I}_Z^2 \), which is an isomorphism as the both sides are locally free \( \mathcal{O}_Z \)-modules of the same rank.

2. This is clear from the construction. \( \square \)

In the situation as above, we define a natural isomorphism (3.4) for \( \mathcal{F} \in \text{Qcoh}(X) \) as follows.

Consider a Koszul complex

(3.1)

\[ K^\bullet: (\det \mathcal{E})^\vee \xrightarrow{\text{id} \otimes (s \wedge)} (\det \mathcal{E})^\vee \otimes \mathcal{E} \xrightarrow{\text{id} \otimes (s \wedge)} (\det \mathcal{E})^\vee \otimes \wedge^2 \mathcal{E} \xrightarrow{\text{id} \otimes (s \wedge)} \cdots \xrightarrow{\text{id} \otimes (s \wedge)} (\det \mathcal{E})^\vee \otimes \det \mathcal{E} \cong \mathcal{O}_X. \]

Here \( \det \mathcal{E} \) denotes the highest exterior power \( \wedge^n \mathcal{E} \). As the section \( s \) gives an \( \mathcal{O}_X \)-regular sequence, (3.1) is a locally free resolution of \( i_* \mathcal{O}_Z \). Let \( \mathcal{F} \) be a quasi-coherent \( \mathcal{O}_X \)-module. The resolution (3.1) defines a quasi-isomorphism \( R\mathcal{H}om_{\mathcal{O}_X}(i_* \mathcal{O}_Z, \mathcal{F}) \cong \mathcal{H}om_{\mathcal{O}_X}(K^\bullet, \mathcal{F}) \) by definition. The \( m \)-th component of the right-hand side is \( \mathcal{H}om_{\mathcal{O}_X}(K^{-m}, \mathcal{F}) \) and the differential \( \mathcal{H}om_{\mathcal{O}_X}(K^{-m}, \mathcal{F}) \to \mathcal{H}om_{\mathcal{O}_X}(K^{-m-1}, \mathcal{F}) \) is given by \( \varphi \mapsto (-1)^{m+1} \varphi d_{K^{-m-1}} \)

where we write \( d_{K^{-m-1}}: K^{-m-1} \to K^{-m} \). Then the canonical map

(3.2)

\[ \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(K^\bullet, \mathcal{O}_X) \to \mathcal{H}om_{\mathcal{O}_X}(K^\bullet, \mathcal{F}) \]

is a morphism of complexes, hence an isomorphism of complexes as \( K^\bullet \) is a bounded complex of locally finite free \( \mathcal{O}_X \)-modules.

The complex \( \mathcal{H}om_{\mathcal{O}_X}(K^\bullet, \mathcal{O}_X) \) is of the form

(3.3)

\[ \mathcal{O}_X \to \det \mathcal{E} \otimes \wedge^{n-1} \mathcal{E}^\vee \to \cdots \to \det \mathcal{E} \]
Lemma 3.4.  1. In the situation as above, The map (3.4) is a quasi-isomorphism.

2. Let \((E_1, s_1)\) be as in Lemma 3.3.2 with an isomorphism \(\alpha : E \to E_1\) sending \(s\) to \(s_1\). Then the diagram

\[
\begin{array}{c}
\text{RHom}_{O_X}(i_*O_Z, F) \xrightarrow{\text{3.4}} \mathcal{F} \otimes_{O_X} \text{Hom}_{O_X}(K^*, O_X) \\
\xrightarrow{\text{i*}} \mathcal{F} \otimes_{O_X} i_*i^* \det E[-n] \xrightarrow{\cong} i_*(L_i^* F \otimes_{O_Z} i^* \det E)[-n]
\end{array}
\]

in the derived category. Here we write \(\otimes\) instead of \(\otimes_{O_Z}\) \(i^*\) \(\det E\) instead of \(\otimes_{O_Z} i^* \det E\) in the right-hand side since \(i^* \det E\) is a locally free \(O_Z\)-module.

3. ([17, III 7.3]) Let \(j : W \to Y\) be a regular closed immersion of schemes of codimension \(n\). For a quasi-coherent \(O_Y\)-module \(F\) which is \(Lj^*\)-acyclic, we have a natural quasi-isomorphism

\[
R\text{Hom}_{O_Y}(j_*O_W, F) \cong j_*((j^* F \otimes_{O_W} \omega_{W/Y})[-n])
\]

with the following properties. Here \(\omega_{W/Y}\) is the determinant of the conormal sheaf.

(a) Suppose that we are given a locally free sheaf \(E\) of rank \(n\) on \(Y\) and a section \(s\) of \(E\) whose vanishing locus is \(W\). Then, we have a commutative diagram

\[
\begin{array}{c}
\text{RHom}_{O_Y}(j_*O_W, F) \xrightarrow{\text{3.5}} j_*((j^* F \otimes_{O_W} \omega_{W/Y})[-n]) \\
\xleftarrow{\text{3.4}} j_*(j^* F \otimes_{O_W} j^* \det E)[-n]
\end{array}
\]

where the horizontal arrow is the one induced from the isomorphism \(j^* \mathcal{E}^\vee \to \mathcal{I}_W/\mathcal{I}_W^2\) in Lemma 3.3.

(b) For an open immersion \(U \hookrightarrow Y\), the restriction of (3.4) to \(U\) is equal to the one defined for \(F|_U\) on \(U\).

(c) Further assume that \(F\) is \(O_Y\)-flat. Let \(f : Y' \to Y\) be a morphism of schemes such that \(W' := W \times_Y Y' \to Y'\) is a regular immersion of codimension \(n\). Set \(j' : W' \to Y'\) and \(f_W : W' \to W\). The diagram

\[
\begin{array}{c}
Lf^* R\text{Hom}_{O_Y}(j_*O_W, F) \xrightarrow{\text{3.5}} Lf^* j_*((j^* F \otimes_{O_W} \omega_{W/Y})[-n]) \\
\xrightarrow{\text{3.6}} R\text{Hom}_{O_Y}(j'_*O_{W'}, f'^*F) \xleftarrow{\text{3.5}} j'_*(j'^* F \otimes_{O_{W'}} \omega_{W'/Y'})[-n]
\end{array}
\]

is commutative.
The flatness assumption in 3 is apparently superfluous, and indeed one can construct (3.5) which satisfies (a), (b) for general quasi-coherent sheaves, which is done in [17, III, 7]. However, it seems difficult to verify the property (c) in this full generality, for which the author could not find a reference.

Proof. 1. The only non-trivial part is that the middle arrow in (3.4) is a quasi-isomorphism. Since (3.3) is a locally free resolution of $i_*i^*\det E$ as $\mathcal{O}_X$-modules, the assertion follows.

2. The exterior powers of $\alpha$ induce an isomorphism between the Koszul complexes (3.1) defined from $(E, s), (E_1, s_1)$. The assertion is verified directly from the construction of (3.4).

3. Let $F$ be a quasi-coherent $\mathcal{O}_Y$-module which is $Lj^*$-acyclic. Locally on $Y$, we can take $(E, s)$ as in (a). Then, by 1, we have a quasi-isomorphism $R\text{Hom}_{\mathcal{O}_Y}(j_*\mathcal{O}_W, F) \to j_*(j^*F \otimes_{\mathcal{O}_W} j^*\det E)[-n]$. Composing $j^*\det E \xrightarrow{\simeq} \omega_{W/Y}$ defined from Lemma 3.3.1, we have a quasi-isomorphism $R\text{Hom}_{\mathcal{O}_Y}(j_*\mathcal{O}_W, F) \to j_*(j^*F \otimes_{\mathcal{O}_W} \omega_{W/Y})[-n]$. By Lemma 3.3.2 and 2, this isomorphism is independent of the choice of $(E, s)$. Since the complexes under consideration are concentrated on one degree, the isomorphisms glue to an isomorphism on the whole of $Y$, which satisfies the properties (a), (b) by the construction.

We verify the property (c). We show the commutativity of (3.6). By the assumption on $f$, the canonical map $Lf^*j_* \to j'_*Lf^*_W$ is a quasi-isomorphism. Hence the right vertical arrow of (3.6) is a quasi-isomorphism. Also the horizontal arrows are quasi-isomorphisms. Consequently the complexes in the diagram are concentrated on degree $n$. Therefore we may replace $Y$ and $Y'$ by arbitrary open coverings to check the commutativity. Hence we may assume that we are given a pair $(E, s)$ on $Y$ as in (a). Then, the assertion follows from the compatibility of the construction of (3.4).

Let $S$ be a scheme and $f: X \to S$ be a smooth morphism of schemes purely of relative dimension $n$. Let $E$ be a locally free $\mathcal{O}_X$-module of rank $n$ on $X$ and $s \in \Gamma(X, E)$ be a global section with the vanishing locus $Z$. Let $i: Z \to X$ be the closed immersion and $g = f \circ i: Z \to S$ be the structure morphism. We apply Lemma 3.3.3 to describe $g^!$.

**Lemma 3.5.** Assume that $Z$ is quasi-finite over $S$.

1. The immersion $i: Z \to X$ is transversally regular of codimension $n$ relative to $S$ ([13, (19.2.2)]). $Z$ is flat of finite presentation over $S$.

2. Assume further that $Z$ is finite over $S$. Then there is a canonical isomorphism

\begin{equation}
(3.7) \quad \Psi_s: g^! \to g^*(-) \otimes \omega_{X/S} \otimes \det E
\end{equation}

of functors. In particular, the counit $\text{Tr}_g: g_*g^!\mathcal{O}_S \to \mathcal{O}_S$ is canonically identified with a map $g_*i^*(\omega_{X/S} \otimes \det E) \to \mathcal{O}_S$. This map is denoted by $\text{Res}_S[i]$, or $\text{Res}_S[i]$ if we want to specify the base scheme, and called the residue symbol (cf. [17, III 9]).

3. Let $E_1$ be another locally free sheaf of rank $n$ on $X$ with a global section $s_1$ whose vanishing locus equals to $Z$. For an isomorphism $\alpha: E \to E_1$ which sends $s$ to $s_1$, the composition of

\[ i^*(\omega_{X/S} \otimes \det E) \xrightarrow{\Psi_{s_1}^{-1}} g^!\mathcal{O}_S \xrightarrow{\Psi_{s_1}} i^*(\omega_{X/S} \otimes \det E_1) \]

equals to $i^*(\text{id} \otimes \det \alpha)$.
Proof. 1. Since the assertion is local on $X$, we may assume that $S = \text{Spec}(A), X = \text{Spec}(B)$ are affine and that $\mathcal{E} \cong \mathcal{O}_X^m$ is free. Let $s_j \in B$ be the $j$-th coordinate of $s$. We need to show that $(s_j)_j$ is a $B$-regular sequence around $Z$ and that $B/\sum_j s_j B$ is $A$-flat.

By a limit argument, we may assume that $A$ and $B$ are noetherian. In this case, \cite[Corollary to Theorem 22.5]{22} is applicable. Using this, we reduce the assertion to the case where $A$ is a field. Take $x \in Z \subset \text{Spec}(B)$. By the assumption that $Z$ is finite over $S$, the local ring $B_x/\sum_j s_j B_x$ is 0-dimensional, which implies that $s_1, \ldots, s_n$ is a system of parameters. As $B$ is regular, hence Cohen-Macaulay, $(s_j)_j$ is a $B$-regular sequence.

2. The proof goes similarly as in \cite{17}. Since we need a concrete description for $g^! \mathcal{O}_S$, we give a proof for this case. The general case can be deduced from this particular case, as is explained in \cite[5.4]{23}.

We define a canonical isomorphism $g^! \mathcal{O}_S \cong i^*(\omega_{X/S} \otimes_{\mathcal{O}_X} \det \mathcal{E})$. Consider a commutative diagram

$$
\begin{array}{ccc}
Z' & \xrightarrow{\Gamma_i} & Z \times_S X \\
\downarrow{\text{id}} & & \downarrow{pr_X} \\
Z & \xrightarrow{pr_Z} & S
\end{array}
$$

where the square is cartesian. As $pr_Z^* g^! \mathcal{O}_Z$ is $\Gamma_i^*$-acyclic, we can apply Lemma \ref{3.4.3} to $(\Gamma_i, pr_Z^* g^! \mathcal{O}_Z)$ to get an isomorphism

$$
R\text{Hom}_{\mathcal{O}_{Z \times X}}(\Gamma_i \mathcal{O}_Z, pr_Z^* g^! \mathcal{O}_S) \cong \Gamma_i (g^! \mathcal{O}_S \otimes_{\mathcal{O}_X} i^* \omega_{X/S}^{-1})[-n].
$$

Here we use a canonical isomorphism $\omega_{Z/Z \times Z} \cong i^* \omega_{X/S}^{-1}$. From this, we compute

$$
\Gamma_i (g^! \mathcal{O}_S) \cong R\text{Hom}_{\mathcal{O}_{Z \times Z \times X}}(\Gamma_i \mathcal{O}_Z, pr_Z^* g^! \mathcal{O}_S \otimes pr_X^* \omega_{X/S}^n) \cong R\text{Hom}_{\mathcal{O}_{Z \times Z \times Z}}(\Gamma_i \mathcal{O}_Z, pr_X^* \omega_{X/S}^n).
$$

Taking $pr_{X*}$, we have

$$
i_* g^! \mathcal{O}_S \cong pr_{X*} R\text{Hom}_{\mathcal{O}_{Z \times Z \times X}}(\Gamma_i \mathcal{O}_Z, pr_X^* \omega_{X/S}^n) \cong R\text{Hom}_{\mathcal{O}_X}(i_* \mathcal{O}_Z, \omega_{X/S}^n) \cong i_* i^*(\omega_{X/S} \otimes \det \mathcal{E})
$$

where the second isomorphism is the derived version of the adjunctions and the last one is due to Lemma \ref{3.4.1}.

3. The exterior powers of $\alpha$ induce an isomorphism of Koszul complexes \cite{3.1}. The assertion follows from the construction in 2 and Lemma \ref{3.4.2}.

We collect some of functorialities of the construction.

**Lemma 3.6.** Let $h: X' \to S$ be a morphism of schemes. Let $X', \mathcal{E}'$ be the pullbacks of $X, \mathcal{E}$ to $S'$. Let $s' \in \Gamma(X', \mathcal{E}')$ be the pullback of $s$.

1. The vanishing locus $Z'$ of $s'$ is the pullback of $Z$ by $X' \to X$.
2. Assume that $Z$ is finite over $S$. Then the coherent sheaf $g^! \mathcal{O}_S$ is $\mathcal{O}_Z$-flat.
3. Assume that $Z$ is finite over $S$. Let $g' : Z' \to S'$ be the structure map and $h_Z : Z' \to Z$ be the projection. Let $i' : Z' \to X'$ be the immersion. The base change map $h_Z^* g^! \to g'^* h^*$ fits into a commutative diagram

$$
\begin{array}{ccc}
    h_Z^* g^! & \longrightarrow & g'^* h^* \\
    \downarrow & & \downarrow \\
    h_Z^* (g^*(-) \otimes i^*(\omega_{X/S} \otimes \det \mathcal{E})) & \longrightarrow & g'^* h^*(-) \otimes i'^*(\omega_{X'/S'} \otimes \det \mathcal{E}').
\end{array}
$$

Here the vertical arrows are the ones constructed in Lemma 3.5.2.

4. Assume that $Z$ is finite over $S$. The diagram

$$
\begin{array}{ccc}
    h^* g_* i^*(\omega_{X/S} \otimes \det \mathcal{E}) & \stackrel{h^* \text{Res}_{S'[z]}}{\longrightarrow} & \mathcal{O}_{S'} \\
    \downarrow & & \downarrow \text{Res}_{S'[z]} \\
    g'_* h_Z^* i^*(\omega_{X/S} \otimes \det \mathcal{E}) & \longrightarrow & g'_* i'^*(\omega_{X'/S'} \otimes \det \mathcal{E}').
\end{array}
$$

is commutative.

**Proof.** The assertion 1 is clear from the definition.

To apply Lemma 3.4.3.(c) to the assertion 3, we need to prove 2, which is verified as follows. By a limit argument, we may assume that $S$ is noetherian. By the local criterion of flatness, we further reduce it to the case where $S$ is the spectrum of an algebraically closed field $k$. Shrinking $X$, we also assume that $Z$ is local. In this case, $\mathcal{O}_Z$ is a Gorenstein local ring. In particular, $\mathcal{O}_Z$ has a unique non-zero minimal ideal. Dually, $g^! \mathcal{O}_Z = \mathcal{H}om_k(\mathcal{O}_Z, k)$ has a unique $\mathcal{O}_Z$-linear surjection $g^! \mathcal{O}_Z \to k$ up to scalar multiplication. This implies that $g^! \mathcal{O}_Z \otimes_{\mathcal{O}_Z} k$ is one-dimensional. Hence there exists an $\mathcal{O}_Z$-linear surjection $\mathcal{O}_Z \to g^! \mathcal{O}_Z$, which must be an isomorphism as they have the same lengths. The assertion 2 follows.

We show 3. Lemma 3.4.3.(c) ensures the commutativity of the first isomorphism in (3.9) with base change. For the second one, it follows from the compatibility of the adjunction with base change, which can be verified directly in this case. The commutativity of the last one follows from the construction of (3.4).

The assertion 4 follows from 3 and the functoriality of the counits. □

**Lemma 3.7.** Let $f : X \to S'$ be a smooth morphism of schemes purely of relative dimension $n$ and let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank $n$. Let $s$ be a global section of $\mathcal{E}$ whose vanishing locus $Z$ is finite over $S'$.

Let $h : S' \to S$ be a finite étale morphism. Then, the residue symbol $\text{Res}_{S'[z]}$ equals to $\text{Tr}_{S'/S} \circ \text{Res}_{S'[z]}$.

**Proof.** Regard $X$ as a smooth scheme over $S$ and consider a commutative diagram

$$
\begin{array}{cccc}
    Z & \longrightarrow & Z' & \longrightarrow & X' & \longrightarrow & S' \\
    \downarrow \text{id} & & h_Z & & h_X & & h \\
    Z & \longrightarrow & X & \longrightarrow & S
\end{array}
$$
where the squares are cartesian. Write \( g: Z \to S \), \( g': Z' \to S' \). We have a commutative diagram

\[
\begin{array}{c}
h_{Z*}g^1\mathcal{O}_S^\text{Tr} \\
\downarrow \text{Tr} \\
h_{Z*}g^1\mathcal{O}_S^\text{Tr}
\end{array}
\begin{array}{c}
\cong \\
\downarrow \text{Tr} \\
h_{Z*}g^1\mathcal{O}_S^\text{Tr}
\end{array}
\begin{array}{c}
g^1\mathcal{O}_S \\
\downarrow \text{Tr} \\
g^1\mathcal{O}_S
\end{array}
\]

Applying Lemma 3.6, the statement follows by restricting \( g^1\mathcal{O}_{S'} \) to the open and closed subscheme \( Z \) of \( Z' \).

\[\square\]

**Lemma 3.8.** Let \( X_j \) \((j = 1, 2)\) be smooth \( S\)-schemes purely of relative dimensions \( n_j \) and let \( \mathcal{E}_j \) be locally free sheaves of rank \( n_j \) on \( X_j \). Let \( s_j \) be global sections of \( \mathcal{E}_j \) whose vanishing loci \( Z_j \) are finite over \( S \). Set \( g_j: Z_j \to S \) and \( i_j: Z_j \to X_j \). Let \( X := X_1 \times_X X_2 \) be the projections from the fibered product.

Let \( \mathcal{E} := \mathcal{E}_1 \boxtimes \mathcal{E}_2 = \text{pr}_1^*\mathcal{E}_1 \oplus \text{pr}_2^*\mathcal{E}_2 \) be the external product on \( X \). Write \( s \) for \( \text{pr}_1^*s_1 + \text{pr}_2^*s_2 \in \Gamma(X, \mathcal{E}) \). Then the vanishing locus of \( s \) equals to \( Z := Z_1 \times_S Z_2 \). We write \( i: Z \to X \) for the inclusion. Under the identification \( i^*(\omega_{X/S} \otimes \det \mathcal{E}) \cong \text{pr}_1^*i_1^*(\omega_{X_1/S} \otimes \det \mathcal{E}_1) \otimes \text{pr}_2^*i_2^*(\omega_{X_2/S} \otimes \det \mathcal{E}_2), \) \( \text{Res}_s \) corresponds to \( \text{Res}_{s_1} \otimes \text{Res}_{s_2} \).

**Proof.** The first assertion on the vanishing locus is clear, as the ideal sheaf of \( Z \) is generated by those of \( Z_1, Z_2 \).

Let \( g: Z \to S \). For the second assertion, we have a diagram

\[
\begin{array}{c}
g_*g^1\mathcal{O}_S \\
\downarrow c \\
g_*\text{pr}_1^*i_1^*(\omega_{X_1/S} \otimes \det \mathcal{E}_1) \otimes \mathcal{O}_Z \downarrow d \\
g_*\text{pr}_1^*i_1^*(\omega_{X_1/S} \otimes \det \mathcal{E}_1) \otimes \mathcal{O}_Z \downarrow e \\
g_*i^*(\omega_{X/S} \otimes \det \mathcal{E}) \\
\downarrow k \\
g_*\text{pr}_2^*i_2^*(\omega_{X_2/S} \otimes \det \mathcal{E}_2) \otimes \mathcal{O}_Z \downarrow h \\
g_*\text{pr}_2^*i_2^*(\omega_{X_2/S} \otimes \det \mathcal{E}_2) \otimes \mathcal{O}_Z \downarrow f \\
g_*\text{pr}_1^*i_1^*(\omega_{X_1/S} \otimes \det \mathcal{E}_1) \downarrow g \\
g_*\text{pr}_1^*i_1^*(\omega_{X_1/S} \otimes \det \mathcal{E}_1) \downarrow l \\
\mathcal{O}_S
\end{array}
\]

Here \( a, k \) are the canonical ones, \( b, h \) are the counits of adjunctions, \( c, d, e \) are the isomorphisms in (3.7). \( f \) is induced from \( \text{Res}_{s_2} \). \( l \) is \( \text{Res}_{s_1} \otimes \text{Res}_{s_2} \). It is straightforward to check that it is commutative and that the map \( g_*g^1\mathcal{O}_S \to \mathcal{O}_S \) from the upper left to the lower right is the counit.

\[\square\]

At the end of this preliminary subsection, we give a way to compute the isomorphism (3.7) in Lemma 3.6. To make the notation compatible with the one in the subsection 3.3, let us suppose that \( X \) is a smooth \( S\)-scheme purely of relative dimension \( n + 1 \). Let \( \mathcal{E} \) be a locally free \( \mathcal{O}_X\)-module of rank \( n + 1 \). Suppose that we are given a global section \( s \) of \( \mathcal{E} \) whose vanishing locus \( Z \) is finite over \( S \). Further assume that \( X \) is equipped with local parameters \( t_0, \ldots, t_n \).

In this setting, we define a map \( \Phi_s: g_*i^*(\omega_{X/S} \otimes \mathcal{O}_X \det \mathcal{E}) \to g_*g^1\mathcal{O}_S \) as follows. Let \( K^\bullet(\xi) \) be the Koszul complex defined by \( \xi_i = 1 \otimes t_i - t_i \otimes 1 \in \mathcal{O}_{Z \times_S X} \). We regard \( \mathcal{O}_{Z \times_S X} \) as a right \( \mathcal{O}_X\)-module by the right component of the tensor product. Take a morphism of
resolutions of $\mathcal{O}_Z$ by locally free $\mathcal{O}_X$-modules

\begin{align*}
K^\bullet(\xi) : \mathcal{O}_{Z \times_S X} & \xrightarrow{\eta} \mathcal{O}_{Z \times_S X}^{n+1} \xrightarrow{\eta} \cdots \xrightarrow{\eta} \mathcal{O}_{Z \times_S X} \\
(3.10) & \xrightarrow{\zeta} \mathcal{O}_X.
\end{align*}

Namely, it is a morphism of complexes such that the composition $\mathcal{O}_{Z \times_S X} \xrightarrow{\zeta} \mathcal{O}_X \to \mathcal{O}_Z$ coincides with the quotient map.

We identify the $\mathcal{O}_S$-sheaves $g_*i^*(\omega_{X/S} \otimes \mathcal{E})$, $g_*\text{Hom}_{\mathcal{O}_Z}(i^*(\mathcal{E})^\vee, i^*\omega_{X/S})$ under the isomorphism $g_*i^*(\omega_{X/S} \otimes \mathcal{E}) \ni x \otimes y \mapsto (x \mapsto \varphi(y)x)$. For an element $\bar{\alpha} \in g_*i^*(\omega_{X/S} \otimes \mathcal{E})$, take an $\mathcal{O}_X$-linear map $\alpha : (\mathcal{E})^\vee \to \omega_{X/S}$ which is a lift of $\bar{\alpha}$ in $g_*i^*(\omega_{X/S} \otimes \mathcal{E}) \cong g_*\text{Hom}_{\mathcal{O}_S}(i^*(\mathcal{E})^\vee, i^*\omega_{X/S})$.

Composing $\alpha$ and $\eta$ in (3.10), we get a map $\alpha \circ \eta : \mathcal{O}_{Z \times_S X} \to \omega_{X/S}$. Trivializing $\omega_{X/S}$ by $1 \mapsto \wedge_i dt_i$, we identify this map with an element in $\text{Hom}_{\mathcal{O}_X}(\mathcal{O}_{Z \times_S X}, \mathcal{O}_X)$, for which we use the same symbol $\alpha \circ \eta$.

Under the canonical isomorphism

$$\text{Hom}_{\mathcal{O}_X}(\mathcal{O}_{Z \times_S X}, \mathcal{O}_X) \cong \text{Hom}_{\mathcal{O}_S}(\mathcal{O}_Z, \mathcal{O}_S) \otimes_{\mathcal{O}_S} \mathcal{O}_X,$$

suppose that the map $\alpha \circ \eta$ in the left-hand side corresponds to $\sum_i \phi_i \otimes a_i$ in the right-hand side. Then we get an element $\Phi_s(\alpha) \in g_*g^!\mathcal{O}_S = \text{Hom}_{\mathcal{O}_S}(\mathcal{O}_Z, \mathcal{O}_S)$ by setting $\Phi_s(\alpha) = \sum_i \bar{a}_i \phi_i = \sum_i \phi_i(\bar{a}_i, -)$ where $\bar{a}_i$ is the image of $a_i$ in $\mathcal{O}_Z$.

**Lemma 3.9.** The map $\Phi_s : g_*i^*(\omega_{X/S} \otimes \mathcal{E}) \to g_*g^!\mathcal{O}_S$ constructed above coincides with $g_*\Psi^{-1}$ in (3.7).

**Proof.** By Lemma 3.4.3, the isomorphism (3.8) in Lemma 3.5 is identified with the one (3.11) constructed for $(\mathcal{E}, s) = (\mathcal{O}_{Z \times_S X}^{n+1}, (1 \otimes t_i - t_i \otimes 1)_i)$, under the identification $\det(\mathcal{E})^\vee = \mathcal{O}_{Z \times_S X} \to \omega_{X/S}, 1 \mapsto \wedge_i dt_i$. Then the construction of $\Phi_s$ is just a restatement of the construction in Lemma 3.5.2. \hfill \Box

### 3.2 Bilinear forms of isolated singularities

In this subsection, we construct non-degenerate symmetric bilinear forms from isolated singularities.

First let us give the notion of isolated singularities in a manner which suits our purposes.

**Definition 3.10.**

1. Let $T$ be a henselian trait. A function with an isolated singularity is a map of schemes $f : X \to T$ which can be obtained as the henselization of a $T$-scheme $X'$ flat of finite type at a closed point $x \in X'$ with the following properties. $X'$ is regular, $x$ maps to the closed point of $T$, and $X' \setminus \{x\}$ is smooth over $T$. In particular, $X$ is regular and henselian local.

2. A family of isolated singularities is a commutative diagram of schemes

\[ \begin{tikzcd}
Z' \arrow{r}{i} \arrow{d}{g} & X \arrow{r}{f} \arrow{d}{S} & C \\
\end{tikzcd} \]
where $X$ is smooth over $S$, $C$ is a smooth $S$-curve, and $Z$ is the singular locus of $f$ which is assumed to be quasi-finite over $S$.

Let $S$ be a scheme and $X$ be a smooth scheme purely of relative dimension $n$ over $S$. Consider a commutative diagram of $S$-schemes

$$
\begin{align*}
X & \xrightarrow{f} C \\
\downarrow & \downarrow \\
S & \quad S
\end{align*}
$$

where $C$ is a smooth $S$-curve. The map $f^*: f^*\Omega^1_{C/S} \to \Omega^1_{X/S}$ defines a section of the locally free $\mathcal{O}_X$-module $\mathcal{H}om(f^*\Omega^1_{C/S}, \Omega^1_{X/S})$, which we also denote by $f^*$. Recall that the singular locus $Z$ of $f$ is the closed subscheme of $X$ defined by vanishing of $f^*$.

We record Lemma 3.5 again in the form which fits into the current situation.

**Lemma 3.11.** Assume that $Z$ is quasi-finite over $S$.

1. $Z$ is flat of finite presentation over $S$.
2. Assume further that $Z$ is finite over $S$. Then, the $!$-pull-back $g^!$ on quasi-coherent sheaves is canonically isomorphic to $g^*(-) \otimes i^*(\omega^2_{X/S} \otimes f^*\omega^{\otimes(-n)})$.
3. Assume further that $Z$ is finite over $S$ and that $C$ admits an everywhere non-zero differential $\omega$. Then the isomorphism

$$
g^! \cong g^*(-) \otimes i^*\omega^2_{X/S}
$$

which is given by applying Lemma 3.5 to $(\mathcal{E}, s) = (\Omega^1_{X/S}, f^*\omega)$ coincides with the isomorphism in 2 followed by the trivialization $f^*\omega^2_{X/S} \cong \mathcal{O}_X, f^*\omega^{\otimes(-n)} \mapsto 1$.

**Proof.** For 1, 2, apply Lemma 3.5 to $(\mathcal{E}, s) = (\mathcal{H}om(f^*\Omega^1_{C/S}, \Omega^1_{X/S}), f^*)$. The assertion 3 follows from Lemma 3.5.3.

From now on, we assume that $Z$ is finite over $S$ and that an everywhere non-zero differential $\omega \in \Omega^1_{C/S}$ is given. In this case, the residue symbol

$$
\text{Res}[\frac{\omega^2_{X/S}}{f^*\omega}] : g_*i^*\omega^2_{X/S} \to \mathcal{O}_S
$$

is given by the composition of $g_*i^*\omega^2_{X/S} \cong g_*g^!\mathcal{O}_S \xrightarrow{\text{Tr}} \mathcal{O}_S$ where the first isomorphism is the one in Lemma 3.11.3.

Set $\varphi_f := g_*i^*\omega_{X/S}$. Let us write $B_{f,\omega}$ for the composition of

$$
\varphi_f \otimes_{\mathcal{O}_S} \varphi_f \to g_* (i^*\omega_{X/S} \otimes_{\mathcal{O}_S} i^*\omega_{X/S}) = g_* i^*\omega^2_{X/S} \xrightarrow{\text{Res}[\frac{\omega^2_{X/S}}{f^*\omega}]} \mathcal{O}_S.
$$

This defines an $\mathcal{O}_S$-bilinear form on the coherent sheaf $\varphi_f$ on $S$.

**Definition 3.12.** Assume that $Z$ is finite over $S$. The bilinear form attached to $(f, \omega)$ is the pair $(\varphi_f, B_{f,\omega})$ constructed above. We also write $(\varphi_{f,S}, B_{f,\omega,S})$ for it to specify the base scheme.

**Lemma 3.13.** Let the notation be as above. Assume that $Z$ is $S$-finite.
1. The $\mathcal{O}_S$-module $\varphi_f$ is locally free with the rank equal to $\text{rk}_{\mathcal{O}_S}\mathcal{O}_Z$.

2. $B_{f,\omega}$ is symmetric and $\mathcal{O}_Z$-equivariant, i.e. we have $B_{f,\omega}(\alpha x, y) = B_{f,\omega}(x, \alpha y)$ for $\alpha \in g_*\mathcal{O}_Z$ and $x, y \in \varphi_f$.

3. $B_{f,\omega}$ is non-degenerate, i.e. the induced map $\varphi_f \to \varphi_f^\vee$ is an isomorphism.

4. For a unit $\alpha \in \Gamma(C, \mathcal{O}_C)$, we have $B_{f,\omega}(-, -) = B_{f,\omega}(-, \alpha^n -)$, where $n$ is the relative dimension of $X/S$.

5. Let $h: S' \to S$ be a morphism of schemes. Let $f', \omega'$ be the pullbacks of $f, \omega$ by $h$. Then we have a canonical isomorphism

$$h^*(\varphi_f, B_{f,\omega}) \cong (\varphi_{f'}, B_{f',\omega'}).$$

Proof. 1. It follows since $i^*\omega_{X/S}$ is an invertible $\mathcal{O}_Z$-module and $g: Z \to S$ is finite flat of finite presentation (Lemma 3.11.1).

2. Note that, since $i^*\omega_{X/S}$ is of rank 1, the map $i^*\omega_{X/S} \otimes i^*\omega_{X/S} \to i^*\omega_{X/S} \otimes i^*\omega_{X/S}$ defined by switching the components $x \otimes y \mapsto y \otimes x$ equals to the identity. The assertion follows from this observation.

3. Let $x \in \varphi_f$ be an element in the kernel of $\varphi_f \to \varphi_f^\vee$. Let $\mathcal{M}' := \mathcal{O}_Z \cdot x$ be the quasi-coherent subsheaf of $i^*\omega_{X/S}$ generated by $x$ and let $\mathcal{M} := \mathcal{M}' \otimes_{\mathcal{O}_Z} i^*\omega_{X/S}$, which is a quasi-coherent subsheaf of $i^*\omega_{X/S}^\otimes \cong g'_*\mathcal{O}_S$. By the assumption, we have $\text{Res}_x[i^*\omega] = B_{f,\omega}(x, y) = 0$ for any element $y \in i^*\omega_{X/S}$. Since $x \otimes y$ generates $g_*(\mathcal{M} \otimes_{\mathcal{O}_Z} i^*\omega_{X/S})$, the composition

$$g_*(\mathcal{M} \otimes_{\mathcal{O}_Z} i^*\omega_{X/S}) \to g_*g'_*\mathcal{O}_S \xrightarrow{\text{Tr}} \mathcal{O}_S$$

is zero. Thus, by Lemma 3.2.3, we know that $\mathcal{M}' \otimes_{\mathcal{O}_Z} i^*\omega_{X/S}$ is zero, which implies $x = 0$ since $i^*\omega_{X/S}$ is invertible.

Therefore we know that the map $\varphi_f \to \varphi_f^\vee$ induced by $B_{f,\omega}$ is injective. Since this holds after any base change $S' \to S$, the map is an isomorphism.

4. It follows from Lemma 3.11.3.

5. It follows from Lemmas 3.6.3, 4. Lemma 3.14.

Consider a family of isolated singularities

$$\xymatrix{ X \ar[rr]^f \ar[dr] & & C \ar[dl] \\ & S' & }$$

with the singular locus $Z$ finite over $S'$. Let $h: S' \to S$ be a finite étale morphism. Then the bilinear form $(\varphi_{f, S}, B_{f,\omega, S})$ is canonically isomorphic to $(h_*\varphi_{f, S'}, \text{Tr}_{S'/S} \circ B_{f,\omega, S'})$.

Proof. It follows from Lemma 3.7.
Lemma 3.15. Let $X_j$ ($j = 1, 2$) be smooth $S$-schemes and let $f_j: X_j \to \mathbb{A}_S^1$ be $S$-morphisms whose singular loci $Z_j$ are finite over $S$. Let $X := X_1 \times_S X_2$ and let $f: X \to \mathbb{A}_S^1$ be the map $(x, y) \mapsto f_1(x) + f_2(y)$. Then the singular locus of $f$ is $Z_1 \times_S Z_2$. We have a canonical isomorphism

$$(\varphi_f, B_{f, \omega}) \cong (\varphi_{f_1}, B_{f_1, \omega}) \otimes_{\mathcal{O}_S} (\varphi_{f_2}, B_{f_2, \omega})$$

for an everywhere non-zero differential $\omega$.

Proof. This follows from Lemma 3.8. Indeed, by the projections $\text{pr}_j: X \to X_j$, we have canonically $\Omega^1_{X/S} \cong \text{pr}_1^*\Omega^1_{X_1/S} \oplus \text{pr}_2^*\Omega^1_{X_2/S}$. Under this identification, the pullback $f^*: \Omega^1_{k_S/S} \to \Omega^1_{X/S}$ equals to $f_1^* + f_2^*$. The assertion follows.

Next we recall the notion of Milnor number and its continuity. Let $S$ be a scheme and $X$ be a smooth $S$-scheme. Consider an $S$-morphism $f: X \to C$ to a smooth $S$-curve. Let $Z$ be an open and closed subscheme of the singular locus of $f$ which is finite over $S$. Recall then that $Z$ is finite locally free over $S$ (Lemma 3.11.1).

Definition 3.16. Let the notation be as above. We call the rank of $\mathcal{O}_Z$ as locally free $\mathcal{O}_S$-module the Milnor number of $f$ along $Z$ and write $\mu(f, Z)$ for it. When $Z$ stands for the underlying set of such an open and closed subscheme of the singular locus, we also write $\mu(f, Z)$ for it, as this convention does not lead to confusions.

When $S$ is the spectrum of a field and $Z$ consists of only one point $x$, we also write $\mu(f, x)$ for $\mu(f, Z)$.

Proposition 3.17. Consider a family of isolated singularities (Definition 3.10.2)

$$\xymatrix{ Z \ar^i[r] & X \ar^f[r] & C \ar^g[dl] \ar^S[dr] }$$

Let $\bar{z}$ be a geometric point of $Z$ and let $\bar{s}$ be the geometric point of $S$ induced from $\bar{z}$ by composing $g$. Let $\bar{t}$ be a geometric point of $S$ which specializes to $\bar{s}$. Then we have

$$\mu(f, \bar{z}) = \sum_{z' \in Z(\bar{z}) \times_S (\bar{s}, \bar{t})} \mu(f, z').$$

Let $\bar{s}$ be a geometric point of $S$, $f_\bar{s}$, etc., see the end of the introduction.

Proof. We may replace $S$ by the strict henselization $S(\bar{s})$. Then we replace $X$ by an open neighborhood around (the image of) $\bar{z}$ so that $Z$ equals to $Z(\bar{z})$. In this case, $\mathcal{O}_Z$ is a finite free $\mathcal{O}_S$-module and the left (resp. right) hand side in (3.11) is the rank of $\mathcal{O}_Z \otimes_{\mathcal{O}_S} k(\bar{s})$ (resp. $\mathcal{O}_Z \otimes_{\mathcal{O}_S} k(\bar{t})$).

For the notation $S(\bar{s})$, $f_\bar{s}$, etc., see the end of the introduction.
3.3 Ordinary quadratic singularities

In this subsection, we recall the notion of ordinary quadratic singularities and compute various invariants of such singularities, which is necessary for the proofs of the main theorems.

Definition 3.18. ([8 1.2])

1. Let $Y$ be a scheme of finite type over a field $k$. Let $y \in Y$ be a closed point.

   (a) When $k$ is algebraically closed, we say that $y$ is an ordinary (resp. non-degenerate) quadratic point if the completion $\widehat{O}_{Y,y}$ is $k$-isomorphic to $k[[x_0, \ldots, x_n]]/(f)$ where $f$ starts in degree 2 and the homogeneous part of degree 2 is an ordinary (resp. non-degenerate) quadratic form ([7, 1.1]).

   (b) In general, we say that $y$ is an ordinary (resp. non-degenerate) quadratic point if the following holds: Let $Y \times_k \bar{k}$ be the base change to $\bar{k}$, where $\bar{k}$ is an algebraic closure of $k$. For some (hence all) $\bar{y} \in Y \times_k \bar{k}$ which projects to $y$, $\bar{y}$ is an ordinary (resp. non-degenerate) quadratic point in the sense of (a).

2. Let $f : X \to T$ be a flat morphism of schemes of finite type. We say that a point $x \in X$ is an ordinary (resp. non-degenerate) quadratic point if so is $x$ in the fiber $X \times_T f(x)$.

Let $y \in Y$ be an ordinary quadratic point on a $k$-scheme $Y$ of finite type. Then $y$ is non-degenerate if and only if $\text{char}(k)$ is odd or $Y$ has odd dimension at $y$ [8 1.2.2].

First we recall results on deformations of ordinary quadratic singularities ([8]). Let $(R, \mathfrak{m}_R)$ be a henselian local ring with the residue field $k$ and $S = \text{Spec}(R)$ be its affine spectrum. We write $s$ for the closed point. Let $X$ be a smooth $S$-scheme of relative dimension $n + 1$ and let $f : X \to C$ be an $S$-morphism to a smooth $S$-curve $C$ with the singular locus $Z$ finite over $S$. We assume that $Z$ is local. Thus $Z$ has only one closed point, which we denote by $x$.

To state the lemma, let us introduce the following notation. Let $(A, \mathfrak{m}_A)$ be a henselian local ring with a local homomorphism $R \to A$. An element $a \in \mathfrak{m}_A$ gives a unique morphism of local $R$-algebras $R[t] \to A$, $t \mapsto a$, where $R[t]$ is the henselization of $R[t]$ at $(\mathfrak{m}_R, t)$. We write $f(a)$ for the image of $f \in R\{t\}$ under this map.

Lemma 3.19. ([8 Proposition 1.3.1]) Let the notation be as above. We assume that $k(x)$ is isomorphic to $k$.

1. Suppose that $k$ is of odd characteristic or that $n + 1$ is even. Then there is an $R$-isomorphism $R\{t\} \cong O_{C,(f(x))}$ of henselizations such that the henselization of $O_{X,x}$ is isomorphic to

   $$R\{t, t_0, \ldots, t_n\}/(Q-t)$$

as algebras over $O_{C,(f(x))} = R\{t\}$. Here $Q \in R[t_0, \ldots, t_n]$ is a non-degenerate quadratic form. The Milnor number $\mu(f_s, x)$ of the closed fiber $f_s : X_s \to C_s$ equals to 1. The singular locus $Z$ equals to the closed subscheme of $X_{(x)}$ defined by $t_0 = \cdots = t_n = 0$. In particular, $Z \to S$ is isomorphic.
2. Suppose that $k$ is of characteristic 2 and that $n+1$ is odd. Then, if we choose appropriately an $R$-isomorphism $R\{t\} \cong \mathcal{O}_{\mathbb{C}, \{f(x)\}}$, an element $b(t) \in R\{t\}$ in the maximal ideal whose reduction $\bar{b}(t)$ in $k\{t\}$ is non-zero, and a non-degenerate quadratic form $Q \in R[t_1, \ldots, t_n]$ in variables $t_1, \ldots, t_n$, the henselization $\mathcal{O}_{X, (x)}$ is $R\{t\}$-isomorphic to

\[(3.12) \quad \mathcal{O}_{X, (x)} \cong R\{t, t_0, t_1, \ldots, t_n\}/(Q + t_0^2 + b(t)t_0 - t).\]

The Milnor number $\mu(f, x)$ of the special fiber equals to $2\text{ord}_t(\bar{b}(t))$, twice the normalized valuation of the reduction $\bar{b}(t)$.

Further if $R$ is over $\mathbb{F}_2$ and if $\bar{b}(t) \in k\{t\}$ is a uniformizer, the map $Z \to S$ is finite flat of finite presentation of degree 2 and is a universal homeomorphism.

3. In the situation 2, the $R\{t\}$-algebra (3.12) is isomorphic to the henselization of

\[(3.13) \quad R\{t_1, \ldots, t_n\} \otimes_R R\{u\}\]

at $(m_R, u, t_1, \ldots, t_n)$ which is regarded as an $R\{t\}$-algebra via the map

$t \mapsto Q + t := Q \otimes 1 + 1 \otimes t.$

Here $\bar{t}$ is an element of $R\{u\}$ which satisfies the relation $u^2 + b(\bar{t})u - \bar{t} = 0$.

**Proof.** For 1, the assertion on the presentation of $\mathcal{O}_{X, (x)}$ is given in §1.3.1. Note that $b$ in §1.3.1(i) can be taken as $b = t$ as in 1 because the fiber $X_s$ is regular. For 2, the presentation is given in §1.3.1(ii). We can take $b, c$ in loc. cit. as in 2 because $X_s$ is regular and $x$ is an isolated singular point. The computation on the Milnor numbers are given in §1.13 or can be verified directly from the definition in this case.

We verify the assertions on $Z$ in each case 1 and 2. For 1, the assertion follows since $Q$ is non-degenerate, hence the coefficients of $dQ$ span the ideal $(t_0, \ldots, t_n)$. For 2, assume that $R$ is an $\mathbb{F}_2$-algebra. Then we have

\[d(Q + t_0^2 + b(t)t_0 - t) = dQ + b(t)dt_0 - (1 - b'(t)t_0)dt\]

where $b'(t) = \frac{d b}{d t}$. Hence we have

\[\mathcal{O}_Z = \mathcal{O}_{X, (x)}/(t_1, \ldots, t_n, b(t)) \cong R\{t, t_0\}/(t_0^2 + b(t)t_0 - t, b(t)) \cong R\{t_0\}/(b(t_0^2)).\]

As $R$ is of characteristic 2, we have $b(t_0^2) = b(t_0)^2$. By the assumption that $\bar{b}(t)$ is a uniformizer of $k\{t\}$, the map $R\{b\} \to R\{t_0\}$ sending $b$ to $t_0$ is an isomorphism. The assertion follows.

We show 3. Let $A$ be the henselization of (3.13) at $(m_R, t_1, \ldots, t_n, u)$. We show that the polynomial $Q(x) + x_0^2 + b(Q(t) + \bar{t})x_0 - (Q(t) + \bar{t}) \in A[x_0, \ldots, x_n]$ has a solution $(u', t'_1, \ldots, t'_n) \in A^{n+1}$ which equals to $(u, t_1, \ldots, t_n) \mod (u, t_1, \ldots, t_n)^2 + m_R A$. Such a solution gives an étale map $A \to R\{t, x_0, \ldots, x_n\}/(Q(x) + x_0^2 + b(t)x_0 - t)$ of henselian $R\{t\}$-algebras, which is an isomorphism.

We find such a solution applying [11 5.11] to $y^e = (u, t_1, \ldots, t_n), a = Au$, and $f = Q(x) + x_0^2 + b(Q(t) + \bar{t})x_0 - (Q(t) + \bar{t})$ with the notation given there. We have

\[Q(t) + u^2 + b(Q(t) + \bar{t})u - (Q(t) + \bar{t}) = u^2 + b(Q(t) + \bar{t})u - \bar{t} \equiv u^2 + b(\bar{t})u - \bar{t} = 0 \mod (t_1, \ldots, t_n)^2 u.\]
As $Q$ is non-degenerate, the ideal $\Delta$ in \textit{loc. cit.} equals to $(t_1, \ldots, t_n, 2u + b(Q(t) + \tilde{t}))$. Thus, by [1, 5.11], we find a solution $(u', t_1', \ldots, t_n')$ as desired. 

We compute the bilinear form $(\varphi_f, B_{f,\omega})$ for certain quadratic singularities in mixed characteristic.

\textbf{Proposition 3.20.} 1. Let $R$ be a ring and $Q \in R[t_0, \ldots, t_n]$ be a non-degenerate quadratic form. Let

$$f : X := \text{Spec}(R[t_0, \ldots, t_n]) \to \mathbb{A}^1_R = \text{Spec}(R[t])$$

be the $R$-morphism defined by $t \mapsto Q$. Then the singular locus $Z$ of $f$ equals to \{t_0, \ldots, t_n = 0\} \subset X$ as a subscheme. The bilinear form $(\varphi_f, B_{f,\omega})$ for a differential $\omega$ on $\mathbb{A}^1_R$ which is non-zero along \{t = 0\} is isomorphic to $(R, B)$ where $B : R \otimes R \to R$ is defined by $1 \otimes 1 \to \frac{1}{\text{disc} Q} \cdot (\frac{\omega}{dt}|_{t=0})^{n+1}$.

2. Let $k$ be a perfect field of characteristic 2. Let $R = W_{m+1}(k) = W(k)/p^{m+1}$ be the ring of Witt vectors of length $m + 1$. Let $A$ be an étale $R[t]$-algebra such that $A/(t)$ is isomorphic to $R$. Let $Q \in R[t_1, \ldots, t_n]$ be a non-degenerate quadratic form (hence $n$ is assumed even) and $b \in A$ be an element such that its image in $A/p$ is non-zero and is contained in the ideal $t \cdot A/p$. Consider

$$f : X' := \text{Spec}(A[t_0, t_1, \ldots, t_n]/(Q + t_0^2 + bt_0 - t)) \to \mathbb{A}^1_R = \text{Spec}(R[t]).$$

Let $x \in X'$ be the $k$-rational point defined by $\{p, t_0, \ldots, t_n = 0\}$. Then $X'$ is $R$-smooth around $x$, and $x$ is an isolated singular point with respect to $f$.

Let $X$ be an open neighborhood of $x$ in $X'$ so that $X$ is $R$-smooth and the singular locus $Z$ of $f$ consists of only $x$. Further assume that the Milnor number $\mu(f \otimes_R k, x)$ equals to 2. Then the discriminant of the bilinear form $(\varphi_f, B_{f,dt})$ equals to $-1$ modulo $(W_m(k)^*)^2$.

In the situation 2, we will prove the equality $\text{disc}B_{f,dt} \equiv (-1)^{(n+1)\mu(f,x)/2}$ without the assumption on the Milnor number in Proposition 4.14.

\textbf{Proof.} Put $S = \text{Spec}(R)$. We apply Lemma 4.9 to $(\mathcal{E}, s) = (\Omega^1_{X/S}, f^*\omega)$ in each case. We follow the notations given there. In particular, we write $\xi_i$ for $1 \otimes t_i - t_i \otimes 1 \in \mathcal{O}_{Z \times_S X}$.

1. In this case, $Z$ equals to the 0-section $S \hookrightarrow X = \mathbb{A}^{n+1}_S$ and $\xi_i = 1 \otimes t_i$ in $\mathcal{O}_{Z \times_S X}$. We trivialize $\Omega^1_{X/S}$ by $dt_0, \ldots, dt_n$ and identify the complex (3.1) defined for $s = f^*dt$ with the Koszul complex $K^*(f)$ defined by $f_0, \ldots, f_n$ where we set $f^*dt = \sum f_i dt_i$.

Let $F : \mathcal{O}^{n+1}_X \to \mathcal{O}^{n+1}_X$ be the $\mathcal{O}_X$-linear map defined by the hessian $(\frac{\partial^2 Q}{\partial t_i \partial t_j})_{i,j}$. Its exterior powers give an isomorphism $K^*(F)^* : K^*(f) \to K^*(f)$. The map $\frac{1}{\text{disc} Q} K^*(F)$ then gives a morphism of resolutions of $\mathcal{O}_Z$. Using this, we can identify the residue symbol $\text{Res}_{\mathcal{O}_S}^{\mathcal{O}_S}[\frac{1}{\text{disc} Q} K^*(F)]$ with $\mathcal{O}_S \xrightarrow{\text{min}} \mathcal{O}_S$. The assertion for $\omega$ comes from that for $dt$ and Lemma \textbf{3.13} 4.

2. First we treat the case $n = 0$. Hence we assume that $X' = \text{Spec}(A[u]/(u^2 + bu - t))$. We compute

$$d(u^2 + bu - t) = (2u + b)du - (1 - ub')dt$$
where \( b' = \frac{db}{dt} \). As \( u, t \) are local parameters on \( \text{Spec}(A[u]) \), \( X' \) is smooth at \( x \). Around \( x \in X' \), we have

\[
(3.14) \quad f^*dt = \frac{2u + b}{1 - ub'} du.
\]

Hence, in the special fiber, we have \( f^*dt = \frac{b}{1 - ub'} du \). Thus there exists an open neighborhood \( X \) around \( x \) such that \( f|_X \) has only one singular point at \( x \).

In the special fiber, \( X_s \to \mathbb{A}^1_k \) is totally wildly ramified over \( \{ t = 0 \} \) and the kernel of the map \( \mathcal{O}_{X_s,x} \to \mathcal{O}_{Z_s,x} \) is generated by \( b \) by \((3.14)\). Thus \( \mathcal{O}_Z \otimes_R k \) has a basis \( 1, u, \ldots, u^{2\text{ord}_t(b) - 1} \) as a \( k \)-vector space. By Nakayama’s lemma, \( \mathcal{O}_Z \) itself has a basis \( 1, u, \ldots, u^{2\text{ord}_t(b) - 1} \) as a free \( R \)-module.

Assume further that \( \mu(f_s, x) = 2 \). We identify \( \mathcal{O}_Z \) with the free \( R \)-module \( R \oplus Ru \). In order to compute \( (\varphi_f, B_{f,dt}) \), we express several elements in \( \mathcal{O}_Z \) as linear combinations of \( 1 \) and \( u \), which are summarized from 1 to 5 below. As \( b \) is a uniformizer of \( k\{t\} \), the map \( W_2(k)[b]/(b^2) \to R\{t\}/(4,b^2) \) is an isomorphism. Thus we find a relation of the form

\[
(3.15) \quad t \equiv 2c + ab \mod (4,b^2)
\]

where \( c \in R \) and \( a \in R^\times \). Up to replacing \( \mathbb{A}^1_k \) by an étale neighborhood \( C \) around 0, we assume that we have a relation as \((3.15)\) in \( \mathcal{O}_C \).

We express \( u^2 \in \mathcal{O}_Z \) as the form \( u^2 = m + nu \) where \( m, n \in R \). Then the ratio \( \frac{u^2 - m - nu}{2u + b} \) is defined around \( x \) in \( X \). We compute various elements in \( \mathcal{O}_Z/4\mathcal{O}_Z \) as follows. Here \( \equiv \) indicates the congruence modulo 4.

1. \( b = -2u \) in \( \mathcal{O}_Z \).
2. \( t \equiv 2c - 2au \) in \( \mathcal{O}_Z/4\mathcal{O}_Z \).
3. \( b' \equiv \frac{1}{a} \) in \( \mathcal{O}_Z/4\mathcal{O}_Z \).
4. \( m \equiv 2c, n \equiv 2a \) in \( R/4R \).
5. \( \frac{u^2 - m - nu}{2u + b} \equiv -a + u \) in \( \mathcal{O}_Z/4\mathcal{O}_Z \).

The first one follows since \( 2u + b \) is a generator of the ideal sheaf defining \( \mathcal{O}_Z \). The second one follows from (1) and \((3.15)\). Differentiating \((3.15)\) by \( t \), we have

\[ 1 \equiv ab' \mod (4,2bb',b^2). \]

As we have \( b = -2u \) in \( \mathcal{O}_Z \), (3) holds in \( \mathcal{O}_Z/4\mathcal{O}_Z \).

We have

\[
(u - a)(2u + b) - u^2 = u^2 + bu - 2au - ab = t - 2au - ab \equiv 2c - 2au \mod (4,b^2)
\]

by \((3.15)\). Since we have \((4,b^2) = (4,(2u + b)^2) \), the assertions (4) and (5) follow.

To compute \((\varphi_f, B_{f,dt})\), we construct a morphism of complexes as \((3.10)\). Define \( \zeta: \mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{O}_X \to \mathcal{O}_X \) to be the \( \mathcal{O}_X \)-linear map sending \( 1 \otimes 1 \mapsto 1, u \otimes 1 \mapsto u \). We
need to find an $O_X$-linear map $\eta: O_Z \otimes_{O_S} O_X \to O_X$ which makes the diagram

$$
\begin{array}{c}
O_Z \otimes_{O_S} O_X \\
\downarrow \eta \\
O_X
\end{array}
\begin{array}{c}
\cong
\downarrow \zeta \\
\cong
\begin{array}{c}
O_Z \otimes_{O_S} O_X \\
\downarrow 2u+b \\
n_{1-u}b
\end{array}
\end{array}
\begin{array}{c}
O_X
\end{array}
$$

commutative. The image of $1 \otimes 1$ by $\nu: O_Z \otimes_{O_S} O_X \to O_Z \otimes_{O_S} O_X$ is $\zeta$ which corresponds to $O_X$ is zero. Thus we can set $\eta(1 \otimes 1) = 0$. The image $\nu(u \otimes 1)$ equals to $\zeta(u \otimes u - (m + nu) \otimes 1) = u^2 - m - nu$. As this can be divided by $2u + b$, we can define $\eta(u \otimes 1)$ to be $\frac{u^2 - m - nu}{2u + b}(1 - ub')$. Set $\beta := \frac{u^2 - m - nu}{2u + b}(1 - ub')$. Under the identification

$$
\text{Hom}_{O_X}(O_Z \times O_X, O_X) \cong \text{Hom}_{O_S}(O_Z, O_S) \otimes_{O_S} O_X,
$$

the map $\alpha \eta$ with $\alpha \in O_X$ corresponds to

$$u^* \otimes \beta \alpha$$

where $u^*: O_Z \to O_S$ maps $1 \otimes 1 \mapsto 0$, $u \otimes 1 \mapsto 1$. Therefore, we can identify the bilinear form $B_{f,dt}$ with

$$B: O_Z \times O_Z \to O_S, (r_1, r_2) \mapsto u^*(r_1 r_2).$$

From now on, we compute everything modulo 4. By 3, 4, 5 above, we have

$$\beta \equiv (-a + u)(1 - \frac{u}{a}) = 2u - a - \frac{u^2}{a} \equiv 2u - a - 2c \mod 4.$$ 

Thus we have

$$B(r_1, r_2) \equiv -(a + 2c)u^*(r_1 r_2).$$

Therefore, we have $B(1, 1) \equiv 0$, $B(u, u) \in 2O_S$. Hence we have

$$\text{disc} B_{f,dt} = B(1, 1)B(u, u) - B(1, u)^2 \equiv -B(1, u)^2 \mod 8.$$ 

This completes the proof, as $B$ is non-degenerate and $1 + 8W_m(k)$ is contained in $(W_m(k)^x)^2$ by Lemma \ref{2.12}. For general $n \geq 0$, we reduce it to the case $n = 0$. By Lemma \ref{3.19}, the henselization $O_{X_t}(x)$ is $R\{t\}$-isomorphic to $R\{t_1, \ldots, t_n\} \otimes_R R\{u\}$ by the map sending $t \mapsto Q \otimes 1 + 1 \otimes \bar{t}$ where $\bar{t} \in R\{u\}$ satisfies the relation $u^2 + b(\bar{t})u - \bar{t} = 0$. As the Milnor number of $\text{Spec}(A[u]/(u^2 + bu - t)) \to \mathbf{A}^1_k$ at $(t, u) = (0, 0)$ is 2, Lemma \ref{3.15} and 1 imply the assertion.

\begin{corollary}
In the situation in 1 of Proposition \ref{3.20}, assume that $R = W_3(k) = W(k)/8W(k)$ for a perfect field $k$ of characteristic 2. Then $(-1)^{\frac{n+1}{2}}$ times the discriminant of $B_{f,dt}$ is contained in the image of $1 + 4W_3(k) \to W_3(k)^x/(W_3(k)^x)^2$ and equals to $1 + 4[\text{Arf}(Q \otimes_R k)]$ where $\text{Arf}(\cdot)$ denotes the Arf invariant of a non-degenerate quadratic form.
\end{corollary}
Proof. The discriminant of $B_{f,dt}$ equals to $\frac{1}{\text{disc}(Q)} \equiv \text{disc}(Q) \mod (W_3(k)^\times)^2$. Then the assertion follows from [3, 1.4]. Indeed, let $Q'$ be a quadratic form on a free $W(k)$-module which is a lift of $Q$. It is proved in loc. cit. that the discriminant of $Q'$ equals to $(-1)^{\frac{p-1}{2}} (1 + 4 \text{Arf}(Q' \otimes_{W(k)} k))$ in $W(k)^\times/(W(k)^\times)^2$. Since we have $W(k)^\times/(W(k)^\times)^2 \cong W_3(k)^\times/(W_3(k)^\times)^2$ by Lemma 2.12.1, the assertion is verified.

We give a corresponding result of Proposition 3.20 for local epsilon factor. The following is used in the proof of Theorem 5.4.

Let $p$ be a prime number. Fix a prime number $\ell \neq p$ and take an algebraic closure $\overline{\mathbb{Q}_\ell}$ of the $\ell$-adic field $\mathbb{Q}_\ell$. We fix a non-trivial additive character $\psi: \mathbb{F}_p \to \overline{\mathbb{Q}_\ell}^\times$. For a finite extension $k/\mathbb{F}_p$, we write $\psi_k$ for the composition $k \xrightarrow{\text{Tr}_{k/\mathbb{F}_p}} \mathbb{F}_p \xrightarrow{\psi} \overline{\mathbb{Q}_\ell}^\times$. We use these characters to define local epsilon factors in equal-characteristic case, as explained in [20, (3.1.5.4)]. We follow the notation for local epsilon factor in loc. cit.

Let $f: X \to \mathbb{A}_k^1$ be a $k$-morphism of smooth $k$-schemes with a $k$-rational isolated singular point $x \in X$. We write $R\Phi_f(\mathbb{Q}_\ell)_x$ for the vanishing cycles complex supported at $x$. This is a bounded complex of finite dimensional $\mathbb{Q}_\ell$-vector spaces with $G_\eta$-action, where $G_\eta$ is the absolute Galois group of generic point $\eta$ of the henselization $\mathbb{A}_k^1_{\eta,(f(\eta))}$. Hence the local epsilon factor $\varepsilon_0(\mathbb{A}_k^1_{\kappa,(f(x))}, R\Phi_f(\mathbb{Q}_\ell)_x, dt)$ is defined.

Lemma 3.22. Let $k$ be a finite extension of $\mathbb{F}_p$ with $q$ elements.

1. Assume that $p$ is odd. For $a \in k^\times$, define

$$f: \mathbb{A}_k^1 \to \mathbb{A}_k^1$$

by $t \mapsto at^2$. Then we have

$$\varepsilon_0(\mathbb{A}_k^1_{(0)}, R\Phi_f(\mathbb{Q}_\ell)_x, dt) = (\frac{-a}{k}) \sum_{x \in k} \psi_k(x^2).$$

2. Assume that $p = 2$. Let $a \in k$ and consider a map

$$f: \mathbb{A}_k^2 \to \mathbb{A}_k^1$$

defined by $(x, y) \mapsto x^2 + xy + ay^2$. Then the product

$$-\varepsilon_0(\mathbb{A}_k^1_{(0)}, R\Phi_f(\mathbb{Q}_\ell)_x, dt)q$$

is $\pm 1$. This is $1$ if and only if $a \in \varphi(k)$.

3. Assume that $p = 2$. Let $t: C \to \mathbb{A}_k^1$ be an étale morphism of smooth $k$-curves such that $C \times_{\mathbb{A}_k^1} 0$ equals to a $k$-rational point, which we also denote by 0. Let $b \in \Gamma(C, \mathcal{O}_C)$ be an element which gives a uniformizer at 0. Let

$$f: X := \text{Spec}(\mathcal{O}_C[u]/(u^2 + bu - t)) \to C.$$

Then $X$ is smooth around $\{u = 0\}$ and we have

$$\varepsilon_0(C(0), R\Phi_f(\mathbb{Q}_\ell)_x, dt) = q.$$
Proof. Although they are well-known, we include a proof for completeness. Set $F := k((t))$ and let $\mathcal{O}_F$ be the ring of integers in $F$.

1. The quadratic extension $E := k((t))[u]/(u^2 - at)$ of $F$ corresponds to a tame character $\chi$ of order 2 of the absolute Galois group of $F$. Then the vanishing cycles complex $R\Phi_f(\mathcal{O}_\ell)_0$ is isomorphic to $\chi$. We have (cf. [6, 5.10])

$$
\varepsilon_0(\mathcal{A}_{k(0)}^1, \chi, dt) = \chi(t) \sum_{x \in k^*} \chi(x)\psi_k(\text{Res}(x \frac{dt}{t})))
= \chi(-a) \sum_{x \in k^*} \chi(x)\psi_k(x).
$$

As $E/F$ is a totally tamely ramified quadratic extension, $\chi|_{k^*}$ is the unique non-trivial character of order 2, i.e. it equals to the Legendre symbol $(\frac{x}{k})$. The assertion follows.

2. Let $X := \text{Proj}(\mathcal{O}_F[x, y, z]/(x^2 + xy + ay^2 - tz^2))$. This is regarded as a closed subscheme of $\mathbb{P}^2_{\mathbb{O}_\ell}$. By the Jacobian criterion, the structure map $f : X \to \text{Spec}(\mathcal{O}_F)$ is smooth except at the point $(x : y : z) = (0 : 0 : 1)$ in the special fiber. By the local acyclicity of smooth morphism and the proper base change theorem, we have a distinguished triangle

$$R\Gamma(X_s, \mathcal{Q}_\ell) \to R\Gamma(X_{\bar{\eta}}, \mathcal{Q}_\ell) \to R\Phi_f(\mathcal{O}_\ell)_0 \to$$

of $G_{\mathcal{F}}$-representations. As $X_{\bar{\eta}}$ is isomorphic to a projective line, the cohomology groups are $H^0(X_{\bar{\eta}}, \mathcal{Q}_\ell) = \mathcal{Q}_\ell$, $H^2(X_{\bar{\eta}}, \mathcal{Q}_\ell) \cong \mathcal{Q}_\ell(-1)$ and 0 in the other degrees. The geometric special fiber $X_s$ is the union of two projective lines glued at the origins. The action of $G_k$ on the irreducible components is non-trivial if and only if the special fiber $X_s$ is irreducible, which happens if and only if $a$ is non-zero in $k/\wp(k)$. Let $\chi_a$ be the corresponding character of $G_k$ via Artin–Schreier theory. We have $R\Phi_f(\mathcal{O}_\ell)_0 \cong \chi_a \otimes \mathcal{Q}_\ell(-1)[-1]$. Since we have $R\Phi_f(\mathcal{O}_\ell)_0 \cong R\Phi_f(\mathcal{O}_\ell)_0$, the assertion follows in this case.

3. We are treating the quadratic extension $E := F[u]/(u^2 + bu - t)$. Let $\chi$ be the character of order 2 corresponding to it. The Artin conductor of $\chi$ is 2. Take $\gamma \in F$ with $\text{ord}_F(\gamma) = 2$. Let $dx$ be the Haar measure on the additive topological group $F$ normalized as $\mathcal{O}_F$ has volume 1. We have (cf. [6, 5.8])

$$
\varepsilon_0(\mathcal{O}_F, \chi, dt) = \int_{\gamma^{-1} \mathcal{O}_F^x} \chi(x)\psi_k(\text{Res}(xdt)) dx
= q^2 \int_{\mathcal{O}_F^x} \chi(\gamma x)\psi_k(\text{Res}(\frac{xdt}{\gamma})) dx
= \chi(\gamma)q^2 \sum_{a \in k^*} \sum_{b \in (1+\gamma(t))/(1+\gamma(t)^2)} \int_{1+\gamma(t)^2} \chi(abx)\psi_k(\text{Res}(\frac{abxdt}{\gamma})) dx
= \chi(\gamma)q^2 \sum_{a, b} \chi(ab) \int_{(t)^2} \psi_k(\text{Res}(\frac{abdt}{\gamma}))\psi_k(\text{Res}(\frac{abxdt}{\gamma})) dx.
$$

Note that the term $\psi_k(\text{Res}(\frac{abxdt}{\gamma})) = 1$ since $x \in (t)^2$. We proceed

$$
\chi(\gamma)q^2 \sum_{a, b} \chi(ab) \int_{(t)^2} \psi_k(\text{Res}(\frac{abdt}{\gamma}))\psi_k(\text{Res}(\frac{abxdt}{\gamma})) dx = \chi(\gamma) \sum_{a, b} \chi(ab)\psi_k(\text{Res}(\frac{abdt}{\gamma})).
$$

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Take \( \alpha \in \kx \) so that \( \chi(1 + z) = \psi_k(\text{Res}(\alpha \frac{zdt}{\gamma})) \) for \( z \in (t)/(t)^2 \). We have

\[
\chi(\gamma) \sum_{a,b} \chi(ab) \psi_k(\text{Res}(\frac{abdt}{\gamma})) = \chi(\gamma) \sum_{a \in k^\times, z \in (t)/(t)^2} \chi(a(1 + z)) \psi_k(\text{Res}(\frac{a(1 + z)dt}{\gamma}))
\]

\[
= \chi(\gamma) \sum_{a \in k^\times} \chi(a) \psi_k(\text{Res}(\frac{adt}{\gamma})) \sum_{z \in (t)/(t)^2} \psi_k(\text{Res}(\frac{(a + z)dt}{\gamma})).
\]

The last sum \( \sum_{z \in (t)/(t)^2} \psi_k(\text{Res}(\frac{(a + z)dt}{\gamma})) \) is 0 unless \( a = \alpha \). Thus we obtain

\[
\varepsilon_0(O_F, \chi, dt) = \chi(\gamma) \chi(\alpha) \psi_k(\text{Res}(\frac{adt}{\gamma})) q.
\]

From now on, we take \( \gamma \) to be \( t^2 \). For this choice, the terms \( \chi(\gamma), \psi_k(\text{Res}(\frac{adt}{\gamma})) \) are trivial. The term \( \chi(\alpha) \) is also trivial since the orders of elements in \( k^\times \) are odd. Thus the assertion follows.

### 3.4 Deforming isolated singularities

Here we explain that any function with an isolated singularity of equal characteristic can be embedded into a family of isolated singularities which generically contains ordinary quadratic singularities (cf. \([9, 2.5]\)).

**Lemma 3.23.** Let \( k \) be a field. Let \( f : X \to \text{Spec}(O) \) be a function with an isolated singularity (Definition 3.10.1) such that \( O \) is isomorphic to the henselization \( \mathcal{O}_{k^1,0} \). Let \( x \in X \) be the singular point. Assume that the residue field \( k(x) \) is isomorphic to \( k \). Then there exists a family of isolated singularities (Definition 3.10.2)

\[
Y \xrightarrow{j} \mathbb{A}^1_S \xleftarrow{\pi} S
\]

such that

1. \( S \) is a smooth connected \( k \)-curve. The singular locus \( Z \) is finite over \( S \).
2. There exists a \( k \)-rational point \( s \in S(k) \) such that
   
   (a) \( Z \times_S s \) consists of only one point \( x_0 \). The point \( x_0 \) maps to the origin of \( \mathbb{A}^1_{k(s)} = \mathbb{A}^1_S \times_S s \).
   
   (b) There is a \( k \)-homomorphism \( \mathcal{O}_{k^1,(s)} \to \mathcal{O} \) which preserves uniformizers such that the henselization of \( Y \times_{\mathbb{A}^1_S} \mathcal{O} \) at \( x_0 \) is \( \mathcal{O} \)-isomorphic to \( X \).
3. There exists an open dense subset \( U \subset S \) which has a section \( \iota : U \to Z \) such that, for any \( u \in U \), \( \iota(u) \) is an ordinary quadratic point of \( \hat{f} \) with Milnor number 1 or 2.
4. \( Y, S \) are affine.
Proof. Let \( \dim X = n + 1 \). Let \( J^n(X/\mathcal{O}) \) be the jacobian ideal. Namely, this is the annihilator of \( \Omega_{X/\mathcal{O}}^{n+1} \). As \( X \) is \( \mathcal{O} \)-smooth outside \( x \), \( J^n(X/S) \) contains \( \mathfrak{m}_x^n \), a power of the ideal sheaf \( \mathfrak{m}_x \) of \( x \).

Fix a uniformizer \( t \in \mathcal{O} \), hence identifications \( \mathcal{O} \cong k\{t\} \) and \( \hat{\mathcal{O}} \cong k[[t]] \). Since \( \mathcal{O}_{X,x} \) is regular, there is a presentation as a \( k[[t]] \)-algebra

\[
\hat{\mathcal{O}}_{X,x} \cong k[[t, t_0, \ldots, t_n]]/(h).
\]

Let \( M \) be an integer \( \geq 2N + 1 \). Set \( h_1 \) to be the degree \( \leq M \) part of \( h \). As \( J^n(X/\mathcal{O}) \supset \mathfrak{m}_x^n \), \([11, 5.11]\) implies that the natural isomorphism \( \mathcal{O}_{X,x}/\mathfrak{m}_x^M \cong k[t, t_0, \ldots, t_n]/(h_1, (t, t_0, \ldots, t_n)^M) \) extends to an \( \mathcal{O} \)-isomorphism

\[
\mathcal{O}_{X,(x)} \cong k\{t, t_0, \ldots, t_n\}/(h_1).
\]

We take a polynomial \( \tilde{Q} \in k[t, t_0, \ldots, t_n] \) of the following form depending on which cases we are dealing with:

1. If \( \text{char}(k) \) is odd or \( n + 1 \) is even, set \( \tilde{Q} = Q + t \) for some non-degenerate quadratic form \( Q \) over \( k \) in variables \( t_0, \ldots, t_n \).

2. Otherwise, set \( \tilde{Q} = Q + t_0^2 + tt_0 + t \) for some non-degenerate quadratic form \( Q \) in variables \( t_1, \ldots, t_n \) (note that \( n \) is even in this case).

We construct a “homotopy” which connects \( h_1 \) and \( \tilde{Q} \), adding an extra coordinate \( a \). We consider a commutative diagram

\[
\begin{array}{ccc}
Y' & \xrightarrow{\bar{f}} & \mathbb{A}_S^1 = \text{Spec}(k[a, t]) \\
\pi \downarrow & & \downarrow \\
S := \text{Spec}(k[a]) & & \\
\end{array}
\]

where \( Y' \) is defined for now by \( Y' := \text{Spec}(k[a, t, t_0, \ldots, t_n]/(h_1 + a(\tilde{Q} - h_1))) \). Let \( x_0 = (0, 0, \ldots, 0) \), \( x_1 = (1, 0, \ldots, 0) \) be points of \( Y' \). Since \( Y' \) is Cohen–Macaulay and the fibers of \( \pi \) over \( a = 0, 1 \) are of codimension 1 in \( Y' \), \( \pi \) is flat over \( a = 0, 1 \). By the local criterions of smoothness and flatness, we know that \( \pi \) is smooth and \( \bar{f} \) is flat at \( x_0, x_1 \). Shrinking \( Y' \) around \( x_0, x_1 \), we may assume that \( \pi \) is smooth and that \( \bar{f} \) is flat.

Let \( Z \) be the singular locus of \( \bar{f} \). Then \( x_0, x_1 \) are isolated in the fibers \( Z \times_S \{a = 0\} \) and \( Z \times_S \{a = 1\} \) respectively. Replacing \( Y' \) by an open neighborhood around \( x_0, x_1 \) if necessary, we may assume that \( Z \) is quasi-finite (by \([12, (13.1.3)]\)\)), hence flat, over \( S \) and that \( Z \times_S \{a = 0\} \) consists of only one point \( x_0 \). We write \( Y \) for such an open neighborhood.

Let \( Z_0 \) be the closed subscheme of \( Y \) defined by \( t, t_0, \ldots, t_n = 0 \). This defines a section of \( Z \to S \) which passes through \( x_0 \) and \( x_1 \). Note that \( Z_0 \) is an irreducible component of \( Z \) as \( Z \) is \( S \)-finite.

Note that \( x_1 \) is an ordinary quadratic point of \( \bar{f} \) with the Milnor number 1 or 2. Therefore Lemma \([3, 3.19]\) can be applicable to \( Y_{x_1} \to S(1) \). In particular, the map \( Z \to S \) is universally injective around \( x_1 \). On the other hand, \( Z_0 \) gives a section of \( Z \to S \). Hence there exists an open neighborhood \( U' \) of \( x_1 \) in \( Z \) which is contained in \( Z_0 \). By \([8, 1.3.4]\), we may assume that \( U' \) only consists of ordinary quadratic points. Applying Proposition \([3, 3.17] \)
to $U'$, we know that the Milnor number at any point in $U'$ equals to 1 or 2, accordingly to the division into cases made above.

After replacing $S$ by an irreducible étale neighborhood around $a = 0$ and then shrinking $Y$ by an open neighborhood around $x_0$, we can make $S, Y$ affine, and $Z$ finite over $S$. As $Z_0$ is irreducible and intersects with $Z$ at $x_0$, $U'$ remains non-empty after this replacement. Then such $Y, \pi, \tilde{f}$ satisfy the conditions; we can take $U$ in the condition 3 to be the image of $U'$ by the map $Z \to S$.

This lemma gives us a following well-known result in characteristic 2.

**Proposition 3.24.** Let $k$ be a perfect field of characteristic 2. Let $X$ be a smooth $k$-scheme of odd dimension $n$. Let $f: X \to C$ be a $k$-morphism to a smooth curve which has an isolated singular point at $x \in X$. Then $\mu(f, x)$ is even.

**Proof.** Taking the base change to an algebraic closure, we may assume that $k$ is algebraically closed, hence $x$ is $k$-rational.

Applying Lemma 3.23 to the map of the henselizations $X(x) \to C(f(x))$, we find a family (3.16) as in the lemma. We follow the notation there. By the condition 2 in the lemma, we need to show that the $O_S$-rank of $O_Z$ is even. Let $\bar{t}$ be a geometric generic point of $S$. Then the fiber $Z_{\bar{t}}$ consists of an ordinary quadratic point with the Milnor number 1 or 2. However, by Lemma 3.19, the Milnor number must be 2 in this case. Therefore $Z_{\bar{t}}$ consists of points $z_1, \ldots, z_m$ with $\mu(\tilde{f}_{\bar{t}}, z_1) = 2$. By the continuity (Proposition 3.17), we have

$$\mu(f, x) = \sum_i \mu(\tilde{f}_{\bar{t}}, z_i) = 2 + \sum_{i \neq 1} \mu(\tilde{f}_{\bar{t}}, \bar{t}_i).$$

In particular, the Milnor numbers $\mu(\tilde{f}_{\bar{t}}, z_i)$ for $i = 2, \ldots, m$ is strictly less than $\mu(f, x)$ and we conclude by induction on $\mu(f, x)$. \hfill $\square$

### 4 $\mathbb{Z}/2$-coverings associated with isolated singularities

#### 4.1 Discriminants of symmetric bilinear forms

We recall that one can define the discriminants of non-degenerate symmetric bilinear forms as $\mu_2$-torsors. Let $S$ be a scheme. Let $(\varphi, B)$ be a pair of a locally free $O_S$-module $\varphi$ of finite rank and a non-degenerate symmetric bilinear form $B$ on $\varphi$. The form $B$ induces an isomorphism $\varphi \to \varphi^\vee$. Taking the determinant, it induces an isomorphism $\det B \to \det \varphi^\vee$, hence an isomorphism $\det \varphi^{\otimes 2} \to O_S$, which is denoted by $\det B$.

Let $(\text{Sch}/S)$ denote the category of $S$-schemes. For an $S$-scheme $T$, we use the same symbol $\det B$ for the base change $(\det \varphi \otimes_{O_S} O_T)^{\otimes 2} \to O_T$ of $\det B$. A presheaf on $S$ means a functor $(\text{Sch}/S)^{\text{op}} \to (\text{Set})$ to the category of sets.

For a non-degenerate symmetric bilinear form $(\varphi, B)$ and an integer $N$, define $\text{disc}_N B$ to be the presheaf on $S$ sending an $S$-scheme $T$ to the set

$$\text{disc}_N B(T) := \{ x \in \det \varphi \otimes_{O_S} O_T | \det B(x, x) = (-1)^N \}. \tag{4.1}$$

Let $\mu_2$ be the presheaf on $S$ sending an $S$-scheme $T$ to the group

$$\{ a \in \Gamma(T, O_T) | a^2 = 1 \}.$$
This functor is represented by $\text{Spec}(\mathcal{O}_S[t]/(t^2 - 1))$ and is isomorphic to the kernel of $\mathbb{G}_{m,S} \xrightarrow{a \mapsto a^2} \mathbb{G}_{m,S}$. We consider the action of $\mu_2$ on $\text{disc}_N B$ defined by

$$\mu_2(T) \times \text{disc}_N B(T) \ni (a, x) \mapsto ax \in \text{disc}_N B(T).$$

**Lemma 4.1.**

1. Assume that $\det \varphi$ is monogenic. Set $\alpha = \det B(x, x)$ for a basis $x \in \det \varphi$. Then $\text{disc}_N B$ is represented by $\text{Spec}(\mathcal{O}_S[u]/(u^2 - (-1)^N \alpha))$.

2. The presheaf $\text{disc}_N B$ is representable by an $S$-scheme of finite flat of finite presentation. The action of $\mu_2$ gives $\text{disc}_N B$ a structure of a $\mu_2$-torsor over $S$ in the fppf topology.

3. Let $h: S' \to S$ be a morphism of schemes. Let $(\varphi', B')$ be the pullback of $(\varphi, B)$ by $h$. Then we have $\text{disc}_N B' \cong \text{disc}_N B \times_S S'$.

**Proof.**

1. The isomorphism $\mathcal{O}_S \to \det \varphi, 1 \mapsto x$ identifies $\det B$ with $\mathcal{O}_S \otimes \mathcal{O}_S \to \mathcal{O}_S, a \otimes b \mapsto aba$. This identification gives the identification of $\text{disc}_N B$ with the presheaf

$$T \mapsto \{a \in \Gamma(T, \mathcal{O}_T)|a^2 = (-1)^N a^{-1}\},$$

which is represented by $\text{Spec}(\mathcal{O}_S[u]/(u^2 - (-1)^N \alpha))$.

2. As $\text{disc}_N B$ is a sheaf in the Zariski topology, we can work Zariski-locally on $S$. Thus we may assume that $\det \varphi$ is free of rank $1$. Then the assertion follows from 1.

3. It is obvious from the definition. \qed

The Kummer short exact sequence

$$0 \to \mu_2 \to \mathbb{G}_m \xrightarrow{a \mapsto a^2} \mathbb{G}_m \to 0$$

gives an injection $\Gamma(S, \mathbb{G}_m)/\Gamma(S, \mathbb{G}_m)^2 \to H^1(S, \mu_2)$. Lemma 4.1 means that, if $\det \varphi$ has a global basis $x$, the $\mu_2$-torsor $\text{disc}_N B$ is equal to the image of $\det B(x, x)$ under this injection.

**Definition 4.2.**

1. For a non-degenerate symmetric bilinear form $(\varphi, B)$ and $N \in \mathbb{Z}$, we call the presheaf $\text{disc}_N B$ defined by $\text{(4.1)}$ the $N$-signed discriminant of $(\varphi, B)$. When $N = 0$, we omit $N$ in the symbol and write $\text{disc} B$ for it, which is simply called the discriminant of $(\varphi, B)$.

2. When $\det \varphi$ is monogenic, we also use the same symbol $\text{disc}_N B$ (resp. $\text{disc} B$) for the element in $\Gamma(S, \mathbb{G}_m)/\Gamma(S, \mathbb{G}_m)^2$ mapping to $\text{disc}_N B$ (resp. $\text{disc} B$) under the injection $\Gamma(S, \mathbb{G}_m)/\Gamma(S, \mathbb{G}_m)^2 \to H^1(S, \mu_2)$.

### 4.2 The case where $2$ is invertible

Let $S$ be a scheme. Let $X$ be a smooth $S$-scheme purely of relative dimension $n$ and $f: X \to C$ be an $S$-morphism to a smooth $S$-curve $C$. Assume that the singular locus $Z$ is finite over $S$. We also fix an everywhere non-zero differential $\omega \in \Omega^1_C/S$.

Assume that $2$ is invertible in $S$. In this case, $\mu_2$ is canonically isomorphic to the constant sheaf $\mathbb{Z}/2$ and we are allowed to give the following definition.
Definition 4.3. For an integer $N$, define the element
\[ \rho_{g,\omega,N}^f \in H^1(S, \mathbb{Z}/2) \]
to be the cohomology class corresponding to the $N$-signed discriminant $\text{disc}_N B_{f,\omega}$ (Definition 4.2) under the identification $H^1(S, \mathbb{Z}/2) \cong H^1(S, \mu_2)$.

When $S$ is noetherian and connected, $H^1(S, \mathbb{Z}/2)$ is identified with the group of continuous group homomorphisms $\pi_1^{ab}(S) \to \{\pm 1\}$. In this case we also use the same symbol $\rho_{g,\omega,N}^f$ for the corresponding group homomorphism.

Lemma 4.4. Let the notation be as above. Assume that 2 is invertible in $S$.

1. For a morphism $h: S' \to S$ of schemes, we have
\[ h^* \rho_{g,\omega,N}^f = \rho_{g',\omega',N}^{f'} \in H^1(S', \mathbb{Z}/2). \]
Here $f'$ and $\omega'$ are the pullbacks of $f$ and $\omega$ to $S'$.

2. Assume that $\det \varphi_f$ is monogenic. Put $\alpha := \det B_{f,\omega}(x,x)$ for a basis $x$. Then $\rho_{g,\omega}^f$ is the character of the square root of $(-1)^N \alpha$.

Proof. 1. This follows from the isomorphism of functors $\text{disc}_N B_{f',\omega'} \cong \text{disc}_N B_{f,\omega} \times_S S'$ by Lemma 4.1.3.

2. This is a restatement of Lemma 4.1.1 in this particular case. \qed

4.3 The case of characteristic 2

In this subsection, we construct finite étale $\mathbb{Z}/2$-coverings from isolated singularities in characteristic 2, which can be regarded as an étale analogue of the construction in the subsection 4.2.

As we consider a lift to Witt rings, we slightly change the notation as follows. Let $S_0$ be an $\mathbb{F}_2$-scheme, $X_0$ be a smooth $S_0$-scheme, and $f_0: X_0 \to \mathbb{A}^1_{S_0}$ be an $S_0$-morphism. The singular locus $Z_0 \subset X_0$ of $f_0$ is assumed to be finite over $S_0$. Write $t$ for the standard coordinate of $\mathbb{A}^1_{S_0}$. From these data, we construct a $\mathbb{Z}/2$-covering over $S_0$ in the case where $S_0, X_0$ are affine or $S_0$ is the spectrum of a field.

First we treat the case where $S_0$ and $X_0$ are affine. Let $S_0^{\text{perf}}$ be the perfection of $S_0$, i.e. the projective limit of the absolute Frobenius $\Phi: S_0 \to S_0$

\[ \cdots \to S_0 \xrightarrow{\Phi} S_0^{\text{perf}} \to S_0 \]
indexed by non-negative integers. Note that the 0-th projection $\pi_0: S_0^{\text{perf}} \to S_0$ induces an equivalence between their étale topoi. Therefore we can replace $S_0$ by $S_0^{\text{perf}}$ to construct finite étale coverings.

Thus we further assume that $S_0$ is perfect from now on. Let $A := \Gamma(S_0, \mathcal{O}_{S_0})$. Let $S_n := \text{Spec}(W_{n+1}(A)) = \text{Spec}(W(A)/p^{n+1})$ be the spectrum of the ring of Witt vectors of length $n + 1$.

We collect some basic terminologies of lifts which are needed.

Definition 4.5. Let $m \geq n \geq 0$ be integers.
1. Let \( X_n \) be a smooth \( S_n \)-scheme. A lift of \( X_n \) to \( S_m \) is a pair \((X_m, \iota_m)\) where \( X_m \) is a smooth \( S_m \)-scheme together with an \( S_n \)-isomorphism \( \iota_m : X_n \to X_m \times_{S_m} S_n \).

2. Let \( X_n, Y_n \) be smooth \( S_n \)-schemes and \( f_n : X_n \to Y_n \) be an \( S_n \)-morphism. Let \((X_m, \iota_m), (Y_m, \kappa_m)\) be lifts of \( X_n, Y_n \) to \( S_m \). An \( S_m \)-morphism \( f_m : X_m \to Y_m \) is called a lift of \( f_n \) if we have \( f_m \circ \iota_m = \kappa_m \circ f_n \).

**Lemma 4.6.** \([14]\) Let \( m \geq n \geq 0 \) be integers. Let \( X_n \) be a smooth \( S_n \)-scheme. Assume that \( X_n, S_n \) are affine.

1. Let \((X_m, \iota_m)\) be a lift to \( S_m \). Let \( Y_n \) be another affine smooth \( S_n \)-scheme with a lift \((Y_m, \kappa_m)\) to \( S_m \). Let \( f_n : X_n \to Y_n \) be an \( S_n \)-morphism. Then there exists at least one lift of \( f_n \) to \( S_m \). If \( f_n \) is an isomorphism, any lift is an isomorphism.

2. There exists at least one lift of \( X_n \) to \( S_m \). Such lifts are isomorphic to each other.

**Proof.** 1. The existence of a lift of \( f_n \) follows from the smoothness of \( Y_m \). The second assertion is a special case of \([14, 4.2]\).

2. This is \([14, 6.7]\): since \( X_n \) is affine, the obstructions to the existences of lifts always vanish. The latter assertion follows from 1. □

Let us go back to our situation. Hence \( X_0 \) is an affine smooth \( S_0 \)-scheme and \( f_0 : X_0 \to \mathbb{A}^1_{S_0} \) is an \( S_0 \)-morphism with the singular locus \( Z_0 \) finite over \( S_0 \). By Lemma 4.6, we take and fix a lift \( X_2 \) of \( X_0 \) to \( S_2 \) and a lift \( f_2 : X_2 \to \mathbb{A}^1_{S_2} \) of \( f_0 \). Let \( Z_2 \) be the singular locus of \( f_2 \). By Lemma 3.11, we know that \( Z_2 \times_{S_2} S_0 \) is isomorphic to \( Z_0 \). In particular, \( Z_2 \) is finite over \( S_2 \). Applying the construction in Definition 3.12, we have a symmetric bilinear form \((\varphi_{f_2}, B_{f_2, dt})\) on \( W_3(A) \).

Following is the main theorem in this section.

**Theorem 4.7.** Let the notation be as above. Set \( N = \frac{\dim(X_0/S_0) \mu(f_0, Z_0)}{2} \), which is an integer by Proposition 3.24.

1. (étaleness) The class \( \text{disc}_N B_{f_2, dt} \in H^1(S_2, \mu_2) \) belongs to the image of the injective map \( H^1(S_2, \mathbb{Z}/2) \to H^1(S_2, \mu_2) \) in Lemma 2.13.2. We write \( \rho_{f_2, dt}^\# \) for the class in \( H^1(S_2, \mathbb{Z}/2) \) mapping to \( \text{disc}_N B_{f_2, dt} \).

2. (compatibility with base change) Let \( h : S'_0 \to S_0 \) be a morphism of affine perfect schemes. Then we have

\[
h^* \rho_{f_2, dt}^\# = \rho_{f'_2, dt}^\# \quad \text{in} \quad H^1(S'_0, \mathbb{Z}/2).
\]

Here \( f'_2 \) denotes the pullback of \( f_2 \).

3. (independence) Assume that \( A \) is normal. Then \( \rho_{f_2, dt}^\# \) is independent of the choice of lifts \( X_2, f_2 \).

**Remark 4.8.** Suppose that \( X_2 \) lifts to a formal \( \text{Spf}(W(A)) \)-scheme \( X \) formally smooth formally of finite type and that \( f_2 \) lifts to a morphism \( f : X \to \mathbb{A}^1_{W(A)} \) of formal \( W(A) \)-schemes where \( \mathbb{A}^1_{W(A)} \) is the formal \( p \)-adic completion of \( \mathbb{A}^1_{W(A)} \). Such lifts can be taken using Lemma 4.6 inductively. For \( n \geq 0 \), set \( X_n = X \otimes_{W(A)} W_{n+1}(A) \) and \( f_n = f \otimes_{W(A)} W_{n+1}(A) \). We have a projective system \( \{ (\varphi_{f_n}, B_{f_n, dt}) \}_n \) of bilinear forms on finite projective \( W_{n+1}(A) \)-modules. Its limit \((\varphi, B) := \lim_{n \to \infty} (\varphi_{f_n}, B_{f_n, dt})\) gives a non-degenerate symmetric
bilinear form on $W(A)$. Consider the $N$-signed discriminant $\text{disc}_N B$ defined in Definition 4.2.1 for $N = \dim(X_0/O_S)\rho(f_0, Z_0)$. Then, by Lemmas 2.13.1, 2.13.2, the statement 1 in the theorem is equivalent to saying that the normalization of $\text{Spec}(W(A))$ in $\text{disc}_N B \otimes_{W(A)} W(A)[\frac{1}{2}]$ is finite étale over $W(A)$. Indeed, the image of $\rho_{f_2,dt}^g$ in $H^1(S_\infty[\frac{1}{2}], \mathbb{Z}/2)$ in the notation of Lemma 2.13 is equal to $\text{disc}_N B \otimes_{W(A)} W(A)[\frac{1}{2}]$.

We generalize the definition of $\rho_{f_2,dt}^g$ to an affine normal $\mathbb{F}_2$-scheme $S$ as follows. Here we say that an affine scheme $S$ is normal if $\Gamma(S, \mathcal{O}_S)$ is the finite product of normal domains.

Let $S$ be an affine normal $\mathbb{F}_2$-scheme. Let $f: X \to \mathbb{A}^n_S$ be a morphism of smooth affine $S$-schemes with the singular locus $Z$ finite over $S$. We write $S^{\text{perf}}$ for the perfection of $S$.

Applying Theorem 4.7 to the base change of $f$ to $S^{\text{perf}}$, we obtain a quadratic character $\rho_{f_2,dt}^g$ of $S^{\text{perf}}$, which is independent of the choice of lifts $f_2$ by Theorem 4.7.3. Since the étale topoi of $S$ and $S^{\text{perf}}$ are canonically equivalent, the following definition is allowed.

**Definition 4.9.** Let $S$ be an affine normal $\mathbb{F}_2$-scheme. We define $\rho_{f,dt}^g \in H^1(S, \mathbb{Z}/2)$ to be the cohomology class corresponding to $\rho_{f_2,dt}^g$ for any lift $f_2$ of $f \times_S S^{\text{perf}}$, via the equivalence of the étale topoi of $S$ and $S^{\text{perf}}$.

Recall that the Artin–Schreier exact sequence gives an isomorphism of groups

$$A/\varphi(A) \to H^1(\text{Spec}(A), \mathbb{Z}/2)$$

for an $\mathbb{F}_2$-algebra $A$. Using this, we can define a generalization of Arf invariant as follows.

**Definition 4.10.** Let $S$ be an affine normal $\mathbb{F}_2$-scheme. Let $A$ denote $\Gamma(S, \mathcal{O}_S)$. Let $X$ be a smooth $S$-scheme purely of dimension $n$. Let $f: X \to \mathbb{A}^n_S$ be an $S$-morphism whose singular locus $Z$ is finite over $S$. We define Arf($f$, $Z$) to be the element in $A/\varphi(A)$ which corresponds to $\rho_{f,dt}^g$ in Definition 4.9 via the isomorphism (4.2).

This terminology is justified by Corollary 3.21: in this corollary, we see that the invariant $\text{Arf}(Q, 0)$ so obtained from a non-degenerate quadratic form $Q$ over a perfect field of characteristic 2 coincides with the Arf invariant of $Q$ in the usual sense.

With an extra assumption that $\det \varphi_f$ is monogenic, we have a following explicit definition of $\text{Arf}(f, Z)$ (compare Corollary 3.21).

**Proposition 4.11.** Let the notation be as in Definition 4.10. Further assume that $A$ is perfect and that $\det \varphi_f$ is monogenic. Take a lift $f_2$ of $f$ to $W_3(A)$. Then, $\det \varphi_{f_2}$ is monogenic and we have the discriminant $\text{disc}B_{f_2,dt}$ in $W_3(A)^\times/(W_3(A)^\times)^2$ (Definition 4.2.2). Then we have an equality

$$(-1)^N \text{disc}B_{f_2,dt} \equiv 1 + 4[\text{Arf}(f, Z)]$$

in $W_3(A)^\times/(W_3(A)^\times)^2$. Here $N = \frac{uv(f, Z)}{2}$.

The equality uniquely characterizes $\text{Arf}(f, Z)$ by Lemma 2.12.2.

**Proof.** We follow the notations in Lemma 2.13. Let $\alpha \in W(A)^\times$ be an element in the class $\text{disc}_N B_{f,dt} = (-1)^N \text{disc}B_{f,dt} \in W(A)^\times/(W(A)^\times)^2 = W_3(A)^\times/(W_3(A)^\times)^2 \subset H^1(S_2, \mu_2)$. Consider the sequence of maps

$$H^1(S_2, \mathbb{Z}/2) \to H^1(S_2, \mu_2) \to H^1(S_\infty[\frac{1}{2}], \mathbb{Z}/2)$$

39
in Lemma 2.13.2. As its composition $H^1(S_2, \mathbb{Z}/2) \to H^1(S_\infty[\frac{1}{2}], \mathbb{Z}/2)$ can be identified with the canonical one $H^1(S_\infty, \mathbb{Z}/2) \to H^1(S_\infty[\frac{1}{2}], \mathbb{Z}/2)$ by Lemma 2.13, the statement 1 in Theorem 4.7 implies that $W(A)[\frac{1}{2}][u]/(u^2 - \alpha)$ extends to a finite étale $W(A)$-algebra, which is equivalent to saying that $\alpha$ is congruent to an element of the form $1 + 4[a]$ modulo $(W(A)^*)^2$ by Lemma 2.12.3. The equality $a = \text{Arf}(f, Z)$ in $A/\varphi(A)$ follows from Lemma 2.12.

In the rest of this subsection, we explain how to reduce Theorem 4.7 to the case of perfect fields. The compatibility 2 follows from Lemma 4.1.3. For the assertion 1, we may replace $A$ by the residue field at each generic point by Lemma 2.13.3. The assertion 3 in Theorem 4.7 can be checked at the generic point $\eta$ of $A$ by Lemma 2.13.4.

In this way, we reduce Theorem 4.7 to the case where $A = k$ is a perfect field, which we treat in the next subsection.

### 4.4 The case of a field of characteristic 2

Let $k$ be a perfect field of characteristic 2. Let $X_0$ be a smooth $k$-scheme purely of dimension $n$ and $f_0: X_0 \to \mathbb{A}^1_k$ be a $k$-morphism. Assume that the singular locus of $f_0$ consists of one point $x$. Shrinking $X_0$ around $x$, we assume that $X_0$ is affine.

Let $X_2$ be a lift of $X_0$ to $W_3(k)$ and $f_2: X_2 \to \mathbb{A}^1_{W_3(k)}$ be a lift of $f_0$ (Definition 4.5). Theorem 4.7 follows from the following.

**Theorem 4.12.** Set $N = \frac{n\mu(f_0, x)}{2}$.

1. $(-1)^N \text{disc} B_{f_2, dt}$ belongs to the image of $1 + 4W_3(k) \to W_3(k)^\times/(W_3(k)^\times)^2$.

2. $(-1)^N \text{disc} B_{f_2, dt}$ is independent of the choice of lifts $X_2$ and $f_2$.

**Proof.** First we treat the case where $x$ is $k$-rational. In this case, we prove the assertion by induction on the Milnor number $\mu(f_0, x)$. If $\mu(f_0, x) = 0$, there is nothing to prove as $f_0$ is smooth at $x$.

Suppose that $\mu(f_0, x) > 0$. Take a family

$$
\begin{array}{ccc}
Y & \xrightarrow{\tilde{f}} & \mathbb{A}^1_S \\
\pi \downarrow & & \downarrow \\
S & \xrightarrow{} & \\
\end{array}
$$

which satisfies the conditions in Lemma 3.23 for $(X \to \Spec(O)) = (X_0, (x) \to \mathbb{A}^1_{k, (f_0(x))})$. We follow the notation given there. In particular, we are given a $k$-rational point $s \in S$ such that the henselizations of the fibers $(Y \times_S s)_{(x_0)} \to \mathbb{A}^1_{k, (0)}$ is isomorphic to the map of the henselizations $f_0: X_{0, (x)} \to \mathbb{A}^1_{k, (f_0(x))}$. Let $Z$ be the singular locus of $\tilde{f}$.

Let $S_0$ be the perfection of $S$. Note that $s \to S$ uniquely lifts to $s \to S_0$. Let us put the symbol $(-)_{0}$ to indicate base change by $S_0 \to S$. Write $A$ for $\Gamma(S_0, \mathcal{O}_{S_0})$. Take a lift $\tilde{f}_2: Y_2 \to \mathbb{A}^1_{W_3(A)}$ to $W_3(A)$ of $Y_0$ and $\tilde{f}_0$.

First we show that the class $\text{disc}_\mathcal{N} B_{f_2, dt} \in H^1(S_2, \mu_2)$ is contained in the image of $H^1(S_2, \mathbb{Z}/2) \to H^1(S_2, \mu_2)$ and that it is independent of the choice of $\tilde{f}_2$. To do this, we may replace $S_0$ by its generic point $\eta$ by Lemmas 2.13.3,4. Hence we assume that $A$ is a perfect field. By the condition 3 in Lemma 3.23 $Z_0 \times_{S_0} \eta$ consists of points $z_1, \ldots, z_m$ where
z_1 is an ordinary quadratic point with \( \mu(f_0 \times_{S_0} \eta, z_1) = 1, 2 \). For the points \( z_2, \ldots, z_m \), we can apply the induction hypothesis. For \( z_1 \), the assertion is proved in Corollary 3.21 if \( \mu(f_0 \times_{S_0} \eta, z_1) = 1 \). If \( \mu(f_0 \times_{S_0} \eta, z_1) = 2 \), it is proved in Proposition 3.22.

We prove the theorem in the case where \( x \) is \( k \)-rational. The base changes \( X_2 := Y_2 \times_{W_3(k)} W_3(k), f_2 := \tilde{f}_2 \times_{W_3(k)} W_3(k) \) by \( s \to S_0 \) give lifts of \( X_0, f_0 \) respectively. The assertion for one already proved for these lifts. Thus it suffices to show that, for other lifts \( X, f \) of \( X_0, f_0 \), we have the equality \( \text{disc}_N B_{f_2,dt} = \text{disc}_N B_{f,dt} \). First, by Lemma 4.6.2, we may replace \( X_2 \) by \( X_2 \). Then, since the reduction of \( f_2, f'_2 \) to \( k \) are the same, there exists a function \( g : X_2 \to \mathbb{A}^1_{W_3(k)} \) such that \( f'_2 = f_2 + 2g \). Take a map \( \tilde{g} : Y_2 \to \mathbb{A}^1_{W_3(k)} \) whose base change by \( W_3(A) \to W_3(k) \) equals to \( g \). Then \( \tilde{f}_2 + 2\tilde{g} \) gives another lift of \( \tilde{f}_0 \). The assertion follows since we already know \( \text{disc}_N B_{f_2,dt} = \text{disc}_N B_{f,dt} \).

We reduce the case where \( x \) is not necessarily \( k \)-rational to the case treated above. First we show that \( \text{disc}_N B_{f,dt} \) is independent of the choice of \( f_2 \). Replacing \( f_0 \) and \( f_2 \) by \( f_0 \times_k k(x) \) and \( f_2 \times_k W_3(k(x)) \), we may assume that the structure map \( X_2 \to \text{Spec}(W_3(k)) \) factors through \( \text{Spec}(W_3(k(x))) \). Then the assertion follows from the case over \( k \) and Lemma 4.13 below, as we have \( \text{disc}_N B_{f_{2},dt} = (-1)^{N} \text{disc}B_{f_{2},dt} \).

We show that \( \text{disc}_N B_{f_{2},dt} \) belongs to the image of \( 1 + 4W_3(k) \). This is equivalent to showing that \( F[u]/(u^2 - \alpha) \) is unramified where \( F = W(k)[\frac{1}{2}] \). Hence we may take the base changes to an algebraic closure and reduce it to the case where \( x \) splits into rational points, which case is already treated.

**Lemma 4.13.** Let \( R \) be a commutative ring. Let \( R' \) be an \( R \)-algebra which is finite free as an \( R \)-module. Let \( (V, B') \) be a finite free \( R' \)-module \( V \) with an \( R' \)-bilinear form \( B' \). Let \( B \) be the composition

\[
V \times V \xrightarrow{B'} R' \xrightarrow{\text{Tr}_{R'/R}} R,
\]

which we view as an \( R \)-bilinear form on the finite free \( R \)-module \( V \). Then the discriminant of \((V, B)\) equals to \( \text{disc}(R'/R)^{rk_{R'}V} N_{R'/R}(\text{disc}(V, B')) \) in \( R^\times/(R^\times)^2 \). Here \( \text{disc}(R'/R) \) denotes the discriminant of the form \( \text{Tr}_{R'/R} \).

**Proof.** See [5, Section 2, Proposition 9].

We note that the Arf invariant is trivial if the variety is of odd dimension, as the following proposition shows.

**Proposition 4.14.** Let \( k \) be a perfect field of characteristic 2. Let \( f : X \to \mathbb{A}^1_k \) be a morphism of smooth \( k \)-schemes with an isolated singularity \( x \in X \). Assume that \( \dim X \) is odd. Then we have \( \text{Arf}(f, x) = 0 \).

**Proof.** We prove the assertion by induction on \( \mu(f, x) \).

First we reduce it to the case where \( x \) is \( k \)-rational. Replacing \( X \) by \( X \times_k k(x) \) and \( x \) by the image of the diagonal \( x \to X \times_k k(x) \), we may assume that \( X \to \text{Spec}(k) \) factors through \( k(x) \). Then the assertion for \((X, x)\) over \( k \) follows from that for \((X \times_k k(x), x)\) over \( k(x) \) by Lemma 4.13. Note that, as the Milnor number is even in this case, the part \( \text{disc}(R'/R)^{rk_{R'}V} \) in the lemma is trivial.

Take a family \((3.16)\) as in Lemma 3.23. We follow the notation given there. Then, by Definition 4.9 we have a character \( \rho^g_{f,dt} : \pi_0^{ab}(S) \to \{ \pm 1 \} \) such that, for a morphism \( h : \text{Spec}(k') \to S \) from the spectrum of a perfect field, the composition \( \rho^g_{f,dt} \circ h_{*} \) corresponds to \( \sum_{y \in Z_{k'}} \text{Arf}(\tilde{f}_{k'}, y) \) via the Artin–Schreier theory.
We take such a \( k' \) as the perfection of the generic point of \( S \). By the condition 3 in Lemma 3.23, \( Z_{k'} \) consists of \( z_1, \ldots, z_m \) where \( z_1 \) is ordinary quadratic of the Milnor number 1 or 2. However, as \( \dim X \) is odd, it cannot be 1 and we can apply Proposition 3.20 to \( z_1 \). For \( z_2, \ldots, z_m \), the Milnor numbers are strictly less than \( \mu(f, x) \) as we have

\[
\mu(f, x) = \sum_{i} \mu(f'_k, z_i).
\]

Hence the induction hypothesis is applied.

Combining this proposition with Theorem 5.4 in the next section, we see that the local epsilon factors of isolated singularities in odd dimension are always trivial. This triviality is also deduced from the continuity of local epsilon factors (Theorem 5.1) together with Lemmas 3.23, 3.22 without referring to the Arf invariants.

5 Milnor formula for local epsilon factors

Throughout this section, we fix a perfect field \( k \) of characteristic \( p > 0 \) and a prime number \( \ell \) different from \( p \). We fix an algebraic closure \( \overline{\mathbb{Q}}_{\ell} \) of \( \mathbb{Q}_{\ell} \). We also fix a non-trivial additive character \( \psi: \mathbb{F}_p \rightarrow \overline{\mathbb{Q}}_{\ell}^\times \), which we use to define local epsilon factors as in [20, (3.1.5.4)].

We give some notation which is used in this section.

- For a field \( k \), we write \( G_k \) for its absolute Galois group.
- Let \( T \) be a scheme and \( t \in T \) be a point. For a morphism of schemes \( z \rightarrow t \) which comes from a finite separable field extension of \( k \), we write \( T(z) \) for the unramified extension of the henselization \( T(t) \) whose residue field is \( k(z) \).

5.1 Preliminaries on local epsilon factors

Let \( T \) be a henselian trait which is isomorphic to the henselization of \( \mathbb{A}^1_k \) at the origin. Let \( s \) and \( \eta \) be the closed point and the generic point of \( T \). Fix a non-zero rational differential \( \omega \in \Omega^1_k(\eta) \). In [32], [15], Yasuda and Guignard independently give generalizations of the theory of local epsilon factors to general perfect residue fields. They attach, in a canonical way, a character \( \varepsilon_{0,k}(T, V, \omega): G^\ab_k \rightarrow \overline{\mathbb{Q}}_{\ell}^\times \) of the absolute Galois group of the residue field, to a finite dimensional \( \mathbb{Q}_{\ell} \)-representation \( V \) of \( G_k(\eta) \). Their theories coincide by [33, 8.3] (for finite field cases) and [15, 11.8]. In this paper, we choose the settings and notation for local epsilon factors similar to [15], as it fits our purposes well. An explanation to translate the results in [32] to our settings is given in [29, 3.1].

We recall the relation of their theories with the classical one. When \( k \) is finite, the Galois group \( G_k \) is topologically generated by the geometric Frobenius \( \text{Frob}_k \). The value \((-1)^{\dim_{\text{tot}} V} \varepsilon_{0,k}(T, V, \omega)(\text{Frob}_k)\) equals to the classical local epsilon factor in [6], [20 (3.1.5.4)], where \( \dim_{\text{tot}} \) denotes the sum of the Swan conductor and the actual dimension of \( V \). This equality follows from the cohomological interpretation by Laumon [20 (3.5.1.1)] and [33, 8.3], [15, 11.8].

We usually identify in the sequel the absolute Galois group of \( \eta \) and that of the completion with respect to the discrete valuation. The identification depends on the choice of the embedding of an algebraic closure of \( k(\eta) \) into that of the completion. When we use this identification, we fix one of such embeddings. Since such an isomorphism is unique up to the conjugation, local epsilon factors do not depend on its choice.

To recall the continuity of local epsilon factors in [29], we give necessary notation. Let \( T \) be a henselian trait with the residue field \( k \) as above. Recall that, for a finite separable
field extension $k(z)/k$, we write $T(z)$ for the unramified extension of the henselization of $T$ whose residue field is $k(z)$. We also write $\eta_z$ for the generic point of $T(z)$.

Let $X$ be a $T$-scheme of finite type and let $f : X \to T$ be the structure morphism. For a constructible complex $F$ of $\mathbb{Q}_{\ell}$-sheaves on $X$, the vanishing cycles complex $R\Phi_f(F)$ is defined to be a constructible complex on the geometric special fiber $X_{\bar{s}}$. Let $Z \subset X$ be the closed subset outside of which $f$ is universally locally acyclic relatively to $F$. Then the complex $R\Phi_f(F)$ is supported on $Z_{\bar{s}}$ and it admits an action of the absolute Galois group $G_{\eta}$ equivariantly to the action on $Z_{\bar{s}}$. Therefore, for a closed point $z \in Z$, the restriction of $R\Phi_f(F)$ to $z \times_{\bar{s}} \bar{s} \subset Z_{\bar{s}}$ gives a bounded complex of finite dimensional $\mathbb{Q}_{\ell}$-representations of $G_{k(z)}$, which we denote by $R\Phi_f(F)_z$.

Let $S$ be a noetherian $\mathbb{F}_p$-scheme. Let $Z \twoheadrightarrow X \twoheadrightarrow \mathbb{A}^1_S \twoheadrightarrow S$ be a family of isolated singularities. Namely,

1. $Z$ is the singular locus of $f$ which we assume finite over $S$.
2. $\pi$ is smooth.

For a finite separable extension $k'/k$ of fields, we denote by $\text{tr}_{k'/k} : G_k^{ab} \to G_{k'}^{ab}$ the transfer morphism induced by the inclusion $G_k \hookrightarrow G_{k'}$. The determinant character of the induced representation $\text{Ind}_{G_k}^{G_{k'}} 1_{G_k'}$ of the trivial representation is denoted by $\delta_{k'/k}$. The following is an arithmetic analogue to Proposition 3.17.

**Theorem 5.1.** (A special case of [29, 4.8]) Let the notation and assumption be as above. Assume that $S$ is connected. From these data given above, we can attach a character

$$\rho_{f,dt}^a : \pi_1(S)^{ab} \to \overline{\mathbb{Q}}_{\ell}$$

in such a way that the following hold.

1. The formation of $\rho_{f,dt}^a$ commutes with base change $S' \to S$.
2. When $S$ is the spectrum of a perfect field $k$, we have

$$\rho_{f,dt}^a = \prod_{z \in Z} \delta_{k(z)/k} \cdot \varepsilon_{k(z)}(R\Phi_f(\mathbb{Q}_{\ell})_z, dt) \circ \text{tr}_{k(z)/k}.$$

When $S$ is noetherian normal, the character $\rho_{f,dt}^a$ is uniquely determined by these properties.

We also recall the compatibility with additive convolution. Let $f_1 : X_1 \to \mathbb{A}^1_k$ and $f_2 : X_2 \to \mathbb{A}^1_k$ be $k$-morphisms of finite type. For each $i = 1, 2$, we assume that $X_i$ is smooth over $k$ and that isolated singular points $x_i \in X_i$ of $f_i$ are given. We further assume that $x_i$ are $k$-rational and that the image $f_i(x_i)$ equal to the origin. Let $X := X_1 \times_k X_2$ and denote by $f : X \to \mathbb{A}^1_k$ the map sending $(x, y) \mapsto f_1(x) + f_2(y)$.
Lemma 5.2. \((3.18)\) Let the notation be as above. Let \(x := (x_1, x_2) \in X\) be the \(k\)-rational point over \(x_1\) and \(x_2\). Then we have the equality
\[
\varepsilon_{0, k}(A^1_{k,(0)}, R\Phi_f(\mathbb{Q}_l)_x, dt)^{-1} = \varepsilon_{0, k}(A^1_{k,(0)}, R\Phi_f(\mathbb{Q}_l)_x, dt)^{\text{dim}_{\mathbb{Q}_l}} \cdot \varepsilon_{0, k}(A^1_{k,(0)}, R\Phi_{f_2}(\mathbb{Q}_l)_{x_2}, dt)^{\text{dim}_{\mathbb{Q}_l}},
\]
as characters of \(G_k\).

5.2 Main result

To state the main theorem, we introduce some notations.

We are fixing a non-trivial character \(\psi: F \to \mathbb{C}\). Let the notation be as above. Let \(\psi\) be a finite extension of \(F\) and \(a \in A^1_{\mathbb{F}_p}(k) \cong k\) be a \(k\)-rational point. Then the geometric Frobenius of \(G_k\) acts on the stalk \(\mathcal{L}_{\psi,a}\) by the multiplication of \(\psi_k(a)\).

To the additive character \(\psi_k\), we attach the quadratic Gauss sum \(\tau_{\psi,k}\) by the following
\[
\tau_{\psi,k} := -\sum_{a \in k} \psi_k(a^2) = -\sum_{a \in k} (\frac{a}{k}) \psi_k(a).
\]

Here the symbol \((\frac{\cdot}{k})\) denotes the Legendre symbol. Note that its square \(\tau^2_{\psi,k}\) equals to \((\frac{1}{k})q\) where \(q\) is the cardinality of \(k\).

Lemma 5.3. Assume that \(p\) is odd. Let \(f: A^1_{\mathbb{F}_p} \to A^1_{\mathbb{F}_p}\) be a morphism of schemes defined by \(t \mapsto t^2\). Then the étale cohomology group \(H^1(A^1_{\mathbb{F}_p}, f^*\mathcal{L}_\psi)\) vanishes except in degree 1 and \(H^1\) is a one dimensional \((\mathbb{Q}_l)\)-vector space on which the geometric Frobenius of any finite extension \(k/\mathbb{F}_p\) acts by the multiplication of \(\tau_{\psi,k}\).

For a connected noetherian \(\mathbb{F}_p\)-scheme \(S\), we write \(\rho_{\psi}\) for the composition of the map \(\pi^0_1(S) \to \pi^0_1(\text{Spec}(\mathbb{F}_p))\) and the character \(\pi^0_1(\text{Spec}(\mathbb{F}_p)) \to \mathbb{Q}_l^\times\) corresponding to \(H^1(A^1_{\mathbb{F}_p}, f^*\mathcal{L}_\psi)\).

Proof. For the first assertion, \(H^0\) vanishes as the Artin–Schreier covering stays connected after the base change by \(\mathbb{F}_p \to \overline{\mathbb{F}}_p\). For higher \(H^1\), it vanishes in degree \(\geq 2\) as \(A^1_{\mathbb{F}_p}\) is affine and 1-dimensional.

The assertion on the dimension of \(H^1\) comes from the Grothendieck–Ogg–Shafarevich formula applied to \(f^*\mathcal{L}_\psi\), as the Swan conductor of the covering \(X \times A^1_{\mathbb{F}_p} f A^1_{\mathbb{F}_p} \to A^1_{\mathbb{F}_p}\) at the infinity is 2. The last assertion follows from the Lefschetz trace formula.  

Here is the main theorem of this article.

Theorem 5.4. Let \(S\) be a noetherian affine connected normal \(\mathbb{F}_p\)-scheme and let
\[
\xymatrix{ Z \ar[r] & X \ar[r]^f & A^1_S \ar[d]^\pi \ar[l]_\psi & S }
\]
be a commutative diagram of $S$-schemes of finite type where $\pi$ is smooth purely of relative dimension $n$ and $Z$ is the singular locus of $f$. We assume that $Z$ is finite over $S$.

In this case, the character $\rho_{f,dt}^a : \pi_1(S)^{ab} \to \overline{\mathbb{Q}_\ell}^\times$ is given in Theorem 5.1. Let $\mu(f,Z)$ be the Milnor number along $Z$ (Definition 3.16).

1. Assume that $p$ is odd. Let $\rho_\psi$ be as in Lemma 5.3. Then we have an equality

$$\rho_{f,dt}^a = \rho_{f,-dt,0}^g \cdot \rho_\psi^{(-1)^{n+1}n\mu(f,Z)}$$

of characters $\pi_1(S)^{ab} \to \overline{\mathbb{Q}_\ell}^\times$. For the definition of $\rho_{f,-dt,0}^g$, see Definition 4.3.

2. Assume that $p = 2$. Let $\chi_{cyc}$ denotes the inverse of the character of the Tate twist, so that the value at the geometric Frobenius equals to $q$ if $S$ is the spectrum of a finite field with $q$ elements. We have

$$\rho_{f,dt}^a = \rho_{f,dt}^g \cdot \chi_{cyc}^{(-1)^{n+1}n\mu(f,Z)}.$$ 

For the definition of $\rho_{f,dt}^g$, see Definition 4.9.

As a corollary, we have the following result, as promised in the introduction. Recall our convention: $\varepsilon_0$ denotes the classical local epsilon factors and we have $\varepsilon_{0,k}(\text{Frob}_k) = (-1)^{\dim_{\text{tot}}\varepsilon_0}$.

**Corollary 5.5.** Let $k$ be a finite field. Let $X$ be a smooth $k$-scheme of dimension $n$ and let $f : X \to \mathbb{A}^1_k$ be a $k$-morphism with an isolated singular point $x \in X$. We write $\mathbb{A}^1_{k,(x)}$ for the unramified extension of $\mathbb{A}^1_{k,(f(x))}$ with the residue field $k(x)$.

1. Assume that $k$ is of odd characteristic. Then we have

$$(-1)^{[k(x):k]\dim_{\text{tot}}R\Phi_f(Q_\ell)x} \varepsilon_0(\mathbb{A}^1_{k,(x)}, R\Phi_f(Q_\ell)_x, dt) = \tau_\psi^{(-1)^{n+1}n\mu(f,x)} \cdot \left( \frac{\text{disc}B_{f,-dt}}{k} \right).$$

2. Assume that $k$ is of characteristic 2. Write $q$ for the cardinality of $k$. Then the ratio

$$(-1)^{[k(x):k]\dim_{\text{tot}}R\Phi_f(Q_\ell)x} \varepsilon_0(\mathbb{A}^1_{k,(x)}, R\Phi_f(Q_\ell)_x, dt)/q^{(-1)^{n+1}n\mu(f,x)}$$

is $\pm 1$. This is 1 if and only if $\text{Arf}(f,x) \in \wp(k)$.

**Proof.** The statements are the special case of Theorem 5.4 where $S = \text{Spec}(k)$. □

Actually the proof of Theorem 5.4 is reduced to the finite field case. We prepare an auxiliary lemma, so that we can freely assume that a singular point in consideration is $k$-rational. Before doing so, we recall a well-known result on trace forms. For a finite separable field extension $L/F$, write $\delta_{L/F}$ for the determinant character of the induced representation $\text{Ind}_{G^F_L}^{G^F} 1_{G^F_L}$ of the trivial representation.

**Lemma 5.6.** Let $F$ be a field in which 2 is invertible. For a finite separable field extension $L/F$, write $\text{disc}(L/F)$ for the discriminant of the non-degenerate quadratic form $(L, \text{Tr}_{L/F}(x^2))$ over $F$. Then the quadratic character $\delta_{L/F}$ is defined by the square root of $\text{disc}(L/F)$.
Lemma 5.7 and that

In the left-hand side, we have \((\mu, \lambda)\) with \((\mu, \lambda)\). As the characteristic is 2, lifting

such that

\(k\) find a commutative diagram of smooth

\(k\) spectrum of a finite field

\(k\) with

\(k\) are consequences of Lemma 4.13.

Consider the situation in Corollary 5.5. Let

\(F\) closure

Proof. We have \(\mu(f, x) = \mu(f, x)[k(x) : k]\) and \(\mu[f, k] = \tau_{\psi, k}\) when \(k\) is of odd characteristic (cf. Lemma 5.3). As the local epsilon factors do not depend on the base fields, it remains to check the equality of signs.

If the characteristic is odd, it remains to show that

\[(-1)^{[k(x):k]-1}\mu(f, x)\left(\frac{\text{disc}B_{f, -dt, k(x)}}{k(x)}\right) = \left(\frac{\text{disc}B_{f, -dt, k}}{k}\right).\]

In the left-hand side, we have \((-1)^{[k(x):k]-1} = \left(\frac{\text{disc}(k(x)/k)}{k}\right)\) by Lemma 5.6. Then the assertion follows from the equality

\[\text{(5.1) } \text{disc}(k(x)/k)^{\mu(f, x)}N_{k(x)/k}(\text{disc}B_{f, -dt, k(x)}) = \text{disc}B_{f, -dt, k} \text{ in } k^x/(k^x)^2.\]

If the characteristic is 2, lifting \(f\) to \(W(k)\), the assertion is reduced to the similar statement as \((5.1)\) with \(k\) replaced with the fraction field of \(W(k)\). By Lemma 3.14 the assertions are consequences of Lemma 4.13.

(Proof of Theorem 5.4)

As the formations of \(\rho_{f, dt}^0, \rho_{f, -dt, 0}^0\), and \(\rho_{f, dt}^0\) commute with base change, we may assume that \(S\) is of finite type over \(Fbar\). By Cebotarev density theorem, it suffices to show the equality of the values of the characters at the geometric Frobenius of every closed point. Again by the commutativity of the formations, we reduce it to the case where \(S\) is the spectrum of a finite field \(k\). We also assume that \(Z\) consists of a \(k\)-rational point \(x\) by Lemma 5.7 and that \(x\) maps to the origin of \(A^1_k\).

We prove the statement by induction on \(\mu(f, x)\). Applying Lemma 3.23 to \((f, x)\), we find a commutative diagram of smooth \(k\)-schemes

\[\begin{array}{ccc}
Y & \xrightarrow{j} & A^1_{S'} \\
\downarrow & & \downarrow \\
S' & &
\end{array}\]

such that
1. $Y \to S'$ is smooth. The singular locus $Z'$ of $\tilde{f}$ is finite over $S'$.

2. There is a $k$-rational point $s \in S'(k)$ such that $Z'_s = Z' \times_{S'} s$ consists of one point $x_0$ mapping to the origin by $\tilde{f}$ and the map of the henselizations of the fibers $\tilde{f}_s: Y_{s,(x_0)} \to \mathbb{A}^1_{k,(0)}$ is isomorphic to $X_{(x)} \to \mathbb{A}^1_{k,(0)}$.

3. There is an open dense subset $U' \subset S'$ such that, for any closed point $t \in U'$, the fiber $Z'_t$ has a $k$-rational ordinary quadratic point $z_1$ with $\mu(\tilde{f}_t, z_1) = 1, 2$.

By Cebotarev density, it is enough to show the equality on the geometric Frobeniuses at points on $U'$. Let $t \in U'$ be a closed point and let $z_1, \ldots, z_m$ be the points in $Z'_t$ where $z_1$ is such an ordinary quadratic singularity. For $z_i$ ($i = 2, \ldots, m$), we apply the induction hypothesis on Milnor number.

It remains to treat the case where $x$ is $k$-rational and ordinary quadratic with $\mu(f, x) = 1, 2$. By Lemma 3.19, the henselization $\mathcal{O}_{X,(x)}$ is isomorphic to the following as $k\{t\}$-algebras.

1. If $p$ is odd or $n$ is even,

$$\mathcal{O}_{X,(x)} \cong k\{t, t_0, \ldots, t_{n-1}\}/(Q - t)$$

for some non-degenerate quadratic form $Q \in k[t_0, \ldots, t_{n-1}]$.

2. If $p = 2$ and $n$ is odd,

$$\mathcal{O}_{X,(x)} \cong k\{t_1, \ldots, t_{n-1}\} \otimes k\{u\}$$

which is regard as $k\{t\}$-algebra by the map $t \mapsto Q \otimes 1 + 1 \otimes \tilde{t}$ where $Q \in k[t_1, \ldots, t_{n-1}]$ is a non-degenerate quadratic form and $\tilde{t}$ satisfies $u^2 + b(\tilde{t})u - \tilde{t} = 0$ for some uniformizer $b \in k\{t\}$.

Up to changing the coordinates, we can take $Q$ to be the form $a_0t_0^2 + \cdots + a_{n-1}t_{n-1}^2$ ($a_i \in k$) if $p$ is odd, and $(t_0^2 + t_0t_1 + a_0t_1^2) + (t_2^2 + t_2t_3 + a_2t_3^2) + \cdots$ ($a_i \in k$) if $p = 2$. By the compatibility with additive convolution (Lemmas 3.19 5.2), we may assume one of the following:

1. $\mathcal{O}_{X,(x)} \cong k\{t, t_0\}/(at_0^2 - t)$ for $a \in k^\times$.

2. $\mathcal{O}_{X,(x)} \cong k\{t, t_0, t_1\}/(t_0^2 + t_0t_1 + at_1^2 - t)$ for $a \in k$.

3. $\mathcal{O}_{X,(x)} \cong k\{t, u\}/(u^2 + bu - t)$ for a uniformizer $b \in k\{t\}$.

Each case is treated in Proposition 3.20 and Lemma 3.22. The proof is completed. \hfill \Box

5.3 Example: discriminants of homogeneous polynomials

At the end of this paper, we compute the discriminant of the bilinear form associated with a homogeneous polynomial $F$ in terms of its divided discriminant [26].

To start with, we recall a relation between conditions on smoothness of a homogeneous polynomial.
Lemma 5.8. Let $F \in k[T_0, \ldots, T_{n+1}]$ be a homogeneous polynomial of degree $d > 1$ with coefficients in a field $k$. Consider the following conditions.

1. The map $F: \mathbb{A}^{n+2}_k \to \mathbb{A}^1_k$ has an isolated singular point at the origin.
2. The map $F: \mathbb{A}^{n+2}_k \to \mathbb{A}^1_k$ is smooth outside the origin.
3. The hypersurface $Y := \{F = 0\} \subset \mathbb{P}^{n+1}_k$ is smooth.

We have $1 \iff 2 \implies 3$. If $d$ is invertible in $k$, they are equivalent.

Proof. We may assume that $k$ is algebraically closed. The implication $2 \implies 1$ is obvious. For the other direction, suppose that $F$ has a singular point $x \in \mathbb{A}^{n+2}_k$ outside the origin. Since $F$ is homogeneous, the map $\mathbb{A}^1_k \to \mathbb{A}^{n+2}_k$, $t \mapsto tx$ factors through the singular locus. Hence, the origin is not isolated.

By the jacobian criterion, the condition 2 is equivalent to saying that the polynomials $\frac{\partial F}{\partial T_i}$ do not have a non-zero common root. On the other hand, 3 is equivalent to saying that those polynomials together with $F$ do not have a non-zero common root, hence the implication $2 \implies 3$. The last assertion follows from Euler’s identity $F = \frac{1}{d} \sum_i T_i \frac{\partial F}{\partial T_i}$. \hfill $\square$

Let $n$ be an integer $\geq 0$. For a multi-index $I = (i_0, \ldots, i_{n+1}) \in \mathbb{N}^{n+2}$, write $T^I := \prod_j T_j^{i_j}$ and $|I| := \sum_j i_j$, as usual. Consider an affine space $S = \text{Spec}(\mathbb{Z}[[C_I]])$ with indeterminates $C_I$ indexed by the multi-indices $I$ with $|I| = d$, which we view as the moduli space of homogeneous polynomials of degree $d$ with the universal homogeneous polynomial $F = \sum_I C_I T^I$. Its divided discriminant $\text{disc}_d(F) \in \mathbb{Z}[[C_I]]$ is defined in [26, Section 2]. It is an irreducible homogeneous polynomial and fits into the equality

\[
(5.2) \quad \text{Res}(\frac{\partial F}{\partial T_0}, \ldots, \frac{\partial F}{\partial T_{n+1}}) = d^{a(n,d)} \text{disc}_d(F)
\]

where $\text{Res}(\frac{\partial F}{\partial T_0}, \ldots, \frac{\partial F}{\partial T_{n+1}})$ is the resultant [11, 13.1.A] of the partial derivatives and $a(n,d) = \frac{(d-1)^{n+2} \cdots (-1)^{n+2}}{d}$. The integer $d^{a(n,d)}$ is the greatest common divisor of the coefficients of the resultant.

For a homogeneous polynomial $F$ of degree $d$ with coefficients in a ring $R$, we define the divided discriminant $\text{disc}_d(F) \in R$ to be the specialization by the map $\mathbb{Z}[[C_I]] \to R$ defining $F$. As is explained in [26], the discriminant $\text{disc}_d(F) \in R$ is invertible if and only if the hypersurface $Y := \{F = 0\}$ in $\mathbb{P}^{n+1}_R = \text{Proj}(R[T_0, \ldots, T_{n+1}])$ is smooth. On the other hand, the resultant $\text{Res}(\frac{\partial F}{\partial T_0}, \ldots, \frac{\partial F}{\partial T_{n+1}}) \in R$ is invertible if and only if the map $F: \mathbb{A}^{n+2}_R \to \mathbb{A}^1_R$ is smooth outside the origin [11, 13.1.A]. By (5.2), the latter condition is equivalent to the former one only when $d = 2$ as $a(n,d)$ is zero only in this case. When $d > 2$, it also requires that $d$ is invertible in $R$. Taking this into account, the open subscheme $\bar{U} := \text{Spec}(\mathbb{Z}[[C_I]], 1/\text{disc}_d(F))$ is regarded as the moduli of homogeneous polynomials of degree $d$ whose hypersurfaces are smooth, whereas the open subscheme $\tilde{U}_d := \bar{U} \times_{\mathbb{Z}} \mathbb{Z}[1/d]$ (when $d > 2$) or $\bar{U}$ (when $d = 2$) is considered as the moduli of homogeneous polynomials with isolated singularities at the origin. To simplify the situation, let us focus on $\tilde{U}_{2d} := \bar{U} \times_{\mathbb{Z}} \mathbb{Z}[1/2d]$; we also invert 2 so that the square roots of invertible elements define étale $\mathbb{Z}/2$-coverings.

Set $\bar{U}_{2d} = \bar{U} \times_{\mathbb{Z}} \mathbb{Z}[1/2d]$ and $R_{2d} = \mathbb{Z}[[C_I]], 1/2d, 1/\text{disc}_d(F)) = \Gamma(\bar{U}_{2d}, \mathcal{O}_{\bar{U}_{2d}})$. Since $\mathbb{Z}[[C_I]]$ is a unique factorization domain and $\text{disc}_d(F)$ is irreducible, the $\mathbb{F}_2$-vector space $R_{2d}^\times/(R_{2d}^\times)^2$ is spanned freely by the classes of $-1, \text{disc}_d(F), \ell$ (where $\ell$ runs through the
prime divisors of $2d$). For an element $a \in R_{2d}^\times$, we write $[a]$ for the quadratic character of the square root of $a$ in $H^1(\tilde{U}_{2d}, \mathbb{Z}/2)$. Since the map $a \mapsto [a]$ gives an isomorphism $R_{2d}^\times/(R_{2d}^\times)^2 \to H^1(\tilde{U}_{2d}, \mathbb{Z}/2)$, the latter has a basis $[-1], [\text{disc}_d(F)], [\ell]$ as $\mathbb{F}_2$-vector space.

As the map $F: \mathbb{A}^{n+2} \to \mathbb{A}^1$ on $\tilde{U}_{2d}$ is smooth outside the origin, its singular locus is finite over $\tilde{U}_{2d}$. Therefore we have a non-degenerate symmetric bilinear form $(\varphi_F, B_{F,dt})$ on $R_{2d}$ and its discriminant $\text{disc}B_{F,dt} \in R_{2d}^\times/(R_{2d}^\times)^2 \cong H^1(\tilde{U}_{2d}, \mathbb{Z}/2)$. The purpose of this subsection is to express $\text{disc}B_{F,dt}$ via the basis $[-1], [\text{disc}_d(F)], [\ell]$.

The quadratic character corresponding to $[\text{disc}_d(F)]$ relates with the determinants of the $\ell$-adic cohomologies of middle degree of the hypersurfaces $Y$ when $n$ is even (\cite[Theorem 3.5]{26}) and those of double covers of $\mathbb{P}^{n+1}$ ramified along $Y$ when $n$ is odd and $d$ is even (\cite[Theorem 2.3]{31}).

We recall the result in \cite{26}, as it is necessary for our proof. For a proper smooth $k$-variety $Z$ of even dimension $m$, the cup product induces a symmetric perfect pairing on $H^m(Z_k, \mathbb{Q}_\ell(\frac{m}{2}))$. Hence its determinant character is of order at most 2. The same construction is also carried out on a proper smooth scheme over a general scheme by using $\ell$-adic sheaves for a prime $\ell$ invertible in the scheme.

When $Z = Y$ is a hypersurface, we have

**Theorem 5.9.** (\cite[Theorem 3.5]{26})

Let $Y \subset \mathbb{P}^{n+1}_{\tilde{U}_{2d}}$ be the universal smooth hypersurface defined by the equation $F = 0$ over $\tilde{U}_{2d}$. Suppose that $n$ is even. Then the quadratic character $\det R^n f_* \mathbb{Q}_\ell(\frac{n}{2})$, where $f: Y \to \tilde{U}_{2d}$ is the structure map and $\ell$ is a prime divisor of $2d$, is independent of $\ell$ and is defined by the square root of $\varepsilon(n,d) \cdot \text{disc}_d(F)$ with a sign $\varepsilon(n,d) = \pm 1$. The sign is $(-1)^{\frac{d-1}{2}}$ when $d$ is odd and $(-1)^{\frac{n+1}{2}}$ when $d$ is even.

The following is the main result in this subsection.

**Proposition 5.10.** Let $\tilde{U}_{2d} = \text{Spec}(\mathbb{Z}[\{C_i\}, 1/2d, 1/\text{disc}_d(F)]) = \text{Spec}(R_{2d})$. Let $F = \sum_i C_i T^i$ be the universal homogeneous polynomial on $\tilde{U}_{2d}$ and let $\text{disc}B_{F,dt} \in R_{2d}^\times/(R_{2d}^\times)^2 \cong H^1(\tilde{U}_{2d}, \mathbb{Z}/2)$ be the discriminant of $(\varphi_F, B_{F,dt})$. Then we have the following equalities in $H^1(\tilde{U}_{2d}, \mathbb{Z}/2)$.

1. Suppose that $d$ is even.
   
   (a) When $n$ is odd, we have
   
   $$\text{disc}B_{F,dt} = [\text{disc}_d(F)] + \frac{d-2}{2}[-1] + [d].$$

   (b) When $n$ is even, we have
   
   $$\text{disc}B_{F,dt} = [\text{disc}_d(F)].$$

2. Suppose that $d$ is odd.
   
   (a) When $n$ is odd, we have
   
   $$\text{disc}B_{F,dt} = 0.$$

   (b) When $n$ is even, we have
   
   $$\text{disc}B_{F,dt} = [d] + [\text{disc}_d(F)].$$
We record a corresponding result in characteristic 2 as a corollary, in the following form.

**Corollary 5.11.** Let $\mathbb{F}_q$ be a finite field of characteristic 2 with $q$ elements. Let $F \in \mathbb{F}_q[T_0, \ldots, T_{n+1}]$ be a homogeneous polynomial of odd degree $d \geq 3$. Assume that $n$ is even and that $F: \mathbb{A}^{n+2}_{\mathbb{F}_q} \to \mathbb{A}^1_{\mathbb{F}_q}$ has an isolated singularity at the origin. Write $Y \subset \mathbb{P}^{n+1}_{\mathbb{F}_q}$ for the smooth hypersurface defined by $F = 0$. Then the sign

\[(5.3) \quad \epsilon(\mathbb{F}_q, d) \cdot \det(\text{Frob}_q, H^n(Y_{\mathbb{F}_q}, \mathbb{Q}_d(\frac{n}{2})))
\]

is 1 if and only if the Arf invariant $\text{Arf}(F, 0)$ is trivial in $\mathbb{F}_q/\wp(\mathbb{F}_q)$. Here $\epsilon(\mathbb{F}_q, d)$ is a sign which is $(-1)^{\frac{d^2 - 1}{8}}$ if $\mathbb{F}_q$ does not contain $\mathbb{F}_4$ and is 1 otherwise.

**Proof.** Let $W = W(\mathbb{F}_q)$ be the Witt ring with $\mathbb{F}_q$-coefficients and $K_0$ be its fraction field. Choose a homogeneous polynomial $\tilde{F} \in W[T_0, \ldots, T_{n+1}]$ which is a lift of $F$ to $W$. Then $\tilde{F}$ defines a map $\mathbb{A}^{n+2}_W \to \mathbb{A}^1_W$ which is smooth outside the origin, as the non-zero locus of $\text{Res}(\frac{\partial F}{\partial T_0}, \ldots, \frac{\partial F}{\partial T_{n+1}})$ is open. Specializing the equality in Proposition 5.10.2, we have

\[(5.4) \quad \text{disc } B_{\tilde{F},dt} = [(-1)^{\frac{d+1}{2}}d] + [(-1)^{\frac{d+1}{2}}\text{disc}_d(F)]
\]

in $H^1(K_0, \mathbb{Z}/2)$. By Theorem 5.9, the right-hand side in (5.4) is trivial if and only if the sign (5.3) is 1. Note that the Milnor number $\mu(F, 0)$ is equal to $(d - 1)^{n+2}$, which is even. This follows from Lemma 5.12.3 below and the continuity of Milnor numbers (Proposition 3.17). Hence the discriminant $\text{disc } B_{\tilde{F},dt}$ is trivial if and only if $\text{Arf}(F, 0)$ is zero in $\mathbb{F}_q/\wp(\mathbb{F}_q)$ (cf. Remark 3.8). The assertion follows.

First, considering homogeneous polynomials of Fermat type, we show that the proposition is true when $d$ is even and that $\text{disc } B_{\tilde{F},dt}$ is either 0 or $[d] + [\text{disc}_d(F)]$ when $d$ is odd.

**Lemma 5.12.** Set $R_{a^*, 2d} = \mathbb{Z}[a_0^{+1}, \ldots, a_{n+1}^{+1}, 1/2d]$. This ring admits a universal polynomial $F_{a^*} := \sum_i a_i T_i^d$.

1. The divided discriminant of $F_{a^*}$ is equal to

   \[d^{(n+2)(d-1)^{n+1} - a(n,d)} \cdot (a_0 \cdots a_{n+1})(d-1)^{n+1}.\]

2. The discriminant of $(\varphi_{F_{a^*}}, B_{F_{a^*},dt})$ is equal to

   \[(-1)^{(d-2)(d-1)^{n+2}a(n+2) + 2d} \cdot q^{d(d-1)^{n+2}a(n+2)} \cdot (a_0 \cdots a_{n+1})(d-1)^{n+2}.\]

3. The Milnor number $\mu(F_{a^*}, 0)$ is equal to $(d - 1)^{n+2}$.

**Proof.** 1. By (5.2), we have

   \[\text{disc}_d(F_{a^*}) = d^{-a(n,d)} \text{Res}(a_0dT_0^{d-1}, \ldots, (a_{n+1}d)T_{n+1}^{d-1}).\]

   The assertion follows since the resultant is homogeneous in each coefficient $a_i$ of degree $(d - 1)^{n+1}$ and is normalized by $\text{Res}(T_0^{d-1}, \ldots, T_{n+1}^{d-1}) = 1$. 

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Let the situation be as in Proposition 5.10.

1. When \( d \) is even, the assertion 1 in Proposition 5.10 holds.

2. When \( d \) is odd, \( \text{disc} B_{F,dt} \) is equal to either 0 or \([d] + \text{disc}_d(F)\).

Proof. In the sequel, we frequently use Lemmas 5.12.1,2 to compute the images of \( \text{disc} B_{F,dt}, [\text{disc}_d(F)] \) by the map (5.5).

1. Suppose that \( d \) is even. The image of \([\text{disc}_d(F)]\) by (5.5) is \( \sum_i [a_i] \), as \((n + 2)(d - 1)^{n+1} - a(n,d)\) is even. Consequently, the map (5.5) is injective in this case. Hence, it suffices to show the equalities in (a) and (b) as characters in \( H^1(\hat{U}_{a,2d}, \mathbb{Z}/2) \). Then the assertion follows from Lemma 5.12.

2. When \( d \) is odd, the map (5.5) sends \([\text{disc}_d(F)]\) to \([d]\). Hence its kernel consists of the two elements 0 and \([d] + [\text{disc}_d(F)]\). As \( \text{disc} B_{F,dt} \) is contained in the kernel, it is equal to 0 or \([d] + [\text{disc}_d(F)]\) in \( H^1(\hat{U}_{2d}, \mathbb{Z}/2) \) in this case.

To determine \( \text{disc} B_{F,dt} \) when \( d \) is odd, we use the following lemma.

Lemma 5.14. Assume that \( d \) is odd. Let \( k \) be a finite field whose characteristic is prime to \( 2d \). We assume that \( k \) contains a primitive \( d \)-th power root of unity. Then we have the following equalities.

1. When \( n \) is odd, we have

\[
\left( \frac{\text{disc} B_{F,dt}}{k} \right) = 1.
\]

2. When \( n \) is even, we have

\[
\left( \frac{\text{disc} B_{F,dt}}{k} \right) = \det H^n(Y_k, \mathbb{Q}_\ell(\frac{n}{2}))(\text{Frob}_k).
\]
Let us postpone the proof of this lemma and complete the proof of Proposition 5.10

(Proof of Proposition 5.10 admitting Lemma 5.14)

The assertion 1 in the proposition is proved in Lemma 5.13. By the same lemma, we already know that disc_{B,F,dt} = 0 or \([d] + [\text{disc}_d(F)]\) when \(d\) is odd. For a finite field \(k\) with the characteristic prime to 2\(d\), set \(\tilde{U}_k := \tilde{U} \times_k k\). As the divided discriminant \(\text{disc}_d(F) \in k[\{C_1\}]\) with \(k\)-coefficient is geometrically irreducible ([26 Proposition 2.12]), the map \(H^1(\tilde{U}_{2d}, \mathbb{Z}/2) \to H^1(\tilde{U}_k, \mathbb{Z}/2)\) is injective on the subspace \(\{0, [d] + [\text{disc}_d(F)]\}\).

Hence it is enough to determine the image of \(\text{disc}_{B,F,dt}\).

In the sequel of the proof, We choose \(k\) which contains a primitive \(d\)-th power root of unity and also the square roots of \(-1, d\). For such a \(k\), we have equalities in \(H^1(\tilde{U}_k, \mathbb{Z}/2)\)

\[ [d] + [\text{disc}_d(F)] = [\text{disc}_d(F)] = \det Rf_*\mathbb{Q}_\ell(\frac{n}{2}), \]

where \(f: Y \to \tilde{U}_k\) is the structure map of the universal hypersurface. The second equality is the main result of [26], which we recall in Theorem 5.9.

By the Chebotarev density, the assertion 2 in Proposition 5.10 is reduced to the following: for any finite extension \(k'/k\) and any \(k'\)-valued point \(z \in \tilde{U}_k(k')\),

(a)' when \(n\) is odd, we have \(\left(\frac{\text{disc}_{B,F,dt}}{k'}\right) = 1\).

(b)' when \(n\) is even, we have \(\left(\frac{\text{disc}_{B,F,dt}}{k'}\right) = \det H^n(Y_{z,k}, \mathbb{Q}_\ell(\frac{n}{2}))(\text{Frob}_{k'})\).

Here \(F_z\) is the homogeneous polynomial defined by the point \(z\) and \(Y_z\) is the hypersurface defined by \(F_z\). Hence Lemma 5.14 completes the proof.

The rest of this subsection is devoted to the proof of Lemma 5.14. For this purpose, we compute the local epsilon factors of the vanishing cycles complexes. The following method by blowing up is suggested to the author by T. Saito.

Let \(n \geq 0, d \geq 2\) be integers and \(k\) be a perfect field whose characteristic is prime to 2\(d\). For a homogeneous polynomial \(F = \sum_{|I| = d} C_IT^I \in k[T_0, \ldots, T_{n+1}]\) satisfying the equivalent conditions in Lemma 5.8, we describe the vanishing cycles complex \(R\Phi_F(Q_\ell)_{T_0}\) of \(F: \mathbb{A}^{n+2}_k \to \mathbb{A}^1_k\) as follows. Let \(S = \text{Spec}(k[t])\) be the henselization of \(\mathbb{A}^1_k\) at the origin, whose closed and generic points are denoted by \(s\) and \(\eta\). Let \(X\) be the base change \(\mathbb{A}^{n+2}_k \times_{\mathbb{A}^1_k} S\). We use the same symbol \(F: X \to S\) for the base change by abuse of notation.

Let \(\bar{X} \to X\) be the blow-up at the origin and let \(\tilde{F}\) be the structure map \(\bar{X} \to S\). We write \(E\) for the exceptional divisor of the blow-up and \(D_F\) for the strict transform of the special fiber \(X_s\) in \(\bar{X}\). Using the coordinate functions \(T_0, \ldots, T_{n+1}\), we identify \(E\) with \(\mathbb{P}^{n+1}_k\) and \(E \cap D_F\) with the hypersurface \(Y = \{F = 0\} \subset \mathbb{P}^{n+1}_k = E\).

Let \(S' := \text{Spec}(O_S[u]/(u^d - t))\) be a tame ramified extension of \(S\). We define \(\hat{X}'\) to be the normalization of \(\bar{X} \times_S S'\). Let \(E'\) be the pullback of \(E \subset \bar{X}\) to \(\hat{X}'\). For each \(0 \leq i \leq n+1\), \(\hat{X}\) has the open locus where we have \(T_i \cdot \mathcal{O}_{\hat{X}} = \mathcal{O}_{\hat{X}}(-E)\). Over this open subset, \(\hat{X}'\) is defined by the \(d\)-th power root \(\sqrt[d]{F(T_0/T_i, \ldots, T_{n+1}/T_i)}\). Therefore, the finite coverings \(\hat{X}' \to \hat{X}\) and \(E' \to E\) are tamely ramified along \(D_F\) and \(Y = E \cap D_F\) respectively and finite étale elsewhere. Moreover, the inclusion \(D_{F'} \hookrightarrow \hat{X}\) lifts to an inclusion \(D_{F'} \hookrightarrow \hat{X}'\), which defines a smooth divisor \(D'_{F'}\) of \(\hat{X}'\). The divisor \(E' \cup D'_{F'}\) is simple normal crossings in \(\hat{X}'\) with \(E' \cap D'_{F'}\) isomorphic to \(Y\).

Write \(s'\) for the closed point of \(S'\). Since the special fiber \(\hat{X}'_{s'}\) is equal to \(E' \cup D'_{F'}\) as divisors, \(\hat{X}'\) is strictly semi-stable over \(S'\), by which we mean that, Zariski locally on \(\hat{X}'\),

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it is étale over \( \text{Spec}(O_S[t_1, \ldots, t_m]/(t_1 \cdots t_r - \pi)) \) for a uniformizer \( \pi \) of \( O_S \) and integers \( 1 \leq r \leq m \). For a strictly semi-stable scheme over a trait, its nearby cycles complex is computed in \[25\], which we recall for \( X' \) in Lemma 5.15.

To proceed, we give necessary definitions and notations. Let \( \bar{k} \) be an algebraic closure of \( k \) with its affine spectrum denoted by \( \bar{s} \). Write \( S^{\text{sur}} \) for the strict henselization of \( S \) at \( s \rightarrow S \). Write \( S^{\text{sur}} := S' \times_S S^{\text{sur}} \) and let \( s' \) be its closed point defined by the diagonal map \( \bar{s} \rightarrow S' \times_S S^{\text{sur}} \). We write \( \bar{X}'_{s'} = \bar{X}' \times_S s' \) for the geometric special fiber of \( \bar{X}' \) over \( S' \). Let \( \bar{s} \) be the spectrum of a separable closure of the function field of \( S^{\text{sur}} \), where \( k(\bar{s}) \) is also regarded as a separable closure of the function field \( k(\eta) \) of \( S \). We have canonical morphisms

\[
(5.6) \quad \bar{X}' \times_S s' \xrightarrow{\bar{\theta}} \bar{X}' \times_S S^{\text{sur}} \xrightarrow{\bar{\psi}} \bar{X}'_{s'}.
\]

The Galois group \( \text{Gal}(\bar{s}/\eta) \) acts on \( \bar{X}' \times_S S^{\text{sur}} \) as follows. It acts on \( \bar{X} \times_S S^{\text{sur}} \) via the right component of the product, hence acts on its normalization \( \bar{X}' \times_S S^{\text{sur}} \). Under the isomorphisms \( \bar{X}' \times_S s' \cong (\bar{X}' \times S^{\text{sur}}) \times_{S^{\text{sur}}} \bar{s}, \bar{X}'_{s'} \cong (\bar{X}' \times S^{\text{sur}}) \times_{S^{\text{sur}}} s' \), these schemes also admit Galois actions, by which \( (5.6) \) are \( \text{Gal}(\bar{s}/\eta) \)-equivariant. Let us write \( \bar{\psi}' : (\bar{X}' \times_S S^{\text{sur}}) \setminus \bar{X}'_{s'} \rightarrow \bar{X}' \times_S S^{\text{sur}} \) for the open complement of \( \bar{\psi}' \).

For an immersion \( \iota : Z \rightarrow T \) of schemes, \( \Omega_{\ell,Z} \) denotes the 0-extension \( \iota_0 \iota^* \Omega_{\ell} \). Set \( \Omega_{\ell,E'_k,D'_{F,k}} := \Omega_{\ell,E'_k} \oplus \Omega_{\ell,D'_{F,k}} \) which is an \( \ell \)-adic sheaf on \( \bar{X}'_{s'} \). Let \( \theta'_1 : \Omega_{\ell,E'_k,D'_{F,k}} \rightarrow \bar{\psi}' R^{1}\hat{j}'_s \Omega_{\ell}(1) \) be the map defined by the divisors \( E'_k, D'_{F,k} \) \((25.1.1)\). By cup product, it induces a map \( \theta'_m : \wedge^m \Omega_{\ell,E'_k,D'_{F,k}} \rightarrow \hat{\psi}' R^m \hat{j}'_s \Omega_{\ell}(m) \) for \( m \geq 1 \).

**Lemma 5.15.** Let \( R^\psi \Omega_{\ell} = \hat{\psi}' R^m \hat{j}'_s \Omega_{\ell} \) be the nearby cycles complex of \( \bar{X}' \) as an \( S' \)-scheme.

1. \((25. \text{Proposition 1.1.1.2.1})\) The map \( \theta'_m \) is an isomorphism for \( m \geq 1 \).

2. \((25. \text{Proposition 1.1.2.2})\) Consider the composition of

\[
\theta : \Omega_{\ell,\bar{X}'_{s'}} \xrightarrow{\delta} \Omega_{\ell,E'_k,D'_{F,k}} \xrightarrow{\theta'_1} \hat{\psi}' R^1 \hat{j}'_s \Omega_{\ell}(1)
\]

where \( \delta \) is the canonical map. Then, for \( m \geq 0 \), the cup product \( \theta' \cup : \hat{\psi}' R^m \hat{j}'_s \Omega_{\ell}(m) \rightarrow \hat{\psi}' R^{m+1} \hat{j}'_s \Omega_{\ell}(m+1) \) factors as \( \hat{\psi}' R^m \hat{j}'_s \Omega_{\ell}(m) \rightarrow R^m \psi \Omega_{\ell}(m) \rightarrow \hat{\psi}' R^{m+1} \hat{j}'_s \Omega_{\ell}(m+1) \) where the former map is the canonical one, and this factorization identifies \( R^m \psi \Omega_{\ell}(m) \) with the image of the map \( \theta' \).

3. We have \( \Omega_{\ell,\bar{X}'_{s'}} \cong R^0 \psi \Omega_{\ell}, \Omega_{\ell,E'_k,D'_{F,k}}(-1) \cong R^1 \psi \Omega_{\ell}, \) and \( R^m \psi \Omega_{\ell} = 0 \) for \( m \neq 0, 1 \) as sheaves with equivariant \( \text{Gal}(\bar{s}/\eta) \)-actions.

**Proof.** The assertions 1, 2 are proved in \[25\] Proposition 1.1.2. Then the assertion 3 readily follows from the commutativity of the diagram

\[
\begin{array}{c}
\wedge^m \Omega_{\ell,E'_k,D'_{F,k}} \\
\downarrow \theta'_m \quad \downarrow \theta'_{m+1}
\end{array}
\xrightarrow{\delta \wedge} \\
\begin{array}{c}
\wedge^{m+1} \Omega_{\ell,E'_k,D'_{F,k}} \\
\end{array}
\xrightarrow{\hat{\psi}' R^m \hat{j}'_s \Omega_{\ell}(m)} \\
\xrightarrow{\hat{\psi}' R^{m+1} \hat{j}'_s \Omega_{\ell}(m+1)},
\]

where \( \theta'_m \) for \( m = 0 \) means the canonical isomorphism \( \Omega_{\ell,\bar{X}'_{s'}} \rightarrow \hat{\psi}' \hat{j}'_s \Omega_{\ell} \). \qed
Let $R\Phi_F(\mathbb{Q}_\ell)_0$ be the vanishing cycles complex of $X$ with respect to $F$, supported on the origin. Using Lemma [5.15] we describe $R\Phi_F(\mathbb{Q}_\ell)_0$ as follows. The morphisms (5.6) fit into the commutative diagram

$$
\begin{align*}
&X' \times_{S'} \overline{\eta} \xrightarrow{j} \tilde{X}' \times_{S'} S'^{ur} \xrightarrow{i} \tilde{X}'_S \\
&\qquad \cong \\
&\tilde{X} \times_S \overline{\eta} \xrightarrow{j} \tilde{X} \times_S S^{ur} \xrightarrow{i} \tilde{X}_S \\
&\qquad \cong \\
&X \times_S \overline{\eta} \xrightarrow{j} X \times_S S^{ur} \xrightarrow{i} X_S,
\end{align*}
$$

on which $\text{Gal}(\overline{\eta}/\eta)$ acts equivariantly. Here the left vertical arrows are isomorphisms since the blow-up has its center in the special fiber and $\tilde{X} \times_S S'$ is regular on its generic fiber.

**Lemma 5.16.** In the Grothendieck group of $\ell$-adic representations of $\text{Gal}(\overline{\eta}/\eta)$, we have an equality

$$
[R\Phi_F(\mathbb{Q}_\ell)_0] = [R\Gamma(E'_k, \mathbb{Q}_\ell)] - [R\Gamma(Y_k, \mathbb{Q}_\ell)(-1)] - [\mathbb{Q}_\ell].
$$

Here the Galois action on the cohomology groups in the right-hand side is given as follows.

1. Let $\mu_d$ be the group of $d$-th power roots of unity in $\overline{k}$. The inertia subgroup acts on $R\Gamma(E'_k, \mathbb{Q}_\ell)$ through the quotient $\mu_d$, whose action is induced from the geometric action on $E'_k$ given by $d$-th power roots of $F(T_0/T_1, \ldots, T_{n+1}/T_i)$. Let $\eta'$ be the generic point of $S'$. The action of the subgroup $\text{Gal}(\overline{\eta}/\eta')$ is given by the composition of the unramified quotient $\text{Gal}(\overline{\eta}/\eta') \to \text{Gal}(\overline{k}/k)$ and the canonical action of $\text{Gal}(\overline{k}/k)$ on $R\Gamma(E'_k, \mathbb{Q}_\ell)$.

2. The Galois group $\text{Gal}(\overline{\eta}/\eta)$ acts on $R\Gamma(Y_k, \mathbb{Q}_\ell)(-1)$ through the unramified quotient $\text{Gal}(\overline{k}/k)$.

**Proof.** First, we decompose $\alpha_* R\psi_{\mathbb{Q}_\ell} \cong i^* R\overline{\pi}_j; \mathbb{Q}_\ell =: R\Psi_F(\mathbb{Q}_\ell)$ in the Grothendieck group of $\ell$-adic sheaves on $\tilde{X}_S$ with equivariant $\text{Gal}(\overline{\eta}/\eta)$-actions. By Lemma [5.15] we have $[\alpha_* R^0 \psi_{\mathbb{Q}_\ell}] = [\alpha_* Q_{\ell,E'_k}] + [Q_{\ell,D_{E'_k}}] - [Q_{\ell,Y_k}]$, $[\alpha_* R^1 \psi_{\mathbb{Q}_\ell}] = [Q_{\ell,Y_k}(1)]$, and $\alpha_* R^i \psi_{\mathbb{Q}_\ell} = 0$ for the others. Hence we have

$$
[R\Psi_F(\mathbb{Q}_\ell)] = [\alpha_* R\psi(\mathbb{Q}_\ell)] = [\alpha_* Q_{\ell,E'_k}] + [Q_{\ell,D_{E'_k}}] - [Q_{\ell,Y_k}],
$$

The right squares in (5.7) are cartesian up to nilpotent thickening. Hence the proper base change theorem gives us $R\pi_* R\Psi_F(\mathbb{Q}_\ell) \cong R\Psi_F(\mathbb{Q}_\ell)$. Noting the isomorphism $R\pi_* Q_{\ell,D_{E'_k}}(1) \cong Q_{\ell,F^{-1}(0),Y_k}$, we have

$$
[R\Phi_F(\mathbb{Q}_\ell)_0] = [R\Psi_F(\mathbb{Q}_\ell)] - [Q_{\ell,F^{-1}(0)}] = [R(\pi\alpha)_* Q_{\ell,E'_k}] - [R\pi_* Q_{\ell,Y_k}(1)] - [\mathbb{Q}_\ell,0]
$$

in the Grothendieck group of $\ell$-adic sheaves on $X_S$. Taking the stalk at the origin, we obtain the desired equality. The Galois actions on $E'_k, Y_k$ are induced from the Galois action on (5.7). Hence the assertions 1, 2 are verified.

To compute the local epsilon factor of $R\Gamma(E'_k, \mathbb{Q}_\ell)$, we need to recall well-known results on the cohomology groups of a variety with a finite group action.
Lemma 5.17. Let \( k \) be a finite field with \( q \) elements. For elements \( a, b \in \overline{\mathbb{Q}_\ell}^\times \), let us write \( a \equiv b \) if \( ab^{-1} \in \mathbb{Q}_\ell^\times \).

Let \( Z \) be a proper smooth \( k \)-variety of dimension \( m \) with an admissible action of a finite group \( G \). Write \( Z_k/G \) for the quotient scheme of \( Z_k \) by \( G \). For an irreducible \( \mathbb{Q}_\ell \)-representation \( \rho \) of \( G \), let \( H^i_\rho \) denote the \( \rho \)-isotypic component of \( H^i(Z_k, \mathbb{Q}_\ell) \).

1. The cup product induces an isomorphism \( H^i_\rho \otimes (H^i_\rho)^! \cong (H^i_\rho)^! \otimes (-m) \) of Galois representations. Consequently, we have

\[
\text{det}(\pm \text{Frob}_k, H^i_\rho) \cdot \text{det}(\pm \text{Frob}_k, H^{2m-i}_\rho) = q^{mb_{\rho,i}},
\]

where the signs are taken to be equal to each other and we set \( b_{\rho,i} = \text{rk} H^i_\rho = \text{rk} H^{2m-i}_\rho \).

2. Suppose that an irreducible representation \( \rho \) is self-dual. Then we have

\[
\text{det}(\text{Frob}_k, H^m_\rho) = \pm q^{mb_{\rho,m}}.
\]

Further if \( m \) is odd, then \( b_{\rho,m} \) is even and the sign is 1.

3. For the trivial representation \( \rho = 1 \), we have an isomorphism \( H^i_1 \cong H^i(Z_k/G, \mathbb{Q}_\ell) \) of Galois representations.

4. If \( G \) is abelian of odd order, we have

\[
\text{det}(\text{Frob}_k, R\Gamma(Z_k, \mathbb{Q}_\ell)) \equiv \text{det}(\text{Frob}_k, H^m_\rho(Z_k/G, \mathbb{Q}_\ell))^{(-1)^m}.
\]

Proof. The pairing

\[
(5.10) \quad H^i_\rho \times H^{2m-i}_\rho \rightarrow H^2m(Z_k, \mathbb{Q}_\ell) \rightarrow \mathbb{Q}_\ell(-m)
\]

is perfect by the Poincaré duality and the fact that the cup product is \( G \)-equivariant, which shows the assertion 1. Applying 1 to a self-dual \( \rho \), the equality in 2 follows. When \( m \) is odd and \( \rho \) is self-dual, \( (5.10) \) equips \( H^m_\rho \) with an alternating perfect pairing, which shows the remaining part of 2.

Let \( f: Z \rightarrow Z/G \) be the quotient map. The isomorphism \( R\Gamma(G, R\Gamma(Z_k/G, -)) \cong R\Gamma(Z_k/G, R\Gamma(G, -)) \) of derived functors gives

\[
R\Gamma(Z_k, \mathbb{Q}_\ell)^G \cong (R\Gamma(Z_k/G, f_! \mathbb{Q}_\ell))^G \cong R\Gamma(Z_k/G, (f_! \mathbb{Q}_\ell)^G).
\]

As the canonical map \( \mathbb{Q}_\ell \rightarrow (f_! \mathbb{Q}_\ell)^G \) is an isomorphism, which can be verified stalk-wise, the assertion 3 follows.

We show 4. The assumption on \( G \) implies that any irreducible representation of \( G \) (which is one dimensional as \( G \) is abelian) is not self-dual unless it is trivial. Hence by 1, we have

\[
\text{det}(\text{Frob}_k, R\Gamma(Z_k, \mathbb{Q}_\ell)) \equiv \text{det}(\text{Frob}_k, H^1_1^{(1)}(-1)^m)
\]

in \( \overline{\mathbb{Q}_\ell}^\times / q^\mathbb{Z} \). Then the assertion 4 follows from 3. \( \square \)

Let us go back to our situation. Let \( k \) be a finite field in which \( 2d \) is invertible. Let \( F: \mathbb{A}_k^{n+2} \rightarrow \mathbb{A}_k^{1} \) be a homogeneous function of degree \( d \) which is smooth outside the origin. We collect necessary results on \( Y, E' \) as follows.
Lemma 5.18. Assume that $d$ is odd.

1. The integer $\chi(\mathbb{P}_k^{n+1}) + \chi(Y_k)$ is odd.

2. Let $R\Gamma(E'_k, \mathbb{Q}_\ell)$ be the complex of tamely ramified $\text{Gal}(\overline{\eta}/\eta)$-representations considered in Lemma 5.16. Further assume that $k$ contains $\mu_d$. We have

$$\varepsilon_0(A^1_{k,(0)}, R\Gamma(E'_k, \mathbb{Q}_\ell), dt) \equiv (-1)^{\chi(\mathbb{P}_k^{n+1})}.$$ 

Here we write $a \equiv b$ for $a, b \in \overline{\mathbb{Q}_\ell}^\times$ if $ab^{-1} \in q^Z$.

Proof. 1. We write $T_X$ for the tangent bundle of a smooth variety $X$. Let $h := c_1(\mathcal{O}_{\mathbb{P}_k^{n+1}}(1)) \in \text{CH}^1(\mathbb{P}_k^{n+1})$ be the first Chern class of the very ample sheaf. Let $\iota : Y \to \mathbb{P}_k^{n+1}$ be the closed immersion. The canonical exact sequence

$$0 \to T_Y \to \iota^*T_{\mathbb{P}_k^{n+1}} \to \iota^*\mathcal{O}_{\mathbb{P}_k^{n+1}}(d) \to 0$$

and the projection formula give a formula for the total Chern class $\iota_*c(T_Y)$ ([10, 3.2.12]):

$$\iota_*c(T_Y) = \left( (1 + h)^{n+2}/(1 + dh) \right) \cap [Y] = (1 + h)^{n+2}dh/(1 + dh)$$

in the Chow ring of $\mathbb{P}_k^{n+1}$. As $d$ is odd, taking modulo 2 gives $\iota_*c(T_Y) \equiv (1 + h)^{n+1}h$, hence $\chi(Y_k) \equiv n + 1$. On the other hand, we have $\chi(\mathbb{P}_k^{n+1}) = n + 2$. The assertion 1 follows.

2. Note that the action of $\mu_d$ on $E'_k$ is defined over $k$ by the assumption $\mu_d \subset k$. Hence we can apply Lemma 5.17 to $(Z, G) = (E', \mu_d)$.

By [6, (5.4)], we have

$$\varepsilon_0(A^1_{k,(0)}, R\Gamma(E'_k, \mathbb{Q}_\ell), dt) \equiv \det(R\Gamma(E'_k, \mathbb{Q}_\ell))(t) \cdot \varepsilon_0(A^1_{k,(0)}, R\Gamma(E'_k, \mathbb{Q}_\ell), \frac{dt}{t}).$$

Here the determinant in the product is considered as a character of the function field of $S$ by the local class field theory. Note that the norm map $\mathcal{O}_S' \to \mathcal{O}_S$ sends $u = \sqrt{t}$ to $t$. Hence the functoriality of the local class field theory and Lemma 5.16 show that $\det(R\Gamma(E'_k, \mathbb{Q}_\ell))(t)$ is equal to $\det(\text{Frob}_k, R\Gamma(E'_k, \mathbb{Q}_\ell))$, which is equal to $\det(\text{Frob}_k, H^{n+1}((\mathbb{P}_k^{n+1}, \mathbb{Q}_\ell))^{-1} \equiv 1$ by Lemma 5.17.

It remains to show $\varepsilon_0(A^1_{k,(0)}, R\Gamma(E'_k, \mathbb{Q}_\ell), \frac{dt}{t}) \equiv (-1)^{\chi(\mathbb{P}_k^{n+1})}$. For a character $\chi : \mu_d \to \overline{\mathbb{Q}_\ell}^\times$, set $V_{\chi} = (R\Gamma(E'_k, \mathbb{Q}_\ell) \otimes \chi^{-1})^{\mu_d}$. By [6, 5.10], we have

$$\varepsilon_0(A^1_{k,(0)}, R\Gamma(E'_k, \mathbb{Q}_\ell), \frac{dt}{t}) \equiv \prod_{\chi} (-\tau(\chi_k, \psi))^{rkV_{\chi}}.$$ 

Here we set $\chi_k(a) = \chi(a^{1/k})$ for $a \in k^\times$ and set $\tau(\chi_k, \psi) = -\sum_{a \in k^\times} \chi_k(a)^{-1} \psi(a)$. For $\chi \neq 1$, we have [6, 5.7]

$$\tau(\chi_k, \psi) \cdot \tau(\chi_k^{-1}, \psi) = \chi_k(-1)q = q.$$ 

The equality $\chi_k(-1) = 1$ holds as $\chi$ is of odd order. Since we have $rkV_{\chi} = rkV_{\chi^{-1}}$ and $rkV_{1} = \chi(\mathbb{P}_k^{n+1})$ by Lemma 5.17, we have

$$\varepsilon_0(A^1_{k,(0)}, R\Gamma(E'_k, \mathbb{Q}_\ell), \frac{dt}{t}) \equiv (-\tau(1_k, \psi))^{rkV_1} = (-1)^{\chi(\mathbb{P}_k^{n+1})}.$$ 

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The assertion follows. \hfill \qed

(Proof of Lemma 5.14)

Since there only appear signs in the lemma, we can work in \( \mathbb{Q}_\ell^\times / q^\mathbb{Z} \), rather than \( \mathbb{Q}_\ell^\times \).

As in Lemmas 5.17, 5.18 we write \( a \equiv b \) for \( a, b \in \mathbb{Q}_\ell^\times \) if \( ab^{-1} \in q^\mathbb{Z} \).

From the continuity of Milnor numbers (Proposition 3.17) and Lemma 5.12.3, the Milnor number \( \mu(F, 0) \) is equal to \((d - 1)^n + 2\), which is also equal to \( \dim \text{tot} R\Phi_F(\mathbb{Q}_\ell)_0 \) up to sign, by the Milnor formula (1.1). Hence they are multiples of 4 as \( d \) is odd. Then Corollary 5.5.1 takes the following form in this case:

\[
\varepsilon_0(\mathbb{A}_k^{1}, R\Phi_F(\mathbb{Q}_\ell)_0, dt) = q^{(-1)^n + 3(\frac{n+2}{2})\mu(F,0)} \cdot (\text{disc} B_F dt)_k,
\]

where we use \( \tau^2_0 = (-\frac{1}{k})q \). We compute the local epsilon factor modulo \( q^\mathbb{Z} \). By Lemma 5.16 and the multiplicativity of local epsilon factor, \( \varepsilon_0(\mathbb{A}_k^{1}, R\Phi_F(\mathbb{Q}_\ell)_0, dt) \) is equal to

\[
\varepsilon_0(\mathbb{A}_k^{1}, R\Gamma(E_k', \mathbb{Q}_\ell), dt) \cdot \det(-\text{Frob}_k, R\Gamma(Y_k, \mathbb{Q}_\ell)(-1))^{-1} \cdot (-1),
\]

where we use, for an unramified representation \( V \), \( \varepsilon_0(\mathbb{A}_k^{1}, V, dt) = \det(-\text{Frob}_k, V)^{-1} \). By Lemma 5.18.2, and the fact that the cohomology groups of projective spaces are either 0 or Tate twists, we proceed

\[
\varepsilon_0(\mathbb{A}_k^{1}, R\Phi_F(\mathbb{Q}_\ell)_0, dt) \equiv (-1)^{\chi(\mathbb{P}^{n+1}) + \chi(Y_k) + 1} \cdot \det(\text{Frob}_k, H^n(Y_k, \mathbb{Q}_\ell))^{-1}
\]

\[
= \det(\text{Frob}_k, H^n(Y_k, \mathbb{Q}_\ell))^{-1},
\]

as \( \chi(\mathbb{P}^{n+1}) = \chi(Y_k) + 1 \) is even (Lemma 5.18.1). Hence the assertion follows by applying Lemmas 5.17.1,2 to \( (Z, G) = (Y, \{1\}) \).

The proof of Proposition 5.10 is completed.

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