Phase-shift inversion in oscillator systems with periodically switching couplings

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(Dated: February 28, 2012)

A system’s response to external periodic changes can provide crucial information about its dynamical properties. We investigate the synchronization transition, an archetypical example of a dynamic phase transition, in the framework of such a temporal response. The Kuramoto model under periodically switching interactions has the same type of phase transition as the original mean-field model. Furthermore, we see that the signature of the synchronization transition appears in the relative delay of the order parameter with respect to the phase of oscillating interactions as well. Specifically, the phase shift becomes significantly larger as the system gets closer to the phase transition so that the order parameter at the minimum interaction density can even be larger than that caused by the diverging relaxation time, in a similar way to the resonance near the critical point in the kinetic Ising model. Our result, based on exhaustive simulations on globally coupled systems as well as scale-free networks, shows that an oscillator system’s phase transition can be manifested in the temporal response to the topological dynamics of the underlying connection structure.

PACS numbers: 05.45.Tp, 05.45.Xt, 64.60.an

I. INTRODUCTION

A collection of interacting coupled oscillators is one of the most intensively studied systems showing a dynamic phase transition called a synchronization transition [1–3]. Such systems have been used as models to describe various phenomena such as the pacemakers of the heart, the collection of amoeba, large-scale ecosystems [4–9], and fiber-optic networks of optoelectronic oscillators [10]. As the coupling strength increases, such systems undergo a phase transition from the desynchronized (disordered) state to the synchronized (ordered) state characterized by critical phenomena. The most well-known theoretical formalism is the Kuramoto model [1] where the oscillators interact with each other by a sinusoidal coupling strength with respect to the phase differences of the two oscillators.

Aside from the steady-state behavior after the initial transient behavior, the dynamical aspect of the oscillators has also been studied recently [11, 12]. Especially, the effects of temporally varying interaction structures [13] themselves are worthwhile to investigate since we can systematically analyze the response of systems to such structural changes of interactions [14, 13]. For instance, the mobile oscillator is considered in Refs. [16–18] as an example of temporally switching interactions. In this Brief Report, we take the Kuramoto model with periodically switching interactions and analyze the temporal response in the collective phase of oscillators to such periodicity in interactions. Such periodically switching interactions may have implications to biological or social communications among compartments under natural circadian rhythms, for instance [19].

In terms of the temporally averaged order parameter and its amplitude of oscillation, we verify that the model undergoes the same synchronization transition as the mean-field (MF) Kuramoto model [2] with the finite-size scaling (FSS) analysis. Furthermore, with numerical simulations and analogy to the kinetic Ising model [20], we show that the signature of the synchronization transition is revealed in the relative phase shift of the order parameter with respect to the density of interactions, as a resonance-like phase-shift inversion. The result is also consistent with the diverging relaxation time at the critical coupling strength [11, 12]. Therefore, we claim that the system’s temporal response to the temporally changing interactions can be used as an indicator of a phase transition.

II. MODEL

We consider the Kuramoto-type oscillator dynamics [2] composed of $N$ oscillators as

$$\frac{d\phi_i}{dt} = \omega_i + \frac{2K}{N} \sum_{j=1}^{N} a_{ij} T_{ij}(t) \sin(\phi_i - \phi_j), \quad (1)$$

where \{\phi_i\} is the set of oscillators’ phases, \{\omega_i\} is the set of natural frequencies of oscillators given by the Gaussian distribution with average 0 and unit variance, and the symmetric adjacency matrix $a_{ij} = 1$ if $i$ and $j$ are connected and 0 otherwise. The interaction $a_{ij} T_{ij}(t)$ mediated by the edge between the oscillators $i$ and $j$ is subject to the periodic function as $T_{ij}(t) = H[\sin(\Omega t + \theta_{ij})]$, where $H$ is the Heaviside step function.
where $H$ is the Heaviside step function, $\Omega$ is the frequency of switching common to all the edges, and $\{\theta_{ij}\}$ for $i$–$j$ pairs with $a_{ij} = 1$ is the set of edges’ intrinsic phases, which is randomly assigned from the uniform distribution of the interval $[0, \pi]$. By choosing such an interval, the edge densities change periodically with the exact triangle wave subject to the frequency $\Omega$, from fully connected edges to isolated oscillators without interactions. Note that the factor 2 in Eq. (1) is to directly compare the coupling strength $K$ of our model with $T_{ij} = 1/2$ to the $K$ value in the original Kuramoto model, where $\overline{r}$ refers to the temporal average of $x(t)$ over a period.

In addition to the original control parameter of coupling strength $K$ in the Kuramoto model, the edge frequency $\Omega$ driving the system is a crucial parameter as well. The response of oscillators to the toggling interactions is measured by the temporal behavior of conventional phase order parameter $\Delta(t)$ where

$$\Delta(t) \exp[i\psi(t)] = \frac{1}{N} \sum_{i=1}^{N} \exp[i\phi_i(t)]. \quad (2)$$

The order parameter’s dynamics depends on the periodic change of interactions with the density $\rho(t) = \sum_{i<j} a_{ij} T_{ij}(t)/\sum_{i<j} a_{ij}$. We provide the java applet simulating this model system in case of globally connected oscillators, for readers to observe the behavior of a system of relatively small sizes [2]. From now on, we present our numerical simulation results and their implications.

III. RESULTS

A. Globally coupled oscillators

First, we consider the globally coupled case, i.e., $a_{ij} = 1 - \delta_{ij}$. A typical temporal behavior of oscillators is illustrated in Fig. 1 in case of $\Omega = 1$. Note that the oscillating behavior of the order parameter $\Delta(t)$ driven by switching edges (with the same frequency) is shown even in case of quite low values of $K$, for which the original MF Kuramoto model corresponds to the disordered phase. Another interesting aspect is the relative phase shift of $\Delta(t)$ with respect to $\rho(t)$. In other words, the maximum or minimum of $\Delta(t)$ occurs after some time when the maximum of minimum of $\rho(t)$ is reached, as shown in Fig. 1(a). Moreover, as $K$ is increased, the delay for maximum values and the delay for minimum values become more asymmetric, as clearly shown in Fig. 1(b).

In spite of such temporal oscillations of $\Delta(t)$, the temporally averaged value $\overline{\Delta}$ (averaged after the transient time $t \simeq 20$) shows exactly the same universality class of synchronization transition as the MF Kuramoto model with $\beta = 1/2$, and $\nu = 5/2$, as described in the FSS scaling collapse with $\overline{\Delta} = N^{-N/2} f ((K - K_c(\Omega)) N^{1/\nu}; r)$ (the scaling function $f$ itself, as well as the critical coupling strength $K_c$, depends on $\Omega$) shown in Fig. 2. We suggest the slightly larger value of $K_c(\Omega)$ than that of the MF model $K_c^{MF} = 2\sqrt{2/\pi} \simeq 1.596$ [2] stem from additional quenched randomness caused by $\{\theta_{ij}\}$ for edges, similarly to the lower $T_c$ (the disordered phase extended) for the kinetic Ising model under the external oscillating field [20]. Since we have confirmed the MF transition for $\overline{\Delta}$, from now on we focus the temporal behavior of $\Delta(t)$, especially with respect to the oscillating interac-

![Figure 1](image1.png)

**FIG. 1:** A typical behavior of the order parameter $\Delta(t)$ and the interaction density $\rho(t)$. The system size $N = 800$, the edge frequency $\Omega = 1$, and the time series of $\Delta(t)$ depending on $K$ are shown as smooth curves in (a) and $\rho(t)$ (independent of $K$) as black triangle waves in (b). The $\Delta$–$\rho$ diagram for $t > 20$ after the transient behavior is shown in (b). All the curves are results averaged over 50 samples.

![Figure 2](image2.png)

**FIG. 2:** FSS scaling collapse of the temporally averaged order parameter $\overline{\Delta}$ for various system sizes with the critical exponents of the original MF Kuramoto model [$K_c = 1.62(1)$, $\beta = 1/2$, and $\nu = 5/2$]. The edge frequency is $\Omega = 1$.
the “linear velocity” of Δ(t), which is shown to be conserved for a given value of K. More importantly, the synchronization transition is clearly shown in the steep change of A and S near the critical coupling strength K_{MF} for the original MF Kuramoto model, as shown in Figs. 3 and 4. Therefore, we conclude that the signature of the MF synchronization transition of Kuramoto model is resurfaced in terms of A and S.

Aside from the amplitude of Δ and enclosed area in the Δ–ρ diagram, a notable property of the Δ–ρ diagram in Fig. 3(b) is successive change in the shapes of enclosed regions, as K is increased. First, the desynchronization process of Δ(t) after the onset of decreasing phase of ρ(t) is slower than the synchronization process of Δ(t) after the onset of increasing phase of ρ(t) in general. This asymmetry is natural, since the response of the order parameter to such linear functions is integrated form, i.e., quadratic functions depending on the initial values.

We show that the synchronization transition is also observed in the phase shift of Δ(t) with respect to ρ(t) as well. To systematically analyze such a phase shift caused by the temporal delay of Δ(t), we check the relative height difference of Δ at the right and left ends in Fig. 3(b), corresponding to Δ(ρ_{max}) and Δ(ρ_{min}) respectively, also affects the shape of enclosed regions. In Fig. 3(a), we plot such height differences in the K–Ω space, and the height difference occurs mainly in small Ω and large K regimes. Interestingly, near K ≈ 1.6 in relatively large Ω regimes, the difference becomes negative, which corresponds to Δ(ρ_{max}) < Δ(ρ_{min}). We call this phenomenon phase-shift inversion. K ≈ 1.6 is close to the MF critical coupling constant K_{MF} again, and we suggest that it is related to critical slowing down near the critical point in a phase transition, as shown in the diverging relaxation time \( \tau \) at \( K_{MF} \) given by \( 1 = \sqrt{\pi/8K \exp[1/(2\tau^2)]} \). This resonance-like phase-shift inversion phenomenon occurs similarly to the frequency matching condition around the phase transition in the MF kinetic Ising model [20].

### B. Scale-Free Networks

To validate our conclusion on the phase transition in terms of phase-shift inversion, we use substrate network topologies other than the globally coupled one. The static scale-free network (SFN) model [22] with the tunable degree distribution \( \rho(k) \sim k^{-\gamma} \) provides various underlying interaction structures such as the pure MF case (\( \gamma > 5 \)), the MF with critical exponents depend on \( \gamma (3 < \gamma < 5) \), and eventually the one where critical coupling strength vanishes as \( K_c \rightarrow 0 \) (\( \gamma < 3 \)) [23]. We use the SFN model for \{\eta_j\} in Eq. 1 and observe the phase-shift inversion diagram for the three representative cases: \( \gamma = 6.5, 3.75, \) and \( 2.2 \), as shown in Figs. 3(b)–(d).
One can observe the phase-shift inversion in SFN cases with the same system size $N = 800$ is used for all the cases. The rescaled coupling strength $KN/\langle k \rangle$, where $\langle k \rangle$ is set as 20, is used for (b)–(d), to compare with the globally coupled case (a) with the same Eq. [1]. The same mean-field synchronization transition as the original Kuramoto model with the frequency matching condition around the phase transition [20] and would be worth investigating further. Since periodicity plays important roles in many parts of the nature, especially for systems under natural circadian rhythms or external stimuli, our results suggest that the temporal response to such periodicity can be an important cue to characterize the system.

**IV. SUMMARY AND DISCUSSIONS**

We have studied the Kuramoto model under periodically switching interactions, checked that it shows the same mean-field synchronization transition as the original Kuramoto model with the finite-size scaling analysis, and found that the phase transition is observed in terms of dynamical properties such as the amplitude of oscillation of the order parameter and its relative phase shift with respect to the overall strength of interactions. In particular, the latter causes the phase-shift inversion phenomenon that the significantly large phase shift near the phase transition let the order parameter at the minimum interaction density to be larger than that at the maximum interaction density. The conclusion does not only hold for the globally connected substrate structure, but also holds for other types of substrate structures such as scale-free networks. Such observations strongly suggest that our model is related to the mean-field kinetic Ising model with the frequency matching condition around the phase transition [20] and would be worth investigating further. Since periodicity plays important roles in many parts of the nature, especially for systems under natural circadian rhythms or external stimuli, our results suggest that the temporal response to such periodicity can be an important cue to characterize the system.

**Acknowledgments**

We greatly appreciate B. J. Kim, H.-H. Jo, M. Gastner, T. Gross, and S. Dorogovtsev for their valuable comments. This work was supported by the Swedish Research Council (S.H.L. and P.H.), the Wenner-Gren Foundation (S.L.), and the WCU program through NRF Korea funded by MEST R31-2008-10029 (P.H.). Computation was partially carried out using the cluster in CSSPL, KAIST.

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