Action Minimizing Solutions of The One-Dimensional $N$-Body Problem $^*$

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Abstract. In recent years, a lot of results on the $N$-body problem have been gotten by the action minimizing methods. Since the potential of the $N$-body problem is singular at collision configurations, the main problem is how to avoid collisions when we use variational methods. In particular, Christian Marchal got a remarkable advance, that is, a path minimizing the Lagrangian action functional between two given configurations is always a true (collision-free) solution of the $N$-body problem, so long as the dimension $d$ of physical space $\mathbb{R}^d$ satisfies $d \geq 2$. Unfortunately, the idea of Christian Marchal can’t apply to the case of the one-dimensional physical space, thus, in this paper, we will study the fixed-ends (Bolza) problem for the one-dimensional Newtonian $N$-body problem with equal masses. More precisely, we will prove that the path, which minimizes the Lagrangian action functional between two given configurations, is always a true (collision-free) solution of the one-dimensional $N$-body problem, if the particles at two endpoints have the same order; otherwise, there must be collisions for any path, however, we claim that there are at most $N! - 1$ collisions for any action minimizing orbit.

Key Words: $N$-body problems; Collisions; Variational minimization methods; Central configurations; The fixed-ends problem

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1 Introduction and Main Results

In Euclidean space $\mathbb{R}^d$, we consider $N \geq 2$ particles with positive masses, affected by their gravitational interactions. The equation of motion of the $N$-body problem is written as

$$m_k \ddot{q}_k = \sum_{1 \leq j \leq N, j \neq k} \frac{m_j m_k (q_j - q_k)}{|q_j - q_k|^3},$$

(1.1)

where $m_k$ is the mass and $q_k$ the position of the $k$-th body. Since these equations are invariant by translation, we can assume that the center of masses is at the origin. Firstly, we set some notations and describe preliminary results that will be needed later. Let $X_d$ denote the space of configurations for $N$ point particles in Euclidean space $\mathbb{R}^d$ with dimension $d$, whose center of masses is at the origin, that is, $X_d = \{ q = (q_1, \cdots, q_N) \in (\mathbb{R}^d)^N : \sum_{k=1}^N m_k q_k = 0 \}$. For each pair of indices $j, k \in \{1, \ldots, N\}$, let $\Delta_{j,k}$ denote the collision set of the $j$-th and $k$-th particles $\Delta_{j,k} = \{ q \in X_d : q_j = q_k \}$. Let $\Delta = \bigcup_{j,k} \Delta_{j,k}$ be the collision set in $X_d$. The space of collision-free configurations $X_d \setminus \Delta$ is denoted by $\hat{X}_d$. Let $\mathbb{T}$ denote the time interval $[T_1, T_2]$. By the path space $\Lambda$, we mean the Sobolev space $\Lambda = H^1(\mathbb{T}, X_d)$; we use $\Lambda(q_i, q_f)$ to denote the space of paths $q(t) \in \Lambda$ beginning in the configuration $q_i$ at moment $T_1$ and ending in the configuration $q_f$ at moment $T_2$. For a motion $q(t)$ of the $N$-body problem, we say there is a collision at time $t_0$ if, for at least two indices, say $j$ and $k$, $q_k(t) \to c_k$, $q_i(t) \to c_i$ as $t \to t_0$, and $c_j = c_k$. We now ‘cluster’ the particles according to their limit points, that is, according to which particles are colliding each other. So, let the different limit points be $c_1, \ldots, c_n$, and let

$$S_k = \{ j \in \{1, \ldots, N\} : q_j(t) \to c_k \text{ as } t \to t_0 \}, \quad k = 1, \ldots, n.$$

We consider the opposite of the potential energy (force function) defined by

$$U(q) = \sum_{k<j} \frac{m_k m_j}{|q_k - q_j|}.$$  \hspace{1cm} (1.2)

The kinetic energy is defined (on the tangent bundle of $X_d$) by $K = \sum_{j=1}^N \frac{1}{2} m_j |\dot{q}_j|^2$, the total energy is $E = K - U$ and the Lagrangian is $L(q, \dot{q}) = L = K + U = \sum_{j} \frac{1}{2} m_j |\dot{q}_j|^2 + \sum_{k<j} \frac{\overline{m_k m_j}}{|q_k - q_j|}$. Given the Lagrangian $L$, the positive definite functional $A : \Lambda \to \mathbb{R} \cup \{ +\infty \}$ defined by

$$A(q) = \int_{\mathbb{T}} L(q(t), \dot{q}(t)) dt$$ \hspace{1cm} (1.3)

is termed as action functional (or the Lagrangian action).

The action functional $A$ is of class $C^1$ on the collision-free space $\hat{\Lambda}(q_i, q_f) \subset \Lambda(q_i, q_f)$. Hence the critical point of $A$ in $\hat{\Lambda}(q_i, q_f)$ is a classical solution (of class $C^2$) of Newtonian
equations

\[ m_j \ddot{q}_j = \frac{\partial U}{\partial q_j}. \tag{1.4} \]

From the viewpoint of the Least Action Principle, the variational minimal solution of the N-body problem is the most important and the simplest, so it is natural to search for minimizers of the Lagrangian action joining two given configurations in a fixed time. It’s worth noticing that a lot of results have been gotten by the action minimization methods just in recent years, please see [1] [2] [3] [4] [6] [7] [8] [9] [10] [11] [13] [14] [15] [18] [19] [20] and the references therein. Recently, the interest in this problem has grown considerably due to the discovery of the figure eight solution [9].

Since the potential of the N-body problem is singular at collision configurations, the main problem involved in variational minimizations is that the minimizer could well be such that, for a non-empty set of instants (measure zero), the system undergoes a collision of two or more bodies, which prevents it from being a true solution. Some techniques are created to overcome the difficulty, ultimately, one got a major advance (essentially due to Christian Marchal) in this subject. More specifically, the advance is the following remarkable theorem [15] [8] [11].

**Theorem 1.1** (Marchal) Given the initial moment \( T_1 \), the final moment \( T_2 \) (\( T_2 > T_1 \)) and two corresponding N-body configurations \( q_i = (q_{i1}, \ldots, q_{iN}) \), \( q_f = (q_{f1}, \ldots, q_{fN}) \) in \( \mathbb{R}^d \) (\( d > 1 \)), an action minimizing path joining \( q_i \) to \( q_f \) in time \( T_2 - T_1 \) is collision-free for \( t \in (T_1, T_2) \).

This theorem, together with the lower semicontinuity of the action, implies in particular that there always exists a collision-free minimizing solution joining two given collision-free N-body configurations in a given time.

Unfortunately, the idea of Christian Marchal can’t apply to the case of the one-dimensional physical space. Since, roughly speaking, Marchal’s idea is as following : let \( a = 2 \), by \( \int_0^{2\pi} \frac{1}{|a + e^{\sqrt{-1}\theta}|} - \frac{1}{|a|} < 0 \), there must be some \( \theta \) satisfying \( \frac{1}{|a + e^{\sqrt{-1}\theta}|} < \frac{1}{|a|} \); but in the case of \( d = 1 \), we have \( \frac{1}{|a + 1|} + \frac{1}{|a - 1|} > \frac{2}{|a|} \). Thus, in this paper, by using a different approach, we will study the fixed-ends (Bolza) problem for the one-dimensional Newtonian N-body problem. More precisely, we will prove that the path minimizing the Lagrangian action functional between two given configurations is always a true (collision-free) solution of the one-dimensional N-body problem, if the particles at two endpoints have the same order, where, we say that the particles at configurations \( q_i = (q_{i1}, \ldots, q_{iN}) \)
and \( q_f = (q_{f1}, \ldots, q_{fN}) \) have the same order if \( q_{ij} - q_{ik} \geq 0 \Leftrightarrow q_{fj} - q_{fk} \geq 0 \) for any \( j \neq k \), in other words, the relations \( q_{ij} > q_{ik} \) and \( q_{fj} < q_{fk} \) can’t hold for any \( j \neq k \) at the same time. In particular, if \( q_{j1} < q_{j2} < \cdots < q_{jN} \), we call \((j_1, j_2, \cdots, j_N)\) is the order of the configuration \((q_1, q_2, \cdots, q_N)\). This requirement is necessary, since it is obvious that there must be collisions for any path if the particles at two endpoints have different order.

In this paper, our main results are the following Propositions.

**Theorem 1.2** Suppose a motion \( q(t) \) of the one-dimensional Newtonian \( N \)-body problem has a collision at moment \( t_0 \), every corresponding colliding cluster \( S_k \) has \( n_k \) elements, then we have the following results for some right neighborhood or left neighborhood of \( t_0 \)

- if \( n_k = 1 \), let \( j \in S_k \), then \( q_j(t) = q_j(t_0) + \dot{q}_j(t_0)(t - t_0) + o(t - t_0) \);
- if \( n_k \geq 2 \), let \( j \in S_k \), then \( q_j(t) = q_j(t_0) + s_j(t - t_0)^{\frac{k}{2}} + o((t - t_0)^{\frac{k}{2}}) \), where \( s_j \cdot j \in S_k \) is a central configuration for the particles corresponding to the colliding cluster \( S_k \).

**Corollary 1.1** Suppose the motion \( q(t) \) of the one-dimensional Newtonian \( N \)-body problem has a collision at moment \( t_0 \), then the collision is isolated at time \( t_0 \), that is, there exists some \( \varepsilon > 0 \), \( q(t) \) is collision-free in \((t_0 - \varepsilon, t_0 + \varepsilon)\) except at time \( t_0 \). Hence there are at most finitely many collision moments for the fixed-ends (Bolza) problem.

**Theorem 1.3** For the one-dimensional \( N \)-body problem with equal masses, given the initial moment \( T_1 \), the final moment \( T_2 \) \((T_2 > T_1)\) and two corresponding \( N \)-body configurations \( q_i = (q_{i1}, \ldots, q_{iN}) \), \( q_f = (q_{f1}, \ldots, q_{fN}) \) in \( \mathbb{R}^1 \), if \( q_i \), \( q_f \) have the same order in \( \mathbb{R}^1 \), then the action minimizing path of the fixed-ends problem joining \( q_i \) to \( q_f \) in time \( T_2 - T_1 \) is collision-free for \( t \in (T_1, T_2) \).

**Corollary 1.2** If the given two configurations \( q_i \), \( q_f \) have the different order in \( \mathbb{R}^1 \), then the action minimizing path of the fixed-ends problem with equal masses joining \( q_i \) to \( q_f \) in time \( T_2 - T_1 \) has some collisions for some \( t \in (T_1, T_2) \), but there are at most \( N! - 1 \) collision moments in \((T_1, T_2)\).

**Remark.** Our results and methods remain valid for more general force function defined by \( U(q) = \sum_{k<j} \frac{m_k m_j}{|q_k - q_j|^\alpha} \), where \( \alpha \) is any positive real number.

It is natural to ask the following questions.

**Question.** 1. Does the Theorem 1.3 hold for the problem with any masses? 2. Given two configurations which have the different order in \( \mathbb{R}^1 \) and a time \( T = T_2 - T_1 > 0 \),
0, what is the largest number of collision times in \((T_1, T_2)\)? Is the largest number of collision times in \((T_1, T_2)\) one? The similar questions can be asked for the fixed-ends problem with any masses.

We hope that the answers of these questions are all positive.

The paper is structured as follows. Section 2 introduces some definitions and some classical results, Section 3 gives the proofs of the main results by using the concepts and results introduced in Section 1 and Section 2.

2 Some Definitions and Some Classical Results

In this section, we give some definitions and recall some classical results.

The first one is the important concept of the central configuration [17],

Definition 2.1 A configuration \(q = (q_1, \cdots, q_N) \in X_d \setminus \Delta_d\) is called a central configuration if there exists a constant \(\lambda \in \mathbb{R}\) such that

\[
\sum_{j=1, j \neq k}^{N} \frac{m_j m_k}{|q_j - q_k|^3} (q_j - q_k) = -\lambda m_k q_k, 1 \leq k \leq N, \tag{2.1}
\]

the value of \(\lambda\) in \((2.1)\) is uniquely determined by

\[
\lambda = \frac{U(q)}{I(q)}, \tag{2.2}
\]

where

\[
I(q) = \sum_{1 \leq j \leq N} m_j |q_j|^2. \tag{2.3}
\]

Let us recall that, for a motion \(q(t)\) of \(N\)-body problem, we say there is a collision at time \(t_0\) if as \(t \to t_0, q_j(t) \to c_j, j \in \{1, \cdots, N\}\) and for at least two different indices, say \(j\) and \(k\) such that \(c_j = c_k\). And without loss of generality, we can assume that the time \(t\) approach \(t_0\) from the right of \(t_0\), that is, we think \(t \to t_0^+\). Denote the different limit points by \(c_1, \cdots, c_n\), and classify the indices according to particles colliding each other, let \(S_k = \{j \in \{1, \cdots, N\} : q_j(t) \to c_k as t \to t_0^+\}\), and assume \(S_k\) has \(n_k\) elements for \(k = 1, \cdots, n\); then we say that every \(S_k\) is a colliding cluster of particles. If \(j \in S_k\), let \(r_{(k)}(t) = \frac{q_j - c_k}{(t-t_0)^2}\), then we call \(r_{(k)}(t) = (r_{(k))l_1(t), \cdots, r_{(k))l_{n_k}(t))\) be the normalized configuration corresponding to the colliding cluster \(S_k\), where \(\{l_1, \cdots, l_{n_k}\} = S_k\).
Let

$$\text{CC}_k := \{ r(k) : \sum_{j \in S_k, j \neq i} \frac{m_j}{r(k)_{ij} - r(k)_{ii}} (r(k)_j - r(k)_i) = \frac{2}{9} r(k)_i, i \in S_k \}$$ (2.4)

be the set of the central configuration corresponding to colliding cluster $S_k$, where we assume the value of $\lambda$ which only affects the size of the central configuration to be $\frac{2}{9}$.

Before giving the proofs of the main results of this paper, we recall some classical results concerning collision solutions (see [16, 11, 5] for a proof).

The first one says that all collision orbits have the property that $r(t) \to \text{CC}_k$ as $t \to t_0$, where $r(t)$ and $\text{CC}_k$ are respectively the normalized configuration of the collision orbit and the set of the central configuration corresponding to colliding cluster $S_k$.

**Theorem 2.1** Suppose a colliding cluster $S_k$ have $n_k \geq 2$ elements, let $r_j(t) = \frac{q_j - c_k}{(t - t_0)^3}$ for any $j \in S_k$, be the normalized configuration. Then for every converging sequence $r(t_j) = (r_1(t_j), \ldots, r_{n_k}(t_j))$, where $l_1, \ldots, l_{n_k} \in S_k$, $t_j$ belong to some neighborhood of $t_0$ ($j \in \mathbb{N}$), the limit $\lim_{j \to \infty} r(t_j) := s$ is a central configuration; and for any $\lambda \in (0, 1)$, we have

$$\lim_{j \to \infty} r(\lambda t_j) = \lim_{j \to \infty} r(t_j) = s$$ (2.5)

The second one says that all collinear central configurations are non-degenerate.

**Theorem 2.2** All collinear central configurations are non-degenerate in $\mathbb{R}^1$.

Then, in the following, we have the important result which says that, for a collision of particles, not only does $r(t) \to \text{CC}_k$ as $t \to t_0$, but also there is a central configuration $s \in \text{CC}_k$ so that $r(t) \to s$ as $t \to t_0$, so long as all central configurations are non-degenerate.

**Theorem 2.3** For the one-dimensional $N$-body problem, suppose a colliding cluster $S_k$ have $n_k \geq 2$ elements, let $r_j(t) = \frac{q_j - c_k}{(t - t_0)^3}$ for any $j \in S_k$, be the normalized configuration. Then $\lim_{t \to t_0} r(t)$ exists, the limit $s := \lim_{t \to t_0} r(t)$ is a central configuration, furthermore, $s$ and $r(t)$ have the same order.

**Proof of Theorem 2.3:**

It’s similar to a particular case of the results of Saari [16], we can get **Theorem 2.3** by using the unstable manifold theorem for a normally hyperbolic invariant set (Hirsch
et al. [12]) and Theorem 2.2.

We will give the main results of this paper in the next section.

3 The Proofs of Main Results

In this section, we give the proofs of main results in this paper.

Proof of Theorem 1.2:

This result easily comes from Theorem 2.3.

□

Proof of Corollary 1.1

Suppose the motion $q(t) \to c$ as $t \to t_0^+$, we now ‘cluster’ the particles according to their limit points, that is, we classify the indices according to particles colliding each other. Let the different limit points be $c_1, \ldots, c_n$, and let $S_k = \{j \in \{1, \ldots, N\} : q_j(t) \to c_k$ as $t \to t_0^+ \}$, $k = 1, \ldots, n$. For any $j \in S_k$ let $r_{(k)}(t) = \frac{q_j - c_k}{(t-t_0)^{\frac{1}{3}}}$ be the normalized configuration, then $\lim_{t \to t_0^+} r(t)$ exists. Let $s_{(k)} := \lim_{t \to t_0^+} r_{(k)}(t)$ be a central configuration, $s_{(k)}$ and $r_{(k)}(t)$ have the same order, then for any $j \in S_k$, we have

$$q_j(t) = c_k + s_{(k)}(t-t_0)^{\frac{1}{3}} + o((t-t_0)^{\frac{1}{3}}).$$

(3.1)

It’s easy to know that the distance of any two particles $m_i, m_j$ is not zero when $t-t_0 > 0$ and $t-t_0$ is sufficiently small. So there exists some positive number $\varepsilon$ such that $q(t)$ is collision-free in $(t_0, t_0 + \varepsilon)$.

Similarly, one can get the same result for $t \to t_0^-$. Hence we have proved that the collision is isolated for the one-dimensional Newtonian $N$-body problem. Since the time interval of the fixed-ends problem is compact, we know there are at most finitely many collision times for the fixed-ends problem.

□

Proof of Theorem 1.3:

First of all, let’s establish some lemmas to simplify the proof.

Lemma 3.1 Given the initial moment $T_1$, the final moment $T_2$ ($T_2 > T_1$) and two corresponding N-body configurations $q_i = (q_{i1}, \ldots, q_{iN})$, $q_f = (q_{f1}, \ldots, q_{fN})$ which have the same order in $\mathbb{R}^1$, then there is some path $q(t)$ which has the same order with $q_i$ and $q_f$ in $\mathbb{R}^1$, furthermore, $q(t)$ is collision-free for $t \in (T_1, T_2)$. 

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**Lemma 3.2** Let \( q(t) \) be a path in \((T_1, T_2)\), randomly permute the position of the particles, suppose the new path is \( h(t) \), if all the particles have the same masses, then the action of the new path \( q(t) \) in \((T_1, T_2)\) and the action of the original path \( q(t) \) in \((T_1, T_2)\) are equal. More precisely, suppose \( \tau \) is a permutation of \((1, 2, \cdots, N)\), let \( h(t) = (h_1(t), h_2(t), \cdots, h_N(t)) = (q_{\tau(1)}(t), q_{\tau(2)}(t), \cdots, q_{\tau(N)}(t)) \), if \( m_1 = m_2 = \cdots = m_N \), then

\[
\int_{T_1}^{T_2} L(q(t), \dot{q}(t))dt = \int_{T_1}^{T_2} L(h(t), \dot{h}(t))dt. \tag{3.2}
\]

The validity of **Lemma 3.1** and **Lemma 3.2** is obvious, so we don’t give their proofs here.

**Lemma 3.3** Given the initial moment \( T_1 \), the final moment \( T_2 \) \((T_2 > T_1)\) and two corresponding N-body configurations \( q_i = (q_{i1}, \cdots, q_{iN}), q_f = (q_{f1}, \cdots, q_{fN}) \) which have the same order in \( \mathbb{R}^1 \), suppose the path \( q(t) \in \Lambda(q_i, q_f) \) has collision in \((T_1, T_2)\), and the collision moments in \((T_1, T_2)\) are respectively \( t_1, t_2, \cdots, t_n \) \((T_1 < t_1 < \cdots < t_n < T_2)\). Then there is some path \( h(t) \in \Lambda(q_i, q_f) \) such that \( \{t_1, \cdots, t_n\} \) are collision moments in \((T_1, T_2)\) and the order of \( h(t) \) are the same for all the time \( t \in [T_1, T_2] \). Furthermore, if all the particles have the same masses, then

\[
\int_{T_1}^{T_2} L(q(t), \dot{q}(t))dt = \int_{T_1}^{T_2} L(h(t), \dot{h}(t))dt. \tag{3.3}
\]

**Proof of Lemma 3.3**

From **Lemma 3.1**, there is some orbit \( g(t) \) which has the same order with \( q_i \) and \( q_f \) in \( \mathbb{R}^1 \) and \( g(t) \) is collision-free for \( t \in (T_1, T_2) \), suppose the order of the orbit \( g(t) \) for \( t \in (T_1, T_2) \) is \((j_1, \cdots, j_N)\), that is, \( g_{j_1}(t) < \cdots < g_{j_N}(t) \), where \( \{j_1, \cdots, j_N\} = \{1, \cdots, N\} \). Without loss of generality, we can assume that \((j_1, j_2, \cdots, j_N) = (1, 2, \cdots, N)\). Let \( t_0 = T_1 \) and \( t_{n+1} = T_2 \), suppose the order of the orbit \( q(t) \) for \( t \in (t_k, t_{k+1}) \) is \((j_{k1}, \cdots, j_{kN})\), that is, \( q_{j_{k1}}(t) < \cdots < q_{j_{kN}}(t) \), where \( k \in \{0, \cdots, n\} \). Suppose \( \tau_k \) is a permutation from \((j_{k1}, \cdots, j_{kN})\) to \((1, 2, \cdots, N)\), let

\[
h^{(k)}(t) = (h^{(k)}_1(t), h^{(k)}_2(t), \cdots, h^{(k)}_N(t)) = (q_{\tau_k(1)}(t), q_{\tau_k(2)}(t), \cdots, q_{\tau_k(N)}(t)) \tag{3.4}
\]

for \( t \in (t_k, t_{k+1}) \). Firstly, it is easy to know that

\[
\lim_{t \to t_0^+} h^{(0)}(t) = q_i, \quad \lim_{t \to t_{n+1}^-} h^{(n)}(t) = q_f \tag{3.5}
\]
In the following, we prove that
\[
\lim_{t \to t_{k+1}^-} h_j^{(k)}(t) = \lim_{t \to t_{k+1}^+} h_j^{(k+1)}(t) \quad (3.6)
\]
for every \( j \in \{1, \cdots, N \} \) and \( k \in \{0, \cdots, n-1 \} \).

In fact, from \( h_j^{(k)}(t) = q_{\tau_k(j)}(t) \) for \( t \in (t_k, t_{k+1}) \) and \( h_j^{(k+1)}(t) = q_{\tau_{k+1}(j)}(t) \) for \( t \in (t_{k+1}, t_{k+2}) \), it is easy to know that we only need to prove the relation \( q_{\tau_k(j)}(t_{k+1}) = q_{\tau_{k+1}(j)}(t_{k+1}) \). For the sake of a contradiction, we can suppose that \( q_{\tau_k(j)}(t_{k+1}) > q_{\tau_{k+1}(j)}(t_{k+1}) \) or \( q_{\tau_k(j)}(t_{k+1}) < q_{\tau_{k+1}(j)}(t_{k+1}) \). If \( q_{\tau_k(j)}(t_{k+1}) > q_{\tau_{k+1}(j)}(t_{k+1}) \), from \( h_j^{(k)}(t) > h_j^{(k)}(t) \) for \( N \geq l > j, t \in (t_k, t_{k+1}) \), we have \( q_{\tau_k}(t_{k+1}) = \lim_{t \to t_{k+1}^-} h_j^{(k)}(t) \geq \lim_{t \to t_{k+1}^-} h_j^{(k)}(t) = q_{\tau_k}(t_{k+1}) > q_{\tau_{k+1}(j)}(t_{k+1}) \). Hence \( h_j^{(k+1)}(t_{k+1}) = q_{\tau_k}(t_{k+1}) > q_{\tau_{k+1}(j)}(t_{k+1}) = h_j^{(k+1)}(t_{k+1}) \) for every \( l \) such that \( N \geq l > j, t \in (t_k, t_{k+1}+\epsilon) \), \( \epsilon \) is some sufficiently small positive number. So we have \( \tau_{k+1}(l) > j \) for every \( l \) such that \( N \geq l > j \), but there are at most \( N-j \) number larger than \( j \) in \( \{1,2,\cdots,N\} \), this is a contradiction. If \( q_{\tau_k(j)}(t_{k+1}) < q_{\tau_{k+1}(j)}(t_{k+1}) \), it is similar to get a contradiction. So we have
\[
\lim_{t \to t_{k+1}^-} h_j^{(k)}(t) = \lim_{t \to t_{k+1}^+} h_j^{(k+1)}(t) \quad (3.7)
\]
for every \( j \in \{1, \cdots, N \} \) and \( k \in \{0, \cdots, n-1 \} \).

Let \( h(t) = h^{(k)}(t) \) for \( t \in (t_k, t_{k+1}) \), \( h(t_k) = \lim_{t \to t_k^+} h^{(k)}(t) \) for \( 1 \leq k \leq n \), \( h(T_1) = \lim_{t \to t_0^+} h^{(0)}(t) \), \( h(T_2) = \lim_{t \to t_{n+1}^-} h^{(n)}(t) \), then \( h(t) \in \Lambda(q_i, q_j) \) and \( \{t_1, \cdots, t_n\} \) are collision moments in \( (T_1, T_2) \) and the order of \( h(t) \) are the same for all the time \( t \in [T_1, T_2] \).

Furthermore, if all the particles have the same masses, then, from Lemma 3.2, we have
\[
\int_{T_1}^{T_2} L(q(t), \dot{q}(t))dt = \int_{T_1}^{T_2} L(h(t), \dot{h}(t))dt. \quad (3.8)
\]

Thus Lemma 3.3 holds.

In the following, we prove Theorem 1.3 by using Lemma 3.3.

By using reduction to absurdity, suppose the action minimizing orbit \( q(t) \) has collision moments in \( (T_1, T_2) \), the collision moments in \( (T_1, T_2) \) are respectively \( t_1, t_2, \cdots, t_n (T_1 < t_1 < \cdots < t_n < T_2) \). Furthermore, we can assume that \( q_1(t) < q_2(t) < \cdots < q_N(t) \) for \( t \in (T_1, T_2) \setminus \{t_1, \cdots, t_n\} \) and \( q_1(t_k) \leq q_2(t_k) \leq \cdots \leq q_N(t_k) \) for \( k \in \{1, \cdots, n\} \) by using Lemma 3.3.

Let \( x_k(t) = q_{k+1}(t) - q_k(t) \) for \( k \in \{1, \cdots, N-1\} \) and \( m_1 = m_2 = \cdots = m_N = m \), then by Lagrangian identity, \( x(t) = (x_1(t), x_2(t), \cdots, x_{N-1}(t)) \) is an action minimizing
orbit of the action functional

\[ \mathcal{F}(x) = \int_T \sum_{1 \leq l < k \leq N} \left( \frac{m}{2N} \right) \left( \sum_{l \leq j \leq k-1} \dot{x}_j \right)^2 + \frac{m^2}{|\sum_{l \leq j \leq k-1} x_j|} dt \]  (3.9)

In fact, we have

\[ \mathcal{A}(q) = \int_T L(q(t), \dot{q}(t)) dt \]
\[ = \int_T \left[ \frac{1}{2} \sum_{1 \leq l < k \leq N} m_k m_l |\dot{q}_k - \dot{q}_l|^2 + \sum_{1 \leq l < k \leq N} \frac{m_k m_l}{|q_k - q_l|} \right] dt \]
\[ = \int_T \sum_{1 \leq l < k \leq N} \left( \frac{m}{2N} \right) \left( \sum_{l \leq j \leq k-1} \dot{x}_j \right)^2 + \frac{m^2}{|\sum_{l \leq j \leq k-1} x_j|} dt \]

In the following, we will construct another path \( y(t) \) satisfies the same boundary conditions with \( x(t) \), but the value of \( \mathcal{F}(y) \) is smaller than the value of \( \mathcal{F}(x) \).

Suppose, for the sake of convenience, \( x_1(t) \to 0 \) when \( t \to t_1 \), then we have \( x_1(t) = \alpha(t_1-t)^\frac{3}{2} + o((t_1-t)^\frac{3}{2}) \) for some left neighborhood of \( t_1 \) and \( x_1(t) = \beta(t-t_1)^\frac{3}{2} + o((t-t_1)^\frac{3}{2}) \) for some right neighborhood of \( t_1 \) from Theorem 1.2 where \( \alpha, \beta \) are appropriate positive number. For sufficiently small positive number \( \delta \), there are two sufficiently small positive numbers \( \epsilon, \varepsilon \) such that \( x_1(t_1-\epsilon) = x_1(t_1+\varepsilon) = \delta, x_1(t) \leq \delta \) for \( t \in [t_1-\epsilon, t_1+\varepsilon] \) and \( |\dot{x}_1(t)| \) is sufficiently large for \( t \in [t_1-\epsilon, t_1+\varepsilon] \). Let \( y_1(t) = \delta \) for \( t \in [t_1-\epsilon, t_1+\varepsilon] \), \( y_1(t) = x_1(t) \) for \( t \in [T_1, T_2] \setminus [t_1-\epsilon, t_1+\varepsilon] \), and \( y_j(t) = x_j(t) \) for \( t \in [T_1, T_2] \) and \( 2 \leq j \leq N-1 \). Let \( y(t) = (y_1(t), y_2(t), \ldots, y_{N-1}(t)) \), then

\[ \mathcal{F}(x) - \mathcal{F}(y) = \int_{t_1-\epsilon}^{t_1+\epsilon} \sum_{1 \leq l < k \leq N} \left( \frac{m}{2N} \right) \left( \sum_{l \leq j \leq k-1} \dot{x}_j \right)^2 + \frac{m^2}{|\sum_{l \leq j \leq k-1} x_j|} dt \]
\[ - \int_{t_1-\epsilon}^{t_1+\epsilon} \sum_{1 \leq l < k \leq N} \left( \frac{m}{2N} \right) \left( \sum_{l \leq j \leq k-1} \dot{y}_j \right)^2 + \frac{m^2}{|\sum_{l \leq j \leq k-1} y_j|} dt \]
\[ = \int_{t_1-\epsilon}^{t_1+\epsilon} \left[ A\dot{x}_1^2 + 2B\dot{x}_1 + \sum_{3 \leq k \leq N} \left( \frac{m^2}{x_1 + \sum_{2 \leq j \leq k-1} x_j} \right) \right] dt \]
\[ - \int_{t_1-\epsilon}^{t_1+\epsilon} \sum_{3 \leq k \leq N} \left( \frac{m^2}{\delta + \sum_{2 \leq j \leq k-1} x_j} \right) dt \]
\[ > 0 \]

where \( A = \frac{m(N-1)}{2N}, B = \frac{m}{2N} \sum_{3 \leq k \leq N} \sum_{2 \leq j \leq k-1} \dot{x}_j \), since

\[ A\dot{x}_1^2 + 2B\dot{x}_1 > 0 \]  (3.10)
and

\[
\frac{1}{|x_1 + \sum_{2 \leq j \leq k-1} x_j|} \geq \frac{1}{|\delta + \sum_{2 \leq j \leq k-1} x_j|} \tag{3.11}
\]

t \in [t_1 - \epsilon, t_1 + \epsilon]$. In fact, from Theorem 1.2 we know that

- if $x_j(t) \to 0$ when $t \to t_1$ for some $j \in \{2, \cdots, N-1\}$, then $B = \tilde{\alpha}(t_1 - t)^{\frac{4}{3}} + o((t_1 - t)^{\frac{4}{3}})$ for some left neighborhood of $t_1$ and $B = \tilde{\beta}(t-t_1)^{\frac{4}{3}} + o((t-t_1)^{\frac{4}{3}})$ for some right neighborhood of $t_1$, where $\tilde{\alpha}, \tilde{\beta}$ are appropriate positive numbers;

- if $x_j(t) > 0$ for some neighborhood of $t_1$ and any $j \in \{2, \cdots, N-1\}$, then $B = a + b(t-t_1) + o(|t_1-t|)$ for some neighborhood of $t_1$, where $a, b$ are appropriate real number.

It is easy to know that the inequality 3.10 holds for any case.

If there is some $k \geq 2$ such that $x_k(t) \to 0$ when $t \to t_1$, we can get similar result. So we know that, for the N-body problem with equal masses, given two moments and corresponding configurations which have the same order in $\mathbb{R}^1$, the action minimizing path of the fixed-ends problem joining two configurations is collision-free for $t \in (T_1, T_2)$.

□

Proof of Corollary 1.2

Suppose the action minimizing orbit $q(t)$ has collision in $(T_1, T_2)$, the collision moments in $(T_1, T_2)$ are respectively $t_1, t_2, \cdots, t_n$ $(T_1 < t_1 < \cdots < t_n < T_2)$, let $t_0 = T_1$ and $t_{n+1} = T_2$. Let us investigate $n + 1$ collision-free path sections: $q(t), t \in (t_k, t_{k+1})$, $0 \leq k \leq n$. If $n > N! - 1$, then there are two sections which have the same order, suppose the corresponding time intervals are respectively $(t_j, t_{j+1})$ and $(t_l, t_{l+1})$, $j < l$. Let us choose two moments $s_1 \in (t_j, t_{j+1})$ and $s_2 \in (t_l, t_{l+1})$, then it is easy to know that the path $q(t), t \in [s_1, s_2]$ is an action minimizing orbit of the fixed-ends problem for two moments $s_1, s_2$ and corresponding configurations $q(s_1), q(s_2)$. However, from Theorem 1.3, $q(t)$ is collision-free in $(s_1, s_2)$, this contradicts with $t_{j+1}, t_l \in (s_1, s_2)$.

□

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