LAGRANGE-DIRAC SYSTEMS FOR CHARGED PARTICLES IN GAUGE FIELDS

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ABSTRACT. In this work, we use the Sternberg phase space (which may be considered as the classical phase space of particles in gauge fields) in order to explore the dynamics of such particles in the context of Lagrange-Dirac systems and their associated Hamilton-Pontryagin variational principles. For this, we develop an analogue of the Pontryagin bundle in the case of the Sternberg phase space. Moreover, we show the link of this new bundle to the so-called magnetized Tulczyjew triple, which is an analogue of the link between the Pontryagin bundle and the usual Tulczyjew triple. Taking advantage of the symplectic nature of the Sternberg space, we induce a Dirac structure on the Sternberg-Pontryagin bundle which leads to the Lagrange-Dirac structure that we are looking for. We also analyze the intrinsic and variational nature of the equations of motion of particles in gauge fields in regards of the defined new geometry. Lastly, we illustrate our theory through the case of a $U(1)$ gauge group, leading to the paradigmatic example of a electricaly charged particle in an electromagnetic field.

1. INTRODUCTION

In the Hamiltonian formalism, many classical mechanical systems are described by a manifold, which plays the role of phase space, endowed with a symplectic structure and a choice of Hamiltonian function. More concretely, if $S$ is a smooth manifold equipped with a symplectic two-form $\Omega_S$, i.e $(S, \Omega_S)$, the dynamics induced by a smooth Hamiltonian function $H : S \to \mathbb{R}$, embodied in its Hamiltonian vector field $X_H : S \to TS$, is determined by the well-known Hamiltonian equations

$$i_{X_H} \Omega_S = dH.$$

As can be noticed, these equations are global and may be derived from the pure geometry of the phase space. Particularly, the dynamics of a particle with configuration manifold $Q$ is determined by its cotangent bundle $(T^*Q, \Omega_{T^*Q})$, the usual phase space in classical mechanics, and a given Hamiltonian function $H : T^*Q \to \mathbb{R}$. The Lagrangian counterpart of mechanics is not as geometrical as the Hamiltonian side, say the Euler-Lagrange equations for a given Lagrangian function $L : TQ \to \mathbb{R}$ cannot be obtained from the geometry of the tangent bundle $TQ$. Nevertheless, both approaches may be described intrinsically under the same framework when one combines the theory of Lagrangian submanifolds (see [33, 34]) with the so-called Tulczyjew’s triple (see [29, 30, 31]): namely, both Hamiltonian and Lagrangian dynamics are described by suitable Lagrangian submanifolds of the double vector bundle $TT^*Q$. Roughly speaking, a Lagrangian submanifold is a maximally isotropic submanifold of a given symplectic manifold, while the Tulczyjew triple is the set made out of the double vector bundles $T^*T^*Q$, $TT^*Q$, $T^*TQ$ and two symplectomorphisms among them, say $\alpha_Q$, $\beta_Q$. This is a powerful mechanism and it has been widely applied in modern Geometric Mechanics, from continuous to discrete
systems or from unconstrained to variationally constrained (meaning *vakonomically* constrained) systems, as can be seen in the recent references [2, 3, 4, 8, 9, 14].

Mathematically speaking, in a gauge theory with gauge group $G$ formulated over a manifold $Q$, a gauge field is a connection of the $G$−principal bundle $P \to Q$. The addition of a gauge field into the classical particle dynamics is non-trivial, specially when the group is non-abelian. From a symplectic perspective, the description of the phase space of a particle on a gauge field was initiated by Sternberg in [27], giving rise to the so-called *Sternberg phase space* $F^\#$, following the initial ideas in [35], where the equations of motion of the particle and the gauge field are obtained taking advantage of a Poisson approach; further developments on this subject may be found in [22, 32]. From the physical point of view, the dynamics of a classical particle in interaction with a gauge field is interesting in few cases, being the paradigmatic one the case of a charged particle evolving in space and coupled to an electromagnetic field. Of course, this instance is important for its own sake, but recently some attention has been put upon the magnetized Kepler problems [1, 19, 20], problems that fit in the setup presented in this work. On the other hand, it is mandatory to mention that gauge fields acquire crucial importance at a quantum level, for instance in Yang-Mills theories [36] such as the Standard Model of particle physics, which is a quantum field theory where the gauge fields play the role of the intermediate bosons of fundamental interactions (see [24] for a theoretical perspective on the Standard Model).

Again at a classical level, to obtain the equations of motion of a charged particle subject to a gauge field is not easy, and usually it is achieved in the physical literature through the so called *minimal coupling* procedure (which consists on shifting the classical momenta by the gauge field). In a more elegant and geometrical way, it has been accomplished in the recent work [21] the task of deriving these equations in the context of a generalization of the Tulczyjew triple (called the *magnetized Tulczyjew triple* (where the role of the cotangent bundle $T^*Q$ is played by the *Sternberg phase space* $F^\#$) and the Lagrangian submanifold theory.

Although symplectic manifolds are the appropriate spaces to describe Hamiltonian systems and have great importance in modern mathematics, they are not suitable to describe *all* classical systems. Mechanical systems with symmetries are described by Poisson structures and systems with constraints are described by closed (but not exact, therefore *presymplectic*) two-forms. Systems with both symmetries and constraints are described using Dirac structures, introduced by Courant in the early 1990s [6]. The original idea was to formulate the dynamics of constrained systems, including constraints induced from degenerate Lagrangians, as in [11, 12]. As a matter of fact, Hamiltonian systems can be formulated in the context of Dirac structures, and their application to electric circuits and mechanical systems with nonholonomic constraints (namely constraints depending on the configuration and velocity variables which, moreover, are not integrable) was studied in detail in [25] where they called the associated Hamiltonian systems with Dirac structures *implicit Hamiltonian systems*. On the other hand, in [37, 38] it was explored the Lagrangian side of this framework, developing the notion of *implicit Lagrangian system* (or *Lagrange-Dirac system*) as a Lagrangian analogue of implicit Hamiltonian systems. This kind of structures was designed to account for the link between Dirac structures in the cotangent bundle and a degenerate Lagrangian system with nonholonomic constraints. Moreover, the suitable space to derive their equations of motion in a variational fashion, through the *Hamilton-Pontryagin principle*, is the so-called *Pontryagin bundle* $TQ \oplus T^*Q$. Besides succeeding in the description of electric circuits and nonholonomic mechanics, the Lagrange-Dirac systems can be also applied to constrained variational dynamics as lately shown in [16].
In this work we follow the ideas just introduced and obtain, employing the already defined Sternberg phase space and magnetized Tulczyjew triple, new geometrical structures providing the dynamics of a charged classical particle subject to a gauge field. Particularly, we will apply a generalized notion of Lagrange-Dirac systems to such particles. Our formulation is general, and accounts for a non-abelian Lie group $G$. For this, we construct an analogue of the Pontryagin bundle in the case of the Sternberg phase space, which we will name as the Sternberg-Pontryagin bundle, and, furthermore, a Dirac structure there, taking advantage of a suitable presymplectic struture. Moreover, we will prove that the Sternberg-Pontryagin bundle is the appropriate space to derive variationally the equations of motion of the Lagrange-Dirac system under consideration. We put emphasis on the local properties of these geometrical structures, performing most of the computations in local coordinates. We enclose our main results in theorem 6.3 developed in §6.

The paper is structured as follows:

§2 is devoted to introduce the Sternberg phase space and to carefully describe its local expression and associated symplectic two-form. In §3 we describe both the usual Tulczyjew triple and its magnetized version. Moreover, the equations of motion of a charged particle in a gauge field are introduced, while they are put in the context of §21 in proposition 3.1. §4 accounts for the description of Dirac structures and Lagrange-Dirac systems. We employ the Pontryagin bundle to illustrate the Lagrange-Dirac systems in proposition 4.2, result which, despite quite natural, is original to the extent of our knowledge. In §5 the Sternberg-Pontryagin bundle is defined and its relationship with the magnetized Tulczyjew triple shown; moreover we present the Sternberg-Pontryagin Lagrange-Dirac system. §6 contains our main result, split into the propositions 6.1, 6.2 and 6.3, where the desired equations of motion are obtained in the context of the Sternberg-Pontryagin bundle from variational, intrinsic and Dirac points of view, respectively. Finally, our theory is illustrated in §7 through the paradigmatic example of a electrically charged particle in an electromagnetic field.

Regarding the repeated indices, we will employ Einstein’s summation convention in this paper unless otherwise noted.

2. The Sternberg phase space

Throughout this work we assume that $Q$ is a smooth manifold, $G$ is a compact connected Lie group with Lie algebra $\mathfrak{g}$, $\pi : P \rightarrow Q$ is a principal $G$–bundle with a fixed principal connection form $\Theta$, and $F$ is a Hamiltonian $G$–space with symplectic form $\Omega_F$ and equivariant moment map $\Phi : F \rightarrow \mathfrak{g}^*$ (meaning commutative with respect to the $G$–action), where $\mathfrak{g}^*$ is the dual of the algebra. By Hamiltonian $G$–space we mean that $F$ is a symplectic manifold with symplectic form $\Omega_F$, that $G$ acts on $F$ as a group of symplectomorphisms, so that there is a homomorphism of the Lie algebra $\mathfrak{g}$ into the algebra of Hamiltonian vector fields, and that we are given a lifting of this homomorphism to a homomorphism of $\mathfrak{g}$ into the Lie algebra of functions on $F$ (where the Lie algebra structure is given by Poisson bracket). Assuming that $Q$ is $n$–dimensional while $F$ is $m$–dimensional, we denote $(q^i)$, $i = 1,...,n$, and $(z^\alpha)$, $\alpha = 1,...,m$ (with $m$ an even number since $F$ is a symplectic manifold), as their local coordinates respectively (we will use $(q, z)$ with some abuse of notation).
Let $\mathcal{F} := P \times_G F$, and $\mathcal{F}^\sharp$ be defined through the commutativity of the diagram
\[
\begin{array}{ccc}
\mathcal{F}^\sharp & \xrightarrow{\tilde{\pi}_Q} & \mathcal{F} \\
\rho^\sharp & & \rho \\
T^*Q & \xrightarrow{\pi_Q} & Q
\end{array}
\]  

where $\pi_Q$ is the canonical projection. It was proven in [27] that there is a correct substitute $\Omega_\Theta$ on $\mathcal{F}$ for $\Omega_F$ on $F$, in the sense that it is a closed two-from on $\mathcal{F}$ and it’s equal to $\Omega_F$ when $P \to Q$ is a trivial bundle with the product connection. Furthermore, if $\Omega_{T^*Q}$ is the canonical symplectic form on $T^*Q$, then
\[
\Omega^\sharp := \Omega_{T^*Q} + \Omega_\Theta
\]
is a symplectic two-form on $\mathcal{F}^\sharp$ (which we will name henceforth as the Sternberg symplectic form). For sake of simplicity, we shall use the same notation for both the differential form (or a map) and its pullback under a fibre bundle projection map (for instance, in (2) both the symplectic two-form on $T^*Q$ and its pullback by $\rho^\sharp$ are denoted by $\Omega_{T^*Q}$, while $\Omega_\Theta$ denotes both a two-form on $\mathcal{F}$ and its pullback through $\tilde{\pi}_Q$; therefore the sum of both two-forms makes sense). Now, we characterize carefully these elements in the next subsection.

2.1. Description of $\mathcal{F}^\sharp$ and $\Omega^\sharp$. Consider a local trivialization $\phi$ of the principal bundle $\pi : P \to Q$, namely a local diffeomorphism $\phi : Q \times G \to P$. On the other hand, locally we have the isomorphism $Q \times F \cong (Q \times G) \times_G F$, which establishes the local trivialization
\[
\phi_F : Q \times F \to \mathcal{F},
\]
where we recall that $\mathcal{F} := P \times_G F$; note that this trivialization can be described in terms of $\phi$ as $\phi_F = \phi \times_G \text{Id}_F$, where $\text{Id}_F : F \to F$ is the identity mapping. Considering $\phi_F$, $(q, z)$ are local coordinates of $\mathcal{F}$. Moreover, let $(q, p)$ be local coordinates of $T^*Q$ (where obviously $p$ stands for $p_i$). Thus, the commutativity of diagram (1) establishes $(q, p, z)$ as local coordinates of $\mathcal{F}^\sharp$ and the following local expression of the projections:
\[
\begin{align*}
\tilde{\pi}_Q : \mathcal{F}^\sharp & \to \mathcal{F}; & \tilde{\pi}_Q : (q, p, z) & \mapsto (q, z), \\
\rho^\sharp : \mathcal{F}^\sharp & \to T^*Q; & \rho^\sharp : (q, p, z) & \mapsto (q, p), \\
\rho : \mathcal{F} & \to Q; & \rho : (q, z) & \mapsto (q).
\end{align*}
\]

To describe the Sternberg symplectic form, we consider the principal connection form $\Theta$, which is a $g$–valued one-form on $P$ satisfying
\[
\begin{align*}
(1) & \quad R^*_a \Theta = \text{Ad}_a \Theta, \text{ for any } a \in G, \text{ where } R_a \text{ denotes the right action of } a \text{ on } P \text{ and } \text{Ad}_a \text{ the adjoint action of } a \text{ on } g, \\
(2) & \quad \Theta(X_\xi) = \xi, \text{ where } X_\xi \in \mathfrak{X}(P) \text{ is the infinitesimal right action of } \xi \in g \text{ on } P.
\end{align*}
\]

From the local point of view, we employ the trivialization $\phi$ to construct a $g$–valued differential one-form on $Q \times G$ by
\[
\phi^* \Theta = g^{-1} A_\phi g + g^{-1} dg
\]
for a unique $g$–valued differential one-form $A_\phi$ on $Q$. Since $G$ is a compact connected Lie group, we can assume that it is a Lie subgroup of $SO(N)$ for some positive integer $N$. In the last equation, $g$ denotes the inclusion map of $G$ into
the vector space of all real square matrices of order $N$. Note that, in terms of $g$, the Maurer-Cartan form on $G$ can be written as $g^{-1}dg$, here, the product between $g^{-1}$ and $dg$ is the matrix multiplication. To see how $A_\phi$ and $A_{\phi'}$ are related (corresponding to two different trivializations $\phi$ and $\phi'$) we note that the bundle isomorphism $\lambda$ defined by the commutative diagram

$$
\begin{array}{ccc}
Q \times G & \xrightarrow{\lambda} & Q \times G \\
\phi & \downarrow & \phi' \\
F & \xrightarrow{\phi} & F
\end{array}
$$

can be written as $\lambda : (q, h) \mapsto (q, a(q)h)$ for a unique smooth map $a : Q \to G$. The commutativity of the previous diagram yields $\phi^* \Theta = \lambda^*(\phi'^* \Theta)$, and therefore we arrive at

$$A_\phi = a^{-1}A_{\phi'}a + a^{-1}da, \quad A_{\phi'} = aA_\phi a^{-1} + ada^{-1}.$$  

As in regards of the Sternberg symplectic form defined in (2), we next consider its local form. First, we define a closed two-form in $F$ in terms of $\Theta$ and $\Phi$ introduced in (3). Namely, let $\Omega := \Omega_F - d(A_\phi, \Phi)$ be a two form on $Q \times F$. Here $\langle \cdot, \cdot \rangle$ denotes the pairing between $g$ and $g^*$ and the momentum map $\Phi$ is understood to be evaluated on $F$ and consequently is an element of the dual algebra. Thus, we define the **Sternberg two-form** $\Omega_\Theta$ on $F$ by

$$\Omega_\Theta := (\phi_F^{-1})^* \Omega_\phi.$$  

(4)

Note that $\Omega_\Theta$ is closed by construction since $\Omega_F$ is symplectic (therefore also closed) and $d^2 = 0$; more concretely $d\Omega_\Theta = (\phi_F^{-1})^* d\Omega_\phi = (\phi_F^{-1})^* d(\Omega_F - d(A_\phi, \Phi)) = (\phi_F^{-1})^* (d\Omega_F - d^2(A_\phi, \Phi)) = 0$. Note as well that $\Omega_\Theta$ as defined in (4) depends on the trivialization $\phi$ in (3). Thus, to show that $\Omega_\Theta$ is well-defined on $F$, it is needed to prove that $(\phi_F^{-1})^* \Omega_\phi = (\phi_F^{-1})^* \Omega_{\phi'}$ for two different trivializations $\phi$ and $\phi'$ according to the diagram

$$
\begin{array}{ccc}
Q \times F & \xrightarrow{\lambda_F} & Q \times F \\
\phi & \downarrow & \phi' \\
F & \xrightarrow{\phi_F} & F
\end{array}
$$

where $\lambda_F(q, f) = (q, a(q)f)$. The condition $(\phi_F^{-1})^* \Omega_\phi = (\phi_F^{-1})^* \Omega_{\phi'}$ may be translated into $\lambda_F^* \Omega_F = \Omega_F - d(a^{-1}da, \Phi)$. Both facts, namely the equivalence of the two conditions and the validity of the latter, are proved in [21], lemma 2.1. After this development we arrive at the following conclusion:

**Proposition 2.1.** There exists a closed real differential well-defined two-form $\Omega_\Theta$ on $F$ defined by $\Omega_\Theta := \Omega_F - d(A, \Phi)$ under a local trivialization of $P \to Q$, where the connection $\Theta$ is represented by the $g$-valued differential one-form $A$ on $Q$.

In this proposition we already drop the subscript $\phi$ of $A$ for sake of simplicity. Finally, the two-form $\Omega^2$ defined in (2) is established as a symplectic form on $F^\sharp$ through the following proposition:

**Proposition 2.2.** The differential two-form $\Omega^2$ is a symplectic form on $F^\sharp$.

*Proof.* It has been already proved that $\Omega_\Theta$ is closed. Besides, $\Omega_{F^\sharp} Q$ is closed as a symplectic two-form, making $\Omega^2$ also closed.
On the other hand, consider the local coordinates of \( \mathcal{F}^\sharp \), \((q,p,z)\), and the local form of \( \Omega^\sharp \), namely
\[
\Omega^\sharp = dq^i \wedge dp_i + \frac{1}{2} \Omega_{\alpha\beta} dz^\alpha \wedge dz^\beta + \frac{1}{2} \left( \partial_i A_j - \partial_j A_i, \Phi \right) dq^i \wedge dq^j - \left( A_i, \partial_\alpha \Phi \right) dq^i \wedge dz^\alpha,
\]
where \( \partial_i = \frac{\partial}{\partial q^i} \), \( \partial_\alpha = \frac{\partial}{\partial z^\alpha} \) and \( \Omega_{\alpha\beta} \) is the local expression of the symplectic form \( \Omega_F \) on the Hamiltonian space \( F \). Employing the matrix form
\[
\Omega^\sharp = \begin{pmatrix}
\delta_i^j & -\delta_{ij}
\end{pmatrix}
\begin{pmatrix}
\partial_\alpha A_i - \partial_\beta A_i, \Phi
\end{pmatrix}
\begin{pmatrix}
\delta_{ij} & \Omega_{\alpha\beta}.
\end{pmatrix}
\]

it is easy to check that \( \Omega^\sharp \) is non-degenerate everywhere by block reduction. This makes the claim hold.

The symplectic manifold \((\mathcal{F}^\sharp, \Omega^\sharp)\) is referred as the Sternberg phase space. In \[32\] it was introduced a symplectic space out of the principal \( G \)-bundle \( P \) \( Q \) and the Hamiltonian \( G \)-space \( F \), and showed that a connection \( \Theta \) yields a symplectomorphism to the Sternberg phase space.

Similarly, we define the space \( \mathcal{F}_\sharp \) through the commutativity of the diagram
\[
\begin{array}{ccc}
\mathcal{F}_\sharp & \xrightarrow{\tilde{\tau}_Q} & \mathcal{F} \\
\rho_1 & & \rho \\
TQ & \xrightarrow{\tau_Q} & Q,
\end{array}
\]
where \( \tau_Q : TQ \rightarrow Q \) is the canonical projection of the tangent bundle. Using the trivialization \[3\] and introducing local coordinates \((q,v)\) for \( TQ \) (where \( v \) stands for \( v' \)), we may describe locally \( \mathcal{F}_\sharp \) by \((q,v,z)\) and the projections in (7) by:
\[
\tilde{\tau}_Q : \mathcal{F}_\sharp \rightarrow \mathcal{F}; \quad \tilde{\tau}_Q : (q,v,z) \mapsto (q,z),
\rho_1 : \mathcal{F}_\sharp \rightarrow TQ; \quad \rho_2 : (q,v,z) \mapsto (q,v).
\]

3. The magnetized Tulczyjew triple

Taking advantage of the symplectic structure of the Sternberg phase space \((\mathcal{F}^\sharp, \Omega^\sharp)\) described in \[2\] and the relationship between \( \mathcal{F} \) and \( \mathcal{F}_\sharp \), namely \( \tilde{\tau}_Q : \mathcal{F}_\sharp \rightarrow \mathcal{F} \), it has been elegantly introduced in \[21\] an analogue of the usual Tulczyjew triple made out of these spaces, named as the magnetized Tulczyjew triple. We introduce both notions and some other useful results for our purposes.

3.1. The Tulczyjew triple. The spaces \( TT^*Q \), \( T^*TQ \) and \( T^*T^*Q \) are naturally double vector bundles (see \[13, 23\]) over \( T^*Q \) and \( TQ \). In \[30\] and \[31\], Tulczyjew established two symplectomorphisms among these spaces, the first one between \( TT^*Q \) and \( T^*TQ \) (namely \( \alpha_Q \)) and the second one between \( TT^*Q \) and \( T^*T^*Q \) (namely \( \beta_Q \)). As cotangent bundles, \( T^*TQ \) and \( T^*T^*Q \) are naturally equipped with symplectic two-forms, \( \Omega_{TT^*Q} \) and \( \Omega_{T^*T^*Q} \) respectively. On the other hand, it may be also proven that \( TT^*Q \) is a symplectic manifold, equipped with the symplectic two-form \( \Omega_{TT^*Q} := dq_T \Omega_{T^*Q} \), where \( dq_T \Omega_{T^*Q} \) is the tangent lift of \( \Omega_{T^*Q} \), which is the usual symplectic form of the cotangent bundle \( T^*Q \) (see \[10\] for more
details.) In the following diagram, known as the Tulczyjew triple, we show the different relationships among these bundles:

\[
\begin{array}{c}
T^*T^*Q \xrightarrow{\kappa_Q} T^*Q \\
\downarrow\pi_{T^*Q} & \quad \downarrow\tau_{T^*Q} & \quad \downarrow\pi_{TQ} \\
T^*Q & \xrightarrow{\beta_Q} TT^*Q & \xrightarrow{\alpha_Q} T^*TQ
\end{array}
\]

(8)

where \(\kappa_Q := \beta_Q \circ \alpha_Q^{-1}\).

**Remark 3.1.** We have introduced the Tulczyjew triple in terms of the canonical symplectic structures corresponding to the double vector bundles \(T^*TQ, T^*T^*Q\). Nevertheless, in a more general geometric landscape, one can always establish the isomorphism \(T^*E \cong T^*E^*\), for any vector bundle \(E \to X\), in terms of the canonical pairings \[15\].

Chosing \((q,p,\dot{q},\dot{p})\) as local coordinates of \(TT^*Q\) and \((q,v,\alpha_q,\alpha_v)\) for \(T^*TQ\), these symplectomorphisms locally read

\[
\begin{align*}
\alpha_Q : (q,p,\dot{q},\dot{p}) & \mapsto (q,\dot{q},\dot{p},p), \\
\beta_Q : (q,p,\dot{q},\dot{p}) & \mapsto (q,p,-\dot{p},\dot{q}), \\
\kappa_Q : (q,v,\alpha_q,\alpha_v) & \mapsto (q,\alpha_v,-\alpha_q,v).
\end{align*}
\]

In order to show the importance of this construction in Geometric Mechanics (and also to describe the procedure employed in the next subsection to obtain geometrically the equations of motion of charged particles in gauge fields), now we briefly discuss how to describe intrinsically both Lagrangian and Hamiltonian mechanics through the Tulczyjew triple, employing as well the notion of Lagrangian submanifold. We use a rather pedestrian definition of the latter concept since a deeper analysis on this subject is not the purpose of this work. Let \((S,\Omega_S)\) be a symplectic manifold and \(N \subset S\) a smooth submanifold with inclusion map \(i\). We say that \(N\) is a Lagrangian submanifold of \(S\) if the following conditions hold:

1) \(\dim N = \frac{1}{2}\dim S\) and 2) \(i^*\Omega_S = 0\).

Consider a Lagrangian (Hamiltonian) function \(L : TQ \to \mathbb{R}\) \((H : T^*Q \to \mathbb{R})\) generating the differential map \(dL : TQ \to T^*TQ\) \((dH : T^*Q \to T^*T^*Q)\). It can be proven that \(dL(TQ) \subset T^*TQ\) \((dH(T^*Q) \subset T^*T^*Q)\) is a Lagrangian submanifold. Employing \(\alpha_Q^{-1} (\beta_Q^{-1})\) we can therefore generate a Lagrangian submanifold of \(TT^*Q\) from \(dL\) \((dH)\), submanifold which determines a system of implicit differential equations whose integrable part can be obtained by applying the integrability constraint algorithm (see \[9\] \[13\] for more details). Of course, these implicit differential equations represent the Lagrangian dynamics, i.e. they are equivalent to the Euler-Lagrange equations (respectively the Hamiltonian dynamics and the usual Hamiltonian equations).

### 3.2. The magnetized Tulczyjew triple.

In the next diagram, in analogy to \[8\], we introduce the magnetized Tulczyjew triple (see \[21\] for more details):
where $\kappa_T := \beta_T \circ \alpha_T^{-1}$ and the projection $T\mathcal{F}$, for coordinates $(q, p, z, \dot{q}, \dot{p}, \dot{z})$ of $T^*\mathcal{F}$, is locally defined by $T\mathcal{F} : (q, p, z, \dot{q}, \dot{p}, \dot{z}) \mapsto (q, z)$. The symplectic structures on $T^*\mathcal{F}$ and $T^*\mathcal{F}_{\tau}$ are provided by their cotangent structure ($\Omega_{T^*\mathcal{F}_T}$ and $\Omega_{T^*\mathcal{F}_{\tau}}$ respectively), while $\Omega_{T^*\mathcal{F}_T}$ is defined from $\Omega_\Theta$, this is $\Omega_{T^*\mathcal{F}_T} := dT\Omega_\Theta$, where again $dT$ represents the tangent lift. Considering $(q, v, z, \alpha_q, \alpha_v, \alpha_z)$ as local coordinates of $T^*\mathcal{F}_{\tau}$, next we display the local form of the symplectomorphisms in $[9]$: \[
\begin{align*}
\alpha_T & : (q, p, z, \dot{q}, \dot{p}, \dot{z}) \mapsto (q, \dot{q}, \dot{p}, \dot{z}, p, z), \\
\beta_T & : (q, p, z, \dot{q}, \dot{p}, \dot{z}) \mapsto (q, p, z, \dot{q}, \dot{p}, \dot{z}), \\
\kappa_T & : (q, v, z, \alpha_q, \alpha_v, \alpha_z) \mapsto (q, \alpha_v, v, \alpha_q, -v, \alpha_z).
\end{align*}
\]

This triple is used in [21] in order to obtain the equations of motion of a charged particle in the presence of gauge fields (role globally played by the connection $\Theta$ and locally by its local expression under trivialization, say $A$). For this, in analogy with how the Lagrangian dynamics is obtained employing the usual Tulczyjew triple, a smooth Lagrangian submanifold of $(T^*\mathcal{F}, \Omega_{T^*\mathcal{F}})$ is considered, in particular the submanifold generated by a Lagrangian function $L_\tau : \mathcal{F}_\tau \to \mathbb{R}$ through the following diagram:

Let $c : \mathbb{R} \to \mathcal{F}$ be a parametrized curve on $\mathcal{F}$ and let $(c(t), \frac{d}{dt}(\rho \circ c(t)))$, where $\rho$ is the projection map defined in diagrams [4] and [7], be the lifted curve to $\mathcal{F}_\tau$ (note that the local coordinates of $(c(t), \frac{d}{dt}(\rho \circ c(t)))$ may be considered, with some abuse of notation, $(q(t), \dot{q}(t), z(t))$). Now, considering the differential map $dL_\tau : \mathcal{F}_\tau \to T^*\mathcal{F}_\tau$, we employ the magnetized Tulczyjew triple [9] to obtain the Lagrangian submanifold $\alpha_T^{-1}(dL_\tau((c(t), \frac{d}{dt}(\rho \circ c(t)))))$. Finally, taking into account the local expression of
where, in the last equation, $\alpha \in \Omega^{\alpha_\beta}$ are equivalent:

\begin{equation}
\frac{d}{dt} z^\alpha = \Omega^{\alpha_\beta} \left( \frac{\partial L_\zeta}{\partial z^\beta} - \langle \dot{q}^k A_k^\alpha, \frac{\partial \Phi}{\partial z^\beta} \rangle \right),
\end{equation}

(10)

\begin{equation}
\frac{d}{dt} \left( \frac{\partial L_\zeta}{\partial \dot{q}^i} \right) = \frac{\partial L_\zeta}{\partial q^i} + \dot{q}^j \frac{\partial A_i^\alpha}{\partial q^k} - \frac{\partial A_j^\alpha}{\partial q^i} \Phi + \dot{z}^\alpha \langle A_i, \frac{\partial \Phi}{\partial z^\alpha} \rangle,
\end{equation}

(11)

where $\{\Omega^{\alpha_\beta}\} = (\Omega_{\alpha_\beta})^{-1}$ exists, since $\Omega_F$ is full-rank.

**Remark 3.2.** As it is well known, the Darboux’s theorem ensures that, for any point in $F$, there exists an open neighbourhood in which the local coordinates $z^\alpha$ may be split into $z^\alpha = (z^a, z^\delta)$, where $a, \delta = 1, ..., m/2$, such that $\Omega_F = \Omega_{\alpha_\beta} dz^a \wedge dz^\delta = \delta_{\alpha \delta} dz^a \wedge dz^\delta$, where $\delta_{\alpha \delta}$ is the usual Kronecker delta. Using this particular local representation, the equations (10) read

\begin{equation}
\frac{d}{dt} z^a = -\delta a^a \left( \frac{\partial L_\zeta}{\partial z^a} - \langle \dot{q}^k A_k^\alpha, \frac{\partial \Phi}{\partial z^a} \rangle \right),
\end{equation}

(11)

\begin{equation}
\frac{d}{dt} z^\delta = \delta a^a \left( \frac{\partial L_\zeta}{\partial z^a} - \langle \dot{q}^k A_k^\alpha, \frac{\partial \Phi}{\partial z^a} \rangle \right),
\end{equation}

(11)

where, in the last equation, $\alpha = (a, \delta)$ and $\delta a^a$ is the inverse of $\delta_{\alpha \delta}$. In general, we shall use equally the expressions (10) and (11), preferring the latter in some proofs for convenience.

As shown by this procedure, the equations above may be obtained from a geometrical condition. On the other hand, they can be obtained by usual calculus of variations (as mentioned, but not proved, in [21]). We enclose this result in the following proposition, which must be understood as a rephrasing of part of the main theorem in [21]:

**Proposition 3.1.** Let $L_\zeta : T\zeta \rightarrow \mathbb{R}$ be a smooth Lagrangian function and $\tilde{c} : \mathbb{R} \rightarrow T\zeta$ a smooth curve. For a charged particle with configuration space $Q$, internal space $F$, gauge field $\Theta$ and Lagrangian $L_\zeta$, its equations of motion are locally written as (10)–(11), equations that can be obtained from the next two statements (which are equivalent):

1. $\tilde{c}(t) \in \alpha^{-1}_F (dL_\zeta(T\zeta))$,

2. let $L_\zeta$ be a extended Lagrangian defined by

\begin{equation}
L_\zeta := L_\zeta - \langle \dot{q} A_i^\alpha, \Phi \rangle + \Omega_F(z, \dot{z}),
\end{equation}

(12)

where we set $\Omega_F(z, \dot{z}) := \Omega_F(z^a \frac{\partial}{\partial z^a}, z^\delta \frac{\partial}{\partial z^\delta}) = \delta_{\alpha \delta} \dot{z}^\alpha z^\alpha$. Then, the stationary condition for the action functional

\begin{equation}
\int_{t_1}^{t_2} L_\zeta((q(t), z(t), \dot{q}(t), \dot{z}(t)) dt
\end{equation}

where the endpoints of $(q(t), z(t))$ are fixed, sigles out a curve obeying the equations (10), (11).
Note that the extended Lagrangian $\mathcal{L}_4$ is degenerate on $T\mathbb{F}_2$, i.e. if we define the function $\mathcal{L}_4 : T\mathbb{F}_2 \to \mathbb{R}$ is easy to see that $\frac{\partial \mathcal{L}_4}{\partial \dot{z}} = 0$ using the local coordinates $(q,v,z,q,v,z)$ for $T\mathbb{F}_2$. Our task in the subsequent sections, which is the main purpose of this paper, is to reobtain the equations $[10],[11]$ from a new variational principle and a Lagrange-Dirac condition in the Sternberg-Pontryagin bundle, which will be introduced in $[14]$.

4. DIRAC STRUCTURES AND LAGRANGE-DIRAC SYSTEMS

4.1. Dirac Structures. We first recall the definition of a Dirac structure on a vector space $V$, say finite dimensional for simplicity (see $[5]$ and $[7]$). Let $V^*$ be the dual space of $V$, and $\langle \cdot, \cdot \rangle$ be the natural paring between $V^* \times V$. Define the symmetric paring $\langle \langle \cdot, \cdot \rangle \rangle$ on $V \oplus V^*$ by

$$\langle \langle (v,\alpha), (\bar{v}, \bar{\alpha}) \rangle \rangle = \langle \alpha, \bar{v} \rangle + \langle \bar{\alpha}, v \rangle,$$

for $(v,\alpha), (\bar{v}, \bar{\alpha}) \in V \oplus V^*$. A Dirac structure on a vector space $V$ is a subspace $D \subset V \oplus V^*$ such that $D = D^\perp$, where $D^\perp$ is the orthogonal of $D$ relative to the pairing $\langle \langle \cdot, \cdot \rangle \rangle$.

Now let $M$ be a smooth manifold and let $TM \oplus T^*M$ denote the Whitney sum bundle over $M$, namely, the bundle over the base $M$ and with fiber over the point $x \in M$ equal to $T_xM \times T^*_xM$. In this paper, we shall call a subbundle $D_M \subset TM \oplus T^*M$ a Dirac structure on the manifold $M$, or a Dirac structure on the bundle $\tau_M : TM \to M$, when $D_M(x)$ is a Dirac structure on the vector space $T_xM$ at each point $x \in M$. A given two-form $\omega$ on $M$ together with a distribution $\Delta_M$ on $M$ determines a Dirac structure on $M$ as follows:

**Proposition 4.1.** The two-form $\omega$ determines a Dirac structure $D_M$ on $M$ whose fiber is given for each $x \in M$ as

$$D_M(x) = \{(v_x, \alpha_x) \in T_xM \times T^*_xM \mid v_x \in \Delta_M(x), \quad \alpha_x(w_x) = \omega_{\Delta_M}(v_x, w_x) \text{ for all } w_x \in \Delta_M(x)\},$$

where $\Delta_M \subset TM$ and $\omega_{\Delta_M}$ is the restriction of $\omega$ to $\Delta_M$.

We present the proof for convenience of the reader (see $[27]$ for more details).

**Proof.** The orthogonal of $D_M \subset TM \oplus T^*M$ at the point $x \in M$ is given by

$$D_M^\perp(x) = \{(u_x, \beta_x) \in T_xM \times T^*_xM \mid \alpha_x(u_x) + \beta_x(v_x) = 0, \quad \forall v_x \in \Delta_M \text{ and } \langle \alpha_x, w_x \rangle = \omega_{\Delta_M}(x)(v_x, w_x) \text{ for all } w_x \in \Delta_M\}.$$

In order to prove that $D_M \subset D_M^\perp$, let $(v_x, \alpha_x), (v'_x, \alpha'_x)$ belong to $D_M(x)$. Then

$$\langle \alpha_x, v'_x \rangle + \langle \alpha'_x, v_x \rangle = \omega_{\Delta_M}(x)(v_x, v'_x) + \omega_{\Delta_M}(x)(v'_x, v_x) = 0,$$

since $\omega_{\Delta_M}(x)$ is skew-symmetric. Therefore, $D_M \subset D_M^\perp$.

To conclude the proof we shall check that $D_M^\perp \subset D_M$. Let $(u_x, \beta_x) \in D_M(x)^\perp$. By definition of $D_M^\perp$, we have that

$$\langle \alpha_x, u_x \rangle + \langle \beta_x, v_x \rangle = 0$$

for all $v_x \in \Delta_M$ and $\langle \alpha_x, w_x \rangle = \omega_{\Delta_M}(x)(v_x, w_x)$ for all $w_x \in \Delta_M$. Choose $v_x, u_x \in \Delta_M$ arbitrary vectors. From $\langle \alpha_x, u_x \rangle + \langle \beta_x, v_x \rangle = 0$ and the fact that $u_x \in \Delta_M$ is an arbitrary vector we have that

$$\omega_{\Delta_M}(x)(v_x, u_x) + \beta_x(v_x) = 0$$

for all $v_x \in \Delta_M$, that is $\beta_x(v_x) = -\omega_{\Delta_M}(x)(u_x, v_x)$ due to the skew-symmetry of $\omega_{\Delta_M}$. Thus, $(u_x, \beta_x) \in D_M(x)$ and hence $D_M^\perp \subset D_M$, as required. Consequently, $D_M^\perp = D_M$ and the claim holds.
Of course, this proposition is also valid when $\Delta_M = TM (\omega_{\Delta_M} = \omega)$, which is the case in this work since we do not consider restricted systems, and, furthermore, either for pre-symplectic or symplectic two-forms since the key property to accomplish the result is their skew-symmetry. On the other hand, throughout this work we shall define the Dirac structures in a different but equivalent way to proposition 4.1. Namely, each two-form $\omega$ on $M$ defines a bundle map $\omega^\flat : TM \to T^*M$ by $\omega^\flat(v) = \omega(v, \cdot)$. Consequently, we may equivalently define $D_M(x)$ in (13) as

$$D_M(x) = \{ (v_x, \alpha_x) \in T_xM \times T^*_xM \mid v_x \in \Delta_M(x), \text{ and } \alpha_x - \omega^\flat(x)(v_x) \in \Delta_M^\flat(x) \},$$

or in other words $D_M(x) := \text{graph}(\omega^\flat)|_{|x}.$

4.2. Lagrange-Dirac systems. As shown just above, the Dirac structures can be given by the graph of the bundle map associated with the canonical symplectic structure, and hence it naturally provides a geometric setting for Hamiltonian mechanics. On the other hand, as mentioned in the introduction, the Dirac systems are also useful in the Lagrangian side when one considers degenerate Lagrangian functions and restricted systems [16, 37, 38].

Based on the ideas of these references, we next present a rather general definition of a Lagrange-Dirac dynamical system and its equations of motion; afterwards, we give a significative example.

**Definition 4.1.** Consider a Dirac structure $D_M$ on $M$, a curve $x : \mathbb{R} \to M$ and the exterior differential $d\gamma : M \to T^*M$, where $\gamma : M \to \mathbb{R}$ is a smooth function. We define the Lagrange-Dirac dynamical system (or implicit Lagrangian system) induced by the Dirac structure $D_M$ and the curve $\gamma$ as the pair $(D_M, \gamma)$. Its equations of motion are given by

$$(\dot{x}(t), d\gamma(x(t))) \in D_M(x(t)).$$

Any curve $x(t) \subset M$, $t_1 \leq t \leq t_2$ satisfying this condition is called a solution curve of the Lagrange-Dirac system.

We illustrate the Lagrange-Dirac systems by means of the Pontryagin bundle $TQ \oplus T^*Q$ over a manifold $Q$, that is the Whitney sum of the tangent bundle and the cotangent bundle over $Q$, whose fiber at $q \in Q$ is the product $T_qQ \times T^*_qQ$. If $(q, p)$ and $(q, v)$ are local coordinates for $T^*Q$ and $TQ$ respectively, the Pontryagin bundle is locally described by $(q, v, p)$, and these three projections are naturally defined:

$$\begin{align*}
p_{TQ} & : TQ \oplus T^*Q \to TQ; \quad (q, v, p) \mapsto (q, v), \\
p_{T^*Q} & : TQ \oplus T^*Q \to T^*Q; \quad (q, v, p) \mapsto (q, p), \\
p_Q & : TQ \oplus T^*Q \to Q; \quad (q, v, p) \mapsto (q).
\end{align*}$$
The Pontryagin bundle and its projections fits in the Tulczyjew triple as in the next diagram:

Consider now the presymplectic two-form $\Omega$ on $T^*Q \oplus TQ$ (where we denote by $\Omega_{T\cdot Q}$ its pullback under the projection $pr_{T\cdot Q}$). Thus, employing the proposition \[ \text{(E.1)} \], we can define the Dirac structure

$$D_{PB}(y) = \{ (v_y, \alpha_y) \in T_y(TQ \oplus T^*Q) \times T_y^*\pi^*Q \mid v_y \in T_y(TQ \oplus T^*Q),$$

and $\alpha_y(w_y) = \Omega_{T\cdot Q}(y)(v_y, w_y)$ for all $w_y \in T_y(TQ \oplus T^*Q)$,

where $y = (q, v, p) \in TQ \oplus T^*Q$, or in the simpler form $D_{PB}(y) = \text{graph}(\Omega_{T\cdot Q})\big|_{y'}$.

Given a Lagrangian $L : TQ \to \mathbb{R}$ (possibly degenerate) and its associated generalized energy $E_L : TQ \oplus T^*Q \to \mathbb{R}$, $E_L := (p, v) - L(q, v)$, according to definition \[ \text{(E.1)} \], we can state the following proposition:

**Proposition 4.2.** The equations of motion of the Lagrange-Dirac system $(D_{PB}, E_L)$ are locally given for each $y = (q, v, p) \in TQ \oplus T^*Q$ by

$$((\dot{q}, \dot{v}, \dot{p}), dE_L(q, v, p)) \in D_{PB}(q, v, p). \quad \text{(14)}$$

These equations are equivalent to the usual Euler-Lagrange equations.

**Proof.** The Dirac structure $D_{PB} \subset T(TQ \oplus T^*Q) \oplus T^*(TQ \oplus T^*Q)$ is locally defined by

$$D_{PB}(y) = \{ ((\dot{q}, \dot{v}, \dot{p}), (\alpha, \beta, u)) \mid -\dot{p} = \alpha, 0 = \beta, \dot{q} = u \},$$

where $\alpha d\dot{q} + \beta d\dot{v} + u dt = T^*(TQ \oplus T^*Q)$. Setting $(\alpha, \beta, u) = dE_L$, we arrive at $\alpha = -\frac{\partial L}{\partial \dot{q}}$, $\beta = p - \frac{\partial L}{\partial v}$ and $u = v$, and, therefore, at the coordinate equations of motion of the Lagrange-Dirac system

$$\dot{\dot{q}} = \frac{\partial L}{\partial \dot{q}}, \quad \dot{p} = \frac{\partial L}{\partial v} = 0, \quad \dot{q} = v,$$

which are, after a straightforward computation, the usual Euler-Lagrange equations of a Lagrangian system, namely

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) = \frac{\partial L}{\partial q}.$$

From the variational point of view, it is easy to prove, employing usual calculus of variations, that these equations can be also obtained from the stationary condition of the action functional

$$\int_{t_1}^{t_2} \left[ [\dot{p}(t), \dot{q}(t)] - E_L(q(t), v(t), p(t)) \right] dt$$

with fixed endpoints of $q(t)$. This is known as the Hamilton-Pontryagin principle.
We use the spaces $\mathcal{F}^\sharp$ and $\mathcal{F}_2$ defined in §2 in order to introduce, in analogy to the usual Pontryagin bundle $TQ \oplus T^*Q$, what we define as the Sternberg-Pontryagin bundle.

**Definition 5.1.** Consider the bundle $\mathcal{F}^\sharp \oplus \mathcal{F}_2$ over $\mathcal{F}$, whose fiber at $(q, z) \in \mathcal{F}$ is the product $\mathcal{F}^\sharp \times_{(q, z)} \mathcal{F}_2$. We call the bundle $\mathcal{F}^\sharp \oplus \mathcal{F}_2$ the Sternberg-Pontryagin bundle.

Under this definition, the local coordinates of $\mathcal{F}^\sharp \oplus \mathcal{F}_2$ are written $(q, v, p, z)$, while the following three projections are naturally defined:

- $\text{pr}_{\mathcal{F}^\sharp} : \mathcal{F}^\sharp \oplus \mathcal{F}_2 \to \mathcal{F}^\sharp$; $(q, v, p, z) \mapsto (q, p, z)$,
- $\text{pr}_{\mathcal{F}_2} : \mathcal{F}^\sharp \oplus \mathcal{F}_2 \to \mathcal{F}_2$; $(q, v, p, z) \mapsto (q, v, z)$,
- $\text{pr}_{\mathcal{F}} : \mathcal{F}^\sharp \oplus \mathcal{F}_2 \to \mathcal{F}$; $(q, v, p, z) \mapsto (q, z)$.

All the previous developments may be summarized into the following diagram, where (1) and (7) have been taken into account:

Similarly to the case of the usual Pontryagin bundle $TQ \oplus T^*Q$, the Sternberg-Pontryagin bundle $\mathcal{F}^\sharp \oplus \mathcal{F}_2$ fits in the magnetized Tulczyjew triple (8) as shown in the following diagram:

As proven in §2, $\mathcal{F}^\sharp$ is equipped with a symplectic form $\Omega^\sharp$. Taking advantage of the projection $\text{pr}_{\mathcal{F}^\sharp} : \mathcal{F}^\sharp \oplus \mathcal{F}_2 \to \mathcal{F}_2$, we can induce a presymplectic two-form in the Pontryagin-Sternberg bundle $\mathcal{F}^\sharp \oplus \mathcal{F}_2$, namely $(\text{pr}_{\mathcal{F}^\sharp})^* \Omega^\sharp$ (which we will also denote $\Omega^\sharp$). Furthermore, this two-form induces the bundle map

$$(\Omega^\sharp)^\flat : T(\mathcal{F}^\sharp \oplus \mathcal{F}_2) \to T^*(\mathcal{F}^\sharp \oplus \mathcal{F}_2),$$

and consequently, according to proposition 4.1, the Dirac structure

$$D^\sharp(x) = \text{graph} (\Omega^\sharp)^\flat |_x,$$
where \( x = (q, v, p, z) \in \mathcal{F}^7 \oplus \mathcal{F}_z \). We name \( D^7 \) the Pontryagin-Sternberg Dirac structure. On the other hand, consider a Lagrangian function (possibly degenerate) \( L_2 : \mathcal{F}_z \to \mathbb{R} \) and define its associated generalized energy function \( E_{L_2} : \mathcal{F}^7 \oplus \mathcal{F}_z \to \mathbb{R} \) in local coordinates by

\[
E_{L_2}(q, v, p, z) := \langle p, v \rangle - L_2(q, v, z),
\]

where \( \langle \cdot, \cdot \rangle \) denotes the natural pairing between \( TQ \) and \( T^*Q \). With all these ingredients and according to definition \([4.1]\) we introduce the following Lagrange-Dirac system:

**Definition 5.2.** Consider the Dirac structure \( D^7 \) on \( \mathcal{F}^7 \oplus \mathcal{F}_z \), a Lagrangian function (possibly degenerate) \( L_2 : \mathcal{F}_z \to \mathbb{R} \), its associated generalized energy function \( E_{L_2} : \mathcal{F}^7 \oplus \mathcal{F}_z \to \mathbb{R} \) \([17]\) and a curve \( x(t) = (q(t), v(t), p(t), z(t)) \in \mathcal{F}^7 \oplus \mathcal{F}_z \). We define the Pontryagin-Sternberg Lagrange-Dirac system by \((D^7, E_{L_2})\) and its equations of motion by

\[
(\dot{x}(t), dE_{L_2}(x(t))) \in D^7(x(t)).
\]

6. Main Theorem

In this section we split our main result into three propositions, enclosing them in a compact way in the final theorem. The two statements in proposition \([3.1]\) might be also included (since they are all equivalent) but we prefer to keep them out in order to emphasize the new results.

First we establish a variational principle providing the equations of motion of a charged particle in a gauge field \([10]\). For that, we present some useful definitions. As above, let \( x = (q, v, p, z) \) be local coordinates of \( \mathcal{F}^7 \oplus \mathcal{F}_z \); therefore \( (x, \dot{x}) = (q, v, p, z, \dot{q}, \dot{v}, \dot{p}, \dot{z}) \) are the local coordinates of \( T(\mathcal{F}^7 \oplus \mathcal{F}_z) \). Furthermore, consider \( r = (q, v, z) \) local coordinates for \( \mathcal{F}_z \) and therefore \( (r, \dot{r}) = (q, v, z, \dot{q}, \dot{v}, \dot{z}) \) for \( T\mathcal{F}_z \).

Define the extended generalized energy function \( E_{\mathcal{L}_4} : T(\mathcal{F}^7 \oplus \mathcal{F}_z) \to \mathbb{R} \), locally given by

\[
E_{\mathcal{L}_4}(x, \dot{x}) := \langle p, v \rangle - \mathcal{L}_4(r, \dot{r}),
\]

where \( \langle \cdot, \cdot \rangle \) is the pairing between \( TQ \) and \( T^*Q \) and \( \mathcal{L}_4 \) is the extended Lagrangian defined in \([12]\). Note that \( E_{\mathcal{L}_4} \) is also degenerate by definition due to its \((\dot{v}, \dot{p})\)-independence.

**Proposition 6.1.** Let \( \mathcal{L}_4 : T\mathcal{F}_z \to \mathbb{R} \) be a degenerate Lagrangian function defined by \([12]\) and \( E_{\mathcal{L}_4} : T(\mathcal{F}^7 \oplus \mathcal{F}_z) \to \mathbb{R} \) the degenerate extended generalized energy in \([19]\). Define the action functional

\[
\int_{t_1}^{t_2} \left[ p(t) \dot{q}(t) - v(t) \right] dt + \mathcal{L}_4(r(t), \dot{r}(t)) \right] dt
\]

\[
= \int_{t_1}^{t_2} \left[ \langle p(t), \dot{q}(t) \rangle - E_{\mathcal{L}_4}(x(t), \dot{x}(t)) \right] dt.
\]

Then, keeping the endpoints of \( (q(t), z(t)) \in \mathcal{F} \) fixed, whereas the endpoints of \( v(t) \) and \( p(t) \) are allowed to be free, the stationary condition for this action functional induces the equations \([10]\)–\([11]\).
Proof. By direct computations, the variation of (20) reads
\[
\delta \int_{t_1}^{t_2} [\langle p, \dot{q} - v \rangle + \mathcal{L}_q(r, \dot{r})] \, dt = \int_{t_1}^{t_2} \left[ (\delta p, \dot{q} - v) + \left( \frac{\partial \mathcal{L}_q}{\partial v} \right)_{\delta v} \right] \, dt
\]
\[
+ \left( \frac{\partial \mathcal{L}_q}{\partial \dot{q}} \right)_{\delta q} + \left( \frac{\partial \mathcal{L}_q}{\partial v} \right)_{\delta v} + \left( \frac{\partial \mathcal{L}_q}{\partial z} \right)_{\delta z}
\]
\[
- \left( \delta q^i A_i, \Phi \right) - \langle \dot{q}^i \partial_j A_i, \delta q^j \rangle - \langle \dot{q}^i A_i, \partial_\alpha \Phi \delta z^\alpha \rangle
\]
\[
+ \delta_\alpha^a z^a \delta z^\alpha + \delta_\alpha^a z^a \delta z^\alpha \right] dt,
\]
where the particular form of (12), i.e. \( \mathcal{L}_q = \mathcal{L}_q - \langle \dot{q}^i A_i, \Phi \rangle + \Omega_F(z, \dot{z}) \), has been taken into account (note in the last two terms the difference between the Kronecker’s delta and the variation of the coordinates) as long with the splitting of coordinates \( \alpha = (a, \bar{a}) \). Moreover, in the first four terms \( \langle \cdot, \cdot \rangle \) means the pairing between \( T^*Q \) and \( T^*Q \), in the next three ones the pairing between \( T^*\mathcal{F}_2 \) and \( T\mathcal{F}_2 \) and, finally, in the next three ones the pairing between \( \mathfrak{g}^* \) and \( \mathfrak{g} \). Now, reordering the terms and performing integration by parts we arrive at
\[
\delta \int_{t_1}^{t_2} [\langle p, \dot{q} - v \rangle + \mathcal{L}_q(r, \dot{r})] \, dt = \int_{t_1}^{t_2} \left[ (\delta p, \dot{q} - v) + \left( \frac{\partial \mathcal{L}_q}{\partial v} \right)_{\delta v} \right] \, dt
\]
\[
+ \left( -\dot{p}^i + \frac{\partial \mathcal{L}_q}{\partial \dot{q}^i} + \dot{q}^i \left( \frac{\partial A_i}{\partial q^j} - \frac{\partial A_j}{\partial q^i} \right) + \langle A_i, z^\alpha \partial_\alpha \Phi, \delta q^i \rangle \right)
\]
\[
+ \left( \frac{\partial \mathcal{L}_q}{\partial z^\alpha} - \langle \dot{q}^i A_i, \partial_\alpha \Phi \rangle - \delta_\alpha^a z^a, \delta z^\alpha \rangle \right] dt
\]
\[
+ \left( \frac{\partial \mathcal{L}_q}{\partial z^a} - \langle \dot{q}^i A_i, \partial_a \Phi \rangle + \delta_\alpha^a z^a, \delta z^\alpha \rangle \right] dt
\]
\[
+ \langle p, \delta q \rangle \bigg|_{t_1}^{t_2} - \delta q^i \langle A_i, \Phi \rangle \bigg|_{t_1}^{t_2} + \delta_\alpha^a z^a \delta z^\alpha \bigg|_{t_1}^{t_2},
\]
where we have used that
\[
\int_{t_1}^{t_2} \delta q^i \langle A_i, \Phi \rangle \, dt = \delta q^i \langle A_i, \Phi \rangle \bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \delta q^i \frac{d}{dt} \langle A_i, \Phi \rangle \, dt
\]
\[
= \delta q^i \langle A_i, \Phi \rangle \bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \delta q^i \left[ (\dot{q}^i \partial_j A_j, \Phi) + \langle A_i, \dot{z}^\alpha \partial_\alpha \Phi \rangle \right] dt
\]
under integration by parts.

Now, taking into account that \( \delta q(t_1) = \delta q(t_2) = \delta z(t_1) = \delta z(t_2) = 0 \) the last three terms vanish. Moreover, considering that \( (\delta q, \delta v, \delta p, \delta z) \) are free, the stationary condition above provides the following equations.

\[
\dot{q} = v,
\]
\[
p = \frac{\partial L_q}{\partial \dot{v}},
\]
\[
\dot{p}_i = \frac{\partial L_q}{\partial \dot{q}^i} + \dot{q}^i \left( \frac{\partial A_j}{\partial q^i} - \frac{\partial A_i}{\partial q^j} \right) + \langle A_i, z^\alpha \partial_\alpha \Phi \rangle,
\]
\[
\dot{z}^a = -\delta_\alpha^a \left( \frac{\partial L_q}{\partial z^\alpha} - \langle \dot{q}^i A_i, \partial_\alpha \Phi \rangle \right),
\]
\[
\dot{z} = \delta_\alpha^a \left( \frac{\partial L_q}{\partial z^a} - \langle \dot{q}^i A_i, \partial_a \Phi \rangle \right).
\]
These are obviously the equations (11) as claimed. \( \square \)

Now, taking advantage of the geometry introduced in the diagrams (15) and (16), we attempt to obtain an intrinsic expressions of the action functional (20) and the
equations (10). As a first guess, considering (15) we notice that the Poincaré-Cartan one form $\Theta_{T\otimes Q}$ on $T^*Q$ (with local form $\Theta_{T\otimes Q} = p_1dq^i$) can be pulled-back to $T(F^\otimes \mathcal{T})$ through the chain

$$T(F^\otimes \mathcal{T}) \xrightarrow{pr_{\otimes}} F^\otimes \mathcal{T} \xrightarrow{pr_{T^*}} T^*Q,$$

where $\tau_{\mathcal{T}\oplus \mathcal{T}}: T(F^\otimes \mathcal{T}) \rightarrow F^\otimes \mathcal{T}$ is obviously the canonical tangent projection, inducing a one-form $\rho_1^*\circ pr_{\mathcal{T}} \circ \tau_{\mathcal{T}\oplus \mathcal{T}})\ast \Theta_{T\otimes Q}$ on $T(F^\otimes \mathcal{T})$ (denoted $\Theta_{T\otimes Q}$ as well). Denoting $\tilde{x} = (x, \dot{x}) \in T(F^\otimes \mathcal{T})$, the action functional

$$\int_{t_1}^{t_2} \left(\Theta_{T\otimes Q}(\tilde{x}(t)), \dot{\tilde{x}}(t) \right) - E_{L_1}(\tilde{x}(t)) dt$$

is a fair global expression of (20), fact that can be easily proven by direct computations in coordinates. Nevertheless, taking variations and integrating by parts we arrive at

$$\int_{t_1}^{t_2} \left[\left(-1_{\tilde{x}(t)}d\Theta_{T\otimes Q}(\tilde{x}(t)) - dE_{L_1}(\tilde{x}(t)), \delta \tilde{x}(t)\right)\right] dt + \left(\Theta_{T\otimes Q}(\tilde{x}(t)), \delta \tilde{x}(t)\right)|_{t_1}^{t_2} = 0,$$

which fixing the endpoints of $q(t)$ yields $1_{\tilde{x}(t)}\Omega_{T\otimes Q}(\tilde{x}(t)) = dE_{L_1}(\tilde{x}(t))$, with $\Omega_{T\otimes Q} = -\partial Q$. After some calculations, we realize that this is not a global representation of (10) (we skip the details for sake of short). This fact points out that the usual symplectic geometry, pulled-back to the new space $F^\otimes \mathcal{T}$, is not enough to describe the equations of a charged particle in a gauge field. Consequently, we reorient our attention to the Sternberg-Pontryagin bundle in order to construct a meaningful one-form there. Indeed, noting that the connection $A$ is a $g^*$-valued one-form on $T^*Q$ and appealing to the considerations in remark 3.2 we define (using the Darboux’s coordinates $(z^a, z^a)$ for $F$):

$$\Theta^2 := (p_i - \langle A_i, \Phi \rangle)\, dq^i + z^a dz^a,$$

where the one-form in $F$, i.e. $z^a dz^a =: \Theta_F$, is defined such that $-d\Theta_F = \Omega_F = \delta_{ab}dz^a \wedge dz^b$. Taking into account that $\Omega^2 = \Omega_{T\otimes Q} - d(A, \Phi) + \Omega_F$, it is easy to check that $\Omega^2 = -d\Theta^2$. Pulling-back $\Theta^2$ to $F^\otimes \mathcal{T}$ through $pr_{\mathcal{T}}$ (note that $\Omega^2$ will be presymplectic in $F^\otimes \mathcal{T}$) and taking into account the generalized energy (17) we define the action functional

$$\int_{t_1}^{t_2} \left(\Theta^2(x(t)), \dot{x}(t)\right) - E_{L_1}(x(t)) dt,$$

where again $x = (q, v, p, z) \in F^\otimes \mathcal{T}$, which is as well a fair global expression of (20), as can be easily proven by direct computations in coordinates. We show in the next proposition that this new action functional provides also a global representation of (10).

**Proposition 6.2.** Under the endpoints $(q(t), z(t)) = pr_F(x(t))$ fixed, the stationary condition of the action functional (23) singles out a critical curve $x(t)$ that satisfies the intrinsic equations of motion of a charged particle in a gauge field:

$$1_{\tilde{x}(t)}\Omega^2(\tilde{x}(t)) = dE_{L_1}(x(t)).$$

Moreover, these equations are equivalent to (10).

**Proof.** To prove the first statement, we take variations over (23), which yields:

$$\delta \int_{t_1}^{t_2} \left[\left(\Theta^2(x(t)), \dot{x}(t)\right) - E_{L_1}(x(t))\right] dt$$

$$= \int_{t_1}^{t_2} \left[\left(-1_{\tilde{x}(t)}d\Theta^2(x(t)) - dE_{L_1}(x(t)), \delta x(t)\right)\right] dt + \left(\Theta^2(x(t)), \delta x(t)\right)|_{t_1}^{t_2} = 0,$$
where integration by parts has been performed. For all variations $\delta x(t)$ and fixed endpoints $(q(t), z(t)) = pr_T(x(t))$, one arrives straightforwardly at

$$dE_{L^1}(x(t)) = \delta \{ L^2(x(t)) \} = \delta \{ H(x(t)) \}$$

To prove the second, we consider the local form of $\Omega^2$ on $\mathcal{F}^2 \oplus \mathcal{F}_2$, particularly (recall (6))

$$\Omega^2 = \begin{pmatrix}
\langle \partial_i A_j - \partial_j A_i, \Phi \rangle & \delta_j^i & -\langle A_i, \partial_\alpha \Phi \rangle & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-\delta_j^i & 0 & 0 & 0 & 0 & 0 \\
\langle A_i, \partial_\alpha \Phi \rangle & 0 & 0 & \Omega_{\alpha \beta} & 0 & 0 \\
\end{pmatrix},$$

which leads to

$$i_x \Omega^2(x) = \langle q' (\partial_i A_j - \partial_j A_i, \Phi) - \hat{p}_i + \hat{z}^\alpha (A_i, \partial_\alpha \Phi), dq^j \rangle + \langle dp, \hat{q} \rangle + \langle -q' (A_i, \partial_\alpha \Phi) + \Omega_{\alpha \beta} \hat{z}^\beta, dz^\alpha \rangle.$$  

On the other hand

$$dE_{L^1} = \langle \frac{\partial E_{L^1}}{\partial q}, dq \rangle + \langle \frac{\partial E_{L^1}}{\partial v}, dv \rangle + \langle dp, \frac{\partial E_{L^1}}{\partial p} \rangle + \langle \frac{\partial E_{L^2}}{\partial z}, dz \rangle$$

$$= \langle -\frac{\partial L_z}{\partial q}, dq \rangle + \langle p - \frac{\partial L_q}{\partial v}, dv \rangle + \langle dp, v \rangle + \langle -\frac{\partial L_q}{\partial z}, dz \rangle.$$  

(25)

Equating both expressions we arrive at equations (21), and therefore the claim holds.

**Remark 6.1.** Roughly speaking, in the definition of the one-form $\Theta^2$ we have performed a sort of minimal coupling condition: namely we have established the substitution $p_i \to p_i - \langle A_i, \Phi \rangle$, where $p_i$ are the coordinates of the momentum in $T^*Q$. The minimal coupling is the standard procedure in the physics literature to derive the Lorentz equations in a relativistically invariant manner. More concretely, the substitution $p \to p - eA$ is made in the Hamiltonian function (where $p$ is the four-momentum and $A$ is a four-potential of the electromagnetic field, while $e$ is the electric charge). As observed in [26] [28], this procedure is equivalent to leaving the Hamiltonian invariant and adding $e \, dA$ to the symplectic form in the original phase space. This is the beginning point by Sternberg himself when constructing the Stenbg’s phase space in [27].

By means of this proposition we have proven that the suitable space to intrinsically describe the equations of motion of a charged particle in a gauge field is the Pontryagin-Sternberg bundle $\mathcal{F}^2 \oplus \mathcal{F}_2$.

Finally, we employ the Pontryagin-Sternberg Lagrange-Dirac system to reobtain (10).

**Proposition 6.3.** Consider the Pontryagin-Sternberg Lagrange-Dirac system $(D^2, E_{L^2})$ defined in 5.2. Its equations of motion, namely

$$\{ \dot{x}(t), dE_{L^2}(x(t)) \} \in D^2(x(t)),$$

are equivalent to (10).

**Proof.** To prove this, we provide the local expression of $D^2 = \text{graph} \{ \Omega^2 \}$, which is obtained by considering the local form of $\Omega^2$ (24). Namely

$$D^2(x) = \{ (q', \dot{v}, \dot{z}, (\alpha, \beta, u, \mu)) \mid q' (\partial_i A_j - \partial_j A_i, \Phi) - \hat{p}_i + \hat{z}^\alpha (A_i, \partial_\alpha \Phi) = \alpha_i, \langle \alpha_i, dq^i + \beta_i dv^i + udq^i + \mu_i dz^\alpha \rangle \in T^*(\mathcal{F}^2 \oplus \mathcal{F}_2) \},$$

where $\alpha_i dq^i + \beta_i dv^i + udq^i + \mu_i dz^\alpha \in T^*(\mathcal{F}^2 \oplus \mathcal{F}_2)$. When we set $(\alpha, \beta, u, \mu) = dE_{L_2}$, which is accomplished by taking into account the local expression (25), we obtain
the equations of motion of the Pontryagin-Sternberg Lagrange-Dirac, equations which are obviously equivalent to (21), as claimed.

\[ \square \]

**Remark 6.2.** The definition of the extended Lagrangian \( L_\sharp \) is crucial in propositions 6.1 and 6.2, where we construct the variational principle and its intrinsic expression in \( F^2 \oplus F^p \). Despite the particular form of \( L_\sharp \) is highly influenced by the Sternberg symplectic structure and therefore quite natural (note that \( L_\sharp = L_2 + \langle \Theta^2, (q, \dot{z}) \rangle - \langle \Theta_{\Gamma^2}, \dot{q} \rangle \), where \( \Theta^2 \) is defined in (22)), it is completely unnecessary from the Lagrange-Dirac point of view. In fact, we only need \( L_2 \) in order to construct the generalized energy \( E_{L_2} \), function which forms the Lagrange-Dirac system \( (D^{1}, E_{L_2}) \). The Sternberg symplectic structure is only present in the definition of the Dirac structure \( D^2 \), and consequently in the dynamical condition \( (\dot{x}, dE_{L_2}(x)) \in D^2(x) \). In other words, the symplectic structure influences the geometry of the space under study, but it does not influence its dynamical function, following somehow the Sternberg’s program sketched in remark 6.1.

We enclose the results obtained in this section in our main theorem:

**Theorem 6.3.** The following statements are equivalent:

1. The Sternbert-Hamilton-Pontryagin principle for the following action integral
   \[ \int_{t_1}^{t_2} \left[ (p(t), \dot{q}(t)) - E_{L_2}(x(t), \dot{x}(t)) \right] dt, \]
   holds for \( (q(t), z(t)) \) with fixed endpoints.

2. The curve \( x(t) = (q(t), v(t), p(t), z(t)) \in \mathcal{F}^2 \oplus \mathcal{F}^p, t \in [t_1, t_2], \) satisfies the implicit equations
   \[ i_{\dot{x}(t)}\Omega^\sharp(x(t)) = dE_{L_2}(x(t)), \]
   whose local expression is
   \[ \dot{q} = v, \]
   \[ p = \frac{\partial L_2}{\partial v}, \]
   \[ \dot{p}_i = \frac{\partial L_2}{\partial q^i} + \dot{q}^j \left( \frac{\partial A_i}{\partial q^j} - \frac{\partial A_j}{\partial q^i} \right) \Phi + \langle A_i, \dot{z}^\alpha \partial_\alpha \Phi \rangle, \]
   \[ \dot{z}^\alpha = \Omega^{\alpha\beta} \left( \frac{\partial L_2}{\partial z^\beta} - \langle \dot{q}^i A_i, \partial_\beta \Phi \rangle \right). \]

3. The curve \( x(t) = (q(t), v(t), p(t), z(t)) \in \mathcal{F}^2 \oplus \mathcal{F}^p, t \in [t_1, t_2], \) is a solution of the Pontryagin-Sternberg Lagrange-Dirac system \( (D^{1}, E_{L_2}) \), whose equations of motion are
   \[ (\dot{x}(t), dE_{L_2}(x(t))) \in D^2(x(t)). \]

7. Example

As mentioned in the introduction, the paradigmatic example in classical physics of a charged particle subject to a gauge field is an electric charged particle evolving in space and coupled to an electromagnetic field (other interesting examples as the Wong’s equations or the magnetized Kepler problems may be found in [35] and [19] respectively). We shall consider the autonomous case, i.e. the electromagnetic field does not depend on time, and denote \( \mathbf{E} := \{ E^i \}, \) \( \mathbf{B} := \{ B^i \}, \) using the vector notation of physics literature, the electric and magnetic fields, respectively, in the three space coordinates corresponding to \( Q = \mathbb{R}^3 \) (with local coordinates
\[ q^\mu = \{ x, y, z \}. \] The textbook equations of motion of a charged particle (charge=\( e \) and unit mass \( m = 1 \)) coupled to an electromagnetic field \((E, B)\) are:

\[ \ddot{q} = e \left[ E + \frac{\dot{q}}{c} \times B \right], \]

where \( q := \{ q^\mu \}, \times \) denotes the curl operation, \( \cdot \) the scalar product in \( \mathbb{R}^3 \), \( \mathcal{E} \) the energy of the particle and \( c \) is the speed of light. Moreover, as it is well-known, both fields may be obtained from the so-called scalar and vector potentials, \( \varphi \) and \( A \) respectively, by

\[ \mathbf{E} = -\nabla \varphi \quad \text{and} \quad \mathbf{B} = \nabla \times \mathbf{A}. \] 

In the context of this work, the equation \((26a)\) can be obtained by taking into account the following setup: \( Q = \mathbb{R}^3 \), \( G = U(1) \) is the one-dimensional unitary group, \( F \) is a coadjoint orbit of \( G \) (consequently a point \(-e \in \mathbb{R}\) with \( \Phi \) the inclusion map. Needless to say, the connection \( \Theta \) is determined locally by the vector potential \( \mathbf{A} \); furthermore \( L_4 = \frac{1}{2} \dot{q}^i \dot{q}^i - e \varphi(q) \). In this case, the first equation in \((10)\), i.e. \( \dot{z}^\alpha = \Omega^{\alpha\beta} \left( \frac{\partial z^i}{\partial z^\beta} - \langle q^i, A_k, \frac{\partial \varphi}{\partial z^k} \rangle \right) \), leads to \( 0 = 0 \), while the second reads

\[ \dot{q}^i = -e \partial_\alpha \varphi - e \dot{q}^j (\partial_i A_j - \partial_j A_i), \]

which according to \((27)\) is nothing but equation \((26a)\). In the context of special relativity theory, both equations \((26)\) can be elegantly enclosed in the same condition by redefining the configuration manifold as the Minkowski space-time \( Q = \mathbb{R}^{1,3} \), this is \( \mathbb{R}^4 \) endowed with a flat pseudo-Riemannian metric of Lorentz signature \((-+, +, +)\). The rest of the setup remains the same, this is \( G = U(1), F = \{-e\} \) and \( \Phi \) the inclusion map. In this new case, we establish the coordinates \( q^\mu = (ct, q^i) \) for the configuration manifold (where \( c \) is the speed of light), while the momentum \( T^*Q \) is determined locally by \( p_\mu = (\mathcal{E}/c, p_i) \). We fix the connection by the local expression \( A_\mu = (\varphi/c, A_i) \) (where the components are the potentials in \((27)\) and the new Lagrangian function reads

\[ L_4 = \frac{1}{2} \eta(q_i, w_q) = \frac{1}{2} \eta_{\mu\nu} \frac{dq^\mu}{dt} \frac{dq^\nu}{dt}, \]

where \( \tau \) is re-scaling of the usual time \( t \) by \( e \) (in the following we will set \( e = 1 \) for simplicity), \( w_q = \frac{dq^\mu}{dt} \frac{\partial}{\partial q^\mu} \in T_qQ \) and \( \eta : TQ \otimes TQ \to \mathbb{R} \) is the pseudo-Riemannian metric with local form \( \eta(\partial/\partial q^\mu, \partial/\partial q^\nu) = \eta_{\mu\nu} = \text{diag} \(-+, +, +, +\). To fix the notation, we shall denote \( \frac{dq^\mu}{dt} = \dot{q}^\mu = (1, \dot{q}^i) \) and therefore \( L_4 = \frac{1}{2} \eta_{\mu\nu} \dot{q}^\mu \dot{q}^\nu \); besides \( p_\mu = \eta_{\mu\nu} \dot{q}^\nu \) since the metric provides us with an isomorphism between \( TQ \) and \( T^*Q \). In this new setup the first equation in \((10)\) is again \( 0 = 0 \) while the second reads

\[ \frac{d}{dt} p_\mu = -e \dot{q}^\nu \left( \frac{\partial A_\mu}{\partial q^\nu} - \frac{\partial A_\nu}{\partial q^\mu} \right). \]

Recalling that \( A_\mu \) is independent of time and that \( q^0 = t \), this equation may be decomposed as

\[ \dot{p}_i = -e \dot{q}^\nu \left( \frac{\partial A_i}{\partial q^\nu} - \frac{\partial A_\nu}{\partial q^i} \right) = e \dot{q}^0 \left( \frac{\partial A_i}{\partial q^0} - \frac{\partial A_0}{\partial q^i} \right) - e \dot{q}^i \left( \frac{\partial A_i}{\partial q^i} - \frac{\partial A_j}{\partial q^j} \right), \]

\[ \dot{p}_0 = -e \dot{q}^\nu \left( \frac{\partial A_0}{\partial q^\nu} - \frac{\partial A_\nu}{\partial q^0} \right) = e \dot{q}^0 \frac{\partial A_0}{\partial q^0} - e \dot{q}^i \frac{\partial A_0}{\partial q^i} = -e \dot{q}^i \frac{\partial \varphi}{\partial q^i}, \]
from which, considering that $p_0 = \mathcal{E}$ and $p_i = \delta_{ij} \dot{q}^j$ and taking into account equation (27), we recover the equations (26), i.e.

$$\delta_{ij} \ddot{q}^j = e \delta_{ij} E^j + e \epsilon_{ijk} \dot{q}^j B^k$$

and

$$\dot{\mathcal{E}} = e \dot{q} \cdot \mathbf{E},$$

where $\epsilon_{ijk}$ is the Levi-Civita tensor.

Now, we employ the approach developed in this work to reobtain these equations. First, consider the Lagrangian function (28), which in the space $\mathcal{F}_x$ and its coordinates $(q^\mu, v^\mu, z^\alpha)$ is redefined by

$$\mathcal{L} = \frac{1}{2} \eta_{\mu\nu} v^\mu v^\nu.$$ Therefore, the equations of motion obtained from the Hamilton-Sternberg-Pontryagin principle (21) read in this case

$$\frac{d}{d\tau} q^\mu = \dot{q}^\mu, \quad p_\mu = \frac{\partial \mathcal{L}}{\partial v^\mu} = \eta_{\mu\nu} v^\nu$$

and

$$\frac{d}{d\tau} p_\mu = -e \dot{q}^\nu \left( \frac{\partial A_\mu}{\partial q^\nu} - \frac{\partial A_\nu}{\partial q^\mu} \right);$$

thus we recover (29). On the other hand, regarding the Dirac structure $D^\sharp$ and the Lagrange-Dirac system $(D^\sharp, E_L^\sharp)$, the generalized Energy $E_L : \mathcal{F}_x \oplus \mathcal{F}_x \to \mathbb{R}$ (17) reads

$$E_L = \langle p, v \rangle - \mathcal{L}(q, v, z) = p_\mu v^\mu - \frac{1}{2} \eta_{\mu\nu} v^\nu v^\nu.$$ Taking into account the two-form $\Omega^\sharp$, the equations of motion of the Pontryagin-Sternberg Lagrange-Dirac system defined in proposition 6.3 are written as

$$\begin{pmatrix} \dot{q}^\nu & \dot{v}^\nu & \dot{p}_\nu \\ 0 & 0 & 0 \\ -\delta^\nu_\mu & 0 & 0 \end{pmatrix} = \begin{pmatrix} e(\partial_\nu A_\mu - \partial_\mu A_\nu) & 0 & \delta^\mu_\nu \\ 0 & 0 & 0 \\ -\delta^\nu_\mu & 0 & 0 \end{pmatrix},$$

which after a straightforward computation leads to (29).

8. Conclusions

In this paper, we have explored the construction of Lagrange-Dirac structures in the defined Pontryagin-Sternberg bundle, which we show is the suitable space to obtain, from different points of view, the equations of motion for charged particles in gauge fields. We apply the theory to a charged particle coupled to an electromagnetic field, field represented by a connection in a $U(1)$ principal bundle. However, our setting is general enough to cover also non-abelian groups. Our beginning point is the symplectic Sternberg phase space $(\mathcal{F}_x, \Omega_x)$, upon which we have constructed an analogue of the Pontryagin bundle $TQ \oplus T^*Q$, that we have named the Sternberg-Pontryagin bundle $\mathcal{F}_x \oplus \mathcal{F}_x$. We have related this bundle to the magnetized Tulczyjew triple [21] analogously to how the Pontryagin bundle is related to the usual Tulczyjew triple. Then, we have shown that this is the suitable space to derive the equations of motion of particles in gauge fields from variational and intrinsic points of view (in the Lagrangian side). Moreover, we also show that it is necessary to define a (degenerate) extended Lagrangian function when deriving the equations in these contexts, extended Lagrangian which is highly influenced by the geometry of the Sternberg phase space. On the other hand, we have employed the Dirac structures theory to induce a Lagrange-Dirac system on $\mathcal{F}_x \oplus \mathcal{F}_x$ whose dynamical equations are equivalent to the equations under study. We have proved that this Dirac space generates naturally the desired dynamics and, furthermore, the needed Lagrangian function (which can be also degenerate) is simpler than the extended one proposed previously, i.e. it does not need to be extended.

Acknowledgements: I would like to thank Hiroaki Yoshimura for introducing me to Lagrange-Dirac systems, and Carlos Navarrete-Benlloch for reading part of this manuscript.
REFERENCES

[1] Bai Z, Meng G and Wang E, “On the orbits of magnetized Kepler problems in dimension $2k+1$”, Journal of Geometry and Physics, 73, pp. 260–269, (2013).
[2] Barbero-Lián M, Farías Puiggali M and Martín de Diego D, “Isotropic submanifolds and the inverse problem for mechanical constrained systems”, Preprint, arXiv:1404.1961 (2014).
[3] Barbero-Lián M, de León M and Martín de Diego D, “Lagrangian submanifolds and Hamilton-Jacobi equation”, Monatschefte für Mathematik, 171(1-3), pp. 269–290, (2013).
[4] Campos CM, Guzmán E and Marrero JC, “Classical field theories of first order and Lagrangian submanifolds of premultisymplectic manifolds,” J. Geom. Mech. 4(1), pp. 1–26, (2012).
[5] Cortés J, de León M, Martín de Diego D and Martínez S, “Geometric description of vakonomic and nonholonomic dynamics. Comparison of solutions”, SIAM J. Control Optim., 41(5), pp. 1389–1412, (2003).
[6] Courant TJ, “Dirac manifolds,” Trans. Amer. Math. Soc. 319(2), pp. 631–661, (1990).
[7] Courant TJ and Weinstein A, “Beyond Poisson structures’” Action hamiltoniennes de groupes. Troisieme théorème de Lie (Lyon, 1986), volume 27 of Travaux en Cours, pp. 39–49, (1988).
[8] García-Torano E, Guzmán E, Marrero JC and Mestdag T, “Reduced dynamics and Lagrangian submanifolds of symplectic manifolds” J. Phys. A, 47(22), 24pp., (2014).
[9] de León M, Jiménez, F and Martín de Diego D, “Hamiltonian dynamics and constrained variational calculus: continuous and discrete settings”, Journal of Physics A, 45, 29 pp., (2012).
[10] de León N and Rodrigues PR, “Methods of Differential Geometry in Analytical Mechanics”, North-Holland, Amsterdam (1989).
[11] Dirac PAM, “Generalized Hamiltonian dynamics”, Canadian J. Math., 2, pp. 129–148, (1950).
[12] Dirac PAM, “Lectures on Quantum Mechanics”, Belfer Graduate School of Science, Yeshiva University, New York, (1964).
[13] Godbillon, “Géometrie différentielle et mécanique analytique”. Hermann, Paris (1969).
[14] Grabowska K and Grabowski J, “Variational calculus with constraints on general algebroids”, Journal of Physics A, 41, (2008).
[15] Grabowska K, Grabowski J and Urbański, “Geometrical Mechanics on algebroids”, J. Geom. Meth. Mod. Physics., 3, pp. 559-575, (2006).
[16] Jiménez F and Yoshimura H, “Dirac Structures in Vakonomic Mechanics”, Preprint, arXiv:1405.5394 (2014).
[17] Leok M and Ohsawa, “Variational and Geometric Structures of Discrete Dirac Mechanics” Foundations of Computational Mathematics, 11(5), pp. 529–562, (2011).
[18] Mendella M, Marín M and Tulczyjew WM, “Integrability of implicit differential equations” Journal of Physics A: Mathematical and General, 28(1), pp. 149–164, (1995).
[19] Meng G, “The Poisson realization of so(2, 2k + 2) on magnetic leaves and generalized MICZ-Kepler problems”, Journal of Mathematical Physics, 54, 052902, (2013).
[20] Meng G, “The classical magnetized Kepler problems in higher odd dimensions”, J. Geom. Symmetry Physics, 32, pp. 15–32, (2013).
[21] Meng G, “Tulczyjew’s approach for particles in gauge fields”, Preprint, arXiv:1405.0748 (2014).
[22] Montgomery R, “Canonical formulation of a classical particle in a Yang-Mills field and Wong’s equations”, Lett. Math. Phys. 8, pp. 59–67, (1984).
[23] Pradines J, “Fibrés vectoriels doubles et calcul des jets non holonomes”, Amiens (1974).
[24] Robinson M, Bland K, Cleaver G and Dittmann J, “A Simple Introduction to Particle Physics”, Review, arXiv:0810.3528 (2009).
[25] van der Schaft AJ and Maschke BM, “The Hamiltonian formulation of energy conserving physical systems with external ports”, Archiv für Elektronik und Übertragungstechnik, 49, pp. 362–371, (1995).
[26] Śniatycki J, “Geometric Quantization and Quantum Mechanics”, University of Calgary, Alberta, (1977).
[27] Stienberg S, “Minimal coupling and the symplectic mechanics of a classical particle in the presence of a Yang-Mills field”, Proc. Nat. Acad. Sci. 74, pp. 5253–5254, (1977).
[28] Souriau JM, “Structure des Systemes Dynamiques”, Dunod, Paris, (1970).
[29] Tulczyjew WM, “Hamiltonian systems, Lagrangian systems, and the Legendre transformation”, Symposia Mathematica vol XIV (Convegno di Geometria Simplicetta e Fisica Matematica, INDAM, Rome), pp. 247–268, (1973).
[30] Tulczyjew WM, “Les sous-variétés lagrangiennes et la dynamique hamiltonienne”, C. R. Acad. Sc. Paris 283 Série A, pp. 15–18, (1976).
[31] Tulczyjew WM, “Les sous-variétés lagrangiennes et la dynamique lagrangienne”, C. R. Acad. Sc. Paris 283 Série A, pp. 675–678, (1976).
[32] Weinstein A, “A universal phase space for particles in Yang-Mills fields”, Lett. Math. Phys. 2, pp. 417–420, (1978).
[33] Weinstein A, “Symplectic manifolds and their Lagrangian submanifolds”, Advances in Mathematics, 6(3), pp. 329–346, (1971).
[34] Weinstein A, “Lectures on symplectic manifolds”, CBMS Regional Conference Series in Mathematics, 29. American Mathematical Society, Providence, R.I., (1979).
[35] Wong SK, “Field and particle equations for the classical Yang-Mills fields and particles with isotropic spin”, Il Nuovo Cimento A, 65(4), pp. 689–694, (1970).
[36] Yang CN and Mills RL, “Conservation of Isotropic Spin and Isotropic Gauge Invariance”, Physical Review, 96(1), pp. 191–195, (1954).
[37] Yoshimura H and Marsden JE, “Dirac structures in Lagrangian mechanics Part I: Implicit Lagrangian systems”, Journal of Geometry and Physics, 57, pp. 133–156, (2006).
[38] Yoshimura H and Marsden JE, “Dirac structures in Lagrangian mechanics Part II: Variational structures”, Journal of Geometry and Physics, 57, pp. 209–250, (2006).

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