Characterization of optimal binary linear codes with
one-dimensional hull

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Abstract

The hull of a linear code over finite fields is the intersection of the code and its
dual. Linear codes with small hulls have been widely studied due to their applica-
tions in computational complexity and information protection. In this paper, we
study some properties of binary linear codes with one-dimensional hull, and estab-
lish their relation with binary LCD codes. Some interesting inequalities are thus
obtained. We determine the exact value of \( d_{\text{one}}(n, k) \) for \( k \in \{1, 3, 4, n - 5, n -
4, n - 3, n - 2, n - 1\} \) or \( 14 \leq n \leq 24 \), where \( d_{\text{one}}(n, k) \) denotes the largest minimum
weight among all binary linear \([n, k]\) codes with one-dimensional hull. We partially
determine the value of \( d_{\text{one}}(n, k) \) for \( k = 5 \) or \( 25 \leq n \leq 30 \). As an application, we
construct some entanglement-assisted quantum error-correcting codes (EAQ ECCs).

Keywords: Hull, binary LCD codes, building-up construction, EAQ ECCs.

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1 Introduction

The hull of a linear \([n, k]\) code \( C \) is defined as \( \text{Hull}(C) = C \cap C^\perp \), which was first
introduced in 1990 by Assmus and Key [4] to classify finite projective planes. Suppose
that the dimension of \( \text{Hull}(C) \) is \( l \). If \( l = 0 \), that is to say, \( C \cap C^\perp = \{0\} \), then the code
\( C \) is a linear complementary dual (LCD) code. If \( l = k \), that is to say, \( C \subseteq C^\perp \), then
the code \( C \) is a self-orthogonal (SO) code. It has been shown that the hull determines
the complexity of the algorithms for checking permutation equivalence of two linear codes
and for computing the automorphism group of a linear code [23,32,34]. These algorithms
are very effective if the dimension of the hull is small. They also have been employed to
construct entanglement-assisted quantum error-correction codes (see [16,21,26,27,37,38]).

The smallest dimension of the hull of a linear code is 0, i.e., an LCD code, which
was introduced by Massey [29]. In 2004, Sendrier [33] showed that LCD codes meet the

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asymptotic Gilbert-Varshamov bound. In 2016, Carlet and Guilley [8] investigated an application of binary LCD codes against Side-Channel Attacks (SCA) and Fault Injection Attack (FIA). The study of LCD codes has thus become a hot research topic and the reader is referred to [10, 17, 35] for recent papers. A surprising result is that Carlet et al. [11] showed that any code over $\mathbb{F}_q$ is equivalent to some Euclidean LCD code for $q > 3$. This motivates us to study LCD codes, especially LCD codes over small fields.

The second smallest dimension of the hull of a linear code is 1, i.e., a linear code with one-dimensional hull. Li and Zeng [24], Carlet, Li and Mesnager [9] have made a lot of contributions. More precisely, Li and Zeng [24] gave sufficient conditions for linear codes with one-dimensional hull and constructed linear codes with one-dimensional hull by employing quadratic number fields, partial difference sets, and difference sets. Carlet, Li and Mesnager [9] constructed linear codes with one-dimensional hull by employing character sums in semi-primitive case from cyclotomic fields and multiplicative subgroups of finite fields. For more related work, readers are refereed to [30, 31].

There are several constructions for binary linear codes with small hulls. Let $d_{\text{LCD}}(n, k)$ denote the largest minimum distance among all binary LCD $[n, k]$ codes and $d_{\text{one}}(n, k)$ denote the largest minimum distance among all binary linear $[n, k]$ codes with one-dimensional hull. The exact value of $d_{\text{LCD}}(n, k)$ for $n \leq 24$ was determined by Galvez et al. [14], Harada et al. [18] and Araya et al. [1]. The exact value of $d_{\text{LCD}}(n, k)$ for $25 \leq n \leq 40$ was partially determined by Fu et al. [13], Bouyuklieva [6], Ishizuka et al. [20] and Li et al. [25]. Dougherty et al. [12], Galvez et al. [14], Harada et al. [18], Araya et al. [1] and Araya et al. [2, 3] determined the exact value of $d_{\text{LCD}}(n, k)$ for $k \in \{1, 2, 3, 4, n-1, n-2, n-3, n-4, n-5\}$ and any $n$. Araya et al. [1] and Araya et al. [3] also partially determined the exact value of $d_{\text{LCD}}(n, 5)$. Less results have been known for binary linear codes with one-dimensional hull. Li and Zeng [24] constructed some binary linear codes with one-dimensional hull for $n = 8, 9, 10$. Kim [22] determined the exact value of $d_{\text{one}}(n, k)$ for $1 \leq k \leq n \leq 13$ by a building-up construction. Mankean and Jitman [28] determined the exact value of $d_{\text{one}}(n, 2)$. Therefore, it is an interesting problem to study and construct binary linear codes with one-dimensional hull.

In this paper, we study some properties of binary linear codes with one-dimensional hull, and establish the connection between binary LCD codes and binary linear codes with one-dimensional hull. Some interesting inequalities for $d_{\text{LCD}}(n, k)$ and $d_{\text{one}}(n, k)$ are obtained. Using the building-up construction in [22] and these inequalities, we extend Kim’s results to lengths up to 30. We completely determine the value of $d_{\text{one}}(n, k)$ for $k \in \{1, 3, 4, n-5, n-4, n-3, n-2, n-1\}$ and partially determine the value of $d_{\text{one}}(n, k)$ for $k = 5$. As an application, we construct some EAQECCs with different parameters compared with [15, 27, 37].

This paper is organized as follows. In Section 2, we give some notations and preliminaries. In Section 3, we establish the connection between binary LCD codes and binary linear codes with one-dimensional hull. In Section 4, we study some properties of binary linear codes with one-dimensional hull. In Section 5, we introduce a complete building-up construction for binary linear codes with one-dimensional hull. In Section 6, we characterize the value of $d_{\text{one}}(n, k)$ for $k \in \{1, 3, 4, 5, n-1, n-2, n-3, n-4, n-5\}$. In Section 7, we construct some EAQECCs. In Section 8, we conclude the paper.


## 2 Preliminaries

### 2.1 Binary linear codes and some bounds

Let $\mathbb{F}_2$ denote the finite field with 2 elements. For any $x \in \mathbb{F}_2^n$, the support of $x = (x_1, x_2, \ldots, x_n)$ is defined as follows:

$$\text{supp}(x) = \{i \mid x_i = 1\}.$$ 

The (Hamming) weight $\text{wt}(x)$ of $x$ is the number of nonzero coordinates of $x$, so $\text{wt}(x) = |\text{supp}(x)|$. The distance between two vectors $x$ and $y$ is $d(x, y) = \text{wt}(x, y)$. The minimum distance (or minimum weight) of $C$ is $\min\{\text{wt}(x) \mid 0 \neq x \in C\}$. A binary linear $[n, k, d]$ code $C$ is a $k$-dimensional subspace of $\mathbb{F}_2^n$ with its minimum distance $d$.

The dual code $C^\perp$ of a binary linear $[n, k]$ code $C$ is defined as

$$C^\perp = \{y \in \mathbb{F}_2^n \mid \langle x, y \rangle = 0, \text{ for all } x \in C\},$$

where $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$ for $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{F}_2^n$. The hull of a binary linear code $C$ is defined as

$$\text{Hull}(C) = C \cap C^\perp.$$ 

A generator matrix for an $[n, k]$ code $C$ is any $k \times n$ matrix $G$ whose rows form a basis for $C$. A parity-check matrix for a linear code $C$ is a generator matrix for the dual code $C^\perp$. For any set of $k$ independent columns of a generator matrix $G$, the corresponding set of coordinates forms an information set for $C$.

It is well-known that the Griesmer bound [19, Chap. 2, Section 7] on a binary linear $[n, k, d]$ code is given by

$$n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d_i}{2} \right\rceil,$$

where $[a]$ is the least integer greater than or equal to $a$. A binary $[n, k, d]$ code $C$ is said to be a Griesmer code if $C$ meets the Griesmer bound, i.e., $n = \sum_{i=0}^{k-1} \left\lceil \frac{d_i}{2} \right\rceil$. The sphere-packing bound on a binary $[n, k, d]$ linear code is given by

$$2^k \leq \sum_{i=0}^{\lfloor d/2 \rfloor} \binom{n}{i}.$$ 

A vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{F}_2^n$ is even-like if $\sum_{i=1}^n x_i = 0$ and is odd-like otherwise. A binary code is said to be even-like if it has only even-like codewords, and is said to be odd-like if it is not even-like.

### 2.2 Characterization of binary linear codes with small hulls

Carlet et al. [10] presented a new characterization of binary LCD codes, and solved a conjecture proposed by Galvez et al. [14] on the minimum distance of binary LCD codes. We introduce the new characterization of binary LCD codes as follows.
Theorem 2.1. [10, Theorem 3] Let $C$ be an odd-like binary $[n,k]$ code. Then $C$ is LCD if and only if there exists a basis $c_1, c_2, \ldots, c_k$ of $C$ such that for any $i, j \in \{1,2,\ldots,k\}$, $c_i \cdot c_j$ equals 1 if $i = j$ and equals 0 if $i \neq j$.

Theorem 2.2. [10, Lemma 7] Let $C$ be an even-like binary $[n,k]$ code. Then $C$ is LCD if and only if $k$ is even and there exists a basis $c_1, c_1', \ldots, c_k, c_k'$ of $C$ such that for any $i, j \in \{1,2,\ldots,k\}$, the following conditions hold:

(i) $c_i \cdot c_i = c_i' \cdot c_i' = 0$;
(ii) $c_i \cdot c_j = 0$, for $i \neq j$;
(iii) $c_i \cdot c_i' = 1$.
(iv) $c_{i,1} = c_{i,1}'$, where $c_i = (c_{i,1}, \ldots, c_{i,n})$ and $c_i' = (c_{i,1}', \ldots, c_{i,n}')$.

Lemma 2.3. [24, Proposition 1] Let $C$ be a binary $[n,k]$ linear code with generator matrix $G$. Then $C$ has $l$-dimensional hull if and only if $l = k - \text{rank}(GG^T)$.

Theorem 2.4. Let $C$ be a binary linear $[n,k]$ code. Then $C$ is an odd-like (resp. even-like) code with one-dimensional hull if and only if there exists a basis $c_1, c_2, \ldots, c_k$ of $C$ such that the code generated by $c_1, c_2, \ldots, c_{k-1}$ is an odd-like (resp. even-like) binary LCD $[n,k-1]$ code and $c_k \cdot c_i = 0$ for $1 \leq i \leq k$.

Proof. Let $C$ be a binary linear $[n,k]$ code. Assume that there exists a basis $c_1, c_2, \ldots, c_k$ of $C$ such that the code generated by $c_1, c_2, \ldots, c_{k-1}$ is an odd-like (resp. even-like) binary LCD $[n,k-1]$ code and $c_k \cdot c_i = 0$ for $1 \leq i \leq k$. Let $G$ be a matrix whose rows are $c_1, c_2, \ldots, c_k$. Then $\det(GG^T) = 0$, which implies $\text{rank}(GG^T) \leq k - 1$. Since $GG^T$ contains a $(k-1) \times (k-1)$ submatrix of rank $k - 1$, $\text{rank}(GG^T) = k - 1$. By Lemma 2.3, we obtain that $C$ is an odd-like (resp. even-like) binary linear code with one-dimensional hull.

Conversely, if $C$ is a binary code with one-dimensional hull, then there exists a basis $c_1, c_2, \ldots, c_k$ of $C$ such that $\text{Hull}(C) = \{0, c_k\}$. From Lemma 22 in [10], the code $C'$ generated by $c_1, c_2, \ldots, c_{k-1}$ is a binary LCD $[n,k-1]$ code. Moreover, it is easy to check that $C$ is odd-like (resp. even-like) if and only if $C'$ is odd-like (resp. even-like).

In the following, we give a necessary condition for a binary linear code with one-dimensional hull to be even-like.

Lemma 2.5. If there exists an even-like binary $[n,k]$ linear code with one-dimensional hull, then $k$ is odd.

Proof. Let $C$ be an even-like binary $[n,k]$ linear code with one-dimensional hull and a basis $c_1, c_2, \ldots, c_k$ such that $\text{Hull}(C) = \{0, c_k\}$. Let $G'$ be a matrix whose rows are $c_1, c_2, \ldots, c_{k-1}$. According to [11, Lemma 22], the linear code $C'$ with the generator matrix $G'$ is an even-like binary LCD $[n,k-1]$ code. It follows from Theorem 2.2 that $k - 1$ is even. Hence $k$ is odd.
2.3 The shortened codes and the punctured codes

Let $C$ be a binary $[n, k, d]$ linear code, and let $T$ be a set of $t$ coordinate positions in $C$. We puncture $C$ by deleting all the coordinates in $T$ in each codeword of $C$. The resulting code is still linear and has length $n - t$. We denote the punctured code by $C^T$. Consider the set $C(T)$ of codewords which are 0 on $T$; this set is a subcode of $C$. Puncturing $C(T)$ on $T$ gives a binary code of length $n - t$ called the code shortened on $T$ and denoted $C_T$.

**Lemma 2.6.** [19, Theorem 1.5.7] Let $C$ be a binary linear $[n, k, d]$ code. Let $T$ be a set of $t$ coordinates. Then:

1. $(C^\perp)_T = (C^T)^\perp$ and $(C^\perp)^T = (C_T)^\perp$, and
2. if $t < d$, then $C^T$ and $(C^\perp)_T$ have dimensions $k$ and $n - t - k$, respectively.

The following lemma is a simple generalization of [10, Lemma 22].

**Lemma 2.7.** Let $C$ be a binary $[n, k]$ linear code. Let $s$ and $t$ be two integers such that $s \geq t$. Then $C$ has $s$-dimensional hull if and only if there are $C_1$ and $C_2$ such that

\[ C = C_1 \oplus C_2, \quad C_1 \subseteq C_2^\perp, \]

where $C_1$ is a binary self-orthogonal $[n, t]$ code and $C_2$ is a binary $[n, k - t]$ code with $(s - t)$-dimensional hull.

**Proof.** If there are a binary self-orthogonal $[n, t]$ code $C_1$ and a binary $[n, k - t]$ code $C_2$ with $(s - t)$-dimensional hull such that

\[ C = C_1 \oplus C_2, \quad C_1 \subseteq C_2^\perp, \]

then $\text{Hull}(C) = C_1 \oplus \text{Hull}(C_2)$. It turns out that

\[ \dim(\text{Hull}(C)) = \dim(C_1) + \dim(\text{Hull}(C_2)) = t + s - t - s. \]

Hence, $C$ has $s$-dimensional hull.

Conversely, assume that $C$ has $s$-dimensional hull. Let $\{\alpha_1, \ldots, \alpha_s, \alpha_{s+1}, \ldots, \alpha_k\}$ be a basis of $C$ such that $\{\alpha_1, \ldots, \alpha_s\}$ is a basis of $\text{Hull}(C) = C \cap C^\perp$. Then since $t \leq s$, the code $C_1$ generated by $\alpha_1, \alpha_2, \ldots, \alpha_t$ is self-orthogonal. Let $C_2$ be a linear code generated by $\alpha_{t+1}, \alpha_{t+2}, \ldots, \alpha_k$. Thus $C_1 \subseteq C_2^\perp$ and $C = C_1 \oplus C_2$. It turns out that $C_2$ is an $[n, k - t]$ code with an $(s - t)$-dimensional hull. Otherwise, $\dim(\text{Hull}(C)) = \dim(C_1) + \dim(\text{Hull}(C_2)) > t + (s - t) = s$, which is a contradiction. This completes the proof. \qed

Bouyuklieva [6] established a relation between $C$ and a shortened code of $C$. We will further subdivide this result.

**Proposition 2.8.** Let $C$ be a binary linear $[n, k]$ code with $\dim(\text{Hull}(C)) = s$. For $1 \leq i \leq n$, let $C_{(t)}$ be the shortened code and the punctured code of $C$ on $t$-th coordinate, respectively. Then we have the following result.

1. Assume that $s \geq 1$ and $t \in T$ for some information set $T$ of $\text{Hull}(C)$. Then

\[ \dim(\text{Hull}(C_{(t)})) = \dim(\text{Hull}(C_{(t)})) = s - 1. \]
(2) Assume that \( t \notin T \) for any information set \( T \) of \( \text{Hull}(C) \). If \( s \geq 1 \), then
\[
s - 1 \leq \dim \left( \text{Hull}(C^{(t)}) \right), \dim \left( \text{Hull}(C_{(t)}) \right) \leq s + 1.
\]

If \( s = 0 \), then
\[
\dim \left( \text{Hull}(C^{(t)}) \right) \leq 1 \text{ and } \dim \left( \text{Hull}(C_{(t)}) \right) \leq 1.
\]

Proof. (1) Let \( C \) be a binary \([n, k, d]\) linear code with \( s \)-dimensional hull and generator matrix \( G \). Without loss of generality, we may assume that \( t = 1 \) and
\[
G = (I_k|A) = (e_{k,i}|a_i)_{1 \leq i \leq k},
\]
where \( e_{k,i} \) and \( a_i \) are the \( i \)-th row of \( I_k \) (the identity matrix) and \( A \), respectively.

Since \( t \in T \) for some information set of \( \text{Hull}(C) \), there exists \( r_j \in \text{Hull}(C) \) such that \( r_{j,1} = 1 \), where \( r_j = (r_{j,1}, \ldots, r_{j,n}) \). Then we know that \( \{r_j\} \cup \{e_{k,i}|a_i\}_{2 \leq i \leq k} \) is a basis of \( C \). By Lemma 2.7, the code \( C' \) with the generator matrix \( (e_{k,i}|a_i)_{2 \leq i \leq k} \) is an \([n, k - 1]\) linear code with \((s - 1)\)-dimensional hull. By deleting the zero column of \( (e_{k,i}|a_i)_{2 \leq i \leq k} \), we obtain the following matrix
\[
G_{(1)} = (e_{k-1,i}|a_{i+1})_{1 \leq i \leq k-1},
\]
where \( e_{k-1,i} \) is the \( i \)-th row of \( I_{k-1} \) and \( a_i \) is the \( i \)-th row of \( A \) for \( 1 \leq i \leq k - 1 \). This is a generator matrix of the shortened code \( C_{(1)} \) of \( C \) on the first coordinate.

Since \( C' \) is a linear code with \((s - 1)\)-dimensional hull, \( C_{(1)} \) be an \([n - 1, k - 1, d^* \geq d] \) linear code with \((s - 1)\)-dimensional hull. This completes the proof.

(2) It follows from Proposition 5 in [6] that \( s - 1 \leq \dim \left( \text{Hull}(C_{(t)}) \right) \leq s + 1 \) for \( s \geq 1 \) and \( \dim \left( \text{Hull}(C_{(t)}) \right) \leq 1 \) for \( s = 0 \). It turns out that \( s - 1 \leq \dim \left( \text{Hull}(C^{(t)}) \right) \leq s + 1 \) for \( s \geq 1 \) and \( \dim \left( \text{Hull}(C^{(t)}) \right) \leq 1 \) for \( s = 0 \). By Lemma 2.6,
\[
s - 1 \leq \dim \left( \text{Hull}(C^{(t)}) \right) = \dim \left( \text{Hull}(C^{(t)})^\perp \right) = \dim \left( \text{Hull}(C^{(t)}_1) \right) \leq s + 1 \text{ if } s \geq 1,
\]
\[
\dim \left( \text{Hull}(C^{(t)}) \right) \leq 1 \text{ if } s = 0.
\]

This completes the proof. \( \square \)

A similar result for the inverse of the punctured codes is given as follows.

**Proposition 2.9.** Let \( C \) be a binary \([n, k]\) linear code with generator matrix \( G \) and \( \dim(\text{Hull}(C)) = s \). Let \( C' \) be a binary \([n + 1, k]\) linear code with the generator matrix \( (v^T, G) \), where \( v \in \mathbb{F}_2^n \). Then \( s - 1 \leq \dim(\text{Hull}(C')) \leq s + 1 \) for \( s \geq 1 \) and \( \dim(\text{Hull}(C')) \leq 1 \) for \( s = 0 \).

Proof. Let \( s \geq 0 \). It is easy to see that \( C \) is the punctured code of \( C' \) on the first coordinate. Let \( s' = \dim(\text{Hull}(C')) \). We claim that \( s' \leq s + 1 \). Otherwise \( \dim(\text{Hull}(C)) > s \) by Proposition 2.8, which is a contradiction.

If \( s \geq 1 \), then we claim that \( s' \geq s - 1 \). Otherwise \( \dim(\text{Hull}(C)) < s \) by Proposition 2.8, which is a contradiction. This completes the proof. \( \square \)
3 Linear codes with one-dimensional hull from LCD codes

We recall that \( d_{\text{LCD}}(n, k) \) is the largest minimum distance among all binary LCD \([n, k]\) codes and \( d_{\text{one}}(n, k) \) is the largest minimum distance among all binary \([n, k]\) codes with one-dimensional hull for a given pair \((n, k)\). By Proposition 2.8, we give two upper bounds for \( d_{\text{one}}(n, k) \).

**Lemma 3.1.** Suppose that \( 1 \leq k \leq n-1 \) and \( d_{\text{one}}(n, k) \geq 2 \). Then we have \( d_{\text{one}}(n, k) \leq d_{\text{LCD}}(n-1, k-1) \) and \( d_{\text{one}}(n, k) \leq d_{\text{LCD}}(n-1, k) + 1 \).

**Proof.** Let \( C \) be a binary \([n, k, d_{\text{one}}(n, k)]\) linear code with one-dimensional hull. By Proposition 2.8 and Lemma 2.6, there are binary LCD \([n-1, k-1, \geq d_{\text{one}}(n, k)]\) and \([n-1, k, \geq d_{\text{one}}(n, k) - 1]\) codes. Hence \( d_{\text{one}}(n, k) \leq d_{\text{LCD}}(n-1, k-1) \) and \( d_{\text{one}}(n, k) \leq d_{\text{LCD}}(n-1, k) + 1 \). \( \square \)

An interesting relationship between \( d_{\text{LCD}}(n+1, k) \) and \( d_{\text{one}}(n, k) \) is given as follows.

**Lemma 3.2.** Suppose that \( 2 \leq k \leq n-1 \). Then we have \( d_{\text{one}}(n, k) \leq d_{\text{LCD}}(n+1, k) \).

**Proof.** Let \( \mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_k \) be a basis of a binary \([n, k, d]\) linear code with one-dimensional hull \( C \) such that \( \text{Hull}(C) = \{ \mathbf{0}, \mathbf{c}_k \} \). Let \( G_1 \) be a matrix whose rows are \( \mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_{k-1} \). According to [11, Lemma 22], \( G_1 G_1^T \) is nonsingular. Let \( C' \) be a binary linear \([n+1, k]\) code with the generator matrix \( G' \) whose rows are \((0, \mathbf{c}_1), (0, \mathbf{c}_2), \ldots, (0, \mathbf{c}_{k-1}), (1, \mathbf{c}_k)\). Then \( C' \) has the minimum distance at least \( d \), and we have

\[
G'G'^T = \begin{pmatrix}
G_1 G_1^T & 0 \\
0 & 1
\end{pmatrix}
\]

Hence \( G'G'^T \) is nonsingular, which implies that \( C' \) is a binary LCD \([n+1, k]\) code. Thus, the result holds. \( \square \)

**Proposition 3.3.** If \( C \) is a binary even-like LCD \([n, k, d]\) code with \( d \geq 2 \) and \( d^\perp \geq 2 \), then the shortened code of \( C \) on any coordinate is a linear code with one-dimensional hull.

**Proof.** It follows from Proposition 2 in [6] that the punctured code of \( C \) on any coordinate is again LCD. According to Lemma 2 in [6], exactly one of the codes \( C^{(i)} \) and \( C_{(i)} \) is LCD on any coordinate. By (2) of Proposition 2.8, the shortened code of \( C \) on any coordinate is either an LCD code or a linear code with one-dimensional hull. It turns out that the shortened code of \( C \) on any coordinate is a linear code with one-dimensional hull. \( \square \)

**Corollary 3.4.** If there is a binary even-like LCD \([n, k, d]\) code with \( d \geq 2 \), then there is a binary even-like \([n-1, k-1, \geq d]\) code with one-dimensional hull.

**Proof.** If \( d^\perp \geq 2 \), then the result follows from Proposition 3.3. If \( d^\perp = 1 \), then we obtain a binary even-like LCD \([n-i, k, d]\) code with the dual distance at least 2 by deleting all zero columns of \( C \). By Proposition 3.3, there is a binary even-like \([n-i-1, k-1, d]\) code with one-dimensional hull. By adding zero-column, a binary even-like \([n-1, k-1, \geq d]\) code with one-dimensional hull is constructed. \( \square \)
Proposition 3.5. Let $C$ be an odd-like LCD $[n, k, d]$ code with $d \geq 2$ and $d^+ \geq 2$. If $1 \in C$, then the punctured code of $C$ on any coordinate is a linear code with one-dimensional hull. If $1 \notin C$, then there exist $1 \leq i, j \leq n$ such that the shortened code $C \cap_{i,j}$ of $C$ on the $i$-th coordinate and the punctured code $C^{(j)}$ of $C$ on the $j$-th coordinate are linear codes with one-dimensional hull.

Proof. If $C$ contains the all-ones vector, then the dual $C^\perp$ is even-like. By Proposition 3.3, the shortened code of $C^\perp$ on any coordinate is a linear code with one-dimensional hull. By Lemma 2.6, $C^{(t)} = (C^\perp \cap_{i,t})^\perp$. Hence the punctured code of $C$ on any coordinate is a linear code with one-dimensional hull.

If $C$ does not contain the all-ones vector, then it follows from [6, Proposition 3] that there exist $1 \leq i, j \leq n$ such that the punctured code $C \cap_{i,j}$ of $C$ on the $i$-th coordinate and the shortened code $C^{(j)}$ of $C$ on the $j$-th coordinate are LCD codes. According to Lemma 2 in [6], exactly one of the codes $C^{(t)}$ and $C \cap_{i,j}$ is LCD on any coordinate. By (2) of Proposition 2.8, the shortened code $C \cap_{i,j}$ of $C$ on the $i$-th coordinate and the punctured code $C^{(j)}$ of $C$ on the $j$-th coordinate are linear codes with one-dimensional hull.

Corollary 3.6. If there is a binary odd-like LCD $[n, k, d]$ code with $d \geq 2$, then there is a binary $[n - 1, k, \geq d - 1]$ code with one-dimensional hull.

Proof. The proof is similar to that of Corollary 3.4, the main difference is that we use Proposition 3.5 instead of Proposition 3.3.

Corollary 3.7. If $k$ is odd and $d_{LCD}(n + 1, k) \geq 2$, then $d_{one}(n, k) = d_{LCD}(n + 1, k)$ or $d_{LCD}(n + 1, k) - 1$.

Proof. Let $C$ be a binary LCD $[n + 1, k, d_{LCD}(n + 1, k)]$ code. Since $k$ is odd, $C$ is odd-like. By Corollary 3.6, there is a binary $[n, k, \geq d_{LCD}(n + 1, k) - 1]$ linear code with one-dimensional hull. Combined with Lemma 3.2, we can obtain the desired result.

Proposition 3.8. If there is a binary LCD $[n, k, d]$ code for odd $k$, then there is a binary even-like $[n + 1, k, d$ or $d + 1]$ linear code with one-dimensional hull.

Proof. Let $C$ be a binary LCD $[n, k, d]$ code. Since $k$ is odd, $C$ is odd-like by Theorem 2.2. By Theorem 2.1, there exists a basis $c_1, c_2, \ldots, c_k$ of $C$ such that for any $i, j \in \{1, 2, \ldots, k\}$, $c_i \cdot c_j$ equals 1 if $i = j$ and equals 0 if $i \neq j$. Let $C'$ be a binary $[n + 1, k]$ code with the generator matrix $G'$ whose rows are the codewords $(1, c_1), (1, c_2), \ldots, (1, c_k)$. Then we have

$$G'G'^\top = J_k - I_k,$$

where $J_k$ is the all-ones matrix and $I_k$ is the $k \times k$ identity matrix. It is not difficult to calculate that $\det(G'G'^\top) = 0$ since $k$ is odd. Implying that $C'$ is not LCD. It turns out that $C'$ is an even-like $[n + 1, k]$ code with one-dimensional hull from Proposition 2.9.

Corollary 3.9. Suppose that $1 \leq k \leq n - 1$, $k$ is odd and $d_{one}(n, k) \geq 2$. Then $d_{one}(n, k) = d_{LCD}(n - 1, k)$ or $d_{LCD}(n - 1, k) + 1$. In particular, if $d_{LCD}(n - 1, k)$ is odd, then $d_{one}(n, k) = d_{LCD}(n - 1, k) + 1$. 

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Proof. Let $C$ be a binary LCD $[n-1,k,d_{LCD}(n-1,k)]$ code. By Proposition 3.8, there is a binary $[n,k,d_{LCD}(n-1,k)$ or $d_{LCD}(n-1,k) + 1$ code with one-dimensional hull. Combined with Lemma 3.1, we have

$$d_{LCD}(n-1,k) \leq d_{\text{one}}(n,k) \leq d_{LCD}(n-1,k) + 1.$$  

If $d_{LCD}(n-1,k)$ is odd. By Proposition 3.8, there is a binary $[n,k,d_{LCD}(n-1,k) + 1]$ code with one-dimensional hull. Hence $d_{\text{one}}(n,k) = d_{LCD}(n-1,k) + 1$. \qed

4 Some properties of binary linear codes with one-dimensional hull

Let $d(n,k)$ be the largest minimum distance among all binary $[n,k]$ linear codes. It is well-known that $d(n,k) \leq d(n,k-1)$. Carlet et al. [10] proved that $d_{LCD}(n,k) \leq d(n,k-1)$ for any $k \geq 2$ using a new characterization of binary LCD codes, which solved the conjecture on the minimum distance of binary LCD codes proposed by Galvez et al. [14]. This conclusion is no longer valid for $d_{\text{one}}(n,k)$.

Theorem 4.1. Suppose that $2 \leq k \leq n-1$. If $k$ is even or $n$ is odd, then $d_{\text{one}}(n,k) \leq d_{\text{one}}(n-1,k)$.

Proof. Let $C$ be a binary linear $[n,k,d_{\text{one}}(n,k)]$ code with one-dimensional hull.

If $k$ is even, then $C$ is odd-like. By Theorem 2.4, there exists a basis $c_1,c_2,\ldots,c_k$ of $C$ such that the code generated by $c_1,c_2,\ldots,c_{k-1}$ is an odd-like binary LCD $[n,k-1]$ code and $c_k \cdot c_i = 0$ for $1 \leq i \leq k$. Without loss of generality, we assume that $c_1,c_2,\ldots,c_{k-1}$ satisfy the conditions in Theorem 2.1. Let $C'$ be the code generated by $c_1,c_2,\ldots,c_{k-2},c_k$. According to Theorems 2.1 and 2.4, $C'$ is a binary $[n,k-1]$ linear code with one-dimensional hull and the minimum distance at least $d_{\text{one}}(n,k)$.

Let $n$ be odd. If $C$ is odd-like, then the result follows from a similar discussion in the case $k$ odd. In the following, assume that $C$ is even-like. From Theorems 2.4 and 2.2, $k$ is odd and there exists a basis $c_1,c'_1,\ldots,c_{k-1},c'_{k-1}$ of $C$ such that the code generated by $c_1,c'_1,\ldots,c_{k-1},c'_{k-1}$ is an even-like binary LCD $[n,k-1]$ code, $c_k \cdot c_i = c_k \cdot c'_i = 0$ for $1 \leq i \leq k-1$, and for any $i,j \in \{1,2,\ldots,k-1\}$, the following conditions hold (i) $c_i \cdot c_i = c'_i \cdot c'_i = 0$; (ii) $c_i \cdot c'_j = 0$, for $i \neq j$; (iii) $c_i \cdot c_i' = 1$; (iv) $c_{i,1} = c'_{i,1}$, where $c_i = (c_{i,1},\ldots,c_{i,n})$ and $c'_i = (c'_{i,1},\ldots,c'_{i,n})$. Since $n$ is odd, $c_k \neq (1,\ldots,1)$. Without loss of generality, assume that $c_{k,1} = 0$, where $c_k = (c_{k,1},c_{k,2},\ldots,c_{k,n})$.

Suppose that $c_{i,1} = c'_{i,1} = 0$ for $1 \leq i \leq k-1$. According to Theorem 8 in [10], the code generated by $S_1 = \{c_1,c'_1,\ldots,c_{k-1},c'_{k-1},e_1\}$ is a binary LCD $[n,k-2]$ code, where $e_1 = (1,0,\ldots,0)$. Let $C'$ be a binary linear code generated by $\{c_k\} \cup S_1$. By Theorem 2.4, $C'$ is a binary $[n,k-1]$ linear code with one-dimensional hull and the minimum distance at least $d_{\text{one}}(n,k)$.

If for some $1 \leq i \leq k-1$ such that $c_{i,1} \neq 0$. Without loss of generality, assume that $c_{i,1} = c'_{i,1} = 1$ for $1 \leq i \leq l$ and $c_{j,1} = c'_{j,1} = 0$ for $l+1 \leq j \leq k-1$, where $l$ is some positive integer. According to Theorem 8 in [10], the code generated by $S_2 =...
\{c_1 + c'_i + e_i\} \cup \{c_j + c_1, c'_j + c_1 | 2 \leq i \leq l\} \cup \{c_j, c'_j | l + 1 \leq j \leq \frac{k-1}{2}\} is a binary LCD \([n, k-2]\) code, where \(e_i = (1, 0, \ldots, 0)\). Let \(C'\) be a binary code generated by \(\{c_k\} \cup S_2\).

By Theorem 2.4, \(C'\) is a binary \([n, k-1]\) linear code with one-dimensional hull and the minimum distance at least \(d_{one}(n, k)\). This completes the proof.

**Remark 4.2.** When \(k\) is odd and \(n\) is even, the above theorem may not be true. For example, \(d_{one}(18, 8) = 5, d_{one}(18, 9) = 6\) (see Table 1).

**Proposition 4.3.** If there is an odd-like (resp. even-like) binary linear \([n, k, d]\) code with one-dimensional hull for an odd \(k\), then there is an even-like (resp. odd-like) binary linear \([n + 1, k, d\) or \(d + 1]\) code with one-dimensional hull.

**Proof.** Let \(C\) be an odd-like binary linear \([n, k, d]\) code with one-dimensional hull. From Theorems 2.1 and 2.4, there exists a basis \(c_1, c_2, \ldots, c_k\) of \(C\) such that for any \(i, j \in \{1, 2, \ldots, k - 1\}\), \(c_i \cdot c_j\) equals 1 if \(i = j\) and equals 0 if \(i \neq j\), \(c_k \cdot c_i = 0\) for \(1 \leq i \leq k\). Let \(C'\) be a binary linear code with the generator matrix \(G'\) whose rows are \((1, c_1), \ldots, (1, c_{k-1}), (0, c_k)\). Then \(C'\) is an even-like binary code with the minimum distance at least \(d\). According to [6, Proposition 1], the code generated by \((1, c_1), \ldots, (1, c_{k-1})\) is an LCD code. By Theorem 2.4, \(C'\) is an even-like binary linear \([n + 1, k, d\) or \(d + 1]\) code with one-dimensional hull.

Let \(C\) be an even-like binary linear \([n, k, d]\) code with one-dimensional hull. By Theorems 2.2 and 2.4, there exists a basis \(c_1, c'_1, \ldots, c_{\frac{k-1}{2}}, c'_\frac{k-1}{2}, c_k\) of \(C\) such that \(c_1, c'_1, \ldots, c_{\frac{k-1}{2}}, c'_{\frac{k-1}{2}}, c_k \cdot c_i = c_k \cdot c'_i = 0\) for \(1 \leq i \leq \frac{k-1}{2}\). Similar to the discussion above, the code generated by \((1, c_1), (1, c'_1), \ldots, (1, c_{\frac{k-1}{2}}), (1, c'_{\frac{k-1}{2}}), (0, c_k)\) is an odd-like binary linear \([n + 1, k, d\) or \(d + 1]\) code with one-dimensional hull.

**Corollary 4.4.** If \(k\) is odd and \(d_{one}(n - 1, k)\) is odd, then \(d_{one}(n, k) \geq d_{one}(n - 1, k) + 1\). **Proof.** The proof is straightforward by Proposition 4.3, so we omit it here.

The following propositions show some properties of the shortened and punctured codes of binary linear codes with one-dimensional hull.

**Proposition 4.5.** Let \(C\) be an even-like binary linear \([n, k]\) code with \(Hull(C) = \{0, c_k\}\).
If \(t \notin Supp(c_k)\), then the punctured code \(C^{(t)}\) of \(C\) on the \(t\)-th coordinate is a binary linear code with one-dimensional hull. If \(t \in Supp(c_k)\), then the punctured code \(C^{(t)}\) and the shortened code \(C_{\{t\}}\) of \(C\) on the \(t\)-th coordinate are binary LCD codes.

**Proof.** Let \(C\) be an even-like binary linear \([n, k]\) code with one-dimensional hull. From Theorems 2.2 and 2.4, there exists a basis \(c_1, c'_1, \ldots, c_{\frac{k-1}{2}}, c'_{\frac{k-1}{2}}, c_k\) of \(C\) such that \(c_1, c'_1, \ldots, c_{\frac{k-1}{2}}, c'_{\frac{k-1}{2}}, c_k \cdot c_i = c_k \cdot c'_i = 0\) for \(1 \leq i \leq \frac{k-1}{2}\). Let \(C'\) be the code generated by \(c_1, c'_1, \ldots, c_{\frac{k-1}{2}}, c'_{\frac{k-1}{2}}\). Let \(c'_k = (c_{k,1}, \ldots, c_{k,t-1}, c_{k,t+1}, \ldots, c_{k,n})\), where \(c_k = (c_{k,1}, \ldots, c_{k,n})\). Hence \(C^{(t)} = (C')^{(t)} \oplus \langle c'_k \rangle\).

Assume that \(t \notin Supp(c_k)\), i.e., \(c_{k,t} = 0\). It follows from [6, Proposition 2] that the punctured code \((C')^{(t)}\) of \(C'\) on the \(t\)-th coordinate is again LCD. Since \(t \notin Supp(c_k)\),
Let \( C \) be a binary linear code with one-dimensional hull. By Lemma 2.6, if \( t \in \text{Supp}(c_k) \), then we can obtain the desired result by (1) of Proposition 2.8.

**Corollary 4.6.** If \( k \) is odd, \( n \) is even and \( d_{\text{LCD}}(n, k) \) is odd, then \( d_{\text{one}}(n, k) \geq d_{\text{LCD}}(n, k) \).

**Proof.** If there exists a binary LCD \([n, k, d_{\text{LCD}}(n, k)]\) code, then it follows from Proposition 3.8 that there is an even-like binary \([n + 1, k, d_{\text{LCD}}(n, k) + 1]\) code \( C \) with one-dimensional hull. Since \( n \) is even, \( n + 1 \) is odd. It turns out that \( 1 \notin C \). By Proposition 4.5, there is a binary \([n, k, d_{\text{LCD}}(n, k)]\) code with one-dimensional hull. Hence \( d_{\text{one}}(n, k) \geq d_{\text{LCD}}(n, k) \).

**Corollary 4.7.** If \( k \) is odd, \( n \) is even and \( d_{\text{one}}(n, k) \) is odd, then \( d_{\text{one}}(n, k) \leq d_{\text{LCD}}(n, k) \).

**Proof.** If there exists a binary linear \([n, k, d_{\text{one}}(n, k)]\) code with one-dimensional hull, then it follows from Proposition 4.3 that there is an even-like binary \([n + 1, k, d_{\text{one}}(n, k) + 1]\) linear code \( C \) with one-dimensional hull. Since \( n \) is even, \( n + 1 \) is odd. This implies that \( 1 \notin C \). By Proposition 4.5, there is a binary LCD \([n, k, d_{\text{one}}(n, k)]\) code. Hence \( d_{\text{one}}(n, k) \leq d_{\text{LCD}}(n, k) \).

**Proposition 4.8.** Let \( C \) be an odd-like binary linear \([n, k]\) code with \( \text{Hull}(C) = \{0, c_k\} \) and even-like dual. If \( t \notin \text{Supp}(c_k) \), then the shortened code \( C_{\{t\}} \) of \( C \) on the \( t \)-th coordinate is a binary linear code with one-dimensional hull.

**Proof.** Obviously, \( \text{Hull}(C^\perp) = \text{Hull}(C) = \{0, c_k\} \). If \( t \notin \text{Supp}(c_k) \), it follows from Proposition 4.5 that the punctured code \((C^\perp)_{\{t\}}\) of \( C^\perp \) on the \( t \)-th coordinate is a binary linear code with one-dimensional hull. By Lemma 2.6,

\[
\text{Hull}(C_{\{t\}}) = \text{Hull}((C_{\{t\}})^\perp) = \text{Hull}(((C^\perp)_{\{t\}})^\perp),
\]

which implies that the shortened code \( C_{\{t\}} \) of \( C \) on the \( t \)-th coordinate is a binary linear code with one-dimensional hull.

**Proposition 4.9.** Let \( C \) be a binary linear \([n, k]\) code with one-dimensional hull and a generator matrix \( G \). Let \( C' \) be a binary linear \([n, k]\) code with the generator matrix \((v^T, v^T, G)\), where \( v \in \mathbb{F}_2^k \). Then \( C \) has one-dimensional hull if and only if \( C' \) has one-dimensional hull.

**Proof.** It is easy to check that \( GG^T = G'G'^T \). Hence the result follows.

**Corollary 4.10.** If \( d_{\text{one}}(n, k) \) is odd, then \( d_{\text{one}}(n + 2, k) \geq d_{\text{one}}(n, k) + 1 \).

**Proof.** The proof is straightforward by Proposition 4.9, so we omit it here.

### 5 Building-up construction for binary linear codes with one-dimensional hull

In this section, we introduce a complete building-up construction for linear codes with one-dimensional hull as follows.
Theorem 5.1. [22, Theorem 1] Let $C$ be a binary LCD $[n, k]$ code. Let $G$ be a generator matrix for $C$. Suppose that $x = (x_1, x_2, \ldots, x_n) \in \mathbb{F}_2^n$ satisfies $x \cdot x = 1$. Let $y_i = x \cdot r_i$ for $1 \leq i \leq k$ where $r_i$ is the $i$-th row of $G$. The following matrix
\[
G_1 = \begin{bmatrix}
1 & 0 & x_1 & \ldots & x_n \\
y_1 & y_1 & \ & \ & r_1 \\
y_2 & y_2 & \ & \ & r_2 \\
\vdots & \vdots & \ & \ & \vdots \\
y_k & y_k & \ & \ & r_k
\end{bmatrix}
\]
generates an $[n + 2, k + 1]$ linear code $C_1$ with one-dimensional hull.

Example 5.2. We start from a binary LCD $[12, 2, 6]$ code. By applying Theorem 5.1, we can construct a binary $[14, 3, 7]$ code with one-dimensional hull and generator matrix
\[
G = \begin{bmatrix}
10 & 100110010111 \\
11 & 111111000000 \\
00 & 000111111100
\end{bmatrix}
\]

The converse of the building-up construction is also true in the following sense.

Theorem 5.3. Any binary linear $[n, k]$ code $C$ with one-dimensional hull, $\text{Hull}(C) \neq \langle 1 \rangle$ and minimum weight $d > 2$ can be obtained from some binary LCD $[n - 2, k - 1]$ code $C_0$ using the above building-up construction.

Proof. Let $G$ be a generator matrix of $C$ with one-dimensional hull. Without loss of generality, we may assume that
\[
G = \begin{bmatrix}
10 & b_1 & a_1 \\
01 & 0 & a_2 \\
00 & e_3 & a_3 \\
\vdots & \vdots & \vdots \\
00 & e_k & a_k
\end{bmatrix}
\]
where $c = (1, 0, b_1, a_1) \in \text{Hull}(C)$ and $e_i$ is the $(i-2)$-th row of $I_{k-2}$ (the identity matrix).

It is not difficult to check that the following matrix
\[
\begin{bmatrix}
11 & b_1 & a_1 + a_2 \\
00 & e_3 & a_3 \\
\vdots & \vdots & \vdots \\
00 & e_k & a_k
\end{bmatrix}
\]
generates an LCD $[n, k - 1]$ code.

It suffices to prove that there exist a vector $x = (x_1, \ldots, x_{n-2})$ and an LCD code $C_0$ of length $n - 2$ whose extended code $C_1$, by Theorem 5.1, is a code equivalent to $C$. To
do that, first consider a linear code $C_0$ with the following generator matrix:

$$G_0 = \begin{bmatrix} b_1 & a_1 + a_2 \\ e_3 & a_3 \\ \vdots & \vdots \\ e_k & a_k \end{bmatrix},$$

which is an LCD $[n - 2, k - 1]$ code by [6, Proposition 4].

Using the row $x = (b_1|a_1)$ of length $n - 2$ and $G_0$, we get a generator matrix $G_1$ of a linear $[n, k]$ code $C_1$ by Theorem 5.1, in this case, $\text{wt}(x)$ is odd.

$$G_1 = \begin{bmatrix} 10 & b_1 \\ 11 & b_1 \\ 00 & e_3 \\ \vdots & \vdots \\ 00 & e_k \end{bmatrix} a_1 + \begin{bmatrix} 10 & b_1 \\ 01 & 0 \\ 00 & e_3 \\ \vdots & \vdots \\ 00 & e_k \end{bmatrix} a_2 \sim = G.$$

Thus the given code $C$ is equivalent to $C_1$, as desired. This completes the proof. \qed

**Theorem 5.4.** Let $C$ be any binary linear $[n, k]$ code $C$ with $\text{Hull}(C) = \langle 1 \rangle$. Then $n$ is even, $k$ is odd and there exists an even-like binary LCD $[n, k - 1]$ code $C_0$ such that $C = C_0 \oplus \langle 1 \rangle$.

**Proof.** Since $\text{Hull}(C) = \langle 1 \rangle$, $C$ and $C^\perp$ are even-like, which implies that $n$ is even. Note that $k$ is odd by Lemma 2.5. Let $c_1, c_2, \ldots, c_{k-1}, 1$ be a basis of $C$. Then it follows from [10, Lemma 22] that the code $C_0$ generated by $c_1, c_2, \ldots, c_{k-1}$ is an even-like binary LCD $[n, k - 1]$ code. Hence $C = C_0 \oplus \langle 1 \rangle$. This completes the proof. \qed

Using Theorem 5.1, we can obtain the following corollary.

**Corollary 5.5.** Let $C$ be a binary LCD $[n, k]$ code with generator matrix $G$. Suppose that $x \in C^\perp$ and $\text{wt}(x)$ is odd. Then the following matrix

$$\begin{bmatrix} 1 & x \\ 0 & G \end{bmatrix}$$

generates a binary $[n + 1, k + 1]$ linear code with one-dimensional hull.

**Proof.** The code $C_3$ constructed from Theorem 5.1 is a binary $[n + 2, k + 1]$ code with one-dimensional hull. Since $x \in C^\perp$, $y_i = 0$ for $i = 1, \ldots, k$. Therefore, by puncturing $C_3$ on the second coordinate, we obtain the matrix

$$\begin{bmatrix} 1 & x \\ 0 & G \end{bmatrix},$$

which also generates a binary $[n + 1, k + 1]$ code with one-dimensional hull. \qed
Example 5.6. We start from a binary LCD $[13, 5, 5]$ code. By applying Corollary 5.5 we can construct a binary $[14, 6, 5]$ linear code with one-dimensional hull and generator matrix

$$G = \begin{bmatrix} 1 & 1011100010111 \\ 0 & 1000111010111 \\ 0 & 0100011000010 \\ 0 & 0010010001110 \\ 0 & 0001000111011 \\ 0 & 0000101111101 \end{bmatrix}.$$

Theorem 5.7. Any binary $[n, k, d]$ linear code with one-dimensional hull can be obtained from some binary LCD $[n - 1, k - 1, \geq d]$ code by the construction of Corollary 5.5.

Proof. Let $C$ be a binary $[n, k, d]$ linear code with one-dimensional hull. By Proposition 2.8, there is at least one coordinate position $i$ such that the shortened code $C_{(i)}$ of $C$ on the $i$-th coordinate is a binary LCD $[n - 1, k - 1, \geq d]$ code. Without loss of generality, we consider that $i = 1$. Assume that $C_{(1)}$ has the generator matrix $G_1$. Then $C$ has the generator matrix $G = \begin{bmatrix} 1 & x' \\ 0 & G_1 \end{bmatrix}$ for some $x' = (x'_1, \ldots, x'_{n-1}) \in \mathbb{F}_2^{n-1}$. Since $C_{(1)}$ is a binary LCD code, $\mathbb{F}_2^{n-1} = C_{(1)} \oplus (C_{(1)})^\perp$. So there are $x = (x_1, \ldots, x_{n-1}) \in (C_{(1)})^\perp$ and $y = (y_1, \ldots, y_{n-1}) \in C_{(1)}$ such that $x' = x + y$. Hence the following matrix

$$G_0 = \begin{bmatrix} 1 & x \\ 0 & G_1 \end{bmatrix}$$

is also the generator matrix of $C$. It turns out that $wt(x)$ is odd, otherwise $C \cap C^\perp = \{0\}$, which is a contradiction. This completes the proof. \qed

Remark 5.8. If we would like to obtain all binary $[n, k, d]$ linear codes with one-dimensional hull, then we can start from all binary LCD $[n - 1, k - 1, \geq d]$ codes. This theorem may be very useful in classification.

Example 5.9. According to [18], there exist a unique inequivalent binary LCD $[15, 7, 5]$ code. By applying Corollary 5.5 we cannot construct a binary $[16, 8, 5]$ linear code with one-dimensional hull. So $d_{\text{one}}(16, 8) \leq 4$.

Harada and Saito [18] gave a complete classification of optimal binary LCD $[n, k]$ codes for $1 \leq k \leq n \leq 16$. Bouyuklieva [6] gave a partial classification of optimal binary LCD $[n, k]$ codes for $1 \leq k \leq n \leq 40$. A complete classification of optimal binary LCD $[n, 3]$ codes was given in [2, 18].

Proposition 5.10. There are no binary linear $[16, 8, 5]$, $[16, 10, 4]$, $[17, 9, 5]$, $[18, 8, 6]$, $[20, 4, 10]$, $[20, 8, 7]$, $[20, 10, 6]$, $[22, 4, 11]$, $[22, 8, 8]$, $[23, 6, 10]$, $[24, 4, 12]$, $[25, 6, 11]$, $[26, 8, 10]$, $[27, 6, 12]$, $[28, 8, 11]$, $[29, 6, 13]$ codes with one-dimensional hull.
Proof. We start from all binary LCD \([15, 7, 5], [15, 9, 4]\) and \([16, 8, 5]\) codes (see \([18]\)). By applying Corollary 5.5, we cannot construct binary \([16, 8, 5], [16, 10, 4]\) and \([17, 9, 5]\) linear code with one-dimensional hull.

We start from all binary LCD \([n, k, d]\) codes, where \((n, k, d) \in \{(17, 7, 6), (19, 7, 7), (19, 9, 6), (21, 7, 8), (22, 5, 10), (24, 5, 11), (25, 7, 10), (26, 5, 12), (27, 7, 11), (28, 5, 13)\}\) (see \([6]\)). By applying Corollary 5.5, we cannot construct a binary \([n + 1, k + 1, d]\) linear code with one-dimensional hull.

We start from all binary LCD \([n, k, d]\) codes, where \((n, k, d) \in \{(19, 3, 10), (21, 3, 11), (23, 3, 12)\}\) (see \([2, 18]\)). By applying Corollary 5.5, we cannot construct a binary \([n + 1, k + 1, d]\) linear code with one-dimensional hull. This completes the proof. 

**Proposition 5.11.** There is no binary linear \([23, 14, 5]\) code with one-dimensional hull.

Proof. There exists a unique inequivalent binary \([23, 14, 5]\) linear code \([36]\), which has 3-dimensional hull by MAGMA \([5]\). 

Remark 5.12. For fixed \(n\) and \(k\), there are two upper bounds on \(d_{\text{one}}(n, k)\):

\[
d_{\text{one}}(n, k) \leq d_{\text{LCD}}(n - 1, k - 1) \quad \text{(Lemma 3.1)} \quad \text{and} \quad d_{\text{one}}(n, k) \leq d(n, k).
\]

For the upper bound of \(d_{\text{LCD}}(n, k)\), we refer to \([1–3, 6, 13, 14, 17, 18, 20, 25]\). For the upper bound of \(d(n, k)\), we refer to \([15]\).

| \(n/k\) | \(1\) | \(2\) | \(3\) | \(4\) | \(5\) | \(6\) | \(7\) | \(8\) | \(9\) | \(10\) | \(11\) | \(12\) | \(13\) | \(14\) | \(15\) |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 14    | 14  | 8   | 7   | 6   | 6   | 5   | 4   | 4   | 3   | 2   | 2   | 1   | 2   |
| 15    | 14  | 9   | 8   | 7   | 6   | 5   | 5   | 4   | 4   | 3   | 3   | 2   | 2   | 1   |
| 16    | 16  | 9   | 8   | 7   | 6   | 6   | 6   | 4   | 4   | 3   | 3   | 2   | 2   | 1   |
| 17    | 16  | 11  | 9   | 8   | 7   | 7   | 6   | 5   | 4   | 4   | 4   | 3   | 3   | 2   |
| 18    | 18  | 11  | 10  | 8   | 8   | 7   | 6   | 5   | 6   | 4   | 4   | 3   | 3   |
| 19    | 18  | 12  | 10  | 9   | 8   | 7   | 7   | 6   | 6   | 5   | 4   | 4   | 3   |
| 20    | 20  | 12  | 10  | 9   | 8   | 8   | 6   | 6   | 5   | 5   | 4   | 4   | 3   |
| 21    | 20  | 13  | 11  | 10  | 10  | 8   | 8   | 7   | 6   | 6   | 5   | 4   | 4   |
| 22    | 22  | 13  | 12  | 10  | 10  | 9   | 8   | 7   | 7   | 6   | 6   | 5   | 4   |
| 23    | 22  | 15  | 12  | 11  | 10  | 9   | 8   | 8   | 7   | 6   | 6   | 4   | 4   |
| 24    | 24  | 15  | 13  | 11  | 11  | 10  | 10  | 8   | 7   | 7   | 6   | 6   | 5   |
| 25    | 24  | 16  | 14  | 12  | 12  | 10  | 10  | 9   | 8   | 8   | 7   | 6   | 5-6 |
| 26    | 26  | 16  | 14  | 13  | 12  | 11  | 10  | 9   | 8   | 8   | 7   | 6   |
| 27    | 26  | 17  | 14  | 13  | 12  | 11  | 11  | 10  | 8-9 | 8   | 8   | 7   |
| 28    | 28  | 17  | 15  | 14  | 13  | 12  | 12  | 10  | 9   | 8   | 8   | 7   |
| 29    | 28  | 19  | 16  | 14  | 14  | 12  | 12  | 11  | 10  | 9-10 | 9   | 8   | 8   |
| 30    | 30  | 19  | 16  | 15  | 14  | 13  | 12  | 11-12| 11  | 10  | 10   | 9   | 8   |

Table 1: \(d_{\text{one}}(n, k)\), where \(14 \leq n \leq 30, 1 \leq k \leq 15\)
Table 2: $d_{\text{one}}(n, k)$, where $17 \leq n \leq 30, 16 \leq k \leq 29$

| $n/k$ | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
|-------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 17    | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 18    | 1  | 2  |    |    |    |    |    |    |    |    |    |    |    |    |
| 19    | 2  | 2  | 1  |    |    |    |    |    |    |    |    |    |    |    |
| 20    | 2  | 2  | 1  | 2  |    |    |    |    |    |    |    |    |    |    |
| 21    | 3  | 2  | 2  | 2  | 1  |    |    |    |    |    |    |    |    |    |
| 22    | 3  | 3  | 2  | 2  | 1  | 2  |    |    |    |    |    |    |    |    |
| 23    | 4  | 4  | 3  | 2  | 2  | 2  | 1  |    |    |    |    |    |    |    |
| 24    | 4  | 4  | 3  | 2  | 2  | 1  | 2  |    |    |    |    |    |    |    |
| 25    | 4  | 4  | 4  | 3  | 2  | 2  | 1  | 2  |    |    |    |    |    |    |
| 26    | 5  | 4  | 4  | 3  | 3  | 2  | 2  | 1  | 2  |    |    |    |    |    |
| 27    | 5-6| 5  | 4  | 4  | 4  | 3  | 2  | 2  | 2  | 1  |    |    |    |    |
| 28    | 6  | 5  | 4  | 4  | 4  | 3  | 2  | 2  | 1  | 2  |    |    |    |    |
| 29    | 6  | 6  | 5-6| 5  | 4  | 4  | 3  | 2  | 2  | 2  | 1  |    |    |    |
| 30    | 6  | 6  | 6  | 6  | 5  | 4  | 4  | 3  | 2  | 2  | 2  | 1  | 2  |    |

Remark 5.13. The value in Tables 1 and 2 denotes the minimum distance of an optimal binary $[n, k]$ code with one-dimensional hull. All computations in this paper have been done by MAGMA [5]. To save the space, the codes in Tables 1, 2 can be obtained from one of the authors’ website, namely, https://cicagolab.sogang.ac.kr/.

6 Optimal binary linear codes with one-dimensional hull

In this section, we characterize the optimal binary $[n, k]$ linear codes with one-dimensional hull for $k \in \{1, 2, 3, 4, 5, n - 1, n - 2, n - 3, n - 4, n - 5\}$.

6.1 Optimal binary $[n, 1]$ and $[n, n - 1]$ linear codes with one-dimensional hull

Theorem 6.1. If $n$ is odd, then $d_{\text{one}}(n, 1) = n - 1$ and $d_{\text{one}}(n, n - 1) = 1$. If $n$ is even, then $d_{\text{one}}(n, 1) = n$ and $d_{\text{one}}(n, n - 1) = 2$.

Proof. By the Griesmer bound, we have $d_{\text{one}}(n, 1) \leq n$ and $d_{\text{one}}(n, n - 1) \leq 2$. Assume that $n$ is odd. The repetition $[n, 1, n]$ code is not a linear code with one-dimensional hull. The code $C$ generated by $[011\ldots1]$ is a linear code with one-dimensional hull. So $d_{\text{one}}(n, 1) = n - 1$. The dual code $C^\perp$ of $C$ is a linear code with one-dimensional hull and the minimum weight 1. If $d_{\text{one}}(n, n - 1) = 2$, then the corresponding code $C'$ is the even $[n, n - 1, 2]$ code. The dual of $C'$ is the repetition $[n, 1, n]$ code, which is not a linear code with one-dimensional hull. Thus $d_{\text{one}}(n, n - 1) = 1$.

Assume that $n$ is even. The repetition $[n, 1, n]$ code and its dual code are linear codes with one-dimensional hull. Hence $d_{\text{one}}(n, 1) = n$ and $d_{\text{one}}(n, n - 1) = 2$. \qed
6.2 Optimal binary \([n, 2]\) and \([n, n - 2]\) linear codes with one-dimensional hull

Mankean and Jitman [28] determined the exact value of \(d_{\text{one}}(n, 2)\).

**Theorem 6.2.** [28] Let \(n > 2\) be an integer. Then we have

\[
d_{\text{one}}(n, 2) = \begin{cases} 
\left\lfloor \frac{2n}{3} \right\rfloor, & \text{for } n \equiv 1, 5 \pmod{6}, \\
\left\lfloor \frac{2n}{3} \right\rfloor - 1, & \text{for } n \equiv 0, 3, 4 \pmod{6}.
\end{cases}
\]

Next, we consider the exact value of \(d_{\text{one}}(n, k)\) for \(k = n - 2\).

**Theorem 6.3.** Let \(n > 2\) be an integer. Then we have

\[
d_{\text{one}}(n, n - 2) = \begin{cases} 
2, & \text{if } n \text{ is odd}, \\
1, & \text{if } n \text{ is even}.
\end{cases}
\]

**Proof.** By the Griesmer bound, \(d_{\text{one}}(n, n - 2) \leq 2\). Hence \(d_{\text{one}}(n, n - 2) = 2\) or 1. Let \(x, y, z, s\) be four integers. Consider the code \(C\) of length \(n\) with parity-check matrix

\[
H = \begin{bmatrix}
1 & \ldots & 1 & 1 & \ldots & 1 & 0 & \ldots & 0 \\
0 & \ldots & 0 & 1 & \ldots & 1 & 1 & \ldots & 1 \\
x & y & z & s
\end{bmatrix}.
\]

It is easy to see that the code \(C\) has minimum weight 2 (resp. 1) if and only if \(s = 0\) (resp. \(s > 0\)). Let \(n\) be an odd integer, i.e., \(n = 2m + 1\) for some positive integer \(m\). If \(x = m, y = 0, z = m + 1, s = 0\), then the code \(C\) is a linear \([2m + 1, 2m - 1, 2]\) code with one-dimensional hull. Therefore, \(d_{\text{one}}(n, n - 2) = 2\) if \(n\) is odd.

Let \(n\) be an even integer. Assume that \(s = 0\).

- If \(x\) is odd, then \(y + z\) is odd. Whether \(y\) is odd or even, \(C\) is an LCD code.
- If \(x\) is even, then \(y + z\) is even.
  - If \(y\) is odd, then it is not difficult to check that \(C\) is an LCD code.
  - If \(y\) is even, then it is not difficult to check that \(C^\perp \subset C\).

This implies that \(s > 0\) when \(C\) is a binary linear code with one-dimensional hull. Therefore, \(d_{\text{one}}(n, n - 2) = 1\) if \(n\) is even.

6.3 Optimal binary \([n, 3]\) and \([n, n - 3]\) linear codes with one-dimensional hull

Assume that \(S_k\) is a matrix whose columns are all nonzero vectors in \(\mathbb{F}_2^k\). It is well-known that \(S_k\) generates a binary simplex code, which is an one-weight self-orthogonal \([2^k - 1, k, 2^{k-1}]\) Griesmer code for \(k \geq 3\) (see [19]).
Lemma 6.4. Assume that $S_k$ is a matrix whose columns are all nonzero vectors in $\mathbb{F}_2^k$ for $k \geq 3$. Let $C$ be a binary $[n, k, d]$ linear code with the generator matrix $G$. Then $C$ has one-dimensional hull if and only if $C'$ with the following matrix

$$G' = [S_k|\cdots|S_k|G]$$

is a binary $[m(2^k - 1) + n, k, m2^{k-1} + d]$ linear code with one-dimensional hull.

Proof. It is well-known that $S_k$ generates a binary simplex code, which is an one-weight self-orthogonal $[2^k - 1, k, 2^{k-1}]$ Griesmer code. So

$$G'G'^T = GG^T.$$  

Therefore, $C$ has one-dimensional hull if and only if $C'$ has one-dimensional hull. Since $C$ has the minimum distance $d$, $C'$ has the minimum distance at least $d + 2^{k-1}m$. Since the simplex code is an one-weight code, there is at least a codeword of weight $d + 2^{k-1}m$ in $C'$. The converse is also true. This completes the proof. \[]

Let $h_{k,i}$ be the $i$-th column of the matrix $S_k$. Let $G_k(m)$ be a $k \times \sum_{i=1}^{2^k-1} m_i$ matrix which consists of $m_i$ columns $h_{k,i}$ for each $i$ as follows:

$$G_k(m) = [h_{k,1}, \ldots, h_{k,1}, h_{k,2^k-1}, \ldots, h_{k,2^k-1}],$$

where $m = (m_1, \ldots, m_{2^k-1})$ and $m_i$ is a nonnegative integer. For a binary $[n, k, d]$ linear code with $d(C') \geq 2$, there exists a vector $m = (m_1, \ldots, m_{2^k-1})$ such that $C$ is equivalent to the code $C_k(m)$ with the generator matrix $G_k(m)$.

Proposition 6.5. Let $m = (m_1, m_2, \ldots, m_{2^k-1})$ and $m = \min\{m_1, m_2, \ldots, m_{2^k-1}\}$. Let $C$ be a binary $[n, k, d]$ linear code with the generator matrix $G_k(m)$. Let $C'$ be a binary linear code with the generator matrix $G_k(m')$, where $m' = (m_1 - m, m_2 - m, \ldots, m_{2^k-1} - m)$. If $d > m2^{k-1}$, then $C'$ is a binary $[n - m(2^k - 1), k, d - m2^{k-1}]$ linear code.

Proof. We just verify that $C'$ has $2^k$ codewords, i.e., rank($G_k(m')$) = $k$. Without loss of generality, let

$$G_k(m) = [S_k, \ldots, S_k, G_k(m')].$$

Assume that rank($G_k(m')$) < $k$. Since $S_k$ generates an one-weight code, we obtain $d = m2^{k-1}$, which is a contradiction. This completes the proof. \[]

The following is an interesting and useful result proposed by Araya et al. [2].

Lemma 6.6. [2] Suppose that $(q, k_0) = (2, 3)$ and $k \geq k_0$. If the code $C_k(m)$ has minimum weight at least $d$, then

$$2d - n \leq m_i \leq n - \frac{2^{k-1} - 1}{2^{k-2}}d,$$

for each $i \in \{1, 2, \ldots, 2^k - 1\}$, where $m = (m_1, \ldots, m_{2^k-1})$ and $n = \sum_{i=1}^{2^k-1} m_i$.  

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Proposition 6.7. There is no binary \([7m+6, 3, 4m+3]\) linear code with one-dimensional hull for \(m \geq 0\).

*Proof.* Suppose that \(C\) is a binary \([7m+6, 3, 4m+3]\) linear code with one-dimensional hull. Then \(C\) is a Griesmer code and \(d(C^\perp) \geq 2\). Hence there is a vector \(m = (m_1, \ldots, m_7)\) such that \(C\) is equivalent to \(C_3(m)\). By Lemma 6.6, \(m_i \geq m\). Let \(m' = (m_1 - m, \ldots, m_7 - m)\). By Proposition 6.5 and Lemma 6.4, the code \(C_3(m')\) is a binary \([6, 3, 3]\) linear code with one-dimensional hull, which contradicts \(d_{one}(6, 3) = 2\) (see [22, Table 1]). Hence \(d_{one}(7m + 6, 3) \leq 4m + 2\).

By the Griesmer bound and some known results, we obtain the following theorem.

**Theorem 6.8.** Let \(n > 3\) be an integer. Then we have

\[
\begin{align*}
d_{one}(n, 3) &= \left\{ \begin{array}{ll}
\left\lfloor \frac{4n}{7} \right\rfloor, & \text{for } n \equiv 1, 3, 4, 5 \pmod{7}, \\
\left\lfloor \frac{4n}{7} \right\rfloor - 1, & \text{for } n \equiv 0, 2, 6 \pmod{7}.
\end{array} \right.
\end{align*}
\]

*Proof.* By the Griesmer bound, we have

\[
d_{one}(n, 3) \leq \left\{ \begin{array}{ll}
\left\lfloor \frac{4n}{7} \right\rfloor, & \text{if } n \equiv 0, 1, 3, 4, 5, 6 \pmod{7}, \\
\left\lfloor \frac{4n}{7} \right\rfloor - 1, & \text{if } n \equiv 2 \pmod{7}.
\end{array} \right.
\]

(i) Assume that \(n \equiv 4 \pmod{7}\), i.e., \(n = 7m + 4\) for some integer \(m\). Applying Lemma 6.4 to the binary \([4, 3, 2]\) linear code with one-dimensional hull (see [22, Table 1]), we have

\[
d_{one}(7m + 4, 3) \geq 4m + 2 = \left\lfloor \frac{4(7m + 4)}{7} \right\rfloor.
\]

Combined with the Griesmer bound, we have \(d_{one}(n, 3) = \left\lfloor \frac{4n}{7} \right\rfloor\) for \(n \equiv 4 \pmod{7}\). A similar argument works for \(n \equiv 1, 3, 5 \pmod{7}\).

(ii) Assume that \(n \equiv 0 \pmod{7}\), i.e., \(n = 7m\) for some integer \(m\). By Lemma 3.2 and [18, Theorem 5.1],

\[
d_{one}(7m, 3) \leq d_{LCD}(7m + 1, 3) = \left\lfloor \frac{4(7m + 1)}{7} \right\rfloor - 1 = 4m - 1 = \left\lfloor \frac{4 \times 7m}{7} \right\rfloor - 1.
\]

On the other hand, applying Lemma 6.4 to the binary \([7, 3, 3]\) linear code with one-dimensional hull (see [22, Table 1]), we have

\[
d_{one}(7m, 3) \geq 4m - 1 = \left\lfloor \frac{4 \times 7m}{7} \right\rfloor - 1.
\]

This implies that \(d_{one}(n, 3) = \left\lfloor \frac{4n}{7} \right\rfloor - 1\) for \(n \equiv 0 \pmod{7}\).

(iii) Assume that \(n \equiv 2 \pmod{7}\), i.e., \(n = 7m + 2\) for some integer \(m\). Applying Lemma 6.4 to the binary \([9, 3, 4]\) linear code with one-dimensional hull (see [22, Table 1]), we have

\[
d_{one}(7m + 2, 3) \geq 4m = \left\lfloor \frac{4(7m + 2)}{7} \right\rfloor - 1.
\]
Combined with the Griesmer bound, we have \( d_{\text{one}}(n, 3) = \left\lfloor \frac{4n}{7} \right\rfloor - 1 \) for \( n \equiv 2 \pmod{7} \).

(iv) Assume that \( n \equiv 6 \pmod{7} \), i.e., \( n = 7m + 6 \) for some integer \( m \). Applying Lemma 6.4 to the binary [6, 3, 2] linear code with one-dimensional hull (see [22, Table 1]), we have

\[
d_{\text{one}}(7m + 6, 3) \geq 4m + 2 = \left\lfloor \frac{4(7m + 6)}{7} \right\rfloor - 1.
\]

Combined with Proposition 6.7, we have \( d_{\text{one}}(n, 3) = \left\lfloor \frac{4n}{7} \right\rfloor - 1 \) for \( n \equiv 6 \pmod{7} \).

Combining (i), (ii), (iii) and (iv), the result holds.

\[\square\]

**Lemma 6.9.** Let \( k \geq 3 \) and \( n \geq 2^k \). Then \( d_{\text{one}}(n, n - k) = 2 \).

**Proof.** If \( d(n, n - k) \geq 3 \), then

\[
2^{n-k} > \frac{2^n}{1+n},
\]

which contradicts the sphere-packing bound. Hence \( d_{\text{one}}(n, n - k) \leq d(n, n - k) \leq 2 \).

Consider the code \( C \) with the following matrix

\[
G = \begin{bmatrix}
1 & 0 & \ldots & 0 & 1 & 1 & 1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 & 1 & 1 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1 & 1 & 1 & 0 & 0 & \ldots & 0
\end{bmatrix}_{(n-k)\times n}
\]

Then \( C \) has parameters \([n, n - k, 2] \) and

\[
GG^T = \begin{bmatrix}
0 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1
\end{bmatrix}_{(n-k)\times(n-k)}.
\]

It turns out that \( \text{rank}(GG^T) = n - k - 1 \). This implies that \( C \) has one-dimensional hull. Hence \( d_{\text{one}}(n, n - k) = 2 \). This completes the proof. \[\square\]

Next, we consider the exact value of \( d_{\text{one}}(n, k) \) for \( k = n - 3 \).

**Theorem 6.10.** Let \( n \geq 4 \) be an integer. Then

\[
d_{\text{one}}(n, n - 3) = \begin{cases} 
4, & \text{if } n = 4, \\
3, & \text{if } n = 5, \\
2, & \text{if } n \geq 6.
\end{cases}
\]

**Proof.** From [22, Table 1], \( d_{\text{one}}(4, 1) = 4, d_{\text{one}}(5, 2) = 3, d_{\text{one}}(6, 3) = d_{\text{one}}(7, 4) = 2 \). It follows from Lemma 6.9 that \( d_{\text{one}}(n, n - 3) = 2 \) for \( n \geq 8 \). This completes the proof. \[\square\]
6.4 Optimal binary \([n, 4]\) and \([n, n - 4]\) linear codes with one-dimensional hull

**Proposition 6.11.** There is no binary \([n, 4, \left\lfloor \frac{8n}{15} \right\rfloor]\) linear code with one-dimensional hull for \(n \equiv 0, 1, 5, 7, 8, 9, 12, 14 \pmod{15}\) and \(n \geq 7\).

**Proof.** Assume that \(n \equiv 0, 5, 7, 8, 12, 14 \pmod{15}\). If there is a binary \([n, 4, \left\lfloor \frac{8n}{15} \right\rfloor]\) linear code \(C\) with one-dimensional hull, then it can be checked that \(C\) is a Griesmer code. It turns out that \(d(C^\perp) \geq 2\).

Assume that \(n \equiv 7 \pmod{15}\), i.e., \(n = 15m + 7\) for some integer \(m\). If \(C\) is a binary \([15m + 7, 4, 8m + 3]\) linear code with one-dimensional hull for \(m \geq 2\), then \(d(C^\perp) \geq 2\) and there is a vector \(m = (m_1, \ldots, m_{15})\) such that \(C\) is equivalent to \(C_4(m)\). By Lemma 6.6, we have \(m_i \geq m - 1\). Let \(m' = (m_1 - m + 1, \ldots, m_{15} - m + 1)\). Combining Proposition 6.5 and Lemma 6.4, the code \(C_4(m')\) is a binary \([22, 4, 11]\) linear code with one-dimensional hull, which contradicts that \(d_{\text{one}}(22, 4) = 10\) (see Table 1). Hence there is no binary \([n, 4, \left\lfloor \frac{8n}{15} \right\rfloor]\) linear code with one-dimensional hull for \(n \equiv 7 \pmod{15}\). A similar argument works for \(n \equiv 0, 5, 8, 12, 14 \pmod{15}\).

Assume that \(n \equiv 1 \pmod{15}\), i.e., \(n = 15m + 1\) for some integer \(m\). Suppose that \(C\) is a binary \([15m + 1, 4, 8m]\) linear code with one-dimensional hull for \(m \geq 1\). If \(d(C^\perp) = 1\), then there is a binary \([15m, 4, 8m]\) linear code with one-dimensional hull, which contradicts that \(d_{\text{one}}(15m, 4) < 8m\). Hence \(d(C^\perp) \geq 2\). Then there is a vector \(m = (m_1, \ldots, m_{15})\) such that \(C\) is equivalent to \(C_4(m)\). By Lemma 6.6, we have \(m_i \geq m - 1\). Let \(m' = (m_1 - m + 1, \ldots, m_{15} - m + 1)\). Combining Proposition 6.5 and Lemma 6.4, the code \(C_4(m')\) is a binary \([16, 4, 8]\) linear code with one-dimensional hull, which contradicts that \(d_{\text{one}}(16, 4) = 7\) (see Table 1). Hence there is no binary \([n, 4, \left\lfloor \frac{8n}{15} \right\rfloor]\) linear code with one-dimensional hull for \(n \equiv 1 \pmod{15}\). A similar argument works for \(n \equiv 9 \pmod{15}\). This completes the proof. □

**Theorem 6.12.** Let \(n \geq 7\) be an integer. Then we have

\[
\begin{align*}
d_{\text{one}}(n, 4) &= \begin{cases} 
\left\lfloor \frac{8n}{15} \right\rfloor, & \text{if } n \equiv 11, 13 \pmod{15}, \\
\left\lfloor \frac{8n}{15} \right\rfloor - 1, & \text{if } n \equiv 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14 \pmod{15}.
\end{cases}
\end{align*}
\]

**Proof.** By the Griesmer bound, we have

\[
\begin{align*}
d_{\text{one}}(n, 4) &\leq \begin{cases} 
\left\lfloor \frac{8n}{15} \right\rfloor, & \text{if } n \equiv 0, 1, 5, 7, 8, 9, 11, 12, 13, 14 \pmod{15}, \\
\left\lfloor \frac{8n}{15} \right\rfloor - 1, & \text{otherwise}.
\end{cases}
\end{align*}
\]

(i) Assume that \(n \equiv 11 \pmod{15}\), i.e., \(n = 15m + 11\) for some integer \(m\). Applying Lemma 6.4 to the binary \([11, 4, 5]\) linear code with one-dimensional hull (see [22, Table 1]), we have

\[
d_{\text{one}}(15m + 11, 4) \geq 8m + 5 = \left\lfloor \frac{8(15m + 11)}{15} \right\rfloor.
\]

Combined with the Griesmer bound, we have \(d_{\text{one}}(n, 4) = \left\lfloor \frac{8n}{15} \right\rfloor\) for \(n \equiv 11 \pmod{15}\). A similar argument works for \(n \equiv 13 \pmod{15}\).
(ii) Assume that \( n \equiv 10 \pmod{15} \), i.e., \( n = 15m + 10 \) for some integer \( m \). Applying Lemma 6.4 to the binary \([10, 4, 4]\) linear code with one-dimensional hull (see [22, Table 1]), we have
\[
d_{\text{one}}(15m + 10, 4) \geq 8m + 4 = \left\lfloor \frac{8(15m + 10)}{15} \right\rfloor - 1.
\]
Combined with the Griesmer bound, we have \( d_{\text{one}}(n, 4) = \left\lfloor \frac{8n}{15} \right\rfloor - 1 \) for \( n \equiv 10 \pmod{15} \). A similar argument works for \( n \equiv 2, 3, 4, 6 \pmod{15} \).

(iii) Assume that \( n \equiv 7 \pmod{15} \), i.e., \( n = 15m + 7 \) for some integer \( m \). Applying Lemma 6.4 to the binary \([7, 4, 2]\) linear code with one-dimensional hull (see [22, Table 1]), we have
\[
d_{\text{one}}(15m + 7, 4) \geq 8m + 2 = \left\lfloor \frac{8(15m + 7)}{15} \right\rfloor - 1.
\]
Combined with Proposition 6.7, we have \( d_{\text{one}}(n, 4) = \left\lfloor \frac{8n}{15} \right\rfloor - 1 \) for \( n \equiv 7 \pmod{15} \). A similar argument works for \( n \equiv 0, 1, 5, 8, 9, 12, 14 \pmod{15} \).

Combining (i), (ii) and (iii), we can obtain the desired result.

Next, we consider the exact value of \( d_{\text{one}}(n, k) \) for \( k = n - 4 \).

**Theorem 6.13.** Let \( n \geq 5 \) be an integer. Then
\[
d_{\text{one}}(n, n - 4) = \begin{cases} 
4, & \text{if } n = 5, \\
3, & \text{if } 6 \leq n \leq 12, \\
2, & \text{if } n \geq 13.
\end{cases}
\]

**Proof.** According to Table 1 and [22, Table 1], \( d_{\text{one}}(5, 1) = 4 \), \( d_{\text{one}}(n, n - 4) = 3 \) for \( 6 \leq n \leq 12 \), \( d_{\text{one}}(n, n - 4) = 2 \) for \( 13 \leq n \leq 15 \). It follows from Lemma 6.9 that \( d_{\text{one}}(n, n - 4) = 2 \) for \( n \geq 16 \). This completes the proof.

### 6.5 Optimal binary \([n, 5]\) and \([n, n - 5]\) linear codes with one-dimensional hull

First, we recall some known results on \( d_{\text{LCD}}(n, 5) \), which can be found in [1, 3].

| \( n \)     | \( d_{\text{LCD}}(n, 5) \) |
|------------|------------------|
| 31m + 1    | 16m + 2          |
| 31m + 2    | 16m + 3          |
| 31m + 3    | 16m + 4          |
| 31m + 4    | 16m + 5          |
| 31m + 5    | 16m + 6          |
| 31m + 6    | 16m + 7          |
| 31m + 7    | 16m + 8          |
| 31m + 8    | 16m + 9          |
| 31m + 9    | 16m + 10         |

| \( n \)     | \( d_{\text{LCD}}(n, 5) \) |
|------------|------------------|
| 31m + 10   | 16m + 11         |
| 31m + 11   | 16m + 12         |
| 31m + 12   | 16m + 13         |
| 31m + 13   | 16m + 14         |
| 31m + 14   | 16m + 15         |
| 31m + 15   | 16m + 16         |
| 31m + 16   | 16m + 17         |
| 31m + 17   | 16m + 18         |
| 31m + 18   | 16m + 19         |
| 31m + 19   | 16m + 20         |
| 31m + 20   | 16m + 21         |

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Combining Corollary 3.9 and Table 3, we have the following table.

| $n$ | $d_{one}(n, 5)$ | $n$ | $d_{one}(n, 5)$ | $n$ | $d_{one}(n, 5)$ |
|-----|----------------|-----|----------------|-----|----------------|
| $31m + 2$ | $16m$ | $31m + 14$ | $16m + 6$ | $31m + 25$ | $16m + 12$ |
| $31m + 6$ | $16m + 2$ | $31m + 18$ | $16m + 8$ | $31m + 26$ | $16m + 12$ |
| $31m + 7$ | $16m + 2$ | $31m + 21$ | $16m + 10$ | $31m + 29$ | $16m + 14$ |
| $31m + 10$ | $16m + 4$ | $31m + 22$ | $16m + 10$ | $31m + 30$ | $16m + 14$ |

Assume that $n \geq 7$. By the Griesmer bound, we have

$$d_{one}(n, 5) \leq \begin{cases} \left\lfloor \frac{16n}{31} \right\rfloor, & \text{if } n \equiv 0, 1, 9, 13, 15, 16, 17, 21, 23, 24, 25, 27, 28, 29, 30 \pmod{31}, \\ \left\lfloor \frac{16n}{31} \right\rfloor - 1, & \text{if } n \equiv 2, 3, 5, 6, 7, 8, 10, 11, 12, 14, 18, 19, 20, 22, 26 \pmod{31}, \\ \left\lfloor \frac{16n}{31} \right\rfloor - 2, & \text{if } n \equiv 4 \pmod{31}. \end{cases}$$

**Proposition 6.14.** There is no binary $[n, 5, \left\lfloor \frac{16n}{31} \right\rfloor]$ linear code with one-dimensional hull for $n \equiv 9, 24, 17, 28 \pmod{31}$ and $n \geq 7$.

**Proof.** Assume that $n \equiv 9 \pmod{31}$, i.e., $n = 31m + 9$ for some integer $m$. If $C$ is a binary $[31m + 9, 5, 16m + 4]$ linear code with one-dimensional hull for $m \geq 0$, then $C$ is a Griesmer code with a minimum distance multiple of 4. By [12], $C$ is self-orthogonal. Hence there is no binary $[n, 5, \left\lfloor \frac{16n}{31} \right\rfloor]$ linear code with one-dimensional hull for $n \equiv 9 \pmod{31}$. A similar argument works for $n \equiv 17 \pmod{31}$.

Assume that $n \equiv 24 \pmod{31}$, i.e., $n = 31m + 24$ for some integer $m$. If $C$ is a binary $[31m + 24, 5, 16m + 12]$ linear code with one-dimensional hull for $m \geq 1$, then $d(C^\perp) \geq 2$ and there is a vector $\mathbf{m} = (m_1, \ldots, m_{31})$ such that $C$ is equivalent to $C_5(\mathbf{m})$. By Lemma 6.6, we have $m_5 \geq m$. Let $\mathbf{m}' = (m_1 - m, \ldots, m_{31} - m)$. Combining Proposition 6.5 and Lemma 6.4, the code $C_5(\mathbf{m}')$ is a binary $[24, 5, 12]$ linear code with one-dimensional hull, which contradicts that $d_{one}(24, 5) = 11$ (see Table 1). Hence there is no binary $[n, 5, \left\lfloor \frac{16n}{31} \right\rfloor]$ linear code with one-dimensional hull for $n \equiv 24 \pmod{31}$. A similar argument works for $n \equiv 28 \pmod{31}$. This completes the proof. \qed

**Theorem 6.15.** Let $n \geq 7$ be an integer. Then we have

$$d_{one}(n, 5) = \begin{cases} \left\lfloor \frac{16n}{31} \right\rfloor, & \text{if } n \equiv 21, 25, 29 \pmod{31}, \\ \left\lfloor \frac{16n}{31} \right\rfloor - 1, & \text{if } n \equiv 2, 3, 5, 6, 7, 9, 10, 11, 14, 17, 18, 19, 20, 22, 24, 26, 28, 30 \pmod{31}, \\ \left\lfloor \frac{16n}{31} \right\rfloor - 2, & \text{if } n \equiv 4 \pmod{31}. \end{cases}$$

**Proof.** (i) Assume that $n \equiv 3 \pmod{31}$, i.e., $n = 31m + 3$ for some integer $m$. Then we have

$$d_{one}(31m + 3, 5) \geq d_{one}(31m + 2, 5) = 16m = \left\lfloor \frac{16(31m + 3)}{31} \right\rfloor - 1.$$

Combined with the Griesmer bound, we have $d_{one}(n, 5) = \left\lfloor \frac{16n}{31} \right\rfloor - 1$ for $n \equiv 3 \pmod{31}$. \hfill 23
(ii) Assume that \( n \equiv 4 \pmod{31} \), i.e., \( n = 31m + 4 \) for some integer \( m \). By Table 4, there is a binary \([37, 5, 18]\) linear code \( C \) with one-dimensional hull. Let \( G \) be the generator matrix of \( C \). Since \( 37 > 32 \), \( G \) has the same two columns. By Proposition 4.9, there is a binary \([35, 5, 16]\) linear code with one-dimensional hull. Applying Lemma 6.4 to the binary \([35, 5, 16]\) linear code with one-dimensional hull, we have

\[
d_{\text{one}}(31m + 4, 5) \geq 16m = \left\lfloor \frac{16(31m + 4)}{31} \right\rfloor - 2.
\]

Combined with the Griesmer bound, we have \( d_{\text{one}}(n, 5) = \left\lfloor \frac{16n}{31} \right\rfloor - 2 \) for \( n \equiv 4 \pmod{31} \).

(iii) Assume that \( n \equiv 5 \pmod{31} \), i.e., \( n = 31m + 5 \) for some integer \( m \). Applying Lemma 6.4 to the binary \([36, 5, 17]\) linear code with one-dimensional hull (see BKLC [15]), we have

\[
d_{\text{one}}(31m + 5, 5) \geq 16m + 1 = \left\lfloor \frac{16n}{31} \right\rfloor - 1.
\]

Combined with the Griesmer bound, we have \( d_{\text{one}}(n, 5) = \left\lfloor \frac{16n}{31} \right\rfloor - 1 \) for \( n \equiv 5 \pmod{31} \).

(iv) Assume that \( n \equiv 9 \pmod{31} \), i.e., \( n = 31m + 9 \) for some integer \( m \). Applying Lemma 6.4 to the binary \([9, 5, 3]\) linear code with one-dimensional hull (see [22, Table 1]), we have

\[
d_{\text{one}}(31m + 9, 5) \geq 16m + 3 = \left\lfloor \frac{16n}{31} \right\rfloor - 1.
\]

Combined with Proposition 6.14, we have \( d_{\text{one}}(n, 5) = \left\lfloor \frac{16n}{31} \right\rfloor - 1 \) for \( n \equiv 9 \pmod{31} \).

(v) Assume that \( n \equiv 11 \pmod{31} \), i.e., \( n = 31m + 11 \) for some integer \( m \). Applying Lemma 6.4 to the binary \([11, 5, 4]\) linear code with one-dimensional hull (see [22, Table 1]), we have

\[
d_{\text{one}}(31m + 11, 5) \geq 16m + 4 = \left\lfloor \frac{16n}{31} \right\rfloor - 1.
\]

Combined with the Griesmer bound, we have \( d_{\text{one}}(n, 5) = \left\lfloor \frac{16n}{31} \right\rfloor - 1 \) for \( n \equiv 11 \pmod{31} \).

(vi) Assume that \( n \equiv 17 \pmod{31} \), i.e., \( n = 31m + 17 \) for some integer \( m \). Applying Lemma 6.4 to the binary \([17, 5, 7]\) linear code with one-dimensional hull (see Table 1), we have

\[
d_{\text{one}}(31m + 17, 5) \geq 16m + 7 = \left\lfloor \frac{16n}{31} \right\rfloor - 1.
\]

Combined with Proposition 6.14, we have \( d_{\text{one}}(n, 5) = \left\lfloor \frac{16n}{31} \right\rfloor - 1 \) for \( n \equiv 17 \pmod{31} \).

(vii) Assume that \( n \equiv 19 \pmod{31} \), i.e., \( n = 31m + 19 \) for some integer \( m \). Then we have

\[
d_{\text{one}}(31m + 19, 5) \geq d_{\text{one}}(31m + 18, 5) = 16m + 8 = \left\lfloor \frac{16(31m + 19)}{31} \right\rfloor - 1.
\]

Combined with the Griesmer bound, we have \( d_{\text{one}}(n, 5) = \left\lfloor \frac{16n}{31} \right\rfloor - 1 \) for \( n \equiv 19 \pmod{31} \).

(viii) Assume that \( n \equiv 20 \pmod{31} \), i.e., \( n = 31m + 20 \) for some integer \( m \). Applying Lemma 6.4 to the binary \([20, 5, 9]\) linear code with one-dimensional hull (see Table 1), we have

\[
d_{\text{one}}(31m + 20, 5) \geq 16m + 9 = \left\lfloor \frac{16n}{31} \right\rfloor - 1.
\]
Combined with the Griesmer bound, we have \( d_{\text{one}}(n, 5) = \left\lfloor \frac{16n}{31} \right\rfloor - 1 \) for \( n \equiv 20 \pmod{31} \).

(ix) Assume that \( n \equiv 24 \pmod{31} \), i.e., \( n = 31m + 24 \) for some integer \( m \). Applying Lemma 6.4 to the binary \([24, 5, 11]\) linear code with one-dimensional hull (see Table 1), we have
\[
d_{\text{one}}(31m + 24, 5) \geq 16m + 11 = \left\lfloor \frac{16n}{31} \right\rfloor - 1.
\]
Combined with Proposition 6.14, we have \( d_{\text{one}}(n, 5) = \left\lfloor \frac{16n}{31} \right\rfloor - 1 \) for \( n \equiv 24 \pmod{31} \). A similar argument works for \( n \equiv 28 \pmod{31} \).

Combining (i)-(ix) and Table 4, we obtain the desired result. \( \square \)

**Theorem 6.16.** Let \( n \geq 7 \) be an integer. Then we have
\[
d_{\text{one}}(n, 5) \geq \begin{cases} 
\left\lfloor \frac{16n}{31} \right\rfloor - 1, & \text{if } n \equiv 13, 15, 23, 27 \pmod{31}, \\
\left\lfloor \frac{16n}{31} \right\rfloor - 2, & \text{if } n \equiv 0, 1, 8, 12, 16 \pmod{31}.
\end{cases}
\]

**Proof.** By Table 1 and [22, Table 1], there is a binary \([n, 5, d]\) linear code with one-dimensional hull for \((n, d) \in S = \{(8, 2), (12, 4), (13, 5), (15, 6), (16, 6), (23, 10), (27, 12), (31, 14), (32, 14)\} \). For \((n_0, d_0) \in S\), applying Lemma 6.4 to the binary \([n_0, 5, d_0]\) linear code with one-dimensional hull, we can obtain the desired result. \( \square \)

Next, we consider the exact value of \( d_{\text{one}}(n, k) \) for \( k = n - 5 \).

**Theorem 6.17.** Let \( n \geq 6 \) be an integer. Then
\[
d_{\text{one}}(n, n-5) = \begin{cases} 
6, & \text{if } n = 6, \\
4, & \text{if } n \in \{7, 8, 10, 12\}, \\
3, & \text{if } n \in \{9, 11, 13, 14, \ldots, 27\}, \\
2, & \text{if } n \geq 28.
\end{cases}
\]

**Proof.** By Table 1 and [22, Table 1], \( d_{\text{one}}(6, 1) = 6, d_{\text{one}}(n, n-5) = 4 \) for \( n \in \{7, 8, 10, 12\} \), \( d_{\text{one}}(n, n-5) = 3 \) for \( n \in \{9, 11, 13, 14, \ldots, 27\} \), \( d_{\text{one}}(n, n-5) = 2 \) for \( n \in \{28, 29, 30\} \). By Lemma 3.1, \( d_{\text{one}}(31, 26) \leq d_{\text{LCP}}(30, 25) = 2 \) (see [3]). By the proof of Lemma 6.9, there is a binary \([31, 26, 2]\) linear code with one-dimensional hull. Thus \( d_{\text{one}}(31, 26) = 2 \). It follows from Lemma 6.9 that \( d_{\text{one}}(n, n-5) = 2 \) for \( n \geq 32 \). This completes the proof. \( \square \)

**7 Applications to EAQECCs**

In this section, we introduce some definitions about EAQECCs and construct some EAQECCs with new parameters. We use \([[n, k, d; c]]_2\) to denote a binary entanglement-assisted quantum error-correcting code that encodes \( k \) information qubits into \( n \) channel qubits with the help of \( c \) pre-shared entanglement pairs. EAQECCs were introduced by Brun et al. in [7], which include the standard quantum stabilizer codes as a special case.

**Proposition 7.1.** [16] Let \( C \) be a binary \([n, k, d]\) linear code and \( C^\perp \) be its dual code with parameters \([n, n-k, d^\perp]\), where \( d^\perp \) is the minimum distance of \( C^\perp \). Assume that \( \dim(Hull(C)) = 1 \). Then, there exist an \([[n, k-1, d; n-k-1]]_2\) EAQECC and an \([[n, n-k-1, d^\perp; k-1]]_2\) EAQECC.

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Example 7.2. The method we construct EAQECCs is the same as that of [37], which is to construct EAQECCs from binary linear codes. Therefore, we compare our results with [37]. By Table 1, there is a binary [20, 6, 8] linear code with one-dimensional hull. By Proposition 7.1, we obtain an EAQECC with parameters [[20, 5, 7; 13]]_2, which has a better minimum distance than the best known EAQECC [[20, 5, 7; 13]]_2 (see [37, Table 4]). More EAQECCs with different parameters are given in Table 5.

We also compare our results with [15] and [27]. By Table 1, there is a binary [26, 11, 8] linear code with one-dimensional hull. By Proposition 7.1, we obtain an EAQECC with parameters [[26, 10, 8; 14]]_2, which has a smaller amount of entanglement than the best known EAQECC [[26, 10, 8; 16]]_2 (see [15]). More EAQECCs with different parameters are given in Table 6.

| The known EAQECCs [37] | Our parameters (Tables 1-2) | The related EAQECCs |
|------------------------|-------------------------------|---------------------|
| [17, 5, 5; 8]_2 (Table 10) | [17, 8, 5]_2 | [17, 7, 5; 8]_2 |
| [18, 4, 7; 12]_2 (Table 10) | [18, 5, 8]_2 | [18, 4, 8; 12]_2 |
| [19, 3, 7; 11]_2 (Table 10) | [18, 6, 7]_2 | [18, 5, 7; 11]_2 |
| [19, 4, 6; 10]_2 (Table 10) | [18, 7, 6]_2 | [18, 6, 6; 10]_2 |
| [19, 4, 6; 8]_2 (Table 10) | [18, 9, 6]_2 | [18, 8, 6; 8]_2 |
| [19, 7, 5; 10]_2 (Table 10) | [19, 8, 6]_2 | [19, 7, 6; 10]_2 |
| [19, 7, 5; 8]_2 (Table 10) | [19, 10, 5]_2 | [19, 9, 5; 8]_2 |
| [20, 5, 7; 13]_2 (Table 4) | [20, 6, 8]_2 | [20, 5, 8; 13]_2 |
| [20, 8, 5; 10]_2 (Table 10) | [20, 9, 6]_2 | [20, 8, 6; 10]_2 |
| [21, 9, 5; 10]_2 (Table 5) | [21, 10, 6]_2 | [21, 9, 6; 10]_2 |
| [21, 14, 3; 5]_2 (Table 5) | [21, 15, 4]_2 | [21, 14, 4; 5]_2 |
| [22, 2, 11; 18]_2 (Table 4) | [22, 3, 12]_2 | [22, 2, 12; 18]_2 |
| [22, 3, 9; 17]_2 (Table 4) | [22, 4, 10]_2 | [22, 3, 10; 17]_2 |
| [22, 5, 8; 15]_2 (Table 4) | [22, 6, 9]_2 | [22, 5, 9; 15]_2 |
| [22, 6, 7; 14]_2 (Table 4) | [22, 7, 8]_2 | [22, 6, 8; 14]_2 |
| [22, 8, 6; 12]_2 (Table 4) | [22, 9, 7]_2 | [22, 8, 7; 12]_2 |
| [25, 18, 3; 5]_2 (Table 4) | [25, 19, 4]_2 | [25, 18, 4; 5]_2 |

8 Conclusion

We have studied some properties of binary linear codes with one-dimensional hull, and have established the connection between binary LCD codes and binary linear codes with one-dimensional hull. We have completely determined the exact value of \( d_{ome}(n, k) \) for \( k \in \{1, 3, 4, n - 5, n - 4, n - 3, n - 2, n - 1\} \) and partially determined the exact value of \( d_{ome}(n, 5) \), which generalized a result of Mankean and Jitman [28]. Furthermore, we have extended Kim’s result [22] on \( d_{ome}(n, k) \) (1 \( \leq k \leq n \leq 13 \)) to lengths up to 30. Finally, as an application, we constructed some EAQECCs with different parameters compared with [15, 27, 37].

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Table 6: \([n, k; d; c]\), EAQECCs

| The known EAQECCs | Our parameters (Table 4) | The related EAQECCs |
|-------------------|--------------------------|---------------------|
| \([25, 10, 5; 0]\) | \([25, 11, 8]\)         | \([25, 10, 8; 13]\) |
| \([25, 10, 6; 3]\) |                          |                     |
| \([25, 10, 7; 6]\) |                          |                     |
| \([25, 8, 6; 10]\) | \([26, 9, 9]\)          | \([26, 8, 9; 16]\)  |
| \([25, 8, 7; 4]\) |                          |                     |
| \([25, 8, 8; 8]\) |                          |                     |
| \([25, 11, 8]\)   |                          |                     |
| \([26, 10, 8; 16]\) | \([26, 11, 8]\)         | \([26, 10, 8; 14]\) |
| \([27, 6, 10; 12]\) | \([27, 7, 11]\)        | \([27, 6, 11; 19]\) |
| \([27, 7, 9; 11]\) | \([27, 8, 10]\)        | \([27, 7, 10; 18]\) |
| \([27, 8, 9; 16]\) | \([27, 9, 10]\)        | \([27, 8, 10; 17]\) |
| \([27, 12, 7; 7]\) | \([27, 13, 8]\)        | \([27, 12, 8; 13]\) |
| \([28, 13, 12; 12]\) | \([28, 4, 14]\)      | \([28, 3, 14; 23]\) |
| \([28, 4, 12; 12]\) | \([28, 5, 13]\)        | \([28, 4, 13; 22]\) |
| \([28, 6, 10; 12]\) | \([28, 7, 12]\)        | \([28, 6, 12; 20]\) |
| \([29, 4, 12; 4]\) | \([29, 5, 14]\)        | \([29, 4, 14; 23]\) |
| \([29, 13, 7; 6]\) | \([29, 14, 8]\)        | \([29, 13, 8; 14]\) |
| \([30, 5, 12; 16]\) | \([30, 6, 13]\)        | \([30, 5, 13; 23]\) |
| \([30, 18, 5; 4]\) | \([30, 19, 6]\)        | \([30, 18, 6; 10]\) |

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