Korn inequality on irregular domains

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Abstract. In this paper, we study the weighted Korn inequality on some irregular domains, e.g., s-John domains and domains satisfying quasi-hyperbolic boundary conditions. Examples regarding sharpness of the Korn inequality on these domains are presented. Moreover, we show that Korn inequalities imply certain Poincaré inequality.

1 Introduction

Let $\Omega$ be a bounded domain of $\mathbb{R}^n$, $n \geq 2$. For each vector $\mathbf{v} = (v_1, \cdots, v_n) \in W^{1,p}(\Omega)^n$, let $D\mathbf{v}$ denotes its gradient matrix, and $\epsilon(\mathbf{v})$ denotes the symmetric part of $D\mathbf{v}$, i.e., $\epsilon(\mathbf{v}) = (\epsilon_{i,j}(\mathbf{v}))_{1 \leq i, j \leq n}$ with

$$\epsilon_{i,j}(\mathbf{v}) = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right).$$

Korn’s (second) inequality states that, if $\Omega$ is sufficient regular (e.g., Lipschitz), then there exists $C > 0$ such that

$$(K_p) \quad \int_{\Omega} |D\mathbf{v}|^p \, dx \leq C \left\{ \int_{\Omega} |\epsilon(\mathbf{v})|^p \, dx + \int_{\Omega} |\mathbf{v}|^p \, dx \right\}.$$

The Korn inequality $(K_p)$ is a fundamental tool in the theory of linear elasticity equations; see [1, 2, 6, 8, 9, 11, 18, 23] and the references therein. Notice that Korn inequality $(K_p)$ fails for $p = 1$ even on a cube; see the example from [5].

On $\mathbb{R}^2$ and $p = 2$, several different inequalities (including the Friedrichs’ inequality) are actually equivalent to Korn’s inequality $(K_2)$ on simply connected Lipschitz domains; see [12, 23] for example.

Friedrichs [9] proved the Korn inequality $(K_p)$ for $p = 2$ on domains with a finite number of corners or edges on $\partial \Omega$. Nitsche [21] proved the Korn inequality $(K_p)$ for $p = 2$ on Lipschitz domains, while Ting [24] proved $(K_p)$ for all $p \in (1, \infty)$ by using Calderón-Zygmund theory; Kon-dratiev and Oleinik [18] studied the Korn inequality $(K_2)$ on star-shaped domains. Recently,
Acosta, Durán and Muschietti [1] proved the Korn inequality \((K_p)\) holds for all \(p \in (1, \infty)\) on John domains.

Weighted Korn inequality on irregular domains (in particular, Hölder domains) have received considerable interest recently; see [1, 2, 3, 6, 18] and references therein. Motivated by this, in this paper, we study weighted Korn inequality on some irregular domains including \(s\)-John domains \((s \geq 1)\) and domains satisfying quasihyperbolic boundary conditions.

We use the divergence equation as the main tool, which is intimately connected to the weighted Poincaré inequality; for the recent progress see [1, 4, 7, 15]. Our approach is in particular motivated by [7]. Let \(\Omega\) be a bounded domain in \(\mathbb{R}^n\). From [7, Theorem 4.1] the validity of Poincaré inequality

\[
\int_{\Omega} |u(x) - u_{\Omega}|^p \, dx \leq C \int_{\Omega} |\nabla u(x)|^p \, \text{dist} \,(x, \partial \Omega)^b \, dx,
\]

implies certain regularity of solutions to the divergence equation \(\text{div} \, u = f\). Then by using duality, one gets the (weighted) Korn inequality; see [6] for instance. We in Section 2 will generalize the arguments to more general settings to obtain the (weighted) Korn inequality.

We in Section 3 will go to \(s\)-John domains and domains satisfying quasi-hyperbolic boundary conditions, respectively. By using Poincaré inequalities on these domains, we deduce the (weighted) Korn inequalities on them. Moreover, we will show the obtained (weighted) Korn inequalities are sharp by presenting some counter-examples in Section 4.

Another interesting question is what is the geometric counterpart of the Korn inequality. In general the Korn inequality \((K_p)\) does not imply any Poincaré inequality. Indeed, if \(\Omega_1, \Omega_2 \subset \mathbb{R}^n, \Omega_1 \cap \Omega_2 = \emptyset\), are two domains that support the Korn inequality \((K_p)\), then \(\Omega := \Omega_1 \cup \Omega_2\) admits the Korn inequality \((K_p)\) also. However, Poincaré inequality does not have this property.

The paper is organized as follows. In Section 2, we will show that, abstractly, weighted Poincaré inequality implies a weighted Korn inequality; conversely, Korn inequality \((\tilde{K}_p)\) also implies a Poincaré inequality. In Section 3, we establish the Korn inequality on \(s\)-John domains and domains satisfying quasihyperbolic boundary conditions, and present examples for the sharpness of the Korn inequality in Section 4.

Throughout the paper, we denote by \(C\) positive constants which are independent of the main parameters, but which may vary from line to line. For \(p \in [1, n)\), denote its Sobolev conjugate \(\frac{np}{n-p}\) by \(p^*\). Corresponding to to a function space \(X\), we denote its \(n\)-vector valued spaces by \(X^{n^*}\). We will usually omit the superscript \(n\) or \(n \times n\) for simplicity.
2 Korn inequality and Poincaré inequality

In this section, we show that, abstractly, Poincaré inequality implies Korn inequality; and conversely, certain Korn inequality implies a Poincaré inequality.

Throughout the paper, let \( \rho(x) \) be the distance from \( x \) to the boundary \( \partial \Omega \), i.e., \( \rho(x) := \text{dist}(x, \partial \Omega) \). Let \( a, b \in \mathbb{R} \) and \( p \in [1, \infty) \), the weighted Lebesgue space \( L^p(\Omega, \rho^a) \) is defined as set of all measurable functions \( f \) in \( \Omega \) such that

\[
\|f\|_{L^p(\Omega, \rho^a)} := \left( \int_{\Omega} |f(x)|^p \rho(x)^a \, dx \right)^{1/p} < \infty.
\]

We denote by \( L^p_0(\Omega, \rho^a) \) the set of functions \( f \in L^p(\Omega, \rho^a) \) with \( \int_{\Omega} f(x) \rho(x)^a \, dx = 0 \).

The weighted Sobolev space \( W^{1,p}(\Omega, \rho^a, \rho^b) \) is defined as

\[
W^{1,p}(\Omega, \rho^a, \rho^b) := \left\{ u \in L^p(\Omega, \rho^a) : \nabla u \in L^p(\Omega, \rho^b) \right\}
\]

with the norm

\[
\|u\|_{W^{1,p}(\Omega, \rho^a, \rho^b)} := \|u\|_{L^p(\Omega, \rho^a)} + \|\nabla u\|_{L^p(\Omega, \rho^b)}.
\]

We denote \( W^{1,p}(\Omega, \rho^a, \rho^b) \) by \( W^{1,p}(\Omega, \rho^a) \), and denote \( W^{1,p}(\Omega, \rho^a) \) by \( W^{1,p}(\Omega) \) if \( a = 0 \).

Notice that as \( \rho^a \) and \( \rho^b \) are continuous positive functions in \( \Omega \), smooth functions \( C^\infty(\Omega) \cap W^{1,p}(\Omega, \rho^a, \rho^b) \) is dense in \( W^{1,p}(\Omega, \rho^a, \rho^b) \); see [10, Theorem 3].

Let \( p \geq 1 \) and \( a \geq 0 \). We say that the \( (P_{a,b}) \)-Poincaré inequality holds, if there exists \( C > 0 \) such that for every \( u \in W^{1,p}(\Omega, \rho^a, \rho^b) \), it holds

\[
(P_{a,b}) \quad \int_{\Omega} |u(x) - u_{\Omega,a}|^p \rho(x)^a \, dx \leq C \int_{\Omega} |\nabla u(x)|^p \rho(x)^b \, dx,
\]

where we denote by \( u_{\Omega,a} := \frac{1}{\lambda_{\Omega,a}^\rho^a} \int_{\Omega} u \rho^a \, dx \) and \( u_{\Omega} := u_{\Omega,a} \) for \( a = 0 \).

2.1 Korn inequality from Poincaré inequality

In this subsection we will prove that Poincaré inequality implies Korn inequality and in the following Section 3 we will provide examples which show sharpness of our results.

**Theorem 2.1.** Let \( \Omega \) be a bounded domain of \( \mathbb{R}^n \), \( n \geq 2 \). Let \( p > 1 \), \( a \geq 0 \) and \( b \in \mathbb{R} \). Suppose the \( (P_{a,b}) \)-Poincaré inequality holds on \( \Omega \). Then for an arbitrarily fixed cube \( Q \subset \subset \Omega \), there exists \( C = C(p, a, b, \Omega, Q) \) such that for every \( v \in W^{1,p}(\Omega, \rho^a)^n \), the following inequality holds

\[
(\overline{K}_{p,a,b-p}) \quad \int_{\Omega} |Dv(x)|^p \rho(x)^a \, dx \leq C \left\{ \int_{\Omega} |\epsilon(v)(x)|^p \rho(x)^{b-p} \, dx + \int_{Q} |Dv(x)|^p \rho(x)^a \, dx \right\}.
\]
**Remark 2.1.** If \( a = 0 \) and \( b = p \), then \( (\overline{K}_{p,a,b-\rho}) \) implies \((K_p)\); see Kondratiev and Oleinik [18]. Indeed, as \( Q \subset \Omega \), it holds \( \text{dist}(Q, \partial \Omega) \leq \rho(x) \leq \text{diam}(\Omega) \) for each \( x \in Q \). Since the Korn inequality \((K_p)\) holds on cubes, we always have

\[
\int_Q |Dv|^p \rho^a \, dx \leq C(a,\Omega,Q) \int_Q |Dv|^p \, dx \leq C(a,\Omega,Q) \left( \int_Q |e(v)|^p \, dx + \int_Q |v|^p \, dx \right)
\]

\[
\leq C(p,a,b,\Omega,Q) \left( \int_Q |e(v)|^p \rho^{b-p} \, dx + \int_Q |v|^p \rho^a \, dx \right).
\]

Thus \((\overline{K}_{p,a,b-\rho})\) above implies that

\[
\|Dv\|_{L^p(\Omega,\rho^a)} \leq C(p,a,b,\Omega,Q) \left( \|e(v)\|_{L^p(\Omega,\rho^{b-p})} + \|v\|_{L^p(\Omega,\rho^a)} \right)
\]

and hence

\[
(K_{p,a,b-\rho}) \quad \|Dv\|_{L^p(\Omega,\rho^a)} \leq C(p,a,b,\Omega,Q) \left( \|e(v)\|_{L^p(\Omega,\rho^{b-p})} + \|v\|_{L^p(\Omega,\rho^a)} \right),
\]

which is the usual Korn inequality \((K_p)\) if \( a = 0 \) and \( b = p \).

We employ the divergence equation to prove the previous theorem.

Let \( p,q \in (1,\infty) \) satisfying \( 1/q + 1/p = 1 \), and \( \Omega \) be a bounded domain in \( \mathbb{R}^n \). A vector function \( u \) is called a solution to the divergence equation \( \nabla \cdot u = f \) for some \( f \in L^p_0(\Omega,\rho^a) \), if for every \( \phi \in W^{1,q}(\Omega,\rho^a,\rho^b) \) it holds that

\[
(\nabla_{p,a,b}) \quad \int_{\Omega} u(x) \cdot \nabla \phi(x) \, dx = \int_{\Omega} f(x)\phi(x)\rho^a(x) \, dx.
\]

Recall that \( C^\infty(\Omega) \cap W^{1,q}(\Omega,\rho^a,\rho^b) \) is dense in \( W^{1,q}(\Omega,\rho^a,\rho^b) \).

**Proposition 2.1.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \), \( p,q \in (1,\infty) \) with \( 1/p + 1/q = 1 \), \( a \geq 0 \) and \( b \in \mathbb{R} \). Suppose that \( \Omega \) supports a \((P_{p,a,b})\)-Poincaré inequality, then for each \( f \in L^p_0(\Omega,\rho^a) \), there exists \( u \in W^{1,q}(\Omega,\rho^{-q/b}p,\rho^{a-q/b}p) \) such that

\[
\nabla \cdot u = f \quad \text{in} \quad D'(\Omega,\rho^a)
\]

and

\[
\|D_{\rho^a}u\|_{L^q(\Omega,\rho^{-q/b}p)} \leq C\|f\|_{L^p(\Omega,\rho^a)}.
\]

where \( C = C(n, p, a, b) > 0 \).

**Proof.** The case \( a = 0 \) is obtained in [7, Theorem 4.1]; the proof of the case \( a > 0 \) is essentially the same as the case \( a = 0 \) in [7], we outline the proof here.

For \( f \in L^p_0(\Omega,\rho^a) \), by using the \((P_{p,a,b})\)-Poincaré inequality, similarly as [7, Proposition 3.2], we conclude that there exists a solution \( u \) to the equation \( \nabla \cdot u = f \) in \( D'(\Omega,\rho^a) \) such that

\[
\|u\|_{L^p(\Omega,\rho^{-q/b}p)} \leq \|f\|_{L^p(\Omega,\rho^a)}.
\]
Let \( \{Q_j\}_j \) be a Whitney decomposition of \( \Omega \). Similar to [7, Proposition 4.2], we obtain a decomposition of \( f \) as
\[
f(x) \rho(x)^a = \sum_{j \in I} f_j(x),
\]
where \( \{f_j\} \) satisfies:
(i) \( \text{supp} \ f_j \subset 2Q_j \);
(ii) \( \int_{2Q_j} f_j(x) \, dx = 0 \);
(iii) \( \sum_j \int_{2Q_j} |f_j(x)|^q \rho(x)^{q-b/p} \, dx \leq C \int_\Omega |f(x)|^q \rho(x)^a \, dx \) for some \( C = C(\Omega, p, a, b) \).

For each \( j \), by [4, Theorem 2], there exists \( u_j \in W^{1,q}_0(2Q_j)^n \) such that \( \text{div} \, u_j = f_j \) and
\[
\|Du_j\|_{L^q(2Q_j)} \leq C(q)\|f_j\|_{L^q(2Q_j)}.
\]

Denote \( u := \sum_j u_j \). Since the dilations of Whitney cubes have bounded overlap, one easily see that \( \text{div} \, u = f \) holds in \( \Omega \). Indeed, for each \( \phi \in C^\infty(\Omega) \),
\[
\int_\Omega \text{div} \, u \phi \, dx = \sum_j \int_\Omega u_j(x) \cdot \nabla \phi \, dx = \int_\Omega \sum_j f_j(x) \cdot \phi \, dx = \int_\Omega f(x) \phi \rho(x)^a \, dx.
\]
Moreover, by using the property of Whitney decomposition again, i.e., \( \rho(x) \sim \ell(Q_j) \) for each \( x \in 2Q_j \) and each \( j \), we further deduce that
\[
\|Du\|_{L^q(\Omega, \rho^{-q-qb/p})} \leq \sum_j \int_{2Q_j} |Du_j(x)|^q \rho(x)^{q-b/p} \, dx
\leq C \sum_j \ell(Q_j)^{q-b/p} \int_{2Q_j} |Du_j(x)|^q \, dx
\leq C \sum_j \ell(Q_j)^{q-b/p} \int_{2Q_j} |f_j(x)|^q \, dx
\leq C \sum_j \int_{2Q_j} |f_j(x)|^q \rho(x)^{q-b/p} \, dx
\leq C \int_\Omega |f(x)|^q \rho(x)^a \, dx,
\]
which completes the proof. \( \Box \)

**Proof of Theorem 2.1.** Recall that \( Dv = (\frac{\partial v_i}{\partial x_j})_{1 \leq i, j \leq n} \), \( 1 \leq i, j \leq n \), and \( e(v) = (e_{i,j}(v))_{1 \leq i, j \leq n} \) with
\[
e_{i,j} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)
\]
and the identity
\[
\frac{\partial^2 v_i}{\partial x_j \partial x_k} = \frac{\partial e_{i,k}}{\partial x_j} + \frac{\partial e_{i,j}}{\partial x_k} - \frac{\partial e_{j,k}}{\partial x_i}.
\]
By this and using the properties of solutions to the divergence equations (Proposition 2.1), we see that for each \( f \in L^q_0(\Omega, \mu^p) \) following holds

\[
\left| \int_{\Omega} f(x)\rho(x)^p \left( \frac{\partial v_j}{\partial x_i} - \left( \frac{\partial v_j}{\partial x_i} \right)_\Omega \right) dx \right| = \left| \int_{\Omega} \text{div} \, u(x) \left( \frac{\partial v_j}{\partial x_i} - \left( \frac{\partial v_j}{\partial x_i} \right)_\Omega \right) dx \right|
\]

\[
= \left| \int_{\Omega} u(x) \cdot \nabla v_j(x) dx \right|
\]

\[
\leq \|Du\|_{L^q(\Omega, \mu^p)} \|\epsilon(v)\|_{L^p(\Omega, \mu^p)}
\]

\[
\leq C \|f\|_{L^q(\Omega, \mu^p)} \|\epsilon(v)\|_{L^p(\Omega, \mu^p)},
\]

which implies that

\( (2.1) \)

\[
\left\| \frac{\partial v_j}{\partial x_i} - \left( \frac{\partial v_j}{\partial x_i} \right)_\Omega \right\|_{L^p(\Omega, \mu^p)} \leq C \|\epsilon(v)\|_{L^p(\Omega, \mu^p)},
\]

Now for an arbitrarily fixed cube \( Q \subseteq \Omega \), we choose a \( \psi \in C_0^\infty(Q) \) such that \( \text{supp} \, \psi \subseteq Q, \int_Q \psi \, dx = 1 \) and \( |\nabla \psi| \leq C/\ell(Q)^{a+1} \). Write

\( (2.2) \)

\[
\frac{\partial v_j}{\partial x_i} = \frac{\partial v_j}{\partial x_i} - \left( \frac{\partial v_j}{\partial x_i} \right)_\Omega + \int_Q \left[ \frac{\partial v_j}{\partial x_i} - \frac{\partial v_j}{\partial x_i} \right] \psi \, dx + \int_Q \frac{\partial v_j}{\partial x_i} \psi \, dx.
\]

Then from the Hölder inequality, we obtain

\[
\left| \int_Q \left[ \frac{\partial v_j}{\partial x_i} - \frac{\partial v_j}{\partial x_i} \right] \psi \, dx \right| \leq C(a, p, Q, \Omega) \|\epsilon(v)\|_{L^p(\Omega, \mu^p)}
\]

and

\[
\left| \int_Q \frac{\partial v_j}{\partial x_i} \psi(x) \, dx \right| \leq C(a, p, Q, \Omega) \left\| \frac{\partial v_j}{\partial x_i} \right\|_{L^p(Q, \mu^p)}.
\]

Combining (2.1), (2.2) and the above estimates, we obtain that

\[
\left\| \frac{\partial v_j}{\partial x_i} \right\|_{L^p(\Omega, \mu^p)} \leq C(p, a, b, \Omega, Q) \left\{ \|\epsilon(v)\|_{L^p(\Omega, \mu^p)} + \left\| \frac{\partial v_j}{\partial x_i} \right\|_{L^p(Q, \mu^p)} \right\},
\]

which is

\[
(\overline{K}_{p,a,b-p}) \quad \|Dv\|_{L^p(\Omega, \mu^p)} \leq C(p, a, b, \Omega, Q) \left\{ \|\epsilon(v)\|_{L^p(\Omega, \mu^p)} + \|Dv\|_{L^p(Q, \mu^p)} \right\}.
\]

The proof is completed. \( \square \)
2.2 Korn inequality implies Poincaré inequality

From the previous subsection, we know that the Poincaré inequality implies Korn inequality, and in this section we will prove a partial converse result.

**Theorem 2.2.** Let \( \Omega \) be a bounded domain of \( \mathbb{R}^n \), \( n \geq 2 \). Let \( p > 1 \) and \( Q \subset \Omega \) be a closed cube. Suppose that for all \( v \in W^{1,p}(\Omega)^n \) it holds that

\[
\|Dv\|_{L^p(\Omega)} \leq C \left( \|\epsilon(v)\|_{L^p(\Omega)} + \|Dv\|_{L^p(Q)} \right),
\]

then there exists \( C > 0 \) such that for all \( u \in W^{1,p}(\Omega) \), it holds

\[
(\tilde{P}_p) \quad \int_{\Omega} |u(x) - u_\Omega|^p \, dx \leq C \int_{\Omega} |\nabla u(x)|^p \, dx.
\]

Notice that the Poincaré inequality \((\tilde{P}_p)\) is weaker than \((P_p)\).

We will need the following characterization of weighted Poincaré inequality from Hajłasz and Koskela [10, Theorem 1] (for non-weighted cases see Maz'ya [25]). A subset \( A \subset \Omega \) is admissible if \( A \) is open and \( \partial A \cap \Omega \) is a smooth submanifold.

**Theorem 2.3** ([10]). Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \), \( n \geq 2 \). Let \( p \geq 1 \) and \( a, b \in \mathbb{R} \). Then the following conditions are equivalent.

(i) There exists a constant \( C > 0 \) such that, for every \( u \in C^\infty(\Omega) \) it holds that

\[
\int_{\Omega} |u(x) - u_\Omega|^p \rho(x)^a \, dx \leq C \int_{\Omega} |\nabla u(x)|^p \rho(x)^b \, dx.
\]

(ii) For an arbitrary cube \( Q \subset \subset \Omega \), there exists a constant \( C = C(Q) > 0 \) such that

\[
(2.3) \quad \int_A \rho(x)^a \, dx \leq C \inf_u \int_{\Omega} |\nabla u(x)|^p \rho(x)^b \, dx
\]

for every admissible set \( A \subset \Omega \) with \( A \cap Q = \emptyset \). Here the infimum is taken over the set of all \( u \in C^\infty(\Omega) \) that satisfy \( u|_A = 1 \) and \( u|_Q = 0 \).

We next prove Theorem 2.2.

**Proof of Theorem 2.2.** We only need to verify that the second condition of Theorem 2.3 holds. Assume that \((\tilde{K}_p)\) holds. Fix a \( y = (y_1, y_2, \cdots, y_n) \in \Omega \).

Let \( A \subset \Omega \) with \( A \cap Q = \emptyset \) be an admissible set, and \( u \in C^\infty(\Omega) \) that satisfies \( u|_A = 1 \) and \( u|_Q = 0 \).

For each \( x = (x_1, \cdots, x_n) \in \Omega \) let \( v = (v_1, v_2, 0, \cdots, 0) \) with

\[
\begin{align*}
  v_1(x_1, \cdots, x_n) &= (x_2 - y_2)u(x_1, \cdots, x_n), \\
  v_2(x_1, \cdots, x_n) &= (y_1 - x_1)u(x_1, \cdots, x_n),
\end{align*}
\]

and...
Then for each $x = (x_1, \cdots, x_n) \in A$,

$$Dv(x) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

and for $x \in Q$, $Dv(x) = 0$. These imply that

$$\|Dv\|_{L^p(\Omega)}^p \geq \int_A dx$$

and $\|D(v)\|_{L^p(Q)} = 0$. On the other hand, for every $x = (x_1, x_2, \cdots, x_n) \in \Omega$, it holds

$$\left| (x_2 - y_2) \frac{\partial v}{\partial x_2} - u + (x_2 - y_2) \frac{\partial u}{\partial x_2} + (x_2 - y_2) \frac{\partial v}{\partial x_i} \right| = \left| \frac{\partial u}{\partial x_2} v \right| + \left| \frac{\partial v}{\partial x_2} u \right| \leq \|\nabla u\|_{L^p(\Omega)} \|\nabla v\|_{L^p(\Omega)} \|v\|_{L^p(\Omega)}$$

which implies that

$$\|v\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)} \|v\|_{L^p(\Omega)} \leq C \text{diam}(\Omega) \|\nabla u\|_{L^p(\Omega)}.$$
3 Korn inequality on general domains

In this section, we are going to study the Korn inequality on some irregular domains. If \( \Omega \) is an \( \alpha \)-Hölder domain for some \( \alpha \in (0, 1] \), it is then proved in [2] that there is a constant \( C = C(n, p, \Omega, \alpha) > 0 \) such that for every \( v \in W^{1,p}(\Omega, \rho^a) \), it holds

\[
\int_{\Omega} |Dv|^p \rho^a \, dx \leq C \left\{ \int_{\Omega} |\epsilon(v)|^p \rho^{b-p} \, dx + \int_{\Omega} |v|^p \rho^a \, dx \right\},
\]

where \( 0 \leq a = b - \alpha p \). See [2, 3] for more on this aspect and the counterexample for sharpness.

We next focus on two kinds of irregular domains: \( s \)-John domains and quasi-hyperbolic domains.

3.1 \( s \)-John domains

Let us first recall the definition of \( s \)-John domain.

**Definition 3.1** (\( s \)-John domain). A bounded domain \( \Omega \subset \mathbb{R}^n \) with a distinguished point \( x_0 \in \Omega \) called an \( s \)-John domain, \( s \geq 1 \), if there exists a constant \( C > 0 \) such that for all \( x \in \Omega \), there is a curve \( \gamma : [0, l] \to \Omega \) parametrised by arclength such that \( \gamma(0) = x \), \( \gamma(l) = x_0 \), and \( d(\gamma(t), \mathbb{R}^n \setminus \Omega) \geq Ct^s \).

If \( s = 1 \) then we say that \( \Omega \) is a John domain for simplicity. John domains were introduced by Martio and Sarvas [20], F. John [16] had earlier considered a similar class of domains.

The following Poincaré inequality is a special case from [10, 17].

**Theorem 3.1.** Suppose that \( \Omega \) is an \( s \)-John domain, \( s \geq 1 \) and \( a \geq 0 \). Then there is a constant \( C = C(n, p, \Omega, a, b) > 0 \) such that

\[
\left( \int_{\Omega} |u - u_{\Omega,a}|^p \rho^a \, dx \right)^{1/p} \leq C \left( \int_{\Omega} |\nabla u|^p \rho^b \, dx \right)^{1/p}
\]

for each \( u \in C^\infty(\Omega) \), where \( n + a \geq s(n + b - 1) - p + 1 \).

We have the corresponding Korn inequality on \( s \)-John domains.

**Theorem 3.2.** Suppose that \( \Omega \) is an \( s \)-John domain with \( s \geq 1 \), and \( a \geq 0 \). Then there is a constant \( C = C(n, p, \Omega, a, b) > 0 \) such that for every \( v \in W^{1,p}(\Omega, \rho^a) \), it holds

\[
(K_{p,a,b-p}) \quad \int_{\Omega} |Dv|^p \rho^a \, dx \leq C \left\{ \int_{\Omega} |\epsilon(v)|^p \rho^{b-p} \, dx + \int_{\Omega} |v|^p \rho^a \, dx \right\},
\]

where \( a \geq 0 \) and \( n + a \geq s(n + b - 1) - p + 1 \).

Moreover, for \( a \geq 0 \) and \( n + a < s(n + b - 1) - p + 1 \), there exists a domain \( \Omega \) which does not support the Korn inequality \((K_{p,a,b-p})\).
Proof. By using Theorem 2.1 and the Poincaré inequality (Theorem 3.1), we see that the Korn inequality \((K_p,a,b-p)\) holds if \(n + a \geq s(n + b - 1) - p + 1\). The converse part follows from the Example 4.1(1) in Section 4.\(\square\)

Remark 3.1. For the case \(s = 1\), i.e., on the John domain, we can then take \(a = 0\) and \(b = p\), and obtain the usual Korn inequality. This gives an another proof of [1, Theorem 4.2]

3.2 Quasihyperbolic domains

Let \(\Omega\) be a proper domain in \(\mathbb{R}^n, n \geq 2\). By quasihyperbolic metric we mean that for all \(x, y \in \Omega\),

\[k(x, y) := \inf_\gamma \int_\gamma \frac{1}{\text{dist}(z, \partial \Omega)} \, ds(z),\]

where the infimum is taken over all curves \(\gamma\) joining \(x\) to \(y\) in \(\Omega\). The quasihyperbolic metric arises naturally in the theory of conformal geometry and plays an important role for example in the study of the boundary behavior of quasiconformal maps.

Our domain \(\Omega\) is said to satisfy a \(\beta\)-quasihyperbolic boundary condition (for short, \(\beta\)-QHBC), if for some fixed base point \(x_0\) there exists \(C_0 < \infty\) such that for every \(x \in \Omega\)

\[k(x, x_0) \leq \frac{1}{\beta} \log \frac{\text{dist}(x_0, \partial \Omega)}{\text{dist}(x, \partial \Omega)} + C_0.\]

Changing the base point \(x_0\) changes the constant \(C_0\).

We first establish the following weighted Poincaré inequality on these domains; for non-weighted cases see [22, 19, 14], and recent paper [13] for \((q, p)\)-Poincaré inequality with \(q < p\).

Let \(\mathcal{W}\) be a Whitney decomposition of \(\Omega\). We may and do assume that the basepoint \(x_0\) is the center of some \(Q \in \mathcal{W}\). For each \(Q \in \mathcal{W}\), we choose a quasihyperbolic geodesic \(\gamma\) joining \(x_0\) to the center of \(Q\) and let \(P(Q)\) denote the collection of all of Whitney cubes that intersect \(\gamma\). The shadow of the cube \(Q \in \mathcal{W}\) is the set

\[S(Q) := \bigcup_{Q_1 \in P(Q)} Q_1.\]

We have the following estimate for the shadow of a cube from [14].

Lemma 3.1 ([14]). Let \(\Omega\) satisfy the \(\beta\)-quasihyperbolic boundary condition, for some \(\beta \leq 1\). There exists a constant \(C = C(n, C_0)\) such that for all \(Q \in \mathcal{W}\)

\[\text{diam}(S(Q)) \leq C \text{dist}(x_0, \partial \Omega)^{\frac{1-\beta}{1-\beta/n}} \text{diam}(Q)^{\frac{2\beta}{1-\beta/n}}.\]

Theorem 3.3. Let \(\Omega \subset \mathbb{R}^n\) be a proper subdomain satisfying a \(\beta\)-quasihyperbolic boundary condition, for some \(\beta \leq 1\). Then there is a constant \(C = C(n, p, q, \beta, \Omega) > 0\) such that

\[\left(\int_{\Omega} |u - u_{\Omega, a}|^q \rho^a \, dx\right)^{1/p} \leq C \left(\int_{\Omega} |\nabla u|^p \rho^b \, dx\right)^{1/p}.\]
for each \( u \in C^\infty(\Omega) \), where \( 1 \leq p \leq q < \infty \), \( a \geq 0 \),

\[
\frac{a + n}{q} \frac{2\beta}{1 + \beta} + \frac{p - n - b}{p} > 0;
\]

additionally, \( q \leq \frac{np}{n-p} \) if \( p < n \).

**Proof.** For \( p = 1 \), the same proof as [10, Proof of theorem 7] applies with Lemma 3.1 replacing the \( s \)-John condition there.

For \( p > 1 \), the proof is similar to the proof of theorem 3.2 in [19] with small modifications from [14]. We will verify condition (ii) of Theorem 2.3. Let \( \mathcal{W} \) be a Whitney decomposition of \( \Omega \). Let \( A \) be an admissible set and \( Q_0 \) some fixed cube. Let \( u \) be a smooth test function which equals 1 on \( A \) and 0 on \( Q_0 \).

We split our set \( A \) to two parts

\[
A_g = \{ x \in A : u_Q \leq \frac{1}{2} \text{ for some Whitney cube } Q \ni x \}
\]

and \( A_b = A \setminus A_g \). For all points \( x \in A_g \) with \( x \in Q \in \mathcal{W} \), from the properties of the Whitney decomposition, we have \( \rho(x) \sim \ell(Q) \), and hence

\[
\frac{1}{2} \left( \int_{Q \cap A} \rho(x)^a \, dx \right)^{\frac{2}{a}} \leq C \ell(Q)^{\frac{ap}{q}} \left( \int_Q |u(x) - u_Q|^q \, dx \right)^{\frac{2}{q}}
\]

\[
\leq C \ell(Q)^{\frac{ap}{q} + 1 - n + \frac{mn}{q} - b} \int_Q |

\nabla u(x)|^p \rho(x)^b \, dx
\]

\[
\leq C \text{ diam } (\Omega)^{\frac{mp}{q} + 1 - n + \frac{mn}{q} - b} \int_Q |

\nabla u(x)|^p \rho(x)^b \, dx,
\]

where \( \frac{ap}{q} + 1 - n + \frac{mn}{q} - b \geq p \left( \frac{a + n}{q} \frac{2\beta}{1 + \beta} + \frac{p - n - b}{p} \right) \geq 0 \).

Summing over all such cubes \( Q \), as \( q \geq p \), we obtain

\[
(3.1) \quad \int_\Omega |\nabla u|^p \rho(x)^b \, dx \geq C^{-1} \left( \int_{A_g} \rho(x)^a \, dx \right)^{\frac{2}{a}}.
\]

Next we estimate the integral over the bad set. For each \( x \in A_b \), let \( P(Q(x)) \) consist of the collection of all of the Whitney cubes which intersect the quasihyperbolic geodesic joining \( x_0 \) to the center of \( Q(x) \), then a straightforward chaining argument shows that

\[
C \sum_{Q \in P(Q(x))} \text{ diam } Q \int_Q |\nabla u(y)| \, dy \geq 1.
\]

Hence, by using the H"older inequality, we have

\[
\int_{A_b} \rho(x)^a \, dx
\]
Lemma 3.2.
The proof is completed by combining the above estimate together with (3.1).

\[
\begin{align*}
\leq C \int_{A_b} \rho(x)^a \sum_{Q \in P(A_b)} (\text{diam } Q) \left( \int_Q |\nabla u(y)|^p \, dy \right)^{1/p} \, dx \\
= C \sum_{Q \in W} \int_{S(Q) \cap A_b} \rho(x)^a \, dx \left( \int_Q |\nabla u(y)|^p \, dy \right)^{1/p} \\
\leq C \sum_{Q \in W} \int_{S(Q) \cap A_b} \rho(x)^a \, dx \left( \text{diam } Q \right)^{1-p/p} \left( \int_Q |\nabla u(y)|^p \, dy \right)^{1/p} \\
\leq C \left( \sum_{Q \in W} \left( \int_{S(Q) \cap A_b} \rho(x)^a \, dx \right)^{p'} |Q|^{\left( \frac{1\cdot q}{p} - \frac{a}{q} \right)p'} \right)^{\frac{1}{p'}} \left( \int_\Omega |\nabla u(y)|^p \, dy \right)^{\frac{1}{p}}.
\end{align*}
\]

This together with the following Lemma 3.2 gives that

\[
\left( \int_{A_b} \rho(x)^a \, dx \right)^{1/q} \leq C \left( \int_\Omega |\nabla u(y)|^p \, dy \right)^{1/p}.
\]

The proof is completed by combining the above estimate together with (3.1). \hfill \Box

**Lemma 3.2.** With the assumptions of Theorem 3.3 and \( p > 1 \), we have

\[
\sum_{Q \in W} \left( \int_{S(Q) \cap A_b} \rho(x)^a \, dx \right)^{p'} |Q|^{\left( \frac{1\cdot q}{p} - \frac{a}{q} \right)p'} \leq C \left( \int_{A_b} \rho(x)^a \, dx \right)^{\frac{p'}{q'}}.
\]

**Proof.** Since \( q \geq p > 1 \), we have \( p' - 1 - \frac{p'}{q} \geq 0 \) and hence

\[
\sum_{Q \in W} \left( \int_{S(Q) \cap A_b} \rho(x)^a \, dx \right)^{p'} |Q|^{\left( \frac{1\cdot q}{p} - \frac{a}{q} \right)p'} \\
\leq \left( \int_{A_b} \rho(x)^a \, dx \right)^{p' - 1 - \frac{p'}{q}} \sum_{Q \in W} \left( \int_{S(Q)} \rho(x)^a \, dx \right)^{\frac{p'}{q'}} \int_{S(Q) \cap A_b} \rho(x)^a \, dx |Q|^{\left( \frac{1\cdot q}{p} - \frac{a}{q} \right)p'} \\
= \left( \int_{A_b} \rho(x)^a \, dx \right)^{p' - 1 - \frac{p'}{q}} \sum_{Q \in W} \left( \int_{S(Q)} \rho(x)^a \, dx \right)^{\frac{1}{q'}} |Q|^{\left( \frac{1\cdot q}{p} - \frac{a}{q} \right)p'} \int_{S(Q) \cap A_b} \rho(x)^a \, dx \\
= \left( \int_{A_b} \rho(x)^a \, dx \right)^{p' - 1 - \frac{p'}{q}} \sum_{Q \in W} \sum_{Q \in S(Q)} \left( \int_{S(Q)} \rho(x)^a \, dx \right)^{\frac{1}{q'}} |Q|^{\left( \frac{1\cdot q}{p} - \frac{a}{q} \right)p'} \int_{Q \cap A_b} \rho(x)^a \, dx.
\]
Above in estimating the last inequality, we use Lemma 3.1 to see that

\[
\begin{align*}
&= \left( \int_{A_b} \rho(x)^a \, dx \right)^{p'-1} \sum_{Q \in W} \sum_{Q' \in P(Q')} \left( \frac{\left( \int_{S(Q)} \rho(x)^a \, dx \right)^{\frac{1}{p'}}}{|Q|^{\frac{1}{p} - \frac{1}{p'}}} \right) \int_{Q \cap A_b} \rho(x)^a \, dx \\
&\leq C \left( \int_{A_b} \rho(x)^a \, dx \right)^{p'-1} = C \left( \int_{A_b} \rho(x)^a \, dx \right)^{p'.}
\end{align*}
\]

and [19, Lemma 2.6] to obtain

\[
\sum_{Q \in P(Q')} \left( \frac{\left( \int_{S(Q)} \rho(x)^a \, dx \right)^{\frac{1}{p'}}}{|Q|^{\frac{1}{p} - \frac{1}{p'}}} \right) \le C \sum_{Q \in P(Q')} |Q|^{\left( \frac{1}{p} - \frac{1}{p'}\right) + \frac{1}{n}} \le C(a, b, p, q, \beta, \Omega, n),
\]

as

\[
\frac{a}{n} + 1 = \frac{2\beta}{(1 + \beta)q} + \frac{1}{n} - \frac{1}{p} - \frac{b}{np} > 0.
\]

The proof is completed. \(\square\)

**Remark 3.2.** If \(a = b = 0\), then the Poincaré inequality obtained above coincides with [14, Theorem 1]. One can modify [19, Example 5.5] to show that the Poincaré inequality from Theorem 3.3 is sharp, in the sense that the inequality

\[
\left( \int_{\Omega} |u - u_{\Omega, a}|^q \rho^a \, dx \right)^{1/p} \leq C \left( \int_{\Omega} |\nabla u|^p \rho^b \, dx \right)^{1/p}
\]

does not hold if \(\frac{a+n}{q} \frac{2\beta}{1+\beta} + \frac{b-n-b}{p} < 0\).

We have the following Korn inequality for domain satisfying a \(\beta\)-QHBC.

**Theorem 3.4.** Let \(\Omega \subset \mathbb{R}^n\) be a proper subdomain satisfying a \(\beta\)-QHBC, for some \(\beta \leq 1\). Let \(p > 1\). Then there is a constant \(C = C(n, p, q, \beta, \Omega) > 0\) such that for every \(v \in W^{1,p}(\Omega, \rho^a)^n\), it holds

\[
(K_{p,a,b}) \quad \int_{\Omega} |Dv|^p \rho^a \, dx \leq C \left\{ \int_{\Omega} |v|^p \rho^b \, dx + \int_{\Omega} |v|^p \rho^a \, dx \right\},
\]

where \(a \geq 0\), \(b \in \mathbb{R}\) satisfying \((a+n) \frac{2\beta}{1+\beta} > n + b - p\).

Moreover, for \(a \geq 0\) and \((a+n) \frac{2\beta}{1+\beta} < n + b - p\), the Korn inequality \((K_{p,a,b-p})\) fails on \(\Omega\).
**Proof.** By using Theorem 2.1 and the Poincaré inequality (Theorem 3.3) with \( p = q \), we see that the Korn inequality \((K_{p,a,b-p})\) holds if \((a + n)(\frac{2\beta}{1+\beta}) + p - n - b > 0\).

The converse part follows from Example 4.1(2) in Section 4. \(\square\)

**Remark 3.3.** Notice that in the Poincaré inequality (Theorem 3.3) and the Korn inequality (Theorem 3.4), there are no result for the borderline case \((a + n)(\frac{2\beta}{1+\beta}) + p - n - b = 0\). However, we believe the Poincaré inequality and the Korn inequality is true at the borderline.

**4 Examples**

We next give examples to indicate the sharpness of Theorems 3.2 and 3.4 for \( n = 2 \). It is easy to check that the example works also for higher dimension.

**Example 4.1.** Let \( \Omega \) be a domain of the union of sequences of rectangles

\[ \Omega = Q_0 \cup C_1 \cup Q_1 \cup C_2 \cup Q_2 \cup C_3 \cup \ldots. \]

The rectangles are arranged as in Figure 1. This is possible if the sidelengths converge to 0 fast enough. The sidelength of \( Q_0 \) is one and that of square \( Q_i \) is \( r_i \). The height of the rectangle \( C_i \) is \( r_i^\tau \) and width is \( r_i^\sigma \) for all \( i \geq 1 \), where \( \sigma, \tau \geq 1 \) is a fixed real number. The domain is called “A rooms-and-corridors domain”.

![Figure 1: A rooms-and-corridors domain.](image)

We can control the boundary accessibility by choosing the constants \( \sigma \) and \( \tau \). Here are two relevant choices.

(i) \( \Omega \) is an \( s \)-John domain if \( s = \sigma \) and \( \Omega \) is not an \( s \)-John domain if \( s < \sigma \) (independent of \( \tau \)); see [19, Example 5.5].
The above estimates imply that if the Korn inequality (4.1) holds, then for each $Korn inequality$

$$\Omega$$ is a $\beta$-QHBC domain if $\sigma \leq \tau$, $\beta = \frac{1}{2r_{i}^\tau}$ and is not a $\beta$-QHBC domain for any $\beta > 0$ if $1 \leq \tau < \sigma$; see [19, Example 5.5] and [14].

For each $i \in \mathbb{N}$, define the vector function $u_i(x, y)$ on $\Omega$ as follows:

$$u_i(x, y) = \begin{cases} (2y + r_i^\tau, -2(x - x_i)), & \forall (x, y) \in Q_i \\ (-\frac{r_i^\tau}{2}, \frac{r_i^\tau}{2}(x - x_i)y), & \forall (x, y) \in C_i \\ (0, 0), & \forall (x, y) \in \Omega \setminus (C_i \cup Q_i). \end{cases}$$

Above $(x_i, -r_i/2 - r_i^\tau)$ is the center of the cube $Q_i$. It is immediate that $u_i$ is Lipschitz continuous in $\Omega$.

Direct computation gives that when $(x, y) \in Q_i$,

$$Du_i(x, y) = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix},$$

and hence $\epsilon(u_i)(x, y) = 0$; when $(x, y) \in C_i$,

$$Du_i(x, y) = \begin{pmatrix} 0 & -2y/r_i^\tau \\ 2y/r_i^\tau & 2(x - x_i)/r_i^\tau \end{pmatrix},$$

and

$$\epsilon(u_i)(x, y) = \begin{pmatrix} 0 & 0 \\ 0 & 2(x - x_i)/r_i^\tau \end{pmatrix}.$$

Meanwhile, for $(x, y) \in \Omega \setminus (C_i \cup Q_i)$, $\epsilon(u_i)(x, y) = Du_i(x, y) = 0$.

From the above calculations, we deduce that

$$\int_{\Omega} |Du_i|^p \rho^\sigma \, dx \, dy \geq \int_{Q_i} \rho^\sigma \, dx \, dy \sim r_i^{\sigma + 2},$$

and since $\rho(x) = r_i^\sigma - |x - x_i|$ in the corridor $C_i$, we see that

$$\int_{\Omega} |\epsilon(v)|^p \rho(x)^{b-p} \, dx \, dy \sim \int_{C_i} (r_i^\sigma - |x - x_i|)^{b-p} \left(\frac{|x - x_i|}{r_i^\tau}\right)^p \, dx \, dy \sim r_i^\sigma (b+1) + \tau (1-p).$$

Moreover,

$$\int_{\Omega} |u_i|^p \rho^\sigma \, dx \, dy \sim \int_{Q_i} r_i^p \rho^\sigma \, dx \, dy \sim r_i^{\sigma + p + 2}.$$
(1) **Sharpness of Theorem 3.2.** Let \( 1 = \tau \leq \sigma \), from Example 4.1 (i) we know \( \Omega \) is a \( s \)-John domain and \( s = \sigma \).

In this case, if the Korn inequality \((K_{p,a,b-p})\) holds on \( \Omega \), then (4.1) becomes

\[
\frac{r_i^{\sigma+2}}{r_i^{\sigma b+1-p}} \lesssim r_i^{\sigma b+1-p} + r_i^{a+p+2}.
\]

This is true for all \( i \). Thus \( a + 2 \geq \sigma(b + 1) + 1 - p \) and we see that Korn inequality \((K_{p,a,b-p})\) fails if \( a + 2 < s(b + 1) + 1 - p \), therefore our Theorem 3.2 is sharp.

(2) **Sharpness of Theorem 3.4.** Let \( 1 \leq \tau = \sigma \), then \( \Omega \) is a \( \beta \)-QHBC domain with \( \beta = \frac{1}{2\sigma - 1} \) according to Example 4.1 (ii).

Suppose the Korn inequality \((K_{p,a,b-p})\) holds on \( \Omega \), then (4.1) becomes

\[
\frac{r_i^{\sigma+2}}{r_i^{\sigma b-p}} \lesssim r_i^{\sigma b+1-p} + r_i^{a+p+2}
\]

for each \( i \). This implies that \( a + 2 \geq \sigma(b + 2 - p) \). We see that Korn inequality \((K_{p,a,b-p})\) fails if \( \frac{2a}{b-p}(a + 2) = \frac{1}{\beta}(a + 2) < b + 2 - p \), which implies that Theorem 3.4 is sharp.

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