Exact Geometries from Boundary Gravity

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Abstract

We show that the extremal Reissner-Nordström type multi-black holes in an emergent scenario are exact in General Relativity. It is shown that an axion in the bulk together with a geometric torsion ensures the required energy momentum to source the \((3+1)\) geometry in the Einstein tensor. Analysis reveals a significant role of dark energy in curved space-time.

Keywords

Geometric torsion · Emergent gravity · Black holes · Dark energy · Higher forms

1 Introduction

Einstein gravity elegantly ensures the deformation geometries of space-time underlying the Riemann curvature tensor \(R_{\mu\nu\lambda\rho}\). It is a classical theory of metric \(g_{\mu\nu}\) field and delves into the continuum of space-time. Furthermore, the \(R_{\mu\nu\lambda\rho}\) is reducible and is re-expressed with three irreducible curvature tensors, i.e. Conformal-Weyl \(C_{\mu\nu\lambda\rho}\), trace-less Ricci \(S_{\mu\nu}\) and Ricci scalar \(\mathcal{R}\). Thus the curved space-time in Einstein’s gravity is contained in one or more of the three irreducible curvatures. However, Einstein’s gravity was elegantly described by the Einstein-Hilbert action which is expressed only in terms of the scalar \(\mathcal{R}\). The metric field equations turn out to be non-linear and they are beautifully expressed in terms of the Einstein tensor \(G_{\mu\nu} = \kappa T_{\mu\nu}\), where \(\kappa\) denotes the coupling and \(T_{\mu\nu}\) ensures the energy-momentum tensor. Importantly the \(T_{\mu\nu}\) is believed to source the intrinsic curvatures of space-time. There were numerous attempts in the past hundred years to apprehend the \(T_{\mu\nu}\) which could...
describe the geometry in $G_{\mu\nu}$ but none of them were completely successful [1–4]. In fact, a firm belief leading to a conjecture seeking an appropriate $T_{\mu\nu}$ remains unanswered.

The unresolved issue in Einstein’s gravity provokes thought to believe in an alternate formulation [5–8] to go beyond General Relativity (GR). This was with a plausible aim to find a satisfactory $T_{\mu\nu}$. Along the line, higher dimensional theories underlying the Riemann curvature tensor have been explored in the past [9–11]. In this paper, we attempt to address an apparently unresolved problem in GR but with a higher dimensional gauge theoretic framework [12–14]. Research ensures that an order two anti-symmetric $B_{\mu\nu}$ field underlying a $U(1)$ gauge symmetry in $(4 + 1)$ may serve as an insightful tool to find the required $T_{\mu\nu}$. Interestingly a geometric torsion (GT) theory is known to ensure an open geometry which is unlike the closed geometries in GR. The observed perihelion precision [15] reconfirms the open geometry in presence of an extra dimension to GR. Interestingly the proposed bulk/boundary correspondence resembles an established open/closed string duality [16].

In this article we revisit a GT description underlying a 5D bulk gauge theory with a renewed interest in a boundary gravity [17, 18]. In particular, we show that the dynamical sector in the boundary gravity ensures an extremal charged black hole which turns out to be exact in GR. We may mention a test particle in a GT theory undergoes a spiral motion and reconfirms an open trajectory in the bulk. Thus a GT in bulk naturally incorporates the boundary gravity description. In fact, a topological $BF$ term is argued to incorporate the winding modes in gravity. It is believed to ensure a UV finite quantum correction to GR. It was shown that a torsion connection modifies a covariant derivative $\nabla_{\mu} \rightarrow D_{\mu}$ in a $U(1)$ gauge theory [12, 19]. The non-Riemann curvature tensors were derived in the modified theory which has been checked to ensure the space-time beyond GR. It was demonstrated that an emergent metric underlying the known black hole geometries in Einstein gravity is a low energy phenomenon derived from a symmetric matrix comprising the quantum fluctuations of $B_{\mu\nu}$ field. Thus the modified gauge theory with the Minkowski signature of space-time has been shown to incorporate intrinsic curvatures in presence of a GT connection.

It was argued that a gravitational pair of membrane/anti-membrane across the horizon of a background black hole can be created by the quantum of a GT in a gauge theory. This is analogous to the pair creation process ($\gamma \rightarrow e^+e^-$) known in quantum electrodynamics. Similarly, the $B_{\mu\nu}$ field quantum is believed to vacuum create a pair of string/anti-string [20]. In fact, a vacuum pair creation is a powerful quantum tool and is believed to provide an enhanced vision to explore more in the quantum gravity domain. Interestingly the idea has been exploited to explain the Hawking radiation phenomenon at the horizon of a black hole [21]. We may mention that the tool was also beautifully explored to address a cosmic pair [22].

2 Aspects of Multi Black Holes in GR

Reissner-Nordström (RN) black hole is known as exact in GR. It is characterized by two unequal deformation parameters ($M > |Q_e|$) and they source two independent potentials within the RN black hole. The instability with two horizons in RN black hole stabilizes to an extremal geometry with an equipotential. Thus a charged, or more than one parameter, a black hole is likely to provide a clue to unfold some aspects of quantum gravity. Interestingly a drastic change in feature from an extremal RN black hole to multi black holes in isotropic coordinates [23] may be perceived with a notion of cloning in a quantum scenario. In fact, the cloning of black holes is believed to be associated with the non-locality on the event
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horizon [14]. Intuitively a non-local horizon in an extremal RN black hole may be viewed as a collection of closely spaced horizons that ensure the exact multi-black holes in GR.

In addition, an extremal black hole is believed to provide a clue to a quantum tunneling phenomenon between the multiple black holes. Intuitively, the non-zero integer values of the conserved charge \( M = |Q_e| \) reconfirm the multiple black holes. We recall that \( M \) takes a continuum range of values and a typical electric charge can take non-zero integer values in a non-extremal RN black hole. However, the RN black hole shrinks \((r_+ \leftrightarrow r_-)\) to ensure a stable configuration for a positive integer value of \( M \). This in turn would describe an extremal RN multi-black hole as \( M \) takes \( M_i \) positive integer values. A similar analysis in an emergent quantum gravity scenario [13] would like to incorporate the multi-black hole scenario even for non-extremal RN black holes. We postpone a detailed analysis to Section 4.

In the context diverse perspectives on the charged multi-black hole had been elegantly investigated in the folklore of theoretical physics [24–27]. In the recent past, the coupling of \( U(1) \) gauge theory along with scalar (dilaton-axion) to Einstein’s gravity has been crucial to the analysis of dark matter and black hole thermodynamics [28–33]. Further investigation has led to the effect of quantum matter halo on a supermassive black hole [34].

3 Boundary Quantum Gravity

A promising theoretical idea relating a bulk \( B_{\mu\nu} \) field theory to a boundary gravity [18] is believed to be insightful to a theory of quantum gravity. The boundary action comprises a local sector GR and a global sector which turns out to be topological. In this case, the holographic correspondence naturally ensures that an emergent gravity is described by a closed theory and the bulk is an open theory. At this point, we may recall the established \( AdS_5 \) bulk/boundary \( CFT_4 \) correspondence [35], where the gravity in bulk describes an isolated system due to the \( AdS \) radius. However, this feature is not feasible with an arbitrary geometry in bulk.

Now we begin with a \( B_{\mu\nu} \) field dynamics ensuring a \( U(1) \) flux \( H_{\mu\nu\lambda} = 3V_{[\mu} B_{\nu\lambda]} \) in a perturbation gauge theory. A covariant constant \( \tilde{B}_{\mu\nu} \) field is considered in addition to the \( B_{\mu\nu} \) field dynamics. Interestingly a constant \( \tilde{B}_{\mu\nu} \) has been explored as a perturbation parameter. Remarkably \( \tilde{B}_{\mu\nu} \) theory has been shown to incorporate higher order perturbation corrections in \( \tilde{B}_{\mu\nu} \) field. In fact, the corrections were vital to envisage the gravitational interactions in a plausible quantum theory. An iteration replaces \( H_{\mu\nu\lambda} \rightarrow \mathcal{H}_{\mu\nu\lambda} \). Then a term in GT is given by

\[
\mathcal{D}_\lambda \tilde{B}_{\mu\nu} = \nabla_\lambda \tilde{B}_{\mu\nu} + \frac{1}{2} H_{\lambda\mu}{}^\rho \tilde{B}_{\rho\nu} - \frac{1}{2} H_{\lambda\nu}{}^\rho \tilde{B}_{\rho\mu} \tag{1}
\]

Then the GT flux is given by

\[
\mathcal{H}_{\mu\nu\lambda} = H_{\mu\nu\lambda} + H_{\mu\nu}{}^\alpha \tilde{B}_{\alpha\lambda} + H_{\nu\lambda}{}^\alpha \tilde{B}_{\alpha\mu} + H_{\lambda\mu}{}^\alpha \tilde{B}_{\alpha\nu} + O(\tilde{B}^2) \tag{2}
\]

With an appropriate coupling or length scale \( l \) the \( U(1) \) gauge transformation becomes

\[
\delta \tilde{B}_{\mu\nu} = l (\partial_\mu A_\nu - \partial_\nu A_\mu). \]

Explicitly a variation of the action yields

\[
\delta \mathcal{H}^2_{\mu\nu\lambda} = 6l \left( \mathcal{H}_{\mu\nu}{}^\alpha \mathcal{H}_{\nu\lambda}{}^\alpha - \mathcal{H}_{\mu\nu}{}^\alpha \mathcal{H}_{\nu\lambda}{}^\alpha \right) \nabla_\alpha A_\lambda \tag{3}
\]

The broken \( U(1) \) gauge symmetry may be restored [12] with a symmetric matrix \( f_{\mu\nu} = \pm l^2 (\mathcal{H}_{\mu\alpha} H^{\alpha\beta} \mathcal{H}_{\nu\beta}) \). It shows that a non-linear GT and a linear gauge theoretic torsion together
ensure the matrix fluctuations and they have been argued to source an emergent quantum gravity potential $V_q$ in ref. [15]. Analysis reveals that $V_q$ should be a non-Newtonian potential and hence is believed to be defined with a lower cutoff on the radial coordinate. Alternately, the required UV cutoff can also be governed by a non-point (or extended) charge. It was argued that a non-point charge, naturally modifies a point notion, and incorporates winding modes into the theory. In a large (length) scale limit, the symmetric matrix reduces to a non-trivial metric correction to the otherwise flat metric $g_{\mu\nu}$. It is given by

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} \pm f_{\mu\nu} \mid_{\text{large scale}}$$  (4)

With the Minkowski signature of space-time and for on-shell $\tilde{B}_{\mu\nu}$, the generic curvature tensor has been shown to describe a 4-flux $F_{\mu\nu\lambda\rho} = 4i\nabla_{[\mu}H_{\nu\lambda\rho]}$ in addition to $K_{\mu\nu\lambda\rho}$. The reduced effective curvatures in the case are given by

$$[D_\mu, D_\nu] A_\lambda = (K_{\mu\nu\lambda\rho} + F_{\mu\nu\lambda\rho}) A_\rho$$

$$+ (F_{\nu\mu} + H_{\nu\mu\lambda}) D_\lambda$$

where

$$K_{\mu\nu\lambda\rho} = \frac{1}{4} (H_{\mu\lambda}^\sigma H_{\nu\sigma\rho} - H_{\nu\lambda}^\sigma H_{\mu\sigma\rho})$$  (5)

Then the effective action [18] underlying an open geometry (say $R \times S_4$ topology) becomes

$$S_V = \frac{-1}{48l^3} \int d^5x \sqrt{-\tilde{g}} \left( I^2 F^2_{\mu\nu\lambda\rho} + 4H^2_{\mu\nu\lambda} + 4H^2_{\mu\nu\lambda} \right)$$  (6)

A priori, the first term reconfirms a non-interacting GT in bulk with one propagating degree of freedom (PDF). It can be checked to confirm a dynamical axion in the quantum theory. An axion is known to source an instanton and is believed to ensure the quantum tunneling among the multiple vacua. The second term in (6) ensures a non-canonical interaction in the perturbation theory. The third term describes free field dynamics. Altogether the generic bulk dynamics is governed by two interacting fields (GT and $\tilde{B}_{\mu\nu}$) and a non-interacting $B_{\mu\nu}$ in the perturbation theory. The background $\tilde{B}_{\mu\nu}$ independence of the flux $H_{\mu\nu\lambda}$ allows the exterior calculus to re-express the torsion as a 3-form $H_3 = dB_2$. The third term in (6) is a total derivative and hence ensures a boundary term $B_2 \wedge F_2$. In this case the open geometry with bulk GT dynamics associates a boundary $S_4$ with a propagating axion along $R$. The vacuum expectation $< \chi > = \chi_0 = (\text{cosmological})$ constant $\Lambda$ ensures that the boundary space-time curvature is primarily sourced by an interacting second term in (6).

Remarkably the precise curvature on the boundary theory has been argued to map to $\mathcal{R}$ under a proposed correspondence between the bulk GT and boundary GR [17, 18]. In fact, the bulk/boundary gravity correspondence was primarily based on two key shreds of evidence and they are (i) $K_{\mu\nu\lambda\rho}$ shares both the symmetry properties of $\mathcal{R}_{\mu\nu\lambda\rho}$ under the interchange of Lorentz indices independently and in pairs though they are respectively sourced by $B_{\mu\nu}$ field and metric field [12] and (ii) generically the PDF of a $B_{\mu\nu}$ field in bulk is always one degree higher than that of a metric field on the boundary [18]. Thus with a bulk/boundary correspondence, the second term ensures a Ricci-like scalar $\mathcal{K}$ and may be identified with $\mathcal{R}$. With subtlety the boundary action becomes

$$S_{\partial V} = \frac{1}{2\kappa} \int d^4x \sqrt{-\tilde{g}} \left( R - 2\Lambda \right) - \int B_2 \wedge F_2$$  (7)

It reconfirms that the boundary (quantum) theory is generically described by GR with a topological correction. In this article, we show that the boundary dynamics in the emergent scenario is exact. A nonzero topological invariant in GR such as Euler characteristics $\chi$ or signature $\tau$ is believed to unfold the presence of winding modes. However, an extremal RN
ensures $\chi = 0 = \tau$, and the quantum correction is apparently decoded with a non-local horizon. This in turn describes multi-black holes as an exact in GR. At a first sight, these two alternate descriptions are believed to be in agreement with the non-unique nature of quantum gravity.

At this point, we formally compare the emergent boundary quantum gravity scenario with the string landscape and swampland idea [36]. Firstly the topological correction in (7) may see to ensure an upper bound on the number of vacua which is unlike that in the landscape. Further analysis reveals that a large but finite number of extremal black holes underlie the quantum gravity scenario discussed in this article. We believe that the swampland description underlying the effective (quantum) field theory may formally be identified with the bulk gauge theory and the string landscape may as well be compared with the boundary gravity in the case. Interestingly the swampland distance conjecture leading to an infinite tower of massless particles may be seen to be governed by the Kaluza-Klein curvatures (\(R\) and \(F_{\mu\nu}\) with any fixed value of the scalar field) derived from the bulk \(AdS_5\) theory coupled to gauge theoretic torsion \(H_{\mu\nu\lambda}\). Thus the boundary gravity conjecture formally appears to be in agreement with the landscape but with a (winding) UV-cutoff on the number of vacua.

4 RN Type Multi Black Holes

We begin with the gauge ansatz in \((3 + 1)\) dimensions \([13]\) for \((b, p) > 0\). For simplicity, we consider \(l = 1\) and then the ansatz becomes

\[
B_t \theta = B_r \phi = b, \quad \tilde{B}_\theta \phi = p \sin^2 \theta, \\
A_t = \frac{Q_e}{r}, \quad A_\phi = -Q_m \cos \theta.
\]

They have been used to obtain the line element for an RN-type geometry in a large length scale limit. Then the line element becomes

\[
ds^2 = -\left(1 - \frac{b^2}{r^2} + \frac{Q_e^2}{r^4}\right) dt^2 + \left(1 - \frac{b^2}{r^2} + \frac{Q_e^2}{r^4}\right)^{-1} dr^2 + r^2 \left(1 - \frac{Q_m^2}{r^4}\right) d\Omega^2.
\]

where the integers \((b, Q_e, Q_m)\) are the conserved charges. The emergent line-element empirically confirms a \((3 + 1)\) dimensional dyonic black hole, with a mass \(b^2\), embedded in \((4 + 1)\). Since the contribution to geometry from the mass, electric and magnetic charges take on positive numbers \((1, 4, 9, 16 \ldots)\), their spectrum describes bound states of non-extremal black holes for \(b^2 > \pm 2Q_e^2\) and with horizon radii \(r_\pm\) satisfying \(2r_\pm = b^2 \pm \sqrt{b^4 - 4Q_e^2}\). However, each bound state describes a continuum of space-time within and is separated from all other bound states by classically forbidden regimes. Nevertheless, the quantum gravity scenario allows tunneling between the black holes. It is a new feature associated with the non-extremal RN-type black hole which is unlike that in GR.

In addition, the multi-RN type black holes are defined with a reduced radius of \(S_2\) due to the magnetic charge. The \(S_2\) area becomes singular at \(r^4 = Q_m^2\). With an imposed self-duality and in a limit \(r^4 \to Q_e^2\), the emergent RN identifies with the causal patch of a Schwarzschild black hole with a reduced effective mass \((b^2/2)\) embedded in a higher dimension. This is due to the fact that the action, for a \(U(1)\) gauge field, vanishes for a self-dual electromagnetic field which ensures a vacuum solution. Interestingly some of the
emerging features share with that of a \((3 + 1)\) dimensional charged black hole in string theory \([37]\).

Since the causal patch is independent of the magnetic charge, one may set \(Q_m = 0\) in the extremal RN-type black hole without any loss of generality. With a shift in the square of the radial coordinate \(\rho^2 = (r^2 - Q_e)\) the line element in isotropic coordinates become

\[
ds^2 = -H^{-2}dt^2 + H \left( d\rho^2 + \rho^2 d\Omega^2 \right),
\]

where \(H = \left(1 + \frac{Q_e}{\rho^2}\right)\)

It is straightforward to check that a time-independent \(H\) satisfies Laplace’s equation \(\nabla^2 H = 0\) in \((4 + 1)\) dimensions. Its solution is given by

\[
H = 1 + \sum_{i=1}^{N} \frac{m_i}{|y - y_i|^2}
\]

The harmonic function \(H\) is well-behaved at the spatial infinity. Each value of \(i\) defines a horizon and hence the solution \((10)\) describes an extremal RN multi-black hole in near horizon geometry even in an emergent gravity scenario. It provokes one to believe in a nonlocal horizon which in turn is a collection of a multi horizon along a radial coordinate \(\rho\). In fact, an extremal RN type is not derived as an exact solution in GR. The multi-black holes are obtained in the limit for a large-scale structure of space-time in a quantum gravity theory.

In this context, it is interesting to recall the weak gravity conjecture \([38–40]\) formally for the boundary quantum gravity \((7)\). Preliminary analysis reveals that the macroscopic extremal multi-black holes \((10)\) are unstable due to the quantum fluctuations at the extremality \(r_+ \leftrightarrow r_-\). They decay to microscopic black holes when the ratio of charges \(\frac{b^2}{|Q_e|} > 2\). On the other hand, the large black holes decay to elementary charged particles when the same ratio becomes less than two. Thus the analysis ensures that the quantum instability is removed with an emergent bunch of smaller and stable black holes in the boundary gravity scenario.

5 Boundary GR from the Bulk \(T_{\mu\nu}\)

In this section, we would like to review the validity of the conjectured GT in bulk/boundary gravity correspondence. In particular, we check if the Einstein tensor type in an emergent scenario is exact in GR. The non-vanishing components of the Einstein type tensor \(\tilde{G}_{\mu\nu} = (\tilde{\mathcal{R}}_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{\mathcal{R}})\) are worked out with near horizon coordinate to yield

\[
\tilde{G}_{\theta\theta} = \frac{2Q_e (2Q_e - \rho^2)}{(\rho^2 + Q_e)^2}, \quad \tilde{G}_{\phi\phi} = \frac{2Q_e (2Q_e - \rho^2)}{(\rho^2 + Q_e)^2} \sin^2 \theta
\]

\[
\tilde{G}_{tt} = \frac{\rho^4 Q_e (Q_e - 2\rho^2)}{(\rho^2 + Q_e)^5}, \quad \tilde{G}_{rr} = -\frac{Q_e (Q_e - 2\rho^2)}{\rho^2 (\rho^2 + Q_e)^2}
\]

\[%]
The components are re-expressed in terms of the radial coordinate and they become

\[ \tilde{G}_{tt} = \frac{3Q^4_e}{r^{10}} - \frac{8Q^3_e}{r^8} + \frac{7Q^2_e}{r^6} - \frac{2Q_e}{r^4} \]
\[ \tilde{G}_{rr} = -\frac{Q^2_e}{(r^2 - Q_e)} r^4 + \frac{2Q_e}{r^4} \]
\[ \tilde{G}_{\theta\theta} = \frac{6Q^2_e}{r^4} - \frac{2Q_e}{r^2} \]
\[ \tilde{G}_{\phi\phi} = \frac{6Q^2_e \sin^2 \theta}{r^4} - \frac{2Q_e \sin^2 \theta}{r^2} \]  
(13)

It is important to note that the conserved charges on a brane are opposite in sign to that on an anti-brane [12]. The components of Einstein-type tensor on a vacuum-created gravitational pair of (33)-brane are worked out to yield

\[ \tilde{G}_{tt} = \frac{6Q^4_e}{r^{10}} + \frac{14Q^2_e}{r^6} , \quad \tilde{G}_{rr} = -\frac{2Q^4_e}{r^{10}} - \frac{2Q^2_e}{r^6} , \]
\[ \tilde{G}_{\theta\theta} = \frac{12Q^2_e}{r^4} \quad \text{and} \quad \tilde{G}_{\phi\phi} = \frac{12Q^2_e \sin^2 \theta}{r^4} \]  
(14)

Now it is interesting to check the validity of Einstein-like tensor for the exactness i.e. \( \tilde{G}_{\mu\nu} = \tilde{\kappa} T_{\mu\nu} \), where \( \tilde{\kappa} \) is believed to be an effective gravitational coupling. Alternately, at least the \( \tilde{G}_{\mu\nu} \) should be proportional to \( T_{\mu\nu} \) if the emergent gravity scenario corresponds to Einstein’s gravity. In fact, we would like to examine the above-mentioned conjecture by explicitly computing the components of \( T_{\mu\nu} \) derived from a bulk gauge theory in the case. It is important to realize that the perturbation gauge theory in the case incorporates the dynamics of \( \tilde{B}_{\mu\nu} \) as well as the three index GT which respectively sources \( T_{\mu\nu}(1) \) and \( T_{\mu\nu}(2) \). Together they define the total \( T_{\mu\nu} = T_{\mu\nu}(1) + T_{\mu\nu}(2) \). Explicitly they are given by

\[ T_{\mu\nu}(1) = H_{\mu\alpha\beta} H^{\alpha\beta\nu} - \frac{g_{\mu\nu}}{6} H_{\alpha\beta\gamma} H^{\alpha\beta\gamma} \]
\[ T_{\mu\nu}(2) = F_{\mu\alpha\beta\gamma} F^{\alpha\beta\gamma\nu} - \frac{g_{\mu\nu}}{8} F_{\alpha\beta\gamma\delta} F^{\alpha\beta\gamma\delta} \]  
(15)

Invoking the gauge ansatz [12] we write

\[ B_t\psi = B_r\psi = b , \quad B_{\theta\psi} = \tilde{B}^3 \sin^2 \psi \cot \theta , \]
\[ \tilde{B}_{\psi\phi} = P^3 \sin^2 \psi \cos \theta \]  
(16)

The non-vanishing components of GT are

\[ H_{\theta\phi} = \frac{P^3}{r^2} \sin^2 \psi \sin \theta , \]
\[ H_{\theta\phi} = -H_{\theta\phi} = \frac{b P^3}{r^2} \sin^2 \psi \sin \theta \]  
(17)

Subsequently the nontrivial components of the 4-index flux [18] become

\[ F_{t\psi\theta\phi} = -F_{r\psi\theta\phi} = \frac{-2b P^3}{r^2} \sin 2\psi \sin \phi , \]
\[ F_{r\theta\phi} = \frac{2b P^3}{r^3} \sin^2 \psi \sin \theta \]  
(19)
Explicitly the components of both the energy-momentum tensors are given by

\[
\begin{align*}
T_{tt}(1) &= \left(1 + \frac{2b^2}{r^2}\right) \frac{p^6}{r^6}, \quad T_{rr}(1) = -\left(1 - \frac{2b^2}{r^2}\right) \frac{p^6}{r^6} \\
T_{\psi\psi}(1) &= \frac{p^6}{r^4}, \quad T_{\theta\theta}(1) = \frac{p^6}{r^4} \sin^2 \psi \\
T_{\phi\phi}(1) &= \frac{p^6}{r^4} \sin^2 \psi \sin^2 \theta \\
T_{tt}(2) &= -\frac{12b^2}{r^{10}} \left(8 \cot^2 \psi + 3\right) \\
T_{rr}(2) &= \frac{12b^2}{r^{10}} \left(-8 \cot^2 \psi + 3\right) \\
T_{\psi\psi}(2) &= \frac{12b^2}{r^8}, \quad T_{\theta\theta}(2) = \frac{36b^2}{r^8} \sin^2 \psi \\
T_{\phi\phi}(2) &= \frac{36b^2}{r^8} \sin^2 \psi \sin^2 \theta
\end{align*}
\] (20)

We consistently fix an angular coordinate \(\psi = \frac{\pi}{2}\) to arrive at a \((3+1)\)-dimensional embedded regime within a \((4+1)\) bulk. Then the energy-momentum tensors in a specific combination of their components have been checked to satisfy the field equations. A priori they are defined with eight constants \((\kappa_i, \tilde{\kappa}_i)\) for \(i = (0, 1, 2, 3)\) and the field equations are given by

\[
\begin{align*}
\tilde{G}_{tt} &= \kappa_0 T_{tt}(1) - \tilde{\kappa}_0 T_{tt}(2) - \tilde{\kappa}_3 T_{\phi\phi}(2) \\
\tilde{G}_{rr} &= \kappa_1 T_{rr}(1) - \kappa_1 T_{rr}(2) - \kappa_2 T_{\theta\theta}(2) \\
\tilde{G}_{\theta\theta} &= \kappa_2 T_{\theta\theta}(1) \quad \text{and} \quad \tilde{G}_{\phi\phi} = \kappa_3 T_{\phi\phi}(1)
\end{align*}
\] (21)

The emergent equations take the usual form of Einstein field equations. The conditions are worked out to yield

\[
\begin{align*}
\kappa_0 &= 7 \kappa_1 = \frac{7}{6} \kappa_2 = \frac{7}{6} \kappa_3 = 126 \tilde{\kappa}_2 = 18 \tilde{\kappa}_3 \sin^2 \theta \\
\tilde{\kappa}_0 &= \frac{1}{6} \quad \text{and} \quad \tilde{\kappa}_1 = \frac{1}{18}
\end{align*}
\] (22)

They ensure only one independent constant \(\kappa\) among the eight \((\kappa_i, \tilde{\kappa}_i)\). All the resulting field equations (21) can be checked to be defined with a nontrivial coupling as is believed in GR. The result reconfirms our conjecture that the RN type in an emergent gravity is exact in GR. Analysis presumably identifies an emergent gravity scenario with that of GR at least for the RN black hole.

In addition the complete \(T_{\mu\nu}\) has been shown to be sourced by two independent fields in a perturbation theory. Importantly a combination of energy-momentum tensor components for both fields was shown to source the Riemann curvature for the extremal RN black hole. The result is remarkable and is believed to strengthen our proposed correspondence [17] between a gauge theory in bulk and a boundary quantum gravity underlying the Einstein gravity.
6 Concluding Remarks

In this article, we have revisited a correspondence between the $5D$ bulk $U(1)$ gauge theory and a boundary gravity [18] with a renewed interest. It was shown that the boundary gravity dynamics comprises GR and a topological $BF$ term which in turn describes a mass dipole correction to GR. It is argued that the topological correction incorporates winding mode(s) into the otherwise point particle notion in GR. Arguably the winding modes ensure a UV finite (non-perturbation) quantum correction to GR.

We have shown that the extremal RN-type multi-black holes are exact in GR. In particular, the energy-momentum tensor in the boundary gravity was shown to be a special case of the bulk dynamics ensured by two completely anti-symmetric fluxes $H_{\mu\nu\lambda}$ and $F_{\mu\nu\lambda\rho}$ in a perturbation theory. It was important to note that the non-Riemannian geometry, sourced by the gauge theory, has ensured a plausible scenario of a charged black hole of RN type which has been identified with the elegant and established RN black hole in GR. A striking difference between the two becomes prominent with the emergent quantum scenario which in turn has been shown to describe multiple microscopic black holes. Furthermore, the multi nature of a black hole has been argued generically with a topological $B_2 \wedge F_2$ correction in RN type. Intuitively the extremal RN type black holes may equivalently be viewed as a single (overlapping) microscopic black hole(s) described with a nonlocal horizon which is very different from the extremal RN black hole. This in turn has led to a notion of a degenerate [14] horizon in the emerging quantum gravity scenario.

Our analysis is believed to reveal the quantum tunneling between the multi-vacua in the emergent scenario sourced by a geometric torsion in a gauge theoretic framework. The presence of a 4-form flux has been shown to source a dynamical axionic scalar in the bulk gauge theory which in turn is likely to ensure the enormous dark energy in Universe. In this context, authors become aware of an interesting recent development [33] leading to the testing of dynamical torsion effects in multiple charged black holes. Importantly the electric/magnetic charges, in the RN type [13], take nontrivial positive integer values and hence naturally bring in the notion of multiple black holes in the emergent scenario. In fact, the emergent Schwarzschild may be seen to describe multiple microscopic black holes which is unlike that in GR.

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References

1. Ashtekar, A., Magnon-Ashtekar, A.: Phys. Rev. Lett. 43, 181 (1979)
2. Murchadha, N.Ó.: J. Math. Phys. 27, 2111–2128 (1986)
3. Sharif, M.: Il Nuovo Cimento B118, 669 (2004)
4. Gümrukçüoğlu, A.E., Namba, R.: Phys. Rev. D100, 124064 (2019)
5. Wilczek, F.: Phys. Rev. Lett. 80, 4851 (1998)
6. BLAGOJEVIĆ, M., CVETKOVIĆ, B., VASILIĆ: Phys. Rev. D88, 101501(R) (2013)
7. Anastasiou, A., Borsten, L., Duff, M.J., Nagy, S., Zoccali, M.: Phys. Rev. Lett. 121, 211601 (2018)
8. Benisty, D., Guendelman, E.I., van de Venn, A., Vasak, D., Struckmeier, J., Stoecker, H.: Eur. Phys. J. C82, 264 (2022)
9. Cho, Y.: J. Math. Phys. 16, 2029 (1975)
10. Troncoso, R., Zanelli, J.: Class. Quant. Grav. 17, 4451 (2000)
11. Wesson, P.S.: Int. J. Mod. Phys. D24, 1530001 (2015)
12. Singh, A.K., Pandey, K.P., Singh, S., Kar, S.: J. High Energy Phys. **05**, 33 (2013)
13. Singh, A.K., Pandey, K.P., Singh, S., Kar, S.: Phys. Rev. **D88**, 066001 (2013)
14. Singh, S., Singh, K.P., Pandey A.K., Kar, S.: Nucl. Phys. **B879**, 216 (2014)
15. Nitish, R., Gupta, R.K., Kar, S.: Int. J. Mod. Phys. **D29**, 2050074 (2020)
16. Sen, A.: Phys. Rev. Lett. **91**, 181601 (2003)
17. Nitish, R., Singh, D., Kar, S.: Phys. Scr. **94**, 075301 (2019)
18. Gupta, R.K., Kar, S., Nitish, R.: Int. J. Mod. Phys. **D29**, 2050019 (2020)
19. Nitish, R., Kar, S.: Int. J. Mod. Phys. **D30**, 2150011 (2021)
20. Bachas, C., Porrati, M.: Phys. Lett. **B296**, 77 (1992)
21. Hawking, S.W.: Comm. Math. Phys. **43**, 199 (2022)
22. Majumdar, M., Davis, A.C.: J. High Energy Phys. **03**, 056 (1975)
23. Carroll, S.M.: Spacetime and Geometry. Cambridge University Press, Cambridge (2019)
24. Maki, T., Shiraishi, K.: Class. Quant. Grav. **10**, 2171 (1993)
25. Kastor, D., Traschen, J.: Phys. Rev. **D47**, 5370 (1993)
26. Gregory, R., Lim, Z.L., Scoins, A.: Front. Phys. **9**, 666041 (2021)
27. Astorino, M., Viganò, A.: Eur. Phys. J. **C82**, 829 (2022)
28. Javed, W., Babar, R., Övgün, A.: Phys. Rev. **D100**, 104032 (2019)
29. Javed, W., Abbas, J., Övgün, A.: Phys. Rev. **D100**, 044052 (2019)
30. Fernandes, P.G.S.: Phys. Lett. **B805**, 135468 (2020)
31. Övgün, A.: Phys. Lett. **B820**, 136517 (2021)
32. Pantig, R.C., Övgün, A.: J. Cosmol. Astropart. Phys. **08**, 056 (2022)
33. Pantig, R.C., Övgün, A.: Ann. Phys. **448**, 169197 (2023)
34. Pantig, R.C., Övgün, A.: Fortschr. Phys. **71**, 2200164 (2023)
35. Maldacena, J.: Int. J. Theor. Phys. **38**, 1113 (1999)
36. Ooguri, H., Vafa, C.: Nucl. Phys. **B766**, 21 (2007)
37. Garfinkle, D., Horowitz, G.T., Strominger, A.: Phys. Rev. **D43**, 3140 (1991)
38. Arkani-Hamed, N., Motl, L., Nikolis, A., Vafa, C.: J. High Energy Phys. **06**, 060 (2007)
39. Cheung, C., Liu, J., Remmen, G.N.: J. High Energy Phys. **10**, 04 (2018)
40. Loges, G.J., Noumi, T., Shiu, G.: Phys. Rev. **D102**, 046010 (2020)

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