Comparison between Rigid and Overconvergent Cohomology with Coefficients

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Abstract

For a smooth scheme over a perfect field of characteristic $p > 0$, we generalise a definition of Bloch and introduce overconvergent de Rham-Witt connections. This provides a tool to extend the comparison morphisms of Davis, Langer and Zink between overconvergent de Rham-Witt cohomology and Monsky-Washnitzer respectively rigid cohomology to coefficients.

Résumé

Pour un schéma lisse sur un corps parfait de caractéristique $p > 0$, on généralise une définition de Bloch et introduit des connexions surconvergentes de de Rham-Witt. Cela fournit une possibilité d’étendre à des coefficients convenable les morphismes de comparaison de Davis, Langer et Zink entre la cohomologie surconvergente de de Rham-Witt et la cohomologie de Monsky-Washnitzer respectivement la cohomologie rigide.

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Introduction

Let $X$ be a smooth variety over a perfect field $k$ of positive characteristic $p$. In [3] Spencer Bloch studies connections on the de Rham–Witt complex of $X$ and establishes the following equivalence of categories.

**Theorem** (Bloch 2001, [3, Thm. 1.1]). The functor $E \mapsto (E, \nabla)$ defines an equivalence of categories between locally free $F$-crystals on $X$ and locally free $W_X$-modules with quasi-nilpotent integrable connections and a Frobenius structure.

The objects considered in this statement form a suitable family of coefficients for crystalline cohomology. Yet, we have to bear in mind, that crystalline cohomology is only a good integral model for rigid cohomology in the case of a proper variety $X$. The appropriate cohomology theory for non-proper varieties was constructed by Davis, Langer and Zink in [7] in form of the overconvergent de Rham–Witt complex. In an attempt to define coefficients for this overconvergent integral cohomology theory, we realise that Bloch’s equivalence of categories provides an interesting approach to obtain overconvergent crystals.

We define a subcategory of the de Rham-Witt connections from above’s theorem, by taking modules over $W^\dagger X$ – the overconvergent Witt vectors – together with connections that take values in the overconvergent de Rham–Witt complex. It turns out, that this is a reasonable category of coefficients for the overconvergent cohomology theory. In particular, using this definition we are able to generalise the comparison theorems of Davis, Langer and Zink between overconvergent cohomology and Monsky–Washnitzer, respectively rigid cohomology to coefficients.

In Section 1 we start out by recalling some facts about crystals and their relation to de Rham–Witt connections as studied by Etesse and Bloch. In Section 1.3 we define overconvergent de Rham–Witt connections and prove that they form a sub-category of the usual de Rham–Witt connections.

Section 2 is devoted to the comparison between Monsky–Washnitzer and overconvergent cohomology with coefficients. Thence, we recall Monsky–Washnitzer algebras and define overconvergent connections on modules over them. We follow closely the argumentation of Davis, Langer and Zink in Section 2.2 to compare Monsky–Washnitzer cohomology to overconvergent cohomology using the mentioned connections on the Monsky–Washnitzer side and overconvergent de Rham–Witt connections on the overconvergent side.

This is also an important ingredient for the comparison between rigid cohomology and overconvergent cohomology with coefficients to which we turn in Section 3. We recall Berthelot’s notion of overconvergent isocrystals in Section 3.1. In Section 3.2 we verify that the methods used by Davis, Langer and Zink for the comparison theorem without coefficients, carry over to the case of coefficients. Thus we obtain the following main result.

**Theorem.** Let $X$ be a smooth quasi-projective scheme over $k$, and $E \in \text{Isoc}^\dagger(X \subset X/W(k)$ a locally free isocrystal. Then there is a natural quasi-isomorphism

$$R^\alpha \Gamma_{\text{rig}}(X, E) \to R^\alpha \Gamma(X, E \otimes (W^\dagger \Omega_{X/k} \otimes \mathbb{Q})).$$

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1 Crystals and de Rham–Witt connections

Let $X$ be a smooth variety over a perfect field $k$ of positive characteristic $p$. As usual denote by $W(k)$ the ring of $p$-typical Witt vectors of $k$. A classical result in crystalline cohomology is the comparison with the hypercohomology of the de Rham–Witt complex.

**Theorem 1.0.1** (Illusie 1979, [15 II.1.4]). For each $n \in \mathbb{N}_0$ there is a canonical isomorphism between the crystalline cohomology ring $H^\bullet(X/W_n)$ and the hypercohomology of the de Rham–Witt complex $\mathbb{H}^\bullet(X, W_n \Omega)$ which is compatible with the Frobenius action on both sides. More generally, there is a quasi-isomorphism

$$R u_* E_{X/W_n} \to W_n^\dagger \Omega_X.$$
of sheaves of \( W_n(k) \)-modules which is functorial in \( X/k \). Here \( u : (X/W_n)_{\text{cris}} \to X_{zar} \) is the natural projection of topoi.

This induces an isomorphism
\[
H^*(X/W) \cong H^*(X, W\Omega).
\]

It is natural to generalise this to crystals.

### 1.1 Crystals on \( X/W(k) \)

Let \( S \) be a PD-scheme. A crystal is a sheaf on the crystalline site which is, in the spirit of Grothendieck, characterised by two properties: it is rigid and it grows. Berthelot and Ogus give in [1] the following definition.

**Definition 1.1.1.** A crystal \( E \) on \( X/S \) is a sheaf of \( \mathcal{O}_{X/S} \)-modules on the crystalline site \( \text{Cris}(X/S) \), such that for any morphism \( u : (U', T', \varepsilon') \to (U, T, \delta) \) in \( \text{Cris}(X/S) \) the induced map
\[
u^* E_{(U', T', \varepsilon')} \to E_{(U, T, \delta)}
\]
is an isomorphism.

Here \( \mathcal{O}_{X/S} \) denotes the structure sheaf of the crystalline site of \( X/S \). It is also a trivial example of a crystal. A less trivial example is given by a closed immersion \( i : Y \to X \) of \( S \)-schemes. The immersion \( i \) induces naturally a morphism of crystalline topoi \( i_{\text{cris}} : (Y/S)_{\text{cris}} \to (X/S)_{\text{cris}} \). Then the functor \( i_{\text{cris}}^* \) is exact, and \( i_{\text{cris}}^* (\mathcal{O}_{Y/S}) \) is a crystal of \( \mathcal{O}_{X/S} \)-algebras. This is not obvious, for a proof of this see [1 Proposition 6.2].

For \( Y \hookrightarrow X \) as above, denote by \( \mathcal{D}_Y(X) \) the PD-envelope of \( Y \) in \( X \), and by \( \mathcal{D}_{X/S}(\nu) \) the PD-envelope of \( X \) in \( X/S(\nu+1) \). There are canonical maps
\[
\mathcal{D}_Y(X) \otimes \mathcal{D}_{X/S}(1) \to \mathcal{D}_Y(X \times X)
\]
\[
\mathcal{D}_{X/S}(1) \otimes \mathcal{D}_Y(X) \to \mathcal{D}_Y(X \times X)
\]
which by the crystalline property are isomorphisms as shown in [1 Corollary 5.3]. Hence, one can deduce a natural isomorphism
\[
\mathcal{D}_{X/S}(1) \otimes \mathcal{D}_Y(X) \cong \mathcal{D}_Y(X) \otimes \mathcal{D}_{X/S}(1)
\]
which is essentially a hyper PD-stratification, meaning
1. it is \( \mathcal{D}_{X/S} \)-linear,
2. reduces to the identity modulo the associated PD-ideal and
3. satisfies the cocycle condition.

Such a stratification corresponds to an integrable, quasi-nilpotent connection
\[
\nabla : \mathcal{D}_Y(X) \to \mathcal{D}_Y(X) \otimes \Omega^1_{X/S}.
\]

**Definition 1.1.2.** Let \( X/S \) as before and \( p^m \mathcal{O}_S = 0 \) and \( \{ x_i \} \) a set of local coordinates for \( X/S \). A connection \( \nabla \) on an \( \mathcal{O}_X \)-module \( E \) is quasi-nilpotent with respect to the coordinate system, if and only if for each open \( U \subset X \) and all sections \( s \in \Gamma(U, E) \) there exists an open covering \( \{ U_\alpha \} \) of \( U \) and integers \( \{ e_{i, \alpha} \} \), such that
\[
\left[ \nabla \left( \frac{\partial}{\partial x_i} \right) \right]^{e_{i, \alpha}} \left( s|_{U_\alpha} \right) = 0
\]
for all \( i, \alpha \).

A similar statement holds true for crystals. In fact, let \( X \hookrightarrow Y \) be a closed immersion of \( S \)-schemes with \( Y/S \) smooth, then the following categories are equivalent:
1. The category of crystals of \( \mathcal{O}_{X/S} \)-modules on \( \text{Cris}(X/S) \).
2. The category of \( \mathcal{D}_X(Y) \)-modules together with a hyper PD-stratification compatible with the canonical one explained above.
3. The category of \( \mathcal{D}_X(Y) \)-modules together with an integrable, quasi-nilpotent connection compatible with the one described in the previous paragraph.

This equivalence is established in [1 Theorem 6.6].
1.2 Towards de Rham–Witt connections

The morphism mentioned at the beginning of this section was generalised by Etesse in [11] to coefficients.

Let $E$ be a locally free crystal on $X/W(k)$ and $E$ the induced $W_X$-module, which will be denoted by $E_n$ if seen as $W_n(X)$ module. Furthermore, let

$$\nabla_n : E_n \to E_n \otimes \Omega^1_{W_n(\mathcal{O}_X)}$$

be the connection associated for each $n \in \mathbb{N}_0$. The surjective morphism $\Omega^1_{W_n(\mathcal{O}_X)} \to W_n\Omega^1_{\mathcal{O}_X}$ induces naturally a surjection

$$E_n \otimes \Omega^1_{W_n(\mathcal{O}_X)} \to E_n \otimes W_n\Omega^1_{\mathcal{O}_X}$$

which gives rise to a connection

$$\nabla_n : E_n \to E_n \otimes W_n\Omega_X$$

for every $n$.

**Definition 1.2.1.** The complex $E_n \otimes W_n\Omega_X$ is called the de Rham–Witt complex of length $n$ with coefficients in the crystal $E$. The induced pro-complex $E_\bullet \otimes W_\bullet\Omega_X$ is the de Rham–Witt pro-complex $E \otimes W\Omega_X$ with coefficients in the crystal $E$. Taking limits, we obtain the de Rham–Witt complex with coefficients in the crystal $E$.

Etesse shows in [11, Proposition 1.2.7] that this is compatible with the tensor product of $E$ and $W\Omega_X$ over $\mathcal{O}_X$ as expected. Thus the notation makes sense. The connections $\nabla_n$ induce a connection

$$\nabla_n : E \otimes W_1\Omega^1_{X/k} \to E \otimes W_2\Omega^2_{X/k} \to \cdots$$

His main result is the following.

**Theorem 1.2.2** (Etesse 1988, [11]). Let $u_n : (X/W_n)_{\text{cris}} \to X_{\text{zar}}$ be the canonical projection of topoi. Then the morphism

$$R u_n(E) \to E_n \otimes W_n\Omega^*$$

is an isomorphism.

If $X/k$ is in addition proper and $E$ of finite type, the previous theorem induces an isomorphism

$$H^\bullet (X/W, E) \cong H^\bullet (X, E \otimes W\Omega).$$

Bloch’s discussion of the functor $E \mapsto (E, \nabla)$ establishes an equivalence of categories between locally free crystals on $X/W(k)$ and locally free $W_X$-modules with a quasi-nilpotent integrable connection. In the next section, we will introduce a sub-category of the latter category.

1.3 Overconvergent connections

In [3] C. Davis, A. Langer and T. Zink define for a finitely generated $k$-algebra $A$ the ring of overconvergent Witt vectors $W^\dagger(A) \subset W(A)$ as subring of the usual Witt vectors. In the subsequent paper [7] they extend this to the definition of an overconvergent de Rham–Witt complex $W^\dagger\Omega_{A/k} \subset W\Omega_{A/k}$ which can in fact be globalised to a sheaf on a smooth variety $X$ over $k$, and so can of course $W^\dagger$.

Let $E$ be a $W^\dagger_X$-module on $X$.

**Definition 1.3.1.** An overconvergent de Rham–Witt connection on $E$ will be a map

$$\nabla : E \to E \otimes W^\dagger\Omega^1_{X/k}$$

which satisfies the Leibniz rule

$$\nabla(\omega e) = \omega \nabla(e) + e \otimes d\omega.$$
This definition makes sense, as by construction, the differential of the de Rham–Witt complex takes overconvergent elements to overconvergent elements. As usual, we say that an overconvergent connection is integrable if

\[ \nabla^2 = 0. \]

A morphism

\[ \phi : (E, \nabla_E) \to (F, \nabla_F) \]

of \( W_X^+ \)-modules with connections is a morphism of \( W_X^+ \)-modules compatible on each side with the connection, or in other words, it is a morphism of complexes

\[ E \otimes W^+ \Omega_X \to F \otimes W^+ \Omega_{X/k} \]

where the boundary maps are given by the connections.

**Lemma 1.3.2.** The set of overconvergent de Rham–Witt connections \((E, \nabla)\) together with the morphisms as indicated above form a full subcategory of the category of de Rham–Witt connections as defined by Bloch.

**Proof.** For a \( W_X^+ \)-module \( E \) the scalar extension

\[ E \mapsto E \otimes_{W_X^+} W_X \]

yields a functor from \( W_X^+ \)-modules to \( W_X \)-modules and it is clear that the first one is a full subcategory of the second one. To extend this to the notion of overconvergent connections on such a module \( E \), we start by looking at the overconvergent de Rham–Witt complex. As mentioned above, the differential of the overconvergent de Rham–Witt complex is the restriction of the differential of the usual de Rham–Witt complex. In particular, it takes overconvergent elements to overconvergent elements. Therefore the scalar extension functor is compatible with the de Rham–Witt differential. This extends to the overconvergent de Rham–Witt complex with coefficients in a \( W_X^+ \)-module in the obvious way.

We restrict now to locally free \( W_X^+ \)-modules with overconvergent integrable, quasi-nilpotent connections \((E, \nabla)\).

**Remark 1.3.3.** Quasi-nilpotent is meant in the sense of [1 Def. 4.10 and 4.14]. As the notions of being integrable and quasi-nilpotent are independent of the overconvergence property, it follows from the lemma, that this is a subcategory of the category of locally free \( W_X \)-modules with a quasi-nilpotent integrable connection, which was the category studied by Bloch [3 Theo. 1.1]. Taking into account that he established an equivalence of categories between the latter one and the category of locally free crystals on \( X \), one would like to have a suitable notion of overconvergent crystals on \( X \).

## 2 Overconvergent and Monsky–Washnitzer cohomology with coefficients

Let \( X/k \) be as before. The goal of this section is to extend the comparison morphism of Davis, Langer and Zink between overconvergent de Rham–Witt and Monsky–Washnitzer algebra to coefficients. This lies the ground work for the comparison between overconvergent and rigid cohomology.

### 2.1 Monsky–Washnitzer algebras and overconvergent connections

For a complete discrete valuation ring \( R \) and a quotient of a polynomial algebra \( B = R[t_1, \ldots, t_n]/(f_1, \ldots, f_m) \) the associated Monsky–Washnitzer algebra is according to [23]

\[ B^\dag = R[t_1, \ldots, t_n]/(f_1, \ldots, f_m), \]

where \( R[t_1, \ldots, t_n]/(f_1, \ldots, f_m) \) consists of those power series \( \sum_a a_\alpha t^\alpha \) in \( R[t_1, \ldots, t_n] \) such that for some \( C > 0 \) and \( 0 < \rho < 1 \)

\[ |a_\alpha| \leq C \rho^{|\alpha|} \]

for all coefficients. The elements are called overconvergent power series and can be characterised by converging in a poly-disc with radii \( \rho_i > 1 \).
The algebra $R(t_1, \ldots, t_n)^{\dagger}$ satisfies Weierstraß preparation and division and has the Artin approximation property.

We turn now to the special case, when $R = W(k)$ for a perfect field $k$ of characteristic $p$. The notation is as in [6]. Let $\text{Spec}(\overline{B})$ be a smooth affine variety over $k$. There exists a lift of the form $B = W(k)[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and we consider the associated Monsky–Washnitzer algebra

$$B^{\dagger} := W(k)(x_1, \ldots, x_n)^{\dagger}/(f_1, \ldots, f_m).$$

Davis gives a description of the overconvergent property in this case, which is especially suited for the comparison map. It is possible to write an element $a \in B^{\dagger}$ uniquely in the form $a = \sum_{k=0}^{\infty} p^k \sum_{j=0}^{n_k} a_{ij} x^j$ with $a_{ij} \in W(k)^* \cup 0$ and $a_{ij,k} \neq 0$ unless $n_k = 0$ and for $j$ fixed $a_{jk} \neq 0$ for at most one $k$. Then an element is overconvergent, if there is $C > 0$ such that $n_k \leq C(k + 1)$ for $k \geq 0$.

Let $\Omega_{B^{\dagger}/W(k)}$ be the module of continuous differentials of $B^{\dagger}$ over $W(k)$. The Monsky–Washnitzer cohomology of $\text{Spec} B$ is then calculated by the rational complex $\Omega_{B^{\dagger}/W(k)} \otimes \mathbb{Q}$

$$H^i_{MW}(\text{Spec} B/K) = H^i(\Omega_{B^{\dagger}/W(k)} \otimes \mathbb{Q}).$$

These notions are well-defined, independent of the choice of representation and functorial.

To introduce coefficients to this cohomology theory, in the spirit of overconvergent crystals, we make the following definition.

**Definition 2.1.1.** Let $\tilde{E}$ be a free $B^{\dagger}$-module. A connection on $\tilde{E}$ is a map

$$\tilde{\nabla} : \tilde{E} \to \tilde{E} \otimes_{B^{\dagger}} \Omega_{B^{\dagger}/W(k)}$$

satisfying the Leibniz rule $\tilde{\nabla}(we) = w\tilde{\nabla}(e) + e \otimes dw$. It is said to be integrable if $\tilde{\nabla}^2 = 0$ as usual.

The Monsky–Washnitzer cohomology of $\text{Spec} B$ with coefficients in $\tilde{E}$ is then

$$H^i_{MW}(\text{Spec} B/K, \tilde{E}) = H^i(\tilde{E} \otimes \Omega_{B^{\dagger}/W(k)} \otimes \mathbb{Q}).$$

### 2.2 Comparison with overconvergent cohomology

Let $W^{\dagger}(\Omega_{\overline{B}/k})$ be the overconvergent subcomplex of the de Rham–Witt complex as defined in [6] Chapter 3 or [7] Section 3. For a smooth lift of the Frobenius $F : B^{\dagger} \to B^{\dagger}$, which always exists, there is a map

$$t_F : B^{\dagger} \to W(\overline{B})$$

which lands in fact in the overconvergent subring $W^{\dagger}(\overline{B}) \subset W(\overline{B})$ and induces by the universal property of Kähler differentials and functoriality a comparison map

$$t_F : \Omega_{B^{\dagger}/W(k)} \to W^{\dagger}(\Omega_{\overline{B}/k}).$$

Although this depends a priori on the choice of Frobenius lift, Davis proves in [6] Corollary 4.1.11 that for standard étale affines it is independent in the derived category.

The main result of Davis, Langer and Zink [7] Proposition 3.24 regarding this comparison morphism is that for an arbitrary smooth algebra $B$ the kernel and cokernel of the induced homomorphism on cohomology is annihilated by $p^{2\kappa}$, where $\kappa = \lfloor \log_p(\dim B) \rfloor$. As a corollary they obtain the following result.

**Corollary 2.2.1** (Davis, Langer, Zink 2011 [7]).

(a) Let $\dim B < p$. Then the induced map $t_{F*}$ is an isomorphism.

(b) In general, there is a rational isomorphism

$$H^i_{MW}(B/K) \cong H^i(W^{\dagger}(\Omega_{B/k}) \otimes_{W(k)} K)$$

with $K = W(k)[1/p]$ between Monsky–Washnitzer cohomology and overconvergent cohomology.
This is proved using a spectral sequence argument.

We want to generalise this to a comparison between Monsky–Washnitzer cohomology and overconvergent cohomology with coefficients. For this let \((\tilde{E}, \tilde{\nabla})\) be a free \(B^1\)-module with integrable connection as defined in (2.1.1). Then using the map \(t_F : B^1 \to W^+(\overline{B})\) consider the tensor product

\[ E := \tilde{E} \otimes_{t_F} W^+(\overline{B}) . \]

This is a free \(W^+(\overline{B})\)-module and the connection \(\tilde{\nabla}\) gives rise to an integrable overconvergent de Rham–Witt connection \(\nabla\): Since the map \(t_F\) is by construction via the functoriality of Kähler differentials compatible with the differential maps on \(\Omega^1_{B^1/W(k)}\) and \(W^1\Omega^1_{\overline{B}/k}\) respectively, the composition

\[ \tilde{E} \otimes_{t_F} W^+(\overline{B}) \xrightarrow{\tilde{\nabla} \otimes \text{id} + \text{id} \otimes d} \tilde{E} \otimes_{B^1} \Omega^1_{B^1/W(k)} \otimes_{t_F} W^+(\overline{B}) \oplus \tilde{E} \otimes_{t_F} W^1\Omega^1_{\overline{B}/k} \xrightarrow{\text{id} \otimes t_F \otimes \text{id}} \tilde{E} \otimes_{t_F} W^1\Omega^1_{\overline{B}/k} , \]

where the object on the left is \(E\) and the object on the right is \(E \otimes_{W^1(\overline{B})} W^1\Omega^1_{\overline{B}/k}\), yields a map

\[ \nabla : E \to E \otimes W^1\Omega^1_{\overline{B}/k} . \]

and the diagram

\[ \begin{array}{ccc}
\nabla & : & E \\
\downarrow & & \downarrow \\
\tilde{E} \otimes_{B^1} \Omega^1_{B^1/W(k)} & \longrightarrow & E \otimes_{W^1(\overline{B})} W^1\Omega^1_{\overline{B}/k} ,
\end{array} \]

where the horizontal maps are induced by \(t_F\) taking basis to basis, is commutative.

**Proposition 2.2.2.** The above constructed map \(\nabla : E \to E \otimes W^1\Omega^1_{\overline{B}/k}\) is an integrable connection.

**Proof.** As the original connection \(\tilde{\nabla}\) satisfies the Leibniz rule, it is clear by the definition and the commutativity of diagram (2.2.1), that \(\nabla\) satisfies the Leibniz rule as well.

Since \(t_F\) is compatible with the differentials on either side, we might extend the diagram to higher differentials and it still stays commutative. In order to prove integrability, we use that \(\tilde{\nabla}\) and \(d\) are integrable connections themselves. For \(e \in \tilde{E}\) and \(w \in W^1(\overline{B})\) we calculate

\[ \nabla^2 (e \otimes w) = \nabla \left( \tilde{\nabla} e \otimes w + e \otimes dw \right) = \tilde{\nabla}^2 e \otimes w + \tilde{\nabla} e \otimes dw + \tilde{\nabla} e \otimes dw + e \otimes d^2 w = 0 \]

with the correct sign convention. \(\square\)

**Lemma 2.2.3.** With the notation as before, if \(\nabla\) is quasinilpotent, so is \(\nabla\).

**Proof.** Quasinilpotence is a local property which can be checked in local coordinates since by [1, Remark 4.11] it is independent of the choice of coordinate system. It is clear that by this definition the differential \(d\) on the de Rham–Witt complex of \(\overline{B}\) is quasinilpotent. Thus the quasinilpotence of \(\nabla\) induce quasinilpotence of \(\nabla\). \(\square\)

For different choices of Frobenius lifts on \(B^1\), we obtain different \(W^+(\overline{B})\)-modules \(E\), which are however canonically isomorphic. On the other hand, the comparison map

\[ t_F : \tilde{E} \otimes_{B^1} \Omega^1_{B^1/W(k)} \to E \otimes_{W^1(\overline{B})} W^1\Omega^1_{\overline{B}/k} \]

still depends on the choice of Frobenius lift \(F\) and it is not clear a priori that two different choices of Frobenius induce the same map on cohomology.

We will follow closely the argumentation of Davis, Langer and Zink in [7, Section 3], exploring first the case of standard étale affines and then generalising to arbitrary smooth affines.

Therefore, let \( \overline{B} \) and \( \overline{C} \) be standard étale affines (for the definition see [8, Definition 4.1.3]). Let \( \psi_1, \psi_2 : B^1 \to W^1(\overline{C}) \) denote two ring homomorphisms such that for every \( b \in B^1 \), \( \psi_2(b) - \psi_1(b) \in \text{Im}(V) \). Davis shows in [8, Theorem 4.1.6] that the induced maps on differential graded algebras

\[ p^\ast \psi_1, p^\ast \psi_2 : \Omega^1_{B^1/W(k)} \to W^1\Omega^1_{\overline{C}} \]
are chain homotopic. Assume further that $\psi_1, \psi_2$ seen as extension of scalars for the free $B^1$-module $\tilde{E}$ induce isomorphic complexes 

$$\alpha : \tilde{E} \otimes_{\psi_1} W^1 \Omega_{\overline{\mathbb{C}}} \cong \tilde{E} \otimes_{\psi_2} W^1 \Omega_{\overline{\mathbb{C}}}$$

then we claim the following.

**Proposition 2.2.4.** The induced maps 

$$p^s \psi_1, p^s \psi_2 : \tilde{E} \otimes_{B^1} \Omega_{B^1 / W(k)} \to \tilde{E} \otimes_{\psi_1, 2} W^1 \Omega_{\overline{\mathbb{C}}}$$

are chain homotopic.

**Proof.** Following the argumentation of [6, Theorem 4.1.6] Let $D'(\overline{\mathbb{C}})$ be the differential graded algebra given by 

$$D'(\overline{\mathbb{C}})^i = W^i \Omega_{\overline{\mathbb{C}}} \left[ \frac{1}{p} \right] \oplus W^i \Omega_{\overline{\mathbb{C}}} \left[ \frac{1}{p} \right] \wedge dT.$$ 

Davis constructs a homomorphism of differential graded $W(k)$-algebras 

$$\varphi : \Omega_{B^1 / W(k)} \to D'(\overline{\mathbb{C}})$$

such that reduction modulo $(T)$ yields $\psi_1$ and reduction modulo $(T - p)$ yields $\psi_2$. Since restriction to degree zero provides a map $B^1 \to D'(\overline{\mathbb{C}})$, it makes sense to consider the tensor product $\tilde{E} \otimes_{B^1} D'(\overline{\mathbb{C}})$ and $\varphi$ induces a morphism 

$$\varphi : \tilde{E} \otimes_{B^1} \Omega_{B^1 / W(k)} \to \tilde{E} \otimes_{B^1} D'(\overline{\mathbb{C}})$$

which modulo $(T)$ gives the corresponding map $\psi_1$ on $\tilde{E} \otimes_{B^1 / W(k)}$ and modulo $(T - p)$ gives $\psi_2$. Denote these reductions by $h_0$ and $h_p$ respectively. 

As in Davis’ proof, the homotopy factors through $D'(\overline{\mathbb{C}})$, meaning it is de facto a homotopy between $p^s h_0$ and $p^s h_p$. We take his homotopy $L$, which is given for each $i$ on the image of $\varphi$ in $D'(\overline{\mathbb{C}})^i$ to $W^1 \Omega_{\overline{\mathbb{C}}}^{-1}$ by 

$$L(T^i \omega_i) = 0 \quad L(T^i dT \wedge \omega_i) = \frac{p^{i+1}}{j+1} \omega_i$$

for $\omega_i \in W^i \Omega_{\overline{\mathbb{C}}} \left[ \frac{1}{p} \right]$. One can easily check that this is indeed a homotopy, i.e. 

$$Ld + dL = p^s h_p - p^s h_0$$

on the image of $p^s \varphi$. Consequently, $h := L \circ \varphi$ is a homotopy between $p^s \psi_1$ and $p^s \psi_2$:

$$p^s \psi_2 - p^s \psi_1 = p^s h_p \varphi - p^s h_0 \varphi = Ld \varphi + dL \varphi = L \varphi d + dL \varphi = hd + dh. \quad (2.2.2)$$

We now generalise this construction of homotopy to the case of coefficients in $\tilde{E}$. We have to bear in mind that we assumed $\tilde{E}$ to be a free $B^1$-module, and thus $\tilde{E} \otimes W^1(\overline{\mathbb{C}})$ is a free $W^1(\overline{\mathbb{C}})$-module regardless of the connecting morphism. As a consequence, the above construction can be carried out component wise. In fact we have the following diagram

\[
\begin{array}{ccc}
\tilde{E} \otimes_{B^1} \Omega_{B^1 / W(k)} & \xrightarrow{id \otimes \varphi} & \tilde{E} \otimes_{\varphi} D'(\overline{\mathbb{C}}) \\
\downarrow{id \otimes h_0} & & \downarrow{\sim} \\
\tilde{E} \otimes_{\psi_1} W^1 \Omega_{\overline{\mathbb{C}}} & & \tilde{E} \otimes_{\psi_2} W^1 \Omega_{\overline{\mathbb{C}}} \\
\downarrow{id \otimes h_p} & & \\
\tilde{E} \otimes_{\psi_2} W^1 \Omega_{\overline{\mathbb{C}}}
\end{array}
\]

Define maps on the image of $id \otimes \varphi$ in $\tilde{E} \otimes_{\varphi} D'(\overline{\mathbb{C}})^i$ into $\tilde{E} \otimes_{\psi_2} W^1 \Omega_{\overline{\mathbb{C}}}$, serving as a homotopy between $id \otimes p^s h_0$ and $id \otimes p^s h_p$ by 

$$L^E(e \otimes_{\varphi} \omega) = e \otimes_{\psi_2} L(\omega) = \alpha(e \otimes_{\psi_1} L(\omega))$$

$$L^E((\nabla e) \otimes_{\varphi} \omega) = -((\nabla e) \otimes_{\psi_2} L(\omega) = \alpha \left( (\nabla e) \otimes_{\psi_1} L(\omega) \right)$$
for a generator $e$ of $\tilde{E}$ over $B^!$ and $\omega \in D'(C)$, where $L$ is as described above. This is well-defined as $e$ is a generator and $\tilde{\nabla}$ is the connection defined over $B^!$ compatible with the differential $d$. Consequently we have the relation

$$L^F \nabla (e \otimes_\varphi \omega) - \nabla L^F (e \otimes_\varphi \omega) = L^F (\tilde{\nabla} \otimes_\varphi \omega + e \otimes_\varphi d\omega) + \nabla (e \otimes_\psi_2 \omega)$$

$$= -\tilde{\nabla} e \otimes_\psi_2 L \omega + e \otimes_\psi_2 Ld\omega + \tilde{\nabla} e \otimes_\psi_2 \omega + e \otimes_\psi_2 dL\omega$$

$$= e \otimes_\psi_2 (Ld\omega + dL\omega)$$

$$= e \otimes_\psi_2 (p^k h_\omega) - \alpha (e \otimes_\psi_1 p^k h_\omega)$$

$$= (id \otimes_\psi p^k h_\omega - \alpha \circ id \otimes_\psi p^k h_0) (e \otimes_\omega).$$

It follows that the composition $h^E = L^F \circ (id \otimes_\varphi)$ provides a homotopy between $\alpha \circ (id \otimes_\psi_1)$ and $id \otimes_\psi_2$. 

Corresponding to [6] we obtain the subsequent corollaries.

**Corollary 2.2.5.** Let $\overline{B}$ be a standard étale affine and let $F, F'$ denote two different lifts of Frobenius to $B^!$. Then the induced maps

$$p^* t_F, p^* t_{F'} : \tilde{E} \otimes_{B^!} \Omega_{B^!/W(k)} \to E \otimes_{W^1(\overline{B})} W^! \Omega_{\overline{B}/k}$$

are chain homotopic.

**Corollary 2.2.6.** Let $\overline{F} : \overline{B} \to \overline{C}$ be a morphism of localisations of standard étale affine polynomial algebras. Let $F_1, F_2$ denote Frobenius lifts on $B^!$ and $C^!$ respectively. Fix a lift $g : B^! \to C^!$ of $\overline{F}$. Then the two maps

$$p^* t_{F_2} \circ g, p^* \overline{F} \circ t_{F_1} : \tilde{E} \otimes_{B^!} \Omega_{B^!/W(k)} \to E \otimes_{W^1(\overline{C})} W^! \Omega_{\overline{C}}$$

are chain homotopic.

If the dimension of $\overline{B}$ is “small enough”, we obtain a better result. In the case that $\kappa = \lfloor \log_p (\dim \overline{B}) \rfloor = 0$, in other words, if the dim $\overline{B} < p$ the above statements compare $t_F$ and $t_{F'}$ directly.

**Corollary 2.2.7.** Let $\overline{B}$ be a standard étale affine of dimension $\dim \overline{B} \leq p - 1$. Let $F, F'$ denote two different lifts of Frobenius to $B^!$. Then the induced maps

$$t_F, t_{F'} : \tilde{E} \otimes_{B^!} \Omega_{B^!/W(k)} \to E \otimes_{W^1(\overline{B})} W^! \Omega_{\overline{B}/k}$$

are chain homotopic.

The next step is to show that the comparison map $t_F : \tilde{E} \otimes_{B^!} \Omega_{B^!/W(k)} \to E \otimes_{W^1(\overline{B})} W^! \Omega_{\overline{B}/k}$ is a quasi-isomorphism. For the moment, we still assume that $\overline{B}$ is a standard étale affine. The complex $W^! \Omega_{\overline{B}/k}$ has a decomposition in terms of integral and rational weights of the basic Witt differentials

$$W^! \Omega_{\overline{B}/k} = W^! \text{int} \Omega_{\overline{B}/k} \otimes W^! \text{frac} \Omega_{\overline{B}/k}$$

which is compatible with the differential graded structure. Davis shows in [6] Section 4.2 that on the one hand $t_F$ maps in fact into the integral part of the overconvergent complex and moreover $t_F : \Omega_{B^!/W(k)} \to W^! \text{int} \Omega_{\overline{B}/k}$ is an isomorphism, and on the other hand, that $W^! \text{frac} \Omega_{\overline{B}/k}$ is acyclic.

We wish to apply the same strategy but have to be a bit careful tensoring with $\tilde{E}$.

**Proposition 2.2.8.** Let $\overline{B}$ be a standard étale affine. Then

$$t_F : \tilde{E} \otimes \Omega_{B^!/W(k)} \to E \otimes W^! \Omega_{\overline{B}/k}$$

is a quasi-isomorphism.
Proof. The map of complexes $t_F$ can be represented by a commutative diagram of which we provide the first square

$$
\begin{array}{ccc}
\tilde{E} & \xrightarrow{\psi} & \tilde{E} \otimes W^\dagger,\text{int}(\tilde{B}) \oplus \tilde{E} \otimes W^\dagger,\text{frac}(\tilde{B}) \\
\tilde{E} \otimes \Omega^1_{B^+/W(k)} \otimes_{t_F} W^\dagger(\tilde{B}) & \xrightarrow{(\nabla \otimes \text{id} + \text{id} \otimes \text{id})(\text{id} \otimes \text{id})} & \tilde{E} \otimes_{t_F} W^\dagger,\text{int} \Omega^1_{\tilde{B}/k} \\
\tilde{E} \otimes \Omega^1_{B^+/W(k)} & \xrightarrow{\text{id} \otimes \text{id}} & \tilde{E} \otimes_{t_F} W^\dagger,\text{frac} \Omega^1_{\tilde{B}/k} \\
\end{array}
$$

From this diagram we see that the decomposition of the vertical right hand map in $(\text{id} \otimes t_F + \text{id}) \circ (\nabla \otimes \text{id} + \text{id} \otimes \text{id})$ and $\text{id} \circ (\text{id} \otimes \text{id})$ respects the decomposition of the complex in integral and rational part. In particular, the term including $\nabla$ acts only on the integral part. As the fractional part is acyclic and $\tilde{E}$ is free it follows that $t_F : \tilde{E} \otimes \Omega^1_{B^+/W(k)} \to E \otimes W^\dagger,\text{int} \Omega^1_{\tilde{B}}$ is an isomorphism and as a consequence the claim follows. \hfill \Box

It remains to generalise this result to an arbitrary smooth algebra $\overline{A}$. Consider an overconvergent Witt lift

$$
\psi = t_F : A^\dagger \to W^\dagger(\overline{A})
$$

uniquely determined by a Frobenius lift to $A^\dagger$. By abuse of notation we denote also by $\psi$ the induced map of complexes

$$
\psi : \Omega^1_{A^+/W(k)} \to W^\dagger \Omega^1_{\overline{A}/k}
$$

and with the same notation as above for a free $A^\dagger$-module with integrable connection $(\tilde{E}, \nabla)$

$$
\psi : \tilde{E} \otimes_{A^\dagger} \Omega^1_{A^+/W(k)} \to E \otimes W^\dagger \Omega^1_{\overline{A}/k}
$$

Passing to cohomology, we generalise the result of [7] Proposition 3.24.

**Proposition 2.2.9.** Let $\kappa = [\log_p \dim \overline{A}]$. Then the kernel and cokernel of the induced map on cohomology

$$
\psi_* : H^i(\Omega^1_{A^+/W(k)}) \to H^i(W^\dagger \Omega^1_{\overline{A}/k})
$$

are annihilated by $p^{2\kappa}$.

Proof. We note first that the map $\psi : \tilde{E} \otimes_{A^\dagger} \Omega^1_{A^+/W(k)} \to E \otimes W^\dagger \Omega^1_{\overline{A}/k}$ induces a map of complexes of Zariski sheaves on $\text{Spec } \overline{A}$:

$$
\tilde{\psi} : \tilde{E} \otimes_{A^\dagger} \Omega^1_{A^+/W(k)} \to E \otimes W^\dagger \Omega^1_{\text{Spec } \overline{A}/k}.
$$

Recall from [7] Proposition 1.2 and [21] Lemma 7 that for Zariski topology $H^i(\text{Spec } \overline{A}, \tilde{E} \otimes_{A^\dagger} \Omega^1_{A^+/W(k)}) = H^i(\text{Spec } \overline{A}, W^\dagger \Omega^1_{\text{Spec } \overline{A}/k}) = 0$. Since $\tilde{E}$ is a free $A^\dagger$-module and $E$ is a free $W^\dagger(\overline{A})$-module, the same equalities hold true with coefficients. Thus we have

$$
\begin{align*}
\text{R}^i(\text{Spec } \overline{A}, \tilde{E} \otimes_{A^\dagger} \Omega^1_{A^+/W(k)}) &= \tilde{E} \otimes_{A^\dagger} \Omega^1_{A^+/W(k)} \\
\text{R}^i(\text{Spec } \overline{A}, E \otimes W^\dagger \Omega^1_{\text{Spec } \overline{A}/k}) &= E \otimes W^\dagger \Omega^1_{\overline{A}/k}
\end{align*}
$$

which means, as point out Davis, Langer and Zink, that we can reconstruct $\psi$ from $\tilde{\psi}$.

Let $\{U_i = \text{Spec } \overline{B}_i\}$ be a finite cover of $\text{Spec } \overline{A}$ in standard étale affines and such that their intersections are again standard étale affine, which always exists by a result of Kedlaya in [19]. Let

$$
\psi_i : \tilde{E} \otimes \Omega^1_{B^+/W(k)} \to E \otimes W^\dagger \Omega^1_{\overline{B}_i/k}
$$
be the localisation of $\psi$ to $\mathcal{B}$. By Proposition \ref{pro:overconvergent Witt lift}, the maps $p^*\psi_1$ and $p^*\sigma$, where $\sigma$ is induced by an overconvergent Witt lift $\sigma: B^1 \to W^1(\mathcal{B})$ uniquely determined by a lift of Frobenius, are chain homotopic. Consequently, from Proposition \ref{pro:overconvergent Witt lift} we see that the kernel and cokernel of $(p^*\psi_1)_*$ are annihilated by $p^*$ and so are $\text{Ker}((\psi_1)_*) \subseteq \text{Ker}((p^*\psi_1)_*)$ and $\text{Coker}((\psi_1)_*)$ as a subquotient of $\text{Coker}((p^*\psi_1)_*)$.

Denote by $C^*$ the cokernel of $\tilde{\psi}$. Then one has a short exact sequence of Zariski sheaves

$$0 \to \tilde{E} \otimes \Omega^*_{A^1/W(k)} \to E \otimes \Omega^*_{\text{Spec} A/k} \to C^* \to 0.$$ 

Because it follows from the previous paragraph that for all $i \in \mathbb{N}_0$ the cohomology sheaves $\mathcal{H}^i(C^*)$ are annihilated by $p^{2i}$, the map of multiplication by $p^i$ on the complex $C^*$ induces the zero map on cohomology. Hence it is zero in the derived category and applying the functor $R\Gamma$ does not change this fact.

This proves the statement about the cokernel. For the one about the kernel, one uses a similar strategy with the obvious changes.

From this result we also conclude the following.

**Corollary 2.2.10.** (a) Let $\dim \mathcal{A} < p$. Then $\psi_*$ from the proposition is an isomorphism.

(b) There is a rational isomorphism

$$H^*_MW(A/K, \tilde{E}) \cong \mathbb{H}^*(E \otimes W^\dagger_\mathcal{A} \otimes W(k), K)$$

between Monsky–Washnitzer cohomology and overconvergent cohomology with coefficients, where $K = W(k)[\frac{1}{p}]$.

**Example 2.2.11.** Consider the affine line $A^1_k = \text{Spec} A$ where $A = k[x]$ and let $A^\dagger = W(k)[x]^\dagger$ be the associated overconvergent algebra. An integrable connection on a free module $\tilde{E}$ in one generator $e$ over $A^\dagger$ is determined by an element $f \in A^\dagger$ via

$$\tilde{\nabla}(e) = f(x)e \otimes dx.$$ 

Let $E = \tilde{E} \otimes W^\dagger(A)$, which is generated by $e \otimes 1$. The by $\tilde{\nabla}$ induced connection on $E$ is given by

$$\nabla = (\text{id} \otimes t_F \otimes \text{id}) \circ (\tilde{\nabla} \otimes \text{id + id} \otimes d).$$ 

It is determined by

$$e = e \otimes 1 \quad \mapsto \quad \tilde{\nabla}(e) \otimes 1 + e \otimes dx = f(x)e \otimes dx \otimes 1 \mapsto f(x)e \otimes t_F(x).$$ 

If the chosen lift of Frobenius gives $F(x) = x^p$, then $t_F(x) = [x]$ is simply the Teichmüller lift \cite[Prop. 2.2.2]{Davis} and the last expression becomes $f(x)e \otimes d[x]$.

**Remark 2.2.12.** In a recent paper, Davis and Zureick-Brown prove, after showing that integral Monsky–Washnitzer cohomology groups are indeed well defined for a non-singular affine variety over a perfect field of characteristic $p > 0$, that there is an integral comparison isomorphism between Monsky–Washnitzer cohomology and overconvergent cohomology in low degrees compared to the characteristic of the ground field. See \cite{Davis}. The author thinks that it should be possible to extend this morphism to cohomology groups with coefficients using similar methods to the ones in this paper.

## 3 Overconvergent and rigid cohomology with coefficients

The goal of this section is to globalise the result of the previous one to a quasiprojective variety $X$ over $k$. 

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3.1 Overconvergent isocrystals

We will briefly recall the definition of rigid cohomology after Berthelot in [22] and point out different methods to calculate it. A good description can be found in [20].

Let $X$ be any smooth $k$-scheme, $Z$ a closed subscheme and set $U = X - Z$. By a result of Nagata, there is always an open immersion of $X$ into a proper smooth scheme $\overline{X}$ which we denote by $j_X$. Assume that there is a closed immersion of $\overline{X}$ into a formal scheme $\mathcal{Y}$ over $\text{W}(k)$ which is smooth in a neighbourhood of $X$. Consider the generic fibre $Y$ of $\mathcal{Y}$ and let $\text{sp} : Y \to \mathcal{Y}$ be the specialisation map. Let $\mathcal{Y}_0$ be the reduction of $\mathcal{Y}$ over $k$, then for any subscheme $T \subset \mathcal{Y}_0$ the set

$$[T] := \text{sp}^{-1}(T)$$

is called the the tube of $T$. A strict neighbourhood $V$ of $[X]$ in $[\overline{X}]$ is an open subvariety of $[\overline{X}]$ such that the cover $(V, \overline{X} - \overline{X})$ is admissible in the usual sense (of Grothendieck topology). For a sheaf $\mathcal{E}$ on $[\overline{X}]$ set

$$j^\dagger \mathcal{E} := \lim_{\alpha} \alpha_{V*} \alpha_{V}^* \mathcal{E},$$

the limit taken over the strict neighbourhoods $\alpha_V : V \hookrightarrow [\overline{X}]$ of $[X]$. Rigid cohomology is then defined as

$$H^i_\text{rig}(X/K) := H^i([\overline{X}], j^\dagger \Omega^i_{[\overline{X}]}).$$

We can also calculate rigid cohomology using so called dagger spaces [12]. The idea is to associate a smooth scheme a rigid analytic space endowed with an overconvergent structure sheaf. With the notation from above, the tube $[\overline{X}]$ seen as a partially proper rigid space [12, Theorem 5.1] is equivalent to a partially proper dagger space $\mathcal{Y}$ (see [12, Theorem 2.26]). Let $Z$ be the open subspace of $Q$ whose underlying set can be identified with $[X]$. Then Grosse-Klöllner proves in [12, Theorem 5.1] that for a coherent $\mathcal{O}_Q$-module $\mathcal{F}$ and the associated coherent $\mathcal{O}_{[\overline{X}]}$-modules $\mathcal{F}'$ there is a canonical isomorphism

$$H^i(Z, \mathcal{F}) \cong H^i([\overline{X}], j^\dagger \mathcal{F}').$$

In particular this is true for the de Rham complex, thus giving a comparison to rigid cohomology $H^i_{\text{dR}}(Z) = H^i_{\text{rig}}(X/K)$.

As we want to generalise the comparison map of [7] to rigid respectively overconvergent cohomology with coefficients, we introduce the following notion of overconvergent isocrystals proposed by Berthelot, which was picked up by Le Stum [20, Section 2.5].

**Definition 3.1.1.** Let $X \subset \overline{X} \subset \mathcal{Y}$ be a formal embedding as above. An overconvergent isocrystal on $(X \subset \overline{X} \subset \mathcal{Y})$ is a coherent $j^\dagger \mathcal{O}_{[\overline{X}]}$-module with an integrable connection relative to the base whose Taylor series converges on a strict neighbourhood of the diagonal. They form a category denoted by $\text{Isoc}^1(X \subset \overline{X} \subset \mathcal{Y}/\text{W}(k))$.

This is independent of $\mathcal{Y}$ by results of Le Stum if $\mathcal{Y}$ is smooth at $X$ over $\text{W}(k)$, which is the case in our setting. Thus we omit $\mathcal{Y}$ in the notation.

**Lemma 3.1.2.** Keep the setting from above. Let $Q$ be the partially proper dagger space associated to $[\overline{X}]$. An overconvergent isocrystal $\mathcal{E} \in \text{Isoc}^1(X \subset \overline{X}/\text{W}(k))$ is equivalent to a coherent $\mathcal{O}_Q$-module with integrable connection.

**Proof.** By assumption the morphism $X \to \text{Spec } k$ is realisable, meaning that it extends to a morphism $\mathcal{Y} \to \text{W}(k)$ which is proper and smooth at $X$ (cf. Le Stum’s book on the overconvergent site [20]). Consequently, the overconvergent isocrystal $\mathcal{E}$ has a realisation $\mathcal{E}_\mathcal{Y}$ on $\mathcal{Y}$, which is a coherent $j^\dagger \mathcal{O}_{[\overline{X}]}$-module. Then there exists a good open neighbourhood $V$ of $[X]$ in $[\overline{X}]$ and a coherent module with integrable connection $\mathcal{F}$ on $V$ such that $\mathcal{E}_\mathcal{Y} = i_{\overline{X}}^\dagger \mathcal{F}$ (cf. [20, Proposition 3.5.8]). According to [2] (2.1.1.3)) this can be seen as a coherent $\mathcal{O}_{[\overline{X}]}$-module with integrable connection.

By Grosse-Klöllner there is an equivalence of categories between coherent $\mathcal{O}_{[\overline{X}]}$-modules and coherent $\mathcal{O}_Q$-modules [12, Theorem 2.26]. The claim follows. \qed

**Corollary 3.1.3.** Let $\mathcal{E} \in \text{Isoc}^1(X \subset \overline{X}/\text{W}(k))$. Then there is a canonical isomorphism

$$H^i_{\text{dR}}(Z, \mathcal{E}) \cong H^i_{\text{rig}}(X/K, \mathcal{E})$$

of cohomology groups with coefficients.
3.2 Comparison with overconvergent cohomology

Let $X/k$ be a smooth quasiprojective scheme over a perfect field of characteristic $p > 0$. Davis, Langer and Zink in [7] construct a comparison map

$$R\Gamma_{rig}(X) \rightarrow R\Gamma(X, W^1\Omega_{X/k}) \otimes \mathbb{Q} \quad (3.2.1)$$

and show that it is indeed an isomorphism. The goal of this section is to introduce coefficients in this morphism. Therefore we consider a locally free isocrystal $\mathcal{E} \in \text{Isoc}^1(X \subset \mathbb{X}/W(k))$ and let $\nabla$ be the connection that comes with it. We follow closely the proceeding of [7, Section 4].

Let $X = \text{Spec} \mathbb{A}$ be smooth and affine over $k$, $F = \text{Spec} A$ a Witt lift together with a morphism $\varpi : B \rightarrow W^1(\mathbb{A})$ which lifts $B \rightarrow \mathbb{A}$ such that $(X, F, \varpi)$ is an overconvergent Witt frame as defined in [7, Definition 4.1]. Denote by $\hat{F}$ the $p$-adic completion of $F$ and $|X[\hat{F}]$ the tubular neighbourhood of $X$ in the rigid analytic space $\hat{F}_K$. Furthermore let $V \subset F_K^{an}$ be a strict neighbourhood of $|X[\hat{F}]$. The rigid cohomology of $X$ is then $R\Gamma_{rig}(X) = R\Gamma(X, j^!\Omega_V)$ which is independent of the choice of the strict neighbourhood $V$.

Davis, Langer and Zink construct a map

$$\psi : \Gamma(V, j^!\Omega_V) \rightarrow W^1\Omega_{\mathbb{A}/k} \otimes \mathbb{Q} \quad (3.2.2)$$

and show that it factors canonically through $R\Gamma(V, j^!\Omega_V)$. More precisely, they construct a morphism in degree zero

$$\psi : \Gamma(V, j^!\mathcal{E}_V) \rightarrow W^1(\mathbb{A})$$

and use the universal property of Kähler differentials. This enables us to copy the method we used in Section 2, by reason that it makes now sense to consider the object

$$\mathcal{E}' = \mathcal{E} \otimes (W^1(\mathbb{A}) \otimes \mathbb{Q})$$

as a locally free $W^1(\mathbb{A}) \otimes \mathbb{Q}$-module and endow it with the connection

$$\nabla' : \mathcal{E} \otimes (W^1(\mathbb{A}) \otimes \mathbb{Q}) \rightarrow \nabla' = \nabla \otimes \mathbb{Q}.$$

Then the diagram

$$\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\nabla} & \mathcal{E}' \\
\downarrow & & \downarrow \\
\mathcal{E} \otimes \Omega^1_V & \xrightarrow{\nabla'} & \mathcal{E}' \otimes (W^1\Omega^1_{\mathbb{A}}) \\
\end{array} \quad (3.2.3)$$

where the horizontal maps are induced by $\psi$, is commutative.

**Lemma 3.2.1.** The pair $(\mathcal{E}', \nabla')$ is a locally free $W^1(\mathbb{A}) \otimes \mathbb{Q}$-module with integrable connection.

**Proof.** It remains only to show the Leibniz rule and integrability for the connection $\nabla'$. Since by hypothesis the original connection $\nabla$ satisfies the Leibniz rule, it is clear from the construction of $\nabla'$ and the commutative diagram (3.2.3) that this is also the case for $\nabla'$.

Because the morphism $\psi$ is compatible with the differentials on each side of the diagram (3.2.3), it is possible to extend it to the whole complex. As in the previous section we make use of the fact that $\nabla$ and $d$ are integrable. For $e \in \mathcal{E}'$ and $W \in W^1(A) \otimes \mathbb{Q}$ we calculate

$$\nabla' (e \otimes w) = \nabla' (\nabla e \otimes w + e \otimes dw) = \nabla'^2 e \otimes w - \nabla e \otimes dw + \nabla e \otimes dw + e \otimes d^2 w = 0.$$  

Notice the sign convention displayed in the last line. This shows integrability.

This result enable us to extend the comparison morphism (3.2.2) to coefficients

$$\psi : \Gamma(V, j^!(\mathcal{E}_V \otimes \Omega_V)) \rightarrow \mathcal{E} \otimes_{\mathbb{Q}} (W^1\Omega_{\mathbb{A}/k} \otimes \mathbb{Q}) \quad (3.2.4)$$

and we show following [7] that it factors through the derived global section functor.
Lemma 3.2.2. The last morphism factors canonically through a morphism

\[ \psi : R\Gamma (V, j^!(\mathcal{E}_V \otimes \Omega_V)) \to \mathcal{E} \otimes_{\psi} \left( W^!\Omega_{\mathcal{A}/k} \otimes \mathbb{Q} \right) \].

Proof. Fix the strict neighbourhood \( V \) of \( |X|_{\mathbb{F}} \) in \( F^\text{an}_K \). The argumentation of [7, (4.29)] can be transferred to the current situation verbatim. The morphism (3.2.4), as well as the original one, is defined using a construction which is functorial in the triple \((\mathcal{A}, \mathcal{A}, \kappa)\), and thus the choice of a Witt frame, albeit the nature of its construction depends on the choice of a Witt frame, albeit the construction is functorial in the triple \((X, F, \kappa)\).

Proposition 3.2.3. The morphism from Lemma 3.2.2 for overconvergent Witt frames is a quasiisomorphism. The induced morphism (3.2.5) is independent of the choice of the overconvergent Witt frame.

Proof. First we show independence for (3.2.5) of the choice of the overconvergent Witt frame. Note that the morphism

\[ \lim_{\to} R\Gamma (V', (\mathcal{E}'_{V'} \otimes \Omega_{V'})) \cong \lim_{\to} \Gamma (W', (\mathcal{E}'_{W'} \otimes \Omega_{W'})) \to \mathcal{E} \otimes_{\psi} \left( W^!\Omega_{\mathcal{A}/k} \otimes \mathbb{Q} \right) \],

where the first morphism is induced by restriction. Together with the canonical map

\[ \lim_{\to} \Gamma (V', (\mathcal{E}'_{V'} \otimes \Omega_{V'})) = \Gamma (V, j^!(\mathcal{E}_V \otimes \Omega_V)) \to R\Gamma (V, j^!(\mathcal{E}_V \otimes \Omega_V)) \]

this shows the claim.

As a consequence, we have defined for each overconvergent Witt frame \((X, F, \kappa)\) a morphism

\[ R\Gamma_{\text{rig}}(X, \mathcal{E}) \to \mathcal{E} \otimes (W^!\Omega_{\mathcal{A}/k} \otimes \mathbb{Q}). \] (3.2.5)

A priori, this depends by the very nature of its construction on the choice of a Witt frame, albeit the construction is functorial in the triple \((X, F, \kappa)\).

Remark 3.2.4. 1. In the proof of the previous proposition we use that for the “right” choice of overconvergent Witt frame rigid and rational Monsky–Washnitzer cohomology (with coefficients) coincide.

2. This proof uses explicitly functoriality in a rational setting. At this point, it doesn’t seem to be clear yet, how to prove functoriality in an integral setting.

We follow the suggestion of [7] to use dagger spaces to globalise this result. They argue that an overconvergent Witt frame (in fact any special frame) \((X, F)\) gives rise to a dagger space structure on the rigid analytic space \(|X|_{\mathbb{F}}\). We denote it by \(|X|_{\mathbb{F}}^\dagger\). According to Corollary 3.1.3 we have for an overconvergent isocrystal \(\mathcal{E}\)

\[ R\Gamma (|X|_{\mathbb{F}}^\dagger, \mathcal{E} \otimes \Omega_{|X|_{\mathbb{F}}^\dagger}) = R\Gamma_{\text{rig}} (X, \mathcal{E}). \]

Moreover, Davis, Langer and Zink point out that the dagger space \(|X|_{\mathbb{F}}^\dagger\) is functorial in the special frame \((X, F)\) and therefore allows to glue using a Čech spectral sequence argument.

Hence we can rewrite the comparison map in terms of dagger spaces

\[ \Gamma (|X|_{\mathbb{F}}^\dagger, \mathcal{E} \otimes \Omega_{|X|_{\mathbb{F}}^\dagger}) \to \mathcal{E} \otimes (W^!\Omega_{\mathcal{A}/k} \otimes \mathbb{Q}) \]
and obtain simultaneously a local version via the specialisation map
\[
\text{sp}_e \otimes \Omega^\bullet_{X/k} \rightarrow \Delta \otimes (W^t \Omega_{X/k} \otimes \mathbb{Q}).
\]

We come now to the generalisation to an arbitrary smooth quasiprojective scheme \(X\) over \(k\). This works essentially in the same way as without coefficients. Because \(X\) is quasiprojective, we may choose a finite covering \(X = \bigcup_{i \in I} X_i\) of \(X\) by standard smooth affine schemes over \(k\), such that the intersections
\[
X_J = X_{i_1} \cap \ldots \cap X_{i_\ell}
\]
for a subset \(J = \{i_1, \ldots, i_\ell\} \subset I\) are again standard smooth in the sense of [7] Definition 4.33. Denote \(X_J = \text{Spec} \mathcal{A}_J\). It is possible to lift a standard smooth algebra over \(k\) to an algebra over \(W(k)\) which is again standard smooth. Thus, choose for each \(\mathcal{A}_J\) a standard smooth lift \(B_i\) over \(W(k)\) and set \(F_i = \text{Spec} B_i\) to obtain a special frame \((X_i, F_i)\). Then for any subset \(J \subset I\) the closed embedding
\[
X_J \rightarrow \prod_{i \in J} F_i =: F_J
\]
is a special frame.

**Proposition 3.2.5.** Let \(Q = |X_J|_{\mathcal{F}_J}\) be the dagger space associated to the special frame \((X_J, F_J)\) and \(\text{sp}: Q \rightarrow X_J\) the specialisation map. Then the induced morphism
\[
\text{sp}_e \otimes \Omega^\bullet_{Q/k} \rightarrow R\text{sp}_e \otimes \Omega^\bullet_{Q/k}
\]
is a quasiisomorphism.

**Proof.** According to the prove of [7] Proposition 4.35] the strong fibration theorem for dagger spaces implies that for two liftings \(F_1\) and \(F_2\) of an affine smooth scheme \(Z\) over \(k\) such that there is a morphism of frames \(\nu : (Z, F_1) \rightarrow (Z, F_2)\) which restricts to the identity on \(Z\), the dagger spaces associated to the frames are isomorphic. This ultimately allows us to restrict to the special case, where \(F_J\) is of the form \(F'_{i_\ell} \times \mathbb{A}^n_{W(k)}\) with \(F'_{i_\ell}\) a localisation of a lift \(F_{i_\ell}\) of \(X_{i_\ell}\).

Therefore let \((X_J, F_J) = (Z, F \times \mathbb{A}^n_{W(k)})\) be of this particular form. The associated dagger space to this frame is \(Q \times \mathcal{D}^n\) where \(\mathcal{D}^n\) designates the open unit ball of dimension \(n\) with its natural dagger space structure. Let \(\text{sp}: Q \times \mathcal{D}^n \rightarrow Z\) be the specialisation map. As in [7] Corollary 4.38 we see that the natural morphism
\[
\text{sp}_e \otimes \Omega^\bullet_{Q \times \mathcal{D}^n} \rightarrow R\text{sp}_e \otimes \Omega^\bullet_{Q \times \mathcal{D}^n}
\]
is a quasiisomorphism. Indeed, consider the spectral sequence of hypercohomology
\[
\mathcal{H}^p(R^q \text{sp}_e \otimes \Omega^\bullet_{Q \times \mathcal{D}^n}) \Rightarrow R^{p+q} \text{sp}_e \otimes \Omega^\bullet_{Q \times \mathcal{D}^n}.
\]
Because the de Rham complex is locally free, as is by hypothesis the overconvergent isocrystal \(\delta\), we may choose an open affine subset \(U \subset Z\) small enough such that for the pre-image \(\mathcal{U} \subset Q\) under the specialisation morphism, which is an affinoid dagger space, \(\delta_{\mathcal{U}} \otimes \Omega^\bullet_{\mathcal{U}}\) is free. Then the subsequent Lemma 3.2.6 shows that the complex \(H^p(\mathcal{U} \times \mathcal{D}^n, \delta_{\mathcal{U}} \otimes \Omega^\bullet_{\mathcal{U} \times \mathcal{D}^n})\) is acyclic for \(p \geq 1\). Thus the same holds for the complex \(R^p \text{sp}_e \otimes \Omega^\bullet_{Q \times \mathcal{D}^n}\) and the spectral sequence degenerates.

This proves that the morphism in question is indeed a quasiisomorphism and finishes the proof of the proposition. \(\square\)

It remains to show the following statement.

**Lemma 3.2.6.** Let \(\mathcal{D}\) be the one dimensional open unit ball with natural dagger space structure. Let \(Q = \text{Sp}^1 A\) be a smooth affinoid dagger space, such that \(\Omega^\bullet_{Q}\) is a free \(\text{O}_Q\)-algebra and \(\delta\) is a free \(\text{O}_Q\)-module.

(a) The canonical morphism
\[
H^0(Q, \Omega^\bullet_{Q}) \rightarrow H^0(Q \times \mathcal{D}^n, \Omega^\bullet_{Q \times \mathcal{D}^n})
\]
is a quasiisomorphism of complexes.
(b) The complex $H^1(Q \times \tilde{D}^n, \Omega^*_{Q \times D^n})$ is acyclic.

(c) $H^i(Q \times \tilde{D}^n, \Omega^q_{Q \times D^n}) = 0$ for $i \geq 2$ and all $q$.

Proof. The last statement is proved in [7, Proposition 4.35] for an arbitrary abelian sheaf which is a coherent $\mathcal{O}_{Q \times D^n}$-module.

For the remaining two assertions, we may replace the complex $\Omega^q_{Q \times D^n}$ respectively $\Omega_Q$ in Davis,Langer and Zink’s argument by the complex $\mathcal{E} \otimes \Omega^q_{Q \times D^n}$ respectively $\mathcal{E} \otimes \Omega_Q$ which are both coherent locally free modules with integrable connections as we have seen above, and then proceed accordingly.

We are now in a position to finish the proof of the main result.

Theorem 3.2.7. Let $X$ be a smooth quasiprojective scheme over $k$, and $\mathcal{E} \in \text{Isoc}^1(X \subset \mathfrak{X}/W(k)$ a locally free isocrystal. Then there is a natural quasiisomorphism

$$R\Gamma_{rig}(X, \mathcal{E}) \rightarrow R\Gamma(X, \mathcal{E} \otimes (W^1\Omega_{X/k} \otimes \mathbb{Q})).$$

Proof. Choose a finite covering $\{X_i\}_{i \in I}$ as above, and consider the associated simplicial scheme $\mathfrak{X}_\bullet = (X_J)_{J \subset I}$ together with the augmentation $\epsilon : \mathfrak{X}_\bullet \rightarrow X$. Upon choosing liftings of the $X_i$ over $W(k)$ one obtains a simplicial object of frames $(X_J, F_J)$ and as a consequence a simplicial object of dagger spaces $\Omega_\bullet = \{\Omega_J\}_{J \subset I}$. As seen above, we have for each $J \subset I$ a comparison morphism

$$\text{sp}_\bullet \mathcal{E} \otimes \Omega_{Q_J} \rightarrow \mathcal{E} \otimes (W^1\Omega_{X_J/k} \otimes \mathbb{Q})$$

which glues to a morphism of simplicial sheaves

$$\text{sp}_\bullet \mathcal{E} \otimes \Omega_{\mathfrak{X}_\bullet} \rightarrow \mathcal{E} \otimes (W^1\Omega_{\mathfrak{X}_\bullet/k} \otimes \mathbb{Q})$$  \hspace{1cm} (3.2.6)

and by Propositions 3.2.5 and 3.2.3 it gives rise to a quasiisomorphism

$$R \text{sp}_\bullet \mathcal{E} \otimes \Omega_{\mathfrak{X}_\bullet} \rightarrow \mathcal{E} \otimes (W^1\Omega_{\mathfrak{X}_\bullet/k} \otimes \mathbb{Q}).$$

Because of functoriality in Witt frames, we may now apply the functor $R\epsilon_\bullet$ to obtain

$$R\epsilon_\bullet R \text{sp}_\bullet \mathcal{E} \otimes \Omega_{\mathfrak{X}_\bullet} \cong R\epsilon_\bullet \mathcal{E} \otimes (W^1\Omega_{\mathfrak{X}_\bullet/k} \otimes \mathbb{Q}) \cong \mathcal{E} \otimes (W^1\Omega_{X/k} \otimes \mathbb{Q}).$$  \hspace{1cm} (3.2.7)

As pointed out in the proof of [7, Theorem 4.40], in the case without coefficients a result by Chiarellotto and Tsuzuki [4] implies that the left-hand side of the equality (3.2.7) is a complex on $X$ whose hypercohomology is rigid cohomology. However, Chiarellotto’s and Tsuzuki’s results are a lot more general dealing with coherent sheaves with integrable connections. Thus they apply to our situation as well.

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