SPECIFICATION AND THERMODYNAMICAL PROPERTIES OF SEMIGROUP ACTIONS

FAGNER B. RODRIGUES AND PAULO VARANDAS

ABSTRACT. In the present paper we study the thermodynamical properties of finitely generated continuous subgroup actions. We propose a notion of topological entropy and pressure functions that does not depend on the growth rate of the semigroup and introduce strong and orbital specification properties, under which, the semigroup actions have positive topological entropy and all points are entropy points. Moreover, we study the convergence and Lipschitz regularity of the pressure function and obtain relations between topological entropy and exponential growth rate of periodic points in the context of semigroups of expanding maps. The specification properties for semigroup actions and the corresponding one for its generators and the action of push-forward maps is also discussed.

1. Introduction

The thermodynamical formalism was brought from statistical mechanics to dynamical systems by the pioneering works of Sinai, Ruelle and Bowen [8, 9, 38] in the mid seventies. The correspondence between one-dimensional lattices and uniformly hyperbolic maps allowed to translate and introduce several notions of Gibbs measures and equilibrium states in the realm of dynamical systems. The present study of the thermodynamical formalism for non-uniformly hyperbolic dynamical systems is now parallel to the development of a thermodynamical formalism of gases with infinitely many states, a hard subject not yet completely understood. Moreover, the notion of entropy constitutes one of the most important in the study of dynamical systems (we refer the reader to Katok [23] and references therein for a survey on the state of the art).

An extension of the thermodynamical formalism for continuous finitely generated group actions has revealed fundamental difficulties and the global description of the theory is still incomplete. A first attempt was to consider continuous actions associated to finitely generated abelian groups. The statistical mechanics of expansive $\mathbb{Z}^d$-actions satisfying a specification property was studied by Ruelle [33], where he introduced a pressure function, defined on the space of continuous functions, and discussed its relations with measure theoretical entropy and free energy. The notion of specification was introduced in the seventies as a property of uniformly hyperbolic basic pieces and became a characterization of complexity in dynamical
systems. The crucial fact that continuous $\mathbb{Z}^d$-actions on compact spaces admit probability measures invariant by every continuous maps associated to the group action allowed Ruelle to prove a variational principle for the topological pressure and to build equilibrium states as the class of pressure maximizing invariant probability measures. This duality between topological and measure theoretical complexity of the dynamical system is very fruitfull, e.g. was used later by Eizenberg, Kifer and Weiss [17] to establish large deviations principles to $\mathbb{Z}^d$-actions satisfying the specification property. Other specification properties of interest have been introduced recently (see e.g. [15]).

A unified approach to the thermodynamical formalism of continuous group actions is still unavailable, while still few definitions of topological pressure exists and most of them unrelated. In many cases the existent definitions for topological entropy take into account either abelianity, amenability or growth rate of the corresponding group. A non-extensive list of contributions by many authors include important contributions by Ghys, Langevin, Walczak, Friedland, Lind, Schmidt, Bufetov, Biś, Urbanski, Ma, Wu, Miles, Ward, Chen, Zheng and Schneider among others (see e.g. [21, 19, 26, 11, 3, 6, 28, 4, 5, 29, 14, 36] and references therein).

While the connection between topological and ergodic properties of group actions still fails to provide a complete description the complexity of the dynamical system, our approach is to extend some notions originally from the statistical mechanics, used much successfully by Ruelle [33] for $\mathbb{Z}^d$-actions, for the more general context of finitely generated semigroup actions. No commutativity or conditions on the semigroup growth rate will be required. Inspired by several forms of the specification property for discrete time transformations with some hyperbolicity (see e.g. [37, 32, 35, 31, 42]), we introduce the notions of strong and orbital specification properties for continuous actions associated to finitely generated (not necessarily abelian) groups and deduce some of its relation with exponential growth of periodic points and positive topological entropy. The notion of topological entropy and pressure used here is strongly motivated by the topological entropy Bufetov [11] and does not take into account the semigroup growth rate. In the case of semigroup actions of expanding transformations we prove not only that both topological pressure and entropy can not only be computed as true limits of the increasing topological complexity of the dynamics at a given scale, but also that the topological pressure function on the space of Hölder continuous observables can be shown to exhibit a Lipschitz regularity (we refer the reader to the theorems for precise statements).

The relation between strong and orbital specification properties for the group action is also discussed and a class of examples of group actions is given where orbital specification properties present a flavor of the non-uniform versions arising in non-uniformly hyperbolic dynamics. For the convenience of the reader, we describe briefly the beginning of each section the main results to be proved there. Except when we mention explicit otherwise, we shall consider the context of semigroup actions and, in case the existence of inverse elements is needed, we shall make precise mention to that fact.

This paper is organized as follows. In Section 2 we introduce both the strong specification property and some orbital specification properties for finitely generated semigroups actions and discuss the relation between these notions and the specification property for the generators. The connections between specification
properties for group actions, for the push-forward group actions and hyperbolic-
ity are also discussed. In Section 3 we introduce a notion of topological entropy
and pressure for continuous semigroup actions and study group actions that ex-
hibit some forms of specification. In particular, we prove that these have positive
topological entropy and every point is an entropy point. In Section 4 we study
the semigroup action induced by expanding maps. We prove that these semigroups
satisfy the previous notions of specification and that topological entropy is a lower
bound for the exponential growth rate of periodic orbits. We also deduce that the
pressure function acting on the space of Hölder continuous potentials is Lipschitz,
 hence almost everywhere differentiable along families $t\varphi$ with $t \in \mathbb{R}$ and \varphi Hölder
continuous. Finally, in Section 5 we provide several examples where we discuss
the specification properties and establish a comparison between some notions of
topological entropy.

2. Specification for a finitely generated semigroup actions

In this section we introduce the notions of specification and orbital specification
properties for the context of group and semigroup actions. The specification prop-
erty for the group action implies that all generators satisfy the specification property
(Lemma 2.2) and also that the push-forward group action satisfies the specifica-
tion property (Theorem A). Moreover, $C^1$-robust specification implies structural
stability (Corollary 2.9).

2.1. Strong specification property. The specification property for a continuous
map on a compact metric space $X$ was introduced by Bowen [7]. A continuous map
$f: X \to X$ satisfies the specification property if for any $\delta > 0$ there exists an integer
$p(\delta) \geq 1$ such that the following holds: for every $k \geq 1$, any points $x_1, \ldots, x_k$, and
any sequence of positive integers $n_1, \ldots, n_k$ and $p_1, \ldots, p_k$ with $p_i \geq p(\delta)$ there
exists a point $x \in X$ such that

$$d(f^j(x), f^j(x_1)) \leq \delta, \quad \forall 0 \leq j \leq n_1$$

and

$$d(f^{j+n_1+p_1+\cdots+n_{i-1}+p_{i-1}}(x), f^j(x_i)) \leq \delta$$

for every $2 \leq i \leq k$ and $0 \leq j \leq n_i$. This property means that pieces of orbits of $f$
can be $\delta$-shadowed by a individual orbit provided that the time lag between each
shadowing is larger than some prefixed time $p(\delta)$.

The notion of specification was extended to the context of continuous $\mathbb{Z}^d$-actions
on a compact metric space $X$ by Ruelle motivated by statistical mechanics. Let
$(\mathbb{Z}^d, +)$ be endowed with the distance $d_{\mathbb{Z}^d}(a, b) = \sum_{i=1}^{p} |a_i - b_i|$. Following [33],
the group action $\mathbb{Z}^d \times X \to X$ satisfies the specification property if for any $\delta > 0$
there exists $p(\delta) > 0$ such that for any finite families $(\Lambda_i)_{i \in I}, (x_i)_{i \in I}$ satisfying if
$i \neq j$, the distance of $\Lambda_i$, $\Lambda_j$ (as subsets of $\mathbb{Z}^d$) is $> p(\delta)$, there is $x \in X$ such that
d$(m_i, x, m_i, x_i) < \delta$, for all $i \in I$, and all $m_i \in \Lambda_i$. This notion clearly extends to
group actions associated to finitely generated abelian groups.

Specification property for groups and its generators. In this article we shall ad-
dress the specification properties and thermodynamical formalism to deal both
with finitely generated group actions and semigroup actions. For simplicity, we shall
state our results in the more general context of semigroups actions whenever the re-
sults do not require the existence of inverse elements. More precisely, given a finitely
generated semigroup \((G, \circ)\) with a finite set of generators \(G_1 = \{id, g_1, g_2, \ldots, g_m\}\) one can write \(G = \bigcup_{n \in \mathbb{N}_0} G_n\) where \(G_0 = id\) and

\[(1) \quad g \in G_n \text{ if and only if } g = g_{i_n} \cdots g_{i_1}g_{i_1} \text{ with } g_{i_j} \in G_1\]

(where we use \(g_jg_i\) instead of \(g_j \circ g_i\) for notational simplicity). If, in addition, the elements of \(G_1\) are invertible, the finitely generated group \((G, \circ)\) is defined by 
\[G = \bigcup_{n \in \mathbb{N}_0} G_n\] where \(G_0 = id, G_1 = \{id, g_1^\pm, g_2^\pm, \ldots, g_m^\pm\}\) and the elements \(g \in G_n\) are defined by \([1]\). In both settings, \(G_n\) consists of those group elements which are concatenations of at most \(n\) elements of \(G_1\). Since \(id \in G_n\) then \((G_n)_{n \in \mathbb{N}}\) defines an increasing family of subsets of \(G\). Moreover, \(G\) is a finite semigroup if and only if \(G_n\) is empty for every \(n\) larger than the cardinality of the group. Given a semigroup \(G\) we say \(g \in G\) has finite order if there exists \(n \geq 1\) so that \(g^n = id\). If the later property does not hold then an element \(g \in G\) is said to have infinite order. Denote by \(G_1^* = G_1 \setminus \{id\}\) and \(G_n^* = \{g = g_{i_n} \cdots g_{i_1}g_{i_1} \mid g_{i_j} \in G_1^*\}\). For shortness we shall write \(|g| = n\) instead of \(g \in G_n^*\).

Motivated by applications by actions of non-abelian semigroups we first introduce some generalizations of the previous specification property for group actions. Let \((G, \circ)\) be a finitely generated group endowed with the distance \(d_G(h, g) = |h^{-1}g|\), for \(h, g \in G\). It is not difficult to check that it is a metric in the group \(G\) and that \(d_G(h, g) = n\) if and only if there exists \(g_n \in G_n\) so that \(g = h g_n\). We are unaware of a natural notion of metric for semigroups. The following notion extends of the specification property introduced by \([33]\) to more general group actions.

**Definition 2.1.** Let \(G\) be a finitely generated group, \(X\) be a compact metric space and let \(T : G \times X \to X\) be a continuous action. We say that the group action \(T\) has the **specification property** if for any \(\delta > 0\) there exists \(p(\delta) > 0\) such that for any finite families \((\Lambda_i)_{i \in I}, (x_i)_{i \in I}\) so that the \(d_G(\Lambda_i, \Lambda_j) > p(\delta)\) for every \(i \neq j\), then there is \(x \in X\) such that \(d(g_i x, g_i x_i) < \delta\) for every \(i \in I\) and \(g_i \in \Lambda_i\).

The later notion implies on a strong topological indecomposability of the group action. Given a continuous action \(T : G \times X \to X\) we say that \(T\) is topologically transitive if there exists a point \(x \in X\) such that the orbit \(O_G(x) := \{g(x) : g \in G\}\) is dense in \(X\). We say that \(T\) is topologically mixing if for any open sets \(A, B \subseteq X\) there exists \(N > 1\) such that for any \(n \geq N\) there is \(g \in G\) with \(|g| = n\) satisfying \(g(A) \cap B \neq \emptyset\). It is easy to check that any continuous action with the specification property is topologically mixing, hence topologically transitive. For a survey on several mixing properties for group actions we refer the reader to the survey \([13]\) and references therein.

Given a continuous action \(T : G \times X \to X\) of a group \(G\) on a compact metric space \(X\) we denote, by some abuse of notation, \(g : X \to X\) to be the continuous map \(x \mapsto T(g, x)\). Given \(g \in G\) we say that \(x \in X\) is a **fixed point** for \(g\) if \(g(x) = x\) and use the notation \(x \in Fix(g)\). We say that \(x \in M\) is a **periodic point of period** \(n\) if there exists \(g_n \in G_n\) so that \(g_n(x) = x\). In other words, \(x \in \bigcup_{g \in G_n} Fix(g)\). We let \(Per(G_n)\) denote the set of periodic points of period \(n\) and set \(Per(G) = \bigcup_{n \geq 1} Per(G_n)\). If the tracing orbit in the specification property can be chosen periodic we will say that the action satisfies the **periodic specification** property. It is not hard to check that an invertible transformation \(f : X \to X\) satisfies the specification property if and only if the group action on \(X\) associated to the group \(G = \{f^n : n \in \mathbb{Z}\}\) (isomorphic to \(\mathbb{Z}\)) satisfies the specification property.
The next lemma asserts that this specification property for group actions implies all generators to satisfy the corresponding property.

**Lemma 2.2.** Let $G$ be a finitely generated with generators $G_1 = \{g_1^\pm, g_2^\pm, \ldots, g_k^\pm\}$. If the group action $T : G \times X \to X$ satisfies the specification property then every $g \in G_1$ with infinite order has the specification property.

**Proof.** Let $\delta > 0$ be fixed and let $p(\delta) > 0$ be given by the specification property for the group action $T$. Take arbitrary $k \geq 1$, points $x_1, \ldots, x_k$, and positive integers $n_1, \ldots, n_k$ and $p_1, \ldots, p_k$ with $p_i \geq p(\delta)$. Since $g \in G_1$ is a generator then for any $i = 1 \ldots k$ the set

$$\Lambda_i = \left\{ g^j : \sum_{s=0}^{i-1} (p_s + n_s) \leq j \leq n_i + \sum_{s=0}^{i-1} (p_s + n_s) \right\}$$

is finite and connected (assume $n_0 = p_0 = 0$). Moreover, since $g$ has infinite order it is not hard to check that $d_{\omega}(\Lambda_i, \Lambda_j) \geq p(\delta)$ for any $i \neq j$. Let $\bar{x}_j = g^{-\sum_{s=1}^{i} p_s + n_s}(x_j)$, for $1 \leq j \leq k$. Thus, by the specification property there exists a point $x \in X$ such that $d(hx, h\bar{x}_i) < \delta$, for all $i = 1 \ldots k$ and all $h \in \Lambda_i$ which are reduced in this case to

$$d\left( g^i(x), g^j(x_1) \right) \leq \delta, \quad \forall 0 \leq j \leq n_1$$

and

$$d\left( g^{i+n_1+p_1+\ldots+n_{i-1}+p_{i-1}}(x), g^i(x_1) \right) \leq \delta$$

for every $2 \leq i \leq k$ and $0 \leq j \leq n_i$. This proves that the map $g$ has the specification property and finishes the proof of the lemma. □

Let us mention that the existence of elements of generators of finite order is not an obstruction for the group action to have the specification (e.g. the $\mathbb{Z}^2$-action on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ whose generators are a hyperbolic automorphism and the reflection on the real axis). We refer the reader to Section 3 for a simple example of a $\mathbb{Z}^2$-action for which the converse implication is not necessarily true.

**The push-forward group action.** Given a compact metric space $X$ let $\mathcal{P}(X)$ denote the space of probability measures on $X$, endowed with the weak*-topology. It is well known that $\mathcal{P}(X)$ with the weak* topology is a compact set. We recall that the weak*-topology in $\mathcal{P}(X)$ is metrizable and a metric that generates the topology can be defined as follows. Given a countable dense set of continuous functions $(\phi_k)_{k \geq 1}$ in $C(X)$ and $\mu, \nu \in \mathcal{P}(X)$ define

$$d_{\mathcal{P}}(\mu, \nu) = \sum_{k \geq 1} \frac{1}{2^k \|\phi_k\|} \left| \int \phi_k \, d\mu - \int \phi_k \, d\nu \right| .$$

For a continuous map $f : X \to X$, the space of $f$-invariant probability measures correspond to the fixed points of the push-forward map $f^\#: \mathcal{P}(X) \to \mathcal{P}(X)$, which is a continuous map. For that reason the push-forward $f^\#$ reflects the ergodic theoretical aspects of $f$. Moreover, the dynamics of $f$ is embedded in the one of $f^\#$ since it corresponds to the restriction of $f^\#$ to the space $\{\delta_x : x \in X\} \subset \mathcal{P}(X)$ of Dirac measures on $X$. This motivates the study of specification properties for the group action of the push-forward maps.

Given a finitely generated group $G$ and a continuous group action $T : G \times X \to X$ let us denote by $T^\#: G \times \mathcal{P}(X) \to \mathcal{P}(X)$ denote the group action defined by
It implies that the probability measure \( T \) continuous group action satisfying the specification property. Then the group action \( T_g : G \times \mathcal{P}(X) \to \mathcal{P}(X) \) satisfies the specification property.

The following lemma will play an instrumental role in the proof of the theorem.

**Lemma 2.3.** Given probability measures \( \mu_1, ..., \mu_k \in \mathcal{P}(X) \) and \( \delta > 0 \), there are \( N \in \mathbb{N} \) and points \( (x_1', ..., x_N') \in X^N \) such that the probabilities \( \mu_i' = \frac{1}{N} \sum_{j=1}^{N} \delta_{x_j} \) satisfy \( d(\mu_i, \mu_i') < \delta \) for \( 1 \leq i \leq k \).

**Proof.** It is well known that the finitely supported atomic measures are dense in \( \mathcal{P}(X) \). Then, for \( \delta > 0 \), there are \( \bar{\mu}_1, ..., \bar{\mu}_k \), with \( \bar{\mu}_j = \sum_{j=1}^{M} \alpha_j \delta_{x_j} \in \mathcal{P}(X) \), so that \( d(\mu_k, \bar{\mu}_k) < \delta/2 \). Let \( p_j'/q_j' \) be a positive rational such that \( |\alpha_j' - p_j'/q_j'| < \delta/10 \). Let \( N = \prod_{i,j=1}^{M} q_i' \) and \( N'_i = p_i' \prod_{i,j=1, i \neq k}^{M} q_i' \). Notice that \( |N'_i/N - \alpha_i'| < \delta/10 \) and

\[
\mu_i' = \frac{1}{N} \left( \sum_{i=1}^{N'_i} \delta_{x_j} \right),
\]

satisfies \( d(\mu_j', \bar{\mu}_j) < \delta/2 \), and by triangular inequality, \( d(\mu_j, \mu_j') < \delta \). \( \Box \)

**Proof of the Theorem 2.4.** Assume that the action \( T : G \times X \to X \) has the specification property. Clearly, if \( T \) satisfies the specification property then for any \( N \geq 1 \) the continuous action \( T^{(N)} : G \times X^N \to X^N \) on the product space \( X^N \) endowed with the distance \( d_N((x_i)_i, (y_i)_i) = \max_{1 \leq i \leq N} d(x_i, y_i) \) and given by \( g \cdot (x_1, ..., x_N) = (g x_1, ..., g x_N) \) also satisfies the specification. In fact, for any \( \delta > 0 \) just take \( p(\delta) > 0 \) as given by the specification property for \( T \).

Let us proceed with the proof of the theorem. Take \( \delta > 0 \) and let \( p(\delta/2) \) be given by the specification property. Take \( \mu_1, ..., \mu_k \in \mathcal{P}(X) \) and \( \Lambda_1, ..., \Lambda_k \) finite subsets of \( G \) with \( d(\Lambda_i, \Lambda_j) > p(\delta/2) \). Let \( \mu'_i = \frac{1}{N} \sum_{j=1}^{N} \delta_{x_j} \), such that \( d(\mu'_i, g \mu_i) < \delta/2 \) for all \( g \in \Lambda_i \). By considering the finite sequence \( (x_1', ..., x_N') \in X^N \) and the sets \( \Lambda_1, ..., \Lambda_k \), there exists a point \( (x_1', ..., x_N') \in X^N \) in the product space such that

\[
d(g \cdot (x_1, ..., x_N), g \cdot (x_1', ..., x_N')) < \frac{\delta}{2} \text{ for all } g \in \Lambda_i.
\]

It implies that the probability measure \( \mu = \frac{1}{N} \sum_{j=1}^{N} \delta_{x_j} \) satisfies

\[
d(g \cdot \mu, g \cdot \mu) < \frac{\delta}{2}, \text{ for all } g \in \Lambda_i.
\]

This completes the proof of the theorem. \( \Box \)

The converse implication in the previous theorem is not immediate. In fact, given the specification property for \( T_g \) and any specified pieces of orbit by \( T_g \) it is not clear that this can be shadowed by the \( T_{l_g} \)-orbit of a Dirac probability measure \( \delta_l \). Nevertheless this is indeed the case for the dynamics of continuous interval maps.

**Corollary 2.4.** Let \( f \) be a continuous interval map. Then \( f \) satisfies the specification property if and only if \( f_2 \) satisfies the specification property.
Proof. It follows from Theorem A that the specification property for \( f \) implies the specification property for \( f_\# \), and so we are reduced to prove the other implication. First we observe that the specification property implies the topologically mixing one. By \([2]\), \( f \) is topologically mixing if and only if \( f_\# \) is topologically mixing. Moreover, Blokh \([10]\) proved that any continuous topologically mixing interval map satisfies the specification property, thus these are equivalent properties for continuous interval maps. This proves the corollary. □

It is not clear to us if \([10]\) can be extended to group actions, and so the previous equivalence does not have immediate counterpart for group actions of continuous interval maps.

2.2. Orbital specification properties. In this subsection we introduce weaker notions of specification. In opposition to the notion introduced in Definition 2.1, which takes into account the existence of a metric in the group, the following orbital specification properties are most suitable for semigroups actions. A first problem to define orbital specification properties is that group elements \( g \in G \) may have different representations as concatenation of the generators. For that reason one should explicitly mention what is the ‘path’, or concatenation of elements, that one is interested in tracing.

Definition 2.5. We say that the continuous semigroup action \( T : G \times X \to X \) associated to the finitely generated semigroup \( G \) satisfies the strong orbital specification property if for any \( \delta > 0 \) there exists \( p(\delta) > 0 \) such that for any \( h_{p_j} \in G_{p_j}^* \) (with \( p_j \geq p(\delta) \) for \( 1 \leq j \leq k \)) any points \( x_1, \ldots, x_k \in X \) and any natural numbers \( n_1, \ldots, n_k \), any semigroup elements \( g_{n_j, j} = g_{i_{n_j, j}} \cdots g_{i_2, j} g_{i_1, j} \in G_{n_j} \) \((j = 1 \ldots k)\) there exists \( x \in X \) so that \( d( g_{n_1, 1}(x) : g_{n_1, 1}(x_1) ) < \delta \) for every \( \ell = 1 \ldots n_1 \) and

\[
d( g_{n_1, 1} h_{p_{j-1}} \cdots g_{n_2, 2} h_{p_1} g_{n_{1, 1}}(x) : g_{n_{1, 1}}(x_{j})) < \delta
\]

every \( j = 2 \ldots k \) and \( \ell = 1 \ldots n_j \) (here \( g_{n_{1, 1}} := g_{i_{1, 1}} \cdots g_{i_{1, 1}} \)).

Remark 2.6. The previous notion demands that every ‘long word’ semigroup element \( h_{p_j} \) can be used to shadow the pieces of orbits. Here, ‘long word’ means that the element has at least one representation that is obtained by concatenation of a large number \((\geq p_j)\) of generators, the identity not included. In the case of finitely generated free semigroups the representation of every element as a concatenation of generators is unique and it makes sense to notice that the size \( |h_{p_j}| \) of an element \( h_{p_j} \) is well defined and coincides with \( p_j \). However, the later property holds for
group actions if and only if $X$ is a unique point, since in the case that $G$ is a group then $id \in G_n$ for every $n \geq 2$. This is one of the reasons to choose $G_n^*$ instead of $G_n$.

We also introduce a weaker notion of orbital specification for semigroups inspired by some nonuniform versions for maps.

**Definition 2.7.** We say that the continuous semigroup action $T : G \times X \to X$ associated to the finitely generated semigroup $G$ satisfies the weak orbital specification property if for any $\delta > 0$ there exists $p(\delta) > 0$ so that for any $p \geq p(\delta)$, there exists a set $\bar{G}_p \subset G^*$ satisfying $\lim_{p \to \infty} \frac{\# \bar{G}_p}{p} = 1$ and for which the following holds: for any $h_{p_j} \in \bar{G}_{p_j}$ with $p_j \geq p(\delta)$, any points $x_1, \ldots, x_k \in X$, any natural numbers $n_1, \ldots, n_k$ and any concatenations $g_{n_1, j} = g_{i_1, j} \cdots g_{i_j, j}$ with $1 \leq j \leq k$ there exists $x \in X$ so that $d(\mu_{x, j}(x), \mu_{x, 1}(x)) < \delta$ for every $j = 1 \ldots n_1$ and for every $j = 2 \ldots k$ and $\ell = 1 \ldots n_j$.

This definition weakens the later one by allowing a set of admissible elements (whose proportion increases among all possible semigroup elements) for the shadowing. It is not hard to check that the later notions do not depend on the set of generators for the semigroup. Non-uniform versions of the previous orbital specification properties can be defined in the same spirit as [33, 31, 42, 32, 41], but we shall not need or use this fact here.

In Section 5 we provide examples satisfying the orbital specification property but not the usual specification property. The following proposition is the counterpart of Theorem [X] for orbital specification properties.

**Proposition 2.8.** Let $G$ be a finitely generated group. If a continuous group action $T : G \times X \to X$ satisfies the strong (resp. weak) orbital specification property then the push-forward group action $T_x^* : G \times \mathcal{P}(X) \to \mathcal{P}(X)$ satisfies the strong (resp. weak) orbital specification property.

**Proof.** Since the proofs of the two claims in the proposition are similar we shall prove the first one with detail and omit the other. By Lemma 2.3 it is enough to prove the proposition for probabilities that lie on the set $\mathcal{M}_N(X) = \{ \frac{1}{N} \sum_{\ell=1}^N \delta x_{\ell} : x_{\ell} \in X \}$, for any $N \in \mathbb{N}$. Observe that if $T$ satisfies the strong orbital specification property then the same property holds for the induced action $T^{(N)}$ on the product space $X^N$. Let $\delta > 0$ and take $p(\delta) \in \mathbb{N}$ given by the strong orbital specification property of the induced action on $X^N$. Let $\mu_1, \ldots, \mu_k \in \mathcal{M}_N(X)$ with $\mu_j = \frac{1}{N} \sum_{\ell=1}^N \delta x_{\ell}$ and $\mu_{n_1, j} \in G_{n_1}$ (1 $\leq j \leq k$) be given. If we consider $\bar{x}_j = (x_1^j, \ldots, x_N^j)$, for any $h_{p_j} \in G_{n_j}$ such that $d(g_{\ell, j}(x), g_{\ell, 1}(\bar{x}_1)) < \delta$ for every $\ell = 1, \ldots, n_1$ and for every $j = 2, \ldots, k$ and $\ell = 1, \ldots, n_j$. Let $\mu = \frac{1}{N} \sum_{\ell=1}^N \delta x_{\ell}$. In particular $\mu$ satisfies $d(\mu_{\ell, 1} \cdot \mu, \mu_{\ell, 1} \cdot \mu_{1}) < \delta$ for every $\ell = 1, \ldots, n_1$ and for every $j = 2, \ldots, k$ and $\ell = 1, \ldots, n_j$, which finishes the proof of the proposition. □
2.3. Specification and hyperbolicity. The relation between specification properties, uniform hyperbolicity and structural stability has been much studied in the last decades, a concept that we will recall briefly. The content of this subsection is of independent interest and will not be used later on along the paper. Given a $C^1$ diffeomorphism $f$ on a compact Riemannian manifold $M$ and an $f$-invariant compact set $\Lambda \subset M$ (that is $f(\Lambda) = \Lambda$) we say that $\Lambda$ is uniformly hyperbolic if there exists a $Df$-invariant splitting $T_\Lambda M = E^s \oplus E^n$ and constants $C > 0$, $0 < \lambda < 1$ so that $\|Df^n(x)\|_{E^s} \leq C\lambda^n$ and $\|(Df^n(x)\)|_{E^n}^{-1} \leq C\lambda^n$ for every $x \in \Lambda$ and $n \geq 1$. If $\Lambda = M$ is a hyperbolic set for $f$ then $f$ is called an Anosov diffeomorphism.

Originally the notion of specification was introduced by Bowen [7] for uniformly hyperbolic dynamics but fails dramatically in the complement of uniform hyperbolicity (even partially hyperbolic dynamical systems with period points of different index do not satisfy the specification property, see [39, 40] for more details). On the other hand Sakai, Sumi and Yamamoto [35] proved that if the specification property holds in a $C^1$-open set of diffeomorphisms then the dynamical systems are Anosov. It is well know that every $C^1$ Anosov diffeomorphism $f$ is structurally stable, that is, there exists a $C^1$-open neighborhood $\mathcal{U}$ of $f$ in $\text{Diff}^1(M)$ so that for every $g \in \mathcal{U}$ there is an homeomorphism $h_g : M \to M$ satisfying $g \circ h_g = h_g \circ f$. Thus the $C^1$-robust specification implies rigidity of the underlying dynamical systems.

The previous results can be extended for finitely generated group actions acting on a compact Riemannian manifold $M$ in a more or less direct way as we now describe. Let $G$ be a finitely generated subgroup of $\text{Diff}^1(M)$ with generators $G_1 = \{g_1^\pm, \ldots, g_k^\pm\}$. We will say that the group action $G \times M \to M$ is structurally stable if all the generators are structurally stable. In other words, there are $C^1$-neighborhoods $\mathcal{U}_i$ of the generators $g_i$ ($1 \leq i \leq k$) such that for any choice $\tilde{g}_i \in \mathcal{U}_i$ there exists a homeomorphism $h_i$ such that $\tilde{g}_i \circ h_i = h_i \circ g_i$. In the case that $G$ is abelian one can require the conjugacies to coincide (c.f. definition of structural stability by Sad [34]). We say that the group action $T : G \times M \to M$ satisfies the $C^1$-robust specification property if there exists a $C^1$-neighborhood $\mathcal{V}$ of $T$ such that any $C^1$-action $\tilde{T} \in \mathcal{V}$ satisfies the specification property. As a byproduct of the previous results we deduce the following consequence:

**Corollary 2.9.** Let $G$ be a finitely generated subgroup of $\text{Diff}^1(M)$ such that group action $T : G \times M \to M$ satisfies the $C^1$-robust specification property. Then every generator is an Anosov diffeomorphism and the group action is structurally stable.

**Proof.** Since the group action $T : G \times M \to M$ satisfies the $C^1$-robust specification property there exists a $C^1$-neighborhood $\mathcal{V}$ of $T$ such that any $C^1$-action $\tilde{T} \in \mathcal{V}$ satisfies the specification property. Moreover, from Lemma 2.2 any such $T$ can be identified with a group action associated to a subgroup $G$ of $\text{Diff}^1(M)$ whose generators $G_1 = \{g_1^\pm, \ldots, g_k^\pm\}$ satisfy the specification property. This proves that the generators $g_i \in \text{Diff}^1(M)$ satisfy the $C^1$-robust specification property and, by [35], are Anosov diffeomorphisms, hence structurally stable. This proves the corollary.

The previous discussion raises the question of whether the $C^1$-smoothness assumption is necessary in the previous characterization. For instance, one can ask if a homeomorphism satisfying the specification property $C^0$-robustly has some form of hyperbolicity. In the remaining of this subsection we shall address some comments
on this problem taking as a simple model the push-forward dynamics, which is continuous and acts on the compact metric space of probability measures. Roughly, we will look for some hyperbolicity of the push-forward dynamics assuming that it has the specification property. Clearly, if \( f \) is a topologically mixing subshift of finite type then it satisfies the specification property and so does \( f^t \). On the other hand, the set of \( f \)-invariant measures are (non-hyperbolic) fixed points for \( f^t \) and, consequently, this map does not present global hyperbolicity. For that reason we will focus on the fixed points for the continuous map \( f \) of finite type then it satisfies the specification property and so does \( f \). Clearly, if \( f \) is a hyperbolic fixed point of saddle type for \( f \) then \( f \) has the specification property. Therefore, to analyze the existence of hyperbolic fixed points of saddle type for \( f \), we will mostly be interested in hyperbolic fixed points of saddle type for \( f \).

It follows from the definition that hyperbolic fixed points for \( f \) then we will mostly be interested in hyperbolic fixed points of saddle type for \( f \) if it is a fixed point and there exists \( \delta > 0 \) and constants \( C > 0 \) and \( 0 < \lambda < 1 \) so that:

\[
\begin{align*}
(\text{i}) & \quad d_p(f^j_\eta(\mu), f^j_\eta(\eta)) < C\lambda^j \quad \text{for every } j \geq 1 \text{ and } \eta \in W^s_\delta(\mu) \\
(\text{ii}) & \quad d_p(f^{-j}_\eta(\mu), f^{-j}_\eta(\eta)) < C\lambda^j \quad \text{for every } j \geq 1 \text{ and } \eta \in W^u_\delta(\mu)
\end{align*}
\]

We say that the hyperbolic fixed point is of saddle type if both stable and unstable sets are non-trivial. Since the specification implies the topologically mixing property then we will mostly be interested in hyperbolic fixed points of saddle type for \( f \).

It follows from the definition that hyperbolic fixed points for \( f \) are isolated. The following properties follow from the definitions and Lemma 2.3:

1. \( f \) is an affine map, that is, \( f(t\eta + s\mu) = tf_\eta(\eta) + sf_\mu(\mu) \) for every \( t, s \geq 0 \) with \( t + s = 1 \) and \( \eta, \mu \in \mathcal{P}(X) \)
2. \( \mu \) is an isolated fixed point for \( f \) if and only if the set of \( f \)-invariant probability measures satisfies \( \mathcal{M}_f(X) = \{ \mu \} \) (i.e. \( f \) is uniquely ergodic),
3. \( \mathcal{M}_n(X) = \{ \frac{1}{n} \sum_{i=1}^n \delta_{x_i} : x_i \in X \} \subset \mathcal{P}(X) \) is a closed \( f \)-invariant set, and
4. \( \bigcup_{n \geq 1} \mathcal{M}_n(X) \) is a dense subset of \( \mathcal{P}(X) \).

Therefore, to analyze the existence of hyperbolic fixed points of saddle type for \( f \) that satisfies the specification property we are reduced to the case that \( f \) is uniquely ergodic. If \( f \) is a contraction on a compact metric space then Banach’s fixed point theorem implies the existence of a unique fixed point that is a global attractor and, consequently, the Dirac measure at the attractor is the unique hyperbolic (attractor) fixed point for \( f \), which is incompatible with transitivity. However, it is nowadays well known that \( C^0 \)-generic maps have a dense set of periodic points (see e.g. [24]) and, in particular, \( C^0 \)-generic homeomorphisms \( f \) are not uniquely ergodic. In conclusion, there is no open set of homeomorphisms \( f \) so that \( f \) has a unique hyperbolic fixed point of saddle type.

3. Specification properties and the entropy of semigroup actions

The notion of entropy is one of the most important in dynamical systems, either as a topological invariant or as a measure of the chaoticity of the dynamical system. For that reason several notions of entropy and topological pressure have been introduced for group actions in an attempt to describe its dynamical characteristics. As discussed in the introduction, some of the previously introduced definitions take into account the growth rate of the (semi)group, that is, the growth of \( |G_n| \) as \( n \) increases (see e.g. [3] and references therein). We refer the reader to [24, 25] for a detailed description about growth rates for groups and geometric group theory.
In this section we characterize entropy points of semigroup actions with specification (Theorem [B]) and prove that these actions have positive topological entropy (Theorems [D] and [E]).

3.1. Entropy points. Let $X$ be a compact metric space and $G$ be a semigroup. First we shall introduce the notion of dynamical balls. Given $\varepsilon > 0$ and $g := g_{i_1} \cdots g_{i_n} \in G_n$ we define the dynamical ball $B(x, g, \varepsilon)$ by

$$B(x, g, \varepsilon) := B(x, g_{i_1} \cdots g_{i_n}, \varepsilon)$$

(2)

$$= \left\{ y \in X : d(g_j(y), g_j(x)) \leq \varepsilon, \text{ for every } 0 \leq j \leq n \right\}$$

where, by some abuse of notation, we set $g_j := g_{i_j} \cdots g_{i_1} \in G_n$ for every $1 \leq j \leq n - 1$ and $g_0 = id$. We also assign a metric $d_g$ on $X$ by setting

$$d_g(x_1, x_2) := d_{g_{i_1} \cdots g_{i_n}}(x_1, x_2) = \max_{0 \leq j \leq n} d(g_j(x_1), g_j(x_2)).$$

It is important to notice that here both the dynamical ball and metric are adapted to the underlying concatenation of generators $g_{i_1} \cdots g_{i_n}$ instead of the group element $g$, since the later one may have distinct representations. For notational simplicity we shall use the condensed notations $B(x, g, \varepsilon)$ and $d_g(\cdot, \cdot)$ when no confusion is possible. In the case that $g = f^n$ the later notions coincide with the usual notion of dynamical ball $B_f(x, n, \delta)$ and dynamical distance $d_f(\cdot, \cdot)$ with respect to the dynamical system $f$, respectively.

Now, we recall a notion of topological entropy introduced by Ghys, Langevin, Walczak [21] and the notion of entropy point introduced by Biś [5]. Two points $x, y$ in $X$ are $(n, \varepsilon)$-separated by $G$ if there exists $g \in G_n$ such that $d(g(x), g(y)) \geq \varepsilon$. Given $E \subset X$, let us denote by $s(n, \varepsilon, E)$ the maximal cardinality of $(n, \varepsilon)$-separated set in $E$. The limit

$$h((G, G_1), E) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log s(n, \varepsilon, E)$$

is well defined by monotonicity on $\varepsilon$. The entropy of $(G, G_1)$ is defined by the previous expression with $E = X$. In this setting of a semigroup $G$ we define by $B_G(x, n, \varepsilon) := \bigcap_{g = g_{i_1} \cdots g_{i_n} \in G_n} B(x, g, \varepsilon)$ the dynamical ball for the semigroup $G$ associated to $x$, length $n$ and size $\varepsilon$ centered at $x$, where the intersection is over all concatenations that lead to elements in $G_n$. This corresponds to consider points that are $\varepsilon$-close along the orbit of $x$ by all the trajectories arising from concatenations of generators. We say that the finitely generated semigroup $(G, G_1)$ acting on a compact metric space $X$ admits an entropy point $x_0$ if for any open neighbourhood $U$ of $x_0$ the equality

$$h((G, G_1), U) = h((G, G_1), X)$$

holds. Entropy points are those for which local neighborhoods reflect the complexity of the entire dynamical system. In [5], Biś proved remarkably that any finitely generated group $(G, G_1)$ acting on a compact metric space $X$ admits an entropy point $x_0$. We prove that the orbital specification property for continuous semigroup actions is enough to prove that all points are entropy points. More precisely,

**Theorem B.** Let $G \times X \to X$ be a continuous finitely generated semigroup action on a compact Riemannian manifold $X$ so that every element $g \in G_1$ is a local
homeomorphism. If the semigroup action satisfies the weak orbital specification property then every point of \(X\) is an entropy point.

**Proof.** First we notice that following the proof of \([5, \text{Theorem 2.5}]\) *ipsis litteris* we get the existence of an entropy point \(x_0 \in X\) for any finitely generated semigroup of continuous maps on \(X\) (the proof does not require invertibility). Hence, for any open neighborhood \(U\) of \(x_0\) it holds that \(h((G, G_1), X) = h((G, G_1), \overline{U})\). Let \(\zeta > 0\) be arbitrary and take \(\varepsilon = \varepsilon_0(\zeta) > 0\) such that

\[
\limsup_{n \to \infty} \frac{1}{n} \log s(n, \varepsilon, \overline{U}) \geq h((G, G_1), X) - \zeta
\]

for every \(0 < \varepsilon \leq \varepsilon_0\).

Given any \(z \in X\) and \(V\) any open neighborhood of \(z\) we claim that \(h((G, G_1), \overline{V}) = h((G, G_1), X)\). Fix \(0 < \varepsilon \leq \varepsilon_0\) let \(p(\varepsilon) \geq 1\) be given by the strong orbital specification property. Since there are finitely many elements in \(G_{p(\varepsilon)}\), finitely many of its concatenations and the local inverse branches of elements \(y : X \to X\) are uniformly continuous there exists a uniform constant \(C_\varepsilon > 0\) (that tends to zero as \(\varepsilon \to 0\)) so that \(\text{diam}(h^{-1}(B(y, \varepsilon))) \leq C_\varepsilon\) for every \(h \in G_{p(\varepsilon)}\) and \(y \in X\). Take \(n \geq 1\) arbitrary, let \(E = \{x_1, \ldots, x_l\} \subset \overline{U}\) be a maximal \((n, \varepsilon, \overline{U})\)-separated set and consider the open set \(W \subset V\) defined by the set of points \(y \in V\) so that \(d(y, \partial V) > C_{\varepsilon_0}\). Assume that \(0 < \varepsilon \ll \varepsilon_0\) satisfies \(\varepsilon + C_\varepsilon < C_{\varepsilon_0}\).

Let \(g := g_{i_1} \cdots g_{i_l} \in G_n\) be fixed. Given a maximal \((\varepsilon, \overline{W})\)-separated set \(F = \{z_1, \ldots, z_m\}\), by the weak specification property there exists \(h = h_{i(p)} \cdots h_{i_l} \in G_{p(\overline{\varepsilon})}\) so that for any \(x_i \in E\) and \(z_j \in F\), there exists \(y_i^j \in B(z_j, \frac{\varepsilon}{2}) \cap h^{-1}(B(x_i, g, \frac{\varepsilon}{4}))\). Since \(\text{diam}(h^{-1}(B(x_i, \frac{\varepsilon}{4}))) \leq C_\varepsilon\), this implies that \(d(h^{-1}(B(x_i, g, \frac{\varepsilon}{4})), \partial V) \geq C_{\varepsilon_0} - \frac{\varepsilon}{4}\). By construction, the dynamical balls \((B_G(x_i, n, \frac{\varepsilon}{4}))\) are pairwise disjoint and consequently the number of \((n+p(\overline{\varepsilon}), \frac{\varepsilon}{4})\)-separated points in \(\overline{V}\) is at least \(s(n, \varepsilon, \overline{U})\). In other words, \(s(n+p(\overline{\varepsilon}), \frac{\varepsilon}{4}, \overline{V}) \geq s(n, \varepsilon, \overline{U})\) and, consequently,

\[
\limsup_{n \to \infty} \frac{1}{n+p(\overline{\varepsilon})} \log s(n+p(\overline{\varepsilon}), \frac{\varepsilon}{4}, \overline{V}) \geq \limsup_{n \to \infty} \frac{1}{n+p(\overline{\varepsilon})} \log s(n, \varepsilon, \overline{U})
= \limsup_{n \to \infty} \frac{1}{n} \log s(n, \varepsilon, \overline{U}).
\]

The last inequalities show that \(h((G, G_1), X) \geq h((G, G_1), \overline{V}) \geq h((G, G_1), X) - \zeta\). Since \(\zeta\) was chosen arbitrary this completes the proof of the theorem.

The previous result indicates that the specification properties are powerful tools to prove the local complexity of semigroup actions. Observe that the previous result clearly applies for individual transformations.

We now use the notion of topological entropy introduced by Bufetov \([11]\). This notion measures the mean cardinality of separated points among possible trajectories generated by the semigroup. Although one can expect that most finitely generated semigroups are free and so to have exponential growth (c.f. proof of Proposition 4.5 by Ghys \([20]\) implying that for a Baire generic set of pairs of homeomorphisms the generated group is a free group on two elements) the notion of average entropy that we consider seems suitable for wider range of semigroups.
Let $E \subset X$ be a compact set. Given $g = g_i \cdots g_1 \in G_n$, we say a set $K \subset E$ is $(g, n, \varepsilon)$-separated set if $d_{x_i}(x_1, x_2) > \varepsilon$ for any distinct $x_1, x_2 \in K$. When no confusion is possible with the notation for the concatenation of semigroup elements, the maximum cardinality of a $(g, \varepsilon, n)$-separated sets of $X$ will be denoted by $s(g, n, E, \varepsilon)$. We now recall the notion of topological entropy introduced by Bufetov [11].

**Definition 3.1.** Given a compact set $E \subset X$, we define

$$
(5) \quad h_{top}((G, G_1), E) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log Z_n((G, G_1), E, \varepsilon),
$$

where

$$
(6) \quad Z_n((G, G_1), E, \varepsilon) = \frac{1}{m^n} \sum_{|g|=n} s(g, n, E, \varepsilon),
$$

where the sum is taken over all concatenation $g$ of $n$-elements of $G_1 \setminus \{id\}$ and $m = |G_1 \setminus \{id\}|$. The topological entropy $h_{top}((G, G_1), X)$ is defined for $E = X$.

In the case that $E = X$, for simplicity reasons, we shall use simply the notations $s(g, n, \varepsilon)$ and $Z_n((G, G_1), \varepsilon)$. It is easy to check that $h_{top}((G, G_1), X) \leq h((G, G_1), X)$. Moreover, this notion of topological entropy corresponds to the exponential growth rate of the average cardinality of maximal separated sets by individual dynamical systems $g$. This average is taken over elements that are, roughly, in the “ball of radius $n$ in the semigroup $G$”, corresponding to $G_n$. Notice that for any finite semigroup $G$, every element $g \in G$ has finite order. In this special case, we notice that every continuous map in the generated semigroup action has zero topological entropy, which is also coherent with the definition of entropy presented in [11].

In this context, and similarly to before, we say that $x \in X$ is an entropy point if for any neighborhood $U$ of $x$ one has $h_{top}((G, G_1), \overline{U}) = h_{top}((G, G_1), X)$. Our next theorem asserts that, under the (crucial) strong orbital specification property all points are also entropy points for this notion of entropy. More precisely,

**Theorem C.** Let $G \times X \to X$ be a continuous finitely generated semigroup action on a compact Riemannian manifold $X$ so that every element $g \in G_1$ is a local homeomorphism. If the semigroup action satisfies the strong orbital specification then every point is an entropy point.

**Proof.** Given any point $z \in X$ and $V$ any open neighborhood of $z$ we claim that $h_{top}((G, G_1), \overline{V}) = h_{top}((G, G_1), X)$. Let $\zeta > 0$ be arbitrary and take $\varepsilon_0 = \varepsilon_0(\zeta) > 0$ such that

$$
\limsup_{n \to \infty} \frac{1}{n} \log s(n, \varepsilon, \overline{U}) \geq h((G, G_1), X) - \zeta
$$

for every $0 < \varepsilon \leq \varepsilon_0$. Let $p(\varepsilon) \geq 1$ be given by the strong orbital specification property. Since there are finitely many elements in $G_{p(\varepsilon)}$, finitely many of its concatenations and the local inverse branches of elements $g: X \to X$ are uniformly continuous there exists a uniform constant $C_{\varepsilon} > 0$ (that tends to zero as $\varepsilon \to 0$) so that $diam(B(B(y, \varepsilon))) \leq C_{\varepsilon}$ for every $B \in G_{p(\varepsilon)}$ and $y \in X$.

Fix $h = h_{p(n)} \cdots h_{i_1} \in G_{p(n)}$. Take $n \geq 1$ and $g := g_{n} \cdots g_{1} \in G_n$ arbitrary, let

$E = \{x_1, \ldots, x_l\} \subset X$ be a maximal $(g, n, \varepsilon)$-separated set and consider the open set $W \subset V$ defined by the set of points $y \in V$ so that $d(y, \partial V) > C_{\varepsilon_0}$. Given a
maximal \((\varepsilon, \mathbb{W})\)-separated set \(F = \{z_1, \ldots, z_m\}\), by the specification property, for any \(x_i \in E\) and \(z_j \in F\) there exists

\[ y^j_i \in B(z_j, \frac{\varepsilon}{4}) \cap h^{-1}(B(x_i, \frac{\varepsilon}{4})). \]

Similarly as before we deduce that \(h^{-1}(B(x_i, \frac{\varepsilon}{4})) \subset V\) for every \(i\). By construction, the dynamical balls \(B(x_i, \frac{\varepsilon}{4})\) are pairwise disjoint and the points \(y^j_i\) are \((g, h, \frac{\varepsilon}{4}, \mathbb{V})\)-separated. This proves that

\[ s(g, h, \frac{\varepsilon}{4}, \mathbb{V}) \geq s(g, n, X, \varepsilon) s(id, 0, \mathbb{V}, \varepsilon) \geq s(g, n, X, \varepsilon). \]

Since the elements \(g\) and \(h\) were chosen arbitrary then, summing over all possible concatenations, we deduce

\[
\limsup_{n \to \infty} \frac{1}{n+p\left(\frac{\varepsilon}{4}\right)} \log \left[ \sum_{|g|=n+p\left(\frac{\varepsilon}{4}\right)} s(g, n + p\left(\frac{\varepsilon}{4}\right), \mathbb{V}, \varepsilon) \right] \\
\geq \limsup_{n \to \infty} \frac{1}{n+p\left(\frac{\varepsilon}{4}\right)} \log \left( \frac{1}{m^{n+p\left(\frac{\varepsilon}{4}\right)}} \sum_{|g|=n} s(g, n, X, \varepsilon) \right) \\
= \limsup_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{m^n} \sum_{|g|=n} s(g, n, X, \varepsilon) \right).
\]

The last inequalities show that \(h_{\text{top}}((G, G_1), X) \geq h_{\text{top}}((G, G_1), \mathbb{V}) \geq h_{\text{top}}((G, G_1), X) - \zeta\). Since both \(z \in X\) and \(\zeta > 0\) were chosen arbitrary this completes the proof of the theorem.  

3.2. **Positive topological entropy.** We now prove that orbital specification properties are enough to guarantee that the semigroup action has positive topological entropy.

**Theorem D.** Let \(G\) be a finitely generated semigroup with set of generators \(G_1\) and assume that \(G \times X \to X\) is a continuous semigroup action on a compact metric space \(X\). If \(G \times X \to X\) satisfies the strong orbital specification property then \(h_{\text{top}}((G, G_1), X) > 0\). In consequence, \(h((G, G_1), X) > 0\).

**Proof.** Since the expression in the right hand side of (5) is increasing as \(\varepsilon \to 0\) then it is enough to prove that there exists \(\varepsilon > 0\) small so that

\[
\limsup_{n \to \infty} \frac{1}{n} \log \frac{1}{m^n} \sum_{g \in G_n} s(g, n, \varepsilon) > 0.
\]

Let \(\varepsilon > 0\) be small and fixed so that there are at least two distinct \(2\varepsilon\)-separated points \(x_1, x_2 \in X\). Take \(p(\frac{\varepsilon}{2}) \geq 1\) given by the strong orbital specification property. Taking \(g_{n_1,1} = g_{n_2,2} = id\) and \(h = h_{p\left(\frac{\varepsilon}{2}\right)}\), \(h_1 \in G_{p\left(\frac{\varepsilon}{2}\right)}\), there are \(x_{i,j} \in B(x_i, \frac{\varepsilon}{2})\), with \(i, j \in \{1, 2\}\), such that \(h(x_{i,j}) \in B(x_j, \frac{\varepsilon}{2})\). In particular it follows that \(s(h, p(\frac{\varepsilon}{2}), \varepsilon) \geq 2^2\).

By a similar argument, given \(g := g_{i_1} \cdots g_{i_k} \in G_n\) with \(n = k.p(\frac{\varepsilon}{2})\), it can be written as a concatenation of \(k\) elements in \(G_{p(\frac{\varepsilon}{2})}\). In other words, \(g = h_k \cdots h_1\) with \(h_i \in G_{p(\frac{\varepsilon}{2})}\) and repeating the previous reasoning it follows that \(s(g, n, \varepsilon) \geq 2^k\).
Thus,

$$\limsup_{n \to \infty} \frac{1}{n} \log Z_n((G, G_1), \varepsilon) \geq \limsup_{k \to \infty} \frac{1}{k p(\frac{\varepsilon}{2})} \log \left( \frac{1}{m^{k p(\frac{\varepsilon}{2})}} \sum_{|g|=k p(\frac{\varepsilon}{2})} s(g, k p(\frac{\varepsilon}{2}), \varepsilon) \right)$$

$$\geq \frac{1}{p(\frac{\varepsilon}{2})} \log 2.$$

This proves that the entropy is positive and finishes the proof of the theorem. □

Let us observe that in [18] the author obtained a lower bound for the topological entropy of $C^1$-maps on smooth orientable manifolds. Here we require continuity of the semigroup action and a specification property (which most likely can be weakened) for deducing that topological entropy is strictly positive. One could expect that the weak orbital specification property could imply the semigroup action to have positive entropy. In fact this is the case whenever the semigroup satisfies additional conditions on the growth rate which hold e.g. for free semigroups.

**Theorem E.** Assume that $G$ is a finitely generated semigroup and that the continuous action $G \times X \to X$ on a compact metric space $X$ satisfies the weak orbital specification property with

$$(H) \quad \limsup_{p \to \infty} \frac{|G^*_p \setminus \hat{G}_p|}{m^{\gamma p}} < 1 \text{ for every } 0 < \gamma < 1.$$

Then the semigroup action $G \times X \to X$ has positive topological entropy.

In Subsection 5 we give some examples of semigroups combining circle expanding maps and rotations that satisfies the weak orbital specification property and for which $|G^*_p \setminus \hat{G}_p|$ is finite, hence (H) holds.

**Proof of Theorem E.** Given $\varepsilon > 0$, let $p(\varepsilon) \geq 1$ be given by the specification property. For any $p \geq p(\varepsilon)$ let $\hat{G}_p \subset G^*_p$ be given by the weak orbital specification property. Take $n = kp$ with $p \geq p(\frac{\varepsilon}{2})$ and assume that (H) holds.

For any $g \in G^*_p$ one can write it as a concatenation of $k$ elements in $G^*_p$, that is, $g = h_k \ldots h_1$ with $h_i \in G^*_p$. If this is the case, given $0 < \gamma < 1$ we will say that $g = h_k \ldots h_1 \in G^*_p$ is $\gamma$-acceptable if $\sharp\{0 \leq j \leq k : h_j \in \hat{G}_p\} > \gamma k$. Notice that

$$\sharp\{g = h_k \ldots h_1 \in G^*_p : g \text{ not } \gamma\text{-acceptable}\} \leq \sum_{l \geq \lceil \gamma k \rceil} \sharp\{g \in G^*_p : \sharp\{0 \leq j \leq k : h_j \in G^*_p \setminus \hat{G}_p\} = l\}.$$

In consequence,

$$\frac{\sharp\{g \in G^*_p : g \text{ is not } \gamma\text{-acceptable}\}}{m^{kp}} \leq \sum_{l \geq \lceil \gamma k \rceil} \left( \frac{k}{l} \right) |G^*_p|^{k-l} |G^*_p \setminus \hat{G}_p|^l$$

$$\leq \frac{k}{\lceil \gamma k \rceil} |G^*_p|^k |G^*_p \setminus \hat{G}_p|^k$$

$$\leq \frac{1}{m^{kp}} k\left( \frac{|G^*_p \setminus \hat{G}_p|}{m^{\gamma p}} \right)^k.$$

(7)
By assumption (H), given $0 < \gamma_0 < 1$ let $0 < \delta \ll \log 2$ be small so that
\[ \limsup_{p \to \infty} \frac{|G^*_p \setminus \tilde{G}_p|}{m^{\gamma p}} < e^{-2\delta} < 1. \]
Then by monotonicity of the later limsup in $\gamma$, it is clear that
\[ \limsup_{p \to \infty} \frac{|G^*_p \setminus \tilde{G}_p|}{m^{\gamma p}} < e^{-2\delta} < 1 \]
for every $\gamma \in (\gamma_0, 1)$. Up to consider larger $\gamma$ sufficiently close to 1 so that
\[ k \left( \frac{k}{[\gamma k]} \right) \leq e^{\delta k} \] for every $k$ large. The later implies that
\[ \# \{ g \in G^*_k : g \text{ is not } \gamma \text{-acceptable} \} \leq e^{\delta k} \left( \frac{|G^*_k \setminus \tilde{G}_p|}{m^{\gamma p}} \right)^k \leq e^{-\delta k} \]
which decreases exponentially fast in $k$ (provided that $p$ is large enough). Moreover,
given $p \gg p(\frac{1}{2})$ one can proceed as in the proof of the previous theorem and prove
that $s(g, kp, \varepsilon) \geq 2^\gamma$ for any $\gamma$-admissible $g \in G^*_k$. Consequently,
\[ \limsup_{n \to \infty} -\frac{1}{n} \log Z_n((G, G^*_1), \varepsilon) \geq \limsup_{k \to \infty} \frac{1}{k p(\frac{1}{2})} \log \left( \frac{\# \{ g \in G^*_k : g \text{ is } \gamma \text{-acceptable} \}}{m^{kp(\frac{1}{2})}} \right) \cdot 2^{\gamma k} \]
\[ \geq \frac{1}{p(\frac{1}{2})} \log 2^{\gamma} + \limsup_{k \to \infty} \frac{1}{k p(\frac{1}{2})} \log (1 - e^{-\delta k}) \]
\[ \geq \frac{\gamma}{p(\frac{1}{2})} \log 2 - \frac{\delta}{p(\frac{1}{2})} \]
which is strictly positive, by the choice of $\delta$ and $\gamma$. This proves the theorem. \(\square\)

4. Thermodynamics of expansive semigroup actions with specification

In this section we study thermodynamical properties of positively expansive semigroup actions satisfying specification and also semigroups of uniformly expanding maps. First we prove that semigroups of expanding maps satisfy the orbital specification properties (Theorems F). Then we obtain conditions for the convergence of topological pressure (Theorem G). Finally we prove a strong regularity of the topological pressure function (Theorem H) and prove that topological entropy is a lower bound for the exponential growth rate of periodic points (Theorem I).

4.1. Semigroup of expanding maps and specification. Throughout this subsection we shall assume that $X$ is a compact Riemannian manifold. We say that a $C^1$-local diffeomorphism $f : M \to M$ is an expanding map if there are constants $C > 0$ and $0 < \lambda < 1$ such that $\|(Df^n(x))^{-1}\| \leq CA^n$ for every $n \geq 1$ and $x \in X$.

**Theorem F.** Let $G_1 = \{g_1, g_2, \ldots, g_k\}$ be a finite set of expanding maps and let $G$ be the generated semigroup. Then $G$ satisfies the strong orbital specification property.

The following two lemmas will be instrumental in the proof of Theorem F.

**Lemma 4.1.** Let $g_1, \ldots, g_k$ be $C^1$-expanding maps on the compact manifold $X$. There exists $\delta_0 > 0$ so that $g(B(x, \delta)) = B(g(x), \delta)$ for any $0 < \delta \leq \delta_0$, any $x \in X$ and any $\overline{g} \in G$. 

Proof. Let \( d_i = \deg(g_i) \) be the degree of the map \( g_i \). Since \( g_i \) is a local diffeomorphism there exists \( \delta > 0 \) (depending on \( g_i \)) so that for every \( x \in X \) setting \( g_i^{-1}(x) = \{x_{i,1}, \ldots, x_{i,d_i}\} \) there are \( d_i \) well defined inverse branches \( g_i^{-1}: B(x, \delta) \to V_{x_{i,j}} \) onto an open neighborhood of \( x_{i,j} \). Since there are finitely many maps \( g_i \) there exists an uniform constant \( \delta_0 > 0 \) so that all inverse branches for \( g_i \) are defined in balls of radius \( \delta_0 \). Furthermore, since all \( g_i \) are uniformly expanding all inverse branches are \( \lambda \)-contracting for some uniform \( 0 < \lambda < 1 \), meaning that \( d((g_i^{-1}(y),g_i^{-1}(z))) \leq \lambda d(y,z) \) for any \( x \in X \), any \( y, z \in B(x, \delta_0) \) and \( i = 1 \ldots k \). In particular \( (g_i^{-1}(B(x, \delta_0))) \subset B(x_{i,j}, \delta_0) \) and so

\[
V_{x_{i,j}} = \{ y \in X : d(y, x_{i,j}) < \delta_0 \text{ for } i = 1, \ldots, k \}
\]

Using this argument recursively, every \( g_j = g_{i_1} \ldots g_{i_p} \in G_j \) is a contraction and we get that the dynamical ball \( B(x, \delta) = \bigcap_{j=0}^{\delta} (g_j^{-1}(B(g_j(x), \delta))) \) is mapped diffeomorphically by \( g \) onto \( B(g(x), \delta) \), proving the lemma. \( \square \)

**Lemma 4.2.** Let \( g_1, \ldots, g_k \) be \( C^1 \)-expanding maps on the compact manifold \( X \). For any \( \delta > 0 \) there exists \( N = N(\delta) \in \mathbb{N} \) so that \( g_N(B(x, \delta)) = X \) for all \( x \in X \) and every \( g_N \in G_N \).

Proof. There exists a uniform \( 0 < \lambda < 1 \) so that all inverse branches for \( g_i \) are \( \lambda \)-contracting for any \( i \). Fix \( \delta > 0 \). Using the compactness of \( X \) it is enough to prove that for any \( x \in X \) there exists \( N \geq 1 \) so that \( g_N(B(x, \delta)) = X \) for all \( g_N \in G_N \). Take \( N = N(\delta) \geq 1 \) be large and such that \( \lambda^N(1 + \text{diam } X) < \delta \). Let \( g_N \in G_N \) be arbitrary and assume, by contradiction, that \( g_N(B(x, \delta)) \neq X \). Then there exists a curve \( \gamma_N \) with diameter at most \( \text{diam } X \) connecting the points \( x \) and \( y \in X \setminus g_N(B(x, \delta)) \). Consider a covering of \( \gamma_N \) by balls of radius \( \delta \) and consider \( \gamma \) the image of \( \gamma_N \) by the inverse branches, such that \( \gamma \) connects \( x \) to some point \( z \not\in B(x, \delta) \) so that \( g_N(z) = y \). Using that \( y \not\in g_N(B(x, \delta)) \) one gets that \( z \not\in B(x, \delta) \).

Since \( g_N \) is a \( \lambda^N \)-contraction then \( \delta < d(x, z) \leq \text{length}(\gamma) \leq \lambda^N(1 + \text{diam } X) < \delta \), which is a contradiction. Thus the lemma follows. \( \square \)

**Proof of Theorem 4.3**. The proof of the theorem follows from the previous lemmas. In fact, let \( \delta > 0 \) be fixed and consider \( x_1, x_2, \ldots, x_k \in X \), natural numbers \( n_1, n_2, \ldots, n_k \) and group elements \( g_{n_1,j} = g_{n_2,j} \ldots g_{n_j,j} \in G_{n_j} \) for \( j = 1 \ldots k \).

By Lemma 4.1 there exists \( \varepsilon_0 \) such that for \( \varepsilon \leq \varepsilon_0 \)

\[
g_{n_i}(B(x_j, g_{n_j}(x_j), \varepsilon)) = B(g_{n_j}(x_j), \varepsilon), \forall 1 \leq j \leq k.
\]

We may assume without loss of generality that \( \delta < \varepsilon_0 \). Let \( p(\delta) = N(\delta) \) be given by Lemma 4.2. Given \( p_1, \ldots, p_k \geq p(\delta) \), for \( h_{p_j} \in C^p_{p_j} \) we have that \( h_{p_j}(B(g_{n_{k+1,j}}(x_{k+1,j}), \delta)) = X \). It implies that given \( x_k \in B(x_k, g_{n_{k+1,j}}(x_{k+1,j})) \), one has \( \tilde{x}_k = h_{p_{k+1}}(x_{k+1,j}) \), with \( x_{k+1,j} \in B(g_{n_{k+1,j}}(x_{k+1,j}), \varepsilon) \), and then \( \tilde{x}_k = g_{n_{k-1,j}} \cdot h_{p_{k+1}}(\tilde{x}_{k-2}) \), for some \( \tilde{x}_{k-2} \in B(x_{k-2}, g_{n_{k-1,j}}(x_{k-1,j}), \varepsilon) \). By induction, there exists \( x \in B(x_1, g_{n_1,j}, \varepsilon) \), such that

\[
g_{n_j}(x) \in B(x_j, g_{n_j}(x_j), \varepsilon)
\]

for every \( j = 1 \ldots k \) and \( \ell = 1 \ldots n_j \). This completes the proof of the theorem. \( \square \)

For completeness, let us mention that the results in this subsection hold also for general topologically mixing distance expanding maps on compact metric spaces.
(\(X,d\)). Recall \(f\) is a distance expanding map if there are \(\delta > 0\) and \(0 < \lambda < 1\) so that \(d(f(x), f(y)) \geq \lambda^{-1}d(x, y)\) for every \(d(x,y) < \delta\). Our motivation to focus on smooth maps comes from the fact free semigroups can be constructed and shown to be robust in this context (c.f. Section 5).

4.2. Convergence and regularity of entropy and the pressure function. In what follows we shall introduce a notion of topological pressure. For notational simplicity, given \(g \in G_n\) and \(U \subset X\) we will use the notation \(S_g^\varphi(x) = \sum_{i=0}^{n-1} \varphi(g_i(x))\) and \(S_g^\varphi(U) = \sup_{x \in U} S_g^\varphi(x)\).

**Definition 4.3.** For any continuous observable \(\varphi \in C(X)\) we define the topological pressure of \((G,G_1)\) with respect to \(\varphi\) by

\[
P_{\text{top}}((G,G_1), \varphi, X) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log Z_n((G,G_1), \varphi, \varepsilon),
\]

where

\[
Z_n((G,G_1), \varphi, \varepsilon) = \frac{1}{m^n} \sum_{|g| = n} \sup_E \left\{ \sum_{x \in E} e^{\sum_{i=0}^{n-1} \varphi(g_i(x))} \right\}
\]

and the supremum is taken over all sets \(E = E_{g,n,\varepsilon}\) that are \((\varepsilon,n,\varepsilon)\)-separated.

Observe that in the case that \(G\) has only one generator \(f\) then \(|G_n| = |\{f^n\}| = 1\) and \(P_{\text{top}}((G,G_1), \varphi)\) coincides with the classical pressure \(P_{\text{top}}(f, \varphi)\). The case that the potential is constant to zero corresponds to the notion of topological entropy introduced in Definition 3.1. We proceed to prove that the topological pressure of expansive semigroup actions with the specification property can be computed as a limit. For that purpose we provide an alternative formula to compute the topological pressure using open covers. Given \(\varepsilon > 0\), \(n \in \mathbb{N}\) and \(g \in G_n\), we say that an open cover \(\mathcal{U}\) of \(X\) is an \((\varepsilon, n, \varepsilon)\)-cover if any open set \(U \in \mathcal{U}\) has \(d_{\varepsilon}\) diameter smaller than \(\varepsilon\), where \(d_{\varepsilon}\) is the metric introduced in [3]. Let \(\text{cov}(g, n, \varepsilon)\) be the minimum cardinality of a \((\varepsilon, n, \varepsilon)\)-cover of \(X\). To obtain a characterization of the topological pressure using open covers of the space we need the continuous potential to satisfy a regularity condition. Given \(\varepsilon > 0\) and \(g := g_{n_1} \cdots g_{n_t} \in G\) we define the variation of \(S_g^\varphi\) in dynamical balls of radius \(\varepsilon\) by

\[
\text{Var}_g(\varphi, \varepsilon) = \sup_{d_{\varepsilon}(x,y) < \varepsilon} |S_g^\varphi(x) - S_g^\varphi(y)|.
\]

We say that \(\varphi\) has bounded distortion property (in dynamical balls of radius \(\varepsilon\)) if there exists \(C > 0\) so that

\[
\sup_{g \in G} \sup_{\varepsilon \in X} \text{Var}_g(\varphi, \varepsilon) \leq C.
\]

For short we denote by \(BD(\varepsilon)\) the space of continuous potentials that have bounded distortion in dynamical balls of radius \(\varepsilon\) and we say that \(\varphi\) has bounded distortion property if there exists \(\varepsilon > 0\) so that \(\varphi\) has bounded distortion on dynamical balls of radius \(\varepsilon\). In what follows we prove that Hölder potentials have bounded distortion for semigroups of expanding maps.

**Lemma 4.4.** Let \(G\) be a finitely generated semigroup of expanding maps on a compact metric space \(X\) with generators \(G_1 = \{g_1, \ldots, g_m\}\). Then any Hölder continuous observable \(\varphi : M \to \mathbb{R}\) satisfies the bounded distortion property.
is less than $U$ distortion condition. Then the topological pressure

\[ P_{\text{top}}((G,G_1),\varphi,X) = \lim \limsup_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{m^n} \sum_{|g|=n} \inf_{U \in \mathcal{U}} \sum_{U \in \mathcal{U}} e^{S_g \varphi(U)} \right), \]

where the infimum is taken over all open covers $\mathcal{U}$ of $X$ such that $\mathcal{U}$ is a $(g,n,\varepsilon)$-open cover.

\textbf{Proof.} Although the proof of this proposition follows a classical argument we include it here for completeness. Take $\varepsilon > 0$, $n \in \mathbb{N}$ and $g \in G_n$. To simplify the notation we denote

\[ C_n((G,G_1),\varphi,\varepsilon) = \frac{1}{m^n} \sum_{|g|=n} \inf_{U \in \mathcal{U}} \sum_{U \in \mathcal{U}} e^{S_g \varphi(U)} \]

where the infimum are taken over all $(g,n,\varepsilon)$-open covers and let $Z_n((G,G_1),\varphi,\varepsilon)$ be given by equation (9). Given a $(g,n,\varepsilon)$-maximal separated set $E$ it follows that $\mathcal{U} = \{ B(x,g,\varepsilon) \}_{x \in E}$ is a $(g,n,2\varepsilon)$-open cover. By the bounded distortion assumption, $S_g \varphi(B(x,g,\varepsilon)) = \sup_{z \in B(x,g,\varepsilon)} S_g \varphi(z) \leq S_g \varphi(x) + C$ for some constant $C > 0$, depending only on $\varepsilon$. Consequently,

\[ \limsup_{n \to \infty} \frac{1}{n} \log C_n((G,G_1),\varphi,2\varepsilon) \leq \limsup_{n \to \infty} \frac{1}{n} \log Z_n((G,G_1),\varphi,\varepsilon). \]

On the other hand, if $\mathcal{U}$ is $(g,n,\varepsilon)$-open cover, for any $(g,n,\varepsilon)$-separated set $E \subset X$ we have that $\varepsilon E \leq \sharp \mathcal{U}$, since the diameter of any $U \in \mathcal{U}$ in the metric $d_g$ is less than $\varepsilon$. By the bounded distortion condition we get that

\[ \limsup_{n \to \infty} \frac{1}{n} \log Z_n((G,G_1),\varphi,\varepsilon) \leq \limsup_{n \to \infty} \frac{1}{n} \log C_n((G,G_1),\varphi,\varepsilon). \]

Now, combining equations (10) and (11) we get that
\( \limsup_{n \to \infty} \frac{1}{n} \log Z_n((G, G_1), \varphi, \varepsilon) \leq \limsup_{n \to \infty} \frac{1}{n} \log C_n((G, G_1), \varphi, \varepsilon) \)
\[ \leq \limsup_{n \to \infty} \frac{1}{n} \log Z_n((G, G_1), \varphi, \varepsilon/2) \]

and then the result follows.

In the next lemma we provide a condition under which the topological pressure can be computed as a limit.

**Proposition 4.6.** Let \( \varphi : X \to \mathbb{R} \) be a continuous potential. Given \( \varepsilon > 0 \), the limit superior
\[ \limsup_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{m^n} \sum_{|g|=n} \inf_{U \in \mathcal{U}} \sum_{V \in \mathcal{V}} e^{S_g \varphi(U)} \right) \]
is indeed a limit.

**Proof.** Since \( \varphi \) is continuous then it is bounded from below. Assume without loss of generality that \( \varphi \) is non-negative, otherwise we just consider a translation \( \varphi + C \) since it will affect the \( \limsup \) by a translation of \( C \).

Given \( \varepsilon > 0 \), recall that the infimum is taken over all \( (g, m + n, \varepsilon) \)-open covers \( \mathcal{U} \) of \( X \). For any element \( g = h/k \in G_{\ell+n}^* \) with \( h \in G^{*}_{\ell} \), \( k \in G_{n,*}^* \), and any \( (h, \ell, \varepsilon) \)-cover \( \mathcal{U} \) and \( (k, \ell, \varepsilon) \)-cover \( \mathcal{V} \) then \( W := k^{-1}(\mathcal{U}) \cup \mathcal{V} \) is a \( (g, \ell + n, \varepsilon) \)-cover, and
\[ \sum_{W \in k^{-1}(\mathcal{U}) \cup \mathcal{V}} e^{S_g \varphi(W)} \leq \left( \sum_{V \in \mathcal{V}} e^{S_k \varphi(V)} \right) \left( \sum_{U \in \mathcal{U}} e^{S_h \varphi(U)} \right) \]
Taking the infimum over the open covers \( \mathcal{U} \) and \( \mathcal{V} \) we deduce that
\[ \inf_{W} \left\{ \sum_{W \in \mathcal{W}} e^{S_g \varphi(W)} \right\} \leq \inf_{V} \left\{ \sum_{V \in \mathcal{V}} e^{S_k \varphi(V)} \right\} \inf_{U} \left\{ \sum_{U \in \mathcal{U}} e^{S_h \varphi(U)} \right\}, \]
where the first infimum can be taken over all \( (g, m + n, \varepsilon) \)-open covers \( \mathcal{W} \). Summing over every elements \( g = h/k \in G_{\ell+n}^* \),
\[ \sum_{|g|=\ell+n} \inf_{W} \left\{ \sum_{W \in \mathcal{W}} e^{S_g \varphi(W)} \right\} \leq \left( \sum_{|k|=\ell} \inf_{V} \sum_{V \in \mathcal{V}} e^{S_{k} \varphi(V)} \right) \left( \sum_{|h|=n} \inf_{U} \sum_{U \in \mathcal{U}} e^{S_{h} \varphi(U)} \right). \]
Thus, the sequence of real numbers \( \{a_n\}_{n \in \mathbb{N}} \) given by
\[ a_n = \log \left( \sum_{|g|=n} \inf_{W} \left\{ \sum_{W \in \mathcal{W}} e^{S_g \varphi(W)} \right\} \right) \]
is subadditive and \( \{a_n/n\}_{n \in \mathbb{N}} \) is convergent. Since the term \( \frac{1}{n} \log \frac{1}{m^n} \) is clearly constant this completes the proof of the proposition.

From the previous results, the topological pressure can be computed as the limiting complexity of the group action as the size scale \( \varepsilon \) approaches zero. In what follows we will be mostly interested in providing conditions for the topological pressure of group actions to be computed as a limit at a definite size scale. Let us introduce the necessary notions. Let \( X \) be a compact metric space and \( G \times X \to X \) be a continuous action associated to the finitely generated semigroup \( (G, G_1). \)
Definition 4.7. Given $\delta^* > 0$, the semigroup action $G \times X \to X$ is $\delta^*$-expansive if for every $x, y \in X$ there exists $k \geq 1$ and $g \in G_k$ such that $d(g(x), g(y)) > \delta^*$. The semigroup action $G \times X \to X$ is strongly $\delta^*$-expansive if for any $\gamma > 0$ and any $x, y \in X$ with $d(x, y) \geq \gamma$ there exists $k \geq 1$ (depending on $\gamma$) such that $d(g(x, y)) > \delta^*$ for all $g \in G_k$.

Remark 4.8. By compactness of the phase space $X$, a continuous action is strongly $\delta^*$-expansive satisfies the following equivalent formulation: given $\gamma > 0$ and $x, y \in X$ with $d(x, y) \geq \gamma$ there exists $k_0 \geq 1$ (depending on $\gamma$) such that $d(g(x, y)) > \delta^*$ for all $g \in G_{k_0}$ and $k \geq k_0$.

In what follows we prove that the topological entropy of expansive semigroup actions can be computed as the topological complexity that is observable at a definite scale. More precisely,

Theorem G. Assume the continuous action of $G$ on the compact metric space $X$ is strongly $\delta^*$-expansive. Then, for every continuous potential $\varphi : X \to \mathbb{R}$ satisfying the bounded distortion condition and every $0 < \varepsilon < \delta^*$

$$P(\varphi) := P_{\text{top}}((G, G_1), \varphi, X) = \limsup_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{m^n} \sum_{|g| = m} \sup_{E} \sum_{x \in E} e^{S_\varphi(x)} \right)$$

where the supremum is taken over all $(g, n, \varepsilon)$-separated sets $E \subset X$.

We just observe, before the proof, that in view of the previous characterization given in Proposition 4.3, the same result as above also holds if we consider open covers instead of separated sets.

Proof of Theorem G. Since $X$ is compact and $\varphi : X \to \mathbb{R}$ is continuous we assume, without loss of generality, that $\varphi$ is non negative. Fix $\gamma$ and $\varepsilon$ with $0 < \gamma < \varepsilon < \delta^*$. We want to show that

$$\limsup_{n \to \infty} \frac{1}{n} \log Z_n((G, G_1), \varphi, \gamma) \leq \limsup_{n \to \infty} \frac{1}{n} \log Z_n((G, G_1), \varphi, \varepsilon).$$

The other inequality is clear. By strong $\delta^*$-expansiveness and Remark 4.8 for any two distinct points $x, y \in X$ with $d(x, y) \geq \gamma$ there exists $k_0 \geq 1$ (depending on $\gamma$) so that $d(g(x, y)) \geq \delta^* > \varepsilon$ for any $g \in G_{k_0}$ and $k \geq k_0$. Take $n \geq k_0$ and $g \in G_{n+k}$ arbitrary and write $g = h_{1,2} h_{1,1}$ with $h_{1,2} \in G_n$ and $h_{1,1} \in G_k$. Given any $(h_{1,1}, n, \gamma)$-separated set $E$ we claim that the set $E$ is $(g, n + k, \varepsilon)$-separated. In fact, given $x, y \in E$ there exists a decomposition $h_{1,2} \equiv h_{1,2} h_{1,1} \in G_{n+k}$ so that $d(h_{1,1}(x), h_{1,1}(y)) > \gamma$. Using that $h_{1,2} h_{1,2} h_{1,1} \in \bigcup_{k \geq k} G_k^*$ and Remark 4.8 it follows that $d(g(x, y)) \geq d(h_{1,1}(x), h_{1,1}(y)) > \varepsilon$ proving the claim. Now, using that $\varphi$ is non-negative,

$$e^{S_\varphi(x)} = e^{S_\varphi(h_{1,1}(x))} e^{S_\varphi((h_{1,1}(x))} \leq e^{S_\varphi(x)},$$

which implies that $Z_n((G, G_1), \varphi, \gamma) \leq m^k Z_n((G, G_1), \varphi, \varepsilon)$ because

$$Z_n((G, G_1), \varphi, \gamma) = \frac{1}{m^n} \sum_{|g| = m} \sup_{E} \sum_{x \in E} e^{S_\varphi(x)} \leq \frac{1}{m^n} \frac{1}{m^{n+k}} \sum_{|g| = n+k} \sup_{E} \sum_{x \in E} e^{S_\varphi(x)} = m^k Z_n((G, G_1), \varphi, \varepsilon).$$
Thus it follows that
\[ \limsup_{n \to \infty} \frac{1}{n} \log Z_n((G, G_1), \varphi, \gamma) \leq \limsup_{n \to \infty} \frac{1}{n + k} \log Z_{n+k}((G, G_1), \varphi, \varepsilon), \]
as we wanted to prove. This completes the proof of the theorem. \[ \square \]

Some comments on our assumptions are in order. It is clear that if some generator for the group is an expansive map then the group is itself expansive. Clearly, expanding maps are expansive. Moreover, the semigroup \( G \) generated by \( G_1 = \{g_1, \ldots, g_k\} \) that admits some expansive generator is clearly expansive. In Lemma 4.9 below we prove that semigroups of expanding maps are strongly expansive semigroups.

**Lemma 4.9.** Let \( G \) be a finitely generated semigroup of expanding maps on a compact metric space \( X \) with generators \( G_1 \). Then there exists \( \delta^* > 0 \) so that \( G \) is strongly \( \delta^* \)-expansive.

**Proof.** Let \( G_1 = \{g_1, \ldots, g_m\} \) be the set of generators of \( G \). Following the proof of Lemma 1.1 there are uniform constants \( \delta_0 > 0 \) and \( 0 < \lambda < 1 \) so that all inverse branches \( g_{i,j}^{-1} \) for \( g_i \) are defined in balls of radius \( \delta_0 \) and \( d(\gamma_{i,j}^{-1}(y), \gamma_{i,j}^{-1}(z)) \leq \lambda d(y, z) \) for any \( x \in X \), any \( y, z \in B(x, \delta_0) \) and \( i = 1 \ldots m \). Take \( \delta_1 = \delta_0/2 \). Given \( \gamma > 0 \) take \( k \geq 1 \) (depending on \( \gamma \)) so that \( \lambda^k \delta < \gamma \). We claim that for any \( x_1, y_1 \in X \) with \( d(x_1, y_1) \geq \gamma \) and \( g \in G_k \) we have \( d_g(x_1, y_1) > \delta^* \). Assume, by contradiction, that there exists \( g = g_{i_1} \ldots g_{i_k} \in G_k \) with \( d(g(x_1), g(y_1)) \leq \delta^* \). Then \( d(g_{i_1} \ldots g_{i_k}(x_1), g_{i_1} \ldots g_{i_k}(y_1)) \leq \lambda^{k-1} d(g_{i_1} \ldots g_{i_k}(x_1), g_{i_1} \ldots g_{i_k}(y_1)) \) for every \( 1 \leq j \leq k \) and so \( d(x_1, y_1) \leq \lambda^k d(g(x_1), g(y_1)) < \gamma \), which is a contradiction. This finishes the proof of the lemma. \[ \square \]

**Theorem H.** Let \( G \) be a finitely generated semigroup with generators \( G_1 \). If the semigroup action induced by \( G \) on the compact metric space \( X \) is strongly \( \delta^* \)-expansive and the potentials \( \varphi, \psi : X \to \mathbb{R} \) are continuous and satisfy the bounded distortion property then

1. \( P_{\text{top}}((G, G_1), \varphi + c, X) = P_{\text{top}}((G, G_1), \varphi, X) + c \) for every \( c \in \mathbb{R} \)
2. \( |P_{\text{top}}((G, G_1), \varphi, X) - P_{\text{top}}((G, G_1), \psi, X)| \leq \|\varphi - \psi\|_{t \varphi, X} \), and
3. the pressure function \( t \mapsto P_{\text{top}}(G, G_1), t \varphi, X \) is an uniform limit of differentiable maps.

Moreover, \( t \mapsto P_{\text{top}}((G, G_1), t \varphi, X) \) is differentiable Lebesgue-almost everywhere.

**Proof.** We start by observing that property (1) follows directly from the definition of the topological pressure. By hypothesis let \( \varepsilon_0 > 0 \) be so that \( \varphi, \psi \in BD(\varepsilon_0) \). On the one hand, by Theorem \[ G \] together with equation \[ 12 \] it follows that for any \( 0 < \varepsilon < \delta^* \),

\[ P(\varphi) := P_{\text{top}}((G, G_1), t \varphi, X) = \limsup_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{m^n} \sum_{|g|=n} \inf_{U \in \mathcal{U}} \sum_{U \in \mathcal{U}} e^{tS_{g}^\varphi(U)} \right) \]

where the infimum is taken over all \((g, n, \varepsilon)-open\) covers \( \mathcal{U} \). On the other hand, by Proposition \[ 4.6 \] the right hand side above is actually a true limit. Thus, for any
\[ t \in \mathbb{R} \] we have that
\[
P_{\text{top}}((G, G_1), t\varphi, X) = \lim_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{m_n} \sum_{|g|=n} \inf_{U \in \mathcal{U}} \sum_{U \in \mathcal{U}} e^{tS_2 g(U)} \right),
\]
where the infimum is taken over all \((g, n, \varepsilon)\)-covers \(\mathcal{U}\) for any \(0 < \varepsilon < \min\{\delta^*, \varepsilon_0\}\).

It means that the map \(t \mapsto P_{\text{top}}((G, G_1), t\varphi, X)\) is a pointwise limit of real analytic functions. We claim that the convergence is indeed uniform. To prove this we will prove that the sequence of real functions \(\left( P_n(t\varphi) \right)_{n \geq 1}\) defined by
\[
t \mapsto P_n(t\varphi) := \frac{1}{n} \log C_n((G, G_1), t\varphi, \varepsilon)
\]
is equicontinuous in compact intervals, i.e., given \(\varepsilon > 0\) there exists \(\delta > 0\) such that if \(|t_1 - t_2| < \delta\) then \(|P_n(t_1\varphi) - P_n(t_2\varphi)| < \varepsilon\), for every \(n \in \mathbb{N}\). Let \(\varepsilon > 0\) be fixed and take \(0 < \delta < \varepsilon/\|\varphi\|\). Given \(t_1, t_2\) arbitrary with \(|t_1 - t_2| < \delta\) it holds that
\[
|P_n(t_1\varphi) - P_n(t_2\varphi)| = \frac{1}{n} \log \left[ \frac{\sum_{|g|=n} \inf_{U \in \mathcal{U}} \left\{ \sum_{U \in \mathcal{U}} e^{t_1 S_2 g(U)} \right\} }{\sum_{|g|=n} \inf_{U \in \mathcal{U}} \left\{ \sum_{U \in \mathcal{U}} e^{t_2 S_2 g(U)} \right\} } \right] 
\leq \frac{1}{n} \log \left[ \frac{e^{\delta\|\varphi\|} \sum_{|g|=n} \inf_{U \in \mathcal{U}} \left\{ \sum_{U \in \mathcal{U}} e^{t_2 S_2 g(U)} \right\} }{\sum_{|g|=n} \inf_{U \in \mathcal{U}} \left\{ \sum_{U \in \mathcal{U}} e^{t_1 S_2 g(U)} \right\} } \right] = \delta\|\varphi\| < \varepsilon.
\]

Hence the sequence is equicontinuous. Since \(\left( P_n(t\varphi) \right)_{n \in \mathbb{N}}\) converges pointwise, we have that the sequence converges uniformly on compact intervals and so \(t \mapsto P_{\text{top}}((G, G_1), t\varphi, X)\) is a continuous function. Furthermore, for any \(n \in \mathbb{N}\) the function \(t \mapsto P_n(\varphi + t\psi)\) is differentiable and
\[
\left| \frac{d P_n(\varphi + t\psi)}{dt} \right| = \frac{1}{C_n((G, G_1), t\varphi, \varepsilon)} \frac{1}{n} \left( \frac{1}{m_n} \sum_{|g|=n} \inf_{U \in \mathcal{U}} \left\{ \sum_{U \in \mathcal{U}} S_2 g(U) e^{S_2 (\varphi + t\psi)(U)} \right\} \right)
\]
is bounded from above by \(\|\psi\|\) (here the infimum is taken over all \((g, n, \varepsilon)\)-covers \(\mathcal{U}\) as in (13)). This proves property (3). Moreover, by the mean value inequality
\[
|P_n(\varphi) - P_n(\psi)| \leq \sup_{0 \leq t \leq 1} \left| \frac{d P_n(\varphi + t(\psi - \varphi))}{dt} \right| \leq \|\varphi - \psi\|.
\]
Taking \(n \to \infty\) we get that \(|P_{\text{top}}((G, G_1), \varphi, X) - P_{\text{top}}((G, G_1), \psi, X)| \leq \|\varphi - \psi\|\) and so the pressure function \(P_{\text{top}}((G, G_1), \cdot, X)\) acting on the space of potentials with bounded distortion is Lipschitz continuous with Lipschitz constant equal to one. This proves property (2). The later implies that \(t \mapsto P_{\text{top}}((G, G_1), t\varphi, X)\) is Lebesgue-almost everywhere differentiable, which concludes the proof of the theorem. \(\square\)
4.3. Topological entropy and growth rate of periodic points. In the remaining of this section we prove that the topological entropy is a lower bound for the exponential growth rate of periodic points for semigroup of expanding maps. Clearly the theorems of the previous section apply to the topological entropy since it corresponds to the constant to zero potential.

**Theorem I.** Let $G$ be the semigroup generated by a set $G_1 = \{g_1, \ldots, g_k\}$ of uniformly expanding maps on a Riemannian manifold $X$. Then:

(a) $G$ satisfies the periodic orbital specification property,

(b) periodic points $\text{Per}(G)$ are dense in $X$, and

(c) the mean growth of periodic points is bounded from below as

$$0 < h_{top}((G, G_1), X) \leq \limsup_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{m^n} \sum_{|g|=n} \| \text{Fix}(g) \| \right).$$

**Proof.** Take $n \geq 1$ arbitrary and fixed. It follows from Lemmas 4.1 and 4.2 that there exists $\delta_0 > 0$ satisfying: for any $0 < \delta \leq \delta_0$ there exists a uniform $N(\delta) \geq 1$ so that for any $x \in X$, any $g_n \in G_n$ and $g_N \in G_N^*$ with $N \geq N(\delta)$ it holds

$$g_n(g_N(B(x, g, \delta))) = X.$$ 

Consider $\delta > 0$, $x_1, x_2, \ldots, x_k \in X$, natural numbers $n_1, n_2, \ldots, n_k$ and group elements $g_{n_1, j} = g_{i_1}, j \ldots g_{i_k}, j$, $g_{i_1}, j \in G_{n_1}$ ($j = 1 \ldots k$) be given and let us prove that $G$ satisfies the periodic orbital specification property, that is, there exists a periodic orbit shadowing the previously defined pieces of orbit. For that let us define $x_{k+1} = x_1$ and $g_{n_{k+1}} = g_1 \in G_{n_1}$.

By the proof of Theorem F, there exists $p(\delta) \geq 1$ so that for any $p_1, \ldots, p_k \geq p(\delta)$, for $h_{p_i} \in G_{p_i}^*$ we have that $h_{p_i}(B(g_{n_i}(x_i), \delta)) = X$. Hence, there is a well defined inverse branch (which we denote by $g_{n_i}^{-1}h_{p_i}^{-1}$ for simplicity) so that

$$g_{n_i}^{-1}h_{p_i}^{-1}(B(x_{i+1}, g_{n_{i+1}}, \delta)) \subset B(x_i, g_{n_1}, \delta)$$

and $g_{n_i}^{-1}h_{p_i}^{-1}|_{B(x_{i+1}, g, \delta)}$ is a contraction. Since, $B(x_{k+1}, g_{n_{k+1}}, \delta) = B(x_1, g_{n_1}, \delta)$,

$$g_{n_1}^{-1}h_{p_1}^{-1} \ldots g_{n_k}^{-1}h_{p_k}^{-1}(B(x_{k+1}, g_{n_{k+1}}, \delta)) \subset B(x_1, g_{n_1}, \delta)$$

and the composition $g_{n_1}^{-1}h_{p_1}^{-1} \ldots g_{n_k}^{-1}h_{p_k}^{-1}$ is a uniform contraction, then there exists a unique repelling fixed point for $h_{p_k}^{-1}g_{n_k}^{-1} \ldots h_{p_1}^{-1}g_{n_1}^{-1}$ in the dynamical ball $B(x_1, g_{n_1}, \delta)$. By construction, the fixed point for $h_{p_k}^{-1}g_{n_k}^{-1} \ldots h_{p_1}^{-1}g_{n_1}^{-1}$ shadows the specified pieces of orbits. This proves that $G$ satisfies the periodic orbital specification property in (a). Clearly (b) is a consequence of the first claim (a).

Now, take $g \in G_n^*$ and observe that for any maximal $(g, n, 2\delta)$-separated set $E$, the dynamical balls $\{B(x, g, \delta) : x \in E\}$ form a pairwise disjoint collection. Let $p(\delta)$ be given by the previous periodic orbital specification property. For any arbitrary $k \in G_{n+p(\delta)}^*$ one can write $k = h_g g$ for $g \in G_n^*$ and $h_g \in G_{p(\delta)}^*$. Notice that, proceeding as before,

$$k(B(x, g, \delta)) = h_g(B(g(x), \delta)) = X$$

for every $x \in E$ and so there is a unique fixed point for $k$ on the dynamical ball $B(x, g, \delta)$. This yields $\text{Fix}(k) \geq \sharp E$ and so...
\[
\sum_{|k|=n+p(\delta)} \# \text{Fix}(k) \geq \sum_{|g|=n} \# \text{Fix}(h_g g) \geq \sum_{|g|=n} s(g, n, 2\delta).
\]

Therefore,
\[
\limsup_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{m^n} \sum_{|k|=n} \# \text{Fix}(k) \right) = \limsup_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{m^{n+p(\delta)}} \sum_{|k|=n+p(\delta)} \# \text{Fix}(k) \right)
\]
\[
= \limsup_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{m^n} \sum_{|k|=n+p(\delta)} \# \text{Fix}(k) \right)
\]
\[
\geq \limsup_{n \to \infty} \frac{1}{n} \log \left( \frac{1}{m^n} \sum_{|g|=n} s(g, n, 2\delta) \right).
\]

Taking \( \delta \to 0 \) in the left hand side the previous inequality and recalling Theorem D this proves (c) and finishes the proof of the theorem.

Some comments are in order. Firstly it is not hard to check that an analogous result holds for the notion of entropy \( h((G, G_1), X) \), leading to
\[
h((G, G_1), X) \leq \limsup_{n \to \infty} \frac{1}{n} \log \# \text{Per}(G_n).
\]

Secondly, since any expanding map satisfies the periodic specification property then periodic measures are dense in the space of invariant probability measures (see e.g. [16, Proposition 21.8]). Hence, given a finitely generated semigroup of expanding maps \( G \) it is clear that whenever the set \( \mathcal{M}(G) \) of probability measures invariant by every element \( g \in G \) is non-empty then the set of periodic measures
\[
\mathcal{P}_{\text{per}}(G) = \bigcup_{n \geq 1} \bigcup_{g \in G_n} \left\{ \frac{1}{n} \sum_{j=0}^{n-1} \delta_{\sigma^j g} : x \in \text{Fix}(g) \right\}
\]
is dense in the set of probability measures \( \mathcal{M}(G) \). Finally, weighted versions of the previous theorem for potentials with bounded distortion are also very likely to hold.

5. Applications

In this section we provide some classes of examples of semigroup actions that combine hyperbolicity and specification properties. We also provide some examples for which while we compare the notions of topological entropy used here with some others previously introduced and available in the literature, and discuss the relation between entropy, periodic points and specification properties.

The following example illustrates that in the notion of specification some ‘linear independence condition’ on the set of generators must be assumed in order to obtain that the group has the specification property.

Example 5.1. Consider the integer valued matrix
\[
A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},
\]
which induces a linear (topologically mixing) Anosov \( f_A \) on \( T^2 = \mathbb{R}^2/\mathbb{Z}^2 \) that satisfies the specification property. Hence, the \( \mathbb{Z} \) action \( \mathbb{Z} \times T^2 \to T^2 \) given by \((n,x) \mapsto f^n_A(x)\) satisfies the specification property.

Now, take \( B = A^{-2} \in SL(2, \mathbb{Z}) \) which also induces a linear Anosov \( f_B \) on the torus and satisfies the specification property. Nevertheless, the \( \mathbb{Z}^2 \)-action \( \mathbb{Z}^2 \times T^2 \to T^2 \) given by \((\langle m,n \rangle, x) \mapsto f^n_{A^m} f^m_{B}(x) = f^{m-2n}_{A} (x)\) clearly does not satisfy the specification property because every element in the (unbounded) subgroup \( \{(2n,n) : n \in \mathbb{Z}\} \subset \mathbb{Z}^2 \) induces the identity map. This indicates that generators should be taken in an irreducible way, that is, that there are \( n_1, \ldots, n_k \in \mathbb{Z} \) not all simultaneously zero so that \( g_1^{n_1} \cdots g_k^{n_k} = \text{Id}_G \).

The next modification of the previous example illustrates that the irreducibility of the generators in the sense that two generators \( A \) and \( B \) satisfy \( A^m B^n \neq \text{Id} \) for all \( m, n \in \mathbb{Z} \) not simultaneously zero is not the unique obstruction.

**Example 5.2.** Let \( A, B \) be the two matrices in \( SL(4, \mathbb{Z}) \) given by
\[
A = \begin{pmatrix} A & 0 \\ \mathbb{T}_2 & A \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \quad \text{where} \quad A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in SL(2, \mathbb{Z}),
\]
\( \mathbb{T}_2 \in M_{2 \times 2}(\mathbb{Z}) \) denotes the identity matrix and \( 0 \in M_{2 \times 2}(\mathbb{Z}) \) is the null matrix. It is not difficult to see that \( A \) and \( B \) are hyperbolic matrices (hence the diffeomorphisms induced by \( A \) and \( B \) satisfy the specification property), these commute but \( B \neq A^n \) for all \( m \in \mathbb{Z} \). Consider the \( \mathbb{Z}^2 \)-action \( T : \mathbb{Z}^2 \times T^4 \to T^4 \) of \( \mathbb{Z}^2 \) on the torus \( T^4 \) defined by \((\langle m,n \rangle, x) \mapsto A^m B^n (x)\). Since the element
\[
A^{-1} B = \begin{pmatrix} \mathbb{T}_2 & 0 \\ \mathbb{T}_2 & \mathbb{T}_2 \end{pmatrix}
\]
does not satisfy the specification property one can deduce from Lemma\textsuperscript{22} that this group action does not satisfy the specification property. Similarly, it is not hard to check that this group action does not satisfy neither of the orbital specification properties.

It follows from the discussion on the previous section that \( C^1 \)-robust specification property implies that the corresponding generators are uniformly hyperbolic and, in particular, the action is structurally stable. Our twofold purpose in the next example is: (i) to exhibit broad families of non-hyperbolic smooth maps that satisfy orbital specification properties although generators do not necessarily have the specification property; (ii) present examples where the weak orbital specification property holds while the strong orbital property does not.

**Example 5.3.** Let \( f : S^1 \to S^1 \) be a \( C^1 \)-expanding map of the circle and \( R_\alpha : S^1 \to S^1 \) be the rotation of angle \( \alpha \). Let \( G \) be the semigroup generated by \( G_1 = \{ id, f, R_\alpha \} \). A physical motivation for this class of examples arises from the consideration of ‘random’ orbits as follows. If one combines the Bowen-Series expanding map on the circle arising from the geodesic flow on compact surface of negative curvature after identification of the boundary with \( S^1 \) (see e.g. \textsuperscript{27}) and admit that each time the points hit the boundary the same can move by a fixed translation displacement. Hence, orbits correspond to all possibilities of iterations by an expanding map combined with a fixed rotation on \( S^1 \). This example can be modified for the semigroup \( G \) to be free (e.g. by taking an irrational rotation and an expanding map with trivial centralizer c.f. discussion in the Example\textsuperscript{5.4}).
Claim 1: The action induced by the semigroup $G$ on the unit circle $S^1$ does not satisfy the strong orbital specification property.

Proof of Claim 1. Take $\delta > 0$ and $x_1, x_2 \in$ in the circle, $n_1 = n_2 = n \geq 1$ and the maps $g_{x_1} = f^n$ and $g_{x_2} = f^{n_2}$. For any $p \geq 1$ take $b_p = R_p = R_{\alpha p}$ the rotation of angle $\alpha$. If $n$ is large then the dynamical balls $B_f(x_1, n_1, \delta)$ and $B_f(x_2, n_2, \delta)$ are disjoint and small. In particular, there exists $p \geq 1$ so that $b_p(\{x_1, n_1, \delta\}) \cap B_f(x_2, n_2, \delta) = \emptyset$. In particular the semigroup action $G$ on $S^1$ does not satisfy the strong specification orbital property. □

Claim 2: The action induced by the semigroup $G$ on the unit circle $S^1$ satisfies the weak orbital specification property.

Proof of Claim 2. Since $f$ is $C^1$-expanding, by the proof of Lemma 4.1 there exists $\delta_0 > 0$ so that for any $0 < \delta \leq \delta_0$, any $x \in X$ and any $n \in \mathbb{N}$ it follows that $f^n(B_f(x, n, \delta)) = B_f(x, n, \delta)$. Moreover, there exists $N = N(\delta) \geq 1$ so that any ball of radius $\delta$ is mapped onto $S^1$ by $f^N$. We can now prove the claim. Given $\delta > 0$ take $p(\delta) = N(\delta) \geq 1$. For any $p \geq p(\delta)$ let $G_p \subset G^*$ denote the set of elements $b_p \in G^*$ for which the following holds: given arbitrary points $x_1, \ldots, x_k \in X$, any positive integers $n_1, \ldots, n_k \geq 1$, any elements $g_n_{j, i} = g_{n_{j, i}} \in G^*$ and any elements $h_{p_j} \in G^*$ with $p_j \geq p(\delta)$ there exists $x \in X$ so that $d(g_{n_{j, i}}(x), g_{n_{j, i}}(x)) < \delta$ for every $\ell = 1 \ldots n_1$ and

$$d(g_{n_{j, i}}(x), g_{n_{j, i}}(x)) < \delta$$

for every $j = 2 \ldots k$ and $\ell = 1 \ldots n_j$. We claim that $\lim_{p \to +\infty} |\tilde{G}_p|/|G^*| = 1$. We notice that $g_{n_{j, i}}(B(x, g_{n_{j, i}}, \delta)) = B(g_{n_{j, i}}(x), \delta)$ is a ball of radius $\delta$ for any $1 \leq j \leq k$. So, if the expanding map $f$ is combined at least $p(\delta)$ times in any way in the words $b_p$ we get $b_p(B(y, \delta)) = S^1$ for any $y$ which clearly implies that $b_p \in \tilde{G}_p$. Thus for any $p \geq p(\delta)$

$$G^* \setminus \tilde{G}_p = \{h_{p_j} = h_{p_1} \ldots h_{p_j}, h_{i_1} \in G_p : 1 \leq j \leq p : h_{i_j} = f\} < p(\delta)$$

Clearly, for any $0 < \gamma < 1$

$$\frac{|G^* \setminus \tilde{G}_p|}{2^{-p}} \leq 2^{-\gamma p} \sum_{k=0}^{p(\delta)-1} \binom{p}{k} \leq p(\delta) 2^{-\gamma p} p^{\delta} \to 0$$

as $p$ tends to infinity, which proves our claim. □

Since the assumption (H) in Theorem 5.1 is a direct consequence of the previous equation (15) then we deduce that this semigroup action has positive topological entropy.

Clearly we can modify the previous strategy to deal with other different kind of semigroups with more generators. Our next purpose is to provide an example of a semigroup with exponential growth that is not a free semigroup but still satisfy the assumptions of Theorem 5.1.

Example 5.4. Let $X = S^1$ be the circle and consider the expanding maps on $S^1$ given by $g_1(x) = 2x$ (mod 1), that $g_2(x) = 3x$ (mod 1). It is clear that these maps commute (that is, $g_1 \circ g_2 = g_2 \circ g_1$) and that $g^k_1 \neq g^k_2$ for every $k, \ell \in \mathbb{Z}_+$ (since 2 and
3 are relatively prime). Now, consider another $C^1$-expanding map $g_3$ such that its centralizer $Z(g_3)$ is trivial, meaning

$$Z(g_3) := \{ h : S^1 \to S^1 \text{ expanding } : h \circ g_3 = g_3 \circ h \} = \{ g_3^\ell : \ell \in \mathbb{Z}_+ \}.$$ 

In particular the subgroup generated by $g_2$ and $g_1$ is disjoint from $Z(g_3)$. In other words, $g_3 \circ g_2^k \circ g_1^k \neq g_2^k \circ g_1^k \circ g_3$ for every $\ell, k \in \mathbb{Z}_+$. The existence of such $g_3$ is guaranteed by [1]. Let $G$ be the semigroup of expanding maps with generators $G_1 = \{ g_1, g_2, g_3 \}$. By construction, the subgroup $\tilde{G}$ of $G$ generated by $G_1 = \{ g_1, g_3 \}$ is a free semigroup then

$$\lim_{n \to \infty} \frac{1}{n} \log |G_n| \geq \log 2 > 1$$

and the semigroup has exponential growth. Since the generators do not have finite order then any elements $g \in G_n$ is a concatenation $g = g_{i_n} \ldots g_{i_1}$ with $g_{i_i} \in G_1$. By commutativity, all concatenations of $j$ elements $g_1$ and $k$ elements $g_2$ coincide with the expanding map $g_1^j g_2^k$ and consequently there are exactly $n+1$ elements in $G_n$ obtained as concatenations of the elements $g_1$ and $g_2$. This semigroup has exponential growth and is not abelian but still satisfies the conditions of Theorem [1] for every Hölder continuous potential $\varphi : X \to \mathbb{R}$ and, in particular, the pressure function $t \mapsto P_{\text{top}}((G, G_1), t\varphi, X)$ is differentiable Lebesgue-almost everywhere.

In what follows we shall provide a simple example of a $\mathbb{Z}^d$-semigroup action where we can already discuss the relation between the notion of topological entropy that we introduced in comparison with some of the previous ones. We focus on the case of semigroups of expanding maps for simplicity of computations while we notice that an example of actions of total automorphisms as considered in Example 5.1 could be constructed analogously.

**Example 5.5.** Let $X = S^1$ be the circle and the $\mathbb{Z}^3$-group action $T : \mathbb{Z}^3 \times S^1 \to S^1$ defined by $((m, n, k), x) \mapsto g_1^m g_2^n g_3^k(x)$, where $g_1(x) = 2x$ (mod 1), $g_2(x) = 3x$ (mod 1) and $g_3(x) = 5x$ (mod 1) are commuting expanding maps of the circle. By commutativity and the fact that the numbers $2, 3, 5$ are relatively prime it is easy to check that $|G_n| = (n+1)(n+2)/2$. First we shall compute the topological pressure as considered by Bis in [3]. If $s(n, \delta)$ denotes the number of $(n, \delta)$-separated sets by $G$ the topological entropy in [3] is defined by

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{|G_{n-1}|} \log s(n, \delta).$$

In our context, for any $\delta > 0$

$$\limsup_{n \to \infty} \frac{1}{|G_{n-1}|} \log s(n, \delta) \leq \limsup_{n \to \infty} \frac{2}{n^2} \log(5^n) = 0$$

proving that the entropy in (16) is zero. For the sake of completeness let us mention that it is remarked in [3] that having positive topological entropy with this definition does not depend on the generators. Ruelle [33] considered a slightly different but similar notion of topological entropy but that does coincide with (16) in this context.

Let us now proceed to compute the notion of topological entropy considered by Ghys, Langvin, Walczak [21] and Bis [5]. According to their definition entropy is computed as

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s(n, \delta) = \log 5$$
and it measures the maximal entropy rate in the semigroup. Finally we observe that it follows from [12] that the topological entropy of the semigroup action, according to Definition 3.1, in the case the generators are expanding is given by

$$h_{\text{top}}((G, G_1), X) = \log \left( \frac{\deg g_1 + \deg g_2 + \deg g_3}{3} \right) = \log \left( \frac{10}{3} \right) > 0.$$ 

Finally let us mention that this semigroup action satisfies the strong orbital specification properties and, consequently, it follows from Theorems [13] and [14] that every point in the circle is an entropy point with respect to both entropy notions.

**Acknowledgements.** F.G. is supported by BREUDS and P.V. is supported by a postdoctoral fellowship by CNPq-Brazil and are grateful to Faculdade de Ciências da Universidade do Porto for the excellent research conditions.

**References**

[1] C. Arteaga Centralizers of expanding maps of the circle. *Proc. Amer. Math. Soc.*, 114: 1 (1992), 263–267.

[2] W. Bauer and K. Sigmund. Topological dynamics of transformations induced on the space of probability measures. *Monatsh. Math.*, 79, 81–92, 1975.

[3] A. Bisi, Partial variational principle for finitely generated groups of polynomial growth and some foliated spaces, *Colloq. Math.* 110 (2008), 431-449.

[4] A. Bisi and P. Walczak. Entropy of distal groups, pseudogroups, foliations and laminations. *Ann. Polon. Math.* 100 (2011), no. 1, 45–54.

[5] A. Bisi, An analogue of the variational principle for group and pseudogroup actions., *Ann. Inst. Fourier. Grenoble.*, 63, 839–863.

[6] A. Bisi and M. Urbanski. Some remarks on topological entropy of a semigroup of continuous maps. Cubo 8 (2006), no. 2, 63–71.

[7] R. Bowen, Periodic points and measures for Axiom A diffeomorphisms, *Trans. Amer. Math. Soc.*, 154, (1971), 377–397.

[8] R. Bowen. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, volume 470 of *Lect. Notes in Math.* Springer Verlag, 1975.

[9] R. Bowen and D. Ruelle. The ergodic theory of Axiom A flows. *Invent. Math.*, 29:181–202, 1975.

[10] A.M. Blokh, Decomposition of dynamical systems on an interval, *Russ. Math. Surv.* 38 (1983), 133-134.

[11] A. Bufetov, Topological entropy of free semigroup actions and skew-product transformations, *Journal of dynamical and control systems*, 5, No. 1, 137-142, 1999.

[12] A. Bufetov. Teor. Verojatnost. i Primenen. 44 (1999), 696–697. English transl. in Theory Probab. Appl. 44 (1999).

[13] G. Cairns, A. Kolganova, and A. Nielsen Topological Transitivity and Mixing Notions for Group Actions *Rocky Mountain J. Math.* 37, 2 (2007), 371–397.

[14] E. Chen and D. Zheng Bowen entropy for action of amenable groups. Preprint [arXiv:1410.4645](https://arxiv.org/abs/1410.4645)

[15] N.-P. Chung and H. Li. Homoclinic groups, IE groups, and expansive algebraic actions *Invent. Math.*, 199-3 (2015) 805–858.

[16] M. Denker, C. Grillenberger and K. Sigmund. Ergodic theory on compact spaces. *Lecture Notes in Mathematics, Vol. 527*, . Springer-Verlag, Berlin-New York, 1976.

[17] A. Eizenberg, Yu. Kifer and B. Weiss, Large deviations for $\mathbb{Z}^d$ actions, *Comm. Math. Phys.*, 164, (1994), 433–454.

[18] A Yu Fishkin. An analogue of the Misiurewicz-Przytycki theorem for several maps. *Russ. Math. Surv.* 56, (2001), 158–159.

[19] S. Friedland. “Entropy of graphs, semigroups and groups”, in Ergodic theory of $\mathbb{Z}^d$ actions (Warwick. 1993 1991), London Math. Soc. Lecture Note Ser., vol. 228, Cambridge Univ. Press. Cambridge, 1996. p. 319–343.

[20] E. Ghys. Groups acting on the circle. *Enseign. Math.* (2) 47 (2001), no. 3–4, 329–407.
[21] E. Ghys, R. Langevin and P. Walczak. Entropie géométrique des feuilletages. *Acta Math.* 160 (1988), no. 1-2, p. 105–142.
[22] R. I. Grigorchuk, Degrees of growth of finitely generated groups and the theory of invariant means *Akad. Nauk SSSR*, 48:5, (1984), 939–985.
[23] A. Katok. Fifty years of entropy in dynamics: 1958?2007. *J. Mod. Dyn.* 1:4, 545–596, (2007).
[24] P. Kościelniak and M. Mazur. Chaos and the shadowing property. *Topology Appl.* 154 (2007), no. 13, 2553–2557.
[25] P. de la Harpe Topics in Geometric Group Theory. Univ. of Chicago Press, 2000.
[26] D. Lind and K. Schmidt Symbolic and algebraic dynamical systems Handbook of Dynamical Systems, Volume 1, Edited by H.W. Broer, B. Hasselblatt and F. Takens
[27] A. Lopes and Ph. Thieullen. Mather measures and the Bowen-Series transformation. *Ann. Inst. H. Poincaré - Anal. Non Linéaire* 23., no. 5, 663–682, 2006.
[28] D. Ma and M. Wu. Topological pressure and topological entropy of a semigroup of maps. *Discrete Contin. Dyn. Syst.* 31 (2011), no. 2, 545–557.
[29] R. Miles and T. Ward. Directional uniformities, periodic points and entropy. Preprint arXiv:1411.5295
[30] K. Moriyasu, K. Sakai and K. Yamamoto, Regular maps with the specification property, *Discrete and Continuous Dynam. Sys.*, 33, (2013), 2991–3009.
[31] K. Oliveira and X. Tian. Nonuniform hyperbolicity and nonuniform specification *Trans. Amer. Math. Soc.* 365, no. 8, 4371–4392, 2013.
[32] C.E. Pfister and W. Sullivan, Large Deviations Estimates for Dynamical Systems without the Specification Property Application to the Beta-Shifts, *Nonlinearity*, 18, (2005), 237–261.
[33] D. Ruelle, Statistical mechanics on a compact set with $\mathbb{Z}^p$ action satisfying expansiveness and specification, *Trans. Amer. Math. Soc.*, Vol. 185, 237–251, 1973.
[34] D. Ruelle, A $\mathbb{Z} \times \mathbb{Z}$ structurally stable action *Trans. Amer. Math. Soc.*, 260:2 515–525, 1980.
[35] K. Sakai and N. Sumi and K. Yamamoto, Diffeomorphisms satisfying the specification property. *Proc. Amer. Math. Soc.*, 138, 315–321, 2010
[36] F. M. Schneider Topological entropy of continuous actions of compactly generated groups Preprint arXiv:1502.03080
[37] K. Sigmund, *On dynamical systems with the specification property*, *Tran. Amer. Math. Soc.*, 190 (1974), 285–299.
[38] Ya. Sinai. Gibbs measures in ergodic theory. *Russian Math. Surveys*, 27:21–69, 1972.
[39] N. Sumi and P. Varandas and K. Yamamoto. Partial hyperbolicity and specification. Preprint arXiv:1307.1182
[40] N. Sumi and P. Varandas and K. Yamamoto. Absence of specification for partially hyperbolic flows. Preprint arXiv:1502.00745
[41] D. Thompson *Irregular sets, the $\beta$-transformation and the almost specification property*, *Trans. Amer. Math. Soc.*, 364 (2012), 5395–5414.
[42] P. Varandas Non-uniform specification and large deviations for weak Gibbs measures. *J. Stat. Phys.* 146, no. 2, 330–358, 2012.
[43] K. Yamamoto On the weaker forms of specification property and their applications. *Proc. Amer. Math. Soc.*, 137, 11: 3807–3814, 2009.

**DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DO RIO GRANDE DO SUL, BRAZIL.**
& **CMUP, UNIVERSITY OF PORTO, PORTUGAL**

*E-mail address: fagnerbernardin1@gmail.com*

**DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DA BAHIA, AV. ADÉMAR DE BARROS S/N, 40170-110 SALVADOR, BRAZIL.** & **CMUP, UNIVERSITY OF PORTO, PORTUGAL**

*E-mail address: paulo.varandas@ufba.br*