THE BORSUK-ULAM THEOREM FOR CLOSED 3-MANIFOLDS HAVING GEOMETRY $S^2 \times \mathbb{R}$

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Abstract. Let $M$ be a closed 3-manifold which admits the geometry $S^2 \times \mathbb{R}$. In this work we determine all the free involutions $\tau$ on $M$, and the Borsuk-Ulam index of $(M, \tau)$.

1. Introduction

The theorem now known as the Borsuk-Ulam theorem seems to have first appeared in a paper by Lyusternik and Schnirel’man [13] in 1930, then in a paper by Borsuk [9] in 1933 (where a footnote mentions that the theorem was posed as a conjecture by S. Ulam). One of the most familiar statements (Borsuk’s Satz II) is that for any continuous map $f : S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ such that $f(x) = f(-x)$. The theorem has many equivalent forms and generalizations, an obvious one being to replace $S^n$ and its antipodal involution $\tau(x) = -x$ by any finite-dimensional connected CW-complex $X$ equipped with some fixed point free involution $\tau$, and ask whether $f(x) = f(\tau(x))$ must hold for some $x \in X$. The original theorem and its generalizations have many applications in topology, and also – since Lovász’s [12] and Bárány’s [1] pioneering work in 1978 – in combinatorics and graph theory. An excellent general reference is Matoušek’s book [14]. For more about some new developments in the subject, as well the terminology used here, see [5].

In recent years, the following generalization of the question raised by Ulam has been studied for many families of pairs $(M, \tau)$, where $\tau$ is a free involution on the space $M$: Given $(M, \tau)$, determine all positive integers $n$ such that for every map $f : M \to \mathbb{R}^n$, there is an $x \in M$ for which $f(x) = f(\tau(x))$. When $n$ belongs to this family, we say that the pair $(M, \tau)$ has the Borsuk-Ulam property with respect to maps into $\mathbb{R}^n$. Not surprisingly, examples of spaces are known for which the value of $n$ depends not only on the space but also on the free involution (assuming such exists), cf. [10], [11], or [5], as well as the present note.

The Borsuk-Ulam property has been studied for several families of closed (compact without boundary) manifolds, the spheres being the first such family. The problem of existence and classification of free involutions arises naturally for a manifold $M$. In particular various results have been obtained for 3-manifolds, cf. [5].

In this note we shall consider closed 3-manifolds admitting the geometry $S^2 \times \mathbb{R}$. Our problem, then, is to classify the free involutions $\tau$ (up to equivalence, see Definition 2 below) on each $M$ under consideration, and to determine, for such a pair $(M, \tau)$, the integers $n$ for which the pair has the Borsuk-Ulam property with respect to maps into $\mathbb{R}^n$. 
Remark 1. Since the seven 3-manifolds admitting the geometry \( S^2 \times \mathbb{R} \) are precisely the manifolds covered by \( S^2 \times \mathbb{R} \) [16, p. 458], it follows that for any covering projection \( M \to N \) of 3-manifolds, if one of the two admits the geometry \( S^2 \times \mathbb{R} \) then so does the other.

Remark 2. Among these seven 3-manifolds, exactly four are closed [17], namely:

1. \( S^2 \times S^1 \),
2. the non-orientable \( S^2 \)-bundle \( E \) over \( S^1 \),
3. \( \mathbb{R}P^2 \times S^1 \), and
4. \( \mathbb{R}P^3 \# \mathbb{R}P^3 \), the connected sum of two \( \mathbb{R}P^3 \).

These four manifolds are naturally Seifert-fibred by taking a suitable group of isometries of \( S^2 \times \mathbb{R} \) [16, p. 457].

The goal of the present work is to answer the problem above for the four Seifert manifolds of Remark 2, in the same spirit as we did for flat- and nil-geometries [3], [4]. For most of the remaining Seifert manifolds\(^1\) the same problem can be solved similarly, using [6] and [7].

For the closed manifolds which admit Sol geometry, some partial results are known, see [2]. For hyperbolic 3-manifolds not much is presently known about the Borsuk-Ulam property.

In order to determine the pairs \((M, \tau)\), where \( M \) runs over the double coverings of a given manifold \( N \), we compute \( \pi_1(M) = \ker \varphi \), for all possible epimorphisms \( \varphi : \pi_1(N) \to \mathbb{Z}_2 \). The involution \( \tau \) is then the involution associated to the double cover, and \( M/\tau \) is homeomorphic to \( N \). The characteristic class \( [\varphi] \in H^1(N; \mathbb{Z}_2) \) of the fibre bundle \( M \to N \) determines the answer to the Borsuk-Ulam problem for \((M, \tau)\). For the four closed 3-manifolds supporting \( S^2 \times \mathbb{R} \) geometry, the main result is Theorem 4, which is also summarized in Figures 1, 2. The cases A) and B) of Theorem 4 were solved recently, using a slightly different approach, in [8, Theorem 17 and Proposition 18].

This work contains two sections besides the present Introduction. In Section 2 we give more details about the Borsuk-Ulam property and the methods used to study it, as well as some details about Seifert manifolds. In Section 3 we solve the problem for the closed manifolds having geometry \( S^2 \times \mathbb{R} \).

2. Preliminaries

Let us recall some known results that will be used throughout this paper. We henceforth assume that \( M \) is a connected 3-manifold, \( \tau \) a free involution on it, \( N \) the quotient manifold \((N = M/\tau)\), and \( \varphi : \pi_1(N) \to \mathbb{Z}_2 \) the associated epimorphism classifying the principal \( \mathbb{Z}_2 \)-bundle \( M \to N \).

Definition 1.\(^1\)

- The pair \((M, \tau)\) satisfies the Borsuk-Ulam property for \( \mathbb{R}^n \) if for any continuous map \( f : M \to \mathbb{R}^n \), there is at least one point \( x \in M \) such that \( f(x) = f(\tau(x)) \).
- The \( \mathbb{Z}_2 \)-index of \((M, \tau)\), denoted by \( \text{ind}_{\mathbb{Z}_2}(M, \tau) \), is the greatest integer \( n \) such that \((M, \tau)\) satisfies the Borsuk-Ulam property for \( \mathbb{R}^n \).

\(^1\)The only presently unknown case is that of manifolds which admit geometry \( \mathbb{H}^2 \times \mathbb{R} \), with an involution such that the quotient is not a Seifert manifold in the sense of Seifert’s definition.
Corollary 3. With $M, N, \tau, \varphi$ as above, $\text{ind}_{\mathbb{Z}_2}(M, \tau)$ equals:

- 1 if $N = S^2 \times S^1$ or $E$, or if $N = \mathbb{R}P^2 \times S^1$ and $\varphi(v) = 0$;
- 2 if $N = \mathbb{R}P^2 \times S^1$ or $\mathbb{R}P^3 \# \mathbb{R}P^3$ and $\varphi(h) = 0$;
- 3 if either $N = \mathbb{R}P^2 \times S^1$ and $\varphi(h) + \varphi(v) \neq 0$, or $N = \mathbb{R}P^3 \# \mathbb{R}P^3$ and $\varphi(h) \neq 0$.

From [13] and [11], it is known that $1 \leq \text{ind}_{\mathbb{Z}_2}(M, \tau) \leq \dim(M)$, and from Theorems 3.1 and 3.2 proved in [11], we have:

Theorem 1. 

(i) $\text{ind}_{\mathbb{Z}_2}(M, \tau) = 1$ if and only if the epimorphism $\varphi : \pi_1(N) \to \mathbb{Z}_2$ factors through the projection $\mathbb{Z} \to \mathbb{Z}_2$.

(ii) $\text{ind}_{\mathbb{Z}_2}(M, \tau) = 3$ if and only if the cup-cube $[\varphi]^3$ is non-zero, where $[\varphi] \in H^1(N; \mathbb{Z}_2)$ is the characteristic class of the double covering (principal $\mathbb{Z}_2$-bundle) $M \to N$ associated to $\varphi$.

Next, let us define an equivalence relation on the set of pairs $(M, \tau)$.

Definition 2. For $i = 1$ or $2$, let $M_i$ be a connected 3-manifold, $\tau_i$ a free involution on it, $N_i$ the quotient manifold $(N_i = M_i/\tau_i)$, and $\varphi_i$ the associated epimorphism $(\varphi_i : \pi_1(N_i) \to \mathbb{Z}_2)$.

$(M_1, \tau_1)$ and $(M_2, \tau_2)$ are equivalent – or $\varphi_1$ and $\varphi_2$ are equivalent – if the two bundles $M_i \to N_i$ are isomorphic.

In the sequel, most pairs will be trivially non-equivalent because the base spaces of their associated bundles are non-homeomorphic.

Henceforth, $M$ is one of the four manifolds $S^2 \times S^1$, $E$, $\mathbb{R}P^2 \times S^1$, or $\mathbb{R}P^3 \# \mathbb{R}P^3$ of Remark 2. Following the notation of Orlik [15], $M$ will be described by a list of Seifert invariants written in the normal form $\{b; (\epsilon, g)\}$, hence without exceptional fibres (the type $\epsilon$ reflects the orientations of the base surface and the total space of the Seifert fibration of $M$, and the integer $g$ is the genus of the base surface – the orbit space obtained by identifying each $S^1$ fibre of $M$ to a point). These invariants $\{b; (\epsilon, g)\}$ provide the Seifert presentations of the fundamental groups of our four particular manifolds:

- $\pi_1(S^2 \times S^1) = \pi_1(0; (o_1, 0)) = \langle h \rangle \approx \mathbb{Z}$,
- $\pi_1(E) = \pi_1(1; (n_1, 1)) = \langle v, h | v^2h^{-1} = \langle v \rangle \approx \mathbb{Z}$,
- $\pi_1(\mathbb{R}P^2 \times S^1) = \pi_1(0; (n_1, 1)) = \langle v, h | v^2, vh^{-1}h^{-1} \rangle \approx \mathbb{Z}_2 \times \mathbb{Z}$,
- $\pi_1(\mathbb{R}P^3 \# \mathbb{R}P^3) = \pi_1(0; (n_2, 1)) = \langle v, h | v^2, (vh)^2 \rangle \approx \mathbb{Z}_2 \times \mathbb{Z}_2$.

We extract from [5] Proposition 4.2] the restricted cases needed here:

Proposition 2. Let $N$ be one of the four manifolds $S^2 \times S^1, E, \mathbb{R}P^2 \times S^1$, or $\mathbb{R}P^3 \# \mathbb{R}P^3$, and $\varphi, [\varphi]$ as in Theorem 1 above. Then, $[\varphi]^3 = 0$ except in the two following cases:

- $N = \mathbb{R}P^2 \times S^1$ and $\varphi(h) + \varphi(v) \neq 0$;
- $N = \mathbb{R}P^3 \# \mathbb{R}P^3$ and $\varphi(h) \neq 0$.

Using this proposition and our presentation of $\pi_1(N)$, Theorem [11] gives:

Corollary 3. With $M, N, \tau, \varphi$ as above, $\text{ind}_{\mathbb{Z}_2}(M, \tau)$ equals:

- 1 if $N = S^2 \times S^1$ or $E$, or if $N = \mathbb{R}P^2 \times S^1$ and $\varphi(v) = 0$;
- 2 if $N = \mathbb{R}P^2 \times S^1$ or $\mathbb{R}P^3 \# \mathbb{R}P^3$ and $\varphi(h) = 0$;
- 3 if either $N = \mathbb{R}P^2 \times S^1$ and $\varphi(h) + \varphi(v) \neq 0$, or $N = \mathbb{R}P^3 \# \mathbb{R}P^3$ and $\varphi(h) \neq 0$. 

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Remark 1 and Remark 2 imply that the four manifolds under consideration are exactly the 3-manifolds covered by $S^2 \times S^1$ (this is also illustrated in Figure 1), and also that this family is closed under taking finite covers or finite quotients.

We now state the main result (recall that “unique” here means up to the equivalence of Definition 2).

**Theorem 4.** Up to equivalence,

A) The space $S^2 \times S^1$ admits four free involutions $\tau_1, \tau_2, \tau_3, \tau_4$ such that:

1. The quotient $(S^2 \times S^1)/\tau_1$ is homeomorphic to $S^2 \times S^1$, and $\text{ind}_{\mathbb{Z}_2}(S^2 \times S^1, \tau_1) = 1$.
2. The quotient $(S^2 \times S^1)/\tau_2$ is homeomorphic to $E$, and $\text{ind}_{\mathbb{Z}_2}(S^2 \times S^1, \tau_2) = 1$.
3. The quotient $(S^2 \times S^1)/\tau_3$ is homeomorphic to $\mathbb{R}P^2 \times S^1$, and $\text{ind}_{\mathbb{Z}_2}(S^2 \times S^1, \tau_3) = 2$.
4. The quotient $(S^2 \times S^1)/\tau_4$ is homeomorphic to $\mathbb{R}P^3 \# \mathbb{R}P^3$, and $\text{ind}_{\mathbb{Z}_2}(S^2 \times S^1, \tau_4) = 2$.

B) The space $E$ admits a unique free involution $\tau_5$. The quotient $E/\tau_5$ is homeomorphic to $\mathbb{R}P^2 \times S^1$, and $\text{ind}_{\mathbb{Z}_2}(E, \tau_5) = 3$.

C) The space $\mathbb{R}P^2 \times S^1$ admits a unique free involution $\tau_6$. The quotient $(\mathbb{R}P^2 \times S^1)/\tau_6$ is homeomorphic to $\mathbb{R}P^2 \times S^1$, and $\text{ind}_{\mathbb{Z}_2}(\mathbb{R}P^2 \times S^1, \tau_6) = 1$.

D) The space $\mathbb{R}P^3 \# \mathbb{R}P^3$ admits a unique free involution $\tau_7$. The quotient $(\mathbb{R}P^3 \# \mathbb{R}P^3)/\tau_7$ is homeomorphic to $\mathbb{R}P^3 \# \mathbb{R}P^3$, and $\text{ind}_{\mathbb{Z}_2}(\mathbb{R}P^3 \# \mathbb{R}P^3, \tau_7) = 3$.

The rest of this section is devoted to the proof of this theorem, and the involutions $\tau_1, \ldots, \tau_7$ will be specified in Propositions 5-8.

3.1. Coverings and $\mathbb{Z}_2$-index. The purpose of this subsection is to present the graph below where the arrows are directed from the covering to the base. The number associated with each arrow is the $\mathbb{Z}_2$-index of the covering. This graph is a summary of Theorem 4. For example, the arrow from $S^2 \times S^1$ to $E$ represents Theorem 4 (A2), and the 1 along the arrow corresponds to $\text{ind}_{\mathbb{Z}_2}(S^2 \times S^1, \tau_2) = 1$.

![Figure 1: Graph for the geometry $S^2 \times \mathbb{R}$](image-url)
As before, let \( \varphi : \pi_1(N) \to \mathbb{Z}_2 \) be an epimorphism. In order to obtain the 2-covering \( M \) of \( N \) determined by \( \varphi \), we compute \( \pi_1(M) = \ker \varphi \). Each case of Theorem 3 is proved by one of the following propositions; this correspondence is given in Figure 2 below. In all cases the presentation of \( \pi_1(N) \) is as in Section 2. We shall generally denote points of \( S^1 \) as \( z \in \mathbb{C}, |z| = 1 \), points of \( S^2 \) as \( x \in \mathbb{R}^3, \|x\| = 1 \), and points of \( \mathbb{R}P^2 \) as \( [x] = \{\pm x\}, x \in \mathbb{R}^2 \).

**Proposition 5.**

1. There is a unique epimorphism from \( \pi_1(S^2 \times S^1) \) onto \( \mathbb{Z}_2 \).
2. The associated 2-covering and free involution form the pair \( (S^2 \times S^1, \tau_1) \), where \( \tau_1(x, z) = (x, -z) \).
3. Its \( \mathbb{Z}_2 \)-index is 1.

**Proof.**

1. There is a unique epimorphism from \( Z \) onto \( \mathbb{Z}_2 \).
2. Clearly, \( \tau_1 \) is a free involution on \( S^2 \times S^1 \), and \( (S^2 \times S^1)/\tau_1 \) is homeomorphic to \( S^2 \times S^1 \).
3. This follows from Corollary 3.

**Proposition 6.**

1. There is a unique epimorphism from \( \pi_1(E) \) onto \( \mathbb{Z}_2 \).
2. The associated 2-covering and free involution form the pair \( (S^2 \times S^1, \tau_2) \), where \( \tau_2(x, z) = (-x, -z) \).
3. Its \( \mathbb{Z}_2 \)-index is 1.

**Proof.** Same arguments as in the previous proposition.

**Proposition 7.**

1. There are three epimorphisms from \( \pi_1(\mathbb{R}P^2 \times S^1) \) onto \( \mathbb{Z}_2 \) defined by:
   \[
   \varphi_1(v) = 0, \quad \varphi_1(h) = 1 \\
   \varphi_2(v) = 1, \quad \varphi_2(h) = 0 \\
   \varphi_3(v) = 1, \quad \varphi_3(h) = 1.
   \]
2. (i) The 2-covering and free involution associated to \( \varphi_1 \) form the pair \( (\mathbb{R}P^2 \times S^1, \tau_6) \), where \( \tau_6([x], z) = ([x], -z) \). Its \( \mathbb{Z}_2 \)-index is 1.
   (ii) The 2-covering and free involution associated to \( \varphi_2 \) form the pair \( (S^2 \times S^1, \tau_3) \), where \( \tau_3(x, z) = (-x, z) \). Its \( \mathbb{Z}_2 \)-index is 2.
   (iii) The 2-covering and free involution associated to \( \varphi_3 \) form the pair \( (E, \tau_5) \), where \( \tau_5([x, y]) = [-x, y] \) (using the representation of \( E \) as \( S^2 \times S^1/\tau_2 \)). Its \( \mathbb{Z}_2 \)-index is 3.

**Proof.**

1. This is clear.
2. (i) One has \( \ker \varphi_1 = \langle v, h^2 \rangle \approx \mathbb{Z}_2 \times \mathbb{Z} \) and \( (\mathbb{R}P^2 \times S^1)/\tau_6 \) is homeomorphic to \( \mathbb{R}P^2 \times S^1 \). By Corollary 3 \( \text{ind}_{\mathbb{Z}_2}(\mathbb{R}P^2 \times S^1, \tau_6) = 1 \).
   (ii) One has \( \ker \varphi_2 \approx \mathbb{Z}_2 \), generated by the element \((0, 1) \in \mathbb{Z}_2 \times \mathbb{Z} \). This corresponds to the product by \( S^1 \) of the orientation cover of \( \mathbb{R}P^2 \). The correctness of \( (M, \tau_5) \) follows. By Corollary 3 \( \text{ind}_{\mathbb{Z}_2}(\mathbb{R}P^2 \times S^1, \tau_3) = 2 \).
   (iii) It is straightforward to see that \( E/\tau_5 \) is homeomorphic to \( \mathbb{R}P^2 \times S^1 \). The only remaining possibility for the classifying map of the bundle \( E \to \mathbb{R}P^2 \times S^1 \) is \( \varphi_3 \), whence by Corollary 3 \( \text{ind}_{\mathbb{Z}_2}(E, \tau_5) = 3 \).
Proposition 8.  
(1) There are three epimorphisms from $\pi_1(\mathbb{R}P^3\#\mathbb{R}P^3)$ onto $\mathbb{Z}_2$ defined by:

$$
\varphi_1(v) = 1, \quad \varphi_1(h) = 0 \\
\varphi_2(v) = 0, \quad \varphi_2(h) = 1 \\
\varphi_3(v) = 1, \quad \varphi_3(h) = 1.
$$

(i) The 2-covering and free involution associated to $\varphi_1$ form the pair $(S^2 \times S^1, \tau_4)$, where $\tau_4(x, z) = (-x, z)$. Its $\mathbb{Z}_2$-index is 2.

(ii) The epimorphisms $\varphi_2$ and $\varphi_3$ are equivalent (cf. Definition 2). For both of them, the associated 2-covering and free involution form the same pair $(\mathbb{R}P^3\#\mathbb{R}P^3, \tau_7)$, where $\tau_7([x, z]) = [x, -z]$ (using the representation of $\mathbb{R}P^3\#\mathbb{R}P^3$ as $S^2 \times S^1/\tau_4$). Its $\mathbb{Z}_2$-index is 3.

Proof. (1) This is clear.

(2) First notice that by Corollary 3, $\text{ind}_{\mathbb{Z}_2}(S^2 \times S^1, \tau_4) = 2$ and $\text{ind}_{\mathbb{Z}_2}(\mathbb{R}P^3\#\mathbb{R}P^3, \tau_7) = 3$.

Also note that since $N$ is orientable, so are its double covers.

(i) One has $\ker \varphi_1 = \langle h \rangle \cong \mathbb{Z}$, hence $M = S^2 \times S^1$.

The correctness of $\tau_4$ stems from the fact that $(S^2 \times S^1)/\tau_4$ is homeomorphic to $\mathbb{R}P^3\#\mathbb{R}P^3$, which is a simple consequence of [16, p. 457].

(ii) One has $\ker \varphi_2 = \langle v, h^2 \rangle$ and $\ker \varphi_3 = \langle hv, h^2 \rangle$. These two subgroups of $\pi_1(N)$ are isomorphic to the group itself, hence the total space $M$ of both associated bundles over $N$ is $N$.

One easily checks that $\tau_7$ is a fixed point free involution and $M/\tau_7$ is homeomorphic to $N$. Representing $N = \mathbb{R}P^3\#\mathbb{R}P^3$ as $S^2 \times S^1/\tau_4$, the homeomorphism $\sigma$ of this quotient given by $\sigma([x, z]) = [x, \overline{z}]$ exchanges the two copies of $\mathbb{R}P^3$, hence it gives rise to two (obviously isomorphic) fibre bundles over $N$ : if $p : M \to N$ is one of them, the other one is $\sigma \circ p$. They correspond to $\varphi_2$ and $\varphi_3$ (in this order or in reverse order, depending on which homeomorphism between $M/\tau_7$ and $N$ was chosen to define $p$).

$\square$

In addition to Figure 1 above, the following Figure 2 also gives a summary of Propositions 5 to 8 from a slightly different perspective.

| Theorem | Proposition | $M$ | $N$ | Involution | Index |
|---------|-------------|-----|-----|------------|-------|
| (A1)    | Proposition 5 | $S^2 \times S^1$ | $S^2 \times S^1$ | $\tau_1$ | 1 |
| (A2)    | Proposition 6 | $S^2 \times S^1$ | $E$ | $\tau_2$ | 1 |
| (A3)    | Proposition 7 (2)(ii) | $S^2 \times S^1$ | $\mathbb{R}P^2 \times S^1$ | $\tau_3$ | 2 |
| (A4)    | Proposition 8 (2)(i) | $S^2 \times S^1$ | $\mathbb{R}P^3\#\mathbb{R}P^3$ | $\tau_4$ | 2 |
| (B)     | Proposition 9 (2)(iii) | $E$ | $\mathbb{R}P^2 \times S^1$ | $\tau_5$ | 3 |
| (C)     | Proposition 10 (2)(i) | $\mathbb{R}P^2 \times S^1$ | $\mathbb{R}P^2 \times S^1$ | $\tau_6$ | 1 |
| (D)     | Proposition 11 (2)(ii) | $\mathbb{R}P^3\#\mathbb{R}P^3$ | $\mathbb{R}P^3\#\mathbb{R}P^3$ | $\tau_7$ | 3 |

Figure 2: Summary of Propositions 5 to 8.
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