Multi-peakon solutions of the Degasperis–Procesi equation

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Abstract. We present an inverse scattering approach for computing $n$-peakon solutions of the Degasperis–Procesi equation (a modification of the Camassa–Holm (CH) shallow water equation). The associated non-self-adjoint spectral problem is shown to be amenable to analysis using the isospectral deformations induced from the $n$-peakon solution, and the inverse problem is solved by a method generalizing the continued fraction solution of the peakon sector of the CH equation.

Degasperis and Procesi [2] showed, using the method of asymptotic integrability, that the PDE

$$u_t - u_{xxx} + (b + 1)uu_x = bu_xu_{xx} + uu_{xxx}$$

(1)
cannot be completely integrable unless $b = 2$ or $b = 3$. The case $b = 2$ is the Camassa–Holm (CH) shallow water equation [1], which is well known to be integrable and to possess multi-soliton (weak) solutions with peaks, so called multi-peakons. Degasperis, Holm and Hone [3, 4] proved that the case $b = 3$, which they called the Degasperis–Procesi (DP) equation, is also integrable and admits multi-peakon solutions. They found the two-peakon solution explicitly by direct computation.

The purpose of this note is to briefly describe an inverse scattering procedure for obtaining $n$-peakon solutions of the DP equation. Full details will be published elsewhere in a longer paper [8]. Our approach is similar to that used by Beals, Sattinger and Szmigielski to obtain $n$-peakon solutions of the CH equation [5, 6], but the present case does involve substantially new features; in particular, the spectral problem is of third order instead of second, and consequently is not self-adjoint.

The DP equation can be written as a system for $u(x,t)$ and $m(x,t)$:

$$m_t + m_x u + 3mu_x = 0,$$  

(2)

$$m = u - u_{xx},$$  

(3)
As shown in [8], this is the compatibility condition for the overdetermined linear system
\[
(\partial_x - \partial_x^2)\psi = zm\psi, \quad (\partial_x - \partial_x^2)\psi = z \psi \quad (4)
\]
\[
\psi = [z^{-1}(c - \partial_x^2) + u_x - u \partial_x] \psi \quad (5)
\]
for a wave function \(\psi(x, t)\). The constant \(c\) is arbitrary; for our purposes \(c = 1\) is the appropriate choice.

The \(n\)-peakon solution has the form
\[
u(x, t) = \sum_{k=1}^n m_k(t) e^{-|x-x_k(t)|}, \quad m(x, t) = \sum_{k=1}^n 2 m_k(t) \delta(x - x_k(t)), \quad (6)
\]
where \(\delta\) is the Dirac delta distribution. This satisfies (4) by construction, while (2) is satisfied if and only if the functions \(\{x_k(t), m_k(t)\}_{k=1}^n\), which describe the positions and heights of the peakons, evolve according to the following system of ODE (where \(\text{sgn} 0 = 0\)):
\[
\dot{x}_k = \sum_{i=1}^n m_i e^{-|x_i-x_k|}, \quad \dot{m}_k = 2 \sum_{i=1}^n m_k m_i \text{sgn}(x_k - x_i) e^{-|x_k-x_i|}. \quad (7)
\]
The case \(n = 1\) is trivial: \(m_1 = \text{constant}, x_1 = x_1(0) + m_1 t\). Also when \(n = 2\), the solution can be found by straightforward integration [8].

In this note we will always assume that all \(m_k > 0\) and \(x_1 < \ldots < x_n\). To show that this property is preserved by the flow, assume that it holds for some value of \(t\). It can be verified directly that \(M_1 = \sum_{k=1}^n m_k\) and \(M_n = \left(\prod_{k=1}^n m_k\right) \left(\prod_{k=1}^{n-1} (1 - e^{x_k-x_{k+1}})^2\right)\) are constants of motion. If all \(m_k\) are positive, then \(M_1\) and \(M_n\) are also positive, which implies that there is a constant \(m_0\) such that \(0 < m_0 < m_j(t) < M_1\) for all \(t\) and \(1 \leq j \leq n\), and that \(x_k(t) - x_{k+1}(t)\) can never become zero; hence \(x_1(t) < \ldots < x_n(t)\) must hold for all \(t\).

Since \(\dot{x}_k = \sum_{i=1}^n m_i e^{-|x_i-x_k|} > m_k e^0 > m_0\), we see that \(x_k \to \pm \infty\) as \(t \to \pm \infty\). Even more is true: the peakons scatter, that is \(|x_j - x_k| \to \infty\) as \(t \to \pm \infty\) for all \(j \neq k\), and the particles behave asymptotically like free particles moving with velocities \(m_k(\pm \infty) = \lim_{t \to \pm \infty} m_k(t)\) as \(t \to \pm \infty\). Since mutual distances between particles grow indeﬁnitely, the asymptotic velocities are distinct, rendering \(m_{ij}(\pm \infty) \neq m_k(\pm \infty)\) for all \(j \neq k\). In this sense the DP peakons belong to the same class of mechanical systems as the ﬁnite Toda lattice [7] and CH peakons [5]. A complete proof of the scattering properties will be presented elsewhere [8].

Now consider equation (4) in the case when \(m\) is a discrete measure as in [7].
With the \(t\) dependence suppressed, the equation reads
\[
\psi_x(x) - \psi_{xxx}(x) = z \left(\sum_{k=1}^n 2 m_k \delta(x - x_k)\right) \psi(x). \quad (8)
\]
Let \(x_0 = -\infty\) and \(x_{n+1} = +\infty\). Since \(\psi_x - \psi_{xxx} = 0\) away from the support of \(m\), the wave function is piecewise given by expressions of the form
\[
\psi(x) = A_k e^{x} + B_k + C_k e^{-x}, \quad x \in (x_k, x_{k+1}) \quad (k = 0, 1, \ldots, n). \quad (9)
\]
By [5], \(\psi\) and \(\psi_x\) are continuous at each point \(x_k\), while \(\psi_{xx}\) has a jump discontinuity of \(-2z m_k \psi(x_k)\). This gives, with \(I\) denoting the \(3 \times 3\) identity matrix,
\[
\begin{bmatrix} A_k \\ B_k \\ C_k \end{bmatrix} = S_k(z) \begin{bmatrix} A_{k-1} \\ B_{k-1} \\ C_{k-1} \end{bmatrix}, \quad S_k(z) = I - z m_k \begin{bmatrix} e^{-x_k} \\ -2 \\ e^{x_k} \end{bmatrix} \begin{bmatrix} e^{x_k}, 1, e^{-x_k} \end{bmatrix}. \quad (10)
\]
Consider the particular wave function satisfying $\psi(x) = e^x$ for $x < x_1$; that is, $[A_0, B_0, C_0] = [1, 0, 0]$. For $x > x_n$, we then have $\psi(x) = A_n(z)e^x + B_n(z) + C_n(z)e^{-x}$, where $[A_n, B_n, C_n] = S_n(z) \cdots S_2(z)S_1(z) [1, 0, 0]^t$, so $A_n$, $B_n$ and $C_n$ are polynomials in $z$ of degree $n$, with coefficients depending on $m_1, \ldots, m_n$ and $e^{x_1}, \ldots, e^{x_n}$. For $z = 0$, the right-hand side of (8) together with the above boundary conditions are given by the zeros of the degree polynomial $\lambda_n$. This holds iff $\lambda_n$ are positive and simple. Denoting them by $\lambda_1, \ldots, \lambda_n$, we have $A_n(z) = \prod_{k=1}^n (1 - z/\lambda_k)$.

As in [5] it will prove useful to consider an equivalent spectral problem on the finite interval $[-1, 1]$. Let $y = \tanh(x/2)$ and define $\phi(y)$ by $\psi(x) = \frac{2}{y^2} \phi(y)$. This maps $\psi(x)$ into a piecewise quadratic function: $\phi(y) = \frac{1}{2}(A_k(1 + y)^2 + B_k(1 - y^2) + C_k(1 - y^2))$ for $(y_k, y_{k+1})$ (where $y_k = \tanh(x_k/2)$, $y_0 = -1$, $y_n+1 = 1$). The spectral problem (5) with boundary conditions $B_0 = C_0 = 0, A_n = 0$ is equivalent to
\begin{equation}
- \phi_{yy}(y) = z g(y) \phi(y), \quad \phi(-1) = \phi(1) = 0, \quad \phi(1) = 0,
\end{equation}
where
\begin{equation}
g(y) = \sum_{k=1}^n g_k \delta(y - y_k), \quad g_k = \frac{8 m_k}{(1 - y_k^2)^2},
\end{equation}
which generalizes the string equation approach used in [4]. At each $y_k$, the second derivative has a jump: $\phi_{yy}(y_k+) = \phi_{yy}(y_k-) - z g_k \phi(y_k)$. We define a pair of Weyl functions $W(z)$ and $Z(z)$, and let $b_k$ and $c_k$ be the residues in their partial fractions decompositions:
\begin{equation}
W(z) = \frac{\phi_y(1)}{z \phi(1)} = \frac{1}{z} - \frac{B_n(z)}{2zA_n(z)} = \sum_{k=0}^n \frac{b_k}{z - \lambda_k},
\end{equation}
\begin{equation}
Z(z) = \frac{\phi_{yy}(1)}{z \phi(1)} = \frac{1}{2z} - \frac{B_n(z)}{2zA_n(z)} + \frac{C_n(z)}{2zA_n(z)} = \sum_{k=0}^n \frac{c_k}{z - \lambda_k},
\end{equation}
where we have set $\lambda_0 = 0$ (so $b_0 = 1$ and $c_0 = 1/2$). We will see below that the second Weyl function $Z(z)$ is actually determined by the first Weyl function $W(z)$, a fact that is not obvious from the definition.

We now derive the time evolution of the scattering data $\{\lambda_j, b_j\}$ defined by the first Weyl function $W(z)$, when $m(x, t)$ evolves as described by (7). Then $\psi(x, t)$ evolves according to [3]. For $x < x_1$, equation (10) shows that $u_x = u$, so in that interval $\psi(x, t) = e^{-x}$ does indeed satisfy (7) for all $t$ (with our choice of $c = 1$). For $x > x_n$ we have $u_x = -u$, which implies that $\psi(x, t) = A_n(z, t)e^x + B_n(z, t) + C_n(z, t)e^{-x}$ satisfies (8) in that interval if and only if
\begin{equation}
\dot{A}_n = 0, \quad \dot{B}_n = B_n/z - 2A_n M_+, \quad \dot{C}_n = -B_n M_+, \quad \dot{M}_+ = \sum_{k=1}^n m_k e^{z_k},
\end{equation}
where $M_+ = \sum_{k=1}^n m_k e^{z_k}$. By computing the matrix product $S_n \cdots S_1$, it is not hard to see that $A_n(z) = 1 - M_1 z + \ldots + (-1)^n M_n z^n$, which proves that $M_1$ and $M_n$ are constants of motion, as claimed above. Moreover, by analyzing how the coefficients of $A_n$ depend on positions $x_j$, and exploiting the scattering property of the system, it can be seen that as $t \to \infty$ the coefficients tend to the elementary symmetric functions of $m_1(\infty) < \ldots < m_n(\infty)$, implying $A_n(z) = \lim_{t \to \infty} A_n(z) = \prod_{k=1}^n (1 - zm_k(\infty))$. 

Multi-peakon solutions of the Degasperis–Procesi equation
Thus the scattering property of the DP peakons manifest itself in the spectrum of the Dirichlet-like problem \((14)\) being real and simple.

The evolution equations \((14)\) readily imply that the scattering data flows according to

\[
\lambda_k = \text{constant}, \quad b_0(t) = 1, \quad b_k(t) = b_k(0)e^{t/\lambda_k} \quad (k \geq 1).
\]

To see how the scattering data determines \(c_k\), and hence \(Z(z)\), we proceed as follows. We always have \(c_0 = 1/2\). Let \(\hat{W}(z) = -B_n/2zA_n = \sum_{k=1}^n b_k/(z - \lambda_k)\) and \(\hat{Z}(z) = C_n/2zA_n = \sum_{k=1}^n (c_k - b_k)/(z - \lambda_k)\). From \((10)\) it follows that \(B_n(z) = 2zM_+ + O(z^2)\), which implies \(\hat{W}(0) = -M_+\). Then \((14)\) gives \(\hat{Z}(z) = \sum_{j=1}^n b_j/(z - \lambda_j)\) for \(k \geq 1\). The polynomial \(C_n\) vanishes as \(t \to -\infty\) (which is again seen by analyzing how its coefficients depend on \(x_j\)'s), hence \(c_k - b_k\) vanishes. By \((15)\) we have \(b_j b_k = b_j(0)b_k(0)\exp(1/\lambda_j + 1/\lambda_k)t\), so integration from \(-\infty\) to \(t\) yields

\[
c_k = \lambda_k b_k \sum_{j=0}^n \frac{b_j}{\lambda_j + \lambda_k} \quad (k \geq 1).
\]

Finally we show that the inverse spectral problem has a unique solution, i.e., that the \(y_j\)'s and \(g_j\)'s are uniquely determined by the scattering data. For \(0 \leq j \leq n\) we define \((1, w_{2j}, z_{2j-1}) = \{\phi_{yy}, \phi_y, \phi\}_y = y_{j+1} + (1, w_{2j}, z_{2j}) = \sum_{j=1}^n \phi_{yy}(y_j, y_j + 1)\). These quantities are analogs of remainders in the theory of one dimensional continued fractions. Since on the interval \((y_j, y_{j+1})\) the solution to \((14)\) takes the form

\[
\phi(y) = \phi(y_{j+1}) + \phi_y(y_{j+1})(y - y_{j+1}) + \phi_{yy}(y_{j+1}) (y - y_{j+1})^2/2,
\]
we obtain the following descending fractional linear transformations for the remainders, where \(l_j = y_{j+1} - y_j\) is the length of the interval:

\[
\begin{align*}
w_{2j-1} &= \frac{w_{2j}}{z_{2j}} - l_j, \quad z_{2j-1} = \frac{1}{z_{2j}} - l_j \frac{w_{2j}}{z_{2j}} + \frac{l_j^2}{2}, \\
w_{2j-2} &= \frac{w_{2j-1}}{z_{2j-1}}, \quad z_{2j-2} = \frac{1}{z_{2j-1}} + zg_j;
\end{align*}
\]

the iteration starts at \(w_{2n} = \hat{Z}(W(z))\), \(z_{2n} = \hat{Z}(z)\) (which are known in terms of scattering data) and stops at \(w_{-1}, z_{-1}\). The unknown quantities \(\{l_j, g_j\}\) are determined in each step from the large \(z\) asymptotics of remainders known from the previous step as a result of the following property: all even remainders \(w_{2j}\) and \(z_{2j}\) are \(O(1)\) at \(z = \infty\), while all odd remainders \(w_{2j-1}\) and \(z_{2j-1}\) are \(O(1/z)\) there. In particular, denoting by \(a^{(m)}\) the coefficient of \(z^{-m}\) in the expansion of a holomorphic function \(a(z)\) at \(z = \infty\) we obtain the recovery formulas

\[
l_j = \frac{w_{2j}^{(0)}}{z_{2j}^{(0)}}, \quad g_j = \frac{-1}{z_{2j-1}^{(1)}}.
\]

Mapping this back to the original variables, we have a recursive way of deriving formulas for the \(x_k\)'s and \(m_k\)'s in the \(n\)-peakon solution in terms of the scattering data. Together with \((15)\) this gives the solution \(\{x_k(t), m_k(t)\}\) of \((7)\). In a longer paper \([8]\), we derive closed form expressions for these quantities, and analyze the
Multi-peakon solutions of the Degasperis–Procesi equation

dynamics in more detail. Here we merely state the results for the three-peakon case ($n = 3$):

\[ x_3(t) = \ln(b_1 + b_2 + b_3), \]

\[ x_2(t) = \ln \left( \frac{(\lambda_1 - \lambda_2)^2 b_1 b_2 + (\lambda_1 - \lambda_3)^2 b_1 b_3 + (\lambda_2 - \lambda_3)^2 b_2 b_3}{\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3} \right), \]

\[ x_1(t) = \ln \left( \frac{(\lambda_1 - \lambda_2)^2(\lambda_1 - \lambda_3)^2(\lambda_2 - \lambda_3)^2 b_1 b_2 b_3}{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)} \right), \]

\[ m_3(t) = \frac{(b_1 + b_2 + b_3)^2}{\lambda_1 b_1^2 + \lambda_2 b_2^2 + \lambda_3 b_3^2 + \frac{\lambda_1 \lambda_2 \lambda_3}{\lambda_1 + \lambda_2 + \lambda_3} b_1 b_2 + \frac{\lambda_1 \lambda_2 \lambda_3}{\lambda_1 + \lambda_2 + \lambda_3} b_1 b_3 + \frac{\lambda_1 \lambda_2 \lambda_3}{\lambda_1 + \lambda_2 + \lambda_3} b_2 b_3}, \]

with similar (but more involved) expressions for $m_1(t)$ and $m_2(t)$.

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