Per-capita GDP and nonequilibrium wealth-concentration in a model for trade

Cristian F Moukarzel
CINVESTAV del IPN, Depto. de Física Aplicada, 97310 Mérida, Yucatán, México.
E-mail: cfmoukarzel@mda.cinvestav.mx

Abstract. Data describing the per-capita Gross Domestic Product of around two hundred countries in years 1960 to 2008 are analyzed and found to decay approximately exponentially with rank. We discuss this experimental fact in the context of a wealth exchange model called Yard-Sale exchange, in which pairs of agents (i.e. nations) ‘bet’ for a fraction \( f \) of the wealth of the poorest of them. If the chances for this poorest agent to win the bet are not large enough, this model presents a ‘wealth condensation’ phase, in which one lucky agent gets to own the whole wealth in the end. In a recent study of this model [1], it was found that, in the condensed phase, the typical wealth of an agent with rank \( R \) decays exponentially with \( R \). By establishing a parallel between wealth of a nation and its per-capita GDP, these observations suggest that international trade rules are such that strong wealth concentration is favored. Possible extensions of this work, that also consider endogenous factors affecting the evolution of GDP, are also discussed.

1. Wealth exchange model
The existence of large wealth inequalities in human societies[2], which are themselves composed of individuals possessing not so different abilities, certainly deserves explanation [3, 4]. One important factor driving the appearance of inequalities is the fact that wealthier individuals are in a better position to influence their social environment to their advantage, thus producing a self-reinforcing cascade. In other words, rich agents have an explicit advantage. Angle [5] pioneered the use of conservative wealth exchange models in order to explain wealth inequalities arising among equally able “agents”. In Angle’s initial proposal, wealth concentration is a consequence of an explicit advantage, or edge favoring the richer agent in each pairwise exchange \(^1\). However, it is now recognized [1, 6] that an explicit advantage favoring the rich is not necessary for wealth concentration to happen. This is a remarkable result that only recently has been stressed in the Econophysics literature. For certain realistic wealth-exchange rules, in the long run all wealth may end up in the hands of just one agent, even if the poor agent has an edge over the rich one in each pairwise transaction. The key ingredient for this rather counterintuitive phenomenon is the fact that the amount at stake in each transaction is proportional to the poorest agent’s wealth [1, 6, 7, 8, 9, 10]. In other words, multiplicative stochastic transfer of wealth whose scale is dictated by the wealth of the poorest intervening agent (“poorest-scheme” models) implies a “hidden” bias favoring the rich.

\(^1\) An agent is said to have an edge when the expectation value of a single trade favors him.
Stochastic multiplicative “poorest scheme” transfer rules constitute an appropriate simple model of the wealth exchange process occurring during commercial interaction, or trade [8, 11, 10, 9, 6, 1]. Wealth transfer occurs in a trade operation not because money changes hands, which is not necessarily always the case, but as a consequence of the difference in values between the items swapped. This makes the interaction stochastic in character, as none of the agents is perfectly aware of the true values of the items being interchanged. The total amount involved in the trade, and therefore the amount of wealth transferred among agents, must be proportional to the wealth of the poorest agent, since otherwise the richest agent could be up for loosing many times more than he can win, which cannot be considered realistic in a consensual trade process [8, 10]. Additionally, it is reasonable to assume that the poor agent may have a larger probability to accrue wealth in each transaction, since it is him who has the strongest incentive to bargain aggressively [11].

1.1. The model

Trade interactions are modeled as a process in which wealth is stochastically transferred within a population of $N$ agents, according to the following rules. In each transaction a pair of agents is chosen at random and the poorest one, initially with wealth $w_{\text{poor}}$, receives a gain $\kappa w_{\text{poor}}$, where $-1 < \kappa < 1$ is a random return with distribution $\pi(\kappa)$ ($\kappa < 0$ indicates that the poor agent looses wealth). The richest agent’s wealth changes by $-\kappa w_{\text{poor}}$, so the transaction is wealth-conserving. The condition $|\kappa| < 1$ furthermore ensures that both agents have a positive wealth after the trade. The sign of the expectation value $\langle \kappa \rangle$ determines whether the poor agent ($\langle \kappa \rangle > 0$) or the rich agent ($\langle \kappa \rangle < 0$) has an advantage in each transaction.

Yard-Sale [8, 11, 9, 6] is a particular simple case of this process that can be described as a “bet” for a fraction $f$ of the wealth of the poorest agent, and where the poorest agent has a probability $p$ to win. Therefore $\kappa = +f$ with probability $p$ and $\kappa = -f$ with probability $q = (1 - p)$, so $\pi_{\text{YS}}(\kappa) = p\delta(\kappa - f) + q\delta(\kappa + f)$. In this case, the poor agent has an explicit advantage, or edge, whenever $p > 1/2$.

1.2. Wealth Condensation

Depending on $\pi(\kappa)$, long term evolution may give rise either to a stable wealth distribution $P(w)$ or to wealth condensation [9, 6, 1]. The surface consisting of distributions $\pi(\kappa)$ separating these two cases is called condensation interface. The location of this interface can be derived [6, 1] as follows: Consider an agent who has become so poor that, in most subsequent trades he will be the poorest. His own wealth will thus almost always evolve according to

$$w_{t+1} = w_t (1 + \kappa_t),$$

i.e. it will undergo a Random Multiplicative Process [12] with multiplier $\eta_t = (1 + \kappa_t)$ at each timestep. After a large number $N$ of timesteps, the appropriate central tendency estimator for $w$ is the geometric average

$$e^{\langle \ln w_N \rangle} = w_0 e^{N\langle \ln (1 + \kappa) \rangle},$$

Clearly the wealth of a poor agent will diminish steadily if $\langle \ln (1 + \kappa) \rangle < 0$, in which case there is a sustained transference of wealth from poorer to richer agents, the system is in a condensing phase, and the whole wealth ends up in the hands of one agent in the long run [6, 1]. This catastrophic collapse of the wealth distribution is called wealth condensation, and can happen even if the poorer agent has an edge in each trade, i.e. if $\langle \kappa \rangle > 0$. By the heuristic arguments above, the condensation interface is therefore defined by

$$\langle \ln (1 + \kappa) \rangle = 0.$$
1.3. Wealths by rank in the condensing phase

We now describe the dynamics in the condensing phase [1]. Let \( w_R \) denote, at a given fixed time, the wealths of the \( N \) agents ordered by rank \( R \), so that \( w_1 > w_2 > \ldots > w_N \). When an agent with rank \( R \) interacts with an agent with rank \( S \),

\[
\begin{align*}
  w_R^{(t+1)} &= \begin{cases} 
    w_R^{(t)} + \kappa w_I^{(t)} = w_R^{(t)} \eta & \text{if } R > S \\
    w_R^{(t)} - \kappa w_S^{(t)} & \text{if } R < S,
  \end{cases} 
\end{align*}
\]

Eq. (4) is not valid in general, since it disregards rank changes resulting from interactions. Its validity is restricted to the case in which agents keep fairly constant ranks, i.e. there is no “social mobility”. This condition is approximately satisfied in the condensing phase, but not in the stable phase, therefore our analysis in the following is only valid in the condensing phase.

In the condensing phase, furthermore, one has \( w_S/w_R << 1 \) for any \( S > R \), so that the interaction with poorer agents can be neglected altogether to write

\[
\ln \left( \frac{w_R^{(t+1)}}{w_R^{(t)}} \right) \approx \begin{cases} 
  \ln \eta & \text{if } R > S \\
  0 & \text{if } R < S.
\end{cases}
\]

Averaging over \( \pi(\kappa) \), over all \((N-1)\) possible choices of \( S \), and defining the relative rank \( r = (R-1)/(N-1) \) so that \( r = 0 \) indicates the richest and \( r = 1 \) the poorest agents,

\[
\langle \ln \left( \frac{w_R^{(t+1)}}{w_R^{(t)}} \right) \rangle_{\pi} = r \langle \ln \eta \rangle_{\pi}
\]

\[
\Rightarrow \langle \ln w_R^{(t)} \rangle_{\pi} = \ln w_R^{(0)} + rt \langle \ln \eta \rangle_{\pi}
\]

The typical value of \( w(r, t) \) therefore satisfies

\[
w(r, t) \sim e^{-rt\phi},
\]

where we have defined \( \phi = -\langle \ln \eta \rangle_{\pi} > 0 \) in the condensing phase. Normalization for a system of \( N \) agents with a total wealth \( W \) then results in

\[
w(r, t) = W \frac{(1-e^{-t\phi}/N)}{(1-e^{-t\phi})} e^{-rt \phi}
\]
Now since \( r = P_>(w(r)) \) we have that \( P(w) = -1/(\partial w(r)/\partial r) \). From Equation (8) we thus obtain

\[
P(w) \sim \frac{1}{w}, \tag{10}
\]

At finite times in the condensing phase. The validity of Equations (8) and (10) was verified numerically in previous work [1]. We next discuss the fact that the per-capita Gross Domestic Product (GDP) of countries are distributed in a manner that is consistent with Equation (8).

2. Per capita GDP distribution

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figures.png}
\caption{(a) Slope \( \Phi \). (b) Largest product \( G \) in thousands of dollars, as obtained from fitting (Equation 11) to data in Fig. 1b.}
\end{figure}

In this section, a time-dependent analysis of the ranked distribution of \( g \), the per-capita Gross Domestic Product (GDP) for several countries is presented\(^2\). Fig. 1a shows per-capita GDP (in year 2000 usd) for several countries and several years, \( g_R \), plotted versus rank \( R \) in log-linear scale. The behavior of the data in Fig. 1 is found to be roughly compatible with an exponential decay versus rank.

Inequalities in per-capita GDPs, average income and wealth among countries, as well as their relations to trade and other factors have been analyzed by several authors recently [13, 14, 15, 16, 17]. One of the issues in discussion is whether international wealth or output indicators converge over time, i.e. evolve towards similar values [18, 19]. A naive interpretation of the data in Fig. 1a could give the false impression that inequalities in \( g \) are decreasing in time, since the slopes of these plots are seen to diminish in time. Notice that Equation (8) predicts a slope that, on the contrary, grows linearly in time (if \( \phi \) is assumed constant), which reflects the fact that inequalities increase steadily if an exchange system is in a condensing phase. The observed decrease of the slopes in Fig. 1a with time is due to the fact that the number of countries reported in this particular data bank increases with time. When normalized ranks \( r = (R - 1)/N \) are used instead for the plot, Fig. 1b is obtained, where slopes are seen to increase slowly in time. In order to make these observations more quantitative, we next fit the data in Fig. 1b using

\[
\log g_r = \log G - \Phi r, \tag{11}
\]

\(^2\) Data downloaded from http://data.worldbank.org/indicator/NY.GDP.PCAP.KD
with two free parameters: the largest product $G$, and the slope $\Phi$ in a log-lin plot. The estimated slopes $\Phi$ (a rough measure of economic inequality) are displayed in Fig. 2a. It is seen in this figure that economic inequality increases roughly linearly in time as Equation (8) would require, but with differing speeds in the periods 1960-1977 and 1978-2010. In this later period, the rate of wealth concentration by the richest countries appears to be slower.

![Figures 2a and 2b](image)

(a) Dispersion $\sigma$ (a measure of inequality).  
(b) Geometric average $\mu$ of $g$, as obtained from assuming that data in Fig. 1b stem from $g$ values that are lognormally distributed, and fitting them using Equation (13).

The time evolution of the largest domestic product $G$, as obtained from fits of Equation (11) to the data in Fig. 1b, is shown in Fig. 2b. This quantity has a similar time dependence as the slope $\Phi$ shown in Fig. 1b, showing two clearly differentiated time periods. Notice that sometimes a lognormal distribution is assumed to be a good description of available data for per-capita income and per-capita GDP \(^3\):

$$P(g) = \frac{1}{g\sqrt{2\pi}\sigma^2} e^{-\frac{(\mu-\log g)^2}{2\sigma^2}}.$$  \hspace{1cm} (12)

If this were the case, the ranked values $g_r$ would not decrease exactly exponentially with normalized $r$, but would instead satisfy

$$\log g_r = \mu + \sigma \text{erfc}^{-1}(1-2r).$$  \hspace{1cm} (13)

If $\sigma$ is small, Equation (13) differs notably from Equation (11) only near the ends of its range $r \to 0, 1$. In the middle range, $r \approx 1/2$, one has $1-2r \approx 0$ and therefore $\text{erfc}^{-1}(x) \approx \sqrt{\pi}/2x$, so that Equation (13) behaves approximately as

$$\log g_r = \mu + \frac{\sigma \sqrt{\pi}}{2} - \sigma \sqrt{\pi} r,$$  \hspace{1cm} (14)

that is, Eq. (11) with $\log G = \mu + \sigma \sqrt{\pi}/2$ and $\Phi = \sigma \sqrt{\pi}$.

\(^3\) See for example, \texttt{http://en.wikipedia.org/wiki/International_inequality}; \textquotedblleft The skewed world GDP distribution and the interdependence of national institutions\textquotedblright, Alex Coad, DIME Final Conference, 6-8 April 2011 (\texttt{final.dime-eu.org/files/Coad_C9.pdf}); \textquoteleft A normal relationship?\textquoteright: Poverty, Growth, and Inequality\textquoteright, J. H. López and L. Servén, World Bank Policy Research Working Paper 3814, January 2006.
Figure 4: Assuming that $P(g)$ is lognormally distributed, leads to a ranked $g$ distribution that is given by Equation (13). Fits (continuous line) to the data used in this work (crosses) using Eq. (13) show strong deviations near the ends of the range, making the lognormality hypothesis dubious.

We tested the hypothesis of lognormality for the data presented here by fitting the ranked $g$ data on Fig. 1b using Equation (13). Some examples of these fits are shown in Fig. 4. The resulting parameters $\mu$ and $\sigma$ for each year are shown in Fig. 3. While the lognormality hypothesis for $P(g)$ cannot be rejected too clearly, we feel that, for the ranked data discussed in this work, it produces a less satisfactory fit than a simple exponential (See Fig. 4).

3. Discussion
Multiplicative “poorest-scheme” asset-exchange models with an arbitrary return distribution $\pi(\kappa)$, which describe conservative exchange of wealth due to trade, produce wealth condensation in the long run (the whole system’s wealth “condenses” onto one agent) whenever $\langle \ln (1 + \kappa) \rangle_\pi < 0$, in which case richer agents are systematically favored in exchange interactions.

In the condensing phase, and during the transient period leading to full wealth condensation, the ranked wealths $w_r$ are exponentially distributed according to Equation (8) [1], while $P(w) \sim 1/w$. In this nonequilibrium state, a systematic transference of wealth from poorer to richer agents occurs due to asymmetric exchange interactions.

As shown in Section 2, the per-capita GDP of several countries along many years, when plotted vs rank, are found to behave in a manner consistent with Equation (8), i.e. they decay approximately exponentially with rank, and the decay speed becomes progressively steeper...
in time (linearly). In the light of the exchange model \[1, 6\] described in Section 1.1, this observation seems to suggest that the per-capita GDP of countries are in a condensing phase because international trade rules systematically favor richer countries.

Notice, however, that the reasoning behind this conclusion depends on the somewhat loose analogy that relates trade among countries to commercial exchange among individuals, and per-capita GDP of countries to wealth of individual agents. The second part constitutes the less obvious analogy. It is clear that the concepts of ‘wealth’ and GDP of a nation are not interchangeable, although the latter is often used as an estimator for the former. Moreover, we do not need them to be equivalent. The applicability of our description only depends on the fact that the per-capita GDP of a country determines the scale of its interchanges, and is in turn affected by them, in the same was as the wealth of an individual fixes its trade volumes and is affected by them.

Therefore, one should first discuss in which way the wealth exchange due to international trade affects the respective countries’ per-capita GDP, and in which way the intervening countries’ per-capita GDP determine the amounts traded.

In the first place, it is clear that international trade does have a direct incidence on GDP, since one of the terms in the definition of GDP is precisely the trade balance \( E - I \), where \( E \) is the total exports and \( I \) the total imports of the given country in the given year.

As to how GDP determines the amounts traded, notice that several recent works \[13, 14, 20\] in fact describe statistical relationships between trade flows among countries and their respective GDPs. Additionally, note that a successful empirical econometric model for trade among countries, the so-called Gravity Model \[21, 22, 23\], poses that the aggregate trade flow between a given pair of countries is proportional to the product of their GDPs. This supports the notion that the amount of trade is determined by GDP at the end countries. In our exchange model \[1, 6\], however, an important difference is that only the smallest wealth of the two intervening agents, or countries, is relevant to determine the exchange flow for this given pair. We remark that recent statistical investigations of correlations between trade flows and GDPs indicate a stronger correlation of the trade flow with the wealth of the importing country \[14\], showing that a simple product of GDPs is not always appropriate. On top of that, the Gravity Model assumes equilibrium, i.e. a stationary flow distribution, whereas we have shown that per-capita GDPs seem to be in a condensing phase, which is essentially a nonequilibrium situation.

To conclude, if one assumes that (instantaneous, not aggregate) trade among countries can be described by a stochastic exchange model as the one studied in Refs. \[6, 1\], where wealth is replaced by per-capita GDP \( g \), then it follows that the observation of exponentially distributed ranked per-capita GDPs \( g_r \) might mean that the international trade system is in a condensing phase. In other words, the exchange processes, characterized by the distribution of returns \( \pi(\kappa) \) of the poorest countries in each interaction (See Ref. \[1\]) is such that \( \langle \ln (1 + \kappa) \rangle_\pi < 0 \), meaning that international trade produces a systematic transference of wealth from poorer to richer countries.

Notice, however, that the data on which the observations in this work are based are not entirely inconsistent with a totally different hypothesis: that per-capita GDP \( g \) have a lognormal distribution. In this case, ranked values \( g_r \) satisfy Equation \(13\) instead of Equation \(11\). While the difference between these two functional forms is not large in practice for values of \( r \) around 0.5 (so it is hard to decide which describes the data better), the contexts under which one or the other are valid are entirely different. While an exponential decay of ranked \( g_r \) occurs for a strongly interacting system, lognormally distributed \( g \) values are the result of the evolution of each country’s GDP under the influence of independent random multiplicative factors, without interactions among them.

\footnote{http://en.wikipedia.org/wiki/Gross_domestic_product}
Of course, it is still possible for both scenarios indeed to coexist, since a general model for economic evolution must consider that each agent evolves under the simultaneous effects of random multiplicative noise (e.g. describing endogenous factors) and of wealth-conserving interactions (e.g. describing trade) [24]. A study of the joint effects of investment plus Yard-Sale trading is underway and will be published somewhere else.

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