CHIRAL VECTOR BUNDLES:
A GEOMETRIC MODEL FOR CLASS AIII TOPOLOGICAL QUANTUM SYSTEMS

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ABSTRACT. This paper focuses on the study of a new category of vector bundles. The objects of this category, called chiral vector bundles, are pairs given by a complex vector bundle along with one of its automorphisms. We provide a classification for the homotopy equivalence classes of these objects based on the construction of a suitable classifying space. The computation of the cohomology of the latter allows us to introduce a proper set of characteristic cohomology classes: some of those just reproduce the ordinary Chern classes but there are also new odd-dimensional classes which take care of the extra topological information introduced by the chiral structure. Chiral vector bundles provide a geometric model for topological quantum systems in class AIII, namely for systems endowed with a (pseudo-)symmetry of chiral type. The classification of the chiral vector bundles over sphere and tori (explicitly computable up to dimension 4), recovers the commonly accepted classification for topological insulator of class AIII which is usually based on the K-group $K$. However, our classification turns out to be even richer since it takes care also for possible non-trivial Chern classes.

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The mathematical investigation of the nature of the topological phases of the matter is a recent hot topic in mathematical physics. Starting from the seminal work by Kane and Mele \[KM2\], where for the first time $\mathbb{Z}_2$ topological phases were predicted, a big effort has been devoted to the understanding of a general scheme for the complete classification of all possible phases of topological insulators. The first considerable contribution to this program was provided by Kitaev’s “Periodic Table” for topological insulators and superconductors \[K\]. This is a $K$-theoretical-based procedure which organizes into a systematic scheme the stable topological phases of symmetry-protected gapped free-fermion systems. A certain amount of valuable foundational works devoted to the justification and the generalization of Kitaev’s table appeared in recent years: \[SRFL, RSFL, FM, Th1, KZ\] just to mention few.

A quite different approach to the study of the topological phases of quantum systems consists in focusing on the rigorous geometrical analysis of each single topological class. In fact, if from one side the various fundamental symmetries (or better pseudo-symmetries as pointed out in \[KZ\]) can be used to organize the different classes of topological insulators according to the “Bott-clock” induced by the underlying Clifford structure, on the other side each class is described by its own geometric model. The latter is usually specified by the set of symmetries defining the class itself along with a proper notion of ground state, often identified with a Fermi projection in a gap (a different but equivalent point of view has been recently considered in \[KZ\]). For periodic fermionic systems, or more in general for topological quantum systems (in the sense of \[DG1, DG2, DG3\]), these geometric models are realizable as vector bundles enriched by the presence of extra structures. For the case of systems with an even or odd time-reversal symmetry (class A\text{I} and A\text{II}, respectively) a complete analysis of the related geometric theories has been recently performed in \[DG1, DG2\]. Some of the main achievements obtained through this specific study are: the possibility to access the unstable regime which is usually out of the domain of the K-theory (there is a recent interest on that \[KG, KZ\]); the possibility to consider also systems adiabatically perturbed by external fields (meaning that one can replace the Brillouin torus or sphere with quite general manifolds); the identifications of proper cohomological classes capable to completely classify the topological phases. An even more relevant aspect is the possibility to describe the topological classification in terms of differential-geometric invariants, a point recently investigated in \[DG3\].

This work is devoted to the study of a geometric model for the class A\text{III} or, said differently, for the class of quantum systems protected by a chiral symmetry (in the jargon of topological insulators \[SRFL, HK\]). There is a renewed recent interest for the study of the topology of these models (see e. g. \[PS\]). On the one hand it is a common belief that systems in class A\text{III} are classified by the $K$-group $K_1$. On the other hand a closer look at these systems reveals ambiguities and deficiencies in the various definitions, as recently pointed out in \[Th2\] (see also the introduction of \[Th1\]). The main open questions seem to be summarized by the following list:

- Q.1) Which is the class of transformations which preserve the chiral phases?
- Q.2) The topological indices which label the various chiral phases are absolute or relative?
- Q.3) Is $K_1$ enough to completely describe the topology of chiral systems?

It is our conviction that a complete answer to the previous questions (and to the observations raised in \[Th2\]) needs a precise definition of a geometric model for the description of systems in class A\text{III}. In this work we will show how to construct such a model, a fact which is important by itself for a better understanding of the geometry underlying chiral systems. Moreover, we will provide a rigorous study of the related topology in order to provide a finer classification for the class A\text{III}.

From a mathematical point of view the geometric object of our interest is a new class of vector bundles that we decided to call chiral vector bundles, or $\chi$-vector bundles for short. These are formally defined as follow:
Definition 1.1 (Chiral vector bundle). A chiral vector bundle over $X$ is a pair $(\mathcal{E}, \Phi)$ where $\pi: \mathcal{E} \to X$ is a complex hermitian vector bundle and $\Phi \in \text{Aut}(\mathcal{E})$ is an automorphism of $\mathcal{E}$. The rank of the chiral vector bundle $(\mathcal{E}, \Phi)$ is the complex dimension of the fibers of $\mathcal{E}$.

We point out that we are not only interested in the classification over spheres or tori (as usual in the business of topological insulators). Our requirements on the nature of the base space $X$ are quite weak. Hereinafter in this paper we will assume that:

Assumption 1.2. $X$ is a compact and path connected Hausdorff space with a CW-complex structure.

The particular choice of the nomenclature is justified by the fact that $\chi$-vector bundles provide a suitable model for the study of the topology of ground states of topological quantum systems endowed with a chiral symmetry. This connection will be briefly discussed at the end of this introduction and rigorously established in Section 6.

Chiral vector bundles over a topological space $X$ form a category when endowed with a proper notion of morphisms. These technical aspects will be discussed in full detail in Section 2. The only important information for the aims of this introductory discussion is that in this category the notion of isomorphism does not coincide with that of homotopy equivalence, a fact which makes chiral vector bundles quite different from complex vector bundles. The pure notion of isomorphism turns out to be too strong to set up a reasonable classification theory of topological nature. For this reason one needs to weaken the notion of equivalence by looking at the homotopy (see Example 2.5 for more details).

With this precaution the problem of the classification of chiral phases can be translated in the problem of the enumeration of the set

$$\text{Vec}_m^\chi(X) := \{\text{equivalence classes of homotopy equivalent rank } m \text{ chiral vector bundles over } X\}.$$ 

For a precise definition of the notion of homotopy equivalence we refer to Definition 2.4. The virtue of this definition is that $\text{Vec}_m^\chi(X)$ can be classified by homotopy classe of maps from $X$ to a classifying space $B_m^\chi$, i.e.

$$\text{Vec}_m^\chi(X) \simeq [X, B_m^\chi]. \quad (1.1)$$

The isomorphism (1.1) is established in Corollary 3.4 while the space $B_m^\chi$ is described in Section 3.1. Both homotopy and cohomology groups of $B_m^\chi$ can be explicitly determined (cf. Section 4.1 and Section 4.2, respectively) and along with (1.1) this allows for a classification of $\text{Vec}_m^\chi(X)$. For instance, for the classification of the topological phases of free-fermion systems the base space can be identified with a “Brillouin” sphere (see the discussion in [DG1, Section 2]) and so, after setting $X = S^d$ in (1.1) one obtains:

Theorem 1.3 (Classification over spheres). Rank $m$ chiral vector bundles over spheres are classified by the following bijections

$$\text{Vec}_m^\chi(S^d) \simeq \pi_d(B_m^\chi) \simeq \pi_d(U(m)) \oplus \pi_{d-1}(U(m)).$$

In the case of low-dimensional spheres the result of the classification is displayed in Table 1.1 along with the standard classification for complex vector bundles (see Section 5.3 for a more detailed discussion).

The content of Table 1.1 leads to an immediate observation: the topology of chiral vector bundles seems to be just richer than the topology of complex vector bundles. This is indeed true and not surprising at all! From its very definition it turns out that a chiral vector bundle is, in particular, a complex vector bundle plus an extra structure. Then, there is a “forgetting” map morphism which associates to a given $\chi$-bundle $(\mathcal{E}, \Phi)$ just its underlying complex vector bundle $\mathcal{E}$. This map is evidently compatible with the notions of isomorphisms of complex vector bundles and homotopy equivalence of $\chi$-bundles (which is weaker than the isomorphism equivalence) and leads to a well-defined morphism

$$j: \text{Vec}_m^\chi(X) \leftrightarrow \text{Vec}_m^\chi(X) \quad (1.2)$$
\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{VB} & \text{AZC} & d = 1 & d = 2 & d = 3 & d = 4 \\
\hline
\text{Vec}_C(S^d) & \text{A} & 0 & \mathbb{Z} & 0 & 0 \quad (m = 1) \\
& & & & \mathbb{Z} \quad (m > 2) & \\
\text{Vec}_C(S^d) & \text{AIII} & \mathbb{Z} & \mathbb{Z} & 0 \quad (m = 1) & 0 \quad (m = 1) \\
& & & & \mathbb{Z} \quad (m > 2) & \mathbb{Z} \oplus \mathbb{Z}_2 \quad (m = 2) \\
\hline
\end{array}
\]

Table 1.1. The column VB lists the symbols for the sets of equivalence classes of vector bundles in the complex and chiral category, respectively. The related Altland-Zirnbauer-Cartan labels \([SRFL]\) are displayed in column AZC. In the complex case the classification is specified by the first Chern class \(c_1\) in dimension \(d = 2\) and by the second Chern class \(c_2\) in dimension \(d = 4\) (cf. \([DG1]\) Section 4). In the chiral case the classification is richer and in addition to the usual Chern classes one has new odd-dimensional characteristic classes called chiral classes (cf. Section 5.1). The extra invariants introduced by the chiral structure are displayed by the “boxed” entries. In dimension \(d = 1\) and \(d = 3\) the \(\mathbb{Z}\) classification is provided by the first chiral class \(w_1\) and by the second chiral class \(w_2\), respectively. In dimension \(d = 4\) there is an unstable chiral invariant \(\mathbb{Z}_2\) only in rank \(m = 2\) which is related to the homotopy group \(\pi_4(S^3)\) and it is not detectable via characteristic classes (cf. Section 5.3).

which is indeed an injection. In particular the image of \(\text{Vec}_C^m(X)\) in \(\text{Vec}_C^m(X)\) under \(j\) is described by (homotopy) equivalence classes of the type \([\langle \phi, \text{Id}_\phi \rangle]\) which are classified, at least in low dimension, by the help of Chern classes. On the opposite side there is another interesting subclass of \(\chi\)-bundles \(\text{Vec}_C^m(X) \subset \text{Vec}_C^m(X)\) which consists of homotopy equivalence classes of the type \([\langle X \times \mathbb{C}^m, \Phi \rangle]\) where the underlying vector bundle is trivial. Since the automorphisms of a trivial product bundle \(X \times \mathbb{C}^m\) (endowed with a Hermitian metric) are described by maps \(\phi : X \rightarrow U(m)\), one has that

\[
\text{Vec}_C^m(X) \approx [X, U(m)].
\]

This is the part of \(\text{Vec}_C^m(X)\) which is “purely chiral” in the sense that \(\chi\)-bundles in \(\text{Vec}_C^m(X)\) have vanishing Chern classes due to the triviality of the underlying vector bundle and hence they are classified only by pure chiral invariants. This subset, which under stabilization can by classified by \(K_4(X)\), describes the systems of class AIII usually considered in the physical literature concerning topological insulators. However, at least in our opinion, this seems to be a strong physical restriction which is not at all necessary. As a matter of fact, elements in \(\text{Vec}_C^m(X)\) describe chiral systems which posses a ground state (or Fermi projection) with trivial Chern topology and this is not at all the general situation in the physics of band operators. On the other side the full set \(\text{Vec}_C^m(X)\) contains also models of chiral systems associated to ground states with a non-trivial Chern charge and for this reason we believe that a deeper understanding of the chiral phase requires the study of the full set \(\text{Vec}_C^m(X)\) rather than of its “Chern-trivial” subset \(\text{Vec}_C^m(X)\). This interpretative aspect will be properly developed and deepened in Section 6.

The classification of chiral vector bundles becomes more complicated when one replaces spheres with a generic base space \(X\). In this situation, by taking a cue from the similar problem for complex vector bundles, one finds advantageous to construct suitable characteristic classes. As usual these classes can be defined as the pullback with respect to the classifying maps in (1.1) of the integral cohomology of the space \(\mathbb{B}_2^m\), the latter being completely computable. In this way one can associate a set of cohomology classes to each \(\chi\)-bundle (cf. Section 5.1) and, at least in low-dimension, these classes suffice for a complete characterization of the topology. More precisely, one can prove the following result.

**Theorem 1.4** (Classification via characteristic classes). Let \(X\) be as in Assumption 1.2 and let \(d \in \mathbb{N}\) be the maximal dimension of its cells.

(i) The Picard group of chiral line bundles is classified by the isomorphism

\[ (w_1, c_1) : \text{Vec}^1(X) \xrightarrow{\sim} H^1(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \]

induced by the first chiral class \( w_1 \) and the first Chern class \( c_1 \).

(ii) In the case of a generic rank \( m \geq 2 \) and for low dimensional base spaces \( 1 \leq d \leq 3 \) there are bijections of sets

\[ (w_1, c_1, w_2) : \text{Vec}^m(X) \xrightarrow{\sim} H^1(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^3(X, \mathbb{Z}) \]

induced by the first two chiral classes \( w_1, w_2 \) and the first Chern class \( c_1 \).

(iii) If \( d = 4 \) and in the stable range \( m \geq 3 \) there are bijections of sets

\[ (w_1, c_1, w_2, c_2) : \text{Vec}^m(X) \xrightarrow{\sim} H^1(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^3(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z}) \]

induced by the first two chiral classes \( w_1, w_2 \) and the first two Chern classes \( c_1, c_2 \).

Since \( H^k(X, \mathbb{Z}) = 0 \) if \( k > d \), equation (1.4) has to be decorated by the obvious relation \( c_1 = w_2 = 0 \) if \( d = 1 \) and \( w_2 = 0 \) if \( d = 2 \). Item (i) is proved in Proposition 5.5 while the proof of items (ii) and (iii) is described in Proposition 5.7. The first interesting unstable case \( d = 4, m = 2 \) is not covered by the above theorem. A general analysis of this case reserves considerable mathematical difficulties as discussed in Section 5.5. With the help of Theorem 1.4 one can for instance classify \( \chi \)-bundles over tori of low dimension. The latter provide the geometric model for the study of the topological phases for systems of fermions which interact with a periodic background (see the discussion in [DG1, Section 2]). In the case of “Brillouin” tori \( X = T^d \) the classification is displayed in Table 1.2.

| VB | AZC | \( d = 2 \) | \( d = 3 \) | \( d = 4 \) |
|----|-----|---------|---------|---------|
| Vec\(^m\)(\( T^d \)) | A | \( \mathbb{Z} \) | \( \mathbb{Z}^3 \) | \( \mathbb{Z}^6 \oplus \mathbb{Z} \) (\( m = 1 \)) \( \mathbb{Z}^6 \oplus \mathbb{Z} \) (\( m \geq 2 \)) |
| Vec\(^m\)(\( T^d \)) | AIII | \( \mathbb{Z}^2 \oplus \mathbb{Z} \) | \( \mathbb{Z}^3 \oplus \mathbb{Z}^3 \) (\( m = 1 \)) \( \mathbb{Z}^4 \oplus \mathbb{Z}^6 \) (\( m \geq 2 \)) |
| Vec\(^m\)(\( T^d \)) | AIII | \( \mathbb{Z}^2 \oplus \mathbb{Z} \) | \( \mathbb{Z}^3 \oplus \mathbb{Z}^3 \oplus \mathbb{Z} \) (\( m \geq 2 \)) |

Table 1.2. Notations and abbreviations are the same already used in the caption of Table 1.1. The case \( d = 1 \) is already covered in Table 1.1 since \( T^1 = S^1 \). In the complex case the classification is specified by the first Chern class \( c_1 \) up to dimension \( d = 3 \) and by the first and second Chern classes \( (c_1, c_2) \in \mathbb{Z}^2 \oplus \mathbb{Z} \) in dimension \( d = 4 \) (cf. [DG1, Section 4]). The extra invariants introduced by the chiral symmetry are listed by the “boxed” entries. The splitting in the direct sum of the groups is order by the sequence of characteristic classes \( (w_1, c_1, w_2, c_2) \) modulo the obvious conditions of triviality given by \( H^k(T^d, \mathbb{Z}) = 0 \) if \( k > d \). The unstable case \( d = 4, m = 2 \), not covered by Theorem 1.4 is discussed in Section 5.6.

Before ending this introduction, let us briefly justify our basic assumption that chiral vector bundles, as introduced in Definition 1.1, provide the proper geometric models for the topology of the ground state of quantum systems protected by a chiral symmetry. This aspect will be dealt with in depth in Section 6 and will be completed with a comparison with the existing literature. Let us start with the following definition which generalizes the usual notion of (translational invariant) topological insulator in class AIII.
Definition 1.5 (Topological quantum systems with chiral symmetry). Let $X$ be a locally compact, path-connected, Hausdorff space. Let $\mathcal{H}$ be a separable complex Hilbert space and denote by $\mathcal{K}(\mathcal{H})$ the algebra of compact operators on $\mathcal{H}$, respectively. A topological quantum system is a self-adjoint map

$$X \ni x \mapsto H(x) = H(x)^* \in \mathcal{K}(\mathcal{H})$$

(1.6)

continuous with respect to the norm-topology of $\mathcal{K}(\mathcal{H})$. Let $\sigma(H(x)) = \{\lambda_j(x) \mid j \in J \subseteq \mathbb{Z} \subset \mathbb{R}\}$ be the sequence of eigenvalues of $H(x)$ ordered according to $\ldots < \lambda_2(x) \leq \lambda_1(x) < 0 \leq \lambda_4(x) \leq \lambda_3(x) \leq \ldots$. The map $x \mapsto \lambda_j(x)$ (which is continuous by standard perturbative arguments [Kat]) is called $j$-th energy band. An isolated family of energy bands is any (finite) collection $\Omega := \{\lambda_{j_1}(\cdot), \ldots, \lambda_{j_m}(\cdot)\}$ of energy bands such that

$$\min_{x \in X} \text{dist} \left( \bigcup_{j=1}^{m} (\lambda_{j_1}), \bigcup_{j \in J \setminus \{j_1, \ldots, j_m\}} (\lambda_j) \right) = C_{\Omega} > 0.$$ 

(1.7)

Inequality (1.7) is usually called gap condition. We say that the systems is subjected to a chiral symmetry if there is a continuous unitary-valued map $x \mapsto \chi(x)$ on $X$ such that

$$\begin{cases}
\chi(x) H(x) \chi(x)^* = -H(x) \\
\chi(x)^2 = 1_{\mathcal{H}}
\end{cases}$$

(1.8)

where $1_{\mathcal{H}}$ is the identity operator.

A standard construction, which is described in some detail in Section 6.2, associates to each topological quantum system (1.6) with an isolated family of $m$ negative energy bands (1.7) a complex vector bundle $\mathcal{E}_- \to X$ of rank $m$ which is usually called Bloch-bundle. Due to the chiral symmetry the system posses also a family of $m$ positive bands which define a “twin” Bloch-bundle $\mathcal{E}_+ \to X$. These two vector bundles are in general non-trivial and the chiral symmetry induces an isomorphism $\Theta_\chi : \mathcal{E}_- \to \mathcal{E}_+$. These data are enough to univocally specify (up to isomorphisms) a Clifford vector bundle over $X$ of type $(0, 1)$ (cf. Section 6). Objects of this type provide a geometric model for the construction of $K_1(X)$, as showed by Karoubi in [Kar, Chapter III, Section 4]. This is the point of view adopted in [FM, Th] for the classification of system in class AIII. However, although on the one hand the K-theory introduced by Karoubi fits perfectly with the need to explain the “Bott-clock”, on the other hand this K-theory does not seems to take into account the topological content associated with the Chern classes of the underlying vector bundle. A more precise discussion on this delicate point is postponed to the end of Section 6.3. The link between topological quantum systems of type AIII, or equivalently Clifford vector bundles, and chiral vector bundles depends on the choice of a reference isomorphism $h_{\text{ref}} : \mathcal{E}_- \to \mathcal{E}_+$. After that such a $h_{\text{ref}}$ has been chosen, one defines a rank $m$ chiral vector bundle $(\mathcal{E}, \Phi)$ just by setting $\mathcal{E} := \mathcal{E}_-$ and $\Phi := h_{\text{ref}}^{-1} \circ \Theta_\chi$. In our opinion, is exactly the topology of $(\mathcal{E}, \Phi)$ which completely describes the topological phase of the original topological quantum system with chiral symmetry. The arbitrariness inherent in the choice of $h_{\text{ref}}$ seems to be an inescapable aspect and explicitly [Th1, Th2] or implicitly [SRFL, RSFL, QZ, BuT, PS] this problem seems to be common to all attempts to define properly the chiral invariants. We refer to Section 6.3 for a more precise discussion.

At the end of this long introduction, let us summarize our point of view about the description of the topology of systems with a chiral symmetry as follows:

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1The setting described in Definition 1.5 can be generalized to unbounded operator-valued maps $x \mapsto H(x)$ by requiring the continuity of the resolvent map $x \mapsto R(x) := (H(x) - zI)^{-1} \in \mathcal{K}(\mathcal{H})$. Another possible generalization is to replace the norm-topology with the open-compact topology as in [FM] Appendix D. However, these kind of generalizations have no particular consequences for the purposes of this work.

2We notice that the second condition in (1.8) can be replaced by the equivalent constraint $\chi'(x)^2 = -1_{\mathcal{H}}$ under the identification $\chi = \chi'$. 

Assumption 1.6 (Topological phase for chiral topological quantum systems). Let $X$ be a topological space which fulfills Assumption 1.2 and $X \ni x \mapsto H(x)$ a topological quantum system with chiral symmetry $X \ni x \mapsto \chi(x)$ as in Definition 1.5. Assume that there exists a ground state described by a system $\Omega_-$ of $m$ strictly negative energy bands (zero energy gap condition, cf. Assumption 6.8) which are separated from the rest of the spectrum in the sense of (1.7). With these data one can build a pair of “twin” rank $m$ Bloch-bundles $\mathcal{E}_\pm \to X$ and a chiral isomorphism $\Theta_{\chi} : \mathcal{E}_- \to \mathcal{E}_+$. Let $h_{\text{ref}} : \mathcal{E}_- \to \mathcal{E}_+$ be a reference isomorphism arbitrarily chosen. We assume that the topological phase of the chiral system, relatively to the reference map $h_{\text{ref}}$, is detected by the equivalence class 

$$[(\mathcal{E}, \Phi)] \in \text{Vec}^m_{\chi}(X)$$

where $\mathcal{E} := \mathcal{E}_-$ and $\Phi := h_{\text{ref}}^{-1} \circ \Theta_{\chi}$.

The virtue of our Assumption 1.6 is that it allows us to answer the questions Q.1) - Q.3) stated at the very beginning of this introduction, from our own point of view.

R.1) The chiral phases are homotopy invariant of the system. Isomorphisms (chiral isometries) are too strong and generic unitary transforms as in [Th2] are too weak to set up a proper topological theory for chiral systems.

R.2) The notion of topological phase of a chiral system is not absolute but depends on an arbitrary choice of a reference isomorphism. Only the notion of relative phase between two different states of the system has an absolute interpretation. This is equivalent to establish a priori the “reference” trivial chiral phase of “reference”.

R.3) In the classification of the topological phase of chiral systems also the topology of the ground state, described by the topology of the underlying Bloch-bundle, plays a role. The K-group $K_1$ is not enough. For instance it does not contain information about the Chern classes.

In our opinion R.2) is worth of a final comment. In the axiomatic theory of characteristic classes the fact that the topological invariants are defined only relatively to a given choice seems to be the rule rather than the exception. For instance this is true also for the familiar Chern classes which can be defined from the Hirzebruch’s system of axioms only up to a normalization given by fixing the Chern class of the tautological line bundle over $\mathbb{C}P^\infty$ [Hu Chapter 17, Section].

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2. Chiral vector bundles

Before starting the formal study of $\chi$-bundles let us recall some basic facts of the theory of complex vector bundles. Given a complex vector bundle $\pi : \mathcal{E} \to X$ one can always endowed $\mathcal{E}$ with a hermitian fiber-metric $m : \mathcal{E} \times_\pi \mathcal{E} \to \mathbb{C}$ which turns out to be unique up to isomorphisms (see e. g. [Kar Chapter I, Theorem 8.7]). Here $\mathcal{E} \times_\pi \mathcal{E}$ denotes the set of pairs $(p_1, p_2) \in \mathcal{E} \times \mathcal{E}$ such that $\pi(p_1) = \pi(p_2)$. This means that without loss of generality we can restrict our attention to hermitian complex vector bundles of rank $m \in \mathbb{N}$. The latter are characterized by their structure group (the group in which the transition functions take value) which is the unitary group $U(m)$. An automorphism of a hermitian rank $m$ complex vector bundle $\pi : \mathcal{E} \to X$ is a vector bundle isomorphism

$$\begin{array}{ccc}
\mathcal{E} & \overset{\Phi}{\longrightarrow} & \mathcal{E} \\
\pi \downarrow & & \downarrow \pi \\
X & & X
\end{array}$$

which is also an isometry with respect to the hermitian fiber-metric, i. e. $m(p_1, p_2) = m(\Phi(p_1), \Phi(p_2))$ for all $p_1, p_2 \in \mathcal{E} \times_\pi \mathcal{E}$. We denote with $\text{Aut}(\mathcal{E})$ the automorphism group of $\mathcal{E}$. Let us remark that automorphisms in $\text{Aut}(\mathcal{E})$ can be locally identified with maps $X \ni \mathcal{U} \to U(m)$ where $\mathcal{U}$ is a trivializing open set for $\mathcal{E}$. 
The construction of a coherent topological theory of chiral vector bundles requires the introduction of a proper notion of isomorphism. Quite naturally one can say that two rank $m$ $\chi$-bundles $(E_1, \Phi_1)$ and $(E_2, \Phi_2)$ over $X$ are isomorphic if there exists a homeomorphism $f : E_1 \to E_2$ which makes the following diagram

\[
\begin{array}{ccc}
E_1 & \xrightarrow{\Phi_1} & E_1 \\
\downarrow{\pi_1} & & \downarrow{\pi_1} \\
X & \xrightarrow{f} & X \\
\downarrow{\pi_2} & & \downarrow{\pi_2} \\
E_2 & \xleftarrow{\Phi_2} & E_2
\end{array}
\]

commutative. We use the notation $(E_1, \Phi_1) \approx (E_2, \Phi_2)$ to say that $(E_1, \Phi_1)$ and $(E_2, \Phi_2)$ are isomorphic. The trivial rank $m$ $\chi$-bundle is, by definition, the pair $(X \times \mathbb{C}^m, \text{Id}_{X \times \mathbb{C}^m})$ where $X \times \mathbb{C}^m \to X$ is the standard product vector bundle (endowed with the first factor projection) and $\text{Id}_{X \times \mathbb{C}^m}$ is the identity map which fixes each point $(x,v) \in X \times \mathbb{C}^m$. A $\chi$-bundle $(E, \Phi)$ is said to be strongly trivial if $(E, \Phi) \approx (X \times \mathbb{C}^m, \text{Id}_{X \times \mathbb{C}^m})$. The notion of isomorphism $\approx$ induces an equivalence relation and one can define in a usual way the set of isomorphism classes

\[
\text{Vec}_m^X := \left\{ \text{rank } m \text{ chiral vector bundles over } X \right\} / \approx.
\]

**Remark 2.1 (Weak local triviality).** Chiral vector bundles are not “genuine” locally trivial objects. Indeed, a notion of local triviality for a $\chi$-bundle $(E, \Phi)$, which is also coherent with the above definitions of isomorphism $\approx$ and strong triviality, should require the existence of a cover $\{U_i\}$ of open subsets of $X$ and subordinate trivializing maps establishing isomorphisms $(E|_{U_i}, \Phi|_{U_i}) \approx (U \times \mathbb{C}^m, \text{Id}_{U \times \mathbb{C}^m})$. However, if from one side the existence of isomorphisms $h_{U_i} : E|_{U_i} \to U \times \mathbb{C}^m$ is guaranteed by the fact that the underlying complex vector bundle $E \to X$ is locally trivial, from the other side is not generally possible to choose the $h_{U_i}$ in such a way that $h_{U_i} \circ \Phi|_{U_i} \circ h_{U_i}^{-1} = \text{Id}_{U \times \mathbb{C}^m}$. As an example the reader can just consider the complex product line bundle $S^1 \times \mathbb{C} \to S^1$ endowed with the endomorphism $\Phi_p(\theta, v) = (\theta, e^{i\theta}v)$ for all $(\theta, v) \in S^1 \times \mathbb{C}$. Since the structure group $U(1)$ is commutative there is no way to change globally, or just locally, the endomorphism $\Phi_p$ in the identity map. Nevertheless, a closer look to the structure of a $\chi$-bundle $(E, \Phi)$ shows that $\Phi$ can be locally identified with maps $\phi_{U_i} : U \to U(m)$ through the assignment $h_{U_i} \circ \Phi|_{U_i} \circ h_{U_i}^{-1} = \text{Id}_{U \times \mathbb{C}^m}$. This observation allows us to affirm that chiral vector bundles are locally trivial in a weak sense, meaning that each $\chi$-bundle admits local isomorphisms of the type $(E|_{U_i}, \Phi|_{U_i}) \approx (U \times \mathbb{C}^m, \text{Id}_{U \times \Phi|_{U_i}})$ which provide a local “twisted” product structure. □

The notion of homotopy is an extremely important ingredient of the theory of complex vector bundle. Just as a reminder let us recall that if $\pi : E \to X$ is a vector bundle and $\varphi : Y \to X$ is a map then we can construct a complex vector bundle $\varphi^*E \to Y$ called the pullback of $E$ over $Y$ via $\varphi$. The fibers of $\varphi^*E$ are explicitly described by [Hu, Chapter 2, Section 5]

\[
\varphi^*E|_y := \left\{ (y, p) \in Y \times E \mid \varphi(y) = \pi(p) \right\} \approx E|_{\varphi(y)}, \quad \forall \ y \in Y.
\]

When $\varphi_1, \varphi_2 : Y \to X$ are homotopic maps (and $Y$ is paracompact) the pullbacks $\varphi_1^*E$ and $\varphi_2^*E$ turn out to be isomorphic in the category of complex vector bundles over $Y$ [Hu, Chapter 3, Theorem 4.7]. One of the major consequences of this result is that one can reinterpret the notion of isomorphism between complex vector bundles in terms of homotopy equivalences. More precisely one has that $E_1 \to X$ and $E_2 \to X$ are isomorphic as complex vector bundles if and only if there is a vector bundle $E \to X \times [0, 1]$ such that $E|_{X \times \{1\}}$ (resp. $E|_{X \times \{2\}}$) is isomorphic to $E_1$ (resp. $E_2$). From one side the isomorphism between $E|_{X \times \{1\}}$ and $E|_{X \times \{2\}}$ [Hu, Chapter 3, Corollary 4.4] implies by transitivity the isomorphism between $E_1$ and $E_2$. On the other side if $E_1$ and $E_2$ are isomorphic it is enough to consider $E := E_1 \times [0, 1]$ as a vector bundle over $X \times [0, 1]$ endowed with the natural projection.
Remark 2.2. There is a different homotopy characterization of the notion of isomorphism: Two complex vector bundles \( E_1 \to X \) and \( E_2 \to X \) turn out to be isomorphic (in the proper category) if and only if there exists a map \( \varphi : X \to X \) such that \( \varphi \) is homotopic to \( \text{Id}_X \) and \( \varphi^*E_1 \) is isomorphic to \( E_2 \). This claim is essentially a consequence of the fact \( \text{Id}_X^*E \cong E \) [Hu, Chapter 2, Proposition 5.7].

Unfortunately, the situation is less simple in the context of chiral vector bundles. Of course, the notion of pullback is still well defined. Indeed if \( (E, \Phi) \) is a \( \chi \)-bundle over \( X \) and \( \varphi : Y \to X \) is a map we can define the pullback \( \varphi^*E \to Y \) described fiberwise by \( \text{Remark 2.1} \). By \( \varphi \)-line bundles considered in Remark 2.1. Given a map \( f : \varphi^*E \to \varphi^*F \) which restricts to a linear isomorphism \( f_y \) on each pair of fibers, i.e.

\[
E|_{\varphi_1(y)} \cong \varphi_1^*E|_y \cong \varphi_2^*E|_y \cong E|_{\varphi_2(y)}, \quad \forall y \in Y.
\]

Since \( \Phi \) is an automorphism of \( E \) the map \( \varphi^*\Phi \) is an isomorphism on each fiber \( \varphi^*E|_y \) and this is enough to affirm that \( \varphi^*\Phi \in \text{Aut}(\varphi^*E) \) [Hu, Chapter 3, Theorem 2.5]. Let us now consider two homotopic maps \( \varphi_1, \varphi_2 : Y \to X \) (\( Y \) paracompact) and the related pullback \( \chi \)-bundles \( \varphi_1^*(E, \Phi) \) and \( \varphi_2^*(E, \Phi) \) over \( Y \). Since \( \varphi_1^*E \) and \( \varphi_2^*E \) are isomorphic as complex vector bundles there exists a map \( f : \varphi_1^*E \to \varphi_2^*E \) which restricts to a linear isomorphism \( f_y \) on each pair of fibers, i.e.

\[
f_y : \varphi_1^*E|_y \to \varphi_2^*E|_y, \quad \forall y \in Y.
\]

Let \((y, p) \in \varphi_1^*E|_y \) and \( f(y, p) = (y, f_y(p)) \) the corresponding point in \( \varphi_2^*E|_y \) (with an abuse of notation, we are identifying \( f_y \) with the linear isomorphism \( f_y : \varphi_1^*E|_y \to \varphi_2^*E|_y \)). A straightforward calculation shows that

\[
f \circ \varphi_1^*\Phi : (y, p) \mapsto (y, f_y(\Phi(p))), \quad \varphi_1^*\Phi \circ f : (y, p) = (y, \Phi(f_y(p)))
\]

hence the condition \( f \circ \varphi_1^*\Phi = \varphi_2^*\Phi \circ f \) for a \( \chi \)-bundle isomorphism between \( \varphi_1^*(E, \Phi) \) and \( \varphi_2^*(E, \Phi) \) in the sense of \( \text{Remark 2.1} \) necessitates that validity of

\[
f_y \circ \Phi|_{E|_{\varphi_1(y)}} = \Phi|_{E|_{\varphi_2(y)}} \circ f_y, \quad \forall y \in Y.
\]

The set of equations \( \text{(2.6)} \) represents the obstruction for a homotopy between \( \chi \)-bundles to be an isomorphism.

Example 2.3. Let \( (S^1 \times \mathbb{C}, \Phi_n) \) be the \( \chi \)-line bundle considered in Remark 2.1. Given a map \( \varphi : S^1 \to S^1 \) one has that \( \varphi^*(S^1 \times \mathbb{C}) \) is isomorphic to \( S^1 \times \mathbb{C} \) as complex vector bundles. However, the pullback automorphism \( \varphi^*\Phi_n \) acts on each point \((\theta, v) \in S^1 \times \mathbb{C}\) as \( \varphi^*\Phi_n(\theta, v) = (\theta, e^{in\varphi(\theta)}v) \). Hence, if \( \varphi_1, \varphi_2 : S^1 \to S^1 \) are two distinct maps one has that \( \varphi_1^*\Phi_n \) and \( \varphi_2^*\Phi_n \) cannot be related by the equation \( \text{(2.6)} \), even in the case one assumes \( \varphi_1 \) and \( \varphi_2 \) homotopic. This is just because the underlying structure group \( U(1) \) is commutative. On the other side seems natural to consider that all the relevant topology of the \( \chi \)-line bundle \((S^1 \times \mathbb{C}, \Phi_n)\) has to be contained in the map \( \theta \mapsto e^{in\theta} \), which is topologically characterized by its winding number \( n \). In the case the map \( \varphi \) is homotopic to the identity map \( \text{Id}_{S^1} \) one has that also \( \theta \mapsto e^{in\varphi(\theta)} \) is completely specified by the winding number \( n \). Then, the two \( \chi \)-line bundle \((S^1 \times \mathbb{C}, \Phi_n)\) and \( \varphi^*(S^1 \times \mathbb{C}, \Phi_n) \) are to be considered “topologically equivalent” even though they are not isomorphic in the strong sense of diagram \( \text{(2.2)} \).

Example 2.3 suggests that it is necessary to relax the notion of equivalence of \( \chi \)-bundles in \( \text{(2.3)} \) in order to have a classification theory which is preserved under homotopy equivalences.

Definition 2.4 (Equivalence of chiral vector bundles). Let \((E_1, \Phi_1)\) and \((E_2, \Phi_2)\) be two rank \( m \) chiral vector bundles over the same base space \( X \). We say that \((E_1, \Phi_1)\) and \((E_2, \Phi_2)\) are equivalent, and we write \((E_1, \Phi_1) \sim (E_2, \Phi_2)\), if and only if there is a rank \( m \) chiral vector bundle \((E, \Phi)\) over \( X \times [0, 1] \) such that

\[
(E_1, \Phi_1) \approx (E|_{X \times [0]} , \Phi|_{X \times [0]}) \quad \text{and} \quad (E_2, \Phi_2) \approx (E|_{X \times [1]} , \Phi|_{X \times [1]})
\]

where \( \approx \) stands for the isomorphism relation between \( \chi \)-bundles established by the diagram \( \text{(2.2)} \).
From Definition 2.4 one immediately deduces that ~ is an equivalence relation on the set of chiral vector bundles. This differs from the isomorphism relation ≈ and, in particular, it turns out to be weaker. Indeed, just by choosing \( \mathcal{E} = \mathcal{E}_1 \times [0, 1] \) and \( \Phi = \Phi_1 \times \text{Id}_{[0, 1]} \), one can verify that \((\mathcal{E}_1, \Phi_1) \approx (\mathcal{E}_2, \Phi_2)\) implies \((\mathcal{E}_1, \Phi_1) \sim (\mathcal{E}_2, \Phi_2)\). The set of equivalence classes of rank \( m \) \( \chi \)-bundles with respect to \( \sim \) will be denoted by

\[
\text{Vec}_m^\chi(X) := \{ \text{rank } m \text{ chiral vector bundles over } X \}/ \sim.
\] (2.7)

We will show in the rest of this work that \( \text{Vec}_m^\chi(X) \) admits a homotopic classification which can be characterized in term of proper characteristic classes, at least in low dimension.

**Example 2.5.** Let us consider two \( \chi \)-line bundles \((S^1 \times \mathbb{C}, \Phi_i), i = 1, 2\), defined by \( \Phi_i(\theta, v) = (\theta, e^{i\phi_i(\theta)v}) \) with \( \phi_1, \phi_2 : S^1 \to U(1) \) two continuous maps. As showed in Example 2.3 the equality \( \phi_1 = \phi_2 \) is condition necessary and sufficient for the isomorphism (in the strong sense of diagram (2.2)) between the two \( \chi \)-line bundles. On the other side the equivalence relation of Definition 2.4 is less restrictive and requires only the homotopy equivalence between the maps \( \phi_1 \) and \( \phi_2 \). Indeed, let us assume the existence of a homotopy \( \hat{\phi} : S^1 \times [0, 1] \to U(1) \) such that \( \hat{\phi}(\theta, 0) = \phi_1(\theta) \) and \( \hat{\phi}(\theta, 1) = \phi_2(\theta) \) for all \( \theta \in S^1 \). Then one can check that \((S^1 \times \mathbb{C}, \Phi_1) \sim (S^1 \times \mathbb{C}, \Phi_2)\). On the other hand, let us consider two \( \chi \)-line bundles \((S^1 \times \mathbb{C}, \Phi_i), i = 1, 2\), and \( \mathcal{E} \) a chiral bundle. On the one hand, if \( \hat{\phi} \) is a homotopy between \( \phi_1 \) and \( \phi_2 \), then \( \Phi_i = \hat{\phi} \circ \phi_i \). On the other hand, if \( \hat{\phi} \) is a homotopy between \( \phi_1 \) and \( \phi_2 \), then \( \Phi_i = \hat{\phi} \circ \phi_i \). Summarizing, one has that

\[
\overline{\text{Vec}_1^\chi(S^1)} \cong \text{Map}(S^1, U(1)), \quad \text{Vec}_1^\chi(S^1) \cong \pi_1(S^1) \cong \mathbb{Z}, \quad \text{Vec}_1^\chi(S^1) \cong \pi_1(CP^\infty) \cong 0
\]

and this clearly indicates that the equivalence relation \( \sim \) introduced in Definition 2.4 provides a proper structure for a topological classification of chiral vector bundles. Conversely, the notion of strong isomorphism given by (2.2) results too strong (it distinguishes too much) while the notion of isomorphism in the complex category is too weak (it does not distinguish anything).

The following result extends the homotopy property of complex vector bundle [Hu, Chapter 3, Theorem 4.7] to the class of chiral vector bundles, the right notion of equivalence being the one introduced in Definition 2.4.

**Proposition 2.6 (Homotopy property).** Let \((\mathcal{E}, \Phi)\) be a rank \( m \) chiral vector bundle over the base space \( X \) and \( \varphi_1, \varphi_2 : Y \to X \) two homotopic maps \((Y \text{ is assumed to be paracompact})\). Then \( \varphi_1(\mathcal{E}_1, \Phi_1) \sim \varphi_2(\mathcal{E}_2, \Phi_2) \) as chiral vector bundles over \( Y \).

**Proof.** Let \( \hat{\varphi} : Y \times [0, 1] \to X \) be the homotopy between \( \varphi_1 \) and \( \varphi_2 \) such that \( \hat{\varphi}(y, 0) = \varphi_1(y) \) and \( \hat{\varphi}(y, 1) = \varphi_2(y) \) for all \( y \in Y \). Consider the vector bundle \( \pi'_1 : \mathcal{E}'_1 \to Y \times [0, 1] \) with fibers \( \pi'_1^{-1}(y, t) = \hat{\varphi}_y^* \mathcal{E} \) where we used the short notation \( \hat{\varphi}_y^* \mathcal{E} \) for all \( y \in Y \). Each point in \( \mathcal{E}'_1 \) is identified by a triple \((y, t, p)\) such that \( \pi'_1(y, t, p) = (y, t) \) and \( p \in \mathcal{E}|_{\hat{\varphi}_y(0)} \). We can endow \( \mathcal{E}'_1 \) with a chiral structure \( \Phi' \) as follows: \( \Phi'(y, t, p) = (y, t, \Phi(p)) \). This is enough to show that \((\mathcal{E}'_1|_{Y \times [0, 1]}, \Phi|_{Y \times [0, 1]}) \sim \varphi_1(\mathcal{E}, \Phi) \) and \((\mathcal{E}'_1|_{Y \times [0, 1]}, \Phi|_{Y \times [0, 1]}) \sim \varphi_2(\mathcal{E}, \Phi) \).

3. Homotopy classification for chiral vector bundles

The celebrated Brown’s representability theorem [Bri, Hat, Theorem 4E.1] may ensure abstractly (upon a verification of few structural axioms) the existence of a classifying space \( B^m_X \) for chiral vector bundles of rank \( m \) in the sense that

\[
\text{Vec}_m^\chi(X) \cong [X, B^m_X]
\] (3.1)

where on the left-hand side there is the set of equivalence classes of \( \chi \)-bundles in the sense of (2.7) while on the right-hand side there is the set of homotopy classes of maps of the base space \( X \) into \( B^m_X \). The aim of this section is to provide a concrete geometric model for \( B^m_X \).
3.1. Geometric model for the classifying space. For each pair of integers \(1 \leq m \leq n\) we introduce the Grassmann manifold

\[ G_m(\mathbb{C}^n) \cong \mathbb{U}(n)/(\mathbb{U}(m) \times \mathbb{U}(n-m)) \]

of \(m\)-dimensional (complex) subspaces of \(\mathbb{C}^n\). Here \(\mathbb{U}(n)\) indicates the unitary group acting on \(\mathbb{C}^n\). Each \(G_m(\mathbb{C}^n)\) can be endowed with the structure of a finite CW-complex, making it into a closed (i.e., compact without boundary) manifold of (real) dimension \(2m(n-m)\). The inclusions \(\mathbb{C}^n \subset \mathbb{C}^{n+1} \subset \ldots\) given by \(v \mapsto (v, 0)\) yield inclusions \(G_m(\mathbb{C}^n) \subset G_m(\mathbb{C}^{n+1}) \subset \ldots\) and one can equip

\[ G_m(\mathbb{C}^\infty) := \bigcup_{n=m}^{\infty} G_m(\mathbb{C}^n), \]

with the direct limit topology. The resulting space has the structure of an infinite CW-complex which is, in particular, paracompact and path-connected. The inclusions \(\mathbb{C}^n \subset \mathbb{C}^{n+1} \subset \ldots\) also yield inclusions \(\mathbb{U}(n) \subset \mathbb{U}(n+1) \subset \ldots\) and the space

\[ \mathbb{U}(\infty) := \bigcup_{n=1}^{\infty} \mathbb{U}(n), \]

endowed with the direct limit topology, is sometimes called Palais unitary group \([Pa]\). Elements in this \(\mathbb{U}(\infty)\) can be interpreted as infinite unitary matrices whose entries differ from the identity matrix in only finitely many places. A classical result by R. Bott states that \(\pi_k(\mathbb{U}(\infty)) = 0\) if \(k\) is even and \(\pi_k(\mathbb{U}(\infty)) = \mathbb{Z}\) if \(k\) is odd (cf. Appendix \([A]\)).

Let us introduce the following family of sets parametrized by \(1 \leq m \leq n\):

\[ \chi_m(\mathbb{C}^n) := \left\{ (\Sigma, u) \in G_m(\mathbb{C}^n) \times \mathbb{U}(n) \mid u(\Sigma) = \Sigma \right\}. \tag{3.2} \]

Each element in \(\chi_m(\mathbb{C}^n)\) is a pair \((\Sigma, u)\) given by a subspace \(\Sigma \subset \mathbb{C}^n\) of complex dimension \(m\) and a unitary matrix \(u \in \mathbb{U}(\mathbb{C}^n)\) which preserves \(\Sigma\). The inclusions \(\mathbb{C}^n \subset \mathbb{C}^{n+1} \subset \ldots\) also yield the obvious inclusions \(\chi_m(\mathbb{C}^n) \subset \chi_m(\mathbb{C}^{n+1}) \subset \ldots\) and this suggests the following

**Definition 3.1** (Classifying space for \(\chi\)-bundles). For each pair of integers \(1 \leq m \leq n\) let \(\chi_m(\mathbb{C}^n)\) be the space given by (3.2). The space

\[ \mathbb{B}_\chi^m := \bigcup_{n=m}^{\infty} \chi_m(\mathbb{C}^n), \]

equipped with the direct limit topology, will be called the classifying space for \(\chi\)-bundles.

In order to justify this name we need to prove that this space \(\mathbb{B}_\chi^m\) really provides a model for the realization of the isomorphism (3.1).

3.2. The chiral universal vector bundle. Each manifold \(G_m(\mathbb{C}^n)\) is the base space of a canonical rank \(m\) complex vector bundle \(\mathcal{T}_m^\infty \rightarrow G_m(\mathbb{C}^n)\) with total space \(\mathcal{T}_m^\infty\) given by all pairs \((\Sigma, v)\) with \(\Sigma \in G_m(\mathbb{C}^n)\) and \(v\) a vector in \(\Sigma\). The bundle projection is defined by the natural restriction \((\Sigma, v) \mapsto \Sigma\). Now, when \(n\) tends to infinity, the same construction leads to the tautological \(m\)-plane bundle \(\mathcal{T}_m^\infty \rightarrow G_m(\mathbb{C}^\infty)\). This vector bundle is the universal object which classifies complex vector bundles in the sense that any rank \(m\) complex vector bundle \(\mathcal{E} \rightarrow X\) can be realized, up to isomorphisms, as the pullback of \(\mathcal{T}_m^\infty\) with respect to a classifying map \(\varphi : X \rightarrow G_m(\mathbb{C}^\infty)\), that is \(\mathcal{E} \cong \varphi^* \mathcal{T}_m^\infty\). Since pullbacks of homotopic maps yield isomorphic complex vector bundles (homotopy property), the classification of \(\mathcal{E}\) only depends on the homotopy class of \(\varphi\). This leads to the fundamental result

\[ \text{Vec}_C^m(X) \cong \left[ X, G_m(\mathbb{C}^\infty) \right] \tag{3.3} \]

where in the right-hand side there is the set of the equivalence classes of homotopic maps of \(X\) into \(G_m(\mathbb{C}^\infty)\) \([Al]\) \([MS]\) \([Hu]\).
Based on the definition (3.2) one can consider the composition
\[ \chi_m(C^n) \xrightarrow{i} G_m(C^n) \times U(n) \xrightarrow{pr_1} G_m(C^n) \]
where \( i \) is the inclusion given by \( \chi_m(C^n) \subset G_m(C^n) \times U(n) \) and \( pr_1 \) is the first factor projection. The pullback \( \chi_m(C^n) := (pr_1 \circ i)^* \mathcal{F}_m^n \) defines a rank \( m \) vector bundle over \( \chi_m(C^n) \) that can be endowed with a natural automorphism which can be easily constructed after a close inspection to the structure of the total space
\[
\chi_m(C^n) := \left\{ ((\Sigma, u), (\Sigma, v)) \in \chi_m(C^n) \times \mathcal{F}_m^n \mid u(\Sigma) = v, \Sigma \right\}
\]
of the vector bundle \( \pi : \chi_m(C^n) \to \chi_m(C^n) \). The second representation provides two important information: first of all the bundle projection \( \pi \) can be equivalently described as \( \pi(\Sigma, u, v) = (\Sigma, u) \); second the vector bundle \( \chi_m(C^n) \) is identiafiable with a vector subbundle of the product vector bundle \( (G_m(C^n) \times U(n)) \times C^m \). On this product bundle, the usual action of \( U(n) \) on \( C^m \) defines the automorphism
\[
\Phi_m^n : (\Sigma, u, v) \mapsto (\Sigma, u, u \cdot v), \quad \forall (\Sigma, u, v) \in (G_m(C^n) \times U(n)) \times C^m.
\]
By construction this automorphism preserves the fibers of \( \chi_m(C^n) \). Hence, with a slight abuse of notation, equation (3.6) defines an element \( \Phi_m^n \in \text{Aut}(\chi_m(C^n)) \).

Under the inclusions \( C^n \subset C^{n+1} \subset \ldots \) given by \( v \mapsto (v, 0) \) one has related inclusions \( \chi_m(C^n) \subset \chi_m(C^{n+1}) \subset \ldots \) which are compatible with the bundle projections and the automorphisms \( \Phi_m^n \). The natural generalization of the idea of a universal vector bundle for \( \chi \)-bundles is described in the following:

**Definition 3.2 (Universal \( \chi \)-bundle).** For each pair of integers \( 1 \leq m \leq n \) let \( \pi : \chi_m(C^n) \to \chi_m(C^n) \) be the rank \( m \) complex vector bundle (3.3) endowed with the automorphism \( \Phi_m^n \in \text{Aut}(\chi_m(C^n)) \) defined by (3.6). The space
\[
\chi_m(C^n) := \bigcup_{n=m}^{\infty} \chi_m(C^n),
\]
equipped with the direct limit topology, provides the total space for a rank \( m \) complex vector bundle \( \pi : \chi_m(C^n) \to \mathbb{B}_m^n \) over the classifying space introduced in Definition 3.1. Moreover, the map \( \Phi_m^n \) defined by \( \Phi_m^n | \chi_m(C^{n+1}) = \Phi_m^n \) provides an automorphism \( \Phi_m^n \in \text{Aut}(\chi_m(C^n)) \). The pair \( (\chi_m(C^n), \Phi_m^n) \) has the structure of a rank \( m \) chiral vector bundle and it will be called the universal \( \chi \)-bundle.

Notice that \( \mathbb{B}_m^n \) is a subspace of \( G_m(C^{\infty}) \times U(\infty) \) and \( \chi_m(C^n) \) can be identified as the pullback of the universal bundle \( \mathcal{F}_m^n \) for complex vector bundles under the mappings
\[
\mathbb{B}_m^n \xrightarrow{i} G_m(C^n) \times U(n) \xrightarrow{pr_1} G_m(C^n).
\]

### 3.3. The homotopy classification.

The main aim of this section is the proof of the following result:

**Theorem 3.3.** Let \( X \) be a compact Hausdorff space. Every rank \( m \) chiral vector bundle \( (\mathcal{E}, \Phi) \) over \( X \) admits a map \( \varphi : X \to \mathbb{B}_m^n \) such that \( (\mathcal{E}, \Phi) \) is isomorphic (in the sense of the relation \( \approx \) given in diagram (2.2)) to the pullback \( \varphi(\chi_m(C^n), \Phi_m^n) := (\varphi^* \chi_m(C^n), \varphi^* \Phi_m^n) \) of the universal \( \chi \)-bundle.

Before proving this result, we first present the main consequence of Theorem 3.3.

**Corollary 3.4 (Homotopy classification).** Let \( X \) be a compact Hausdorff space. Then there is a natural bijection
\[
\text{Vec}_X^m(X) \cong \left[ X, \mathbb{B}_m^n \right]
\]
where on the left-hand side there is the set of equivalence classes of \( \chi \)-bundles in the sense of (2.7) while on the right-hand side there is the set of homotopy classes of maps of the base space \( X \) into the classifying space \( \mathbb{B}_m^n \) described in Definition 3.1.
Proof. Proposition 2.6 says that the map $\kappa : [X, \mathbb{B}^n_{\chi}] \to \text{Vec}_{\chi}^n(X)$ which associates to each homotopy class $[\varphi]$ the equivalence class $[\varphi^*(\mathcal{F}_m^\chi, \Phi^\chi_m)]$ of $\chi$-bundles is well-defined and Theorem 3.3 implies that $\kappa$ is surjective. The injectivity of $\kappa$ is a consequence of Definition 2.4 for the equivalence of $\chi$-bundles. Let $(\mathcal{E}_i, \Phi_i) := \varphi_i^*(\mathcal{F}_m^\chi, \Phi^\chi_m)$, $i = 1, 2$ be two $\chi$-bundles obtained by pullbacking with respect to the maps $\varphi_i : X \to \mathbb{B}^n_{\chi}$ and assume that $(\mathcal{E}_1, \Phi_1) \sim (\mathcal{E}_2, \Phi_2)$. This implies the existence of a $\chi$-bundle $(\mathcal{E}, \Phi)$ over $X \times [0, 1]$ such that $(\mathcal{E}_1, \Phi_1) \sim (\mathcal{E}|_{X \times 0}, \Phi|_{X \times 0})$ and $(\mathcal{E}_2, \Phi_2) \sim (\mathcal{E}|_{X \times 1}, \Phi|_{X \times 1})$. Theorem 3.3 assures that $(\mathcal{E}, \Phi)$ can be identified, up to an isomorphism, with $\hat{\varphi}^*(\mathcal{F}_m^\chi, \Phi^\chi_m)$ for some map $\hat{\varphi} : X \times [0, 1] \to \mathbb{B}^n_{\chi}$. The isomorphisms $\varphi^*(\mathcal{F}_m^\chi, \Phi^\chi_m) \approx \hat{\varphi}^*(\mathcal{F}_m^\chi, \Phi^\chi_m)$ and $\varphi^*(\mathcal{F}_m^\chi, \Phi^\chi_m) \approx \hat{\varphi}^*(\mathcal{F}_m^\chi, \Phi^\chi_m)$ and the definition of pullback imply that $\varphi_1 = \hat{\varphi}_0 := \hat{\varphi}(-, 0)$ and $\varphi_2 = \hat{\varphi}_1 := \hat{\varphi}(-, 1)$, namely $\hat{\varphi}$ is an isomorphism between $\varphi_1$ and $\varphi_2$. Hence, $\varphi_1$ and $\varphi_2$ define the same class in $[X, \mathbb{B}^n_{\chi}]$ and $\kappa$ turns out to be injective.

Proof of Theorem 3.3. Let $X$ compact and Hausdorff and $(\mathcal{E}, \Phi)$ a $\chi$-bundle over $X$. Then, there is a positive integer $n$ and a map $\psi : X \to G_m(\mathbb{C}^n)$ such that $\mathcal{E}'' := \psi^* \mathcal{F}^n_m$ is isomorphic to the vector bundle $\mathcal{E}$ (see e.g. [Hu, Chapter 3, Proposition 5.8]). Let $f : \mathcal{E} \to \mathcal{E}''$ be such an isomorphism and consider the automorphism $\Phi' \in \text{Aut}(\mathcal{E}'')$ defined by $\Phi' := f \circ \Phi \circ f^{-1}$. By construction

$$\mathcal{E}''|_x := \{(x, (\Sigma, v)) \in X \times \mathcal{F}^n_m \mid \psi(x) = \Sigma, \ v \in \Sigma\} \cong \Sigma.$$ This shows that $\mathcal{E}''$ is a vector subbundle of the product bundle $X \times \mathbb{C}^n$ and one has the direct sum decomposition $\mathcal{E}' \oplus \mathcal{E}'' = X \times \mathbb{C}^n$ for some complement vector bundle $\mathcal{E}'' \to X$. Let $\text{Id}_{\mathcal{E}''}$ be the identity map on $\mathcal{E}''$ and consider the automorphism of $X \times \mathbb{C}^n$ induced by $\Phi' \oplus \text{Id}_{\mathcal{E}''}$. This implies the existence of a map $g : X \to U(n)$ such that

$$\Phi' \oplus \text{Id}_{\mathcal{E}''} : (x, v) \mapsto (x, g(x) \cdot v), \quad \forall (x, v) \in X \times \mathbb{C}^n.$$ The image of the map $(\psi, g) : X \to G_m(\mathbb{C}^n) \times U(n)$ is contained in $\chi_m(\mathbb{C}^n)$ since $\psi(x) = \Sigma \cong \mathcal{E}''|_x$ and $g(x)(\mathcal{E}''|_x) = \mathcal{E}'|_x$ for all $x \in X$. Hence, the following map is well-defined:

$$\varphi : X \to \chi_m(\mathbb{C}^n)$$

$$x \mapsto (\psi(x), g(x)).$$ By construction

$$\varphi^* \mathcal{F}^n_m|_x := \{(x, (\Sigma, u, v)) \in X \times \mathcal{F}^n_m \mid \psi(x) = \Sigma, \ u \in \Sigma, \ v \in \Sigma\} \cong \Sigma \quad (3.8)$$

which proves the isomorphism $\varphi^* \mathcal{F}^n_m \cong \mathcal{E}'$ of complex vector bundles. Moreover,

$$\varphi^* \Phi^m_m : (x, (\Sigma, u, v)) \mapsto (x, \Phi^m_m(\Sigma, u, v)) = (x, (\Sigma, u, u \cdot v))$$

by definition, hence $\varphi^* \Phi^m_m$ agrees with $\Phi'$ under the isomorphism $(3.8)$. Therefore, $\varphi^* (\mathcal{F}^n_m, \Phi^m_m) \cong (\mathcal{E}, \Phi)$. To conclude the proof it is enough to recall that $\mathbb{B}^n_{\chi}$ is defined as the direct limit of $\chi_m(\mathbb{C}^n)$.

Remark 3.5. The condition of compactness in the statement of Theorem 3.3 and consequently also in Corollary 3.4 can be relaxed by requiring only paracompactness. In this case the proof follows along the same lines apart for the fact that $n$ has to be replaced by $\infty$ as suggested in the analogous argument of [Hu, Chapter 3, Theorem 5.5]. However, we will not need this kind of generalization in the following.

4. Topology of the classifying space

In this section we investigate the topology of the classifying space $\mathbb{B}^n_{\chi}$ by computing its homotopy and cohomology. These results are strongly based on the fact that $\mathbb{B}^n_{\chi}$ can be identified with the total space of a fibration

$$U(m) \to \mathbb{B}^n_{\chi} \xrightarrow{\pi} G_m(\mathbb{C}^\infty) \quad (4.1)$$

where the fiber projection $\pi = \text{pr}_1 \circ i$ is defined in (3.7). The proof of this technical but important fact is postponed in Appendix B.
4.1. **Homotopy of the classifying space.** We start with the case \( m = 1 \). In this situation we can use the identification
\[
B^1_\chi \simeq C P^{\infty} \times U(1),
\]
proved in Corollary [B.3] in order to compute the complete set of homotopy groups of \( B^1_\chi \).

**Proposition 4.1.**
\[
\pi_k(B^1_\chi) \simeq \begin{cases} Z & k = 1, 2 \\ 0 & k \in \mathbb{N} \cup \{0\}, \; k \neq 1, 2 \end{cases}.
\]

**Proof.** The above identification leads to
\[
\pi_k(B^1_\chi) \simeq \pi_k(C P^{\infty}) \oplus \pi_k(U(1)).
\]
The proof is completed by \( \pi_k(U(1)) \simeq \delta_{k,1}Z \) and \( \pi_k(C P^{\infty}) \simeq \delta_{k,2}Z \) [Hat, Chapter 4, Example 4.50]. \( \blacksquare \)

The general case \( m > 1 \) can be studied by considering the fiber sequence (4.1) which induces a long exact sequence of homotopy groups
\[
\ldots \pi_k(U(m)) \rightarrow \pi_k(B^m_\chi) \xrightarrow{\pi_*} \pi_k(G_m(C^{\infty})) \rightarrow \pi_{k-1}(U(m)) \rightarrow \ldots
\]

**Theorem 4.2.** For all \( m \in \mathbb{N} \) there are isomorphisms of groups
\[
\pi_k(B^m_\chi) \simeq \begin{cases} \pi_k(U(m)) \oplus \pi_{k-1}(U(m)) & k \in \mathbb{N} \\ 0 & k = 0 \end{cases}.
\]

**Proof.** The proof of the fiber sequence (4.1) is based on the identification of \( B^m_\chi \) with the total space \( Ad(\mathcal{S}_m^{\infty}) \) of an adjoint bundle (cf. Proposition [B.1]). Sections of the adjoint bundle \( \pi : Ad(\mathcal{S}_m^{\infty}) \rightarrow G_m(C^{\infty}) \) are in one to one correspondence with automorphisms of the principal \( U(m) \)-bundle \( \mathcal{S}_m^{\infty} \rightarrow G_m(C^{\infty}) \) [Hu, Chapter 7, Section 1]. Therefore, the identity automorphism Id_{\mathcal{S}_m^{\infty}} identifies a section \( s : G_m(C^{\infty}) \rightarrow Ad(\mathcal{S}_m^{\infty}) \simeq B^m_\chi \). The existence of this section induces a homomorphism \( s_* : \pi_k(G_m(C^{\infty})) \rightarrow \pi_k(B^m_\chi) \) such that \( \pi_* \circ s_* = Id_{\pi_k(G_m(C^{\infty}))} \). It follows that \( \pi_* \) is surjective so that the long exact sequence (4.3) splits into several short exact sequences
\[
0 \rightarrow \pi_k(U(m)) \rightarrow \pi_k(B^m_\chi) \xrightarrow{\pi_*} \pi_k(G_m(C^{\infty})) \rightarrow 0
\]
which are still splitting. Hence, \( \pi_k(B^m_\chi) \simeq \pi_k(U(m)) \oplus \pi_k(G_m(C^{\infty})) \) and the result follows by a comparison with (A.5). Finally \( \pi_0(B^m_\chi) = 0 \) is equivalent to the connectedness of \( B^m_\chi \). \( \blacksquare \)

A comparison with the values in Table A.1 provides an explicit determination of the low dimensional homotopy groups of \( B^m_\chi \):
\[
\pi_1(B^m_\chi) \simeq \pi_2(B^m_\chi) \simeq Z, \quad \forall \; m \in \mathbb{N}
\]
and
\[
\pi_3(B^m_\chi) \simeq \begin{cases} 0 & m = 1 \\ Z & m \geq 2 \end{cases}, \quad \pi_4(B^m_\chi) \simeq \begin{cases} 0 & m = 1 \\ Z_2 \oplus Z & m = 2 \\ Z & m \geq 3 \end{cases}.
\]

The following result will be relevant in the sequel.

**Proposition 4.3.** The action of \( \pi_1(B^m_\chi) \) on \( \pi_k(B^m_\chi) \) is trivial for all \( k \in \mathbb{N} \). In particular one has the isomorphisms
\[
\pi_k(B^m_\chi) \simeq [S^k, B^m_\chi], \quad k \in \mathbb{N}
\]
which allow to neglect the role of base points in the computation of the homotopy groups.
Proof. The split exact sequence (4.4) suggests that the action of $\pi_1(B^m) = \pi_1(U(m)) \oplus 0$ on $\pi_k(B^m) \simeq \pi_k(U(m)) \oplus \pi_k(G_m(\mathbb{C}^\infty))$ reduces just to the action of $\pi_1(U(m))$ on $\pi_k(U(m))$. The latter is trivial due to Lemma [A]. The last part of the claim is a consequence of the general isomorphism

$$[S^k, X] \simeq \pi_k(X)/\pi_1(X), \quad k \in \mathbb{N}$$

which is valid for any path-connected space $X$ [BT Proposition 17.6.1].

### 4.2. Cohomology of the classifying space

By construction $B^m$ is a subset of $G_m(\mathbb{C}^\infty) \times U(\infty)$ with an inclusion map $i$ as in (5.7). We also know from (4.1) that $B^m$ is also identifiable with the total space of a fiber bundle over $G_m(\mathbb{C}^\infty)$ with bundle map $\pi$ and fiber $U(m)$. Moreover the inclusion $i$ and the projection $\pi$ are compatible as shown by the following commutative diagram

$$\begin{array}{ccc}
B^m & \xrightarrow{i} & G_m(\mathbb{C}^\infty) \times U(\infty) \\
\downarrow{\pi} & & \downarrow{pr_2} \\
G_m(\mathbb{C}^\infty) & & U(\infty)
\end{array}$$

(4.5)

where $pr_1$ and $pr_2$ are the first and second component projection, respectively (cf. Appendix [B]). These facts are basic for the proof of the next result.

**Theorem 4.4.** For all $m \in \mathbb{N}$ the cohomology ring of the classifying space $B^m$ is the tensor product of an integer coefficient polynomial ring in $m$ even-degree free generators $c^k$ and an exterior algebra generated by $m$ odd-degree classes $w^k$. The even-degree generators $c^k := (pr_1 \circ i)^* c_k$ are the pullbacks of the universal Chern classes $c_k$ which generate the cohomology ring $H^*(G_m(\mathbb{C}^\infty), \mathbb{Z})$. Similarly, the odd-degree generators $w^k := (pr_2 \circ i)^* w_k$ are the pullbacks of the universal odd Chern classes $w_k$ which generate $H^*(U(\infty), \mathbb{Z})$ (cf. Appendix [A]).

Our proof of Theorem 4.4 requires the application of the Leray-Serre spectral sequences associated to the fibrations $\pi$ and $pr_1$. For more details on this technique we refer to [DG3 Appendix D] and references therein. Since $i$ is a fiber bundle map, we have also an induced map $i'$ between the spectral sequences. For the $E_2$-pages of these spectral sequence this map reads

$$E_2^{p,q}(pr_1) = H^p(G_m(\mathbb{C}^\infty), H^q(U(\infty), \mathbb{Z})) \simeq H^p(G_m(\mathbb{C}^\infty), \mathbb{Z}) \otimes_{\mathbb{Z}} H^q(U(\infty), \mathbb{Z})$$

$$E_2^{p,q}(\pi) = H^p(G_m(\mathbb{C}^\infty), H^q(U(m), \mathbb{Z})) \simeq H^p(G_m(\mathbb{C}^\infty), \mathbb{Z}) \otimes_{\mathbb{Z}} H^q(U(m), \mathbb{Z}).$$

(4.7)

The last two isomorphisms are a consequence of $\pi_1(G_m(\mathbb{C}^\infty)) = 0$ which implies that the effect of any local system of coefficients on the cohomology can be factorized with a tensor product (see e.g. [DK Section 5.2]). For proving Theorem 4.4 we need to anticipate a technical result.

**Lemma 4.5.** For all $p, q \in \mathbb{N} \cup \{0\}$ the map $i'^*: E_2^{p,q}(pr_1) \to E_2^{p,q}(\pi)$ is surjective.

**Proof.** It is enough to show that at each point $\Sigma \in G_m(\mathbb{C}^\infty)$ the fiberwise restriction

$$i|_{\Sigma} : B^m|_{\Sigma} \simeq U(m) \to \{\Sigma\} \times U(\infty) \simeq U(\infty)$$

induces a surjection $i^*|_{\Sigma}$ from $H^*(U(\infty), \mathbb{Z})$ onto $H^*(U(m), \mathbb{Z})$ (see e.g. [FFG Section 21]). From its very definition it is clear that $i|_{\Sigma}$ agrees with the standard inclusion $U(m) \hookrightarrow U(\infty)$, hence its pullback induces the surjection in cohomology. The later is determined in terms of generators by the relations $i^*|_{\Sigma}(w_k) = w_k$ if $k = 1, \ldots, m$ and $i^*|_{\Sigma}(w_k) = 0$ for $k > m$ (cf. Appendix [A]).

■
Proof of Theorem 4.2. Since \( pr_1 : G_m(\C^n) \times U(\infty) \to G_m(\C^n) \) is a product bundle the associated spectral sequence degenerates at the \( E_2 \)-page, i.e. all the differentials \( \delta_r \) are trivial for \( r \geq 2 \). This implies that \( E_2^{p,q}(pr_1) = E_3^{p,q}(pr_1) = \ldots = E_\infty^{p,q}(pr_1) \). By Lemma 4.5 the maps \( i^* : E_2^{p,q}(pr_1) \to E_2^{p,q}(\pi) \) are surjective. In particular all the non-trivial generators \( \zeta_k \otimes w_j \in E_2^{2k,2j-1}(\pi) \) come from the related generators \( \zeta_k \otimes w_j \in E_2^{2k,2j-1}(pr_1) \). Hence the triviality of the differentials \( \delta_r : E_r^{p,q}(pr_1) \to E_r^{p+q, q-r+1}(pr_1) \) implies the triviality of the differentials \( \delta_r : E_r^{p,q}(\pi) \to E_r^{p+q, q-r+1}(\pi) \). Because of that, one gets \( E_\infty^{p,q}(\pi) = E_\infty^{p,q}(\pi) \). In order to recover the cohomology of \( \B^m_\chi \) from \( E_\infty^{p,q}(\pi) \) one has to solve the extension problems. However, in the present case all the abelian groups \( E_\infty^{p,q}(\pi) \) are free, so that all the extensions are split. As a result, we have the following isomorphism of abelian groups:

\[
H^k(\B^m_\chi, \Z) \cong \bigoplus_{p+q=k} E_\infty^{p,q}(\pi) \cong H^k(G_m(\C^n) \times U(\infty), \Z),
\]

where the last isomorphism is a consequence of the Künneth formula for cohomology. This isomorphism gives rise to a ring isomorphism. Indeed, as a consequence of Lemma 4.5 we can infer that the ring map

\[
i^* : H^*(G_m(\C^n) \times U(\infty), \Z) \to H^*(\B^m_\chi, \Z)
\]

is surjective and acts on the generators of \( H^*(G_m(\C^n) \times U(\infty), \Z) \) as follows: \( i^*(\zeta_k) = \zeta_k \) and \( i^*(w_k) = w_k \) for all \( k = 1, \ldots, m \) and \( i^*(w_k) = 0 \) for \( k > m \).

Remark 4.6 (The winding number). Let us give a closer look at the generator \( w_1^\chi \) of \( H^1(\B^m_\chi, \Z) \cong \Z \). We recall that the first integral cohomology group of a space \( X \) has the realization \( H^1(X, \Z) \cong [X, U(1)] \). This fact is discussed in [Hat, Section 3.1, Exercise 13] but the reader can also prove it by applying the technique describe in Section 5.4. Due to Theorem 4.2 we know that the generator \( w_1^\chi \) is the pullback with respect to the map \( pr_1 \circ i \) of the first universal odd Chern classes \( w_1 \) which generate \( H^1(U(\infty), \Z) \) (cf. Appendix A). This generator is identifiable with the class \([\det] \in [U(\infty), U(1)]\) associated to the “limit” determinant map \( \det : U(\infty) \to U(1) \). Thus the composition

\[
\det \circ pr_1 \circ i : \B^m_\chi \to U(1)
\]

\[
(\Z, u) \mapsto \det(u)
\]

provides a representative for the generator \([\det \circ pr_1 \circ i] \in [\B^m_\chi, U(1)] \cong H^1(\B^m_\chi, \Z) \). Finally, the winding number of this map provides the identification with \( \Z \).

5. Classification of chiral vector bundles

In this section we will combine the knowledge of the topology of the classifying space \( \B^m_\chi \) developed in Section 4 with the criterion of the homotopy classification proved in Corollary 3.4 in order to define the appropriate family of characteristic classes that classify chiral vector bundles. In effect, we will show that these classes provide a complete classification scheme in the case the base space \( X \) is a CW-complex of low dimension \( d \leq 4 \). The special cases of \( X = S^d \) and \( X = \mathbb{T}^d \) will be discussed in detail.

5.1. Characteristic classes. Usually one uses the cohomology of the classifying space of a given category of vector (or principal) bundles to define the associated characteristic classes by pullback with respect to the classifying maps. We apply the same strategy here to define the characteristic classes for chiral vector bundles.

Definition 5.1 (Chiral characteristic classes). Let \((\mathcal{E}, \Phi)\) be a rank \( m \) chiral vector bundle over \( X \) and \( \varphi : X \to \B^m_\chi \) a representative for the classifying map which classifies \([\mathcal{E}, \Phi]\). We define the \( k \)-th even chiral class of \((\mathcal{E}, \Phi)\) to be

\[
c_k(\mathcal{E}, \Phi) := \varphi^\chi \mathcal{E}_k \in H^{2k}(X, \Z), \quad k = 1, \ldots, m.
\]
In much the same way, we refer to
\[ w_k(\mathcal{E}, \Phi) := \varphi^*w_k^\chi \in H^{2k-1}(X, \mathbb{Z}), \quad k = 1, \ldots, m. \]
as the \( k \)-th odd chiral class of \((\mathcal{E}, \Phi)\).

**Proposition 5.2.** The even chiral classes \( c_k(\mathcal{E}, \Phi) \) of the chiral vector bundle \((\mathcal{E}, \Phi)\) coincide with the Chern classes \( c_k(\mathcal{E}) \) of the underlying complex vector bundle \( \mathcal{E} \).

**Proof.** As is shown in the proof of Theorem 3.3 if the chiral vector bundle \((\mathcal{E}, \Phi)\) is classified by a map \( \varphi : X \to B_m^\mathbb{C} \), then the map \( \psi : X \to G_m(\mathbb{C}^n) \) (for some \( n \) big enough), given by \( \psi := (\text{pr}_1 \circ \iota) \circ \varphi \) classifies \( \mathcal{E} \) as complex vector bundle. Now, by the very definition of chiral Chern class, \( c_k(E, \Phi) = \varphi^*c_k = \psi^*c_k = c_k(\mathcal{E}). \)

**Remark 5.3** (Simplified nomenclature). We can take advantage from the previous proposition to simplify the nomenclature concerning the characteristic classes for chiral vector bundles. Indeed, the equality \( c_k(\mathcal{E}, \Phi) = c_k(\mathcal{E}) \) shows that we can properly refer to the even chiral classes simply as Chern classes. In this way we can reserve the name chiral classes only for the odd classes \( w_k(\mathcal{E}, \Phi) \). This choice of names presents the advantage to make clear that the extra topological information induced on the vector bundle \( \mathcal{E} \) by a chiral structure \( \Phi \) is only contained in the odd classes \( w_k(\mathcal{E}, \Phi) \) and not in the Chern classes.

The chiral classes possess a kind of additive behavior induced by the composition of automorphisms.

**Proposition 5.4.** Let \((\mathcal{E}, \Phi_1), (\mathcal{E}, \Phi_2)\) and \((\mathcal{E}, \Phi_1 \circ \Phi_2)\) three chiral vector bundles which share the same underlying complex vector bundle \( \mathcal{E} \to X \). Then, the related chiral classes obey to
\[ w_k(\mathcal{E}, \Phi_1 \circ \Phi_2) = w_k(\mathcal{E}, \Phi_1) + w_k(\mathcal{E}, \Phi_2). \]

The above relations are completed by
\[ w_k(\mathcal{E}, \text{Id}_E) = 0. \]

**Proof.** Let \( \varphi_1, \varphi_2, \varphi : X \to B_m^\mathbb{C} \) be the classifying maps for \((\mathcal{E}, \Phi_1), (\mathcal{E}, \Phi_2)\) and \((\mathcal{E}, \Phi_1 \circ \Phi_2)\) respectively. From the very definition of the classifying maps one can verify that \((\text{pr}_2 \circ \iota) \circ \varphi : X \to U(\infty)\) agrees with the pointwise product of the maps \((\text{pr}_2 \circ \iota) \circ \varphi_i : X \to U(\infty), i = 1, 2\). Therefore, one has
\[ w_k(\mathcal{E}, \Phi_1 \circ \Phi_2) = \varphi^* \circ (\text{pr}_2 \circ \iota)^*w_k = (\varphi_1 \varphi_2)^* \circ (\text{pr}_2 \circ \iota)^*w_k = \varphi_1^* \circ (\text{pr}_2 \circ \iota)^*w_k + \varphi_2^* \circ (\text{pr}_2 \circ \iota)^*w_k = w_k(\mathcal{E}, \Phi_1) + w_k(\mathcal{E}, \Phi_2). \]

Finally, we can classify \((\mathcal{E}, \text{Id}_E)\) by a map \( \varphi : X \to B_m^\mathbb{C} \) such that \((\text{pr}_2 \circ \iota) \circ \varphi = 1\). This immediately leads to (5.2).

### 5.2. The Picard group of chiral line bundles.

Given two rank \( m \) chiral vector bundles \((\mathcal{E}_1, \Phi_1), (\mathcal{E}_2, \Phi_2)\) over \( X \) we can define their tensor product \((\mathcal{E}_1, \Phi_1) \otimes (\mathcal{E}_2, \Phi_2) := (\mathcal{E}_1 \otimes \mathcal{E}_2, \Phi_1 \otimes \Phi_2)\) as the rank \( m^2 \) chiral vector bundle with underlying complex vector bundle \( \mathcal{E}_1 \otimes \mathcal{E}_2 \to X \) given by the tensor product of \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) (in the sense of [Hui, Chapter 6, Section 6]) and automorphism \( \Phi_1 \otimes \Phi_2 \in \text{Aut}(\mathcal{E}_1 \otimes \mathcal{E}_2) \) defined by \( \Phi_1 \otimes \Phi_2(p_1 \otimes p_2) := \Phi_1(p_1) \otimes \Phi_2(p_2) \) for all \( p_1 \in \mathcal{E}_1 \) and \( p_2 \in \mathcal{E}_2 \). Just by considering tensor product homotopies of chiral vector bundles one can easily show that the operation \( \otimes \) is compatible with the notion of equivalence in Definition 2.4. In particular, one has that
\[ \otimes : \text{Vec}^1_X(\mathcal{E}_1) \times \text{Vec}^1_X(\mathcal{E}_2) \to \text{Vec}^1_X(\mathcal{E}_1 \otimes \mathcal{E}_2) \]
endows \( \text{Vec}^1_X(\mathcal{E}) \) with an abelian group structure. According to a standard terminology, we refer to \( \text{Vec}^1_X(\mathcal{E}) \) as the chiral Picard group.

**Proposition 5.5** (Classification of chiral line bundles). There is a group isomorphism
\[ (w_1, c_1) : \text{Vec}^1_X(\mathcal{E}) \to H^1(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \]
induced by the first chiral class \( w_1 \) and the first Chern class \( c_1 \).
Proof. As a consequence of the identification (4.2) one has

\[ \text{Vec}_d^1(X) \cong \{X, B_1\} \cong \{X, U(1)\} \times \{X, CP^\infty\}. \]

By combining Remark 4.6 and Proposition 5.4 one can show that \( w_1 \) sets the group isomorphism \([X, U(1)] \cong H^1(X, \mathbb{Z})\). The proof is completed by recalling the well-known group isomorphism \([X, CP^\infty] \cong \text{Vec}_d^1(X) \cong H^d(X, \mathbb{Z})\) induced by \( c_1 \) (cf. [DG1] Section 3.2 and references therein).

When we apply the above result to spheres of dimension \( d \) we get the following isomorphisms

\[
\begin{align*}
  w_1 & : \text{Vec}_d^1(S^1) \rightarrow H^1(S^1, \mathbb{Z}) \cong \mathbb{Z} \\
  c_1 & : \text{Vec}_d^1(S^2) \rightarrow H^2(S^2, \mathbb{Z}) \cong \mathbb{Z} \\
  \text{Vec}_d^1(S^d) & \cong 0 \quad \text{if} \quad d \geq 3.
\end{align*}
\]

Instead, for tori of dimension \( d \) one has the isomorphisms

\[
(w_1, c_1) : \text{Vec}_d^1(T^d) \rightarrow \mathbb{Z}^d \oplus \mathbb{Z}^{d(d-1)}.
\]

where we used \( H^k(T^d, \mathbb{Z}) \cong \mathbb{Z}^d \).

5.3. Chiral vector bundles over spheres. The classification of chiral vector bundles over spheres (cf. Theorem 1.3) is a direct consequence of the homotopy classification described in Corollary 3.4 combined with Proposition 4.3 and Theorem 4.2.

In the case of low-dimensional spheres the classification can be described in terms of characteristic classes by the following bijections

\[
\begin{align*}
  w_1 & : \text{Vec}_d^m(S^1) \rightarrow H^1(S^1, \mathbb{Z}) \cong \mathbb{Z} \\
  c_1 & : \text{Vec}_d^m(S^2) \rightarrow H^2(S^2, \mathbb{Z}) \cong \mathbb{Z} \\
  w_2 & : \text{Vec}_d^m(S^3) \rightarrow H^3(S^3, \mathbb{Z}) \cong \mathbb{Z} \quad (m > 1).
\end{align*}
\]

When \( d = 1 \) one has \( \pi_1(B_1^\infty) \cong \pi_1(U(m)) \cong \pi_1(U(1)) \cong H^1(S^1, \mathbb{Z}) \) where the last isomorphism has been discussed in Remark 4.6. Similarly, for \( d = 3 \) one has \( \pi_3(B_1^\infty) \cong \pi_3(U(m)) \cong \pi_3(SU(2)) \cong \pi_3(S^3) \cong H^3(S^3, \mathbb{Z}) \) where we used the standard identification between \( SU(2) \) and \( S^3 \) and the last isomorphism can be understood in terms of the Brouwer’s degree [DG1 Remark 5.8]. In the even case \( d = 2 \) one has \( \pi_2(B_1^\infty) \cong \pi_2(G_m(C^{\infty})) \cong [S^2, G_m(C^{\infty})] \cong \text{Vec}_C(S^2) \cong H^2(S^2, \mathbb{Z}) \) where the last isomorphism is given, as usual, by the first Chern class \( c_1 \).

The case \( d = 4 \) is more involved and hence interesting. A direct computation shows that

\[
\text{Vec}_4^m(S^4) \cong \pi_4(U(m)) \oplus \pi_4(G_m(C^{\infty})) \cong \begin{cases} 
\mathbb{Z}_2 \oplus \mathbb{Z} & \text{if} \quad m = 2 \\
\mathbb{Z} & \text{if} \quad m > 3
\end{cases}
\]

where the \( \mathbb{Z} \) summand is given by \( \pi_4(G_m(C^{\infty})) \cong [S^4, G_m(C^{\infty})] \cong \text{Vec}_C(S^4) \cong H^4(S^4, \mathbb{Z}) \cong \mathbb{Z} \) (if \( m > 1 \)) and is described by the second Chern class \( c_2 \). The \( \mathbb{Z}_2 \) summand is given by the unstable group \( \pi_4(U(2)) \cong \pi_4(SU(2)) \cong \mathbb{Z}_2 \) (cf. Table A.1) which is non-trivial only when \( m = 2 \). This torsion summand is not accessible by the “primary” characteristic classes. One way to understand this invariant is to look at the map \( f : S^4 \rightarrow S^3 \) which generates \( \pi_4(S^3) \cong \pi_4(SU(2)) \). This map agrees with the (reduced or unreduced) suspension of the Hopf’s map \( h : S^3 \rightarrow S^2 \) which generates \( \pi_3(S^2) \).

An explicit realization of \( f \) is given by

\[
f(k_0, k_1, k_2, k_3, k_4) := \frac{2}{1 + k_0} \left( k_0, k_1 k_3 - k_2 k_4, k_1 k_4 + k_2 k_3, k_1^2 + k_2^2 - k_3^2 - k_4^2 \right)
\]

for all \((k_0, k_1, k_2, k_3, k_4) \in \mathbb{R}^5\) which verify the constraint \( \sum_{i=0}^{4} k_i^2 = 1 \). One can check that when \( k_0 = 0 \) the map \( f \) restricts exactly to the Hopf’s map. A differential geometric approach to the study of \( \pi_4(SU(2)) \cong \mathbb{Z}_2 \) is discussed in the final part of [Wi1] (see also [Ko, Section 1.4]). Given a map \( f : S^4 \rightarrow SU(2) \) and the standard inclusion \( j : SU(2) \hookrightarrow SU(3) \) one can consider the composition
Given $j \circ f : S^4 \to SU(3)$. Since $\pi_4(SU(3)) = 0$ there exists a map $F : D^5 \to SU(3)$ defined on the unit ball $D^5 \subset \mathbb{R}^5$ such that $F|_{D^4} = j \circ f$. In [Wi] E. Witten showed that the integral (a pure gauge Chern-Simons form)

$$CS_5(f) := \frac{i}{240\pi^3} \int_{D^5} \text{Tr}C^3 \left[(F^{-1} \, dF)^5\right]$$

depends only on the homotopy class $[f] \in \pi_4(SU(2))$ modulo $\mathbb{Z}$. Moreover, the homotopy invariant $e(f) := e^{2\pi CS_5(f)} \in \{-1\}$ takes the value $-1$ when $f$ is a representative for the non-trivial element of $\pi_4(SU(2))$. See also [Ko, Proposition 1.11] for more details.

### 5.4. Chiral vector bundles in low dimension.

We proved in Section 4.1 that $\pi_0(B_\chi^m) = 0$ and $\pi_1(B_\chi^m) \simeq \mathbb{Z}$ acts trivially on each $\pi_k(B_\chi^m)$, $k > 1$. Moreover, $B_\chi^m$ has a CW-complex structure. All these facts imply that $B_\chi^m$ is a simple space, a fact which assures that $B_\chi^m$ admits a Postnikov tower of principal fibrations [Hat, Theorem 4.69] (see also [Arl, Section 7] or [Ark, Chapter 7]). More precisely this means that there exists a sequence of spaces $\mathcal{B}_j^m$ along with maps $\alpha_j : B_\chi^m \to \mathcal{B}_j^m$ and $p_{j+1} : \mathcal{B}_{j+1}^m \to \mathcal{B}_j^m$ such that the diagram

$$\begin{align*}
K(\pi_{j+2}, j+3) & \twoheadrightarrow K(\pi_{j+1}, j+2) & \to & \to K(\pi_j, j+1) & \to & \to K(\pi_j, 1) \\
\cdots & \downarrow{p_{j+2}} & & \downarrow{p_{j+1}} & & \downarrow{\alpha_{j+1}} & \downarrow{\alpha_j} & \downarrow{\alpha_{j-1}} & \downarrow{\alpha_1} & \downarrow{\alpha_0} & \downarrow{1} & \cdots & \downarrow{1} \quad (5.6)
\end{align*}$$

is commutative, i.e., $p_{j+1} \circ \alpha_{j+1} = \alpha_j$ for all $j \geq 1$. The maps $p_j$ and $\kappa_{j+1}$ define (principal) fibration sequences

$$K(\pi_j, j) \twoheadrightarrow \mathcal{B}_j^m \xrightarrow{p_j} \mathcal{B}_{j-1}^m \xrightarrow{\kappa_{j+1}} K(\pi_j, j+1) \quad (5.7)$$

where the symbols $K(\pi_j, n)$ denote the Eilenberg-MacLane spaces associated to $B_\chi^m$. In particular $\mathcal{B}_j^m$, usually called $j$-th Postnikov section, turns out to be (up to weak homotopy equivalence) the homotopy fiber of the map $\kappa_{j+1}$. We recall that $K(\pi_j, n)$ is a connected space (with a uniquely specified homotopy type) defined by the following property [Hat, Section 4.2]

$$\pi_k(K(\pi_j, n)) \simeq \begin{cases} 
\pi_k(B_\chi^m) & \text{if } k = n \\
0 & \text{if } k \neq n.
\end{cases}$$

Moreover, one has the isomorphisms [Hat, Theorem 4.57]

$$[X, K(\pi_j, n)] \simeq H^n(X, \pi_j(B_\chi^m)) \quad (5.8)$$

given by $[f] \mapsto f^*(\xi)$ with $\xi \in H^0(K(\pi_j, n), \pi_j(B_\chi^m))$ being the basic of fundamental class [Ark, Definition 5.3.1]. In our particular case we can choose $K(\pi_1, 1) = S^1$. The map $\alpha_j : B_\chi^m \to \mathcal{B}_j^m$ is a $(j + 1)$-equivalence in the sense that it induces isomorphisms $\pi_k(B_\chi^m) \simeq \pi_k(\mathcal{B}_j^m)$ for all $k \leq j$ and one has that $\pi_k(\mathcal{B}_j^m) = 0$ for all $k > j$. Said differently, the “auxiliary” spaces $\mathcal{B}_j^m$ (which still have the homotopy type of CW-complexes) approximate the system of homotopy groups of $B_\chi^m$ up to the level $j$. The collection of maps $\alpha_j$ induces a weak homotopy equivalence between $B_\chi^m$ and the inverse limit generated by the “auxiliary” spaces $\mathcal{B}_j^m$ (the inverse limit generally will not have the homotopy type of a CW-complex). The “$k$-invariants” $\kappa_{j+1} \in H^{j+1}(\mathcal{B}_{j+1}^m, \pi_j(B_\chi^m))$ are to be regarded as cohomology classes. These classes together with the homotopy groups $\pi_k(B_\chi^m)$ specify the weak homotopy type of $B_\chi^m$. In particular $\kappa_{j+1} = 0$ as cohomological class means that $\kappa_{j+1}$ is homotopic to the constant map in the fibration sequences (5.7) and, in this case, one has that the induced fibration $p_j$ is trivial, i.e., $\mathcal{B}_j^m \simeq \mathcal{B}_{j-1}^m \times K(\pi_j, j)$ (for more details see [Arl, Lemma 7.3]).

The importance of the Postnikov tower for the classification of chiral vector bundles lies in the following general result.
Proposition 5.6. Let $X$ be as in Assumption 1.2 and assume that the maximal dimension of its cells is $d$. Then

$$\text{Vec}^m_\chi(X) \cong [X, \mathcal{B}^m_d] .$$

Proof. The proof of the claim is just a combination of the homotopy classification for chiral vector bundles provided by Corollary 5.4 with the classical result [JT, Lemma 4.1].

The concrete utility of Proposition 5.6 is related to the ability to compute the auxiliary spaces $\mathcal{B}^m_j$, a problem which usually is of difficult solution. However, if one restricts the interest to low dimensions a rigorous computation becomes reasonably doable.

Proposition 5.7 (Classification in low dimension). Let $X$ be as in Assumption 1.2 and assume that the maximal dimension of its cells is $d$.

(i) If $1 \leq d \leq 3$, there are bijections of sets

$$\text{Vec}^m_\chi(X) = \bigoplus_{j=1}^d H^j(X, \mathbb{Z}) , \quad m \geq 2$$

induced by the characteristic classes $w_1, c_1, w_2$ up to the suitable dimension.

(ii) If $d = 4$, there are bijections of sets

$$\text{Vec}^m_\chi(X) = \bigoplus_{j=1}^4 H^j(X, \mathbb{Z}) , \quad m \geq 3$$

induced by the characteristic classes $w_1, c_1, w_2, c_2$.

Proof. We know that $\mathcal{B}^m_1 = K(\pi_1, 1) = S^1$ and from Proposition 5.6 and the isomorphism $[X, S^1] \cong H^1(X, \mathbb{Z})$ (see eq. 5.8) or Remark 4.6 we can conclude the proof for the case $d = 1$. In particular, as a consequence of Lemma A.3 and Remark A.4 we know that the basic class of $H^1(S^1, \mathbb{Z})$, which determines the isomorphism 5.8, can be identified with the generator $w_1$ of $H^*(\mathbb{U}(\infty), \mathbb{Z})$ and consequently with the first chiral generator $w_1$ of $H^*(\mathbb{B}^m_1, \mathbb{Z})$ according to Theorem 4.4.

The study of the case $d = 2$ in Proposition 5.6 needs the computation of the Postnikov section $\mathcal{B}^m_2$. Equation 5.7 says that $\mathcal{B}^m_2$ is the total space of a principal fibration over $\mathcal{B}^m_1 = S^1$ with fiber $K(\pi_2, 2)$. Since $\pi_2(\mathbb{B}^m_1) = \mathbb{Z}$ we can use the identification $K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$. Moreover, $H^3(S^1, \mathbb{Z}) = 0$ implies the vanishing of the $\kappa$-invariant $\kappa^3 = 0$. This assures the triviality of the fibration 5.7, namely $\mathcal{B}^m_2 = S^1 \times \mathbb{C}P^\infty$. At this point Proposition 5.6 provides

$$\text{Vec}^m_\chi(X) \cong [X, S^1] \times [X, \mathbb{C}P^\infty] \cong H^1(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) .$$

and the basic class in $H^2(K(\mathbb{Z}, 2), \mathbb{Z})$ which induces the isomorphism $[X, K(\mathbb{Z}, 2)] \cong H^2(X, \mathbb{Z})$ as in 5.8 can be identified with the first universal Chern class $c_1 \in H^2(\mathbb{G}_m(\mathbb{C}^\infty), \mathbb{Z})$ as discussed in Remark A.2.

The case $d = 3$ is more involved. The Postnikov section $\mathcal{B}^m_3$ is the total space of the fibration

$$K(\mathbb{Z}, 3) \rightarrow \mathcal{B}^m_3 \xrightarrow{p_3} \mathcal{B}^m_2 = S^1 \times \mathbb{C}P^\infty$$

where we used $\pi_3(\mathbb{B}^m_1) \cong \mathbb{Z}$ when $m \geq 2$. This section is determined by the invariant $\kappa^4 \in H^4(S^1 \times \mathbb{C}P^\infty, \mathbb{Z}) \cong H^4(\mathbb{C}P^\infty, \mathbb{Z}) \cong \mathbb{Z}$ and we want to prove that $\kappa^4 = 0$. From one hand we know that the map $\alpha_3 : \mathbb{B}^m_1 \rightarrow \mathcal{B}^m_3$ is a 4-equivalence so that $H^j(\mathbb{B}^m_1, \mathbb{Z}) \cong H^j(\mathcal{B}^m_3, \mathbb{Z})$ for all $j \leq 3$ and $\alpha_3^* : H^4(\mathcal{B}^m_3, \mathbb{Z}) \rightarrow H^4(\mathbb{B}^m_1, \mathbb{Z})$ is injective (cf. Note 3 in Remark A.4). Our knowledge of the cohomology of $\mathbb{B}^m_1$ implies that

$$H^1(\mathcal{B}^m_3, \mathbb{Z}) \cong H^2(\mathcal{B}^m_3, \mathbb{Z}) \cong \mathbb{Z} , \quad H^3(\mathcal{B}^m_3, \mathbb{Z}) \cong \mathbb{Z}^2$$

and $H^4(\mathcal{B}^m_3, \mathbb{Z})$ is a subgroup of $\mathbb{Z}^2$ with no torsion. On the other hand we can compute this cohomology by means of the Leray-Serre spectral sequence associated with the fibration 5.11 (we refer
As a consequence of [St, Theorem 16.9] one has that the group
\[ \pi_1(\mathcal{R}_2^m) \approx \pi_1(S^1) \approx \mathbb{Z} \] acts trivially on higher homotopy groups
\[ \pi_k(\mathcal{R}_2^m) = \pi_k(S^1) \times \pi_k(K(\mathbb{Z}, 2)), \quad k \geq 1. \]
This fact implies that the system of coefficients in \( \text{(5.12)} \) is constant and not local. In particular, by
using that \( \mathcal{R}_2^m \) is path-connected one obtains from \( \text{(5.12)} \) the following isomorphisms
\[ E_2^{0,q} \approx H^q(K(\mathbb{Z}, 3), \mathbb{Z}). \] (5.13)
The explicit knowledge of the cohomology of \( K(\mathbb{Z}, 3) \) (see e.g. [BT, Section 18])
leads to the computation of \( E_2^{0,q} \) showed in the following table:

| \( k \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-------|---|---|---|---|---|---|---|---|---|
| 0     | \( \mathbb{Z} \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1     | \( \mathbb{Z} \) | \( \mathbb{Z} \) | \( \mathbb{Z} \) | \( \mathbb{Z} \) | \( \mathbb{Z} \) | \( \mathbb{Z} \) | \( \mathbb{Z} \) | \( \mathbb{Z} \) | \( \mathbb{Z} \) |

Since the sequence is concentrate in the first quadrant (i.e. \( E_r^{p,q} = 0 \) if \( p < 0 \) or \( q < 0 \)) one obtains by
a recursive application of the formula
\[
E^{p,q}_{r+1} = \frac{\text{Ker}(\delta_r : E_{r}^{p,q} \to E_{r}^{p+r,q-r+1})}{\text{Im}(\delta_r : E_{r}^{p-r,q+r-1} \to E_{r}^{p,q})}
\] (5.14)
that \( E_4^{0,3} \approx E_2^{0,3} \approx E_2^{0,3} \) and \( E_4^{4,0} \approx E_3^{4,0} \approx E_2^{4,0} \). Moreover, one has that
\[ E_6^{0,3} \approx \ldots \approx E_6^{0,3} \approx E_5^{0,3} = \text{Ker}(\delta_4 : E_4^{0,3} \to E_4^{4,0}) \]
and the isomorphisms \( E_4^{0,3} \approx E_2^{0,3} \approx H^3(K(\mathbb{Z}, 3), \mathbb{Z}) \) and \( E_4^{4,0} \approx E_2^{4,0} \approx H^4(\mathbb{C}P^\infty, \mathbb{Z}) \) allow to write
\[ E_\infty^{0,3} \approx \text{Ker}(\delta_4 : H^3(K(\mathbb{Z}, 3), \mathbb{Z}) \to H^4(\mathbb{C}P^\infty, \mathbb{Z})). \]
The map \( \delta_4 \) relates the basic class \( w_2 \in H^2(K(\mathbb{Z}, 3), \mathbb{Z}) \) with the invariant \( k^4 \in H^4(\mathbb{C}P^\infty, \mathbb{Z}) \) according to the formula \( \delta_4(w_2) = -k^4 \) (cf. [Ark] Remark 7.2.6 (4)) or [MP] Lemma 3.4.2]]. The convergence of
the spectral sequence provides the following short exact sequences:

\[
0 \rightarrow F^1H^3(\mathcal{B}_3^m, \mathbb{Z}) \rightarrow H^3(\mathcal{B}_3^m, \mathbb{Z}) \rightarrow E_\infty^{0,3} \rightarrow 0
\]

\[
0 \rightarrow F^2H^3(\mathcal{B}_3^m, \mathbb{Z}) \rightarrow F^1H^3(\mathcal{B}_3^m, \mathbb{Z}) \rightarrow E_\infty^{1,2} \rightarrow 0
\]

\[
0 \rightarrow F^3H^3(\mathcal{B}_3^m, \mathbb{Z}) \rightarrow F^2H^3(\mathcal{B}_3^m, \mathbb{Z}) \rightarrow E_\infty^{2,1} \rightarrow 0
\]

Since \(E_\infty^{2,1} = \ldots = E_2^{2,1} = 0\) and similarly \(E_\infty^{1,2} = \ldots = E_2^{1,2} = 0\) one gets that

\[
F^1H^3(\mathcal{B}_3^m, \mathbb{Z}) \cong F^2H^3(\mathcal{B}_3^m, \mathbb{Z}) \cong F^3H^3(\mathcal{B}_3^m, \mathbb{Z})
\]

By observing that \(F^3H^3(\mathcal{B}_3^m, \mathbb{Z}) \cong H^3_\infty\), one ends up with the following short exact sequence

\[
0 \rightarrow E_\infty^{3,0} \rightarrow H^3(\mathcal{B}_3^m, \mathbb{Z}) \rightarrow E_\infty^{0,3} \rightarrow 0
\]

(5.15)

Since we know that \(H^3(\mathcal{B}_3^m, \mathbb{Z}) \cong \mathbb{Z}^2\) and we can compute that \(E_\infty^{3,0} = \ldots = E_3^{3,0} = E_2^{3,0} = \mathbb{Z}\) we can immediately conclude from the above exact sequence that \(E_\infty^{0,3} = \mathbb{Z}\) which in turn implies that \(\delta_4 = 0\) acts as the trivial map. Finally the vanishing of the Postnikov invariant \(\kappa^4 = 0\) assures that \(\mathcal{B}_3^m = S^1 \times CP^\infty \times K(\mathbb{Z}, 3)\). Proposition 5.6 provides

\[
\text{Vec}_c^m(\mathcal{X}) \cong [X, S^1] \times [X, CP^\infty] \times [X, K(\mathbb{Z}, 3)] \cong H^1(\mathcal{X}, \mathbb{Z}) \oplus H^2(\mathcal{X}, \mathbb{Z}) \oplus H^3(\mathcal{X}, \mathbb{Z})
\]

and the basic class in \(H^3(K(\mathbb{Z}, 3), \mathbb{Z})\) which induces the isomorphism \([X, K(\mathbb{Z}, 3)] \cong H^3_\infty(X, \mathbb{Z})\) described in (5.8) can be identified with the second generator \(w_2 \in H^3_\infty(U(\infty), \mathbb{Z})\), as discussed in Lemma A.3 and Remark A.4, and consequently, with the second chiral generator \(w_2^{\mathcal{X}}\) of \(H^*_C(\mathcal{B}_3^m, \mathbb{Z})\) according to Theorem 4.4.

The next case \(d = 4, m \geq 3\) can be discussed along the same lines as those of the previous case. The Postnikov section \(\mathcal{B}_4^m\) is the total space of the fibration

\[
K(\mathbb{Z}, 4) \rightarrow \mathcal{B}_4^m \xrightarrow{p_4} \mathcal{B}_3^m = S^1 \times CP^\infty \times K(\mathbb{Z}, 3)
\]

(5.16)

where we used \(p_4(\mathcal{B}_3^m) \cong \mathbb{Z}\) when \(m \geq 3\). We point out that for \(m = 2\) the group \(\pi_4(\mathcal{B}_3^2)\) has a torsion part and (5.16) needs to be modified (see Section 5.5). This section is determined by the invariant

\[
\kappa^5 \in H^5(S^1 \times CP^\infty \times K(\mathbb{Z}, 3), \mathbb{Z}) \cong \left( H^2(CP^\infty, \mathbb{Z}) \otimes \mathbb{Z} H^3(K(\mathbb{Z}, 3), \mathbb{Z}) \right) \oplus \left( H^4(S^1, \mathbb{Z}) \otimes \mathbb{Z} H^4(CP^\infty, \mathbb{Z}) \right)
\]

and we want to prove that \(\kappa^5 = 0\). Since the map \(\alpha_4 : \mathcal{B}_4^m \rightarrow \mathcal{B}_3^m\) is a 5-equivalence we can deduce from our knowledge of the cohomology of \(\mathcal{B}_4^m\) that

\[
H^1(\mathcal{B}_4^m, \mathbb{Z}) \cong H^2(\mathcal{B}_4^m, \mathbb{Z}) \cong \mathbb{Z}, \quad H^3(\mathcal{B}_4^m, \mathbb{Z}) \cong \mathbb{Z}^2, \quad H^4(\mathcal{B}_4^m, \mathbb{Z}) \cong \mathbb{Z}^3.
\]

On the other hand we can compute this cohomology by means of the Leray-Serre spectral sequence associated with the fibration (5.16). The 2-page of this spectral sequence is given by

\[
E_2^{p,q} = H^p(\mathcal{B}_3^m, H^q(K(\mathbb{Z}, 4), \mathbb{Z}))
\]

(5.17)

with constant system of coefficients since \(\pi_1(\mathcal{B}_3^m) \cong \pi_1(S^1) \cong \mathbb{Z}\) acts trivially on higher homotopy groups

\[
\pi_k(\mathcal{B}_3^m) \cong \pi_k(S^1) \times \pi_k(K(\mathbb{Z}, 2)) \times \pi_k(K(\mathbb{Z}, 3)), \quad k \geq 1.
\]

In particular, since \(\mathcal{B}_3^m\) is path-connected one obtains from (5.17) the following isomorphisms

\[
E_2^{0,q} \cong H^q(K(\mathbb{Z}, 4), \mathbb{Z}).
\]

(5.18)

---

For the precise derivation of the short exact sequences the reader can refer to [LGG3] eq. D8 (and reference therein). Here we used the shorter notational convention

\[
F^iH^q(X, \mathbb{Z}) := \text{Ker}(H^q(X, \mathbb{Z}) \rightarrow H^q(X_{j-1}, \mathbb{Z})) \subset H^q(X, \mathbb{Z})
\]

where \(X_1 \subset X_0 \subset X_1 \subset \ldots \subset X_\rho \subset \ldots \subset X\) is any filtration for the space \(X\).
The cohomology of \(K(\mathbb{Z}, 4)\) can be computed as in [BT] Section 18 (see also [Per] Table 1) and one obtains:

\[
\begin{array}{cccccccc}
  k = 0 & k = 1 & k = 2 & k = 3 & k = 4 & k = 5 & k = 6 & k = 7 & k = 8 \\
  H^k(K(\mathbb{Z}, 4), \mathbb{Z}) & \mathbb{Z} & 0 & 0 & 0 & \mathbb{Z} & 0 & 0 & \mathbb{Z}_2 & \mathbb{Z} \\
\end{array}
\]

This allows us to compute the values of the 2-page \(E_2^{p,q}\) as showed in the following table:

| \(q\)  | 0    | 0    | 0    | 0    | 0    | 0    | 0    |
|--------|------|------|------|------|------|------|------|
| \(q = 6\) | 0    | 0    | 0    | 0    | 0    | 0    | 0    |
| \(q = 5\) | 0    | 0    | 0    | 0    | 0    | 0    | 0    |
| \(q = 4\) | \(\mathbb{Z}\) | \(\mathbb{Z}\) | \(\mathbb{Z}\) | \(\mathbb{Z}^2\) | \(\mathbb{Z}^2\) | \(\mathbb{Z}^2\oplus\mathbb{Z}_2\) |
| \(q = 3\) | 0    | 0    | 0    | 0    | 0    | 0    | 0    |
| \(q = 2\) | 0    | 0    | 0    | 0    | 0    | 0    | 0    |
| \(q = 1\) | 0    | 0    | 0    | 0    | 0    | 0    | 0    |
| \(q = 0\) | \(\mathbb{Z}\) | \(\mathbb{Z}\) | \(\mathbb{Z}\) | \(\mathbb{Z}^2\) | \(\mathbb{Z}^2\) | \(\mathbb{Z}^2\oplus\mathbb{Z}_2\) |
| \(E_2^{p,q}\) | \(p = 0\) | \(p = 1\) | \(p = 2\) | \(p = 3\) | \(p = 4\) | \(p = 5\) | \(p = 6\) |

A recursive application of the formula (5.14) provides

\[
E_\infty^{0,4} \cong \cdots \cong E_0^{0,4} \cong E_6^{0,4} := \text{Ker}(\delta_5 : E_5^{0,4} \to E_5^{5,0})
\]

and the isomorphisms

\[
\begin{align*}
E_5^{0,4} & \cong E_4^{0,4} \cong E_3^{0,4} \cong E_2^{0,4} \cong H^4(K(\mathbb{Z}, 4), \mathbb{Z}) \cong \mathbb{Z} \\
E_5^{5,0} & \cong E_4^{5,0} \cong E_3^{5,0} \cong E_2^{5,0} \cong H^5(\mathcal{B}_3^m, \mathbb{Z}) \cong \mathbb{Z}_2
\end{align*}
\]

allow us to write

\[
E_5^{0,4} \cong \text{Ker}(\delta_5 : H^4(K(\mathbb{Z}, 4), \mathbb{Z}) \to H^5(\mathcal{B}_3^m, \mathbb{Z})).
\]

The map \(\delta_5\) relates the basic class \(c_2 \in H^4(K(\mathbb{Z}, 4), \mathbb{Z})\) with the invariant \(\kappa^5 \in H^5(\mathcal{B}_3^m, \mathbb{Z})\) according to the formula \(\delta_5(c_2) = -\kappa^5\) (cf. [Ark] Remark 7.2.6 (4) or [MP] Lemma 3.4.2). The convergence of the spectral sequence provides in exactly the same way as above the short exact sequence

\[
0 \to E_\infty^{4,0} \to H^4(\mathcal{B}_4^m, \mathbb{Z}) \to E_\infty^{0,4} \to 0. \tag{5.19}
\]

Since we know that \(H^4(\mathcal{B}_4^m, \mathbb{Z}) \cong \mathbb{Z}_2^3\) and we can compute that \(E_\infty^{4,0} \cong \cdots \cong E_3^{4,0} \cong E_2^{4,0} \cong \mathbb{Z}_2^2\) we can immediately conclude from (5.19) that \(E_\infty^{0,4} \cong \mathbb{Z}^3\) which in turn implies that \(\delta_5 = 0\) acts as the trivial map. Finally the vanishing of the Postnikov invariant \(\kappa^5 = 0\) assures that \(\mathcal{B}_4^m \cong S^1 \times \mathbb{C}P^\infty \times K(\mathbb{Z}, 3) \times K(\mathbb{Z}, 4)\) if \(m \geq 3\). Proposition 5.6 provides

\[
\text{Vec}_4^m(X) \cong [X, S^1] \times [X, \mathbb{C}P^\infty] \times [X, K(\mathbb{Z}, 3)] \times [X, K(\mathbb{Z}, 4)] = \bigoplus_{j=1}^4 H^j(X, \mathbb{Z})\, , \quad m \geq 3
\]

and the basic class in \(H^4(K(\mathbb{Z}, 4), \mathbb{Z})\) which induces the isomorphism \([X, K(\mathbb{Z}, 4)] \cong H^4(X, \mathbb{Z})\) described in (5.8) can be identified with the second universal Chern class \(c_2 \in H^4(G_m(\mathbb{C}^\infty), \mathbb{Z})\) as discussed in Remark [A, 2].
Remark 5.8. In the statement of Proposition 5.7 we pointed out that the equations (5.9) and (5.10) have to be understood as bijections of sets rather than group isomorphisms. This is because the standard group structure of direct sums of cohomology groups does not coincide \textit{a priori} with a possible group structure on classes of chiral vector bundles induced by some geometric operations such as the Whitney sum. For this reason it should be better to replace expressions like \( H^1(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \times \ldots \) instead of \( H^1(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus \ldots \) in (5.9) and (5.10). However, in accordance with a modern (maybe inaccurate) custom in the classification of topological phases we chose to keep with the use of the symbol \( \oplus \) instead of \( \times \), always keeping in mind that the identification is at level of sets and not of groups. The main reason and the first advantage of this choice of notation is that Table 1.1 and Table 1.2 result directly from Proposition 5.9.

Remark 5.9. The next step in the construction of the Postnikov tower of \( B^m_\chi \) is the determination of the invariant \( \kappa^6 \). However, this invariant turns out to be non-trivial even in the stable range \( m \geq 3 \) for a reason that we will sketch below. First of all form the group structure of direct sums of cohomology groups does not coincide with the existing literature. In any case this “conflict” in the notation opens the necessity for a deeper investigation of the group structure induced on the classes of chiral vector bundles by Whitney sum. We plan to investigate this aspect in an incoming work.

The next step in the construction of the Postnikov tower of \( B^m_\chi \) is the determination of the invariant \( \kappa^6 \). However, this invariant turns out to be non-trivial even in the stable range \( m \geq 3 \) for a reason that we will sketch below. First of all form the group structure of direct sums of cohomology groups does not coincide with the existing literature. In any case this “conflict” in the notation opens the necessity for a deeper investigation of the group structure induced on the classes of chiral vector bundles by Whitney sum. We plan to investigate this aspect in an incoming work.
In particular one has that
\[ k^5 \in H^5(\mathcal{B}_3^2, \mathbb{Z} \oplus \mathbb{Z}) \cong \mathbb{Z}_2^3 \oplus \mathbb{Z}^2. \]

We can obtain information about the cohomology of \( \mathcal{B}_4^2 \) by recalling that the map \( \alpha_4 : B_4^2 \to B_4^2 \) is a 5-equivalence. Our knowledge of the cohomology of \( B_4^2 \) provides
\[
H^1(\mathcal{B}_4^2, \mathbb{Z}) \cong H^2(\mathcal{B}_4^2, \mathbb{Z}) \cong \mathbb{Z}, \quad H^3(\mathcal{B}_4^2, \mathbb{Z}) \cong \mathbb{Z}^2 \quad H^4(\mathcal{B}_4^2, \mathbb{Z}) \cong \mathbb{Z}^3.
\]
Moreover \( H^5(\mathcal{B}_4^2, \mathbb{Z}) \) is injected in \( H^5(B_4^2, \mathbb{Z}) \cong \mathbb{Z}^3 \) and so it has no torsion. This also implies information about homology (cf. [Hat Corollary 3.3]) and with the help of the universal coefficient theorem one obtains
\[ H^k(\mathcal{B}_4^2, \mathbb{Z} \oplus \mathbb{Z}) \cong \text{Hom}_{\mathbb{Z}}(H^k(\mathcal{B}_4^2, \mathbb{Z}), \mathbb{Z}_2) \oplus H^k(\mathcal{B}_4^2, \mathbb{Z}), \quad k = 1, 2, 3, 4. \]
In particular for \( k = 4 \) one has
\[ H^4(\mathcal{B}_4^2, \mathbb{Z} \oplus \mathbb{Z}) \cong \mathbb{Z}_2^3 \oplus \mathbb{Z}^3. \]

We can study the cohomology of \( \mathcal{B}_4^2 \) with coefficients \( \mathbb{Z}_2 \oplus \mathbb{Z} \) also from the help of the spectral sequence associated to the principal fibration
\[ K(\mathbb{Z}_2 \oplus \mathbb{Z}, 4) \to P_4 \to P_3. \quad (5.22) \]

Since we know that \( \pi_1(\mathcal{B}_3^2) \) acts trivially on higher homotopy groups the 2-page of this spectral sequence is given by
\[ E_2^{p,q} = H^p(\mathcal{B}_3^2, H^q(K(\mathbb{Z}_2 \oplus \mathbb{Z}, 4), \mathbb{Z}_2 \oplus \mathbb{Z})) \quad (5.23) \]
with constant system of coefficients. In particular the path-connectedness of \( \mathcal{B}_3^m \) implies
\[ E_2^{0,q} \cong H^q(K(\mathbb{Z}_2 \oplus \mathbb{Z}, 4), \mathbb{Z}_2 \oplus \mathbb{Z}). \quad (5.24) \]
The computation of the spectral sequence requires the cohomology of \( K(\mathbb{Z}_2 \oplus \mathbb{Z}, 4) \). One can use the relation
\[ K(G_1 \oplus G_2, j) \cong K(G_1, j) \times K(G_2, j) \]
and the Künneth formula for cohomology to reduce the problem to the computation of the cohomology of \( K(\mathbb{Z}, 4) \) and \( K(\mathbb{Z}_2, 4) \). Since we already used the cohomology of \( K(\mathbb{Z}, 4) \) in the proof of Proposition 5.7 we need compute only the cohomology of \( K(\mathbb{Z}_2, 4) \). This can be done along the lines sketched in [BT Section 18] or with the “Eilenberg-MacLane machine” described in [C]. The results are shown in the following table:

| \( k \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| \( H^k(\mathbb{Z}, 4), \mathbb{Z} \) | \( \mathbb{Z} \) | 0 | 0 | 0 | \( \mathbb{Z} \) | 0 | 0 | \( \mathbb{Z}_2 \) |
| \( H^k(\mathbb{Z}_2 \oplus \mathbb{Z}, 4), \mathbb{Z} \) | \( \mathbb{Z} \) | 0 | 0 | 0 | \( \mathbb{Z}_2 \) | 0 | \( \mathbb{Z}_2 \) |
| \( H^k(\mathbb{Z}_2 \oplus \mathbb{Z}, 4), \mathbb{Z}_2 \oplus \mathbb{Z} \) | \( \mathbb{Z}_2 \oplus \mathbb{Z} \) | 0 | 0 | 0 | \( \mathbb{Z}_2^2 \oplus \mathbb{Z} \) | \( \mathbb{Z}_2^2 \) | \( \mathbb{Z}_2^3 \) |
With these results we can compute the values of $E^{p,q}_2$ showed in the following table:

| $q = 5$ | $\mathbb{Z}_2^2$ | $\ldots$ |   |   |
|-------|-----------------|----------|---|---|
| $q = 4$ | $\mathbb{Z}_2^2 \oplus \mathbb{Z}$ | $\mathbb{Z}_2^2 \oplus \mathbb{Z}$ | $\ldots$ |   |
| $q = 3$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $q = 2$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $q = 1$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $q = 0$ | $\mathbb{Z}_2 \oplus \mathbb{Z}$ | $\mathbb{Z}_2 \oplus \mathbb{Z}$ | $\mathbb{Z}_2 \oplus \mathbb{Z}$ | $\mathbb{Z}_2^2 \oplus \mathbb{Z}^2$ | $\mathbb{Z}_2^2 \oplus \mathbb{Z}^2$ | $\mathbb{Z}_2^3 \oplus \mathbb{Z}^2$ |
| $E^{p,q}_2$ | $p = 0$ | $p = 1$ | $p = 2$ | $p = 3$ | $p = 4$ | $p = 5$ |

A recursive application of the formula (5.14) provides

$$E^{0,4}_\infty \simeq \ldots \simeq E^{0,4}_7 \simeq E^{0,4}_6 := \text{Ker}(\delta_5 : E^{0,4}_5 \to E^{5,0}_5)$$

and the isomorphisms

$$E^{0,4}_5 \simeq E^{0,4}_4 \simeq E^{0,4}_3 \simeq E^{0,4}_2 \simeq H^4(K(\mathbb{Z}_2 \oplus \mathbb{Z},4), \mathbb{Z}_2 \oplus \mathbb{Z}) \simeq \mathbb{Z}_2^2 \oplus \mathbb{Z}$$

$$E^{5,0}_5 \simeq E^{5,0}_4 \simeq E^{5,0}_3 \simeq E^{5,0}_2 \simeq H^5(\mathbb{B}_3^2, \mathbb{Z}_2 \oplus \mathbb{Z}) \simeq \mathbb{Z}_2^3 \oplus \mathbb{Z}^2$$

allow to write

$$E^{0,4}_\infty \simeq \text{Ker}(\delta_5 : H^4(K(\mathbb{Z}_2 \oplus \mathbb{Z}, 4), \mathbb{Z}_2 \oplus \mathbb{Z}) \to H^5(\mathbb{B}_3^2, \mathbb{Z}_2 \oplus \mathbb{Z}))$$.

The map $\delta_5$ relates the basic class $\mathfrak{d} \in H^4(K(\mathbb{Z}_2 \oplus \mathbb{Z}, 4), \mathbb{Z}_2 \oplus \mathbb{Z})$ with the invariant $\kappa^5 \in H^5(\mathbb{B}_3^2, \mathbb{Z}_2 \oplus \mathbb{Z})$ according to the formula $\delta_5(\mathfrak{d}) = -\kappa^5$ (cf. [Ark], Remark 7.2.6 (4)) or [MP, Lemma 3.4.2]). The convergence of the spectral sequence provides in exactly the same way as in the proof of Proposition 5.7 the short exact sequence

$$0 \to E^{4,0}_\infty \to H^4(\mathbb{B}_3^2, \mathbb{Z}_2 \oplus \mathbb{Z}) \to E^{0,4}_\infty \to 0.$$  

Since we know that $H^4(\mathbb{B}_3^2, \mathbb{Z}_2 \oplus \mathbb{Z}) \simeq \mathbb{Z}_2^3 \oplus \mathbb{Z}^3$ and we can compute that $E^{4,0}_\infty \simeq \ldots \simeq E^{4,0}_3 \simeq E^{4,0}_2 \simeq \mathbb{Z}_2^2 \oplus \mathbb{Z}^2$ we can conclude that:

**Lemma 5.10.** $E^{0,4}_\infty \simeq \mathbb{Z}_2 \oplus \mathbb{Z}$.

**Proof.** The key observation is that the spectral sequence $E^{p,q}_r$ induced by the 2-page (5.23) is the direct sum of two spectral sequences $E^{p,q}_r(\mathbb{Z}_2)$ and $E^{p,q}_r(\mathbb{Z})$ induced by the natural splitting $E^{p,q}_2 \simeq E^{p,q}_2(\mathbb{Z}_2) \oplus E^{p,q}_2(\mathbb{Z})$ where

$$E^{p,q}_2(\mathbb{Z}_2) := H^p(\mathbb{B}_3^2, H^q(K(\mathbb{Z}_2 \oplus \mathbb{Z}, 4), \mathbb{Z}_2)) \Rightarrow H^*(\mathbb{B}_3^2, \mathbb{Z}_2)$$

$$E^{p,q}_2(\mathbb{Z}) := H^p(\mathbb{B}_3^2, H^q(K(\mathbb{Z}_2 \oplus \mathbb{Z}, 4), \mathbb{Z})) \Rightarrow H^*(\mathbb{B}_3^2, \mathbb{Z}).$$

Consequently, also the short exact sequence (5.26) splits as

$$0 \to E^{4,0}_\infty(\mathbb{Z}_2) \to H^4(\mathbb{B}_3^2, \mathbb{Z}) \to E^{0,4}_\infty(\mathbb{Z}_2) \to 0$$

$$0 \to E^{4,0}_\infty(\mathbb{Z}) \to H^4(\mathbb{B}_3^2, \mathbb{Z}_2) \to E^{0,4}_\infty(\mathbb{Z}) \to 0$$

and from $E^{4,0}_\infty(\mathbb{Z}_2) \simeq \mathbb{Z}_2^2$, $E^{0,4}_\infty(\mathbb{Z}) \simeq \mathbb{Z}^2$ and $H^4(\mathbb{B}_3^2, \mathbb{Z}_2) \simeq \mathbb{Z}_2^3$, $H^4(\mathbb{B}_3^2, \mathbb{Z}) \simeq \mathbb{Z}^3$ one immediately gets $E^{0,4}_\infty(\mathbb{Z}_2) \simeq \mathbb{Z}_2$ and $E^{0,4}_\infty(\mathbb{Z}) \simeq \mathbb{Z}$. The proof is concluded by $E^{0,4}_\infty \simeq E^{0,4}_\infty(\mathbb{Z}_2) \oplus E^{0,4}_\infty(\mathbb{Z})$.  

$\blacksquare$
The claim follows by setting Proposition 5.13.

As a consequence of Proposition 5.12 (and Proposition 5.6) one has that rank 2 chiral vector bundles

Let us recall that

Thus, as a consequence of Corollary 5.11 one has that the invariant \( \kappa^5 \) behaves as a map

where we used 0 for the constant map.

**Proposition 5.12.**

\[
\mathcal{B}_4^2 = \left( \mathcal{B}_3^2 \times_{\xi^5} K(\mathbb{Z}_2, 4) \right) \times K(\mathbb{Z}, 4).
\]  

(5.27)

where the “twisted” product \( \mathcal{B}_3^2 \times_{\xi^5} K(\mathbb{Z}_2, 4) \) is defined in the proof and \( \xi^5 \) can be identified with a non trivial class in \( H^3(\mathcal{B}_3^2, \mathbb{Z}_2) \cong \mathbb{Z}_2^3 \).

**Proof.** The Postnikov section \( \mathcal{B}_4^2 \) is a principal fibration which is built from the following diagram

where \( \varphi = (\varphi_1, \varphi_2) \) is the path fibration and the bottom square is the homotopy pull-back. Therefore, one has that

\[
\mathcal{B}_4^2 \cong \left\{ (x, y, z) \in \mathcal{B}_3^2 \times PK(\mathbb{Z}, 5) \times PK(\mathbb{Z}_2, 5) \mid \varphi_1(y) = 0(x) = z, \varphi_2(z) = \xi^5(x) \right\}
\]

\[
\cong \varphi_2^{-1}(\ast) \times \left\{ (x, z) \in \mathcal{B}_3^2 \times PK(\mathbb{Z}_2, 5) \mid \varphi_2(z) = \xi^5(x) \right\}.
\]

The claim follows by setting \( \mathcal{B}_3^2 \times_{\xi^5} K(\mathbb{Z}_2, 4) := \left\{ (x, z) \in \mathcal{B}_3^2 \times PK(\mathbb{Z}_2, 5) \mid \varphi_2(z) = \xi^5(x) \right\} \) and by observing that \( \varphi_2^{-1}(\ast) \cong K(\mathbb{Z}, 4) \).

As a consequence of Proposition 5.12 (and Proposition 5.6) one has that rank 2 chiral vector bundles over a CW-complex \( X \) of dimension \( d = 4 \) are classified by

\[
\text{Vec}_2^2(X) \cong \left[ X, \mathcal{B}_3^2 \times_{\xi^5} K(\mathbb{Z}_2, 4) \right] \oplus H^4(X, \mathbb{Z})
\]

(5.28)

where the contribution of \( H^4(X, \mathbb{Z}) \) is given by the second Chern class. Unfortunately, the determination of the space of homotopic equivalent maps \( \left[ X, \mathcal{B}_3^2 \times_{\xi^5} K(\mathbb{Z}_2, 4) \right] \) is difficult in general and at the moment it seems to be beyond our capabilities. However, under some extra assumptions on \( X \), a description of the above set seems possible.

**Proposition 5.13.** Let \( X \) be such that \( H^5(X, \mathbb{Z}_2) = 0 \). Then, there is a bijection of sets

\[
\left[ X, \mathcal{B}_3^2 \times_{\xi^5} K(\mathbb{Z}_2, 4) \right] \cong \left[ X, \mathcal{B}_3^2 \right] \times \left[ X, K(\mathbb{Z}_2, 4) \right].
\]
Proof. The fibration projection \( \pi : (B_2^3 \times \xi^5 K(\mathbb{Z}_2, 4)) \to B_3^2 \) induces a map

\[
\pi_* : [X, (B_2^3 \times \xi^5 K(\mathbb{Z}_2, 4))] \to [X, B_3^2].
\]

To prove the claim we need to show that: (i) the inverse image \( \pi_*^{-1}([f]) \) of every element \([f] \in [X, B_3^2]\) can be identified with the set \([X, K(\mathbb{Z}_2, 4)]\), (ii) \( \pi_* \) is surjective. First of all we notice that \( \pi_*^{-1}([f]) \) is homotopy equivalent to the homotopy classes of lifts of \( f : X \to B_3^2 \), i.e.

\[
\pi_*^{-1}([f]) \simeq \{ \tilde{f} : X \to (B_2^3 \times \xi^5 K(\mathbb{Z}_2, 4)) \mid \pi \circ \tilde{f} = f \} / \text{homotopy}.
\]

Let us consider the pullback fibration \( \tilde{\pi} : f^* (B_2^3 \times \xi^5 K(\mathbb{Z}_2, 4)) \to X \). Sections \( s \) of this fibration are in one-to-one correspondence with the lifts \( \tilde{f} \) by means of the formula \( s(x) := (x, \tilde{f}(x)) \) for all \( x \in X \). Hence

\[
\pi_*^{-1}([f]) \simeq \text{Sections}(f^* (B_2^3 \times \xi^5 K(\mathbb{Z}_2, 4)) \big/ \text{homotopy}.
\]

As a principal fibration \( f^* (B_2^3 \times \xi^5 K(\mathbb{Z}_2, 4)) \) is characterized by \( f^* \xi^5 \in H^5(X, \mathbb{Z}_2) = 0 \). Therefore, there is a homotopy equivalence \( f^* (B_2^3 \times \xi^5 K(\mathbb{Z}_2, 4)) \simeq X \times K(\mathbb{Z}_2, 4) \). This implies that the sections of \( f^* (B_2^3 \times \xi^5 K(\mathbb{Z}_2, 4)) \) can be identified with maps on \( X \) into \( K(\mathbb{Z}_2, 4) \) and so from (5.29) one gets

\[
\pi_*^{-1}([f]) \simeq [X, K(\mathbb{Z}_2, 4)]
\]

as desired for point (i). The point (ii) which concerns the surjectivity of \( \pi_* \) follows from the exact sequence of pointed sets

\[
[X, (B_2^3 \times \xi^5 K(\mathbb{Z}_2, 4))]_s \to [X, B_3^2]_s \xrightarrow{\xi^5} [X, K(\mathbb{Z}_2, 5)]_s \to H^5(X, \mathbb{Z}_2) = 0
\]

induced by the fibration \( B_2^3 \times \xi^5 K(\mathbb{Z}_2, 4) \to B_3^2 \to K(\mathbb{Z}_2, 5) \). We used \([\ , \ ]_s\) for the homotopy classes of base point preserving maps. The fact that \( B_3^2 \) has an H-space structure implies the bijection \([X, B_3^2]_s \simeq [X, B_3^2]\) (cf. [Hat, Example 4A.3]). Finally, the surjectivity of \( \pi_* \) is a consequence of the obvious inclusion \([X, (B_2^3 \times \xi^5 K(\mathbb{Z}_2, 4))]_s \subset [X, (B_2^3 \times \xi^5 K(\mathbb{Z}_2, 4))]\].

Let us point out that the bijection (5.30) depends on a choice of a homotopy equivalence between \( f^* (B_2^3 \times \xi^5 K(\mathbb{Z}_2, 4)) \) and \( X \times K(\mathbb{Z}_2, 4) \). This suggests that, a priori, different equivalences will lead to different bijections. Said differently the bijection claimed in Proposition 5.13 could not be natural.

5.6. Chiral vector bundles over tori. Proposition 5.7 applies, in particular, to the torus \( T^d \) for \( 1 \leq d \leq 4 \). However, the unstable case of rank 2 chiral vector bundles over \( T^4 \) necessitates of an extra argument.

Corollary 5.14. Let \( X \) be a closed oriented 4-dimensional manifold. The following bijection of sets

\[
\text{Vec}_X^2 \simeq \text{Vec}_{\chi X}^{m>2}(X) \oplus \mathbb{Z}_2
\]

holds true.

Proof: The key argument is contained in Proposition 5.13 that can be applied here since \( H^5(X, \mathbb{Z}_2) = 0 \). The result then follows just by combining equation (5.28) with the classification of \( \text{Vec}_{\chi X}^{m>2}(X) \) in Proposition 5.7 and the fact that \([X, K(\mathbb{Z}_2, 4)] \simeq H^4(X, \mathbb{Z}_2) \simeq \mathbb{Z}_2\).

Since the torus \( T^4 \) fulfills all the conditions of Corollary 5.14 one immediately gets

\[
\text{Vec}_X^2(T^4) \simeq \mathbb{Z}_2^{14} \oplus \mathbb{Z}_2.
\]
6. From topological quantum systems of class AIII to chiral Bloch-bundles

6.1. Chiral vector bundles vs. Clifford vector bundles. Vector bundles endowed with a Clifford action provide a geometric model for the $K$-theoretical version of the “Bott-clock” [Kar, Chapter III]. For this reason, these objects are usually taken as base for the justification of the Kitaev’s “Periodic Table” for topological insulators [FM, Th1]. In this section we investigate how vector bundles endowed with a Clifford action can be related to $\chi$-bundles.

Let us start by introducing some fundamental facts, and a little of terminology, about Clifford algebras. For a more complete introduction we refer to [Kar, Chapter III, Section 3] or [Le]. A Clifford algebra $\text{Cl}(V, \Omega)$ is a unital associative algebra that contains and is generated by a vector space $V$ over a (commutative) field $K$, where $V$ is equipped with a quadratic form $\Omega$. More properly $\text{Cl}(V, \Omega)$ is specified by a homomorphism $i : V \rightarrow \text{Cl}(V, \Omega)$ on the underlying vector space and by the conditions $i(v)^2 = \Omega(v)\mathbb{I}$, for all $v \in V$, where $\mathbb{I}$ denotes the unit element in the algebra. Clifford algebras turn out to be universal objects in the following sense: given any unital associative algebra $A$ over $K$ and any linear map $j : V \rightarrow A$ such that $j(v)^2 = \Omega(v)\mathbb{I}_A$ (where $\mathbb{I}_A$ denotes the multiplicative identity of $A$), there is a unique algebra homomorphism $\varphi : \text{Cl}(V, \Omega) \rightarrow A$ such that $\varphi \circ i = j$.

The case we are more interested is when $K = \mathbb{R}$ and $V = \mathbb{R}^n$ endowed with the quadratic form $Q_{p,n-p} := \text{diag}(-1, \ldots, -1, +1, \ldots, +1)$ with $p$ negative entries and $n - p$ positive entries. We introduce the short notation $\mathbb{R}^{p,n-p}$ for the pair $(\mathbb{R}^n, Q_{p,n-p})$ and we denote by $\text{Cl}(p,n-p)$ the corresponding Clifford algebra. According to a classical result [Kar, Chapter III, Corollary 3.11] the algebra $\text{Cl}(p,n-p)$ is generated over $\mathbb{R}$ by a collection of symbols $e_1, \ldots, e_n$ subjected to the relations $(e_i)^2 = -1$ if $1 \leq i \leq p$; $(e_i)^2 = +1$ if $p + 1 \leq i \leq n$ and $e_i e_j = -e_j e_i$ if $i \neq j$. Clifford algebras over $\mathbb{R}^{p,n-p}$ have been completely classified. For our purpose, we need only the isomorphisms $\text{Cl}^{1,0} = \mathbb{C}$ and $\text{Cl}^{0,1} = \mathbb{R} \oplus \mathbb{R}$.

Clifford algebras are compatible with the structure of (complex) vector bundles in the following sense:

**Definition 6.1** (Clifford vector bundle). Let $\mathcal{E} \rightarrow X$ be a rank $m$ complex vector bundle over $X$ and denote with $\text{End}(\mathcal{E})$ the (vector bundle) endomorphisms of $\mathcal{E}$. We can endow $\mathcal{E}$ by a Clifford action by means of an $\mathbb{R}$-algebra homomorphism $\rho : \text{Cl}^{p,q} \rightarrow \text{End}(\mathcal{E})$. We call the pair $(\mathcal{E}, \rho)$ a Clifford vector bundle of type $(p, q)$. A morphism between two Clifford vector bundles of the same type $(\mathcal{E}, \rho)$ and $(\mathcal{E}', \rho')$ is a vector bundle morphism $f : \mathcal{E} \rightarrow \mathcal{E}'$ such that $f \circ \rho(a) = \rho'(a) \circ f$ for all $a \in \text{Cl}^{p,q}$.

We want to investigate the connection between chiral vector bundles and Clifford vector bundles of type $(0, 1)$. Let us remark that $\text{Cl}^{0,1}$ is generated over $\mathbb{R}$ by a unique symbol $e$ such that $e^2 = +1$. This implies that given an object $\mathcal{A}$ in a Banach category (e.g., a vector bundle) and an endomorphism $g \in \text{End}(\mathcal{A})$ such that $g^2 = \text{Id}_{\mathcal{A}}$, there is a unique $\mathbb{R}$-algebra homomorphism $\rho : \text{Cl}^{0,1} \rightarrow \text{End}(\mathcal{A})$ completely specified by $\rho(e) = g$ and $\rho(+1) = \text{Id}_{\mathcal{A}}$.

**Lemma 6.2.** For a given rank $m$ chiral vector bundle $(\mathcal{E}, \Phi)$ over $X$, the pair $(\hat{\mathcal{E}}, \hat{\rho})$ given by

\[
\hat{\mathcal{E}} := \mathcal{E} \oplus \mathcal{E}, \quad \hat{\rho}(e) := \begin{pmatrix} 0 & \Phi \\ \Phi^{-1} & 0 \end{pmatrix}
\]

defines a rank $2m$ Clifford vector bundle of type $(0, 1)$ over $X$. Moreover, this assignment is compatible with isomorphisms in the sense that isomorphic chiral vector bundles define isomorphic Clifford vector bundles.

**Proof.** Let $g = \rho(e) \in \text{End}(\mathcal{E})$ as specified by (6.1). For the proof that $(\hat{\mathcal{E}}, \hat{\rho})$ is a Clifford vector bundle with a $\text{Cl}^{0,1}$-action it is enough to observe that $g^2 = \text{Id}_{\mathcal{E}}$. Now, let us consider two isomorphic $\chi$-bundles $(\mathcal{E}_1, \Phi_1) \simeq (\mathcal{E}_2, \Phi_2)$ with isomorphism given by a map $f$ in the sense of diagram (2.2). Let $(\hat{\mathcal{E}}_1, \hat{\rho}_1)$ and $(\hat{\mathcal{E}}_2, \hat{\rho}_2)$ be the Clifford vector bundles associated to $(\mathcal{E}_1, \Phi_1)$ and $(\mathcal{E}_2, \Phi_2)$, respectively. The map

\[
\hat{f} := \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}
\]
provides an isomorphism between \((\mathcal{E}_1, \Phi_1)\) and \((\mathcal{E}_2, \Phi_2)\) in the category of Clifford vector bundles since 
\[ f_+ \circ \rho_1(e) = \rho_2(e) \circ f_\].

The Clifford vector bundle \((\hat{\mathcal{E}}, \hat{\rho})\) built from a \(X\)-bundle \((\mathcal{E}, \Phi)\) according to the prescription in Lemma [6.2] has more structure. Indeed, it is endowed with an \textit{odd symmetric gradation} \(\Gamma \in \text{End}(\hat{\mathcal{E}})\) defined by
\[
\Gamma := \begin{pmatrix} \text{Id}_\mathcal{E} & 0 \\ 0 & -\text{Id}_\mathcal{E} \end{pmatrix}.
\]  

A direct computation shows that
\[
\Gamma^2 = \text{Id}_\hat{\mathcal{E}}, \quad \Gamma \circ \rho(e) = -\rho(e) \circ \Gamma
\]
and this two relations agree with the following general definition.

**Definition 6.3** (Odd symmetric gradation). Let \(\mathcal{E} \to X\) be a rank \(m\) complex vector bundle over \(X\). A gradation of \(\mathcal{E}\) is a \(\Gamma \in \text{End}(\mathcal{E})\) such that \(\Gamma^2 = \text{Id}_\mathcal{E}\). The two endomorphisms \(\Pi_\pm \in \text{End}(\mathcal{E})\) defined by
\[
\Pi_\pm := \frac{1}{2}(\text{Id}_\mathcal{E} \pm \Gamma)
\]
are idempotent, i.e. \(\Pi_\pm^2 = \Pi_\pm\). Let \(X\) be as in Assumption [2.2]. This assures that the numbers \(n_\pm := \dim \ker(\Pi_\pm)\) do not depend of the choice of \(x \in X\) (see e.g. the argument in [GBVF, Theorem 2.10]) and \(n_+ + n_- = m\). The numbers \(n_\pm\) are called the indeces of \(\Gamma\). The two vector subbundles \(\mathcal{E}_\pm := \ker(\Pi_\pm)\) [Hu, Chapter 3, Theorem 8.2] provide a splitting
\[
\mathcal{E} = \mathcal{E}_+ \oplus \mathcal{E}_-.
\]
The gradation \(\Gamma\) is called symmetric when \(n_+ = n_-\) (which implies that \(m\) has to be even). If \((\mathcal{E}, \rho)\) is a Clifford vector bundle of type \((p, q)\) we say that \(\Gamma\) is an odd gradation for \((\mathcal{E}, \rho)\) if in addition \(\Gamma \circ \rho(e_i) = -\rho(e_i) \circ \Gamma\) for all \(1 \leq i \leq p + q\).

**Lemma 6.4.** Let \((\hat{\mathcal{E}}, \hat{\rho})\) be a rank \(2m\) Clifford vector bundle of type \((0, 1)\) over a space \(X\) that verifies Assumption [2.2]. Let \(\Gamma \in \text{End}(\hat{\mathcal{E}})\) be an odd symmetric gradation and \(\mathcal{E}_\pm := \ker(\Pi_\pm)\) the two rank \(m\) vector bundles defined by the idempotents \([6.4]\). Then \(\mathcal{E}_+\) and \(\mathcal{E}_-\) are isomorphic as complex vector bundles and there is an isomorphisms \(\Theta : \mathcal{E}_- \to \mathcal{E}_+\) such that
\[
\hat{\mathcal{E}} = \mathcal{E}_+ \oplus \mathcal{E}_-,
\rho(e) = \begin{pmatrix} 0 & \Theta \\ \Theta^{-1} & 0 \end{pmatrix}.
\]
Moreover, each pair of isomorphisms \(h_\pm : \mathcal{E}_\pm \to \mathcal{E}\) defines a rank \(m\) chiral vector bundle \((\mathcal{E}, \Phi_h)\) with \(\Phi_h := h_+ \circ \Theta \circ h_-^{-1}\).

**Proof.** The first part follows by observing that \(\rho(e) \circ \Pi_\pm \circ \rho(e) = \Pi_\pm\) which implies that \(\Pi_+ \circ \rho(e) \circ \Pi_-\) is an endomorphism of \(\hat{\mathcal{E}}\) which restricts to a linear isomorphism between corresponding fibers of \(\mathcal{E}_-\) and \(\mathcal{E}_+\). This assures by [Hu, Chapter 3, Theorem 2.5] that the restriction of \(\Pi_\pm \circ \rho(e) \circ \Pi_-\) defines an isomorphism \(\Theta : \mathcal{E}_- \to \mathcal{E}_+\) with inverse given by \(\Pi_- \circ \rho(e) \circ \Pi_+\). The second part of the claim is just a consequence of the fact that \(\Phi_h\) is an endomorphisms of \(\mathcal{E}\) by construction.

**Remark 6.5.** Lemma [6.4] deserves some comments.

(a) There is a natural notion of isomorphism between two Clifford vector bundles of type \((0, 1)\) endowed with symmetric gradations \((\mathcal{E}, \rho, \Gamma)\) and \((\mathcal{E}', \rho', \Gamma')\). This is given by an isomorphism of complex vector bundles \(\hat{f} : \hat{\mathcal{E}} \to \hat{\mathcal{E}}'\) such that \(\hat{f} \circ \Gamma = \Gamma' \circ \hat{f}\) and \(\hat{f} \circ \rho(e) = \rho'(e) \circ \hat{f}\). In terms of the splitting \([6.5]\) this is equivalent to the existence of two isomorphisms \(f_\pm : \mathcal{E}_\pm \to \mathcal{E}'_\pm\) such that
\[
\hat{f} = \begin{pmatrix} f_+ & 0 \\ 0 & f_- \end{pmatrix},
\]
\(f_+ \circ \Theta = \Theta' \circ f_-\).
(b) There is an asymmetry between Lemma 6.2 and Lemma 6.4. The first proves that a chiral vector bundle uniquely specifies (up to isomorphisms) a Clifford vector bundle of type (0, 1) endowed with an odd symmetric gradation \( (\hat{E}, \rho, \Gamma \rangle \). Conversely, Lemma 6.4 shows that in order to reconstruct a chiral vector bundle \((\hat{E}_h, \Phi_h)\) from a Clifford vector bundle of type (0, 1) with odd symmetric gradation \((\hat{E}, \rho, \Gamma \rangle \) one needs to make a choice of a pair of isomorphisms \( h_\pm : \hat{E}_\pm \to \hat{E} \). However, the new Clifford vector bundle of type (0, 1) with odd symmetric gradation \((\hat{E}_h, \rho_h, \Gamma_h)\) obtained from \((\hat{E}_h, \Phi_h)\) by following the construction in Lemma 6.2 turns out to be isomorphic (in the sense of point (a) above) to the original triplet \((\hat{E}, \rho, \Gamma \rangle \) through the isomorphism

\[
\hat{h} : (\hat{E}, \rho, \Gamma \rangle \to (\hat{E}_h, \rho_h, \Gamma_h \rangle , \quad \hat{h} = \begin{pmatrix} h_+ & 0 \\ 0 & h_- \end{pmatrix}.
\]

(c) The construction of the \( \chi \)-bundle \((\hat{E}_h, \Phi_h)\) is based on the choice of a pair of isomorphisms \( h_\pm : \hat{E}_\pm \to \hat{E} \). However, what is really important is it only the difference between \( h_+ \) and \( h_- \). More precisely, let us consider the pair of isomorphisms \( h_\pm^{(\pm)} : \hat{E}_\pm \to \hat{E}_\pm \) defined by \( h_+^{(\pm)} = \text{Id}_{\hat{E}} \) and \( h_-^{(\pm)} = h_-^{-1} \circ h_+ \) and the related chiral vector bundle \((\hat{E}_\pm, \Phi_h)\) with \( \Phi_h := \Theta \circ h_-^{-1} \circ h_+ \).

Then, \((\hat{E}_+, \Phi_h) \approx (\hat{E}, \Phi_h)\) as chiral vector bundles with isomorphism given by the map \( h_\pm \). Similarly, one can consider the pair of maps \( h_\pm^{(\pm)} : \hat{E}_\pm \to \hat{E}_\pm \) defined by \( h_\pm^{(\pm)} = h_\pm^{-1} \circ h_\pm \) and \( h_-^{(\pm)} = \text{Id}_{\hat{E}} \) and the associated chiral vector bundle \((\hat{E}_\pm, \Phi_h)\). Then the map \( h_\pm \) provides a \( \chi \)-bundle isomorphism \((\hat{E}_\pm, \Phi_h) \approx (\hat{E}, \Phi_h)\). Finally, by transitivity one gets

\[
(\hat{E}_+, \Phi_h) \approx (\hat{E}, \Phi_h) \approx (\hat{E}_-, \Phi_h).
\]

In conclusion, in order to realize a \( \chi \)-bundle from a triplet \((\hat{E}, \rho, \Gamma \rangle \) one only needs to specify a reference map \( h_\ref : \hat{E}_\pm \to \hat{E}_\pm \) and to consider the associated standard \( \chi \)-bundle \((\hat{E}_+, \Phi)\) with \( \Phi := h_\ref^{-1} \circ \Theta \). Let \( h_\ref' : \hat{E}_- \to \hat{E}_+ \) be a second reference map. Then \((\hat{E}_-, \Phi) \approx (\hat{E}_+, \Phi')\) as chiral vector bundles if and only if there is a \( f \in \text{Aut}(\hat{E}_-) \) such that \( \Phi' = f \circ \Phi \circ f^{-1} \), namely if and only if

\[
h_\ref' = \Theta \circ f \circ \Theta^{-1} \circ h_\ref \circ f^{-1}.
\]

This equation shows that the freedom in the choice of \( h_\ref \) is measured by \( \text{Aut}(\hat{E}_-) \).

(d) Consider now two isomorphic triplets \((\hat{E}, \rho, \Gamma \rangle \) and \((\hat{E}', \rho', \Gamma' \rangle \) with isomorphism \( \hat{f} \) as in (6.6). Let \( h_\ref : \hat{E}_- \to \hat{E}_+ \) and \( h_\ref' : \hat{E}'_- \to \hat{E}'_+ \) be two reference maps and \((\hat{E}_-, \Phi)\) and \((\hat{E}', \Phi')\) the two associated standard \( \chi \)-bundles described in (c). If it happens that

\[
f_+ \circ h_\ref = h_\ref' \circ f_-
\]

then \( f_- : (\hat{E}_-, \Phi) \to (\hat{E}', \Phi') \) is an isomorphism of chiral vector bundles. Indeed the condition \( \Phi = f_-^{-1} \circ \Phi' \circ f_- \) follows by exploiting the relations (6.6) and (6.7). This observation is, to some extent, an inverse of the content of Lemma 6.2.

(e) The effect of the reference map \( h_\ref : \hat{E}_- \to \hat{E}_+ \) on the triplet \((\hat{E}, \rho, \Gamma \rangle \) can be also understood as the introduction of a second \( \text{Cl}^{0,1} \) Clifford-action

\[
\rho_{\text{ref}}(e) = \begin{pmatrix} 0 & h_\ref^{-1} \\ h_\ref & 0 \end{pmatrix}
\]

which evidently anti-commute with the gradation \( \Gamma \).

The sequence (a)-(e) of the above remarks suggests a strong relation between \( \chi \)-bundles and Clifford vector bundle with a double \( \text{Cl}^{0,1} \)-action endowed with an odd symmetric gradation.
complex vector bundles over the base space.

The set $\text{Vec}$ (see structure) they are topologically classified by the set $\text{Vec}$ for all concrete applications of the vector bundles and continuous families of projections by $\text{Cl}$. Since topological quantum systems in class (1.6) and (1.7) as topological quantum systems in class $\text{Cl}$ of the Landau operator.

Construction is necessary for the Harper operator [DL] which describes the strong magnetic field limit. In this case a quotient procedure of the type [At, Proposition 1.6.1] is required. For instance, this kind of associated with any isolated family of energy bands $\text{Cl}$. Theorem 6.7. There is a one-to-one correspondence between homotopic classes $[(\hat{\mathcal{E}}, \Phi)]$ of rank $m$ chiral vector bundles and homotopic classes $[(\hat{\mathcal{E}}, \rho_1, \rho_2, \Gamma)]$ of rank $2m$ Clifford vector bundles with a double $\text{Cl}$-action and odd symmetric gradation.

6.2. Construction of the chiral Bloch-bundle. A standard construction associates to each topological quantum system $\text{Cl}$ with an isolated family of $m$ energy bands $\text{Cl}$ a complex vector bundle of rank $m$ over $X$ which, to some extent, can be named Bloch-bundle. The first step of this construction is the realization of a continuous map of rank $m$ spectral projections $X \ni x \mapsto P(x)$ by means of the Riesz-Dunford integral:

$$P_\Omega(x) := \frac{i}{2\pi} \int_C dz \left( H(x) - z \mathbb{1} \right)^{-1}$$

(6.9)

associated with any isolated family of energy bands $\Omega$ which verifies $\text{Cl}$. Here $C \subset \mathbb{C}$ is any regular closed path which encloses, without touching, the spectral range $\Omega$. The second step turns out to be a concrete application of the Serre-Swan Theorem [Sw] (see also [GBVF, Theorem 2.10]) which relates vector bundles and continuous families of projections by $\hat{\mathcal{E}}_\Omega := \left\{ x \in X \right\} \text{Ran} P_\Omega(x)$.

(6.10)

Sometimes, $x \mapsto P_\Omega(x)$ is equivariant with respect to a free action of a topological group $G$ over $X$. In this case a quotient procedure of the type [A] Proposition 1.6.1] is required. For instance, this kind of construction is necessary for the Harper operator [DL] which describes the strong magnetic field limit of the Landau operator.

By borrowing the accepted terminology for topological insulators we can refer to systems which verify only $\text{Cl}$ and $\text{Cl}$ as topological quantum systems in class A (see e.g. [SRFL]). In this case the (Cartan) label A expresses the absence of any kind of (pseudo-)symmetry or other extra structures. Since topological quantum systems in class A lead to complex vector bundles (without any other extra structure) they are topologically classified by the set $\text{Vec}^m(X)$ of the isomorphism classes of rank $m$ complex vector bundles over the base space $X$. The classification of complex vector bundles is a classical, well-studied, problem in topology. Under rather general assumptions on the base space $X$, the set $\text{Vec}^m(X)$ can be classified by using homotopy theory techniques and, for dimension $d \leq 4$ (and for all $m$) a complete description can be done in terms of cohomology groups and Chern classes [Pet] (see e.g. [DGI, Section 3] for a review of these standard results).

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This name is justified by the fact that the Bloch theory for electrons in a crystal (see e.g. [AM]) provides some of the most interesting examples of topological quantum systems.
The presence of a chiral symmetry \( (1.8) \) introduces more structure. The first consequence is that the spectrum of the operator \( H(x) \) is symmetric with respect to the zero energy, i.e.
\[
\lambda(x) \in \sigma(H(x)) \iff -\lambda(x) \in \sigma(H(x)) \quad \forall x \in X . \tag{6.11}
\]
Indeed, if \( \psi(x) \in \mathcal{H} \) is an eigenvector of \( H(x) \) with eigenvalue \( \lambda(x) \) then \( \psi'(x) := \chi(x)\psi(x) \) is still an eigenvector but with eigenvalue \( -\lambda(x) \). This symmetry justifies the following assumption which is usually verified in the most common physical situations.

**Assumption 6.8** (Zero energy gap). Let \( X \ni x \mapsto H(x) \) be a topological quantum system with chiral symmetry \( X \ni x \mapsto \chi(x) \) as in Definition 1.3. We will assume that \( H(x) \) as an energy gap in zero, i.e. \( 0 \notin \sigma(H(x)) \) for all \( x \in X \).

Due to this assumption we can select an isolated family of energy bands of type \( \Omega = \Omega_- \cup \Omega_+ \) where the sets \( \Omega_\pm := \{ \lambda(x,j) \}, \ldots, \lambda(x,j) \} \) contain energy bands of fixed sign and pair of bands \( \lambda(x,j), \lambda(x,j) \) are related by the chiral symmetry \( \chi \). We refer to \( \Omega_+ \) and \( \Omega_- \) as the positive and negative energy sector, respectively. The spectral projection associated to \( \Omega \) by (6.9) splits accordingly in a positive and a negative part
\[
P_\Omega(x) = P_{\Omega_+}(x) + P_{\Omega_-}(x), \quad P_{\Omega_+}(x) P_{\Omega_-}(x) = 0
\]
since the zero energy gap assumption. Moreover, the chiral symmetry intertwines the two energy sectors,
\[
\chi(x) P_{\Omega_+}(x) \chi(x) = P_{\Omega_-}(x) . \tag{6.12}
\]
The splitting of \( P_\Omega \) induces a splitting of the Bloch-bundle
\[
\hat{\Theta}_\Omega = \hat{\Theta}_\Omega_+ \oplus \hat{\Theta}_\Omega_-, \quad \hat{\Theta}_\Omega_+ := \bigoplus_{x \in X} \text{Ran} P_{\Omega_+}(x)
\]
and the relation (6.12) implies that the two summands are isomorphic as complex vector bundles, \( \hat{\Theta}_\Omega_+ \cong \hat{\Theta}_\Omega_- \). The operator
\[
\Gamma_\Omega(x) := P_\Omega(x) \chi(x) P_\Omega(x) , \quad \Gamma_\Omega(x)^2 = P_\Omega(x)
\]
has the structure of an involution on the space \( \text{Ran} P_\Omega(x) \) and so it identifies a gradation
\[
\Gamma_\Omega \in \text{End}(\hat{\Theta}_\Omega), \quad \Gamma_\Omega^2 = \text{Id}_{\hat{\Theta}_\Omega} \text{ of the rank } 2m \text{ Bloch-bundle } (6.13). \text{ This gradation turns out to be symmetric.}
\]

**Lemma 6.9.** Let \( \Gamma_\Omega \) be the gradation of the Bloch-bundle \( \hat{\Theta}_\Omega \to X \) defined fiberwise by (6.14). Define the two idempotent endomorphisms \( \Pi_{\Omega,\pm} \in \text{End}(\hat{\Theta}_\Omega) \) as in (6.4). Then \( \dim \text{Ker}(\Pi_{\Omega,\pm}) = m \) showing that \( \Gamma_\Omega \) is a symmetric gradation.

**Proof.** Since \( \Pi_{\Omega,\pm} \) are idempotent endomorphisms of \( \hat{\Theta}_\Omega \), the dimension of their kernels are locally constant on \( X \) (see e.g. the argument in [GBVF, Theorem 2.10]). The connectedness of \( X \) assures that the dimension of the kernels are globally constant. This means that we can make the computation just by looking to a single fiber over a point \( x \in X \). Here the idempotents \( \Pi_{\Omega,\pm} \) have the expression
\[
\Pi_{\Omega,\pm}(x) = \frac{1}{2} \left( P_\Omega(x) \pm P_\Omega(x) \chi(x) P_\Omega(x) \right) . \tag{6.15}
\]
Let \( \{ \psi_j \}, j = 1, \ldots, m, \) be a basis for the negative energy sector eigenspace \( P_{\Omega_-}(x)\mathcal{H} \) and \( \{ \psi_j \} \) the related basis for the positive energy sector eigenspace \( P_{\Omega_+}(x)\mathcal{H} \). The relation between the two basis is fixed by \( \psi_j = \chi(x)\psi_j \). Let \( \{ \phi_j \}, j = 1, \ldots, m, \) be the new basis for the full space \( P_\Omega(x)\mathcal{H} \) given by
\[
\phi_j := \frac{1}{2} \left( \psi_j \pm \psi_j \right) . \tag{6.16}
\]
Since \( \chi(x)\phi_j = \pm \phi_j \) and \( P_\Omega(x)\phi_j = \phi_j \) one checks that \( \Pi_{\Omega,\pm}(x)\phi_j = \phi_j \) and \( \Pi_{\Omega,\pm}(x)\phi_j = 0 \). This concludes the proof.
The evident relations \( P(x) = \Pi_{\Omega^+}(x) + \Pi_{\Omega^-}(x) \) and \( \Pi_{\Omega^+}(x)\Pi_{\Omega^-}(x) = 0 \) imply a different splitting of the Bloch-bundle
\[
\hat{E}_\Omega = E_+ \oplus E_-, \quad E_\pm := \bigcup_{x \in X} \text{Ran } \Pi_{\Omega^\pm}(x).
\] (6.17)
The flattened (restricted) Hamiltonian
\[
\rho_\Omega(x) := P_\Omega(x) \frac{H(x)}{|H(x)|} P_\Omega(x) = P_{\Omega^+}(x) - P_{\Omega^-}(x)
\] turns out to be well-defined due to Assumption 6.8 and the two relations
\[
\rho_\Omega(x)^2 = P_\Omega(x), \quad \Gamma_\Omega(x) \rho_\Omega(x) = -\rho_\Omega(x) \Gamma_\Omega(x)
\] prove that the operator \( \rho_\Omega \) introduces a \( C^\theta_{\!0.1} \)-action on the total Bloch-bundle \( \hat{E}_\Omega \) and the gradation \( \Gamma_\Omega \) is odd with respect to this action. The relations
\[
\rho_\Omega(x) \Pi_{\Omega^\pm}(x) \rho_\Omega(x) = \Pi_{\Omega^\pm}(x), \quad \Gamma_\Omega(x) = \Pi_{\Omega^+}(x) - \Pi_{\Omega^-}(x)
\]
imply the isomorphism \( E_+ \cong E_- \) of the complex vector bundles in the splitting (6.17). Moreover, with respect to such a gradation the Clifford action \( \rho_\Omega(e) \) and the gradation \( \Gamma_\Omega \) are represented as
\[
\Gamma_\Omega = \begin{pmatrix} +\text{Id}_{E_-} & 0 \\ 0 & -\text{Id}_{E_+} \end{pmatrix}, \quad \rho_\Omega(e) = \begin{pmatrix} 0 & \Theta_\chi^1 \\ \Theta_\chi^{-1} & 0 \end{pmatrix}.
\]
The isomorphism \( \Theta_\chi : E_-^\chi \rightarrow E_+^\chi \) and its inverse \( \Theta_\chi^{-1} \) are described fiberwise by
\[
\Theta_\chi(x) := \Pi_{\Omega^+}(x) \rho_\Omega(x) \Pi_{\Omega^-}(x) \quad \Theta_\chi^{-1}(x) := \Pi_{\Omega^-}(x) \rho_\Omega(x) \Pi_{\Omega^+}(x).
\]

Summarizing, up to now we have shown how from a topological quantum system with chiral symmetry and zero energy gap one can associate a rank 2m Bloch-bundle \( \hat{E}_\Omega \) with a \( C^\theta_{\!0.1} \)-action \( \rho_\Omega \) and an symmetric odd gradation \( \Gamma_\Omega \), provided that there exists a family of \( m \) negative (resp. positive) energy bands \( \Omega^- \) (resp. \( \Omega^+ \)) separated from the rest of the spectrum. However, as discussed at the end of Section 6.1 the data contained in the triplet \( (\hat{E}_\Omega, \rho_\Omega, \Gamma_\Omega) \) are not enough alone to determinate a chiral vector bundle. We need a second reference isomorphism \( \rho_{\text{ref}} : E_- \rightarrow E_+ \) or equivalently a second \( C^\theta_{\!0.1} \)-action \( \rho_{\text{ref}} \) compatible with the gradation \( \Gamma_\Omega \). Once such an \( \rho_{\text{ref}} \) has been given we can uniquely specify the \( \chi \)-bundle \( (E_-, \Phi_\chi) \) with \( \Phi_\chi := h_{\text{ref}}^{-1} \circ \Theta_\chi \). This construction justifies the Assumption 1.6 given in the introduction.

**Remark 6.10.** In order to complete the picture of the relation between topological quantum systems with chiral symmetry and \( \chi \)-bundles a couple of remarks are still necessary.

(a) The choice of the reference isomorphism \( \rho_{\text{ref}} \) corresponds to the choice of a (local) system of coordinates for the vector bundles \( \pi_\pm : E_\pm \rightarrow X \). As a matter of fact \( E_- \) and \( E_+ \) are isomorphic hence they can be trivialized on the same open covering \( \{U_\alpha\} \) of \( X \). Let \( h_{\alpha,\pm} : \pi_-^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{C}^m \) be a choice of a local trivialization. On the overlappings \( U_\alpha \cap U_\beta \) both systems of local trivializations transform with the same system of transition functions \( g_{\beta \alpha} : U_\alpha \cap U_\beta \rightarrow U(m) \) according to \( h_{\beta,\pm} = g_{\beta \alpha} \circ h_{\alpha,\pm} \). This implies that the collection of local isomorphisms \( h_{\alpha,\text{ref}} := h_{\alpha,\pm}^{-1} \circ h_{\alpha,-} : \pi_-^{-1}(U_\alpha) \rightarrow \pi_+^{-1}(U_\alpha) \) “glues” together and defines a global isomorphism \( \rho_{\text{ref}} : E_- \rightarrow E_+ \).

(b) The Clifford \( C^\theta_{\!0.1} \)-action \( \rho_{\text{ref}} \) associated to the reference isomorphisms \( \rho_{\text{ref}} \) according to (6.8) has fiberwise the expression
\[
\rho_{\text{ref}}(x) = \Pi_{\Omega^+}(x) h_{\text{ref}}(x) \Pi_{\Omega^-}(x) + \Pi_{\Omega^-}(x) h_{\text{ref}}^{-1}(x) \Pi_{\Omega^+}(x).
\]
After expressing the projections \( \Pi_{\Omega^\pm}(x) \) in terms of the Fermi projection \( P_\Omega(x) \) according to (6.13) one gets after some algebra
\[
\rho_{\text{ref}}(x) = P_\Omega(x) \left( \frac{1 + \chi(x)}{2} h_{\text{ref}}(x) \frac{1 - \chi(x)}{2} + \frac{1 - \chi(x)}{2} h_{\text{ref}}^{-1}(x) \frac{1 + \chi(x)}{2} \right) P_\Omega(x).
\]
By choosing the map \( x \mapsto h_{\text{ref}}(x) \) to be unitary-valued (this implies no loss of generality) one can see that \( \rho_{\text{ref}}(x) \) turns out to be a self-adjoint operator acting on the relevant spectral subspace \( P_{\Omega}(x)\mathcal{H} \) and endowed with the (evident) chiral symmetry \( \chi(x)\rho_{\text{ref}}(x)\chi(x) = -\rho_{\text{ref}}(x) \). In this sense \( \rho_{\text{ref}}(x) \) can be considered as a reference chiral Hamiltonian for the initial chiral Hamiltonian \( H(x) \).

Some of the observations in (a) and (b) above will acquire a certain importance inside the contest of Section 6.3.

We conclude this section with a comparison between the energy splitting (6.13) and the chiral splitting (6.17) of the Bloch-bundle \( \hat{\varepsilon}_{\Omega} \).

**Lemma 6.11.** Let \( \varepsilon_{\Omega} \) be the two rank \( m \) vector bundles which provide the energy splitting (6.13) of the Bloch-bundle \( \hat{\varepsilon}_{\Omega} \). Similarly, let \( \varepsilon_{\pm} \) be the two rank \( m \) vector bundles which provide the chiral splitting (6.17) of \( \hat{\varepsilon}_{\Omega} \). Then one has

\[
\varepsilon_{\Omega^{-}} \cong \varepsilon_{\Omega^{+}} \cong \varepsilon_{\chi^{+}} \cong \varepsilon_{\chi^{-}}
\]

where all the isomorphisms are meant in the category of complex vector bundles.

**Proof.** The isomorphisms \( \varepsilon_{\Omega^{-}} \cong \varepsilon_{\Omega^{+}} \) and \( \varepsilon_{\pm} \cong \varepsilon_{\chi} \) has already been discussed in the text. At any rate all the isomorphisms can be checked directly by showing that all the vector bundles have a same system of transition functions \( g_{\beta\alpha} \) subordinate to an open covering \( \{ U_{\alpha} \} \) of the base space \( X \) [Hu Chapter 5, Theorem 2.7]. Let \( U_{\alpha} \) and \( U_{\beta} \) be two elements of the covering with non trivial intersection \( U_{\alpha} \cap U_{\beta} \neq \emptyset \). Let \( \{ \psi_{+j}^{(\alpha)} \} \) and \( \{ \psi_{-j}^{(\beta)} \} \), \( j = 1, \ldots, m \) be two local frames for \( \varepsilon_{\Omega^{-}} \) supported on \( U_{\alpha} \) and \( U_{\beta} \), respectively, and related by \( \psi_{+j}^{(\beta)} = \sum_{k=1}^{m} g_{\beta\alpha}^{jk} \psi_{+k}^{(\alpha)} \) on the intersection \( U_{\alpha} \cap U_{\beta} \). The local systems \( \{ \psi_{-j}^{(\alpha)} := \chi \psi_{+j}^{(\alpha)} \} \) and \( \{ \psi_{-j}^{(\beta)} := \chi \psi_{+j}^{(\beta)} \} \) provide a local trivialization for \( \varepsilon_{\Omega^{-}} \). On the intersection \( U_{\alpha} \cap U_{\beta} \) we have that

\[
\psi_{-j}^{(\beta)} = \chi \psi_{+j}^{(\beta)} = \chi \left( \sum_{k=1}^{m} g_{\beta\alpha}^{jk} \psi_{+k}^{(\alpha)} \right) = \sum_{k=1}^{m} g_{\beta\alpha}^{jk} (\chi \psi_{+k}^{(\alpha)}) = \sum_{k=1}^{m} g_{\beta\alpha}^{jk} \psi_{-k}^{(\alpha)}
\]

and this shows that \( g_{\beta\alpha} \) provides a system of transition functions also for \( \varepsilon_{\Omega^{-}} \). Local frames for \( \varepsilon_{\pm} \) and \( \varepsilon_{\chi} \) can be realized as linear combinations of \( \{ \psi_{+j}^{(\beta)} \} \) and \( \{ \psi_{-j}^{(\beta)} \} \) according to equation (6.16). Since \( \{ \psi_{+j}^{(\beta)} \} \) and \( \{ \psi_{-j}^{(\beta)} \} \) transform according to the same system of transition functions also their linear combinations will transforms according to the same system just by linearity.

### 6.3. Comparison with the literature

The aim of this section is to compare our point of view about the topological phases of chiral topological quantum systems (as given in Definition 1.6) with various aspects typically accepted in the literature concerning the topology of the chiral class AIII.

**Topological insulators of class AIII.** We will briefly sketch the usual derivation of the topological invariants for topological insulators of class AIII as presented in the classical literature on the subject [SRFL, RSFL] as well as in more recent reviews [QZ, BuT] or new research works [PS].

One start with a matrix-valued map of the type

\[
X \ni x \mapsto H(x) = H(x)^* \in \text{Mat}_{\mathbb{C}}(2m)
\]

where \( X \) is usually a \( d \)-dimensional sphere \( S^d \) (free fermions system) or a \( d \)-dimensional torus \( T^d \). Then one assumes the existence of a constant involutive unitary matrix \( \Gamma \in \text{Mat}_{\mathbb{C}}(2m) \) such that

\[
\Gamma H(x) + H(x) \Gamma = 0, \quad \Gamma = \Gamma^* = \Gamma^{-1}.
\]

The existence of a gap at zero energy (in the same spirit of Assumption 6.3) assures that the spectrum of \( H(x) \) is symmetric with respect to the zero energy with equal number of positive and negative
eigenvalues $n_+ = n_- = m$. The gradation $\Gamma$ turns out to be symmetric and, up to a choice of a splitting $\mathbb{C}^{2m} = \mathcal{H}_+ \oplus \mathcal{H}_-$ (which does not fix uniquely a basis!), it is represented by

$$\Gamma = \begin{pmatrix} \mathbb{1}_m & 0 \\ 0 & -\mathbb{1}_m \end{pmatrix}$$

(6.22)

where $\mathbb{1}_m$ is the $m \times m$ identity matrix. At this point one considers the the flattened Hamiltonian

$$\rho(x) = \frac{H(x)}{|H(x)|} = P_+(x) - P_-(x)$$

(6.23)

which is well defined since the zero gap condition. The spectral projections $P_\pm(x)$ provide pointwise a splitting of the ambient space $\mathbb{C}^{2m}$ in the positive and negative energy sector. One of the consequences of (6.23) is that

$$\rho(x)^2 = \mathbb{1}_{2m}.$$  

(6.24)

With respect to the basis which provides the grading of $\Gamma$ in (6.22) one has that

$$\rho(x) = \begin{pmatrix} 0 & U(x) \\ U(x)^* & 0 \end{pmatrix}$$

(6.25)

where the anti-diagonal structure is given by the anti-commutation relation with $\Gamma$ and the presence of the adjoint operator $U(x)^*$ in the lower left corner is due to the self-adjointness of $\rho(x)$. Finally the constraint (6.24) implies that $U(x)^* = U(x)^{-1}$ which means that $x \mapsto U(x) \in \mathbb{U}(m)$ is a unitary-valued map. At this point one defines the topological phase of the system as

$$[U] \in [X, \mathbb{U}(m)]$$

namely as the homotopy class of the map $U : X \to \mathbb{U}(m)$. When $X = \mathbb{S}^d$ the phase is classified by $\pi_d(\mathbb{U}(m))$ and $[U]$ coincides with the degree (or winding number) of the map.

This derivation seems to be quite close to our arguments in Section (6.2). However there are two important differences which make the above approach less general than our point of view.

1) **Non-trivial Fermi projection** - Model of the type (6.20) with a fiber of finite dimension are usually introduced as effective operators which describe the physics related to a relevant energy regime (finite number of filled bands) of a realistic, generally unbounded, operator. Said differently $H(x)$ has to be interpreted as the restriction $P_\Omega(x)\hat{H}(x)P_\Omega(x)$ where $\hat{H}(x)$ is an operator living in a bigger, possibly infinite dimensional, Hilbert space and $P_\Omega(x)$ is the Fermi projection on the relevant set $\Omega = \Omega_+ \cup \Omega_-$ of the $m$ positive and $m$ negative energy bands. In effect, this is the precise mechanisms which allows to justify the use of tight-binding models as the diagonalization on the basis of the Wannier functions [Wa] of a full Hamiltonian restricted on a finite range of energies (see also [Te, ST]) for a more modern and general point of view. The hypothesis that $H(x)$ is an element of an $x$-independent space $\text{Mat}_{C}(2m)$ is equivalent to assume that the Fermi projection $P_\Omega(x)$ is constant or that it can be at least continuously deformed to a constant projection. This implies that topology of the Bloch-bundles $\hat{\partial}_\Omega$ associated to $P_\Omega(x)$ is trivial. This fact can also be derived from the pointwise splitting of the Fermi projection in its positive and negative energy parts $P_\Omega(x) = P_+(x) + P_-(x)$. Equations (6.24) and (6.23) implies that $1_{2m} = [P_+(x) - P_-(x)]^2 = P_\Omega(x)$. Moreover, $P_+(x)$ and $P_-(x)$ are related by a unitary transformation induced by the gradation $\Gamma$ hence they are topologically equivalent. Since their sum gives a trivial bundle they have to be singly trivial. In conclusion, the definition (6.20) introduces since from the beginning the “hidden” hypothesis that the system exhibits a collection of $m$ negative (or positive) energy bands with a total trivial Chern charge. This is a hypothesis which is in general false (see e.g. the case of the Harper operator in the limit of a strong magnetic field [DL]) and absolutely non necessary as showed in Section (6.2).

2) **Arbitrary choice of the eigenbasis of the gradation** - Even tough one accept the restrictions included in definition (6.20), the construction of the topological invariant for the chiral system based on the decomposition (6.25) turns out to be ambiguously defined. Indeed, the diagonal form of $\Gamma$ in (6.22)
is preserved by every (pointwise) reshuffling of the basis of the two eigenspaces \( \mathbb{C}^{2m} = \mathcal{H}_+(x) \oplus \mathcal{H}_-(x) \) of \( \Gamma \), namely by the action of any unitary-valued map of the form
\[
x \mapsto W(x) = \begin{pmatrix} W_1(x) & 0 \\ 0 & W_2(x) \end{pmatrix}, \quad W_1(x), W_2(x) \in U(m).
\]
Since
\[
\rho(x) \mapsto W(x) \rho(x) W(x)^* = \begin{pmatrix} 0 & W_1(x)U(x)W_2(x)^* \\ W_2(x)U(x)^*W_1(x)^* & 0 \end{pmatrix}
\]
and \([U] \neq [W_1 U W_2^*]\) it is evident that the definition of the topological phase is affected by the ambiguity in the choice of the pointwise basis of the eigenspaces of \( \Gamma \). Of course, one can notice that the ambiguity can be removed in the case the map \( W(x) = W \) is constant-valued a fact which is equivalent to affirm that the positive and negative eigenspaces of \( \Gamma \) can be fixed independently of the base points. However, this is possible only if \( \Gamma \) is a constant gradation acting on the trivial bundle \( X \times \mathbb{C}^{2m} \), and, as discussed in (1) above, this is a non-innocent restrictive assumption which excludes many interesting situations.

**The Karoubi-Thiang approach.** In a series of two recent brilliant papers [Th1, Th2] G. Thiang commented about some problematic aspects of the usually accepted definition of topological phases for topological insulators in class AIII. He recognized an incompatibility between the notions of isomorphism and homotopic equivalence by showing how certain diagonal unitary transforms can change the value of the winding number. He also noticed that the notion of phase can be well defined only in a relative sense. These kind of criticisms are in effect also contained in questions Q.1) - Q.3) that we posed (and we answered) in the introduction. In the approach by Thiang the topological phases of topological insulators in class AIII are defined as the \( K \)-theoretical classes of a \( K \)-theory introduced by M. Karoubi to classify gradations acting on abelian category endowed with a Clifford action [Kar, Chapter III, Section 4]. Element of this \( K \)-theory are equivalence classes \( [\Gamma_1, \Gamma_2] \) of gradations acting on an element \( \mathcal{E} \) of an abelian category (e.g. a complex vector bundle) endowed with the action \( \rho \) of a Clifford algebra and the equivalence is meant in the sense of homotopy deformations inside the space of gradations. The meaning reserved by Thiang to \([\Gamma_1, \Gamma_2]\) is that of a relative topological phase between the flattened Hamiltonians \( \Gamma_1 \) and \( \Gamma_2 \). Modulo a change of roles between gradations and Clifford actions the point of view by Thiang is very closed with our Definition 1.6. However, we notice that there is an evident “loss of information” in the classification scheme constructed by Thiang on the basis of the Karoubi’s \( K \)-theory. In fact, according to Karoubi-Thiang the triviality of the class \([\Gamma_1, \Gamma_2] = 0\) is equivalent to the homotopy \( \Gamma_1 \sim \Gamma_2 \). In this way one completely erases from the theory possible non-trivial topological aspects related to underlying vector bundle \( \mathcal{E} \). Our point of view, summarized in Definition 1.6, can be understood as a generalization of the Karoubi-Thiang theory in which one requires that \([\Gamma_1, \Gamma_2] \in K_0(\mathcal{E})\) whenever \( \Gamma_1 \sim \Gamma_2 \).

**The Atiyah-Hopkins-Witten \( K_\pm \)-theory.** In 1998, in relation to \( K \)-theoretic classification of D-brane charges, E. Witten introduced a variant of the \( K \)-theory, denoted by \( K_\pm \) [Wi2]. The geometric situation concerns a manifold \( X \) with an involution \( \tau \) having a fixed point submanifold \( X^\tau \). On \( X \) one wants to study a pair of complex vector bundles \((\mathcal{E}_+, \mathcal{E}_-)\) with the property that \( \tau \) is covered by a map \( q \) which interchanges them. In terms of the virtual vector bundle \( \mathcal{E}_+ - \mathcal{E}_- \), then \( q \) takes this into its negative, and \( K_\pm(X, \tau) \) is meant to be the appropriate \( K \)-theory of this situation. This twisted \( K \)-theory has been precisely defined and studied by M. Atiyah and M. Hopkins in a subsequent work [AH]. More recently some new aspects have been investigated by K. Gomi in [Go]. In the case of a trivial involution \( \tau = \text{Id}_X \) the map \( q \) can be described in terms of an isomorphism \( \Theta : \mathcal{E}_- \rightarrow \mathcal{E}_+ \) as in (6.3). This makes evident the link between the \( K_\pm \)-theory and the classification of chiral vector bundles. Following [AH] Section
2], the \( K_\ast \)-theory of the space \( X \) with trivial action verifies the exact sequence

\[
\begin{align*}
R(\mathbb{Z}_2) \otimes K^1(X) \xrightarrow{\phi} K^1(X) \xrightarrow{\delta=0} K^1_\ast(X)
\end{align*}
\]

\[
\begin{align*}
K^0_\ast(X) \xleftarrow{\delta=0} K^0(X) \xleftarrow{\phi} R(\mathbb{Z}_2) \otimes K^0(X)
\end{align*}
\]

where \( R(\mathbb{Z}_2) \simeq \mathbb{Z} \oplus \mathbb{Z} \) is the representation ring (over complex) of \( \mathbb{Z}_2 \) and it is generated by the trivial representation \( 1 : \mathbb{Z}_2 \to \{1\} \subset \mathbb{C} \) and the sign representation \( e : \mathbb{Z}_2 \to \{-1,+1\} \subset \mathbb{C} \). The two homomorphisms \( \phi \), which forget the \( R(\mathbb{Z}_2) \) component, are surjective with kernels \((1-e)\otimes K^\ast(X)\) and so it turns out that \( \delta = 0 \). This implies that \( K^1_\ast(X) \simeq \text{Ker} \phi = K^{j+1} \ast(X) \) where \( j \) is meant modulo 2. If from one side the isomorphism \( K^0_\ast(X) \simeq K^1(X) \) indicates that \( K^0 \ast \) contains the topological information about the automorphism group of vector bundles over \( X \), from the other side the \( \delta : K^0(X) \to \{0\} \subset K^0_\ast(X) \) shows that \( K^0_\ast(X) \) contains no kind of information about the topology of the underlying vector bundles themselves. Then, the description provided by the \( K_\ast \)-theory covers only partially the phenomenology of chiral systems as described in Section 6.2.

**Appendix A. Topology of unitary groups and Grassmann manifolds**

**Homotopy.** The complete determination of the homotopy groups of the unitary groups \( U(m) \) is still an open problem. Probably the most general result in this direction is the observation by R. Bott that the homotopy groups \( \pi_k(U(m)) \) stabilize when \( m \) becomes sufficiently large. More in detail, one has

\[
\pi_k(U(m)) = \begin{cases} 
0 & \text{if } k \text{ even or } 0, \ 2m > k \\
\mathbb{Z} & \text{if } k \text{ odd, } 2m > k \\
\mathbb{Z}_m! & \text{if } k = 2m
\end{cases}
\]

which is a result nowadays known as Bott periodicity \cite{Bot1, Bot2}. The condition \( m > 2k \) is called stable regime. The Bott periodicity provides a complete solution for the computation of the homotopy groups of the Palais unitary group \( U(\infty) \) which is the “infinite” unitary group

\[
U(\infty) := \bigcup_{m=1}^{\infty} U(m) ,
\]

obtained as the inductive limit of the inclusions \( U(m) \subset (n+1) \subset \ldots \) \cite{Pa}. Elements in this \( U(\infty) \) can be interpreted as infinite unitary matrices whose entries differ from the identity matrix in only finitely many places. The Bott periodicity immediately yields

\[
\pi_k(U(\infty)) = \begin{cases} 
0 & \text{if } k \text{ even or } 0 \\
\mathbb{Z} & \text{if } k \text{ odd}
\end{cases}
\]

The computation of the homotopy groups in the non stable regime \( 2m < k \) requires ad hoc techniques for each specific case and many values are already tabulated in the literature.

The fiber sequence \( SU(m) \to U(m) \to U(1) \) shows that \( U(m) \) can be seen as a principal \( SU(m) \)-bundle over \( U(1) \simeq S^1 \) with bundle projection given by the determinant. Since \( SU(m) \)-bundles, as well as \( U(m) \)-bundles, are classified by the same space \( G_m(\mathbb{C}^m) \) one has that \( SU(m) \)-bundles over \( S^1 \) are automatically trivial due to \([S^1,G_m(\mathbb{C}^m)] \simeq \pi_0(U(m)) = 0\). Then, by regarding \( U(m) \) as the total space of the \( SU(m) \)-bundle one gets the identification \( U(m) \simeq S^1 \times SU(m) \) and the isomorphisms

\[
\pi_k(U(m)) = \pi_k(S^1) \oplus \pi_k(SU(m)) \quad k \in \mathbb{N} \cup \{0\} .
\]

Equation (A.1), along with the computation of the homotopy groups \( \pi_1(S^1) = \mathbb{Z} \) and \( \pi_1(SU(m)) = 0 \) for all \( k \neq 1 \) \cite[Proposition 4.1]{Hat}, provides

\[
\pi_k(SU(m)) = \pi_k(U(m)) , \quad k \geq 2
\]

and

\[
\pi_1(U(m)) = \mathbb{Z} , \quad \pi_1(SU(m)) = 0 .
\]
Table A.1. First homotopy groups for the unitary groups. The entries enclosed in a box represent the values of the homotopy groups in the stable regime. Groups of the type \(\pi_{2n+1}(U(m))\) with \(r = 1, 2\) have been computed in [Ka]. A larger table with the related references can be found in [Lu].

Table A.1 can be completed with

\[
\pi_0(U(m)) = \pi_0(SU(m)) = 0 \quad (A.3)
\]

which simply means that the groups \(U(m)\) and \(SU(m)\) are path-connected.

In general, the fundamental group of a path-connected space \(X\) acts on its higher homotopy groups by automorphisms, i.e. \(\pi_1(X) \ni \gamma \mapsto \beta_\gamma \in \text{Aut}(\pi_k(X))\) (see [St, Section 16] or [BT, Warning 17.6] for a precise definition).

**Lemma A.1.** The action of \(\pi_1(U(m))\) (resp. \(\pi_1(\infty)\)) on \(\pi_k(U(m))\) (resp. \(\pi_k(\infty)\)) is trivial for all \(k \in \mathbb{N}\). In particular this implies the isomorphisms

\[
\pi_k(U(m)) \cong [S^k, U(m)], \quad \pi_k(U(\infty)) \cong [S^k, U(\infty)], \quad k \in \mathbb{N}
\]

which allow to neglect the role of base points in the computation of the homotopy groups.

**Proof.** The triviality of the action of \(\pi_1\) on higher-dimensional groups \(\pi_k\) is a classical result about the homotopy theory of topological groups [St, Theorem 16.9]. The last part of the claim is a consequence of the general isomorphism

\[
[S^k, X] \cong \pi_k(X)/\pi_1(X), \quad k \in \mathbb{N}
\]

which is valid for any path-connected space \(X\) [BT, Proposition 17.6.1].

The computation of the homotopy groups of the Grassmann manifold \(G_m(\mathbb{C}^\infty)\) can be reduced to the problem of the computation of the homotopy groups of \(U(m)\) by means of the fiber sequence associated to the universal classifying principal \(U(m)\)-bundle

\[
U(m) \to S^\infty_m \to G_m(\mathbb{C}^\infty) \quad (A.4)
\]

where \(S^\infty_m := U(n)/U(n-m)\) is the Stiefel variety and \(S^\infty_m := \bigcup_{n \geq 1} S^\infty_m\) is, as usual, the inductive limit. The universality of the space \(S^\infty_m\) implies its contractibility, i.e. \(\pi_k(S^\infty_m) = 0\) for all \(k\) [Hu, Chapter 8, Theorem 5.1]. This fact applied to the homotopy exact sequence induced by \((A.4)\) gives

\[
\pi_k(G_m(\mathbb{C}^\infty)) \cong \pi_{k-1}(U(m)), \quad k \in \mathbb{N}. \quad (A.5)
\]

The connectedness of the Grassmann manifold also implies \(\pi_0(G_m(\mathbb{C}^\infty)) = 0\).

**Cohomology.** The computation of the cohomology ring of the Grassmann manifold \(G_m(\mathbb{C}^\infty)\) is an extremely important result in the theory of (complex) vector bundles. It is well-known that

\[
H^*(G_m(\mathbb{C}^\infty), \mathbb{Z}) \cong \mathbb{Z}[\epsilon_1, \ldots, \epsilon_m], \quad \epsilon_k \in H^{2k}(G_m(\mathbb{C}^\infty), \mathbb{Z}) \quad (A.6)
\]

is the ring of polynomials with integer coefficients and \(m\) generators \(\epsilon_k\) of even degree [MS, Theorem 14.5]. These generators \(\epsilon_k\) are called *universal* Chern classes and there are no polynomial relationships
between them. By using the fact that isomorphism classes of rank \( m \) complex vector bundles over \( X \) are classified by maps \([\varphi] \in [X, \text{Gr}(m, \mathbb{C})]\), one defines the Chern classes of a given vector bundle \( \mathcal{E} \to X \) by the pullback of the universal Chern classes via the classifying map \( \varphi \). More precisely, one uses the induced homomorphisms in cohomology \( \varphi^* : H^k(\text{Gr}(m, \mathbb{C}), \mathbb{Z}) \to H^k(X, \mathbb{Z}) \) to define

\[
c_k(\mathcal{E}) := \varphi^*(c_k) \in H^{2k}(X, \mathbb{Z}) \quad k \in \mathbb{N}
\]

as the (topological) Chern classes of the vector bundle \( \mathcal{E} \). Since the homomorphism \( \varphi^* \) only depends on its homotopy class \([\varphi]\), isomorphic vector bundles possess the same family of Chern classes.

**Remark A.2** (Postnikov sections of the Grassmann manifold). The problem of the construction of the Postnikov tower for the spaces \( G_m(\mathbb{C}^\infty) \) has been firstly studied in \([\text{Pet}]\). With reference to the content of Section 5.4 let \( \alpha_j : G_m(\mathbb{C}^\infty) \to \mathcal{G}_m^j \) be the \( j \)-th Postnikov section of the Grassmann manifold \( G_m(\mathbb{C}^\infty) \). This section is obtained from the previous one \( \mathcal{G}_m^{j-1} \) according to the (principal) fibration sequences

\[
K(\pi_j, j) \longrightarrow \mathcal{G}_m^j \xrightarrow{p_j} \mathcal{G}_m^{j-1} \xrightarrow{\kappa_{j+1}} K(\pi_j, j+1)
\]

where \( p_j := \pi_j(\mathcal{G}_m^j) \) and \( \kappa_{j+1} \) define the related Postnikov invariant. Since \( \pi_j(\mathcal{G}_m^j) = 0 \) for \( j \) odd and \( j \leq 2m \) one has that \( \mathcal{G}_m^{2j-1} = \mathcal{G}_m^{2j-2} \) for \( j \leq m \). Thus, one has to compute \( \kappa_{2j+1} \in H^{2j+1}(\mathcal{G}_m^{2j-1},\pi_{2j}) \) for \( j \leq m \). Now \([\text{Pet}]\) Lemma 4.4 states that for \( j \leq m \) the invariant \( \kappa_{2j+1} \) has order \((j-1)!\). With this information we can immediately conclude that \( \mathcal{G}_m^m = \{\ast\} \) which implies \( \kappa_3 = 0 \) and so \( \mathcal{G}_m^m \simeq K(\mathbb{Z}, 2) \simeq \mathbb{C}P^\infty \) for all \( m \geq 1 \). For the determination of the next section one needs to compute \( \kappa_5 \in H^5(\mathbb{C}P^\infty, \pi_4) \simeq \mathbb{Z} \) for \( m \geq 2 \). However, according to \([\text{Pet}]\) Lemma 4.4 the invariant \( \kappa_5 \) needs to be of order 1, hence \( \kappa_5 = 0 \) when \( m \geq 2 \). In conclusion one has

\[
\mathcal{G}_m^m \simeq K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 4), \quad \forall \ m \geq 2.
\]

Lemma 4.1 in \([\text{Pet}]\) assures that \( H^k(\mathcal{G}_4^m, \mathbb{Z}) \simeq H^k(G_m(\mathbb{C}^\infty), \mathbb{Z}) \) for all \( k \leq 4 \) and with the help of the Künneth formula for cohomology and the explicit knowledge of the cohomology groups of \( K(\mathbb{Z}, 2) \) and \( K(\mathbb{Z}, 4) \) one obtains that for all \( m \geq 2 \)

\[
\begin{align*}
H^0(G_m(\mathbb{C}^\infty), \mathbb{Z}) &\simeq H^0(K(\mathbb{Z}, 2), \mathbb{Z}) \otimes \mathbb{Z} H^0(K(\mathbb{Z}, 4), \mathbb{Z}) \simeq \mathbb{Z} \\
H^1(G_m(\mathbb{C}^\infty), \mathbb{Z}) &\simeq 0 \\
H^2(G_m(\mathbb{C}^\infty), \mathbb{Z}) &\simeq H^2(K(\mathbb{Z}, 2), \mathbb{Z}) \simeq \mathbb{Z} \\
H^3(G_m(\mathbb{C}^\infty), \mathbb{Z}) &\simeq H^3(K(\mathbb{Z}, 2), \mathbb{Z}) \simeq 0 \\
H^4(G_m(\mathbb{C}^\infty), \mathbb{Z}) &\simeq H^4(K(\mathbb{Z}, 2), \mathbb{Z}) \oplus H^4(K(\mathbb{Z}, 4), \mathbb{Z}) \simeq \mathbb{Z}^2.
\end{align*}
\]

The above computations show that the first two universal Chern classes \( c_1 \) and \( c_2 \) can be identified with the basic class of \( H^2(K(\mathbb{Z}, 2), \mathbb{Z}) \) and \( H^4(K(\mathbb{Z}, 4), \mathbb{Z}) \), respectively (for a definition of basic class see e. g. \([\text{Ark}]\) Definition 5.3.1)).

Also the cohomology ring of the unitary group \( U(m) \) is well-known. It is a classical result that

\[
H^*(U(m), \mathbb{Z}) \simeq \bigwedge_{\mathbb{Z}} [w_1, \ldots, w_m], \quad w_k \in H^{2k-1}(U(m), \mathbb{Z})
\]

is the exterior algebra generated by \( m \) odd-degree classes \( w_k \) \([\text{Bor}]\) Théorème 19.1 or \([\text{Bor}]\) Section 10). By adhering to a modern terminology (see e. g. \([\text{PS}]\)) we will refer to the \( w_k \)’s as the universal odd Chern classes. The description \((A.9)\) can be generalized to the infinite unitary group \( U(\infty) \).

**Lemma A.3.** The cohomology ring of the infinite unitary group \( U(\infty) \)

\[
H^*(U(\infty), \mathbb{Z}) \simeq \bigwedge_{\mathbb{Z}} [w_1, w_2, \ldots], \quad w_k \in H^{2k-1}(U(\infty), \mathbb{Z}), \quad k \in \mathbb{N}
\]

is the exterior algebra generated by a countable family of odd-degree classes \( w_k \).
The Milnor exact sequence \([\text{Mi}, \text{Lemma } 2]\) states that

\[
\text{Notice that each } i_i \text{.}
\]

The relation between the inverse system \((A.11)\) and the cohomology \(H^*(\mathbb{U}(\infty), \mathbb{Z})\) is specified by the \textit{Milnor exact sequence}. Let us define a cochain complex \((C^*, \delta)\) by

\[
C^k := \left\{ \prod_j H^*(\mathbb{U}(j), \mathbb{Z}) \middle| \begin{align*}
& j = 0, 1 \\
& j \geq 2
\end{align*} \right\}
\]

where \(\prod\) is the direct product of abelian groups and the (only non-trivial) differential \(\delta : C^0 \to C^1\) is defined by

\[
\delta(a_1, a_2, a_3, \ldots) := (a_1 - i_1'(a_2), a_2 - i_2'(a_3), a_3 - i_3'(a_4), \ldots), \quad a_j \in H^*(\mathbb{U}(j), \mathbb{Z}).
\]

By definition the inverse limit \((A.11)\) arises as the 0-th cohomology group of the cochain complex \((C^*, \delta)\), i.e.,

\[
\lim_{\leftarrow} H^*(\mathbb{U}(j), \mathbb{Z}) := H^0(C^*, \delta) = \text{Ker}(\delta).
\]

The cokernel of \(\delta\) is usually called “limit 1”:

\[
\lim_{\leftarrow} H^*(\mathbb{U}(j), \mathbb{Z}) := H^1(C^*, \delta) = C^1 / \text{Im}(\delta).
\]

The Milnor exact sequence \([\text{Mi}, \text{Lemma } 2]\) states that

\[
0 \to \lim_{\leftarrow} H^*(\mathbb{U}(j), \mathbb{Z}) \to H^*(\mathbb{U}(\infty), \mathbb{Z}) \to \lim_{\leftarrow} H^*(\mathbb{U}(j), \mathbb{Z}) \to 0.
\]

Notice that each \(i_j^*: H^*(\mathbb{U}(j + 1), \mathbb{Z}) \to H^*(\mathbb{U}(j), \mathbb{Z})\) is surjective since \(i_j^*(w_k) = w_k\) for all \(k = 1, \ldots, j\) and \(i_j^*(w_{j+1}) = 0\). Then, a simple argument shows that also the differential \(\delta : C^0 \to C^1\) is surjective and so the “limit 1” is trivial. This implies that \(H^*(\mathbb{U}(\infty), \mathbb{Z}) \simeq \text{Ker}(\delta)\) and the kernel \(\text{Ker}(\delta)\) is generated by elements of the form \((w_k, w_k, w_k, \ldots)\) which can be identified with the generators \(w_k\) for all \(k \in \mathbb{N}\). This concludes the proof.

\[\text{Remark A.4} (\text{Postnikov sections of the infinite unitary group}). \] The Postnikov resolution of the infinite unitary group \(\mathbb{U}(\infty)\) can be used to provide a different description of the universal odd Chern classes \(w_k\), at least in low dimension. With reference to the technological apparatus described in Section \(5.4\) we want to compute the Postnikov sections \(a_j : \mathbb{U}(\infty) \to \mathbb{U}^\infty_j\). From \(\pi_1(\mathbb{U}(\infty)) = \mathbb{Z}\) we immediately get \(\mathbb{U}^\infty_1 = K(\mathbb{Z}, 1) \simeq S^1\). Since we know that \(\pi_2(\mathbb{U}(\infty)) = 0\) and by using the fact that \(K(0, j) = \ast\) (along with \([\text{Hat}, \text{Corollary } 4.63]\)) we obtain from \((5.7)\) that \(\mathbb{U}^\infty_j \simeq \mathbb{U}^\infty_{j-1}\). Hence, we need to compute only the odd sections. For the computation of \(\mathbb{U}^\infty_3\) we need the knowledge of the Postnikov invariant \(\kappa^4\) in the fiber sequence

\[
K(\mathbb{Z}, 3) \to \mathbb{U}^\infty_3 \overset{p_4}{\to} \mathbb{U}^\infty_2 \simeq \mathbb{U}^\infty_1 \overset{\kappa^4}{\to} K(\mathbb{Z}, 4).
\]

From its very definition we know that \(\kappa^4 \in H^4(\mathbb{U}^\infty_2, \mathbb{Z}) = H^4(S^1, \mathbb{Z}) = 0\). The vanishing of \(\kappa^4\) immediately yields

\[
\mathbb{U}^\infty_4 \simeq \mathbb{U}^\infty_3 \simeq K(\mathbb{Z}, 1) \times K(\mathbb{Z}, 3).
\]

The next step requires the computation of the Postnikov invariant \(\kappa^5\) in the fiber sequence

\[
K(\mathbb{Z}, 5) \to \mathbb{U}^\infty_5 \overset{p_5}{\to} \mathbb{U}^\infty_4 \simeq \mathbb{U}^\infty_3 \overset{\kappa^5}{\to} K(\mathbb{Z}, 6).
\]
In this case the Postnikov invariant is an element of $\kappa^6 \in H^6(\mathcal{U}^\infty_4, \mathbb{Z}) \simeq H^6(S^1 \times K(\mathbb{Z}, 3), \mathbb{Z})$. The knowledge of the non-trivial cohomology $H^k(S^1, \mathbb{Z}) \simeq \mathbb{Z}$ if $k = 0, 1$ and the use of the K"unneth formula for cohomology provide

$$H^k(\mathcal{U}_4^\infty, \mathbb{Z}) = \left( \mathbb{Z} \otimes_{\mathbb{Z}} H^0(K(\mathbb{Z}, 3), \mathbb{Z}) \right) \oplus \left( \mathbb{Z} \otimes_{\mathbb{Z}} H^2(K(\mathbb{Z}, 3), \mathbb{Z}) \right) \simeq \mathbb{Z} \oplus \mathbb{Z} \simeq \mathbb{Z}_2 .$$

where we used the explicit results $H^2(K(\mathbb{Z}, 3), \mathbb{Z}) = 0$ and $H^6(K(\mathbb{Z}, 3), \mathbb{Z}) \simeq \mathbb{Z}_2$ computed in [BT Section 18]. This is compatible with the more general result [AB Lemma 5] which states that $\kappa^6$ is a (non-trivial) element of order 2. This implies that

$$\mathcal{U}_6^\infty \simeq \mathcal{U}_5^\infty \simeq \left( K(\mathbb{Z}, 1) \times K(\mathbb{Z}, 3) \right) \times_{\kappa^6} K(\mathbb{Z}, 5) .$$

where the last product is “twisted” by the action of the non-trivial class $\kappa^6$. Evidently the construction becomes more and more involved when the degree of the Postnikov section increases. Let us consider now the implication of (A.14) for the interpretation of the first two generators of $U(\infty)$. By construction the map $\alpha_4 : U(\infty) \to \mathcal{U}_4^\infty$ is a 5-equivalence, hence the Whitehead’s second theorem\(^5\) assures that $H^k(U(\infty), \mathbb{Z}) \simeq H^k(\mathcal{U}_4^\infty, \mathbb{Z})$ for all $k \leq 4$. With the help of the K"unneth formula for cohomology and the explicit knowledge of cohomology of $K(\mathbb{Z}, 1)$ and $K(\mathbb{Z}, 3)$ one computes

$$H^0(U(\infty), \mathbb{Z}) \simeq H^0(K(\mathbb{Z}, 1), \mathbb{Z}) \otimes_{\mathbb{Z}} H^0(K(\mathbb{Z}, 3), \mathbb{Z}) \simeq \mathbb{Z}$$

$$H^1(U(\infty), \mathbb{Z}) \simeq H^1(K(\mathbb{Z}, 1), \mathbb{Z}) \otimes_{\mathbb{Z}} H^0(K(\mathbb{Z}, 3), \mathbb{Z}) \simeq H^1(K(\mathbb{Z}, 1), \mathbb{Z}) \simeq \mathbb{Z}$$

$$H^2(U(\infty), \mathbb{Z}) \simeq 0$$

$$H^3(U(\infty), \mathbb{Z}) \simeq H^0(K(\mathbb{Z}, 1), \mathbb{Z}) \otimes_{\mathbb{Z}} H^3(K(\mathbb{Z}, 3), \mathbb{Z}) \simeq H^3(K(\mathbb{Z}, 3), \mathbb{Z}) \simeq \mathbb{Z}$$

$$H^4(U(\infty), \mathbb{Z}) \simeq H^1(K(\mathbb{Z}, 1), \mathbb{Z}) \otimes_{\mathbb{Z}} H^3(K(\mathbb{Z}, 3), \mathbb{Z}) \simeq \mathbb{Z} .$$

The above result shows that $w_1$ and $w_2$ can be identified with the basic class of $H^1(K(\mathbb{Z}, 1), \mathbb{Z})$ and $H^3(K(\mathbb{Z}, 3), \mathbb{Z})$, respectively (for a definition of basic class see e. g. [Ark, Definition 5.3.1]). As a final comment we can observe that (A.14) also describes the 3-rd and 4-th Postnikov sections of $U(m)$ for all $m \geq 2$ while (A.16) describes the 5-rd and 6-th Postnikov sections of $U(m)$ for all $m \geq 3$ (stable regime).

### Appendix B. Fiber bundle identification of the classifying space

The definition (3.2) says that the spaces $X_m(C^n)$ are subspaces of $G_m(C^n \times U(n))$. These spaces can be also identified with the total spaces of suitable fiber bundles. For this aim, let us recall a standard construction: For any Lie group $G$ and any principal $G$-bundle $\pi : \mathcal{P} \to X$ we use the adjoint action of $G$ on $G$ to get the associated fiber bundle

$$Ad(\mathcal{P}) := \mathcal{P} \times_{Ad} G = (\mathcal{P} \times G)/G$$

where $g \in G$ acts on $(p, h) \in \mathcal{P} \times G$ by $(p, h) \mapsto (p \cdot g^{-1}, g \cdot h \cdot g^{-1})$. It is a well-known fact that sections of $Ad(\mathcal{P}) \to X$ are in one to one correspondence with automorphisms of $\mathcal{P}$ [Hu, Chapter 7, Section 1].

Let us also recall that the Grassmann manifold and its tautological $U(m)$-frame bundle (Stiefel variety) $\mathcal{S}_m^n \to G_m(C^n)$ can be realized as quotient spaces:

$$G_m(C^n) = U(n)/(U(m) \times U(n-m)) , \quad \mathcal{S}_m^n \simeq U(n)/U(n-m) , \quad \text{for } \mathcal{S}_m^n \simeq U(n)/U(n-m) , \quad \text{for } \mathcal{S}_m^n \simeq U(n)/U(n-m) , \quad \text{B.1}$$

where $U(m)$ and $U(n-m)$ act (on the right) on $U(n)$ through the inclusions of $C^n$ and $C^{n-m}$ into $C^n = C^m \oplus C^{n-m}$. Let $\Sigma \in G_m(C^n)$ be a subspace of $C^n$ of dimension $m$. Given orthonormal basis

---

\(^5\) More precisely the Whitehead’s second theorem [Ark Theorem 6.4.15] (see also [Sp Chapter I, Section 8, Theorem 9]) says that $\alpha_4$ induces a 5-homotopy equivalence (see [Ark, Definition 6.4.10]). At this point the proof that a $n$-homotopy equivalence implies a $n$-cohomology equivalence is provided by the use of the universal coefficient theorem as in [Pet, Lemma 4.1].
$\nu := (v_1, \ldots, v_m)$ and $\omega := (w_1, \ldots, w_{n-m})$ of $\Sigma \subset \mathbb{C}^n$ and $\Sigma^\perp \subset \mathbb{C}^n$, respectively one can form an element $(\nu, \omega) \in \mathbb{U}(n)$ by arraying the vectors as columns, i.e.

$$
(\nu, \omega) := \begin{pmatrix}
v_1 & \cdots & v_m & w_1 & \cdots & w_{n-m} \\
\end{pmatrix}.
$$

(B.2)

According to this notation $\nu$ and $\omega$ can be considered an $n \times m$ and an $n \times (m-n)$ matrices, respectively. The pair $(\nu, \omega)$ provides a representative for $([\nu], [\omega]) = \Sigma \in G_m(\mathbb{C}^n)$ according to the quotient description of the Grassmann manifold in (B.1). Here, the equivalence relations are naturally given by $\nu \sim \nu \cdot a$ and $\omega \sim \omega \cdot b$ for some $a \in \mathbb{U}(m)$ and $b \in \mathbb{U}(n-m)$. Similarly, we can use $(\nu, [\omega]) \in \mathcal{S}_m$ for a point in the Stiefel variety according to the quotient representation in (B.1) and an element $c \in \mathbb{U}(m)$ in the structure group can be represented by a block diagonal matrix

$$
\begin{pmatrix}
c & 0 \\
0 & I_{n-m}
\end{pmatrix} \subset \mathbb{U}(n).
$$

In this way points in $\text{Ad}(\mathcal{S}_m)$ are given by equivalence classes $[([\nu], \omega), c]$ with respect to the equivalence relation $([\nu, \omega \cdot b], c) \sim ([\nu \cdot a^{-1}, \omega], a \cdot c \cdot a^{-1})$ for some $a \in \mathbb{U}(m)$ and $b \in \mathbb{U}(n-m)$.

The core of this appendix is the proof of the following identifications:

**Proposition B.1.** There are natural homeomorphisms

$$
\chi_m(\mathbb{C}^n) \simeq \text{Ad}(\mathcal{S}_m), \quad \text{and} \quad \mathbb{B}_x^m \simeq \text{Ad}(\mathcal{S}_m),
$$

where, as usual, $\mathcal{S}_m$ denotes the inductive limit obtained by the inclusions $\mathcal{S}_m \subset \mathcal{S}_{m+1} \subset \ldots$

However, we start first with a technical preliminary result.

**Lemma B.2.** For each pair of integers $1 \leq m \leq n$ there is a bijective continuous map

$$
\vartheta : \text{Ad}(\mathcal{S}_m) \longrightarrow \chi_m(\mathbb{C}^n).
$$

(B.3)

**Proof.** Given an element $(\nu, \omega) \in \mathbb{U}(n)$ as in (B.2) and a $c \in \mathbb{U}(m)$ there is a unique unitary matrix $u(\nu; \omega; c) \in \mathbb{U}(n)$ such that $u(\nu; \omega; c) \cdot (\nu, \omega) = (\nu \cdot c, \omega)$. Such a matrix is explicitly given by

$$
u(\nu; \omega; c) := (\nu, \omega) \begin{pmatrix}
c & 0 \\
0 & I_{n-m}
\end{pmatrix} \cdot (\nu, \omega)^{-1}.
$$

By construction $u(\nu; \omega; c)$ preserves the $m$-dimensional subspace $([\nu], [\omega]) = \Sigma \subset \mathbb{C}^n$ spanned by the columns of $\nu$. Therefore, the pair $(\Sigma, u) := (([\nu], [\omega]), u(\nu; \omega; c))$ provides a point in $\chi_m(\mathbb{C}^n)$. This allows to define the map

$$
\vartheta : \mathcal{S}_m \times \mathbb{U}(m) \longrightarrow \chi_m(\mathbb{C}^n)
$$

$$
((\nu, [\omega]), c) \longmapsto (\Sigma, u).
$$

(B.4)

This map factors through the equivalence relation that defines $\text{Ad}(\mathcal{S}_m)$ since $([\nu], [\omega]) = ([\nu \cdot a], [\omega]) = \Sigma$ and

$$
u(\nu \cdot a; \omega; a \cdot c \cdot a^{-1}) = u(\nu; \omega; c) = u(\nu; \omega \cdot b; c), \quad \forall \ a \in \mathbb{U}(m), \ b \in \mathbb{U}(n-m)
$$

Hence, the prescription (B.4) defines a map like (B.3). The map $\vartheta$ is also continuous. Indeed, the topology of $\text{Ad}(\mathcal{S}_m)$ is induced from $\mathbb{U}(n) \times \mathbb{U}(m)$ by a quotient, and that of $\chi_m(\mathbb{C}^n)$ is induced from $G_m(\mathbb{C}^n) \times \mathbb{U}(n)$ by an inclusion. Thus, to prove that $\vartheta$ is continuous, it suffices to prove that the composition of the following maps is continuous:

$$
\begin{array}{c}
\mathbb{U}(n) \times \mathbb{U}(m) & \xrightarrow{\mathbb{P}} & G_m(\mathbb{C}^n) \times \mathbb{U}(n) \\
\text{pr} & & \downarrow \\
\text{Ad}(\mathcal{S}_m) & \xrightarrow{\vartheta} & \chi_m(\mathbb{C}^n)
\end{array}
$$

(B.5)
The natural projection $\text{pr}$ and the inclusion $i$ are continuous by construction. Moreover, the composition of the three maps is easily computed to be $((\nu, [w]), c) \mapsto (\langle \nu, [w] \rangle, u(\nu; w; c))$ which is evidently continuous. This, in turn, implies the continuity of $\vartheta$.

In order to prove that $\vartheta$ is bijective we construct its inverse $\varrho$. Let us start with a point $(\Sigma, u) \in \chi_m(C^n(\mathbb{C}))$ and choose orthonormal basis $\nu$ and $w$ such that $\langle [\nu], [w] \rangle = \Sigma$ as above. Since $u \in \mathbb{U}(n)$ preserves $\Sigma$ there is a unique $c(\nu; u) \in \mathbb{U}(m)$ such that $u \cdot \nu = \nu \cdot c(\nu; u)$. Such a matrix is explicitly given by

$$c(\nu; u) := \langle \nu \cdot c(\nu; u) \rangle$$

where $\langle \cdot \rangle$ denotes the transpose of the matrix $\langle \cdot \rangle$, complex conjugated of $\nu$. Now we define

$$\varrho : \chi_m(C^n(\mathbb{C})) \to \text{Ad}(\mathcal{S}_m)$$

$$(\Sigma, u) \mapsto (\langle \nu, [w] \rangle, c(\nu; u)),$$ \hspace{1cm} (B.6)

where $(\nu, [w]) \in \mathbb{U}(n)/\mathbb{U}(n-m)$ identifies an element of $\mathcal{S}_m$. The map $\varrho$ does not depend on the choice of a frame $w$ for $\Sigma^\perp$ (as denoted by the square brackets). Moreover, if $\nu' = \nu \cdot a$ with $a \in \mathbb{U}(m)$ is a new frame for $\Sigma$, a direct computation shows $\varrho(\nu'; u) = a^{-1} \cdot c(\nu; u) \cdot a$. These two facts prove that $\varrho$ really maps into $\text{Ad}(\mathcal{S}_m)$. The proof that $\varrho = \vartheta^{-1}$ follows now from a direct, as well trivial, verification. \hfill \blacksquare

**Proof of Proposition [B.7]** Lemma [B.3] proves the existence of a continuous bijection $\vartheta : \text{Ad}(\mathcal{S}_m^n) \to \chi_m(C^n(\mathbb{C}))$. Since $\text{Ad}(\mathcal{S}_m^n)$ is compact (it is the quotient of a compact group) and $\chi_m(C^n(\mathbb{C}))$ is Hausdorff (it is the subspace of a Hausdorff space), $\vartheta$ turns out to be a homeomorphism (see e.g. [Ja] Chapter I, Section 8). This homeomorphism is compatible with taking the direct limits, so that it induces a homeomorphism $\vartheta : \text{Ad}(\mathcal{S}_\infty) \to \mathbb{B}_m^\infty$. \hfill \blacksquare

From the construction of the identifications in Proposition [B.1] it follows that the mapping $\chi_m(C^n(\mathbb{C})) \to G_m(C^n(\mathbb{C}))$ in (3.4) and $\mathbb{B}_m^\infty \to G_m(C^n(\mathbb{C}))$ in (3.7) agree with the bundle maps $\text{Ad}(\mathcal{S}_m^n) \to G_m(C^n)$ and $\text{Ad}(\mathcal{S}_\infty) \to G_m(C^n)$, respectively. Summarizing, one has the fiber sequence

$$\mathbb{U}(m) \to \text{Ad}(\mathcal{S}_m^n) \simeq \mathbb{B}_m^\infty \xrightarrow{\pi} G_m(C^n)$$

with projection $\pi$ given by (3.7).

When $m = 1$, just by exploiting the fact that $\mathbb{U}(1)$ is abelian, one gets the following immediate consequence of Proposition [B.1].

**Corollary B.3.**

$$\mathbb{B}_1^\infty \simeq \text{Ad}(\mathcal{S}_1^n) \simeq \mathbb{C}P^\infty \times \mathbb{U}(1).$$

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