A multiplicity result for the problem  
\[ \delta d\xi = f'(\langle \xi, \xi \rangle)\xi \]

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Abstract

In this paper we consider the nonlinear equation involving differential forms on a compact Riemannian manifold \( \delta d\xi = f'(\langle \xi, \xi \rangle)\xi \). This equation is a generalization of the semilinear Maxwell equations recently introduced in a paper by Benci and Fortunato. We obtain a multiplicity result both in the positive mass case (i.e. \( f'(t) \geq \varepsilon > 0 \) uniformly) and in the zero mass case (\( f'(t) \geq 0 \) and \( f'(0) = 0 \)) where a strong convexity hypothesis on the nonlinearity is assumed.

Keywords  
Semilinear Maxwell equations; Strongly indefinite functional; Strong convexity

Introduction

Let \((M, g)\) be a compact Riemannian \(n\)-manifold, where \(n \geq 3\), and \(\Lambda^k(M)\) be the set of regular \(k\)-forms on \(M\). We consider the following equation

\[
\begin{aligned}
\delta d\xi &= f'(\langle \xi, \xi \rangle)\xi, \\
\xi &\in \Lambda^k(M), \quad 1 \leq k \leq n - 1,
\end{aligned}
\]

where \(f : \mathbb{R} \to \mathbb{R}\) is a \(C^2\) map, \(d\) is the exterior differential, \(\delta\) is its adjoint with respect to the inner product

\[
(\eta, \xi)_2 = \int_M \langle \eta, \xi \rangle \omega = \int_M (\eta \wedge \ast \xi) \omega
\]

1
* is the Hodge operator and ω is a volume n-form.

In this paper we are looking for weak solutions of (1), namely for solutions

\[
\begin{cases}
\xi \in H^1_k(M), \text{(see section 1.2 for the definition)} \\
\int_M \langle d\xi, d\eta \rangle \omega = \int_M f' \langle \langle \xi, \xi \rangle \rangle \langle \xi, \eta \rangle \omega, \quad \forall \eta \in H^1_k(M).
\end{cases}
\]

If we set

\[F(\xi) := \int_M f(\langle \xi, \xi \rangle) \omega,\]

then, assuming a suitable condition on the growth of \(f'\), by standard arguments we have that \(F \in C^1(H^1_k(M))\), so in order to solve (3) we find critical points of the functional

\[J(\xi) = \int_M \langle d\xi, d\xi \rangle \omega - F(\xi)\]

defined for all \(\xi \in H^1_k(M)\).

The strongly indefinite nature of the functional \(J\), largely discussed in [4], doesn’t allow us to approach this problem in a standard way. In other words, the functional \(J\) doesn’t present the geometry of the mountain pass in any space with finite codimension.

Assume that

1. \(f(0) = 0, \) and \(\exists \varepsilon > 0 \) s.t. \(\forall t \geq 0 : f'(t) \geq \varepsilon,\)
2. \(f\) is strictly convex,

and for \(p \in ]2, \frac{2n}{n-2}[\)

3. \(\exists a > 0, b > 0 \) s.t. \(|f'(t)| \leq a t^{\frac{p}{2}-1} + b, \forall t \geq 0,\)
4. \(\exists R > 0 \) s.t. \(0 < \frac{p}{2} f(t) \leq f'(t)t\) for \(t > R.\)

We have the following result

**Theorem 1.** If \(f_1)\ldots-f_4)\) hold, then the problem (3) has infinitely many solutions.

Moreover the same conclusion holds if \(f_1\) and \(f_2\) are substituted respectively by

\[\tilde{f}_1) \ f(0) = f'(0) = 0, \) and \(\forall t \geq 0 : f'(t) \geq 0,\]
\( \tilde{f}_2 \) \( \exists c > 0 \) s.t. \( \forall \xi, \eta \in \Lambda^k(M) \)

\[
f(\langle \xi, \xi \rangle) - f(\langle \eta, \eta \rangle) - 2f'(\langle \eta, \eta \rangle)\langle \eta, \xi - \eta \rangle \geq c\langle \xi - \eta, \xi - \eta \rangle^p
\]

pointwise in \( M \).

**Remark 2.** As in [5], in the sequel we shall refer to the hypotheses \( f_1 \) and \( \tilde{f}_1 \) respectively as the “positive mass” and “zero mass” case.

Moreover we want to point out the fact that \( \tilde{f}_2 \) is just a pointwise convexity condition. In fact for every \( q \in M \) we can define the scalar product \( \langle \cdot , \cdot \rangle_q \) on the vector space \( \Lambda^k(M) \) and the functional

\[
I_q(\xi) = f(\langle \xi, \xi \rangle_q).
\]

Since \( I_q'(\eta) = 2f'(\langle \eta, \eta \rangle)\langle \eta, \cdot \rangle \), \( \tilde{f}_2 \) implies that for all \( \xi, \eta \in \Lambda^k(M) \) s.t. \( \eta \neq \xi \)

\[
I_q(\xi) - I_q(\eta) - \langle I_q'(\eta), \xi - \eta \rangle > 0,
\]

and then \( I_q \) is strictly convex. In the Appendix we shall show that \( \tilde{f}_2 \) is satisfied when \( f(t) = t^p \).

In order to prove Theorem 1, as in [4] we shall use Hodge decomposition to split \( \xi \) in this way

\[
\xi = d\alpha + \beta \tag{6}
\]

where \( \delta\beta = 0 \).

The functional (5) formally becomes

\[
J(\alpha, \beta) = \int_M \langle d\beta, d\beta \rangle \omega - \int_M f(\langle d\alpha + \beta, d\alpha + \beta \rangle) \omega. \tag{7}
\]

If we set

\[
J_\alpha : \beta \mapsto J_\alpha(\beta) = J(\alpha, \beta) \tag{8}
\]

and

\[
J_\beta : \alpha \mapsto J_\beta(\alpha) = J(\alpha, \beta), \tag{9}
\]

then we can define the partial derivative of \( J \) as follows

\[
\frac{\partial J}{\partial \alpha}(\alpha, \beta) : = dJ_\beta(\alpha) \tag{10}
\]

\[
\frac{\partial J}{\partial \beta}(\alpha, \beta) : = dJ_\alpha(\beta). \tag{11}
\]
We are interested in finding critical points of (7), i.e. the couples \((\alpha, \beta)\) such that

\[
\frac{\partial J}{\partial \alpha}(\alpha, \beta) = 0 \quad (12)
\]

and

\[
\frac{\partial J}{\partial \beta}(\alpha, \beta) = 0. \quad (13)
\]

Set

\[
F_{\beta} := \alpha \mapsto F(d\alpha + \beta) \quad (14)
\]

and note that by the convex nature of (14), the problem (12) is actually a minimizing problem.

In section 1, where we assume \(f_1 \ldots f_4\), we introduce some preliminary results and the definition of the spaces \(V\) and \(W\) which respectively \(\alpha\) and \(\beta\) belong to. These spaces are constructed in such a way we have, for any \(\beta \in W\), a unique solution \(\Phi(\beta) \in V\) for the minimizing problem (12).

Then in Theorem 5 we’ll show that in order to solve the system (12) and (13) we are reduced to study the critical points of the functional

\[
\hat{J}(\beta) = J(\Phi(\beta), \beta), \quad \beta \in W. \quad (15)
\]

Differently from \(J\), \(\hat{J}\) doesn’t exhibit strong indefiniteness, so the proof of the existence of infinitely many critical points is carried out by using a well known multiplicity result for even functionals.

In section 2 we replace \(f_1\) and \(f_2\) by \(\tilde{f}_1\) and \(\tilde{f}_2\). Differently from the previous situation, we can’t give any proof about the regularity of the functional (15). To overcome this difficulty we work in an indirect way. In fact we perturb the problem (11) by adding a linear “mass term” \(m\xi, m > 0\), to the nonlinearity on the right hand side. For this perturbed problem we have infinitely many solutions since it satisfies the hypothesis \(f_1\).

Then we study the behaviour of the solutions of the perturbed problem when the perturbation goes to zero.
1 Positive mass case

1.1 The functional framework

For any $q > 1$ and $k \in \mathbb{N}$, let $H_{k}^{1,q}(M)$, $H_{k}^{1}(M)$ and $L_{k}^{q}(M)$ be defined as follows

\begin{align*}
H_{k}^{1,q}(M) &:= \Lambda^{k}(M)^{1,q}, \\
H_{k}^{1}(M) &:= \Lambda^{k}(M)^{1}, \\
L_{k}^{q}(M) &:= \Lambda^{k}(M)^{1,q},
\end{align*}

where, for every $\xi \in \Lambda^{k}(M),$

\begin{align*}
\|\xi\|_{1,q}^{q} &:= \int_{M} \langle d\xi, d\xi \rangle^{q/2} \omega + \int_{M} \langle \delta\xi, \delta\xi \rangle^{q/2} \omega + \int_{M} \langle \xi, \xi \rangle^{q/2} \omega \\
\|\xi\|^{2} &:= \int_{M} \langle d\xi, d\xi \rangle \omega + \int_{M} \langle \delta\xi, \delta\xi \rangle \omega + \int_{M} \langle \xi, \xi \rangle \omega \quad (16) \\
|\xi|^{q} &:= \int_{M} \langle \xi, \xi \rangle^{q/2} \omega.
\end{align*}

Since

\begin{equation}
H_{k}^{1}(M) \hookrightarrow L_{k}^{q}(M) \quad \text{for } 1 \leq q \leq \frac{2n}{n-2}, \tag{17}
\end{equation}

by $f_3$ we have that $J(\alpha, \beta) < +\infty$ for $\alpha \in H_{k-1}^{1,p}(M)$ and $\beta \in H_{k}^{1}(M)$.

Observe that $F_{\beta} : H_{k-1}^{1,p}(M) \to \mathbb{R}$ defined by (14) is not coercive, since it is constant on the space

\begin{equation}
C := \{ \eta \in H_{k-1}^{1,p}(M) \mid d\eta = 0 \}. \tag{18}
\end{equation}

However we have the following result

**Lemma 3.** For all $\beta \in L_{k}^{p}(M)$, $F_{\beta}$ is coercive on

\begin{equation}
\mathcal{V} := \left\{ \alpha \in H_{k-1}^{1,p}(M) \mid \forall \eta \in C : \int_{M} \langle \alpha, \eta \rangle \omega = 0 \right\}. \tag{19}
\end{equation}

**Proof.** First observe that, by $f_4$,

\begin{equation}
\exists c > 0, d > 0 \text{ s.t. } ct^{p} \leq f(t) + d, \forall t \geq 0. \tag{20}
\end{equation}

Let $\beta \in L_{k}^{p}(M)$. By Lemma 6 in [2], we know that the norm on $H_{k}^{1}(M)$ defined by

\begin{equation}
\|\xi\|_{2}^{2} := \int_{M} \left( \langle d\xi, d\xi \rangle + \langle \delta\xi, \delta\xi \rangle + \langle \xi^{0}, \xi^{0} \rangle \right) \omega \tag{21}
\end{equation}

and
where $\xi^0$ is the orthogonal projection of $\xi$ on $\ker(-\Delta)$, is equivalent to the norm defined by \ref{16}.
In particular, in the space $\mathcal{V}$ the norm \ref{21} becomes
\[
\|\xi\|^2 = \int_M \langle d\xi, d\xi \rangle \omega.
\]
Indeed, if $\alpha \in \mathcal{V}$, then
\[
\int_M \langle \delta \alpha, \delta \alpha \rangle \omega = \int_M \langle d\delta \alpha, \alpha \rangle \omega = 0 \tag{22}
\]
because $d\delta \alpha \in \mathcal{C}$.
Moreover, since $\alpha^0 \in \ker(-\Delta)$, then $\alpha^0 \in \mathcal{C}$. But $\alpha^0 \in \mathcal{V}$ and then
\[
\int_M \langle \alpha^0, \alpha^0 \rangle \omega = 0. \tag{23}
\]
By \ref{22} and \ref{23} we can conclude that $|d\alpha|_2$ is a norm on the space $\mathcal{V}$ equivalent to $\|\alpha\|$, i.e.
\[
\exists \tilde{c} > 0 \text{ s.t. } \forall \alpha \in \mathcal{V}: \|\alpha\| \leq \tilde{c}|d\alpha|_2. \tag{24}
\]
By \ref{24} and since $H^{1}_{k-1}(M) \hookrightarrow L^p_k(M)$,
\[
\|\alpha\|_{1,p} = |d\alpha|_p + |\alpha|_p \leq |d\alpha|_p + c_1\|\alpha\|^p \leq |d\alpha|_p + c_2|d\alpha|_2 \tag{25}
\]
and then
\[
\|\alpha\|_{1,p} \leq c_3(|d\alpha|_2 + |d\alpha|_2) \leq c_4|d\alpha|_p.
\]
Now, consider $(\alpha_n)_{n \geq 1}$ in $\mathcal{V}$ s.t. $\|\alpha_n\|_{1,p} \to +\infty$. By \ref{25}
\[
\int_M \langle d\alpha_n + \beta, d\alpha_n + \beta \rangle^{\frac{p}{2}} \omega \to +\infty, \tag{26}
\]
so by \ref{20} we have
\[
c \int_M \langle d\alpha_n + \beta, d\alpha_n + \beta \rangle^{\frac{p}{2}} \omega \leq d \text{ meas}(M) + \int_M f(\langle d\alpha_n + \beta, d\alpha_n + \beta \rangle) \omega. \tag{27}
\]
The coerciveness of $F_\beta$ is a consequence of \ref{26} and \ref{27}.

\begin{theorem}
For every $\beta \in L^p_k(M)$ there exists a unique minimizer of $F_\beta|_{\mathcal{V}}$.
\end{theorem}

\begin{proof}
Let $\beta \in L^p_k(M)$. By $f_2$ and $f_3$ the functional $F : L^p_k(M) \to \mathbb{R}$ defined by \ref{4} is strictly convex and continuous. Obviously, also $F_\beta$ has the same properties, so it is weakly lower semicontinuous. Since $F_\beta$ is also coercive in $\mathcal{V}$ by Lemma \ref{5}, certainly it possesses a minimizer $\Phi(\beta) \in \mathcal{V}$. The uniqueness is a consequence of the strict convexity.
\end{proof}
1.2 Regularity, symmetry and compactness

Assume the following definitions:

\[ \Phi : L^p_k(M) \to V \text{ s.t. } \Phi(\beta) \text{ is the minimizer of } F_{\beta|V} \]

\[ \hat{J} : W \to \mathbb{R} \text{ s.t. } \forall \beta \in W : \hat{J}(\beta) = J(\Phi(\beta), \beta) \]

where \[ W := \{ \beta \in H^1_k(M) \mid \delta \beta = 0 \} \].

**Theorem 5.** If \( \Phi \in C^1(W, V) \), then \( \hat{J} \in C^1(W) \) and its critical points are solutions of (12), (13).

**Proof.** Suppose \( \Phi \in C^1(W, V) \), then certainly \( \hat{J} \in C^1(W) \) since it is the composition of \( C^1 \) maps.

Now let \( \beta_0 \in W \) be a critical point of \( \hat{J} \). We have that for any \( \beta \in W \)

\[ 0 = \langle \hat{J}'(\beta_0), \beta \rangle = \langle \frac{\partial J}{\partial \alpha}(\Phi(\beta_0), \beta_0), \Phi'(\beta_0)(\beta) \rangle + \langle \frac{\partial J}{\partial \beta}(\Phi(\beta_0), \beta_0), \beta \rangle, \]

that is

\[ \frac{\partial J}{\partial \beta}(\Phi(\beta_0), \beta_0) = -\frac{\partial J}{\partial \alpha}(\Phi(\beta_0), \beta_0) \circ \Phi'(\beta_0). \]

But

\[ \frac{\partial J}{\partial \alpha}(\Phi(\beta_0), \beta_0) = 0 \]

because \( \Phi(\beta_0) \) is a minimizer of \( F_{\beta|V} \), so also

\[ \frac{\partial J}{\partial \beta}(\Phi(\beta_0), \beta_0) = 0. \]

\[ \square \]

In order to study the functional \( \hat{J} \), we need to investigate the properties of the map \( \Phi \). Then, in the next theorem we are going to prove some regularity, symmetry and compactness properties of the map \( \Phi \). To get regularity, in particular, we will use the implicit function theorem on

\[ \frac{\partial J}{\partial \alpha} : V \times W \to V'. \]

where \( V' \) is the dual of \( V \).

Observe that \( J \in C^2(V \times W) \). Moreover we have the following
Lemma 6. Set
\[ \tilde{F} := (\alpha, \beta) \in \mathcal{V} \times \mathcal{W} \mapsto F(d\alpha + \beta). \] (34)

Then for all \( \beta \in L^p_k(M) \), \( \tilde{F}(\cdot, \beta) \in C^2(\mathcal{V}) \) is uniformly convex on \( \mathcal{V} \) with respect to the norm \( \| \cdot \| \), i.e. there exists \( C > 0 \) such that for all \( \alpha, \overline{\alpha} \in \mathcal{V} \)
\[ \frac{\partial^2 \tilde{F}}{\partial \alpha^2}(\alpha, \beta)[\overline{\alpha}, \overline{\alpha}] := d^2 \tilde{F}(\cdot, \beta)(\alpha)[\overline{\alpha}, \overline{\alpha}] \geq C\|\overline{\alpha}\|^2. \] (35)

Proof. Let \( \beta \in L^p_k(M) \) and \( \alpha \in \mathcal{V} \). If \( \overline{\alpha} \in \mathcal{V} \), then
\[ \frac{\partial^2 \tilde{F}}{\partial \alpha^2}(\alpha, \beta)[\overline{\alpha}, \overline{\alpha}] = 4 \int_M f''((d\alpha + \beta, d\alpha + \beta)((d\alpha + \beta, d\overline{\alpha}))^2 \omega \\
+ 2 \int_M f'(d\alpha + \beta) d\overline{\alpha}) \omega \\
\geq \varepsilon \int_M (d\overline{\alpha}, d\overline{\alpha}) \omega, \]
where the last inequality follows from the convexity of \( f \) and assumption \( f_1 \).

Now we are ready to prove

Theorem 7. The following properties hold
1. \( \Phi \in C^1(\mathcal{W}) \);
2. \( \Phi \) is odd;
3. \( \Phi \) is compact.

Proof. 1. Since \( \Phi(\beta) \) is a minimizer of \( F_\beta \)
\[ \forall \beta \in \mathcal{W} : \frac{\partial \tilde{F}}{\partial \alpha}(\Phi(\beta), \beta) = 0, \] (36)
so, by Lemma 6 and the implicit function theorem, we have that \( \Phi \in C^1(\mathcal{W}, \mathcal{V}) \) and for any \( \beta \in \mathcal{W} \):
\[ \Phi'(\beta) = - \left( \frac{\partial^2 \tilde{F}}{\partial \alpha^2}(\Phi(\beta), \beta) \right)^{-1} \circ \frac{\partial^2 \tilde{F}}{\partial \alpha \beta}(\Phi(\beta), \beta). \]
2. If $\beta \in \mathcal{W}$, then some calculations show that

$$\frac{\partial \tilde{F}}{\partial \alpha}(-\Phi(\beta), -\beta) = \frac{\partial \tilde{F}}{\partial \alpha}(\Phi(\beta), \beta) = 0,$$

so, by uniqueness, $-\Phi(\beta) = \Phi(-\beta)$.

3. By the same arguments as above, we can prove that $\Phi$ is also in $C^1(\mathcal{L}_k^p(M), \mathcal{V})$ so, if $(\beta_n)_n$ is a sequence in $\mathcal{W}$ and

$$\beta_n \rightharpoonup \beta \in \mathcal{W}$$

with respect to the norm $\| \cdot \|$, then

$$\beta_n \rightarrow \beta \text{ in } \mathcal{L}_k^p(M)$$

and so

$$\Phi(\beta_n) \rightarrow \Phi(\beta) \text{ in } \mathcal{V}.$$ 

\[\square\]

1.3 Main Theorem (first part)

We introduce some results on the Laplace Beltrami operator $-\Delta$.

It is well known that $-\Delta$ is a self adjoint operator on $L^2_k(M)$ with a nonnegative, discrete and divergent spectrum $\sigma(\Delta)$.

Set

$$\lambda_1 \leq \lambda_2 \leq \ldots$$

the sequence of the eigenvalues different from zero repeated according to their finite multiplicity. The corresponding eigenvectors

$$\eta_1, \eta_2, \ldots$$

constitute an orthonormal basis of $(\ker(-\Delta)^\perp)$. Take a basis $h_1, h_2, \ldots, h_N$ of $\ker(-\Delta)$ so that, if $\beta \in \mathcal{L}_k^p(M)$, we have

$$\beta = \sum_{i \geq 1} \beta_i \eta_i + \beta^0,$$

where $(\beta_i)_{i \geq 1}$ are the Fourier components of $\beta$ corresponding to $\eta_1, \eta_2 \ldots$, and $\beta^0$ is its projection on $\ker(\Delta)$.

For every $s \in \mathbb{R}$ define the Sobolev space

$$W^s_{k, 2}(M) := \left\{ \beta \in \mathcal{L}_k^p(M) \mid \|\beta\|_{s, 2} < \infty \right\},$$

where
\[ \|\beta\|_{s,2}^2 := \sum_{i \geq 1} (\lambda_i^s + 1)\beta_i^2 + |\beta_0^0|_2^2. \] (39)

It is easy to prove that \((W_{s,2}^k(M), \| \cdot \|_{s,2})\) is a Banach space and \(W_{1,2}^k(M) \equiv H_k^1(M)\). Since \(p < \frac{2n}{n-2}\), by Sobolev embedding theorem, there exist \(s < 1\) and \(\tilde{c} > 0\) such that
\[ |\beta|^2_p \leq \tilde{c}\|\beta\|_{s,2}^2, \quad \forall \beta \in W_{k}^{s,2}(M). \] (40)

Now we recall the following abstract multiplicity theorem whose proof can be found in \([3]\) (see also \([1]\)).

**Theorem 8.** Let \(H\) be an Hilbert space and \(I\) be a \(C^1\) even functional on \(H\) such that

1. \(I(0)=0\),
2. \(I\) satisfies (P-S) condition i.e. any sequence \((x_n)\) such that
\[ I(x_n) \text{ is bounded} \]
\[ I'(x_n) \to 0, \]

admits a convergent subsequence,
3. there exist \(H^-, H^+\) two closed subspaces of \(H\) such that
   
   (a) \( \text{codim}(H^+) < \dim(H^-) < +\infty \)
   
   (b) \( \exists c_0, \rho > 0 \text{ s.t. } I(x) \geq c_0, \forall x \in \partial S_\rho(0) \cap H^+, (\text{where } \partial S_\rho(0) := \{x \in H \mid \|x\| = \rho\}) \)
   
   (c) \( \exists c_1 > 0 \text{ s.t. } \forall x \in H^- : I(x) < c_1 \)

then \(I\) possesses at least \(\dim(H^-) - \text{codim}(H^+)\) couples of critical points whose corresponding critical values are in \([c_0, c_1]\).

In the next lemmas we shall verify the hypotheses of the previous theorem for the functional \(\hat{J}\) on the Hilbert space \(\mathcal{W}\).

**Lemma 9.** \(\hat{J}\) is a \(C^1\) even functional satisfying the (P.S.) condition

**Proof.** The regularity and symmetry properties of \(\hat{J}\) are an immediate consequence of the structure of \(J\), and Theorem 7.
As to (P.S.) condition, let \((\beta_n)_n\) be a sequence in \(W\) such that
\[
\hat{J}(\beta_n) = |d\beta_n|_2^2 - \int_M f(b_n) \omega \leq M, \ M \geq 0 \tag{41}
\]
and
\[
\hat{J}'(\beta_n) \rightarrow 0 \tag{42}
\]
where we have set \(b_n = (\beta_n + d\Phi(\beta_n), \beta_n + d\Phi(\beta_n))\) to simplify the notations.

We want to show that \(\{\beta_n\}\) is precompact. By using \(f_3\) and 3 of Theorem 7 it can be easily seen that \(\hat{J}'\) is the sum of an homeomorphism and a compact map, so, by standard arguments, we are reduced to prove that \((\beta_n)_n\) is bounded.

If we set
\[
\varepsilon_n := \frac{1}{2} \langle \hat{J}'(\beta_n), \frac{\beta_n}{\|\beta_n\|} \rangle, \tag{43}
\]
from (42) we deduce that \(\varepsilon_n \rightarrow 0\).

Rendering (43) explicit, we obtain
\[
|d\beta_n|_2^2 - \int_M f'(b_n) \langle \beta_n + d\Phi(\beta_n), \beta_n + d\Phi'(\beta_n) \rangle \omega = \varepsilon_n \|\beta_n\|. \tag{44}
\]

By (36) we have
\[
\int_M f'(b_n) \langle \beta_n + d\Phi(\beta_n), d\Phi'(\beta_n) \rangle \omega = 0 \tag{45}
\]
\[
\int_M f'(b_n) \langle \beta_n + d\Phi(\beta_n), d\Phi(\beta_n) \rangle \omega = 0 \tag{46}
\]
so, comparing (44), (45) and (46) we have
\[
|d\beta_n|_2^2 - \int_M f'(b_n) \langle \beta_n + d\Phi(\beta_n), \beta_n + d\Phi(\beta_n) \rangle \omega = \varepsilon_n \|\beta_n\|. \tag{47}
\]
Comparing (41) and (47) we get,
\[
\frac{p - 2}{2} |d\beta_n|_2^2 - \int_M \left( \frac{p}{2} f(b_n) - f'(b_n) b_n \right) \omega \leq K_1 + K_2 \|\beta_n\| \tag{48}
\]
and
\[
\int_M \left( f'(b_n) b_n - f(b_n) \right) \omega \leq K_3 + K_4 \|\beta_n\| \tag{49}
\]
where $K_1, \ldots, K_4$ and the following $\{(K_i) \mid i \geq 5\}$ are positive constants. By $f_4$, there exists $L > 0$ such that

$$-L \leq f'(t)t - \frac{d}{2}f(t), \quad \forall t \geq 0,$$

(50) so by (20),

$$f'(t)t - f(t) \geq -L + \frac{d-2}{2}f(t) \geq -K_5 + K_6 t^2, \quad \forall t \geq 0.$$

(51)

From (51) and (49) we have

$$|\hat{\beta}_n + d\Phi(\beta_n)|_p \leq K_7 + K_8\|\beta_n\|^\frac{1}{2},$$

(52) so, since $L_k(M) \hookrightarrow L_k^2(M)$,

$$|\beta_n|^2 \leq |\beta_n|^2 + |d\Phi(\beta_n)|^2 = |\beta_n + d\Phi(\beta_n)|^2 \leq K_9|\beta_n + d\Phi(\beta_n)|_p \leq K_9 + K_11\|\beta_n\|^\frac{2}{2}.$$  

(53)

From (48) and (50) we also derive

$$|d\beta_n|^2 \leq K_{12} + K_{13}\|\beta_n\|.$$  

(54)

Inequalities (53) and (54) imply that the sequence $(\|\beta_n\|)_n$ is bounded. \hfill \Box

Now, for any $\mu > 0$ and $\rho > 0$, we set

$$\partial S_\rho := \{ \beta \in W \mid \|\beta\| = \rho \},$$

(55)

$$H^+(\mu) := \bigoplus_{\lambda_i > \mu} M_{\lambda_i},$$

(56)

$$H^-(\mu) := (H^+(\mu))^\perp \oplus M_{\lambda_k},$$

(57)

where $(M_{\lambda_i})_{i \geq 1}$ are the spaces of the eigenfunctions corresponding to the eigenvalues $(\lambda_i)_{i \geq 1}$ and $k := \min \{ i \in \mathbb{N} \mid \lambda_i > \mu \}$. Observe that, since every eigenspace has finite dimension, from (56) and (57) we deduce

$$\dim H^-(\mu) = \text{codim } H^+(\mu) + \dim M_{\lambda_k} < +\infty.$$  

(58)

Moreover we have that

**Lemma 10.** There exist a strictly increasing sequence $(C_i)_{i \geq 1}$ and two positive numbers sequences $(\rho_i)_{i \geq 1}$ and $(\mu_i)_{i \geq 1}$ such that for all $i \geq 1$ we have

3b) $\hat{J}(\beta) \geq C_{2i}, \quad \forall \beta \in \partial S_{\rho_i} \cap H^+(\mu_i),$
Proof. We will prove that for every $C > 0$ there exist $\mu > 0$ and $\rho > 0$ such that

$$\widehat{J}(\beta) \geq C, \quad \forall \beta \in \partial S_\rho \cap H^+(\mu) \quad (59)$$

$$\sup_{\beta \in H^-(\mu)} \widehat{J}(\beta) < +\infty. \quad (60)$$

Set $C > 0$.

By $f_3$ and (28), we have that there exists $b' > 0$ such that

$$\int_{M} f(\langle \beta + d\Phi(\beta), \beta + d\Phi(\beta) \rangle) \omega \leq \int_{M} f(\langle \beta, \beta \rangle) \omega \leq a|\beta|^p + b'|\beta|^2 \leq a|\beta|^p + b'\tilde{c}^2|\beta|^2. \quad (61)$$

Set $\mu = \rho^{\frac{2}{1-s}}$, and $K = \min_{\lambda_i > \mu} \lambda_i > 0$ where $\rho$ is a suitable real number that we are going to evaluate and $s \in (0, 1)$ is defined as in (40).

Let $\beta \in H^+(\mu) \cap S_\rho$. Using (39), (40) and (17), we have

$$|d\beta|^2 = \sum_{\lambda_i > \mu} \lambda_i |\beta_i|^2 \geq \mu^{1-s} \sum_{\lambda_i > \mu} \lambda_i |\beta_i|^2 \geq \mu^{1-s} \|\beta\|^2_{s,2} \geq K\tilde{c}^{-1}\mu^{1-s}|\beta|^2_p \quad (62)$$

and then by (61) and (62),

$$\widehat{J}(\beta) = |d\beta|^2 - \int_{M} f(\langle \beta + d\Phi(\beta), \beta + d\Phi(\beta) \rangle) \omega \geq |d\beta|^2 - a|\beta|^p - b'|\beta|^2 \geq \rho^2 - a\left(\frac{\tilde{c}}{K\mu^{1-s}}\right)^{\frac{p}{2}} - b'\tilde{c}^2K = \rho^2 - a\left(\frac{\tilde{c}}{K}\right)^{\frac{p}{2}} - b'\tilde{c}K.$$

Since

$$\lim_{\rho \to +\infty} K = 1$$

from the previous chain of inequalities we get (59) for $\rho$ large enough.

Now take $\beta \in H^-(\mu)$. Observe that

$$|d\beta|^2 \leq \lambda_k |\beta|^2. \quad (63)$$
Moreover, since $\beta$ and $d\Phi(\beta)$ are orthogonal in the space $L^2_k(M)$, there exists $K_1 > 0$ s.t.

$$|\beta + d\Phi(\beta)|_p^p \geq K_1(|\beta + d\Phi(\beta)|_2^2)^{\frac{p}{2}}$$

$$\geq K_1(|\beta|^2 + |d\Phi(\beta)|_2^2)^{\frac{p}{2}}$$

$$\geq K_1|\beta|^p.$$  \hfill (64)

By (20), (63) and (64), since in $H^-(\mu)$ all the norms are equivalent, there exists $K_2, K_3 > 0$ such that

$$\tilde{J}(\beta) = |d\beta|^2 - \int_M f(\langle \beta + d\Phi(\beta), \beta + d\Phi(\beta) \rangle) \omega$$

$$\leq \lambda_k |\beta|^2 - c|\beta + d\Phi(\beta)|_p^p + d \text{meas}(M)$$

$$\leq \lambda_k |\beta|^2 - K_2|\beta|_p^p + K_3$$

$$\leq \sup_{t>0}(\lambda_k t^2 - K_2 t^p + K_3).$$

So we are ready to give the following

**Proof (of the first part of Theorem 1).** By Lemma 9, Lemma 10 and (58), using Theorem 8, we find infinitely many couples of critical points. In fact, for all $i \geq 1$ there exist at least $\dim M_{\lambda_k}$ critical points for $\tilde{J}$, whose critical values are in the interval $[C_{2i}, C_{2i+1}]$. Since the sequence $(C_i)_{i \geq 1}$ is strictly increasing, we obtain a countable set of critical points.

**2 Zero mass case**

In this section we consider again the problem (1), replacing condition $f_1$ by $\tilde{f}_1$.

Moreover we replace $f_2$ by the technical hypothesis $\tilde{f}_2$ that, as already seen, implies for all $q \in M$ the strict convexity of the functional

$$I_q(\xi) = f(\langle \xi, \xi \rangle_q).$$

Observe that, integrating in $\tilde{f}_2$, by density we also deduce that

$F$ and $F_\beta$ are strictly convex on $L^p_k(M).$ \hfill (65)

Now, in order to prove the second part of Theorem 1 we consider the perturbed equation

$$\delta d\xi = f_\varepsilon'(\langle \xi, \xi \rangle)\xi$$  \hfill (66)
where \( f_\varepsilon(t) = f(t) + \varepsilon t, \varepsilon > 0. \)

We set
\[
F_\varepsilon : \xi \in L^p_k(M) \mapsto \int_M f_\varepsilon(\langle \xi, \xi \rangle) \omega
\]
and for all \( \beta \in L^p_k(M) \)
\[
(F_\varepsilon)_\beta : = \alpha \in \mathcal{V} \mapsto F_\varepsilon(d\alpha + \beta).
\]

The function \( f_\varepsilon \) satisfies \( f_1, f_3 \) and \( f_4 \), and, by (65), \( F_\varepsilon \) and \((F_\varepsilon)_\beta\) are uniformly convex respectively on \( L^p_k(M) \) and \( \mathcal{V} \). From the first part, we conclude that the equation (66) possesses infinitely many \( \varepsilon \)-solutions of the type \( \xi_\varepsilon = \beta_\varepsilon + d\Phi_\varepsilon(\beta_\varepsilon) \) where
\[
\Phi_\varepsilon : L^p_k(M) \to \mathcal{V} \quad \text{s.t.} \quad \Phi_\varepsilon(\beta) \text{ is the minimizer of } (F_\varepsilon)_\beta
\]
and \( \beta_\varepsilon \) is a critical point of the functional
\[
\tilde{J}_\varepsilon(\beta) = \int_M \langle d\beta, d\beta \rangle \omega - \int_M f_\varepsilon(\langle d\Phi_\varepsilon(\beta) + \beta, d\Phi_\varepsilon(\beta) + \beta \rangle) \omega
\]
\[
= \int_M \langle d\beta, d\beta \rangle \omega - \int_M f(\langle d\Phi_\varepsilon(\beta) + \beta, d\Phi_\varepsilon(\beta) + \beta \rangle) \omega
\]
\[
- \varepsilon \int_M \langle d\Phi_\varepsilon(\beta) + \beta, d\Phi_\varepsilon(\beta) + \beta \rangle \omega.
\]

Now we construct infinitely many sequences \( (\beta_n) \) of \( \varepsilon_n \)-solutions of (66), in such a way, passing to the limit, we get solutions for the non-perturbed problem (1).

Of course, assuming that two different sequences converge, we have no chance to prove that the corresponding limits are different without any a-priori estimate on the critical values.

So we need a separation property on the sequences and, with reference to this, we introduce the following

**Definition 1.** Let \( (\varepsilon_n)_{n \geq 1} \) be a sequence of real numbers and set \( (\beta^1_n)_{n \geq 1} \) and \( (\beta^2_n)_{n \geq 1} \) two sequences of k-forms. We say that \( (\beta^1_n)_{n \geq 1} \) and \( (\beta^2_n)_{n \geq 1} \) are well-separated if there exist \( k_1, k_2, k_3, k_4 \in \mathbb{R} \) such that
\[
k_1 \leq \tilde{J}_{\varepsilon_n}(\beta^1_n) \leq k_2 < k_3 \leq \tilde{J}_{\varepsilon_n}(\beta^2_n) \leq k_4, \quad \forall n \geq 1.
\]

Actually we have this result
Theorem 11. Let $\varepsilon_n \downarrow 0^+$. Then there exists a countable set of sequences $((\beta_i^n)_{n \geq 1})_{i \geq 1}$ of $\varepsilon_n$-solutions that are each other well-separated.

Proof. Observe that, using Theorem 8 and its notations, we get the conclusion if we prove that there exist two sequences $(\mu_i)_{i \geq 0}$ and $(\rho_i)_{i \geq 0}$, and a strictly increasing sequence $(C_i)_{i \geq 0}$ such that 3b and 3c of Lemma 10 hold $\varepsilon_n$—uniformly, i.e.

$$C_{2i} \leq \hat{J}_{\varepsilon_n}(\beta), \quad \forall \beta \in \partial S_{\rho_i} \cap H^+(\mu_i), \forall \varepsilon_n$$  \hspace{1cm} (71)

$$\hat{J}_{\varepsilon_n}(\beta) < C_{2i+1}, \quad \forall \beta \in H^-(\mu_i), \forall \varepsilon_n.$$  \hspace{1cm} (72)

We shall prove that for every $C > 0$ there exist $\rho > 0$ and $\mu > 0$ such that

$$\inf_{\beta \in \partial S_{\rho} \cap H^+}(\mu) \hat{J}_{\varepsilon_n}(\beta) \geq C,$$  \hspace{1cm} (73)

$$\sup_{\beta \in H^-(\mu)} \hat{J}_{\varepsilon_n}(\beta) < +\infty.$$  \hspace{1cm} (74)

By $f_3$ and (69), using the embedding $L^p_k(M) \hookrightarrow L^2_k(M)$, and since $\varepsilon_n$ is decreasing we have

$$\varepsilon_n |\beta + d \Phi_{\varepsilon_n}(\beta)|_2^2 + \int_M f(\langle \beta + d \Phi_{\varepsilon_n}(\beta), \beta + d \Phi_{\varepsilon_n}(\beta) \rangle) \omega \leq \varepsilon_n |\beta|_2^2 + \int_M f(\langle \beta, \beta \rangle) \omega \leq \varepsilon_1 |\beta|_2^2 + a |\beta|_p^p + b' |\beta|_p^2 \leq a |\beta|_p^p + b'' |\beta|_p^2,$$  \hspace{1cm} (75)

where $b''$ is a suitable positive constant.

Set $\mu = \rho^{\frac{1}{p-2}}$, and take $\beta \in H^+(\mu) \cap S_\rho$. By (75) and using (62),

$$\hat{J}_{\varepsilon_n}(\beta) = |d \beta|_2^2 - \varepsilon_n |\beta + d \Phi_{\varepsilon_n}(\beta)|_2^2 - \int_M f(\langle \beta + d \Phi_{\varepsilon_n}(\beta), \beta + d \Phi_{\varepsilon_n}(\beta) \rangle) \omega \geq |d \beta|_2^2 - a |\beta|_p^p - b'' |\beta|_p^2 \geq \rho^2 - a \left( \frac{\hat{c}}{K} \right)^\frac{2}{p} - b'' \hat{c} \geq C$$  \hspace{1cm} (76)

uniformly for $n \geq 1$, for $\rho$ large enough.

Moreover, if $\beta \in H^-(\mu)$, then by (20)

$$\int_M f(\langle \beta + d \Phi_{\varepsilon_n}(\beta), \beta + d \Phi_{\varepsilon_n}(\beta) \rangle) \omega \geq c |\beta + d \Phi_{\varepsilon_n}(\beta)|_p^p - d \text{ meas}(M).$$  \hspace{1cm} (77)
So, if we set \( \lambda_k = \min\{\lambda_j \mid \lambda_j > \mu\} \), then, for suitable \( c_1, c_2 > 0 \), by (77) we have

\[
\hat{J}_{\epsilon_n}(\beta) = |d\beta|_2^2 - \varepsilon_n|\beta + d\Phi_{\epsilon_n}(\beta)|_2^2 - \\
- \int_M f((\beta + d\Phi_{\epsilon_n}(\beta), \beta + d\Phi_{\epsilon_n}(\beta)) \omega \leq \\
\leq \lambda_k|\beta|_2^2 - c_1|\beta|_2^2 + |d\Phi_{\epsilon_n}(\beta)|_2^2 + c_2 \leq \\
\leq \sup_{t \geq 0} (\lambda_k t^2 - c_1 t^2 + c_2) < +\infty
\]

uniformly for \( n \geq 1 \).

\[
2.1 \quad \text{Some preliminary results}
\]

**Lemma 12.** Let \((\xi_n)_{n \geq 1}\) a sequence of forms in \( L^p_k(M) \) and \( \xi \in L^p_k(M) \).

If \( f \) satisfies \( \tilde{f}_2, f_3 \) and

\[
\xi_n \rightharpoonup \xi \text{ in } L^p_k(M) \quad (78)
\]

\[
\int_M f(\langle \xi_n, \xi_n \rangle) \omega \rightharpoonup \int_M f(\langle \xi, \xi \rangle) \omega \quad (79)
\]

then

\[
\xi_n \to \xi \text{ in } L^p_k(M). \quad (80)
\]

**Proof.** First suppose \( \xi_n \) and \( \xi \) in \( \Lambda_k(M) \). By \( \tilde{f}_2 \) we have

\[
f(\langle \xi_n, \xi_n \rangle) - f(\langle \xi, \xi \rangle) - 2 f'(\langle \xi, \xi \rangle) \langle \xi_n, \xi_n - \xi \rangle \geq \tau \langle \xi_n - \xi, \xi_n - \xi \rangle, \]

so, integrating, we obtain,

\[
\int_M \left( f(\langle \xi_n, \xi_n \rangle) - f(\langle \xi, \xi \rangle) \right) \omega - 2 \Psi_\xi(\xi_n - \xi) \geq \\
\geq \tau \int_M \langle \xi_n - \xi, \xi_n - \xi \rangle \omega \quad (81)
\]

where \( \Psi_\xi \) represents the map

\[
\Psi_\xi : \eta \in L^p_k(M) \mapsto \int_M f'(\langle \xi, \xi \rangle) \langle \xi, \eta \rangle \omega.
\]

Observe that \( \Psi_\xi \) is linear and continuous by \( f_3 \) so we get (80) from (78), (79) and (81).

If \( \xi_n \) and \( \xi \) are in \( L^p_k(M) \), then we get the same conclusion by density. \( \square \)
Lemma 13. \( \forall \beta \in \mathcal{W}, \forall \varepsilon > 0 : \)
\[
0 \leq \int_M f(\langle \beta + d\Phi(\beta), \beta + d\Phi(\beta) \rangle) \omega - \int_M f(\langle \beta + d\Phi(\beta), \beta + d\Phi(\beta) \rangle) \omega \leq \varepsilon (|\beta + d\Phi(\beta)|_2^2 - |\beta + d\Phi(\beta)|_2^2)
\] (82)

**Proof.** Consider \( \beta \in \mathcal{W} \) and \( \varepsilon > 0 \), and set \( b = \langle \beta + d\Phi(\beta), \beta + d\Phi(\beta) \rangle \) and \( b_\varepsilon = \langle \beta + d\Phi_\varepsilon(\beta), \beta + d\Phi_\varepsilon(\beta) \rangle \). By definition of \( \Phi \) and \( \Phi_\varepsilon \), we have that
\[
F(\beta + d\Phi(\beta)) \leq F(\beta + d\Phi_\varepsilon(\beta)) \quad (83)
\]
\[
F_\varepsilon(\beta + d\Phi_\varepsilon(\beta)) \leq F_\varepsilon(\beta + d\Phi(\beta)) \quad (84)
\]
that is
\[
\int_M f(b) \omega \leq \int_M f(b_\varepsilon) \omega \quad (85)
\]
and
\[
\int_M f(b_\varepsilon) \omega + \varepsilon |\beta + d\Phi_\varepsilon(\beta)|_2^2 \leq \int_M f(b) \omega + \varepsilon |\beta + d\Phi(\beta)|_2^2. \quad (86)
\]
Combining (85) and (86) together, we get (82). \( \square \)

2.2 Main Theorem (second part)

The following lemma holds

**Lemma 14.** Let \( \varepsilon_n \searrow 0^+ \) and \( (\beta_n)_{n \geq 1} \) a sequence in \( \mathcal{W} \) such that
a) \( \exists k_2 > k_1 > 0 \) s.t. \( k_1 \leq \hat{J}_{\varepsilon_n}(\beta_n) \leq k_2, \forall n \geq 1, \)
b) \( \hat{J}_{\varepsilon_n}(\beta_n) = 0, \forall n \geq 1. \)

Then there exists \( \beta \in \mathcal{W} \) and a subsequence relabelled \( (\beta_n)_{n \geq 1} \) such that
i) \( d\Phi_{\varepsilon_n}(\beta_n) \longrightarrow d\Phi(\beta) \) in \( L^p_k(M) \)
ii) \( \beta_n \longrightarrow \beta \) in \( H^1_k(M) \)
iii) \( k_1 \leq \hat{J}(\beta) \leq k_2. \)

**Proof.** To simplify the notations, set \( b_n = \langle \beta_n + d\Phi_{\varepsilon_n}(\beta_n), \beta_n + d\Phi_{\varepsilon_n}(\beta_n) \rangle \).

By a) we have
\[
k_1 \leq |d\beta_n|_2^2 - \varepsilon_n|\beta_n + d\Phi_{\varepsilon_n}(\beta_n)|_2^2 - \int_M f(b_n) \omega \leq k_2 \quad (87)
\]
and, on the other hand, by \( b \) and using (36),

\[
|d\beta_n|_2^2 - \varepsilon_n|\beta_n + d\Phi_{\varepsilon_n}(\beta_n)|_2^2 - \int_M f'(b_n)b_n\omega = 0. \tag{88}
\]

Using arguments similar to those in the proof of Theorem 9, we have that \((\beta_n)_{n\geq1}\) and \((\Phi_{\varepsilon_n}(\beta_n))_{n\geq1}\) are bounded respectively in \(W\) and \(V\), so there exist \(\beta \in W\) and \(\eta \in V\) such that (up to a subsequence)

\[
\beta_n \rightharpoonup \beta \quad \text{in} \quad H^1_k(M) \tag{89}
\]

\[
\Phi_{\varepsilon_n}(\beta_n) \rightharpoonup \eta \quad \text{in} \quad H^{1,p}_{k-1} \tag{90}
\]

and by compactness

\[
\beta_n \rightarrow \beta \quad \text{in} \quad L^p_k(M). \tag{91}
\]

Now applying (82) to \(\beta_n\) for every \(n \geq 1\), we obtain

\[
0 \leq \int_M f(b_n)\omega - \int_M f(\langle \beta_n + d\Phi(\beta_n), \beta_n + d\Phi(\beta_n) \rangle)\omega \leq \varepsilon_n(|\beta_n + d\Phi(\beta_n)|_2^2 - |\beta_n + d\Phi_{\varepsilon_n}(\beta_n)|_2^2). \tag{92}
\]

We claim that

\[
\varepsilon_n(|\beta_n + d\Phi(\beta_n)|_2^2 - |\beta_n + d\Phi_{\varepsilon_n}(\beta_n)|_2^2) \xrightarrow{n \to +\infty} 0. \tag{94}
\]

In fact, suppose by contradiction that (94) is not true. Since (87) and (88) imply that

\[
|\beta_n + d\Phi_{\varepsilon_n}(\beta_n)|_2 \text{ is bounded (see proof of Theorem 9),} \tag{95}
\]

by the contradiction hypothesis we have that, up to a subsequence,

\[
|\beta_n + d\Phi(\beta_n)|_2 \rightarrow +\infty
\]

and then

\[
|\beta_n + d\Phi(\beta_n)|_p \rightarrow +\infty. \tag{96}
\]

By (20) and (96)

\[
\int_M f(\langle \beta_n + d\Phi(\beta_n), \beta_n + d\Phi(\beta_n) \rangle)\omega \rightarrow +\infty. \tag{97}
\]

Comparing (92) and (97) we deduce that

\[
\int_M f(b_n)\omega \rightarrow +\infty. \tag{98}
\]
On the other hand, from (87) and (88) we have that
\[ |d\Phi_{\epsilon_n}(\beta_n) + \beta_n|_p \] is bounded (see proof of Theorem 9),
(99)
so, considering \( f_3 \), we get
\[ \int_M f(b_n) \omega \] is bounded
that contradicts (98).

Now observe that (92), (93) and (94) imply that
\[ \int_M f(b_n) \omega - \int_M f(\langle \beta_n + d\Phi(\beta_n), \beta_n + d\Phi(\beta_n) \rangle) \omega \to 0. \]
(100)
Since the map
\[ \beta \in \mathcal{W} \mapsto \int_M f(\langle \beta + d\Phi(\beta), \beta + d\Phi(\beta) \rangle) \omega \]
is weakly continuous (see [3] or [2]), (89) implies that
\[ \int_M f(b_n) \omega - \int_M f(\langle \beta_n + d\Phi(\beta_n), \beta_n + d\Phi(\beta_n) \rangle) \omega \to 0. \]
(101)
and then, by (100),
\[ \int_M f(b_n) \omega \to \int_M f(\langle \beta + d\Phi(\beta), \beta + d\Phi(\beta) \rangle) \omega. \]
(102)
Since \( F \) is weakly lower semicontinuous, from (90), (91) and (102) we have
\[
F_\beta(\eta) = F(\beta + d\eta) \\
\leq \liminf_n F(\beta_n + d\Phi_{\epsilon_n}(\beta_n)) = F(\beta + d\Phi(\beta)) = F_\beta(\Phi(\beta))
\]
and then, by the uniqueness of the minimizer of \( F_\beta \),
\[ \eta = \Phi(\beta). \]
(103)
Now from (90), (91) and (103) we have that
\[ \beta_n + d\Phi_{\epsilon_n}(\beta_n) \rightharpoonup \beta + d\Phi(\beta) \text{ in } L^p_k(M) \]
(104)
so, by (102), (104) and Lemma 12, we have that
\[ \beta_n + d\Phi_{\epsilon_n}(\beta_n) \rightharpoonup \beta + d\Phi(\beta) \text{ in } L^p_k(M), \]
(105)
and then, taking (91) into account,

\[ d\Phi_{\epsilon_n}(\beta_n) \to d\Phi(\beta) \text{ in } L^p_k(M), \tag{106} \]

that corresponds to the assertion i).

Now we pass to the proof of ii).

From \( b) \) we have that

\[ 0 = \tilde{J}_{\epsilon_n}(\beta_n) = L(\beta_n) - \epsilon_n(\beta_n + d\Phi_{\epsilon_n}(\beta_n)) - K(\beta_n, d\Phi_{\epsilon_n}(\beta_n)) \tag{107} \]

where \( L \) is the Riesz isomorphism between \( \mathcal{W} \) and its dual and

\[ K : (\xi, \eta) \in H^1_k(M) \times L^p_k(M) \to K(\xi, \eta) \in (H^1_k(M))'. \]

From (107) and considering (95) we have

\[ L(\beta_n) - K(\beta_n, d\Phi_{\epsilon_n}(\beta_n)) \to 0 \]

and then

\[ \beta_n - L^{-1}(K(\beta_n, d\Phi_{\epsilon_n}(\beta_n))) \to 0. \tag{108} \]

Now observe that

\[ K \text{ is compact with respect to } \xi \tag{109} \]

\[ K \text{ is continuous with respect to } \eta \tag{110} \]

so, by (99) and (106), from (108) we get (up to a subsequence)

\[ \beta_n \to L^{-1}(K(\beta, d\Phi(\beta))) \]

and hence ii).

Finally, note that from ii), (95) and (102) we have

\[ \tilde{J}_{\epsilon_n}(\beta_n) \to \tilde{J}(\beta) \tag{111} \]

so iii) is a consequence of a) and (111).

And now we are ready for the following

**Proof (of the second part of Theorem 1).** Let \( \epsilon_n \searrow 0^+ \). By Theorem \([\text{11}] \) there exist infinitely many well separated sequences of the type described in a) and \( b) \) of the Lemma \([\text{14}] \).

Certainly each of these sequences (up to a subsequence) converges in \( H^1_k(M) \) by ii) and each limit is different from another by iii).

Say \((\beta_n)_{n \geq 1}\) one of these sequences and \( \beta \) its limit. If we show that \( \beta + d\Phi(\beta) \)
is a solution for (3), then we have finished.
Let \( \eta \in \Lambda^k(M) \). For every \( n \geq 1 \), certainly
\[
\langle L(\beta_n), \eta \rangle = \varepsilon_n \int_M \langle \beta_n + d\Phi_{\varepsilon_n}(\beta_n), \eta \rangle \omega + \langle K(\beta_n, d\Phi_{\varepsilon_n}(\beta_n)), \eta \rangle
\]
where \( L \) and \( K \) are those defined in Lemma 14.
By continuity, \( \text{ii}) \) of Lemma 14 implies that
\[
\langle L(\beta_n), \eta \rangle \longrightarrow \langle L(\beta), \eta \rangle
\]
while (109) and (110) together with \( \text{i}) \) and \( \text{ii}) \) of Lemma 14 imply
\[
\langle K(\beta_n, d\Phi_{\varepsilon_n}(\beta_n)), \eta \rangle \longrightarrow \langle K(\beta, d\Phi(\beta)), \eta \rangle.
\]
Since trivially
\[
\varepsilon_n \int_M \langle \beta_n + d\Phi_{\varepsilon_n}(\beta_n), \eta \rangle \omega \longrightarrow 0,
\]
by (112), (113) and (114) we obtain
\[
\int_M \langle d\beta, d\eta \rangle \omega = \int_M f'(\langle \beta + d\Phi(\beta), \beta + d\Phi(\beta) \rangle) \langle \beta + d\Phi(\beta), \eta \rangle \omega
\]
and then the conclusion.

\section*{Appendix}

In this appendix we want to show that assumption \( \tilde{f}_2 \) is satisfied by the function \( f(t) = t^{\frac{p}{2}} \). We will prove it by the following more abstract result

\textbf{Lemma A.1.} Let \( (H, \langle \cdot, \cdot \rangle) \) be an Hilbert space, and \( \| \cdot \| \) the induced norm. If \( p > 2 \), then there exists \( \overline{c} > 0 \) s.t. for every \( x, y \in H \) the following inequality holds
\[
\|x\|^p - \|y\|^p - p\|y\|^{p-2}(y|x - y) \geq \overline{c}\|x - y\|^p.
\]
\textbf{Proof.} In [6] the following inequality has been proved for all \( a, b \in \mathbb{R} \)
\[
|a|^p - |b|^p - p|b|^{p-2}b(a - b) \geq K|a - b|^p,
\]
where \( K > 0 \) does not depend on \( a \) and \( b \). Now we fix \( y \in H \) and distinguish the following cases:
\begin{itemize}
  \item \( x = ty, t \geq 0; \)
\end{itemize}
- $x = ty$, $t < 0$;
- $x \neq ty$, $t \in \mathbb{R}$.

If $x = ty$ for $t \geq 0$, then $(x|y) = \|x\|\|y\|$ and $(x - y|x - y) = (\|x\| - \|y\|)^2$. So (116) can be written as follows

\[ \|x\|^p - \|y\|^p - p\|y\|^{p-1}(\|x\| - \|y\|) \geq \bar{\tau} \|x\| - \|y\|^p, \tag{118} \]

that corresponds to (117) for $a = \|x\|$ and $b = \|y\|$.

If $x = ty$ for $t < 0$, then $(x|y) = -\|x\|\|y\|$ and $(x - y|x - y) = (\|x\| + \|y\|)^2$.

In this case, (116) becomes

\[ \|x\|^p - \|y\|^p + p\|y\|^{p-1}(\|x\| + \|y\|) \geq \bar{\tau} \|x\| + \|y\|^p, \tag{119} \]

that corresponds to (117) for $a = \|x\|$ and $b = -\|y\|$.

Finally, if $x \notin \{ty|t \in \mathbb{R}\}$, then $x = x_1 + x_2$ where $x_1 \in \{ty|t \in \mathbb{R}\}$ and $(x_2|y) = 0$. Since (116) holds for $x_1$, we have that there exist three positive constants $c_1, c_2$ and $c_3$ s.t.

\[
\|x\|^p - \|y\|^p - p\|y\|^{p-2}(y|x - y) = (\|x_1\|^2 + \|x_2\|^2)^{\frac{p}{2}} - \|y\|^p - p\|y\|^{p-2}(y|x_1 - y) \\
\geq \|x_1\|^p - \|y\|^p - p\|y\|^{p-2}(y|x_1) + \|x_2\|^p \\
\geq c_1(\|x_1 - y\|^p + \|x_2\|^p) \\
\geq c_2(\|x_1 - y\|^2 + \|x_2\|^2)^{\frac{p}{2}} \\
= c_3(\|x_1 - y + x_2\|^2)^{\frac{p}{2}} = c_3\|x - y\|^p.
\]

\[ \Box \]

Now, since $M$ is a compact Riemannian manifold, then for every $q \in M$ the space $\Lambda^k(T_q(M))$ of the $k$–forms at $q$ is an Hilbert space with the scalar product $\langle \cdot , \cdot \rangle_q$. So, by Lemma A.1, there exists $\bar{\tau} > 0$ s.t. for every $\xi_q, \eta_q \in \Lambda^k(T_q(M))$ we have that

\[
\langle \xi_q, \xi_q \rangle_q^{\frac{p}{2}} - \langle \eta_q, \eta_q \rangle_q^{\frac{p}{2}} - p\langle \eta_q, \eta_q \rangle_q^{\frac{p}{2}-1} \langle \eta_q, \xi_q - \eta_q \rangle_q \geq \bar{\tau} \langle \xi_q - \eta_q, \xi_q - \eta_q \rangle_q. \tag{120}
\]

Since (120) holds pointwise, then for $\xi, \eta \in \Lambda^k(M)$ the following inequality holds globally

\[
\langle \xi, \xi \rangle^{\frac{p}{2}} - \langle \eta, \eta \rangle^{\frac{p}{2}} - p\langle \eta, \eta \rangle^{\frac{p}{2}-1} \langle \eta, \xi - \eta \rangle \geq \bar{\tau} \langle \xi - \eta, \xi - \eta \rangle^{\frac{p}{2}}. \tag{121}
\]

23
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