DATALOG with constraints — an answer-set programming system

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Abstract

Answer-set programming (ASP) has emerged recently as a viable programming paradigm well attuned to search problems in AI, constraint satisfaction and combinatorics. Propositional logic is, arguably, the simplest ASP system with an intuitive semantics supporting direct modeling of problem constraints. However, for some applications, especially those requiring that transitive closure be computed, it requires additional variables and results in large theories. Consequently, it may not be a practical computational tool for such problems. On the other hand, ASP systems based on nonmonotonic logics, such as stable logic programming, can handle transitive closure computation efficiently and, in general, yield very concise theories as problem representations. Their semantics is, however, more complex. Searching for the middle ground, in this paper we introduce a new nonmonotonic logic, DATALOG with constraints or DC. Informally, DC theories consist of propositional clauses (constraints) and of Horn rules. The semantics is a simple and natural extension of the semantics of the propositional logic. However, thanks to the presence of Horn rules in the system, modeling of transitive closure becomes straightforward. We describe the syntax and semantics of DC, and study its properties. We discuss an implementation of DC and present results of experimental study of the effectiveness of DC, comparing it with the cnsat satisfiability checker and smodels implementation of stable logic programming. Our results show that DC is competitive with the other two approaches, in case of many search problems, often yielding much more efficient solutions.

Content Areas: constraint satisfaction, search, knowledge representation, logic programming, nonmonotonic reasoning.

Introduction

Many important computational problems in combinatorial optimization, constraint satisfaction and artificial intelligence can be cast as search problems. Answer-set programming (ASP) [Marek & Truszczynski 1999; Niemela 1998] was recently identified as a declarative programming paradigm appropriate for such applications. Logic programming with the stable-model semantics (stable logic programming, for short) was proposed as an embodiment of this paradigm. Disjunctive logic programming with the answer-set semantics is another implementation of ASP currently under development [Eiter et al. 1998]. Early experimental results demonstrate the potential of answer-set programming frameworks such as planning and constraint satisfaction [Niemela 1998; Lifschitz 1999a; Lifschitz 1999b].

In this paper we describe another formalism that implements the ASP approach. We call it DATALOG with constraints and denote by DC. Our goal is to design an ASP system with a semantics more readily understandable than the semantics of stable models. We seek a semantics that would be as close as possible to propositional satisfiability yet as expressive and as effective, especially from the point of view of conciseness of representations and time performance, as the stable logic programming. We argue that DC has a potential to become a practical declarative programming tool. We show that it yields intuitive and small-size encodings, we characterize its complexity and expressive power and present computational experiments demonstrating its effectiveness.

Answer-set programming is a paradigm in which programs are built as theories in some formal system $F$ with a well-defined syntax, and with a semantics that assigns to a theory $P$ in the system a collection of subsets of some domain. These subsets are referred to as answer sets of $P$ and specify the results of computation based on $P$. To solve a problem $\Pi$ in an ASP formalism, we find a program $P$ so that the solutions to $\Pi$ can be reconstructed, in polynomial (ideally, linear) time, from the answer sets to $P$.

The definition of the answer-set programming given above is very general. Essentially any logic formalism can be a basis for an answer-set programming system. For instance, the propositional logic gives rise to an ASP system: programs are collections of propositional...
clauses, their models are answer sets. To solve, say, a planning problem, we encode the constraints of the problem as propositional clauses in such a way that legal plans are determined by models of the resulting propositional theory. This approach, called *satisfiability planning*, received significant attention lately and was shown to be quite effective (Kautz & Selman 1992; Kautz & Selman 1996). Recently, several implementations of the ASP approach were developed that are based on nonmonotonic logics such as smodels (Niemela & Simons 1996), for stable logic programming; div (Eiter et al. 1998), for disjunctive logic programming with answer-set semantics, and deres (Cholewinski, Marek, & Truszczynski 1999), for default logic with Reiter’s extensions. All these systems have been extensively studied. Promising experimental results concerning their performance were reported (Cholewinski et al. 1999; Eiter et al. 1998; Niemela 1998).

The question arises which formal logics are appropriate as bases of answer-set program implementations. To discuss such a general question one needs to formulate quality criteria with respect to which ASP systems can be compared. At the very least, these criteria should include:

1. expressive power
2. time performance
3. simplicity of the semantics
4. ease of coding, conciseness of programs.

We will discuss these criteria in detail elsewhere. We will make here only a few brief comments on the matter. From the point of view of the expressive power all the systems that we discussed are quite similar. Propositional logic and stable logic programming are well-attuned to the class NP (Schlipf 1995). Disjunctive logic programming and default logic capture the class \( \Sigma_2 \) (Eiter & Gottlob 1993). However, this distinction is not essential as recently pointed out in (Janhunen et al. 2000). The issue of time performance can be resolved only through comprehensive experimentation and this work is currently under way.

As concerns inherent complexity of the system and intuitiveness of the semantics, ASP systems based on the propositional logic seem to be clear winners. However, propositional logic is monotone and modeling indefinite information and phenomena such as the frame problem is not quite straightforward. In applications involving the computation of transitive closures, as in the problem of existence of hamilton cycles, it leads to programs that are large and, thus, difficult to process. In this respect, ASP systems based on nonmonotonic logics have an edge. They were designed to handle incomplete and indefinite information. Thus, they often yield more concise programs. However, they require more elaborate formal machinery and their semantics are more complex.

Searching for the middle ground between systems such as logic programming with stable model semantics and propositional logic, we propose here a new ASP formalism, DC. Our guiding principle was to design a system which would lead to small-size encodings, believing that small theories will lead to more efficient solutions. We show that DC is nonmonotonic, has the same expressive power as stable logic programming but that its semantics stays closer to that of propositional logic. Thus, it is arguably simpler than the stable-model semantics. We present experimental results that demonstrate that DC is competitive with ASP implementations based on nonmonotonic logics (we use smodels for comparison) and those based on propositional logics (we use csat (Dubois et al. 1996) in our experiments). Our results strongly indicate that formalisms which provide smaller-size encodings are more effective as practical search-problem solvers.

**DATALOG with constraints**

A DC theory (or program) consists of constraints and Horn rules (DATALOG program). This fact motivates out choice of terminology — DATALOG with constraints. We start a discussion of DC with the propositional case. Our language is determined by a set of atoms \( At \). We will assume that \( At \) is of the form \( At = At_C \cup At_H \), where \( At_C \) and \( At_H \) are disjoint.

A DC theory (or DC program) is a triple \( T = (T_C, T_H, T_{PC}) \), where:

1. \( T_C \) is a set of propositional clauses \( \neg a_1 \lor \ldots \lor \neg a_m \lor b_1 \lor \ldots \lor b_n \) such that all \( a_i \) and \( b_j \) are from \( At_C \).
2. \( T_H \) is a set of Horn rules \( a_1 \land \ldots \land a_m \rightarrow b \) such that \( b \in At_H \) and all \( a_i \) are from \( At \).
3. \( T_{PC} \) is a set of clauses over \( At \).

By \( At(T) \), \( At_C(T) \) and \( At_{PC}(T) \) we denote the set of atoms from \( At \), \( At_C \) and \( At_{PC} \), respectively, that actually appear in \( T \).

With a DC theory \( T = (T_C, T_H, T_{PC}) \) we associate a family of subsets of \( At_C(T) \). We say that a set \( M \subseteq At_C(T) \) satisfies \( T \) (is an answer set of \( T \)) if:

1. \( M \) satisfies all the clauses in \( T_C \), and
2. the closure of \( M \) under the Horn rules in \( T_H \), \( M^c = LM(T_H \cup M) \) satisfies all clauses in \( T_{PC} \) (\( LM(P) \) denotes the least model of a Horn program \( P )). Intuitively, the collection of clauses in \( T_C \) can be thought of as a representation of the constraints of the problem, Horn rules in \( T_H \) can be viewed as a mechanism to compute closures of sets of atoms satisfying the constraints in \( T_C \), and the clauses in \( T_{PC} \) can be regarded as constraints on closed sets (we refer to them as *post-constraints*). A set of atoms \( M \subseteq At_C(T) \) is a model if it (propositionally) satisfies the constraints in \( T_C \) and if its closure (propositionally) satisfies the constraints in \( T_{PC} \). Thus, the semantics of DC retains much of the simplicity of the semantics of propositional logic.

DC can be used as a computational tool to solve search problems. We define a search problem \( \Pi \) to be determined by a set of finite instances, \( D_\Pi \), such that for each instance \( I \in D_\Pi \), there is a finite set \( S_\Pi(I) \) of all
solutions to \( \Pi \) for the instance \( I \). For example, the problem of finding a hamilton cycle in a graph is a search problem: graphs are instances and for each graph, its hamilton cycles (sets of their edges) are solutions. A DC theory \( T = (T_C, T_H, T_{PC}) \) solves a search problem \( \Pi \) if solutions to \( \Pi \) can be computed (in polynomial time) from answer sets to \( T \). Propositional logic and stable logic programming are used as problem solving formalisms following the same general paradigm. To illustrate all the concepts introduced here and show how DC programs can be built by modeling problem constraints, we will now present a DC program that solves the hamilton-cycle problem.

Consider a directed graph \( G \) with the vertex set \( V \) and the edge set \( E \). Consider a set of atoms \( \{hc(a,b):(a,b) \in E\} \). An intuitive interpretation of an atom \( hc(a,b) \) is that the edge \((a,b)\) is in a hamilton cycle. Include in \( T_C \) all clauses of the form \( \neg hc(b,a) \lor \neg hc(c,a) \), where \( a, b, c \in V \), \( b \neq c \) and \( (b,a),(c,a) \in E \). In addition, include in \( T_C \) all clauses of the form \( \neg hc(a,b) \lor \neg hc(a,c) \), where \( a, b, c \in V \), \( b \neq c \) and \( (a,b),(a,c) \in E \). Clearly, the set of propositional variables of the form \( \{hc(a,b):(a,b) \in F\} \), where \( F \subseteq E \), satisfies all clauses in \( T_C \) if and only if no two distinct edges in \( F \) end in the same vertex and no two distinct edges in \( F \) start in the same vertex. In other words, \( F \) spans a collection of paths and cycles in \( G \).

To guarantee that the edges in \( F \) define a hamilton cycle, we must enforce that all vertices of \( G \) are reached by means of the edges in \( F \) if we start in some (arbitrarily chosen) vertex of \( G \). This can be accomplished by means of a simple Horn program. Let us choose a vertex, say \( s \), in \( G \). Include in \( T_H \) the Horn rules \( hc(s,t) \rightarrow vstd(t) \), for every edge \((s,t) \in G\). In addition, include in \( T_H \) Horn rules \( vstd(t),hc(t,u) \rightarrow vstd(u) \), for every edge \((t,u) \in G \) not starting in \( s \). Clearly, the least model of \( F \cup T_H \), where \( F \) is a subset of \( E \), contains precisely these variables of the form \( vstd(t) \) for which \( t \) is reachable from \( s \) by a nonempty path spanned by the edges in \( F \). Thus, \( F \) is the set of edges of a hamilton cycle of \( G \) if and only if the least model of \( F \cup T_H \), contains variable \( vstd(t) \) for every vertex \( t \) of \( G \). Let us define \( T_{PC} = \{vstd(t):t \in V\} \) and \( T_{ham}(G) = (T_C, T_H, T_{PC}) \). It follows that hamilton cycles of \( G \) can be reconstructed (in linear time) from answer sets to the DC theory \( T_{ham}(G) \). In other words, to find a hamilton cycle in \( G \), it is enough to find an answer set for \( T_{ham}(G) \).

This example illustrates the simplicity of the semantics — it is only a slight adaptation of the semantics of propositional logic to the case when in addition to propositional clauses we also have Horn rules in the theories. It also illustrates the power of DC to generate concise encodings. All known propositional encodings of the hamilton-cycle problem require that additional variables are introduced to “count” how far from the starting vertex an edge is located. Consequently, propositional encodings are much larger and lead to inefficient computational approaches to the problem. We present experimental evidence to this claim later in the paper.

The question arises which search problems can be represented (and solved) by means of finding answer sets to appropriate DC programs. In general, the question remains open. We have an answer, though, if we restrict our attention to the special case of decision problems. Consider a DC theory \( T = (T_C, T_H, T_{PC}) \), where \( T_H = T_{PC} = \emptyset \). Clearly, \( M \) is an answer set for \( T \) if and only if \( M \) is a model of the collection of clauses \( T_C \). Thus, the problem of existence of an answer set is at least as hard as the propositional satisfiability problem. On the other hand, for every DC theory \( T \) and for every set \( M \subseteq At_C(T) \), it can be checked in linear time whether \( M \) is an answer set for \( T \). Thus, we obtain the following complexity result.

**Theorem 1** The problem of existence of an answer set for a finite propositional DC theory \( T \) is NP-complete.

It follows that every problem in NP can be polynomially reduced to the problem of existence of an answer set for a propositional DC program. Thus, given a problem \( \Pi \) in NP, for every instance \( I \) of \( \Pi \), \( \Pi \) can be decided by deciding the existence of an answer set for the DC program corresponding to \( \Pi \) and \( I \).

Propositional DC can be extended to the predicate case. It is important as it significantly simplifies the task of developing programs for solving problems with DC. In the example discussed above, the theory \( T_{ham}(G) \) depends heavily on the input. Each time we change the input graph, a different theory has to be used. However, when constructing predicate DC-based solutions to a problem \( \Pi \), it is often possible to separate the representation of an instance (input) to \( \Pi \) from that of the constraints that define \( \Pi \). As a result only one (predicate) program describing the constraints of \( \Pi \) needs to be written. Specific input for the program, say \( I \), can be described separately as a collection of facts (according to some uniform schema). Both parts together can be combined to yield a DC program whose answer sets determine solutions to \( \Pi \) for the input \( I \). Such an approach, we will refer to it as uniform, is often used in the context of DATALOG, DATALOG+ or logic programming to study complexity of these systems as query languages. The part representing input is referred to as the extensional database. The part representing the query or the problem is called the intensional database or program. Due to the space limitations we do not discuss the details of the predicate case here. They will be given in the full version of the paper. We only state a generalization of Theorem 1.

**Theorem 2** The expressive power of DC is the same as that of stable logic programming. In particular, a decision problem \( \Pi \) can be solved uniformly in DC if and only if \( \Pi \) is in the class NP.

**Implementation**

Some types of constraints appear frequently in applications. For instance, when defining plans we may want
to specify a constraint that says exactly what one action from the set of allowed actions be selected at each step. Such constraints can be modeled by *collections* of clauses. To make sure DC programs are as easy to write and as concise as possible we have extended the syntax of DC by providing explicit ways to model constraints of the form “select at least (at most, exactly) $k$ elements from a set”. Having these constraints results in shorter programs which, as we believe, has a significant positive effect of the performance of our system.

An example of a select constraint with a short explanation is presented here. Let $PRED$ be the set of predicates occurring in the IDB. For each variable $X$ declared in the IDB the range $R(X)$ of $X$ is determined by the EDB.

$Select(n, m; X_1, X_2, \ldots, X_k; P_1, P_2, \ldots, P_l) = \{t \in \mathbb{R}^k : \forall i (1 \leq i \leq k) \exists X_i \in P_i \}$

Clearly, if all unassigned literals were tested it would prune the most search space. At the same time, the savings might not be large enough to compensate for the increase in the running time caused by extensive lookahead. Thus, we select only a portion of all unassigned literals for lookahead. The number of literals to consider in the lookahead and which literals to select in the lookahead and which literals to select in the lookahead are determined by the previous best literal already considered.

We implemented DC in the predicate setting. Thus, our system consists of two main modules. The first of them, referred to as *grounder*, converts a predicate DC program (consisting of both the extensional and intentional parts) into the corresponding propositional DC program. The second module, DC solver, denoted $dcs$, finds the answer sets to propositional DC programs. Since we focus on the propositional case here, we only describe the key ideas behind the DC solver, $dcs$.

The DC solver uses a Davis-Putnam type approach, with backtracking, propagation and lookahead (also called literal testing), to deal with constraints represented as clauses, *select* constraints and Horn rules, and to search for answer sets. The lookahead in DC is similar to local processing performed in $csat$ (Dubois et al. 1996). However, we use different methods to determine how many literals to consider in the lookahead phase. Other techniques, especially propagation and search heuristics, were designed specifically for the case of DC as they must take into account the presence of Horn rules in programs.

The lookahead procedure selects a number of literals which have not yet been assigned a value. For each such literal, the procedure tries both truth values: true and false. For each assignment, the truth value is evaluated using propagation. If in both cases a contradiction is reached, then it is necessary to backtrack. If only one evaluation a conflict is reached, then the literal is assigned the other truth value and we proceed to the next step. If neither evaluation results in a contradiction, we cannot assign a truth value to this literal but we save the data such as the number of forced literals and the number of clauses satisfied, computed during propagation.

**Experimentation**

We compared the performance of DC solver $dcs$ with $smodels$, a system for computing stable models of logic programs (Niemela & Simons 1996), and $csat$, a system for testing propositional satisfiability (Dubois et al. 1996). In the case of $smodels$ we used version 2.24 in conjunction with the grounder $lparse$, version 0.99.41. These versions of $lparse$ and $smodels$ implement the expressive rules described by (Simons 1999). The expressive rules were used whenever applicable during the testing. The programs were all executed on a Sun SparcStation 20. For each test we report the cpu user times for processing the corresponding propositional program or theory. We tested all three system to compute hamilton cycles and colorings in graphs, to solve the $N$-queens problem, to prove that the pigeonhole problem has no solution if the number of pigeons exceeds the number of holes, and to compute Schur numbers.
The Hamilton cycle problem has already been described. We randomly generated one thousand graphs with the edge-to-vertex ratio such that \( \approx 50\% \) of the graphs contained Hamilton cycles (crossover region). The number of vertices ranged from 30 to 80. We used encodings of the problem as a DC program, logic program (in \texttt{smodels} syntax) and as a propositional theory. \texttt{dcs} performed better than \texttt{smodels} and \texttt{smodels} performed significantly better than \texttt{csat} (Fig. 1). We believe that a major factor behind poorer performance of \texttt{csat} is that all known propositional encodings of the hamilton cycle problem are much larger than those possible with DC or logic programs (under the stable model semantics). Propositional encodings, due to their size, rendered \texttt{csat} not practical to execute for graphs with more than 40 vertices.

The \( N \)-queens problem consists of finding a placement of \( N \) queens on an \( N \times N \) board such that no queen can remove another. Both \texttt{csat} and \texttt{dcs} execute in much less time than \texttt{smodels} for these problems (Fig. 2). Again the size of the encoding seems to be a major factor. One thing to consider in this case is that the number of rules for \texttt{smodels} is approximately five times that for DC and more than twice that of propositional encodings.

The Schur problem consists of placing \( N \) numbers \( 1, 2, \ldots, N \) in \( B \) bins such that no bin is closed under sums. That is, for all numbers \( x, y, z \), \( 1 \leq x, y, z \leq N \), if \( x \) and \( y \) are the same bin, then \( z \) is not \((x \text{ and } y \text{ need not be distinct})\). The Schur number \( S(B) \) is the maximum number \( N \) for which such a placement is still possible. It is known to exist for every \( B \geq 1 \). We considered the problem of the existence of the placement for \( B = 3 \) and \( N = 13 \) and 14, and for \( B = 4 \) and \( N = 43, 44 \) and 45. In each case we used all three systems to process the corresponding encodings. The results are shown in Fig. 3. It follows that \( S(3) = 13 \) and \( S(4) = 44 \). Again, \texttt{dcs} outperforms both \texttt{smodels} and \texttt{csat}.

Results for graph 3-coloring for graphs with the number of vertices ranging from 50 to 300 are shown in Fig. 4 (for every choice of the number of vertices, 100 graphs from the crossover region were randomly generated). Both \texttt{dcs} and \texttt{csat} performed better than \texttt{smodels}. Again the size of the theory seems to be a factor. The CNF theory for coloring is smaller than a logic program encoding the same problem. The sizes of propositional and DC encodings are similar.

Results for the pigeonhole placement problem show a similar performance of all three algorithms, with \texttt{csat} doing slightly better than the others and \texttt{dcs} outperforming (again only slightly) \texttt{smodels}.

**Conclusions**

We described a new system, DC, for solving search problems. We designed DC so that its semantics was as close as possible to that of propositional logic. Our goal was to design a system that would result in short prob-

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**Figure 1:** Hamilton cycle problem; times on the log scale as function of the number of vertices.

**Figure 2:** \( N \)-queens problem; log scale.

**Figure 3:** Schur problem; times and the number of choice points.

**Figure 4:** 3-coloring problem; log scale.
len encodings. Thus, we provided constructs for some frequently occurring types of constraints and we built into DC elements of nonmonotonicity by including Horn rules in the syntax. As a result, DC programs encoding search problems are often much smaller than those possible with propositional theories. Experimental results show that $dcs$ often outperforms systems based on propositional satisfiability as well as systems based on nonmonotonic logics, and that it constitutes a viable approach to solving problems in AI, constraint satisfaction and combinatorial optimization. We believe that our focus on short programs is the key to the success of DC and its reasoning engine $dcs$. Our results show that when building general purpose solvers of search problems, the size of encodings should be a key design factor.

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