We continue our study of the emergence of Non-Equilibrium Steady States in quantum integrable models focusing on the expansion of a Lieb-Liniger gas for arbitrary repulsive interaction. As a first step towards the derivation of the asymptotics of observables in the thermodynamic and large distance and time limit, we derive an exact multiple integral representation of the time evolved many-body wave-function. Starting from the known but complicated expression for the overlaps of the initial state of a geometric quench, which are derived from the Slavnov formula for scalar products of Bethe states, we eliminate the awkward dependence on the system size and distinguish the Bethe states into convenient sectors. These steps allow us to express the rather impractical sum over Bethe states as a multiple rapidity integral in various alternative forms. Moreover, we examine the singularities of the obtained integrand and calculate the contribution of the multivariable kinematical poles, which is essential information for the derivation of the asymptotics of interest.
I. INTRODUCTION

In an earlier work [1], we have studied the expansion of a Tonks-Girardeau gas from one half to the entire confining box and demonstrated how the emergence of Non-Equilibrium Steady States (NESS) can be derived from the asymptotics of the many-body wave-function in the combined thermodynamic and large distance and time limit. We avoided using the effectively free fermionic nature of the system, aiming to develop an exact method for the derivation of the asymptotics in the many-body context that would be suitable for generalisation to the genuinely interacting case of the Lieb-Liniger gas at arbitrary repulsive interaction. A crucial intermediate step in this method is the expression of the many-body wave-function in the form of a multiple rapidity integral, in which the integrand is well-defined in the thermodynamic limit.

The method is analogous to a general method for the derivation of asymptotics of equilibrium correlation functions in Bethe Ansatz solvable models [2–9] ([10] for a recent review). The main challenge in this type of problems is that, even though all of the ingredients that are necessary for a formal expression of the observable of interest are known, at least in some implicit form (Bethe states and Bethe roots, matrix elements of the observables in the Bethe state basis), expanding in a thermodynamically large eigenstate basis and extracting the asymptotics of the resulting sum is very difficult. The general method to tackle this problem consists of two main steps. The first step is to express the observable of interest in an exact multiple integral representation [2–4, 11–14]. The second step is to derive the asymptotics from the multiple integral representation [4, 6–8, 15–19]. This method has been successfully applied to the study of asymptotics of correlations in XXZ and Lieb-Liniger model at thermal equilibrium.

The first step is crucial as the expansion of the observable in the Bethe eigenstate basis is not suitable for asymptotic analysis. On the contrary, the multiple integral representation is especially suitable, since by identifying the locations of poles of the integrand in the complex rapidity plane and performing a convenient deformation of the integration contours, it is possible to reduce the derivation of the asymptotics to the evaluation of multivariable pole residues. The integrand of such multiple integral expressions generally has the form of a determinant of a thermodynamically large matrix or products and ratios of such determinants.

In the present work, we are interested in the asymptotics of observables in an out-of-equilibrium problem, more specifically, a geometric quench. The above outlined method has not been applied to out-of-equilibrium problems so far. Here, our goal is to generalise the method in the form shown in [1] to the genuinely interacting case of the Lieb-Liniger model at arbitrary interaction $c > 0$. In particular, we focus on the time evolved many-body wave-function after the quench and derive a multiple integral representation for it, thus accomplishing the first step of the above general approach. Deriving the asymptotics of this quantity in the combined thermodynamic and large distance and time limit would allow us to establish the emergence of the NESS and extract its dependence on the ray ratio $x/t = \xi$, providing a way to verify the predictions of the recently introduced Generalised HydroDynamics theory [20, 21]. It should be noted that performing this first step for the many-body wave-function is generally simpler.
than for correlation functions, because the analysis of the latter comes with additional complications related to the complex form of the matrix elements of field operators. On the other hand, in the application of this method to a quantum quench problem, a different type of complications emerge due to the fact that the initial state overlaps, when known exactly, typically have a very complex functional form: they are not necessarily smooth functions of the rapidity variables but may instead be non-vanishing only for special classes of states [22–24].

In the case of a geometric quench, in particular, even though the initial state overlaps are exactly known [25] being derivable from the Slavnov formula for scalar products of Bethe states, their complicated form does not allow for a direct expression of the many-body wave-function in the form of a multiple rapidity integral. By eliminating the system size dependence using the Bethe equations, we observe that the overlaps exhibit an alternating-sign behaviour. Because of this feature, a suitable splitting of the Bethe states into sectors is required to arrive at the desired multiple integral representation. At the same time, the analyticity properties of the resulting integrands may be severely restricted. Evidently, this endangers the possibility of performing the contour deformations that may be necessary for bringing the integral in a form convenient for the evaluation of its asymptotics. It is therefore crucial to examine and identify the singularities of the integrand of the obtained multiple integral formula and, if possible, manipulate it so as to bring it into a convenient form. For these reasons, we express the above formula in alternative forms and study the analytical properties of the integrands, observing the presence of branch cuts and calculating the residue of its multivariable kinematical poles. Based on the calculation presented in [1] and the above discussion, this information is expected to be important for the derivation of the asymptotics we are interested in.

The paper is organised as follows. We first introduce the necessary preliminaries to our calculation: the definition of the Lieb-Liniger model, its solution by Bethe Ansatz, the Slavnov and Gaudin formulas for scalar products and norms of Bethe states, respectively, as well as the definition of the quench protocol and statement of our objectives (sec. II). Short proof outlines and alternative forms of some useful formulas are included for use in later discussions. We then present in detail the derivation of a multiple integral representation for the time evolved wave-function of the system after the quench (sec. III I), starting with a manipulation of the initial state overlaps (subsec. III A), using a suitable variant of a standard complex analysis trick for passing from the sum over Bethe states to multiple rapidity integrals and further manipulating the resulting formula to obtain an alternative version (subsec. III B). We derive a simple formula for the kinematical pole residue of the integrand, a result important for subsequent steps (sec. IV), and discuss the role of branch cut singularities of the integrand. Lastly, we underline aspects of the derivation that are likely to be generally valid and their relevance for a mathematically rigorous solution of the problem of quantum dynamics in integrable models (sec. V). A heuristic derivation of a crucial intermediate formula is presented in the appendix (app. A).

II. PRELIMINARIES

A. Bethe Ansatz solution for the Lieb-Liniger model

The Lieb-Liniger model for a system of $N$ interacting bosons in a box of length $L$ with periodic boundary conditions $\Psi(+L/2) = \Psi(-L/2)$ is described by the Hamiltonian

$$H = \int_{-L/2}^{+L/2} dx \left[ -\Psi^\dagger(x)\partial_2^2\Psi(x) + c \Psi^\dagger(x)\Psi(x)\Psi(x) \right]$$

where $c$ is the interaction strength and the particle mass has been set to $m = 1/2$. In this work we will consider only the repulsive case $c > 0$. The particle number and momentum operators are respectively

$$N = \int_{-L/2}^{+L/2} dx \Psi^\dagger(x)\Psi(x)$$

$$P = -i \int_{-L/2}^{+L/2} dx \Psi^\dagger(x)\partial_x\Psi(x)$$

The Lieb-Liniger model is integrable i.e. its eigenstates are given by the Bethe Ansatz. The coordinate space wave-functions of the Bethe eigenstates are

$$\langle x|\Phi(\lambda)\rangle := \langle x_1, x_2, \ldots, x_N|\Phi(\lambda_1, \lambda_2, \ldots, \lambda_N)\rangle$$

$$\propto \frac{1}{\sqrt{N!\prod_{j>i}(\lambda_j - \lambda_i)^2 + c^2}} \sum_{\text{all perm. } \pi} (-1)^{\sigma(\pi)} \exp\left(i \sum_{i=1}^{N} \lambda_{\pi_i} x_i \right) \prod_{j>i} (\lambda_{\pi_j} - \lambda_{\pi_i} - ic \text{sign}(x_j - x_i)) \tag{1}$$
where in order to satisfy the boundary conditions, the rapidities \( \lambda := \{ \lambda_j \}_{j=1}^N \) must satisfy the Bethe equations (BA)

\[
\exp (i \lambda_j L) = \prod_{i \neq j}^{N} \frac{\lambda_j - \lambda_i + ic}{\lambda_j - \lambda_i - ic} \quad \text{for all } j = 1, 2, \ldots, N
\]  

The latter can also be written in the following equivalent form

\[
\exp (i Q_i(\lambda)) = 1
\]

where we defined

\[
Q_i(\lambda) := \lambda_i L + \sum_{j \neq i}^{N} \theta(\lambda_i - \lambda_j)
\]

with

\[
\theta(\lambda) := -i \log \left( \frac{\lambda - ic}{\lambda + ic} \right) = -i \log S(\lambda)
\]

and

\[
S(\lambda) := \frac{\lambda - ic}{\lambda + ic}
\]

the two-particle S-matrix of the Lieb-Liniger model. In the repulsive case considered here all solutions of the Bethe equations correspond to real rapidities. The energy and momentum eigenvalues corresponding to an eigenstate with rapidities \( \lambda \) are respectively

\[
E(\lambda) = \sum_{i=1}^{N} c(\lambda_i) = \sum_{i=1}^{N} \lambda_i^2
\]

\[
P(\lambda) = \sum_{i=1}^{N} p(\lambda_i) = \sum_{i=1}^{N} \lambda_i
\]

In the hard-core boson limit \( c \rightarrow \infty \) the Bethe equations simplify to

\[
\exp (i \lambda_j L) = (-1)^{N-1} \quad \text{for all } j = 1, 2, \ldots, N
\]

with solutions

\[
\lambda_j = \begin{cases} \frac{2\pi}{L} n_j & \text{for } N \text{ odd} \\ \frac{2\pi}{L} (n_j + \frac{1}{2}) & \text{for } N \text{ even} \end{cases} \quad n_j \in \mathbb{Z}
\]

and the eigenfunctions are

\[
\langle x | \Phi(\lambda) \rangle = \frac{1}{\sqrt{N!}} \det \left[ \exp (i \lambda_j x_i) \right] \prod_{j > i} \text{sign} (x_j - x_i)
\]

This is the Tonks-Girardeau limit considered in [1], which will be used as a consistency check.

**B. Bethe state scalar products: Slavnov formula**

Overlaps between Bethe states are given by the Slavnov formula [26]. The latter gives the scalar product \( S_N(\mu; \lambda) := \langle \Phi(\mu) | \Phi(\lambda) \rangle \) between two Bethe states, one of which has rapidities \( \mu \) satisfying Bethe equations while those of the other one, \( \lambda \), are left unconstrained. The Slavnov formula is

\[
S_N(\mu; \lambda) = G_N(\mu, \lambda) \det [M_N(\mu; \lambda)]_{ik}
\]
where

\[ G_N(\mu, \lambda) := \prod_{j > k}^N g(\lambda_j, \lambda_k)g(\mu_k, \mu_j) \prod_{j, k}^N h(\mu_j, \lambda_k) \]  

(10)

\[ [M_N(\mu; \lambda)]_{lk} := \frac{g(\mu_k, \lambda_l)h(\mu_l, \lambda_k)}{h(\mu_k, \lambda_k)} - r(\lambda_l) \frac{g(\lambda_l, \mu_k)}{h(\lambda_l, \mu_k)} \prod_{n=1}^N f(\lambda_l, \mu_n) \]  

(11)

and the functions \( g(\lambda, \lambda') \), \( f(\lambda, \lambda') \), \( h(\lambda, \lambda') \) and \( r(\lambda) \) are defined as

\[
g(\lambda, \lambda') := \frac{ic}{\lambda - \lambda'}
\]

\[
f(\lambda, \lambda') := \frac{\lambda - \lambda' + ic}{\lambda - \lambda'} = g(\lambda, \lambda') \left( 1 + \frac{1}{g(\lambda, \lambda')} \right)
\]

(12)

\[
h(\lambda, \lambda') := \frac{f(\lambda, \lambda')}{g(\lambda, \lambda')} = \frac{\lambda - \lambda' + ic}{ic}
\]

and

\[
r(\lambda) := e^{-i\lambda L}
\]

(13)

The Bethe equations (2) can be expressed in terms of the above functions as

\[
r(\lambda_j) \prod_{i(\neq j)}^N \frac{f(\lambda_j, \lambda_i)}{f(\lambda_i, \lambda_j)} = 1 \quad \text{for all} \quad j = 1, 2, \ldots, N
\]

(14)

Note that the overlaps as given by the Slavnov formula are symmetric under permutations of either set of rapidities \( \lambda \) and \( \mu \), as they should be. To see this we first notice that if two rapidities are exchanged then \( G_N(\mu, \lambda) \) changes sign due to the fact that \( g(\lambda', \lambda) = -g(\lambda, \lambda') \). If we do an arbitrary permutation, \( G_N(\mu, \lambda) \) will pick up a sign equal to the signature of the permutation. On the other hand, a rapidity permutation is equivalent to a permutation of the rows or columns of the matrix \( [M_N(\mu; \lambda)]_{lk} \), and so it also results in the same sign change for the determinant \( \det [M_N(\mu; \lambda)]_{lk} \), which therefore cancels out with the sign change of \( G_N(\mu, \lambda) \).

The Slavnov formula (9) can be written in the following equivalent form

\[
S_N(\mu; \lambda) = \frac{1}{\det [g(\mu_k, \lambda_l)]_{lk}} \det \left[ g^2(\mu_k, \lambda_l) \left( \prod_{m(\neq k)}^N f(\mu_m, \lambda_l) - r(\lambda_l) \prod_{m(\neq k)}^N f(\lambda_l, \mu_m) \right) \right]_{lk}
\]

(15)

To see this we first write \( G_N \) as

\[
G_N(\mu, \lambda) = \frac{\prod_{j > k}^N g(\lambda_j, \lambda_k)g(\mu_k, \mu_j)}{\prod_{j, k}^N g(\mu_j, \lambda_k)} \prod_{j, k}^N f(\mu_j, \lambda_k)
\]

\[= \frac{1}{\det [g(\mu_j, \lambda_k)]_{jk}} \prod_{j, k}^N f(\mu_j, \lambda_k)
\]

(16)

where we used the Cauchy determinant formula

\[
\det \left[ \frac{1}{\mu_j - \lambda_k} \right]_{jk} = \prod_{j > k}^N (\lambda_j - \lambda_k)(\mu_k - \mu_j) \prod_{j, k}^N (\mu_j - \lambda_k)
\]

Next we absorb the product \( \prod_{j, k}^N f(\mu_j, \lambda_k) \) in \( \det [M_N(\mu; \lambda)]_{lk} \) using the formula

\[
\det [B_iA_{ij}]_{ij} = \left( \prod_{i=1}^N B_i \right) \det A_{ij}
\]
The system lies in the ground state or any other eigenstate of the Lieb-Liniger Hamiltonian restricted in this interval

\[ H_0 = \int_0^{L/2} dx \left[ -\Psi^\dagger(x)\partial_x^2 \Psi(x) + c \Psi^\dagger(x)\Psi^\dagger(x)\Psi(x)\Psi(x) \right] \quad (20) \]
with periodic boundary conditions $\Psi(L/2) = \Psi(0)$. The rapidities $\mu$ characterising the initial state $|\Phi_0(\mu)\rangle$ which is an eigenstate of $H_0$, satisfy Bethe equations corresponding to system length equal to $L/2$ (BA$_0$)

$$\exp(ib_jL/2) = \prod_{i(\neq j)}^{N} \frac{\mu_j - \mu_i + ic}{\mu_j - \mu_i - ic} \quad \text{for all } j = 1, 2, \ldots, N$$

We then time evolve the initial state under the Lieb-Liniger Hamiltonian $H$ defined in the entire interval of length $L$ again with periodic boundary conditions $\Psi(L) = \Psi(0)$. Generally we are interested in the asymptotics of local observables $\hat{O}(x, t)$ in the thermodynamic limit where both $N$ and $L$ tend to infinity with fixed non-zero density $N/L = n$, followed by the limit of large time $t$ and distance $x$ keeping the ratio $x/t$ fixed. As discussed in more detail in [1], our strategy is to focus on the quantity

$$\mathcal{K}(\mu; z; x, t) := \langle z|e^{-iP_{x-t}Ht}||\Phi_0(\mu)\rangle$$

i.e. the time evolved many-body wave-function projected onto a local basis $\langle z |$, and to evaluate it in the above combined limit. Our objective, in the present work, is to derive an exact multiple integral formula for $\mathcal{K}(\mu; z; x, t)$ as a first step of this program. Similar multiple integral representations of many-body wave-functions in integrable models under special initial conditions have been also studied in [29, 30], even though those works focus on the many-body propagator or Green’s function which formally solves the time-dependent problem when the initial positions of the particles are given, while here we consider instead the case where the initial state is an eigenstate of the pre-quench Hamiltonian $H_0$.

## III. DERIVATION OF MULTIPLE INTEGRAL REPRESENTATION FOR THE TIME EVOLVED MANY-BODY WAVE-FUNCTION

We start by formally expanding the initial state in the post-quench basis, introducing a resolution of the identity in terms of Bethe eigenstates

$$e^{iP_{x-t}Ht}|\Phi_0(\mu)\rangle = \sum_{\lambda: \text{BA}} e^{iP_{x}Ht}\langle \Phi(\mu)\rangle(\lambda)\phi(\lambda)_{\mu} \langle \Phi(\mu)|\Phi(\lambda)\rangle \langle \Phi(\lambda)\rangle$$

$$= \sum_{\lambda: \text{BA}} e^{iP_{x}Ht}\langle \Phi(\mu)\rangle(\lambda)\frac{M(\lambda; \mu)}{N(\lambda)} \langle \Phi(\lambda)\rangle$$

where the sum runs over all solutions of the Bethe equations for $N$ particles in length $L$, the eigenstate overlaps and norms are

$$M(\lambda; \mu) := \langle \Phi(\lambda)\rangle\phi(\mu)$$

$$N(\lambda) := \langle \Phi(\lambda)\rangle\phi(\lambda)$$

and the energy and momentum eigenvalues are given by (7) and (8).

### A. Initial state overlaps

The first step of our study is the calculation of the overlaps $M(\lambda; \mu)$. Even though for a general quantum quench this is typically a hard task, luckily the initial state overlaps for the geometric quench are exactly known and given by the Slavnov formula with different Bethe equations imposed on the two Bethe states, as was observed in [25, 31].

Let us first write the overlaps in terms of the coordinate space wave-functions of the Bethe Ansatz eigenstates

$$M(\lambda; \mu) = \int_{0}^{L/2} dx \langle \Phi(\lambda)|x\rangle\langle x|\Phi(\mu)\rangle$$

Next we observe that the functional form (1) of the Bethe wave-functions is independent of the system size: the Bethe Ansatz eigenstates depend on the latter only implicitly through the rapidities and because these are solutions of the Bethe equations (2) which are the ones that depend explicitly on the system size. Therefore both pre- and post-quench eigenstates are given by the same functions of the rapidity variables, the difference being that the rapidities $\mu$ and $\lambda$
take different values as they satisfy different equations, more precisely, the Bethe equations corresponding to system sizes \(L/2\) and \(L\), respectively.

This means that in order to calculate \(\mathcal{M}(\lambda; \mu)\) we can use the Slavnov formula (9) that gives the scalar product \(S_N\) between two Bethe states, one with rapidities satisfying Bethe equations and the other with unconstrained rapidities. For our purposes we set the state described by the fixed rapidities to be an eigenstate of the half system, therefore we impose the corresponding Bethe equations (BA) for the rapidities \(\mu\)

\[
e^{-i\mu L/2} \prod_{j \neq i}^{N} \left( \frac{\mu_i - \mu_j + i\epsilon}{\mu_i - \mu_j - i\epsilon} \right) = e^{-i\mu L/2} \prod_{j \neq i}^{N} \frac{f(\mu_i, \mu_j)}{f(\mu_j, \mu_i)} = \exp \left(-iQ^{(0)}_i(\mu)\right) = 1
\]  

(24)

where we define \(Q^{(0)}_i(\lambda)\) as in (4) but for the half system i.e. replacing \(L\) by \(L/2\)

\[
Q^{(0)}_i(\lambda) := \lambda_i L/2 + \sum_{j \neq i}^{N} \theta(\lambda_i - \lambda_j)
\]

(25)

The rapidities \(\lambda\) of the other state, which are generally unconstrained, are set to satisfy the Bethe equations for the entire system of length \(L\) (BA)

\[
e^{-i\lambda L} \prod_{j \neq i}^{N} \left( \frac{\lambda_i - \lambda_j + i\epsilon}{\lambda_i - \lambda_j - i\epsilon} \right) = e^{-i\lambda L} \prod_{j \neq i}^{N} \frac{f(\lambda_i, \lambda_j)}{f(\lambda_j, \lambda_i)} = \exp (-iQ_i(\lambda)) = 1
\]

(26)

In particular, note that choosing the fixed rapidities \(\mu\) to satisfy (24) means that the function \(r\) in (9) is equal to \(r(\mu) = e^{-i\mu L/2}\), because the system size entering the definition (13) is now \(L/2\).

Noting that the fixed rapidities \(\mu\) enter in (9) in the bra-state and \(\lambda\) in the ket-state, we have

\[
\mathcal{M}(\lambda; \mu)^* = S_N(\mu; \lambda) := \prod_{j > k} g(\lambda_j, \lambda_k) g(\mu_k, \mu_j) \prod_{j,k} h(\mu_j, \lambda_k) \det \left[ \frac{g(\mu_k, \lambda_l)}{h(\mu_k, \lambda_l)} \right] \frac{g(\lambda_i, \mu_k)}{h(\lambda_i, \mu_k)} e^{-i\lambda L/2} \prod_{m=1}^{N} \frac{f(\lambda_i, \mu_m)}{f(\mu_m, \lambda_i)}
\]

(27)

with \(\mu\) and \(\lambda\) satisfying (24) and (26) respectively. This result was found in [25].

The above expression is highly inconvenient for expressing the time evolved wave-function in a multiple integral form, which is important for passing to the thermodynamic limit. The reason is that the overlaps are not continuous functions of the rapidities \(\lambda\) and in their original form (27) it is not clear how to obtain a convenient reformulation. At this point we should recall from [1] that, in order to express the overlaps \(\mathcal{M}(\lambda; \mu)\) as functions of continuous variables \(\lambda\), it was crucial to first eliminate the \(L\) dependence which enters through the factors \(e^{-i\lambda L/2}\). We do this by replacing these factors using the Bethe equations (26). Taking the square root of both sides of the equation, we find that

\[
e^{-i\lambda L/2} = \pm \sqrt{\left( \prod_{n \neq l}^{N} \frac{f(\lambda_l, \lambda_n)}{f(\lambda_n, \lambda_l)} \right)^{-1}}
\]

(28)

with either a plus or a minus sign, depending on whether in a given solution \(\lambda\) of the Bethe equations, \(Q_i(\lambda)\) is an even or odd integer multiple of \(2\pi\) respectively. We therefore introduce discrete indices \(s_i\) to distinguish the two cases, exactly as we did in the Tonks-Girardeau case [1]. More explicitly, we define

\[
s_i := e^{-iQ_i(\lambda)/2} = \begin{cases} +1 & \text{if } Q_i(\lambda)/(2\pi) \text{ even} \\ -1 & \text{if } Q_i(\lambda)/(2\pi) \text{ odd} \end{cases}, \text{ for } \lambda : \text{BA}
\]

(29)

for any solution \(\lambda\) of the Bethe equations (26). We can now redefine \(\mathcal{M}(\lambda; \mu)\) to be a function of continuous rapidity variables\(\lambda\) and discrete indices \(s\). Substituting (28) into (27) and taking the complex conjugate using the definitions of \(g, f, h\) (12,13) we finally find

\[
\mathcal{M}_s(\lambda; \mu) = \prod_{j > k}^{N} g(\lambda_j, \lambda_k) g(\mu_k, \mu_j) \prod_{j,k}^{N} h(\lambda_k, \mu_j) \det \left[ \frac{g(\lambda_l, \mu_k)}{h(\lambda_l, \mu_k)} - s_l \left( \prod_{n \neq l}^{N} \frac{f(\lambda_l, \lambda_n)}{f(\lambda_n, \lambda_l)} \right)^{1/2} \frac{g(\mu_k, \lambda_l)}{h(\mu_k, \lambda_l)} \prod_{m=1}^{N} \frac{f(\mu_m, \lambda_l)}{f(\mu_m, \lambda_l)} \right]^{1/k}
\]

(30)
Writing $S_N(\mu; \lambda)$ in the alternative form (15) we also have

$$M_a(\lambda; \mu) = \frac{1}{\det [g(\lambda_l, \mu_k)]_{lk}} \det \left[ g^2(\lambda_l, \mu_k) \left( \prod_{m(\neq k)} f(\lambda_l, \mu_m) - s_l \prod_{n(\neq l)} \left( \frac{f(\lambda_l, \lambda_n)}{f(\lambda_n, \lambda_l)} \right)^{1/2} \prod_{m(\neq k)} f(\mu_m, \lambda_l) \right) \right]_{lk}$$

To lighten the notation, we now introduce the functions $\tilde{Q}_i(\lambda)$ and $\tilde{Q}(\lambda; \mu)$ defined as follows

$$\exp \left( -i \tilde{Q}_i(\lambda) \right) := \prod_{j(\neq i)} ^N \frac{f(\lambda_i, \lambda_j)}{f(\lambda_j, \lambda_i)}$$

$$\exp \left( -i \tilde{Q}(\lambda; \mu) \right) := - \prod_{j(\neq i)} ^N \frac{f(\lambda_i, \mu_j)}{f(\mu_j, \lambda_i)}$$

The relation of $\tilde{Q}_i(\lambda)$ with $Q^{(0)}_i(\lambda)$ and $Q(\lambda; \mu)$ defined in (25) and (4) is

$$\exp \left( -i Q^{(0)}_i(\lambda) \right) = e^{-i \lambda L/2} \exp \left( -i \tilde{Q}_i(\lambda) \right)$$

$$\exp \left( -i Q(\lambda) \right) = e^{-i \lambda L} \exp \left( -i \tilde{Q}_i(\lambda) \right)$$

respectively. Moreover, notice that

$$\lim_{\lambda_i \to \mu_i} \exp \left( -i \tilde{Q}(\lambda; \mu) \right) = \prod_{j(\neq i)} ^N \frac{f(\mu_i, \mu_j)}{f(\mu_j, \mu_i)} = \exp \left( -i \tilde{Q}(\mu) \right)$$

Using these definitions we can write (31) in the more compact form

$$M_a(\lambda; \mu) = G_N(\mu, \lambda) \det \left[ \frac{g(\lambda_i, \mu_k)}{f(\lambda_i, \mu_k)} + s_l \frac{g(\mu_k, \lambda_i)}{h(\mu_k, \lambda_i)} \exp \left( -i \tilde{Q}(\lambda)/2 + i \tilde{Q}(\mu; \lambda) \right) \right]$$

Let us now focus on the analyticity properties of the above overlaps for real rapidities $\lambda$. From (12) we notice that the function $g(\lambda, \lambda')$ has a pole at $\lambda = \lambda'$, the function $f(\lambda, \lambda')$ has also a pole at the same point, but the ratio $f(\lambda, \lambda')/f(\lambda', \lambda)$ does not, and lastly the function $h(\lambda, \lambda')$ has neither poles nor zeros for real $\lambda$. Therefore $G_N(\lambda, \mu)$ has no poles, while each matrix element $[M_N(\lambda; \mu)]_{lk}$ has a pole at $\lambda_l = \mu_k$ due to the $g$ functions. We therefore find that the overlaps $M_a(\lambda; \mu)$ exhibit simple poles when any of the rapidities $\lambda_i$ tends to any of the $\mu_j$ i.e. they exhibit $N$-dimensional poles at $\lambda \to \mu$ and permutations thereof, all of which have the same residue, since as explained in Sec. II B the Slavnov formula is symmetric under rapidity permutations. In addition, there are singularities coming from the square root factors $\exp(-i \tilde{Q}(\lambda)/2)$. Indeed, due to the factors $f(\lambda_i, \lambda_j)/f(\lambda_j, \lambda_i)$ in (32), the function $\exp(-i \tilde{Q}(\lambda))$ has poles and zeros when $\lambda_i$ approaches the points $\lambda_j \pm ic$ where $\lambda_j$ is any other rapidity. When the rapidities $\lambda$ are in the neighbourhood of the real axes, these points are all distant from them, lying on lines parallel to the real axes at distance equal to $c$. However, the square roots in $\exp(-i \tilde{Q}(\lambda)/2)$ give rise to branch cuts in the complex $\lambda_i$-plane that start from and connect the pairs of points $\lambda_j \pm ic$, and therefore cross the real $\lambda_i$-axis at the positions $\lambda_i = \lambda_j$ for each index $j$. These are all the singularities of $M_a(\lambda; \mu)$.

Analogously to the non-interacting case, the poles at $\lambda = \mu$ and permutations are of “kinematical” type and they reflect the elasticity of particle scattering in integrable models. Their presence is a characteristic property of scalar products of Bethe states and, together with recursion relations satisfied by their residues and other basic requirements (permutation symmetry and decay properties with respect to the rapidity arguments), they are sufficient to completely determine the scalar product. In fact this is precisely the way the Slavnov formula was derived in [26].

The branch cut singularities, on the other hand, are a consequence of the initial splitting of the system in two halves, which results in the different functional form of the overlaps in the odd and even sectors, in combination with the fact that the S-matrix (6) is negative at small rapidity differences, which is a general property of Bethe Ansatz solvable models.

B. Eigenstate summation through multivariable Cauchy’s integral formula

The second problem we encounter is the summation over post-quench eigenstates. The discrete allowed values of post-quench eigenstate rapidities $\lambda$ in (23) are solutions of the non-linear highly complicated Bethe equations,
therefore they are not explicitly known as in the Tonks-Girardeau case with periodic boundary conditions. Luckily also this problem can be circumvented, using essentially the same complex analysis trick used in the Tonks-Girardeau case [1]: The sum over eigenstates can be still transformed into a (multiple) contour integral over complex rapidities by introducing a suitable function of the rapidities that has simple poles of residue 1/(2\pi i) precisely at the roots of Bethe equations, and distinguishes between odd and even integer sectors for each rapidity \( \lambda_j \) through the corresponding discrete index \( s_j \).

Such a function is indeed possible to construct, despite the fact that the Bethe roots are only implicitly known [2–4, 10]. As we explain in detail in App. A, the suitable function is

\[
F_s(\lambda) := \frac{1}{(4\pi)^N} g_{N,L}(\lambda) \prod_{i=1}^{N} \frac{1}{1 - s_i e^{-iQ_i(\lambda)/2}}
\]

(38)

where \( g_{N,L}(\lambda) \) is defined in (18).

Attention must again be paid to the singularities of this function, since we need to ensure that not only the right function has poles at the right points, but also that it is otherwise analytic in the region of the complex plane scanned by the contours when deformed as required for the evaluation of the asymptotics. We observe that the function \( F_s(\lambda) \) has the same branch cut singularities as the overlaps \( M_s(\lambda; \mu) \), due to the square roots \( \exp(-iQ_i(\lambda)/2) \) appearing in (38), which are the same as those of \( \exp(-iQ_i(\lambda)/2) \) in (37). In the Tonks-Girardeau limit studied in [1], these square roots were not a real problem, because in that limit the function \( \exp(-iQ_i(\lambda)) \) reduces to \(-1)^{N-1} \exp(-i\lambda_j L)\) which has neither poles nor zeroes. On the contrary, in the present general case the function \( \exp(-iQ_i(\lambda)) \), like \( \exp(-iQ_i(\lambda)) \), has poles and zeroes when \( \lambda_i \) is at the points \( \lambda_j \pm ic \), due to the factors \( f(\lambda_i, \lambda_j)/f(\lambda_j, \lambda_i) \), and the square root in \( \exp(-iQ_i(\lambda)/2) \) gives rise to branch cuts crossing the real axes at the positions \( \lambda_i = \lambda_j \), i.e. whenever two rapidities are equal to each other.

Even though these singularities can be removed by a redefinition of \( F_s(\lambda) \) such that the branch cuts are diverted to imaginary infinity instead of crossing the real axis (which can be done by introducing \(-1\) factors in the arguments of the square roots, compensated by an equal number of imaginary unit factors outside of them), such a change would not really solve the problem. First, the cost of this change would be a modification of the signs that the corrected summation over Bethe states in (38) refers to ordered rapidities and passing from sum to integral is also restricted by construction to the domain \( \lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_N \). Since the summand (and corresponding integrand) is symmetric under permutation of the rapidities, it is possible to raise the ordering constraint and extend the rapidity domain to the entire real axis dividing by \( N! \), which is generally convenient but not necessary. In the present case, we note that we do not aim at deforming the integration contours away from the real rapidity axis, since for the asymptotic analysis we need to focus on the contribution of the kinematical poles, which also lie on the real axis like the Bethe roots themselves. Therefore, it is not required to extend the integration to the entire real rapidity axes and there is no real obstacle to contour deformation due to the branch cut singularities of the integrand at the boundaries of this domain \( \lambda_i = \lambda_{i+1} \) for any index \( i \), as soon as we can keep the boundaries of the integration region fixed.

Recall now the multivariable version of the residue theorem [1, 10], which states that, given a function \( F(\lambda) = 1/\prod_{i=1}^{N} f_i(\lambda) \) such that \( f_i(\lambda) \) has simple zeroes at the collection of points \( \{ \lambda^* \} \), a multi-dimensional contour \( C = C_1 \times C_2 \times \cdots \times C_N \) encircling these points, and a function \( g(\lambda) \) of the rapidities \( \lambda \) that is analytic inside the contour \( C \) except at the points \( \{ \lambda_p \} \) which are all different from \( \{ \lambda^* \} \), we have

\[
\oint_C \frac{d^N\lambda}{(2\pi i)^N} F(\lambda) g(\lambda) = \sum_{\lambda^*} \frac{g(\lambda^*)}{\det \left( \frac{\partial f_i}{\partial \lambda_j} \right)_{\lambda^*}} + \sum_{\lambda_p} F(\lambda_p) \frac{\text{Res} g(\lambda)}{\lambda = \lambda_p}
\]

(39)

This allows us to write the sum over Bethe states as

\[
\sum_{\lambda: BA} \cdots = \frac{1}{N!} \sum_s \oint_C d^N\lambda F_s(\lambda) \cdots
\]

where the kinematical poles should be excluded from the contour \( C \).
Using the above defined function (38) and substituting (37) and (17), the eigenstate sum in (23) can be written in integral form as

\[ e^{+iP_x-iHt}\langle \Phi_0(\mu) \rangle = \sum_{\lambda \in \Lambda A} e^{+iP(\lambda x-iE(\lambda)t)} \frac{M(\lambda; \mu)}{N(\lambda)} \langle \Phi(\lambda) \rangle \]

\[ = \sum_s \oint_C dN \lambda F_s(\lambda) e^{+iP(\lambda x-iE(\lambda)t)} \frac{M_s(\lambda; \mu)}{N(\lambda)} \langle \Phi(\lambda) \rangle \]

\[ = \oint_C \frac{d^N \lambda}{(4\pi)^N} e^{+iP(\lambda x-iE(\lambda)t)} \frac{G_N(\lambda, \mu)}{c^N} \prod_{m \neq \ell} f(\lambda_m, \lambda_\ell) \times \sum_s \prod_{i=1}^N \frac{1}{1 - s_i e^{-iQ_i(\lambda)/2}} \det \left[ \frac{g(\lambda_i, \mu_k)}{h(\lambda_i, \mu_k)} + s_i \frac{g(\mu_k, \lambda_i)}{h(\mu_k, \lambda_i)} e^{-iQ_i(\lambda)/2 + i\tilde{Q}_i(\lambda; \mu)} \right]_{lk} \langle \Phi(\lambda) \rangle \]

(40)

where the multi-contour \( C \) in the above integral is defined as

\[ C = C_1 \times C_2 \times \cdots \times C_N \]

(41)

\[ C_i = C_i^{(p)} \]

\[ C_i^{(r)} = (\lambda_{i-1} + \epsilon - i\epsilon, \lambda_{i+1} - \epsilon - i\epsilon, \lambda_{i+1} - \epsilon + i\epsilon, \lambda_{i-1} + \epsilon - i\epsilon) \]

\[ C_i^{(p)} = (\mu_i, \eta) \]

In more detail, each of the contours \( C_i \) should enclose the interval \((\lambda_{i-1}, \lambda_{i+1})\) of the real rapidity axis, where all Bethe roots lie in the repulsive case, but it should exclude the kinematical pole of the overlap \( M_s(\lambda; \mu) \) at \( \lambda_i \to \mu_i \). This is achieved by choosing \( C_i \) to consist of a thing rectangle \( C_i^{(r)} \) composed by two straight lines one just above and one just below this real axis interval, and subtracting a small circle \( C_i^{(p)} \) around the the value \( \mu_i \). Note that this pole is between the Bethe roots and does not coincide with any of them in general. This is because the roots of the half-system Bethe equations are not roots of the full-system Bethe equations, since otherwise by dividing (24) and (26) we would have that \( e^{i\mu_i L/2} = 1 \) i.e. the corresponding rapidities are integer multiples of \( 4\pi/L \). This means that the occasion of coinciding pre- and post-quench momenta for even eigenstates that occurs in the Tonks-Girardeau case for odd \( N \) [1] is exceptional and not present in the interacting case. Also note that all other constituents of the integrand in (40) i.e. the functions \( e^{+iP(\lambda x-iE(\lambda)t)} \), \( \det^{-1}(\lambda) \) and the state \( |\Phi(\lambda)\rangle \) itself, are analytic functions for real rapidities and do not introduce any other singularities.

In addition, the summation over the discrete indices \( s \) can be perfomed at this step. By absorbing the product of (38) and the sum over \( s \) into the determinant of (40), we can write the expression of the last line of (40) as

\[ \sum_s \prod_{i=1}^N \frac{1}{1 - s_i e^{-iQ_i(\lambda)/2}} \det \left[ \frac{g(\lambda_i, \mu_k)}{h(\lambda_i, \mu_k)} + s_i \frac{g(\mu_k, \lambda_i)}{h(\mu_k, \lambda_i)} e^{-iQ_i(\lambda)/2 + i\tilde{Q}_i(\lambda; \mu)} \right]_{lk} \]

\[ = \det \left[ \sum_s \frac{g(\lambda_i, \mu_k)}{h(\lambda_i, \mu_k)} + s_i \frac{g(\mu_k, \lambda_i)}{h(\mu_k, \lambda_i)} e^{-iQ_i(\lambda)/2 + i\tilde{Q}_i(\lambda; \mu)} \right]_{lk} \]

\[ = \det \left[ \frac{2}{1 - e^{-iQ_i(\lambda)}} \right]_{lk} \]

\[ = 2^N \prod_{i=1}^N \frac{1}{1 - e^{-iQ_i(\lambda)}} \det \left[ \frac{g(\lambda_i, \mu_k)}{h(\lambda_i, \mu_k)} + \frac{g(\mu_k, \lambda_i)}{h(\mu_k, \lambda_i)} e^{-iQ_i(\lambda)/2 + i\tilde{Q}_i(\lambda; \mu)} \right]_{lk} \]

(42)
Substituting back to (40), we obtain the alternative formula
\[
e^{iP_x-iHt}\langle \Phi_0(\mu) \rangle = \int_{C} \frac{d^N \lambda}{(2\pi)^N} e^{iP(\lambda) x-iE(\lambda)t} \frac{G_N(\lambda, \mu)}{c^N} \left( \prod_{m \neq \ell} I(\lambda_m, \lambda_l) \right) \times \prod_{i=1}^{N} \frac{1}{1-e^{-iQ_i(\lambda)}} \det \left[ \frac{g(\lambda_i, \mu_k)}{h(\lambda_i, \mu_k)} + \frac{g(\mu_k, \lambda_i)}{h(\mu_k, \lambda_i)} \right] e^{-i\Delta_1(\lambda)/2-i\Delta_2(\lambda)/2+i\Delta(\lambda; \mu)}_{ik} |\Phi(\lambda)| \]
\[(43)\]

Note that in passing from (40) to (43) using (42), we have implicitly used that the eigenstates $|\Phi(\lambda)\rangle$ are essentially independent of the indices $s$. This is true as long as we are interested in the dynamics of the system in the bulk in the thermodynamic limit. Indeed, the coordinate space wave-functions of the Bethe eigenstates (1) are continuous functions of the rapidities $\lambda$ in the bulk of the system, independent of the quantisation conditions (2), therefore, independent of whether they correspond to odd or even quantum numbers. The dependence on the indices $s$ is only evident close to the boundaries, which move away to infinity in the thermodynamic limit.

Eqs. (40) and (43) together with the definition (41) of the multi-contour $C$ are our main results for the time evolved many-body wave-function expressed in a multiple integral representation. The integrands in these expressions involve determinants originating from the Slavnov formula with adjustments and manipulation specific to the geometric quench problem.

**IV. MULTIVARIABLE KINEMATICAL POLE RESIDUE**

Having identified the analyticity properties and the location of singularities of the integrands of (40) and (43) paves the way for the derivation of the asymptotics in the combined thermodynamic and large distance and time limit. As an additional step towards this direction, we will now focus on the contribution of the multivariable kinematical pole at $\lambda \to \mu$ to the integral, calculating its residue. As anticipated based on the Tonks-Girardeau calculation [1], this is expected to give the only non-vanishing contribution in the thermodynamic and large time and distance limit.

Explicitly, we split the integral in (40) into two parts, one corresponding to the $N$-dimensional kinematical pole residue and the remainder $\mathcal{R}$ consisting of all other contributions, i.e. all cross terms corresponding to products of lower-order pole residues and integrals
\[
e^{iP_x-iHt}\langle \Phi_0(\mu) \rangle = (-2\pi i)^N \sum_s F_s(\lambda) e^{iP(\mu) x-iE(\mu)t} \frac{\text{Res}_s M_s(\lambda; \mu)}{\mathcal{N}(\mu)} |\Phi(\mu)| + \mathcal{R} \]
\[(44)\]

Note that the presence of the minus sign in $(-2\pi i)^N$ is due to the pole contribution being subtracted from each of the rapidity integrals.

We start by evaluating $F_s(\lambda)$ at $\lambda = \mu$. Since the rapidities $\mu$ satisfy the Bethe equations in the half system (24), we have
\[
\exp(-iQ_j(\mu)) = \exp(-\mu_j L) \exp \left( -i\tilde{Q}_j(\mu) \right) \\
= \exp(-\mu_j L/2) \exp \left( -i\tilde{Q}_j(\mu) \right) \\
= \exp(-\mu_j L/2) \]
\[(45)\]

from which we find
\[
\exp(-iQ_j(\mu)/2) = \rho_j \exp(-\mu_j L/4) 
\]
where, similarly to $s_i$, we have introduced the sign $\rho_i := e^{-i\tilde{Q_i}(\mu)/2} = \pm 1$ corresponding to the $i$-th rapidity of the set $\mu$ considered as a solution of the Bethe equations for the half system. Therefore, from (38) we obtain
\[
F_s(\mu) = \frac{1}{(4\pi)^N} \left( \prod_{i=1}^{N} \frac{1}{1-s_i e^{-i\tilde{Q}_i(\mu)/2}} \right) \varphi_{N,L}(\mu) \\
= \frac{1}{(4\pi)^N} \left( \prod_{i=1}^{N} \frac{1}{1-s_i \rho_i e^{-i\mu_i L/4}} \right) \varphi_{N,L}(\mu) 
\]
\[(46)\]
Next we evaluate the residue of the overlaps \( M_s(\lambda; \mu) \) at \( \lambda = \mu \) from (30). As mentioned earlier, the \( N \)-dimensional pole of \( M_s(\lambda; \mu) \) at this point is due to the presence of the functions \( g(\mu, \lambda) = ic/(\mu - \lambda) \) in the matrix \( M_N(\mu; \lambda) \).

By expanding the \( \det [M_N(\mu; \lambda)]_{lk} \) as a sum over permutations of products of matrix elements, we see that there is exactly one term that contains all \( \lambda \). By expanding the determinant appearing in (37) when considered as a function of continuous rapidities \( \lambda \). More specifically, in the present case we have

\[
\text{Res}_{\lambda=\mu} \det \left[ \frac{g(\lambda_l, \mu_k)}{h(\lambda_l, \mu_k)} + s_l \frac{g(\mu_k, \lambda_l)}{h(\mu_k, \lambda_l)} e^{-iQ_l(\lambda_l)/2+iQ_l(\lambda_l;\mu)} \right]_{lk} = \text{Res}_{\lambda=\mu} \prod_{k=1}^{N} \left[ \frac{g(\lambda_k, \mu_k)}{h(\lambda_k, \mu_k)} + s_k \frac{g(\mu_k, \lambda_k)}{h(\mu_k, \lambda_k)} e^{-iQ_k(\lambda_k)/2+iQ_k(\lambda_k;\mu)} \right] = \prod_{k=1}^{N} \frac{1}{h(\mu_k, \mu_k)} \left[ 1 - s_k e^{-iQ_k(\mu)/2} \lim_{\lambda_k \rightarrow \mu_k} e^{iQ(\lambda_k;\mu)} \right] \text{Res}_{\lambda=\mu} g(\lambda_k, \mu_k) = (ic)^N \prod_{k=1}^{N} \left[ 1 - s_k e^{-iQ_k(\mu)/2+iQ_k(\mu)} \right]
\]

In the third line we used that \( g(\mu, \lambda) = -g(\lambda, \mu) \), while in the last line we used (36) and replaced \( \text{Res}_{\lambda=\mu} g(\lambda, \mu) = ic \) and \( h(\mu, \mu) = 1 \). Using once again the half system Bethe equations (24) from which we obtained (45), we have

\[
e^{+iQ(\mu)} = e^{-i\mu/L/2}
\]

and

\[
e^{+iQ(\mu)/2} = \rho e^{-i\mu/L/4}
\]

so that we can rewrite the last result as

\[
\text{Res}_{\lambda=\mu} \det \left[ \frac{g(\lambda_l, \mu_k)}{h(\lambda_l, \mu_k)} + s_l \frac{g(\mu_k, \lambda_l)}{h(\mu_k, \lambda_l)} e^{-iQ_l(\lambda_l)/2+iQ_l(\lambda_l;\mu)} \right]_{lk} = (ic)^N \prod_{k=1}^{N} \left( 1 - s_k e^{+iQ_k(\mu)/2} \right) = (ic)^N \prod_{k=1}^{N} \left( 1 - s_k \rho e^{-i\mu_k L/4} \right)
\]

Finally, using also (19) for \( G_N(\mu, \mu) \), we obtain the following result for the kinematical pole residue of the overlaps

\[
\text{Res}_{\lambda=\mu} M_s(\lambda; \mu) = (ic)^N \prod_{i \neq j} f(\mu_i, \mu_j) \prod_{k=1}^{N} \left( 1 - s_k \rho e^{-i\mu_k L/4} \right)
\]

Substituting (46), (47) and the norm formula (17), we obtain

\[
F_s(\mu) = \frac{\text{Res}_{\lambda=\mu} M_s(\lambda; \mu)}{N(\mu)} = \left( \frac{i}{4\pi} \right)^N \prod_{k=1}^{N} \left( \frac{1 - s_k \rho e^{-i\mu_k L/4}}{1 - s_k \rho e^{-i\mu_k L/4}} \right) = \left( \frac{i}{4\pi} \right)^N
\]

We notice that the value of the residue is the same independently of the signs \( s_k \) and \( \rho_k \). Using the last result, we finally conclude that the contribution of the kinematical pole residue in (44) is

\[
(-2\pi i)^N \sum_s F_s(\mu) e^{+iP(\mu)x-iE(\mu)t} \frac{\text{Res}_{\lambda=\mu} M_s(\lambda; \mu)}{N(\mu)} \langle \Phi(\mu) \rangle = e^{+iP(\mu)x-iE(\mu)t} \langle \Phi(\mu) \rangle
\]
Compared to the long and complicated form of intermediate formulas, this is a surprisingly simple and elegant result. Note especially that, despite the necessary step of decomposition into odd and even sectors, the final formula for the residue does not contain any sign of this. It can be readily verified that the above result is consistent with that of [1] for the Tonks-Girardeau limit.

V. DISCUSSION

In this work we have derived a multiple integral representation for the time evolved wave-function after a geometric quench in the Lieb-Liniger model for arbitrary interaction $c > 0$. We have also discussed the analyticity properties of the integrand, identified its singularities and calculated the kinematical pole residue. These results serve as a first step towards the derivation of the asymptotics of observables in the thermodynamic and large distance and time limit, and therefore the complete characterisation of the emergent NESS.

The formulas derived here have been verified numerically for two and three particles. Even though partial, these tests are useful, especially since the formulas involve alternating signs and all steps of the calculation are very sensitive to sign errors. More specifically, we have verified all of the alternative forms (30), (31) and (37) of the initial state overlaps (27). We have also verified that the function (38) is the correct function for rewriting the sum over Bethe states as a multiple integral. Moreover, we have verified the equivalence between the original sum over Bethe states (23) and each of the multiple integral formulas (40) and (43). To this end, we considered closed and finite complex plane contours encircling several Bethe roots, evaluated the resulting integrals numerically and compared them with the sums over the specified encircled Bethe roots. This test was performed both in the case of the kinematical pole being located inside and outside of the chosen contour. Lastly, we have verified the value of the kinematical pole residue (49), especially the independence of (48) from the signs $s$ and $\rho$ and the pre-quench rapidities $\mu$, both by numerical integration along small contours encircling them and by numerical evaluation of the limit formula.

As discussed in the introduction, a key point in the derivation of the asymptotics of observables using a multiple integral representation is whether the integrand is analytic to an extent sufficient to allow for the necessary deformations of the integration contours. In the case of the geometric quench studied here we observe that, owing to the form of the initial state overlaps, the integrand exhibits inevitable singularities (branch cuts) that obstruct extensive contour deformations. Nevertheless, the kinematical poles of the integrand that are expected to contain all information about the asymptotics we are interested in lie infinitesimally close to the original contours, specifically, within the locus of the Bethe roots. As a result, the necessary contour deformations are only infinitesimally small and therefore are not obstructed by the presence of the branch cut singularities. Furthermore, by calculating the kinematical pole residue of the integrand, we demonstrated that it is completely independent of the fine details of the initial state overlaps. This is an indication that, despite the complicated form of the initial state overlaps, detailed information about them is irrelevant in the thermodynamic and large distance and time limit. This suggests that different initial states that correspond to the same rapidity density remain close to each other also during the time evolution and result in the same asymptotic behaviour. This observation is consistent with general statistical physics arguments that are the basis of the Quench Action approach.

Appendix A: Construction of the meromorphic function with poles at even/odd Bethe roots

An essential step in our calculation is the transformation of the sum over Bethe states into a multiple integral, which can be done using a meromorphic function $F_s(\lambda)$ having poles of equal residue at the Bethe roots and selecting all those corresponding to the sector with parity indices $s$. In this appendix we discuss in detail how the suitable function can be heuristically constructed.

From the Bethe equations (BA) in the full length system (26), we can see that a simple function having poles at all Bethe roots is

$$\prod_{i=1}^{N} \frac{1}{\exp(iQ_i(\lambda)) - 1}$$

To ensure that the residue of the required function at each of these poles equals $1/(2\pi i)$, we should multiply with
\[ \det \left( \frac{\partial}{\partial \lambda_j} e^{iQ_i(\lambda)/2} \right), \] therefore obtaining
\[
\frac{1}{(2\pi)^N} \prod_{i=1}^{N} \left( \frac{1}{\exp(iQ_i(\lambda)) - 1} \right) \det \left( \frac{\partial}{\partial \lambda_j} e^{iQ_i(\lambda)} \right) 
= \frac{1}{(2\pi)^N} \prod_{i=1}^{N} \left( \frac{1}{1 - e^{-iQ_i(\lambda)/2}} \right) \det \left( \frac{\partial Q_i(\lambda)}{\partial \lambda_j} \right) 
\tag{A1}
\]

However, we also need to distinguish the roots between odd and even integer sectors for each rapidity. Writing the fraction appearing in the product of the last expression as
\[
\frac{1}{1 - e^{-iQ_i(\lambda)/2}} = \left( \frac{1}{1 - e^{-iQ_i(\lambda)/2}} \right) \left( 1 + e^{-iQ_i(\lambda)/2} \right)
\]
we see that the first factor corresponds to the even sector and the second to the odd. The last expression can also be written as
\[
\frac{1}{(1 - e^{-iQ_i(\lambda)/2}) (1 + e^{-iQ_i(\lambda)/2})} = \frac{1}{2} \left( \frac{1}{1 - e^{-iQ_i(\lambda)/2}} + \frac{1}{1 + e^{-iQ_i(\lambda)/2}} \right) 
= \frac{1}{2} \sum_{s=\pm 1} \frac{1}{1 - se^{-iQ_i(\lambda)/2}}
\]

Therefore the function \((A1)\) that has poles at all Bethe roots can be equivalently written as
\[
F(\lambda) := \frac{1}{(4\pi)^N} \prod_{i=1}^{N} \sum_{s_i=\pm 1} \frac{1}{1 - s_i e^{-iQ_i(\lambda)/2}} \det \left( \frac{\partial Q_i(\lambda)}{\partial \lambda_j} \right)
= \sum_s \frac{1}{(4\pi)^N} \prod_{i=1}^{N} \frac{1}{1 - s_i e^{-iQ_i(\lambda)/2}} \varphi_{N,L}(\lambda)
\tag{A2}
\]
where we also used \((18)\). From the last expression it is clear that the right function that selects the Bethe roots belonging to the sector corresponding to a given vector of discrete indices \(s\) is
\[
F_s(\lambda) := \frac{1}{(4\pi)^N} \varphi_{N,L}(\lambda) \prod_{i=1}^{N} \frac{1}{1 - s_i e^{-iQ_i(\lambda)/2}}
\]
which is the function used in the main text. Summing the roots of all possible sectors, we recover the entire set of Bethe roots, \(F(\lambda) = \sum_s F_s(\lambda)\).

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