Impedance measurement technique for quantum systems

S.N. Shevchenko

B. Verkin Institute for Low Temperature Physics and Engineering, 47 Lenin Ave., Kharkov 61103, Ukraine

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Abstract. The impedance measurement technique consists in that the phase-dependent (parametric) inductance of the system is probed by the classical tank circuit via measuring the voltage. The notion of the parametric inductance for the impedance measurement technique is revisited for the case when a quantum system is probed. Measurement of the quantum state of the system of superconducting circuits (qubits) is studied theoretically. It is shown that the result of the measurement is defined by the partial energy levels population in the qubits and by its derivative.

PACS. 85.25.Am Superconducting device characterization, design, and modeling – 85.25.Cp Josephson devices

1 Introduction

The supercurrent $I$, flowing through a weak link between two bulk superconductors with the phase difference $\phi$, has the properties as a nonlinear inductor. This can be described by introducing the phase-dependent (parametric) inductance $L = (\Phi_0/2\pi)(\partial I/\partial \phi)^{-1}$. If the weak link is included in the ring, then the phase $\phi$ is related to the magnetic flux $\Phi$, piercing the ring, $\phi = 2\pi\Phi/\Phi_0$. The parametric inductance can be measured [1] and being inductively coupled to the resonant $LC$ tank circuit provides the tool to measure the flux $\Phi$ [2,3]. The effective inductance of the tank circuit depends on the parametric inductance $L(\phi)$ and current $I(\phi)$ and thus the measurement in the tank circuit can be used for finding the inductance $L$ [4], which is the so-called impedance measurement technique.

The impedance measurement technique was recently applied for the measurement of the small currents in mesoscopic samples [5] and was proposed for the description of the currents in superconducting qubits [6-9]; the series of the experimental results were obtained [10]. However the theoretical works in this field consider mostly the ground state. If the superconducting qubit is excited to the upper state, then the current in it has the probabilistic character, and in this way the parametric inductance depends not only on the clockwise (counter-clockwise) current value but also on the probabilities of the respective states. This consideration was used for the description of the phase-biased charge qubit [11]. And in this paper we study in detail the specifics of the impedance measurement technique when the quantum system is probed. The detailed presentation is aimed to show how the tank circuit is influenced by the parametric inductance, and how this inductance have to be treated for the system of coupled superconducting circuits (qubits). For concreteness we consider the superconducting circuits to be either flux or phase-biased charge qubits. The flux qubit [12] consists of a loop with three Josephson junctions. The phase-biased charge qubit [6,7] consists of a loop with two closely situated Josephson junctions and with the gate, which controls the charge on the island between the junctions.

2 Tank circuit coupled to quantum object

2.1 Equations for tank circuit

The quantum system (coupled superconducting qubits) is considered to be weakly coupled via a mutual inductance $M$ to the classical tank circuit. The circuit consists of the inductor $L_T$, capacitor $C_T$, and the resistor $R_T$ connected in parallel. The tank circuit is biased by the current $I_{\text{bias}}$, and the voltage on it $V_T$ can be measured. To obtain the equation for the voltage, we write down the system of equations, for the current in the three branches, namely, through the inductor ($I_L$), the capacitor ($I_C$), and the resistor ($I_R$) (see e.g. in the Chapter 14 of Ref. [2]):

$$I_{\text{bias}} = I_L + I_C + I_R,$$
$$I_C = \dot{\epsilon}, \quad \epsilon = C_T V_T,$$
$$I_R = V_T/R_T,$$
$$V_T = L_T I_L - \Phi_e,$$

where $\epsilon$ is the charge at the capacitor plate, the dot stands for the time derivative, $\Phi_e$ is the flux through the tank.
circuit. This flux is the response of the quantum system to the flux, induced in it by the current \( I_L \), and its time derivative equals (see below for details):
\[
\Phi_e = \bar{L} I_L, \tag{5}
\]
and thus equation (4) can be rewritten by introducing the effective inductance of the tank circuit \( L_{\text{eff}} \):
\[
V_T = L_{\text{eff}} I_L, \tag{6}
\]
\[
L_{\text{eff}} = L_T - \bar{L}. \tag{7}
\]
Then from the system of equations (1)–(4) we derive the equation for the voltage in the tank circuit:
\[
C_T \ddot{V}_T + R_T \dot{V}_T + L_{\text{eff}}^{-1} V_T = i_{\text{bias}}. \tag{8}
\]

### 2.2 Effective inductance of qubits

Now we derive the relation (5): consider the flux \( \Phi_e = \sum_i \Phi^{(i)}_e \), where \( \Phi^{(i)}_e \) is the flux induced by ith qubit in the tank circuit: \( \Phi^{(i)}_e = M_{iT} I^{(i)}_{qb} \). Here \( M_{iT} \) is the mutual inductance of the qubit and the circuit, \( i^{(i)}_{qb} \) is the current in the ith qubit which equals to the expectation value of the current operator: \( i^{(i)}_{qb} = \langle \hat{I}_i \rangle = \hat{S} \hat{\rho} \hat{I}_i \), where \( \hat{\rho} \) is the reduced density matrix of the system of qubits. (Note that for one qubit, substituting \( \Phi_e = M I_{qb} \), equation (8) coincides with Eq. (15) in Ref. [8].)

The total flux that threads the loop of the ith qubit \( \Phi^{(i)} \) consists of the external magnetic flux \( \Phi^{(i)}_x \) and the self-induced flux \( -L_i i^{(i)}_{qb} \) (\( L_i \) is the geometrical inductance of the loop):
\[
\Phi^{(i)} = \Phi^{(i)}_x - L_i i^{(i)}_{qb}. \tag{9}
\]
This equation can be rewritten by introducing the parametric inductance,
\[
L_i^{-1} = \frac{\partial i^{(i)}_{qb}}{\partial \Phi^{(i)}_x}, \tag{10}
\]
to relate the variations of the external flux through the qubit \( \delta \Phi^{(i)}_x \) and of the current in it \( \delta i^{(i)}_{qb} \), as following:
\[
\delta \Phi^{(i)}_x = \delta \Phi^{(i)} + L_i \delta i^{(i)}_{qb} = (L_i + L_i) \delta i^{(i)}_{qb}. \tag{11}
\]
Thus, we obtain the variation of the flux induced by the qubit in the tank circuit:
\[
\delta \Phi^{(i)} = M_{iT} \delta i^{(i)}_{qb} = \frac{M_{iT}}{L_i + L_i} \delta \Phi^{(i)}_x. \tag{12}
\]
The flux \( \Phi^{(i)}_x \) in the ith qubit is considered to consist of the fluxes induced by the tank circuit, \( M_{iT} I_L \), by the microwave source, \( \Phi^{(i)}_{\text{ac}} \sin \omega t \), and by additional lines and by other qubits, \( \Phi^{(i)}_{\text{shift}} \) [13]:
\[
\Phi^{(i)}_x(t) = M_{iT} I_L(t) + \Phi^{(i)}_{\text{shift}} + \Phi^{(i)}_{\text{ac}} \sin \omega t. \tag{13}
\]
Now let us recall that we consider the variation of the flux in order to calculate the derivative in time \( \Phi_e \) which have to be substituted in equation (4), which describe the tank circuit. We note that usually the dynamics of the tank circuit (with the frequency \( \omega_{\text{rf}} \) close to the resonant frequency \( \omega_T = (L_T C_T)^{-1/2} \)) is significantly slower than the dynamics of a qubit driven by the microwave source at frequency \( \omega \gg \omega_T \). Hence, being interested in the response of the measurement system (that is of the tank circuit), we average equations over the period \( 2\pi/\omega \). After this averaging the component \( \Phi^{(i)}_{\text{ac}} \sin \omega t \) tends to zero and we have: \( \delta \Phi^{(i)}_x \approx M_{iT} \delta I_L \). Here and below the time-averaging is assumed. (Note that here it is assumed that the expectation value for the current \( I^{(i)}_{qb} \) in equation (12) weakly depend on time during the time interval of the order of \( 2\pi/\omega \) and its time dependence is defined by the tank circuit dynamics only.) Then it follows
\[
\Phi_e = \sum_i \frac{\delta \Phi^{(i)}_x}{\delta t} = \sum_i \frac{M_{iT}^2}{L_i + L_i} \dot{I}_L \tag{14}
\]
and equation (5) is obtained with the inductance \( \bar{L} \), which describes the response of the quantum system to the tank circuit signal, given by:
\[
\bar{L} = \sum_i \frac{M_{iT}^2}{L_i + L_i}. \tag{15}
\]
Thus, we have obtained the system of equations (7), (8), (10), and (15), which describe the interaction of the classical tank circuit and the quantum circuits (qubits). More accurate (quantum-mechanical) analysis would start the description from the Hamiltonian of the whole system in terms of the operators for both the tank circuit and the qubits with averaging the equations afterwards, as in reference [14] (see also the discussion about similar systems in [15] and [16]). Since this analysis would yield the same equations for the observable values \( \langle V_T \rangle = \langle \dot{V}_T \rangle \), etc., we do not consider this procedure here in detail.

### 3 Analysis of the response of the tank circuit

The measurement consists in biasing the tank circuit with the current \( I_{\text{bias}} = I_A \cos \omega_{\text{rf}} t \) and measuring both the phase shift \( \alpha \) and amplitude \( V_A \) of the voltage \( V_T = V_A \cos((\omega_{\text{rf}} t + \alpha)) \). (The oscillations can be considered close to the harmonic form due to the small losses in the high-quality tank circuit which is weakly coupled to the nonlinear qubits’ inductances, see (19) and (20) below.) Substituting these expressions for \( I_{\text{bias}} \) and \( V_T \) in equation (8) and equating coefficients before \( \sin \omega_{\text{rf}} t \) and \( \cos \omega_{\text{rf}} t \), we obtain:
\[
\tan \alpha = \frac{R_T}{\omega_{\text{rf}}} \left( \frac{1}{L_{\text{eff}}^{-1} - \frac{\omega_{\text{rf}}^2}{\omega_T^2} L_T^{-1}} \right), \tag{16}
\]
\[
V_A = R_T I_A \cos \alpha. \tag{17}
\]