A GENERAL FORMULA FOR INDEFINITE FALSE THETA FUNCTIONS

ERIC T. MORTENSON

Abstract. In recent work where Matsusaka generalizes the relationship between Habiro-type series and false theta functions after Hikami, five families of Hecke-type double-sums of the form

$$\left( \sum_{r,s \geq 0} - \sum_{r,s < 0} \right) (-1)^{r+s} x^r y^s q^{a(r) + b r s + c(s)},$$

where $b^2 - ac < 0$, are decomposed into sums of products of theta functions and false theta functions. Here we obtain a general formula for such double-sums in terms of theta functions and false theta functions, which subsumes the decompositions of Matsusaka. Our general formula is similar in structure to the case $b^2 - ac > 0$, where Mortenson and Zwegers obtain a decomposition in terms of Appell functions and theta functions.

1. Introduction

Let $q$ be a nonzero complex number with $|q| < 1$. We recall the $q$-Pochhammer notation:

$$(x)_n = (x; q)_n := \prod_{i=0}^{n-1} (1 - q^i x), \quad (x)_{\infty} = (x; q)_{\infty} := \prod_{i \geq 0} (1 - q^i x).$$

We also recall the basic definition of a theta function:

$$\Theta(x) := (x)_{\infty} (q/x)_{\infty} (q)_{\infty} = \sum_{n=-\infty}^{\infty} (-1)^n q^{(n^2)/2} x^n;$$

where the last equality is the Jacobi triple product identity. False theta functions are theta functions but with the wrong signs [2]. Partial theta functions can be thought of as half of a theta function, in that we only sum over non-negative $n$.

Matsusaka [10] recently expressed false theta functions in terms of Hecke-type double-sums:

Definition 1.1. Let $x, y \in \mathbb{C} \backslash \{0\}$ and define $\text{sg}(r) := 1$ for $r \geq 0$ and $\text{sg}(r) := -1$ for $r < 0$. Then

$$f_{a,b,c}(x, y; q) := \sum_{r,s \in \mathbb{Z}} \text{sg}(r,s)(-1)^{r+s} x^r y^s q^{a(r) + b r s + c(s)}, \quad \text{sg}(r,s) := \left( \frac{\text{sg}(r) + \text{sg}(s)}{2} \right). \quad (1.1)$$

We note that we can also write

$$f_{a,b,c}(x, y; q) = \left( \sum_{r,s \geq 0} - \sum_{r,s < 0} \right) (-1)^{r+s} x^r y^s q^{a(r) + b r s + c(s)}. \quad (1.2)$$

The motivation for Matsusaka’s work is a family of $q$-series originating from Habiro’s work on Witten–Reshetikhin–Turaev invariants. Matsusaka [10] generalizes the relation between five Habiro-type series [5] and false theta functions after Hikami [7]. In [10], five families of Habiro-type series

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are obtained that express the double-sums in terms of Appell functions, i.e. the building blocks of $H$ where $b$ sums where instead there is a plus sign between the summation symbols in (1.2) and we do not have the restriction $b^2 - ac < 0$. As an example, one family reads [10] Theorem 2.15:

$$H_p^{(2)}(q) = \frac{1}{(q)_\infty} \left( \sum_{a,b\geq 0} - \sum_{a,b<0} \right) (-1)^{a+b} q^{(p+1) a} q^{2b} q^{(2a+1)(\frac{q}{2}) + 2ab + 3\left(\frac{q}{2}\right)},$$

where $H_p^{(2)}(q)$ is a Habiro-type series, see Section [2]. Matsusaka then decomposes the five families into sums of products of theta functions and false theta functions [10] Theorems 3.12, 3.13, 3.15. The decompositions are then used to compute radial limits [9], [10, Theorem 3.21].

This setting contrasts with [3] [15], where false theta functions are expressed in terms of double-sums where instead there is a plus sign between the summation symbols in (1.2) and we do not have the restriction $b^2 - ac < 0$. It also contrasts with the setting where $b^2 - ac > 0$.

In [6] [16], double-sums of the form (1.2) where $b^2 - ac > 0$ are extensively studied. Expansions are obtained that express the double-sums in terms of Appell functions, i.e. the building blocks of Ramanujan’s mock theta functions, and theta functions. Here, Appell functions are defined

$$m(x, z; q) := \frac{1}{\Theta(z; q)} \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{r(\frac{3}{2})} z^r}{1 - q^{r+1} x z}.$$

The results in [6] are for general double-sums which enjoy certain symmetries. For example, with one type of symmetry we have $b > a = c$ [6, Theorem 1.3], of which a special case reads

$$f_{1,2,1}(x, y; q) = \Theta(y; q) m \left( \frac{q^2 x}{y^2}, -1; q^3 \right) + \Theta(x; q) m \left( \frac{q^2 y}{x^2}, -1; q^3 \right) - y \cdot \frac{(q^3; q^3)_{\infty}}{\Theta(-1; q^3) \Theta(-q y^2/x; q^3) \Theta(-q x^2/y; q^3)}.$$

The above expansion has the immediate corollaries:

$$f_{1,2,1}(q, q; q) = (q)_{\infty}^2, \quad f_{1,2,1}(q, -q; q) = \Theta(-q; q^4) \phi(q),$$

where

$$\phi(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} (q; q^2)_n}{(-q)_{2n}} = 2m(q, -1; q^3)$$

is a sixth-order mock theta function [11].

In the course of resolving an open question on the modularity of certain duals of generalized quantum modular forms of Hikami and Lovejoy [8], Mortenson and Zwegers [16] obtained a decomposition for the general form (1.2), where $b^2 - ac > 0$. To state the decomposition, we first define the following expression involving Appell functions [6] [17]:

**Definition 1.2.** Let $a$, $b$, and $c$ be positive integers with $D := b^2 - ac > 0$. Then

$$m_{a,b,c}(x, y, z; q; z_0; q) := \sum_{t=0}^{a-1} (-y)^t q^{t(\frac{1}{2})} \Theta(q^{ht}; x; q^a) m \left( -q^{a(\frac{b+1}{2}) - c(\frac{a+1}{2}) - tD \left( \frac{-y}{a} \right)^a; z_0, q^aD \right)$$

$$+ \sum_{t=0}^{c-1} (-x)^t q^{t(\frac{1}{2})} \Theta(q^{ht}; y; q^c) m \left( -q^{c(\frac{b+1}{2}) - a(\frac{c+1}{2}) - tD \left( \frac{-x}{c} \right)^c; z_1, q^cD \right).$$

**Theorem 1.3.** [16] Corollary 4.2] Let $a$, $b$, and $c$ be positive integers with $D := b^2 - ac > 0$. For generic $x$ and $y$, we have

$$f_{a,b,c}(x, y; q) = m_{a,b,c}(x, y, -1, -1; q) + \frac{1}{\Theta(-1; q^aD) \Theta(-1; q^cD)} \vartheta_{a,b,c}(x, y; q),$$
where
\[ \vartheta_{a,b,c}(x, y; q) := \sum_{d^* = 0}^{b} \sum_{e^* = 0}^{b} q^{a(d-c)/2 + b(d-c/2)(e+a/2)+c(e+a/2)/2} (-x)^{d-c/2} (-y)^{e+a/2} \]
\[ \sum_{f=0}^{b-1} q^{ab^2} f(-y)^{af} \Theta(-q^c(ad+be+a(b-1)/2+abf) (-x)^{c} q^{cb^2}) \]
\[ \Theta(-q^a((d+b(b+1)/2+bf)/b^2-ac+c(a-b)/2) (-x)^{-ac} (-y)^{ab} q^{ab^2 D}) \]
\[ \Theta(q^{Dc+a(c-b)/2} (-x)^{b(-y)^{-a}} q^{D(a-b)/2} (-y)^{b(-x)^{-c}} q^{bD}) \]

Here \( d := d^* + \{c/2\} \) and \( e := e^* + \{a/2\} \), with \( 0 \leq \{\alpha\} < 1 \) denoting fractional part of \( \alpha \).

When \( b^2 - ac > 0 \), one can obtain an Appell function expression such as (1.3) by first determining the appropriate functional equation for (1.1) and iterating it. If one starts to see divergent partial theta functions, one uses a heuristic that relates divergent partial theta functions and Appell functions in order to express (1.3) in terms of Appell functions up to a theta function. See for example [6, Section 3], [12, Section 4.1], [14, Section 4], [13, Section 8]. Determining the theta function is a difficult task. Sometimes, one can obtain the theta function in the course of a direct proof [11, 16].

When \( b^2 - ac < 0 \), iterating the appropriate functional equation for (1.3) yields partial theta functions that do not diverge. This makes the situation straightforward and also leads us to our main result.

**Theorem 1.4.** Let \( a, b, \) and \( c \) be positive integers with \( D := b^2 - ac < 0 \). For generic \( x \) and \( y \), we have that
\[ f_{a,b,c}(x, y; q) = \frac{1}{2} \left( \sum_{t=0}^{a-1} (-y)^t q^{c(t)} \Theta(q^{bD}; x; q^a) \sum_{r \in \mathbb{Z}} \sum_{c=0}^{c-1} (-x)^t q^{c(t)} \Theta(q^{bD}; y; q^c) \sum_{r \in \mathbb{Z}} \sum_{c=0}^{c-1} (-x)^t q^{c(t)} \right) \]
\[ \sum_{t=0}^{a-1} (-y)^t q^{c(t)} \Theta(q^{bD}; x; q^a) \sum_{r \in \mathbb{Z}} \sum_{c=0}^{c-1} (-x)^t q^{c(t)} \Theta(q^{bD}; y; q^c) \sum_{r \in \mathbb{Z}} \]
\[ \Theta(q^{bD}; x; q^a) \Theta(q^{bD}; y; q^c) \Theta(q^{bD}; z; q^d) \]
\[ \sum_{r \in \mathbb{Z}} \sum_{c=0}^{c-1} (-x)^t q^{c(t)} \Theta(q^{bD}; x; q^a) \sum_{r \in \mathbb{Z}} \sum_{c=0}^{c-1} (-y)^t q^{c(t)} \Theta(q^{bD}; y; q^c) \sum_{r \in \mathbb{Z}} \sum_{c=0}^{c-1} (-z)^t q^{c(t)} \Theta(q^{bD}; z; q^d) \]

Matsusaka [10, Theorems 3.12, 3.13, 3.15] only computes compositions for the case \((a, b, c) = (2p + 1, 2, 3)\) and certain values of \( x \) and \( y \). Our Theorem 1.4 is for \((a, b, c)\) with \( b^2 - ac < 0 \). For another example where iterating the functional equation yields convergent partial theta functions, see [14, Section 7], which discusses triple-sum partial theta function identities found in [4].

In Section 2 we discuss examples related to results in [7, 10]. In Section 3 we prove Theorem 1.4.

### 2. Examples

In [10], Matsusaka generalized Hikami’s examples of Habiro-type series [7] to five infinite families that are of \( q \)-hypergeometric multi-sum form:
\[ H_p^{(1)}(q) := \sum_{s_p \geq \cdots \geq s_1 \geq 0} q^{s_p(s_p+1)} \prod_{i=1}^{p-1} q^{s_i(s_i+1)} \left[ \frac{s_{i+1}}{s_i} \right]_q, \]
\[ H_p^{(2)}(q) := \sum_{s_p \geq \cdots \geq s_1 \geq 0} q^{s_p(s_p+1)} \prod_{i=1}^{p-1} q^{s_i(s_i+1)} \left[ \frac{s_{i+1}}{s_i} \right]_q, \]
When easily derived properties for theta functions. We recall that Matsusaka then computes the false theta function decompositions for the above five families in [10, 4 ERIC T. MORTENSON b] Matsusaka then employs Bailey’s Lemma to express each family in terms of the Hecke-type double-
Habiro-type series [7], [10, Section 2.3]: Specifically, Hikami and Matsusaka obtain the following Hecke-type expansions of the generalized Habiro-type series [7, 10 Section 2.3]:

\[ H_p^{(1)}(q) := \sum_{s_p \geq \cdots \geq s_1 \geq 0} q^{2sp}(q^{s_p+1}) \prod_{i=1}^{p-1} q^{s_i(s_i+1)} \left[ \frac{s_i+1}{s_i} \right]_q, \]

\[ H_p^{(2)}(q) := \sum_{s_p \geq \cdots \geq s_1 \geq 0} q^{s_p}(q^{s_p+1}) \prod_{i=1}^{p-1} q^{s_i(s_i+1)} \left[ \frac{s_i+1}{s_i} \right]_q, \]

\[ H_p^{(3)}(q) := \sum_{s_p \geq \cdots \geq s_1 \geq 0} q^{s_p}(q^{s_p+1}) \prod_{i=1}^{p-1} q^{s_i(s_i+1)} \left[ \frac{s_i+1}{s_i} \right]_q. \]

Matsusaka then computes the false theta function decompositions for the above five families in [10, Theorems 3.12, 3.13, 3.15]. In [10 Theorem 3.21], he computes the radial limits.

We demonstrate our results by computing a few examples, but first remind the reader of some easily derived properties for theta functions. We recall that

\[ \Theta(q/x; q) = \Theta(x; q) \] and \[ \Theta(q^2; q^3) = \Theta(q; q^3) = (q)_\infty. \]

When \( n \in \mathbb{Z} \) we have \( \Theta(q^n; q) = 0 \) and the elliptic transformation property:

\[ \Theta(q^n x; q) = (-1)^n x^{-n} q^{-\frac{n}{2}} \Theta(x; q). \quad (2.1) \]

For \( H_1^{(2)}(q) \), our Theorem [1,4] yields

\[ H_1^{(2)}(q) = \frac{1}{(q)_\infty} f_{3,2,3}(q^2, q^2; q) \]

\[ = \frac{1}{(q)_\infty} \sum_{t=0}^{2} (-q^2)^t q^{3\binom{t}{2}} \Theta(q^{2t+2}; q^3) \sum_{r \in \mathbb{Z}} \text{sg}(r)(-q^{-7+5t} q^{15}) q^{\binom{r+1}{2}} \]

\[ = \frac{1}{(q)_\infty} \left( \Theta(q^2; q^3) \sum_{r \in \mathbb{Z}} \text{sg}(r)(-1)^r q^{-7r} q^{15} \right) \]

\[ - q^2 \Theta(q^4; q^3) \sum_{r \in \mathbb{Z}} \text{sg}(r)(-1)^r q^{-2r} q^{15} \]
where we used (3.1), substituted $n$ argument for the second summand on the right-hand side of (1.4) is analogous.

Using the (2.1), we have

For $H_2^{(2)}(q)$, our Theorem 1.4 yields

$$H_2^{(2)}(q) = \frac{1}{(q)_\infty} f_{5,2,3}(q^3; q^2; q)$$

$$= \frac{1}{2(q)_\infty} \left( \Theta(q^3; q^5) \sum_{r \in \mathbb{Z}} \sgn(r)(-1)^r q^{-26r} q^{55(r+1)} - q^5 \Theta(q^2; q^5) \sum_{r \in \mathbb{Z}} \sgn(r)(-1)^r q^{-4r} q^{55(r+1)} 
+ q^{11} \Theta(q^4; q^5) \sum_{r \in \mathbb{Z}} \sgn(r)(-1)^r q^{7r} q^{55(r+1)} + q^{19} \Theta(q; q^5) \sum_{r \in \mathbb{Z}} \sgn(r)(-1)^r q^{18r} q^{55(r+1)} 
+ \Theta(q^2; q^3) \sum_{r \in \mathbb{Z}} \sgn(r)(-1)^r q^{-16r} q^{33(r+1)} + q^2 \Theta(q; q^3) \sum_{r \in \mathbb{Z}} \sgn(r)(-1)^r q^{-5r} q^{33(r+1)} \right).$$

3. Proof of Theorem 1.4

For reference, we recall that

$$f_{a,b,c}(x, y, q) = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \left( \frac{\sgn(m) + \sgn(n)}{2} \right) (-1)^{r+s} x^m y^n q^{a(m)} + bmn + c(n).$$

Using the (2.1), we have

$$\Theta(q^{b(ar+t)} x; q^a) = \Theta(q^{abr} xq^{bt}; q^a) = (-x)^{-br} q^{-br^2} \Theta(q^{-a(2^r)} q^{bt}; q^a). \quad (3.1)$$

Proof of Theorem 1.4. We want to demonstrate that the right-hand side of (1.4) equals the left-hand side of (1.4). We consider the first summand on the right-hand side. We have

$$\sum_{t=0}^{a-1} (-y)^t q^{c(2)} \Theta(q^{bt} x; q^a) \sum_{r \in \mathbb{Z}} \sgn(r) \left( q^{a(b+1)} - c^{a+1} - tD \right) \left( (-x)^b \right) r q^{-aD(r+1)}$$

$$= \sum_{t=0}^{a-1} q^{c(2)} \Theta(q^{bt} x; q^a) \sum_{r \in \mathbb{Z}} \sgn(ar + t) \left( q^{a(b+1)} - c^{a+1} - tD \right) \left( (-x)^b \right) r q^{-aD(r+1)}$$

$$= \sum_{t=0}^{a-1} q^{c(2)} \sum_{r \in \mathbb{Z}} \sgn(ar + t) \Theta(q^{b(ar+t)} x; q^a) (-x)^{br^2} q^{b(2)}$$

$$\cdot \left( q^{a(b+1)} - c^{a+1} - tD \right) \left( (-x)^b \right) r q^{-aD(r+1)}$$

$$= \sum_{t=0}^{a-1} \sum_{r \in \mathbb{Z}} \sgn(ar + t) \Theta(q^{b(ar+t)} x; q^a) q^{c(a+1)} (-y)^{ar+t}$$

$$= \sum_{n \in \mathbb{Z}} \sgn(n) \Theta(q^{bn} x; q^a) q^{c(n)} (-y)^n$$

$$= \sum_{n \in \mathbb{Z}} \sgn(n) q^{c(n)} (-y)^n \sum_{m \in \mathbb{Z}} (-x)^m q^{bmn} q^{a(m)},$$

where we used (3.1), substituted $n = ar + t$, and then used the Jacobi triple product identity. The argument for the second summand on the right-hand side of (1.4) is analogous.
tbd.

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