Introducing Geometric Algebra to Geometric Computing Software Developers: A Computational Thinking Approach

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Abstract

Designing software systems for Geometric Computing applications can be a challenging task. Software engineers typically use software abstractions to hide and manage the high complexity of such systems. Without the presence of a unifying algebraic system to describe geometric models, the use of software abstractions alone can result in many design and maintenance problems. Geometric Algebra (GA) can be a universal abstract algebraic language for software engineering geometric computing applications. Few sources, however, provide enough information about GA-based software implementations targeting the software engineering community. In particular, successfully introducing GA to software engineers requires quite different approaches from introducing GA to mathematicians or physicists. This article provides a high-level introduction to the abstract concepts and algebraic representations behind the elegant GA mathematical structure. The article focuses on the conceptual and representational abstraction levels behind GA mathematics with sufficient references for more details. In addition, the article strongly recommends applying the methods of Computational Thinking in both introducing GA to software engineers, and in using GA as a mathematical language for developing Geometric Computing software systems.

Keywords: Computational Thinking, Geometric Algebra, Geometric Computing, Software Engineering

1 Introduction

Geometric Algebra (GA) is an expressive algebraic framework capable of unifying many mathematical tools that engineers and scientists use to model their ideas [1, 2, 3]. GA can be used for unified algebraic representation and manipulation of multidimensional Euclidean and non-Euclidean geometries in a consistent manner [4, 5, 6, 7]. Many good sources exist that explain the mathematics behind GA and explore some of its possible applications [8, 9, 3, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. These sources vary in their scope, intended audience, goals, level of details, and mathematical rigor. Few sources investigate the concepts, options, and issues software engineers need to understand and study when designing practical GA-based software systems for Geometric Computing applications [8, 20, 21, 3, 22, 23, 24, 25, 26, 27]. This led to less attention given to GA-based models simply because software engineers don’t have enough GA material targeting their domain of knowledge. The software engineering domain has quite different thought process characteristics from that of non-software oriented engineers, mathematicians, and physicists typically producing the GA models. Without sufficient attention from the developers of Geometric Computing software implementations, many of the good GA models would be trapped inside the limited academic circle of the GA community.

1.1 Geometric Algebra and Geometric Computing

In many areas of computer science, engineering, mathematics, physics, biology, and chemistry we find common geometric ideas defining, relating, and manipulating objects in space and time. In addition, there is a prevalent use of modern computing environments to implement geometric algorithms and to process geometric information [28]. Many researchers informally use the term “Geometric Computing”
1.2 GA as a Language for Computational Thinking

Computational Thinking (CT) complements critical thinking as a way of reasoning to understand and solve problems, take proper actions, and interact with our surroundings. The concepts and techniques of CT are drawn from computer and information science while having broad application in the arts, sciences, engineering, humanities and social sciences [40]. One definition of CT is as follows [41]:

(1) GA as a Language for Computational Thinking

To the best of my knowledge there is no solid definition of this term in modern literature. Some researchers even use the term Geometric Computing to actually refer to Computational Geometry [29, 30], which is just one application area that requires GC. As an attempt to make the meaning of this term clear as I understand and use it in this work, I will adopt the following definition, which is a modification of the term “Computing” in the 1989 ACM report on “Computing as a Discipline” [31]:

**Definition 1.** The discipline of Geometric Computing is the systematic study of algorithmic processes that describe and transform geometric information: their theory, analysis, design, efficiency, implementation, and application. The fundamental question underlying all geometric computing is “What (and how) geometric processes can be efficiently automated?”

An essential ingredient in creating GC applications is the use of symbolic algebraic tools, in the mathematical sense, to express and manipulate abstract geometric objects, spaces, and processes. Many such tools exist from diverse areas of mathematics; for example matrix algebra, 3D vector algebra, quaternions, complex numbers, several kinds of hyper-complex numbers, and many more. The use of so many conceptually and computationally incompatible algebraic tools to express geometric ideas results in various problems. Such problems manifest in multiple levels and forms including:

- The difficulty of expressing geometrically intuitive ideas in an algebraically consistent manner.
- The need to learn many distinct algebraic representations in order to model the geometry of relatively complex problems.
- The need for many conversions between algebraic frameworks within the context of the same problem domain.
- The awkward isolation of people working in areas of research that essentially depend on the same set of geometric ideas primarily because such groups tend to use isolated algebraic frameworks.

The prevalent state in developing GC applications is to rely on software abstractions [32] to unify the interface between the users and the GC software infrastructure. For example, in a typical GC software implementation the software engineer creates a set of classes, implementing a unified software interface, to represent primitive geometric objects like points, lines, spheres, circles, planes, etc. The software engineer would then implement transformations on all these geometric objects using specialized hand-written subroutines for each class; an exhausting and difficult task for large systems. The situation gets even worse when implementing geometric operations involving multiple objects like an intersections, collision detection, or distance computations [33, 34]. Such approach eventually creates many problems in GC software design, complexity, maintenance, and cost. A much better approach is to rely instead on higher-level algebraic abstractions to unify the mathematical base of many geometric objects. This is partially done in computer graphics and robotics, for example, when implementing 3D affine transformations using $4 \times 4$ homogeneous matrices [35].

There has been a search going on for decades to find a unifying algebraic framework capable of expressing geometric ideas in a universal, consistent, dimension-independent, and coordinates-independent manner. Recent research and numerous applications have proven Geometric Algebra to be a powerful algebraic framework that is capable of providing such features. GA-based algebraic abstractions enable domain specific optimizations, provide unification of geometric representations, and clarify expression of geometric ideas [3, 36, 37]. In addition, GA can replace and extend most of the distinct algebraic frameworks we use in practice. Thus we can learn a single algebraic framework and uniformly apply it to more domains with minimum need for representational conversions. This would also remove many of the communication boundaries between scientific and engineering fields that have a common base of geometric ideas. For more information about the historical developments that led to modern GA the reader can refer to [38, 39].
Figure 1.1: Main GA abstractions and their relations
Definition 2. Computational Thinking is a brain-based activity that enables problems to be resolved, situations better understood, and values better expressed through systematic application of abstraction, decomposition, algorithmic design, generalization, and evaluation in the production of an automation implementable by a digital or human computing device.

CT relies on using abstraction and decomposition when attacking a large complex task or designing a large complex system; it requires thinking at multiple levels of abstraction [42]. Geometric Algebra can be a valuable mathematical language to acquire and develop such CT skills for handling Geometric Computing problems. As illustrated in Figure 1.1, a Geometric Algebra is an abstract, elegant, and sophisticated mathematical structure with many integrating components. In order to fully appreciate all aspects of GA-based software implementations, the team containing GA-model developers and software engineers should collectively think on 3 integrating levels of abstraction, as required by sound CT and shown in Figure 1.2:

1. The Conceptual Level. On the most abstract level, we find mathematical algebraic concepts like scalar fields, linear spaces, subspaces, linear maps, orthogonal operators, metrics, duality, and space embeddings. Each of these concepts have particular roles to play, and must have a specific set of features to be able to play its roles. Fully understanding these concepts, in the most abstract sense possible, is essential to avoid many bad design decisions, and to take full advantage of GA’s powerful unifying language. The team should also appreciate the strong relations between abstract linear algebra and projective and Cayley-Klein geometries [43, 44].

2. The Representational Level. On an intermediate level of abstraction, we find that each concept of the above can have many representations in less abstract mathematical domains. For example, a linear space can be an abstraction of signals (as in electrical engineering), polynomials, geometric directions in Euclidean space, or tuples of coordinate vectors. An orthogonal map can be represented using an orthogonal matrix, a GA versor, or a Discrete Fourier Transform. Understanding the commonalities and relations between such representations and the limitations of each is essential for the selection of the best representations for a particular application domain. GA can provide many advantages over matrix algebra in this level, while still integrating will with established matrix representations.

3. The Implementational Level. On the lowest level of abstraction we find software abstractions for representing all the above inside a computational environment. Here we find elements such as floating point numbers, combinatorial representations (for example classes and structures), programming paradigms, software interfaces, subroutines, memory hierarchies, and computer networks.
Computers impose many physical constraints on the above two levels of abstraction that must be taken into consideration when addressing practical GA-based software implementations. Many GA-based software tools are currently present to be used at this level including numerical, symbolic, and Generative Programming-based systems.

These three levels are familiar to software engineers in other domains of application. For example, in database systems design we find three analogous levels of Conceptual Design, Logical Design, and Physical Design [45]. The role GA plays in Geometric Computing applications can be thought to be analogous to the role of Relational Algebra in relational database systems design. The study of the mathematics behind Relational Algebra alone is not sufficient to produce successful database applications, however. Software engineers must address other complementary aspect of the design related to user interaction with data (using SQL as a Domain Specific Language for example), physical storage and transfer of data, optimization of data query executions, data visualization and presentation, scalability, and many more. Without addressing such aspects, Relational Algebra wouldn't have become a basic part of computer science curricula worldwide. We must address similar complementary aspects for Geometric Algebra in order to achieve its rightful place in the scientific, educational, and industrial fields.

Whenever possible, expressing our ideas at the top level of abstraction is very powerful conceptually. At this level we can understand and relate many application areas at a fundamental level. We can communicate ideas and transfer knowledge between them more easily. Sadly, many people don’t have access to this level of abstraction in practice. We are taught to think about our mathematical tools starting from the second intermediate level of abstraction, not the first top level. The benefits of eliminating this serious problem appears in all areas in which GA can be applied; for example:

- Many transformations we apply in signal and image processing are just instances of abstract orthogonal linear maps, with more unifying common properties than initially perceived. Such transforms include continuous and discrete Fourier transforms, Laplace, z-, Walsh-Hadamard, slant, Haar, Karhunen-Loeve, and wavelet transforms [46]. Using GA to represent and apply these transforms can lead to new applications and insights [47, 48], and eventually to new unified architectures for multi-dimensional signal processing software systems with modeling and processing capabilities well beyond the current systems.

- In geometric modeling and geometric reasoning, Euclidean, Hyperbolic, and Elliptic geometries have a common algebraic foundation within GA. This enables us to create GA-based universal geometric constructions and apply them to specific problems with any desired geometry of these three [4, 5, 6, 7]. Some dynamic geometry software systems already apply this approach, like Cinderella [49, 50] that internally models the general Cayley-Klein geometry using complex numbers [44].

- Many algebras that are very useful in practice are actually sub-algebras of some GA. The list include the algebra of real numbers, complex numbers, n-D Euclidean vectors, quaternions, dual quaternions, spinors, Clifford’s dual numbers, and Grassmann numbers. GA can unify and convert these numbers within the context of a single problem, engineering discipline, or scientific field.

Another anti-CT pattern facing most software engineers in designing GC applications results from not having a clear separation between those three levels of abstraction. In many cases, intermediate representational abstractions are incorrectly perceived to be identical to the conceptual abstractions. As an example, consider the default use of matrices to represent linear maps in GC applications. There are other intermediate representations that are better than matrices in modeling certain geometric aspects with better computational properties. For example, it’s much easier to extract the axis and angle of rotation of a 3D general rotation linear map if we use a quaternion to represent the linear map. Quaternions require less memory, less processing, and are numerically more stable compared to rotation matrices [3]. As another example for incorrectly mixing levels of abstraction, many programmers blindly use floating point numbers as a perfect representation of real numbers, not taking into consideration some of their problematic features [51].

Clear separation of the first two abstraction levels can result from studying a course in projective geometry [44], abstract algebra [52], and abstract linear spaces [53, 54] in addition to the classical coordinate based linear algebra courses [55]. GA can be very helpful in this regard as it contains enough mathematical abstraction and generality to provide clear understanding and separation of abstract levels of thinking. This skill is typically available to mathematicians and physicists, but less so for computer scientists and engineers. Separation of the third level requires careful study of the physical limitations of
representing and communicating information inside computational environments. In addition, a through understanding of capabilities of modern programming languages and programming paradigms is necessary to design better implementations [56, 57]. This skill is typically available to computer scientists and software engineers, but less so for mathematicians and physicists.

From another angle, learning GA can take much time and effort. Applying Computational Thinking to the GA learning problem can reduce time and effort considerably. Because GA is relevant to so many areas in science and engineering, its presentation should be formulated to each specific discipline. A very good example for presenting GA to electrical and electronic engineers, for example, is [17]. Similar efforts are needed to properly introduce GA to software engineers and software developers. Presenting GA to a software engineer is different from presenting to a mathematician or physicist. The mindset of a software engineer prefers dealing with diagrams, specifications, relations, and algorithms rather than axioms, theorems, proofs, and equations [8]. Such efforts also include designing easy to use domain-specific GA-based software systems for educational and prototyping purposes in addition to production purposes for Geometric Computing applications.

1.3 Aims and Contents

This article is intended as a Computational Thinking driven exposition of GA for software engineers interested in creating GA-based GC software systems. I attempt to emphasize the conceptual and representational abstraction levels related to each mathematical element of Geometric Algebra, leaving the implementational level to future articles. The conceptual level is purely mathematical and is independent of any particular software implementation. The representational level is also mathematical but typically defines the high-level design of the GA-based software system. My main intention here is to provide a unified entry point for facilitating further study of the mathematics behind the concepts summarized here that is suitable for software engineers.

The main body of this article consists of 3 parts. In the first part of this article in section 2, I summarize the main abstract and algebraic concepts of Metric Linear Spaces, the base on which GAs are constructed. In the second part in section 3, I build on the concepts of section 2 to explain the elegant mathematical structure of Geometric Algebra with references to additional information sources for the interested reader. Since I’m mainly interested here in the most computationally-significant algebraic constructions of GA, I will not discuss GA’s numerous geometric interpretations found in the literature. In the third part in section 4, I focus on defining GA Coordinate Frames and how to use them for computing linear operations, products, and maps on GA multivectors. This is the mathematical base for the symbolic computations infrastructure layer in GMac, a universal GA-based implementation generator system I designed [58, 59]. Finally, in section 5 I provide some concluding remarks and suggestions.

2 Metric Linear Spaces

2.1 Scalar Fields

Many number systems exist in mathematics with varying properties and applications. In practice, however, we tend to concentrate on a few of them: rational numbers $\mathbb{Q}$, real numbers $\mathbb{R}$, and complex numbers $\mathbb{C}$. Such numbers are also called scalars to distinguish them from vectors in linear spaces. There are common properties of these number systems that, when abstracted into algebraic relations, give us the concept of a scalar field [52]. On the top conceptual level of abstraction, a field $\mathbb{F}$ is a set of “scalars” closed under two operations called addition and multiplications satisfying some familiar properties like associativity and commutativity of addition and multiplication, presence of unique additive and multiplicative identities and inverses, and distributivity of multiplication over addition. From these simple properties many features, theorems, and operations can be defined and deduced based on these abstract concepts without having to give concrete examples like the real or complex number systems. Some roles of scalars in Geometric Computing applications include:

- Used as abstractions of physical measurements like mass, velocity, length, area, etc..
- Used to encode, quantify, sort, and compare geometric objects and their properties.
- Used as construction elements in Linear Combinations and other related combinations over Vectors.

Mathematically, we can construct a linear space, hence a GA, over any scalar field; including finite fields [60]. Linear spaces over finite fields have interesting properties that could be investigated using Geometric
Algebra especially for digital and discrete geometry applications [61, 62]. In the GA literature there exist strong assertions that only real numbers should be used as a base for constructing GAs [39]. This point of view is mainly based on the existence of isomorphisms between complex numbers-based GAs and real numbers-based GAs; so the use of complex-based GAs is mathematically redundant and geometrically more complex for modeling the physical space and time we live in. This is certainly a respectable point of view, especially in physics. From a software engineering and educational point of view, however, I recommend to leave the door open for using the most suitable number system for a particular problem at hand. I believe many problems can be more easily transformed from the classical representations into GA-based representations if we are flexible about the choice of the number system we use [11].

2.2 Linear Combinations and Abstract Vectors

At the base of the elegant GA mathematical structure we find the abstract concept of Linear Spaces; also commonly called Vector Spaces [53, 54, 55]. Many study linear spaces because of their basic role in encoding the Superposition Principle; a cornerstone in modern science and engineering. Typical mathematical introductions to linear spaces concentrate on the abstract algebraic properties of vectors and their two main operations of vector addition and scalar multiplication. From a computational point of view, however, the central concept in linear spaces is the Linear Combination. A linear combination is an expression of the form $a_1v_1 + a_2v_2 + \cdots + a_kv_k \equiv \sum_{i=1}^{k} a_i v_i$ where $v_i$ are “vectors” and $a_i$ are scalars not all zero. A linear space is simply any set of “vectors” that is closed under linear combinations over a given scalar field; i.e. any linear combination of any collection of vectors is also a vector in the same set. The familiar algebraic properties of vector addition and scalar multiplication are necessary to perform linear combinations consistently. This very abstract concept has so many manifestations in science and engineering that it is a central concept in many applications. All other main concepts of linear spaces are derived from linear combinations; for example:

- **Span**: The span of a given set of vectors $span \{v_1, v_2, \ldots, v_k\}$ is the set of vectors resulting from all possible linear combinations of these vectors. Here the vectors $v_i$ are fixed while the scalars $a_i$ can take any possible values from their field.

- **Subspace**: A linear subspace $W$ of a larger linear space $V$, denoted here as $W \leq V$, is a subset of the linear space $V$ that is closed under linear combinations. The span of any set of vectors from $V$ is always a subspace of $V$.

- **Linear Independence**: A collection of vectors are called Linearly Dependent when we can express any of them as a linear combination of the others; else they are Linearly Independent (LID) vectors. These two are basic conceptual relations among any given collection of vectors.

- **Basis**: A basis is a LID set of vectors $\{e_1, e_2, \ldots, e_n\}$ that spans the whole linear space. Any vector in the linear space can be expressed as a unique linear combination of the basis vectors. A linear space can have an infinite number of basis sets, but they all contain the same number of vectors $n$. This number $n$ is the dimension of the linear space denoted by $dim(V)$. In all the following discussions, the basis is assumed to be an ordered set, not a general set; denoted here as $\{e_1, e_2, \ldots, e_n\}$.

- **Coordinate Vector**: Given a fixed ordered basis $E = \{e_1, e_2, \ldots, e_n\}$ we can express any abstract vector $v$ as a linear combination of the basis vectors $v = \sum_{i=1}^{k} a_i e_i$. The scalar coefficients $a_i \in F$ can be written as a tuple $v_E = (a_1, a_2, \ldots, a_n) \in F^n$ that is called the coordinate vector representation of $v$. The abstract vector $v$ and its coordinate vector $v_E$ are two conceptually distinct entities, but have a linear isomorphism between them; so we can compute with coordinate vectors and interpret the results in the context of the abstract linear space. Sometimes we prefer to express the coordinate vector in matrix form as a column vector holding the same scalars. I will denote the column vector representation of an abstract vector $v$ on the basis $E$ as: $[v]_E = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}^T$.

- **Linear Map**: A linear map is a map between two linear spaces $f : V \rightarrow W$ that preserves linear combinations $f[\sum_{i=1}^{k} a_i v_i] = \sum_{i=1}^{k} a_i f[v_i]$. When the two linear spaces are the same, its is called a linear operator.

- **Other Combinations**: Imposing constraints on the scalar coefficients of linear combinations leads to theoretically and practically significant concepts with many important geometric interpretations.
like Affine Combinations \( \sum_{i=1}^{k} a_i = 1 \) \([35]\), Conical Combinations \( a_i \geq 0 \), and Convex Combinations \( a_i \geq 0 \), \( \sum_{i=1}^{k} a_i = 1 \) \([63]\).

It is important to note that we are not yet talking about distances and angles between vectors or orthogonality of vectors because such concepts require the more fundamental concept of metric defined later. The main relation between vectors in non-metric abstract linear spaces is the Linear Dependence/Independence relation. The main construction operation is the Linear Combination. We can “divide” two vectors (i.e. compare their relative scale) but only if one of them is a linear combination (i.e. a scaled version) of the other. Generally, this is not how engineers are usually taught linear spaces in undergraduate courses, but a clear understanding and separation of these fundamental concepts is necessary to correctly understand and use the mathematical structure of Geometric Algebra that is based on abstract linear spaces.

### 2.3 Abstract Vectors and Coordinate Vectors

In order to use computers for dealing with abstract concepts of linear spaces, we need an equivalent intermediate representation that only uses numbers and their basic operations of addition and multiplication. Mathematics provide a base for such representation through coordinate vectors. Without loss of generality I will concentrate on the field of real numbers \( \mathbb{R} \) as the scalar field for all the following discussions. Having an \( n \)-dimensional abstract linear space \( V \) on \( \mathbb{R} \) with basis \( E = \{e_1, e_2, \ldots, e_n\} \) we can set up a linear isomorphism (i.e. one-to-one linear map) \( \phi \) defined with its inverse map \( \phi^{-1} \) as follows:

\[
\begin{align*}
\phi : \mathbb{R}^n &\rightarrow V, \phi : (a_1, a_2, \ldots, a_n) \mapsto \sum_{i=1}^{n} a_ie_i; \\
\phi^{-1} : V &\rightarrow \mathbb{R}^n, \phi^{-1} : v \mapsto v_E.
\end{align*}
\]

(2.1)

(2.2)

This way, linear combinations on the coordinate vectors of the real linear space \( \mathbb{R}^n \) are equivalent representations of the same linear combinations on the abstract linear space \( V \). Now we can add two vectors in \( V \) by simply adding the real components of their coordinate vectors in \( \mathbb{R}^n \) and apply the linear isomorphism to get the final result in \( V \). We can do the same for scalar multiplication by multiplying the scalar with the components of the coordinate vector. All derived linear operations on \( V \) can be formulated “numerically” on the equivalent real linear space \( \mathbb{R}^n \). This is the playground of matrix algebra \([55]\), the typical starting point where most engineers learn about linear spaces. The \( n \)-dimensional real coordinate vectors space \( \mathbb{R}^n \) is a linear space that is equivalent to all \( n \)-dimensional abstract linear spaces; \( \mathbb{R}^n \) is a universal intermediate representation for all abstract linear spaces.

One important point to realize is that by changing the basis of \( V \) we are also changing the linear isomorphism \( \phi \) because the same abstract vector has a different linear combination on a different basis. To make our computations consistent we must use the same basis for all related computations. In addition, some facts should remain the same regardless of the used basis and isomorphism. For example, linear independence of a set of vectors should remain the same regardless of the selected basis. Such properties are called coordinate-independent or basis-independent. GA can provide many coordinate-independent formulations for properties of linear spaces and at the same time act as an excellent intermediate representation through its multivectors and products. Because a GA is itself a linear space, as will be explained later, we can always represent all GA multivectors and operations using matrix algebra. This is the approach used in some GA software systems like the Clifford Multivector Toolbox for MATLAB \([64, 65]\) for example.

### 2.4 Metrics and Their Representations

A metric linear space is just a linear space with an additional bilinear map, called the metric, that associates a scalar with each pair of vectors \([66]\). The objective of defining a metric is to enable comparing vectors and subspaces of different attitude in space using scalars. Many familiar concepts we use are actually based on the more fundamental metric concept. Such concepts include distance, length, area, angle, orthogonality, orthogonal maps, projections, rotations, and many others. In GA the definition of a metric is based on the concept of a symmetric bilinear form and the associated concept of a quadratic form. A symmetric bilinear form \( B \) on the real linear space \( V \) is a mapping \( B : V \times V \rightarrow \mathbb{R} \) that is linear in both arguments (i.e. bilinear) and symmetric \( B(u, v) = B(v, u) \) \( \forall u, v \in V \). A related concept
is the quadratic form that is related to a symmetric bilinear form by: \( Q(u) = \frac{1}{2}B(u, u), \ B(u, v) = Q(u+v)-Q(u)-Q(v) \forall u, v \in V. \) The quadratic form satisfies the relation \( Q(av) = a^2Q(v) \forall v \in V, a \in \mathbb{R}. \)

The metric also associates each vector in the linear space with some scalar by putting the vector in both inputs of the metric. This scalar is called the norm \( \|v\| \equiv v^2 \equiv B(v, v) \) of the vector \( v \in V \) and is equal to double the quadratic form of the vector \( \|v\| = 2Q(v) \). If two vectors are associated with the same scalar they are of equal norm, and null vectors are vectors having zero norm. In this context the norm is any general real number; even zero and negative numbers are allowed for non-zero vectors in GA. This is one important generalization different from metrics in classical linear algebra that are usually restricted to being positive definite. One of the common interpretations of vector norm in the special case of Euclidean linear spaces is the the squared length of a direction vector.

If the linear space has the basis \( \{e_1, e_2, \ldots, e_n\} \) then we can construct a bilinear form matrix \( A_B = [a_{ij}], a_{ij} = B(e_i, e_j) \), also called the metric matrix on this basis. This matrix is a real symmetric matrix that we can use to compute the bilinear form of any two vectors \( u, v \in V \) given their representation on the basis as follows:

\[
\begin{align*}
  u & = u_1e_1 + \cdots + u_ne_n, \\
  v & = v_1e_1 + \cdots + v_ne_n \\
\Rightarrow B(u, v) & = (u_1 \cdots u_n) A_B (v_1 \cdots v_n)^T
\end{align*}
\]

Using bilinear forms the concept of orthogonality of vectors can be defined as follows: two vectors \( u, v \) are called orthogonal iff \( B(u, v) = 0 \). The inner product of two vectors is simply the bilinear form of the vectors \( u \cdot v \equiv B(u, v) \), and the norm is the inner product of a vector with itself \( v^2 = v \cdot v \); thus justifying the use of the name Inner Product Matrix (IPM) for the symmetric bilinear form matrix. The IPM \( A_B \), being a real symmetric matrix, can be diagonalized using a Change of Basis Matrix (CBM) \( P \) to obtain a diagonal matrix \( D = P^T A_B P \) where \( P \) is an orthogonal matrix \( P^{-1} = P^T \). The columns of \( P \) are orthogonal eigen vectors for \( A_B \). The diagonalization can always be performed such that the numbers on the diagonal (called the eigen values) are either -1, 0, or +1. The number of eigen values that are 1,-1, and 0 are characteristics for a given metric and define what is called the signature of the bilinear form. A bilinear form is said to have the signature \( (p,q,r) \) if there exists a diagonalization of the IPM having \( p \) eigen values with value 1, \( q \) eigen values with value -1 and \( r \) eigen values with value zero. If the IPM is singular (i.e. has no inverse which is equivalent to \( r > 0 \)) the bilinear form is called degenerate. If all the eigen values are positive the IPM is positive definite and the space is a Euclidean space; there exists a basis with all basis vectors norms equal to +1. A mixed-signature metric space has some non-zero vectors with norm equal to zero. Such vectors are called null vectors and only exist in mixed-signature spaces (spaces having a bilinear form with \( p > 0 \) and \( q > 0 \)) in addition to degenerate spaces. The signature of the IPM extends to the signature of the whole GA that we construct using the IPM. By combining the concept of metric and the concept of space embedding, discussed later, we can consistently model Euclidean and non-Euclidean geometries using GAs of various signatures.

To illustrate how a metric effects the geometry of the space, Table 1 shows some possible metrics of a 2D linear space with basis \( \{e_1, e_2\} \). Using this general definition of the unit circle “The set of position vectors having unit norm \( \{v : v = xe_1 + ye_2, \|v\| = 1, x, y \in \mathbb{R}\} \)” we get the general equation \( x^2e_1^2 + 2xy(e_1 \cdot e_2) + y^2e_2^2 = 1 \). We see that only in Euclidean space \( e_1 \cdot e_2 = \delta^1_2 \) where we get the familiar circle equation, where in other metrics we get totally different geometries. The same goes for the geometric meanings of other metric-dependent concepts like orthogonality, angle, rotation, distance, area, projection, etc.

2.5 Linear Maps and Their Representations

Linear maps are a central concept for creating Geometric Computing applications. One of the main reasons is that linear maps have a direct relation to multi-dimensional Projective Geometry [44, 36], which is the base for all Euclidean and non-Euclidean geometries, and has many applications in computer graphics, computer vision, robotics, and image processing, for example. I will denote the effect of a linear map \( f : V \rightarrow W \) on a vector \( x \in V \) and on a subspace \( X = \text{span} (x_1, x_2, \ldots, x_k) \leq V \) as \( f[x] \in W \) and \( f[X] \leq W \) respectively. Classically the concept of a linear map is associated with matrix algebra through the following construction: assuming the real linear spaces \( V, W \) with bases \( A = \langle a_1, a_2, \cdots, a_n \rangle, B = \langle b_1, b_2, \cdots, b_m \rangle \)
Table 1: The concept of a unit circle in different metrics on 2D linear space

| IPM   | Equation | Geometric Shape |
|-------|----------|-----------------|
| $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ | $ax^2 + by^2 = 1$ | Circle $a = b > 0$ |
|       |          | Imaginary Circle $a = b < 0$ |
|       |          | Ellipse $a > 0, b > 0$ |
|       |          | Imaginary Ellipse $a < 0, b < 0$ |
|       |          | Hyperbola $ab < 0$ |
| $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ | $x^2 = 1$ | Straight Line |
| $\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ | $x^2 = -1$ | Imaginary Straight Line |
| $\begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}$ | $xy = \frac{1}{2a}$ | Rectangular Hyperbola |

$\langle b_1, b_2, \ldots, b_n \rangle$ respectively; we can define a linear map $f : V \to W$ such that the effect of $f$ on each basis vector in $A$ is known and expressed as a linear combination of the basis vectors in $B$ (i.e. the column vectors $m_i = [f(a_i)]_B$ are known for all $a_i$), the matrix of $f$ acting on $A$ with respect to $B$ is defined as the $m \times n$ matrix:

$$M_f = [f]_{A,B} \equiv \begin{bmatrix} m_1 & m_2 & \cdots & m_n \end{bmatrix}, \ m_i = [f(a_i)]_B \quad (2.4)$$

We can then compute a coordinate representation of the transformation $f[x] \in W$ of any vector $x = \sum_{i=1}^n x_i a_i \in V$ through a simple matrix multiplication operation:

$$[f[x]]_B = [x_1 f[a_1] + x_2 f[a_2] + \cdots + x_n f[a_n]]_B$$

$$= [f_1 f[a_1]]_B + x_2 [f[a_2]]_B + \cdots + x_n [f[a_n]]_B$$

$$= [f]_{A,B} [x]_A \quad (2.5)$$

When the two linear spaces are the same $V = W$ then $f$ is a linear operator on $V$. When the two basis are also the same $A = B$ these relations become:

$$M_f = [f]_A \equiv \begin{bmatrix} m_1 & m_2 & \cdots & m_n \end{bmatrix}, \ m_i = [f(a_i)]_A, \ [f[v]]_A = [f]_A [v]_A \forall v \in V \quad (2.6)$$

The unique matrix $[f]_{A,B}$ is called the matrix representation of $f$ with respect to basis $A$ and $B$. We can then find many properties of the linear map by applying matrix algebra operations on its matrix. For example:

- The dimensions of the domain $V$ and co-domain $W$ of a linear map $f : V \to W$ are respectively equal to the number of columns and rows of its matrix $[f]_{A,B}$ on any two basis. In addition, we can apply a composition of linear maps between linear spaces using matrix multiplication of their matrices. This is extensively used in computer graphics and robotics for composing sequences of motions expressed as linear maps.

- The adjoint linear operator $f^T$ associated with a linear operator $f$ defined on a real metric linear space $V$ with bilinear form $B$ is the operator that satisfies $B(f[x], y) = B(x, f^T[y]) \forall x \in V$. The matrix representation of $f^T$ is the transpose of the matrix representation of $f$, but only on the same basis: $[f^T]_A = [f]^T_A$.

- An isomorphism is a linear map defined on two linear spaces of the same dimension that is also invertible. The matrix of an isomorphism on any two bases is non-singular. In addition, the matrix of an invertible linear operator $f$ and the matrix of its inverse $f^{-1}$ are inverses to each other, but only on the same basis $[f]_A \ [f^{-1}]_A = I$.

- Any isomorphism has a unique basis-independent scalar associated with it called its determinant $|f|$ [67]. The determinant of the isomorphism is equal to the determinant of its matrix on any bases.
2.6 Oriented Subspaces

2 METRIC LINEAR SPACES

|f| = |f|_{A,B}. The geometric significance of the determinant is more apparent within the context of GA’s outermorphisms discussed later.

- We can define a unique Change of Basis isomorphism between two linear spaces of the same dimension \( g : V \rightarrow W \) that takes \( A \) to \( B \) such that \( g[a_i] = b_i \forall i = 1,2,\ldots,n \). This isomorphism also has a unique invertible matrix \([g]_{A,B}\) called a Change of Basis Matrix (CBM). This means that the same invertible matrix may represent an invertible linear operator on the same basis, or a change of basis linear map on two different bases. This is one instance of mixing of conceptual abstractions that is common in matrix algebra formulations. Such issue might lead to confusions in algebraic formulations when using matrix algebra to represent abstract linear maps.

- Two matrices \( M, N \) are called similar \( M \sim N \) if they represent the same linear operator on different bases, or equivalently if there is a CBM \( C \) such that \( M = C^{-1}NC \). Similarity between square matrices is an equivalence relation. Many invariant properties of similar matrices are actually properties of their common abstract linear map. Most notably, spectral analysis and invariant subspace techniques in linear algebra [68] depend on this relation between an abstract linear map and its infinite number of representation matrices. These techniques are very important in many scientific and engineering applications.

- An invertible linear operator \( f \) that satisfies \( B(f[x], f[y]) = B(x, y) \forall x, y \in V \), where \( B \) is the bilinear form on \( V \), is called an orthogonal linear operator; it preserves the metric between vectors. This means that \( f \) preserves many metric-dependent properties and operations like the inner product, norm, orthogonality, and angle between vectors. In addition, its adjoint is equal to its inverse: \( f^{-1} = T \). For non-degenerate metrics, the matrix of an orthogonal operator is invertible, has ±1 determinant, and has columns that represent orthonormal vectors; i.e., each two column vectors are orthogonal and have unit (i.e., ±1) norm. These matrices are called orthogonal matrices and are very important in many practical applications. We can analyze construct any such map as a composition of a series of geometric reflections in homogeneous hyper-planes (i.e., \((n-1)\)-dimensional subspaces) of the linear space. The Householder operator [69, 70, 71], one of the most important computational tools in numerical matrix algebra, is based on this conceptual construction. GA provide a better algebraic alternative using its Versors and Versor Product.

- The Kernel \( \ker f \) or Null Space of a linear map \( f : V \rightarrow W \) is the set of vectors that transform to the zero vector of \( W \) under \( f \): \( \ker f = \{ v : v \in V, f[v] = 0_W \} \). The range \( \text{range}_f \) or Image of \( f \) is the set of vectors in \( W \) that are transformations under \( f \) of vectors in \( V \): \( \text{range}_f = \{ w : \exists v \in V \, \forall f[v] = w \} \). These two sets are subspaces satisfying the relations \( \ker f \subseteq V, \text{range}_f \subseteq W, \) and \( \dim(V) = \dim(\ker f) + \dim(\text{range}_f) \) where \( \dim \equiv \dim(\ker f) \) is also called the nullity off and \( \dim(\text{range}_f) \) is called the rank off. We can use matrix algebra to find the kernel of \( f \) using its representation matrix \([f]_{A,B} \) on any two bases by solving the matrix equation \( [f]_{A,B}v = 0 \) for coordinate vectors \( v \) to find a set of LID spanning coordinate vectors for the kernel. We can also represent the range of a linear map using \([f]_{A,B} \) by viewing its column vectors as representing abstract vectors in \( W \) that span \( \text{range}_f \). This leads to the familiar matrix rank of \([f]_{A,B} \) being equal to the rank of the linear map it represents \( \text{rank}([f]_{A,B}) = \text{rank}_f \). These linear spaces and their relations are an important part of the Fundamental Theorem of Linear Algebra [72] usually expressed using matrices not abstract linear maps.

It is very important when designing Geometric Computing applications in a Computational Thinking sound manner to have clear conceptual distinction between an abstract linear map and its infinite number of possible matrix representations. In GC applications it is typical that the choice of basis is not arbitrary or even unique. The same problem may need many bases to be used, as in the case of robotics and computer graphics for example. Because matrices can also represent subspaces (as lists of column vectors) and metrics (as IPMs), matrix algebra formulations can hide the abstract geometric meaning behind the clutter of its less abstract and basis-implicit representations. The use of GA formulations instead of matrix algebra can, in many cases, enforce a clear separation of basic abstract concepts from their representations.

2.6 Oriented Subspaces

When we use matrix algebra to represent linear spaces, we have a well-developed set of tools to algebraically represent and manipulate abstract vectors. In many applications in science and engineering,
Some common subspace manipulations we use include:

- To construct a subspace given a set of vectors that spans the subspace; the vectors may or may not be linearly independent.
- We may need to extract information about a subspace such as its dimensionality, its relation to fixed subspaces in the problem, and the “best” basis of vectors we can use to span the subspace. Here the word “best” is context-dependent. We may prefer a basis for getting more numerically-stable computations, or perhaps for having a better correspondence with actual physical elements of our model.
- To apply a linear map to a whole subspace and get another.
- To operate on two or more subspaces in order to get another subspace as output. For example, to find the common subspace of two subspaces, to find the smallest subspace containing two subspaces, to project one subspace on another, to find a subspace that complements another into a bigger subspace, and to reflect one subspace on another.
- To compare two subspaces having different attitudes in space. This includes, for example, finding the angle of a single rotation operation that takes one subspace into another, or finding if two subspaces are orthogonal to each other in the sense that each vector in the first is orthogonal to all vectors in the second subspace.

Any single vector \( v \) actually represents a 1-dimensional subspace \( \langle v \rangle \) through its span: \( \langle v \rangle = \text{span}(v) \). Extending this to more dimensions we can use the span of \( k \) LID vectors to represent their \( k \)-dimensional subspace \( W = \text{span}(v_1, v_2, \ldots, v_k) \). A matrix \( A_W \) can represent an ordered set of vectors by putting their equivalent coordinate representations on some basis \( E \) as rows or columns in the matrix:

\[
A_W = \begin{bmatrix}
[v_1]_E & [v_2]_E & \cdots & [v_k]_E
\end{bmatrix}.
\]

This way we can use matrix algebra and matrix operations to manipulate this “list of coordinate vectors” as an indirect (and mostly awkward) computational representation of abstract linear subspaces. This kind of representation has disadvantages for practical Geometric Computing applications. Matrix algebra is a suitable mathematical abstraction for low-level computations inside machines, but is not an intuitive modeling abstraction when designing GC models and algorithms. Much geometric information get scattered among the numbers of the matrix, and we need significant effort to extract such information. In addition, matrix algebra-based formulations are often basis-dependent and metric-dependent. As I will explain in the next section, Geometric Algebra can provide more powerful and geometrically significant representations for subspaces using GA’s Blades. GA-based formulations are found to be significantly more compact and basis-independent for many applications.

While the set intersection \( U \cap W \) of two subspaces \( U, W \leq V \) is also a subspace in \( V \), their set union \( U \cup W \) is not guaranteed to be a linear space. An analogous operation to set union that guarantees a subspace result is called the sum of subspaces defined as \( W + U = \{x : x = w + u; \ w \in W, u \in V\}, W, V \leq V \). Having a set of mutually disjoint subspaces \( W_1, W_2, \ldots, W_k \leq V \) (i.e. \( W_i \cap W_j = \{0\} \forall i, j = 1, 2, \ldots, k, i \neq j \)) the subspace sum of \( W_i \) is called the direct sum of the disjoint subspaces and is denoted by \( \bigoplus_{i=1}^{k} W_i \equiv W_1 \oplus W_2 \oplus \cdots \oplus W_k \). The dimension of the direct sum of disjoint subspaces is equal to the numerical sum of their respective dimensions \( \dim \left( \bigoplus_{i=1}^{k} W_i \right) = \sum_{i=1}^{k} \dim(W_i) \). We often use this notation to construct a larger linear space, like the linear Grassmann space of multivectors, out of a number of mutually disjoint linear spaces. This conceptual construction is metric-independent and basis-independent.

Another important concept is the orthogonal complement of a metric subspace \( W \leq V \) defined by \( W^\perp = \{x \in V : \ y \perp x \forall y \in W\} \). The orthogonal complement of a subspace \( W \leq V \) has the following properties:

\[
V = W \oplus W^\perp
\]
\[
x \perp y \forall x \in W, y \in W^\perp
\]
\[
(W^\perp)^\perp = W
\]

The classical treatment of subspaces in linear algebra mostly deals with un-oriented subspaces, were a subspace is just a set of vectors closed under linear combinations. In many practical scientific and engineering applications, however, we need to distinguish between two opposite orientations for any
subspace. This orientation concept is particularly useful in applications involving Projective and Cayley-Klein Geometries [43]. We can mathematically define the concept of orientation for linear spaces as follows [67]: Let $V$ be a finite-dimensional real linear space and let $E = (e_1, e_2, \ldots, e_n)$ and $F = (f_1, f_2, \ldots, f_n)$ be two ordered bases for $V$ with a Change of Basis isomorphism $g : V \rightarrow V$. The bases $E$ and $F$ are said to have the same orientation iff $g$ has a positive determinant; otherwise they have opposite orientations, meaning that $g$ involves a geometric reflection. The property of having the same orientation defines an equivalence relation on the set of all ordered bases for $V$. There are only two equivalence classes determined by this relation. An orientation on $V$ is an assignment of $+1$ to one equivalence class and $-1$ to the other. Blades in Geometric Algebra can naturally represent oriented subspaces as I will explain later in the next section.

From the previous discussions we can see that matrix algebra is a good intermediate representation capable of representing metrics, linear maps, and subspaces; but we need to be extra careful about the selection of basis and abstract meanings behind matrix operations. However, GA provides better basis-independent, metric-independent, and dimension-independent alternatives for studying and extending oriented linear subspaces and linear maps without the explicit need to use matrices. Most notably here, GA Blades can naturally represent not only oriented subspaces, but weighted oriented subspaces as I will explain in the next section.

### 2.7 Space Embeddings

The abstract concepts I described in earlier subsections are necessary tools that enable the use of the powerful conceptual idea of Space Embedding [73]. In the study of 3D Euclidean space, for example, simple geometric concepts like points, general lines, and planes can’t be mathematically represented as elements of a 3D linear space; they simply don’t satisfy the abstract axioms of 3D linear spaces. In 1827, August Ferdinand Möbius introduced homogeneous coordinates, or projective coordinates, to solve this problem by embedding 3D Euclidean space into a 4D projective space. Using this embedding we could easily model additional geometric concepts as 4D vectors and subspaces. This algebraic construction has greatly impacted many applications in engineering and computer science including robotics, computer graphics, computer vision, computer-aided design, and more. By extending this idea to larger dimensions and using various metrics, we can embed a smaller space of interest, linear or not, into a larger metric linear space. Then we can use the algebraic tools of the larger linear space to represent and manipulate the objects of the smaller space. This is one kind of linearization that scientists and engineers should exploit more in their work. Expressing this is possible, in principle, using matrix algebra; but it’s much better to use Geometric Algebra to express Space Embeddings. Many GAs are already applied for representing mathematical and geometric spaces in this way including:

- Among the first, and most important GAs comes the Space-Time Algebra (STA) [74, 75], a GA of signature $(1,3,0)$ that provides a unified, coordinate-free mathematical framework for both classical and quantum physics. STA is particularly important for electrical engineers as it combines the electric and magnetic fields into a single complex and frame-independent bivector field, and reduces electrodynamics to a single Maxwell equation on multivectors with explicit kinship to Dirac’s equation.

- The 3D Euclidean GA with metric of signature $(3,0,0)$ is a simple space to express rotations on homogeneous lines and planes [3, 18]. The algebra of quaternions is a sub-algebra of this GA.

- The 4D Homogeneous GA with metric of signature $(4,0,0)$ is a GA extension of Möbius’s homogeneous coordinates mentioned above [3, 18]. Some of the Euclidean transforms are linear orthogonal maps in this space, while others are non-orthogonal linear maps.

- Most notably, the 5D Conformal GA (CGA) [3, 22, 76] is the most applied GA with too many practical GC applications to reference here. This space has a metric with the signature $(4,1,0)$. Some of the objects we can linearize with CGA vectors and subspaces include spheres, circles, point-pairs, general lines and planes, tangent lines and planes, and many more. All conformal and similarity transforms (translations, reflections, rotations, uniform scalings, inversions, etc.) are linear orthogonal maps in this space. In addition, perspective projection could be represented using rotations of this space [77].

- Projective GA (PGA) is a class of degenerate GAs of signatures $(n,0,1)$ that provides a powerful efficient model for n-dimensional Euclidean geometry. [36, 78, 7, 79], especially for applications
in kinematics and rigid body mechanics. For classical flat euclidean geometry, PGA exhibits distinct advantages over the alternative approaches. PGA serves also as an ideal stepping-stone both scientifically and pedagogically to more complex GAs such as CGA.

- Recently, the 10D Double Conformal GA (DCGA) with metric of signature (8,2,0) is used to represent points and general (quartic) Darboux cyclide surfaces in Euclidean 3D space, including circular tori and all quadrics, and all surfaces formed by their inversions in spheres. In addition to representing Dupin cyclides, which are quartic surfaces formed by inversions in spheres of torus, cylinder, and cone surfaces; and parabolic cyclides which are cubic surfaces formed by inversions in spheres that are centered on points of other surfaces. All DCGA entities can be transformed by orthogonal maps of this space, and reflected in spheres and planes.

More GAs are also under study for other purposes. The list will probably grow over time requiring efficient software implementations to computationally realize the potentials of such GA-based space embeddings.

3 Geometric Algebras

The previous discussion about scalars and metric linear spaces introduced many familiar concepts of linear algebra in a way to be suitable for constructing Geometric Algebras. The cornerstone in the GA structure is the concept of Blade and the operation of Outer Product. All concepts in metric linear spaces can be generalized, in geometrically significant ways, to handle blades rather than just vectors.

Blades are excellent representations for oriented linear subspaces, and adding them to metrics and space embeddings gives GA its representational and computational power. To really understand and appreciate the power of GA as a mathematical language, a software engineer, as a good Computational Thinker, has to investigate GA on 3 levels:

- The abstract level including the defining mathematical axioms and main algebraic properties. Understanding this level is more important to GA model developers, but it’s also important for GC software engineers for communicating with the developers of GA models, and for having a solid mathematical base for GA-based computations. I recommend starting with simple GA introductions, for example.

- The representational level where GC software engineers study examples for geometric entities and processes they can represent and manipulate with elements of GA. The GA literature is the best place to develop a good understanding of GA at this level for any particular fields of study.

- The computational level including how to use elements of GA Coordinate Frames to perform and interpret useful computations. I will provide more details on this level in section 4. The best way to appreciate GA on this level is to learn by doing: by selecting some GA software system, like CLUViz, and actually computing with and visualizing GA elements.

In this section, I attempt to briefly discuss the mathematical GA structure through a gradual construction Computational Thinking-based process. My intention is not to provide much mathematical details, but to prepare for the discussion about the last computational level in the following section about GA Coordinate Frames. The mathematics in this section mostly follows the first 7 chapters of which contains more mathematical details, discussions, and very good practical programming examples.

3.1 Blades and The Outer Product

In 3D Euclidean space we are taught a number of products involving Euclidean direction vectors expressed on an orthogonal basis:

- The scalar multiplication between a scalar and a vector \( av \) that changes the length of the vector \( v \) by the scalar factor \( a \).

- The dot product of two vectors \( u \cdot v \) that produces a scalar proportional to cosine the angle between two vectors and their lengths \( u \cdot v = \|u\|\|v\|\cos(\theta) \).

- The cross product of two vectors \( u \times v \) that produces a third vector orthogonal to both vectors with a length proportional to the sine of the angle between them and their lengths \( \|u \times v\| = \|u\|\|v\|\sin(\theta) \).
These operations along with vector addition construct the core of classical vector algebra [35], a basic mathematical tool in science and engineering historically emerging from a war among mathematicians and engineers [38]. In mathematics, however, there are many other products between vectors with significant geometric interpretations and much better universal representative capabilities. One such products is called the Exterior or Outer Product of vectors $x \wedge y$, a cornerstone of the structure of Geometric Algebra [5, 15].

We can use an abstract vector in a n-dimensional linear space with Euclidean metric $(n,0,0)$ as a representation of an nD Euclidean direction vector $v$. We can also think about $v$ as a generator of all the points on a homogeneous line; a line passing through the origin of Euclidean nD space, parallel to $v$. We can generate all the points on that line by a scalar multiplication $av$ with any real number $a$. Then $v$ represents a 1-dimensional subspace having some attitude in space, a length or weight, and an orientation. The outer product can algebraically extend this simple geometric construction to more dimensions. If $x$ and $y$ are two LID vectors, then $x \wedge y$ is a distinct algebraic entity that can represent a Directed Area in nD Euclidean space. This directed area, called a bivector, has an attitude determined by the combined attitudes of its 2 vectors, a weight proportional to their lengths, and an orientation resulting from their order in the outer product. In addition, this bivector represents a homogeneous plane in nD Euclidean space.

We can extend this even more by taking the outer product of $k$ LID vectors, where $k \leq n$, to obtain a new class of algebraic entities called Blades. As we can represent the same homogeneous line using many vectors differing only by their lengths or orientations, we can represent any k-dimensional subspace using an infinite number of blades differing only in their weights or orientations. This construction also has similar representational roles in other metric spaces, but the metric defines the “geometric shape” that the blade represents. This is where the concept of subspace with non-Euclidean metric differs from our intuitive flat hyperplane geometry of multi-dimensional Euclidean spaces. One important characteristic of the outer product is that it’s a metric-independent concept. The algebraic axioms of the outer product do not depend on the selected metric of the linear space, only the interpretation of the resulting blades do.

In other space embeddings, Blades have a surprising capability to linearly represent many geometric objects we need in practical applications. For example in the 5D Conformal GA, 4-blades can represent points, spheres, and general planes. This unifies the geometry of points, spheres, and planes by algebraically treating a plane as a sphere with infinite radius, and a point as a sphere with zero radius, enabling interesting interpolations between them. In addition, we can represent spheres with positive or negative squared radii using 4-blades in CGA, i.e. we can represent a sphere with imaginary radius. This adds more geometric freedom and algebraic consistency to many CGA-based models by removing many special cases that we need to explicitly address while developing GA-based geometric models.

Because the outer product is metric-independent, without loss of generality I will concentrate in this section on the simple real Euclidean linear spaces $\mathbb{R}^n$ with a basis $\{e_1, e_2, \ldots, e_n\}$ as they are isomorphic to all other real Euclidean linear spaces of the same dimension. The focus is on all subspaces of $\mathbb{R}^n$ of dimensions $k$ where $0 \leq k \leq n$. The geometric meaning of any such subspace is a k-dimensional homogeneous flat (the origin, a line through the origin, a plane through the origin, etc.) in $\mathbb{R}^n$. The Outer Product of an ordered set of $k$ LID vectors $\langle a_1, a_2, \cdots, a_k \rangle$ is used to define algebraic objects, called k-blades in GA, that can be used to represent subspaces algebraically with four main characteristics for each subspace:

1. The dimensionality of a subspace $k$: This is represented by the Grade $k$ of the k-blade, the number of LID vectors in the outer product producing the blade.

2. The attitude of the subspace: this is equivalent to the traditional un-oriented span in classical linear algebra of the set of vectors $\{a_1, a_2, \cdots, a_k\}$.

3. The orientation of the subspace: which is a sign (+1 or -1) associated with the subspace to define the relative orientation or handedness of its basis.

4. The weight of the subspace: which is a real number associated with the attitude (and it also includes the sign i.e. the orientation/handedness of the subspace).

The simplest subspace is the 0-dimensional subspace spanned by no vectors (i.e. it only contains the zero vector) with a corresponding 0-blade that is simply a scalar $\lambda \in \mathbb{R}$; this subspace will be denoted by $B_0^n = \mathbb{R}$. Any vector $x \in \mathbb{R}^n$ is a 1-blade by definition and it corresponds to a 1-dimensional subspace spanned by that vector alone; the space of 1-blades will be denoted by $B_1^n = \mathbb{R}^n$. The set of k-blades for
any value of \( k \in \{0, 1, \cdots, n\} \) is denoted by \( B^n_k \) and the set of all blades is denoted here by \( B^n = \bigcup_{k=0}^{n} B^n_k \), the set union of blades of all grades. I will use the notation \( \text{grade}(A) \) to refer to the grade of a blade \( A \). I will denote that a blade \( A \) represents an oriented subspace \( W \) as: \( A \propto W \). I will also use the notation \( \hat{A} \) to indicate the oriented subspace spanned by a blade \( A \).

The Outer Product is an associative grade-rising bilinear map used to construct higher-grade blades from lower-grade ones: \( \land: B^n_r \times B^n_s \to B^n_{r+s}, r, s, r+s \in \{0, 1, \cdots, n\} \). The basic properties of the outer product of scalars (0-blades), vectors (1-blades), bivectors (2-blades), and general k-blades are as follows:

\[
\begin{align*}
\alpha \land \beta &= \alpha \beta \\
\alpha \land x &= x \land \alpha = \alpha x \\
x \land y &= -y \land x \\
X \land (Y + Z) &= X \land Y + X \land Z \\
A \land (B \land C) &= (A \land B) \land C \\
A \land (\alpha B) &= \alpha(A \land B)
\end{align*}
\]

\[ (\alpha, \beta) \in B^n_0, \]
\[ x, y, z \in B^n_1, \]
\[ X, Y, Z, (Y + Z) \in B^n_k, \]
\[ A, B, C \in B^n \]

In addition, the anti-symmetry property (3.3) is a special case of a more general relation \( X \land Y = (-1)^{s \times r} Y \land X, X \in B^n_r, Y \in B^n_s \). The anti-symmetry property (3.3) leads to the important relation \( x \land x = -x \land x = 0 \). This means that the Outer Product of a collection of linearly dependent vectors is always zero. A non-zero blade algebraically encodes the relation of linear independence among vectors. This is one major difference between the use of Blades vs matrices for representing linear subspaces that has many conceptual, representational, and computational consequences. Having the r-vectors. This is one major difference between the use of Blade s vs matrices for representing linear subspaces.

In d Euclidean spaces, we can define useful geometric operations on vectors using the inner product. For example the squared length of a vector \( ||x|| = x \cdot x \) and the angle between two vectors \( \cos(\theta) = \frac{u \cdot v}{||u|| ||v||} \). We can extend the bilinear form of any metric linear space to operate on k-blades of any grade, not just vectors. We can use this extended bilinear form as a product to define similar geometrically significant operations for higher-grade blades. This product is called the Scalar Product of blades [86, 3]. The scalar product can be defined as follows:

3.2 Generalizing the Inner Product

In d Euclidean spaces, we can define useful geometric operations on vectors using the inner product. For example the squared length of a vector \( ||x|| = x \cdot x \) and the angle between two vectors \( \cos(\theta) = \frac{u \cdot v}{||u|| ||v||} \). We can extend the bilinear form of any metric linear space to operate on k-blades of any grade, not just vectors. We can use this extended bilinear form as a product to define similar geometrically significant operations for higher-grade blades. This product is called the Scalar Product of blades [86, 3]. The scalar product can be defined as follows:
3.2 Generalizing the Inner Product

For two blades of different grades, the scalar product has a zero value by definition; it can only be zero when they are algebraically orthogonal in the sense of being independent; i.e., not having enough in common in terms of dimension or attitude such that there is no single rotation with any angle that can make them identical. For two blades of different grades, the scalar product has a zero value by definition; it can only relate subspaces of the same dimension.

To compare subspaces of different dimensions another bilinear product is required that should be universally applicable to all blades. The Left Contraction Product [86, 3] is one such product having geometrically significant interpretations. The Left Contraction Product is denoted by \(A \mid B\) and pronounced “\(A\) contracted on \(B\)’’ where \(\{ : B^n \times B^m \to B^r \mid r, s, r \in \{0, 1, \ldots, n\}\) is a grade-lowering bilinear map on blades. This product was introduced by Lounesto as the adjoint of the Outer Product under the extended bilinear form expressed here as the Scalar Product [87]. The Left Contraction Product is bilinear and distributive over addition, but not associative; this is apparent from comparing the grade of \((A \mid B)\mid C\) and \(A \mid (B \mid C)\) that are generally not equal. The Left Contraction is identical to the Scalar Product of two same-grade blades \(A \mid B = A \ast B\), \(\forall A, B \in B^n\). Having \(A \propto \tilde{A}\), \(B \propto \tilde{B}\), \(A \in B^n\), \(B \in B^n\), the geometric meaning of \(A \mid B\) is the \((s-r)\)-blade \(C \propto \tilde{B} \cap (\tilde{A})^\perp\). If the subspace \(C\) has a dimension other than \(s - r\) the result of \(A \mid B\) is considered zero by definition to preserve its linearity. A constructive explicit definition of the left contraction is as follows [3]:

\[
\begin{align*}
\alpha \mid \beta &= \alpha \beta \\
\alpha \mid A &= \alpha A \\
A \mid B &= 0, \text{ grade}(A) > \text{grade}(B) \\
a \mid b &= B(a, b) = a \cdot b \\
(a \mid (B \wedge C)) &= (a \mid B) \wedge C + (-1)^{\text{grade}(B)}B \wedge (a \mid C) \\
(A \mid B) \mid C &= A \mid (B \mid C) \\
\alpha, \beta &\in B_0^n, \\
A, B, C &\in B^n
\end{align*}
\]

The relation (3.13) is valid for any three blades \(A, B, C\) whereas the following relation of the three blades is only valid under a certain condition:

\[
(A \mid B) \mid C = A \wedge (B \mid C), \quad A \leq C
\]
Equations (3.13) and (3.14) are called the duality formulas that link the Outer and Contraction products on blades. One more useful property of the contraction is given by:

\[ x \langle a_1 \wedge a_2 \wedge \cdots \wedge a_k \rangle = \sum_{i=1}^{k} a_1 \wedge a_2 \wedge \cdots \wedge (x) a_i \wedge \cdots \wedge a_k \]  \hspace{1cm} (3.15)

\[ \Rightarrow x \langle a \wedge b \rangle = (x \cdot a)b - (x \cdot b)a \]  \hspace{1cm} (3.16)

Geometrically when \( A, B \) are blades, \( A \rangle B \) is another blade contained in \( B \) and perpendicular to \( A \) with a norm proportional to the norms of \( A, B \), and the projection of \( A \) on \( B \). In addition, the following relation between a vector and a blade is important: \( x \langle A \rangle = 0 \Leftrightarrow x \perp y, \forall y \leq A \); meaning that \( x \langle A \rangle = 0 \) iff \( x \) is orthogonal to all vectors contained in the subspace \( \sum A \). Another computationally useful version of the Left Contraction can be defined that is called the **Right Contraction** product, denoted by \( B \rangle A \) and pronounced as “\( B \) contracted by \( A \)” where \( \langle \rangle: B^n \times B^n \rightarrow B^{n-r}, r, s, r - s = \{0, 1, \cdots, n\} \). The right contraction is related to the left contraction by:

\[ B \rangle A = \left( \tilde{A} \right) \tilde{B} = (-1)^{a(b+1)} A \rangle B, \]  \hspace{1cm} (3.17)

\[ a = \text{grade} (A), b = \text{grade} (B) \]

The duality formulas (3.13) and (3.14) can be written for the right contraction as:

\[ C[(B \wedge A)] = (C[B]) \langle A \rangle, \forall A, B, C \in B^n \]  \hspace{1cm} (3.18)

\[ C[(B \langle A \rangle)] = (C[B]) \wedge A \forall A, B, C \in B^n, A \leq C \]  \hspace{1cm} (3.19)

### 3.3 Orthogonality and Duality of Blades

Any non-null blade \( A \in B^n_k, \|A\| \neq 0 \) can have an inverse blade \( A^{-1} \) with respect to the left contraction product (i.e. \( A \langle A^{-1} \rangle = 1 \)) defined as:

\[ A^{-1} = \tilde{A} \frac{(-1)^{k(k-1)/2}}{\|A\|}, A = \text{grade} (A) \]  \hspace{1cm} (3.20)

This inverse is not unique with respect to the left contraction, but is always present for non-null blades. A special case is the inverse of a non-null vector given by \( a^{-1} = \frac{a}{\|a\|} \). When combined with the geometric product in the next subsection, this inverse defines a geometrically meaningful “division” by non-null blades and vectors for the first time. For any blade with unit norm like the pseudo-scalar of a Euclidean space the inverse of the blade is its reverse \( I^{-1} = I^\perp, \|I\| = 1 \). For a mixed-signature metric space with signature \( (p, q, 0) \) the inverse of the pseudo-scalar is given by \( I^{-1} = (-1)^q I^\perp \). For degenerate metric spaces the inverse of the pseudo-scalar is not defined.

Using the inverse of a blade a very important operation on blades can be defined that is called the dual of a blade \( A \in B^n_k \) with respect to a larger containing blade \( X \in B^n_s, A \leq X \) that is a linear mapping \( \ast: B^n_s \times B^n_s \rightarrow B^n_{s-r} \) that acts as follows:

\[ A^{\ast X} = A \langle X^{-1} \rangle, \forall A \leq X \]  \hspace{1cm} (3.21)

When the larger blade is the space pseudo-scalar \( I \) the dual is simply written as \( A^\ast = A \rangle I^{-1} \). The geometric meaning of the dual \( A^\ast \) is simply a blade orthogonal to the original blade \( A \) such that they together complete the space; i.e. \( A \propto \tilde{A} \iff A^\ast \propto (\tilde{A})^\perp \). This means that any blade \( A \in B^n_k \) can computationally represent two subspaces \([3, 10]\):

- The \( r \)-Blade \( A \) directly represents the \( r \)-dimensional subspace \( X = \{ x: x \wedge A = 0 \} \); this is denoted here as \( A \propto X \). In this case, the subspace \( X \) is called the Outer Product Null Space (OPNS) of the blade \( A \).
- The \( r \)-Blade \( A \) dually represents the \((n - r)\)-dimensional subspace \( Y = \{ y: y \mid A = 0 \} \); this is denoted here as \( A \quad \tilde{\propto} \quad Y \). In this case, the subspace \( Y \) is called the Inner Product Null Space (IPNS) of the blade \( A \).
These two representation methods will need special attention when consistently applying linear maps on subspaces using outermorphisms of blades in subsection 3.6. In 3D Euclidean spaces we use the IPNS in the form of normal vectors computed from the cross product. We can then replace and generalize the cross product using the relation \( u \times v = (u \wedge v)^* \in B_{n-2}^n \).

By applying relation (3.14), we find that taking the dual of a blade two times results in the same blade with a weight change:

\[
(A^*X)^*X = (-1)^{(s−1)/2} \frac{1}{||X||} A, \quad \forall A \in B^n_x, X \in B^n_x, A \leq X
\]  

Another related operation on a blade \( A \leq X \) called the un-dualization of the blade \( A \) with respect to the blade \( X \) can be defined as follows:

\[
A^{\odot X} = A |X, \forall A \leq X
\]  

Applying the un-dualization after the dualization (and similarly applying the dualization after the un-dualization) results in the original blade with no weight change: \((A^*X)^{\odot X} = (A^{\odot X})^*X = A\). Using the duality formulas a duality relation can be found between the contraction products and the outer product for any two blades:

\[
(A \wedge B)^*X = A^*B^*X, \quad (A \langle B \rangle)^*X = A \wedge B^*X \forall A, B \leq X
\]  

A useful application on the concepts in this subsection is the typical need is to express a vector \( x \in \mathbb{R}^n \) as a linear combination of general (i.e. not necessarily orthogonal) basis vectors \( \langle b_1, b_2, \cdots, b_n \rangle \) [3]. First an association of each basis vector \( b_i \) with a reciprocal vector is done, defined as \( c_i = (-1)^{i-1}(b_1 \wedge b_2 \wedge \cdots \wedge b_{i-1} \wedge b_{i+1} \wedge \cdots \wedge b_n)I^{-1}, i = 1, 2, \cdots, n, I = b_1 \wedge b_2 \wedge \cdots \wedge b_n \). The basis \( \langle b_1, b_2, \cdots, b_n \rangle \) and \( \langle c_1, c_2, \cdots, c_n \rangle \) are easy to be shown mutually orthogonal \( b_i \cdot c_j = \delta_{ij}, \forall i, j = 1, 2, \cdots n \). The geometric meaning of a reciprocal basis vector \( c_i \) is the orthogonal complement of the span of all basis vectors except the basis vector \( b_i \). To determine the coefficients \( x_i \) such that \( x = x_1b_1 + x_2b_2 + \cdots + x_nb_n \) the relation \( x_i = x \cdot c_i \) (i.e. \( x = \sum_{i=1}^{n}(x \cdot c_i) b_i \)) is used. If the linear space is Euclidean with orthonormal basis then all basis vectors have a norm of \( ||b_i|| = 1 \) hence the reciprocal basis vector \( c_i \) is the same as the basis vector \( b_i \). Generally, two reciprocal basis vectors are not co-linear \( b_i \wedge c_i \neq 0 \) however the following relation holds: \( \sum_{i=1}^{n} b_i \wedge c_i = 0 \).

### 3.4 Multivectors and The Geometric Product

Having a mathematical structure consisting of an n-dimensional real linear space \( V \) with basis \( E = \langle e_1, e_2, \cdots, e_n \rangle \), and associated bilinear form \( B \) with signature \( (p, q, r) \), up until this point we can perform the following algebraic operations using the scalars and vectors of this structure:

1. Create vectors using linear combinations of other vectors. This involves the operations of scalar multiplication and vector addition. We can also represent any vector as a linear combination of the basis vectors \( e_i \).
2. Apply the bilinear form to vectors as an inner product \( x \cdot y \) to get a geometrically significant scalar value.
3. Construct k-blades from LID vectors using the outer product where \( k = 0, 1, \cdots, n \).
4. Extend the bilinear form to blades as a Scalar Product \( s = A \wedge B \) having a geometrically significant scalar value \( s \).
5. Apply the Left Contraction as a dual operation to the Outer Product on blades to obtain a geometrically significant blade \( C \) from two blades \( C = A \wedge B \).

What remains to reach the full Geometric Algebra structure is the following steps. These steps are easy to formulate mathematically, but they create the surprisingly elegant and universal GA structure:

1. Create a total of \( 2^n \) different Basis Blades by taking all possible non-zero outer products of the basis vectors in \( E \).
Table 2: Example for constructing $n + 1$ basis sets for $k$-vectors $E_k^n$ from the set of basis vectors $E = \langle e_1, e_2, \ldots, e_n \rangle$ using the outer product

| Grade | Dimension | Name | Basis Blades |
|-------|-----------|------|--------------|
| 0     | 1         | $E_0^1$ | $\langle 1 \rangle$ |
| 1     | 4         | $E_1^4$ | $\langle e_1, e_2, e_3, e_4 \rangle$ |
| 2     | 6         | $E_2^6$ | $\langle e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4 \rangle$ |
| 3     | 4         | $E_3^4$ | $\langle e_1 \wedge e_2 \wedge e_3, e_1 \wedge e_2 \wedge e_4, e_1 \wedge e_3 \wedge e_4, e_2 \wedge e_3 \wedge e_4 \rangle$ |
| 4     | 1         | $E_4^1$ | $\langle e_1 \wedge e_2 \wedge e_3 \wedge e_4 \rangle$ |

2. Create linear combinations of the basis blades to get new algebraic entities called k-vectors and multivectors. This leads to the construction of a non-metric graded linear Grassmann Space $\Lambda^n$ from the base linear space $\mathbb{R}^n$.

3. Extend the metric of the base linear space $V$ to act on multivectors. This leads to a metric graded linear Geometric Algebra $G^{p,q,r}$.

4. Define a universal bilinear Geometric Product (GP) between multivectors based on the Outer Product and the Bilinear Form between vectors. This product actually contains all other bilinear products as special cases. Physicists and pure mathematicians usually start with this step backwards and deduce the other products from the GP. However, for software developers this construction sequence could be more suitable for their create refactor Computational Thinking mental process.

**In the first step** of this construction, the 0-grade basis blade is the scalar 1 by definition. There are $n$ 1-blades that are the basis vectors themselves $e_i$. We can create $\binom{n}{2} = n(n-1)$ basis 2-blades (bivectors) using the basis vectors $e_i$. Note that $e_i \wedge e_j = -e_j \wedge e_i \forall i \neq j$, so we can only consider one of them to be a basis 2-blade and the other just one of its scalar multiples. I will select the basis 2-blade such that $i < j$ to get a canonical ordering of the basis 2-blades based on the ordering of the basis vectors. Generally, for all k-blades we can extend this construction to obtain canonically ordered $\binom{n}{k}$ basis k-blades of the form $e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_k}$, $j_1 < j_2 < \cdots < j_k$ for each $k = 0, 1, 2, \ldots, n$. This leads to a total of $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ basis blades. I will denote the $n + 1$ sets of basis k-blades as $E_k^n$ and the set of all basis blades as $E^n = \bigcup_{k=0}^{n} E_k^n$. Table 2 shows an example for constructing the basis blades of the 4D Euclidean Geometric Algebra.

**Going to the second step**, it is now natural to try to apply linear combinations to basis blades to get other elements. Taking a linear combination of basis k-blades in $E_k^n$ doesn’t generally produce a k-blade. For example, the algebraic element $3e_1 \wedge e_2 - 2e_1 \wedge e_3 = e_1 \wedge (3e_2 - 2e_3)$ is a 2-blade since it is the outer product of two vectors $e_1$ and $3e_2 - 2e_3$, while $e_1 \wedge e_2 + e_3 \wedge e_4$ can never be expressed as an outer product of vectors so it’s not a blade. This new kind of element is actually called a k-vector, or a Homogeneous Multivector. Such element is a member of the Clifford Algebra $[1]$ on which the GA is based. This simply means that basis k-blades span the $\binom{n}{k}$ dimensional linear space of k-vectors denoted here by $\Lambda_k^n$, so that all k-blades are k-vectors $B_k^n \subseteq \Lambda_k^n$, but not the other way around. The only 4 values for $k$ where both k-blades and k-vectors are identical are $k = 0, 1, n-1, n$. The elements of these four spaces are called scalars, vectors, pseudo-vectors, and pseudo-scalars respectively.

We can now complete this step by taking linear combinations of basis blades of different grades and identifying the zero scalar with all zero k-vectors as a single algebraic entity for convenience. This is the most general case by which we get a full $2^n$-dimensional linear space called the Grassmann Space of Multivectors and denoted by $\Lambda^n = \bigoplus_{k=0}^{n} \Lambda_k^n$. In this way scalars, vectors, k-blades, and k-vectors are all special cases of these multivectors.

A useful metric-independent operator to define on multivectors is the Grade Extraction operator $\langle \rangle_k : \Lambda^n \to \Lambda^n_k$ that extracts the k-vector component from any multivector. For example, if $A = e_3 + 3e_1 \wedge e_2 - 2e_3 \wedge e_4 - e_1 \wedge e_3 \wedge e_4 \in \Lambda^4$ is a multivector, then $\langle A \rangle_0 = 0$, $\langle A \rangle_1 = e_3$, $\langle A \rangle_2 = 3e_1 \wedge e_2 - 2e_3 \wedge e_4$, $\langle A \rangle_3 = 0$, and $\langle A \rangle_4 = -e_1 \wedge e_2 \wedge e_3 \wedge e_4$. If $A \in \Lambda_k^n$ is a k-vector then $\langle A \rangle_k = A$, $\langle A \rangle_r = 0 \forall r \neq k$. This way, we can symbolically express any multivector $A \in \Lambda^n$ as the sum of its k-vectors: $A = \sum_{k=0}^{n} \langle A \rangle_k$. I will also denote the sum of even grade k-vectors in a multivector $A$ as $A_{\text{even}} = \sum_{r} \langle A \rangle_{2r}$, and the
3.4 Multivectors and The Geometric Product

sum of its odd grade k-vectors as \( \langle A \rangle_{\text{odd}} \equiv \sum r \langle A \rangle_{2r+1} \) so that any multivector can also be expressed as \( A = \langle A \rangle_{\text{even}} + \langle A \rangle_{\text{odd}} \). If the multivector only contains k-vectors of even grade \( A = \langle A \rangle_{\text{even}} \) it is called an even multivector. If it only contains k-vectors of odd grade \( A = \langle A \rangle_{\text{odd}} \) it is called an odd multivector.

We can then define a useful Grade Parity operator on multivectors as:

\[
\text{grade}(A) = \begin{cases} 
1 & A = \langle A \rangle_{\text{odd}} \\
0 & A = \langle A \rangle_{\text{even}} \\
\text{undefined} & \text{otherwise}
\end{cases}
\]

\( \forall A \in \mathbb{R}^n \)

Now for the third step to construct a Geometric Algebra \( G^{p,q,r} \) from a base \( n \)-dimensional metric linear space \( \mathbb{R}^n \) with signature \( (p, q, r) \), we just need to generalize the linear products and operations of \( \mathbb{R}^n \) to multivectors to obtain a full Geometric Algebra \( G^{p,q,r} \) out of the non-metric Grassmann Space of multivectors \( \wedge^n \) where \( n = p + q + r \). Because all basic algebraic products and operations are linear the generalizations are straightforward as follows:

\[
\tilde{A} = \sum_{r=0}^{n} (-1)^{(r-1)/2} \langle A \rangle_r
\]

(3.25)

\[
\hat{A} = \sum_{r=0}^{n} (-1)^r \langle A \rangle_r
\]

(3.26)

\[
A \wedge B = \sum_{r=0}^{n} \sum_{s=0}^{n-r} \langle A \rangle_r \wedge \langle B \rangle_s
\]

(3.27)

\[
A \star B = \sum_{r=0}^{n} \langle A \rangle_r \star \langle B \rangle_r
\]

(3.28)

\[
A \mathbin{\upharpoonright} B = \sum_{r=0}^{n} \sum_{s=0}^{r} \langle A \rangle_r \mathbin{\upharpoonright} \langle B \rangle_s
\]

(3.29)

\[
A \mathbin{\downharpoonright} B = \sum_{r=0}^{n} \sum_{s=0}^{r} \langle A \rangle_r \mathbin{\downharpoonright} \langle B \rangle_s
\]

\( \forall A, B \in G^{p,q,r} \)

The above relations are mathematically useful, but computationally inefficient for computing with multivectors. I will give much better formulations in the section 4 when talking about computing with GA Coordinate Frames. In addition, not all multivectors have a geometrically significant meaning in a given problem domain. We must be careful to clearly distinguish between algebraic computations on multivectors from the actual geometric meaning they represent. This issue is generally less severe in GA than in matrix algebra due to the richer and more geometric significant structure of GA.

The fourth and final step is to define the closed bilinear universal Geometric Product of multivectors, the following is not an axiomatic definition, but more like a listing of the main properties of the GP. First of all, the GP is associative (3.31), bilinear (3.32, 3.33), and distributive over addition (3.34, 3.35):

\[
X(YZ) = (XY)Z \quad (3.31)
\]

\[
(aX + bY)Z = a(XZ) + b(YZ) \quad (3.32)
\]

\[
Z(aX + bY) = a(ZX) + b(ZY) \quad (3.33)
\]

\[
(X + Y)Z = XZ + YZ \quad (3.34)
\]

\[
Z(X + Y) = ZX + ZY \quad (3.35)
\]

\( \forall X, Y, Z \in G^{p,q,r}, a, b \in \mathbb{R} \)

On scalars and vectors the GP is defined using the multiplication of real numbers and the scalar multiplication of vectors and scalars:
\[ ab = ba \quad \equiv \quad \text{The same as real numbers multiplication} \quad \text{(3.36)} \]
\[ ax = xa \quad \equiv \quad \text{The same as scalar multiplication} \quad \text{(3.37)} \]
\[ \forall x \in \mathbb{R}^n, \ a, b \in \mathbb{R} \]

By assuming an orthonormal basis \( E = \langle e_1, e_2, \ldots, e_n \rangle \) for \( \mathbb{R}^n \), i.e. \( e_i \cdot e_j = 0 \forall i \neq j, e_i^2 \in \{1, -1, 0\} \) then on vectors the GP is defined using the outer and inner products as follows:

\[ xx = x^2 \quad \equiv \quad \text{B}(x, x) = x \cdot x = \|x\| \quad \text{(3.38)} \]
\[ xy = x \cdot y + x \wedge y \quad \text{(3.39)} \]
\[ e_i e_j = -e_j e_i \quad \text{(3.40)} \]
\[ \forall x, y \in \mathbb{R}^n, e_i e_j \in E, i \neq j \]

Using relation (3.40) we can now compute the GP of any two basis blades easily. Then we can use the other relations to compute the GP on general multivectors of any kind as long as they are expressed on the orthonormal basis \( E \). Note that the GP is neither commutative nor anti-commutative for general multivectors. With the GP any non-null vector \( a \in \mathbb{R}^n \) has the unique inverse:

\[ a^{-1} = \frac{1}{\|a\|} a. \] The inverse \( a^{-1} \) is a vector in the same direction of \( a \) but properly scaled to make \( aa^{-1} = 1 \). We can prove that the main products, with one vector argument, are related to the GP using the following relations on Blades, then extend them by linearity to multivectors:

\[ v \wedge X = \frac{1}{2}(vX + \bar{X}v) \quad \text{(3.41)} \]
\[ X \wedge v = \frac{1}{2}(Xv + v\bar{X}) \quad \text{(3.42)} \]
\[ v\lfloor X = \frac{1}{2}(vX - Xv) \quad \text{(3.43)} \]
\[ X\lfloor v = \frac{1}{2}(Xv - v\bar{X}) \quad \text{(3.44)} \]
\[ \forall v \in V, X \in \mathbb{G}^{p,q,r} \]

We can also compute the main products on blades using the GP:

\[ A \wedge B = \langle AB \rangle_{r+s} \quad r + s \leq n \quad \text{(3.45)} \]
\[ A\rfloor B = \langle AB \rangle_{s-r} \quad 0 \leq s - r \leq n \quad \text{(3.46)} \]
\[ A\lfloor B = \langle AB \rangle_{r-s} \quad 0 \leq r - s \leq n \quad \text{(3.47)} \]
\[ \forall A \in B^n_r, B \in B^n_s \]
\[ A * B = \langle AB \rangle_0 \quad \forall A \in B^n_k, B \in B^n_k \quad \text{(3.48)} \]

Then we can use linearity to generalize these relations to multivectors:

\[ A \wedge B = \sum_{r=0}^{n} \sum_{s=0}^{n-r} \langle \langle A \rangle_r \langle B \rangle_s \rangle_{r+s} \quad \text{(3.49)} \]
\[ A\rfloor B = \sum_{s=0}^{r} \sum_{r=0}^{s} \langle \langle A \rangle_r \langle B \rangle_s \rangle_{s-r} \quad \text{(3.50)} \]
\[ A\lfloor B = \sum_{r=0}^{n} \sum_{s=0}^{r} \langle \langle A \rangle_r \langle B \rangle_s \rangle_{r-s} \quad \text{(3.51)} \]
\[ A * B = \sum_{r=0}^{n} \langle \langle A \rangle_r \langle B \rangle_s \rangle_0 \quad \forall A, B \in \mathbb{G}^{p,q,r} \quad \text{(3.52)} \]
These last relations are useful mathematically for expressing the bilinear products using the GP, but they are also computationally inefficient. I will explain the more efficient method for computing the GP and all the bilinear products in section 4.

### 3.5 Linear Maps on Multivectors

The construction of a GA is based on a linear Grassmann Space. When we use GA to model some practical GC problem, we might need several GA spaces each representing one aspect of the problem. We could also need to define several linear maps to transform multivectors between the GAs. In many practical GC applications, however, we need to impose more restrictions on general linear maps.

Some of the most applied restrictions are:

- The preservation of the Outer Products $T[A \wedge B] = T[A] \wedge T[B] \forall A, B \in \Lambda^n$. Such linear maps are called Outermorphisms. An important class of outermorphisms are invertible outermorphisms, which can be used as Change of Basis Outermorphisms (CBO) between GA Coordinate Frames as discussed later in subsection 4.1.

- The preservation of the Geometric Products $T[AB] = T[A]T[B] \forall A, B \in G^{p,q,r}$. These linear maps are called Automorphisms. Every automorphism $T$ also preserves all bilinear products on multivectors including the outer and inner products $T[A \star B] = T[A] \star T[B] \forall A, B \in G^{p,q,r}$. This means that an automorphism is also an outermorphism and an orthogonal linear map on vectors and multivectors in general.

### 3.6 Outermorphisms

The concept of a linear map on vectors $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ can be extended to act on a whole subspace $S = \text{span}(x_1, x_2, \ldots, x_k) \leq \mathbb{R}^n$ by applying $f$ to the spanning vectors of the subspace and reconstructing the transformed subspace afterwards $f[S] = \text{span}(f[x_1], f[x_2], \ldots, f[x_k]) \leq \mathbb{R}^m$. An alternative approach is possible using the algebraic constructions of GA through extending the linear map to act on arbitrary blades, by constructing what is called an Outermorphism $\overline{T}$ based on $f$ as follows:

\[
\overline{T} : \Lambda^n \rightarrow \Lambda^m \quad (3.53)
\]

\[
\overline{T}[a] = a \quad (3.54)
\]

\[
\overline{T}[x] = f[x] \quad (3.55)
\]

\[
\overline{T}[aX + bY] = a\overline{T}[X] + b\overline{T}[Y] \quad (3.56)
\]

\[
\overline{T}[X \wedge Y] = \overline{T}[X] \wedge \overline{T}[Y] \quad (3.57)
\]

\[
\forall a, b \in B^n_0, x \in B^n_1, X, Y \in \Lambda^n
\]

An extension of a map “vectors to vectors” in this manner to the whole of the Grassmann Algebra is called extension as a linear outermorphism, since its last property shows that a morphism (i.e., a mapping) is obtained that commutes with the outer product. Outermorphisms have nice algebraic properties that are essential to their geometrical usage [3]:

- **Blades Remain Blades**: Geometrically, oriented subspaces are transformed to oriented subspaces of the same grade: $\text{grade}(A) = \text{grade}(\overline{T}[A]) \forall A \in B^n$. This means that the dimensionality of subspaces do not change under a linear transformation.

- **Preservation of Factorization**: If two blades $A, B$ have a blade $C$ in common then the blades $\overline{T}[A], \overline{T}[B]$ have $\overline{T}[C]$ in common.
The determinant of a linear operator \( f \) is a fundamental scalar property of \( f \) defined using its outer-morphism as: \( \overline{f}[I] = \det (f) I^2 \). It signifies the change in weight between the pseudo-scalar of the space \( I \) and its transformed version under \( \overline{f} \) which is the original definition of determinants in abstract linear algebra. Using this definition it is easy to show properties of determinants of linear transforms such as \( \det (g \circ f) = \det (g) \det (f) \) without using matrices and coordinates as usually done in linear algebra texts.

Another important concept in linear algebra is the adjoint of a linear operator the following relations for all blades:

\[
\overline{f}^T : \mathbb{R}^n \rightarrow \mathbb{R}^n \\
\overline{f}^T [x] = \sum_{i=1}^{n} (x \cdot \overline{f} [b_i]) c_i \forall x \in \mathbb{R}^n
\]

(3.58)

The outer-morphism of the adjoint can be constructed as above. The adjoint outer-morphism satisfies the following relations for all blades:

\[
\overline{f}[A] \ast B = A \ast \overline{f}^T [B] \quad \forall A, B \in B^n
\]

(3.59)

\[
\left( \overline{f}^T \right)^T = \overline{f}
\]

(3.60)

\[
\left( \overline{f}^T \right)^{-1} = \left( \overline{f}^{-1} \right)^T \equiv \overline{f}^{-T}
\]

(3.61)

Applying an outer-morphism to the scalar product is simple since it always produces a scalar: \( \overline{f}[A \ast B] = A \ast B \). For the left contraction product the relation is: \( \overline{f}[A]B = \overline{f}^T [A] \overline{f}[B] \), and in the case that \( f \) is an orthogonal operator the relation becomes simpler: \( \overline{f}[A]B = \overline{f}[A] \overline{f}[B] \) because \( \overline{f}^{-T} = \overline{f} \) in this case. Actually, for orthogonal isomorphisms \( \overline{f}^{-T} = \overline{f} \) any bilinear product on multivectors \( \ast \) satisfies the relation \( \overline{f}[A \ast B] = \overline{f}[A] \ast \overline{f}[B] \) including the outer and geometric products.

We know that in 3D Euclidean space transforming a normal vector \( w = u \times v \) using any linear map \( f \) will generally not preserve its orthogonality property \( f [w] \cdot f [u] \neq 0 \), \( f [w] \cdot f [v] \neq 0 \) \( \forall u, v \in \mathbb{R}^n \), \( w = u \times v \). To correctly transform \( w \) as a normal vector we need to use \( \overline{f}^{-T} [w] \) not \( f [w] \). This is because \( w \) is a dual representation for the subspace \( \text{span} (u, v) \) not a direct representation like \( u \wedge v \). This idea can be generalized in GA for any blade \( A \propto X \). When applying an outer-morphism to \( A \) we need to use \( Y = \det (f) \overline{f}^{-T} [A] \propto f [X] \), not the usual \( \overline{f}[A] \), in order to ensure the consistency of transforming the represented subspace under the linear map \( f \). Only for orthogonal outer-morphisms \( \overline{f}^{-T} = \overline{f} \) with \( \det (f) = 1 \) that we can use \( \overline{f}[A] \propto f [X] \) for consistently transforming a blade \( A \propto X \). This also means that only invertible linear maps can be used to consistently transform blades like \( A \). If the linear map is not invertible we can only consistently transform blades \( A \propto X \) but not \( B \perp X \). This includes normal vectors in 3D Euclidean geometry as a special case.

We can write an expression for the inverse of an outer-morphism, if it exists, as follows:

\[
\overline{f}^{-1} [A] = \frac{1}{\det (f)} \left( \overline{f}^T [A^\dagger] \right)^{\odot}
\]

\[
= \frac{1}{\det (f)} \overline{f}^T [A J^{-1}] I
\]

(3.62)

Although this expression uses metric-dependent dualities it is actually a metric-independent expression because the two dualities cancel each other. Hence any metric can be assumed for computing the inverse outer-morphism, preferably a simple Euclidean metric. In section 4 I will explain how to represent and compute with outer-morphisms on GA Coordinate Frames using matrices.

[^2]: Here I use \( \det (f) \) instead of \( \det (\overline{f}) \) because the determinant is a property of \( f \) that can be defined through its extension \( \overline{f} \).
3.7 Representing Orthogonal Operators with Versors

Using the geometric product of non-null vectors, a definition for a powerful GA-based representation for linear orthogonal maps can be made. This representation, alternative to real orthogonal matrices, is called a Versor. According to the Cartan-Dieudonné Theorem [88], any orthogonal transformation in \( \mathbb{R}^n \) is equivalent to a composition of simple reflections on \((n-1)\)-dimensional subspaces. Algebraically, a reflection of a single vector \( v \in \mathbb{R}^n \) on a \((n-1)\)-dimensional subspace dually represented by a non-null vector \( v \in \mathbb{R}^n \) can be defined using the geometric product as the simple linear expression \(-vav^{-1}\). In this expression the actual norm of \( v \) is irrelevant since it is canceled by the inverse in \( v^{-1} \). We can simply extend this to an outermorphism on \( r \)-blades as:

\[
L_v[A] = (-1)^r vAv^{-1} = vAv^{-1} \quad \forall A 
\]

(3.63)

We can further extend this as a composition of simple reflections for the blade \( A \propto X \) on \( k \) \((n-1)\)-dimensional subspaces dually represented by non-null vectors \( v_1, v_2, \ldots, v_k \) can be written as:

\[
L_V[A] = (-1)^{kr} v_1 \cdots v_2v_1Av_1^{-1}v_2^{-1} \cdots v_k^{-1}
= (-1)^{kr} VAV^{-1}
\]

(3.65)

\[
\Rightarrow L_V[a] = (-1)^k Vav^{-1}
\]

(3.66)

(3.67)

The multivector \( V = v_1 \cdots v_2v_1 \) is called a Versor and is essentially an even or odd multivector created by the geometric product of the non-null vectors \( v_i \). I will denote the set of even versors as \( V_+ \in G^{p,q,r} \), the set of odd versors as \( V_- \in G^{p,q,r} \), and the set of all versors as \( V = V_+ \cup V_- \in G^{p,q,r} \). In addition, an important class of versors is the set of non-null blades \( B = \{ A : A \in B^n, \|A\| \neq 0 \} \subseteq V \), as any non-null blade can be expressed as the geometric product of non-null orthogonal vectors.

Using this construction we can define a new bilinear product \( V \odot A \equiv L_V[A] = (-1)^{kr} VAV^{-1} \forall V \in \mathbb{V}^{p,q,r}, A \in B^n \) called the Versor Product. For some fixed \( V \), the versor product is an orthogonal outermorphism extending the orthogonal linear map on vectors in equation (3.67). We can extend the versor product to handle any general multivector \( X = (X)_{even} + (X)_{odd} \in G^{p,q,r} \) as follows:

\[
V \odot X = V \langle X \rangle_\text{even} V^{-1} + (-1)^k V \langle X \rangle_\text{odd} V^{-1}
\]

(3.68)

(3.69)

Versors and the versor product construct a very powerful representational component of Geometric Algebra. For example, we can use the versor product to orthogonally transform other orthogonal maps \( V \odot X \in \mathbb{V} \forall V, X \in \mathbb{V} \). Orthogonal maps are themselves objects to be transformed by other orthogonal maps using versors. We can then create an arbitrary hierarchy of orthogonal maps acting on subspaces to express a sophisticated geometric process on subspaces. In addition, this naturally leads to a powerful algebraic representation for Orthogonal Groups [89].

Any even versor \( V \in V_+ \) represents a rotation, which is an orthogonal map that has a determinant of 1 and preserves orientation (handedness) of a subspace it transforms. Any odd versor \( V \in V_- \) represents an anti-rotation (i.e., a composition of a rotation and a single reflection), which is an orthogonal map that has a determinant of \(-1\) and changes orientation of a subspace it transforms. This result is independent of the used metric, basis, or space dimension. If an orthogonal outermorphism \( L \) is represented by a versor \( V \), the inverse outermorphism \( L^{-1} \) is represented by \( V^{-1} \). In addition, the composition of two orthogonal outermorphisms \( L_{V_2}, L_{V_1} \) respectively represented by versors \( V_2, V_1 \) is represented by the geometric product of the two versors \( (L_{V_2} \circ L_{V_1})[X] \equiv L_{V_2} [L_{V_1} [X]] = L_{V_2,V_1} [X] \).

The versor product \( V \odot X \), being both an even and an innermorphism, preserves all GA bilinear products \(*\) including the outer and geometric products:
Table 3: Values of the sign factor $s$ in the expression $(-1)^s AXA^{-1}$ used for computing the reflection of an oriented subspace $W$ in an oriented subspace $V$

| Case | Blade Representing $V$ | Blade Representing $W$ | Sign Factor $s$ |
|------|------------------------|------------------------|-----------------|
| 1    | $A \land V$           | $X \land W$           | $x (a + 1)$     |
| 2    | $A \land V$           | $X \lor W$            | $(x + 1) (a + 1) + n - 1$ |
| 3    | $A \lor V$            | $X \land W$           | $xa$            |
| 4    | $A \lor V$            | $X \lor W$            | $(x + 1) a$     |

$$V \land (aX + bY) = a (V \land X) + b (V \land Y) \quad (3.70)$$

$$V \land (X \ast Y) = (V \land X) \ast (V \land Y) \quad (3.71)$$

\[
\forall V \in V, X, Y \in G^{p,q,r}, a, b \in \mathbb{R}
\]

This is a very important property of the versor product. Any algebraic construction based on the above operations can be transformed directly under an orthogonal map in a structure-preserving manner. Meaning that transforming the components and then creating the structure is equivalent to creating the structure and then applying the orthogonal map to the whole geometric structure; may it be an oriented subspace or an orthogonal map by itself.

### 3.8 Computing with Oriented Subspaces

The above discussion on versors is based on a single type of reflections: to reflect an oriented subspace directly represented by some blade in a $(n - 1)$-dimensional subspaces dually represented by a non-null vector. We can also study reflections of arbitrary oriented subspaces in other oriented subspaces. We can assume any of the two kinds of representations for the reflected subspace $W$ and the reflection subspace $V$ resulting in 4 computational possibilities. The mathematical details are presented in [3] and I will only show the final results here. The reflection formulas in the 4 cases take the general form $F_A [X] = (-1)^s AXA^{-1}$ where $s$ is an integer dependent on the case and the grades of the blades $A \in B^n_a$ and $X \in B^n_n$ representing $V$ and $W$ respectively as shown in Table 3. In all 4 cases for a fixed $A$ this expression defines an invertible outermorphism $F_A [X]$ on blades that can be extended to act on general multivectors $X \in G^{p,q,r}$. The 3rd case is where we can extend $A$ to be a versor, not just a non-null blade, and obtain a geometrically significant interpretation using the versor product and the the Cartan-Dieudonné Theorem. In addition, the sign factor can be ignored if the orientation of the resulting subspace is not relevant for a particular problem so we can just use $AXA^{-1}$ in all 4 cases.

We can also use a blade $A \in B^n_a$ to construct a projection outermorphisms $P_A [X]$ using the following equivalent relations:

$$P_A [X] = (-1)^{r(a+1)} A [[X], A^{-1}] \quad (3.72)$$

$$= (X [A]) A^{-1} \quad (3.73)$$

$$= (X [A]) A^{-1} \quad (3.74)$$

In this case the blade $A$ directly represents an oriented subspace $A \land V$ on which we can project another subspace $W$ directly represented by $X \in B^n_n$. One important difference between a reflection outermorphism $F_A [X]$ and a projection outermorphism $P_A [X]$ is that $F_A [F_A [X]] = X$ (i.e. a double reflection is an identity map) while $P_A [P_A [X]] = P_A [X]$ meaning that applying the same projection is equivalent to a single projection of the projected subspace. These constructions add more representational power to blades. A blade can directly or dually represent a weighted oriented subspace. In addition, a non-null blade $A$ can represent reflection outermorphisms $F_A [X] = (-1)^s AXA^{-1}$, a projection outermorphism $P_A [X] = (X [A]) A^{-1}$, a dualization outermorphism $X^\ast A = X [A^{-1}] \forall X \leq A$, or an orthogonal outermorphism $L_A [X] = A \land X$.

We can define additional computations on oriented subspaces using blades. Having two disjoint subspaces $V \cap W = \{ \phi \}$ directly represented by two blades $A \land V, B \land W$ we can construct the smallest subspace containing both of them, called their Join, as $A \cup B \equiv A \cap B \land V \cup W$. This is mainly because $A$ and $B$ have no vectors in common so their outer product is not zero. If the two subspaces are not
disjoint this expression will give a zero blade and can’t be used to compute the geometric Join of the subspaces.

There exists a related difference between the geometric meaning of a projection outermorphism and the classical geometric meaning of projection of subspaces. For example in 3D Euclidean space, if we geometrically project a homogeneous line on a homogeneous plane the result is not always a line in the projection plane but sometimes a point. This degenerate case means that geometric projections do not preserve the dimensionality of the projected subspace like projection outermorphisms do. For an important class of geometric operations on subspaces, outermorphisms are not suitable representations, and we generally need an algorithmic approach for computing them. Such geometric operations include:

- Factoring a given blade $A$ into a set of vectors $v_i$ such that $A = v_1 \wedge v_1 \wedge \cdots \wedge v_r$. This may be a metric-dependent or independent operation according to the conditions we assume on $v_i$.
- Factoring a given versor $V$ into a set of non-null vectors $v_i$ such that $V = v_r \cdots v_2 v_1$. His is a metric-dependent operation by nature.
- Finding the blade $J$ that directly represents the smallest subspace containing two blades $J = A \sqcup B \propto \{ x : x = a + b; a \in \mathbb{A}, b \in \mathbb{B} \}$. This operation is called the Join of two blades and can be a metric-dependent or independent operation on subspaces according to the properties we need $J$ to satisfy.
- Finding the blade $M$ that directly represents the largest subspace common to two blades $M = A \cap B \propto \mathbb{A} \cap \mathbb{B}$. This operation is called the Meet of two blades and can be a metric-dependent or independent operation on subspaces according to the properties we need $M$ to satisfy.
- Geometrically projecting a subspace on another using the blades they are represented by.

The interested reader can find detailed information on how to algorithmically perform these subspace computations using GA operations in many sources including [90, 20, 3, 10, 91].

4 Computing with GA Coordinate Frames

When introducing Geometric Algebra to software developers it is much better to follow a method that builds gradual construction of concepts as done in the previous two sections. From a computational point of view, however, the opposite approach is much more suitable. In this section I explain the mathematics behind practical computing with a GA Coordinate Frame (GACF). This explanation is an extension and reformulation of the additive representation of multivectors described in [20, 21, 3]. The symbolic computations layer in GMac [58, 59] is mainly based on this formulation.

4.1 Components of a GACF

A GACF $\mathcal{F} (\mathcal{F}_n^k, \mathcal{A}_\mathcal{F})$ is the mathematical structure used to define all basic computations of a Geometric Algebra $\mathbb{G}^{p,q,r}$ in terms of the more basic scalar coordinates often used to write a program on a computer. A GACF has contains several components, can be of several types, and can be used to perform GA computations as illustrated in Figure 4.1. A GACF can be completely defined using two components:

1. An ordered set of $n$ basis vectors that determine the dimensionality of the GACF’s base vector space: $\mathcal{F}_n^k = \langle f_0, f_1, \cdots, f_{n-1} \rangle$.

2. A symmetric real bilinear form $\mathcal{B} : \mathcal{F}_n^k \times \mathcal{F}_n^k \to \mathbb{R}$, $\mathcal{B} (f_i, f_j) = \mathcal{B} (f_j, f_i) = f_i \cdot f_j$ to determine the inner product of basis vectors usually given by the symmetric $n \times n$ bilinear form matrix $\mathcal{A}_\mathcal{F} = [f_i \cdot f_j]$; also called the Inner Product Matrix (IPM) of the GACF. According to the general structure of the IPM $\mathcal{A}_\mathcal{F}$ a GACF $\mathcal{F}$ can be of any of the types listed in Table 4.

From these two components, we can automatically construct three additional ones to serve important purposes for GA computations within the GACF:

1. The ordered set of $2^n$ basis blades of all grades $\mathcal{F}^n = \langle F_0, F_1, \cdots, F_{2^n-1} \rangle$. This set is automatically determined by the set of basis vectors $\mathcal{F}_n^k$. This component is independent of the metric represented by $\mathcal{A}_\mathcal{F}$.
Table 4: Classification of GA Coordinate Frames According to their IPM

| GACF Type       | IPM Form                                                                 |
|-----------------|--------------------------------------------------------------------------|
| Euclidean       | Identity matrix                                                          |
|                 | \( f_i \cdot f_j = 1, f_i \cdot f_j = 0 \forall i \neq j \)             |
| Orthogonal      | Invertible, diagonal, with \( \pm 1 \) entries                           |
|                 | \( f_i \cdot f_j = \pm 1, f_i \cdot f_j = 0 \forall i \neq j \)         |
| Orthogonal      | Diagonal                                                                 |
|                 | \( f_i \cdot f_j = d_i, f_i \cdot f_j = 0 \forall i \neq j \)           |
| Degenerate      | Non-invertible, diagonal, with some zeros on diagonal                    |
|                 | \( f_i \cdot f_j = d_i \Xi d_j = 0, f_i \cdot f_j = 0 \forall i \neq j \) |
| Non-Orthogonal  | Invertible, symmetric, non-diagonal                                      |
|                 | \( f_i \cdot f_j = f_j \cdot f_i = b_{ij} \exists i \neq j : b_{ij} \neq 0 \) |

2. The bilinear multivector coordinates map \( G_F : F^n \times F^n \rightarrow \mathcal{G}^{p,q,r} \) that defines the geometric product of basis blades as a multivector expressed on the same basis blades \( G_F(F_i, F_j) = F_i F_j = \sum_{k=0}^{2^n-1} m_k F_k, m_k \in \mathbb{R} \). This bilinear map is automatically determined by the set of basis vectors \( F_1^n \) and the bilinear form \( B \).

3. If the bilinear form is not orthogonal (i.e. \( A_F \) is not diagonal), a base orthogonal GACF \( \mathcal{E} (E_1^n, A_\mathcal{E}) \) of the same dimension is needed, in addition to an orthogonal Change-of-Basis Matrix (CBM) \( C \). The orthogonal CBM is used to express the basis vectors of \( F \) as linear combinations of the basis vectors of \( \mathcal{E} \), and defines a Change of Basis Automorphism (CBA) \( \mathcal{C} \) that can safely transform linear operations on multivectors between \( \mathcal{E} \) and \( F \). This component is required for the computation of the geometric product of basis blades \( G_F \) for non-orthogonal bilinear forms. We can either define \( C \) implicitly from the orthonormal eigen vectors of \( A_F \), or the user can directly supply \( \mathcal{E} (E_1^n, A_\mathcal{E}) \) and \( C \) to define the IPM of \( F \). The details of this component are described in subsections 4.5 and 4.6.

Using these five components any multivector \( X \) can be represented by a column vector of real coefficients \( [x_i]_F \) where \( X = \sum_{k=0}^{2^n-1} x_k F_k, x_k \in \mathbb{R} \) and the geometric product of two multivectors \( X, Y \) can be easily computed as:

\[
XY = \sum_{r=0}^{2^n-1} \sum_{s=0}^{2^n-1} x_r y_s G_F(F_r, F_s) \tag{4.1}
\]

We can then formulate the remaining GA bilinear products using a basis-selection mechanism from the general geometric product expression (4.1).

### 4.2 Representing GACF Basis Blades

Basis vectors and blades are abstract mathematical entities defined only by their relations to each other. To represent such abstract entities inside computers we usually use symbolic representations like assigning a unique ID for each blade. We then implement computational processes that are closely analogous to the abstract relations between these entities. In order to define the basis blades \( F^n = \langle F_0, F_1, \ldots, F_{2^n-1} \rangle \) for a GACF of any type, a canonical ID representation is defined based on the basis vectors \( F_1^n = \langle f_0, f_1, \ldots, f_{n-1} \rangle \). First we introduce the general Ordered Subset Selection (OSS) operator \( \prod_{\oplus} (S, i) \) that applies any associative binary operator \( \oplus \) with the identity element \( I_{\oplus} \) to a subset of an ordered set of elements \( S = \langle s_0, s_1, \ldots, s_{k-1} \rangle \) selected according to the integer index \( i \) as follows:

\[
\prod_{\oplus} (S, i) = \begin{cases} 
I_{\oplus}, & i = 0 \\
S_m, & i = 2^m, m \in \{0, 1, \ldots, k-1\} \\
S_{s_{i_1}} \oplus S_{s_{i_2}} \oplus \cdots \oplus S_{s_{i_r}}, & i_i < i_2 < \cdots < i_r
\end{cases}, \quad \text{where } i = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_r} 
\tag{4.2}
\]

The OSS operator basically expresses the integer \( i \) as a binary number \( (i)_2 \) and selects elements from \( S \) based on the 1s positions in \( (i)_2 \). The OSS then applies the associative binary operator \( \oplus \) to the selected elements. Using the OSS operator, we can define the basis blades from basis vectors as follows:
Figure 4.1: Elements of a GA Coordinate Frame
illustrates this correspondence on a 5D linear space with basis vectors $f_0$ to $f_4$.

Any multivector can be stored in computer memory as an array (or perhaps for efficiency reasons as a dictionary or hash table) of metric independent properties of a basis blade $F$. The subset selection operator $\{i\}^n_F$ uniquely defines the structure of the basis blade $F_i$. This n-bit binary pattern is called the ID of the basis blade $id(F_i)$ and its n-bits binary form is denoted as $id(F_i)_2$. The grade of the basis blade $g = grade(F_i)$ is then equal to the number of 1s in $id(F_i)_2$. Table 5 illustrates this correspondence on a 5D linear space with basis vectors $F^0 = \{f_0, f_1, f_2, f_3, f_4\}$. Any multivector can be stored in computer memory as an array (or perhaps for efficiency reasons as a dictionary or hash table) of 2^n scalars representing the coefficients of the basis blades with respect to the given GACF. A pair $(ID, scalar)$ is called a Term, and represents a weighted basis blade. A multivector is represented as a sum of terms with different IDs ranging from 0 to $2^n - 1$.

Another important property is the order of the basis blade among its g-vector basis blades of the same grade. This property is called the Index of the basis blade $index(F_i)$ and can be defined as $id(g,k) = \{id(F_i) = i\}$, $g = grade(F_i)$, $k = index(F_i)$. The $id(g,k)$ operator is useful when defining outermorphisms as I will describe later. In addition, we can use the $id(g,k)$ operator to describe the ordered set of basis k-vectors of the same grade $g \in \{0,1,\cdots,n\}$ which is a subset of the basis blades set $F^n$ as follows:

$$F^n_g = \langle F_{i_0}, F_{i_1}, \cdots, F_{i_{n-1}} \rangle \subseteq F^n$$

$$i_k = id(g,k) \forall k \in \{0,1,\cdots,r-1\}, r = \left(\begin{array}{c}n \\ g \end{array}\right)$$

These integer operators create a symbolic metric-independent representation for basis blades having an important property of being representationally consistent across multiple metrics and dimensions. Having two basis sets $\mathcal{F}(F^n_m, A_F)$ and $\mathcal{E}(E^n_n, A_E)$ for two different GAs of dimensions $m$ and $n$ prescriptively with $m < n$, we find that $id(F_i) = i = id(E_i) \forall i \in \{0,1,\cdots,2^n-1\}$. We can directly compute metric-independent and dimension-independent properties of a basis blade $F_i$ only using information about its ID $i$, grade $g$, and index $k$. For example, we can compute the signs associated with its reversal $\tilde{F}_i$ and grade involution $\tilde{F}_i$ respectively as $sign(F_i) = \frac{\tilde{F}_i}{F_i} \cdot (-1)^{g(g-1)/2}$ and $sign(\tilde{F}_i) = \frac{\tilde{F}_i}{F_i} \cdot (-1)^g$. We can automatically create a universal lookup table like the following one to store all these metric independent
4.3 The Geometric Product of Euclidean Basis Blades

The geometric product of any two vectors \( u, v \) is \( uv = u \cdot v + u \land v \). For a single vector \( u \land u = 0 \iff u^2 = u \cdot u \). When the two vectors are orthogonal then \( u \cdot v = 0 \iff uv = u \land v = -v \land u = -vu \). A Euclidean GACF \( F(F_{r1}^n, A_x) \) has an IPM \( A_x \) equal to the identity matrix with basis vectors satisfying \( f_i \cdot f_i = 1 \) and \( f_i \land f_j = 0 \forall i \neq j \). This leads to the geometric product of Euclidean basis vectors satisfying \( f_i^2 = 1 \) and \( f_i \land f_j = -f_j \land f_i \forall i \neq j \). For such GACF it is straightforward to compute the geometric product of any two basis blades \( G_F(F_r, F_s) \) as a signed basis blade in the form \( G_F(F_r, F_s) = F_r F_s = \pm F_q \). We only need to find the value of \( q \) and the sign \( \text{Sign}_{EGP}(r, s) \) associated with the resulting basis blade \( F_q \) given the two integers \( r, s \). As an example, take the geometric product of two basis blades \( F_{19} = (f_0 \land f_2 \land f_3)(f_0 \land f_1 \land f_3) = (f_0 f_2 f_3)(f_0 f_1 f_3) \). We can use the associativity of the geometric product to apply a series of swaps between basis vectors to reach the canonical form of the final basis blade as follows:

\[
F_q = + (f_0 \land f_2 \land f_3)(f_0 \land f_1 \land f_3) \\
= + f_0 f_2 (f_3 f_0) f_1 f_3 \\
= - f_0 f_2 (f_0 f_3) f_1 f_3 \\
= - f_0 (f_2 f_0) f_3 f_1 f_3 \\
= + f_0 (f_0 f_2) f_3 f_1 f_3 \\
= + (f_0 f_0) f_2 f_3 f_1 f_3 \\
= + f_2 f_3 f_1 f_3 \\
= - f_2 f_1 f_3 f_3 \\
= + f_1 f_2 (f_3 f_3) \\
= + f_1 \land f_2 = F_6
\]

Using the corresponding IDs we note that:

| \( i = \text{id}(F_i) \) | \( F_i \) | \( \text{id}(F_i)_j \) | \( \text{grade}(F_i) \) | \( \text{index}(F_i) \) | \( \text{sign}(\hat{F_i}) \) | \( \text{sign}(\hat{F_i}) \) |
|---|---|---|---|---|---|---|
| 0 | 1 | 0000 | 0 | 0 | +1 | +1 |
| 1 | \( f_0 \) | 0001 | 1 | 0 | +1 | -1 |
| 2 | \( f_1 \) | 0010 | 1 | 1 | +1 | -1 |
| 3 | \( f_0 \land f_1 \) | 0011 | 2 | 0 | -1 | +1 |
| 4 | \( f_2 \) | 0100 | 1 | 2 | +1 | -1 |
| 5 | \( f_0 \land f_2 \) | 0101 | 2 | 1 | -1 | +1 |
| 6 | \( f_1 \land f_2 \) | 0110 | 2 | 2 | -1 | +1 |
| 7 | \( f_0 \land f_1 \land f_2 \) | 0111 | 3 | 0 | -1 | -1 |
| 8 | \( f_3 \) | 1000 | 1 | 3 | +1 | -1 |
| 9 | \( f_0 \land f_3 \) | 1001 | 2 | 3 | -1 | +1 |
| 10 | \( f_1 \land f_3 \) | 1010 | 2 | 4 | -1 | +1 |
| 11 | \( f_0 \land f_1 \land f_3 \) | 1011 | 3 | 1 | -1 | -1 |
| 12 | \( f_2 \land f_3 \) | 1100 | 2 | 5 | -1 | +1 |
| 13 | \( f_0 \land f_2 \land f_3 \) | 1101 | 3 | 2 | -1 | -1 |
| 14 | \( f_1 \land f_2 \land f_3 \) | 1110 | 3 | 3 | -1 | -1 |
| 15 | \( f_0 \land f_1 \land f_2 \land f_3 \) | 1111 | 4 | 0 | +1 | +1 |
Algorithm 1 $\text{Sign}_{\text{EGP}}(r,s)$: Computes the sign of the geometric product $F_r F_s$ of two Euclidean basis blades $F_r, F_s \in G^{n,0,0}$

1. Initialize the sign variable $S \leftarrow +1$ and the ID variables $id_r \leftarrow id(F_r)$, $id_s \leftarrow id(F_s)$
2. For increasing $i$ from 0 to $n - 1$ do steps 3-6:
3. If bit $i$ in $(id_r)^n_2$ is a 1 do:
4. For decreasing $j$ from $n - 1$ to $i + 1$ do step 5:
5. If bit $j$ in $(id_r)^n_2$ is a 1 Then set $S \leftarrow -S$
6. If bit $i$ in $(id_r)^n_2$ is a 1 Then set it to 0 Else set it to 1
7. Return final result in $S$

\[
\begin{align*}
\text{id}(f_1 \wedge f_2)^4_2 &= 0110 \\
&= 1011 \text{ XOR } 1101 \\
&= \text{id}(f_0 \wedge f_1 \wedge f_3)^4_2 \text{ XOR id}(f_0 \wedge f_2 \wedge f_3)^4_2 \\
\end{align*}
\]

This is not a coincidence because if the same basis vector $f_i$ is present or absent in both input basis blades it will always be absent in the final basis blade due to the property $f_i^2 = 1$, and if a basis vector is only present in one of the input basis blades it’s always present in the final basis blade. Hence we can find the ID of the final basis blade $F_q$ by a bit-wise XOR operation between the IDs of the input basis blades $F_r, F_s$:

\[
F_r F_s = \text{Sign}_{\text{EGP}}(r, s) F_q
\]

\[
(q)_2 = \text{id}(F_q)^n_2 = \text{id}(F_r)^n_2 \text{ XOR id}(F_s)^n_2
\]

\[
= (r)^n_2 \text{ XOR } (s)^n_2
\]

\[
\forall r, s \in \{0, 1, \ldots, 2^n - 1\}
\]

We can compute the sign of the final geometric product term using Algorithm 1 or a similar variant.

Using such algorithm, we can construct a Euclidean Geometric Product Sign lookup table having $2^n - 1$ rows and $2^n - 1$ columns where each cell at row $i$ and column $j$ contains the number $\text{Sign}_{\text{EGP}}(F_i, F_j)$. Although this table is specific to Euclidean metric of dimension $n$, we can use it to compute the Euclidean geometric product of basis blades of any dimension $m \leq n$ because of the universal property if this method of representation. In addition, we can compute the geometric product of basis blades having other metrics based on the signs in this Euclidean table, as I will show shortly. An important property for $\text{Sign}_{\text{EGP}}(i,i)$ is:

\[
F_i^2 = F_i F_i
= F_i (\bar{F_i})^\sim
= (-1)^{g(g-1)/2} F_i F_i
= (-1)^{g(g-1)/2}
\]

\[
\Rightarrow \text{Sign}_{\text{EGP}}(i,i) = (-1)^{g(g-1)/2}, g = \text{grade}(F_i)
\]

\[
\forall i \in \{0, 1, \ldots, 2^n - 1\}
\]

4.4 The Geometric Product of Orthogonal Basis Blades

An orthogonal GACF $\mathcal{F}(F^n_o, A_\mathcal{F})$ has a diagonal IPM $A_\mathcal{F}$ with basis vectors satisfying $f_i : f_i = d_i$ and $f_i : f_j = 0 \forall i \neq j$ leading to the geometric product of orthogonal basis vectors satisfying $f_i^2 = d_i$ and $f_i f_j = -f_j f_i \forall i \neq j$. The only difference between a Euclidean GACF and an orthogonal GACF is that the square of a basis vector can be any real number $d_i$, including negative numbers and zero.
The same algorithm applied for a Euclidean GACF can thus be used to deduce a geometric product for such GACF with a single change to step 5 to become: “If bit \( i \) in \((id)_2\) is a 1 Then set it to 0 and set \( S \leftarrow d_i S \) Else set it to 1”. We could then create a similar lookup table for each orthogonal GACF in our problem. There is a better alternative in this case, however, by using the geometric product for a Euclidean GACF \( E(\mathbf{E}_1^1, \mathbf{A}_E) \) with the same dimension having basis blades \( \mathbf{E}^n = \langle E_0, E_1, \cdots, E_{2^n-1} \rangle \). If \( E_r E_s = \text{Sign}_{\text{EGP}}(r, s) E_k \) then \( F_r F_s = \text{Sign}_{\text{EGP}}(r, s) \lambda_k F_k \) where \( \lambda_k = \prod (E_d E_{d_1} \cdots E_{d_{n-1}} k) = F_k F_k \). The geometric product of \( \mathbf{E}_1^1 \) is \( \| F_k \| \), called the signature of \( F_k \), is the multiplication of all \( d_i \) having a corresponding 1-bit in the bit pattern \((k)_2 = \text{id}(F_k)_2 = \text{id}(E_k)_2 = (r)_2 \). XOR \((s)_2 \). This leads to a save in memory by only storing \( 2^n \) scalar values \( \lambda_k = F_k F_k \), \( k \in \{0, 1, \cdots, 2^n - 1\} \) for each orthogonal GACF, then the Euclidean Geometric Product Sign lookup table is used to compute \( F_r F_s \) as follows:

\[
F_r F_s = \text{Sign}_{\text{EGP}}(r, s) \lambda_k F_k
\]

\[
(k)_2 = \text{XOR}(s)_2
\]

\[
\forall r, s \in \{0, 1, \ldots, 2^n - 1\}
\]

When the orthogonal GACF is degenerate we have some null basis vectors with \( d_i = 0 \) and subsequently we find the basis blade signatures \( \lambda_k \) computed from these null basis vectors will also equal zero. For degenerate orthogonal GACFs we have to be careful when computing with null basis blades in some GA operations; for example when we need to divide by the norm of a blade we must take care not to use null blades.

### 4.5 Constructing a Derived GACF

Having a general GACF \( E(\mathbf{E}_1^n, \mathbf{A}_E) \) with basis vectors \( \mathbf{E}_1^n = \langle e_0, e_1, \cdots, e_{n-1} \rangle \) we can use an invertible Change-of-Basis Matrix \( \mathbf{C} = [c_{ij}] \) to define a new derived set of basis vectors \( \mathbf{F}_1^n = \langle f_0, f_1, \cdots, f_{n-1} \rangle \) for the same linear space as \( f_i = \sum_{j=0}^{n-1} c_{ij} e_j \forall i \in \{0, 1, \cdots, n-1\} \). If a vector \( x \) is represented on the basis \( \mathbf{E}_1^n \) by the column vector \( [x]_{\mathbf{E}_1^n} = [x_0, x_1, \cdots, x_{n-1}]^T \) and on the basis \( \mathbf{F}_1^n \) by the column vector \( [x]_{\mathbf{F}_1^n} = [x_0, x_1, \cdots, y_{n-1}]^T \) we find that:

\[
[x]_{\mathbf{E}_1^n} = \mathbf{C}^T [x]_{\mathbf{F}_1^n}
\]

\[
[x]_{\mathbf{F}_1^n} = \mathbf{C}^{-T} [x]_{\mathbf{E}_1^n}
\]

In the special case that \( \mathbf{C} \) is orthogonal \( \mathbf{C}^{-1} = \mathbf{C}^T \) we get \( [x]_{\mathbf{F}_1^n} = \mathbf{C} [x]_{\mathbf{E}_1^n} \). The elements of the derived IPM \( A_\mathcal{F} = [f_i \cdot f_j] \) can be easily calculated from the IPM \( A_\mathcal{E} = [e_i \cdot e_j] \) as follows for any invertible \( \mathbf{C} \):

\[
f_i \cdot f_j = \left( \sum_{r=0}^{n-1} c_{ir} e_r \right) \cdot \left( \sum_{s=0}^{n-1} c_{js} e_s \right)
\]

\[
= \sum_{s=0}^{n-1} \sum_{r=0}^{n-1} c_{ir} c_{js} (e_r \cdot e_s)
\]

\[
= \sum_{s=0}^{n-1} \sum_{r=0}^{n-1} c_{ir} (e_r \cdot e_s) c_{sj}^T
\]

\[
\Rightarrow A_\mathcal{F} = \mathbf{C} A_\mathcal{E} \mathbf{C}^T
\]

Using \( \mathbf{F}_1^n \) and \( A_\mathcal{F} \) we can then construct a derived GACF \( \mathcal{F}(\mathbf{F}_1^n, A_\mathcal{F}) \) relative to the given base GACF \( \mathcal{E}(\mathbf{E}_1^n, A_\mathcal{E}) \) by means of the invertible CBM \( \mathbf{C} \). To compute the geometric product of multivectors on the derived GACF we have 3 separate cases:

- In the case when \( A_\mathcal{F} \) is diagonal then \( \mathcal{F} \) is an orthogonal GACF and the geometric product of two multivectors represented on \( \mathcal{F} \) can be computed using the method in the previous subsection.

- When \( A_\mathcal{F} \) is not diagonal but the base GACF \( \mathcal{E} \) is orthogonal and \( \mathbf{C} \) is an orthogonal CBM \( \mathbf{C}^{-1} = \mathbf{C}^T \), the geometric product of the derived basis blades \( \mathbf{F}^n \) can be computed by extending \( \mathbf{P} = \mathbf{C}^T = \mathbf{C}^{-1} \) and \( \mathbf{P}^T = \mathbf{C}^{-T} = \mathbf{C} \) as two adjoint orthogonal outermorphisms \( \mathbf{P} \) and \( \mathbf{P}^T \) respectively.
These two outermorphisms preserve all bilinear products including the geometric product. We can safely use \( \overline{P} \) and \( \overline{P}^T \) to transform bilinear products of multivectors back and forth between the base GACF \( \mathcal{E} \) and the derived GACF \( \mathcal{F} \). Any bilinear product \( \ast \) of two multivectors \( X, Y \) can be computed on the derived GACF \( \mathcal{F} \) as:

\[
XY = \overline{P}\left[\overline{P}^T[X \ast \overline{P}^T[Y]\right], \quad \overline{P} = \overline{C}^T, \overline{P}^T = \overline{P}^{-1}
\]  

(4.15)

- When \( A_{\mathcal{F}} \) is not diagonal and either \( \mathcal{E} \) is not orthogonal or \( C \) is not an orthogonal CBM, another method for computing the geometric product is needed, which is explained in the following subsection.

### 4.6 Constructing a Non-Orthogonal GACF

We can directly define a non-orthogonal GACF \( \mathcal{F}(F_1^n, A_{\mathcal{F}}) \) using a given non-diagonal symmetric real IPM \( A_{\mathcal{F}} \). For a non-orthogonal GACF \( \mathcal{F} \) the geometric product of any two basis blades is not guaranteed to be a term (i.e. a weighted basis blade) but is generally a multivector (i.e. the sum of terms of different basis blades). If we try to make a geometric product lookup table for such GACF, each cell in the lookup table would then be a full multivector that may contain up to \( 2^n \) terms. This is a lot to store in memory for a single GACF: \( 2^{3n} \) terms many of which are typically zeros. A better alternative is to use a diagonalization technique on the IPM \( A_{\mathcal{F}} \) to express the non-orthogonal GACF as a derived GACF from a base orthogonal GACF \( \mathcal{E}(E_1^n, A_{\mathcal{E}}) \) with basis vectors \( E_1^n = \{e_0, e_1, \cdots, e_{n-1}\} \). This is done by finding the IPM \( A_{\mathcal{E}} \) of the base orthogonal GACF and the orthogonal CBM \( C = [c_{ij}] \), \( C^{-1} = C^T \) that expresses the basis vectors in \( F_1^n \) as linear combinations of the orthogonal basis vectors in \( E_1^n \) using \( f_i = \sum_{j=0}^{n-1} c_{ij} e_j \forall i \in \{0, 1, \cdots, n-1\} \) as explained in the previous subsection. This time we already have \( A_{\mathcal{E}} \) and we need to compute \( A_{\mathcal{F}} \) and \( C \).

Noting that the IPM \( A_{\mathcal{F}} \) is a symmetric real matrix, it is easy to find the real eigen values \( d_i \) and \( n \) corresponding orthonormal eigen column vectors \( V_i \) of \( A_{\mathcal{F}} \) that satisfy \( A_{\mathcal{F}}V_i = d_i V_i, V_i^T V_j = 0 \forall i, j \in \{0, 1, \cdots, n-1\} \). We can then create an orthogonal matrix \( P = \begin{bmatrix} V_1 & V_2 & \cdots & V_n \end{bmatrix} \), \( P^{-1} = P^T \) as a concatenation of the orthonormal column vectors \( V_i \). The matrix \( A_{\mathcal{E}} = P^T A_{\mathcal{F}} P \) is actually a diagonal matrix containing the eigen values \( d_i \) on its diagonal. Hence \( A_{\mathcal{E}} \) can be considered the IPM of a base orthogonal GACF from which we derive the non-orthogonal GACF \( F(\mathcal{F}_1^n, A_{\mathcal{F}}) \). Now we can use equation (4.15) to compute any bilinear product on two multivectors as before. This means that for each non-orthogonal GACF \( \mathcal{F} \) it is necessary to construct and store the orthogonal outermorphisms \( \overline{P}^T \) and \( \overline{P} \) created through an eigen analysis of \( A_{\mathcal{F}} \).

### 4.7 Constructing a Reciprocal GACF

Having a general non-degenerate GACF \( \mathcal{E}(E_1^n, A_{\mathcal{E}}) \) with basis vectors \( E_1^n = \{e_0, e_1, \cdots, e_{n-1}\} \) and non-null basis blades, it is possible to create a special type of derived GACF called the Reciprocal GACF \( \mathcal{F}(F_1^n, A_{\mathcal{F}}) \) having non-null basis vectors \( F_1^n = \{f_0, f_1, \cdots, f_{n-1}\} \) using the relations [3]:

\[
f_i = (-1)^{i-1}(e_1 \wedge e_2 \wedge \cdots \wedge e_{i-1} \wedge e_{i+1} \wedge \cdots \wedge e_n)I^{-1}
\]

(4.16)

\[
I = e_1 \wedge e_2 \wedge \cdots \wedge e_n,
\]

\[
I^{-1} = (-1)^{n(n-1)/2}I
\]

\[
\Rightarrow f_i \cdot e_j = \delta_{ij} \quad (4.17)
\]

\[
f_i \cdot f_j = e_i \cdot e_j \quad (4.18)
\]

\[
\forall i, j \in \{0, 1, \cdots, n-1\}
\]

(4.19)

If the base GACF \( \mathcal{E} \) is orthogonal the derived reciprocal GACF \( \mathcal{F} \) is also orthogonal and the above relations reduce to the simpler form:

\[
f_i = \frac{1}{e_i \cdot e_i} e_i \forall i \in \{0, 1, \cdots, n-1\}
\]

(4.20)

\[
\Leftrightarrow A_{\mathcal{F}} = A_{\mathcal{E}}^{-1} = A_{\mathcal{E}}^{-1}
\]

(4.21)
4.8 Computing Bilinear Products on a GACF

Starting with orthogonal GACFs, any bilinear product \( \star \) of two multivectors \( X, Y \in \mathcal{G}^{p,q,r} \) performed on their representations \([X]_F, [Y]_F\) in an orthogonal GACF \( F(F^0, A_F) \) with basis blades \( P^n = \{ F_0, F_1, \cdots, F_{2^n-1} \} \) can be implemented as:

\[
X \star Y = \sum_{r=0}^{2^n-1} \sum_{s=0}^{2^n-1} x_r y_s (F_r \star F_s)
\]  

(4.22)

The goal is to find the value of \( F_r \star F_s \) for all \( r, s \in \{0, 1, \cdots, 2^n - 1\} \). Due to the properties of the geometric product on orthogonal frames and the definitions of the bilinear products, the bilinear product of any two basis blades \( F_r \star F_s \) is either a zero or a single term \( \lambda^F F_k \), but never more than a single term. Actually when the value of \( F_r \star F_s = \lambda^F F_k \) the term is equal to the geometric product of the two basis blades \( \lambda^F F_k = G_F (F_r, F_s) = F_r F_s \). Assuming \( a = grade(F_r) \) and \( b = grade(F_s) \), the following relations list some useful GA bilinear products and their relations with the geometric product on the basis blades:

**Scalar Product:**

\[
F_r \star F_s = \langle F_r F_s \rangle_0
\]  

(4.23)

\[
= \begin{cases} 
0 & (r)_2 \ XOR \ (s)_2 \neq (0)_2 \\
F_r F_s & \text{otherwise}
\end{cases}
\]  

(4.24)

**Left Contraction Product:**

\[
F_r ] F_s = \langle F_r F_s \rangle_{b-a}
\]  

(4.25)

\[
= \begin{cases} 
0 & (r)_2 \ AND \ NOT \ (s)_2 \neq (0)_2 \\
F_r F_s & \text{otherwise}
\end{cases}
\]  

(4.26)

**Right Contraction Product:**

\[
F_r [ F_s = \langle F_r F_s \rangle_{a-b}
\]  

(4.27)

\[
= \begin{cases} 
0 & (s)_2 \ AND \ NOT \ (r)_2 \neq (0)_2 \\
F_r F_s & \text{otherwise}
\end{cases}
\]  

(4.28)

**Fat-Dot Product:**

\[
F_r \bullet F_s = \langle F_r F_s \rangle_0 + \langle F_r F_s \rangle_{b-a} + \langle F_r F_s \rangle_{a-b}
\]  

(4.29)

\[
= \begin{cases} 
0 & a = b, \ (r)_2 \ XOR \ (s)_2 \neq (0)_2 \\
0 & a < b, \ (r)_2 \ AND \ NOT \ (s)_2 \neq (0)_2 \\
0 & a > b, \ (s)_2 \ AND \ NOT \ (r)_2 \neq (0)_2 \\
F_r F_s & \text{otherwise}
\end{cases}
\]  

(4.30)

**Hestenes Inner Product:**

\[
F_r \bullet_H F_s = \begin{cases} 
F_r \bullet F_s & ab > 0 \\
0 & ab = 0 \\
0 & ab > 0, a = b, \ (r)_2 \ XOR \ (s)_2 \neq (0)_2 \\
0 & ab > 0, a < b, \ (r)_2 \ AND \ NOT \ (s)_2 \neq (0)_2 \\
0 & ab > 0, a > b, \ (s)_2 \ AND \ NOT \ (r)_2 \neq (0)_2 \\
0 & ab = 0 \\
F_r F_s & \text{otherwise}
\end{cases}
\]  

(4.31)
4.9 Computing Linear Maps on GACFs

**Commutator Product:**

\[
F_r \times F_s = \frac{1}{2} (F_r F_s - F_s F_r) = \begin{cases} 
0 & \text{Sign}_{\text{EGP}}(r, s) = \text{Sign}_{\text{EGP}}(s, r) \\
F_r F_s & \text{otherwise}
\end{cases} \tag{4.33}
\]

**Anti-Commutator Product:**

\[
F_r \overline{\times} F_s = \frac{1}{2} (F_r F_s + F_s F_r) = \begin{cases} 
0 & \text{Sign}_{\text{EGP}}(r, s) \neq \text{Sign}_{\text{EGP}}(s, r) \\
F_r F_s & \text{otherwise}
\end{cases} \tag{4.34}
\]

One exception to this pattern is the **Outer Product** that is metric-independent, and can’t be computed from the metric-dependent geometric product. Based on the discussion in section 4.3 we can assume a Euclidean GACF with the same dimension \( N \) \((N^n, I_n)\) with no loss of generality and then compute the outer product \( F_r \wedge F_s \) from the geometric product on the Euclidean GACF \( N \):

\[
F_r \wedge F_s = N_r \wedge N_s = \langle N_r N_s \rangle_{a+b} = \begin{cases} 
0 & (r)_2 \ \text{AND} \ (s)_2 \neq (0)_2 \\
N_r N_s & \text{otherwise}
\end{cases} \tag{4.35}
\]

Another exception is the **Regressive Product** that we can compute given two multivectors \( X, Y \) using other bilinear products. This particular product has several definitions in the literature. The following is just one of them:

\[
X \triangledown Y = (X^* \wedge Y^*)^\odot = (XI^{-1} \wedge YI^{-1}) I \tag{4.36}
\]

For a non-orthogonal GACF \( \mathcal{F} \) a bilinear product of two basis blades is not guaranteed to produce a single term, except for the outer product. All the above computational relations become invalid in this case. We can use equation (4.15) to compute any metric-dependent bilinear product from the two orthogonal outermorphisms \( \mathbf{F}^T \) and \( \mathbf{F} \) associated with \( \mathcal{F} \).

4.9 Computing Linear Maps on GACFs

Any Grassmann Space \( \Lambda^n \) is a linear space with \( 2^n \) basis blades. We can define and use general linear maps between two Grassmann spaces \( \mathbf{T} : \Lambda^n \rightarrow \Lambda^m \) and use any given bases \( \mathcal{E} (E^n_1, A_\mathcal{E}) \) and \( \mathcal{F} (F^m_1, A_\mathcal{F}) \) on the two spaces to create a \( (2^m - 1) \times (2^n - 1) \) representation matrix \( M_T = [M_0 \ M_1 \ \cdots \ M_{2^m-1}] \) for \( \mathbf{T} \) on the two bases where the column vectors \( M_k = [T[E_k]]_\mathcal{F} \) represent the transformed basis blades in \( \mathcal{E} \) using the basis blades in \( \mathcal{F} \). We can then transform any multivector \( A = \sum_{i=0}^{2^n-1} a_i E_i \in \mathcal{G}^{p_1,q_1,r_1} \) by representing it using a column vector \( [A]_\mathcal{E} = [a_0 \ a_1 \ \cdots \ a_{2^n-1}]^T \) and computing its transformation using simple matrix multiplication:

\[
[T[A]]_\mathcal{E} = M_T [A]_\mathcal{E} \tag{4.37}
\]

A special kind of linear maps on multivectors are the outermorphisms discussed earlier. We can fully define an outermorphism \( \mathbf{T} : \Lambda^n \rightarrow \Lambda^m \) as an extension of a linear map \( \mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m \) using two GACFs \( \mathcal{E} (E^n_1, A_\mathcal{E}) \) and \( \mathcal{F} (F^m_1, A_\mathcal{F}) \) by applying the outer product preservation property to basis vectors \( \mathbf{T}[e_i \wedge e_j] = \mathbf{T}[e_i] \wedge \mathbf{T}[e_j] = \mathbf{f}[e_i] \wedge \mathbf{f}[e_j] \) \( \forall e_i, e_j \in E^n_1 \). I will denote the vectors \( \mathbf{f}[e_i] \in W \) as \( h_i \) and their ordered set as \( H^n_1 = \langle h_0, h_1, \ldots, h_{n-1} \rangle \). Note that the vectors in \( H^n_1 \) are not guaranteed to be LID or even span the whole of \( W \), so the outer product of any subset of \( H^n_1 \) may be a zero blade in \( \Lambda^m \).
This way, we can represent an outermorphism using a large sparse \((2^n - 1) \times (2^n - 1)\) representation matrix \(M_f = [ M_0 \ M_1 \ \cdots \ M_{2^n-1} ]\) where \(M_k = [\prod_i (H_i^k, k)]_F\) and use (4.42) for transforming multivectors.

Computationally, we can exploit the sparsity of \(M_f\) to reach more efficient representations for outermorphisms. First we note that any multivector \(h^g = \prod_i (H_i^g, id(g, k))\) is actually a blade of grade \(g\) in \(\bigwedge^m\) and generally only needs \(\binom{m}{g}\) non-zero coefficients to be represented as a linear combination of basis blades of grade \(g\) on the basis \(F^m_g\). In addition, we can express any multivector \(A \in \bigwedge^n\) using its k-vectors decomposition: \(A = \sum_{g=0}^n \langle A \rangle_g\) and transform each k-vector \(\langle A \rangle_g\) to \(\langle E_{id(g, k)} \rangle_P\), \(r = \binom{n}{g}\) separately using:

\[
\underBrace{\prod_{g=0}^{n-1} \langle A \rangle_g}_{M^T} = \sum_{k=0}^{r-1} a_{id(g, k)} \prod_{g=0}^k \langle E_{id(g, k)} \rangle_P
\]

(4.43)

Using this method we need to create a set of \(n+1\) transformation matrices \(M^T = [ M_0^T \ M_1^T \ \cdots \ M_{r-1}^T ]\), \(r = \binom{n}{g}\), one matrix per grade \(g\), with column vectors \(M^g_k = [ h^g_k \prod_{g=0}^k F^m_g \) representing the g-vectors \(h^g_k\) on the g-vectors basis \(F^m_g\). We can then represent the transformations of \(\langle A \rangle_g\) under the \(\prod\) using the following relation, and finally recombine them into \([\prod [\prod A]_F]_F\):

\[
[\prod_{g=0}^{n-1} \langle A \rangle_g]_{F^m_g} = \prod_{g=0}^{n-1} \langle A \rangle_g]_{F^m_g}
\]

(4.44)

Because the base linear map on vectors \(f\) has rank \(\text{rank}_f \leq \min(m, n)\), we will only need to transform g-vectors of grades \(0 \leq g \leq \text{rank}_f \leq \min(m, n)\) because all \(h^g_k = 0 \forall k > \text{rank}_f\). This is due to the guaranteed linear dependence of any set of \(k > \text{rank}_f\) vectors in \(\mathbb{R}^m\) that are images of vectors in \(\mathbb{R}^n\) under \(f\). If we can’t compute \(\text{rank}_f\) in advance we can simply use \(\min(m, n)\) as an upper limit. For each outermorphism we need to compute and store the matrices \(M^T_f\). This approach is summarized in Algorithm 2.

If the outermorphism is invertible, all its matrices are also invertible. This approach has some benefits for representing a related outermorphism like the inverse \(\prod^{-1}\) or the adjoint \(\prod^T\) on the same GACF. Taking the adjoint as an example, we can either compute \(M^T_f\) and apply Algorithm 2 to get the outermorphism matrices, or we could directly apply the transpose operations to all matrices \(M^T_f\) already computed for \(\prod\) to get the outermorphism matrices \((M^T_f)^T\) for \(\prod^T\). Similar options exist for:

- The inverse of an outermorphism \(\prod^{-1}\) and its adjoint \(\prod^{-T}\) for which we compute the inverse and transposed inverse of the matrices respectively.
- An outermorphism that extends a linear map \(f : \mathbb{R}^n \rightarrow \mathbb{R}^m\) that is a linear combination of other linear maps: \(f[x] = \sum a_i f_i[x]\), where \(f_i : \mathbb{R}^n \rightarrow \mathbb{R}^m\) are linear maps on the same linear spaces. In this case we apply the same linear combination to the matrices of \(\prod_i\).
- An outermorphism that is the composition of other outermorphisms: \(f[x] = f_2 [f_1 [x]]\), where we use matrix multiplication between the matrices of \(\prod_i\).

Within the same GACF, any Outer Product-preserving linear operation on vectors can be converted to an outermorphism matrix representation. For example expressions like the projection of a vector on a blade \(L[x] = (x)B\) or the versor product on a vector \(L[x] = (-1)^{\text{grade}(A)} A x A^{-1}\) can be extended as outermorphisms. Given some GACF \(E_i, A, E\), we can construct the column vectors \(m_i\) of a linear transformation matrix \(M_L\) for any such expression by applying the expression \(L[x]\) to the basis vectors of \(E_i\):

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Algorithm 2. Computes the k-vector transformation matrices $M^g_f$ of an outermorphism $\bar{F}: \bigwedge^n \rightarrow \bigwedge^m$ given a $m \times n$ transformation matrix $M_f$ representing its base linear map $f$ on two GACFs $E(E_1^n, A\varepsilon)$ and $F(F_1^m, A\wedge)$.

1. Set $M^0_f \leftarrow I_1$, the $1 \times 1$ identity matrix.
2. Set $M^1_f \leftarrow M_f$.
3. Either set $K \leftarrow \text{rank}(M_f)$ or set $K \leftarrow \text{min}(n, m)$
4. Construct $n$ column vector representations $v_i^1$ on $E$ using the column vectors of $M_f$. Each $v_i^1$ is a sparse $(2^n - 1) \times 1$ column vector containing non-zero entries only at rows $1, 2, 4, 8, \ldots, 2^{n-1}$.
5. For increasing $g$ from 2 to $K$ do steps 6-11:
6. Set $r_g \leftarrow \binom{n}{g}$
7. For increasing $k$ from 0 to $r_g - 1$ do steps 8-10:
8. Select any two integers $r, s$ such that $(\text{id}(g, k))_2 = (\text{id}(g - 1, r))_2$ OR $(\text{id}(g - 1, s))_2$.
9. Compute the outer product $v^g_{k} = v^{g-1}_r \wedge v^{g-1}_s$ as discussed in Section 4.8.
10. Construct the column vector $m^g_k$ from $(v^g_{k}|_g)$ by selecting coefficients of basis g-blades in $v^g_{k}$ into rows of $m^g_k$ in their canonical order.
11. Set $M^g_f \leftarrow [m_0^g \quad m_1^g \quad \cdots \quad m_{r_g-1}^g]$.
12. Return final result as the matrices $M^g_f$.

\begin{equation}
mi = [L[v_i]|_{E^g_1}^g] \quad \forall i \in \{0, 1, \ldots, n - 1\}
\end{equation}

We can then construct the matrix representation using Algorithm 2. For an Automorphism (an orthogonal outermorphism) we can either use the above outermorphism matrix representation that would then have orthogonal matrices, or we can use the Versor multivector representation $A = \prod_{i=1}^{k} a_i$ where $a_i$ are $k$ non-null vectors, and the Versor Product $A \otimes X$ described earlier to compute the automorphism, the latter being more efficient in many cases. If we have the versor as a multivector $A$ we can find the column vectors $m_i$ of the corresponding linear map representation matrix $M_f$ using:

\begin{equation}
m_i = \text{(-1)}^\text{grade}(A) [Ae_iA^{-1}]_{E^g_1}^g \quad \forall i \in \{0, 1, \ldots, n - 1\}
\end{equation}

We can also find the versor multivector $A$ given an orthogonal matrix that represents an orthogonal linear map. This can be done using Householder Operators to find the Householder vectors $a_i$ \cite{69, 70, 71} then compute their geometric product to get the desired versor $A = \prod_{i=1}^{k} a_i$.

5 Summary and Conclusions

Software developers and engineers are natural Computational Thinkers. Introducing an elegant and sophisticated mathematical language like Geometric Algebra to software developers requires initially to focus on the abstract concepts and their relations more than the mathematics. To really understand the structure of Geometric Algebra the software developer should be familiar with some important conceptual abstractions of metric linear spaces not commonly taught in linear algebra courses. Only then that the software developer can use GA-related constructs like the outer product and the contraction to understand the elegant GA structure and the role of each of its components. Software developers better learn by
REFERENCES

doing; they need to watch abstract mathematical ideas come to life on computer displays. Creating a GA-based software library is the best way for a software developer to learn the mathematical details of GA. This article provided a Computational Thinking-based introduction to Geometric Algebra targeting software developers. The main three parts of this article introduced concepts of metric linear systems, and then used them to construct the main structural elements of GA in the second part. The third part aimed at providing enough mathematics to implement a GA-based software library either for learning, prototyping, or production purposes. In addition, the interested reader can find enough resources in the references for more information on the concepts and techniques presented in this article.

I believe the future of widely accepting GA as a universal mathematical language for Geometric Computing depends on how the scientific computing and software engineering communities appreciate GA as a powerful language for developing Geometric Computing software systems. Making GA implementations into valuable and enjoyable software systems for the public domain is possible only through the efforts of good software developers who understand and use GA in their own creative Computational Thinking way. Targeting these communities should be a top priority for the GA community to gain more popularity for their GA-based models. I recommend for the GA community to communicate more with software developers on both academic and practical levels. This would also make the GA community more aware of the practical problems facing GA-based software implementations that would require more research into GA-based algorithms and techniques.

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