The Complexity of Modified Instances

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Abstract

In this paper we study the complexity of solving a problem when a solution of a similar instance is known. This problem is relevant whenever instances may change from time to time, and known solutions may not remain valid after the change. We consider two scenarios: in the first one, what is known is only a solution of the problem before the change; in the second case, we assume that some additional information, found during the search for this solution, is also known. In the first setting, the techniques from the theory of NP-completeness suffice to show complexity results. In the second case, negative results can only be proved using the techniques of compilability, and are often related to the size of considered changes.

1 Introduction

Many complexity results in the literature are about simple decision problems, such as: is a propositional formula consistent? is there a vertex cover of a graph containing at most $k$ nodes? is there a plan to achieve this goal? etc. These problems are often simplification of real-world problems. In practice, what is needed is not only a “yes/no” answer. For instance, from the problem of planning what is really needed is a plan to achieve the goal, not only an answer to the question of existence.

Another property of these simplified problem is the absence of any hint in their specification. Constraints and hints are different:

**constraint:** affects the specification of the problem: if a constraint is added or removed, the set of solutions of the problem may change;

**hint:** something that may help finding the solution; it can be neglected without changing the set of solutions.
In this paper we consider some problems in which both constraints and hints are present. Namely, if the available data is composed of an instance of a problem, and a solution for a similar instance, we have both constraints (the instance) and a hint (the solution of the similar instance). This scenario may occur in practice, e.g., a solution for a specific instance is known, and then the instance is changed. In the planning domain, for example, it may be that, during the execution of a plan, some conditions that were previously assumed to hold, are indeed false, and some actions in the plan cannot be performed any more. In such a case, a new plan (from a different initial state) is needed. In this case, besides the instance specification (the description of initial state, goal, and actions) an hint is also available: the plan for the previous instance. In some cases, this may be of help, because some parts of the plan can be reused for the modified instance.

A similar definition can be given for almost all search problem, although in some cases the analysis is only of theoretical interest. For instance, we can consider what happens if we want to solve the satisfiability problem when a solution for a similar instance is known, but this problem does not seem to have a practical relevance. We study it nevertheless, as its analysis is very simple to carry on, allowing for a clear explanation of definitions and methods used for other problems for which changes are of more practical interest.

Besides satisfiability and planning, we also consider the problem of vertex cover. Namely, given an optimal vertex cover for a graph, we consider the problem of finding a vertex cover for a graph obtained by adding one or more edges to the previous graph.

In all those cases, we consider the complexity of the corresponding decision problem. For example, for the case of vertex cover, we want to determine the complexity of deciding whether there exists a vertex cover composed of at most \( k \) nodes, for the graph \((N, E \cup \{e\})\), if an optimal vertex cover for \((N, E)\) is known.

The analysis is done using quite standard complexity technique (and is in many cases very simple), that is, the theory of NP-completeness. However, there is a natural generalization of these problems that cannot be analyzed using standard complexity classes. Consider the case in which we want to find a plan, and we are already alerted that the plan may fail because of a change in the specification of the initial state. What we can do is to “save” some intermediate results which may allow for updating the plan when a new initial state is defined.

It is quite easy to see that the classes of the polynomial hierarchy and the polynomial many-one reduction are useless for solving this problem. Indeed, the problem is not completely specified, since there is no way of formalizing a concept such as “an intermediate result” of an algorithm, where moreover the algorithm
itself is not specified. In practice, the problem is “how to reduce the problem of satisfiability to the problem of updating a plan when some intermediate results of some algorithm are known?” If we consider that anything can be an intermediate result, and any algorithm is a possible candidate, we see that giving a polynomial reduction from e.g., satisfiability, seems to be very hard.

This problem can be solved very easily using compilability classes. Indeed, we can generalize further the notion of “intermediate result”, saying that the hint we have is any polynomial-sized data structure depending only on the original instance. Since compilability allows for giving theorems of negative results in this case, these negative theorems also extend to the case in which only intermediate results of algorithms are considered.

2 Propositional Satisfiability

In order to show the concepts used in the paper, we begin analyzing the problem of solving a modified instance for the problem of propositional satisfiability. The proofs of this sections are quite straightforward, and the results are completely unsurprising. This simplicity allows for an easier presentation of the definitions and techniques used to prove our results.

2.1 Fixed Model

Let us consider a set of propositional clauses $F$, over an alphabet $X$. Assume that a model $I \in \text{Mod}(F)$ is known. For some reason, this set $F$ is changed, for instance a clause is added, and we want to determine a model for this new instance $F \cup \{\gamma\}$. Disregarding the model $I$, finding a model of a formula $F \cup \{\gamma\}$ is NP-equivalent. Knowing a model of a similar formula cannot make the problem harder, so it is still NP: in general, any upper bound for the problem with no hint extends to the problem with hints.

The knowledge of the model $I$ may help. For instance, if $I \models \gamma$, then $I$ is itself a solution of $F \cup \{\gamma\}$. In this case, finding a model for the modified instance only takes linear time. However, if $I \not\models \gamma$, then $I$ may be useless for finding a model of $F \cup \{\gamma\}$.

In order to formally prove our intuition that a model cannot help in general for solving a modified instance, we consider the related decision problem, and establish its complexity. Given a model $I$ of a set of clauses $F$, and a clause $\gamma$, we want to decide whether $F \cup \{\gamma\}$ is satisfiable. Note that $F$ is satisfiable, since
it has at least a model. This problem is clearly in NP (disregard $I$, and check satisfiability of $F \cup \{\gamma\}$). This is an upper bound of the problem.

Finding a lower bound is slightly more complex. The data of the problem is not only composed of a set of clauses, but rather of a triple $\langle F, \gamma, I \rangle$, where $F$ is a set of clauses, $\gamma$ is another clause, and $I$ is a model of $F$. The problem is that the third part of the input (the model $I$) does not put any additional constraint (that is, the answer can be determined from $F$ and $\gamma$). Rather, it is an hint, that is, a suggestion that can be taken or not.

A proof of NP-hardness can be obtained by reduction from sat. We prove that there exists a polynomial reduction from the problem of checking satisfiability of a set of clauses $G$ to the problem under consideration. Namely, if $\langle F, \gamma, I \rangle$ is the instance obtained by applying the reduction to $G$, then $G$ is satisfiable if and only if $\langle F, \gamma, I \rangle$ is a valid instance. Validity of $\langle F, \gamma, I \rangle$ means that $I$ is a model of $F$, and $F \cup \{\gamma\}$ is satisfiable. Such a reduction is very easy to give:

$$G \Rightarrow \langle a \lor G, \neg a, \{a\} \rangle$$

where $a \lor G$ is a shorthand for $\{a \lor \gamma \mid \gamma \in G\}$. Indeed, $\{a\}$ is a model of $a \lor G$, but $(a \lor G) \cup \{\neg a\}$ is equivalent to $G \cup \{\neg a\}$, thus it is satisfiable if and only if $G$ is satisfiable (note that $\{a\}$ is not a model of this formula).

We have therefore proved that the problem of checking, given $\langle F, \gamma, I \rangle$, whether $I$ is a model of $F$, and $F \cup \{\gamma\}$ is satisfiable, is NP-hard (a quite trivial result). The next subsection shows that such hardness proof may not be suitable. This calls for a formal definition of this kind of reductions.

We express a search problem in NP by a polynomial binary relation over pairs of strings: the problem is that of finding a string $y$ such that $\langle x, y \rangle \in B$, given $x$. This is the problem for which we want to check whether the solution of an instance may help for solving modified instances.

To formalize instance changes, we use a function $c(x, y)$ that takes two strings $x$ and $y$, and gives a new string $z$ obtained by “merging” in some way $x$ and $y$. Intuitively, $c$ is the function used to modify an instance. In the case of satisfiability, $x$ is a set of clauses, $y$ is a clause, and $c(x, y) = x \cup \{y\}$. For other problems, $c$ may have a different definition.

The relation $B$ and the function $c$ define the problem and the way it is modified, so they suffice to formally define the problem of modified instances. The hardness result used for satisfiability is actually the NP-hardness of the following problem $C$.

$$C = \{\langle x, y, z \rangle \mid \langle x, z \rangle \in B \text{ and } \exists z'. \langle c(x, y), z' \rangle \in B\}$$
Informally, $C$ is the problem whose instances are triples $\langle x, y, z \rangle$, and the “yes” instances are those such that $z$ is a solution of $x$, and $c(x, y)$ is satisfiable (this is the instance obtained by modifying $x$ with $y$).

In order to show the NP-hardness of this problem, we take an already known NP-hard problem $A$, and reduce it to $C$ in polynomial time. In other words, we find three polynomial functions $f$, $g$, and $h$, such that:

$$
\forall x . x \in A \Leftrightarrow \langle f(x), g(x), h(x) \rangle \in C
$$

Summarizing, from the search problem $B$ we define a decision problem $C$, and then show a reduction from an NP-hard problem $A$ to $C$, thus proving that $C$ is NP-hard.

### 2.2 Arbitrary Model

The proof given in the previous section has a drawback. Let us consider the reduction used for $\text{sat}$. What we proved is only that, in general, there exists a model of $a \lor G$ that is useless for solving the satisfiability of $(a \lor G) \cup \{\neg a\}$. However, there may be other, more useful, models. Let us assume, for example, that the model of $a \lor G$ is chosen in the following “unfair” way:

1. if $G$ is satisfiable, then $I$ is a model of $G$;
2. otherwise, $I = \{a\}$.

Clearly, such model $I$ allows for solving the satisfiability of $(a \lor G) \cup \{\neg a\}$ in linear time. This is not in contradiction with the NP-hardness result. Indeed, hardness only proves the existence of hard instances, and not that all instances of a problem are hard. In this case, hardness shows that there are models of the original instance that do not help in solving the modified one.

This example shows that, while there is some model of $F$ that does not help in solving satisfiability of $F \cup \{\gamma\}$, some other model may help. If we could prove that, for each formula $F$ there is a model $I$ that makes the satisfiability of $F \cup \{\gamma\}$ polynomial, we would have proved that a careful choice of the solution of the original instance actually helps the solution of the modified one.

In order to prove the impossibility of such method, we would need to modify the reduction so that, for any model of $F$, verifying satisfiability of $F \cup \{\gamma\}$ is hard.

It is easy to show that, if only one clause is added, such a reduction does not exist. Indeed, we can always set $I$ to be a model of $F \cup \{\gamma\}$ if it is satisfiable,
and thus there is a model that helps solving the modified instance. This proves that proving hardness this way is impossible. However, it does not provide a method for finding a model that helps solving any possible modified instance, as $I$ cannot be determined without knowing the clause $\gamma$ to be added. As a result, we do not have an efficient solution method, and we know we cannot prove its unfeasibility by using hardness in the way we planned.

If we allow clauses to be either added or deleted, we can really prove that no model of a formula helps checking the satisfiability a modified formula. Let us assume that a formula may be modified by adding a clause $\gamma$ and removing another clause $\delta$. Given a model $I$ of $F$, we want to check satisfiability of $F \cup \{\gamma\}\{\delta\}$.

A proof of NP-hardness is the following one. Let $G$ be a propositional formula. We define $F$ to be $\{a\} \cup ((X \cup \{a\}) \lor (G \cup \{\neg a\}))$, where $F \lor G$ is a shorthand for $\{\gamma \lor \delta \mid \gamma \in F, \delta \in G\}$. This formula has a single model: $\{a\} \cup X$. Furthermore, if we set $\gamma = \{\neg a\}$ and $\delta = \{a\}$, we have that $F \cup \{\gamma\}\{\delta\}$ is equivalent to $G \cup \{\neg a\}$, which is satisfiable if and only if $G$ is satisfiable.

This way, we have proved that, given any model of $F$, checking satisfiability of $F \cup \{\gamma\}\{\delta\}$ is NP-hard. For a generic search problem expressed by a binary relation $B$, the technique is to prove that there exists an NP-hard problem $A$ and two polynomial functions $f$ and $g$ such that $(C$ is defined as in the previous section):

$$\forall x, z \cdot x \in A \iff (\langle f(x), g(x), z \rangle \in C \iff \langle f(x), z \rangle \in B)$$

This proves that the problem remain hard no matter how the solution of the original instance $z$ is chosen. In practice, the only way for proving the claim is by giving a reduction in such a way $f(x)$ has exactly one solution. This statement is slightly stronger than the previous one: it cannot be proved for $\text{sat}$ in the assumption that the only possible change is the addition of a clause. Changes involving removing of clauses were needed.

### 2.3 Arbitrary Polynomial Data Structure

We now consider the most general possible setting. In the analysis done so far, we assumed that what is known is only a solution of a given instance. However, it is reasonable to assume that some other information is given. For instance, we may save some intermediate results obtained during the search of the first solution, and this information may be of help in looking for further solutions.
In the case of satisfiability, we may have information about the satisfiability of some subformulae, etc.

In general, we may assume the the size of this piece of information is bounded by a polynomial in the size of the input. Since we want results that are independent from the algorithms used, we have to make the fewer possible assumptions. As a result, we assume that what is known is simply a piece of information related to the formula $F$. In other words, we are given the formula $F$, a list of changes $D$ (additions and deletions of clauses), and some piece of information $I$, whose size is polynomial in the size of $F$. We want to check whether the formula obtained by modifying $F$ with $D$ is satisfiable or not.

The case in which $D$ is composed only of a fixed number of changes (either additions or deletions of clauses) is easy. Indeed, if $D$ is bounded to be composed of at most $k$ changes, the total number of possible changes is $O(2^k)$, and this is a constant as well. Therefore, there are only a constant number of possible changes.

A data structure $I$ that allows for the solution of instances after the change is a table that reports, for each possible change, the solution of the resulting instance. Note, indeed, that we have not given any restriction on $I$, besides it must be of polynomial size. Given $I$ and any possible change of size bounded by $k$, finding a solution of the modified instance is just a matter of a table lookup.

A polynomially-sized data structure is therefore more useful than having simply a model of the original instance, as an hint. Indeed, the former helps in finding a solution after a constant number of changes, while the latter does not, even for the case of two changes only (as proved in the last section).

If the set of changes $D$ is not composed of a fixed number of changes, we can prove that any polynomial-sized piece of information is in general useless. This result is not surprising, as in practice we can obtain any set of clauses $F'$ by a number of changes over any other set of clauses $F$. Therefore, having a data structure $I$ that depends on $F$ is equivalent of having a constant data structure, and this cannot help in solving the general problem of satisfiability. Formally, if the original set is $F = \emptyset$, then its size does not depends on the size of $D$. Therefore, the size of $I$ is unrelated to the size of $D$ as well. Now, any clause can be expressed as the addition of a suitable number of clauses to $\emptyset$. Considering the satisfiability of this set of clauses, the size of $I$ is a constant. As a result, if this problem is polynomial, then the problem of satisfiability is polynomial as well, as it can be solved in polynomial time using a constant (independent from the set of clauses) data structure $I$.

On the other hand, it is clear that, if any change is allowed, no gain can be
achieved, as any possible instance of the problem can be obtained by suitably changing a trivial instance (like $F = \emptyset$). In practice, as for example in the planning case that will be shown in the sequel, the kind of possible changed are often limited to a part of the instance. The most interesting case, in the satisfiability problem, is that of changing a set of clauses without adding new variables. This restrict the set of changes, as for example no changes are possible to $F = \emptyset$. Given a set of clauses, we can change it only if the resulting set of clauses has the same set of variables, or a subset of the original one.

The set of possible changes is now restricted, but it nevertheless allows for proving that, in general, a polynomial data structure depending on a single instance will not help in solving similar instance. Intuitively, this is due to the fact that any set of clauses on the same set of variables can be obtained from a given one. There are exponentially many of such sets, while the data structure is only polynomially large. It looks therefore unlikely that a polynomial data structure can help in solving exponentially many instances. However, this is not a complete proof, and it is also easy to construct problems for which useful data structures exist.

The formal proof of impossibility for sat is based on the following problem nsat:

$$nsat = \{ (x, y) \mid y \in sat \text{ and } x = 1^{\|Var(y)\|} \}$$

In words, the first part of the input is a string whose only information is the number of variables of the formula represented by the varying part. It is easy to show that $nsat$ is $\parallel \text{NP}$-complete [CDLS96, CDLS02].

Adding and removing an arbitrary number of clauses from a set $F$ we can obtain any set of clauses with the same set of variables. This means that the existence of a data structure $I$ of polynomial size, that makes the solution of the changed instance polynomial is the same as the existence of an $I$, depending only on the number of variable, that makes the solution of the satisfiability problem for all sets of clauses over the given alphabet polynomial.

As a result, if there exists a way of determining some piece of information $I$ such that solving any modified instance is polynomial, then $nsat$ would be solvable in polynomial time, once a preprocessing step (of unbounded time length, but producing a polynomially-sized output). This implies that $nsat$ is in $\parallel \text{P}$. The membership of a $\parallel \text{NP}$-hard problem to $\parallel \text{P}$ implies the collapse of the polynomial hierarchy to its second level [CDLS96, CDLS02].
2.4 Summary

The analysis of sat, from the point of view of instance modification, is quite straightforward. However, it allows for showing that different possible formal definition of the problem are possible. In particular, the instance modification problem is:

1. given an instance and a change, is there at least a model of the instance that helps solving the modified instance? (Subsection 2.1: fixed model);

2. given an instance and a change, does any model of the instance help in solving the modified instance? (Subsection 2.2: arbitrary model);

3. given an instance, and a change, is there any polynomially-sized data structure, depending only on the instance (not on the change), that makes the modified instance easier to solve? (Subsection 2.3: polynomial data structure).

For sat, the first is impossible even if the change is the simple addition of a unit clause; the second can be proved impossible only allowing a unit clause to be added and another one to be deleted. The third case depends on additional constraints on the change:

1. if only a constant number of additions/deletions is allowed, then the modified instance problem becomes polynomial;

2. if any change is possible, the problem can be proved to be as hard as the original one with a very simple proof (using only classes of the polynomial hierarchy);

3. if changed are restricted to be about clauses on the same alphabet of the original formula, then the modified instance problem can be proved to be as hard as the original one, but the proof requires the use of compilability classes.

3 Vertex Cover

We consider the problem of vertex cover, and assume that the possible changes are addition of edges. The case with also removal of edges is similar.
Let $G = (N, E)$ be a graph. We assume that an optimal vertex cover $V$ is known, and a set of edges $E'$ is added. The modified instance is thus $G' = (N, E \cup E')$. As in the case of satisfiability, we are looking for an optimal vertex cover for the new instance, but we only consider a related decision problem: deciding whether there exists a vertex cover for $G'$ of size at most $k$, the latter being part of the input. Clearly, if an optimal vertex cover for $G'$ can be found, then this decision problem can be easily solved.

We show that there exists two polynomial functions $f$ and $g$ such that, given a set of clauses $F$, $f(F)$ is a graph, $g(F)$ is an integer, and it holds:

1. $F$ is satisfiable if and only if there exists a vertex cover of $f(F)$ of size at most $g(F)$;
2. adding or removing a unit clause to $F$ is equivalent to adding an edge to $f(F)$ and modifying $k$ (formally, it is possible to add an edge to $f(F)$ in such a way the resulting graph has a vertex cover of size $g(F)$ if and only if $F \cup \{\gamma\}$ is satisfiable, and the same for removing of clauses, adding 1 to $k$).

This can be proved as follows. We consider the usual reduction from 3SAT to vertex cover. Each variable $x_i$ is translated into a pair of nodes joined by an edge. One of the two nodes is associated with $x_i$, the other with $\neg x_i$. Each clause $\gamma_i$ composed of $j$ literals is mapped into a clique of $j$ nodes. If $\gamma_i = \bigvee l_j$, we call these nodes $n^j_i$. Each node of this clique is then joined to the node associated with the corresponding literal. In order to allow for adding/removing of unit clauses, we add two nodes for each literal, say $l'_i$ and $l''_i$, even if the corresponding unit clause is not in $F$. If the unit clause $l_i$ is in $F$, we join $l_i$ with $l'_i$.

Below is the graph corresponding to the set of clauses $F = \{\gamma_1, \gamma_2\}$, where $\gamma_1 = x_1 \lor \neg x_2$, while $\gamma_2 = \neg x_1$.

![Graph Diagram]

It is not difficult to prove that there exists a vertex cover of $n + m - r$ nodes if and only if the set of clauses is satisfiable, where $n$ is the number of variables, $m$ is the total number of occurrences of literals and $r$ is the number of clauses.
Moreover, adding a unit clause $l_i$ is equivalent to adding an edge joining the node $l_i$ and the node $l_i'$. Namely, $F \cup \{l\}$ is satisfiable if and only if the graph obtained by adding an edge between $l_i$ and $l_i'$ has a vertex cover of size $n + m - r$. Below is the graph modified by the addition of the unit clause $\gamma_3 = \neg x_2$.

The operation of removal of unit clauses is also simple. Indeed, if $l_i$ has to be removed from $F$, we can add an edge between $l_i'$ and $l_i''$. The resulting graph has a vertex cover of $n + m - r + 1$ if and only if $F \setminus \{l\}$ is satisfiable. Below, we see the graph obtained by the removal of the unit clause $\gamma_2 = \neg x_1$.

Assuming that $l \in F$, we have an edge between $l$ and $l'$. By construction, either $l$ or $\neg l$ are in the vertex cover. Now, the edge between $l$ and $l'$ enforces $l$ to be in the cover. However, if we add an edge between $l'$ and $l''$, and increase the size of the cover by 1, we can add $l'$ to the cover, and then we are free to choose $l$ or $\neg l$ to be in the optimal cover.

This proves the first part of the claim: given a graph, an optimal vertex cover, and an edge to add, deciding whether there exists a vertex cover for the modified graph composed of at most $k$ nodes is NP-complete. Since removal of unit clauses corresponds to removal of edges, we also have that given any optimal vertex cover of the graph, if we are allowed to add two edges, the problem remains NP-complete.

The third possible definition of the problem of modified instance is that using a polynomial data structure. As for the case of satisfiability, if a constant number
of changes is allowed, then we can create a data structure that simplifies the problem of modified instance.

If changes are not bounded in size by a constant, we can prove that such simplification is impossible. Note that the reduction above cannot be used, as it only works for addition and deletions of unit clauses. The impossibility proof for sat, on the other hand, uses arbitrary changes, that is, additions and deletions of arbitrary clauses.

In this case, we show a direct proof. From the \(|\leq|\)-NP-hardness of the problem we prove that polynomially-sized data structures are useless. It is possible to modify the reduction above in such a way non-unit clauses are translated to edges of the final graph. However, such proof would imply the \(|\leq|\)-NP-hardness of the vertex cover problem, and this is much easier to be proved directly.

The proof of \(|\leq|\)-NP-hardness is very simple: indeed, given a set of clauses \(F\) over an alphabet \(X\), we consider the graph that corresponds to the set \(F'\) of all clauses of three literals over \(X\). From this graph we remove all edges between clause nodes and literal nodes for all clauses that are not in \(F\). It is easy to see that this is a polynomial reduction from satisfiability to vertex cover. Moreover, since the number of nodes in the graph does not depend on the set of clauses, but only on the number of variables, this reduction is monotonic (the varying part of the vertex cover problem is composed of the set of edges).

The impossibility of reducing the complexity using optimal vertex covers of similar instances can be proved using this compilability result. Indeed, suppose that, given a graph, there exists a polynomially-sized piece of information that makes polynomial the problem of vertex cover for any graph obtained by adding a set of edges. Since any graph with \(n\) nodes can be obtained by adding an appropriate number of edges to the graph \((N, \emptyset)\), where \(N\) is a set of \(n\) nodes, it follows that such piece of information makes polynomial the problem of solving the vertex cover problem for any graph with the given number of nodes, that is, the problem of vertex cover would be compilable to \(P\), when the set of nodes is the fixed part of the instances.

Some final comments. In the analysis of the sat problem, we gave two separate proofs for the case in which changes may involve the creation of new variables, and the case in which this is forbidden. For the vertex cover problem, only one proof is needed: by construction, changes can only add new edges, and new nodes cannot be created. Finally, note that changes not only involve the addition of edges, but also the bound \(k\) on the number of nodes in the vertex cover.
4 Planning

The problems analyzed in the previous sections are rather simple, and the results are quite unsurprising. Indeed, it is clear that adding or removing a literal from a set of clauses may lead to a set of clauses that is completely different from the original one, and thus a solution for the first instance does not help in finding the solution of the new one. What is interesting in the satisfiability problem is that it shows that three possible measures of complexity can be defined when hints are given: complexity when a specific solution is given, complexity when an arbitrary solution is given, and compilability (generic hint of polynomial size).

The problem of vertex cover is more interesting, as it may be not immediately clear how solutions of similar instances can be used. If only a solution is given, the proofs of complexity were based on that used for satisfiability. The proof of \(|\rightarrow|\text{-NP}-hardness was almost straightforward.

In this section we consider a problem of practical interest: automated planning. As we will show, the proofs for the case in which an instance is given are very similar to those of satisfiability, while the compilability analysis is more interesting. This is due to the fact that the fixed part of the problem is in this case the set of actions that can be performed to achieve the goal. While in the case of vertex cover was obvious that the set of nodes gives not much information, in the case of planning the set of actions is a significantly large part of the instance.

The planning formalism we consider in this paper is STRIPS. In STRIPS, an instance of the planning problem is a 4-tuple \(⟨P, O, I, G⟩\), where \(P\) is a set of conditions, \(O\) is the set of operators, \(I\) is the initial state, and \(G\) is the goal.

Conditions are facts that can be true or false in the world of interest. A state \(S\) is a set of conditions, and represents the state of the world in a certain time point. The conditions in \(S\) are those representing facts that are true in the world, while those not in \(S\) represent facts currently false.

The initial state is a state, thus a set of conditions. The goal is specified by giving a set of conditions that should be achieved, and another set specifying which conditions should not be made true. Thus, a goal \(G\) is a pair \(⟨M, N⟩\), where \(M\) is the set of conditions that should be made true, while \(N\) is the set of conditions that should be made false.

The operators are actions that can be performed to achieve the goal. Each operator is a 4-tuple \(⟨\phi, \eta, \alpha, \beta⟩\), where \(\phi, \eta, \alpha, \) and \(\beta\) are sets of conditions. When executed, such an operator makes the conditions in \(\alpha\) true, and those in \(\beta\) false, but only if the conditions in \(\phi\) are currently true and those in \(\eta\) are currently false. The conditions in \(\phi\) and \(\eta\) are called the positive and negative
preconditions of the operator. The conditions in $\alpha$ and $\beta$ are called the positive and negative effects or postconditions of the operator.

Given an instance of a STRIPS planning problem $\langle P, O, I, G \rangle$, we define a plan for it as a sequence of operators that, when executed in sequence from the initial state, lead to a state where all the conditions in $M$ are true and all those in $N$ are false. More details about the definition of STRIPS can be found in [FN71] and [Byl91].

For the sake of simplicity, we consider a restriction of STRIPS. Namely, we assume that all operators have only positive postconditions. The problem of deciding whether there exists a plan, under this condition, is denoted as $\text{PLANSA T}^+$, and is known to be NP-complete [Byl91].

The problem of replanning arises when a plan is known, but the scenario is modified by some external cause. In such cases, we have a plan that is no more valid, so a new plan is needed. Let $\langle P, O, I, G \rangle$ be the original planning instance. We have a plan for it, but, at some point, we discover that something we initially believed to be true is instead false. Clearly, some operations cannot be performed any more, so the plan may no longer be valid. We formalize the change by assuming that we have a modified planning instance $\langle P, O, I', G \rangle$. The new initial state reflects both the fact that some actions of the initial plan may have already been executed, and the fact that our knowledge of the world is changed (some facts are not true/false any more).

The first option to investigate is whether what remains to be executed of the old plan is still a plan of the new instance. This can be done in linear time, and a positive outcome means that we have found the plan for the modified instance. However, it may be that the plan is no longer valid.

The idea behind replanning is that the old plan may be useful in the search for the new plan, that is, it may make the search for a new plan easier. Formally, we are given an instance $\langle P, O, I, G \rangle$, one of its (optimal) plans, and a similar instance $\langle P, O, I', G \rangle$ with a different initial state. We want an optimal plan for this second instance. This is the formalization of the case in which only a solution of the original instance is available, like when we consider satisfiability and had a model of the original formula as an hint.

We will prove that, even if $I$ and $I'$ differs from one condition only, the problem is NP-hard, even if an optimal plan for $\langle P, O, I, G \rangle$ is given.

**Theorem 1** Given a planning instance $\langle P, O, I, G \rangle$, one of its plans $P$, and a condition $a$, deciding whether there exists a plan for $\langle P, O, I\{a\}, G \rangle$ is NP-complete.
Proof. The problem is in NP, as we can find the new plan just modifying the planning instance and finding the new plan from it, and the latter problem is in NP, while modifying the instance is polynomial.

The problem of deciding whether there exists a plan, given the old instance, the update, and the old plan, is NP-hard. We show that the problem of satisfiability of a set of clauses $F$ can be reduced to the problem of finding a plan for an updated STRIPS instance. Let $F = \{\gamma_1, \ldots, \gamma_k\}$ be the set of clauses, and $X = \{x_1, \ldots, x_n\}$ be its alphabet. The set of conditions of the corresponding STRIPS instance is:

$$P = \{a\} \cup \{t_1, \ldots, t_n\} \cup \{f_1, \ldots, f_n\} \cup \{c_1, \ldots, c_k\}$$

There are two conditions $t_i$ and $f_i$ for each propositional variable $x_i$, and one condition for each clause in the set $F$. These conditions create a correspondence between states of the planning problem and propositional interpretations. The conditions $t_i$ and $f_i$ are used to represent the truth value of a variable $x_i$, while $c_j$ holds if and only if the clause $\gamma_j$ is true.

The initial state and the goal are specified as follows.

$$I = \{a\}$$
$$G = \langle \mathcal{M}, \mathcal{N} \rangle$$
$$\mathcal{M} = \{c_1, \ldots, c_k\}$$
$$\mathcal{N} = \emptyset$$

The set of operators is as follows.

$$O = \{pl_1, \ldots, pl_n\} \cup \{nl_1, \ldots, nl_n\} \cup \{pc_1, \ldots, pc_k\} \cup \{nc_1, \ldots, nc_k\} \cup \{e\}$$

where

$$pl_i = \langle \emptyset, \{f_i, a\}, \{t_i\}, \emptyset \rangle$$
$$nl_i = \langle \emptyset, \{t_i, a\}, \{f_i\}, \emptyset \rangle$$
$$pc_i = \langle \{t_i\}, \emptyset, \{c_j\}, \emptyset \rangle$$
$$nc_i = \langle \{f_i\}, \emptyset, \{c_j\}, \emptyset \rangle$$
$$e = \langle \{a\}, \emptyset, \{c_1, \ldots, c_k\}, \emptyset \rangle$$

These sets define the old instance of the planning problem. The plan for the original instance is $\langle e \rangle$. It can be easily verified that this is a plan for the above instance (namely, it is the only minimal and irredundant one).
The considered update is the deletion of the condition \( a \) from the initial state. Since there is no operator that has \( a \) among its postconditions, it follows that \( a \) does not hold in any state reached from the initial state. As a result, the operator \( e \) have no effect, thus it can be deleted from the instance, as well as \( a \).

The obtained instance is equivalent to that given in [Byl91] to prove that PLANSAT+ is NP-complete. What is proved in Bylander’s paper is that there exists a plan for the considered instance if and only if the set of clauses \( F \) is satisfiable.

Taking a closer look to the instance considered in the proof, we notice that the old plan does not help because the result of the change is an instance on which the only action of the old plan cannot be executed at all. This means that the old plan is now completely useless.

As for the satisfiability and the vertex cover problems, a possible objection is that the hardness result proves is only that there are useless plans, not that all plans are useless. The question of whether there exists plans that help in solving similar planning instance can be answered with the very same proof of the previous theorem.

**Theorem 2** Given any minimal or any irredundant plan for a planning instance, finding a plan for an updated instance is NP-complete, even if the update is the deletion of a condition from the initial state.

**Proof.** In the planning instance used in proof of Theorem 1, there is exactly one irredundant plan, which is also the only minimal plan: the sequence \( \langle e \rangle \). As a result, given any set of propositional clauses \( F \), we build the planning instance and the update as in the proof of Theorem 1. Now, given any irredundant (or minimal) plan for this instance, we proved that \( F \) is satisfiable if and only if there exists a plan for the updated instance.

To complete the analysis, only one step is missing: the case in which a polynomially-sized data structure is available. This encodes also the case in which some intermediate result, found during the search for the previous plan, is saved for later use with modified planning instances. This data structure can only depend on the original instance.

If a constant number of changes is allowed (that is, the initial state can only be modified for a constant number of conditions that are either added or deleted), then a data structure that helps finding a plan for the modified problem is easy to define.
We therefore consider the case in which the initial states of the original and the new instance can be different. Note, however, that the two instances have the same conditions, operators, and goals. This is a constraint over the possible change, and loosely corresponds to the case of satisfiability when no new variables can be added. Indeed, the impossibility proof make use of compilability classes.

**Theorem 3** If there exists a polynomial data structure such that deciding the existence of plans for updated instances is polynomial, then $\Sigma^p_2 = \Pi^p_2$, i.e., the polynomial hierarchy collapses.

**Proof.** We use a result on non-compilability of planning given in [Lib98]. The result is: given a set of conditions $\mathcal{P}$ and operators $\mathcal{O}$, there is no polynomial data structure that allows the solving of a generic instance $\langle \mathcal{P}, \mathcal{O}, \mathcal{I}, \mathcal{G} \rangle$ in polynomial time. This implies that it is not possible to compile $\mathcal{P}$ and $\mathcal{O}$ in such a way the solving for any initial state and goal is polynomial.

We prove that the same holds when the part of the instance that is compiled is composed by $\mathcal{P}$, $\mathcal{O}$, and $\mathcal{G}$, that is, only the initial state is not compiled. This is shown by showing a $\parallel\rightarrow$-reduction from the problem of planning (in which $\mathcal{P}$ and $\mathcal{O}$ are the fixed part of the instance) to the problem of planning (in which only $\mathcal{I}$ is the varying part of the input).

Let $\langle \mathcal{P}, \mathcal{O}, \mathcal{I}, \mathcal{G} \rangle$ be an instance of the original planning problem, where $\mathcal{G} = \langle \mathcal{N}, \mathcal{M} \rangle$. We derive a new instance $\langle \mathcal{P} \cup \{g\}, \mathcal{O} \cup \{o\}, \mathcal{I}, \{\{g\}, \emptyset\} \rangle$ that has a plan if and only if the original one has. The new instance has one more condition $g$, used to formalize the reachability of the goal in the original one, and a new action $o$ used for the same aim, and defined as $o = \langle \mathcal{N}, \mathcal{M}, g, \emptyset \rangle$. In words, this action can be used, whenever the goal of the original problem is satisfied, to make $g$ true. As a result, a plan for the original problem exists if and only if the new instance admit a plan. In the new instance, the goal is constant (does not depend at all from the original instance). Therefore, its membership to the fixed or varying part of the input is irrelevant. This proves that the problem of planning, where the initial state is the only varying part of the input, is $\parallel\rightarrow$-NP-complete.

We use this result for proving that it is not possible to find, given a planning instance, a polynomially-sized data structure that allows solving similar problems. Suppose that such a structure exists. Then, it would be possible to compile $\mathcal{P}$, $\mathcal{O}$, and $\mathcal{G}$ just by finding such data structure for $\langle \mathcal{P}, \mathcal{O}, \emptyset, \mathcal{G} \rangle$. Since any STRIPS instance $\langle \mathcal{P}, \mathcal{O}, \mathcal{I}, \mathcal{G} \rangle$ can be obtained by updating $\langle \mathcal{P}, \mathcal{O}, \emptyset, \mathcal{G} \rangle$, the data structure makes any such instance solvable in polynomial time, and this contradicts the result given above. \(\square\)
5 Conclusions

This paper report on the study of how the complexity of a problem changes when a solution of a similar instance is given. The most interesting result is that three possible characterizations can be given for this problem. The simplest one is simply the one in which we assume that a given solution is part of the input. While the hardness proof of this case is quite straightforward, it only proves that there are solutions that do not help in solving similar instances. We therefore considered the more interesting case in which any solution is considered. Finally, we considered the case in which the available hint is not only composed of a solution of a similar instance, but may be any polynomially-sized data structure depending on the similar instance only. This formalizes, for example, the case in which some information, derived from the search of the solution of the similar instance, is saved for use on changed instances.

The hardness results of this paper are proven in two ways: either we provide a direct proof of the claim (for example, that satisfiability is not simplified, if a model of a similar instance is given), or obtained by employing a special kind of reductions, in which small changes to the original instance corresponds to small changes of the resulting one. Finally, for the case in which polynomially-sized data structures are available, and there are constraints on the kind of possible changes (like, new variables cannot be introduced), the compilability classes have been used to prove impossibility results.

The case in which a polynomially-sized data structure comes from generalizing the case in which solving an instance of a problem we find some intermediate results that may be useful for solving similar instance. On the other hand, this may look as an over-generalization. In other words, not all polynomially-sized data structures are reasonable partial results of algorithms. In particular, if the changes are small, we used some data structures containing the solution for any possible modified instance. It is clearly unlikely that an algorithm produces such table by accident. However, some facts are to be considered:

1. while solving an instance, we are aware that it may be later modified; therefore, such table may be the actual result of an algorithm that is designed taking this into account;

2. the restricted case in which the additional information is something found during the solution of the original instance is hard to analyze; indeed, the kind of additional information depends on the specific algorithm employed;
3. in some cases, the first instance may be solved without finding any additional information; in the case of satisfiability, the DPLL algorithm may find the model immediately, by making (by accident) the right choice of the first sign of the literal to expand at each branching point; a local search algorithm may find the model as it is the first one it tries; in both cases, no other information is available;

Note that such cases are not that unlikely: for example, DPLL may find a model of $a \lor G$ because $a$ is the literal occurring most often, and is chosen as the first branching literal: if the positive sign is chosen first, then a model is found immediately; for local search, the formula $a \lor G$ is satisfied by all models with $a$, so we have at least 50% probability that the first model tried by the algorithm (chosen at random) contains $a$; in both cases, we have no additional information that can help us if $\neg a$ is added to the set.

Finally, note that proof complexity can hardly be used in this context, as it is mainly oriented to proving unsatisfiability, and mainly when exponentially large proofs are needed.

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