Monomiality and a New Family of Hermite Polynomials

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Abstract

In this article we go deeply into the formulation and meaning of the monomiality principle and employ it to study the properties of a set of polynomials, which, asymptotically, reduce to the ordinary two variable Kampé-de-Fériét family. We derive the relevant differential equations and discuss the associated orthogonality properties, along with the relevant generalized forms.

Keywords

Special functions 33C52, 33C65, 33C99, 33B10, 33B15; Hermite polynomials 33C45; operators theory 44A99, 47B99, 47A62.

1 Introduction

The Hermite polynomials belong to the Appell family [1] and the relevant properties can be conveniently framed within the context of the monomiality principle [2, 3]. This is a modern formulation of a point of view, tracing back to Steffensen [4, 5, 6], but even to older researches by Jeffery (for a recent account see Ref. [7]), Boole [8] and to other speculations developed almost two hundred years ago. These researches deepened their roots into the calculus of differences [9], the first to be recognized as amenable for a symbolic interpretation. The rules underlying monomiality are fairly simple and can be formulated as reported below [2, 3, 11, 12, 13].

Properties 1. \( \forall x \in \mathbb{R}, \forall n \in \mathbb{N}, \) If a couple of operators \( \hat{P}, \hat{M} \) are such that:

a) they do exist along with a differential realization,
b) they can be embedded to form a Weyl algebra \([10, 14, 15]\), namely if commutator is such that 
\[ [\hat{P}, \hat{M}] = 1, \]

c) it is possible to univocally define a polynomial set such that:
\begin{align*}
  i) & \quad p_0(x) = 1, \\
  ii) & \quad \hat{P} p_0(x) = 0, \\
  iii) & \quad p_n(x) = \hat{M}^n 1, \tag{1}
\end{align*}

then it follows that
\begin{align*}
  d) & \quad \hat{M} p_n(x) = \hat{M}^{n+1} = p_{n+1}(x), \tag{2} \\
  e) & \quad \hat{P} p_n(x) = \hat{P} \hat{M}^n 1 = n p_{n-1}(x) \tag{3}
\end{align*}

and the polynomials \( p_n(x) \) are said Quasi-Monomials.

Proof. Eq. (3) needs few lines of comment. We rearrange the operator product 
\( \hat{P} \hat{M}^n \) as \( [16] \)
\[
\hat{P} \hat{M}^n = (\hat{M} \hat{P} + 1) \hat{M}^{n-1} = \hat{M} \hat{P} \hat{M}^{n-1} + \hat{M}^{n-1} = \\
= \hat{M}^2 \hat{P} \hat{M}^{n-2} + 2 \hat{M}^{n-1} = \cdots = \hat{M}^n \hat{P} + n \hat{M}^{n-1},
\]

which eventually yields
\[
\hat{P} \hat{M}^n 1 = \left( \hat{M}^n \hat{P} + n \hat{M}^{n-1} \right) 1 = \hat{M}^n \hat{P} 1 + n \hat{M}^{n-1} 1. \tag{5}
\]

Being \( \hat{M}^n \hat{P} 1 = 0 \) as a consequence of the \( \text{ii}) \) of Eqs. (1), and using property \( \text{iii}) \) too, we state the correctness of Eq. (3). \( \square \)

Remark 1. The important point we like to convey is that the essence of the discussion on Monomiality is the existence of the operators \( \hat{M} \) (multiplicative), which univocally define the set of polynomials \( p_n(x) \) (not vice-versa), and \( \hat{P} \) acting on the polynomials as an ordinary derivative.

According to the above statement polynomial set like Appell, Sheffer [17], Boas Buck [18]... can be ascribed to the monomial family, while others like e. g. Legendre, Chebyshev, Jacobi... [19, 20] are not be framed within such a context.

After these remarks, aimed at clarifying the frame in which we are going to develop our speculations, we remind that the \( \hat{M}, \hat{P} \) operators, defining the Appell family, are defined by
\[
\hat{M} = x + \frac{A'(\sigma)}{A(\sigma)} \big|_{\sigma = \partial_x}, \quad \hat{P} = \partial_x, \tag{6}
\]

\(^1\)We remind that \([\hat{P}, \hat{M}] = 1 \Rightarrow \hat{P} \hat{M} - \hat{M} \hat{P} = 1\)
where $A(\sigma)$ is an analytic function.

According to our introductory remarks, the explicit form of the Appèl polynomials is obtained from the identity (property iii) of Eq. (1)

\[
a_n(x) = \left( x + \frac{A'(\partial_x)}{A(\partial_x)} \right)^n 1. \tag{7}
\]

The use of standard operational rules allows to cast Eq. (7) in a more convenient form.

**Corollary 1.** We note indeed that \[21, 22\]

\[
a_n(x) = \left( x + \frac{A'(\partial_x)}{A(\partial_x)} \right) \left( x + \frac{A'(\partial_x)}{A(\partial_x)} \right)^{n-1} 1 \tag{8}
\]

and noting that

\[
x + \frac{A'(\partial_x)}{A(\partial_x)} = A(\partial_x)x (A(\partial_x))^{-1}, \tag{9}
\]

we can write, by iteration

\[
a_n(x) = \left( A(\partial_x)x (A(\partial_x))^{-1} \right)^n = A(\partial_x)x^n \tag{10}
\]

According to Eq. (10), the generating function of Appèl polynomials reads

\[
\sum_{n=0}^{\infty} \frac{t^n}{n!} a_n(x) = A(\partial_x)e^{tx} = A(t)e^{tx}. \tag{11}
\]

It is evident that it consists of two contributions: the exponential term and $A(t)$, which will be defined as the “amplitude”.

**Corollary 2.** In the case of the two variable Hermite polynomials ($HP$), we have that the amplitude is specified by

\[
A(t) = e^{yt^2} \tag{12}
\]

with the multiplicative operator being explicitly defined by

\[
\hat{M} = x + 2y\partial_x \tag{13}
\]

The associated polynomial family is, accordingly, provided by \[26\]

\[
H_n(x, y) = (x + 2y\partial_x)^n 1. \tag{14}
\]

The use of the Crofton identity \[22\]

\[
e^{y\partial_x^m} f(x) = f \left( x + m \ y \partial_x^{m-1} \right) e^{y\partial_x^m} \tag{15}
\]

3
(or of identities $\text{(10)}$-$\text{(11)}$ as well) allows to cast Eq. $\text{(14)}$ in the form

$$H_n(x, y) = e^{y\partial_x^2}x^n.$$  \hfill (16)

The expansion of the exponential operator in Eq. $\text{(16)}$, along with the relevant action on the monomial $x^n$, yields the explicit form of the two variable Hermite polynomials $\text{[10]}$, namely

$$H_n(x, y) = n! \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{x^{n-2r}y^r}{(n-2r)!r!}. \hfill (17)$$

The operational identity in Eq. $\text{(16)}$ is particularly pregnant from the mathematical point of view. It states that the two variable Hermite $\text{(17)}$ are solutions of the heat equation and can be used as a pivotal tool to prove the orthogonal properties of this polynomial family $\text{[10]}$-$\text{[26]}$.

In this article we consider the polynomial family generated by

$$A(p) = \left(1 + \frac{y}{N^2}\partial_x^2\right)^N, \quad \forall N \in \mathbb{N},$$  \hfill (18)

study the relevant properties and look at the possibility of defining an associated orthogonal set.

2 Quasi-Hermite and Appéll Sequences

In this section we exploit the general properties of the Appéll polynomials, discussed in the introductory remarks, to state the properties of the associated polynomials.

**Definition 1.** Appéll polynomials with amplitude $\text{(18)}$, are explicitly defined$^2$ through the identity

$$H_n(x, y; N) = \left(1 + \frac{y}{N^2}\partial_x^2\right)^N x^n \hfill (19)$$

and they will be called Quasi-Hermite-Polynomials (QHP).

**Properties 2.** The relevant recurrences of QHP are obtained after noting that, for this specific case, we get

$$\frac{A'(\partial_x)}{A(\partial_x)} = \frac{2y\partial_x}{\left(1 + \frac{y}{N^2}\partial_x^2\right)}, \hfill (20)$$

$^2$Definition comes according to the discussion of the previous section. It should be noted that \[
\lim_{N \to \infty} H_n(x, y; N) = e^{y\partial_x^2}x^n = H_n(x, y).
\]
so

1) \( \partial_x H_n(x, y; N) = n \left( 1 + \frac{y}{N} \frac{\partial^2}{\partial x^2} \right)^N x^{n-1} = n H_{n-1}(x, y; N) \),

2) \( H_{n+1}(x, y; N) = \left( x + \frac{2y}{1 + \frac{y}{N} \frac{\partial^2}{\partial x^2}} \right) H_n(x, y; N) \),

3) \( H_{n+1}(x, y; N) - x H_n(x, y; N) - 2n y H_{n-1}(x, y; N) = \frac{y}{N} n (n - 1) (x H_{n-2}(x, y; N) - H_{n-1}(x, y; N)) \).

Proof. Properties 1) and 2) are obtained from the realization of the derivative and multiplicative operators given in Eqs. (6). About the third one, it is the result of some algebraic steps:

i) From property 2) we write

\[
\left( 1 + \frac{y}{N} \frac{\partial^2}{\partial x^2} \right) H_{n+1} = \left( 1 + \frac{y}{N} \frac{\partial^2}{\partial x^2} \right) x H_n + 2y \partial_x H_n
\]

which provides, from property 1),

ii)

\[
H_{n+1} + \frac{y}{N} n(n+1) H_{n-1} = x H_n + \frac{y}{N} (2n H_{n-1} + n(n - 1) x H_{n-2}) + 2n y H_{n-1}
\]

and finally

iii)

\[
H_{n+1} - x H_n - 2n y H_{n-1} = \frac{y}{N} n ((2 - (n + 1)) H_{n-1} + (n - 1) x H_{n-2}) = \frac{y}{N} n (n - 1) (x H_{n-2} - H_{n-1}) .
\]

Proposition 1. The explicit form of the QHP is inferred from Eq. (19), which yields

a) \( H_n(x, y; N) = \sum_{r=0}^{\min[N, \lfloor \frac{N}{2} \rfloor]} \binom{N}{r} \left( \frac{y}{N} \right)^r \frac{n!}{(n-2r)!} x^{n-2r} \), \( \forall x, y \in \mathbb{R}, \forall n, N \in \mathbb{N} \) \label{eq:22}

and the relevant differential equation is

b) \( \left( x + \frac{2y \partial_x}{1 + \frac{y}{N} \frac{\partial^2}{\partial x^2}} \right) \partial_x H_n(x, y; N) = n H_n(x, y; N) \). \label{eq:23}

\footnote{We simplify the writing for brevity by omitting the arguments of the Hermite’s.}
Proof. a) \( \forall x, y \in \mathbb{R}, \forall n, N \in \mathbb{N} \), we use binomial Newton to write

\[
H_n(x, y; N) = \sum_{r=0}^{N} \binom{N}{r} \left( \frac{y}{N} \right)^r x^r y^{n-r} = \sum_{r=0}^{\min\{N, \lfloor \frac{n}{2} \rfloor \}} \binom{N}{r} \left( \frac{y}{N} \right)^r \frac{n!}{(n-2r)!} x^{n-2r}.
\]

b) The relevant differential equation is easily obtained by applying Eqs. (21) in Properties 2.

**Corollary 3.** After a few algebraic manipulations, Eq. (23) can be reduced to the following third order ODE

\[
\frac{y}{N} x z''' + y \left( 2 - \frac{n-2}{N} \right) z'' + x z' = nz, \quad z = H_n(x, y; N) \tag{24}
\]

which, evidently, tends to the ordinary (two variables) Hermite equation, for large \( N \) values.

**Proof.** By starting from Eq. (23) we write

\[
\left(1 + \frac{y}{N} \partial_x^2\right) x + 2y \partial_x \partial_x z = \left(1 + \frac{y}{N} \partial_x^2\right)nz \rightarrow \\
x \partial_x z + \frac{y}{N} \partial_x^2 \partial_x z + 2y \partial_x^2 z = nz + \frac{y}{N} \partial_x^2 nz \rightarrow \\
x z' + \frac{y}{N} \partial_x (z' + x z'') + 2y z'' - \frac{y}{N} nz'' = nz \rightarrow \\
x \frac{y}{N} x z''' + y \left( 2 - \frac{n-2}{N} \right) z'' + x z' = nz
\]

**Corollary 4.** The PDE satisfied by the QHP (expected to be an extension of the heat equation) is obtained by keeping the partial derivative with respect to \( y \) of both sides of Eq. (19), namely

\[
\partial_y H_n(x, y; N) = \partial_x^2 \left(1 + \frac{y}{N} \partial_x^2\right)^{N-1} x^n. \tag{25}
\]

**Example 1.** Eq. (25) can eventually be written as

\[
\begin{align*}
\partial_y H_n(x, y; N) &= \frac{\partial^2}{1 + \frac{y}{N} \partial_x^2} H_n(x, y; N) \\
H_n(x, 0; N) &= x^n
\end{align*} \tag{26}
\]

The relevant (formal) solution can be obtained as

\[
H_n(x, y; N) = \hat{U}_{y,N} x^n, \quad \hat{U}_{y,N} = \exp \left\{ \int_0^y \frac{\partial^2}{1 + \frac{y}{N} \partial_x^2} d\xi \right\}. \tag{27}
\]
where $\hat{U}$ is a kind of evolution operator. To be eventually written as in Eq. (19), after explicitly working out the integral in the exponent of Eq. (27), we find

$$\hat{U}_{y,N} = \exp \left\{ N \log \left( 1 + \frac{y}{N} \partial_x^2 \right) \right\} = \left( 1 + \frac{y}{N} \partial_x^2 \right)^N. \quad (28)$$

According to the previous definition, the QHP satisfies the composition rule

$$\hat{U}_{y,N} \hat{U}_{z,N} = \left( 1 + \frac{y + z}{N} \partial_x^2 + \frac{yz}{N^2} \partial_x^4 \right)^N. \quad (29)$$

Therefore, unlike the two variables HP specified by an amplitude that is an exponential, the composition property $\hat{U}_{y,N} \hat{U}_{z,N} = \hat{U}_{y+z,N}$ does not hold, therefore

$$\hat{U}_{y,N} \hat{U}_{z,N} \neq \hat{U}_{y+z,N}. \quad (30)$$

An important (albeit naïve) consequence of Eq. (29) is the following composition rule

$$\hat{U}_{-y,N} \hat{U}_{y,N} x^n = \left( 1 - \frac{y^2}{N^2} \partial_x^4 \right)^N x^n, \quad (31)$$

which suggests the necessity of a suitable extension of QHP, possibly involving higher order forms, as discussed in the forthcoming section.

**Observation 1.** The non exponential nature of the QHP amplitude determines the further worth to be noted consequence

$$\hat{U}_{-y,N} \neq \hat{U}_{-1y,N} = \frac{1}{\Gamma(N)} \int_0^\infty \left( N e^{-s(1 + \frac{y}{N} \partial_x^2)} \right) ds, \quad (32)$$

where the r.h.s. has been obtained after exploiting standard Laplace transform methods.

We will see in the following that Eq. (32) is of pivotal importance for the definition of the orthogonal properties of the QHP.

**Observation 2.** Before closing this section, we notice that Eq. (23) can be generalized $\forall m \in \mathbb{N}$ such that

$$\left( x + \frac{m y \partial_x^m}{1 + \frac{y}{N} \partial_x^2} \right) \partial_x H_n^{(m)}(x, y; N) = n H_n^{(m)}(x, y; N). \quad (33)$$

and, by following the same procedure provided in the Corollary, it is possible to deduce the relative differential equantion.

### 3 Multivariable QHP

Higher order Hermite polynomials (also called Lacunary HP) are defined through the operational rule

$$H_n^{(m)}(x, y) = e^{y \partial_x^m} x^n, \quad \forall m \in \mathbb{N} \quad (34)$$
and, in analogy, the Higher order QHP are specified by

\[ H_n^{(m)}(x, y; N) = \hat{U}_{y,N}^{(m)} x^n, \quad \hat{U}_{y,N}^{(m)} = \left( 1 + \frac{y}{N} \partial_x^m \right)^N. \quad (35) \]

**Example 2.** According to Eq. (31), we find

\[ \hat{U}_{-y,N} \hat{U}_{y,N} x^n = \hat{U}_{-y^2,N} x^n = H_n^{(4)}(x, -y^2; N) \quad (36) \]

and, more in general,

\[ \hat{U}_{-y,N} \hat{U}_{y,N} x^n = \hat{U}_{-y^2,N} x^n = H_n^{(2m)}(x, -y^2; N) \quad (37) \]

**Example 3.** Before going further, we consider the definition of the QHP of order one, which will be referred as Quasi Binomial Polynomials (QBP), namely

\[ H_n^{(1)}(x, y; N) = \left( 1 + \frac{y}{N} \partial_x \right)^N x^n. \quad (38) \]

For large \( N \) they reduce to \( (x + y)^n \) hence the name. The explicit form of this family of polynomials, writes

\[ H_n^{(1)}(x, y; N) = \sum_{r=0}^{N} \binom{N}{r} \left( \frac{y}{N} \right)^r \partial_x^r x^n = \sum_{r=0}^{\min\{N, n\}} \binom{N}{r} \left( \frac{y}{N} \right)^r \frac{n!}{(n-r)!} x^{n-r}. \quad (39) \]

The same strategy adopted in Corollary 3 by exploiting Eq. (33), yields for the QBP the ODE

\[ \frac{y}{N} xz'' + [(x + y) - (n - 1) \frac{y}{N}] z' = nz, \quad z = H_n^{(1)}(x, y; N) \quad (40) \]

and the PDE

\[ \begin{cases} 
\partial_y F(x, y) = \frac{\partial_x}{1 + \frac{y}{N} \partial_x} F(x, y) \\
F(x, 0) = x^n 
\end{cases} \quad (41) \]

The last identity can also be cast in the integro-differential form

\[ \partial_y F(x, y) = \partial_x \int_0^\infty e^{-s} F \left( x - \frac{y}{N} s, y \right) ds \quad (42) \]

indeed, the Laplace transform provides the integral representation

\[ \frac{1}{1 + \frac{y}{N} \partial_x} = \int_0^\infty e^{-s(1 + \frac{y}{N} \partial_x)} ds \quad (43) \]

which, one inserted in Eq. (41), yields

\[ \partial_y F(x, y) = \partial_x \int_0^\infty e^{-s} e^{-\frac{y}{N} \partial_x} ds F(x, y) \quad (44) \]

and, after exploiting the shift operator identity \( e^{a \partial_x} f(x) = f(x + a) \) [10], we obtain Eq. (42).
Example 4. We can combine the various definition given before to introduce three variables QHP as

\[ H_n(x, y_1, N) = \left( 1 + \frac{y_1}{N}\partial_x \right)^N \left( 1 + \frac{y_2}{N}\partial_x \right)^N x^n = \binom{N}{r} \frac{y_1^r}{N^r} \frac{H_{n-r}(x, y_2; N)}{(n-r)!}. \]  
(45)

Further generalizations can easily be obtained. For example the m-th variable extension reads

\[ H_n^{(1,...,m)}(x, y_1, \ldots, y_m; N) = \left( \prod_{s=1}^{m} \hat{U}_{ys; N} \right) x^n. \]  
(46)

The examples we have just touched on in this section yields an idea of the possible extensions of this family of polynomials, which will be more carefully discussed in a forthcoming research.

4 Final Comments

We have already mentioned the possible orthogonal nature of the QHP, in this section we address the problem by the use of the techniques developed in Refs. 24 25.

Proposition 2. We assume that a given function \( f(x) \) can be expanded in terms of QHP, according to the identity

\[ f(x) = \sum_{n=0}^{\infty} a_n H_n(x, -|y|; N) \]  
(47)

which can be inverted, thus yielding

\[ \frac{1}{\left( 1 - \frac{|y|}{N}\partial_x^2 \right)^N} f(x) = \sum_{n=0}^{\infty} a_n x^n. \]  
(48)

The use of Eq. (32) allows to cast the l.h.s. of Eq. (48) in the form

\[ \frac{1}{\Gamma(N)} \int_0^{\infty} s^{N-1} e^{-s(1+\frac{|y|}{N}\partial_x^2)} ds f(x) = \sum_{n=0}^{\infty} a_n x^n. \]  
(49)

The reasons of “−|y|” will be clarified below.
Corollary 5. Eq. [49] can be so further elaborated:

1. We apply the Gauss-Weierstrass transform \[26\] to write

\[
e^{\frac{|y|}{N} \partial^2_x} f(x) = \frac{1}{2 \sqrt{\pi} s \frac{|y|}{N}} \int_{-\infty}^{\infty} \exp \left\{ -\frac{(x - \xi)^2}{4 s \frac{|y|}{N}} \right\} f(\xi) d\xi =
\]

\[
= \frac{1}{2 \sqrt{\pi} s \frac{|y|}{N}} \int_{-\infty}^{\infty} e^{-\frac{1}{4 s \frac{|y|}{N}} \xi^2} e^{\frac{s}{4 s \frac{|y|}{N}}} e^{-\frac{x^2}{4 s \frac{|y|}{N}}} d\xi.
\]

It holds for \( s \frac{|y|}{N} \geq 0 \) (hence the choice of the sign in the expansion [47]).

2. We use the two variable Hermite generating function \[10\] to write

\[
e^{\frac{|y|}{N} \partial^2_x} f(x) = \frac{1}{2 \sqrt{\pi} s \frac{|y|}{N}} \sum_{n=0}^{\infty} \frac{x^n}{n!} \int_{-\infty}^{\infty} H_n \left( \frac{\xi}{2 s \frac{|y|}{N}}, -\frac{1}{4 s \frac{|y|}{N}} \right) e^{-\frac{\xi^2}{4 s \frac{|y|}{N}}} f(\xi) d\xi.
\]

3. We insert the result of Eq. [51], in Eq. [49] and compare the like \( x \) powers, thus eventually finding

\[
a_n = \frac{1}{\Gamma(N)n!} \int_{0}^{\infty} s^{N-\frac{3}{2}} e^{-s} n G_{y,N}(s) ds,
\]

\[
n G_{y,N}(s) = \int_{-\infty}^{\infty} H_n \left( \frac{\xi}{2 s \frac{|y|}{N}}, -\frac{1}{4 s \frac{|y|}{N}} \right) e^{-\frac{\xi^2}{4 s \frac{|y|}{N}}} f(\xi) d\xi.
\]

According to the above results the expansion holds only if the integrals appearing in Eq. [52] are converging. In order to provide an example we consider the generalization of the Glaisher formula \[27\]. Namely Eq. [48], for \( f(x) = e^{-x^2} \), becomes

\[
F(x, y; N) = \frac{1}{\Gamma(N)} \int_{0}^{\infty} s^{N-1} e^{-s} \left( 1 - \frac{|y|}{N} \partial^2_x \right) e^{-x^2} ds =
\]

\[
= \frac{1}{\Gamma(N)} \int_{0}^{\infty} s^{N-1} e^{-s} \frac{e^{-\frac{s^2}{1 + \frac{4 |y|}{N}}}}{\sqrt{1 + \frac{4 |y|}{N} s}} ds.
\]

\[\text{\textsuperscript{6}}\text{We have } \sum_{n=0}^{\infty} \frac{x^n}{n!} H_n(x, y) = e^{x+y^2}.\]
For very large $N$, \( \frac{1}{N}\frac{\partial^2}{\partial x^2} \) reduces to the ordinary Glaisher identity

\[
\lim_{N \to \infty} F(x, y; N) = e^{y\partial_x^2}e^{-x^2} = \frac{1}{\sqrt{1 + 4y}}e^{-y^2/4y}.
\]

In Figs. we have reported $F(x, y; N)$ vs. $x$ for different values of $N$ and $y$.

The definition of higher order $(QHP)$ is not unique and another possibility is offered by the relation

\[
H_n^{(q,p)}(x, y, z; N) = \left(1 + \frac{y}{N}\frac{\partial^2}{\partial x^2} + \frac{z}{N}\frac{\partial^2}{\partial y^2}\right)^N x^n
\]

where $q < p$ are relatively prime integers. The definition in Eq. (55) allows to write the composition identity as

\[
\hat{U}_{y,N}\hat{U}_{z,N}x^n = H_n^{(2,4)}(x + yz; N)^N x^n.
\]

In this paper we have gone through different aspects of the theory of Hermite polynomials, which has allowed the introduction of a family of Quasi Hermite polynomials. The relevant properties have been studied with the help of the formalism of Monomiality.

In a forthcoming paper we will extend the present analysis to the definition of two variables Quasi Laguerre polynomials.

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