Thermodynamics, phase transitions and Ruppeiner geometry
for Einstein-dilaton Lifshitz black holes
in the presence of Maxwell and Born-Infeld electrodynamics

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In this paper, we first obtain the higher dimensional dilaton-Lifshitz black hole solutions in the presence of Born-Infeld (BI) electrodynamics. We find that there are two different solutions for \(z = n + 1\) and \(z \neq n + 1\) cases where \(z\) is dynamical critical exponent and \(n\) is the number of spatial dimensions. Calculating the conserved and thermodynamical quantities, we show that the first law of thermodynamics is satisfied for both cases. Then, we turn to study different phase transitions for our Lifshitz black holes. We start with Hawking-Page phase transition and explore the effects of different parameters of our model on it for both linearly and BI charged cases. After that, we discuss the phase transitions inside the black holes. We present the improved Davies quantities and prove that the phase transition points shown by them are in coincident with Ruppeiner ones. We show that the zero temperature phase transitions are transitions on radiance properties of black holes by using Landau-Lifshitz theory of thermodynamic fluctuations. Next, we turn to study Ruppeiner geometry (thermodynamic geometry) for our solutions. We investigate thermal stability, interaction type of possible black hole molecules and phase transitions of our solutions for linearly and BI charged cases separately. For linearly charged case, we show that there are no phase transition at finite temperature for the case \(z \geq 2\). For \(z < 2\), it is found that the number of finite temperature phase transition points depends on the value of black hole charge and is not more than two. When we have two finite temperature phase transition points, there are no thermally stable black hole between these two points and we have discontinues small/large black hole phase transitions. As expected, for small black holes, we observe finite magnitude for Ruppeiner invariant which shows the finite correlation between possible black hole molecules while for large black holes, the correlation is very small. Finally, we study the Ruppeiner geometry and thermal stability of BI charged Lifshitz black holes for different values of \(z\). We observe that small black holes are thermally unstable in some situations. Also, the behavior of correlation between possible black hole molecules for large black holes is the same as linearly charged case. In both linearly and BI charged cases, for some choices of parameters, the black hole systems behave like a Van der Waals gas near transition point.

I. INTRODUCTION

It has been over forty years since Bekenstein and Hawking first disclosed that black hole can be considered as a thermodynamic system, with characteristic temperature and entropy [1–4]. Taking into account the fact that black holes have no hair, there are no classical degrees of freedom to account for such thermodynamic properties. It is a general belief that thermodynamic properties of a system may reflect the statistical mechanics of underlying relevant microscopic degrees of freedom. But the detailed nature of these microscopic gravitational states has remained as a mystery. The Bekenstein-Hawking entropy, \(S = A/(4\hbar G)\), depends on both Planck’s constant as well as Newtonian gravitational constant, implying that thermodynamics of black holes may relate quantum mechanics and gravity. Recently, there have been some progresses on understanding the microscopic degrees of freedom of the black hole entropy, for example in string theory [5–7] as well as loop quantum gravity [8–10]. But the accounts of the black hole entropy are not complete and they only work within some particular models and some special domains where string theory and loop quantum gravity can apply. Besides, despite counting very different states, many inequivalent approaches to quantum gravity obtain identical results and it is not clear why any counting of microstates should...
reproduce the same Bekenstein-Hawking entropy [11]. The statistical mechanical description of black hole entropy is still not elegant.

On the other side, black hole can be heated or cooled through absorption and evaporation processes. According to Boltzmann’s insight, if a system can be heated, it must have microscopic structures. Recently, in [12], possible microscopic structures of a charged anti-de Sitter black hole have been studied and some kind of interactions between possible micromolecules have been investigated by an interesting physical tool, the Ruppeiner geometry. Derived from the thermodynamic fluctuation theory, the Ruppeiner geometry [13, 14] is considered powerful to explore the possible interactions between black hole microscopic structures. The sign of the Ruppeiner invariant $R$ (the Ricci scalar of Ruppeiner geometry) was argued to be useful for identifying the physical systems similar to the Fermi (Bose) ideal gas when $R > 0$ ($R < 0$) or the classical ideal gas when $R = 0$ [15]. Besides, the sign of the Ruppeiner invariant $R$ can further be used to interpret the type of the dominated interaction between molecules of a thermodynamic system. When $R > 0$, there is a repulsive interaction between molecules, when $R < 0$ the interaction is attractive, and for $R = 0$ there is no interaction in the microstates [16–18]. Moreover, the magnitude of the Ruppeiner invariant $|R|$ measures the average number of correlated Planck areas on the event horizon for a black hole system [19]. For a review on the description of the Ruppeiner geometry in black hole systems, we refer to [20, 21] and references therein. Further studies on molecular interactions of black holes, based on the Ruppeiner geometry, have been carried out in [12, 22, 23].

Phase transition is another interesting topic in black holes thermodynamics. Davies discussed thermodynamic phase transition of the black holes by looking at the behavior of the heat capacity [24–26]. He claimed that the discontinuity of the heat capacity marks the second order phase transition in black holes. However, it was argued that physical properties do not show any speciality at this discontinuity point if compared with other heat capacity values, for example the regularity of the event horizon is not lost and the black hole internal state remains uninfluenced [27]. Thus, it is hard to accept the discontinuity point of the heat capacity as a true physical point of the phase transition. Employing the Landau-Lifshitz theory of thermodynamic fluctuations [28, 29], Pavon and Rubi gave a deep understanding of the black hole phase transition [30, 31]. They found that some second moments in the fluctuation of relevant thermodynamic quantities diverge when the black hole becomes extreme. This divergence shows that the thermodynamic fluctuation is tremendous and the rigorous meaning of thermodynamical quantities is broken down. This is exactly the characteristic of the thermodynamic phase transition point. At this phase transition point, the Hawking temperature is zero which indicates that for the extreme black hole there is only super-radiation but no Hawking radiation, which is in sharp difference from that of the non-extreme black holes. Black holes phase transition in the context of Landau-Lifshitz theory have been investigated in [32, 33]. Recently, further differences in dynamical properties before and after the black hole thermodynamical phase transition has been disclosed in [34–36]. A question now arises: how we can further understand this macroscopic thermodynamic phase transition in black hole physics? for example whether there is a microscopic explanation of this thermodynamic phase transition. The Ruppeiner geometry is a possible tool we can use to investigate the thermodynamic phase transitions from microscopic point of view. This method is safer to determine true phase transitions than other methods since regardless of microscopic model, $R$ has a unique status in identifying microscopic order (which is at foundation of phase transitions at microscopic level) from thermodynamics [20, 21]. Some attempts, in this direction, have been reported in [37–48]. In a recent work [40], it was found that the divergence of the Ruppeiner invariant coincides with the critical point in the phase transition in a holographic superconductor model. It is interesting to investigate whether the Ruppeiner geometry [20, 21] can present us further reason to determine which of the thermodynamical discussions mentioned above is valid for describing the thermodynamical phase transition. In particular, we would like to explore whether the Davies phase transition conjecture can reflect some special properties in microstructures and be in consistent with the Ruppeiner geometry description. If the Davies conjecture does not have the microscopic explanation, we will further think about how to improve the Davies conjecture to describe the black hole phase transition.

We will employ the black hole in Lifshitz spacetime as a configuration to study our physical problems mentioned above. This spacetime was first introduced in [49], which respects the anisotropic conformal transformation $t → \Lambda^z t$, $\vec{x} → \Lambda \vec{x}$, where $z$ is dynamical critical exponent. For the Lifshitz spacetime, it is necessary to include some matter sources such as massive gauge fields [50–54] or higher-curvature corrections [55] to guarantee the asymptotic behavior of the Lifshitz black hole. It is difficult to find an analytic Lifshitz black hole solution for arbitrary $z$, although some attempts have been performed [56]. This makes the discussion of thermodynamics for such a black hole difficult. Fortunately, in Einstein-dilaton gravity with a massless gauge field, it is possible to find an exact Lifshitz black solution for arbitrary $z$ [57, 58]. This model is suggested in the low energy limit of string theory [59]. While thermodynamical behaviors of uncharged and charged Einstein-dilaton-Lifshitz black holes have been revealed in [57, 60] and [58], respectively, thermodynamics of uncharged Gauss-Bonnet-dilaton-Lifshitz solution has been studied in [61]. It is also interesting to study Lifshitz black hole solutions in the presence of other gauge fields such as the power-law Maxwell field [62], the logarithmic [63] and exponential [64] nonlinear electrodynamics. For example, thermodynamics and thermal and dynamical stabilities of Einstein-dilaton-Lifshitz solutions in the presence of power-law Maxwell field
have been studied in \cite{65}. In the context of AdS/CFT \cite{66-68} application, the electrical conductivity were explored for exponentially \cite{69} and logarithmic \cite{70} charged Lifshitz solutions. In the present work, we shall consider the Born-Infeld (BI) nonlinear electrodynamics in the context of Einstein-dilaton-Lifshitz black holes. The motivation for considering BI-dilaton action comes from the fact that dynamics of D-branes and some soliton solutions of supergravity is governed by the Born-Infeld (BI) action \cite{71-76}. Besides, the low energy limit of open superstring theory suggest the BI electrodynamics action coupled to dilaton field \cite{71-73}. It is surprising to mention that, many years before the appearance of BI action in superstring theory, in 1930’s, this nonlinear electrodynamics was introduced for the first time, with the aim of solving the infinite self-energy problem of a point-like charged particle by imposing a maximum strength for the electromagnetic field \cite{77}.

In this paper, we will first look for a general \((n+1)\)-dimensional Lifshitz black hole solution in the context of Einstein-dilaton gravity in the presence of BI electrodynamics. We will show that the general metric function has different solutions for \(z = n + 1\) and \(z \neq n + 1\) cases. It is important to note that the difference in the metric function has not been observed in the previous studies on Lifshitz-dilaton black holes \cite{65, 69, 70}. Based on this general solution, we will study thermodynamics of Lifshitz-dilaton black holes coupled to linear Maxwell field and BI nonlinear electrodynamics. We will disclose that the Hawking-Page phase transition \cite{78} exists both in the presence of linear and nonlinear electrodynamics. There are some attempts on study phase transitions of uncharged Lifshitz solutions for fixed \(z\) \cite{79} or in three \cite{80} and four \cite{81} dimensions. The disclosed Hawking-Page phase transition in this paper is interesting, since it depends on different values of \(z\) in different spacetime dimensions in the presence of linear Maxwell and nonlinear BI electric and magnetic fields. We will further concentrate our attention to understand the thermodynamic phase transition from microstructures. We shall examine the relation between the Ruppeiner geometry and thermodynamical descriptions of the phase transition such as the Davies conjecture and the Landau-Lifshitz method. We try to give more microscopic understanding of the thermodynamical phase transitions in the black hole system. We explore the thermodynamic geometry (Ruppeiner geometry) for linearly and nonlinearly charged Lifshitz solutions separately and disclose the properties of interactions between possible black hole molecules. Up to our best knowledge, there is no study of thermodynamic geometry on Lifshitz solutions in literature. Interestingly enough, by studying Ruppeiner geometry, we have found that our solutions show the Van der Waals like behavior near critical point in some cases.

The layout of the paper is as follows. In the next section, we provide the basic field equations and obtain the BI charged Lifshitz-dilaton black hole solutions. In section \(\text{III}\), we first explore the satisfaction of the thermodynamics first law for Lifshitz-dilaton black holes in the presence of BI electrodynamics. Then, we study different phase transitions including Hawking-Page phase transition and phase transition at zero temperature for linearly and BI charged cases. In section \(\text{IV}\), we investigate thermodynamic geometry of the obtained solutions for linearly and nonlinearly BI charged cases by adopting the Ruppeiner approach. We finish with summary and closing remarks in section \(\text{V}\).

\section{Action and Asymptotic Lifshitz Solutions}

In this section, we intend to obtain exact \((n+1)\)-dimensional dilaton-Lifshitz black holes in the presence of BI nonlinear electrodynamics. Our ansatz for the line elements of the spacetime is \cite{58, 82}

\begin{equation}
 ds^2 = -\frac{r^{2z} f(r)}{r^{2z}} dr^2 + \frac{l^2 dr^2}{r^2 f(r)} + r^2 d\Omega^2_{n-1}, \tag{1}
\end{equation}

where \(z(\geq 1)\) is dynamical critical exponent and

\[ d\Omega^2_{n-1} = d\theta_1^2 + \sum_{i=2}^{n-1} d\theta_i^2 \prod_{j=1}^{i-1} \sin^2(\theta_j), \]

is an \((n-1)\)-dimensional hypersurface with constant curvature \((n-1)(n-2)\) and volume \(\omega_{n-1}\). As \(r \to \infty\), the line elements (1) reduce asymptotically to the Lifshitz spacetime,

\begin{equation}
 ds^2 = -\frac{r^{2z} dt^2}{l^{2z}} + \frac{l^2 dr^2}{r^2} + r^2 d\Omega^2_{n-1}. \tag{2}
\end{equation}

On the other side, as it is pointed out above, we would like to consider BI nonlinear electrodynamics. In the absence of dilaton field, BI Lagrangian density is written as \cite{77}

\begin{equation}
 L = 4\beta^2 \left( 1 - \sqrt{1 + \frac{F}{2\beta^2}} \right), \tag{3}
\end{equation}

where \(\beta\)
where $\beta$ is the Born-Infeld parameter related to the Regge slope $\alpha'$ as $\beta = 1/(2\pi \alpha')$. $F = F_{\mu\nu}F^{\mu\nu}$ is Maxwell invariant in which $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$ where $A_{\mu}$ is electromagnetic potential. One of the effects of presence of dilaton field is its coupling with electromagnetic field. Thus, in the presence of dilaton field we deal with a modified form for BI Lagrangian density including its coupling with dilaton scalar field $\Phi$ [83, 84]

$$L(F, \Phi) = 4\beta^2 e^{4\lambda \Phi/(n-1)} \left( 1 - \sqrt{1 + \frac{e^{-8\lambda \Phi/(n-1) F}}{2\beta^2}} \right),$$

(4)

where $\lambda$ is a constant. The Lagrangian density of string-generated Einstein-dilaton model [59] with two Maxwell gauge fields [58] in the presence of BI electrodynamics can be written in Einstein frame as

$$\mathcal{L} = \frac{1}{16\pi} \left\{ \mathcal{R} - \frac{4}{n-1}(\nabla \Phi)^2 - 2\Lambda - \sum_{i=1}^{2} e^{-4\lambda \Phi_{i}/(n-1)} H_i + L(F, \Phi) \right\},$$

(5)

where $\mathcal{R}$ is Ricci scalar and $\Lambda$ and $\lambda_i$’s are some constants. In Lagrangian (5), $H_i = (H_i)_{\mu\nu} (H_i)^{\mu\nu}$ where $(H_i)_{\mu\nu} = \partial_{\mu} (B_i)_{\nu}$ and $(B_i)_{\mu}$ is gauge potential. In the large $\beta$ limit, $\mathcal{L}$ recovers the Einstein-dilaton-Maxwell Lagrangian in its leading order [58, 65]

$$\lim_{\beta \to \infty} 16\pi \mathcal{L} = \cdots - e^{-4\lambda \Phi/(n-1) F} + \frac{e^{-12\lambda \Phi/(n-1) F^2}}{8\beta^2} + O \left( \frac{1}{\beta^4} \right).$$

(6)

Varying the action $S = \int_M d^{n+1}x \sqrt{-g} \mathcal{L}$ with respect to the metric $g_{\mu\nu}$, the dilaton field $\Phi$ and electromagnetic potentials $A_{\mu}$ and $(B_i)_{\mu}$, leads us to the following field equations

$$\mathcal{R}_{\mu\nu} - \frac{g_{\mu\nu}}{n-1} \left\{ 2\Lambda + 2L_F F - L(F, \Phi) - \sum_{i=1}^{2} e^{-4\lambda \Phi_{i}/(n-1)} H_i \right\}$$

$$- \frac{4}{n-1} \partial_{\mu} \Phi \partial_{\nu} \Phi + 2L_F F_{\mu\lambda} F_{\nu}^{\lambda} - 2 \sum_{i=1}^{2} e^{-4\lambda \Phi_{i}/(n-1)} (H_i)_{\mu\lambda} (H_i)_{\nu}^{\lambda} = 0,$$

(7)

$$\nabla^2 \Phi + \frac{n-1}{8} L_\Phi + \sum_{i=1}^{2} \frac{\lambda_i}{2} e^{-4\lambda_i \Phi/(n-1)} H_i = 0,$$

(8)

$$\nabla_{\mu} (L_F F^{\mu\nu}) = 0,$$

(9)

$$\nabla_{\mu} \left( e^{-4\lambda \Phi/(n-1)} (H_i)^{\mu\nu} \right) = 0,$$

(10)

where we use the convention $X_Y = \partial X/\partial Y$. Using the metric ansatz (1), electromagnetic field equations (9) and (10) can be solved immediately as

$$F_{rt} = \frac{q\beta e^{4\lambda \Phi/(n-1)} r^{z-n}}{\Upsilon},$$

(11)

$$(H_i)_{rt} = \frac{q\beta e^{4\lambda \Phi/(n-1)}}{r^{z-n}},$$

(12)

where $\Upsilon = \sqrt{1 + q^2 r^{2z-2}/(\beta^2 r^{2z-2})}$, and $\Phi(r)$ can be obtained by subtracting $(tt)$ and $(rr)$ components of Eq. (7) and solving the resulting equation. We find

$$\Phi(r) = \frac{(n-1)\sqrt{z-1}}{2} \ln \left( \frac{r}{\beta} \right).$$

(13)

Substituting Eqs. (11), (12) and (13) in field equations (7) and (8), one can solve the equations for $f(r)$ to obtain

$$f(r) = \begin{cases} 
1 - \frac{m}{r^{n-z-1}} + \frac{(n-2)^2 r}{(n+z-3) r^{n+z-2}} + \frac{4\beta^2 \beta^2 r^{2z-2}}{(n-1)(n+z-1)} - \frac{4\beta^2 \beta^2 r^{2z-2}}{(n-1)(n+z-1)} \int \Upsilon r^{n-z} dr, & \text{for } z \neq n+1, \\
1 - \frac{m}{r^{n-z-1}} + \frac{(n-2)^2 r}{4(n-1) r^{2z}} - \frac{4\beta^2 \beta^2 r^{2z}}{4(n-1) r^{2z+2}} \left[ 1 - \Upsilon + (\frac{1+\Upsilon}{2}) \right], & \text{for } z = n+1,
\end{cases}$$

(14)
where we should set
\[
\lambda = -\sqrt{z-1}, \quad \lambda_1 = \frac{n-1}{\sqrt{z-1}}, \quad \lambda_2 = \frac{n-2}{\sqrt{z-1}},
\]
\[
q_1^2 = -\frac{\Lambda (z-1)^2}{(z+n-2)(2z(n-1))}, \quad q_2^2 = \frac{(n-1)(n-2)(z-1)^2}{2(z+n-3)(2z(n-1))},
\]
\[
\Lambda = -\frac{(n+z-1)(n+z-2)}{2l^2}, \quad \beta = \frac{1}{\sqrt{z-1}}, \quad \beta' = \frac{1}{\sqrt{z-1}},
\]
so that the field equations are fully satisfied. In the solution (14), m is a constant which is related to the total mass of black brane as we will see in next section. The integral of the last term of \( f(r) \) for \( z \neq n+1 \) can be done in terms of hypergeometric function. Thus, \( f(r) \) can be written as
\[
f(r) = 1 - \frac{m}{r^{n+z-1}} + \frac{(n-2)^2}{(n+z-3)^2} + \frac{4b^2(2-2z)^2(1-\gamma)}{(n-1)(n-1+z-2)} + \frac{4q^2b^2(2z-2z-\gamma)}{(n+z-3)(n-1+z-2)} \text{ and } O\left(\frac{1}{r}\right), \quad \text{for } z \neq n+1,
\]
\[
1 - \frac{m}{r^{n+z-1}} + \frac{(n-2)^2}{(n+z-3)^2} + \frac{2q^2b^2(2z-2z-\gamma)}{(n-1)(n-1+z-2)} + O\left(\frac{1}{r}\right), \quad \text{for } z = n+1.
\]
(15)

Note that solution (16) obviously satisfies the fact that \( f(r) \to 1 \) as \( r \to \infty \) (note that \( F(a, b, c, 0) = 1 \)). The behavior of \( f(r) \) for large \( \beta' \) is
\[
\begin{align*}
1 - \frac{m}{r^{n+z-1}} + \frac{(n-2)^2}{(n+z-3)^2} + \frac{2q^2b^2(2z-2z-\gamma)}{(n-1)(n-1+z-2)} + O\left(\frac{1}{\beta'}\right),
\end{align*}
\]
(17)

which reproduces the result of [65] for every \( z \) in linear Maxwell case. The behaviors of the metric function for \( z = n+1 \) and \( z \neq n+1 \) have been depicted in Figs. 1(a) and 1(b) respectively. It is notable to mention that in the case of \( z = n+1 \), there is no Schwartzchild-like black hole since in this case \( f(r) \) goes to positive infinity as \( r \) goes to zero. However, for \( z \neq n+1 \), we may have Schwartzchild-like black hole (dash-dotted line in Fig. 1(a)) in addition to nonextreme (solid line) and extreme (dotted line) black holes and naked singularity (dashed line). For nonextreme case, there are two inner (Cauchy) and outer (event) horizons. In both Figs. 1(a) and 1(b), we see that the larger the nonlinearity parameter \( \beta \) is, the smaller the distance between two inner and outer horizons is so that for large enough \( \beta' \)s, we have just one horizon (extreme case) or naked singularities. The Schwartzchild-like case occurs for lower \( \beta' \)s in the case of \( z \neq n+1 \) as Fig. 1(a) shows.

As one can see in (17), the fourth term in expansions for both \( z = n+1 \) and \( z \neq n+1 \) cases reproduce the charge term of [65] in linear Maxwell case as one expects. The temperature of the black hole horizon can be obtained via
\[
T = \frac{1}{2\pi} \sqrt{-\frac{1}{2} \nabla b \chi_a \nabla b \chi^a} \bigg|_{r=r_+}
\]
(18)
hypergeometric function as
\[ T = \frac{r^{z+1}f'}{4\pi l^{z+1}} \bigg|_{r=r_+} = \frac{(n+z-1)r_+^{z}}{4\pi l^{z+1}} + \frac{(n-2)^2 l^{1-z}}{4\pi(n+z-3)r_+^{z}} + \frac{\beta^{2}b^{2z-2}r_{+}^{2-z}}{\pi(n-1)l^{z-1}}, \]  
(19)

where prime denotes the derivative with respect to \( r \) and \( \Upsilon_+ = \Upsilon (r = r_+) \). Temperature has the same formula (19) for both \( z = n+1 \) and \( z \neq n+1 \) cases. One can check that for large \( \beta \), (19) reduces to the temperature of Einstein-Maxwell-dilaton Lifshitz black holes [65], namely
\[ T = \frac{(n+z-1)r_+^{z}}{4\pi l^{z+1}} + \frac{(n-2)^2 l^{1-z}}{4\pi(n+z-3)r_+^{z}} + \frac{q^2 l^{z-1}b^{2z-2}}{2\pi(n-1)r_+^{2n+z-4}} + \frac{q^4 l^{3z-3}b^{2z-2}}{8\pi(n-1)r_+^{2n+z-6} \beta^2} + O \left( \frac{1}{\beta^4} \right). \]  
(20)

The entropy of the black holes can be calculated by using the area law of the entropy [2, 85, 86] which is applied to almost all kinds of black holes in Einstein gravity including dilaton black holes [87–90]. Therefore, the entropy of the black brane per unit volume \( \omega_{n-1} \) becomes
\[ S = \frac{r_+^{n-1}}{4}. \]  
(21)

Having Eqs. (11), (13) and (15) at hand, we can find electromagnetic gauge potential \( A_t = \int F_{t\tau} dr \) in terms of hypergeometric function as
\[ A_t (r) = -\frac{qb^{2z-2}}{(n+z-3)r^{n+z-3}} F \left( \frac{1}{2}, \frac{n+3-z}{2n-2}, \frac{3n+z-5}{2n-2}; 1 - Y^2 \right). \]  
(22)

The large \( \beta \) behavior of gauge potential is in agreement with [65]
\[ A_t (r) = -\frac{qb^{2z-2}}{(n+z-3)r^{n+z-3}} + \frac{q^3 b^{2z-2} l^{2z-2}}{(3n+z-5)r^{3n+z-5} \beta^2} + O \left( \frac{1}{\beta^4} \right). \]  
(23)

In next section, we will study thermodynamics of dilaton Lifshitz black holes in the presence of BI electrodynamics by seeking for satisfaction of thermodynamics first law through calculation of conserved and thermodynamic quantities. We also show that our Lifshitz solutions can exhibit the Hawking-Page phase transition. Then, we discuss the inside phase transitions of our Lifshitz black holes.

III. THERMODYNAMICS OF LIFSHITZ BLACK HOLES

A. First law of thermodynamics

This subsection is devoted to study the thermodynamics first law for Lifshitz-dilaton black hole solutions in the presence of BI nonlinear electrodynamics. As the first step, we calculate the fundamental quantity for thermodynamics discussions namely mass. For this purpose, we apply the modified subtraction method of Brown and York (BY) [91–93]. In order to use this method, the metric should be written in the form
\[ ds^2 = -X(R)dt^2 + \frac{dR^2}{Y(R)} + R^2 d\Omega_{n-1}^2. \]  
(24)

For our case, it is clear that \( R = r \) and thus
\[ X(R) = \frac{r(R)^{2z} f(r(R))}{l^{2z}}, \quad Y(R) = \frac{r(R)^{2} f(r(R))}{l^{2}}. \]  
(25)

The metric of background is chosen to be the Lifshitz metric (24) i.e.
The quasilocal conserved mass can be obtained through

\[
M = \frac{1}{8\pi} \int_B d^2\varphi \sqrt{\sigma} \left\{ (K_{ab} - K h_{ab}) - (K^0_{ab} - K^0 h^0_{ab}) \right\} n^a \xi^b,
\]

where \(\sigma\) is the determinant of the boundary \(\mathcal{B}\) metric, \(K^0_{ab}\) is the background extrinsic curvature, \(n^a\) is the timelike unit normal vector to the boundary \(\mathcal{B}\) and \(\xi^b\) is a timelike Killing vector field on the boundary surface. Performing the above modified BY formalism, the mass of the space time per unit volume \(\omega_{n-1}\) can be calculated as

\[
M = \frac{(n-1)m}{16\pi l^{z+1}},
\]

where the mass parameter \(m\) can be obtained from the fact that \(f(r_+) = 0\) as

\[
m(r_+) = \begin{cases} 
\frac{r_+^{n+z-1} + (n-2)^2 r_+^{n+z-3}}{(n+z-3)^2} + \frac{4k^{z-2}r_+^2}{(n-1)(n-z+1)}r_+^{-n}, & \text{for } z \neq n+1 \\
\frac{4l^{2z-2}r_+^2}{(n+z-3)(n+z-1)r_+^{z+1}}F \left( \frac{1}{2}, \frac{2n+z-4}{2n-z}, \frac{3n+z-5}{2n-z}, 1 - \frac{2}{n-2} \right) & \text{for } z = n+1 
\end{cases}
\]

Now, we turn to calculate the electric charge of the solution. Using the Gauss law, we can calculate the electric charge via

\[
Q = \frac{1}{4\pi} \int r^{n-1} L F_{\mu\nu} n^\mu u^\nu d\Omega,
\]

where

\[
n^\mu = \frac{1}{\sqrt{-g_{tt}}} dt = \frac{l^z}{\sqrt{f(r)}} dr, \quad u^\nu = \frac{1}{\sqrt{g_{rr}}} dr = \frac{r \sqrt{f(r)}}{l} dr,
\]

are respectively the unit spacelike and timelike normals to the hypersurface of radius \(r\). Using (30), the charge per unit volume \(\omega_{n-1}\) can be computed as

\[
Q = \frac{ql^{z-1}}{4\pi}.
\]

The electrostatic potential difference \((U)\) between the horizon and infinity is defined as

\[
U = A_\mu \chi^\mu |_{r=\infty} - A_\mu \chi^\mu |_{r=r_+},
\]

Using Eqs. (22) and (32), one can obtain the electric potential

\[
U = \frac{q b^{2z-2}}{(n+z-3)r_+^{n+z-3}} F \left( \frac{1}{2}, \frac{n+z-3}{2n-2}, \frac{3n+z-5}{2n-2}, 1 - \frac{2}{n-2} \right),
\]

which is the same for both \(z = n+1\) and \(z \neq n+1\) cases. In order to investigate the first law of black hole thermodynamics, we should obtain the Smarr-type formula for mass (28). With Eqs. (29), (31) and (21) at hand, the mass can be written as a function of extensive thermodynamic quantities \(S\) and \(Q\) in the form of

\[
M(S, Q) = \begin{cases} 
\frac{(n-1)(4S)^{(n-1)}/(n-1)}{16\pi l^{z+1}} + \frac{(n-1)(n-2)^2(4S)^{(n+z-3)}/(n-1)}{16\pi l^{z+1}(n+z-3)^2} + \frac{H^{2z-2}r_+}{4l^{2z-2}r_+^2} \left( \frac{1}{2}, \frac{2n+z-4}{2n-z}, \frac{3n+z-5}{2n-z}, 1 - \frac{2}{n-2} \right), & \text{for } z \neq n+1, \\
\frac{(n-1)(4S)^{2n/(n-1)}}{16\pi l^{z+1}} + \frac{(n-2)^2 S^2}{4\pi(n-1)^2} - \frac{b^2 h^{2n}}{4\pi(n-1)^2} \left[ 1 - \Gamma + \ln \left( \frac{1+\Gamma}{2} \right) \right], & \text{for } z = n+1,
\end{cases}
\]
where \( \Gamma = \sqrt{1 + \pi^2 Q^2 / (\beta^2 S^2)} \). Calculations show that intensive quantities

\[
T = \left( \frac{\partial M}{\partial S} \right)_Q \quad \text{and} \quad U = \left( \frac{\partial M}{\partial Q} \right)_S,
\]

(35)

coincide with those computed by Eqs. (19) and (33). Therefore, the thermodynamics quantities satisfy the first law of thermodynamics

\[
dM = TdS + UdQ,
\]

(36)

for both solutions for \( z = n + 1 \) and \( z \neq n + 1 \).

In next part of this section, we will discuss the Hawking-Page and inside black hole phase transitions for our Lifshitz solutions.

**B. Black hole phase transitions**

1. *Hawking-Page phase transition*

As it is clear from Fig. 1, there are some parameter choices for which we have extreme black holes and therefore zero temperature. In addition, as one can see from Fig. 2, there are some other choices of parameters that show a non-zero positive minimum for temperature \( T_{\text{min}} \). The influences of different parameters on \( T_{\text{min}} \) can be seen from Fig. 2. When we increase the dimension \( n \), \( T_{\text{min}} \) increases too, while it decreases with increasing \( z \). Comparing Figs. 2(a) and 2(b), one finds out that the effect of nonlinearity implies increasing in \( T_{\text{min}} \). The behaviors illustrated in Fig. 2 present a Hawking-Page phase transition for the obtained solutions. Let us have a closer look on Fig. 2. In the first part of \( T - S \) curves where we have small black holes (note that \( S = r^{n-1}_+ / 4 \)), \( \partial T / \partial S < 0 \) which implies negative heat capacity and therefore small black holes are thermally unstable. But, in the large black holes part of the curves we have a positive heat capacity and therefore large black holes are thermally stable. In addition to small and large black holes, we have a thermal Lifshitz or radiation solution too. Since the small black holes are thermally unstable, system has two choices between large black hole and thermal Lifshitz that chooses to be on one of them according to the Gibbs free energy. The Gibbs free energy

\[
G(T, U) = M - TS - QU,
\]

(37)

can be obtained by using (19), (21), (28), (31) and (33). Figs. 3 and 4 show the behavior of Gibbs free energy for some choices of parameters. The two up and bottom branches correspond to small and large black holes, respectively. The positive Gibbs free energy shows that the system is in radiation phase while there is a Hawking-Page phase transition at intersection point of bottom branch and \( G = 0 \). This fact that the Gibbs free energy of large black holes always have the lower energy in comparison to small ones confirms the above arguments about the thermal stability of them. As one moves rightward on temperature axis in \( G - T \) diagram, first experiences radiation regime or thermal Lifshitz
solution for which $G > 0$. At $G = 0$, the Hawking-Page phase transition between thermal Lifshitz and large black holes occurs and for $G < 0$, we are at large black hole phase. The temperature at which phase transition occurs is called Hawking-Page temperature $T_{HP}$. Effects of change in electric potential $U$, critical exponent $z$ and nonlinearity parameter $\beta$ can be seen from Figs. 3 and 4. Increase in electric potential $U$ and critical exponent $z$ makes $T_{HP}$ lower. Also, the lower the nonlinearity parameter $\beta$ is, the lower Hawking-Page temperature $T_{HP}$ is. Note that lower $\beta$ makes the electrodynamics more affected by nonlinearity.

2. Phase transitions inside the black hole

There are at least three well-known ways to discuss the phase transitions inside the black hole. Two of these ways are based on macroscopic point of view and one of them is based on microscopic viewpoint. The two macroscopic ways are Davies [24] and Landau-Lifshitz [28, 29] methods that discuss, respectively, the behavior of heat capacities and thermodynamic fluctuations. Thermodynamic geometry or Ruppeiner geometry [16, 20, 21] is the microscopic way which discusses the phase transitions in addition to type and strength of interactions. In what follows, we discuss the relation between the phase transitions predicted by Ruppeiner geometry and Davies method. Next, we will turn to Landau-Lifshitz theory of thermodynamic fluctuations.

Ruppeiner and Davies phase transitions

In order to discuss thermodynamic geometry, one should study the divergences, sign and magnitude of Ricci scalar corresponding to Ruppeiner metric (usually called Ruppeiner invariant) to determine phase transitions and strength
and type of dominated interaction between possible black hole molecules [16, 20, 21]. To do that, we define the Ruppeiner metric in \((M, Q)\) space where the entropy \(S\) is thermodynamic potential as

\[
g_{\alpha\beta} = -\frac{\partial^2 S}{\partial X^\alpha \partial X^\beta}, \quad X^\alpha = (M, Q). \tag{38}\]

The above metric can also be rewritten in the Weinhold form

\[
g_{\alpha\beta} = \frac{1}{T} \frac{\partial^2 M}{\partial Y^\alpha \partial Y^\beta}, \quad Y^\alpha = (S, Q). \tag{39}\]

The Ruppeiner invariant corresponding to (39) can be expressed in a general form as

\[
\mathcal{R} = \frac{\mathfrak{M}(S, Q)}{\mathfrak{D}(S, Q)}, \tag{40}\]

where \(\mathcal{R}\) and \(\mathfrak{D}\) stand for numerator and denominator of \(\mathfrak{R}\). The divergences of Ruppeiner invariant is determined by roots of \(\mathfrak{D}\) which is equal to \(T [\mathbf{H}^M_{S,Q}]^2\) where \(\mathbf{H}^M_{S,Q} = M_{SS}M_{QQ} - M_{SQ}^2\) is determinant of Hessian matrix and \(XYZ = \partial^2 X/\partial Y \partial Z\). Of course, at these divergence points the numerator \(\mathcal{R}\) should be finite. These divergences show both zero temperature and vanishing \(\mathbf{H}^M_{S,Q}\). The root of \(\mathbf{H}^M_{S,Q}\) may show us the boundary between thermal stability and instability. For thermal stability, in addition to positivity of determinant of Hessian matrix, \(M_{QQ}\) and \(M_{SS}\) should be positive too [94, 95].

It is remarkable to note that at the point where \(M_{SS}\) vanishes or equivalently heat capacity at constant charge \(C_Q\) diverges, we have a thermally unstable system due to negativity of \(\mathbf{H}^M_{S,Q}\) if \(M_{SQ} \neq 0\) (which occurs in many of black hole systems). Thus, the heat capacity at constant charge \(C_Q\) cannot be suitable thermodynamic quantity to show phase transition of such systems when we have two changing thermodynamic parameters, for instance \(S\) and \(Q\). There are some works in literature (for instance [96]) in which the correctness of Ruppeiner method for recognizing the phase transitions has been judged by comparing the Ruppeiner and \(C_Q\) transition points. This procedure is of course seems to be incorrect according to what we pointed out above. Also, as we discussed in the introduction, divergences of \(\mathcal{R}\) are safer in order to determine phase transitions. On the other hand, in [37] and [42], authors have suggested some suitable thermodynamic quantities to show the phase transitions predicted by Ruppeiner invariant. These quantities are specific heat at constant electrical potential, \(C_U\), analog of volume expansion coefficient, \(\alpha\), and analog of isothermal compressibility coefficient \(\kappa_T\) defined as

\[
C_U = T \left( \frac{\partial S}{\partial T} \right)_U, \quad \alpha = \frac{1}{Q} \left( \frac{\partial Q}{\partial T} \right)_U, \quad \kappa_T = \frac{1}{Q} \left( \frac{\partial Q}{\partial U} \right)_T. \tag{41}\]

As one can see in appendix A, these thermodynamic quantities have the forms

\[
C_U = T \frac{M_{SS}}{H^M_{S,Q}}, \quad \alpha = -\frac{1}{Q} \frac{M_{SQ}}{H^M_{S,Q}}, \quad \kappa_T = -\alpha \frac{\partial T}{\partial U} \bigg|_Q. \tag{42}\]

It is obvious that these quantities show the same phase transitions as the Ruppeiner geometry because all of them diverge at roots of \(H^M_{S,Q}\) and \(C_U\) vanishes at zero temperature where \(\mathcal{R}\) diverges. To show the coincidence of Ruppeiner phase transitions and \(C_U\) divergences, some proofs have been presented in [44, 48]. The above quantities can be considered as improved Davies quantities [24] which present the phase transitions coincided with Ruppeiner ones. In the next part, we study the Landau-Lifshitz theory of thermodynamic fluctuations to explore the possible signature of black hole phase transitions on properties of black hole radiance.

**Landau-Lifshitz theory (nonextreme/extreme phase transition)**

Here, we seek for any possible effect of transition on black hole radiance by using Landau-Lifshitz theory of thermodynamic fluctuations [28, 29]. We focus on \((3 + 1)\)-dimensional linearly charged case. The extension to higher dimensional or nonlinearly charged cases is trivial and give no novel result. Based on Landau-Lifshitz theory [28, 29], in a fluctuation-dissipative process, the flux \(\dot{X}_i\) of a given thermodynamic quantity \(X_i\) is given by

\[
\dot{X}_i = -\sum_j \Gamma_{ij} \chi_j, \tag{43}\]
where the mean value with respect to the steady state is denoted by the angular brackets and the fluctuations \( \delta X_i \) are the spontaneous deviations from the value of steady state \( \langle X_i \rangle \). To guarantee that correlations are zero when two fluxes are independent, the Kronecker \( \delta_{ij} \) is put in Eq. (45).

According to [65], the mass \( M \), electric potential energy \( U \) and temperature \( T \) can be obtained for (3+1)-dimensional linearly charged case as

\[
M = \frac{(4S)(z+2)/2}{8\pi l^{z+1}} + \frac{(4S)^{z/2}}{8\pi z^{2l-1}} + \frac{2\pi Q^2 b^{2z-2}}{z^{l-1}(4S)^{2z/2}},
\]

and

\[
U = \frac{\pi b^{2z-2}Q}{z^{2z-2l-1}S^{z/2}} \quad \text{and} \quad T = \frac{2z^{-4}X}{\pi z^{l-1}S^{z/2+1}},
\]

where

\[
\Xi = S^z + 4z(z+2)S^{z+1}l^{-2} - 4^{2-z}z^2b^{2z-2}Q^2.
\]

We know that in extreme black hole case, the Hawking temperature on the event horizon vanishes and therefore in this case we have \( \Xi = 0 \). Using Eq. (46), we can obtain the entropy production rate as

\[
\dot{S}(M,Q) = \chi_M \dot{M} - \chi_Q \dot{Q},
\]

where

\[
\chi_M = \frac{\pi z l^{-1}S^{z/2+1}}{2z^{-4}X} \quad \text{and} \quad \chi_Q = \frac{\pi^2 b^{2z-2}QS}{4z^{-2}X}.
\]

The mass loss rate is given by [97]

\[
\frac{dM}{dt} = -b\alpha\sigma T^4 + U \frac{dQ}{dt}.
\]

The first term on the right side of Eq. (49) is the thermal mass loss corresponding to Hawking radiation which is just the Stefan-Boltzmann law, with \( b = \pi^2/15 \) (we set \( h = 1 \)) as the radiation constant. The constant \( \alpha \) depends on the number of species of massless particles and the quantity \( \sigma \) is the cross-section of geometrical optics. The second term on the right side of Eq. (49) is responsible for the loss of mass corresponding to charged particles. In fact, it is \( UdQ \) term which rises in first law of black hole mechanics.

With references to what explained and computed above, one can calculate the second moments or correlation functions of the thermodynamical quantities

\[
\langle \delta M \delta M \rangle = -\frac{2z^{-3}X}{\pi z l^{-1}S^{z/2+1}} M, \quad \langle \delta Q \delta Q \rangle = \frac{2z^{-5}b^{2z-2}X}{\pi^2 S Q}, \quad \langle \delta M \delta Q \rangle = U \langle \delta Q \delta Q \rangle,
\]

\[
\langle \delta S \delta S \rangle = \frac{\pi^2 z^2 b^{2z-2}S^z + 2}{4z^{-4}X^2} \left[ \langle \delta M \delta M \rangle + \frac{\pi^2 b^{4z-4}Q^2}{4z^{-2z}z^{2z-2}S^z} \langle \delta Q \delta Q \rangle - \frac{\pi b^{2z-2}Q}{2z^{-3}z^{l-1}S^{z/2}} \langle \delta M \delta Q \rangle \right] = -\frac{\pi z l^{-1}S^{z/2+1}}{2z^{-5}X} \left[ M + \frac{\pi b^{2z-2}Q}{z^{2z-2l-1}S^{z/2}} \dot{Q} \right]
\]

(51)
\[ \langle \delta \tilde{S} \delta T \rangle = \frac{4\pi^2b^{4z-4}Q^2}{\pi l^{2z-2}} \left[ (z-2)S^2 + 4z^2(z + 2)l^{-2}S^{z+1} + \pi^2 z(z + 2)4z^2 - 4z^2 - 2Q^2 \right] \langle \delta \tilde{M} \delta \tilde{M} \rangle \]

\[ + \frac{4z^2 - 4z^2 - 2Q^2}{\pi l^{2z-2}} \left[ (z-1)S^2 + 4z^2(z + 2)l^{-2}S^{z+1} + \pi^2 z(z + 2)4z^2 - 4z^2 - 2Q^2 \right] \langle \delta \tilde{Q} \delta \tilde{Q} \rangle \]

\[ - \frac{\pi b^{2z-2}Q}{2z^{-5}l^{2z-2}} \left[ (z-2)S^2 + 4z^2(z + 2)l^{-2}S^{z+1} + \pi^2 z(z + 2)4z^2 - 4z^2 - 2Q^2 \right] \langle \delta \tilde{M} \delta \tilde{Q} \rangle \]

\[ - \frac{\pi b^{2z-2}Q}{2z^{-5}l^{2z-2}} \left[ (z-2)S^2 + 4z^2(z + 2)l^{-2}S^{z+1} + \pi^2 z(z + 2)4z^2 - 4z^2 - 2Q^2 \right] \langle \delta \tilde{Q} \delta \tilde{M} \rangle \]

\[ = - \frac{\pi S^{2z/2} + 4z^2(z + 2)l^{-2}S^{z+1} + \pi^2 z(z + 2)4z^2 - 4z^2 - 2Q^2}{2z^{-5}l^{2z-2}} \langle \delta \tilde{M} \delta \tilde{M} \rangle \]

\[ + \frac{\pi^3 b^{4z-4}Q^2}{\pi l^{2z-2}} \left[ (z-1)S^2 + 4z^2(z + 2)l^{-2}S^{z+1} + \pi^2 z(z + 2)4z^2 - 4z^2 - 2Q^2 \right] \langle \delta \tilde{Q} \delta \tilde{Q} \rangle \]

\[ - \frac{\pi^2 b^{2z-2}Q}{2z^{-5}l^{2z-2}} \left[ (3z - 4)S^2 + 12z^2(z + 2)l^{-2}S^{z+1} + \pi^2 z(z + 2)4z^2 - 4z^2 - 2Q^2 \right] \langle \delta \tilde{M} \delta \tilde{Q} \rangle \]

\[ - \frac{(z-2)S^2 + 4z^2(z + 2)l^{-2}S^{z+1} + \pi^2 z(z + 2)4z^2 - 4z^2 - 2Q^2}{2z^{-5}l^{2z-2}} \left[ \dot{M} + \frac{\pi b^{2z-2}l^{z-1}Q}{2z^{-2}lS^{z}} \right] \cdot \dot{\tilde{M}} \]

\[ \]
in which $x^{-1} \ll 1$.

In the following section, we turn to study thermodynamic geometry of our black hole solutions to figure out the behavior of black hole possible molecules and phase transitions.

IV. RUPPEINER GEOMETRY

In this section we study thermodynamic geometry of the Lifshitz-dilaton black holes for linear Maxwell and nonlinearly BI gauge fields, separately. We have introduced this method in subsection III B 2 with focus on the study of the phase transitions which occur at divergence of Ruppeiner invariant $\mathcal{R}$. In addition to divergences, $\mathcal{R}$ has other properties which give us information about thermodynamic of the system. The sign of $\mathcal{R}$ gives us the information about the dominated interaction between possible black hole molecules while its magnitude measures the average number of correlated Planck areas on the event horizon [16, 19-21]. $\mathcal{R} > 0$ means the domination of repulsive interaction, $\mathcal{R} < 0$ shows the attraction dominated regime and when $\mathcal{R}$ vanishes the system behaves like ideal gas i.e. there is no interaction. In continue, we first study thermodynamic geometry in the presence of linear Maxwell electrodynamics. Then, we extend our study to nonlinearly charged black holes where BI electrodynamics has been employed. There is just a necessary comment. As we stated before in subsection III B 2, for thermal stability, $M_{QQ}$, $M_{SS}$ and $\mathbf{H}^M_{S,Q} = M_{SS} M_{QQ} - M_{S,Q}^2$ should be positive [94, 95]. One can show that the positivity of $\mathbf{H}^M_{S,Q}$ and $M_{QQ}$ $(M_{SS})$ imposes the positivity of $M_{SS}$ (M_{QQ}). Therefore, we just turn to study the signs of $\mathbf{H}^M_{S,Q}$ and $M_{QQ}$ in our following discussions to guarantee the thermal stability.

A. Linear Maxwell case

The mass and Hawking temperature of black holes in the presence of linear Maxwell (LM) electrodynamics are

$$T_{LM} = \frac{(n + z - 1)r^+_z}{4\pi l^{z+1}} + \frac{(n - 2)^2 l^{1-z}}{4\pi(n + z - 3)r^{2-z}_+} - \frac{q^2 l^{z-1} b^{2z-2}}{2\pi(n - 1)r^{2n+z-4}_+},$$

$$M_{LM}(S, Q) = \frac{(n - 1)(4S)^{(n+z-1)/(n-1)}}{16\pi l^{z+1}} + \frac{(n - 1)(n - 2)^2 (4S)^{(n+z-3)/(n-1)}}{16\pi(n + z - 3)^2 l^{z-1}} + \frac{2\pi Q^2 b^{2z-2}(4S)^{(3-n-z)/(n-1)}}{(n + z - 3) l^{z-1}}.$$  (61)

As we mentioned above, for investigating thermal stability we need to check the signs of $M_{QQ}$ and $\mathbf{H}^M_{S,Q}$. In our case

$$M_{QQ} = \frac{\pi b^{2z-2} S^{-(n+z-3)/(n-1)}}{(n + z - 3) l^{z-1} 2^{z-(z-2)/(n-1)}} > 0.$$  (62)

Thus, in order to disclose the thermal stability of system, we need to study the sign of determinant of Hessian matrix. We find

$$\mathbf{H}^M_{S,Q} = \begin{cases} \frac{(z-2)(n-2)^2 l^{2z-2} S^{-2+2(z-2)/(n-1)}}{4(n-1)(n+z-3)^2 l^{z-2}} \mathbf{\mathcal{D}}(S, Q) z \neq 2, \\ \frac{(n+1)b^{2z^2(n-5)/(1-n)} S^{2(n-2)/(1-n)}}{(n-1)^2 l^4}, z = 2, \end{cases}$$  (63)

where

$$\mathbf{\mathcal{D}}(S, Q) = \left[ S^{2[1+(z-2)/(n-1)]} + \frac{(n + z - 3)(n + z - 1)S^{2[1+(z-2)/(n-1)]}}{2^{-4/(n-1)(z-2)(n-2)^2l^2}} - \frac{\pi^2(n + z - 3)Q^2 S^{2(n-4z+7)/(n-1)}}{(n - 1)(n-2)^2b^{2(1-z)}} \right]^{-n-1}.$$  (64)

The numerator $\mathcal{R}$ of (40) is a complicated finite function of $S$ and $Q$ in this case, including long terms that we do not express it explicitly for economic reasons. However, as it was mentioned in subsection III B 2, one can find the denominator $\mathbf{D}$ in the form of

$$\mathbf{D}(S, Q) = T_{LM} \left[ \mathbf{H}^M_{S,Q} \right]^2,$$  (65)

where $T_{LM}$ and $\mathbf{H}^M_{S,Q}$ have been give in (60) and (63) respectively.

Having Eqs. (63) and (65) at hand, we are in the position to investigate the divergences of $\mathcal{R}$, which play the central role in thermodynamic geometry discussions and also thermal stability of system. As one can see from Eqs.
and (65), for $z = 2$, the divergences occur just in the case of the extremal black holes where $T_{LM} = 0$. For $z \neq 2$, in addition to extremal black hole case, $\mathcal{R}$ diverges in zeros of (63). In the latter case, we can calculate the corresponding temperature by solving $\mathfrak{F} = 0$ for $Q$ and then putting this $Q$ in Eq. (60) to arrive at

$$ T = \frac{(n + z - 1)^2(2z - n + 1)/(n-1)S^z/(n-1)}{\pi(2 - z)^{n+1}}. $$

(66)

The above temperature is negative for $z > 2$, i.e. there is no black hole at this diverging point and therefore the divergences of $\mathcal{R}$ occur just for extremal black hole case when $z > 2$. However, for $z < 2$ when $T > 0$, we can see an upper limit in entropy and charge of system. The largest entropy $S$ for which $\mathfrak{F} = 0$ (which we call it critical entropy $S_c$) can be calculated by finding the extremum point where $\partial^2 \mathfrak{F}/\partial S = 0$ as

$$ S_c^{2/(1-n)} = \frac{z(n + z - 1)(n + z - 2)^{2z/(n-1)}}{(2-z)(n-2)^2l^2}, $$

(67)

at which

$$ Q_c^2 = \frac{(n-2)^2(n-z-2)}{T^{2(n+z-3)l^2(z-1)x^2}} \frac{(n-1)(n + z - 2)^{2z-n-3}(n + z - 1)^{3-n-z}}{2^3(n + z - 3)} \left(\frac{2}{z - 1}\right)^{n+z-3}, $$

(68)

and

$$ \frac{\partial^2 \mathfrak{F}}{\partial S^2} |_{S=S_c} = -\frac{(n + z - 3)4^{(n-2z+3)/(n-1)}}{(n-1)^2} \left[ \frac{(n + z - 1)(n + z - 2)z}{(2-z)(n-2)^2l^2} \right]^{2z} < 0. $$

(69)

One should note that the absolute value of $Q_c$ is also the largest charge value which satisfies $\mathfrak{F} = 0$. Another remark to be mentioned is that (67) imposes an upper limit on the size of black hole too (see (21)). At this point, the corresponding temperature can be obtained as

$$ T_c = \frac{(n - 2)^2}{2\pi l \sqrt{(z(n + z - 2))}} \left(\frac{n + z - 1}{2} \right)^{(2-z)/2}. $$

(70)

For charges greater than $Q_c$, the Ruppeiner invariant diverges only in the case of extremal black holes. For $Q = Q_c$, in addition to $T_{LM} = 0$, we have one other divergence in $\mathcal{R}$ specified by (67) and (70). For $Q < Q_c$, in addition to $T_{LM} = 0$, we have at most two other divergences since the order of polynomial in term of $S$ is always lower than 3 for $n \geq 3$ and $z < 2$. One should note that, in latter case, the temperature region between two divergences is not allowed since $H_{S,Q}^M < 0$ (Fig. (5)).

We have summarized the above discussion in Figs. 6 and 7. These figures also show the sign of Ruppeiner invariant for different choices of parameters that determines the type of interaction between black hole molecules [16, 20, 21]. Fig. 6(a) is depicted for RN-AdS case ($n = 3, z = 1$). In this case, it can be seen that for $Q > Q_c$, the Ruppeiner
invariant diverges only for extremal black holes. As Fig. 6(a) shows, there is also a range of $T$ for which $R < 0$, namely the dominated interaction between black hole molecules is attractive. Furthermore, the interactions near zero temperature is the same as interactions of Fermi gas molecules near zero temperature [16]. According to Eq. (5(a)), for $Q > Q_c$, $H^M_{S,Q}$ is positive (also $M_{QQ} > 0$ (see (62))), and therefore the system is stable for all $T$ region. For $Q = Q_c$, in addition to zero temperature, we have another temperature ($T_c$), that divergence of $R$ occurs at it (see Fig. 6(a)). At zero temperature, the Ruppeiner invariant goes to $+\infty$ while at $T_c$, it goes to $-\infty$. The latter case is similar to the Van der Waals gas phase transition at critical point in this sense that in phase transition temperature, $R$ goes to $-\infty$ [14, 21]. For $Q = Q_c$, $R$ becomes positive when we get away from second divergence ($T_c$) on temperature axis. In $Q = Q_c$ case, $H^M_{S,Q}$ is positive and just vanishes at $T_c$ (Fig. 5(a)), so, the system is always thermally stable. For $Q < Q_c$, there are three divergences; one at $T = 0$, one at $T < T_c$ and one at $T > T_c$. In this case, according to Fig. 5(a), $H^M_{S,Q}$ is negative in the temperature region between two roots and show instability. This not-allowed region is equivalent to the temperature region between two divergences of Ruppeiner invariant for $Q < Q_c$ (Fig. 6(a)). Figures 5(b) and 6(b) show the same properties for black holes with different parameters. In this case, $T_c$ is greater than one of previous case while $Q_c$ is lower. Fig. 7 shows the behavior of Ruppeiner invariant for $z \geq 2$. As this figure shows, there are just divergences at $T = 0$. The properties of black hole molecular interactions ($R > 0$: Repulsion, $R = 0$: No interaction and $R < 0$: Attraction) depend on parameters such as dimension of space time and charge, in this case. According to Eq. (63), for $z = 2$, $H^M_{S,Q}$ is always positive. For $z > 2$, we can find $Q$ from $T_{LM} = 0$ and put it in $H^M_{S,Q}$ to receive

$$H^M_{S,Q,T=0} = \frac{(n + z - 1)b^2z-22^{(n-5)/(1-n)}}{(n-1)(n + z - 3)l^{2z}S(n-z)/(n-1)} > 0.$$  \hfill (71)
Thus, since $H_{S,Q}$ nowhere vanishes for $z > 2$ (see discussions below (66)) and is positive at $T = 0$ according to above equation, it is positive throughout the temperature region and therefore system is always thermally stable for $z > 2$.

Regarding the nature of phase transition occurred at zero temperature where Ruppeiner invariant diverges, we discussed in previous section via Landau-Lifshitz theory of thermodynamic fluctuations. However, regarding the phase transitions occurred at divergences of $\mathcal{R}$ at finite temperatures, we can give some comments here. We have seen two kinds of phase transitions here for $z < 2$ (see Fig. 6) namely continues (for $Q = Q_c$ where $\mathcal{R}$ diverges at just one finite temperature or entropy) and discontinues (for $Q < Q_c$ where $\mathcal{R}$ diverges at two finite temperatures or entropies and we have a jump between these two points since there is no thermally stable black hole between them). Both of these phase transitions can be considered as small/large black holes phase transitions. The first reason for this argument is that as temperature increases, entropy or equivalently size of black hole increases (note that $\partial S/\partial T = M^{z-1}_{S,Q} > 0$). Therefore, the left side of phase transition points where temperature is lower, we have small size black holes and the right side where temperature is higher we have large size ones. This fact can also be seen from the behavior of Ruppeiner invariant magnitude in two sides of phase transitions. For small black holes, we expect the finite correlation between possible black hole molecules (of course far from phase transition points) because those are close to each other. For large black holes, we expect the correlation between possible molecules to tend to a small value near zero since molecules become approximately free. These expected behaviors can be seen in Fig. 6.

**B. Born-Infeld case**

For Born-Infeld case, we can calculate the Ruppeiner invariant by using Eqs. (19), (34) and (39). The Ruppeiner invariant in this case is very complicated due to the presence of hypergeometric functions. Therefore, in this case
we discuss the thermodynamic geometry non-analytically and by looking at plots. We study $z < 2$, $z > 2$, $z = 2$ and $z = n + 1$ cases separately. First, we study $z < 2$ case. Fig. 8(a) shows that changing $\beta$ can cause change in dominated interaction. For instance, in a range of $T$, we have negative $\mathcal{R}$ (attraction) for $\beta = 1$ (note that in this range system is thermally stable as one can see from Figs. 9(a) and 10(a)). For $\beta = 1$, the system behaves like Fermi gas in zero temperature namely $\mathcal{R}$ goes to positive infinity at zero temperature [16]. For $\beta = 0.82$, Ruppeiner invariant diverges at two points that one of them is zero temperature. According to Fig. 9(a), for temperatures lower than the
second divergence point, $H_{S,Q}^M$ is negative and therefore system is thermally unstable. Since $M_{QQ} > 0$ (Fig. 10(a)), the system is thermally stable just for temperatures greater than the temperature of second divergence for $\beta = 0.82$. Fig. 8(a) shows that there is no extremal black hole for $\beta = 0.5$ i.e. we have a black hole with just single horizon. The allowed temperature region is temperatures greater than the temperature of divergence according to Figs. 9(a) and 10(a). In Figs. 8(b), 9(b) and 10(b), respectively Ruppeiner invariant, $H_{S,Q}^M$ and $M_{QQ}$ are depicted for different choices of parameters. It is remarkable to mention that, in the case of $\beta = 0.046$, Fig. 8(b) shows that the behavior of system looks like Van der Waals gas at phase transition temperature i.e. $R$ goes to negative infinity at this point \cite{14, 21}. For $z > 2$, the behavior of $R$ is depicted in Fig. 11(a). It can be seen that the type of dominated interaction changes for different $\beta$’s and we have negative $R$ for some cases. In this case, we have a behavior like Fermi gas at zero temperature for extremal black holes. For $\beta = 0.13$, there is a divergence at non-zero temperature that for temperatures lower than it, system is unstable (Figs. 11(b) and 11(c)). In the case of $z = 2$, $H_{S,Q}^M$ and $M_{QQ}$ are

$$H_{S,Q}^M|_{z=2} = \frac{(n + 1)b^2}{2^{n-5/2}/(n-1)b^4/n^2S / (n-1)}$$

(72)

and

$$M_{QQ}|_{z=2} = \frac{b^2\pi}{(n-1)/|S|}.$$

which are always positive and therefore system is always stable and $R$ experiences no divergence (Fig. 12). In this case, for different values of nonlinear parameter $\beta$, we have different dominated interaction. For this case, possible
molecules of black hole behave like Fermi gas at zero temperature. The last case is \( z = n + 1 \). In this case \( M_{QQ} \) is

\[
M_{QQ}\big|_{z=n+1} = \frac{b^{2n} \beta^2 (\Gamma - 1)}{4\pi(n-1)Q^2 T^n \Gamma},
\]

which is positive for all temperatures. The behavior of Ruppeiner invariant and \( H_{S,Q}^M \) are depicted in Figs. 13(a) and 13(b) for this case, respectively. As one can see the type of interaction is \( \beta \)-dependent for some temperatures. For \( \beta = 0.04 \), \( H_{S,Q}^M \) is positive just for temperatures greater than the finite temperature of divergence (Fig. 13(b)) and therefore system is thermally stable for this range of temperatures.

Most of phase transitions discovered above in the presence of BI electrodynamics at finite temperatures, cannot be interpreted as small/large black hole phase transitions because in these cases small size black holes are unstable. Further studies to disclose the nature of these phase transitions are called for.

V. SUMMARY AND CLOSING REMARKS

In many condensed matter systems, fixed points governing the phase transitions respect dynamical scaling \( t \to \lambda^z t \), \( \mathbf{x} \to \lambda \mathbf{x} \) where \( z \) is dynamical critical exponent. The gravity duals of such systems are Lifshitz black holes. In this paper, we first sought for the \((n+1)\)-dimensional Born-Infeld (BI) charged Lifshitz black hole solutions in the context of dilaton gravity. We found out that these solutions are different for the cases \( z = n + 1 \) and \( z \neq n + 1 \). We obtained both these solutions and showed that the solution for the case \( z = n + 1 \) can never be Schwartzschild-like. Then, we studied thermodynamics of both cases by calculating conserved and thermodynamical quantities and checking the satisfaction of first law of thermodynamics. After that, we looked for the Hawking-Page phase transition for our solutions, both in the cases of linearly and BI charged black holes. We studied this phenomenon and effects of different parameters on it by presenting the behaviors of temperature \( T \) with respect to entropy \( S \) at fixed electrical potential energy \( U \) and also Gibbs free energy \( G \) with respect to \( T \). Then, we turned to discuss the phase transitions inside the black holes. In this part, we first presented the improved Davies quantities that show the phase transition points coincided with ones of Ruppeiner geometry. This coincidence has been proved directly in appendix A. All of our solutions provided that those are thermally stable at zero temperature show the divergence at this point both from Ruppeiner and Davies points of view. Using Landau-Lifshitz theory of thermodynamic fluctuations, we showed that this phase transition is a transition on radiance properties of black holes. At zero temperature, an extreme black hole can just radiate through superradiant scattering whereas a nonextreme black hole at finite temperature can give off particles and radiation via both spontaneous Hawking radiation and superradiant scattering.

Next, we turned to study Ruppeiner geometry for our solutions. We investigated thermal stability, interaction type of possible black hole molecules and phase transitions of our solutions for linearly and nonlinearly BI charged cases separately. For linearly charged case, we showed that there are no diverging points for Ricci scalar of Ruppeiner geometry (Ruppeiner invariant) at finite temperature for the case \( z \geq 2 \). For \( z < 2 \), it was found that the number of divergences (which show the phase transitions) at finite temperatures depend on the value of charge \( Q \). We introduced a critical value for charge \( Q_c \) that for values greater than it there is no divergence at finite temperature, for values lower than it there are at most two divergences and for \( Q = Q_c \), there is just one diverging point for Ruppeiner invariant. For the case of \( Q < Q_c \), there is a thermally unstable region for systems between two divergences at finite temperatures. So, this phase transition can be claimed as a discontinuous phase transition between small and large black holes. For small black holes not close to transition point, we observed finite magnitude for Ruppeiner invariant \( \mathcal{R} \). This is reasonable since the magnitude of \( \mathcal{R} \) shows the correlation of possible black hole molecules. Also, for large black holes the magnitude of Ruppeiner invariant tends to a very small value as expected. For \( Q = Q_c \), the solutions show a continues small/large black holes phase transition at finite temperature. In the case of BI charged solutions, we investigated the Ruppeiner geometry and thermal stability for \( z < 2 \), \( z > 2 \), \( z = 2 \) and \( z = n + 1 \) separately. In some of these cases, small black holes were thermally unstable. So, more studies are called for to discover the nature of phase transitions at diverging points of \( \mathcal{R} \). In both linearly and nonlinearly charged cases, for some choices of parameters, the black hole system behaves like a Van der Waals gas near transition point.

Finally, we would like to suggest some related interesting issues which can be considered for future studies. It is interesting to repeat the studies here such as Hawking-Page phase transition, Ruppeiner geometry and Landau-Lifshitz theory for black branes to discover the effect of different constant curvatures of \((n-1)\)-dimensional hypersurface on those phenomena. One can also seek for any signature of different phase transitions discovered here such as Hawking-Page phase transition, and phase transitions determined by Ruppeiner geometry, in dynamical properties of solutions by investigating quasi-normal modes. Some of these works are in progress by authors.
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Appendix A: Suitable thermodynamic quantities to determine phase transitions

In [37] and [42], authors have shown that the divergences of specific heat at constant electrical potential, $C_U$, analog of volume expansion coefficient, $\alpha$, and analog of isothermal compressibility coefficient $\kappa_T$ are in coincident with the phase transitions specified by Ruppeiner invariant. The definition of these quantities are

$$C_U = T \left( \frac{\partial S}{\partial T} \right)_U, \quad \alpha = \frac{1}{Q} \left( \frac{\partial Q}{\partial T} \right)_U \quad \text{and} \quad \kappa_T = \frac{1}{Q} \left( \frac{\partial Q}{\partial U} \right)_T. \quad (A1)$$

Here we will prove that these quantities are exactly suitable ones to characterize phase transitions shown by Ruppeiner invariant. We showed in section IV that divergences of Ruppeiner invariant occurs at roots of determinant of Hessian matrix $H_{S,Q}^M = M_{SS}M_{QQ} - M_{SQ}^2$ and also zero temperature. In our proof, we will show that $H_{S,Q}^M$ exactly exist at denominator of all above suitable thermodynamic quantities.

Let us start with $C_U$. We have

$$\left. \frac{\partial T}{\partial S} (S, Q (U, S)) \right|_U = \left. \frac{\partial T}{\partial S} \right|_Q + \left. \frac{\partial T}{\partial Q} \right|_S \left. \frac{\partial Q}{\partial S} \right|_U. \quad (A2)$$

On the other hand we know that

$$\left. \frac{\partial Q}{\partial S} \right|_U = - \left. \frac{\partial Q}{\partial U} \right|_S \left. \frac{\partial U}{\partial S} \right|_Q. \quad (A3)$$

With above relations in hand, one can show that

$$\left. \frac{\partial T}{\partial S} (S, Q (U, S)) \right|_U = \left. \frac{\partial T}{\partial S} \right|_Q - \left. \frac{\partial T}{\partial Q} \right|_S \left. \frac{\partial U}{\partial S} \right|_Q = \left. \frac{\partial T}{\partial S} \right|_Q - \left. \frac{\partial T}{\partial Q} \right|_S \left. \frac{\partial U}{\partial S} \right|_Q = \left. \frac{\partial T}{\partial S} \right|_Q - \left. \frac{\partial T}{\partial Q} \right|_S \left. \frac{\partial U}{\partial S} \right|_Q = M_{QQ}M_{SS} - M_{SQ}^2 M_{SS} = \frac{H_{S,Q}^M}{M_{SS}}. \quad (A4)$$

In the last line of (A4), we have used (35). Eq. (A4) shows that $H_{S,Q}^M = M_{SS}M_{QQ} - M_{SQ}^2$ is in denominator of $C_U = T (\partial S/\partial T)_U$ and therefore it exactly diverges at the point where Ruppeiner invariant diverges. To show this fact for $\alpha$, we should obtain

$$\left. \frac{\partial T}{\partial Q} (Q, S (U, Q)) \right|_U = \left. \frac{\partial T}{\partial Q} \right|_S + \left. \frac{\partial T}{\partial S} \right|_Q \left. \frac{\partial S}{\partial Q} \right|_U. \quad (A5)$$

As we know

$$\left. \frac{\partial S}{\partial Q} \right|_U = - \left. \frac{\partial S}{\partial U} \right|_Q \left. \frac{\partial U}{\partial Q} \right|_S, \quad (A6)$$

and therefore we have
\[
\frac{\partial T(Q,S(U,Q))}{\partial Q} \bigg|_U = \frac{\partial T}{\partial Q} - \frac{\partial T}{\partial S} \frac{\partial S}{\partial U} \frac{\partial U}{\partial Q} \bigg|_S = \frac{\partial T}{\partial Q} - \frac{\partial T}{\partial S} \frac{\partial S}{\partial Q}\bigg|_Q \frac{\partial U}{\partial Q} \bigg|_S
\]

\[
= \frac{\frac{\partial T}{\partial Q} \frac{\partial U}{\partial Q} - \frac{\partial T}{\partial S} \frac{\partial S}{\partial Q} \frac{\partial U}{\partial Q} \bigg|_S}{M_{QQ}M_{SS} - M_{QS}^2} = \frac{\mathbf{H}_M^V}{M_{QQ}}.
\]

(A7)

The above relation shows that \(\alpha = Q^{-1} \left(\partial Q/\partial T\right)_U\) diverges at the point where Ruppeiner invariant does. At final, to receive similar result for \(\kappa_T\), we obtain

\[
\frac{\partial U}{\partial Q} \bigg|_T = \frac{\partial U}{\partial T} \bigg|_Q \frac{\partial T}{\partial Q} \bigg|_U = -\frac{1}{Q_\alpha \frac{\partial T}{\partial Q}}.
\]

(A8)

Above relation shows that \(\kappa_T = Q^{-1} \left(\partial Q/\partial U\right)_T\) is proportional to \(\alpha\) and therefore diverges at the same points as Ruppeiner invariant.

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