Demonstration of Two-Qubit Algorithms with a Superconducting Quantum Processor

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(Dated: May 1, 2009)

By harnessing the superposition and entanglement of physical states, quantum computers could outperform their classical counterparts in solving problems of technological impact, such as factoring large numbers and searching databases. A quantum processor executes algorithms by applying a programmable sequence of gates to an initialized register of qubits, which coherently evolves into a final state containing the result of the computation. Simultaneously meeting the conflicting requirements of long coherence, state preparation, universal gate operations, and qubit readout makes building quantum processors challenging. Few-qubit processors have already been shown in nuclear magnetic resonance, cold ion traps and optical systems, but a solid-state realization has remained an outstanding challenge. Here we demonstrate a two-qubit superconducting processor and the implementation of the Grover search and Deutsch–Jozsa quantum algorithms. We employ a novel two-qubit interaction, tunable in strength by two orders of magnitude on nanosecond time scales, which is mediated by a cavity bus in a circuit quantum electrodynamics (cQED) architecture. This interaction allows generation of highly-entangled states with concurrence up to 94%. Although this processor constitutes an important step in quantum computing with integrated circuits, continuing efforts to increase coherence times into a two-qubit device. This allows sufficient time to concatenate ~10 gates, realizing simple algorithms with fidelity greater than 80%.

Our processor, shown in Fig. 1a, is a 4-port superconducting device comprising two transmon qubits inside a microwave cavity bus, and flux-bias lines proximal to each qubit. The cavity, normally off-resonance with the qubit transition frequencies, couples the qubits by virtual photon exchange and shields them from the electromagnetic continuum. As previously demonstrated, microwave pulses resonant with $f_L$ or $f_R$ applied to the cavity input port provide frequency-multiplexed single-qubit $x$- and $y$-rotations with high fidelity and selectivity. Pulsed measurement of the homodyne voltage $V_H$ on the output port of the cavity provides qubit readout. The remaining two ports create local magnetic fields that tune the qubit transition frequencies.

Over the last decade, superconducting circuits have made considerable progress on all the requirements necessary for an electrically-controlled, solid-state quantum computer. Coherence times have risen by three orders of magnitude to ~1 μs, single-qubit gates have reached error rates of 1%, engineered interactions have produced two-qubit entanglement at a level of 60% concurrence, and qubit readout has attained measurement fidelities ~90%. However, combining these achievements in a single device remains challenging. One approach to integration is the quantum bus architecture, which uses an on-chip transmission line cavity to couple, control, and measure qubits.

We augment the architecture in Ref. with flux-bias lines that tune individual qubit frequencies, permitting single-qubit phase gates. By pulsing the qubit frequencies to an avoided crossing where a $\sigma_x \otimes \sigma_x$ interaction turns on, we are able to realize a two-qubit conditional phase (c-Phase) gate. Operation in the strong-dispersive regime of cQED allows joint readout that can efficiently detect two-qubit correlations. Combined with single-qubit rotations, this enables tomography of the two-qubit state. Through an improved understanding of spontaneous emission and careful microwave engineering, we are now able to combine state-of-the-art ~1 μs coherence times into a two-qubit device. This allows sufficient time to concatenate ~10 gates, realizing simple algorithms with fidelity greater than 80%.

The spectrum of single excitations (Fig. 1b) shows the essential features of the cavity-coupled two-qubit Hamiltonian and allows a determination of the relevant system parameters (see Methods). When the qubits are tuned to their maximum frequencies, point I, they are far detuned from the cavity and from each other, so that interactions are small. This point is therefore used for state prepa-
Microwave pulses at the qubit transition frequencies $f_L$ and $f_R$ drive single-qubit rotations, and a pulsed measurement of the cavity homodyne voltage $V_R$ (at frequency $f_N$) provides two-qubit readout. The flux-bias lines (bottom-left and top-right ports) are coplanar waveguides with short-circuit termination next to their target qubit. The termination geometry allows current on the line to couple flux through the split junctions (b, inset).

Grey scale images of cavity transmission and of qubit spectroscopy as a function of $V_R$, showing local tuning of $Q_R$ across the avoided crossing with $Q_L$ (point III) and across the vacuum Rabi splitting with the cavity (point IV). Semi-transparent lines are theoretical best fits obtained from numerical diagonalization of a generalized Tavis–Cummings Hamiltonian. Points I and II are the operating points of the processor. Preparation, single-qubit operations and measurements are performed at point I, and a c-Phase gate is achieved by pulsing into point II.

For both qubits, the cavity-mediated, transverse interaction between these levels, producing a frequency shift $\Omega$, is apparent on examining the one-excitation manifold. The negative $\Omega$-shift is the mechanism at the heart of our conditional phase gate. Flux pulses, adiabatic with respect to the $|1, 1\rangle \leftrightarrow |0, 2\rangle$ avoided crossing, produce phase gates in the computational Hilbert space. Here, $\phi_{tr} = 2\pi \int \delta f_{tr}(t) \, dt$ is the dynamical phase acquired by $|l, r\rangle$, and $\delta f_{tr}$ is the deviation of $f_{tr}$ from its value at point I. A $V_R$ pulse into point II such that $\int \zeta(t) \, dt = (2n + 1)\pi$ with integer $n$ implements a c-Phase, because $\phi_{11} = \phi_{01} + \phi_{10} - \int \zeta(t) \, dt$. This method of realizing a c-Phase by adiabatically using the avoided crossing between computational and non-computational states is generally applicable to any qubit implementation with finite anharmonicity, such as transmons or phase qubits. The negative anharmonicity permits the phase gate at point II to occur before the onset of transverse coupling at point III.

Control of $\zeta$ by two orders of magnitude provides an excellent on-off ratio for the c-Phase gate.
in Fig. 2, measurements of $\zeta$ obtained from spectroscopy and from time-domain experiments show very good agreement. The time-domain method measures the difference in the precession frequency of $Q_L$ in two Ramsey-style experiments where a $V_R$-pulse of varying duration (0–100 ns) is inserted between $\pi/2$ rotations of $Q_L$, with $Q_R$ either in the ground state $|0\rangle$ or excited into state $|1\rangle$. Using the time-domain approach, we measure a residual $\zeta/2\pi \approx 1.2$ MHz at point I (star). The theoretical $\zeta$ obtained by numerical diagonalization shows reasonable agreement with the data, except for a scale factor that is likely due to higher modes of the cavity, not included in the calculation.

The controlled phase interaction allows universal two-qubit gates. As an example, we produce high-fidelity entangled states on demand (Fig. 3). The pulse sequence in Fig. 3 generates any of the four Bell states,

$$|\Psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle \pm |1\rangle),$$

$$|\Phi\rangle = \frac{1}{\sqrt{2}} (|0\rangle \pm |1\rangle),$$

depending on the choice of c-Phase gate $cU_{ij}$ applied ($cU_{ij} |l, r\rangle = (-1)^{i\delta_{l,r}} |l, r\rangle$). These gates are realized through fine control of the dynamical phases $\phi_0$ and $\phi_1$ in a 30 ns $V_R$-pulse close to point II and back. We tune $\phi_0$ over $2\pi$ by making small adjustments to the rising and falling edges of the $V_R$-pulse, and $\phi_1$ with a simultaneous weak $V_L$-pulse.

To detect the entanglement, we first reconstruct the two-qubit density matrix $\rho$ by quantum state tomography using joint dispersive readout. A pulsed measurement of the cavity homodyne voltage $V_H$ measures the operator

$$M = \beta_1 \sigma_1^L + \beta_2 \sigma_1^R + \beta_{12} \sigma_2^L \otimes \sigma_2^R,$$

where the $\sigma$ are two-qubit Pauli operators. Operation in the strong-dispersive regime makes $|\beta_{12}| \sim |\beta_1|, |\beta_2|$, enhancing sensitivity to two-qubit correlations. A complete set of 15 linearly independent measurement operators is built using single-qubit rotations prior to measuring $M$. An ensemble average of each operator is obtained by executing the sequence in Fig. 3 450,000 times. The 15 measured values are then input to a maximum likelihood estimator of $\rho$ (see Supplementary Information).

The inferred density matrices $\rho_{ml}$ reveal highly-entangled states in all four cases (Fig. 3c–e). We quantify performance using the metrics of purity, $P(\rho) = \text{Tr}(\rho^2)$, fidelity to the target state $\langle \psi | \rho | \psi \rangle$, $F(\rho, \psi) = \langle \psi | \rho | \psi \rangle$, and concurrence $C$, computable from $\rho_{ml}$. Note that there are several common definitions of fidelity in the literature, and our definition is the square of the fidelity used in Refs. 22 and 28. Values for $P$, $F$ and $C$ for the four cases are given in the caption to Fig. 3. These values significantly extend the current state of the art for solid-state entanglement, and provide evidence that we have a high-fidelity universal set of two-qubit gates.

One- and two-qubit gates can be concatenated to realize simple algorithms, such as Grover’s quantum search algorithm.
entangler outputs include purity and $c_U \in \{0, 1, 2, 3\}$ such that $f(x) = 1$ except at some $x_0$, where $f(x_0) = -1$. This well-known algorithm can determine $x_0 = 2i + j$ with a single call of an oracle $O = cU_{ij}$, which encodes $f(x)$ in a quantum phase.

We can examine the functioning of the algorithm by interrupting it after each step and performing state tomography. Figure 3 clearly shows all the features of a quantum processor, namely the use of maximally superposed states to exploit quantum parallelism (Fig. 3a), the encoding of information in the entanglement between qubits (Fig. 3b, e), and the interference producing an answer represented in a final computational basis state. The fidelity of the final state (Fig. 3g) to the expected output $(|1, 0\rangle$ for the case $O = cU_{10}$ shown) is 85%. Similar performance is obtained for the other three oracles (Table I).

We have also programmed and executed the Deutsch–Jozsa algorithm.9,30 The two-qubit version of this algorithm determines whether an unknown function $f_i(x)$, mapping a one-bit input to a one-bit output, is constant ($f_0(x) = 0$ or $f_1(x) = 1$) or balanced ($f_2(x) = x$ or $f_3(x) = 1 - x$), doing so with a single call of the function. The algorithm applies the function once to a superposition of the two possible inputs and employs the concept of quantum phase kick-back2 to encode the result in the final state of one qubit (here, $Q_L$) while leaving the other untouched ($Q_R$). The gate sequence realizing the algorithm and the output tomographs for the four cases are shown in Supplementary Fig. S1.

The performance of both algorithms is summarized in Table I. Although there are undoubtedly significant systematic errors remaining, the overall fidelity is nonetheless similar to that expected from the ratio ($\sim 100$ ns/1 $\mu$s) of the total duration of gate sequences to the qubit coherence times.

In summary, we have demonstrated the experimental realization of two-qubit quantum algorithms using a superconducting circuit. The incorporation of local flux control and joint-dispersive readout into cQED, together with a ten-fold increase in qubit coherence over previous two-qubit devices, has enabled on-demand generation and detection of entanglement and the implementation of the Grover and Deutsch–Jozsa algorithms. Superconducting circuits could eventually perform more complex quantum algorithms on many qubits, provided that coherence lifetimes and the resulting gate fidelities can be further improved.

I. METHODS

A. Device fabrication

A 180 nm film of Nb was dc-magnetron sputtered on the epi-polished surface of an R-plane corundum ($\alpha$-Al$_2$O$_3$) wafer (2" diameter, 430 $\mu$m thickness). Coplanar waveguide structures (cavity and flux-bias lines) were patterned by optical lithography and fluorine-based reactive ion etching of Nb. Transmon features (interdigitated capacitors and split junctions) were patterned on individual 2 mm x 7 mm chips using electron-beam lithography, double angle evaporation of Al (20/90 nm) with intermediate oxidation (15% O$_2$ in Ar at 15 Torr for 12 min), and lift-off.

A completed device was cooled to 13 mK in a $^3$He-$^4$He dilution refrigerator. A diagram of the refrigerator wiring...
The Tavis-Cummings Hamiltonian generalized to multi-level transmon qubits is
\[ H = \omega_C a^\dagger a + \sum_{q \in \{L,R\}} \left( \sum_{j=0}^N \omega_{qj}^0 |j\rangle_q \langle j|_q + (a + a^\dagger) \sum_{j,k=0}^N g_{jk}^q \langle j|_q \langle k|_q \right). \]

Here, \( \omega_C \) is the bare cavity frequency, \( \omega_{qj}^0 = \omega_{0j} (E_{Cq}, E_{1q}) \) is the transition frequency for qubit \( q \) from ground to excited state \( j \), and \( g_{jk}^q = g_q n_{jk} (E_{Cq}, E_{1q}) \), with \( g_q \) a bare qubit-cavity coupling and \( n_{jk} \) a level-dependent coupling matrix element. The dependence of these parameters on qubit charging energy \( E_{Cq} \) and Josephson energy \( E_{1q} \) is indicated. The flux control enters through \( E_{1q} = E^\text{max} (\cos(\Phi_q - \Phi_0)) \), with \( \Phi_q \) the flux through the qubit loop, and a linear flux-voltage relation \( \Phi_q = \alpha_q V_L + \alpha_{0R} V_L + \Phi_{q0} \), accounting for crosstalk (\( \sim 30\% \)) and offsets. The above parameters are tightly constrained by the combination of spectroscopy and transmission data shown (Figs. 1b, 2a and 2b) and transmission data (not shown) for the \( Q_L \)-cavity vacuum Rabi splitting. By simultaneously fitting the spectra given by numerical diagonalization of the Hamiltonian (truncated to \( N = 5 \) qubit levels and 5 cavity photons) to these data, we obtain \( E_{1L(R)}/h = 317 \) (297) MHz, \( g_{L(R)}/2\pi = 199 \) (183) MHz, and \( \omega_C/2\pi = 6.902 \) GHz. The cavity linewidth is \( \kappa/2\pi = 1 \) MHz.

**Acknowledgements** We thank V. Manucharyan and E. Boaknin for experimental contributions, and M. H. Devoret, I. L. Chuang and A. Nunnenkamp for discussions. This work was supported by LPS/NSA under ARO Contract No. W911NF-05-1-0365, and by the NSF under Grants No. DMR-0653377 and No. DMR-0603369. We acknowledge additional support from CIFAR, MRI, MITACS, and NSERC (JMG), NSERC, CIFAR, and the Alfred P. Sloan Foundation (AB), and from CNR-Istituto di Cibernetica, Pozzilli, Italy (LF).

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Table I: Summary of algorithmic performance.

| Element          | Grover oracle  | Deutsch–Jozsa function |
|------------------|----------------|------------------------|
|                  | $f_0$ | $f_1$ | $f_{10}$ | $f_{11}$ | $f_0$ | $f_1$ | $f_2$ | $f_3$ |
| $\langle 0,0\rangle$ | Ideal | 1    | 0   | 0   | 0   | 0     | 0    | 1   | 1   |
|                  | Measured | 0.81(1) | 0.08(1) | 0.07(2) | 0.065(7) | 0.010(3) | 0.014(5) | 0.909(6) | 0.841(9) |
| $\langle 0,1\rangle$ | Ideal | 0    | 1   | 0   | 0   | 1     | 1    | 0   | 0   |
|                  | Measured | 0.066(7) | 0.802(9) | 0.05(1) | 0.054(8) | 0.012(4) | 0.008(4) | 0.031(8) | 0.04(2)  |
| $\langle 1,0\rangle$ | Ideal | 0    | 0   | 1   | 0   | 0     | 0    | 0   | 0   |
|                  | Measured | 0.08(1) | 0.05(1) | 0.82(2) | 0.07(1) | 0.93(1) | 0.93(1) | 0.05(1) | 0.04(1)  |
| $\langle 1,1\rangle$ | Ideal | 0    | 0   | 0   | 1   | 0     | 0    | 0   | 0   |
|                  | Measured | 0.05(2) | 0.07(1) | 0.06(1) | 0.81(1) | 0.05(1) | 0.04(1) | 0.012(9) | 0.07(2)  |

*Uncertainties are based on 10 repetitions. ? Uncertainties are based on 8 repetitions.

Fidelity of the reconstructed output states of the Grover and Deutsch–Jozsa algorithms to their ideal outputs. These results suggest that, if combined with single-shot readout, the two algorithms executed with this processor would give the correct answer with probability far exceeding the 50% success probability of the best classical algorithms limited to single calls of the oracle or function.

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Supplementary Material for
‘Demonstration of Two-Qubit Algorithms with a Superconducting Quantum Processor’

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I. SUPPLEMENTARY INFORMATION

A. Perturbation Theory

To gain additional insight on the large on-off ratio of the frequency shift $\zeta$, observed both in experiment and numerical diagonalization of the Hamiltonian (Fig. 2), we perform a perturbative analysis in the rotating-wave approximation. Truncating Eq. (1) at three transmon excitations and assuming $n_{12} \simeq \sqrt{2}$ (valid for $E_1/E_C \gg 1$), we obtain the fourth-order result

$$\zeta = -2 g_{1L} g_{1R} \left( \frac{1}{\delta_1 |\Delta_L|^2} + \frac{1}{\delta_2 |\Delta_R|^2} + \frac{1}{\Delta_L |\Delta_R|^2} + \frac{1}{\Delta_R |\Delta_L|^2} \right).$$

Here, $\Delta_q = \omega_{01} - \omega_C$, $\delta_1 = \omega_{R1} - \omega_{L1}$, and $\delta_2 = \omega_{01} - \omega_{12}$. This expression diverges as the $0 \leftrightarrow 1$ transition of one transmon lines up with the $1 \leftrightarrow 2$ transition of the other. Assuming instead two-level qubits, the expression simplifies to

$$\zeta = -2 g_{1L} g_{1R} \left( \frac{1}{\Delta_1 |\Delta_R|^2} + \frac{1}{\Delta_R |\Delta_L|^2} \right).$$

Both perturbative expressions are shown with numerical diagonalization of the Hamiltonian in Fig. 2c. The three-level expression shows reasonable agreement away from the divergence, while the two-level expression is incorrect in both magnitude and sign.

B. State Tomography

The goal of quantum state tomography is to estimate the density matrix $\rho$ describing a quantum mechanical state. For any two-qubit quantum state we can choose a set of 16 linearly independent operators $\{M_i\}$ such that $\rho$ can be decomposed as

$$\rho = \sum_{i=1}^{16} c_i M_i,$$

where the set $\{c_i\}$ are the 16 parameters to be estimated. If the operators are observables, then the 16 expectation values $m_i = \text{Tr}[M_i \rho]$ determine $c_j$ by

$$m_i = \sum_{j=1}^{16} \text{Tr}[M_i M_j] c_j.$$

Previous work has shown that in cQED a homodyne measurement of the cavity is a faithful measurement of $\sigma_z$. For a quantum bus with two qubits the measurement operator is

$$M = \frac{1}{\tau} \int_0^\tau Q(t) \, dt = \beta_1 \sigma_z^L + \beta_2 \sigma_z^R + \beta_{12} \sigma_L^t \otimes \sigma_R^t.$$

Here, $Q$ is the measured quadrature amplitude, $\tau$ is an averaging window, and the $\beta$ are calibrated coefficients. For this experiment, $\tau = 450$ ns and $(\beta_1, \beta_2, \beta_{12}) \approx (60, 50, 40) \mu$V.

Since the measurement contains both one- and two-qubit operators, a complete set of linearly independent operators $M_i$ can be made by applying only single-qubit rotations prior to measurement. The set of 15 pre-rotations used in this experiment is all combinations of $I$, $R_z^\pi$, $R_x^{\pi/2}$, $R_y^{\pi/2}$ on left and right qubits, except that $R_z^\pi \otimes R_z^\pi$ is not used. Only 15 measurements are needed to determine $\rho$ because of the constraint of trace normalization, $\text{Tr} \rho = 1$ (equivalently we choose $M_{16} = I$, which always gives $m_{16} = 1$).

Experimental averages $m_i$ are obtained by recreating the quantum state (executing the gate array), pre-rotating and measuring 450,000 times. While ideally $\rho$ could be obtained from the experimental $m_i$ by inversion of $\text{Tr}[M_i M_j]$, this method pays no attention to the properties $\rho$ must have: hermiticity and positive semi-definiteness (trace normalization is included by the choice of decomposition). These physical constraints are automatically included by a parametrization

$$\rho = \frac{T^T T}{\text{Tr}[T^T T]},$$

where $T$ is a lower triangular matrix. For two qubits,

$$T = \begin{pmatrix} t_1 & 0 & 0 & 0 \\ t_5 + it_6 & t_2 & 0 & 0 \\ t_{11} + it_{12} & t_7 + it_8 & t_3 & 0 \\ t_{15} + it_{16} & t_{13} + it_{14} & t_9 + it_{10} & t_4 \end{pmatrix}.$$
The $t_i$ are found by standard Maximum likelihood Estimation with a likelihood function
\[
\mathcal{L} = \prod_{i=1}^{16} \alpha_i (m_i - \text{Tr}[M_i \rho])^2,
\]
where the $\alpha_i$ are weighting factors. We weight all measurements equally since amplifier noise dominates the error in all the measurements.

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Figure S1: Implementation of Deutsch–Jozsa algorithm. a. Gate sequence solving the Deutsch–Jozsa problem. The two-qubit gates $U_i$ performing the transformation $|l, r \rangle \rightarrow |l, r \oplus f_i(l)\rangle$ ($\oplus$ denotes addition modulo 2) for $f_0(x) = 0$, $f_1(x) = 1$, $f_2(x) = x$, and $f_3(x) = 1-x$ are $U_0 = I \otimes I$, $U_1 = I \otimes R_x^{\pi/2}$, $U_2 = (I \otimes R_y^{\theta/2} R_x^{\theta}) c U_{00} (I \otimes R_y^{\theta/2})$, and $U_3 = (I \otimes R_y^{\theta/2} R_x^{\theta}) c U_{11} (I \otimes R_y^{\theta/2})$, respectively. b–e, Real part of the inferred density matrix $\rho_{\text{ml}}$ of the algorithm output in the four cases (imaginary elements of $\rho_{\text{ml}}$ are less than 0.05, 0.03, 0.05, 0.06, respectively). For the constant (balanced) functions $f_0$ and $f_1$ ($f_2$ and $f_3$), $\rho_{\text{ml}}$ reveals high fidelity to $|1, 0\rangle$ ($|0, 0\rangle$), as expected. For the tomographs shown, the fidelity to the ideal output state is $F = 0.94$, 0.95, 0.92, and 0.85, respectively. Statistics for 8 runs of each of the four cases are given in Table I.
Figure S2: Experimental setup and wiring. a. Block diagram of room-temperature electronics. Arbitrary waveform generators, with 1 ns sampling rate and 10-bit resolution, produce voltages $V_L$ and $V_R$ directly applied to the flux-bias lines, the I-Q modulation envelopes for the microwave tones driving single-qubit $x$- and $y$-rotations, and the pulse that modulates the cavity measurement. On the output side, an I-Q mixer and a two-channel averager (2 ns, 8-bit sampling) complete the readout chain performing homodyne detection of the cavity quadratures. The arbitrary waveform generators, microwave synthesizers and acquisition card are clocked with a Rubidium frequency standard (SRS FS725, not shown). b. Schematic of the microwave wiring inside the dilution refrigerator, showing heavily-attenuated input lines and an output chain with $\sim 100$ dB gain in the 4–8 GHz range.

Figure S3: Microwave and flux pulses realizing the gates. a. An example sequence, executing the Grover search algorithm with oracle $O = cU_{10}$ and measuring $M_{13} = -\beta_1 \sigma^L_z + \beta_2 \sigma^R_y - \beta_1 \sigma^L_z \otimes \sigma^R_y$. b. Illustration of the microwave and flux pulses realizing the operations directly above. All microwave pulses implementing the $x$- and $y$-rotations have Gaussian envelopes, with standard deviation $\sigma = 2$ ns, truncated at $\pm 2\sigma$. The rotation axis is set using I-Q (vector) modulation (see Fig. S2), and rotation angle is controlled by pulse amplitude. A buffer of 5 ns is inserted between all microwave and flux pulses to avoid any overlap.