Far-field heat and angular momentum radiation of the Haldane model

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Abstract

We investigate the radiation of energy and angular momentum from 2D topological systems with broken inversion symmetry and time reversal symmetry. A general theory of far-field radiation is developed using the linear response of 2D materials to the thermal fluctuation of electric currents. Applying the theory to the Haldane model, we verify that the heat radiation complies with Planck’s law only at low temperature and deviates from it as temperature becomes high. In contrast to normal metals, angular momentum radiation is possible for this system and exhibits saturation as temperature increases. Parameters crucial to the radiation are investigated and optimized. This research enlightens the possibility of transposing the quantum information to the angular momentum degree of freedom.

Keywords: far-field radiation, fluctuation-electrodynamics, Haldane model, angular momentum radiation

1. Introduction

All material bodies are surrounded by a fluctuating electromagnetic field because of the thermal and quantum fluctuations of the current density inside them. The fluctuating field is responsible for important phenomena such as radiative heat transfer [1], the van der Waals interaction, and the van der Waals friction between bodies [2–4].

There have been a lot of research in near-field radiation. Polder and van Hove (PvH) [5] were the first to give a quantitative theory of near-field radiation using the Rytov formulation of fluctuating electromagnetic fields [6, 7]. Many works verified that thermal radiation can be substantially enhanced in the near-field scenario due to the tunneling of evanescent waves [8–12].

While near-field radiation has been extensively studied over the past decades, relatively little attention has been paid to far-field radiation in connection with materials properties. Very recently it has become clear that the thermal radiation properties can dramatically violate Planck’s law even in the far-field regime, which opens new opportunities for the field of thermal radiation [13, 14]. Furthermore, radiative thermal photons carry not only energy but also angular momentum [15–18], which is important in information processing. Modern optical communication systems exploit various electromagnetic wave properties to increase the bit rate per unit carrier frequency [19, 20]. Recently, orbital optical angular momentum (OAM) has shown promise as an additional degree of freedom to increase spectral efficiency. Because there is no physical limit on the OAM order that one can radiate, in theory OAM provides an unlimited number of additional communication channels for any given system [18, 20]. For normal materials such as metal, there is no specific chirality of the radiative field, as a result, the net angular momentum radiation is zero. Topological insulators are a new class of materials which are insulating in bulk but their surface states have unconventional properties. If the time-reversal symmetry is weakly broken, the topological insulator exhibits a topological magneto-electric and magneto–optical effect. The radiation of hot photons to the environment will carry away angular momentum [17, 21].
In this paper we investigate a two-dimensional (2D) material of the Haldane model which has a broken time-reversal symmetry by pure imaginary hopping between next nearest neighbour (NNN) sites [22–24]. The Haldane model is the most typical and simplest model for topological insulators. We exploit the radiation of thermal photons carrying angular momentum. A far-field radiation theory is developed and then applied to the Haldane model. Although the Haldane model is originally a toy model, there are some practical material such as silicene or phosphorene has Hamiltonian similar to the Haldane model [25–27]. Buckled structure provides on-site potential which breaks inversion symmetry. The spin–orbit interaction provides imaginary NNN coupling with a positive or negative sign when electrons jumping clockwise or anticlockwise directions in hexagonal rings.

The far-field radiation is contributed by propagation mode. For an object with a large distance to other objects, the evanescent waves do not give any contribution to energy radiation and angular momentum radiation. The normal component of the wave vector, which in the vacuum region is given by \( \gamma = \sqrt{\omega / \varepsilon - q^2} \), will be purely imaginary for \( q > \omega / \varepsilon \), where \( \omega \) is the electromagnetic wave frequency. This means that only photons with \( q < \omega / \varepsilon \) can escape from the body and propagate in the vacuum to far distance. This implies long-wave approximation is sufficiently accurate to simplify complex details.

2. Theory of far-field radiation

We consider a 2D material such as the Haldane model for light radiation. The system consists of the 2D spinless electrons, radiation field, and the coupling between the material and the radiation field. We ignore the ionic degrees of freedom. Using a tight-binding method combined with Peierls’ substitution [28], the Hamiltonian of the system is written as

\[
\hat{H} = \hat{H}_\gamma + \sum_k \hbar c_j c_k e^{i \gamma / \hbar \hat{\mathbf{A}}_j \cdot \mathbf{r}},
\]

The first term is the radiation field \( \hat{H}_\gamma = \int \psi^\dagger \psi dV \) with the energy density \( u = \frac{1}{2} (\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2) \).

The second term in equation (1) represents the 2D material as a collection of electrons on a lattice and its coupling with the radiation field, in which \( \hbar c_k \) is the hopping parameter between neighbours. \( c_j \) is the creation (annihilation) operator at site \( j \). \( \mathbf{A} \) is the vector potential of the radiation field, and \( \mathbf{r} \) is electron charge. Using trapezoidal rule of integration and expanding the exponent to the first order of \( \hat{A} \), this term can be separated into two terms, the free material \( \hat{H}_0 \) and interaction between material and radiation field \( \hat{H}' \). Both of these two terms rely on material.

We develop the theory of far-field radiation by the approach of fluctuational electrodynamics [3, 29]. Due to thermal and quantum fluctuation in any material, fluctuational electromagnetic field is generated by fluctuations of current \( J'(r, \omega) \),

\[
\nabla \times \hat{H}(r, \omega) = -i \omega \varepsilon_0 \hat{E}(r, \omega) + \hat{J}'(r, \omega),
\]

The average of fluctuational current is zero, but the correlation is not zero. The contribution to energy radiation is obtained by solving Maxwell’s equations. With Lorenz gauge \( \nabla \cdot A = 0 \), the relation between vector potential \( \hat{A}(r, \omega) \) and fluctuational current is [3]

\[
\hat{A}(r, \omega) = \mu_0 \int_V g(r, r', \omega) \hat{J}'(r', \omega) dV'.
\]

Here \( g(r, r', \omega) \) is the Green’s function with \( \bar{r} \) and \( \bar{r}' \) denoting a field and source point. The electric and magnetic field at \( \bar{r} \) due to a source located at \( \bar{r}' \) are given by

\[
\hat{E}(\bar{r}, \omega) = i \omega \mu_0 \int_V \nabla \times \hat{A}(r, \omega) \cdot \hat{J}'(r', \omega) dV',
\]

\[
\hat{H}(\bar{r}, \omega) = \frac{1}{\mu_0} \nabla \hat{A}(\bar{r}, \omega) = \int_V \nabla \times g(r, r', \omega) \cdot \hat{J}'(r', \omega) dV'.
\]

Here we give the relation with \( \hat{B} \) just to be clear. In the above equations, \( \hat{G}(\bar{r}, \bar{r}', \omega) = [I + \frac{1}{\hbar^2} \nabla \nabla] g(r, r', \omega) \) and \( \hat{G}^m(r, r', \omega) = \nabla \times g(r, r', \omega) I \) are electric and magnetic dyadic Green’s functions, respectively, with \( I \) being the identity matrix. In order to calculate the Poynting expectation value efficiently, we perform Fourier transform in the transverse direction (\( x \) and \( y \) directions, and \( z \) direction is normal to the surface) to dyadic Green’s functions. Finally, the electric and magnetic Green’s dyadic are as follows (\( k = \omega / c \))

\[
\hat{G}^e(q, z) = \left( \begin{array}{ccc} (1 - \frac{q^2}{k^2}) & \frac{q_e q_i}{k^2} & -i \frac{q_e}{k^2} \\ \frac{q_i q_e}{k^2} & (1 - \frac{q_i^2}{k^2}) & -i \frac{q_i}{k^2} \\ -i \frac{q_e}{k^2} & -i \frac{q_i}{k^2} & (1 - \frac{q_e^2}{k^2}) \end{array} \right) \cdot \frac{i}{2\gamma} e^{i|z|},
\]

\[
\hat{G}^m(q, z) = \left( \begin{array}{ccc} 0 & -\gamma & q_e \\ -\gamma & 0 & -q_i \\ q_i & q_e & 0 \end{array} \right) \cdot \frac{i}{2\gamma} e^{i|z|}.
\]

Here, \( \bar{q}_e = (q_e, q_i) \) is the wave vector in the material plane and \( \gamma = \sqrt{k^2 - q_e^2} \) is the component along \( z \) direction. Electric and magnetic fields in \( \bar{q}_e, z \) mixed representation are expressed by

\[
\hat{E}(q, z, \omega) = i \omega \mu_0 \hat{G}^e(q, z) \cdot \hat{J}(q, \omega),
\]

\[
\hat{H}(q, z, \omega) = \hat{G}^m(q, z) \cdot \hat{J}(q, \omega).
\]

Now we are in a position to calculate expectation value of Poynting vector \( \mathcal{S}(\bar{r}, \omega) = 4 \times \mathcal{J} \left\{ (\hat{E}(\bar{r}, \omega) \times \hat{H}(\bar{r}, \omega)) \right\} \) for energy transport. In the Fourier transform from time-dependent fields to frequency-dependent quantities, the frequency covers the whole domain, i.e. from \( -\infty \) to \( \infty \). Here we use the factor 4 to simplify computation that only positive frequencies are considered in
the formula [3]. For planar geometry, \( \langle S' \rangle = \langle S'' \rangle = 0 \), with only \( \langle S' \rangle \neq 0 \).

\[
\langle S'(q_z,z,\omega) \rangle = 2 \text{Re} \left\{ i \omega \mu_0 \left[ (G_x G_y)_{11} \left( J'_x J'_y \right) \right. \right.
\]
\[ + \left. \left. - (G_x G_y)_{11} \left( J'_y J'_x \right) \right) + (G_x G_y)_{11} \right\}.
\]

(10)

Owing to the symmetric property of the dyadic and correlation function, the cross terms cancel each other. The current–current correlation function is related to temperature through the fluctuation–dissipation theorem [5, 30, 31]

\[
\langle J'_x J'_x \rangle = 2 \hbar \omega N(\omega) \text{Re}[\sigma_{xx}(\omega)].
\]

(11)

The other component of current–current correlation \( \langle J'_y J'_y \rangle \) is the same owing to symmetry of \( \sigma_{xx}(\omega) = \sigma_{yy}(\omega) \). Putting these correlations into equation (10) and using equations (A.19) and (A.20), the Poynting expectation is obtained as

\[
\langle S'(q_z,z,\omega) \rangle = \text{Re} \left\{ 4 \mu_0 \hbar \omega^3 N(\omega) \text{Re}[\sigma_{xx}(\omega)] \left( 2 - \frac{q^2}{k^2} \right) \frac{1}{4 \gamma} \right\}
\]

(12)

Here \( \gamma = \sqrt{k^2 - q^2} \) is the longitudinal wave vector and \( N(\omega) = 1/(e^{\omega/\hbar T} - 1) \) is the Bose distribution function at temperature \( T \). Here we have used the long-wave (or local) approximation so that the conductivity is independent of the wave vectors. For a far field radiation, because of the exponential decay, there is energy transmission to the far field only if \( q < \frac{\omega}{c} \). While \( c \) is large, this means \( q \) must be small, and wavelength is long. So long wave, small \( q \) is more important for thermal energy radiation. Since we are dealing with the far-field radiation, this is an excellent approximation. The total energy radiation is obtained by integrating over all wave vectors and frequencies,

\[
\langle S' \rangle = \frac{2}{3 \pi \varepsilon_0 c^3} \int_0^\infty \frac{d\omega}{2\pi} \hbar \omega^3 N(\omega) \text{Re}[\sigma_{xx}(\omega)].
\]

(13)

Fluctuational electromagnetic field radiates not only energy, but also angular momentum [17, 32]. In the next step, let us derive the angular momentum radiation. The angular momentum flux is \( \dot{M} = \vec{T} \times \vec{\rho} \) [10, 32] with Maxwell tensor \( T_{ij} = \varepsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \frac{1}{\varepsilon_0} E_i^2 + \frac{1}{\mu_0} B_i^2 \delta_{ij} \). The total angular momentum radiation along the direction perpendicular to the 2D material in the positive \( z \) direction on one side is

\[
N_z = \int dx dy \left( T_{zy} - T_{zy} \right),
\]

(14)

\[
N_z = \int dx dy \left( \varepsilon_0 E_x E_x + \frac{1}{\mu_0} B_x B_x \right) y
\]

\[ - \left( \varepsilon_0 E_y E_y + \frac{1}{\mu_0} B_y B_y \right) x \].

(15)

The integration is over a surface located at \( z \to \infty \). Firstly, let us look at the contribution of \( EE^* \) terms by substituting the electric field in equation (4) into equation (15),

\[
N_z = \int dx dy \varepsilon_0 \mu_0 (\mu_0 \omega)^2 \int d^2 \vec{q} \int d^2 \vec{q}' \left[ (G_{zz} J'_x + G_{zy} J'_y) \times \right.
\]

\[ \left. (G_{zz} J'_x + G_{zy} J'_y) \right] \varepsilon_0 \mu_0 \omega (\mu_0 \omega)^2 \left[ (G_{zz} J'_x + G_{zy} J'_y) \right] \varepsilon_0 \mu_0 \omega (\mu_0 \omega)^2
\]

\[
\times \left( G_{zz} J'_x + G_{zy} J'_y \right) \varepsilon_0 \mu_0 \omega (\mu_0 \omega)^2 \left( G_{zz} J'_x + G_{zy} J'_y \right) x \right].
\]

(16)

In order to calculate efficiently, we perform a Fourier transform to the electric dyadic Green’s functions. A key step is to transfer the factor \( y \) into a derivative to \( q_y' \), that is \( y e^{-i q_y' x} = \frac{\partial}{\partial q_y} e^{-i q_y' x} \). In a similar way we transfer the factor \( x \) into a derivative of \( q_x' \), \( x e^{-i q_x' z} = \frac{i}{q_x} e^{-i q_x' z} \) (for details, refer to appendix B). In order to do this, let us consider an arbitrary torque term as an example \( N_z = \frac{dx}{d\omega} = \int dx dy \langle \dot{E}^2 \rangle \). Here \( E \) or \( B \) is some arbitrary component of \( \vec{E} \) or \( \vec{B} \). With the factor \( y \) transposed to a derivative to \( q_y' \), this torque term becomes

\[
N_z = 2 \int_0^\infty \frac{d\omega}{2\pi} \varepsilon_0 (\mu_0 \omega)^2 \left( \langle J'_x J'_x \rangle \right)_{q_y = 0} \times \int d^2 q \left[ \frac{1}{2\pi} \right] \varepsilon_0 \mu_0 \omega \left( \frac{i q_y^2}{4 \gamma} \right)
\]

(17)

Based on this algorithm, let us look at contribution of \( EE^* \) terms. We substitute dyadic elements and take differentiation of \( q_y' \) into equation (16). Because of the parity of the function, diagonal terms become zero after integration over \( \vec{q} \). Only cross terms are left.

\[
N_z = 2 \int_0^\infty \frac{d\omega}{2\pi} \varepsilon_0 (\mu_0 \omega)^2 \left( \langle J'_x J'_y \rangle \right)_{q_y = 0} \times \int d^2 q \frac{1}{2\pi} \varepsilon_0 \mu_0 \omega \left( \frac{i q_y^2}{4 \gamma} \right)
\]

(18)

The current–current correlation function is related to temperature through the fluctuation-dissipation theorem

\[
\langle J'_x J'_y \rangle = i \hbar \cdot 2 \omega N(\omega) \cdot \text{Im}[\sigma_{xy}(\omega)].
\]

(19)

So the angular momentum radiation is

\[
N_z = \int_0^\infty \frac{d\omega}{2\pi} \varepsilon_0 (\mu_0 \omega)^2 \left( \frac{1}{6 \pi} \cdot \hbar \omega N(\omega) \right) \cdot \text{Im}[\sigma_{xy}(\omega)]
\]

(20)

Taking into consideration of contribution of \( J'_x J'_y \) term and contribution of \( BB^* \) terms, the total angular momentum radiation should be

\[
N_z = \frac{1}{3 \pi \varepsilon_0 c^3} \int_0^\infty \frac{d\omega}{2\pi} \hbar \omega N(\omega) \text{Im}[\sigma_{xy}(\omega)].
\]

(21)

Equations (13) and (21) are the general formulas which can be used to any materials. However, the conductivity is strictly dependent on material structure and electric properties. It is obviously seen that the radiation property depends on alternating current (ac) conductivity of materials. Now let us look at the conductivity property of the Haldane model.
3. Application to the Haldane model

In a landmark paper [22], Haldane introduced a simple tight-binding model demonstrating the possibility of a nonzero Chern number in the 2D Brillouin zone (BZ). The model describes spinless electrons hopping between sites as sketched in figure 1. There are two inequivalent sites (called ‘A’ and ‘B’) shown as blue and brown circles respectively on a honeycomb lattice.

The Hamiltonian of the Haldane model written in second quantized notation is [22, 23]

\[
\hat{H}_0 = \Delta \sum_i \left( -1 \right)^i \epsilon_i \hat{c}_i + \sum_{\langle ij \rangle} (c_i^\dagger c_j + h.c.) + t_2 \sum_{\langle\langle ij\rangle\rangle} (i c_i^\dagger c_j + h.c.).
\]  

(22)

Where \(i\) and \(j\) run over all sites, \(\tau_i = \{1, 2\}\) corresponds to sublattice A and B respectively. Real parameter \(\Delta\) is the hopping strength between nearest neighbour (NN) sites labelled as \(\langle ij \rangle\). The on-site potential \(\Delta\) breaks inversion symmetry and the pure imaginary hopping \(it_2\) (or \(-it_2\)) between NNN \(\langle\langle ij\rangle\rangle\) breaks time reversal symmetry. The model given in the form of equation (22) is perhaps the simplest possible model having a topological phase.

According to equation (1), the interaction between material and the radiation field is

\[
\hat{H}' = \sum_{\tilde{\mathbf{F}}} \sum_{\tilde{\mathbf{F}}'} \sum_{\alpha} \sum_{\tilde{\mathbf{F}}\tilde{\mathbf{F}}'} \hat{M}_{\tilde{\mathbf{F}}\tilde{\mathbf{F}}'}^{\alpha} \hat{c}_{\alpha,\tilde{\mathbf{F}}} \hat{A}_{\alpha,\tilde{\mathbf{F}}}'^\dagger.
\]

(23)

Where \(\hat{A}\) is the vector potential of the radiation field and \(\hat{M}\) is the interaction matrix. The index \(l, l', l''\) run over all Bravais sites, \(j, f, f'\) run over sublattice A or B. \(\alpha\) is Cartesion index. In reciprocal space, the interaction Hamiltonian is

\[
\hat{H}' = \frac{1}{\sqrt{N}} \sum_{k,q} \tilde{\epsilon}(\tilde{k} + \tilde{q}) \frac{e}{2} \left[ \tilde{V}(\tilde{k} + \tilde{q}) + \tilde{V}(\tilde{k}) \right] c^\dagger(\tilde{k}) \cdot \hat{A}(\tilde{q}).
\]

(24)

In which \(\tilde{c} = (c_A^\dagger \ c_B^\dagger)\) is creation operator for both sublattices A and B. \(\tilde{V}(\tilde{k}) = \frac{i e \hbar}{\alpha} \frac{\partial \tilde{H}(\tilde{k})}{\partial \tilde{k}}\) is the electron group velocity [23, 33, 34]. The current in the lattice is

\[
\tilde{I}_q = \frac{1}{\sqrt{N}} \left( -\frac{e}{2} \right) \sum_{\tilde{k}} \tilde{c}(\tilde{k} + \tilde{q}) \left[ \tilde{V}(\tilde{k} + \tilde{q}) + \tilde{V}(\tilde{k}) \right] \cdot \hat{c}(\tilde{k}).
\]

(25)

Introduce new operators \(\tilde{c}(\tilde{k}) = S_l^\dagger(\tilde{k}) \tilde{c}_l(\tilde{k}) S_l(\tilde{k})\) to transform the interaction to mode space

\[
\hat{H}' = \sum_{l,q} \sum_{\alpha,\tilde{m},\tilde{n}} \tilde{c}_m(\tilde{k} + \tilde{q}) \left( -\frac{e}{2} \right) \frac{1}{\sqrt{N}} \varphi_{\alpha}^+(\tilde{k} + \tilde{q}) \frac{e}{2} \left[ \tilde{V}^{\alpha}(\tilde{k} + \tilde{q})
\]

\[
+ \tilde{V}^{\alpha}(\tilde{k}) \right] \varphi_{\alpha}(\tilde{k}) \tilde{c}_n(\tilde{k}) \tilde{A}_n^\dagger(\tilde{q}). \]

(26)

In which, \(m\) and \(n\) denote the conduction or valence bands, respectively. The interaction matrix in mode space is \(g_{\alpha m}^{\alpha n}(\tilde{k}) = \left( \varphi_{\alpha}^+(\tilde{k} + \tilde{q}) [\tilde{V}^{\alpha}(\tilde{k} + \tilde{q}) + \tilde{V}^{\alpha}(\tilde{k})] \right) \varphi_{\alpha}(\tilde{k})\). We introduce current–current correlation Green’s function \(\pi^e\) and express it in mode space

\[
\pi^e(q, \omega) = g_{\alpha m}^{\alpha n}(\tilde{k} + \tilde{q}) g_{\alpha n}^{\alpha m}(\tilde{k})
\]

\[
\times \frac{[f(\varepsilon_n(\tilde{k} + \tilde{q})) - f(\varepsilon_m(\tilde{k}))]}{\omega} + i \eta.
\]

(27)

Where \(\varepsilon_n(\tilde{k})\) is the energy of band \(n\) and wave vector \(\tilde{k}\). \(\eta\) is the inverse of duration lifetime of quasiparticle, \(f(\varepsilon_n(\tilde{k})) = \frac{1}{\varphi_{\alpha}(\tilde{k})^{\alpha n}(\tilde{k} + \tilde{q})}\) is Fermi distribution function at temperature \(T\) for band \(n\). The chemical potential is set to zero. Using long-wave approximation \(\tilde{q} \to 0\) and denoting the wave function by the Dirac notation, \(\varphi_{\alpha}(\tilde{k}) = [\tilde{k}, n]\), the current density correlation is obtained,

\[
\pi^e,\alpha(\tilde{q} \to 0, \omega) = \frac{e}{A} \sum_{mnk} \langle \tilde{k}, n \mid V^{\alpha}(\tilde{k}) \mid \tilde{k}, m \rangle \langle \tilde{k}, m \mid V^{\alpha}(\tilde{k}) \mid \tilde{k}, n \rangle
\]

\[
\times \left[ \frac{f_m(\tilde{k}) - f_n(\tilde{k})}{\varepsilon_m(\tilde{k}) - \varepsilon_n(\tilde{k}) - (\hbar \omega + i \eta)} \right].
\]

(28)

Here \(A\) is the area of the 2D material (we will take it very large). The connection between the spectral density of the fluctuating current sources and the local temperature of a body is provided by the FDT. In this way the conductivity is bridged to local temperature [35], thus

\[
\sigma^{\alpha\beta}(\omega) = \frac{ie^2}{A} \omega \sum_{mnk} \langle \tilde{k}, n \mid V^{\alpha}(\tilde{k}) \mid \tilde{k}, m \rangle \langle \tilde{k}, m \mid V^{\alpha}(\tilde{k}) \mid \tilde{k}, n \rangle
\]

\[
\times \left[ \frac{f_m(\tilde{k}) - f_n(\tilde{k})}{\varepsilon_m(\tilde{k}) - \varepsilon_n(\tilde{k}) - (\hbar \omega + i \eta)} \right].
\]

(29)

This is one important result of this paper. With the conductivity, energy and angular momentum radiation can be calculated.
4. Numerical results of conductivity and radiation

The conductivity of the Haldane model has the general properties of topological insulators. The longitudinal components are identical while the transverse components are antisymmetric. In the limit of $T \rightarrow 0$ and $\omega \rightarrow 0$, the only left conductivity component is transverse conductivity $\sigma_{xy}$, which equals the conductance quantum $e^2/h$ only when hopping between next nearest neighbours (NNN) $|t_2| > |t_{2c}|$, while remains zero when $|t_2| < |t_{2c}|$ (see figure 2(c)). This fact reveals the transition of the Haldane model from topological trivial state to topological nontrivial state. The critical value of NNN coupling is determined by band structure $t_{2c} = \Delta/\sqrt{3}$ [23]. At finite frequency, conductivity has peaks at specific frequency which corresponds to the resonant transition between the valance and conduction bands (see figure 2(b)). There are roughly two peaks when $|t_2|$ is not very large. The left peak corresponds to minimum band gap, while the highest peak corresponds to transition between highest density of states (see figure 2(c)). When $|t_2|$ is less than the critical absolute value $|t_{2c}|$, the conductivity threshold frequency decreases with the increasing of $|t_2|$, because of narrowed band gap at one $K$ point in the Brillouin zone. When $|t_2|$ increases further to overpass the critical value, peaks move to high frequencies because band gaps are broadened. Positions of peaks and troughs of transverse component are the same as those of longitudinal component (see figure 2(c) and (d)).

Conductivity components vary with temperatures. $\text{Re}(\sigma_{xx})$ and $\text{Im}(\sigma_{xy})$ are specifically considered, since they are particularly relevant to energy radiation and angular momentum radiation, respectively. It can be seen in figure 3 that conductivity decreases when temperature increases. Conductivity peaks are more pronounced at low temperatures but still survive as temperature goes high. Peak positions do not move as temperature changes. This manifests that peaks are determined by energy band structures which are not relevant to temperature.

The energy and angular momentum radiation as functions of temperature with fixed $\Delta$ and $t_1$ is displayed in figure 4. Both energy and angular momentum radiation increase slowly and then rapidly with temperature when temperature is not very high. The energy radiation is approximately $T^3$ (inset of figure 4(a)), implying consistency with Planck’s law. However, the fitting coefficient is approximately three orders of magnitude smaller than the Stefan–Boltzmann constant. When temperature is high, heat radiation deviates from $T^4$ tendency and presents a saturation with temperature. The mechanics here is that any concrete material has specific band structure which cannot emit photons with unlimited frequency. Similar conclusion can be drawn for angular momentum radiation. However, the power law at low temperature for angular momentum radiation is $T^3$ (inset of figure 4(b)). The value of the factor of $S_z$ is $\frac{4\pi^2t_0^2}{c^3} \sim 10^{-7}\text{SI}$, where $t_0 = 1\text{ meV}$. The value of the factor of $N_z$ is $\frac{4\pi^2t_0^2}{c^3} \sim 10^{-18}\text{SI}$. We can estimate the ratio of energy radiation to angular momentum radiation at room temperature is approximately $t_0/h \sim 10^{14}$. This value corresponds to room temperature dominant photon frequency. It reveals that each photon carries angular momentum of the magnitude of $h$. 

\[\text{Figure 2. Conductivity components } \sigma_{\alpha\beta} \text{ as a function of frequency } \omega \text{ for several NNN couplings } t_1. \text{ Some fixed parameters } \Delta = 4 \text{ meV}, t_1 = -10 \text{ meV}, T = 0.1\Delta. \text{ The critical value of NNN coupling } t_2 \text{ is displayed in figure 4(b)}. \]
Conductivity components vary with frequency at different temperatures. Fixed parameters $\Delta = 4$ meV, $t_1 = -10$ meV, $t_2 = -0.5\Delta$.

Energy radiation (a) and angular momentum radiation (b) as a function of temperature with fixed parameters $\Delta = 4$ meV, $t_1 = -10$ meV. Black squares, red triangles and green triangles correspond to $t_2 = -0.1\Delta$, $-0.5\Delta$, $-1.0\Delta$, respectively. Insets in (a) and (b) are radiation of $t_2 = -2.0$ meV versus $T^4$ and $T^3$ at very low temperatures. $t_0 = 1$ meV is energy unit in the calculation. $\alpha$ is fine structure constant.

On-site potential $\Delta$ and strength of imaginary coupling $t_2$ between NNN are important parameters to entitle the Haldane model topological insulator properties by breaking inversion symmetry and time reversal symmetry. Both of them are determinant factors to energy band structure, hence have great effect on the property of radiation. In figure 5 we investigate how they mutually affect energy and angular momentum radiation. The left panel is energy radiation contour of $\Delta$ and $t_2$. The right panel is the angular momentum radiation contour of $\Delta$ and $t_2$. It can be seen that the effects of $\Delta$ and $t_2$ on energy radiation and angular momentum are almost concurrent. The small patterns in both panels exhibit strong radiation for both energy and angular momentum. Most patterns are located on the left parts of the two panels, indicating topological nontrivial states for the Haldane model. This fact reveals that the strength of NNN coupling is more efficient in producing the radiation of the Haldane model. These patterns help us to optimize the two parameters. Obviously, the patterns located at $(t_2, \Delta) \approx (-3.3, 3.0)$ meV and $(t_2, \Delta) \approx (-3.4, 2.5)$ meV are examples of optimal combinations to generate high energy radiation as well as high angular momentum radiation.

The above result is obtained at room temperature with $t_1 = -10$ meV. It is verified that optimal values of these two
Figure 5. Energy and angular momentum radiation contour of $\Delta$ and $t_2$ at room temperature ($T = 27$ meV). Fixed parameter is $t_1 = -10$ meV. There is a factor of $8\alpha t_0^4/(3\pi c^2\hbar^3)$ for energy radiation. The factor for angular momentum radiation is $4\alpha t_0^3/(3\pi c^2\hbar^2)$.

Figure 6. Energy and angular momentum radiation vs $t_2$ with constant $\Delta$ and $t_1$ at two different temperatures ($T = 27$ meV, $T = 54$ meV). Fixed parameters are $\Delta = 2.8$ meV, $t_1 = -10$ meV. There is a factor of $8\alpha t_0^4/(3\pi c^2\hbar^3)$ for energy radiation. The factor for angular momentum radiation is $4\alpha t_0^3/(3\pi c^2\hbar^2)$.

parameters remain unchanged when temperature changes. Figure 6 is radiation vs $t_2$ with different temperatures.

There are distinct peaks in both the energy radiation curves and the angular momentum radiation curves. With fixed temperature and on-site potential, appropriate value of $t_2$ can greatly enhance both energy radiation and angular momentum radiation. It can also be seen that peaks are much higher when the absolute value of $t_2$ is large, which corresponds to nontrivial state of the material. There are two apparent features for these peak positions. One is that peak positions are independent of temperatures. The other is that peak positions of the two radiations are identical. These features imply that both energy radiation and angular momentum radiation are determined by energy band structure of the material, which is not relevant to temperature. If the on-site potential takes other values, there will be corresponding peaks at different places along $t_2$ axis. These discrete peaks explain the irregularly distributed patterns in the radiation contours in figure 5. Among
Because of the random fluctuational current \( J'(\vec{r}, \omega) \) in the material, Amperé’s law is written as
\[
\nabla \times \vec{H}(\vec{r}, \omega) = -i\omega \varepsilon_0 \vec{E}(\vec{r}, \omega) + \vec{F}(\vec{r}, \omega). \tag{A.1}
\]

The current density \( J'(\vec{r}, \omega) \) causes thermal fluctuations of the field. The average of fluctuational current is not zero, but the current–current correlation is not zero. Therefore, the radiative heat flux is not zero and can be obtained by solving the stochastic Maxwell’s equations. The vector potential \( \vec{A}(\vec{r}, \omega) \) is related to the electromagnetic field by (using the identity \( \nabla \times (\nabla \phi) = 0 \))
\[
\vec{B}(\vec{r}, \omega) = \nabla \times \vec{A}(\vec{r}, \omega), \tag{A.2}
\]
\[
\vec{E}(\vec{r}, \omega) = i\omega \vec{A}(\vec{r}, \omega) - \nabla \phi. \tag{A.3}
\]

The Faraday law is written as
\[
\nabla \times (\nabla \times \vec{A}(\vec{r}, \omega)) = -i\omega \varepsilon_0 \vec{E}(\vec{r}, \omega) + i\omega\varepsilon_0 \mu_0 \nabla \phi + \mu_0 \vec{F}(\vec{r}, \omega). \tag{A.4}
\]

Using the vector identity \( \nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - \nabla \nabla \cdot \vec{A} \), and the fact that \( k^2 = \omega^2 \varepsilon_0 \mu_0 = \frac{\omega^2}{c^2} \) (we will set the relative dielectric constant to 1 as we assume the field propagates in vacuum)
\[
(\nabla^2 + k^2) \vec{A}(\vec{r}, \omega) = \nabla \nabla \cdot \vec{A}(\vec{r}, \omega) - \mu_0 \vec{J} (\vec{r}, \omega) - i\omega\varepsilon_0 \mu_0 \nabla \phi. \tag{A.5}
\]

Using Lorenz gauge \( \nabla \cdot \vec{A} = -i\omega\varepsilon_0 \mu_0 \phi = 0 \), the solution of the vector potential is
\[
\vec{A}(\vec{r}, \omega) = \mu_0 \int_V \left[ \frac{\varepsilon_0 \mu_0}{k^2} g(\vec{r}, \vec{r'}, \omega) \vec{J}(\vec{r'}, \omega) \right] dV', \tag{A.6}
\]
in which the Green’s function is \( g(\vec{r}, \vec{r'}, \omega) = \frac{e^{-ik|\vec{r}-\vec{r'}|}}{4\pi|\vec{r}-\vec{r'}|} \).

Physically, this equation means that the solution for the field due to the source \( \vec{J} \) is the convolution of Green’s function with that source.

With the vector potential, we obtain the electric and magnetic field as
\[
\vec{E}(\vec{r}, \omega) = i\omega \mu_0 (I + \frac{1}{k^2} \nabla \nabla) \int_V g(\vec{r}, \vec{r'}, \omega) \cdot \vec{J}(\vec{r'}, \omega) dV', \tag{A.7}
\]
\[
\vec{H}(\vec{r}, \omega) = \nabla \times \int_V \left[ g(\vec{r}, \vec{r'}, \omega) I \cdot \vec{J}(\vec{r'}, \omega) \right] dV'. \tag{A.8}
\]

Defining electric and magnetic dyadic Green’s functions
\[
G^e(\vec{r}, \vec{r'}, \omega) = I + \frac{1}{k^2} \nabla \nabla \int_V g(\vec{r}, \vec{r'}, \omega) \cdot \vec{J}(\vec{r'}, \omega) dV', \tag{A.9}
\]
\[
G^m(\vec{r}, \vec{r'}, \omega) = \nabla \times \int_V (g(\vec{r}, \vec{r'}, \omega) I), \tag{A.10}
\]
the electric and magnetic fields are written as a function of dyadic Green’s functions,
\[
\vec{E}(\vec{r}, \omega) = i\omega \mu_0 \int_V G^e(\vec{r}, \vec{r'}, \omega) \cdot \vec{J}(\vec{r'}, \omega) dV', \tag{A.11}
\]
\[
\vec{H}(\vec{r}, \omega) = \nabla \times \int_V (G^m(\vec{r}, \vec{r'}, \omega) I) \cdot \vec{J}(\vec{r'}, \omega) dV'. \tag{A.12}
\]
\[ \tilde{H}(\mathbf{r}, \omega) = \int_V \tilde{G}_m(\mathbf{r}, \mathbf{r}', \omega) \cdot \mathbf{J}^*(\mathbf{r}', \omega) dV'. \]  

(A.12)

Time-averaged Poynting vector is expressed as [3]

\[ \langle \mathbf{S}(\mathbf{r}, \omega) \rangle = 4 \times \frac{1}{2} \text{Re} \left\{ \mathbf{E}(\mathbf{r}, \omega) \times \tilde{H}^*(\mathbf{r}, \omega) \right\}. \]  

(A.13)

The factor 4 comes from the fact that only positive frequencies are considered in the formula while Fourier transform from time-dependent fields to frequency-dependent quantities contains frequency in the whole domain. The radiation medium is time-dependent, so the Fourier transforms are considered in the formula while Fourier transform from real space to mixed representation by \[ \mathbf{S}(\mathbf{r}, \omega) = \int_V \mathbf{E}(\mathbf{r}, \omega) \times \mathbf{H}^*(\mathbf{r}, \omega) dV'. \]  

Here we consider the system as three dimensional. According to the fluctuation–dissipation theorem (FDT) \[ \langle J'_x(\mathbf{r}, \omega) J'_y(\mathbf{r}', \omega) \rangle = 2 \hbar \omega^2 \varepsilon_0 N(\omega) \text{Im} \{ \varepsilon(\omega) \} \delta(\mathbf{r}' - \mathbf{r}') \delta_{\epsilon,\gamma} \]  

Since the material geometry is a plane, we specialize the FDT to 2D by replacing the bulk current with surface current and take \( \varepsilon(\omega) \) in place of \( \varepsilon_r \) in the dielectric constant. The current–current correlation is related to material conductivity by

\[ \left\langle J'_x(\mathbf{r}, \omega) J'_y(\mathbf{r}', \omega) \right\rangle = 2 \hbar \omega N(\omega) \varepsilon_0 \text{Re} \{ \sigma_{\alpha \beta}(\omega) \} \delta(\mathbf{r}' - \mathbf{r}'). \]  

(A.15)

Here \( \mathbf{r}' \) and \( \mathbf{r} \) are any in-plane position vectors. In order to calculate the Poynting expectation value, let us do Fourier transform to dyadic Green’s functions. First, from real space to reciprocal space,

\[ [G'(\mathbf{q})]_{ij} = [\delta_{ij} - \frac{q_i q_j}{k^2}] g(q), \]  

(A.16)

where \( g(q) \) is obtained by solving Green’s function of Helmholtz equation Fourier transformed from real space to reciprocal space

\[ g(q_1, q_2) = \frac{1}{2 \pi} \frac{1}{q^2 - (\gamma + i\eta)^2}, \]

\[ \gamma^2 = k^2 - q^2, \]

\[ k^2 = \omega^2 \varepsilon_0 \mu_0 = \frac{\omega^2}{c^2}. \]  

(A.17)

Then we need to perform Fourier transform from reciprocal space to a mixed representation by \[ [G'(q_1, q_2)]_{ij} = [\delta_{ij} - \frac{q_i q_j}{k^2}] \int \frac{d^2 q'}{2 \pi} e^{i \mathbf{q}' \cdot \mathbf{r}} \]  

and obtain

\[ G'_{\alpha \beta}(q_1, q_2) = [\delta_{\alpha \beta} - \frac{q_\alpha q_\beta}{k^2}] \int \frac{d^2 q'}{2 \pi} e^{i \mathbf{q}' \cdot \mathbf{r}}. \]  

(A.18)

Eventually the electric Green’s dyadic is \( (k = \omega/c) \)

\[ G'(q_1, q_2) = \begin{pmatrix} 1 - \frac{q_x q_y}{k^2} & \frac{q_x q_y}{k^2} & -q_x \gamma \frac{k}{k^2} \\ \frac{q_x q_y}{k^2} & (1 - \frac{q_x q_y}{k^2}) & -q_y \gamma \frac{k}{k^2} \\ -q_x \gamma \frac{k}{k^2} & -q_y \gamma \frac{k}{k^2} & (1 - \frac{\gamma^2}{k^2}) \end{pmatrix} \cdot \frac{i}{2 \gamma} e^{i |\mathbf{q}|}. \]  

(A.19)

The magnetic Green’s dyadic is derived in a similar way

\[ G'(q_1, q_2) = \begin{pmatrix} 0 & -\gamma q_x & q_y \\ q_x & 0 & -q_z \\ -q_y & q_z & 0 \end{pmatrix} \cdot \frac{i}{2 \gamma} e^{i |\mathbf{q}|}. \]  

(A.20)

With the fact of symmetry of conductivity \( \sigma_{\varepsilon \gamma}(\omega) = \sigma_{\gamma \varepsilon}(\omega) \), expectation value of Poynting vector is calculated by substituting equation (A.19), (A.20) and (A.15) into equation. (A.14) and integrating over \( q_1 \), so that

\[ \langle \mathbf{S}(\mathbf{r}, \omega) \rangle = \int_0^\infty \frac{d^2 q}{2 \pi} \frac{2 \mu_0}{3 \pi} \hbar \omega^3 N(\omega) \text{Re} [\sigma_{\varepsilon \gamma}(\omega)]. \]  

(A.21)

This is the radiation of monochromatic electro–magnetic field. Total radiation includes contributions of all photons with different frequencies

\[ \langle \mathbf{S} \rangle = \frac{2}{3 \pi \varepsilon_0 c^3} \int_0^\infty d\omega \frac{2 \mu_0}{2 \pi} \hbar \omega^3 N(\omega) \text{Re} [\sigma_{\varepsilon \gamma}(\omega)]. \]  

(A.22)

**Appendix B. Derivation of angular momentum radiation \( \mathbf{N}_2 \)**

Poynting vector of electro–magnetic field \( \mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \) is energy flux per area per unit time. It can also be written as \( \mathbf{S} = \mathbf{u} \mathbf{c} \) where \( u = c \varepsilon(\varepsilon_0 \mathbf{E}^2 + \frac{1}{\mu_0} \mathbf{B}^2) \) is the energy density and \( c \) is speed of light. From the photons point of view, the energy and momentum are related by \( \varepsilon = c \nu \) since photons are massless. From this we can write the momentum density, or momentum per unit volume as \( \mathbf{u} \mathbf{c} \) or \( \frac{1}{\mu_0} \mathbf{B}^2 / 2 \). Taking the origin as reference point, the angular momentum density is \( \mathbf{I} = \mathbf{r} \times \frac{\mathbf{E} \times \mathbf{B}}{\mathbf{r}} \).

\[ \mathbf{I} = \mathbf{r} \times (\mathbf{E} \times \mathbf{B}) \varepsilon_0. \]  

(B.1)

Having identified the angular momentum density \( \mathbf{I} \), we then determine the angular momentum flux, i.e. for what tensor \( \mathbf{M} \) is away from the source, so that \( \frac{d \mathbf{M}}{dt} + \nabla \cdot \mathbf{M} = 0 \).

To find the tensor \( \mathbf{M} \), we use sourceless Maxwell’s equations. Taking derivation of equation (B.1) and using vector analysis identity, we have

\[ \frac{d}{dt} \mathbf{I} = \nabla \cdot \{ \mathbf{r} \times [\varepsilon_0 \mathbf{E} \mathbf{E} + \frac{1}{\mu_0} \mathbf{B} \mathbf{B} - \mathbf{u} \mathbf{I}] \}, \]  

(B.2)

where dot \( \mathbf{I} \) means time differentiation. The Maxwell tensor is

\[ \mathbf{T} = \frac{\mathbf{E} \mathbf{E} + \frac{1}{\mu_0} \mathbf{B} \mathbf{B} - \mathbf{u} \mathbf{I}}{\mu_0}. \]  

(B.3)
Then the conservation of angular momentum is \( \frac{\partial}{\partial t} + \nabla \cdot (T \times \vec{r}) = 0 \). The angular momentum flux is \( \dot{M} = \vec{T} \times \vec{F} \). This agrees with O Keller’s equations (2.140) and (2.141) [10].

Having worked out the angular momentum transfer flux, we apply it to a planar geometry. The matter is located at \( z = 0 \) in the \( x-y \) plane. We find the total flux by integrating on a surface at \( z \to \infty \) in the \( x-y \) plane, which is \( \vec{N} = \frac{d\vec{r}}{dt} = \int dxdy \vec{z} \cdot \dot{\vec{M}} \). By symmetry, either average of \( \vec{z} \) component or average of \( y \) component is zero. The \( \vec{z} \) component is

\[
N_z = \int dxdy \left( T_{xz} - T_{zy} \right),
\]

(B.4)

In which \( T_{xz} = \varepsilon_0 E_x E_z + \frac{1}{\mu_0} B_z B_x, T_{zy} = \varepsilon_0 E_z E_y + \frac{1}{\mu_0} B_y B_z \).

A key step is to transfer factor \( y \) into a derivative of \( \vec{q}' \), that is \( y e^{-i\vec{q}' \cdot \vec{r}_z} = \int_0^\infty \frac{d\vec{q}'}{(2\pi)^2} e^{-i\vec{q}' \cdot \vec{r}_z} \). The factor \( x \) is also needed to transfer to a derivative as \( x e^{-i\vec{q}' \cdot \vec{r}_z} = \int_0^\infty \frac{d\vec{q}'}{(2\pi)^2} e^{-i\vec{q}' \cdot \vec{r}_z} \). In order to do this, let us consider an arbitrary torque term as an example

\[
N_z = \frac{dL_z}{dt} = \int dxdy \langle EB' \rangle y.
\]  

(B.5)

Here \( E \) or \( B \) is some arbitrary component of \( \vec{E} \) or \( \vec{B} \). Electric and magnetic fields equations (A.11) and (A.12) are substituted into this equation,

\[
N_z = \int dxdy \int G' \left( \vec{r} - \vec{r}_z', z \right) j(\vec{r}_z') d^2r_z'
\]

\[
\times \int G'' \left( \vec{r} - \vec{r}_z'', z \right) j(\vec{r}_z'') d^2r_z'' \times y.
\]

(B.6)

Do Fourier transform for Green’s functions and integrate over \( x, y \) first, we get

\[
N_z = \int \frac{d^2r_z'}{(2\pi)^2} \int \frac{d^2r_z''}{(2\pi)^2} \int \frac{d^2q}{(2\pi)^2} \int \frac{d^2q}{(2\pi)^2} G' (\vec{q}, z) e^{-i\vec{q}' \cdot \vec{r}_z'} j(\vec{r}_z') f_2(\vec{r}_z'') G'' (\vec{q}, z)
\]

\[
\times \int dxdy e^{i\vec{q}' \cdot \vec{r}_z} \left[ \frac{\partial}{\partial q_x} e^{-i\vec{q}' \cdot \vec{r}_z} \right] e^{i\vec{q}' \cdot \vec{r}_z''}.
\]  

(B.7)

Exchange the order of \( \int dx \int dy \int \frac{d\vec{q}_x}{(2\pi)^2} \) to \( \int \frac{d\vec{q}_x}{(2\pi)^2} \int dx \int dy \), the equation becomes

\[
N_z = \int \frac{d^2r_z'}{(2\pi)^2} \int \frac{d^2r_z''}{(2\pi)^2} \int \frac{d^2q}{(2\pi)^2} \int \frac{d^2q}{(2\pi)^2} G' (\vec{q}, z)
\]

\[
\times \left( j(\vec{r}_z') f_2(\vec{r}_z'') \right) e^{-i\vec{q}' \cdot \vec{r}_z'} e^{i\vec{q}' \cdot \vec{r}_z''} \times G'' (\vec{q}', z) \delta (\vec{q} - \vec{q}').
\]  

(B.8)

We use the fact that current–current correlation is translationally invariant, that is \( \int d^2r_z' \left\langle j(\vec{r}_z') f_2(\vec{r}_z'') \right\rangle e^{-i\vec{q}' \cdot \vec{r}_z'} = \left\langle j(\vec{r}_z') \right\rangle e^{-i\vec{q}' \cdot \vec{r}_z} \). In the real space Fourier transform of \( jj \) is independent of \( \vec{r}_z \).

Substitute dyadic elements and take differentiation of \( q_x \) or \( q_y \), into equation (B.14). Because the parity of the function, diagonal terms become zero after integration over \( \vec{q} \). Only cross terms are left,
of the Haldane model. In general formulation. Now, let us look at the conductivity property

\[ N_\varepsilon(E) = 2 \int_0^\infty \frac{d\omega}{2\pi} \varepsilon_0(\mu\omega)^2 \langle j_x j_y \rangle_{q=-i(\frac{\pi}{\gamma})} \]

\[ \times \int \frac{d^2q}{(2\pi)^2} \left( -\frac{i\gamma}{4} \right) \]

\[ = 2 \int_0^\infty \frac{d\omega}{2\pi} \varepsilon_0(\mu\omega) \frac{-1}{c} \sum_{\mathbf{k}} \langle j_x j_y \rangle_{q=-i(\frac{\pi}{\gamma})} \]

(B.15)

According to fluctuation-dissipation theorem and linear respond theory \( \langle j_x j_y \rangle = \hat{\eta} \cdot \omega N(\omega) \cdot \Im[\sigma_{xy}(\omega)] \) [39], the angular momentum is related to the conductivity of the material,

\[ N_\varepsilon = \frac{h}{3\pi} \int_0^\infty \frac{d\omega}{2\pi} \varepsilon_0^2 \omega N(\omega) \Im[\sigma_{xy}(\omega)]. \]

(B.16)

Taking into consideration of contributions of \( j_x, j_y \) term and terms of exchange \( x \leftrightarrow y \), the overall contribution of \( EE' \) terms to angular momentum radiation is

\[ N_\varepsilon = \frac{1}{3\pi} \int_0^\infty \frac{d\omega}{2\pi} \varepsilon_0^2 \omega N(\omega) \Im[\sigma_{xy}(\omega)]. \]

(B.17)

Further calculations verify that all \( BB' \) terms contribute zero to the angular momentum radiation.

Equations (A.22) and (B.17) are the general formulas of far-field radiation. It is obviously seen that the radiation property ultimately depends on AC conductivity of materials. Our result equations (A.22) and (B.17) agrees with that obtained by Maghrebi et al. in reference [14], where they have use path integral formulation. Now let us look at the conductivity property of the Haldane model.

**Appendix C. Derivation of conductivity of the Haldane model \( \sigma_{\alpha\beta}(\omega) \)**

The Haldane model is defined on a graphene-like honeycomb structure. When interacting with radiation field, the Hamiltonian is written as

\[ \hat{H} = \sum_{ij} \sum_{\mathbf{k}} c_i^\dagger \mathcal{H}_{\mathbf{k}ij} \mathcal{A} \cdot \hat{x} \hat{y} e^{\pm i \mathbf{k} \cdot \mathbf{r}}, \]

(C.1)

where \( \hat{A} \) is the vector potential of radiation field. Using a trapezoidal rule of integration and expanding the exponent to the first order of \( \hat{A} \), the Hamiltonian is divided into two parts, the free Hamiltonian of the Haldane model and the interaction with the radiation field. In \( \hat{\mathbf{k}} \) space, the free material Hamiltonian is

\[ \hat{H}_0 = \sum_{\mathbf{k}} c_\mathbf{k}^\dagger \hat{\mathbf{H}}(\hat{\mathbf{k}}) c_{\mathbf{k}}, \]

(C.2)

in which, \( c_\mathbf{k} = (c_{\mathbf{k}x}^\dagger \ c_{\mathbf{k}y}^\dagger)^T \) are row and column vectors of the creation and annihilation operators, \( \hat{\mathbf{H}}(\hat{\mathbf{k}}) = \sum_{\mathbf{j}} [\mathbf{t}_1 \cos(\hat{\mathbf{k}} \cdot \hat{\mathbf{r}}) \sigma_z + t_1 \sin(\hat{\mathbf{k}} \cdot \hat{\mathbf{r}}) \sigma_y + \Delta - 2t_2 \sum_{\mathbf{j}'} \sin(\hat{\mathbf{k}} \cdot \hat{\mathbf{r}}) \sigma_z, ] \) \( j \) represents the displacement vector of nearest neighbors \( (j = 1, 2, 3) \) or NN (\( j = 4, 5, 6 \)).

The interaction Hamiltonian is

\[ \hat{H}^\prime = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}, j, i, j'} c_\mathbf{k}^\dagger \hat{M}_{ij}(\mathbf{k}) c_{\mathbf{k}'} \mathcal{A}_{ij}(\mathbf{q}). \]

(C.3)

For simplicity, we use long wave approximation \( r_j^\prime \approx 0, \mathcal{A}_{ij}(\mathbf{k}) \approx \mathcal{A}_0(\mathbf{k}), \sum_{\mathbf{j}} \delta_{\mathbf{j}, \mathbf{j}'} = 1 \). Considering electron velocity \( \mathcal{V}_{ij}(\mathbf{q}) = \frac{\hbar}{\varepsilon_0(\mu\omega)} = \sum_i H_{ij}(\varepsilon_i) \varepsilon_i^{-\frac{1}{2}} R_i e^{i\mathbf{q} \cdot \mathbf{R}_i}, \) the interaction matrix can be expressed as \( \hat{M}_{ij} = \delta(\mathbf{k} - \mathbf{k}' + \mathbf{q}) \mathcal{V}_{ij}(\mathbf{q}) \mathcal{A}_{ij}(\mathbf{q}). \) The interaction Hamiltonian becomes

\[ \hat{H}^\prime = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}, j, i, j'} c_\mathbf{k}^\dagger \hat{M}_{ij}(\mathbf{k}) c_{\mathbf{k}'} \mathcal{A}_{ij}(\mathbf{q}). \]

(C.4)

Then we need to transform the Hamiltonian to mode space. In reciprocal space,

\[ \hat{H}_0 = \sum_{\mathbf{k}} c_\mathbf{k}^\dagger \hat{\mathbf{H}}(\mathbf{k}) c_{\mathbf{k}} = \sum_{\mathbf{k}} c_\mathbf{k}^\dagger \hat{S_1}(\mathbf{k}) \frac{\varepsilon_1}{0} \frac{0}{\varepsilon_2} S_1^\dagger c_{\mathbf{k}}. \]

(C.5)

Introducing new operators, \( \tilde{c}(\mathbf{k}) = S_1^\dagger c_{\mathbf{k}}, \tilde{c}_1(\mathbf{k}) = \tilde{c}(\mathbf{k}) S_1, \) the free electron Hamiltonian is transformed to mode space,

\[ \hat{H}_0 = \sum_{\mathbf{k}} c_{\mathbf{k}}^\dagger \tilde{c}_1(\mathbf{k}) \frac{\varepsilon_1}{0} \frac{0}{\varepsilon_2} \tilde{c}_1(\mathbf{k}) = \sum_{\mathbf{k}} \sum_{n=1}^2 \varepsilon_n c_n^\dagger c_n. \]

(C.6)

In the same way, we transform the interaction part to mode space,

\[ \hat{H}^\prime = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}, j, i, j'} c_{\mathbf{k}}^\dagger \hat{M}_{ij}(\mathbf{q}) c_{\mathbf{k}'} \mathcal{A}_{ij}(\mathbf{q}). \]

(C.7)

With an interaction matrix \( \hat{M}_{ij}(\mathbf{q}) = \tilde{c}(\mathbf{k} + \mathbf{q}) \mathcal{V}_{ij}(\mathbf{q}) \mathcal{A}_{ij}(\mathbf{q}). \)

The current in the mode space is

\[ I_0^\prime(\mathbf{q}) = -\frac{1}{\sqrt{N}} \sum_{m,n} c_{\mathbf{k}}^\dagger (\mathbf{k} + \mathbf{q}) \mathcal{V}_{mn}^\dagger(\mathbf{q}) \mathcal{A}_{mn}(\mathbf{q}) c_{\mathbf{k}}(\mathbf{q}). \]

(C.8)

Introduce current–current correlation Green’s function \( \pi^\alpha \) and derive the expression in mode space. In real space, the current–current correlation is defined as the Green’s function

\[ \pi^\alpha(\mathbf{r}, t) = -\frac{1}{\hbar} \langle I^\alpha_0(0) I^\beta_0(t) \rangle. \]

(C.9)

Using Wick’s theorem, and expressing the result in terms of the electron Green’s function \( \mathcal{G}(\mathbf{q}, t) \), the result is

\[ \pi^\alpha(\mathbf{r}, t) = -\frac{1}{\hbar} \text{Tr} \left[ M^\alpha M^\beta(\mathbf{q}) \mathcal{G}(\mathbf{q}, t) \mathcal{G}^\dagger(\mathbf{q}, t) \mathcal{G}^\dagger(\mathbf{q}, t) \mathcal{G}(\mathbf{q}, t) \right]. \]

(C.10)

The trace above is over the electron degrees of freedom. Do Fourier transform and diagonalize electronic Green’s
functions $G^F$ and $G^<$

$$\pi'(q, \tau) = (-i\hbar) \text{Tr}_{\mathcal{R}} \left( g_{nm}^{\alpha\beta}(k + q) \left( \{ \xi^\alpha_n(k + q, \tau) \} \{ \xi^\beta_m(k, -\tau) \} \right) + g_{nm}^{\alpha\beta}(k + q) \left( \{ \xi^\beta_m(k + q, \tau) \} \{ \xi^\alpha_n(k, -\tau) \} \right) \right).$$

(C.11)

As electron Green’s function is substituted into (C.11), the correlation function becomes

$$\pi'(q, \tau) = \frac{1}{\hbar} \text{Tr}_{\mathcal{R}} \left( g_{nm}^{\alpha\beta}(k + q) g_{mn}^{\alpha\beta}(k) f(\xi^\alpha_n(k)) \right) - f(\xi_m(k + q)) e^{-\frac{i}{\hbar} \int [v_{\alpha}(k + q) - \xi^\alpha_n(k - q)]} \pi. \quad (C.12)$$

Fourier transform this correlation function to frequency domain and let $\varphi_m(k) = \langle \vec{k}, n \rangle$.

$$\pi^{J,J';\omega}(\vec{q}, \omega) = \sum_{nm} \epsilon \langle \vec{k}, n | V^{\alpha}(\vec{k}) + V^{\alpha}(\vec{k} + \vec{q}) | \vec{k} + \vec{q}, m \rangle \times \epsilon \langle \vec{k} + \vec{q}, m | V^{\beta}(\vec{k} + \vec{q}) + V^{\beta}(\vec{k}) | \vec{k}, n \rangle \cdot \left[ f_m(\vec{k} + \vec{q}) - f_n(\vec{k}) \right] - f_m(\vec{k} + \vec{q}) - f_n(\vec{k}) - (h\omega + i\eta) \right]. \quad (C.13)$$

Finally, we use long-wave approximation $\vec{q} \rightarrow 0$. Furthermore, with $J_0 = I_{xx}/I_0$ and material area $A = I_{xx}/I_0$, we transfer current correlation to current density correlation,

$$\pi^{J,J';\omega}(\vec{q}, \omega) = \frac{\epsilon^2}{A} \sum_{nm} \langle \vec{k}, n | V^{\alpha}(\vec{k}) | \vec{k}, m \rangle \langle \vec{k}, m | V^{\beta}(\vec{k}) | \vec{k}, n \rangle \times \left[ f_m(\vec{k}) - f_n(\vec{k}) \right] - f_m(\vec{k}) - (h\omega + i\eta) \right]. \quad (C.14)$$

According to linear electronic response $\pi_{\omega}^{J,J';\omega}(\vec{q}, \omega) = -i\omega \sigma^{\alpha\beta}(\vec{q}, \omega)$, we obtain the conductivity

$$\sigma^{\alpha\beta}(\omega) = \frac{\epsilon^2}{A} \sum_{nm} \langle \vec{k}, n | V^{\alpha}(\vec{k}) | \vec{k}, m \rangle \langle \vec{k}, m | V^{\beta}(\vec{k}) | \vec{k}, n \rangle \times \left[ f_m(\vec{k}) - f_n(\vec{k}) \right] - f_m(\vec{k}) - (h\omega + i\eta) \right] \quad (C.15)$$

In the limit of $\omega \rightarrow 0$ and $T \rightarrow 0$, it can be verified that $\sigma_{xx} = 0$ and $\sigma_{xy} = \frac{\epsilon^2}{\pi} C$ [23]. Here $C$ is the Chern number. For the Haldane model $C = 1, 0, -1$, depends on the value of NNN hopping $t_2$.

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