Exact solution of the time-dependent harmonic plus an inverse harmonic potential with a time-dependent electromagnetic field

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Abstract

In this article, the problem of the charged harmonic plus an inverse harmonic oscillator with time-dependent mass and frequency in a time-dependent electromagnetic field is investigated. It is reduced to the problem of the inverse harmonic oscillator with time-independent parameters and the exact wave function is obtained.

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I. INTRODUCTION

A time-dependent quantum system with a Hamiltonian which contains a singular term

\[ \hat{H} = \frac{\hat{p}^2}{2m(t)} + \frac{m(t)}{2} \omega^2(t) \hat{q}^2 + \frac{C}{m(t)\hat{q}^2}, \] (1)

is one of the rare examples admitting exact solution and has been subject of great interest in a variety of contexts [1–10], because of its various applications in different areas of physics. For example, it is used for constructing exactly solvable models of interacting many-body systems [1,2], for modelling polyatomic-molecules [3] and for coherent-state formalism [11]. The Hamiltonian in Eq. (1) has been studied and the exact solution was obtained by using the Lewis and Riesenfeld invariant method and by making canonical and unitary transformations [5–9]. Although the exact solution of the time-dependent charged harmonic oscillator in a time-varying electromagnetic field is known [12], the charged harmonic plus inverse harmonic oscillator problem in a time-varying electromagnetic field has not yet been solved exactly.

In this letter we take the following Hamiltonian into account

\[ \hat{H} = \frac{[\hat{p} - q\hat{A}(t)]^2}{2m(t)} + \frac{m(t)}{2} \omega^2(t) \hat{q}^2 + \frac{C}{m(t)\hat{q}^2} + q\hat{\phi}(t) = i \frac{\partial}{\partial t}, \] (2)

where we have set \( \hbar = 1 = c \). By making some transformations, Eq. (2) is reduced to

\[ \frac{\hat{p}'^2}{2} + \frac{C}{\hat{q}^2} = if(t') \frac{\partial}{\partial t'}, \] (3)

where \( f(t) \) is a time-dependent function. Finally, Eq. (3) is integrated.

II. METHOD

The Hamiltonian for the charged harmonic plus inverse harmonic oscillator constrained to move in the \( x - y \) plane is given

\[ \hat{H} = \frac{[\hat{p} - q\hat{A}(t)]^2}{2m(t)} + \frac{m(t)}{2} \omega^2(t) (\hat{x}^2 + \hat{y}^2) + \frac{C}{m(t)(x^2 + y^2)} + q\hat{\phi}(t) = i \frac{\partial}{\partial t}. \] (4)
where \( C \) is any constant and \( \mathbf{p} = -i \frac{\partial}{\partial x} \mathbf{e}_1 - i \frac{\partial}{\partial y} \mathbf{e}_2 \).

In this letter, we restrict ourselves to the case where the electromagnetic potentials are given by

\[
\mathbf{A} = \frac{B(t)}{2} (\dot{y} \mathbf{e}_1 - \dot{x} \mathbf{e}_2), \quad \phi(t) = 0,
\]

where \((\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)\) are the unit vectors of the cartesian basis. With this choice, the magnetic and electric fields turn out to be

\[
\mathbf{B} = B(t) \mathbf{e}_3, \quad \mathbf{E} = \frac{1}{2} \rho dB(t) \frac{d\phi}{dt} \mathbf{e}_\phi,
\]

where \((\hat{\rho}, \hat{\phi}, \hat{\mathbf{e}_3})\) form the cylindrical basis. Hereafter, we will denote \(\omega(t), m(t)\) and \(B(t)\) by \(\omega, m\) and \(B\), respectively, for brevity. By substituting Eq. (5) into Eq. (4), the Hamiltonian becomes

\[
\frac{\hat{p}_x^2 + \hat{p}_y^2}{2m} - \frac{qB}{4m} (\dot{y} \hat{p}_x - \dot{x} \hat{p}_y) + \frac{m}{2} \left( \frac{\omega^2 + \frac{q^2B^2}{4m^2}}{m(\dot{x}^2 + \dot{y}^2)} \right) (\ddot{x}^2 + \ddot{y}^2) + \frac{C}{m(\dot{x}^2 + \dot{y}^2)} = i \frac{\partial}{\partial t}
\]

Now we want to make a transformation to get rid of the cross-term in Eq. (8). To do this, an orthogonal transformation is introduced [12].

\[
\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \cos \beta(t) & -\sin \beta(t) \\ \sin \beta(t) & \cos \beta(t) \end{pmatrix} \begin{pmatrix} \dot{x}' \\ \dot{y}' \end{pmatrix},
\]

The time-dependent transformation (9) implies the following transformations for the quadratic terms

\[
\hat{p}_x^2 + \hat{p}_y^2 \rightarrow \hat{p}_x'^2 + \hat{p}_y'^2, \quad \dot{x}^2 + \dot{y}^2 \rightarrow \dot{x}'^2 + \dot{y}'^2,
\]

and for the cross-term and time derivative

\[
\dot{y} \hat{p}_x - \dot{x} \hat{p}_y \rightarrow \dot{y}' \hat{p}_x' - \dot{x}' \hat{p}_y',
\]

\[
\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} + i\beta(t) \left[ \dot{\beta}(t) - \dot{x}' \hat{p}_y' - \dot{y}' \hat{p}_x' \right].
\]
To eliminate the cross-term from the Eq. (8), $\beta$ is chosen as

$$\dot{\beta}(t) = \frac{qB}{4m},$$

(13)

where dot denotes time derivation as usual. In this case, Eq. (8) becomes

$$\frac{\hat{p}_x^2 + \hat{p}_y^2}{2m} + \frac{m}{2} \left( \omega^2 + \frac{q^2B^2}{4m^2} \right) (\hat{x}^2 + \hat{y}^2) + \frac{C}{m(\hat{x}^2 + \hat{y}^2)} = i \frac{\partial}{\partial t}. $$

(14)

Eq. (14) does not depend on the cross-term and the square of the angular frequency is shifted by an amount of $q^2B^2/(4m^2)$.

To remove the harmonic part, we make the transformation

$$\hat{p}_x' \rightarrow \hat{P}_1 = \hat{p}_x' + i\alpha(t)\hat{x}', \quad \hat{p}_y' \rightarrow \hat{P}_2 = \hat{p}_y' + i\alpha(t)\hat{y}',$$

(15)

$$\hat{x}' \rightarrow \hat{Q}_1 = \hat{x}', \quad \hat{y}' \rightarrow \hat{Q}_2 = \hat{y}',$$

(16)

where $\alpha(t)$ is a time-dependent function to be determined later. Since $\hat{p} = -i \frac{\partial}{\partial x} \hat{e}_1 - i \frac{\partial}{\partial y} \hat{e}_2$, we observe that the transformation (15) is equivalent to the multiplication of the wave function by $\exp[-\alpha(t)(x'^2 + y'^2)/2]$.

Under the transformation (15), the time derivative operator becomes

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} - \dot{\alpha}(t) \left( \frac{\hat{Q}_1^2 + \hat{Q}_2^2}{2} \right),$$

(17)

Substituting Eqs. (15), (16) and (17) in (14), we get

$$\frac{\hat{P}_1^2 + \hat{P}_2^2}{2m} + \Omega^2(t) \left[ \hat{Q}_1^2 + \hat{Q}_2^2 \right] + \frac{C}{m(\hat{Q}_1^2 + \hat{Q}_2^2)} + i \frac{\alpha(t)}{m} (\hat{Q}_1\hat{P}_1 + \hat{Q}_2\hat{P}_2) = i \frac{\partial}{\partial t} + \frac{\alpha(t)}{m},$$

(18)

where

$$\Omega^2(t) = \frac{m}{2} \left( \omega^2 + \frac{q^2B^2}{4m^2} \right) - \frac{\alpha^2(t)}{2m} + i \frac{\dot{\alpha}(t)}{2}. $$

The time-dependent functions $\alpha(t)$ should be chosen so that harmonic term in the Hamiltonian vanishes.

$$\frac{m}{2} \left( \omega^2 + \frac{q^2B^2}{4m^2} \right) - \frac{\alpha^2(t)}{2m} + i \frac{\dot{\alpha}(t)}{2} = 0.$$

(19)
Eq. (18) is simplified after the elimination of the harmonic part.

\[
\frac{\hat{P}_1^2 + \hat{P}_2^2}{2m} + \frac{C}{m} \left( \frac{\hat{Q}_1^2}{\hat{Q}_1^2 + \hat{Q}_2^2} \right) + i\alpha(t) \left[ \hat{Q}_1 \hat{P}_1 + \hat{Q}_2 \hat{P}_2 \right] = i \frac{\partial}{\partial t} + \frac{\alpha(t)}{m}. \tag{20}
\]

Now a velocity-dependent interaction term is contained in Eq. (20). To get rid of this term, we need another transformation.

\[
\hat{P}_1 \rightarrow \hat{P}_1' = \mu(t) \hat{P}_1, \quad \hat{P}_2 \rightarrow \hat{P}_2' = \mu(t) \hat{P}_2 \tag{21}
\]

\[
\hat{Q}_1 \rightarrow \hat{Q}_1' = \frac{\hat{Q}_1}{\mu(t)}, \quad \hat{Q}_2 \rightarrow \hat{Q}_2' = \frac{\hat{Q}_2}{\mu(t)}, \tag{22}
\]

then the time derivative operator becomes

\[
\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} - i \frac{\dot{\mu}(t)}{\mu(t)} (\hat{Q}_1' \hat{P}_1' + \hat{Q}_2' \hat{P}_2'). \tag{23}
\]

If \( \mu(t) \) is chosen so that

\[
\alpha(t) = -\frac{\dot{\mu}(t)}{\mu(t)}, \tag{24}
\]

then the velocity dependent interaction term vanishes. The new equation is just the equation of a particle under an inverse harmonic potential and with a time-dependent mass.

\[
\hat{P}_1'^2 + \hat{P}_2'^2 + \frac{2C}{\hat{Q}_1'^2 + \hat{Q}_2'^2} = i2m\mu^2(t) \frac{\partial}{\partial t} + 2\mu^2(t)\alpha(t) = k^2. \tag{25}
\]

Since left-hand side and right-hand side of (25) depend on different parameters, the two sides are equal to a constant, \( k^2 \). We can rewrite Eq. (25) as a wave equation rather than an operator equation

\[
-\frac{\partial^2 \Psi(x, y, t)}{\partial Q_1'^2} - \frac{\partial^2 \Psi(x, y, t)}{\partial Q_2'^2} + \frac{2C}{Q_1'^2 + Q_2'^2} \Psi(x, y, t) = i2m\mu^2 \frac{\partial \Psi(x, y, t)}{\partial t} + 2\mu^2 \alpha \Psi(x, y, t). \tag{26}
\]

Using the separation of variables technique, we can solve Eq. (26).

Let us write \( \Psi = R(Q_1', Q_2')T(t) \)

\[
T(t) = \exp \left[ -i \int_0^t \frac{k^2 + 2\mu^2(t')\alpha(t')}{2m(t')\mu^2(t')} dt' \right]. \tag{27}
\]
Coordinate-dependent part of Eq. (26) can be decomposed in polar coordinates

\[ Q^2 \frac{d^2Z}{dQ^2} + Q \frac{dZ}{dQ} + \left( k^2 Q^2 - 2C - n^2 \right) Z = 0 \]  
\( (28) \)

\[ \frac{d^2 \Theta}{d\theta^2} = -n^2 \Theta, \]  
\( (29) \)

where \( Q^2 = Q_1^2 + Q_2^2 \), \( Q'_1 = Q \cos \theta \), \( Q'_2 = Q \sin \theta \) and \( R(Q'_1, Q'_2) = Z(Q) \Theta(\theta) \).

The relation between \( \theta \) and \( x, y \) is given

\[ \tan \theta = \frac{Q'_2}{Q'_1} = \frac{\cos \beta x + \sin \beta y}{-\sin \beta x + \cos \beta y} \]  
\( (30) \)

where \( \beta \) was defined in Eq. (13).

If we introduce \( \nu^2 = 2C + n^2 \), we see that Eq. (28) is just the Bessel differential equation of order \( \nu \) [13]. The solutions of (28,29)

\[ Z(Q) = AJ_\nu(kQ) + BN_\nu(kQ) \]  
\( (31) \)

\[ \Theta = C \exp(\pm in\theta), \]  
\( (32) \)

where \( J_\nu(kQ') \), \( N_\nu(kQ') \) are the Bessel function of the first and the second kind respectively. The constants \( A, B \) and \( C \) are determined from the boundary conditions. Note that \( \nu \) and \( n \) are not the same. If the inverse harmonic potential term goes to zero, they become equal to each other.

Finally, the complete exact solution of Eq. (4) is given

\[ \Psi = \left[ AJ_\nu \left( k \frac{\sqrt{x^2 + y^2}}{\mu(t)} \right) + BN_\nu \left( k \frac{\sqrt{x^2 + y^2}}{\mu(t)} \right) \right] \exp[\alpha(t)(x^2 + y^2) \pm in\theta - if] \]  
\( (33) \)

where \( f(t) = \int_0^1 \frac{k^2 + 2\mu^2(t')\alpha(t')}{2m(t')\mu^2(t')} dt' \) and \( \alpha, \mu, \theta \) were defined Eqs. (19), (24) and (30) respectively.

In summary, the exact solution of the problem of the charged harmonic plus an inverse harmonic oscillator with time-dependent mass and frequency in a time-dependent electromagnetic field is obtained by making some appropriate transformation on momentum and coordinate operators.
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