Quantum Legendre-Fenchel Transform

David Sutter$^1$, Giacomo Nannicini$^2$, Tobias Sutter$^3$ and Stefan Woerner$^1$

$^1$IBM Quantum, IBM Research – Zurich
$^2$IBM Quantum, IBM T.J. Watson Research Center
$^3$Risk Analytics and Optimization Chair, EPFL

Abstract

We present a quantum algorithm to compute the discrete Legendre-Fenchel transform. Given access to a convex function evaluated at $N$ points, the algorithm outputs a quantum-mechanical representation of its corresponding discrete Legendre-Fenchel transform evaluated at $K$ points in the transformed space. For a fixed regular discretization of the dual space the expected running time scales as $O(\sqrt{\kappa}\text{polylog}(N,K))$, where $\kappa$ is the condition number of the function. If the discretization of the dual space is chosen adaptively with $K$ equal to $N$, the running time reduces to $O(\text{polylog}(N))$. We explain how to extend the presented algorithm to the multivariate setting and prove lower bounds for the query complexity, showing that our quantum algorithm is optimal up to polylogarithmic factors. For certain scenarios, such as computing an expectation value of an efficiently-computable observable associated with a Legendre-Fenchel-transformed convex function, the quantum algorithm provides an exponential speedup compared to any classical algorithm.

1 Introduction

Quantum algorithms utilize intrinsic features of quantum mechanics to perform certain tasks asymptotically faster than any known classical method. The design of quantum algorithms is challenging; as a result, most currently known quantum algorithms use a few basic building blocks as subroutines. Examples of these building blocks are: the quantum Fourier transform [6, 25], Grover’s algorithm for unstructured search [10, 11], or quantum phase estimation [16]. It is a major scientific challenge to identify more quantum algorithms that outperform their classical counterparts. In particular, the discovery of novel quantum subroutines could help the development of further quantum algorithms.

In this paper we present a quantum algorithm for a task that is known in the classical world, but it had not been quantized before: the computation of the Legendre-Fenchel transform (LFT). The LFT, also known as convex conjugate or simply as Legendre transform, is used in a variety of different disciplines, such as: thermodynamics, where the LFT is used to switch between potentials [26]; classical mechanics, where the LFT serves as a bridge between the Hamiltonian and the Lagrangian formalism [9]; and large deviation theory, where the LFT provides the link between the moment generating function, and a corresponding rate function via Cramér’s theorem [8, Section 2.2]. In convex analysis, the LFT has an importance similar to that of the Fourier transform in signal processing [28]. One of the main reasons behind the relevance of the LFT in convex analysis is the fact that the infimal convolution (also called epi-sum) is equivalent to a simple addition in the Legendre-Fenchel space [28, Theorem 16.4], whereas the analogous property for the Fourier transform is that the convolution operator is equivalent to a product in the Fourier space.

For a function $f : \mathbb{R}^d \to \mathbb{R}$, its LFT is denoted $f^* : \mathbb{R}^d \to \mathbb{R}$ and defined as

$$f^*(s) := \sup_{x \in \mathbb{R}^d} \{ \langle s, x \rangle - f(x) \}. \quad (1)$$
The LFT is a nonlinear operation that has many desirable properties, some of which are summarized in Section 2.3. For certain functions the LFT can be computed in closed-form; however, in general the supremum in (1) may not allow a simple closed-form solution, hence it is not straightforward to evaluate. Furthermore, in some applications the function \( f \) may not be known analytically, but only observed via a finite number of data points.

This motivates the definition of a discrete Legendre–Fenchel transform. Let \( \mathcal{X}_N^d = \{x_0,\ldots,x_{N-1}\} \subseteq \mathbb{R}^d \) and \( \mathcal{S}_K^d = \{s_0,\ldots,s_{K-1}\} \subseteq \mathbb{R}^d \) be discrete primal and dual spaces, respectively. Then, the discrete LFT of the function \( f \) is defined by the mapping \((f(x_0),\ldots,f(x_{N-1})) \mapsto (f^*(s_0),\ldots,f^*(s_{K-1}))\),

\[
f^*(s_j) = \max_{x \in \mathcal{X}_K^d} \{ (s_j,x) - f(x) \}, \quad \text{for } j = 0,\ldots,K-1.
\]

If \( f \) is continuous, the discrete LFT converges to the (continuous) LFT when \( N,K \to \infty \), if we assume that \( f \) and \( f^* \) have a bounded and fixed domain [7, Theorem 2.1]. The discrete LFT plays a fundamental role in discrete convex analysis [23], where it defines a conjugacy relationship between some well-studied classes of functions (\( L \)-convex and \( M \)-convex, \( L^\blacklozenge \)-convex and \( M^\blacklozenge \)-convex) obtained by discretizing different characterizations of continuous convex functions. For example, the discrete LFT can be used to define primal/dual optimality criteria for submodular flow problems [14, 22].

Even though the LFT is defined for an arbitrary function \( f \), in this work we impose some mild regularity assumptions that will be crucial for the quantum algorithm presented in Section 4. The precise assumptions are discussed in Section 2.2; arguably the strongest one is that we restrict ourselves to convex functions. Furthermore, we restrict ourselves to functions with a compact domain, assumed to be \([0,1]^d\) without loss of generality.

From (2) we immediately see that a brute force calculation of the discrete LFT has complexity \( O(NK) \). A more refined algorithm that requires \( O(N + K) \) operations only is given in [18]. We explain this algorithm in detail in Section 3. Since writing down the input \((f(x_0),\ldots,f(x_{N-1}))\) and output \((f^*(s_0),\ldots,f^*(s_{K-1}))\) takes \( O(N) \) and \( O(K) \) time, respectively, the classical algorithm is asymptotically optimal. In multi-dimensional scenarios (i.e., \( d > 1 \)) the number of discretization points \( N \) typically needs to be exponentially large in \( d \), i.e., \( N \propto \exp(d) \), to ensure a good approximation of the function’s behaviour over \([0,1]^d\). This is one of the sources of the well-known “curse of dimensionality”, leading to a running time for the LFT algorithm that is exponential in \( d \).

**Results** For any convex function \( f : [0,1] \to \mathbb{R} \) satisfying some mild regularity assumptions (see Section 2.2) we present two algorithms to compute a quantum-mechanical representation of the discrete LFT. Given access to \( f \) on a regular discretization in the form of samples \((f(x_0),\ldots,f(x_{N-1}))\), these algorithms calculate the quantum state

\[
\frac{1}{\sqrt{K}} \sum_{j=0}^{K-1} |j\rangle |f^*(s_j)\rangle |s_j\rangle.
\]

The two algorithms differ in the sense that:

(a) one uses a regular discretization of the dual space and its expected running time is given by \( O(\sqrt{\kappa \text{polylog}(N,K)}) \), where \( \kappa \) is the condition number of \( f \);\(^2\)

(b) the other uses an adaptive discretization of the dual space of size \( N \), chosen by the algorithm and hence a-priori unknown, and runs in time \( O(\text{polylog}(N)) \).

We refer to Theorems 4.7 and 4.11 for more precise statements.

\(^1\)We use the two different asterisk symbols \( * \) and \( \ast \) to distinguish between the continuous and the discrete LFT.

\(^2\)In the running time expression there are no products between \( N \) and \( K \).
Our algorithms can be extended to the multi-dimensional case. For \( n \in \mathbb{N} \) we use the notation \([n] := \{0, 1, \ldots, n-1\} \). For any convex function \( f : [0, 1]^d \to \mathbb{R} \) satisfying some mild regularity assumptions (see Section 2.2), known via \( N = \prod_{\ell=0}^{d-1} N_\ell \) discrete samples \( \{f(x_{i_0}, \ldots, x_{i_{d-1}})\}_{i_\ell \in [N_\ell]} \) for \( \ell \in [d] \) on a regular grid, we can compute the state

\[
\frac{1}{\sqrt{K}} \sum_{j_0 = 0}^{K_0-1} \cdots \sum_{j_{d-1} = 0}^{K_{d-1}-1} |j_0, \ldots, j_{d-1}\rangle | f^*(s_{j_0}, \ldots, s_{j_{d-1}}) \rangle | s_{j_0}, \ldots, s_{j_{d-1}} \rangle ,
\]

where \( K = \prod_{\ell=0}^{d-1} K_\ell \) denotes the discretization of the dual space. As for the one-dimensional case, the two algorithms are different in the following sense:

(a) one uses a regular discretization of the dual space and its expected running time is given by \( O(\kappa^d/2 \text{polylog}(N, K)) \);

(b) the other uses an adaptive discretization of the dual space with size \( N \), chosen by the algorithm, and runs in \( O(\text{polylog}(N)) \).

We refer to Corollaries 5.2 and 5.3 for more precise statements in the multi-dimensional case.

We further show that the quantum LFT algorithm is optimal, up to polylogarithmic factors, for the task of calculating the discrete LFT at a specific point (see Corollary 6.2). Furthermore, the expected running time of the algorithm features the optimal scaling with respect to a certain parameter (related to the condition number), characterizing for which functions the quantum LFT can be computed efficiently (see Proposition 6.3 and Corollary 6.5). As a consequence our algorithm cannot be substantially improved.

For certain tasks, such as calculating computing an expectation value of an efficiently-computable observable associated with a Legendre-Fenchel-transformed convex function, the quantum algorithm provides an exponential speedup in the size of the primal and dual space compared to any classical algorithm.

Given that the LFT is a fundamental mathematical tool with applications in various fields, we believe that the quantum algorithm for the LFT presented here may serve as a novel building block in the design of future quantum algorithms.

Outline  After discussing some preliminaries in Section 2, we start by reviewing the classical algorithm for the one-dimensional LFT in Section 3, which is then extended to the quantum case in Section 4. We distinguish two different cases, depending on choosing a regular or an adaptive discretization on the dual space. In Section 5 the quantum algorithm is extended to the multi-dimensional case. We discuss its optimality in Section 6 by proving lower bounds on the oracle complexity of two tasks: calculating the LFT at a specific point, and computing the discrete LFT for functions with a certain property. We also show that our quantum algorithm achieves these lower bounds up to polylogarithmic factors.

2 Preliminaries

2.1 Notation and definitions

For a vector \( x = (x_0, \ldots, x_{N-1}) \) its Euclidean norm is denoted by \( \|x\| \). A function \( f : \mathbb{R}^d \to \mathbb{R} \) is called Lipschitz continuous with constant \( L \geq 0 \) if \( |f(x) - f(y)| \leq L \|x - y\| \) for all \( x, y \in \mathbb{R}^d \). The function is said to be \( \mu \)-strongly convex if for all \( x, y \in \mathbb{R}^d \) and \( t \in [0, 1] \) we have \( f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \frac{\mu}{2} t(1-t) \|x - y\|^2 \). A function that is differentiable and its gradient is \( L \)-Lipschitz continuous is also called \( L \)-smooth. The condition number of \( f \) is defined by \( \kappa := L/\mu \) [24, Section 2.1.3]. We note that by definition \( \kappa \geq 1 \). For a multivariate quadratic function \( f : [0,1]^d \to \mathbb{R} \) given by \( x \mapsto x^TQx + (a,x) + b \) for some positive definite \( Q \in \mathbb{R}^{d \times d}, a \in \mathbb{R}^d, \) and \( b \in \mathbb{R} \), the condition number of \( f \) coincides with the condition number of the matrix \( Q \), i.e., \( \kappa = \lambda_{\max}(Q)/\lambda_{\min}(Q) \), where
\(\lambda_{\text{max}}\) and \(\lambda_{\text{min}}\) denote the largest and smallest eigenvalues. The indicator function is defined by \(1\{X\} := 1\) if \(X = \text{true}\) and 0 otherwise. The logical 'and' and 'or' operations are denoted by \(\land\) and \(\lor\), respectively.

### 2.2 Regularity assumptions

Throughout the entire manuscript we assume the following regularity assumptions of the function \(f\):

**Assumption 2.1** (Regularity assumptions). We assume that the function \(f : [0, 1]^d \to \mathbb{R}\) is:

(i) differentiable at the boundary of \([0, 1]^d\);

(ii) (jointly) convex and \(\exists \nu < \infty\) such that for all \(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{d-1} \in [0, 1]\) and for all \(i \in [d]\)

\[
\nabla_x f(x_0, \ldots, x_{i-1}, 1, x_{i+1}, x_{d-1}) - \nabla_x f(x_0, \ldots, x_{i-1}, 0, x_{i+1}, x_{d-1}) \geq \nu;
\]

(iii) such that for any \(\delta > 0\) \(\exists \bar{L}_\delta < \infty\) such that for all \(x_i \in [\delta, 1-\delta], x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{d-1} \in [0, 1]\) and for all \(i \in [d]\)

\[
\begin{align*}
&f(x_0, \ldots, x_{i-1}, x_i + \delta, x_{i+1}, x_{d-1}) - 2f(x_0, \ldots, x_{i-1}, x_i, x_{i+1}, x_{d-1}) \\
&+ f(x_0, \ldots, x_{i-1}, x_i - \delta, x_{i+1}, x_{d-1}) \leq \bar{L}_\delta \delta^2;
\end{align*}
\]

(iv) lower semi-continuous.

Assumptions (i), (ii), and (iii) above can be slightly strengthened so that the overall analysis gets less technical and easier to read. However, we emphasize that the stronger assumptions on \(f\) (stated below) are not necessary.

**Assumption 2.2** (Stronger regularity assumptions). We assume that the function \(f : [0, 1]^d \to \mathbb{R}\) is:

(i+) differentiable on \([0, 1]^d\);

(ii+) (jointly) \(\mu\)-strongly convex with \(\mu \in \mathbb{R}_+\);

(iii+) such that \(\nabla f\) is \(L'\)-Lipschitz continuous with \(L' \in \mathbb{R}_+\).

As suggested by the naming, Assumption 2.2 implies Assumption 2.1 for \(\nu = \mu\) and \(\bar{L}_\delta = L'\). We recall that the condition number of \(f\) is defined as [24, Section 2.1.3]

\[
\kappa := \frac{L'}{\mu}.
\]

In convex optimization, the condition number oftentimes shows up in worst-case running times for several algorithms [24]. Note that by definition \(\kappa \in [1, \infty)\). A function with a small condition number is called “well-conditioned”, whereas a large condition number indicates that the function is “ill-conditioned”, usually resulting in slower algorithmic performance. As we will see in Section 4 the running time of the quantum algorithm for computing the LFT of \(f\) scales as \(\sqrt{\kappa}\).

### 2.3 Properties of the Legendre-Fenchel transform

The LFT features several desirable properties. For a detailed overview we refer the interested reader to [27, Section 11]. Here we discuss only properties that are relevant for our work. The LFT is an involution for convex and lower semi-continuous functions, i.e., we have \((f^*)^* = f\). By construction \(f^*\) is a jointly convex function, irrespective of the function \(f\), since it is the pointwise supremum of a family of affine functions [3, Section 3.3]. It is known [13] that \(\nabla f\) is \(L'\)-Lipschitz continuous if, and
only if \( f^* \) is \( 1/L' \)-strongly convex. Because the LFT is an involution for convex functions, this directly implies that the LFT does not change the condition number, i.e., \( \kappa_f = \kappa_{f^*} \). For certain classes of functions \( C \) the LFT is closed with respect to this class, meaning that \( f \in C \) implies \( f^* \in C \). Examples of such classes are

(i) piecewise linear functions [27, Theorem 11.14];

(ii) piecewise linear-quadratic functions [27, Definition 10.20 and Theorem 11.14].

3 One-dimensional classical Legendre-Fenchel transform

Before explaining the classical algorithm [18] that computes the discrete LFT in time \( O(N + K) \), we state basic assumptions on the discretization of the primal and dual spaces. For the dual space we distinguish between a regular discretization, where the grid points are equidistant, and an adaptive discretization, chosen by the algorithm depending on the properties of the function to be transformed.

3.1 Regular discretization

A regular discretization of a one-dimensional continuous space is such that all distances between any two nearby points are equal. Furthermore we assume that the points are sorted.

Assumption 3.1 (One-dimensional regular discretization). The discrete sets \( X_N^1 = \{x_0, \ldots, x_{N-1}\} \) and \( S_K^1 = \{s_0, \ldots, s_{K-1}\} \) are such that:

(i) The discretization is sorted, i.e., \( x_i \leq x_{i+1} \) for all \( i \in [N-1] \) and \( s_j \leq s_{j+1} \) for all \( j \in [K-1] \);

(ii) The discretization is regular, i.e., \( x_{i+1} - x_i = \gamma_x \) for all \( i \in [N-1] \) and \( s_{j+1} - s_j = \gamma_s \) for all \( j \in [K-1] \).

Under these assumptions on the discretization and the regularity conditions of the function \( f \) mentioned in Assumption 2.1, the classical algorithm to compute the discrete LFT presented in [18] is particularly simple. Define the discrete gradients

\[
c_i := \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} = \frac{f(x_{i+1}) - f(x_i)}{\gamma_x}, \quad \text{for} \quad i \in [N-1],
\]

and \( c_{-1} := c_0 - \varepsilon, \ c_{N-1} := c_{N-2} + \varepsilon \) for an arbitrarily chosen \( \varepsilon > 0 \). Because \( f \) is convex we have \( c_{-1} \leq c_0 \leq \ldots \leq c_{N-1} \). The optimizer \( x^* \) for the discrete LFT (2) is then given the following rule: Set \( x_0^* = x_0, x_{K-1}^* = x_{N-1} \) and

\[
\text{for each} \quad j \in \{1, \ldots, K-2\} \quad \text{find} \quad i \in [N] \quad \text{such that} \quad c_{i-1} < s_j \leq c_i \implies x_j^* := x_i.
\]

We refer to [18, Lemma 3] for a proof that \( x^* \) defined as above is indeed the correct optimizer.

With the help of (5) we obtain a linear-time algorithm to compute \( f^*(s) \), because sorting two increasing sequences of length \( N \) and \( K \) takes \( O(N + K) \) time. For completeness, the procedure is summarized in Algorithm 1.

Remark 3.2 (Range of dual space). The nontrivial domain of the discrete LFT is \( s_j \in [c_0, c_{N-2}] \), because for all \( s_j < c_0 \) and \( s_j > c_{N-2} \) the corresponding optimizers are always \( x_j^* = x_0 \) and \( x_j^* = x_{N-1} \), respectively. In other words, for \( s_j \notin [c_0, c_{N-2}] \) computing \( f^*(s_j) \) is trivial.

Remark 3.3 (Sufficient precision). Algorithm 1 assumes sufficient numerical precision so that basic arithmetic operations do not introduce any errors: this assumption makes the analysis of the algorithm significantly simpler. Indeed, numerical errors on the discrete gradients \( c_i \) could result in a wrong optimizer computed via the rule (5), which would then yield an error in the LFT values that depends on \( \gamma_x \), due to the discrete nature of the optimizers.
The mapping \( i \mapsto j \) such that \( x_i = x_j^* \) (as defined in (5)) can be represented by a binary \( N \times K \) matrix \( M \) defined by \( M_{i+j+1} = 1 \{ x_i = x_j^* \} \) for \( i \in [N] \) and \( j \in [K] \). In the following we want to quantify how many \( s_j \) share the same optimizer. This will be relevant for the quantum algorithm presented in Section 4. To do so, we introduce the parameter

\[
W := \max_{i \in \{1, \ldots, N-2\}} g(i) = \max_{i \in \{1, \ldots, N-2\}} \left\{ \frac{c_i - c_{i-1}}{\gamma_s} \right\} \leq \left[ \frac{\bar{L}_{\gamma_s} \gamma_s}{\gamma_s} \right],
\]

(6)

where we used Assumption 2.1 (iii) to derive the upper bound. In case \( \nabla f \) is \( L \)-Lipschitz continuous, we can replace \( \bar{L}_{\gamma_s} \) by \( L' \) and hence bound \( W \) by

\[
W \leq L' \frac{\gamma_s}{\gamma_s} = L' \frac{K(x_{N-1} - x_0)}{N(s_{K-1} - s_0)}.
\]

(7)

It can be seen that each column in \( M \) contains exactly a single one, because each \( s_j \) has a single optimizer \( x_j^* \in \mathcal{X}_N \). Furthermore, the matrix \( M \) contains at most \( W \) ones per row and at most \( W \) consecutive rows can be entirely zero. For later use, we also define the parameter

\[
\nu := \frac{c_{N-2} - c_0}{x_{N-1} - x_0} \geq \mu,
\]

(8)

where \( \mu \) denotes the strong convexity parameter of \( f \).

Assuming that the dual space is chosen to include only the nontrivial domain of the discrete LFT (see Remark 3.2), i.e. \( S_K^1 = \{ s_0, c_0 + \gamma_s, \ldots, c_{N-2} - \gamma_s, c_{N-2} \} = \{ s_0, \ldots, s_{K-1} \} \), we define the set

\[
\mathcal{A} := \left\{ (i, m) \in [N] \times [W] : \left( \frac{c_i - c_{i-1}}{\gamma_s} \right) \geq m + 1 \land i \in \{1, \ldots, N-2\} \right\} \cup (m = 0 \land i = 0) \lor (m = 0 \land i = N-1),
\]

(9)

which contains, for each \( i \in [N] \), an element for each \( j \in [K] \) such that \( x_j^* = x_i \). Note that this implies \( |\mathcal{A}| = K \). Finally, the function

\[
j(i, m, c_{i-1}) := \begin{cases} 
\emptyset & \text{if } (i, m, c_{i-1}) \notin \mathcal{A} \\
0 & \text{if } i = 0 \land m = 0 \\
k - 1 & \text{if } i = N - 1 \land m = 0 \\
\min_{\ell \in [K]} \{ \ell + m : c_i - 1 < s_\ell \} & \text{otherwise}
\end{cases}
\]

(10)
is such that \( x^*_i(x_{i-1},c_{i-1}) = x_i \) and for fixed discrete gradients \( c_{j-1}, c_0, \ldots, c_{N-2} \) we have \( \{ j(i,m,c_{i-1}) : i \in [N], m \in [W] \} = [K] \).

**Remark 3.4** (Extension to nonconvex functions). The discrete LFT of a nonconvex function \( f \) can be computed by determining the convex hull of \( f \) before starting Algorithm 1 [18]. The convex hull of \( N \) points in \( \mathbb{R}^2 \) can be computed in time \( O(N \log h) \), where \( h \) is the number of points comprising the hull, which is known to be optimal [15]. We note that this additional convex hull step will not affect the overall complexity for the computation of the discrete LFT.\(^{3}\)

### 3.2 Adaptive discretization

We consider the same regular discretization of the primal space as above, i.e., the set \( X_N^1 \) satisfies Assumption 3.1. The idea of an adaptive discretization in the dual space is to choose \( K = N \) and

\[
S^1_{N, \text{adaptive}} = \left\{ \frac{c_{-1} + c_0}{2}, \frac{c_0 + c_1}{2}, \ldots, \frac{c_{N-3} + c_{N-2}}{2}, \frac{c_{N-2} + c_{N-1}}{2} \right\}.
\] (11)

This choice implies that according to (5), the optimizer in (2) satisfies \( x^*_i = x_i \). We have a unique optimizer of each discretization point in the dual space: this is the major difference between adaptive discretization and regular discretization. The classical algorithm for the adaptive discretization may not be clear in the classical setting discussed here. However, its relevance will be justified in Section 4.3, where we discuss a quantum algorithm to compute the LFT: in that case the adaptive discretization may lead to a better running time compared to the regular discretization.

**Algorithm 2** One-dimensional discrete LFT with adaptive discretization

**Input:** \( N \in \mathbb{N}, X^1_N = \{ x_0, \ldots, x_{N-1} \} \) satisfying Assumption 3.1, and samples \( (f(x_0), \ldots, f(x_{n-1})) \), where \( f \) is convex;

**Output:** \( \{ f^*(s_0), \ldots, f^*(s_{K-1}) \} \) for \( S^1_{N, \text{adaptive}} = \{ \frac{c_{-1} + c_0}{2}, \frac{c_0 + c_1}{2}, \ldots, \frac{c_{N-3} + c_{N-2}}{2}, \frac{c_{N-2} + c_{N-1}}{2} \} \) with \( c_i \) defined in (4);

Do the following three steps

1. Compute \( (c_{-1}, \ldots, c_{N-1}) \) with \( c_i \) defined in (4);
2. Let \( (s_0, s_1, \ldots, s_{N-2}, s_{N-1}) = (\frac{c_{-1} + c_0}{2}, \frac{c_0 + c_1}{2}, \ldots, \frac{c_{N-3} + c_{N-2}}{2}, \frac{c_{N-2} + c_{N-1}}{2}) \);
3. Evaluate \( f^*(s_i) = x_i s_i - f(x_i) \) for all \( i \in [N] \);

Output \( \{ f^*(s_0), \ldots, f^*(s_{N-1}) \} \);

**Remark 3.5.** There are other possibilities than (11) to define the adaptive dual set, such as, for example

\[
S^1_{K, \text{adaptive.right}} = \{ c_0, c_1, \ldots, c_{N-2}, c_{N-2} \} \quad \text{or} \quad S^1_{K, \text{adaptive.left}} = \{ c_0, c_0, c_1, \ldots, c_{N-2} \}.
\]

The choice \( S_{K, \text{adaptive}} \) has the advantage that for piecewise linear functions, the dual points are (intuitively) potentially located “in the middle of the linear pieces”, which can provide useful information about the shape of \( f^* \). This is illustrated in the examples in Section 3.3 next.

\(^{3}\)This is not the case for the quantum LFT discussed in the next section: it is not clear how to extend the algorithm to nonconvex functions without increasing the running time. We refer to Remark 4.9 for a detailed explanation.
3.3 Examples

In this section we discuss three examples that illustrate the LFT algorithm, and in particular the differences between regular and adaptive discretization. We discuss one quadratic function and two different piecewise linear functions. These examples will be useful also in the next section, where we develop a quantum algorithm for the LFT. We will see that the success probability of the quantum algorithm depends on the intrinsic structure of the function.

**Example 3.6** (Quadratic function). Consider the convex quadratic function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = x^2 - \frac{3}{4}x + \frac{1}{2}.$$ 

This function satisfies Assumptions 2.1 and 2.2; more precisely, it is 2-strongly convex, its gradient is 2-Lipschitz continuous and hence $\kappa = 1$. The continuous LFT can be computed analytically for this example as

$$f^*(s) = \begin{cases} 
-1/2 & s < -3/4 \\
-3/4 & -3/4 \leq s \leq 5/4 \\
1/2 & s > 5/4.
\end{cases}$$

For a regular discretization $\mathcal{X}_S^3 = \{0, 1/4, 1/2, 3/4, 1\}$, we compute $(c_-, \ldots, c_4) = (-1/2 - \varepsilon, -1/2, 0, 1/2, 1, 1 + \varepsilon)$, where $\varepsilon > 0$ is an arbitrary constant. We thus see that the nontrivial range of the dual space is $[c_0, c_3] = [-1/2, 1]$ as explained in Remark 3.2. We may choose a regular dual set $\mathcal{S}_4^1 = \{-1/2, 0, 1/2, 1\}$ of size 4. Algorithm 1 then gives $(x_0, x_1, x_2, x_3) = (0, 1/4, 1/2, 1)$ and finally $(f^*(s_0), f^*(s_1), f^*(s_2), f^*(s_3)) = (-1/2, -3/8, -1/8, 1/4)$, as illustrated in Figure 1.4 The adaptive Algorithm 2 outputs $(f^*(s_0), f^*(s_1), f^*(s_2), f^*(s_3)) = (-1/2, 7/16, -1/4, 1/16, 1/4)$ for $\mathcal{S}_{5, \text{adaptive}}^1 = \{-1/2, -1/4, 1/4, 3/4, 1\}$, and $(f^*(s_0), f^*(s_1), f^*(s_2), f^*(s_3)) = (-1/2, -3/8, -1/8, 1/4, 1/4)$ for $\mathcal{S}_{5, \text{adaptive}, \text{right}}^1 = \{-1/2, 0, 1/2, 1\}$.

![Graphical visualization of Example 3.6](image)

Figure 1: Graphical visualization of Example 3.6 for the discrete sets (a) $\mathcal{X}_S^3 = \{0, 1/4, 1/2, 3/4, 1\}$, (b) $\mathcal{S}_4^1 = \{-1/2, 0, 1/2, 1\}$, (c) $\mathcal{S}_{5, \text{adaptive}}^1 = \{-1/2, -1/4, 1/4, 3/4, 1\}$, and (c) $\mathcal{S}_{5, \text{adaptive}, \text{right}}^1 = \{-1/2, 0, 1/2, 1\}$. The dual space is plotted in the nontrivial domain $[-1/2, 1]$, as explained in Remark 3.2.

**Example 3.7** (Piecewise linear function). Consider the convex and piecewise linear function $f : (a) x \in \mathcal{X}_S^3 = \{0, 1/4, 1/2, 3/4, 1\}$, (b) $\mathcal{S}_4^1 = \{-1/2, 0, 1/2, 1\}$, (c) $\mathcal{S}_{5, \text{adaptive}}^1 = \{-1/2, -1/4, 1/4, 3/4, 1\}$, and (c) $\mathcal{S}_{5, \text{adaptive}, \text{right}}^1 = \{-1/2, 0, 1/2, 1\}$. The dual space is plotted in the nontrivial domain $[-1/2, 1]$, as explained in Remark 3.2.

---

4We will see that because the condition number is $\kappa = 1$, the quantum algorithm described in Section 4 succeeds with probability 1.
with $\nu = 3/4$ and $\bar{L}_{1/4} = 1$; it is depicted in Figure 2. We consider a regular discretization $\mathcal{X}_5^1 = \{0,1/4,1/2,3/4,1\}$. Its continuous LFT can be computed analytically as

$$f(x) = \begin{cases} 
0 & 0 \leq x < 1/4 \\
3x/4 - 6/16 & 3/4 \leq x \leq 1.
\end{cases}$$

This function satisfies Assumption 2.1 with $\nu = 3/4$ and $\bar{L}_{1/4} = 1$; it is depicted in Figure 2. We may choose the regular dual set $S(b) = \{0,3/16,6/16,9/16,13/16\}$. We next use Algorithm 1 to compute $(x_0^*, x_1^*, x_2^*, x_3^*, x_4^*) = (x_0, x_1, x_2, x_3, x_4) = (0, 1/4, 1/2, 3/4, 1)$ and $(f^*(s_0), f^*(s_1), f^*(s_2), f^*(s_3), f^*(s_4)) = (0, 3/16, 1/8, 5/16, 6/16)$, as illustrated in Figure 2. The adaptive Algorithm 2 outputs the vector $(f^*(s_0), f^*(s_1), f^*(s_2), f^*(s_3), f^*(s_4)) = (0, 1/32, 1/8, 9/32, 6/16)$ for $S_{\text{adapt,left}} = \{0, 1/8, 3/8, 5/8, 6/8\}$, and $(f^*(s_0), f^*(s_1), f^*(s_2), f^*(s_3), f^*(s_4)) = (0, 1/16, 3/16, 6/16, 6/16)$ for $S_{\text{adapt,centered}} = \{0, 1/4, 1/2, 3/4, 3/4\}$.

![Graphical visualization of Example 3.7](image)

(a) regular primal space (b) regular dual space (c) adaptive dual space

Figure 2: Graphical visualization of Example 3.7 for the discrete sets (a) $\mathcal{X}_3^1 = \{0,1/4,1/2,3/4,1\}$, (b) $S^1_5 = \{0,3/16,6/16,9/16,3/4\}$, (c) $S^1_{\text{adapt,centered}} = \{0,1/8,3/8,5/8,6/8\}$, and (c) $S^1_{\text{adapt,centered}} = \{0,1/4,1/2,3/4,3/4\}$. We see that the adaptive (centered) discretization leads to points in the middle of the linear segments. The dual space is plotted in the nontrivial domain $[0, 3/4]$, as explained in Remark 3.2.

**Example 3.8** (Piecewise linear function). Consider the convex and piecewise linear function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 
0 & 0 \leq x < 1/4 \\
x/2 - 1/8 & 1/4 \leq x < 3/4 \\
x - 1/2 & 3/4 \leq x \leq 1.
\end{cases}$$

We note that the parameter $W = [(1/4)/(3/16)] = [4/3] = 1$, as defined in (6), hence $K/(NW) = 1$. As a result, the quantum Algorithm presented in Section 4 succeeds with probability 1.
This function satisfies Assumption 2.1 with $\nu = 1$ and $L_{1/4} = 2$: it is depicted in Figure 3. Let $X_3^1 = \{0, 1/4, 1/2, 3/4, 1\}$ be a regular discretization of the primal space. The continuous LFT of $f$ can be computed analytically as

$$f^*(s) = \begin{cases} 
0 & s < 0 \\
\frac{s}{4} & 0 \leq s < 1/2 \\
3s/2 - 1/4 & 1/2 \leq s < 1 \\
s - 1/2 & s \geq 1.
\end{cases}$$

To study the discrete case, we compute $(c_-, \ldots, c_4) = (0 - \varepsilon, 0, 1/2, 1, 1 + \varepsilon)$ for an arbitrary constant $\varepsilon > 0$. We see that the nontrivial range of the dual space is $[c_0, c_3] = [0, 1]$; hence, we may choose $S_3^1 = \{0, 1/4, 1/2, 3/4, 1\}$ as a regular dual set. We next use Algorithm 1 to compute $(x_0^1, x_1^1, x_2^1, x_3^1) = (x_0, x_1, x_3, x_4) = (0, 1/4, 1/4, 3/4, 1)$ and $(f^*(s_0), f^*(s_1), f^*(s_2), f^*(s_3), f^*(s_4)) = (0, 1/16, 1/8, 5/16, 1/2)$, as illustrated in Figure 2.\footnote{We note that the parameter $W = [(1/2)/(1/4)] = 2$, as defined in (6), hence $K/(NW) = 1/2$. This implies that the quantum Algorithm presented in Section 4 succeeds with probability 1/2.}

The adaptive Algorithm 2 outputs $(f^*(s_0), f^*(s_1), f^*(s_2), f^*(s_3), f^*(s_4)) = (0, 1/16, 1/8, 5/16, 1/2)$ for $S_{5,\text{adaptive}}^1 = \{0, 1/4, 1/2, 3/4, 1\}$, and $(f^*(s_0), f^*(s_1), f^*(s_2), f^*(s_3), f^*(s_4)) = (0, 1/8, 1/8, 1/2, 1/2)$ for $S_{5,\text{adaptive, right}}^1 = \{0, 1/2, 1/2, 1, 1\}$.

Figure 3: Graphical visualization of Example 3.8 for the discrete sets (a) $X_3^1 = \{0, 1/4, 1/2, 3/4, 1\}$, (b) $S_3^1 = \{0, 1/4, 1/2, 3/4, 1\}$, (c) $S_{5,\text{adaptive}}^1 = \{0, 1/4, 1/2, 3/4, 1\}$, and (c) $S_{5,\text{adaptive, right}}^1 = \{0, 1/2, 1/2, 1, 1\}$. The dual space is plotted in the nontrivial domain $[0, 1]$, as explained in Remark 3.2.

4 One-dimensional quantum Legendre-Fenchel transform

Let $f : [0, 1] \to \mathbb{R}$ satisfy Assumption 2.1 and consider discrete sets $X_N^K = \{x_0, \ldots, x_{N-1}\} \subseteq [0, 1]$ and $S_K^i = \{s_0, \ldots, s_{K-1}\} \subseteq \mathbb{R}$. The quantum Legendre-Fenchel transform (QLFT) is defined by the mapping

$$\frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle |f(x_i)\rangle \quad \rightarrow \quad \frac{1}{\sqrt{K}} \sum_{j=0}^{K-1} |j\rangle |f^*(s_j)\rangle .$$

(12)

Remark 4.1 (Connections to Fourier transform). Unlike the Fourier transform, the LFT is a nonlinear operation. In the $max$-$plus$ algebra with the semiring $\mathbb{R}_{\text{max}} = [\mathbb{R} \cup \{-\infty\}, \oplus, \otimes]$, where $a \oplus b := \max\{a, b\}$ and $a \otimes b := a + b$, the LFT is a linear operation that corresponds to the Fourier transform in the standard algebra \cite{19}. 

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Figure 3: Graphical visualization of Example 3.8 for the discrete sets (a) $X_3^1 = \{0, 1/4, 1/2, 3/4, 1\}$, (b) $S_3^1 = \{0, 1/4, 1/2, 3/4, 1\}$, (c) $S_{5,\text{adaptive}}^1 = \{0, 1/4, 1/2, 3/4, 1\}$, and (c) $S_{5,\text{adaptive, right}}^1 = \{0, 1/2, 1/2, 1, 1\}$. The dual space is plotted in the nontrivial domain $[0, 1]$, as explained in Remark 3.2.

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Given that the discrete Fourier transform features an efficient quantum-mechanical implementation [6, 25], Remark 4.1 above raises hope that the same happens for the discrete LFT. This is indeed the case, and we show that there exists a quantum algorithm for the QLFT that is exponentially faster than the classical discrete LFT algorithm. We achieve this by generalizing the Algorithms 1 and 2 to the quantum case.

### 4.1 Input and output model

For \( N = 2^n \) with \( n \in \mathbb{N} \), we can encode a vector \( x \in \mathbb{R}^N \) into a \( n \)-qubit state \( \frac{1}{\|x\|} \sum_{i=0}^{N-1} x_i |i\rangle \) via the following unitary transformation \( U_A \):

\[
U_A(x)|0\rangle = \frac{1}{\|x\|} \sum_{i=0}^{N-1} x_i |i\rangle.
\] (13)

Whereas in (13) the classical data is encoded into the amplitudes of the quantum state (this is usually referred to as analog or amplitude encoding) we can also encode classical data into the basis vectors of a quantum state (this is called digital or binary encoding). Let \( d = (d_0, \ldots, d_{N-1}) \) denote a digital approximation of \( x \), where \( d_i \) are \( q \)-bit strings. Then, we define a unitary transformation \( U_D \) by

\[
U_D|i\rangle|0\rangle = |i\rangle|d_i\rangle \quad \text{for all} \quad i \in [N].
\]

In the following we state two assumptions ensuring that we can load the problem efficiently. We note that both assumptions are fairly standard.

**Assumption 4.2** (Efficient loading of primal space). Let \( X^d_N = \{x_0, \ldots, x_{N-1}\} \) be a regular primal space satisfying Assumption 3.1. We assume that the operation \( |i\rangle|0\rangle \mapsto |i\rangle|x_i\rangle \) can be done in \( O(\text{polylog}(N)) \) for all \( i \in [N] \).

As for the discrete LFT in the classical case, we do not need to know the function \( f \) on the entire domain \([0, 1]\), but only at \( N \) points. In the quantum case this is achieved by the following assumption.

**Assumption 4.3** (Access to function). We assume having access to a unitary \( U_f \) such that

\[
U_f(x_i|0\rangle) = |x_i\rangle|f(x_i)\rangle \quad \text{for all} \quad i \in [N].
\]

Furthermore, the cost of running \( U_f \) is \( O(\text{polylog}(N)) \).

The second part of the assumption is well justified because for every classically efficiently computable function, i.e., computable in \( O(\text{polylog}(N)) \), we can use quantum arithmetic to construct \( U_f \) with \( O(\text{polylog}(N)) \) gates [25].

**Assumption 4.4** (Sufficient precision). We assume to have sufficient precision such that all basic quantum arithmetic operations can be executed without any errors.

We note that this assumption is also necessary for the classical discrete LFT algorithm (see Remark 3.3). Since this paper discusses discrete algorithms, this is a natural assumption to avoid complications that would arise with any finite-precision computer, and is not specific to the quantum algorithms. The next remark discusses a way to bypass Assumption 4.4.

**Remark 4.5** (Circumventing Assumption 4.4). As discussed in Remark 3.3, Assumption 4.4 is needed in the LFT algorithm because numerical errors could lead to an incorrect optimizer via (5), which would then be amplified because of the discrete nature of the optimizers. Suppose we have a limited precision such that the the discrete gradients \( c_{i-1}, \ldots, c_{N-1} \) all have an error of at most \( \varepsilon > 0 \). If we assume that \( \varepsilon \) is sufficiently small that each interval \( [(c_i - c_{i-1})/\gamma_s] - 2\varepsilon, [(c_i - c_{i-1})/\gamma_s] + 2\varepsilon] \) does not contain
any integer, then rule (5) chooses the correct optimizer despite the presence of the precision errors.\footnote{Notice that if \((c_i - c_{i-1}))/\gamma_s\ is an integer, this assumption never holds. A solution would be to change the primal discretization in such a way that the \(c_i\) values change.} Alternatively, we would have to accept errors that scale as \(\gamma_s\) in the output of the discrete (quantum or classical) LFT.

The QLFT, as defined in (12), stores the classical information of the LFT, in form of the values \(f^*(s_0), \ldots, f^*(s_{K-1})\), in quantum registers using a digital representation. For certain applications it may be useful to have them represented as amplitudes. This can be achieved by an efficient probabilistic operation called “quantum digital-analog conversion”, described in Remark 4.6. In technical terms, we can efficiently perform the operation

\[
\frac{1}{\sqrt{K}} \sum_{j=0}^{K-1} |j\rangle f^*(s_j) \rightarrow \frac{1}{\alpha} \sum_{j=0}^{K-1} f^*(s_j) |j\rangle,
\]

where \(\alpha := \sum_{j=0}^{K-1} f^*(s_j)^2\) is a normalization constant.

**Remark 4.6 (Quantum digital-analog conversion).** It can be useful to transform digitally encoded data into the analog representation. This is achieved by a so-called quantum digital-analog conversion, which is used in existing quantum algorithms such as in [12].\footnote{We refer the interested reader to [21] for more information about digital-analog conversions. The rough idea of the algorithm is rotate the function value to an ancilla system and performing a measurement which is repeated until we see a specific outcome. The process can be sped up by using amplitude amplification.} There is a probabilistic quantum algorithm for the transformation (14) with an expected running time \(O(\sqrt{\omega_f} \text{polylog}(K))\), where the probability of success of the algorithm is given by

\[
\omega_f := \frac{1}{K} \sum_{j=0}^{K-1} \left(\frac{f^*(s_j)}{\max_{\ell \in [K]} |f^*(s_\ell)|}\right)^2.
\]

Using the equivalence of norms we see that a worst case bound is given by \(\omega_f \geq 1/K\), which holds with equality in case the vector \((f^*(s_0), \ldots, f^*(s_{K-1}))\) is zero everywhere except at one entry. This bound will destroy the exponential speedup in \(K\). Fortunately, for better behaved functions, whose entries are more uniformly distributed, \(\omega_f\) can be independent of \(K\); this is the case, for example, for quadratic functions.\footnote{It has been observed in different quantum algorithms that data needs to be sufficiently uniformly distributed in order to obtain a quantum speedup [1].} For the function discussed in Example 3.6, for \(s \in [-1/2, 1]\) we see that in the continuous case (where \(K \to \infty\)) we have \(\omega_f = \frac{\pi}{4} \|f^*\|_2^2 / \|f^*\|_\infty^2 = 1841/9920\).\footnote{For a multi-dimensional example consider the a function \(f\) such that its (continuous) LFT is given by \(f^* : [-L, L]^d \to \mathbb{R}\) with \(s \to s^T\) As for \(A = \text{diag}(a_0, \ldots, a_{d-1})\) where \(a_i \geq 0\) for all \(i \in [d]\). A straightforward calculation reveals that \(\omega_f = (2L)^{-d} \|f^*\|_2^2 / \|f^*\|_\infty^2 = (2L)^{-d} L^d 3/4\), which decays exponentially in \(d\). We see that this is inline with the running time for the \(d\)-dimensional QLFT (see Corollary 5.2).}

### 4.2 Regular discretization

In this section we assume that the sets \(X_N^1\) and \(S_K^1\) are regular, i.e., they fulfill Assumption 3.1. Algorithm 3 below presents an overview of the steps required to compute the QLFT. The details can be found in the proof of Theorem 4.7, which also proves the correctness and the worst-case running time of the algorithm.

**Theorem 4.7 (Performance of Algorithm 3).** Let \(n, k \in \mathbb{N}\), \(N = 2^n\), \(K = 2^k\), \(X_N^1 = \{x_0, \ldots, x_{N-1}\}\) satisfying Assumption 3.1 and \(f\) satisfying Assumptions 2.1 and 4.3. Algorithm 3 is successful with probability

\[
\frac{K}{NW} \geq \frac{1}{N} \frac{1}{L_{\gamma_n} \gamma_s / \gamma_{\nu_s}} \geq \frac{\nu}{L} \geq \frac{1}{K},
\]
Algorithm 3 One-dimensional QLFT with regular discretization

**Input:** $n, k \in \mathbb{N}, N = 2^n, K = 2^k, \mathcal{X}_N = \{x_0, \ldots, x_{N-1}\}$ satisfying Assumption 3.1, and a function $f$ satisfying Assumptions 2.1 and 4.3.

**Output:** State $\frac{1}{\sqrt{K}} \sum_{j=0}^{K-1} |j\rangle |f^*(s_j)\rangle |\text{Garbage}(j)\rangle$ where $S_K = \{s_0, \ldots, s_{K-1}\}$ satisfies Assumption 3.1 with $s_0 = c_0$ and $s_{K-1} = c_{N-2}$ as defined in (4);

Compute $c_0$ and $c_{N-2}$ as defined in (4), and let $(s_0, \ldots, s_{K-1}) = (c_0, c_0 + \gamma_s, \ldots, c_{N-2} - \gamma_s, c_{N-2})$ for constant $\gamma_s = (c_{N-2} - c_0)/2$. Afterwards, do the following:

1. Compute $\frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle |x_i-1, x_i, x_{i+1}\rangle |f(x_i-1), f(x_i), f(x_{i+1})\rangle$;
2. Compute $\frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle |x_i-1, x_i, x_{i+1}\rangle |f(x_i-1), f(x_i), f(x_{i+1})\rangle |c_{i-1}, c_i\rangle$;
3. Compute $\frac{1}{\sqrt{K}} \sum_{j=0}^{K-1} |j\rangle |x_j^\star\rangle |f(x_j^\star)\rangle |\text{Garbage}(j)\rangle$, where $x_j^\star$ is defined in (5);
4. Compute $|v\rangle = \frac{1}{\sqrt{K}} \sum_{j=0}^{K-1} |j\rangle |f^*(s_j)\rangle |\text{Garbage}(j)\rangle$.

Output $|v\rangle$;

As explained in the proof of Theorem 4.7, Step 3 is probabilistic and may be repeated until it succeeds.

where $W$ is defined in (6), $\tilde{L}_{\nu}$ and $\nu$ are defined in Assumption 2.1, $\gamma_x, \gamma_s$ are defined in Assumption 3.1, and $\kappa$ denotes the condition number of $f$. The penultimate inequality holds only if $f$ is differentiable, its derivative is $L'$-Lipschitz continuous and the final inequality is valid if $f$ is strongly convex. Given Assumption 4.4, the output $|v\rangle$ of a successful run of Algorithm 3 is equal to

$$\frac{1}{\sqrt{K}} \sum_{j=0}^{K-1} |j\rangle |f^*(s_j)\rangle |\text{Garbage}(j)\rangle,$$

where $(s_0, \ldots, s_{K-1}) = (c_0, c_0 + \gamma_s, \ldots, c_{N-2} - \gamma_s, c_{N-2})$, for $c_0$ and $c_{N-2}$ as defined in (4), and $\text{Garbage}(j)$ denotes the content of a working register that depends on $j$, defined more precisely in the proof. Combined with amplitude amplification and given Assumption 4.2, the expected running time of the algorithm is

$$O\left(\sqrt{NW}/K \polylog(N,K)\right) = O\left(\sqrt{K} \polylog(N,K)\right).$$

**Proof.** The correctness of Algorithm 3 follows straightforwardly from the correctness of the classical algorithm [18]. The nontrivial part is that all the steps can be done efficiently. Because of Assumption 4.4, quantum arithmetic operations can be carried out without introducing any errors. The initialization step can be done in time $O(\polylog(N))$, as $c_0$ and $c_{N-2}$ can be computed via $U_f$. We next analyze each step separately:

1. This step has complexity $O(\polylog(N))$. By Assumption 4.2 we can load the classical vector $x$ efficiently, i.e., we can prepare $\frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle |x_i\rangle$ in $O(\polylog(N))$. Next, apply the unitary $U_f$ to obtain

$$\frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle U_f(|x_i\rangle |0\rangle) = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle |x_i\rangle |f(x_i)\rangle.$$

Because we can do the mapping $|0\rangle |i\rangle |0\rangle \leftrightarrow |i-1\rangle |i\rangle |i+1\rangle$, we can repeat the construction above to create

$$\frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle |x_{i-1}, x_i, x_{i+1}\rangle |f(x_{i-1}), f(x_i), f(x_{i+1})\rangle,$$
where \( x_{-1}, x_N, f(x_{-1}), \) and \( f(x_N) \) are irrelevant for the calculation and can be set to an arbitrary value.

2. This step can be done in time \( O(\text{polylog}(N)) \) by evaluating the function \( f(x_{i-1}) - f(x_i) \) to compute \( c_{i-1} \) and \( c_i \). We note that the value of \( c_{-1} \) is irrelevant, and can be filled with an arbitrary number.

3. The complexity for this step is \( O(\text{polylog}(N, K)) \). As explained in Section 3, at most \( W \) different discrete points in \( s_j \) can have the same optimizer \( x_j^* \), where \( W \) is defined in (6). This is correct under the assumption that \( s_j \in [c_0, c_N - 2] \), which is the range where the LFT is nontrivial. From the state prepared in Step 2 we can create

\[
\frac{1}{\sqrt{NW}} \sum_{i=0}^{N-1} \sum_{m=0}^{W-1} |i|x_{i-1}, x_i, x_{i+1}| f(x_{i-1}), f(x_i), f(x_{i+1})| c_{i-1}, c_i | m | \mathbb{I}\{ (i, m) \in A \} | j(i, m, c_{i-1}) ,
\]

where \( A \) and \( j(i, m, c_{i-1}) \) are defined in (9) and (10), respectively. We next uncompute the registers \( | x_{i-1}, x_{i+1}| f(x_{i-1}), f(x_{i+1})| c_{i-1}, c_i \), which gives

\[
\frac{1}{\sqrt{NW}} \sum_{i=0}^{N-1} \sum_{m=0}^{W-1} |i|x_i| f(x_i)| m | \mathbb{I}\{ (i, m) \in A \} | j(i, m, c_{i-1}) \).
\]

(15)

If we perform a measurement on the register with the indicator function, then, conditioned on seeing the outcome “1” and after a relabelling of the sum, we obtain

\[
\frac{1}{\sqrt{K}} \sum_{j=0}^{K-1} |j|x_j^*| f(x_j^*)| m(j)| i(j) ,
\]

(16)

where \( x_j^* \) denotes the optimizer given in (5). (Recall that \( |A| = K \).) This step is probabilistic, and succeeds with probability

\[
\frac{K}{NW} \geq \frac{K}{N} \frac{1}{L^'x_j^* \gamma_{x_j^*/\gamma_{x_j^*}}},
\]

where we used the fact that the indicator function in (15) maps \( N \times W \) nonzero indices to \( K \) nonzero indices. The inequality then follows from (6). If \( f \) is differentiable and its gradient is \( L' \) Lipschitz continuous, we can further simplify this probability as

\[
\frac{K}{NW} \geq \frac{s_{K-1} - s_0}{L'(x_{N-1} - x_0)} = \frac{c_{N-2} - c_0}{L'(x_{N-1} - x_0)} = \frac{\mu}{L'} = \frac{1}{\kappa},
\]

where the first inequality uses (7), and the second inequality follows from (8) and holds only if \( f \) is strongly convex.

4. This step can be done in \( O(\text{polylog}(K)) \). To see this, note we can compute \( sx^* - f(x^*) \) from the state (16) to obtain

\[
\frac{1}{\sqrt{K}} \sum_{j=0}^{K-1} |j| s_j x_j^* - f(x_j^*)| f(x_j^*)| m(j)| i(j) = \frac{1}{\sqrt{K}} \sum_{j=0}^{K-1} |j| f^*(s_j)| x_j^*| f(x_j^*)| m(j)| i(j).
\]

Because we are given \( |x_j^*\rangle \), we can uncompute \( |f(x_j^*)\rangle \) and find

\[
\frac{1}{\sqrt{K}} \sum_{j=0}^{K-1} |j| f^*(s_j)| x_j^*| m(j)| i(j).
\]

(17)

where \( |x_j^*\rangle | m(j)| i(j) \) may be viewed as garbage that depends on \( j \).
We can use amplitude amplification [4] in Step 3 so that \(O(\sqrt{NW/K}) = O(\sqrt{\kappa})\) repetitions of Algorithm 3 are sufficient to succeed with constant probability. As a result the overall expected running time is given by \(O(\sqrt{NW/K} \text{polylog}(N, K)) = O(\sqrt{\kappa} \text{polylog}(N, K))\).

**Remark 4.8** (Algorithm 3 can succeed with probability 1). As stated in Theorem 4.7 the probability of success of Algorithm 3 depends on intrinsic properties of the function \(f\). For well-behaved functions this probability can be 1 which is of particular relevance in a multi-dimensional scenario — as discussed in Section 5. Examples of such functions are

(i) quadratic functions of the form \(f(x) = ax^2 + bx + c\) (they have condition number \(\kappa = 1\), since the second derivative is constant);

(ii) certain piecewise linear functions, e.g., the one discussed in Example 3.7. We note that there are also piecewise linear functions where the probability of success of Algorithm 3 is strictly less than 1 (see Example 3.8).

In case Algorithm 3 does not succeed with probability 1, one can always switch to an adaptive dual set which enforces the algorithm to be deterministic. This is discussed in detail in Section 4.3.

**Remark 4.9** (QLFT for nonconvex functions). Following Remark 3.4, it is natural to ask if the QLFT can be computed efficiently for nonconvex functions. Classically, this can be achieved by running Algorithm 1 on the convex hull of the function. There is a quantum algorithm that computes the convex hull of \(N\) points in \(\mathbb{R}^2\) in \(O(\sqrt{Nh})\), where \(h\) is the number of points comprising the hull [17]; this is obtained using Grover search to achieve a quadratic speedup in \(N\) compared to a classical algorithm. Furthermore, a heuristic version of the quantum algorithm, that often works in practice, runs in time \(O(\sqrt{Nh})\) [17]. Since the QLFT algorithm works on a uniform superposition of points, one may ask if there is hope for an “implicit” quantum convex hull algorithm that does not output a classical description of the convex hull; rather, an algorithm that takes as input a quantum state encoding of a superposition of points, and outputs a uniform superposition of only the points in the convex hull. However, it is easily established that this cannot be done in polylogarithmic time. Indeed, suppose we are given oracle access to a function \(g : [N] \to \{0, 1\}\) that takes value 1 only at the point \(z\). By preparing the superposition \(\frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle|g(i)\rangle\), applying the “implicit” quantum convex hull on the second register, then measuring the second register, we would observe \(|z\rangle\) in the first register with probability \(1/2\) because the convex hull of the second register is \([0, 1]\), and \(z\) is the only one point whose second-register coordinate is 1. It is well known that \(\Omega(\sqrt{N})\) queries to \(g\) are necessary to recover \(z\) [30], hence an “implicit” convex hull algorithm that runs in time \(O(\text{polylog}(N))\) is ruled out.

### 4.3 Adaptive discretization

As discussed in Section 3.2, for the classical algorithm we can choose the dual set \(S^*_K\) in an adaptive way, so that we have a unique optimizer in (2). This has the advantage that the quantum algorithm succeeds with probability 1, can be described by a unitary evolution, and does not accumulate any garbage. The downside is that we cannot control the discretization resolution in the dual space.

Algorithm 4 below gives an overview of the steps required to compute the QLFT for an adaptively chosen dual space. The details can be found in the proof of Theorem 4.7, which also proves the correctness and the worst-case running time of the algorithm.

**Remark 4.10.** If desired, we can uncompute the \(|x_i\rangle\) register so that the output of Algorithm 4 is an approximation to

\[
\frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle|s_i\rangle|f^*(s_i)\rangle.
\]
Algorithm 4 One-dimensional QLFT with adaptive discretization

**Input:** $n \in \mathbb{N}$, $N = 2^n$, $\mathcal{X}_N^1 = \{x_0, \ldots, x_{N-1}\}$ satisfying Assumption 3.1, and a function $f$ satisfying Assumptions 2.1 and 4.3;

**Output:** State $\frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle|x_i\rangle|s_i\rangle|f^*(s_i)\rangle$ for $\mathcal{S}_{N,\text{adaptive}} = \{s_0, \ldots, s_{N-1}\}$ as defined in (11);

Do the following:

1. Compute $\frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle|x_{i-1}, x_i, x_{i+1}\rangle|f(x_{i-1}), f(x_i), f(x_{i+1})\rangle$;
2. Compute $\frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle|x_i\rangle|f(x_i)\rangle|s_i\rangle$;
3. Compute $|v\rangle = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle|x_i\rangle|s_i\rangle|f^*(s_i)\rangle$;

Output $|v\rangle$;

---

**Theorem 4.11 (Performance of Algorithm 4).** Let $n \in \mathbb{N}$, $N = 2^n$, $\mathcal{X}_N^1 = \{x_0, \ldots, x_{N-1}\}$ satisfying Assumption 3.1, and $f$ satisfying Assumptions 2.1 and 4.3. Then Algorithm 4 is successful with probability 1 and, given Assumption 4.4, its output $|v\rangle$ is equal to

$$\frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle|x_i\rangle|s_i\rangle|f^*(s_i)\rangle,$$

where $(s_0, \ldots, s_{N-1}) = \left(\frac{c_{i-1}+c_i}{2}, \frac{c_{i-1}+c_i}{2}, \ldots, \frac{c_{N-2}+c_{N-1}}{2}, \frac{c_{N-2}+c_{N-1}}{2}\right)$ with $c_i$ defined in (4). Furthermore, given Assumption 4.2, Algorithm 4 runs in time $O(\text{polylog}(N))$.

**Proof.** Recalling that by Assumption 4.4, quantum arithmetic operations can be carried out without introducing any errors. The state

$$\frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle|x_{i-1}, x_i, x_{i+1}\rangle|f(x_{i-1}), f(x_i), f(x_{i+1})\rangle$$

(18)

can be constructed in time $O(\text{polylog}(N))$, as discussed in Step 1 of the proof of Theorem 4.7. Since for the (centered) adaptive discretization we have $s_i = (c_{i-1} + c_i) / 2$ for $i \in [N]$, we can first construct the state given in Step 2 of Algorithm 3, and from there compute

$$\frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle|x_i\rangle|f(x_i)\rangle|s_i\rangle,$$

(19)

where we uncomputed all the registers that are not needed anymore.\(^\dagger\) At this point we can use the fact that for the adaptive protocol the dual discretization is chosen such that $x_i^* = x_i$, yielding $f^*(s_i) = s_i x_i - f(x_i)$. This allows us to create

$$\frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle|x_i\rangle|s_i\rangle|f^*(s_i)\rangle,$$

(20)

where we again uncompute registers that are no longer needed.

\(^\dagger\)We are flexible to also consider slight variants of the (centered) adaptive discretization as discussed in Remark 3.5.

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5 Multi-dimensional quantum Legendre-Fenchel transform

For \( d \in \mathbb{N} \), consider a multivariate function \( f : [0,1]^d \to \mathbb{R} \) that satisfies Assumption 2.1. The factorization property of the LFT ensures that for \( s \in \mathbb{R}^d \) we have

\[
f^*(s) = \sup_{x \in [0,1]^d} \{ \langle s, x \rangle - f(x) \}
\]

\[
= \max_{x_0 \in [0,1]} \left\{ s_0 x_0 + \max_{x_1 \in [0,1]} \{ s_1 x_1 + \ldots + \max_{x_{d-1} \in [0,1]} \{ s_{d-1} x_{d-1} - f(x) \} \ldots \} \right\}.
\]

From this we see that we can use \( d \)-times the one-dimensional discrete LFT algorithm to efficiently compute the discrete LFT in a \( d \)-dimensional setting.

To improve the readability of this manuscript we sometimes restrict ourselves to \( d = 2 \), however, all the statements can be extended to an arbitrary dimension \( d \in \mathbb{N} \). Furthermore in this section we will use a function \( g \) defined as the negative LFT of \( f(x_0, x_1) \) in the variable \( x_1 \) for fixed \( x_0 \), i.e.,

\[
g(x_0, s_1) := - \max_{x_1 \in [0,1]} \{ s_1 x_1 - f(x_0, x_1) \}.
\]

With the help of \( g \) we see that to compute the LFT of \( f^* \) it suffices to first compute the one-dimensional LFT of \( f(x_0, x_1) \) in the variable \( x_1 \), while keeping \( x_0 \) fixed which introduces the function \( g \). Afterwards we compute the LFT of the function \( g \) in the variable \( x_0 \) while keeping \( s_1 \) fixed. In technical terms this can be expressed as

\[
f^*(s_0, s_1) = \max_{x_0 \in [0,1]} \{ s_0 x_0 + \max_{x_1 \in [0,1]} \{ s_1 x_1 - f(x_0, x_1) \} \} = \max_{x_0 \in [0,1]} \{ s_0 x_0 - g(x_0, s_1) \}.
\]

For any constant \( s_1 \) the function \( x_0 \mapsto g(x_0, s_1) \) is convex, because \( f \) is by assumption jointly convex [29, Proposition 5]. Furthermore, \( f^*(s_0, s_1) \) is the LFT of \( g(x_0, s_1) \) in the variable \( x_0 \).

5.1 Regular discretization

As discussed in the one-dimensional case in Section 3.1, we start by defining the regular discrete primal and dual sets. Let \( N = \prod_{\ell=0}^{d-1} N_\ell \) and \( K = \prod_{\ell=0}^{d-1} K_\ell \), where \( N_\ell \) and \( K_\ell \) denote the number of grid points per dimension.

Assumption 5.1 (Two-dimensional regular discretization). The sets \( X^2_N = \{x_{i_0,i_1} \}_{i_0 \in [N_0], i_1 \in [N_1]} \) and \( S^2_K = \{s_{j_0,j_1} \}_{j_0 \in [K_0], j_1 \in [K_1]} \) are such that:

(i) the discretization is sorted, i.e.,

\[
\begin{align*}
&\text{\quad } x_{i_0+1,i_1} - x_{i_0,i_1} \leq \gamma_x \text{ for all } i_0 \in [N_0] \text{ and } i_1 \in [N_1] \\
&\text{\quad } s_{j_0+1,j_1} - s_{j_0,j_1} \leq \gamma_s \text{ for all } j_0 \in [K_0] \text{ and } j_1 \in [K_1];
\end{align*}
\]

(ii) the discretization is regular, i.e.,

\[
\begin{align*}
&\text{\quad } x_{i_0+1,i_1} - x_{i_0,i_1} = \gamma_x \text{ for all } i_0 \in [N_0] \text{ and } x_{i_0,i_1+1} - x_{i_0,i_1} = \gamma_x \text{ for all } i_1 \in [N_1] \\
&\text{\quad } s_{j_0+1,j_1} - s_{j_0,j_1} = \gamma_s \text{ for all } j_0 \in [K_0] \text{ and } s_{j_0,j_1+1} - s_{j_0,j_1} = \gamma_s \text{ for all } j_1 \in [K_1].
\end{align*}
\]

Algorithm 5 gives an overview of how to compute a multi-dimensional QLFT; the details are given in the proof of Corollary 5.2, which also discusses correctness and running time. The \( d \)-dimensional QLFT is reduced to the task of performing \( d \)-times a one-dimensional QLFT, which we know how to do using Algorithm 3.\(^{12}\)

\(^{12}\)The regularity assumption can be slightly relaxed so that \( \gamma_x \) and \( \gamma_s \) are different in each dimension. For the sake of readability, we work under the assumption of uniform discretization steps.
Algorithm 5 Two-dimensional QLFT with regular discretization

**Input:** \(n_0, n_1, k_0, k_1 \in \mathbb{N}, N_0 = 2^{n_0}, N_1 = 2^{n_1}, K_0 = 2^{k_0}, K_1 = 2^{k_1}, N = N_0 N_1, \) set \(X_N^2\) satisfying Assumption 5.1, and \(f\) satisfying Assumptions 2.1, 2.2, and 4.3.

**Output:** State \(\frac{1}{\sqrt{N_0 N_1}} \sum_{j_0=0}^{K_0-1} \sum_{j_1=0}^{K_1-1} (j_0, j_1) |f^*(s_{j_0}, s_{j_1})| |s_{j_0}, s_{j_1}| |\text{Garbage}(j_0, j_1)|\) where \((s_{j_0}, s_{j_1})\) are regular grid points spanning the nontrivial domain of \(f^*\);

Do the following:

1. Compute \(\frac{1}{\sqrt{N_0 N_1}} \sum_{i_0=0}^{N_0-1} \sum_{i_1=0}^{N_1-1} |j_0, i_1| |x_{i_0, i_1-1}, x_{i_0, i_1}, x_{i_0, i_1+1}, x_{i_0+1, i_1}| f(x_{i_0, i_1-1}), f(x_{i_0, i_1}), f(x_{i_0, i_1+1}), f(x_{i_0+1, i_1})|\); 
2. Perform 1-dimensional QLFT (see Algorithm 3) to obtain \(\frac{1}{\sqrt{N_0 N_1}} \sum_{i_0=0}^{N_0-1} \sum_{i_1=0}^{N_1-1} |j_0, i_0| |g(x_{i_0-1, s_{j_1}}, g(x_{i_0, s_{j_1}}, g(x_{i_0, s_{j_1}+1}, s_{j_1}))| |s_{j_1}(x_{i_0})| |\text{Garbage}(i_0, j_1)|\), where \(g()\) is defined in (21); 
3. Perform 1-dimensional QLFT (see Algorithm 3) to obtain \(|v| = \frac{1}{\sqrt{N_0 N_1}} \sum_{j_0=0}^{K_0-1} \sum_{j_1=0}^{K_1-1} (j_0, j_1) |f^*(s_{j_0}, s_{j_1})| |s_{j_0}, s_{j_1}| |\text{Garbage}(j_0, j_1)|\); 

Output |\(v|\);

Corollary 5.2 (Performance of Algorithm 5). Let \(d, n, k, \ell \in \mathbb{N}, n = 2^{n_1}, k = 2^{k_1}, \) for all \(\ell \in [d], N = \prod_{\ell=0}^{d-1} N_\ell, K = \prod_{\ell=0}^{d-1} K_\ell, X_N^d\) satisfying Assumption 5.1, \(f\) satisfying Assumptions 2.1, 2.2, and 4.3. Then Algorithm 5 applied to the \(d\)-dimensional case succeeds with probability \(1/\kappa^d\), where \(\kappa\) is the condition number of \(f\). Given Assumption 4.4, its output |\(v)| of a successful run is equal to

\[
\frac{1}{\sqrt{K}} \sum_{j_0=0}^{K_0-1} \sum_{j_1=0}^{K_1-1} (j_0, \ldots, j_{d-1}) |f^*(s_{j_0}, \ldots, s_{j_{d-1}})| |s_{j_0}, \ldots, s_{j_{d-1}}| |\text{Garbage}(j_0, \ldots, j_{d-1})|
\]

where \((s_{j_0}, \ldots, s_{j_{d-1}})\) are regular grid points spanning the nontrivial domain of \(f^*\). \(\text{Garbage}(j_0, \ldots, j_{d-1})\) denotes the content of a working register that depends on \((j_0, \ldots, j_{d-1})\), defined more precisely in the proof. Combined with amplitude amplification and given Assumption 4.2, the expected running time of the algorithm is \(O(\kappa^{d/2} \text{polylog}(N, K))\).

**Proof.** The proof is discussed for the case \(d = 2\), however the extension to arbitrary dimensions \(d \in \mathbb{N}\) (as stated in the corollary) can be easily derived from the two-dimensional case.

1. Step 1 is equivalent to the first step of Algorithm 3 which can be done in \(O(\text{polylog}(N))\). In \(d\)-dimensions we need \(O(d)\)-registers to create the required state.
2. To see how Step 2 works, note that we add two points of redundancy to the 1-dimensional QLFT, by choosing the \(s\)-vector for a fixed \(i_0\) as \(s_{i_0, 0} = c_{i_0, 0} - \gamma_{s, i_0}, s_{i_0, 1} = c_{i_0, 0}, c_{i_0, K_1 - 2} = c_{i_0, N_1 - 2}\), and \(s_{i_0, K_1 - 1} = c_{i_0, N_1 - 2} + \gamma_{s, i_0}\) (see Remark 3.2). With this convention we find that

\[
s_{j_1}(x_{i_0}) = \left[ \frac{f(x_{i_0}, \gamma_x) - f(x_{i_0}, 0)}{\gamma_x}, \frac{f(x_{i_0}, 1) - f(x_{i_0}, 1 - \gamma_x)}{\gamma_x} \right]
\]

which is increasing in \(j_1 \in [K_1]\) in \(K_1\) regular steps. Algorithm 3 can be used to do the QLFT of \(f(x_{i_0}, x_{i_1})\) in the \(x_{i_1}\) variable for fixed \(x_{i_0}\). More precisely, we can create the state

\[
\frac{1}{\sqrt{K_1 N_0}} \sum_{j_1=0}^{K_1 - 1} \sum_{i_0=0}^{N_0 - 1} |j_1, i_0| |g(x_{i_0-1, s_{j_1}}, g(x_{i_0, s_{j_1}}, g(x_{i_0, s_{j_1}+1}, s_{j_1}))| s_{j_1}(x_{i_0}) |\text{Garbage}(i_0, j_1)|
\]

where \(g()\) is defined in (21). This step has a probability of success given by \(1/\kappa\).
3. For the final step we note that the dual variable $s_{j_0}$ is monotonically increasing in equidistant steps, so that

$$s_{j_0} \in \left[ \frac{f(\gamma x,0) - f(0,0)}{\gamma x} \gamma x, \frac{f(1,1) - f(0 - \gamma x, 1)}{\gamma x} \right],$$

for $j_0 \in [K_0]$, where we used the facts that $g(0, s_{j_1}) - g(\gamma x, s_{j_1}) = f(\gamma x, 0) - f(0, 0)$ and $g(1 - \gamma x, s_{j_1}) - g(\gamma x, s_{j_1}) = f(1, 1) - f(1 - \gamma x, 1)$. Algorithm 3 can then be used to perform the QLFT of $g(x_{i_0}, s_{j_1})$ in the $x_{i_0}$ variable for a fixed $s_{j_1}$, to obtain

$$\frac{1}{\sqrt{K_0 N_1}} \sum_{j_0 = 0}^{K_0 - 1} \sum_{j_1 = 0}^{K_1 - 1} |j_0, j_1| f^*(s_{j_0}, s_{j_1}) |s_{j_0}, s_{j_1} \rangle \langle \text{Garbage}(j_0, j_1)|.$$

This step is successful with probability $1/\kappa$, decreasing the overall success probability to $1/\kappa^2$.

We can use amplitude amplification [4] so that $O(\kappa^{d/2})$ repetitions (rather than $O(\kappa^d)$) are sufficient to secure with constant probability. As a result the overall expected running time is given by $O(\kappa^{d/2} \text{ polylog}(N, K))$.

**5.2 Adaptive discretization**

We can perform a multi-dimensional QLFT with an adaptive dual discretization which is the generalization of the method explained in Section 4.3 for the one-dimensional case. The benefit of an adaptively chosen dual space is that the algorithm succeeds with probability 1. Note that in case the condition number of $f$ satisfies $\kappa > 1$, Algorithm 5 needs to be repeated on average $\kappa^{d/2}$ times which scales exponentially in the dimension $d$.

**Corollary 5.3** (Performance of Algorithm 6). Let $d, n_\ell \in \mathbb{N}$, $N_\ell = 2^{n_\ell}$, for all $\ell \in [d]$, $N = \prod_{\ell=0}^{d-1} N_\ell$, $\mathcal{X}_N^d$ satisfying Assumption 5.1, $f$ satisfying Assumptions 2.1 and 4.3; 

Then Algorithm 6 applied on a $d$-dimensional function succeeds with probability 1. Given Assumption 4.4, its output $|v\rangle$ is equal to

$$\frac{1}{\sqrt{N}} \sum_{i_0=0}^{N_0-1} \ldots \sum_{i_d=0}^{N_d-1} |i_0, \ldots, i_{d-1}| f^*(s_{i_0}, \ldots, s_{i_{d-1}}) |s_{i_0}, \ldots, s_{i_{d-1}}\rangle.$$

**Algorithm 6** Two-dimensional QLFT with adaptive discretization

**Input**: $n_0, n_1 \in \mathbb{N}$, $N_0 = 2^{n_0}$, $N_1 = 2^{n_1}$, $N = N_0 N_1$, set $\mathcal{X}_N^2$ satisfying Assumptions 5.1, $f$ satisfying Assumptions 2.1 and 4.3;

**Output**: State $\frac{1}{\sqrt{N_0 N_1}} \sum_{i_0=0}^{N_0-1} \sum_{i_1=0}^{N_1-1} |i_0, i_1| f^*(s_{i_0}, s_{i_1}) |s_{i_0}, s_{i_1}\rangle$ where $(s_{i_0}, s_{i_1})$ are adaptively chosen grid points spanning the nontrivial domain of $f^*$.

Do the following:

1. Compute $\frac{1}{\sqrt{N_0 N_1}} \sum_{i_0=0}^{N_0-1} \sum_{i_1=0}^{N_1-1} |i_0, i_1| |x_{i_0-1,i_1}, x_{i_0,i_1-1}, x_{i_0,i_1}, x_{i_0,i_1+1}, x_{i_0+1,i_1}| f(x_{i_0-1,i_1}), f(x_{i_0,i_1-1}), f(x_{i_0,i_1}), f(x_{i_0,i_1+1}), f(x_{i_0+1,i_1})$;

2. Perform 1-dimensional QLFT (see Algorithm 4) to obtain

$$\frac{1}{\sqrt{N_0 N_1}} \sum_{i_0=0}^{N_0-1} \sum_{i_1=0}^{N_1-1} |i_0, i_1| g(x_{i_0-1,i_1}, s_{i_1}), g(x_{i_0,i_1}, s_{i_1}), g(x_{i_0+1,i_1}) |s_{i_1} (x_{i_0})\rangle,$$

where $g(\cdot)$ is defined in (21);

3. Perform 1-dimensional QLFT (see Algorithm 4) to obtain

$|v\rangle = \frac{1}{\sqrt{N_0 N_1}} \sum_{i_0=0}^{N_0-1} \sum_{i_1=0}^{N_1-1} |i_0, i_1| f^*(s_{i_0}, s_{i_1}) |s_{i_0}, s_{i_1}\rangle$;

Output $|v\rangle$;
where \((s_{i_0}, \ldots, s_{i_{d-1}})\) are adaptively chosen grid points spanning the nontrivial domain of \(f^*\). The time complexity for a successful run is \(O(\text{polylog}(N))\).

**Proof.** The proof follows by the same lines as the proof of Corollary 5.2, with the only difference that we use Algorithm 4 instead of Algorithm 3 as a building block. \qed

**Remark 5.4.** Suppose we want to compute \(f^*(s)\) for a specific \(s \in \mathbb{R}^d\). Combining Algorithm 6 with Grover adaptive search \([5, 2]\) allows us find \(f^*(s')\) in time \(O(\sqrt{N}\ \text{polylog}(N))\), where \(s'\) is the best approximation to \(s\) on the adaptively chosen dual space. \(13\)

### 5.3 Overview of QLFT algorithms

In the previous sections we have seen different algorithms to compute the QLFT. Table 1 summarizes their differences. The main distinction is that we have the flexibility to choose between a regular and an adaptive discretization for the dual space. The regular discretization may be advantageous in scenarios where we want to exploit a regular dual space, e.g., if we want to perform operations between two functions in the dual space, and require the same discretization grid for both. The adaptive discretization, on the other hand, has the benefit of being deterministic; this results in a better running time, and is of particular importance in multi-dimensional scenarios where the dimension is large. Furthermore, the adaptive QLFT is a unitary transformation, whereas the QLFT with a regular dual space and postselection is not (although as indicated in Thm. 4.7, we can eliminate the postselection step and perform amplitude amplification to obtain constant success probability, at the expense of the running time).

| Algorithm   | Dimension | Primal space | Dual space   | Expected complexity  | Process     |
|-------------|-----------|--------------|--------------|----------------------|-------------|
| Algorithm 3 | 1         | \(N\)-regular | \(K\)-regular | \(\sqrt{\kappa}\ \text{polylog}(N, K)\) | stochastic  |
| Algorithm 4 | 1         | \(N\)-regular | \(N\)-adaptive | \(\text{polylog}(N)\) | unitary     |
| Algorithm 5 | \(d\)     | \(N\)-regular | \(K\)-regular | \(\kappa^{d/2}\ \text{polylog}(N, K)\) | stochastic  |
| Algorithm 6 | \(d\)     | \(N\)-regular | \(N\)-adaptive | \(\text{polylog}(N)\) | unitary     |

Table 1: Overview of different algorithms for the QLFT.

### 6 Optimality

It is a natural question to ask if the QLFT algorithms presented above (see Table 1 for an overview) can be further improved. In this section we present two lower bounds showing that the regular QLFT algorithm presented above is optimal. First, we prove that for the task of evaluating the LFT of a function at a specific point on a \(d\)-dimensional space any quantum algorithm requires at least \(\Omega(\sqrt{2^d/d})\) queries to the function — a lower bound that is achieved (up to polylogarithmic factors) by Algorithm 5 combined with amplitude amplification. Second, we show that the success rate for the regular QLFT algorithm above cannot be further improved (i.e., it is optimal up to polylogarithmic factors) under reasonable assumptions. This is important because the success rate parameter determines for which problems the QLFT can be done efficiently.

**Proposition 6.1.** Let \(\mathcal{X}_N^d \subseteq [0, 1]^d\) be a finite set. Any quantum algorithm that outputs the value of the discrete LFT of a function \(f\) satisfying Assumption 2.1 over \(\mathcal{X}_N^d\) at an arbitrary dual value \(s \in \mathcal{X}_N^d\) requires \(\Omega(\sqrt{2^d/d})\) queries to \(f\) in general.

\(13\) Alternatively, we could find the \(d\) best approximations, which takes time \(O(\sqrt{N}\ \text{polylog}(N))\), and then do a linear approximation.
Proof. Let \( z \in \{0,1\}^d \), and let \( g : \{0,1\}^d \to \{0,1\} \) be such that \( g(z) = 1 \) and \( g(y) = 0 \) for all \( y \in \{0,1\}^d \setminus \{z\} \). It is known that \( \Omega(\sqrt{2^d}) \) quantum queries to \( g \) are necessary to determine \( z \) [30]. Let

\[
f(x) := \max_{i=1,\ldots,n} |x_i - z_i|,
\]

which is a jointly convex function. To see this we recall that a function is jointly convex if the epigraph is a convex set [28]. All the functions \( |x_i - z_i| \) have a convex epigraph and hence taking the intersection remains a convex set. Notice that on the vertices of the hypercube \([0,1]^d\), the equality \( f(x) = 1 - g(x) \) holds, and therefore we can simulate a call to \( f(x) \) with a single query to \( g(x) \).

Fix \( X^d_N = S^d_N = \{0,1\}^d \) and let \( e_j \in \{0,1\}^d \) be the string with a 1 in position \( j \) and 0 otherwise. Consider the value of the discrete LFT at \( e_j \), i.e.,

\[
f^*(e_j) = \max_{x \in X^d_N} \{ (e_j, x) - f(x) \} = \max_{x \in X^d_N} \{ x_j - 1 + g(x) \} = \begin{cases} 1 & \text{if } z_j = 1 \\ 0 & \text{if } z_j = 0, \end{cases}
\]

which implies that \( f^*(e_j) = z_j \). It follows that if we are able to output the value of \( f^* \) at arbitrary values of the dual variables, we simply need to evaluate \( f^*(e_j) \) for \( j = 1,\ldots,d \) to be able to determine \( z \). As stated earlier, determining \( z \) requires \( \Omega(\sqrt{2^d}) \) queries to \( f \). Thus, to output the value of \( f^* \) at a single point, must require \( \Omega(\sqrt{2^d}/d) \) queries to \( f \). \( \square \)

The lower bound from Proposition 6.1 above can be utilized to prove that the multi-dimensional regular QLFT algorithm is essentially optimal for the task of evaluating the discrete LFT at a specific point.

Corollary 6.2 (Optimality of regular QLFT algorithm for well-behaved functions). Assume that \( f \) is a \( d \)-dimensional function with constant \( W \). To output the value of the discrete LFT of a function \( f \) at a single point, using Algorithm 5 followed by amplitude amplification uses the optimal number of queries to \( f \) up to polylogarithmic factors.

Proof. We use the same function \( f \) and discretizations \( X^d_N = S^d_N \) as for Proposition 6.1 above. Corollary 5.2 ensures that the running time of the multi-dimensional QLFT on \( f \), given by Algorithm 5, is \( O(\text{polylog}(N)) \) and the probability of success is 1, because \( N = K = 2^d \) and \( W = 1 \) as defined in (6), due to the fact that the slope of the function is at most 1 along every edge of the hypercube, and \( \gamma_a = 1 \). Furthermore, the multi-dimensional QLFT makes \( d \) queries to \( f \). Using amplitude amplification [4] to output the value of \( f^*(e_j) \) with high probability requires \( O(\sqrt{2^d}) \) repetitions of the multi-dimensional QLFT algorithm, increasing the number of queries to \( f \) to \( O(\sqrt{2^d} \text{poly}(d)) \). By Proposition 6.1, the number of queries is \( \Omega(\sqrt{2^d}/d) \), which concludes the proof. \( \square \)

So far we considered classes of instances that are well-behaved, but with similar arguments we can also show that every quantum algorithm to compute the \( d \)-dimensional regular QLFT has to scale as \( \sqrt{W} \), up to polylogarithmic factors, where \( W \) is defined in (6). This shows that \( W \) is a fundamental quantity that determines under which conditions an efficient QLFT calculation is possible — depending on how \( W \) scales with the problem size. We start by showing that for certain convex functions calculating the QLFT cannot be done efficiently. These are functions for which \( W \) scales badly with the problem size.

Proposition 6.3 (QLFT calculation can be hard). Let sets \( X^d_N = \{x_0,\ldots,x_{N-1}\} \subseteq \mathbb{R}^d \) and \( S^d_N = \{s_0,\ldots,s_{N-1}\} \subseteq \mathbb{R}^d \) satisfy Assumption 5.1 and let \( f \) be convex. Then any quantum algorithm to compute

\[
|v⟩ = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} |j⟩|f^*(s_j)⟩ \tag{22}
\]

with constant probability requires \( \Omega(\sqrt{2^d}/d) \) queries to \( f \) in general.
Proof. We use the same function \( f \) as for Proposition 6.1, but we now set \( X_N^d = \{0, 1\}^d, S_N^d = \{0, 2^{-d}\}^d \). This yields \( \gamma_\mathbf{x} = 1, \gamma_s = 2^{-d} \), and \( N = W = 2^d \), according to its definition in (6). Notice that

\[
    f^*(s_j) = \max_{x \in \{0,1\}^d} \{\langle x, s_j \rangle - f(x)\} = \langle z, s_j \rangle - f(z) = \langle z, s_j \rangle,
\]

because \( \langle x, s_j \rangle < 1 \) for all \( x \in \{0, 1\}^d \), and therefore the optimum of the max is always obtained by setting \( z = z \), yielding a value of at least 0 (for \( x \neq z \), the expression has a negative value). Suppose we are able to create the state \( |v\rangle \) defined in (22) by performing \( T_v \) queries to \( f \). Then any measurement yields a pair \( (j, f^*(s_j)) \), and because \( S_N^d \) is known we can recover the corresponding \( s_j \). If we perform \( h \) measurements, thereby obtaining a set of pairs \( (s_j, \langle z, s_j \rangle) \), we can set up a system of linear equations:

\[
    \langle s_j, x \rangle = \langle z, s_j \rangle, \quad \text{for } j = 1, \ldots, h.
\]

We can multiply through by \( 2^d \) so that this is a set of equations in the unknown \( x \) with 0-1 coefficients. Whenever this system admits a unique solution, that solution has to be \( z \). Notice that with each measurement we obtain an equation \( \langle s_j, x \rangle = \langle z, s_j \rangle \) where \( s_j \) is chosen uniformly at random among all \( s_j \in S_N^d \). We need to determine what value of \( h \) (i.e., how many equations) is needed, so that the system admits a unique solution. It is known that after sampling \( d + t \) such equations, the probability that the system contains at least \( d \) linearly independent equations is at least \( 1 - \frac{1}{d} \) [20, Appendix G]. Hence, for a given maximum probability of failure \( \delta \), the algorithm that we just described identifies the string \( z \) requiring \( d + \log \frac{1}{\delta} \) samples from the state \( |v\rangle \). Due to [30], \( \Omega(\sqrt{2d}) \) queries to \( f \) are necessary in general to compute \( z \) with any fixed (constant) probability; our algorithm requires \( O(d) \) samples from \( |v\rangle \) to solve this problem. Thus, we must have \( \sqrt{2d} \leq dT_v \), from which the claimed result follows.

Next, we show that the parameter \( W \), used to characterize the probability of success of the regular QLFT algorithm, is a scale-independent parameter: for any given problem instance there is an infinite family of instances that share the same \( W \) and have LFT values that can be mapped one-to-one. This will be useful to show that \( W \) is the right fundamental parameter to look at, when determining difficulty of an instance.

Lemma 6.4. Let sets \( X_N^d = \{x_0, \ldots, x_{N-1}\} \subseteq \mathbb{R}^d \) and \( S_K^d = \{s_0, \ldots, s_{K-1}\} \subseteq \mathbb{R}^d \) satisfy Assumption 5.1 with grid discretization parameters \( \gamma_\mathbf{x}, \gamma_s \). Let \( f \) be convex. Let \( \xi := \max_{i \in \{1, \ldots, N-2\}} \{c_i - c_{i-1}\}/\gamma_\mathbf{x} \), yielding \( W = (\xi \gamma_\mathbf{x} / \gamma_s) \). Then we can construct a function \( \tilde{f} \) and discretizations \( \tilde{X}_N^d = \{\tilde{x}_0, \ldots, \tilde{x}_N\} \) and \( \tilde{S}_K^d = \{\tilde{s}_0, \ldots, \tilde{s}_{K-1}\} \) with \( \tilde{\gamma}_x = 1, \tilde{\gamma}_s = \gamma_s/(\xi \gamma_\mathbf{x}) \) such that \( f^*(s_j) = \tilde{f}^*(\tilde{s}_j) \) for all \( j \in [K] \). Furthermore, the value of \( \tilde{W} \) for \( \tilde{f} \) is equal to \( W \).

Proof. Define \( \tilde{f}(x) := f(x)/\xi, \) and choose \( \tilde{X}_N^d = \{x_i/\gamma_x \mid i \in [N]\} \), \( \tilde{S}_K^d = \{s_j/(\xi \gamma_x) \mid j \in [K]\} \). By construction, \( \tilde{\gamma}_x = 1 \) and \( \tilde{\gamma}_s = \gamma_s/(\xi \gamma_x) \). Using the scaling properties of the LFT [27, Section 11.A] gives \( \tilde{f}^*(\tilde{s}_j) = f^*(s_j/(\xi \gamma_x)) = f^*((\xi \gamma_x s_j)/(\xi \gamma_x))/\xi \). For the final part of the statement, we simply note that the parameter \( W \) remains unchanged, i.e.,

\[
    \tilde{W} = \left[\frac{\tilde{\gamma}_x}{\tilde{\gamma}_s}\right] \left[\frac{1}{\tilde{\gamma}_x}\right] = \left[\frac{\xi \gamma_x}{\gamma_x}\right] = W.
\]

The main consequence of Lemma 6.4 is the fact that we can always assume \( \gamma_x = 1 \) and \( \xi = 1 \) when analyzing a problem instance: the instance can always be rescaled as a preprocessing step if necessary. This plays a role in the next result. To improve its readability let \( \underline{x} := \min_{i \in [N]} x_i, \) \( \overline{x} := \max_{i \in [N]} x_i, \) \( \underline{s} := \min_{j \in [K]} s_j, \) \( \overline{s} := \max_{j \in [K]} s_i, \) \( \underline{c} := \min_{i \in [N]} c_i, \) and \( \overline{c} := \max_{i \in [N]} c_i. \)

Corollary 6.5 (Fundamental importance of parameter \( W \)). Let sets \( X_N^d = \{x_0, \ldots, x_{N-1}\} \subseteq \mathbb{R}^d \) and \( S_K^d = \{s_0, \ldots, s_{K-1}\} \subseteq \mathbb{R}^d \) satisfy Assumption 5.1. Let \( f \) be convex with discrete gradients \( c_0, \ldots, c_N \) defined in (4). Suppose that a quantum algorithm computes the state \( |v\rangle \) defined in (22)
with constant probability of success in time \( \mathcal{C}(N, K, d, x, \tau, s, \xi, \gamma, \tau) \) for some function \( \mathcal{C} \). Then if \( \mathcal{C}(N, K, d, x, \tau, s, \xi, \gamma, \tau) = \text{poly}(\log N, \log K, d)\mathcal{C}(x, \tau, s, \xi, \gamma, \tau) \) for some function \( \mathcal{C} \), we have \( \hat{\mathcal{C}}(x, \tau, s, \xi, \gamma, \tau) = \Omega(\sqrt{W}/\text{poly}(\log N, \log K, d)) \) in general.

Proof. From the proof of Proposition 6.3 (using the same notation), such an algorithm requires \( \Omega(\sqrt{2^d}/d) \) queries. By assumption, the running time is \( \text{poly}(\log N, \log K, d)\hat{\mathcal{C}}(x, \tau, s, \xi, \gamma, \tau) \) for some function \( \hat{\mathcal{C}} \). Hence, we must have \( \hat{\mathcal{C}}(x, \tau, s, \xi, \gamma, \tau) = \Omega(\sqrt{2^d}/\text{poly}(\log N, \log K, d)) \). Furthermore, notice that for the instance constructed in Proposition 6.3, we can rewrite \( \hat{\mathcal{C}}(x, \tau, s, \xi, \gamma, \tau) = \mathcal{C}(\gamma, \gamma, \xi) = \mathcal{C}_W(W) \) for some functions \( \mathcal{C} \), and \( \mathcal{C}_W \), where the first equality is due to the fact that \( \gamma, \gamma, \xi \) subsume the arguments of \( \hat{\mathcal{C}} \), and the second equality is due to Lemma 6.4. Recalling that \( W = 2^d \) for the considered function, we have \( \mathcal{C}_W(2^d) = \Omega(\sqrt{2^d}/\text{poly}(\log N, \log K, d)) \), which concludes the proof.

We remark that the \( d \)-dimensional regular QLFT algorithm, whose running time is characterized in Theorem 4.7 and Corollary 5.2, achieves the optimal scaling of \( \sqrt{W} \) in terms of the parameter \( W \) when combined with amplitude amplification [4]. As shown by Corollary 6.5, the parameter \( \gamma \) determines if an efficient QLFT is possible. We note that in case the function \( f \) satisfies Assumption 2.2 then \( W \leq \kappa K/N \), i.e., in case \( K/N = O(1) \) the condition number \( \kappa \) describes the optimal scaling of any QLFT algorithm. The following two examples show how \( W \) may behave for different convex function \( f : [0, 1]^d \to \mathbb{R} \):

(i) For a multivariate quadratic function \( f : [0, 1]^d \to \mathbb{R} \) given by \( x \mapsto x^TQx + \langle a, x \rangle + b \) for some positive definite \( Q \in \mathbb{R}^{d \times d} \), \( a \in \mathbb{R}^d \), and \( b \in \mathbb{R} \) we have \( W \leq \kappa K/N = O(\kappa) = O(\lambda_{\text{max}}(Q)/\lambda_{\text{min}}(Q)) \), under the assumption that \( K/N = O(1) \) where \( \lambda_{\text{max}}(Q) \) and \( \lambda_{\text{min}}(Q) \) denote the maximal and minimal eigenvalue of \( Q \). We used that for multivariate quadratic case the condition number of \( f \) coincides with the condition number of the matrix \( Q \).

(ii) For a piecewise linear convex function, we have \( W \leq \eta \gamma_s/\gamma_s = O(\eta) \), under the assumption that \( \gamma_s/\gamma_s = O(1) \), where \( \eta \) denotes the maximal difference of two consecutive slopes in one direction.

7 Discussion and open problems

For any sufficiently well-behaved convex function \( f \) known at \( N \) discrete points \( f(x_0), \ldots, f(x_{N-1}) \) that can be loaded efficiently, we present a quantum algorithm to compute a quantum-mechanical representation of its discrete LFT, either in digital or an analog representation. More precisely, we can prepare the states

\[
\frac{1}{\sqrt{K}} \sum_{j=0}^{K-1} |j\rangle |f^*(s_j)\rangle \quad \text{and} \quad \frac{1}{\sqrt{\alpha}} \sum_{j=0}^{K-1} f^*(s_j) |j\rangle = : |f^*\rangle ,
\]

in time \( O(\text{polylog}(N, K)) \), where \( \alpha := \sum_{j=0}^{K-1} f^*(s_j)^2 \) is a normalization constant.

For scenarios in which explicit knowledge of all the elements \( f^*(s_0), \ldots, f^*(s_{K-1}) \) is not required the QLFT transform presented in this paper can achieve an exponential speedup compared to classical algorithms. For example, if \( H \in \mathbb{C}^{K \times K} \) is an observable that can be implemented with complexity \( O(\text{polylog}(K)) \), we can approximately compute the expectation value

\[
\langle f^*|H|f^*\rangle ,
\]

in time \( O(\text{polylog}(N, K)) \). An open question is to determine applications where (23) is a quantity of interest. We can also use the quantum-mechanical representation of the the LFT for two functions to compute their infimal convolution, by performing addition in the dual space [28]. Based on the broad applicability of the LFT, we believe that the QLFT could become a useful tool in the future development of quantum algorithms.
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