CONSTRUCTING NON-COMPUTABLE JULIA SETS

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Abstract. We completely characterize the conformal radii of Siegel disks in the family

\[ P_\theta(z) = e^{2\pi i \theta} z + z^2, \]

corresponding to computable parameters \( \theta \). As a consequence, we constructively produce quadratic polynomials with non-computable Julia sets.

The purpose of this note is to completely characterize the conformal radii of Siegel disks in the family

\[ P_\theta(z) = e^{2\pi i \theta} z + z^2, \]
corresponding to computable parameters \( \theta \). As one consequence, we derive the following statement:

Theorem. There exist computable complex parameters \( c \), such that the Julia set \( J_c \) of the quadratic polynomial \( f_c(z) = z^2 + c \) is non-computable.

In the end of the note we discuss the computational complexity of constructing such values of \( c \), and show that it is poly-time, assuming a certain widely believed conjecture holds.

1. Computability of subsets of \( \mathbb{R}^n \)

We recall the relevant definitions of constructive analysis very briefly. The reader is referred to our earlier work [BY] for a more detailed exposition.

Denote by \( \mathbb{D} \) the set of the dyadic rationals, that is, rationals of the form \( \frac{p}{2^m} \). The classical definition of a computable real number may be formulated as follows:

Definition 1.1. A number \( x \in \mathbb{R} \) is computable, if there exists a Turing Machine (further abbreviated as TM) \( M(n) \) with a positive integer input \( n \), such that for each \( n \), this TM terminates and outputs a number \( d_n \in \mathbb{D} \) with the property \( |x - d_n| < 2^{-n} \).

The definition is generalized to points in \( \mathbb{R}^k \) in an obvious way.

Sometimes, it is desirable to use a real number as a parameter of a computation, without regard to its computable properties. This is achieved with the use of oracles.

Definition 1.2. We say that \( \phi : \mathbb{N} \to \mathbb{D} \) is an oracle for a real number \( x \), if \( |x - \phi(n)| < 2^{-n} \) for all \( n \in \mathbb{N} \). We say that a TM \( M^\phi \) is an oracle machine, if at every step of the computation \( M \) is allowed to query the value \( \phi(n) \) for any \( n \).

Now, for instance, we may define computable functions of a real variable (compare with the discussion in [Brv]).

Definition 1.3. We say that a function \( f : [a, b] \to [c, d] \) is computable, if there exists an oracle TM \( M^\phi(m) \) such that if \( \phi \) is an oracle for \( x \in [a, b] \), then on input \( m \), \( M^\phi \) outputs a \( y \in \mathbb{D} \) such that \( |y - f(x)| < 2^{-m} \).

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Recall that the Hausdorff metric is a metric on compact subsets of $\mathbb{R}^n$ defined by

$$d_H(X, Y) = \inf\{\epsilon > 0 \mid X \subset U_\epsilon(Y) \text{ and } Y \subset U_\epsilon(X)\},$$

where $U_\epsilon(S)$ is defined as the union of the set of $\epsilon$-balls with centers in $S$.

We introduce a class $\mathcal{C}$ of sets which is dense in metric $d_H$ among the compact sets and which has a natural correspondence to binary strings. Namely $\mathcal{C}$ is the set of finite unions of dyadic balls:

$$\mathcal{C} = \left\{ \bigcup_{i=1}^{n} B(d_i, r_i) \mid \text{where } d_i, r_i \in \mathbb{D} \right\}.$$

We now define the notion of computability of subsets of $\mathbb{R}^n$ (see [Wei]).

**Definition 1.4.** We say that a compact set $K \subseteq \mathbb{R}^k$ is computable, if there exists a TM $M(n), n \in \mathbb{N}$ which outputs a set $C_n \in \mathcal{C}$ such that $\text{dist}_H(C_n, K) < 2^{-n}$.

It is not difficult to see that:

**Proposition 1.1.** A set $K \subseteq \mathbb{R}^k$ is computable if and only if the distance function $d_K(x) = \inf\{|x - y| \mid y \in K\}$ is a computable function of a real variable.

In this note we are concerned with computability of quadratic Julia sets $J_c = J(z^2 + c)$. Define the function $J : \mathbb{C} \to K^*$ ($K^*$ is the set of all compact subsets of $\mathbb{C}$) by $J(c) = J(f_c)$. In a complete analogy to Definition 1.3 we can define

**Definition 1.5.** We say that a function $\kappa : S \to K^*$ for some bounded set $S$ is computable, if there exits an oracle TM $M^\phi(m)$ with $\phi$ representing $x \in S$ such that on input $m$, $M^\phi$ outputs a $C \in \mathcal{C}$ such that $d_H(C, \kappa(x)) < 2^{-m}$.

In the case of Julia sets:

**Definition 1.6.** We say that $J_c$ is computable if the function $J : d \mapsto J_d$ is computable on the set $\{c\}$.

We have established the existence of non-computable quadratic Julia sets in [BY], based on the analysis of Julia sets with Siegel disks. The next chapter summarizes the relevant tools of Complex Dynamics.

## 2. Conformal radii of Siegel disks and Yoccoz’s Brjuno function

Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of the Riemann sphere. For a periodic point $z_0 = R^p(z_0)$ of period $p$ its multiplier is the quantity $\lambda = \lambda(z_0) = DR^p(z_0)$. We may speak of the multiplier of a periodic cycle, as it is the same for all points in the cycle by the Chain Rule. In the case when $|\lambda| \neq 1$, the dynamics in a sufficiently small neighborhood of the cycle is governed by the Intermediate Value Theorem: when $0 < |\lambda| < 1$, the cycle is attracting (super-attracting if $\lambda = 0$), if $|\lambda| > 1$ it is repelling. Both in the attracting and repelling cases, the dynamics can be locally linearized:

$$\psi(R^p(z)) = \lambda \cdot \psi(z)$$

where $\psi$ is a conformal mapping of a small neighborhood of $z_0$ to a disk around 0. By a classical result of Fatou, a rational mapping has at most finitely many non-repelling periodic orbits.
In the case when $\lambda = e^{2\pi i \theta}$, $\theta \in \mathbb{R}$, the simplest to study is the parabolic case when $\theta = n/m \in \mathbb{Q}$, so $\lambda$ is a root of unity. In this case $R^p$ is not locally linearizable; it is not hard to see that $z_0 \in J(R)$. In the complementary situation, two non-vacuous possibilities are considered: Cremer case, when $R^p$ is not linearizable, and Siegel case, when it is. In the latter case, the linearizing map $\psi$ from (2.1) conjugates the dynamics of $R^p$ on a neighborhood $U(z_0)$ to the irrational rotation by angle $\theta$ (the rotation angle) on a disk around the origin. The maximal such neighborhood of $z_0$ is called a Siegel disk.

Let us discuss in more detail the occurrence of Siegel disks in the quadratic family. For a number $\theta \in [0, 1)$ denote $[r_0, r_1, \ldots, r_n, \ldots]$, $r_i \in \mathbb{N} \cup \{\infty\}$ its possibly finite continued fraction expansion:

$$[r_0, r_1, \ldots, r_n, \ldots] \equiv \frac{1}{r_0 + \frac{1}{r_1 + \frac{1}{\cdots + \frac{1}{r_n + \cdots}}}}$$

Such an expansion is defined uniquely if and only if $\theta \notin \mathbb{Q}$. In this case, the rational convergents $p_n/q_n = [r_0, \ldots, r_{n-1}]$ are the closest rational approximants of $\theta$ among the numbers with denominators not exceeding $q_n$. In fact, setting $\lambda = e^{2\pi i \theta}$, we have

$$|\lambda^h - 1| > |\lambda^{q_n} - 1|$$

for all $0 < h < q_{n+1}$, $h \neq q_n$.

The difference $|\lambda^{q_n} - 1|$ lies between $2/q_{n+1}$ and $2\pi/q_{n+1}$, therefore the rate of growth of the denominators $q_n$ describes how well $\theta$ may be approximated with rationals.

We recall a theorem due to Brjuno (1972):

**Theorem 2.1** ([Bru]). Let $R$ be an analytic map with a periodic point $z_0 \in \hat{\mathbb{C}}$. Suppose that the multiplier of $z_0$ is $\lambda = e^{2\pi i \theta}$, and

$$B(\theta) = \sum \log(q_{n+1}) q_n < \infty.$$  

Then $z_0$ is a Siegel point.

Note that a quadratic polynomial with a fixed Siegel disk with rotation angle $\theta$ after an affine change of coordinates can be written as

$$P_\theta(z) = z^2 + e^{2\pi i \theta}z.$$  

In 1987 Yoccoz [Yoc] proved the following converse to Brjuno’s Theorem:

**Theorem 2.2** ([Yoc]). Suppose that for $\theta \in [0, 1)$ the polynomial $P_\theta$ has a Siegel point at the origin. Then $B(\theta) < \infty$.

The numbers satisfying (2.3) are called Brjuno numbers; the set of all Brjuno numbers will be denoted $\mathcal{B}$. It is a full measure set which contains all Diophantine rotation numbers. In particular, the rotation numbers $[r_0, r_1, \ldots]$ of bounded type, that is with sup $r_i < \infty$ are in $\mathcal{B}$. The sum of the series (2.3) is called the Brjuno function. For us a different characterization of $\mathcal{B}$ will be more useful. Inductively define $\theta_1 = \theta$ and $\theta_{n+1} = 1/\theta_n$. In this way,

$$\theta_n = [r_{n-1}, r_n, r_{n+1}, \ldots].$$
We define the Yoccoz’s Brjuno function as
\[ \Phi(\theta) = \sum_{n=1}^{\infty} \theta_1 \theta_2 \cdots \theta_{n-1} \log \frac{1}{\theta_n}. \]

One can verify that
\[ B(\theta) < \infty \iff \Phi(\theta) < \infty. \]

The value of the function \( \Phi \) is related to the size of the Siegel disk in the following way.

**Definition 2.1.** Let \( (U,u) \) be a simply-connected subdomain of \( \mathbb{C} \) with a marked interior point. Consider the unique conformal isomorphism \( \phi : \mathbb{D} \to U \) with \( \phi(0) = u \), and \( \phi'(0) > 0 \). The **conformal radius of** \( (U,u) \) **is** the value of the derivative \( r(U,u) = \phi'(0) \).

Let \( P(\theta) \) be a quadratic polynomial with a Siegel disk \( \Delta_\theta \ni 0 \). The **conformal radius of the Siegel disk** \( \Delta_\theta \) **is** \( r(\theta) = r(\Delta_\theta,0) \). For all other \( \theta \in [0,\infty) \) we set \( r(\theta) = 0 \), and \( \Delta_\theta = \{0\} \).

By the Koebe 1/4 Theorem of classical complex analysis (see e.g. [Ahl]), the radius of the largest Euclidean disk around \( u \) which can be inscribed in \( U \) is at least \( r(U,u)/4 \).

We note that one has the following direct consequence of the Carathéodory Kernel Theorem (see e.g. [Pom]):

**Proposition 2.3.** The conformal radius of a quadratic Siegel disk varies continuously with respect to the Hausdorff distance on Julia sets.

Yoccoz [Yoc] has shown that the sum
\[ \Phi(\theta) + \log r(\theta) \]
is bounded below independently of \( \theta \in \mathcal{B} \). Recently, Buff and Chéritat have greatly improved this result by showing that:

**Theorem 2.4 ([BC]).** The function \( \theta \mapsto \Phi(\theta) + \log r(\theta) \) extends to \( \mathbb{R} \) as a 1-periodic continuous function.

In [BBY1] we obtain the following result on computability of quadratic Siegel disks:

**Theorem 2.5.** The following statements are equivalent:

(I) the Julia set \( J(P_\theta) \) is computable by a TM with an oracle access to \( \theta \);

(II) the conformal radius \( r(\theta) \) is computable by a TM with an oracle access to \( \theta \);

(III) the inner radius \( \inf_{z \in \partial \Delta_\theta} |z| \) is computable by a TM with an oracle access to \( \theta \).

We note that when \( \theta \) is not a Brjuno number, the quantities in (II) and (III) are each equal to zero, and the claim is simply that \( J(P_\theta) \) is computable in this case.

We will make use of the following Lemma which bounds the variation of the conformal radius under a perturbation of the domain. It is a direct consequence of the Koebe Theorem (see e.g. [RZ] for a proof).

**Lemma 2.6.** Let \( U \) be a simply-connected subdomain of \( \mathbb{C} \) containing the point 0 in the interior. Let \( V \subset U \) be a subdomain of \( U \). Assume that \( \partial V \subset B_\varepsilon(\partial U) \). Then
\[ 0 < r(U,0) - r(V,0) \leq 4 \sqrt{r(U,0)} \sqrt{\varepsilon}. \]
3. Computing noble Siegel discs

We will make use of a computability result for noble Siegel disks. The term “noble” is applied in the literature to rotation numbers of the form \([a_0, a_1, \ldots, a_k, 1, 1, 1, \ldots]\). The noblest of all is the golden mean \(\gamma_* = [1, 1, 1, \ldots]\).

**Lemma 3.1.** There is a Turing Machine \(M\), which given a finite sequence of numbers \([a_0, a_1, \ldots, a_k]\) computes the conformal radius \(r_\gamma\) for the noble number \(\gamma = [a_0, \ldots, a_k, 1, \ldots]\).

Noble (or more generally, bounded type) Siegel quadratic Julia sets may be constructed by means of quasiconformal surgery (cf. [Dou1]) on a Blaschke product

\[ f_\gamma(z) = e^{2\pi i \tau(\gamma)} z^2 \frac{z - 3}{1 - 3z}. \]

This map homeomorphically maps the unit circle \(\mathbb{T}\) onto itself with a single (cubic) critical point at 1. The angle \(\tau(\gamma)\) can be uniquely selected in such a way that the rotation number of the restriction \(\rho(f_\gamma|_\mathbb{T}) = \gamma\).

For each \(n\), the points \(\{1, f_\gamma(1), f_\gamma^2(1), \ldots, f_\gamma^{k+1}(1)\}\) form the \(n\)-th dynamical partition of the unit circle. We have (cf. Theorem 3.1 of [dFdM]) the following:

**Theorem 3.2 (Universal real a priori bound).** There exists an explicit constant \(B > 1\) independent of \(\gamma\) and \(n\) such that the following holds. Let \(\gamma \in \mathbb{R} \setminus \mathbb{Q}\) and \(n \in \mathbb{N}\). Then any two adjacent intervals \(I\) and \(J\) of the \(n\)-th dynamical partition of \(f_\gamma\) are \(B\)-commensurable:

\[ B^{-1} |I| \leq |J| \leq B |I|. \]

**Proposition 3.3 ([He]).** For each noble \(\gamma = [a_0, \ldots, a_k, 1, \ldots]\) the Blaschke product \(f_\gamma\) is \(K_1\)-quasisymmetrically conjugate to the rotation \(R_\gamma: x \mapsto x + \gamma \mod \mathbb{Z}\). The quasisymmetric constant may be taken as \(K_1 = (2 \max a_i)^{10B^2}\).

Let us now consider the mapping \(\Psi\) which identifies the critical orbits of \(f_\gamma\) and \(P_\gamma\) by

\[ \Psi: f_\gamma^i(1) \mapsto P_\gamma^i(c_\gamma). \]

We have the following (see, for example, Theorem 3.10 of [YZ]):

**Theorem 3.4 (Douady, Ghys, Herman, Shishikura).** The mapping \(\Psi\) extends to a \(K\)-quasiconformal homeomorphism of the plane \(\mathbb{C}\) which maps the unit disk \(\mathbb{D}\) onto the Siegel disk \(\Delta_\gamma\). The constant \(K\) may be taken as the quasiconformal dilatation of any global quasiconformal extension of the \(K_1\)-qs conjugacy of Proposition 3.3. In particular, \(K \leq 2K_1\).

Elementary combinatorics implies that each interval of the \(n\)-th dynamical partition contains at least two intervals of the \((n + 2)\)-nd dynamical partition. This in conjunction with Theorem 3.2 implies that the size of an interval of the \((n + 2)\)-nd dynamical partition of \(f_\gamma\) is at most \(\tau^n\) where

\[ \tau = \sqrt{\frac{B}{B + 1}}. \]
We now complete the proof of Lemma 3.1. Denote $W_n$ the connected component containing 0 of the domain obtained by removing from the plane a closed disk of radius $2K\tau^n$ around each point of
\[ \Omega_n = \{ P_i^\gamma(c), \; i = 0, \ldots, q_n+2 \}. \]
By Theorem 3.4,
\[ \text{dist}_H(\Omega_n, \partial\Delta_\gamma) < K\tau^n, \]
and we have
\[ W_n \subset \Delta_\gamma \text{ and } \text{dist}_H(\partial\Delta_\gamma, \partial W_n) \leq \epsilon_n = 2K\tau^n. \]

Any constructive algorithm for producing the Riemann mapping of a planar region (e.g. that of [BB]) can be used to estimate the conformal radius $r(W_n, 0)$ with precision $\epsilon_n$. Denote this estimate $r_n$. Elementary estimates imply that the Julia set $J(P_\gamma)$ is contained in $B(0,2)$. By Schwarz Lemma this implies $r(\Delta_\gamma, 0) < 2$. By Lemma 2.6 we have
\[ |r(\Delta_\gamma, 0) - r_n| \leq |r(\Delta_\gamma, 0) - r(W_n, 0)| + |r(W_n, 0) - r_n| < 4\sqrt{\epsilon_n} + \epsilon_n \rightarrow 0, \]
and the proof is complete.

4. Main result

Definition 4.1. (cf. [Wei]) A real number $r$ is right-computable (or right r.e.) if there exists a TM $M(n)$ which outputs a non-increasing sequence of dyadic numbers $r_n \geq r$ such that $r_n \rightarrow r$.

A right computable number does not have to be computable, as the definition does not require the existence of a constructive bound on the difference $r_n - r$. We will give a specific example of a non-computable $r$ in §5.

The following result gives a precise characterization of the conformal radii of quadratic Siegel Julia sets $J_{z^2+c}$ with a computable $c$.

Theorem 4.1. Let $r \in [0,0.1]$ be a real number. Then $r = r(c)$ is the conformal radius of a Siegel disc of the Julia set $J_{z^2+c}$ for some computable number $c$ if and only if $r$ is right-computable.

Proof. We prove the “only if” direction here. The remainder of the section is dedicated to proving the “if” direction.

We assume that $c$ is computable, and show that $r(c)$ is right-computable. The case $r(c) = 0$ is obvious, we assume that $r(c) > 0$ for the remainder of the proof. Recall that periodic orbits are dense in the Julia set $J_c$. Let $H_n$ be the union of the first $n$ repelling periodic orbits. Let $\alpha$ be the center of the Siegel disc we are considering. Note that $\alpha$ is a computable real number.

We can algorithmically find a strictly increasing sequence $\{n_l\} \subset \mathbb{N}$ such that $\mathbb{C} \setminus U_l$ has a simply-connected component $W_l$ containing $\alpha$, where
\[ B_{2^{-(l+1)}}(H_{n_l}) \subset U_l \subset B_{2^{-l}}(H_{n_l}). \]

Using any constructive algorithm for computing the conformal radius [BB] we can approximate the $k$-th term of the sequence
\[ R_k = r(B_{2^{-(k-1)}}(W_k), \alpha) + 5 \cdot 2^{4^{-\frac{l+1}{2}}} \]
By Lemma 2.6, $R_k \rightarrow r(c)$, and moreover, \{R_k\} is a non-increasing sequence. Let $\rho_k$ be a dyadic approximation of $R_k$ that we compute so that $|\rho_k - R_k| < 2^{-k}$. Let

\[ r_k = \rho_k + 3 \cdot 2^{-k}. \]

Then \{r_k\} is a computable sequence of dyadic numbers. We have

\[ \lim_{k \to \infty} r_k = \lim_{k \to \infty} \rho_k = \lim_{k \to \infty} R_k = r(c), \]

and for each $k$,

\[ r_k = \rho_k + 3 \cdot 2^{-k} \geq R_k + 2 \cdot 2^{-k} \geq R_{k+1} + 4 \cdot 2^{-(k+1)} \geq \rho_{k+1} + 3 \cdot 2^{-(k+1)} = r_{k+1}. \]

This shows that $r(c)$ is right-computable. \hfill \Box

We now want to prove the “if” direction of Theorem 4.1. Given a computable sequence \{r_n\} such that $r_n \searrow r$ we claim that we can construct a $c$ such that $r = r(c)$. It is easy to do in the case when $r = 0$, as any parabolic parameter would do. From now on, we assume that $r > 0$. We will be using the following two lemmas. The first one is Lemma 3.1 of [BY], and the second one is Lemma 4.2 of [BBY2].

**Lemma 4.2.** For any initial segment $I = [a_0, a_1, \ldots, a_n]$, write $\omega = [a_0, a_1, \ldots, a_n, 1, 1, 1, \ldots]$. Then for any $\varepsilon > 0$, there is an $m > 0$ and an integer $N$ such that if we write $\beta = [a_0, a_1, \ldots, a_n, 1, 1, \ldots, 1, N, 1, 1, \ldots]$, where the $N$ is located in the $n + m$-th position, then

\[ \Phi(\omega) + \varepsilon < \Phi(\beta) < \Phi(\omega) + 2\varepsilon. \]

**Lemma 4.3.** For $\omega$ as above, for any $\varepsilon > 0$ there is an $m_0 > 0$, which can be computed from $(a_0, a_1, \ldots, a_n)$ and $\varepsilon$, such that for any $m \geq m_0$, and for any tail $I = [a_{n+m}, a_{n+m+1}, \ldots]$ if we denote

\[ \beta^I = [a_1, a_2, \ldots, a_n, 1, 1, \ldots, 1, a_{n+m}, a_{n+m+1}, \ldots], \]

then

\[ \Phi(\beta^I) > \Phi(\omega) - \varepsilon. \]

Using Lemma 3.1, we can get a computable version of Lemmas 4.2 and 4.3.

**Lemma 4.4.** For any given initial segment $I = [a_0, a_1, \ldots, a_n]$ and $m_0 > 0$, write $\omega = [a_0, a_1, \ldots, a_n, 1, 1, 1, \ldots]$. Then for any $\varepsilon > 0$, we can uniformly compute $m > m_0$, an integer $t$ and an integer $N$ such that if we write $\beta = [a_0, a_1, \ldots, a_n, 1, 1, \ldots, 1, N, 1, 1, \ldots]$, where the $N$ is located in the $n + m$-th position, we have

\[ r(\omega) - 2\varepsilon < r(\beta) < r(\omega) - \varepsilon, \]

\[ \Phi(\beta) > \Phi(\omega), \]

and for any

\[ \gamma = [a_0, a_1, \ldots, a_n, 1, 1, \ldots, 1, N, 1, 1, \ldots, 1, c_{n+m+t+1}, c_{n+m+t+2}, \ldots], \]

\[ \Phi(\gamma) > \Phi(\omega) - 2^{-n}. \]
Proof. We first show that such \( m \) and \( N \) exist, and then give an algorithm to compute them. By Lemma 4.2 we can increase \( \Phi(\omega) \) by any controlled amount by modifying one term arbitrarily far in the expansion.

By Theorem 2.4, \( f : \theta \mapsto \Phi(\theta) + \log r(\theta) \) extends to a continuous function. Hence for any \( \varepsilon_0 \) there is a \( \delta \) such that \( |f(x) - f(y)| < \varepsilon_0 \) whenever \( |x - y| < \delta \). In particular, there is an \( m_1 \) such that \( |f(\beta) - f(\omega)| < \varepsilon_0 \) whenever \( m \geq m_1 \).

This means that if we choose \( m \) large enough, a controlled increase of \( \Phi \) closely corresponds to a controlled drop of \( r \) by a corresponding amount, hence there are \( m > m_0 \) and \( N \) such that (4.1) holds. (4.2) is satisfied almost automatically. The only problem is to computably find such \( m \) and \( N \).

To this end, we apply Lemma 3.1. It implies that for any specific \( m \) and \( N \) we can compute \( r(\beta) \). This means that we can find the suitable \( m \) and \( N \), by enumerating all the pairs \((m, N)\) and exhaustively checking (4.1) and (4.2) for all of them. We know that eventually we will find a pair for which (4.1) and (4.2) hold.

Finally, \( t \) exists and can be computed by Lemma 4.3. \( \square \)

We are now ready to give an algorithm for computing a rotation number \( \theta \) for which \( r(\theta) = \lim \gamma_k \). \( c \) is easily computed from \( \theta \). The algorithm works as follows. On stage \( k \) it produces a finite initial segment \( I_k = [a_0, \ldots, a_{m_k}] \) such that the following properties are maintained:

1. \( I_0 = [\] ;
2. \( I_k \) has at least \( k \) terms, i.e. \( m_k \geq k \);
3. for each \( k \), \( I_{k+1} \) is an extension of \( I_k \);
4. for each \( k \), denote \( \gamma_k = [I_k, 1, 1, \ldots] \), then \( r_k + 2^{-(k+1)} < \gamma_k < r_k + 2^{-k} \);
5. for each \( k \), \( \Phi(\gamma_k) > \Phi(\gamma_{k-1}) \);
6. for each \( k \), for any extension \( \beta = [I_k, b_{m_k+1}, b_{m_k+2}, \ldots] \),

\[ \Phi(\beta) > \Phi(\gamma_k) - 2^{-k}. \]

The first three properties are very easy to assure. The last three are maintained using Lemma 4.4. By this Lemma we can decrease \( r(\gamma_{k-1}) \) by any given amount (possibly in more than one step) by extending \( I_{k-1} \) to \( I_k \). Here we use the facts that the \( r_k \)'s are computable and non-increasing.

Denote \( \theta = \lim_{k \to \infty} \gamma_k \). The continued fraction expansion of \( \theta \) is the limit of the initial segments \( I_k \). This algorithm gives us at least one term of the continued fraction expansion of \( \theta \) per iteration, hence we would need at most \( O(n) \) iterations to compute \( \theta \) with precision \( 2^{-n} \) (in fact, much fewer iterations would suffice). It remains to prove that, in fact, \( \theta \) is the rotation number we are looking for.

**Lemma 4.5.** The following equalities hold:

\[ \Phi(\theta) = \lim_{k \to \infty} \Phi(\gamma_k) \text{ and } r(\theta) = \lim_{k \to \infty} r(\gamma_k) = r. \]

Proof. By the construction, the limit \( \theta = \lim \gamma_k \) exists. We also know that the sequence \( r(\gamma_k) \) converges to the number \( r = \lim \gamma_k \), and that the sequence \( \Phi(\gamma_k) \) is monotone non-decreasing, and hence converges to a value \( \psi \) (a priori we could have \( \psi = \infty \)). By the Carathéodory Kernel Theorem, we have \( r(\theta) \geq r > 0 \), so \( \Phi(\theta) < \infty \). On the other hand, by
the property we have maintained through the construction, we know that \( \Phi(\theta) > \Phi(\gamma_k) - 2^{-k} \) for all \( k \). Hence \( \Phi(\theta) \geq \psi \). In particular, \( \psi < \infty \).

From [BC] we know that

\[
\psi + \log r = \lim (\Phi(\gamma_k) + \log r(\gamma_k)) = \Phi(\theta) + \log r(\theta).
\]

Hence we must have \( \psi = \Phi(\theta) \), and \( r = r(\theta) \), which completes the proof. \( \square \)

5. Constructing non-computable Julia sets

Theorem 4.1 provides us with a tool for constructing explicit parameters \( c \) for which \( J_{z^2+c} \) is non-computable. In fact, for any right-computable \( r \) which is not a computable number, we obtain such a set.

In particular, we can give an explicit construction of a Julia set parameter, for which computing the Julia set with an arbitrarily high precision would allow us to solve the halting problem.

Let \( R(x,t) \) be the predicate which evaluates to 1 if and only if \( x \) is a valid encoding of a Turing Machine which terminates after exactly \( t \) steps. Then for any fixed \( x \), \( R(x,\bullet) \) evaluates to 1 at most once, and

\[
H(x) = \exists t R(x,t)
\]

is the non-computable Halting predicate. Let \( \{r_k\} \) be the following computable sequence of dyadic numbers

\[
r_k = \frac{1}{16} \left( 1 - \sum_{x=1}^{k} \sum_{t=1}^{k} 4^{-x} \cdot R(x,t) \right).
\]

\( r_k \) is obviously a non-increasing sequence, and it converges to

\[
r = \lim_{k \to \infty} r_k = \frac{1}{16} \left( 1 - \sum_{x=1}^{\infty} 4^{-x} \cdot H(x) \right).
\]

Hence \( r \) is a right-computable number, and there is a computable parameter \( c \) such that \( r(c) = r \). Now note that computing \( r = r(c) \) with precision \( 4^{-(x+3)} \) would require evaluating \( H(1), H(2), \ldots, H(x) \), which is impossible.

6. The complexity of \( \theta \)'s for which \( J_\theta \) is non-computable

In this note we have demonstrated that one can explicitly compute a parameter \( \theta \) such that the corresponding Julia set \( J_\theta = J_{P_\theta} \) is non-computable. In fact, computing \( J_\theta \) would allow us to solve the Halting Problem. It is a natural question to ask whether such a parameter \( \theta \) can be computed fast. That is, how much time is required to produce a \( 2^{-n} \)-approximation of \( \theta \). The algorithm for generating \( \theta \) presented above does not yield any explicit bound on the computational complexity of \( \theta \). In this section we show that under a reasonable conjecture \( \theta \) can be shown to be \textit{poly-time} computable. That is, there is a polynomial \( p(n) \) such that it takes at most \( p(n) \) steps to produce a \( 2^{-n} \) approximation of \( \theta \). The conjecture is a stronger version of Theorem 2.4.

\textbf{Conjecture 6.1.} The 1-periodic continuous function \( f : \theta \mapsto \Phi(\theta) + \log r(\theta) \) has a computable modulus of continuity. In other words, there is a computable function \( \mu : \mathbb{N} \to \mathbb{N} \) such that \( |f(\theta_1) - f(\theta_2)| < 2^{-n} \) whenever \( |\theta_1 - \theta_2| < 2^{-\mu(n)} \).
Note that in Conjecture 6.1 we do not put any restrictions on the growth of \( \mu \) besides it being computable. \( \mu(n) \) can grow extremely fast, as some computable integer functions do. In fact, the following much stronger conjecture has been put forward by Marmi, Moussa, and Yoccoz in [MMY], and is generally supported by the available evidence:

**Marmi-Moussa-Yoccoz Conjecture.** [MMY] The function \( f: \theta \mapsto \Phi(\theta) + \log r(\theta) \) is Hölder of exponent 1/2.

Assuming the weaker Conjecture 6.1, we show that the \( \theta \)'s from Theorem 4.1 (and hence the \( c \)'s) can be made poly-time computable. Of course, the strengthening of Theorem 4.1 only comes in the “if” direction. We only state that direction here.

**Theorem 6.2 (Conditional).** Suppose Conjecture 6.1 holds, and suppose there is a computable sequence \( r_1, r_2, \ldots \) of dyadic numbers such that

1. \( \{r_i\} \) is non-increasing, \( r_1 \geq r_2 \geq \ldots \), and
2. \( \lim_{i \to \infty} r_i = r \).

Then there is a poly-time computable \( \theta \) (and hence a poly-time computable \( c = c(\theta) \)) such that \( r(c) = r \).

**Proof.** The proof goes along the lines of the proof of the “if” direction of Theorem 4.1. We outline the modifications made to the proof here and leave the details to the reader. The key difference is that in the proof of Theorem 4.1 we used Lemma 4.4 to perform a step in decreasing the conformal radius from \( r(\gamma_{k-1}) \) to \( r(\gamma_k) \). The algorithm there is basically an exhaustive search, which of course could take much more than a polynomial time in the precision of \( \gamma_k \) to compute. By assuming Conjecture 6.1 we can deal with \( \Phi(\gamma_{k-1}) \) and \( \Phi(\gamma_k) \) instead of the \( r(\bullet) \)'s. We have an explicit formula for \( \Phi \) that converges well, and we can compute the continued fractions coefficient to make \( \Phi(\gamma_k) \) close to whatever we want relatively fast.

The step of going from \( \gamma_{k-1} \) to \( \gamma_k \) is as follows. First, we do the following computations:

- compute \( d_k \) which is the “drop” in \( r \) we are trying to achieve; we want \( d_k/2 < \log(r(\gamma_{k-1})) - \log(r(\gamma_k)) < d_k \);
- compute using the function \( \mu \) a value \( \delta_k \) such that \( |f(x) - f(y)| < d_k/8 \) whenever \( |x - y| < \delta_k \).

We have no a priori bound on how long these computations would take, but we would still like to be computing \( \theta \) in polynomial time. To achieve this, we use 1’s in the continued fraction expansion of \( \theta \) to “pad” the computation.

When asked about the value of \( \theta \) with precision \( 2^{-n} \) which is higher than what the known terms of the expansion \([I_{k-1}]\) can provide, we do the following:

- try to compute \( d_k \) and \( \delta_k \) as above, but run the computation for at most \( n \) steps;
- if the computation does not terminate, output an answer consistent with the initial segment \([I_{k-1}, 1, 1, \ldots, 1]^{2^n}\);

- if the computation terminates in less than \( n \) steps proceed as described below.

Note that so far the computation is polynomial in \( n \). For some sufficiently large \( n \) the computation will terminate in \( n \) steps, at which point we will have computed \( d_k \) and \( \delta_k \). We then add more 1’s to the initial segment to assure that \( |\gamma_{k-1} - \gamma_k| < \delta_k \).
Recall that our goal is to assure that
\[ d_k/2 < \log(r(\gamma_{k-1})) - \log(r(\gamma_k)) < d_k. \]
With the current initial segment for \( \gamma_k \) we have \(|\gamma_{k-1} - \gamma_k| < \delta_k \), and hence in the difference
\[ \log(r(\gamma_{k-1})) - \log(r(\gamma_k)) = \Phi(\gamma_k) - \Phi(\gamma_{k-1}) + (f(\gamma_{k-1}) - f(\gamma_k)) \]
the last term is bounded by \( d_k/8 \). This means that for the current step it suffices to increase \( \Phi(\gamma_k) \) relative to \( \Phi(\gamma_{k-1}) \) by between \( \delta_k/5d_k \) and \( \delta_k/7d_k \).

Let \( M \) be the total length of \( I_{k-1} \) and the 1’s we have added, and let us extend the continued fraction by putting \( N \in \mathbb{N} \) in the \( M+1 \)-st term, and all 1’s further. Increasing \( M \) if necessary, we can ensure an approximate equality
\[ \Phi(\gamma_k) \approx \Phi(\gamma_{k-1}) + \alpha(N) \log N \]
up to an error of \( \frac{1}{32}d_k \). Let \( p_M/q_M \) be the \( M \)-th convergent of the resulting continued fraction. Recall that on an input \( n \) we need to compute \( \theta \) with precision \( 2^{-n} \) in time polynomial in \( n \).

If \( 2^{-n} > 1/\sqrt{q_M} \), then we do not need to know anything about \( N \) to compute the required approximation. Suppose \( 2^{-n} < 1/\sqrt{q_M} \), which means \( n > \log q_M/2 \). And we have time polynomial in \( \log q_M \) to perform the computation.

Note that \( M = O(\log q_M) \). It is also not hard to see that \( \alpha(N) < 2^{M/2} \), so in order to have a change by \( \approx 3d_k/4 \) we must have \( N > e^{\Omega(2^M)} \), hence by making \( M \) sufficiently large (depending on the value of \( d_k \)), we can guarantee that \( N > e^{2^{M/3}} \). This means that we can approximate \( \alpha(N) \) with the truncated function \( \Phi \) at the \( M \)-th convergent of the continued fraction. Write \( p_M/q_M = [a_1, a_2, \ldots, a_M] \), and denote \( \beta = [a_1, a_2, \ldots, a_M] \cdot [a_2, a_3, \ldots, a_M] \cdot \ldots \cdot [a_{M-1}, a_M] \cdot [a_M] \).

Then \( \beta \) approximates \( \alpha(N) \) within a very small relative error. In particular, we can assure that
\[ \beta \cdot \left( 1 - \frac{1}{32} \right) < \alpha(N) < \beta \cdot \left( 1 + \frac{1}{32} \right). \]
In time polynomial in \( \log q_M \) we can compute the exact expression for \( \beta \) using rational arithmetic: \( \beta = p/q \). Now we can estimate \( N \) and write it as \( e^{6d_k/83} \) in time polynomial in \( \log(q_M) \). From there we can continue by adding enough 1’s to get \( I_k \) and \( \gamma_k = [I_k, 1, 1, \ldots] \).

By the construction, it would give us the necessary decrease in the value of \( r(\gamma_k) \).

\[ \square \]

**References**

[Ahl] L. Ahlfors, *Complex Analysis*, McGraw-Hill, 1953

[BB] E. Bishop, D.S. Bridges, *Constructive Analysis*, Springer-Verlag, Berlin, 1985.

[BC] X. Buff, A. Chéritat, *The Yoccoz Function Continuously Estimates the Size of Siegel Disks*, Annals of Math., to appear.

[BBY1] I. Binder, M. Braverman, M. Yampolsky, *Filled Julia sets with empty interior are computable*. e-print, math.DS/0410580.

[BBY2] I. Binder, M. Braverman, M. Yampolsky, *On computational complexity of Siegel Julia sets*. *Commun. Math. Phys.*, to appear.

[Brv] M. Braverman, *On the Complexity of Real Functions*. e-print, cs.CC/0502066, 2005.

[BY] M. Braverman, M. Yampolsky, *Non-computable Julia sets*. *Journ. Amer. Math. Soc.*, to appear.

[Bru] A.D. Brijuno, *Analytic forms of differential equations*, *Trans. Mosc. Math. Soc.* 25(1971)

[dFdM] E. de Faria and W. de Melo, *Rigidity of critical circle mappings I*. *J. Eur. Math. Soc. (JEMS)* 1(1999), no. 4, 339-392.
[Dou1] A. Douady. Disques de Siegel et anneaux de Herman, Sem. Bourbaki, Astérisque, 152-153(1987), 151-172.

[Dou2] A. Douady. Does a Julia set depend continuously on the polynomial? In Complex dynamical systems, Proc. Symp. Appl. Math., vol. 49

[He] M. Herman. Conjugaison quasi symétrique des homéomorphismes du cercle à des rotations, Manuscript, 1986. and Conjugaison quasi symétrique des difféomorphismes du cercle à des rotations et applications aux disques singuliers de Siegel, Manuscript, 1986. Available from http://www.math.kyoto-u.ac.jp/~mitsu/Herman/index.html

[Lav] P. Lavaurs. Systèmes dynamiques holomorphes, Explosion de points périodiques paraboliques. Thése, Orsay, 1989.

[Mil] J. Milnor. Dynamics in one complex variable, 3rd ed. Annals of Math. Studies, Princeton University Press, 2006.

[Pom] C. Pommerenke, Boundary behavior of conformal maps, Springer-Verlag, 1992.

[MMY] S. Marmi, P. Moussa, J.-C. Yoccoz, The Brjuno functions and their regularity properties, Commun. Math. Phys. 186(1997), 265-293.

[RZ] S. Rohde, M. Zinsmeister, Variation of the conformal radius, J. Anal. Math., 92 (2004), pp. 105-115.

[Wei] K. Weihrauch, Computable Analysis, Springer, Berlin, 2000.

[YZ] M. Yampolsky, S. Zakeri, Mating Siegel quadratic polynomials, Journ. Amer. Math. Soc., 14(2000), 25-78.

[Yoc] J.-C. Yoccoz, Petits diviseurs en dimension 1, S.M.F., Astérisque, 231(1995).