Dynamic Responses of a Pile with a Cap under the Freezing and Thawing Processes of a Saturated Porous Media Considering Slippage between Pile and Soil

Qiang Li 1,*, Xinyi Li 1,*, Minjie Wen 2,3, Ling Hu 1, Weiwei Duan 1 and Jiaxing Li 1

1 Department of Civil Engineering, Zhejiang Ocean University, Zhoushan 316022, China; z19085222014@zjou.edu.cn (L.H.); weiweiduan@zjou.edu.cn (W.D.); z20086100052@zjou.edu.cn (J.L.)
2 Research Center of Coastal Urban Geotechnical Engineering, Zhejiang University, Hangzhou 310027, China; 0620577@zj.edu.cn
3 Key Laboratory of Soft Soils and Geoenvironmental Engineering of Ministry of Education, Zhejiang University, Hangzhou 310027, China
* Correspondence: qiangli@zjou.edu.cn (Q.L.); z19085222024@zjou.edu.cn (X.L.)

Abstract: The freezing/thawing stratification effect of seasonal factors or artificial disturbances in frozen soil regions has an important influence on the vertical vibration of the pile–soil–cap system. Taking into account the slippage between the pile and soil, a simplified layered analytical model of the vertical vibration of the pile–soil–cap system in a double-layered stratum under the freezing and thawing processes of a saturated porous medium was established, and the analytical solution of the dynamic response on the top of the pile cap was obtained. In this model, frozen saturated porous media and Biot’s porous media theory were used to simulate frozen soil and unfrozen soil, respectively. The validation of the slippage model was first verified by comparison with the results of the existing model tests. This was followed by a dynamic model test of the pile–soil–cap system in a self-made, ground-freezing system. In comparison with the analytical results and the experimental results of model tests under the freezing/thawing processes, the validation of the present model is further verified. A comprehensive parametric study reveals that the parameters of the frozen or thawed soil layer have significant effects on the amplitude–frequency curve of the vertical vibration of the pile foundation.

Keywords: frozen porous medium; pile foundation; slippage; dynamic response; resonance frequency; model test

1. Introduction

The theory of dynamic pile–soil–cap interaction is an important work in earthquake engineering, transportation engineering and machinery foundation design [1]. Over the past several decades, the dynamic interaction between pile and soil has been widely investigated, especially in the research of the low-strain reflection theory of a single pile and its application in the field of pile integrity detection. Large amounts of work have been conducted on the technology of detecting defective piles, large-diameter piles and pipe piles [2–13]. However, the dynamic interaction between a pile with cap and the soil has not been sufficiently studied due to the complexity of the pile–soil–cap interaction. There are three methods for the studies on the dynamic responses of piles with caps including theoretical analysis [14–20], numerical simulations [21,22] and in situ or indoor model tests [23–34].

The analytical method, despite its difficulty and challenges, is still widely used by researchers as it helps to understand the inherent mechanisms in the dynamic interaction of the pile–soil–cap system. The vertical and lateral vibrations are extensively investigated to give insight into the dynamic responses of pile foundations [35–43]. The theoretical
Investigations of the vertically dynamic response of a pile foundation involve two keys: soil simulation and pile–soil interface description. In the early stage, a single-phase elastic medium was used to simulate the soil. In recent years, the theory of saturated or unsaturated porous medium was gradually developed to simulate the dynamic interaction between the pile and soil [44–53].

However, with the development of engineering in high-latitude and cold regions, the application of pile foundation has been highly popularized in permafrost or seasonal permafrost areas. Frozen soil is a special kind of soil that is a complex multiphase structure with solid particles, ice, water, and gas. A prominent feature of frozen ground engineering is that there is a freezing/thawing phenomenon with the change in temperature that will have a significant impact on the dynamic responses of the pile foundation [54–56]. Based on the theories of frozen porous media established by Leclaire [57] and Carcione [58], Li et al. [59] and Cao et al. [60] obtained the analytical solution of the vertical vibration of a single pile in frozen porous media. Immediately after, the analytical solution for pile–soil system vertical vibrations in a double-layered stratum under the freezing and thawing processes of a saturated porous medium was derived by Li et al. [61]. However, the effects of the slippage between the pile and soil and the mass of pile cap were not considered, and it is worth noting the lack of validation of the relevant model test data. In reality, the slippage between the pile and soil is a very common phenomenon when the pile is subjected to large external excitations. Kwon and Yoo [62] investigated the dynamic interaction of pile–soil structure in liquefied sand by FLAC3D. They proposed an interface model to simulate the dynamic response between the pile and soil, which in effect was that the non-linearity of the soil was considered by using hysteretic damping. Kim et al. [63] considered the slip contact between the pile and soil and established a numerical FEM model of the vertically dynamic responses of a pile foundation. Numerical results also revealed that the resonant frequencies were overestimated, and their corresponding amplitudes were underestimated when the perfect bonded model was adopted. Although the numerical method has developed rapidly in recent years, there are some difficulties in parameter acquisition and constitutive relationship determination. As another important verification method for theoretical analysis, experimental investigation of pile foundation is generally used to verify the validity of the theoretical results. In situ tests of pile foundations are expensive, and the experimental conditions in a field test are difficult to control. Model tests present the advantages of controllable conditions, timesaving, and great economy compared to in situ tests. Therefore, it usually exists as a reliable means of verification. Manna and Baidya [64] performed a series of dynamic tests on the model piles under several levels of vertical harmonic excitation. Additionally, a comprehensive study involving both vertical vibration testing and theoretical analysis of the soil–pile–cap were presented by Manna and Baidya [65]. Biswas et al. [66] performed dynamic field tests and Novak’s continuum approach analysis on a full-scale single pile under coupled vibrations. Elkasabgy and El Naggar [67] carried out a large number of vertically dynamic loaded tests of the pile foundation and used the nonlinear weak-zone model to analyze the vertical vibration of the pile foundation. Their results showed that the weak-zone surrounding the pile had a significant effect on the vibration of the pile foundation. Ralli et al. [68] conducted vertical vibration tests on an inclined pile with different angles. The results demonstrated that the axial load capacity utilization of an inclined pile was lower compared to a vertical pile. These experimental results provide strong support for the vertically dynamic analysis of pile foundations. Although there are some research results of the vertical vibration of a pile with the cap, there have been few investigations on the vertical vibration of pile foundation in the freezing/thawing stratum taking into account the slippage between the pile and soil.

In this paper, the effect of the freezing/thawing processes on the vertical vibration of a pile–soil–cap system is studied. A simplified vertical vibration slippage model of a pile foundation in a double-layered stratum is established by using a frozen saturated porous medium and a saturated porous medium to simulate frozen soil and unfrozen or thawed soil, respectively. The dynamic response of a pile foundation in a freezing/thawing
double-layered stratum under the vertical vibration load is obtained. The validity of
the model is verified by examining the results of an experimental study of the vertical
vibration of pile–soil–cap in freeze–thaw ground. In addition, the need for the model to
take into account the effects of pile–soil slippage is also illustrated. Then, the effects
of some crucial parameters of the freezing/thawing stratum on the dynamic responses of
the pile foundation are investigated.

2. Model Establishing

2.1. Schematic Diagram

The vertical vibration of the pile–soil–cap system under the action of an arbitrary
excitation force on the top of the pile cap is studied. According to the influence of the
environmental temperature on the upper active layer, the permafrost and seasonal frozen
soil foundation can be divided into four cases. The first and second cases assume that
a pile foundation is applied in seasonally frozen soil regions. In the summer, the soil
layer is unfrozen, which is simulated by a double-layered saturated stratum. When the
atmospheric temperature decreases in winter, the upper active layer is frozen, and the
lower layer is still an unfrozen soil layer, forming a double-layered freezing/thawing
stratum. The third and fourth cases assume that a pile foundation is applied in permafrost
regions. In the winter, the soil layers are frozen to full depth, which are used to simulate
by a double-layered frozen, saturated stratum. Similarly, in the summer, the increase in
temperature causes the upper active layer of permafrost to melt and form a double-layered
freezing/thawing stratum. The schematic model of the double-layered stratum under the
freezing/thawing processes is shown in Figure 1. The soil layer is divided into two layers,
with the lower layer marked as 1 and the upper layer marked as 2. Compared with the
vertical displacements, the radial displacements have little effect on the soil layer, which
are ignored for simplification. The interaction between the soil layers is simplified as a
Winkler elastic foundation. The distributed spring coefficient \( k_{st1} \) or \( k_{st2} \) is represented as
the interaction between the soil layers, and \( k_{sb1} \) is represented as the spring coefficient
of the foundation at the bottom of the soil layer. The upper surface of the upper active layer is
a free boundary, and the normal stress is zero (equivalent to \( k_{st2} = 0 \)). For the description of
the soil layer, the unfrozen soil layer (or thawed soil layer) is described by Biot’s saturated
porous medium [69], and the frozen soil layer is described by the frozen saturated porous
medium proposed by Leclaire et al. [57]. Both of the soil layers are homogeneous and
isotropic. The pile is treated as a one-dimensional rod with pile length \( H_0 \) and pile diameter \( a \). The density and elastic modulus of the pile are \( \rho_b \) and \( E_b \), respectively. The bottom is
simplified as an elastic foundation, and \( k_{pb1} \) is represented as the elastic support coefficient
of the underlying soil layer to the pile. There is relative slippage between the pile and soil,
and the friction resistance of the pile shaft is recorded as \( f(z) \). The pile cap is at the top of
the pile, in contact with the ground. The contact action is represented by viscoelastic support,
with the elastic support coefficient of \( k_{ms} \) and the cohesive coefficient of \( c_{ms} \). A force \( F(t) \)
acts on the pile cap; the vibration of the pile–soil–cap system is small deformation.

2.2. Governing Equations

2.2.1. Dynamic Equations of the Soil Layer

The governing equation of a porous medium can be expressed as

\[
\mathbf{R} \nabla \cdot \mathbf{u} - p \nabla \times \nabla \times \mathbf{u} + \mathbf{A} \cdot \frac{\partial \mathbf{u}}{\partial t} = \mathbf{F}
\]

where \( t \) is the time parameter; \( \nabla \) is the Hamilton operator; \( \mathbf{u} \) is the displacement vector;
and \( \mathbf{u} = \begin{pmatrix} \mathbf{u}(s) \mathbf{u}(f) \mathbf{u}(i) \end{pmatrix}^T \) and \( \mathbf{u} = \begin{pmatrix} \mathbf{u}(s) \mathbf{u}(f) \mathbf{u}(i) \end{pmatrix}^T \) correspond to the Biot’s saturated
porous medium and Leclaire’s frozen saturated porous medium, respectively. \( \mathbf{u}(s) \), \( \mathbf{u}(f) \),
and \( \mathbf{u}(i) \) are the solid displacement, liquid displacement and ice displacement, respectively.
\( \bar{\mathbf{K}}, \bar{\mathbf{m}}, \bar{\mathbf{p}} \) and \( \bar{\mathbf{A}} \) represent the coefficients in the governing equation of a porous medium as shown below.

For Biot's saturated porous medium,

\[
\bar{\mathbf{K}} = \begin{bmatrix}
\lambda_c + 2\mu & aM \\
aM & M
\end{bmatrix}, \quad \bar{\mathbf{m}} = \begin{bmatrix}
\mu & 0 \\
0 & 0
\end{bmatrix}, \quad \bar{\mathbf{p}} = \begin{bmatrix}
\rho & \rho_f \\
\rho_f & m
\end{bmatrix}, \quad \bar{\mathbf{A}} = \begin{bmatrix}
0 & 0 \\
0 & b
\end{bmatrix}.
\]

For Leclaire's frozen saturated porous medium,

\[
\bar{\mathbf{K}} = \begin{bmatrix}
R_{11} & R_{12} & 0 \\
R_{12} & R_{22} & R_{23} \\
0 & R_{23} & R_{33}
\end{bmatrix}, \quad \bar{\mathbf{m}} = \begin{bmatrix}
\mu_{11} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \mu_{33}
\end{bmatrix}, \quad \bar{\mathbf{p}} = \begin{bmatrix}
\rho_{11} & \rho_{12} & 0 \\
\rho_{12} & \rho_{22} & \rho_{23} \\
0 & \rho_{23} & \rho_{33}
\end{bmatrix}, \quad \bar{\mathbf{A}} = \begin{bmatrix}
b_{11} & -b_{11} & 0 \\
b_{11} & b_{11} + b_{33} & -b_{33} \\
0 & -b_{33} & b_{33}
\end{bmatrix}.
\]

where \( \lambda_c = \lambda + \alpha^2 M \), and \( \lambda \) and \( M \) are Biot's parameters. \( \alpha = 1 - K_b/K_s, K_d = K_s [1 + n (K_b/K_f - 1)] \), and \( M = K_s^2/(K_d - K_b) \). \( n \) is the porosity of the saturated soil. \( K_s, K_f \) and \( K_b \) are the bulk moduli of the solid grains, fluid and soil skeleton, respectively. \( \mu \) is the shear modulus. \( \rho = (1 - n) \rho_s + n \rho_f \), \( \rho_s \) is the fluid density, \( \rho_f \) is the soil particle density, and \( p \) is the soil density. \( b = \rho_f g/\kappa_b \) is the permeability coefficient, and \( g \) denotes the gravity acceleration. \( R_{11} = K_1 + \frac{1}{\mu_{11}}, R_{12} = R_{21} = C_{12}, R_{22} = K_2, R_{23} = R_{32} = C_{23}, R_{33} = K_3 + \frac{1}{\mu_{33}} \). \( K_1, K_2 \) and \( K_3 \) represent the bulk moduli of the solid, the pore fluid and the ice, respectively. \( \mu_{11} \) and \( \mu_{33} \) represent the shear moduli of the soil skeleton and ice. \( C_{12} \) and \( C_{23} \) represent the elastic coupling coefficients of the pore fluid with the solid and ice, respectively. \( \rho_{11}, \rho_{22} \) and \( \rho_{33} \) denote the densities of the solid particle, pore fluid and ice, respectively. \( \rho_{12} \) and \( \rho_{23} \) denote the quality densities induced by the inertial coupling between solid and fluid and between ice and fluid, respectively. \( b_{11} \) and \( b_{33} \) denote the viscous coupling coefficients between the soil skeleton and the ice skeleton between the pore fluid, respectively.

2.2.2. Governing Equations of Pile Vibration

The equation governing vertical vibration of the \( i \)-th pile segment was expressed as:

\[
E_{bi} \pi a^2 \frac{d^2 w_{bi}}{dz^2} - f_i(z) = \rho_{bi} \pi a^2 \frac{d^2 w_{bi}}{dt^2} \tag{2}
\]

where \( E_{bi} \) and \( \rho_{bi} \) denote the elastic modulus and the density of the \( i \)-th pile segment, respectively. \( w_{bi} \) is the vertical displacement of the \( i \)-th pile segment, and \( f_i(z) \) is the friction force of the \( i \)-th pile segment. \( z \) represents the length of the pile along the \( z \) coordinate. \( d^2 \) and \( d \) denote differential symbols.

2.2.3. Governing Equations of Pile Cap Vibration

\[
M \frac{d^2 w(z,t)}{dt^2} + c_{ms} \frac{dw(z,t)}{dt} + k_{ms} w(z,t) + p(t) \pi a^2 = F(t) \tag{3}
\]

where \( w \) is the vibration displacement of the pile cap, and \( M \) is the mass of the pile cap. \( c_{ms} \) and \( k_{ms} \) are the bearing stiffness and damping of the pile cap, respectively. \( p(t) \) is the reaction force of the pile top, and \( F(t) \) is the excitation force of the pile cap.

2.3. Boundary and Initial Conditions of the Pile–Soil–Cap System

2.3.1. Boundary Conditions of the Soil Layer in Four Cases

In the previous section, permafrost and seasonally frozen soil are divided into four cases according to the freezing/thawing processes caused by seasonal factors or artificial activities, in which the upper part is the active layer, and the lower part is the inactive layer, as shown in Figure 1.
The boundary conditions of the inactive layer

The boundary condition of the soil layer is expressed as follows in an axisymmetric coordinate system:

\[
E_{s1}^{(av)} \frac{\partial u_{s1}^{(s)}}{\partial z}(r, H_0) + k_{sb1} u_{s1}^{(s)}(r, H_0) = 0
\]  
(4)

\[
- \frac{E_{s1}^{(av)}}{r} \frac{\partial u_{s1}^{(s)}}{\partial r}(r, H_1) + k_{at1} u_{s1}^{(s)}(r, H_1) = 0
\]  
(5)

where \(E_{s1}^{(av)}\) represents the elastic modulus of solid in the inactive layer. Subscript 1 indicates the lower layer of the soil. For seasonally frozen soil, the inactive layer is thawed soil. For permafrost, the inactive layer is frozen soil; \(E_{s1}^{(av)} = R_{11}\). \(u_{s1}^{(s)}\) represents the vertical displacement of solid phase in the inactive layer.

The radial displacement of the soil layer at infinity is zero; there is no lateral displacement, and the pile shaft is impermeable.

For seasonally frozen soil,

\[
u_{s1}^{(s)}(a, z) = 0, \quad u_{s1}^{(f)}(a, z) = 0
\]  
(6)

where \(u_{s1}^{(s)}\) and \(u_{s1}^{(f)}\) are the radial displacements of the solid and liquid in the inactive layer, and subscript 1 represents the inactive layer (i.e., the lower layer).
For permafrost,

\[ u_1^{(s)}(a, z) = 0, \quad u_1^{(f)}(a, z) = 0, \quad u_1^{(f)}(a, z) = 0 \]  

where \( u_1^{(i)} \) is the radial displacements of the ice phases in the inactive layer, and subscript 1 represents the inactive layer (i.e., the lower layer).

The shear stresses at the interface between the pile and soil remain equal. A slippage interface with a spring and damper is used to simulate the dynamic stiffness and damping of the pile–soil interface. It is assumed that the dynamic stiffness is independent of the frequency, and the damping ratio is proportional to the frequency within a certain range. The corresponding pile–soil contact conditions of seasonally frozen soil are expressed as follows:

\[ \tau_{zr}^{(s)}(a, z) = -f_{s1}(z)/2\pi a, \quad f_{s1} = k_{s1}\Delta w_1 + D_{s1}\Delta \dot{w}_1 \]  

The pile–soil contact conditions of permafrost are as follows:

\[ \tau_{zr}^{(s)}(a, z) + \tau_{zr}^{(i)}(a, z) = -[f_{s1}(z) + f_{i1}(z)]/2\pi a, \quad f_{i1} = k_{i1}\Delta w_1 + D_{i1}\Delta \dot{w}_1, \]

where \( \tau_{zr}^{(s)} \) and \( \tau_{zr}^{(i)} \) are the solid and ice phase shear stress acting on a plane perpendicular to the z axis but along the r axis, respectively. \( \Delta w_i = (w_{bi} - u_2^{(i)}) \) represents the relative movement between the pile and soil; the overdot denotes the derivative with respect to time. \( k_{si} \) (or \( k_{ii} \)) and \( D_{si} \) (or \( D_{ii} \)) represent the dynamic stiffness and damping coefficient of the pile–soil interface, respectively. Their first subscript indicates the solid or ice phase, and the second subscript indicates the label of the soil layer, with the subsoil being labeled as 1 and the upper layer as 2. The damping coefficient is directly proportional to frequency, and the dot symbol represents the derivative of time.

(2) The boundary conditions of the active layer

The boundary condition of the soil layer is expressed as follows:

\[ E_{s2}^{(sv)} \frac{\partial u_2^{(s)}}{\partial z} (r, H_1) + k_{s2} u_2^{(s)} (r, H_1) = 0 \]  

where \( E_{s2}^{(sv)} \) represents the elastic modulus of solid in the active layer. In summer, the upper active layer is thawed soil or unfrozen soil: \( E_{s2}^{(sv)} = \mu (3\lambda + 2\mu) / (\lambda + \mu) \). In winter, the upper active layer is frozen soil: \( E_{s2}^{(sv)} = R_{11} \). Subscript 2 indicates the upper layer of soil.

The active layer satisfies the following conditions:

(1) The upper surface is a free boundary, the normal stress is zero, and the pore pressure is zero;

(2) There is no lateral displacement on the pile shaft, and it meets the impervious condition;

(3) The shear stresses at the interface between the pile and soil remain equal, and the slippage contact is kept between the pile and soil.

In summer, these conditions can be expressed as

\[ \sigma_2^{(s)}(r, 0) = 0, \sigma_2^{(f)}(r, 0) = 0 \]

\[ u_2^{(s)}(a, z) = 0, \quad u_2^{(f)}(a, z) = 0 \]

\[ \tau_{zr}^{(s)}(a, z) = -f_{s2}(z)/2\pi a, \quad f_{s2} = k_{s2}\Delta w_2 + D_{s2}\Delta \dot{w}_2 \]

where \( \sigma_2^{(s)} \) is the vertical positive stress in the solid phase in the upper soil layer; \( \sigma_2^{(f)} \) is the positive stress in the liquid phase in the upper soil layer.
In winter, it can be expressed as:

\[ \sigma_{iz}^{(s)}(r, 0) = 0, \quad r_{iz}^{(f)}(r, 0) = 0, \quad \sigma_{iz}^{(f)}(r, 0) = 0 \] (14)

\[ u_{iz}^{(s)}(a, z) = 0, \quad u_{iz}^{(f)}(a, z) = 0, \quad u_{iz}^{(f)}(a, z) = 0 \] (15)

\[ r_{iz}^{(s)}(a, z) + r_{iz}^{(f)}(a, z) = -\left[f_{s2}(z) + f_{z2}(z)\right]/2\pi a, \quad f_{s2} = k_{s2}\Delta w_2 + D_{s2}\Delta \dot{w}_2, \] (16)

2.3.2. The Boundary Conditions of the Single Pile

The boundary conditions of the pile are expressed as:

\[ \frac{\partial w_{bi}}{\partial z} \bigg|_{z=H_i} = \frac{P_i(t)}{E_{bi}a^2} \left( E_{bi}a^2 \frac{\partial w_{bi}}{\partial z} + k_{ph}w_{bi} \right) \bigg|_{z=H_{i-1}} = 0 \] (17)

where \( P_i(t) \) is the force exerted by the \((i+1)\)-th pile segment on the top of the \(i\)-th pile segment, and \( k_{ph} \) is the foundation reaction coefficient at the bottom of the \(i\) pile segment.

2.3.3. The Initial Conditions of the Pile–Soil–Cap Vibration System

At the initial moment, the pile–soil–cap system is stationary:

\[ \frac{\partial w_{bi}}{\partial t} \bigg|_{t=0} = 0, \quad \frac{\partial^2 w_{bi}}{\partial t^2} \bigg|_{t=0} = 0, \quad w_{bi} \bigg|_{t=0} = 0, \quad \frac{\partial w_{bi}}{\partial t} \bigg|_{t=0} = 0 \] (18)

3. Solutions of the Pile Foundation

According to the four cases described in Section 2, there are four models. Two models are the lower part of which is the thawed soil layer, the upper part of which is the thawed soil layer or the frozen soil layer. The other two models are the lower part of which is the frozen soil layer, and the upper part is the frozen soil or the thawed soil layer. In this section, the governing equations of the soil layer are transformed into wave equations in a dimensionless form by introducing the vector Helmholtz decomposition. Then, the axisymmetric dynamic solutions of frozen saturated porous medium and saturated porous medium are obtained by the separation of variables method, which can be found in the literature [59,61].

3.1. Solution to Dynamic Response of the Single Pile with Slippage

In the pile–soil–cap model, the lower soil layer is the inactive layer, which does not change with seasonal factors. It is a frozen soil layer for permafrost. However, it is a thawed soil layer for seasonally frozen soil. On the contrary, the upper soil layer of the model is the active layer, which changes with atmospheric temperature. For permafrost, it is frozen in winter and thawed in summer. However, for seasonally frozen soil, it is unfrozen in summer and frozen in winter.

3.1.1. Inactive Layer Solution

Whether the inactive layer is unfrozen or frozen, the upper and lower boundaries of the layer are the same, satisfying Equations (4) and (5). The eigenvalue \( h_{n1} \) of the soil layer satisfies the following characteristic equation:

\[ (h_{n1} + \bar{k}_{sb1})e^{h_{n1}\theta_0} - (h_{n1} - \bar{k}_{sb1})e^{-h_{n1}\theta_0} = 0 \] (19)

where \( h_{n1} \) represents the eigenvalue of the subsoil. \( \delta_{1n1} = \frac{(h_{n1} - \bar{k}_{sb1})}{(h_{n1} + \bar{k}_{sb1})}2\theta_1, \theta_0 = H_0/a, \theta_1 = H_1/a, \bar{k}_{sb1} = k_{sb1}a/E_{sz}, \) and \( \bar{k}_{sb1} = k_{sb1}a/E_{sz}. \)

For seasonally frozen soil, the displacement of the first pile segment is...
\[ \varpi_{b1}(z) = A_1 \left[ e^{\kappa z} + \sum_{n=1}^{\infty} \frac{-2\eta_{1n}E_{n1}(e^{\kappa_{1n}z} + \delta_{1n1}e^{-\kappa_{1n}z})}{E_b^*(h_{n1}^2 - \kappa^2)} \right] + B_1 \left[ e^{-\kappa z} + \sum_{n=1}^{\infty} \frac{-2\eta_{1n}E_{n1}(e^{\kappa_{1n}z} + \delta_{1n1}e^{-\kappa_{1n}z})}{E_b^*(h_{n1}^2 - \kappa^2)} \right] \]  

where \( \varpi_{b1} = \varpi_{b1}/a, z = z/a, \kappa = \sqrt{\rho_b^*/E_b^*} \delta, \delta = \sqrt{\rho/\mu} a, \) and \( s \) is the Laplace transform of time. \( \rho_b^* = \rho_b/\rho, E_b^* = E_b/\mu, \) and \( A_1 \) and \( B_1 \) are undetermined coefficients.

Therefore, the complex impedance of the pile segment top is defined as

\[ Z_{u1} = \frac{1}{A_1^*[e^{\kappa \theta_1} + \sum_{n=1}^{\infty} \frac{-2\eta_{1n}E_{n1}(e^{\kappa_{1n}\theta_1} + \delta_{1n1}e^{-\kappa_{1n}\theta_1})}{E_b^*(h_{n1}^2 - \kappa^2)}] + B_1^* [e^{-\kappa \theta_1} + \sum_{n=1}^{\infty} \frac{-2\eta_{1n}E_{n1}(e^{\kappa_{1n}\theta_1} + \delta_{1n1}e^{-\kappa_{1n}\theta_1})}{E_b^*(h_{n1}^2 - \kappa^2)}] \]  

For permafrost, the displacement of the first pile segment is

\[ \varpi_{b2}(z) = A_2 \left[ e^{\kappa z} + \sum_{n=1}^{\infty} \frac{-2T_{n1}E_{n2}(e^{\kappa_{1n}z} + \delta_{1n1}e^{-\kappa_{1n}z})}{E_b^*(h_{n1}^2 - \kappa^2)} \right] + B_2 \left[ e^{-\kappa z} + \sum_{n=1}^{\infty} \frac{-2T_{n1}E_{n2}(e^{\kappa_{1n}z} + \delta_{1n1}e^{-\kappa_{1n}z})}{E_b^*(h_{n1}^2 - \kappa^2)} \right] \]  

where \( A_2 \) and \( B_2 \) are undetermined coefficients.

Therefore, the complex impedance of the first pile segment top is defined as

\[ Z_{u2} = \frac{1}{A_2^*[e^{\kappa \theta_1} + \sum_{n=1}^{\infty} \frac{-2T_{n1}E_{n2}(e^{\kappa_{1n}\theta_1} + \delta_{1n1}e^{-\kappa_{1n}\theta_1})}{E_b^*(h_{n1}^2 - \kappa^2)}] + B_2^* [e^{-\kappa \theta_1} + \sum_{n=1}^{\infty} \frac{-2T_{n1}E_{n2}(e^{\kappa_{1n}\theta_1} + \delta_{1n1}e^{-\kappa_{1n}\theta_1})}{E_b^*(h_{n1}^2 - \kappa^2)}] \]  

For the convenience of the reader, the parameter descriptions in Equations (20)–(23) are listed in Appendix A.

3.1.2. Active Layer Solution

The upper layer of the double-layered stratum is an active layer, which is thawed soil (or unfrozen soil) in summer or frozen soil in winter. The bottom of the active layer is simplified as an elastic support, which satisfies the boundary condition (10), and the free surface satisfies the boundary condition Formulas (11) and (14). The eigenvalue \( h_{n2} \) of the soil layer satisfies the following characteristic equation:

\[ (h_{n2} + \kappa_{n2})e^{\kappa_{n2} \theta_1} - (h_{n2} - \kappa_{n2})e^{-\kappa_{n2} \theta_1} = 0 \]  

The formula is consistent with Equation (19) considering \( \delta_{1n2} = 1 \) at the upper layer. For seasonally frozen soil, the displacement of the upper pile segment is

\[ \varpi_{b3}(z) = A_3 \left[ e^{\kappa z} + \sum_{n=1}^{\infty} \frac{-2T_{n2}E_{n3}(e^{\kappa_{1n}z} + e^{-\kappa_{1n}z})}{E_b^*(h_{n2}^2 - \kappa^2)} \right] + B_3 \left[ e^{-\kappa z} + \sum_{n=1}^{\infty} \frac{-2T_{n2}E_{n3}(e^{\kappa_{1n}z} + e^{-\kappa_{1n}z})}{E_b^*(h_{n2}^2 - \kappa^2)} \right] \]  

where \( A_3 \) and \( B_3 \) are undetermined coefficients.

Thus, the displacement of the pile segment is determined, and the complex impedance of the pile top can be defined as:

\[ Z_{u3} = \frac{1}{A_3^*[1 + \sum_{n=1}^{\infty} \frac{-4\eta_{1n2}E_{n3}}{E_b^*(h_{n2}^2 - \kappa^2)}] + B_3^*[1 + \sum_{n=1}^{\infty} \frac{-4\eta_{1n2}E_{n3}}{E_b^*(h_{n2}^2 - \kappa^2)}]} \]  

In fact, this formula is the same as the complex impedance Formula (21) when the inactive layer is saturated soil, given that \( \theta_0 = \theta_1; \theta_1 = 0; k_{a2} = 0; \) and \( k_{a3} = k_{a1}. \) \( \theta \) is the dimensionless quantity of elevation, and the subscripts 0, 1, and 2 represent the elevation of the lowest boundary, middle boundary and uppermost boundary, respectively.

For permafrost,
\[ \tau_{b4}(z) = A_4 \left\{ e^{\pi z} + \sum_{n=1}^{\infty} \frac{-2Tn_2F_{n4}}{E_b(h_{n2}^2 - \kappa^2)} \left( e^{\pi z} + e^{-\pi z} \right) \right\} + B_4 \left\{ e^{-\pi z} + \sum_{n=1}^{\infty} \frac{-2Tn_2F_{n4}}{E_b(h_{n2}^2 - \kappa^2)} \left( e^{\pi z} + e^{-\pi z} \right) \right\} \] (27)

where \( A_4 \) and \( B_4 \) are undetermined coefficients.

Thus, the displacement of the pile segment is determined, and the complex impedance of the pile top can be defined as

\[ Z_{n4} = \frac{1}{A_4 \left[ 1 + \sum_{n=1}^{\infty} \frac{-2Tn_2F_{n4}}{E_b(h_{n2}^2 - \kappa^2)} \right] + B_4 \left[ 1 + \sum_{n=1}^{\infty} \frac{-2Tn_2F_{n4}}{E_b(h_{n2}^2 - \kappa^2)} \right]} \] (28)

For the convenience of the reader, the parameter descriptions are also listed in Appendix A.

### 3.2. Solution to Dynamic Response of the Single Pile with Slippage

The vertical vibration equation of a rigid pile cap can be obtained by Laplace transformation and the dimensionless method:

\[ \rho_m^s \delta^2 \tilde{w}(z, s) + c_m^s \delta \tilde{w}(z, s) + k_m^s \tilde{w}(z, s) + Z_{u3/4}(0) A^* \tilde{w}(z, s) = \tilde{F}(s) \] (29)

The following expression can be obtained:

\[ \tilde{w}(z, s) = \frac{\tilde{F}(s)}{\left( \rho_m^s \delta^2 + c_m^s \delta + k_m^s \right) + Z_{u3/4}(0) A^*} \] (30)

where \( \tilde{w} \) is the dimensionless vertical amplitude of the pile cap by Laplace transformation and nondimensionalization; \( \rho_m^s \) is the density of the cap. \( \rho_m^s = \frac{\rho_m}{\rho_m^c}, c_m^s = \frac{c_m}{\sqrt{\varphi \rho_m^c}}, k_m^s = \frac{k_m}{\varphi \rho_m^c}. \)

\( A^* = \frac{\varphi \rho_m^c}{A_m}, \tilde{F}(s) = \frac{\tilde{f}(s)}{\varphi \rho_m^c A_m}, \) and \( \tilde{f}(s) \) is the excitation force of the pile cap by Laplace transformation. \( A_m \) denotes the equivalent area of the pile cap. \( Z_{u3/4} \) is the complex impedance of the pile top, where the subscripts 3 and 4 denote seasonally frozen soil and permafrost, respectively.

### 4. Model Validation

In this section, the pile–soil–cap mathematical model is degraded to compare with the existing solutions in saturated soil. A corresponding model test was also carried out to demonstrate the validity of the theoretical model. In the calculation, the parameters are listed as follows unless otherwise specified. The dimensionless thicknesses of the freeze–thaw layers were \( \theta_2 = h_2 / a = 3.75, 7.5 \) and 11.25. The porosity of soil was \( \varphi = 0.45. \) The density of soil grain was \( \rho_s = 2650 \text{ kg/m}^3, \) the density of water was \( \rho_w = 1000 \text{ kg/m}^3, \) and the ice density was \( \rho_i = 920 \text{ kg/m}^3. \) The bulk modulus of the grain, water and ice were 38.7 GPa, 2.25 GPa and 8.58 GPa, respectively. The shear moduli of the grain and ice were 36.0 GPa and 3.7 GPa, respectively. The bulk modulus and the shear modulus of the dry skeleton of soil were 14.4 GPa and 42.92 MPa. The permeabilities of the soil and ice were \( 1.07 \times 10^{-13} \text{ m}^2 \) and \( 5 \times 10^{-4} \text{ m}^2, \) and the viscosity of water was \( 1.798 \times 10^3 \text{ Pa-s}. \) The length of pile was 8 m and the radius was 0.4 m. The density of pile was \( \rho_p = 2500 \text{ kg/m}^3, \) and the elastic modulus was \( E_b = 36.1 \text{ GPa}. \) The mass of the pile cap was \( M_b = 1000 \text{ kg}. \) The bearing of the pile cap was simplified by \( k_{ms} = M_b \times 2500 \pi^2 \text{ N/(m/s)} \) and \( c_{ms} = 1 \times 104 \text{ N/m}. \)

The dimensionless bearing parameters between the soil layers were \( k_{sb2} = 0.1 \) and \( k_{sb1} = 1. \) The bearing of pile toe was \( k_{sb1} = 0.03. \) The stiffness between the pile and frozen soil were \( k_3 = 0.05 \) and \( k_4 = 0.05, \) respectively.

As shown in Figure 1, the lower soil layer was taken as the first soil layer, and the bottom layer was simplified by a Winkler foundation, with \( k_{sb1} \) as the bearing coefficient of the soil layer. The continuous condition between layers was also simulated by a sim-
plified Winkler foundation, with $k_{sb2}$ as the elastic bearing coefficient between layers. A recently published paper by the present authors analyzed the impact of this simplification and verified the rationality of a single pile in the double-layered model with perfect contact [61]. To clarify the validation of the present model of a pile cap foundation with the slippage between the pile and soil, two aspects of further verifications will be given in the following subsections.

4.1. Degradation of the Vertical Vibration of a Single-Pile Foundation in a Freezing–Thawing Stratum

For this paper, a pile foundation in a double-layered stratum was established to simulate the vertical vibration of a single-pile foundation under the freezing/thawing processes of the soil layer. Biot’s porous medium theory [69] was used for the saturated soil layer, and Leclaire’s frozen porous medium theory [57] was used for the frozen saturated soil layer in the double-layered model. It can reflect the change in ice content in the soil layer with the change in temperature and thus the effect of temperature on the vertical vibration of the pile foundation. When the temperature is close to the freezing point, the ice content is close to zero, which can be reduced to the dynamic model of a pile foundation in a saturated soil layer. To verify the validation of the slippage model, the degenerate solutions ($T = -0.15 \, ^\circ C$) are compared for the four cases described in Section 2.3, with the model parameters following the literature data (Static load ($W_s$) = 10 kN, $W_e = 0.278$ N.m) of Manna et al. [70]. In order to convert the vertical amplitude to a dimensioned value, the receptance of the pile cap is first multiplied by a in the MATLAB program, and the vertical displacement of the pile cap is subsequently obtained. The corresponding parameters have been taken with reference to the measured data in the literature and are shown in Appendix B. As can be seen from Figure 2a, the resonant frequencies and amplitudes of the degenerate solutions are in general identical. It indicates that the present model is reasonable. Due to a small amount of ice still present in the frozen soil at the temperature of $-0.15 \, ^\circ C$ in Case 2, a slight frequency shift is observed. Figure 2b shows the difference among the amplitude–frequency curves of the slippage model, perfect contact (non-slippage) model in Case 1 and the experimental results of Manna et al. [70]. The result reveals that the slippage model can agree with the experimental results better than the non-slippage model. In addition, a comparison between the numerical simulation curve of Kim et al. [63] considering the spring stiffness at the interface and the slippage model is included in Figure 2c, where $F_a$ is a factor that determines the strength of the bond between the pile and the soil. A higher value of $F_a$ ($F_a = 5$) will make the pile-soil interface behave as a perfect bonded one. It can be seen from Figure 2c that the results of the present analytical solution with the slippage model are agree well with the Kim’s numerical solution at $F_a = 1$. Figure 2d shows the three amplitude frequency curves for Li’s model [61], with slippage parameters $k_s$ and $k_i$ of 10 and 0.05. Similarly, the slippage model is verified to have a lower resonant frequency and a greater amplitude compared to the fully bonded model of Li et al. When the slippage parameters, both $k_s$ and $k_i$, are set to 10, the curves of the model are in good agreement with the curves of Li’s model, which well explains the relationship between the slip model and the non-slip model.

4.2. Comparison of the Pile Foundation of the Slippage Model with the Model Tests

In this section, an artificial frozen model test system was built for the vertical vibration simulation of a pile foundation in a frozen–thawed stratum. Similarity criterion is derived based on similarity theory [71], and harmonic frequency sweep model tests of the pile–soil–cap in the freezing/thawing ground were carried out using PVC pipes as model piles.
Figure 2. Verification of the simplified layered model with the existing results (Reprinted/adapted with permission from Refs. [61,63,70]. 2020; 2004; 2009, Elsevier; Korean Society of Civil Engineers; Taylor & Francis). (a) Four cases of degradation validation; (b) comparison with Manna experimental data for validation; (c) comparison with Kim numerical simulation data for validation and (d) comparison with Li’s theoretical model.

4.2.1. Model Materials

The sand used in the model test was taken from the Coast of Zhou Shan Island. In this study, a model box with a steel cylinder of 1500 mm diameter and height was used, as shown in Figure 3a. The schematic diagram of the model test can be seen in Figure 3b, which shows the distribution of the upper artificially frozen soil and the lower thawed soil, with the location of the shaker installation and the buried position of the temperature sensor. Two sets of temperature sensors were set at 50 mm intervals along the depth. The inside of the model box was laminated with a circle of wave-absorbing cotton, which can eliminate the effect of reflected waves. The filter layer at the bottom of the model box consists of gravel of different grain sizes.

The test system consists of a self-made ground-freezing system, a vibration loading unit and a signal acquisition section. The artificial ground-freezing system consists of a one-way refrigeration plate and a DC-2030 refrigeration compressor, as shown in Figure 3c,d. The vibration loading unit consists of an electrodynamic shaker, a HEAS-50 power amplifier and a DF1405 digital synthesized function signal generator, as shown in Figure 3e–g. The signal acquisition section mainly consists of a KD1100LC acceleration sensor, a KD5201 constant current adapter and a WS-5921U data-acquisition instrument, as shown in Figure 3h–j.
3c,d. The vibration loading unit consists of an electrodynamic shaker, a HEAS-50 power amplifier and a DF1405 digital synthesized function signal generator, as shown in Figure 3e–g. The signal acquisition section mainly consists of a KD1100LC acceleration sensor, a KD5201 constant current adapter and a WS-5921U data-acquisition instrument, as shown in Figure 3h–j.

Figure 3. Schematic diagram of the artificial frozen experimental system for the model pile tests. (a) Model box; (b) schematic diagram of the model test; (c) one-way refrigeration plate; (d) DC-2030 refrigeration compressor; (e) shaker; (f) HEAS-50 power amplifier; (g) DF1405 digital; (h) KD1100LC acceleration sensor; (i) KD5201 constant current adapter and (j) WS-5921U data-acquisition system.

After reviewing the modulus of elasticity of various materials, PVC pipe was selected as the material for the model pile. An overview of the model test parameters is listed in Table 1.

| Pile                        | \( \rho_b \) (kg/m\(^3\)) | \( E_b \) (GPa) | \( H \) (m) | \( r \) (m) |
|-----------------------------|-----------------------------|----------------|------------|------------|
| Frozen soil layer (0.25 m)  | \( 920 \) | \( 2.5 \times 10^9 \) | \( 0.9 \) | \( 0.04 \) |
| Soil layer (0.90 m)         | \( 1850 \) | \( 0.258 \) | \( 0.2 \); \( 0.2 \); \( 0.2 \); \( 0.001 \) | \( 50 \) |

After reviewing the modulus of elasticity of various materials, PVC pipe was selected as the material for the model pile. An overview of the model test parameters is listed in Table 1.
4.2.2. Comparison of the Results from the Model Tests

Experimental tests were carried out after the upper one-way refrigeration plate had been in operation for approximately 36 hours, and the lowest temperature sensor attached −0.5 °C. Then, the refrigerator was shut down, and the soil layer started to warm up gradually. When the average temperature of the upper soil layer had reached −0.5 °C, the freezing depth was approximately equal to 0.25 m (The average temperature was obtained by averaging 10 temperature sensors). The shaker applied a steady-state sinusoidal load to the top of the pile cap in a sweeping manner to obtain an image of the acceleration as a function of time. The amplitude–frequency curves for the model test were transformed after the time–frequency conversion and filtering process.

Figure 4a shows the difference between the harmonic response of the vertical amplitudes at several frequencies of the model test and the analytical results using the model parameters listed in Table 1, and the vertical amplitudes of the experimental and analytical results agree well at the resonance frequency ($f = 270$ Hz). Figure 4b shows the amplitude–frequency curves for model test and analytical model. The resonant frequencies obtained from the analytical solution of the pile foundation with the slippage between the pile and soil are in good agreement with the experimental results of the model pile foundation, thereby verifying the correct implementation of the solution.

![Figure 4a: Comparison of dynamic response of the pile cap using the slippage model with the measured results from the model pile tests.](image1)

![Figure 4b: Amplitude–frequency curves.](image2)

**Figure 4.** Comparison of dynamic response of the pile cap using the slippage model with the measured results from the model pile tests. (a) Amplitude–time history at different frequencies; (b) amplitude–frequency curves.

5. Parameter Study and Discussion

This section discusses the dynamic responses of the vertical vibration of the pile foundation under the freezing/thawing processes of porous medium. Effects of several key parameters of the pile–soil–cap system are studied. The model parameters are given in the previous section.

5.1. Influence of the Parameters of the Freezing/Thawing Stratum on the Dynamic Responses

5.1.1. Comparison of the Vertical Vibrations of a Rigid Cap with/without the Mass

In the existing research results, some scholars use the rigid massless pile cap to analyze the vibration of pile foundations, which is feasible for the analysis of piles but needs to be treated with caution for the dynamic analysis of pile caps. The single-pile foundation with a rigid massless pile cap is equivalent to that of a massless rigid body attached to the pile foundation. Therefore, the vertical amplitude at the top of the massless pile cap is equivalent to the response at the top of the pile. Figure 5 compares the amplitude curves of the top of the pile caps with/without mass ($M = 1000$ kg) for the four cases, reflecting...
that the mass of the pile cap has considerable influence on the resonant frequency and amplitude, and neglecting the mass of the pile cap overestimates the resonant frequency of the pile foundation. In addition, the vertical coordinates in the figures indicate the dimensionless vertical amplitude ($\tilde{\varphi}$).

**Figure 5.** Comparison of the amplitude–frequency curves of the pile foundation with/without the mass of the pile cap. (a) Case 1; (b) Case 2; (c) Case 3 and (d) Case 4.

### 5.1.2. Effect of the Freezing and Thawing Depth

With the change in seasons, the active layer will freeze or melt, and the depth of the active layer will change with the temperature. Three different depths of the active layer, $h_2 = 1.5$ m, 3.0 m and 4.5 m, were selected to investigate the influence of the depth of the active layer on the amplitude of the pile foundation.

Figure 6 shows the amplitude–frequency curves of the vertical vibration of the pile foundation under different freezing and thawing depths, taking $T = -0.35 \, ^\circ$C. Figure 6a shows that, in Case 2, with increasing freezing depth, the resonant frequency of the pile cap increases slightly, while the amplitude decreases significantly. In Case 4, with increasing thawing depth, the amplitude of the pile cap increases considerably, while the resonant frequency decreases. Because the resonance frequencies of the permafrost with the thickness of the upper active thawing layer (Case 4, Figure 6b) decrease and the amplitudes increase, it is not conducive to the reduction of the resonance of the pile foundation when the thawed depth of the active layer increases due to climate warming. Nevertheless, since the resonance frequencies of the seasonally frozen soil with the thickness of the upper active freezing layer (Case 2, Figure 6a) increase and the amplitudes decrease, the increase in the frozen depth tends to be beneficial to structural vibration due to the drop in temperature. In short, the existence of a frozen soil layer can restrain the foundation vibration. Whether
seasonally frozen soil or permafrost, it is necessary to keep the frozen state as long as possible to prevent the active layer from thawing.

![Graph](image1.png)

**Figure 6.** Effects of the depth of the active layer on the amplitude–frequency curves of the pile foundation. (a) Freezing depth of seasonally frozen soil and (b) thawing depth of permafrost.

5.1.3. Effects of Temperature on the Dynamic Response of the Single-Pile Foundation

The lower the temperature is, the more water will be transformed into ice, which will cause a change in soil properties, and the dynamic response of the pile foundation will be significantly different. Figure 7 shows the influence of temperature in cases 2 and 4 on the vertical vibration of a single-pile foundation in a freezing/thawing ground. To simplify the calculation, the average temperature of the frozen layer is used for calculation. In the Figure, the freezing or thawing depth is 1.5 m, and the amplitude–frequency curves of the pile foundation at five different temperatures (T = −0.15 °C, −0.20 °C, −0.25 °C, −0.30 °C and −0.35 °C) are selected.

![Graph](image2.png)

**Figure 7.** Effect of temperature on the amplitude–frequency curves of the single-pile foundation in (a) Case 2; (b) Case 4.

It can be seen from Figure 7 that in cases 2 and 4, with the temperature decreasing, the ice content gradually increases, and the pile cap resonance frequency increases for Case 2 and decreases for Case 4 while both the amplitudes decrease. The results also show that in permafrost or seasonally frozen soil, once the active layer melts due to the change in the external environment, the potential hazard exists for the single-pile foundation in the project.
5.1.4. Effect of the Dynamic Shear Modulus of Unfrozen Soil

The dynamic shear moduli of unfrozen soil are \( \mu_{sm0} = 42.92, 76.30, \) and 119.22 MPa. Figure 8 shows the influence of the soil modulus on the amplitude–frequency curves of the pile caps in the two cases of the upper layer freezing and the upper layer thawing (Case 2 and Case 4). With an increasing soil modulus, the resonance frequencies and amplitudes of Case 2 increase slightly. This indicates that in the case of a frozen upper soil layer, the hardness of the foundation has a minor effect on the dynamic response of the single pile embedded in soil. In contrast, with an increasing soil modulus, the resonance frequencies and amplitudes of Case 4 decrease significantly. This shows that the pile foundations vibrate more strongly in the upper soft ground than the pile foundations in the harder ground when the upper layer is thawing. Therefore, it is advantageous to increase the dynamic shear modulus of unfrozen soil for the dynamic response of the pile foundation in Case 4.

![Figure 8](image1.png)

**Figure 8.** Effects of the soil modulus on the amplitude–frequency curves of the pile foundation on (a) Case 2 and (b) Case 4.

5.2. Influence of the Single-Pile Foundation Parameters

5.2.1. Influence of the Pile–Soil Interface Parameters

It can be seen from Figure 9 that with the decrease in pile–soil contact stiffness, the resonance frequency decreases and the amplitude increases. The slippage between the pile and soil caused by the excitation may be an important factor affecting the reduction in the resonance frequency. Therefore, a dynamic analysis of the pile foundations in perfect contact that does not taking account of the slippage between the pile and soil overestimates the resonant frequency and underestimates the amplitude.

![Figure 9](image2.png)

**Figure 9.** Effects of \( k_s \) and \( k_i \) on the amplitude–frequency curves of the pile foundations.
5.2.2. Influence of Pile Bottom-Support Stiffness

Figure 10 shows that the resonance frequencies and amplitudes of the pile cap increase with increasing pile bottom-support stiffness. This depicts that the harder the underlying soil layer, the stronger the supporting effect on the pile bottom, and the greater the resonance frequency and amplitude of the pile foundation.

![Figure 10](image)

**Figure 10.** Influence of pile bottom-support stiffness ($k_{pb1}$) on the amplitude–frequency curves of the single-pile foundation.

5.2.3. Influence of the Pile Cap Mass

Seen from Figure 11, the resonance frequencies and amplitudes increase slightly in Case 2, while both the resonance frequencies and amplitudes increase noticeably in Case 4 with the decrease in the pile cap mass. This shows that the mass of the pile cap will have a greater influence on the resonant frequency and amplitude when the permafrost is thawing.

![Figure 11](image)

**Figure 11.** Influence of the pile cap mass on the amplitude–frequency curves of the single-pile foundation. (a) Case 2 and (b) Case 4.

5.2.4. Influence of the Support Parameters at the Pile Cap Bottom

Generally, the pile cap is embedded in soil. Figure 12 shows the influence of the support stiffness and damping at the pile cap bottom when the pile cap is placed on the surface. Figure 12a shows the influence of different support stiffness values on the amplitude–frequency curve of the pile foundation under the same damping condition. The resonance frequencies and amplitudes of the pile foundation increase with increasing support stiffness. Figure 12b shows the influence of different damping coefficients on the amplitude–frequency curve of the pile foundation under the same support stiffness. The
damping coefficient does not affect the resonance frequency, but as the damping coefficient increases, the amplitude decreases significantly.

**Figure 12.** Effects of the support coefficients of the pile cap on the vertical amplitude–frequency curves. (a) $k_{ms}$ and (b) $c_{ms}$.

6. Conclusions

In this study, a new pile–soil–cap model was developed to investigate its vertical vibration in a double-layered stratum under the freezing and thawing processes of a saturated porous medium, taking into account the slippage effect between the pile and soil. Based on the theoretical model, the degradation of the four conditions described in the paper was verified, and the comparison between the existing solutions illustrated the need to take the pile–soil slippage effect. Then, a dynamic model test of the pile–soil–cap in a self-made ground-freezing system was performed. The reasonableness of the model was further validated by comparing the proposed solution with the experimental results. Parametric analyses were also carried out to discuss the effects of the cap mass, freezing and thawing depth, soil temperature, dynamic shear modulus of unfrozen soil, pile bottom-support stiffness and support parameters at the pile cap bottom on the amplitude–frequency curve of pile–soil–cap. The main findings can be summarized below, as follows:

- Based on the theory of composite porous media, a simplified axisymmetric model of the vertical vibration of a single-pile foundation in a frozen/thawed stratum is established, and the analytical solution of the dynamic response of the pile cap is obtained. The present solution taking into account the slippage between the pile and soil is more reasonable than the perfect contact model, which overestimates the resonant frequency and underestimates the amplitude of the single-pile foundation;
- Both the comparisons of the present solution with the existing experimental results of a vertical vibration model test for a pile cap foundation in unfrozen soil, and the present experiment in an artificially freezing/thawing ground show that the slippage model for pile foundation can agree well with the experimental results;
- The mass of the pile cap has an obvious influence on the vertical vibration of the pile foundation, which should not be ignored. The thickness of the active layer, the freezing temperature, the shear modulus of the unfrozen soil and the contact parameters of the pile and soil also have a significant influence on the vibration of vertical the pile foundation;
- The slippage parameters in the proposed method can be calibrated according to different pile and soil types, which needs to be further studied through a large number of experiments.
Author Contributions: Q.L.: methodology, supervision, validation, resources, writing—review and editing and funding acquisition. X.L.: methodology, formal analysis, investigation, data curation, software and writing—original draft. M.W.: methodology, supervision, resources and funding acquisition. L.H.: conceptualization and resources. W.D.: writing—review. J.L.: data curation. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the National Natural Science Foundation of China [grant numbers 52108347]. The authors are grateful to the editorial board and reviewers of this paper.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

Appendix A.1 Inactive Layer Solution of Seasonally Frozen Soil

\[
A_1 = \frac{\mathcal{P}(s)[(\bar{K}_{pb1} - \kappa) e^{-s\theta_0} + G_F]}{(k_F e^{-s\theta_1} - F')[(\bar{K}_{pb1} + \kappa) e^{s\theta_0} + G_E] + (\kappa e^{s\theta_1} + E')[(\bar{K}_{pb1} - \kappa) e^{-s\theta_0} + G_F]} \quad (A1)
\]

\[
B_1 = \frac{-\mathcal{P}(s)[(\bar{K}_{pb1} + \kappa) e^{-s\theta_0} + G_E]}{(k_F e^{-s\theta_1} - F')[(\bar{K}_{pb1} + \kappa) e^{s\theta_0} + G_E] + (\kappa e^{s\theta_1} + E')[(\bar{K}_{pb1} - \kappa) e^{-s\theta_0} + G_F]} \quad (A2)
\]

\[
\eta_{1n1} = (1 + \frac{h_n^{2}}{h_n^{2}}) \frac{\lambda_1 - \lambda_2}{\lambda_2 - \lambda_3} \sum_{n=1}^{\infty} g_{1n} h_{1n} k_{1}(g_{1n}, \bar{K}_{pb1}) = \frac{k_{pb1} a}{E_b \pi} \quad (A3)
\]

\[
E_{n1} = \frac{\{ (h_n - \kappa) [e^{(x+h_n)\theta_0} - e^{(x+h_n)\theta_1} - \delta_{1n1}(h_n + \kappa) [e^{(x-h_n)\theta_0} - e^{(x-h_n)\theta_1}] \} \}}{[\eta_{2n1}(h_n^{2} - \kappa^{2}) + \frac{2\eta_{1n1}}{E_b} \sum_{n=1}^{\infty} \frac{g_{1n}(h_n^{2} - \kappa^{2})}{g_{1n}(\delta_{1n1}(g_{1n})k_{1}(g_{2n}))}] \} \} \quad (A5)
\]

\[
F_{n1} = \frac{\{ (h_n + \kappa) [e^{-(x-h_n)\theta_0} - e^{-(x-h_n)\theta_1}] - \delta_{1n1}(h_n + \kappa) [e^{-(x+h_n)\theta_0} - e^{-(x+h_n)\theta_1}] \} \}}{[\eta_{2n1}(h_n^{2} - \kappa^{2}) + \frac{2\eta_{1n1}}{E_b} \sum_{n=1}^{\infty} \frac{g_{1n}(h_n^{2} - \kappa^{2})}{g_{1n}(\delta_{1n1}(g_{1n})k_{1}(g_{2n}))}] \} \} \quad (A6)
\]

\[
E^* = \sum_{n=1}^{\infty} \frac{-2\eta_{1n1} E_{n1} h_{1n} (e^{h_{1n} \theta_1} - \delta_{1n1} e^{-h_{1n} \theta_1})}{E_{b}(h_n^{2} - \kappa^{2})}, \quad F^* = \sum_{n=1}^{\infty} \frac{-2\eta_{1n1} F_{n1} h_{1n} (e^{h_{1n} \theta_1} - \delta_{1n1} e^{-h_{1n} \theta_1})}{E_{b}(h_n^{2} - \kappa^{2})} \quad (A7)
\]

\[
G_E = \sum_{n=1}^{\infty} \frac{-2\eta_{1n1} E_{n1} G_{n1}}{E_{b}(h_n^{2} - \kappa^{2})}, \quad G_F = \sum_{n=1}^{\infty} \frac{-2\eta_{1n1} F_{n1} G_{n1}}{E_{b}(h_n^{2} - \kappa^{2})} \quad (A8)
\]

\[
G_{n1} = (h_n + \bar{K}_{pb1}) e^{h_{n} \theta_0} - \delta_{1n1}(h_n - \bar{K}_{pb1}) e^{-h_{n} \theta_0} \quad (A9)
\]

where \(\mathcal{P}(s)\) is the Laplace-transformed dimensionless form of the force at the top of the pile; other unspecified parameters are parameter symbols set by the operator for writing convenience.

Appendix A.2 Inactive Layer Solution of Permafrost

\[
A_2 = \frac{\mathcal{P}(s)[(\bar{K}_{pb1} - \kappa) e^{-s\theta_0} + G_{F2}]}{(k_F e^{-s\theta_1} - F'')[\bar{K}_{pb1} + \kappa) e^{s\theta_0} + G_{E2}] + (\kappa e^{s\theta_1} + E'')[\bar{K}_{pb1} - \kappa) e^{-s\theta_0} + G_{F2}]} \quad (A10)
\]
\[ B_2 = \frac{-\mathcal{P}(s)(\mathcal{F}_{pb1} + \kappa)e^{\phi_0} + G_{E2}}{(\kappa e^{-x_1} - \alpha - F)(\mathcal{F}_{pb1} + \kappa)e^{\phi_0} + G_{E2} + (\kappa e^{x_1} + F)}[\mathcal{F}_{pb1} - \kappa e^{\phi_0} + G_{F2}] \]  

(\text{A11})

\[ T_{n1} = \frac{(\eta_{1n1} + \eta_{3n1})/(\mathcal{H}_{4n} - \mathcal{H}_{2n}) + (\eta_{2n1} + \eta_{4n1})/(\mathcal{H}_{1n} - \mathcal{H}_{3n})}{\mathcal{H}_{1n}\mathcal{H}_{4n} - \mathcal{H}_{2n}\mathcal{H}_{3n}} \]  

(\text{A12})

\[ E_{n2} = \frac{\frac{e^{(x-h_1)[\xi - (x-h_1)]}}{(k-n_1)} + \delta_{1n1}\frac{e^{(x-h_1)[\xi - (x-h_1)]}}{(k-n_1)}}{[2\delta_{1n1}(\theta_0 - \theta_1) + \frac{\delta_{1n1}}{2\mathcal{H}_{4n}} - \frac{\delta_{1n1}(e^{x-h_1} + e^{-x-h_1})}{2\mathcal{H}_{4n}}]} \]  

(\text{A13})

\[ F_{n2} = -\frac{\frac{e^{(x-h_1)[\xi - (x-h_1)]}}{(k-n_1)} + \delta_{1n1}\frac{e^{(x-h_1)[\xi - (x-h_1)]}}{(k-n_1)}}{[2\delta_{1n1}(\theta_0 - \theta_1) + \frac{\delta_{1n1}}{2\mathcal{H}_{4n}} - \frac{\delta_{1n1}(e^{x-h_1} + e^{-x-h_1})}{2\mathcal{H}_{4n}}]} \]  

(\text{A14})

\[ E'' = \sum_{n=1}^{\infty} -\frac{2T_{n1}E_{n1}h_{1n}(e^{\phi_0} + 1) - \delta_{1n}e^{-\phi_0}h_{1n}}{E_{n1}(h_{1n}^2 - \kappa^2)} \]  

(\text{A15})

\[ F'' = \sum_{n=1}^{\infty} -\frac{2T_{n1}F_{n1}h_{1n}(e^{\phi_0} + 1) - \delta_{1n}e^{-\phi_0}h_{1n}}{E_{n1}(h_{1n}^2 - \kappa^2)} \]  

(\text{A16})

\[ G_{E2} = \sum_{n=1}^{\infty} -\frac{2T_{n1}E_{n1}G_{n1}}{E_{n1}(h_{1n}^2 - \kappa^2)}, G_{F2} = \sum_{n=1}^{\infty} -\frac{2T_{n1}F_{n1}G_{n1}}{E_{n1}(h_{1n}^2 - \kappa^2)} \]  

(\text{A17})

\[ \eta_{1n} = [-2g_{11}g_{12}k_{1}(g_{11})\xi - 2g_{21}g_{22}k_{1}(g_{21})\xi - 21 + (g_{41} - g_{41}g_{42})k_{1}(g_{41})\xi + (g_{33} - g_{33}g_{32})k_{1}(g_{33})\xi + 71] \]  

(\text{A18})

\[ \eta_{2n} = [-2g_{31}g_{32}k_{1}(g_{31})\xi - 2g_{21}g_{22}k_{1}(g_{21})\xi - 23 + (g_{33} - g_{33}g_{32})k_{1}(g_{33})\xi + (g_{33} - g_{33}g_{32})k_{1}(g_{33})\xi + 73] \]  

(\text{A19})

\[ \eta_{3n} = [-2g_{31}g_{32}k_{1}(g_{31})\xi - 2g_{31}g_{32}k_{1}(g_{31})\xi - 21 + (g_{33} - g_{33}g_{32})k_{1}(g_{33})\xi + (g_{33} - g_{33}g_{32})k_{1}(g_{33})\xi + 71] \]  

(\text{A20})

\[ \eta_{4n} = [-2g_{31}g_{32}k_{1}(g_{31})\xi - 2g_{31}g_{32}k_{1}(g_{31})\xi - 21 + (g_{33} - g_{33}g_{32})k_{1}(g_{33})\xi + (g_{33} - g_{33}g_{32})k_{1}(g_{33})\xi + 71] \]  

(\text{A21})

\[ \eta_{5n} = [g_{12}k_{0}(g_{11})\xi + g_{22}k_{0}(g_{21})\xi - 21 + (g_{41} - g_{41}g_{42})k_{1}(g_{41})\xi + (g_{33} - g_{33}g_{32})k_{1}(g_{33})\xi + 71] \]  

(\text{A22})

\[ \eta_{6n} = [g_{32}k_{0}(g_{31})\xi + g_{22}k_{0}(g_{21})\xi - 21 + (g_{41} - g_{41}g_{42})k_{1}(g_{41})\xi + (g_{33} - g_{33}g_{32})k_{1}(g_{33})\xi + 71] \]  

(\text{A23})

\[ \eta_{7n} = [g_{12}k_{0}(g_{11})\xi - 21 + (g_{41} - g_{41}g_{42})k_{1}(g_{41})\xi + (g_{33} - g_{33}g_{32})k_{1}(g_{33})\xi + 71] \]  

(\text{A24})

\[ \eta_{8n} = [g_{32}k_{0}(g_{31})\xi - 21 + (g_{41} - g_{41}g_{42})k_{1}(g_{41})\xi + (g_{33} - g_{33}g_{32})k_{1}(g_{33})\xi + 71] \]  

(\text{A25})

\[ a_{21} = \frac{[\xi(\xi - \xi) - \xi(\xi - \xi) + \xi(\xi - \xi)]g_{11}k_{1}(g_{11})}{[\xi(\xi - \xi) - \xi(\xi - \xi) + \xi(\xi - \xi)]g_{21}k_{1}(g_{21})} \]  

(\text{A26})

\[ a_{23} = \frac{[\xi(\xi - \xi) - \xi(\xi - \xi) + \xi(\xi - \xi)]g_{31}k_{1}(g_{31})}{[\xi(\xi - \xi) - \xi(\xi - \xi) + \xi(\xi - \xi)]g_{21}k_{1}(g_{21})} \]  

(\text{A27})

\[ a_{61} = \frac{g_{31}k_{1}(g_{11})\xi - 21 + g_{21}k_{1}(g_{21})\xi - 21}{g_{41}g_{42}k_{1}(g_{41})(\xi - 6)}a_{21} \]  

(\text{A28})

\[ a_{63} = \frac{g_{31}k_{1}(g_{31})\xi - 21 + g_{21}k_{1}(g_{21})\xi - 21}{g_{41}g_{42}k_{1}(g_{41})(\xi - 6)}a_{23} \]  

(\text{A29})

\[ a_{71} = \frac{g_{31}k_{1}(g_{11})\xi - 21 + g_{21}k_{1}(g_{21})\xi - 21}{g_{31}g_{32}k_{1}(g_{31})(\xi - 6)}a_{21} \]  

(\text{A30})
\[ \alpha_{73} = \frac{g_{31}k_1(g_{31})(\xi_3 - \zeta_6) + g_{21}k_1(g_{21})(\xi_2 - \zeta_6)\eta_{23}}{g_{51}g_{25}k_1(g_{51})(\xi_6 - \zeta_7)} \]  
\[ (A31) \]

\[ \bar{P}_{4n} - \bar{P}_{2n} = (\eta_{8n1} - \eta_{6n1}) - \left[ \frac{\eta_{4n1}}{k_1(1 + iD_1)} - \frac{\eta_{2n1}}{k_1(1 + iD_3)} \right] \]  
\[ (A32) \]

\[ \bar{P}_{1n} - \bar{P}_{3n} = (\eta_{5n1} - \eta_{7n1}) - \left[ \frac{\eta_{1n1}}{k_1(1 + iD_1)} - \frac{\eta_{3n1}}{k_1(1 + iD_3)} \right] \]  
\[ (A33) \]

\[ \bar{P}_{1n}\bar{P}_{4n} - \bar{P}_{2n}\bar{P}_{3n} = |\eta_{6n1} + \frac{2(\eta_{8n1} + \eta_{6n1})}{E_p(k_{31} - k_{32}^2)}| |\eta_{8n1} + \frac{2(\eta_{8n1} + \eta_{6n1})}{E_p(k_{31} - k_{32}^2)} - \eta_{6n1} - \frac{2(\eta_{8n1} + \eta_{6n1})}{E_p(k_{31} - k_{32}^2)}| \]  
\[ (A34) \]

**Appendix A.3 Active Layer Solution of Seasonally Frozen Soil**

\[ A_3 = \frac{\bar{P}_{ix}(s)}{E_p \kappa} \left\{ \left[ (k_{ph2} - \kappa)e^{-\kappa \theta_1} + \sum_{n=1}^{\infty} \frac{-2\eta_{i2n}F_{2n}G_{2n}}{E_p(k_{n2} - k_{n2}^2)} \right] \right\} \]  
\[ (A35) \]

\[ B_3 = -\frac{\bar{P}_{ix}(s)}{E_p \kappa} \left\{ (k_{ph2} + \kappa)e^{\kappa \theta_1} + \sum_{n=1}^{\infty} \frac{-2\eta_{i2n}F_{2n}G_{2n}}{E_p(k_{n2} - k_{n2}^2)} \right\} \]  
\[ (A36) \]

\[ E_{n3} = \left\{ \left[ (h_{n2} - \kappa)[e^{(\kappa - h_{n2})\theta_1} - 1] - (h_{n2} + \kappa)[e^{(\kappa - h_{n2})\theta_1} - 1] \right] \right\} \]  
\[ (A37) \]

\[ F_{n3} = \left\{ \left[ \left[ \eta_{2n2}(h_{n2} - \kappa)^2 + 2\eta_{2n2} - \eta_{2n2}(h_{n2} - \kappa^2) \right] - \frac{h_{n2} - \kappa}{h_{n2}} \right] \right\} \]  
\[ (A38) \]

\[ G_{n2} = (h_{n2} + k_{ph2})e^{h_{n2}\theta_1} - (h_{n2} - \kappa_2)e^{-h_{n2}\theta_1} \]  
\[ (A39) \]

\[ \eta_{1n2} = (1 + \frac{h_{n2}^2}{h_{n2}^2}) \frac{h_{n2} \lambda_1 - \lambda_2}{h_{n2}^2 - \lambda_2} \frac{\lambda_1 - \lambda_2 \xi_{12}h_{n2}k_1}{\xi_{12}^2} \]  
\[ (A40) \]

\[ \eta_{2n2} = h_{n2}k_0(g_{1n}) - \frac{\lambda_1 - \lambda_2}{\lambda_2 - \lambda_3 \xi_{12}^2} h_{n2}k_1(g_{1n})k_0(g_{2n}) \]  
\[ (A41) \]

**Appendix A.4 Active Layer Solution of Permafrost**

\[ A_4 = \frac{\bar{P}_{i}(s)}{E_p \kappa} \left\{ (k_{ph2} - \kappa)e^{-\kappa \theta_1} + \sum_{n=1}^{\infty} \frac{-2T_{2n}F_{2n}G_{2n}}{E_p(k_{n2} - k_{n2}^2)} \right\} \]  
\[ (A42) \]

\[ B_4 = -\frac{\bar{P}_{i}(s)}{E_p \kappa} \left\{ (k_{ph2} + \kappa)e^{\kappa \theta_1} + \sum_{n=1}^{\infty} \frac{-2T_{2n}F_{2n}G_{2n}}{E_p(k_{n2} - k_{n2}^2)} \right\} \]  
\[ (A43) \]

\[ E_{n4} = \frac{e^{(\kappa - h_{n2})\theta_1} - 1 + e^{(\kappa - h_{n2})\theta_1} - 1}{2\theta_1 + \sinh(2\theta_1 h_{n2})}, F_{n4} = -\frac{e^{(\kappa - h_{n2})\theta_1} - 1 + e^{(\kappa - h_{n2})\theta_1} - 1}{2\theta_1 + \sinh(2\theta_1 h_{n2})} \]  
\[ (A44) \]
\[
T_{n2} = \frac{(\eta_{1n2} + \eta_{3n2})(\overline{F}_{4n} - \overline{F}_{2n}) + (\eta_{2n2} + \eta_{4n2})(\overline{F}_{1n} - \overline{F}_{3n})}{\overline{F}_{1n}\overline{F}_{4n} - \overline{F}_{2n}\overline{F}_{3n}} \tag{A45}
\]

\[
F_{n2} = -\left(\frac{e^{-(\xi+\eta_0)\theta} - e^{-(\xi+\eta_0)\theta}}{\theta} + \delta_{1n1} e^{-(\xi+\eta_0)\theta} - e^{-(\xi+\eta_0)\theta}}{\theta} + \delta_{2n1} e^{-(\xi+\eta_0)\theta} - e^{-(\xi+\eta_0)\theta}}{\theta} + \right) \left(2\delta_{1n1}(\theta - \theta_1) + \frac{\eta_{2n2}}{2\theta_1 - \theta} - \frac{\delta_{2n1}}{2\theta_1 - \theta} - \frac{\delta_{2n2}}{2\theta_1 - \theta} \right)
\]

\[
G_{n3} = h_{n2} \sinh(h_{n2}\theta_1) + \frac{\theta_{n2}}{\sin(h_{n2}\theta_1)} \tag{A47}
\]

\[
\eta_{1n2} = -2g_{11}g_{12}k_{1}(g_{11})\xi_1 - 2g_{21}g_{22}k_{1}(g_{21})\xi_2 + (g_{31}^2 - g_{41}g_{42})k_{1}(g_{41})\xi_6 + (g_{31}^2 - g_{51}g_{52})k_{1}(g_{51})\xi_7 \tag{A48}
\]

\[
\eta_{2n2} = -2g_{13}g_{12}k_{1}(g_{31})\xi_3 - 2g_{21}g_{22}k_{1}(g_{21})\xi_2 + (g_{31}^2 - g_{41}g_{42})k_{1}(g_{41})\xi_6 + (g_{31}^2 - g_{51}g_{52})k_{1}(g_{51})\xi_7 \tag{A49}
\]

\[
\eta_{3n2} = -2g_{13}g_{12}k_{1}(g_{11})\xi_1 - 2g_{21}g_{22}k_{1}(g_{21})\xi_2 + (g_{31}^2 - g_{41}g_{42})k_{1}(g_{41})\xi_6 + (g_{31}^2 - g_{51}g_{52})k_{1}(g_{51})\xi_7 \tag{A50}
\]

\[
\eta_{4n2} = -2g_{13}g_{12}k_{1}(g_{31})\xi_3 - 2g_{21}g_{22}k_{1}(g_{21})\xi_2 + (g_{31}^2 - g_{41}g_{42})k_{1}(g_{41})\xi_6 + (g_{31}^2 - g_{51}g_{52})k_{1}(g_{51})\xi_7 \tag{A51}
\]

\[
\eta_{n2} = [g_{12}k_{0}(g_{11})\xi_1 + g_{22}k_{0}(g_{21})\xi_2 - g_{41}k_{0}(g_{41})\xi_6 + g_{51}k_{0}(g_{51})\xi_7] \tag{A52}
\]

\[
\eta_{n2} = [g_{13}k_{0}(g_{31})\xi_3 + g_{22}k_{0}(g_{21})\xi_2 - g_{41}k_{0}(g_{41})\xi_6 + g_{51}k_{0}(g_{51})\xi_7] \tag{A53}
\]

\[
\eta_{n2} = [g_{13}k_{0}(g_{11})\xi_1 + g_{22}k_{0}(g_{21})\xi_2 - g_{41}k_{0}(g_{41})\xi_6 + g_{51}k_{0}(g_{51})\xi_7] \tag{A54}
\]

\[
\eta_{n2} = [g_{32}k_{0}(g_{31})\xi_3 + g_{22}k_{0}(g_{21})\xi_2 - g_{41}k_{0}(g_{41})\xi_6 + g_{51}k_{0}(g_{51})\xi_7] \tag{A55}
\]

\[
\alpha_{21} = \left\{ \frac{\zeta_1(g_{11}g_{06} - g_{06}) - \zeta_0(g_{06})}{g_{11}k_1(g_{11})} + \frac{\zeta_6(g_{11}g_{06} - g_{06})}{g_{11}k_1(g_{11})} \right\} \tag{A56}
\]

\[
\alpha_{23} = \left\{ \frac{\zeta_1(g_{11}g_{06} - g_{06}) - \zeta_0(g_{06})}{g_{11}k_1(g_{11})} + \frac{\zeta_6(g_{11}g_{06} - g_{06})}{g_{11}k_1(g_{11})} \right\} \tag{A57}
\]

\[
\alpha_{61} = \frac{g_{11}k_1(g_{11})(\xi_1 - \xi_7) + g_{21}k_1(g_{21})(\xi_2 - \xi_6)}{g_{41}g_{42}k_1(g_{41})(\xi_6 - \xi_6)} \tag{A58}
\]

\[
\alpha_{63} = \frac{g_{31}k_1(g_{31})(\xi_3 - \xi_6) + g_{21}k_1(g_{21})(\xi_2 - \xi_6)}{g_{41}g_{42}k_1(g_{41})(\xi_6 - \xi_6)} \tag{A59}
\]

\[
\alpha_{71} = \frac{g_{11}k_1(g_{11})(\xi_1 - \xi_6) + g_{21}k_1(g_{21})(\xi_2 - \xi_6)}{g_{51}g_{52}k_1(g_{51})(\xi_6 - \xi_6)} \tag{A60}
\]

\[
\alpha_{73} = \frac{g_{31}k_1(g_{31})(\xi_3 - \xi_6) + g_{21}k_1(g_{21})(\xi_2 - \xi_6)}{g_{51}g_{52}k_1(g_{51})(\xi_6 - \xi_6)} \tag{A61}
\]

\[
\overline{F}_{n} - \overline{F}_{2n} = \left(\eta_{8n2} - \eta_{6n2}\right) - \frac{\eta_{4n2}}{k_{2}(1 + iD_{2})} - \frac{\eta_{2n2}}{k_{2}(1 + iD_{2})} \tag{A62}
\]

\[
\overline{F}_{1n} - \overline{F}_{3n} = \left(\eta_{8n2} - \eta_{7n2}\right) - \frac{\eta_{1n2}}{k_{2}(1 + iD_{2})} - \frac{\eta_{3n2}}{k_{2}(1 + iD_{2})} \tag{A63}
\]

\[
\eta_{n2} + \frac{2(\eta_{n2} + \eta_{6n2})}{k_{2}(1 + iD_{2})} = \frac{\eta_{n2} + \eta_{6n2}}{2k_{2}(1 + iD_{2})} \tag{A64}
\]

where the unspecified parameters such as \(E_{\eta 1}, E_{\eta 1}, E_{n2}, E_{n3}, \) etc., can be seen in the literature [61] for details.
Appendix B

Table A1. Parameters in Figure 2b.

| Experimental Parameter | $H_0$ | $a$ | $p_b$ | $E_b$ | Static Load ($W_s$) | $W_e$ | $\mu$ |
|-------------------------|-------|-----|-------|-------|---------------------|-------|-------|
| Value                   | 22 m  | 0.45 m | 2500 kg/m$^3$ | 30 GPa | 10 kN | 0.278 N-m | 4-15 MPa |
| Analytical parameter    | $k_{sb}$ | $k_{st1}$ | $k_{st1}$ | $\mu_{st0}$ | $u$ | $k_s$ | $k_i$ |
| Value                   | 0.1   | 0.1 | 1 | 20 MPa | 0.35 | 0.001 | 0.001 |

Table A2. Parameters in Figure 2c.

| Experimental Parameter | $H_0$ | $a$ | $p_b$ | $E_b$ | Static Load ($W_s$) | $W_e$ | $\mu$ |
|-------------------------|-------|-----|-------|-------|---------------------|-------|-------|
| Value                   | 2.75 m | 0.0953 m | 7850 kg/m$^3$ | 210 GPa | 9.2 kN | 0.0245 N-m | 12-62 MPa |
| Analytical parameter    | $k_{sb}$ | $k_{st1}$ | $k_{st1}$ | $\mu_{st0}$ | $u$ | $k_s$ | $k_i$ |
| Value                   | 0.018 | 0.1 | 1 | 42 MPa | 0.3 | 0.15 | 0.01 |

References

1. Deb Roy, S.; Pandey, A.; Saha, R. Shake table study on seismic soil-pile foundation-structure interaction in soft clay. *Structures* 2021, 29, 1229–1241. [CrossRef]
2. Zhang, Y.; Jiang, G.; Wu, W.; El Naggar, M.H.; Liu, H.; Wen, M.; Wang, K. Analytical solution for distributed torsional low strain integrity test for pipe pile. *Int. J. Numer. Anal. Methods Geomech.* 2022, 46, 47–67. [CrossRef]
3. Liu, X.; El Naggar, M.H.; Wang, K.; Wu, W. Dynamic soil resistance to vertical vibration of pipe pile. *Ocean Eng.* 2021, 220, 108381. [CrossRef]
4. Zheng, C.; Liu, H.; Kouretzis, G.P.; Sloan, S.W.; Ding, X. Vertical response of a thin-walled pipe pile embedded in viscoelastic soil to a transient point load with application to low-strain integrity testing. *Comput. Geotech.* 2015, 70, 50–59. [CrossRef]
5. Loseva, E.; Lozovsky, I.; Zhostkov, R. Identifying Small Defects in Cast-in-Place Piles Using Low Strain Integrity Testing. *Indian Geotech. J.* 2022, 52, 270–279. [CrossRef]
6. Tu, Y.; Wang, K.; Liu, X.; Rizvi, S.M.F.; Qu, X. Wave reflection characteristics at the pile tip-soil interface under low strain dynamic condition. *Comput. Geotech.* 2021, 130, 103916. [CrossRef]
7. Wu, J.; El Naggar, M.H.; Wang, K.; Liu, X. Analytical Study of Employing Low-Strain Lateral Pile Integrity Test on a Defective Extended Pile Shaft. *J. Eng. Mech.* 2020, 146, 4020103. [CrossRef]
8. Liu, H.; Wu, W.; Yang, X.; Jiang, G.; El Naggar, M.H.; Mei, G.; Liang, R. Detection sensitivity analysis of pipe pile defects during low-strain integrity testing. *Ocean Eng.* 2019, 194, 106627. [CrossRef]
9. Wu, J.; El Naggar, M.H.; Wang, K.; Wu, W. Lateral vibration characteristics of an extended pile shaft under low-strain integrity test. *Soil Dyn. Earthq. Eng.* 2019, 126, 105812. [CrossRef]
10. Gao, L.; Wang, K.; Wu, J.; Xiao, S.; Wang, N. Analytical solution of the dynamic response of a pile with a variable-section interface in low-strain integrity testing. *J. Sound Vib.* 2017, 395, 328–340. [CrossRef]
11. Lu, Z.; Wang, Z.; Liu, D. Study on low-strain integrity testing of pipe-pile using the elastodynamic finite integration technique. *Int. J. Numer. Anal. Methods Geomech.* 2013, 37, 536–550. [CrossRef]
12. Chai, H.-Y.; Phoon, K.-K.; Zhang, D.-J. Effects of the Source on Wave Propagation in Pile Integrity Testing. *J. Geotech. Geoenviron. Eng.* 2010, 136, 1200–1208. [CrossRef]
13. Ding, X.; Liu, H.; Liu, J.; Chen, Y. Wave Propagation in a Pipe Pile for Low-Strain Integrity Testing. *J. Eng. Mech.* 2011, 137, 598–609. [CrossRef]
14. Varghese, R.; Boominathan, A.; Banerjee, S. Stiffness and load sharing characteristics of piled raft foundations subjected to dynamic loads. *Soil Dyn. Earthq. Eng.* 2020, 133, 106117. [CrossRef]
15. Bhownik, D.; Baidya, D.K.; Dasgupta, S.P. A numerical and experimental study of hollow steel pile in layered soil subjected to vertical dynamic loading. *Soil Dyn. Earthq. Eng.* 2016, 85, 161–165. [CrossRef]
16. Liming, Q.; Xuanming, D.; Chongjie, Z.; Chongrong, W.; Guangwei, C. Numerical and test study on vertical vibration characteristics of pile group in slope soil topography. *Earthq. Eng. Eng. Vib.* 2021, 20, 377–390. [CrossRef]
17. Bhownik, D.; Baidya, D.K.; Dasgupta, S.P. A numerical and experimental study of hollow steel pile in layered soil subjected to lateral dynamic loading. *Soil Dyn. Earthq. Eng.* 2013, 53, 119–129. [CrossRef]
18. Biswas, S.; Manna, B. Three dimensional finite element nonlinear dynamic analysis of full-scale piles under vertical excitations. In *Proceedings of the Proceedings of the 18th International Conference on Soil Mechanics and Geotechnical Engineering*, Paris, France, 2–6 September 2013.
19. Yang, X.; Wang, L.; Wu, W.; Liu, H.; Jiang, G.; Wang, K.; Mei, G. Vertical Dynamic Impedance of a Viscoelastic Pile in Arbitrarily Layered Soil Based on the Fictitious Soil Pile Model. *Energies* 2022, 15, 2087. [CrossRef]
49. Wang, L. Vertical response of a pile embedded in highly-saturated soil with compressible pore fluid and anisotropic permeability. *Comput. Geotech.* **2021**, *140*, 104462. [CrossRef]

50. Guan, W.; Wu, W.; Jiang, G.; Liang, R.; Liu, H. Study on vertical vibration characteristics of incompletely bonded pipe pile in saturated soil. *J. Hum. Univ. (Nat. Sci.)* **2021**, *48*, 46–58. [CrossRef]

51. Li, T.; Su, Q.; Kaewunruen, S. Saturated Ground Vibration Analysis B Modifiedased on a Three-Dimensional Coupled Train-Track-Soil Interaction Model. *Appl. Sci.* **2019**, *9*, 4991. [CrossRef]

52. Xiao, S.; Wang, K.; Gao, L.; Wu, J. Dynamic characteristics of a large-diameter pile in saturated soil and its application. *Int. J. Numer. Methods Geomech.* **2018**, *42*, 1255–1269. [CrossRef]

53. Li, T.; Su, Q.; Kaewunruen, S. Saturated Ground Vibration Analysis B Modifiedased on a Three-Dimensional Coupled Train-Track-Soil Interaction Model. *Appl. Sci.* **2019**, *9*, 4991. [CrossRef]

54. He, P.; Mu, Y.; Yang, Z.; Ma, W.; Dong, J.; Huang, Y. Freeze-thaw cycling impact on the shear behavior of frozen soil-concrete interface. *Cold Reg. Sci. Technol.* **2021**, *173*, 103024. [CrossRef]

55. Xiao, F.; Chen, G.S.; Hulsey, J.L.; Davis, D.; Yang, Z. Characterization of the viscoelastic effects of thawed frozen soil on pile by measurement of free response. *Cold Reg. Sci. Technol.* **2018**, *145*, 229–236. [CrossRef]

56. Zhang, X.; Yang, Z.; Chen, X.; Guan, J.; Pei, W.; Luo, T. Experimental study of frozen soil effect on seismic behavior of bridge pile foundations in cold regions. *Structures* **2021**, *32*, 1752–1762. [CrossRef]

57. Leclaire, P.; Cohen-Ténoudji, F.; Aguirre-Puente, J. Extension of Biot's theory of wave propagation to frozen porous media. *J. Acoust. Soc. Am.* **1994**, *96*, 3753–3768. [CrossRef]

58. Carcione, J.M.; Gurevich, B.; Cavallini, F. A generalized Biot–Gassmann model for the acoustic properties of shaley sandstones1. *Geophys. Prospect.* **2000**, *48*, 539–557. [CrossRef]

59. Li, Q.; Shu, W.; Cao, L.; Duan, W.; Zhou, B. Vertical vibration of a single pile embedded in a frozen saturated soil layer. *Soil Dyn. Earthq. Eng.* **2019**, *122*, 185–195. [CrossRef]

60. Cao, L.; Zhou, B.; Li, Q.; Duan, W.; Shu, W. Vertically Dynamic Response of an End-Bearing Pile Embedded in a Frozen Saturated Porous Medium under Impact Loading. *Shock Vib.* **2019**, *2019*, 8983128. [CrossRef]

61. Li, Q.; Shu, W.; Duan, W.; Cao, L. Vertical vibration of a pile in a double-layered stratum under the freezing and thawing processes of saturated porous media. *Cold Reg. Sci. Technol.* **2020**, *169*, 102891. [CrossRef]

62. Kwon, S.Y.; Yoo, M. Study on the Dynamic Soil-Pile-Structure Interactive Behavior in Liquefiable Sand by 3D Numerical Simulation. *Appl. Sci.* **2020**, *10*, 2723. [CrossRef]

63. Kim, M.K.; Lee, J.; Kim, M.K. Vertical vibration analysis of soil-pile interaction systems considering the soil-pile interface behavior. *KSCE J. Civ. Eng.* **2004**, *8*, 221–226. [CrossRef]

64. Manna, B.; Baidya, D. Dynamic vertical response of model piles—Experimental and analytical investigations. *Int. J. Geotech. Eng.* **2009**, *3*, 271–287. [CrossRef]

65. Manna, B.; Baidya, D.K. Dynamic nonlinear response of pile foundations under vertical vibration—Theory versus experiment. *Soil Dyn. Earthq. Eng.* **2010**, *30*, 456–469. [CrossRef]

66. Biswas, S.; Manna, B.; Kumar Baidya, D. Experimental and theoretical study on the nonlinear response of full-scale single pile under coupled vibrations. *Soil Dyn. Earthq. Eng.* **2017**, *94*, 109–115. [CrossRef]

67. Elkasabgy, M.; El Naggar, M.H. Dynamic response of vertically loaded helical and driven steel piles. *Can. Geotech. J.* **2013**, *50*, 521–535. [CrossRef]

68. Ralli, R.; Manna, B.; Datta, M. Field testing of single instrumented steel driven batter piles under vertical vibrations. *Acta Geotech.* **2021**, *12*, 1–12. [CrossRef]

69. Biot, M. Mechanics of deformation and acoustic propagation in porous media. *J. Appl. Phys.* **1962**, *33*, 1483–1498. [CrossRef]

70. Manna, B.; Baidya, D.K. Vertical Vibration of Full-Scale Pile—Analytical and Experimental Study. *J. Geotech. Geoenviron. Eng.* **2009**, *135*, 1452–1461. [CrossRef]

71. Zhu, W.; Gu, L.; Mei, S.; Nagasaki, K.; Chino, N.; Zhang, F. 1g model tests of piled-raft foundation subjected to high-frequency vertical vibration loads. *Soil Dyn. Earthq. Eng.* **2021**, *141*, 106486. [CrossRef]