Eigenvalues of some signed graphs with negative cliques

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Abstract
In a signed graph $G$, a negative clique is a complete subgraph having negative edges only. In this article, we give characteristic polynomial expressions, and eigenvalues of some signed graphs having negative cliques. This includes signed cycle graph, signed path graph, a complete graph with disjoint negative cliques, and star block graph with negative cliques.

1 Introduction
In 1956, the American mathematician Frank Harary modeled the cognitive structure of balance in signed networks by introducing the concept of signed graphs [5, 9]. This study was motivated by with Heider’s concept of balance in signed networks [10]. In a signed graph, the vertices represent individuals and a positive edge (the edge with a positive sign) between two vertices reflects the existence of liking relationship whereas a negative edge (the edge with a negative sign) represents disliking. After the introduction of signed graphs, several attempts have been made for investigating a possible connection between spectral properties and balancedness of signed graphs, for example, see [1][11].

A graph $G$ consists of a finite set of vertices $V(G)$ and set of edges $E(G)$ consisting of distinct, unordered pairs of vertices. Thus, $(i, j)$ or $(j, i)$ represent an edge between vertices $i, j \in V(G)$, and $i, j$ are called adjacent vertices. The number of vertices in $G$ is called its order. A signed graph is a graph equipped with a weight function $f : E(G) \rightarrow \{-1, 0, 1\}$. Thus, signed graph may have positive, negative edges with weights $1, -1$, respectively. Let $G$ be a signed graph on $n$ vertices. Then, the adjacency matrix $A$ of order $n \times n$ associated with $G$ is defined by

$$A_{i,j} = \begin{cases} 1 & \text{if the vertices } i, j \text{ are linked with a positive edge} \\ -1 & \text{if the vertices } i, j \text{ are linked with a negative edge} \\ 0 & \text{if the vertices } i, j \text{ are not linked} \end{cases}$$

where, $1 \leq i, j \leq n$ [8]. The degree of a vertex $i$ in a signed graph $G$ is defined as $d_i = \sum_j |A_{i,j}|$. Thus, it equals to the number of incident edges to $i$, irrespective of its signs. A subgraph of $G$ is a graph $H$, such that, $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The subgraph $H$ is called spanning subgraph of $G$ if it covers all the vertices in $G$. Subgraph $H$ is an induced subgraph of $G$ if $i, j \in V(H)$ and $(i, j) \in E(G)$ indicate $(i, j) \in E(H)$. Two subgraphs $H_1$ and $H_2$ are called vertex-disjoint subgraphs if $V(H_1) \cap V(H_2) = \emptyset$. A path of length $k$ between two vertices $v_1$ and $v_k$ is a sequence of distinct vertices $v_1, v_2, \ldots, v_{k-1}, v_k$, such that, for all $i = 1, 2, \ldots, k-1$, $(v_i, v_{i+1}) \in E(G)$.

A signed cycle graph on $n$ vertices is a signed graph having an equal number of vertices and edges with each vertex has degree equals to two. We denote a signed cycle graph by $C_n$ or $n$-cycle. The adjacency matrix $A$ of $C_n$ is given by $A_{i,i+1} = A_{i+1,i} \in \{1, -1\}$, $i = 1, 2, \ldots, n-1$ and $A_{n,1} = A_{1,n} \in \{1, -1\}$, all other

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For each pair of such vertices \(i, j\), called a 1-factor, is a matching that covers all vertices of \(G\). We call it the matching number of \(G\), denoted by \(\phi(G)\). A signed tree is a signed graph which does not have any cycle graph as its subgraph.

Similarly, if each edge of a clique is negative then, we call it a negative clique. We denote a complete graph \(K_n\) on \(n\) vertices, having each edge positive, by \(A^+\). If diagonal elements of \(A\) are zero. Moreover, the sign of \(C_n\) is defined as the product of signs for positive +1 and for negative −1 of its edges. If the sign of \(C_n\) is positive it is called balanced cycle graph otherwise, it is called an unbalanced cycle graph \([6]\). Examples of a balanced and an unbalanced \(C_4\) are shown in figure (1a),(1b), respectively. A signed tree is a connected signed graph which does not have any cycle graph as its subgraph. We denote a signed tree on \(n\) vertices by \(T_n\). The signed path graph \(P_n\) on \(n\) vertices is a tree in which two vertices are having degree 1, and the remaining \((n−2)\) vertices are having degree 2.

A signed complete graph is a signed graph where each distinct pair of vertices is connected by an edge. A signed clique in signed graph \(G\) is an induced subgraph which is a signed complete graph. In cycle graph \(C_n\), path graph \(P_n\) edges are cliques. When each edge of a clique is negative we call it a negative clique. Similarly, if each edge of a clique is positive then, we call it a positive clique. We denote a complete graph on \(n\) vertices, having each edge positive, by \(K_n^+\). By \(K_n^{m,r}\), we denote a signed complete graph on \(n\) vertices, having a \(m\) number of vertex-disjoint negative cliques each of order \(r\), and all the other edges positive except those are in cliques. Example of a \(K_n^{2,3}\) graph is given in figure 1c, where two vertex-disjoint negative cliques, each of order 3 are on vertex-sets \{2,3,4\}, and \{6,7,8\} respectively. A block in a signed graph \(G\) is a maximal subgraph which has no cut-vertex. If each block of \(G\) is complete graph then \(G\) is called block graph. For block graphs without negative edges see \([3]\). If block graph \(G\) has at most one cut vertex then we call it star block graph. We consider a star block graph having \(k\) number of blocks each having \(r\) number of vertices. We call it \(r\)-regular star block graph. An example of a 3-regular star block graph is given in figure 1d.

The characteristic polynomial of a square matrix \(A\) of order \(n\) is the polynomial defined by \(\det(A−\lambda I)\) where, \(I\) denotes the \(n\times n\) identity matrix. We denote the characteristic polynomial of \(A\) by \(\phi(A)\). The characteristic polynomial of signed graph \(G\), denoted by \(\phi(G)\), is characteristic polynomial of its adjacency matrix \(A\) that is \(\phi(G) = \phi(A)\). Eigenvalues of a matrix \(A\) are roots of characteristic polynomial \(\det(A−\lambda I)\). The spectrum of a signed graph \(G\) is set of eigenvalues of its adjacency matrix along with their multiplicities. For convenience, we can relabel the vertices in graph \(G\). In graph theory, these relabelling are captured by permutation similarity of adjacency matrix \(A\). The determinant of permutation matrices is equal to 1. Thus, relabelling on vertex-set keep the determinant, and characteristic polynomial unchanged.

1.1 Matchings and Coates digraph

First, we modify some preliminaries from \([4]\) for signed graphs. A matching in a signed graph \(G\) is a collection of edges no two of which have a vertex in common. The largest number of edges in a matching in \(G\) is the matching number \(m(G)\). A matching with \(k\) edges is called a \(k\)-matching. A perfect matching of \(G\), also called a 1-factor, is a matching that covers all vertices of \(G\).

The Coates digraph \(D(A)\) generated from a matrix \(A\) of order \(n\) has \(n\) vertices labelled by \(1,2,\ldots,n\) and for each pair of such vertices \(i, j\) a directed edge exists from \(j\) to \(i\) of weight \(A_{i,j}\) \([4]\). The elements of the main diagonal of \(A\) corresponds to loops at vertices in \(D(A)\). If diagonal elements of \(A\) are zero, then no loops are considered on corresponding vertices of \(D(A)\). A linear subdigraph of \(D(A)\) is a spanning subdigraph of \(D(A)\) in which each vertex has indegree 1 and outdegree 1 that is exactly one edge into each vertex and exactly one (possibly the same, in the case of the loop) out of each vertex. Thus a linear subdigraph consists

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Examples: Dark line shows positive edge(weight +1), dotted line shows negative edge(weight -1).}
\end{figure}
of a spanning collection of pairwise vertex-disjoint cycles. The weight of a linear subdigraph is the product of the weights of edges in it. For example, the Coates digraph representation of the matrix \( A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \) is given in figure 2a.

By the Coates digraph of a signed graph, we mean the Coates digraph corresponding to the adjacency matrix of the signed graph. Consider a signed graph \( G \) and denote its Coates digraph by \( D(G) \). For an edge between vertices \( i, j \) in \( G \), there are two directed edges of equal weights in \( D(G) \), one from \( i \) to \( j \) and other from \( j \) to \( i \). This forms a directed cycle of length 2 which we call a directed 2-cycle. In a linear subdigraph of \( D(G) \), if there are \( k \) such directed 2-cycles then these appear due to the \( k \) matchings in \( G \). So, there is a one-one correspondence between matchings in \( G \) and directed 2-cycles in a linear subdigraph of \( D(G) \) if it exists. Thus, by \( k \)-matching in linear subdigraphs we mean, the existence of \( k \) vertex-disjoint directed 2-cycles. For example, the Coates digraph of balanced \( C_4 \) in figure 1a is depicted in figure 2b. Note that, there are two 2-matchings in balanced \( C_n \) in figure 1a. These are \( \{(1,2),(3,4)\} \), and \( \{(2,3),(1,4)\} \). In figure 2b, corresponding to these matchings, there are two directed 2-cycles in linear subdigraph \( L_3, L_4 \), respectively, in Coates digraph of the balanced \( C_n \). Now we recall the definition of the determinant of the adjacency matrix \( A \) of \( G \) in terms of its linear subdigraphs in \( D(G) \).

**Theorem 1.1.** [4] Let \( A \) be square matrix of order \( n \). Then

\[
\det A = (-1)^n \sum_{L \in \mathcal{L}(A)} (-1)^{c(L)} w(L),
\]

where, \( w(L) \) is the weight of linear subdigraph \( L \) of the Coates digraph \( D(A) \), \( c(L) \) is the number of directed cycles in \( L \), and \( \mathcal{L}(A) \) denotes the set of all linear subdigraphs of \( D(A) \).

The article is organized as follows: In the section 2, we calculate characteristic polynomials, eigenvalues of signed cycle and path graphs using the concept of linear subdigraphs, and matching. In section 3, we calculate characteristic polynomial and eigenvalues of complete graphs having disjoint negative cliques of the same order. In section 4, we give bounds of eigenvalues of complete graphs having disjoint negative cliques of different orders. Finally in section 5 we calculate eigenvalues of regular star block graphs.

## 2 Characteristic Polynomial of \( C_n \) and \( P_n \)

We denote the weight of signed cycle graph \( C_n \) by \( \delta \). When \( C_n \) is balanced \( \delta = 1 \), otherwise, \( \delta = -1 \). Coates digraph corresponding to adjacency matrix \( \left( A(C_n) - \lambda I_n \right) \), is a directed graph on \( n \) vertices with

1. Loop of weight \(-\lambda\) at each vertex.
2. For every adjacent vertices in cycle \(C_n\), there are two opposite directed edges, connecting these adjacent vertices in Coates digraph.

Next, we require number of \(k\)-matchings in \(C_n\), which is used to find linear subdigraphs of Coates digraph of \(A(C_n) - \lambda I_n\). We state following standard result [12].

**Proposition 2.1.** The number of \(k\)-matching in cycle graph \(C_n\) is equal to

\[
\frac{n}{n-k} \binom{n-k}{k},
\]

(1)

For cycle graphs \(m(G) = |n/2|\) so number of all possible matching in \(G\) is given by

\[
\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k} \binom{n-k}{k},
\]

(2)

where, \(k = 0\) corresponding to no matching. Each \(k\)-matching in cycle graph corresponds to \(k\) vertex-disjoint directed 2-cycles in its Coates digraph covering \(2k\) vertices. These \(k\) directed 2-cycles along with loops at remaining \(n - 2k\) vertices form linear subgraphs in Coates digraph of \(C_n\).

**Theorem 2.2.** Characteristic polynomial \(\phi(C_n)\) of signed cycle graph \(C_n\), having weight \(\delta \in \{-1, 1\}\) is given by

\[
\phi(C_n) = \begin{cases} 
(-1)^n \left( \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{n-1}{n-k} \binom{n-k}{k} \times (-1)^{n-k} \times (-\lambda)^{n-2k} \right) + 2(-1)^{\frac{n}{2}} - 2\delta & \text{if } n \text{ is even,} \\
(-1)^n \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{n-1}{n-k} \binom{n-k}{k} \times (-1)^{n-k} \times (-\lambda)^{n-2k} - 2\delta & \text{if } n \text{ is odd,}
\end{cases}
\]

**Proof.** In Coates digraph of matrix \(A(C_n) - \lambda I_n\) there will be following two type of linear subdigraphs along with their contribution to \(\phi(C_n)\)

1. Two directed \(n\)-cycles; one clockwise and another anticlockwise respectively, each having weight \(\delta\). Using theorem 1.1 their contribution to \(\phi(C_n)\) is

\[
(-1)^n \left( 2(-1)^{\frac{n}{2}} \right) = (-1)^n (-2\delta).
\]

2. Linear subgraph having \(k\)-matching covering \(2k\) vertices, and loops at remaining \(n - 2k\) vertices for \(k = 1, 2, \ldots \lfloor n/2 \rfloor\). Weight of each \(k\)-matching is 1, and weight of \(n - 2k\) loops is \((-\lambda)^{n-2k}\). Total number of cycles are \(k + n - 2k = n - k\). If,

(a) \(n\) is even: for \(k = \frac{n}{2}\), there will be two linear subdigraphs having \(\frac{n}{2}\) directed 2-cycles. Thus, no loop will be selected in these two linear subdigraphs. Their contribution is

\[
(-1)^n 2(-1)^{\frac{n}{2}}.
\]

(b) \(n\) is odd: there will be no linear subdigraphs having \(\frac{n}{2}\) directed 2-cycles.

Thus, using proposition 2.1 and combining 1. and 2. the result follows.

**Corollary 2.3.** Determinant of signed cycle \(C_n\), having weight \(\delta \in \{-1, 1\}\) is given by

\[
\det(C_n) = \begin{cases} 
2 - 2\delta & \text{if } n \text{ is even and even multiple of 2} \\
-2 - 2\delta & \text{if } n \text{ is even and odd multiple of 2} \\
2\delta & \text{if } n \text{ is odd}
\end{cases}
\]

**Proof.** To calculate the determinant we need to set \(\lambda = 0\) in characteristic polynomial. Hence, the result directly follows from theorem 2.2.
2.1 Eigenvalues of signed $C_n$

Let us consider a matrix $Q$ of order $n \geq 2$ such that, the $Q_{i,i+1} \in \{1,-1\}$, $i = 1,2,...,n-1$, the $Q_{n,1} \in \{1,-1\}$ and remaining entries of $Q$ are zero. The Coates digraph $D(Q-\lambda I)$ is a digraph having directed $n$-cycle with a loop of weight $-\lambda$ at each of its vertices. Thus, Coates digraph $D(Q-\lambda I)$ has only two linear subgraphs. One having the directed $n$-cycle without loops, and another consisting of the $n$ loops only. Weight of directed $n$-cycle is either 1 or $-1$. It follows from (1.1) that characteristic equation of $Q$ is given by:

$$(-1)^n \left((-1)^n(\lambda n + (-1)^i \delta) = 0 \implies \lambda^n - \delta = 0, \right)$$

which means eigenvalues of $Q$ are $1, \omega, \omega^2, ..., \omega^{n-1}$, where,

$$\omega = \begin{cases} 
e 2\pi \frac{k}{n} & \text{if } C_n \text{ is balanced} \\ 2\cos\left(\frac{\pi + 2\pi k}{n}\right) & \text{if } C_n \text{ is unbalanced} \end{cases}$$

$k = 1, 2...n$.

**Proof.** It is clear that eigenvalues of $A(C_n)$ are $\omega^k + \omega^{n-k}, k = 1...n$. To derive adjacency matrix of balanced $C_n$ from $Q$, value of $\delta$ has to be 1. Similarly, to derive adjacency matrix of unbalanced $C_n$ from $Q$, value of $\delta$ has to be $-1$. Now, when $\delta = 1$

$$\omega^k + \omega^{n-k} = \omega^k + \omega^{-k} = e^{2\pi i k/n} + e^{-2\pi i k/n} = 2 \cos\left(\frac{2\pi k}{n}\right).$$

And, when $\delta = -1$

$$e^{\pi i + \frac{2\pi i k}{n}} + e^{-\pi i + \frac{2\pi i k}{n}} = 2 \cos\left(\frac{\pi + 2\pi k}{n}\right)$$

For $k = 1, 2...n$.

**Theorem 2.5.** Let $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ be eigen values of balanced signed cycles and $\beta_1 \geq \beta_2 \geq ... \geq \beta_n$ be eigenvalues of unbalanced signed cycles of length $n > 2$. Then,

$$|\lambda_i - \beta_i| = |\lambda_{n-i+1} - \beta_{n-i+1}|.$$

**Proof.** 1. If $n$ is even: cos function lie in range [-1,1]. Eigenvalues of the balanced and unbalanced $C_n$ are $2 \cos\left(\frac{2\pi k}{n}\right)$ and $2 \cos\left(\frac{\pi + 2\pi k}{n}\right)$, respectively for $k = 1,2,...,n$. To get $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ and $\beta_1 \geq \beta_2 \geq ... \geq \beta_n$ we need to sort values of $2 \cos\left(\frac{2\pi k}{n}\right)$ and $2 \cos\left(\frac{\pi + 2\pi k}{n}\right)$ in descending order. Also, $2 \cos\left(\frac{2\pi k}{n}\right) = 2 \cos\left(\frac{2\pi (n-k)}{n}\right)$, and $2 \cos\left(\frac{\pi + 2\pi k}{n}\right) = 2 \cos\left(\frac{\pi - 2\pi k}{n}\right) = 2 \cos\left(\frac{\pi + 2\pi (n-k)}{n}\right)$. Sorted order of eigenvalues of balanced $C_n$ is for sequence $k = n, (n-1), 2, (n-2),...,i,(n-i),...,1$ and for unbalanced $C_n$, ordered order is for sequence $k = n, (n-1), 2, (n-2),...,i,(n-i),...,1$. Now, consider $\lambda_i$ and $\lambda_{n-i+1}$. As, their corresponding $k$ indices are at difference of $n/2$. We have, $2 \cos\left(\frac{\pi (i \pm n/2)}{n}\right) = -2 \cos\left(\frac{2\pi k}{n}\right)$. Hence, $\lambda_{n-i+1} = -\lambda_i$. Corresponding $k$ indices of $\beta_i$ and $\beta_{n-i+1}$ are also at difference of $n/2$. Thus, $2 \cos\left(\frac{\pi + 2\pi (k \pm n/2)}{n}\right) = -2 \cos\left(\frac{\pi + 2\pi k}{n}\right)$. Hence, $\beta_{n-i+1} = -\beta_i$, and $|\lambda_i - \beta_i| = |\lambda_{n-i+1} - \beta_{n-i+1}|$.

2. If $n$ is odd: following similar steps as the case for even $n$, in this case to get $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$, we need the sequence $k = n, (n-1), 2, (n-2),...,i,(n-i),...(n-1)/2,(n+1)/2$, and to get $\beta_1 \geq \beta_2 \geq ... \geq \beta_n$ we need the sequence $k = n, (n-1), 1, (n-1-1), 2, (n-2-1),...,i,(n-i-1),...(n+1)/2,(n-1)/2$. The difference between $k$ indices for $\beta_i$ and $k$ indices for $\alpha_{n-i+1}$ is $\pm n/2-1/2$. We have, $2 \cos\left(\frac{\pi + 2\pi (k \pm n/2 - 1/2)}{n}\right) = -2 \cos\left(\frac{2\pi k}{n}\right)$. Hence, $\lambda_i = -\beta_{n-i+1}$. Similarly, $\beta_i = -\alpha_{n-i+1}$. Thus, $|\lambda_i - \beta_i| = |\lambda_{n-i+1} - \beta_{n-i+1}|$. 

\[\square\]
2.2 Characteristic Polynomial of $P_n$

Coates digraph corresponding to adjacency matrix $A(P_n)$ of signed path graph $P_n$, is a directed graph having $n$ vertices with

1. Loop of weight $-\lambda$ at each vertex.
2. For every adjacent vertices in path $P_n$, there are two opposite directed edges, connecting these adjacent vertices in Coates digraph.

We state the following standard result [12].

**Proposition 2.6.** The number of $k$-matching in signed path graph $P_n$ is equal to

$$\binom{n-k}{k}. \quad (4)$$

Thus, for path graphs $m(G) = \lfloor n/2 \rfloor$ number of all possible matching in $G$ is given by:

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}. \quad (5)$$

**Theorem 2.7.** Characteristic polynomial $\phi(P_n)$ of signed $P_n$ is given by

$$\phi(P_n) = \begin{cases} (-1)^n \left( \sum_{k=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-k}{k} \times (-1)^{n-k} \times (-\lambda)^{n-2k} \right) + (-1)^{\frac{n}{2}} & \text{if } n \text{ is even} \\ (-1)^n \sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-k}{k} \times (-1)^{n-k} \times (-\lambda)^{n-2k} & \text{if } n \text{ is odd} \end{cases}$$

**Proof.** In Coates digraph of matrix $\left( A(P_n) - \lambda I \right)$ there will be following linear subdigraph along with their contribution to $\phi(P_n)$

1. Subdigraph having $k$-matching covering $2k$ vertices and loops at remaining $n - 2k$ vertices for $k = 1, 2, \ldots \lfloor n/2 \rfloor$. Weight of $k$-matching is 1, and weight of $n - 2k$ loops is $(-\lambda)^{n-2k}$. Total number of cycles are $k + n - 2k = n - k$. If

   (a) $n$ is even: for $k = \frac{n}{2}$, there will be one linear subdigraphs having $\frac{n}{2}$ directed 2-cycles. Thus, no loop will be selected in this linear subdigraph. Its contribution is
   $$(-1)^n(-1)^{\frac{n}{2}}.$$

   (b) $n$ is odd: There will be no linear subdigraphs having $\frac{n}{2}$ directed 2-cycles.

   Thus, using proposition 2.6, and combining 1. and 2. the result follows.

As characteristic polynomial of all signed path graphs $P_n$ for a given $n$ is same, their eigenvalues are same.

**Corollary 2.8.** Determinant of signed path $P_n$ is given by

$$\det(P_n) = \begin{cases} 1 & \text{if } n \text{ is even and even multiple of } 2 \\ -1 & \text{if } n \text{ is even and odd multiple of } 2 \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

**Proof.** Proof directly follows from theorem 2.7, after setting $\lambda = 0.$
3 Characteristic polynomial of $K_{n}^{m,r}$

In this section we derive characteristic polynomial of $K_{n}^{m,r}$. Here, determinant, and eigenvalues are readily follows from characteristic polynomials hence, they are stated as corollaries without proofs. We first derive the result for the case when, $n = mr$, that is, when all $m$ negative cliques each of order $r$ cover all the $n$ vertices of complete graph.

**Theorem 3.1.** Characteristic polynomial of $A(K_{mr}^{m,r})$ is given by

$$
\phi(K_{mr}^{m,r}) = (1 - \lambda)^{m(r-1)}(1 - 2r - \lambda)^{m-1}\left(1 + r(m - 2) - \lambda\right).
$$

**Proof.** With suitable relabelling of vertices in $K_{mr}^{m,r}$ we have,

$$
A(K_{mr}^{m,r}) = \begin{bmatrix}
-A(K_r) & J & J & \cdots & J \\
J & -A(K_r) & J & \cdots & J \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
J & J & J & \cdots & -A(K_r)
\end{bmatrix}_{mr \times mr},
$$

where, $A(K_r)$ denotes adjacency matrix of a positive clique $K_r$. Also, $J$ is all-one matrix of order $r$.

Then,

$$
A(K_{mr}^{m,r}) - \lambda I_{mr} = \begin{bmatrix}
Y & X & X & \cdots & X \\
X & Y & X & \cdots & X \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
X & X & X & \cdots & Y
\end{bmatrix}_{mr},
$$

where,

$$
Y = -A(K_r) - \lambda I_r, \quad X = J_r,
$$

and $J_r$ is all-one matrix of order $r$.

In the above matrix, $A(K_{mr}^{m,r}) - \lambda I_{mr}$, subtract the last row from all the other rows. This produces

$$
\begin{bmatrix}
Y - X & O & O & \cdots & O & X - Y \\
O & Y - X & O & \cdots & O & X - Y \\
O & O & Y - X & \cdots & O & X - Y \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
O & O & O & \cdots & Y - X & X - Y \\
X & X & X & \cdots & X & Y
\end{bmatrix},
$$

Now, add first $r - 1$ columns to the last column. This produce the following lower triangular matrix,

$$
\begin{bmatrix}
Y - X & O & O & \cdots & O & O \\
O & Y - X & O & \cdots & O & O \\
O & O & Y - X & \cdots & O & O \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
O & O & O & \cdots & Y - X & O \\
X & X & X & \cdots & X & Y + (m - 1)X
\end{bmatrix}
$$

Hence,

$$
\det \left(A(K_{mr}^{m,r}) - \lambda I_{mr}\right) = \det(Y - X)^{m-1} \det \left(Y + (m - 1)X\right). \quad (6)
$$

Also,

$$
Y - X = -2A(K_r) - (\lambda + 1)I_r.
$$
The eigenvalues of $A(K_r)$ are given by $-1, (r-1)$ with the multiplicity $(r-1)$, $1$, respectively \cite{[2]}. Hence, eigenvalues of the matrix, $Y - X$, are $(1 - \lambda), (-2r + 1 - \lambda)$ with multiplicities $(r-1)$, $1$, respectively. As the determinant of a matrix is product of its eigenvalues with the multiplicities thus,

$$\det(Y - X) = (1 - \lambda)^{r-1}(1 - 2r - \lambda).$$

Next,

$$Y + (m-1)X = (m-2)A(K_r) + (m-1-\lambda)I_r.$$  

Eigenvalues of $Y + (m-1)X$ are $(1 - \lambda), (1 + r(m-2) - \lambda)$ with multiplicity $(r-1), 1$, respectively. Hence,

$$\det\left(Y + (m-1)X\right) = (1 - \lambda)^{r-1}\left(1 + r(m-2) - \lambda\right).$$

Thus, from equation (6)

$$\phi(K_{mr}^{m,r}) = \left((1 - \lambda)^{r-1}(1 - 2r - \lambda)\right)^{m-1}(1 - \lambda)^{r-1}\left(1 + r(m-2) - \lambda\right)$$

$$= (1 - \lambda)^{(m-1)(1 - 2r - \lambda)^{m-1}}(1 + r(m-2) - \lambda).$$

\hfill $\square$

**Corollary 3.2.** Determinant of $K_{mr}^{m,r}$ is given by

$$(1 - 2r)^{(m-1)}\left(1 + r(m-2)\right).$$

**Corollary 3.3.** Eigenvalues of $K_{mr}^{m,r}$ are $1, (1 - 2r)$, and $\left(1 + r(m-2)\right)$ with multiplicity $m(r-1), m-1, and 1$ respectively.

Next we give inverse of the matrix $A(K_{mr}^{m,r}) - \lambda I_{mr}$. It is used to get characteristic polynomial of general case $A(K_n^{m,r})$.

**Lemma 3.4. Inverse of $A(K_{mr}^{m,r}) - \lambda I_{mr}$ is given by**

$$\frac{1}{\lambda + 2r - 1}\left(\frac{1}{\lambda - 1}I_m \otimes \left(2A(K_r) - (\lambda + 2r - 3)I_r\right) - \frac{1}{\lambda + r(2 - m) - 1}J\right),$$

where, $\lambda \neq 1, (1 - 2r)$, and $\left(1 + r(m-2)\right)$. Also, $J$ is all-one matrix of order $mr$, and $\otimes$ denotes tensor product of matrices.

**Proof.** Using the same construction as in theorem 3.1, we can write,

$$A\left(K_{mr}^{m,r}\right) - \lambda I_{mr} = \left(I_m \otimes (Y - X)\right) + (1_{m \times m} \otimes X) = \left(I_m \otimes (Y - X)\right) + mr1_{mr}^T.$$

Let $A_1 = \left(I_m \otimes (Y - X)\right)$. Now, recall the Sherman–Morrison formula: If $A$ is a nonsingular square matrix and $1 + v^T A^{-1} u \neq 0$ for some column vectors $u, v$ then

$$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^T A^{-1}}{1 + v^T A^{-1} u}.$$  

In order to find $A_1^{-1}$ we need to find $(Y - X)^{-1}$. By symmetry let, $\alpha, \beta$ be diagonal, non-diagonal entries of $(Y - X)^{-1}$, respectively. On solving following two equations we get the values of $\alpha, \beta$.

$$-\alpha(\lambda + 1) - 2\beta(r - 1) = 1,$$

$$-\beta(\lambda + 1) - 2\alpha - 2\beta(r - 2) = 0.$$  

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we get,
\[\alpha = \frac{-(\lambda + 2r - 3)}{(\lambda - 1)(\lambda + 2r - 1)}, \quad \beta = \frac{2}{(\lambda - 1)(\lambda + 2r - 1)}\]
Thus, \(A^{-1}_1\) can be written as,
\[A^{-1}_1 = \frac{1}{(\lambda - 1)(\lambda + 2r - 1)} \left( I_m \otimes \left( 2A(K_r) - (\lambda + 2r - 3)I_r \right) \right).
\]
Also,
\[A^{-1}_1 J_m A^{-1}_1 = \frac{1}{(\lambda + 2r - 1)^2} \times J,
\]
and
\[1 + J_m A^{-1}_1 J_m = \frac{\lambda + r(2 - m) - 1}{\lambda + 2r - 1},\]
where, \(J\) is all-one matrix of order \(mr\).
Hence,
\[\left( A(K^{m,r}_m) - \lambda I_{mr} \right)^{-1} = \frac{1}{(\lambda - 1)(\lambda + 2r - 1)} \left( I_m \otimes \left( 2A(K_r) - (\lambda + 2r - 3)I_r \right) \right) - \frac{1}{(\lambda + 2r - 1)(\lambda + r(2 - m) - 1)} J
\]
\[= \frac{1}{\lambda + 2r - 1} \left( \frac{I}{\lambda - 1} \right) \left( \frac{I}{\lambda + r(2 - m) - 1} \right) \left( \frac{I}{\lambda - 1} \right) \left( \frac{I}{\lambda + r(2 - m) - 1} \right)
\]
\[\text{Theorem 3.5. Characteristic polynomial of } A(K^{m,r}_m) \text{ is given by}
\]
\[(1 - \lambda)^{m(r-1)}(1 - 2r - \lambda)^{m-1} \left( -\frac{\lambda^2 - r(2 + \lambda(2 - m) - m) + 1}{\lambda + r(2 - m) - 1} \right) ^{n-mr-1}
\times \left( n(1 - 2r - \lambda) + 2r(1 + m(r - 1) + \lambda) - 1 + \lambda^2 \right).
\]
\textbf{Proof.} With suitable relabelling of vertices in } \(K^{m,r}_n\), matrix \(A(K^{m,r}_n) - \lambda I_n\) can be written in the form
\[A(K^{m,r}_n) = \begin{bmatrix} A_1 - \lambda I_{mr} & J \\ J^T & A_2 - \lambda I_{n-mr} \end{bmatrix},\]
where, \(A_1 = A(K^{m,r}_m), A_2 = A(K_{n-mr}).\) Also, \(J\) is all-one matrix of order \((mr) \times (n - mr)\), and \(J^T\) is the transpose of \(J\). By Schur complement formula ([2],p.4) we have,
\[\det \left( A(K^{m,r}_n) - \lambda I_n \right) = \det(A_1 - \lambda I_{mr}) \times \det \left( (A_2 - \lambda I_{n-mr}) - J^T(A_1 - \lambda I_{mr})^{-1} J \right)
\]
Using Lemma 3.4
\[J^T(A_1 - \lambda I_{mr})^{-1} J = \frac{-mr}{\lambda + r(2 - m) - 1} J_1,
\]
and,
\[(A_2 - \lambda I_{n-mr}) - J^T(A_1 - \lambda I_{mr})^{-1} J = \left( \frac{\lambda + 2r - 1}{\lambda + r(2 - m) - 1} \right) K_{n-mr} + \left( \frac{mr}{\lambda + r(2 - m) - 1} \right) I_{n-mr}.
\]
Eigenvalues of above matrix are
\[-\frac{-\lambda^2 - r(2 + \lambda(2 - m) - m) + 1}{\lambda + r(2 - m) - 1}, \quad \frac{n(\lambda + 2r - 1) - 2r(1 + \lambda - m + mr) - \lambda^2 + 1}{\lambda + r(2 - m) - 1}\]
with multiplicity \( n - mr - 1, 1 \), respectively.
From theorem 3.1
\[
\det(A_1 - \lambda I_{mr}) = (1 - \lambda)^{m(r-1)}(1 - 2r - \lambda)^{m-1}(1 + r(m - 2) - \lambda).
\]
Hence,
\[
\phi\left(A(K_{n,r}^m)\right) = (1 - \lambda)^{m(r-1)}(1 - 2r - \lambda)^{m-1}\left(-\lambda^2 - r(2 + \lambda(2 - m) - m) + 1\right)\frac{\lambda + r(2 - m) - 1}{n - mr - 1} \\
\times \left(n(1 - 2r - \lambda) + 2r(1 + m(r - 1) + \lambda) - 1 + \lambda^2\right)
\]
\[
\square
\]

**Corollary 3.6.** Determinant of \( A(K_{n,r}^m) \) is given by
\[
(1 - 2r)^{m-1}(-1)^{n-mr-1}\left(n(1 - 2r) + 2r(1 + m(r - 1)) - 1\right).
\]

**Corollary 3.7.** Eigenvalues of \( A(K_{n,r}^m) \) are
\[
1, (1 - 2r), \frac{(n - 2r) \pm \sqrt{8mr - 8r - 4n - 8mr^2 + 4 + (n + 2r)^2}}{2}
\]
and roots of polynomial
\[
\left(-\lambda^2 - r\left(2 + \lambda(2 - m) - m\right) + 1\right)\frac{\lambda + r(2 - m) - 1}{n - mr - 1}
\]
with multiplicity \( m(r - 1), (m - 1), 1, n - mr - 1 \) respectively.

## 4 Complete graph with negative cliques of different order

In this section we consider the complete graph \( G \) having disjoint negative cliques of different orders which cover the vertex-set of \( G \). Let \( G \) have \( k \) number of negative cliques with order \( n_1, n_2, \ldots, n_k \), respectively. Let \( n_1 \leq n_2 \leq \ldots \leq n_k \). Thus, the adjacency matrix of such a graph \( G \) can be written as
\[
A(G) = \begin{bmatrix}
-A(K_{n_1}) & J_{12} & \ldots & J_{1k} \\
J_{12}^T & -A(K_{n_2}) & \ldots & J_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
J_{1k}^T & J_{2k}^T & \ldots & -A(K_{n_k})
\end{bmatrix}, \tag{7}
\]
where, \( A(K_{n_i}) \) denotes the adjacency matrix of the positive clique \( K_{n_i}, i = 1, \ldots, k \) and \( J_{pq} \) denotes the all-one matrix of order \( n_p \times n_q \). To calculate eigenvalues we use approach similar to in [7] for complete multipartite graph. Note that, it is enough to investigate the eigenvalues of \( A(G) - I_n \) in order to investigate eigenvalues of \( A(G) \). Indeed, \( \lambda \) is an eigenvalue of \( A(G) - I_n \) corresponding to an eigenvector \( X \in \mathbb{R}^n \) if and only if \( \lambda + 1 \) is an eigenvalue of \( G \) corresponding to the eigenvector \( X \). Observe that the diagonal blocks of \( A(G) - I_n \) are \(-J_{n_i n_i}, i = 1, \ldots, k \), and the off diagonal blocks are same as that of \( A(G) \).

We first prove the following lemma which is used in the sequel.

**Lemma 4.1.** Let
\[
N = \begin{bmatrix}
-n_1 & n_2 & \ldots & n_k \\
n_1 & -n_2 & \ldots & n_k \\
\vdots & \vdots & \ddots & \vdots \\
n_1 & n_2 & \ldots & -n_k
\end{bmatrix}, \tag{8}
\]
be a matrix of order \( k \times k \). Let \( N_\lambda = N - \lambda I_k \). Then
\[
\det(N_\lambda) = \left[ \prod_{i=1}^{k} (-2n_i - \lambda) + \sum_{i=1}^{k} n_i \prod_{j=1,j\neq i}^{k} (-2n_j - \lambda) \right].
\]

Proof. Let \( n = [n_1 \ n_2 \ldots \ n_k]^T \in R^k \). Then,
\[
\det(N_\lambda) = \det \left( \begin{bmatrix} 1 & -n^T \\ 0_k & -n^T \end{bmatrix} \right) = \det \left( \begin{bmatrix} 1 & -n^T \\ 1_k & -2 \text{diag}(n) - \lambda I_k \end{bmatrix} \right).
\]

Expanding the right hand side, the desired result follows. \( \square \)

Now, we have the following theorem which completely characterizes the eigenvalues of \( A(G) - I_n \), and hence the eigenvalues of \( G \).

**Theorem 4.2.** Let \( G \) be a complete graph on \( n \) vertices with \( k \) disjoint negative cliques of order \( n_1, n_2, \ldots, n_k \) such that \( n_1 + n_2 + \ldots + n_k = n \). Suppose \( \pi_i, i = 1, \ldots, t, \ t \leq k \) be the distinct numbers in the set \( \{n_1, \ldots, n_k\} \). Then,

(a) \( 0 \) is an eigenvalue of \( A(G) - I_n \) with algebraic multiplicity \( n - k \) corresponding to eigenvectors \( X = [X_1 \ X_2 \ldots \ X_k]^T, X_i \in R^{n_i} \) such that \( \Sigma X_i = 0 \) for all \( i \).

(b) \( 2\pi_i, i = 1, \ldots, t \) are nonzero eigenvalues of \( A(G) - I_n \) with multiplicity \( m_i - 1 \) where \( m_i \) is the number of distinct clusters in \( G \) of order \( \pi_i \). The other nonzero eigenvalues are the roots of the polynomial \( 1 + p(\lambda) \) where
\[
p(\lambda) = \sum_{i=1}^{t} \frac{m_i \pi_i}{-2\pi_i - \lambda}.
\]

Moreover, the eigenvectors corresponding to the nonzero eigenvalues of \( A(G) - I_n \) are of the form \( X = [\alpha_1 I_{n_1}^T \alpha_2 I_{n_2}^T \ldots \alpha_k I_{n_k}^T]^T \) where \( 0_k \neq \alpha = [\alpha_1 \alpha_2 \ldots \alpha_k]^T \) satisfies \( N_\lambda \alpha = 0 \). Such an \( \alpha \) determines an eigenvector corresponds to the eigenvalue \( \lambda \) for which \( \lambda(\alpha_i - \alpha_j) = 2(n_j \alpha_j - n_i \alpha_i), i, j = 1, \ldots, k \).

Proof. (a) Let \( X = [X_1 \ X_2 \ldots \ X_k]^T, X_i \in R^{n_i} \) such that \( (A(G) - I_n)X = 0 \). Then for \( i, j \in \{1, \ldots, k\} \),
\[
\sum_{r \neq i, r=1}^{k} \Sigma X_r - \Sigma X_i = \sum_{r \neq j, r=1}^{k} \Sigma X_r - \Sigma X_j = 0.
\]

This yields \( \Sigma X_i = 0 \) for all \( i = 1, \ldots, k \). Since dimension of the vector space \( \{X_i \in R^{n_i} : \Sigma X_i = 0\} \) over \( R \) is \( n_i - 1 \), the desired result follows.

(b) Let \( \lambda \neq 0 \) and \( (A(G) - I_n)X = \lambda X \) where \( X = [X_1 \ X_2 \ldots \ X_k]^T, X_i \in R^{n_i} \). For any \( i \), consider the vector \( X_i \), any two entries of \( X_i \), say \( x_{p(i)} \), \( x_{q(i)} \), satisfy
\[
\lambda x_{p(i)} = \sum_{r \neq i, r=1}^{k} \Sigma X_r - \Sigma X_i = \lambda x_{q(i)}.
\]

Since, \( \lambda \neq 0 \), \( X_i = \alpha_i 1_{n_i} \) for some constant \( \alpha_i \) for all \( i = 1, \ldots, k \). Setting \( X = [\alpha_1 1_{n_1}^T \alpha_2 1_{n_2}^T \ldots \alpha_k 1_{n_k}^T]^T \), by equation (9) we have
\[
\lambda \alpha_i = \sum_{r \neq i, r=1}^{k} n_r \alpha_r - n_i \alpha_i.
\]

For any \( j \neq i \), similarly, we have
\[
\lambda \alpha_j = \sum_{r \neq j, r=1}^{k} n_r \alpha_r - n_j \alpha_j.
\]
Adding these above two equations, we obtain \( \lambda (\alpha_i - \alpha_j) = 2(n_j \alpha_j - n_i \alpha_i) \) for any \( i, j \in \{1, \ldots, k\} \).

In order to find all \( \alpha_i, i = 1, \ldots, k \) which satisfy equation (10) for each \( i \), it gives the linear system \( N_i \alpha = 0 \). Note that both \( \lambda \) and \( \alpha \) are unknown in this linear system and for the existence of a nonzero solution vector \( \alpha \), we must have \( \det(N_i) = 0 \). Thus, the nonzero eigenvalues of \( A(G) - I_n \) are the roots of the polynomial \( \det(N_i) \). Now from Lemma 4.1, we have

\[
\det(T_\lambda) = \prod_{i=1}^{t} (-2n_i - \lambda)^{m_i} + \sum_{i=1}^{t} \frac{m_i n_i}{-2n_i - \lambda} \prod_{j=1}^{t} (-2n_j - \lambda)^{m_j}
\]

\[
= \prod_{i=1}^{t} (-2n_i - \lambda)^{m_i-1} \left[ \prod_{i=1}^{t} (-2n_i - \lambda) + \sum_{i=1}^{t} m_i n_i \prod_{j=1, j \neq i}^{t} (-2n_j - \lambda) \right].
\]

Hence the proof.

\[
\textbf{Lemma 4.3.} \text{ Let } \lambda_1^* > \lambda_2^* > \ldots > \lambda_{t-1}^* > \lambda_t^* \text{ be the roots of polynomial } 1 + p(\lambda). \text{ Then }
\lambda_1^* > -2n_1 > \lambda_2^* > -2n_2 > \ldots > \lambda_{t-1}^* > -2n_{t-1} > \lambda_t^* > -2n_t.
\]

In general, if \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{k-1} \geq \lambda_k \) are the nonzero eigenvalues of \( A(G) - I_n \), then

\[
\lambda_1 \geq -2n_1 \geq \lambda_2 \geq -2n_2 \geq \lambda_{k-1} \geq -2n_{k-1} \geq \lambda_k \geq -2n_k.
\]

\[
\text{Proof.} \quad \text{Polynomial } p(\lambda) \text{ is continuous and strictly increasing in interval } (-2n_{i+1}, -2n_i). \text{ Also, } \lim_{\lambda \to \infty} p(\lambda) = +\infty \text{ and } \lim_{\lambda \to -2n_i} p(\lambda) = -\infty \text{ for } i = 1, 2, \ldots, t - 1. \text{ Hence using intermediate value theorem there exists a root } \lambda_i^* \text{ of equation } 1 + p(\lambda) = 0 \text{ in interval } (-2n_{i+1}, -2n_i) \text{ for } i = 1, 2, \ldots, t - 1, \text{ satisfying } -2n_i > \lambda_i^* > -2n_{i+1}. \text{ For } i = 1 \lim_{\lambda \to -2n_i} p(\lambda) = -\infty \text{ and } \lim_{\lambda \to \infty} p(\lambda) = 0. \text{ Again, using intermediate value theorem } \lambda_1^* > -2n_1 \text{ which proves (12). Similarly, (13) follows from Theorem 4.2.}
\]

\[
\textbf{Corollary 4.4.} \text{ Let } \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n \text{ be eigenvalues of } A, \text{ and } \alpha_1^* > \alpha_2^* > \ldots > \alpha_{t-1}^* > \alpha_t^* \text{ be its non-zero non-integer eigenvalues. Then,}
\]

1. \[
\alpha_1^* > -2n_1 + 1 > \alpha_2^* > -2n_2 + 1 > \ldots > \alpha_{t-1}^* > -2n_{t-1} + 1 > \alpha_t^* > -2n_t + 1.
\]

2. \[
\alpha_1 \geq -2n_1 + 1 \geq \alpha_2 \geq -2n_2 + 1 \geq \ldots \geq \alpha_{k-1} \geq -2n_{k-1} + 1 \geq \alpha_k \geq -2n_k + 1.
\]

\[
\text{Proof.} \text{ It directly follows from the fact that } \alpha_i = \lambda_i + 1 \forall i \text{ and Lemma 4.3.}
\]

## 5 Regular Star Block Graph

In this section we calculate eigenvalues of \( r \)-regular signed star block graph.

\[
\textbf{Theorem 5.1.} \text{ Let } G \text{ be a } r \text{-regular star block graph having } k \text{ blocks. If } l \text{ number of blocks are negative cliques for } l \leq k, \text{ then}
\]

\[
\phi(G) = l \left( \phi(K_r) \phi(K_{r-1})^{l-1} \phi(K_{r-1})^{k-l} \right) + (k - l) \left( \phi(K_r) \phi(K_{r-1})^{l-1} \phi(K_{r-1})^{k-l-1} \right),
\]

where, \( \phi(K_r) \) denotes characteristic polynomial of a negative clique of order \( r \).

\[
\text{Proof.} \text{ In each linear subdigraphs corresponding Coates digraph of } G, \text{ the cut vertex } v \text{ will associate with linear subdigraphs of exactly one clique. From theorem 1.1}
\]

\[
\phi(G) = l \left( \phi(K_r) \phi(K_{r-1})^{l-1} \phi(K_{r-1})^{k-l} \right) + (k - l) \left( \phi(K_r) \phi(K_{r-1})^{l-1} \phi(K_{r-1})^{k-l-1} \right)
\]
Eigenvalues of $K_n$ are $-1, (n-1)$ while eigenvalues of $\tilde{K}_n$ are $1^{n-1}, 1-n$, with multiplicities $(n-1), 1$, respectively. Hence,

$$\phi(K_n) = (-1 - \lambda)^{n-1}(n - 1 - \lambda), \quad \phi(\tilde{K}_n) = (1 - \lambda)^{n-1}(1 - n - \lambda)$$

$$\phi(G) = l \left( \phi(\tilde{K}_r)\phi(\tilde{K}_{r-1})^{l-1}\phi(K_{r-1})^{k-l-1} \right) + (k - l) \left( \phi(K_r)\phi(K_{r-1})^{k-l-1}\phi(\tilde{K}_{r-1})^{l} \right)$$

$$\phi(G) = \left( (1 - \lambda)^{r-2}(2 - r - \lambda) \right)^{(l-1)} \left( (-1 - \lambda)^{r-2}(r - 2 - \lambda) \right)^{k-l-1}$$

$$l \left( (1 - \lambda)^{r-1}(1 - r - \lambda)(-1 - \lambda)^{r-2}(r - 2 - \lambda) \right) + (k - l) \left( (-1 - \lambda)^{r-1}(r - 1 - \lambda)(1 - \lambda)^{r-2}(2 - r - \lambda) \right)$$

Thus, eigenvalues of $G$ are $1, 2 - r, -1, r - 2$ with multiplicities $(r-2)(l-1)$, $(l-1)$, $(r-1)(k-l-1)$, $(k-l-1)$, respectively and rest of the eigenvalues are given by roots of polynomial $\varphi$.

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