Bounds for the minimum distance and covering radius of orthogonal arrays via their distance distributions

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Abstract—We study the packing and covering properties of orthogonal arrays (OAs) as we propose new approaches for obtaining estimations on the minimum distance and covering radius of orthogonal arrays (designs) via examination of their distance distributions. First, we use some special representation of a linear system of equations which allows us to analyse extreme solutions (distance distributions). Second, we show how databases with (feasible) distance distributions can be used for obtaining sharp bounds for the minimum distance and covering radius of OAs. Correspondingly, new bounds are presented either in analytic form and as products of an ongoing project for computation and investigation of the possible distance distributions of OAs.

Index Terms—Orthogonal Arrays, Distance distributions, Minimum distance, Covering radius

I. INTRODUCTION

The investigation on the covering radius problem for orthogonal arrays was started by Tietäväinen [24], [25] (see also [21], [22]) with obtaining upper bounds for the covering radius of a binary OA as a function of its strength. The relations between the covering radius and the strength (i.e., the dual distance) were further investigated in [16], [17] in 1990’s. The Tietäväinen bound was generalized by Fazekas-Levenshtein [24], [25] with obtaining upper bounds for the covering radius of OAs. Theorem III.2 explains out methodology for obtaining estimations for the targeted parameters. Bounds obtained by investigation of analytical expressions for suitably chosen distance distributions are considered in Section 4. Section 5 presents an approach for derivation of new bounds (and exact values in many cases) based on knowledge of possible sets of distance distributions.

II. PRELIMINARIES

Let $H_q$ be an alphabet of $q$ letters and $H_q^n$ be the Hamming space over $H_q$. Here $q$ is not necessarily a prime power, i.e. we do not need any structure of the alphabet.

Definition II.1. An orthogonal array (OA) of strength $t$ and index $\lambda$ in $H_q^n$ consists of the rows of an $M \times n$ matrix $C$ with the property that every $M \times t$ submatrix of $C$ contains all ordered $t$-tuples of $H_q^n$ each one exactly $\lambda = M/q^t$ times as rows. We denote $C$ by $OA(M, n, q, t)$.

Example II.2. The shortened Hamming code of length 5 is an OA(8, 5, 2, 2) of index $\lambda = 2$:

\[
\begin{align*}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0
\end{align*}
\]

Example II.3. The following is one of the three nonisomorphic versions of OA(18, 7, 3, 2) given in [12]:

The paper is organized as follows. In Section 2 we give the necessary definitions, some examples, and a characterization result by Delsarte which sets the base for our approach. In Section 3 we describe the relations between the distance distributions and the minimum distance and covering radius of OAs. Theorem III.2 explains out methodology for obtaining estimations for the targeted parameters. Bounds obtained by investigation of analytical expressions for suitably chosen distance distributions are considered in Section 4. Section 5 presents an approach for derivation of new bounds (and exact values in many cases) based on knowledge of possible sets of distance distributions.
and the distance distribution w.r.t. points in Example II.3 give the same distance distribution, namely w.r.t. to a point from \( w \)

the cardinality \( C \) computed for given \( M, n, q, t \) and \( t \) as shown next.

**Definition II.4.** Let \( C \) be an \( OA(M, n, t, q) \) and \( x \in H_q^n \). The distance distribution of \( C \) with respect to \( x \) is the \((n+1)\)-tuple

\[
w = w(x) = [w_0(x), w_1(x), \ldots, w_n(x)],
\]

where \( w_i(x) = 1 \{ y \in C \mid d(x, y) = i \} \), \( i = 0, \ldots, n \), and \( d(x, y) \) is the Hamming distance.

The distance distributions of the OAs from Examples II.2 and II.3 are as follows.

**Example II.5.** The eight points of \( C = OA(8, 5, 2, 2) \) give the same distance distribution, namely \([1, 0, 2, 4, 1, 0] \). Eight of the points in \( H_2^8 \setminus C \) give \([0, 1, 4, 2, 0, 1] \), and the remaining sixteen points in \( H_2^8 \setminus C \) give \([0, 2, 2, 2, 2, 0] \). In particular, the distance distribution w.r.t. \((1, 0, 0, 0, 0) \in H_2^8 \setminus C \) is \([0, 1, 4, 2, 0, 1] \) and the distance distribution w.r.t. \((0, 1, 0, 0, 0) \in H_2^8 \setminus C \) is \([0, 2, 2, 2, 2, 0] \).

**Example II.6.** All points of \( C = OA(18, 7, 3, 2) \) from Example II.3 give the same distance distribution, namely \([1, 0, 0, 0, 3, 12, 2, 0] \). There are sixteen distance distributions w.r.t. a point from \( H_3^7 \setminus C \) as follows:

\[
\begin{align*}
0, 0, 0, 0, 14, 0, 0, 4, &; [0, 0, 0, 5, 2, 6, 4, 1], [0, 0, 0, 4, 6, 2, 2], \\
0, 0, 0, 4, 5, 3, 5, 1, &; [0, 0, 0, 3, 7, 3, 3, 2], [0, 1, 0, 3, 2, 2, 1, 1], \\
0, 0, 2, 0, 6, 4, 6, 0, &; [0, 0, 1, 4, 0, 5, 0], [0, 0, 1, 3, 5, 6, 0], \\
0, 0, 1, 3, 2, 8, 3, 1, &; [0, 0, 1, 2, 6, 7, 0], [0, 0, 1, 2, 5, 4, 1], \\
[0, 1, 0, 0, 7, 6, 3, 1], &; [0, 1, 0, 0, 6, 9, 0, 2], [0, 1, 0, 1, 5, 6, 5, 0], \\
0, 1, 0, 1, 4, 9, 2, 1, &; [0, 1, 0, 1, 4, 9, 2, 1].
\end{align*}
\]

In particular, the point \((0, 1, 1, 1, 0, 0) \in H_3^7 \setminus C \) gives

\[
\begin{align*}
0, 0, 0, 0, 14, 0, 0, 4, &; [0, 1, 0, 0, 3, 11, 0, 0], [0, 1, 0, 0, 4, 6, 2, 2], \\
0, 0, 0, 4, 5, 3, 5, 1, &; [0, 0, 0, 3, 7, 3, 3, 2], [0, 1, 0, 0, 3, 2, 2, 1], \\
0, 0, 2, 0, 6, 4, 6, 0, &; [0, 0, 1, 4, 0, 5, 0], [0, 0, 1, 3, 5, 6, 0], \\
0, 0, 1, 3, 2, 8, 3, 1, &; [0, 0, 1, 2, 6, 7, 0], [0, 1, 0, 1, 5, 6, 5, 0], \\
0, 1, 0, 1, 4, 9, 2, 1, &; [0, 1, 0, 1, 4, 9, 2, 1].
\end{align*}
\]

For given parameters \( M, n, q, t \), the distance distributions can be computed in various ways, based on classical results of Delsarte [8], [9] (see also [10], [18]).

**Theorem II.7** (Delsarte [8], [9]). Let \( C \) be an \( OA(M, n, q, t) \) and \( x \in H_q^n \). If \( w(x) = [w_0, w_1, \ldots, w_n] \) is the distance distribution of \( C \) with respect to \( x \), then

\[
\sum_{i=0}^{n} w_i K_k^{(n,q)}(i) = 0, \quad k = 1, \ldots, t,
\]

where

\[
K_k^{(n,q)}(z) := \sum_{j=0}^{i} (-1)^j(q-1)^{i-j} \binom{n-z}{i-j} \in \mathbb{Q}[z],
\]

\( i = 0, 1, \ldots, n \), are the Krawtchouk polynomials [23].

Krawtchouk polynomials are among the standard tools for investigations in coding theory. In packing and covering problems, the universal bounds (13], [16], [17], [24], [25], etc.) from 90’s rely on the properties of the zeros of Krawtchouk and their related polynomials. In our approach, we first utilize the values of Krawtchouk polynomials in the (possible) distances of OAs and then play with integrality conditions.

Assuming the existence of an \( OA(M, n, q, t) \), we observe that the Vandermonde-type system (1) from Theorem II.7 has \( t+1 \) linear equations with rational coefficients and \( n+1 \) unknowns. Therefore, the set of its solutions is a subspace of \( \mathbb{Q}^{n+1} \). Moreover, all admissible values of the entries of the solutions are integers from the set \([0, 1, \ldots, q^n] \). Therefore, there are finitely many possible distance distributions, which can be found from (1) and further investigated. This idea was first applied in [5] and followed in [3], [4], [6], [7], [20] and others. For this investigation, one apply combinatorial relations between an original OA and its derived. These in turn imply further systems (1) already with new parameters. Then the interplay between the old and new distance distributions (plus the integrality conditions) makes difference. In many cases, only a few (sometimes just one) possible distance distributions remain implying tight bounds or even exact values for the minimum distance and the covering radius of the OA under consideration.

### III. DISTANCE DISTRIBUTIONS, MINIMUM DISTANCE AND COVERING RADIUS OF ORTHOGONAL ARRAYS

We need to make difference between the distance distributions with respect to internal \( x \in C \) or external \( x \in H_q^n \setminus C \) points. It is easy to see that if \( x \in C \), then \( w_0(x) \geq 1 \) (with an equality if and only if \( x \) is not repeated in \( C \)) and if \( x \in H_q^n \setminus C \), then \( w_0(x) = 0 \). We denote by

\[
P(C) := \{ p = [p_0(x) \geq 1, p_1(x), \ldots, p_n(x)] : x \in C \}
\]

the set of all distance distributions of \( C \) with respect to \( x \in C \), calling the elements of \( P(C) \) internal distance distributions, and, similarly, by

\[
R(C) := \{ r = [r_0(x) = 0, r_1(x), \ldots, r_n(x)] : x \in H_q^n \setminus C \}
\]

\(^1\)The strength is the maximal possible \( t \).
the set of all distance distributions of $C$ with respect to $x \not\in C$, calling the elements of $R(C)$ external distance distributions. Thus, we use $p$ and $r$ to replace $w$ in order to underline the difference between the internal (in $P(C)$) and external (in $R(C)$) distance distributions. It is clear that $W(C) = P(C) \cup R(C)$ is a disjoint union.

Switching to the existence/classification problem for OAs, in what follows we will consider the sets of distance distributions for given parameters $M, n, q, t$ instead of these of a particular OA. Thus, we will denote and consider the sets $P_f(M, n, q, t)$, $R_f(M, n, q, t)$, and $W_f(M, n, q, t)$, analogs of $P(C)$, $R(C)$, and $W(C)$, as the sets of all feasible distance distributions of (sometimes only putative) OAs having these parameters. The subscript $f$ is set for "feasible" and means that the actual sets of any $OA(M, n, q, t)$ could be subsets not necessarily coinciding with $P_f(M, n, q, t)$, $R_f(M, n, q, t)$, and $W_f(M, n, q, t)$, respectively. The feasible sets of distance distributions can be further reduced by investigations of the relations between hypothetical $OA(M, n, q, t)$ and its related orthogonal arrays as shown in [5]–[7] and others.

Example III.1. For $(M, n, q, t) = (18, 7, 3, 2)$ (see Examples II.3 and II.6), the computation via Theorem II.7 shows that $P_f(18, 7, 3, 2) = \{[1, 0, 0, 0, 3, 12, 2, 0], [1, 0, 0, 1, 0, 15, 1, 0]\}$ and the set $R_f(18, 7, 3, 2)$ contains 41 distance distributions (too many to be shown here). Further investigations show that $[1, 0, 0, 1, 0, 15, 1, 0]$ can be excluded from $P_f(18, 7, 3, 2)$ and, similarly, 9 out of the 41 distributions from the initial $R_f(18, 7, 3, 2)$ cannot be realized.

For an orthogonal array $C$ in $H_q^n$ its minimum distance is

$$d(C) := \min_{x, y \in C, x \neq y} d(x, y),$$

and its covering radius is

$$\rho(C) := \max_{x \in H_q^n} \min_{y \in C} d(x, y),$$

respectively. Given $M, n, q, t, x$ we define the quantities

$$MinD(M, n, q, t) := \min\{d(C) : C \text{ is an OA}(M, n, q, t)\},$$

$$MaxD(M, n, q, t) := \max\{d(C) : C \text{ is an OA}(M, n, q, t)\},$$

and

$$CR(M, n, q, t) = \min\{\rho(C) : C \text{ is an OA}(M, n, q, t)\}.$$}

Simple (i.e., without repeated rows) OAs are also (often very good) error-correcting codes. Thus, it makes sense to consider the quantities like $MaxD(M, n, q, t)$ and $CR(M, n, q, t)$ whose importance is well known from the coding theory. Since OAs are expected to be well distributed in the ambient space $H_q^n$, the investigation of $MinD(M, n, q, t)$ makes sense as well along with $CR(M, n, q, t)$.

As mentioned above, bounds for the minimum distance and covering radius of orthogonal arrays were considered since 1990 in many papers, for example [11], [16], [17], [21], [22], [24], [25], as Tietjävän [24], [25] was the first to investigate the relations between the covering radius and the strength. OAs with the smallest possible covering radius for fixed parameters $M, n, q, t$ are of special interest and were called maximin distance designs and investigated in [11], [15] and others. More references can be found in the book [14].

However, it seems that systematic investigations of the relation between minimum distance and covering radius and the data about distance distributions, given by Delsarte’s Theorem II.7, is not claimed yet. In our tutorial paper [2] we presented many examples and announced some of the bounds from this paper.

We next explain the basic relations between the distance distributions and the minimum distance and covering radius of OAs. It follows immediately from the definitions that if $C$ is an OA with sets of distance distributions $P(C)$ and $R(C)$, then

$$d(C) = 1 + \min_{p \in P(C)} \{i : p_1(x) = \cdots = p_i(x) = 0\},$$

$$\rho(C) = 1 + \max_{r \in R(C)} \{j : r_1(x) = \cdots = r_j(x) = 0\},$$

If the existence of $C$ is undecided, or if there are many possible OAs with the same main parameters, the quantities in the right hand sides of the last two equalities serve as a lower bound for $d(C)$ and a upper bound for $\rho(C)$, respectively. More precisely, we have the following statement.

Theorem III.2. Let $M, n, q, t$ be feasible parameters for orthogonal arrays in $H_q^n$ and let $P_f(M, n, q, t)$ and $R_f(M, n, q, t)$ be sets of feasible internal and external distance distributions of $C$, respectively. Then the following inequalities are valid:

$$MinD(M, n, q, t) \geq 1 + \min\{i : p_1(x) = \cdots = p_i(x) = 0\},$$

(2)

$$MaxD(M, n, q, t) \leq 1 + \max\{i : p_1(x) = \cdots = p_i(x) = 0\},$$

(3)

where both the minimum and maximum are taken over all distributions $p \in P_f(M, n, q, t)$, and

$$CR(M, n, q, t) \leq 1 + \max\{j : r_1(x) = \cdots = r_j(x) = 0\},$$

(4)

where the maximum is taken over all distributions $r \in R_f(M, n, q, t)$.

Proof. Assume that $C \subset H_q^n$ is an OA $(M, n, q, t)$ and $x$ and $y$ are points of $C$ at distance $d(C)$. Then $p_{d(C)−1}(x) = 0$ since there are no points of $C$ at distance less than $d(C)$. This proves (2) and, since it is true also for all points $x \in C$ which do not have $y \in C$ at distance $d(C)$, proves (3) as well.

Considering a point $x \in H_q^n \setminus C$ at distance $\rho(C)$ to $C$, we see that $r_1(x) = \cdots = r_{\rho(C)−1}(x) = 0$, which proves (4). □

Assuming the existence of an $OA(M, n, q, t)$ we can apply Theorem III.2 in two ways. First, we can derive and investigate analytical expressions for specific entries (close to the maximal possible number of initial zeros) of the distance distributions.
Second, we can investigate the data of feasible distance distributions (i.e. sets $P_f(M, n, q, t)$ and $R_f(M, n, q, t)$) to determine minimums and maximums in Theorem III.2. To this end we find some initial $P_f(M, n, q, t)$ and $R_f(M, n, q, t)$ (given by (1) and apply, in the next stages, the techniques from [5]–[7] to reduce the sets $P_f(M, n, q, t)$ and $R_f(M, n, q, t)$, implying this way better bounds via Theorem III.2.

**Example III.3.** It follows from the explanations in Example III.1 that $d(C) = 4$ for every $C = O_A(18, 7, 3, 2)$. Indeed, the only remaining distance distribution in $P_f(18, 7, 3, 2)$ implies that $d(C) = 4$. Therefore

$$MinD(18, 7, 3, 2) = MaxD(18, 7, 3, 2) = 4.$$ Further, the distance distribution $[0, 0, 0, 0, 14, 0, 0, 4] \in R_f(18, 7, 3, 2)$ is realized and gives $d(C) = 4$ for the OA from Example II.6. Since this is the distance distribution in $R_f(18, 7, 3, 2)$ with most zeros in the beginning, it also implies $CR(18, 7, 3, 2) \leq 4$.

In the next two sections we apply these two approaches and obtain some bounds.

**IV. Bounds via analytical expressions of distance distributions**

A. Preliminaries

Manev [20] obtained several different systems all equivalent to the system from Theorem II.7.

**Theorem IV.1.** [20, Theorem 7] Let $C$ be an OA($M, n, q, t$). All distance distributions from the sets $W(C) = P(C) \cup R(C)$ satisfy the system

$$q^n \sum_{i=0}^{n} \binom{i-s}{m} w_i = M \sum_{i=0}^{n} \binom{n}{i} \binom{i-s}{m} (q-1)^i,$$ (5) $m = 0, 1, \ldots, t$, where $s \in \{0, 1, n-t\}$ is fixed.

Suitable choices of $s$ in (5) allow good expressions for important entries of the distance distributions. Using (5) with $s = n-t$ and multiplying both sides by the $(t+1) \times (t+1)$ nonsingular matrix $\left(\left(-1\right)^{i+j} \binom{i}{j}\right)$, $i, j \in \{0, 1, \ldots, t\}$, we obtain the system

$$Bw^T = b := (b_0, b_1, \ldots, b_t)^T.$$ (6)

The entries of the matrix $B = (b_{ij})$ and the vector $b$ were explicitly computed in [3], [4]. These expressions can be used for deriving general upper bounds for the quantities $MaxMD(M, n, q, t)$ and $CR(M, n, q, t)$.

B. Upper bounds for the covering radius

The results in this subsection are not new, but reformulated from [4], where (4) from Theorem III.2 was applied.

It is easy to see that the system (6) has a unique solution whose first $n-t$ coordinates are zeros, namely

$$r = [0, \ldots, 0, b_0, b_1, \ldots, b_t] \in R_f(M, n, q, t).$$ In particular, the exact values of $b_0$ and $b_1$ allow one to derive the upper bound

$$CR(M, n, q, t) \leq n-t$$ (7)

and its strengthening

$$CR(M, n, q, t) \leq n-t-1,$$ (8)

provided $n-t > q-1$ [4, Theorems III.4, III.5]. In the next step, the investigation of the distance distributions of OAs of related parameters, gives the stronger bound

$$CR(M, n, q, t) \leq n-t-2$$ (9)

provided $n > 2(t + q - 1)$ [4, Theorem IV.2]. For each bound (7)-(9), there are cases of attaining.

C. Upper bounds for the minimum distance

Theorem III.2 was applied in [6] for the case $(M, n, q, t) = (96, 10, 2, 4)$, where the set $P_f(96, 10, 2, 4)$ was consecutively reduced to contain only three (out of seventeen initially) distance distributions, all beginning with 1 followed by two zeros, i.e. $MinMD(96, 10, 2, 4) = MaxMD(96, 10, 2, 4) = 3$. However, this gives a contradiction to the Hamming bound and therefore proves that there exist no OA$(96, 10, 2, 4)$.

The bound

$$MaxMD(M, n, q, t) \leq n-t+1$$ (10)

is well known (see, for example, [19]) and is attained by the MDS codes. More precisely, the OAs of index $\lambda = 1$ are exactly the MDS codes (see Theorems 4.20 and 4.21 in [14]). Our first result in this section shows that our technique provides this classical bound.

Again, the most convenient choice of $s$ in Theorem IV.1 is $s = n-t$, giving the system (6). In this case, we are interested in the solutions beginning by 1 followed by several zeros.

**Theorem IV.2.** Let $C$ be an OA($M, n, q, t$) of index $\lambda > 1$ and minimum distance $d(C)$. Then

$$d(C) \leq n-t.$$ (11)

Proof. It is enough to prove that every distance distribution

$$p(x) = [1, 0, \ldots, 0, p_{n-t}, \ldots, p_n], \quad x \in C,$$

has $p_{n-t} \neq 0$. In fact, in this case the system (6) has a unique solution of this type and it is given by

$$p(x) = [1, 0, \ldots, 0, b_0 - b_{0,0}, b_1 - b_{1,0}, \ldots, b_t - b_{t,0}].$$ (12)

Since

$$b_0 - b_{0,0} = \binom{n}{t} (\lambda - 1) > 0,$$

the proof is completed. \[\square\]

**Remark IV.3.** In our interpretation, Theorem IV.2 shows that the condition $\lambda = 1$ is necessary and sufficient for $d(C) = n-t+1$, and otherwise one has $d(C) \leq n-t$. 


Corollary IV.4. Let $M$, $n$, $q$, and $t$ be feasible parameters for OAs in $H_q^n$ of index $\lambda > 1$. Then
\[ \text{MaxMD}(M,n,q,t) \leq n-t. \]

The investigation of the unique solution from the proof of Theorem IV.2 allows obtaining better bounds.

Theorem IV.5. Let $C$ be an $OA(M,n,q,t)$ of index $\lambda > 1$ and minimum distance $d(C)$. If $n-t > \frac{\lambda(q-1)}{\lambda-1}$, then
\[ d(C) \leq n-t-1. \]

Proof. Suppose that $d(C) = n-t$, i.e. the bound of Theorem IV.2 is achieved. Then the only solution of (6) (given by (11))
\[ \text{MinMD}(M,n,q,t) = \text{MaxMD}(M,n,q,t) \]
\[ \text{MinMD}(20,16,2,2) = \text{MaxMD}(20,16,2,2) = 7 \]
\[ \text{MinMD}(128,15,3,4) = \text{MaxMD}(128,15,3,4) = 6. \]

As mentioned above, our bounds for the covering radius are compared to the Fazekas-Levenshtein bounds [13, Theorem 2].

In all completed cases we obtain the same or better bound.

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