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Continuum absorption in the vicinity of the toroidicity-induced Alfvén gap

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Abstract
Excitation of Alfvén modes is commonly viewed as a concern for energetic particle confinement in burning plasmas. The 3.5 MeV alpha particles produced by fusion may be affected as well as other fast ions in both present and future devices. Continuum damping of such modes is one of the key factors that determine their excitation thresholds and saturation levels. This work examines the resonant dissipative response of the Alfvén continuum to an oscillating driving current when the driving frequency is slightly outside the edges of the toroidicity-induced spectral gap. The problem is largely motivated by the need to describe the continuum absorption in the frequency sweeping events. A key element of this problem is the negative interference of the two closely spaced continuum crossing points. We explain why the lower and upper edges of the gap can have very different continuum absorption features. The difference is associated with an eigenmode whose frequency can be arbitrarily close to the upper edge of the gap whereas the lower edge of the gap is always a finite distance away from the closest eigenmode.

1. Introduction
The physics of continuum absorption derives from the classical resonant absorption problem, in which a driven mechanical oscillator absorbs energy efficiently when the frequency of the driving force matches the oscillator’s natural frequency. Continuum absorption is very common for electromagnetic waves in nonuniform media, where the wave frequency can locally match another natural frequency of the medium. A typical example is the magnetic beach [1]. When a low frequency magnetohydrodynamic (MHD) wave propagates along the weakening magnetic field, the local ion cyclotron frequency eventually becomes equal to the wave frequency, resulting in complete absorption of the wave due to ion cyclotron resonance. Another well-known example is the resonant absorption of laser light in a nonuniform plasma target [2].

In fusion devices, continuum absorption is one of the key damping mechanisms that determine excitation thresholds and saturation levels for Alfvén modes driven by energetic ions such as fusion-product alpha particles or fast ions generated via neutral beam injection and rf heating. The potentially unstable Alfvén waves cause undesirable losses of the fast ion population. A careful evaluation of continuum absorption is an essential element of the fast ion stability assessment. It is also essential for understanding nonlinear consequences of the fast ion driven instabilities. The absorption takes place near a magnetic surface where the driving frequency matches the local shear Alfvén frequency $\omega_k = k_\parallel v_A$, where $v_A$ is the local Alfvén velocity. Figure 1 shows a typical radial profile of the shear Alfvén frequency in a tokamak; this profile represents the so-called Alfvén continuum.

The studies of Alfvénic instabilities are largely focused on discrete spectral lines within the frequency gap in the continuum [3–5]. Continuum absorption can then occur when the “tails” of such Toroidicity-induced Alfvén Eigenmodes (TAEs) cross the continuum. In this case, the continuum spectrum near the crossing point can be approximated by a linear function, and the resulting absorption introduces a small damping rate for the mode.
This damping rate can be calculated similarly to Landau damping of plasma oscillations [6], and it is known to be inversely proportional to the slope of the continuum at the crossing point.

However, the constant slope approximation breaks down at the edges of the toroidicity induced gap. The gap forms at \( r = r_m \), where the local dispersion relation is satisfied for the \( m \) and \( m+1 \) poloidal components simultaneously, so that \( \omega = -k_{m+1}v_A = k_mv_A \). At the edges of the gap, the continuum is nearly flat and form two tips. This aspect needs special attention, because the situation is now different from the constant slope case. The need to evaluate continuum absorption at the tips becomes apparent when energetic-particle-driven modes chirp away from the TAE frequency and hit one of the tips. Recalling the constant slope picture, one might then expect a very strong continuum absorption at the tip. Yet, a more careful investigation presented herein shows that this is actually not the case.

In order to solve the tip absorption problem, we use a formalism that probes the MHD response of the plasma to an external current. The external current enters the linearized MHD equations as an oscillating source term, and we examine the response as a function of the source frequency. This source mimics the energetic particle current in the chirping event. In addition, we introduce a small dissipative term that prevents singularity in the MHD response. The dissipative term can be viewed as a friction force acting on the plasma flow. The resulting dissipative power is quite informative: it has a narrow peak inside the gap at the eigenmode frequency, and it represents continuum absorption at other frequencies provided that the friction force is sufficiently small.

We have modified the ideal MHD eigenvalue code adaptive eigenfunction independent solution (AEGIS) [7] to implement this approach numerically. The adaptive grid used in AEGIS and the iterative scheme to search for the continuum crossings assures proper resolution near the tip frequency. Analytically, we choose a low shear setup and calculate the dissipative power by solving the MHD equations for shear Alfvén perturbations via asymptotic matching. Our result shows that continuum absorption vanishes at the lower tip and scales as a square root of the frequency deviation from the tip when the frequency is slightly below the tip. By comparison, the absorption near the upper tip can vary considerably due to an eigenmode that can form arbitrarily close to the upper tip depending on system parameters. These findings agree with our numerical results, and they resolve the outstanding mystery that the two tips have very different absorption features [8].

The paper is organized as follows. Section 2 introduces our basic equations and a reduced version of these equations in the limit of large aspect ratio and low magnetic shear. Section 3 presents an analytical consideration of continuum absorption within the reduced model. The numerical scheme and benchmark of the modified AEGIS code is described in section 4, followed by numerical solution of the unabridged equations. Section 5 summarizes our results.

2. Basic equations and \( \epsilon \) versus \( s \) ordering

Assuming zero compressibility, we use a linearized ideal MHD equation for a cold plasma:

\[
\frac{d^2\mathbf{\xi}}{dt^2} = \left[ \nabla \times \nabla \times (\mathbf{\xi} \times \mathbf{B}) \right] \times \mathbf{B} + \left( \nabla \times \mathbf{B} \right) \times \left[ \nabla \times (\mathbf{\xi} \times \mathbf{B}) \right],
\]

where \( \mathbf{\xi} \) is the perturbed plasma displacement, and \( \mathbf{B} \) is the equilibrium magnetic field. To mimic the energetic particle drive, we introduce an external current \( \delta J_0 \) with a tunable frequency \( \omega \). In what follows, we assume this
current to be localized on a single magnetic surface away from the toroidicity-induced gap. We also add a dissipative term $\mu_d \rho \partial_t \mathbf{d} \xi / \partial t$ to resolve the singularity at the continuum crossing, so that equation (1) now describes a forced oscillating system with frictional damping:

$$\mu_d \rho \partial_t^2 \xi + \mu_d \rho \partial_y^2 \xi = \left[ \nabla \times \nabla \times (\xi \times \mathbf{B}) \right] \times \mathbf{B} + \left[ \nabla \times (\xi \times \mathbf{B}) \right] - \left( \delta \gamma e^{-i \omega t} + c. c. \right) \times \mathbf{B}.$$  

(2)

We use a Fourier representation of the plasma displacement, $\xi_m$, so that

$$\xi = \frac{1}{2} \left( \xi_e e^{-i \omega t} + \xi_s e^{i \omega t} \right).$$

This gives the following expression for the time-averaged power $Q$ dissipated in the plasma volume due to the friction force:

$$Q = \int \rho \gamma v \cdot dV = \int \frac{1}{2} \rho \gamma \omega^2 \xi_e^2 \cdot \xi_s dV.$$  

(3)

Although the power is formally proportional to $\gamma$, it actually remains finite in the limit as $\gamma \to 0$ because $\xi_s$ is large at the continuum crossing points. This allows us to choose a sufficiently small $\gamma$ and scan the frequency to study the continuum absorption near the tip as the frequency changes.

In order to examine the absorption analytically, we consider equation (2) in the large-aspect-ratio ($\epsilon = r/R \ll 1$) and low-magnetic-shear ($s = d(\ln q)/d(\ln r) \ll 1$) limit, which is a common approximation for tokamaks. The asymptotic matching technique of TAE theory [9] will then allow us to evaluate the continuum damping rate for TAEs as well as absorption away from the eigenmode frequency.

To start with, we use the following plasma displacement representation:

$$\xi = \xi_b + \frac{1}{B^2} [\mathbf{B} \times \nabla \Phi] + \frac{1}{B} \nabla \psi - \frac{1}{B} (\mathbf{b} \cdot \nabla \psi),$$

in which $\Phi$ represents the shear Alfvén perturbation and dominates in the shear Alfvén frequency range. The potential, $\Phi$, can be expressed as $\Phi = \exp(-i \omega t + i n q \gamma / \omega) \sum \phi_{m} e^{-i m \theta}$, where $m$ and $n$ are the poloidal and toroidal mode numbers. In the limit of large aspect ratio, low shear and high toroidal mode number, equation (2) reduces to a set of coupled equations for $\phi_m$ and $\phi_{m+1}$, the two dominant poloidal components of $\Phi$, near the gap location $r_m$ [11]:

$$\frac{d}{dy} \left[ \Omega \left( \Omega + i \bar{\gamma} \right) - (y + 1/2)^2 \right] \phi_m - \frac{1}{s^2} \left[ \Omega \left( \Omega + i \bar{\gamma} \right) - (y + 1/2)^2 \right] \phi_m = - \eta \frac{d^2 \phi_{m+1}}{dy^2},$$

$$- \frac{\epsilon - \Delta'}{s} \frac{d \phi_{m+1}}{dy} - \frac{\Delta'}{2s} \phi_{m+1} + \gamma_{m+1} \left( y_0 + 1/2 \right) \delta \left( y - y_0 \right).$$

$$\frac{d}{dy} \left[ \Omega \left( \Omega + i \bar{\gamma} \right) - (y - 1/2)^2 \right] \phi_{m+1} - \frac{1}{s^2} \left[ \Omega \left( \Omega + i \bar{\gamma} \right) - (y - 1/2)^2 \right] \phi_{m+1} = - \eta \frac{d^2 \phi_m}{dy^2},$$

$$+ \frac{\epsilon - \Delta'}{s} \frac{d \phi_m}{dy} - \frac{\Delta'}{2s^2} \phi_m + \gamma_m \left( y_0 - 1/2 \right) \delta \left( y - y_0 \right).$$  

(4)

Here $y = n [q(r) - q(r_m)]$ is the radial variable, $\Omega = \omega / \omega_0$ is the normalized frequency and $\bar{\gamma} = \gamma / \omega_0$ is the normalized friction rate, with $\omega_0 \equiv v_A (r_m) / q(r_m) R_0$. The quantities $\epsilon$, $\Delta'$, and $\eta$ are evaluated at $r_m$, and all three of them have the same order of magnitude: $\epsilon = r / R$ is the inverse aspect ratio; $\Delta'$ is the radial derivative of the Shafranov shift, and $\eta = (\epsilon + \Delta')/2$. Without loss of generality, the external current is assumed to be localized at $y = y_0$ with $y_0 > 0$, and we also assume that the external current flows along the equilibrium magnetic field, with $\delta_{m+1}$ representing the $m$th poloidal component of the current.

It is easy to see that $\Omega = \pm \eta + 1/2$ are the lower and upper edges of the frequency gap in the Alfvén continuum (i.e. the tip frequencies), and that both tips are located at $y = 0$. In the absence of the source terms, these wave equations describe bound states (TAEs) within the gap. We now recall [10–12] and remind some features of the TAEs that are essential for our subsequent steps. The wave equations contain two-dimensionless parameters: the inverse aspect ratio $\epsilon$ and the magnetic shear $s$. Depending on their relative values, the gap accommodates one, two, or multiple eigenmodes. There is only one TAE mode in the gap when $\epsilon \ll s^2$. This mode is symmetric ($\phi_m \approx \phi_{m+1}$) and its frequency is slightly above the lower tip of the gap [11]. The second mode appears when $\epsilon$ becomes comparable to $s^2$. The frequency of this mode lies slightly below the upper tip of the gap, and the mode is antisymmetric ($\phi_m \approx -\phi_{m+1}$) [12]. For even larger values of $\epsilon$, the gap contains multiple (more than two) modes. This happens when $\epsilon$ is comparable to or greater than $s$ [10].

To simplify the subsequent analysis, we restrict ourselves to the case when $\epsilon \ll s$. We thereby exclude multiple modes from our consideration. However, we still intend to consider the $\epsilon \sim s^2$ range, which means...
that we need to take two modes into account: the ever-present symmetric mode near the lower tip and the antisymmetric mode that may emerge near the upper tip. We note that the condition \( \epsilon \ll s \) makes it allowable to neglect the first derivative terms on the righthand side (RHS) of equation (4). Following [11], we introduce the symmetric and antisymmetric combinations: \( S \equiv \phi_m + \phi_{m+1} \) and \( A \equiv \phi_m - \phi_{m+1} \). We also take into account that \( y \ll 1 \), which simplifies equation (4) to

\[
\frac{d}{dy} \left[ \eta (g+1) + iv \right] \frac{dS}{dy} - \frac{1}{s^2} \left[ \eta g + iv + \Delta ' \right] S - y \frac{dS}{dy} = 0.
\]

(5a)

\[
\frac{d}{dy} \left[ \eta (g-1) + iv \right] \frac{dA}{dy} - \frac{1}{s^2} \left[ \eta g + iv - \Delta ' \right] A - y \frac{dA}{dy} = 0.
\]

(5b)

where \( g = \left( 4 \Omega^2 - 1 \right) / 4 \eta \) is the frequency parameter, \( \nu \equiv \Omega i, \beta_j \equiv y_0 (\beta_{j_m} + \beta_{j_{m+1}}) + (\beta_{j_m} - \beta_{j_{m+1}})/2 \), and \( \beta_j \equiv y_0 (\beta_{j_m} - \beta_{j_{m+1}}) + (\beta_{j_m} + \beta_{j_{m+1}})/2 \).

These equations describe two eigenmodes within the gap. The frequency parameter of the lower (nearly symmetric) mode is

\[
g_1 = -1 + \frac{\pi^2 s^2}{8} \left( \frac{2\eta - 2\Delta}{s^2} + 1 \right)^2.
\]

(6)

This mode is always present since \( 2\eta - 2\Delta = \epsilon - \Delta ' \) is positive. By comparison, the antisymmetric eigenmode near upper tip can only exist when \( \epsilon - \Delta ' > s^2 \), and its frequency parameter is:

\[
g_2 = 1 - \frac{\pi^2 s^2}{8} \left( \frac{2\eta - 2\Delta}{s^2} - 1 \right)^2.
\]

(7)

These expressions for \( g_1 \) and \( g_2 \) follow from the discussion of TAEs in [11, 12].

3. Analytical consideration

The limiting case of \( \epsilon \ll s \) involves a separation of scales in the solution of equation (5). The large difference between the outer region (\( |y| \sim s \)) and the inner region (\( |y| \sim \epsilon \)) allows us to connect the outer and inner solutions via asymptotic matching.

We first consider the vicinity of the lower tip (\( |g + 1| \ll 1 \)), and only keep the dominant terms of equation (5) in the inner region (the first term on the LHS and the first term on the RHS). This simplification enables equation of equation (5) to obtain:

\[
\frac{dS}{dy} = \frac{R_1}{2\eta} \frac{y}{\eta (g + 1)} + \frac{R_2}{\eta (g + 1)} + \frac{y}{\eta (g + 1)} + \frac{\nu^2/2\eta}{R_2/2\eta} + \frac{2\eta y}{\eta (g + 1)} + \frac{\nu^2/2\eta}{R_2/2\eta} - \frac{2\eta y}{\eta (g + 1)} + \frac{\nu^2/2\eta}{R_2/2\eta}.
\]

(8)

where \( R_1 \) and \( R_2 \) are the integration constants that remain to be expressed in terms of the external current by means of asymptotic matching. The inner solution gives the dominant contribution to total dissipative power so that one can substitute expressions (8) for \( dS/dy \) and \( dA/dy \) in equation (3) and then evaluate the integral using the residue theorem. The characteristic value of \( y \) in the inner region is \( y \sim \eta \sqrt{|g + 1|} \), which shows that the contribution of \( A \) to \( Q \) can be neglected since \( dS/dy \gg dA/dy \). We thus obtain:

\[
Q \approx \int_{-\infty}^{\infty} \gamma \left| \frac{dS}{dy} \right|^2 dy.
\]

(9)

It follows from equation (8) that \( S \) and \( A \) have jumps, \( \Delta S \) and \( \Delta A \), across the inner layer. Integration of \( dS/dy \) and \( dA/dy \) over a wide symmetric interval covering the layer gives:
\[ \Delta S = \frac{R_2}{y_1}, \]
\[ \Delta A = \frac{\pi R_1}{\sqrt{2\eta}} y_1, \]

where \( y_1 = \sqrt{\eta (g + 1) + iv} \) and the branch of the square root is specified by the condition that the imaginary part of \( y_1 \) is positive.

In the outer region (\( \rho \sim s \)), the first term on the LHS in equation (5a) scales as \( [\eta (g + 1)/y^2] S \), and we observe that this term is much smaller than the second term, which scales as \( (\eta/s^2) S \) when \( |g + 1| \ll 1 \).

Equation (5a) can therefore be simplified to:
\[
\frac{d}{dy} y A - \frac{1}{s^2} y A = \frac{1}{s^2} (\eta - \Delta') S - y \frac{dS}{dy} + \delta_{j_1} \delta (y - y_0). \tag{10}
\]

It now follows from equation (10) that \( S \) is much greater than \( A \), i.e. \( A/S = O(\epsilon/s) \). The LHS of equation (5b) can be estimated as \( (\eta/s^2) A \), and we observe that it can be dropped compared to the RHS, which roughly scales as \( y/S^2 \).

The resulting simplified equation (5b) is
\[
\frac{d}{dy} y S = \frac{1}{s^2} y S = \delta_{j_1} \delta (y - y_0). \tag{11}
\]

The homogenous solutions for equation (11) are the zeroth-order Macdonald's function \( K_0 \) and Bessel function \( I_0 \). Considering the inner solution for \( S \), the coefficient in front of \( K_0 \) must be \( R_1 \) to match \( S y_j y y \) near zero to the inner solution. We also take into account that \( S \) should vanish at infinity. The resulting outer solution for \( S \) is
\[
S = \begin{cases} R_1 K_0(\rho|/s), & y < 0; \\
C_0(\rho|/s) + R_1 K_0(\rho|/s), & 0 < y < y_0; \\
R_1 K_0(\rho|/s) + C K_0(y_0/s), & y > y_0, \end{cases}
\]

where
\[
C = \delta_{j_1} \frac{y_0}{s} K_0(y_0/s).
\]

Note that \( S \) is an almost even function, except for the jump near the origin due to the source: \( \Delta S = C \).

Given the solution for \( S \), we treat the RHS of equation (10) as a source term, \( \hat{L}(S) \), and use the Green's function method to find \( A \) in terms of \( S \):
\[
A = \begin{cases} I_0(\rho|/s) \int_{-\infty}^0 K_0(|x|/s) \hat{L}(S) dx - K_0(\rho|/s) \int_0^\infty K_0(|x|/s) \hat{L}(S) dx - R_2 K_0(\rho|/s), & y > 0; \\
I_0(\rho|/s) \int_{-\infty}^0 K_0(|x|/s) \hat{L}(S) dx - K_0(\rho|/s) \int_0^\infty K_0(|x|/s) \hat{L}(S) dx - R_2 K_0(\rho|/s), & y < 0. \end{cases} \tag{12}
\]

The \( +\infty \) and \( -\infty \) integration limits in these expressions ensure that \( A \) vanishes at infinity, whereas the coefficient \( R_2 \) in front of \( K_0 \) is chosen to match \( dS/dy \) at small values of \( y \) to the inner solution. As seen from equation (12), the jump in \( A \) across the origin is
\[
\Delta A = R_1 \frac{\pi^2 s}{4} \left[ \frac{2\eta - 2\Delta'}{s^2} + 1 \right] + j_-, \]

where
\[
j_- = C \int_0^{y_0} K_0(|x|/s) \left[ \frac{1}{s^2} (\eta - \Delta') - x \frac{d}{dx} \right] \left[ I_0(|x|/s) - \frac{I_0(y_0/s)}{K_0(y_0/s)} K_0(|x|/s) \right] dx
\]
\[
+ \frac{I_0(y_0/s)}{K_0(y_0/s)} \frac{\pi^2 s}{8} \left[ \frac{2\eta - 2\Delta'}{s^2} + 1 \right] + \delta_{j_1} K_0(y_0/s)
\]
describes the contribution of the source.

By equating the jumps in \( S \) in the outer and inner solutions, we find
\[
R_2 = y_1 C.
\]
We also match the values of $\Delta A$ in the inner and outer solutions, which gives

$$R_0 = \frac{\pi^2 s}{4} \left( \frac{2\eta - 2\Delta}{s^2} + 1 \right) - \frac{\pi}{2\eta} \gamma_1.$$  

Substitution of these expressions for $R_1$ and $R_2$ into equation (9) yields

$$Q = \left[ \pi \sqrt{2\eta} C^2 + \frac{j_+^2}{\sqrt{\eta(\gamma_1 + 1) - \gamma_2}} \right] \sqrt{\eta^2 (g + 1)^2 + \nu^2 - \eta(g + 1)},$$  

(13)

where $g_1$, defined by equation (6), is the frequency parameter of the lower TAE.

As seen from equation (13), the dissipative power has a sharp peak inside the gap at the TAE frequency $(g = g_1)$ with $Q \sim 1/\nu$. This feature is characteristic for a simple forced oscillator with small friction, since there is no continuum absorption in the gap.

When the frequency of the source current is somewhat below the gap, the corresponding value of $g$ is less than $-1$, and the quantity $\gamma_1$ is predominantly imaginary. We can then simplify equation (13) to

$$Q \approx \frac{\sqrt{|\eta(g + 1)|}}{\eta(\gamma_1 + 1) + |\eta(g + 1)|} C^2.$$  

(14)

As we scan the source frequency downward from the tip, the quantities $j_+$ and $C$ remain nearly constant (they do not change significantly in the vicinity of the tip). For the first term in equation (14), the denominator contains a constant contribution and the $|\eta(g + 1)|$ term. The constant part is finite at the lower tip, which shows that the total dissipative power $Q$ scales as $\sqrt{|\eta(g + 1)|}$ downward from the lower tip. We can roughly estimate the range of $g$ values for the $\sqrt{|\eta(g + 1)|}$ scaling as:

$$|\eta(g + 1)| \sim \eta(g_1 + 1),$$

$$g > g_L = -1 - \frac{\pi^2 s^2}{8} \left( \frac{2\eta - 2\Delta}{s^2} + 1 \right)^2,$$

and we observe that this range is of the same order as the distance from the lower TAE to the tip. The technique of solving equation (3) for the frequencies near the upper tip is very similar to the lower tip case, but the solution near the upper tip is now dominated by the antisymmetric combination $A$. The resulting dissipative power near the upper tip is given by

$$Q = \left[ \pi \sqrt{2\eta} C^2 + \frac{j_+^2}{\sqrt{\eta(\gamma_1 + 1) - \gamma_2}} \right] \sqrt{\eta^2 (g - 1)^2 + \nu^2 + \eta(g - 1)},$$  

(15)

where

$$C' = \delta j \frac{\nu_0}{s} K_0(\nu_0/s);$$

$$j_+ = C' \int_0^\infty K_0(|x|/s) \left[ \frac{1}{s^2} (\eta - \Delta') + x \frac{d}{dx} \left[ I_0(|x|/s) - \frac{I_0(\nu_0/s)}{K_0(\nu_0/s)} K_0(|x|/s) \right] \right] dx$$

$$+ C' \frac{I_0(\nu_0/s)}{K_0(\nu_0/s)} \frac{\pi^2 s}{8} \left( \frac{2\eta - 2\Delta'}{s^2} - 1 \right) + \delta j K_0(\nu_0/s)$$

and $\gamma_2 = \sqrt{-\eta(g - 1) + \nu^2}$ is chosen in such a way that $\text{Im}(\gamma_2) > 0$. This dissipative power exhibits a peak at $g = g_2$, provided that there is an upper TAE mode in the gap. The absorption otherwise vanishes below the upper tip in the limit of $\nu \to 0$.

Above the upper tip ($g > 1$), equation (15) simplifies to

$$Q \approx \frac{\sqrt{\eta(g - 1)}}{\pi^2 \nu^2} \left( \frac{2\eta - 2\Delta'}{s^2} - 1 \right) j_+^2 + \sqrt{\eta(g - 1)} C^2.$$  

(16)
As we scan the source frequency upward from the upper tip, the range for the $g_1 \langle \rangle h^{-1}$ scaling of $Q$ is

$$g < g_{UL} = 1 + \frac{\pi^2 s^2}{8} \left( \frac{2\eta - 2\Delta}{s^2} - 1 \right)^2,$$

which is of the same order as the frequency difference between the upper TAE and the tip. In contrast with the lower tip case, the threshold frequency can be very close to the tip because $s_2 \langle \rangle h^{-1} \Delta$ may change sign and become very small as parameter changes. In the limiting case when $s_2 \langle \rangle h^{-1} \Delta = 1$, the first term in equation (16) scales as $1/\sqrt{|\eta(g - 1)|}$, and results in a large divergent part $1/\sqrt{|\eta(g - 1)|}$ in the total dissipative power. Figure 2 shows the behavior of the first term in equation (16) versus frequency for various parameters, which demonstrates that the absorption can be large at the tip, and is sensitive to the parameters when $s_2 \langle \rangle h^{-1} \Delta \sim 1$.

We can now summarize the different scenarios for continuum absorption near the two tips in the $\epsilon \ll s$ limit. Near the lower tip, where there is an ever-present neighboring eigenmode that never touches the tip, the absorption always vanishes at the tip. When $\epsilon$ increases, the separation between the eigenfrequency and the lower tip will increase as well as the range for the $\sqrt{|\eta(g - 1)|}$ scaling of $Q$ below the lower tip. For the upper tip, where there is no neighboring eigenmode until $\epsilon$ becomes comparable to $s^2$, the range of the $\sqrt{|\eta(g - 1)|}$ scaling shrinks when $\epsilon$ approaches the mode-existence threshold; consequently, the $\sqrt{|\eta(g - 1)|}$ scaling breaks down and we may expect a large continuum absorption at the tip when the upper eigenmode is just about to appear. As we further increase $\epsilon$, the range of the $\sqrt{|\eta(g - 1)|}$ scaling of $Q$ grows in step with the distance between the eigenmode frequency and the tip. These features are responsible for significant asymmetry in continuum absorption at the tips.

4. Numerical results

4.1. Numerical scheme and benchmark

We use the AEGIS code to study continuum absorption numerically. AEGIS is a linear MHD eigenvalue code with an adaptive mesh in the radial direction (here toroidal magnetic flux $\psi$ is used as the radial coordinate), and Fourier decomposition in the poloidal ($\theta$) and toroidal ($\zeta$) directions. The plasma displacement vector ($\xi$), which is orthogonal to the equilibrium magnetic field under incompressibility condition, is represented by two functions ($\xi_s$ and $\xi_u$) as

$$\xi \times B = \xi_s \nabla \psi + \xi_u \chi'(\nabla \zeta - q \nabla \theta),$$

where $2\pi \chi$ is the poloidal magnetic flux. The perturbed quantities are Fourier transformed as:

$$\xi_{s, u} e^{-im\zeta} = \sum_{m=-\infty}^{\infty} \xi_{s, u} e^{i(m\theta - n\zeta)}.$$

We then use equation (2) to introduce the external current $\delta I_\lambda$ with frequency $\omega$ in AEGIS. Assuming the current to flow on a single magnetic surface $\psi_0$ and being divergence free ($\nabla \cdot \delta I_\lambda = 0$), we express $\xi_s$ in terms of $\xi_u$ and obtain the following set of equations for $\xi_{s, u}$.
\[ \left( F \xi^e_{nm} + K \xi^e_{nm} \right)' - \left( K^\dagger \xi^e_{nm} + G \xi^e_{nm} + n S_m \delta (\psi - \psi_0) \right) = 0, \]  

(17)

where

\[ S_m = \chi' (m - n \eta) \delta_{mn}, \]

represents the external current. The definitions of matrices \(F\), \(K\), and \(G\) are given in [7]. We add a small positive imaginary part \((i \gamma/2)\) to the frequency \(\omega\) in the expressions for \(F\), \(K\), and \(G\), to capture the effect of friction. To solve for \(m_x y\) in equation (17), AEGIS divides the radial computational domain into multiple regions, and matches the independent solutions at the interfaces of these regions. Consequently, \(0_y\) can be set as one of the interfaces between the adjacent regions in AEGIS so that the source contribution will only affect the matching condition across \(0_y\). The adaptive mesh in AEGIS allows us to input the resonance point and pack the nearby grid points exponentially, which ensures proper resolution near the continuum crossing for small values of \(\gamma\).

With the modified code, we first study the continuum absorption of the \(n = 1\) TAE as a test case. Since the continuum crossing is away from the gap in this case, toroidicity-induced coupling can be neglected near the crossing, which means that the matrices \(F\), \(K\), and \(G\) are almost diagonal there. Suppose \(m\)th equation in (17) has singularity \((F_{mm} \sim 0)\) near the crossing. We can then keep the highest derivative term of \(m_x y\) and simplify the equation to

\[ F_{mm} \xi^e_{nm} = C_1, \]

where \(C_1\) is an integration constant. By linear expansion of \(F_{mm}\) at the crossing \(\psi_0\), we find

\[ \xi^e_{nm} = \frac{C_1}{F'_{mm}(\psi - \psi_0) + i \gamma^*}. \]

(18)

Here \(\gamma^*\) is the imaginary part of \(F_{mm}\), which is proportional to \(\gamma\). The total dissipative power is

\[ Q = \int \frac{1}{2} \rho \gamma^2 \xi^e \cdot \xi I d\psi d\zeta d\theta \sim \int \pi \rho \gamma^2 \frac{C_1^2}{F'_{mm}^2(\psi - \psi_0)^2 + \gamma^*^2} d\psi \sim C_i^2 \left( \frac{dF_{mm}}{d\psi} \right)^{-1}, \]

which shows that \(Q\) is inversely proportional to the slope of the continuum spectrum near the crossing when \(\gamma\) is sufficiently small.

We choose a low beta tokamak equilibrium that has nearly circular cross section (see figure 3), with the equilibrium pressure and safety factor plotted in figure 4. By slightly varying the density profile near the plasma edge, the gap can be either open or closed without changing the TAE frequency significantly. Figure 5 shows two density profiles we used (‘density profile I’ and ‘density profile II’) and the corresponding continuum spectra. For ‘density profile I’, the gap is open and we find the \(n = 1\) TAE (whose frequency is labeled in figure 5). On the other hand, ‘density profile II’ closes the gap and introduces continuum absorption at the edge for the TAE. We scan the source frequency for ‘density profile II’ for different values of \(\gamma\) and plot the total dissipative power in figure 6. This figure shows that the dissipative power has a peak near the eigenfrequency, and it converges for small values of \(\gamma\) away from the eigenfrequency, as we expect analytically.
Figure 4. The safety factor and bulk plasma pressure profiles used in AEGIS. The radial variable $\psi$ is the normalized toroidal magnetic flux.

Figure 5. Upper panel: two density profiles used in the simulations. Lower panel: the corresponding $n = 1$ continuum spectra, where the frequency is normalized by the core Alfvén frequency $v_{A}/R_{0}$. The dotted line marks the TAE frequency for 'density profile I'. 
To check how well the mode structure is resolved at the continuum crossing, we plot the plasma response when dissipative power has a peak and compare it to the TAE structure when the gap is open in figure 7. We see that the plasma displacement in the two cases agrees quite well away from the continuum crossing. In addition, we compare the calculated plasma response near the continuum crossing with the analytical solution, which is obtained after integration of equation (18):

$$\frac{\xi_{cm}(\psi^*)}{\xi_{cm}^0} = 2 \left( \frac{dF_{rin}}{d\psi} \right)^{-1} \tan^{-1} \left( \frac{\Delta}{\gamma} \right)$$

The code output is found to be in close agreement with this expression.

Near the eigenfrequency, the bulk plasma response is large and it can still contribute considerably to the total dissipative power even when $\gamma$ is relatively small. As a result, the total dissipative power is greater than the continuum absorption at the crossing. To single out the continuum absorption at eigenfrequency, we define the...
Here we use the fact that the total energy of the mode, \( E_{\text{tot}} \), is contributed equally from the bulk plasma kinetic energy and potential energy, and we choose \( \delta \) to determine the integration limits. For small values of \( \gamma \), the bulk plasma energy is well-separated from the continuum crossing and \( \gamma \). We calculate the continuum damping rate for different values of \( \gamma \) for the \( n = 1 \) TAE. The continuum damping rate converges as \( \gamma \) decreases, and its value agrees with the result obtained via analytic continuation in [13].

### 4.2. Tip absorption results

In the studies of continuum absorption at the tips, there are two closely spaced crossings that are equally important. To resolve the field structure at the two crossings, \( \gamma \) must be very small and the points for grid setting need to be chosen carefully. To meet the numerical requirement, we search for the continuum crossings iteratively. Previous studies of tip absorption in a reversed-shear configuration [8] provide a good guidance for our simulation since the numerical requirements are similar despite the differences in the physics picture.

For better comparison with the analytical result, we study the \( n = 5 \) case using the same tokamak equilibrium (with ‘density profile II’) as in section 4.2. Figure 8 is the corresponding \( n = 5 \) continuum spectrum. We set the boundary at \( \psi = 0.4 \) to focus on the tip absorption in the first gap. In this way the gap is open and TAEs appear both near the upper and the lower tip (labeled in figure 8). The two TAEs are strongly asymmetric (the upper TAE is much closer to the upper tip than the lower TAE to the lower tip).

We scan the source frequency from below the lower tip to the upper tip and plot the total dissipative power in figure 9. Inside the gap, the continuum absorption is almost zero except for the peaks at the TAE eigenfrequencies. Outside the gap, the dissipative power shows good convergence for different values of \( \gamma \), and we observe a significant difference between the upper and lower tips. Figure 10 is a zoom-in of the dissipative power near the lower tip. It shows that the absorption almost vanishes at the tip and follows the \( \sqrt{\Delta \omega} \) scaling below the lower tip, where \( \Delta \omega \) measures the frequency difference from the lower tip and is proportional to the
Figure 9. Plots of the total dissipative power $Q$ versus frequency for different values of $\gamma$ for the $n = 5$ case, from below the lower tip to above the upper tip. The dissipative power is negligible in the gap, except for the TAE peaks. The peaks at the lower TAE ($\omega = 0.437$) are outside the frame. The peak values are $Q = 1333$ for $\gamma = 1 \times 10^{-6}$, $Q = 745$ for $\gamma = 2 \times 10^{-6}$ and $Q = 327$ for $\gamma = 5 \times 10^{-6}$ (they roughly scale as $1/\gamma$). Note that $Q$ is insensitive to $\gamma$ outside the gap.

Figure 10. A zoom of the total dissipative power $Q$ versus frequency near the lower tip for $\gamma = 1 \times 10^{-6}$. The plot shows a clear $\sqrt{\Delta \omega}$ scaling when the frequency is below the lower tip.

Figure 11. A zoom of the total dissipative power $Q$ versus frequency near the upper tip for $\gamma = 1 \times 10^{-6}$, in which the absorption peak in the gap is well-resolved. Above the upper tip, the power has a small range of $\sqrt{\Delta \omega}$ scaling and quickly grows to a finite value.

$|\eta(g + 1)|$ in section 3. Similar zoom-in of the upper tip absorption is presented in figure 11, in which there is a well-resolved absorption peak at the upper TAE frequency. The continuum absorption in this figure has only a very narrow range of validity for the $\sqrt{\Delta \omega}$ scaling (in this case $\Delta \omega$ is the frequency difference from the upper tip and is proportional to the $|\eta(g - 1)|$), and this range is comparable to the frequency separation of the TAE from the tip.

To study the tip absorption as system parameters change, we investigate a set of eight equilibrium cases, which are generated by varying the strength of the equilibrium current. All these equilibrium cases have
negligible pressure and similar safety factor profile. Yet, the safety factor value decreases monotonically from case 1 to case 8, which can be seen from figure 12. Thus, as one moves from case 1 to case 8, the position of the gap moves outwards and the ratio $s^2$ will also decrease monotonically.

With these equilibria, we observe the expected sensitivity of the upper tip absorption to variation of the aspect ratio and magnetic shear. From case 1 to case 8, the frequency interval between the lower TAE and the lower tip doesn’t change significantly. Consequently, the lower tip absorption turns out to be nearly the same. Yet, the upper TAE and the upper tip absorption are much more sensitive. Figure 13 shows the absorption near the upper tip as equilibrium changes. It is apparent that the TAE frequency moves closer to the upper tip and the range of the $\omega_D$ scaling narrows from case 1 to case 4. As the $q$ profile decreases from case 5 to case 8, the TAE disappears and the range of the $\sqrt{\Delta \omega}$ scaling grows. This feature of the upper tip absorption agrees with the analytical expectation presented in figure 2, and it also appears in our simulations for the $n = 1$ case.

5. Summary and discussion

To summarize, we have examined the continuum absorption around the edges of the toroidicity-induced gap in the Alfvén continuum spectrum. To solve the problem, we introduce a driving current and a small dissipative term to the ideal MHD equations.
For an analytical assessment, we reduce the basic MHD equations to a two-poloidal-mode set. Based on earlier work, we show how the inverse aspect ratio $\alpha$ and magnetic shear $s$ cause an asymmetry of the lower and upper TAE frequencies within the gap. Compared to the lower TAE, the existence of the upper TAE is strongly sensitive to $\alpha$ and $s$. This high sensitivity offers an interesting possibility of measuring the relationship between $\alpha$ and $s$ through the presence of the upper TAE.

Using the simplified equation, we show that the continuum absorption always vanishes at the lower tip, and scales as the square root of the downward frequency shift $(\sqrt{\Delta \omega}^D)$ from the lower tip. On the other hand, the continuum absorption above the upper tip usually has a smaller range of $\sqrt{\Delta \omega}$ scaling, and may not vanish at the tip when the neighbouring TAE frequency in the gap is very close to the tip frequency.

We have modified the AEGIS code to study the continuum absorption numerically. With a nearly circular tokamak equilibrium, we vary $\alpha$ and $s$ by changing the safety factor profile. The resulting lower tip absorption does not change dramatically, whereas the upper tip absorption can vary significantly when the $q$ profile changes, which is consistent with our analytical calculation.

These absorption patterns result from an interference of the two poloidal harmonics of the perturbation, a feature that is absent when the tip of the continuum is due to shear reversal (see [8]). The reversed shear tip is formed where $q'_0 = 0$ with finite $q''$, and the continuum absorption is very large at the tip, scaling as $1/\sqrt{\Delta \omega}$ as frequency approaches the tip within the continuum.

It should be noted that we draw our conclusion from the simplified shear Alfvén equations, which assume high $n$, $\epsilon < s$, and low pressure. The pressure gradient tends to shift the TAE frequencies towards the continuum (see [12, 14]), and should affect dissipation at both tips. However, the asymmetry between the lower and upper eigenmodes should still be present in this case, as well as the resulting differences between the upper and lower tip absorption patterns.

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