EFFECT ALGEBRAS ARE THE EILENBERG-MOORE CATEGORY FOR THE KALMBACH MONAD

GEJZA JENČA

Abstract. The Kalmbach monad is the monad that arises from the free-forgetful adjunction between bounded posets and orthomodular posets. We prove that the category of effect algebras is isomorphic to the Eilenberg-Moore category for the Kalmbach monad.

1. Introduction

In [6], Kalmbach proved the following theorem.

Theorem 1. Every bounded lattice $L$ can be embedded into an orthomodular lattice $K(L)$.

The proof of the theorem is constructive, $K(L)$ is known under the name Kalmbach extension or Kalmbach embedding. In [10], Mayet and Navara proved that Theorem 1 can be generalized: every bounded poset $P$ can be embedded in an orthomodular poset $K(P)$. In fact, as proved by Harding in [5], this $K$ is then left adjoint to the forgetful functor from orthomodular posets to bounded posets. This adjunction gives rise to a monad on the category of bounded posets, which we call the Kalmbach monad.

For every monad $(T, \eta, \mu)$ on a category $C$, there is a standard notion Eilenberg-Moore category $C^T$ (sometimes called the category of algebras or the category of modules for $T$). The category $C^T$ comes equipped with a canonical adjunction between $C$ and $C^T$ and this adjunction gives rise to the original monad $T$ on $C$.

In the present paper we prove that the Eilenberg-Moore category for the Kalmbach monad is isomorphic to the category of effect algebras.

2. Preliminaries

We assume familiarity with basics of category theory, see [9, 11] for reference.

2.1. Bounded posets. A bounded poset is a structure $(P, \leq, 0, 1)$ such that $\leq$ is a partial order on $P$, $0, 1 \in P$ are the bottom and top elements of $(P, \leq)$, respectively.

Let $P_1, P_2$ be bounded posets. A map $\phi : P_1 \to P_2$ is a morphism of bounded posets if and only if it satisfies the following conditions.

- $\phi(1) = 1$ and $\phi(0) = 0$.
- $\phi$ is isotone.

The category of bounded posets is denoted by BPos

1991 Mathematics Subject Classification. Primary: 03G12, Secondary: 06F20, 81P10.

Key words and phrases. effect algebra, Kalmbach extension, orthomodular poset, monad.

This research is supported by grants VEGA G-1/0297/11, G-2/0059/12 of MS SR, Slovakia and by the Slovak Research and Development Agency under the contracts APVV-0073-10, APVV-0178-11.
2.2. **Effect algebras.** An effect algebra is a partial algebra \((E; \oplus, 0, 1)\) with a binary partial operation \(\oplus\) and two nullary operations 0, 1 satisfying the following conditions.

(E1) If \(a \oplus b\) is defined, then \(b \oplus a\) is defined and \(a \oplus b = b \oplus a\).

(E2) If \(a \oplus b\) and \((a \oplus b) \oplus c\) are defined, then \(b \oplus c\) and \(a \oplus (b \oplus c)\) are defined and \((a \oplus b) \oplus c = a \oplus (b \oplus c)\).

(E3) For every \(a \in E\) there is a unique \(a' \in E\) such that \(a \oplus a'\) exists and \(a \oplus a' = 1\).

(E4) If \(a \oplus 1\) is defined, then \(a = 0\).

Effect algebras were introduced by Foulis and Bennett in their paper [3].

In an effect algebra \(E\), we write \(a \leq b\) if and only if \(a \oplus c = b\). It is easy to check that for every effect algebra \(E\), \(\leq\) is a partial order on \(E\). Moreover, it is possible to introduce a new partial operation \(\ominus\): \(b \ominus a\) is defined if and only if \(a \leq b\) and then \(a \oplus (b \ominus a) = b\). It can be proved that, in an effect algebra, \(a \oplus b\) is defined if and only if \(a \leq b'\) if and only if \(b \leq a'\). In an effect algebra, we write \(a \perp b\) if and only if \(a \oplus b\) exists.

Let \(E_1, E_2\) be effect algebras. A map \(\phi : E_1 \to E_2\) is called a morphism of effect algebras if and only if it satisfies the following conditions.

- \(\phi(1) = 1\).
- If \(a \perp b\), then \(\phi(a) \perp \phi(b)\) and \(\phi(a \oplus b) = \phi(a) \oplus \phi(b)\).

The category of effect algebras is denoted by \(\mathbf{EA}\). There is an evident forgetful functor \(U : \mathbf{EA} \to \mathbf{BPos}\).

2.3. **D-posets.** In their paper [8], Chovanec and Köpka introduced a structure called D-poset. Their definition is an abstract algebraic version the D-poset of fuzzy sets, introduced by Köpka in the paper [7].

A D-poset is a system \((P; \leq, \ominus, 0, 1)\) consisting of a partially ordered set \(P\) bounded by 0 and 1 with a partial binary operation \(\ominus\) satisfying the following conditions.

(D1) \(b \ominus a\) is defined if and only if \(a \leq b\).

(D2) If \(a \leq b\), then \(b \ominus a \leq b\) and \(b \ominus (b \ominus a) = a\).

(D3) If \(a \leq b \leq c\), then \(c \ominus b \leq c \ominus a\) and \((c \ominus a) \ominus (c \ominus b) = b \ominus a\).

Let \(D_1, D_2\) be D-posets. A map \(\phi : D_1 \to D_2\) is called a morphism of D-posets if and only if it satisfies the following conditions.

- \(\phi(1) = 1\).
- If \(a \leq b\), then \(\phi(a) \leq \phi(b)\) and \(\phi(b \ominus a) = \phi(b) \ominus \phi(a)\).

The category of D-posets is denoted by \(\mathbf{DP}\).

There is a natural, one-to-one correspondence between D-posets and effect algebras. Every effect algebra satisfies the conditions (D1)-(D3). When given a D-poset \((P; \leq, \ominus, 0, 1)\), one can construct an effect algebra \((P; \ominus, 0, 1)\): the domain of \(\ominus\) is given by the rule \(a \perp b\) if and only if \(a \leq 1 \ominus b\) and we then have \(a \ominus b = 1 \ominus (1 \ominus a) \ominus b\). The resulting structure is then an effect algebra with the same \(\ominus\) as the original D-poset. It is easy to see that this correspondence is, in fact, an isomorphism of categories \(\mathbf{DP}\) and \(\mathbf{EA}\).

Another equivalent structure was introduced by Giuntini and Greuling in [4]. We refer to [2] for more information on effect algebras and related topics.
The following lemma collects some well-known properties connecting the $\oplus$, $\ominus$ and $'$ operations in effect algebras (or D-posets). Complete proofs can be found, for example, in Chapter 1 of [2]. We shall use these facts without an explicit reference.

**Lemma 2.**

(a) \( a \leq b' \text{ iff } b \leq a' \text{ iff } a \oplus b \text{ exists and then } (a \oplus b)' = a' \ominus b = b' \ominus a \).

(b) \( a \leq b \text{ iff } a \oplus b' \text{ exists and then } (b \ominus a)' = a \oplus b \).

(c) \( a \leq c \ominus b \text{ iff } b \leq c \ominus a \text{ iff } a \oplus b \leq c \text{ and then } c \ominus (a \oplus b) = (c \ominus a) \ominus b = (c \ominus b) \ominus a \).

(d) \( a \leq b \leq c \text{ iff } a \oplus (c \ominus b) \text{ exists and then } c \ominus (b \ominus a) = a \oplus (c \ominus b) \).

2.4. **Orthomodular posets.** An orthomodular poset is a structure \((A, \leq, \ominus, 0, 1)\) such that \((A, \leq, 0, 1)\) is a bounded poset and $'$ is a unary operation (called orthocomplementation) satisfying the following conditions.

- \( x \leq y \) implies \( y' \leq x' \).
- \( x'' = x \).
- \( x \wedge x' = 0 \).
- If \( x \leq y' \), then \( x \lor y \) exists.
- If \( x \leq y \), then \( x \lor (x \lor y)' = y \).

If \( x \leq y' \), we say that \( x, y \) are **orthogonal**.

Let \( A_1, A_2 \) be orthomodular posets. A map \( \phi : A_1 \to A_2 \) is called a morphism of orthomodular posets if and only if it satisfies the following conditions.

- \( \phi(1) = 1 \).
- If \( a \leq b' \), then \( \phi(a) \leq \phi(b)' \) and \( \phi(a \lor b) = \phi(a) \lor \phi(b) \).

Alternatively, we may define a morphism of orthomodular posets as order preserving, preserving the orthocomplementation, and preserving joins of orthogonal elements.

The category of orthomodular posets is denoted by \( \text{OMP} \). An orthomodular lattice is an orthomodular poset that is a lattice. We remark that the usual category of orthomodular lattices, with morphisms preserving joins and meets is not a full subcategory of \( \text{OMP} \).

If \( A \) is an orthomodular poset, then we may introduce a partial $\oplus$ operation on \( A \) by the following rule: \( x \oplus y \) exists iff \( x \leq y' \) and then \( x \oplus y := x \lor y \). The resulting structure is then an effect algebra. This gives us the object part of an evident full and faithful functor \( \text{OMP} \to \text{EA} \).

2.5. **Kalmbach construction.** If \( C = \{x_1, \ldots, x_n\} \) is a finite chain in a poset \( P \), we write \( C = [x_1 < \cdots < x_n] \) to indicate the partial order.

**Definition 3.** [6] Let \( P \) be a bounded poset, write

\[ K(P) = \{ C : C \text{ is a finite chain in } P \text{ with even number of elements} \} \]

Define a partial order on \( K(P) \) by the following rule:

\[ [x_1 < x_2 < \cdots < x_{2n-1} < x_{2n}] \leq [y_1 < y_2 < \cdots < y_{2k-1} < y_{2k}] \]
if for every $1 \leq i \leq n$ there is $1 \leq j \leq k$ such that

$$y_{2j-1} \leq x_{2i-1} < x_{2i} \leq y_{2j}.$$  

Define a unary operation $C \mapsto C^\perp$ on $K(P)$ to be the symmetric difference with the set $\{0, 1\}$.

Originally, Kalmbach considered the construction only for lattices. If $P$ is a bounded lattice, then $K(P)$ is a lattice as well. Moreover, $(K(P), \land, \lor, ', 0, 1)$ is an orthomodular lattice. However, as observed by Harding in [5], $K$ is not an object part of a functor from the category of bounded lattices to the category of orthomodular lattices.

On the positive side, for any bounded poset $P$, $K(P)$ is an orthomodular poset (see [10]) and $K$ can be made to a functor $K : \text{BPos} \to \text{OMP}$. Indeed, let $f : P \to Q$ be a morphism in $\text{BPos}$ and define $K(f) : K(P) \to K(Q)$ by the following rule.

For an arrow $f : P \to Q$ in $\text{BPos}$, write

$$K(f)([x_1 < x_2 < \cdots < x_{2n-1} < x_{2n}]) = \{y \in Q : \text{card}\{1 \leq i \leq n : f(x_i) = y\} \text{ is odd}\}.$$  

A more elegant way how to write the same rule is

$$K(f)([x_1 < x_2 < \cdots < x_{2n-1} < x_{2n}]) = \Delta_{i=1}^{2n}\{f(x_i)\},$$

where $\Delta$ is the symmetric difference of sets.

Then $K$ is a functor. Moreover, as proved by Harding in [5], $K$ is left-adjoint to the forgetful functor $U : \text{OMP} \to \text{BPos}$. Since every functor has (up to isomorphism) at most one adjoint, this can be viewed as an alternative definition of the Kalmbach construction.

The unit of the $K \dashv U$ adjunction is the natural transformation $\eta : \text{id}_{\text{BPos}} \to UK$, given by the rule

$$\eta_P(a) = \begin{cases} 0 < a & a > 0 \\ \emptyset & a = 0 \end{cases}$$

and the counit of the adjunction is the natural transformation $\epsilon : KU \to \text{id}_{\text{OMP}}$ given by the rule

$$\epsilon_L([x_1 < \cdots < x_{2n}]) = (x_1^+ \land x_2) \lor \cdots \lor (x_{2n-1}^+ \land x_{2n}).$$

3. Kalmbach monad

Let $C$ be a category. A monad on $C$ can be defined as a monoid in the strict monoidal category of endofunctors of $C$. Explicitly, a monad on $C$ is a triple $(T, \eta, \mu)$, where $T : C \to C$ is an endofunctor of $C$ and $\eta, \mu$ are natural transformations $\eta : \text{id}_C \to T$, $\mu : T^2 \to T$ satisfying the equations $\mu \circ T\mu = \mu \circ \mu T$ and $\mu \circ T\eta = \mu \circ \eta T = 1_T$.

Every adjoint pair of functors $F : C \rightleftarrows D : G$, with $F$ being left adjoint, gives rise to a monad $(FG, \eta, \mu G)$ on $C$.

Let $(T, \eta, \mu)$ be a monad on a category $C$. Recall, that the Eilenberg-Moore category for $(T, \eta, \mu)$ is a category (denoted by $C^T$), such that objects (called algebras for that monad) of $C^T$ are pairs $(A, \alpha)$, where $\alpha : T(A) \to A$, such that the
diagrams

(1) \[
\begin{array}{ccc}
A & \xrightarrow{\eta_A} & T(A) \\
& _{1_A} \searrow & \downarrow \alpha \\
& & A
\end{array}
\]

(2) \[
\begin{array}{ccc}
T^2(A) & \xrightarrow{T(\alpha)} & T(A) \\
\mu_A & \searrow & \downarrow \alpha \\
T(A) & \xrightarrow{\alpha} & A
\end{array}
\]

commute. A morphism of algebras \(h : (A_1, \alpha_1) \to (A_2, \alpha_2)\) is a \(C\)-morphism such that the diagram

\[
\begin{array}{ccc}
T(A) & \xrightarrow{T(h)} & T(B) \\
\alpha_1 & \downarrow & \alpha_2 \\
A & \xrightarrow{h} & B
\end{array}
\]

commutes.

Consider now the adjunction \(K \dashv U\) between the categories, \(\text{BPos}\) and \(\text{OMP}\) from the preceding section. This adjunction gives rise to a monad on \(\text{BPos}\), which we will denote \((T, \eta, \mu)\). Explicitly, \(T = U \circ K\), \(\eta\) remains the same and \(\mu = U\epsilon K\) turns out to be given by the following rule

\[
\mu_P\left([C_1 < C_2 < \cdots < C_{2n}]\right) = \Delta^{2n}_{i=1} C_i,
\]

where \([C_1 < C_2 < \cdots < C_{2n}]\) is a chain of even length of chains of even length, that is, an element of \(T^2(P)\), and \(\Delta\) is the symmetric difference of sets.

**Theorem 4.** The category of effect algebras is isomorphic to the Eilenberg-Moore category for the Kalmbach monad.

**Proof.** From now on, let \(U\) be the forgetful functor \(U : \text{EA} \to \text{BPos}\). Let us define a functor \(G : \text{EA} \to \text{BPos}^T\). For an effect algebra \(A\), define \(m_A : T(U(A)) \to U(A)\) by the rule

\[
m_A\left([x_1 < x_2 < \cdots x_{2n-1} < x_{2n}]\right) = (x_2 \oplus x_1) \oplus \cdots \oplus (x_{2n} \oplus x_{2n-1}).
\]

We claim that \(G(A) = (U(A), m_A)\) is an algebra for the Kalmbach monad. We need to prove that the diagrams (1) and (2) commute. Clearly, for every \(x \in U(A)\),

\[
(m_A \circ \eta_U(A))(x) = m_A([0 < x]) = (x \oplus 0) = x,
\]

and we see that the triangle diagram (1) commutes. Consider now the square diagram (2): the elements of \(T^2(U(A))\) are chains of chains of elements of \(U(A)\); let \([C_1 < C_2 < \cdots < C_{2n}] \in T^2(U(A))\). Note that \(C_i < C_j\) implies that \(m_A(C_i) < m_A(C_j)\), so the elements of the sequence \((m_A(C_1), \ldots, m_A(C_{2n}))\) are pairwise distinct. Therefore,

\[
(m_A \circ T(m_A))\left([C_1 < C_2 < \cdots < C_{2n}]\right) = m_A\left([m_A(C_1) < m_A(C_2) < \cdots m_A(C_{2n})]\right) = (m_A(C_2) \oplus m_A(C_1)) \oplus \cdots \oplus (m_A(C_{2n}) \oplus m_A(C_{2n-1}))
\]
Note that, if $C < D$ in $T(U(A))$, then $C \Delta D < D$, $m_A(C) < m_A(D)$ and $m_A(D) \ominus m_A(C) = m_A(C \triangle D)$. Using these facts,
\[
(m_A \circ \mu_{U(A)})\{(C_1 < C_2 < \cdots < C_{2n})\} = m_A(C_1 \Delta C_2 \Delta \cdots \Delta C_{2n}) = \\
m_A((C_1 \Delta C_2 \Delta \cdots \Delta C_{2n-1}) \Delta C_{2n}) = \\
m_A(C_{2n}) \ominus m_A(C_1 \Delta C_2 \Delta \cdots \Delta C_{2n-1}) = \\
m_A(C_{2n}) \ominus (m_A(C_{2n-1}) \ominus m_A(C_1 \Delta C_2 \Delta \cdots \Delta C_{2n-2})) = \\
(m_A(C_{2n}) \ominus m_A(C_{2n-1})) \ominus m_A(C_1 \Delta C_2 \Delta \cdots \Delta C_{2n-2}).
\]

The desired equality now follows by a simple induction.

If $f : A \to B$ is a morphism of effect algebras, we define $G(f) = U(f)$. We need to prove that the diagram
\[
\begin{array}{ccc}
TU(A) & \xrightarrow{TU(f)} & TU(A) \\
m_A & \downarrow & m_B \\
U(A) & \xrightarrow{U(f)} & U(B)
\end{array}
\]

commutes. After some simple steps, this reduces to the following equality in $B$:
\[
(3) \quad (f(x_2) \ominus f(x_1)) \ominus \cdots \ominus (f(x_{2n}) \ominus f(x_{2n-1})) = m_B(\Delta_{i=1}^n\{f(x_i)\}),
\]
for each $[x_1 < x_2 < \cdots < x_{2n}] \in T(U(A))$.

Let us define an auxiliary function $k : T(U(A)) \to \mathbb{N}$: for $C = [x_1 < x_2 < \cdots < x_{2n}] \in T(U(A))$, $k(C)$ is the number of equal consecutive pairs in the sequence $(f(x_1), \ldots, f(x_n))$, that means, $k(C)$ is the cardinality of the set $\{i : f(x_i) = f(x_{i+1})\}$.

To prove the equality (3), we use induction with respect to $k(C)$. If $k(C) = 0$, then the equality (3) clearly holds.

If $k(C) > 0$, then let us pick some $i$ with $f(x_i) = f(x_{i+1})$. Then $\{f(x_i)\} \Delta \{f(x_{i+1})\} = \emptyset$ and we may skip them on the right hand side of (3).

If $i$ is odd, then $f(x_{i+1}) \ominus f(x_i) = 0$ and we may delete that term from the left-hand side of (3). If $i$ is even, then
\[
(f(x_i) \ominus f(x_{i-1})) \ominus (f(x_{i+2}) \ominus f(x_{i+1})) = f(x_{i+2}) \ominus f(x_{i-1})
\]
and we may simplify the left-hand side of (3) accordingly.

So (3) is true if and only if it is true for the chain $C - \{x_i, x_{i+1}\}$. Clearly, $k(C - \{x_i, x_{i+1}\}) = k(C) - 1$ and we have completed the induction step.

Let $(A, \alpha)$ be an algebra for the Kalmbach monad. Let us define a partial operation $\ominus$ on the bounded poset $A$ given by this rule: $b \ominus a$ is defined if and only if $a \leq b$ and
\[
\begin{array}{c|c}
b \ominus a & a = b \\
& a < b
\end{array}
\]
We claim that $E(A, \alpha) = (A, \leq, \ominus, 0, 1)$ is then a D-poset, hence an effect algebra.

The axiom (D1) follows by definition.

Before we prove the other two axioms, let us note that for all $a \in A$, $a \ominus 0 = a$. Indeed, if $0 < a$ then the triangle diagram (1) implies that $a = \alpha([0 < a]) = a \ominus 0$ and for $a = 0$ we obtain $a \ominus 0 = 0 \ominus 0 = 0$ by definition of $\ominus$.

To prove (D2), let $a, b \in A$ be such that $a \leq b$. 

}\]
Let us prove that $b \circ a \leq b$. If $a < b$, then $[a < b] \leq [0 < b]$ in the poset $T(A)$ and
\[ b \circ a = \alpha([a < b]) \leq \alpha([0 < b]) = b \circ 0 = b. \]
If $a = b$ then $b \circ a = 0 \leq b$.

Let us prove that $b \circ (b \circ a) = a$. There are three possible cases.

(D2.1) Suppose that $0 < a < b$. Then, $[a < b] < [0 < b]$ in $T(A)$ and hence $[a < b] \cdot [0 < b] \in T^2(A)$. Suppose that $\alpha([a < b]) = \alpha([0 < b])$. From the commutativity of the square we obtain
\[
\begin{array}{c}
[a < b] < [0 < b] \xrightarrow{T(\alpha)} 0 \\
\mu_A \\
\end{array}
\]
However, $0 < a = \alpha([0 < a]) = 0$ is false and we have proved that $\alpha([a < b]) < \alpha([0 < b])$. Chasing the element $[a < b] < [0 < b]$ around the square
\[
\begin{array}{c}
[a < b] < [0 < b] \xrightarrow{T(\alpha)} [\alpha([a < b]) < \alpha([0 < b])] \\
\mu_A \\
\end{array}
\]
gives us the equality in the bottom right corner, meaning that $b \circ (b \circ a) = a$.

(D2.2) Suppose that $0 = a$. We already know that $b \circ 0 = b$ and we may compute
\[ b \circ (b \circ a) = b \circ (b \circ 0) = b \circ b = 0 = a. \]

(D2.3) Suppose that $a = b$. Then
\[ b \circ (b \circ a) = b \circ 0 = b = a. \]

To prove (D3), let $a, b, c \in A$ be such that $a \leq b \leq c$.

Let us prove that $c \circ b \leq c \circ a$. If $a = b$, there is nothing to prove. If $b = c$, then $b \circ c = 0 \leq c \circ a$. Assume that $a < b < c$. Then $[b < c] < [a < c]$ and
\[ c \circ b = \alpha([b < c]) \leq \alpha([a < c]) = c \circ a. \]

Let us prove that $b \circ a = (c \circ a) \circ (c \circ b)$.

(D3.1) Suppose that $a < b < c$ and assume that $\alpha([b < c]) = \alpha([a < c])$. The square
\[
\begin{array}{c}
[b < c] < [a < c] \xrightarrow{T(\alpha)} 0 \\
\mu_A \\
\end{array}
\]
gives us $\alpha([a < b]) = 0$, so $b \circ a = 0$. However, using only the properties of $\circ$ we already proved,
\[ b = b \circ 0 = b \circ (b \circ a) = a < b, \]
which is false. Thus, assuming \( \alpha([b < c]) < \alpha([a < c]) \) the square

\[
\begin{array}{ccc}
[b < c] < [a < c] & \xrightarrow{T(\alpha)} & \alpha([b < c]) < \alpha([a < c]) \\
\mu_A & \downarrow & \downarrow \alpha \\
[a < b] & \xrightarrow{\alpha} & \alpha([a < b]) = \alpha\left(\alpha([b < c]) < \alpha([a < c])\right)
\end{array}
\]

gives us the equality in the bottom right corner meaning that

\[
b \ominus a = (c \ominus a) \ominus (c \ominus b).
\]

(D3.2) Suppose that \( b = c \). Then

\[
(c \ominus a) \ominus (c \ominus b) = (c \ominus a) \ominus 0 = c \ominus a = b \ominus a.
\]

(D3.3) If \( a = b \), then there is nothing to prove.

If it is now easy to check that an arrow \( h : (A, \alpha) \rightarrow (B, \beta) \) in \( \text{BPos}^T \) is, at the same time, a morphism of D-posets (and thus, a morphism of effect algebras) \( E(A, \alpha) \rightarrow E(B, \beta) \). Indeed, \( h(0) = 0 \) and \( h(1) = 1 \). If \( a < b \) and \( h(a) < h(b) \), then

\[
h(b \ominus a) = h(\alpha([a < b])) = \beta(T(h)([a < b])) = \beta([h(a) < h(b)]) = h(b) \ominus h(a).
\]

If \( a < b \) and \( h(a) = h(b) \) then

\[
h(b \ominus a) = h(\alpha[a < b]) = \beta(T(h)([a < b])) = \beta(\emptyset) = 0 = h(b) \ominus h(a).
\]

Therefore \( E \) is a functor from \( \text{BPos}^T \) to \( \text{EA} \).

It remains to prove that \( E, G \) are mutually inverse functors. Let \( A \) be an effect algebra. We claim that \( EG(A) = A \). The underlying poset of \( EG(A) \) and \( A \) is the same. For all \( a < b \),

\[
b \ominus_{EG(A)} a = m_A([a < b]) = b \ominus_A a,
\]

hence \( EG(A) = A \). It is obvious that for every morphism \( f : A \rightarrow B \) of effect algebras \( EG(f) = f \), since both \( E \) and \( G \) preserve the underlying poset maps.

Let \( (A, \alpha) \) be an algebra for the Kalmbach monad. We claim that \( (A, \alpha) = GE(A, \alpha) \), that means, for all \( [x_1 < \cdots < x_{2k}] \in T(A) \),

\[
(4) \quad \alpha([x_1 < x_2 < \cdots < x_{2k-1} < x_{2k}]) = (x_2 \ominus x_1) \oplus \cdots \oplus (x_{2k} \ominus x_{2k-1}),
\]

where the \( \oplus, \ominus \) on the right-hand side are taken in \( E(A, \alpha) \).

To prove (4) we need an auxiliary claim: for every \( C \in T(A) \) and an upper bound \( u \) of \( C \) with \( u \notin C \), \( \alpha(C \Delta [0 < u]) = \alpha([\alpha(C) < u]) \). This is easily seen by chasing the element \( [C < [0 < u]] \in T^2(A) \) around the square (4).

Clearly, equality (4) is true for \( k = 0 \). Suppose it is valid for some \( k = n \in \mathbb{N} \). Then, for \( k = n + 1 \), equality (4) is then equivalent to

\[
\alpha([x_1 < x_2 < \cdots < x_{2n-1} < x_{2n} < x_{2n+1} < x_{2n+2}]) = \\
\alpha([x_1 < x_2 < \cdots < x_{2n-1} < x_{2n}]) \oplus (x_{2n+2} \ominus x_{2n+1})
\]
Put $C = [x_1 < x_2 < \cdots < x_{2n-1} < x_{2n}]$. Using the definition of $\ominus$ in $E(A, \alpha)$ and applying the auxiliary claim twice we obtain

\[
\begin{align*}
\alpha(C) \oplus (x_{2n+2} \ominus x_{2n+1}) &= \\
x_{2n+2} \ominus (x_{2n+1} \ominus \alpha(C)) &= \\
x_{2n+2} \ominus \left( \alpha([\alpha(C) < x_{2n+1}]) \right) &= \\
x_{2n+2} \ominus \left( \alpha(C \Delta [0 < x_{2n+1}]) \right) &= \\
\alpha\left( \left[ \alpha(C \Delta [0 < x_{2n+1}]) < x_{2n+2} \right] \right) &= \\
\alpha\left( [C \Delta [0 < x_{2n+1}] \Delta [0 < x_{2n+2}]] \right) &= \\
\alpha\left( [C \Delta [x_{2n+1} < x_{2n+2}]] \right) &= \\
\alpha([x_1 < x_2 < \cdots < x_{2n-1} < x_{2n} < x_{2n+1} < x_{2n+2}]) &=
\end{align*}
\]

It is obvious that for every morphism $f$ in $\text{BPos}^T$, $GE(f) = f$. □

References

[1] Awodey, S.: Category theory. No. 49 in Oxford Logic Guides. Oxford University Press (2006)
[2] Dvurečenskij, A., Pulmannová, S.: New Trends in Quantum Structures. Kluwer, Dordrecht and Ister Science, Bratislava (2000)
[3] Foulis, D., Bennett, M.: Effect algebras and unsharp quantum logics. Found. Phys. 24, 1325–1346 (1994)
[4] Giuntini, R., Greuling, H.: Toward a formal language for unsharp properties. Found. Phys. 19, 931–945 (1989)
[5] Harding, J.: Remarks on concrete orthomodular lattices. International Journal of Theoretical Physics 43(10), 2149–2168 (2004)
[6] Kalmbach, G.: Orthomodular lattices do not satisfy any special lattice equation. Archiv der Mathematik 28(1), 7–8 (1977)
[7] Kôpka, F.: D-posets of fuzzy sets. Tatra Mt. Math. Publ. 1, 83–87 (1992)
[8] Kôpka, F., Chovanec, F.: D-posets. Math. Slovaca 44, 21–34 (1994)
[9] Mac Lane, S.: Categories for the Working Mathematician. No. 5 in Graduate Texts in Mathematics. Springer-Verlag (1971)
[10] Mayet, R., Navara, M.: Classes of logics representable as kernels of measures. Contributions to General Algebra 9, 241–248 (1995)

Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak Technical University, Radlinského 11, Bratislava 813 68, Slovak Republic
E-mail address: gejza.jenca@stuba.sk