Remarks on an Article by Rabern et al. *

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Abstract

We show that conjecture 15 in [RRM13] is wrong, comment on theorem 24 in [RRM13], and conclude with some remarks on structures similar to the Yablo construction.

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1 Introduction

This paper is a footnote to [RRM13]. [RRM13] is perhaps best described as a graph theoretical analysis of Yablo’s construction, see [Yab82]. We continue this work.

To make the present paper self-contained, we repeat the definitions of [RRM13]. To keep it short, we do not repeat ideas and motivations of [RRM13]. Thus, the reader should probably be familiar with or have a copy of [RRM13] ready.

All graphs etc. considered will be assumed to be cycle-free, unless said otherwise.

1.1 Overview

(1) Section 1.2 (page 2) contains most of the definitions we use, many are taken from [RRM13].

(2) In Section 2 (page 4), we show that conjecture 15 in [RRM13] is wrong. This conjecture says that a directed graph $G$ is dangerous iff every homomorphic image of $G$ is dangerous. (The definitions are given in Definition 1.1 (page 2), (3) and (11).)

To show that the conjecture is wrong, we modify the Yablo construction, see Definition 1.3 (page 3), slightly in Example 2.1 (page 4), illustrated in Diagram 2.1 (page 6), show that it is still dangerous in Fact 2.2 (page 5), and collaps it to a homomorphic image in Example 2.2 (page 5). This homomorphic image is not dangerous, as shown in Fact 2.1 (page 4).

(3) In Section 3 (page 7), we discuss implications of Theorem 24 in [RRM13] - see the paragraph immediately after the proof of the theorem in [RRM13]. This theorem states that an undirected graph $G$ has a dangerous orientation iff it contains a cycle. (See Definition 1.1 (page 2) (4) for orientation.)

We show that for any simply connected directed graph $G$ - i.e., in the underlying undirected graph $U(G)$, from any two vertices $X, Y$, there is at most one path from $X$ to $Y$, see Definition 3.2 (page 7) - and for any denotation $d$ for $G$, we find an acceptable valuation for $G$ and $d$.

The proof consists of a mixed induction, successively assigning values for the $X$, and splitting up the graph into ever smaller independent subgraphs. The independence of the subgraphs relies essentially on the fact that $G$ (and thus also all subgraphs of $G$) is simply connected.

(4) In Section 4 (page 8), we discuss various modifications and generalizations of the Yablo structure. Remark 4.1 (page 9) illustrates the argument in the Yablo structure, Example 4.1 (page 9) considers trivial modifications of the Yablo structure. In Fact 4.2 (page 9) we show that infinite branching is necessary for a graph being dangerous, and Example 4.2 (page 10) shows why infinitely many finitely branching points cannot replace infinite branching - there is an infinite “procrastination branch”.

Finally, we define a generalization of the Yablo structure in Definition 4.1 (page 9), a transitive graph, with all $d(X)$ of the form $\bigwedge\{\neg X_i : i \in I\}$.

Our main result here is in Fact 4.3 (page 12), where we show that in Yablo-like structures, the existence of an acceptable valuation is strongly related to existence of successor nodes, where $X'$ is a successor of $X$ in a directed graph $G$, iff $X \rightarrow X'$ in $G$, or, written differently, $XX' \in E(G)$, the set of edges in $G$.

1.2 Some definitions

Notation and definitions are taken mostly from [RRM13].

Definition 1.1

(1) Given a (directed or not) graph $G$, $V(G)$ will denote its set of vertices, $E(G)$ its set of edges. In a directed graph, $xy \in E(G)$ will denote an arrow from $x$ to $y$, which we also write $x \rightarrow y$, if $G$ is not directed, just a line from $x$ to $y$.

We often use $x, y$, or $X, Y$, etc. for vertices.
(2) A graph $G$ is called transitive iff $xy, yz \in E(G)$ implies $xz \in E(G)$.

(3) Given two directed graphs $G$ and $H$, a homomorphism from $G$ to $H$ is a function $f : V(G) \rightarrow V(H)$ such that, if $xy \in E(G)$, then $f(x)f(y) \in E(H)$.

(4) Given a directed graph $G$, the underlying undirected graph is defined as follows: $V(U(G)) := V(G)$, $xy \in E(U(G))$ iff $xy \in E(G)$ or $yx \in E(G)$)), i.e., we forget the orientation of the edges. Conversely, $G$ is called an orientation of $U(G)$.

(5) $S$, etc. will denote the set of propositional variables of some propositional language $\mathcal{L}$, $S^+$, etc. the set of its formulas. $\top$ and $\bot$ will be part of the formulas.

(6) Given $\mathcal{L}, v$ will be a valuation, defined on $S$, and extended to $S^+$ as usual - the values will be $\{0, 1\}$, $\{\top, \bot\}$, or so. $[s]_v$, $[\alpha]_v$ will denote the valuation of $s \in S$, $\alpha \in S^+$, etc. When the context is clear, we might omit the index $v$.

(7) $d$ etc. will be a denotation assignment, or simply denotation, a function from $S$ to $S^+$.

(8) A valuation $v$ is acceptable on $S$ relative to $d$, iff for all $s \in S$, $[s]_v = [d(s)]_v$, i.e., iff $[s \leftrightarrow d(s)]_v = \top$. (When $S$ and $d$ are fixed, we just say that $v$ is acceptable.)

(9) A system $(S, d)$ is called paradoxical iff there is no $v$ acceptable for $S$, $d$.

(10) Given $S, d$, we define $G_{S,d}$ as follows: $V(G_{S,d}) := S$, $ss' \in E(G_{S,d})$ iff $s' \in S$ occurs in $d(s)$.

(11) A directed graph $G$ is dangerous iff there is a paradoxical system $(S, d)$, such that $G$ is isomorphic to $G_{S,d}$.

**Definition 1.2**

Let $G$ be a directed graph, $x, x' \in V(G)$.

1. $x'$ is a successor of $x$ iff $xx' \in E(G)$.

   $\text{succ}(x) := \{x' : x'$ is a successor of $x\}$,

2. Call $x'$ downward from $x$ iff there is a path from $x$ to $x'$, i.e. $x'$ is in the transitive closure of the succ operator.

3. Let $[x \rightarrow]$ be the subgraph of $G$ generated by $\{x\} \cup \{x' : x'$ is downward from $x\}$, i.e. $V([x \rightarrow]) := \{x\} \cup \{x' : x'$ is downward from $x\}$, and $x' \rightarrow x'' \in E([x \rightarrow])$ iff $x', x'' \in V([x \rightarrow])$, and $x'x'' \in E(G)$.

**Definition 1.3**

For easier reference, we define the Yablo structure, see e.g. [RRM13].

Let $V(G) := (Y_i : i < \omega)$, $E(G) := \{Y_i Y_j : i, j < \omega, i < j\}$, and $d(Y_i) := \bigwedge\{\neg Y_j : i < j\}$.

($Y_i \in S$ for a suitable language.)

**Definition 1.4**

1. Call a denotation $d \bigwedge \neg$ or $\bigwedge \neg$ iff all $d(X)$ have the form $d(X) = \bigwedge\{-X_i : i \in I\}$ - as in the Yablo structure.

2. The dual notation $\bigwedge +$ expresses the analogous case with $+$ instead of $\neg$, i.e. $d(X) = \bigwedge\{X_i : i \in I\}$.

3. We will use $\neg$ and $-$ for negation, and $+$ when we want to emphasize that a formula is not negated.

**Remark 1.1**

Note that we interpret $\bigwedge$ in the strict sense of $\forall$, i.e., $\neg \bigwedge \{-X_i : i \in I\}$ means that there is at least one $X_i$ which is true. In particular, if $d(X) = \bigwedge\{-X_i : i \in I\}$, and $[X] = [d(X)] = \bot$, then $d(X)$ must contain a propositional variable, i.e. cannot be composed only of $\bot$ and $\top$, so there is some arrow $X \rightarrow X'$ in the graph.
Thus, if in the corresponding graph \( \text{succ}(X) = \emptyset \), \( d \) is of the form \( \bigwedge \), \( v \) is an acceptable valuation for \( d \), then \([X]_v = \top\).

Dually, for \( \bigvee +, \neg \bigwedge \{X_i : i \in I\} \) means that there is at least one \( X_i \) which is false.

Thus, if in the corresponding graph \( \text{succ}(X) = \emptyset \), \( d \) is of the form \( \bigvee \), \( v \) is an acceptable valuation for \( d \), then \([X]_v = \bot\).

2 A comment on conjecture 15 in [RRM13]

We show in this section that conjecture 15 in [RRM13] is wrong.

Definition 2.1

Call \( X \subseteq \mathbb{Z} \) (the integers) contiguous iff for all \( x, y, z \in \mathbb{Z} \), if \( x < y < z \) and \( x, z \in X \), then \( y \in X \), too.

Fact 2.1

Let \( G \) be a directed graph, \( V(G) = X \) for some contiguous \( X \), and \( x_i x_j \in E(G) \) iff \( x_j \) is the direct successor of \( x_i \).

Then for any denotation \( d \):

(1) \( d(x) \) may be (equivalent to) \( x + 1 \), \( \neg(x + 1) \), \( \bot \), or \( \top \).

If \( d(x) = \bot \) or \( \top \), we abbreviate \( d(x) = c, c' \), etc. (\( c \) for constant).

If \( v \) is acceptable for \( d \), then:

(1) If \( d(x) = c \), then \( d(x - 1) = c' \) (if \( x - 1 \) exists in \( X \)).

(2) if \( d(x) = (x + 1) \), then \([x]_v = [x + 1]_v\)

if \( d(x) = \neg(x + 1) \), then \([x]_v = \neg[x + 1]_v\)

(3) Thus:

(3.1) If \( d(x) = c \) for some \( x \), then for all \( x' < x \) \( d(x') = c' \) for some \( c' \).

(3.2) We have three possible cases:

(3.2.1) \( d(x) = c \) for all \( x \in X \),

(3.2.2) \( d(x) = c \) for no \( x \in X \),

(3.2.3) there is some maximal \( x' \) s.t. \( d(x') = c \), so \( d(x'') \neq c' \) for all \( x'' > x' \).

- In the first case, for all \( x \), if \( d(x) \) is \( \bot \) or \( \top \), then the valuation for \( x \) starts anew, i.e. independent of \( x + 1 \), and continues to \( x - 1 \) etc. according to (2).
- in the second case, there is just one acceptable valuation: we chose some \( x \in X \), and \([x]_v \) and propagate the value up and down according to (2)
- in the third case, we work as in the first case up to \( x' \), and treat the \( x'' > x' \) as in the second case.
- Basically, we work downwards from constants, and up and down beyond the maximal constant. Constants interrupt the upward movement.

(4) Consequently, any \( d \) on \( X \) has an acceptable valuation \( v_d \), and the graph is not dangerous.

(The present fact is a special case of Fact 3.2 (page 7), but it seems useful to discuss a simple case first.)

Example 2.1

We define now a modified Yablo graph \( YG' \), and a corresponding denotation \( d \), which is paradoxical.

We refer to Fig.3 in [RRM13], and Diagram 2.1 (page 6).
We keep all $Y_i$ of Fig. 3 in [RRM13], and introduce new vertices $(Y_i, Y_j, Y_k)$ for $i < k < j$. (When we write $(Y_i, Y_j, Y_k)$, we tacitly assume that $i < k < j$.)

(2) The arrows:
All $Y_i \rightarrow Y_{i+1}$ as before. We “factorize” longer arrows through new vertices:

(2.1) $Y_i \rightarrow (Y_i, Y_j, Y_{i+1})$
(2.2) $(Y_i, Y_j, Y_k) \rightarrow (Y_i, Y_j, Y_{k+1})$
(2.3) $(Y_i, Y_j, Y_{j-1}) \rightarrow Y_j$

See Diagram 2.1 (page 5).

We define $d$ (instead of writing $d((x, y, z))$ we write $d(x, y, z)$ - likewise $[x, y, z]_v$ for $[(x, y, z)]_v$ below):

(1) $d(Y_i) := \neg Y_{i+1} \land \{\neg(Y_i, Y_j, Y_{i+1}) : i + 2 \leq j\}$

(This is the main idea of the Yablo construction.)
(2) $d(Y_i, Y_j, Y_k) := (Y_i, Y_j, Y_{k+1})$ for $i < k < j - 1$
(3) $d(Y_i, Y_j, Y_{j-1}) := Y_j$

Obviously, $YG'$ corresponds to $S$ and $d$, i.e. $YG' = G_{S,d}$.

**Fact 2.2**

$YG'$ and $d$ code the Yablo Paradox:

**Proof**

Let $v$ be an acceptable valuation relative to $d$.

Suppose $[Y_1]_v = \top$, then $[Y_2]_v = \bot$, and $[Y_1, Y_k, Y_2]_v = \bot$ for $2 < k$, so $[Y_k]_v = \bot$ for $2 < k$, as in Fact 2.1 (page 4), (2). By $[Y_2]_v = \bot$, there must be $j$ such that $j = 3$ and $[Y_3]_v = \top$, or $j > 3$ and $[Y_2, Y_j, Y_3]_v = \top$, and as in Fact 2.1 (page 4), (2) again, $[Y_j]_v = \top$, a contradiction.

If $[Y_1]_v = \bot$, then as above for $[Y_2]_v$, we find $j \geq 2$ and $[Y_j]_v = \top$, and argue with $Y_j$ as above for $Y_1$.

Thus, $YG'$ with $d$ as above is paradoxical, and $YG'$ is dangerous.

$\Box$

**Example 2.2**

We first define $YG''$: $V(YG'') := \{Y_i : i < \omega\}$, $E(YG'') := \{\langle Y_i \rangle \rightarrow \langle Y_{i+1} \rangle : i < \omega\}$.

We now define the homomorphism from $YG'$ to $YG''$. We collapse for fixed $k Y_k$ and all $(Y_i, Y_j, Y_k)$ to $\langle Y_k \rangle$, more precisely, define $f$ by $f(Y_k) := f(Y_i, Y_j, Y_k) := \langle Y_k \rangle$ for all suitable $i, j$.

Note that $YG'$ only had arrows between “successor levels”, and we have now only arrows from $\langle Y_k \rangle$ to $\langle Y_{k+1} \rangle$, so $f$ is a homomorphism, moreover, our structure $YG''$ has the form described in Fact 2.1 (page 4), and is not dangerous, contradicting conjecture 15 in [RRM13].
Diagram YG’

This is just the start of the graph, it continues downward through $\omega$ many levels.

The lines stand for downward pointing arrows. The lines originating from the $Y_i$ correspond to the negative lines in the original Yablo graph, all others are simple positive lines, of the type $d(X) = X'$.

The left part of the drawing represents the graph YG’, the right hand part the collapsed graph, the homomorphic image YG”.

Compare to Fig.3 in [RRM13].
3 A comment on Theorem 24 of [RRM13]

We comment in this section on the meaning of theorem 24 in [RRM13].

Definition 3.1
Fix a denotation \(d\).
Let \(s(X) := s(d(X))\) be the set of \(s \in S\) which occur in \(d(X)\).
Let \(r(X) \subseteq s(X)\) be the set of relevant \(s\), i.e. which influence \([d(X)]_v\) for some \(v\). E.g., in \((\alpha \lor \neg \alpha) \land \alpha'\), \(\alpha\) is relevant, \(\alpha'\) is not.

Definition 3.2
(1) Let \(G\) be a directed graph. For \(X \in G\), let the subgraph \(C(X)\) of \(G\) be the connected component of \(G\) which contains \(X: X \in V(C(X))\), and \(X' \in V(C(X))\) iff there is a path in \(U(G)\) from \(X\) to \(X'\), together with the induced edges of \(G\), i.e., if \(Y, Y' \in V(C(X))\), and \(YY' \in E(G)\), then \(YY' \in E(C(X))\).

(2) \(G\) is called a simply connected graph iff for all \(X, Y\) in \(G\), there is at most one path in \(U(G)\) from \(X\) to \(Y\).
(One may debate if a loop \(X \rightarrow X\) violates simple connectedness, as we have the paths \(X \rightarrow X\) and \(X \rightarrow X\rightarrow X\) - we think so. Otherwise, we exclude loops.)

(3) Two subgraphs \(G', G''\) of \(G\) are disconnected iff there is no path from any \(X' \in G'\) to any \(X'' \in G''\) in \(U(G)\).

Fact 3.1
Let \(G, d\) be given, \(G = G_{S,d}\).
If \(G', G''\) are two disconnected subgraphs of \(G\), then they can be given truth values independently.

Proof
Trivial, as the subgraphs share no propositional variables. \(\square\)

Fact 3.2
Let \(G\) be simply connected, and \(d\) any denotation, \(G = G_{S,d}\). Then \(G, d\) has an acceptable valuation.

Proof
This procedure assigns an acceptable valuation to \(G\) and \(d\) in several steps.
More precisely, it is an inductive procedure, defining \(v\) for more and more elements, and cutting up the graph into disconnected subgraphs. If necessary, we will use unions for the definition of \(v\), and the common refinement for the subgraphs in the limit step.
The first step is a local step, it tries to simplify \(d(X)\) by looking locally at it, propagating \([X]\) to \(X'\) with \(X' \rightarrow X\) if possible, and erasing arrows from and to \(X\), if possible. Erasing arrows decomposes the graph into disconnected subgraphs, as the graph is simply connected.
The second step initializes an arbitrary value \(X\) (or, in step (4), uses a value determined in step (2)), propagates the value to \(X'\) for \(X' \rightarrow X\), erases the arrow \(X' \rightarrow X\). Initializing \(X\) will have repercussions on the \(X''\) for \(X \rightarrow X''\), so we chose a correct possibility for the \(X''\) (e.g., if \(d(X) = X'' \land X'''\), setting \([X] = \top\), requires to set \([X''] = [X'''] = \top\), too), and erase the arrows \(X \rightarrow X''\). As \(G\) is simply connected, the only connection between the different \(C(X'')\) is via \(X\), but this was respected and erased, and they are now independent.

(1) Local step
(1.1) For all \( X' \in s(X) - r(X) \):

(1.1.1) replace \( X' \) in \( d(X) \) by \( \top \) (or, equivalently, \( \bot \)), resulting in logically equivalent \( d'(X) \) (\( s(X') \) might now be empty),

(1.1.2) erase the arrow \( X \rightarrow X' \).

Note that \( C(X') \) will then be disconnected from \( C(X) \), as \( G \) is simply connected.

(1.2) Do recursively:

If \( s(d(X)) = \emptyset \), then \( d(X) \) is equivalent to \( \top \) (or \( \bot \)) (it might also be \( \top \land \bot \) etc.), so \( [X]_v = [d(X)]_v = \top \) (or \( \bot \)) in any acceptable valuation, and \([d(X)]_v\) is independent of \( v \).

(1.2.1) For \( X' \rightarrow X \), replace \( X \) in \( d(X') \) by \( \top \) (or \( \bot \)) (\( s(X') \) might now be empty),

(1.2.2) erase \( X' \rightarrow X \) in \( G \).

\( X \) is then an isolated point in \( G \), so its truth value is independent of the other truth values (and determined already).

(2) Let \( G'' \) be a non-trivial (i.e. not an isolated point) connected component of the original graph \( G \), chose \( X \) in \( G'' \). If \( X \) were already fixed as \( \top \) or \( \bot \), then \( X \) would have been isolated by step (1). So \([X]_v\) is undetermined so far. Moreover, if \( X \rightarrow X' \) in \( G'' \), then \( d(X') \) cannot be equivalent to a constant value either, otherwise, the arrow \( X \rightarrow X' \) would have been eliminated already in step (1).

Chose arbitrarily a truth value for \( d(X) \), say \( \top \).

(2.1) Consider any \( X' \) s.t. \( X' \rightarrow X \) (if this exists)

(2.1.1) Replace \( X \) in \( d(X') \) with that truth value, here \( \top \).

(2.1.2) Erase \( X' \rightarrow X \)

As \( G'' \) is simply connected, all such \( C(X') \) and \( C(X) \) are now mutually disconnected.

(2.2) Consider simultaneously all \( X'' \) s.t. \( X \rightarrow X'' \). (They are not constants, as any \( X'' \in V(G'') \) must be a propositional variable.)

(2.2.1) Chose values for all such \( X'' \), corresponding to \([X]_v = [d(X)]_v \) (= \( \top \) here).

E.g., if \( d(X) = X'' \land X''' \), and the value for \( X \) was \( \top \), then we have to chose \( \top \) also for \( X'' \) and \( X''' \).

This is possible independently by Fact 5.1 (page 13), as the graph \( G'' \) is simply connected, and \( X \) is the only connection between the different \( X'' \).

(2.2.2) Erase all such \( X \rightarrow X'' \).

\( X \) is now an isolated point, and as \( G'' \) is simply connected, all \( C(X'') \) are mutually disconnected, and disconnected from all \( C(X') \) with \( X' \rightarrow X \), considered in (2.1).

The main argument here is that we may define \([X'']_v\) and \([X''']_v\) for all \( X \rightarrow X'' \) and \( X \rightarrow X''' \) independently, if we respect the dependencies resulting through \( X \).

(3) Repeat step (1) recursively on all mutually disconnected fragments resulting from step (2).

(4) Repeat step (2) for all \( X'' \) in (2.2), but instead of the free choice for \([X]_v \) in (2), the choice for the \( X'' \) has already been made in step (2.2.1), and work with this choice.

\( \square \)

4 Various remarks on the Yablo structure

We comment in this section on various modifications and generalizations of the Yablo structure. We think that the transitivity of the graph, and the form of the \( d(X) = \bigwedge \{ \neg X_i : i \in I \} \) are the essential properties of “Yablo-like” structures.

We make this official:
Definition 4.1
A structure $G, d$ is called Yablo-like iff $G$ is transitive, and $d$ of the $\land -$ form.
(See Remark 1.1 (page 3) for our interpretation of $\land$.)

Remark 4.1
This remark is for illustration and intuition.
In the Yablo structure, after some $Y_i$ which is true, all $Y_j, j > i$ have to be false. After some $Y_i$ which is false, there has to be some $Y_j, j > i$ which is true.

(1) We can summarize this as the following two rules:

(1.1) After $\land$, only $\land$ may follow, abbreviating:
after some $Y_i$ with $[Y_i]_v = \top$, all $Y_j$ with $j > i$ have to be $[Y_j]_v = \bot$
(1.2) After $\land$, there has to be some $\land$.

- This has only finite solutions: $\ldots - \land$, a sequence of $\land$, ending with last element $\land$.
  Last-but-one $\land$ does not work, as then the last is $\land$, but we need an $\land$ after this one.
- If we have an infinite sequence, there has to be a $\land$ somewhere, followed by $\land$ only, contradiction.
  (1.1) if we start with $\land$, then the first $\land$ imposes a $\land$ somewhere, contradiction
  (1.2) if we start with $\land$, then there has to be $\land$ somewhere, say at element $i$, so $i + 1$ has to be $\land$, so
  some $j > i + 1$ has to be $\land$, contradiction.

(2) An alternative view is the following:
$\forall$ (or $\land$) constructs defensive walls, $\exists$ (or $\land$) attacks them.
The elements of the walls ($\land$) themselves are attacks on later parts of the walls, the attacks attack earlier constructions ($\land$) of the walls.

Example 4.1
We discuss here some very simple examples, all modifications of the Yablo structure.
Up to now, we considered graphs isomorphic to (parts of) the natural numbers with arrows pointing to bigger numbers. We consider now other cases.

(1) Consider the negative numbers (with 0), arrows pointing again to bigger numbers. Putting $\land$ at 0, and $\land$ to all other elements is an acceptable valuation.

(2) Consider a tree with arrows pointing to the root. The tree may be infinite. Again $\land$ at the root, $\land$ at all other elements is an acceptable valuation.

(3) Consider an infinite tree, the root with $\omega$ successors $x_i, i < \omega$, and from each $x_i$ originating a chain of length $i$ as in Fig. 10 of [RRM13], putting $\land$ at the end of the branches, and $\land$ everywhere else is an acceptable valuation.

(4) This trivial example shows that an initial segment of a Yablo construction can again be a Yablo construction.
Instead of considering all $Y_i, i < \omega$, we consider $Y_i, i < \omega + \omega$, extending the original construction in the obvious way.

Fact 4.2
Let $G$ be loop free and finitely forward branching, i.e. for any $s$, there are only finitely many $s'$ such that $s \rightarrow s'$ in $G$. Then $G$ is not dangerous.
($d$ may be arbitrary, not necessarily of the $\land -$ form.)
Proof

Let $d$ be any assignment corresponding to $G$. Then $d(s)$ is a finite, classical formula. Replace $[s]_v = [d(s)]_v$ by the classical formula $\phi_s := s \leftrightarrow d(s)$. Then any finite number of $\phi_s$ is consistent.

Proof: Let $\Phi$ be a finite set of such $\phi_s$, and $S_\Phi$ the set of $s$ occurring in $\Phi$. As $G$ is loop free, and $S_\Phi$ finite, we may initialise the minimal $s \in S_\Phi$ (i.e. there is no $s'$ such that $s \rightarrow s'$ in the part of $G$ corresponding to $\Phi$) with any truth values, and propagate the truth values upward according to usual valuation rules. This shows that $\Phi$ is consistent, i.e. we have constructed a (partial) acceptable valuation for $d$.

Extend $\Phi$ by classical compactness, resulting in a total acceptable valuation for $d$.

(In general, in the logics considered here, compactness obviously does not hold: Consider $\{\neg \bigwedge \{Y_i : i < \omega\} \cup \{Y_i : i < \omega\}$.)

\square

The following modification of the Yablo structure has only one acceptable valuation for $Y_1$:

Example 4.2

Let $Y_i, i < \omega$ as usual, and introduce new $X_i, 3 \leq i < \omega$.

Let $Y_i \rightarrow Y_{i+1}, Y_i \rightarrow X_{i+2}, X_i \rightarrow Y_i, X_i \rightarrow X_{i+1}$, with $d(Y_i) := \neg Y_{i+1} \land X_{i+2}, d(X_i) := \neg Y_i \land X_{i+1}$.

If $Y_1 = \top$, then $\neg Y_2 \land X_3$, by $X_3$, $\neg Y_3 \land X_4$, so generally, if $Y_i = \top$, then $\{\neg Y_j : i < j\}$ and $\{X_j : i+1 < j\}$.

If $\neg Y_1$, then $Y_2 \lor \neg X_3$, so if $\neg X_3, Y_3 \lor \neg X_4$, etc., so generally, if $\neg Y_i$, then $\exists j(i < j, Y_j)$ or $\forall j(\neg X_j : i+1 < j)$.

Suppose now $Y_1 = \top$, then $X_j$ for all $2 < j$, and $\neg Y_j$ for all $1 < j$. By $\neg Y_2$ there is $j, 2 < j$, and $Y_j$, a contradiction, or $\neg X_j$ for all $3 < j$, again a contradiction.

But $\neg Y_1$ is possible, by setting $\neg Y_i$ and $\neg X_i$ for all $i$.

Thus, replacing infinite branching by an infinite number of finite branching does not work for the Yablo construction, as we can always chose the “procrastinating” branch.

See Diagram 4.1 (page 11).
Diagram 4.1

The lines stand again for downward pointing arrows. Crossed lines indicate negations.
Fact 4.3

Let $G$ be transitive, and $d$ be of the type $\bigwedge \neg \cdot$

1. If $\exists X. (\text{succ}(X) \neq \emptyset \text{ and } \forall X' \in \text{succ}(X).\text{succ}(X') \neq \emptyset)$, then $d$ has no acceptable valuation.

Let acceptable $v$ be given, $[\cdot]$ is for this $v$.

Case 1: $[X] = \top$. So for all $X' \in \text{succ}(X).[X'] = \bot$, and there is such $X'$, so (either by the prerequisite $\text{succ}(X') \neq \emptyset$, or by Remark 1.1 (page 3)) $\exists X'' \in \text{succ}(X').[X''] = \top$, but $\text{succ}(X') \subseteq \text{succ}(X)$, a contradiction.

In abbreviation: $X^+ \rightarrow \bigwedge \neg \cdot X'^{-} \rightarrow \bigwedge \neg \cdot X''^+$

Case 2: $[X] = \bot$. So $\exists X' \in \text{succ}(X).[X'] = \top$, so $\forall X'' \in \text{succ}(X').[X''] = \bot$, and by prerequisite $\text{succ}(X') \neq \emptyset$, so there is such $X''$, so by Remark 1.1 (page 3) $\text{succ}(X'') \neq \emptyset$, so $\exists X''' \in \text{succ}(X'').[X'''] = \top$, but $\text{succ}(X'') \subseteq \text{succ}(X')$, a contradiction.

$X^- \rightarrow \bigwedge \neg \cdot X'^{+} \rightarrow \bigwedge \neg \cdot X''^{-} \rightarrow \bigwedge \neg \cdot X'''^+$
(Here we need Remark 1.1 (page 3) for the additional step from $X''$ to $X'''$.)

2. Conversely:

Let $\forall X (\text{succ}(X) = \emptyset \text{ or } \exists X' \in \text{succ}(X).\text{succ}(X') = \emptyset)$.

By Remark 1.1 (page 3), if $\text{succ}(Y) = \emptyset$, then for any acceptable valuation, $[Y] = \top$. Thus, if there is $X' \in \text{succ}(X)$, $\text{succ}(X') = \emptyset$, $[X'] = \top$, and $[X] = \bot$.

Thus, the valuation defined by $[X] = \top$ iff $\text{succ}(X) = \emptyset$, and $\bot$ otherwise is an acceptable valuation.
(Obviously, this definition is free from contradictions.)

\qed
References

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