ALMOST ELEMENTARINESS AND FIBERWISE AMENABILITY
FOR ÉTALE GROUPOIDS

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Abstract. In this paper, we introduce two new types of approximation properties for étale groupoids, almost elementariness and (ubiquitous) fiberwise amenability, inspired by Matui’s and Kerr’s notions of almost finiteness. In fact, we show that, in their respective scopes of applicability, both notions of almost finiteness are equivalent to the conjunction of our two properties. Our new properties stem from viewing étale groupoids as coarse geometric objects in the spirit of geometric group theory. Fiberwise amenability is a coarse geometric property of étale groupoids that is closely related to the existence of invariant measures on unit spaces and corresponds to the amenability of the acting group in a transformation groupoid. Almost elementariness may be viewed as a better dynamical analogue of the regularity properties of $C^*$-algebras than almost finiteness, since, unlike the latter, the former may also be applied to the purely infinite case. To support this analogy, we show almost elementary minimal groupoids give rise to tracially $Z$-stable reduced groupoid $C^*$-algebras. In particular, the $C^*$-algebras of minimal amenable almost finite groupoids in Matui’s sense are $Z$-stable.

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1. Introduction

Groupoids are a generalization of groups where the multiplication operation is allowed to be only partially defined. The study of topological groupoids lies at the crossroads of group theory, dynamics, geometry, topology, mathematical physics, and operator algebras, largely thanks to their Swiss-knife-like ability to handle many mathematical objects, such as groups, group actions, equivalence relations on topological spaces, nonperiodic tilings, etc., through a unifying framework.

A recurring theme in many of these mathematical topics is the various ways in which infinite structures may be approximated by finite structures. Analysis
often enters the picture this way. An influential poster child of this theme is the notion of amenability in group theory, together with its many reincarnations in other fields, such as injectivity in von Neumann algebra theory, nuclearity in $C^*$-algebraic theory, topological amenability in topological dynamics, and metric amenability in coarse geometry, etc. For topological groupoids —— usually assumed to be locally compact, $\sigma$-compact and Hausdorff, and sometimes also étale, which is a groupoid generalization of having discretized time, as opposed to continuous time, in a classical topological dynamical system —— the richness of their structures is reflected in a number of different yet intricately related approximation properties that realize this theme.

Among the strongest approximation properties for topological groupoids is the notion of \textit{AF groupoids} \cite{Re80}. The terminology AF was borrowed from operator algebras and was originally short for \textit{approximately finite}. This property applies to \textit{ample} (i.e., totally disconnected) étale groupoids and demands that every compact subset of a topological groupoid is contained in a subgroupoid that is \textit{elementary} —— namely, it is isomorphic to a principal (i.e., not containing non-trivial subgroups) finite groupoid, typically “fattened up” with topological spaces. The way these elementary groupoids embed reminds one of Kakutani-Rokhlin towers, a fundamental tool in measure-theoretic and topological dynamics. Thanks to the transparent and rigid structure of elementary groupoids, AF groupoids are well studied and classified \cite{Kri80, GPS04, GMPS08}, though the notion is relatively restrictive.

Inspired by the Følner set approach to (group-theoretic) amenability, Matui \cite{Ma12} introduced a more general notion called \textit{almost finite groupoids} \cite{Phi05}, which, like AF groupoids, also applies to ample étale groupoids, but it only demands that every compact subset of a topological groupoid is \textit{almost} contained in an elementary subgroupoid in a Følner-like sense. Almost finiteness strikes a remarkable balance between applicability and utility: it was shown to be enjoyed by \textit{transformation groupoids} arising from free actions on the Cantor set by $\mathbb{Z}^n$ (later generalized in \cite{KS20}; see below), as well as those arising from aperiodic tilings \cite{IWZ12}; on the other hand, this notion has found applications in the homology theory of ample groupoids, topological full groups, and the structure theory and $K$-theory of reduced groupoid $C^*$-algebras, and deep connections to an increasing number of other important properties have been established (see, for example, \cite{Ma13, Ma17, Nek19, Ker20, Suz20}).

Focusing on the case of transformation groupoids (arising from \textit{topological dynamical systems}, that is, groups acting on topological spaces), Kerr \cite{Ker20} presented an insightful perspective that links almost finiteness with \textit{regularity properties} in the classification and structure theory of $C^*$-algebras, extending the link between AF groupoids and AF $C^*$-algebras, as well as paralleling the link between \textit{hyperfiniteness} for ergodic probability-measure-preserving equivalence relations and hyperfiniteness for $\text{II}_1$ factors. Here, “regularity properties” refers to a handful of natural and intrinsic properties of $C^*$-algebras pioneered by Winter that arose from Elliott’s classification program of simple separable nuclear $C^*$-algebras and played pivotal roles in its eventual success: in short, they were the missing piece needed to characterize the $C^*$-algebras classifiable via the Elliott invariant \cite{GLN14, EGLN15, TWW17, Phi00}.

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\footnote{Almost finiteness should not be abbreviated as AF or confused with \textit{almost AF groupoids} \cite{Phi05}!}
Moreover, as predicted by Toms and Winter, this handful of regularity properties turn out to be (mostly) equivalent for simple separable nuclear \( C^* \)-algebras (\cite{Ror04, MS12, Win12, SWW15, LW17, CET19}); thus having one of them is good enough for classification. Among these regularity properties, it was Hirshberg and Orlovitz’s tracial \( \mathcal{Z} \)-stability (\cite{HO13}) that Kerr found to be the closest in spirit to Matui’s almost finiteness.

Furthermore, Kerr did not merely translate Matui’s almost finiteness into the language of group actions — this would just mean that we have a group action on a totally disconnected space that admits partitions into open (in fact, clopen) Rokhlin towers modelled on Følner sets. He installed an upgrade to its applicability: while the elementary subgroupoids in Matui’s almost finiteness are required to cover the entire unit space, Kerr’s almost finiteness allows a “small” remainder to be left uncovered. This slight relaxation has immense consequences: while for actions on a totally disconnected (i.e., zero-dimensional) space, this “small” remainder can always be absorbed into the Rokhlin towers and we thus recover Matui’s definition, yet Kerr’s version now also applies to actions on higher-dimensional spaces (though in this case we also need to explicitly require that the levels of the Rokhlin towers all have tiny diameters). This idea of approximation modulo a small remainder is well aligned with the understanding of tracial \( \mathcal{Z} \)-stability as an analogue of the McDuff property that allows for a tracially small error in a way similar to Lin’s earlier definition of tracially AF \( C^* \)-algebras (\cite{Lin01b, Lin01a}).

This pivotal upgrade, however, depends on the precise meaning of “small” sets. There are two natural ways to describe them: The first is measure-theoretic: namely, a set is “small” if, with regard to every invariant measure, its measure is smaller than a predetermined positive number \( \varepsilon \). The other is topological: roughly speaking, a set is “small” if it is dynamically subequivalent to a (predetermined) nonempty open set, i.e., roughly speaking, the “small” set is able to be disassembled and then translated, piece by piece via the group action, into “non-touching” positions inside the nonempty open set, where “non-touching” means the closures of the translated pieces do not intersect. It is clear that the second method yields a stricter sense of smallness, and it turns out to be a desirable property of a topological dynamical system for the two methods to agree. This is the essence of what Kerr dubbed dynamical (strict) comparison, after the analogous property of strict comparison of positive elements in a \( C^* \)-algebra, which is also among the aforementioned handful of \( C^* \)-algebraic regularity properties.

Thus Kerr’s almost finiteness comes in two flavors for higher-dimensional spaces: the ordinary one uses dynamical subequivalence to express smallness of the remainder and an auxiliary notion called almost finiteness in measure, which uses invariant measures instead. Kerr and Szabó \cite{KS20} showed that the former condition is equivalent to the conjunction of the latter and dynamical strict comparison, while the latter condition is equivalent to the small boundary property (which, in turn, is closely related to mean dimension zero). To cement the link to \( C^* \)-algebraic regularity properties, Kerr \cite{Ker20} proved that a minimal almost finite action on a compact space by an amenable group gives rise to a tracially \( \mathcal{Z} \)-stable crossed product \( C^* \)-algebra. This was applied in \cite{KS20} to show that any free minimal

\textsuperscript{2} Kerr’s definition actually requires the remainder to be dynamically subequivalent to a small portion of the unit space of the open elementary subgroupoid, instead of a predetermined nonempty open set. This makes it work better in the non-minimal setting.
action on a finite-dimensional metric space by a group with subexponential growth produces a classifiable crossed product $C^*$-algebra. Based on these evidences, Kerr suggested, at least in the case of actions by amenable groups, almost finiteness may be understood as a dynamical regularity property.

This great confluence of ideas from topological dynamicals, topological groupoid theory, and operator algebras opened the gate to a plethora of new connections and applications. At the same time, it also left us with a number of unanswered questions and problems.

Q1. Do almost finite groupoids in Matui’s sense always give rise to groupoid $C^*$-algebras satisfying regularity properties such as tracial $\mathcal{Z}$-stability?

Kerr’s result above answered this in the affirmative for transformation groupoids, but it remained largely unclear beyond that case. Kerr’s method appears difficult to generalize, for it makes use of the fact that the levels in the Rokhlin towers witnessing almost finiteness are labeled by group elements, which allows one to apply Ornstein and Weiss’ theory of quasi-tilings to carefully manipulate the towers. Ito, Whittaker and Zacharias [IWZ12] managed to extend Kerr’s method and result to the case of étale groupoids from aperiodic tilings, exploiting the fact that étale groupoids from aperiodic tilings are reductions of transformation groupoids associated to $\mathbb{R}^n$-actions. Nevertheless, this method appears unsuitable for groupoids without obvious underlying group structures.

Q2. Is there a groupoid regularity property that works both for groupoids with invariant measures on their unit spaces and for those without?

Both Matui’s and Kerr’s notions of almost finiteness imply the existence of invariant measures on the unit space of a groupoid and thus the existence of traces in their groupoid $C^*$-algebras. While this is often a useful feature, it does not line up with the fact that $C^*$-algebraic regularity properties also applies to purely infinite algebras, which do not have any traces. Thus one would hope a more relaxed notion could include groupoids without invariant measures on their unit spaces. We point out that the correspondence between dynamical strict comparison and pure infiniteness was established by the first author [Ma12].

Q3. Related to the previous item: Can we isolate from Matui’s definition of almost finiteness an “amenability property” that is responsible for the existence of invariant measures?

Both Kerr’s and Matui’s definitions made explicit use of Følner sets or a Følner-type condition —— this is the immediate reason for the existence of invariant measures. While Kerr’s notion forces the acting group to be amenable, it suggested an analogous kind of amenability property for étale groupoids might hide behind Matui’s notion. A possible candidate was topological amenability, but this would be the wrong target, as it does not imply the existence of invariant measures in general, and a number of examples have shown almost finiteness does not imply topological amenability ([ABL12, Ele12]).

The present paper is intended to address these questions. We summarize our results, with the following standing assumption.

Assumption: The groupoids below are $\sigma$-compact, locally compact, Hausdorff, étale topological groupoids.
In addition, many of our results focus on minimal groupoids, as it appears to us at the moment there lacks a clear vision of the landscape of regularity properties for non-simple \(C^*-\)algebra and non-minimal groupoids.

Motivated by \([3]\), we introduce a notion termed fiberwise amenability for \(\acute{e}tale\) groupoids (Definition \([5.4]\) see also Proposition \([5.5]\)). It simply demands, for any \(\varepsilon\) and any compact subset \(K\) in our \(\acute{e}tale\) groupoid \(G\), there is a nonempty finite subset \(F\) in \(G\) such that

\[\frac{|KF|}{|F|} \leq \varepsilon.\]

Essentially, \(F\) is what one may call a \(\text{Følner set}\). This notion satisfies the following basic properties:

S1 (Remark \([5.6]\)). In the case of transformation groupoids, fiberwise amenability is equivalent to the amenability of the acting groups (rather than the topological amenability of the actions).

S2 (Proposition \([5.9]\)). When the unit space is compact, it implies the existence of invariant probability measures on it.

While this notion is often easy to verify in concrete examples, yet given the typically non-homogeneous structure of groupoids, it appears too weak by itself for many purposes — for example, it is not hard to see that \(\acute{e}tale\) groupoids with noncompact unit spaces are always fiberwise amenable. This is what led us to introduce a stronger variant termed ubiquitous fiberwise amenability (Definition \([5.4]\)), which requires, in addition to the existence of a single Følner set for each choice of \((K,\varepsilon)\) as above, that such Følner sets can be found in every source fiber and near every element of \(G\) in a uniform sense. Although one may quickly see that this is strictly stronger than fiberwise amenability in general, we show:

S3 (Theorem \([5.13]\)). For minimal groupoids, ubiquitous fiberwise amenability is equivalent to fiberwise amenability.

To prove the above statement and apply these notions, we develop a way to view \(\acute{e}tale\) groupoids from the lens of coarse geometry, akin to how countable groups are treated as metric spaces in geometric group theory.

S4 (Theorem \([4.9]\) and Definition \([4.11]\)). Up to coarse equivalence, there is a canonical right-invariant extended metric on an \(\acute{e}tale\) groupoid \(G\), which is induced from a proper continuous length function on \(G\).

This extended metric may be characterized by the description that a prototypical bounded neighborhood of an arbitrary set \(E\) in \(G\) looks like (a subset of) the product set \(KE\) for some compact subset \(K\) in \(G\). Here “extended” means points may have infinite distances. In this case, two elements of \(G\) have a finite distance if and only if they are on the same source fiber. This entails that our canonical extended metric rarely induces the topology of \(G\) in the usual sense; rather, they are related in that the metric changes continuously from one source fiber to another. In fact, we prove a local slice lemma (Lemma \([5.10]\) that details how the canonical extended metric can be locally trivialized. We then observe:

S5 (Proposition \([5.5]\)). Both fiberwise amenability and ubiquitous fiberwise amenability are coarse geometric properties of \(\acute{e}tale\) groupoids.
More precisely, these notions only depend on the coarse geometry of the canonical right-invariant extended metric. This enables us to use metric techniques to show that ubiquitous fiberwise amenability has an a priori stronger reformulation: for each choice of \((K, \varepsilon)\), we can convert any finite subset of \(\mathcal{G}\) into a \((K, \varepsilon)\)-Følner set by enlarging it within a uniformly bounded distance (Theorem 5.15). This stronger form of ubiquitous fiberwise amenability plays an important role in our discussion of groupoid strict comparison (Definition 6.2), the natural generalization of dynamical strict comparison to the groupoids setting.

Motivated by \([Q2]\), we introduce, for (minimal) étale groupoids with compact unit spaces, a new regularity property and approximation property called almost elementariness, which generalizes both Matui’s and Kerr’s almost finiteness (c.f., Section 7). In fact, we show:

**S6** (Theorem 7.6). For transformation groupoids, Kerr’s almost finiteness is equivalent to the conjunction of almost elementariness and ubiquitous fiberwise amenability (i.e., the amenability of the acting group).

**S7** (Theorem 7.4). For groupoids with totally disconnected unit spaces, Matui’s almost finiteness is also equivalent to the conjunction of almost elementariness and ubiquitous fiberwise amenability.

Our definition replaces the Følner-type conditions in Matui’s and Kerr’s almost finiteness by a new condition that requires the elementary subgroupoids in the approximation to be extendable to a larger elementary subgroupoid. This condition again draws inspiration from coarse geometry, e.g., from how, in the definition of the asymptotic dimension of a metric space, we ask for a cover that is able to grow or shrink by a predetermined large distance without losing its desired structure, in this case, the chromatic number as well as the property of being a cover. Similarly, in our definition of almost elementary groupoids, we would like our elementary subgroupoid to be able to grow by a predetermined distance (as measure by compact subsets in the groupoid) without losing its elementariness (alternatively, it should be able to shrink without jeopardizing the smallness of the remainder). Replacing the Følner-type conditions by this new condition allows our notion to break free from fiberwise amenability and apply to groupoids both with and without invariant probability measures on their unit spaces. This inclusiveness with regard to invariant measures underlies the fact that the groupoid \(C^*\)-algebras of almost elementary groupoids include both stably finite algebras and purely infinite ones, a fact that makes almost elementariness a candidate for a closer analogue of \(C^*\)-algebraic regularity properties.

To support this analogy with \(C^*\)-algebraic regularity properties, we establish several connections. First of all, we have

**S8** (Remark 6.15). For minimal almost elementary groupoids, almost elementariness implies groupoid strict comparison.

As indicated above, ubiquitous fiberwise amenability and the coarse geometry of étale groupoids play important roles in this result. As a consequence of this interaction between these properties, we have the following link between the measure structure and the geometric structure of a minimal almost elementary groupoid:

\[\text{This is one reason why we decide not to use the word “finite” in the name of our new notion.} \]
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S9 (Remark 6.15). For minimal almost elementary groupoids, fiberwise amenability is equivalent to the existence of invariant probability measures on the unit spaces.

We also establish a direct link to tracial $\mathcal{Z}$-stability, providing an affirmative answer to [Q1]. Recall that to show a $C^*$-algebra $A$ is tracially $\mathcal{Z}$-stable, we need to produce order zero maps from arbitrarily large matrix algebras into $A$ with approximately central images and with only “tracially small” defects from being unital.

S10 (Theorem 9.7). Let $\mathcal{G}$ be a second countable étale minimal groupoid with a compact unit space. Suppose $\mathcal{G}$ is almost elementary (e.g., almost finitn). Then the reduced groupoid $C^*$-algebra $C^r_*(\mathcal{G})$ is tracially $\mathcal{Z}$-stable.

This generalizes Kerr’s result on almost finite actions of amenable groups ([Ker20]) in several ways: our theorem can be applied to groupoids without obvious underlying group structures and without topological amenability; even when restricted to transformation groupoids, our result can now deal with actions by nonamenable groups.

Moreover, as indicated in [Q1] our proof necessarily takes an approach different from Kerr’s. In place of the Ornstein-Weiss tiling theory of amenable groups, we develop a “nesting” form of almost elementariness, which is an approximation (modulo a small remainder) of the groupoid $\mathcal{G}$ by not one, but two open elementary subgroupoids in a nested position, a notion reminiscent of how multi-matrix algebras embed into each other. Thus passing from approximation by one elementary subgroupoid to approximation by a nesting of two (or perhaps more) is akin to how, from the local definition of an AF algebra, one can produce a tower of nested multi-matrix algebras organized by a Bratteli diagram. Indeed, an open elementary subgroupoid of $\mathcal{G}$ will induce an order zero map from a multi-matrix algebra into $C^r_*(\mathcal{G})$, and if another open elementary subgroupoid is nested in the first one, then we have an embedding between two multi-matrix algebras. By arranging the nesting to have a large multiplicity, we can ensure this embedding has a large relative commutant. This will essentially be the source of the desired large matrix algebra together with an order zero map into $C^r_*(\mathcal{G})$ with an approximately central image. The existence of the remainders in these approximations unfortunately makes the proof appear technically complicated, but their smallness will eventually guarantee that the resulting order zero map only has a “tracially small” defect from being unital.

The following is a direct consequence of S10 by combining results in [CET+19, EGLN15, GLNT14, HO13, Phi00, TWW17].

S11 (Corollary 9.9). Let $\mathcal{G}$ be a second countable amenable minimal étale groupoid with a compact unit space. Suppose $\mathcal{G}$ is almost elementary. Then $C^r_*(\mathcal{G})$ is unital simple separable nuclear and $\mathcal{Z}$-stable and thus has nuclear dimension one. In addition, in this case $C^r_*(\mathcal{G})$ is classified by its Elliott invariant. Finally, if $M(\mathcal{G}) \neq \emptyset$, then $C^r_*(\mathcal{G})$ is quasidiagonal; if $M(\mathcal{G}) = \emptyset$, then $C^r_*(\mathcal{G})$ is a unital Kirchberg algebra.

We will also provide several explicit examples in the last section.

2. Preliminaries

In this section we recall some basic backgrounds on coarse geometry, étale groupoids and $C^*$-algebras.
In this paper, there are two types of metric spaces under consideration. One concerns usual topological metrizable spaces focus on local behavior while another are coarse metric spaces from large scale geometric point of view. However, even these two types have different nature, as metric spaces, they share some same notations. Let \((X, d)\) be a metric space equipped with the metric \(d\). We denote by \(B_d(x, R)\) the open ball \(B_d(x, R) = \{y \in X : d(x,y) < R\}\) and by \(\overline{B}_d(x, R)\) the closed ball \(\overline{B}_d(x, R) = \{y \in X : d(x,y) \leq R\}\). Let \(A\) be a subset of \(X\). We write \(B_d(A, R)\) and \(\overline{B}_d(A, R)\) for analogous meaning. If the metric is clear, we write \(B(A, R)\) and \(\overline{B}(A, R)\) instead for simplification. We refer to [NY12] as a standard reference for topics of large scale geometry.

We refer to [Ren80] and [Sim12] as references for groupoids and we record several fundamental definitions and results for locally compact Hausdorff étale groupoids here.

**Definition 2.1.** A groupoid \(\mathcal{G}\) is a set equipped with a distinguished subset \(\mathcal{G}^{(2)} \subset \mathcal{G} \times \mathcal{G}\), called the set of *composable pairs*, a product map \(\mathcal{G}^{(2)} \to \mathcal{G}\), denoted by \((\gamma, \eta) \mapsto \gamma \eta\) and an inverse map \(\mathcal{G} \to \mathcal{G}\), denoted by \(\gamma \mapsto \gamma^{-1}\) such that the following hold

1. If \((\alpha, \beta) \in \mathcal{G}^{(2)}\) and \((\beta, \gamma) \in \mathcal{G}^{(2)}\) then so are \((\alpha \beta, \gamma)\) and \((\alpha, \beta \gamma)\). In addition, \((\alpha \beta) \gamma = \alpha (\beta \gamma)\) holds in \(\mathcal{G}\).
2. For all \(\alpha \in \mathcal{G}\) one has \((\gamma, \gamma^{-1}) \in \mathcal{G}^{(2)}\) and \((\gamma^{-1})^{-1} = \gamma\).
3. For any \((\alpha, \beta) \in \mathcal{G}^{(2)}\) one has \(\alpha^{-1}(\alpha \beta) = \beta\) and \((\alpha \beta) \beta^{-1} = \alpha\).

Every groupoid is equipped with a subset \(\mathcal{G}^{(0)} = \{\gamma \gamma^{-1} : \gamma \in \mathcal{G}\}\) of \(\mathcal{G}\). We refer to elements of \(\mathcal{G}^{(0)}\) as *units* and to \(\mathcal{G}^{(0)}\) itself as the *unit space*. We define two maps \(s, r : \mathcal{G} \to \mathcal{G}^{(0)}\) by \(s(\gamma) = \gamma^{-1} \gamma\) and \(r(\gamma) = \gamma \gamma^{-1}\), respectively, in which \(s\) is called the *source map* and \(r\) is called the *range map*.

When a groupoid \(\mathcal{G}\) is endowed with a locally compact Hausdorff topology under which the product and inverse maps are continuous, the groupoid \(\mathcal{G}\) is called a locally compact Hausdorff groupoid. A locally compact Hausdorff groupoid \(\mathcal{G}\) is called *étale* if the range map \(r\) is a local homeomorphism from \(\mathcal{G}\) to itself, which means for any \(\gamma \in \mathcal{G}\) there is an open neighborhood \(U\) of \(\gamma\) such that \(r(U)\) is open and \(r|_U\) is a homeomorphism. In this case, since the map of taking inverses is an involutive homeomorphism on \(\mathcal{G}\) that intertwines \(r\) and \(s\), thus the source map \(s\) is also a local homeomorphism. A set \(B\) is called a *bisection* if there is an open set \(U\) in \(\mathcal{G}\) such that \(B \subset U\) and the restriction of the source map \(s|_U : U \to s(U)\) and the range map \(r|_U : U \to r(U)\) on \(U\) are both homeomorphisms onto open subsets of \(\mathcal{G}^{(0)}\). It is not hard to see a locally compact Hausdorff groupoid is étale if and only if its topology has a basis consisting of open bisections. We say a locally compact Hausdorff étale groupoid \(\mathcal{G}\) is *ample* if its topology has a basis consisting of compact open bisections.

**Example 2.2.** Let \(X\) be a locally compact Hausdorff space and \(\Gamma\) be a discrete group. Then any action \(\Gamma \acts X\) by homeomorphisms induces a locally compact Hausdorff étale groupoid

\[
X \times \Gamma := \{(\gamma x, \gamma, x) : \gamma \in \Gamma, x \in X\}
\]
equipped with the relative topology as a subset of \(X \times \Gamma \times X\). In addition, \((\gamma x, \gamma, x)\) and \((\beta y, \beta, y)\) are composable only if \(\beta y = x\) and

\[
(\gamma x, \gamma, x)(\beta y, \beta, y) = (\gamma \beta y, \gamma \beta y, y).
\]
One also defines \((\gamma x, \gamma, x)^{-1} = (x, \gamma^{-1}, \gamma x)\) and announces that \(G^{(0)} := \{(x, e\Gamma, x) : x \in X\}\). It is not hard to verify that \(s(\gamma x, \gamma, x) = x\) and \(r(\gamma x, \gamma, x) = \gamma x\). The groupoid \(X \times \Gamma\) is called a transformation groupoid.

The following are several basic properties of locally compact Hausdorff étale groupoids whose proofs could be found in \cite{Sim12}.

**Proposition 2.3.** Let \(G\) be a locally compact Hausdorff étale groupoid. Then \(G^{(0)}\) is a clopen set in \(G\).

**Proposition 2.4.** Let \(G\) be a locally compact Hausdorff étale groupoid. Suppose \(U\) and \(V\) are open bisections in \(G\). Then \(UV = \{\alpha\beta : (\alpha, \beta) \in G^{(2)} \cap U \times V\}\) is also an open bisection.

It is also convenient to define, for \(n = 1, 2, \ldots\), the set of composable \(n\)-tuples

\[G^{(n)} = \{(x_1, \ldots, x_n) \in G^n : s(x_i) = r(x_{i+1}) \text{ for } i = 1, 2, \ldots, n-1\}\]

and the \(n\)-ary multiplication map

\[\delta^{(n)} : G^{(n)} \to G, \quad (x_1, \ldots, x_n) \mapsto x_1 \cdots x_n.\]

**Corollary 2.5.** Let \(G\) be a locally compact Hausdorff étale groupoid. Then for any \(n \in \{1, 2, \ldots\}\), the \(n\)-ary multiplication map is a local homeomorphism.

We also record the following useful fact about local homeomorphisms.

**Lemma 2.6.** Let \(f : X \to Y\) be a local homeomorphism between topological spaces with \(Y\) being Hausdorff. Then for any \(y \in Y\) and any compact subset \(K \subseteq X\), there are an open neighborhood \(U\) of \(y\) in \(Y\) and a finite family of open subsets \(V_1, \ldots, V_n\) in \(X\) such that

1. the map \(f\) restricts to a homeomorphism between \(V_i\) and \(U\), for any \(i \in \{1, \ldots, n\}\), and
2. we have \(f^{-1}(U) \cap K \subseteq V_1 \cup \ldots \cup V_n\).

**Proof.** Since \(f\) is a local homeomorphism, we know for any \(x \in f^{-1}(y)\), there are open neighborhoods \(V_x\) of \(x\) and \(U_y\) of \(y\) such that \(f\) restricts to a homeomorphism between \(V_x\) and \(U_y\). Since the collection \(\{f^{-1}(Y \setminus \{y\}), V_x : x \in f^{-1}(y)\}\) form an open cover of \(K\), by compactness, there are \(x_1, \ldots, x_n \in f^{-1}(y)\) such that \(\{f^{-1}(Y \setminus \{y\}), V_{x_i} : i = 1, \ldots, n\}\) form a finite open cover of \(K\). Let \(L = K \setminus \bigcup_{i=1}^n V_{x_i}\), which is a closed subset of \(K\) and thus also compact; so is the image \(f(L)\). Observe that \(L \subseteq f^{-1}(Y \setminus \{y\})\), i.e., \(y \notin f(L)\). Since a Hausdorff space has separation between a point and a compact set, there is an open neighborhood \(W\) of \(y\) in \(Y\) such that \(W \cap f(L) = \emptyset\). Let \(U = W \cap (\bigcap_{i=1}^n U_{x_i})\) and let \(V_i = \left(f |_{V_{x_i}}\right)^{-1}(U)\), for \(i = 1, \ldots, n\). They clearly satisfy the first condition. As for the second condition, we observe that \(f^{-1}(U) \cap L = \emptyset\) and thus \(f^{-1}(U) \cap K = f^{-1}(U) \cap (L \cup V_1 \cup \ldots \cup V_n) \subseteq V_1 \cup \ldots \cup V_n\). \(\square\)

For any set \(D \subset G^{(0)}\), Denote by

\[G_D := \{\gamma \in G : s(\gamma) \in D\}, \quad G^D := \{\gamma \in G : r(\gamma) \in D\}, \quad \text{and} \quad G^{D}_D := G^D \cap G_D.\]

For the singleton case \(D = \{u\}\), we write \(G_u, G^u\) and \(G^u_u\) instead for simplicity. In this situation, we call \(G_u\) a source fiber and \(G^u\) a range fiber. In addition, each \(G^u_u\) is a group, which is called the isotropy at \(u\). We say a groupoid \(G\) is principal if all
isotropy groups are trivial, i.e., $G_u^{\alpha} = \{u\}$ for all $u \in G^{(0)}$. We say a groupoid $G$ is topologically principal if the set $\{u \in G^{(0)} : G_u^\alpha = \{u\}\}$ is dense in $G^{(0)}$. A subset $D$ in $G^{(0)}$ is called $G$-invariant if $r(D) = D$, which is equivalent to the condition $G^D = G_D$. Note that $G|_D := G_D^0$ is a subgroupoid of $G$ with the unit space $D$ if $D$ is a $G$-invariant set in $G^{(0)}$. A groupoid $G$ is called minimal if there are no proper non-trivial closed $G$-invariant subsets in $G^{(0)}$.

Let $G$ be a locally compact Hausdorff étale groupoid. We define a convolution product on $C_c(G)$ by

$$(f * g)(\gamma) = \sum_{\alpha \beta = \gamma} f(\alpha)g(\beta)$$

and an involution by

$$f^*(\gamma) = \overline{f(\gamma^{-1})}.$$ 

These two operations make $C_c(G)$ a *-algebra. Then the reduced groupoid $C^*$-algebra $C^*_r(G)$ is defined to be the completion of $C_c(G)$ with respect to the norm $\|\cdot\|_r$ induced by all regular representation $\pi_u$ for $u \in G^{(0)}$, where $\pi_u : C_c(G) \to B(\ell^2(G_u))$ is defined by $\pi_u(f)\eta = f * \eta$ and $\|f\|_r = \sup_{u \in G^{(0)}} \|\pi_u(f)\|$. It is well known that there is a $C^*$-algebraic embedding $\iota : C_0(G^{(0)}) \to C^*_r(G)$. On the other hand, $E_0 : C_c(G) \to C_c(G^{(0)})$ defined by $E_0(a) = a|_{G^{(0)}}$ extends to a faithful canonical conditional expectation $E : C^*_r(G) \to C_0(G^{(0)})$ satisfying $E(\iota(f)) = f$ for any $f \in C_0(G^{(0)})$ and $E(\iota(f)a(\gamma)g) = \overline{f}E(a)g$ for any $a \in C^*_r(G)$ and $f, g \in C_0(G^{(0)})$.

As a typical example, it can be verified that for the transformation groupoid in Example 2.2, the reduced groupoid $C^*$-algebra is isomorphic to the reduced crossed product $C^*$-algebra of the dynamical system. The following are some standard facts on reduced groupoid $C^*$-algebras that could be found in [Sim12]. Throughout the paper, the notation $\text{supp}_o(f)$ for a function $f$ on a topological space $X$ denotes the open support $\{x \in X : f(x) \neq 0\}$ of $f$. In addition, we write $\text{supp}(f)$ the usual support $\text{supp}_o(f)$ of $f$. We say an open set $O$ in a topological space $X$ is precompact if $\overline{O}$ is compact.

**Proposition 2.7.** Let $G$ be a locally compact Hausdorff étale groupoid. Any $f \in C_c(G)$ can be written as a sum $f = \sum_{i=0}^n f_i$ such that there are precompact open bisections $V_0, \ldots, V_n$ such that $V_0 \subset G^{(0)}$ and $V_i \cap G^{(0)} = \emptyset$ for all $0 < i \leq n$ as well as $\text{supp}(f_i) \subset V_i$ for any $0 \leq i \leq n$.

**Proposition 2.8.** Let $G$ be a locally compact Hausdorff étale groupoid. Suppose $U, V$ are open bisections and $f, g \in C_c(G)$ such that $\text{supp}(f) \subset U$ and $\text{supp}(g) \subset V$. Then $\text{supp}(f * g) \subset U \cdot V$ and for any $\gamma = \alpha \beta \in U \cdot V$ one has $(f * g)(\gamma) = \overline{f(\alpha)}g(\beta)$.

**Proposition 2.9.** Let $G$ be a locally compact Hausdorff étale groupoid. For $f \in C_c(G)$, one has $\|f\|_{\infty} \leq \|f\|_r$. If $f$ is supported on a bisection, then one has $\|f\|_{\infty} = \|f\|_r$.

Let $G$ be a locally compact Hausdorff étale groupoid. Suppose $U$ is an open bisection and $f \in C_c(G)_+$ such that $\text{supp}(f) \subset U$. Define functions $s(f), r(f) \in C_0(G^{(0)})$ by $s(f)(\gamma) = f(\gamma)$ and $r(f)(\gamma) = f(\gamma)$ for $\gamma \in \text{supp}(f)$. Since $U$ is a bisection, so is $\text{supp}(f)$. Then the functions $s(f)$ and $r(f)$ are well-defined functions supported on $s(\text{supp}(f))$ and $r(\text{supp}(f))$, respectively. Note that $s(f) = (f^* f)^{1/2}$ and $r(f) = (f \ast f^*)^{1/2}$. 
The Jiang-Su algebra $\mathcal{Z}$, introduced in [JS99] by Jiang and Su, is an infinite dimensional unital nuclear simple separable $C^*$-algebra, but $KK$-equivalent to $\mathbb{C}$ in the sense of Kasparov. We say a $C^*$-algebra $A$ is $\mathcal{Z}$-stable if $A \otimes \mathcal{Z} \simeq A$.

Finally, throughout the paper, we write $B \sqcup C$ to indicate that the union of sets $B$ and $C$ is a disjoint union. In addition, we denote by $\lceil \cdot \rceil$ the ceiling function and by $\lfloor \cdot \rfloor$ the floor function from $[0, \infty)$ to $\mathbb{N}$.

### 3. Amenability of Extended Coarse Spaces

In this section, we recall and study the amenability of (uniformly locally finite) extended metric spaces from a coarse geometric point of view. In particular, we introduce a strengthening of the notion of metric amenability called ubiquitous (metric) amenability, which will be a central tool in our investigation of coarse structures of groupoids. In particular, we prove a pair of lemmas at the end of the section that display how ubiquitous amenability and non-amenability lead to contrasting behaviors on bounded enlargements of arbitrary finite subsets in metric spaces.

**Definition 3.1.** Recall an extended metric space is a metric space in which the metric is allowed to take the value $\infty$. An extended metric space admits a unique partition into ordinary metric spaces, called its coarse connected components, such that two points have finite distance if and only if they are in the same coarse connected component. An extended metric space is called locally finite if any bounded set has finite cardinality.

Let $(X, d)$ be a locally finite extended metric space and $A$ be a subset of $X$. For any $R > 0$ we define the following boundaries of $A$:

(i) outer $R$-boundary: $\partial^+_R A = \{ x \in X \setminus A : d(x, A) \leq R \}$;

(ii) inner $R$-boundary: $\partial^-_R A = \{ x \in A : d(x, X \setminus A) \leq R \}$;

(iii) $R$-boundary: $\partial_R A = \{ x \in X : d(x, A) \leq R \text{ and } d(x, X \setminus A) \leq R \}$.

**Remark 3.2.** Let $(X, d)$ be an extended metric space. Suppose $A \subset X$ and $R > 0$. It is straightforward to see $\partial^+_R A \subset B(\partial^-_R A, R)$ and $\partial^-_R A \subset B(\partial^+_R A, R)$.

The following concept of amenability of extended metric spaces was introduced in [BW92] by Block and Weinberger and further studied in [ALLW18] by Ara, Li, Lledò, and the second author.

**Definition 3.3.** Let $(X, d)$ be a locally finite extended metric space. For $R > 0$ we define the following boundaries of $A$:

(i) For $R > 0$ and $\epsilon > 0$, a finite non-empty set $F \subset X$ is called $(R, \epsilon)$-Følner if it satisfies

$$\frac{|\partial_R F|}{|F|} \leq \epsilon.$$

We denote by $\text{Føl}(R, \epsilon)$ the collection of all $(R, \epsilon)$-Følner sets.

(ii) The space $(X, d)$ is called amenable if, for every $R > 0$ and $\epsilon > 0$, there exists an $(R, \epsilon)$-Følner set.

The following elementary lemma shows that Følner sets can always be “localized” to a single coarse connected component.

**Lemma 3.4.** Let $(X, d)$ be a extended locally finite metric space and let $X_i, i \in I$, be its coarse connected components. Fix $R, \epsilon > 0$ and let $F$ be an $(R, \epsilon)$-Følner set of $X$. Write $F_i = F \cap X_i$ for each $i \in I$. Then there is $i_0 \in I$ such that $F_{i_0}$ is also an $(R, \epsilon)$-Følner set.
Lemma 3.7. Suppose that \( \partial N \) exist distinct points \( x \). The proof is straightforward by observing that for any \( d(N) \) tended metric spaces is straightforward but useful.

The following criteria for establishing amenability for uniformly locally finite extended metric spaces is straightforward but useful.

Definition 3.5. An extended metric space \( (X,d) \) is called ubiquitously amenable (or ubiquitously metrically amenable) if, for every \( R > 0 \) and \( \epsilon > 0 \), there exists an \( S > 0 \) such that for any \( x \in X \), there is an \( (R,\epsilon) \)-Følner set \( F \) in the ball \( B(x,S) \).

An extended metric space \( (X,d) \) is called uniformly locally finite \( ^4 \) if for any \( R > 0 \), there is a uniform finite upper bound on the cardinalities of all closed balls with radius \( R \). i.e.,

\[
\sup_{x \in X} |B(x,R)| < \infty.
\]

To simplify the notation, for uniformly locally finite space \( (X,d) \), we define an function \( \mathfrak{N} : \mathbb{R}^+ \to \mathbb{N} \) by

\[
\mathfrak{N}(r) = \sup_{x \in X} |B(x,r)|.
\]

The following criteria for establishing amenability for uniformly locally finite extended metric spaces is straightforward but useful.

Proposition 3.6. Let \( (X,d) \) be a uniformly locally finite extended metric space. The following are equivalent.

(i) For any \( R, \epsilon > 0 \) there is a finite set \( F \subset X \) such that \( |\partial_R F| \leq \epsilon |F| \).

(ii) For any \( R, \epsilon > 0 \) there is a finite set \( F \subset X \) such that \( |\partial^+_R F| \leq \epsilon |F| \).

(iii) For any \( R, \epsilon > 0 \) there is a finite set \( F \subset X \) such that \( |\partial^-_R F| \leq \epsilon |F| \).

(iv) For any \( R, \epsilon > 0 \) there is a finite set \( F \subset X \) such that \( |\partial F, R| \leq (1+ \epsilon) |F| \).

Proof. The proof is straightforward by observing that for any \( R > 0 \) and \( F \subset X \) one has \( \partial_R F = \partial^+_R F \cup \partial^-_R F \) and \( \partial_R^+ F = \partial (F, R) \setminus \partial F \) as well as the facts \( |\partial_R^+ F| \leq \mathfrak{N}(R) |\partial_R F| \) and \( |\partial_R^- F| \leq \mathfrak{N}(R) |\partial_R^+ F| \) by Remark \( \ref{remark} \).

The following lemma is useful in establishing Proposition \( \ref{prop3.6} \) and \( \ref{prop3.7} \).

Lemma 3.7. Suppose that \( (X,d) \) is a uniformly locally finite extended metric space. Let \( s > 0 \) and \( n \in \mathbb{N} \). Then for any finite set \( F \subset X \) satisfying \( |F| \geq n \cdot \mathfrak{N}(s) \), there exist distinct points \( x_1, \ldots, x_n \in F \) such that, for all \( i, j \in \{1, \ldots, n\} \) with \( i \neq j \), one has \( d(x_i, x_j) > s \). In particular there exists at least \( n \) many disjoint balls \( \overline{B}(x_i, s/2) \) for \( i = 1, \ldots, n \) in \( B(F, s/2) \).

Proof. We choose points \( x_1, \ldots, x_n \) by induction. First pick \( x_1 \in F \). Suppose that \( x_1, \ldots, x_k \) has been defined such that \( d(x_i, x_j) > s \) for all \( 1 \leq i \neq j \leq k \). Then observe that

\[
|\bigcup_{i=1}^k \overline{B}(x_i, s)| \leq \sum_{i=1}^k |\overline{B}(x_i, s)| \leq k \cdot \mathfrak{N}(s).
\]

\footnote{This notion also appeared as bounded geometry in the literature.}
Then choose \( x_{k+1} \in F \setminus \bigcup_{i=1}^{k} B(x_i, s) \) whose cardinality satisfies

\[
|F \setminus \bigcup_{i=1}^{k} B(x_i, s)| \geq (n - k) \cdot \mathfrak{N}(s).
\]

This finishes the proof. \( \square \)

The following proposition shows that Følner sets appear quite repetitively in ubiquitously amenable and uniformly locally finite spaces.

**Proposition 3.8.** Let \((X, d)\) be a ubiquitously amenable and uniformly locally finite extended metric space. Then for all \( r, \epsilon > 0 \) there exists an \( S > 0 \) such that for all finite sets \( M \subset X \) there exists a finite set \( F \) with \( M \subset F \subset \bar{B}(M, S) \) and

\[
\frac{|\partial^+_r F|}{|F|} \leq \epsilon.
\]

**Proof.** Suppose the contrary, i.e., there exist \( r, s > 0 \) such that the statement does not hold. Since \((X, d)\) is ubiquitously amenable, for the \( r, \epsilon > 0 \) choose an \( s' > 0 \) such that for all \( x \in X \) there is an finite set \( F_x \subset \bar{B}(x, s') \) such that

\[
\frac{|\partial^+ r F_x|}{|F_x|} \leq \frac{\epsilon}{2}.
\]

Now define

\[
n = \left\lfloor \frac{2\mathfrak{N}(r)}{\epsilon} \right\rfloor \cdot \mathfrak{N}(2s'), \quad \hat{S} = \left\lfloor \log(1+\epsilon) n \right\rfloor \cdot r \quad \text{and} \quad S = \hat{S} + s' + 1.
\]

Then for the \( S \) there is a finite set \( M \) in \( X \) such that for every finite \( F \) with \( M \subset F \subset \bar{B}(M, S) \) always satisfies

\[
\frac{|\partial^+ r F|}{|F|} > \epsilon.
\]

Thus in particular one has

\[
\frac{|\bar{B}(F, r)|}{|F|} > 1 + \epsilon.
\]

Then by induction we define \( F_0 = M \) and \( F_{k+1} = \bar{B}(F_k, r) \) for \( k \in \mathbb{N} \). This implies that \(|F_{k+1}| = |\bar{B}(F_k, r)| > (1 + \epsilon)|F_k|\) whenever \( F_k \subset \bar{B}(M, S) \). Thus if \( F_k \subset \bar{B}(M, S) \) one has

\[
\frac{|\bar{B}(M, kr)|}{|M|} \geq \frac{|F_k|}{|F_0|} > (1 + \epsilon)^k.
\]

In particular, by the definition of \( \hat{S} \) one has

\[
\frac{|\bar{B}(M, \hat{S})|}{|M|} \geq (1 + \epsilon)^{\left\lfloor \log(1+\epsilon) n \right\rfloor} > n = \left\lfloor \frac{2\mathfrak{N}(r)}{\epsilon} \right\rfloor \cdot \mathfrak{N}(2s').
\]

Now write \( m = \left\lfloor \frac{2\mathfrak{N}(r)}{\epsilon} \right\rfloor \cdot |M| \). Then Lemma 3.7 implies that there are distinct points \( x_1, \ldots, x_m \in \bar{B}(M, \hat{S}) \) such that \( d(x_i, x_j) > 2s' \). Then we write \( F = M \cup (\bigcup_{i=1}^{m} F_{x_i}) \subset \bar{B}(M, S) \) because all \( F_{x_i} \subset \bar{B}(x_i, s') \). Then we have

\[
\frac{|\partial^+ r (F)|}{|F|} \leq \frac{|\partial^+ r (M)| + \sum_{i=1}^{m} |\partial^+ r (F_{x_i})|}{\sum_{i=1}^{m} |F_{x_i}|}.
\]
Because $|\partial_r^+(M)| \leq |B(M, r)| \leq 2\mathfrak{N}(r)|M|$ and
$$\frac{|M|}{\sum_{i=1}^m |F_x|} \leq \frac{|M|}{m} \leq \frac{\epsilon}{2\mathfrak{N}(r)},$$
one has
$$\frac{|\partial_r^+(M)|}{|M|} \frac{|M|}{\sum_{i=1}^m |F_x|} \leq \frac{\epsilon}{2}.$$
On the other hand, by the definition of all $F_x$, one has
$$\sum_{i=1}^m \frac{|\partial_r^+(F_x)|}{|F_x|} \cdot \frac{|F_x|}{\sum_{i=1}^m |F_x|} \leq \frac{\epsilon}{2} \sum_{i=1}^m \frac{|F_x|}{\sum_{i=1}^m |F_x|} = \frac{\epsilon}{2}.$$
This implies that
$$\frac{|\partial_r^+(F)|}{|F|} \leq \epsilon,$$
which is a contradiction. This finishes the proof. \square

In contrast, we show below a paradoxical phenomenon in non-amenable extended metric spaces that can be considered as the polar opposite of the above.

**Proposition 3.9.** Let $(X, d)$ be a uniformly locally finite extended metric space, which is not amenable. For all $n \in \mathbb{N}$ and $R > 0$ there exists an $S > 0$ such that for all finite set $M$ in $X$ there are at least $n|M|$ many disjoint $R$-balls, $B(x_i, R) : i = 1, \ldots, n|M|$, contained in $B(M, S)$.

**Proof.** Let $n \in \mathbb{N}$ and $R > 0$. Since $(X, d)$ is not amenable, there is an $\epsilon > 0$ and $r > 0$ such that for all finite set $F$ in $X$ one has
$$\frac{|B(F, r)|}{|F|} > 1 + \epsilon.$$
Choose a $k \in \mathbb{N}$ such that $(1 + \epsilon)^k \geq n \cdot \mathfrak{N}(2R)$. Then define $F_0 = M$ and $F_{k+1} = B(F_k, r)$ for $k \in \mathbb{N}$. For all finite set $M \subset X$ one has
$$\frac{|B(M, kr)|}{|M|} \geq (1 + \epsilon)^k \geq n \cdot \mathfrak{N}(2R).$$
We write $S = kr + R$. Then Lemma 3.7 implies that there exist distinct points $x_1, \ldots, x_n|M| \in B(M, kr)$ such that $d(x_i, x_j) > R$ for all $i \neq j \in \{1, \ldots, n|M|\}$. In particular there exists at least $n|M|$ disjoint balls $B(x_i, R)$ for $i = 1, \ldots, n$ in $B(M, S)$. \square

4. **Coarse geometry of étale groupoids**

A fundamental and motivating fact in coarse geometry is that one can always assign a length function to a countable discrete group, which induces a (right-)invariant proper metric on the group, in a way unique up to coarse equivalence - for a finitely generate group, this amounts to taking the graph metric of the Cayley graph (after fixing a set of generators). In this procedure, the amenability of the group itself is equivalent to the metric amenability mentioned in last section of the resulting metric space.
Motivated by this, one may establish a similar framework for locally compact \(\sigma\)-compact Hausdorff étale groupoids, realizing them as extended metric spaces by equipping metrics to all the source (or range) fibers in a uniform and invariant manner. We start our discussion without the topological structure.

**Definition 4.1.** An extended metric on a groupoid \( G \) is

- **invariant** (or, more precisely, right-invariant) if, for any \( x, y, z \in G \) with \( s(x) = s(y) = r(z) \), we have \( \rho(x, y) = \rho(xz, yz) \);
- **fiberwise** (or, more precisely, source-fiberwise) if, for any \( x, y \in G \), we have \( \rho(x, y) = \infty \) if and only if \( s(x) \neq s(y) \).

Just as in the case of groups, it is more efficient to encode invariant metrics by length functions. To the best knowledge of the authors, the discussion of length functions on étale groupoids first appeared in [OY19, Definition 2.21], with ideas from J.-L. Tu. Our terminology differs slightly.

**Definition 4.2.** Recall a length function on a groupoid \( G \) is a function \( \ell : G \to [0, \infty) \) satisfying, for any \( x, y \in G \),

(i) \( \ell(x) = 0 \) if and only if \( x \in G^{(0)} \),
(ii) (symmetry) \( \ell(x) = \ell(x^{-1}) \), and
(iii) (subadditivity) \( \ell(xy) \leq \ell(x) + \ell(y) \) if \( x \) and \( y \) are composable in \( G \).

On a groupoid \( G \), there is a canonical one-to-one correspondence between length functions and invariant fiberwise extended metrics. On the one hand, given any length function \( \ell \) on \( G \), we associate an extended metric \( \rho_{\ell} \) by declaring, for \( x, y \in G \),

\[
\rho_{\ell}(x, y) = \begin{cases} 
\ell(xy^{-1}), & s(x) = s(y) \\
\infty, & s(x) \neq s(y)
\end{cases}.
\]

On the other hand, given any invariant fiberwise extended metric \( \rho \) on \( G \), we associate a function \( \ell_{\rho} : G \to [0, \infty), \ g \mapsto \rho(g, s(g)) \), which does not take the value \( \infty \) since \( \rho \) is fiberwise.

**Lemma 4.3.** On a groupoid \( G \), the above assignments give rise to a pair of bijections between length functions and invariant fiberwise extended metrics.

**Proof.** It is routine to verify that \( \rho_{\ell} \) as defined above is indeed an extended metric, where positive definiteness and symmetricity of \( \ell \) lead to those of \( \rho_{\ell} \) and subadditivity leads to the triangle inequality. It is also clear that \( \rho_{\ell} \) is invariant and fiberwise. On the other hand, to verify that \( \ell_{\rho} \) is a length function, we see, with the help of invariance, the same correspondence between the conditions in the opposite direction. \( \square \)

Now we focus on étale groupoids. We show, in analogy with the case of groups, that a locally compact \(\sigma\)-compact Hausdorff étale groupoid determines, up to coarse equivalence, a canonical invariant fiberwise extended metric that enjoys the following properties.

**Definition 4.4.** Let \( \ell : G \to [0, \infty) \) be a length function on an étale groupoid \( G \). For any subset \( K \subseteq G \), we write

\[
\bar{\ell}(K) = \sup_{x \in K} \ell(x).
\]

We say \( \ell \) is
Lemma 4.7. equivalent to each other.

where we adopt the convention that \( f \) is nondecreasing unbounded functions

observe that for any \( r, s \) and

Under the correspondence of Lemma 4.3, coarse equivalence of two

Remark 4.6. i.e.,

It is straightforward to see that coarse equivalence of length functions is indeed

Lemma 4.5. Two length functions \( \ell_1, \ell_2 \) are coarsely equivalent if and only if there are non-decreasing unbounded functions \( f_+, f_- : [0, \infty) \to [0, \infty) \) (sometimes referred to as control functions) such that

\[
(4.1) \quad f_- (\ell_1(x)) \leq \ell_2(x) \leq f_+ (\ell_1(x)) \quad \text{for any} \ x \in \mathcal{G}.
\]

Moreover, we may also assume \( f_+(0) = f_-(0) = 0 \) in the above.

Proof. Assuming there are non-decreasing unbounded functions \( f_+, f_- : [0, \infty) \to [0, \infty) \) satisfying (4.1), then for any \( r > 0 \), we have

\[
\sup \{ \ell_2(x) : \ell_1(x) \leq r \} \leq \sup \{ f_+ (\ell_1(x)) : \ell_1(x) \leq r \} \leq f_+(r) < \infty,
\]

and

\[
\sup \{ \ell_1(x) : \ell_2(x) \leq r \} \leq \sup \{ \ell_1(x) : f_- (\ell_1(x)) \leq r \} \leq \sup \{ s : f_- (s) \leq r \} < \infty,
\]

thanks to the unboundedness of \( f_- \).

On the other hand, assuming \( \ell_1, \ell_2 \) are coarsely equivalent as above, we may define

\[
f_+ (r) = \sup \{ r, \ell_2(x) : \ell_1(x) \leq r \} \quad \text{and} \quad f_- (r) = \inf \{ r, \ell_2(x) : \ell_1(x) \geq r \}
\]

for \( r \in [0, \infty) \). It is immediate that both functions are nondecreasing, \( f_+ \) is unbounded, \( f_+(0) = f_-(0) = 0 \), and (4.1) is satisfied. To see \( f_+ \) is unbounded, we observe that for any \( r, s \geq 0 \), we have \( f_-(r) < s \) if and only if either \( r < s \) or there is \( x \in \mathcal{G} \) such that \( \ell_2(x) < s \) but \( \ell_1(x) \geq r \), the latter possibility implying \( r \leq \sup \{ \ell(x) : \ell_2(x) < s \} \). This shows that \( f_-^{-1}([0, s]) \) is bounded for any \( s \geq 0 \), i.e., \( f_- \) is unbounded. \( \square \)

Remark 4.6. Under the correspondence of Lemma 4.3, coarse equivalence of two length functions \( \ell_1 \) and \( \ell_2 \) translates to coarse equivalence of their induced extended metrics \( \rho_{\ell_1} \) and \( \rho_{\ell_2} \), that is, we have

\[
\sup \{ \rho_{\ell_1}(x, y) : \rho_{\ell_2}(x, y) \leq r \} < \infty \quad \text{and} \quad \sup \{ \rho_{\ell_2}(x, y) : \rho_{\ell_1}(x, y) \leq r \} < \infty,
\]

or equivalently, there are non-decreasing unbounded functions \( f_+, f_- : [0, \infty) \to [0, \infty) \) (sometimes referred to as control functions) such that

\[
f_- (\rho_{\ell_1}(x, y)) \leq \rho_{\ell_2}(x, y) \leq f_+ (\rho_{\ell_1}(x, y)) \quad \text{for any} \ x, y \in \mathcal{G},
\]

where we adopt the convention that \( f_+(\infty) = f_-(\infty) = \infty \).

Lemma 4.7. Any two coarse length functions on an \( \text{étale} \) groupoid \( \mathcal{G} \) are coarsely equivalent to each other.
\textbf{Proof.} Given two coarse length functions $\ell_1, \ell_2$ on $G$, we see that for any $r > 0$, by the properness of $\ell_1$, the set \( \{ g \in G \setminus G^{(0)} : \ell_1(g) \leq r \} \) is precompact, and thus by the facts that $\ell_2$ is controlled and $\ell_2(G^{(0)}) = \{0\}$, we have $\sup \{ \ell_2(g) : \ell_1(g) \leq r \} < \infty$. Similarly, we have $\sup \{ \ell_1(g) : \ell_2(g) \leq r \} < \infty$, as desired. \( \square \)

To prove the existence of coarse continuous length functions, we make use of the following simple topological fact. We include the proof for completeness.

\textbf{Lemma 4.8.} Let $X$ be a $\sigma$-compact, locally compact and Hausdorff space. Then there exists a continuous proper map $g : X \to [0, \infty)$.

\textbf{Proof.} Choose a sequence of compact subsets $K_0 \subset K_1 \subset \ldots \subset X$ with $K_i \subseteq K_{i+1}$ for each $i$ and $X = \bigcup_{i=0}^{\infty} K_i$. It follows that the closed sets $K_0, \partial K_1, \partial K_2, \ldots$ are disjoint, allowing us to define $g(K_0) = \{0\}$ and $g(\partial K_i) = \{i\}$ for $i = 1, 2, \ldots$. Applying the Tietze extension theorem to the compact Hausdorff spaces $K_{i+1} \setminus K_i^n$, for $i = 0, 1, 2, \ldots$, we obtain a continuous function $g : X \to [0, \infty)$ mapping $K_{i+1} \setminus K_i^n$ into $[i, i+1]$, for $i = 0, 1, 2, \ldots$, which implies it is proper. \( \square \)

\textbf{Theorem 4.9.} Up to coarse equivalence, any $\sigma$-compact locally compact Hausdorff étale groupoid has a unique coarse continuous length function.

\textbf{Proof.} Uniqueness up to coarse equivalence follows from Lemma 4.8. It remains to show existence. To get started, we choose a continuous function $f : G \to \{0\} \cup [1, \infty)$ such that $f^{-1}(0) = G^{(0)}$, $f(x) = f(x^{-1})$ for any $x \in G$, and $f |_{G \setminus G^{(0)}}$ is proper, i.e., for any $r \geq 1$, the inverse image $f^{-1}([1, r])$ is compact. Indeed, to construct $f$, we first observe that since $G^{(0)}$ is a clopen subset of $G$, the complement $G \setminus G^{(0)}$ is also $\sigma$-compact, locally compact and Hausdorff, which enables us to apply Lemma 4.8 to obtain a proper continuous function $g : G \setminus G^{(0)} \to [0, \infty)$ and then define

$$f(x) = \begin{cases} 0, & x \in G^{(0)} \\ 1 + \frac{g(x) + g(x^{-1})}{2}, & x \in G \setminus G^{(0)}, \end{cases}$$

which clearly satisfies all the requirements.

We define a function $\ell : G \to [0, \infty)$ by

$$\ell(x) = \inf \left\{ \sum_{j=1}^{k} f(y_j) : k \in \mathbb{N} \text{ and } y_1, \ldots, y_k \in G \setminus G^{(0)} \text{ such that } x = y_1 \ldots y_k s(x) \right\}$$

for $x \in G$, where the degenerate case of $k = 0$ corresponds to $x = s(x)$ and $\ell(x) = 0$. It is immediate that the function $\ell$ defined above is a length function on $G$.

To study $\ell$, we describe an equivalent definition of it in terms of the sets $G^{(n)}$ of composable $n$-tuples and the $n$-ary multiplication maps $\delta^{(n)}$. For $n = 1, 2, \ldots$, define

$$G^{(n)} = G^{(n)} \cap \left( G \setminus G^{(0)} \right)^n = \left\{ (x_1, \ldots, x_n) \in G^{(n)} : x_1, \ldots, x_n \in G \setminus G^{(0)} \right\},$$

$$\delta^{(n)} = \delta^{(n)} \mid_{G^{(n)}} : G^{(n)} \to G, \quad (x_1, \ldots, x_n) \mapsto x_1 \cdots x_n,$$

$$f^{(n)} : G^{(n)} \to [0, \infty), \quad (x_1, \ldots, x_n) \mapsto \sum_{j=1}^{k} f(x_j),$$

where $\partial G^{(n)}$ is the set of all boundary elements in $G^{(n)}$. The length function $\ell$ can be expressed as

$$\ell(x) = \sup \left\{ \ell(x_1, \ldots, x_n) : \delta^{(n)}(x_1, \ldots, x_n) = x \right\}$$

for $x \in G$.
Moreover, observe that for \( n = 1, 2, \ldots \), the range of \( \hat{f}^{(n)} \) is contained in \([n, \infty)\) and, for any \( r \geq 1 \), we have

\[
\left( \hat{f}^{(n)} \right)^{-1} ([1, r]) \subseteq (f^{-1}([1, r]))^n.
\]

Hence for any \( x \in G \) and \( N \in \mathbb{N} \) satisfying \( \ell(x) \leq N \), we may remove large values that do not affect the infimum and obtain

\[
(4.2) \quad \ell(x) = \inf \left( \{ f(x) \} \cup \bigcup_{j=2}^{N} \left( \left( f^{-1}([1, N]) \right)^j \cap \left( \delta^{(n)} \right)^{-1} (x) \right) \right).
\]

To show that \( \ell \) is proper, let \( K \subset G \setminus G^{(0)} \) and suppose \( \ell(K) < N \) for some \( N \in \mathbb{N} \). Then for any \( x \in K \), since \( f(x) \geq 1 \), it follows from (4.2) that there is \( n \in 1, \ldots, N \) such that the set

\[
\left( f^{-1}([1, N]) \right)^n \cap \left( \delta^{(n)} \right)^{-1} (x)
\]

is non-empty. Therefore we have

\[
K \subseteq \bigcup_{j=1}^{N} f^{-1}([1, N]) \cdots f^{-1}([1, N]),
\]

the latter set being a finite union of products of compact sets, and thus compact. This shows that \( \ell \) defined above is proper.

To show that \( \ell \) is continuous, it suffices to prove that for any \( x \in G \), there is an open neighborhood \( U \) of \( x \) such that \( \ell \) is continuous when restricted to \( U \). To this end, we let \( N = \lceil \ell(x) + 1 \rceil \). Thus for any \( y \) in the open neighborhood \( f^{-1}([0, N]) \) of \( x \), the formula (4.2) applies with \( y \) in place of \( x \). Observe that for \( j = 1, 2, \ldots, N \), by Corollary 2.6 and the fact that \( G^{(j)} \) is a clopen subset of \( G^{(j)} \), we know \( \delta^{(j)} \) is a local homeomorphism. Applying Lemma 2.6 with \( \delta^{(j)} \), \( x \), and \( (f^{-1}([1, N]))^j \) in place of \( f, y, \) and \( K \), we may find an open neighborhood \( U^{(j)} \) of \( x \) inside \( f^{-1}([0, N]) \) and a finite family of open subsets \( V^{(j)}_1, \ldots, V^{(j)}_{m_j} \) in \( G^{(j)} \) such that \( \delta^{(j)} \) restricts to a homeomorphism between \( V^{(j)}_i \) and \( U^{(j)} \), for any \( i \in 1, \ldots, m_j \), and we have

\[
(f^{-1}([1, N]))^j \cap \left( \delta^{(j)} \right)^{-1} \left( U^{(j)} \right) \subseteq V^{(j)}_1 \cup \ldots \cup V^{(j)}_{m_j}.
\]

Writing \( \eta^{(j)}_i : U^{(j)} \to V^{(j)}_i \) for the inverse of \( \delta^{(j)} \mid V^{(j)}_i \), for \( i = 1, \ldots, m_j \). Then for any \( y \in U^{(j)} \), we have

\[
(f^{-1}([1, N]))^j \cap \left( \delta^{(j)} \right)^{-1} (y) = (f^{-1}([1, N]))^j \cap \left\{ \eta^{(j)}_i (y) : i = 1, \ldots, m_j \right\}.
\]

Let \( U = U_1 \cap \ldots \cap U_N \). Then, for any \( y \in U \), we may rewrite (4.2) as

\[
(4.3) \quad \ell(y) = \min \left\{ f(y), \left( \hat{f}^{(n)} \circ \eta^{(j)}_i \right) (y) : j = 1, \ldots, N, \ i = 1, \ldots, m_j \right\}.
\]
Hence on the open neighborhood $U$ of $x$, $\ell$ is equal to the minimum of finite number of continuous functions, and is thus itself continuous, as desired.

The controlledness of $\ell$ follows directly from its continuity. Therefore $\ell$ is a coarse continuous length function. □

**Remark 4.10.** If a $\sigma$-compact locally compact Hausdorff étale groupoid $\mathcal{G}$ is also ample, then we can choose a coarse length function $\ell$ that is locally constant. Indeed, when carrying out the proof of Theorem 4.9, we observe that we can choose the function $g$ and thus also the function $f$ to be locally constant, by choosing an increasing sequence of open compact subsets $K_0 \subset K_1 \subset \ldots \subset \mathcal{G}\setminus \mathcal{G}^{(0)}$ with $\mathcal{G}\setminus \mathcal{G}^{(0)} = \bigcup_{i=0}^{\infty} K_i$ and then defining $g(x) = \min \{i : x \in K_i\}$ for all $x \in \mathcal{G}\setminus \mathcal{G}^{(0)}$. It then follows from (4.3) that $\ell$ is locally equal to the minimum of a finite collection of locally finite functions, and thus is itself locally constant.

**Definition 4.11.** Let $\mathcal{G}$ be a $\sigma$-compact locally compact Hausdorff étale groupoid. The unique-up-to-coarse-equivalence invariant fiberwise extended metric induced by a coarse continuous length function on $\mathcal{G}$ will be called a canonical extended metric. We will abuse notation and denote any such metric by $\rho$ or $\rho_{\mathcal{G}}$.

**Example 4.12.** Let $X$ be a $\sigma$-compact locally compact Hausdorff space and $\Gamma$ be a countable group that acts on $X$ by homeomorphisms. Then the transformation groupoid $X \rtimes \Gamma$ (c.f., Example 2.2) is also $\sigma$-compact. To construct a coarse continuous length function on $X \rtimes \Gamma$, we may fix a proper length function $\ell_{\Gamma}$ on $\Gamma$ and a continuous proper function $g : X \rightarrow [0, \infty)$, and then define

$$\ell_{X \rtimes \Gamma} : X \rtimes \Gamma \rightarrow [0, \infty), \quad (\gamma x, \gamma, x) \mapsto \ell_{\Gamma}(\gamma) (1 + \max \{g(\gamma x), g(x)\}).$$

Note that when $X$ is compact, then we may simply choose $g = 0$ and thus $\ell_{X \rtimes \Gamma}(\gamma x, \gamma, x) = \ell_{\Gamma}(\gamma)$ for any $(\gamma x, \gamma, x) \in X \rtimes \Gamma$.

**Lemma 4.13.** Any canonical extended metric $\rho$ on a $\sigma$-compact locally compact Hausdorff étale groupoid $\mathcal{G}$ is uniformly locally finite.

**Proof.** Let $\ell$ be the coarse length function that induces $\rho$ by Lemma 4.3. Fix $R > 0$. Since $\ell$ is proper, the set $L_R = \{z \in \mathcal{G}\setminus \mathcal{G}^{(0)} : \ell(z) \leq R\}$ is precompact. Then there is a finite family $\{V_1, \ldots, V_m\}$ of precompact open bisections such that $L_R \subset \bigcup_{i=1}^{m} V_i$.

Then, for every $x \in \mathcal{G}$, one has

$$\bar{B}(x, R) = \{y \in \mathcal{G} : \rho(y, x) \leq R\} = \{y \in \mathcal{G} : \ell(y x^{-1}) \leq R\} \subset \{x\} \cup L_R x \subset \{x\} \cup \bigcup_{i=1}^{m} V_i x.$$ 

This implies that $|\bar{B}(x, R)| \leq m$. Since $R$ was arbitrarily chosen and $m$ only depends on $R$, thus $\rho$ is uniformly locally finite. □

**Remark 4.14.** For readers familiar with abstract coarse spaces in terms of entourages (c.f., [Roe03, Chapter 2]), we point out that the coarse structure on $\mathcal{G}$ (as a set) determined by any canonical extended metric can be directly defined as follows: a subset $E$ of $\mathcal{G} \times \mathcal{G}$ is an entourage if and only if there is a precompact subset $K$ of $\mathcal{G}$ such that for any $(x, y) \in E$, we have either $x = y$ or $x \in Ky$. This
construction is different from, but related to, the notion of coarse structures on a groupoid (c.f., [HPR97, TWY18]), in that our entourages are subsets of $G \times G$ with $G$ embedded as the diagonal, instead of subsets of $G$ with $G^{(0)}$ playing the role of a diagonal, but on the other hand, our coarse structure can be viewed as induced from the smallest coarse structure on the groupoid $G$ (generated by the relatively compact subsets and $G^{(0)}$) via the canonical translation action of $G$ on itself.

A lot of the contents in this paper may be handled with this abstract coarse structure, which would have the advantage of circumventing the somewhat inconvenient fact that the canonical extended metric is only unique up to coarse equivalence. The definition and some basic properties even extend beyond the case of $\sigma$-compact Hausdorff étale groupoids. However, we opt to stick to the language of metrics since it is more intuitive while $\sigma$-compact Hausdorff groupoids are prevalent in the main applications we have in mind (indeed, it is necessary to ensure that $C^\ast_r(G)$ is separable).

5. Fiberwise amenability

In this section, we introduce the notions of fiberwise amenability and ubiquitous fiberwise amenability for étale groupoids, with inspirations from (uniform) metric amenability (c.f., Definition 3.3 and 3.5). Fiberwise amenability is closely related to the existence of invariant measures on the unit space of an étale groupoid. As a motivating example, a transformation groupoid is (uniformly) fiberwise amenable if and only if the acting group is amenable. Ubiquitous fiberwise amenability will play an important auxiliary role when we discuss groupoid strict comparison and almost elementariness for minimal groupoids in the later sections. For this purpose, we show in the later half of this section that for minimal groupoids, fiberwise amenability is also equivalent to the a priori stronger notion of ubiquitous fiberwise amenability.

We first define boundary sets in groupoids in analogy with Definition 3.1.

**Definition 5.1.** Let $G$ be a groupoid. For any subsets $A, K \subseteq G$, we define the following boundary sets:

(i) left outer $K$-boundary: $\partial^+ KA = (KA) \setminus A = \{yx \in G \setminus A : y \in K, x \in A\}$;
(ii) left inner $K$-boundary: $\partial^- KA = A \cap (K^{-1}G \setminus A) = \{x \in A : yx \in G \setminus A \text{ for some } y \in K\}$;
(iii) left $K$-boundary: $\partial K A = \partial^+ KA \cup \partial^- KA$.

Observe that if $A$ as above is contained in a single source fiber, then $KA$ and all these boundary sets are also contained in this source fiber. This is the reason for the terminology fiberwise amenability.

**Remark 5.2.** For any subsets $A, K \subseteq G$, it is straightforward to see $\partial^+ KA \subset K\partial^- KA$ and $\partial^- KA \subset K^{-1}\partial^+ KA$.

The following concept is analogous to the metric case, too.

**Definition 5.3.** Let $G$ be a locally compact étale groupoid. For any subset $K \subseteq G$ and $\epsilon > 0$, a finite non-empty set $F \subset X$ is called $(K, \epsilon)$-Følner if it satisfies

$$\frac{|\partial_K F|}{|F|} \leq \epsilon.$$ 

We denote by $\text{Fol}(K, \epsilon)$ the collection of all $(K, \epsilon)$-Følner sets.
This leads to a natural definition of fiberwise amenability.

**Definition 5.4.** Let \( \mathcal{G} \) be a locally compact étale groupoid.

1. We say \( \mathcal{G} \) is **fiberwise amenable** if for any compact subset \( K \) of \( \mathcal{G} \) and any \( \epsilon > 0 \), there exists a \((K, \epsilon)\)-Følner set.
2. We say \( \mathcal{G} \) is **ubiquitously fiberwise amenable** if and only if for any compact subset \( K \) of \( \mathcal{G} \) and any \( \epsilon > 0 \), there exists a compact subset \( L \) of \( \mathcal{G} \) such that for any unit \( u \in \mathcal{G}^{(0)} \), there is a \((K, \epsilon)\)-Følner set in \( Lu \cup \{u\} \).

Since the groupoids we focus on are \(\sigma\)-compact and come equipped with extended metric structure in a somewhat canonical way (c.f., Definition 4.11), we may also reformulate Definition 5.4 using the canonical extended metric. This will establish a connection with Section 3 and enable us to apply the results there.

**Proposition 5.5.** Let \( \mathcal{G} \) be a \(\sigma\)-compact locally compact Hausdorff étale groupoid and let \((\mathcal{G}, \rho)\) be the extended metric space induced by a coarse length function \(\ell\).

1. The groupoid \( \mathcal{G} \) is fiberwise amenable if and only if for any compact subset \( K \) of \( \mathcal{G} \) and any \( \epsilon > 0 \), there exists a nonempty finite subset \( F \) in \( \mathcal{G} \) satisfying
   \[
   \frac{|KF|}{|F|} \leq 1 + \epsilon,
   \]
   if and only if \((\mathcal{G}, \rho)\) is amenable in the sense of Definition 3.3.

2. The groupoid \( \mathcal{G} \) is ubiquitously fiberwise amenable if and only if for any compact subset \( K \) of \( \mathcal{G} \) and any \( \epsilon > 0 \), there exists a compact subset \( L \) of \( \mathcal{G} \) such that for any unit \( u \in \mathcal{G}^{(0)} \), there is a nonempty finite subset \( F \) in \( Lu \cup \{u\} \) satisfying
   \[
   \frac{|KF|}{|F|} \leq 1 + \epsilon,
   \]
   if and only if \((\mathcal{G}, \rho)\) is ubiquitously amenable in the sense of Definition 3.5.

**Proof.** We prove the second statement, the first being similar. To prove the three conditions are equivalent, we first observe that the equivalence of the first two follows from Remark 5.2.

To prove the last condition is equivalent to the rest, we first observe that by Proposition 5.4, the ubiquitous fiberwise amenability of \( \mathcal{G} \) is equivalent to that for any \( R > 0 \) and \( \epsilon > 0 \), there is \( S > 0 \) such that for any \( x \in \mathcal{G} \), there is a nonempty finite subset \( F \) in \( B_\rho(x, S) \) such that \( |B_\rho(F, R)| \leq (1 + \epsilon)|F| \). Making use of the right-invariance of \( \rho \), we replace \( x \) by \( r(x) \) and \( F \) by \( Fx^{-1} \) if necessary, we see that, without loss of generality, we may replace, in the above, the quantifier \( x \in \mathcal{G} \) by \( x \in \mathcal{G}^{(0)} \).

Now, to prove the ubiquitous fiberwise amenability of \( \mathcal{G} \) implies ubiquitous amenability of \((\mathcal{G}, \rho)\), we fix arbitrary \( R > 0 \) and \( \epsilon > 0 \), from which we define \( K \) to be the closure of \( \{x \in \mathcal{G} \setminus \mathcal{G}^{(0)} : \ell(x) \leq R\} \), which is precompact by the properness of \( \ell \), and then our assumption provides us a compact subset \( L \) of \( \mathcal{G} \) such that for any unit \( u \in \mathcal{G}^{(0)} \), there is a nonempty finite subset \( F \) in \( Lu \cup \{u\} \) satisfying \( |KF| \leq (1 + \epsilon)|F| \); thus setting \( S = \sup_{x \in L} \ell(x) \), which is finite as \( \ell \) is controlled, we see that for any \( x \in \mathcal{G}^{(0)} \), there is a nonempty finite subset \( F \) in \( B_\rho(x, S) \) such that \( |B_\rho(F, R)| \leq (1 + \epsilon)|F| \), as desired. The reverse direction follows the same arguments. \(\square\)
Remark 5.6. Let $\alpha : \Gamma \curvearrowright X$ be an action of a countable discrete group $\Gamma$ on a compact space $X$. We denote by $X_\times \alpha \Gamma$ the transformation groupoid of this action $\alpha$. When we equip $\Gamma$ with a proper length function $\ell_\Gamma$ and $X_\times \alpha \Gamma$ with the induced length function $\ell_{X_\times \alpha \Gamma} : (\gamma x, \gamma x) \mapsto \ell_\Gamma(\gamma)$ (c.f., Example 4.12), each source fiber 

\[ (X_\times \alpha \Gamma)_x = \{ (\gamma x, \gamma x) : \gamma \in \Gamma \}, \]

for $x \in X$, becomes isometric to $\Gamma$. Therefore $X_\times \alpha \Gamma$ is fiberwise amenable if and only if $\Gamma$ is amenable.

In particular, the group $\Gamma$ as a groupoid is fiberwise amenable if and only if it is ubiquitously fiberwise amenable.

Remark 5.7. Fiberwise amenability is not an interesting property for groupoids $G$ with noncompact unit spaces, for it is automatically satisfied in this case. Indeed, for any compact subset $K$ of $G$, if we choose an arbitrary point $u$ in $G^{(0)} \setminus s(K)$, then $Ku = \emptyset$ and thus $\{ u \}$ becomes a $(K, 0)$-Følner set. Ubiquitous fiberwise amenability may still fail; an easy example being the disjoint union of two groupoids, the first having a noncompact unit space and the second lacking ubiquitous fiberwise amenability.

Focusing on the case of compact unit spaces, we next show that fiberwise amenability implies the existence of invariant probability measures on unit spaces. This directly generalizes the case of actions by amenable groups on compact spaces.

Definition 5.8. A measure on the unit space of a locally compact Hausdorff étale groupoid $G$ is invariant if $\mu(\rho(U)) = \mu(s(U))$ for any measurable bisection $U$. We write $M(G)$ for the collection of all invariant Borel probability measures on $G^{(0)}$.

Proposition 5.9. Let $G$ be a fiberwise amenable, $\sigma$-compact, locally compact Hausdorff étale groupoid with a compact unit space. Then $M(G) \neq \emptyset$.

Proof. It suffices to show that there is a Borel probability measure $\mu$ on $G^{(0)}$ such that $\mu(\rho(f)) = \mu(s(f))$ for all function $f \in C_c(G)_+$ whose support $\text{supp}(f)$ is a compact bisection. We may also assume $\| f \| \leq 1$. Write $K = \text{supp}(f)$ for simplicity. Note that $r(f), s(f)$ are functions supported on $r(K)$ and $s(K)$, respectively. Now, we work in the metric space $(G, \rho)$ defined above. First, define

$$ R = \sup_{y \in K \cup K^{-1}} \ell(y) < \infty. $$

Then one has $(K \cup K^{-1})x \subset \bar{B}_\rho(x, R)$ for all $x \in G$.

Since $G$ is fiberwise amenable, for a decreasing sequence $\{ \epsilon_n : n \in \mathbb{N} \}$ converging to 0, we can choose a sequence of finite sets $\{ F_n \subset G : n \in \mathbb{N} \}$ such that for all $n \in \mathbb{N}$ one has

$$ |\bar{B}_\rho(F_n, n)| < (1 + \epsilon_n)|F_n|. $$

Now for each $n \in \mathbb{N}$ we define

$$ \mu_n = \frac{1}{|F_n|} \sum_{x \in F_n} \delta_{\gamma(x)} $$

which are probability measures on $G^{(0)}$. Suppose that $\mu$ is a $w^*$-cluster point of $\{ \mu_n : n \in \mathbb{N} \}$ and in fact we may assume $\mu_n \to \mu$ in the $w^*$-topology by passing to subsequences. We show that $\mu \in M(G)$ by estimating the following

$$ |\mu(\rho(f)) - \mu(s(f))| \leq |\mu(\rho(f)) - \mu_n(\rho(f))| + |\mu_n(\rho(f)) - \mu_n(s(f))| + |\mu_n(s(f)) - \mu(s(f))|. $$
Since \( K \) is a bisection, one has
\[
|\mu_n(r(f)) - \mu_n(s(f))| = \frac{1}{|F_n|} \left( \sum_{x \in F_n} r(f)(r(x)) - \sum_{x \in F_n} s(f)(r(x)) \right)
= \frac{1}{|F_n|} \left( \sum_{x \in F_n} r(f)(r(x)) - \sum_{x \in F_n} r(f)(Kx) \right)
= \frac{1}{|F_n|} \left( \sum_{x \in F_n} r(f)(r(x)) - \sum_{x \in Kn} r(f)(r(x)) \right).
\]
In addition, note that the function \( \kappa : F_n \setminus K F_n \to K^{-1} F_n \setminus F_n \) by \( \kappa(x) = K^{-1} x \) is bijective because \( K \) is a bisection. Then for all \( n > R \), one has \( (K \cup K^{-1}) F_n \subset \bar{B}_\rho(F_n, R) \subset \bar{B}_\rho(F_n, n) \), which implies that
\[
|(K \cup K^{-1}) F_n \setminus F_n| \leq \epsilon_n |F_n|.
\]
This shows that
\[
|KF_n \Delta F_n| = |KF_n \setminus F_n| + |F_n \setminus KF_n|
= |KF_n \setminus F_n| + |K^{-1} F_n \setminus F_n|
\leq 2 \epsilon_n |F_n|.
\]
Now for every \( \epsilon > 0 \) we choose an \( n > R \) big enough such that \( |\mu(r(f)) - \mu_n(r(f))| < \epsilon/3 \), \( |\mu(s(f)) - \mu_n(s(f))| < \epsilon/3 \) and \( \epsilon_n < \epsilon/6 \). This implies that
\[
|\mu(r(f)) - \mu(s(f))| < \epsilon.
\]
This establishes \( \mu(s(f)) = \mu(r(f)) \) as desired. \( \square \)

In the rest of the section, we show that for minimal groupoids, fiberwise amenability is equivalent to the a priori stronger notion of ubiquitous fiberwise amenability. The strategy to show the former implies the latter, roughly speaking, is: on the one hand, a Følner set on a single source fiber, is always able to “permeate” horizontally to nearby fibers; on the other hand, the recurrence behavior guaranteed by minimality allows every source fiber to “pick up” a Følner set from this permeation every so often, thus resulting in ubiquitous fiberwise amenability.

To explain how this “permeation” arises, it is convenient to use the following result about the existence of local trivializations that almost preserve the metric.

Lemma 5.10 (Local Slice Lemma). Let \( G \) be a \( \sigma \)-compact locally compact Hausdorff étale groupoid and let \( \rho \) be a canonical extended metric on \( G \) induced by a coarse continuous length function \( \ell \) as in Definition 4.11. Let \( u \in G^{(0)} \). Then for any \( R, \epsilon > 0 \), there are a number \( S \in [R, R + \epsilon] \), an open neighborhood \( V \) of \( u \) in \( G^{(0)} \), an open set \( W \) in \( G \), and a homeomorphism \( f : \bar{B}_\rho(u, S) \times V \to W \) such that
1. \( f(u, v) = v \) for any \( v \in V \),
2. \( f(x, u) = x \) for any \( x \in \bar{B}_\rho(u, S) \),
3. \( f(\bar{B}_\rho(u, S) \times \{v\}) = \bar{B}_\rho(v, S) \) for any \( v \in V \), and
4. \( |\rho(x, y) - \rho(f(x, v), f(y, v))| < \epsilon \) for any \( x, y \in \bar{B}_\rho(u, S) \) and \( v \in V \).
Proof. By Lemma 4.13, the “open” ball $B_p(u, R + \varepsilon)$, i.e., the set $\{x \in \mathcal{G}_u : \ell(x) < R + \varepsilon\}$, is finite, and thus
$$\overline{\ell} (B_p(u, R + \varepsilon)) = \max \{\ell(x) : x \in B_p(u, R + \varepsilon)\} < R + \varepsilon.$$ Hence we may choose $S \in [R, R + \varepsilon] \cap (\overline{\ell} (B_p(u, R + \varepsilon)), R + \varepsilon)$, e.g.,
$$S = \max \left\{R, \frac{\overline{\ell} (B_p(u, R + \varepsilon)) + R + \varepsilon}{2} \right\},$$
which guarantees $\overline{B}_p(u, S) = B_p(u, R + \varepsilon)$ and thus $S > \overline{\ell} (B_p(u, S))$.

For each $x \in \overline{B}_p(u, S)$, choose an open bisection $U_x$ containing $x$ and let $f_x : s(U_x) \rightarrow U_x$ be the inverse of the homeomorphism $s |_{U_x}$. With out loss of generality, we may assume $U_u = \mathcal{G}^{(0)}$ and $f_u$ is the identity map. Define
$$L = \ell^{-1}([0, S]) \setminus \left( \bigcup_{x \in B_p(u, S)} U_x \right)$$
and $U = \mathcal{G}^{(0)} \setminus s(L)$. Unpacking the definition and using the fact $\overline{B}_p(v, S) = \ell^{-1}([0, S]) \cap s^{-1}(v)$ for any $v \in \mathcal{G}^{(0)}$, we have
$$U = \left\{ v \in \mathcal{G}^{(0)} : \overline{B}_p(v, S) \subseteq \bigcup_{x \in B_p(u, S)} U_x \right\} \quad (5.1)$$
and, in particular, $u \in U$. Since $\ell$ is proper and continuous, we see that $L$ is compact and hence $U$ is an open neighborhood of $u$ in $\mathcal{G}^{(0)}$. Define a continuous map
$$f : \overline{B}_p(u, S) \times U \rightarrow \mathcal{G}, \quad (x, v) \mapsto f_x(v).$$
It follows from the construction of the $f_x$’s that

1. $f(u, v) = v$ for any $v \in U$, and
2. $f(x, u) = x$ for any $x \in \overline{B}_p(u, S)$.

We also have $(s \circ f)(x, v) = v$ for any $(x, v) \in \overline{B}_p(u, S) \times U$. It then follows from (5.1) that
$$f (\overline{B}_p(u, S) \times \{v\}) \supseteq \overline{B}_p(v, S) \quad \text{for any} \ v \in U. \quad (5.2)$$

Now we define a finite collection of continuous maps
$$g_{xy} : U \rightarrow [0, \infty), \quad v \mapsto \rho (f(x, v), f(y, v)),$$ for $x, y \in \overline{B}_p(u, S)$ and define
$$\eta = \min \left\{\varepsilon, S - \overline{\ell} (\overline{B}_p(u, S)), \frac{\rho(x, y)}{2} : x, y \in \overline{B}_p(u, S) \text{ with } x \neq y \right\}.$$ Note that $\eta > 0$ by our choice of $S$. By continuity, there exists an open neighborhood $V$ of $u$ inside $U$ such that $|g_{xy}(u) - g_{xy}(v)| < \eta$ for any $v \in V$ and $x, y \in \overline{B}_p(u, S)$. This choice implies the following:

3. For any $x, y \in \overline{B}_p(u, S)$ and $v \in V$, since $g_{xy}(u) = \rho(x, y)$, we have
$$|\rho (x, y) - \rho (f(x, v), f(y, v))| < \varepsilon.$$
(4) For any $x \in \bar{B}_\rho(u, S)$ and $v \in V$, we have $\ell(f(x, v)) = \rho(f(x, v), f(u, v)) = g_{xu}(v) < g_{xu}(u) + \eta = \ell(x) + \eta \leq \ell(\bar{B}_\rho(u, S)) + \eta \leq S$, and thus combined with (5.2), we have
\[
f(\bar{B}_\rho(u, S) \times \{v\}) \subseteq \bar{B}_\rho(v, S).
\]

\[\bullet\] For any $v \in V$ and any $x, y \in \bar{B}_\rho(u, S)$ with $x \neq y$, we have $\rho(f(x, v), f(y, v)) = g_{xu}(v) > g_{xu}(u) - \eta = \rho(x, y) - \eta > 0$ and thus $f(x, v) \neq f(y, v)$. This implies that the collection $\{f_{x}(V) : x \in \bar{B}_\rho(u, S)\}$ of open sets is disjoint and $f$ is a homeomorphism onto its image when restricted to $\bar{B}_\rho(u, S) \times V$.

Defining $W = f(\bar{B}_\rho(u, S) \times V)$ and restricting $f$ to $\bar{B}_\rho(u, S) \times V$ thus completes the construction. \hfill \Box

The existence of local slices as in Lemma 5.10 allows us to “clone” a Følner set in every nearby source fiber.

**Lemma 5.11.** Let $G$ be a $\sigma$-compact locally compact Hausdorff étale groupoid and let $\rho$ be a canonical extended metric on $G$ induced by a coarse continuous length function $\ell$ as in Definition 4.11. Let $R, \varepsilon > 0$ and $u \in G(0)$. Let $F \subset G_u$ be an $(R, \varepsilon)$-Følner set. Then there is an open neighborhood $V$ of $u$ in $G(0)$ such that for any $v \in V$, there is an $(R, \varepsilon)$-Følner set in $\bar{B}_\rho(v, \ell(F) + \varepsilon)$.

**Proof.** Let $R' = \ell(F) + R + \varepsilon$ and $\eta = \min \{\varepsilon, \rho(x, y) - R : x, y \in \bar{B}_\rho(u, R') \}$ with $\rho(x, y) > R$. Applying Lemma 5.10 with $u$, $R'$, and $\eta$ in place of $u$, $\rho$, $\varepsilon$, we obtain a number $S \in [R', R' + \eta]$, an open neighborhood $V$ of $u$ in $G(0)$, an open set $W \subset G$, and a homeomorphism $f : \bar{B}_\rho(u, S) \times V \to W$ such that

1. $f(v, u) = v$ for any $v \in V$,
2. $f(x, u) = x$ for any $x \in \bar{B}_\rho(u, S)$,
3. $f(\bar{B}_\rho(u, S) \times \{v\}) = \bar{B}_\rho(v, S)$ for any $v \in V$, and
4. $|\rho(x, y) - \rho(f(x, v), f(y, v))| < \eta$ for any $x, y \in \bar{B}_\rho(u, S)$ and $v \in V$.

Observing that $F \subseteq \bar{B}_\rho(u, S)$, we then define, for any $v \in V$, the bijection
\[
\tau_v : \bar{B}_\rho(u, S) \to \bar{B}_\rho(v, S), \quad x \mapsto f(x, v)
\]
and the set
\[
F_v = \tau_v(F).
\]

For any $v \in V$, we claim that $F_v$ is the desired $(R, \varepsilon)$-Følner set in $\bar{B}_\rho(v, \ell(F) + \varepsilon)$. Indeed, it follows from condition (4) that $\ell(F_v) < \ell(F) + \eta \leq \ell(F) + \varepsilon$ and thus $F_v \subseteq \bar{B}_\rho(v, \ell(F) + \varepsilon)$. On the other hand, to see $F_v$ is an $(R, \varepsilon)$-Følner set just like $F$, it suffices to show that
\[
\partial^+_{R} F_v \subseteq \tau_v(\partial^+_{R} F) \quad \text{and} \quad \partial^{-}_{R} F_v \subseteq \tau_v(\partial^{-}_{R} F).
\]

To prove the former containment, we observe that for any $y \in \partial^+_{R} F_v$, since $\ell(y) \leq \ell(F_v) + R \leq \ell(F) + R + \varepsilon = R' \leq S$, it is in the range of $\tau_v$. Let $x = (\tau_v)^{-1}(y)$. Since $y \notin F_v$, we have $x \notin F$. It remains to show that $\rho(x, F) \leq R$. Suppose this were not the case, i.e., for any $z \in F$, we have $\rho(x, z) > R$. Then by our choice of $\eta$, we would have
\[
\rho(y, F_v) = \min \{\rho(\tau_v(x), \tau_v(z)) : z \in F\} \geq \min \{\rho(x, z) - \eta : z \in F\} \geq R,
\]
contradictory to the fact that $y \in \partial^+_{R} F_v$. This shows $\partial^+_{R} F_v \subseteq \tau_v(\partial^+_{R} F)$. \hfill \Box
Lemma 5.12. Let $G$ be a minimal locally compact Hausdorff étale groupoid. Let $K$ and $V$ be subsets of $G^{(0)}$ such that $K$ is compact and $V$ is non-empty and open. Then there are precompact open bisections $V_1, \ldots, V_n$ such that $\bigcup_{i=1}^n r(V_i) \subseteq V$ and $K \subseteq \bigcup_{i=1}^n s(V_i)$.

Proof. Since $G$ is minimal, for any $u \in K$, there is a $v \in V$ and $x \in G$ such that $r(x) = v$ and $s(x) = u$. Then since $G$ is locally compact étale, there is a precompact open bisection $V_x$ such that $x \in V_x \subseteq r^{-1}(V)$. This implies that $v = r(x) \in r(V_x) \subseteq V$ and $u = s(x) \in s(V_x)$. In addition, all such $s(V_x)$'s form an open cover of $K$. By compactness, there are finitely many precompact open bisection $V_1, \ldots, V_n$ such that $K \subseteq \bigcup_{i=1}^n s(V_i)$. In addition, our construction also implies $\bigcup_{i=1}^n r(V_i) \subseteq V$. □

Now we are ready to establish the equivalence of fiberwise amenability and ubiquitously fiberwise amenability for minimal second countable groupoids.

Theorem 5.13. Let $G$ be a $\sigma$-compact locally compact Hausdorff étale groupoid. Suppose $G$ is minimal. Then $G$ is fiberwise amenable if and only if it is ubiquitously fiberwise amenable.

Proof. The “if” direction follows directly from the definitions. To show the “only if” direction, we let $\rho$ be a canonical extended metric on $G$ induced by a coarse continuous length function $\ell$ as in Definition 3.11. By Proposition 5.5, it suffices to show, assuming the extended metric space $(G, \rho)$ is amenable, that it is also ubiquitously amenable, i.e., for every $R > 0$ and $\varepsilon > 0$, there exists an $S > 0$ such that for any $x \in G$, there is a $(R, \varepsilon)$-Følner set $F$ in the ball $B_\rho(x, S)$. To this end, given $R, \varepsilon > 0$, since we assume $(G, \rho)$ is amenable, we know there exists an $(R, \varepsilon)$-Følner set $F_0$ in $G$. By Lemma 3.11, we may assume without loss of generality that $F_0$ is contained in a single source fiber $G_u$ for some $u \in G^{(0)}$. By Lemma 5.11 there is an open neighborhood $V$ of $u$ in $G^{(0)}$ such that for any $v \in V$, there is an $(R, \varepsilon)$-Følner set $F_v$ in $B_\rho(v, \bar{\ell}(F_0) + \varepsilon)$. Let

$$K = s \left( \ell^{-1}([0, R]) \setminus G^{(0)} \right),$$

which is a compact subset of $G^{(0)}$, as $\ell$ is a continuous proper length function. By Lemma 5.12, there are precompact open bisections $V_1, \ldots, V_n$ such that $\bigcup_{i=1}^n r(V_i) \subseteq V$ and $K \subseteq \bigcup_{i=1}^n s(V_i)$. Let

$$S = \bar{\ell}(F_0) + \varepsilon + \max \{ \bar{\ell}(V_i) : i = 1, \ldots, n \},$$

To prove the latter containment, we observe that any $y \in \partial_R F_v$ is $F_v$ and thus we may define $x = (\tau_v)^{-1}(y)$ in $F$. It remains to show that $\rho(x, G \setminus F) \leq R$. Suppose this were not the case. Then by the decomposition $G \setminus F_v = (G \setminus B_\rho(v, S)) \cup (B_\rho(v, S) \setminus F_v)$ and our choice of $S$ and $\eta$, we would have

$$\rho(y, G \setminus F_v) = \inf \{ \rho(y, w) : w \in G \setminus F_v \} = \inf \{ \rho(y, w), \rho(\tau_v(x), \tau_v(z)) : w \in G \setminus B_\rho(v, S), z \in B_\rho(u, S) \setminus F \} > \min \{ S - \ell(y), \rho(x, z) - \eta : z \in B_\rho(u, S) \setminus F \} \geq R,$$

contradictory to the fact that $y \in \partial_R F_v$. This shows $\partial_R F_v \subseteq \tau_v(\partial_R F)$ and completes the proof. □

The following lemma underlies the recurrence behavior of minimal groupoids with compact unit spaces.

Lemma 5.12. Let $G$ be a minimal locally compact Hausdorff étale groupoid. Let $K$ and $V$ be subsets of $G^{(0)}$ such that $K$ is compact and $V$ is non-empty and open. Then there are precompact open bisections $V_1, \ldots, V_n$ such that $\bigcup_{i=1}^n r(V_i) \subseteq V$ and $K \subseteq \bigcup_{i=1}^n s(V_i)$.

Proof. Since $G$ is minimal, for any $u \in K$, there is a $v \in V$ and $x \in G$ such that $r(x) = v$ and $s(x) = u$. Then since $G$ is locally compact étale, there is a precompact open bisection $V_x$ such that $x \in V_x \subseteq r^{-1}(V)$. This implies that $v = r(x) \in r(V_x) \subseteq V$ and $u = s(x) \in s(V_x)$. In addition, all such $s(V_x)$’s form an open cover of $K$. By compactness, there are finitely many precompact open bisection $V_1, \ldots, V_n$ such that $K \subseteq \bigcup_{i=1}^n s(V_i)$. In addition, our construction also implies $\bigcup_{i=1}^n r(V_i) \subseteq V$. □

Now we are ready to establish the equivalence of fiberwise amenability and ubiquitously fiberwise amenability for minimal second countable groupoids.
which is finite since all the sets involved are precompact. Now, for any \( x \in \mathcal{G} \), we need to construct an \((R, \varepsilon)\)-Følner set \( F \) in \( \bar{B}_\rho(x, S) \). There are two cases:

- If \( r(x) \not\in K \), then \( \bar{B}_\rho(x, R) = \bar{B}_\rho(r(x), R) = (\mathcal{G}_{r(x)} \cap \ell^{-1}([0, R])) x = \{ r(x)x \} = \{ x \} \) by our choice of \( K \), and thus we may set \( F = \{ x \} \), which is an \((R, 0)\)-Følner set.

- If \( r(x) \in K \), then we may choose \( i_x \in \{ 1, \ldots, n \} \) such that \( r(x) = s(V_{i_x}) \). Let \( z \in V_{i_x} \) be such that \( r(x) = s(z) \). Note that \( r(z) \in V \) and thus we have an \((R, \varepsilon)\)-Følner set \( F_{r(z)} \in \bar{B}_\rho(r(z), \ell(F_0) + \varepsilon) \). Let \( F = F_{r(z)}x \), which is also an \((R, \varepsilon)\)-Følner set by the right-invariance of \( \rho \) (see Lemma 4.4).

Finally, since for any \( y \in F_{r(z)} \), we have \( \rho(yzzx, x) = \rho(yz, r(x)) = \ell(yz) \leq \ell(F_{r(z)}) + \ell(V_{i_x}) \leq \ell(F_0) + \varepsilon + \ell(V_{i_x}) \leq S \), we conclude that \( F \) is in the ball \( \bar{B}_\rho(x, S) \).

\[ \square \]

**Corollary 5.14.** Let \( \mathcal{G} \) be a \( \sigma \)-compact locally compact Hausdorff étale groupoid. Suppose \( \mathcal{G} \) is minimal and \( \mathcal{G}^{(0)} \) is noncompact. Then \( \mathcal{G} \) is ubiquitously fiberwise amenable.

**Proof.** This follows from Theorem 5.13 and Remark 5.7. \( \square \)

The following theorem, as an application of Theorem 5.13, shows a dichotomy on amenability against paradoxicality for locally compact Hausdorff second countable étale minimal groupoids on compact spaces as metric spaces.

**Theorem 5.15.** Let \( \mathcal{G} \) be a locally compact Hausdorff second countable minimal étale groupoid. Equip \( \mathcal{G} \) with the metric \( \rho \) as in Definition 4.7. Then we have the following dichotomy.

1. If \( \mathcal{G} \) is fiberwise amenable then for all \( R, \varepsilon > 0 \) there is a compact set \( K \subset \mathcal{G} \) such that for all compact set \( L \subset \mathcal{G} \) and all unit \( u \in \mathcal{G}^{(0)} \), there is a finite set \( F_u \) satisfying

   \[ Lu \subset F_u \subset KLu \text{ and } \bar{B}_\rho(F_u, R) \leq (1 + \varepsilon)|F_u|. \]

2. If \( \mathcal{G} \) is not fiberwise amenable then for all compact set \( L \subset \mathcal{G} \) and \( n \in \mathbb{N} \) there is a compact set \( K \subset \mathcal{G} \) such that for all compact set \( M \subset \mathcal{G} \), all unit \( u \in \mathcal{G}^{(0)} \), the set \( KMu \) contains at least \( n|M u| \) many disjoint sets of the form \( L\gamma u \), i.e., there exists a disjoint family \( \{ L\gamma_i u \subset KM u : i = 1, \ldots, n|M u| \} \).

**Proof.** Suppose that \( \mathcal{G} \) is fiberwise amenable. Theorem 5.13 shows that \( \mathcal{G} \) is in fact ubiquitously fiberwise amenable. Let \( R, \varepsilon > 0 \). Proposition 3.3 shows that there is an \( S > 0 \) such that for all compact set \( L \subset \mathcal{G} \) and \( u \in \mathcal{G}^{(0)} \) there is a finite set \( F_u \) satisfying

\[ Lu \subset F_u \subset B_{\rho}(Lu, S) \text{ and } |B_{\rho}(F_u, R)| \leq (1 + \varepsilon)|F_u|. \]

On the other hand, For this \( S > 0 \), define a compact set \( K = \{ z \in \mathcal{G} : \ell(z) \leq S \} \). It is straightforward to see for all \( x \in \mathcal{G} \) one has \( \bar{B}_{\rho}(x, S) \subset Kx \). This implies that \( \bar{B}_{\rho}(Lu, S) \subset KLu \) for all compact set \( L \subset \mathcal{G} \) and \( u \in \mathcal{G}^{(0)} \). This establishes (1).

Now suppose that \( \mathcal{G} \) is not fiberwise amenable. Let \( L \) be a compact subset of \( \mathcal{G} \) and \( n \in \mathbb{N} \). Define \( R = \sup_{y \in L} \ell(y) \prec \infty \) and thus \( Lx \subset \bar{B}_{\rho}(x, R) \) for all \( x \in \mathcal{G} \). Then Proposition 3.2 shows that there is an \( S > 0 \) such that for any finite set \( F \) in \( \mathcal{G} \) there are at least \( n|F| \) many \( R \)-balls contained in \( \bar{B}_{\rho}(F, S) \). In particular, this holds for the finite set \( F = Mu \) whenever \( M \subset \mathcal{G} \) is compact and \( u \in \mathcal{G}^{(0)} \).
On the other hand, for $K = \{z \in \mathcal{G} : \ell(z) \leq S\}$, one has $KM_u$ contains at least $n|M_u|$ many disjoint $R$-balls, say, $\{B_p(\gamma_i, R) : i = 1, \ldots, n|M_u|\}$. Now since $L\gamma_i u = L\gamma_i \subset B_p(\gamma_i, R)$ for each $i \leq n|M_u|$, the family \{L$\gamma_i u : i \leq n|M_u|$\} is disjoint and $KM_u$ contains $L\gamma_i u$ for all $i = 1, \ldots, n|M_u|$. This establishes (2). \hfill \square

6. Almost elementary étale groupoids and groupoid strict comparison

In this section, we introduce two regularity properties of étale groupoids, groupoid strict comparison and almost elementariness. Both are central to our analysis.

**Definition 6.1.** Let $\mathcal{G}$ be a locally compact Hausdorff étale groupoid.

(i) Let $K$ be a compact subset of $\mathcal{G}^{(0)}$ and $V$ an open subset of $\mathcal{G}^{(0)}$. We write $K \prec \mathcal{G} V$ if there are open bisections $A_1, \ldots, A_n$ such that $K \subset \bigcup_{i=1}^n s(A_i)$, $\bigcup_{i=1}^n r(A_i) \subset V$.

(ii) Let $U, V$ be open subsets of $\mathcal{G}^{(0)}$. We write $U \not\prec \mathcal{G} V$ if $K \prec \mathcal{G} V$ for every compact subset $K \subset U$.

(iii) If $X \rtimes_\alpha \Gamma$ is a transformation groupoid for an action $\alpha : \Gamma \curvearrowright X$ of countable discrete group $\Gamma$ on a compact metrizable space $X$, we write $\prec_\alpha$ instead of $\prec_{X \rtimes_\alpha \Gamma}$, and $\not\prec_\alpha$ instead of $\not\prec_{X \rtimes_\alpha \Gamma}$, for the sake of simplicity.

We remark that if $U$ is compact and open and $V$ is open, then $U \not\prec \mathcal{G} V$ if and only if $U \not\prec \mathcal{G} V$. We also point out that for any open sets $U, V$ in $\mathcal{G}^{(0)}$, it is not hard to verify that if $U \not\prec \mathcal{G} V$ then $\mu(U) \leq \mu(V)$ holds for all $\mu \in M(\mathcal{G})$ (c.f. Definition 5.8). The notion of groupoid strict comparison below is a partial converse of this condition.

**Definition 6.2.** Let $\mathcal{G}$ be a locally compact Hausdorff étale groupoid. We say $\mathcal{G}$ has groupoid strict comparison (or simply groupoid comparison or comparison) if, for any open sets $U, V$ in $\mathcal{G}^{(0)}$, we have $U \not\prec \mathcal{G} V$ whenever $\mu(U) < \mu(V)$ for all $\mu \in M(\mathcal{G})$. If a transformation groupoid $X \rtimes_\alpha \Gamma$ of an action $\alpha : \Gamma \curvearrowright X$ described in Definition 6.1(iii) has groupoid strict comparison, we say $\alpha$ has dynamical strict comparison.

**Remark 6.3.** It is not hard to see our dynamical strict comparison defined above for transformation groupoid $X \rtimes_\alpha \Gamma$ of an action; an action $\alpha : \Gamma \curvearrowright X$ described in Definition 6.1(iii) is equivalent to the dynamical strict comparison (c.f. Ker20 Definition 3.2) defined directly for the action $\alpha : \Gamma \curvearrowright X$.

On the other hand, for ample groupoids, it is helpful to work with a simplified version of groupoid strict comparison.

**Definition 6.4.** Let $\mathcal{G}$ be a locally compact Hausdorff étale ample groupoid. We say $\mathcal{G}$ has groupoid strict comparison for compact open sets for any compact open sets $U, V$ in $\mathcal{G}^{(0)}$ one has $U \not\prec \mathcal{G} V$ whenever $\mu(U) < \mu(V)$ for any $\mu \in M(\mathcal{G})$.

The following lemma proved by Kerr in Ker20 Lemma 3.3 is very useful. We record this here for completeness.

**Lemma 6.5 (Kerr).** Let $X$ be a compact metrizable space with a compatible metric $d$ and let $\Omega$ be a weak* closed subset of $M(X)$, which is the set consisting of all Borel regular probability measures on $X$. Let $A$ be a closed set and $O$ be an open
set in \(X\) such that \(\mu(A) < \mu(O)\) for all \(\mu \in \Omega\). Then there exists an \(\eta > 0\) such that the sets

\[
O_{-\eta} = \{x \in X : d(x, X \setminus O) > \eta\},
\]

and

\[
A_{+\eta} = \overline{B}(A, \eta) = \{x \in X : d(x, A) \leq \eta\}
\]
satisfy \(\mu(A_{+\eta}) + \eta \leq \mu(O_{-\eta})\) for all \(\mu \in \Omega\).

When the groupoid \(G\) is ample and \(G^{(0)}\) is compact metrizable, the two comparison properties introduced in Definitions 6.2 and 6.4 coincide.

**Proposition 6.6.** Let \(G\) be a locally compact Hausdorff étale ample groupoid with a compact metrizable unit space. Then \(G\) has groupoid strict comparison if and only if \(G\) has groupoid strict comparison for compact open sets.

**Proof.** It suffices to show the “if” part. Suppose \(G\) has groupoid strict comparison for compact open sets. Now, let \(O, W\) be open sets in \(G^{(0)}\) such that \(\mu(O) < \mu(W)\) for any \(\mu \in M(G)\). Now since \(G\) is ample, for any compact set \(K \subset O\) there is a compact open set \(N\) such that \(K \subset N \subset O\) and also satisfies that \(\mu(N) < \mu(W)\) for all \(\mu \in M(G)\). Then Lemma 6.3 allows us to find an \(\eta > 0\) and open set \(W_{-\eta}\) such that \(W_{-\eta} \subset W_{-\eta} \subset W_{-(\eta/2)} \subset W\) and \(\mu(N) < \mu(W_{-\eta})\) for all \(\mu \in M(G)\). Now, choose another compact open set \(P\) such that \(W_{-\eta} \subset P \subset W\). Note that one has \(\mu(N) < \mu(P)\) for any \(\mu \in M(G)\), which implies \(N \not\subset P\) since we have assumed that \(G\) has groupoid strict comparison for compact open sets. In addition, this establishes \(O \not\subset W\) since \(K \subset N \subset O\) and \(P \subset W\). \(\square\)

The following definition of **multisections** was introduced by Nekrashevych in [Nek19, Definition 3.1]

**Definition 6.7.** A finite set of bisections \(\mathcal{T} = \{C_{i,j} : i, j \in F\}\) with a finite index set \(F\) is called a **multisection** if it satisfies

1. \(C_{i,j}C_{j,k} = C_{i,k}\) for \(i, j, k \in F\);
2. \(\{C_{i,i} : i \in F\}\) is a disjoint family of subsets of \(G^{(0)}\).

We call all \(C_{i,i}\) the levels of the multisection \(\mathcal{T}\). All \(C_{i,j}\) (\(i \neq j\)) are called ladders of the multisection \(\mathcal{T}\).

We say a multisection \(\mathcal{T} = \{C_{i,j} : i, j \in F\}\) open (closed) if all bisections \(C_{i,j}\) are open (closed). In addition, we call a finite disjoint family of multisections \(\mathcal{C} = \{\mathcal{T}_l : l \in I\}\) a **castle**, where \(I\) is a finite index set. If all multisections in \(\mathcal{C}\) are open (closed) then we say the castle \(\mathcal{C}\) is open (closed).\(^5\) We also explicitly write \(\mathcal{C} = \{C_{i,j}^l : i, j \in F_l, l \in I\}\), which satisfies the following

1. \(\{C_{i,j}^l : i, j \in F_l\}\) is a multisection;
2. \(C_{i,j}^l C_{i,j}^{l'} = \emptyset\) if \(l \neq l'\).

Let \(\mathcal{C} = \{C_{i,j}^l : i, j \in F_l, l \in I\}\) be a castle. Any certain level in a multisection in \(\mathcal{C}\) is usually referred to as a \(\mathcal{C}\)-level. Analogously, any ladder in in a multisection in \(\mathcal{C}\) is usually referred as a \(\mathcal{C}\)-ladder. We remark that the disjoint union

\[
\mathcal{H}_\mathcal{C} = \bigcup \mathcal{C} = \bigcup_{l \in I} \bigcup_{i,j \in F_l} C_{i,j}^l
\]

\(^5\) We point out that in [Nek19, Definition 3.1], the author worked with ample groupoids and assumed multisections are clopen; we do not make such an assumption.
of bisections in \( C \) is an elementary groupoid. From this point of view, we denote by 
\[ C^{(0)} = \{ C_{i,i}^l : i \in F_l, l \in I \}. \]
Sometimes we will talk about multisections inside \( C \).

We denote by \( C^l = \{ C_{i,j}^l : i, j \in F_l \} \) for each index \( l \in I \). Similarly, each \( C \)-ladder \( C_{i,j}^l \) in \( C^l \) for \( i \neq j \) is also called a \( C \)-ladder and any \( C \)-level \( C_{i,i}^l \) is also referred as a \( C \)-level. Finally, we write \( (C_l)^{(0)} = \{ C_{i,i}^l : i \in F_l \} \).

Let \( C = \{ C_{i,j}^l : i, j \in F_l, l \in I \} \) be a castle and \( K \) be a compact set in \( G \) with 
\( G^{(0)} \subset K \). We say that \( C \) is \( K \)-extendable if there is another castle \( D = \{ D_{i,j}^l : i, j \in E_l, l \in I \} \) such that

\[
K \cdot \bigcup_{i,j \in F_l} C_{i,j}^l \subset \bigcup_{i,j \in E_l} D_{i,j}^l
\]
where \( E_l \subset F_l \) and \( C_{i,j}^l = D_{i,j}^l \) if \( i, j \in E_l \) for all \( l = 1, \ldots, m \). In this case, we also say that \( C \) is \( K \)-extendable to \( D \).

**Definition 6.8.** Let \( G \) be a locally compact Hausdorff étale groupoid with a compact unit space. We say that \( G \) is almost elementary if for any compact set \( K \) satisfying \( G^{(0)} \subset K \subset G \), any non-empty open set \( O \) in \( G^{(0)} \) and any open cover \( V \) there are open castles \( \mathcal{C} = \{ C_{i,j}^l : i, j \in F_l, l \in I \} \) and \( \mathcal{D} = \{ D_{i,j}^l : i, j \in E_l, l \in I \} \) satisfying

(i) \( \mathcal{C} \) is \( K \)-extendable to \( \mathcal{D} \);
(ii) every \( \mathcal{D} \)-level is contained in an open set \( V \in \mathcal{V} \);
(iii) \( \mathcal{G}^{(0)} \setminus \bigcup_{l \in I} \bigcup_{i,j \in E_l} C_{i,j}^l \prec \mathcal{G} \).

Now we show the first property of almost elementary groupoids when it is minimal.

**Proposition 6.9.** Let \( G \) be a minimal locally compact Hausdorff étale groupoid on a compact space. Suppose that \( G \) is almost elementary. Then \( G \) is effective.

**Proof.** Suppose the contrary that \( \text{Iso}(G)^o \setminus G^{(0)} \neq \emptyset \). Then it is an open set because \( G \) is étale. Then there is a precompact bisection \( V \) such that \( V \subset \overline{V} \subset \text{Iso}(G)^o \setminus G^{(0)} \) because \( G \) is locally compact Hausdorff étale. Then define an open set \( O = s(V) = r(V) \subset G^{(0)} \). Since \( G \) is additionally assumed to be minimal, there are precompact open bisections \( V_1, \ldots, V_n \) such that

(i) all \( \overline{V}_k \) are also bisections;
(ii) \( \bigcup_{k=1}^n r(V_k) \subset O \)
(iii) \( V = \{ s(V_1), \ldots, s(V_n) \} \) is an open cover of \( G^{(0)} \).

Note that (ii) above implies that \( \overline{V} \cdot V_k \neq \emptyset \). In addition, for all \( k \leq m \) and \( \lambda \in \overline{V}_k \) and \( \gamma \in V \) with \( r(\lambda) = s(\gamma) \), observe that \( 0 \neq \gamma \lambda \neq \lambda \). Otherwise, \( \gamma \lambda = \lambda \) implies that

\[
0 = \gamma r(\lambda) = \gamma \lambda \lambda^{-1} = \lambda \lambda^{-1} = r(\lambda) \in G^{(0)}.
\]

But this is a contradiction for our original assumption on \( \gamma \in V \subset \text{Iso}(G)^o \setminus G^{(0)} \).

Now we define a compact set \( K = (\bigcup_{k=1}^n \overline{V} \cdot V_k) \cup (\bigcup_{k=1}^n \overline{V}_k) \cup G^{(0)} \). Then since \( G \) is almost elementary, for \( K \) and \( V \), there are open castles \( \mathcal{C} = \{ C_{i,i}^l : i, j \in F_l, l \in I \} \) and \( \mathcal{D} = \{ D_{i,j}^l : i, j \in E_l, l \in I \} \) satisfying

(i) \( \mathcal{C} \) is \( K \)-extendable to \( \mathcal{D} \) and
(ii) every \( \mathcal{D} \)-level is contained in an open set \( s(V_k) \in V \).

Now consider a \( C \)-level \( C_{i,i}^l \) and a unit \( u \in C_{i,i}^l \) for an \( l \) and an \( i \in F_l \). First we have \( D_{i,i}^l = C_{i,i}^l \subset s(V_k) \) for some \( k \leq n \). Let \( \lambda \in \overline{V}_k \) and \( \gamma \in V \) such that \( s(\lambda) = u \) and
On the other hand, observe that \( r(\gamma) = s(\gamma) \). Then observe that
\[
\{ \gamma u, \lambda u \} \subset (V : \overrightarrow{V_k} \cup \overleftarrow{V_k}) u \subset Ku \subset \bigcup_{j \in E_l} D^l_{j,i} u.
\]
Since \( \gamma \lambda \neq \lambda \) and the subgroupoid \( \mathcal{H}_D = \bigcup D \) is principal, there are different \( j_1 \neq j_2 \in E_l \) such that
\[
\{ \lambda \} = \{ \lambda u \} = D^l_{j_1,i} u
\]
and
\[
\{ \gamma \lambda \} = \{ \gamma \lambda u \} = D^l_{j_2,i} u.
\]
On the other hand, observe that \( r(\gamma \lambda) = r(\gamma) \). But this implies that \( D_{j_1,j_1} \cap D_{j_2,j_2} \neq \emptyset \), which is a contradiction since they are different \( D \)-levels. \( \square \)

If the unit space \( \mathcal{G}^{(0)} \) is also metrizable, we then remark that there is an equivalent definition of almost elementariness by considering closed bisections.

**Proposition 6.10.** Let \( \mathcal{G} \) be a locally compact Hausdorff étale groupoid on a compact metrizable unit space. \( \mathcal{G} \) is almost elementary if and only for any open compact set \( K \) satisfying \( \mathcal{G}^{(0)} \subset K \subset \mathcal{G} \), any non-empty open set \( O \) and any open cover \( \mathcal{V} \) of \( \mathcal{G}^{(0)} \) there is an open castle \( C = \{ C_{i,j}^l : i,j \in F_l, l \in I \} \) and a closed castle \( A = \{ A_{i,j}^l : \} \)

\[i,j \in F_l, l \in I \] satisfying

1. \( A_{i,j}^l \) and \( C_{i,j}^l \) are open and \( A_{i,j}^l \subset C_{i,j}^l \) for all \( i,j \in F_l \) and all \( l \in I \);
2. \( C \) is \( K \)-extendable to an open castle \( D = \{ D_{i,j}^l : i,j \in E_l, l \in I \} \);
3. every \( D \)-level is contained in an open set \( V \in \mathcal{V} \);
4. \( \mathcal{G}^{(0)} \setminus \bigcup_{i=1}^{m} \bigcup_{l \in F_l} A_{i,l}^l \subset \mathcal{G}^O \).

**Proof.** It is not hard to see that if a groupoid \( \mathcal{G} \) satisfies the conditions (i)-(iv) above then \( \mathcal{G} \) is almost elementary. Thus it suffices to show the converse.

Now suppose that \( \mathcal{G} \) is almost elementary. Write \( T = \mathcal{G}^{(0)} \setminus \bigcup_{l \in I} \bigcup_{i \in F_l} C_{i,l}^l \) for simplicity. Since one has \( T \subset O \), there are bisections \( \{ U_1, \ldots, U_n \} \) such that \( T \subset \bigcup_{k=1}^{n} s(U_k) \) and \( \bigcup_{k=1}^{n} r(U_k) \subset O \). Now fix a compatible metric \( d \) on \( \mathcal{G}^{(0)} \). Then there is a \( \delta > 0 \) such that \( B_d(T, \delta) \subset \bigcup_{k=1}^{n} s(U_k) \). For any \( \eta > 0 \) and any open set \( P \) we denote by \( P_{-\eta} \) the open set \( \{ u \in \mathcal{G}^{(0)} : d(u, \mathcal{G}^{(0)} \setminus \overline{P}) > \eta \} \) as in Lemma 6.5

Now for each \( l \) and \( j \in F_l \) choose an \( \eta > 0 \) such that \( (C_{j,l}^l)_{-\eta} \subset C_{j,l}^l \) and for all \( l \) and \( j \in F_l \) one has
\[
(C_{j,l}^l) \setminus (C_{j,l}^l)_{-\eta} \subset B_d(T, \delta).
\]
Fix an \( l \in I \) and an \( i \in F_l \). Denote by \( B_{l}^{j,i} = (C_{j,l}^l)_{-\eta} \) for simplicity. Define
\[
A_{l,i,i}^l = \bigcup_{j \in F_l} r(C_{l,j,i}^l B_{l,j,i}^l),
\]
which is a subset of \( C_{l,i,i}^l \). In addition for \( i,j \in F_l \) we define bisections
\[
A_{l,i,j}^l = C_{l,i,j}^l A_{l,i,i}^l,
\]
and
\[
A_{l,i,j}^l = A_{l,i,i}^l \cdot (A_{l,i,j}^l)^{-1}
\]
Observe that
\[
\overline{A_{l,i,j}^l} = \bigcup_{j \in F_l} r(C_{l,j,i}^l B_{l,j,i}^l) \subset C_{l,0,0}^l \]
and
\[ A_{j,i}^l = (s_{\rho_{j,i}}^{-1})(A_{j,i}^l). \]
In addition, one has
\[ A_{j,i}^l = A_{j,i}^0 \cdot (A_{j,i}^0)^{-1} \]
Now we claim the cast \( \mathcal{A} = \{A_{j}^{l} : i,j \in F_{l}, l \in I \} \) satisfies the condition (i)-(iv) above. In fact, by our construction, it suffices to verify (iv) for \( \mathcal{A} \). By our definition, one has \( B_{i,i}^l \subset A_{i,i}^l \) and thus
\[ C_{i,i}^l \setminus A_{i,i}^l \subset C_{i,i}^l \setminus (C_{i,i}^l)_{=\eta} \subset B_{d}(T,\delta). \]
Now write \( T' = G^{(0)} \setminus \bigcup_{l \in I} \bigcup_{i \in F_{l}} A_{i,i}^l \) and thus
\[ T' \subset T \cup \bigcup_{l \in I} \bigcup_{i \in F_{l}} (C_{i,i}^l \setminus (C_{i,i}^l)_{=\eta}) \subset B_{d}(T,\delta) \subset \bigcup_{k=1}^{n} s(U_k) \]
while one has
\[ \bigcup_{k=1}^{n} r(U_k) \subset O. \]
This establishes (iv) for \( \mathcal{A} \).

The following is a preliminary general result having the similar flavor to Lemma 6.5 established by the first author in [Ma19, Lemma 3.2]. See also [Ker20, Lemma 9.1].

**Lemma 6.11.** Let \( X \) be a compact metrizable space with a compatible metric \( d \) and \( \Omega \) a weak*-closed subset of \( M(X) \). Suppose \( \lambda > 0 \) and \( A \) is a closed subset of \( X \) such that \( \mu(A) < \lambda \) for all \( \mu \in \Omega \). Then there is a \( \delta > 0 \) such that \( \mu(A+\delta) < \lambda \) for all \( \mu \in \Omega \) where \( A+\delta = B_{d}(A,\delta) = \{x \in X : d(x,A) \leq \delta \} \).

**Remark 6.12.** In Definition 6.8 when \( G^{(0)} \) is metrizable, we remark that for any \( \epsilon > 0 \) one can choose the castle \( \mathcal{C} = \{C_{i,j}^l : i,j \in F_{l}, l \in I \} \) satisfying \( \mu(\bigcup_{l \in I} \bigcup_{i \in F_{l}} C_{i,j}^l) > 1-\epsilon \) as well. This is because one can choose a non-empty open set \( O \) with \( \mu(O) < \epsilon \) for any \( \mu \in M(\mathcal{G}) \) by Lemma 6.11 and make \( G^{(0)} \setminus \bigcup C^{(0)} \prec G O \) a priori. The same argument shows that one can also ask the castle \( \{A_{i,j}^{l} : i,j \in F_{l}, l \in I \} \) in Proposition 6.10 above satisfying that \( \mu(\bigcup_{l \in I} \bigcup_{i,j \in F_{l}} A_{i,j}^{l}) > 1-\epsilon \).

In the remaining part of this section, when \( G^{(0)} \) is metrizable, we will show that if \( G \) is almost elementary and minimal then it has groupoid strict comparison. There are two metrics involved in the proof of the following propositions. When the unit space \( G^{(0)} \) is metrizable, we usually fix an metric \( d \) on it. On the other hand, we may view \( G \) as a coarse metric space with a canonical extended metric \( \rho \) by a coarse length function \( \ell \) as in Definition 4.11. We begin with the fiberwise amenable case.

**Proposition 6.13.** Let \( G \) be a fiberwise amenable minimal locally compact Hausdorff étale groupoid with a compact metrizable unit space. If \( G \) is almost elementary then \( G \) has groupoid strict comparison.

**Proof.** It suffices to show \( M \prec G N \) for all compact set \( M \) and open set \( N \) in \( G^{(0)} \) satisfying \( \mu(M) < \mu(N) \). Let \( d \) be a compatible metric on \( G^{(0)} \). First one has \( M(G) \neq \emptyset \) by Proposition 7.9. Then the function from \( M(G) \) to \([0,\infty)\) defined by \( \mu \mapsto \mu(N) - \mu(M) \) is lower semi-continuous. Then the compactness of \( M(G) \) shows
that there is an $\eta > 0$ such that $\mu(N) - \mu(M) \geq \eta$ for all $\mu \in \mathcal{M}(\mathcal{G})$. This implies $N \setminus M$ is open and non-empty. Let $y \in N \setminus M$ and choose a $\delta > 0$ such that $C = \bar{B}_d(y, \delta) \subset N \setminus M$ and $\mu(C) < \eta/2$ for all $\mu \in \mathcal{M}(\mathcal{G})$ by Lemma 6.11. Define $O = N \setminus C$ and thus one has

$$\mu(O) = \mu(N) - \mu(C) > \mu(M) + \eta/2$$

for all $\mu \in \mathcal{M}(\mathcal{G})$. Then choose $\eta' \leq \eta$ and define

$$O_{-\eta'} = \{u \in \mathcal{G}^{(0)} : d(u, \mathcal{G}^{(0)} \setminus O) > \eta'\}$$

and

$$M_{+\eta'} = \bar{B}_d(M, \eta')$$

such that $M_{+\eta'} \cap C = O_{-\eta'} \cap C = \emptyset$ and $\mu(M_{+\eta'}) + \eta' \leq \mu(O_{-\eta'})$ by Lemma 6.15.

Now we claim that there are $R_0, \epsilon_0 > 0$ such that for all $u \in \mathcal{G}^{(0)}$ and all finite $F \subset \mathcal{G}_u$ one has that if $|\bar{B}_d(F, R_0)| \leq (1 + \epsilon_0)|F|$ then one has

$$(\star) \quad \frac{1}{|F|} \sum_{x \in F} 1_{M_{+\eta'}}(r(x)) + \eta'/2 \leq \frac{1}{|F|} \sum_{x \in F} 1_{O_{-\eta'}}(r(x)).$$

Suppose the contrary, for all $n \in \mathbb{N}$ and $\epsilon_n > 0$ with $\{\epsilon_n\}$ decreasing to 0, there are $u_n \in \mathcal{G}^{(0)}$ and finite set $F_n \subset \mathcal{G}_{u_n}$ such that $|\bar{B}_d(F_n, u_n)| \leq (1 + \epsilon_n)|F_n|$ and

$$\mu_n(M_{+\eta'}) + \eta'/2 \leq \mu_n(O_{-\eta'})$$

where $\mu_n = \frac{1}{|F_n|} \sum_{x \in F_n} \delta_{r(x)}$. Then the argument in Proposition 5.9 shows that any cluster point $\mu$ of $\{\mu_n\}$ is an invariant probability measure. Then the Portmanteau Theorem shows that

$$\mu(O_{-\eta'}) + \eta'/2 \leq \liminf_{n \to \infty} \mu_n(O_{-\eta'}) + \eta'/2 \leq \limsup_{n \to \infty} \mu_n(M_{+\eta'}) + \eta' \leq \mu(M_{+\eta'}) + \eta'.$$

This is a contradiction. Therefore, the claim above holds.

Now since $\mathcal{G}$ is fiberwise amenable, for $R_0, \epsilon_0 > 0$ obtained above, there is a compact set $\mathcal{G}^{(0)} \subset K \subset \mathcal{G}$ satisfying the first part of Theorem 5.15. Then since $\mathcal{G}^{(0)}$ is compact, we fix an open cover $\mathcal{V}$ of $\mathcal{G}^{(0)}$ containing open sets with diameter less than min{$\eta', \delta/2$}. Because $\mathcal{G}$ is also almost elementary, for the compact set $K$, the open cover $\mathcal{V}$ and open ball $B_d(y, \delta) \subset C$, Proposition 6.10 implies that there is an open castle $\mathcal{C} = \{C_{i,j} : i, j \in F_l, l \in I\}$ and a closed castle $\mathcal{A} = \{A_{i,j} \subset C_{i,j} : i, j \in F_l, l \in I\}$ satisfying

1. $A_{i,j}$ and $C_{i,j}$ are open and $A_{i,j} \subset C_{i,j}$ for all $i, j \in F_l$ and all $l = 1, \ldots, m$;
2. $\mathcal{C}$ is $K$-extendable to an open castle $\mathcal{D} = \{D_{i,j} \subset E_l, l \in I\}$;
3. every $\mathcal{D}$-level is contained in an open set $V \in \mathcal{V}$; and
4. $\mathcal{G}^{(0)} \setminus \bigcup_{l \in I} \bigcup_{i \in F_l} A_{i,i} \prec_\mathcal{G} B_d(y, \delta)$.

For each $l \in I$ we define a compact set $L_{il} = \bigcup_{i,j \in F_l} A_{i,j}$ in $\mathcal{G}$. Then Theorem 5.15(1) shows that for all $u \in \mathcal{G}^{(0)}$ there is an finite set $T_u$ such that

$$L_{il}u \subset T_u \subset K \mathcal{L}_l u$$

and $|\bar{B}_d(T_u, R_0)| < (1 + \epsilon_0)|T_u|$.

Now for each $l \in I$ we fix an $i_l \in F_l$ and choose a $u_l \in A_{i_l,i_l}$. In addition, we write $T_l = T_{u_l}$ for simplicity. Then note that $L_{il}u_l \subset T_l \subset K \mathcal{L}_l u_l \subset \bigcup_{i \in E_l} D_{i,i} \cap T_{il} \neq \emptyset$. Then for each $l \in I$ we define

$$S_l = \{D_{i,i} \in \mathcal{D}^{(0)} : i \in E_l \text{ and } T_l \cap D_{i,i} \neq \emptyset\}$$
and also define

(i) \( P_{i,1} = \{ D_{i,i}^l \in S_i : D_{i,i}^l \cap M \neq \emptyset \} \) and

(ii) \( P_{i,2} = \{ D_{i,i}^l \in S_i : D_{i,i}^l \cap O_{-\eta} \neq \emptyset \} \).

Since every member \( V \in \mathcal{V} \) has diameter less than \( \min\{\eta', \delta/2\} \), this shows that \( \operatorname{diam}_d(D_{i,i}^l) \leq \min\{\eta', \delta/2\} \) for all \( D\)-levels \( D_{i,i}^l \). Then one has

(i) \( P_{i,1} \subset Q_{i,1} := \{ D_{i,i}^l \in S_i : D_{i,i}^l \subset M_{+\eta} \} \);

(ii) \( P_{i,2} \subset Q_{i,2} := \{ D_{i,i}^l \in S_i : D_{i,i}^l \subset O \} \);

Then, for all \( l \in I \) we claim \( |P_{i,1}| < |P_{i,2}| \). Suppose the contrary. Then there is an \( l \) such that \( |P_{i,1}| \geq |P_{i,2}| \) and thus \( |Q_{i,1}| \geq |P_{i,2}| \). Then we have

\[ |\{ r(x) : x \in T_l \text{ and } r(x) \in M_{+\eta'} \}| \geq |Q_{i,1}| \geq |P_{i,2}| \geq |\{ r(x) : x \in T_l \text{ and } r(x) \in O_{-\eta} \}|, \]

and thus one has

\[ \sum_{x \in T_l} 1_{M_{+\eta'}}(r(x)) \geq \sum_{x \in T_l} 1_{O_{-\eta}}(r(x)) \]

which is a contradiction to the inequality \([\star]\) above because \( T_l \) satisfies

\[ |\bar{B}_p(T_l, R_0)| < (1 + \epsilon_0) |T_l|. \]

This establishes our claim \( |P_{i,1}| < |P_{i,2}| \).

Now, since \( |P_{i,1}| < |P_{i,2}| \leq |Q_{i,2}| \) holds for each \( l \in I \), we choose an injection \( \phi_l : P_{i,1} \to Q_{i,2} \). Observe that all sets in \( P_{1,1} \) and \( Q_{1,2} \) are \( D \)-levels in the same multisection \( \mathcal{D}^l \). This implies that for any \( U \in P_{i,1} \) there is a bisection \( \hat{U} \) such that \( s(\hat{U}) = U \) and \( r(\hat{U}) = \phi_l(U) \). This implies that

\[ \bigcup_{l \in I} P_{i,1} \triangleleft \bigcup_{l \in I} Q_{i,2} \subset O \]

On the other hand, recall that

\[ \bigcup_{i \in F_l} C_{i,i} \cdot u_l = \bigcup_{i \in F_l} \overline{A}_{i,i} \cdot u_l = L_l \cdot u_l \subset T_l \cdot u_l. \]

Then one has \( \{ C_{i,i}^l : i \in F_l \} \subset S_l \). Thus

\[ \mathcal{G}^{(0)} \setminus \bigcup_{l \in I} \bigcup_{i \in F_l} S_l \subset \mathcal{G}^{(0)} \setminus \bigcup_{l \in I} \bigcup_{i \in F_l} C_{i,i}^l \subset \mathcal{G}^{(0)} \setminus \bigcup_{l \in I} \bigcup_{i \in F_l} A_{i,i}^l \triangleleft_B \overline{B}_d(x, \hat{y}) \subset C. \]

We write \( R = \mathcal{G}^{(0)} \setminus \bigcup_{l=1}^m \bigcup_{i \in F_l} S_l \) for simplicity. So one has

\[ R \cap M \triangleleft_B \overline{B}_d(x, \hat{y}) \subset C. \]

Recall that \( C \cap O = \emptyset \). Finally since

\[ M \cap (\mathcal{G}^{(0)} \setminus R) = M \cap \left( \bigcup_{l \in I} \bigcup_{i \in F_l} S_l \right) \subset \bigcup_{l \in I} \bigcup_{i \in F_l} P_{i,1} \]

holds and

\[ \bigcup_{l \in I} P_{i,1} \triangleleft \bigcup_{l \in I} Q_{i,2} \subset O \]

obtained above, we have verified that \( M \triangleleft_B N \). This shows that \( \mathcal{G} \) has groupoid strict comparison. \( \square \)
We say a topological space $X$ is perfect if $X$ has no isolated points. Let $\mathcal{G}$ be a locally compact Hausdorff étale minimal groupoid with an infinite compact unit space. Then $\mathcal{G}^{(0)}$ is necessarily perfect. Suppose the contrary, let $\{u\}$ be an open set in $\mathcal{G}^{(0)}$. Since $\mathcal{G}^{(0)}$ is compact and $\mathcal{G}$ is minimal, one can find finitely many open bisections $O_1, \ldots, O_n$ such that $\mathcal{G}^{(0)} = \bigcup_{i=1}^n s(O_i)$ and $\bigcup_{i=1}^n r(O_i) \subset \{u\}$. But this implies $\mathcal{G}^{(0)}$ is finite, which is a contradiction to our assumption. The following proposition shows that if $\mathcal{G}$ stated above is almost elementary but not fiberwise amenable then it has paradoxical flavor by using Theorem 6.10 in the sense that every two non-empty open sets in the unit space can be compared in the sense of groupoid strict comparison.

**Proposition 6.14.** Let $\mathcal{G}$ be a non-fiberwise amenable minimal locally compact étale groupoid with a compact metrizable unit space. Suppose $\mathcal{G}$ is almost elementary then $\mathcal{G}$ has groupoid strict comparison and $M(\mathcal{G}) = \emptyset$.

**Proof.** We first show $\mathcal{G}^{(0)} \prec_\mathcal{G} O$ for any non-empty open set $O \subset \mathcal{G}^{(0)}$. Since $\mathcal{G}^{(0)}$ is perfect, one can choose disjoint non-empty open subsets $U, V$ of $O$. Let $\eta > 0$ such that $V_{-\eta} = \{u \in \mathcal{G}^{(0)} : d(u, \mathcal{G}^{(0)} \setminus V) > \eta\}$ is non-empty. Since $\mathcal{G}$ is minimal there are precompact open bisections $\{W_1, \ldots, W_n\}$ such that

(i) $W_k$ is also a bisection for each $k = 1, \ldots, n$;
(ii) $\bigcup_{k=1}^n s(W_k) = \mathcal{G}^{(0)}$;
(iii) $\bigcup_{k=1}^n r(W_k) \subset V_{-\eta}$.

We write $\mathcal{W}$ for the cover $\{s(W_1), \ldots, s(W_n)\}$ of $\mathcal{G}^{(0)}$. Define $L = \bigcup_{k=1}^n W_k \cup \mathcal{G}^{(0)}$. Then Theorem 6.10 shows that there is a compact set $\mathcal{G}^{(0)} \subset K \subset \mathcal{G}$ such that for all compact set $M \subset \mathcal{G}$ and all $u \in s(M)$ one has $KMu$ contains at least $|Mu|$ many disjoint non-empty sets of form $L\gamma u$. We choose another open cover $\mathcal{V}$ of $\mathcal{G}^{(0)}$ finer than $\mathcal{W}$ and contains open sets with diameter less than $\eta$. Because $\mathcal{G}$ is almost elementary, Proposition 6.10 implies that there is an open castle $\mathcal{C} = \{C^l_{i,j} : i, j \in F_l, l \in I\}$ and a closed castle $\mathcal{A} = \{A^l_{i,j} : i, j \in F_l, l \in I\}$ satisfying

(i) $A^l_{i,j}$ and $C^l_{i,j}$ are open and $A^l_{i,j} \subset C^l_{i,j}$ for all $i, j \in F_l$ and all $l \in I$;
(ii) $\mathcal{C}$ is $K$-extendable to an open castle $\mathcal{D} = \{D^l_{i,j} : i, j \in E_l, l \in I\}$;
(iii) every $D$-level is contained in a member of $\mathcal{V}$;
(iv) $\mathcal{G}^{(0)} \setminus \bigcup_{l \in I} \bigcup_{i \in F_l} A^l_{i,j} \prec_\mathcal{G} U$.

Define $M = \bigcup_{l=1}^m \bigcup_{i \in F_l} A^l_{i,j}$. For each $l$ we choose $i_l \in F_l$ and $u_l \in A_{i_l,i_l}$. Note that $|Mu_l| = F_l$ for each $l \in I$. Thus, by our choice of $K$, for each $i \in I$, there is a family $\{\gamma^l_i \in \mathcal{G} : i \in F_l, r(\gamma^l_i) = u_l\}$ with the following properties

(i) $\{L\gamma^l_i u_l : i \in F_l\}$ is a disjoint family;
(ii) $\bigcup_{i \in F_l} L\gamma^l_i u_l \subset KMu_l \subset \bigcup_{i \in E_l} D^l_{i,j} u_l = \bigcup_{i \in E_l} D^l_{i,j} u_l$.

Then for each $l \in I$ we choose a bijection as follows.

\[ \varphi^l : \{C^l_{i,j} : i \in F_l\} \rightarrow \{\gamma^l_i \in \mathcal{G} : i \in F_l\}. \]

Therefore, in particular, one has

\[ \bigcup_{i \in F_l} L\varphi^l(C^l_{i,j}) u_l \subset \bigcup_{j \in E_l} D^{j}_{i,j} u_l. \]

Now since $\mathcal{G}^{(0)} \subset L$, for each $i \in F_l$ there is a $j_l \in E_l$ such that $\varphi^l(C^l_{i,j_l}) u_l = D^{j}_{i,j_l} u_l$. Observe that $r(D^{j}_{i,j_l}) = D^{j}_{i,j_l} \subset s(W_{j_l})$ for some $k_l \leq n$ because the cover $\mathcal{V}$ is
finer than $W$. Thus, one has $r(W_{k_{j_i}} D_{j_{i},i_{i}}^l) \subset V_{-\eta}$, which implies that
\[ r(W_{k_{j_i}} \varphi_l(C_{i,i}^l)u_l) = r(W_{k_{j_i}} D_{j_{i},i_{i}}^l u_l) \subset V_{-\eta}. \]
On the other hand, since $W_{k_{j_i}} \varphi_l(C_{i,i}^l)u_l \subset L\varphi_l(C_{i,i}^l)u_l \subset \bigcup_{l \in E_l} D_{l,i}^l u_l$, there is a $t_{k_{j_i}} \in E_l$ such that
\[ W_{k_{i}} \varphi_l(C_{i,i}^l)u_l = D_{t_{k_{j_i}},i}^l u_l \]
and thus one has
\[ D_{t_{k_{j_i}},i}^l \cap V_{-\eta} \neq \emptyset. \]

For all $i \in F_l$ we write $f_l(i)$ for $t_{k_{j_i}} \in E_l$ obtained above for simplicity. Then we have $D_{f_l(i),f_l(i)}^l \subset V$ since the diameter of all $D$-levels are less than $\eta$. In addition, since the family $\{ L\varphi_l(C_i)u_l : i \in F_l \}$ is disjoint for each $l$, the member in $\{ f_l(i) : i \in F_l \}$ are distinct. This shows that

\[ (*) \quad \bigcup_{l=1}^{m} \bigcup_{i \in F_l} C_{i,i}^l \preceq_{G} \bigcup_{l=1}^{m} \bigcup_{i \in F_l} D_{f_l(i),f_l(i)}^l \subset V. \]

On the other hand, one has

\[ (**) \quad G^{(0)} \setminus \bigcup_{l \in F_l} C_{i,i}^l \subset G^{(0)} \setminus \bigcup_{l \in F_l} A_{i,i}^l \preceq_{G} U \]

Recall that $U$ and $V$ are disjoint subset of $O$. Then combining $(*)$ and $(**)$, one has $G^{(0)} \preceq_{G} O$ as desired.

Then it is straightforward to see $M(G) = \emptyset$. Otherwise, suppose $\mu \in M(G)$ and let $O_1$ and $O_2$ be two disjoint non-empty sets in $G^{(0)}$. Then for each $i = 1, 2$, since $G^{(0)} \preceq_{G} O_i$, one has $1 = \mu(G^{(0)}) \leq \mu(O_i) \leq 1$. But this is a contradiction because $2 = \mu(O_1 \cup O_2) \leq 1$. Then since one also has $U \subset G^{(0)} \preceq_{G} O$ for any non-empty open sets $U, O$, the groupoid $G$ has groupoid strict comparison.

**Remark 6.15.** Let $G$ be a minimal locally compact étale groupoid on a compact metrizable unit space. We remark that Proposition 6.14 above shows that if $G$ is almost elementary and non-fiberwise amenable then $G$ is purely infinite in the sense of [Ma12] Definition 3.5. See also Lemma 3.10 and the discussion before it in [Ma12]. Therefore, we know in priori that the reduced groupoid $C^*$-algebra $C^*_r(G)$ in this case is strongly purely infinite by Corollary 1.1 in [Ma12]. In particular it is $\mathcal{Z}$-stable. On the other hand, combining Proposition 6.9 and 6.14, we obtain that if $G$ is almost elementary, we have $M(G) \neq \emptyset$ if and only if $G$ is fiberwise amenable.

We end this section by listing the following theorem as a direct corollary of Proposition 6.13 and 6.14.

**Theorem 6.16.** Let $G$ be a minimal locally compact étale groupoid on a compact metrizable unit space. Suppose that $G$ is almost elementary. Then $G$ has groupoid strict comparison.

7. Almost elementariness and almost finiteness

In this section we show the relation between almost finiteness and our almost elementariness. Recall that the original notion of almost finiteness was introduced by Matui in [Mat12] for ample groupoid. In [Ker20], Kerr generalized this notion in the case of transformation groupoid generating by actions of amenable groups.
on compact metrizable spaces. We begin by recalling Matui’s notion as follows (see [Matt, Definition 6.2]).

**Definition 7.1** (Matui). Let $G$ be a locally compact Hausdorff étale ample groupoid on a compact unit space. $G$ is called almost finite if for any compact set $K$ in $G$ and $\epsilon > 0$ there is a compact open elementary subgroupoid $H$ of $G$ with $H^{(0)} = G^{(0)}$ such that for any $u \in G^{(0)}$, one has

$$|K \cdot H \cdot u | < \epsilon |Hu|.$$

It was proved by Matui in [Matt, Lemma 6.7] that almost finite second countable locally compact Hausdorff étale ample groupoid has groupoid strict comparison for compact open sets. In fact, his proof still works in the setting that the groupoid is only $\sigma$-compact. Thus we have the following result.

**Proposition 7.2** (Matui). Let $G$ be a locally compact $\sigma$-compact Hausdorff étale ample groupoid with a compact unit space. If $G$ is almost finite then $G$ has groupoid strict comparison for compact open sets.

We first show that almost finiteness implies fiberwise amenability.

**Proposition 7.3.** Let $G$ be a locally compact Hausdorff étale ample groupoid. If $G$ is almost finite then $G$ is fiberwise amenable with respect to the metric $\rho$.

**Proof.** Let $\ell$ be a coarse continuous length function for $G$ and $\rho$ be the metric induced by $\ell$ (cf., Definition 1.11). Let $u \in G^{(0)}$ and $R, \epsilon > 0$. Define $K = \{ z \in G : \ell(z) \leq R \}$, which is a compact set in $G$. In general, for any $y \in G$ one has

$$\bar{B}_{\rho}(y,R) = \{ x \in G : \rho(x,y) \leq R \} = \{ x \in G : \ell(xy^{-1}) \leq R \} \subset Ky$$

and therefore for any finite set $F \subset G$ one has

$$\bar{B}_{\rho}(F,R) = \bigcup_{y \in F} \bar{B}_{\rho}(y,R) \subset \bigcup_{y \in F} Ky = KF.$$

Now, since $G$ is almost finite, for the $K$ and the $\epsilon$ above, there is a compact open elementary subgroupoid $H$ of $G$ with $H^{(0)} = G^{(0)}$ such that for any $v \in G^{(0)}$, one has

$$|K \cdot H \cdot v | < \epsilon |Hv|.$$ Now define $F = H \cdot u$, which is a finite set in $G$. Then one has

$$|\bar{B}_{\rho}(F,R)| \leq |KF| < (1 + \epsilon)|F|.$$ This shows that $(G,\rho)$ is fiberwise amenable by Proposition 5.6. \qed

Let $K$ be a compact set in $G$ and $F \subset G_u$ for some $u \in G^{(0)}$. Recall $\partial_K^+ F = KF \setminus F$ and $\partial_K^- F = \{ x \in F : Kx \cap (G \setminus F) \neq \emptyset \}$, which satisfy $\partial_K^+ F \subset K \cdot \partial_K^- F$ and $\partial_K^- F \subset K^{-1} \cdot \partial_K^+ F$. Now we show that almost finiteness implies almost elementariness. Also recall the function $\mathcal{R} : [0, \infty) \to \mathbb{N}$ given by $\mathcal{R}(r) = \sup_{x \in G} |\bar{B}_{\rho}(x,r)|$.

**Theorem 7.4.** Let $G$ be a locally compact $\sigma$-compact Hausdorff almost finite minimal ample groupoid on a compact space. Then $G$ is almost elementary.

**Proof.** Let $K$ be a compact set in $G$ with $G^{(0)} \subset K$, $O$ a non-empty open set in $G^{(0)}$ and $V$ an open cover of $G^{(0)}$. First, choose a non-empty clopen set $U \subset O$. In addition since $U$ is non-empty and $G$ is minimal, there is an $\epsilon > 0$ such that $\mu(U) > \epsilon$ for any $\mu \in M(G)$. Furthermore, by choosing a finer cover, one can also assume all members in $V$ are clopen.
Let $\rho$ be the extended metric induced by the length function $\ell$ as usual. Denote by

$$R = \sup \{ \ell(y) : y \in K \cup K^{-1} \} < \infty.$$  

Note that for any $x \in \mathcal{G}$ one has $(K \cup K^{-1})x \subset \bar{B}_p(x, R)$ because $\rho(yx, x) = \ell(y)$ for any $y \in K \cup K^{-1}$. To establish the almost elementariness, without loss of generality, we may assume $K = \bigcup_{k=0}^n O_k$ where each of $O_k$ is a compact open bisection and $O_0 = \mathcal{G}^{(0)}$. Let $\mathcal{H}$ be the compact open principal elementary subgroupoid satisfying almost finiteness for $(K, \rho/\rho(R))$ with the fundamental domain decomposition $\mathcal{H} = \bigsqcup_{l \in I} \bigsqcup_{i,j \in F_l} C^l_{i,j}$ such that for any $u \in \mathcal{G}^{(0)}$ one has $|K\mathcal{H}u \setminus \mathcal{H}u| < (\epsilon/\rho(R))|\mathcal{H}u|$. In addition, by a standard chopping process for clopen sets, one may assume that the partition $\{C^l_{i,j} : i \in F_l, l \in I\}$, as an open cover of $\mathcal{G}^{(0)}$, is finer than $\mathcal{V}$.

Fix a $u \in \mathcal{G}^{(0)}$ and define $F = \mathcal{H}u$, which satisfies $|\partial_K^u(F)| < (\epsilon/\rho(R))|F|$. Define $M = \{ \gamma \in F : K \cdot \gamma \subset F \}$. Note that $M = F \setminus \partial_K^u F$. Then because $\partial_K^u F \subset K^{-1} \cdot \partial_K^u F \subset \bar{B}_p(\partial_K^u F, R)$, one has $|\partial_K^u F| \leq \rho(R)|\partial_K^u F|$ and thus

$$|M| = |F| - |\partial_K^u F| \geq |F| - \rho(R)|\partial_K^u F| \geq (1 - \epsilon)|F|.$$

Now we claim $M = \bigcap_{k=0}^n (O_k^{-1} \cdot F)$. Indeed, for any $\eta \in \bigcap_{k=0}^n (O_k^{-1} \cdot F)$ one has for any $0 \leq k \leq n$ there is an $\gamma_k \in O_k$ and $\alpha_k \in F$ such that $\eta = \gamma_k^{-1} \alpha_k$. Then because all $O_k$ are bisections, one has

$$K\eta = \bigcap_{k=0}^n O_k \gamma_k^{-1} \alpha_k = \{ \alpha_0, \ldots, \alpha_n \} \subset F.$$

This shows $\bigcap_{k=0}^n (O_k^{-1} \cdot F) \subset M$. For the reverse direction, for any $\gamma \in M$, the definition of $M$ implies $K\gamma = \bigcup_{k=0}^n O_k \gamma \subset F$. Then for each $0 \leq k \leq n$, since each $O_k$ is a bisection, there is a unique $\eta_k \in O_k$ such that $\eta_k \gamma \subset F$. This implies that $\gamma \in O_k^{-1} \cdot F$ for all $0 \leq k \leq n$ and thus $M = \bigcap_{k=0}^n (O_k^{-1} \cdot F)$. This establishes the claim. Now define $T = \bigcap_{k=0}^n (O_k^{-1} \cdot \mathcal{H}) \subset \mathcal{H}$. Since $\mathcal{G}$ is ample and all $O_k^{-1} \cdot C^l_{i,j}$ are still compact open bisections, one can choose finitely many compact open bisections $N_1, \ldots, N_m$ such that

1. $T = \bigcup_{p=1}^m N_p$;
2. for any $1 \leq p_1, p_2 \leq m$ either $s(N_{p_1}) = s(N_{p_2})$ or $s(N_{p_1}) \cap s(N_{p_2}) = \emptyset$ and
3. for any $1 \leq p \leq m$ one has $N_p \subset C^l_{i,j}$ for some $l \in I$ and $i,j \in F_l$.  

On the other hand, by the analysis above, for any $u \in \mathcal{G}^{(0)}$ one has $K Tu \subset \mathcal{H}u$ and

$$|Tu| = \bigcap_{i=0}^n |(O_i^{-1} \cdot \mathcal{H}u)| \geq (1 - \epsilon)|\mathcal{H}u|.$$  

In particular, one has $s(T) = \mathcal{G}^{(0)}$. Now, for any $u \in \mathcal{G}^{(0)}$, denoted by $C^l_{i,u,i,u}$ the unique level in $\mathcal{H}$ such that $u \in C^l_{i,u,i,u}$. Define a function $f : \mathcal{G}^{(0)} \rightarrow \bigsqcup_{l \in I} \mathcal{P}(F_l)$ by

$$f(u) = \{ i \in F_u : Tu \cap C^l_{i,u,i,u} \neq \emptyset \},$$

where $\mathcal{P}(F_l)$ denotes the power set of $F_l$. This is equivalent to say $Tu = \bigsqcup_{i \in f(u)} C^l_{i,u,i,u} u$. We claim that $f$ is locally constant. Indeed, let $u \in C^l_{i,u,i,u}$ and denote by $J_u = \{ p \leq m : u \in s(N_p) \}$. For each $p \in J_u$, by (iii) above for $N_p$, one can choose a unique $i_p \in F_u$ such that $N_p \subset C^l_{i_p,u,i,u}$. Note that $f(u) = \{ i_p : p \in J_u \}$. Define
W = s(N_p) for some p ∈ J_u, which is a compact open neighborhood of u. For any
w ∈ W ⊂ C_{i,u} one has
\[ Tw = \bigcup_{p=1}^{m} N_p w = \bigcup_{p \in J_u} N_p w = \bigcup_{i \in f(u)} C_{i,p,u} w = \bigcup_{i \in f(u)} C_{i,u} w. \]
This implies that \( f(w) = f(u) \) holds for any \( w \in W \) and thus \( f \) is locally constant
and thus continuous.

Now for each \( l \in I \) fix an \( i_l \in F_l \). For any \( S \subset F_l \) define a compact open set
\( W_{i_l,i_l}^S = f^{-1}(\{S\}) \cap C_{i_l,i_l}^l \) (could be empty). Then the collection \( \{W_{i_l,i_l}^S : S \subset F_l\} \)
forms a compact open partition of \( C_{i_l,i_l}^l \). Then for any \( S \subset F_l \) and \( i \in S \) define
\( W_{i,i}^S = C_{i,i}^l W_{i,i}^S \). In addition, for any \( i,j \in S \) define \( W_{i,j}^S = W_{i,j}^S \). It is
obvious to see that the collection \( \{W_{i,j}^S : i,j \in S\} \) is a multisection. Furthermore,
one has that
\[ \mathcal{W} = \bigcup_{l \in I} \bigcup_{S \subset F_l} \bigcup_{i,j \in S} W_{i,j}^S \]
is an elementary groupoid. By our construction, For each \( S \subset F_l \) the level \( W_{i,i}^S \subset C_{i,i}^l \)
and thus \( W_{i,i}^S \) is a subset of a member of \( \mathcal{V} \).

Now, we claim \( \mathcal{W} \) is \( K \)-extendable to \( \mathcal{H} \). It suffices to show \( KWu \subset H \) for any
\( u \in W^{(0)} \). First assume \( u \in W_{i,i}^S \) for some \( l \in I \) and \( S \subset F_l \). Then one has
\[ Wu = \bigcup_{i \in S} W_{i,i}^S u = \bigcup_{i \in S} C_{i,i}^l u. \]
Since \( u \in W_{i,i}^S = f^{-1}(\{S\}) \cap C_{i,i}^l \) one has \( f(u) = S \) and thus \( Wu = Tu \). Therefore,
\( KWu = KTu \subset Hu \subset H \). Now if \( v \in W_{i,i}^S \subset C_{i,i}^l \) for some \( i \in S \), \( S \subset F_l \) and \( l \in I \). Then there is a \( \gamma \in W_{i,i}^S \) such that \( s(\gamma) = v \) and \( r(\gamma) \in W_{i,i}^S \). Then
\[ KWv = K(\bigcup_{j \in S} W_{j,j}^S) = K(\bigcup_{j \in S} W_{j,i}^S W_{i,i}^S) = K(\bigcup_{j \in S} W_{j,i}^S r(\gamma) \gamma) = KT \cdot r(\gamma) \cdot \gamma \subset H \]
since \( KT \cdot r(\gamma) \subset H \) by the argument above and the fact \( \gamma \in W_{i,i}^S \subset C_{i,i}^l \subset H \). This
establishes that \( \mathcal{W} \) is \( K \)-extendable to \( \mathcal{H} \).

In addition, since \( \{W_{i,j}^S : S \subset F_l\} \) forms a compact open partition of \( C_{i,j}^l \), for
any \( \mu \in M(\mathcal{G}) \) one has
\[ \mu(C_{i,j}^l) = \sum_{S \subset F_l} \mu(W_{i,j}^S). \]
Note that for any \( l \in I \) and \( S \subset F_l \), if \( f^{-1}(\{S\}) \neq \emptyset \) then there is a \( u \) such that
\[ |S| = |f(u)| = |Tu| > (1 - \epsilon)|Hu| = (1 - \epsilon)|F_l|. \]
This implies that for any \( \mu \in M(\mathcal{G}) \) one has
\[ \mu\left( \bigcup_{S \subset F_l} W_{i,i}^S \right) = \sum_{S \subset F_l} \mu(W_{i,i}^S) = \sum_{S \subset F_l} |S| \mu(W_{i,i}^S) \geq (1 - \epsilon)|F_l| \mu(C_{i,i}^l). \]
Thus for any \( \mu \in M(\mathcal{G}) \) one has
\[ \mu\left( \bigcup_{l \in I} \bigcup_{S \subset F_l} W_{i,i}^S \right) \geq (1 - \epsilon) \sum_{l \in I} |F_l| \mu(C_{i,i}^l). \]
A tower is called open if \( V \) is almost finiteness in the sense of Kerr (see [Ker20, Definition 8.2]). Let \( Γ \) be a set in \( X \) and \( V \) be a set in \( X \). We say \( (S, V) \) is a tower if \( \{sV : s \in S\} \) is a disjoint family. A tower is called open if \( V \) is open. Similar to the groupoid case, a finite family \( \{(S_i, V_i) : i \in I\} \) of towers is called a castle if \( sV_i \cap tV_j = \emptyset \) for any \( s \in S_i \) and \( t \in S_j \) and different \( i, j \in I \). A castle is called open if all towers inside are open.

**Definition 7.5** (Kerr). We say a free action \( \alpha : Γ \curvearrowright X \) of a countable discrete amenable group \( Γ \) on a compact metrizable space \( X \) is almost finite if for every \( n \in \mathbb{N}, \) finite set \( K \subseteq Γ \) and \( \delta > 0 \) there are

(i) an open castle \( \{(S_i, V_i) : i \in I\} \) in which all shapes \( S_i \) are \((K, δ)\)-invariant in the sense that \( |\bigcap_{t \in K} t^{-1}S_i| \geq (1 - \epsilon)|S_i| \) and all levels \( sV_i \) for \( s \in S_i \) have diameter less than \( δ \),

(ii) sets \( S'_i \subseteq S_i \) such that \( |S'_i| < |S_i|/n \) and

\[
X \setminus \bigcup_{i \in I} S_i V_i \prec_{\alpha} \bigcup_{i \in I} S'_i V_i.
\]

**Theorem 7.6.** Let \( X \times_\alpha Γ \) be the transformation groupoid of a minimal free action \( \alpha : Γ \curvearrowright X \) of a countable discrete amenable group \( Γ \) on a compact metrizable space \( X \). Then \( X \times_\alpha Γ \) is fiberwise amenable and almost elementary if and only if \( \alpha \) is almost finite.

**Proof.** Suppose \( X \times_\alpha Γ \) describe above is fiberwise amenable and almost elementary. Then Remark 5.6 shows that \( Γ \) is amenable, which is necessary for \( \alpha \) to be almost finite. In addition, since \( X \times_\alpha Γ \) is minimal and almost elementary, Theorem 6.16 shows that \( X \times_\alpha Γ \) has dynamical strict comparison. Then Remark 5.6 shows that the action \( \alpha : Γ \curvearrowright X \) has dynamical strict comparison in the sense of [Ker20, Definition 3.2]. On the other hand, Proposition 3.8 in [Ma19] shows that \( \alpha : Γ \curvearrowright X \) has the small boundary property. Therefore, Theorem A in [KS20] shows that \( \alpha \) is almost finite.

For the inverse direction, suppose \( \alpha \) is almost finite. Let \( O \) be an non-empty open set in \( X \), \( K \) a compact set in \( X \times_\alpha Γ \) and \( U \) an open cover of \( X \). Denote by \( l_U \) the Lebesgue number for \( U \) and choose finitely many group element \( γ_1, \ldots, γ_n \in Γ \) such that \( K \subseteq K' := \bigcup_{i=1}^n \{(γ_i x, γ_i, x) : x \in X\} \). Write \( F = \{γ_1, \ldots, γ_n\} \) for simplicity. Now choose \( \delta > 0 \) such that \( μ(O) > δ \) for all \( μ \in M(Γ) \). Choose \( 0 < \epsilon < l_U \) and \( n \in \mathbb{N} \) such that \( (1 - \epsilon)(1 - 1/n) > 1 - \delta \). Then almost finiteness of \( \alpha \) implies that there are

(i) an open castle \( S = \{(S_i, V_i) : i \in I\} \) whose shapes \( S_i \) are \((F, \epsilon)\)-invariant and all levels \( sV_i \) for \( s \in S_i \) have diameter less than \( \epsilon \), and
(ii) sets $S'_i \subset S_i$ such that $|S'_i| < |S_i|/n$ and
\[ X \setminus \bigsqcup_{i \in I} S_i V_i \prec_{\alpha} \bigsqcup_{i \in I} S'_i V_i. \]
Now, for each $i \in I$, define $T_i = \bigcap_{t \in F} t^{-1} S_i$, which satisfies $F T_i \subset S_i$. Since $S_i$ is $(F, \epsilon)$-invariant, one has $|T_i| \geq (1 - \epsilon)|S_i|$. In addition, since each $|S'_i| \leq |S_i|/n$, (ii) above implies that
\[ \mu(X \setminus \bigsqcup_{i \in I} S_i V_i) \leq \mu\left( \bigsqcup_{i \in I} S'_i V_i \right) \leq (1/n) \mu\left( \bigsqcup_{i \in I} S_i V_i \right) \leq 1/n \]
for all $\mu \in M(X \rtimes_{\alpha} \Gamma)$. This implies that $\mu(\bigsqcup_{i \in I} S_i V_i) \geq 1 - 1/n$ for any $\mu \in M(X \rtimes_{\alpha} \Gamma)$ and thus one has
\[ \mu(\bigsqcup_{i \in I} T_i V_i) \geq (1 - \epsilon)(1 - 1/n) > 1 - \delta \]
for any $\mu \in M(X \rtimes_{\alpha} \Gamma)$. Therefore, one has $\mu(X \setminus \bigsqcup_{i \in I} T_i V_i) < \mu(O)$ for any $\mu \in M(X \rtimes_{\alpha} \Gamma)$ and this implies $X \setminus \bigsqcup_{i \in I} T_i V_i \prec_{\alpha} O$ since $\alpha$ has dynamical strict comparison by Theorem 9.2 in Ker20.

In addition, by our definition, $\mathcal{T} = \{(T_i, V_i) : i \in I\}$ is $K'$-extendable to $\mathcal{S} = \{(S_i, V_i) : i \in I\}$ and thus $K$-extendable to $\mathcal{S}$. Finally, since each level $s V_i$ in $\mathcal{S}$ has diameter $\epsilon < 1/d$ and thus $s V_i$ is contained in some member of $\mathcal{U}$. Thus, we have established that $X \rtimes_{\alpha} \Gamma$ is almost elementary. Finally, since $\Gamma$ is amenable, Remark 6.3 shows that $X \rtimes_{\alpha} \Gamma$ is fiberwise amenable.

To end this section, we record an example due to Gabor Elek. This example indicates that our ubiquitous fiberwise amenability in general not necessarily implies (topologically) amenability of groupoids. However, in the transformation groupoid cases, this is a well-known fact that any action of an amenable group is (topologically) amenable.

Example 7.7. In Ele12 Theorem 6], Elek constructed a class of groupoids, called geometric groupoid by using so-called stable actions. They are locally compact Hausdorff (second countable) étale minimal principal almost finite ample groupoids but not (topologically) amenable. However, Proposition 7.3 and Theorem 5.13 implies that Elek’s geometric groupoids are ubiquitous fiberwise amenable.

8. SMALL BOUNDARY PROPERTY AND A NESTING FORM OF ALMOST ELEMENTARINESS

In this section, we establish a nesting version of the almost elementariness, which is the main tool in investigating the structure of reduced groupoid $C^*$-algebras. We start with the following lemmas.

Lemma 8.1. Let $\mathcal{G}$ be a locally compact Hausdorff étale groupoid on a compact metrizable space. Let $K$ be a compact set in $\mathcal{G}$ and $M$ be a precompact open bisection such that $\overline{M} \subset K$. Let $\mathcal{C} = \{C_{i, j} : i, j \in F\}$ be an open multisection that is $K$-extendable to an open multisection $\mathcal{D} = \{D_{i, j} : i, j \in E\}$ and $C_{k, k}$ be a $C$-level such that $C_{k, k} \subset s(M)$. Then there are open castles $\mathcal{A} = \{A^l_{i, j} : i, j \in F, l \in E\}$ and $\mathcal{B} = \{B^l_{i, j} : i, j \in E, l \in E\}$ satisfying

(1) index sets $F_l = F$ and $E_l = E$ for every $l \in E$;
(2) $A^l_{i, j} \subset C_{i, j}$ and $B^l_{i, j} \subset D_{i, j}$ for all $i, j \in E_l$ and $l \in E$. 
(3) \(A\) is \(K\)-extendable to \(B\).

(4) \(C_{k,k} = \bigcup_{E} A_{k,k}^l\).

(5) For any \(A_{k,k}^l\) one has \(MA_{k,k}^l = B_{l,k}^l \in B\).

(6) \(\bigcup A^{(0)} = \bigcup C^{(0)}\) and \(\bigcup B^{(0)} = \bigcup D^{(0)}\).

Proof. Since \(C\) is \(K\)-extendable to \(D\), because \(\overline{M} \subset K\), one has

\[M \cdot \bigcup_{i,j \in F} C_{i,j} \subset \bigcup_{i,j \in E} D_{i,j}\]

In particular, we have

\[MC_{k,k} \subset \bigcup_{j \in E} D_{j,k}\]

Now for any \(u \in C_{k,k}\), since each \(D_{l,k}\) is a bisection and \(C_{k,k} \subset s(M)\), there is a unique \(j_u \in E\) such that \(Mu \in D_{j_u,k}\). Then for each \(l \in E\), define \(O_l = \{u \in C_{k,k} : Mu \in D_{l,k}\}\) (may be the empty set).

We claim that \(O_l\) is open for any \(l \in E\). Indeed, for a non-empty \(O_l\), let \(u \in O_l\) and \(\gamma = Mu \in D_{l,k}\). Then one can choose an open bisection \(N \subset M \cap D_{l,k}\) such that \(\gamma \in N\). Note that \(s(N) \cap C_{k,k}\) is a open neighborhood of \(u\). For any \(v \in s(N) \cap C_{k,k}\), because \(N\) and \(M\) are bisections, one has \(Mv = Nv = D_{l,k}v\) and thus \(u \in s(N) \cap C_{k,k} \subset O_l\). This shows that \(O_l\) is open.

On the other hand, since \(M\) and all \(D_{l,k}\) are bisections, one has if \(l_1 \neq l_2 \in E\) then \(O_{l_1} \cap O_{l_2} = \emptyset\). This implies that \(C_{k,k} = \bigcup_{l \in E} O_l\). Define \(C_{k,k} = O_l\). Then for any \(i, j \in F\) define \(A_{i,k}^l = C_{i,k} \cdot A_{k,k}^l\) and \(A_{i,j}^l = A_{i,k}^l \cdot (A_{j,k}^l)^{-1}\). By our definition, it is not hard to see \(A = \{A_{i,j}^l : i, j \in F, l \in E\}\) is a bisection. Similarly, for any \(i \in E\) define \(B_{i,k}^l = D_{i,k} \cdot A_{k,k}^l\) and \(B_{i,j}^l = B_{i,k}^l \cdot (B_{j,k}^l)^{-1}\). Observe that \(B = \{B_{i,j}^l : i, j \in E, l \in E\}\) is also a bisection. By our construction, it is not hard to see \(1\), \(2\), \(3\), \(4\) and \(6\) above hold. Now for any \(A_{k,k}^l \subset C_{k,k}\) one has

\[MA_{k,k}^l = D_{i,k}A_{k,k}^l = B_{i,k}^l\]

This establishes (5).

Then we take a groupoid version of the small boundary property into our picture to select levels of a castle. The small boundary property for dynamical systems was first introduced by Lindenstrauss and Weiss in [LW00]. The following is a direct groupoid analogue.

Definition 8.2. Let \(\mathcal{G}\) be a locally compact Hausdorff étale groupoid on a compact metrizable space. \(\mathcal{G}\) is said to have groupoid small boundary property (GSBP for short) if for any \(u \in \mathcal{G}^{(0)}\) and any open neighborhood \(U\) of \(u\) with \(u \in U \subset \mathcal{G}^{(0)}\) there is another open neighborhood \(V\) such that \(u \in V \subset \overline{V} \subset U\) and \(\mu(\partial V) = 0\) for any \(\mu \in \mathcal{M}\).

We then shows that the almost elementariness implies the groupoid small boundary property. First if \(\mathcal{M}(\mathcal{G}) = \emptyset\) then \(\mathcal{G}\) satisfies the groupoid small boundary property trivially. Therefore, it suffices to show the case that \(\mathcal{M}(\mathcal{G}) \neq \emptyset\). We begin with the following concept.

Definition 8.3. Let \(X\) be a compact metrizable space with a compatible metric \(d\). Let \(\Omega\) be a weak*-closed subset of \(\mathcal{M}(X)\). We say \(X\) has \(\Omega\)-small boundary property
if for any \( x \in X \) and open neighborhood \( U \) of \( x \) there is another open neighborhood \( V \) of \( x \) such that \( x \in V \subset \overline{V} \subset U \) and \( \mu(\partial V) = 0 \) for any \( \mu \in \Omega \).

Note that if \( \alpha : \Gamma \curvearrowright X \) be an action of a countable discrete group \( \Gamma \) on \( X \) then the small boundary property in the sense of Lindenstrauss and Weiss is nothing but \( M_\Gamma(X) \)-small boundary property. The following result is an equivalent approximation form of \( \Omega \)-small boundary property. This result is first established in the case of the original small boundary property for \( \Omega = M_\Gamma(X) \) by Gárdó Szabó (c.f., [Ma19, Proposition 3.8]). However, the same proof would establish the case for general weak*-closed set \( \Omega \) in \( M(X) \) and thus we omit the proof.

**Proposition 8.4.** Let \( X \) be a compact metrizable space with a compatible metric \( d \). Let \( \Omega \) be a weak*-closed subset of \( M(X) \). Suppose for any \( \epsilon, \delta > 0 \) there is a disjoint collection \( \mathcal{U} \) of open sets such that \( \max_{U \in \mathcal{U}} \text{diam}(U) < \delta \) and \( \mu(X \setminus \bigcup \mathcal{U}) < \epsilon \) for any \( \mu \in \Omega \). Then \( X \) has \( \Omega \)-small boundary property.

In our groupoid case, note that \( M(\mathcal{G}) \) is a weak*-closed set in \( M(\mathcal{G}(0)) \). Then using Proposition 8.4 we have the following result.

**Theorem 8.5.** Let \( \mathcal{G} \) be a locally compact Hausdorff étale minimal groupoid on a compact metrizable space. Then if \( \mathcal{G} \) is almost elementary then \( \mathcal{G} \) has groupoid small boundary property.

**Proof.** As we mentioned above, it suffices to show the case that \( M(\mathcal{G}) \neq \emptyset \). Let \( \epsilon, \delta > 0 \). Choose an open cover \( \mathcal{V} \) of \( \mathcal{G}(0) \) such that each member \( U \in \mathcal{U} \) has the diameter less than \( \delta \). In addition, as usual, choose an open set \( O \) such that \( \mu(O) < \epsilon \) for any \( \mu \in M(\mathcal{G}) \) by Lemma 8.11. Since \( \mathcal{G} \) is almost elementary, for the \( \mathcal{V} \) and \( O \) one has a castle \( \mathcal{C} = \{C_{i,j}^l : i,j \in F, l \in I\} \) such that

(i) each \( \mathcal{C} \)-level \( C_{i,j}^l \) is contained in some \( V \in \mathcal{V} \)

(ii) \( \mathcal{G}(0) \setminus \bigcup \mathcal{C}(0) \prec_\mathcal{G} O \).

For the disjoint collection \( \mathcal{C}(0) \) of \( \mathcal{C} \)-levels, one has \( \max_{l \in F, i,j \in I} \text{diam}(C_{i,j}^l) < \delta \) and \( \mu(\mathcal{G}(0) \setminus \mathcal{C}(0)) < \epsilon \) for any \( \mu \in M(\mathcal{G}) \). Then Proposition 8.4 shows that \( \mathcal{G} \) has \( \mathcal{G}(\mathcal{G}) \)-small boundary property, which is exactly the groupoid small boundary property.

Then based on Lemmas above, we have the following characterization of almost elementariness.

**Theorem 8.6.** Let \( \mathcal{G} \) be a locally compact Hausdorff minimal étale groupoid with a compact metrizable unit space. Then \( \mathcal{G} \) is almost elementary if and only if for any precompact open bisections \( U_0, \ldots, U_n \) satisfying \( U_0 = \mathcal{G}(0) \) and \( \mu(\partial s(U_i)) = 0 \) for any \( i = 0, \ldots, n \) and any \( \mu \in M(\mathcal{G}) \), non-empty open set \( O \) in \( \mathcal{G}(0) \) and open cover \( \mathcal{U} \) there are open castles \( \mathcal{C} \) and \( \mathcal{D} \) satisfying

1. \( \mathcal{C} \) is \( K \)-extendable to \( \mathcal{D} \), where \( K = \bigcup_{k=0}^{n} U_k \);
2. every \( \mathcal{D} \)-level is contained in an open set \( U \in \mathcal{U} \);
3. for any \( \mathcal{C} \)-level \( C \) and \( 0 \leq p \leq n \) either \( C \subset s(U_p) \) or \( C \cap s(U_p) = \emptyset \) and if \( C \subset s(U_p) \) then there exists a \( \mathcal{D} \in \mathcal{D} \) such that \( U_p \cdot C = D \);
4. \( \mathcal{G}(0) \setminus \bigcup \mathcal{C}(0) \prec_\mathcal{G} O \).

**Proof.** Fix a compatible metric \( d \) on \( \mathcal{G}(0) \). For “if” part, suppose \( \mathcal{G} \) satisfies assumptions above. First note that the same proof of Theorem 8.5 implies that \( \mathcal{G} \) has the
GSBP. Now for any compact set $K$ with $G^{(0)} \subset K' \subset \mathcal{G}$, there are precompact open bisections $G^{(0)} = V_0, \ldots, V_n$ such that $K' \subset \bigcup_{i=0}^n V_i$. Then the GSBP implies that there are precompact open bisections $G^{(0)} = U_0, \ldots, U_n$ such that $U_i \subset \bigcup_{i=0} V_i$ and $µ(\partial s(U_i)) = 0$ for any $i = 0, \ldots, n$ and $µ \in M(\mathcal{G})$ as well as $K' \subset \bigcup_{i=0}^n U_i$. Now, take castles $\mathcal{C}$ and $\mathcal{D}$ satisfying (1)-(4) above. Then $\mathcal{C}$ is also $K'$-extendable to $\mathcal{D}$. This shows that $\mathcal{G}$ is almost elementary.

For “only if” part, first suppose $K = \bigcup_{k=0}^n \mathcal{M}_k$ where all $U_i$ are precompact open bisections satisfying $U_0 = G^{(0)}$ and $µ(\partial s(U_i)) = 0$ for any $i = 0, \ldots, n$ and any $µ \in M(\mathcal{G})$. Let $O$ be an open set in $G^{(0)}$ and $\mathcal{U}$ is an open cover $G^{(0)}$. Minimality of $\mathcal{G}$ implies that there is an $ε > 0$ such that $µ(O) > ε$ for all $µ \in M(\mathcal{G})$.

Define $S = \bigcup_{i=0}^n \partial s(U_i)$ Now Lemma 6.11 implies that there is a $δ > 0$ such that $µ(\bar{B}(S, δ)) < ε$ for any $µ \in M(\mathcal{G})$. Now choose an open cover $\mathcal{V}$ of $G^{(0)}$, which is finer than $\mathcal{U}$ and each member in $\mathcal{V}$ is of diameter less than $δ$. Now for each $0 \leq i \leq n$ define

$$T_i = \{u \in G^{(0)} : d(u, G^{(0)} \setminus s(U_i)) \geq δ\} \cup \{u \in G^{(0)} : d(u, \bar{s}(U_i)) \geq δ\}$$

In addition, define $R = \bigcap_{i=0}^n T_i$. Note that $G^{(0)} \setminus R \subset \bar{B}(S, δ)$ and thus $µ(R) \geq 1 - ε$ for any $µ \in M(\mathcal{G})$. Now since $\mathcal{G}$ is almost elementary, there are open castles $\mathcal{A}$ and $\mathcal{B}$ such that

(i) $\mathcal{A}$ is $K$-extendable to $\mathcal{B}$;
(ii) every $\mathcal{B}$-level is contained in an open set $V \in \mathcal{V}$;
(iii) $G^{(0)} \setminus \bigcup \mathcal{A}^{(0)} \prec_\mathcal{G} O$.

Then for any $\mathcal{A}$-level $A \in \mathcal{A}^{(0)}$, since diam$_d(A) < δ$, if $A \cap R \neq \emptyset$ then $A \cap S = \emptyset$ and thus either $A \subset s(U_i)$ or $A \cap s(U_i) = \emptyset$ for any $0 \leq i \leq n$.

Now define $\mathcal{A}_0^{(0)} = \{A \in \mathcal{A}^{(0)} : A \cap S = \emptyset\}$. Observe that $µ(\bigcup \mathcal{A}_0^{(0)}) \geq 1 - 2ε$ for any $µ \in M(\mathcal{G})$. Now we define $\mathcal{A}_0 = \{A \in \mathcal{A} : s(A), r(A) \in \mathcal{A}_0^{(0)}\}$. Then sinc $\mathcal{A}_0 \subset \mathcal{A}$ one has $\mathcal{A}_0$ is $K$-extendable to $\mathcal{B}$ as well. In addition, observe that $µ(G^{(0)} \setminus \bigcup \mathcal{A}_0^{(0)}) \prec_\mathcal{G} O$ for any $µ \in M(\mathcal{G})$. This implies that $G^{(0)} \setminus \bigcup \mathcal{A}_0^{(0)} \prec_\mathcal{G} O$ by Theorem 6.11. We proceed by induction on bisections $U_0, \ldots, U_n$ to establish desired castles. First, for $U_0 = G^{(0)}$, observe that (1)-(4) holds trivially for $\mathcal{A}_0$ and $\mathcal{B}$ and multisection $U_0$. Define $\mathcal{A}_1 = \mathcal{A}_0$ and $\mathcal{B}_1 = \mathcal{B}$. Now suppose we have defined castles $\mathcal{A}_k$ and $\mathcal{B}_k$ for $0 \leq k < n$ such that

(i) $\mathcal{A}_k$ is $K$-extendable to $\mathcal{B}_k$, where $K = \bigcup_{p=0}^n \mathcal{U}_p$;
(ii) every $\mathcal{B}_k$-level is contained in an open set $V \in \mathcal{V}$;
(iii) for any $\mathcal{A}_k$-level $A$ and $p \leq n$ either $A \subset s(U_p)$ or $A \cap s(U_p) = \emptyset$;
(iv) if $A \subset s(U_p)$ for some $p \leq k$ then there exists a $\mathcal{B}_k$-ladder $B$ such tat $U_p \cdot A = B$ with $s(B) = A$;
(v) $\bigcup \mathcal{A}_k^{(0)} = \bigcup \mathcal{A}_0^{(0)}$

Now for $\mathcal{A}_{k+1}$, we write $\mathcal{A}_k = \{A_{i,j}^m : i, j \in S_m, m \in J\}$ and $\mathcal{B}_k = \{B_{i,j}^m : i, j \in T_m, m \in J\}$ explicitly. First (iii) above says that for any $A_{i,j}^m \subset s(U_p)$ for some $p \leq k$ there is a $B_{i,j}^m \in \mathcal{B}$ such that

$$(\star \star \star) \quad U_p A_{i,j}^m = B_{i,j}^m.$$

Note that by our assumption one has either $A_{i,j}^m \subset s(U_{k+1})$ or $A_{i,j}^m \cap s(U_{k+1}) = \emptyset$ for each $m \in J$ define $F_m = \{i \in S_m : A_{i,j}^m \subset s(U_{k+1})\}$. Denoted by $N_m = |F_m|$ and we fix an enumeration $F_m = \{i_1, \ldots, i_{N_m}\}$. 

For any $m \in J$, fix the multisects $A_k^m = \{A_{i,j}^m : i, j \in S_m\}$ and $B_k^m = \{B_{i,j}^m : i, j \in T_m\}$ inside $A_k$ and $B_k$. Then for any $i_1 \in F_m$, apply Lemma 8.1 to multisects $A_{i_1,i_1}^m$, $B_{i_1,i_1}^m$ and $A_{i_1,i_1} \subset U_{k+1}$ to decompose $A_{i_1,i_1}$ and $B_{i_1,i_1}$ to castles $A_{k,i_1}^m = \{A_{i,j}^m : i, j \in S_m, i_1 \in T_m\}$ and $B_{k,i_1}^m = \{B_{i,j}^m : i, j \in T_m, i_1 \in T_m\}$ satisfying the corresponding properties (1)-(6) in Lemma 8.1 and in particular note that $U_{k+1}A_{i_1,i_1} = B_{i_1,i_1}^m$ and $\bigcup(A_{i_1,i_1}) = \bigcup(A_{k,i_1}^m)$ as well as $\bigcup(B_{k,i_1}^m) = \bigcup(B_{k,i_1}^m)$. In addition, for each $i_1 \in T_m$ and $i \in S_m$, one still has each either $A_{i_1,i_1} \subset s(U_{k+1})$ or $A_{i_1,i_1} \cap s(U_{k+1}) = \emptyset$. Now apply Lemma 8.1 to any multisect $A_{i_1,i_1}^m = \{A_{i,j}^m : i, j \in S_m\}$ and $B_{i_1,i_1}^m = \{B_{i,j}^m : i, j \in T_m\}$ as well as $A_{i_1,i_1}^m \subset s(U_{k+1})$ with $j_0 \in F_m$ and $i_2 \neq i_1$ to decompose multisects $A_{i_1,i_1}^m$ and $B_{i_1,i_1}^m$ to castles $A_{k,i_1}^m = \{A_{i,j}^m : i, j \in S_m, i_2 \in T_m\}$ and $B_{k,i_1}^m = \{B_{i,j}^m : i, j \in T_m, i_2 \in T_m\}$ satisfying the corresponding properties (1)-(6) in Lemma 8.1. In particular, one has that $U_{k+1}A_{i_1,i_1}^m = B_{i_1,i_1}^m$ and $\bigcup(A_{i_1,i_1}^m) = \bigcup(A_{k,i_1}^m)$ as well as $\bigcup(B_{k,i_1}^m) = \bigcup(B_{k,i_1}^m)$. In addition, for each $i_2 \in T_m$ one has $U_{k+1}A_{i_1,i_1}^m = U_{k+1}A_{i_1,i_1}^m A_{i_1,i_1}^m = B_{i_1,i_1}^m A_{i_1,i_1}^m = A_{i_1,i_1}^m$. Then we can do the same decomposition process for all multisects in $A_{k,i_1}^m$ and $B_{k,i_1}^m$ for another index $i_3 \in F_m$ such that $i_3 \neq i_1, i_2$. In addition, we repeat this process by induction, for all $l_1, \ldots, l_{N_m} \in T_m$, to obtain disjoint multisects $A_{l_1,l_{N_m}}^m = \{A_{l_{N_m},i}^m : i, j \in S_m\}$ and $B_{l_1,l_{N_m}}^m = \{B_{l_{N_m},i}^m : i, j \in T_m\}$ such that for any $m \in J$ and $l_1, \ldots, l_{N_m} \in T_m$ one has

(i) $A_{l_1,l_{N_m}}^m \subset A_{i,j}^m$ for any $i, j \in S_m$ and $B_{l_1,l_{N_m}}^m \subset B_{i,j}^m$ for any $i, j \in T_m$;

(ii) $A_{l_1,l_{N_m}}^m$ is $K$-extendable to $B_{l_1,l_{N_m}}^m$;

(iii) every $B_{l_1,l_{N_m}}^m$-level is contained in an open set $V \in \mathcal{V}$;

(iv) $U_{k+1}A_{l_1,l_{N_m}}^m = B_{l_1,l_{N_m}}^m$ for any $i_p \in F_m$;

(v) $\bigcup_{l_1,l_{N_m} \in T_m} \bigcup(A_{l_1,l_{N_m}}^m) = \bigcup(A_{k,l_{N_m}}^m)$ and

(vi) $\bigcup_{l_1,l_{N_m} \in T_m} \bigcup(B_{l_1,l_{N_m}}^m) = \bigcup(B_{k,l_{N_m}}^m)$

Now define $A_{k+1} = \bigcup_{m \in J} \bigcup_{l_1,l_{N_m} \in T_m} A_{l_1,l_{N_m}}^m$ and $B_{k+1} = \bigcup_{m \in J} \bigcup_{l_1,l_{N_m} \in T_m} B_{l_1,l_{N_m}}^m$. By our definition of $A_{k+1}$ and $B_{k+1}$, it is straightforward to see that $A_{k+1}$ is $K$-extendable to $B_{k+1}$. In addition, each $B_{k+1}$-level is contained in a member of the cover $\mathcal{V}$ and thus a member in $\mathcal{U}$. Furthermore, by our construction, since each $A_{k+1}$-level $A$ is contained in a $A_k$-level, then for any $p \leq n$ either $A \subset s(U_p)$ or $A \cap s(U_p) = \emptyset$. Finally one still has $\bigcup A_{k+1} = \bigcup A_0$. 45

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Now suppose a \( A_{k+1} \)-level \( A_{m,l}^{m_1,\ldots,l_1,m} \) satisfies \( A_{i,i}^{m_1,\ldots,l_1,m} \subset s(U_p) \) for some \( p \leq k \). Then using (i) above there is a \( j \in T_m \) such that
\[
U_p A_{i,i}^{m_1,\ldots,l_1,m} = U_p A_{j,i}^{m_1,\ldots,l_1,m} = B_j^{m_1,\ldots,l_1,m} = B_{j,i}^{m_1,\ldots,l_1,m}.
\]
Therefore, combing (iv) above one actually has that if a \( A_{k+1} \)-level \( A \subset s(U_p) \) for some \( p \leq k + 1 \) then there is a \( B_{k+1} \)-ladder \( B \) such that \( U_p A = B \). This finishes the inductive step for \( U_{k+1} \).

Now define \( C = \mathcal{A}_n \) and \( D = \mathcal{B}_n \). The argument above has established conditions (1)-(3). Finally since \( \bigcup C^{(0)} = \bigcup A_0^{(0)} \) and \( \mathcal{G}^{(0)} \backslash A_0^{(0)} \prec \mathcal{O} \), one has \( \mathcal{G}^{(0)} \backslash \mathcal{A}_0^{(0)} \prec \mathcal{G} \mathcal{O} \) as well.

Another equivalent condition established in the following theorem for almost elementariness is called the nesting form of the almost elementariness. We need the following definition.

**Definition 8.7.** Let \( \mathcal{C} = \{ C_{i,j} : i, j \in F \} \) and \( \mathcal{D} = \{ D_{m,n} : m, n \in E \} \) be two multisections. Let \( N \in \mathbb{N} \). We say \( \mathcal{C} \) is nesting in \( \mathcal{D} \) with multiplicity at least \( N \) if the following holds.

(i) For any \( C \)-level \( C_{i,i} \in C^{(0)} \) there is a \( D \)-level \( D_{i,i} \in D^{(0)} \) with \( C_{i,i} \subset D_{i,i} \).

(ii) \( \{ r(D_{m,n}C) : C \subset D_{m,n}, C \in C^{(0)} \} = \{ C \in C^{(0)} : C \subset D_{m,n} \} \) holds for any \( m, n \in E \).

(iii) \( |\{ C \in C^{(0)} : C \subset D_{m,n} \}| > N \) holds for one \( m \in E \).

We remark that (ii) and (iii) in fact imply \( |\{ C \in C^{(0)} : C \subset D_{m,n} \}| > N \) holds for any \( m \in E \).

**Definition 8.8.** Let \( \mathcal{C} = \{ C^l : l \in I \} \) and \( \mathcal{D} = \{ D^k : k \in J \} \) be two castles in which \( C^l \) and \( D^k \) are multisections. Let \( N \in \mathbb{N} \). We say \( \mathcal{C} \) is nesting in \( \mathcal{D} \) with multiplicity at least \( N \) if the following holds.

(i) For any multisection \( C_l \in \mathcal{C} \) there is a unique multisection \( D_k \in \mathcal{D} \) such that \( C_l \subset D_k \) with multiplicity at least \( N \).

(ii) For any multisection \( D_k \in \mathcal{D} \) there is at least one multisection \( C_l \in \mathcal{C} \) such that \( D_k \subset C_l \) with multiplicity at least \( N \).

**Theorem 8.9.** Let \( \mathcal{G} \) be a locally compact second countable minimal étale groupoid on a compact metrizable unit space. Then \( \mathcal{G} \) is almost elementary if and only if for any compact set \( K \) satisfying \( \mathcal{G}^{(0)} \subset K \subset \mathcal{G} \), any non-empty open set \( O \) in \( \mathcal{G}^{(0)} \), any open cover \( \mathcal{V} \) and any integer \( N \in \mathbb{N} \) there are open castles \( \mathcal{A}, \mathcal{B}, \mathcal{C} \) and \( \mathcal{D} \) such that

1. Both \( \mathcal{C} = \{ C : C \in \mathcal{C} \} \) and \( \mathcal{D} = \{ D : D \in \mathcal{D} \} \) are compact castles;
2. \( \mathcal{A} \) is \( K \)-extendable to \( \mathcal{B} \) and \( \mathcal{C} \) is \( K \)-extendable to \( \mathcal{D} \);
3. \( \mathcal{B} \) is nested in \( \mathcal{D} \) with multiplicity at least \( N \);
4. \( \mathcal{A} \) is nested in \( \mathcal{C} \) with multiplicity at least \( N \);
5. any \( \mathcal{D} \)-level is contained in a member of \( \mathcal{V} \);
6. \( \mathcal{G}^{(0)} \backslash \bigcup \mathcal{A}^{(0)} \prec \mathcal{G} \mathcal{O} \).

**Proof.** It is straightforward to see that if \( \mathcal{G} \) satisfies the conditions above then \( \mathcal{G} \) is almost elementary. Now we establish the converse. Fix a compatible metric \( d \) on \( \mathcal{G}^{(0)} \). Since \( \mathcal{G} \) is minimal and \( \mathcal{G}^{(0)} \) is compact, one can choose an \( \epsilon > 0 \) such that \( \mu(O) > 2\epsilon \) for all \( \mu \in M(\mathcal{G}) \). Now because \( \mathcal{G} \) is almost elementary, Proposition 6.10 implies that there are open castles \( \mathcal{C}_+ = \{ C_{m,n}^k : m, n \in T_k, k \in J \} \) and \( \mathcal{D}_+ = \{ D_{m,n}^k : m, n \in S_k, k \in J \} \) such that
(i) $\mathcal{C}_+ = \{ C_{k,m,n} : m, n \in T_k, k \in J \}$ and $\mathcal{D}_+ = \{ D_{m,n} : m, n \in S_k, k \in J \}$ are compact castles;
(ii) $\mathcal{C}_+$ is $K$-extendable to $\mathcal{D}_+$;
(iii) every $\mathcal{D}_+$-level is contained in an open set $V \in \mathcal{V}$.
(iv) $\mu(\bigcup_{k \in J} \bigcup_{m \in T_k} C_{k,m,n}) \geq 1 - \epsilon$ for any $\mu \in M(\mathcal{G})$ by remark \ref{remark:extendability}.

In addition, since $\mathcal{G}$ is almost elementary, applying the GSBP in the proof of Proposition \ref{proposition:gsbp} one can make $\mathcal{C}_+$ and $\mathcal{D}_+$ additionally satisfy
(v) $\mu(\partial D_{m,n}^k) = 0$ for any $\mu \in M(\mathcal{G})$, $m \in S_k$ and $k \in J$.

Write $F = G^{(0)} \setminus \bigcup \mathcal{C}_+^{(0)}$ for simplicity. By Lemma \ref{lemma:deltaExtension} one can choose a $\delta > 0$ such that the open set $M = B_d(F, \delta)$ satisfying $\mu(M) < \epsilon$ for any $\mu \in M(\mathcal{G})$. Then the collection $\mathcal{U} = \{ C_{k,m} : m \in T_k, k \in J \} \cup \{ M \}$ forms an open cover of $G^{(0)}$. Let $N \in \mathbb{N}$.

Let $C_{m,m}^k$ be a $\mathcal{C}_+$-level and $u \in G^{(0)}$. Define $H_{k,u,m} = \{ \gamma \in \mathcal{G} : s(\gamma) = u, r(\gamma) \in C_{m,m}^k \}$. Because $\mathcal{G}$ is minimal, the unit space $G^{(0)}$ is perfect in this case by the discussion after Proposition \ref{proposition:gsbp}. This implies that each $H_{k,u,m}$ is infinite. Thus, one can choose a $P_{k,u,m} \subset H_{k,u,m}$ such that $|P_{k,u,m}| > N$ and $r(P_{k,u,m})$ consists distinct units in $\mathcal{G}^{(0)}$. Since $\mathcal{G}$ is Hausdorff, there are open bisections $\{ U_{k,u,m,\gamma} : \gamma \in P_{k,u,m} \}$ such that $\gamma \in U_{k,u,m,\gamma}$ for each $\gamma \in P_{k,u,m}$ and the collection $\{ r(U_{k,u,m,\gamma}) : \gamma \in P_{k,u,m} \}$ is a disjoint family of compact subsets of $C_{m,m}^k$. In addition, since $\mathcal{G}$ is almost elementary and thus has the GSBP, by shrinking each $U_{k,u,m,\gamma}$, one may assume $\mu(s(U_{k,u,m,\gamma})) = 0$ for any $\mu \in M(\mathcal{G})$.

Then since all $\mathcal{C}_+$-levels are disjoint, note that actually $r(U_{k,u,m,\gamma}) : \gamma \in P_{k,u,m}, m \in F_k, k \in J$ is a disjoint family. Define

$$O_u = \bigcap_{k \in J} \bigcap_{m \in T_k} \bigcap_{\gamma \in P_{k,u,m}} (U_{k,u,m,\gamma}),$$

which is an open neighborhood of $u$. Then $\{ O_u : u \in G^{(0)} \}$ forms a cover of $G^{(0)}$. Then the compactness of $G^{(0)}$ implies there is a finite subcover $O = \{ O_{u_1}, \ldots, O_{u_p} \}$.

Denoted by $\mathcal{W} = O \cup \mathcal{U}$. Define a compact set

$$H = \bigcup_{k \in J} \bigcup_{m, n \in S_k} D_{m,n}^k \cdot \bigcup_{k \in J} \bigcup_{m \in T_k} P \bigcup_{q = 1}^p \bigcup_{\gamma \in P_{k, u_q, m}} U_{k,u_q,m,\gamma} \cup G^{(0)} \cup G^{(0)}$$

$$= \bigcup_{k \in J} \bigcup_{m, n \in S_k} P \bigcup_{q = 1}^p \bigcup_{\gamma \in P_{k, u_q, m}} D_{m,n}^k \cdot U_{k,u_q,m,\gamma} \cup \bigcup_{k \in J} \bigcup_{m, n \in S_k} D_{m,n}^k \cup G^{(0)}.$$ 

Note that for any bisection $D_{n,m}^k U_{k,u_q,m,\gamma} \subset H$ and $\mu \in M(\mathcal{G})$ one also has

$$\mu(s(D_{n,m}^k U_{k,u_q,m,\gamma})) = \mu(s(U_{k,u_q,m,\gamma})) = 0.$$ 

Now, for $H$ and the cover $\mathcal{W}$, Theorem \ref{theorem:almostElementary} implies that there are open castles $\mathcal{A}' = \{ A_{i,j}^l : i, j \in F_l', l \in I' \}$ and $\mathcal{B}' = \{ B_{i,j}^l : i, j \in E_l', l \in I' \}$ such that

(i) $\mathcal{A}'$ is $H$-extendable to $\mathcal{B}'$;
(ii) each $\mathcal{B}'$-level is contained in a member of $\mathcal{W}$;
(iii) if a $\mathcal{A}'$-level $A \subset s(D_{n,m}^k U_{k,u_q,m,\gamma})$ then there is a $B_1 \in \mathcal{B}'$ such that $D_{n,m}^k U_{k,u_q,m,\gamma} A = B_1$;
(iv) if a $\mathcal{A}'$-level $A \subset s(D_{n,m}^k)$ then there is a $B_2 \in \mathcal{B}'$ such that $D_{n,m}^k A = B_2$. 

Now we define required castles $A$ and $B$ as sub-castles of $B'$. First define $R_i = \{i \in F_l' : A'_{i,i}^l \text{ is contained in a } C_{+}\text{-level}\}$ and a castle $A'' = \{A''_{i,j}^l : i,j \in R_l, l \in I'\}$. Because each $B'$-level is contained in a member of $W$, which is a refinement of $U$, for any $\mu \in M(G)$, one has 
\[
\mu\left(\bigcup_{i \in I} \bigcup_{l \in F_l' \setminus R_l} A_{i,i}^l\right) \leq \mu(M) < \epsilon.
\]
Therefore, for any $\mu \in M(G)$, one has 
\[
\mu(\bigcup_i (A''(0))) = \mu\left(\bigcup_{i \in I} \bigcup_{l \in R_l} A_{i,i}^l\right) > 1 - 2\epsilon.
\]
On the other hand, Since any $B'$-level is contained in a member of $W$ and thus also in one member in $\mathcal{O}$. Then for any $l \in I$ and $A''$-level $A_{i,i}^l$ there is an $O_{uq}$ such that $A_{i,i}^l \subset O_{uq}$. This shows that $A_{i,i}^l \subset s(U_{k,uq,m}) \subset s(B_{k,uq,m})$ for any $k \in J$, $m \in T_k$, $n \in S_k$ and $\gamma \in P_{k,uq,m}$ such that $r(U_{k,uq,m}) \subset C_{m,m}$. Therefore, in these case, by (iii) for $A'$ and $B'$ above one may assume there is a $j_1 \in E'_{l}$ such that

(\ast)
\[
D_{n,m} \cdot U_{k,uq,m,\gamma} A_{i,i}^l = B_{j_1,i}^l.
\]

On the other hand, any $A''$-level $A_{i,i}^l$ is contained in some $C_\ast$-level $C_{m,m}^k$. Then (iv) for $A'$ and $B'$ above shows that for any $n \in S_k$ there is a $j_2 \in E'_{l}$ such that

(\ast\ast)
\[
D_{n,m} A_{i,i}^l = B_{j_2,i}^l.
\]

Now, for any multisection $D_{n,m} = \{D_{n,m} : m,n \in S_k\}$ together with its sub-multisection $C_{n,m} = \{C_{n,m} : m,n \in T_k\}$ and any $l \in I'$, by \[\text{ and }\]\[\text{ and }\], one can define index sets $Q_{l,k}^A$ and $Q_{l,k}^B$ as subsets of $E'_{l}$ by claiming for any $j \in E'_{l}$, $j \in Q_{l,k}^A$ if there exists $i \in R_l$, $m,n \in T_k$, $q \leq p$, $\gamma \in P_{k,uq,m}$ with $A_{i,i}^l \subset C_{m,m} \cap O_{uq}$ such that $B_{j,i}^l = C_{m,m} A_{i,i}^l$ or $B_{j,i}^l = C_{m,m} U_{k,uq,m,\gamma} A_{i,i}^l$ and $j \in Q_{l,k}^B$ if there exists $i \in R_l$, $m,n \in T_k$, $q \leq p$, $\gamma \in P_{k,uq,m}$ with $A_{i,i}^l \subset C_{m,m} \cap O_{uq}$ such that $B_{j,i}^l = D_{n,m} A_{i,i}^l$ or $B_{j,i}^l = D_{n,m} U_{k,uq,m,\gamma} A_{i,i}^l$.

Now for $j_1,j_2 \in Q_{l,k}^A \subset E'_{l}$ define $A_{j_1,j_2}^l = B_{j_1,j_2}^l$. Similarly, $j_1,j_2 \in Q_{l,k}^B \subset E'_{l}$ define $B_{j_1,j_2}^l = B_{j_1,j_2}^l$. Then we define following multisections for $k \in J$ and $l \in I'$ by
\[
A_{l}^k = \{A_{j_1,j_2}^l : j_1,j_2 \in Q_{l,k}^A\}
\]
and
\[
B_{l}^k = \{B_{j_1,j_2}^l : j_1,j_2 \in Q_{l,k}^B\}.
\]
Note that some $A_{l,k}$ and $B_{l,k}$ may be empty. Thus, we refine our castle $D_{+}$ by first defining the index set
\[
I = \{k \in J : \text{there exist an } m \in T_k \text{ and a } A'' \text{-level } A_{i,i}^l \text{ such that } A_{i,i}^l \subset C_{m,m}^k\}
\]
and define
\[
C = \{C_{n,m} \subset C_\ast : m,n \in T_k, k \in I\}.
\]
and
\[ \mathcal{D} = \{ D_{n,m}^k : m, n \in S_k, k \in I \}. \]

Then we define
\[ A = \{ A_{j_1,j_2}^{l,k} : j_1, j_2 \in Q_{l,k}^A, k \in I, l \in I' \} \]
and
\[ B = \{ B_{j_1,j_2}^{l,k} : j_1, j_2 \in Q_{l,k}^B, k \in I, l \in I' \}. \]

Now, we prove the castle \( \mathcal{A}, \mathcal{B}, \mathcal{C} \) and \( \mathcal{D} \) satisfying our requirements. First (1) and (5) are clear by the definition of these castles. For (2), note that \( \mathcal{C} \) and \( \mathcal{D} \) consists of multisections in \( \mathcal{C}_+ \) and \( \mathcal{D}_+ \) with the same index set \( I \subset J \), respectively. Then since \( \mathcal{C}_+ \) is \( K \)-extendable to \( \mathcal{D}_+ \), the castle \( \bar{\mathcal{C}} \) is \( K \)-extendable to \( \bar{\mathcal{D}} \) as well. In order to show that \( \mathcal{A} \) is \( K \)-extendable to \( \mathcal{B} \), first fix a \( \theta \in A_{j_1,j_2}^{l,k} \). By definition, \( A_{j_1,j_2}^{l,k} = B_{j_1,j_2}^{l,k} \) where \( j_1, j_2 \in Q_{l,k}^A \subset E_l \). Note that first \( A_{j_1,j_2}^{l,k} = B_{j_1,j_2}^{l,k} \) for some \( i_1, i_2 \in R_l \) such that \( B_{j_1,i_1}^{l,k} = C_{n_1,m_1}^{l,k} U_{k,u_{q_1},m_1} A_{i_1,i_1}^{l,k} \) or \( B_{j_1,i_1}^{l,k} = C_{n_1,m_1}^{l,k} A_{i_1,i_1}^{l,k} \) for some \( n_1, m_1 \in T_k, q_1 \leq p, \gamma_1 \in P_{k,u_{q_1},m_1} \) with \( A_{i_1,i_1}^{l,k} \subset C_{m_1,m_1}^{l,k} \cap O_{u_{q_1}} \) by the definition of \( Q_{l,k}^B \). Then \( r(\theta) \in C_{n_1,m_1}^{l,k} \) and since \( \mathcal{C} \) is \( K \)-extendable to \( \mathcal{D} \), suppose \( B_{j_1,i_1}^{l,k} = C_{n_1,m_1}^{l,k} U_{k,u_{q_1},m_1} A_{i_1,i_1}^{l,k} \) then there is an \( n \in S_k \) such that
\[ K \theta = K r(\theta) = D_{n,m_1}^{l,k} C_{n_1,m_1}^{l,k} U_{k,u_{q_1},m_1} A_{i_1,i_1}^{l,k} \]
\[ = D_{n,m_1}^{l,k} U_{k,u_{q_1},m_1} \gamma_1 A_{i_1,i_1}^{l,k} (B_{j_1,i_2}^{l,k})^{-1} \]
\[ = B_{j_1,i_1}^{l,k} A_{i_1,i_2}^{l,k} (B_{j_1,i_2}^{l,k})^{-1} \]
for some \( j_3 \in Q_{l,k}^B \). On the other hand, if \( B_{j_1,i_1}^{l,k} = C_{n_1,m_1}^{l,k} A_{i_1,i_1}^{l,k} \) then same argument above shows that there is a \( j_4 \in Q_{l,k}^B \) such that \( K \theta = B_{j_4,j_2}^{l,k} \). This shows that \( \mathcal{A} \) is \( K \)-extendable to \( \mathcal{B} \) and establishes (2).

We now establish (3) and (4). We begin with showing \( \mathcal{A} \) is nested in \( \mathcal{C} \) with multiplicity at least \( N \). First, fix a \( k \in I, l \in I' \) in \( \mathcal{A} \). Look at a multisection \( A_{j_1,j_2}^{l,k} \). By the definition of \( \mathcal{A} \), it is straightforward that any \( A_{j_1,j_2}^{l,k} \)-level is contained in a \( C_l \)-level, where the multisection \( C_l = C_{m,n}^{l,k} : m, n \in T_k \). Then for a \( C_l \)-level \( C_{m,n}^{l,k} \) such that there is an \( A_l \)-level \( A_{i_0,i_0}^{l,k} \subset C_{m,n}^{l,k} \) by definition of \( I \). Then one has
\[ \{ A \in (A_{j_1,j_2}^{l,k})^{(0)} : A \subset C_{m,n}^{l,k} \} = \{ r(B_{j_1,i}^{l,k}) : B_{j_1,i}^{l,k} = C_{m,n}^{l,k} U_{k,u_{q_1},m_1} \}
\]
\[ \gamma_1 \in P_{k,u_{q_1},m_1} \cap O_{u_{q_1}}, \gamma_1 \leq p, \text{ and } m \in T_k, i \in R_l \}, \]
and in particular contains the set
\[ \{ r(B_{j_1,i_0}^{l,k}) : B_{j_1,i_0}^{l,k} = A_{i_0,i_0}^{l,k} \text{ or } B_{j_1,i_0}^{l,k} = U_{k,u_{q_1},m_1} A_{i_0,i_0}^{l,k} \cap O_{u_{q_1}}, q \leq p, \gamma_1 \in P_{k,u_{q_1},m_1} \}. \]
This thus implies
\[ \{ A \in (A_{j_1,j_2}^{l,k})^{(0)} : A \subset C_{m,n}^{l,k} \} > N \]
because \( |P_{k,u_{q_1},m_1}| \geq N \) and \( \{ r(U_{k,u_{q_1},m_1}) : \gamma_1 \in P_{k,u_{q_1},m_1} \} \) is a disjoint family of subsets of \( C_{m,n}^{l,k} \) for any \( q \leq p \) such that \( A_{i_0,i_0}^{l,k} \subset O_{u_{q_1}} \). Now for another \( m \in T_k \), by our definition of \( \mathcal{A} \) and (4), one has
\[ \{ A \in (A_{l,k}^{l,k})^{(0)} : A \subset C_{m,n}^{l,k} \} = \{ r(C_{m,n}^{l,k}) : A \in (A_{l,k}^{l,k})^{(0)}, A \subset C_{m,n}^{l,k} \}. \]
This shows that multisection $A^{k,l}$ is nested in $C^k$ with multiplicity at least $N$ for $k \in I$ and $l \in I'$. Therefore, one has that the castle $A$ is nested in the castle $C$ by the definition of the index set $I$.

Similarly, to show $B^{l,k} = \{B^{l,k}_{j_1,j_2} : j_1, j_2 \in Q_{l,k}^E \}$ is nested in $D^k = \{D^k_{m,n} : m, n \in S_k \}$ for the $k \in I$ and $l \in I'$, first note that any $B^{l,k}$-level is contained in a $D^k$-level by definition. Then since the level $D^k_{m,n} = C^k_{m,n}$ for any $n \in T_k$ above and any $A$-level is also a $B$-level by $K$-extendability of $A$ to $B$ and $C$ to $D$, respectively, one has

$$\| \{B \in B^{l,k}_{i,k} : B \subset D^k_{n,n} \} \| > N$$

for the $n \in T_k$ established in 9(I) above. Finally, it suffices to observe by definition of $B$ and 9(I) again that

$$\{B \in B^{l,k}_{i,k} : B \subset D^k_{m,m} \} = \{r(D^k_{m,n}B) : B \in A^{l,k}_{i,k}, B \subset D^k_{n,n} \}$$

for any $m \in S_k$. This shows that multisection $B_{i,k}$ is nested in $D_k$ with multiplicity at least $N$ for the $k \in I$ and $l \in I'$. Thus the castle $B$ is nested in the castle $D$. This establishes (3) and (4) as desired.

Finally, we establish (6). Note that $A^\mu \subset A$ because $R_t \subset \bigcup_{k \in I} Q^{A}_{l,k}$. Thus for any $\mu \in M(\mathcal{G})$ one has

$$\mu(\bigcup A^0) \geq \mu(\bigcup (A^\mu)^0) > 1 - 2\epsilon.$$ 

This implies that $\mu(O) > 2\epsilon \geq \mu(G^0 \setminus \bigcup A^0)$ for any $\mu \in M(\mathcal{G})$. Then Theorem 6.16 shows that $G^0 \setminus \bigcup A^0 \subseteq G$. This establishes (6).

**Remark 8.10.** We remark that if the compact set $K$ in Theorem 8.9 is a union of compact bisections, say, $K = \bigcup_{t=0}^n \overline{O}_t$, where each $O_t$ is a precompact open bisection with $O_0 = \mathcal{G}^0$ and $\mu(s(O_t)) = 0$ for any $0 \leq i \leq n$ and $\mu \in M(\mathcal{G})$, Theorem 8.6 implies that the castles $C$ and $D$ can be chosen furthermore satisfying that

1. for any $0 \leq i \leq n$ and $C \subset \mathcal{G}^0$ either $C \subset s(O_i)$ or $C \cap s(O_i) = \emptyset$ and
2. whenever a $C$-level $C \subset s(O_i)$ for some $i \leq n$ then there is a $D \in D$ such that $s(D) = C$ and $\mathcal{G} \cap C = D$.

Indeed, we do this by adjusting the beginning of the proof of Theorem 8.9. First, for $\epsilon > 0$ and the cover $V$ there, using Theorem 8.9 one obtains open castles $C'$ and $D'$ satisfying

(i) $C'$ is $K$-extendable to $D'$, where $K = \bigcup_{t=0}^n \overline{O}_t$;
(ii) every $D'$-level is contained in an open set $V \subset V$;
(iii) for any $C'$-level $C'$ and $0 \leq i \leq n$ either $C' \subset s(O_i)$ or $C' \cap s(O_i) = \emptyset$ and if $C' \subset s(O_i)$ then there exists a $D' \in D'$ such that $O_i \cdot C = D'$;
(iv) $\mu(\bigcup C^0) > 1 - \epsilon$ for any $\mu \in M(\mathcal{G})$.

Then write $C' = \{C^{k}_{m,n} \cap T_k, k \in J \}$ and $D' = \{D^{k}_{m,n} \cap T_k, k \in J \}$ explicitly and use the shrinking technique in Proposition 6.10 and the GSBP for $C'$ and $D'$, one obtains castles $C_+ = \{C^{k}_{m,n} \cap T_k, k \in J \}$ and $D_+ = \{D^{k}_{m,n} \cap T_k, k \in J \}$ such that

(i) $\tilde{C}_+ = \{C^{k}_{m,n} \cap T_k, k \in J \}$ and $\tilde{D}_+ = \{D^{k}_{m,n} \cap T_k, k \in J \}$ are compact castles;
(ii) $\tilde{D}^{k}_{m,n} \subset D^{k}_{m,n}$ for any $k \in J$ and $m, n \in S_k$;
(iii) $\tilde{C}_+$ is $K$-extendable to $\tilde{D}_+$;
(iv) every $\tilde{D}$-level is contained in an open set $V \subset V$;
Therefore, by (ii) for $D_+$ and thus $C_+$, for any $C_+$-level $C$ and $m \leq n$, $(C \cap s(O_1)) = \emptyset$. Write $C = C^k_{m,m}$ explicitly and suppose $C^k_{m,m} \subseteq C^k_{s(O_1)}$ for some $i \leq n$. Then necessarily $C^k_{m,m} \subseteq C^k_{s(O_1)}$ by (ii) for $C_+$ and (iii) for $C'$. Therefore there is a $D^k_{n,m} \in D'$ such that $O_i C^k_{m,m} = D^k_{n,m}$. This implies that $O_i C^k_{m,m} = O_i C^k_{m,m} C^k_{m,m} = D^k_{n,m} C^k_{m,m} = D^k_{n,m}$. This thus establishes original conditions for $C_+$ and $D_+$ in Theorem 8.3 and additional conditions (1) and (2) above. Therefore, we may arrange the castles $\mathcal{C}$ and $\mathcal{D}$ satisfying (1) and (2) because they are subcastles of $\mathcal{C}_+$ and $\mathcal{D}_+$, respectively.

9. Tracial $Z$-stability

In this section, we investigate structure properties of the reduced $C^*$-algebra $C_p^r(\mathcal{G})$ of an almost elementary groupoid $\mathcal{G}$. In particular, we will show $C_p^r(\mathcal{G})$ is tracially $Z$-stable in the sense of [HO13 Definition 2.1].

Let $A, B$ be $C^*$-algebras. Denote by $A_+$ the set of all positive elements in $A$. Recall a c.p.c. map $\psi : A \to B$ is said to be order zero if for any $a, b \in A_+$ with $ab = 0$, we have $\psi(a)\psi(b) = 0$ as well. For $a, b \in A_+$, $a$ is said to be $Cuntz$-subequivalent to $b$, denoted by $a \preceq b$, if there is a sequence $\{x_n \in A : n \in \mathbb{N}\}$ such that $\lim_{n \to \infty} ||a - x_n b_x^*|| = 0$. We write $a \sim b$ if $a \preceq b$ and $b \preceq a$. The following concept of tracial $Z$-stability was introduced by Hirshberg-Orovitz in [HO13 Definition 2.1].

**Definition 9.1.** [Hirshberg-Orovitz] A unital $C^*$-algebra $A$ is said to be **tracially $Z$-stable** if $A \neq \mathbb{C}$ and for any finite set $F \subset A$, $\epsilon > 0$, any non-zero positive element $a \in A_+$ and $n \in \mathbb{N}$, there is an order zero c.p.c. map $\psi : M_n(\mathbb{C}) \to A$ such that the following hold:

1. $1_A - \psi(1_n) \preceq a$;
2. for any $x \in M_n(\mathbb{C})$ with $||x|| = 1$ and any $y \in F$ one has $||\psi(x), y|| < \epsilon$.

In addition, it was proved in [HO13] that the tracial $Z$-stability is equivalent to $Z$-stability in the case that the $C^*$-algebra $A$ under consideration is unital simple separable nuclear. For our case, we will use the nesting form of almost elementariness established in Theorem 8.3. We begin with the following result, which is a groupoid version of [Phi12 Lemma 7.9].

**Lemma 9.2.** Let $\mathcal{G}$ be a locally compact Hausdorff étale effective groupoid on a compact metrizable space. For any non-zero element $a \in C_p^r(\mathcal{G})_+$, there is a non-zero function $g \in C(\mathcal{G}(0))_+$ such that $g \preceq a$.

**Proof.** Let $a \in C_p^r(\mathcal{G})_+ \setminus \{0\}$. Without loss of generality, one may assume $||a|| = 1$. Let $\epsilon \leq (1/6)||E(a)||$ where $E : C_p^r(\mathcal{G}) \to (C(\mathcal{G}))_+$ is the canonical faithful expectation. Then there is an $h \in C_{\epsilon}(\mathcal{G})_+$ such that $||a^{1/2} - h|| < \epsilon$ and $||h|| \leq 1$. Then one has $||h^* h^* - a|| < 2\epsilon$ and $||h^* h - a|| < 2\epsilon$. Then for $h^* h \in C_{\epsilon}(\mathcal{G})_+$, Lemma 4.2.5 in [Sim12] shows that there is a function $f \in C(\mathcal{G}(0))_+$ such that $||f|| = 1$, $f \ast h^* h \ast f = f(E(h^* h) f)$ and $||f \ast h^* h \ast f|| \geq ||E(h^* h)|| - \epsilon$. Then one has $||f \ast h^* h \ast f|| > ||E(h^* h)|| - \epsilon \geq ||E(a)|| < 3\epsilon$. Therefore, by (ii) for $D_+$ and thus $C_+$, for any $C_+$-level $C$ and any $m \leq n$, $(C \cap s(O_1)) = \emptyset$. Write $C = C^k_{m,m}$ explicitly and suppose $C^k_{m,m} \subseteq C^k_{s(O_1)}$ for some $i \leq n$. Then necessarily $C^k_{m,m} \subseteq C^k_{s(O_1)}$ by (ii) for $C_+$ and (iii) for $C'$. Therefore there is a $D^k_{n,m} \in D'$ such that $O_i C^k_{m,m} = D^k_{n,m}$. This implies that $O_i C^k_{m,m} = O_i C^k_{m,m} C^k_{m,m} = D^k_{n,m} C^k_{m,m} = D^k_{n,m}$.
Then define \( g = (f \ast h \ast h \ast f - 2\varepsilon)_+ = (fE(h \ast h)f - 2\varepsilon)_+ \in C(G^{(0)})_+ \setminus \{0\} \). Then using Lemma 1.6 and 1.7 in [12], one has

\[
g \sim (h \ast f^2 \ast h^* - 2\varepsilon)_+ \preceq (h \ast h^* - 2\varepsilon)_+ \preceq \eta
\]
as desired. □

**Lemma 9.3.** Let \( G \) be a locally compact Hausdorff étale effective groupoid on a compact metrizable space. Suppose \( G \) has the GSBP, and for any \( \varepsilon > 0 \), \( n \in \mathbb{N} \), non-zero positive element \( g \in C(G^{(0)})_+ \) and finite collection \( F \subset C_c(G) \) in which the support \( \text{supp}(f) \) for any \( f \in F \) is a compact bisection contained in an open bisection \( V_f \) and satisfying \( \mu(\partial s(\text{supp}(f))) = 0 \) for any \( \mu \in M(G) \), there is an order zero c.p.c. map \( \psi : M_n(\mathbb{C}) \to C^*_r(G) \) such that the following hold:

1. \( 1_{C^*_r(G)} - \psi(1_n) \preceq g \).
2. For any \( x \in M_n(\mathbb{C}) \) with \( \|x\| = 1 \) and any \( f \in F \) one has \( \|\psi(x), f\|_r < \varepsilon \).

Then the \( C^* \)-algebra \( C^*_r(G) \) is tracially \( \mathcal{Z} \)-stable.

**Proof.** Let \( a \in C^*_r(G) \setminus \{0\}, \varepsilon > 0 \), \( F \) a finite set in \( C^*_r(G) \) and \( n \in \mathbb{N} \). We aim to find a c.p.c. map \( \psi \) satisfies Definition [74]. Since \( C_c(G) \) is dense in \( C^*_r(G) \), without loss of any generality, we may assume \( F \subset C_c(G) \). Then Proposition [27] implies that one may assume further that each support \( \text{supp}(f) \) of \( f \in F \) is a compact set contained in an open bisection \( V \). Let \( f \in F \) and write \( K = \text{supp}(f) \subset V \). Now since \( G \) has the GSBP, there is an open set \( O \) such that \( s(K) \subset O \subset s(V) \) with \( \mu(\partial O) = 0 \) for any \( \mu \in M(G) \). Then choose a function \( g \in C(G^{(0)}) \) such that \( g = 1 \) on \( s(K) \), \( 0 < g \leq 1 \) on \( O \) and \( g = 0 \) on \( G^{(0)} \setminus O \). Now define \( h = g(s(f)) + \varepsilon/3 \in C(G^{(0)}) \). Observe that \( \text{supp}^o(h) = O \) and \( \|h - s(f)\|_\infty \leq \varepsilon/3 \). Now define \( f' = C_c(G) \) by \( f'(x) = h(s(x)) \) for \( x \in V \) and \( f' = 0 \) on \( G \setminus V \). Note that \( \text{supp}^o(f') = (s|_V)^{-1}(O) \) and thus \( \text{supp}(f') = (s|_V)^{-1}(O) \). In addition, since \( f \sim f' \) is supported on \( (s|_V)^{-1}(O) \), which is a bisection. Then Proposition [29] implies that

\[
\|f - f'\|_r = \|f - f'\|_\infty = \sup_{x \in (s|_V)^{-1}(O)} |f(x) - f'(x)| = \sup_{u \in O} |s(f)(u) - h(u)| \leq \varepsilon/3.
\]

Denote by \( F' = \{f' : f \in F\} \) obtained by the argument above. Now choose a non-zero positive function \( f_0 \in C(G^{(0)})_+ \) such that \( f_0 \preceq a \) by Lemma [92]. Then for \( \varepsilon > 0 \), finite set \( F' \), \( n \in \mathbb{N} \) and \( f_0 \), by assumption, there is an order zero c.p.c. map \( \psi : M_n(\mathbb{C}) \to C^*_r(G) \) such that the following hold:

1. \( 1_{C^*_r(G)} - \psi(1_n) \preceq f_0 \).
2. For any \( x \in M_n(\mathbb{C}) \) with \( \|x\| = 1 \) and any \( f' \in F' \) one has \( \|\psi(x), f'\|_r < \varepsilon/3 \).

Then first, one has \( 1_{C^*_r(G)} - \psi(1_n) \preceq f_0 \preceq a \) by our choice of \( f_0 \). In addition, for each \( f \in F \), one has

\[
\|\psi(x), f\|_r \leq \|\psi(x), f'\|_r + 2\|\psi(x)\|_r \cdot \|f - f'\|_r \leq \varepsilon.
\]

This shows that \( C^*_r(G) \) is tracially \( \mathcal{Z} \)-stable. □

Then we generically show how to construct c.p.c. order zero map from \( M_n(\mathbb{C}) \) to \( C^*_r(G) \) to establish the tracial \( \mathcal{Z} \)-stability.

**Remark 9.4.** Let \( G \) be a locally compact Hausdorff étale groupoid on a compact space. Let \( n \in \mathbb{N} \) and \( \varepsilon > 0 \). In addition, let \( N \in \mathbb{N} \) such that \( N > 2/\varepsilon \) and \( K \) be a compact set in \( G \). Suppose \( A, B, C \) and \( D \) are open castles such that

1. \( A \) is \( K \)-extendable to \( B \) and \( C \) is \( K \)-extendable to \( D \).
(ii) \( B \) is nested in \( D \) with multiplicity at least \( nN \).

(iii) \( A \) is nested in \( C \) with multiplicity at least \( nN \).

(iv) \( \mu(\bigcup \mathcal{A}^{(0)}) > 1 - \varepsilon/2 \) for any \( \mu \in M(\mathcal{G}) \).

If we write \( A \) and \( B \) explicitly, say, by \( A = \{ A_{i,j}^l : i, j \in F_l, l \in I \} \) and \( B = \{ B_{i,j}^l : i, j \in E_l, l \in I \} \) where \( F_l \subset E_l \) for each \( l \in I \) then Proposition 6.14 implies that there are compact castle \( A' = \{ A_{i,j}^l : i, j \in F_l, l \in I \} \) and \( B' = \{ B_{i,j}^l : i, j \in E_l, l \in I \} \) such that

(i) all \( A_{i,j}^l \) and \( B_{i,j}^l \) are compact sets;

(ii) for each \( l \in I \), one has \( A_{i,j}^l \subset A_{i,j}^l \) for any \( i, j \in F_l \) and \( B_{i,j}^l \subset B_{i,j}^l \) for any \( i, j \in E_l \);

(iii) \( A' \) is \( K \)-extendable to \( B' \) and

(iv) \( \mu(\bigcup \mathcal{A}^{(0)}) > 1 - \varepsilon/2 \) for any \( \mu \in M(\mathcal{G}) \).

Now for each \( l \in I \) fix an \( i \in E_l \). We first define a function \( h_{B_{i}^{l}} \) by choosing a continuous function in \( C(\mathcal{G}^{(0)}) \) such that \( \text{supp}(h_{B_{i}^{l}}) \subset B_{i}^{l} \) and \( h_{B_{i}^{l}} \equiv 1 \) on the compact subset \( B_{i}^{l} \). Then for \( j, k \in E_l \), we define \( h_{B_{j,k}^{l}} \) be the function such that \( s(h_{B_{j,k}^{l}}) = h_{B_{j,k}^{l}} \) and define \( h_{B_{j,k}^{l}} = r(h_{B_{j,k}^{l}}) \). By this process, we have a collection \( \{ h_B : B \in \mathcal{B} \} \) of functions in \( C_c(\mathcal{G}) \) such that

(i) \( s(h_B) = h_s(B) \) and \( r(h_B) = h_r(B) \) for each \( B \in \mathcal{B} \).

(ii) \( h_r(B) \ast 1_B = h_B \) and \( 1_B \ast h_s(B) = h_B \) for each \( B \in \mathcal{B} \).

(iii) \( \text{supp}(h_B) \subset B \) for each \( B \in \mathcal{B} \).

(iv) \( h_B \equiv 1 \) on \( B' \) for any \( B \in \mathcal{B} \) where \( B' \subset B \) is the compact bisection contained in \( B' \).

In this case we say the collection \( \{ h_B : B \in \mathcal{B} \} \) above is \( \mathcal{B} \)-compatible.

Now we write \( \mathcal{C} \) and \( \mathcal{D} \) explicitly by \( \mathcal{C} = \{ C_{s,t}^{p} : t, s \in T_p, p \in J \} \) and \( \mathcal{D} = \{ D_{t,s}^{p}, t, s \in S_p, p \in J \} \). Let \( \mathcal{H}^{(0)} \subset \mathcal{D}^{(0)} \) be a subset containing \( \mathcal{C}^{(0)} \). Now, since \( \mathcal{C}^{(0)} \subset \mathcal{H}^{(0)} \), one has that \( \mathcal{H}^{(0)} \) contains some \( \mathcal{D} \)-levels from multisection \( (\mathcal{D}^{p})^{(0)} \) for any \( p \in J \). Denote by \( (\mathcal{H}^{(0)})^{p} = (\mathcal{D}^{p})^{(0)} \cap \mathcal{H}^{(0)} \). Now, for each \( p \in J \) and let \( l \in I \) such that \( B^l \) is nested in \( D^p \) with multiplicity at least \( nN \). Fix a level \( D^p \), where \( t \in S_p \) and define \( P_{p,t,l} = \{ B \in (B^l)^{(0)} : B \subset D^p_{t,l} \} \). Note that \( |P_{p,t,l}| \geq nN \). Then for \( m = 1, \ldots, n \) choose a subset \( P_{p,t,l,m} \subset P_{p,t,l} \) such that \( |P_{p,t,l,m}| = |P_{p,t,l}|/n \).

In addition, choose a bijection \( \Theta_{p,t,l,m} : P_{p,t,l,1} \to P_{p,t,l,m} \) and define \( \Theta_{p,t,l,k,m} = \Theta_{p,t,l,k} \circ \Theta_{p,t,l,m}^{-1} \). From this construction, for any \( p \in J \) and \( l \in I \) such that \( B^l \) is nested in \( D^p \), we actually have the following configuration.

(i) \( P_{D,l} = \{ B \in (B^l)^{(0)} : B \subset D \} \) has the cardinality \( |P_{D,l}| > nN \) for any \( D \in (\mathcal{H}^{(0)})^{p} \).
(ii) There are collections $P_{D,l,m} \subset P_{D,l}$ such that $|P_{D,l,m}| = |P_{D,l}|/n$ for any $D \in (\mathcal{H}^p)^{(0)}$ and $1 \leq m \leq n$.

(iii) There are bijective maps $\Theta_{D,l,m} : P_{D,l,m} \to P_{D,l,k}$ for any $D \in (\mathcal{H}^p)^{(0)}$ and $1 \leq m, k \leq n$. For any $1 \leq k, m, p \leq n$, these functions also satisfy

(a) $\Theta_{D,l,m,m}$ is the identity map;
(b) $\Theta_{D,l,m,k}^{-1} = \Theta_{D,l,k,m}$;
(c) $\Theta_{D,l,m} \Theta_{D,l,m,p} = \Theta_{D,l,k,p}$.

(iv) For any $D \in D$ such that $s(D), r(D) \in (\mathcal{H}^p)^{(0)}$ one has

$$r(D \Theta_{s(D),l,k,m}(B)) = \Theta_{r(D),l,k,m}(r(DB))$$

for any $B \in P_{s(D),l,k,m}$ and $1 \leq k, m \leq n$.

In this case, we call such a collection of all sets $P_{D,l,m}$ together with all maps $\Theta_{D,l,m,k}$ for any $p \in J, l \in I$ such that $B^l$ is nested in $D^p$, $D \in (\mathcal{H}^p)^{(0)}$, $1 \leq m, k \leq n$, a $\mathcal{H}^{(0)}$-nesting system.

Now for $D \in (\mathcal{H}^p)^{(0)}$, $l \in I$ such that $B^l$ is nested in $D^p$ and $1 \leq k, m \leq n$, we denote by

$$R_{D,l,k,m} = \{B \in B^l : s(B) \in P_{D,l,m} \text{ and } r(B) = \Theta_{D,l,k,m}(s(B))\}.$$

For each $p \in J$ write $I_p = \{l \in I : B^l \text{ is nested in } D^p \text{ with multiplicity at least } nN\}$ and for each $D \in (\mathcal{H}^p)^{(0)}$ define

$$Q_{k,m,D} = \bigcup_{l \in I_p} R_{D,l,k,m}.$$

In addition, we fix an arbitrary function $\kappa : \mathcal{H}^{(0)} \to [0, 1]$. Denote by $e_{km}$ the matrix in $M_n(\mathbb{C})$ whose $(k, m)$-entry is 1 while other entries are zero. Now we define a map $\psi : M_n(\mathbb{C}) \to C^*_r(\mathcal{G})$ by

$$\psi(e_{km}) = \sum_{D \in \mathcal{H}^{(0)}} \sum_{B \in Q_{k,m,D}} \kappa(D) h_B$$

and is linearly extended to define on the whole $M_n(\mathbb{C})$. We will show in the following lemma that the map $\psi$ is a c.p.c. order zero map.

On the other hand, note that for each $p \in J$ the index set $I_p$ consists exactly all $l \in I$ such that $A^l$ is nested in $C^p$ with multiplicity at least $nN$. Then for any $C^p$-level $C^p_{l,i}$ and $l \in I_p$, there are at most $n - 1$ levels $A \in (A^l)^{(0)}$ with $A < C$ so that $\psi(1_n)$ is not supported on. Then choose one of them, denoted by $A_{p,l}$. Now, for any $\mu \in M(\mathcal{G})$, first the fact that $A^l$ is nested in $C^p$ with multiplicity at least $nN$ implies that

$$\sum_{p \in J} \sum_{l \in I_p} nN |T_p| \mu(A_{p,l}) \leq \mu(\bigcup A^{(0)}) \leq 1.$$

Then one has

$$\mu(\bigcup \{ A \in A^{(0)} : \psi(1_n) \equiv 0 \text{ on } A \}) \leq \sum_{p \in J} \sum_{l \in I_p} (n - 1) |T_p| \mu(A_{p,l}) \leq 1/2.$$  

and thus

$$\mu(\bigcup \{ A' \in A^{(0)} : \psi(1_n) \equiv 1 \text{ on } A' \}) \geq 1 - \epsilon/2 - 1/2 \geq 1 - \epsilon.$$

Write $f = 1_{C^*_r(\mathcal{G})} - \psi(1_n)$. Then one has $\mu(\text{supp}(f)) < \epsilon$ for any $\mu \in M(\mathcal{G})$. 

Lemma 9.5. Let $G$ be a locally compact Hausdorff second countable étale groupoid on a compact space. The map $\psi$ defined in Remark 9.4 is a c.p.c. order zero map.

Proof. Let $A$, $B$, $C$ and $D$ be open castles defined above. Let $H^{(0)}$, $P_D$ and $Q_{k,m,D}$ be specific sets defined in Remark 9.4 above as well. Now define $\varphi : M_n(\mathbb{C}) \to C^*_r(G)^{**}$ by

$$\varphi(e_{km}) = \sum_{D \in H^{(0)}} \sum_{B \in Q_{k,m,D}} 1_B$$

and extending linearly where $1_B$ is the indicator function on the open set $B$. It is straightforward to see $\varphi$ above is a homomorphism by using (a), (b) and (c) of (iii) in the configuration of $H^{(0)}$, $B^{(0)}$-nesting system. Define $h_0 \in C_c(G)$ by

$$h_0 = \psi(1_n) = \sum_{i=1}^n \sum_{D \in H^{(0)}} \sum_{B \in Q_{i,i,D}} \kappa(D) h_B$$

where $\psi$ is the map defined in Remark 9.4. Then consider

$$h_0 * \varphi(e_{km}) = \left( \sum_{i=1}^n \sum_{D' \in H^{(0)}} \sum_{B' \in Q_{i,i,D'}} \kappa(D') h_{B'} \right) * \left( \sum_{D \in H^{(0)}} \sum_{B \in Q_{k,m,D}} 1_B \right)$$

$$= \sum_{i=1}^n \sum_{D' \in H^{(0)}} \sum_{B' \in Q_{i,i,D'}} \sum_{D \in H^{(0)}} \sum_{B \in Q_{k,m,D}} \kappa(D') h_{B'} * 1_B.$$ 

Let $B' \in Q_{i,i,D'}$ and $B \in Q_{k,m,D}$. Note that $\kappa(D') h_{B'} * 1_B = \kappa(D) h_B$ if $B' = r(B)$, $D = D'$ and $i = k$. Otherwise, $\kappa(D') h_{B'} * 1_B = 0$. Thus one has

$$h_0 * \varphi(e_{km}) = \sum_{D \in H^{(0)}} \sum_{B \in Q_{k,m,D}} \kappa(D) h_B = \psi(e_{km}).$$

Similarly, one has

$$\varphi(e_{km}) * h_0 = \left( \sum_{D \in H^{(0)}} \sum_{B \in Q_{k,m,D}} 1_B \right) \left( \sum_{i=1}^n \sum_{D' \in H^{(0)}} \sum_{B' \in Q_{i,i,D'}} \kappa(D') h_{B'} \right)$$

$$= \sum_{D \in H^{(0)}} \sum_{B \in Q_{k,m,D}} \sum_{i=1}^n \sum_{D' \in H^{(0)}} \sum_{B' \in Q_{i,i,D'}} \kappa(D') 1_B * h_{B'}. $$

Let $B' \in Q_{i,i,D'}$ and $B \in Q_{k,m,D}$. Note that $\kappa(D') 1_B * h_{B'} = \kappa(D) h_B$ if $B' = s(B)$, $D = D'$ and $i = m$. Otherwise, $\kappa(D') h_{B'} * 1_B = 0$. This implies that

$$\varphi(e_{km}) * h_0 = \sum_{D \in H^{(0)}} \sum_{B \in Q_{k,m,D}} \kappa(D) h_B = \psi(e_{km}).$$

This shows that the homomorphism $\varphi$ in fact maps $M_n(\mathbb{C})$ into $C^*_r(G)^{**} \cap \{h_0\}'$ and $\varphi(a) \psi(1_n) = \psi(a)$ for any $a \in M_n(\mathbb{C})$. Then Theorem 3.3 in [WZ09] shows that $\psi$ is a c.p.c. order zero map. 

Lemma 9.6. Let $G$ be a locally compact Hausdorff étale groupoid on a compact space. Let $U_1$, $U_2$, $O_1$ and $O_2$ be precompact open bisections such that $\overline{U}_i \subset O_i$ for $i = 1, 2$ and $\mu(\partial s(U_i)) = 0$ for any $\mu \in M(G)$. Then $\mu(\partial s(U_1 U_2)) = \mu(\partial r(U_1 U_2)) = 0$ for any $\mu \in M(G)$. 


Proof. For any \( \mu \in M(G) \), one first has \( \mu(\partial r(U_i)) = \mu(\partial s(U_i)) = 0 \) for \( i = 1, 2 \). Then \( \mu(\partial (r(U_i) \cap s(U_2))) \leq \mu(\partial r(U_1)) + \mu(\partial s(U_2)) = 0 \). Thus one has \( \mu(\partial s(U_1U_2)) = \mu(r(O_1) \cap s(U_2))) = 0 \) and in the similar way one also has \( \mu(\partial r(U_1U_2)) = \mu(r(O_2) \partial r(U_1) \cap s(U_2))) = 0 \).

Now, we are ready to establish the following theorem.

**Theorem 9.7.** Let \( G \) be a locally compact Hausdorff second countable étale minimal groupoid on a compact space. Suppose \( G \) is almost elementary. Then \( C^*_v(G) \) is simple separable and tracially \( \mathcal{Z} \)-stable.

**Proof.** First fix a metric \( d \) on \( G^{(0)} \) and an integer \( n \in \mathbb{N} \). Since \( G \) is almost elementary, Proposition 6.9 and Theorem 8.5 imply that \( G \) is effective and has groupoid small boundary property. Thus we prove this theorem by using Lemma 9.3. Now let \( \epsilon > 0, n \in \mathbb{N}, q \) be a non-zero positive function in \( C(G^{(0)})_+ \) and \( F \) a finite collection of functions in \( C(G) \) such that for any \( f \in F \), \( \text{supp}(f) \), of \( f \) is contained in an open bisection \( V_f \) and satisfies \( \mu(\text{supp}(f)) = 0 \) for any \( \mu \in M(G) \). Write \( m = |F| \). Since \( G \) is minimal, there is an \( \eta > 0 \) such that \( \mu(\text{supp}(g)) > 2 \eta \) for any \( \mu \in M(G) \) Note that \( \eta < 1/2 \) necessarily. In addition, choose an integer \( N \in \mathbb{N} \) such that \( N \geq \max\{1/2n^2\epsilon, 1/\eta\} \). Denote by \( O_f = \text{supp}(f) \). Since \( r(V_f \partial s(O_f)) = \partial r(O_f) \), one also has \( \mu(\partial r(O_f)) = 0 \) for any \( \mu \in M(G) \).

Define

\[
S = ( \bigcup_{f \in F} \partial s(O_f)) \cup ( \bigcup_{f \in F} \partial r(O_f)) \).
\]

Now Lemma 6.11 implies that there is a \( \delta > 0 \) such that \( \mu(B_d(S, \delta)) < \frac{\epsilon}{2(n+1)^{N+1}} \) for any \( \mu \in M(G) \). Define a compact set

\[
K = \bigcup_{i=1}^{N+1} \left( \bigcup_{f_1, \ldots, f_i \in F} U_{f_1} \circ \cdots \circ U_{f_i} \bigcup G^{(0)} \right)
\]

where \( U_f = O_f \) or \( O_f^{-1} \) for any \( f \in F \). Choose an open cover \( V \) of \( G^{(0)} \) in which any member \( V \in V \) has diameter less than \( \delta \) and for any \( u, v \in V \) and \( f \in F \) one has

\[
|s(f)(u) - s(f)(v)| < \epsilon/2n^2 \text{ and } |r(f)(u) - r(f)(v)| < \epsilon/2n^2.
\]

Since \( G \) is almost elementary, For the compact set \( K \), the cover \( V \), and the integer \( n \), Theorem 8.9, Remark 8.10 and Lemma 9.3 imply that there are open castles \( A, B, C \) and \( D \) such that

(i) both \( \mathcal{C} = \{ C : C \in C \} \) and \( \mathcal{D} = \{ D : D \in D \} \) are compact castles;
(ii) \( A \) is \( K \)-extendable to \( B \) and \( \mathcal{C} \) is \( K \)-extendable to \( D \);
(iii) For any \( i \leq N + 1 \) and \( f_1, \ldots, f_i \in F \) if a \( C \)-level \( C \subset s(U_{f_1} \cdot U_{f_2} \cdots \cdot U_{f_i}) \)

where \( U_{f_k} = O_{f_k} \) or \( O_{f_k}^{-1} \) for any \( 1 \leq k \leq i \) then \( U_{f_1} \cdot U_{f_2} \cdots \cdot U_{f_i} C = D \)

for some \( D \in \mathcal{D} \).
(iv) \( B \) is nested in \( D \) with multiplicity at least \( nN \)
(v) \( A \) is nested in \( C \) with multiplicity at least \( nN \);
(vi) any \( D \)-level is contained in a member of \( V \);
(vii) \( \mu(G^{(0)} \setminus \bigcup A^{(0)}) < \eta \) for any \( \mu \in M(G) \).

Now for each \( f \in F \) define

\[
T^1_f = \{ u \in G^{(0)} : d(u, G^{(0)} \setminus s(O_f)) \geq \delta \} \cup \{ u \in G^{(0)} : d(u, \overline{O_f}) \geq \delta \}
\]
Thus one has
\[ m \in G(0) : d(u, G(0) \setminus r(O_f)) \geq \delta \} \cup \{ u \in G(0) : d(u, r(O_f)) \geq \delta \}.
\]
In addition, define \( R = \bigcap_{f \in F} (T_f^1 \cap T_f^2) \). Note that \( G(0) \setminus R \subset B(S, \delta) \) and thus
\[ \mu(R) \geq 1 - \frac{\eta}{(2m)^{N+1}} \] for any \( \mu \in M(G) \). Then for any \( D \)-level \( D \in D(0) \), since
\[ \text{diam}_d(D) < \delta \), if \( D \cap R \neq \emptyset \) then \( D \cap S = \emptyset \) and thus either \( D \subset s(O_f) \) or \( D \cap s(O_f) = \emptyset \) and either \( D \subset r(O_f) \) or \( D \cap r(O_f) = \emptyset \) for any \( f \in F \).

Now define \( C_{0}^{(0)} = \{ C \in C^{(0)} : C \cap S = \emptyset \} \). Observe that for any \( \mu \in M(G) \) one has
\[ \mu(\bigcup \{ A \in A^{(0)} : A \subset C, C \in C_{0}^{(0)} \}) \geq 1 - \eta - \eta/(2mN+1)^{N+1}\]
and for each \( f \in F \) and \( C \in C_{0}^{(0)} \) one has either \( O_f C \in D \) or \( O_f C = \emptyset \) and either \( O_f^{-1} C \in D \) or \( O_f^{-1} C = \emptyset \). Define
\[ C_{0}^{(1)} = \{ C \in C_{0}^{(0)} : \text{there exists } D \in D(0) \text{ such that } D = O_f C \text{ or } D = O_f^{-1} C \text{ for some } f \in F \text{ and } D \cap S \neq \emptyset \} \]
and \( C_{1}^{(0)} = C_{0}^{(0)} \setminus C_{0}^{(0)} \). Observe that for any \( D \in D(0) \) with \( D \cap S \neq \emptyset \) one has \( D \subset B(S, \delta) \) and that at most \( 2m \) levels \( C \in C^{(0)} \) such that \( O_f C = D \) or \( O_f^{-1} C = D \) for some \( f \in F \). This implies that
\[ \mu(\bigcup \{ C_{0}^{(0)} \}) \leq \frac{\eta}{(2m)^N} \] for any \( \mu \in M(G) \). Thus one has
\[ \mu(\bigcup \{ A \in A^{(0)} : A \subset C, C \in C_{1}^{(0)} \}) \geq 1 - \eta - \eta/(2m)^{N+1} - \eta/(2m)^N. \]

In addition, for any \( C \in C_{1}^{(0)} \) and \( f, g \in F \), by \( K \)-extendability of \( C \) and Remark S.10 either \( U_f U_g C = \emptyset \) or \( U_f U_g C = D \) where \( U_f = O_f \) or \( O_f^{-1} \) and \( U_g = O_g \) or \( O_g^{-1} \).

Then by induction, suppose we have \( C_{k}^{(0)} \) for \( k < N \) such that
(i) \( U_{f_1} \ldots U_{f_k} C \in D \) or they are the empty set for any \( f_1, \ldots, f_k \in F \) and \( i \leq k \), where \( U_{f_i} = O_{f_i} \) or \( O_{f_i}^{-1} \) for any \( i \leq k \).
(ii) if \( D = U_{f_1} \ldots U_{f_k} C \in D \) for some \( f_1, \ldots, f_k \in F \) and \( i \leq k \) then \( r(D) \cap S = \emptyset \).
(iii) if \( f_{k+1} \ldots U_{f_k} C \in D \) for any \( f_1, \ldots, f_k \in F \) they are not empty by (ii) above, \( K \)-extendability of \( C \) and Remark S.10.
(iv) for any \( \mu \in M(G) \) one has \( \mu(\bigcup \{ C^{(0)}_{k} \}) \geq \mu(\bigcup \{ A \in A^{(0)} : A \subset C, C \in C_{k}^{(0)} \}) \geq 1 - \eta - \frac{\eta}{(2m)^{N+1}} - \sum_{i=1}^{k} \frac{\eta}{(2m)^{N+1-i}}. \)

Define
\[ C_{k}^{(0)} = \{ C \in C_{k}^{(0)} : \text{there exists } D \in D(0) \text{ such that } D = r(U_{f_{k+1}} \ldots U_{f_1} C), \]
\[ U_{f_i} = O_{f_i} \text{ or } U_{f_i} = O_{f_i}^{-1} \} \in F \text{ and } D \cap S \neq \emptyset \} \]
and \( C_{k+1}^{(0)} = C_{k}^{(0)} \setminus C_{k}^{(0)} \). Then, similarly, for any \( D \in D(0) \) and \( D \cap S \neq \emptyset \), there are at most \( (2m)^{k+1} \) levels \( C \in C_{k}^{(0)} \) such that \( D = r(U_{f_1} \ldots U_{f_{k+1}} C) \). Then \( \mu(\bigcup \{ C_{k}^{(0)} \}) < (2m)^{k+1} \cdot \frac{\eta}{(2m)^{N+1}} < \eta/(2m)^{N+1} \) holds and thus one has
\[ \mu(\bigcup \{ A \in A^{(0)} : A \subset C, C \in C_{k}^{(0)} \}) \geq 1 - \eta - \frac{\eta}{(2m)^{N+1}} - \sum_{i=1}^{k+1} \frac{\eta}{(2m)^{N+1-i}}. \]
for any $\mu \in M(\mathcal{G})$. In addition, by definition of $C_{k+1}^{(0)}$, it is straightforward to verify the corresponding properties (i)-(iii) above for $k + 1$. This finishes our inductive definition for $k = 0, \ldots, N$. Now we look at $C_N^{(0)}$, which satisfies corresponding properties (i)-(iv) for $k = N$. In particular, one has

$$\mu(C_N^{(0)}) \geq \mu\left(\bigcup\{A \in A^{(0)} : A \subset C, C \in C_N^{(0)}\}\right) \geq 1 - \eta - \frac{\eta}{(2m)^{N+1}} - \sum_{i=1}^{N} \frac{\eta}{(2m)^{N+i}} > 1 - 2\eta > 0.$$ 

and thus in particular $C_N^{(0)}$ is not empty. Now define $D_0^{(0)} = C_N^{(0)}$ and inductively define

$$D_k^{(0)} = \{D \in D^{(0)} : D = r(U_{f_k} \ldots U_{f_1} C), U_{f_i} = O_{f_i}, \text{ or } O_{f_i}^{-1}, \text{ for } i = 1, \ldots, k, f_1, \ldots, f_k \in F \text{ and } C \in C_N^{(0)}\} \setminus \bigcup_{i=1}^{k-1} D_i^{(0)}$$

for $k = 1, \ldots, N + 1$ (some $D_k^{(0)}$ may be empty). Define $\mathcal{H}^{(0)} = \bigcup_{k=0}^{N+1} D_k^{(0)}$, which is a subset of $\mathcal{D}^{(0)}$ and contains $C_N^{(0)} = D_0^{(0)}$. Now we define a c.p.c. order zero map by using Remark 9.4 via choosing a $\mathcal{B}$-compatible functions $\{h_B \in C_c(\mathcal{G}) : B \in \mathcal{B}\}$ and a $\mathcal{H}^{(0)}$-$\mathcal{B}^{(0)}$-nesting system. Note that $C_N^{(0)}$ here plays the role as $C^{(0)}$ in Remark 9.4. Then we define a function $\kappa : \mathcal{H}^{(0)} \to [0, 1]$ by $\kappa(D) = 1 - k/(N + 1)$ if $D \in D_k^{(0)}$ for $k = 0, \ldots, N + 1$. Finally, we define $\psi : M_n(\mathbb{C}) \to C^*_c(\mathcal{G})$ by

$$\psi(e_{ij}) = \sum_{D \in \mathcal{H}^{(0)}} \sum_{B \in Q_{i,j,D}} \kappa(D)h_B$$

and extending linearly. Lemma 9.3 implies that $\psi$ is a c.p.c. order zero map. In addition, by Remark 9.4 for function $h = 1_{C^*_c(\mathcal{G})} - \psi(1_n)$ one has

$$\mu(\text{supp}(h)) < 2\eta < \mu(\text{supp}^\eta(g))$$

for any $\mu \in M(\mathcal{G})$. Then since $\mathcal{G}$ has groupoid strict comparison by Theorem 5.16 one has $\text{supp}^\eta(h) \preceq_\mathcal{G} \text{supp}^\eta(g)$, which implies that

$$1_{C^*_c(\mathcal{G})} - \psi(1_n) = h \preceq g$$

by Proposition 6.1 in [Ma12].

Now, for any $f \in F$, $e_{ij} \in M_n(\mathbb{C})$, define sets

$$S_f = \{D \in \mathcal{H}^{(0)} : D \subset s(O_f), r(O_f D) \in \mathcal{H}^{(0)}\}$$

and

$$R_f = \{D \in \mathcal{H}^{(0)} : D \subset r(O_f), r(O_f^{-1} D) \in \mathcal{H}^{(0)}\}.$$ 

Observe that the map $\sigma_f : S_f \to R_f$ defined by $\sigma_f(D) = r(O_f D)$ is bijective. Define a map $\pi_f : S_f \to D$ by $s(\pi_f(D)) = D$ and $r(\pi_f(D)) = \sigma_f(D)$. Then define another bijective map $\theta_{i,j,f,D} : Q_{i,j,D} \to Q_{i,j,\sigma_f(D)}$ in the following way. For any $B \in Q_{i,j,D}$, define $\theta_{i,j,f,D}(B) \in B$ such that $s(\theta_{i,j,f,D}(B)) = r(\pi_f(D) s(B))$ and $r(\theta_{i,j,f,D}(B)) = r(\pi_f(D) r(B))$. The map $\theta_{i,j,f,D}$ is well-defined because the property (iv) of the definition of $\mathcal{H}^{(0)}$-$\mathcal{B}^{(0)}$-nesting system in Remark 9.4. Note that

$$\pi_f(D)B = \theta_{i,j,f,D}(B)\pi_f(D)s(B) \in B.$$
Now one has
\[
[\psi(e_{ij}), f] = \sum_{D \in \mathcal{H}(0)} \sum_{B \in \mathcal{Q}_{i,j,D}} \kappa(D) f \ast h_B - \sum_{D \in \mathcal{H}(0)} \sum_{B \in \mathcal{Q}_{i,j,D}} \kappa(D) h_B \ast f
\]
\[
= \sum_{D \in \mathcal{H}(0)} \sum_{B \in \mathcal{Q}_{i,j,D}} \kappa(D) f \ast h_B - \sum_{D \in \mathcal{H}(0)} \sum_{B \in \mathcal{Q}_{i,j,D}} \kappa(D) h_B \ast f
\]
\[
= \sum_{D \in \mathcal{H}(0)} \sum_{B \in \mathcal{Q}_{i,j,D}} \kappa(D) f \ast h_B - \sum_{D \in \mathcal{H}(0)} \sum_{B \in \mathcal{Q}_{i,j,D}} \kappa(D) h_B \ast f
\]
\[
= \sum_{D \in \mathcal{Q}_{i,j,D}} \kappa(D) f \ast h_B - \sum_{D \in \mathcal{Q}_{i,j,D}} \kappa(D) h_B \ast f
\]

The second equality above is due to the fact that either \(D \subset s(O_f)\) or \(D \cap s(O_f) = \emptyset\) and either \(D \subset r(O_f)\) or \(D \cap r(O_f) = \emptyset\) for any \(D \in \mathcal{H}(0)\). On the other hand, if \(\emptyset \neq D \in \mathcal{H}(0)\) with \(D \subset s(O_f)\) but \(r(O_f) \notin \mathcal{H}(0)\) (note that in this case \(r(O_f)\) may not even be a \(\mathcal{D}\)-level), then \(D \in \mathcal{D}_{N+1}\) necessarily. In this case, observe that \(\kappa(D) = 0\). In the same way, if \(\emptyset \neq D \in \mathcal{H}(0)\) with \(D \subset r(O_f)\) but \(r(O_f^{-1}) \notin \mathcal{H}(0)\) then \(\kappa(D) = 0\). This establishes the third equality above. Finally, the fourth equality above is to use bijections \(\sigma_f\) and \(\theta_{i,j,f,D}\) defined above. Now for fixed \(i, j, f\) write
\[
a_{D,B} = \kappa(D) f \ast h_B - \kappa(\sigma_f(D)) h_{\theta_{i,j,f,D}(B)} \ast f
\]
for simplicity. Note that \(a_{D,B} \in C_c(G)\) and supported on the bisection \(\pi_f(D)B \in \mathcal{B}\) and thus \(\|a_{D,B}\|_r = \|a_{D,B}\|_{\infty}\) by Proposition 2.9. Now for any \(D \in \mathcal{S}_f\) with \(D \in \mathcal{D}_k\), if \(k = 0\), then \(\sigma_f(D) \in \mathcal{D}_0(0) \cup \mathcal{D}_1(0)\). If \(1 \leq k \leq N\) one has \(\sigma_f(D) \in \mathcal{D}_k(0) \cup \mathcal{D}_{k+1}(0)\). Finally, if \(k = N + 1\) then necessarily one has \(\sigma_f(D) \in \mathcal{D}_N(0) \cup \mathcal{D}_{N+1}(0)\). Therefore, in any case, for \(D \in \mathcal{S}_f\), one has
\[
|\kappa(D) - \kappa(\sigma_f(D))| < 1/N < \epsilon/2n^2.
\]

On the other hand, for any \(\gamma \in B' = \pi_f(D)B = \theta_{i,j,f,D}(B)\pi_f(D)B\), there is a unique decomposition of \(\gamma\) by \(\gamma = \alpha_1 \beta_1 = \beta_2 \alpha_2\), where \(\alpha_1 \in \pi_f(D)r(B)\), \(\alpha_2 \in \pi_f(D)s(B)\), \(\beta_1 \in B\) and \(\beta_2 \in \theta_{i,j,f,D}(B)\). In addition, by \(\mathcal{B}\)-compatibility of \(h_B\), one has
\[
h_B(\beta_1) = h_{s(B)}(\beta_1) = h_{s(B')}\gamma) = h_B(\gamma)
\]
\[
= h_{r(B')}(\gamma) = h_{r(\theta_{i,j,f,D}(B))}(r(\beta_2)) = h_{\theta_{i,j,f,D}(B)}(\beta_2).
\]

Finally, since \(D \subset V\) for some \(V \in \mathcal{V}\), then \((\ast \ast \ast)\) implies that
\[
|f(\alpha_1) - f(\alpha_2)| = |s(f)(s(\alpha_1)) - s(f)(s(\alpha_2))| < \epsilon/2n^2.
\]

This implies that for any \(\gamma = \alpha_1 \beta_1 = \beta_2 \alpha_2 \in B'\) as decomposed above one has
\[
|f * h_B(\gamma) - (h_{\theta_{i,j,f,D}(B)} * f)(\gamma)| = |f(\alpha_1)h_B(\beta_1) - h_{\theta_{i,j,f,D}(B)}(\beta_2)f(\alpha_2)|
\]
\[
= |h_B(\beta_1)||f(\alpha_1) - f(\alpha_2)| \leq \epsilon/2n^2.
\]

This implies that
\[
\|f * h_B - h_{\theta_{i,j,f,D}(B)} * f\|_{\infty} = \sup_{\gamma \in B'} |f * h_B(\gamma) - (h_{\theta_{i,j,f,D}(B)} * f)(\gamma)| \leq \epsilon/2n^2
\]
and thus one has
\[
\|a_{D,B}\|_r = \|a_{D,B}\|_{\infty}
\]
etale minimal groupoid on a compact space. Suppose $G$ is almost elementary. Then $C^*_r(G)$ has the strict comparison for positive elements.

Proof. Theorem 9.7 implies that $C^*_r(G)$ is simple unital tracially $\mathcal{Z}$-stable $C^*$-algebras. Then Theorem 3.3 in [HO13] shows that $C^*_r(G)$ has the strict comparison for positive elements.

We emphasize that Corollary 9.8 does not assume the nuclearity of $C^*_r(G)$. Then if we assume that $G$ is amenable, then we have the following much stronger result.

Corollary 9.9. Let $G$ be a locally compact Hausdorff amenable second countable etale minimal groupoid on a compact space. Suppose $G$ is almost elementary. Then $C^*_r(G)$ is unital simple separable nuclear and $\mathcal{Z}$-stable and thus has nuclear dimension one. In addition, in this case $C^*_r(G)$ is classified by its Elliott invariant. Finally, if $M(G) \neq \emptyset$, then $C^*_r(G)$ is quasidiagonal and if $M(G) = \emptyset$ then $C^*_r(G)$ is a unital Kirchberg algebra.

Proof. Since $G$ is assumed to be amenable, Theorem 9.7 implies $C^*_r(G)$ is unital simple separable nuclear and trivially $\mathcal{Z}$-stable and thus $\mathcal{Z}$-stable by Theorem 4.1 in [HO13]. In this case, the nuclear dimension $\dim_{\text{nuc}}(C^*_r(G)) = 1$ by Theorem A and Corollary C in [CET+19]. Therefore, $C^*_r(G)$ is classified by Elliott invariant by the recent progress of classification theorem for unital simple nuclear separable $C^*$-algebras having finite nuclear dimension and satisfying the UCT via combining [EGLN14, GLN14, TWW17, Phi00]. Finally, if $M(G) \neq \emptyset$ then there is a non-zero tracial state on $C^*_r(G)$. This implies that $C^*_r(G)$ is stably finite and thus quasidiagonal by Corollary 6.1 in [TWW17]. On the other hand, if $M(G) = \emptyset$, then $C^*_r(G)$ is traceless and thus $C^*_r(G)$ is purely infinite by Corollary 5.1 in [Ror01]. Therefore, in this case $C^*_r(G)$ is a unital Kirchberg algebra.
Now we apply our results to almost finite groupoids in Matui’s sense and obtain the following result.

**Corollary 9.10.** Let $G$ be a locally compact Hausdorff second countable ample étale minimal groupoid on a compact space. Suppose $G$ is almost finite in Matui’s sense. Then $C^*_r(G)$ is tracially $\mathcal{Z}$-stable and thus has the strict comparison for positive elements. If we assume $G$ is also amenable then $C^*_r(G)$ is $\mathcal{Z}$-stable and quasidiagonal.

**Proof.** Proposition 7.3 and Theorem 7.4 shows that $G$ is fiberwise amenable and almost elementary. Then Proposition 5.9 implies that $M(G) \neq \emptyset$. Now Corollary 9.8 and 9.9 shows the result. □

Then we may recover the following result due to Kerr in [Ker20].

**Corollary 9.11.** Let $\alpha : \Gamma \curvearrowright X$ be a minimal free action of a countable discrete amenable group $\Gamma$ on a compact metrizable space $X$. Suppose $\alpha$ is almost finite in Kerr’s sense. Then the crossed product $C(X) \rtimes_r \Gamma$ is $\mathcal{Z}$-stable and quasidiagonal.

**Proof.** Theorem 7.6 shows that the transformation groupoid $X \rtimes_\alpha \Gamma$ is fiberwise amenable and almost elementary. In addition, amenability of $\Gamma$ implies that $X \rtimes_\alpha \Gamma$ is amenable. Then Corollary 9.9 shows the result. □

We finally provide several applications of our result on $\mathcal{Z}$-stability of almost finite groupoids.

**Example 9.12.** Recently, in [IWZ12], Ito, Whittaker and Zacharias established $\mathcal{Z}$-stability of Kellendonk’s $C^*$-algebra of an aperiodic and repetitive tiling with finite local complexity through generalizing the approach for group actions in [Ker20] to groupoid actions. In addition, they showed that such a $C^*$-algebra is a reduced $C^*$-algebra of a locally compact Hausdorff étale second countable principal almost finite tiling groupoid. Thus, their result is a direct application of our Corollary 9.10.

Recall a geometric groupoid $G$ in Example 7.7, constructed by Elek is a locally compact Hausdorff second countable étale principal almost finite tiling groupoid $G$, which is not amenable. Therefore, Corollary 9.10 implies that $C^*_r(G)$ is not nuclear but tracially $\mathcal{Z}$-stable.

**References**

[ABBL12] Pere Ara, Christian Bönicke, Joan Bosa, and Kang Li. Strict comparison for $C^*$-algebras arising from almost finite groupoids. preprint, arXiv:2002.12221, 2012.

[ALLW18] Pere Ara, Kang Li, Fernando Lledó, and Jianchao Wu. Amenability of coarse spaces and $K$-algebras. *Bull. Math. Sci.*, 8(2):257–306, 2018.

[BW92] Jonathan Block and Shmuel Weinberger. Aperiodic tilings, positive scalar curvature and amenability of spaces. *J. Amer. Math. Soc.*, 5(4):907–918, 1992.

[CET19] Jorge Castillejos, Samuel Evington, Aaron Tikuisis, Stuart White, and Wilhelm Winter. Nuclear dimension of simple $C^*$-algebras. *arXiv preprint arXiv:1901.05853*, 2019.

[EGLN15] George A. Elliott, Guihua Gong, Huaxin Lin, and Zhuang Niu. On the classification of simple $C^*$-algebras with finite decomposition rank, II. *arXiv:1507.03437*, 2015.

[Ele12] Gábor Elek. Qualitative graph limit theory. Cantor dynamical systems and constant-time distributed algorithms. *arXiv:1812.07511*, 2012.

[GLN14] Guihua Gong, Huaxin Lin, and Zhuang Niu. Classification of simple amenable $\mathcal{Z}$-stable $C^*$-algebras. *arXiv:1501.00135*, 2014.

[GMPS08] Thierry Giordano, Hiroki Matui, Ian F. Putnam, and Christian F. Skau. The absorption theorem for affable equivalence relations. *Ergodic Theory Dynam. Systems*, 28(5):1509–1531, 2008.
[GPS04] Thierry Giordano, Ian Putnam, and Christian Skau. Affable equivalence relations and orbit structure of Cantor dynamical systems. *Ergodic Theory Dynam. Systems*, 24(2):441–475, 2004.

[HO13] Ilan Hirshberg and Joav Orovitz. Tracially $Z$-absorbing $C^*$-algebras. *J. Funct. Anal.*, 265(5):765–785, 2013.

[HPR97] Nigel Higson, Erik Kjær Pedersen, and John Roe. $C^*$-algebras and controlled topology. *K-Theory*, 11(3):209–239, 1997.

[IWZ12] Luke Ito, Michael Whittaker, and Joachim Zacharias. Classification of tiling $C^*$-algebras. preprint, arXiv:1408:5546, 2012.

[JS99] Xinhui Jiang and Hongbing Su. On a simple unital projectionless $C^*$-algebra. *Amer. J. Math.*, 121(2):359–413, 1999.

[Ker20] David Kerr. Dimension, comparison, and almost finiteness. *J. Eur. Math. Soc. (JEMS)*, 22(11):3697–3745, 2020.

[Kri80] Wolfgang Krieger. On a dimension for a class of homeomorphism groups. *Math. Ann.*, 252(2):87–95, 1979/80.

[KS20] David Kerr and Gábor Szabó. Almost finiteness and the small boundary property. *Comm. Math. Phys.*, 374(1):1–31, 2020.

[Lin01a] Huaxin Lin. *An introduction to the classification of amenable $C^*$-algebras*. World Scientific Publishing Co. Inc., River Edge, NJ, 2001.

[Lin01b] Huaxin Lin. Tracially AF $C^*$-algebras. *Trans. Amer. Math. Soc.*, 353(2):693–722, 2001.

[LW00] Elon Lindenstrauss and Benjamin Weiss. Mean topological dimension. *Israel J. Math.*, 115:1–24, 2000.

[Ma12] Xin Ma. Purely infinite locally compact hausdorff étale groupoids and their $C^*$-algebras. preprint, arXiv:2001.03706, 2012.

[Ma19] Xin Ma. Invariant ergodic measures and the classification of crossed product $C^*$-algebras. *J. Funct. Anal.*, 276(4):1276–1293, 2019.

[Mat12] Hiroki Matui. Homology and topological full groups of étale groupoids on totally disconnected spaces. *Proc. Lond. Math. Soc. (3)*, 104(1):27–56, 2012.

[Mat15] Hiroki Matui. Topological full groups of one-sided shifts of finite type. *J. Reine Angew. Math.*, 705:35–84, 2015.

[Mat17] Hiroki Matui. Topological full groups of étale groupoids. In *Operator algebras and applications—the Abel Symposium 2015*, volume 12 of *Abel Symp.*, pages 203–230. Springer, [Cham], 2017.

[MS12] Hiroki Matui and Yasuhiko Sato. Strict comparison and $\mathcal{Z}$-absorption of nuclear $C^*$-algebras. *Acta Math.*, 209(1):179–196, 2012.

[Nek19] Volodymyr Nekrashevych. Simple groups of dynamical origin. *Ergodic Theory Dynam. Systems*, 39(3):707–732, 2019.

[NY12] Piotr W. Nowak and Guoliang Yu. *Large scale geometry*. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2012.

[OOY19] Hervé Oyono-Oyono and Guoliang Yu. Quantitative $K$-theory and the Künneth formula for operator algebras. *J. Funct. Anal.*, 277(7):2003–2091, 2019.

[Phi00] N. Christopher Phillips. A classification theorem for nuclear purely infinite simple $C^*$-algebras. *Doc. Math.*, 5:49–114, 2000.

[Phi05] N. Christopher Phillips. Crossed products of the Cantor set by free minimal actions of $Z^d$. *Comm. Math. Phys.*, 256(1):1–42, 2005.

[Phi12] N. Christopher Phillips. Large subalgebras. preprint, arXiv:1408:5546, 2012.

[Ren80] Jean Renault. *A groupoid approach to $C^*$-algebras*, volume 793 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.

[Roe03] John Roe. *Lectures on coarse geometry*, volume 31 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2003.

[Rør04] Mikael Rørdam. The stable and the real rank of $\mathcal{Z}$-absorbing $C^*$-algebras. *Internat. J. Math.*, 15(10):1065–1084, 2004.

[Sim12] Aidan Sims. Hausdorff étale groupoids and their $C^*$-algebras. preprint, arXiv:1710.10897, 2012.

[Suz20] Yuhei Suzuki. Almost Finiteness for General Étale Groupoids and Its Applications to Stable Rank of Crossed Products. *Int. Math. Res. Not. IMRN*, (19):6007–6041, 2020.
[SWW15] Yasuhiko Sato, Stuart White, and Wilhelm Winter. Nuclear dimension and \( \mathcal{Z} \)-stability. *Invent. Math.*, 202(2):893–921, 2015.

[TWW17] Aaron Tikuisis, Stuart White, and Wilhelm Winter. Quasidiagonality of nuclear \( C^* \)-algebras. *Ann. of Math. (2)*, 185(1):229–284, 2017.

[TWY18] Xiang Tang, Rufus Willett, and Yi-Jun Yao. Roe \( C^* \)-algebra for groupoids and generalized Lichnerowicz vanishing theorem for foliated manifolds. *Math. Z.*, 290(3-4):1309–1338, 2018.

[Win12] Wilhelm Winter. Nuclear dimension and \( \mathcal{Z} \)-stability of pure \( C^* \)-algebras. *Invent. Math.*, 187(2):259–342, 2012.

[WZ09] Wilhelm Winter and Joachim Zacharias. Completely positive maps of order zero. *Münster J. Math.*, 2:311–324, 2009.

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