A NEW INTERPRETATION OF THE RACAH-WIGNER
6j-SYMBOL AND THE CLASSIFICATION OF UNISERIAL
$\mathfrak{sl}(2) \ltimes V(m)$-MODULES

LEANDRO CAGLIERO AND FERNANDO SZECHTMAN

Abstract. All Lie algebras and representations will be assumed to be finite
dimensional over the complex numbers. Let $V(m)$ be the irreducible $\mathfrak{sl}(2)$-
module with highest weight $m \geq 1$ and consider the perfect Lie algebra $g =$
$\mathfrak{sl}(2) \ltimes V(m)$. Recall that a $g$-module is uniserial when its submodules form
a chain. In this paper we classify all uniserial $g$-modules. The main family of
uniserial $g$-modules is actually constructed in greater generality for the perfect
Lie algebra $g = s \ltimes V(\mu)$, where $s$ is a semisimple Lie algebra and $V(\mu)$ is the
irreducible $s$-module with highest weight $\mu \neq 0$. The fact that the members of
this family are, but for a few exceptions of lengths 2, 3 and 4, the only uniserial
$\mathfrak{sl}(2) \ltimes V(m)$-modules depends in an essential manner on the determination of
certain non-trivial zeros of Racah-Wigner 6j-symbol.

1. Introduction

All Lie algebras and representations considered in this paper are assumed to be
finite dimensional over the complex numbers.

The problem of classifying all indecomposable modules of a given Lie algebra (or
a family of Lie algebras) is usually hard. Very serious difficulties are encountered
even for Lie algebras of very low dimensionality. Two notoriously difficult examples
are furnished by the 2-dimensional abelian Lie algebra (see [GP], Corollary 1) and
the 3-dimensional Euclidean Lie algebra $\mathfrak{e}(2) = so(2) \ltimes \mathbb{C}^2$ (see [Sa], Theorem 4.3).

As is well-known, the classification problem has a satisfactory answer for the class
of all semisimple Lie algebras, and perfect Lie algebras possess favorable properties
that make them suitable for consideration in this problem. Indeed, it is precisely the
class of Lie algebras that enjoy an abstract Jordan decomposition [CS1], a crucial
tool needed for the classification of the irreducible representations of semisimple Lie
algebras. Additionally, a Lie algebra $\mathfrak{g}$ is perfect if and only if its solvable radical
$\mathfrak{r}$ coincides with its nilpotent radical $[\mathfrak{g}, \mathfrak{r}]$. Thus, a perfect Lie algebra $\mathfrak{g}$ has a
Levi decomposition $\mathfrak{g} = s \ltimes \mathfrak{r}$, where $s$ is semisimple and the solvable radical $\mathfrak{r}$ acts
trivially on every irreducible $\mathfrak{g}$-module, and hence nilpotently on every $\mathfrak{g}$-module.
Further positive features of perfect Lie algebras, from the representation theory
point of view, can be found in [22].

But even for the easiest perfect Lie algebra (other than semisimple), namely
$\mathfrak{g} = \mathfrak{sl}(2) \ltimes \mathbb{C}^2$, the classification of the indecomposable representations is far from

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being achieved (see [DR] and [P]). Therefore a natural approach to this problem is to identify a distinguished class of indecomposable representations for which one could expect to obtain a reasonable classification.

This line of research has been followed in a number of papers. For instance, some authors have considered embeddings of a given \( g \) into a semisimple Lie algebra \( \tilde{g} \) and considered the highest weight modules of \( \tilde{g} \) to construct or classify indecomposable \( g \)-modules obtained by restriction. See, for instance, [Ca], [CMS], [Dd], [DP], [DR], [P]. On the other hand, A. Piard [P] obtained a classification of all indecomposable \( g \)-modules \( V \), where \( g = sl(2) \ltimes r \), \( r = C^2 \) and \( V/rV \) is irreducible.

In this paper we focus our attention on the class of uniserial representations. Recall that a module \( V \) is uniserial if its submodules form a chain. We think that, within this class, a classification can be achieved for certain families of Lie algebras and, moreover, that the members of this class might be viewed as building blocks to understand more general classes of indecomposable representations. Indeed, the following two facts support this belief.

First, one of the main results of this paper gives a complete classification of all uniserial \( g \)-modules for \( g = sl(2) \ltimes V(m) \), where \( V(m) \) is the irreducible \( sl(2) \)-module with highest weight \( m \geq 1 \). As far as we know, this is the first time that a structurally defined class of indecomposable modules, other than the irreducible ones, has been simultaneously classified for all members of an infinite family of Lie algebras. Moreover, in contrast to the case of the indecomposable modules of the abelian and Euclidean Lie algebras, a classification of all uniserial modules for these and many other solvable Lie algebras is attained in [CS2].

Secondly, uniserial modules are also considered (see [BH], [HZ], [HZ2]) as a starting point in terms of classification and as building blocks of other indecomposable modules in the case of certain finite dimensional associative algebras (notice that the Jordan Normal Form Theorem states that any \( \mathbb{C}[x] \)-module is a direct sum of uniserial modules). In the context of Lie algebras, all of Piard’s indecomposable \( sl(2) \ltimes V(1) \)-modules mentioned above, as well as further indecomposable modules for more general perfect Lie algebras, can be constructed as a series of extensions of uniserial modules (see [CS3]).

A crucial step in the proof of our classification requires the determination of non-trivial zeros of the (classical) Racah-Wigner 6j-symbol within certain parameters. The 6j-symbol is a real number \[ \begin{bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{bmatrix} \] associated to six non-negative half-integer numbers \( j_1, j_2, j_3, j_4, j_5 \) and \( j_6 \), originally studied because it plays a central role in angular momentum theory (see for instance [CFS], [E], [RBMW]).

Our results provide a bridge between two related but different lines of research and we think that very interesting connections of this kind will appear by considering other families of Lie algebras.

1.1. **Main results.** Let \( g \) be a Lie algebra with solvable radical \( r \) and Levi decomposition \( g = s \ltimes r \).

Let \( V \) be a \( g \)-module and let \( 0 = V_0 \subset V_1 \subset \cdots \subset V_n = V \) be a composition series of \( V \). By Lie’s theorem \( r \) acts via scalar operators on each composition factor \( W_i = V_i/V_{i-1} \), \( 1 \leq i \leq n \). In particular every \( W_i \) is an irreducible \( s \)-module.
Theorem 1.2. Let order, the following are the only admissible sequences for $g$ and we spend considerable effort in $g$ the uniserial modules of $g$ cyclic indecomposable $Z$ uniserial $Z$ exceptional modules, together with the modules $V$ and $V$

Theorem 1.1. Let $g$ and $r$

As indicated earlier, $V$ is uniserial if it has only one composition series, i.e., if the socle series of $V$ has irreducible factors. A uniserial module is clearly indecomposable. A sequence $W_1, ..., W_n$ of irreducible $g$-modules is said to be admissible if there is a uniserial $g$-module with socle factors $s$-isomorphic to $W_1, ..., W_n$. By considering dual modules, it is clear that a sequence $W_1, ..., W_n$ is admissible if and only if so is $W^*_n, ..., W^*_1$.

In general the classification of uniserial $g$-modules breaks down into two steps:

Step 1. Determine all admissible sequences.

Step 2. Given an admissible sequence, find all uniserial modules giving rise to it.

As mentioned earlier, perfect Lie algebras are well suited for consideration in this problem. Indeed, from $g = s \ltimes r$ we obtain the Levi decomposition $[g, g] = s \ltimes [g, r]$. Thus $g = [g, g]$ if and only if $r = [g, r]$. Here $[g, r] = r \cap [g, g]$ is not only the solvable radical of $[g, g]$ but also the nilpotent radical of $g$, i.e. the ideal of all $x \in g$ such that $xV = 0$ for every irreducible $g$-module $V$. It follows that $g$ is perfect if and only if $r$ annihilates every irreducible $g$-module. Thus if $g$ is perfect an irreducible $g$-module is nothing but an irreducible $s$-module annihilated by $r$. More generally, if $g$ is perfect then the terms of the socle series of $V$ can be intrinsically obtained from $r$ as follows: $soc^i(V)/soc^{i-1}(V)$, for $1 \leq i \leq n$, is simply the is the $0$-weight space for the action of $r$ on $V/soc^{i-1}(V)$.

We begin our paper in $(2)$ and $(3)$ by furnishing general criteria to recognize, construct and classify uniserial modules for perfect Lie algebras with abelian radical. These results turn out to be fundamental for the rest of the paper. As a first application we prove in $(4)$ the following theorem.

**Theorem 1.1.** Let $s$ be a non-zero semisimple Lie algebra. Let $b \in \mathbb{Z}_{\geq 0}$ and let $\lambda$ and $\mu$ be dominant integral weights of $s$, where $\mu \neq 0$ and $\mu^*$ is the highest weight of $V(\mu)^*$. Consider the perfect Lie algebra $g = s \ltimes V(\mu)$. Then, up to isomorphism, there exists one and only one uniserial $g$-module, say $Z(\lambda, b)$, with socle factors $V(\lambda), V(\lambda + \mu^*), ..., V(\lambda + \mu b^*)$.

It is then clear that the dual $g$-module $Z(\lambda, b)^*$ is, up to isomorphism, the only uniserial $g$-module with socle factors $V(\lambda^* + b\mu), ..., V(\lambda^* + \mu), V(\lambda^*)$.

In $(5)$ we exhibit explicit matrix realizations of these modules for $g = sl(2) \ltimes V(m)$ and we spend considerable effort in $(6)$ to achieve an axiomatic characterization of $Z(\lambda, b)$ and $Z(\lambda, b)^*$, which presents them as a particular subclass of the class of cyclic indecomposable $g$-modules.

Other uniserial $g$-modules are possible for $g = s \ltimes V(\mu)$, as indicated in $(8)$. These exceptional modules, together with the modules $Z(\lambda, b)$ and $Z(\lambda, b)^*$, comprise all the uniserial modules of $g = sl(2) \ltimes V(m)$. In fact, our main result reads as follows.

**Theorem 1.2.** Let $g = sl(2) \ltimes V(m)$, where $m \geq 1$. Then, up to a reversing of the order, the following are the only admissible sequences for $g$:
Length 1. \( V(a) \).

Length 2. \( V(a), V(b) \), where \( a + b \equiv m \mod 2 \) and \( 0 \leq b - a \leq m \leq a + b \).

Length 3. \( V(a), V(a + m), V(a + 2m) \); or
\( V(0), V(m), V(c) \), where \( c \equiv 2m \mod 4 \) and \( c \leq 2m \).

Length 4. \( V(a), V(a + m), V(a + 2m), V(a + 3m) \); or
\( V(0), V(m), V(m), V(0) \), where \( m \equiv 0 \mod 4 \).

Length \( \geq 5 \). \( V(a), V(a + m), \ldots, V(a + sm) \), where \( s \geq 4 \).

Moreover, each of these sequences arises from only one isomorphism class of uniserial \( g \)-modules, except for the sequence \( V(0), V(m), V(m), V(0) \), \( m \equiv 0 \mod 4 \). The isomorphism classes of uniserial \( g \)-modules associated to this sequence are parametrized by the complex numbers.

Explicit matrix realizations illustrating this theorem are given in [4] and [8].

One major step towards the proof of the above theorem is the determination of all admissible sequences of length 3 for \( g = \mathfrak{sl}(2) \times V(m) \) and this is done in [8]. From the results of [2] and [3] it follows that \( V(a), V(b), V(c) \) is an admissible sequence of length 3 for \( g \) if and only if \( V(m) \) enters \( V(a) \otimes V(b) \) and \( V(b) \otimes V(c) \), and \( \mathcal{L} \) is abelian, where \( \mathcal{L} \) is the Lie subalgebra of \( \mathfrak{gl}(V) \), with \( V = V(a) \oplus V(b) \oplus V(c) \), generated by \( f(r) + g(r) \), \( r \in V(m) \), and \( f : V(m) \rightarrow \text{Hom}(V(b), V(a)) \) as well as \( g : V(m) \rightarrow \text{Hom}(V(c), V(b)) \) are \( \mathfrak{sl}(2) \)-embeddings. In terms of matrices, \( \mathcal{L} \) is the Lie subalgebra of \( \mathfrak{gl}(a + b + c + 3; \mathbb{C}) \) generated by \( \{ \begin{pmatrix} f(r) & 0 & 0 \\ 0 & 0 & g(r) \\ 0 & 0 & 0 \end{pmatrix} : r \in V(m) \} \).

The determination of all \( a, b, c \) and \( m \) for which \( \mathcal{L} \) is abelian requires the following theorem from [11].

**Theorem 1.3.** Let \( a, b, c, p, q, k \) be non-negative integers for which there exist \( \mathfrak{sl}(2) \)-embeddings
\[ V(k) \xrightarrow{f_0} \text{Hom}(V(c), V(a)), \quad V(k) \xrightarrow{f_1} V(p) \otimes V(q), \]
\[ V(p) \xrightarrow{f_2} \text{Hom}(V(b), V(a)), \quad V(q) \xrightarrow{f_3} \text{Hom}(V(c), V(b)) \]
and let \( \left\{ \begin{pmatrix} a & b & c \\ \frac{a}{2} & \frac{b}{2} & \frac{c}{2} \end{pmatrix} \right\} \) be the Racah-Wigner \( 6j \)-symbol associated to them. If
\[ f_4 = \left\{ \begin{pmatrix} a & b & c \\ \frac{a}{2} & \frac{b}{2} & \frac{c}{2} \end{pmatrix} \right\} f_0 \]
then, after a suitable normalization of \( f_i \), \( i = 0, 1, 2, 3, 4 \), the following diagram of \( \mathfrak{sl}(2) \)-morphisms is commutative, where \( g \) sends \( \alpha \otimes \beta \rightarrow \alpha \beta \):
\[ \begin{array}{ccc}
V(k) & \xrightarrow{f_4} & \text{Hom}(V(c), V(a)) \\
| f_1 \downarrow & & \uparrow g \\
V(p) \otimes V(q) & \xrightarrow{f_2 \otimes f_3} & \text{Hom}(V(b), V(a)) \otimes \text{Hom}(V(c), V(b)).
\end{array} \]
In particular, \( V(k) \) appears in the image of \( g(f_2 \otimes f_3) \) if and only if \( \left\{ \begin{pmatrix} a & b & c \\ \frac{a}{2} & \frac{b}{2} & \frac{c}{2} \end{pmatrix} \right\} \neq 0 \).
Recall that the (classical) Racah-Wigner 6j-symbol is a real number \( \{j_1, j_2, j_3\} \) \( \{j_4, j_5, j_6\} \) associated to six non-negative half-integer numbers \( j_1, j_2, j_3, j_4, j_5, j_6 \) and, originally, it is implicitly defined (see [CFS]) in terms of the transition matrix between the following two basis of \( \text{Hom}_{\mathfrak{sl}(2)}(V(k), V(a) \otimes V(b) \otimes V(c)) \)

\[
\{ V(k) \rightarrow V(p) \otimes V(c) \rightarrow V(a) \otimes V(b) \otimes V(c) \}_{p \geq 0}
\]

and 

\[
\{ V(k) \rightarrow V(a) \otimes V(q) \rightarrow V(a) \otimes V(b) \otimes V(c) \}_{q \geq 0}.
\]

In fact, the 6j-symbol can be defined as above in a more general context, in particular for any semisimple multitensor category, see for instance [FFK].

As far as we know, Theorem 1.3 is not known and gives a new and clean definition of the Racah-Wigner 6j-symbol in terms of the transition matrix between the above theorem might have applications to it.

Then the following conditions are equivalent:

1. \( \mathcal{L} \) is abelian.
2. \( \mathcal{J} = 0 \).
3. \( \left\{ \begin{array}{c} \frac{a}{2} \ b \ c \\ \frac{b}{2} \ c \ a \\ \frac{c}{2} \ a \ b \end{array} \right\} = 0 \) for all non-negative integers \( k \) satisfying \( k \equiv 2m - 2 \mod 4 \).
4. Up to a swap of \( a \) and \( c \) we have: \( c = 0, b = m, a \equiv 2m \mod 4 \) and \( a \leq 2m \); or \( b = c + m \) and \( a = c + 2m \).
5. \( \Lambda^2(V(m)) \) is disjoint from \( \text{Hom}(V(c), V(a)) \).
6. There is a uniserial \( \mathfrak{sl}(2) \times V(m) \)-module with socle factors \( V(a), V(b), V(c) \).

The proof of this theorem requires the determination of non-trivial zeros of the 6j-symbol within certain parameters. Finding non-trivial zeros of the 6j-symbol, in general, a very difficult problem (see for instance [L], [R] or [ZR]) and we think that the above theorem might have applications to it.

2. Matrix recognition of uniserial modules

Let \( \mathfrak{g} \) be a Lie algebra with solvable radical \( \mathfrak{r} \) and Levi decomposition \( \mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r} \), and fix a representation \( T : \mathfrak{g} \rightarrow \mathfrak{gl}(V) \). Given a basis \( B \) of \( V \) we let \( M_B : \mathfrak{g} \rightarrow \mathfrak{gl}(d) \), \( d = \dim(V) \), stand for the corresponding matrix representation.

By an \( \mathfrak{s} \)-basis of \( V \) of type 1 we understand a basis of the form \( B = B_1 \cup \cdots \cup B_n \), where each \( B_i \) is a basis of an \( \mathfrak{s} \)-submodule \( W_i \) of \( V \), and

\[
0 < W_1 \subset W_1 \oplus W_2 \subset W_1 \oplus W_2 \oplus W_3 \subset \cdots \subset W_1 \oplus \cdots \oplus W_n = V
\]
is the socle series of $V$. We likewise define an $s$-basis of type 2 by requiring that $\square$ be a composition series of $V$. Since $V$ is completely reducible as an $s$-module it is clear that bases of both types exist. In either case $B$ gives rise to a sequence $V_0, V_1, ..., V_n$ of $g$-modules defined by $V_0 = 0$ and $V_i = W_1 \oplus \cdots \oplus W_i$ for $1 \leq i \leq n$.

**Lemma 2.1.** The ideal $[g, t]$ annihilates every irreducible $g$-module.

*Proof.* An elementary proof can be found, for instance, in Lemma 2.4 of [CS1]. □

**Corollary 2.2.** If $B$ is any $s$-basis of $V$ then $M_B(s)$ is block diagonal and $M_B(r)$ is strictly block upper triangular for all $s \in s$ and $r \in [g, t]$.

**Lemma 2.3.** If $B$ is an $s$-basis of type 1 then none of the blocks in the first superdiagonal of $M_B(r)$ is identically 0.

*Proof.* Let $B$ and $W_1, ..., W_n, V_0, V_1, ..., V_n$ be as in $\square$. Let $2 \leq i \leq n$ and suppose, if possible, that the block $(i - 1, i)$ of $M_B(r)$ is identically 0. It follows easily that $W_i \oplus V_{i-2}$ is a submodule of $V$. Clearly $(W_i \oplus V_{i-2})/V_{i-2}$ and $V_{i-1}/V_{i-2}$ are non-zero submodules of $V/V_{i-2}$ having trivial intersection. This contradicts the fact that the socle of $V/V_{i-2}$ is $V_{i-1}/V_{i-2}$. □

**Theorem 2.4.** The $g$-module $V$ is uniserial if and only if given any $s$-basis $B$ of type 2 none of the blocks in the first superdiagonal of $M_B(r)$ is identically 0. If $g$ is perfect and there exists one $s$-basis $B$ of type 2 such that none of the blocks in the first superdiagonal of $M_B(r)$ is identically 0 then $V$ is uniserial.

*Proof.* If $V$ is uniserial an $s$-basis of type 2 is also of type 1, so Lemma 2.3 applies. If $V$ is not uniserial then some factor of its socle series is not irreducible. This factor is a completely reducible $g$-module, which easily yields an $s$-basis $B$ of type 2 with at least one block in the first superdiagonal of $M_B(r)$ identically 0.

Suppose next $g$ is perfect and let $B$ be an $s$-basis of type 2 such that none of the blocks in the first superdiagonal of $M_B(r)$ are identically 0. As indicated above, $B$ gives rise to a series of $s$-modules $W_1, ..., W_n$ and $g$-modules $V_0, V_1, ..., V_n$ in such a way that $\square$ composition series of $V$. We will show that $\square$ is in fact the socle series of $V$.

Arguing by induction, it suffices to show that $\text{soc}(V) = W_1$. Let $U$ be a non-zero submodule of $V$. We wish to show that $W_1 \subseteq U$. Since $U \cap V \neq 0$ there exists a smallest index $1 \leq j \leq n$ such that $U \cap V_j \neq 0$. If $j = 1$ we are done by the irreducibility of $W_1$. Suppose, if possible, that $1 < j \leq n$. The definition of $j$ ensures the existence of $u \in U$ such that $u = w_1 + \cdots + w_j$, where $w_i \in W_i$ for $1 \leq i \leq j$ and $w_j \neq 0$. Let $r \in r$. Then $r \in [g, t]$, since $g$ is perfect, so $ru = rw_2 + \cdots + rw_j$. Since $rw_i \in V_{i-1}$ for all $2 \leq i \leq j$ by Lemma 2.1 In particular $ru \in U \cap V_{j-1}$. The choice of $j$ forces forces $ru = 0$, so $ru_j \subseteq V_{j-2}$. Let $T$ be 0-weight space for the action of $r$ on $V_j/V_{j-2}$. As $r$ is an ideal of $g$, the subspace $T$ is $s$-invariant. Since $w_j + V_{j-2} \subseteq T$ the $s$-submodule of $V_j/V_{j-2}$ generated by $w_j + V_{j-2}$ is contained in $T$, i.e. $(W_j + V_{j-2})/V_{j-2} \subseteq T$, which means $tw_j \subseteq V_{j-2}$, a contradiction. □

**Note 2.5.** Let $g$ be any imperfect Lie algebra. Then there is a non-uniserial $g$-module with an $s$-basis $B$ of type 2 such that none of the blocks in the first superdiagonal of $M_B(r)$ is identically 0. It suffices to find a counterexample when $g = \mathbb{C}x$ is one dimensional, in which case we can take $x \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. 

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Lemma 2.6. If $U$ is a subspace of $V$ let $U^0 = \{ f \in V^* \mid f(U) = 0 \}$. Then $U \mapsto U^0$ is an inclusion reversing bijective correspondence from the $\mathfrak{g}$-submodules of $V$ to those of $V^*$. Moreover, if $U \subseteq W$ are $\mathfrak{g}$-submodules of $V$ then $U^0/W^0 \cong (W/U)^\circ$ via $f + W^0 \mapsto \tilde{f}$, where $\tilde{f}(w + U) = f(w)$. In particular, if $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ is a (resp. the only) composition series of $V$, with composition factors $X_1, \ldots, X_n$, then $0 = V_n^0 \subset \cdots \subset V_1^0 \subset V_0^0 = V^*$ is a (resp. the only) composition series of $V^*$, with composition factors $X_n^*, \ldots, X_1^*$.

**Proof.** Use the natural isomorphism of $\mathfrak{g}$-modules $V \to V^{**}$.

**Note 2.7.** Lemma [2.1] through Theorem [2.4] are valid, mutatis mutandis, for an arbitrary finite dimensional complex associative algebra $A$. Both $\mathfrak{r}$ and $[\mathfrak{g}, \mathfrak{r}]$ are to be replaced by the Jacobson radical $\mathfrak{J}$ of $A$, and $\mathfrak{s}$ by a semisimple subalgebra of $A$ complementing $\mathfrak{J}$, whose existence is ensured by the Wedderburn-Malcev theorem.

Lemma [2.6] is also valid if $A$ has an involution $a \mapsto a^*$ and we make $V^*$ into an $A$-module via $(af)(v) = f(a^*v)$.

3. Admissible sequences

In this section $\mathfrak{g} = \mathfrak{s} \times \mathfrak{r}$, where $\mathfrak{s}$ and $\mathfrak{r}$ are arbitrary Lie algebras.

Let $W_1, \ldots, W_n$ be $\mathfrak{s}$-modules and set $V = W_1 \oplus \cdots \oplus W_n$. Let $T_i : \mathfrak{s} \to \mathfrak{gl}(W_i)$ and $T : \mathfrak{s} \to \mathfrak{gl}(V)$ stand for the associated representations. As is well known, $\mathfrak{gl}(V)$ becomes an $\mathfrak{s}$-module via $s \cdot f = [T(s), f]$, for $s \in \mathfrak{s}$ and $f \in \mathfrak{gl}(V)$.

Suppose first that $X : \mathfrak{s} \to \mathfrak{gl}(V)$ is a representation extending $T$ and denote by $Y : \mathfrak{r} \to \mathfrak{gl}(V)$ the restriction of $X$ to $\mathfrak{r}$. Then

$$Y(s \cdot r) = Y([s, r]) = X([s, r]) = [X(s), X(r)] = [T(s), Y(r)] = s \cdot Y(r),$$

i.e. $Y$ is a homomorphism of $\mathfrak{s}$-modules.

Suppose conversely that $Y : \mathfrak{r} \to \mathfrak{gl}(V)$ is a homomorphism of $\mathfrak{s}$-modules and let $X : \mathfrak{s} \to \mathfrak{gl}(V)$ be the linear extension of $T$ and $Y$ to $\mathfrak{g}$. Then

$$X([s, r]) = Y([s, r]) = Y(s \cdot r) = s \cdot Y(r) = [T(s), Y(r)] = [X(s), X(r)],$$

i.e. $X$ preserves all brackets $[s, r]$. As $T$ is a Lie homomorphism, $X$ also preserves all brackets $[s_1, s_2]$. But $Y$, and hence $X$, need not preserve the brackets $[r_1, r_2]$.

In any case, we will identify $\mathfrak{gl}(V)$ with $\bigoplus_{1 \leq i, j \leq n} \text{Hom}(W_i, W_j)$ as $\mathfrak{s}$-modules by interpreting each linear map $W_i \to W_j$ as a linear map $V \to V$ that is 0 on all summands $W_k$ of $V$ with $k \neq i$. Suppose we are given $n(n-1)/2$ $\mathfrak{s}$-homomorphisms $f_{i,j} : \mathfrak{r} \to \text{Hom}(W_i, W_j)$, where $i > j$, and let $Y : \mathfrak{r} \to \mathfrak{gl}(V)$ be the $\mathfrak{s}$-homomorphism corresponding to them. At this point we make the simplifying assumption that $\mathfrak{r}$ be abelian. Then $Y([r, t]) = 0$ for all $r, t \in \mathfrak{r}$. On the other hand, in order to have $[Y(r), Y(t)] = 0$ it is necessary that the following maps vanish:

$$(3.1) \quad f_{i+1,i}(r)f_{i+2,i+1}(t) - f_{i+1,i}(t)f_{i+2,i+1}(r) \in \text{Hom}(W_{i+2}, W_i), \quad 1 \leq i \leq n - 2.$$
The necessity of this condition is examined in [9]. Taking into account the preceding discussion and Theorem 2.4, the following criterion is established.

**Proposition 3.1.** Let \( \mathfrak{g} = \mathfrak{s} \times \mathfrak{r} \) be a Levi decomposition and let \( W_1, \ldots, W_n \) be a sequence of irreducible \( \mathfrak{s} \)-modules.

(a) Suppose \( W_1, \ldots, W_n \) is admissible (as defined in [7]). Then \( \mathfrak{r} \) is not disjoint with any of the \( \mathfrak{s} \)-modules \( W_2^* \otimes W_1, \ldots, W_n^* \otimes W_{n-1} \). In particular, if \( \mathfrak{r} \) is irreducible, it must be a constituent of \( W_2^* \otimes W_1, \ldots, W_n^* \otimes W_{n-1} \).

(b) Assume \( \mathfrak{g} \) is perfect and \( \mathfrak{r} \) is abelian. Then \( W_1, \ldots, W_n \) is admissible if and only if \( \mathfrak{r} \) is not disjoint with any of the \( \mathfrak{s} \)-modules \( W_2^* \otimes W_1, \ldots, W_n^* \otimes W_{n-1} \) and for some choice of non-zero \( \mathfrak{s} \)-homomorphisms \( f_{i+1,i} : \mathfrak{r} \to \text{Hom}(W_{i+1}, W_i) \) the Lie subalgebra of \( \text{gl}(V), V = W_1 \oplus \cdots \oplus W_n \), generated by \( f_{2,1}(r) + \cdots + f_{n,n-1}(r), r \in \mathfrak{r} \), is abelian. In particular, if \( \Lambda^2 \mathfrak{r} \) is disjoint with the \( \mathfrak{s} \)-modules \( W_2^* \otimes W_1, \ldots, W_n^* \otimes W_{n-1} \) and \( \mathfrak{r} \) is not disjoint with any of the \( \mathfrak{s} \)-modules \( W_2^* \otimes W_1, \ldots, W_n^* \otimes W_{n-1} \) then \( W_1, \ldots, W_n \) is admissible.

In regards to uniqueness, we have the following criterion.

**Proposition 3.2.** Suppose \( \mathfrak{g} \) is perfect, with Levi decomposition \( \mathfrak{g} = \mathfrak{s} \times \mathfrak{r} \). Let \( W_1, \ldots, W_n \) be irreducible \( \mathfrak{s} \)-modules satisfying:

(a) \( \dim(\text{Hom}(\mathfrak{r}, W_{i+1}^* \otimes W_i)) = 1 \) if \( 1 \leq i < n \).

(b) \( \dim(\text{Hom}(\mathfrak{r}, W_j^* \otimes W_i)) = 0 \) if \( j - i \geq 2 \).

Then there exists at most one \( \mathfrak{g} \)-module \( V \), up to isomorphism, with socle factors \( W_1, \ldots, W_n \).

**Proof.** Let \( V \) be one such a module. Since its socle factors are irreducible, \( V \) is uniserial. Let \( B \) be an \( \mathfrak{s} \)-basis of \( V \). Since \( \mathfrak{g} \) is perfect, Corollary 2.2 ensures that \( M_B(\mathfrak{r}) \) is strictly block upper triangular. By Lemma 2.3, none of the blocks in the first superdiagonal of \( M_B(\mathfrak{r}) \) are identically 0, while (b) guarantees that all other strictly upper triangular blocks of \( M_B(\mathfrak{r}) \) are identically 0. By (a) the blocks in the first superdiagonal of \( M_B(\mathfrak{r}) \) are uniquely determined up to a non-zero scalar (which depends only on the position of the block). Conjugating all \( M_B(x), x \in \mathfrak{g} \), by a suitable block diagonal matrix, with each block a scalar matrix, we can arbitrarily scale all blocks in the first superdiagonal. This yields the desired result. \( \square \)

4. Existence and uniqueness of the uniserial module \( Z(\lambda, b) \)

The notation introduced here will be kept for the remainder of the paper. Let \( \mathfrak{s} \) be a non-zero semisimple Lie algebra with Cartan subalgebra \( \mathfrak{h} \), associated root system \( \Phi \), and fixed system of simple roots \( \Pi \). The coroots \( h_\alpha \in \mathfrak{h} \) associated to the simple roots \( \alpha \in \Pi \) form a basis of \( \mathfrak{h} \). The basis of \( \mathfrak{h}^* \) dual to \( \{ h_\alpha \ | \ \alpha \in \Pi \} \) consists of the fundamental weights \( \{ \lambda_\alpha \ | \ \alpha \in \Pi \} \). Let \( \Lambda^+ \) stand for the dominant integral weights of \( \mathfrak{h} \) associated to \( \Pi \), i.e. the non-negative integral linear combinations of the fundamental weights \( \lambda_\alpha \). Given \( \lambda, \mu \in \mathfrak{h}^* \) we declare \( \lambda \leq \mu \) if \( \mu - \lambda \) is a non-negative rational linear combination of simple roots. It is well-known that the inverse of the Cartan matrix has non-negative rational coefficients. It follows that all fundamental weights are strictly positive. Therefore, all non-zero dominant integral weights are strictly positive. This fact will be repeatedly and implicitly used below.

Let \( W \) stand for the Weyl group of \( \Phi \) and write \( w_0 \) for the longest element of \( W \), i.e. the one sending \( \Pi \) to \( -\Pi \).
We fix \( \mu \in \Lambda^+ \) and let \( V(\mu) \) stand for an irreducible \( \mathfrak{s} \)-module with highest weight \( \mu \). Define the dual weight \( \mu^* = -w_0 \mu \in \Lambda^+ \), noting that \( V(\mu)^* \cong V(\mu^*) \). We assume henceforth that that \( \mu \neq 0 \) and consider the perfect Lie algebra \( \mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r} \), where \( \mathfrak{r} = V(\mu) \). By the special case we mean the case \( \mathfrak{g} = \mathfrak{sl}(2) \ltimes V(m) \), \( m \geq 1 \).

**Theorem 4.1.** Let \( \lambda \in \Lambda^+ \) and \( b \geq 0 \). Then, up to isomorphism, there exists one and only one uniserial \( \mathfrak{g} \)-module, say \( Z(\lambda, b) \), that has length \( b + 1 \) and socle factors \( V(\lambda), V(\lambda + \mu^*), \ldots, V(\lambda + b \mu^*) \).

**Proof.** In light of Proposition 3.1 existence follows if we can prove that \( V(\mu) \) is not disjoint with any of \( V(\lambda^* + (i + 1)\mu) \otimes V(\lambda + i\mu^*) \) while \( \Lambda^2 V(\mu) \) is disjoint with all \( V(\lambda^* + (i + 2)\mu) \otimes V(\lambda + i\mu^*) \). Now \( \text{Hom}(V(\mu), V(\lambda^* + (i + 1)\mu) \otimes V(\lambda + i\mu^*)) \cong \text{Hom}(V(\lambda^* + (i + 1)\mu^*), V(\mu^*) \otimes V(\lambda + i\mu^*)) \) as vector spaces. It is clear that the latter space is not only non-zero but in fact one dimensional.

Reasoning as above and using that fact that \( (\Lambda^2 V(\mu))^* \cong \Lambda^2 V(\mu^*) \) we see that the vector space \( \text{Hom}_{\mathfrak{g}}(\Lambda^2 V(\mu), V(\lambda^* + (i + 2)\mu) \otimes V(\lambda + i\mu^*)) \) is isomorphic to \( \text{Hom}_{\mathfrak{g}}(V(\lambda^* + (i + 2)\mu^*), \Lambda^2 V(\mu^*) \otimes V(\lambda + i\mu^*)) \). But the latter is 0 since all weights of \( \Lambda^2 V(\mu^*) \otimes V(\lambda + i\mu^*) \) are strictly less than \( \lambda + (i + 2)\mu^* \).

In regards to uniqueness, note that \( V(\lambda^* + j\mu) \otimes V(\lambda + i\mu^*) \) and \( V(\mu) \) are disjoint provided \( j - i \geq 2 \). This follows as above by observing that all weights of \( V(\mu^*) \otimes V(\lambda + i\mu^*) \) are strictly less than \( \lambda + j\mu^* \). Now apply Proposition 3.2. \( \square \)

**Corollary 4.2.** In the notation of Theorem 4.1, there exists one and only one uniserial \( \mathfrak{g} \)-module, namely \( Z(\lambda, b)^* \), that has length \( b + 1 \) and whose socle factors are \( V(\lambda^* + b\mu), \ldots, V(\lambda^* + \mu), V(\lambda^*) \).

**Proof.** Immediate consequence of Theorem 4.1 and Lemma 2.6. \( \square \)

**Note 4.3.** It is clear that all modules constructed in Theorem 4.1 and Corollary 4.2 are non-isomorphic from each other, except in the obvious case \( b = 0 \) and \( \lambda = \lambda^* \), when \( Z(\lambda, b) = V(\lambda) \cong V(\lambda^*) = Z(\lambda, b)^* \).

**Note 4.4.** In the special case \( \mathfrak{g} = \mathfrak{sl}(2) \ltimes V(m) \) these results read as follows. Given integers \( \ell, b \geq 0 \) there exists one and only one uniserial \( \mathfrak{g} \)-module having socle factors either \( V(\ell), V(\ell + m), \ldots, V(\ell + bm) \) or \( V(\ell + m), V(\ell + m), V(\ell) \). These modules will be respectively denoted by \( Z(\ell, b) \) and \( Z(\ell, b)^* \).

**Note 4.5.** Suppose \( \mathfrak{g} \) is perfect with Levi decomposition \( \mathfrak{s} \ltimes \mathfrak{r} \) such that \([\mathfrak{s}, \mathfrak{r}] = 0\). Let \( W_1, W_2 \) be irreducible \( \mathfrak{s} \)-modules. By Proposition 3.2 there exists a uniserial \( \mathfrak{g} \)-module \( V \) with socle factors \( W_1, W_2 \) if and only if \( \mathfrak{r} \) is not disjoint with \( W_2 \otimes W_1 \).

Let \( H = \text{Hom}_{\mathfrak{g}}(\mathfrak{r}, W_2 \otimes W_1) \) and set \( P = P(H) \), the associated projective space (i.e. the points of \( P \) are the the lines of \( H \) through the origin). It is not difficult to see that the isomorphism classes of such modules \( V \) are parametrized by the points of \( P \). In particular, if \( \dim(H) > 1 \) there are infinitely many such classes.

In the special case \( \mathfrak{g} = \mathfrak{sl}(2) \ltimes V(m) \) the possibility \( \dim(H) > 1 \) never arises. Indeed, we have \( W_1 = V(a), W_2 = V(b) \), with \( 0 \leq a \leq b \) (otherwise consider \( V^* \)). Our previous comments and the Clebsch-Gordan formula ensure that such \( V \) exists

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if and only if m is in the list of numbers b − a, b − a + 2, ..., b + a − 2, b + a. But then V will be unique since V(b) \otimes V(a) is multiplicity free and Proposition 3.2 applies.

5. AN EXPLICIT MATRIX REALIZATION OF THE sl(2) \times V(m)-MODULE Z(ℓ, b)

In this section we consider the special case g = sl(2) \times V(m) and construct a matrix realization of Z(ℓ, b), where ℓ \geq 0 and b \geq 0. Taking the opposite transpose of our representation yields a matrix version of Z(ℓ, b)*.

The Lie algebra g has basis e, h, f, v_0, v_1, ..., v_m, subject to the following relations:

\begin{align*}
[&h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h, \\
[v_i, v_j] &= 0, \quad 0 \leq i, j \leq m, \\
[h, v_i] &= (m - 2i)v_i, \quad [e, v_i] = (m - (i - 1))v_i, \quad [f, v_i] = (i + 1)v_{i+1},
\end{align*}

where 0 \leq i \leq m and v_{-1} = 0 = v_{m+1}.

For a \geq 0 let H(a), E(a), F(a) be the matrices in gl(a + 1) all of whose entries are 0, except that the diagonal of H(a) is (a, a - 2, ..., a + 2, -a) and, if a \geq 1, the first superdiagonal of E(a) is (a, ..., 2, 1) and the first subdiagonal of F(a) is (1, 2, ..., a). Set H(ℓ, b) = H(ℓ) \oplus \cdots \oplus H(ℓ + bm), E(ℓ, b) = E(ℓ) \oplus \cdots \oplus E(ℓ + bm) and F(ℓ, b) = F(ℓ) \oplus \cdots \oplus F(ℓ + bm).

For a \geq 0 we define the m + 1 rectangular matrices W_0(a), ..., W_m(a), all of size (a + 1) \times (a + m + 1), as follows:

\begin{align*}
W_0(a) = (0_{(a+1) \times m}I_{a+1}), & \quad W_1(a) = (0_{(a+1) \times (m-1)}I_{a+1}0_{(a+1) \times 1}), \\
W_2(a) = (0_{(a+1) \times (m-2)}I_{a+1}0_{(a+1) \times 2}), & \quad ..., W_m(a) = (I_{a+1}0_{(a+1) \times m}).
\end{align*}

We next scale these matrices to produce

V_i(a) = (-1)^i \binom{m}{i} W_i(a), \quad 0 \leq i \leq m.

For 0 \leq i \leq m let V_i(ℓ, b) be the block partitioned matrix all of whose blocks are equal to 0, except that if b \geq 1 the blocks along the first superdiagonal are V_i(ℓ), V_i(ℓ + m), ..., V_i(ℓ + (b - 1)m).

Lemma 5.1. The map h \mapsto H(ℓ, b), e \mapsto E(ℓ, b), f \mapsto F(ℓ, b), v_i \mapsto V_i(ℓ, b), where 0 \leq i \leq m, defines a matrix representation of g with associated module Z(ℓ, b).

Proof. Since H(ℓ, b), E(ℓ, b), F(ℓ, b) satisfy (5.1) we see that the map h \mapsto H(ℓ, b), e \mapsto E(ℓ, b), f \mapsto F(ℓ, b) defines a matrix representation of sl(2) whose associated module decomposes as V(ℓ) \oplus V(ℓ + m) \oplus \cdots \oplus V(ℓ + bm).

Given a \geq 0 let U_i(a) be the matrix partitioned into 4 blocks, whose (1,2) block is W_i(a) and all other blocks are 0. Direct calculation shows that

\begin{align*}
[H(a) \oplus H(a + m), U_0(a)] &= mU_0(a), \quad [E(a) \oplus E(a + m), U_0(a)] = 0
\end{align*}

and

\begin{align*}
[F(a) \oplus F(a + m), U_i(a)] &= -(m - i)U_{i+1}(a), \quad 0 \leq i \leq m.
\end{align*}

Thus U_0(a), ..., U_m(a) is a basis for an sl(2)-module, say S_a, isomorphic to V(m). From (5.4) we get

f^i U_0(a) = (-1)^i m(m-1) \cdots (m-(i-1)) U_i(a) = (-1)^i m!/(m-i)! U_i(a), \quad 0 \leq i \leq m,
It follows that
\[ f^i U_\alpha(a)/i! = (-1)^i \binom{m}{i} \] 

is a basis for \( S_a \) upon which \( h, e, f \) act as in (5.3). Hence \( V_0(\ell, b), \ldots, V_m(\ell, b) \) is a basis of an \( \mathfrak{sl}(2) \)-module upon which \( h, e, f \) act via \( H(\ell, b), E(\ell, b), F(\ell, b) \) as in (5.2).

Next we verify that the relations (5.2) are preserved. By means of the actions of \( e \) and \( f \) on \( V_0(\ell, b), \ldots, V_m(\ell, b) \), it suffices to verify \( [V_0(\ell, b), V_m(\ell, b)] = 0 \). This easily reduces to the case \( b = 2 \), which is confirmed through a simple calculation.

We thus have a matrix representation of \( \mathfrak{g} \). By Theorem 2.4 the associated module, say \( V \), is uniserial with socle factors \( V(\ell), V(\ell + m), \ldots, V(\ell + bm) \), so \( V \cong Z(\ell, b) \) by Proposition 3.2.

Here we present a matrix realization for \( m = 2 \) and \( V \cong Z(1, 2) \).

\[
\begin{array}{cccccc}
\hline
h & e & v_2 & -2v_1 & v_0 & 0 \\
\hline
f & -h & 0 & 0 & 0 & v_2 \\
\hline
3h & 3e & 0 & 0 & 0 & v_2 \\
f & h & 2e & 0 & 0 & -2v_1 \\
0 & 2f & -h & e & 0 & v_0 \\
0 & 0 & 3f & -3h & e & v_0 \\
\hline
\end{array}
\]

A matrix realization of \( Z(1, 2)^* \) is obtained by taking the opposite transpose of the above matrix. A suitable change of basis presents a realization of \( Z(1, 2)^* \) as the following block upper triangular matrices:

\[
\begin{array}{cccccc}
\hline
5h & 5e & 0 & 0 & 0 & 0 \\
f & 3h & 4e & 0 & 0 & 0 \\
0 & 2f & h & 3e & 0 & 0 \\
0 & 0 & 3f & -h & 2e & 0 \\
0 & 0 & 0 & 4f & -3h & e \\
0 & 0 & 0 & 0 & 5f & -5h \\
\hline
10v_0 & 0 & 0 & 0 & 0 & 0 \\
4v_1 & 6v_0 & 0 & 0 & 0 & 0 \\
v_2 & 6v_1 & 3v_0 & 0 & 0 & 0 \\
0 & 0 & 6v_2 & 4v_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 5f & -5h \\
0 & 0 & 0 & 0 & 10v_0 & 0 \\
\hline
\end{array}
\]

6. Characterization of the uniserial modules \( Z(\lambda, b) \) and \( Z(\lambda, b)^* \)

We adhere to the notation introduced in (2.4) Let \( V \) be a \( \mathfrak{g} \)-module. By a weight vector we mean a non-zero common eigenvector for the action of \( h \) on \( V \). A highest weight vector, or just a maximal vector, is a weight vector that is annihilated by all \( e_\alpha, \alpha \in \Pi \). The weight spaces \( r_\mu \) and \( u_{w_\mu} \) as well as the root spaces \( s_\alpha, \alpha \in \Pi \), are all one dimensional and we fix a spanning vector for each of them, say \( u_\mu \in r_\mu, u_{w_\mu} \in u_{w_\mu} \) and \( e_\alpha \in s_\alpha \).

**Lemma 6.1.** Let \( V \) be a \( \mathfrak{g} \)-module and let \( v \in V \).

(a) Let \( \alpha \in \Pi \). Then \( e_\alpha u_{w_\mu} v = 0 \) for all \( i \geq 0 \) if and only if \( e_\alpha v = 0 \) and \([e_\alpha, u_{w_\mu}]v = 0\).

(b) Let \( i \geq 0 \). If \( e_\alpha u_{w_\mu} v = 0 \) for all \( \alpha \in \Pi \) then \( u_{w_\mu}^i u_\mu v \) is included in the \( \mathfrak{s} \)-submodule of \( V \) generated by \( u_{w_\mu}^{i+1} v \).

(c) Let \( \alpha \in \Pi \). If \( e_\alpha v = 0 \) then \( e_\alpha u_\mu v = 0 \) for all \( i \geq 0 \).
Proof. (a) Necessity is obvious. As for sufficiency, we argue by induction. The base case $i = 0$ is given. Suppose the result is true for some $i \geq 0$. Since $\tau$ is abelian,
\[ e_\alpha u^{i+1}_\mu v = u_{\omega\mu} e_\alpha u^i_{\omega\mu} v + [e_\alpha, u_{\omega\mu}] u^i_{\omega\mu} v = 0 + u^i_{\omega\mu} [e_\alpha, u_{\omega\mu}] v = 0. \]
(b) It suffices to prove this for elements of $\tau$ of the form $[e_\alpha, u_{\omega\mu}], [e_\alpha, [e_\alpha, u_{\omega\mu}]], \ldots$ as these span $\tau$. If $s = 0$ the multibracket reduces to $u_{\omega\mu}$ and the result is true by definition. This case and the stated hypothesis yield the result by induction.
(c) Since $[e_\alpha, u_{\omega}] = 0$ we obtain $e_\alpha u^i_\mu v = u^i_\mu e_\alpha v = 0$. □

**Theorem 6.2.** Let $V$ be a $g$-module, $\lambda \in \Lambda^+$, $b \geq 0$ and let $u = u_{\omega\mu}$ (resp. $u = u_\mu$). Then $V \cong Z(\lambda, b)$ (resp. $V \cong Z(\lambda, b^*)$) if and only if there is a vector $v \in V$ satisfying conditions (C1),(C2),(C3) (resp. (C1),(C2),(C3)*) stated below.

(C1) $v$ is a maximal vector of weight $\lambda + b\mu^*$ (resp. $\lambda^*$) that generates $V$ as a $g$-module.
(C2) $u^0 v \neq 0$, $u^{b+1} v = 0$.
(C3) $[e_\alpha, u_{\omega\mu}] v = 0$ for all $\alpha \in \Pi$.
(C3)* $u_\mu^i v$ is included in the $s$-module generated by $u_\mu^{i+1} v$ for all $0 \leq i \leq b$.

Moreover, in such case $v$ is unique up to scaling.

Proof. This naturally breaks into two parts.

**SUFFICIENCY.** Let $W_i$ be the $s$-submodule of $V$ generated by $u^i v$ for $i = 0, \ldots, b$. Then (C1),(C3) and Lemma 6.1 ensure that $u^i v$ is a maximal vector of weight $\lambda + (b-i)\mu^*$ (resp. $\lambda^* + i\mu$), so $W_i \cong V(\lambda + (b-i)\mu^*)$ (resp. $W_i \cong V(\lambda^* + i\mu)$). Let $V_{b+1} = 0$ and set
\[ V_{b-i} = W_b \oplus \cdots \oplus W_{b-i}, \quad 0 \leq i \leq b. \]
We claim that $V_{b-i}$ is a $g$-submodule of $V$ for all $i = 0, \ldots, b$. It suffices to show that $r V_{b-i} \subset V_{b-(i-1)}$. Since the 0-weight space of $\tau$ acting on any $g$-module is $s$-invariant and $V_{b-i}/V_{b-(i-1)}$ is $s$-irreducible, it suffices to prove that $\tau$ has a common 0-eigenvector in $V_{b-i}/V_{b-(i-1)}$. We contend that $u^{b-i} v + V_{b-(i-1)} \subset V_{b-i}/V_{b-(i-1)}$ is non-zero and annihilated by $\tau$. That $u^{b-i} v \notin V_{b-(i-1)}$ follows from the fact that $u^{b-i} v$ is a maximal vector and none of the maximal vectors of $V_{b-(i-1)}$ has the same weight as $u^{b-i} v$. It follows from (C2) and Lemma 6.1 (resp. (C2) and (C3)*) that $ru^{b-i} v \in V_{b-(i-1)}$. This proves our contention and hence the claim.

The $g$-invariance of $V_0$ and (C1) now yield $V = V_0$. We have shown that
\[ 0 = V_{b+1} \subset V_b \subset \cdots \subset V_0 = V \]
is a composition series of the $g$-module $V$, with composition factors
\[ V_{b-i}/V_{b-(i-1)} \cong V(\lambda + i\mu^*) \quad (\text{resp. } V_{b-i}/V_{b-(i-1)} \cong V(\lambda^* + (b-i)\mu)). \]
Since $u u^{b-i} v \in W_{b-(i-1)}$, 0 < $i$ < $b$, Theorem 2.4 ensures that $V$ is uniserial with socle factors $V(\lambda), V(\lambda + \mu^*), \ldots, V(\lambda + b\mu^*)$ (resp. $V(\lambda + b\mu), \ldots, V(\lambda^* + \mu), V(\lambda^*)$). From the uniqueness part of Theorem 1.1 we conclude that $V \cong Z(\lambda, b)$ (resp. $V \cong Z(\lambda, b^*)$).

**NECESSITY.** By assumption the socle series of $V$, say
\[ 0 = V_{b+1} \subset V_b \subset \cdots \subset V_0 = V, \]
has irreducible factors
\[ V_{b-i}/V_{b-(i-1)} \cong V(\lambda + i\mu^*) \quad (\text{resp. } V_{b-i}/V_{b-(i-1)} \cong V(\lambda^* + (b-i)\mu)). \]
Up to scaling $V$ has a unique maximal vector, say $v_i$, of weight $\lambda + i\mu^*$ (resp. $\lambda^* + (b-i)\mu$). In any uniserial module, a vector belonging only to the last term of the socle series generates the entire module. Hence $V$ is generated by $v = v_b \notin V_1$.

We know from Lemma 2.1 that $V_{b-i} \subseteq V_{b-(i-1)}$. In particular $u^{b+1}v = 0$.

Suppose next $V \cong Z(\lambda, b)$. If $0 \leq i \leq b$ and $u^i v \neq 0$ then $u^i v$ is a maximal vector since its weight, namely $\lambda + (b-i)\mu^*$, is the highest in $V_i$. Since $v \neq 0$ and $u^{b+1}v = 0$ there is an index $i$ satisfying $0 \leq i \leq b$, $u^i v \neq 0$ and $u^{i+1}v = 0$. Our preceding comment implies $e_\alpha u^i v = 0$ for all $\alpha \in \Pi$. This and $uu^i v = 0$ yield $vu^i v = 0$, that is, $u^i v \in \text{soc}(V) \cong V(\lambda)$. But the highest weight in $V(\lambda)$ is $\lambda$ and $u^i v$ has weight $\lambda + (b-i)\mu^*$, so $i = b$. Thus $u^i v$ is a maximal vector for all $0 \leq i \leq b$.

Suppose finally $V \cong Z(\lambda, b)^*$. Thus $V$ has soc factors $V(\lambda^* + b\mu), \ldots, V(\lambda^*)$ and we view these as $s$-submodules of $V$. We claim that $u$ sends a maximal vector of $V(\lambda^* + i\mu)$ into one of $V(\lambda^* + (i+1)\mu)$ for all $0 \leq i \leq b$. By assumption $uV(\lambda^* + i\mu)$ is non-zero and included in $V(\lambda^* + (i+1)\mu)$. Since $r$ is an irreducible $s$-module, it follows that $u\mu V(\lambda^* + i\mu) \neq 0$, so there is a weight vector $w$ in $V(\lambda^* + i\mu)$ not annihilated by $u\mu$. Since $[e_\alpha, u\mu] = 0$ for all $\alpha \in \Pi$ by repeatedly applying the $e_\alpha$ to $u\mu w$ we may assume that $u\mu w$ is a maximal vector of $V(\lambda^* + (i+1)\mu)$, in which case $w$ must have weight $\lambda^* + i\mu$. This proves our claim. Since $v$ is a maximal vector of $V(\lambda^*)$ we deduce that $u^i v$ is a maximal vector of $V(\lambda^* + i\mu)$ for all $0 \leq i \leq b$. Moreover, $ru^i v \subseteq rV(\lambda^* + b\mu) = 0$ and $vu^i v \subseteq rV(\lambda^* + i\mu) \subseteq V(\lambda^* + (i+1)\mu)$, which is $s$-generated by $u^{i+1} v$ for all $0 \leq i < b$. \hfill \Box

7. A natural construction of the $s \ltimes V(\mu)$-module $Z(0, b)$

As mentioned in the Introduction, there have been recent constructions of indecomposable modules for a Lie algebra $\mathfrak{g}$ by embedding $\mathfrak{g}$ into a semisimple Lie algebra $\mathfrak{t}$ and restricting an irreducible $\mathfrak{t}$-module to $\mathfrak{g}$. In this section we use the characterization given in Theorem 6.2 to produce $Z(0, b)$ in the spirit just described.

We adopt the notation introduced at the beginning of 6.1 and 6.2. In particular, $\mathfrak{g} = s \ltimes \mathfrak{r}$, where $\mathfrak{r} = V(\mu)$. Consider the $s$-module $W = \mathfrak{r}^* \oplus \mathbb{C}w$, where $\mathfrak{r}$ acts trivially on $\mathbb{C}w$. We make $W$ into a $\mathfrak{g}$-module as follows:

\begin{equation}
(s + r)(f + aw) = sf + f(r)w, \quad s \in s, r \in \mathfrak{r}, f \in \mathfrak{r}^*, a \in \mathbb{C}.
\end{equation}

This gives a representation $\mathfrak{g} \rightarrow \mathfrak{sl}(W)$ (which is an embedding if $\mathfrak{r}$ is faithful and, in particular, if $\mathfrak{s}$ is simple).

Let $f \in \mathfrak{r}^*$ be the only linear functional such that $f(u_{w_{\mu k}}) = 1$ and $f(z) = 0$ for any $z$ belonging to a weight space in $\mathfrak{r}$ of weight different from $w_{\mu k}$. It is clear from this definition that

\begin{equation}
f \in (\mathfrak{r}^*) - w_{\mu k} = (\mathfrak{r}^*) - w_{\mu k}.
\end{equation}

Fix $b \geq 0$ and let $X = S^b(W)$, the $b$th symmetric power of $W$. This is an irreducible $\mathfrak{sl}(W)$-module. We view $X$ as a $\mathfrak{g}$-module via the Lie homomorphism $\mathfrak{g} \rightarrow \mathfrak{sl}(W)$.

Let $V$ be the $\mathfrak{g}$-submodule of $X$ generated by $f^b$.

**Theorem 7.1.** The $\mathfrak{g}$-module $V$ is isomorphic to $Z(0, b)$. Moreover, the $\mathfrak{g}$-module $X$ is indecomposable, with trivial socle, full socle series

\begin{equation}
0 = \text{soc}^0(X) \subset \text{soc}^1(X) \subset \text{soc}^2(X) \subset \cdots \subset \text{soc}^{b+1}(X) = X,
\end{equation}

and socle factors

\begin{equation}
\text{soc}^{i+1}(X)/\text{soc}^i(X) \cong S^i(\mathfrak{r}^*) \cong S^i(V(\mu^*)), \quad 0 \leq i \leq b.
\end{equation}
Thus the socle factors of \( V \), namely \( V(0), V(\mu^*), \ldots, V(\mu^*) \), are precisely the top summands of the socle factors of \( X \). In particular, \( X \) itself need not be uniserial, but it is so in the very special case \( g = \mathfrak{sl}(2) \ltimes V(1) \), when \( X = V \).

**Proof.** The first assertion follows at once from Theorem 6.2 applied to \( v = f^b \). Indeed, \( v \) is clearly a maximal vector of \( V \) of weight \( \mu^* \) that generates \( V \) as a \( g \)-module. Moreover,

\[
u^b f^b = b! w^b \neq 0 \quad \text{and} \quad u^{b+1} f^b = 0, \quad u = u_{\omega g \mu},
\]

and the very definition of \( f \) gives

\[
[e_\alpha, u] f = f([e_\alpha, u]) w = 0, \quad \text{so} \quad [e_\alpha, u]^b f = 0, \quad \alpha \in \Pi, \quad u = u_{\omega g \mu}.
\]

As remarked in the Introduction, \( \text{soc}^{i+1}(X)/\text{soc}^i(X) \) is the 0-weight space for the action of \( r \) on \( X/\text{soc}^i(X) \). The formula (7.1) makes it clear that

\[
\text{soc}^{i+1}(X) = w^b S^0(\tau^*) \oplus w^{b-1} S^1(\tau^*) \oplus \cdots \oplus w^{b-i} S^i(\tau^*), \quad 0 \leq i \leq b
\]

which gives the isomorphisms of \( g \)-modules

\[
\text{soc}^{i+1}(X)/\text{soc}^i(X) \cong w^{b-i} S^i(\tau^*), \quad 0 \leq i \leq b.
\]

The remaining assertions now follow immediately. \( \square \)

The special case \( g = \mathfrak{sl}(2) \ltimes V(m) \) can be translated as follows. Let \( g \to \mathfrak{gl}(m+2) \) be the matrix representation defined in 13 for the uniserial \( g \)-module with socle factors \( V(0), V(m) \).

Let \( S \) be the algebra of polynomials in \( m + 2 \) variables \( X_1, \ldots, X_{m+2} \). This is a module for \( \mathfrak{gl}(m + 2) \), where each basic matrix \( E_{ij} \) acts via derivations on \( S \) by means of \( M_{ij} \circ \partial/\partial X_j \), i.e. partial differentiation with respect to \( X_j \) followed by multiplication by \( X_i \).

Given \( b \geq 0 \), the subspace \( X \) of \( S \) of all homogeneous polynomials of degree \( b \) is \( \mathfrak{gl}(m + 2) \)-stable. We may thus view \( X \) as a \( g \)-module via \( g \to \mathfrak{gl}(m + 2) \), and consider the \( g \)-submodule \( V \) of \( X \) generated by \( X_2^b \). It follows immediately from Theorem 6.2 that \( V \cong Z(0, b) \).

Note finally that in the very special case \( m = 1 \) we have \( X = V \). In this case, by factoring the terms of the socle series of \( V \) we obtain all \( g \)-modules \( Z(\ell, b), \ell \geq 0 \).

### 8. Other Uniserial Modules

The uniserial modules \( Z(\lambda, b) \) and their duals are not the only possible ones, even in the special case \( g = \mathfrak{sl}(2) \times V(m) \). We already noted this when dealing with uniserial modules of length in Note 4.6. In this section we produce further exceptions, in this case of lengths 3 and 4. It is be shown in 10 that no other exceptions exist.

We maintain throughout the notation introduced in 13. In particular, \( g = \mathfrak{s} \ltimes r \), where \( r = V(\mu) \).

**Lemma 8.1.** Let \( \lambda \in \Lambda^+ \). There is a unique uniserial \( g \)-module with socle factors \( V(0), V(\mu^*), V(\lambda) \) provided \( V(\lambda) \) occurs once \( V(\mu^*) \otimes V(\mu^*) \) but not in \( \Lambda^2(V(\mu^*)) \). Equivalently, when \( V(\lambda) \) occurs once in \( S^2(V(\mu^*)) \) and \( V(\mu^*) \otimes V(\mu^*) \).

**Proof.** This follows easily from Propositions 3.1 and 3.2 except for uniqueness when \( \lambda = \mu^* \). In this case there is a uniserial \( g \)-module \( V \) with socle factors \( V(0), V(\mu^*), V(\mu^*) \) and we need to establish the uniqueness of \( V \) up to isomorphism.
Let $B$ be an $\mathfrak{s}$-basis of $V$ which yields identical matrix representations of $\mathfrak{s}$ on $W_2, W_3$ in the notation of [2.1]. Our hypotheses ensure that each of the blocks (1,2), (2,3), (1,3) of $M_B(\tau)$ is completely determined, up to a scalar, once $B$ is fixed. Moreover, this scalar must be non-zero in the first two cases. The (1,3) block of $M_B(\tau)$ is a scalar multiple, say by $a \in \mathbb{C}$, of the (1,2) block. Conjugating by the block matrix

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & a \\
0 & 0 & 1
\end{pmatrix}
$$

we obtain a new matrix representation which is identical to the first except that the (1,3) block of $M_B(\tau)$ is now surely 0. The result follows. \hfill \Box

Under the hypotheses of Lemma 8.1, there is a unique uniserial module with socle factors $V(\lambda^*), V(\mu), V(0)$, dual to the above. Clearly, there is at most one uniserial module of both types, namely the self-dual module with socle factors $V(0), V(\mu), V(0)$, where $\mu = \mu^*$. We next find explicit conditions for the existence of such a module. Let $\mu = \mu^*$. Then, up to scaling, there is one and only one non-zero $\mathfrak{s}$-invariant bilinear form $\phi : V(\mu) \times V(\mu) \to \mathbb{C}$, necessarily non-degenerate. By our discussion in [3] there is a uniserial $\mathfrak{g}$-module with socle factors $V(0), V(\mu), V(0)$ if and only if the $\mathfrak{s}$-homomorphism $F : \Lambda^2 V(\mu) \to \text{Hom}(\mathbb{C}, \mathbb{C})$ associated to 8.1 is trivial. But $F_{v\wedge w}(a) = a(\phi(v, w) - \phi(w, v))$ for all $a \in \mathbb{C}$ and $v, w \in V(\mu)$. This is 0 if and only if $\phi$ is symmetric. We have proven

**Lemma 8.2.** If $\mu = \mu^*$ there is a uniserial module with socle factors $V(0), V(\mu), V(0)$ if and only if the non-zero $\mathfrak{s}$-invariant bilinear form on $V(\mu)$ is symmetric.

**Note 8.3.** Suppose that $\mu = \mu^*$ and the hypotheses of Lemma 8.1 are met for $\lambda = \mu$, i.e., $V(\mu)$ occurs once in $S^2(V(\mu))$ and $V(\mu) \otimes V(\mu)$. Then, clearly, there is a uniserial module with socle factors $V(0), V(\mu), V(\mu), V(0)$. However, in this case the isomorphism classes of such modules are parametrized by the complex numbers. Indeed, once all diagonal blocks of $M_B(\mathfrak{s})$ as well as the first superdiagonal blocks $M_B(\tau)$ have been fixed and the block (1,3) of $M_B(\tau)$ has been cleared, there is no way to modify the block (2,4) of $M_B(\tau)$.

We next adapt the above observations to the special case $\mathfrak{g} = \mathfrak{sl}(2) \ltimes V(m)$. Recall that $V(m) \otimes V(m) = V(2m) \oplus V(2m - 2) \oplus \cdots \oplus V(2) \oplus V(0)$, where $S^2(V(m)) = V(2m) \oplus V(2m - 4) \oplus \cdots$ and $\Lambda^2(V(m)) = V(2m - 2) \oplus V(2m - 6) \oplus \cdots$. Thus $V(\ell)$ appears in $S^2(V(m))$ if and only if $\ell \leq 2m$ and $2m \equiv \ell \mod 4$. In particular, $V(0)$ appears in $S^2(V(m))$ (that is, the non-zero $\mathfrak{sl}(2)$-invariant bilinear form on $V(m)$ is symmetric) if and only if $m$ is even. We have shown

**Lemma 8.4.** Let $\ell \geq 0$. Then there is a unique uniserial module with socle factors $V(0), V(m), V(\ell)$ if $\ell \leq 2m$ and $2m \equiv \ell \mod 4$, in which case the dual module is uniserial with socle factors $V(0), V(m), V(0)$. Moreover, there is a unique uniserial module with socle factors $V(0), V(m), V(0)$ if and only if $m$ is even. Furthermore, if $m \equiv 0 \mod 4$ there is a parametrization by $\mathbb{C}$ of the isomorphism classes of uniserial modules with the same socle factors $V(0), V(m), V(m), V(0)$. 

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Example 8.5. A matrix realization of a uniserial $\mathfrak{sl}(2) \ltimes V(3)$-module with socle factors $V(0), V(3), V(2)$ is:

\[
\begin{array}{cccc|ccc}
0 & v_3 & 3v_2 & -3v_1 & v_0 & 0 & -3v_1 & 3v_0 & 0 \\
f & h & 2e & 0 & -2v_2 & v_1 & v_0 & 0 \\
0 & 2f & -h & e & -v_3 & -v_2 & 2v_1 & 0 \\
0 & 0 & 3f & -3h & 0 & -3v_3 & 3v_2 & 0 \\
\end{array}
\]

The one parameter family, parametrized by $z \in \mathbb{C}$, of non-isomorphic uniserial $\mathfrak{sl}(2) \ltimes V(4)$-modules with socle factors $V(0), V(4), V(4), V(0)$ is given by:

\[
\begin{array}{cccc|ccc}
0 & v_4 & -4v_3 & 6v_2 & -4v_1 & v_0 & 0 & -4v_1 & v_0 \\
f & 2h & 3e & 0 & 3v_3 & -3v_2 & -3e_1 & 3v_0 & 0 \\
0 & 2f & 0 & 2e & 0 & v_4 & 2v_3 & -6v_2 & 2v_1 & v_0 & 0 \\
0 & 0 & 3f & -2h & e & 0 & 3v_4 & -3e_3 & -3v_2 & 3v_1 & v_3 \\
0 & 0 & 0 & 4f & -4h & 0 & 0 & 6v_4 & -12v_3 & 6v_2 & 0 \\
\end{array}
\]

Let $\mathfrak{g} = \mathfrak{sl}(3) \ltimes \mathbb{C}^3$ and let $\lambda_1$ and $\lambda_2$ be the fundamental weights of $\mathfrak{sl}(3)$. We now show that there exists a unique uniserial $\mathfrak{g}$-module with socle factors $V(2\lambda_2), V(\lambda_1 + \lambda_2), V(2\lambda_1)$.

Notice that in contrast to the other examples considered so far, none of the differences between the highest weights of these three $\mathfrak{sl}(3)$-modules is a dominant weight.

Let $\mathfrak{r} = \mathbb{C}^3$. According to Propositions 3.1 and 3.2, it suffices to prove that

$$\text{Hom}_{\mathfrak{sl}(3)}(\mathfrak{r}, V(\lambda_1 + \lambda_2)^* \otimes V(2\lambda_2)) \neq 0, \text{ Hom}_{\mathfrak{sl}(3)}(\mathfrak{r}, V(2\lambda_1)^* \otimes V(\lambda_1 + \lambda_2)) \neq 0,$$

and

$$\text{Hom}_{\mathfrak{sl}(3)}(\mathfrak{r}, V(2\lambda_1)^* \otimes V(2\lambda_2)) = 0 = \text{Hom}_{\mathfrak{sl}(3)}(\Lambda^2 \mathfrak{r}, V(2\lambda_1)^* \otimes V(2\lambda_2)).$$

Since $\lambda_1^* = \lambda_2$ and $\mathfrak{r} \cong V(\lambda_1), \Lambda^2 \mathfrak{r} \cong V(\lambda_2)$, the above conditions follow from the following tensor product decompositions:

$$V(\lambda_1 + \lambda_2)^* \otimes V(2\lambda_2) = V(\lambda_1 + \lambda_2) \otimes V(2\lambda_2) = V(\lambda_1 + 3\lambda_2) \oplus V(2\lambda_1 + \lambda_2) \oplus V(2\lambda_2) \oplus V(\lambda_1),$$

$$V(2\lambda_1)^* \otimes V(2\lambda_2) = V(2\lambda_2) \otimes V(2\lambda_2) = V(4\lambda_2) \oplus V(\lambda_1 + 2\lambda_2) \oplus V(2\lambda_1).$$

Here is an example with $\mathfrak{s} = \mathfrak{so}(m)$, $m \geq 3$, and $\mu = \lambda_1$, the first fundamental weight. A matrix representation of $\mathfrak{so}(m) \ltimes V(\lambda_1)$ that is uniserial with socle factors $V(0), V(\lambda_1), V(0)$ can be obtained as follows.

Let $U$ be a vector space of dimension $m$, and let $f : U \times U \to \mathbb{C}$ be a non-degenerate symmetric bilinear form. The subalgebra of $\mathfrak{gl}(U)$ preserving $f$ is $\mathfrak{so}(m)$, and $U$ is the natural module for $\mathfrak{so}(m)$. 
Set $n = m + 2$ and let $J$ be the $n \times n$ matrix with 1’s along the secondary diagonal and 0’s elsewhere. The $n \times n$ matrices $X$ satisfying $X'J + JX = 0$, where $X'$ indicates the transpose of $X$, form $\mathfrak{so}(n)$. The appearance of such $X$ is

$$X = \begin{pmatrix} a & x_1 & \cdots & x_m & 0 \\ y_1 & * & * & * & -x_m \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ y_m & * & * & * & -x_1 \\ 0 & -y_m & \cdots & -y_1 & -a \end{pmatrix}$$

where the inner * follow the same pattern as the outer entries, i.e. $X$ is skew-symmetric relative to the secondary diagonal. The subalgebra of $\mathfrak{so}(n)$ formed by all $X$ having 0’s in the outer rows/columns is clearly isomorphic to $\mathfrak{so}(m)$. Moreover, the subspace of all $X$ having 0’s in the first column, last row, and all inner entries, is normalized by $\mathfrak{so}(m)$, in such a way that together they form a subalgebra isomorphic to $\mathfrak{so}(m) \ltimes U$. Theorem 2.4 ensures that this is a uniserial representation, whose socle factors are clearly $V(0), U, V(0)$.

In particular when $m = 3$ we obtain a uniserial module for $\mathfrak{sl}(2) \ltimes V(2)$ with socle factors $V(0), V(2), V(0)$. Explicitly, we have the embedding of $\mathfrak{sl}(2) \ltimes V(2)$ into $\mathfrak{so}(5)$

$$\begin{pmatrix} 0 & a & b & c & 0 \\ 0 & h & e & 0 & -c \\ 0 & f & 0 & -e & -b \\ 0 & 0 & -f & -h & -a \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

which makes $\mathbb{C}^5$ into a uniserial module with socle factors $V(0), V(2), V(0)$.

Thus the isomorphism $\mathfrak{sl}(2) \cong \mathfrak{so}(3)$ yields a uniserial module for $\mathfrak{sl}(2) \ltimes V(2)$ with socle factors $V(0), V(2)$, $V(0)$, where $V(2)$ is the natural module for $\mathfrak{so}(3)$. But when we identify $\mathfrak{sl}(2)$ with $\mathfrak{sp}(2)$ the invariant form on $V(1)$ is skew-symmetric and no uniserial module for $\mathfrak{sl}(2) \ltimes V(1)$ with socle factors $V(0), V(1), V(0)$ exists, as indicated above.

It is perhaps worth noting that when we pass to a perfect Lie algebra whose radical is nilpotent of class 2, in addition to all modules arising from the abelian case, we may obtain some new ones as well. As an illustration, let $m \geq 1$, let $U$ be a vector space of dimension $2m$ and let $f : U \times U \rightarrow \mathbb{C}$ be a non-degenerate skew-symmetric bilinear form. The subalgebra of $\mathfrak{gl}(U)$ preserving $f$ is $\mathfrak{sp}(2m)$ and $U$ is the natural module for $\mathfrak{sp}(2m)$. The Heisenberg algebra $\mathfrak{h}(2m+1)$ can be defined on the vector space $U \oplus \mathbb{C}$ by declaring $[u+a, v+b] = f(u, v)$. Then $\mathfrak{sp}(2m)$ acts via derivations on $\mathfrak{h}(2m+1)$ by $[x, u+a] = x(u)$, and we may then form the perfect Lie algebra $\mathfrak{sp}(2m) \ltimes \mathfrak{h}(2m+1)$. We have a natural Lie epimorphism $\mathfrak{sp}(2m) \ltimes \mathfrak{h}(2m+1) \rightarrow \mathfrak{sp}(2m) \ltimes U$, which allows us to view every uniserial module for $\mathfrak{sp}(2m) \ltimes U$ as one for $\mathfrak{sp}(2m) \ltimes \mathfrak{h}(2m+1)$ in which the center acts trivially. We wish to construct a uniserial module $V$ for $\mathfrak{sp}(2m) \ltimes \mathfrak{h}(2m+1)$ with socle factors $V(0), U, V(0)$. Since the $\mathfrak{sp}(2m)$-invariant form on $U$, namely $f$, is skew-symmetric, our earlier comments ensure that no such module exists for $\mathfrak{sp}(2m) \ltimes U$. The reader will also note that the center of $\mathfrak{sp}(2m) \ltimes \mathfrak{h}(2m+1)$ does not act trivially on $V$. 
Set \( n = m + 1 \) and let \( J \) be the \( n \times n \) matrix with 1’s along the secondary diagonal and 0’s everywhere else. Let \( S \) be the \( 2n \times 2n \) skew-symmetric invertible matrix

\[
S = \begin{pmatrix}
0 & J \\
-J & 0
\end{pmatrix}.
\]

The \( 2n \times 2n \) matrices \( X \) satisfying \( X'S + SX = 0 \) form \( \text{sp}(2n) \). The appearance of such \( X \) is

\[
X = \begin{pmatrix}
a & x_1 & \cdots & x_m & y_1 & \cdots & y_m & z \\
b_1 & * & * & * & * & * & * & y_m \\
& \vdots & * & * & * & * & * & \vdots \\
b_m & * & * & * & * & * & * & y_1 \\
c_1 & * & * & * & * & * & * & -x_m \\
& \vdots & * & * & * & * & * & \vdots \\
c_m & * & * & * & * & * & * & -x_1 \\
d & c_m & \cdots & c_1 & -b_m & \cdots & -b_1 & -a
\end{pmatrix},
\]

where the inner * follow the same pattern as the outer entries. More explicitly, if we partition \( X \) into 4 blocks of size \( n \times n \), that is, \( X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \), then \( B, C \) are symmetric relative to the secondary diagonal and \( D \) is the opposite of the transpose of \( A \) relative to the secondary diagonal. In particular, the inner * form a subalgebra isomorphic to \( \text{sp}(2m) \), which together with the first row, and last column, with \( a = 0 \), form a subalgebra isomorphic to \( \text{sp}(2m) \times \mathfrak{h}(2m + 1) \). This makes the column space \( V = \mathbb{C}^{2n} \) into a uniserial module for \( \text{sp}(2m) \times \mathfrak{h}(2m + 1) \) with socle factors \( V(0), U, V(0) \).

In particular when \( m = 1 \) we get a uniserial module for \( \mathfrak{sl}(2) \times \mathfrak{h}(3) \) whose socle factors are \( V(0), V(1), V(0) \). Explicitly, this is obtained through the following embedding of \( \mathfrak{sl}(2) \times \mathfrak{h}(3) \) into \( \text{sp}(4) \):

\[
\begin{pmatrix}
0 & x & y & z \\
0 & a & b & y \\
0 & c & -a & -x \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

9. Admissible sequences of length 3 for \( g = \mathfrak{sl}(2) \times V(m) \)

Let \( g \) be an arbitrary Lie algebra and let \( X, Y, Z \) be \( g \)-modules. Then the map \( \text{Hom}(Y, X) \otimes \text{Hom}(Z, Y) \to \text{Hom}(Z, X) \), defined by \( \alpha \otimes \beta \mapsto \alpha \beta \), is an epimorphism \( g \)-modules.

Let \( M, N \) be \( g \)-modules and let \( f : M \to \text{Hom}(Y, X) \), \( g : N \to \text{Hom}(Z, Y) \) be homomorphisms of \( g \)-modules. They give rise to the homomorphism of \( g \)-modules

\[
M \otimes N \xrightarrow{f \otimes g} \text{Hom}(Y, X) \otimes \text{Hom}(Z, Y) \to \text{Hom}(Z, X).
\]

We are interested in the image, say \( Z \), of this map. Here is a matrix interpretation. Let \( d_X, d_Y, d_Z \) be the dimensions of \( X, Y, Z \) and fix bases \( B_X, B_Y, B_Z \) on them. Let \( M_X : g \to \mathfrak{gl}(d_X) \), \( M_Y : g \to \mathfrak{gl}(d_Y) \), \( M_Z : g \to \mathfrak{gl}(d_Z) \) be the matrix representations associated to the modules \( X, Y, Z \) relative to the bases \( B_X, B_Y, B_Z \). Consider the \( g \)-module \( U = X \oplus Y \oplus Z \) of dimension \( d = d_X + d_Y + d_Z \) and basis \( B = B_X \cup B_Y \cup B_Z \).
Theorem 9.1. Let $f$ be the choice of $sl(9.2) \rightarrow V$. We wish to determine the $g$-submodules of $sl(9.2)$ and, in this case, $J$ is, relative to the bases $B_Y, B_X$ and $B_Z, B_Y$. Then $I$ is, relative to the bases $B_Z, B_X$, the subspace generated by all matrices

$$
\left\{ \begin{pmatrix} 0 & f(m)g(n) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} : m \in M, n \in N \right\}.
$$

Note that $A = D \oplus I$ is the associative algebra generated by $D$.

We are also interested in the case $M = N$. In this case we have the $g$-module decomposition $M \otimes M = \Lambda^2(M) \oplus S^2(M)$. Let $J$ be the image of $\Lambda^2(M)$ under $D$. Then $J$ is, relative to the bases $B_Z, B_X$, the subspace generated by all matrices

$$
\left\{ \begin{pmatrix} 0 & 0 & f(m)g(n) - f(n)g(m) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : m, n \in M \right\}.
$$

and, in this case, $L = D_{\text{diag}} \oplus J$ is the Lie algebra generated by

$$
D_{\text{diag}} = \left\{ \begin{pmatrix} 0 & f(m) \\ 0 & 0 \\ g(m) & 0 \\ 0 & 0 \end{pmatrix} : m \in M \right\}.
$$

Let $a, b, c, p, q$ be non-negative integers. We next focus attention on the case:

$$
g = sl(2), \quad M = V(p), \quad N = V(q), \quad X = V(a), \quad Y = V(b), \quad Z = V(c).
$$

We wish to determine the $g$-module structure of $I$ (resp. $J$ when $p = q$). Since the tensor product of irreducible $sl(2)$-modules is multiplicity free, $I$ is independent of the choice of $f$ and $g$ (as long as they are non-zero) and determining $I$ is equivalent to finding all $k \geq 0$ such that $V(k)$ is a submodule of $I$.

**Theorem 9.1.** Let $a, b, c, p, q, k$ be non-negative integers. Assume the existence of $sl(2)$-embeddings $V(p) \rightarrow \text{Hom}(V(b), V(a))$ and $V(q) \rightarrow \text{Hom}(V(c), V(b))$, and let $I$ be the image of the corresponding $sl(2)$-homomorphism

$$
(9.2) \quad V(p) \otimes V(q) \rightarrow \text{Hom}(V(b), V(a)) \otimes \text{Hom}(V(c), V(b)) \rightarrow \text{Hom}(V(c), V(a)).
$$
Then $V(k)$ appears in $\mathcal{I}$ if and only if the Wigner-Racah 6j-symbol (see [12])

\begin{equation}
\begin{pmatrix} \frac{q}{2} & \frac{k}{2} & \frac{c}{2} \\ \frac{a}{2} & \frac{b}{2} & \frac{d}{2} \end{pmatrix} \neq 0.
\end{equation}

**Proof.** Let $V(p) \to V(a) \otimes V(b)$ and $V(q) \to V(b) \otimes V(c)$ be $\mathfrak{sl}(2)$-embeddings, and let $j : V(b) \to V(b)^*$ be an $\mathfrak{sl}(2)$-isomorphism. Let $\mathcal{K}$ be the image of the corresponding $\mathfrak{sl}(2)$-homomorphism

$$V(p) \otimes V(q) \to V(a) \otimes V(b) \otimes V(b) \otimes V(c) \xrightarrow{j} V(a) \otimes V(c),$$

where

\begin{equation}
\mu(x \otimes y_1 \otimes y_2 \otimes z) = (j(y_1))(y_2)x \otimes z, \quad x \in V(a), \ y_1, y_2 \in V(b), \ z \in V(c).
\end{equation}

It is not difficult to see that $\mathcal{K}$ is independent of the choice of $j$ and the given embeddings, and that $\mathcal{I} \cong \mathcal{K}$. Hence, it suffices to prove the result for $\mathcal{K}$.

If $V(k)$ does not occur in $V(p) \otimes V(q)$ or $V(a) \otimes V(c)$ then $V(k)$ does not occur in $\mathcal{K}$ and, moreover, the left hand side of (9.3) is 0. Thus, we may assume that $V(k)$ appears in $V(p) \otimes V(q)$ and $V(a) \otimes V(c)$. In [11] we furnish a concrete embedding $\iota^{k} : V(r) \to V(s) \otimes V(t)$ for any non-negative integers such that $|t-s| \leq r + s$ and $t + s \equiv r \mod 2$, as well as a fixed isomorphism $j_{+} : V(r) \to V(r)^*$ for any $r \geq 0$. These data yield a specific $\mathfrak{sl}(2)$-homomorphism $\phi : V(k) \to V(a) \otimes V(c)$

\begin{equation}
V(k) \xrightarrow{\iota^{k}} V(p) \otimes V(q) \xrightarrow{\iota^{a,b,c}} V(a) \otimes V(b) \otimes V(b) \otimes V(c) \xrightarrow{\mu} V(a) \otimes V(c).
\end{equation}

But, up to scaling, the only $\mathfrak{sl}(2)$-homomorphism $V(k) \to V(a) \otimes V(c)$ is $\iota_{k}^{a,c}$. Hence

$$\phi = \lambda \iota_{k}^{a,c}$$

for a unique scalar $\lambda$. A long and technical calculation (performed independently in [11]) shows that

$$\lambda = C \begin{pmatrix} \frac{q}{2} & \frac{k}{2} & \frac{c}{2} \\ \frac{a}{2} & \frac{b}{2} & \frac{d}{2} \end{pmatrix},$$

where $C$ is a non-zero scalar explicitly defined in [11]. The result now follows. □

**Corollary 9.2.** Keep the hypothesis of Theorem 9.1 and suppose that $p = q$. Let $\mathcal{J}$ be the image of $\Lambda^{2}(V(p))$ under $\mathfrak{sl}(2)$. Then $V(k)$ appears in $\mathcal{J}$ if and only if $k \equiv 2p - 2 \mod 4$ and (9.3) holds.

**Proof.** Clearly $V(k)$ appears in $\mathcal{J}$ if and only if $V(k)$ appears in $\mathcal{I}$ and $\Lambda^{2}(V(p))$. Recalling that $V(k)$ appears in $\Lambda^{2}(V(p))$ if and only if $0 \leq k \leq 2p$ and $2p - 2 \equiv k \mod 4$, the result follows from Theorem 9.1. □

**Example 9.3.** Let us consider the $\mathfrak{sl}(2)$-homomorphism

$$V(4) \otimes V(4) \to \text{Hom}(V(6), V(4)) \otimes \text{Hom}(V(4), V(6)) \to \text{Hom}(V(4), V(4)),$$

i.e. $p = q = a = c = 4$ and $b = 6$. It turns out that

$$\begin{pmatrix} \frac{4}{2} & \frac{0}{2} & \frac{4}{2} \\ \frac{4}{2} & \frac{6}{2} & \frac{4}{2} \end{pmatrix} = -\frac{1}{5}, \quad \begin{pmatrix} \frac{4}{2} & \frac{4}{2} & \frac{4}{2} \\ \frac{4}{2} & \frac{4}{2} & \frac{4}{2} \end{pmatrix} = 0, \quad \begin{pmatrix} \frac{4}{2} & \frac{6}{2} & \frac{4}{2} \\ \frac{4}{2} & \frac{4}{2} & \frac{4}{2} \end{pmatrix} = \frac{1}{17}, \quad \begin{pmatrix} \frac{4}{2} & \frac{8}{2} & \frac{4}{2} \\ \frac{4}{2} & \frac{6}{2} & \frac{4}{2} \end{pmatrix} = \frac{1}{10}.$$
we have $\mathcal{I} = V(0) \oplus V(2) \oplus V(4) \oplus V(8)$ and thus $\mathcal{J} = V(2)$.

**Definition 9.4.** Given three non-negative integers $a$, $b$ and $c$, we will say that the triple $(a, b, c)$ satisfies the triangle condition if $a$, $b$ and $c$ are the side lengths of a (possibly degenerate) triangle and $a + b + c$ is even.

From the Clebsch-Gordan formula for the decomposition of the tensor product of two $\mathfrak{sl}(2)$-modules, we know that $V(k)$ is a submodule of $V(a) \otimes V(b)$ if and only if $|a - b| \leq k \leq a + b$ and $k \equiv a + b \mod 2$. It is clear that this is the same as saying that $(a, b, k)$ satisfies the triangle condition.

In terms of Theorem 9.1, it is clear that a necessary condition for $V(k)$ to appear in the image $\mathcal{I}$ of (9.2) is that the four triples

$$(9.6) \quad (k, p, q), \quad (p, a, b), \quad (q, b, c), \quad (k, a, c)$$

satisfy the triangle condition. These four triangle conditions can be depicted by the following labeled tetrahedron:

![tetrahedron diagram]

We point out, however, that the above four triangle conditions do not imply the existence of an euclidean metric tetrahedron with side lengths $a$, $b$, $c$, $p$, $q$ and $k$ (as indicated in the above picture); it is known that an additional condition on the Cayley-Menger determinant is required for that (see, [Bl], [GV] or [WD]).

Note also that the four triangle conditions (9.6) are not sufficient for $V(k)$ to appear in the image of (9.2), as shown in Example 9.3. According to Theorem 9.1,

$V(k)$ will not appear in the image of (9.2) if and only if $\left\{ \frac{a - 1}{2} \frac{a}{2} \frac{b}{2} \frac{c}{2} \right\} = 0$. We recall that finding the non-trivial zeros of the $6j$-symbol is a well studied and very difficult problem (see, for instance, [L], [R] and the references therein, or more recently [ZR]). In particular it is known that

$$\left\{ \frac{a - 1}{2} \frac{a}{2} \frac{b}{2} \frac{c}{2} \right\} = 0 \quad \text{and} \quad \left\{ \frac{j}{2} \frac{j-2}{2} \frac{j-6}{2} \frac{j}{2} \right\} = 0$$

for all integers $a \geq 2$ and $j \geq 4$ (see equations (4.14) and (4.15) in [L]).

We can now state the following theorem which is important for the classification of the uniserial modules of the Lie algebra $\mathfrak{sl}(2) \ltimes V(m)$.

**Theorem 9.5.** Let $a, b, c$ and $m$ be non-negative integers such that $(a, b, m)$ and $(b, c, m)$ satisfy the triangle condition, and let $\mathcal{J}$ be the image of $\Lambda^2(V(m))$ under (9.2) when $p = m = q$.

Let $f : V(m) \rightarrow \text{Hom}(V(b), V(a))$ and $g : V(m) \rightarrow \text{Hom}(V(c), V(b))$ be non-zero $\mathfrak{sl}(2)$-homomorphisms and consider the Lie subalgebra, say $\mathcal{L}$, of $\mathfrak{gl}(V)$, where $V = V(a) \oplus V(b) \oplus V(c)$, generated by $f(r) + g(r)$, $r \in V(m)$.

Then the following conditions are equivalent:

1. $\mathcal{L}$ is abelian.
2. $\mathcal{J} = 0$. 
(3) \( \left\{ \frac{m}{2}, \frac{b}{2}, \frac{m}{2} \right\} \) = 0 for all non-negative integers \( k \) satisfying \( k \equiv 2m - 2 \mod 4 \).

(4) Up to a swap of \( a \) and \( c \) we have: \( c = 0 \), \( b = m \), \( a \equiv 2m \mod 4 \) and \( a \leq 2m \); or \( b = c + m \) and \( a = c + 2m \).

(5) \( \Lambda^2(V(m)) \) is disjoint from \( \text{Hom}(V(c), V(a)) \).

(6) There is a uniserial \( \mathfrak{sl}(2) \ltimes V(m) \)-module with socle factors \( V(a), V(b), V(c) \).

Proof. (1)\( \Rightarrow \) (2) We already noted that \( \mathcal{L} = \mathcal{D}_{\text{diag}} \oplus \mathcal{J} \) is generated by \( \mathcal{D}_{\text{diag}} \), so \( \mathcal{J} = 0 \) if \( \mathcal{L} \) is abelian.

(2)\( \Rightarrow \) (3) If \( 0 \leq k \leq 2m \) and \( (a, c, k) \) satisfies the triangle condition then (2) and Corollary 9.2 imply that \( \left\{ \frac{b}{2}, \frac{a}{2}, \frac{m}{2} \right\} = 0 \). For all other \( k \equiv 2m - 2 \mod 4 \), this \( \mathcal{J} \)-symbol is zero by definition.

(3)\( \Rightarrow \) (4) As the \( \mathcal{J} \)-symbol is invariant under the permutation of any two columns ([CFS]), we may assume that \( a \geq c \).

Since \((a, b, m)\) and \((b, c, m)\) satisfy the triangle condition it follows that \( m \geq a - b \), \( m \geq b - c \) and thus \( 2m \geq a - c \).

We claim that

\( a - c \equiv 2m \mod 4 \).

Otherwise, \( k = a - c \equiv 2m - 2 \mod 4 \) and, since \( (a, c, k) \) is a degenerate triangle, it follows from Property (iii) in the list of properties of the \( \mathcal{J} \)-symbol in §12 that \( \left\{ \frac{m}{2}, \frac{b}{2}, \frac{m}{2} \right\} \neq 0 \) which contradicts (3).

If \( a - c = 2m \), the triangle conditions on \((a, b, m)\) and \((b, c, m)\) imply that \( b = m + c \) and \( a = c + 2m \). Otherwise, \( a - c < 2m \) and we will prove that \( c = 0 \) and thus \( b = m \), \( a = 2m \mod 4 \) and \( a \leq 2m \). Since \( a - c < 2m \), it follows from (9.7) that \( a - c \leq 2m - 4 \). If \( c \geq 1 \) then

\( (h, m, m), (h, a, c), (m, a, b), (m, b, c) \)

satisfy the triangle condition for \( h = a - c \) and \( h = a - c + 2 \) and we are in a position to apply Lemma 12.1 to

\[ j_1 = \frac{a - c}{2}, \quad j_2 = j_3 = \frac{m}{2}, \quad j_4 = \frac{b}{2}, \quad j_5 = \frac{c}{2}, \quad j_6 = \frac{a}{2}. \]

We obtain that \( \left\{ \frac{m}{2}, \frac{b}{2}, \frac{m}{2} \right\} = \left\{ \frac{b}{2}, \frac{c}{2}, \frac{a}{2} \right\} \neq 0 \) for either \( k = a - c + 2 \) or \( k = a - c + 6 \).

Since both \( k \) satisfy \( k \equiv 2m - 2 \mod 4 \), this contradicts (3).

(4)\( \Rightarrow \) (5) Immediate from the decompositions of \( \Lambda^2(V(m)) \) and \( V(a) \otimes V(c) \).

(5)\( \Rightarrow \) (1) This is obvious.

(6)\( \iff \) (1) If \( m \geq 1 \) this follows from Proposition 3.1. If \( m = 0 \) then \( a = b = c \) and it is easy to see that conditions (1) and (6) are both true.

\[ \square \]

10. Classification of uniserial \( \mathfrak{sl}(2) \ltimes V(m) \)-modules

We are finally in a position to show that, except for a few exceptions of lengths 2, 3 and 4, every uniserial \( \mathfrak{sl}(2) \ltimes V(m) \)-module is isomorphic to \( Z(\ell, b) \) or its dual.
Theorem 10.1. Let \( g = \mathfrak{sl}(2) \ltimes V(m) \), where \( m \geq 1 \). Then, up to a reversing of
the order, the following are the only admissible sequences for \( g \):

Length 1. \( V(a) \).

Length 2. \( V(a), V(b) \), where \( a + b \equiv m \mod 2 \) and \( 0 \leq b - a \leq m \leq a + b \).

Length 3. \( V(a), V(a + m), V(a + 2m) \); or \( V(0), V(m), V(c) \), where \( c \equiv 2m \mod 4 \) and \( c \leq 2m \).

Length 4. \( V(a), V(a + m), V(a + 2m), V(a + 3m) \); or \( V(0), V(m), V(m), V(0) \), where \( m \equiv 0 \mod 4 \).

Length \( \geq 5 \). \( V(a), V(a + m), \ldots, V(a + sm) \), where \( s \geq 4 \).

Moreover, each of these sequences arises from only one isomorphism class of uniserial \( g \)-modules, except for the sequence \( V(0), V(m), V(m), V(0) \), \( m \equiv 0 \mod 4 \). The isomorphism classes of uniserial \( g \)-modules associated to this sequence are parametrized by the complex numbers, as described in Note 5.5.

Proof. That the stated sequences are admissible is proven in [4] and [8] while the uniqueness, up to isomorphism, of the uniserial modules arising from such sequences follows from Proposition 8.2, except for the sequence \( V(0), V(m), V(m), V(0) \), where \( m \equiv 0 \mod 4 \), which is handled in Note 8.3.

It remains to prove that, up to a reversing of the order, the only admissible sequences are as indicated. Those of length 2 are considered in Note 1.5. Let \( V(a), V(b), V(c) \) be an admissible sequence of length 3, which is condition (1) of Theorem 9.5. This is equivalent to condition (4) of Theorem 9.3, so all admissible sequences of length 3 are as stated. Next let \( V(a_1), \ldots, V(a_n) \) be an admissible sequence of length \( n \geq 4 \). By Lemma 2.6 we may assume that \( a_1 \leq a_n \). Since any submodule or quotient of a uniserial module is also uniserial, we see that \( V(a_{i-1}), V(a_i), V(a_{i+1}) \) is also admissible for any \( 1 < i < n \). Applying this fact in combination with our determination of all admissible sequences of length 3, we deduce the following: \( a_i \neq 0 \) for \( 1 < i < n \); either \( a_1, \ldots, a_n \) is strictly increasing, in which case it does so by a fixed increment of \( m \), or else \( n = 4 \) and \( a_1 = 0, a_2 = m, a_3 = m, a_4 = 0 \).

11. A new interpretation of the Wigner-Racah 6j-symbol and the calculation of the scalar \( \lambda \)

Let \( k \) be a non-negative integer and let \( e_k \) denote a highest weight vector of the irreducible \( \mathfrak{sl}(2) \)-module \( V(k) \) of highest weight \( k \). If \( \{ H, E, F \} \) is the standard basis of \( \mathfrak{sl}(2) \) then \( \mathcal{B}_k = \{ F^r e_k : r = 0, \ldots, k \} \) is a basis of \( V(k) \) and

\[
H F^r e_k = (k - 2r) F^r e_k, \quad E F^r e_k = r(k + 1 - r) F^{r-1} e_k.
\]

We know that \( V(k) \) is isomorphic to the dual \( \mathfrak{sl}(2) \)-module \( V(k)^* \). In fact, if \( \{(F^r e_k)^* : r = 0, \ldots, k\} \) is the dual basis of \( \mathcal{B}_k \), then \( (F^k e_k)^* \) is the highest weight vector of \( V(k)^* \) and the map \( j_k : V(k) \to V(k)^* \), given by

\[
F^r e_k \mapsto (-1)^r (F^{k-r} e_k)^*
\]
is, up to a scalar, the unique \( \mathfrak{sl}(2) \)-module isomorphism between \( V(k) \) and \( V(k)^* \). Suppose \( (a, b, k) \) satisfies the triangle condition. In this case \( V(k) \) occurs with
multiplicity one in \( V(a) \otimes V(b) \) and if

\[
x_{k}^{a,b} = \frac{a + b - k}{2}
\]

then

\[
(11.2) \quad v_{k}^{a,b} = \sum_{r=0}^{z} (-1)^r \left( \frac{x_{r}}{x_{r}^{a,b} + k} \right) F^{r} e_{a} \otimes F^{x_{r}^{a,b} - r} e_{b}
\]
is, up to a scalar, the unique highest weight vector of weight \( k \) in \( V(a) \otimes V(b) \). We denote by

\[
t_{k}^{a,b} \in \text{Hom}_{s(2)}(V(k), V(a) \otimes V(b))
\]
the unique \( s(2) \)-module homomorphism sending \( e_{k} \) to \( v_{k}^{a,b} \).

Let us assume that the four triples

\[
(k, p, q), \quad (p, a, b), \quad (q, b, c), \quad (k, a, c)
\]
satisfy the triangle condition. Let \( \phi \) be the map defined by \([11.3]\), that is

\[
\phi = \mu \circ (t_{p}^{a,b} \otimes t_{q}^{b,c}) \circ t_{k}^{p,q},
\]
where \( \mu : V(a) \otimes V(b) \otimes V(c) \otimes V(d) \) is defined in \([11.3]\) by means of the isomorphism \( j = j_{b} : V(b) \rightarrow V(b)^{*} \) given in \([11.1]\). Explicitly, we have

\[
\mu(x \otimes F^{r} e_{b} \otimes F^{s} e_{b} \otimes z) = (-1)^{r} \delta_{r,s} x \otimes z, \quad x \in V(a), z \in V(c), 0 \leq r, s \leq b.
\]
As noted at the end of the proof of Theorem \([9.1]\) we have \( \phi = \lambda v_{k}^{a,c} \). Thus

\[
\phi(e_{k}) = \lambda v_{k}^{a,c},
\]
where \( v_{k}^{a,c} \) is defined in \([11.2]\). We will now compute \( \lambda \). First we have

\[
t_{k}^{p,q} e_{k} = \sum_{r_{1}=0}^{z} (-1)^{r_{1}} \left( \frac{x_{r_{1}}^{p,q}}{x_{r_{1}}^{p,q} + k} \right) F^{r_{1}} e_{p} \otimes F^{x_{r_{1}}^{p,q} - r_{1}} e_{q}.
\]

\[
v_{p}^{a,b} e_{p} = t_{p}^{a,b} e_{p} = \sum_{r_{2}=0}^{z} (-1)^{r_{2}} \left( \frac{x_{r_{2}}^{a,b}}{x_{r_{2}}^{a,b} + p} \right) F^{r_{2}} e_{a} \otimes F^{x_{r_{2}}^{a,b} - r_{2}} e_{b},
\]

\[
v_{q}^{b,c} e_{q} = t_{q}^{b,c} e_{q} = \sum_{r_{3}=0}^{z} (-1)^{r_{3}} \left( \frac{x_{r_{3}}^{b,c}}{x_{r_{3}}^{b,c} + q} \right) F^{r_{3}} e_{b} \otimes F^{x_{r_{3}}^{b,c} - r_{3}} e_{c}.
\]
Thus

\[
\phi(e_{k}) = \mu \sum_{r_{1}=0}^{z} (-1)^{r_{1}} \left( \frac{x_{r_{1}}^{p,q}}{x_{r_{1}}^{p,q} + k} \right) F^{r_{1}} \left( \sum_{r_{2}=0}^{z} (-1)^{r_{2}} \left( \frac{x_{r_{2}}^{a,b}}{x_{r_{2}}^{a,b} + p} \right) F^{r_{2}} e_{a} \otimes F^{x_{r_{2}}^{a,b} - r_{2}} e_{b} \right)
\]

\[
\otimes F^{x_{r_{1}}^{p,q} - r_{1}} \left( \sum_{r_{3}=0}^{z} (-1)^{r_{3}} \left( \frac{x_{r_{3}}^{b,c}}{x_{r_{3}}^{b,c} + q} \right) F^{r_{3}} e_{b} \otimes F^{x_{r_{3}}^{b,c} - r_{3}} e_{c} \right).
\]
Applying Leibniz rule to compute the action of $F^{r_1}$ and $F^{x_k^{a,b} - r_1}$ on tensors we get

$$
\phi(e_k) = \sum (-1)^{r_1+r_2+r_3} \frac{(r_1)!}{(x_k^{a,b} - r_1)! (x_k^b)^{r_1}} \frac{(r_2)!}{(x_k^{b,c} - r_2)! (x_k^c)^{r_2}} \frac{(r_3)!}{(x_k^{c,d} - r_3)! (x_k^d)^{r_3}} \\
\times F^{r_2+r_3} e_a \otimes \mu \left(F^{x_k^{a,b} - r_2 + r_1} e_b \otimes F^{r_3+r_4} e_d \right) \otimes F^{x_k^{b,c} - r_3 + x_k^{c,d} - r_5} e_c.
$$

Here the sum runs over all $(r_1, r_2, r_3, r_4, r_5)$ allowed by the binomial coefficients in the numerator. In what follows, all the sums will run over the indicated indices with the restriction that the factorial numbers involved are non-negative. In order to compute $\mu$ we need to consider the case when

$$x_k^{a,b} - r_2 + r_1 - r_4 + r_3 + r_5 = b$$

and thus

$$\phi(e_k) = (-1)^{x_k^{a,b}} \sum_{r_1, r_2, r_3, r_4} (-1)^{r_4+r_3} \frac{(r_1)!}{(x_k^{a,b} - r_1)! (x_k^b)^{r_1}} \frac{(r_2)!}{(x_k^{b,c} - r_2)! (x_k^c)^{r_2}} \frac{(r_3)!}{(x_k^{c,d} - r_3)! (x_k^d)^{r_3}} \\
\times F^{r_2+r_3} e_a \otimes \mu \left(F^{x_k^{a,b} - r_2 + r_1} e_b \otimes F^{r_3+r_4} e_d \right) \otimes F^{x_k^{b,c} - r_3 + x_k^{c,d} - r_5} e_c.$$

We know that $\phi(e_k) = \lambda v_k^{a,c}$. Comparing coefficients at $e_a \otimes F^{x_k^{a,c}} e_c$ yields

$$(-1)^{x_k^{a,b}} \sum_{r_1, r_2, r_3} (-1)^{r_3} \frac{(r_1)!}{(x_k^{a,b} - r_1)! (x_k^b)^{r_1}} \frac{(r_2)!}{(x_k^{b,c} - r_2)! (x_k^c)^{r_2}} \frac{(r_3)!}{(x_k^{c,d} - r_3)! (x_k^d)^{r_3}} = \frac{\lambda}{(x_k^{a,b} + k)}.$$ 

Replacing $x_k^{a,b} - r_3$ and using $\binom{s}{t} = \binom{s-1}{t-1}$ at appropriate places gives

$$(-1)^{x_k^{a,b} + x_k^{b,c}} \sum_{r_1, r_3} (-1)^{r_3} \frac{(r_1)!}{(x_k^{a,b} - r_1)! (x_k^b)^{r_1}} \frac{(r_3)!}{(x_k^{b,c} - r_3)! (x_k^c)^{r_3}} = \frac{\lambda (x_k^{a,b} + p)}{(x_k^{a,b} + k + p)}.$$ 

We next concentrate on the inner part of the above double sum.

**Lemma 11.1.**

$$\sum_{r_1} \frac{(x_k^{a,b} - r_1)!}{(x_k^{a,b} + k)!} \frac{(x_k^{b,c} - r_1)!}{(x_k^{b,c} + k)!} = \frac{x_k^{p,q} + k + 1}{x_k^{p,q} + k + p} \frac{(x_k^{a,b} - r_1)!}{(x_k^{a,b} + k)!}.$$ 

**Proof.** Note that if $x$, $y$, $z$ are non-negative integers and $y \geq z$ then

$$\sum_{r=0}^{z} \binom{x+r}{r} \binom{y-r}{z-r} = \binom{x+y+1}{z}.$$ 

This is easily seen by induction on $y$ by repeatedly using $\binom{s}{t} = \binom{s-1}{t-1} + \binom{s-1}{t}$. Using \[11.3\] in the equivalent form

$$\sum_{r=0}^{z} \frac{(x+r)!}{r!} \frac{(y-r)!}{(z-r)!} = \frac{x! (y-z)!}{z! (x+y+z+1)!}.$$
Theorem 11.2.

as well as the meaning of $\Delta$ and $\square$

where

$$\sum_{r_1} \left( \frac{x_k^{p,q-r_1}}{x_k^{p,q-r_1}} \right) \left( x_k^{p,q-r_1} \right)^{-2} = \sum_{r_1} \frac{(x_k^{p,q})!(x_k^{p,q} + k - p + r_1)!(p - r_1)!}{(x_k^{p,q})!(x_k^{p,q} + k + r_1)!(p - r_1)!}$$

$$= \sum_{r_1} \frac{(x_k^{p,q})!(x_k^{p,q} + k - p + r_1)!(p - r_1)!}{(x_k^{p,q})!(x_k^{p,q} + k + r_1)!(p - r_1)!} = \sum_{r_1} \frac{(x_k^{p,q})!(x_k^{p,q} + k - p + r_1)!(p - r_1)!}{(x_k^{p,q})!(x_k^{p,q} + k + r_1)!(p - r_1)!} = \sum_{r_1} \frac{(x_k^{p,q})!(x_k^{p,q} + k - p + r_1)!(p - r_1)!}{(x_k^{p,q})!(x_k^{p,q} + k + r_1)!(p - r_1)!}$$

$$= \sum_{r_1} \frac{(x_k^{p,q})!(x_k^{p,q} + k - p + r_1)!(p - r_1)!}{(x_k^{p,q})!(x_k^{p,q} + k + r_1)!(p - r_1)!} = \sum_{r_1} \frac{(x_k^{p,q})!(x_k^{p,q} + k - p + r_1)!(p - r_1)!}{(x_k^{p,q})!(x_k^{p,q} + k + r_1)!(p - r_1)!}$$

as we wanted to prove. \(\square\)

From the above lemma we obtain

$$(-1)^{x_p^{a,b} + x_q^{b,c}} \frac{x_k^{p,q} + k + 1}{x_k^{p,q} + k - 1 - p} \sum_{r_3} (-1)^{r_3} \frac{(x_k^{p,q})!(x_k^{p,q} + k - r_3)!}{(x_k^{p,q} + k + 1 - r_3)!} = \frac{\lambda}{(x_k^{p,q} + 1)}$$

We refer the reader to [12] for the definition and basic properties of the $6j$-symbol as well as the meaning of $\Delta$ and $R$ as used below.

Theorem 11.2.

$$\lambda = C \left\{ \begin{array}{ccc} a & k & p \\ \frac{a}{p} & t & \frac{a}{t} \end{array} \right\},$$

where

$$C = \frac{(-1)^{x_k^{a,c} + x_k^{b,c}}(p+q+k+2)(a+b+p+2)(b+c+q+2)\Delta(\frac{p}{k},\frac{q}{k}) \Delta(\frac{q}{k},\frac{p}{k}) \Delta(\frac{p}{k},\frac{p}{k})}{4(a+c+k+2)\Delta(\frac{p}{k},\frac{p}{k})}$$

Proof. From the identity above the theorem we have that

$$\lambda = (-1)^{x_k^{a,c} + x_k^{b,c}} \frac{x_k^{p,q} + k + 1}{x_k^{p,q} + k - 1 + p} \sum_{r_3} (-1)^{r_3} \frac{(x_k^{p,q})!(x_k^{p,q} + k - r_3)!}{(x_k^{p,q} + k + 1 - r_3)!} \frac{(x_k^{p,q})!(x_k^{p,q} + k - r_3)!}{(x_k^{p,q} + k + 1 - r_3)!}$$

If we replace $r_3$ by $x_k^{a,c} - t$, the above sum is

$$\sum_{r_3} (-1)^{r_3} \frac{(x_k^{p,q})!(x_k^{p,q} + k - r_3)!}{(x_k^{p,q} + k + 1 - r_3)!} \frac{(x_k^{p,q})!(x_k^{p,q} + k - r_3)!}{(x_k^{p,q} + k + 1 - r_3)!}$$

$$= (-1)^{x_k^{a,c} + x_k^{b,c}} \frac{x_k^{p,q}!x_k^{b,c}!(x_k^{p,q} + k + 1 - p)!}{(x_k^{p,q} + k + 1 - p)!}$$

$$\times \sum_{r_3} \frac{(x_k^{p,q})!(x_k^{p,q} + k - r_3)!}{(x_k^{p,q} + k + 1 - r_3)!} \frac{(x_k^{p,q})!(x_k^{p,q} + k - r_3)!}{(x_k^{p,q} + k + 1 - r_3)!} \frac{(x_k^{p,q})!(x_k^{p,q} + k - r_3)!}{(x_k^{p,q} + k + 1 - r_3)!}$$

$$= \frac{\lambda}{(x_k^{p,q} + 1)}$$

$$\times \sum_{r_3} \frac{(x_k^{p,q})!(x_k^{p,q} + k - r_3)!}{(x_k^{p,q} + k + 1 - r_3)!} \frac{(x_k^{p,q})!(x_k^{p,q} + k - r_3)!}{(x_k^{p,q} + k + 1 - r_3)!} \frac{(x_k^{p,q})!(x_k^{p,q} + k - r_3)!}{(x_k^{p,q} + k + 1 - r_3)!}.$$
Define \( j_1, j_2, j_3, j_4, j_5, j_6 \) by

\[
ap = 2j_1, \quad \quad p = 2j_2, \quad \quad b = 2j_3, \quad \quad q = 2j_4, \quad \quad c = 2j_5, \quad \quad k = 2j_6.
\]

Then above sum together with (12.2) and \( a + c \equiv k \mod 2 \) yield

\[
\sum_{r_3} (-1)^{r_3} \left( \frac{x_k^{r_3}}{x_k^{r_3} + 1 - r_3} \right) \left( \frac{x_k^{r_3}}{x_k^{r_3} + 1 - p} \right) \left( \frac{x_k^{r_3}}{x_k^{r_3} + 1 - r_3} \right) = (-1)^{j_1 + j_6 - j_6} \left( \frac{(j_1 + j_4 + j_5)!(j_3 - j_4 + j_5)!(j_6 - j_4 + j_6 + 1)!}{(j_3 + j_4 + j_5)!} \right)
\]

\[
\times \sum_{i} (-1)^{i} \left( \frac{(j_2 + j_4 + j_5 + i)!(j_1 + j_4 + j_5 + i)!}{(j_3 + j_4 + j_5 + i)!} \right) \left( \frac{R_{i,4}^1 R_{i,6}^4}{R_{i,3}^5 R_{i,5}^3} \right) \binom{j_1, j_2, j_3}{j_4, j_5, j_6}.
\]

Therefore

\[
\lambda = (-1)^{j_1 + 2j_3 + j_5 + j_6} \left( \frac{(j_2 + j_4 + j_5 + 1)!(j_1 + j_4 + j_5 + 1)!}{(j_3 + j_4 + j_5 + 1)!} \right) \left( \frac{R_{i,4}^1 R_{i,6}^4}{R_{i,3}^5 R_{i,5}^3} \right) \binom{j_1, j_2, j_3}{j_4, j_5, j_6}.
\]

The theorem now follows from the symmetry \( \binom{j_1, j_2, j_3}{j_4, j_5, j_6} = \binom{j_4, j_6, j_2}{j_1, j_3, j_5} \). \( \square \)

12. Appendix. The Clebsch-Gordan coefficients and the 6j-symbol

In this appendix we recall the basic facts about the 6j-symbol we needed in this paper. We will mainly follow [VMK].

Let \( 2j_1, 2j_2 \) and \( 2j_3 \) be three non-negative integers and define (see [VMK] §8.2, eq.(1))

\[
\Delta(j_1, j_2, j_3) = \sqrt{\frac{(j_1 + j_2 - j_3)! (j_1 - j_2 + j_3)! (-j_1 + j_2 + j_3)!}{(j_1 + j_2 + j_3 + 1)!}}
\]

if \( (2j_1, 2j_2, 2j_3) \) satisfies the triangle condition (see Definition 4.4); otherwise set \( \Delta(j_1, j_2, j_3) = 0 \).

If \( 2m_1, 2m_2 \) and \( 2m_3 \) are three integers such that \( |m_i| \leq j_i \) for \( i = 1, 2, 3 \), we recall that the corresponding Clebsch-Gordan coefficient is zero, if \( m_1 + m_2 \neq m_3 \), and otherwise is (see [VMK] §8.2, eq.(3))

\[
C^{j_3, m_3}_{j_1, m_1; j_2, m_2} = \Delta(j_1, j_2, j_3) \sqrt{(2j_3 + 1)} \times \sqrt{(j_1 + m_1)!(j_1 - m_1)!(j_2 + m_2)!(j_2 - m_2)!(j_3 + m_3)!(j_3 - m_3)!}
\]

\[
\times \sum_r (-1)^r \frac{(j_1 + j_2 - j_3 - r)!(j_3 - m_1 - r)!(j_3 - m_2 - r)!}{(j_2 + m_1 + r)!(j_3 - j_1 - m_2 - r)!}
\]

where the sum runs over all \( r \) such that all the numbers under the factorial symbol are non-negative.
Let \( a = 2j_1, b = 2j_2 \) and \( k = 2j \). If we define (see [VMK, §3.1.1])
\[
\mathcal{M}_{j,m} = \sqrt{\frac{(j+m)!}{(j-m)!}} F_{j-m}^j e_k
\]
then (see [VMK, §8.2, eq.(10)])
\[
\{ \mathcal{M}_{j1} \otimes \mathcal{M}_{j2} \}_{j,m} := \sum_{m_1 + m_2 = m} C^{j,m}_{j_1,j_2,m_1,j_2,m_2} \mathcal{M}_{j1,m_1} \otimes \mathcal{M}_{j2,m_2} : m = -j, -j + 1, \ldots, j
\]
is a basis of the unique \( \mathfrak{sl}(2) \)-submodule of \( V(a) \otimes V(b) \) isomorphic to \( V(k) \), and in fact, the map
\[
V(k) \to V(a) \otimes V(b)
\]
\[
\mathcal{M}_{j,m} \mapsto \{ \mathcal{M}_{j1} \otimes \mathcal{M}_{j2} \}_{j,m}
\]
is exactly \( \Delta(a,b) \sqrt{\frac{2j+1}{j_1+j_2+j+1}} \frac{a,b}{j_1,j_2,j} \).

The (classical) Racah-Wigner 6j-symbol is a real number \( \{ j_1 j_2 j_3 \} \) associated to six non-negative half-integer numbers \( j_1, j_2, j_3, j_4, j_5 \) and \( j_6 \). The 6j-symbol plays a central role in angular momentum theory since they describe the recoupling of three angular momenta. Some classical references to them, other than [VMK], are [CFS], [Ed], [RBMW], etc. Let us recall its definition in terms of the representation theory of \( \mathfrak{sl}(2) \).

If one of following four triples
\[
(2j_1,2j_2,2j_3), \ (2j_1,2j_5,2j_6), \ (2j_4,2j_2,2j_6), \ (2j_4,2j_5,2j_3)
\]
does not satisfy the triangle condition then \( \{ j_1 j_2 j_3 \} \) is zero by definition. If all the above four triples do satisfy the triangle condition, which may be depicted by the following tetrahedron,

\[
\begin{vmatrix}
2j_4 \\
2j_1 \\
2j_5 \\
2j_6
\end{vmatrix}
\]

let
\[
\begin{align*}
a &= 2j_1, & b &= 2j_2, & p &= 2j_3, \\
c &= 2j_4, & k &= 2j_5, & q &= 2j_6.
\end{align*}
\]

The 6j-symbols \( \{ j_1 j_2 j_3 \} \) are the coefficients needed to express the \( \mathfrak{sl}(2) \)-module homomorphism
\[
(\iota_p^{a,b} \otimes 1) \circ \iota_k^{b,c} : V(k) \to V(p) \otimes V(c) \to V(a) \otimes V(b) \otimes V(c)
\]
as a linear combination of the \( \mathfrak{sl}(2) \)-module homomorphisms
\[
(1 \otimes \iota_q^{b,c}) \circ \iota_k^{a,q} : V(k) \to V(a) \otimes V(q) \to V(a) \otimes V(b) \otimes V(c)
\]
as \( q \) varies. More precisely, the following sets
\[
\{ (l_p^{a,b} \otimes 1) \circ i_k^{p,c} : p \in \mathbb{Z} \text{ and } (a,b,c) \neq 0 \}
\]
and
\[
\{ (1 \otimes l_q^{b,c}) \circ i_k^{q,a} : q \in \mathbb{Z} \text{ and } (a,b,c) \neq 0 \}
\]
are two different bases of \( \text{Hom}_{\mathfrak{su}(2)}(V(k), V(a) \otimes V(b) \otimes V(c)) \) and the \( 6j \)-symbol describe the transition matrix between these two bases, that is \( \{ j_1 j_2 j_3 \}
\]
which is defined by the following identity (see \[12.1\])
\[
\frac{(-1)^{j_1-j_2-j_3+j_5+j_6}}{\sqrt{2j_3+1}} C_{j_1,j_2,j_3}^{j_1,m_3} j_1; j_2,m_2 C_{j_3,j_4,j_5}^{j_5,m_5} j_3,m_3; j_4,m_4
\]
\[
= \sum_q (-1)^{2j_6} \sqrt{2j_6+1} \begin{pmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{pmatrix} C_{j_2,m_2; j_4,m_4}^{j_6,m_6} C_{j_3,m_3; j_6,m_6}^{j_5,m_5}
\]
for all \( m_i, i = 1, \ldots, 6 \), such that \( |m_i| \leq j_i \) and \( m_1 + m_2 = m_3, m_3 + m_4 = m_5, m_2 + m_4 = m_6 \) and \( m_1 + m_6 = m_5 \). This identity is derived from \[VMK\ §8.7.5, eq.(36)] and the symmetry properties of the Clebsch-Gordan coefficients \[VMK\ §8.4.2, eq.(5)].

If we set \( \Delta_{y,z} = \Delta(j_x,j_y,j_z) \), then the \( 6j \)-symbol can be explicitly expressed as (see \[VMK\ §9.2.1, eq.(1)\])
\[
\begin{pmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{pmatrix} = \Delta_{1,2,3} \Delta_{3,4,5} \Delta_{2,4,6} \Delta_{1,5,6}
\]
\[
\times \sum_t \frac{(-1)^t(t+1)!}{(t-\alpha_0)!(t-\alpha_1)!(t-\alpha_2)!(t-\alpha_3)!(\beta_1-t)!(\beta_2-t)!(\beta_3-t)!}
\]
where \( t \) runs from \( \max\{\alpha_0,\alpha_1,\alpha_2,\alpha_3\} \) to \( \min\{\beta_1,\beta_2,\beta_3\} \) and
\[
\begin{align*}
\alpha_0 &= j_1 + j_2 + j_3, \\
\alpha_1 &= j_1 + j_5 + j_6, \\
\alpha_2 &= j_4 + j_2 + j_6, \\
\alpha_3 &= j_4 + j_5 + j_3.
\end{align*}
\]
\[
\beta_1 = j_2 + j_3 + j_5 + j_6, \\
\beta_2 = j_1 + j_3 + j_4 + j_6, \\
\beta_3 = j_1 + j_2 + j_4 + j_5.
\]

Also, if
\[
R_x^{y,z} = \sqrt{\frac{(j_x^2+2j_x+1)!}{(j_y^2+2j_y+1)!}}
\]
then (see \[VMK\ §9.2.1, eq.(5)\])
\[
(12.2)
\]
\[
\begin{pmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{pmatrix} = \frac{(-1)^{j_1+j_2+j_4+j_5} R_{1,2,3}^{1,4,5} R_{4,5,6}^{3,4,6}}{R_{1,6}^{1,6} R_{2,6}^{2,6}}
\]
\[
\times \sum_t \frac{(-1)^t ((j_1+j_5+j_6+t)!)(j_2-j_4+j_6+t)!}{t! ((j_1+j_5-j_6-t)!)(j_2+j_4-j_6-t)! (2j_6+1+t)!}
\]
We need the following three properties of the \( 6j \)-symbol (see \[VMK\ §9.4.2\]):
Lemma 12.1. The 6j-symbol is invariant under the permutation of any two columns.

(ii) The 6j-symbol is invariant if upper and lower arguments are interchanged in any two columns.

(iii) If all the triples

\[
(2j_1, 2j_2, 2j_3), (2j_1, 2j_5, 2j_6), (2j_4, 2j_2, 2j_6), (2j_4, 2j_5, 2j_3)
\]

satisfy the triangle condition, but one of them is a degenerate triangle, then \( \{j_1, j_2, j_3\} \neq 0 \). Indeed, if one of the above triples corresponds to a degenerate triangle, then (i) and (ii) imply that we may assume \( j_6 = j_1 + j_5 \). Now it follows from (12.2) that

\[
\frac{R_{1,3}^4 R_{3,5}^4}{R_{1,4,6}^4 R_{2,6}^4} \left( -j_1 + j_5 + j_6 \right)! (j_2 - j_4 + j_6)! (j_1 + j_3 + j_4 - j_6)! (j_2 + j_4 - j_6)! (j_1 - j_3 - j_4 + j_6)! (2j_6 + 1)! \neq 0.
\]

The following lemma shows that, under certain additional conditions, other 6j-symbols are non-zero.

**Lemma 12.1.** Let \( j_1, j_2, j_3, j_4, j_5 \) and \( j_6 \) be non-negative half-integer such that \( j_6 = j_1 + j_5 \), \( j_2 = j_3 \) and all the triples

\[
(2h, 2j_2, 2j_3), (2h, 2j_5, 2j_6), (2j_4, 2j_2, 2j_6), (2j_4, 2j_5, 2j_3)
\]

satisfy the triangle condition for \( h = j_1 \) and \( h = j_1 + 1 \). If \( \{j_1 + 1, j_2, j_3\} \neq 0 \) then

\[
\{j_1 + 2, j_2, j_3\} \neq 0 \text{ and } \{j_1 + 3, j_2, j_3\} \neq 0.
\]

**Proof.** The Biedenharn-Elliott identity yields, in particular, the following three-term recurrence relation (see [SG] pag. 1963)

\[
(12.3)
\]

\[
i_1 E(i_1 + 1) \left\{ \frac{i_1 + 1}{i_4} \frac{i_2}{i_5} \frac{i_3}{i_6} \right\} + F(i_1) \left\{ \frac{i_1}{i_4} \frac{i_2}{i_5} \frac{i_3}{i_6} \right\} + (i_1 + 1) E(i_1) \left\{ \frac{i_1 - 1}{i_4} \frac{i_2}{i_5} \frac{i_3}{i_6} \right\} = 0
\]

where

\[
F(i_1) = (2i_1 + 1)(i_1(i_1 + 1)(-i_1(i_1 + 1) + i_2(i_2 + 1) + i_3(i_3 + 1))
\]

\[
+ i_5(i_5 + 1)(i_1(i_1 + 1) + i_2(i_2 + 1) - i_3(i_3 + 1))
\]

\[
+ i_6(i_6 + 1)(i_1(i_1 + 1) - i_2(i_2 + 1) + i_3(i_3 + 1))
\]

\[
- 2i_1(i_1 + 1)i_4(i_4 + 1)
\]

and

\[
E(i_1) = \sqrt{(i_1^2 - (i_2 - i_3)^2)((i_2 + i_3 + 1)^2 - i_7^2)(i_7^2 - (i_5 - i_6)^2)((i_5 + i_6 + 1)^2 - i_7^2)}.
\]

If we fix \( (i_2, i_3, i_4, i_5, i_6) = (j_2, j_3, j_4, j_5, j_6) \) we obtain

\[
E(i_1) = \sqrt{i_1^2((2j_2 + 1)^2 - i_7^2)((j_1 + 2j_5 + 1)^2 - i_7^2)}
\]

\[
F(i_1) = -(2i_1 + 1)i_1(i_1 + 1)
\]

\[
\times (i_1(i_1 + 1) - 2j_2(j_2 + 1) - j_5(j_5 + 1) - j_6(j_6 + 1) + 2j_4(j_4 + 1)),
\]
and we point out that the triangle conditions satisfied by $(2(j_1 + 1), 2j_5, 2j_6)$ and $(2(j_1 + 1), 2j_2, 2j_3)$ imply that
\[ E(j_1 + 1) \neq 0. \]

We also claim that $F(j_1 + 2) \neq 0$, and this will be proved later by considering separately the cases $j_1 = 0$ and $j_1 > 0$.

Assume that \( j_1 + j_2 + j_3 = 0 \). Since \( j_6 = j_5 + j_1 \) it follows that \( j_1 + j_2 + j_3 \) \( \neq 0 \) (see (iii) above) and \( j_1 + j_2 + j_3 = 0 \) (the triple \( (2(j_1 - 1), 2j_5, 2j_6) \) does not satisfy the triangle condition). Since $E(j_1 + 1) \neq 0$, it follows, from the recurrence relation \( (12.3) \) applied to $i_1 = j_1 + 1$, that \( j_1 + j_2 + j_3 \) \( \neq 0 \).

Now, accepting that $F(j_1 + 2) \neq 0$, the recurrence relation \( (12.3) \) applied to $i_1 = j_1 + 2$ implies that \( j_1 + j_2 + j_3 \) \( \neq 0 \) as we wanted to prove.

It remains to be proved that $F(j_1 + 2) \neq 0$. From the recurrence relation \( (12.3) \) applied to $i_1 = j_1$ we obtain that $F(j_1) = 0$ and, if $j_1 > 0$, it follows that

\[ j_1(j_1 + 1) - 2j_2(j_2 + 1) - j_3(j_3 + 1) - j_4(j_4 + 1) = 0, \]

which implies that $F(j_1 + 2) \neq 0$. If $j_1 = 0$ then

\[
0 = \begin{bmatrix} j_1 + 1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{bmatrix} = \begin{bmatrix} 1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{bmatrix} = (-1)^{j_1 + j_2 + j_3 + j_6} \frac{R_{3,3}^1 R_{4,5}^1}{R_{5,6}^1 R_{2,6}^1} 
\times \sum_{t=0}^{\infty} \frac{(-1)^t}{t!} \frac{(1 + 2j_6 + t)!}{(j_2 + j_4 + j_6 + t)! (j_2 + j_4 + j_6 + t)!} \frac{(1 + j_2 + j_4 - j_6 - t)!}{(j_2 + j_4 - j_6 - t)! (j_2 + j_4 - j_6 - t)!} \frac{(1 + j_2 + j_4 - j_6 - t)!}{(j_2 + j_4 - j_6 - t)! (j_2 + j_4 - j_6 - t)!}.
\]

and hence

\[
\frac{(-1 + 2j_6)!}{(j_2 + j_4 - j_6)!} \frac{(j_2 + j_4 - j_6)!}{(1 + j_2 + j_4 - j_6)!} \frac{(1 + j_2 + j_4 - j_6)!}{(2j_6 + 1)!} - \frac{(2j_6)!}{(j_2 + j_4 - j_6)!} \frac{(j_2 + j_4 - j_6 + 1)!}{(j_2 + j_4 + j_6)!} \frac{(j_2 + j_4 + j_6)!}{(2j_6 + 2)!} = 0,
\]

or

\[
(j_2 - j_4 + j_6)(1 + j_2 + j_4 - j_6)(j_6 + 1) - (j_2 - j_4 + j_6 + 1)(j_2 + j_4 - j_6)j_6 = 0,
\]

which implies $j_2(j_2 + 1) + j_6(j_6 + 1) - j_4(j_4 + 1) = 0$ and therefore

\[ F(2) = -30(6 - 2j_2(j_2 + 1) - j_5(j_5 + 1) - j_6(j_6 + 1) + 2j_4(j_4 + 1)) = -180. \]
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RACAH-WIGNER 6j-SYMBOL AND UNISERIAL \( \mathfrak{sl}(2) \times V(m) \)-MODULES

CIEM-CONICET, FAMAF-UNIVERSIDAD NACIONAL DE CÓRDOBA, CóRDOBA, ARGENTINA.

Current address: Math. Department, MIT, Cambridge, MA 02139-4307, USA.

E-mail address: caglierofamaf.unc.edu.ar

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF REGINA, CANADA

E-mail address: fernando.szechtman@gmail.com