Transport coefficients of Quark-Gluon Plasma in a Kinetic Theory approach

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Abstract. One of the main results of heavy ions collision at relativistic energy experiments is the very small shear viscosity to entropy density ratio of the Quark-Gluon Plasma, close to the conjectured lower bound $\eta/s = 1/4\pi$ for systems in the infinite coupling limit. Transport coefficients like shear viscosity are responsible of non-equilibrium properties of a system: Green-Kubo relations give us an exact expression to compute these coefficients. We computed shear viscosity numerically using Green-Kubo relation in the framework of Kinetic Theory solving the relativistic transport Boltzmann equation in a finite box with periodic boundary conditions. We investigated different cases of particles, for one component system (gluon matter), interacting via isotropic or anisotropic cross-section in the range of temperature of interest for HIC. Green-Kubo results are in agreement with Chapman-Enskog approximation while Relaxation Time approximation can underestimates the viscosity of a factor 2. Another transport coefficient of interest is the electric conductivity $\sigma_{el}$ which determines the response of the QGP to the electromagnetic fields present in the early stage of the collision. We study the $\sigma_{el}$ dependence on microscopic details of interaction and we find also in this case that Relaxation Time Approximation is a good approximation only for isotropic cross-section.

1. Introduction

High energy Heavy Ion Collisions (HIC) represent the only way to have experimental access in the high temperature and small baryon density region of the nuclear matter phase diagram. Relativistic Heavy Ion Collider (RHIC) at BNL and Large Hadron Collider (LHC) at CERN have produced a new state of matter, the Quark-Gluon plasma, in which quarks and gluons are not confined. The azimuthal asymmetry in momentum space [1], the elliptic flow $v_2$, has shown that the QGP has a very small shear viscosity to entropy density ratio $\eta/s$ [2], very close to the conjectured lower-bound limit $\eta/s = 1/4\pi$ [3] for a strongly interacting system in the limit of infinite coupling.

Calculation of transport coefficients is of great interest to characterize the non-equilibrium dynamics and dissipative effects of QGP created in HIC.

While shear and bulk viscosity are studied enough, only very recently electric conductivity $\sigma_{el}$ has captured large attention due to the strong electric field created in the collision region [4, 5]. Electric conductivity represents the response of the system to the applied electric field: this coefficients is responsible for the production of an electric current generated by the charged particles (quarks) in the early stage of the collision and for collective flow due to the presence of such fields [6].
The paper is organized as follows. In section 2, we discuss the method for calculating the shear viscosity $\eta$ using the Green-Kubo relation in the framework of transport theory solving the relativistic Boltzmann equation and, after a brief overview of analytical approximation like Relaxation Time Approximation (RTA) and Chapman-Enskog (CE) scheme [7, 8], we show the comparison between our results using the Green-Kubo relation and these schemes of approximation for anisotropic cross-section and for a more realistic case of anisotropic and energy-dependent cross-section. The knowledge of the exact analytical formula for shear viscosity leads to the development of a transport approach at fixed $\eta/s$ in order to have a comparison with experimental data in the same way of hydrodynamics models [9, 10, 11]. In section 3, we discuss how to compute the electric conductivity with experimental data in the same way of hydrodynamics models [9, 10, 11].

2. Green-Kubo relation for shear viscosity

Green-Kubo relation gives an exact expression to compute transport coefficient like shear and bulk viscosity, electric and thermal conductivity [12]. Each coefficient can be expressed in terms of correlation function of a flux in thermal equilibrium. For shear viscosity Green-Kubo relation assumes the following form [8]

$$\eta = \frac{1}{T} \int_{0}^{\infty} dt \int_{V} d^{3}x \langle \pi^{xy}(x, t)\pi^{xy}(0, 0) \rangle$$

(1)

where $T$ is the temperature, $\pi^{xy}$ is the $xy$ component of the energy-momentum tensor, while $\langle \ldots \rangle$ denotes the ensemble average. In this work we compute numerically the correlation function $\langle \pi^{xy}(t)\pi^{xy}(0) \rangle$ solving the ultrarelativistic Boltzmann transport equation for a particle system in a static box of volume $V$ with periodic boundary conditions at thermal equilibrium. The parton cascade developed [13] solves the Relativistic Boltzmann-Vlasov equation:

$$p^{\mu} \partial_{\mu} f(x, p) + M(x) \partial_{\mu} M(x) \partial_{p}^{\mu} f(x, p) = C(x, p)$$

(2)

where $f(x, p)$ is the distribution function for on-shell particles with mass $M(x)$, $C(x, p)$ is the collision integral that for a one-component system can be written as

$$C(x, p) = \int f_{2} \int f_{1} \int f_{2}' \int f_{1}' (f_{1}' f_{2}' - f_{1} f_{2}) |M_{1, 1' \rightarrow 1, 1}| \delta^{4}(p_{1} + p_{2} - p_{1}' p_{2}')$$

(3)

with $\int f = \int d^{3}p / (2\pi)^{3} 2E_{J}$, $M$ denotes the transition matrix for the elastic process linked to the differential cross-section $|M|^{2} = 16\pi s(s - 4M^{2})d\sigma / dt$.

In this work we solve numerically the collision integral to evaluate shear viscosity using Green-Kubo relation and compare this one with the results of the RTA and CE approximations. The dynamic of the system is simulated via Monte Carlo methods based on the stochastic interpretation of the transition [14] according to which collision probability of two particles in a cell of volume $\Delta V_{cell}$ and time-step $\Delta t_{step}$ is

$$P = v_{rel} \frac{\sigma_{tot}}{N_{test}} \Delta t / \Delta V_{cell}$$

(4)

where $\sigma_{tot}$ is the total cross-section and $v_{rel} = \sqrt{s(s - 4M^{2})/2E_{1}E_{2}}$ is the relative velocity of incoming particles. The number of test-particles $N_{test}$ is chosen to reproduce known quantities, such energy density $\epsilon$ and pressure $P$, and reduces statistical fluctuations.

The shear component of the energy-momentum tensor is given by

$$\pi^{xy}(x, t) = T^{xy}(x, t) = \int \frac{d^{3}p}{(2\pi)^{3}} \frac{p^{x}p^{y}}{E} f(x, p; t).$$

(5)
It is useful to write the time correlation function \( \langle \pi^{xy}(t)\pi^{xy}(0) \rangle \) as follows
\[
\langle \pi^{xy}(t)\pi^{xy}(0) \rangle = \left\langle \lim_{T_{\text{max}} \to \infty} \frac{1}{T_{\text{max}}} \int_{0}^{T_{\text{max}}} dt' \pi^{xy}(t + t')\pi^{xy}(t') \right\rangle
\]
where \( T_{\text{max}} \) is the maximum time chosen in simulation, \( N_{T_{\text{max}}} = T_{\text{max}}/\Delta t \) is the maximum number of time steps, \( i\Delta t = t \) while \( \langle \ldots \rangle \) denotes the average over numerically generated events.

In figure 1 is shown an example of time fluctuations of \( \pi^{xy}(t) \) for a system in thermal equilibrium: the average \( \langle \pi^{xy}(t) \rangle \) = 0 red dashed line. In figure 2 we show the correlation function \( \langle \pi^{xy}(t)\pi^{xy}(0) \rangle \) as a function of time for different values of total cross-section \( \sigma_{\text{tot}} \). In these calculations the particles are distributed uniformly in coordinate space while in momentum space according to thermal distribution function \( f(p) \sim e^{-E/T} \) (\( T = 0.4 \) GeV). In the particular case of figure 2 the particles are massless and interact via an isotropic energy-independent differential cross-section. As we can see, \( \langle \pi^{xy}(t)\pi^{xy}(0) \rangle \) is a decreasing exponential function \( \langle \pi^{xy}(t)\pi^{xy}(0) \rangle = \langle \pi^{xy}(0)\pi^{xy}(0) \rangle e^{-t/\tau} \) so Green-Kubo relation equation (1) simplifies to
\[
\eta = \frac{V}{T}\langle \pi^{xy}(0)\pi^{xy}(0) \rangle \tau
\]
where \( \tau \) is determined performing a fit on the temporal range in which the time correlation function is a decreasing exponential [13, 15, 16, 17].

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{figure1.png}
\caption{(Color online) \( xy \) component of energy momentum tensor as a function of time at thermal equilibrium. Average value of \( \langle \pi^{xy} \rangle = 0 \) at thermal equilibrium (red dashed line).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{figure2.png}
\caption{(Color online) Correlation function \( \langle \pi^{xy}(t)\pi^{xy}(0) \rangle \) as a function of time for different value of total cross-section \( \sigma_{\text{tot}} \). These results are for massless gluons interacting via isotropic cross-section; temperature is fixed to \( T = 0.4 \) GeV.}
\end{figure}

2.1. Shear viscosity for anisotropic scatterings

In this section we compare Green-Kubo results for shear viscosity \( \eta \), for a system of massless gluon interacting with an anisotropic cross-section, with analytical expressions already present in Literature: RTA and CE scheme.

We choose a typical pQCD-inspired cross-section with the infrared singularity regularized by a Debye thermal mass \( m_D \):
\[
\frac{d\sigma}{dt} = \frac{9\pi\alpha_s^2}{2} \frac{1}{(t - m_D^2)^2} \left( 1 + \frac{m_D^2}{s} \right)
\]
where $s, t$ are the Mandelstam variables. This kind of cross-section is typically used in parton cascade approaches [14, 18, 19, 20, 21, 22]. The total cross-section is $\sigma_{tot} = 9\pi\alpha_s^2/m_D^2$, which is energy and temperature independent. In equation (8) $m_D$ is a parameter that regulates the anisotropy of the scattering: we change it to modify the anisotropy but keeping constant the total cross-section.

The transport cross-section is:

$$\sigma_{tr}(s) = \int dt \frac{d\sigma}{dt} \sin^2 \Theta = \sigma_{tot} h(a) \quad (9)$$

where $h(a) = 4a(1 + a)[(2a + 1)\log(1 + 1/a) - 2]$. For $m_D \rightarrow \infty$ the function $h(a) \rightarrow 2/3$ and we recover the isotropic limit $\sigma_{tr} = (2/3)\sigma_{tot}$.

The difference between the two schemes of approximation, RTA and CE, resides in the different way of approximating the collision integral (3). The starting hypothesis of the RTA is that the collision integral can be approximated by $C[f] = - (f - f^{eq})/\tau$, where $\tau$ is the so called relaxation time, and one can obtain the following relation:

$$\eta_{RTA} = \frac{4}{15} \langle p \rangle \tau = \frac{\langle p \rangle}{15} \sigma_{tot}, \quad \eta_{CTA} = \frac{4}{15} \langle p \rangle \tau_{tr} = \frac{\langle p \rangle}{15} \frac{\langle h(a) \sigma_{tr} \rangle}{\sigma_{tot}} \quad (10)$$

where $\langle p \rangle$ is the average momentum, $\tau^{-1} = \langle \rho \sigma_{tr} \rangle$ is the relaxation time while $\tau_{tr}^{-1} = \langle \rho \sigma_{tr} \rangle$ is the transport relaxation time. The modified RTA $\eta_{RTA}$ differs from the simple one only in considering the transport cross-section $\sigma_{tr}$ instead of the total one.

On the other hand, the CE approach is based on a variational approach to the collision integral [23, 13] and one can obtain the following formula:

$$\eta_{CE} = \frac{4}{15} \rho \sigma_{tr} \tau = \frac{\langle p \rangle}{15} \frac{g(a)}{\sigma_{tot}}, \quad g(a) = \frac{1}{50} \int dy y^6 \left[ (y^2 + \frac{1}{3}) K_2(2y) - y K_2(2y) \right] \frac{a^2}{y^2} \quad (11)$$

with $K_n$ being the Bessel functions and $a = m_D/2T$. Note that equation (11) is valid only for massless particles (see [13] for the general formula). The main difference between the two schemes is that in CE the shear viscosity depends in a natural way on the transport cross-section.

In figure 3 we show shear viscosity $\eta$ as a function of Debye mass $m_D$, that represent the anisotropy of the cross-section, for three different values of temperature $T = 0.3, 0.4, 0.5$ GeV. Circles are Green-Kubo results and they are not affected by any kind of approximations. On the left panel lines represent the modified RTA equation (10) while on the right CE equation (11): as we can see, the modified RTA can underestimate $\eta$ of about a factor 2 for $m_D \leq 1.5$ GeV however it is in agreement with the isotropic limit; CE approximation is able to account for the correct value of $\eta$ for all the range of temperature and anisotropy explored. This means that the correct formula describing the dependence of shear viscosity on microscopic details is just the CE approximation.

2.2. Shear viscosity of a gluon plasma

In this section we investigate the more realistic case of shear viscosity of a gluon plasma. We consider massless gluons in thermal equilibrium interacting via two-body collisions corresponding to direct $u$ and $t$ channels:

$$\frac{d\sigma^{gg-gg}}{dq^2} = 9\pi\alpha_s^2 \frac{1}{(q^2 + m_D^2)^2}, \quad \alpha_s(T) = \frac{4\pi}{11 \log \left( \frac{2\pi T}{\Lambda} \right)^2}, \quad \Lambda = 200 \text{ MeV} \quad (12)$$

where $m_D = T\sqrt{4\pi\alpha_s}$ is Debye mass according to Hard-Thermal-Loop (HTL) calculations, and $\alpha_s$ is the pQCD running coupling constant.
The shear viscosity $\eta$ as a function of the Debye mass $m_D$ for three different values of temperature $T = 0.3$ GeV (blue solid line), $T = 0.4$ GeV (red dashed line) and $T = 0.5$ GeV (green dash-dotted line). Dotted lines represent the isotropic limit.

The total cross-section in this scheme is energy and temperature dependent:

$$\sigma_{tot} = \frac{9\pi\alpha_s^2}{m_D^2} \frac{s}{s + m_D^2}. \quad (13)$$

For this realistic case we have computed the $\eta/s$ as a function of temperature in the range $0.2 \leq T \leq 1.0$ GeV using Green-Kubo formula. Entropy density $s/T^4$ is constant as for free gas. In figure 4 Green-Kubo results (dark circles) are compared with the analytical expression given by: CE (orange dashed line), modified RTA (dark dash-dotted line), simple RTA (green dotted line) and next-to-leading-log order (NLL) (red line) [24]. Ref. [24] is a perturbative calculation which includes also the $s$ channel and all interference terms and considers a gluon propagator with both longitudinal and transverse components. We find again that CE approximation is in excellent agreement with Green-Kubo results while the modified RTA significantly underestimates $\eta/s$ by about a 20\% at $T \sim 0.2$ GeV. We find also a good agreement for $T \geq 0.4$ GeV between Green-Kubo, CE and NLL from [24]: this means that the $u$ and $t$ channels are the dominant contribution to shear viscosity $\eta$.

We note that the agreement between CE and Green-Kubo supplies a relatively simple analytical expression that can allow one to develop a kinetic transport theory at fixed $\eta/s(T)$ with much higher accuracy with respect to first attempt in RTA [18]: the idea is to change locally the total cross-section in equation (11) keeping constant the value $\eta/s[9, 10, 11]$. 

Figure 3. (Color online) Shear viscosity $\eta$ as a function of the Debye mass $m_D$ for three different value of temperature $T = 0.3$ GeV (blue solid line), $T = 0.4$ GeV (red dashed line) and $T = 0.5$ GeV (green dash-dotted line). Dotted lines represent the isotropic limit.

Figure 4. (Color online) Shear viscosity to entropy density ratio $\eta/s$ of a gluon plasma interacting through the differential cross-section in equation (8) as a function of temperature $T$. Green dotted line is the simple RTA, dark dot-dashed line is the modified RTA, orange dashed line is CE while red line represents results of [24].
3. Electric conductivity
Computing transport coefficients of the QGP is mandatory to study non-equilibrium and dissipative effects during the space-time evolution of the fireball created in Heavy Ion Collisions experiments at ultrarelativistic energy. Another transport coefficient of interest is the electric conductivity \( \sigma_{el} \) of the QGP: it represents the response of the system to an applied electric field.

In this section we compute the electric conductivity for a system of massive particles (quark, antiquark and gluons) in a box in thermal equilibrium with an external electric field \( E \). To simulate a constant electric field it is sufficient to modify the equation of motion of each particle as follows [4]:

\[
\frac{d}{dt} p_z^i = q_i e E_z
\]

where \( q_i \) is the charge of the particles and we have chosen the electric field along the \( z \)--direction. The presence of such electric field will develop an electric current whose density is given by:

\[
j_z(t) = \frac{1}{V} \sum_i e q_i p_z^i(t) \frac{m_i}{m_z}
\]

where the sum runs over particles of mass \( m \) and \( V \) is the volume of the system.

In figure 5 we show electric density current \( j_z(t) \) as a function of time for different value of the applied electric field: the current saturates towards the equilibrium value proportional to \( eE \). Electric conductivity can be computed as the ratio between the equilibrium current \( j_z^{eq} \) and the applied electric field:

\[
\sigma_{el} = \frac{j_z^{eq}}{E_z}.
\]

In figure 6 we plot electric conductivity to temperature ratio as a function of the applied electric field: as we can see, \( \sigma_{el}/T \) is a constant in the range of \( eE \) explored, in agreement with other results obtained in [4].

![Figure 5](image1.png)  
**Figure 5.** (Color online) Electric current \( j_z \) as a function of time for different values of the applied electric field.

![Figure 6](image2.png)  
**Figure 6.** Electric conductivity \( \sigma_{el} \) over temperature as a function of the applied electric field.

3.1. Electric conductivity: anisotropic case
In this section our aim is to study the dependence of \( \sigma_{el} \) on microscopic details, in particular on anisotropy of the cross-section, so we consider a quark-gluon plasma interacting via anisotropic
cross-section equation (8): in this simulations we fixed $\sigma_{tot} = 10$ mb, $T = 0.3$ GeV, $eE = 0.06$ GeV/fm and $m_q = m_{\bar{q}} = 0.4$ GeV, typical masses of quasiparticle model for $T = 2.0 T_c$ [25].

The electric conductivity is expected to be described by the well known Drude model in kinetic theory according to:

$$\sigma_{el} = \sum_j e^2 q_j^2 \rho_j \tau_j / m_j$$

where the sum runs over charged species, $\rho$ is the particle density, $\tau$ is the collision time. For a multi-component system, interacting via the same cross-section, one can generalize the transport relaxation time as follows:

$$\tau_{tr,i}^{-1} = \sum_j \langle \rho_j \sigma_{ij} v_{rel}^{ij} \rangle = \sigma_{tot} \langle v_{rel} h(a) \rangle (\rho_q + \rho_{\bar{q}} + \rho_g)$$

where $j$ runs over the species, $\sigma_{ij}$ is the cross-section for a $ij$ collision and $\rho_{q,\bar{q},g}$ is the particle density.

In figure 7 we show numerical results of electric conductivity as a function of Debye mass $m_D$ for a system of only quarks (red circles) and a system of quarks and gluons (dark circles). Dashed lines represent RTA while dotted lines are the isotropic limit that are reached for $m_D \rightarrow \infty$. The presence of gluons decreases the value of electric conductivity because they have not electric charge but contribute to the collision integral so they represent a resistance. As we can see from figure 7 RTA is a good approximation only in the isotropic limit while for $m_D \leq 1$ GeV it underestimates the results of about a factor 2: the deviation between Gree-Kubo and RTA appears to be similar but even larger than the one seen also for the shear viscosity.

4. Conclusions
Recent developments in the physics of strong interactions at finite T are focusing many efforts on the calculation of transport coefficients being interested in the non-equilibrium dynamics of the QGP created in HIC: shear viscosity $\eta$ is linked directly to the physical observable of elliptic flow $v_2(p_T)$. There are several approximation schemes that relate the viscosity $\eta$ to the microscopic details such as cross-section. The validity of such approximations was not yet investigated. We developed a method to solve numerically the Green-Kubo formula for shear viscosity for a relativistic Boltzmann gas. Our objective was to compare the CE and RTAs to Green-Kubo results that in principle should give the correct results.
We have found that, for the realistic case of massless gluons interacting via pQCD-like cross-section, CE approximation is in agreement with Green-Kubo results at the level of 4% [13]: this agreement supplies an analytical expression that can allow one to develop kinetic transport theory at fixed viscosity with very good precision while RTA in general understimates the viscosity even by a factor 2.

We have studied the microscopic dependence of electric conductivity $\sigma_{el}$: investigating the properties of such coefficient is very interesting in order to quantify the response of the QGP to the initial electromagnetic fields present in the central region during the collisions. We have found that RTA understimates the correct values of electric conductivity $\sigma_{el}$ of about 50% for anisotropic cross-section ($m_D \leq 2$ GeV) while it is a good approximation in the isotropic limit $m_D \to \infty$.

In the next future it would be interesting to apply the methods for the calculation of $\eta/s$ and $\sigma_{el}$ in order to explore quasiparticle models, where till now only RTA have been considered [25].

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