Chiral properties of discrete Joyce and Hestenes equations

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Abstract This paper concerns the question of how chirality is realized for discrete counterparts of the Dirac-Kähler equation in the Hestenes and Joyce forms. It is shown that left and right chiral states for these discrete equations can be described with the aid of some projectors on a space of discrete forms. The proposed discrete model admits a chiral symmetry. We construct discrete analogues of spin operators, describe spin eigenstates for a discrete Joyce equation, and also discuss chirality.

Key words: Dirac-Kähler equation, Hestenes equation, Joyce equation, Chirality, Clifford product, Spin eigenstates

1 Introduction

We present some recent results in the discretisation of the Dirac equation in the geometric algebra of spacetime by using the Dirac-Kähler approach. In this approach, a discretisation scheme is geometric in nature and rests upon the use of the differential forms calculus. The general topic of this paper is the description of some discrete constructions in which the chiral properties of the Dirac theory are captured. In the context of the geometric discretisation, it is natural to introduce a Clifford product acting on the space of discrete inhomogeneous forms as was discussed in [20]. This work is a direct continuation of that described in my previous papers [15,16,17,18,19,20]. In [16], on the issue of chirality, special attention to a discrete Hodge star operator has been paid. A central role of the Hodge star to deal with chiral symmetry in the lattice formulation was already pointed out by Rabin [14]. There are several approaches to study of discrete versions of the Dirac-Kähler
equation based on the use of a discrete Clifford calculus framework on lattices. For a review of discrete Clifford analysis, we refer the reader to [5, 6, 7, 13, 21].

We first briefly review some notations and basic facts on the Dirac-Kähler equation [12, 14] and the Dirac equation in the spacetime algebra [8, 9]. Let \( M = \mathbb{R}^{1,3} \) be Minkowski space. Denote by \( \Lambda^r(M) \) the vector space of smooth complex-valued differential \( r \)-forms, \( r = 0, 1, 2, 3, 4 \). Let \( d : \Lambda^r(M) \rightarrow \Lambda^{r+1}(M) \) be the exterior differential and let \( \delta : \Lambda^r(M) \rightarrow \Lambda^{r-1}(M) \) be the formal adjoint of \( d \) with respect to natural inner product in \( \Lambda^r(M) \). We have

\[
\delta = * d *,
\]

where \( * \) is the Hodge star operator \( * : \Lambda^r(M) \rightarrow \Lambda^{4-r}(M) \) with respect to the Lorentz metric. Denote by \( \Lambda(M) \) the set of all differential forms on \( M \). We have

\[
\Lambda(M) = \Lambda^0(M) \oplus \Lambda^1(M) \oplus \Lambda^2(M) \oplus \Lambda^3(M) \oplus \Lambda^4(M).
\]

Let \( \Omega \in \Lambda(M) \) be an inhomogeneous differential form, i.e., \( \Omega = \sum_{r=0}^4 \omega_r \), where \( \omega_r \in \Lambda^r(M) \). The Dirac-Kähler equation for a free electron is given by

\[
i (d + \delta) \Omega = m \Omega,
\]

(1)

where \( i \) is the usual complex unit and \( m \) is a mass parameter.

Let \( \{ \gamma_0, \gamma_1, \gamma_2, \gamma_3 \} \) be a vector basis of the Clifford algebra \( \mathbb{C}^l(1,3) \), namely \( \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = g_\mu \nu \), where \( g_\mu \nu = \text{diag}(1, -1, -1, -1) \) and \( \mu, \nu = 0, 1, 2, 3 \). Hestenes [9] calls this algebra the spacetime algebra. It is known that the vectors \( \gamma_\mu \) can be represented by the \( 4 \times 4 \) Dirac gamma matrices [11, 9]. Through the identification of the basic covectors \( dx^\mu \) and the matrices \( \gamma_\mu \) which arises from representation theory, one connects the differential forms under the Clifford product to the algebra of gamma matrices. In other words, the graded algebra \( \Lambda(M) \) endowed with the Clifford multiplication is an example of a Clifford algebra. It is true that Eq. (1) is equivalent to the four usual Dirac equations (traditional column-spinor equations). Let \( \Lambda_\mathbb{R}(M) \) denote the set of real-valued differential forms and let \( \Lambda^{cv}(M) = \Lambda^0(M) \oplus \Lambda^2(M) \oplus \Lambda^4(M) \). The Dirac equation in the Hestenes form [8, 9] can be written in terms of inhomogeneous forms as

\[
-(d + \delta) \Omega^{cv} \gamma_2 = m \Omega^{cv} \gamma_0, \quad \Omega^{cv} \in \Lambda^{cv}_\mathbb{R}(M).
\]

(2)

We consider also the generalized bivector Dirac equation [10] in the form

\[
i (d + \delta) \Omega^{cv} = m \Omega^{cv} \gamma_0, \quad \Omega^{cv} \in \Lambda^{cv}(M).
\]

(3)

Following Baylis [2] we call Eq. (3) the Joyce equation. This equation is equivalent to two copies of the usual Dirac equation. For a deeper discussion of equivalence of Dirac formulations we refer the reader to [11].

The goal of this work is to establish the chirality of discrete versions of the Dirac equation in the Hestenes and Joyce forms. We show that defined some projectors
on the space of discrete forms one can decompose solutions of Eqs. (1–3) into its left-handed and right-handed parts. Two types of such projectors are introduced and we prove that a discrete Dirac-Kähler operator flips the chirality for both of them. We also construct spin $\pm \frac{1}{2}$ eigenstates for a discrete counterpart of the plane wave solution to a discrete Joyce equation and discuss chirality for such fields.

2 Discrete Dirac-Kähler, Hestenes and Joyce equations

In this section, we recall some discrete constructions concerning the Dirac-Kähler equation and a discrete Clifford calculus. A discretization scheme is based on the language of differential forms and is described in [16]. The approach was originated by Dezin [4]. For the convenience of the reader we briefly repeat the relevant material from [16] without proofs, thus making our presentation self-contained. All details one can find in [15] [16].

Let $K(4) = K \otimes K \otimes K \otimes K$ be a cochain complex with complex coefficients, where $K$ is the 1-dimensional complex generated by 0- and 1-dimensional basis elements $x^{k\mu}$ and $e^{k\mu}$, $k_\mu \in \mathbb{Z}$, respectively. Then an arbitrary $r$-dimensional basis element of $K(4)$ can be written as $s^{(r)} = x^{k_0} \otimes x^{k_3} \otimes x^{k_2} \otimes x^{k_1}$, where $s^{k\mu}$ is either $x^{k\mu}$ or $e^{k\mu}$. $\mu = 0, 1, 2, 3$ and $k = (k_0, k_1, k_2, k_3)$ is a multi-index. The symbol $(r)$ contains the whole required information about the number and position $e^{k\mu} \in K$ in $s^{(r)} \in K(4)$. For example, the 1-dimensional basis elements of $K(4)$ can be written as

$$e_0^0 = e^{k_0} \otimes x^{k_1} \otimes x^{k_2} \otimes x^{k_3}, \quad e_1^0 = x^{k_0} \otimes e^{k_1} \otimes x^{k_2} \otimes x^{k_3}, \quad e_2^0 = x^{k_0} \otimes x^{k_1} \otimes e^{k_2} \otimes x^{k_3}, \quad e_3^0 = x^{k_0} \otimes x^{k_1} \otimes x^{k_2} \otimes e^{k_3}.$$ 

The 2-dimensional basis elements of $K(4)$ have the form

$$e_{01}^0 = e^{k_0} \otimes e^{k_1} \otimes x^{k_2} \otimes x^{k_3}, \quad e_{02}^0 = e^{k_0} \otimes x^{k_1} \otimes e^{k_2} \otimes x^{k_3}, \quad e_{03}^0 = e^{k_0} \otimes x^{k_1} \otimes x^{k_2} \otimes e^{k_3}, \quad e_{12}^0 = x^{k_0} \otimes e^{k_1} \otimes e^{k_2} \otimes x^{k_3}, \quad e_{13}^0 = x^{k_0} \otimes e^{k_1} \otimes x^{k_2} \otimes e^{k_3}, \quad e_{23}^0 = x^{k_0} \otimes x^{k_1} \otimes e^{k_2} \otimes e^{k_3}.$$ 

In the same way one can write down the 3-dimensional basic elements $e_{012}^0$, $e_{013}^0$, $e_{023}^0$ and $e_{123}^0$. Finally, denote by

$$x^k = x^{k_0} \otimes x^{k_1} \otimes x^{k_2} \otimes x^{k_3}, \quad e^k = e^{k_0} \otimes e^{k_1} \otimes e^{k_2} \otimes e^{k_3}$$

the 0- and 4-dimensional basis elements of $K(4)$.

The complex $K(4)$ is a discrete analogue of $\Lambda(M)$ and cochains play a role of differential forms. Let us call them forms or discrete forms to emphasize their relationship with differential forms. Then we have

$$K(4) = K^0(4) \oplus K^1(4) \oplus K^2(4) \oplus K^3(4) \oplus K^4(4),$$

where $K^s(4)$ are the $s$-form elements in $K(4)$.
where $K'(4)$ denotes the set of all discrete $r$-forms, and any $\hat{\omega} \in K'(4)$ can be expressed as

$$0 \omega = \sum_k^0 \omega_k x^k, \quad 2 \omega = \sum_k \sum_{\mu < \nu} \omega_k^{\mu \nu} e^{k}_{\mu \nu}, \quad 4 \omega = \sum_k^4 \omega_k e^k,$$

$$1 \omega = \sum_k^3 \omega_k^\mu e^k_\mu, \quad 3 \omega = \sum_k \sum_{i < \nu} \omega_k^{i \mu \nu} e^k_{i \mu \nu}, \quad (4)$$

where $\omega_k^0, \omega_k^{0 \nu}, \omega_k^\mu$ and $\omega_k^{i \mu \nu}$ are complex numbers.

Let $d_c : K'(4) \rightarrow K'^{-1}(4)$ be a discrete analogue of the exterior derivative $d$ and let $\delta_c : K'(4) \rightarrow K'^{-1}(4)$ be a discrete analogue of the codifferential $\delta$. It is clear that $\delta_c = +d^c \ast$. For more precise definitions of these operators we refer the reader to [16]. In this paper we give only the difference expressions for $d^c$ and $\delta^c$. Let the difference operator $\Delta_\mu$ be defined by

$$\Delta_\mu 0 \omega_k^{(r)} = 0 \omega_k^{(r)} - \omega_k^{(r)}, \quad (6)$$

where $0 \omega_k^{(r)} \in \mathbb{C}$ is a component of $r \omega \in K'(4)$ and $\tau_\mu$ is the shift operator which acts as $\tau_\mu k = (k_0, \ldots, k_\mu + 1, \ldots, k_3)$, $\mu = 0, 1, 2, 3$. For forms (4), (5) we have

$$d^c 0 \omega = \sum_k^3 \sum_{\mu = 0}^3 (\Delta_\mu 0 \omega_k^0) e^k_\mu, \quad d^c 1 \omega = \sum_k \sum_{\mu < \nu} (\Delta_\mu \omega_k^0 - \Delta_\nu \omega_k^0) e^{k}_{\mu \nu}, \quad (7)$$

$$d^c 2 \omega = \sum_k [(\Delta_0 \omega_k^{10} - \Delta_1 \omega_k^{02} + \Delta_2 \omega_k^{01}) e^{k}_{012} + (\Delta_0 \omega_k^{13} - \Delta_1 \omega_k^{03} + \Delta_3 \omega_k^{01}) e^{k}_{013} + (\Delta_0 \omega_k^{23} - \Delta_2 \omega_k^{03} + \Delta_3 \omega_k^{02}) e^{k}_{023} + (\Delta_1 \omega_k^{23} - \Delta_2 \omega_k^{13} + \Delta_3 \omega_k^{12}) e^{k}_{123}], \quad (8)$$

$$d^c 3 \omega = \sum_k [(\Delta_0 \omega_k^{123} - \Delta_1 \omega_k^{023} + \Delta_2 \omega_k^{013} - \Delta_3 \omega_k^{012}) e^k, \quad d^c 4 \omega = 0, \quad (9)$$

$$\delta^c 0 \omega = 0, \quad \delta^c 1 \omega = \sum_k (\Delta_0 \omega_k^0 - \Delta_1 \omega_k^1 - \Delta_2 \omega_k^2 - \Delta_3 \omega_k^3) e^k, \quad (10)$$

$$\delta^c 2 \omega = \sum_k [(\Delta_1 \omega_k^{01} + \Delta_2 \omega_k^{02} + \Delta_3 \omega_k^{03}) e^k_0 + (\Delta_0 \omega_k^{01} + \Delta_2 \omega_k^{02} + \Delta_3 \omega_k^{03}) e^k_1 + (\Delta_0 \omega_k^{02} - \Delta_1 \omega_k^{12} + \Delta_3 \omega_k^{10}) e^k_2 + (\Delta_0 \omega_k^{03} - \Delta_1 \omega_k^{13} - \Delta_2 \omega_k^{23}) e^k_3], \quad (11)$$
The operation is linearly extended to arbitrary discrete forms.

Consider the following unit forms

\[ x = \sum_k x^k, \quad e = \sum_k e^k, \quad e_\mu = \sum_k e_{\mu}^k, \quad e_{\mu \nu} = \sum_k e_{\mu \nu}^k, \quad (17) \]

where \( \mu, \nu = 0, 1, 2, 3 \). Note that the unit 0-form \( x \) plays a role of the unit element in \( K(4) \), i.e., for any r-form \( \tilde{x} \) we have \( x \tilde{x} = \tilde{x} x = \tilde{x} \).

**Proposition 1.** For the unit forms \( x \in K^0(4) \) and \( e_\mu \in K^1(4) \) given by (17) the following holds
\[ e_\mu e_\nu + e_\nu e_\mu = 2g_{\mu\nu}x, \quad \mu, \nu = 0, 1, 2, 3. \quad (18) \]

*Proof.* By the rule (b), it is obvious. \( \square \)

**Proposition 2.** Let \( \Omega \in K(4) \) be an inhomogeneous discrete form. Then we have
\[ (d^c + \delta^c)\Omega = \sum_{\mu=0}^{3} e_\mu \Delta_\mu \Omega, \quad (19) \]
where \( \Delta_\mu \) is the difference operator which acts on each component of \( \Omega \) by the rule (6).

*Proof.* See Proposition 1 in [18]. \( \square \)

Thus the discrete Dirac-Kähler equation (15) can be rewritten in the form
\[ i \sum_{\mu=0}^{3} e_\mu \Delta_\mu \Omega = m\Omega. \]

Let \( K^c(4) = K^0(4) \oplus K^2(4) \oplus K^4(4) \) and let \( \Omega^c \in K^c(4) \) be a real-valued even inhomogeneous form, i.e., \( \Omega^c = \tilde{\omega} + \omega + \tilde{\omega} \). A discrete analogue of the Hestenes equation (2) is defined by
\[ -(d^c + \delta^c)\Omega^c e_1 e_2 = m\Omega^c e_0, \quad (20) \]
or equivalently,
\[ -\sum_{\mu=0}^{3} e_\mu \Delta_\mu \Omega^c e_1 e_2 = m\Omega^c e_0, \]
where \( e_1, e_2 \) and \( e_0 \) are given by (17). A discrete analogue of the Joyce equation (3) is given by
\[ i(d^c + \delta^c)\Omega^c = m\Omega^c e_0, \quad (21) \]
where \( \Omega^c \in K^c(4) \) is a complex-valued even inhomogeneous form. Clearly, Eq. (21) can be rewritten in the form
\[ i \sum_{\mu=0}^{3} e_\mu \Delta_\mu \Omega^c = m\Omega^c e_0. \]

Applying (7)–(13) Eqs. (20) and (21) can be expressed also in terms of difference equations (see [17, 18]).

### 3 Chirality and a discrete Joyce equation

In the continuum Dirac theory, the fifth gamma matrix \( \gamma_5 \) defined by
\[ \gamma_5 = i\gamma_0 \gamma_1 \gamma_2 \gamma_3 \]
plays a central role in formulating chiral fermions. It is known that in the language
of differential forms the Hodge star operator $\ast$ has similar properties, up to sign, as $\gamma_5$. The difficulties in defining a discrete Hodge star operator to deal with chirality on the lattice were discussed by Rabin in [14]. Several discrete versions of the Hodge star operator have been proposed in [3, 16, 22] in which the chiral properties for Dirac-Kähler fermions in the geometric discretisation are captured. In this section, we use a discrete analogue of $\gamma_5$ to describe the chirality of a discrete Dirac field in the Joyce formulation.

Consider the constant 4-form $e_5$ defined by
\[ e_5 = ie_0e_1e_2e_3 = ie, \]  \hspace{1cm} (22)
where $e_\mu \in K^1(4)$ and $e \in K^4(4)$ are given by (17). The form $e_5$ generates the action $e_5 : \hat{\omega} \rightarrow e_5\hat{\omega}$, where $\hat{\omega} \in K'(4)$. Note also that
\[ e_5 : K^r(4) \rightarrow K^{4-r}(4). \]  \hspace{1cm} (23)
Hence the form $e_5 \in K^4(4)$ has exactly the same properties as $\gamma_5$.

**Proposition 3.** For any inhomogeneous form $\Omega \in K(4)$ we have
\[ e_5(d^c + \delta^c)\Omega = -(d^c + \delta^c)e_5\Omega. \]  \hspace{1cm} (24)
**Proof.** By Proposition 2 and (23), the equality (24) follows. $\square$

Consider the following constant forms
\[ P_L = \frac{x - e_5}{2}, \quad P_R = \frac{x + e_5}{2}. \]  \hspace{1cm} (25)
Since
\[ P_L^2 = P_L, \quad P_R^2 = P_R, \quad P_L \circ P_R = P_R \circ P_L, \]
it follows that $P_L$ and $P_R$ are projectors. Let us represent $\Omega \in K(4)$ as
\[ \Omega = \Omega_L + \Omega_R, \]  \hspace{1cm} (26)
where
\[ \Omega_L = P_L\Omega, \quad \Omega_R = P_R\Omega. \]  \hspace{1cm} (27)
It is clear that $e_5\Omega_R = \Omega_R$ and $e_5\Omega_L = -\Omega_L$. Hence we can say that $\Omega$ decomposes into its self-dual and anti-self-dual parts with respect to the action $e_5$. The self-dual and anti-self-dual components of $\Omega$ correspond to the chiral right and chiral left parts of a solution of the discrete Dirac-Kähler equation.

**Proposition 4.** If $\Omega$ is a solution of the massless discrete Dirac-Kähler equation
\[ i(d^c + \delta^c)\Omega = 0, \]  

(28)

then so are both \( \Omega_R \) and \( \Omega_L \).

**Proof.** Let \( \Omega \in K(4) \) be a solution of Eq. (28). Using (24) and (27) we obtain

\[ i(d^c + \delta^c)(\Omega \pm e_5\Omega) = i(d^c + \delta^c)\Omega \mp e_5i(d^c + \delta^c)\Omega = 0. \]

\[ \square \]

From Proposition 4 it follows immediately that the massless discrete Dirac-Kähler equation is invariant under the transformation

\[ \Omega \longrightarrow \Omega \pm e_5\Omega. \]  

(29)

In other words, the discrete model admits the chiral symmetry (29) of Eq. (28) with respect to the action \( e_5 \).

**Proposition 5.** If \( \Omega \) is a solution of the discrete Dirac-Kähler equation (15) then we have

\[ i(d^c + \delta^c)\Omega_L = m\Omega_R, \]

\[ i(d^c + \delta^c)\Omega_R = m\Omega_L. \]

**Proof.** From (24) it follows that

\[ (d^c + \delta^c)P_L\Omega = P_R(d^c + \delta^c)\Omega, \quad (d^c + \delta^c)P_R\Omega = P_L(d^c + \delta^c)\Omega \]  

(30)

for any \( \Omega \in K(4) \). Let \( \Omega \) be a solution of Eq. (15). By (30), we have

\[ i(d^c + \delta^c)\Omega_L = i(d^c + \delta^c)P_L\Omega = P_R(m\Omega) = m\Omega_R \]

and

\[ i(d^c + \delta^c)\Omega_R = i(d^c + \delta^c)P_R\Omega = P_L(m\Omega) = m\Omega_L. \]

\[ \square \]

Hence, just as in the continuum case, the operator \( i(d^c + \delta^c) \) flips the chirality and the massive discrete Dirac-Kähler equation decomposes into two parts.

Let \( \Omega^e \in K^{ev}(4) \) be a complex-valued even inhomogeneous form. Then we have

\[ \Omega^e = P_L\Omega_L^e + P_R\Omega_R^e = \Omega_L^{ev} + \Omega_R^{ev}, \]

where \( P_L \) and \( P_R \) are given by (25). The discrete Joyce equation splits into two parts in the following way.

**Proposition 6.** If \( \Omega^e \) is a solution of the discrete Joyce equation (27) then we have

\[ i(d^c + \delta^c)\Omega_L^e = m\Omega_R^e e_0, \]  

(31)

\[ i(d^c + \delta^c)\Omega_R^e = m\Omega_L^e e_0. \]  

(32)
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Proof. The proof is the same as that for Proposition 5.

Thus the chiral properties are captured for our discrete model.

4 Chirality and a discrete Hestenes equation

Recall that the Hestenes equation is a form of the Dirac equation in the real algebra \( C^{0}_{R}(1, 3) \). The discrete Hestenes equation acts in the space of real-valued even form \( K^{ev}(4) \). Unfortunately, to discuss the chiral properties of this equation the action (22) makes no sense because the form \( e_{5} \) defined by (22) is complex-valued. To make sense of the chiral action one must substitute for \( e_{5} \) a real-valued action. Let us denote by \( *_{5} \) the following transformation

\[
*_{5} : \hat{\omega} \to e^{\hat{\omega}} e_{2} e_{1},
\]

where \( \hat{\omega} \in K^{r}(4) \) and \( e, e_{2}, e_{1} \) are given by (17). It is true that \( *_{5} : K^{ev}(4) \to K^{ev}(4) \).

Proposition 7. For any inhomogeneous form \( \Omega \in K(4) \) we have

\[
(*_{5})^{2} \Omega = \Omega, \quad \text{and} \quad (*_{5} e_{\mu} + e_{\mu} *_{5}) \Omega = 0 \quad \text{for} \quad \mu = 0, 1, 2, 3.
\]

Proof. By definition, \( e = e_{0} e_{1} e_{2} e_{3} \) and \( e^{2} = e e = -x \). Then for any \( \hat{\omega} \in K^{r}(4) \) we have

\[
(*_{5})^{2} \hat{\omega} = *_{5}(*_{5} \hat{\omega}) = e(e^{\hat{\omega}} e_{2} e_{1}) e_{2} e_{1} = x^{\hat{\omega}} x = \hat{\omega}.
\]

Since \( e \in K^{4}(4) \) anticommutes with \( e_{\mu} \in K^{1}(4) \) for \( \mu = 0, 1, 2, 3 \), i.e., \( e e_{\mu} = -e_{\mu} e \), the second equality of (34) follows immediately.

Proposition 8. Let \( \Omega \in K(4) \) be an inhomogeneous form. Then the following holds

\[
(*_{5}(d^{c} + \delta^{c}) + (d^{c} + \delta^{c}) *_{5}) \Omega = 0.
\]

Proof. By (34), the proof repeats the proof of Proposition 3.

From (34) and (35) it follows that to deal with chirality in the case of the discrete Hestenes equation one can take \( *_{5} \).

Proposition 9. The massless discrete Dirac-Kähler equation is invariant under the transformation

\[
\Omega \mapsto \Omega \pm *_{5} \Omega.
\]

Proof. By (35), it is obvious.

It follows that the discrete model admits a chiral symmetry of the type (36).

Let us consider the following operations

\[
P_{L}^{*} = \frac{1 - *_{5}}{2}, \quad P_{R}^{*} = \frac{1 + *_{5}}{2}.
\]
It is easy to check that
\[(P_L^*L)^2 \Omega = P_L^2 \Omega, \quad (P_R^*R)^2 \Omega = P_R^2 \Omega, \quad P_L^* P_R^* \Omega = P_R^* P_L^* \Omega = 0\]
for any \( \Omega \in K(4) \). Hence, the operations \( P_L^* \) and \( P_R^* \) are projectors. Then \( \Omega \in K(4) \) can be represented as \([26]\), where
\[\Omega_L = P_L^* \Omega, \quad \Omega_R = P_R^* \Omega.\]

Let \( \Omega^{ev} \in K^{ev}(4) \) be a real-valued even inhomogeneous form. Then the forms \( \Omega^{ev}_L = P_L^* \Omega^{ev} \) and \( \Omega^{ev}_R = P_R^* \Omega^{ev} \) are even and we have
\[\Omega^{ev} = \Omega^{ev}_L + \Omega^{ev}_R.\]

It should be noted that \( \Omega^{ev}_R \) and \( \Omega^{ev}_L \) are self-dual and anti-self-dual parts of \( \Omega^{ev} \) with respect to the action \( \ast \). They correspond to the chiral right and chiral left parts of a solution of the discrete Hestenes equation. Similarly, as in the case of the Joyce equation, we have the following decomposition of the discrete Hestenes equation.

**Proposition 10.** If \( \Omega^{ev} \) is a solution of the discrete Hestenes equation \([20]\) then we have
\[-(d^c + \delta^c) \Omega^{ev}_L e_1 e_2 = m \Omega^{ev}_R e_0,\]
\[-(d^c + \delta^c) \Omega^{ev}_R e_1 e_2 = m \Omega^{ev}_L e_0.\]

**Proof.** Using \([35]\) and \([37]\) we obtain
\[(d^c + \delta^c) P_L^* \Omega = P_R^* (d^c + \delta^c) \Omega, \quad (d^c + \delta^c) P_R^* \Omega = P_L^* (d^c + \delta^c) \Omega\]
for any \( \Omega \in K(4) \). Therefore the proof repeats the proof of Proposition \(5\). \(\square\)

Let us consider the parity operation \( P : K^r(4) \to K^r(4) \) defined by
\[P \omega = e_0 \hat{\omega} e_0, \quad (38)\]
where \( \hat{\omega} \in K^r(4) \) and \( e_0 \in K^1(4) \) is given by \([17]\). It is clear that \( P^2 \omega = \hat{\omega} \). But the second statement of Proposition \(7\) is not true. The parity operation \([38]\) changes the chirality of discrete forms in the following way.

**Proposition 11.** For any form \( \Omega \in K(4) \) we have
\[P(P^*_L \Omega) = P^*_R (P \Omega), \quad P(P^*_R \Omega) = P^*_L (P \Omega), \quad (39)\]
where \( P_L^* \) and \( P_R^* \) are given by \([37]\).

**Proof.** Since \( e_0 \) commutes with \( e_2 e_1 \) and anticommutes with \( e \) it follows immediately. \(\square\)

Decompose an even inhomogeneous form \( \Omega^{ev} \in K(4) \) as follows
\[ \Omega^{cv} = \Omega_+^{cv} + \Omega_-^{cv}, \]

where \( \Omega_+^{cv} \) commutes with \( e_0 \) and \( \Omega_-^{cv} \) anticommutes with it, i.e.,

\[ e_0 \Omega_+^{cv} = \pm \Omega_+^{cv} e_0. \] (40)

**Proposition 12.** Let \( \Omega_+^{cv} = P_L \Omega_+^{cv} \) and \( \Omega_-^{cv} = P_L \Omega_-^{cv} \). Then we have

\[ \begin{align*}
P \Omega_+^{cv} &= \Omega_+^{cv}, \\
P \Omega_-^{cv} &= -\Omega_-^{cv}, \\
P \Omega_+^{cv} &= \Omega_+^{cv}, \\
P \Omega_-^{cv} &= -\Omega_-^{cv}.
\end{align*} \]

**Proof.** By (38)–(40), we obtain

\[ P \Omega_+^{cv} = P(L \Omega_+^{cv}) = P(L (e_0 \Omega_+^{cv} e_0)) = P(L \Omega_+^{cv} e_0 e_0) = P \Omega_+^{cv}. \]

The same proof remains valid for all other cases. \(\square\)

## 5 Discrete plane wave solutions and spin eigenstates

Discrete versions of the plane wave solutions to discrete Joyce and Hestenes equations are constructed in [19] and [20]. In this section, we study spin properties of these solutions in the case of the discrete Joyce equation and discuss how the chirality is realized for spin eigenstates in our discrete model. Recall a discrete version of the general plane wave solution for the Joyce equation (see for details [19]). Let \( \psi \in K^0(4) \) and let

\[ \psi = \sum_k (ip_0 + 1)^k (ip_1 + 1)^k (ip_2 + 1)^k (ip_3 + 1)^k x^k, \]

where \( i \) is the usual complex unit, \( p_\mu \in \mathbb{R} \) and \( k = (k_0, k_1, k_2, k_3) \) is a multi-index.

Let \( A \) be the even inhomogeneous form given by

\[ A = a_1((m-p_0)x + p_1 e_{01} + p_2 e_{02} + p_3 e_{03}) + a_2((m-p_0)e_{12} + p_2 e_{01} - p_1 e_{02} + p_3 e) + a_3((m-p_0)e_{13} + p_3 e_{01} - p_1 e_{03} + p_2 e) + a_4((m-p_0)e_{23} + p_3 e_{02} - p_2 e_{03} + p_1 e), \]

where \( p_0 = \pm \sqrt{m^2 + p_1^2 + p_2^2 + p_3^2} \), \( a_\mu = \frac{\alpha_\mu}{m-p_0} \) and \( \alpha_\mu \) is an arbitrary complex number for \( \mu = 1, 2, 3, 4 \). Here the even unit forms \( x \in K^0(4) \), \( e \in K^4(4) \) and \( e_{\mu \nu} \in K^2(4) \) are given by [17]. Then the most general plane wave solution of Eq. (21) is

\[ \Omega^{cv} = A \psi. \] (41)
Let consider a particular case of (41), namely $p_2 = p_3 = 0$. This situation corresponds to one in the continuum case in which the plane wave solution is propagating along only one axis, e.g., $x_1$. In the continuum case, such solutions for a Dirac generalized bivector equation are described in [10]. Then we have

$$\psi = \sum_k (ip_0 + 1)k_0 (ip_1 + 1)k_1 x^k$$

and

$$A = a_1 ((m - p_0)x + p_1 e_{01}) + a_2 ((m - p_0)e_{12} - p_1 e_{02}) + a_3 ((m - p_0)e_{13} - p_1 e_{03}) + a_4 ((m - p_0)e_{23} + p_1 e).$$

Let us introduce the following constant 2-forms

$$S_1 = \frac{1}{2} e_{23}, \quad S_2 = -i \frac{1}{2} e_{13}, \quad S_3 = i \frac{1}{2} e_{12}.$$  

By definition, we have $e_{12} e_{12} = e_{13} e_{13} = e_{23} e_{23} = -x$ and one may easily calculate that $S_1^2 + S_2^2 + S_3^2 = \frac{1}{2} (1 + 1) x$. Hence similarly to the continuum case the forms (44) can be interpreted as spin operators for our discrete model and spin eigenstates of $\pm \frac{1}{2}$ along the direction of propagation can be described for the solution (41), where $\psi$ and $\Lambda$ are given by (42) and (43).

An easy computation shows that the equations $S_2 A = \frac{1}{2} A$ and $S_3 A = \frac{1}{2} A$, where $A$ is given by (43), have only trivial solutions, i.e., $a_1 = a_2 = a_3 = a_4 = 0$. However, the equation $S_1 A = \frac{1}{2} A$ has an non-trivial solution. Indeed, applying the spin operator $S_1$ to (43) we obtain

$$S_1 A = \frac{1}{2} (a_1 (m - p_0) e_{23} + a_1 p_1 e + a_2 (m - p_0) e_{13} - a_2 p_1 e_{03}) - a_3 (m - p_0) e_{12} + a_3 p_1 e_{02} - a_4 (m - p_0) x - a_4 p_1 e_{01}).$$

Combining (45) with (43) we conclude that $S_1 A = \frac{1}{2} A$ if and only if $a_1 = -ia_4$ and $a_2 = -ia_3$. It follows that $A$ can be represented as

$$A = a_1 A_1 + a_2 A_2,$$

where

$$A_1 = (m - p_0) x + p_1 e_{01} + i(m - p_0) e_{23} + i p_1 e,$$

$$A_2 = (m - p_0) e_{12} - p_1 e_{02} + i(m - p_0) e_{13} - i p_1 e_{03},$$

and $a_1, a_2$ are arbitrary constant. Since $\psi$ is a 0-form we have $S_1 \Omega^{ev} = \frac{1}{2} \Omega^{ev}$, where $\Omega^{ev}$ is the plane wave solution (41) and $A$ is given by (43). On other words, $\Omega^{ev}$ is an eigenstate corresponding to the eigenvalue $\frac{1}{2}$ of the spin operator $S_1$.

It is clear that if $A \psi$ is a solution of the discrete Joyce equation then $\bar{A} \psi$, where $\bar{A}$ denotes the complex conjugate of $A$, is also a solution. It can also be seen that
\( \bar{A} = a_1 \bar{A}_1 + a_2 \bar{A}_2 \) satisfies the equation \( S_1 \bar{A} = -\frac{1}{2} \bar{A} \). Hence, similarly as in continuum case \cite{10} the solutions \( A \psi \) and \( \bar{A} \psi \), where \( \psi \) and \( A \) are given by \cite{12} and \cite{46}, can be interpreted as spin up and spin down solutions correspondingly.

It should be noted that the chirality is captured for the spin solutions described above. Applying the projectors \cite{25} to the forms \( A_1 \) and \( A_2 \) given by \cite{47} one can calculate

\[
\begin{align*}
\mathcal{P}_R A_1 &= \frac{1}{2} (m - p_0 + p_1) (x + e_{01} + ie_{23} + ie), \\
\mathcal{P}_L A_1 &= \frac{1}{2} (m - p_0 - p_1) (x - e_{01} + ie_{23} - ie), \\
\mathcal{P}_R A_2 &= \frac{1}{2} (m - p_0 - p_1) (e_{12} - e_{02} + ie_{13} - ie_{03}), \\
\mathcal{P}_L A_2 &= \frac{1}{2} (m - p_0 + p_1) (e_{12} + e_{02} + ie_{13} + ie_{03}).
\end{align*}
\]

Thus we have the following two left and two right chiral states

\[
\begin{align*}
\Omega_{1L}^{cv} &= \mathcal{P}_L A_1 \psi, & \Omega_{2L}^{cv} &= \mathcal{P}_L A_2 \psi, \\
\Omega_{1R}^{cv} &= \mathcal{P}_R A_1 \psi, & \Omega_{2R}^{cv} &= \mathcal{P}_R A_2 \psi,
\end{align*}
\]

where \( \psi \) is given by \cite{41}. Obviously, as has already been described in Sect. 3 the forms \cite{48} satisfy Eqs. \cite{31} and \cite{32}.

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