ISOPERIMETRIC INEQUALITY ALONG THE TWISTED KÄHLER-RICCI FLOW

SHOUWEN FANG, TAO ZHENG

Abstract. We prove a uniform isoperimetric inequality for all time along the twisted Kähler-Ricci flow on Fano manifolds.

1. Introduction

The classical isoperimetric inequality states that for Borel set $\Omega \in \mathbb{R}^n (n \geq 2)$ with finite Lebesgue measure $|\Omega|$, the ball with the same measure has a lower perimeter, that is,

$$P(\Omega) \geq n\omega_n^{\frac{1}{n}} |\Omega|^{\frac{n-1}{n}},$$  \hspace{1cm} (1.1)

where $P(\Omega)$ is the distributional perimeter of $\Omega$ which coincides with the classical $n-1$-dimensional area of $\partial \Omega$ if $\Omega$ has smooth boundary and $\omega_n$ is the volume of unit ball in $\mathbb{R}^n$. It is also well-known that equality holds in (1.1) if and only if $\Omega$ is a ball $B$ in $\mathbb{R}^n$. De Giorgi [21] (see also [22] for English version) proved (1.1) for the first time in the general framework of sets with finite perimeter. Then there is a long and complex history of the various kinds of proofs and differential formulations of the isoperimetric inequality (see [2, 4, 6, 16, 31, 41] and references therein).

Indeed, we can understand “isoperimetric inequality” as any inequality relating to two or more geometric and/or physical quantities associated to the same set which is called optimal in the sense that the equality sign holds for some set or in the limit as the set degenerates (see [34]). For example, there is a quantitative version of the isoperimetric inequality so-called Bonnesen type inequality named by Osserman [31] (see also [1, 3, 18, 25, 26, 32]). For the Bonnesen type inequality, there is Hall’s Conjecture solved by [17] which states that for any Borel set $\Omega \subset \mathbb{R}^n (n \geq 2)$ with $0 < |\Omega| < \infty$, there exists a constant $C(n)$ such that

$$\lambda(\Omega) \leq C(n) \sqrt{D(\Omega)}.$$

Here $\lambda(\Omega)$ is the Fraenkel asymmetry of $\Omega$ defined by

$$\lambda(\Omega) := \min \left\{ \frac{d(\Omega, x + rB)}{r^n} \mid x \in \mathbb{R}^n \right\},$$
where $r > 0$ such that $|\Omega| = r^n|B|$ and $d(E, F)$ denotes the measure of the symmetric difference between any two Borel sets $E, F$. $D(\Omega)$ is defined by

$$D(\Omega) := \frac{P(\Omega)}{n\omega_n |\Omega|^{\frac{n-1}{n}}} - 1.$$ 

There are a number of practical uses of isoperimetric inequalities as noted in the preface of Pólya-Szego [35]. By making use of such inequalities, one can deduce estimates of physical quantities in terms of geometric ones, or not easily accessible quantities in terms of more easily computable ones which may be precise enough for practical purposes. Isoperimetric inequalities are also useful in various kinds of initial and/or boundary problems (see for example [33, 34] and references therein).

In the case of geometric flow, Hamilton [24] obtained an isoperimetric estimate for the Ricci flow on the two sphere. For complex 2-dimensional Kähler-Ricci flow, Chen-Wang [7] proved that the isoperimetric constant for $(M, g(t))$ is bounded from below by a uniform constant. Here $g(t)$ is the solution of the Kähler-Ricci flow (see (1.2) with $\theta_J \equiv 0$). Later, Tian-Zhang [42] proved that, for all complex $n$-dimensional Kähler-Ricci flow on Fano manifolds, the isoperimetric constant for $(M, g(t))$ is also bounded from below by a uniform constant.

In this paper, we obtain a uniform estimate of lower bound on isoperimetric constant along the twisted Kähler-Ricci flow on Fano manifolds. To be precise, we need some notations and definitions. Let $M$ be a real $n (= 2m)$ dimensional Fano manifold with Kähler form $\omega_0$ associated to the Kähler metric $g_0$. We consider the twisted Kähler-Ricci flow (See [11, 29, 48] and the references therein)

$$\left\{ \begin{array}{l}
\frac{\partial}{\partial t} g^{\sigma}(x, t) = - R^{\sigma}(x, t) + \theta_i^{\sigma}(x) + g^{\sigma}(x, t), \\
g^{\sigma}(x, 0) = (g_0)^{\sigma}(x),
\end{array} \right. \quad (1.2)$$

where $\theta$ is a closed semi-positive $(1,1)$ form and

$$[2\pi c_1(M)] = [\omega(x, t) + \theta].$$

Here $\omega(x, t) = \sqrt{-1}g^\sigma(x, t)dz^i \wedge d\zbar^j$ associated to the Kähler metric $g(x, t)$. For convenience, we denote

$$S^{\sigma}_i(x, t) = R_i^{\sigma}(x, t) - \theta_i^{\sigma}(x).$$

and

$$S(x, t) = 2 \sum_{i,j=1}^{m} g^\sigma_i(x, t)S^{\sigma}_i(x, t).$$

We know that

**Proposition 1.1.** For the twisted Kähler-Ricci flow (1.2) on Fano manifolds, there exist uniform positive constants $C, \kappa$ and $C_S$ such that

- (a) $|S(x, t)| \leq C$,
- (b) $|\text{diam}(M, g(t))| \leq C$,
(e) $\|h\|_{C^1} \leq C$, where from $\partial \bar{\partial}$-lemma, $h \in C^\infty(M, \mathbb{R})$ satisfies
\[
S g - g = \partial_i \partial_j h,
\] (1.3)

(d) $\text{Vol}_{g(t)}(B(x, r, t)) \geq \kappa r^n$, for any $t > 0$ and $r \in (0, \text{diam}(M, g(t)))$,

(e) $\text{Vol}_{g(t)}(B(x, r, t)) \leq \kappa^{-1} r^n$, for any $t > 0$ and $r > 0$,

(f) for any $f \in W^{1,2}(M)$,
\[
\left( \int_M |f|^{\frac{2n}{n-2}} d\mu(t) \right)^{\frac{n-2}{n}} \leq C_S \left( \int_M \left[ |\nabla f|_{g(t)}^2 + f^2 \right] d\mu(t) \right).
\]

Items (a)-(d) in Proposition 1.1 can be founded in [11, 29] and items (e)-(f) in Proposition 1.1 can be founded in [14]. Since the volume of $(M, g(t))$ is a constant, from item (e) in Proposition 1.1 there exists a uniform constant $\beta > 0$ such that
\[
\text{diam}(M, g(t)) > \beta.
\]

In the case of Kähler-Ricci flow, that is, $\theta \equiv 0$, Items (a)-(d) in Proposition 1.1 is due to Perelman (See [39]). Item (e) in Proposition 1.1 belongs to [8, 47] and item (f) was established by [44, 45, 46].

As a consequence of Proposition 1.1, we can deduce our main theorem as follows.

**Theorem 1.2.** For the twisted Kähler-Ricci flow (1.2) on Fano manifolds, for any $u \in C^\infty(M, \mathbb{R})$, there holds
\[
\left( \int_M u \frac{d\mu(t)}{n} \right)^{\frac{n}{n-1}} \leq S_1 \int_M |\nabla u|_{g(t)} d\mu(t) + \frac{C}{[\text{Vol}_{g(t)}(M)]^\frac{1}{2}} \int_M |u| d\mu(t),
\] (1.4)

where $S_1 > 0$ is a uniform constant depending only on $g_0$ and $C$ is a positive numerical constant. The Sobolev inequality (1.4) implies isoperimetric inequality
\[
\|f(x) - f_M\|_{L^\infty(M)} \leq C_I \|\nabla f\|_{L^1(M)}, \quad f \in C^\infty(M, \mathbb{R}),
\]
where $C_I > 0$ is a uniform constant depending only on $g_0$.

**Remark 1.1.** From Theorem 1.2 we can get a uniform lower bound for the isoperimetric constant in $(M, g(t))$ as follows. There holds
\[
I(M, g(t)) := \inf_{V \subset M} \frac{\text{Area}_{g(t)}(\partial V)}{\left[ \min \{\text{Vol}_{g(t)}(V), \text{Vol}_{g(t)}(M - V)\} \right]^{\frac{1}{n-1}}} \geq \delta,
\]
where $V$ is a subdomain of $M$ such that $\partial V$ is an $n-1$ dimensional submanifold of $M$, and $\delta$ is a positive constant depending only on initial metric $g_0$. A proof can be found in Section 5.1 of [9].

**Remark 1.2.** In the case of Kähler-Ricci flow, the theorem and the isoperimetric inequality above were obtained in Tian-Zhang [42].
2. SOME BASIC GRADIENT ESTIMATES

In this section, we give some gradient estimates of harmonic function and solution to heat equation on $n \geq 3$ dimensional Riemannian manifold $M$ with a fixed Riemannian metric $g$. We need the assumptions as follows.

**Assumption 1.** For $f \in W^{1,2}(M)$, there holds $L^2$ Sobolev inequality:

$$
\left( \int_M |f|^{\frac{2n}{n-2}} \, d\mu \right)^{\frac{n-2}{n}} \leq A \int_M \left[ |\nabla f|^2_g + f^2 \right] \, d\mu,
$$

where $A$ is a positive constant.

**Assumption 2.** There exists a positive constant $\Lambda$ such that

$$
\Lambda r^n \leq \text{Vol}_g(B(x, r)) \leq \Lambda^{-1}r^n, \quad 0 < r < \text{diam}(M) =: d \leq 1.
$$

**Assumption 3.** There exists a symmetric 2-tensor $S$ such that the Ricci curvature $R_{ij} \geq S_{ij}$ and

$$
S_{ij} = P_{ij}^{k\ell} \nabla_k \nabla_\ell L + Q_{ij}^{k\ell} g_{k\ell},
$$

where $L \in C^\infty(M, \mathbb{R})$, $P$ and $Q$ are both $(2,2)$-tensors and $P$ is parallel. Moreover, assume $\|P\|_\infty \leq 1$ and $\|Q\|_\infty \leq 1$.

In particular, for the twisted Kähler-Ricci flow (1.2),

$$
S_{\tau\bar{\tau}} = R_{\tau\bar{\tau}} - \theta_{\tau\bar{\tau}} = g_{\tau\bar{\tau}} + \partial_\tau \partial_{\bar{\tau}} h
$$

is such a kind of tensor.

**Lemma 2.1 (Moser[30]).** Let $(M, g)$ be an $n$-dimensional Riemannian manifold and $u$ be a non-negative solution of parabolic equation

$$
\partial_t u - \Delta u = 0
$$

on $M \times [0, +\infty)$. Then we have

$$
\sup_{t \in [\sigma T, T]} \int_{B(x_0, \sigma r)} u^2 \, d\mu \leq C \left( \frac{1}{(\mu - \sigma)^2 r^2} + \frac{1}{(\theta - \tau) T} \right) \int_{\tau T}^T \int_{B(x_0, \mu r)} u^2 \, d\mu \, dt
$$

and

$$
\int_{\tau T}^T \int_{B(x_0, \sigma r)} |\nabla u|^2 \, d\mu \leq C \left( \frac{1}{(\mu - \sigma)^2 r^2} + \frac{1}{(\theta - \tau) T} \right) \int_{\tau T}^T \int_{B(x_0, \mu r)} u^2 \, d\mu \, dt,
$$

where $C > 0$ is a numerical constant and $0 < \sigma < \mu < 1, 0 < \tau < \theta < 1$.

**Proof.** For any $\varphi \in C^\infty(M \times [0, \infty))$ with support in $B(x_0, \mu r)$, combining (2.2) and the Stokes’ theorem, we have

$$
\frac{1}{2} \int_{\tau T}^T \int_{B(x_0, \mu r)} \varphi^2 \partial_t (u^2) \, d\mu \, dt + \int_{\tau T}^T \int_{B(x_0, \mu r)} |\nabla u|^2 \varphi^2 \, d\mu \, dt = -2 \int_{\tau T}^T \int_{B(x_0, \mu r)} u \varphi \nabla u \cdot \nabla \varphi, \tag{2.5}
$$

where we choose $u \varphi^2$ as a trial function.
From the Cauchy-Schwarz inequality, we get

\[ -2u\varphi \nabla u \cdot \nabla \varphi \leq \frac{1}{2} |\nabla u|^2 \varphi^2 + 2|\nabla \varphi|^2 u^2 \]  

(2.6)

Using (2.5) and (2.6), we arrive at

\[ \int_{\tau T}^{T} \int_{B(x_0, \mu r)} \partial_t (u^2 \varphi^2) d\mu dt + \int_{\tau T}^{T} \int_{B(x_0, \mu r)} |\nabla u|^2 \varphi^2 d\mu dt \]

\[ \leq 4 \int_{\tau T}^{T} \int_{B(x_0, \mu r)} u^2 (|\nabla \varphi|^2 + |\varphi \partial_t \varphi|) d\mu dt. \]  

(2.7)

Now choose \( \varphi(x, t) = \varphi_1(\text{dist}(x_0, x)) \varphi_2(t) \), where

\[ \varphi_1(\rho) \begin{cases} = 1, & 0 \leq \rho \leq \sigma r, \\ = 0, & \rho \geq \mu r, \\ \in [0, 1], & \sigma r \leq \rho \leq \mu r, \end{cases} \]

and

\[ \varphi_2(t) \begin{cases} = 0, & 0 \leq t \leq \tau T, \\ = 1, & t \geq \theta T, \\ \in [0, 1], & \tau T \leq t \leq \theta T, \end{cases} \]

such that

\[ |\varphi'_1| \leq \frac{4}{(\mu - \sigma)r}, \quad |\varphi'_2| \leq \frac{4}{(\theta - \tau)T}. \]

Then we have

\[ \sup_{t \in [\theta T, T]} \int_{B(x_0, \sigma r)} u^2 d\mu := \int_{B(x_0, \sigma r)} [u(\cdot, T')]^2 d\mu \]

\[ = \int_{\tau T}^{T} \int_{B(x_0, \sigma r)} \partial_t (u^2 \varphi^2) d\mu dt \]

\[ \leq 4 \int_{\tau T}^{T} \int_{B(x_0, \mu r)} u^2 (|\nabla \varphi|^2 + |\varphi \partial_t \varphi|) d\mu dt \]

\[ \leq C \left( \frac{1}{(\mu - \sigma)^2 T^2} + \frac{1}{(\theta - \tau)T} \right) \int_{\tau T}^{T} \int_{B(x_0, \mu r)} u^2 d\mu dt, \]

which implies (2.3).

Since

\[ \int_{\tau T}^{T} \int_{B(x_0, \mu r)} \partial_t (u^2 \varphi^2) d\mu dt = \int_{B(x_0, \mu r)} [u(\cdot, T) \varphi(T)]^2 d\mu \geq 0, \]

we have

\[ \int_{\tau T}^{T} \int_{B(x_0, \mu r)} |\nabla u|^2 \varphi^2 d\mu dt \leq 4 \int_{\tau T}^{T} \int_{B(x_0, \mu r)} u^2 (|\nabla \varphi|^2 + |\varphi \partial_t \varphi|) d\mu dt \]

\[ \leq C \left( \frac{1}{(\mu - \sigma)^2 T^2} + \frac{1}{(\theta - \tau)T} \right) \int_{\tau T}^{T} \int_{B(x_0, \mu r)} u^2 d\mu dt, \]
which implies \((2.4)\).

\[\square\]

**Lemma 2.2.** Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold satisfying Assumptions 1, 2, 3. Then

(a) for smooth harmonic function \(u\) in \(B(x_0, r)\) with \(r \leq d\), we have

\[
\sup_{x \in B(x_0, \frac{r}{2})} |\nabla u(x)| \leq C \frac{1}{r} \left( \frac{1}{r^n} \int_{B(x_0, r)} u^2 d\mu \right)^{\frac{1}{2}},
\]

where \(C\) is a positive constant depending only on \(A, n, P, Q\) and \(\|\nabla L\|_{\infty}\).

(b) for a non-negative smooth function \(u\) on \(M \times [0, \infty)\) satisfying

\[
\partial_t u - \Delta u = 0,
\]

we have

\[
|\nabla u(x, t)| \leq \frac{C}{\sqrt{\eta t}} \left( \frac{2}{(\eta \tilde{t})^{\frac{n}{p}}} \int_{t-\tilde{t}}^{t} \int_{B(x, \sqrt{\eta t})} u^2 d\mu dt \right)^{\frac{1}{2}},
\]

where \(0 < \eta \leq 1\) is a parameter, \(\tilde{t} = \min\{t, d^2\}\) and \(C\) depends only on \(n, A\) and \(\|\nabla L\|_{\infty}\).

**Proof.** Proof of Item (a).

Since \(u\) is a harmonic function, it follows from the Bochner’s formula that

\[
\Delta |\nabla u|^2 = 2|\text{Hess} u|^2 + 2\text{Ric}(\nabla u, \nabla u).
\]

Given \(0 < \sigma < \mu \leq 1\), define Lipschitz cut-off function \(\psi(x) \in [0, 1]\) such that

\[
\psi(x) = \begin{cases} 
1, & x \in B(x_0, \sigma r), \\
0, & x \in M - B(x_0, \mu r)
\end{cases}
\]

and

\[
|\nabla \psi| \leq \frac{4}{(\mu - \sigma) r}.
\]

Setting \(f = |\nabla u|^2\), for \(p \geq 1\), from the Stokes’ theorem, we can get

\[
-\int_{B(x_0, \mu r)} (\Delta f) f^{2p-1}\psi^2 = \frac{2p-1}{p^2} \int_{B(x_0, \mu r)} |\nabla (f^p \psi)|^2 - \frac{1}{p^2} \int_{B(x_0, \mu r)} f^{2p} |\nabla \psi|^2
\]

\[
-\frac{2(p-1)}{p^2} \int_{B(x_0, \mu r)} f^p \nabla (f^p \psi) \cdot \nabla \psi.
\]

Using the Cauchy-Schwarz inequality, we deduce

\[
\int_{B(x_0, \mu r)} f^p \nabla (f^p \psi) \cdot \nabla \psi \leq \frac{\epsilon}{2} \int_{B(x_0, \mu r)} |\nabla (f^p \psi)|^2 + \frac{1}{2\epsilon} \int_{B(x_0, \mu r)} f^{2p} |\nabla \psi|^2.
\]
Combining (2.11), (2.12) and (2.13) together, we obtain

\[-2 \int_{B(x_0, \mu r)} |\text{Hess} u|^2 f^{2p-1} \psi^2 - 2 \int_{B(x_0, \mu r)} \text{Ric}(\nabla u, \nabla u) f^{2p-1} \psi^2 \]

\[= \frac{2p - 1}{p^2} \int_{B(x_0, \mu r)} |\nabla (f^p \psi)|^2 - \frac{1}{p^2} \int_{B(x_0, \mu r)} f^{2p} |\nabla \psi|^2 - \frac{2(p - 1)}{p^2} \int_{B(x_0, \mu r)} f^p \nabla (f^p \psi) \cdot \nabla \psi \]

\[\geq \left( \frac{2p - 1}{p^2} - \epsilon(p - 1) \right) \int_{B(x_0, \mu r)} |\nabla (f^p \psi)|^2 - \left( \frac{1}{p^2} + \frac{p - 1}{\epsilon p^2} \right) \int_{B(x_0, \mu r)} f^{2p} |\nabla \psi|^2. \]

Taking

\[\epsilon = \frac{2p - 1}{2(p - 1)},\]

we have

\[\frac{2p - 1}{2p^2} \int_{B(x_0, \mu r)} |\nabla (f^p \psi)|^2 \leq \frac{2p^2 - 2p + 1}{p^2(2p - 1)} \int_{B(x_0, \mu r)} |\nabla \psi|^2 f^{2p} - 2 \int_{B(x_0, \mu r)} |\text{Hess} u|^2 f^{2p-1} \psi^2 \]

\[-2 \int_{B(x_0, \mu r)} \text{Ric}(\nabla u, \nabla u) f^{2p-1} \psi^2 \]

\[\leq \frac{2p^2 - 2p + 1}{p^2(2p - 1)} \int_{B(x_0, \mu r)} |\nabla \psi|^2 f^{2p} - 2 \int_{B(x_0, \mu r)} |\text{Hess} u|^2 f^{2p-1} \psi^2 \]

\[-2 \int_{B(x_0, \mu r)} \mathcal{S}(\nabla u, \nabla u) f^{2p-1} \psi^2 \]

\[= I_1 + I_2 + I_3. \tag{2.14}\]

Due to the Stokes’ theorem and the fact that \( P \) is parallel, we derive

\[I_3 = -2 \int_{B(x_0, \mu r)} \mathcal{S}(\nabla u, \nabla u) f^{2p-1} \psi^2 \]

\[= -2 \int_{B(x_0, \mu r)} P^{k}_{ij} \nabla_k \nabla \ell L \nabla_i u \nabla_j u f^{2p-1} \psi^2 - 2 \int_{B(x_0, \mu r)} Q^{k}_{ij} g_{k \ell} \nabla_i u \nabla_j u f^{2p-1} \psi^2 \]

\[= 2 \int_{B(x_0, \mu r)} P^{k}_{ij} \nabla \ell L \nabla_k \nabla_i u \nabla_j u f^{2p-1} \psi^2 + 2 \int_{B(x_0, \mu r)} P^{k}_{ij} \nabla \ell L \nabla_i u \nabla_j u f^{2p-1} \psi^2 \]

\[+ 2 \int_{B(x_0, \mu r)} P^{k}_{ij} \nabla \ell L \nabla_i u \nabla_j u \nabla_k (f^{2p-1} \psi^2) - 2 \int_{B(x_0, \mu r)} Q^{k}_{ij} g_{k \ell} \nabla_i u \nabla_j u f^{2p-1} \psi^2. \tag{2.15}\]

From the Young’s inequality, we have

\[\int_{B(x_0, \mu r)} P^{k}_{ij} \nabla \ell L \nabla_k \nabla_i u \nabla_j u f^{2p-1} \psi^2 \]

\[\leq \frac{\epsilon_1}{2} \int_{B(x_0, \mu r)} |\nabla_k \nabla_i u|^2 |\nabla_j u|^2 f^{2p-2} \psi^2 + \frac{1}{2\epsilon_1} \int_{B(x_0, \mu r)} |P^{k}_{ij}|^2 |\nabla \ell L|^2 f^{2p} \psi^2 \tag{2.16}\]

\[\leq \frac{\epsilon_1}{2} \int_{B(x_0, \mu r)} |\text{Hess} u|^2 f^{2p-1} \psi^2 + \frac{1}{2\epsilon_1} C \|\nabla L\|_{L^\infty}^2 \int_{B(x_0, \mu r)} f^{2p} \psi^2, \]
and
\[ \int_{B(x_0,\mu r)} P_{ij}^{k\ell} \nabla_i L \nabla_j u \nabla_k \nabla_j u f^{2p-1} \psi^2 \]
\[ \leq \frac{\epsilon_2}{2} \int_{B(x_0,\mu r)} |\text{Hess} u|^2 f^{2p-1} \psi^2 + \frac{1}{2\epsilon_2} C \|\nabla L\|_{L^\infty}^2 \int_{B(x_0,\mu r)} f^{2p} \psi^2. \]  
(2.18)

Moreover, we also have
\[ \int_{B(x_0,\mu r)} P_{ij}^{k\ell} \nabla_i L \nabla_i u \nabla_j u \nabla_k (f^{2p-1} \psi^2) = \frac{2p-1}{p} \int_{B(x_0,\mu r)} P_{ij}^{k\ell} \nabla_i L \nabla_j u \nabla_j u f^{2p-1} \psi \nabla_k (\psi f^p) \]
\[ + \frac{1}{p} \int_{B(x_0,\mu r)} P_{ij}^{k\ell} \nabla_i L \nabla_i u \nabla_j u f^{2p-1} \psi \nabla_k \psi. \]  
(2.19)

By the Cauchy-Schwarz inequality, we can get
\[ \int_{B(x_0,\mu r)} P_{ij}^{k\ell} \nabla_i L \nabla_i u \nabla j u f^{2p-1} \psi \nabla_k (\psi f^p) \]
\[ \leq \frac{\epsilon_3}{2} \int_{B(x_0,\mu r)} |\nabla (\psi f^p)|^2 + \frac{1}{2\epsilon_3} \int_{B(x_0,\mu r)} |P_{ij}^{k\ell}|^2 |\nabla L|^2 f^{2p} \psi^2 \]  
(2.20)
\[ \leq \frac{\epsilon_3}{2} \int_{B(x_0,\mu r)} |\nabla (\psi f^p)|^2 + \frac{1}{2\epsilon_3} C \|\nabla L\|_{L^\infty}^2 \int_{B(x_0,\mu r)} f^{2p} \psi^2 \]

and
\[ \int_{B(x_0,\mu r)} P_{ij}^{k\ell} \nabla_i L \nabla_i u \nabla_j u f^{2p-1} \psi \nabla_k \psi \]
\[ \leq \frac{\epsilon_4}{2} \int_{B(x_0,\mu r)} |\nabla L|^2 f^{2p} |\nabla \psi|^2 + \frac{1}{2\epsilon_4} \int_{B(x_0,\mu r)} |P_{ij}^{k\ell}|^2 f^{2p} \psi^2 \]  
(2.21)
\[ \leq \frac{\epsilon_4}{2} \|\nabla L\|_{L^\infty}^2 \int_{B(x_0,\mu r)} f^{2p} |\nabla \psi|^2 + \frac{1}{2\epsilon_4} C \int_{B(x_0,\mu r)} f^{2p} \psi^2. \]

Using the Cauchy-Schwarz inequality again, we also obtain
\[ \int_{B(x_0,\mu r)} Q_{ij}^{k\ell} g_{kl} \nabla_i u \nabla_j u f^{2p-1} \psi^2 \leq C \int_{B(x_0,\mu r)} f^{2p} \psi^2. \]  
(2.22)

Therefore, combining (2.16)–(2.22), we can deduce
\[ I_3 \leq \frac{2p-1}{4p^2} \int_{B(x_0,\mu r)} |\nabla (f^p \psi)|^2 + \int_{B(x_0,\mu r)} |\text{Hess} u|^2 f^{2p-1} \psi^2 \]
\[ + C \left( \frac{\|\nabla L\|_{L^\infty}^2}{(\mu - \sigma)^2 r^2} + \|\nabla L\|_{L^\infty}^2 + 1 \right) \int_{B(x_0,\mu r)} f^{2p}. \]  
(2.23)

Substituting (2.23) into (2.15) yields
\[ \int_{B(x_0,\mu r)} |\nabla (f^p \psi)|^2 \leq C p^2 \left( \frac{\|\nabla L\|_{L^\infty}^2}{(\mu - \sigma)^2 r^2} + \|\nabla L\|_{L^\infty}^2 + \frac{1}{(\mu - \sigma)^2 r^2} + 1 \right) \int_{B(x_0,\mu r)} f^{2p}, \]

where C is independent of p.
Since $d \leq 1$, it holds that for $p \geq 1$
\[
\int_{B(x_0, \mu r)} |\nabla (f^p \psi)|^2 \leq C p^2 \left( \frac{\|\nabla L\|_\infty^2 + 1}{(\mu - \sigma)^2 r^2} \right) \int_{B(x_0, \mu r)} f^{2p}.
\]

Setting
\[
\chi = \frac{n}{n-2},
\]
from the Sobolev inequality (2.1), we have
\[
\left( \int_{B(x_0, \mu r)} |f^p \psi|^{2\chi} \right)^{\frac{1}{\chi}} \leq A \left( \int_{B(x_0, \mu r)} |\nabla (f^p \psi)|^2 + \int_{B(x_0, \mu r)} \psi^2 |f|^{2p} \right)
\]
\[
\leq C p^2 \left( \frac{\|\nabla L\|_\infty^2 + 1}{(\mu - \sigma)^2 r^2} \right) \int_{B(x_0, \mu r)} f^{2p},
\]
that is, for $p \geq 2$, we get
\[
\left( \int_{B(x_0, \sigma r)} |f|^p \right)^{\frac{1}{p}} \leq C_{\rho p} \frac{1}{p} \left( \frac{(\|\nabla L\|_\infty^2 + 1)}{(\mu - \sigma)^2 r^2} \right)^{\frac{1}{p}} \left( \int_{B(x_0, \mu r)} f^p \right)^{\frac{1}{p}}. \tag{2.24}
\]

Letting
\[
\gamma_i = p\chi^{i-1}, \quad r_i = \frac{r}{2} + \frac{r}{4^i},
\]
we have
\[
\gamma_{i+1} = \gamma_i \chi, \quad r_i - r_{i+1} = \frac{3r}{4^{i+1}}.
\]

It follows from (2.24) that
\[
\|f\|_{L^{\gamma_{k+1}}(B(x_0, r_{k+1}))} = \|f\|_{L^{\gamma_k}(B(x_0, r_k))}
\]
\[
\leq C_{\gamma_k} \frac{\gamma_k}{\gamma_k} \left( \frac{\|\nabla L\|_\infty^2 + 1}{r_{k+1}^2} \right)^{\frac{1}{\gamma_k}} \left( \int_{B(x_0, r_k)} f^{\gamma_k} \right)^{\frac{1}{\gamma_k}}
\]
\[
\leq (C(\|\nabla L\|_\infty^2 + 1))^{\frac{1}{p}} \sum_{i=1}^{k} \frac{1}{\chi} \left( \int_{B(x_0, r_i)} |f|^{p_i} \right)^{\frac{1}{p_i}}
\]
\[
\times 16^{\frac{1}{p}} \sum_{i=1}^{k} \frac{1}{\chi} \left( \int_{B(x_0, \frac{3r}{4^i})} |f|^{p_i} \right)^{\frac{1}{p_i}}.
\]

As $k \to \infty$, we can arrive at
\[
\|f\|_{L^{\infty}(B(x_0, \frac{3r}{4^i}))} \leq C(A, n, p, \|\nabla L\|_\infty) r^{-\frac{n}{p}} \left( \int_{B(x_0, \frac{3r}{4^i})} |f|^p \right)^{\frac{1}{p}}.
\]

Now choosing $p = 2$, we have
\[
\|f\|_{L^{\infty}(B(x_0, \frac{3r}{4^i}))} \leq C(A, n, \|\nabla L\|_\infty) r^{-\frac{n}{2}} \left( \int_{B(x_0, \frac{3r}{4^i})} |f|^2 \right)^{\frac{1}{2}}.
\]
For $q \in (0, 2)$, applying the Young's inequality we deduce
\[
\|f\|_{L^\infty(B(x_0, \frac{3r}{4}))} \leq Cr^{-\frac{n}{2}} \left( \int_{B(x_0, \frac{3r}{4})} |f|^q \right)^{\frac{1}{q}}
\leq Cr^{-\frac{n}{2}} \|f\|_{L^\infty(B(x_0, \frac{3r}{4}))}^{\frac{1}{2}} \left( \int_{B(x_0, \frac{3r}{4})} |f|^q \right)^{\frac{1}{q}}
\leq \frac{1}{2} \|f\|_{L^\infty(B(x_0, \frac{3r}{4}))} + Cr^{-\frac{n}{2}} \left( \int_{B(x_0, \frac{3r}{4})} |f|^q \right)^{\frac{1}{q}}.
\]

Therefore, by Lemma 4.3 in [27] we have
\[
\|f\|_{L^\infty(B(x_0, \frac{3r}{4}))} \leq Cr^{-\frac{n}{2}} \|f\|_{L^q(B(x_0, \frac{3r}{4}))}.
\]

Taking $q = 1$, we get
\[
\sup_{x \in B(x_0, \frac{3r}{4})} |\nabla u|^2 \leq Cr^{-n} \int_{B(x_0, \frac{3r}{4})} |\nabla u|^2.
\]

Since
\[
0 = -\int_{B(x_0, r)} (\Delta u)(u\psi^2) = \int_{B(x_0, r)} \nabla u \cdot \nabla (u\psi^2)
\]
\[
= \int_{B(x_0, r)} \psi^2 \nabla u \cdot \nabla u + \int_{B(x_0, r)} 2u\psi \nabla u \cdot \nabla \psi
\]
\[
\geq \int_{B(x_0, r)} \psi^2 |\nabla u|^2 - \left( \int_{B(x_0, r)} \frac{1}{2} \psi^2 |\nabla u|^2 + 2u^2 \nabla \psi \cdot \nabla \psi \right),
\]
choosing $\mu = 1$, $\sigma = \frac{3}{4}$ in the cut-off function $\psi$, we have
\[
\int_{B(x_0, \frac{3r}{4})} |\nabla u|^2 \leq \int_{B(x_0, r)} \psi^2 \nabla u \cdot \nabla u
\leq 4 \int_{B(x_0, r)} u^2 \nabla \psi \cdot \nabla \psi
\leq \frac{1024}{r^2} \int_{B(x_0, r)} u^2.
\]

Therefore, we can deduce (2.8).

Proof of Item (b). Since $u$ is a non-negative solution of (2.9), using the Bochner's formula again, we have
\[
\Delta |\nabla u|^2 = \partial_t |\nabla u|^2 + 2|\text{Hess}u|^2 + 2\text{Ric}(\nabla u, \nabla u) \quad (2.25)
\]
$\psi$ is also the cut-off function in the proof of Item (a). For $p \geq 1$, also setting $f = |\nabla u|^2$, from (2.25), we can get
\[ \int_{B(x_0, \mu r)} f^{2p-1} \psi^2 f_t \, d\mu - \int_{B(x_0, \mu r)} \psi^2 f^{2p-1} \Delta f \, d\mu \]

\[ = - \int_{B(x_0, \mu r)} (2|\text{Hess}u|^2 + 2\text{Ric}(\nabla u, \nabla u)) \psi^2 f^{2p-1} \, d\mu. \tag{2.26} \]

On the other hand, we also have

\[ \int_{B(x_0, \mu r)} f^{2p-1} \psi^2 f_t \, d\mu = \frac{1}{2p} \int_{B(x_0, \mu r)} \psi^2 (\partial_t f^{2p}) \, d\mu \]

\[ = \frac{1}{2p} \int_{B(x_0, \mu r)} \psi^2 f^{2p} \, d\mu. \tag{2.27} \]

Similar to the process to get (2.15), from (2.26) and (2.27), we derive

\[ \frac{1}{2p} \partial_t \int_{B(x_0, \mu r)} \psi^2 f^{2p} \, d\mu + \frac{1}{p} \int_{B(x_0, \mu r)} |\nabla (f^p \psi)|^2 \, d\mu \]

\[ \leq \frac{1}{p} \int_{B(x_0, \mu r)} |\nabla \psi|^2 f^{2p} - 2 \int_{B(x_0, \mu r)} |\text{Hess}u|^2 f^{2p-1} \psi^2 \]

\[ - 2 \int_{B(x_0, \mu r)} \text{Ric}(\nabla u, \nabla u) f^{2p-1} \psi^2 \]

\[ \leq \frac{1}{p} \int_{B(x_0, \mu r)} |\nabla \psi|^2 f^{2p} - 2 \int_{B(x_0, \mu r)} |\text{Hess}u|^2 f^{2p-1} \psi^2 \]

\[ - 2 \int_{B(x_0, \mu r)} \mathcal{S}(\nabla u, \nabla u) f^{2p-1} \psi^2 \]

\[ =: I_1 + I_2 + I_3. \tag{2.28} \]

Substituting (2.23) into (2.28), we arrive at

\[ \partial_t \int_{B(x_0, \mu r)} \psi^2 f^{2p} \, d\mu + \int_{B(x_0, \mu r)} |\nabla (\psi f^p)|^2 \, d\mu \leq C p^2 \left( \|\nabla L\|_{\infty}^2 + \frac{1}{(\mu - \sigma)^2 r^2} \right) \int_{B(x_0, \mu r)} f^{2p}, \]

which implies

\[ \partial_t \int_{B(x_0, \mu r)} \psi^2 f^{2p} \, d\mu + \int_{B(x_0, \mu r)} (\psi^2 f^{2p} + |\nabla (\psi f^p)|^2) \, d\mu \]

\[ \leq C p^2 \left( \|\nabla L\|_{\infty}^2 + \frac{1}{(\mu - \sigma)^2 r^2} \right) \int_{B(x_0, \mu r)} f^{2p}, \tag{2.29} \]

where we use the facts that \( d \leq 1 \) and \( p \geq 1 \).

Define

\[ \eta(t) = \begin{cases} 
0, & 0 \leq t \leq \tau T, \\
\frac{t - \tau T}{(\theta - \tau)T}, & \tau T \leq t \leq \theta T, \\
1, & \theta T \leq t \leq T.
\end{cases} \]
Multiplying both sides of (2.29) by \( \eta(t) \), it leads to

\[
\partial_t \left( \eta(t) \int_{B(x_0, \mu r)} \psi^2 f^{2p} d\mu \right) + \eta(t) \int_{B(x_0, \mu r)} \left( \psi^2 f^{2p} + |\nabla (\psi f^p)|^2 \right) d\mu \\
\leq \left( \frac{C p^2 (\| \nabla L \|_\infty^2 + 1)}{(\mu - \sigma)^2 r^2} + \eta'(t) \right) \int_{B(x_0, \mu r)} f^{2p} d\mu,
\]

which implies

\[
\sup_{\theta T \leq t \leq T} \int_{B(x_0, \mu r)} \psi^2 f^{2p} d\mu + \int_{\theta T}^T \int_{B(x_0, \mu r)} \left( f^{2p} \psi^2 + |\nabla (\psi f^p)|^2 \right) d\mu dt \\
\leq 2 \left( \frac{C p^2 (\| \nabla L \|_\infty^2 + 1)}{(\mu - \sigma)^2 r^2} + \frac{1}{(\theta - \tau) T} \right) \int_{\tau T}^T \int_{B(x_0, \mu r)} f^{2p} d\mu dt. \tag{2.30}
\]

From the Sobolev inequality (2.1) and (2.30), we have

\[
\int_{\theta T}^T \int_{B(x_0, \sigma r)} f^{2p(1 + \frac{2}{n})} d\mu dt \\
\leq \int_{\theta T}^T \int_{B(x_0, \mu r)} (\psi f^p)^{2(1 + \frac{2}{n})} d\mu dt \\
\leq \int_{\theta T}^T \left( \int_{B(x_0, \mu r)} \psi^2 f^{2p} d\mu \right)^{\frac{2}{n}} \left( \int_{B(x_0, \mu r)} (\psi f^p)^{\frac{2n}{n-2}} d\mu \right)^{\frac{n-2}{n}} dt \\
\leq \left( \sup_{\theta T \leq t \leq T} \int_{B(x_0, \mu r)} \psi^2 f^{2p} d\mu \right)^{\frac{2}{n}} \int_{\theta T}^T \left( \int_{B(x_0, \mu r)} (\psi f^p)^{\frac{2n}{n-2}} d\mu \right)^{\frac{n-2}{n}} dt \\
\leq A \left( \sup_{\theta T \leq t \leq T} \int_{B(x_0, \mu r)} \psi^2 f^{2p} d\mu \right)^{\frac{2}{n}} \int_{\theta T}^T \left( \int_{B(x_0, \mu r)} (\psi^2 f^{2p} + |\nabla (\psi f^p)|^2) d\mu \right) dt \\
\leq C^{1 + \frac{2}{n}} p^{2(1 + \frac{2}{n})} \left( \frac{1}{(\mu - \sigma)^2 r^2} + \frac{1}{(\theta - \tau) T} \right)^{1 + \frac{2}{n}} \left( \int_{\tau T}^T \int_{B(x_0, \mu r)} f^{2p} d\mu dt \right)^{1 + \frac{2}{n}}. \tag{2.31}
\]

Defining

\[
H(p, r, t) = \left( \int_{\theta T}^T \int_{B(x_0, r)} f^p d\mu dt \right)^{\frac{1}{p}}, \quad \chi = \frac{n + 2}{n},
\]

for \( p_0 \geq 2 \) fixed, then (2.31) implies

\[
H(p_0 \chi, r_2, t_2) \leq C^{\frac{1}{p_0}} p_0^{\frac{2}{n}} \left( \frac{1}{(r_1 - r_2)^2} + \frac{1}{t_2 - t_1} \right)^{\frac{1}{p_0}} H(p_0, r_1, t_1). \tag{2.32}
\]

Setting

\[
\gamma_i = p_0 \chi_i^{-1}, \quad r_i = \sigma r + \frac{(\mu - \sigma) r}{2i - 1}, \quad t_i = \tau T + \left( 1 - \frac{1}{2i - 1} \right) (\theta - \tau) T,
\]

for \( i \geq 1 \), which implies

\[
H(p_0 \chi_i, r_i, t_i) \leq C^{\frac{1}{p_0}} p_0^{\frac{2}{n}} \left( \frac{1}{(r_1 - r_2)^2} + \frac{1}{t_2 - t_1} \right)^{\frac{1}{p_0}} H(p_0, r_1, t_1).
\]
from (2.32), we have

\[ H(\gamma_{i+1}, r_{i+1}, t_{i+1}) \leq C_\gamma \sqrt{\gamma_i} \left( \frac{1}{(r_i - r_{i+1})^2} + \frac{1}{t_{i+1} - t_i} \right)^{\frac{1}{4}} H(\gamma_i, r_i, t_i) \]

\[ = C_{p_0} \frac{1}{p_0 \chi^{2(\ell-1)}_{\gamma}} + \frac{1}{p_0 \chi^{2(\ell-1)}_{\gamma}} \times \left( \frac{1}{(\mu - \sigma)^2 r^2 + \frac{1}{(\theta - \tau)T}} \right)^{\frac{1}{p_0 \chi^{2(\ell-1)}_{\gamma}}} H(\gamma_i, r_i, t_i) \]  (2.33)

Letting \( i \to \infty \) in (2.33) yields

\[
\sup_{B(x_0, r\tau) \times [\theta T, T]} f(x, t) = \lim_{i \to \infty} H(\gamma_{i+1}, r_{i+1}, t_{i+1}) \\
\leq C \left( \frac{1}{(\mu - \sigma)^2 r^2} + \frac{1}{(\theta - \tau)T} \right)^{\frac{n+2}{4p_0}} \left( \int_{\tau T}^{T} \int_{B(x_0, \mu r)} f^{p_0} d\mu dt \right)^{\frac{1}{p_0}},
\]

where \( C \) depends on \( p_0, A, n, \|\nabla L\|_{\infty} \) and the upper bound of \( \text{diam}(M) \). Taking \( p_0 = 2 \), from the inequality above, we get

\[
\sup_{B(x_0, r\tau) \times [\theta T, T]} f(x, t) \leq C \left( \frac{1}{(\mu - \sigma)^2 r^2} + \frac{1}{(\theta - \tau)T} \right)^{\frac{n+2}{4}} \left( \int_{\tau T}^{T} \int_{B(x_0, \mu r)} f^2 d\mu dt \right)^{\frac{1}{2}}.  \\ (2.34)
\]

For \( p_0 \in (0, 2) \), there holds

\[
\left( \int_{\tau T}^{T} \int_{B(x_0, \mu r)} f^2 d\mu dt \right)^{\frac{1}{2}} \leq \left( \sup_{B(x_0, \mu r) \times [\tau T, T]} f^{2-p_0} \int_{\tau T}^{T} \int_{B(x_0, \mu r)} f^{p_0} d\mu dt \right)^{\frac{1}{2}} \left( \int_{\tau T}^{T} \int_{B(x_0, \mu r)} f^{2} d\mu dt \right)^{\frac{1}{2}},  \\ (2.35)
\]

Defining

\[
h(\sigma, \theta) := \sup_{B(x_0, r\tau) \times [\theta T, T]} f(x, t),
\]

Combining (2.33) and (2.35), we deduce

\[
h(\sigma, \theta) \leq \frac{1}{2} h(\mu, \tau) + C \left( \frac{1}{(\mu - \sigma)^2 r^2} + \frac{1}{(\theta - \tau)T} \right)^{\frac{n+2}{2p_0}} \left( \int_{\tau T}^{T} \int_{B(x_0, \mu r)} f^{p_0} d\mu dt \right)^{\frac{1}{p_0}}.
\]
Then from Lemma 4.3 in [27], we have

\[
  h(\sigma, \theta) \leq C \left( \frac{1}{(\mu - \sigma)^2 r^2} + \frac{1}{(\theta - \tau)} \right) \int_{T}^{T} \int_{B(x_0, \mu r)} \eta \frac{1}{p_0} f^{p_0} \, d\mu \, dt \right)^{1 \over p_0}.
\]

Now if we choose \( p_0 = 1 \), we obtain

\[
  \sup_{B(x_0, \sigma r) \times [\theta T, T]} |\nabla u|^2 \leq C \left( \frac{1}{(\mu - \sigma)^2 r^2} + \frac{1}{(\theta - \tau)} \right) \int_{T}^{T} \int_{B(x_0, \mu r)} |\nabla u|^2 \, d\mu \, dt.
\]

Taking \( \tilde{t} = \min\{T, d^2\} \), \( \theta T = T - \frac{1}{2} \eta \tilde{t} \), \( \tau T = T - \frac{3}{4} \eta \tilde{t} \), \( \sigma = \frac{1}{2} \), \( \mu = \frac{3}{4} \), we get

\[
  |\nabla u(x_0, T)|^2 \leq \frac{C}{\eta t} \int_{T - \frac{3}{4} \eta t}^{T} \int_{B(x_0, \sqrt[4]{t}\eta t)} |\nabla u|^2 \, d\mu \, dt,
\]

where \( 0 < \eta \leq 1 \) is a parameter. Applying Lemma 2.1, we have

\[
  |\nabla u(x_0, T)|^2 \leq \frac{C}{\eta t} \int_{T - \eta t}^{T} \int_{B(x_0, \sqrt{t}\eta t)} |u|^2 \, d\mu \, dt,
\]

which leads to (2.10) as required.

**Lemma 2.3** (Grigor’yan [23] and Saloff-Coste [36]). Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold satisfying Assumptions 1, 2. Assume that \( u \) is a smooth harmonic function in \( B(x_0, r) \) where \( x_0 \in M, r \in (0, d) \). There exists a positive constant \( C_1 = C_1(A, p) \) such that

\[
  \sup_{x \in B(x_0, \sigma r)} |u(x)| \leq C_1 \left( 1 + \frac{1}{(\mu - \sigma)^2 r^2} \right) \left( \int_{B(x_0, \mu r)} u^p \, d\mu \right)^{1 \over p},
\]

where \( 0 < p < +\infty \) and \( 0 < \sigma < \mu \leq 1 \).

In addition, denote \( \tilde{t} = \min\{t, d^2\} \) and

\[
  Q_\delta = B \left( x_0, \delta \sqrt{\eta t} \right) \times [t - \delta \eta t, t], \quad Q = B \left( x_0, \sqrt{\eta t} \right) \times [t - \eta t, t].
\]

If \( u \) is a solution of heat equation (2.9) in the space time cube \( M \times [0, +\infty] \), then given \( 0 < \delta < 1 \), there holds

\[
  \sup_{(x, s) \in Q_\delta} |u(x, s)|^p \leq C_2 (1 - \delta)^{-\frac{n+2}{2}} (1 + \eta t) \left( \int_{Q} |u(\cdot, s)|^p \, d\mu \, ds \right)^{1 \over p},
\]

where \( C_2 \) depends on \( A \) and \( p \) with \( 0 < p < +\infty \) and \( 0 < \eta \leq 1 \) is a parameter.

**Proof.** Using the Sobolev inequality (2.11), by the standard Moser’s iteration, we can deduce (2.36) and (2.37). Here we omit the details of the proof. (See for example Grigor’yan [23] and Saloff-Coste [36].)

Denote by \( H(x, y, t) \) the heat kernel of heat equation (2.9). We have the following estimates for the heat kernel.
Lemma 2.4. Let \( (M, g) \) be an \( n \)-dimensional Riemannian manifold satisfying Assumptions 1, 2 and 3. Then we have

\[
H(x, y, t) \leq C_1 \left( 1 + \frac{[\text{dist}(x, y)]^2}{t} \right)^{\frac{2}{2} - \frac{n}{4} \epsilon} \eta^{-\frac{n}{4} \epsilon} e^{-\frac{[\text{dist}(x, y)]^2}{4t}}
\]  

(2.38)

and

\[
|\nabla_x H(x, y, t)| \leq C_2 \left( 1 + \frac{[\text{dist}(x, y)]^2}{t} \right)^{\frac{n+1}{2} \epsilon} \eta^{-\frac{n+1}{2} \epsilon} e^{-\frac{[\text{dist}(x, y)]^2}{4t}},
\]

(2.39)

where \( \tilde{t} = \min\{t, d^2\} \), \( C_1 \) depends on \( A, \Lambda \) and \( n \), and \( C_2 \) depends on \( A, \Lambda, n \) and \( \|\nabla L\|_{L^\infty(M)} \).

Proof. Fix \( \lambda \in \mathbb{R} \) and a bounded function \( \psi \) satisfying \( |\nabla \psi| \leq 1 \). For any nice complex function \( f \), set \( f_z(y) = e^{\lambda \psi(y)} \left[ e^{t \Delta} (e^{-\lambda \psi} f) \right] (y) \) for \( z = te^{\sqrt{\eta} \theta} > 0, t > 0, |\theta| \leq \frac{1}{2} \epsilon \), where \( 0 < \epsilon \ll 1 \) is a small parameter. Saloff-Coste [37] proved

\[
\|f_z\|^2 \leq e^{2\lambda(1+\epsilon)t} \|f\|^2.
\]  

(2.40)

Introduce function

\[
u(y, t) = e^{-\lambda \psi(y)} f_z(y) = \left[ e^{t \Delta} (e^{-\lambda \psi} f) \right] (y),
\]

where \( f \in L^2(M) \) is a real function. The function \( u \) satisfies the heat equation (2.37). Thus, from (2.37), we have

\[
|u(x, t)|^2 \leq C(1 + \eta t)^{\frac{n+2}{2} (\eta t)^{-\frac{n+2}{4}} \int_{t-\eta t}^t \int_B \mu ds} |u(z, s)|^2 ds.
\]  

(2.41)

Here for later use, we take \( 0 < \eta \ll 1 \) determined later.

Multiplying both sides of (2.41) by \( e^{2\lambda \psi(z)} \), from (2.40), we get

\[
e^{2\lambda \psi(z)} |u(x, t)|^2 \leq C(1 + \eta t)^{\frac{n+2}{2} (\eta t)^{-\frac{n+2}{4}} \int_{t-\eta t}^t \int_B \mu ds} \|f\|^2.
\]  

(2.42)

Take cut-off function \( \varphi(z) \) such that \( \varphi(z) = 1 \) on \( B(y, \sqrt{\eta t}) \) and \( \varphi(z) = 0 \) on \( M - B(y, (1+\epsilon)\sqrt{\eta t}) \), where \( 0 < \epsilon \ll 1 \) small enough. Choosing

\[
f(z) = \frac{\varphi(z) H(x, z, t)}{\|\varphi(z) H(x, z, t)\|_{L^2(M)}},
\]

we obtain

\[
e^{\lambda (\psi(z) - \psi(y))} \int_M e^{2\lambda \sqrt{\eta t} + \lambda (\psi(z) - \psi(y))} H(x, z, t) \frac{\varphi(z) H(x, z, t)}{\|\varphi(z) H(x, z, t)\|_{L^2(M)}} d\mu(z)
\geq e^{\lambda (\psi(z) - \psi(y))} \int_{B(y, (1+\epsilon)\sqrt{\eta t})} H(x, z, t) \frac{\varphi(z) H(x, z, t)}{\|\varphi(z) H(x, z, t)\|_{L^2(M)}} d\mu(z)
\geq e^{\lambda (\psi(z) - \psi(y))} \|\varphi(z) H(x, z, t)\|_{L^2(B(y, (1+\epsilon)\sqrt{\eta t}))}
\geq e^{\lambda (\psi(z) - \psi(y))} \|H(x, z, t)\|_{L^2(B(y, \sqrt{\eta t}))}.
\]  

(2.43)
Considering $H(x, z, t)$ as a function of $z$, from (2.32) and (2.43), we deduce

$$\|H(x, z, t)\|_{L^2(B(y, \sqrt{\eta_0^2}))} \leq C(1 + \eta \tilde{t})(\eta \tilde{t})^{-\frac{n+2}{4}} \left(\eta \tilde{t} - \frac{\eta}{2} \right)^{\frac{3}{4}} e^{3|\lambda|\sqrt{\eta_0^2 + \lambda^2(1+\varepsilon)t - \lambda(\psi(x) - \psi(y))}}$$

(2.44)

By using (2.37), we have

$$|H(x, y, t)|^2 \leq C(1 + \eta \tilde{t})^{-\frac{n+2}{4}} (1 + \eta \tilde{t})^{-\frac{n+2}{4}} (\eta \tilde{t})^{-\frac{n}{4} \left(\eta \tilde{t} - \frac{\eta}{2} \right)^{\frac{3}{4}} e^{3|\lambda|\sqrt{\eta_0^2 + \lambda^2(1+\varepsilon)t - \lambda(\psi(x) - \psi(y))}}\right)$$

i.e.

$$H(x, y, t) \leq C(1 + \eta \tilde{t})^{-\frac{n+2}{4}} (1 + \eta \tilde{t})^{-\frac{n+2}{4}} (\eta \tilde{t})^{-\frac{n}{4} \left(\eta \tilde{t} - \frac{\eta}{2} \right)^{\frac{3}{4}} e^{3|\lambda|\sqrt{\eta_0^2 + \lambda^2(1+\varepsilon)t - \lambda(\psi(x) - \psi(y))}}\right)$$

(2.45)

Combining (2.10) and (2.44), we can derive

$$|\nabla_x H(x, y, t)| \leq \frac{C}{\sqrt{\eta \tilde{t}}} \left(1 + \frac{1}{(\eta \tilde{t})^{\frac{n+2}{4}}} \int_{-\eta \tilde{t}}^{\tilde{t}} \int_{B(x, \sqrt{\eta \tilde{t}})} [H(y, z, s)]^2 d\mu(z) ds \right)^{\frac{1}{2}}$$

(2.46)

Finally, taking $\psi$ such that $\psi(x) - \psi(y) = \text{dist}(x, y)$ and

$$\lambda = \frac{\text{dist}(x, y)}{2(1+\varepsilon)t}, \quad \eta = \frac{1}{10 \left(1 + \frac{\text{dist}(x, y)^2}{t}\right)};$$

we can deduce (2.38) and (2.39).

**Lemma 2.5.** Let $(M, g)$ be an $n$-dimensional Riemannian manifold satisfying Assumptions 1, 2 and 3. Then we have

$$H(x, y, t) \geq C t^{-\frac{n}{2}} e^{-\frac{\|L\|_{L^\infty(M)}}{Ct}}$$

(2.47)

for $0 < t \leq d^2$ and $C$ depending only on $\Lambda, \Lambda, n$ and $\|\nabla L\|_{L^\infty(M)}$.

**Proof.** Since for $t \in (0, d^2]$, from (2.38), we have

$$H(x, x, t) \leq C t^{-\frac{n}{2}}.$$

By Assumption 2 and Theorem 7.2 in [13], we can deduce

$$H(x, x, t) \geq C t^{-\frac{n}{2}}, \quad 0 < t \leq d^2.$$

(2.48)

Combining (2.39) and (2.48), for $0 < t \leq d^2$, we get

$$|H(x, z, t) - H(y, z, t)| \leq C \text{dist}(x, y) t^{-\frac{n+1}{2}} \leq \frac{C \text{dist}(x, y)}{\sqrt{t}} H(z, z, t).$$
In particular, for $0 < t \leq d^2$, we have

$$|H(z, z, t) - H(y, z, t)| \leq C \frac{\text{dist}(z, y)}{\sqrt{t}} H(z, z, t)$$

and for $a$ small enough

$$|H(z, z, t) - H(y, z, t)| \leq \frac{1}{2} H(z, z, t)$$

if $\text{dist}(y, z) \leq a\sqrt{t} \leq ad$. Hence, we can derive

$$H(y, z, t) \geq C \frac{t^{-\frac{d}{2}}}{\text{Vol}(M)}$$

for $\text{dist}(y, z) \leq a\sqrt{t} \leq ad$. By the standard iteration (see Pages 394-395 of [43]), we can get (2.47). \hfill \Box

**Remark 2.1.** More details about diagonal lower bound of heat kernel can be found in [12].

**Lemma 2.6.** Let $(M, g)$ be an $n$-dimensional Riemannian manifold satisfying Assumptions 1, 2 and 3. Then for any $f \in C^\infty(M, \mathbb{R})$, we have Poincaré inequality

$$\int_M (f(z) - f_M)^2 \, d\mu(z) \leq C \int_M |\nabla f(z)|^2 \, d\mu(z),$$

(2.49)

where $C$ depends on $A, n, \Lambda$ and $\|\nabla L\|_{L^\infty}$ and

$$f_M = \frac{1}{\text{Vol}(M)} \int_M f(z) \, d\mu(z).$$

**Proof.** Applying (2.47) and Assumption 2, we have

$$H(x, y, d^2) \geq \frac{C}{\text{Vol}(M)}.$$ 

For any $f \in C^\infty(M, \mathbb{R})$,

$$u(x, t) = \int_M H(x, z, t)f(z) \, d\mu(z)$$

(2.50)

is the solution of heat equation (2.9) on $M$. By the lower bound of $H$, we can get

$$\int_M H(x, z, d^2) \left( f(z) - u(x, d^2) \right)^2 \, d\mu(z) \geq \frac{C}{\text{Vol}(M)} \int_M \left( f(z) - u(x, d^2) \right)^2 \, d\mu(z)$$

$$\geq \frac{C}{\text{Vol}(M)} \int_M \left( f(z) - f_M \right)^2 \, d\mu(z).$$

(2.51)

Integrating on $M$ for (2.51), we derive

$$\int_M \int_M H(x, z, d^2) \left( f(z) - u(x, d^2) \right)^2 \, d\mu(z) \, d\mu(x) \geq C \int_M \left( f(z) - f_M \right)^2 \, d\mu(z).$$

Since

$$\int_M H(x, z, t) \, d\mu(z) = 1,$$
it follows from (2.50) that
\[
\int_M \int_M H(x, z, d^2) \left( (f(z) - u(x, d^2))^2 d\mu(z) d\mu(x) \right) = \int_M \int_M H(x, z, d^2) \left( \left[ f(z) \right]^2 - 2f(z)u(x, d^2) + \left[ u(x, d^2) \right]^2 \right) d\mu(z) d\mu(x) \\
= \int_M \left[ f(z) \right]^2 d\mu(z) - \int_M \left[ u(x, d^2) \right]^2 d\mu(x) \\
= -\int_0^{d^2} \partial_s \int_M [u(x, s)]^2 d\mu(x) \\
= 2 \int_0^{d^2} \int_M |\nabla u(x, s)|^2 d\mu(x),
\]
where we use integration by parts and the fact that \( u \) is the solution of heat equation.

Noticing that
\[
\partial_s \int_M |\nabla u(x, s)|^2 d\mu(x) = \int_M \nabla u(x, s) \cdot \nabla \Delta u(x, s) d\mu(x) \\
= -\int_M (\Delta u(x, s))^2 d\mu(x) \leq 0,
\]
from (2.52), we have
\[
\int_M \int_M H(x, z, d^2) \left( (f(z) - u(x, d^2))^2 d\mu(z) d\mu(x) \right) \leq d^2 \int_M |\nabla f|^2 d\mu. \tag{2.53}
\]
Since \( d \leq 1 \) in Assumption 2, combining (2.51) and (2.53), we can deduce (2.49). \( \square \)

**Remark 2.2.** For any \( p \in M, 0 < R \leq d \), denote by \( H^R_N(x, y, t) \) and \( H^R_D(x, y, t) \) the heat kernel with Neumann and Dirichlet boundary condition on geodesic ball \( B(p, R) \) respectively. Then following the discussion in Pages 396-397 of [43], from (2.47), there exist constants \( C_1, C_2 \) and \( C_3 \) \( (C_2 \leq \frac{1}{2}) \) depending only on \( A \) and \( \Lambda \) in Assumption 1 and 2 such that
\[
H^R_N(x, y, C_1 R^2) \geq H^R_D(x, y, C_1 R^2) \geq C_3 R^{-n} \tag{2.54}
\]
for all \( x, y \in B(p, C_2 R) \). From (2.51), by standard argument (see for example Pages 142-142 in [46]), we can deduce the weak Poincaré inequality
\[
\int_{B(p, C_2 R)} \left( f - f_{B(p, C_2 R)} \right)^2 d\mu \leq C R^2 \int_{B(p, R)} |\nabla f|^2 d\mu \tag{2.55}
\]
where
\[
f_{B(p, C_2 R)} := \frac{1}{|B(p, C_2 R)|} \int_{B(p, C_2 R)} f(y) d\mu(y).
\]
Then by the trick in [28] (see also [58, 16]), (2.55) implies the Poincaré inequality
\[
\int_{B(p, R)} \left( f - f_{B(p, R)} \right)^2 d\mu \leq C R^2 \int_{B(p, R)} |\nabla f|^2 d\mu, \quad \forall \ p \in M, \ 0 < R \leq d.
\]
In the case of twisted Kähler-Ricci flow (1.2), from (2.55), we can get the Poincaré inequality by the estimate on the first non-zero eigenvalue of self-adjoint elliptic operator

$$
\Delta_h f = \overline{\partial} \partial f - \sum_{i=1}^m \nabla^i f \nabla_i h, \quad \forall \ f \in C^\infty(M, \mathbb{C}),
$$

where $h$ is defined in (1.3).

From (1.3) and Theorem 2.3.4 in Futaki [19], for the first non-zero eigenvalue $\lambda_1$ of $\Delta_h$, we have

$$
\lambda_1 \int_M |\partial f|^2 e^{h(t)} \, d\mu(t) = \int_M \sum_{i,j=1}^m \left( R_{j\bar{i}} f \nabla^j \nabla_j f - \nabla_{i\bar{j}} h \nabla^j \nabla_i f + \nabla^i f \nabla_i \nabla_j f \nabla^j f \right) e^{h(t)} \, d\mu(t)
$$

$$
= \int_M \left( |\partial f|^2 + \sum_{i,j=1}^m \theta_{j\bar{i}} f \nabla^j \nabla_i f + \sum_{i,j=1}^m \nabla^i f \nabla_i \nabla_j f \right) e^{h(t)} \, d\mu(t),
$$

which implies

$$
\lambda_1 \geq 1.
$$

Thus, for any $\varphi \in C^\infty(M, \mathbb{R})$ with $\int_M \varphi e^{h(t)} \, d\mu(t) = 0$, we get

$$
\int_M \varphi^2 e^{h(t)} \, d\mu(t) \leq \int_M |\partial \varphi|^2 e^{h(t)} \, d\mu(t). \quad (2.56)
$$

For any $\psi \in C^\infty(M, \mathbb{R})$ with $\int_M \psi \, d\mu(t) = 0$, denote $a = \int_M \psi e^{h(t)} \, d\mu(t)$. So from (2.56) it holds that

$$
\int_M \left( \psi - \frac{\int_M \psi e^{h(t)} \, d\mu(t)}{\int_M e^{h(t)} \, d\mu(t)} \right)^2 e^{h(t)} \, d\mu(t) \leq \int_M |\partial \psi|^2 e^{h(t)} \, d\mu(t). \quad (2.57)
$$

Since $\|h\|_{C^1} \leq C$, using (2.57), we obtain

$$
C \int_M |\partial \psi|^2 \, d\mu(t) \geq \int_M |\partial \psi|^2 e^{h(t)} \, d\mu(t)
$$

$$
\geq \int_M \left( \psi - \frac{\int_M \psi e^{h(t)} \, d\mu(t)}{\int_M e^{h(t)} \, d\mu(t)} \right)^2 e^{h(t)} \, d\mu(t)
$$

$$
\geq C^{-1} \int_M \left( \psi - \frac{\int_M \psi e^{h(t)} \, d\mu(t)}{\int_M e^{h(t)} \, d\mu(t)} \right)^2 \, d\mu(t)
$$

$$
=C^{-1} \left( \int_M \psi^2 \, d\mu(t) + \left[ \frac{\int_M \psi e^{h(t)} \, d\mu(t)}{\int_M e^{h(t)} \, d\mu(t)} \right]^2 \text{Vol}(t) (M) \right)
$$

$$
\geq C^{-1} \int_M \psi^2 \, d\mu(t),
$$

which implies the Poincaré inequality along the twisted Kähler-Ricci flow (1.2)

$$
\int_M \left( f - \frac{1}{\text{Vol}(t) (M)} \int_M f \, d\mu(t) \right)^2 \, d\mu(t) \leq C \int_M |\nabla f|^2 \, d\mu(t), \quad \forall \ f \in C^\infty(M, \mathbb{R}).
$$
Denote by
\[ G_0(x, y) := \int_0^{+\infty} \left( H(x, y, t) - \frac{1}{\operatorname{Vol}(M)} \right) dt \]
the Green’s function of Laplacian.

**Lemma 2.7.** Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold satisfying Assumptions 1, 2 and 3. Assume also \(d \geq \beta > 0\). Then there hold

(a) \(|G_0(x, y)| \leq \frac{C}{\operatorname{dist}(x,y)^{n-2}}, \quad x, y \in M,\)

(b) \(|\nabla_x G_0(x, y)| \leq \frac{C}{\operatorname{dist}(x,y)^{n-1}}, \quad x, y \in M,\)

where \(C\) depends on \(A, n, \beta, \Lambda\) and \(\|\nabla L\|_{L^\infty(M)}\).

**Proof.** For any \(f_0 \in \mathcal{C}^\infty(M, \mathbb{R})\) with \(\int_M f_0 d\mu = 0\),

\[ f(x, t) = \int_M \left( H(x, z, t) - \frac{1}{\operatorname{Vol}(M)} \right) f_0(z) d\mu(z) \]

is the solution of heat equation (2.9) satisfying

\[ \int_M f(x, t) d\mu(x) = 0, \quad f(x, 0) = f_0(x). \]

From the Poincaré inequality (2.49), we get

\[ \partial_t \int_M [f(x, t)]^2 d\mu(x) = -2 \int_M |\nabla f(x, t)|^2 d\mu(x) \leq -C^{-1} \int_M [f(x, t)]^2 d\mu(x), \]

which implies

\[ \int_M [f(x, t)]^2 d\mu(x) \leq e^{-\frac{t}{2}} \int_M [f_0(x)]^2 d\mu(x), \quad t > 0. \] (2.58)

Combining (2.37) and (2.58), for \(t \geq 10\beta^2\), we have

\[ [f(x, t)]^2 \leq \frac{C_1}{\beta^{n+2}} \int_{t-\beta^2}^t \int_M [f(z, s)]^2 d\mu(z) ds \leq C_1 e^{-\frac{t}{4}} \int_M [f_0(x)]^2 d\mu(x), \]

that is,

\[ \left\{ \int_M \left( H(x, z, t) - \frac{1}{\operatorname{Vol}(M)} \right) f_0(z) d\mu(z) \right\}^2 \leq C_1 e^{-\frac{t}{4}} \int_M [f_0(x)]^2 d\mu(x), \] (2.59)

where \(C_1\) depends on \(A, n, \Lambda, \beta\) and \(\|\nabla L\|_{L^\infty(M)}\).

For \(t \geq 10\beta^2\) and \(x\) fixed, taking \(f_0(z) = H(x, z, t) - \frac{1}{\operatorname{Vol}(M)}\), from (2.59), we can deduce

\[ \int_M \left( H(x, z, t) - \frac{1}{\operatorname{Vol}(M)} \right)^2 d\mu(z) \leq C_1 e^{-\frac{t}{4}}, \quad t \geq 10\beta^2. \] (2.60)
For $x$ fixed, the function $H(x, z, t) - \frac{1}{\text{Vol}(M)}$ of $z$ is also the solution of heat equation (2.9). Thus, from (2.37) and (2.60), for $t \geq 10\beta^2$, we can derive

$$
\left( H(x, y, t) - \frac{1}{\text{Vol}(M)} \right)^2 \leq C_1 \beta^{n+2} \int_{t-\beta^2}^{t} \int_{M} \left( H(x, z, t) - \frac{1}{\text{Vol}(M)} \right)^2 d\mu(z) d\nu \leq C_1 e^{-\frac{t}{\beta^2}},
$$

that is,

$$
\left| H(x, y, t) - \frac{1}{\text{Vol}(M)} \right| \leq C_2 e^{-\frac{t}{\beta^2}}, \quad (2.61)
$$

where $C_2$ and $C$ depend on $A, n, \Lambda, \beta$ and $\|\nabla L\|_{L^\infty(M)}$.

Noticing $G_0(x, y) = \int_{0}^{\infty} \left( H(x, y, t) - \frac{1}{\text{Vol}(M)} \right) dt + \int_{0}^{\infty} \left( H(x, y, t) - \frac{1}{\text{Vol}(M)} \right) dt$, by (2.38) and (2.61), we can prove Item (a).

Item (b) follows from Item (a) and (2.8). $\square$

3. Proof of Theorem 1.2

In this section, for convenience, denote by $|\Omega|$ the volume of the set $\Omega$ with respect to the metric $g$. For any $f \in C^\infty(M, \mathbb{R})$, the Hardy-Littlewood maximal function $Mf$ is defined by

$$(Mf)(x) := \sup_{r>0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(\xi)| d\mu(\xi),$$

and we also define

$$(I_\alpha f)(x) := \int_{M} |f(\xi)| \left[ \frac{\text{dist}(x, \xi)}{|B(x, \text{dist}(x, \xi))|} \right]^\alpha d\mu(\xi), \quad 0 < \alpha < n.$$

Lemma 3.1. Let $(M, g)$ be a real $n$-dimensional Riemannian manifold satisfying Assumption 2. Then for any $f \in L^1(M)$ and $\gamma > 0$, there holds

$$\gamma |\{ x | (Mf)(x) > \gamma \}| \leq C \|f\|_{L^1(M)},$$

where $C$ depends only on $\Lambda$ and $n$.

Proof. The ideas comes from Chapter 3 in [20] (see also [10, 14]). For any $x \in \{ x | (Mf)(x) > \gamma \} =: S_\gamma$, there exists $r_x$ such that

$$\frac{1}{|B(x, r_x)|} \int_{B(x, r_x)} |f(\xi)| d\mu(\xi) > \gamma.$$

Obviously,

$$\{ B(x, r_x) | x \in S_\gamma \}$$

is an open covering of $S_\gamma$. For any $0 < c < |S_\gamma|$, from measure theory (see for example Theorem 2.40 in [20]), there exists a compact set $K$ such that $|K| > c$ and finitely many balls, saying $B(x_1, r_{x_1}), \ldots, B(x_p, r_{x_p})$, cover $K$. Let $B(x_{i_1}, r_{x_{i_1}})$ be the ball with
the largest radius in $B(x_i, r_{x_i})$, let $B(x_{i_2}, r_{x_{i_2}})$ be the ball with the largest radius in $B(x_i, r_{x_i})$'s that are disjoint from $B(x_{i_1}, r_{x_{i_1}})$, $B(x_{i_3}, r_{x_{i_3}})$ the ball with the largest radius in $B(x_i, r_{x_i})$'s that are disjoint from $B(x_{i_1}, r_{x_{i_1}})$ and $B(x_{i_2}, r_{x_{i_2}})$, and so on until the list of $B(x_i, r_{x_i})$ is exhausted. According to the construction above, if $B(x_i, r_{x_i})$ is not the one of the $B(x_{i_j}, r_{x_{i_j}})$s, there is a $j$ such that $B(x_i, r_{x_i}) \cap B(x_{i_j}, r_{x_{i_j}}) \neq \emptyset$, and if $j$ is the smallest integer with this property, the radius of $B(x_i, r_{x_i})$ is at most that of $B(x_{i_j}, r_{x_{i_j}})$.

Therefore $B(x_i, r_{x_i}) \subset B(x_{i_j}, 3r_{x_{i_j}})$ and then

$$K \subset \cup_j B(x_{i_j}, 3r_{x_{i_j}}).$$

Therefore, from Assumption 2, we have

$$c < |K| \leq \sum_j |B(x_{i_j}, 3r_{x_{i_j}})| \leq 3^n \Lambda^2 \sum_j |B(x_{i_j}, r_{x_{i_j}})| \leq 3^n \Lambda^2 \int_{B(x_{i_j}, r_{x_{i_j}})} |f(\zeta)| d\mu(\zeta) \leq \frac{3^n \Lambda^2}{\gamma} \|f\|_{L^1(M)}.$$

Letting $c \to |S_n|$, we can deduce the desired conclusion.

**Lemma 3.2.** Let $(M, g)$ be a real $n$-dimensional Riemannian manifold satisfying Assumption 2. Then for any $f \in L^1(M)$ and $\mu > 0$, there holds

$$\mu^{\frac{n}{n-\alpha}} |\{x \in M | (I_\alpha f)(x) > \mu\}| \leq C\|f\|_{L^1(M)}^{\frac{n}{n-\alpha}}, \quad 0 < \alpha < n,$$

where $C$ depends only on $\Lambda$, $n$ and $\alpha$.

**Proof.** Denote

$$(I_{\alpha,1}f)(x) := \int_{B(x, \varepsilon)} |f(\zeta)| \frac{[\text{dist}(x, \zeta)]^\alpha}{|B(x, \text{dist}(x, \zeta))|} d\mu(\zeta)$$

and

$$(I_{\alpha,2}f)(x) := \int_{M-B(x, \varepsilon)} |f(\zeta)| \frac{[\text{dist}(x, \zeta)]^\alpha}{|B(x, \text{dist}(x, \zeta))|} d\mu(\zeta).$$

Then for $0 < \varepsilon < d$, we have

$$(I_{\alpha,1}f)(x) = \sum_{k=0}^{\infty} \int_{\varepsilon(2^{-k+1}) \leq \text{dist}(x, \zeta) < 2^{-k}\varepsilon} |f(\zeta)| \frac{[\text{dist}(x, \zeta)]^\alpha}{|B(x, \text{dist}(x, \zeta))|} d\mu(\zeta)$$

$$\leq \sum_{k=0}^{\infty} (2^{-k}\varepsilon)^{\alpha} |B(x, 2^{-(k+1)}\varepsilon)|^{-1} \int_{B(x, 2^{-k}\varepsilon)} |f(\zeta)| d\mu(\zeta)$$

$$\leq C \sum_{k=0}^{\infty} (2^{-k}\varepsilon)^{\alpha} |B(x, 2^{-k}\varepsilon)|^{-1} \int_{B(x, 2^{-k}\varepsilon)} |f(\zeta)| d\mu(\zeta)$$

$$\leq C(Mf)(x) \varepsilon^{\alpha},$$

where $C$ depends only on $\Lambda$ and $n$.

Thus, Lemma 3.1 and 3.2 implies

$$|\{x \in M | (I_{\alpha,1}f)(x) > \mu\}| \leq \left| \{x \in M | (Mf)(x) > \frac{\mu}{C\varepsilon^{\alpha}} \} \right| \leq \frac{C\varepsilon^{\alpha}}{\mu} \|f\|_{L^1(M)},$$
where $C$ depends only on $\Lambda$ and $n$.

By Assumption 2 we derive
\[
\sup_{\zeta \in M - B(x, \varepsilon)} \frac{[\text{dist}(x, \zeta)]^\alpha}{|B(x, \text{dist}(x, \zeta))|} \leq \Lambda \sup_{\zeta \in M - B(x, \varepsilon)} [\text{dist}(x, \zeta)]^{\alpha - n} = \Lambda \varepsilon^{\alpha - n},
\]
which implies
\[
(I_{\alpha,2} f)(x) \leq \Lambda \varepsilon^{\alpha - n} \|f\|_{L^1(M)}. \tag{3.3}
\]
From (3.2) and (3.3), taking
\[
\varepsilon = \left(\frac{\|f\|_{L^1(M)}}{(M f)(x)}\right)^\frac{1}{n},
\]
we have
\[
(I_{\alpha} f)(x) \leq C \left(\frac{\|f\|_{L^1(M)}}{(M f)(x)}\right)^\frac{n - \alpha}{n} \|f\|_{L^1(M)}^{\frac{\alpha}{n}}, \tag{3.4}
\]
where $C$ depends only on $\Lambda$ and $n$. We remark that $(I_{\alpha,2} f)(x) = 0$ if $\varepsilon > d$.

Combining Lemma 3.1 and (3.4), we arrive at
\[
\left|\left\{ x \in M \mid (I_{\alpha} f)(x) > \mu\right\}\right| \leq \left|\left\{ x \in M \mid (M f)(x) > \mu \frac{n - \alpha}{n} C \|f\|_{L^1(M)}^{\frac{\alpha}{n}}\right\}\right| \leq \mu \frac{n - \alpha}{n} C \|f\|_{L^1(M)}^{\frac{\alpha}{n}},
\]
which implies (3.1) as desired. \hfill \Box

**Remark 3.1.** Lemma 3.2 can be found in [5] in the case of Euclidean space. Another similar definition so-called Riesz Potential of order 1 can be found in [42].

**Lemma 3.3.** Let $(M, g)$ be a real $n$-dimensional Riemannian manifold satisfying Assumptions 1, 2 and 3. Then for any $\mu > 0$ and $f \in C^\infty(M, \mathbb{R})$ with $\int_M f d\mu = 0$, there holds
\[
\mu \frac{n - \alpha}{n} |\{ x \in M \mid |f(x)| > \mu\} | \leq C \|\nabla f\|_{L^1(M)}^{\frac{n - \alpha}{n}}, \tag{3.5}
\]
where $C$ depends only on $A$, $\Lambda$, $n$ and $\|\nabla L\|_{L^\infty(M)}$.

**Proof.** Since $\Delta f = \Delta f$ and $\int_M f d\mu = 0$, by integration by parts, we have
\[
f(x) = - \int_M G_0(x, z) \Delta f(z) d\mu(z)
\]
\[
= - \lim_{r \to 0} \int_{M - B(x, r)} G_0(x, z) \Delta f(z) d\mu(z)
\]
\[
= \lim_{r \to 0} \int_{M - B(x, r)} \nabla_z G_0(x, z) \nabla f(z) d\mu(z) - \lim_{r \to 0} \int_{\partial B(x, r)} G_0(x, z) \partial_\nu f(z) dS(z)
\]
\[
= \int_M \nabla_z G_0(x, z) \nabla f(z) d\mu(z),
\]
where $\nu$ is the inward normal vector on $B(x, r)$ and we use Assumption 2 and the fact
\[
|G_0(x, z)| \leq \frac{C}{[\text{dist}(x, z)]^{n - 2}}.
\]
Using Assumption 2, Lemma 2.7 and (3.6) together, we have
\[ |f(x)| \leq C \int_M \frac{|\nabla f(z)|}{\text{dist}(x, z)^{n-1}} d\mu(z) \]
\[ \leq C \int_M |\nabla f(z)| \frac{\text{dist}(x, z)}{|B(x, \text{dist}(x, z))|} d\mu(z) \]
\[ = C(I_1|\nabla f|)(x), \tag{3.7} \]
where \( C \) is a constant depending on \( A, \Lambda, n \) and \( ||L||_{L^\infty(M)} \). Thus, the lemma follows from Lemma 3.2 and (3.7).

**Lemma 3.4.** Let \((M, g)\) be a real \( n \)-dimensional Riemannian manifold satisfying Assumption 1, 2 and 3. Then for any \( f \in C^\infty(M, \mathbb{R}) \), there holds
\[ || f ||_{L^\frac{n}{n-1}(M)} \leq C || \nabla f ||_{L^1(M)} + C_1 [\text{Vol}(M)]^{-\frac{1}{n}} || f ||_{L^1(M)}, \tag{3.8} \]
where \( C \) is a constant depending on \( A, \Lambda, n \) and \( ||L||_{L^\infty(M)} \) and \( C_1 \) is a numeral number.

**Proof.** The proof can be found in [42] (see also [5, 15]). For completeness, we rewrite it here. If for any \( f \in C^\infty(M, \mathbb{R}) \) with \( \int_M f d\mu = 0 \), (3.8) holds, then for any \( f \in C^\infty(M, \mathbb{R}) \), by the Minkowski inequality, we have
\[ || f ||_{L^\frac{n}{n-1}(M)} \leq ||f - f_M||_{L^\frac{n}{n-1}(M)} + ||f_M||_{L^\frac{n}{n-1}(M)} \]
\[ \leq C || \nabla f ||_{L^1(M)} + C_1 [\text{Vol}(M)]^{-\frac{1}{n}} || f ||_{L^1(M)} + [\text{Vol}(M)]^{-\frac{1}{n}} || f ||_{L^1(M)}, \]
which implies (3.8). Here \( f_M = \frac{1}{\text{Vol}(M)} \int_M f(z) d\mu(z) \).

So without loss of generality, we concentrate on the case \( f \in C^\infty(M, \mathbb{R}) \) with \( \int_M f d\mu = 0 \). Denote \( p := \frac{n}{n-1} \). For \( k \in \mathbb{Z} \), define \( f_k = \min\{(|f| - 2^k)^+, 2^k\} \), where \(|f| - 2^k)^+ = \max\{|f| - 2^k, 0\} \). Obviously, \( 0 < f_k \leq \frac{|f|}{2} \). For \( k \in \mathbb{Z} \), we have
\[ B_k := \{x \in M | f_k(x) = 2^k\} \]
\[ \subset \{x \in M | f_k(x) > 2^{k-1}\} \]
\[ \subset \{x \in M | |f_k(x) - f_{k, M}| > 2^{k-2}\} \cup \{x \in M | f_{k, M} > 2^{k-2}\}, \tag{3.9} \]
where
\[ f_{k, M} = \frac{1}{\text{Vol}(M)} \int_M f_k(z) d\mu(z) \leq \frac{|| f ||_{L^1(M)}}{2\text{Vol}(M)}. \tag{3.10} \]
From Lemma 3.3, we have
\[ |\{x \in M | |f_k(x) - f_{k, M}| > 2^{k-2}\}| \leq C 2^{-(k-2)p} || \nabla f_k ||_{L^1(M)}^p \]
\[ \leq C 2^{-k p} \left( \int_{2^k < |f| \leq 2^{k+1}} |\nabla f| d\mu \right)^p. \tag{3.11} \]
Using (3.10), we can deduce
\[ |\{x \in M | f_{k, M} > 2^{k-2}\}| \leq \begin{cases} 0, & k \geq \log_2 \frac{|| f ||_{L^1(M)}}{\text{Vol}(M)} + 1 =: k_0 \\ \text{Vol}(M), & k < k_0. \end{cases} \tag{3.12} \]
Combining (3.9), (3.11), (3.12) and the Minkowski inequality together, (3.8) follows from
\[ \|f\|_{L^p(M)} = \left\{ \sum_{k \in \mathbb{Z}} \int_{2^k < |f| \leq 2^{k+1}} |f|^p d\mu \right\}^{\frac{1}{p}} \]
\[ \leq \left\{ \sum_{k \in \mathbb{Z}} 2^{p(k+1)} \mu(\{ |f| > 2^k \}) \right\}^{\frac{1}{p}} \]
\[ \leq \left\{ C \sum_{k \in \mathbb{Z}} \left( \int_{2^k < |f| \leq 2^{k+1}} |\nabla f|^p + \operatorname{Vol}(M) \sum_{k = -\infty}^{[k_0] + 1} 2^{p(k+1)} \right) \right\}^{\frac{1}{p}} \]
\[ \leq \left\{ \sum_{k \in \mathbb{Z}} \left( C \int_{2^k < |f| \leq 2^{k+1}} |\nabla f|^p \right) \right\}^{\frac{1}{p}} + \left\{ \sum_{k = -\infty}^{[k_0] + 1} \left( \frac{[k_0] + 1}{\operatorname{Vol}(M)} \right)^{\frac{p}{2}} 2^{k+1} \right\}^{\frac{1}{p}} \]
\[ \leq C \sum_{k \in \mathbb{Z}} \int_{2^k < |f| \leq 2^{k+1}} |\nabla f| + \frac{1}{\operatorname{Vol}(M)} \left[ \sum_{k = -\infty}^{[k_0] + 1} 2^{k+1} \right] \]
\[ \leq C \int_M |\nabla f| + C_1 \left[ \operatorname{Vol}(M) \right]^{-\frac{1}{n}} \|f\|_{L^1(M)}, \]
where \( \chi_{\{k \leq [k_0] + 1\}} \) is the characteristic function of set \( \{ k \in \mathbb{Z} | k \leq [k_0] + 1 \} \). \( \square \)

**Remark 3.2.** For any \( f \in C^\infty(M, \mathbb{R}) \), from (3.8), we have
\[ \|f - f_M\|_{L^{\frac{n}{n-1}}(M)} \leq C_1 \|\nabla f\|_{L^1(M)} + C_1 \left[ \operatorname{Vol}(M) \right]^{-\frac{1}{n}} \|f - f_M\|_{L^1(M)}. \] (3.13)

By using (3.7), we get
\[ |f(x) - f_M| \leq C \int_M \frac{|\nabla f(z)|}{\left[ \operatorname{dist}(x, z) \right]^{n-1}} d\mu(z). \] (3.14)

Applying the discussion in (3.2), Assumption 2 and (3.14), we can deduce
\[ \|f(x) - f_M\|_{L^1(M)} \leq C d \|\nabla f\|_{L^1(M)}, \]
which, combining with Assumption 2 and (3.13), implies isoperimetric inequality
\[ \|f(x) - f_M\|_{L^{\frac{n}{n-1}}(M)} \leq C \|\nabla f\|_{L^1(M)}, \]
where \( C \) is a constant depending on \( A, \Lambda, n \) and \( \|\nabla L\|_{L^\infty(M)} \).

**Proof of Theorem 1.2.** Noting that we just need to prove the theorem for every fixed time, the conclusion is a special case of Lemma 3.4 and Remark 3.2. \( \square \)

**Acknowledgements** This work was carried out while the authors were visiting Mathematics Department at Northwestern University. We would like to thank Professor Valentino Tosatti and Professor Ben Weinkove for hospitality and helpful discussions.
We also thank Professor Qi S. Zhang for offering us more details about heat kernel and Wenshuai Jiang for some helpful conversations.

References

[1] F. Bernstein, ¨Uber die isoperimetriche Eigenschaft des Kreises auf der Kugeloberflache und in der Ebene, Math. Ann. 60(1905), 117-136.
[2] W. Blaschke, Kreis und Kugel, de Gruyter, Berlin, 2nd edn. (1956).
[3] T. Bonnesen, ¨Uber die isoperimetriche Defizit ebener Figuren, Math. Ann. 91(1924), 252-268.
[4] Yu. D. Burago and V. A. Zalgaller, Geometric Inequalities, Springer-Verlag, New York, 1988; Original russian edition: Geometricheskie neravenstva, Leningrad.
[5] L. Capogna, D. Danielli and N. Garofalo, The geometric Sobolev embedding for vector fields and the isoperimetric inequality, Comm. Anal. Geom. 2(1994), 203-215.
[6] I. Chavel, Isoperimetric Inequalities, Cambridge Tracts in Math. 145, Cambridge Univ. Press, Cambridge, U.K. (2001).
[7] X. Chen and B. Wang, Space of Ricci flow I, Comm. Pure Appl. Math. 65(2012), no. 10, 1399-1457.
[8] X. Chen and B. Wang, On the conditions to extend Ricci flow (III), Internat. Math. Res. Notices IMRN 2013(2013), no. 10, 2349-2367.
[9] B. Chow, P. Lu and L. Ni, Hamilton’s Ricci flow, Grad. Stud. math., Vol. 77, American Mathematical Society, Providence, RI; Science Press, New York, 2006.
[10] R. R. Coifman and G. Weiss, Analyse harmonique non-commutative sur certain spaces homogènes. Etudes de certaines intégraux singulières, in Lecture Notes in Mathematics 242, Springer-Verlag, Berlin, New York 1971.
[11] T. Collins and G. Székelyhidi, The twisted Kähler-Ricci flow, arXiv:1207.5441v1.
[12] T. Coulhon and X. Duong, Riesz transforms for 1 ≤ p ≤ 2, Transactions of the American Mathematical Society, 351(1999), 1151-1169.
[13] T. Coulhon and A. Grigor’yan, On-diagonal lower bounds for heat kernels and Markov chains, Duke Univ. Math. J., 89(1997), 133-199.
[14] S. Fang and T. Zheng, The (logarithmic) Sobolev inequalities along geometric flow and applications, arXiv: 1502.02305.
[15] B. Franchi, S. Gallot and R. L. Wheeden, Sobolev and isoperimetric inequalities for degenerate metrics, Math. Ann. 300(1994), 557-571.
[16] N. Fusco, The classical isoperimetric theorem, Rend. Acc. Sci. Fis. Mat. Napoli 71(2004), 63-107.
[17] N. Fusco, F. Maggi, and A. Pratelli, The sharp quantitative isoperimetric inequality, Annals of Mathematics, 168(2008), 941-980.
[18] B. Fuglede, Stability in the isoperimetric problem for convex or nearly spherical domains in R^n, Trans. Amer. Math. Soc. 314(1989), 619-638.
[19] A. Futaki, Kähler-Einstein metrics and integral invariants, Lecture Notes in Mathematics, 1314, Springer-Verlag, Berlin, 1988.
[20] G. B. Folland, Real analysis, 2nd ed., Pure Appl. Math. (N.Y.), John Wiley and Sons, New York, 1999.
[21] E. De Giorgi, Sulla proprietà isoperimetrica dell’ipersfera, nella classe degli insiemi aventi frontiera orientata di misura finita, Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Nat., Sez. 1, 8(1958), 33-44.
[22] E. De Giorgi, Selected Papers (L. Ambrosio, G. Dal Maso, M. Forti, and S. Spagnolo, eds.), Springer-Verlag, New York(2005).
[23] A. A. Grigor’yan, The heat equation on noncompact Riemannian manifolds, Mat. Sb. 182(1991), no.1, 55-87; translation in Math. USSR-Sb. 72(1992), no.1, 47-77.
[24] R. S. Hamilton, An isoperimetric estimate for the Ricci flow on the two-sphere, Modern Methods in Complex Analysis (Princeton, NJ, 1992), Ann. of Math. Stud., Vol. 137(1995), 191-200.
[25] R. R. Hall, A quantitative isoperimetric inequality in n-dimensional space, J. Reine Angew. Math. 428 (1992), 161-176.
[26] R. R. Hall, W. K. Hayman and A. W. Weitsman, On asymmetry and capacity, J. d’Analyse Math. 56 (1991), 87-123.
[27] Q. Han and F. Lin, Elliptic partial differential equations, Courant Lect. Notes Math. 1, Courant Institute of Mathematical Sciences, New York 1997.
[28] D. Jerison, The Poincaré inequality for vector fields satisfying Hömander’s condition, Duke Math. J. 53 (1986), 503-523.
[29] J. Liu, The generalized Kähler Ricci flow, J. Math. Anal. Appl., 408 (2013) 751-761.
[30] J. Moser, A harnack inequality for parabolic differential equations, Comm. Pure. Appl. Math. 17 (1964), 101-134.
[31] R. Osserman, The isoperimetric inequality, Bull. Amer. Math. Soc. 84 (1978), 1182-1238.
[32] R. Osserman, Bonnesen-style isoperimetric inequalities, Amer. Math. Monthly, 86 (1979), 1-29.
[33] L. E. Payne, Error bounds based on a prior inequalities, Numerical Solutions of Partial Differential Equationos, J. II. Bramble, ed., Academic Press, New York, 1966, 83-94.
[34] L. E. Payne, Isoperimetric inequality and their applications, Siam Review 9 (1967), 453-488.
[35] G. Pólya and G. Szegö, Isoperimetric inequalities in Mathematical Physics, Ann. of Math. Studies 27, Princeton University press, Princeton, 1955.
[36] L. Saloff-Coste, A notes on Poincaré, Sobolev, and Harnack inequalities, Internat. Math. Res. Notices IMRN 1992 (1992), no.2, 27-38.
[37] L. Saloff-Coste, Uniformly elliptic operators on Riemannian manifolds, J. Differential Geometry 36 (1992), 417-450.
[38] L. Saloff-Coste, Aspects of Sobolev-type inequalities, London Mathematical Society Lecture Note Series, 289, Cambridge University Press, Cambridge, 2002.
[39] N. Sesum and G. Tian, Bounding scalar curvature and diameter along the Kähler-Ricci flow (after Perelman), Journal of the Institute of Mathematics of Jussieu, 7 (2008), 575-587.
[40] E. M. Stein, Sigular Integrals and Differentiability Properties of Functions, Princeton Univ. Press, Princeton 1970.
[41] G. Talenti, The standard isoperimetric theorem, in Handbook of Convex Geometry, Vol. A (P.M. Gruber and J. M. Wills, eds.), 73-123, North Holland Publ. Co., Amsterdam(1993).
[42] G. Tian and Q. S. Zhang, Isoperimetric inequality under Kähler-Ricci flow, Amer. J. Math. 136 (2014), no. 5, 1155-1173.
[43] J. Wang, Global heat kernel estimate, Pacific J. Math. 178 (1997), 377-398.
[44] R. Ye, The logarithmic Sobolev inequality along the Ricci flow, arXiv: 070724v4.
[45] Q. S. Zhang, A uniform Sobolev inequality under Ricci flow, Internat. Math. Res. Notices IMRN 2007 (2007), nol 17, Art. ID rnm056, 17, ibidi erratum, addendum.
[46] Q. S. Zhang, Sobolev Inequalities, Heat Kernels under Ricci flow, and the Poincaré Conjecture, CRC Press, Boca Raton, FL, 2011.
[47] Q. S. Zhang, Bounds on volume growth of geodesic ball under Ricci flow, Math. Res. Lett. 19 (2012), no. 1, 245-253.
[48] X. Zhang and X. Zhang, Generalized Kähler-Einstein metrics and energy functionals, Canad. J. Math. 66 (2014), 1413-1435.

Shouwen Fang
School of Mathematical Science, Yangzhou University,
Yangzhou, Jiangsu 225002, P. R. China
E-mail: shwfang@163.com

Tao Zheng
School of Mathematics and Statistics, Beijing Institute of Technology,
Beijing 100081, P. R. China
E-mail: zhengtao08@amss.ac.cn