A Family of Hybrid Random Number Generators with Adjustable Quality and Speed

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Abstract

Conventional random number generators provide the speed but not necessarily the high quality output streams needed for large-scale stochastic simulations. Cryptographically-based generators, on the other hand, provide superior quality output but are often deemed too slow to be practical for use in large simulations. We combine these two approaches to construct a family of hybrid generators that permit users to choose the desired tradeoff between quality and speed for a given application. We demonstrate the effectiveness, performance, and practicality of this approach using a standard battery of tests, which show that high quality streams of random numbers can be obtained at a cost comparable to that of fast conventional generators.

Keywords: random number generator, pseudorandom, stochastic simulation, one-way function, cryptographic transform

AMS Subject Classifications: 65C10, 11K45, 68P25, 68Q17, 68Q85

1 Random Number Generators

Large stochastic simulations, such as those typically run on highly parallel supercomputers, consume vast quantities of random numbers. Designers and users of random number generators face a dilemma: how to achieve the speed necessary to make the computation feasible while avoiding any hint of correlation or bias that would invalidate the results. Conventional random number generators [11, 7, 9], typically based on simple arithmetic recurrences, may possess the requisite speed, but they carry a significant risk of failure due to the questionable quality of the resulting streams, especially when produced in great abundance, thereby inviting more opportunity for detectable patterns to emerge [6, 13, 18].

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This should not be surprising: according to Kolmogorov's definition \cite{8}, a truly random stream is incompressible (i.e., it has no shorter description than simply enumerating its members); a generator algorithm provides a succinct encoding of the stream it produces, which therefore cannot be truly random. Such deterministically generated (hence reproducible) streams are more properly called pseudorandom, but nevertheless a well-designed generator may still produce streams that appear random in that they may pass various statistical tests of randomness \cite{11, 15}.

1.1 Cryptographically-Based Generators

An alternative view of randomness is taken in cryptography, in which random number generators are ubiquitously employed to generate random keys, signatures, and the like \cite{3}. In security applications, quality is paramount and performance secondary, as an easily compromised cryptographic protocol would be useless regardless of its efficiency. Here, a stream is considered random if it is computationally indistinguishable from the desired true distribution, that is, detecting any difference would require a prohibitive amount of computation \cite{4}. Many cryptographic protocols are based on one-way functions, that is, functions that are easy to compute but difficult to invert (i.e, to find an input value that yields a given output value). In principle, a one-way function can be used to construct a “perfect” random number generator, in that resistance to inversion ensures computational indistinguishability of the resulting stream \cite{4, 5}. Alas, there are two difficulties with this putative panacea: the existence of one-way functions is an open theoretical question (closely related to the famously unresolved \( P = \text{NP?} \) problem) \cite{12}, and the known candidates believed to be one-way functions (e.g., factoring products of large primes) are relatively expensive to evaluate, making the resulting generator slow and thus seemingly unsuitable for large stochastic simulations \cite{10}. The first of these issues does not seem to be a problem in practice unless efficient algorithms are unexpectedly discovered for these currently intractable inversion problems, or if it turns out that \( P = \text{NP} \), in which case true one-way functions do not exist. The speed of cryptographically-based generators is an important practical issue, however, which we address next.

Practical cryptographic transforms, such as block ciphers and hash functions, are based on algorithms that are carefully designed and extensively tested to ensure their resistance to inversion. These algorithms employ basic operations (e.g., substitutions, permutations) that obscure any structure in the input by dissipating it over the output. This process is usually repeated multiple (often many) times (called “rounds”) to ensure sufficient diffusion, which is the main reason such transforms are relatively slow. Nevertheless, their superior ability to produce high quality random output is crucial in security applications, and it could also make stochastic simulations more robust if the speed of cryptographically-based generators could be made competitive with conventional generators. Recently, cryptographically-based generators have been used for a few stochastic simulations and computer graphics applications. Crypto-
graphic transforms employed have included AES and Threefish [19], MD5 [21], and TEA [22]. To make the cryptographically-based generators competitive in speed, the corresponding cryptographic transforms were weakened by reducing the number of rounds significantly, and in some cases by making other simplifications to the algorithms as well. The resulting generators are no longer sufficiently secure for cryptographic applications, but still produce streams of sufficient quality for stochastic simulations, as demonstrated by passing standard statistical test batteries. Here we will pursue the same objective but with a different approach, using a full-strength cryptographic transform but combining the output of the resulting generator with that of a conventional generator, which will allow us explicit, quantitative control of the tradeoff between quality and speed.

2 Combined Cryptographic/LCG Generator

To create a fast generator with sufficient randomness, we combine a cryptographically-based generator with a fast conventional generator. Because of their speed, simplicity, and widespread use, we chose a linear congruential generator (LCG) for the conventional generator (which will also serve to illustrate that even a relatively low-quality generator can be rehabilitated using our technique), but any type of conventional generator should also work. Because cryptographic transforms are relatively slow, we amortize the cost of computing random bits by reusing them in the combined generator. We define two parameters relevant to combining the two generators:

- **size**: the number of cryptographically-based 32-bit random numbers (cryptographic values) stored at a given time;
- **repetition**: the number of times each cryptographic value is used before being discarded.

Motivated by its well-known bias-reducing property [2], we use bitwise exclusive-or (XOR) to combine cryptographic values with LCG-based 32-bit random numbers (LCG values) in an alternating fashion. Let $k$ be the value of size and $n$ the value of repetition. The first $k$ LCG values are XORed uniquely with the $k$ cryptographic values. Specifically, the first LCG value is XORed with the first cryptographic value, the second LCG value is XORed with the second cryptographic value, and so on, until the $k$th LCG value is XORed with the $k$th cryptographic value. The next $k$ LCG values are XORed with the same $k$ cryptographic values in a similar fashion. This process is repeated until each cryptographic value has been used to XOR $n$ LCG values. Then the $k$ cryptographic values are discarded and $k$ new cryptographic values are produced. This procedure is repeated until the generator stream is terminated. Figure 1 illustrates the effects of various values of the size and repetition parameters in the combined generator for some small examples.
2.1 Practical Implementation

We chose the Secure Hash Algorithm SHA-256 [17] as the basis for the cryptographic generator due to its exceptional resistance to inversion and its wide acceptance in the cryptographic community. Because each call to SHA-256 produces 256 random bits of output, eight 32-bit cryptographic values are produced by each call. Due to the cryptographic strength of SHA-256, we use \(i = 1, 2, 3, \ldots\) to seed the generator (as in [19]), with no noticeable correlation in the resulting output.

We selected three popular LCGs to combine with SHA-256:

- Super-Duper [14]: \(x_n = x_{n-1} \times 69069 + 1 \pmod{2^{32}}\),
- glibc rand: \(x_n = x_{n-1} \times 1103515245 + 12345 \pmod{2^{32}}\),
- Borland C++ rand: \(x_n = x_{n-1} \times 22695477 + 1 \pmod{2^{32}}\).

Because the low-order bits in power-of-two modulus LCGs have much shorter periods than the high-order bits [16], we call each LCG twice and concatenate the 16 highest-order bits of each iteration to obtain a more reliable 32-bit random number.

2.2 Parallel Generators

One of the key advantages of using a cryptographic transform as part of our combined generator is ease of parallelization. SHA-256 has an exceptionally large \((2^{64} - 1)\)-bit seed space [17], which allows for production of \(8 \times 2^{2^{64} - 1}/n\) independent streams of \(n\) 32-bit random numbers. For example, the SHA-256 generator can easily produce \(2^{1000000}\) independent streams of \(2^{1000000}\) 32-bit random values if desired. To parallelize the SHA-256 generator, simply concatenate the stream number with a counter \(i = 1, 2, 3, \ldots\) for each stream. Because each stream will have a unique seed, the cryptographic strength of
SHA-256 ensures independence of streams. To parallelize the LCG, one can choose any of several suitable ways as outlined by Srinivasan, Mascagni, and Ceperley [20], such as introducing a lag, choosing prime addends, or choosing prime moduli.

3 Test Results

To test our combined generators, we used the test batteries in TestU01 [11], a collection of popular random number generators and tests to measure their effectiveness. TestU01 includes three predefined test batteries for random bit sequences: SmallCrush, Crush, and BigCrush. We used SmallCrush for preliminary testing, but it is insufficiently stringent to detect anything other than gross defects in generators. Crush is a considerably more strenuous battery of tests, applying 96 statistical tests and producing 144 test statistics and p-values while consuming approximately $2^{35}$ random numbers. Crush was applied to all generators in this paper to assess their approximate quality and suggest candidates for further testing. BigCrush was then used to test the most promising generators more rigorously. BigCrush applies 106 statistical tests and produces 160 test statistics and p-values while consuming approximately $2^{38}$ random numbers. All programs were written in C, compiled with gcc, and run on a 2.8 GHz Intel Xeon X5560 processor.

Figures 2 and 3 show results of BigCrush tests for each of the three combined generators for various values of the size and repetition parameters. The column for size=0 gives the number of test failures (out of 160) using only the corresponding LCG, for which repetition is irrelevant. For a given nonzero value of size, increasing repetition (reuse of cryptographic values) degrades reliability (larger numbers of test failures), as expected. Varying the size parameter reveals a more complex relationship, in which increasing size from 1 improves reliability up to an optimal value, then may degrade reliability beyond that point. An explanation for this behavior is given in the discussion below. To avoid running expensive BigCrush tests for cases that were certain to have a relatively large number of failures (and therefore not yield a useful generator), the figures merely indicate (by orange and red shading) repetition factors for which the number of failures is 5 or more (out of 160), except that full results are given for the optimal value of size for each generator (blue shading). The three generators produced generally similar results, although the SHA-256/glibc rand combined generator achieves maximum reliability for size=32, whereas size=16 is optimal for the other two combined generators.

Timing results given in Table 1 show that increasing repetition increases the speed of the combined generator, i.e., the more times the cryptographic values are reused, the faster the generator produces random bits. In Table 1, we see that the SHA-256 generator takes an average of 1.56 seconds to produce $2^{23}$ random values, whereas each of the LCGs alone takes an average of 0.19 seconds to produce the same number of values. The combined generator that uses each cryptographic value only once (repetition=1) slows performance due to the
overhead of creating both cryptographic values and LCG values. However, the speed is improved by producing fewer cryptographic values and increasing the number of times they are used in the combined generator stream. As repetition increases, speed initially increases substantially, but improvement levels off for repetition > 64. The combined generators with repetition ≥ 256 are nearly identical in speed to the LCG generator. Table 1 gives timings only for the optimal value of size for each combined generator, but additional tests showed that for a given fixed repetition, timing results remain nearly identical as the size parameter varies.

3.1 Discussion

Reusing SHA-256 values in the combined generators results in less costly but less reliable streams of random numbers. Table 1 illustrates the relationship between repetition and both speed and failure rate for each of the combined generators. For example, the combined SHA-256/Super-Duper generator with repetition=16 has a more than five-fold increase in speed compared to the SHA-256 generator alone, while still passing all of the statistical tests. At repetition=256, the combined generator is essentially identical in speed to the LCG generator, while failing only 3/160 tests compared to failing 109/160 tests using

![Figure 2: BigCrush failures (out of 160 total) for combined SHA-256/Super-Duper generator.](image)

![Figure 3: BigCrush failures (out of 160 total) for combined SHA-256/glibc `rand` generator (left) and combined SHA-256/Borland C++ `rand` generator (right).](image)
Table 1: Execution times in seconds (averaged over 20 trials) for producing $2^{32}$ 32-bit random numbers for three combined generators, each combining SHA-256 with indicated LCG. Failures indicates number of test statistics failed out of 160 using BigCrush.

| LCG size | SHA | glibc `rand` | Borland C++ `rand` |
|----------|-----|--------------|-------------------|
|          | 16  | 32           | 16                |
| repetition | time | failures | time | failures | time | failures |
| SHA      | 1.56 | 0          | 1.56 | 0        | 1.56 | 0        |
| 1        | 1.70 | 0          | 1.70 | 0        | 1.70 | 0        |
| 2        | 0.95 | 0          | 0.95 | 0        | 0.94 | 0        |
| 4        | 0.57 | 0          | 0.57 | 0        | 0.57 | 0        |
| 8        | 0.38 | 0          | 0.38 | 1        | 0.38 | 0        |
| 16       | 0.28 | 0          | 0.28 | 2        | 0.28 | 1        |
| 32       | 0.24 | 1          | 0.24 | 3        | 0.24 | 3        |
| 64       | 0.21 | 1          | 0.21 | 3        | 0.21 | 3        |
| 128      | 0.20 | 2          | 0.20 | 3        | 0.20 | 4        |
| 256      | 0.19 | 3          | 0.19 | 3        | 0.19 | 4        |
| 512      | 0.19 | 3          | 0.19 | 3        | 0.19 | 4        |
| 1024     | 0.19 | 3          | 0.19 | 3        | 0.19 | 4        |
| 2048     | 0.19 | 3          | 0.19 | 3        | 0.19 | 4        |
| 4096     | 0.19 | 3          | 0.19 | 3        | 0.19 | 4        |
| 8192     | 0.19 | 3          | 0.19 | 3        | 0.19 | 4        |
| 16384    | 0.19 | 5          | 0.19 | 5        | 0.19 | 6        |
| LCG      | 0.19 | 109        | 0.19 | 104      | 0.19 | 88       |

Super-Duper alone.

Varying size produces slightly less intuitive results. It is well known that LCGs exhibit a high degree of serial correlation \[6, 13, 18\]. With a relatively small size parameter, we are able to eliminate correlation between successive LCG values. Unfortunately, correlation remains in each substream that shares a common cryptographic-value buffer. For example, if we have a high-repetition generator with size=2, there will be significant correlation between every second value in the sequence. This is the same type of correlation that lagged LCGs suffer from. Because this correlation stems from the fact that each LCG value is determined uniquely by the previous value (given that the other parameters are fixed), we call this type of correlation \textit{local} correlation (indicated by orange shading in Figures \[2, 3\]).

Increasing size improves generator quality, but only up to a point. Clearly, as size increases, \textit{local} correlation in the resulting stream decreases. But as size becomes large, another type of correlation in the LCG stream becomes visible between long, disjoint sequences of values. This correlation is a result of the simple recursive nature of the LCG, leading to an inherent structure in the stream regardless of the actual values produced \[10\]. We call this type
of correlation global correlation due to its ubiquitous nature (indicated by red shading in Figures 2–3).

Thus, the size parameter should be chosen sufficiently large to hide local correlation, but not so large as to reveal global correlation in the LCG stream. Our results suggest that combined generators with size between 16 and 32 are most suitable in this regard (blue shading in Figures 2–3).

4 Recommendations

Several factors should be considered when choosing a combined random number generator. We will start with the easiest parameter to choose: size. Clearly, it is preferable to choose the size that minimizes failure rate. If several size values produce similarly reliable generators, then the smallest acceptable size should be chosen to minimize the amount of state required, which in turn minimizes the amount of memory required. This is particularly important for highly-parallel applications, for which careful use of memory is often critical. If minimizing state is absolutely essential, a suboptimal size can be chosen at some cost in reliability.

To choose the best repetition factor, users must carefully consider their needs. Consider, for example, the SHA-256/Super-Duper combined generator. In Table 1, we see that the repetition ≤16 generators provide the highest reliability, while the repetition ≥256 generators provide maximum speed. Choosing the repetition parameter may be easy for users who are either very conservative (≤16) or care mainly about speed (≥256), but other users may prefer a compromise between speed and reliability. For example, many users may find the size=128 generator suitable for their needs, as it passes all but 2/160 BigCrush tests, and thus is vastly superior in quality to standard library LCGs, yet runs only 3.2% slower than the pure LCG. Because it is easy to vary the repetition parameter, users can experiment and choose the value that best meets their needs in a given situation, for example, “quick and dirty” preliminary exploration versus more exacting final simulation runs.

When choosing a cryptographic transform as the basis for the cryptographic generator, we recommend SHA-256 for reasons already given, although other cryptographic functions may be equally suitable, and quite possibly faster, which might give somewhat different tradeoffs than we observed in our tests. For relatively reliable and easily parallelizable LCGs, we recommend the SPRNG library [16]. These generators have been extensively studied and are suitable for highly parallel implementation, and based on our results they should perform well as part of a size=16 or size=32 combined generator.

5 Conclusion

In this paper we have shown that the use of a full-strength cryptographic transform in a combined generator can dramatically improve the quality of streams
produced by a conventional linear congruential generator for stochastic simula-
tions, with no significant sacrifice in speed. Moreover, our approach provides
explicit, quantitative control of the tradeoff between speed and reliability, so
that users can easily select the choice that best meets their needs in a particular
situation. Future work along this line could include consideration of other cryp-
tographic transforms besides SHA-256, as well as other types of conventional
generators besides LCGs, such as lagged Fibonacci generators, for the respec-
tive components of a combined generator. Either or both of these options may
produce somewhat different tradeoffs between speed and reliability from those
we observed in our experiments.

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