ANISOTROPIC FLOW OF CONVEX HYPERSURFACES BY THE SQUARE ROOT OF THE SCALAR CURVATURE

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ABSTRACT. We show the existence of a smooth solution for the flow deformed by the square root of the scalar curvature multiplied by a positive anisotropic factor \( \psi \) given a strictly convex initial hypersurface in Euclidean space suitably pinched. We also prove the convergence of rescaled surfaces to a smooth limit manifold which is a round sphere if \( \psi \) has a strict local minimum at the limit point. In dimension two, it is shown that, with a volume preserving rescaling, the limit profile satisfies a soliton equation.

1. Introduction

The evolution of hypersurfaces in Euclidean spaces governed by curvature has been considered in many aspects in geometric analysis and mathematical physics. In this paper, we consider a one parameter family of immersions \( X(\cdot, t) : S^n \to M_t \subset \mathbb{R}^{n+1}, M_t := X(S^n, t), \) and its evolution in time governed by the square root of scalar curvature on \( M_t \) and a given smooth positive function \( \psi \) in \( \mathbb{R}^{n+1} \) with compact support: consider the following initial value problem

\[
\frac{\partial X}{\partial t} = -\psi(X(x, t))R(x, t)^{1/2} \nu,
\]

\( M_0 = X(S^n, 0), \)

where \( \nu \) is the outward unit normal to \( M_t \), and \( M_0 \) is a strictly convex smooth hypersurface in \( \mathbb{R}^{n+1} \). Here \( \psi \), by which we call the anisotropic factor, can be considered as a nonhomogeneous influence on the curvature flow from the underlying manifold \( \mathbb{R}^{n+1} \). Note the dependence of \( \psi \) on the position \( X(x, t) \), not on the normal vector \( \nu \) as considered in [A2] and [A7]. Throughout the paper, we shall call the flow in (1.1) anisotropic scalar curvature flow in short. The flow we concern in this paper is the generalization of that considered by Chow in [C2]. Our aim is to show the smooth convergence of the flow and find a condition on \( \psi \) to have a spherical limit profile of the rescaled flow.

1.1. Notation. In a local coordinates system \( \{x_1, \cdots, x_n\} \), the induced metric and the second fundamental form are given by

\[
g_{ij} = \left( \frac{\partial X}{\partial x^i}, \frac{\partial X}{\partial x^j} \right) \quad \text{and} \quad h_{ij} = -\left( \frac{\partial^2 X}{\partial x^i \partial x^j}, \nu \right),
\]
respectively, where \( \langle \cdot, \cdot \rangle \) is the standard inner product and \( \nu \) is the outward unit normal vector to \( M \). In terms of these, the Weingarten map \( W \) is given by

\[
W = (h^i_j) = (g^{ik} h_{kj}),
\]

with the eigenvalues \( \lambda_1, \ldots, \lambda_n \), and its inverse given by \( W^{-1} = (h^{-1})^i_j = \frac{\mathbb{g}}{\lambda_i}(h)_{kj} \), where \( \mathbb{g} \) is the standard round metric on the \( n \)-dimensional sphere \( S^n \) and \( \nabla \) is the connection of \( \mathbb{g} \) on \( S^n \). Let \( \sigma_k = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k} \) be the \( k \)-th symmetric function of the curvature, and one can write the mean curvature \( H = \text{trace}(h^i_j) = \sigma_1 = \sum_{1 \leq i \leq n} \lambda_i \), the Gauss curvature \( K = \det(h^i_j) = \lambda_1 \lambda_2 \cdots \lambda_n \) and the scalar curvature \( R = \sum_{i \neq j} \lambda_i \lambda_j \) on which we shall focus in this paper.

We consider the following parabolic flow which is expected to converge to a smooth hypersurface \( M^* \):

\[
(1.2) \quad X_t = -F(W, \nu) \nu
\]

where we take

\[
F(W, \nu) = \psi R^{1/2}(\nu, t),
\]

for the flow \( (1.1) \). This can be written in terms of the support function \( S = S(z, t) = \langle z, X(t) \rangle, z \in S^n \), as

\[
S_t = -\psi \left( \frac{s_n}{s_{n-2}} \right)^{1/2} = \Phi(W^{-1}, z)
\]

for \( \Phi(W^{-1}, z) = -F(W, \nu) \) for \( z = \nu \in S^n \).

Throughout the paper, \( C \) denotes a positive constant depending only on the dimension \( n \) and other fixed constants, and we write \( c(a_1, \ldots, a_k) \) for a positive constant depending only on its arguments \( a_1, \ldots, a_k \).

1.2. History. The well known example of evolution of hypersurfaces by extrinsic curvature is the mean curvature flow (see [B], [CGG], [CM], [E1], [EH], [Hu1], [Hu2], [Wh] among many) for which excellent lecture notes ([E2], [Sm], [Wa], [Z]) are available, and others are Gauss curvature flow (see, for example, [A1], [A2], [C1], [CEI], [DH], [DL1], [DL2], [F], [H], [KLR]), the flows evolving with the speed of powers of mean curvature (see [CRS], [Sc1], [Sc2]), the flows by homogeneous functions of the principal curvatures ([A3], [A4], [AM]), and the most notably for our interest, the flow deformed by powers of the scalar curvature ([AS], [C2]). In [C2], Chow proved the short time existence and the long time existence as well as the convergence to a point, and also the convergence of the rescaled flow to a round sphere. The difference between the flow in [C2] and \( (1.1) \) is the presence
of the anisotropic factor $\psi$ and the limit profile is expected to satisfy a non-trivial limit equation. With a further assumption that the perturbation from $\psi$ is relatively small compared with the initial data, we show that the flow under the parabolic rescaling converges to a round sphere. Note that the flow (1.1) is somewhat related to the logarithmic Gauss curvature flow considered in [CW] to solve the Minkowski problem where the evolution equation is given by

$$\frac{\partial X}{\partial t} = -\log \frac{K(v)}{f(v)} v,$$

(1.3)

where $K(v)$ is the Gauss curvature of $M_t$ and $f$ is a positive smooth function on $S^n$. Along this flow, the smoothness and the convexity of the hypersurfaces are preserved, and given that the weighted center of mass is equal to zero and starting from suitably chosen initial data, the limit profile of (1.3) satisfies

$$\log \frac{K(v)}{f(v)} = 0,$$

which is equivalent to have $K = f$ so that the given Borel measure of $S^n$ coincides with the area measure of the convex hypersurface. Likewise, consider the immersions of convex hypersurfaces with the evolution according to

$$\frac{\partial X}{\partial t} = -\left(\frac{F}{f(v)} - 1\right) v,$$

(1.4)

where $F$ is a function depending on the curvature of the hypersurface and $f$ is a function given a priori. Although it is not shown in this paper that under (1.1), the limit hypersurface has its scalar curvature equal to a given smooth function on $S^n$, one may expect that depending on $f$, the flow (1.4) contracts to a point, expand to an asymptotic sphere or converges to a convex hypersurface with its limit profile satisfying $F = f$ under some conditions.

1.3. Main Theorems. We state the main results for the flow (1.1).

Theorem 1.1. Let $M_0 = X(S^n, 0)$ be a compact, connected and strictly convex smooth manifold in $\mathbb{R}^{n+1}$. Suppose that $h_{ij} \geq \epsilon (H + c) g_{ij}$ initially for some $\epsilon > 0$ and $c > 0$ satisfying

$$c \geq \max \left\{ \frac{5n^2 |D\psi|}{\epsilon^2 \psi}, \frac{3n |D^2\psi||1/2}{\epsilon \psi^{1/2}} + \frac{2n^3 |D\psi|}{\psi} \right\},$$

(1.5)

where $D$ is the gradient in $\mathbb{R}^{n+1}$, $|D\psi|^2 = \sum_{k \geq 1} |D_k \psi|^2$ and $|D^2\psi|| = \sup_{w \in S^n} |D^2\psi(w, w)|$. Then there exist a maximal time $T > 0$ and a unique smooth solution $\{M_t = X(S^n, t)\}$ satisfying (1.1) for $t \in [0, T)$, and $M_t$ converges to a point $x_0 = M^*$ as $t$ approaches $T$.

Remark 1.2.

(i) The condition (1.5) can be regarded as the balance between strict convexity and the perturbation of $\psi$ from a constant map.
(ii) If the smallest positive principal curvature is large compared with the perturbation of $\psi$, the initial hypersurface satisfies (1.5). Then a pinching estimate follows and the convexity of the hypersurfaces preserved. Otherwise $\psi$ dominates and the convexity of the hypersurfaces may not be preserved.

(iii) If the smallest positive principal curvature is small, then the perturbation of $\psi$ is required to be small for (1.5) to be satisfied.

In order to observe the behavior of the solution near the maximal time $T$, we rescale the solution and the time parameter by

$$\tilde{X}(x, \tau) = \frac{X(x, t) - X(x, T)}{\sqrt{2(T - t)}},$$

and

$$\tau = -\frac{1}{2} \log \left(\frac{T - t}{T}\right).$$

Then the rescaled equation of (1.2) is

$$(1.7) \quad \frac{\partial \tilde{X}}{\partial \tau} = -\tilde{F}(x, \tau) + \tilde{X}, \quad \tilde{F} = \tilde{\psi}\tilde{R}^{1/2},$$

on $S^n \times [0, \infty)$.

For the rescaled hypersurfaces $\tilde{M}_t$, we obtain the convergence to a smooth manifold:

**Theorem 1.3.** Under the hypotheses in Theorem 1.1, the hypersurfaces $\tilde{M}_t$ under the parabolic rescaling converge in the $C^\infty$-topology to a smooth manifold $\tilde{M}^*$ as $\tau$ approaches infinity. In addition, if $\psi$ has a strict local minimum at $x_0$, then $\tilde{M}^*$ is a round sphere $S^n$. In dimension two, with a volume preserving rescaling, the limit hypersurface $\tilde{M}^*$ satisfies the equation $\tilde{S}^* = C\tilde{\psi}(\tilde{R}^*)^{1/2}$ for some constant $C > 0$, where $\tilde{S}^*$ and $\tilde{R}^*$ are the support function and the scalar curvature of $\tilde{M}^*$, respectively.

1.4. **Outline.** The paper is organized as follows. In Section 2, we find the evolution equations of tensors related to curvature. In Section 3, a pinching estimate for the second fundamental form is shown. In general, the pinching estimate derived from a maximum principle to tensors plays a crucial role to prove the convergence of convex hypersurfaces. In [A6], the second derivative pinching estimates for a class of nonlinear parabolic equations were shown when the function describing the speed satisfies some structural criteria. For the flow (1.2), $F$ is the function composed of the anisotropic factor $\psi$ and second derivatives of $X$ which belongs to the space of concave functions. In order to control the trouble terms that appear in the modification of the maximum principle in [A6], an additional perturbation term is required in the pinching. In Section 4, we show a global $L^p$ estimate of a scale invariant curvature quantity for large $p$ and consequently obtain a global $L^\infty$ estimate using the Moser iteration. This also can be achieved from the De Giorgi method. In Section 5, applying the pinching estimate, we obtain the uniform upper
bound of curvature before the blow up (i.e. \(0 \leq t \leq T - \delta\)) for any small \(\delta > 0\) if \(M_t\) is smooth. For the rescaled flow, we also acquire a uniform curvature bound. Finally, in Section 6, the proofs of Theorem 1.1 and Theorem 1.3 are given: the existence of a smooth limit manifold \(M^*\) and the convergence of the rescaled flow to a round sphere under the conditions in Theorem 1.3. Moreover, in dimension two, by rescaling the hypersurfaces homothetically so that the volume is preserved, it is shown that a soliton equation is realized by the limit profile.

2. Evolution Equations

In this section, we obtain the evolution of the quantities related to the curvature of the hypersurface \(M_t\). Prior to the computation, we introduce the first and the second derivatives of the curvature \(F\) with respect to \(h_{ij}\):

\[
\begin{align*}
\dot{F}^{ij} &:= \psi R^{-1/2}(Hg^{ij} - h^{ij}), \\
\ddot{F}^{ijkl} &:= -\psi R^{-3/2}(Hg^{ij} - h^{ij})(Hg^{kl} - h^{kl}) + \psi R^{-1/2}(g^{ij}g^{kl} - \delta^{ik}\delta^{jl}).
\end{align*}
\]

Lemma 2.1. Under the parabolic flow (1.2), we have

\[
\begin{align*}
(1) \frac{\partial}{\partial t} g_{ij} &= -2Fh_{ij} \\
(2) \frac{\partial}{\partial t} v &= \nabla_i F \frac{\partial X}{\partial x} \\
(3) \frac{\partial}{\partial t} h_{ij} &= \nabla_i \nabla_j F - Fh_{ij}h_j^i \\
(4) \frac{\partial}{\partial t} H &= 2Fh_{ij}h_{ij} + g^{ij} \frac{\partial}{\partial t} h_{ij} \\
(5) \frac{\partial}{\partial t} |A|^2 &= 2(\nabla_i \nabla_j F)h_{ij} + 2F(\text{tr}A^3) \\
(6) \frac{\partial}{\partial t} F &= \dot{F}^{ij} \nabla_i \nabla_j F + \psi^2 |A|^2 - \psi^2 (\text{tr}A^3) - \psi R(D\psi \cdot v),
\end{align*}
\]

where \(\text{tr}A^3 = h_{ij}h_{jk}h_{ki}\), and \(D\) is the gradient in \(\mathbb{R}^{n+1}\).

From Lemma 2.1, the detailed evolutions of the second fundamental form and hence the mean curvature can be derived.

Lemma 2.2. Under the anisotropic scalar curvature flow (1.2), we have

\[
\begin{align*}
\frac{\partial}{\partial t} h_{ij} &= \dot{F}^{ki} \nabla_k \nabla_j h_{ij} - \frac{\psi R^{-\frac{1}{2}}}{H^2}(H
abla_j h_{kl} - h_{kl} \nabla_j H)(H \nabla_j h_{kl} - h_{kl} \nabla_j H) \\
&\quad + \frac{1}{4} \psi H^4 R^{-\frac{3}{2}} \nabla_i \left( \frac{|A|^2}{H^2} \right) \nabla_j \left( \frac{|A|^2}{H^2} \right) \\
&\quad + \psi R^{-\frac{1}{2}} \left[ (H \nabla_j H - A \cdot \nabla_j A) \cdot \nabla_i \psi + (H \nabla_i H - A \cdot \nabla_i A) \cdot \nabla_j \psi \right] \\
&\quad + R^* \nabla_i \nabla_j \psi - (A^3)_{ij} + \psi R^{-\frac{1}{2}} [(H|A|^2 - \text{tr}A^3)h_{ij} - (H^2 - |A|^2)(A^2)_{ij}] \\
&\quad + \frac{\psi}{H^2} \left[ (H \nabla_j H - A \cdot \nabla_j A) \cdot \nabla_i \psi + (H \nabla_i H - A \cdot \nabla_i A) \cdot \nabla_j \psi \right] \\
&\quad + R^* \nabla_i \nabla_j \psi - (A^3)_{ij} + \psi R^{-\frac{1}{2}} [(H|A|^2 - \text{tr}A^3)h_{ij} - (H^2 - |A|^2)(A^2)_{ij}].
\end{align*}
\]
where $\text{tr} A^3 = h^{ij} h_{ij} h_{ij}$. Let $\Box := (Hg^{ij} - h^{ij}) \nabla_i \nabla_j$.

\[
\frac{\partial}{\partial t} H = R^{-1/2} \left[ \psi \Box H - \frac{\psi}{H^2} |\nabla H h| - h_{ij} \nabla^2 H - \frac{\psi}{4R} |H^2 \nabla H + \frac{2R}{\psi} \nabla \psi|^2 \right] + \frac{2R}{H} (\nabla H, \nabla \psi) + \frac{R^{1/2}}{\psi} |\nabla \psi|^2 + R^{1/2} \Delta \psi + \psi R^{-1/2} H |H A|^2 - \text{tr} A^3,
\]

\[
\frac{\partial}{\partial t} |A|^2 = F^{ij} \nabla_k \nabla_j |A|^2 - \frac{2\psi}{R^{1/2}} |\nabla H h_{ig}|^2 - \frac{2\psi}{H R^{1/2}} |H \nabla H h| - h_{ij} |\nabla H|^2 - \frac{H^4 \psi}{2R^{3/2}} |\nabla (|A|^2 / H^2)|^2_{h} + \frac{2\psi}{R^{1/2}} (H |A|^2 - \text{tr} A^3)|A|^2 + \frac{4h^{ij}}{R^{1/2}} (H \nabla H - A \cdot \nabla \psi) \cdot \nabla \psi + 2H^{1/2} h^{ij} \nabla_i \nabla_j \psi,
\]

\[
\frac{\partial}{\partial t} \left( \frac{|A|^2}{H^2} \right) = \psi R^{-1/2} \Box \left( \frac{|A|^2}{H^2} \right) - \frac{2\psi R^{1/2}}{H^3} |H \nabla H h| - h_{ij} |\nabla H|^2 + \frac{2\psi}{R^{1/2} H} (\nabla H, \nabla (|A|^2 / H^2))_{h} - \frac{\psi H}{2R^{3/2}} |\nabla (|A|^2 / H^2)|^2_{h} - \frac{2R^{1/2}}{\psi H^3} |\nabla \psi|^2_{A \cdot H} - \frac{2R^{1/2}}{\psi H} |\nabla \psi|^2_{h \cdot \nabla H} - \frac{4R^{1/2}}{H^4} (|A|^2 / H^2 - h^{ij}) H \nabla \psi_{ij}.
\]

**Remark 2.3.** Depending on the sign of the term $D \psi \cdot \nu$, a lower bound for $F$ does not follow directly from Lemma 2.1 (6) in general.

### 3. Pinching Estimate

In this section, it shall be shown that if all the principal curvatures are of the same order at the initial time, it remains so until the maximal time. This will be used in later sections to deduce the convergence to a point and find the limit of the rescaled solution. Denote $E = R^{1/2}$ so that $F = \psi E$. With the notation (2.1), we have

\[
F^{ij} = \psi \dot{E}^{ij}, \quad F^{ij,kl} = \psi \dot{E}^{ij,kl}.
\]

Define a $(2, 0)$-tensor $W$ by

\[
W_{ij} = h_{ij} - c(H + c) g_{ij},
\]
matrices, we say $F$ is inverse-concave if

$$
\frac{\partial F}{\partial A} = -2\psi E_{ij}
$$

$$
\frac{\partial h_{ij}}{\partial t} = \nabla \nabla (\psi \mathcal{E}) - \psi E^{k}_{ij} h_{kl} h_{lj}
$$

$$
= \psi [E^{k}_{ij} h_{kl} h_{lj} + E^{j}_{ik} h_{kl} h_{lq} + h_{ij} E^{kl} g^{pq} h_{pq} h_{kl} - 2E g^{kl} h_{kl} h_{jl}]
$$

$$
+ EV_{ij} \psi + \nabla_{l} EV_{ij} \psi + \nabla_{i} \psi \nabla_{j} E
$$

$$
\frac{\partial W_{ij}}{\partial t} = \psi E^{k}_{ij} h_{kl} h_{lj} + W_{ij} \psi E^{kl} g^{pq} h_{pq} h_{kl} - 2W_{ij} \psi E g^{kl} h_{kl} + N_{ij} + M_{ij}
$$

where

$$h_{ij} = \nabla \nabla (\psi h_{ij}),$$

$$N_{ij} = \psi E^{k}_{ij} h_{kl} h_{pq} E_{pq} + \psi g_{ij} E^{kl} g^{pq} h_{pq} h_{kl},$$

$$M_{ij} = E [\nabla_{l} EV_{ij} \psi + h_{ij} \psi h_{lq} \nabla_{q} \psi].$$

For a symmetric matrix $A$, we may write $E(A) = f(\lambda(A))$, where $\lambda(A) = (\lambda_1, \cdots, \lambda_n)$ is the map which takes $A$ to its eigenvalues $\lambda_i$. Thus for $A = (h_{ij})$, one has

$$E(h_{ij}) = f(\lambda_1, \cdots, \lambda_n) = R^{1/2} = \left( \sum_{i \neq j} \lambda_i \lambda_j \right)^{1/2}.$$

**Definition 3.1.** For a $C^2$ function $F$ defined on the cone $S_+$ of positive definite symmetric matrices, we say $F$ is inverse-concave if

$$F(A) := -f(\lambda_1^{-1}, \cdots, \lambda_n^{-1}),$$

is concave for any $A \in S_+$.

For a symmetric matrix $A$ with eigenvalues $\lambda_i$'s, $R^{1/2}(A)$, a symmetric homogeneous function of degree one, is concave and inverse concave since the ratio of symmetric functions $\sigma_{k+1}/\sigma_k$, $k = 0, \cdots, n-1$, and their geometric means are concave and inverse-concave as shown in [A6] and [Li]. Let $f^k$ denote the derivative of $f(\lambda(A))$ with respect to $\lambda_k$, and $(\delta_{ij})$ be the diagonal matrix with 1 in the entries. From the definition of inverse concavity, one can obtain the following lemma:

**Lemma 3.1 (Corollary 5.4, [A6]).** Let $A$ be a symmetric $n \times n$ matrix and let $F = F(A)$ be a smooth function of $A$. Then $F$ is concave at $A$ if and only if

$$f^{k} + 2 f^{k - 1} \delta_{kk} \geq 0$$

for all $k \neq l$, and

$$f^{k l} + 2 f^{k - 1} \delta_{kl} \geq 0.$$

Indeed this holds for $A = (h_{ij})$ and $f = R^{1/2}$ whose derivatives are

$$f^{i} = R^{-1/2}(H - h_{ii}),$$

$$f^{ij} = -R^{-3/2}(H - h_{ii})(H - h_{ij}) + \psi R^{-1/2}(1 - \delta^{ij}).$$
In order to apply the maximum principle for \( W_{ij} \), one shall see in Lemma 3.3 that a perturbation term \( c_\varepsilon g_{ij} \) in \( W_{ij} \) nullifies the effect of \( \psi \). Assuming this, one obtains the following pinching estimate adapting Theorem 3.2 in [Aô].

**Theorem 3.2.** Let

\[
W_{ij} := h_{ij} - c(H + c)g_{ij},
\]

for some constants \( c > 0 \) and \( c > 0 \) satisfying

\[
c \geq \max \left\{ \frac{5n^2|D\psi|}{\varepsilon^2 \psi}, \frac{3n||D^2\psi||^{1/2}}{\varepsilon \psi^{1/2}} + \frac{2n^2|D\psi|}{\psi} \right\},
\]

where \( D \) is the gradient in \( \mathbb{R}^{n+1} \), \( |D\psi|^2 = \sum_{k=1}^n |D_k\psi|^2 \) and \( ||D^2\psi|| = \sup_{w \in S^n} |D^2\psi(w, w)| \). Then if \( W_{ij} \) is non-negative everywhere in \( M \) at time \( t = 0 \), then it remains so on \( M \times [0, T] \).

**Proof.** Suppose that \( W_{ij} \) takes its minimum at \( (p, t_0) \in M \times [0, T] \) in the direction say \( v \in T_pM \) where local coordinates \( \{x_1, \ldots, x_n\} \) are chosen to have \( v = \frac{\partial}{\partial x_1} \) and the connection coefficients vanish at \( (p, t_0) \). Taking \( p = 0 \) for convenience, one has

\[
\min_{(x, t)} \min_{\xi} W_{ij} \xi^i \xi^j = W_{11}(0, t_0) = 0.
\]

Then for any \( n \times n \) matrix \( B^{ij} = B^{ij}(x_1, \ldots, x_n) \) and \( \xi = \xi^i \frac{\partial}{\partial x^i} \), where \( \xi^i = \delta_i^1 + B^{ij}x_j \),

\[
W(x, t) := W_{ij} \xi^i \xi^j(x, t) \geq 0 \quad \text{for } t \in [0, t_0],
\]

satisfies \( W(0, t_0) = W_{11}(0, t_0) = 0 \). At \( (0, t_0) \), since \( \xi^i(0) = \delta_i^1 \) and \( \frac{\partial \xi^i}{\partial x^j} = B^{ij} \), one has

\[
\frac{\partial W}{\partial x^k} = \frac{\partial W_{ij}}{\partial x^k} \xi^i \xi^j + 2W_{ij} \frac{\partial \xi^i}{\partial x^k} \xi^j
\]

\[
= \frac{\partial W_{11}}{\partial x^k} + 2W_{1i} B^{ik} = 0
\]

\[
\frac{\partial^2 W}{\partial x^k \partial x^l} = \frac{\partial^2 W_{ij}}{\partial x^k \partial x^l} \xi^i \xi^j + 2 \frac{\partial W_{ij}}{\partial x^k} \frac{\partial \xi^i}{\partial x^l} \xi^j + 2 \frac{\partial W_{ij}}{\partial x^l} \frac{\partial \xi^i}{\partial x^k} \xi^j + 2W_{ij} \frac{\partial \xi^i}{\partial x^k} \frac{\partial \xi^j}{\partial x^l}
\]

\[
= \frac{\partial^2 W_{11}}{\partial x^k \partial x^l} + 2 \frac{\partial W_{1j}}{\partial x^k} B^{ik} + 2 \frac{\partial W_{1i}}{\partial x^l} B^{ik} + 2W_{ij} B^{ik} B^{ij}
\]

so that for \( i > 1 \),

\[
\nabla_i \nabla_j W = \nabla^k \nabla_i \nabla_j W = \nabla^k \frac{\partial^2 W_{11}}{\partial x^k \partial x^l} + 4\nabla^k \frac{\partial W_{11}}{\partial x^k} B^{ik} + 2\nabla^k \frac{\partial W_{ij}}{\partial x^i} B^{ik} + \frac{2f^k}{h_{ii} - h_{11}} \frac{\partial W_{il}}{\partial x^k} \frac{\partial W_{jl}}{\partial x^l} + 2(h_{ii} - h_{11}) f^k \delta_{ii} \delta_{ij}(B^{ik} + \frac{1}{h_{ii} - h_{11}} \frac{\partial W_{il}}{\partial x^k})(B^{ik} + \frac{1}{h_{jj} - h_{11}} \frac{\partial W_{jl}}{\partial x^l}).
\]

By taking \( B^{ik} = -\frac{\partial W_{ik}}{\partial x^k} \), one obtains

\[
\nabla_i \nabla_j W(0, t_0) \geq \frac{2f^k}{h_{ii} - h_{11}} h_{ik}^2,
\]
Proof. Suppose that \( c \) is chosen to satisfy

\[
\frac{\partial W}{\partial t} = \psi \tilde{E}^{\mu \nu}[h_{\mu \nu} - \epsilon h_{\rho \delta} \delta_{\mu \nu}] + \frac{\partial^2 \psi}{\partial \psi^{1/2}} + \frac{2n^2 |D\psi|}{\psi} \leq 0
\]

and that local coordinates are taken as in Theorem \( \text{3.2} \). Then one has

\[
\frac{2\psi f^k}{h_{ij} - h_{11}} h_{11}^2 + N_{11} + M_{11} \geq 0 \quad \text{at } (0, t_0).
\]

By applying a maximum principle, it suffices to show that the right hand side of (3.7) is non-negative. The conclusion follows from Lemma \( \text{3.3} \) below which provides the required null eigenvector condition.

Lemma 3.3. Suppose that \( c \) is chosen to satisfy

\[
c \geq \max \left\{ \frac{5n^2 |D\psi|}{e^2 \psi}, \frac{3n^2 |D^2 \psi|^{1/2}}{e^2 \psi^{1/2}} + \frac{2n^2 |D\psi|}{\psi} \right\},
\]

and that local coordinates are taken as in Theorem \( \text{3.2} \). Then one has

\[
\frac{2\psi f^k}{h_{ij} - h_{11}} h_{11}^2 + N_{11} + M_{11} \geq 0 \quad \text{at } (0, t_0).
\]
where its proof can be found in Theorem 5.1 in [A6]. This implies that (3.9) can be written as

$$Q := \sum_{j,k \geq 1} [F(h_{1kl}, h_{1kl}) - e \sum_{j,k \geq 1} F(h_{jkl}, h_{jkl})] + 2\psi \sum_{j,k \geq 1} \frac{\hat{f}^k}{h_{jkl}} h_{1kl}^2 + R^{1/2} [\psi_{11} - e \Delta \psi]$$

$$+ 2 \sum_{j,k \geq 1} f^j [h_{1jl} \psi_1 - e \sum_{j,k \geq 1} h_{kjl} \psi_1] + c e \psi \sum_{k \geq 1} f^k h_{kk}^2$$

$$= \psi \sum_{j,k \geq 1} f^k h_{1kk} h_{1jl} + 2\psi \sum_{j,k \geq 1} \frac{\hat{f}^k - f^j}{h_{kk} - h_{jk}} h_{1kl}^2 - e \psi \sum_{j,k \geq 1} f^k h_{jkl} h_{1jl} - 2\psi \sum_{j,k \geq 1} \frac{\hat{f}^k - f^l}{h_{kk} - h_{jl}} h_{1kl}^2$$

$$+ 2\psi \sum_{j,k \geq 1} \frac{\hat{f}^k}{h_{jkl}} h_{1kl}^2 + R^{1/2} [\psi_{11} - e \Delta \psi] + 2 \sum_{j,k \geq 1} f^j [h_{1jl} \psi_1 - e \sum_{j,k \geq 1} h_{kjl} \psi_1] + c e \psi \sum_{k \geq 1} f^k h_{kk}^2.$$

Since $W_{11}(0, t_0) = 0$, one has

$$h_{111} = \frac{e}{1 - e} \sum_{j \geq 1} h_{jj}^2, \quad k \geq 1,$$

which shall be used frequently in the computation below. The sum $Q$ of quadratic terms can be decomposed into $Q_k$, $k \geq 1$, ones with repeated indices, and $Q_{jkl}$, ones with distinct indices:

$$Q = Q_0 + \sum_{1 \leq k \leq n} Q_k + \sum_{1 \leq j < k \leq n} Q_{jkl},$$

where

$$Q_0 = R^{1/2} [\psi_{11} - e \Delta \psi] + c e \psi \sum_{1 \leq k \leq n} f^k h_{kk}^2 + \frac{1}{2} c e \psi f^n h_{mn}^2,$$

$$Q_1 = (1 - e) \psi \sum_{j,k \geq 1} f^j h_{1jk} h_{1jl} + 2\psi \sum_{j,k \geq 1} \frac{f^j}{h_{jj} - h_{11}} h_{1kl}^2 - e \psi \sum_{j,k \geq 1} f^j h_{jkl} h_{1jl} - 2\psi \sum_{j,k \geq 1} \frac{f^j}{h_{jj} - h_{11}} h_{1kl}^2$$

$$+ 2(1 - e) \sum_{j \geq 1} \psi_1 f^j h_{1jl} + \frac{1}{2n} c e \psi f^n h_{mn}^2,$$

$$Q_k = -e \psi \sum_{j,k \geq 1} f^j h_{1jk} h_{1kl} + 2\psi \frac{f^j}{h_{kk} - h_{11}} h_{1kl}^2 + 2\psi \frac{f^k - f^l}{h_{kk} - h_{11}} h_{1kl}^2 - 2\psi \sum_{j,k \geq 1} \frac{f^j}{h_{jj} - h_{11}} h_{1kl}^2$$

$$- 2(1 - e) \sum_{j \geq 1} \psi_1 f^j h_{1jl} + \frac{1}{2n} c e \psi f^n h_{mn}^2, \quad \text{for } k > 1,$$

$$Q_{jkl} = 2[(1 - e) \frac{f^j - f^k}{h_{jj} - h_{kl}} + \frac{f^j}{h_{jj} - h_{11}} + \frac{f^k}{h_{kk} - h_{11}} - e\left(\frac{f^j}{h_{jj} - h_{11}} + \frac{f^k}{h_{kk} - h_{11}}\right)] h_{1kl}^2,$$

$$Q_{jkl} = -2e\left[\frac{f^j - f^k}{h_{jj} - h_{kk}} + \frac{f^j}{h_{jj} - h_{11}} + \frac{f^k}{h_{kk} - h_{11}}\right] h_{1kl}^2, \quad 1 < j < k < l \leq n.$$
We will show that each of these is non-negative with appropriate choice of $c$.

For the computation below, the following estimates can be easily obtained from the definition $f^i$ and the pinching condition at $(0, t_0)$:

$$\frac{f^i - f^k}{h_{ii} - h_{kk}} = -R^{-1/2},$$

$$h_{ii} \geq c(h_{jj} + c) \geq cc$$

for $i, j = 1, \ldots, n$,

$$\frac{c}{n} \leq \frac{h_{n-1,n-1}}{n(h_{nn}h_{n-1,n-1})^{1/2}} \leq f^i \leq \frac{h_{nn}}{h_{11}} \leq c^{-1}.$$

(i) From Lemma 3.1, it follows that $Q_{iij} \geq 0$.

(ii) From the concavity of $f$ and Lemma 3.1 one has $Q_{iij} \geq 0$ for $1 < j < k < l$.

(iii) For a fixed $k > 1$,

$$Q_k = -c\psi \sum_{i,j \geq 1} f^{ij} h_{ii} h_{jj} + 2\psi \frac{f^1}{h_{kk} - h_{11}} h_{k11}^2 + 2\psi \frac{f^k}{h_{kk} - h_{11}} h_{k11}^2 - 2c\psi \sum_{j \geq 1} \frac{f^i - f^k}{h_{jj} - h_{kk}} h_{jj}^2$$

$$- 2c \sum_{j \geq 1} \psi h_{jj} f^i h_{jj} + \frac{1}{2n} c \psi f^n h_{nn}^2,$$

$$\geq 2\psi \frac{f^k}{h_{kk} h_{k11}} h_{k11}^2 + 2c \psi \sum_{j \geq 1} R^{-1/2} h_{k11}^2 - 2c \sum_{j \geq 1} \psi f^i h_{jj} + \frac{1}{2n} c \psi f^n h_{nn}^2.$$ 

By Young’s inequality and (5.10),

$$\sum_{j \geq 1} \psi f^i h_{jj} \leq \sum_{j \geq 1} \psi f^j h_{jj} + \psi f^k h_{kk}$$

$$\leq \sum_{j \geq 1} \left( \frac{\psi}{2R^{1/2}} h_{jj}^2 + \frac{2n|\psi|^2}{\psi} |f^i|^2 R^{1/2} \right) + \psi f^k \left( \frac{1 - c}{c} h_{k11} - \sum_{j \geq 1} h_{jj} \right)$$

$$\leq \sum_{j \geq 1} \left( \frac{\psi}{2R^{1/2}} h_{jj}^2 + \frac{2n|\psi|^2}{\psi} |f^i|^2 R^{1/2} \right) + \psi f^k \left( \frac{h_{kk}|\psi|^2}{e^2 \psi} R^{1/2} + \frac{h_{k11}}{h_{kk}} + \frac{|\psi|^2}{e^2 \psi} \right)$$

$$+ \sum_{j \geq 1} \left( \frac{\psi}{2R^{1/2}} h_{jj}^2 + \frac{2n|\psi|^2}{\psi} |f^k|^2 R^{1/2} \right)$$

$$\leq \sum_{j \geq 1} \left( \frac{\psi}{R^{1/2}} h_{jj}^2 + \psi f^k \left( \frac{h_{kk}|\psi|^2}{e^2 \psi} R^{1/2} + \frac{4n|\psi|^2}{e^2 \psi} R^{1/2} \right) \right),$$

which yields

$$Q_k \geq -\frac{2}{e^2 \psi} |\psi|^2 R^{1/2} + \frac{1}{2n} c \psi f^n h_{nn}^2 \geq -\frac{10n|\psi|^2}{e^2 \psi} R^{1/2} + \frac{1}{2n} c e^{3/2} \psi H^2.$$
Therefore since \( H \geq cn \), \( Q_k \geq 0 \) if

\[
(3.11) \quad c \geq \frac{5n^2|D_k \psi|}{e^{7/4} \psi}.
\]

(iv) Consider \( Q_1 \):

\[
Q_1 = (1 - e) \psi \sum_{i,j} f_{ij} h_{1ij} + 2 \psi \sum_{j > 1} \frac{f_j}{h_{1j}} h_{1j}^2 - 2 \epsilon \psi \sum_{j > 1} \frac{f_j - f_1}{h_{1j}} h_{1j}^2
+ 2(1 - e) \sum_{j > i} \psi_1 f_{ij} h_{1ij} + \frac{1}{2n} e \epsilon \psi f_{ij} h_{1j}^2,
\]

Using (3.10), the first sum above can be written as

\[
(1 - e) \sum_{k,l > 1} \left( \frac{f_{kl}}{h_{kk} - h_{11}} + \frac{\epsilon}{1 - \epsilon} \left( \frac{f_{kl}}{h_{kk}} + f_{kl} \right) \right) + \left( \frac{\epsilon}{1 - \epsilon} \right)^2 f_{kl} h_{1k} h_{1l}.
\]

Let \( \phi(x_2, \ldots, x_n) := f \left( \frac{1}{n^2} (x_2 + \ldots + x_n), x_2, \ldots, x_n \right) \) and then its derivatives are

\[
\hat{\phi}^k = f^k + \frac{\epsilon}{1 - \epsilon} f^k,
\]

\[
\hat{\phi}^{kl} = f^{kl} + \frac{\epsilon}{1 - \epsilon} (f^{kl} + f^{kl}) + \left( \frac{\epsilon}{1 - \epsilon} \right)^2 f^{kl}
\]

for \( k, l > 1 \). The coefficients of the second and the third sum in \( Q_1 \) above are

\[
2 \psi \left( \frac{f_{kk}}{h_{kk} - h_{11}} - \epsilon \frac{f_{kk} - f_{kk}^1}{h_{kk} - h_{11}} \right) \geq \frac{2 \psi}{h_{kk} - h_{11}} [1 - (1 - e) f^1 + e f^1] = 2(1 - e) \psi \frac{\hat{\phi}^k}{h_{kk} - h_{11}},
\]

so that

\[
Q_1 \geq (1 - e) \psi \sum_{k,l > 1} \left( \frac{\hat{\phi}^{kl}}{h_{kk} - h_{11}} + 2 \frac{\hat{\phi}^{k}}{h_{kk} - h_{11}} \delta_{kl} \right) T_{1kk} T_{1ll} + 2(1 - e) \sum_{j > i} \psi_1 f_{ij} h_{1ij} + \frac{1}{2n} e \epsilon \psi f_{ij} h_{1j}^2
\]

\[
\geq 2 \psi \sum_{k > 1} \left( \frac{1}{h_{kk} - h_{11}} - \frac{1}{h_{kk}} \right) [1 - (1 - e) f^k + e f^k] h_{1k}^2 + 2(1 - e) \sum_{j > i} \psi_1 f_{ij} h_{1ij}
+ \frac{1}{2n} e \epsilon \psi f_{ij} h_{1j}^2.
\]

where the last inequality can be shown from the fact that the inverse-concavity of \( \phi \) follows from that of \( \hat{f} \) and Lemma (3.1) (ii). By Young's inequality and (3.10) for the terms involving \( \psi_1 \),

\[
2(1 - e) \sum_{j > i} \psi_1 f_{ij} h_{1ij} \geq -2 \psi R^{-1/2} \sum_{k > 1} \frac{h_{11}}{h_{kk}(h_{kk} - h_{11})} H_k h_{1kk} - \frac{2|\psi_1|^2}{R^{1/2} \psi} \sum_{k > 1} \frac{h_{kk}(h_{kk} - h_{11})}{h_{11}} H_k
\]

where \( H_k = H - h_{kk} + e(h_{kk} - h_{11}) \), one has

\[
Q_1 \geq -\frac{2|\psi_1|^2}{R^{1/2} \psi} \sum_{k > 1} \frac{h_{kk}(h_{kk} - h_{11})}{h_{11}} H_k + \frac{e \epsilon \psi}{2n R^{1/2} (H - h_{mn}) h_{mn}^2}
\]

\[
\geq -\frac{2n |\psi_1|^2}{R^{1/2} \psi} h_{11}^2 H + \frac{e \epsilon \psi}{2n R^{1/2} h_{11} h_{mn}^2}.
\]
which implies $Q_1 \geq 0$ if
\begin{equation}
(3.12) \\
c \geq \frac{2n^{3/2}|D_1 \psi|}{\epsilon^2 \psi}.
\end{equation}

(v) The terms involving $D^2 \psi$ in $Q_0$ can be bounded below as
\[ R^{1/2}(V_1 V_1 \psi - \epsilon \Delta \psi) \geq -R^{1/2}(|V_1 V_1 \psi| + \epsilon |\Delta \psi|) \geq -R^{1/2}[3||D^2 \psi|| + \epsilon H|D\psi|] \]
since $|\Delta_M \psi| \leq (n + 1)||D^2 \psi|| + H|D\psi|$. Thus one has
\[ Q_0 \geq -R^{1/2}[3||D^2 \psi|| + \epsilon H|D\psi|] + \frac{c\epsilon\psi}{2R^{1/2}}(H - h_{nn})h_{nn}^2, \]
and hence $Q_0 \geq 0$ if
\begin{equation}
(3.13) \\
c \geq \frac{2R}{\epsilon\psi(H - h_{nn})h_{nn}^2}[3||D^2 \psi|| + \epsilon H|D\psi|].
\end{equation}

The estimate
\[ \frac{R}{(H - h_{nn})h_{nn}^2} \leq \frac{n^2 h_{n-1} - 1 h_{nn}}{(H - h_{nn})h_{nn}^2} \leq \frac{n^3}{H}, \]
implies that $Q_0 \geq 0$ if
\begin{equation}
(3.14) \\
c \geq \frac{3n||D^2 \psi||^{1/2}}{\epsilon\psi^{1/2}} + \frac{2n^3 |D\psi|}{\psi}.
\end{equation}

Finally, from the conditions (3.11), (3.12) and (3.13), one can conclude that $Q$ is non-negative if $c$ satisfies the condition (3.8). \hfill \Box

**Corollary 3.4.** Let $h_{nn}$ and $h_{11}$ be the largest and the smallest nonzero eigenvalues of the Weingarten map respectively. With the conditions in Theorem 3.2, we have the following curvature pinching:
\begin{equation}
(3.14) \\
c \geq \frac{3n||D^2 \psi||^{1/2}}{\epsilon\psi^{1/2}} + \frac{2n^3 |D\psi|}{\psi}.
\end{equation}

In particular, $F \geq c \inf_M \psi$.

**4. Curvature Estimates**

In this section, we obtain the uniform upper bounds for curvatures of the evolving manifolds before the maximal time ($0 < t < T - \delta_0$) from some fixed $\delta_0$. With the Gauss map $\nu$, one can parametrize the hypersurface so that the support function $S$ is written as
\[ S(z, t) = \langle z, X(\nu^{-1}(z), t) \rangle, \quad \text{for } z \in S^n \text{ and } t \in [0, T - \delta_0]. \]

Along the flow \eqref{1.2},
\[ \frac{\partial S}{\partial t} = \left( z, \frac{\partial \nu^{-1}}{\partial t} \cdot \nabla X + \frac{\partial X}{\partial t} \right) = -F. \]
Let $\bar{g}$ be the standard metric on $S^n$ and $\bar{\nabla}$ be its connection. Note that $\bar{g}$ is independent of $t$. From the definition, one can compute
\begin{equation}
\bar{g}_{ij} = h_{ik}h_{jl}g^{kl}, \quad \text{and} \quad |A|^2 = g^{ij}\bar{g}_{ij}.
\end{equation}
Furthermore,
\begin{equation}
h_{ij} = \nabla_i \nabla_j S + S \bar{g}_{ij}
\end{equation}
implies
\begin{equation}
\frac{\partial h_{ij}}{\partial t} = \nabla_i \nabla_j \frac{\partial S}{\partial t} + \frac{\partial S}{\partial t} \bar{g}_{ij}.
\end{equation}

Lemma 4.1. With the metric $\bar{g}_{ij}$, we have the following evolution equation for $F$:
\begin{equation}
\frac{\partial F}{\partial t} = \frac{\psi H g^{ij}}{R^2} \nabla_i \nabla_j F + \psi^2 (H|A|^2 - trA^3) - \psi R (D\psi \cdot \nu).
\end{equation}

Proof. Lemma 2.1 and (4.1) give
\begin{equation*}
H = g^{ij}h_{ij} = (h^{-1})^{ik}\bar{g}_{kl}(h^{-1})^{jl}h_{ij} = \bar{g}_{kl}(h^{-1})^{ik},
\end{equation*}
\begin{equation*}
\frac{\partial H}{\partial t} = -\bar{g}_{kl}(h^{-1})^{ik}\frac{\partial h_{ij}}{\partial t}(h^{-1})^{jl} = g^{ij}(\nabla_i \nabla_j F + F \cdot \bar{g}_{ij}).
\end{equation*}
From these and Lemma 2.1 we can compute the evolution of $F = \psi R^{1/2}$:
\begin{equation*}
\frac{\partial F}{\partial t} = \frac{\psi}{2R^2} (2H \frac{\partial H}{\partial t} - \frac{\partial |A|^2}{\partial t}) - \psi R (D\psi \cdot \nu)
= \frac{\psi H g^{ij}}{R^2} \nabla_i \nabla_j F + \psi^2 (H|A|^2 - trA^3) - \psi R (D\psi \cdot \nu).
\end{equation*}

Let $r_{in}$ and $r_{out}$ be the inner and the outer radii of $\Sigma_t$, respectively, and let $w(z) := S(z) + S(-z)$, $z \in S^n$, be the width on $z$-direction. Then we have the maximum width $w_{\text{max}} = w(z_+)$ and the minimum $w_{\text{min}} = w(z_-)$ for some $z_+$ and $z_-$ in $S^n$.

Lemma 4.2.
If the initial hypersurface is pinched as in Theorem 3.2, we have
\begin{enumerate}[(i)]
\item $w_{\text{max}} \leq Cw_{\text{min}}$,
\item $r_{\text{out}} \leq \frac{w_{\text{max}}}{\sqrt{2}}$ and $r_{\text{in}} \geq \frac{w_{\text{min}}}{\sqrt{n+2}}$, and in particular, $r_{\text{out}} \leq Cr_{\text{in}}$,
\end{enumerate}
for some positive constant $C$.

Proof. (i) For the parametrization of a totally geodesic 2-sphere in $S^n$, choose a pair of angular variables $(\theta_+, \psi_+)$ mapped to $(\cos \theta_+ \sin \psi_+, \sin \theta_+ \sin \psi_+, \cos \psi_+)$ such that $\psi_+ = 0$ is corresponding to $z_+$. Expressing the second fundamental form $\Pi$ in terms of the support function $S$ in the $\psi_+$ direction, one has
\begin{equation*}
w(z_+) = S(z_+) + S(-z_+) = \int_{S^2} \Pi(\partial \psi_+, \partial \psi_+ ) d\mu_{S^2}.
\end{equation*}
See Theorem 5.1 in [A4] for the detail. Similarly, we also have \( w(z_-) = S(z_-) + S(-z_-) = \int \Pi(\partial \psi_{-}, \partial \psi_{-}) d\mu_2 \) for the corresponding parametrization \((\theta_{-}, \psi_{-})\). From Corollary 5.4 we have \( \Pi(\partial \psi_{+}, \partial \psi_{+}) \leq C \Pi(\partial \psi_{-}, \partial \psi_{-}) \) which implies \( w_{\max} \leq C w_{\min} \).

(ii) Consider the intersection of the hypersurface and its largest enclosed sphere. Then the intersection has at most \((n+2)\) elements. For the detailed proof, see Lemma 5.4 in [A4].

\[ \text{Lemma 4.3.} \quad \text{Let} \ M \quad \text{be convex and let} \ \delta_0 > 0 \quad \text{be a fixed constant. Set} \ t_0 = T - \delta_0 \quad \text{and} \ \rho_0 = \frac{1}{2} r_{\min}(t_0). \quad \text{Suppose that the pinching condition holds at} \ t = 0 \quad \text{with the condition (3.6) on} \ c \quad \text{as in Theorem 3.5 and} \ \epsilon \leq \frac{\pi}{3}. \quad \text{Then there is a constant} \ C_F = C_F(t_0) > 0 \quad \text{such that} \]

\[ \sup_{x \in M, \ 0 \leq t \leq t_0} F(x, t) \leq C_F = \max \left( \sup_{x \in M} F(x, 0), \sup_{x \in M} \frac{C}{\rho_0} \right). \]

**Proof.** Reparametrize the hypersurface to be defined on \( S^n \). Let \( F_S = \frac{1}{S - \rho_0} \) where \( \rho_0 = \frac{1}{2} r_{\min}(t_0) \) on \( S^n \times [0, t_0] \), and suppose that \( F_S \) takes its maximum at \((z_1, t_1)\) on \( S^n \times [0, t_0] \). Assume \( t_1 > 0 \) and unless stated otherwise, the computation below is carried out at \((z_1, t_1)\).

\[ 0 = \nabla_i F_S = \frac{\nabla_i F}{S - \rho_0} - \frac{F \nabla_i S}{(S - \rho_0)^2}. \]

\[ 0 \leq \frac{\partial F_S}{\partial t} = \frac{F_t}{S - \rho_0} - \frac{F S_t}{(S - \rho_0)^2}. \]

By Lemma 4.1 this implies that

\[ 0 \leq (S - \rho_0)[H g^{i j} \nabla_i \nabla_j F + F(H |A|^2 - tr A^3) - R^{1/2}(D \psi \cdot \nu)] + FR. \]

From (4.4), one easily gets

\[ 0 \geq \nabla_i \nabla_j F_S = \frac{\nabla_i \nabla_j F}{S - \rho_0} - \frac{F \nabla_i \nabla_j S}{(S - \rho_0)^2}, \]

and using (4.2), one then has

\[ 0 \geq (S - \rho_0)[H g^{i j} \nabla_i \nabla_j F + \nabla_j A^2] + \rho_0 FH |A|^2 - H^2 F. \]

Combining (4.5) and (4.6).

\[ H^2 F + FR \geq (S - \rho_0)[F \cdot tr A^3 + R^{1/2}(D \psi \cdot \nu)] + \rho_0 FH |A|^2. \]

On the other hand, one also has \( H^2 F + FR \leq 2 H^2 F \), and for \( \epsilon \leq \frac{\pi}{3}, \)

\[ (S - \rho_0)[F \cdot tr A^3 + R^{1/2}(D \psi \cdot \nu)] + \rho_0 FH |A|^2 \geq \frac{\rho_0}{n} FH^3, \]

since one has \( ce^2 \psi \geq 5n^2 |D \psi|^4 \) from (3.6) and

\[ F \cdot tr A^3 + R^{1/2}(D \psi \cdot \nu) \geq \psi c e h^3_{mn} - n^3 |D \psi| h^3_{mn}. \]
from the pinching condition. Thus for $\varepsilon \leq \frac{\varepsilon_0}{n}$, (4.7) yields

\begin{equation}
H(z_1, t_1) \leq \frac{2n}{\rho_0}.
\end{equation}

If $F$ takes its maximum at $(z_2, t_1)$, then take $t_0 = t_1$. In any case, one has

\[ F(z_2, t_1) \leq \frac{S(z_2, t_1) - \rho_0}{S(z_1, t_1) - \rho_0} F(z_1, t_1). \]

Also, from Lemma 4.2

\[ \frac{S(z_2, t_1) - \rho_0}{S(z_1, t_1) - \rho_0} \leq \frac{2r_{out}(t_1)}{r_{in}(t_1)} \leq C. \]

By the definition of $F$ and from (4.8),

\[ F(z_1, t_1) \leq \psi(z_1) H(z_1, t_1) \leq \frac{2n}{\rho_0} \psi_1. \]

Thus we have

\[ F(z_2, t_1) \leq Cr_{in}(t_1)^{-1}. \]

If $F$ takes its maximum at $(z_2, t_2)$ where $t_2 < t_1$, then reduce $t_0$ so that $F_S$ takes its maximum at $t_1 = t_2$, and follow the argument above. \[ \square \]

**Lemma 4.4.** Let $M_0$ be convex, and suppose that the initial hypersurface is pinched as in Theorem 3.2. Then there exists a positive constant $C = C(n, \varepsilon, \psi, C_F)$ such that

\[ H \leq Cr_{out}(t_0)^{-1} \]

**Proof.** From (4.3), we have

\[ h_{nn}h_{11} \leq R \leq C_1^2 \psi_0^{-2}, \]

where $h_{nn}$ and $h_{11}$ are the largest and the smallest principal curvatures, respectively, and the pinching estimate (3.14) implies

\[ H^2 \leq n^2 h_{nn}^2 \leq n^2 \varepsilon^{-1} h_{nn} h_{11} \leq n^2 C_1^2 \varepsilon^{-1} \psi_0^{-2}. \]

Then the result follows from Lemma 4.3. \[ \square \]

To obtain the short time existence of the flow (1.2), using a radial function $r(z, t)$, $z \in S^n$, we parametrize the hypersurface by

\[ X(x, t) = r(z, t)z, \]

for $x = \pi^{-1}(z) \in M$, where $\pi : M^n \rightarrow S^n$ is the normalizing map.

**Lemma 4.5.** If the initial hypersurface is pinched as in Theorem 3.2, then one has the short time existence for the flow (1.1).
Proof. One can compute that
\[ g^{ij} = r^{-2}(\delta^{ij} - \frac{\nabla_i \nabla_j r}{r^2 + |\nabla r|^2}), \]
\[ v = \frac{1}{\sqrt{r^2 + |\nabla r|^2}}(rz - \delta^{ij} \nabla_i z \cdot \nabla_j r), \]
\[ h_{ij} = \frac{1}{\sqrt{r^2 + |\nabla r|^2}}(-r \nabla_i \nabla_j r + 2 \nabla_i r \nabla_j r + r^2 g^{ij}), \]
where the detail can be found in Chapter 3 in \[Z\]. Then
\[ \frac{\partial r}{\partial t} = -\frac{F}{r} \sqrt{r^2 + |\nabla r|^2}, \]
and since \( F = \psi((g^{ij} h_{ij})^2 - g^{ik} g^{jl} h_{ij} h_{kl})^{1/2} \), we have
\[ \frac{\partial r}{\partial t} = -\frac{\psi}{r^3} \left[ (g^{ij} - \frac{\nabla_i \nabla_j r}{r^2 + |\nabla r|^2})(-r \nabla_i \nabla_j r + 2 \nabla_i r \nabla_j r + r^2 \delta^{ij}) \right]^2 \]
\[ - \frac{\psi}{r^3} \left[ (g^{ij} - \frac{\nabla_i \nabla_j r}{r^2 + |\nabla r|^2})(-r \nabla_i \nabla_j r + 2 \nabla_i r \nabla_j r + r^2 \delta^{ij}) \right] \]
\[ \times \left[ (g^{ij} - \frac{\nabla_i \nabla_j r}{r^2 + |\nabla r|^2})(-r \nabla_i \nabla_j r + 2 \nabla_i r \nabla_j r + r^2 \delta^{ij}) \right]^{1/2} \]
For \( M^n = S^n \), we have the following evolution equation of the radial function:
\[ -\frac{\sqrt{n^2 - 1}}{r} \psi \leq \frac{d}{dt} \frac{r}{\sqrt{n^2 - 1}} \psi \leq \frac{\sqrt{n^2 - 1}}{r} \psi \]
so that
\[ (4.10) \quad \left[ r(0)^2 - C^{-}(T - t) \right]^{1/2} \leq r \leq \left[ r(0)^2 - C^{+}(T - t) \right]^{1/2}, \]
where \( C^{-} := 2(n^2 - 1)^{1/2} \psi_1 \) and \( C^{+} := 2(n^2 - 1)^{1/2} \psi_0 \). Regarding the hypersurface \( X(t, \tau) \) as a graph in (4.9), the short time existence follows from the standard parabolic theory.

With the parabolic rescaling (1.6), one also has the following typical regularity of the rescaled flow as such is given for the mean curvature flow in Lemma 7.2 of [A4] and in Lemma 3.4 of \[Z\]:

Lemma 4.6.
If the initial hypersurface is pinched as in Theorem 3.2, we have
(i) \( \tilde{C}^{-1} \leq \tilde{r}_{in}(\tau) \leq \tilde{r}_{out}(\tau) \leq \tilde{C} \) for all \( \tau > 0 \),
(ii) \( \sup_{(x, \tau) \in \Sigma \times [0, \infty)} \tilde{H}(x, \tau) \leq \tilde{C}, \) and
(iii) \( \sup_{(x, \tau) \in \Sigma \times [0, \infty)} \left| \nabla^n \tilde{A} \right|^2 \leq \tilde{C} \) for \( m = 0, 1, 2, \cdots \).
Proof. (i) For $t' \in (t, T)$, the hypersurface $X(\cdot, t')$ is enclosed by $\partial B_{r(t')}(p)$ by the maximum principle for some $p \in \mathbb{R}^{n+1}$ where $r_+(t') \leq \left( \frac{r_{\text{out}}^2(t) - C^+(t' - t)}{2} \right)^{1/2}$ from (4.10). Thus we have $r_{\text{out}}(t) \geq (C^+(t' - t))^{1/2}$ and, letting $t'$ tend to $T$, we have $r_{\text{out}}(t) \geq (C^+(T - t))^{1/2}$. With the rescaling, from Lemma 4.2

$$\frac{C^+}{2} \leq \tilde{r}_{\text{out}}(\tau)^2 \leq C\tilde{r}_{\text{in}}(\tau)^2$$

On the other hand, suppose that $X(\cdot, t)$ encloses the ball $B_{r(t)}(p)$ for some $p \in \mathbb{R}^{n+1}$ where

$$r_{\text{in}}(t') \geq r_{\text{out}}(t') \geq \left( \frac{r_{\text{in}}^2(t) - C^-(t' - t)}{2} \right)^{1/2}.$$ 

Recall that $X(\cdot, t)$ shrinks to a point as $t \to T$, we have $r_{\text{in}}^2(t) \leq C^-(T - t)$ and hence $\tilde{r}_{\text{in}}^2(\tau) \leq \frac{1}{2}C^-$. (ii) From Lemma 4.3 and Lemma 4.4, one has $H \leq Cr_{\text{in}}(t_0)^{-1} \leq Cr_{\text{out}}(t_0)^{-1}$. Also the parametrization in (1.6) gives

$$r_{\text{out}}(t_0)^{-1} = \frac{\tilde{r}_{\text{out}}(t_0)^{-1}}{\sqrt{2(T - t_0)}} \leq \frac{C}{\sqrt{2(T - t_0)}}.$$ 

Thus we have

$$\tilde{H}(x, \tau) = \sqrt{2(T - \tau)}H(x, t) \leq \sqrt{2(T - t)} \cdot \frac{C}{\sqrt{2(T - t_0)}} \leq C.$$ 

(iii) Since $\tilde{h}_{11}$ and $\tilde{h}_{\text{in}}$ have uniform lower and upper bound by positive constants, (1.7) is uniformly parabolic (See [T] for the detail). Then, by Schauder estimates (see Theorem 4.9 in [Lib], for example), one has the higher derivatives of the curvature uniformly bounded.

□

5. Global boundedness

In this section we show that the curvature quantity defined below is uniformly bounded with the pinching estimate given. It shall be seen in Section 6 that the limit manifold under the parabolic rescaling is a round sphere. Suppose that $h_{ij} \geq e(H + c)g_{ij}$, for some $e > 0$ and $c > 0$, holds initially. Then from Theorem 3.2 it remains so until the collapsing. This implies that $eH^2 \leq R \leq H^2$ which will be used repeatedly throughout this section. Let $f_\alpha = \frac{eH^{2-\alpha}}{H^2}$ and $f = f_0$. Then, with
the straightforward computation
\[
\frac{H^{1+\sigma}}{2R} |\nabla f| + \frac{2R}{\psi H^2} \nabla \psi |_{\Omega_{G-Hh}}^2
\]
\[
= \frac{H^{1+\sigma}}{2R} |\nabla f_o|_{\Omega_{G-Hh}}^2 - \sigma \frac{f_o}{H^\sigma R} \langle \nabla H, \nabla f_o \rangle_{\Omega_{G-Hh}} + \sigma^2 \frac{f_o^2}{2 \sigma H^{1+\sigma} R} |\nabla H|^2_{|\Omega_{G-Hh}}^2
\]
\[
+ \frac{2}{\psi H^2} \langle \nabla f, \nabla \psi \rangle_{\Omega_{G-Hh}} + \frac{2R}{\psi^2 H^3} |\nabla \psi |^2_{\Omega_{G-Hh}},
\]
the time derivative of \( f_o \) follows from Lemma 2:
\[
(5.1) \quad \frac{\partial f_o}{\partial t} = \psi R^{-1/2} \left[ \langle f_o - \frac{2R}{H^3 - \sigma} H\nabla h |_{\Omega_{G-Hh}} - h_t \nabla H |_{\Omega_{G-Hh}}^2 - \sigma \frac{H^3 f_o}{4R} |\nabla f| + \frac{2R}{\psi H^2} \nabla \psi |_{\Omega_{G-Hh}}^2
\]
\[
n - \sigma \frac{f_o}{H^\sigma R} \langle \nabla H, \nabla f_o \rangle_{\Omega_{G-Hh}} + \sigma^2 \frac{f_o^2}{2 \sigma H^{1+\sigma} R} |\nabla H|^2_{\Omega_{G-Hh}}^2
\]
\[
+ \frac{H^{1-\sigma}}{2R} |\nabla f_o|_{\Omega_{G-Hh}}^2 - \sigma \frac{f_o}{H^\sigma R} \langle \nabla H, \nabla f_o \rangle_{\Omega_{G-Hh}} + \alpha^2 \frac{\hat{\sigma}^2}{2 \sigma H^{1+\sigma} R} |\nabla H|^2_{\Omega_{G-Hh}}^2
\]
\[
+ \sigma f_o \langle H | A^2 - trA^2 \rangle + \frac{2}{R^{1/2} H} \langle \nabla f_o - \frac{f_o}{H} \nabla H, \nabla \psi \rangle_{\Omega_{G-Hh}} + \frac{R^{1/2} f_o}{\psi H} |\nabla \psi |^2_{\Omega_{G-Hh}}^2
\]
\[
+ \frac{R^{1/2}}{H^{3-\sigma}} \left[ (\sigma - 2) |\nabla |^2 - \frac{H^2}{n} \nabla \psi + 2 \Delta \psi | \right],
\]
where \( \Delta h = h^\Omega \nabla |\nabla \psi |. \) Choose \( \sigma \) sufficiently small so that
\[
\sigma (\sigma - 1) \frac{f_o}{H^\sigma R} |\nabla H|^2_{\Omega_{G-Hh}} + \sigma^2 \frac{2R^{1/2} f_o}{2 \sigma H^{1+\sigma} R} |\nabla H|^2_{|\Omega_{G-Hh}} | \leq 0,
\]
which holds if \( \sigma \leq 2/3 \). Prior to computing the time derivative of \( L_p \) norm of \( f_o \), pointwise bounds for some of the terms in (5.1) can be easily obtained:
\[
(5.2) \quad \frac{2R^{1/2}}{H} \langle \nabla f_o - \frac{f_o}{H} \nabla H, \nabla \psi \rangle_{\Omega_{G-Hh}} \leq 2e^{-1/2} (|\nabla f_o| + \sigma H^{\sigma - 1} |\nabla H|) |\nabla \psi |,
\]
\[
\frac{2R^{1/2}}{H^{3-\sigma}} \left[ 2 \langle \nabla H, \nabla \psi \rangle_{\Omega_{G-Hh}} + \sigma f_o H^{\sigma - 3} \langle \nabla H, \nabla \psi \rangle \right] \leq \sigma H^{\sigma - 1} |\nabla H| |\nabla \psi |,
\]
\[
\sigma \frac{R^{1/2} f_o}{\psi H} |\nabla \psi |^2 \leq \sigma H^{\sigma - 1} |\nabla \psi |^2.
\]
These are bounded by
\[
10 e^{-1/2} (|\nabla f_o| + H^{\sigma - 1} |\nabla H|) |\nabla \psi | + \sigma H^{\sigma - 1} |\nabla \psi |^2.
\]

**Lemma 5.1.** Suppose that the flow in (1.1) converges to the limit point \( x_0 \) and such a point is a strict local minimum of a \( C^2 \) function \( \psi > 0 \). Then one has
\[
(5.3) \quad \left[ (\sigma - 2) |\nabla |^2 - \frac{H^2}{n} \nabla \psi + 2 \Delta \psi \right] \leq 0,
\]
in a small neighborhood of \( x_0 \).
Proof. This follows from the relation between the Laplacian of $\psi$ in $M$ and that in $\mathbb{R}^{n+1}$:

$$\Delta_M \psi = \Delta_{\mathbb{R}^{n+1}} \psi - D^2 \psi(\nu, \nu) + H \nu \cdot D \psi,$$

where $D$ is the gradient in $\mathbb{R}^{n+1}$ (see Appendix A in [E2] for the geometry of hypersurfaces). The sum of the first two terms in the right hand side of (5.4) is positive near the strict local minimum $x_0$. Since $\psi$ is locally increasing away from $x_0$ and $M_t$ are strictly convex hypersurfaces, $\nu \cdot D \psi$ is also positive near $x_0$. Moreover, as $M_t$ approaches $x_0$, one has $H \geq 1$ so that the first two terms in (5.3) dominate compared with the last. $\square$

Using $f_\sigma \leq H^\sigma$, the terms involving the gradient of $\psi$ in (5.1) are bounded above by

$$C(\epsilon, \sigma, \psi)(|\nabla f_\sigma| + H^{\alpha-1}|\nabla H| + H^\sigma),$$

where $C(\epsilon, \sigma, \psi) := 10e^{-1/2}|\nabla \psi| + \frac{1}{\epsilon^2}$ for $\epsilon < 1$. As computed in Lemma 2.3 (ii) in [Hu1], one can show that

$$|H\nabla [Mf]| - \nabla, H \cdot h_{kl}|^2 \geq \frac{1}{2}h_{22}^2|\nabla H|^2 \geq \frac{1}{2}e^2(H + \sigma)^2|\nabla H|^2,$$

where the second inequality is obtained by choosing an orthonormal frame with the first element being $\nabla V/H|\nabla H|$. From (5.1), (5.2), (5.5) and (5.6), one then has

$$\frac{\partial f_\sigma}{\partial t} \leq \psi R^{-1/2} \left\{ \square f_\sigma - \epsilon^2 \frac{R}{2H^3 - \sigma} \nabla H|^2 + \frac{2(1 - \sigma)}{H} \langle \nabla H, \nabla f_\sigma \rangle_{H^g - h} 
+ \frac{H^{1-\sigma}}{2R} |\nabla f_\sigma|_{A^g_{H^g - h}}^2 - \sigma \frac{f_\sigma}{H^{\alpha}} \langle \nabla H, \nabla f_\sigma \rangle_{A^g_{H^g - h}} + \sigma f_\sigma (H|A|^2 - trA^2) \right\}
+ C(\epsilon, \sigma, \psi)(|\nabla f_\sigma| + H^{\alpha-1}|\nabla H| + H^\sigma).$$

In order to prove that there exists a positive constant $C$ such that

$$||f_\sigma||_p \leq C,$$

for some large $p$, we generalize the argument in Lemma 5.5 in [Hu1]: it is sufficient to show that near the finite maximal time $T$, say in the time interval $[T - \delta, T)$ for a fixed $\delta$ depending on $\epsilon$ and $\psi$,

$$\frac{\partial}{\partial t} \int_M f_\sigma^p d\mu \leq 0.$$

For any $p$, we have

$$\frac{\partial}{\partial t} \int_M f_\sigma^p d\mu \leq \int_M pf_\sigma^{p-1} \frac{\partial f_\sigma}{\partial t} d\mu,$$

for

$$p \int_M \psi R^{-1/2} f_\sigma^p \square f_\sigma d\mu = \int_M \psi R^{-1/2} \square f_\sigma^p d\mu - p(p - 1) \int_M \psi R^{-1/2} f_\sigma^{p-2} |\nabla f_\sigma|^2_{H^g - h} d\mu.$$
Integrating by parts,
\[ \int_M \psi R^{-1/2} \Box f_\alpha^p d\mu = \frac{1}{2} \int_M \psi R^{-3/2} \left( \nabla (H^2 \frac{R}{H^2}) , \nabla f_\alpha^p \right)_{Hg-h} - \int_M R^{-1/2} \left( \nabla \psi , \nabla f_\alpha^p \right)_{Hg-h}, \]
and from the definition \( f_\alpha \),
\[ H^2 (\nabla \frac{R}{H^2} , \nabla f_\alpha^p ) d\mu = -pH^{2-\sigma} f_\alpha^{p-1} |\nabla f_\alpha|^2 d\mu + \sigma p H^{1-\sigma} f_\alpha^p (\nabla H , \nabla f_\alpha) d\mu. \]
Hence
\[ \int_M \psi R^{-1/2} \Box f_\alpha^p d\mu = \]
\[ -p \int_M \psi R^{-3/2} H^{2-\sigma} f_\alpha^{p-1} |\nabla f_\alpha|^2_{Hg-h} + \sigma p \int_M \psi R^{-3/2} H^{1-\sigma} f_\alpha^p (\nabla H , \nabla f_\alpha)_{Hg-h} \]
\[ + p \int_M \psi R^{-1/2} H^{-1} f_\alpha^{p-1} (\nabla H , \nabla f_\alpha)_{Hg-h} - p \int_M R^{-1/2} f_\alpha^{p-1} (\nabla \psi , \nabla f_\alpha)_{Hg-h}. \]
Then, multiplying the factor \( p f_\alpha^{p-1} \) in (5.7) and integrating over \( M \), together with (5.8), (5.9) and the estimate \( |R^{-1/2} f_\alpha^{p-1} (\nabla \psi , \nabla f_\alpha)_{Hg-h} | \leq e^{-1/2} f_\alpha^{p-1} |\nabla f_\alpha| |\nabla \psi| \), it follows that
\[ \frac{\partial}{\partial t} \int_M f_\alpha^p d\mu \leq \int_M \psi R^{-1/2} \left( \frac{R}{H^2} f_\alpha^{p-1} |\nabla f_\alpha|^2_{Hg-h} - \frac{p}{2} R^{-1} H^{2-\sigma} f_\alpha^{p-1} |\nabla f_\alpha|^2_{Hg-h} \right) \]
\[ + \sigma P \int_M \psi R^{-3/2} H^{1-\sigma} f_\alpha^p (\nabla H , \nabla f_\alpha)_{Hg-h} + p H^{1-\sigma} f_\alpha^{p-1} (\nabla H , \nabla f_\alpha)_{Hg-h} \]
\[ - \frac{p \varepsilon^2}{2} f_\alpha^{p-1} H^{p-3} |\nabla H|^2 + \frac{p \varepsilon^2}{2} f_\alpha^{p-1} 2(1-\sigma) |\nabla f_\alpha|_{\Box g-Hh} \]
\[ + \sigma f_\alpha^{p-1} |(H A^2 - tr A^3)| d\mu \]
\[ + pC(\epsilon, \sigma, \psi) \int_M f_\alpha^{p-1} |\nabla f_\alpha| + H^{1-\sigma} |\nabla H| + H^p d\mu. \]
For the four terms involving \( (\nabla H , \nabla f_\alpha)_{Hg-h} \), noting that \( R \geq c H^2, f_\alpha \leq H^\sigma \) and \( |A|^2 g - Hh \leq H(Hg-h) \), there exists a positive constant \( C(\epsilon, \sigma) \) such that those terms are bounded by
\[ pC(\epsilon, \sigma) \int_M \psi \frac{f_\alpha^{p-1}}{H} |\nabla H| |\nabla f_\alpha| d\mu, \]
where \( C(\epsilon, \sigma) \) can be taken to be \( e^{-1/2}(3 - 2\sigma + \frac{3\sigma}{2}) \) so that if \( \sigma \leq O(\epsilon) \), then \( C(\epsilon, \sigma) = O(e^{-1/2}). \) Similarly the three terms involving \( |\nabla f_\alpha|^2_{Hg-h} \) in (5.10) are bounded by
\[ -p(p - 1) \int_M \psi R^{-1/2} f_\alpha^{p-2} |\nabla f_\alpha|^2_{Hg-h} d\mu. \]
Thus, from (5.10), (5.11) and (5.12),

\[(5.13)\]

\[\frac{\partial}{\partial t} \int_M f_0^p \, d\mu \leq -p(p - 1) \int_M \psi R^{-1/2} f_0^{p-2} |\nabla f_0|^2 \, d\mu + \sigma p \int_M \psi R^{-1/2} H^2 f_0^p \, d\mu + pC(\epsilon, \sigma) \int_M \psi \frac{f_0^{p-1}}{H} |\nabla f_0| \, d\mu - \frac{p^2}{2} \int_M \psi f_0^{p-1} R^{1/2} H^{p-3} |\nabla H|^2 \, d\mu + pC(\epsilon, \sigma, \psi) \int_M f_0^{p-1} (|\nabla f_0| + H^{\alpha-1} |\nabla H| + H^{\alpha}) \, d\mu.\]

By Young’s inequality,

\[(5.14)\]

\[pC(\epsilon, \sigma) \int_M \psi \frac{f_0^{p-1}}{H} |\nabla f_0| \, d\mu \leq \frac{e p(p - 1)}{2} \int_M \psi f_0^{p-2} |\nabla f_0|^2 \, d\mu + \frac{C(\epsilon, \sigma)^2}{2} \int_M \psi H^{2} f_0^p \, d\mu.\]

Note that the pinching estimate in (3.14) implies that \(H_{g_{ij}} - h_{ij} \geq (n - 1)\epsilon(H + c)g_{ij} \geq \epsilon H g_{ij}\) and \(\epsilon H^2 \leq R \leq H^2\). Combining (5.13) and (5.14), we see that

\[\frac{\partial}{\partial t} \int_M f_0^p \, d\mu \leq \left(\frac{e p(p - 1)}{2} - ep(p - 1)\right) \int_M \psi f_0^{p-2} |\nabla f_0|^2 \, d\mu + \frac{p(\frac{C(\epsilon, \sigma)^2}{p - 1} - e^{7/2})}{2} \int_M \psi f_0^{p-1} H^{p-2} |\nabla H|^2 \, d\mu + \frac{\sigma p}{e^{3/2}} \int_M \psi H^2 f_0^p \, d\mu + pC(\epsilon, \sigma, \psi) \int_M f_0^{p-1} (|\nabla f_0| + H^{\alpha-1} |\nabla H| + H^{\alpha}) \, d\mu.\]

Choose \(p \geq p_0\), where \(p_0 := 2C(\epsilon, \sigma)^2 e^{-7/2} + 1 = O(e^{-9/2})\) for \(\sigma \leq \epsilon\) so that

\[(5.15)\]

\[\frac{\partial}{\partial t} \int_M f_0^p \, d\mu \leq -\frac{e}{2} p(p - 1) \int_M \psi f_0^{p-2} |\nabla f_0|^2 \, d\mu - \frac{e^{5/2}}{4} p \int_M \psi f_0^{p-1} H^{p-2} |\nabla H|^2 \, d\mu + \frac{\sigma p}{e^{3/2}} \int_M \psi H^2 f_0^p \, d\mu + pC(\epsilon, \sigma, \psi) \int_M f_0^{p-1} (|\nabla f_0| + H^{\alpha-1} |\nabla H| + H^{\alpha}) \, d\mu.\]

We shall show that the last integral in (5.15) can be controlled by other integrals in the right hand side of (5.15). Note that from the pinching estimate in Theorem 3.2, one has a uniform lower bound for \(H\)

\[H \geq H_0 := c\epsilon.\]

A simple computation, using Young’s inequality, yields

\[pf_0^{p-1} H^\alpha \leq (p - 1)H_0^\alpha H^2 f_0^p + H_0^{-\sigma_1},\]

where \(\sigma_1 = \frac{2\alpha(\alpha-1)}{p-1} < 0\) for \(p \geq 2\) and \(\sigma < \frac{1}{2}\), since

\[H_0^{\alpha-1} f_0^p \leq \frac{p - 1}{p} H_0^\alpha H^2 f_0^p + \frac{1}{p} H_0^{-\sigma_1}.\]
With this bound, the last integral in (5.15) can be estimated: it suffices to consider the case in which
\[ \int_M f_0^p d\mu > O(\frac{1}{p}). \]
Let \( \psi_0 := \inf_{\mathbb{R}^n+1} \psi. \) Then with \( p \) sufficiently large, one has
\[ p \int_M H^p f_0^{p-1} d\mu \leq \frac{p-1}{\psi_0} H_0^{\alpha} \int_M \psi H^p f_0^\beta d\mu + \int_M H_0^{-\alpha} d\mu \]
\[ \leq \frac{2p}{\psi_0} H_0^{\alpha} \int_M \psi H^2 f_0^\beta d\mu, \]
and similarly,
\[ p \int_M f_0^{p-1} H^{\alpha-1} |\nabla H| d\mu \leq \frac{\beta p^2}{\psi_0} \int_M \psi H^{\alpha-2} f_0^{p-1} |\nabla H|^2 d\mu + \frac{1}{\beta \psi_0} \int_M \psi H f_0^{\beta-1} d\mu, \]
\[ \leq \frac{\beta p^2}{\psi_0} \int_M \psi H^{\alpha-2} f_0^{p-1} |\nabla H|^2 d\mu + \frac{2H_0^{\alpha}}{\beta \psi_0} \int_M \psi H^2 f_0^\beta d\mu, \]
where \( \beta \) is a positive constant depending on \( H_0 \) and \( p \). These estimates together with (5.15) give
\[ \frac{\partial}{\partial t} \int_M f_0^p d\mu \leq + \left[ \frac{1}{\psi_0} C(\epsilon, \sigma, \psi) \beta p^2 - \frac{\epsilon}{2} (p^2 - 1) \right] \int_M \psi f_0^{p-2} |\nabla f_0|^2 d\mu \]
\[ + \left[ \frac{1}{\psi_0} C(\epsilon, \sigma, \psi) \beta p^2 - \frac{p}{4} \epsilon^{5/2} \right] \int_M \psi H^{\alpha-2} f_0^{p-1} |\nabla H|^2 d\mu \]
\[ + \left[ \frac{2}{\psi_0} C(\epsilon, \sigma, \psi) (pH_0^{\alpha} + 2H_0^{\alpha} \beta^{-1}) + \frac{2p}{\epsilon^{3/2}} \right] \int_M \psi H^2 f_0^\beta d\mu. \]
Choose \( \delta \), which depends on \( \psi \), small enough that in the time interval \( [T - \delta, T] \) where the volume of \( M \) is sufficiently small, we have \( |\nabla \psi| \leq \delta \) for some small \( \delta \). We choose \( \beta = \frac{C(H_0)}{\rho} \) where \( C(H_0) \) is a constant depending only on \( H_0 \), and for convenience, let \( C(H_0) = 1 \). Take \( \delta \leq \sigma \) and \( \sigma < O(\epsilon^3) \) so that (5.16) yields that in \( [T - \delta, T] \),
\[ \frac{\partial}{\partial t} \int_M f_0^p d\mu \leq - \frac{\epsilon}{4} (p-1) \int_M \psi f_0^{p-2} |\nabla f_0|^2 d\mu - \frac{\epsilon^{5/2}}{8} p \int_M \psi H^{p-2} f_0^{p-1} |\nabla H|^2 d\mu \]
\[ + \frac{2p}{\epsilon^{3/2}} \int_M \psi H^2 f_0^\beta d\mu. \]
To eliminate the last integral above, we apply the following Michael-Simon Sobolev type inequality as given in Lemma 5.4 in [Hu1].
Lemma 5.2. If $H > 0$ and $h_{ij} \geq \varepsilon (H + c) g_{ij}$ for some $\varepsilon > 0$ and $c > 0$ initially, then we have

$$
\int_M \psi f_o^p H^2 d\mu \leq \frac{(2\gamma p + 5)\psi_1}{n \epsilon^2 \psi_0} \int_M \psi f_o^{p-1} |\nabla H|^2 d\mu + \frac{p - 1}{n \epsilon^2 \gamma \psi_0} \int_M \psi f_o^{p-2} |\nabla f_o|^2 d\mu.
$$

for $p \geq 2$, any $\gamma > 0$ and any $0 \leq \sigma \leq 1/2$, where $\psi_1 := \sup_{R^{n+1}} \psi$.

From Lemma $5.2$ and $5.17$, we have, in $[T - \tilde{\delta}, T)$,

$$
\frac{\partial}{\partial t} \int_M f_o^p d\mu \leq \left( \frac{2\psi_0}{n \epsilon^2 \psi_0} (p - 1) - \frac{\epsilon}{4} p(p - 1) \right) \int_M \psi f_o^{p-2} |\nabla f_o|^2 d\mu + \frac{2\psi_0}{n \epsilon^2 \psi_0} \left( 2\gamma p + 5 - \frac{\epsilon^3}{8} \right) \int_M \psi f_o^{p-1} |\nabla H|^2 d\mu.
$$

By choosing $\gamma = \frac{8\psi_0}{n \epsilon^2 \psi_0}$ and $\sigma < O(\epsilon^3)$, one concludes that

$$
\frac{\partial}{\partial t} \int_M f_o^p d\mu \leq 0, \quad \text{for} \quad p_0 \leq p \leq p_o := \frac{n \epsilon^7 \psi_0}{16 \sigma \psi_1} \left( \frac{\epsilon^5}{16 \sigma^2} - 5 \right),
$$

in $[T - \tilde{\delta}, T)$. Therefore one can conclude that if $\psi$ has a local minimum at the limit point $x_0$, there exists a positive constant $C$ such that in $[T - \tilde{\delta}, T)$,

$$
\|f_o\|_p \leq C \quad \text{for} \quad p_0 \leq p \leq p_o.
$$

Let $\eta$ be a smooth test function which will be given explicitly later. The computation similar to (5.9) yields

$$
\int_M \eta^2 \psi R^{-1/2} \Box f_o^p d\mu =
\frac{p}{2} \int_M \eta^2 \psi R^{-3/2} H^{2-\sigma} f_o^p |\nabla f_o|^2 d\mu + \sigma \frac{p}{2} \int_M \eta^2 \psi R^{-3/2} H^{-1-\sigma} f_o^p (\nabla H, \nabla f_o)_{H^{8-h}}
$$

$$
+ p \int_M \eta^2 \psi R^{-1/2} H^{-1} f_o^{p-1} (\nabla H, \nabla f_o)_{H^{8-h}} - p \int_M R^{-1/2} f_o^{p-1} (\nabla (\eta^2 \psi), \nabla f_o)_{H^{8-h}}.
$$

Note that only the last term above involves the derivative of $\eta$. The lower bound $R \geq \epsilon H^2$ implies

$$
p R^{-1/2} f_o^{p-1} (\nabla (\eta^2 \psi), \nabla f_o) \leq \epsilon^{-1/2} \eta^2 |\nabla \psi| |\nabla f_o| + 2 \epsilon^{-1/2} \eta |\nabla \eta| |\nabla f_o|
$$

Multiply (5.7) by $p \eta^2 f_o^{p-1}$ and integrate by parts. Then from (5.10), using the pinching estimate and Young’s inequality, one sees that in $[T - \tilde{\delta}, T)$,

$$
\frac{\partial}{\partial t} \int_M \eta^2 f_o^p d\mu - 2 \int_M f_o^p \eta \frac{\partial \eta}{\partial t} d\mu
$$

$$
\leq - \frac{\epsilon}{4} p(p - 1) \int_M \eta^2 \psi f_o^{p-2} |\nabla f_o|^2 d\mu - \frac{\epsilon^5}{8} p \int_M \eta^2 \psi H^{2-\sigma} f_o^{p-1} |\nabla H|^2 d\mu
$$

$$
+ \frac{2 \epsilon^2 p}{\epsilon^3} \int_M \eta^2 \psi H^2 f_o^p d\mu + 2 \epsilon^{-1/2} \int_M |\nabla \eta||\nabla f_o| d\mu - \epsilon^2 \int_M \eta^2 \psi H^2 f_o^p d\mu.
$$
The integral come from the derivative of the measure \(d\mu\) and that \(R \geq \epsilon H^2\). Note that \(p(p - 1)f_{0}^{p-2}|\nabla f_{0}|^{2} \geq 2|\nabla f_{0}|^{2}\). Let \(\epsilon_i, i = 1, 2, 3,\) be any positive numbers. The following computation
\[
\eta^{2}|\nabla f_{0}|^{2} = |\nabla(\eta f_{0})|^{2} + f_{0}^{p}|\nabla\eta|^{2} - 2f_{0}^{p} \langle \nabla(\eta f_{0}), \nabla\eta \rangle,
\]
\[
2f_{0}^{p} \langle \nabla(\eta f_{0}), \nabla\eta \rangle \leq \epsilon |\nabla(\eta f_{0})|^{2} + \epsilon^{-1}f_{0}^{p}|\nabla\eta|^{2},
\]
yields
\[
-\frac{\epsilon}{4} p(p - 1) \int_{M} \eta^{2} \psi f_{0}^{p-2}|\nabla f_{0}|^{2} d\mu
\leq -\frac{\epsilon(1 - \epsilon_1)}{2} \int_{M} \psi|\nabla(\eta f_{0})|^{2} d\mu - \frac{\epsilon(1 - \epsilon_1)}{2} \int_{M} \eta^{2}|\nabla\eta|^{2} d\mu,
\]
and one also has
\[
\eta \psi|\nabla\eta||\nabla f_{0}^{p} \leq 2\epsilon_2 \psi|\nabla(\eta f_{0})|^{2} + 2\epsilon_2^{-1}\psi f_{0}^{p}|\nabla\eta|^{2}.
\]
Taking \(\epsilon_1 = \frac{1}{2}\) and \(\epsilon_2 = \frac{\epsilon_1^{3/2}}{2}\) with sufficiently small \(\epsilon\), in \([T - \delta, T), (5.21)\) becomes
\[
(5.22) \quad \frac{\partial}{\partial t} \int_{M} \eta^{2} f_{0}^{p} d\mu - 2 \int_{M} f_{0}^{p} \eta \frac{\partial\eta}{\partial t} d\mu
\leq -\frac{\epsilon}{8} \int_{M} \psi|\nabla(\eta f_{0})|^{2} d\mu + \frac{200}{\epsilon^2} \int_{M} \psi|\nabla\eta|^{2} f_{0}^{p} d\mu
- \frac{\epsilon^3}{8} \int_{M} \eta^{2} \psi H^{p-2} f_{0}^{p-1} |\nabla H|^{2} d\mu + \frac{2\sigma p}{\epsilon^{1/2}} \int_{M} \eta^{2} \psi H^{2} f_{0}^{p} d\mu - \epsilon^{1/2} \int_{M} \eta^{2} \psi H^{2} f_{0}^{p} d\mu.
\]
With this parabolic equation, we run the Moser iteration which is also useful for extending mean curvature flow past singular time as shown in \[LS\] and \[XYZ\]. Rescale and translate time \(t\) in \([T - \delta, T)\) by \(\delta^{-1}(t - T + \delta)\) so that the rescaled time, also denoted by \(t\), is in \([0, 1)\). Let
\[
D = \cup_{0 \leq t \leq 1}(B(x_0, 1) \cap M_t), \quad \hat{D} = \cup_{\frac{1}{12} \leq t \leq 1}(B(x_0, \frac{1}{2}) \cap M_t),
\]
where \(x_0\) is the limit point of \(M_t\), and let
\[
r_{k} = \frac{1}{2} + \frac{1}{2^{k+1}}, \quad t_{k} = \frac{1}{12}(1 - \frac{1}{4^k}),
\]
\[
\rho_{k} = r_{k-1} - r_{k} = \frac{1}{2^{k+1}}.
\]
Consider the set
\[
D_{k} = \cup_{t \leq \delta} \cup_{t \leq \delta} (B(x_0, r_{k}) \cap M_t).
\]
Note that \(D_{0} = D\) and \(t_{k} - t_{k-1} = \rho_{k}^{2}\). For convenience, we write \(M\) for \(M_{t_{k}}\). Let \(\eta = \eta_{k}\) be the smooth test function defined on \(M \times [0, 1)\) by
\[
\eta_{k}(x, t) := \psi_{k}(|x - x_{0}|^{2})\phi_{k}(t),
\]
where

$$v_k(s) = \begin{cases} 1 & \text{for } s \leq r_k^s, \\ 0 & \text{for } s \geq r_k^s \end{cases}$$

and $v_k(s) \in [0, 1]$ with $|v_k'(s)| \leq c_n \rho_k^{-2}$ for $r_k^s \leq s \leq r_k^{s-1}$, and

$$\phi_k(t) = \begin{cases} 0 & \text{for } 0 \leq t \leq t_k-1, \\ 1 & \text{for } t_k \leq t \leq 1, \end{cases}$$

and $\phi_k(t) \in [0, 1]$ with $|\phi_k'(t)| \leq c_n \rho_k^{-2}$ for $t_k-1 \leq t \leq t_k$. From (5.22), in the time internal $[0, 1)$,

$$\frac{\delta \frac{d}{dt}}{e^2} \int_{M} \eta^2 f^p_d d\mu + \frac{\varepsilon}{8} \int_{M} \psi |\nabla (\eta f^p_d)|^2 d\mu + \varepsilon^2 \int_{M} \eta^2 |\psi H^2 f^p_d|^2 d\mu$$

$$\leq \frac{200}{e^2} \int_{M} \psi |\nabla \eta^2 f^p_d d\mu + 2\delta \int_{M} f^p_d \eta \frac{\partial \eta}{\partial t} d\mu + \frac{2c}{e^{1/2}} \int_{M} \eta^2 \psi H^2 f^p_d d\mu.$$  

For $u \in W^{1,1}(M)$ and $T_1 \leq T_2$, using the Sobolev inequality and the Schwarz inequality, one has

$$\int_{T_1}^{T_2} \left( \int_{M} u^{2+\delta} d\mu \right)^{\frac{2}{2+\delta}} dt \leq c(n) \left( \sup_{[T_1, T_2]} \int_{M} u^2 d\mu \right)^{\frac{2+\delta}{2}} \left( \int_{T_1}^{T_2} \int_{M} (|\nabla u|^2 + H^2 u^2) d\mu dt \right)^{\frac{\delta}{2}},$$

and using the interpolation inequality,

$$\int_{M} u^{2+\delta} d\mu \leq \left( \int_{M} u^2 d\mu \right)^{\frac{2+\delta}{2}} \left( \int_{M} u^{\frac{2+\delta}{2}} d\mu \right)^{\frac{2}{2+\delta}},$$

one has

$$\int_{T_1}^{T_2} \int_{M} u^{2+\delta} d\mu dt \leq c(n) \left( \sup_{[T_1, T_2]} \int_{M} u^2 d\mu \right)^{\frac{2+\delta}{2}} \left( \int_{T_1}^{T_2} \int_{M} (|\nabla u|^2 + H^2 u^2) d\mu dt \right)^{\frac{\delta}{2}}.$$  

Integrating (5.25) over $[0, 1)$, we have

$$\delta \sup_{t \in [0, 1)} \frac{d}{dt} \int_{M} \eta^2 f^p_d d\mu + \frac{\varepsilon_{0}}{8} \int_{0}^{1} \int_{M} \eta^2 |\nabla (\eta f^p_d)|^2 d\mu dt + \varepsilon \int_{0}^{1} \int_{M} \eta^2 H^2 f^p_d d\mu dt$$

$$\leq \frac{200}{e^2} \int_{0}^{1} \int_{M} |\nabla \eta^2 f^p_d d\mu | \frac{\partial \eta}{\partial t} d\mu dt$$

$$+ \frac{2c}{e^{1/2}} \int_{0}^{1} \int_{M} \eta^2 H^2 f^p_d d\mu dt.$$  

Assuming that $\varepsilon \psi_0 \leq 8$ and $\delta \leq 1$, this implies

$$\int_{\sup \eta} (|\nabla (\eta f^p_d)|^2 + \eta^2 H^2 f^p_d) d\mu dt$$

$$\leq \frac{1600}{e^3 \psi_0} \int_{\sup \eta} f^p_d (|\nabla \eta|^2 + 2|\eta \frac{\partial \eta}{\partial t} + \alpha \eta^2 H^2) d\mu dt,$$
and the same bound also holds for \( \tilde{\sigma} \sup_{[0,1]} \int_0^1 \eta^{\frac{2}{p}} f_0^p \, d\mu \). Note that \( f_0^p H^2 = f_0^p \) where \( \tilde{\sigma} = \sigma + \frac{2}{p} \). We choose \( \tilde{\sigma} \) small enough so that \( H \geq 1 \) in \([0,1)\). This can be achieved applying the pinching estimate in Corollary 5.4 so that \( f_0 \leq f_0 \) for \( \sigma \leq \tilde{\sigma} \).

Substituting \( u \) by \( p \) in (5.27), one obtains

\[
\int_{\text{supp} \eta} (\eta f_0^p) \frac{\partial \eta}{\partial t} \, d\mu dt \leq c(n,e) \left( \int_{\text{supp} \eta} (f_0^p (|\nabla \eta|^2 + \frac{\partial \eta}{\partial t}) + \sigma p \eta f_0^p \, d\mu dt) \frac{\partial \eta}{\partial t} \right) \frac{\partial \eta}{\partial t},
\]

where \( c(n,e) := c(n) \left( \frac{1600\psi h}{e \psi_0} \right) \). Typically, as in \([E1]\) and \([LS]\), one has

\[
|\nabla \eta|^2 + \frac{\partial \eta}{\partial t} \frac{\partial \eta}{\partial t} \leq c(n) p_k^{-2} = \tilde{c}(n) 4^k \leq \sigma \eta 4^k \eta D_k \leq 1, \quad \text{for } \tilde{c}(n) \text{ is a constant depending only on } n. \text{ Thus for } \tilde{\sigma} \geq \sigma, \text{ (5.30) yields}
\]

\[
\int_{\text{supp} \eta} (\eta f_0^p) \frac{\partial \eta}{\partial t} \, d\mu dt \leq \tilde{c}(n,e) \left( \int_{\text{supp} \eta} (4^k p f_0^p) \, d\mu dt \right) \frac{\partial \eta}{\partial t},
\]

where \( \tilde{c}(n,e) := c(n,e) (2\tilde{c}(n)) \). Let \( \lambda = \frac{n+1}{n}, \quad p = \lambda^{1-k} \quad \text{and} \quad \sigma_k = \sigma + 2 \lambda^{k+1}. \) This implies that

\[
\int_{D_k} \eta_k \lambda^k f_0^k \, d\mu dt \leq \tilde{c}(n,e) \left( \int_{D_k} \lambda^{k-1} \lambda_{k-1} \, d\mu dt \right) \lambda_k
\]

since \( \eta_k \equiv 1 \) on \( D_k \) and \( \text{supp} \eta_k \subset D_{k-1} \). That is,

\[
\|f_0\|_{L^2(D_k)} \leq (\tilde{c}(n,e)^{1-k} \lambda^{k-1} \lambda_k^{k+1}) \|f_0\|_{L^2(D_{k-1})}.
\]

Note \( \sum_{k=1}^n k \lambda^{-k} = O(1) \), and \( \sigma \) can be chosen sufficiently small so that

\[
\sigma - 2 \sum_{j=0}^n \lambda^{-k+1-j} = \sigma - 2(n+1) p^{-1} > 0,
\]

where \( p_k = \lambda^{k+1} \), since \( p_k = O(e^{17/2} \sigma^{-2}) \) for \( \sigma < O(e^5) \). Thus from (5.19), we have an iteration relation:

\[
\|f_0\|_{L^2(\hat{\Omega})} \leq c'(n,e) \|f_0\|_{L^2(D_{k_0-1})} \leq C,
\]

for some fixed \( k_0 = k_0(e, \sigma) \) and constants \( c'(n,e) \) and \( C \), where the last inequality follows from (5.19). Therefore,

\[
\sup_{|T_{\tilde{\sigma}'}|} \sup_{M \in \tilde{\beta}(k_0, k_0)} f_0 \leq C,
\]

where \( \tilde{\sigma}' = \frac{\tilde{\sigma}}{12} \). From Lemma 4.3 we conclude that

**Theorem 5.3.** If \( h_{ij} \geq c(H + c)g_{ij} \) for some \( c > 0 \) and \( c > 0 \) initially, then we have

\[
|A|^2 - \frac{H^2}{n} \leq CH^{2-\tilde{\sigma}},
\]

for some small \( \tilde{\sigma} \).
6. Proofs of main theorems

6.1. Proof of Theorem 1.1. From Lemma 4.2 and the containment principle, one can conclude that \( X(\cdot, t) \) converges to a point as \( t \) tends to \( T \) via the regularity theory of uniformly parabolic equations (see, for example, [KS] for the regularity theory).

□

6.2. Limit equation in dimension two. The monotone quantity is a useful tool to analyze the asymptotic behavior of geometric flows. For the mean curvature flow, a monotonicity formula using the backwards heat kernel gives a limit equation which leads to the classification of self-similar solutions (see [Hu2]). Here, we simply use the volume of a convex region with its boundary being \( M_t \) to normalize the hypersurface. In general, without the divergence structure for the speed \( F \) depending on the curvature, it is difficult to deduce a limit equation. However, in dimension two, this can be overcome since \( R = 2K \), where \( K \) is the Gauss curvature, and \( K \) has a quantity that is not quite monotone but enough to obtain the limit behaviour. For this reason, we consider \( X_t = -\psi (2K)^{1/2} \nu \) which coincides with the flow (1.2) in dimension two, and call this the anisotropic Gauss curvature flow (to be precise, (1/2)-Gauss curvature flow).

The (half) volume of a convex region with its boundary \( M_t \) can be written in an integral form using the support function \( S \):

\[
V(t) = \frac{1}{n + 1} \int_{S^n} S \frac{d\sigma_s}{2K},
\]

where \( d\sigma_s \) is the standard measure on \( S^n \). This is used to define a mixed volume of convex regions in [A5] where it is shown that given specific speeds of evolution, some dilation invariant integral quantities monotonically decrease.

Lemma 6.1. Under the flow (1.1), we have

\[
\frac{\partial}{\partial t} V(t) = -\int_{S^n} \frac{\psi}{(2K)^{1/2}} d\sigma_s.
\]

Proof. Denote \( \mathcal{K} = 1/K \). Using integration by parts and the fact that \( \nabla_j \mathcal{K} (h^{-1})^j = 0 \),

\[
\int_{S^n} S \mathcal{K}_i d\sigma_s = \int_{S^n} S \mathcal{K} (h^{-1})^j h_{ij} d\sigma_s = \int_{S^n} S \mathcal{K} (h^{-1})^j (\nabla_i \nabla_j S + S_{ij} \mathcal{S}^i) d\sigma_s = \int_{S^n} S \mathcal{K} (h^{-1})^j (\nabla_i \nabla_j S + S_{ij} \mathcal{S}) d\sigma_s = \int_{S^n} S \mathcal{K} (h^{-1})^j h_{ij} d\sigma_s = n \int_{S^n} S \mathcal{K} d\sigma_s.
\]

Then we have

\[
\frac{\partial}{\partial t} V(t) = \frac{1}{2(n + 1)} \int_{S^n} (\mathcal{K} S + S \mathcal{K}_i) d\sigma_s = -\int_{S^n} \frac{\psi}{(2K)^{1/2}} d\sigma_s.
\]
From this, one can write \( V(t) = V(0) - \int_0^t \eta(s) \, ds \) where \( \eta(t) := \int_{S_t} \frac{\psi}{(2K)^n} \, d\sigma_r \). In order to normalize the volume, rescale the hypersurface by
\[
\hat{X}(\tau) = \frac{X(t)}{V(t)^{1/(n+1)}} \quad \text{and} \quad \tau(t) = -\log \left( \frac{V(t)}{V(0)} \right).
\]
One can easily compute that
\[
(6.1) \quad \frac{d\hat{X}}{d\tau} = -\frac{\hat{\psi}(2\hat{K})^{1/2}}{\hat{\eta}} \hat{\psi} + \frac{1}{n+1} \hat{X}.
\]

**Lemma 6.2.** Let \( \hat{I}(\tau) = \left( \int_{S_t} \frac{\hat{\psi}^2}{\hat{S}} \, d\sigma_r \right)^{-1} \).

In dimension two, under the volume preserving rescaling, with the initial pinching condition satisfying (1.5) given, one has
\[
\frac{d}{d\tau} \hat{I}(\tau) \to 0,
\]
as \( \tau \to \infty \), and the limit profile satisfies \( \hat{S} = C \hat{\psi}(\hat{K})^{1/2} \) for some constant \( C > 0 \), where \( \hat{S} \) and \( \hat{K} \) are the support function and the scalar curvature of the rescaled limit manifold \( \hat{M} \), respectively.

**Proof.** From (6.1), one obtains
\[
\frac{\hat{\psi}^2}{\hat{S}^2} \left( \frac{d\hat{S}}{d\tau} - \frac{1}{n+1} \hat{S} \right) = -\frac{\hat{\psi}^3(2\hat{K})^{1/2}}{\hat{\eta}\hat{S}^2},
\]
which implies
\[
\frac{d}{d\tau} \hat{I}(\tau) = \hat{I}(\tau) \left[ \frac{1}{n+1} \int_{S_t} \frac{\hat{\psi}^2}{\hat{S}} \, d\sigma_r - \frac{1}{\hat{\eta}} \int_{S_t} \frac{\hat{\psi}^3(2\hat{K})^{1/2}}{\hat{S}^2} \, d\sigma_r - 2 \int \frac{\hat{\psi} \hat{\psi}'}{\hat{S}} \, d\tau \right].
\]
Using Hölder’s inequality and the definition of \( \hat{\eta} \) yield
\[
\left( \int_{S_t} \frac{\hat{\psi}^2}{\hat{S}} \, d\sigma_r \right) \left( \int_{S_t} \frac{\hat{\psi}}{(2\hat{K})^{1/2}} \, d\sigma_r \right)^{1/2} \leq \left( \int_{S_t} \frac{\hat{\psi}^3(2\hat{K})^{1/2}}{\hat{S}^2} \, d\sigma_r \right)^{1/2} \left( \int_{S_t} \frac{\hat{\psi}}{(2\hat{K})^{1/2}} \, d\sigma_r \right)
\]
\[
\leq \left( \int_{S_t} \frac{\hat{\psi}^3(2\hat{K})^{1/2}}{\hat{S}^2} \, d\sigma_r \right)^{1/2} \left( \int_{S_t} \frac{\hat{S}}{2\hat{K}} \, d\sigma_r \right)^{1/2} \left( \int_{S_t} \frac{\hat{\psi}^2}{\hat{S}} \, d\sigma_r \right)^{1/2}.
\]
The fact that \( \hat{\psi}(\tau) = \frac{1}{\hat{\eta}} \int_{S_t} \frac{\hat{S}}{2\hat{K}} \, d\sigma_r = 1 \) implies
\[
\left( \int_{S_t} \frac{\hat{\psi}^2}{\hat{S}} \, d\sigma_r \right) \left( \int_{S_t} \frac{\hat{\psi}}{(2\hat{K})^{1/2}} \, d\sigma_r \right) \leq (n+1) \int_{S_t} \frac{\hat{\psi}^3(2\hat{K})^{1/2}}{\hat{S}^2} \, d\sigma_r,
\]
where the equality holds if and only if \( \hat{S} = C \hat{\psi} \hat{R}^{1/2} \) for some constant \( C > 0 \), and therefore,
\[
\frac{d}{d\tau} \hat{I}(\tau) \leq -2 \hat{I}(\tau) \hat{\eta} \int_{\hat{S}} \frac{\hat{\psi}}{\hat{S}} \frac{d\sigma_{\hat{S}}}{d\tau}.
\]
Since \( D\hat{\psi} \to 0 \) as \( \tau \to \infty \) and, in dimension two, the pinching estimate controls \( \hat{K} \) and \( \hat{S} \), one has \( \frac{d\hat{\psi}}{d\tau} \to 0 \) as \( \tau \to \infty \). Also \( \hat{\nu}(\tau) = 1 \) implies that
\[
\int_{\hat{S}} \frac{\hat{\psi}}{\hat{S}} d\sigma_{\hat{S}} \leq C,
\]
for some positive constant \( C \), and hence,
\[
\lim_{\tau \to \infty} \frac{d}{d\tau} \hat{I}(\tau) = 0,
\]
so that the limit profile satisfies \( \hat{S}^* = C \hat{\psi} (\hat{R}^*)^{1/2} \) for some constant \( C > 0 \), where \( \hat{S}^* \) and \( \hat{R}^* \) are the support function and the scalar curvature of \( \hat{M}^* \), respectively.

6.3. **Proof of Theorem 1.3**

Parametrizing the rescaled hypersurface as a graph by
\[
\hat{X}(x, t) = \hat{r}(z, t)z,
\]
where \( x = \pi^{-1}(z) \), \( z \in S^n \) and \( \pi : M^n \to S^n \) is the normalizing map, Lemma 4.6 (i) and the convexity guarantee the uniform boundedness of the first derivative of \( \hat{r} \) in the rescaled version of (4.9). Then the regularity theory of uniformly parabolic equation provide the boundedness of the higher derivatives of \( \hat{r} \). Thus each time slice \( \hat{X}(\cdot, \tau_k) \) has a \( C^\infty \)-convergent subsequence to a smooth strictly convex limit hypersurface \( \hat{M}^* \). In dimension two, the limit hypersurface \( \hat{M}^* \) of the volume preserving anisotropic scalar curvature flow satisfies the equation \( \hat{S}^* = C \hat{\psi} (\hat{R}^*)^{1/2} \) for some \( C > 0 \) by Lemma 6.2.

Suppose that \( h_{ij} \geq c(H + c)g_{ij} \) initially. We follow the argument in Sect.7 in [C2]. Since there is a point \( p_0 \) in \( \hat{M}^* \) with \( \hat{K}(p_0) > 0 \), there is an open neighborhood \( \hat{U} \) containing \( p_0 \) with \( \hat{K} > 0 \) in \( \hat{U} \). However the unnormalized \( H \) blows up in the open neighborhood \( \hat{U} \) corresponding to \( \hat{U} \) and then from Theorem 5.3 and the scale invariance of \( f \), we have \( f = 0 \) in \( \hat{U} \) which implies that \( \hat{U} \) is totally umbilical. Thus \( \hat{K} \) is constant in \( \hat{M}^* \) so that \( \hat{M}^* \) is a round sphere \( S^n \).

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