**Abstract**

In this work we investigate the effect produced by the BCS coupling in spinless fermions in one spatial dimension. Using bosonization techniques our initial model is rewritten in terms of a sine-Gordon field and a free massless scalar field. As a result the Cooper pair in our scenario is made up of soliton and antisoliton particles. We calculate the single particle Green’s function, the pair correlation function and the optical conductivity associated with the physical fermions and we show how they differ from their conventional quasiparticle analogues. Finally, we compare our results with related experimental findings for high temperature superconductors and we display how they fit qualitatively well the related observed effects produced by the anti-nodal quasiparticles in those materials.

**Introduction**

The cuprate superconductors are distinct for having anomalous physical properties above \( T_c \) and a somewhat simpler d-wave superconducting phase at lower temperatures. In both the optimally doped and the underdoped metallic phases, for \( T > T_c \), quasiparticles are not well defined even in the vicinity of the Fermi surface. In contrast, for \( T < T_c \), in the superconducting phase, in spite of the fact that this phase is far from being a conventional BCS-like state, there are Cooper pairs and well-defined quasiparticle states along the nodal directions, the so-called nodal quasiparticles. In the antinodal momentum regions where the d-wave superconducting gap is non-zero [1], quasiparticle-like modes are also observed as soon as one dissociates the existing Cooper pairs. The observation of those low-lying single particle excitations puts into question why they are present in the superconducting state if they are not observed either in the pseudogap state or in the anomalous metal phase above \( T_c \). It is well known that the high \( T_c \) cuprates are planar 2D materials. However it is also true that several of their properties carry a distinguished quasi-one-dimensional (1D) signature [2]. One recent evidence of that is the intertwining of the charge density wave and superconducting order which is also present in a 1D electron gas. Besides, it is interesting by itself to find out how a BCS interaction term directly affects the single particle properties in the Tomonaga–Luttinger Hamiltonian for spinless fermions in 1D.

We formulate a minimal model to carry out such an investigation and we calculate the associated correlation functions which are suitable to describe the nature of the single particle states in the presence of this BCS gap. In order to begin with the discussion of this problem, let us consider spinless fermions in the presence of a BCS condensate which pairs those particles non-trivially in a 1D Luttinger liquid. The basic reference model for a non-Fermi liquid in 1D+1 is, of course, the Tomonaga–Luttinger liquid model [3]. The Hamiltonian density in the vicinities of the left(-) and right(+) Fermi points is given by (we simplify our notation by setting \( \nu = 1 \))

\[
\mathcal{H} = \nu \left[ \sum_{k_y \Delta < p < k_y + \Delta} p \: \psi^\dagger_p \psi_p(p) + \sum_{-k_y - \Delta < p < -k_y + \Delta} p \: \psi^\dagger_p \psi_p(p) \right] \\
+ \frac{1}{2} \sum_{p,p',\Delta} g_{\Delta \alpha}(k) \psi^\dagger_{\alpha}(p + k) \psi^\dagger_{\alpha}(p' - k) \psi^\dagger_{\alpha}(p) \psi_{\alpha}(p')
\]  

(1)

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where $v_F$ is the Fermi velocity, $\psi_\pm$ describes right and left moving fermions and $k$ is such that $|k| \lesssim \Lambda \ll k_F$, with $\Lambda$ being a momentum cutoff.

The $g_{\alpha,\alpha'}$ are forward-like interactions between particles of the same branch or between particles from different branches. Using the so-called 'g-ology' notation [8] we have in this way $g_{++}(k) = g_{--}(k) = g_4(k)$ and $g_{+-}(k) = g_6(k)$. For spinless fermions the main effect of $g_4$ is to increment the Fermi velocity. Since $g_4$ plays no role in a BCS-type mechanism we may as well assume it to be of small magnitude and neglect its effect altogether.

We can then, for simplicity, take $g_6 = g$, from now on. In this way, the Hamiltonian density $\mathcal{H}$ associated with equation (1) can be reduced to be of the form

$$\mathcal{H}(x) = v_F \left( -iv^\phi_+ (x) \partial_k \psi_+(x) + iv^\phi_- (x) \partial_k \psi_-(x) \right) + g(x) \psi^\dagger_+ (x) \psi^\dagger_- (x) \psi_-(x) \psi_+(x).$$

(2)

Using the Nambu two-component spinor field

$$\Psi = \begin{pmatrix} \psi_+(x) \\ \psi_-^\dagger (x) \end{pmatrix},$$

(3)

can be rewritten as

$$\mathcal{H} = -i \Psi^\dagger_+ \partial_k \sigma_3 \Psi_+ + g(x) \Psi^\dagger_+ \sigma_3 \Psi^\dagger_- \sigma_3 \Psi + \text{const.}$$

(4)

Taking into account the existence of a condensate of Cooper pairs, we assume that $\langle \psi_-, \psi_+ \rangle$, $\langle \psi^\dagger_- \psi^\dagger_+ \rangle = 0$ and we adopt the BCS ansatz for our effective Hamiltonian density

$$\mathcal{H} = v_F \left( -iv^\phi_+ (x) \partial_k \psi_+(x) + iv^\phi_- (x) \partial_k \psi_-(x) \right)
+ \Delta \left[ e^{i\frac{\theta}{\hbar} x} \psi^\dagger_+ (x) \psi^\dagger_- (x) + e^{-i\frac{\theta}{\hbar} x} \times \psi_-(x) \psi_+(x) \right] + \mathcal{H}_\theta,$$

(5)

where

$$\Delta e^{i\frac{\theta}{\hbar} x} = g \left\langle \psi_-(x) \psi_+(x) \right\rangle,$$

$$\Delta e^{-i\frac{\theta}{\hbar} x} = g \left\langle \psi^\dagger_- (x) \psi^\dagger_+ (x) \right\rangle,$$

(6)

with $v_\phi$ and $\kappa$ being the superfluid velocity and the compressibility of condensate, respectively. This BCS ansatz for our Hamiltonian density does not stop us from assuming that the superconductivity is produced by some electronic mechanism. In view of that, in this work, we consider that the Fermi velocity and the superfluid velocity are fine tuned to be of the same magnitude, i.e. $v_\phi = v_F$. There is some experimental evidence which indicates that these two velocities are equal in high $T_c$ superconductors [33]. We simplify our notation even further taking $v_F = 1$ from now on.

Here $\mathcal{H}_\theta$ is the Hamiltonian density which describes the fluctuations of the phase of the superconducting order parameter [2]. The field $\theta$ is the bosonic mode which glues the fermions and the condensate together. In this way, we have that

$$\mathcal{H}_\theta = \frac{1}{2} \left( \Pi_\theta + \left( \partial_t \theta \right)^2 \right),$$

(7)

with $\Pi_\theta = \partial_t \theta$ being the momentum field conjugate to $\theta$. The presence of this field $\theta$ in our Hamiltonian model guarantees that there is no symmetry breaking even when the fermions acquire a non-zero gap in their single particle spectrum. Using the notation

$$\Psi = \Psi^\dagger \gamma^0,$$

(8)

where $\gamma^0 = \sigma_0$, $\gamma^1 = -i \sigma_2$, $\gamma^5 = \gamma^0 \gamma_1 = \sigma_3$, with the $\sigma_i$s being simple Pauli matrices, the Lagrangian density in terms of the Nambu fields is

$$\mathcal{L} = \bar{\Psi} i \gamma^\mu \partial_\mu \Psi - \Delta \bar{\Psi} e^{i\frac{\theta}{\hbar} x} \gamma^0 \Psi + \frac{1}{2} \left( \partial_t \theta \right)^2 - \left( \partial_\theta \theta \right)^2.$$

(9)

Note that $\mathcal{L}$ is invariant under the global chiral transformation $\Psi \rightarrow e^{i\alpha(x)} \Psi$, and $\theta \rightarrow \theta + 2\alpha \sqrt{\kappa}$. On top of that it also has a $U(1)$ global symmetry $\Psi \rightarrow e^{-i\alpha} \Psi$.

**Bosonization scheme**

This Lagrangian model can be solved analytically, if we apply bosonization methods [4]. In view of that we introduce the boson fields $\phi_1(x)$ and $\phi_2(x)$ such that
Following standard procedure we introduce next the so-called dual fields \( \varphi \) and \( \vartheta \) such that

\[
\varphi(x) = \phi_+(x) + \phi_-(x), \\
\vartheta(x) = \phi_+(x) - \phi_-(x).
\]  

It is also common to use \( j(x) \) to define the conjugate momentum density of \( \vartheta(x) \). In this way we have that

\[
\partial_x \varphi = -\Pi_\vartheta = -\partial_t \vartheta.
\]

The bosonized Lagrangian density in terms of new fields becomes

\[
\mathcal{L} = \frac{1}{2} \left[ \left( \partial_t \vartheta(x) \right)^2 - \left( \partial_x \vartheta(x) \right)^2 \right] \\
- \frac{\Delta}{\pi} \cos \left[ \frac{\theta(x)}{\sqrt{4\pi}} \right] + \frac{1}{2} \left[ \left( \partial_x \theta \right)^2 - \left( \partial_x \vartheta \right)^2 \right].
\]  

The Lagrangian equation (13) can be simplified further by means of the Bogoliubov-like transformations

\[
\vartheta' = \frac{\vartheta + \frac{\theta}{\sqrt{4\pi}}}{\sqrt{1 + \frac{1}{4\pi\kappa}}},
\]

and

\[
\theta' = \frac{\theta - \frac{\theta}{\sqrt{4\pi}}}{\sqrt{1 + \frac{1}{4\pi\kappa}}}.
\]

Thus, the bosonized Lagrangian density becomes

\[
\mathcal{L} = \frac{1}{2} \left[ \left( \partial_x \theta' \right)^2 - \left( \partial_x \vartheta' \right)^2 \right] - \frac{\Delta}{\pi} \cos \left( \beta \theta' \right) + \frac{1}{2} \left[ \left( \partial_x \theta' \right)^2 - \left( \partial_x \vartheta' \right)^2 \right],
\]  

where

\[
\beta = \sqrt{4\pi + \frac{1}{\kappa}}.
\]

This represents a massive sine-Gordon theory for the \( \theta' \) field in the presence of a decoupled free massless \( \vartheta' \).

The sine-Gordon spectrum is known exactly, and since, clearly, \( \beta > \sqrt{4\pi} \), \( \theta' \) is directly associated with a massive fermionic solitonic particle [6]. Moreover, since \( \beta < \sqrt{8\pi} \) is a necessary condition for the sine-Gordon theory to be well defined in the ultraviolet limit [5], it follows that \( \sqrt{4\pi} < \beta < \sqrt{8\pi} \) in our case. Notice that there are no corresponding breather solutions for this range of \( \beta \) [7]. If we assume that the inverse lattice parameter imposes a natural UV cutoff the inequality \( \beta > \sqrt{8\pi} \) simply reduces to a condition which represents the limit for a gapless Tomonaga–Luttinger liquid model with \( \beta = \sqrt{8\pi} \) being simply a Berezinskii–Kosterlitz–Thouless phase transition point in which the fermions decouple from the bosonic ‘glue.’ In other words the presence of the BCS pairing in this model generates a massive fermionic soliton, together with a completely decoupled and free massless phase mode. This situation resembles a d-wave superconductor, with no spin degrees of freedom, in the absence of other scattering processes besides the Cooper channel. There are in that case two types of quasiparticle modes. The nodal quasiparticles are gapless and unpaired while the antinodal quasiparticles are massive and actively engaged in pairing. In order to find out what is the precise nature of our paired quasiparticles we should refermionize the sine-Gordon field \( \vartheta' \). To do that in a more straightforward manner we define \( \vartheta \) and \( \theta \) such that

\[
\theta' = \tilde{\vartheta}' \left( 1 + \frac{1}{4\pi\kappa} \right) \vartheta = \vartheta,
\]

and we introduce the corresponding new fermion field \( \tilde{\Psi} \) appropriately by means of the identity

\[
\tilde{\Psi} \gamma^\mu \Psi = -\frac{1}{\sqrt{\pi}} \epsilon^{\mu\nu} \partial_\nu \vartheta,
\]

where the new Nambu spinor field \( \tilde{\Psi} \) is such that

\[
\Psi = \begin{pmatrix} \psi_\perp(x, t) \\ \psi_\parallel(x, t) \end{pmatrix},
\]
It follows from this that the partially fermionized Lagrangian becomes
\[
\mathcal{L} = \bar{\psi} \left( i \gamma^\mu \partial_\mu + \gamma^0 \Delta - \Delta \right) \psi + \frac{1}{2} e^2 \left( \bar{\psi} \gamma^\mu \gamma^\alpha \psi \right) \bar{\psi} \gamma^\beta \psi + \frac{1}{2} \left[ \left( \partial_\mu \bar{\psi} \right) \gamma^\mu \psi \right] ^2 ,
\]
(21)
where
\[
g = \frac{\pi}{1 + 4\pi\kappa},
\]
(22)
summation over repeated indices in equation (21) is implied. It is well known that the coupling constant \( g \) is related to \( \beta \) in equation (16) by [5]
\[
g = \frac{\pi}{1 - \frac{4\pi}{\beta^2}}.
\]
(23)

The Lagrangian equation (21) is essentially an attractive SU(2) Thirring model for massive fermions together with a free boson field for the phase mode. This model again describes two different types of quasiparticles: the ones directly involved in pairing with a bare mass determined by BCS gap amplitude \( \Delta \), and massless free particles which are left out in bosonic form for simplicity. The attractive self-interaction for the massive quasiparticles also results from the pairing produced by the condensate. Notice that since \( \bar{\psi} \) is a free massless boson, it can be written as a sum of chiral and anti-chiral components and it is not a Goldstone mode. This explains why there is no symmetry breaking when the mass of the physical fermion field \( \bar{\psi} \) is generated [6]. To make this more transparent let us consider the chiral symmetries of both old and new fermion fields. Since our main interest is to calculate correlation functions and to analyze the physical properties of the novel massive fermion, which might mimic the antinodal quasiparticles, we will not fermionize the bosonic field and we will leave \( \bar{\psi} \) as it is. Here, as opposed to what happens in a conventional BCS superconductor in higher spatial dimension, there is no off-diagonal long range order and the anomalous Green’s function must vanish identically.

Consider then the anomalous propagator \( \langle \psi_+(x) \psi_+(y) \rangle \) and let us see how this nullification takes place. To calculate this propagator, we relate the original \( \psi \) fields to the new mutually independent \( \bar{\psi} \) and \( \bar{\theta} \) fields. The physical fermion field \( \bar{\psi} \) has zero chiral components and the \( \bar{\theta} \) field can be decomposed into right- and left-moving parts, i.e. \( \bar{\theta} = \bar{\theta}_+ + \bar{\theta}_- \). It then follows that
\[
\psi_+(x) = \exp \left( i \sqrt{4 g} \bar{\theta}_+ \right) \bar{\psi}_+(x),
\]
(24)
\[
\psi_-(x) = \exp \left( i \sqrt{4 g} \bar{\theta}_- \right) \bar{\psi}_-(x).
\]
(25)
As a result, we get
\[
\left\{ \psi_+(x) \psi_+(y) \right\} = \left\{ e^{i \sqrt{4 g} \bar{\theta}_+(x)} \bar{\psi}_+(x) e^{i \sqrt{4 g} \bar{\theta}_+(y)} \bar{\psi}_+(x) \right\} = 0,
\]
(26)
since, for free massless bosonic fields, \( e^{i \sqrt{4 g} \bar{\theta}_{(x)}} e^{i \sqrt{4 g} \bar{\theta}_{(y)}} = 0 \) in the Gaussian model [15].

**Pair correlation function and single particle green’s function**

However, to test the presence of an incipient superconducting ordering we must calculate its associated correlation functions. We therefore consider next the pair correlation function \( \langle \psi_+^\dagger(x) \psi_+^\dagger(x) \psi_+(0) \psi_+(0) \rangle \) which checks the existence of electronic pairs. In view of the Mermin–Wagner–Hohenberg theorem [30] it must vanish for large \( |x| \), with a power law behavior. In fact using the same bosonization scheme as before we find that
\[
\left\{ \psi_+^\dagger(x) \psi_+^\dagger(x) \psi_+(0) \psi_+(0) \right\} = \left\{ e^{-i \sqrt{4 g} \bar{\theta}(x)} e^{-i \sqrt{4 g} \bar{\theta}(0)} \right\} \left\{ \psi_+^\dagger(x) \psi_+^\dagger(x) \psi_+(0) \psi_+(0) \right\},
\]
(27)
where again \( \bar{\theta} = \bar{\theta}_+ + \bar{\theta}_- \). The second factor approaches a constant at large \( |x| \) due to the presence of the gap in the fermionic spectrum. Taking into consideration that the \( \bar{\theta}(x) \) is a free boson field, we have that
\[
\left\{ \bar{\theta}(x) \bar{\theta}(0) \right\} = -\frac{1}{2\pi} \ln |x|.
\]
(28)
Using this result we arrive at the expected power law behavior for the pair correlation function, i.e.
\[
\left\{ \psi_+^\dagger(x) \psi_+^\dagger(x) \psi_+(0) \psi_+(0) \right\} \sim |x|^{-2g},
\]
(29)
at large \( |x| \).

Let us consider next the single particle propagator \( \langle \psi_+(x) \psi_+(0) \rangle \) for the right-moving spinless fermions. Following the same bosonization procedure as before we get that
\[
\left\langle \psi_i(x) \psi^\dagger_j(0) \right\rangle = \left\langle e^{i\int_0^L \beta \partial_x} \psi_i(x) e^{-i\int_0^L \beta \partial_x} \right\rangle \left\langle \bar{\psi}_i(x) \bar{\psi}^\dagger_j(0) \right\rangle.
\] (30)

Here we have \( \left\langle e^{i\int_0^L \beta \partial_x} \psi_i(x) e^{-i\int_0^L \beta \partial_x} \right\rangle = |x|^{-\frac{\delta}{2}} \), while \( \left\langle \bar{\psi}_i(x) \bar{\psi}^\dagger_j(0) \right\rangle \) is the fermion propagator of a particle of mass \( \Delta \), and which therefore decays exponentially with the distance as \( \sim e^{-\Delta|x|} \). Thus, we find that
\[
\left\langle \psi_i(x) \bar{\psi}_j(0) \right\rangle \sim |x|^{-\frac{1}{2}} e^{-\Delta|x|}.
\] (31)

Such a power law behavior is a premise of a branch cut. However in the presence of a non-zero superconducting gap, the massive physical fermion contribution removes all non-analyticities of the fermionic propagator in momentum space and it does indeed resemble in this respect a conventional BCS quasiparticle. This can be inferred directly from the calculation of the momentum distribution function \( n_+(k) \), for the right-moving spinless fermions. We find that
\[
n_+(k) = \int_{-\infty}^{\infty} e^{-i(k-k_F)\gamma} \left\langle \psi_i(x) \psi^\dagger_j(0) \right\rangle \, dx
\]
\[
\propto |\Delta + i(k - k_F)|^{-\frac{1}{2}}
\]
\[
\times \cos \left[ \left( \frac{g}{\pi} - \frac{1}{2} \right) \arctan \left( \frac{k - k_F}{\Delta} \right) \right].
\] (32)

Indeed, for \( \Delta = 0 \) there is neither a discontinuity nor any nonanalyticity in \( n_+(k) \) at \( k = + k_F \).

**The full correlation function**

Now, we consider in detail the calculation of the full single particle Green’s function for the Lagrangian equation (16). First, we assume that the operators \( \psi_i(x, \tau) \) and \( \psi^\dagger_i(x, \tau) \) are local fields. This is a necessary ingredient in the standard form factor approach. The Lagrangian (12), as we discussed earlier, is the sum of two parts, the sine–Gordon model and the free boson term. Using bosonization methods we consider the fields \( \psi_i(x, \tau) \) and \( \psi^\dagger_i(x, \tau) \) in Euclidean space-time with, \( \tau = it \). Generalizing our fermion field decomposition in terms of \( \tilde{\theta}(x, \tau) \) and \( \tilde{\bar{\theta}}(x, \tau) \), we find
\[
G_E(x, \tau) = \sum_{\sigma = \pm} e^{\sigma \delta k x} G_{E}^{\sigma},
\] (33)
where
\[
G_{E}^{\sigma} = - \left\langle T \, e^{i\int \beta \partial_x} \psi_i(x, \tau) e^{-i\int \beta \partial_x} \psi^\dagger_j(0, 0) \right\rangle \left\langle T \bar{\psi}_j(x, \tau) \bar{\psi}^\dagger_i(0, 0) \right\rangle.
\]

Here \( T \) is the imaginary time ordering operator. Clearly, \( G_{E}^{\sigma = -} = 0 \) vanishes identically. Consider the case of the chiral component Green’s function \( G_{E}^{++} \). The on-shell chiral component free boson \( \tilde{\theta}_+ \) now gives
\[
\left\langle e^{i\int \beta \partial_x} \psi_i(x, \tau) e^{-i\int \beta \partial_x} \right\rangle = \left( \frac{1}{\tau - i\pi} \right)^{\frac{g}{2}}.
\] (35)

The derivation of the physical fermion \( \left\langle \tilde{\psi} \tilde{\bar{\psi}}^\dagger \right\rangle \) Green’s function benefits from the integrability of the sine–Gordon model and employs a form factor expansion [9, 10]. Let us consider form factors for the right-moving Fermi operator \( \psi_i \). For particles with relativistic dispersion it is useful to parametrize the spectrum in terms of the rapidities \( \{ \theta \} \) (not to be confused with the free boson \( \theta(x, \tau) \)) such that the energy \( E \) and the momentum \( P \) are given by
\[
E = \Delta \cosh(\theta), \quad P = \Delta \sinh(\theta).
\] (36)

Let us now turn to the construction of a basis of scattering states of solitons and antisolitons. A convenient formalism for this is obtained in terms of the so-called Zamolodchikov–Faddeev (ZF) algebra [18, 21]. The ZF algebra can be considered as an extension of the algebra of creation and annihilation operators for free fermions or bosons to the case of interacting solitonic particles with factorizable scattering processes [17]. We define the rapidity–dependent creation and annihilation operators by means of the following algebra,
\[
Z_{\alpha_i}(\theta_i) Z_{\beta_i}(\theta_i) = S^e_{\alpha_i \beta_i}(\theta_i - \theta_i) Z_{\beta_i} Z_{\alpha_i}(\theta_i),
\]
\[
Z^\dagger_{\beta_i}(\theta_i) Z^\dagger_{\alpha_i}(\theta_i) = Z^\dagger_{\alpha_i}(\theta_i) Z^\dagger_{\beta_i}(\theta_i) S^e_{\alpha_i \beta_i}(\theta_i - \theta_i),
\]
\[
Z^\dagger_{\beta_i}(\theta_i) Z_{\alpha_i}(\theta_i) = 2\pi \delta(\theta_i - \theta_i) \delta_{\alpha_i \beta_i} + Z^\dagger_{\alpha_i}(\theta_i) S^e_{\alpha_i \beta_i}(\theta_i - \theta_i) Z_{\alpha_i}(\theta_i).
\] (37)

Here \( S^e_{\alpha_i \beta_i} \) are the factorizable two-particle scattering matrices, which were derived originally in [23, 24] and the internal indexes \( \epsilon_i = \pm \) are used to distinguish solitons and antisolitons respectively. An appropriate Fock space
of states can be constructed using the ZF operators as follows. The vacuum is defined by

$$Z_{fi}(\theta) |0\rangle = 0,$$

(38)

where $|0\rangle$ is the ground state. Acting with a string of creation operators $Z_{fi}^\dagger (\theta)^s$ on the vacuum we obtain the multiparticle states

$$\left[ \theta_1 \ldots \theta_n \right]_{e_i - e_n} = Z_{ei}^\dagger (\theta_1) \ldots Z_{ni}^\dagger (\theta_n) |0\rangle. $$

(39)

In terms of the basis states (36) the resolution of the identity reads

$$id = |0\rangle \langle 0| + \sum_{n=1}^\infty \frac{1}{n!^2} \sum_{k_1, \ldots, k_n} \int_{-\infty}^{\infty} \frac{d\theta_1 \ldots d\theta_n}{(2\pi)^n} \left[ \theta_1 \ldots \theta_n \right]_{e_i - e_n} \langle \theta_1 | \ldots | \theta_n \rangle. $$

(40)

Inserting equation (40) between operators $(\tilde{\psi} \tilde{\psi})^s$ we obtain the following spectral representation

$$\langle \tilde{\psi}_\alpha(x, \tau) \tilde{\psi}_\alpha^\dagger (0, 0) \rangle = \sum_{n=0}^\infty \frac{1}{n!} \sum_{k_1, \ldots, k_n} \int_{-\infty}^{\infty} \frac{d\theta_1 \ldots d\theta_n}{(2\pi)^n} \times \exp \left( \sum_{j=1}^n P(\theta_j) i x - E(\theta_j) \tau \right) \times \left| \langle 0| \tilde{\psi}_\alpha(0,0) | \theta_1 \ldots \theta_n \rangle \right|_{e_i - e_n}^2,$$

(41)

where $f_j(\theta_1 \ldots \theta_n)_{e_i - e_n} \equiv \langle 0| \tilde{\psi}_\alpha(0,0) | \theta_1 \ldots \theta_n \rangle_{e_i - e_n}$ are the form factors of the operator $\tilde{\psi}_\alpha$. Techniques of integrability require this form factor to be a constant [17]. Combining this with Lorentz invariance gives the first nonvanishing form factor for the fermion operator in the series equation (41) for large $x$ and for $|x^2 + \tau^2| \Delta^2 \gg 1$ [22], the on-shell asymptotic value of that reduces to

$$\frac{Z_0}{2\pi} \sqrt{2\pi \Delta} \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \exp[-\Delta \tau \cosh(\theta) + i \Delta x \sinh(\theta)] \approx \frac{Z_0 e^{-\sqrt{\Delta^2 + x^2}}}{2\pi \sqrt{\tau - i x}}, $$

(42)

where $Z_0$ is determined by our choice of $\beta$. In the particular case in which $\beta \rightarrow \sqrt{8\pi}$, which corresponds to the SU(2) Thirring model, $Z_0 \approx 0.9218$ [11]. Combining this with the free boson contribution, we arrive at the final result for the right-moving on-shell two-point-function

$$G_E^+ \approx -\frac{Z_0}{2\pi} \left( \frac{1}{\sqrt{\tau - i x}} \right) \frac{x + \frac{1}{2}}{e^{-\Delta \sqrt{x^2 + \tau^2}}}.$$

(43)

A similar formula with $x \rightarrow -x$ holds for the left-moving fermions. The resulting Green’s functions display no single pole behavior and the quasiparticle is in fact a composite particle made of a soliton or an antisoliton together with their accompanying bosonic massless mode.

**Spectral function**

The spectral function is a very interesting quantity to be calculated because it can be compared directly with angle-resolved photoemission spectroscopy (ARPES) experiments. In Fourier space, the Euclidean Green’s function equation (43) is in a more suitable form for us to determine the retarded Green’s function. We find that

$$G_{E^+}(q, \omega) = -Z_0 \left( \frac{q + i\omega}{q - i\omega} \right)^{\frac{3}{2}} I_\tau,$$

(44)

where

$$I_\tau = \int_0^{\infty} f \left( \frac{q}{2} + \frac{1}{2}, \sigma \right) r^{(\frac{3}{2} - \sigma)} \exp \left[ \frac{-\Delta \tau}{\sqrt{\omega^2 + q^2}} \right] dr,$$

(45)

with $f(\nu, z)$ being the Anger function defined by $f(\nu, z) = \int_0^\pi \cos(\nu \theta - z \sin(\theta)) d\theta$ and $\omega$ the corresponding fermionic Matsubara frequency. We can obtain the solution for this integral in terms of the Gauss’s hypergeometric function. If we proceed with the analytic continuation $\omega \rightarrow \omega + i\delta$ we obtain the retarded Green’s function in the vicinity of the Fermi point $+k_F$, in the form,
\[
G_{RR}^{++}(q, \omega) = -Z_0 \left[ (\omega + q)^{\frac{3}{2}} - \frac{\Delta}{\sqrt{q^2 - \omega^2}} - \frac{i}{2} \right] \frac{\Delta}{\sqrt{q^2 - \omega^2}} \left[ \frac{-4 g^2 - 4 g \pi + 3\pi^2}{(2g + 3\pi)^2} \right].
\]

(46)

This result is valid for \(\text{Re}[g] < 3\pi/2\), which is true in our case, since \(0 < g < \pi\). The two hypergeometric functions on the right-hand side have zero imaginary parts for real arguments smaller than one, i.e. for \(\Delta^2 < q^2 - \omega^2\). We can use a standard result [19] for the analytic continuation of the Gauss’s hypergeometric function to determine the real and the imaginary parts of our 2F1. From equation (46) we then get the following result for the spectral function

\[
A \left( \pm k_F + q, \omega \right) = \frac{1}{\pi} Z_0 \left[ (\omega + q)^{\frac{3}{2}} - \frac{\Delta}{\sqrt{q^2 - \omega^2}} - \frac{i}{2} \right] \frac{\Delta}{\sqrt{q^2 - \omega^2}} \left[ \frac{-4 g^2 - 4 g \pi + 3\pi^2}{(2g + 3\pi)^2} \right].
\]

\[
\times 2F1 \left[ 1, \frac{5}{4} - \frac{g}{2\pi}, \frac{5}{4} + \frac{g}{2\pi}, \frac{q^2 - \omega^2}{\Delta^2} \right] + \left( -4 g^2 + 4g \pi + 3\pi^2 \right) \times 2F1 \left[ 1, \frac{7}{4} - \frac{g}{2\pi}, \frac{7}{4} + \frac{g}{2\pi}, \frac{q^2 - \omega^2}{\Delta^2} \right].
\]

(47)

With that in hand we can compare our result with the measured spectral function for the cuprate superconductors at low doping. Figure 1(a)–(c) display the experimental energy distribution curves (EDCs) for different momenta between the node and the antinode for three different underdoped Bi-2212 samples. Quasiparticles are visible all around the Fermi surface, and most notably in the antinodal region, for all doping ranges. These peaks are always quasiparticle-like [27] (the scattering rate is smaller than the binding energy). In figure 1(d) we plot our results for the spectral function in the vicinity of the Fermi point \(k_F\). The theoretical curves display the same general features as the experimental curves for underdoped Bi-2212. Notice that, in both theory and experiment, the quasiparticle peaks in Bi-2212 are even more pronounced when we consider the materials at low doping regime. The loss of coherence of the antinodal quasiparticles might induce a deviation from a simple d-wave but this can be interpreted as an artifact produced by that effect [20]. In marked contrast, we note here that the underdoped Bi-2212 samples show sharp quasiparticles over the entire Fermi surface.

**Optical conductivity**

In this section, we discuss the optical conductivity which is obtained directly from the current–current correlation function. The optical conductivity is a fundamental tool widely used to probe the dynamical properties of a given material. For the cuprates, time- and angle-resolved photoemission spectroscopy (TR-ARPES) studies demonstrate that, soon after they are excited with a 1.5 eV pump pulse, an excess number of low-energy electron excitations is accumulated in the antinodal region [29]. The density operator \(\rho\) and the current operator \(j\) in terms of the Nambu spinors are \(\rho \equiv \tilde{\Psi} \gamma^5 \Psi\) and \(j \equiv \tilde{\Psi} \gamma^5 \gamma^\mu \Psi\): respectively, where\(\ldots\): denotes the normal ordering of operators. In other words, in the Nambu representation the axial current \(\tilde{\Psi} \gamma^5 \gamma^\mu \Psi\) is the analog of the electromagnetic vector current [32]. For spinless fermions the electrical current operator taking into account only the contribution of the gapped sectors, reduces to

\[
j_e(x, \tau) \sim \frac{e}{\sqrt{\pi}} \partial \tilde{\vartheta},
\]

(48)

with \(\tau = it\) being once again the imaginary time. Besides, the real part of the optical conductivity is related to the imaginary part of the retarded current–current correlation function by [12]

\[4\text{ The field } \vartheta \text{ contributes to the intragap absorption, our main interest here. The massless field } \tilde{\vartheta} \text{ produces the Drude peak } x \vartheta(\omega) \text{ which is the zero frequency response associated with a perfect conductor which is obtained in any Tomonaga–Luttinger liquid model without superconductivity.} \]
We see from the bosonized expression equation (48) that the current operator couples only to the massive fermion degrees of freedom. Thus, leaving the massless $\theta$ aside the conductivity is essentially produced by the massive fermions in the gapped sector. We therefore consider only soliton–antisoliton contributions to the real part of the optical conductivity. At even lower temperatures, where no thermally activated solitons are available, the conductivity is dominated by the soliton–antisoliton pair production induced by the external electric field [28]. As a result, it suffices to consider only intermediate states with the same numbers of soliton and antisoliton particles in the current–current correlation function due to the neutrality of the current operator [9]. Taking these considerations into account, we arrive at the following spectral representation for the optical conductivity [14]

$$\text{Re } \sigma(x) = -\frac{\text{Im } \chi(\omega)}{\omega}, \quad \omega > 0,$$

where

$$\chi(\omega) = -e^2 \int_{\infty}^{\infty} e^{-\tau} d\tau \int_{\infty}^{\infty} dx \langle 0 | [j(x, \tau), j(0, 0)] | 0 \rangle,$$

at $\omega = 0$. We see from the bosonized expression equation (48) that the current operator couples only to the massive fermion degrees of freedom. Thus, leaving the massless $\theta$ aside the conductivity is essentially produced by the massive fermions in the gapped sector. We therefore consider only soliton–antisoliton contributions to the real part of the optical conductivity. At even lower temperatures, where no thermally activated solitons are available, the conductivity is dominated by the soliton–antisoliton pair production induced by the external electric field [28]. As a result, it suffices to consider only intermediate states with the same numbers of soliton and antisoliton particles in the current–current correlation function due to the neutrality of the current operator [9]. Taking these considerations into account, we arrive at the following spectral representation for the optical conductivity [14]
\[
\text{Re } \sigma(\omega) = 2\pi e^2 \omega^2 \sum_{n=0}^{\infty} \sum_{x=\pm} \int \frac{d\theta_1 \cdots d\theta_n}{(2\pi)^n n!} \left| f^i(\theta_1, \ldots, \theta_n) e^{-\omega t} \right|^2
\]
\[
\times \delta \left( \sum_k P\{\theta_k\} \right) \delta \left( \omega - \sum_k \mathcal{E}\{\theta_k\} \right) = \sigma_{ss} + \sigma_{dss} + \ldots
\]  
where
\[
f^i(\theta_1, \ldots, \theta_n) = \left\langle 0 \right| j(0,0) \left\{ \theta_1, \ldots, \theta_n \right\} \right| \omega = \sum_{k} \mathcal{E}\{\theta_k\} \right\rangle,
\]  
are the form factors of the electric current operator with total energy and total momentum
\[
E = \Delta \sum_{k=1}^{n} \cosh(\theta_k), \quad P = \Delta \sum_{k=1}^{n} \sinh(\theta_k),
\]  
as before. Here \(\sigma_{ss}(\omega)\) is the contribution of intermediate states which originate in a soliton–antisoliton sector; the four-particle contribution (two soliton–two antisoliton, \(\sigma_{dss}\)) is negligible at low frequencies \([16]\). In order to determine \(\sigma(\omega)\) we need the form factor of the current operator. There are several integral representations available for those form factors. We follow closely the results obtained for the sine-Gordon model determined in \([9]\). In this way the soliton–antisoliton form factor is given by \([9, 13]\)
\[
|f^i(\theta, \theta)|^2 = |f^i(\theta, \theta)|^2
\]
\[
= \left[ -\Delta \cosh \left( \frac{\theta}{2} \right) \right]^2 |\xi(\theta, \theta)|^2,
\]  
where
\[
\xi(\theta) = \sinh \left( \frac{\theta}{2} \right) \exp \left[ -\int_{0}^{\infty} dx \frac{\sin^2[(\theta + i\pi)x/\pi]}{xe^x \cosh(x) \sinh(2x)} \right].
\]  
We can use that to obtain the soliton–antisoliton contribution in our optical conductivity calculation. Inserting equations \((54)\) and \((55)\) into our equation \((51)\) we arrive at the following result for the \(\sigma_{ss}\)
\[
\text{Re } \sigma(\omega) = e^2 \sqrt{\omega^2 - 4\Delta^2} \Theta(\omega - 2\Delta) \exp \left( -\int_{0}^{\infty} dx \frac{1 - \cos \left( \frac{4\Delta x}{\pi} \right)}{xe^x \cosh x \sinh 2x} \right),
\]  
where
\[
\theta_0 = \text{csch}^{-1} \left( \frac{\omega}{2\Delta} \right).
\]  
with \(\Theta(x)\) being the Heaviside step function. In figure 2(a)-(b) we compare our results with the experiment. Figure 2(a) shows the data for underdoped Bi-2212 \([25]\). It is easy to see from figure 2(b) that one cannot have any absorption for frequencies below the gap so the optical conductivity is zero for \(\omega < 2\Delta\). That is, the optical absorption can only occur at the energy threshold to break a Cooper pair. Soon after this threshold, the real part of the optical conductivity has a peak around \(\approx 2.5\Delta\) and it decays smoothly at large frequencies. In conclusion the full real part of the optical conductivity \(\text{Re } \sigma(\omega)\) is essentially determined by the first term of equation \((51)\) in the frequency interval \([0, 4\Delta]\) \([14]\). The optical conductivity was calculated for different values of \(\beta\) \([16]\). At the Luther–Emery point, \(\beta \rightarrow \sqrt{4\pi}\). For this to happen, in our case, it requires that \(\kappa \rightarrow \infty\). In this limit, the solitons become noninteracting particles and the gapped sector should turn into a conventional band insulator.

**Discussions**

The hole doping temperature phase diagram \([1]\) of the high-\(T_c\) cuprates has been extensively studied in the last two decades. At zero doping, they display long range antiferromagnetic order. At a slightly larger hole concentration the doping breaks down the antiferromagnetic order and they undergo a transition to the so-called pseudogap phase. That phase is notorious for its truncated Fermi surface, for some quasi-one-dimensional features and for the absence of sharp quasiparticles states throughout the Brillouin zone. This scenario changes completely as one lowers the temperature and crosses the boundary towards the d-wave superconducting phase. The quasiparticles are again present and play an active role in pairing. As expected, the low-frequency conductivity increases strongly with doping in the Cu–O planes. Despite the obvious qualitative nature of our analysis and the one-dimensionality of our model, we showed that some of those features are portrayed in our optical conductivity and spectral function results, once the superconductor state sets in.
Nevertheless, since we take no account of spin effects in this work we do not expect our results to be a realistic description of the two-dimensional magnetically induced cuprate superconductors.

Conclusion

We formulate a model that describes a one-dimensional gas of spinless fermions in the presence of a BCS attraction term. Using the bosonization method, a tool widely used for 1D systems, the resulting bosonic model is basically a sine-Gordon theory for the massive fermions and a free theory for the massless boson field which is essentially the glue which binds those fermions together in the SG sector. To further explore the physical meaning of this effective field theory we refermionize the sine-Gordon boson field and we observe that the superconducting gap amplitude is turned into a mass for those physical fermions in spite of the absence of any spontaneous symmetry breaking. We calculate the anomalous propagator and we find that it vanishes identically confirming the non-violation of $U(1)$ gauge symmetry. To examine the role of the BCS gap in more detail we calculate both the fermionic single particle propagator and the corresponding pair correlation function. We show that they display a characteristic power law at large distances. Such a behavior indicates the absence of long-range order. However, such power law behavior of the single particle propagator is weakened by the presence of an exponential factor which carries an explicit dependence on the BCS gap amplitude. The presence of such a non-zero gap reflects itself in the dispersion of the fermionic particle and this produces a quasiparticle character in the 1-particle excitations in the superconducting state, despite the equally important branch cut behavior of the resulting fermionic propagator. For this reason the related spectral function has many features similar to those produced by quasiparticles. However we call attention to the fact that those massive quasiparticles are related either to solitons or to antisolitons and are quite different from the quasiparticles in a Fermi liquid state. The optical conductivity in our model is dominated by the two-particle form factor contribution in the low frequency region. At much larger frequencies, the asymptotic behavior of the optical conductivity can be determined by perturbative methods. While perturbation theory gives a good approximation only at extremely large frequencies, the
perturbative renormalization-group (RG) method works well for a large region of frequencies. Both the anomalous metal and the pseudogap phases of the cuprate superconductors are notable for their many unconventional physical properties and for the absence of well-defined quasiparticle modes for $T > T_c$. In the superconducting phase, the ARPES spectra from the antinodal regions resemble that of a quasi-one-dimensional superconductor. Many features of the ARPES spectra, especially those referring to the antinodal region of the Brillouin zone in BSCCO are unlike anything observed in a conventional metal [31]. In spite of that the quasiparticles are again observed and assumed to be well defined in the superconducting state.

In conclusion, superconductivity in the cuprates results from either hole or electron doping of a Mott insulator. Several earlier unconventional models explored this fact and were disencouraged by the vast experimental evidence in favor of the presence of quasiparticles modes below $T_c$. We call attention to the fact that although they are indeed present in the superconductor the antinodal quasiparticles may not be as ordinary as one might initially expect. Using a very simplified 1D BCS model we show that quasiparticle-like features are also produced in the resulting low lying excitation spectra in spite of the fact that those massive fermions are now soliton or antisoliton particles. Our preliminary results reinforce the assumption of the quasi-one dimensionality of the antinodal single particle states and the idea that the presence of quasiparticle features does not rule out the possibility that the superconducting phase of a Luttinger liquid could still be a reference state for the cuprate superconductors as well.

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