Protected edge modes without symmetry

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We discuss the question of when a gapped 2D electron system without any symmetry has a protected gapless edge mode. While it is well known that systems with a nonzero thermal Hall conductance, $K_H \neq 0$, support such modes, here we show that robust modes can also occur in systems with $K_H = 0$—if they have quasiparticles with fractional statistics. We show that some types of fractional statistics are compatible with a gapped edge, while others are fundamentally incompatible. More generally, we give a criterion for when an electron system with abelian statistics and $K_H = 0$ can support a gapped edge: we show that a gapped edge is possible if and only if there exists a subset of quasiparticle types $M$ such that (1) all the quasiparticles in $M$ have trivial mutual statistics, and (2) every quasiparticle that is not in $M$ has nontrivial mutual statistics with at least one quasiparticle in $M$. We derive this criterion using three different approaches: a microscopic analysis of the edge, a general argument based on braiding statistics, and finally a conformal field theory approach that uses constraints from modular invariance.

I. INTRODUCTION

In two dimensions, some quantum many-body systems with a bulk energy gap have the property that they have gapless edge modes which are extremely robust. These modes cannot be gapped out or localized by very general classes of interactions or disorder at the edge: they are “protected” by the structure of the bulk phase. Examples include quantum Hall states\cite{tknn} topological insulators\cite{zhang2005}\cite{hasan2010}\cite{qi2011} topological superconductors\cite{witten2011} and bosonic SPT phases\cite{read2000} among others.

Although all of the above examples have protected edge modes, it is possible to distinguish between different levels of edge protection. In some systems, the edge excitations are only robust as long as certain symmetries are preserved. For example, in 2D topological insulators, the edge modes are protected by time reversal and charge conservation symmetry. If either of these symmetries are broken (either explicitly or spontaneously), the edge can be completely gapped. In contrast, in other systems, the edge modes are robust to arbitrary local interactions, independent of any symmetries.

While much previous work has focused on symmetry-protected edges, here we will focus on the latter, stronger, form of robustness. The goal of this paper is to answer a simple conceptual question: when does a gapped 2D quantum many-body system without any symmetry have a protected gapless edge mode?

One case in which such protected edge modes are known to occur is if the system has a nonzero thermal Hall conductance\cite{tknn,hasan2010} at low temperatures, i.e. $K_H \neq 0$. This result is particularly intuitive for systems whose edge can be modeled as a collection of chiral Luttinger liquids. Indeed, in this case, $K_H = (n_L - n_R) \cdot \frac{\pi k_F^2 T}{3h}$, where $n_L, n_R$ are the number of left and right moving chiral edge modes. Hence the condition $K_H \neq 0$ is equivalent to $n_L \neq n_R$. It is then clear that $K_H \neq 0$ implies a protected edge: backscattering terms or other perturbations always gap out left and right moving modes in equal numbers, so if there is an imbalance between $n_L$ and $n_R$, the edge can never be fully gapped. Alternatively, we can understand this result by analogy to the electric Hall conductance, $\sigma_H$: just as systems with $\sigma_H \neq 0$ are guaranteed to have a gapless edge as long as charge conservation is not broken, systems with $K_H \neq 0$ are guaranteed to have a gapless edge as long as energy conservation is not broken.

On the other hand, if $K_H = 0$ then there isn’t an obvious obstruction to gapping the edge. Thus, one might guess that systems with $K_H = 0$ do not have protected edge modes. Indeed, this is known to be true for systems of non-interacting fermions\cite{levin2010}.

In this paper, we show that this picture breaks down in the presence of interactions: we find that systems with $K_H = 0$ can also have protected edge modes—if they support quasiparticle excitations with fractional statistics. The basic point is that some (but not all\cite{levin2010}) types of fractional statistics are fundamentally incompatible with a gapped edge. Thus, quasiparticle statistics provides another mechanism for edge protection which is qualitatively different from the more well-known mechanisms associated with electric or thermal Hall response.

Our main result is a criterion for when an electron system with abelian statistics and $K_H = 0$ can support a gapped edge (we discuss bosonic systems in the conclusion). We show that a gapped edge is possible if and only if there exists a collection of quasiparticle “types” $M$ satisfying two properties:

1. The particles in $M$ have trivial mutual statistics: $e^{i m m'} = 1$ for any $m, m' \in M$.

2. Any particle that is not in $M$ has nontrivial mutual statistics with respect to at least one particle in $M$: if $l \notin M$, then there exists $m \in M$ with $e^{i m l} \neq 1$.

Here, two quasiparticle excitations are said to be of same topological “type” if they differ by an integer number of electrons. In this language, a gapped system typically has only a finite set of distinct quasiparticle types, which
we will denote by \( L \); the set \( M \) should be regarded as a subset of \( L \). Following previous terminology,\(^{12}\) we will call any subset \( M \subseteq L \) that obeys the above two properties a “Lagrangian subgroup” of \( L \).

Our analysis further shows that every gapped edge can be associated with a corresponding Lagrangian subgroup \( M \subseteq L \). Physically, the set \( M \) describes the set of quasiparticles that can be “annihilated” at the edge, as explained in section IV. If \( L \) contains more than one Lagrangian subgroup, then the system supports more than one type of gapped edge. In this sense, different types of gapped edges are (at least partially) classified by Lagrangian subgroups \( M \subseteq L \).

We now briefly discuss the relationship with previous work on this topic. A systematic, microscopic analysis of gapped edges was presented in Ref.\(^{13}\) In that work, the authors constructed and analyzed gapped edges for a large class of exactly soluble bosonic lattice models with both abelian and non-abelian quasiparticle statistics. On the other hand, gapped edges were studied from a field theory perspective in Ref.\(^{14}\) In that paper, the authors investigated “topological boundary conditions” for abelian Chern-Simons theory. Both analyses showed that gapped boundaries (or boundary conditions) are classified by an algebraic structure similar to the Lagrangian subgroup \( M \subseteq L \) introduced above. However, neither result implied that this classification scheme is general and includes all abelian gapped edges: indeed, it is not obvious, a priori, that exactly soluble models or topological boundary conditions are capable of describing all types of gapped edges. One of the main contributions of this work is to fill in this hole and to show, in a concrete fashion, that every abelian gapped edge is associated with some Lagrangian subgroup \( M \subseteq L \). It is this generality that allows us to deduce the existence of protected edges in those cases when \( L \) has no Lagrangian subgroup \( M \), i.e. when the criterion is violated.

We will derive the above criterion using three different approaches: a microscopic edge analysis, a general argument based on quasiparticle braiding statistics, and finally an argument that uses constraints from modular invariance. These derivations are complementary to one another. The microscopic argument proves that the criterion is sufficient for having a gapped edge but does not prove that it is necessary (it only provides evidence to that effect). The other two arguments show that the criterion is necessary for having a gapped edge, but do not prove that it is sufficient. Also, while the microscopic edge analysis gives a concrete picture of the protected edge modes, the braiding statistics argument explains the physical meaning of the criterion, and the modular invariance derivation provides another perspective based on conformal field theory.

This paper is organized as follows: in section II we discuss some illustrative examples of the criterion; in section III we present a microscopic derivation; in section IV we describe the braiding statistics argument; in section V we explain the modular invariance approach. The appendix contains some of the more technical derivations.

II. TWO EXAMPLES

Before deriving the criterion, we first discuss a few examples that demonstrate its implications. A particularly illuminating example is the \( \nu = 2/3 \) fractional quantum Hall edge – that is, the particle-hole conjugate of the \( \nu = 1/3 \) Laughlin state. Let us consider a thought experiment in which the edge of the \( \nu = 2/3 \) state is proximity-coupled to a superconductor (Fig. 1a). Then charge conservation is broken at the edge, so the edge does not have any symmetries.\(^{12}\) At the same time, the thermal Hall conductance of this system vanishes since the edge has two modes that move in opposite directions. Thus, one might have guessed that the edge could be gapped by appropriate interactions. However, according to the above criterion, this gapping is not possible: the edge is protected. To see this, note that the \( \nu = 2/3 \) state supports 3 different quasiparticle types which we denote by \( L = \{0, e/3, 2e/3\} \). The mutual statistics of two quasiparticles \( le/3, me/3 \) is \( \theta_{lm} = -2\pi l \cdot m/3 \). Examining this formula, it is clear that \( L \) has no Lagrangian subgroup \( M \): the only set \( M \) that obeys condition (1) is \( M = \{0\} \) and this set clearly violates condition (2).

To underline the surprising nature of this result, it is illuminating to consider a second example: a \( \nu = 8/9 \) state constructed by taking the particle-hole conjugate of the \( \nu = 1/9 \) Laughlin state. (While this state has not been observed experimentally, we can still imagine it as a matter of principle). Again, let us consider a setup in which the edge is proximity-coupled to a superconductor, thereby breaking charge conservation symmetry (Fig. 1b). This system is superficially very similar to the previous one, with two edge modes moving in op-
posite directions, and a vanishing thermal Hall conductance \( K_H = 0 \). However, in this case the above criterion predicts that the edge is not protected. Indeed, the \( \nu = 8/9 \) state has 9 different quasiparticle types which we denote by \( L = \{0, e/9, 2e/9, \ldots, 8e/9\} \). The mutual statistics of two quasiparticles \( le/9, me/9 \) is given by \( \theta_{lm} = -2\pi l \cdot m/9 \). Examining this formula, we can see that the subset of quasiparticles \( M = \{0, 3e/9, 6e/9\} \) obeys both (1) and (2), i.e. it is a valid Lagrangian subgroup.

III. MICROSCOPIC ARGUMENT

A. Analysis of the examples

In this section, we analyze the \( \nu = 2/3 \) and \( \nu = 8/9 \) examples using a microscopic approach. We argue that, despite the superficial similarities between the two states, the \( \nu = 8/9 \) edge can be gapped while the \( \nu = 2/3 \) edge is protected. In section III B, we extend this analysis to general electron systems with abelian statistics. We follow an approach which is similar to that of Refs. [18–20] and also the recent paper, Ref. [21].

We begin with the \( \nu = 8/9 \) state. To construct a consistent edge theory for this state, consider a model in which there is a narrow strip of \( \nu = 1 \) separating the \( \nu = 8/9 \) droplet and the surrounding vacuum. The edge then contains two chiral modes – a forward propagating mode \( \phi_1 \) at the interface between the \( \nu = 1 \) strip and the vacuum, and a backward propagating mode \( \phi_2 \) at the interface between \( \nu = 8/9 \) and \( \nu = 1 \). The mode \( \phi_1 \) can be modeled as the usual \( \nu = 1 \) edge [12]

\[
L_1 = \frac{1}{4\pi} \left[ \partial_x \phi_1 \partial_t \phi_1 - v_1 (\partial_x \phi_1)^2 \right] \quad (1)
\]

Similarly, the mode \( \phi_2 \) can be modeled as the usual \( \nu = 1/9 \) edge, but with the opposite chirality:

\[
L_2 = \frac{1}{4\pi} \left[ -9 \cdot \partial_x \phi_2 \partial_t \phi_2 - v_2 (\partial_x \phi_2)^2 \right] \quad (2)
\]

Here, the two parameters \( v_1, v_2 \) encode the velocities of the two (counter-propagating) edge modes. We use a normalization convention where the electron creation operator corresponding to \( \phi_1 \) is \( \psi_1^\dagger = e^{i\phi_1} \), while the creation operator for \( \phi_2 \) is \( \psi_2^\dagger = e^{-9i\phi_2} \). Combining these two edge modes into one Lagrangian \( L = L_1 + L_2 \) gives

\[
L = \frac{1}{4\pi} \partial_x \phi_1 (K_{1j} \partial_t \phi_j - V_{1j} \partial_e \phi_j) \quad (3)
\]

where \( I = 1, 2 \) and

\[
K = \begin{pmatrix} 1 & 0 \\ 0 & -9 \end{pmatrix}, \quad V = \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix} \quad (4)
\]

In this notation, a general product of electron creation and annihilation operators corresponds to an expression of the form \( e^{i\Lambda^T K \phi} \) where \( \Lambda \) is an integer vector.

Given this setup, the question we would like to investigate is whether it is possible to gap out the above edge theory [3] by adding appropriate perturbations. For concreteness, we focus on perturbations of the form

\[
U(\Lambda) = U(x) \cos(\Lambda^T K \phi - \alpha(x)) \quad (5)
\]

where \( \Lambda = (\Lambda_1, \Lambda_2) \) is an integer vector. These terms give an amplitude for electrons to scatter from the forward propagating mode \( \phi_1 \) to the backward propagating mode \( \phi_2 \). Importantly, we do not require \( U(\Lambda) \) to conserve charge, since we are assuming charge conservation is broken by proximity coupling to a superconductor (Fig. 1b). However, we do require that \( U(\Lambda) \) conserve fermion parity.

We now consider the simplest scenario for gapping the edge: we imagine adding a single backscattering term \( U(\Lambda) \) to the edge theory [3]. In this case, there is a simple condition that determines whether \( U(\Lambda) \) can open up a gap: according to the null vector criterion of Ref. [22] \( U(\Lambda) \) can gap the edge if and only if \( \Lambda \) satisfies

\[
\Lambda^T K \Lambda = 0 \quad (6)
\]

The origin of this criterion is that it guarantees that we can make a linear change of variables \( \phi' = W \phi \) such that in the new variables, the edge theory [3] becomes a standard non-chiral Luttinger liquid, and \( U(\Lambda) \) becomes a backscattering term. It is then clear that the term \( U(\Lambda) \) can gap out the edge, at least if \( U \) is sufficiently large. Conversely, if \( \Lambda \) doesn’t satisfy [6], it is not hard to show that the corresponding term \( U(\Lambda) \) can never gap out the edge, even for large \( U \) (see appendix B).

Substituting the explicit expression for \( K \) [4] into [6] gives:

\[
\Lambda_1^2 - 9\Lambda_2^2 = 0 \quad (7)
\]

By inspection, we easily obtain the solution \( \Lambda = (3, -1) \). It follows that the corresponding scattering term \( U(\Lambda) \) can gap out the edge. We note that this term is not charge conserving, since it corresponds to a process in which one electron is annihilated on one edge mode and three are created on the other. However, it is still an allowed perturbation in the presence of the superconductor, since it conserves fermion parity (in fact, it is not hard to show that solutions to [6] always conserve fermion parity).

Now let us consider the \( \nu = 2/3 \) edge. Following a construction similar to the one outlined above, we model the edge by the theory [3] with

\[
K = \begin{pmatrix} 1 & 0 \\ 0 & -9 \end{pmatrix}, \quad V = \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix} \quad (4)
\]

As before, we ask whether backscattering terms of the form \( U(\Lambda) \) [5] can gap the edge, and as before, we can answer this question by checking whether \( \Lambda \) satisfies the null vector condition [6]. However, in this case, we can see that [6] reduces to

\[
\Lambda_1^2 - 3\Lambda_2^2 = 0 \quad (9)
\]
which has no integer solutions, since \( \sqrt{3} \) is irrational. We conclude that no single perturbation \( U(\Lambda) \) can open up a gap – suggesting that the edge is protected.

We emphasize that the above analysis only shows that the \( \nu = 2/3 \) edge is robust against a particular class of perturbations – namely single backscattering terms of the form \( \Theta \). Hence, the above derivation only gives evidence that \( \nu = 2/3 \) is protected; it does not prove it.

### B. General abelian states

We now extend the above analysis to general electron systems with abelian quasiparticle statistics and with \( K_H = 0 \). As above, we prove that the criterion is sufficient for having a gapped edge, and we give evidence (but do not definitively prove) that it is necessary.

Our analysis is based on the assumption that every abelian state can be described by a multicomponent \( U(1) \) Chern-Simons theory. More specifically, since our system is composed out of electrons and has vanishing thermal Hall conductance, we assume that it can be described by a \( 2N \) component \( U(1) \) Chern-Simons theory

\[
L_B = \frac{K_{ij}}{4\pi} \epsilon^{\lambda \mu \nu} a_i a_{\lambda} \partial_{\mu} a_{\lambda} a_{\nu}
\]

(10)

where \( K \) is a symmetric integer matrix with vanishing signature, non-vanishing determinant, and at least one odd element on the diagonal.

In this formalism, quasiparticle excitations are parameterized by integer vectors \( l \). Excitations composed out of electrons correspond to vectors \( l \) of the form \( l = K\Lambda \).

Two quasiparticle excitations \( l, l' \) are “equivalent” or “of the same type” if they differ by some number of electrons, i.e. \( l - l' = K\Lambda \) for some integer \( \Lambda \). Also, the mutual statistics of two excitations \( l, l' \) is given by

\[
\theta_{l,l'} = 2\pi i Tr K^{-1} l l'
\]

(11)

while the exchange statistics is \( \theta = \theta_{l,l}/2 \).

Let us translate the criterion from the introduction into the \( K \)-matrix language. We note that the set \( M \) corresponds to a collection of (inequivalent) integer vectors, \( M = \{ m \} \). Condition (1) translates to the requirement that \( m^T K^{-1} m' \) is an integer for any \( m, m' \in M \). Condition (2) translates to the requirement that if \( l \) is not equivalent to any element of \( M \), then there exists \( m \in M \) such that \( m^T K^{-1} l \) is not an integer. The criterion states that the edge can be gapped if and only if there exists a set \( M \) satisfying these two conditions.

To derive this result, we use the bulk-edge correspondence for abelian Chern-Simons theory to model the edge as a \( 2N \) component chiral boson theory. As above, we can gap out the edge by adding backscattering terms \( U(\Lambda) \). In order to gap out all \( 2N \) edge modes, we need \( N \) terms, \( \sum_{i=1}^N U(\Lambda_i) \), where the \( \{ \Lambda_i \} \) are all linearly independent. Similarly to Eq. [5] one can show that this perturbation can gap out the edge if and only if the \( \{ \Lambda_i \} \) satisfy

\[
\Lambda_i^T K \Lambda_j = 0
\]

(12)

for all \( i, j \). This result suggests that a system described by a \( 2N \times 2N \) \( K \)-matrix \( K \) can support a gapped edge if and only if one can find \( N \) linearly independent integer vectors \( \{ \Lambda_i \} \) satisfying [12].

To complete the argument, we need to show that [12] is equivalent to the criterion from the introduction. In other words, we need to show that [12] has a solution if and only if there exists a set of integer vectors \( M \) with the two properties described above. This (purely mathematical) result is derived in appendix [A].

### IV. BRAIDING STATISTICS ARGUMENT

The above microscopic derivation leaves several questions unanswered. First, it does not explain the physical connection between bulk braiding statistics and protected edge modes. Instead, this connection emerges from a mathematical relationship between null vectors and braiding statistics. Another problem is that the derivation is not complete since it only analyzes the robustness of the edge with respect to a particular class of perturbations. As a result, we have not proven definitively that the criterion is necessary for having a gapped edge. In this section, we address both of these problems: we give a general argument showing that any system which supports a gapped edge must satisfy the criterion. In addition, this argument reveals the physical meaning of \( M \).

We begin by explaining the notion of “annihilating” quasiparticles at a gapped boundary. The idea is as follows. Consider a general gapped electron system with a gapped boundary. Let us imagine that we take the ground state \( |\Psi\rangle \) and then excite the system by creating a quasiparticle/quasihole pair \( m, m' \) somewhere in the bulk. After creating these excitations, we separate them and then bring them near two points \( a, b \) on the edge (Fig. [2]). Let us denote the resulting state by \( |\Psi_{ex}\rangle \). We will say that \( m, m' \) “can be annihilated at the boundary” if, for arbitrarily distant \( a, b \), there exist operators \( U_a, U_b \) acting in finite regions near \( a, b \), such that

\[
U_a U_b |\Psi_{ex}\rangle = |\Psi\rangle
\]

(13)

Likewise, if no such operators exist then we will say that \( m, m' \) cannot be annihilated at the boundary. Here, \( U_a \) and \( U_b \) can be any operators composed out electron creation and annihilation operators acting in the vicinity of \( a \) and \( b \) such that \( U_a \cdot U_b \) conserves fermion parity. We note that we do not require that \( U_a \) and \( U_b \) individually conserve fermion parity – only that their product \( U_a \cdot U_b \) does so. Thus, according to the above definition, electron-like excitations can always be annihilated at the boundary, e.g. via \( U_a = c_a, U_b = c_b^\dagger \).
Let $M$ be the set of all quasiparticle types that can be annihilated at the edge:

$$M = \{ m : m \text{ can be annihilated at edge} \}$$  \hspace{1cm} (14)

We will now argue that self-consistency requires that $M$ has a very special structure: in particular, for systems with abelian quasiparticle statistics, $M$ must be a Lagrangian subgroup. In other words, we will show that (1) any two quasiparticle types that can be annihilated at the edge must have trivial mutual statistics, and (2) any quasiparticle type that cannot be annihilated must have nontrivial statistics with at least one particle that can be annihilated. This will establish that the criterion is necessary for having a gapped edge.

We establish condition (1) using an argument similar to one given in Ref. [23]. The first step is to consider a three step process in which we create two quasiparticles $m, \bar{m}$ in the bulk, move them along some path $\beta$ to two points on the edge, and then annihilate them. At a formal level, this process can be implemented by multiplying the ground state $|\Psi\rangle$ by an operator of the form

$$\mathbb{W}_{m,\beta} = U_a U_b W_{m,\beta}$$  \hspace{1cm} (15)

Here, $W_{m,\beta}$ is a (string-like) unitary operator that creates the quasiparticles and moves them to the edge, while $U_a U_b$ is an operator that annihilates them (Fig. 3a). Given that the system returns to the ground state at the end of the process, we have the algebraic relation

$$\mathbb{W}_{m,\beta} |\Psi\rangle = |\Psi\rangle$$  \hspace{1cm} (16)

(Here we assume that the phase of the operator $\mathbb{W}_{m,\beta}$ has been adjusted so that there is no phase factor on the right hand side of Eq. [16].)

Now imagine we repeat this process for another quasiparticle $m'$ and another path $\gamma$ with endpoints $c, d$ (Fig. 3b). We denote the corresponding operator by

$$\mathbb{W}_{m',\gamma} = U_c U_d W_{m',\gamma}$$  \hspace{1cm} (17)

FIG. 2. The concept of “annihilating” particles at a gapped boundary. (a) Consider a thought experiment in which we create a pair of quasiparticle excitations $m, \bar{m}$ in the bulk and then bring them near to two points $a, b$ at the edge. We denote the resulting excited state by $|\Psi_{ex}\rangle$. (b) We say that $m, \bar{m}$ can be annihilated at the boundary if there exist operators $U_a, U_b$ acting in the vicinity of $a, b$, such that $U_a U_b |\Psi_{ex}\rangle = |\Psi\rangle$, where $|\Psi\rangle$ is the ground state. Otherwise we say the particles cannot be annihilated.

Since each process returns the system to the ground state, we have:

$$\mathbb{W}_{m',\gamma} \mathbb{W}_{m,\beta} |\Psi\rangle = |\Psi\rangle$$  \hspace{1cm} (18)

Similarly, if we execute the processes in the opposite order, we have

$$\mathbb{W}_{m,\beta} \mathbb{W}_{m',\gamma} |\Psi\rangle = |\Psi\rangle$$  \hspace{1cm} (19)

At the same time, it is not hard to see that $\mathbb{W}_{m,\beta}, \mathbb{W}_{m',\gamma}$ satisfy the commutation algebra

$$\mathbb{W}_{m,\beta} \mathbb{W}_{m',\gamma} |\Psi\rangle = e^{i\theta_{m,m'}} \mathbb{W}_{m',\gamma} \mathbb{W}_{m,\beta} |\Psi\rangle$$  \hspace{1cm} (20)

for any two paths $\beta, \gamma$ that intersect one another at one point (see e.g. Refs. [23] and [24]). Using this result, together with the observation that $W_{\gamma,\gamma}$ commutes with $U_a U_b$ and $W_{\gamma,\gamma}$ commutes with $U_c U_d$ (since they act on non-overlapping regions), equation (20) follows immediately.

In the final step, we compare (20) with (18) and (19). Clearly, consistency requires that $e^{i\theta_{m,m'}} = 1$ for all $m, m' \in M$. Hence $M$ must satisfy condition (1).

Showing that $M$ satisfies condition (2) is more challenging. Here we simply explain the intuition behind this claim; in appendix we give a detailed argument. To begin, we recall a bulk property of systems with fractional statistics known as “braiding non-degeneracy” (appendix E.5 of Ref. [10]). Suppose $l$ is a quasiparticle excitation that cannot be annihilated in the bulk. That is, suppose that if we create $l, \bar{l}$ out of the ground state and the bring them near two widely separated points $a, b$ in the bulk, then we cannot annihilate them by applying appropriate operators $U_a, U_b$ acting in their vicinity. Braiding non-degeneracy is the statement that, for any such $l$, there is always at least one quasiparticle $m$ that has nontrivial mutual statistics with respect to $l$, i.e., $e^{i\theta_{m,l}} \neq 1$ (Fig. 3b).
FIG. 4. (a) The concept of braiding non-degeneracy in the bulk: if \( l \) is a quasiparticle that cannot be annihilated in the bulk, then there must be at least one quasiparticle species \( m \) that has nontrivial mutual statistics with respect to \( l \), i.e. \( e^{i\theta_{lm}} \neq 1 \). (b) The concept of braiding non-degeneracy at a gapped edge: if \( l \) cannot be annihilated at the edge, then there must be at least one quasiparticle \( m \) that can be annihilated at the edge such that \( e^{i\theta_{lm}} \neq 1 \).

The intuition behind braiding non-degeneracy is as follows: if \( l \) cannot be annihilated by applying any operator, then in particular it cannot be annihilated by cutting a large hole around \( l \). Hence it must be possible to detect the presence of this excitation outside any finite disk centered at \( l \). At the same time, it is natural to expect that the only way to detect excitations non-locally is by an Aharonov-Bohm measurement – i.e. braiding quasiparticles around them and measuring the associated Berry phase. Putting these two observations together, we deduce that \( e^{i\theta_{lm}} \neq 1 \) for some \( m \) since otherwise it would not be possible to detect \( l \) in this way.

For the same reason that bulk fractionalized systems obey braiding non-degeneracy, it is natural to expect that the gapped edges of these systems should obey an analogous property. Specifically, we expect that for each quasiparticle \( l \) which cannot be annihilated at the edge, there must be at least one quasiparticle species \( m \) which can be annihilated at the edge and which satisfies \( e^{i\theta_{lm}} \neq 1 \). The physical intuition behind this statement is similar to that of bulk braiding non-degeneracy: we note that if \( l \) cannot be annihilated, it must be detectable by a measurement far from \( l \). Again, it is reasonable to expect that this non-local detection is based on braiding, since both the edge and bulk are gapped and hence \(22\) have a finite correlation length. In an edge geometry, the analogue of conventional braiding is to create a pair of quasiparticles \( m, \overline{m} \) in the bulk, bring them to the edge on either side of \( l \) and annihilate them (Fig. 4b). Assuming that it is possible to detect \( l \) in this way, it follows that there always exists at least one quasiparticle \( m \) which can be annihilated at the edge, and which has \( e^{i\theta_{lm}} \neq 1 \). (See appendix \( C \) for a detailed argument). This result is exactly the statement that \( M \) satisfies condition (2). We conclude that \( M \) is a Lagrangian subgroup, as claimed.

The reader may wonder: at what point in the argument do we use the assumption that the edge is gapped? This assumption enters in several ways. At an intuitive level, it is implicit in the very definition of quasiparticle annihilation: the physical picture of annihilating quasi-

particles with (exponentially) localized operators \( U_a, U_b \) is only sensible if the edge has a finite correlation length. If instead the correlation length were infinite – as is typical for a gapless edge – then we would not expect such a localized annihilation process to be possible in general. At a mathematical level, the gapped edge assumption plays an important role in the derivation of condition (2): only for a gapped edge can one establish an analogue of braiding non-degeneracy (see appendix \( C \)).

V. MODULAR INVARIANCE ARGUMENT

To complete our discussion, we present another argument that shows that the criterion is necessary for gapping the edge. We proceed in the same way as in the previous section: we consider a general gapped electron system that has abelian quasiparticle statistics, has \( K_M = 0 \), and supports a gapped edge. We then show that the set \( M \) of particles that can be annihilated at the edge forms a Lagrangian subgroup, i.e. obeys conditions (1) and (2) of the criterion.

Our approach makes use of constraints from modular invariance and is similar to that of Ref. 26 (See also Refs. 27, 29 for related work). The starting point for the argument is to consider the system in a strip geometry with a large but finite width \( L \) in the \( y \) direction and infinite extent in the \( x \) direction. We then consider a scenario in which the lower edge is gapped while the upper edge is gapless (Fig. 5). More specifically, we assume that the upper edge is described by the model edge theory

\[
L = \frac{1}{4\pi} \partial_x \phi_I (K_{IJ} \partial_t \phi_J - V_{IJ} \partial_x \phi_J)
\]

where \( K_{IJ} \) is the \( 2N \times 2N \) \( K \)-matrix describing the bulk system\(^{30} \).

To proceed further, we make a change of variables to diagonalize the above action. Let \( W \) be a real matrix such that \( WTKW = \Sigma_z \) where \( \Sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) and \( I \) denotes the \( N \times N \) identity matrix. Setting \( \phi_I = W_{IJ} \tilde{\phi}_J \), the edge theory becomes

\[
L = \frac{1}{4\pi} \partial_x \tilde{\phi}_I (\Sigma_z \partial_t \tilde{\phi}_J - \tilde{V}_{IJ} \partial_x \tilde{\phi}_J)
\]

where \( \tilde{V} = W^T V W \). If we tune the interactions at the upper edge appropriately, we can arrange so that \( \tilde{V} \) is of the form \( \tilde{V} = \nu \delta_{IJ} \). Then all the edge modes propagate at the same speed \( |\nu| \) and the low energy, long wavelength physics of the strip is described by a conformal field theory.

Given this setup, our basic strategy will be to use constraints from conformal field theory to derive the criterion from the introduction. To be specific, we focus on a particular constraint known as modular invariance\(^{31} \). The statement of modular invariance is as follows. For any conformal field theory, we can imagine listing all the...
scaling operators $\mathcal{O}$ along with their scaling dimensions $(\Delta, \overline{\Delta})$ defined by

$$\langle \mathcal{O}(0,0)\mathcal{O}(x,t) \rangle \sim \frac{1}{(x-vt)^{2\Delta}} \cdot \frac{1}{(x+vt)^{2\Delta}}$$

Using this list, we can construct the formal sum (“partition function”)

$$Z(\tau) = e^{\pi i c (\tau - \tau'/12} \sum_{\mathcal{O}} e^{2\pi i (\Delta \tau - \overline{\Delta} \tau')}$$

where $c$ is the central charge and $\tau$ is a formal parameter. If we evaluate this expression for a complex $\tau$ with $Im(\tau) > 0$, the sum converges. Modular invariance is the statement that $Z(\tau)$ has to obey the constraint

$$Z(-1/\tau) = Z(\tau)$$

This equation places restrictions on the operator content of the conformal field theory – that is, the set of scaling operators in the theory. (We note that modular invariance also imposes the constraint that $Z(\tau + 1) = Z(\tau)$ for a bosonic system and $Z(\tau + 2) = Z(\tau)$ for a fermionic system, but we will not need this result here).

We now investigate the implications of modular invariance, in particular equation (26), for our system. The first step is to classify all the scaling operators and find their scaling dimensions. Importantly, we should only consider scaling operators which are local in the $x$-direction – that is, operators composed out of products of electron creation and annihilation operators acting within some finite segment of the strip $[x - \Delta x, x + \Delta x]$.

One set of scaling operators is given by expressions of the form

$$e^{il\phi}$$

These operators describe combinations of electron creation and annihilation operators which are charge neutral in each individual edge mode $\phi_j$ (Fig. 5a). Their scaling dimensions are given by

$$\Delta(\{n_{j,k}\}) = \sum_{J=1}^{N} \sum_{k=1}^{\infty} k \cdot n_{j,k}$$

Another set of operators are expressions of the form $e^{il \phi}$ for integer vectors $l$. These operators describe the annihilation (or creation) of a quasiparticle of type $l$ on the upper edge. An important point is that not all $l$ correspond to physical operators. Indeed, in general one cannot annihilate a fractionalized quasiparticle by itself. The only way such an operator can appear in our theory is as a description of a tunneling/annihilation process in which a quasiparticle of type $l$ tunnels from the upper edge to the lower edge and is subsequently annihilated (Fig. 5b). Thus, the allowed values of $l$ correspond to the quasiparticles that can be annihilated at the lower edge.

We now introduce some notation to parameterize these operators. Recall that $l, l'$ are topologically equivalent if $l - l' = K \cdot \Lambda$ for some integer vector $\Lambda$. Let $L$ be a set of vectors $l$ containing one representative from each of the above equivalence classes. Let $M$ be a subset of $L$, consisting of all quasiparticles that can be annihilated at the lower edge. In this notation, the most general scaling operators in our theory are of the form

$$e^{i(m+\Lambda)\phi} \mathcal{O}_{\{n_{j,k}\}}$$

where $m \in M$, and $\Lambda$ is an integer vector.

Given this parameterization of scaling operators, the partition function $Z(\tau)$ (25) can be written as

$$Z(\tau) = \sum_{m \in M} Z_m(\tau)$$

where $Z_m$ denotes the sum (25) taken over all $\Lambda$ and $\{n_{j,k}\}$, with $m$ fixed.

Now that we understand the structure of $Z(\tau)$, the next step is to compute $Z(-1/\tau)$ and find the constraints imposed by modular invariance. To this end, we recall that

$$Z_l(-1/\tau) = \sum_{l' \in L} S_{ll'} Z_{l'}(\tau)$$

where $S$ is the “$S$-matrix”,

$$S_{ll'} = \frac{1}{D} e^{i\theta_{ll'}} , \quad D = \sqrt{\det(K)}$$

(This identity can be derived, very concretely, from the Poisson summation formula). Substituting (32) into (31) gives

$$Z(-1/\tau) = \sum_{m \in M, l \in L} S_{ml} Z_l(\tau)$$
Applying the modular invariance constraint \( \text{[26]} \), we deduce

\[
\sum_{m \in M} Z_m(\tau) = \sum_{m \in M, l \in L} S_{ml} Z_l(\tau)
\]

(35)

implying that

\[
\sum_{m \in M} S_{ml} = \begin{cases} 
1 & \text{if } l \in M \\
0 & \text{otherwise}
\end{cases}
\]

(36)

Equation (36) is the main result of this section. We now use this result to show that \( M \) obeys the two conditions from the criterion. To this end, we first consider the case where \( l \) is the trivial quasiparticle. In this case, the right hand side of (36) is 1 while the left hand side is

\[
\sum_{m \in M} S_{ml} = \sum_{m \in M} \frac{1}{D} = \frac{|M|}{D}
\]

(37)

where \( |M| \) is the number of elements of \( M \). We deduce that \( |M| = D \).

Next, let \( l \in M \) be arbitrary. In this case, the left hand side of (36) is

\[
\sum_{m \in M} S_{ml} = \sum_{m \in M} e^{i \theta_{ml}}
\]

(38)

Thus, the only way that (36) can be satisfied is if all the phase factors \( e^{i \theta_{ml}} \) are equal to 1. In other words, we must have \( e^{i \theta_{ml}} = 1 \) for all \( m, m' \in M \). That is, \( M \) satisfies condition (1).

Finally, we consider the case where \( l \not\in M \). In this case, if \( l \) has non-integer mutual exchange, \( e^{i \theta_{ml}} \) would evaluate to 1 rather than 0. Hence \( M \) must satisfy condition (2) as well. We conclude that \( M \) is a Lagrangian subgroup, as claimed.

VI. CONCLUSION

In this paper, we have derived a general criterion for when an electron system with abelian quasiparticle statistics and \( K_H = 0 \) can support a gapped edge. We have established this criterion with three arguments – one based on a microscopic analysis of the edge, another on constraints from braiding statistics, and the third on modular invariance. Each of these derivations has different strengths and weakness. The microscopic approach shows that the criterion is sufficient for having a gapped edge, but it does not prove that it is necessary. The other two arguments show that the criterion is necessary, but do not prove it is sufficient. The three approaches also yield different insights: the microscopic derivation provides a concrete picture of the edge excitations, the braiding statistics approach explains the physical meaning of \( M \) (i.e. as the set of particles that can be annihilated at the edge), and the modular invariance approach reveals the connection with conformal field theory.

For concreteness, we have focused on systems built out of electrons (i.e. fermions). However, we believe that the criterion for a gapped edge can be generalized to boson systems with only one modification: condition (1) should require that all the particles in \( M \) have trivial exchange statistics, \( e^{i \theta_m} = 1 \), in addition to having trivial mutual statistics, \( e^{i \theta_{mm'}} = 1 \). In another words, in the bosonic case, we require that all the quasiparticles in \( M \) must be bosons. This modified criterion can be derived using arguments similar to the fermionic case.

There are a number of possible directions for future work. One direction is to perform a more concrete analysis of protected edge modes for a particular system. For example, it would be interesting to investigate a specific model of the \( \nu = 2/3 \) edge \( [S] \) in the presence of arbitrary scattering terms \( [3] \) and explicitly verify that the edge has gapless excitations when proximity-coupled to a superconductor. Such a calculation could also shed light on important physical properties of these edge modes such as their robustness to disorder and their ability to transport heat.

Another direction is to generalize the criterion to systems with non-abelian statistics. To formulate such a generalization, it may be helpful to study the classification of exactly soluble gapped edges given in Ref. \( [10] \). Other guidance may be obtained by extending the braiding statistics and modular invariance arguments of sections \( [11] \) and \( [12] \) to non-abelian states; it is less clear how to generalize the microscopic analysis of section \( [13] \).

Finally, it would be interesting to investigate higher dimensional systems. While some three dimensional (3D) systems such as topological insulators have robust surface states, there are no known examples of 3D systems whose surface states are protected independent of any symmetry. An intriguing problem is to find examples of these robust boundary modes in 3D or higher dimensional systems.

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Appendix A: Equivalence between two conditions

In this section, we show that Eq. [12] is equivalent to the criterion given in the introduction. In other words, we show that one can find \( N \) linearly independent integer vectors \( \{\Lambda_1, \ldots, \Lambda_N\} \) satisfying \( \Lambda^T K \Lambda_j = 0 \) if and only if there exists a set of (inequivalent) integer vectors \( M \) satisfying two properties:

1. \( m^T K^{-1} m' \) is an integer for any \( m, m' \in M \).
2. If \( l \) is not equivalent to any \( m, m' \in M \), then \( m^T K^{-1} l \) is non-integer for some \( m \in M \).
Here $K$ is a $2N \times 2N$ symmetric integer matrix with vanishing signature, non-vanishing determinant, and at least one odd element on the diagonal.

We first establish the “if” direction. Suppose $M$ is a set of vectors satisfying the above two properties. We wish to construct $\{\Lambda_1, ..., \Lambda_N\}$ satisfying $\Lambda_i^T K \Lambda_j = 0$. To this end, let us consider the set

$$\Gamma = \{m + K\Lambda : m \in M, \Lambda \in \mathbb{Z}^{2N}\}$$

(A1)

This set forms a $2N$ dimensional integer lattice, and therefore can be represented as $\Gamma = U\mathbb{Z}^{2N}$ where $U$ is some $2N \times 2N$ integer matrix.

Now consider the matrix $P = U^T K^{-1} U$. We claim that $P$ is a symmetric, integer matrix with vanishing signature, determinant $\pm 1$, and at least one odd element on the diagonal. Indeed, the fact that $P$ is symmetric, has vanishing signature, and has at least one odd element on the diagonal, follows from the corresponding properties of $K$. Also, the fact that $P$ is an integer matrix follows from the first property of $M$. Finally, to see that $P$ has determinant $\pm 1$, we use the second property of $M$: we note that if $x \not\in \mathbb{Z}^{2N}$, then $y^T Px$ is non-integer for some $y \in \mathbb{Z}^{2N}$. Hence, if $x \not\in \mathbb{Z}^{2N}$, then $P x \not\in \mathbb{Z}^{2N}$. It follows that $P^{-1}$ must be an integer matrix, so that $P$ has determinant $\pm 1$.

The next step is to use the following theorem, due to Milnor \cite{Milnor}, suppose $A, A'$ are two symmetric, indefinite, integer matrices with determinant $\pm 1$. Suppose in addition that $A, A'$ have the same dimension and same signature and are either both even or both odd – where an “even” matrix has only even elements on the diagonal, and an “odd” matrix has at least one odd element on the diagonal. Milnor’s theorem (Ref. \cite{Milnor}, p. 25) states that there must exist an integer matrix $W$ with unit determinant such that $W^T A W = A'$.

Applying this result to the matrix $P$ (an “odd” matrix with vanishing signature) we deduce that we can always block diagonalize $P$ as

$$W^T PW = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(A2)

where $W$ is an integer matrix with $\det(W) = \pm 1$ and 1 denotes the $N \times N$ identity matrix.

To complete the argument, we define $w_i$ to be the sum of the $i$th and $(i + N)$th columns of $W$. We then define $\Lambda_i = K^{-1} U w_i$. We can see that the $\Lambda_i$ obey the required relation $\Lambda_i^T K \Lambda_j = 0$. Furthermore, while these vectors are not necessarily integer vectors, we can make them an integer by multiplying them by a suitable constant (e.g. $\det(K)$). This is what we wanted to show.

We next prove the “only if” direction. Suppose that one can find $N$ linearly independent integer vectors $\{\Lambda_1, ..., \Lambda_N\}$ such that $\Lambda_i^T K \Lambda_j = 0$. We wish to construct a set $M$ of integer vectors satisfying the two properties listed above. The first step is to choose a basis so that the matrix $K$ has the block diagonal form

$$K = \begin{pmatrix} 0 & A \\ A^T & B \end{pmatrix}$$

(A3)

where $A, B$ are $N \times N$ matrices. In this basis, $K^{-1}$ is given by

$$K^{-1} = \begin{pmatrix} (A^T)^{-1} BA^{-1} & (A^T)^{-1} \\ A^{-1} & 0 \end{pmatrix}$$

(A4)

We then let $M$ be the set of all vectors of the form

$$\begin{pmatrix} 0 \\ v \end{pmatrix}$$

where $v$ is an $N$ component integer vector. (More precisely, we divide this set into equivalence classes modulo $K \mathbb{Z}^{2N}$, and choose one vector from each equivalence class).

We can easily see that $M$ satisfies the two properties listed above. To establish the first property, note that $m^T K^{-1} m' = 0$ for any $m, m' \in M$, so in particular this quantity is always an integer. As for the second property, let $l = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ be an integer vector such that $t^T K^{-1} m$ is an integer for all $m \in M$. Then $u_1 = A w$ for some integer vector $w$, so we can write

$$l = K \cdot \begin{pmatrix} 0 \\ w \end{pmatrix} + \begin{pmatrix} 0 \\ u_2 - B w \end{pmatrix}$$

(A5)

Examining this expression, we see that $l$ is equivalent to an element of $M$. This establishes the second property of $M$ and completes the proof.

**Appendix B: Proof that the null vector criterion is necessary**

In this section, we consider the two component edge theory \cite{2} in the presence of a single scattering term $U(\Lambda)$ \cite{3}. We show that a necessary condition for $U(\Lambda)$ to gap the edge is that $\Lambda$ satisfy the null vector criterion, $\Lambda^T K \Lambda = 0$.

Our basic strategy is to construct a (fictitious) $U(1)$ charge $Q$ which is conserved by $U(\Lambda)$, and then show that the system has a nonzero Hall conductivity with respect to this charge. To this end, we consider a general $U(1)$ charge of the form

$$Q = \frac{1}{2\pi} \int t^T \partial_x \phi$$

(B1)

where $t^T = (t_1, t_2)$ is some two component real vector.

Next, we choose $t_1 = \Lambda_2, t_2 = -\Lambda_1$ so that

$$[Q, U(\Lambda)] = 0$$

(B2)

For this choice of $t$, the charge $Q$ is a conserved quantity so it is sensible to compute the associated Hall conductivity $\sigma_{xy}^Q$. Following the usual $K$-matrix formalism we have:

$$\sigma_{xy}^Q = t^T K^{-1} t = \frac{1}{\det(K)} (\Lambda_2 - \Lambda_1) \cdot \begin{pmatrix} K_{22} & -K_{12} \\ -K_{21} & K_{11} \end{pmatrix} \cdot \begin{pmatrix} \Lambda_2 \\ \Lambda_1 \end{pmatrix} = \frac{1}{\det(K)} \Lambda^T K \Lambda$$

(B3)
We are now finished: we can see that if $Λ$ doesn’t satisfy the null vector criterion \[^6\], then $σ_{xy}^0 \neq 0$. It then follows that $U(Λ)$ cannot gap out the edge, since a system with a nonzero Hall conductivity has a protected edge if the corresponding Hall charge is conserved.\[^{11,12}\] This proves the claim.

We would like to emphasize that the above argument does not rule out the possibility of gapping the edge with other types of perturbations. In fact, it does not even rule out simple perturbations like a sum of two scattering terms $U(Λ_1) + U(Λ_2)$; these terms break all the $U(1)$ symmetries at the edge, thus invalidating the above analysis.

Appendix C: Establishing condition (2) of the criterion

In this section, we consider a general gapped electron system which has abelian quasiparticle statistics and has a gapped edge. For this class of systems, we argue that the set of quasiparticles that can be annihilated at the edge (denoted by $M$) must obey condition (2) of the criterion. In other words, we show that if $l$ is a quasiparticle that cannot be annihilated at a gapped edge, then $l$ must have nontrivial statistics with at least one quasiparticle $m$ that can be annihilated at the edge.

The argument we present is not a rigorous mathematical proof: we do not give precise definitions for all the concepts that we use, and we regularly drop quantities that we expect to vanish in the thermodynamic limit. Despite these limitations, we believe that the argument could be used as a starting point for constructing a rigorous proof.

1. Preliminaries

Our argument relies on the following conjecture about gapped many-body systems:

**Conjecture 1:** Let $|Ψ⟩$ be the ground state of a 2D gapped many-body system defined in a spherical geometry. Let $|Ψ'⟩$ be another state (not necessarily an eigenstate) which has the same energy density outside of two nonoverlapping disk-like regions $A, B$. Then we can write

$$|Ψ⟩ = \sum_k U_k W_k |Ψ⟩ \quad (C1)$$

where $W_k$ is a (string-like) unitary operator that describes a process in which a pair of quasiparticles $k, \overline{k}$ are created and then moved to regions $A, B$ respectively, and where $U_k$ is an operator acting within $A \cup B$. Here, the sum runs over different quasiparticle types $k$.

In more physical language, the above conjecture is the statement that any excited state whose excitations are located in two disconnected regions $A, B$ can be constructed by moving a pair of quasiparticles $k, \overline{k}$ into $A, B$ and then applying an operator $U_k$ acting within $A \cup B$. This claim is reasonable because we expect that the different excited states of a gapped many-body system can be divided into topological sectors parameterized by the quasiparticle type $k$, and that any two excitations in the same sector can be transformed into one another by local operations.

In addition, we will make use of the following lemma:

**Lemma 1:** Consider a 2D gapped many-body system with a gapped edge. Let $|Ψ⟩$ denote the ground state and let $|Ψ_{ex}⟩$ denote an excited state with a quasiparticle $l$ and quasihole $\overline{l}$ located near two points $a, b$ at the boundary. If $l$ cannot be annihilated at the edge then

$$\lim_{|a-b|→∞} \langle Ψ|U_a U_b |Ψ_{ex}⟩ = 0 \quad (C2)$$

for any operators $U_a, U_b$ acting near $a, b$.

To derive this result, let $H$ be a gapped, local Hamiltonian whose ground state is $|Ψ⟩$. Let $H_{ex}$ be a gapped, local Hamiltonian whose ground state is $|Ψ_{ex}⟩$. We can assume without loss of generality that the ground state energies of $H, H_{ex}$ are both 0:

$$H|Ψ⟩ = H_{ex}|Ψ_{ex}⟩ = 0 \quad (C3)$$

We will also assume that $H_{ex}$ can be written as

$$H_{ex} = H + H_a + H_b \quad (C4)$$

where $H_a, H_b$ are local operators acting near $a, b$.

We will now show that if

$$\lim_{|a-b|→∞} \langle Ψ|U_a U_b |Ψ_{ex}⟩ = α \neq 0 \quad (C5)$$

then we can always construct “dressed” operators $U_a, U_b$ such that $\lim_{|a-b|→∞} \langle Ψ|U_a U_b |Ψ_{ex}⟩ = |Ψ⟩$. This will establish the lemma (since the latter equation means that $l$ can be annihilated at the edge).

To do this, we use a trick due to Hastings (Ref. \[^{34}\] and Kitaev (Ref. \[^{10}\] appendix D.1.2). Let $\tilde{f}(ω)$ be a real, smooth function satisfying

$$\tilde{f}(0) = 1, \quad \tilde{f}(ω) = 0 \quad \text{for } |ω| ≥ Δ \quad (C6)$$

where $Δ$ is the energy gap of $H$. Define

$$f(t) = \frac{1}{2π} \int_{−∞}^{∞} dω \tilde{f}(ω)e^{-iωt} \quad (C7)$$

Given that $\tilde{f}(ω)$ is smooth, it follows that $f(t) → 0$ as $t → ∞$ faster than any polynomial. We then define

$$U = \frac{1}{α} \int_{−∞}^{∞} dt \, f(t) \cdot e^{iHt}U_a U_b e^{-iH_{ex}t} \quad (C8)$$
Furthermore, since $U$ is well-localized near $a$ and $b$, also $U$ can be (approximately) factored as $U = U_a \cdot U_b$, up to terms that decay rapidly in the separation between $a$ and $b$. In this way, we can explicitly construct operators $U_a, U_b$ acting near $a, b$ such that \[
abla \big|_{a-b} \to \infty \big| \nabla \big| \Psi_{ex} \big> = \big| \Psi \big>.
\]

Straightforward algebra gives
\[
\nabla |\Psi_{ex} \rangle = \frac{1}{\alpha} \int_{-\infty}^{\infty} dt \ f(t) \cdot e^{iHt} U_a U_b |\Psi_{ex} \rangle
\]
\[
= \frac{1}{\alpha} \int_{-\infty}^{\infty} dt \ f(t) \cdot \sum_n e^{iE_n t} \langle \Psi_n | U_a U_b |\Psi_{ex} \rangle
\]
\[
= \frac{1}{\alpha} \langle \Psi | U_a U_b |\Psi_{ex} \rangle
\]
so that
\[
\lim_{|a-b| \to \infty} \nabla |\Psi_{ex} \rangle = |\Psi \rangle
\]

Furthermore, since $f$ decays rapidly as $t \to \infty$, and $H$ is a local Hamiltonian, it is not hard to see that the region of support of $\nabla$ is well-localized near $a$ and $b$. Also, $\nabla$ can be (approximately) factored as $\nabla = \nabla_a \cdot \nabla_b$, up to terms that decay rapidly in the separation between $a$ and $b$. In this way, we can explicitly construct operators $\nabla_a, \nabla_b$ acting near $a, b$ such that $\lim_{|a-b| \to \infty} \nabla_a \nabla_b |\Psi_{ex} \rangle = |\Psi \rangle$.

\section{Outline of argument}

We consider a system in a square $L \times L$ geometry with a gapped edge. We let $|\Psi \rangle$ denote the ground state, and let $|\Psi_{ex} \rangle$ denote a state with two quasiparticles $l, \bar{l}$ located near two well-separated points at the boundary (Fig. 6). The argument proceeds in three steps:

1. In the limit $L \to \infty$, we show that there exists an operator $\nabla$ acting in region $C$ such that $\nabla |\Psi \rangle = |\Psi \rangle$ and $\nabla |\Psi_{ex} \rangle = 0$ (Fig. 6).

2. We show that we can replace $\nabla$ by an operator of the form $\sum_k \nabla_k \nabla_{\bar{k}}$, where $\nabla_k$ are operators acting near the boundary, and $\nabla_{\bar{k}}$ are (string-like) unitary operators that describe a process in which a pair of quasiparticles $k, \bar{k}$ are created in the bulk and moved near the boundary (Fig. 6).

3. We show that there is at least one quasiparticle $k$ that has nontrivial statistics with respect to $l$ and can be annihilated at the edge. This result proves the claim: the set $M$ of quasiparticles that can be annihilated at the edge obeys condition (2) of the criterion.

\section{Step 1}

The first step is to partition our system into two pieces, $C$ and $D$ (Fig. 6). We note that $D$ has two connected components — one containing $l$ and one containing $\bar{l}$. We will now show that in the limit $L \to \infty$ there exists an operator $\nabla$ acting in region $C$ with $\nabla |\Psi \rangle = |\Psi \rangle$ and $\nabla |\Psi_{ex} \rangle = 0$ (Fig. 6).

To construct $\nabla$, we consider the Schmidt decomposition of $|\Psi \rangle$ corresponding to the bipartition $C, D$:
\[
|\Psi \rangle = \sum_i \lambda_i |\Psi_{C,i} \rangle \otimes |\Psi_{D,i} \rangle
\]

Here $\{|\Psi_{C,i} \rangle \}$ and $\{|\Psi_{D,i} \rangle \}$ are orthonormal many-body states corresponding to regions $C$ and $D$ and $\lambda_i$ are Schmidt coefficients. Since the $\{|\Psi_{C,i} \rangle \}$, $\{|\Psi_{D,i} \rangle \}$ form a complete orthonormal basis for $C$ and $D$, we can also express $|\Psi_{ex} \rangle$ in terms of these states:
\[
|\Psi_{ex} \rangle = \sum_{ij} \lambda_{ij}^* |\Psi_{C,i} \rangle \otimes |\Psi_{D,j} \rangle
\]

We next observe that, in the limit $L \to \infty$, the coefficients have the property that for each $i$, either (1) $\lambda_i = 0$, or (2) all the $\lambda_{ij}^*$ vanish simultaneously. Indeed, if $\lambda_i$ and $\lambda_{ij}'$ were both nonzero for some $i, j$, then the operator $|\Psi_{D,i} \rangle \langle \Psi_{D,j} |$ would have a nonzero matrix element between $|\Psi \rangle$ and $|\Psi_{ex} \rangle$. But such a nonzero matrix element is not possible according to Lemma 1.

Given this observation, we can now construct the desired operator $\nabla$. We define
\[
\nabla = \sum_{\lambda \neq 0} |\Psi_{C,i} \rangle \langle \Psi_{C,i} |
\]

By construction, we have
\[
\nabla |\Psi \rangle = |\Psi \rangle, \ \nabla |\Psi_{ex} \rangle = 0
\]
as required.

\section{Step 2}

To proceed further, we decompose $\nabla$ as
\[
\nabla = \sum_i \nabla_i'' \cdot \nabla_i'
\]
where $V'$ acts in the interior of the system and $V''$ acts near the boundary.

Next, we construct an operator $P$ with several properties. First, $P$ acts within the region $E$ shown in Figure 7b. Second, $P|\Psi\rangle = |\Psi\rangle$. Third, if $P$ is applied to a state that has no excitations outside of $E \cup F_1 \cup F_2$, it returns a state with no excitations outside of $F_1 \cup F_2$.

In general, we need to work a bit harder. Let $H$ be a gapped Hamiltonian whose ground state is $|\Psi\rangle$. We can write

$$H = H_E + H_F + H_0$$  \hspace{1cm} (C17)$$

where $H_E$ contains terms acting in (or near) region $E$, $H_F$ contains terms acting in (or near) region $F_1 \cup F_2$, and $H_0$ contains all the other terms in the Hamiltonian. In general, $H_E, H_F, H_0$ may not commute with one another since they may overlap along the boundaries between the various regions. However, according to a result from Ref. 34 as well as Ref. 10 appendix D.1.2, we can always choose $H_E, H_F, H_0$ so that $|\Psi\rangle$ is a simultaneous eigenstate of all three operators:

$$H_0|\Psi\rangle = H_E|\Psi\rangle = H_F|\Psi\rangle = 0$$  \hspace{1cm} (C18)$$

To see this, note that

$$|\Psi\rangle = W_{l\gamma}|\Psi\rangle$$  \hspace{1cm} (C22)$$

where $W_{l\gamma}$ is a unitary operator that describes a process in which two quasiparticles $l, \tilde{l}$ are created and then moved to regions $F_1, F_2$ respectively, and where $W_{ki}$ is an operator acting within $F_1 \cup F_2$ (Fig. 8). Here the sum runs over different particle types $k$.

We next argue that the same relation holds for $|\Psi_{ex}\rangle$:

$$P V'_i |\Psi_{ex}\rangle = \sum_k U_{ki} W_k |\Psi_{ex}\rangle$$  \hspace{1cm} (C21)$$

Now consider the state $V'_i |\Psi\rangle$. This state has no excitations outside of $E \cup F_1 \cup F_2$ since $V'$ acts entirely within this region (Fig. 7b). It follows that the state $P V'_i |\Psi\rangle$ has no excitations outside of $F_1 \cup F_2$. Therefore, according to Conjecture 1 (C1), we can write

$$P V'_i |\Psi\rangle = \sum_k U_{ki} W_k |\Psi\rangle$$  \hspace{1cm} (C20)$$

where $W_k$ is a (string-like) unitary operator that describes a process in which a pair of quasiparticles $k, \tilde{k}$ are created and then moved to regions $F_1, F_2$ (Fig. 8). Here the sum runs over different particle types $k$. We then multiply both sides of Eq. (C20) by $W_{l\gamma}$, and commute the operators on both sides, the claim (C21) follows immediately.
We then note that
\[\sum_k U_k W_k |\Psi\rangle = \sum_k V''_i U_{ki} W_k |\Psi\rangle = \sum_i V'_i (PV'_i |\Psi\rangle) = P \sum_i V''_i V'_i |\Psi\rangle = PV |\Psi\rangle = |\Psi\rangle\]

By the same reasoning, we have
\[\sum_k U_k W_k |\Psi_{ex}\rangle = PV |\Psi_{ex}\rangle = 0\]

This is what we wanted to show.

Step 3: It follows from equations (C24) and (C25) that
\[\sum_k \langle \Psi | U_k W_k |\Psi\rangle = 1, \quad \sum_k \langle \Psi_{ex} | U_k W_k |\Psi_{ex}\rangle = 0\]

In particular,
\[\sum_k \langle \Psi | U_k W_k |\Psi\rangle \neq \sum_k \langle \Psi_{ex} | U_k W_k |\Psi_{ex}\rangle\]

At the same time, it is easy to see that
\[\langle \Psi_{ex} | U_k W_k |\Psi_{ex}\rangle = \langle \Psi | U_k W_k |\Psi\rangle \cdot e^{i\theta_{kl}}\]

where \(\theta_{kl}\) is the mutual statistics between \(k\) and \(l\). One way to derive this relation is to use the representation \(|\Psi_{ex}\rangle = W_{l}\gamma |\Psi\rangle\) and to choose the path \(\gamma\) so that it intersects the path corresponding to \(W_k\) at one point (Fig. 9b). Equation (C28) then follows immediately from the string commutation algebra (21).

To complete the derivation, we now compare the two relations (C27), (C28). From (C28), we see that \(\langle \Psi | U_k W_k |\Psi\rangle = \langle \Psi_{ex} | U_k W_k |\Psi_{ex}\rangle\) if \(k\) and \(l\) have trivial mutual statistics. Also, \(\langle \Psi | U_k W_k |\Psi\rangle = \langle \Psi_{ex} | U_k W_k |\Psi_{ex}\rangle\) if \(\langle \Psi | U_k W_k |\Psi\rangle = 0\). On the other hand, from (C27), we know that \(\langle \Psi | U_k W_k |\Psi\rangle \neq \langle \Psi_{ex} | U_k W_k |\Psi_{ex}\rangle\) for at least one \(k\). We conclude that there must be at least one particle type \(k\) which has nontrivial statistics with respect to \(l\) and has \(\langle \Psi | U_k W_k |\Psi\rangle \neq 0\). Applying Lemma 1 (C2), we conclude that there is at least one particle \(k\) which has nontrivial statistics with respect to \(l\) and can be annihilated at the boundary. Hence, the set \(M\) of quasiparticles that can be annihilated at the boundary must obey condition (2) of the criterion, as claimed.

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Our assumption here is that it is always possible to choose the interactions at the upper edge so that the low energy modes are given by (22) – by fine-tuning if necessary.

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One way to derive these relations is to use an alternate interpretation of $Z(\tau)$ as a euclidean space-time partition function defined on a torus of shape $\tau$. See Ref. [31] for an introduction.

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