AN EXAMPLE OF A DIFFERENTIABILITY SPACE WHICH IS PI-UNRECTIFIABLE

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Abstract. We construct a (Lipschitz) differentiability space which has at generic points a disconnected tangent and thus does not contain positive measure subsets isometric to positive measure subsets of spaces admitting a Poincaré inequality. We also prove that $l^2$-valued Lipschitz maps are differentiable a.e., but there are also Lipschitz maps taking values in some other Banach spaces having the Radon-Nikodym property which fail to be differentiable on sets of positive measure.

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1. Introduction

1.1. Overview. This paper deals with the foundations of first-order calculus in metric measure spaces. In this work we empirically address the question of whether a Poincaré inequality is needed, at the infinitesimal level, to have a Rademacher-like Theorem on the a.e. differentiability of Lipschitz functions. These results were announced in [Sch15].

In the seminal [HK98] Heinonen and Koskela introduced the notion of PI-spaces, i.e. a class of metric measure spaces which satisfy an abstract version of the Poincaré inequality. In the remarkable [Che99] Cheeger proved that in PI-spaces it is possible to develop first-order calculus; specifically he proved a version of Rademacher’s Theorem on the a.e. differentiability of real-valued Lipschitz functions. Later [CK09] Cheeger and Kleiner were even able, for PI-spaces, to prove the a.e. differentiability of Lipschitz functions which take value in Banach spaces having the Radon-Nikodym property.

In [Kei04] Keith introduced an analytic condition, the Lip-lip inequality (later shown to self-improve to an equality) and used it to prove a Rademacher Theorem on the a.e. differentiability of real-valued Lipschitz functions; the spaces satisfying

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the conclusion of the differentiability theorem of Keith will be called here \textit{differentiability spaces} (the structure was named by Keith \textit{(strong) measurable differentiable structure}, but had been actually singled out without giving it a name by Cheeger in [Che99]).

Keith claimed to have generalized Cheeger’s result. From the technical standpoint his claim was factual: the Lip-lip inequality appears to be a weaker condition than the Poincaré inequality and so his argument is more general than the one given in [Che99]. However, his claim was not supported by empirical evidence: to the best of our knowledge all known examples of differentiability spaces are PI-rectifiable; i.e., they can be decomposed into a countable union of positive-measure subsets of PI-spaces.

In the last five years there has been a surge of work on differentiability spaces: [BS13, Bat15, Sch16a, Sch16b, CKS16, BL15, Eril16]. Despite this theoretical progress there is a substantial gap between the theory and the structural properties exhibited by known examples. This has led to the following question [CKS16] (here rephrased as a conjecture); disclaimer: to the best of my knowledge [CKS16] is the first place where the question has been written down; I learnt it from Bruce Kleiner, but it is also possible to have been considered before by others, for example in [Hei07] Heinonen said it was important to understand the conditions needed to have a Rademacher Theorem.

\textbf{(ConjPIRect):} Any differentiability space is PI-rectifiable; in particular a.e. its tangents/blow-ups are PI-spaces.

The (ConjPIRect) has been recently proved in the beautiful [Eri16] \textit{under the additional assumption} that Lipschitz functions taking values in Banach spaces with the Radon-Nikodym property are differentiable. In this paper our goal is to disprove (ConjPIRect).

\section{1.2. The Result.} In this paper we construct an example of a metric measure space \((X_\infty, \mu_\infty)\) such that:

\textbf{(TritanopeExa):} Any Lipschitz map \(f: X_\infty \to l^2\) is differentiable \(\mu_\infty\)-a.e., but \((X_\infty, \mu_\infty)\) is PI-unrectifiable; moreover at \(\mu_\infty\)-a.e. point it has a tangent which is \textit{not} topologically connected.

The fact that at \(\mu_\infty\)-a.e. there is a topologically disconnected tangent is Theorem 3.57; this immediately implies that \(X_\infty\) cannot contain a positive measure subset \(S\) of a PI-space: at \(\mu_\infty\)-a.e. \(p \in S\) the tangents of \(X_\infty\) would then be PI-spaces, which are known [Che99] to be quasi-convex and hence connected.

For expository reason we first prove the \(\mu_\infty\)-a.e. differentiability of real-valued Lipschitz functions, Theorem 4.56, and reserve the more technical details for \(l^2\)-valued maps to Theorem 5.11.

\section{1.3. Outline.} \(X_\infty\) is constructed as an \textit{inverse limit system}. The basic operation is similar to Example 1.2 of [CK13b] (see [LP01] for the metric properties of the space and [CK13a] for the proof the Poincaré inequality) or the \textit{Laaksofolds} of [LS11]. Here we essentially double a 3-dimensional cell generating a diamond-like space, Construction 3.1. However we do \textit{not} take the path metric, but \textit{squeeze} closely the centers of the two cells: this destroys the connectedness of some tangents.

The proof of differentiability requires new ideas as the usual arguments [Fed69, \# 3.1.6] or[CK09, BL15] require joining pairs of points by quasi-geodesics and constructing the derivative on these curves. Our argument is \textit{functional}: we show that
if \( f : X_\infty \to l^2 \) is Lipschitz, it must eventually collapse the centers of the doubled cells faster than they are separated in the ambient space, compare Theorems 4.30 and 5.8. This collapsing argument is based on some elementary PDE, see Lemmas 4.6 and 5.1 and might be regarded as a tail-recursive version of quantitative differentiation, see [Che12].

1.4. Questions. Here are some questions that hopefully can give the reader some food for thought.

(Q1): Our example has analytic dimension 3, i.e. the gradient has three components. The techniques of the forthcoming [Sch] suggest that one can (with a lot of technical overhead) modify this example to get analytic dimension 1. However, at the moment we only have examples where the Assouad-Nagata dimension [LS05] is 3, can it be lowered to 1?

(Q2): What is the relationship between the \( \mu_\infty \)-a.e. differentiability for \( l^1 \) and \( l^2 \)-valued maps?

Note by [BL15] there are a Banach space \( B \) having the Radon-Nikodym property and a non-a.e. differentiable Lipschitz map \( f : X_\infty \to B \). As remarked in [Sch16b] the construction in [BL15] can be slightly improved to yield non-differentiability in a canonical Banach space having the Radon-Nikodym property:

\[
\text{Sem} = \bigoplus_{n=1}^{\infty} l_1^n \\
\]

the \( l^1 \)-sum of copies of \( \mathbb{R}^n \), with the \( l_1^n \)-norm, whose dimension progressively increases to \( \infty \). Thus differentiability in Sem is stronger than in \( l^2 \), and (Q2) asks how \( l^1 \) stands compared to \( l^2 \).

Notational conventions. We use the convention \( a \simeq b \) to say that \( a/b, b/a \in [C^{-1}, C] \) where \( C \) is a universal constant; we similarly use notations like \( a \lesssim b \) or \( a \gtrsim b \). The notation \( \mathcal{H}^k \) stands for the \( k \)-dimensional Hausdorff measure and \( \mathcal{L}^k \) for the \( k \)-dimensional Lebesgue measure. Given a map \( f : X \to Y \) and a measure \( \mu \) on \( X \), \( f_#\mu \) denotes the push-forward of \( \mu \) to a measure on \( Y \); finally \( \int_A g \, d\mu \) denotes the average of \( g \) on \( A \): \( \int_A g \, d\mu / \mu(A) \).

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2. Background Material

2.1. Functions and Spaces.

Definition 2.1 (Lipschitz map). Given a Lipschitz map \( f : X \to Z \) between the metric spaces \( X \) and \( Z \), we let \( L(f) \) denote its global Lipschitz constant. To extract
local information at $x$ on the Lipschitz constant we use the **big and small** Lipschitz constants:

\[
\operatorname{Lip}_f(x) = \limsup_{r \to 0} \frac{1}{r} \sup_{y \in B(x,r)} d_Z(f(x), f(y))
\]

(2.2)

\[
\operatorname{lip}_f(x) = \liminf_{r \to 0} \frac{1}{r} \sup_{y \in B(x,r)} d_Z(f(x), f(y)).
\]

**Definition 2.3** ($l^2$-valued harmonic functions). Given $u : \Omega \subset \mathbb{R}^n \to l^2$ we say that it is **harmonic** if each component $u_j$ is harmonic. Given enough regularity on $\partial \Omega$ (e.g. $\partial \Omega$ is locally a Lipschitz graph), given a boundary condition $f : \partial \Omega \to l^2$, we can find a harmonic extension $u : \Omega \to l^2$: take a harmonic extension $u_j$ of each component $f_j$ of $f$, then observe that for each $n$:

\[
(2.4) \quad \sum_{j=1}^{n} \|u_j\|^2
\]

is subharmonic and apply the maximum principle to conclude that if we set $u = (u_j)_{j=1}^{\infty}$ then $u$ is $l^2$-valued.

**Definition 2.5** (Inverse Limit Systems). Let $(X_n)_{n=0}^{\infty}$ be a sequence of compact metric spaces with a uniform bound on their diameters:

\[
\sup_{n} \operatorname{diam} X_n < \infty,
\]

(2.6)

and assume that there are surjective 1-Lipschitz maps $\pi_{n+1,n} : X_{n+1} \to X_n$; then the inverse limit $X_\infty$ of $(X_n)_{n=0}^{\infty}$ consists of all the sequences $(x_n)_{n=0}^{\infty}$ satisfying $\pi_{n+1,n}(x_{n+1}) = x_n$, and where the metric is defined by:

\[
d_{X_\infty}(x_\infty, y_\infty) = \limsup_{n, n' \to \infty} d_{X_n}(x_n, y_n).
\]

(2.7)

Moreover, in this case we obtain a 1-Lipschitz $\pi_\infty, n : X_\infty \to X_n$ just letting $(x_n)_{n=0}^{\infty} \mapsto x_n$.

It is useful to get conditions under which $X_\infty$ is also the Gromov-Hausdorff limit of the $X_n$, compare [CK13b, CK13a]. In this work it suffices to consider the following condition which just says that the $\pi_\infty, n$ give the desired Gromov-Hausdorff approximations:

\[
\lim_{n \to \infty} \sup_{x \in X_n} \operatorname{diam} \pi_{\infty, n}^{-1}(x) = 0.
\]

(2.8)

Assume now that on each $X_n$ we have a Radon measure $\mu_n$ and that $\pi_{n+1,n} \# \mu_{n+1} = \mu_n$ and that (2.8) holds. Then a standard compactness argument yields a Radon measure $\mu_\infty$ on $X_\infty$ such that $\pi_\infty, n \# \mu_\infty = \mu_n$ and $(X_n, \mu_n)$ converges to $(X_\infty, \mu_\infty)$ in the measured Gromov-Hausdorff sense. For a more general treatment of inverse limit systems of measure spaces we refer to [Cho58].

**Definition 2.9** (Tangents/blow-ups). Let $X$ be a metric space and $p \in X$. A **tangent/blow-up** of $X$ at $p$ is a pointed metric space $(Y, q)$ which is the pointed Gromov-Hausdorff limit (you can see [Sch16b, BBI01] for a review of the basic properties of Gromov-Hausdorff convergence) of a sequence $(\frac{1}{r_n} X, p)$ where $r_n \searrow 0$ and $\frac{1}{r_n} X$ denotes the metric space $X$ with the rescaled metric:

\[
d_{\frac{1}{r_n} X}(x, y) = \frac{1}{r_n} d_X(x, y).
\]

(2.10)
If $\mu$ is a Radon measure on $X$, a measured tangent/blow-up at $p$ is a pointed metric measure space $(Y, q, \nu)$ such that $(\frac{1}{r_n}X, p, \mu(B_X(p, r_n)))$ converges to $(Y, q, \nu)$ in the measured Gromov-Hausdorff sense.

2.2. Differentiability Spaces. We start with a brief review of differentiability spaces. For more details we refer to the original papers [Che99, Kei04] or to the nice expository paper [KM11]. This structure has several names in the literature: (strong) measurable differentiable structure, differentiable structure (in the sense of Cheeger and Keith), Lipschitz differentiability space, differentiability space. We highlight the features of differentiability spaces; contrary to some earlier papers, we do not assume a uniform bound on the dimension of the charts.

**Definition 2.11.** Let $(X, \mu)$ be a metric measure space; we say that $X$ is a differentiability space if:

- **DiffChart:** There is a countable collection of charts $\{(U_\alpha, \phi_\alpha)\}_\alpha$, where $U_\alpha \subset X$ is Borel and $\phi_\alpha : X \to \mathbb{R}^{N_\alpha}$ is Lipschitz, such that $X \setminus (\cup_\alpha U_\alpha)$ is $\mu$-null, and each real-valued Lipschitz function $f$ admits a first order Taylor expansion with respect to the components of $\phi_\alpha$ at generic points of $U_\alpha$, i.e. there are (a.e. unique) measurable functions $\frac{\partial f}{\partial \phi_\alpha}$ on $U_\alpha$ such that:

$$f(x) = f(x_0) + \sum_{i=1}^{N_\alpha} \frac{\partial f}{\partial \phi_\alpha^i}(x_0)(\phi_\alpha^i(x) - \phi_\alpha^i(x_0)) + o(d(x, x_0))$$

(for $\mu$-a.e. $x_0 \in U_\alpha$).

Equivalently:

$$\text{Lip} \left( f - \left( \frac{\partial f}{\partial \phi_\alpha}(x_0), \phi_\alpha \right) \right)(x_0) = 0.$$

The integer $N_\alpha$ is the dimension of the chart $\{(U_\alpha, \phi_\alpha)\}_\alpha$, and depends only on the set $U_\alpha$, not on the particular choice of the coordinate functions $\phi_\alpha$. If $\sup_\alpha N_\alpha < \infty$, it is called the differentiability or the analytic dimension.

Note that $\left( \frac{\partial f}{\partial \phi_\alpha} \right)^{N_\alpha}_{i=1}$ are the components of the gradient $\nabla f$ with respect to the coordinate system $\{\phi_\alpha \}_{\alpha=1}^{N_\alpha}$.

By [Che99] to each differentiability space there are associated measurable cotangent and tangent bundles $T^*X$ and $TX$; having locally trivialized $T^*X$ and $TX$, forms in $T^*X$ correspond to differentials of Lipschitz functions, and vectors in $TX$ give rise to differential operators called derivations [Wea00, Sch16a].

We now restate (ConjPIRect) in a more formal way.

- **ConjPIRect:** Let $(X, \mu)$ be a differentiability space. Then there is a countable decomposition $X = \bigcup S_i \cup \Omega$ where $\mu(\Omega) = 0$ and there are isometric embeddings $f_i : S_i \to Y_i$ and measures $\nu_i$ on the spaces $Y_i$ such that $f_i \# \mu S_i = \nu_i L f_i(S_i)$ and such that each $(Y_i, \nu_i)$ is a PI-space.

Following Bate and Li [BL15] we define RNP-differentiability.

**Definition 2.14** (RNP-differentiability). An RNP-differentiability space is a differentiability space where (2.12) and (2.13) hold also for any Lipschitz $f : X \to B$ where $B$ is a Banach space having the Radon-Nikodym property.

**Theorem 2.15** (Summary of results on differentiability spaces). This list summarizes relevant results on differentiability spaces:
(Cheeger): [Che99] if \((X, \mu)\) is a PI-space then \((X, \mu)\) is a differentiability space whose analytic dimension is bounded by an expression that depends only on the doubling constant \(C_\mu\) of \(\mu\) and the constants that appear in the Poincaré inequality. Moreover, for each real-valued Lipschitz function \(f\) one has \(\text{Lip}_f = \text{lip}_f\) \(\mu\)-a.e.

(Keith): [Kei04] assume that \((X, \mu)\) is a doubling metric measure space which satisfies the \textbf{Lip-lip inequality}: there is a constant \(C \geq 1\) such that for each real-valued Lipschitz function \(f\) one has \(\text{Lip}_f \leq Clip_f\) \(\mu\)-a.e. Then \((X, \mu)\) is a differentiability space whose analytic dimension is bounded by an expression that depends only on \(C_\mu\) and \(C\).

(Cheeger-Kleiner): [CK09] any PI-space is an RNP-differentiability space.

(Bate–Speight): [BS13] if \((X, \mu)\) is a differentiability space then \(\mu\) is asymptotically doubling in the sense that for \(\mu\)-a.e. \(x\) there are \((C_x, r_x) \in (0, \infty)^2\) such that:

\[
\mu(B(x, 2r)) \leq C_x \mu(B(x, r)) \quad (r \leq r_x).
\]

(Bate): [Bat15] any differentiability space \((X, \mu)\) can be decomposed, up to throwing away a null-set, into a countable union of positive measure subsets \(S_i\) such that for each \((S_i, \mu|_{S_i})\) the Lip-lip inequality holds with a constant \(C_i = C(S_i)\).

(Schioppa): [Sch16a, Sch16b] A metric measure space \((X, \mu)\) is a differentiability space if and only if it satisfies the \textbf{Lip-lip equality}: given any real-valued Lipschitz function \(f\) one has \(\text{Lip}_f = \text{lip}_f\) \(\mu\)-a.e. Moreover, at \(\mu\)-a.e. \(p \in X\) all the measured tangents at \(p\) are differentiability spaces.

(Cheeger-Kleiner-Schioppa): [CKS16] for differentiability spaces a version of metric differentiation [Amb90, Kir94, AK00] holds.

(Bate-Li): [BL15] if \((X, \mu)\) is an RNP-differentiability space at \(\mu\)-a.e. \(p \in X\) the measured tangents of \(X\) satisfy a non-homogeneous Poincaré inequality.

(Ericksson-Bique): [Eri16] RNP-differentiability spaces are PI-rectifiable.

3. The inverse limit system

In this section we focus on the metric measure properties of the example. We first introduce the building blocks of the construction, the non-quasiconvex diamonds, Construction 3.1, and then describe the inverse limit system, Construction 3.5. We then describe the shape of balls, prove that the measures are doubling, Lemma 3.36, and construct the horizontal gradient, Definition 3.41. We also introduce a way to discretize balls, the fundamental configuration of Definition 3.32 which is used in the proof of differentiability. Then we construct horizontal paths with jumps, Lemma 3.50, that connect a point at the center of a ball with points in the fundamental configuration: these paths are a key construction for proving differentiability. The proof of differentiability will consist in controlling the variation of a Lipschitz function in terms of two pieces: (1) the horizontal gradient on the paths, and (2) a “collapsing factor” on the jumps. Finally, we prove the existence of tangents which are not topologically connected, Theorem 3.57.

3.1. Building blocks.

Construction 3.1 (Non-quasiconvex diamonds). Let \([0, 1]^3\) be the standard cube with the Euclidean metric \(d_{\text{Euc}}\) and \(K = [1/2 - 1/6, 1/2 + 1/6]^3\); fix an integer \(n \geq 26\) and replace the subcube \(K\) with two isometric copies \(K_1, K_2\) glued along
the boundary \( \partial K \). Let \( A = [0, 1]^3 \setminus K \) and note that both the spaces \( A \cup K_1 \) and \( A \cup K_2 \) are isometric to \([0,1]^3\).

As set, define \( \text{Dym}_n([0,1]^3) = A \cup K_1 \cup K_2 \). Let \( c_i \) denote the center of \( K_i \); as metric \( d_{\text{Dym}_n([0,1]^3)} \) we take the largest metric which agrees with \( d_{\text{Euc}} \) on each \( A \cup K_1, A \cup K_2 \) and such that:

\[
d_{\text{Dym}_n([0,1]^3)}(c_1, c_2) = \frac{1}{4n}.
\]

A more concrete description of \( d_{\text{Dym}_n([0,1]^3)} \) can be obtained as follows. First define a symmetric function \( \varrho : \text{Dym}_n([0,1]^3) \times \text{Dym}_n([0,1]^3) \to [0, \infty] \):

\[
\varrho(p, q) = \begin{cases} 
\frac{d_{\text{Euc}}(p, q)}{4n} & \text{if } \{p, q\} \subseteq A \cup K_1 \text{ or } \{p, q\} \subseteq A \cup K_2 \\
\infty & \text{otherwise.}
\end{cases}
\]

Given \( p, q \in \text{Dym}_n([0,1]^3) \) a \textit{chain joining} \( p \) to \( q \) is a finite tuple \((p_0, \ldots, p_N)\) with \( p_0 = p, p_N = q \); then:

\[
d_{\text{Dym}_n([0,1]^3)}(p, q) = \inf \left\{ \sum_{i=0}^{N-1} \varrho(p_i, p_{i+1}) : (p_0, \ldots, p_N) \text{ joins } p \text{ to } q \right\}.
\]

From (3.4) it follows that the map \( \pi : \text{Dym}_n([0,1]^3) \to [0, 1]^3 \) which collapses the two copies \( K_1, K_2 \) together is 1-Lipschitz.

We now induce a cube-complex structure on \( \text{Dym}_n([0,1]^3) \); choose \( N = N(n) \in \mathbb{N} \) such that if one subdivides \([0,1]\) into \( 2N + 1 \) intervals of the same length, this length lies in \([1/(128n), 1/(32n)]\), and such that \( 2N + 1 \) is divisible by 3. Taking products of these intervals we obtain a cube-complex structure on \([0,1]^3\) where they all have the same side length. Moreover, one such cube, call it \( K_N \), is centered at \((1/2, 1/2, 1/2)\). As \( 2N + 1 \) is divisible by 3, this cube-complex structure is compatible with the doubling operation we applied to \( K \), i.e. the boundary of \( K \) will consist of 2-cells and we can induce a cube-complex structure on \( \text{Dym}_n([0,1]^3) \) by adding the requirement that \( \pi \) is open and cellular.

Note that \( \pi^{-1}(K_N) = K_{N,1} \cup K_{N,2} \), the two cubes being centered at \( c_1, c_2 \) respectively. We will call these cubes the \textit{gates}, and let \( \text{Gates}([0,1]^3) = \{K_{N,1}, K_{N,2}\} \).

Now to each copy of \( K \) we attach a color, say \( K_1 \) is green and \( K_2 \) is red. We will then have a green gate \( G_{\text{green}}([0,1]^3) \) and a red gate \( G_{\text{red}}([0,1]^3) \). Similarly, we have two copies \( \{C_1, C_2\} \) of \([0,1]^3\) inside \( \text{Dym}_n([0,1]^3) \) depending on whether we choose green or red for the cover of \( K \): we will call these the \textit{chromatic sheets}, and let \( \text{Chr}([0,1]^3) = \{C_1, C_2\} \). Finally let the \textit{cover sheets} \( \text{Cov}([0,1]^3) = \{B_1, B_2\} \) be the same as \( \{C_1, C_2\} \) (chromatic and cover sheets will differ at the next iterations of the construction).

Finally, we let \( J_{pp}([0,1]^3) = \{(c_1, c_2)\} \) which we call the \textit{jump pair} of \( \text{Dym}_n([0,1]^3) \).

The construction described so far will be called the \textit{nqc-diamond} on \([0,1]^3\) with parameter \( n \). We can extend this construction to each cube \( T \), obtaining \( \text{Dym}_n(T) \): just take a similarity and a translation that identify \( T \) with \([0,1]^3\), perform the above construction, and then scale back the metric so that \( \text{diam}(\text{Dym}_n(T)) = \text{diam}(T) \).

Given a measure \( \mu_T \) on \( T \) which is a sum of multiples of the Lebesgue measure on the 3-dimensional cells \( \text{Cell}(T) \) of \( T \), there is a naturally induced measure \( \mu_{\text{Dym}_n(T)} \) on \( \text{Dym}_n(T) \) such that \( \pi_\# \mu_{\text{Dym}_n(T)} = \mu_T \), which is obtained by splitting in 1/2 the measure across pairs of cells of \( \text{Cell}(\text{Dym}_n(T)) \) that \( \pi \) maps to the same cell of \( T \).
Construction 3.5 (Construction of the inverse limit system). Let \( n_0 \geq 100 \) to be determined later, set \( X_0 = [0,1]^3 \) with the standard Euclidean distance and Lebesgue measure \( \mu_{X_0} = \mathcal{L}^3|_{[0,1]^3} \). For each integer \( k \) let \( n_k = n_0 + k \) and let \( \bar{n}_0 = 0 \), \( \bar{n}_1 = n_1 \), \( \bar{n}_k = \sum_{j<k} n_j \).

**Step 1:** The construction of \( X_1, X_2, \cdots, X_{n_0} \).

We simply let \( X_1 = \text{Dym}_{n_1}(X_0) \) and let \( \pi_{1,0} : X_1 \to X_0 \) be the map \( \pi \) as in Construction 3.1 and \( \mu_{X_1} \) the corresponding measure. Let \( \text{slen}(X_1) \) denote the common length of the sides of the elements of \( \text{Cell}(X_1) \). Set \( \text{ToDouble}(X_0) = [0,1]^3 \).

To obtain \( X_2 \), let \( \text{ToDouble}(X_1) = \text{Cell}(X_1) \setminus \text{Gates}(X_1) \) and for each \( Q \in \text{ToDouble}(X_1) \), replace it with \( \text{Dym}_{n_1}(Q) \); on the other hand, subdivide each \( Q \in \text{Gates}(X_1) \) into smaller subcubes so that all the cells of \( \text{Cell}(X_2) \) have the same side-length \( \text{slen}(X_2) \); the set of the cells of \( X_1 \) which were only subdivided will be denoted by \( \text{Subdiv}(X_1) \). Combining the maps \( \pi_Q : \text{Dym}_{n_1}(Q) \to Q \) for \( Q \in \text{ToDouble}(X_1) \) and the identity for \( Q \in \text{Subdiv}(X_1) \), we get a map \( \pi_{2,1} : X_2 \to X_1 \); as in Construction 3.1 we also obtain a measure \( \mu_{X_2} \) with \( \pi_{2,1} \# \mu_{X_2} = \mu_{X_1} \). Note that \( \mu_{X_2} \) is a multiple of Lebesgue measure on each element of \( \text{Cell}(X_2) \).

We now turn to a description of the metric \( d_{X_2} \) of \( X_2 \) by introducing the **chromatic sheets** \( \text{Chr}(X_2) \) and the **cover sheets** \( \text{Cov}(X_2) \). Take \( X_1 \): we can lift it to \( X_2 \) by choosing for each \( Q \in \text{ToDouble}(X_1) \) either the green or the red lift in the construction of \( \text{Dym}_{n_1}(Q) \); the set of all possible lifts of \( X_1 \) is \( \text{Cov}(X_2) \) and we have:

\[
\# \text{Cov}(X_2) = 2 \# \{ Q \in \text{ToDouble}(X_1) \}.
\]

For the moment note that we want \( d_{X_2} \) so that for each \( B \in \text{Cov}(X_2) \) one has that \( \pi_{2,1} : B \to X_1 \) is an isometry. We now want to lift the chromatic sheets \( \text{Chr}(X_1) \): this time for any lift \( C \) of \( C \) we are always choosing, across all \( Q \in \text{ToDouble}(X_1) \) the *same* color, either green or red. In particular, the set of allchromatic sheets \( \text{Chr}(X_2) \) consists of 4 elements, which we can label \( \text{(green, green)}, \text{(red, green)}, \text{(green, red)} \) and \( \text{(red, red)} \). Also, the set of lifts of \( C \) in \( X_2 \), \( \text{Chr}(X_2, \pi_{2,1}^{-1}(C)) \) has cardinality 2. For the moment note that we want \( d_{X_2} \) so that for each \( C \in \text{Chr}(X_2) \) \( \pi_{2,0} : C \to [0,1]^3 \) is an isometry. Then we can restrict Lebesgue measure on each \( C \) and get the representation:

\[
\mu_{X_2} = \frac{1}{4} \sum_{C \in \text{Chr}(X_2)} \mathcal{L}^3 \cup C.
\]

The set of **jump pairs** of \( X_2 \) is:

\[
\text{Jpp}(X_2) = \bigcup_{Q \in \text{ToDouble}(X_1)} \text{Jpp}(Q);
\]

moreover, note that for \( (p,q) \in \text{Jpp}(X_1) \) we did not double the cells containing \( p \) and \( q \) and so we can regard \( \{p,q\} \) as a subset of \( X_2 \) too. Define a symmetric function \( \varrho : X_2 \times X_2 \to [0,\infty] \) by:

\[
\varrho(p,q) = \begin{cases} 
\frac{1}{4 \text{slen}(Q)} & \text{if } \{p,q\} \in \text{Jpp}(Q) \text{ for } Q \in \text{ToDouble}(X_1) \\
\frac{1}{\text{slen}(Q)} + \infty & \text{otherwise.}
\end{cases}
\]
Then $d_{X}(p, q)$ is obtained by minimizing the cost of chains joining $p$ to $q$:

$$d_{X}(p, q) = \inf \left\{ \sum_{i=0}^{N-1} \varrho(p_i, p_{i+1}) : (p_0, \cdots p_N) \text{ joins } p \text{ to } q \right\}.$$ (3.10)

Then $\pi_{2,1}$ becomes $1$-Lipschitz, open and cellular, and satisfies the desiderata above. Finally note that $\text{Gates}(X_1)$ can be identified with a subset of $X_2$ (as we did not apply the diamond construction on those cells of $X_1$ but just subdivided them) and we let:

$$\text{Gates}(X_2) = \bigcup_{Q \in \text{ToDouble}(X_1)} \text{Gates}(Q) \cup \bigcup_{Q \in \text{Cell}(X_2)} Q.$$ (3.11)

Let $2 \leq k < \bar{n}_1$; to obtain $X_{k+1}$, we define $\text{ToDouble}(X_k) = \text{Cell}(X_k) \setminus \text{Gates}(X_k)$ and replace each $Q \in \text{ToDouble}(X_k)$ with $\text{Dym}_{n_1}(Q)$. The other definitions, e.g. $d_{X_{k+1}}, \pi_{k+1,k}$ are as above.

**Step 2: The construction of $X_{\bar{n}_k+1}, X_{\bar{n}_k+2}, \cdots, X_{\bar{n}_k+k}$.**

To obtain $X_{\bar{n}_k+1}$ from $X_{\bar{n}_k}$, we apply the diamond construction to all the cells of $X_{\bar{n}_k}$: we let $\text{ToDouble}(X_{\bar{n}_k}) = \text{Cell}(X_{\bar{n}_k})$ and replace each $Q \in \text{ToDouble}(X_{\bar{n}_k})$ with $\text{Dym}_{n_k+1}(Q)$. We can lift $X_{\bar{n}_k}$ by choosing for each $Q \in \text{ToDouble}(X_{\bar{n}_k})$ either the green or red lift in the construction of $\text{Dym}_{n_k}(Q)$. We set of all possible lifts of $X_{\bar{n}_k}$ will be denoted by $\text{Cov}(X_{\bar{n}_k+1})$ and we have:

$$\# \text{Cov}(X_{\bar{n}_k+1}) = 2^\#(Q \in \text{ToDouble}(X_{\bar{n}_k})).$$ (3.12)

On the other hand, consider a chromatic sheet $C \in \text{Chr}(X_{\bar{n}_k})$; this admits exactly two lifts $\text{Chr}(X_{\bar{n}_k+1}, \pi_{\bar{n}_k+1, \bar{n}_k}(C)) = \{C_{\text{green}}, C_{\text{red}}\}$ where, whenever we choose a lift $Q \in \text{ToDouble}(X_{\bar{n}_k})$, we *always* choose either red or green. We also define the set of all chromatic sheets:

$$\text{Chr}(X_{\bar{n}_k+1}) = \bigcup_{C \in \text{Chr}(X_{\bar{n}_k})} \text{Chr}(X_{\bar{n}_k+1}, \pi_{\bar{n}_k+1, \bar{n}_k}(C)).$$ (3.13)

To construct $d_{X_{\bar{n}_k+1}}$ we proceed as before: we define a symmetric function $\varrho : X_{\bar{n}_k+1} \times X_{\bar{n}_k+1} \to [0, \infty]$ by:

$$\varrho(p, q) = \begin{cases} d_{X_{\bar{n}_k}}(p, q) & \text{if } p, q \in B \in \text{Cov}(X_{\bar{n}_k+1}) \\ \frac{1}{4n_{k+1}} \text{slen}(Q) & \text{if } \{(p, q)\} \text{ or } \{(q, p)\} = \text{Jpp}(Q) \text{ for } Q \in \text{ToDouble}(X_{\bar{n}_k}) \\ +\infty & \text{otherwise}. \end{cases}$$ (3.14)

Then $d_{X_{\bar{n}_k+1}}(p, q)$ is obtained by minimizing the cost of chains joining $p$ to $q$:

$$d_{X_{\bar{n}_k+1}}(p, q) = \inf \left\{ \sum_{i=0}^{N-1} \varrho(p_i, p_{i+1}) : (p_0, \cdots p_N) \text{ joins } p \text{ to } q \right\}.$$ (3.15)

Thus $\pi_{\bar{n}_k+1, \bar{n}_k}$ becomes open, cellular and $1$-Lipschitz. Moreover for each $B \in \text{Cov}(X_{\bar{n}_k+1})$ the map $\pi_{\bar{n}_k+1, \bar{n}_k} : B \to X_{\bar{n}_k}$ is an isometry. Moreover, letting $\pi_{\alpha, \beta} = \pi_{\alpha, \alpha-1} \circ \pi_{\alpha-1, \alpha-2} \circ \cdots \circ \pi_{\beta+1, \beta}$, we have that for each $C \in \text{Chr}(X_{\bar{n}_k+1})$ the map $\pi_{\bar{n}_k+1, \bar{n}_k} : C \to [0, 1]^3$ is an isometry. Note that:

$$\# \text{Chr}(X_{\bar{n}_k+1}) = 2^{\bar{n}_k+1},$$ (3.16)
so that one has the representation:

\[(3.17) \quad \mu_{X_{n_k+1}} = 2^{-\bar{n}_k - 1} \sum_{C \in \text{Chr}(X_{n_k+1})} \mathcal{L}^3 \mathcal{L} C.\]

We also define the set of jump pairs:

\[(3.18) \quad \text{Jpp}(X_{\bar{n}_k+1}) = \bigcup_{Q \in \text{ToDouble}(X_{\bar{n}_k})} \text{Jpp}(Q).\]

An important difference is that for \(l \leq \bar{n}_k\), if \(\{(p, q)\} \in \text{Jpp}(X_l)\) both \(p\) and \(q\) are going to be centers of some \(Q \in \text{ToDouble}(X_{\bar{n}_k})\). Thus \(p\) gets replaced by a pair \(\{p\text{green}, p\text{red}\}\) and \(q\) gets replaced by \(\{q\text{green}, q\text{red}\}\). Moreover, for all choices \(\alpha, \beta \in \{\text{green, red}\}\) we have:

\[(3.19) \quad d_{X_{\bar{n}_k+1}}(p\alpha, q\beta) = d_{X_{n_k}}(p, q).\]

Finally let

\[(3.20) \quad \text{Gates}(X_{\bar{n}_k+1}) = \bigcup_{Q \in \text{ToDouble}(X_{\bar{n}_k})} \text{Gates}(Q).\]

For \(\bar{n}_k+1 \leq l < \bar{n}_k+1\) we explain how to construct \(X_{l+1}\) from \(X_l\). Let \(\text{ToDouble}(X_l) = \text{Cell}(X_l) \setminus \text{Gates}(X_l)\) and for each \(Q \in \text{ToDouble}(X_l)\) replace it with \(\text{Dym}_{n_k}(Q)\); on the other hand, subdivide each \(Q \in \text{Gates}(X_l)\) into smaller subcubes so that all the cells of \(\text{Cell}(X_{l+1})\) have the same side length \(\text{slen}(X_{l+1})\); the set of cells which were only subdivided will be denoted by \(\text{Subdiv}(X_l)\). We then construct \(d_{X_{l+1}}, \pi_{l+1,l}, \text{etc.}\) as we did for \(X_2\) (but we replace \(n_1\) with \(n_k\)). Finally let:

\[(3.21) \quad \text{Gates}(X_{l+1}) = \bigcup_{Q \in \text{ToDouble}(X_l)} \text{Gates}(Q) \cup \bigcup_{Q \in \text{Cell}(X_{l+1})} Q.\]

Let \(\{X_l\}, \{\mu_l\}\) and \(\{\pi_{l+1,l}\}\) denote the resulting inverse system; let \(X_\infty\), denote the inverse limit; then (2.8) holds (compare the discussion in Lemma 3.27) and thus \(X_l \rightarrow X_\infty\) in the Gromov-Hausdorff sense and we also obtain a limit measure \(\mu_\infty\) such that \(\pi_{\infty,l} \# \mu_\infty = \mu_l\). Here we summarize three properties of the inverse system:

\begin{itemize}
  \item [(IsoCov):] For each \(B \in \text{Cov}(X_{l+1})\) the map \(\pi_{l+1,l} : B \rightarrow X_l\) is an isometry.
  \item [(IsoChrom):] For each \(C \in \text{Chr}(X_{l+1})\) the map \(\pi_{l+1,l} : C \rightarrow [0,1]^3\) is an isometry.
  \item [(MuChrom):] One can represent \(\mu_{X_{l+1}}\) as:
\end{itemize}

\[(3.22) \quad \mu_{X_{l+1}} = 2^{-l-1} \sum_{C \in \text{Chr}(X_{l+1})} \mathcal{L}^3 \mathcal{L} C.\]

**Remark 3.23 (Chromatic Labels).** Note that we can assign to chromatic sheets in \(X_l\) a color label of length \(l\) consisting of entries which are either green or red and that in passing to \(X_{l+1}\) each sheet gets doubled: we can either append red or green to the label. We can then induce a chromatic label also on points; some points belong to just one chromatic sheet and so the label is unambiguous; for points belonging to more than one sheet we make all possible labelings valid.
3.2. Bounded local geometry and the horizontal gradient.

**Definition 3.24** (Discrete logarithm). For \( r \in (0, 1/2] \) let \( \lg(r) \) be the integer such that:

\[
(3.25) \quad \text{slen}(X_{\lg(r)+1}) \leq r < \text{slen}(X_{\lg(r)}).
\]

For \( \varepsilon \in (0, 1] \), as \( \text{slen}(X_{l+1}) \leq \frac{1}{2} \text{slen}(X_l) \), one has the estimate:

\[
(3.26) \quad \lg(\varepsilon r) - \lg(r) \leq \log_2 \frac{1}{\varepsilon} + 5.
\]

**Lemma 3.27** (Shape of balls). Let \( j > l \) (\( j = \infty \) being admissible), \( p_j \in X_j \) and \( p_l = \pi_{j,l}(p_j) \). Then:

\[
(3.28) \quad B_{X_j}(p_j, r) \subset \pi_{j,l}^{-1}(B_{X_l}(p_l, r)) \subset B_{X_j}(p_j, r + 4 \times \text{slen}(X_l)).
\]

There is a universal constant \( C \) such that for each \( k \in \mathbb{N}, p_k \in X_k \) and \( r \in [0, \sqrt{2}] \) one can control the number of chromatic sheets intersected by \( B_{X_k}(p_k, r) \) as follows:

\[
(3.29) \quad \# \{ C \in \text{Chr}(X_k) : C \cap B_{X_k}(p_k, r) \neq \emptyset \} \leq C2^{k-\lg(r)}.
\]

**Proof.** As \( \pi_{j,l} \) is 1-Lipschitz and open, \( \pi_{j,l} : B_{X_j}(p_j, r) \to B_{X_l}(p_l, r) \) is surjective, which gives the inclusion:

\[
(3.30) \quad B_{X_j}(p_j, r) \subset \pi_{j,l}^{-1}(B_{X_l}(p_l, r)).
\]

Let \( q \in \pi_{j,l}^{-1}(B_{X_l}(p_l, r)) \setminus B_{X_j}(p_j, r) \); then we can reach \( q \) from \( B_{X_j}(p_j, r) \) by changing chromatic sheets by looking at the colors added between \( X_l \) and \( X_j \) (for \( j = \infty \) one should use a limiting argument). In fact, note that we can assign to chromatic sheets in \( X_s \) a color label of length \( s \) consisting of entries which are either green or red and that in passing to \( X_{s+1} \) each sheet gets doubled: we can either append red or green to the label. Suppose now that in \( X_s \) we are on a chromatic sheet \( C \) and we want to move to the chromatic sheet that differs only on the last entry of the color label. We just need to move to a gate \( \text{Gates}(X_{l+1}) \) and this can be accomplished by traveling a distance at most \( \text{slen}(X_l) \). As \( \text{slen}(X_{s+1}) \leq \frac{1}{2} \text{slen}(X_s) \) we can control, via a geometric series, also the total distance to travel to change colors added between \( X_s \) and \( X_{s+1} \) so we get the inclusion:

\[
(3.31) \quad \pi_{j,l}^{-1}(B_{X_l}(p_l, r)) \subset B_{X_j}(p_j, r + 4 \times \text{slen}(X_l)).
\]

We now pass to the bound on the number of chromatic sheets. Let \( q \in X_k \cap B_{X_k}(p_k, r) \) and for \( s \leq k \) let \( C_s(q) \) be a chromatic sheet of \( X_s \) containing \( \pi_{k,s}(q) \). If for some \( s \leq k \) there is no choice of \( C_s(q) \) such that \( C_s(q) \) passes through \( \pi_{k,s}(p_k) \), choose \( s = s(q) \) as small as possible having this property. Then either \( d(q, p_k) \geq \text{slen}(X_{s(q)}) \) or \( \pi_{k,s(q)}(q) \) lies at distance \( < \text{slen}(X_{s(q)}) \) from \( \partial K \), where \( K \) is the central cube that gets doubled in passing from \( Q \) to \( \text{Dym}_{n=n(s)}(Q) \) for \( Q \in \text{ToDouble}(X_{s-1}) \). Consider the set of those \( s(q) \) such that the second case happens and \( d(q, p_k) < \text{slen}(X_{s(q)})/16 \): this can only happen for one value \( s(q) \) of \( s(q) \) because \( \partial K \) is a distance \( > 1/8 \text{slen}(X_{s-1}) \) from \( \partial Q \). Now for each \( l \) such that \( r \geq \text{slen}(X_l)/16 \) we can change at most two colors in the chromatic sheet and hence the bound (3.29) follows. \( \square \)
The fundamental configurations. Fix \( \varepsilon \in (0, 1/400) \) and let \( \text{Disc}(\varepsilon, r) = \{ j \in \mathbb{Z} \mid j \leq r \} \). Pick \( p_{\infty} \in X_\infty \) and \( r \in (0, 1/2) \). Let \( p_k = \pi_{\infty,k}(p_\infty) \) and let:
\[
\text{Grid}(p_\infty) = \{ p_\infty + (0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1) \} \cap \text{Disc}(\varepsilon, r).
\]
Then \( \text{Grid}(p_\infty) \subset B_{X_\infty}(p_\infty, \sqrt{3}(1 + \varepsilon)r) \) and \( \text{Grid}(p_\infty) \) is \( \varepsilon^2 r \)-dense in \( B_{X_\infty}(p_\infty, r) \). Let \( j_0 = \log(\varepsilon^2 r) \) and denote by \( \text{Chr}(j_0) \) the set of chromatic sheets which intersect \( B_{X_{j_0}}(p_{j_0}, r) \). For each \( C \in \text{Chr}(j_0) \) let \( \text{Grid}(C) = C \cap (\pi_{j_0,0}(C)^{-1}(\text{Grid}(p_\infty))) \). Then set:
\[
\text{Grid}(p_{j_0}, r) = \bigcup_{C \in \text{Chr}(j_0)} \text{Grid}(C),
\]
which is \( \varepsilon^2 r \)-dense in \( B_{X_{j_0}}(p_{j_0}, r) \) and lies in \( B_{X_{j_0}}(p_{j_0}, 2\sqrt{3}r) \).

For \( j > j_0 \) define \( \text{Chr}(j) \) as follows: for each \( C \in \text{Chr}(j - 1) \) choose just the green lift, i.e. take \( \tilde{C} \in \text{Chr}(j) \) such that \( \pi_{j,j-1}(\tilde{C}) = C \) and the label of \( \tilde{C} \) is obtained by appending green to that of \( C \). As \( \pi_{j,0} : \tilde{C} \to [0, 1]^3 \) is an isometry, let \( \text{Grid}(\tilde{C}) = \tilde{C} \cap (\pi_{j,0}(C)^{-1}(\text{Grid}(p_\infty))) \). Finally let:
\[
\text{Grid}(p_j, r) = \bigcup_{C \in \text{Chr}(j)} \text{Grid}(C),
\]
which lies in \( B_{X_j}(p_j, 2\sqrt{3}r) \).

We now show that \( \text{Grid}(p_j, r) \) is \( 5\varepsilon r \)-dense in \( B_{X_j}(p_j, r) \). From a point \( q \in C \) to change color labels at the positions \( s > j_0 \) one needs to travel a distance at most \( 4\varepsilon \text{s len}(X_{j_0}) \leq 4\varepsilon^2 r < \varepsilon r \). As \( \text{Grid}(p_\infty) \) is \( \varepsilon r \)-dense in \( B_{X_\infty}(p_\infty, r) \) we conclude that \( \text{Grid}(p_j, r) \) is \( (5\varepsilon r) \)-dense in \( B_{X_j}(p_j, r) \). Finally by (3.28) there is a uniform bound \( C = C(\varepsilon) \) on the cardinality of \( \text{Grid}(p_j, r) \). We will call \( \text{Grid}(p_j, r) \) a fundamental configuration at \( p_j \), at scale \( r \) and resolution \( \varepsilon \): we will denote it by \( \text{Fund}_{X_j}(p_j, \varepsilon, r) \).

Finally, we can obtain \( \text{Fund}_{X_{j_0}}(p_{\infty}, \varepsilon, r) \) by a limiting procedure. In fact, each \( C \in \text{Chr}(j_0) \) gives rise to a sequence of chromatic sheets \( C^{(j_0)} = C, C^{(j_0+1)}, \ldots \) where we keep appending green to the labels. This yields a limit sheet \( C^{(\infty)} \subset X_\infty \) and we can then let \( \text{Grid}(C^{(\infty)}) = C^{(\infty)} \cap (\pi_{\infty,0}(C^{(\infty)})^{-1}(\text{Grid}(p_\infty))) \) and then proceed as above.

The measures are doubling. The measures \( \mu_j \) (\( j = \infty \) being admissible) are uniformly doubling, i.e. there is a universal constant \( C \) such that for each \( j \in \mathbb{N} \cup \{ \infty \} \), \( p_j \in X_j \) and \( r \in (0, \sqrt{3}) \) one has:
\[
\mu_j(\text{Fund}_{X_j}(p_j, \varepsilon, r)) \leq C \mu_j(\text{Fund}_{X_j}(p_j, r/2)).
\]

Proof. We treat the case \( j < \infty \) as then \( j = \infty \) follows by a limiting argument. Combining (3.29) with (3.17) we deduce:
\[
\mu_j(\text{Fund}_{X_j}(p_j, r)) \leq \frac{4\pi}{3} C r^3 2^{-\log(r)},
\]
\( C \) being the constant from (3.29).

Definition 3.32 (The fundamental configurations). Fix \( \varepsilon \in (0, 1/400) \) and let \( \text{Disc}(\varepsilon, r) = \{ j \in \mathbb{Z} = r \} \). Pick \( p_{\infty} \in X_\infty \) and \( r \in (0, 1/2) \). Let \( p_k = \pi_{\infty,k}(p_\infty) \) and let:
\[
\text{Grid}(p_\infty) = \{ p_\infty + (0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1) \} \cap \text{Disc}(\varepsilon, r).
\]

Fix a chromatic sheet $\mathcal{C}$ containing $p_j$; by traveling a distance $\leq r/4$ we can change all the last $j - \lg(r/16) \geq j - \lg(r) - 9$ entries of the color label of $\mathcal{C}$. Thus:

$$(3.39) \quad \bigcup_{\mathcal{C} \in \mathcal{S}} \tilde{\mathcal{C}} \cap \pi^{-1}_{j,0}(B_{X_0}(\pi_{j,0}(p_0), r/4)) \subset B_{X_j}(p_j, r/2),$$

where $\mathcal{S} \subset \text{Chr}(X_j)$ has cardinality at least $j - \lg(r) - 9$. We then have:

$$(3.40) \quad \mu_j \left( B_{X_j}(p_j, r/2) \right) \geq \frac{4\pi}{3} \frac{r^3}{64} 2^{-\lg(r) - 9}.$$

\[\square\]

**Definition 3.41** (The horizontal gradient). We want to describe the horizontal gradient $\nabla$ in $X_l$ for $l \in \mathbb{N}\cup\{0, \infty\}$. For $l = 0$ we just take the usual gradient as $X_0 = [0, 1]^3$. In general, for $l < \infty$ the measure $\mu_l$ has a 3-rectifiable representation, i.e. it can be represented as an integral of measures associated to 3-rectifiable sets:

$$(3.42) \quad \mu_{X_l} = 2^{-l} \sum_{\mathcal{C} \in \text{Chr}(X_l)} \mathcal{L}^3 \mathcal{L} \mathcal{C};$$

as each $\mathcal{C} \in \text{Chr}(X_l)$ can be identified with $[0, 1]^3$ we can take the standard gradient $\nabla$ on each $\mathcal{C}$ and obtain the horizontal gradient $\nabla$ on $X_l$.

Let $\vec{x}$ be the tuple $(x^1, x^2, x^3)$ of coordinate functions on $[0, 1]^3$; with abuse of notation we will also write $\vec{x}$ for $\vec{x} \circ \pi_{1,0}$; then at each $p \in X_l$ one has $\nabla \vec{x}(p) = \text{Id}_{[0, 1]^3}$.

For $l = \infty$ let $\text{Chr}(X_{\infty})$ denote the set of all the sequences $(\mathcal{C}_i)_{i \geq 0}^\infty$ where $\mathcal{C}_i \in \text{Chr}(X_i)$ and $\pi_{i+1,j}(\mathcal{C}_{i+1}) = \mathcal{C}_i$ for each $i$. Then $(\mathcal{C}_i)_{i \geq 0}^\infty$ admits an inverse limit $\mathcal{C}_{\infty}$, and note that $\mathcal{C}_{\infty}$ also completely determines the sequence $(\mathcal{C}_i)_{i \geq 0}^\infty$ letting $\mathcal{C}_i = \pi_{\infty,i}(\mathcal{C}_{\infty})$. The uniform probability measures $P_l = 2^{-l}$ on $\text{Chr}(X_l)$ pass to the limit to a probability measure $P_{\infty}$ on $\text{Chr}(X_{\infty})$. More concretely, using sequences on green and red, we can identify $\text{Chr}(X_{\infty})$ with the standard Cantor set and $P_{\infty}$ becomes the corresponding standard probability measure. Taking the limit in (3.42) we get:

$$(3.43) \quad \mu_{\infty} = \int_{\text{Chr}(X_{\infty})} \mathcal{L}^3 \mathcal{L} \mathcal{C} \, dP_{\infty}(\mathcal{C});$$

on each $\mathcal{C}$ the operator $\nabla$ is well-defined, and thanks to (3.43) we can combine them to obtain the horizontal derivative on $X_{\infty}$. Note also that $\nabla \vec{x} = \text{Id}_{[0, 1]^3}$ on $X_{\infty}$ where with abuse of notation we have written $\vec{x}$ for $\vec{x} \circ \pi_{\infty,0}$.

### 3.3. Horizontal paths with jumps.

**Definition 3.44** (Horizontal paths). A horizontal segment $\sigma$ in $X_j$ ($j = \infty$ being admissible) is a geodesic segment such that $\pi_{j,0}(\sigma)$ is a segment of $X_0$ parallel to one of the coordinate axes of $[0, 1]^3$. We allow for a segment to be degenerate, i.e. to be just a point.

A horizontal path $\text{ph}$ in $X_j$ is a finite tuple $\text{ph} = (\sigma_1, \ldots, \sigma_N)$ of horizontal segments such that for $1 \leq i < N$ the end point of $\sigma_i$ is the starting point of $\sigma_{i+1}$. The length of $\text{ph}$ is the sum of the lengths of its segments:

$$(3.45) \quad \text{len}(\text{ph}) = \sum_{i=1}^{N} \text{len}(\sigma_i).$$

**Definition 3.46** (The set of total jump pairs). The set $\text{TJpp}(X_j)$ of total jump pairs of $X_j$ ($j = \infty$ being admissible) consists of all $\{q, q'\} \subset X_j$ such that:
Lemma 3.50

(Existence of good horizontal paths with jumps)

Let \( X \) be a fundamental configuration in \( C \) and \( \epsilon, r \) be a universal constant. Let \( \epsilon, r \) and \( \epsilon, r \) such that for each \( q \in F_{\text{Fund}} \), there is either a horizontal path \( \gamma = \gamma \) or a horizontal path with jumps \( \gamma = \gamma \) such that:

- \( (Gd1): \gamma \) starts at \( p_l \) and ends at \( q_l \).
- \( (Gd2): \) If \( \gamma = \gamma \) there is only one jump, i.e. \( \gamma = (p_{l-}, q_{l-}, p_{l+}, q_{l+}) \).
- \( (Gd3): \) \( \text{len}(\gamma) \leq C d_{X_j}(l, q_l) \).
- \( (Gd4): \) With the exception of at most 10 horizontal segments in \( \gamma \), for each other horizontal segment \( \sigma \) one has:

\[
\text{len}(\sigma) \geq \frac{\epsilon^3 r}{400}.
\]

(3.51)

- \( (Gd5): \gamma \) contains at most 15 horizontal segments.

Proof. The construction will be inductive; for \( j < l \) let \( p_j = \pi_{l,j}(p_l) \) and \( q_j = \pi_{l,j}(q_l) \).

Step 1: The construction in \( X_0 \) and \( X_1 \)

\( X_0 \) is just \([0,1]^3\) with the Euclidean metric and we know that for \( i \in \{1,2,3\} \)

either \( x'(p_0) = x'(q_0) \) or \( |x'(p_0) - x'(q_0)| \geq \epsilon^2 r \). We can then find a horizontal path \( \gamma_0 = \gamma_0 \) starting at \( p_0 \), ending at \( q_0 \) and satisfying the following conditions (henceforth referred to as \( (Inv1) \)):

- \( (Inv1:1): \gamma_0 \) consists of at most 3 horizontal paths and \( \text{len}(\gamma_0) \leq 3d_{X_0}(p_0, q_0) \).
- \( (Inv1:2): \) The length of each horizontal segment in \( \gamma_0 \) is \( \geq \epsilon^3 r \).

As \( \pi_{1,0} \) is open we can lift \( \gamma_0 \) to all lifts of \( \gamma_0 \) (i.e. horizontal paths \( \gamma_1 \) that are mapped to \( \gamma_0 \) by \( \pi_{1,0} \)) starting at \( p_1 \). If one such lift \( \gamma_1 \) ends at \( q_1 \) let \( \gamma_1 = \gamma_1 \) and note that it will satisfy \( (Inv1) \) (change the subscripts in \( (Inv1:1–2) \) from 0 to 1).

Assume that this does not happen. As in Construction 3.1 let \( K \) be the subcube that gets doubled and \( \{c_{\text{green}}, c_{\text{red}}\} \) the pair of jump points. Then \( p_1 \) and \( q_1 \) belong
to lifts of $K$ lying in chromatic sheets with different colors. Note that $\partial K$ can be regarded also as a subset of $X_1$ (the metrics $d_{X_1}$ and $d_{X_0}$ agree on it).

We first consider the case in which there is a $p_K \in \partial K$ such that:

$$d_{X_1}(p_1, p_K) + d_{X_1}(p_K, q_1) \leq 8d_{X_1}(p_1, q_1).$$

Consider the 6 quantities:

$$d_{X_1}(p_1, q_1) \geq \epsilon^3 r \text{ at most 5 of the above quantities can be } \leq \epsilon^3 r / 16 \text{ and thus we can find a horizontal path } \gamma_1 = \text{ph}_1 \text{ starting at } p_1 \text{ and ending at } q_1 \text{ and satisfying the following conditions (henceforth referred to as } \text{(Inv2)}):}$$

- **(Inv2:1):** $\gamma_1 = (\text{ph}_-, \text{ph}_+)$, $\text{ph}_-$ joins $p_1$ to $p_K$ and $\text{len}(\text{ph}_-) \leq 3d_{X_1}(p_1, p_K)$; $\text{ph}_+$ joins $p_K$ to $q_1$ and $\text{len}(\text{ph}_+) \leq 3d_{X_1}(p_K, q_1)$; as a consequence $\text{len}(\gamma_1) \leq 24d_{X_1}(p_1, q_1)$.

- **(Inv2:2):** With the exception of at most 5 horizontal segments in $\gamma_1$, for each other horizontal segment $\sigma$ one has $\text{len}(\sigma) \geq \epsilon^3 r / 16$.

If such a $p_K$ does not exist to change color we must use the jump pair $(c_{green}, c_{red})$; moreover, without loss of generality we can assume that:

$$d_{X_1}(p_1, c_{red}) + d_{X_1}(c_{red}, c_{green}) + d_{X_1}(c_{green}, q_1) = d_{X_1}(p_1, q_1).$$

We can then find a horizontal path with jumps $\gamma_1 = \text{jp}_1 = (\text{ph}_-, \text{jp}, \text{ph}_+)$ which satisfies the following conditions (henceforth referred to as **(Inv3)**):

- **(Inv3:1):** $\text{ph}_-$ joins $p_1$ to $c_{red}$, $\text{jp} = (c_{green}, c_{red})$, $\text{ph}_+$ joins $c_{green}$ to $q_1$; moreover $\text{len}(\text{ph}_-) \leq 3d_{X_1}(p_1, c_{red})$, $\text{len}(\text{ph}_+) \leq 3d_{X_1}(c_{green}, q_1)$ and thus $\text{len}(\gamma_1) \leq 3d_{X_1}(p_1, q_1)$.

- **(Inv3:2):** Except for at most 6 horizontal segments in $\gamma_1$, for each other horizontal segment $\sigma$ one has $\text{len}(\sigma) \geq \epsilon^3 r / 16$.

For the record we also note that:

$$d_{X_1}(p_1, q_1) \geq 4 \text{len}(X_1).$$

**Step 2: The construction in } X_2.**

Consider the lifts $\text{Lift}(\gamma_1)$ of $\gamma_1$ in $X_2$ starting at $p_2$. If one such lift $\tilde{\gamma}$ ends at $q_2$ let $\gamma_2 = \tilde{\gamma}$, which will satisfy the same of **(Inv1-3)** that $\gamma_1$ satisfied.

Suppose that there is no such a lift. Then there are $Q_1, Q_2 \in \text{ToDouble}(X_1)$ such that $p_1 \in Q_1$, $q_1 \in Q_2$. Let $K_1$ be the central subcube of $Q_1$, which is doubled in constructing $\text{Dym}_{K_1}(Q_i)$; then $p_1 \in K_1$ and $q_1 \in K_2$ and $p_2, q_2$ have different color labels at position 2. If $\gamma_1$ satisfied **(Inv2)** we should have crossed $\partial K_1$ following $\gamma_1$ and so we would have been able to find a lift that changed the second color label and ended at $q_2$. If $\gamma_1$ satisfied **(Inv3)**, to reach $c_{red}$ in the lift of $K$, we would again have crossed $\partial K_1$ and we would have been able to change the second color label, finding a lift ending at $q_2$. We thus conclude that $\gamma_1$ satisfies **(Inv1)**. Now we can essentially argue as before. One possibility is that there was a $p_K \in \partial K_1 \cup \partial K_2$ (now regarded as a subset of $X_2$) such that:

$$d_{X_2}(p_1, p_K) + d_{X_2}(p_K, q_1) \leq 8d_{X_2}(p_2, q_2).$$

In this case, as in **Step 1** we can produce a $\gamma_2$ starting at $p_2$ and ending at $q_2$ which satisfies **(Inv2)**. Otherwise, we can use either the lifts of the center of $K_1$ or $K_2$.
to change the second color label and argue as in Step 1 to produce a $\gamma_2$ satisfying (Inv3).

**Step 3:** The construction in $X_l$ for $2 < l < \infty$.

Consider the lifts $\text{Lift}(\gamma_{l-1})$ of $\gamma_{l-1}$ in $X_l$ starting at $p_l$. If one such a lift $\tilde{\gamma}$ ends at $q_l$, let $\gamma_l = \tilde{\gamma}$ which will satisfy the same (Inv1–3) that $\gamma_l$ satisfies.

Assume that this is not the case. Then there are $Q_1, Q_2 \in \text{ToDouble}(X_{l-1})$ such that, denoting by $K_1, K_2$ the central subcubes to be doubled, $p_{l-1} \in K_1$ and $q_{l-1} \in K_2$, and the color labels of $p_{l-1}$ and $q_{l-1}$ differ at the $l$-th position. Assume that $\gamma_{l-1}$ satisfied (Inv2) and that the transition from (Inv1) to (Inv2) (note that $\gamma_0$ will always satisfy (Inv1)) occurred at level $j$. Let then $K$ denote the central subcube in $X_{j-1}$ whose boundary was used by $\gamma_j$ to change the color label at position $j$. Then to reach $\partial K$ (which can be regarded as a subset of $X_{l-1}$) from $p_{l-1}$ we would have crossed $\partial K_1$, so we would have been able to lift $\gamma_{l-1}$ to end at $q_l$. Assume that $\gamma_{l-1}$ satisfied invariant (Inv3). Let $\{c_1, c_2\}$ denote the couple of jump points used in $\gamma_{l-1}$. Then there are two cases to consider. One is that $\{c_1, c_2\}$ can be lifted also to change the $l$-th color label. Recalling Step 2 of Construction 3.5 this can happen in constructing $X_{n_k+1}$ because all $Q \in \text{Cell}(X_{n_k})$ get replaced by $\text{Dym}_{n_k+1}(Q)$. If this is not the case, then to reach $c_1$ or $c_2$ we have to cross $\partial K_1 \cup \partial K_2$ and so can change the $l$-th color label. This implies that a lift of $\gamma_{l-1}$ ending at $q_l$ has to exist. We thus conclude that $\gamma_{l-1}$ has to satisfy (Inv1) and we can argue as in Step 2 finding $\gamma_l$ satisfying either (Inv2) or (Inv3).

**Step 4:** The case $l = \infty$.

For $l = \infty$ we use a limiting argument. Consider the sequence $\{\gamma_j\}$: after some $j_0$ all the $\gamma_j$’s have to satisfy the same (Inv1–3). Then for $j > j_0$, $\gamma_j$ is a lift of $\gamma_{j+1}$ and we can obtain $\gamma_\infty$ as the inverse limit of the $\{\gamma_j\}_{j \geq j_0}$.

\[ \square \]

### 3.4. Existence of disconnected tangents.

**Theorem 3.57** (Existence of disconnected tangents). At $\mu_{\infty}$-a.e. $p_\infty \in X_\infty$ there is a tangent/吹升 which is not topologically connected.

**Proof.** For each $p_\infty \in X_\infty$ and $l < \infty$ let $\pi_{\infty,l}(p_\infty) = p_l$. Recall that $\mu_\infty$ is a probability measure and define:

\[ E_k = \left\{ p_\infty : \text{for some } l \in \{n_k + 1, \ldots, n_{k+1}\} \text{ } p_l \in \text{Gates}(X_l) \right\}. \tag{3.58} \]

In Step 2 of Construction 3.5 we replaced each $Q \in \text{Cell}(X_{n_k+1})$ with $\text{Dym}_{n_k+1}(Q)$; this implies that the events $E_k$ and $E_{k+j}$ are independent for $j \geq 1$. Moreover, there is a universal constant $c > 0$ such that for each $k$ one has:

\[ \mu_{n_k+1}(\text{Gates}(X_{n_k+1})) \geq \frac{c}{n_{k+1}^3}; \tag{3.59} \]

consider now $l$ such that $n_k + 1 < l \leq n_{k+1}$: as in Construction 3.5 we do not apply the diamond construction to the gates at the previous levels, we have the estimate:

\[ \mu_l(\text{Gates}(X_l)) \geq \frac{c}{n_{k+1}^3} \mu_l \left( X_l \setminus \bigcup_{j = n_k + 1}^{l-1} \pi_{l,j}^{-1}(\text{Gates}(X_j)) \right). \tag{3.60} \]
We can therefore estimate the measure of $E_k$ from below:

$$\mu_\infty(E_k) \geq c \frac{n_{k+1}^{n_{k+1} - 1}}{n_{k+1}^3} \sum_{t=0}^{n_{k+1}^3} \left(1 - \frac{c}{n_{k+1}^3}\right)^t$$

(3.61)

$$= 1 - \left(1 - \frac{c}{n_{k+1}^3}\right)^{n_{k+1}^3}$$

$$\geq 1 - 2e^{-c},$$

for $k$ sufficiently large. Hence there is a uniform lower bound on $\mu_\infty(E_k)$ and by the Borel-Cantelli Lemma, for $\mu_\infty$-a.e. $p_\infty$ one has $p_\infty \in E_k$ infinitely often.

4. **Differentiability of real-valued Lipschitz maps**

In this section we prove differentiability for real-valued Lipschitz maps, Theorem 4.56. The emphasis is to get an estimate to control how fast a Lipschitz map will collapse together the points appearing in the jump part of the horizontal paths with jumps that we constructed in Section 3. The key estimate is given in Theorem 4.30; this result is based on taking recursive piecewise harmonic approximations of the function, Definition 4.20 and on some elementary PDE 4.6.

4.1. **Harmonic functions.**

**Lemma 4.1** (Lipschitz estimate for harmonic functions). Let $u : U \rightarrow \mathbb{R}$ or $l^2$ be harmonic where $U \subset \mathbb{R}^3$ is open. Assume that $B(p_0,r) \subset U$; then there is a universal constant $C$, independent of $u,U,p_0$ or $r$, such that $u$ is:

$$C \frac{1}{r} \|u\|_{L^1(B(p_0,r))}-Lipschitz$$

in $B(p_0,r/3)$.

**Proof.** The case where $u$ is real-valued is well-explained in [Eva98, Ch. 2, Thm. 7]; here we explain the minor modifications needed for $l^2$-valued harmonic functions. Let $u_j$ be a component of $u$ and let $p \in B(p_0,r/3)$; by the mean value property:

$$\partial_i u_j(p) = \frac{1}{B(p_0,r/3)} \int_{B(p_0,r/3)} \partial_i u_j \, d\mathcal{L}^3 = \frac{\alpha_3}{r} \int_{\partial B(p_0,r/3)} u_j \nu_i \, d\mathcal{H}^2,$$

(4.3)

where we used integration by parts, $\alpha_3$ denotes a universal constant and $\nu$ is the outer normal to $\partial B(p,r/3)$. Then:

$$\sum_{j=1}^{\infty} |\partial_i u_j(p)|^2 \leq \left(\frac{\alpha_3}{r}\right)^2 \sum_{j=1}^{\infty} \int_{\partial B(p,r/3)} |u_j|^2 \, d\mathcal{H}^2,$$

(4.4)
and so:

\[
\|\partial_i u(p)\|_2 \leq \frac{\alpha_3}{r} \max_{q \in \partial B(p, r/3)} \|u(q)\|_2;
\]

then (4.2) follows applying the mean-value property to \(q \in B(q, r/3) \subseteq B(p_0, r)\).

\[\square\]

**Lemma 4.6** (Lower bound on the energy). Let \(Q\) be a cube with sidelength \(\text{slen}(Q)\) and for \(s \in (0, \frac{\text{slen}(Q)}{6}]\) let \(sQ\) denote the cube with the same center as \(Q\) and with sidelength \(s\). Assume that \(F : Q \setminus sQ \rightarrow \mathbb{R}\) is continuous and locally Lipschitz in \(Q \setminus sQ\); for each compact \(K\) contained in the interior of \(Q \setminus sQ\) the restriction \(F|K\) is Lipschitz. Then there is a universal constant \(c_{H\text{ar}}\) (independent of \(Q\) and \(s\)) such that if:

\[
\left| \int_{\partial(sQ)} F \, d\mathcal{H}^2 \right| \geq \eta
\]

and \(F = 0\) on \(\partial Q\) then:

\[
\int_{Q \setminus sQ} \|
abla F\|_2^2 \, d\mathcal{L}^3 \geq c_{H\text{ar}} \eta^2 s.
\]

**Proof.** Step 1: Reducing the problem to balls.

Up to a translation we can assume that \(Q\) is centered at 0. Let:

\[
\Psi : Q \rightarrow B(0, \frac{\text{slen}(Q)}{2})
\]

\[
x \mapsto \frac{\|x\|_\infty}{\|x\|_2} x.
\]

We compute \(d\Psi\) at a generic point \(x\) where \(x_1 \neq x_2 \neq x_3\) and where \(\|x\|_\infty = |x_1|\):

\[
\partial_i \Psi = \frac{\|x\|_\infty}{\|x\|_2} \left( \delta_{ij} - \frac{x_i x_j}{\|x\|_2^2} \right) + \frac{\text{sgn}(x_1) \chi_{j=1}}{\|x\|_2} x_i;
\]

on the one hand:

\[
\left| \left\langle d\Psi(x), \frac{x}{\|x\|_2} \right\rangle \right| = \left\| \frac{\text{sgn}(x_1)}{\|x_2\|_2^2} x_1(x_1, x_2, x_3) \right\|_2 \geq \frac{1}{3};
\]

on the other hand if \(v\) is a unit vector orthogonal to \(x\):

\[
\left| \langle d\Psi(x), v \rangle \right| = \frac{\|x\|_\infty}{\|x\|_2} \|v\|_2 \geq \frac{1}{3}.
\]

and thus

\[
\left| \langle d\Psi(x), v \rangle \right| \geq 1.
\]

We conclude that \(\Psi\) is \((1/16, 16)\)-bi-Lipschitz and maps \(Q\) onto \(B(0, \frac{\text{slen}(Q)}{2})\), \(sQ\) onto \(B(0, \frac{s}{2})\), \(\partial Q\) onto \(\partial B(0, \frac{\text{slen}(Q)}{2})\) and \(\partial(sQ)\) onto \(\partial B(0, \frac{s}{2})\). We can thus reduce to the case in which \(F : B(0, \frac{\text{slen}(Q)}{2}) \rightarrow \mathbb{R}, F = 0\) for \(r = \frac{\text{slen}(Q)}{2}\) (we are using polar coordinates) and

\[
\int_{\partial B(0, \frac{s}{2})} F \, d\mathcal{H}^2 \geq \eta,
\]

up to changing the sign of \(F\) and by replacing the original \(\eta\) with \(\eta/16^4\).

**Step 2:** Symmetrization
Let $\omega \in S^2$ and define:

\[(4.15) \quad \tilde{F}(r, \omega) = \tilde{F}(r) = \int_{S^2} F(r, \tilde{\omega}) \, dH^2(\tilde{\omega}).\]

As $F$ is locally Lipschitz in $B(0, \frac{s \text{len}(Q)}{2}) \setminus B(0, \frac{\tilde{s}}{2})$, so is $\tilde{F}$. We show that $\tilde{F}$ has lower energy than $F$ in $B(0, \frac{s \text{len}(Q)}{2}) \setminus B(0, \frac{\tilde{s}}{2})$, so it suffices to bound the energy of $\tilde{F}$ from below:

\[(4.16) \quad \int_{B(0, \frac{s \text{len}(Q)}{2}) \setminus B(0, \frac{\tilde{s}}{2})} \| \nabla \tilde{F}(r, \omega) \|^2 r^2 \, dr \, d\omega = \int_{s/2}^{\text{len}(Q)/2} r^2 dr \int_{S^2} \int_{S^2} (\partial_r F(r, \tilde{\omega}))^2 d\omega \leq \int_{s/2}^{\text{len}(Q)/2} r^2 dr \int_{S^2} \int_{S^2} (\partial_r F(r, \tilde{\omega}))^2 d\omega.

Note also that $\tilde{F} = 0$ for $r = \frac{s \text{len}(Q)}{2}$ and $\tilde{F} = a \geq \eta$ for $r = \frac{s}{2}$.

**Step 3: Comparison with a harmonic function.**

The minimum energy will be attained by the harmonic function with the same boundary conditions as $\tilde{F}$: the general solution is of the form $A/r + B$ and we get:

\[(4.17) \quad A = \frac{as}{2(\text{len}(Q) - s)} \text{len}(Q), \quad B = -\frac{2A}{\text{len}(Q)}.\]

We can then compute the energy of this function as follows:

\[(4.18) \quad 4\pi \int_{s/2}^{\text{len}(Q)/2} \left( \frac{A}{r^2} \right)^2 r^2 dr = 4\pi A^2 \left[ -\frac{1}{r} \right]_{r=s/2}^{\text{len}(Q)/2} = 4\pi A^2 \frac{2(\text{len}(Q) - s)}{s \text{len}(Q)} = \pi a^2 s \frac{\text{len}(Q)}{\text{len}(Q) - s} \geq \eta^2 s.\]

**Remark 4.19.** For the proof of Lemma 4.6 we made the simplest assumption of $F$ being locally Lipschitz; one might have made a more general one to run the same argument, say assuming the $F$ belonged to the Sobolev space $W^{1,2}(Q \setminus sQ)$ and extended continuously to $Q \setminus sQ$.

**Definition 4.20 (Piecewise harmonic approximations).** We define the **2-skeleton** of $X_j$ ($j < \infty$) as:

\[(4.21) \quad \text{SK}_2(X_j) = \bigcup_{Q \in \text{Cell}(X_j)} \partial Q;\]

note that for $l \geq j$ ($l = \infty$ being admissible) $\text{SK}_2(X_j)$ embeds isometrically in $X_l$. 
We define the 2-harmonic skeleton of $X_j$ ($j < \infty$) as:

\[(4.22) \quad \text{HSK}_2(X_j) = \text{SK}_2(X_{j-1}) \cup \bigcup_{Q \in \text{Gates}(X_j)} \partial Q; \]

note that for $l \geq j$ ($l = \infty$ being admissible) HSK$_2(X_j)$ embeds isometrically in $X_l$.

Let $f : X_{\infty} \rightarrow \mathbb{R}$ be Lipschitz. We define the piecewise harmonic approximations of $f$ as follows. For $j \geq 0$ let $G_j(f) : X_j \rightarrow \mathbb{R}$ be the piecewise harmonic function which is harmonic inside each cell of $X_j$ and agrees with $f$ on SK$_2(X_j)$.

For $j \geq 1$ let $H_j(f) : X_j \rightarrow \mathbb{R}$ be the piecewise harmonic function which agrees with $f$ on HSK$_2(X_j)$ and such that:

1. For $Q \in \text{ToDouble}(X_{j-1})$ let $K_{\text{green}}$ and $K_{\text{red}}$ be the lifts of $K_Q$ in $\text{Dym}_{n=n(j)}(Q)$ and $\tilde{G}_{\text{green}}, \tilde{G}_{\text{red}}$ the corresponding gates; set $Q_{\text{green}} = Q \setminus K_Q \cup K_{\tilde{g}_{\text{green}}} \setminus \tilde{G}_{\text{green}}$ and $Q_{\text{red}} = Q \setminus K_Q \cup K_{\tilde{g}_{\text{red}}} \setminus \tilde{G}_{\text{red}}$. Then $H_j(f)$ is harmonic in the interior of $Q_{\text{green}} \cup Q_{\text{red}}$ and agrees with $f$ on $\partial Q \cup \partial \tilde{G}_{\text{green}} \cup \partial \tilde{G}_{\text{red}}$.

2. For each $Q \in \text{Gates}(X_j)$ $H_j(f)$ is harmonic in the interior of $Q$.

3. Each $Q \in \text{Subdiv}(X_{j-1})$ gets isometrically lifted in $X_j$ and $H_j(f)$ is harmonic in the interior of $Q$ and agrees with $f$ on $\partial Q$.

We prove that $H_j(f)$ and $G_j(f)$ are continuous and have distributional derivatives in $L^2$.

**Lemma 4.23** (Regularity of piecewise harmonic approximations). The function $H_j(f)$ and $G_j(f)$ are continuous, are in $W^{1,2}(X_j, \mu_j)$ and satisfy the energy bounds

\[(4.24) \quad \sup_j \left\{ \int_{X_j} \|\nabla H_j(f)\|_2^2 \, d\mu_j, \int_{X_j} \|\nabla G_j(f)\|_2^2 \, d\mu_j \right\} \leq (L(f))^2. \]

**Proof.** We are gluing functions which are harmonic in the interia of the cells of $X_j$ using compatible boundary conditions. Thus continuity and membership in $W^{1,2}(X_j, \mu_j)$ follow if we show that the problem:

\[(4.25) \quad \begin{cases} \Delta h = 0 & \text{in } \Omega \hfill \\
 h = G & \text{on } \partial \Omega \end{cases} \]

where $\Omega$ is either a cube or a cube with an inner smaller cube with the same center removed (a “cubular annulus”) and where $G/\partial \Omega$ is Lipschitz which is $C^0(\Omega)$, i.e. it is continuous up to the boundary. We use Perron’s Method [GT01, Sec. 2.8]; the desired solution exists as $\Omega$ has the exterior cone property: for each $p \in \partial \Omega$ there is a small cone:

\[(4.26) \quad C_p = \{ p + v : \|v\|_2 \leq r_p, \angle(v, e_p) \leq \alpha_p \} \]

such that $B(p, r_p/2) \cap \Omega \subset B(p, r_p/2) \setminus C_p$ and $C_p \cap \partial \Omega = \{p\}$. Then one has to construct [GT01, Ex. 2.12] a local barrier $w_p$ at $p$:

1. $w_p$ is superharmonic in $\Omega \cap B(p, s)$ for $s > 0$; here we will content with $w_p$ harmonic.

2. $w_p > 0$ in $(\Omega \setminus \{p\}) \cap B(p, s)$ and $w_p(p) = 0$.

We set up a spherical coordinate system $(r, \theta, \phi)$ with origin at $p$ and angle $\theta = 0$ opposite to $e_p$; the Laplacian is given by:

\[(4.27) \quad \Delta = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial}{\partial r}) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta}). \]
We look for $w_p$ harmonic with ansatz $w_p = r^\lambda f(\theta)$ where $\lambda > 0$; we get the Legendre ODE:

\begin{equation}
(4.28) \quad f''(\theta) + \cot \theta f'(\theta) + \lambda(\lambda + 1)f(\theta) = 0,
\end{equation}

and look for $f(\theta) > 0$ when $\theta \in [0, \pi - \alpha_p]$. As solution we choose $f_\lambda(\theta) = P^0_\lambda(\cos \theta)$ where $P^0_\lambda$ is the Legendre function of the first kind (see [dmlf.nist.gov/14]). As $\lambda \searrow 0$ $P^0_\lambda(\cos \theta)$ converges to the constant function $1$, moreover, $P^0_\lambda(\cos \theta)$ is monotonically decreasing from 1 for $\theta = 0$ to $-\infty$ for $\theta = \pi$; depending on $\alpha_p$, $\lambda > 0$ can thus be taken sufficiently small to ensure $f_\lambda(\theta) > 0$ in the desired range of $\theta$.

For further reference we also record the rate of convergence of $h(x)$ to $G(p)$ as $x \to p$, given the barrier $w_p$; for $\|x - p\|_2 \leq \delta$, from [GT01, Lem. 2.13]:

\begin{equation}
(4.29) \quad |h(x) - G(p)| \leq L(G)\|x - p\|_2 + \frac{2\|G\|_\infty}{\inf_{\|y-p\|_2 > \delta} w_p(y)} w_p(x).
\end{equation}

In particular, as we can translate and rotate the same barrier $w_p$ at the different points of $\partial \Omega$ we get a uniform estimate on the convergence of $h$ to $G$.

Finally, harmonic functions minimize the $l^2$-energy in the class of functions satisfying their boundary conditions. We could then have taken a MacShan extension $f$ of $f|\text{SK}_2(X_j)$ or $f|\text{HSK}_2(X_j)$ and thus (4.24) follows. \qed

**Theorem 4.30** (Rate of collapse of the gates). Let $f : X_\infty \to \mathbb{R}$ be Lipschitz. Let $Q \in \text{ToDouble}(X_{l-1})$ where $n_k + 1 \leq l \leq n_{k+1}$ and let $G_{\text{green}}$, $G_{\text{red}}$ be the two gates of $\text{Dym}_{n_{k+1}}(Q)$. Having fixed $\varepsilon > 0$ we say that $Q$ is **bad** and write $Q \in \text{Bad}(X_{l-1})$ if:

\begin{equation}
(4.31) \quad \left| \int_{\partial G_{\text{green}}} f \, d\mathcal{H}^2 - \int_{\partial G_{\text{red}}} f \, d\mathcal{H}^2 \right| \geq \frac{\varepsilon}{256 \times n_{k+1}} \text{diam}(Q).
\end{equation}

Then there is a universal constant $C$, independent of $f$ and $\varepsilon$ such that we have the following estimate on the measure of the bad cubes:

\begin{equation}
(4.32) \quad \sum_{k \geq 1} \sum_{l = n_k + 1}^{\hat{n}_{k+1}} \sum_{Q \in \text{Bad}(X_{l-1})} \frac{\varepsilon^2}{n_{k+1}^3} \mu_{l-1}(Q) \leq C \times (L(f))^2.
\end{equation}

**Proof.** Step 1: Orthogonality relations.

We show that:

\begin{equation}
(4.33) \quad \int_{X_j} (\nabla H_j(f) - \nabla G_j(f)) \cdot \nabla H_j(f) \, d\mu_j = 0.
\end{equation}

Let $Q \in \text{Cell}(X_{j-1})$ and in the first case assume that $Q \in \text{ToDouble}(X_{j-1})$. Let $K_Q$ be the central subcube to be doubled in passing from $Q$ to $\text{Dym}_{n=n(j)}(Q)$ and $K_{\text{red}}$, $K_{\text{green}}$ the two copies of $K_Q$ and $G_{\text{green}}$, $G_{\text{red}}$ the corresponding gates. Write:

\begin{equation}
(4.34) \quad \text{Dym}_{n=n(j)}(Q) = Q \cup K_Q \cup K_{\text{green}} \cup K_{\text{red}} \cup G_{\text{green}} \cup G_{\text{red}};
\end{equation}

define $A_{\text{green}} = Q \setminus K_Q \cup K_{\text{green}} \setminus G_{\text{green}}$ and $A_{\text{red}} = Q \setminus K_Q \cup K_{\text{red}} \setminus G_{\text{red}}$. As both $H_j(f)$ and $G_j(f)$ are harmonic in the interior of $G_{\text{green}} \cup G_{\text{red}}$ and agree on $\partial G_{\text{green}} \cup \partial G_{\text{red}}$, we have:

\begin{equation}
(4.35) \quad \int_{G_{\text{green}} \cup G_{\text{red}}} (\nabla H_j(f) - \nabla G_j(f)) \cdot \nabla H_j(f) \, d\mu_j = 0,
\end{equation}

minding that harmonic functions are determined by their boundary conditions.
Fix a smooth harmonic function $\phi$ on $A\text{green}$ which can be extended to be smooth on a neighborhood of $A\text{green}$. By Lemma 4.23 $G_j(f)$ and $H_j(f)$ are in $W^{1,2}(X_j, \mu_j)$ and so we can integrate by parts; as $Q \setminus K_Q$ appears both in $A\text{green}$ and $A\text{red}$ we halve $\mu_j$ on it, getting $\tilde{\mu}_j$ which is just a constant multiple of Lebesgue measure on the whole of $A\text{green}$; denoting by $\partial \nu$ the normal derivative on the boundary we get:

$$
\int_{A\text{green}} (\nabla H_j(f) - \nabla G_j(f)) \cdot \nabla \phi d\tilde{\mu}_j = - \int_{A\text{green}} (H_j(f) - G_j(f)) \Delta \phi d\tilde{\mu}_j + \frac{\tilde{\mu}_j(A\text{green})}{L^3(A\text{green})} \int_{\partial A\text{green}} (H_j(f) - G_j(f)) \partial \nu \phi d\mathcal{H}^2.
$$

As $\Delta \phi = 0$ in $A\text{green}$ and $H_j(f) = G_j(f)$ on $\partial A\text{green}$ we conclude that:

$$
\int_{A\text{green}} (\nabla H_j(f) - \nabla G_j(f)) \cdot \nabla \phi d\tilde{\mu}_j = 0.
$$

We enlarge the “cubical annulus” $A\text{green}$ to a slightly larger cubical annulus $A\text{green},\varepsilon$ which lies in a $(6\varepsilon)$-neighborhood of $A\text{green}$. We then choose $f_\varepsilon : \partial A\text{green},\varepsilon \rightarrow \mathbb{R}$ to be 1-Lipschitz on $\partial A\text{green},\varepsilon$ and such that the graphs of $f_\varepsilon \partial A\text{green}$ and $f_\varepsilon$ are at Hausdorff distance $\leq 150\varepsilon$. Let $\phi_\varepsilon$ be the harmonic function defined on $A\text{green},\varepsilon$ which agrees with $\tilde{\phi}$ on $\partial A\text{green},\varepsilon$. By the barrier estimate (4.29) and the maximum principle $\phi_\varepsilon \rightarrow H_j(f)$ uniformly on $A\text{green}$ as $\varepsilon \rightarrow 0$ (here in $A\text{green}$ we also include its boundary). Moreover, as the boundary conditions are 1-Lipschitz, the family $\{\nabla \phi_\varepsilon\}_\varepsilon$ is bounded in $L^2(A\text{green}, \tilde{\mu}_j)$ and we conclude that $\nabla \phi_\varepsilon$ converges weakly to $\nabla H_j(f)$ in $L^2(A\text{green}, \tilde{\mu}_j)$ as $\varepsilon \rightarrow 0$. Thus, as each $\phi_\varepsilon$ satisfies the orthogonality (4.37), we get:

$$
\int_{A\text{green}} (\nabla H_j(f) - \nabla G_j(f)) \cdot \nabla H_j(f) d\tilde{\mu}_j = 0,
$$

and similarly:

$$
\int_{A\text{red}} (\nabla H_j(f) - \nabla G_j(f)) \cdot \nabla H_j(f) d\tilde{\mu}_j = 0.
$$

Putting together (4.35), (4.37) and (4.39) we obtain:

$$
\int_{\pi_{j-1}^{-1}(Q)} (\nabla H_j(f) - \nabla G_j(f)) \cdot \nabla H_j(f) d\mu_j = 0.
$$

The second case is when $Q \in \text{Subdiv}(X_{j-1})$; here $Q$ just gets isometrically lifted in $X_j$ and by construction $H_j(f)$ is harmonic in its interior while $H_j(f)$ and $G_j(f)$ agree on $\partial Q$. Using again the integration by parts argument and the smoothing of $H_j(f)$ via $\phi_\varepsilon$ we conclude that:

$$
\int_Q (\nabla H_j(f) - \nabla G_j(f)) \cdot \nabla H_j(f) d\mu_j = 0.
$$

Combining (4.40) and (4.41) (4.33) follows.

We now show that:

$$
\int_{X_j} \nabla(G_{j-1}(f) \circ \pi_{j,j-1}) \cdot (\nabla H_j(f) - \nabla(G_{j-1}(f) \circ \pi_{j,j-1})) d\mu_j = 0.
$$
Let $Q \in \text{Cell}(X_{j-1})$ and consider any lift $\tilde{Q}$ of $Q$ in $X_j$ (if $Q \in \text{ToDouble}(X_{j-1})$ there are two such lifts, a green and a red one, otherwise there is just one). Now $H_j(f)$ and $G_{j-1}(f) \circ \pi_{j,j-1}$ agree on $\partial \tilde{Q} = \partial Q$; let $\phi$ be a smooth harmonic function defined on $\tilde{Q}$ that can be extended to be smooth on a neighborhood of $\tilde{Q}$. By Lemma 4.23 $H_j(f)$ and $G_{j-1}(f) \circ \pi_{j,j-1}$ are in $W^{1,2}(X_j, \mu_j)$ and thus we can integrate by parts:

$$
\int_{\tilde{Q}} \nabla \phi \cdot (\nabla H_j(f) - \nabla (G_{j-1}(f) \circ \pi_{j,j-1})) \, d\mathcal{L}^3
= \int_{\partial \tilde{Q}} (H_j(f) - G_{j-1}(f) \circ \pi_{j,j-1}) \partial_\nu \phi \, d\mathcal{H}^2
- \int_{\tilde{Q}} (H_j(f) - G_{j-1}(f) \circ \pi_{j,j-1}) \Delta \phi \, d\mathcal{L}^3 = 0.
$$

Now enlarge $\tilde{Q}$ to a slightly larger cube $\tilde{Q}_\varepsilon$ contained in the $(6\varepsilon)$-neighborhood of $\tilde{Q}$. We then choose $f_\varepsilon : \partial \tilde{Q}_\varepsilon \to \mathbb{R}$ to be $1$-Lipschitz on $\partial \tilde{Q}_\varepsilon$ and such that the graphs of $f_\varepsilon$ and $f_\varepsilon$ are at Hausdorff distance $\leq 150 \varepsilon$. Let $\phi_\varepsilon$ be the harmonic function on $\tilde{Q}_\varepsilon$ which equals $f_\varepsilon$ on $\partial \tilde{Q}_\varepsilon$. By the barrier estimate (4.29) and the maximum principle $f_\varepsilon \to G_{j-1}(f) \circ \pi_{j,j-1}$ uniformly on $\tilde{Q}$ as $\varepsilon \to 0$ (here in $\tilde{Q}$ we also include its boundary). Moreover, as the boundary conditions are $1$-Lipschitz, the family $\{\nabla \phi_\varepsilon\}_\varepsilon$ is bounded in $L^2(\tilde{Q}, \mathcal{L}^3)$ and we conclude that $\nabla \phi_\varepsilon$ converges weakly to $\nabla G_{j-1}(f) \circ \pi_{j,j-1}$ in $L^2(\tilde{Q}, \mathcal{L}^3)$ as $\varepsilon \to 0$. Thus, as each $\phi_\varepsilon$ satisfies (4.43), we conclude that:

$$
\int_{\tilde{Q}} (\nabla G_{j-1}(f) \circ \pi_{j,j-1}) \cdot (\nabla H_j(f) - \nabla (G_{j-1}(f) \circ \pi_{j,j-1})) \, d\mathcal{L}^3 = 0.
$$

Minding that $\mu_j$ is a constant multiple of Lebesgue measure on $\pi_{j,j-1}^{-1}(Q)$ for $Q \in \text{Subdiv}(X_{j-1})$ and splitting $\mu_{j-1}$ in half on the two lifts of $Q \in \text{ToDouble}(X_{j-1})$ we get that (4.42) follows from (4.44).

**Step 2: The telescopic series.**

For $\psi : X_j \to \mathbb{R}$ define:

$$
E[\psi] = \int_{X_j} \|\nabla \psi\|^2 \, d\mu_j.
$$

Consider the telescopic series:

$$
\sum_{j=n+1}^{\infty} \left\{ E[G_j(f)] - E[H_j(f)] + E[H_j(f)] - E[G_{j-1}(f)] \right\} = \lim_{n \to \infty} \left( E[G_n(f)] - E[G_{n+1}(f)] \right)
\leq (L(f))^2.
$$

Now

$$
E[G_j(f)] = \int_{X_j} \|\nabla G_j(f) - \nabla H_j(f) + \nabla H_j(f)\|^2 \, d\mu_j
$$

and using the orthogonality relation (4.33) we get:

$$
E[G_j(f)] = E[H_j(f)] + \int_{X_j} \|\nabla G_j(f) - \nabla H_j(f)\|^2 \, d\mu_j
\geq E[H_j(f)].
$$
Similarly, using the orthogonality relation (4.42) and that $E[G_{j-1}(f)] = E[G_{j-1}(f) \circ \pi_{j,j-1}]$ we get:

\[(4.49) \quad E[H_j(f)] - E[G_{j-1}(f)] = \int_{X_j} \|\nabla H_j(f) - \nabla G_{j-1}(f) \circ \pi_{j,j-1}\|^2 d\mu_j.\]

Therefore from (4.46) we have:

\[(4.50) \quad \sum_{k \geq 1} \sum_{l=n_k+1}^{n_{k+1}} \sum_{Q \in \text{Cell}(X_{l-1})} \mu_{l-1}(Q) \int_{\pi_{l,j}(Q)} \|\nabla H_j(f) - \nabla G_{l-1}(f) \circ \pi_{l,l-1}\|^2 d\mu_l \leq (L(f))^2.\]

**Step 3: Application of Lemma 4.6.**

Assume that $Q \in \text{Bad}(X_{l-1})$ and as in Step 1 write $\text{Dym}_{n_{k+1}}(Q) = A_{\text{green}} \cup A_{\text{red}} \cup G_{\text{green}} \cup G_{\text{red}}$. Let $F = H_j(f) - G_{j-1}(f) \circ \pi_{j,j-1}$, which is locally Lipschitz in the interior of $A_{\text{green}} \cup A_{\text{red}}$ and such that $F = 0$ on $\partial Q$. As $Q$ is bad:

\[(4.51) \quad \left| \int_{\partial Q_{\text{green}}} f \, d\mathcal{L}^2 - \int_{\partial Q_{\text{red}}} f \, d\mathcal{L}^2 \right| \geq \frac{\varepsilon}{256 \times n_{k+1}} \text{diam}(Q);\]

however, $\int_{\partial Q_{\text{green}}} G_{j-1}(f) \circ \pi_{j,j-1} = \int_{\partial Q_{\text{red}}} G_{j-1}(f) \circ \pi_{j,j-1}$ and $H_j(f) = f$ on $G_{\text{green}} \cup G_{\text{red}}$, and so at least one of the following two must hold

\[(4.52) \quad \int_{\partial Q_{\text{green}}} F \, d\mathcal{L}^2 \geq \frac{\varepsilon}{512 \times n_{k+1}} \text{diam}(Q),\]

\[(4.53) \quad \int_{\partial Q_{\text{red}}} F \, d\mathcal{L}^2 \geq \frac{\varepsilon}{512 \times n_{k+1}} \text{diam}(Q).\]

Without loss of generality assume that (4.52) holds and apply Lemma 4.6 to $Q_{\text{green}} = A_{\text{green}} \cup G_{\text{green}}$ with $\eta = \frac{\varepsilon}{512 \times n_{k+1}} \text{diam}(Q)$ and $s \simeq \text{diam}(Q)/n_{k+1}$ ($\simeq$ implies a uniform constant). We thus have:

\[(4.54) \quad \int_{A_{\text{green}}} \|\nabla F\|^2 \, d\mathcal{L}^3 \geq \frac{c\varepsilon^2}{n_{k+1}^3} (\text{diam } Q)^3\]

where $c$ is a universal constant independent of $\varepsilon$, $k$ and $l$. As $\mu_l$ is doubling and a constant multiple of Lebesgue measure on each cell of $X_l$, we can deflate $c$ to get:

\[(4.55) \quad \int_{\pi_{l,l-1}(Q)} \|\nabla F\|^2 \, d\mu_l \geq \frac{c\varepsilon^2}{n_{k+1}^3}.\]

Plugging (4.55) in (4.50) finishes the proof. \qed

4.2. The proof of differentiability.

**Theorem 4.56 (Differentiation of real-valued Lipschitz functions).** Let $f : X_\infty \to \mathbb{R}$ be Lipschitz and $\nabla f$ its horizontal gradient. Then $f$ is differentiable $\mu_\infty$-a.e. with derivative $\nabla f$: i.e. for $\mu_\infty$-a.e. $p$ one has:

\[(4.57) \quad \text{Lip}(f - \langle \nabla f(p), \vec{x} \rangle) (p) = 0.\]

**Proof.** Step 1: Reduction to a constant derivative. Fix $\varepsilon > 0$ and write $X_\infty = \Omega \cup \bigcup_{i=1}^{\infty} K_i$, where:

1. $\mu_\infty(\Omega) = 0$. 

\[\text{(1) } \mu_\infty(\Omega) = 0.\]
(2) Each K_i is compact with \( \mu_\infty(K_i) > 0 \) and there is a \( c_t \in \mathbb{R}^3 \) such that:

\[
\sup_{p \in K_i} \| \nabla f(p) - c_t \|_2 \leq \varepsilon.
\]

Fix one index \( t \) and drop it from the notation; let \( F = f - \langle c, \vec{x} \rangle \), which is \((5\mathbf{L}(f))\)-Lipschitz. Assume that we are able to find a universal \( C > 0 \) independent of \( K = K_i \) and \( \varepsilon \) such that whenever \( S \subset K \) is compact with \( \mu_\infty(S) > 0 \) one can find \( \tilde{S} \subset S \) with \( \mu_\infty(\tilde{S}) > 0 \) and

\[
\text{Lip}F \leq C \varepsilon \quad \text{on } \tilde{S}.
\]

Then an exhaustion argument and letting \( \varepsilon \downarrow 0 \) will yield (4.57).

**Step 2: Avoiding bad gates.**

Fix \( S \subset K \) with \( \mu_\infty(S) > 0 \), our goal being to prove (4.59), which is to be accomplished in **Step 6**. Given \( Q \in \text{ToDouble}(X_{t-1}) \) let \( G_{\text{green}}(Q), G_{\text{red}}(Q) \) be the corresponding pair of gates and \( G_{\varepsilon}(Q) \) denote the \( \frac{600 \text{slen}(G_{\text{green}}(Q))}{\varepsilon} \) neighborhood of \( G_{\text{green}}(Q) \cup G_{\text{red}}(Q) \). By Theorem 4.30 we have:

\[
\sum_{k \geq 1} \sum_{l=\bar{n}_k+1}^{\bar{n}_{k+1}} \sum_{Q \in \text{Bad}(X_{t-1})} \mu_l(G_{\varepsilon}(Q)) < \infty;
\]

as the \( \mu_l \) are uniformly doubling (Lemma 3.36) we also have:

\[
\sum_{k \geq 1} \sum_{l=\bar{n}_k+1}^{\bar{n}_{k+1}} \sum_{Q \in \text{Bad}(X_{t-1})} \mu_l(G_{\varepsilon}(Q)) < \infty;
\]

in particular, we can find a \( k_0 = k_0(F, S, \varepsilon) \) such that:

\[
\sum_{k \geq k_0} \sum_{l=\bar{n}_k+1}^{\bar{n}_{k+1}} \sum_{Q \in \text{Bad}(X_{t-1})} \mu_l(G_{\varepsilon}(Q)) \leq \frac{\mu_\infty(S)}{5}.
\]

Let:

\[
X_{\text{bad}} = \bigcup_{k \geq k_0} \bigcup_{l=\bar{n}_k+1}^{\bar{n}_{k+1}} \bigcup_{Q \in \text{Bad}(X_{t-1})} \pi_{\infty, l}^{-1}(G_{\varepsilon}(Q))
\]

and note that \( \mu_\infty(S \setminus X_{\text{bad}}) > 0 \); thus let \( \tilde{S} \subset S \) consist of those Lebesgue density points of \( S \setminus X_{\text{bad}} \) which are also approximate continuity points of \( \nabla f \) and hence of \( \nabla F \).

Pick \( p \in \tilde{S} \) and for \( r > 0 \) let \( \text{Fund}(p, \varepsilon, r) \) be a fundamental configuration at \( p \) at scale \( r \) and resolution \( \varepsilon \). To each \( q \in \text{Fund}(p, \varepsilon, r) \) associate a horizontal path (possibly with one jump) \( \gamma_q \) as in Lemma 3.50. We say that \( q \) is **bad** if:

1. \( \gamma_q \) is of the form \( jph = (ph_-, jp, ph_+) \) with \( \text{Gates}(jp) = G_{\text{green}}(Q) \cup G_{\text{red}}(Q) \) for \( Q \in \text{Bad}(X_{t-1}) \) where \( l \in \{ \bar{n}_k + 1, \cdots, \bar{n}_{k+1} \} \).
2. Letting \( jp = (c_1, c_2) \) either \( k < k_0 \) or \( d_{X_\infty}(c_1, c_2) > \varepsilon d_{X_\infty}(p, q) \).

We now argue that we can find \( r_0 = r_0(\varepsilon) > 0 \) such that if \( r \leq r_0 \) then there is no such bad \( q \). For one thing:

\[
d_{X_\infty}(p, q) \geq d_{X_\infty}(c_1, c_2) \simeq \frac{\text{slen}(X_{t-1})}{n_{k+1}},
\]

thus choosing \( r_0 \) sufficiently small we can guarantee \( k \geq k_0 \). Secondly,

\[
d_{X_\infty}(c_1, c_2) > \varepsilon d_{X_\infty}(p, q)
\]
would imply \( p \in G_x(Q) \) contradicting the definition of \( \tilde{S} \). In the following let \( r \leq r_0 \), pick any \( q \in \text{Fund}(p, \varepsilon, r) \) and let \( \gamma = \gamma_q \).

**Step 3:** \( \gamma \) is a horizontal path.

Let \( \gamma = (\sigma_1, \sigma_2, \cdots, \sigma_s) \) and let \( \text{dom} \sigma_i \) denote the domain of \( \sigma_i \); let \( \sigma_i(\text{end}) \) and \( \sigma_i(\text{sta}) \) denote the end and the starting point of \( \sigma_i \).

As \( p \) is an approximate continuity point of \( \nabla F \), we can find \( r_1 = r_1(\varepsilon) \leq r_0 \) such that for each \( \sigma_i \) with \( \text{len}(\sigma_i) \geq \varepsilon^3 r/400 \) there is another horizontal segment \( \tilde{\sigma}_i \) satisfying:

1. \( \pi_{\infty, 0} \circ \tilde{\sigma}_i \) and \( \pi_{\infty, 0} \circ \sigma_i \) are parallel to the same axis.
2. \( \tilde{\sigma}_i \) has the same domain as \( \sigma_i \) and:
   \[
   d_{X_{\infty}}(\tilde{\sigma}_i(\text{sta}), \sigma_i(\text{sta})) \leq 3\varepsilon^3 r
   \]
   \[
   d_{X_{\infty}}(\tilde{\sigma}_i(\text{end}), \sigma_i(\text{end})) \leq 3\varepsilon^3 r
   \]
   \[
   \text{len}(\tilde{\sigma}_i) = \text{len}(\sigma_i).
   \]

3. \( \int_{\text{dom}(\tilde{\sigma}_i)} \|\nabla F\|_2 \circ \tilde{\sigma}_i \, d\mathcal{L}^1 \leq 2\varepsilon \text{len}(\tilde{\sigma}_i) \).

Then:

\[
(4.67) \quad |F(p) - F(q)| \leq \sum_{i=1}^s |F(\sigma_i(\text{end})) - F(\sigma_i(\text{sta}))|
\]

\[
\leq \sum_{\sigma_i : \text{len}(\sigma_i) \geq \frac{\varepsilon^3 r}{400}} \{|F(\sigma_i(\text{end})) - F(\tilde{\sigma}_i(\text{end}))| + |F(\tilde{\sigma}_i(\text{end})) - F(\tilde{\sigma}_i(\text{sta}))|
\]

\[ + |F(\sigma_i(\text{sta})) - F(\tilde{\sigma}_i(\text{end}))| + \sum_{\sigma_i : \text{len}(\sigma_i) < \frac{\varepsilon^3 r}{400}} |F(\sigma_i(\text{sta})) - F(\sigma_i(\text{end}))|
\]

\[ \leq \left(5L(f)\times(6\varepsilon^3 r)\times\#\{\sigma_i : \text{len}(\sigma_i) \geq \frac{\varepsilon^3 r}{400}\}\right)
\]

\[ + \sum_{\sigma_i : \text{len}(\sigma_i) \geq \frac{\varepsilon^3 r}{400}} \int_{\text{dom}(\tilde{\sigma}_i)} \|\nabla F\|_2 \circ \tilde{\sigma}_i \, d\mathcal{L}^1 + \frac{\varepsilon^3 r}{400} \times \left(5L(f)\times\#\{\sigma_i : \text{len}(\sigma_i) < \frac{\varepsilon^3 r}{400}\}\right)
\]

\[ \leq 40\varepsilon^3 r \times L(f) + 2C\varepsilon d_{X_{\infty}}(p, q),
\]

where \( C \) is the constant from (Gd3) in Lemma 3.50; as \( d_{X_{\infty}}(p, q) \geq \varepsilon^2 r \) we finally get:

\[
(4.68) \quad |F(p) - F(q)| \leq (40 \times L(f) + 2C)\varepsilon d_{X_{\infty}}(p, q).
\]

**Step 4:** \( \gamma \) has a bad jump.

Let \( \gamma = (\text{ph}_-, \text{jp}, \text{ph}_+) \) where \( \text{ph}_- = (\sigma_1, \cdots, \sigma_s) \), \( \text{ph}_+ = (\tau_1, \cdots, \tau_t) \) and Gates(jp) = \( G_{\text{green}}(Q) \cup G_{\text{red}}(Q) \) for \( Q \in \text{Bad}(X_{l-1}) \). Let \( \text{jp} = (c_1, c_2) \); on \( \text{ph}_- \) and \( \text{ph}_+ \) we can estimate as in **Step 3** while by **Step 2** we get:

\[
(4.69) \quad d_{X_{\infty}}(c_1, c_2) \leq \varepsilon d_{X_{\infty}}(p, q)
\]

as \( q \) cannot be bad for \( r \leq r_0 \). Thus:

\[
(4.70) \quad |F(p) - F(q)| \leq (2C + 45L(f))\varepsilon d_{X_{\infty}}(p, q).
\]

**Step 5:** \( \gamma \) has a good jump.
In this case $j_p = (c_{\text{green}}, c_{\text{red}})$ and:

$$\int_{\partial G_{\text{green}}(Q)} F \, d\mathcal{H}^2 - \int_{\partial G_{\text{red}}(Q)} F \, d\mathcal{H}^2 \leq \varepsilon d_{X_\infty}(c_{\text{green}}, c_{\text{red}});$$

(4.71)

without loss of generality we may assume that $\phi_-$ ends at $c_{\text{green}}$ and $\phi_+$ starts at $c_{\text{red}}$; moreover, recall that $d_{X_\infty}(c_{\text{green}}, c_{\text{red}}) \leq d_{X_\infty}(p, q)$. As $F$ is real-valued and continuous, we can find by the Intermediate Value Theorem (this is essentially the point were this argument breaks down for $l^2$-valued maps) $z_{\text{green}} \in \partial G_{\text{green}}(Q)$ and $z_{\text{red}} \in \partial G_{\text{red}}(Q)$ such that:

$$\int_{\partial G_{\text{green}}(Q)} F \, d\mathcal{H}^2 = z_{\text{green}},$$

(4.72)

$$\int_{\partial G_{\text{red}}(Q)} F \, d\mathcal{H}^2 = z_{\text{red}}.$$

Now as in Step 1 of Lemma 3.50 we may find horizontal paths $\phi_{\text{green}}$ and $\phi_{\text{red}}$ such that:

1. $\phi_{\text{green}}$ joins $c_{\text{green}}$ to $z_{\text{green}}$, $\phi_{\text{red}}$ joins $z_{\text{red}}$ to $c_{\text{red}}$.
2. $\text{len}(\phi_{\text{green}}) + \text{len}(\phi_{\text{red}}) \leq 64 d_{X_\infty}(c_{\text{green}}, c_{\text{red}})$.
3. $\phi_{\text{green}} \cup \phi_{\text{red}}$ contains at most 6 horizontal segments and, trivially, at most 6 of them can have length $\leq \varepsilon r/10$.

We can then estimate:

$$|F(p) - F(q)| \leq |F(p) - F(c_{\text{green}})| + |F(c_{\text{green}}) - F(z_{\text{green}})|$$

$$+ |F(z_{\text{green}}) - F(z_{\text{red}})| + |F(z_{\text{red}}) - F(c_{\text{red}})| + |F(c_{\text{red}}) - F(q)|.$$ 

(4.73)

On $|F(p) - F(c_{\text{green}})|$, $|F(c_{\text{green}}) - F(z_{\text{green}})|$, $|F(z_{\text{red}}) - F(c_{\text{red}})|$, $|F(c_{\text{red}}) - F(q)|$ we apply the argument of Step 3 to $\phi_-$, $\phi_{\text{green}}$, $\phi_{\text{red}}$ and $\phi_+$. For $|F(z_{\text{green}}) - F(z_{\text{red}})|$ as the jump is not bad, i.e. (4.71):

$$|F(z_{\text{green}}) - F(z_{\text{red}})| \leq \varepsilon d_{X_\infty}(p, q).$$

(4.74)

Thus:

$$|F(p) - F(q)| \leq (800L(f) + 2C + 8)\varepsilon d_{X_\infty}(p, q).$$

(4.75)

Step 6: The proof of (4.59).

By Steps 3--5 (i.e. by (4.68), (4.70) and (4.75)) we have:

$$\sup_{q \in \text{Fund}(p, \varepsilon, r)} |F(p) - F(q)| \leq (800L(f) + 2C + 8)\varepsilon d_{X_\infty}(p, q).$$

(4.76)

Let $\tilde{q} \in B_r(X_\infty, p)$ and find $q \in \text{Fund}(p, \varepsilon, r)$ with $d_{X_\infty}(q, \tilde{q}) \leq 5\varepsilon r$. Then:

$$|F(p) - F(q)| \leq (800L(f) + 2C + 8)\varepsilon d_{X_\infty}(p, q) + 25\varepsilon L(f)r;$$

(4.77)

thus (4.59) holds for a universal constant $C$ independent of $\varepsilon$ and $S$. 

□

5. Differentiability of Hilbert-valued Lipschitz maps

For an $l^2$-valued Lipschitz $f : X_\infty \to l^2$ the argument is more technical. First recall that $l^2$ has the Radon-Nikodym property: any Lipschitz $G : \mathbb{R}^n \to l^2$ is differentiable $\mathcal{L}^n$-a.e., here $n$ being arbitrary. Thus as in Definition 3.41 we can construct the horizontal gradient of $\nabla f$. As in Definition 4.20 and minding the discussion on $l^2$-valued harmonic functions in 2.3 we can construct the piecewise
harmonic approximations $H_j(f)$ and $G_j(f)$. Now the results of Lemma 4.23 extend to this setting. For the boundary regularity we approximate the boundary conditions with ones that take value in finite-dimensional subspaces of $l^2$ and use that if $\phi$ is $l^2$-valued and harmonic, then $\|\phi\|_2^2$ is subharmonic and so we can apply the maximum principle. For the energy estimates (4.24) we use the Kirszbraun extension theorem on the cells of $X_j$.

However, Step 5 in Theorem 4.56 breaks down as we cannot apply the Intermediate Value Theorem to $f$. To fix it we need to change the definition of a bad cube in Theorem 4.30. This leads to the Theorem 5.8 which is proved in the same way (hence the proof is omitted) provided one has a suitable lower bound on the energy as in Lemma 5.1. The proof of this Lemma is a bit more technical than the one 4.6; moreover we get a worse dependence on $\eta$ and we will have to change the exponent of $\varepsilon$ from 2 (in Theorem 4.30) to a 4 (in Theorem 5.8). Finally we are able to prove Theorem 5.11, where we have just to fix Step 5 in Theorem 4.56.

**Lemma 5.1 (Lower bound on the energy).** Let $Q$ be a cube with sidelength $\text{slen}(Q)$ and for $s \in (0, \frac{\text{slen}(Q)}{6})$ let $sQ$ denote the cube with the same center as $Q$ and with sidelength $s$. Assume that $F : Q \setminus sQ \to l^2$ is continuous and locally Lipschitz in $Q \setminus sQ$; assume also that the restriction $F|\partial(sQ)$ is Lipschitz, that $F = 0$ on $\partial Q$ and that there is a $p \in \partial(sQ)$ such that:

\[
\|F(p)\|_2 \geq \eta s.
\]

Then there is a universal constant $c_{\text{Har}}$ that depends only on the Lipschitz constant of $F|\partial(sQ)$ such that

\[
\int_{Q \setminus sQ} \|\nabla F\|^2_2 \, d\mathcal{L}^3 \geq c_{\text{Har}} \eta^4 s^3.
\]

**Proof.** As in Step 1 of Lemma 4.6 we reduce to the case of concentric balls where $F : B(0, \frac{\text{slen}(Q)}{2}) \setminus B(0, \frac{s}{2}) \to l^2$, that $F = 0$ for $r = \frac{\text{slen}(Q)}{2}$ and that for some $p \in \partial B(0, \frac{s}{2})$ one has $\|F(p)\|_2 \geq \eta s$.

**Step 1: Weighted symmetrization.** Now write $p = \frac{s}{2} \omega_0$ where $\omega_0 \in S^2$; using that $F|\partial B(0, \frac{s}{2})$ is Lipschitz we can find a $c > 0$ depending only on the Lipschitz constant of $F|\partial B(0, \frac{s}{2})$ such that if $\|\omega - \omega_0\|_2 \leq c \eta$ then:

\[
\|F\left(\frac{s}{2} \omega\right)\|_2 \leq \frac{\eta s}{2}.
\]

Let $\varphi_\eta$ be a smooth probability distribution on $S^2$ supported in $B_{S^2}(\omega_0, c \eta)$ and such that:

\[
\varphi_\eta \leq 1000 \frac{e^{-\eta^2}}{\mathcal{H}^2(S^2)};
\]

define

\[
\tilde{F}(r, \omega) = \tilde{F}(r) = \int_{S^2} F(r, \omega) \varphi_\eta(\omega) \, d\mathcal{H}^2(\omega);
\]
as $F$ is locally Lipschitz in $B(0, \frac{slen(Q)}{2}) \setminus B(0, \frac{\varepsilon}{2})$ so is $\tilde{F}$. Moreover $\| \tilde{F}(s/2) \|_2 \geq \eta s/2$. We now have:

\[(5.7) \int_{B(0, \frac{slen(Q)}{2}) \setminus B(0, \frac{\varepsilon}{2})} \| \nabla \tilde{F}(r, \omega) \|_2^2 r^2 d\omega = \int_{s/2}^{\frac{slen(Q)}{2}} r^2 dr \int_{S^2} d\omega \left\| \int_{S^2} \partial_r \tilde{F}(r, \tilde{\omega}) \varphi_{\eta}(\tilde{\omega}) d\tilde{\omega} \right\|^2_2 \leq \int_{s/2}^{\frac{slen(Q)}{2}} r^2 dr \int_{S^2} d\omega \left( \| \partial_r \tilde{F}(r, \tilde{\omega}) \|_2^2 \varphi_{\eta}(\tilde{\omega}) d\tilde{\omega} \right) \leq 1000 \varepsilon^{-2} \eta^{-2} \int_{B(0, \frac{slen(Q)}{2}) \setminus B(0, \frac{\varepsilon}{2})} \| \nabla F(r, \omega) \|_2^2 r^2 d\omega.\]

The proof is now completed as in Step 3 of Lemma 4.6. \qed

The following Theorem is proven like Theorem 4.30 using Lemma 5.1 instead of Lemma 4.6.

**Theorem 5.8** (Rate of collapse of the gates). Let $f : X_\infty \to l^2$ be Lipschitz and $Q \in \text{ToDouble}(X_{l-1})$ where $\bar{n}_k + 1 \leq l \leq \bar{n}_{k+1}$, and let $G_{\text{green}}, G_{\text{red}}$ be the two gates of Dyn$_{n_{k+1}}(Q)$. Having fixed $\varepsilon > 0$ we say that $Q$ is bad and write $Q \in \text{Bad}(X_{l-1})$ if for a pair $y_{\text{green}} \in \partial G_{\text{green}}, y_{\text{red}} \in \partial G_{\text{red}}$ with $\pi_l l-1(y_{\text{green}}) = \pi_l l-1(y_{\text{red}})$ one has:

\[(5.9) \| f(y_{\text{green}}) - f(y_{\text{red}}) \|_2 \geq \frac{\varepsilon}{256 \times n_{k+1}} \text{diam}(Q).\]

Then there is a universal constant $C$, which depends only on the Lipschitz constant of $f$ but not $\varepsilon$, such that we have the following estimate on the measure of the bad cubes:

\[(5.10) \sum_{k \geq 1} \sum_{l = \bar{n}_k + 1}^{\bar{n}_{k+1}} \mu_{l-1}(Q) \leq C \times (L(f))^2.\]

**Theorem 5.11** (Differentiation of Hilbert-valued Lipschitz functions). Let $f : X_\infty \to l^2$ be Lipschitz and let $\nabla f$ be its horizontal gradient. Then $f$ is differentiable $\mu_\infty$-a.e. with derivative $\nabla f$: i.e. for $\mu_\infty$-a.e. $p$ one has:

\[(5.12) \text{Lip}(f - \langle \nabla f(p), \tilde{x} \rangle)(p) = 0.\]

**Proof.** From the proof of Theorem 4.56 we have only to modify the argument for Step 5 as we cannot use the intermediate value theorem. However, as $jp$ is not bad we know that for each pair $z_{\text{green}} \in \partial G_{\text{green}}(Q)$, $z_{\text{red}} \in \partial G_{\text{red}}(Q)$ with $\pi_l l-1(z_{\text{green}}) = \pi_l l-1(z_{\text{red}})$ we have:

\[(5.13) \| f(z_{\text{green}}) - f(z_{\text{red}}) \|_2 \leq \frac{\varepsilon}{256 \times n_{k+1}} \text{diam}(Q) \leq \varepsilon d_{X_{\infty}}(c_1, c_2) \leq \varepsilon d_{X_{\infty}}(z_{\text{green}}, z_{\text{red}}),\]

and $d_{X_{\infty}}(z_{\text{green}}, z_{\text{red}}) \leq 27d_{X_{\infty}}(c_1, c_2)$. Having chosen such a pair we can then argue as in Step 5 of Theorem 4.56. \qed

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