Gravitational lensing in the strong field limit for Kar’s metric

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Abstract

In this paper we calculate the strong field limit deflection angle for a light ray passing near a scalar charged spherically symmetric object described by a metric which comes from the low-energy limit of heterotic string theory. Then, we compare the expansion parameters of our results with those obtained in the Einstein’s canonical frame, obtained by a conformal transformation, and we show that, at least at first order, the results do not agree.

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I. INTRODUCTION

Astrophysical black hole candidates are thought to be the Kerr black holes of general relativity, but the actual nature of these objects is still to be verified. The possibility of testing the nature of astrophysical black hole candidates with current and future observations has recently become an active research field, since in the following years the technique of very long baseline interferometry (VLBI) and, in a longer term, the (sub)millimeter VLBI “Event Horizon Telescope” will produce images of the Galactic center emission capable to see the silhouette predicted by general relativistic lensing [1, 2].

The general theory of relativity has passed several experimental tests in the weak gravitational field regime but it has not been yet tested in a strong gravitational field [3, 4]. Testing the gravitational field in the vicinity of a compact massive object, such as a black hole or a neutron star, could be a possible avenue for such investigations. The importance of gravitational lensing in strong fields is highlighted by the possibility of testing the full general relativity in a regime where the differences with non-standard theories would be manifest, helping the discriminations among the various theories of gravitation [5–7]. For this reason, the scientific community has been interested in the lensing properties near the photon sphere i.e. strong field limit.

The strong field limit lensing studies were first defined in [8], where the strong gravitational lensing due to a Schwarzschild black hole was studied, and in [9], where a static and circularly symmetric lens characterized by mass and scalar charge parameters was constructed. Later on, in [7] employing this limit, the authors found the position and the characteristics of all the relativistic images, term defined in [9], and in [10] Bozza provided a general method to extend the strong field limit to a generic static spherically symmetric spacetime inspired by previous works. In that work the author expanded the deflection angle near the photon sphere, i.e. strong field limit, in opposition to the standard weak field limit. This method can be applied to any spacetime in any theory of gravitation, as long as the photons satisfy the geodesic equation. Using Bozza’s method it is possible to discriminate among different types of black holes on the grounds of their strong field gravitational lensing properties. For example, studying the properties of the relativistic images it may be possible to investigate the regions immediately outside of the event horizon because the parameters of the strong field limit expansion are directly connected with observables [10].
Those works showed examples where gravitational lensing must not be just conceived as a weak field phenomenon, since high bending and looping of light rays in strong field is one of the most well-known prediction of General Relativity [7].

In [10] was calculated the deflection angle in the strong field limit for a light ray passing near to a Schwarzschild, Reissner-Nordström (RN) and Janis-Newman-Winicour (JNW) black holes in the Einstein frame. In this paper, using the same formalism, we calculated the deflection angle for the metric proposed by Kar in [11], which is considered equivalent to JNW’s metric under a conformal transformation, since it is possible to rewrite it in the Einstein canonical frame (as written in [12]) by employing the standard relations between the two metrics $g^{str}_{\mu\nu} = e^{2\phi} g^{E}_{\mu\nu}$ [11]. However, Alvarez and Conde in [13] argued that the equivalence of the Einstein frame and string frame for the description of the gravitational effects is only obtained when all functions involved are smooth, condition which is not satisfied by Bozza’s method since the deflection angle in the strong field limit diverges around the photon sphere. Using Bozza’s formalism we compare the expansion’s parameters and the deflection angle obtained in each frame and we show the discrepancies, at least at first order, according to the divergences involved in the related equations.

This paper is organized as follow. In section II we explain Bozza’s method and write down the relevant equations. In section III we apply the strong field limit expansion to Kar’s metric and finally in section IV we compare the results obtained in the string frame with those obtained by Bozza in the Einstein frame.

II. STRONG FIELD EXPANSION OF THE DEFLECTION ANGLE: BOZZA’S METHOD

A spherical symmetric spacetime is described by the line element [14]

$$ds^2 = A(x)dt^2 - B(x)dx^2 - C(x)(d\theta^2 \sin^2 \theta + \theta d\phi^2).$$

In order to obtain the deflection of a light beam it is necessary to consider the motion of a freely falling photon in a static isotropic gravitational field. From the geodesic equation and the line element (11) it is possible to obtain the following quantity

$$\frac{d\phi}{dx} = \sqrt{\frac{B}{C}} \left[ \frac{C(x)}{C(x_0)} \frac{A(x_0)}{A(x)} - 1 \right]^{-\frac{1}{2}}$$
which gives the angular shift of the photon as function of the distance from the center. However, to study the deflecton angle for a spherically symmetric metric it is necessary to calculate it as a function of the closest approach $x_0$ using the following expression \[14\]

$$
\hat{\alpha}(x_0) = 2 \int_{x_0}^{\infty} \sqrt{\frac{B(x)}{B(x_0)}} \left[ \frac{C(x)}{C(x_0)} \frac{A(x)}{A(x_0)} - 1 \right]^{-\frac{1}{2}} dx - \pi. \tag{2}
$$

According to equation (2) a photon coming from infinity with some impact parameter $u = \sqrt{\frac{C(x_0)}{A(x_0)}}$ will be deviated when it is approaching the black hole, it will reach $x_0$ and then emerge in another direction. By reducing $x_0$ the deflection angle increases until it exceeds $2\pi$, when the photon gives a complete loop around the black hole. By decreasing the impact parameter $u$ the photon will wind several times before emerging. Finally, for $x_0 = x_m$, the deflection angle diverges and the photon is captured.

Bozza’s method is used to calculate the deflection angle in the strong field limit for a spherically and symmetric metric taking the photon sphere as the starting point. A photon sphere is the region of spacetime where gravity is strong enough that photons are forced to travel in orbits. This means that Einstein bending angle of a light ray with the closest distance of approach $x_0$ becomes unboundedly large as $x_0$ tends to $x_m$. In this sense the method requires that the photon sphere equation \[15, 16\]

$$
C'(x)A(x) = A'(x)C(x) \tag{3}
$$

has at least one positive solution. In general, equation (3) has several solutions. Therefore, we will take the largest root as the radius of the photon sphere and denote it by $x_m$, as defined in \[10\].

In order to obtain the deflection angle in the strong field limit, Bozza defines two new variables $y = A(x)$ and $z = \frac{y - y_0}{1 - y_0}$. Replacing these new variables, equation (2) takes now the form

$$
\hat{\alpha}(x_0) = \int_{0}^{1} R(z, x_0)f(z, x_0)dz - \pi \tag{4}
$$

$$
R(z, x_0) = 2\sqrt{\frac{By}{CA'}}(1 - y_0)\sqrt{C_0} \tag{5}
$$

$$
f(z, x_0) = \frac{1}{\sqrt{y_0 - [(1 - y_0)z + y_0]C_0/C}} \tag{6}
$$
where all functions with the subscript 0 are evaluated at \( x = x_0 \) and without it are evaluated at \( x = A^{-1}[(1 - y_0)z + y_0] \). The prime \( ' \) is the derivative with respect to \( x \).

The function \( R(z, x_0) \) is regular for values of \( z \) and \( x_0 \). However, the function \( f(z, x_0) \) diverges for \( z \to 0 \). In this sense, to obtain the order of divergence of the integrant it is necessary to expand the argument of the square root in \( f(z, x_0) \) to the second order in \( z \). Therefore, for \( z \to 0 \) the function \( f(z, x_0) \) can be approximate to

\[
f_0(z, x_0) = \frac{1}{\sqrt{\alpha z + \beta z^2}}
\]

where \( \alpha \) and \( \beta \) are expressed by

\[
\alpha = \frac{1 - y_0}{C_0 A_0'} (C_0' y_0 - C_0 A_0')
\]

\[
\beta = \frac{(1 - y_0)^2}{2C_0^2 A_0'^3} [2C_0 C_0'' A_0'^2 + (C_0 C_0'' - 2C_0'^2) y_0 A_0' - C_0 C_0'' y_0 A_0'].
\]

When \( \alpha \) is not zero, the leading order of the divergence in \( f_0 \) is \( z^{-\frac{3}{2}} \), which can be integrate to give a finite result. When \( \alpha \) vanishes, the divergence is \( z^{-1} \), which makes the integral diverge \([10]\). If we examine the form \( \alpha \), we see that it vanish at \( x_0 = x_m \). Each photon having \( x_0 < x_m \) is captured by the central object and can not emerge back.

In order to calculate the deflection angle in the strong field limit, Bozza proposes to split the integral in equation \([4]\) in two pieces: one part containing the divergence, \( I_D(x_0) \), and the other being the original integral with the divergence subtracted, \( I_R(x_0) \). Mathematically this idea is expressed as follows

\[
I(x_0) = \int_0^1 R(z, x_0) f(z, x_0) dz = I_D(x_0) + I_R(x_0)
\]

\[
I_R(x_0) = \int_0^1 [R(z, x_0) f(z, x_0) - R(0, x_m) f_0(z, x_0)] dz = \int_0^1 g(z, x_0) dz
\]

\[
I_D(x_0) = \int_0^1 R(0, x_m) f_0(z, x_0) dz.
\]
The integral $I_D(x_0)$ can be solved exactly and is found to be

$$I_D(x_0) = R(0, x_m) \frac{2}{\sqrt{\beta}} \ln \frac{\sqrt{\beta} + \sqrt{\alpha + \beta}}{\sqrt{\alpha}}. \quad (12)$$

Since the method works with terms up to first order $O(x_0 - x_m)$, it is necessary to expand $\alpha$ as

$$\alpha = \frac{2\beta_m A'_m}{1 - y_m} (x_0 - x_m) + O(x_0 - x_m)^2,$$

where

$$\beta_m = \beta|_{x_0 = x_m} = \frac{C_m(1 - y_m)^2(C'_m y_m - C_m A''(x_m))}{2y_m^2 C''_m}.$$

Now replacing $\alpha$ and $\beta_m$ equation (12) takes the form

$$I_D(x_0) = -a \ln \left[ \frac{x_0}{x_m - 1} \right] + b_D + O(x_0 - x_m) \quad (13)$$

where

$$a = \frac{R(0, x_m)}{\sqrt{\beta_m}},$$

$$b_D = \frac{R(0, x_m)}{\sqrt{\beta_m}} \ln \frac{2(1 - y_m)}{A'_m x_m}. \quad (14)$$

In order to obtain the regular part of equation (9), Bozza expanded $I_R(x_0)$ in powers of $(x_0 - x_m)$ as

$$I_R(x_0) = \sum_{n=0}^{\infty} \frac{1}{n!} (x_0 - x_m)^n \int_0^1 \frac{\partial^n g}{\partial x_0^n} |_{x_0 = x_m} dz \quad (15)$$

and evaluate the single coefficient. Note that if the singular part is not subtracted from $R(z, x_0)f(z, x_0)$ an infinite coefficient would be obtained for $n = 0$. However the function $g(z, x_0)$ is regular in $z = 0, x_0 = x_m$. In this sense, and recalling that the method works with terms up to first order $O(x_0 - x_m)$ Bozza retained the $n = 0$ term and the regular part would be

$$I_R(x_0) = \int_0^1 g(z, x_m) dz + O(x_0 - x_m)$$

and defines $b_R$ as
\[ b_R = I_R(x_m); \quad (16) \]

this term would be added to \( b_D \). Therefore, the deflection angle for the strong field limit would be expressed

\[
\alpha(x_0) = I_D(x_0) + I_R(x_0) - \pi \\
= -a \ln \left[ \frac{x_0}{x_m} - 1 \right] + b_D + b_R - \pi \quad (17)
\]

the last equation can be expressed in terms of \( \theta \) using the conservation of angular momentum. As mentioned before, there is a relation between the closest approach distance and the impact parameter \( u \) expressed by

\[
u = \sqrt{\frac{C_0}{A_0}} \quad (18)
\]

where \( C \) and \( A \) are evaluated at \( x = x_0 \) \[14\]. From equation \[18\], the minimum impact parameter is \( (x_0 = x_m) \)

\[
\nu_m = \sqrt{\frac{C_m}{y_m}} \quad (19)
\]

Expanding equation \[18\] around \( x_m \) up to second order, we can express the impact parameter as

\[
u - \nu_m = \frac{1}{4} \left( \frac{C''_m A_m - C'_m A''_m}{\sqrt{C_m A^3_m}} \right) (x_0 - x_m)^2 \\
= c(x_0 - x_m)^2. \quad (20)
\]

Using this expansion, it is possible to write the deflection angle as a function of \( \theta \). To do so, we have to remember that \( \theta = \frac{\nu}{D_{OL}} \), where \( D_{OL} \) is the distance from the observer to the lens.

Therefore, equation \[20\] takes the form

\[
\theta D_{OL} - \nu_m = c \left( \frac{x_0}{x_m} - 1 \right) x_m^2.
\]

Finally, equation \[17\] in terms of \( \theta \) is \[10\]
\[\alpha(\theta) = -\bar{a} \ln \left[ \frac{\theta D_{OL}}{u_m} - 1 \right] + \bar{b}\]  
(21)

where

\[\bar{a} = \frac{R(0, x_m)}{2\sqrt{\beta_m}}\]

\[\bar{b} = b_R + \bar{a} \ln \left[ \frac{2\beta_m}{y_m} \right] - \pi.\]  
(22)

In this sense, the method to calculate the deflection angle in the strong field limit proposed by Bozza in [10] has four important steps:

1. Since the method takes the photon sphere as the starting point, it is necessary to know the radius of the photon sphere i.e. solve equation (3). However, it is possible to find \(x_m\) by solving \(\alpha(x_m) = 0\) and taking the largest root as the radius of the photon sphere.

2. In order to find the coefficients \(\bar{a}\) and \(\bar{b}\) of equation (21) it is important to write \(\beta_m\) and \(R(0, x_m)\).

3. Compute \(b_R\) numerically or by a proper expansion in the parameters of the metric. This is the crucial step.

4. Compute the coefficient \(u_m\).

### III. STRONG FIELD EXPANSION FOR KAR’S METRIC

In this section we apply Bozza’s formalism presented in [10] to calculate the deflection angle in the strong field limit for the metric proposed by Kar in [11]. This metric comes from the four-dimensional, low-energy effective action of the heterotic string theory compactified on a 6-torus given by [11]

\[S_{eff} = \int d^4x \sqrt{-g} e^{-2\phi} \left[ R + 4(\nabla \phi)^2 - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} - F_{\mu\nu} F^{\mu\nu} \right].\]

Assuming the Maxwell and antisymmetric tensor field to be zero it is possible to obtain the following spherically symmetric, static solution:

\[ds_{str}^2 = \left( 1 - \frac{2\eta}{r} \right)^{\frac{m+\sigma}{\eta}} dt^2 - \left( 1 - \frac{2\eta}{r} \right)^{\frac{(\sigma-m)}{\eta}} dr^2 - \left( 1 - \frac{2\eta}{r} \right)^{1+\frac{\sigma-m}{\eta}} r^2 d\Omega^2\]  
(23)
where \( m \) is the mass, \( \sigma \) is the scalar charge and \( \eta \) is given by \( \eta^2 = m^2 + \sigma^2 \). For \( \sigma = 0 \) this solution reduces to the Schwarzschild solution. Using geometrized units (the gravitational constant \( G = 1 \) and the speed of light in vacuum \( c = 1 \)) and introducing radial distance defined as \( x = \frac{r}{2\eta} \) and \( x_o = \frac{r_0}{2\eta} \) equation (23) takes the form

\[
d s^2_{str} = \left(1 - \frac{1}{x}\right) \frac{m + \sigma}{\eta} dt^2 - \left(1 - \frac{1}{x}\right) \frac{(x - m)}{\eta} dx^2 - \left(1 - \frac{1}{x}\right)^{1+\frac{x-m}{\eta}} x^2 d\Omega^2. \tag{24}
\]

In order to discuss gravitational lensing in the strong field limit for equation (24), it is necessary to express it in terms of a single parameter. Using the relation \( \eta^2 = m^2 + \sigma^2 \) we defined \( \zeta = \frac{\sigma}{\eta} \) and \( \gamma = \frac{m}{\eta} \) (which is the JNW parameter used by Bozza) so that \( \gamma^2 + \zeta^2 = 1 \). If we choose \( \zeta \) as the parameter, the Kar’s metric can be expressed as

\[
d s^2 = \left(1 - \frac{1}{x}\right)^{\zeta+\sqrt{1-\zeta^2}} dt^2 - \left(1 - \frac{1}{x}\right)^{\zeta-\sqrt{1-\zeta^2}} dx^2 - \left(1 - \frac{1}{x}\right)^{1+\zeta-\sqrt{1-\zeta^2}} x^2 d\Omega^2, \tag{25}
\]

were \( \gamma = \sqrt{1-\zeta^2} \). For \( \zeta = 0 \) the metric (25) reduces to Schwarzschild. Therefore in Kar’s metric the functions for a spherically symmetric metric are

\[
A(x) = \left(1 - \frac{1}{x}\right)^{\zeta+\sqrt{1-\zeta^2}}
\]

\[
B(x) = \left(1 - \frac{1}{x}\right)^{\zeta-\sqrt{1-\zeta^2}}
\]

\[
C(x) = \left(1 - \frac{1}{x}\right)^{1+\zeta-\sqrt{1-\zeta^2}} x^2.
\]

From equation (8) the radius of the photon sphere as a function of \( \zeta \) is

\[
x_m = \sqrt{1 - \zeta^2} + \frac{1}{2}. \tag{26}
\]

For \( \zeta = 0 \) equation (26) reduces to \( x_m = \frac{3}{2} \), which is the value of the Schwarzschild’s photon sphere radius.

In order to discuss gravitational lensing in the strong field limit for Kar’s metric (25) it is necessary to find the values of \( \zeta \) where equation (26) has a solution. Using the analysis made in [15] we found that the photon sphere equation has solution only for \( 0 \leq \zeta < \frac{\sqrt{3}}{2} \),
i.e. for $0 \leq \sigma^2 < 3m^2$. It is easy to obtain the same interval for $\zeta$ using $\frac{1}{2} < \gamma \leq 1$ (the interval for JNW) and recalling that $\gamma = \sqrt{1-\zeta^2}$.

The behaviors of the photon sphere for Kar, JNW and RN metrics as a function of $\zeta$, $\gamma$ and $q$ (which is the charge in the RN metric), respectively, are plotted in Fig. 1.

![Figure 1](image)

Figure 1. Behavior of the photon sphere, $x_m$, for Kar, JNW and RN metrics. In all the red dotted horizontal line is the Schwarzschild limit.

From equations (26) and (8) we obtain that

$$R(0,x_m) = \frac{2\sqrt{1-\zeta^2} + 1}{\sqrt{1-\zeta^2} + \zeta} \left\{ \left( \frac{2\sqrt{1-\zeta^2} - 1}{2\sqrt{1-\zeta^2} + 1} \right)^{1+\zeta-\sqrt{1-\zeta^2}} - \left( \frac{2\sqrt{1-\zeta^2} - 1}{2\sqrt{1-\zeta^2} + 1} \right)^{1+3\zeta-\sqrt{1-\zeta^2}} \right\}$$

where

$$y_m = \left( \frac{2\sqrt{1-\zeta^2} - 1}{2\sqrt{1-\zeta^2} + 1} \right)^{\sqrt{1-\zeta^2}+\zeta}$$

$$\beta_m = \frac{1}{4} \left[ \left( \frac{2\sqrt{1-\zeta^2} + 1}{\sqrt{1-\zeta^2}} \right)^{\zeta+\sqrt{1-\zeta^2}} - \left( \frac{2\sqrt{1-\zeta^2} - 1}{\sqrt{1-\zeta^2}} \right)^{\zeta+\sqrt{1-\zeta^2}} \right]^2$$

$$\left( \sqrt{1-\zeta^2} + \zeta \right)^2 (3 - 4\zeta^2) \sqrt{1-\zeta^2+\zeta-1}$$

Thus, from equation (22) we obtain that $\bar{a} = 1$, which is the same value obtained for JNW [10]. In order to obtain $b_R$ we expand equation (16) up to first order in $\zeta$ around $\zeta = 0$ (for $\zeta = 0$ the metric reduces to Schwarzschild). Making this expansion we obtain
\[ b_R = 2 \ln 6(2 - \sqrt{3}) - 2.3980 \zeta, \]

and using equation (22)

\[ \bar{b} = 2 \ln(6(2 - \sqrt{3})) - 2.3980 \zeta - \pi + \ln \left[ \frac{2\beta_m}{y_m} \right]. \]

From equation (19) the impact parameter is

\[ u_m = \frac{1}{2} \left( \frac{2\sqrt{1 - \zeta^2} - 1}{2\sqrt{1 - \zeta^2} + 1} \right)^{-\frac{1}{2}} \sqrt{1 - \zeta^2}. \]

The behavior of \( u_m, \bar{a} \) and \( b \) in terms of \( \zeta \) is plotted in Fig. 2. In the figure we see that \( \bar{a} \) is constant. Note that each parameter reduces to those of Schwarzschild for \( \zeta = 0 \)

Figure 2. Behavior of \( u_m, \bar{a} \) and \( b \) in terms of \( \zeta \). Note that \( u_{ms} \) and \( \bar{b}_s \) are the values for Schwarzschild metric.

Finally, using equation (21), the deflection angle in the strong field limit for Kar’s metric in terms of the parameter \( \zeta \) is

\[ \alpha = -\ln \left[ \frac{u}{u_m} - 1 \right] + 2 \ln(6(2 - \sqrt{3})) - 2.3980 \zeta - \pi + \ln \left[ \frac{2\beta_m}{y_m} \right]. \] (27)

In Fig. 3 is plotted the behavior of the deflection angles as a function of \( \zeta \). Once we fix \( u = u_m + 0.003 \) we see that the deflection angle decreases as the value of \( \zeta \) increases until it reaches the minimum value of 0.6, then the deflection angle increases and for \( \sqrt{3}/2 \) it
diverges. For $\zeta = 0$ the deflection angle reduces to $-\ln \left( \frac{0.006}{3\sqrt{3}} \right) +\ln(216(7-4\sqrt{3}))-\pi \approx 6.364$, which is the Schwarzschild deflection angle in the strong field limit when $u = u_m + 0.003$.

Figure 3. Deflection angles for Kar’s metric evaluated at $u = u_m + 0.003$ as a function of $\zeta$. The dotted horizontal line is the Schwarzschild limit.

IV. DISCUSSION

We begin our analysis with the photon sphere. The value of $x_m$ as a function of $\gamma$ for JNW is \[ x_m = \frac{2\gamma + 1}{2} \] (28)

and the photon sphere equation in terms of $\zeta$ for Kar’s metric is (equation (26)) $x_m = \sqrt{1 - \zeta^2} + \frac{1}{2}$. At a first glance equations (28) and (26) seems to be different; however it is possible to obtain one from the other by using $\gamma^2 + \zeta^2 = 1$ i.e. by making $\gamma = \sqrt{1 - \zeta^2}$ in equation (28) we obtain equation (26). For $\zeta = 0$ ($\gamma = 1$) both reduce to $x_m = \frac{3}{2}$, which is the photon sphere for Schwarzschild. For $\zeta = \frac{\sqrt{3}}{2}$ ($\gamma = \frac{1}{2}$) both diverge.

In Fig. 4-left we plotted $u_m$, $\bar{a}$ and $\bar{b}$ for JNW in terms of $\gamma$. $\bar{a}$ is a constant, $\bar{b}$ decrease as $\gamma$ increases, but $u_m$ decreases until $\gamma$ reaches the value 0.5, where it diverges. For $\gamma = 1$, $u_m$, $\bar{a}$ and $\bar{b}$ reduce to Schwarzschild’s coefficients.

In Fig. 4-right we plotted the same coefficients for JNW as a function of $\zeta$ by making $\gamma = \sqrt{1 - \zeta^2}$. It can be seen that the behavior of $\bar{u}_m$ and $\bar{a}$ are the same in each frame; nevertheless, the behavior of $\bar{b}$ is quite different for each frame; while $\bar{b}$ for Kar has a minimum
at 0.5, $b_{JNW}$ has a minimum near 0.2. This difference between Kar and JNW metrics may arise because equations for $b$ in each metric are not smooth at $\zeta = \frac{\sqrt{3}}{2}$ (Cf. equation (22)) and $\gamma = 0.5$ respectively in agreement with the ideas presented in [13] for the frames.

![Graphs showing behavior of expansion parameters](image)

Figure 4. Behavior of the expansion parameters. *Left*: $u_m$, $\bar{a}$ and $\bar{b}$ for JNW in terms of $\gamma$. *Right*: $u_m$, $\bar{a}$ and $\bar{b}$ for JNW in terms of $\zeta$. The parameters with a 's' subscript are the values for Schwarzschild.

Finally in Fig. 5, we plotted the deflection angle for JNW and Kar in order to compare both frames. In Fig. 5-*left* we plotted the deflection angle in the strong field limit for JNW as a function of $\gamma$ when $u = u_m + 0.003$, as appear in [10]. $\alpha$ increases as $\gamma$ increases and it reduces to Schwarzschild deflection angle when $\gamma = 1$. On the other hand, $\alpha$ decreases until $\gamma$ reaches its minimum ($\gamma \approx 0.5$), then it begins to increases and finally, for $\gamma = 0.5$, the deflection angle diverges.

In figure Fig. 5-*right* we plotted both JNW and Kar deflection angle as a function of $\zeta$. Although both graphics reduce to Schwarzschild for $\zeta = 0$ and both diverge for $\zeta = \frac{\sqrt{3}}{2}$, the behavior of the deflection angle is very different in each frame. For example, when $\zeta$ increases both angles decrease; however while $\alpha$ for Kar’s metric has a minimum at 0.6, the deflection angle for JNW has its minimum near $\frac{\sqrt{3}}{2}$. Thus, at least at first order, these two frames do not agree, according to the divergences involved in the related equations.

According to Alvarez and Conde in [13] there is no doubt of the equivalence of all frames for the description of the gravitational effects of the string theories at a basic level, at least when all the functions involved are smooth. In this sense, and using the strong field limit
Figure 5. Deflection angles for JNW and Kar for \( u = u_m + 0.003 \). Left: \( \alpha \) for JNW as a function of \( \gamma \). Right: \( \alpha \) for JNW and Kar metrics as a function of \( \zeta \). The dotted horizontal lines are the Schwarzschild limits.

expansion, we see the possibility to explore this idea in the particular case of gravitational lensing by comparing the strong field limit coefficients (\( u_m, \bar{a} \) and \( \bar{b} \)) and the deflection angle in each frame; taking into account that the strong field approximation is not a smooth function near the photon sphere.

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