THE POINT IN WEAK SEMIPROJECTIVITY AND AANR COMPACTA

TERRY A. LORING

Abstract. We initiate the study of pointed approximative absolute neighborhood retracts. Our motivation is to generate examples of $C^*$-algebras that behave in unexpected ways with respect to weak semiprojectivity. We consider both weak semiprojectivity (WSP) and weak semiprojectivity with respect to the class of unital $C^*$-algebras (WSP1). For a non-unital $C^*$-algebra, these are different properties.

One example shows a $C^*$-algebra $A$ can fail to be WSP while its unitization $\tilde{A}$ is WSP. Another example shows WSP1 is not closed under direct sums.

1. Introduction
The “with or without a unit” choice in $C^*$-algebras becomes serious in the context of certain approximation problems for $C^*$-algebras. We find that weak semiprojectivity for $C_0(X)$ in the commutative category does not translate to a standard condition on $X$. Pointed approximative absolute neighborhood retracts are introduced, PAANR for short, as these are the $X$ for which $C_0(X)$ is weakly semiprojective. Our main objective is to understand the commutative $C^*$-algebras based on the spaces shown in Figures 1, 2 and 3 and see what this tells us about weak semiprojectivity.

Let $\alpha X$ be the one-point compactification of the locally compact metrizable space $X$ with added point $\infty$. We find that $(\alpha X, \infty)$ is a PAANR exactly when $C_0(X)$ is weakly semiprojective within commutative $C^*$-algebras. It is possible for $\alpha X$ and $\alpha Y$ to be homeomorphic with $(\alpha X, \infty)$ not a PAANR while $(\alpha Y, \infty)$ is a PAANR. This is not a welcome phenomenon as it implies that weak semiprojectivity lacks an expected closure property.

Starting from the more civilized topological condition that $\alpha X$ is to be an approximative absolute neighborhood retract (AANR) we are

2000 Mathematics Subject Classification. 46L85.
Key words and phrases. $C^*$-algebras, compactum, lifting, approximative retract.
lead to the study of weak semiprojectivity with respect to unital $C^*$-algebras. This is a condition that applies to unital and non-unital $C^*$-algebras, devolving to weak semiprojectivity in the unital case. This condition also lacks an expected closure property; it is not closed under direct sums.

The audience for this note is primarily $C^*$-algebraists working in or near classification or shape theory. The section on PAANR spaces is hoping to be attractive to a few topologists. That section and the preceding section that reviews AANR spaces contain no mention of $C^*$-algebras.

To motivate the definition of a PAANR, we take a moment to discuss two important classes of morphisms between $C^*$-algebras and how these can be constructed from pointed maps and proper maps. Recall that a compactum is a compact, metrizable space. We say $(X, x_0)$ is a pointed compactum when $x_0$ is a point in the compact metrizable space $X$. A pointed map $\gamma : (X, x_0) \to (Y, y_0)$ means a continuous function $\gamma$ from $X$ to $Y$ such that $\gamma(x_0) = \gamma(y_0)$. To a compactum $X$ we associate the $C^*$-algebra $C(X)$, which is commutative, separable and unital. Indeed all commutative, separable, unital $C^*$-algebras arise this way, up to isomorphism. If we drop the unital requirement, we get the slightly broader class of $C^*$-algebras of the form

$$C_0(X, x_0) = \{ f : X \to \mathbb{C} \mid f \text{ is continuous, } f(x_0) = 0 \}$$
where \((X, x_0)\) varies over the pointed compacta. Alternately, we can describe these as \(C_0(Y)\) where \(Y\) is locally compact and metrizable. But what of the morphisms?

The default choice of morphisms between \(C^*\)-algebras are the \(*\)-homomorphisms. Between \(C(X)\) and \(C(Y)\) the unital \(*\)-homomorphisms are of the form \(f \mapsto f \circ \gamma\) for a continuous map \(\gamma : Y \to X\). We miss out on the non-unital \(*\)-homomorphisms, but this turns out to be a trivial matter. For example, when \(Y\) is connected, the only non-unital \(*\)-homomorphism is zero. But what of the non-unital case?

**Myth.** The \(*\)-homomorphisms from \(C_0(X, x_0)\) to \(C_0(Y, y_0)\) are all of the form \(f \mapsto f \circ \gamma\) for a continuous, proper map \(\gamma : Y \setminus \{y_0\} \to X \setminus \{x_0\}\).

**Fact.** Only the so-called proper \(*\)-homomorphisms arise from proper maps, and there are lots of non-proper \(*\)-homomorphisms. See [8].

The \(*\)-homomorphisms from \(C_0(X, x_0)\) to \(C_0(Y, y_0)\) are all of the form \(f \mapsto f \circ \gamma\) for a pointed map \(\gamma : (X, x_0) \to (Y, y_0)\). Recall that pointed requires \(\gamma(x_0) = \{y_0\}\) only. If we also required \(\gamma^{-1}(\{y_0\}) = \{x_0\}\) then we would have, by restriction, a proper map from \(Y \setminus \{y_0\}\) to \(X \setminus \{x_0\}\).

We are focused on examples that show some odd behavior in \(C^*\)-algebras and in getting down all the details on the relation between weak semiprojectivity and \((P)\)AANR compacta. The issue of when the AR, ANR, or AANR property for compacta \(X\) is sufficient to make \(C_0(X \setminus \{x_0\})\) projective, or \(C(X)\) semiprojective or weakly semiprojective, has been much researched lately. See the papers of Chigogidze and Dranishnikov [3], Sørensen and Thiel [14], and Enders [9].

All the spaces considered will be one-dimensional. The reason for this is that when \(X\) is two-dimensional, the \(C^*\)-algebra \(C(X)\) will fail to be weakly semiprojective. Indeed, the converse holds. We are considering spaces with potentially very aberant local structure, so even ignoring interaction with \(C^*\)-algebras, the topology of a one-dimensional space can be tricky [2].

Whenever looking at forms of semiprojectivity, a strong motivation is potential applications in shape theory. For a survey of shape theory of spaces, with even some remarks about shape theory for \(C^*\)-algebras, see [13]. For a treatment of shape theory for \(C^*\)-algebras that connects it with \(E\)-theory, see [5].

The definition of PAANR, and some basic results, circulated in an early version of [12]. In final form, that paper focused on a new property for \(C^*\)-algebras, being weakly projective, and the related topological concept of being a pointed approximative absolute retract (PAAR).
Weak semiprojectivity has been studied for some time [7, 10, 15], having been introduced at least as early as 1997 in [11].

The author thanks Adam Sørensen and Hannes Thiel for feedback on the exposition of this work and many discussions related to shape theory.

2. AANR spaces

We start with a careful review of approximative absolute neighborhood retract (AANR) in the sense of [4]. We are especially interested in an equivalent formulation, following ideas from [1], that translates to a lifting problem in $C^*$-algebras.

**Definition 2.1.** A compactum $X$ is an *approximative absolute neighborhood retract (AANR)* if, for every homeomorphic embedding $\theta :$
X → Y of X into a compact metric space (Y, d), and for every ε > 0, there exists δ > 0 and a continuous function r : U_δ → X so that

\[ d(r \circ \theta(x), x) \leq \varepsilon \]

for all x in X, where

(1) \[ U_\delta = \{ y \in Y \mid d(y, \theta(X)) \leq \delta \} . \]

Clapp asks only that r be defined on a neighborhood N(ε) of θ(X), but this is equivalent since every open set containing the compact set θ(X) contains some U_δ, and each U_δ contains the open neighborhood \[ \{ y \in Y \mid d(y, \theta(X)) < \delta \} . \]

We can gain flexibility in applying the AANR property by allowing for more general decreasing sets. We also downplay the metric on X insisting only that we have uniform convergence. It is key here that X be compact so that we have uniform equivalence of any two compatible metrics and so the uniform convergence in (2) does not depend on the choice of metric on X.

**Proposition 2.2.** A compactum X is an AANR if, and only if, for every continuous embedding θ : X → Y of X into a compactum Y, and for every sequence Y_1 ⊇ Y_2 ⊇ · · · of closed subsets of Y with \( \bigcap Y_n = \theta(X) \), there exists a sequence of continuous functions r_n : Y_n → X so that

(2) \[ \lim_{n \to \infty} r_n(\theta(x)) = x \]

uniformly over x in X.

**Proof.** Let d be a compatible metric on Y. Then \( U_{\frac{1}{k}} \) is a decreasing sequence of closed subsets with intersection θ(X) and so this condition easily implies X is an AANR.

Now assume X is an AANR and that the Y_n are given. For a given k > 0 we know there is a δ > 0 and continuous map r : U_δ → X so that

\[ d(r(\theta(x)), x) \leq \frac{1}{k} \]

for all x in X. Since θ(X) and Y_n are compact, we will see that for some n_k we have the inclusion Y_{n_k} ⊆ U_{\frac{1}{k}} and so can define r_{n_k} as the restriction of r to Y_{n_k}. If there is no such inclusion, then we have y_1, y_2, . . . with y_n ∈ Y_n and \( d(y_n, \theta(x)) \geq \frac{1}{k} \) for all x ∈ X. Passing to a subsequence we have \( y = \lim_n y_n \) in θ(X) with \( d(y, \theta(x)) \geq \frac{1}{k} \) for all x ∈ X, and so \( d(y, y) > \frac{1}{k} \), a contradiction. We can arrange that the n_k are increasing, as define r_\ell : Y_\ell → X as the restriction of r_{n_k} whenever
$k$ is between $n_k$ and $n_{k+1}$. Finally, we define the initial $r_1, \ldots, r_{n-1}$ in any way we like. \qed

We get an even more useful characterization, an approximate local extension property.

**Proposition 2.3.** A compactum $X$ is an AANR if, and only if, for every closed subset $Y$ of a compact metrizable space $Z$, for every sequence $Y_1 \supseteq Y_2 \supseteq \cdots$ of closed subsets of $Z$ with $\bigcap Y_n = Y$, and for every continuous function $\lambda : Y \to X$, there is a sequence of continuous functions $\lambda_n : Y_n \to X$ so that

$$\lim_{n \to \infty} \lambda_n(y) = \lambda(y)$$

uniformly over $y$ in $Y$. To summarize in a diagram:

(3)

$$\begin{array}{ccc}
Z & \xrightarrow{\lambda_n} & Y_n \\
\uparrow & & \uparrow \\
X & \xleftarrow{\lambda} & Y
\end{array}$$

**Proof.** Suppose $X$ is an AANR and we are given $Y$, $Z$, $\lambda$ as indicated,

$$\begin{array}{ccc}
Z & \xrightarrow{\lambda} & Y \\
\uparrow & & \uparrow \\
X & \xleftarrow{\lambda} & Y
\end{array}$$

and that there are closed subsets to that $\bigcap Y_n = Y$. Take the pushout,

$$\begin{array}{ccc}
X \cup_Y Z & \xleftarrow{\iota_Z} & Z \\
\iota_X & & \uparrow \\
X & \xleftarrow{\lambda} & Y
\end{array}$$

in which $X \cup_Y Z$ is a compact metrizable space and $\iota_X$ is one-to-one, continuous, and so a homeomorphism onto its image. We can select a compatible metric on $X \cup_Y Z$ and then the corresponding metric on $X$ to make $\iota_X$ an isometry.

Consider the closed sets

$$\iota_X(X) \cup \iota_Z(Y_n)$$

that have intersection

$$\iota_X(X) \cup \iota_Z(Y) = \iota_X(X) \cup \iota_X(\lambda(Y)) = \iota_X(X).$$
Applying Proposition 2.2 we find maps 
\[ \rho_n : \iota_X(X) \cup \iota_Z(Y_n) \to X \]
so that 
\[ \lim_{n \to \infty} \rho_n \circ \iota_X(x) = x \]
uniformly over \( x \) in \( X \). We then let \( r_n \) be defined on \( Y_n \) by 
\[ r_n(y) = \rho_n(\iota_Z(y)) \]
so that when \( y \) is in \( Y \) we have 
\[ \lim_{n \to \infty} r_n(y) = \lim_{n \to \infty} \rho_n(\iota_Z(y)) = \lim_{n \to \infty} \rho_n(\iota_X(\lambda(y))) = \lambda(y) \]
and this convergence in uniform simply because the convergence \( \rho_n \circ \iota_X \to \text{id} \) is uniform.

For the other implication we will use Proposition 2.2. Suppose we are given \( \theta : X \to Y \) with \( Y_n \) decreasing closed sets such that \( \bigcap_n Y_n = Y \). Here \( \theta \) is assumed to be a continuous embedding, but we can go further and select compatible metrics on \( X \) and \( Y \) so that \( \theta \) is an isometry. We apply the assumed condition to \( \theta^{-1} : \theta(X) \to X \) and so find continuous functions \( r_n : Y_n \to X \) with 
\[ \lim_{n \to \infty} r_n(y) = \theta^{-1}(y) \]
uniformly over \( y \) in \( \theta(Y) \). As \( \theta \) is an isometry, this is equivalent to 
\[ \lim_{n \to \infty} r_n(\theta(x)) = x \]
uniformly over \( x \) in \( X \). \( \square \)

Now we head the other way, looking for a very restrictive approximate retraction problem that will be useful for showing a space is an AANR. We use it when, as is so often the case, \( X \) is given to us as a compact subset in Euclidean space and so sits inside a hypercube, or in some other absolute retract (AR).

**Proposition 2.4.** Suppose \( X \) is a closed subset of \( Q \) where \( Q \) is an absolute retract. Suppose \( X_1 \supseteq X_2 \supseteq \cdots \) are closed subset with \( \bigcap_n X_n = X \) and where for each \( n \) the interior of \( X_n \) contains \( X \). Then \( X \) is an AANR if, and only if, there is a sequence of continuous functions \( r_n : X_n \to X \) so that 
\[ \lim_{n \to \infty} r_n(x) = x \]
uniformly over \( x \) in \( X \).
Proof. The only nontrivial implication is the backwards one.
Suppose we are given a continuous embedding \( \theta : X \to Y \) into a compactum \( Y \) and a sequence \( Y_1 \supseteq Y_2 \supseteq \cdots \) of closed subsets of \( Y \) with \( \bigcap Y_n = \theta(X) \). Consider the diagram

\[
\begin{array}{ccc}
Q & \to & Y \\
\downarrow & & \downarrow \\
X_n & \to & Y_k' \\
\downarrow & & \downarrow \\
X & \to & \theta(\tilde{X}) \\
\end{array}
\]

We apply the extension property of AR spaces to the map \( \theta^{-1} \) to get \( \alpha : Y \to Q \) so that \( \alpha(\theta(x)) = x \) for all \( x \) in \( X \). We also have the assumed \( r_n \) which we indicate now as well,

\[
\begin{array}{ccc}
Q & \leftarrow & Y \\
\downarrow & & \downarrow \\
X_n' & \leftarrow & Y_k' \\
\downarrow & & \downarrow \\
X & \leftarrow & \theta(\tilde{X}) \\
\end{array}
\]

where the diagram is commutative except as it involves \( r_n \), where we have \( r_n(x) \to x \) uniformly over \( x \) in \( X \).

Next we calculate the intersection of the \( \alpha(Y_k) \). Easily we see

\[
\bigcap_k \alpha(Y_k) \supseteq \alpha \left( \bigcap_k Y_k \right) = \alpha(\theta(X)) = X
\]

and so suppose \( p \) is in \( \bigcap_k \alpha(Y_k) \). This means \( p = \alpha(y_k) \) for \( y_k \) in \( Y_k \).

The ambient space \( Y \) is compact, so we can pass to a subsequence \( y_{k_\ell} \) so that the limit \( y \) exists. Notice \( y \) must be in \( \theta(X) \) by the assumptions on the \( Y_k \) and so

\[
p = \lim_{\ell} p = \lim_{\ell} \alpha(y_{k_\ell}) = \alpha \left( \lim_{\ell} y_{k_\ell} \right) = \alpha(y) \in \alpha(\theta(X)) = X,
\]

establishing the expected equality

\[
\bigcap_k \alpha(Y_k) = X.
\]

Fix \( n \). The sets \( \alpha(Y_k) \) are compact and decreasing, so are eventually contained the interior of \( X_n \). We can find a subsequence \( Y_{k_1}, Y_{k_2}, \ldots \)
so that \( \alpha(Y_{k_n}) \subseteq X_n \). We define \( \rho_{k_n} : Y_{k_n} \to X \) by
\[
\rho_{k_n}(y) = r_n(\alpha(y)).
\]
We define \( \rho_1 \) though \( \rho_{k_{1-1}} \) at random and for \( \ell \) strictly between \( k_n \) and \( k_{n+1} \) we define \( \rho_{\ell} \) to be the restriction of \( \rho_{k_n} \) to \( Y_{\ell} \) to ensure \( \rho_{\ell}(\theta(x)) \) is the same as \( \rho_{k_n}(\theta(x)) \) except with each term possibly repeated. This will have no effect on the uniformity of the convergence. For any \( x \) in \( X \) we find
\[
\lim_{\ell \to \infty} \rho_{\ell}(\theta(x)) = \lim_{n \to \infty} \rho_{k_n}(\theta(x))
\]
\[
= \lim_{n \to \infty} r_n(\alpha(\theta(x)))
\]
\[
= \alpha(\theta(x))
\]
\[
= x
\]
and the convergence in uniform because \( r_n \to \text{id} \) uniformly and \( \theta \) must be uniformly continuous. \( \square \)

**Example 2.5.** A standard example of an AANR is the topologist’s sine curve \( X \) as illustrated in Figure 1. This is moreover an AAR, meaning an approximative absolute retract, as was observed by Clapp [4]. The essential argument here is that the square in which \( X \) is embedded can be mapped to \( X \) so as to fix the points of \( X \) except for those in a small region on the left of the square. These are to be mapped a little horizontally to a segment in \( X \). The rest of is mapped vertically to \( X \). This approximate retraction is illustrated in Figure 4.

**Example 2.6.** Joining two copies of the topologist’s sine curve at a point, as indicated in Figure 1 leads to a space \( X \) that is not an AANR. Consider the closed neighborhoods \( X_n \) of \( X \) as illustrated in...
Figure 5. In black the space $X$ as in Figure 1, and in gray an example of the neighborhoods $X_n$ used in Example 2.6.

Each $X_n$ consists of a V-shaped bar in the center and a thin strip around the zig-zag away from the center. These $X_n$ are all path connected, and $X = \bigcap_n X_n$. Were $X$ an AANR then Proposition 2.2 would give us maps $r_n : X_n \to X$ that move points in $X$ by no more than a given $\epsilon$. Since $X$ has three path-components, $r_n(X)$ must lie entirely in one of these path-components, so in the left zig-zag of $X$, the right zig-zag of $X$, or the middle V-shape of $X$, all of which have diameter less than the diameter of $X$. This contradicts the fact that the two outermost points of $X$ are moved very little by $r_n$.

Example 2.7. Joining two copies of the topologist’s sine curve at different point, as indicated in Figure 5, leads to a space $X$ that is an AANR, and indeed an AAR. As in Example 2.5 we can approximately retract a rectangle in the surrounding Euclidean space to $X$. This is illustrated in Figure 6.

3. PAANR spaces

We wish to rework Section 2 for pointed compacta. In contrast to the situation regarding ANR spaces, the PAANR property will depend on the choice of point. We already encountered this dependence when studying, in [12], pointed approximative absolute retracts (PANR).

Definition 3.1. A pointed compactum $(X, x_0)$ is a pointed approximative absolute neighborhood retract (PAANR) if, for every homeomorphic embedding $\theta : X \to Y$ of $X$ into a compact metric space $(Y, d)$, and for every $\epsilon > 0$, there exists $\delta > 0$ and a continuous function $r : U_\delta \to X$ so that

$$r(\theta(x_0)) = x_0$$
and
\[ d(r \circ \theta(x), x) \leq \epsilon \]
for all \( x \) in \( X \), where \( U_\delta \) is as in (1).

We could just as well have asked that \((Y, y_0)\) be a pointed compactum with compatible metric \( d \) and that \( \theta \) and \( r \) be pointed maps. As before we wish to replace the \( U_\delta \) with more general closed sets that decrease to \( \theta(X) \). The sets need not be neighborhoods of \( \theta(X) \), although later we will require this when we devise a method for proving that a closed subset, with chosen point, of an AR is a PAANR.

**Theorem 3.2.** A pointed compactum \((X, x_0)\) is a PAANR if, and only if, for every continuous embedding \( \theta : X \to Y \) of \( X \) into a compactum \( Y \), and for every sequence \( Y_1 \supseteq Y_2 \supseteq \cdots \) of closed subsets of \( Y \) with \( \bigcap Y_n = \theta(X) \), there exists a sequence of continuous functions \( r_n : Y_n \to X \) so that
\[ r_n(\theta(x_0)) = x_0 \]
and
\[ \lim_{n \to \infty} r_n(\theta(x)) = x \]
uniformly over \( x \) in \( X \).

**Proof.** The reverse implication is once again trivial.

Assume \( X \) is an PAANR and that the \( Y_n \) are given. Let \( d \) be a compatible metric on \( Y \). For a given \( k \) we know there is a \( \delta > 0 \) and continuous map \( r : U_\delta \to X \) so that \( r(\theta(x_0)) = x_0 \) and
\[ d(r(\theta(x)), x) \leq \frac{1}{k} \]
for all \( x \) in \( X \). The same argument used for Proposition 2.2 shows we have \( Y_{nk} \subseteq U_1 \) for some \( n \) and we can again use the restrictions of \( r \) to various \( Y_{nk} \).

**Theorem 3.3.** A pointed compactum \((X, x_0)\) is a PAANR if, and only if, for every closed subset \( Y \) of a compact metrizable space \( Z \), for every \( y_0 \in Y \), for every sequence \( Y_1 \supseteq Y_2 \supseteq \cdots \) of closed subsets of \( Z \) with \( \bigcap Y_n = Y \), and for every continuous function \( \lambda : Y \to X \) for which \( \lambda(y_0) = x_0 \), there is a sequence of continuous functions \( \lambda_n : Y_n \to X \) so that

\[
\lambda_n(y_0) = \lambda(y_0)
\]

and

\[
\lim_{n \to \infty} \lambda_n(y) = \lambda(y)
\]

uniformly for \( y \) in \( Y \). To summarize in a diagram:

\[
\begin{array}{c}
(Z, y_0) \\
\downarrow \lambda_n \\
(Y_n, y_0) \\
\downarrow \lambda \\
(X, x_0) \leftarrow (Y, y_0)
\end{array}
\]

**Proof.** We need only modify in a few places the proof of Proposition 2.3.

In the proof of the reverse implication, the additional assumption \( \lambda(y_0) = x_0 \) means that \( y_0 \) and \( x_0 \) get identified in the push-out, or more precisely \( \iota_X(x_0) = \iota_Z(y_0) \). Instead of invoking Proposition 2.2 we invoke Theorem 3.2 which gives us

\[
\rho_n : \iota_X(X) \cup \iota_Z(Y_n) \to X
\]

so that

\[
\lim_{n \to \infty} \rho_n \circ \iota_X(x) = x
\]

uniformly over \( x \) in \( X \) and

\[
\rho_n \circ \iota_X(x_0) = x_0.
\]

As before, \( r_n : Y_n \to X \) is defined by

\[
r_n(y) = \rho_n(\iota_Z(y))
\]

and we get the same uniform convergence \( r_n(y) \to \lambda(y) \), but additionally we find

\[
r_n(y_0) = \rho_n(\iota_Z(y_0)) = \rho_n(\iota_X(x_0)) = x_0.
\]
Going in the other direction, we started with $\theta : X \to Y$ an embedding, and now find continuous functions $r_n : Y_n \to X$ with
\[
\lim_{n \to \infty} r_n(y) = \theta^{-1}(y)
\]
and
\[
r_n(\theta(x)) = \theta^{-1}(\theta(x))
\]
and so get the needed additional conclusion $r_n(\theta(x_0)) = x_0$. \hfill \Box

**Theorem 3.4.** Suppose $X$ is a closed subset of $Q$ where $Q$ is an absolute retract, and that $x_0$ is a point in $X$. Suppose $X_1 \supseteq X_2 \supseteq \cdots$ are closed subset with $\bigcap X_n = X$ and where for each $n$ the interior of $X_n$ contains $X$. Then $(X, x_0)$ is an AANR if and only if there is a sequence of continuous functions $r_n : X_n \to X$ so that
\[
r_n(x_0) = x_0
\]
and
\[
\lim_{n \to \infty} r_n(x) = x
\]
uniformly over $x$ in $X$.

**Proof.** The proof of Proposition 2.4 can be modified as follows, where it is again only the backwards implication that involves any work. We are starting with the additional assumption that $r_n(x_0) = x_0$ and so at the end of the proof we can calculate
\[
\rho_\ell(\theta(x_0)) = \rho_{k_n}(\theta(x_0)) = r_n(\alpha(\theta(x_0))) = r_n(x_0) = x_0.
\]
\hfill \Box

For the record, we have an obvious implication.

**Proposition 3.5.** If $(X, x_0)$ is a PAANR then $X$ is an AANR.

The reverse implication fails. The example is the same example that showed in [12] that a pointed compacta can fail to be a pointed approximative absolute retract (PAAR) while the underlying space is AAR.

**Example 3.6.** Consider the topologist’s sine curve $X$ as illustrated in Figure 1(a), and the point $x_1$ as in Figure 1(b). The approximate retractions shown in Figure 4 all fix $x_1$ and so $(X, x_1)$ is a PAANR.

**Example 3.7.** Consider the topologist’s sine curve $X$ with the point $x_0$ from Figure 1(b). Consider the neighborhoods of $X$ indicated in Figure 7. These are path connected, and as our approximate retracts are required to fix $x_0$ all of the neighborhood must be mapped into the left edge of $X$. This is incompatible with the requirement that we approximately fix $x_1$ and so $(X, x_0)$ is not a PAANR.
Figure 7. In black the space $X$ as in Figure 1 and in gray an example of the neighborhoods $X_n$ used in Example 3.7.

An alternative to “ignoring the special point” is to appoint an uninteresting point to fill the “special role.” Starting with compact space $X$ we can go ahead and take the one-point compactification $\alpha X = X \cup \{\infty\}$, which leads to a compact space which is the old space plus a new isolated point. This leads us to the following, which will look a lot more interesting when dualized to be about $C^*$-algebras, in Theorem 4.13.

**Proposition 3.8.** Suppose $(X, x_0)$ is a pointed compactum. Then $X$ is an AANR if, and only if, the extension property in Theorem 3.3 holds in the special case where $y_0$ is an isolated point in $Y$.

**Proof.** Assume first $X$ is an AANR. We are given $\lambda : Y \rightarrow X$ where $Y$ is compact and $y_0$ is isolated, and we are given $\lambda : Y \rightarrow X$ with $\lambda(y_0) = x_0$, and finally have $Y_1 \supseteq Y_2 \supseteq \cdots$ compact sets with $Y = \bigcap_n Y_n$. Since $y_0$ is isolated in $Y$ and $Y$ is a compact subset in the compactum $Z$, there are disjoint sets $U$ and $V$ open in $Z$ with

$$\{y_0\} = Y \cap U$$

and

$$Y \setminus \{y_0\} = Y \cap V.$$ 

The compact sets $Y_n \setminus (U \cap V)$ are decreasing with intersection

$$\bigcap_n Y_n \setminus (U \cap V) = Y \setminus (U \cap V) = \emptyset$$

so for some $N$, when $n \geq N$ we have $Y_n = A_n \cup B_n$ where $A_n = Y_n \cap U$ and $B_n = Y_n \cap V$. Without loss of generality, $N = 1$, so we have $Y_n$ written as the disjoint union of closed subsets $A_n$ and $B_n$ with both
forming decreasing chains and
\[ \bigcap A_n = Y_n \setminus \{y_0\} \]
and
\[ \bigcap B_n = \{y_0\}. \]
We can define \( \lambda_n : Y_n \to X \) by \( \lambda_n(y) = \lambda(y_0) \) for all \( y \) in \( B_n \) and use the fact that \( X \) is AAR to define \( \lambda_n \) on \( A \) so that \( \lambda_n(y) \to \lambda(y) \)
uniformly over \( x \) in \( B_n \). This is the desired approximate extension that is an exact extension on \( y_0 \).

Now assume the specialized version of the approximate extension property holds and that we are given \( \lambda : Y \to X \) for \( Y \) a closed subset of a compactum \( Z \). We also have decreasing closed \( Y_n \) with intersection \( Y \). We add to \( Z \) an isolated point \( \infty \) and consider the closed subsets \( Y \cup \{\infty\} \) and \( Y_n \cup \{\infty\} \) of \( Z \cup \{\infty\} \). We can extend \( \lambda \) to a map \( \check{\lambda} \) from \( Y \cup \{\infty\} \) to \( X \) by arbitrarily selecting \( x_0 \) in \( X \) and setting \( \check{\lambda}(\infty) = x_0 \). Then there are continuous functions \( \check{\lambda}_n : Y_n \cup \{\infty\} \to X \) with \( \check{\lambda}_n(\infty) = x_0 \) and \( \check{\lambda}_n(y) \to \check{\lambda}(y) \) uniformly over \( Y \cup \{\infty\} \). The desired functions are the restrictions of the \( \check{\lambda}_n \) to the sets \( Y_n \). □

4. Two flavors of weak semiprojectivity

The analog of being an ANR compactum for a \( C^* \)-algebra is that it is unital and semiprojective. Indeed, Blackadar’s definition of semiprojectivity [1] is modeled on the non-approximative version of our Proposition 2.3. A \( C^* \)-algebra \( A \) will semiprojective if we can solve the partial lifting problem indicated here:

\[
\begin{array}{ccc}
A & \xrightarrow{\psi_n} & B/J_n \\
\downarrow{\pi_n} & & \downarrow{\rho_n} \\
C_n & \xrightarrow{\check{\lambda}_n} & C
\end{array}
\]

**Definition 4.1.** A separable \( C^* \)-algebra \( A \) is semiprojective (SP) if given a \(*\)-homomorphism \( \varphi : A \to B/J \), with \( B \) a separable \( C^* \)-algebra with ideal \( J = \bigcup_n J_n \) and \( J_1 \triangleleft J_2 \triangleleft \cdots \) increasing ideals in \( B \), there exist for some \( n \) a \(*\)-homomorphism \( \psi_n : A \to B/J_n \) so that \( \pi_n \circ \psi_n(a) = a \) for all \( a \) in \( A \).

We are using the \( \pi_n \) to be the surjection defined by \( \pi_n(b + J_n) = b + J \).

Weak semiprojectivity can be found by weakening this partial lifting problem in two seemingly different ways. We can either restrict the
allowed $B$ and $J_n$ and keep the exact lifting requirement $\pi_n \circ \psi_n(a) = a$, or we can leave the allowed $B$ and $J_n$ alone and only ask that $\pi_n \circ \psi_n(a) \to a$ for all $a$ in $A$.

Remark 4.2. If $A$ is commutative we can define weak semiprojectivity within the commutative category. We can do the same for all the variations on semiprojectivity that follow.

Definition 4.3. A separable $C^*$-algebra $A$ is weakly semiprojective (WSP) if given a $*$-homomorphism $\varphi: A \to B/J$, with $B$ a separable $C^*$-algebra with ideal $J = \bigcup_n J_n$ and $J_1 \triangleleft J_2 \triangleleft \cdots$ increasing ideals in $B$, there exist a sequence of $*$-homomorphism $\psi_n: A \to B/J_n$ so that $\pi_n \circ \psi_n(a) \to a$ for all $a$ in $A$.

It was shown in [6] that this is equivalent to the original definition [11] of weak semiprojectivity. In that formulation, $B$ is always an infinite product $B = \prod B_n$ and $J_n = B_1 \oplus \cdots \oplus B_n$. It then is possible to interleave any sequence of approximate partial liftings into an exact lifting to $B$.

A $C^*$-algebra is semiprojective if and only if its unitization is $\tilde{A}$ is semiprojective, as was shown in [1]. For weak semiprojectivity this fails. We show this in Example 4.7, with the aid of the following lemmas.

Proposition 4.4. If $A$ is separable and WSP then $\tilde{A}$ is WSP.

Proof. Assume $A$ is WSP and that we have $\varphi: \tilde{A} \to B/J$ and the chain of ideals $J_n$. For some $n$ we can lift $\varphi(1)$ to a projection $p$ in $B/J_n$. Here we have used the semiprojectivity of $C$ (Lemma 4.2.2 in [11]) or the usual argument involving functional calculus and lifting the relations $x^* = x^2 = x$. Consider $C = p(B/J_n)p$, which is a unital $C^*$-subalgebra of $B/J_n$, and $K_m = J_m \cap C$ is an ideal of $C$, and the diagram

$$
\begin{array}{ccc}
C/K_m & \longrightarrow & B/J_m \\
\downarrow \pi_m & & \downarrow \pi_m \\
A & \longrightarrow & B/J
\end{array}
$$

where

$$K = \bigcap_{m \geq n} K_m = J \cap C$$

and the horizontal maps are induced by the inclusion of $C$ into $B$. Applying the weak semiprojectivity of $\tilde{A}$ we find $\tilde{\psi}_m : A \to C/K_m$ with $\tilde{\pi}_m \circ \tilde{\psi}_m \to \varphi_0 \circ \iota$. Since $C$ is unital we can extend this to $\tilde{\psi}_m : A \to \tilde{A}$.
\[ \tilde{A} \to C/K_m \text{ with } \tilde{\pi}_m \circ \tilde{\psi}_m \to \varphi. \] Finally, we use \( \alpha \circ \tilde{\psi}_m \) as the needed approximate lifts \( A \to B/J_m \) for \( m \geq n \), filling in with the zero map for \( m < n \). \( \square \)

**Proposition 4.5.** Let \( A \) be a separable \( C^* \)-algebra. If \( A \) is weakly projective then \( A \) is weakly semiprojective.

**Proof.** The definition given in [12] of \( A \) being weakly projective (WP) is that we can approximately solve a lifting problem

\[
\begin{array}{ccc}
B & \xrightarrow{\pi} & B/J \\
\downarrow \alpha & & \\
A & \longrightarrow & B/J
\end{array}
\]

and so we can easily solve the WSP lifting problem. \( \square \)

We trust that the reader has noticed that the diagram (5) is the dual of the diagram (4). There is much that can said about the connection between AANR spaces and WSP \( C^* \)-algebras—see [1, 2.8-9] and [14, Theorem 1.3]—but all that really concerns us at the moment is that if we want \( C_0(X) \) to be WSP then a necessary condition is that \( X \) be a PAANR.

**Theorem 4.6.** Suppose \( X \) is a locally compact, metrizable space.

1. If \( C_0(X) \) is WSP then \( (\alpha X, \infty) \) is a PAANR.
2. If \( (\alpha X, \infty) \) is a PAANR then \( C_0(X) \) is WSP within the commutative category.

**Proof.** Theorem 3.3 tells us that to show \( (\alpha X, \infty) \) is a PAANR, we need to handle the approximate extension as in diagram (4). In terms of the induced \( \ast \)-homomorphisms, \( \lambda \) and the inclusions give us the diagram

(6)

\[
\begin{array}{ccc}
C_0(Z, y_0) & \downarrow & \\
\downarrow \varphi_n & & \\
C_0(Y_n, y_0) & \xrightarrow{\rho_n} & C_0(Y, y_0) \\
\downarrow (\alpha X, \infty) \xrightarrow{\lambda_*} & & \\
C_0(Y, y_0)
\end{array}
\]

where Definition [11] provides us with \( \varphi_n \) as in the diagram with

\[
\lim_{n \to \infty} \| \rho_n \circ \varphi_n(f) - f \circ \lambda \| = 0
\]
for all $f$ in $C_0(\alpha X, \infty)$. Since $\varphi_n$ is induced by some map $\lambda_n$ of pointed compacta, this is saying

$$\lim_{n \to \infty} \sup_{y \in Y} |f(\lambda_n(y)) - f(\lambda(y))| = 0.$$ 

As this is true for all $f$, we conclude $\lambda_n(y) \to \lambda(y)$ uniformly over $y$ in $Y$. For the second claim, we note simply that in the commutative situation, up to isomorphism the only liftings we need are those in (6).

**Example 4.7.** Consider $A_0 = C_0(X \setminus \{x_0\})$ and $A_1 = C_0(X \setminus \{x_0\})$, where $X$ is the topologist’s sine curve and the named points are as in Figure 1(b). The claim is that $A_1$ and $\tilde{\tilde{A}}_0 \cong \tilde{A}_0 \cong C(X)$ are all WSP, while $A_0$ is not WSP. In [12] we showed that $A_1$ is WP, and so Propositions 4.4 and 4.5 imply that $\tilde{\tilde{A}}_0$ is WSP. Of course this means $\tilde{\tilde{A}}_0$ is WSP. Example 3.7 shows $(X, x_0)$ is not a PAANR, so by Lemma 4.6, $A_0$ is not WSP.

We do find that the WSP property behaves well with direct sums.

**Theorem 4.8.** Suppose $A_1$ and $A_2$ are separable $C^*$-algebras. Then $A_1 \oplus A_2$ is WSP if, and only if, both $A_1$ and $A_2$ are WSP.

**Proof.** Suppose $A_1 \oplus A_2$ is weakly semiprojective and that we are given $\varphi : A_1 \to B/J$ where $J$ is an ideal, etc. We utilize the inclusion $i_j: A_j \to A_1 \oplus A_2$ and the projection $\gamma_j: A_1 \oplus A_2 \to A_j$ in considering the diagram

$$\begin{array}{ccc}
B & \to & B/J \\
\downarrow & & \downarrow \\
B/J_n & \to & B/J \\
\pi_n & & \\
A_j & \xrightarrow{i_j} & A_1 \oplus A_2 & \xrightarrow{\gamma_j} & A_j & \xrightarrow{\varphi} & B/J \\
& & & & \\
\end{array}$$

We have $\varphi_n : A_1 \oplus A_2 \to B/J_n$ with $\pi_n \circ \varphi_n(x) \to \varphi \circ \gamma_j(x)$ for all $x$ in $A_1 \oplus A_2$. Therefore

$$\lim \|\varphi(a) - \pi_n \circ \varphi_n \circ i_j(a)\| = \lim \|\varphi \circ \gamma_j(t_j(a)) - \pi_n \circ \varphi_n(t_j(a))\| = 0.$$ 

Now assume $A_1$ and $A_2$ are weakly semiprojective and that we are given $\varphi : A_1 \oplus A_2 \to B/J$ and so forth. Let $h_1$ and $h_2$ be strictly positive elements in $A_1$ and $A_2$ and consider $\varphi((h_1, 0))$ and $\varphi((0, h_2))$. These are orthogonal positive elements, and so can be lifted to orthogonal positive elements $k_1$ and $k_2$ in $B$. The argument here depends on the projectivity of $C_0[0, 1] \oplus C_0[0, 1]$, which is equivalent to the argument
that orthogonal, positive contractions lift to orthogonal, positive contractions, Proposition 10.1.10 in [11]. Inside $B$ we form $B_j = k_jBk_j$, that is the hereditary subalgebra generated by $k_j$, and as these two $C^*$-subalgebras are orthogonal, we have the copy $B_1 + B_2$ of $B_1 \oplus B_2$ in $B$. The image of $B_1$ under $\pi$ includes $\varphi((h_1, 0))(B/J)\varphi((h_1, 0))$ and so all of $\varphi(A_1 \oplus 0)$. Similarly $\pi(B_2)$ contains $\varphi(0 \oplus A_2)$ and so the image of $\varphi$ is contained in the image of $\pi$. If we consider $\pi$ restricted to $B_j$ we find it has kernel $J \cap k_jBk_j = k_jJk_j$.

where for the inclusion of left into right we use the approximate identity $k_j^t$ for $B_j$. We have also a chain of ideals $k_jJ_nk_j$ with intersection $k_jJk_j$. We have then a commutative diagram

\[
\begin{array}{c}
A_1 \oplus A_2 \xrightarrow{\varphi_0} B_1 / k_1Jk_1 \oplus B_2 / k_2Jk_2 \xrightarrow{\alpha} B / J_n \\
\downarrow \pi_n \oplus \pi_n \quad \downarrow \pi_n \\
B_1 / k_1Jk_1 \oplus B_2 / k_2Jk_2 \xrightarrow{\alpha} B / J
\end{array}
\]

and it is evident how we can use approximate lifting of maps from $A_1$ and $A_2$ to create the desired approximate lifting of the maps from $A_1 \oplus A_2$.

\[\square\]

Remark 4.9. It is important to note that we used the fact that $hBh$ was again a $C^*$-algebra when $h$ in $B$ is positive, and will lose this technique when we restrict to lifting problems involving only unital $C^*$-algebras.

Definition 4.10. Let $A$ be a separable $C^*$-algebra $A$, not necessarily unital. We say $A$ is weakly semiprojective with respect to unital $C^*$-algebras (WSP1) if we can solve the partial approximate lifting problem in Definition 4.1 in the special case where $B$ is a unital separable $C^*$-algebra.

There is already a definition of weakly semiprojective with respect to the class of all unital $C^*$-algebras, Definition 5.2 in [6], but it is equivalent to the one given here.

Remark 4.11. Even if $A$ has a unit, we are not requiring $\varphi$ or the $\varphi_n$ to be unital. In particular, if we have $\varphi_n : A \to B/J_n$ with $\pi_n \circ \psi_n(a) \to a$ then we can pad this out with zero maps, and use the intermediate quotient maps $B/J_n \to B/J_{n+1}$, to get the required sequence $\varphi_n : A \to B/J_n$. This was true for weak semiprojectivity. A common
formulation of weak semiprojectivity is that given \(a_1, \ldots, a_r\) and \(\epsilon > 0\) there is \(\psi : A \to B/J_n\) for some \(n\) with \(\|\pi_n \circ \psi(a_j) - \varphi(a_j)\| < \epsilon\).

**Theorem 4.12.** If \(A\) is a separable \(C^*\)-algebra then following are equivalent:

1. \(A\) is WSP1;
2. the partial approximate lifting problem in Definition 4.1 can be solved whenever \(B\) is a unital \(C^*\)-algebra (so \(B\) is not necessarily separable);
3. Given a \(*\)-homomorphism \(\varphi : A \to \prod B_k/\bigoplus B_k\) with \(B_1, B_2, \ldots\) a sequence of unital \(C^*\)-algebras, there is a \(*\)-homomorphism \(\overline{\varphi} : A \to \prod B_k\) so that \(\kappa \circ \overline{\varphi} = \varphi\). Here the sum and products are indexed by \(\mathbb{N}\) and \(\kappa\) is the quotient map.

**Proof.** Clearly (2) implies (1). For the reverse, assume we are facing \(\varphi : A \to B/J\) and so forth with \(B\) not countable. Take a countable dense subset in \(A\), push these forward with \(\varphi\) to the quotient and then take a random lift of this set to a countable set in \(B\). Let \(\hat{B}\) be the \(C^*\)-algebra generated by this set and the unit, and \(\hat{J}_n = J_n \cap \hat{B}\). These nested ideals of \(\hat{B}\) have intersection \(\hat{J} = J \cap \hat{B}\) and we can factor \(\varphi\) through \(\hat{B}/\hat{J}\), which we treat as a subset of \(B/J\), leading us to

\[
\begin{array}{ccc}
\hat{B}/\hat{J}_n & \longrightarrow & B/J_n \\
\downarrow & & \downarrow \\
A & \xrightarrow{\varphi_0} & \hat{B}/\hat{J} \\
\varphi & \longrightarrow & B/J
\end{array}
\]

which commutes and has \(\hat{B}\) separable and unital. The approximate partial lifts of \(\varphi_0\) can be composed with the inclusions of the \(\hat{B}/\hat{J}_n\) into the \(B/J_n\), solving the problem.

For the equivalence of (2) and (3) we note that the proof of Theorem 3.1 in [6] works just as written in the case where the various target \(C^*\)-algebras are unital. What is essential is that the class of unital \(C^*\)-algebras is closed under quotients and countable direct sums. \(\square\)

**Theorem 4.13.** Suppose \(A\) is a separable \(C^*\)-algebra. Then \(A\) is WSP1 if, and only if, \(\tilde{A}\) is WSP.

**Proof.** The proof of the forward implication of Proposition 4.4 works here to give the forward implication. Notice that we started with \(B\) possibly lacking a unit, but then cut down by a projection to create a \(C^*\)-subalgebra \(C\) that was unital.
Now suppose $\widetilde{A}$ is WSP and we are given a map from $A$ over to a unital situation we can extend to $\widetilde{A}$ using the unit in $B$. Lift the bigger $C^*$-algebra and the smaller goes along from the ride.

Assume $\widetilde{A}$ is WSP and we have $\varphi : A \to B/J$ with $B$ unital. Since $B/J$ is also unital, we can extend $\varphi$ to a map $\hat{\varphi} : \widetilde{A} \to B/J$ and so we arrive at this diagram

$$
\begin{array}{ccc}
B & \to & B/J \\
\downarrow & & \downarrow \\
B/J_n & \to & B/J \\
\psi_n & \nearrow & \pi_n \\
A & \to & \widetilde{A} & \to & B/J \\
\phi & \nearrow & \hat{\varphi}
\end{array}
$$

where we use the assumption on $\widetilde{A}$ to find $\psi_n : \widetilde{A} \to B/J_n$ with $\pi_n \circ \psi_n \to \hat{\varphi}$. The desired approximate lifts are the compositions $\iota \circ \psi_n$. □

**Theorem 4.14.** Suppose $A$ and $B$ are separable $C^*$-algebras. If $A \oplus B$ is WSP1 then both $A$ and $B$ are WSP1.

**Proof.** The proof of the forward direction of Theorem 4.8 works here just as well. □

**Theorem 4.15.** If $A$ is unital then $A$ is WSP1 if, and only if, $A$ WSP.

**Proof.** When $A$ is unital, $\widetilde{A} \cong A \oplus \mathbb{C}$, so by Theorem 4.13

$$
A \text{ is WSP1 } \iff A \oplus \mathbb{C} \text{ is WSP.}
$$

By Theorem 4.8

$$
A \oplus \mathbb{C} \text{ is WSP } \iff A \text{ is WSP and } \mathbb{C} \text{ is WSP.}
$$

We are done, since $\mathbb{C}$ is famously SP and so WSP. □

**Theorem 4.16.** Suppose $X$ is a locally compact, metrizable space.

1. If $C_0(X)$ is WSP1 then $\alpha X$ is an AANR.
2. If $\alpha X$ is an AANR then $C_0(X)$ is WSP1 within the commutative category.

**Proof.** This follows from Proposition 2.3 and Definition 4.10 by essentially the argument used for Theorem 4.6. □

**Example 4.17.** Consider $x_0$ in the topologist’s sine curve, as in Figure 1(b). Then $A = C_0(X \setminus \{x_0\})$ is WSP1 since $\widetilde{A} \cong C(X)$. However $(A \oplus A)^\sim \cong C(Y)$ where $Y$ is the space in Figure 2 and $Y$ is not an AANR, so $A \oplus A$ is not WSP1.
References

[1] Bruce Blackadar. Shape theory for $C^*$-algebras. *Math. Scand.*, 56(2):249–275, 1985.

[2] J.W. Cannon and G.R. Conner. On the fundamental groups of one-dimensional spaces. *Topology Appl.*, 153(14):2648–2672, 2006.

[3] A. Chigogidze and AN Dranishnikov. Which compacta are noncommutative ARs? *Topology Appl.*, 157(4):774–778, 2010.

[4] Michael H. Clapp. On a generalization of absolute neighborhood retracts. *Fund. Math.*, 70(2):117–130, 1971.

[5] Marius Dadarlat. Shape theory and asymptotic morphisms for $C^*$-algebras. *Duke Math. J.*, 73(3):687–711, 1994.

[6] Søren Eilers and Terry A. Loring. Computing contingencies for stable relations. *Internat. J. Math.*, 10(3):301–326, 1999.

[7] Søren Eilers, Terry A. Loring, and Gert K. Pedersen. Stability of anticommutation relations: an application of noncommutative CW complexes. *J. Reine Angew. Math.*, 499:101–143, 1998.

[8] Søren Eilers, Terry A. Loring, and Gert K. Pedersen. Morphisms of extensions of $C^*$-algebras: pushing forward the Busby invariant. *Adv. Math.*, 147(1):74–109, 1999.

[9] D. Enders. A characterization of weak (semi-) projectivity for commutative C*-algebras. *Arxiv preprint arXiv:1102.3303*, 2011.

[10] Don Hadwin and Weihua Li. A note on approximate liftings. *Oper. Matrices*, 3(1):125–143, 2009.

[11] Terry A. Loring. *Lifting solutions to perturbing problems in $C^*$-algebras*, volume 8 of *Fields Institute Monographs*. American Mathematical Society, Providence, RI, 1997.

[12] Terry A. Loring. Weakly projective $C^*$-algebras. *Rocky Mountain J. Math.*, to appear.

[13] S. Mardešić and J. Segal. History of shape theory and its application to general topology. *Handbook of the history of general topology*, page 1145, 2001.

[14] A.P.W. Sørensen and H. Thiel. A characterization of semiprojectivity for commutative C*-algebras. *Arxiv preprint arXiv:1101.1856*, 2011.

[15] Jack Spielberg. Weak semiprojectivity for purely infinite $C^*$-algebras. *Canad. Math. Bull.*, 50(3):460–468, 2007.

Department of Mathematics and Statistics, University of New Mexico, Albuquerque, NM 87131, USA.

URL: http://www.math.unm.edu/~loring/