On regular frames near rotating black holes

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We consider the metric of a generic axially symmetric rotating stationary black hole. The general approach is developed that enables us to construct coordinate frame regular near the horizon. As explicit examples, the Kerr and Kerr-Newmann-(anti-)de Sitter metrics are considered. It is shown how the rotational versions of the Painlevé-Gullstrand and Doran coordinates appear in this scheme as particular cases. For the 2+1 version of the metric the direct generalization of the Lemaître coordinate system is obtained. It is shown that the possibility of introducing a regular frame is indirectly related to the constancy of a black hole angular velocity and the rate with which the metric coefficient responsible for the rotation of spacetime, tends to it.

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I. INTRODUCTION

The metric of the Kerr black hole is well known [1]. Nonetheless, its description continues to attract attention since different coordinate frames are relevant in different physical contexts. First of all, it concerns necessity to have well-defined coordinates near the horizon where the standard Boyer-Lindquist ones [2] fail. To this end, new coordinate systems were suggested that can be considered as generalization of the Painlevé-Gusstland coordinates from the Schwarzschild case or their modification [3], [4]. Meanwhile, in real astrophysical circumstances black holes are surrounded by matter, so their metric may differ from the pure

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Kerr solutions thus representing so-called "dirty" black holes. This leads to the question of finding regular frames near quite generic rotating black holes. Also, this problem is of obvious general interest on its own right. In doing so, we face with two different but tightly related problems: (i) how to describe coordinate transformations that brings the metric into a regular form thus generalizing the procedure found previously for the Kerr solution, (ii) for a given metric, how to unify different transformations, so that coordinate frames of Refs. [3], [4] or their counterpart for dirty black holes be particular cases of some general scheme.

Below, we describe approach that resolves both issues. We check it for the Kerr and Kerr-Newman-(anti-)de Sitter metric and show that it agrees with previous findings. We also demonstrate our approach for the rotational analogue of the Bertotti-Robinson metric.

II. FORM OF METRIC

Let us consider the generic stationary axially symmetric metric. Its form can be written as

\[ ds^2 = -N^2 dt^2 + g_\phi (d\phi - \omega dt)^2 + d\eta^2 + g_{zz} dz^2, \]  

(1)

where all metric coefficient do not depend on \( t \) and \( z \) (see, e.g. Sec. II and especially eq. (11) in Ref. [5]). Introducing a new variable according to \( d\eta = \frac{dr}{A(r)} \), one can rewrite it in the form

\[ ds^2 = -N^2 dt^2 + g_\phi (d\phi - \omega dt)^2 + \frac{dr^2}{A(r)} + g_{zz} dz^2. \]  

(2)

Here, \( A = A(r) \) does not depend on \( z \). In principle, it is sufficient to consider further the metric in the form (2). However, the famous Kerr metric and its extension to the presence of the electric charge and cosmological constant do not have such a form in the Boyer-Lindquist coordinates or their generalization. To handle properly relevant generalizations of the Kerr metric, we will consider somewhat different form

\[ ds^2 = -N^2 dt^2 + g_\phi (d\phi - \omega dt)^2 + \frac{dr^2}{A(r, \theta)} + g_{\theta \theta} d\theta^2, \]  

(3)

in which the dependence on \( \theta \) is allowed in . Also, we use the angle variable \( \theta \) instead of \( z \).

We assume that

\[ N^2 = \alpha \Delta_r, \quad A = \frac{\Delta_r}{\rho^2}, \]  

(4)
where \( \Delta_r \) does not depend on \( \theta \), \( \alpha \) and \( \rho \) depend on both \( r \) and \( \theta \). The maximum zero \( r = r_+ \) of \( \Delta_r \) corresponds to the event horizon. It is supposed that \( \alpha \) and \( \rho \) remain finite and nonzero at \( r = r_+ \). Then, the metric can be written as

\[
ds^2 = -N^2 dt^2 + g_\phi(d\phi - \omega dt)^2 + \frac{\rho^2}{\Delta_r} dr^2 + g_\theta d\theta^2. \tag{5}\n\]

This form can be substantiated on the basis of regularity. Indeed, according to the zero law of black hole mechanics \(^6\), the surface gravity

\[
\kappa = \lim_{r \to r_+} \sqrt{(\nabla N)^2} \tag{6}\n\]

should be finite and constant on the event horizon.

For nonextremal black holes, by definition,

\[
N^2 \sim r - r_. \tag{7}\n\]

Let us admit for a moment that the horizon corresponds to \( r_+ (\theta) \). Then, the term \( g^{\theta\theta} \left( \frac{\partial N}{\partial \theta} \right)^2 \) would lead to divergency of the quantity under consideration. To have regular horizon, we exclude the dependence of \( \Delta_r \) on \( \theta \). For extremal horizons, these arguments do not work. However, more refined examination shows that the dependence on \( \theta \) should be excluded in this case as well (see Sec. III C. 2 in \( ^9 \)). It is worth noting that the form (5) with \( \Delta_r = \Delta_r (r) \) is typical of the Kerr, Kerr-Newman-(anti)de Sitter metrics.

### III. COORDINATE TRANSFORMATIONS

Near the horizon the coordinate system (5) fails. Our goal is to find a new one regular in the vicinity of \( r_+ \). To this end, let us make the transformation to new (barred) coordinates according to

\[
dt = \bar{t} + \frac{zdr}{\Delta_r}, \tag{8}\n\]

\[
d\phi = \bar{\phi} + \frac{\xi dr}{\Delta_r} + \delta d\theta, \ r = \bar{r}. \tag{9}\n\]

In principle, one can add the term \( \gamma d\theta \) in (8) but, to avoid unnecessary complication, we put \( \gamma = 0 \). This holds, in particular, in the Painlevé-Gullstrand and Doran coordinate systems in the Kerr metric \(^4\).
As $dt$ should be a total differential, we require that $z$ does not depend on $\theta$, $z = z(r)$.

Now we have
\begin{equation}
    d\phi - \omega dt = d\bar{\phi} - \omega d\bar{t} + h dr + d\theta \delta,
\end{equation}
where by definition
\begin{align}
    h\Delta_r &= \xi - \omega z, \\
    \mu\Delta_r &= \rho^2 - z^2 \alpha,
\end{align}
where we took into account (4). Then,
\begin{equation}
    ds^2 = -N^2 d\bar{t}^2 - 2zadrd\bar{t} + g_\phi(d\bar{\phi} - \omega d\bar{t} + h dr + d\theta \delta)^2 + \mu dr^2 + d\theta^2 g_\theta
\end{equation}
We choose the functions $z$, $h$ to kill divergences in the metric coefficient $g_{rr}$. To this end, we require $h$ and $\mu$ to be bounded on the horizon.

We have from (12) and (13) that
\begin{align}
    \xi &= \omega z + h\Delta_r, \\
    \mu &= \frac{\rho^2 - z^2 \alpha}{\Delta_r}.
\end{align}
The finiteness of $h$ and $\mu$ on the horizon entails the necessity of the conditions
\begin{align}
    (\xi - \omega z)_{r=r^+} &= 0, \\
    (\rho^2 - z^2 \alpha)_{r=r^+} &= 0.
\end{align}
This implies that $\frac{\rho^2}{\alpha}$ does not depend on $\theta$ on the horizon. Also, the integrability condition in (9) requires
\begin{equation}
    \delta_{,r} = \left(\frac{\xi}{\Delta_r}\right)_{,\theta}.
\end{equation}
We can rewrite (14) in the form
\begin{equation}
    ds^2 = -\frac{\alpha \rho^2}{\mu} dr^2 + \mu(dr - V d\bar{t})^2 + g_\phi(d\bar{\phi} - \omega d\bar{t} + h dr + d\theta \delta)^2 + d\theta^2 g_\theta,
\end{equation}
where
\begin{equation}
    V = \frac{\alpha z}{\mu}.
\end{equation}
Thus we see that specifying the functions $z(r)$, and $h$ we obtain a class of coordinate transformation that make the metric regular near the horizon. Other two functions $\xi$ and $\mu$ can be found from (15), (16). Below, we specify some natural choices and analyze important particular cases.
IV. REGULAR FRAMES AND RIGID ROTATION OF THE HORIZON

There are two typical cases. If \( h = 0 \), it follows from (12) that

\[
\xi = \omega z. \tag{22}
\]

This is the generalization of rotational version of the Painlevé-Gillstrand coordinate frame introduced in [4] for the Kerr metric. Now, eq. (19) reduces to

\[
\delta_r = \frac{z}{\Delta} \omega_\theta. \tag{23}
\]

If the horizon is regular in the sense that tetrad component of the curvature tensor measured by free-falling observers remain bounded, the quantity

\[
\omega_\theta = O(N^2) \tag{24}
\]

equal to

near the horizon. This was shown in [9], see Tables I and II there. Then, it is clear from (23) that \( \delta_r \) is finite near the horizon and \( \delta \) can be found by direct integration. As \( \delta \) is finite, frame (20) is regular near the horizon. This agrees with the previous consideration done for the Kerr metric in [4].

The case \( \delta = 0 \) is generalization of the Doran system [3]. In doing so, the function \( \xi \) should depend on \( r \) only since \( d\phi \) (9) has to be exact differential. Then, it follows from (17) that on the horizon, where \( \omega \) has the meaning of a black hole angular velocity \( \omega_H \), this angular velocity must not depend on \( \theta \). But this holds true for regular black holes automatically [12] and agrees with (24).

Thus we see that the regularity of a new coordinate frame not only requires the finiteness of corresponding functions \( \delta \) and \( h \) but, indirectly, relies on rather subtle properties inherent to regular horizons, whatever frame be used. More precisely, the possibility of introducing the frames under discussion is related to the constancy of a black hole angular velocity \( \omega_H \) and, moreover, may depend on the rate with which \( \omega \) tends to \( \omega_H \).

V. PARTICULAR CLASS OF METRICS

The formulas are simplified if one can take

\[
\mu = \alpha r^2. \tag{25}
\]
Then, it follows from (13), (21) that

\[ V = \frac{\sqrt{1 - N^2}}{\rho \sqrt{\alpha}} = \frac{z}{\rho^2}, \]  

(26)

where the combination

\[ \frac{\sqrt{1 - N^2} \rho}{\sqrt{\alpha}} = z(r) \]  

(27)

should depend on \( r \) only.

A. Choice 1

\[ h = 0. \]  

(28)

Taking into account (25), we obtain

\[ ds^2 = -d\bar{t}^2 + \alpha \rho^2 (dr - \frac{z}{\rho^2} d\bar{t})^2 + g_{\phi}(d\bar{\phi} - \omega d\bar{t} + \delta d\theta)^2 + g_{\theta} d\theta^2. \]  

(29)

This can be called the generalized Painlevé-Gullstrand coordinate frame.

According to (19) and (22),

\[ \delta_{,r} = \left( \frac{\omega z}{\Delta_r} \right)_{,\theta} = \frac{z \omega_{,\theta}}{\Delta_r}. \]  

(30)

since \( z \) and \( \Delta_r \) are functions of \( r \) only.

B. Choice 2

Let us choose

\[ \delta = 0. \]  

(31)

In particular, we can take also

\[ h = \frac{\omega \rho^2}{z}, \]  

(32)

then

\[ \xi = \frac{\omega \rho^2}{\alpha z} = \frac{h}{\alpha}. \]  

(33)
Then, with (12) taken into account we have from (20)
\[ ds^2 = -d\bar{t}^2 + g_{rr}(dr - \frac{z}{\rho^2}d\bar{t} + \frac{g_\phi}{g_{rr}}\frac{\omega^2 \rho^2}{z} d\bar{\phi})^2 + g_\phi(1 - \frac{\omega^2 \rho^2}{z^2} \frac{g_\phi}{g_{rr}})d\bar{\phi}^2 + g_\theta d\theta^2, \] (34)
where
\[ g_{rr} = \alpha \rho^2 + \frac{\rho^4 \omega^2}{z^2} g_\phi. \] (35)

This is a generalization of the Doran coordinate system [3].

One important reservation is in order. The above choice (25) implies that the left hand side of eq. (27) is a function of \( r \) only. This is the case for the Kerr metric (see below) but, in general, may or may not be valid. In particular, we will see that for the Kerr-Newman-(anti-)de Sitter metric, another, more complicated ansatz works successfully instead of (25).

**VI. KERR METRIC**

For the Kerr metric in the Boyer-Lindquist coordinates,\n\[ ds^2 = -\frac{\Delta}{\rho^2}(dt - a^2 \sin^2 \theta d\phi)^2 + \frac{\sin^2 \theta}{\rho^2} [adr - (r^2 + a^2)d\phi]^2 + \frac{\rho^2 dr^2}{\Delta r} + \rho^2 d\theta^2, \] (36)
where \( M \) is a black hole mass,\n\[ \rho^2 = r^2 + a^2 \cos^2 \theta, \] (37)
a is the angular momentum per unit mass.

Then, the metric (36) is rewritten in the form (3),
\[ \omega = \frac{2Mr a}{\Sigma \rho^2}, \] (38)
\[ N^2 = \frac{\Delta}{\Sigma}, \] (39)
\[ \alpha = \frac{1}{\Sigma}, \] (40)
\[ \Sigma = r^2 + a^2 + \frac{2Mr a^2}{\rho^2} \sin^2 \theta, \] (41)
\[ \Delta = r^2 - 2Mr + a^2. \] (42)

By substitution into (26) one finds\n\[ z = \sqrt{2Mr(r^2 + a^2)}. \] (43)
One also finds that for choice 2 the expression (33) reads

\[ \xi = \frac{2Mra}{\sqrt{2Mr(r^2 + a^2)}}, \]  

so \( \xi \) is a function of \( r \) only and condition (19) is satisfied.

**VII. KERR-NEWMAN-(ANTI-)DE SITTER**

The metric has the form (see, e.g. Sec. 11.3 in [14])

\[ ds^2 = -\frac{\Delta_r}{\Xi^2 \rho^2} (dt - a \sin^2 \theta d\phi)^2 + \frac{\Delta_\theta \sin^2 \theta}{\Xi^2 \rho^2} [adt - (r^2 + a^2)d\phi]^2 + \frac{\rho^2 dr^2}{\Delta_r} \right. \left. \frac{\rho^2 d\theta^2}{\Delta_\theta} + \rho^2 d\tau^2, \]  

where

\[ \Xi = 1 + \frac{\Lambda a^2}{3}, \]  

\[ \Delta_r = (r^2 + a^2)(1 - \frac{2M^2}{3}) - 2Mr + e^2, \]  

\[ \Delta_\theta = 1 + \frac{2\Lambda a^2}{3} \cos^2 \theta. \]  

\( \Lambda \) is the cosmological constant, \( e \) being the electric charge, \( \rho \) is given by (37). Then, one finds that

\[ N^2 = \frac{\Delta_r \Delta_\theta}{\Xi^2 \Sigma}, \]  

\[ \alpha = \frac{\Delta_\theta}{\Xi^2 \Sigma}, \]  

\[ \omega = \frac{a(r^2 + a^2)\Delta_\theta - \Delta_r}{\rho^2 \Sigma}, \]  

where

\[ \Sigma = \frac{(r^2 + a^2)^2}{\rho^2} \Delta_\theta - \frac{a^2 \sin^2 \theta \Delta_r}{\rho^2}. \]  

It follows from (16) that now

\[ \mu = \frac{\rho^2 \Sigma \Xi^2 - z^2 \Delta_\theta}{\Delta_r \Sigma \Xi^2}. \]  

It is natural to use the ansatz

\[ z = \Xi(r^2 + a^2)k(r), \]  

where \( k(r) \) is a new function. Then,

\[ \mu = \frac{(r^2 + a^2)^2 \Delta_\theta - a^2 \sin^2 \theta \Delta_r - \Delta_\theta (r^2 + a^2)^2 k^2}{\Delta_r \Sigma}. \]
Let us write

\[ k^2 = 1 - \frac{\Delta r}{f^2}, \quad (56) \]

where \( f \neq 0 \) is finite for \( \Delta r = 0 \), so that \( k = 1 \) on the horizon. This ensures the finiteness of \( \mu \) at \( \Delta r = 0 \). Then, \((55)\) takes the form

\[ dt = d\bar{t} + \frac{\Xi(r^2 + a^2)k(r)}{\Delta r} \, dr. \quad (57) \]

This corresponds to the first part of eq. (5) in \([13]\) (with another choice of signs).

In \([13]\), the choice

\[ \xi = a \Xi k \quad (58) \]

was made. Then, we see that according to (12), (51), (54)

\[ h = \frac{\Xi a k}{\Sigma}, \quad (59) \]

so \( h \) is indeed finite on the horizon. Meanwhile, instead of \((58), (59)\) one can take a more general expression according to (12) where \( h \) is a function of \( r \) and \( \theta \) finite on the horizon, otherwise arbitrary.

**VIII. ROTATIONAL ANALOGUE OF BERTOTTI-ROBINSON SPACETIME**

In this section, we consider one more example of rotating metrics with the horizon. This is the analogue of the Bertotti-Robinson metric. For the first time, such a class of metrics was found in \([10]\). It was shown that such metrics appear naturally as the result of the limiting transition from the nonextremal metric to its extremal state when a black hole is enclosed in a cavity. In doing so, in the canonical thermal ensemble the temperature of a system is kept fixed on a boundary. Another version of such metrics corresponding to the extremal horizon was found in \([11]\). For the sake of definiteness, we choose the variant corresponding to eq. (2.6) of \([11]\). The metric takes the form

\[ ds^2 = \frac{(1 + \cos^2 \theta)}{2} \left[ -\frac{r^2}{r_0^2} dt^2 + \frac{r_0^2}{r^2} dr^2 + r_0^2 d\theta^2 \right] + \frac{2r_0^2 \sin^2 \theta}{1 + \cos^2 \theta} (d\phi + \frac{r}{r_0^2} dt)^2, \quad (60) \]

where \( r_0^2 = 2M^2 \), \( M \) being the mass of the original Kerr black hole from which by means of the limiting transition the metric \((60)\) is obtained. In our terms,

\[ \Delta r = r^2, \quad \alpha = \frac{(1 + \cos^2 \theta)}{2r_0^2}, \quad \rho^2 = \frac{(1 + \cos^2 \theta)r_0^2}{2}, \quad \omega = -\frac{r}{r_0^2}, \quad (61) \]
Near the horizon, the coordinate frame (60) fails. Applying our approach, we can choose

\[ z = r_0^2 \sqrt{1 - \frac{r^2}{r_0^2}}, \]  

(62)

\[ \mu = \frac{(1 + \cos^2 \theta)}{2}. \]  

(63)

Further, we can consider choices 1 and 2 from the above.

**A. Choice 1**

\[ h = -\frac{r}{r_0^2} \frac{(1 + \cos^2 \theta)}{2 \sqrt{1 - \frac{r^2}{r_0^2}}}, \]  

(64)

\[ \xi = -\frac{r}{r_0^2} z + hr^2. \]  

(65)

As now \( \omega_\rho = 0 \), it is clear from (30) that we can also take \( \delta = 0 \).

According to (29),

\[
\begin{align*}
\text{ds}^2 &= -dT^2 + \frac{1 + \cos^2 \theta}{4} (dr - \frac{2 \sqrt{1 - \frac{r^2}{r_0^2}}}{1 + \cos^2 \theta} dT)^2 + \frac{2r_0^2 \sin^2 \theta}{1 + \cos^2 \theta} (d\vec{\phi} + \frac{r}{r_0^2} dT)^2 + \frac{(1 + \cos^2 \theta)}{2} r_0^2 d\theta^2. \\
&= -dT^2 + \frac{1 + \cos^2 \theta}{4} (dr - \frac{2 \sqrt{1 - \frac{r^2}{r_0^2}}}{1 + \cos^2 \theta} dT)^2 + \frac{2r_0^2 \sin^2 \theta}{1 + \cos^2 \theta} (d\vec{\phi} + \frac{r}{r_0^2} dT)^2 + \frac{(1 + \cos^2 \theta)}{2} r_0^2 d\theta^2.
\end{align*}
\]

(66)

**B. Choice 2**

\[ \xi = -\frac{r}{r_0^2} z \]  

(67)

\[ h = 0 \]  

(68)

Eq. (34) looks somewhat cumbersome, so we list only the metric component (35)

\[
g_{rr} = \frac{(1 + \cos^2 \theta)^2}{4} \left[ 1 + \frac{2r^2 \sin^2 \theta}{(r_0^2 - r^2)(1 + \cos^2 \theta)} \right].
\]

(69)

Thus, we successfully remove coordinate singularities.
IX. 2+1 SYSTEMS

One more class of metrics for which the approach under discussion works are the metrics in 2+1 gravity. This includes famous BTZ black hole \[15\] and its generalization if matter surrounding the horizon is allowed. Now the original metric in its general form can be obtain by discarding the last term in \[3\], so

$$ds^2 = -N^2 dt^2 + g_\phi (d\phi - \omega dt)^2 + \frac{dr^2}{A}.$$ \hspace{1cm} (70)

Now, the coefficients are supposed to depend on a variable $r$ only, so integrability conditions imposed on functions $z$ and $\xi$ are satisfied automatically.

As an example, we consider the BTZ black hole. Then, $A = N^2 = \Delta_r$,

$$\rho^2 = 1 = \alpha, \quad g_\phi = r^2;$$ \hspace{1cm} (71)

$$\omega = \frac{J}{2r^2};$$ \hspace{1cm} (72)

$$N^2 = -M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2}.$$ \hspace{1cm} (73)

Here, $M$ is the mass, $J$ being the angular momentum, the cosmological constant $\Lambda = -\frac{1}{\ell^2}$. The choice $\mu = 1$ is quite sufficient. Then, it follows from \[13\] that

$$z = \sqrt{1 - N^2}.$$ \hspace{1cm} (74)

Now, the spacetime is not asymptotically flat, $N$ grows with $r$, so our new coordinate frame is not applicable when $N^2 > 1$. However, this does not cause any problem for the vicinity of the horizon that is just the object of our interest. Applying the previous formulas, one finds for choice 1

$$ds^2 = -d\bar{t}^2 + (dr - zd\bar{t})^2 + r^2 (d\bar{\phi} - \omega d\bar{t})^2$$ \hspace{1cm} (75)

and

$$ds^2 = -d\bar{t}^2 + g_{rr}(dr - zd\bar{t} + \frac{J^2}{2g_{rr}r^2}d\bar{\phi})^2 + \frac{r^2}{1 + \frac{J^2}{4z^2r^2}}d\bar{\phi}^2,$$ \hspace{1cm} (76)

$$g_{rr} = 1 + \frac{J^2}{4z^2r^2}$$ \hspace{1cm} (77)

for choice 2.
The approach under discussion applies also to 3+1 metrics if the $\theta$ coordinate is suppressed, so effectively the metric reveals itself as 2+1 dimensional. For example, this happens for the description of particle motion within the equatorial plane $\theta = \frac{\pi}{2}$.

**X. ROTATIONAL ANALOGUE OF LEMAÎTRE FRAME**

Let us consider the 2+1 counterpart of (29). Now, the factor $\rho^2$ is a function of $r$ only, so it can be absorbed by $\Delta_r$ and we may put $\rho = 1$. Introducing a new variable according to $d\bar{r} = \sqrt{\alpha}dr$, one can cast it in the form

$$ds^2 = -d\overline{t}^2 + (d\overline{r} - \overline{v}d\overline{t})^2 + g_\phi(d\overline{\phi} - \omega d\overline{t})^2,$$

(78)

or

$$ds^2 = -dt^2 f + \frac{d\overline{r}^2}{f} + g_\phi(d\overline{\phi} - \omega dt)^2,$$

(79)

where

$$\overline{v} = z\sqrt{\alpha},$$

(80)

$$dt = d\overline{t} + \frac{\overline{v}}{f}d\overline{r},$$

(81)

$$f = 1 - \overline{v}^2.$$  

(82)

Making the transformation

$$\chi = t + \int \frac{d\overline{r}}{f\overline{v}},$$

(83)

$$\tau = t + \int \frac{d\overline{r}\overline{v}}{f},$$

(84)

one can obtain

$$ds^2 = -d\tau^2 + z^2\alpha d\chi^2 + g_\phi[d\overline{\phi} - \omega d\tau] \frac{1}{f} + \frac{2\omega z^2\alpha d\chi^2}{f}$$

(85)

It is easy to show from the equations of motions that coordinate lines $\chi = \text{const}$, $\overline{\phi} = \text{const}$ are geodesics corresponding to particles with the energy $E = m$ and angular momentum $L = 0$ falling from infinity towards a black hole. The relation between such a generalized Lemaître system and the rotational version of the Painlevé-Gullstrand one is extension of the similar relationship valid in the spherically symmetric case [16].
XI. CONCLUSIONS

We considered a quite generic rotating axially symmetric stationary black hole. We have described a general approach that enables one to remove coordinate singularities that are present initially in the standard Boyer-Lindquiste type of coordinates. Our approach involves two functions $z(r)$, $h(r, \theta)$, that obey two equations (12), (13) plus the conditions of finiteness on the horizon (18). There is also one more function $\delta$ that must obey the integrability condition (19). Thus there is a rather large freedom in the choice of a regular coordinate frame. It is traced how some already known coordinate systems for particular metrics (such as the Kerr or Kerr-Newman - (anti-)de Sitter one and the vacuum throat metric) appear as particular cases. The system found in [3] corresponds to $\delta = 0$, the generalization of the Painlevé–Gullstrand system implies that $h = 0$. As a result, a unified picture that encompasses previously known frames is constructed. For the 2+1 systems or problems in which 3+1 metric effectively reduces to 2+1 one, the generalization of the Lemaître coordinate system is constructed. It is related to the generalized version of the Painlevé-Gullstrand frame in a quite simple way.

As coordinates frames under discussion are extendable across the horizon, the formalism developed can be of some use for further investigation of properties of near-horizon orbits [17], [18], including nonequatorial motion. It can also help with studying the geometry inside a black hole, for example in issues related to the measurement of a black hole volume [19]. One more issue is generalization of river-models [20] to the case of "dirty" black holes.

Our consideration reveals also an additional interesting property. It turned out that between the possibility to introduce regular coordinate frames near the horizon and the behavior of the coefficient $\omega$ near the horizon (in particular, the constancy of the black hole angular velocity) there is deep connection.

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