Immiscible and miscible states in binary condensates in the ring geometry

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Abstract

We report detailed investigation of the existence and stability of mixed and demixed modes in binary atomic Bose–Einstein condensates with repulsive interactions in a ring-trap geometry. The stability of such states is examined through eigenvalue spectra for small perturbations, produced by the Bogoliubov–de Gennes equations, and directly verified by simulations based on the coupled Gross–Pitaevskii equations, varying inter- and intra-species scattering lengths so as to probe the entire range of miscibility–immiscibility transitions. In the limit of the one-dimensional (1D) ring, i.e. a very narrow one, stability of mixed states is studied analytically, including hidden-vorticity (HV) modes, i.e. those with opposite vorticities of the two components and zero total angular momentum. The consideration of demixed 1D states reveals, in addition to stable composite single-peak structures, double- and triple-peak ones, above a certain particle-number threshold. In the 2D annular geometry, stable demixed states exist both in radial and azimuthal configurations. We find that stable radially-demixed states can carry arbitrary vorticity and, counter-intuitively, the increase of the vorticity enhances stability of such states, while unstable ones evolve into randomly oscillating angular demixed modes. The consideration of HV states in the 2D geometry expands the stability range of radially-demixed states.

1. Introduction

Superfluid mixtures are currently routinely probed in experiments with ultracold atomic gases. In addition to Bose–Bose mixtures of different isotopes and atomic species [1–18], experimentalists have in the past few years created condensates with a spin degree of freedom [19], also implementing spin–orbit coupling which gives rise to exciting new states [20–24]; moreover, recent achievements have led to the generation of doubly-superfluid Bose–Fermi mixtures [25], in which both components are condensed, a state so far inaccessible in other settings (such as superfluid helium). Although the stability and phase diagrams of such systems have been extensively studied in the course of more than 20 years [26–51], even simple hetero-species binary mixtures still reveal unexpected features, such as the role of the trap sag, atom number and kinetic energy contribution to the extent of miscibility in trapped configurations [42, 50, 52], and nontrivial effects of the expansion on the mixtures’ dynamics [51, 53].

Configurations which keep drawing growing interest in studies of ultracold atomic gases are based on the annular, alias ring-trap, geometry [54–60]. These configurations are interesting as they lead to closed geometries with controlled flows, that are also of potential use to the emerging field of atomtronics [58, 61]. In this context, mixtures of atomic condensates in toroidal traps and the possibility of sustaining stable persistent currents in them have been previously considered in [62–71], and the corresponding experimental observation [72] has helped to clarify some issues, also raising new questions, such as expansion of the stability area for such states. The aim of the present work is to perform a full classification of accessible stable mixture states in such a geometry, including examination of their stability and decay channels of their unstable counterparts, both in the
absence and presence of overall rotation. Given the potential significance of multi-component states in ring-shaped traps for applications such as rotational sensors, such a classification is relevant. It can also assist in developing methods for control of such mixtures in the experimental work which is currently going on in many laboratories.

Specifically, in this work, we construct a binary Bose–Einstein–condensate (BEC) system trapped in an annular geometry, through the mean-field analysis in the presence of inter- and intra-species interactions, whose parameters are varied in broad limits. After analyzing the corresponding one-dimensional (1D) problem, we focus on the more experimentally-relevant 2D annular structure. We implement periodic boundary conditions (b.c.) in the azimuthal direction in the 1D case, and zero b.c. at inner and outer boundaries of the 2D annular structure. The latter b.c. set enables one to study how the annular structure affects density patterns in the repulsive bosonic mixtures, as a result of the existence of different demixed and mixed states, and their stability.

This paper is structured as follows. First, section 2 introduces and analytically considers our basic model for the mixtures in both 1D and 2D geometries, and presents analytical results for spatially uniform 1D mixed solutions, with zero and hidden vorticities (HV), the latter implying opposite topological charges in the two components, which makes it possible to construct stable binary vortex states with zero total angular momentum in nonlinear optics [73–77] and BEC [53, 78–84]. Most essential are analytical results for stability of these states. Section 3 presents the key results, showing various types of mixed and demixed states in 1D, characterized by different numbers of peaks in them, and both mixed and demixed 2D states. The latter ones include both radially-demixed modes, with different vorticities, and their azimuthally-demixed counterparts. Such states are obtained by means of the imaginary-time-propagation method, applied to the coupled Gross–Pitaevskii equations (GPEs). We also address effects of the strength of the repulsive intra-component interaction, annular width, and embedded vorticity on the existence and stability of different states. A noteworthy finding is that the stable radially-demixed states can exist with arbitrary vorticity. Our findings are summarized in section 4.

2. The mean-field models

At low temperatures, a binary condensate mixture is well described by the mean-field theory for the set of wave functions \( \phi \) and \( \psi \) of the two components. Here we address the system (e.g. a heteronuclear one) which does not admit linear interconversion (Rabi and/or spin–orbit coupling) between the components. The wave functions obey the system of GPEs with nonlinear terms accounting for self- (intra-species) and cross- (inter-species) interactions. In the normalized form, the GPE system is written as

\[
\begin{align*}
\imath \dot{\phi} & + \frac{1}{2m_1} \nabla^2 \phi - (g_{11} |\phi|^2 + g_{12} |\psi|^2) \phi = 0, \\
\imath \dot{\psi} & + \frac{1}{2m_2} \nabla^2 \psi - (g_{22} |\psi|^2 + g_{12} |\phi|^2) \psi = 0,
\end{align*}
\]

where \( m_{1,2} \) are scaled atomic masses, \( g_{1,2} \) are coefficients of self-interaction in species \( \phi \) and \( \psi \), and \( g_{12} > 0 \) is the cross-interaction coefficient. In this work, the analysis is restricted to repulsive interactions, with \( g_{1,2,12} > 0 \).

Then, condition \( g_{12} = \sqrt{g_{11} g_{22}} \) separates the mixing (\( g_{11} g_{22} > g_{12}^2 \)) and phase-separation (demixing, \( g_{11} g_{22} < g_{12}^2 \)) regimes in free space [85]. This criterion is modified by the presence of a trapping potential, which tends to enhance the miscibility [42, 50, 86].

Equations (1) are supplemented by b.c. set at rigid edges, \( r = r_{\text{outer}} \) and \( r = r_{\text{inner}} \) of the annular area filled by the condensate \( r \) is the radial coordinate):

\[
\phi(r = r_{\text{outer,inner}}) = \psi(r = r_{\text{outer,inner}}) = 0.
\]

By means of scaling, we fix

\[
r_{\text{inner}} \equiv 1,
\]

and define the annulus’ width

\[
w \equiv r_{\text{outer}} - 1.
\]

These b.c. imply that the annular area is confined by rigid circular potential walls, as in a recent experiment [87] (performed for a gas of fermions).

The total norm of the 2D system is

\[
N = \int \int (|\phi|^2 + |\psi|^2) \, dx \, dy \equiv N_\phi + N_\psi,
\]

where the integration is performed over the annular region, or over the circumference, in the 1D limit, which corresponds to very tight confinement in the radial direction (see equation (33) below). The energy (Hamiltonian) of the coupled system is
Due to the periodic b.c. set by the ring geometry,
where case of circulation. To address the important issue of the stability of the HV-CW state, or the zero-vorticity one in the integer
Choosing the constant amplitudes of the two states as
2.1.1. The analytical approach in the 1D case
leads to the system of four real equations for the amplitudes and phases:
\[\begin{align*}
a_t + \frac{1}{2m_1} a_{xx} + \frac{1}{m_1} a_x = 0, \\
b_t + \frac{1}{2m_2} b_{xx} + \frac{1}{m_2} b_x = 0,
\end{align*}\]
\[\begin{align*}
- a_{xx} x_t + \frac{1}{2m_1} a_{xx} x_x - \frac{1}{2m_1} a_x x^2 - g_1 a + g_2 b^2 - g_3 a^2 b = 0, \\
- b_{xx} x_t + \frac{1}{2m_2} b_{xx} x_x - \frac{1}{2m_2} b_x x^2 - g_2 b + g_3 a^2 b = 0.
\end{align*}\]

2.1. The 1D setting
To define the 1D limit, we assume that the single coordinate, \(x\), running along the ring of radius \(r = 1\) (which is fixed by scaling in agreement with equation (3)), takes values \(0 \leq x \leq 2\pi\). Then, the substitution of solutions in the Madelung form
\[\phi(x, t) = a(x, t) \exp(i\chi(x, t)), \quad \psi(x, t) = b(x, t) \exp(i\eta(x, t)),\]
leads to the system of four real equations for the amplitudes and phases:
\[\begin{align*}
a_t + \frac{1}{2m_1} a_{xx} + \frac{1}{m_1} a_x = 0, \\
b_t + \frac{1}{2m_2} b_{xx} + \frac{1}{m_2} b_x = 0,
\end{align*}\]
\[\begin{align*}
- a_{xx} x_t + \frac{1}{2m_1} a_{xx} x_x - \frac{1}{2m_1} a_x x^2 - g_1 a + g_2 b^2 - g_3 a^2 b = 0, \\
- b_{xx} x_t + \frac{1}{2m_2} b_{xx} x_x - \frac{1}{2m_2} b_x x^2 - g_2 b + g_3 a^2 b = 0.
\end{align*}\]

2.1.1. The analytical approach in the 1D case
Choosing the constant amplitudes of the two states as \(a_0\) and \(b_0\) respectively, we obtain CW (continuous-wave) solutions of the HV type of equations (8)–(11)
\[\begin{align*}
\chi &= -\mu_1 t + sx, \quad \eta = -\mu_2 t - sx, \\
\mu_1 &= g_1 a_0^2 + g_2 b_0^2 + (s^2/2m_1), \quad \mu_2 = g_1 b_0^2 + g_2 a_0^2 + (s^2/2m_2).
\end{align*}\]
Here integer \(s\) determines the opposite vorticities in the two components, without introducing net phase circulation. To address the important issue of the stability of the HV-CW state, or the zero-vorticity one in the case of \(s = 0\), perturbed solutions to equations (8)–(11) are looked for as
\[\begin{align*}
a(x, t) &= a_0 + \delta a \exp(\sigma t + ipx), \\
b(x, t) &= b_0 + \delta b \exp(\sigma t + ipx), \\
\chi(x, t) &= -\mu_1 t + sx + \delta \chi \exp(\sigma t + ipx), \\
\eta(x, t) &= -\mu_2 t - sx + \delta \eta \exp(\sigma t + ipx),
\end{align*}\]
where \(\sigma\) is the instability growth rate (which may be complex), \(p\) is a real wavenumber of the perturbations, while \(\delta a, \delta b\) and \(\delta \chi, \delta \eta\) are their infinitely small amplitudes. The substitution of these expressions in equations (8)–(11) and linearization (i.e. the derivation of the respective Bogoliubov–de Gennes (BdG) equations) yields the following dispersion equation for \(\sigma(p)\):
\[\begin{vmatrix}
\sigma + i\frac{p}{m_1} & 0 & -\frac{p^2}{2m_1} & 0 \\
0 & \sigma - i\frac{p}{m_1} & 0 & -\frac{p^2}{2m_1} \\
-\frac{p^2}{2m_1} - 2g_1 a_0^2 & -2g_2 b_0^2 & -\sigma - i\frac{p}{m_1} & 0 \\
-2g_1 a_0 b_0 & -\frac{p^2}{2m_1} - 2g_2 b_0^2 & 0 & -\sigma + i\frac{p}{m_1}
\end{vmatrix} = 0.\]

Next, we consider two separate cases, depending on the value of \(s\).

2.1.2. Zero-vorticity states, \(s = 0\)
For \(s = 0\), determinant (15) defining the stability takes the explicit form
\[\begin{align*}
p^8 + 4(g_1 a_0^2 m_1 + g_2 b_0^2 m_2)p^6 + 4[\sigma^2 m_1^2 + \sigma^2 m_2^2 - 4a_0^2 b_0^2 m_1 m_2 (g_1^2 - g_2 g_3)]p^4 \\
+ 16(g_1 a_0^2 m_2 + g_2 b_0^2 m_1)m_1 m_2 \sigma^2 p^2 + 16\sigma^4 m_1^2 m_2^2 = 0.
\end{align*}\]
Due to the periodic b.c. set by the ring geometry, \(p\) is quantized
\[p = n/r, \quad n = 0, \pm 1, \pm 2, \ldots\]
recall we here fix $r = 1$ by means of scaling). The onset of the transition to the immiscibility (i.e. instability against the phase separation) is signaled by condition $\sigma(p = \pm 1/r) = 0$. It follows from equation (16) that this instability takes place at

$$g_1 g_2 > (g_1 g_2)_{cr} \equiv \frac{r^{-2} [4(m g_1 a_0^4 m g_2 b_0^4) + r^{-2}]}{16m_2 m_2 a_0^2 b_0^2}. \quad (18)$$

Note that even in the case of $g_1 = g_2 = 0$ (no self-repulsion), equation (18) yields a finite threshold for the onset of the phase-separation instability:

$$g_1 g_2 \big|_{g_1 = g_2 = 0} \equiv \frac{r^{-4}}{16m_2 a_0^6 b_0^2}. \quad (19)$$

This result explicitly demonstrates that periodic b.c. provide for partial stabilization of the mixed state, in comparison with the infinite free space, see [85].

2.1.3. HV states, $s \geq 1$

As defined above, HV states carry opposite angular momenta in the two components of the mixture, while the total momentum is zero. In an explicit form, the corresponding equation (15), which determines their stability, takes a very cumbersome form. It becomes relatively simple in the case of full symmetry in equations (8)–(11) and (12), (13), viz.

$$m_1 = m_2 \equiv m, \quad g_1 \equiv g_2 \equiv g, \quad g_1 g_2 \equiv 1, \quad a_0 = b_0, \quad (20)$$

for which we obtain $\mu_1 = \mu_2 = -(g + 1)a_0^2 + s^2/(2m)$. Then, equation (15) can be explicitly written as

$$16m^4(8s^2 + 32g m a_0^4 + 8p^4) m^2 p^2 \sigma^2 + 16[(g a_0^2 - s^2)^2 - m^2 a_0^4] p^4 + 8(g m a_0^2 - s^2)p^6 + p^8 = 0. \quad (21)$$

Alternatively, the free term in equation (21) (the part which does not contain $\sigma^2$) can be written as

$$16[(g m a_0^2 - s^2)^2 - m^2 a_0^4] p^4 + 8(g m a_0^2 - s^2)p^6 + p^8 \equiv [4(g m a_0^2 - s^2) + p^4]^2 - 16m^2 a_0^4 p^4.$$
The stability condition means that, for all discrete values of \( P \), given by equation (25), equation (22) must produce negative real solutions for \( \Sigma \). Full consideration of the stability conditions following from equation (22) is too cumbersome for the analytical investigation. Nevertheless, for the infinite system \( (r \to \infty) \), i.e. considering \( P \) as a continuous variable, rather than the discrete one, defined by equation (25)), it is easy to obtain the stability condition in the limit of \( P \to 0 \), for which equation (22) amounts to

\[
16\Sigma^2 + 8(4\gamma + 4g)\Sigma + 16[(g - \gamma)^2 - 1]P^2 = 0.
\]

It is easy to see that equation (26) produces stable solutions, i.e. real \( \Sigma < 0 \) (see equations (14) and (23)), under condition \( |g - \gamma| \geq 1 \), i.e. in either of the two cases:

\[
g \geq 1 + \gamma, \quad \text{or} \quad \gamma \geq 1 + g.
\]

According to equation (24), conditions (27) hold in the case of a relatively high nonlinearity (large \( g \), or the atom number), or large hidden vorticity, \( s^2 \), which appears in equation (24), see further details below.

Further, it is possible to find values of \( P \) at which \( \Sigma \) vanishes; substituting \( \Sigma = 0 \) in equation (22), one obtains

\[
P = 0 \quad \text{and} \quad P = 4(\gamma - g \pm 1).
\]

If equation (28) yields \( P \leq 0 \), i.e. \( g \geq 1 + \gamma \), see equation (27), this implies that the HV states are completely stable both for the infinite system and the ring (since, by definition, \( P \) may only be positive).

Note the modulational stability of uniform HV states with periodic b.c. was studied in [79] for the case of the attractive nonlinearity (on the contrary to the repulsive nonlinearity considered here), for which it was found that the HV-CW states can never be stable.

2.2. The two-dimensional setting

Stationary solutions to equation (1) are looked for in the general form:

\[
\phi(r, \theta, t) = \Phi_S(r) \exp(-i\mu_1 t + i\Sigma_1 \theta), \quad \psi(r, \theta, t) = \Psi_S(r) \exp(-i\mu_2 t + i\Sigma_2 \theta),
\]

where \((r, \theta)\) are the polar coordinates, \(\mu_{1,2}\) chemical potentials of the two components, \(S_{1,2} = 0, 1, 2, \ldots\) … their vorticities [88], and real wave functions \(\Phi\) and \(\Psi\) obey the radial equations:

\[
\mu_1 \Phi_S + \frac{1}{2m_1} \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{S_1^2}{r^2} \right) \Phi_S - (g_1 |\Phi_S|^2 + g_{12} |\Psi_S|^2) \Phi_S = 0,
\]

\[
\mu_2 \Psi_S + \frac{1}{2m_2} \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{S_2^2}{r^2} \right) \Psi_S - (g_2 |\Psi_S|^2 + g_{12} |\Phi_S|^2) \Psi_S = 0.
\]

To address its stability, we replace the stationary solutions with perturbed ones:

\[
\phi(r, \theta, t) = (\Phi(r) + u_1 e^{ir\sigma t} + i\Sigma_1 \theta),
\]

\[
\psi(r, \theta, t) = (\Psi(r) + v_1 e^{ir\sigma t} + i\Sigma_2 \theta),
\]

where \(t\) is an integer azimuthal index of the perturbation with components \(u_{1,2}, \quad v_{1,2}, \quad \text{and} \quad \sigma\) is the instability growth rate.

Linearization around the stationary solutions leads to the BdG equations for the two-component condensate:

\[
-\frac{1}{2} \left( \frac{d^2}{dr^2} \right) u_1 + \frac{1}{r} \frac{d}{dr} u_1 - \frac{(S + \Sigma_1^2)}{r^2} u_1 + g_1 \Psi^2(2u_1 + u_2) + g_{12} \Psi_1 u_1 + g_{12} \Psi \Phi(v_1 + v_2) - \mu_1 u_1 = i\sigma u_1,
\]

\[
-\frac{1}{2} \left( \frac{d^2}{dr^2} \right) v_1 + \frac{1}{r} \frac{d}{dr} v_1 - \frac{(S - \Sigma_1^2)}{r^2} v_1 + g_1 \Phi^2(2v_1 + v_2) + g_{12} \Phi_1 u_1 + g_{12} \Phi \Psi(u_1 + u_2) - \mu_1 v_1 = -i\sigma v_1,
\]

\[
-\frac{1}{2} \left( \frac{d^2}{dr^2} \right) u_2 + \frac{1}{r} \frac{d}{dr} u_2 - \frac{(S + \Sigma_1^2)}{r^2} u_2 + g_1 \Psi^2(2u_2 + u_1) + g_{12} \Psi_2 u_2 + g_{12} \Psi \Phi(v_1 + v_2) + \mu_1 u_2 = i\sigma u_2,
\]

\[
-\frac{1}{2} \left( \frac{d^2}{dr^2} \right) v_2 + \frac{1}{r} \frac{d}{dr} v_2 - \frac{(S - \Sigma_1^2)}{r^2} v_2 + g_1 \Phi^2(2v_2 + v_1) + g_{12} \Phi_2 u_2 + g_{12} \Phi \Psi(u_1 + u_2) + \mu_2 v_2 = -i\sigma v_2,
\]

where the prime stands for \(d/dr\). Instabilities are predicted when numerical solution of equation (32) produces eigenvalues with \(\text{Re} (\sigma) \neq 0\). In the 1D version of equation (32), \(d^2/dx^2\) is replaced by \(d^2/dx^2\), and terms \(-1/r\) and \(1/r^2\) are absent.
Previously, BdG equations were addressed in the annular geometry defined not by the rigid boundaries, as per equation (2), but by weak confinement constructed as the sum of Gaussian and harmonic oscillator potentials \[68\]. BdG equations for two-component condensates were also studied in other settings, including free space \[89, 90\], 1D configurations \[63\], and a full analytical solution \[65\].

### 3. Results and discussion

#### 3.1. The 1D regime

Stationary solutions to equations (1) were produced numerically by means of the imaginary-time-evolution method, using different inputs. Then, stability of these solutions was identified through the calculation of their eigenvalue spectra, using the 1D version of equation (32), and further verified by direct numerical simulations of the perturbed evolution. The system conserves the total norm, i.e. scaled number of atoms.

\[
N_{\text{total}} = N_\psi + N_\phi \equiv \int_0^{2\pi} (|\phi|^2 + |\psi|^2) \text{d}x.
\]

Below, we report numerical results obtained for the basic symmetric states, with \(m_1 = m_2 = 1\) (fixed by scaling), \(g_1 = g_2 \equiv g\), \(g_{12} = 1\) (also fixed by scaling, see equation (20)) and equal 1D norms in the two components

\[
\int_0^{2\pi} |\phi(x)|^2 \text{d}x = \int_0^{2\pi} |\psi(x)|^2 \text{d}x \equiv N. \tag{33}
\]

In the miscible phase, the two components of the condensates overlap with each other, whereas they spatially separate in the immiscible phase. A measure to characterize these phases is the overlap integral, which we define here in the 2D form, as it will be used below in the analysis of the 2D setting:

![Figure 2. Numerically generated existence and stability areas for 1D mixed and demixed modes: (a) overlap integral \(\Lambda\) of the \(\phi\) and \(\psi\) components, defined as per equation (34), versus self-repulsion coefficient \(g\), for single-peak modes with fixed norm \(N = 10\), see equation (33); recall that the inter-species repulsion coefficient is fixed to be \(g_{12} = 1\). Stability and existence areas for the single-peak, double-peak, and triple-peak modes in the plane of \((g, N)\) are displayed, respectively, in panels (b)–(d). Black bottom curves in (b)–(d) separate demixed (left) and mixed (right) states, while blue curves in (c) and (d) separate stable and unstable demixed ones. Insets in panels (b)–(d) represent, respectively, typical examples of a stable single-peak mode (with parameters \((N, g, \Lambda) = (10, 0.8, 0.37)\)), unstable double-peak one (for \((N, g, \Lambda) = (10, 0.1, 0.14)\)), and unstable triple-peak state, for \((N, g, \Lambda) = (150, 0.1, 6.34 \times 10^{-4})\).](image)
the reduction of the definition to the 1D limit being obvious. In this work, we identify demixed and mixed states as those with $L = 1$ and $L = 1$, respectively.

As expected [85, 91], demixed states exist only when the cross-repulsion is stronger than the self-repulsion, i.e. $g_2 > g_1$. They are characterized by local density peaks in each component, located so that a peak in one component coincides with a density minimum in the other, see insets to figures 2(b)–(d). Overlap integral (34) for 1D single-peak demixed modes is displayed in figure 2(a), as a function of self-repulsive coefficient $g$, for a fixed norm, $N = 10$. In this figure, the demixed single-peak mode terminates at $g_{cr} = 0.845$, only the uniformly mixed state existing at $g > g_{cr}$. This numerically identified critical value exactly coincides with the analytical prediction given by equation (18). Further, the existence area for demixed single-peak and mixed modes is presented in figure 2(b). The boundary between them, analytically predicted by equation (18), also exactly coincides with the numerically found counterpart, shown by the black curve in figure 2(b). The single-peak demixed modes are completely stable in their existence domain, which is consistent with earlier findings [69].

Stability and existence areas of demixed double- and triple-peak modes are displayed in parameter plane $(g, N)$ in figures 2(c) and (d), respectively, and typical examples of such modes are displayed in their respective insets. An essential finding is that, unlike the single-peak modes which are always stable, states with two and three peaks feature instability areas in figures 2(c) and (d), being stable only for a sufficiently large norm.

The numerical analysis reveals two instability scenarios for the double-peak mode. If it is taken in the area far from the stability boundary in figure 2(c), the real parts of the corresponding eigenvalues $\sigma$ are relatively large (see equation (31)), and the mode spontaneously transforms into an oscillating single-peak state, see figures 3(a1)–(a3). If the unstable mode is selected close to the instability boundary, with smaller real parts of the eigenvalues, it oscillates around itself, rather than transforming into a single-peak state. Similar to the double-peak states, unstable triple-peak ones transform into oscillating single-peak modes far from the corresponding stability boundary, and persistently oscillate around themselves, if taken close to boundary, see figures 3(b1)–(b3).

We also simulated collision between single-peak demixed components, set in motion by applying opposite kicks to them:

$$\phi(x, t = 0) = \phi(x) e^{i k x}, \quad \psi(x, t = 0) = \psi(x) e^{-i k x}. \quad (35)$$
Figure 4 shows that, for the kick small enough ($k = 0.5$), the peaks in the two components periodically bounce back from each other, which is accompanied by some randomization of the patterns. On the other hand, under the action of a strong kick (e.g. $k = 5$), the moving components pass through each other for about five times, but eventually suffer randomization too, as shown in figures 4(b1), (b2). Under the action of a still stronger kick, $k = 10$, the components kept passing through each as long as the simulations were run, see figures 4(c1), (c2).

It is also relevant to simulate evolution of unstable mixed (uniform) states, which is displayed in figure 5. The instability triggers periodic transformations between the mixed state and a single-peak demixed one, with the period $\approx 10$ in this case.

3.2. Two-dimensional regime

Focusing on the phase-separation scenarios, we identify two different types of 2D demixed modes, namely, those which may be defined as demixed in the radial direction (see [67]), and azimuthally demixed ones, see [63, 67, 68]. In previous works, similar scenarios of the phase separation were also reported for other binary systems, which include rotation [63, 69] and spin–orbit coupling [71].
3.2.1. Radially-demixed modes

In the consideration of the 2D setting subject to b.c. (2), we focus, as above, on the scaled symmetric system, with \( m_1 = m_2 = 1 \), \( g_1 = g_2 \equiv 1 \), \( g_{12} = 1 \), and \( r_{\text{inner}} = 1 \), see equations (20) and (3). Stability of 2D modes was identified by the computation of the eigenvalue spectra in the corresponding BdG equations (32), and further verified by direct simulations.

First, we address 2D zero-vorticity states, including mixed and radially-demixed ones, which may be both stable and unstable (at larger and smaller values of the norm, respectively), as shown in figure 6. A typical example of the evolution of unstable 2D radially-demixed states with vorticities \( S_{1,2} = 0 \) is shown in figure 7. It is observed that the unstable state spontaneously evolves into an azimuthally-demixed one.

A noteworthy finding is that the system supports stable 2D radially-demixed states with arbitrarily high vorticities \( S_1 = S_2 \equiv S \). We first analyze 2D demixed states with \( S = 0 \) and \( S = 5 \) in the parameter space of \((g, N, \Lambda)\), see figure 8. It is seen that the solutions are stable (similar to what was found above for other configurations) above a threshold value of the norm, \( N > N_{\text{th}} \). We stress that the stability threshold is much lower for \( S = 5 \) than for \( S = 0 \) (note different scales of vertical axes in figures 8(a) and (b)).

To further explore how the vorticity affects the stability of the 2D demixed states, we define the atomic density,

\[
|\phi|^2 \quad (a1) \quad (b1) \quad (c1) \\
|\psi|^2 \quad (a2) \quad (b2) \quad (c2)
\]

Figure 5. Numerically simulated evolution of an unstable 1D mixed mode, showing periodic transformations between mixed and demixed states. The parameters are \((N, g, \Lambda) = (10, 0.6, 1)\).

Figure 6. Typical examples of 2D zero-vorticity states \((S_{1,2} = 0)\) with \( g = 0.1 \) and width \( w = 2 \). (a1), (a2) An unstable mixed state with \( N = 5 \) and \( \Lambda = 1 \); (b1), (b2) an unstable demixed state with \( N = 60 \) and overlap parameter \( \Lambda = 0.1307 \) (see equation (34)); (c1), (c2) a stable strongly demixed state with \( N = 9000 \) and \( \Lambda = 4.45 \times 10^{-6} \).
Figure 7. Density snapshots of the evolution of an unstable 2D radially-demixed mode shown in figure 6 (middle), revealing spontaneous formation of azimuthally-demixed states.

Figure 8. Existence and stability areas in the $(g, N)$ plane for 2D mixed and radially-demixed states, in the annular domain with width $w = 2$ (see equation (4)). The overall vorticity is $S = 0$ in (b1) and $S = 5$ in (b2). The gray-scaled shading shows the corresponding values of the overlap parameter $\Lambda$, see equation (34).

Figure 9. (a) The threshold value of density (36) of 2D radially-demixed states versus their vorticity $S$, the solutions being stable at $n > n_{th}$. The corresponding parameter set is $(w, g) = (2, 0.1)$, see equations (20) and (34). (b) The stability region for the radially-demixed state with $S = 0$ in the plane of plane $(n, w)$, for $g = 0.1$. The solutions are stable above the solid curve.
n \frac{N}{\pi (r_{\text{outer}}^2 - 1)}
\end{equation}

(recall that the inner radius of the annulus is scaled to be 1, as per equation (3)), and display the stability-threshold value of \( n \) as a function of \( S \) in figure 9(a). A salient feature is the steep drop of \( n_{\text{th}} \) while \( S \) increases from 1 to 2, which is followed by gradual decrease of the threshold with further increase of \( N \). Thus, the vorticity helps to strongly stabilize the axially symmetric states in the annular domain.

It is also relevant to investigate an effect of the annulus’ width \( w \), defined as per equation (4), on the stability. For the zero-vorticity radially-demixed states, the respective stability diagram in parameter panel \((n, w)\) is presented in figure 9(b). It is seen that the stability area strongly broadens with the increase of \( w \), i.e. as it might be expected, stable radially-demixed modes prefer broad annular domains.

3.2.2. Azimuthally-demixed modes (with \( S = 0 \))
The 2D setting supports, as well, stable modes which are phase-separated in the azimuthal direction (with zero vorticity) [69, 71], an example of such modes can be seen in figure 10. These modes are related to their 1D counterparts displayed above in the insets of figures 2(b)–(d), and unstable 2D radially-demixed modes transform into them (in an excited oscillating state), see figure 7.

To illustrate the evolution of those 2D azimuthally-demixed states which are unstable, we display the evolution of an unstable double-peak state with a small total norm, \( N = 20 \) in figure 11 (similar to other states considered here, they tend to be unstable for relatively small values of \( N \)). It first evolves into a pattern with unequal heights of two peaks, and then restored the original configuration with equal peaks. After several cycles of such shape oscillations, it finally settles into an oscillating single-peak state. The same happens with unstable

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**Figure 10.** Typical examples of 2D azimuthally-demixed modes with zero vorticity, for \( g = 0.1 \) and the annulus’ width \( w = 2 \). (a1), (a2) A stable single-peak mode with total norm \( N = 50 \). (b1), (b2) A stable double-peak mode with \( N = 100 \). (c1), (c2) An unstable triple-peak mode with \( N = 100 \).

**Figure 11.** Density snapshots of the evolution of an unstable 2D double-peak azimuthally-demixed mode (only the \(|\phi|^2\) component is displayed, as the complementary evolution of \(|\psi|^2\) is similar) for \((g, w, N) = (0.1, 2, 20)\). The unstable mode spontaneously transforms into a stable single-peak one.
triple-peak 2D states. This kind of dynamics resembles what was observed above for unstable double- and triple-peak states in the 1D geometry, see figure 3.

On the other hand, we have not found any azimuthally-demixed states with nonzero vorticity.

Finally, it makes sense to address demixed modes in the full circle, with $r_0 = 0$, instead of the annulus, see equation (3). It has been found that radially-demixed modes may (quite naturally) exist in the latter case, while no azimuthally-demixed states were found.

3.3. HV states
3.3.1. The 1D setting
A typical example of the HV state, predicted by analytical solution (12), is presented in figure 12. This particular HV state is an unstable one, as illustrated in figure 1(a) by the dependence of its instability growth rate on the perturbation wavenumber, which is predicted by equation (22); for comparison, figure 1(b) exhibits the same analytical result for a stable HV state. Simulations demonstrate that the evolution transforms unstable 1D HVs into stable single-peak demixed states, with some intrinsic oscillations (not shown here in detail).

3.3.2. The 2D setting
We have also numerically produced 2D radially-demixed HV states, example of which, with $S_{1,2} = \pm 1$ and $\pm 5$, are displayed in figures 13(a) and (b), respectively. The same setting may also support 2D mixed HV states, which we do not consider here in detail, as the demixed states seem more interesting. Results for the stability of the 2D radially-demixed HV modes with the same values of $S_{1,2}$ are summarized in figures 13(c) and (d). An obviously interesting conclusion following from the latter plots is that the increase of the hidden vorticity, $|S_{1,2}|$, leads to stabilization of the the HV states (note that difference in the scales of vertical axes in panels (c) and (d)).

Finally, comparing the total energy of different 2D mixed and demixed states (see equation (6)), which share equal values of the total norm and angular momentum, we have concluded that the single-peak azimuthally-demixed states realize the lowest energy, i.e. the system’s ground state, while the totally mixed configuration has the highest energy.

3.4. Physical estimates
To translate the scaled units into the physical ones, we consider the binary condensate of $^{87}$Rb atoms in two different spin states, such as ones with $F = 1$, $m_F = 1$ and $F = 1$, $m_F = 0$, and use the same parameters as experiments performed with the two-components condensate in a ring [72], with the radius $\lesssim 12 \, \mu m$, and the scattering length $a_s \sim 10 \, nm$ [92]. We conclude that the stable effectively 1D modes predicted by the present analysis may have the actual transverse thickness $\sim 3 \, \mu m$, containing up to $\sim 10^4$ atoms, while the stable 2D modes, predicted for the same outer radius, $\lesssim 12 \, \mu m$, and the inner one $\lesssim 4 \, \mu m$, contain $10^4 \sim 10^5$ atoms.

4. Conclusion
We have studied the stability and phase diagram of the two-component BEC loaded in the 2D annular potential box, as well as its 1D limit form corresponding to a ring. The system was analyzed in the framework of the mean-field approximation, based on coupled GPEs with repulsive intra-species and inter-species interactions.

In the 1D setting, the demixed (phase-separated) states are identified as single-, double- and triple-peak modes, with density peaks in one component coinciding with density minima in the other one. The 1D single-peak demixed states are all stable, while the double- and triple-peak ones are stable only above critical values of...
the total norm, $N$. The unstable double- and triple-peak modes oscillate around themselves when they are located close to the instability boundary, or spontaneously transform into stable single-peak states deeper in the unstable domain of the parameter space. Collisions between two components of stable demixed single-peak states were studied too, by applying opposite kicks to the components. The simulations demonstrate that the weakly kicked components repeatedly bounce from each other, suffering gradual chaotization, while fast ones pass through each other. If the kicks are moderately strong, the components originally pass through each other, and then evolve into the bouncing regime. The evolution of unstable 1D mixed (spatially uniform) modes shows periodic transitions between the mixed state and single-peak demixed ones.

In the 2D setting, we have found both radially- and azimuthally-demixed states, with unstable radially-demixed ones found to evolve into their azimuthally-demixed counterparts. An essential finding is that the system supports radially-demixed modes with arbitrarily large overall vorticity $S$, which are stable above the threshold value of the norm, $N_{th}$. The increase of $S$ leads to stabilization of the modes (decrease of $N_{th}$), with a dramatic drop, following the transition from $S = 1$ to $S = 2$, in figure 9(a). The stability area gradually broadens with the increasing of the annulus’ width, $w$, in figure 9(b). Similar to the 1D demixed states, 2D azimuthally-demixed ones are also identified as single-, double- and triple-peak modes. Unstable 2D double- and triple-peak azimuthally-demixed states (those with relatively small norms) evolve into oscillating single-peak modes. In the solid circle, taken instead of the annulus, only radially-demixed modes are found.

Lastly, both 1D and 2D HV states, with opposite vorticities in the two components, have been addressed too. The stability region for 1D HV modes was found analytically, and fully confirmed by the numerical analysis. Unstable 1D HV modes with components vorticities $S_{1,2} = \pm 1$ showed evolve into oscillating single-peak demixed modes. The stability domain for 2D radially-demixed HV modes expands with the increase of the hidden vorticity, $|S_{1,2}|$. 

Figure 13. Typical examples (the density distribution and phase structure) of stable 2D radially-demixed HV states: (a) $S_{1,2} = \pm 1$; (b) $S_{1,2} = \pm 5$. Both examples correspond to the same parameter set, $(g, w, N) = (0.1, 2, 2000)$. Panels (c) and (d) summarize properties of the respective HV states in parameter plane $(g, N)$. In (c) and (d), black curves separate demixed and mixed states (left and right areas, respectively), while blue curves are stability boundaries for demixed states.
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