ON THREEFOLDS ISOGENOUS TO A PRODUCT OF CURVES

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Abstract. A threefold isogenous to a product of curves $X$ is a quotient of a product of three compact Riemann surfaces of genus at least 2 by the free action of a finite group. In this paper we study these threefolds under the assumption that the group acts diagonally on the product. We explain a method to compute their Hodge diamond and present an algorithm to classify them for a fixed value of $\chi(\mathcal{O}_X)$. Running an implementation of this algorithm we achieve the full classification of threefolds isogenous to a product of curves with $\chi(\mathcal{O}_X) = -1$, under the assumption that the group acts faithfully on each factor.

Introduction

A complex algebraic variety $X$ is isogenous to a product of curves if $X$ is a quotient

$$X = (C_1 \times \cdots \times C_n)/G,$$

where the $C_i$’s are smooth projective curves of genus at least two and $G$ is a finite group acting freely on $C_1 \times \cdots \times C_n$.

This class of varieties of general type has been introduced by Catanese in [Cat00]. Since then, a considerable amount of literature appeared, especially in the case of surfaces. Surfaces isogenous to a product of curves are completely classified in the boundary case $\chi(\mathcal{O}_S) = 1$, i.e. $p_g(S) = q(S)$. Recall that the Bogomolov-Miyaoka-Yau inequality $K_S^2 \leq 9\chi(\mathcal{O}_S)$ and the Debarre inequality $K_S^2 \geq 2p_g(S)$ imply $0 \leq q(S) = p_g(S) \leq 4$.

In [Bea82], Beauville classifies all minimal surfaces $S$ of general type with $p_g(S) = q(S) = 4$: they are products of two genus two curves.

A minimal surface $S$ of general type with $p_g(S) = q(S) = 3$ is either the symmetric square of a genus three curve or $S \cong (F_2 \times F_3)/\tau$, where $F_g$ is a curve of genus $g$ and $\tau$ is an involution acting on $F_2$ as
an elliptic involution and on $F_3$ as a fixed-point free involution, see \textsuperscript{CCML98, Pir02, HP02}.

The classification of surfaces isogenous to a product of curves is due to Bauer, Catanese, Grunewald \textsuperscript{BCG08} for $p_g = q = 0$, to Carnovale, Polizzi \textsuperscript{CP09} for $p_g = q = 1$ and to Penegini \textsuperscript{Pen11} for $p_g = q = 2$.

The aim of this paper is to generalize the methods, developed for surfaces by the mentioned authors, to higher dimensions.

Defining

$$G^0 := G \cap (\text{Aut}(C_1) \times \ldots \times \text{Aut}(C_n)),$$

there are two possibilities for the action of $G$ on the product $C_1 \times \ldots \times C_n$:

- **Unmixed**: the group $G$ acts diagonally that is $G = G^0$.
- **Mixed**: when $G^0 \subsetneq G$, there are elements in $G$ permuting some factors of the product.

In this paper we consider the unmixed case in dimension three, while in a forthcoming paper we will investigate the mixed case. We point out that most of the results in this paper can be easily generalized to any dimension, but the notation becomes more and more complicated.

In higher dimensions, there are technical difficulties which do not occur in dimension two: contrary to the surface case, it is not possible to assume that $G^0$ embeds in Aut$(C_i)$. One has to deal with the kernels of the maps $G^0 \to \text{Aut}(C_i)$. This turns out to be a big problem because the classification becomes computationally very expensive.

In the surface case, once $p_g$ and $q$ are fixed, the other invariants are automatically determined. In higher dimensions, even if the invariants $h^i(O_X)$ for $1 \leq i \leq \dim X$ are fixed, it is not clear how to determine the remaining Hodge numbers.

The first main result of the paper is a method to compute the Hodge diamond using representation theory, see Section 3.1.

The second main result of the paper is an algorithm which classifies threefolds isogenous to a product of curves of unmixed type for a fixed value of $\chi(O_X)$, see Section 4.

We implemented this algorithm using the computer algebra system MAGMA \textsuperscript{BCP97} whose commented version can be downloaded from

\url{http://www.staff.uni-bayreuth.de/~bt300503/}

The paper is organized as follows:

Section 1 is about group actions on compact Riemann surfaces: we recall Riemann’s existence theorem and state the Chevalley-Weil formula which describes the induced action on the space of holomorphic 1-forms.

In Section 2 we introduce varieties isogenous to a product of curves and explain some of their basic properties.

In Section 3 we show that the geometry of a threefold isogenous to a product of curves is encoded in the group via an algebraic datum.
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Based on this, we give a method to compute the Hodge diamond and the fundamental group of these varieties. In Section 4 we prove that the classification of threefolds isogenous to a product $X$ with fixed value of $\chi(O_X)$ is a finite problem and we give an algorithm to perform this classification.

Section 5 contains computational results by running an implementation of the algorithm described in Section 4 in the case $\chi(O_X) = -1$. By a result of Miyaoka, this is the maximum value of $\chi(O_X)$ for threefolds of general type with ample canonical class (see Remark 2.4). In particular, we provide the full classification (Theorem 5.1) of threefolds isogenous to a product of curves of unmixed type with $\chi(O_X) = -1$ under the assumption that the group action is faithful on each factor.

Notation. Throughout the paper all varieties are defined over the field of complex numbers and the standard notation from complex algebraic geometry is used, see for example [GH94].

1. Galois coverings of Riemann surfaces

In this section we recall some principles of group actions on compact Riemann surfaces that we use throughout the paper. In the first part we state Riemann’s existence theorem, in the second part the Chevalley-Weil formula which describes the induced group action on the space of holomorphic 1-forms. Due to a lack of modern reference, we present a proof of the Chevalley-Weil formula which is based on the holomorphic Lefschetz fixed-point formula.

1.1. Group actions on Riemann surfaces. Let $C$ be a compact Riemann surface, $G$ be a finite group and

$$
\psi: G \longrightarrow \text{Aut}(C)
$$

be a group action of $G$ on $C$. This homomorphism factors as

$$
G/K \xrightarrow{\psi} \text{Aut}(C),
$$

where $K := \ker(\psi)$ denotes the kernel of the action and $\pi$ the quotient map. Via $\psi$, we consider $G/K$ as a subgroup of $\text{Aut}(C)$. For $p \in C$, $G_p$ (resp. $(G/K)_p$) denotes the stabilizer group of $p$ for the $G$ (resp. $G/K$) action.

Proposition 1.1.

i) The group $(G/K)_p$ is cyclic for any point $p \in C$.

ii) $G_p = \langle g \rangle \cdot K$, for any $g \in G$ such that $\langle \pi(g) \rangle = (G/K)_p$.

iii) $G_{\psi(h)(p)} = hG_p h^{-1} = \langle hgh^{-1} \rangle \cdot K$, for any $h \in G$. 

Proof. The first assertion is well known (see \cite{Mir95}, Proposition III.3.1). The other two statements follow immediately, since $K = \ker(\psi) \triangleleft G$. \hfill $\square$

**Definition 1.2.** Let $2 \leq m_1 \leq \ldots \leq m_r$ and $g' \geq 0$ be integers and $H$ be a finite group. A generating vector for $H$ of type $[g'; m_1,\ldots,m_r]$ is a $(2g'+r)$-tuple $(d_1, e_1,\ldots,d_{g'}, e_{g'}; h_1,\ldots,h_r)$ of elements of $H$ such that:

i) $H = \langle d_1, e_1,\ldots,d_{g'}, e_{g'}, h_1,\ldots,h_r \rangle$,

ii) $\prod_{i=1}^{g'} [d_i, e_i] \cdot h_1 \cdots h_r = 1_H$,

iii) there exists a permutation $\sigma \in \mathfrak{S}_r$ such that $\text{ord}(h_i) = m_{\sigma(i)}$.

The geometry behind this definition is Riemann’s existence theorem (cf. \cite{Mir95}, Sections III.3 and III.4).

**Theorem 1.3** (Riemann’s existence theorem). A finite group $H$ acts faithfully and biholomorphically on some compact Riemann surface $C$ of genus $g(C)$, if and only if there exists a generating vector $(d_1, e_1,\ldots,d_{g'}, e_{g'}; h_1,\ldots,h_r)$ for $H$ of type $[g'; m_1,\ldots,m_r]$ such that Hurwitz’ formula holds:

$$2g(C) - 2 = |H|\left(2g' - 2 + \sum_{i=1}^{r} \frac{m_i - 1}{m_i}\right).$$

In this case $g'$ is the genus of the quotient Riemann surface $C' := C/H$ and the $H$-cover $C \to C'$ is branched in $r$ points $\{x_1,\ldots,x_r\}$ with branching indices $m_1,\ldots,m_r$, respectively. Moreover, the cyclic groups $\langle h_1 \rangle,\ldots,\langle h_r \rangle$ and their conjugates provide the non-trivial stabilizers of the action of $H$ on $C$.

**Definition 1.4.** Let $C$ be a compact Riemann surface, $\psi: G \to \text{Aut}(C)$ be a group action with kernel $K := \ker(\psi)$ and 

$$V := (d_1, e_1,\ldots,d_{g'}, e_{g'}; h_1,\ldots,h_r)$$

be a generating vector associated to $\psi: G/K \to \text{Aut}(C)$. For all $1 \leq i \leq r$, let $g_i \in G$ be a representative of $h_i \in G/K$. The stabilizer set of $V$ is

$$\Sigma_\psi(V) := \left(\bigcup_{g \in G} \bigcup_{i \in \mathbb{Z}} \bigcup_{j=1}^{r} \{gg_jg^{-1}\}\right) \cdot K.$$

By Proposition 1.1 and Riemann’s existence theorem, it follows:

**Corollary 1.5.** The stabilizer set of $V$ is the union of all stabilizer groups:

$$\Sigma_\psi(V) = \bigcup_{p \in C} G_p.$$
1.2. The Chevalley-Weil formula. Let $C$ be a compact Riemann surface, $G$ be a finite group, $\psi: G \to \text{Aut}(C)$ be a group action with kernel $K := \ker(\psi)$ and $\pi: G \to G/K$ be the quotient map. There is a representation of $G$ associated to $\psi$ via pullback of holomorphic 1-forms:

$$\varphi: G \to \text{GL}(H^{1,0}(C)), \quad h \mapsto [\omega \mapsto (\psi(h^{-1}))^*(\omega)].$$

The aim of this section is to study this representation. Recall that a representation of a finite group is determined, up to isomorphism, by its character (we refer to [FH91] for this and other basic properties about complex linear representations of finite groups used in the sequel).

Since the following diagram commutes

$$\begin{array}{ccc}
G/K & \xrightarrow{\varphi} & \text{GL}(H^{1,0}(C)) \\
\pi \downarrow & & \downarrow \\
G & & \\
\end{array}$$

the character of $\varphi$ is determined by the character of $\overline{\varphi}$:

$$\chi_\varphi = \chi_{\overline{\varphi}} \circ \pi.$$  

For this reason it suffices to consider the case $K = \{1\}$ and we identify $G$ with $\psi(G) < \text{Aut}(C)$.

Remark 1.6. The irreducible characters $\chi_1, \ldots, \chi_k$ of a group $G$ form an orthonormal basis of the vector space of class functions of the group $G$ with respect to the Hermitian product $\langle -,- \rangle$ defined as:

$$\langle \alpha, \beta \rangle := \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \beta(g).$$

There exists a unique decomposition

$$\chi_\varphi = n_1 \chi_1 + \ldots + n_k \chi_k, \quad n_i \in \mathbb{N}.$$  

By orthonormality, it holds $n_i = \langle \chi_\varphi, \chi_i \rangle$.

Let $V$ be a generating vector for the action $\psi$. The Chevalley-Weil formula provides a way to compute $n_i$ from $V$ and an irreducible representation $\psi_i: G \to \text{GL}(W_i)$ with character $\chi_{\psi_i} = \chi_i$.

To state and prove it, we need some preliminary results.

**Lemma 1.7.** Let $\psi: H \to \text{GL}(W)$ be a representation of a finite group $H$ and $h \in H$ be an element of order $m$.

Then $\psi(h)$ is diagonalizable and its eigenvalues are of the form $\xi_m^\alpha$, where $\xi_m := \exp\left(\frac{2\pi \sqrt{-1}}{m}\right)$ and $0 \leq \alpha \leq m - 1$.  

**Proof.** The matrix $\psi(h)$ has finite order: $\psi(h)^m = \text{Id}$, whence it is diagonalizable and each eigenvalue $\lambda$ of $\psi(h)$ satisfies $\lambda^m = 1$.  

\[ \square \]
Definition 1.8. Let $X$ be a compact complex manifold and $f : X \to X$ be a holomorphic map. The number

$$L(f) := \sum_{q=0}^{\dim X} (-1)^q \text{Tr}(f^*_{|H^q(X)})$$

is called the holomorphic Lefschetz number of $f$.

The number $L(f)$ can be computed using the holomorphic Lefschetz fixed-point formula [GH94, p. 426]:

**Theorem 1.9** (Holomorphic Lefschetz fixed-point formula). Let $X$ be a compact complex manifold and $f : X \to X$ be a holomorphic automorphism of finite order with isolated fixed points. Then

$$L(f) = \sum_{p \in \text{Fix}(f)} \frac{1}{\det(Id - J_p(f))}.$$ 

**Lemma 1.10.** Let $X$ be a compact Kähler manifold and $f : X \to X$ be a holomorphic map. Then

$$\text{Tr}(f^*_{|H^{p,q}(X)}) = \overline{\text{Tr}(f^*_{|H^{p,q}(X)})}.$$ 

**Proof.** Let $\{[\omega_1], \ldots, [\omega_k]\}$ be a basis of $H^{p,q}(X)$ with $\omega_i$ harmonic, then $\{[\overline{\omega_1}], \ldots, [\overline{\omega_k}]\}$ is a basis of $H^{q,p}(X)$ and the forms $\overline{\omega_i}$ are harmonic. Now we write

$$[f^* \omega_j] = \sum_{i=1}^{k} a_{ij} \cdot [\omega_i] \quad \text{or, equivalently} \quad f^* \omega_j = \sum_{i=1}^{k} a_{ij} \cdot \omega_i + \overline{\eta_j},$$

where $a_{ij} \in \mathbb{C}$ and $\eta_j$ is a form of type $(p, q-1)$. Complex conjugation yields

$$f^* \overline{\omega_j} = \sum_{i=1}^{k} \overline{a_{ij}} \cdot \overline{\omega_i} + \partial \overline{\eta_j}.$$ 

Since $f^* \overline{\omega_j}$ and $\sum_{i=1}^{k} \overline{a_{ij}} \cdot \overline{\omega_i}$ are $d$-closed forms, $\partial \overline{\eta_j}$ is a $d$-closed form which is $\partial$-exact. According to the $\overline{\partial \partial}$-lemma [Huy05 Corollary 3.2.10], there exists a $(q, p-1)$-form $\xi_j$ such that $\partial \overline{\eta_j} = \partial \overline{\xi_j}$. Therefore:

$$[f^* \overline{\omega_j}] = \sum_{i=1}^{k} a_{ij} \cdot [\overline{\omega_i}].$$

□

**Theorem 1.11** (Chevalley-Weil formula, cf. [CW34]). Let $C$ be a compact Riemann surface, $\psi : G \to \text{Aut}(C)$ be a faithful group action of a finite group $G$ and $(d_1, e_1, \ldots, d_{g'}, e_{g'}; h_1, \ldots, h_r)$ be a generating vector of type $[g'; m_1, \ldots, m_r]$ associated to $\psi$. Let $\chi_\varphi$ be the character of the representation $\varphi$ associated to $\psi$ and $\chi_\varphi$
be the character of an irreducible representation \( \rho : G \to \text{GL}(W) \), then 
\[ n_\chi = d(g' - 1) + \sum_{i=1}^r \sum_{\alpha=1}^{m_i-1} \alpha \cdot N_{i,\alpha} + \sigma. \]

Here \( N_{i,\alpha} \) is the multiplicity of \( \xi_{m_i}^\alpha \) as eigenvalue of \( \rho(h_i) \). The integer \( d \) is the degree of \( \rho \) and \( \sigma = 1 \) if \( \rho \) is trivial, otherwise \( \sigma = 0 \).

**Proof.** Let \( \chi_{\text{triv}} \) be the trivial character of \( G \) and \( \chi \) be the class function 
\[ \chi := \chi_{\text{triv}} - \chi_\varphi. \]

Let \( g \in G \), by Lemma 1.10 it follows
\[ \chi(g) = 1 - \chi_\varphi(g) = 1 - \chi_\varphi(g^{-1}) = 1 - \text{tr}(g_{|\mathcal{H}^1,\nu(C)}) = 1 - \text{tr}(g_{|\mathcal{H}^{0,1}(C)}) = L(g). \]

This shows that \( \chi(g) \) coincides with the holomorphic Lefschetz number of \( g \). We define \( \sigma := \langle \chi_\varphi, \chi_{\text{triv}} \rangle \), then \( \sigma = 1 \) if \( \rho \) is trivial, otherwise \( \sigma = 0 \). The bilinearity of \( \langle - , - \rangle \) implies
\[ \langle \chi_\varphi, L \rangle = \langle \chi_\varphi, \chi_{\text{triv}} \rangle - \langle \chi_\varphi, \chi_\varphi \rangle = \sigma - n_\chi. \]

It remains to compute the inner product \( \langle \chi_\varphi, L \rangle \):
\[ \langle \chi_\varphi, L \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_\varphi(g)\overline{L(g)} = \frac{1}{|G|} \chi_\varphi(1_G)\overline{L(1_G)} + \frac{1}{|G|} \sum_{g \in G, g \neq \text{id}} \chi_\varphi(g)\overline{L(g)}. \]

The value \( \chi_\varphi(1_G) \) is the degree \( d \) of \( \rho \) and \( L(1_G) = 1 - g(C) \). We apply the holomorphic Lefschetz fixed-point formula using the fact \( \overline{L(g)} = L(g^{-1}) \):
\[ \frac{1}{|G|} \sum_{g \in G, g \neq \text{id}} \chi_\varphi(g)\overline{L(g)} = \frac{1}{|G|} \sum_{g \in G, p \in \text{Fix}(g)} \sum_{g \neq \text{id}} \chi_\varphi(g) \frac{1}{1 - J_p(g^{-1})} = \frac{1}{|G|} \sum_{p \in \text{Fix}(C)} \sum_{g \in G, g \neq \text{id}} \chi_\varphi(g) \frac{1}{1 - J_p(g^{-1})}, \]

where \( \text{Fix}(C) := \{ p \in C \mid G_p \neq \{ 1_G \} \} \).

According to Riemann’s existence theorem, for all \( 1 \leq i \leq r \) there exists a point \( p_i \in C \) with \( G_{p_i} = \langle h_i \rangle \) and for any \( g \in G \) we have \( G_{g(p_i)} = \langle gh_i g^{-1} \rangle \); moreover, every \( p \in \text{Fix}(C) \) arises in this way.

As shown in [Lam05, 4.7.2 Satz], the element \( h_i \) is the unique element in \( G_{p_i} \) such that \( J_{p_i}(h_i) = \xi_{m_i} = \exp \left( \frac{2\pi i}{m_i} \right) \); hence the Jacobian \( J_{g(p_i)}(gh_i g^{-1}) \) equals \( \xi_{m_i}^l \) for \( l \in \mathbb{Z} \). Since \( \chi_\varphi \) is a class function, the
equality $\chi_\rho(h_i^l) = \chi_\rho(gh_i^lg^{-1})$ holds too. We get
\[
\frac{1}{|G|} \sum_{p \in Fix(C)} \sum_{g \in G_p \ g \neq id} \chi_\rho(g) \frac{1}{1 - J_p(g^{-1})} = \frac{1}{|G|} \sum_{i=1}^r |G| \sum_{l=1}^{m_i-1} \chi_\rho(h_i^l) \frac{1}{1 - \xi_m^{-l}} = \sum_{i=1}^r \frac{1}{m_i} \sum_{l=1}^{m_i-1} \sum_{\alpha=0}^{m_i-1} N_{i,\alpha} \cdot (\xi_m^\alpha)^l.
\]
According to [Rei87, Eq. 8.8], it holds
\[
\sum_{l=1}^{m_i-1} \frac{(\xi_m^l)^\alpha}{1 - \xi_m^{-l}} = \frac{m_i - 1}{2} - \alpha \quad \text{for all} \quad 0 \leq \alpha \leq m_i - 1.
\]
The expression for $\langle \chi_\rho, L \rangle$ simplifies to
\[
\langle \chi_\rho, L \rangle = \frac{1}{|G|} d(1 - g(C)) + \sum_{i=1}^r \sum_{\alpha=0}^{m_i-1} N_{i,\alpha} \left( \frac{m_i - 1}{2} - \alpha \right).
\]
Combining this with Hurwitz’ formula
\[
\frac{1}{|G|} (g(C) - 1) = (g' - 1) + \frac{1}{2} \sum_{i=1}^r \left( \frac{m_i - 1}{m_i} \right),
\]
we get
\[
n_\rho = \sigma + d(g' - 1) + \sum_{i=1}^r \left[ d \left( \frac{m_i - 1}{2m_i} \right) - \sum_{\alpha=0}^{m_i-1} N_{i,\alpha} \left( \frac{m_i - 1}{2m_i} - \alpha \right) \right].
\]
The proof of the Chevalley-Weil formula is finished because the degree of the representation $\rho$ is $d = \sum_{\alpha=0}^{m_i-1} N_{i,\alpha}$, for all $1 \leq i \leq r$.

Remark 1.12. We point out that the version of the Chevalley-Weil formula above differs from the original one. In the original paper, the integers $N_{i,\alpha}$ are defined differently: let $p \in C$ be a ramification point which lies over the branch point $x_i$. Let $G_p \cong \mathbb{Z}_{m_i}$ be the stabilizer group. The local monodromy is the unique element $h_p \in G_p$ which maps to $\xi_{m_i}$ under the cotangent representation
\[
G_p \rightarrow GL(m_p/m_p^2) \cong \mathbb{C}^*, \quad g \mapsto [\omega \mapsto (g^{-1})(\omega)].
\]
The integer $N_{i,\alpha}$ is defined as the multiplicity of $\xi_{m_i}^\alpha$ as eigenvalue of $\rho(h_p)$. If $p \in C$ has stabilizer $G_p = \langle gh_i^l \rangle$ for $h_i$ in the generating vector $(d_1, e_1, \ldots, d_{g'}, e_{g'}; h_1, \ldots, h_r)$, then its local monodromy $h_p$ is $gh_i^{-l}g^{-1}$.

Remark 1.13. For a given finite group $G$, it is in general very difficult to determine all its irreducible representations even using a computer algebra system, e.g. MAGMA ([BCP97]). MAGMA is able to compute
the irreducible representations of $G$ only over fields of positive characteristic and, if $G$ is solvable, over cyclotomic fields. Nevertheless, MAGMA can determine the character table of any finite group.

To apply the Chevalley-Weil formula it is not necessary to know the irreducible representations, the character table is sufficient by the following observation:

**Remark 1.14.** Let $\varrho: H \to \text{GL}(W)$ be a representation of a finite group $H$ and $h \in H$ be an element of order $m$. To determine the eigenvalues $\lambda_1, \ldots, \lambda_n$ (counted with multiplicity) of $\varrho(h)$, it suffices to know the character $\chi_\varrho$.

Using the elementary symmetric polynomials in $\mathbb{Z}[x_1, \ldots, x_n]$: $s_0 := 1, \quad s_k := \sum_{1 \leq j_1 < \ldots < j_k \leq n} x_{j_1} \cdots x_{j_k} \quad 1 \leq k \leq n,$

the coefficients of the characteristic polynomial $f_h(x) := \sum_{k=0}^{n} a_k \cdot x^k$ of $\varrho(h)$ are given by $a_k = (-1)^{n-k} \cdot s_{n-k}(\lambda_1, \ldots, \lambda_n), \quad 0 \leq k \leq n.$

The elementary symmetric polynomials are related to the power sum polynomials $p_j := \sum_{i=1}^{n} x_i^j$ via the Newton identities (see [Mea92]): $k \cdot s_k = \sum_{i=1}^{k} (-1)^{i-1} s_{k-i} \cdot p_i, \quad 1 \leq k \leq n.$

Now, the characteristic polynomial $f_h(x)$ can be easily determined from $\chi_\varrho$ using the key-fact $\chi_\varrho(h^k) = p_k(\lambda_1, \ldots, \lambda_n)$.

Finally, since the roots of $f_h(x)$ are powers of $\xi_m$, one can factorize $f_h(x)$ and determine the spectrum of $\varrho(h)$.

**2. On Varieties Isogenous to a Product of Curves**

In this section we introduce varieties isogenous to a product of curves and explain some of their properties.

**Definition 2.1.** A complex algebraic variety $X$ is isogenous to a product of curves if there exist smooth projective curves $C_1, \ldots, C_n$ of genus at least 2 and a finite group $G \subset \text{Aut}(C_1 \times \ldots \times C_n)$ acting freely on the product $C_1 \times \ldots \times C_n$ such that $X = (C_1 \times \ldots \times C_n)/G$.

It follows from the definition that a variety $X$ isogenous to a product of curves is smooth, projective, of general type (i.e. $\kappa(X) = \dim(X) = n$) and its canonical class $K_X$ is ample. The $n$-fold self-intersection of the canonical class $K_X^n$, the topological Euler number $e(X)$ and the holomorphic Euler-Poincaré-characteristic $\chi(O_X)$ can be expressed in terms of the genera $g(C_i)$ and the group order $|G|$.
Proposition 2.2. Let $X := (C_1 \times \ldots \times C_n)/G$ be a variety isogenous to a product of curves. Then

$$K^n_X = \frac{n! \cdot 2^n}{|G|} \prod_{i=1}^{n} (g(C_i) - 1), \quad e(X) = \frac{(-1)^n \cdot 2^n}{|G|} \prod_{i=1}^{n} (g(C_i) - 1)$$

and

$$\chi(O_X) = \frac{(-1)^n}{|G|} \prod_{i=1}^{n} (g(C_i) - 1).$$

Proof. We define $Y := C_1 \times \ldots \times C_n$ and denote by $F_i$ a fiber of the natural projection $p_i: Y \to C_i$ for $1 \leq i \leq n$. It holds

$$K^n_Y \equiv \text{num} \sum_{i=1}^{n} (2g(C_i) - 2)F_i.$$

The $n$-fold self-intersection of $K^n_Y$ is

$$K^n_Y = n! \cdot 2^n \prod_{i=1}^{n} (g(C_i) - 1).$$

The product properties of $e$ and $\chi$ imply

$$e(Y) = \prod_{i=1}^{n} (2 - 2g(C_i)) \quad \text{and} \quad \chi(O_Y) = \prod_{i=1}^{n} (1 - g(C_i)).$$

Note that the quotient map $\pi: Y \to X$ is unramified, hence $\pi^*c_i(X) = c_i(Y)$ for each Chern class.

For any compact complex manifold $Z$ it holds $c_1(K_Z) = -c_1(Z)$ (cf. [GH94, p. 408-414]), hence

$$K^n_Y = \int_Y (-1)^n c_1(Y)^n = \deg(\pi) \int_X (-1)^n c_1(X)^n = |G| \cdot K^n_X.$$

The degree of the top Chern class equals the topological Euler number (cf. [GH94, p. 416]), hence:

$$e(Y) = \int_Y c_i(Y) = \deg(\pi) \int_X c_i(X) = |G| \cdot e(X).$$

Similarly, the Hirzebruch-Riemann-Roch theorem (cf. [GH94, p. 437]) implies:

$$\chi(O_Y) = \int_Y \text{Td}(Y) = \deg(\pi) \int_X \text{Td}(X) = |G| \cdot \chi(O_X),$$

where $\text{Td}(X)$ (resp. $\text{Td}(Y)$) denotes the Todd class of $X$ (resp. $Y$).

Corollary 2.3. Let $X := (C_1 \times C_2 \times C_3)/G$ be a threefold isogenous to a product of curves. Then

$$\chi(O_X) = \frac{-(g(C_1) - 1)(g(C_2) - 1)(g(C_3) - 1)}{|G|},$$

$$K^3_X = -48 \chi(O_X), \quad \text{and} \quad e(X) = 8 \chi(O_X).$$
Remark 2.4. Since $g(C_i) \geq 2$, we observe that $\chi(O_X) \leq -1$ for a threefold isogenous to a product of curves. We point out that this is true for every smooth projective threefold $X$ with ample canonical class: the second Chern class $c_2(X)$ is numerically positive by a result of Miyaoka (see [Miy87, Section 6]). Applying the Hirzebruch-Riemann-Roch theorem we get

$$
\chi(O_X) = \frac{1}{24} \int_X c_1(X)c_2(X) = -\frac{1}{24} \int_X c_1(K_X)c_2(X) < 0.
$$

Remark 2.5. Let $G \subset \text{Aut}(C_1 \times \ldots \times C_n)$ be a subgroup of the automorphism group of a product of compact Riemann surfaces of genus $g(C_i) \geq 2$. We define the subgroup

$$
G^0 := G \cap (\text{Aut}(C_1) \times \ldots \times \text{Aut}(C_n)).
$$

Generalizing [Cat00, Lemma 3.8], there are two possibilities for the action of $G$:

- The un\text{mixed type}, where $G = G^0$. In this case, for $1 \leq i \leq n$ the natural projection $p_i: \text{Aut}(C_1) \times \ldots \times \text{Aut}(C_n) \to \text{Aut}(C_i)$ induces a homomorphism $\psi_i: G \to \text{Aut}(C_i)$ and $G$ acts on $C_1 \times \ldots \times C_n$ in the following way:

$$
g(x_1, \ldots, x_n) = (\psi_1(g)(x_1), \ldots, \psi_n(g)(x_n)).
$$

We say that $G$ acts diagonal\text{ally} and call the action diagonal.

- The mixed type, where $G^0 \subsetneq G$. In this case there are elements in $G$ permuting some factors of the product.

**Notation.** From now on, we only consider threefolds isogenous to a product of curves of unmixed type and simply write “threefold isogenous to a product”.

3. **Group theoretical description and Invariants of Threefolds Isogenous to a Product**

Let $X := (C_1 \times C_2 \times C_3)/G$ be a threefold isogenous to a product. For $1 \leq i \leq 3$, let $\psi_i: G \to \text{Aut}(C_i)$ be the induced action on the $i$-th factor and let $K_i := \ker(\psi_i)$ be its kernel.

By definition $G \subset \text{Aut}(C_1 \times C_2 \times C_3)$, hence the intersection of the kernels is trivial $K_1 \cap K_2 \cap K_3 = \{1_G\}$.

**Definition 3.1.** The action of $G$ is minimal if

$$
K_i \cap K_j = \{1_G\} \quad \text{for all} \quad 1 \leq i < j \leq 3.
$$

In this case we say that the realization $(C_1 \times C_2 \times C_3)/G$ of $X$ is minimal.

The action of $G$ is absolutely faithful if $K_i = \{1_G\}$ for all $1 \leq i \leq 3$.

**Proposition 3.2.** Every threefold isogenous to a product can be obtained by a minimal realization.
Proof. Let $K_{i,j} := K_i \cap K_j$, for $1 \leq i < j \leq 3$, and consider the normal subgroup $K := K_{1,2} \cdot K_{1,3} \cdot K_{2,3} \triangleleft G$, then
\[ \frac{C_1 \times C_2 \times C_3}{G} = \frac{(C_1 \times C_2 \times C_3)/K}{(G/K)} = \frac{C_1/K_{2,3} \times C_2/K_{1,3} \times C_3/K_{1,2}}{H}, \]
where $H := G/K$.

The Riemann surfaces $C_k/K_{i,j}$ have genera $g(C_k/K_{i,j}) \geq 2$, since $g(C_k) \geq 2$ and $K_{i,j}$ acts freely on $C_k$. The induced action of $H$ on $C_k/K_{i,j}$ has kernel $K_k = \ker (H \to \text{Aut}(C_k/K_{i,j})) \cong (K_k \cdot K)/K$ and we achieve $K_i \cap K_j = \{1_H\}$ for $i \neq j$. \qed

Remark 3.3. According to [Cat08, p. 105], a minimal realization is unique.

Notation. From now on, we consider only minimal realizations.

Corollary 3.4. Let $X$ be a threefold isogenous to a product. Then it holds
\[ K_i \times K_j \simeq K_i \cdot K_j \triangleleft G \quad \text{for all} \quad 1 \leq i < j \leq 3. \]
In particular $|K_i| \cdot |K_j|$ divides $|G|$.

We shall associate to a threefold isogenous to a product $X$ certain algebraic and numerical data. According to Riemann’s existence theorem, to each action $G/K_i \to \text{Aut}(C_i)$ corresponds a generating vector $V_i$ for $G/K_i$ of type $T_i$; note that only the type $T_i$ is uniquely determined. According to Corollary 1.5, it holds
\[ \Sigma_{\psi_i}(V_i) = \bigcup_{p \in C_i} G_p. \]
Since the action of $G$ on $C_1 \times C_2 \times C_3$ is free, the triple $(V_1, V_2, V_3)$ is disjoint, that is
\[ \Sigma_{\psi_1}(V_1) \cap \Sigma_{\psi_2}(V_2) \cap \Sigma_{\psi_3}(V_3) = \{1_G\}. \]
This motivates the following definition.

Definition 3.5. To the threefold $X$ we attach the tuple
\[ (G, K_1, K_2, K_3, V_1, V_2, V_3), \]
and call it an algebraic datum of $X$. The numerical datum of $X$ is the tuple
\[ (|G|, |K_1|, |K_2|, |K_3|, T_1, T_2, T_3). \]

Remark 3.6. Let $G$ be a finite group and $K_i \triangleleft G$ be three normal subgroups such that $K_i \cap K_j = \{1_G\}$ for all $1 \leq i < j \leq 3$. Let $(V_1, V_2, V_3)$ be a disjoint triple of generating vectors, where $V_i$ is a generating vector for $G/K_i$. Then, by Riemann’s existence theorem, there exists a threefold isogenous to a product with algebraic datum $(G, K_1, K_2, K_3, V_1, V_2, V_3)$.
3.1. The Hodge diamond. The aim of this subsection is to explain how to compute the Hodge diamond of a threefold isogenous to a product $X$ from an algebraic datum of $X$. We start with a result on invariant differential forms.

**Lemma 3.7.** Let $Y$ be a compact complex manifold, $G \subset \text{Aut}(Y)$ be a finite group acting freely on $Y$ and $X := Y/G$. Then

$$H^{p,q}(Y)^G \cong H^{p,q}(X).$$

**Proof.** The holomorphic map $\pi : Y \to X$ is an unramified Galois covering. It induces the linear map

$$\pi^* : H^{p,q}(X) \to H^{p,q}(Y)^G.$$  

$[\omega] \mapsto [\pi^* \omega]$

We want to find the inverse of this map. Let $x \in X$, then there exists an open neighborhood $U_x$ of $x$ such that

$$\pi^{-1}(U_x) = \bigcup_{i=1}^{\# G} V_i$$

and $\pi|_{V_i} : V_i \to U_x$ is biholomorphic. Let $s_i : U_x \to V_i$ be the inverse maps and $[\omega] \in H^{p,q}(Y)^G$ be a class represented by a $\overline{\partial}$-closed $(p,q)$-form $\omega$. We define $\pi_* \omega$ locally on $U_x$ as

$$\pi_* \omega|_{U_x} := \sum_{i=1}^{\# G} s_i^* \omega.$$  

These locally defined forms of type $(p,q)$ glue to a global form $\pi_* \omega$. Since $\omega$ is $\overline{\partial}$-closed also $\pi_* \omega$ is $\overline{\partial}$-closed. If $\omega = \overline{\partial} \psi$, then $\pi_* \omega = \overline{\partial} \pi_* \psi$, hence we get a well defined homomorphism $\pi_* : H^{p,q}(Y)^G \to H^{p,q}(X)$. It is a straightforward computation to verify $\pi_* \pi^*[\omega] = |G| \cdot [\omega]$ and $\pi^* \pi_* [\omega] = |G| \cdot [\omega]$; therefore, $\frac{1}{|G|} \pi_*$ is the inverse map of $\pi^*$.

**Definition 3.8.** Let $Y$ be a compact complex manifold and $G$ be a finite subgroup of $\text{Aut}(Y)$. For $0 \leq p, q \leq \dim Y$, there is an induced representation of $G$ via pull-back:

$$\phi_{p,q} : G \to \text{GL}(H^{p,q}(Y)), \ h \mapsto [\omega \mapsto (h^{-1})^*(\omega)].$$

The character of $\phi_{p,q}$ is denoted by $\chi_{p,q}$.

**Theorem 3.9.** Let $Y := C_1 \times C_2 \times C_3$ be a product of smooth projective curves and $G$ be a finite subgroup of $\text{Aut}(Y)$ acting diagonally on $Y$, then:

i) $\chi_{1,0} = \chi_{1,1}^2 + \chi_{1,2} + \chi_{1,3}$,

ii) $\chi_{1,1} = 2\text{Re}(\chi_{1,1} \overline{\chi_{1,2}} + \chi_{1,1} \overline{\chi_{1,3}} + \chi_{1,2} \overline{\chi_{1,3}}) + 3\chi_{\text{triv}}$,

iii) $\chi_{2,0} = \chi_{2,1} \overline{\chi_{2,2}} + \chi_{2,1} \overline{\chi_{2,3}} + \chi_{2,2} \overline{\chi_{2,3}},$

iv) $\chi_{2,1} = \chi_{1,1} \overline{\chi_{2,2}} + \chi_{1,1} \overline{\chi_{2,3}} + \chi_{1,2} \overline{\chi_{2,3}} + 3(\chi_{1,1} + \chi_{1,2} + \chi_{1,3}),$

v) $\chi_{3,0} = \chi_{1,1} \overline{\chi_{3,1}},$

vi) $\chi_{p,q} = \overline{\chi_{q,p}}$ for all $(p,q)$. 


Here $\chi_{\varphi_i}$ is the character of the representation $\varphi_i$ associated to the action $\psi_i : G \to \text{Aut}(C_i)$ (Eq. (1.1)) and $\chi_{\text{triv}}$ is the trivial character of $G$.

Proof. According to Künneth’s formula (cf. [GH94, p.103-104]), the vector space $H^{p,q}(Y)$ decomposes as:

\[(3.1)\quad H^{p,q}(Y) = \bigoplus_{s_1+s_2+s_3=p\atop t_1+t_2+t_3=q} H^{s_1,t_1}(C_1) \otimes H^{s_2,t_2}(C_2) \otimes H^{s_3,t_3}(C_3).\]

For any $\omega = \omega_1 \otimes \omega_2 \otimes \omega_3 \in H^{s_1,t_1}(C_1) \otimes H^{s_2,t_2}(C_2) \otimes H^{s_3,t_3}(C_3)$ and $g \in G$, the tensors $\omega$ and $(g^{-1})^*(\omega)$ are in the same direct summand because

\[(g^{-1})^*(\omega) = (\psi_1(g^{-1})^*\omega_1) \otimes (\psi_2(g^{-1})^*\omega_2) \otimes (\psi_3(g^{-1})^*\omega_3).\]

This implies that every direct summand in Eq. (3.1) is a subrepresentation of $H^{p,q}(Y)$ and the character $\chi_{p,q}$ is the sum of the characters of these subrepresentations. Using Lemma 1.10 and the fact that the character of a tensor product is the product of the characters, the statement follows.

Remark 3.10. Let $X$ be a threefold isogenous to a product. Using the Chevalley-Weil formula, the characters $\chi_{\varphi_i}$ can be determined from an algebraic datum of $X$. Theorem 3.9 allows us to compute the characters $\chi_{p,q}$ from the characters $\chi_{\varphi_i}$. Thus, the representations $\varphi_{p,q}$ are completely described.

As a byproduct the Hodge Diamond of $X$ can be determined from an algebraic datum of $X$.

Theorem 3.11. Let $X := (C_1 \times C_2 \times C_3)/G$ be a threefold isogenous to a product. Then

\[h^{p,q}(X) = \dim \left( H^{p,q}(C_1 \times C_2 \times C_3)^G \right) = \langle \chi_{\text{triv}}, \chi_{p,q} \rangle, \quad 0 \leq p, q \leq 3.\]

Proof. The statement follows from Lemma 3.7 and the first projection formula, see [FH91, Section 2.2].

3.2. The Fundamental Group. In this subsection we briefly explain how to compute the fundamental group of a threefold isogenous to a product $X$ from an algebraic datum of $X$. This topic is treated in greater generality in [DP12] and we refer to it for further details.

Definition 3.12. Let $g' \geq 0$ and $m_1, \ldots, m_r \geq 2$ be integers. The orbifold surface group of type $[g'; m_1, \ldots, m_r]$ is:

\[\mathbb{T}(g'; m_1, \ldots, m_r) := \langle a_1, b_1, \ldots, a_{g'}, b_{g'}, c_1, \ldots, c_r \mid c_1^{m_1}, \ldots, c_r^{m_r}, \prod_{i=1}^{g'} \left[ a_i, b_i \right], \prod_{i=1}^{r} [c_i, \ldots, c_r] \rangle.\]
Lemma 3.13. Let \( X = (C_1 \times C_2 \times C_3)/G \) be a threefold isogenous to a product. The fundamental group \( \pi_1(X) \) sits in the following short exact sequence:

\[ 1 \to \pi_1(C_1) \times \pi_1(C_2) \times \pi_1(C_3) \to \pi_1(X) \to G \to 1. \]

For \( 1 \leq i \leq 3 \), let \( \psi_i : G \to \text{Aut}(C_i) \) be the induced action on the \( i \)-th factor with kernel \( K_i := \ker(\psi_i) \) and \( p_i : G \to H_i := G/K_i \) be the quotient map. According to Riemann’s existence theorem, to each covering \( C_i \to C_i/H_i \) corresponds a generating vector \( V_i \) for \( H_i \) of type \( T_i \), or equivalently a surjective map \( \gamma_i : T(T_i) \to H_i \) whose kernel is isomorphic to \( \pi_1(C_i) \) (see [DP12, Remark 2.2]):

\[ 1 \to \pi_1(C_i) \to T(T_i) \xrightarrow{\gamma_i} H_i \to 1. \]

Let \( \Gamma_i := G \times H_i \cdot T(T_i) \) be the fiber product corresponding to the maps \( \gamma_i \) and \( p_i \) and let \( \zeta_i : \Gamma_i \to G \) be the projection on the first factor.

We define the fiber product

\[ \mathbb{G} := \{ (x, y, z) \in \Gamma_1 \times \Gamma_2 \times \Gamma_3 \mid \zeta_1(x) = \zeta_2(y) = \zeta_3(z) \}. \]

As a direct consequence of [DP12, Proposition 3.3], there is the following description of the fundamental group and the first homology group.

Proposition 3.14. Let \( X \) be a threefold isogenous to a product. Then \( \pi_1(X) \cong \mathbb{G} \) and \( H_1(X, \mathbb{Z}) \cong \mathbb{G}^{ab} \).

4. Combinatorics, Bounds and Algorithm

In this section we show that the classification of threefolds isogenous to a product \( X \) with fixed value of \( \chi(O_X) \) is a finite problem and we give an algorithm to perform this classification.

Definition 4.1. Let \( g' \geq 0 \) and \( m_1, \ldots, m_r \geq 2 \) be integers. We associate to the tuple \( T := [g'; m_1, \ldots, m_r] \) the rational number

\[ \Theta(T) := 2g' - 2 + \sum_{i=1}^{r} \frac{m_i - 1}{m_i}. \]

Remark 4.2. If \( \Theta(T) > 0 \), then \( \Theta(T) \geq f(g') \), where:

\[ f(g') := \begin{cases} 
\frac{1}{42}, & \text{if } \ g' = 0 \\
\frac{1}{7}, & \text{if } \ g' = 1 \\
2g' - 2, & \text{if } \ g' \geq 2
\end{cases} \]

Moreover, \( \Theta(T) = f(g') \) if and only if \( T \in \{ [0; 2, 3, 7], [1; 2], [2g' - 2; -] \} \), see [Mir95, Lemma III.3.8].

Remark 4.3. According to Hurwitz’ theorem, the automorphism group of a Riemann surface \( C \) of genus \( g(C) \geq 2 \) is finite and its order is bounded:

\[ |\text{Aut}(C)| \leq 84(g(C) - 1). \]
It is known that this bound is not sharp in general, however, sharp bounds for $|\text{Aut}(C)|$ can be found in [Bre00, p. 91] for the cases $2 \leq g(C) \leq 48$.

**Proposition 4.4.** Let $X = (C_1 \times C_2 \times C_3)/G$ be a threefold isogenous to a product. Then

$$|G| \leq \sqrt{\frac{-8 \cdot \chi(\mathcal{O}_X) \cdot \prod_{i=1}^{3} |K_i|}{f(g'_i)}} \leq M(\chi(\mathcal{O}_X)) := 84^6 \cdot (\chi(\mathcal{O}_X))^2,$$

where $g'_i$ is the genus of the quotient Riemann surface $C_i/G$.

**Proof.** Since $g(C_i) \geq 2$ Hurwitz’ formula

$$g(C_i) - 1 = \frac{1}{2} \frac{|G_i|}{|K_i|} \cdot \Theta(T_i)$$

implies $\Theta(T_i) > 0$. By Proposition 2.2, it follows

$$-\chi(\mathcal{O}_X) = \frac{(g(C_1) - 1)(g(C_2) - 1)(g(C_3) - 1)}{|G|}$$

$$= \frac{|G|^2}{8} \prod_{i=1}^{3} \frac{\Theta(T_i)}{|K_i|} \geq \frac{|G|^2}{8} \prod_{i=1}^{3} \frac{f(g'_i)}{|K_i|}.$$ 

Now, we write down the formula for $\chi(\mathcal{O}_X)$ in a different way and estimate

$$-\chi(\mathcal{O}_X) = \frac{(g(C_1) - 1)(g(C_2) - 1)(g(C_3) - 1)|K_3|}{|K_3|} \geq \left( \frac{1}{84} \right)^3 |K_3|.$$ 

The inequality is Hurwitz’ theorem which holds because $K_3$ acts faithfully on $C_1$ and $C_2$, by the minimality assumption, and $G/K_3$ acts faithfully on $C_3$. The same bound holds for $|K_1|$ and $|K_2|$ and the statement follows. \hfill \qed

**Corollary 4.5.** Let $X$ be a threefold isogenous to a product. If the action of $G$ is absolutely faithful, then

$$|G| \leq \lfloor 168 \sqrt{-21 \chi(\mathcal{O}_X)} \rfloor.$$ 

**Proposition 4.6.** Let $X = (C_1 \times C_2 \times C_3)/G$ be a threefold isogenous to a product with numerical datum $(|G|, |K_1|, |K_2|, |K_3|, T_1, T_2, T_3)$, where $T_i = [g'_i; m_{i,1}, \ldots, m_{i,r_i}]$. Then

i) $|K_i| \mid (g(C_{i+1}) - 1)(g(C_{i+2}) - 1)$,

ii) $m_{i,j} \mid (g(C_{i+1}) - 1)(g(C_{i+2}) - 1)$,

iii) $(g(C_i) - 1) \mid \chi(\mathcal{O}_X)|_{K_i}$,

iv) $r_i \leq \frac{4(g(C_i) - 1)|K_i|}{|G|} - 4g'_i + 4$,

v) $m_{i,j} \leq 4g(C_i) + 2$, 

respectively.
vi) \( g'_i \leq 1 - \frac{\chi(O_X)|K_i|}{(g(C_{i+1}) - 1)(g(C_{i+2}) - 1)} \leq 1 - \chi(O_X). \)

Here \( [\cdot] \) denotes the residue mod 3.

**Proof.** Throughout the proof we can assume that \( i = 1. \)

i-ii) Let \( V_1 := (d_1, e_1, \ldots, d_{g'}, e_{g'}; h_1, \ldots, h_r) \) be a generating vector of type \( T_1 \) associated to the covering \( f_1: C_1 \to C_1/G_1, \) where \( G_1 := G/K_1. \)

Let \( g_j \in G \) be a representative of \( h_j \in G_1 \) (if \( r = 0, \) then we set \( g_j := 1_G \)). By the minimality of the \( G \)-action, the subgroup \( \langle g_j \rangle \cdot K_1 \subset G \) acts faithfully on \( C_2 \times C_3. \) Furthermore, \( \langle g_j \rangle \cdot K_1 \) is contained in \( \Sigma_v(C_1) \), therefore, it acts freely on \( C_2 \times C_3. \) In other words

\[
S := \frac{C_2 \times C_3}{\langle g_j \rangle \cdot K_1}
\]

is a surface isogenous to a product and its holomorphic Euler-Poincaré characteristic is

\[
\chi(O_S) = \frac{(g(C_2) - 1)(g(C_3) - 1)}{|\langle g_j \rangle \cdot K_1|}.
\]

We conclude that

\[
|K_1| \mid (g(C_2) - 1)(g(C_3) - 1) \quad \text{and} \quad m_{1,j} = \text{ord}(h_j) \mid (g(C_2) - 1)(g(C_3) - 1).
\]

iii) The statement follows from part i) and Proposition \ref{prop:bound}

\[
-\chi(O_X)\frac{|G|}{|K_1|} = (g(C_1) - 1)\frac{(g(C_2) - 1)(g(C_3) - 1)}{|K_1|}.
\]

iv) This is a straightforward consequence of Hurwitz’ formula, using the fact \( m_{i,j} \geq 2. \)

v) For a cyclic group \( H \) acting faithfully on a compact Riemann surface \( C \) of genus \( g(C) \geq 2 \), Wiman’s bound (see \cite{Wim95}) holds: \( |H| \leq 4g(C) + 2. \) In particular, for \( H = \langle h_j \rangle \) we get \( m_{1,j} \leq 4g(C_1) + 2. \)

vi) By Proposition \ref{prop:bound} and Hurwitz’ formula, it follows

\[
g'_i - 1 \leq \frac{\Theta(T_1)}{2} = \frac{|K_1|}{|G|}(g(C_1) - 1) = -\frac{\chi(O_X)|K_1|}{(g(C_2) - 1)(g(C_3) - 1)}.
\]

The second inequality follows now from part i). \( \square \)

**Remark 4.7.** From Proposition \ref{prop:bound} and Proposition \ref{prop:bound}, it follows that the set of numerical data of threefolds isogenous to a product \( X \) with fixed value of \( \chi(O_X) \) is finite.
4.1. The algorithm. Let $\chi \leq -1$ and $n \leq M(\chi)$ (see Proposition 4.4) be integers. We implemented a MAGMA script to find all threefolds isogenous to a product $X$ with $\chi(\mathcal{O}_X) = \chi$ and $|G| = n$.

The full code is rather long and a commented version of it can be downloaded from:

http://www.staff.uni-bayreuth.de/~bt300503/

The algorithm is:

**Input.** Integers $\chi \leq -1$ and $n \leq M(\chi)$.

**Step 1.** Compute the possible orders of the kernels, i.e. the triples

$$(k_1, k_2, k_3) \in \mathbb{N}^3.$$ 

satisfying the conditions imposed by Proposition 4.4 and Corollary 3.4.

**Step 2.** Determine the possible genera of the curves $C_i$ and $C'_i = C_i/G$; for every triple in the output of Step 1, i.e. construct the set of 9-tuples

$$(k_1, k_2, k_3, g_1, g_2, g_3, g'_1, g'_2, g'_3),$$

which fulfill the conditions given by Proposition 4.4, Proposition 4.6 and Remark 4.3.

**Step 3.** For every 9-tuple in the output of Step 2 construct the set of 9-tuples

$$(k_1, k_2, k_3, g_1, g_2, g_3, T_1, T_2, T_3),$$

where $T_i = [g'_i; m_{i,1}, \ldots, m_{i,r_i}]$ are the types which satisfy the conditions of Proposition 4.6.

**Step 4.** For every 9-tuple in the output of the Step 3 search among the groups of order $n$ for groups $G$ containing normal subgroups $K_i$ of order $k_i$ which have pairwise trivial intersection:

$$K_i \cap K_j = \{1_G\}, \quad 1 \leq i < j \leq 3.$$ 

For every triple of subgroups satisfying these conditions search for triples $(V_1, V_2, V_3)$ of disjoint generating vectors, where $V_i$ is a generating vector for $G/K_i$ of type $T_i$.

**Step 5.** For each 7-tuple $(G, K_1, K_2, K_3, V_1, V_2, V_3)$ in the output of Step 4 there exists a threefold $X$ isogenous to a product with this algebraic datum (see Remark 3.6) and $\chi(\mathcal{O}_X) = \chi$ and $|G| = n$. Using the method described in Section 3, compute the Hodge diamond and the fundamental group of $X$.

Note that each algebraic datum of each threefold $X$ isogenous to a product with $\chi(\mathcal{O}_X) = \chi$ and $|G| = n$ appears in the output of Step 4.

**Output.** The occurrences of

$$[G, K_1, K_2, K_3, T_1, T_2, T_3, p_g, q_2, q_1, h^{1,1}, h^{1,2}, H_1(-, \mathbb{Z})].$$
for all threefolds isogenous to a product $X$ with $\chi(\mathcal{O}_X) = \chi$ and $|G| = n$. Note that we store $H_1$ instead of $\pi_1$ since the latter is always infinite and a presentation of this group is long and useless.

Remark 4.8. In Step 4, we search for generating vectors. We point out that different generating vectors may determine threefolds with the same invariants. For example, this happens if (but not only if) they differ by some Hurwitz moves. These moves are described in [CLP12], [Zim87] and [Pen11] and we refer to them for further details.

Remark 4.9. The algorithm works for arbitrary values of $\chi$ and $n$, but the implemented MAGMA version has some technical limitations: if the output of Step 3 is not empty, then in Step 4, the program has to run through all groups of order $n$. Here we use the database of Small Groups which contains all groups of order up to 2000, excluding the groups of order 1024. The execution of Step 4 is not performed in the following cases:

i) If $n = 1024$ because there is no MAGMA database containing all groups of this order.

ii) If $n \geq 2001$ because, in general, there is no MAGMA database containing all groups of order $n$.

iii) If $n$ is contained in $\{256, 384, 512, 768, 1152, 1280, 1536, 1920\}$ because there are too many isomorphism classes of groups of order $n$ and it is not efficient to run among them; e.g., there are 56092 groups of order 256.

To investigate these exceptional cases one can apply the methods in [BCG08, Section 3, 4] (see also [Gle13] and [Fra13]).

5. Computational Results

In this section we give examples of threefolds isogenous to a product for the boundary value $\chi(\mathcal{O}_X) = -1$ (cf. Remark 2.4).

The bound for the group order $M(-1) = 84^6$ is too large to perform a complete classification (cf. Remark 4.9). On the other hand, if the group action is absolutely faithful then the bound drops to 769 and a complete classification is possible. In the first part of this section we present this classification.

In the second part we investigate the other extremal case: $K_3 = G$ and present a complete classification.

5.1. The absolutely faithful case. Running our MAGMA script in the absolutely faithful case for $\chi = -1$ and for all $n \leq 769$. For the exceptional group orders (cf. Remark 4.9) the output of Step 3 is empty and we obtain the following classification theorem.

Theorem 5.1. Let $X = (C_1 \times C_2 \times C_3)/G$ be a threefold isogenous to a product of curves of unmixed type. Assume that the action of $G$ is
absolutely faithful and $\chi(\mathcal{O}_X) = -1$. Then, the tuple 

$$[G, T_1, T_2, T_3, p_g(X), q_2(X), q_1(X), h^{1,1}(X), h^{1,2}(X), H_1(X, Z)]$$

appears in Table 1.

Conversely, each row in Table 1 is realized by a threefold isogenous to a product of curves of unmixed type with $\chi(\mathcal{O}_X) = -1$ and absolutely faithful $G$-action.

### Table 1

| $G$   | Id     | $T_1$     | $T_2$     | $T_3$ | $p_g$ | $q_2$ | $q_1$ | $h^{1,1}$ | $h^{1,2}$ | $H_1$     |
|-------|--------|-----------|-----------|-------|-------|-------|-------|------------|------------|-----------|
| $\mathfrak{A}_3$ | (60, 5) | [0; 2, 5$^2$] | [0; 3$^2$, 5] | [0; 2$^3$, 3] | 2 | 0 | 0 | 3 | 6 | $\mathbb{Z}_2 \times \mathbb{Z}_{30}$ |
| $\mathfrak{S}_4 \times \mathbb{Z}_2$ (48, 48) | [0; 2, 4, 6] | [0; 2, 4, 6] | [0; 2$^5$] | 3 | 1 | 0 | 5 | 9 | $\mathbb{Z}_2^3 \times \mathbb{Z}_4$ |
| $Q \rtimes \varphi \mathfrak{S}_3$ (48, 29) | [0; 2, 3, 8] | [0; 2, 3, 8] | [2; -] | 5 | 5 | 2 | 11 | 17 | $\mathbb{Z}_4$ |
| $Q \rtimes \varphi \mathfrak{S}_3$ (48, 29) | [0; 2, 3, 8] | [0; 2, 3, 8] | [2; -] | 4 | 4 | 2 | 13 | 18 | $\mathbb{Z}_4$ |
| $\mathbb{Z}_2 \rtimes \varphi \mathfrak{S}_3$ (24, 8) | [0; 2, 4, 6] | [0; 2, 4, 6] | [2; -] | 5 | 5 | 2 | 11 | 17 | $\mathbb{Z}_4$ |
| $\mathbb{Z}_2 \rtimes \varphi \mathfrak{S}_3$ (24, 8) | [0; 2, 4, 6] | [0; 2, 4, 6] | [2; -] | 4 | 4 | 2 | 13 | 18 | $\mathbb{Z}_4$ |
| $Q \rtimes \varphi \mathbb{Z}_3$ (24, 3) | [0; 3, 3, 4] | [0; 3, 3, 4] | [2; -] | 5 | 5 | 2 | 13 | 19 | $\mathbb{Z}_4$ |
| $\mathfrak{S}_4$ (24, 12) | [0; 3, 4$^2$] | [0; 2$^3$, 4] | [0; 2$^2$, 3$^2$] | 3 | 1 | 0 | 5 | 9 | $\mathbb{Z}_2 \times \mathbb{Z}_{24}$ |
| $\mathbb{Z}_8 \rtimes \varphi \mathfrak{Z}_2$ (16, 8) | [0; 2, 4, 8] | [0; 2, 4, 8] | [2; -] | 5 | 5 | 2 | 11 | 17 | $\mathbb{Z}_2 \times \mathbb{Z}_4$ |
| $\mathfrak{D}_4 \times \mathbb{Z}_2$ (16, 11) | [0; 2$^3$, 4] | [0; 2$^3$, 4] | [0; 2$^5$] | 4 | 2 | 0 | 7 | 12 | $\mathbb{Z}_2^3 \times \mathbb{Z}_4^2$ |
| $\mathfrak{D}_4 \times \mathbb{Z}_2$ (16, 11) | [0; 2$^3$, 4] | [0; 2$^3$, 4] | [0; 2$^5$] | 3 | 1 | 0 | 5 | 9 | $\mathbb{Z}_2^3 \times \mathbb{Z}_4^2$ |
| $\mathfrak{S}_3 \times \mathbb{Z}_2$ (12, 4) | [0; 2$^3$, 3] | [0; 2$^3$, 3] | [1; 2$^2$] | 4 | 3 | 1 | 9 | 14 | $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_2^2$ |
| $\mathfrak{S}_3 \times \mathbb{Z}_2$ (12, 4) | [0; 2$^3$, 3] | [0; 2$^3$, 3] | [2; -] | 5 | 5 | 2 | 13 | 19 | $\mathbb{Z}_4^3$ |
| $\mathfrak{S}_3 \times \mathbb{Z}_2$ (12, 4) | [0; 2$^3$, 3] | [0; 2$^3$, 3] | [1; 3] | 4 | 3 | 1 | 9 | 14 | $\mathbb{Z}_2^2 \times \mathbb{Z}_4^2$ |
| $\mathbb{Z}_3 \rtimes \varphi \mathfrak{Z}_2$ (12, 5) | [0; 2$^2$, 6$^2$] | [0; 2$^2$, 6$^2$] | [0; 2$^2$, 6$^2$] | 6 | 6 | 2 | 11 | 18 | $\mathbb{Z}_4$ |
| $\mathbb{Z}_3 \rtimes \varphi \mathfrak{Z}_2$ (12, 5) | [0; 2$^2$, 6$^2$] | [0; 2$^2$, 6$^2$] | [0; 2$^2$, 6$^2$] | 5 | 5 | 2 | 11 | 17 | $\mathbb{Z}_4$ |
| $\mathbb{Z}_3 \rtimes \varphi \mathfrak{Z}_2$ (12, 5) | [0; 2$^2$, 6$^2$] | [0; 2$^2$, 6$^2$] | [0; 2$^2$, 6$^2$] | 4 | 4 | 2 | 13 | 18 | $\mathbb{Z}_4$ |
| $\mathbb{Z}_3 \rtimes \varphi \mathfrak{Z}_2$ (12, 5) | [0; 2$^2$, 6$^2$] | [0; 2$^2$, 6$^2$] | [0; 2$^2$, 6$^2$] | 4 | 4 | 2 | 15 | 20 | $\mathbb{Z}_4$ |
| $\mathbb{Z}_3 \rtimes \varphi \mathfrak{Z}_4$ (12, 1) | [0; 3, 4$^2$] | [0; 3, 4$^2$] | [2; -] | 5 | 5 | 2 | 13 | 19 | $\mathbb{Z}_4^3$ |
| $\mathbb{Z}_{10}$ (10, 2) | [0; 2, 5, 10] | [0; 2, 5, 10] | [2; -] | 5 | 5 | 2 | 13 | 19 | $\mathbb{Z}_4^4$ |
| $\mathbb{Z}_{10}$ (10, 2) | [0; 2, 5, 10] | [0; 2, 5, 10] | [2; -] | 6 | 6 | 2 | 11 | 18 | $\mathbb{Z}_4^5$ |
| $\mathbb{Z}_{10}$ (10, 2) | [0; 2, 5, 10] | [0; 2, 5, 10] | [2; -] | 4 | 4 | 2 | 15 | 20 | $\mathbb{Z}_4^6$ |
| $\mathfrak{D}_4$ (8, 3) | [0; 2$^3$, 4] | [1; 2$^2$] | [0; 2$^6$] | 4 | 3 | 1 | 9 | 14 | $\mathbb{Z}_2^2 \times \mathbb{Z}_4^2$ |
| $\mathfrak{D}_4$ (8, 3) | [0; 2$^3$, 4] | [1; 2$^2$] | [1; 2$^2$] | 4 | 4 | 2 | 11 | 16 | $\mathbb{Z}_2 \times \mathbb{Z}_4^2$ |
| $\mathfrak{D}_4$ (8, 3) | [0; 2$^3$, 4] | [0; 2$^3$, 4] | [1; 2$^2$] | 4 | 3 | 1 | 9 | 14 | $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_2^2$ |
| $\mathfrak{D}_4$ (8, 3) | [0; 2$^3$, 4] | [0; 2$^3$, 4] | [2; -] | 5 | 5 | 2 | 13 | 19 | $\mathbb{Z}_2^3 \times \mathbb{Z}_4^2$ |
| $\mathbb{Z}_8$ (8, 1) | [0; 2, 8$^2$] | [0; 2, 8$^2$] | [2; -] | 6 | 6 | 2 | 11 | 18 | $\mathbb{Z}_2^5 \times \mathbb{Z}_4$ |
| $G$ | $Id$ | $T_1$ | $T_2$ | $T_3$ | $p_g$ $q_2$ $q_1$ $h^{1,1}$ $h^{1,2}$ | $H_1$ |
|-----|------|------|------|------|-----------------|------|
| $\mathbb{Z}_8$ | $(8,1)$ | $[0; 2, 8^2]$ | $[0; 2, 8^2]$ | $[2; -]$ | $4$ $4$ $2$ $15$ $20$ | $\mathbb{Z}_2^4 \times \mathbb{Z}_4^4$ |
| $Q$ | $(8,4)$ | $[0; 4^3]$ | $[0; 4^3]$ | $[2; -]$ | $5$ $5$ $2$ $13$ $19$ | $\mathbb{Z}_2^4 \times \mathbb{Z}_4^4$ |
| $\mathbb{Z}_3$ | $(8,5)$ | $[0; 2^5]$ | $[0; 2^5]$ | $[0; 2^5]$ | $5$ $3$ $0$ $9$ $15$ | $\mathbb{Z}_3^2 \times \mathbb{Z}_6^2$ |
| $\mathbb{Z}_3$ | $(8,5)$ | $[0; 2^5]$ | $[0; 2^5]$ | $[0; 2^5]$ | $4$ $2$ $0$ $7$ $12$ | $\mathbb{Z}_3^3 \times \mathbb{Z}_4^4$ |
| $\mathbb{Z}_3$ | $(8,5)$ | $[0; 2^5]$ | $[0; 2^5]$ | $[0; 2^5]$ | $4$ $2$ $0$ $7$ $12$ | $\mathbb{Z}_3^5 \times \mathbb{Z}_4^4$ |
| $\mathfrak{S}_3$ | $(6,1)$ | $[0; 2^3, 3^2]$ | $[1; 3]$ | $[0; 2^6]$ | $4$ $3$ $1$ $9$ $14$ | $\mathbb{Z}_2^3 \times \mathbb{Z}_6 \times \mathbb{Z}_2^2$ |
| $\mathfrak{S}_3$ | $(6,1)$ | $[0; 2^3, 3^2]$ | $[1; 3]$ | $[1; 2^2]$ | $4$ $4$ $2$ $11$ $16$ | $\mathbb{Z}_3^3 \times \mathbb{Z}_4^4$ |
| $\mathfrak{S}_3$ | $(6,1)$ | $[0; 2^3, 3^2]$ | $[0; 2^3, 3^2]$ | $[2; -]$ | $5$ $5$ $2$ $13$ $19$ | $\mathbb{Z}_3^3 \times \mathbb{Z}_4^4$ |
| $\mathbb{Z}_6$ | $(6,2)$ | $[0; 2^2, 3^2]$ | $[0; 2^2, 3^2]$ | $[2; -]$ | $6$ $6$ $2$ $15$ $22$ | $\mathbb{Z}_3^4$ |
| $\mathbb{Z}_6$ | $(6,2)$ | $[0; 2^2, 3^2]$ | $[0; 3, 6^2]$ | $[2; -]$ | $5$ $5$ $2$ $13$ $19$ | $\mathbb{Z}_3^4$ |
| $\mathbb{Z}_6$ | $(6,2)$ | $[0; 3, 6^2]$ | $[0; 3, 6^2]$ | $[2; -]$ | $6$ $6$ $2$ $11$ $18$ | $\mathbb{Z}_3^5 \times \mathbb{Z}_4^4$ |
| $\mathbb{Z}_6$ | $(6,2)$ | $[0; 3, 6^2]$ | $[0; 3, 6^2]$ | $[2; -]$ | $4$ $4$ $2$ $15$ $20$ | $\mathbb{Z}_3^5 \times \mathbb{Z}_4^4$ |
| $\mathbb{Z}_5$ | $(5,1)$ | $[0; 5^3]$ | $[0; 5^3]$ | $[2; -]$ | $6$ $6$ $2$ $11$ $18$ | $\mathbb{Z}_5^5 \times \mathbb{Z}_4^4$ |
| $\mathbb{Z}_5$ | $(5,1)$ | $[0; 5^3]$ | $[0; 5^3]$ | $[2; -]$ | $6$ $6$ $2$ $11$ $18$ | $\mathbb{Z}_5^6 \times \mathbb{Z}_4^4$ |
| $\mathbb{Z}_5$ | $(5,1)$ | $[0; 5^3]$ | $[0; 5^3]$ | $[2; -]$ | $5$ $5$ $2$ $13$ $19$ | $\mathbb{Z}_5^6 \times \mathbb{Z}_4^4$ |
| $\mathbb{Z}_5$ | $(5,1)$ | $[0; 5^3]$ | $[0; 5^3]$ | $[2; -]$ | $4$ $4$ $2$ $15$ $20$ | $\mathbb{Z}_5^6 \times \mathbb{Z}_4^4$ |
| $\mathbb{Z}_5$ | $(4,2)$ | $[0; 2^5]$ | $[0; 2^5]$ | $[1; 2^2]$ | $6$ $5$ $1$ $13$ $20$ | $\mathbb{Z}_5^3 \times \mathbb{Z}_4^2 \times \mathbb{Z}_2^2$ |
| $\mathbb{Z}_5$ | $(4,2)$ | $[0; 2^5]$ | $[0; 2^5]$ | $[1; 2^2]$ | $5$ $4$ $1$ $11$ $17$ | $\mathbb{Z}_5^3 \times \mathbb{Z}_4^2 \times \mathbb{Z}_2^2$ |
| $\mathbb{Z}_5$ | $(4,2)$ | $[0; 2^5]$ | $[0; 2^5]$ | $[1; 2^2]$ | $5$ $5$ $2$ $13$ $19$ | $\mathbb{Z}_5^3 \times \mathbb{Z}_4^2 \times \mathbb{Z}_2^2$ |
| $\mathbb{Z}_5$ | $(4,2)$ | $[0; 2^5]$ | $[0; 2^5]$ | $[1; 2^2]$ | $5$ $5$ $2$ $13$ $19$ | $\mathbb{Z}_5^3 \times \mathbb{Z}_4^2 \times \mathbb{Z}_2^2$ |
| $\mathbb{Z}_5$ | $(4,1)$ | $[0; 2^4, 4^2]$ | $[0; 2^2, 4^2]$ | $[2; -]$ | $6$ $6$ $2$ $11$ $22$ | $\mathbb{Z}_5^4 \times \mathbb{Z}_4^4$ |
| $\mathbb{Z}_3$ | $(3,1)$ | $[0; 3^4]$ | $[0; 3^4]$ | $[2; -]$ | $6$ $6$ $2$ $15$ $22$ | $\mathbb{Z}_3^4 \times \mathbb{Z}_4^4$ |
| $\mathbb{Z}_2$ | $(2,1)$ | $[1; 2^2]$ | $[1; 2^2]$ | $[2; -]$ | $5$ $8$ $4$ $19$ $26$ | $\mathbb{Z}_8^4$ |
| $\mathbb{Z}_2$ | $(2,1)$ | $[0; 2^6]$ | $[1; 2^2]$ | $[2; -]$ | $6$ $7$ $3$ $17$ $24$ | $\mathbb{Z}_2^4 \times \mathbb{Z}_6^6$ |
| $\mathbb{Z}_2$ | $(2,1)$ | $[0; 2^6]$ | $[0; 2^6]$ | $[2; -]$ | $8$ $8$ $2$ $19$ $28$ | $\mathbb{Z}_2^5 \times \mathbb{Z}_4^4$ |
| $\{1\}$ | $(1,1)$ | $[2; -]$ | $[2; -]$ | $[2; -]$ | $8$ $12$ $6$ $27$ $36$ | $\mathbb{Z}_2^{12}$ |

**Notation.** In Table [I] the types $T_i$ are given in a simplified way:

$$[g; a_1^{k_1}, ..., a_r^{k_r}] := [g; a_1; a_1, ..., a_r; a_r].$$

The column $Id$ reports the MAGMA identifier of the group $G$, here $(a, b)$ denotes the $b^{th}$ group of order $a$ in the database of Small Groups. The cyclic group or order $n$ is denoted by $\mathbb{Z}_n$, the symmetric group on $n$ letters by $\mathfrak{S}_n$, the alternating group on $n$ letters by $\mathfrak{A}_n$, the quaternion group by $Q$ and the dihedral group of order $2n$ by $D_n$. 


5.2. The case $G = K_3$. Let $X = (C_1 \times C_2 \times C_3)/G$ be a threefold isogenous to a product such that $K_3 = G$. By assumption the $G$-action is minimal, hence $K_1 = K_2 = \{1_G\}$ and $X = S \times C_3$, where $S := (C_1 \times C_2)/G$ is a surface isogenous to a product of unmixed type. The holomorphic Euler-Poincaré characteristic of $X$ is
\[
\chi(O_X) = \chi(O_S) - \chi(O_{C_3}).
\]
Since we are interested in the case $\chi(O_X) = -1$, we restrict ourselves to this case, but this construction works for any value of $\chi(O_X) \leq -1$.

Since $S$ is of general type, it holds $\chi(O_S) \geq 1$ ([Bea83, Theorem X.4]). To get $\chi(O_X) = -1$, we need $g(C_3) = 2$ and $p_g(S) = q(S)$. The latter condition, together with the Bogomolov-Miyaoka-Yau inequality $K_S^2 \leq 9\chi(O_S)$ and the Debarre inequality $K_S^2 \geq 2p_g(S)$ (assuming $q(S) > 0$), gives $0 \leq q(S) = p_g(S) \leq 4$ and by K"unneth’s formula:
\[
\begin{align*}
q_1(X) &= q(S) + 2, \\
q_2(X) &= 3q(S), \\
p_g(X) &= 2q(S),
\end{align*}
\]
\[
\begin{align*}
h^{1,1}(X) &= 6q(S) + 3, \\
h^{1,2}(X) &= 8q(S) + 4.
\end{align*}
\]

We collect the possible values for these integers in Table 2.

| $p_g(S) = q(S)$ | $p_g(X)$ | $q_2(X)$ | $q_1(X)$ | $h^{1,1}(X)$ | $h^{1,2}(X)$ |
|-----------------|-----------|-----------|-----------|----------------|----------------|
| 0               | 0         | 0         | 2         | 3              | 4              |
| 1               | 2         | 3         | 3         | 9              | 12             |
| 2               | 4         | 6         | 4         | 15             | 20             |
| 3               | 6         | 9         | 5         | 21             | 28             |
| 4               | 8         | 12        | 6         | 27             | 36             |

Instead of running our program, we use the fact that surfaces isogenous to a product with $p_g(S) = q(S)$ are completely classified (cf. Introduction). We conclude that all threefold isogenous to a product of the form $X = S \times C$ with $\chi(O_X) = -1$ are known and the invariants in each row of Table 2 are realized.

Remark 5.2. With no extra conditions on the kernels, the output of the algorithm is enormous. For example, there are already 503 occurrences in the case $\chi(O_X) = -1$ and $|G| \leq 15$.

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