Charged Accelerating BTZ Black Holes

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In this paper, general uncharged accelerating BTZ black hole solutions are extracted and studied some of their properties. The analysis shows that spacetime’s asymptotic behavior depends on four parameters: the cosmological constant, mass, acceleration, and topological constant. Then, the temperature of these black holes are studied and it is found that the temperature is always positive for AdS spacetime. Next, the study is extended for extracting charged accelerating BTZ black hole solutions in the presence of a nonlinear electrodynamics field known as conformally invariant Maxwell. The findings indicate a coupling between the electrical charge and other quantities of the accelerating BTZ black holes. The asymptotic behavior of charged accelerating BTZ black holes depends on five parameters: the cosmological constant, the electrical charge, mass, the acceleration parameter, and the topological constant. Then, the effects of charge, acceleration parameters, and the topological constant on the root of these black holes are studied. Finally, the temperature of these black holes in AdS spacetime is investigated. For these black holes, the temperature depends on the electrical charge, accelerating parameter, and the cosmological constant. The analysis indicates that the temperature of charged accelerating BTZ AdS black holes is always positive.

1. Introduction

The first black hole solution in the (2 + 1)-dimensions with the negative cosmological constant has been obtained by Banados-Teitelboim-Zanelli,(11,2) which is known as BTZ black hole. BTZ black hole helps us find a profound insight into the physics of black hole, the quantum view of gravity, and its relations to string theory.(1–5) This black hole being locally AdS3 without curvature singularity may be considered as a prototype for general AdS/CFT correspondence.(6,7) So, the (2 + 1)—dimensional gravity has no local dynamics, but the black hole horizon induces an effective boundary. Therefore, quantum studies around BTZ black holes can help better understand AdS/CFT correspondence. Another motivation for studying BTZ black holes is related to the salient problems of quantum black holes, such as loss of information and the endpoint of quantum evaporation, which can be more easily understood in some simple low-dimensional models than directly in four dimensions.(8) In addition, BTZ black holes are related via T-duality and U-duality to classes of asymptotically flat black strings.(9,10)

Furthermore, other attractive black hole solutions are related to accelerating black holes (which could evolve into supermassive black holes(11,12)). They have been extracted in the C–metric.(13–16) These black holes have a conical deficit angle along one polar axis, providing the force driving the acceleration. It is noteworthy that the conical singularity pulling the black hole can be replaced by a magnetic flux tube(17) or a cosmic string.(18) One can imagine that something similar to the C–metric with its distorted horizon could explain a black hole that has been accelerated by an interaction with a local cosmological medium. It was indicated that the Hawking temperature of accelerating black holes is more than the Unruh temperature of the accelerated frame.(19) Thermodynamics and phase transitions of accelerating black holes have been evaluated in refs. [20–28]. Considering the charged accelerating black holes, the late-time growth rate of complexity differs from the ordinary charged black holes. The efficiency of heat engine for (an)a (un)charged accelerating non-rotating AdS black hole(20,31) charged accelerating rotating black hole,(22,33) have been studied. The results showed that these black holes’ parameters could influence heat engine efficiency (see refs. [34, 35], for more details). Quasinormal modes and late-time tails of the accelerating black holes have been studied in ref. [36]. The causal structure of accelerating black holes has been discussed in refs. [37, 38]. Geodesics and gravitational lensing in the spacetime of these black holes have been studied in refs. [39, 40]. The effect of acceleration on the orbit of photons and the shadow of black holes have been evaluated in refs. [41, 42]. It was indicated that the circular orbits of the photons deviated from the equatorial plane. Also, the property of the shadow of black holes changed due to the existence of acceleration. The (2 + 1)—dimensional uncharged accelerating black holes have been obtained in refs. [43–45]. Then, the accelerating systems in (2 + 1)—dimensional spacetime have been evaluated in ref. [46], and three exciting classes of geometry by

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studying holographically their physical parameters have been found.

The coupling of GR and nonlinear sources attracted significant attention because of their specific and exciting properties. One of these nonlinear theories in physics is nonlinear electrodynamics (NED). NED comes from the fact that these theories generalize the Maxwell field. Other motivations for considering NED are related to limitations of the Maxwell theory,\[47\] the radiation propagation inside specific materials,\[48,49\] and also, the description of the self-interaction of virtual electron-positron pairs.\[50,51\]

In addition, we study some properties of these solutions. We show that the obtained solutions recover the known BTZ black holes. In other words, by coupling GR and NED, we intend to extract exact charged accelerating BTZ black hole solutions in this paper. In other words, coupling GR and NED, we intend to extract exact charged accelerating BTZ black hole solutions in arbitrary dimensions.\[57,64\]

According to the mentioned important reasons for \((2 + 1)\)-dimensional and accelerating black holes and the exciting properties of NED, we are interested in finding exact charged accelerating BTZ black hole solutions in this paper. In other words, by coupling GR and NED, we intend to extract exact charged accelerating BTZ black hole solutions in the presence of the cosmological constant.

The structure of this paper is as follows: at first, we obtain the \((2 + 1)\)-dimensional uncharged accelerating black holes. Then, we show that the obtained solutions recover the known BTZ black holes. In addition, we study some properties of these solutions. Next, we extract the charged accelerating BTZ black holes in the presence of the PMI field. We indicate that these solutions turn to the known BTZ black holes in Einstein-conformally invariant Maxwell gravity. Then, we study some exciting properties of these solutions. The final section is devoted to concluding remarks.

### 2. Uncharged Accelerating BTZ Black Hole Solutions

This section devotes to the accelerating BTZ black holes without considering the matter field. The \((2 + 1)\)-dimensional action in the Einstein-Λ gravity can be written as

\[
I(g_{μν}, A_μ) = \frac{1}{16\pi} \int_{\partial M} d^3x \sqrt{-g}(R - 2\Lambda),
\]

where \(R\) and \(\Lambda\) are the Ricci scalar and the cosmological constant, respectively.

Varying the action (1), with respect to the gravitational field \(g_{μν}\), yields

\[
G_{μν} + Ag_{μν} = 0.
\]

To obtain the accelerating BTZ black hole, we consider a \((2 + 1)\)-dimensional spacetime introduced in refs. [44, 45]

\[
ds^2 = \frac{1}{K^2(r, θ)} \left[ -f(r, θ)dt^2 + \frac{dr^2}{f(r, θ)} + r^2dθ^2 \right].
\]

where \(K(r, θ) = a r \cosh(\sqrt{m - kθ}) - 1\), which is called the conformal factor. Also, \(a\) is related to the acceleration parameter, and \(k\) is a topological constant that can be \(± 1\) or 0.\[44,65,66\] The coordinates of \(t\) and \(r\), respectively, are in the ranges \(-∞ < t < ∞\) and \(0 ≤ r < ∞\). Due to the lack of translational symmetry in \(θ\), we have to consider the range \(-π ≤ θ ≤ π\). We will see later that in some cases the range for \(θ\) still needs to be adjusted for observers living on one side of the conformal infinity. The necessity of adjusting the range of \(θ\) in certain cases is one of the major points we will discuss in the following section. Also, \(f(r, θ)\) is a metric function, which we have to find it.

Now, we should find a suitable metric function \(f(r, θ)\) to satisfy all components of Equation (2). Considering the metric (3), one can show that Equation (2) turns to

\[
Eq_{tt} = 2a^2r^2 \left( f'''' + \frac{3}{2}f'' \right) \cosh^2 \left( \sqrt{m - kθ} \right)
- 2ar \cosh \left( \sqrt{m - kθ} \right) \left[ αrf \sqrt{m - k} \sinh \left( \sqrt{m - kθ} \right)f''
+ 2ff'''' - 3f''
\right)
+ 2f'''' + f'' = \left( \frac{1}{2}f + k - m \right)\right].
\]

\[
Eq_{θθ} = α^2r^2 \cosh^2 \left( \sqrt{m - kθ} \right) \left[ 2r^2f''f'' - f''''
- 4f''(k - m + rf'' - f') + 2r^2f''f'' - f''''
+ ar \cosh \left( \sqrt{m - kθ} \right) \left[ 4raf^2f' + 2f'' - 4r^2f^2f'\right]
+ 4r^2f^2(\Lambda + (k - m)a^2)^2,\right].
\]

where \(f = f(r, θ), f' = \frac{∂f}{∂r}, f'' = \frac{∂^2f}{∂r^2}, f''' = \frac{∂^3f}{∂r^3}, f'''' = \frac{∂^4f}{∂r^4}\), and \(f'''\) is related to components of \(tt, rr, θθ\) and \(rθ (rθ)\) of the Equation (2). Finding an exact solution from the above equations is very
difficult. For this purpose we consider $\theta = 0$ in the equations (4)–(6). This choice reduces them to the following forms

$$
Eq_{00} = Eq_{rr} = 2\alpha^2 (k - m) r + 2\alpha \left[ f(r) + m - \frac{f'(r)}{2} \right] + 2r\Lambda + f'(r),
$$

(7)

$$
Eq_{\theta\theta} = (ar - 1) (ar - 1)f''(r) - 2af'(r) + 2(a^2 f'(r) + \Lambda),
$$

(8)

where $Eq_{0\theta} = Eq_{\theta 0} = 0$, and also $f'(r) = \frac{df}{dr}$, and $f''(r) = \frac{d^2 f}{dr^2}$.

After some calculations, we find an exact general solution of Equations (7) and (8), in the following form

$$
f(r) = (ar - 1)^2 C(m, k, \Lambda, a) + 2 (ar - 1)(m - k) - \left( \frac{2ar - 1}{\alpha^2} \right) \Lambda,
$$

(9)

where $C(m, k, \Lambda, a)$ is a constant which depends on mass ($m$), the topological parameter ($k$), the cosmological constant ($\Lambda$), and the acceleration parameter ($\alpha$). We should note that the solution (9) satisfies all components of the field equation (2), provided we consider $\theta = 0$.

In order to more investigate the obtained solution, we compare it with the obtained solution in ref. [44]. Considering $C(m, k, \Lambda, a)$ in the following form

$$
C(m, k, \Lambda, a) = m - k - \frac{\Lambda}{\alpha^2},
$$

(10)

the solution (9) reduces to

$$
f(r) = (m - k) \alpha^2 - \Lambda r^2 + m + k,
$$

(11)

in which covers the introduced solution in refs. [44, 45]. Also, in the absence of the accelerating parameter ($\alpha = 0$), the solution (11) turns to the well-known BTZ solutions as

$$
f(r)_{BTZ} = -\Lambda r^2 - m + k,
$$

(12)

Let us recall the explanation of the conformal factor $1/K^2(r, \theta)$ in the metric (3). Considering a static observer in the following form

$$
x^\mu(\lambda) = \left( \lambda \frac{K(r, \theta)}{\sqrt{f(r)}}, r, \theta \right),
$$

(13)

in the spacetime, $\lambda$ is the proper time. Proper velocity is defined by $u^\mu = \frac{dx^\mu}{d\lambda}$, which is given by

$$
u^\mu(\lambda) = \left( \frac{K(r, \theta)}{\sqrt{f(r)}}, 0, 0 \right).
$$

(14)

Straightforward calculations indicate that the proper acceleration $a^\mu = u^\nu \nabla_\nu u^\mu$ (for this kind of observer at $r = 0$) leads to [44, 45]

$$
a^\mu a^\mu = (k - m) \alpha^2,
$$

(15)

in which shows, $\alpha$ is proportional (but not equal) to the magnitude $|a|$ of the proper acceleration at the origin when $m \neq k$.

Therefore, $\alpha$ may be called an acceleration parameter. It is worthwhile to mention that the $t$–component of $a^\mu$ is zero. So, $a^\mu$ is space-like, and we ought to have $a^\mu a^\mu > 0$ in the static region of spacetime. Considering the equation (15) for $m > k$, we see that $r = 0$ is not in the static region. That the region of spacetime containing the origin $r = 0$ is non-static is typical for black holes centered at the origin. For this reason, we infer that an accelerating black hole can exist only for $m > k$ (see ref. [44], for more details).

In the following, we study the properties of uncharged accelerating BTZ black hole solutions (11) such as Ricci and Kretschmann scalars, horizon structure, and temperature of these black holes. In other words, we will evaluate the effect of various parameters of the accelerating BTZ black holes on these quantities.

### 2.1. Curvature Scalars

Curvature invariant is scalar quantities constructed from tensors that represent curvature. Two well-known curvature invariants are the Ricci and Kretschmann scalars. The Ricci scalar curvature is defined as the trace of the Ricci curvature tensor with respect to the metric. Also, the Kretschmann scalar is constructed from the norm of the Riemann curvature tensor $R_{\alpha\beta\gamma\delta}$. To find the curvature singularity(ies) of the spacetime, we calculate the Ricci and Kretschmann scalars. Using the metric (3) and after some algebraic manipulation, one can find the Ricci scalar ($R$) and Kretschmann scalar ($R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}$) in the following forms

$$
R = 4\alpha^2 \left( \left( k - m - \frac{f(r)}{2} + \frac{r f'(r)}{2} - \frac{r^2 f''(r)}{4} \right) \cos^2 \left( \sqrt{m - k} \theta \right) 
+ \frac{3(k - m)}{2} \sin^2 \left( \sqrt{m - k} \theta \right) \right) 
+ 22 \alpha \left( \sqrt{m - k} \theta \right) \left( k - m - f(r) + \frac{r^2 f''(r)}{2} \right) 
- 2f'(r) + r f''(r) 
$$

(16)

$$
R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = 4 \left[ r^2 B_1 + (k - m) B_2 \sin^2 \left( \sqrt{m - k} \theta \right) \right] a^4 
+ 16 \alpha \cos^2 \left( \sqrt{m - k} \theta \right) \left[ \frac{\sqrt{m - k} \theta}{B_1} B_3 - B_4 \right] a^2 
+ \left[ \frac{8 \cos^2 \left( \sqrt{m - k} \theta \right) B_5}{r^2} \right] a^2 
- \frac{4(k - m) \sin^2 \left( \sqrt{m - k} \theta \right) (2f'(r) + r f''(r))}{r^2} a^2 
\right]
$$

(17)
where $B_1, B_2, B_3, B_4, B_5$ and $B_6$ are

$$B_1 = \left( \frac{1}{r^2} + \frac{f''(r) - \frac{2\alpha}{r} f'(r)}{f(r)} \right) + \frac{2(r - \frac{2\alpha}{r})}{f'(r)} + \frac{2(k - m)^2}{r^2},$$

$$B_2 = \left[ 4(m - k) + 2\left( \frac{f''(r) - f'(r)}{f'(r)} - r^2 f''(r) \right) \cosh^2 \left( \sqrt{m - k} \phi \right) \right] + 3(k - m) \sinh^2 \left( \sqrt{m - k} \phi \right),$$

$$B_3 = \left( k - m - f(r) + \frac{r^2 f''(r)}{2} \right),$$

$$B_4 = \left( \frac{f''(r) - \frac{2\alpha}{r} f'(r)}{2} \right) \frac{r^2}{f'(r)} + \frac{r^2 (1 - \frac{2\alpha}{r^3} + \frac{\alpha^2}{r^2} f''(r)) f''(r)}{2},$$

$$B_5 = \left( 1 + \frac{r^2 f''(r) - 2r f'(r)}{2 f'(r)} \right)^2 f^2(r) + \frac{3r^2 \left( 1 - \frac{2\alpha}{r^3} + \frac{\alpha^2}{r^2} f''(r) \right) f''(r)}{4},$$

$$B_6 = r^2 f''(r) - r^2 f''(r)f'(r) + r^2 f''(r) - 2f'(r) \left[ f'(r) + m - k \right]. \quad (18)$$

where in the absence of acceleration parameter (i.e., $\alpha = 0$) the Ricci and Kretschmann scalars reduce to

$$R = -\frac{2f'(r) + rf''(r)}{r}, \quad (19)$$

$$R_{\theta\phi\theta\phi} = f''(r) + \frac{2f'(r)}{r^2}. \quad (20)$$

Considering the metric function (11) and the equations (16) and (17), we obtain the Ricci and Kretschmann scalars which are

$$R = 6\Lambda, \quad (20)$$

$$R_{\theta\phi\theta\phi} = 12\Lambda^2. \quad (21)$$

However, there is no singularity, but we encounter a conical singularity at $r = 0$.

According to Equation (11), the asymptotical behavior of the obtained solution is dependent on the cosmological constant ($\Lambda$), the parameter of acceleration ($\alpha$), mass ($m$) and topological parameter ($k$). As a result, the obtained spacetime is not exactly asymptotically AdS due to the existence of different parameters in the asymptotical behavior of spacetime.

### 2.2. Horizon Structure

Another quantity we are interested in studying is the black hole’s horizon. In the previous section, we found a conical singularity at $r = 0$. Here, our analysis indicates that there are two roots for the obtained metric function $f(r)$, which are in the following form

$$r_\pm = \pm \sqrt{\frac{m - k}{(m - k)a^2 - \Lambda}}. \quad (22)$$

where $r_+$ and $r_-$ belong to positive and negative roots, respectively.

Now we are in a position to evaluate the obtained real root. It is necessary to mention that we must respect to signatures of spacetime (3) and also for recovering the static BTZ black hole, we have to restrict $m > k$, which leads to $\Lambda < (m - k)a^2$. In other words, our calculations indicate that we only encounter one real root for the uncharged accelerating BTZ black hole when $\Lambda < (m - k)a^2$.

To determine the asymptotical de Sitter (dS) or anti-de Sitter (AdS) of an accelerating BTZ black hole, we can re-express the cosmological constant by using Equation (22) in terms of $r_+$ as

$$\Lambda = (m - k) \left( \frac{\alpha^2 r^2 - 1}{r_+^2} \right). \quad (23)$$

where indicates that there are three cases.

i) $\alpha^2 r_+^2 > 1$, we encounter with dS accelerating black holes.

ii) the asymptotical behavior of accelerating BTZ black hole is AdS when $\alpha^2 r_+^2 < 1$.

iii) flat case is $\alpha^2 r_+^2 = 1$.

Recently it was shown that in three-dimensional dS spacetime, classical black holes do not exist, so we consider this point and proceed with our work for the AdS case (i.e., $\Lambda$ is always negative in this work).

One of the important differences between the obtained accelerating BTZ black hole and the well-known BTZ black hole is related to the fact that the asymptotical behavior of the accelerating BTZ black hole completely depends on the acceleration parameter.

To study the obtained accelerating BTZ black holes, we plot the metric function (11) versus $r$ in Figure 1. According to the equation (22) and our results in Figure 1, the $(2 + 1)$–dimensional black holes with low acceleration have large radii. Indeed, by decreasing the acceleration parameter, the black holes have larger radii (see the left panel in Figure 1). Also, by considering the negative values of the cosmological constant, massive black holes have large radii (see the right panel in Figure 1). Our results reveal that large black holes belong to $k = -1$ and small black holes have $k = 1$ when other quantities have the same values. In other words, by considering the same values of parameters, there is a relation as $r_{\text{dS}} < r_{\text{AdS}} < r_{\text{flat}}$ (see Figure 1, for more details).

Now, we discuss the necessity of adjusting the range of $\theta$ in certain cases. According to Equation (3), every zero of $K(r, \theta)$ represents a conformal infinity in the metric. Considering $m > k$ and for $r > 0$, the conformal factor will never have a zero when $\alpha < 0$ (see the left panel in Figure 2). Therefore, there is no worry about conformal infinities, but for $\alpha > 0$, the conformal infinity exists.
Figure 1. The metric function $f(r)$ versus $r$ for $\Lambda = -1$, and different values of the topological parameter. Left panels for $m = 1.5$. Right panels for $\alpha = 1$. 
The conformal infinity separates the spacetime into two patches:

\[
\begin{align*}
\mathcal{K}(r, \theta) &\leq 0 \quad \text{(-) patch} \\
\mathcal{K}(r, \theta) &\geq 0 \quad \text{(+) patch},
\end{align*}
\]

also, each observer can perceive only one of the two patches. So, determination of the correct range for \(\theta\) depends on which patch the observer lives in. For the AdS case with \(\theta_0 \in [0, \pi]\), we need to choose \(\theta \in [-\theta_0, \theta_0]\) in the (-) patch and \(\theta \in [-\pi, \theta_0) \cup (\theta_0, \pi]\) in the (+) patch (see the right panel in Figure 2). For the flat case, it corresponds to \(\theta_0 = 0\), horizon intersects with the conformal infinity at a single point \(\theta = 0\) in the (+) patch, in which case we need to choose \(\theta \in [-\pi, 0) \cup (0, \pi]\) (see the dotted-dashed line in the right panel from Figure 2), and there is no horizon in the (-) patch (see ref. [44], for more details on the correct range of \(\theta\)).

In the following, we study the temperature of these black holes.

\subsection*{2.3. Temperature}

In order to calculate the Hawking temperature, we employ the definition of surface gravity as

\[
T_{\text{H}} = \frac{\kappa}{2\pi},
\]

where \(\kappa\) is the surface gravity and is defined in the form

\[
\kappa = \sqrt{\frac{H_1'(\theta)(\mathcal{K}(r, \theta)+1)^2 - 2H_1(K_1(\theta)(\mathcal{K}(r, \theta)+1)}{4\mathcal{K}^2(r, \theta)}},
\]

and

\[
H_1 = f''(r) - 2f(r).
\]

It is noteworthy that there is a divergence point for the acceleration parameter \(\alpha\). Equating \(f(r) = 0\), we can find \(m\) in the following form

\[
m = \frac{(ka^2 + \Lambda)r^2_+ - k}{a^2r^2_+ - 1},
\]

using the obtained metric function (11), and by substituting the mass (28) within the equation (27), one can calculate the superficial gravity as

\[
\kappa = \frac{\Lambda r_+}{a^2r^2_+ - 1},
\]

where \(r_+\) is the radius of the event horizon (or positive root in Equation (22)). Replacing the obtained metric function (11) into the relation (26), we can obtain the temperature of the uncharged accelerating BTZ black holes as

\[
T_{\text{H}} = \frac{\Lambda r_+}{2\pi(a^2r^2_+ - 1)}.
\]

Our result in Equation (30) indicates that the temperature depends on the cosmological constant and the acceleration parameter. It is noteworthy that there is a divergence point for the
temperature at \( r_+ = \frac{1}{2} \), and this point only depends on the acceleration parameter. On the other hand, we are interested in considering AdS spacetime, so we must respect \( \alpha^2 r_+^2 < 1 \), which leads to \( r_+ < \frac{1}{\alpha} \). Therefore, there is no divergence point for the temperature. According to Equation (30), and applying the AdS case, the temperature is always positive. In the context of black holes, it is argued that the root of temperature \( (T = 0) \) represents a border line between physical \( (T > 0) \) and non-physical \( (T < 0) \) black holes.\(^{[68]}\) To find the physical black holes, we must obtain the temperature’s root. It is clear that there is no root for the temperature, and for \( \Lambda < 0 \), the temperature is always positive, and we encounter the physical case.

### 3. Charged Accelerating BTZ Black Holes

In this section, we want to extract accelerating BTZ black holes in the presence of the electromagnetic field and the cosmological constant. We first evaluate accelerating BTZ black holes in Einstein-Maxwell-A gravity. Our findings indicate that by considering the Maxwell field, the charged accelerating BTZ black hole solutions cannot recover the known charged BTZ black holes (see Appendix A, for more details). For this purpose, we extend the \( (2 + 1) \)-dimensional action (1) in the presence of the PMI NED field, which can be written as

\[
I(g_{\mu\nu}, A_\mu) = \frac{1}{16\pi} \int d^3x \sqrt{-g} [R - 2\Lambda + L(P)],
\]

where \( L(P) = -(\nabla \phi)^2 \) and \( P \) is the Maxwell invariant which is equal to \( F_{\mu\nu} F^{\mu\nu} \). It is noteworthy that \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the electromagnetic tensor field, and \( A_\mu \) is the gauge potential. Also, \( s \) is called the power of PMI NED.

To find the charged accelerating BTZ black hole solutions, we use a NED field. We consider PMI NED for extracting charged accelerating BTZ black hole solution in line with our purpose. In other words, we suppose \( s = d/4 \) in PMI NED (where \( d \) is related to spacetime dimensions), known as conformally invariant Maxwell (CIM) electrodynamics. According to this fact that we want to investigate the \( (2 + 1) \)-dimensional black hole \( (d = 3) \), we have to consider \( s = 3/4 \). So, \( L(P) \) in the action (31) turns to \(-F_{\mu\nu}^4 \) (i.e., \( L(P) = -F_{\mu\nu}^4 \)). Varying the action (31), with respect to the gravitational field \( g_{\mu\nu} \), and the gauge field \( A_\mu \), yields

\[
\mathcal{G}_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}, \tag{32}
\]

\[
\partial_\mu (\sqrt{g} F^{\mu\nu} + F_{\mu\nu}) = 0, \tag{33}
\]

where \( T_{\mu\nu} \) is energy-momentum tensor. In the presence of CIM NED for the \( (2 + 1) \)-dimensional spacetime, it can be written as

\[
T_{\mu\nu} = \frac{3}{2} \tilde{F}_{\mu\nu} (\nabla \phi)^2 + \frac{1}{2} g_{\mu\nu} (\nabla \phi)^2. \tag{34}
\]

To obtain charged accelerating BTZ black holes, we consider a \( (2 + 1) \)-dimensional spacetime which is introduced in Equation (3). Since we look for black hole solutions with a radial electromagnetic field, so the gauge potential is given by \( A_\mu = h(r) \delta_\mu^0 \). Considering the metric (3), one can show that Equation (33) reduces to

\[
rh''(r) + 2h'(r) = 0, \tag{35}
\]

solving Equation (35), one can find \( h(r) = \frac{r^2}{4} \), where \( q \) is an integration constant that is related to the electric charge, and the electric field in the \( (2 + 1) \)-dimensional spacetime is given by \( F_{\rho\theta} = -F_{\theta\rho} = \frac{\alpha}{r^2} \), where it is inverse square of \( r \).

Now, we should find a suitable metric function to satisfy all components of Equation (32). Considering the obtained \( h(r) \), one can show that Equation (32) reduces to

\[
\mathcal{E}_{\theta\theta} = \mathcal{E}_{\rho\rho} = r^2 \left( \frac{1}{2} - \frac{\alpha}{2} \right) f''(r) + af'(r) + r (\Lambda + (k - m)\alpha^2)
\]

\[
+ a (m - k) + \frac{(2q^2)^{1/4} (ar - 1)^3}{4}, \tag{36}
\]

\[
\mathcal{E}_{\theta\rho} = (ar - 1)^2 f''(r) - 2a (ar - 1) f'(r) + 2a^2 f(r)
\]

\[
+ 2\Lambda - \frac{(2q^2)^{1/4} (ar - 1)^3}{r^3}. \tag{37}
\]

Notably, we supposed \( \theta = 0 \) in Equation (32), similar to the uncharged case. Using Equations (36) and (37), we are in a position to obtain the metric function \( f(r) \). After some calculation, we find an exact general solution in the following form

\[
f(r) = \alpha^2 r^2 C(m, k, \Lambda, a, q) + 2ar (m - k - C(m, k, \Lambda, a, q))
\]

\[
+ 2(k - m) + C(m, k, \Lambda, a, q)
\]

\[
+ \frac{\Lambda}{4} - \frac{2\alpha r - 2(q^2)^{1/4} (ar - 1)^2}{2r}. \tag{38}
\]

where \( C(m, k, \Lambda, a, q) \) is a constant which depends \( m, k, \Lambda, a, \) and \( q \). We should note that the solution (38), satisfy all field equations of Equation (32).

It is necessary that the obtained charged accelerating BTZ black hole solution in Equation (38) reduces to known charged BTZ black hole in the Einstein-CIM gravity\(^{[69]}\) when ignoring the acceleration parameter. For this purpose, we consider \( C(m, k, \Lambda, a, q) \) in the following form

\[
C(m, k, \Lambda, a, q) = m - k - \frac{\Lambda}{4} - \frac{(2q^2)^{1/4} \alpha}{4}, \tag{39}
\]

which the solution (38) turns to

\[
f(r) = k - m + \left( (m - k)a^2 - \Lambda \right) r^2
\]

\[
- \frac{(2q^2)^{1/2} (ar + 2)(ar - 1)^2}{4r}. \tag{40}
\]

It is clear that in the absence of the acceleration parameter, the solution (40) reduces to the charged BTZ black hole in the Einstein-CIM gravity\(^{[69]}\)

\[
f(r)_{\text{BTZ-CIM}} = k - m - \Lambda r^2 - \frac{(2q^2)^{1/2}}{2r}. \tag{41}
\]
where the above solution is an exact solution to the Einstein-CIM gravity with cosmological constant $\Lambda$ when the acceleration parameter is zero. Besides, the equation (40) turns to the obtained uncharged accelerating BTZ black holes (Equation (11)) by considering $q = 0$.

### 3.1. Curvature Scalars

In order to look for the singularity(ies) of spacetime for the obtained solutions (40), we evaluate the Ricci and Kretschmann scalars. Using the metric function (40) and the equations (16) and (17), we obtain these scalars which are

$$ R = 6\Lambda, $$

$$ R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = 12\Lambda^2 + \frac{3\sqrt{2}q^3(\alpha r - 1)^6}{r^6}, $$

where the Kretschmann scalar indicates that there is a singularity at $r = 0$ (i.e., $\lim_{r\to0}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \to \infty$). It is notable that the divergence point removes when $\alpha r = 1$. In other words, we can remove the singularity of Kretschmann scalar by considering $\alpha r = 1$.

According to the obtained solutions (40), the electrical charge affects their asymptotical behavior. Indeed, the asymptotical behavior is dependent on $\Lambda$, $\alpha$, $m$, $k$, and $q$. So, the obtained spacetime for the charged case is not exactly asymptotically AdS due to the existence of different parameters in the asymptotical behavior of spacetime.

### 3.2. Horizon Structure

To find the real roots of the obtained metric function (40), we solve $f(r)|_{r=r_+} = 0$, which leads to

$$ \Lambda = \frac{-2(2q^2)^3(\alpha r_+ + 2)(\alpha r_+ - 1)^2 + 4r_+(m-k)(\alpha^2r_+^2 - 1)}{4r_+^2}, $$

where imposes the following conditions to have AdS spacetime

$$ \Lambda < 0 \Rightarrow \begin{cases} \alpha r_+ + 2 > 0 \\ \alpha^2 r_+^2 - 1 < 0 \end{cases}, $$

and by using the above condition we get the AdS limit in following form

$$ -2 < \alpha r_+ < 1, $$

where we obtain the above limit by considering $m > k$.

To more study the effect of topological constant on the obtained charged accelerating BTZ black holes, we consider $k = \pm 1$ and 0.

#### 3.2.1. Case: $k = 1$

Our results indicate that the radius of black hole increases by increasing the electrical charge and mass parameters. Indeed, massive black holes with high electrical charge have large radii (see left and right panels in Figure 3). On the other hand, there is a different behavior for the acceleration parameter. In other words, the charged BTZ black holes with high acceleration have small radii (see the middle panel in Figure 3). As a result, the accelerating BTZ black holes with high mass and strong electrical charges have large radii.

#### 3.2.2. Case $k = -1$

For this case, massive black holes with high electrical charge have large radii (see left and right panels in Figure 4). It means that the horizon increases by increasing mass and the electrical charge. However, the black holes with high acceleration have small radii (see the middle panel in Figure 4). As a result, the largest accelerating BTZ black holes have large masses and strong electrical charges. Our analysis indicates that the behavior of the horizon for $k = -1$ is similar to $k = 1$, when $\Lambda < 0$ (see Figures 3 and 4). Also, comparing the obtained results in Figures 3 and 4, and
by considering the same values for parameters, we find that the black holes with $k = -1$ are larger than $k = 1$.

3.2.3. Case $k = 0$

There are the same behaviors similar to previous cases. In other words, massive black holes with large values of electrical charge are big black holes (see left and right panels in Figure 5). Also, the radius of black holes decreases by increasing the acceleration parameter. In this case, large black holes have small acceleration (see the middle panel in Figure 5).

We evaluate the effects of the electrical charge, the acceleration parameter, and mass on the horizon of charged accelerating BTZ black holes with different topological constants in Table 1. Our analysis indicates that black holes with topological constant $k = -1$ are bigger than $k = 0$, and $k = 1$. In other words, by considering the same values of parameters, black holes have large radii with topological constant $k = -1$ in comparison with other values of the topological constant (i.e., $k = 0$, and $k = 1$). In addition, the smallest black holes have topological constant $k = 1$.

Another interesting quantity that gives us information on the charged accelerating BTZ black holes is related to the Hawking temperature.

3.3. Temperature

To extract the Hawking temperature of the charged accelerating BTZ black holes (40), we express the mass ($m$) in terms of the radius of event horizon $r_+$, the cosmological constant $\Lambda$, the topological constant, the acceleration parameter $\alpha$, and the electrical charge $q$, by equating $f(r) = 0$, which leads to

$$m = \frac{(ka^2 + \Lambda)r_+^2 - k}{a^2r_+^2 - 1} + \frac{(2q^2)^{3/4}(ar_+ + 2)(ar_+ - 1)}{8(ar_+ + 1)r_+^2}, \quad (46)$$

using the obtained metric function (40), and by substituting the mass (46) within the equations (27) and (26), we can calculate the temperature of charged accelerating BTZ black holes in the following form

$$T_{H} = \frac{\Delta r_+}{2\pi (a^2r_+^2 - 1)} - \frac{(2q^2)^{3/4}(2ar_+ + 1)(ar_+ - 1)^2}{8\pi (a^2r_+^2 - 1)r_+^2}, \quad (47)$$

where $r_+$ is the radius of the black hole. The temperature of these black holes depends on the cosmological constant, the acceleration parameter, and the electrical charge. It is notable that in
the absence of the electrical charge Equation (47) turns to Equation (30).

Here, we evaluate temperature behavior. Our analysis indicates that, there is no root for the temperature when $\Lambda < 0$. However, there is a divergence point of temperature at $a^2 r_s^2 = 1$, but we cannot consider it because we have to respect the AdS limit (i.e., $-2 < a r_s < 1$). Therefore, we do not have any divergence point. By looking at the equation (47), it is clear that for the negative value of the cosmological constant, the temperature is always positive. In addition, there is an interesting effect of $a$ on the temperature. Applying the AdS limit ($-2 < a r_s < 1$), the positive area of the temperature decreases by increasing the value of the acceleration parameter. In other words, charged BTZ AdS black holes with high acceleration have a small physical area.

### 4. Closing Remarks

In this paper, we obtained uncharged accelerating BTZ AdS black hole solutions. The asymptotical behavior of this spacetime was not exactly AdS due to different parameters. In order to respect the signature of spacetime, we extracted two conditions, which are $m > k$ and $\Lambda < (m - k) a^2$. These equations revealed that the mass of black holes must be more than the topological constant. Our analysis of the behavior of the metric function indicated that massive BTZ black holes with low acceleration have large radii when $\Lambda < 0$.

We obtained the temperature of these black holes. The temperature was dependent on the acceleration parameter and the cosmological constant. More studies indicated that the temperature was always positive, and the physical black holes were in the radii less than $\frac{1}{a}$ ($r_s < \frac{1}{a}$).

Then, we evaluated accelerating BTZ black holes in the presence of the Maxwell field. Our findings showed that charged accelerating BTZ black holes could not recover the known BTZ black holes without the acceleration parameter. This result motivated us to consider a NED field instead of a Maxwell field. Then, we obtained exact charged accelerating BTZ black hole solutions by coupling GR with CIM NED in the presence of the cosmological constant. The asymptotical behavior of spacetime was dependent on the cosmological constant, the electrical charge, mass, the acceleration parameter, and the topological constant, and it was not exactly asymptotically AdS.

We evaluated the metric function and found one real root. Then, we studied the effects of charge and acceleration parameters on this root in Figure 3. Our findings indicated that; i) strongly charged accelerating BTZ AdS black holes had large radii, ii) the charged accelerating BTZ AdS black holes had small radii when $\alpha$ increased. iii) charged accelerating BTZ AdS black holes with topological constant $k = -1$ were bigger than $k = 0$, and $k = 1$. In other words, by considering the same values of parameters, black holes had large radii with topological constant $k = -1$ in comparison with other values of the topological constant (i.e., $k = 0$, and $k = 1$). Also, the smallest black holes had topological constant $k = 1$.

We extracted the temperature of the obtained charged accelerating BTZ black holes. Our calculations indicated that it depended on $q$, $a$, and $\Lambda$. Also, we found that the temperature was always positive when $\Lambda < 0$, and the positive area of the temperature decreased by increasing the value of the acceleration parameter.

### Appendix A

The $(2 + 1)$-dimensional action in Einstein-Maxwell-$\Lambda$ gravity is given

$$I(g_{\mu \nu}, A_\mu) = \frac{1}{16\pi} \int_{\partial M} d^3x \sqrt{-g} [R - 2\Lambda - F],$$

(A1)

where $F$ is the Maxwell invariant which is equal to $F_{\mu \nu} F^{\mu \nu}$. It is noteworthy that $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic tensor field, and $A_\mu$ is the gauge potential.

The field equations in this theory of gravity are

$$G_{\mu \nu} + \Lambda g_{\mu \nu} = 2 \left[ F_{\mu \rho} F^{\rho \nu} + \frac{1}{4} g_{\mu \nu} (-F) \right],$$

(A2)

$$\partial_\mu (\sqrt{-g} F^{\mu \nu}) = 0.$$  

(A3)

Considering $A_\mu = h(r) \delta^\mu_0$, and the metric (3), and also by considering $\theta = 0$, one can show that Equation (A3) reduces to

$$(2ar - 1) h'(r) + r (ar - 1) h''(r) = 0,$$

(A4)

solving Equation (A4), one can find $h(r)$, and the electric field ($E(r)$) in the following forms

$$h(r) = -q \ln (\frac{ar - 1}{r}),$$

(A5)

$$E(r) = \frac{-q}{r (ar - 1)},$$

(A6)

where $q$ is an integration constant related to the electric charge. It is notable that in the absence of acceleration parameter (i.e., $\alpha = 0$), the electric field turns to standard form ($E(r) = \frac{q}{r}$), as we expected.
Now, we are going to find the metric function $f(r)$. Considering the obtained $h(r)$, one can show that the components of Equation (A2) reduce to

$$E_{\theta\theta} = E_{\phi\phi} = f'(r) + 2\alpha f(r) + 2\left(\alpha^2 (q^2 - m^2 + k) + \Lambda \right) r^2 - 2(r - k + 2q^2)ar + 2q^2, \quad (A7)$$

$$E_{\phi\theta} = (ar - 1)^2 r^2 f''(r) - 2(ar - 1)ar^2 f'(r) + 2a^2 r^2 f(r) + 2(r - k + 2q^2)(2ar - 1). \quad (A8)$$

We find an exact solution of Equations (A7) and (A8), as

$$f(r) = k - m + (m - k)\alpha^2 - \Lambda r^2 + 2(ar - 1)^2 q^2 \ln \left(\frac{ar - 1}{r}\right), \quad (A9)$$

where we used $C(m, k, \Lambda, \alpha) = (m - k) - \frac{\Lambda}{\alpha^2}$, to find the above solution.

Here we want to check the solution (A9). Without the electrical charge, the solution (Equation (A9)) must recover an uncharged accelerating BTZ black hole. In other words, by replacing $q = 0$ in the solution (A9), it reduces to the obtained uncharged accelerating BTZ black hole in the equation (9). But we cannot find a real solution for charged BTZ black hole in the absence of the acceleration parameter, due to the existence of term, $\ln(\frac{ar - 1}{r})$, in the solution (A9). In other words, the term $\ln(\frac{ar - 1}{r})$, turns to $\ln(-\frac{1}{r})$, when $\alpha = 0$. So the obtained charged accelerating BTZ black hole from Einstein-Maxwell-A theory cannot recover the known charged BTZ black hole. This result motivates us to consider the NED field instead of the Maxwell field.

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Conflict of Interest

The authors declare no conflict of interest.

Keywords

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