ONE PROPERTY OF TRAJECTORIES OF TOEPLITZ FLOWS

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Abstract. We consider left shift transform $S$ on the space $X = \Sigma^\mathbb{Z}$ of two-sided sequences over a compact alphabet $\Sigma$. We give an important and sufficient condition on $x \in X$ which guarantees the restriction of $S$ onto orbit closure of $x$ to be a Toeplitz flow.

The notion of Toeplitz flow was introduced in 1969 by Jackobs and Keane in paper [1] as certain class of subshifts of finite type. Later this definition was expanded by S. Williams to the much more wide class of subshifts of Bernoulli shift $S$ on the space $X = \Sigma^\mathbb{Z}$ of two-sided sequences over compact metric alphabet $\Sigma$ (see [2]).

Both in papers [1] and [2] Toeplitz flow is defined as the restriction of $S$ onto orbit closure of so-called Toeplitz sequence.

Let $x = (x_n) \in X$. Say $x_i \in \Sigma$ is in the periodic part of the sequence $x$ if there exists $k \in \mathbb{N}$ such that

$$x_i = x_j \quad \text{for all} \quad j \equiv i \pmod{k}.$$ 

If it is not the case we say that $x_i$ belongs to the aperiodic part of the sequence $x = (x_n)$. Sequence $x$ is called Toeplitz sequence if it has an empty aperiodic part.

In the paper [2] the set of so-called essential periods is introduced for a nonperiodic Toeplitz sequence $x$ and this set induces in turn the periodic structure on $x$. Next, this periodic structure defines a certain supernatural number. It appears (see [3]) that the flow $(\text{Orb} x, S)$ admits almost one-to-one projection onto the odometer which is defined by the same supernatural number (for the classification of odometers by means of supernatural numbers see [3] and [4]).

Toeplitz flows are remarkable since the class of all Toeplitz flows coincides with the class of minimal flows which are symbolic and admit almost one-to-one projection onto an odometer (for references and further development of this result see [5]).

The definition of Toeplitz flow is not "homogeneous" in the following sense. It is known (see [4]) that given a Toeplitz flow $(T, S)$ and an almost one-to-one projection $\pi : (T, S) \to G$ onto an odometer $G$, an arbitrary point $y \in T$ is a Toeplitz sequence if and only if $\pi$ is one-to-one in the point $y$ (i.e. $\pi^{-1}(\pi(y)) = \{y\}$). Hence the set of all Toeplitz sequences in $T$ is a proper massive subset in $T$ (it contains a dense
Gδ subset of T). That’s why phase space of an arbitrary Toeplitz flow contains at least one element which is not a Toeplitz sequence.

In this connection problem arises to determine whether for a given non Toeplitz sequence \( x \in X \) the dynamical system \((\text{Orb } x, S)\) is a Toeplitz flow. In the case of positive answer another problem appears to find the periodical structure of this flow making use only of the sequence \( x \).

We give an important and sufficient condition on \( x \in X \) which guarantees the restriction of \( S \) onto orbit closure of \( x \) to be a Toeplitz flow. Also we show how to derive the periodical structure of this flow from \( x \).

The technique applied to verify the condition allows us to expand results of S. Williams described above (see [2], theorem 2.2, lemma 2.3 and corollary 2.4) to the case of subshifts on the space of two-sided sequences over a Hausdorff compact alphabet (not necessarily metrizable).

1. Definitions and statement of results.

Let \( \Sigma \) be a compact space, \( X = \Sigma^\mathbb{Z} = \prod_{n \in \mathbb{Z}} \Sigma_n \) with the topology of direct product. By Tikhonov theorem \( X \) is also the compact space. We write elements of \( X \) as \( x = (x_n) \).

In the case when \((\Sigma, \rho)\) is a metric space the distance 
\[
d(x, y) = \sum_{n \in \mathbb{Z}} 2^{-|n|} \rho(x(n), y(n))
\]
is known to induce the product topology on \( X \).

In what follows the next property of \( X \) will be useful to us (see. [3]).

**Proposition 1.** A sequence of points \( \{x_i\} \) in a product \( \prod_{n \in \mathbb{Z}} X_n \) of topological spaces converges to \( x \in \prod_{n \in \mathbb{Z}} X_n \) if and only if the sequence \( \{x_i(n)\} \) converges to \( x(n) \) for every \( n \in \mathbb{Z} \).

Let us designate by \( S : X \to X \) the left shift homeomorphism \( S(x(n)) = x(n + 1), n \in \mathbb{Z} \).

For \( x \in X, p \in \mathbb{N} \) and \( \sigma \in \Sigma \) let
\[
\text{Per}_p(x, \sigma) = \{ n \in \mathbb{Z} \mid x(n') = \sigma \text{ for all } n' \equiv n \pmod{p} \},
\]
\[
\text{Per}_p(x) = \bigcup_{\sigma \in \Sigma} \text{Per}_p(x, \sigma),
\]
\[
\text{Aper}(X) = \mathbb{Z} \setminus \left( \bigcup_{p \in \mathbb{N}} \text{Per}_p(x) \right).
\]

By \( p \)-skeleton of \( x \) we shall name that part of a sequence \( (x(n)) \), which has the period \( p \).

Let us designate \( M_p(x) = \max \{ k \in \mathbb{N} \mid \exists n \in \mathbb{Z} : n + i \in \text{Per}_p(x), i = 0, 1, \ldots, k - 1 \} \).
In other words $M_p(x)$ is the maximal length of a block contained in $p$-skeleton of $x$. Note, that $M_p(x) = \infty$ for periodic sequence $x$ with period $p$ and $M_p(x) < p$, if the sequence $x$ is not periodic.

Let us remind some important definitions.

**Definition 1.** Sequence $\eta \in X$ is called **Toeplitz**, if $\text{Aper}(\eta) = \emptyset$ (in this case the dynamic system $(\overline{\text{Orb}(\eta)}, S)$ is also referred as **Toeplitz**).

**Definition 2.** Let $(X, F)$ be a dynamic system with discrete time, $x \in X$. The point $x$ is **recurrent**, if for its arbitrary open neighbourhood $U$ there exists $n(U)$, such that for any $k \in \mathbb{Z}$

$$U \cap \left( \bigcup_{i=k}^{k+n(U)-1} \{ F^i(x) \} \right) \neq \emptyset.$$

**Definition 3.** Let $(X, F)$ be a dynamic system with discrete time, $x \in X$. The point $x$ is said to be **almost periodic**, if for its arbitrary open neighbourhood $U$ we can find $n(U) \in \mathbb{N}$, such that

$$\bigcup_{k \in \mathbb{Z}} \{ F^{kn(U)}(x) \} \subset U.$$

Clearly each periodic sequence $x \in X$ is Toeplitz. It is easy to check that every Toeplitz sequence is almost periodic since each block of Toeplitz sequence $\eta$ is contained in its $p$-skeleton for some $p$. Hence, according to Birkhoff theorem $\overline{\text{Orb}(\eta)}$ is a minimal set of dynamic system $(X, S)$ (see [2, 3]).

Let $\eta \in X$ be an aperiodic Toeplitz sequence. Generally speaking an equality $\text{Aper}(x) = \emptyset$ is not carried out for an arbitrary $x \in \overline{\text{Orb}(\eta)}$.

Consider a special case when $\Sigma$ is metric space. From one hand, every Toeplitz flow $(\overline{\text{Orb}(\eta)}, S)$ in $X$ have to be expansive (see [3]). From the other hand, every odometer is an equicontinuous dynamic system (see remark 8 below). It is known that any Toeplitz flow admits almost one-to-one projection onto odometer and such a projection have to be one-to-one precisely in points which are Toeplitz sequences (see [2]). So, if every point of a certain Toeplitz flow is a Toeplitz sequence, then this Toeplitz flow have to be conjugate to an odometer. In particular it must be expansive and equicontinuous simultaneously, and this is impossible.

Properties of sequences from $\overline{\text{Orb}(\eta)}$ are in details investigated in [2]. However it is not known, what should be properties of the point $x \in X$ the set $\overline{\text{Orb}(x)}$ to contain some Toeplitz sequence.

The answer to this question gives the following

**Proposition 2.** If for some $x \in X$

$$\lim_{p \to \infty} \sup M_p(x) = \infty,$$

(1)
then set $\text{Orb}(x)$ contains a Toeplitz sequence.

If in addition point $x$ is recurrent, then dynamic system $(\text{Orb}(x), S)$ is Toeplitz.

There is a natural question: what additional information it is possible to take about structure of dynamic system $(\text{Orb}(x), S)$ under condition that a point $x$ is recurrent?

Let’s remind definition of periodic structure of a Toeplitz sequence (see [2]).

**Remark 1.** Let $x \in X$. If $p \mid q$, then $\text{Per}_p(x) \subseteq \text{Per}_q(x)$.

**Definition 4.** Let’s call $p$ the essential period of a sequence $x$, $p \in \mathcal{P}(x)$, if for any $q \in \mathbb{N}$

$$\left(\text{Per}_p(x, \sigma) \subseteq \text{Per}_p(x, \sigma) - q \quad \forall \sigma \in \Sigma\right) \Rightarrow \left(p \mid q\right).$$

In other words, $p \in \mathcal{P}(x)$ if and only if $p$-skeleton of $x$ is not periodic for any smaller period.

**Remark 2.** It is easily checked, that if $p$, $q \in \mathcal{P}(x)$, then $\text{lcm}(p, q) \in \mathcal{P}(x)$ (see [2]).

**Definition 5.** Periodic structure of a nonperiodic Toeplitz sequence $\eta$ is the growing sequence \( \{p_i\}_{i \in \mathbb{N}} \) of natural numbers, such that

(i) $p_i \in \mathcal{P}(\eta)$ for all $i \in \mathbb{N}$;
(ii) $p_i \mid p_{i+1}$;
(iii) $\bigcup_{i \in \mathbb{N}} \text{Per}_{p_i}(x) = \mathbb{Z}$.

To within the equivalence relation which we will not describe here periodic structure for a Toeplitz sequence is determined uniquely (see [2]). For our purposes it is enough to know that any subsequence of a sequence from previous definition sets an equivalent periodic structure.

Now we shall determine periodic structure for any recurrent sequence $x \in X$ which satisfies the relation (I).

**Remark 3.** Let $p \mid q$ for some $p$, $q \in \mathbb{N}$. Then $M_p(x) \leq M_q(x)$.

**Definition 6.** Periodic structure of an aperiodic sequence $x$ which satisfies the relation (II) is the growing sequence \( \{p_i\}_{i \in \mathbb{N}} \) of natural numbers, such that

(i) $p_i \in \mathcal{P}(\eta)$ for all $i \in \mathbb{N}$;
(ii) $p_i \mid p_{i+1}$;
(iii′) $\lim_{i \to \infty} M_{p_i}(x) = \infty$.

**Proposition 3.** For each aperiodic sequence $x$, which satisfies to the relation (II), there exists some periodic structure.

For the benefit of such definition of periodic structure speak the following results.
Theorem 1. Suppose the sequence \( \{p_i\} \) determines certain periodic structure (in the sense of definition 4) for a recurrent sequence \( x \in X \) which satisfies relation (4). Then there exists a Toeplitz sequence \( \eta \in X \), such that \( \text{Orb}(x) = \text{Orb}(\eta) \) and the sequence \( \{p_i\} \) evaluates periodic structure on \( \eta \) (in sense of definition 3).

Corollary 1. Let \( x \in X \) be a recurrent sequence satisfying to equality (4). Then the periodic structure for \( x \) is determined uniquely (to within the relation of equivalence from [2]).

2. Proof of the main results.

2.1. Proof of proposition 3. We fix \( x \in X \). Divide the proof into several steps.

1. Suppose \( \text{Per}_p(x) \neq \emptyset \) for some \( p \in \mathbb{N} \). Find minimal \( k \in \mathbb{N} \), such that \( k \mid p \) and \( \text{Per}_p(x) = \text{Per}_k(x) \).

Let us check that \( k \in P(x) \). Two lemmas will be necessary for this purpose.

Lemma 1. Suppose that the following condition
\[
\text{Per}_p(x, \sigma) = \text{Per}_p(x, \sigma) + m_i \quad \forall \sigma \in \Sigma, \ i = 1, 2
\]
is satisfied for some \( m_1, m_2 \in \mathbb{N} \). Let \( b \in \mathbb{N}, 0 \leq b \leq m_2 - 1 \), be a remainder of the division of \( m_1 \) into \( m_2 \). Then
\[
\text{Per}_p(x, \sigma) = \text{Per}_p(x, \sigma) + b \quad \forall \sigma \in \Sigma.
\]

Proof. On a condition \( m_1 = am_2 + b, a \in \mathbb{Z}_+ \). For every \( \sigma \in \Sigma \)
\[
\text{Per}_p(x, \sigma) = \text{Per}_p(x, \sigma) + m_1 = (\text{Per}_p(x, \sigma) + am_2) + b = \text{Per}_p(x, \sigma) + b.
\]

Lemma 2. Let for some \( q \in \mathbb{N} \) following condition is satisfied
\[
\text{Per}_p(x, \sigma) = \text{Per}_p(x, \sigma) + q \quad \forall \sigma \in \Sigma.
\]

Then \( \text{Per}_{\gcd(p,q)}(x) = \text{Per}_p(x) \).

Proof. Consider Euclidean algorithm of a finding of \( \gcd(p, q) \).
\[
p = a_1q + b_1, \quad 0 \leq b_1 < q;
q = a_2b_1 + b_2, \quad 0 \leq b_2 < b_1;
\ldots
\]
\[
a_{n-2} = a_nb_{n-1} + b_n, \quad 0 \leq b_n = \gcd(p, q) < b_{n-1};
a_{n-1} = a_{n+1}b_n.
\]

Applying the previous lemma by turns to each line of (2) we are convinced that for \( i = 1, \ldots, n \)
\[
\text{Per}_p(x, \sigma) = \text{Per}_p(x, \sigma) + b_i \quad \forall \sigma \in \Sigma.
\]
In particular, \( \text{Per}_p(x, \sigma) = \text{Per}_p(x, \sigma) + \gcd(p, q) \) \( \forall \sigma \in \Sigma \).

Hence, \( \text{Per}_p(x) = \bigcup_{\sigma \in \Sigma} \text{Per}_p(x, \sigma) \subseteq \text{Per}_{\gcd(p,q)}(x) \). On the other hand, since \( \gcd(p, q) \mid p \) the opposite inclusion \( \text{Per}_{\gcd(p,q)}(x) \subseteq \text{Per}_p(x) \) is also true. \( \square \)

So, let \( k \) be the minimal from divisors of \( p \), such that \( \text{Per}_k(x) = \text{Per}_p(x) \). Let for some \( q \in \mathbb{N} \) the equality

\[
\text{Per}_k(x, \sigma) = \text{Per}_k(x, \sigma) + q \quad \forall \sigma \in \Sigma
\]

is hold true. Then \( \text{Per}_{\gcd(k,q)}(x) = \text{Per}_k(x) = \text{Per}_p(x) \) on lemma \( 2 \).

Since \( k \mid p \) and \( \gcd(k,q) \mid k \) then \( \gcd(k,q) = k \) by virtue of a choice of \( k \) and \( k \mid q \). That is \( k \in \mathcal{P}(x) \).

**Remark 4.** As \( \text{Per}_k(x) = \text{Per}_p(x) \) on the construction then \( M_k(x) = M_p(x) \).

2. Now we shall proceed directly to the construction of the periodic structure for \( x \).

Taking into account the equality \( (1) \) we shall choose a sequence \( \{p_i\}_{i \in \mathbb{N}} \) of the natural numbers to comply with the relation

\[
\lim_{i \to \infty} M_{p_i}(x) = \infty.
\]

Further, using argument stated before we shall choose the least divisor \( k_i \) of \( p_i \) for every \( i \in \mathbb{N} \) such that \( \text{Per}_{k_i}(x) = \text{Per}_{p_i}(x) \). We shall receive a sequence \( \{k_i\} \) of the essential periods for \( x \) satisfying the relation \( \lim_{i \to \infty} M_{k_i}(x) = \infty \) (see remark \( 4 \)).

Set

\[
q_i = \text{lcm}(k_1, k_2, \ldots, k_i)
\]

for every \( i \in \mathbb{N} \). It is easily verified that \( q_i \mid q_{i+1}, \; i \in \mathbb{N} \). **Remark 2** guarantees that a sequence \( \{q_i\} \) contains only the essential periods for \( x \), and the equality

\[
\lim_{i \to \infty} M_{q_i}(x) = \infty
\]

follows from remark \( 3 \).

Proposition \( 3 \) is completely proved.

2.2. **Proof of theorem \( 1 \) and proposition \( 2 \).** We fix periodic structure \( \{q_i\} \) on \( x \). Passing to subsequence it is possible to suppose that

\[
M_{q_{i+1}}(x) \geq 3q_i + M_{q_i}(x), \quad i \in \mathbb{N}.
\]

(3)

First we shall construct a Toeplitz sequence \( \eta \in \overline{\text{Orb}(x)} \) such that

\[
\mathbb{Z} = \bigcup_{i \in \mathbb{N}} \text{Per}_{q_i}(\eta)
\]

( and thus we shall prove proposition \( 3 \)), and then we shall show that \( q_i \in \mathcal{P}(\eta), \; i \in \mathbb{N} \).
1. We fix a sequence \( \{m_i\}_{i \in \mathbb{N}} \) of integers, such that \( m_i + j \in \text{Per}_{q_i}(x) \) for all \( i \in \mathbb{N} \) and \( j \in \{0,1,\ldots,M_{q_i}(x) - 1\} \), that is for every \( i \in \mathbb{N} \) if \( n \equiv 0 \pmod{q_i} \) then
\[
x(m_i + j) = x(m_i + j + n), \quad j = 0,1,\ldots,M_{q_i}(x) - 1.
\]

From the relation (3) it follows that
\[
[M_{q_{i+1}}(x) - (q_i + M_{q_i}(x))] - q_i \geq q_i,
\]
therefore for every \( i \in \mathbb{N} \) there exists
\[
s_i \in [m_{i+1} + q_i, (m_{i+1} + M_{q_{i+1}}(x)) - (q_i + M_{q_i}(x))],
\]
which complies with the equality \( m_i \equiv s_i \pmod{q_i} \).

Let us designate
\[
d_l(i) = s_i - m_{i+1}, \quad d_r(i) = (m_{i+1} + M_{q_{i+1}}(x)) - (s_i + M_{q_i}(x)).
\]

Note that \( d_l(i) \) and \( d_r(i) \) are the numbers of symbols of the block
\[
x(m_{i+1}), x(m_{i+1} + 1), \ldots, x(m_{i+1} + M_{q_{i+1}}(x) - 1),
\]
standing accordingly at the left and at the right of the block
\[
x(s_i), x(s_i + 1), \ldots, x(s_i + M_{q_i}(x) - 1)
\]
of the sequence \( x = (x(n)) \).

It is not difficult to see that
\[
d_l(i) \geq q_i, \\
d_r(i) \geq (m_{i+1} + M_{q_{i+1}}(x)) - \left( (m_{i+1} + M_{q_{i+1}}(x)) - (q_i + M_{q_i}(x)) + M_{q_i}(x) \right) = q_i.
\]

Consider a sequence of integers
\[
k_1 = m_1, \\
k_2 = k_1 + (s_1 - m_1) = s_1, \\
\ldots \\
k_j = k_{j-1} + (s_{j-1} - m_{j-1}) = m_1 + (s_1 - m_1) + \ldots + (s_{j-1} - m_{j-1}), \quad j > 1.
\]

and a sequence \( z_j = S^{k_j}(x) \), \( j \in \mathbb{N} \), of elements of the set \( \overline{\text{Orb} \ x} \). We shall note obvious equalities
\[
z_j = S^{k_j-k_l} \circ S^{k_l}(x) = S^{k_j-k_l}(z_l),
\]
\[
k_j - k_l = \sum_{i=l}^{j-1} (s_i - m_i), \quad l < j.
\]

Notice that since for all \( j \in \mathbb{N} \) on construction \( q_j \mid (s_j - m_j) \) and \( q_j \mid q_{j+1} \) then
\[
q_l \mid (k_j - k_l), \quad l < j
\]
and for every \( n \in \text{Per}_{q_l}(x) \) and \( j > l \) we have
\[
z_j(n) = z_l(n) = x(n + k_l).
\]

Let us designate
\[
P_l = \text{Per}_{q_l}(z_l) = \text{Per}_{q_l}(x) - k_l. \tag{5}
\]

Above we have already checked up that \( z_j(n) = z_l(n) \) for all \( j > l \) and \( n \in P_l \). We shall show now that \( \bigcup_{l \in \mathbb{N}} P_l = \mathbb{Z} \).

On construction \([m_l, m_l + M_{q_l}(x) - 1] \subset \text{Per}_{q_l}(x) \), \( l \in \mathbb{N} \), hence \([m_l - k_l, m_l + M_{q_l}(x) - 1 - k_l] \subset P_l \).

Notice that for \( l = 1 \)
\[
m_1 - k_1 = 0,
\]
hence \([0, M_{q_1}(x) - 1] \subset P_1 \).

When \( l \geq 2 \) we have
\[
m_l - k_l = -(s_1 - m_1) - \ldots - (s_{l-1} - m_{l-1}) + m_l =
\]
\[
= -(s_1 - m_2) - \ldots - (s_{l-1} - m_l) =
\]
\[
= -d_l(1) - \ldots - d_l(l - 1) \leq
\]
\[
\leq -q_1 - \ldots - q_{l-1};
\]
\[
m_l + M_{q_l}(x) - 1 - k_l =
\]
\[
= m_l + M_{q_l}(x) - 1 - m_1 - \sum_{i=1}^{l-1} (s_i - m_i) =
\]
\[
= M_{q_l}(x) - 1 + \sum_{i=2}^{l} (m_i - s_{i-1}) =
\]
\[
= M_{q_l}(x) - 1 + \sum_{i=2}^{l} [(m_i + M_{q_l}(x)) - (s_{i-1} + M_{q_{i-1}}(x))] =
\]
\[
= M_{q_l}(x) - 1 + \sum_{i=1}^{l-1} d_r(i) \geq M_{q_1}(x) - 1 + \sum_{i=1}^{l-1} q_i.
\]

Hence, for all \( l > 1 \)
\[
\left[-\sum_{i=1}^{l-1} q_i, M_{q_l}(x) - 1 + \sum_{i=1}^{l-1} q_i\right] \subset P_l. \tag{6}
\]

On construction \( q_i \geq 1, i \in \mathbb{N} \), so \( \bigcup_{l \in \mathbb{N}} P_l = \mathbb{Z} \).

Therefore, it is correctly determined \( \eta \in X \) which meets the equality \( \eta(n) = z_l(n) \), if \( n \in P_l \).

It is easy to see that \( P_l \subseteq \text{Per}_{q_l}(\eta) \), \( l \in \mathbb{N} \). Furthermore, from proposition \( \square \) it follows that \( \eta = \lim_{i \to \infty} z_i \).
Remark 5. So, we have constructed the Toeplitz sequence η ∈ \text{Orb} x. In the argument above we have nowhere used recurrence of x.

Let now a point x is recurrent. Under Birkhoff theorem the set \text{Orb} x is minimal, hence \text{Orb} η = \text{Orb} x.

2. Let η ∈ X be a Toeplitz sequence, x, y ∈ \text{Orb} η. Let us prove that \( M_\eta(x) = M_\eta(y) \) for every \( p \in \mathbb{N} \) and \( P(x) = P(y) \).

Lemma 3. Let A be a minimal subset of dynamic system \((X, S)\), \( x, y \in A \). Let \( \text{Per}_\eta(x) \neq \emptyset \) for some \( p \in \mathbb{N} \). Then there exists \( n(p) \in \mathbb{Z} \) which satisfies the conditions

(i) \( \text{Per}_\eta(y) = \text{Per}_\eta(x) - n(p) \);
(ii) \( x(k + n(p)) = y(k) \) for every \( n \in \text{Per}_\eta(x) \);

Proof. 1. First we shall prove that there exists \( n(p) \in \mathbb{Z} \) which satisfies to a condition (ii) (hence for this \( n(p) \) inclusion \( \text{Per}_\eta(x) \subseteq \text{Per}_\eta(y) + n(p) \) is valid).

Since the set A is minimal then \( A = \text{Orb} x = \text{Orb} y \) and there exists a sequence \( \{z_j = S^{k_1}(x)\}_{j \in \mathbb{N}} \) converging to a point y.

Let us say that \( k_i \sim k_j \) if \( k_i \equiv k_j \pmod{p} \). Under this relation the set \{\( k_i \)\} will fall into no more than on p classes of equivalence. Obviously, at least one of these classes contains infinite number of elements. Hence, passing to a subsequence we can assume that \( k_i \equiv k_j \pmod{p} \) for all \( i, j \in \mathbb{N} \).

Then \( \text{Per}_\eta(z_i) = \text{Per}_\eta(z_1) = \text{Per}_\eta(x) - k_1 \) for all \( i \in \mathbb{N} \) (we shall designate \( P(p) = \text{Per}_\eta(x) - k_1 \)). Moreover, \( z_i(k) = z_1(k) = x(k + k_1) \) for all \( k \in P(p) \).

From proposition \( \[ \] \) it follows that \( y(k) = z_1(k) = x(k + k_1) \) for every \( k \in P(p) \). Therefore, \( (\text{Per}_\eta(x) - k_1) \subseteq \text{Per}_\eta(y) \) also it is possible to let \( n(p) = k_1 \).

2. Let us check now a correlation (i).

Assume that \( \text{Per}_\eta(x) \nsubseteq \text{Per}_\eta(y) + n(p) \). Repeating the argument of item 1 and changing roles of x and y, we shall find \( m(p) \in \mathbb{Z} \) such that \( \text{Per}_\eta(y) \subseteq \text{Per}_\eta(x) + m(p) \). Then \( \text{Per}_\eta(y) + n(p) \subseteq \text{Per}_\eta(x) + (m(p) + n(p)) \) and

\[
\text{Per}_\eta(x) \nsubseteq \text{Per}_\eta(x) + (m(p) + n(p)) .
\]

Clearly, \( m(p) + n(p) \neq 0 \).

Obviously, for every \( r \in \mathbb{Z} \)

\[
\text{Per}_\eta(x) + r \nsubseteq \text{Per}_\eta(x) + (m(p) + n(p)) + r . \quad (7)
\]

Let \( s = \text{lcm}(m(p) + n(p), p) \). Then \( s = a(m(p) + n(p)) \) for certain \( a \in \mathbb{Z} \setminus \{0\} \).
Assume that $a < 0$ (the case $a > 0$ is examined similarly). Using a relation (4), we shall receive the following chain of inclusions

$$\text{Per}_p(x) \supsetneq \text{Per}_p(x) - (m(p) + n(p)) \supsetneq \text{Per}_p(x) - 2(m(p) + n(p)) \supsetneq \ldots \supsetneq \text{Per}_p(x) + a(m(p) + n(p)).$$

However, on construction $p \mid s$, hence

$$\text{Per}_p(x) + a(m(p) + n(p)) = \text{Per}_p(x) + s = \text{Per}_p(x)$$

by definition of $\text{Per}_p(x)$.

The received contradiction finishes the proof of lemma.

\textbf{Corollary 2.} $M_p(x) = M_p(y)$ for every $p \in \mathbb{N}$ and $\mathcal{P}(x) = \mathcal{P}(y)$.

Applying now lemma 3 and corollary 2 to the sequence $\{q_i\}$ we verify that $P_i = \text{Per}_{q_i}(\eta)$ and $q_i \in \mathcal{P}(\eta)$, $i \in \mathbb{N}$. For the completion of the proof of theorem 3 it remains to recall the equality $\bigcup_{i \in \mathbb{N}} P_i = \mathbb{Z}$ which we have already checked above.

\section{Toeplitz subshifts on the space of two-sided sequences over a Hausdorff compact alphabet}

\subsection{Odometers and periodic partitions of dynamic systems.}

\textbf{Definition 7.} A non-bounded sequence $\{a_i \in \mathbb{N}\}_{i \in \mathbb{N}}$ is called regular if $a_i$ divides $a_{i+1}$ for every $i \in \mathbb{N}$.

We fix regular sequence $\{n_i \in \mathbb{N}\}_{i \in \mathbb{N}}$ (without loss of generality it is possible to assume that $n_{i+1} \neq n_i$, $i \in \mathbb{N}$).

Let us consider a sequence of finite cyclic groups $\mathbb{Z}_{n_i} = \mathbb{Z}/n_i\mathbb{Z}$ and group homomorphisms

$$\varphi_i : \mathbb{Z}_{n_{i+1}} \to \mathbb{Z}_{n_i},$$

$$\varphi : 1 \mapsto 1.$$

Let us take an inverse limit $A = \text{proj lim}_{i \to \infty} \mathbb{Z}_{n_i}$ of this sequence of groups and homomorphisms. We receive an abelian group $(A, +)$.

Provide each set $\mathbb{Z}_{n_i} = \{0, 1, \ldots, n_i - 1\}$ with discrete topology. Each of maps $\varphi_i$ is continuous in this topology. Space $A$ with topology $\mathcal{T}$ of the inverse limit is homeomorphic to a Cantor set $\Gamma$.

It is easy to see, that in the group $(A, +)$ operation of addition and pass to an opposite element are continuous in the topology $\mathcal{T}$, thus $A$ turns to be a continuous group.

\textbf{Remark 6.} We remind that an inverse limit $A = \text{proj lim}_{i \to \infty} \mathbb{Z}_{n_i}$ could be imagined as a subset

$$A = \{\bar{a} = (a_i \in \mathbb{Z}_{n_i}) | \varphi_i(a_{i+1}) = a_i, i \in \mathbb{N}\}$$ (8)
of the direct product
\( \prod_{i \in \mathbb{N}} \mathbb{Z}_{n_i} \). \( (9) \)

In such notation the operation of addition in \( A \) is defined component-wise, that is \( \vec{a} + \vec{b} = (a_i + b_i) \) for any \( \vec{a} = (a_i) \), \( \vec{b} = (b_i) \in A \).

As is known, the topology of the direct product (9) is set through a basis consisting of so-called cylindrical sets
\[
U(x_{i_1}, \ldots, x_{i_k}) = \{(a_i) \mid a_{i_s} = x_{i_s}, \ s = 1, \ldots, k\};
\]
\( x_{i_s} \in \mathbb{Z}_{n_{i_s}}, \ i_1 < \ldots < i_k, \ k \in \mathbb{N} \).

From definition of the set \( A \) (see relation (8)) it is easy to see that
\[
U(x_{i_1}, \ldots, x_{i_k}) \cap A = U(x_{i_k}) \cap A
\]
for any \( k \in \mathbb{N}, i_1 < \ldots < i_k \) and \( x_{i_s} \in \mathbb{Z}_{n_{i_s}} \). So, the family of sets
\[
V_{x_j} = U(x_j) \cap A = \{(a_i) \in A \mid a_j = x_j\} = \{(a_i) \in A \mid a_j = x_j, \ a_k = \varphi_k \circ \ldots \circ \varphi_{j-1}(x_j) \text{ when } k < j\};
\]
\( j \in \mathbb{N}, \ x_j \in \mathbb{Z}_{n_j} \)
\( (10) \)
is base of the topology of space \( A \).

The natural metric \( d : A \times A \to \mathbb{R}_+ \) on \( A \) associated with the sequence \( \{n_i\} \) is defined as follows
\[
d(\vec{x}, \vec{y}) = \frac{1}{m}, \quad m = \min\{i \in \mathbb{N} \mid x_k = y_k \text{ when } k < i \text{ and } x_i \neq y_i\}.
\]
The correctness of this definition is checked immediately.

Consider an element \( \vec{e} = (1) = (1, \ldots, 1, \ldots) \in A \). This element is called generator of group \( A \) and has the property that the cyclic subgroup \( \langle \vec{e} \rangle \) generated by it is dense in \( A \) in the topology \( T \).

Obviously, shift mapping
\[
g : A \to A, \quad g : \vec{x} \mapsto \vec{x} + \vec{e},
\]
is a homeomorphism.

**Definition 8.** Dynamic system \((A, g)\) is called an odometer.

**Remark 7.** From the fact the subgroup \( \langle \vec{e} \rangle \) is dense in \( A \) it immediately follows that each trajectory of d. s. \((A, g)\) is dense in \( A \), that is odometer always is a minimal dynamic system.

**Remark 8.** It is easy to verify that in the natural metric defined above the mapping \( g \) is isometric. Specially, the family of mappings \( \{g^k\}_{k \in \mathbb{Z}} \) is equicontinuous, so the odometer \((A, g)\) is the equicontinuous dynamic system.

Actually, it is known that odometers are precisely all equicontinuous minimal dynamic systems on the Cantor set.
Assume a compact Hausdorff space $X$ and homeomorphism $f : X \to X$ are given.

**Definition 9.** We call a finite family $\{W_i\}_{i=0}^{n-1}$ of subsets of space $X$ a *periodic partition* of the dynamic system $(X, f)$ of length $m$, if it satisfies to the following requirements:

(i) all $W_i$ are open-closed subsets of $X$;
(ii) $W_i = f(W_{i-1})$, $i = 1, \ldots, n-1$ and $W_0 = f(W_{n-1})$;
(iii) $W_i \cap W_j = \emptyset$ when $i \neq j$;
(iv) $X = \bigcup_{i=0}^{n-1} W_i$.

**Lemma 4.** Assume $(A, g)$ is an odometer built with the help of a regular sequence $\{n_i\}_{i \in \mathbb{N}}$.

For any $k \in \mathbb{N}$ and $x_k \in \mathbb{Z}_{n_k}$ a family of sets $\{W_j^{(n_k)} = V_{x_k+j}\}_{j=0, \ldots, n_k-1}$ forms periodic partition of a dynamic system $(A, g)$ of length $n_k$.

**Proof.** Obviously,

$$A = \bigcup_{s \in \mathbb{Z}_{n_k}} V_s = \bigcup_{j \in \mathbb{Z}_{n_k}} V_{x_k+j}.$$  

Hence, for the family $\{W_j^{(n_k)}\}$ the requirement (iv) of Definition 9 is carried out.

Since all sets $V_{x_k+j}$, $j \in \mathbb{Z}_{n_k}$, are open on definition and pairwise disjoint, family $\{W_j^{(n_k)}\}$ satisfies also to properties (i) and (iii) of a periodic partition.

For completion of the proof we need to verify that $g(V_{a_k}) = V_{a_k+1}$ (here $1 \in \mathbb{Z}_{n_k}$) for every $a_k \in \mathbb{Z}_{n_k}$.

Let $\vec{b} = (b_i) \in V_{a_k}$. Then $b_k = a_k$ and $g(\vec{b}) = \vec{b} + \vec{c} = (b_k + 1) \in V_{a_k+1}$. Hence, $g(V_{a_k}) \subseteq V_{a_k+1}$.

Back, let $\vec{c} = (c_i) \in V_{a_k+1}$. Then $c_k = a_k + 1$ and $g^{-1}(\vec{b}) = \vec{c} - \vec{c} = (c_k - 1) \in V_{a_k}$. Hence, $g(V_{a_k}) \supseteq V_{a_k+1}$. \qed

### 3.2. Toeplitz subshifts and projections onto odometers

Let $\Sigma$ be a compact Hausdorff space, $X = \Sigma^\mathbb{Z}$, $S : X \to X$ is the left shift on $X$.

Assume $x = (x(n)) \in X$ is non–periodic recurrent point, $\{p_i \in \mathcal{P}(x)\}$ is a sequence which complies with all conditions of definition 8.

Let us consider a family of sets

$$A_j^i = \{y(n) \in \overline{\text{Orb}(y)} \mid y(k + j) = x(k) \ \forall k \in \text{Per}_{p_i}(x)\} =$$

$$= \{y(n) \in \overline{\text{Orb}(y)} \mid y(n) = x(k) \ \forall n \equiv k + j \pmod{p_i}, k \in \text{Per}_{p_i}(x)\},$$

$$j \in \{0, 1, \ldots, p_i - 1\}, \quad i \in \mathbb{N}.$$  

(11)

We can describe $A_j^i$ as the set of all points from $\overline{\text{Orb}(x)}$ which have the same $p_i$–skeleton with $S^j(x)$. 


Lemma 5 (compare with lemma 2.3 from [2]). The family of sets \( \{A^i_j\} \) complies with the following properties

(i) For every \( i \in \mathbb{N} \) the family \( \{A^i_j\}_{j=0}^{p_i-1} \) is the periodic partition of the dynamic system \((\text{Orb}(x), S)\) of length \( p_i \).

(ii) \( A^i_n \supset A^i_m \) when \( i < j \) and \( m \equiv n \pmod{p_i} \).

Proof. We mark first that for every \( y \in X \) and for all \( q_1, q_2 \), such that \( \text{Per}_{q_1}(y), \text{Per}_{q_2}(y) \neq \emptyset \), the following implication is valid

\[
(q_1 \text{ divides } q_2) \Rightarrow (\text{Per}_{q_1}(y, \sigma) \subseteq \text{Per}_{q_2}(y, \sigma), \sigma \in \Sigma). \tag{12}
\]

Let \( x \in X \) and \( \{p_i \in \mathcal{P}(x)\}_{i \in \mathbb{N}} \) satisfy to requirements of lemma. Then from theorem [4] it follows that the dynamic system \((\text{Orb}(x), S)\) is Toeplitz and specially it is minimal.

We fix \( i \in \mathbb{N} \).

From lemma [3] it immediately follows that

\[
\text{Orb}(x) = \bigcup_{j=0}^{p_i-1} A^i_j
\]

and the family of sets \( \{A^i_j\}_{j=0}^{p_i-1} \) satisfies to the requirement (iv) of definition [3].

Verify now the validity of requirement (iii) of this definition. Assume that \( A^i_j \cap A^i_k \neq \emptyset \) for some \( j \neq k \). Then from lemma [3] and definition of the set \( \text{Per}_{p_i}(x, \sigma) \) we get \( x(n) = y(n+j) = y(n+k) \) for all \( n \in \text{Per}_{p_i}(x) \) and

\[
\text{Per}_{p_i}(x, \sigma) = \text{Per}_{p_i}(y, \sigma) - j = \text{Per}_{p_i}(y, \sigma) - k \quad \forall \sigma \in \Sigma.
\]

From corollary [3] we have \( p_i \in \mathcal{P}(y) \). Hence, \( p_i \) divides \( |j - k| \) by definition of essential period. And it contradicts to the inequality \( 0 < |j - k| < p_i \).

Let us prove now property (ii) of definition [3].

From definition of sets \( A^i_j \) the relations follow

\[
S(A^i_{j-1}) \subseteq A^i_j, \quad j \in \{1, \ldots, p_i - 1\};
\]

\[
S(A^i_{p_i-1}) \subseteq A^i_0.
\]

With the help of these relations we immediately conclude that

\[
S^{p_i}(A^i_j) \subseteq A^i_j, \quad j \in \{0, 1, \ldots, p_i - 1\}. \tag{14}
\]

The map \( S \) is a homeomorphism. Hence, if even at least one of the inclusions (13) is strict, then

\[
S^{p_i}(A^i_j) \subsetneq A^i_j, \quad j \in \{0, 1, \ldots, p_i - 1\}.
\]

From this remark and property (iii) of definition [3], which we have already verified, we conclude that in this case

\[
S^{p_i} \left( \text{Orb}(x) \right) = S^{p_i} \left( \bigcup_{j=0}^{p_i-1} A^i_j \right) = \bigcup_{j=0}^{p_i-1} S^{p_i} (A^i_j) \subsetneq \bigcup_{j=0}^{p_i-1} \text{Orb}(x).
\]
Since the set $\overline{\text{Orb}(x)}$ is Hausdorff and compact and $S^{p_i}$ is a homeomorphism, then

$$K = \bigcap_{m \geq 0} S^{mp_i} \left( \overline{\text{Orb}(x)} \right) \neq \emptyset$$

is the proper closed invariant subset of the dynamic system $(\overline{\text{Orb}(x)}, S)$ contrary to the minimality of it.

Consider now property (i) of definition 9.

All sets $A_{ij}$ are closed. Really, we fix $j \in \{0, 1, \ldots, p_i - 1\}$ and a convergent sequence $y_k = y_k(n) \in A_{ij}$. Let $y \in \overline{\text{Orb}(x)}$ is a limit of this sequence. Since we have $y_k(m) = x(m - j)$, $k \in \mathbb{N}$ for all $m \in \text{Per}_{p_i}(x) + j$, then proposition 1 guarantees $y(m) = x(m - j)$ for $m \in \text{Per}_{p_i}(x) + j$.

Consequently, $y \in A_{ij}$ and the sets $A_{ij}$ are closed. That is $\{A_{ij}\}_{j=0}^{p_i-1}$ is the closed finite partition of the dynamic system $(\overline{\text{Orb}(x)}, S)$. Therefore, each set $A_{ij}$ is open–closed in $\overline{\text{Orb}(x)}$ in the induced topology.

The property (ii) of lemma immediately follows from definition of sets $A_{ij}$, relations (12) and (14) and from lemma 3.

Let an odometer $(A, g)$ is built with the help of the sequence $\{p_i\}$.

Assume

$$\vec{a} = (n_i) \in \prod_{i \in \mathbb{N}} \mathbb{Z}_{p_i}.$$  

We denote

$$A_{\vec{a}} = \bigcap_{i \in \mathbb{N}} A_{n_i}^{i}.$$  

From the condition (ii) of lemma it immediately follows that

$$(A_{\vec{a}} \neq \emptyset) \iff \left( \vec{a} \in A \subset \prod_{i \in \mathbb{N}} \mathbb{Z}_{p_i} \right). \quad (15)$$

The condition (i) of lemma mentioned guarantees that the family of sets $\{A_{\vec{a}}, \vec{a} \in A\}$ is partition of the space $\overline{\text{Orb}(x)}$ and

$$S(A_{\vec{a}}) = A_{\vec{a}+\vec{e}} \quad (16)$$

for every $\vec{a} \in A$.

Consider the following correspondence

$$\pi : \overline{\text{Orb}(x)} \to A;$$

$$\pi : A_{\vec{a}} \mapsto \vec{a}, \quad \vec{a} \in A.$$  

From correlation (13) we consequence the correctness of this definition and formula (16) guarantees the equality $\pi \circ S = g \circ \pi$.

Mark that the map $\pi$ is continuous since $\pi^{-1}(V_{x_j}) = A_{x_j}^{j}$ for all $n \in \mathbb{N}$, $x_j \in \mathbb{Z}_{p_j}$. In other words all sets from the family (10), which as
we know that the base of topology of the space $A$, have open–closed preimages in $\text{Orb}(x)$ according to lemma 5.

**Theorem 2.** Assume that a point $x \in X$ is recurrent and a sequence $\{p_i\}_{i \in \mathbb{N}}$ is a periodic structure on $x$ in sense of definition 6.

Then the odometer $(A, g)$ built with the help of the sequence $\{p_i\}$ is an almost one-to-one factor of the flow $(\text{Orb}(x), S)$ under the mapping $\pi$.

Moreover, two following conditions are equivalent:

1) a sequence $y \in \overline{\text{Orb}(x)}$ is Toeplitz;
2) $\pi^{-1}(\pi(y)) = \{y\}$.

**Proof.** Theorem is proved similarly to theorem 2.2 from [2] (the single change is that the above lemma 5 must be referred to instead of lemma 2.3 from [2]).

**References**

[1] Jacobs, Konrad; Keane, Michael 0 − 1-sequences of Toeplitz type 
Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 13 (1969), pp. 123–131;

[2] Williams S. Toeplitz minimal flows which are not uniquely ergodic 
Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 67 (1984), pp. 95–107;

[3] Glimm J. On a certain class of operator algebra Trans. Amer. Mat. Soc., 95 (1960), N 2, pp. 318–340;

[4] G. Barat, T. Downarowicz, A. Iwanik & P. Liardet Propriétés topologiques et 
combinatoires des échelles de numération Colloq. Math., 84/85, part 2 (2000), 
pp. 285-306;

[5] Downarowicz T., Durand F. Factors of Toeplitz flows and other almost 1-1 
extensions over group rotations Math. Scand. (to appear) 
(the preliminary version is available at http://www.im.pwr.wroc.pl/ downar/publ.html);

[6] Kelley John L. General topology . D. Van Nostrand Company, Inc., Toronto- 
New York-London, 1955;

[7] Alekseev, V. M. Symbolic dynamics. (Russian) Eleventh Mathematical School 
(Summer School, Kolomyya, 1973) (Russian), pp. 5–210. Izdanie Inst. Mat. 
Akad. Nauk Ukrain. SSR, Kiev, 1976;

[8] Morse, Marston; Hedlund, Gustav A. Symbolic dynamics I Amer. J. Math. 60 
(1938), pp. 1–42;

[9] Gjerde R., Johansen O. Bratteli–Vershik models for Cantor minimal systems: 
applications to Toeplitz flows Ergod. Th. & Dynam. Sys., 20 (2000), pp. 1687– 
1710.

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