Chiral Supergravitons Interacting with a 0-Brane
$N$-Extended NSR Super-Virasoro Group

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ABSTRACT

We continue the development of $S_{AFF}$ by examining the cases where there are $N$ fermionic degrees of freedom associated with a 0-brane. These actions correspond to the interaction of the $N$-extended super Virasoro algebra with the supergraviton and the associated SO($N$) gauge field that accompanies the supermultiplet. The superfield formalism is used throughout so that supersymmetry is explicit.

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Superscription:

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1 Introduction

In the literature there has been some focus on the role that diffeomorphisms play in the development of both classical and quantum gravity through string theories. One particular approach uses the Virasoro algebra and its dual (the coadjoint representation) as a guide to these developments [1, 2]. This approach differs from theories of gravity that are based purely on geometry since it is the Lie derivative, as opposed to the covariant derivative, which is at the center stage. Since the Lie derivative exists in any dimension, this approach allows one to get a foothold on gravitation on the two-dimensional string worldsheet. From this viewpoint, the Virasoro algebra corresponds to the time independent Lie derivative of contravariant vector fields, while the isotropy equations on the coadjoint orbits serve as extensions of the Gauss’ Law constraints from Yang-Mills theories. Actions that admit these constraints when evaluated on two-dimensional surfaces were constructed in [1, 2]. Recently [3], these actions were extended to the super Virasoro algebra for one supersymmetry ($N = 1$). Since the two genders of particles, fermions and bosons, are placed on equal footing through smooth continuous transformations, these actions were dubbed affirmative actions.

Another way to conceptualize this work is to view our problem as an attempt to couple a NSR 0-brane with $N$-extended supersymmetry to a background containing 2D, $(N,0)$ supergravity fields. This is an extension of the well-known conformal $\sigma$-model technique. There one couples the massless compositions of a string theory via a world sheet action to the string. Conformal invariance of the world sheet $\sigma$-model then imposes the equations of motion on the massless composites.

In our study, we introduce an NSR 0-brane described by coordinates $\tau$ and $\zeta^I$ ($I = 1, \ldots, N$) such that vector fields constructed from these coordinates naturally carry an arbitrary $N$ and model-independent representation of the super-Virasoro algebra. These vector fields provide a representation of the centerless super-Virasoro algebra. As a benefit of such a geometrical approach, the generators of the super-Virasoro algebra naturally possess a set of transformation laws under the action of the group. Consequently, the co-adjoint elements to the generators also naturally obtain a set of transformation law. By demanding that these co-adjoint elements be embedded within background fields and preserve the symmetries of the co-adjoint elements, restrictions on the backgrounds are derived. Thus a major problem for us is to find constructions that respect these symmetries within this approach.

The purpose of this work is to show that we can extend these actions to $N$ supersymmetries for chiral two dimensional models using the results found in Ref.[4] as a guide. However, in this work we will focus on a superfield formulation as opposed to the component construction. Two superfields are used in the construction, one corresponds to the $N$-extended prepotentials associated with the graviton while the other is the prepotential associated with the gauge superfield that accompanies the internal $SO(N)$ symmetry that
is now local.

2 \( N = 1 \) Affirmative Actions

For the sake of continuity we quickly review some of the results of Ref.[3]. Consider the super Virasoro algebra which contains the bosonic Virasoro generators \( L_m, m \in \mathbb{Z} \), and fermionic generators \( G_\mu, \mu \in \mathbb{Z} \) or \( \mathbb{Z} + \frac{1}{2} \), and a central charge \( \hat{c} \). Its algebra is given by

\[
\begin{align*}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{1}{8}\hat{c}(m^3 - m)\delta_{m+n,0} I , \\
[L_m, G_\mu] &= (\frac{1}{2}m-\mu)G_{m+\mu} , \\
\{G_\mu, G_\nu\} &= -i 4 L_{\mu+\nu} - i\frac{\hat{c}}{2} \left( \mu^2 - \frac{1}{4} \right) \delta_{\mu+\nu,0} I .
\end{align*}
\]

One can show [7] that this algebra can be represented with superfields by writing,

\[
A(z, \zeta) = \sum_{m=-\infty}^{\infty} (A^m z^{m+1}) + 2 \zeta \sum_{\mu=-\infty}^{\infty} (A^\mu z^{\mu + \frac{1}{2}}) ,
\]

where for now \( \zeta \) is simply a one dimensional Grassmann coordinate (later we use \( \theta \) to denote the space-time Grassmann supercoordinate variable). The generic element of the algebra written in Eq.(2) has an equivalent representation as a doublet \((A(Z), a)\) with \( Z = \{z, \zeta\} \).

Then, in terms of derivations on the superfield the commutation relations appear as [6, 7]

\[
[[A,a)(B,b)]] = ((\partial A)B - A\partial B - i\frac{1}{2}(\partial A)(\partial B), \oint dZ (\partial^2 \partial A) B) ,
\]

where \( A \) and \( B \) are adjoint elements and

\[
D = \frac{\partial}{\partial z} + \frac{1}{2} \zeta \frac{\partial}{\partial \zeta} , \quad dZ = \frac{dz}{2\pi i} d\zeta .
\]

\( D \) implies \( D^2 = \frac{1}{2} \{D,D\} \).

Now we can take an adjoint superfield, say \( F \), and act on a coadjoint element \( B^* \) as a Lie derivative and supersymmetry transformation to find [3, 4]

\[
\delta_F B^* = -FD^2 B^* - \frac{1}{2}FD DB^* - \frac{3}{2}D^2 FB^* + q D^5 F ,
\]

where \( F \) has the decomposition \( F = \xi + i\zeta \epsilon \) and \( B^* = (u + i \zeta D, b^*) = (u, D, b^*) \). The isotropy equation (stability equation) for the coadjoint element \( B^* \) is given by setting Eq.(5) to zero. This determines the subalgebra that will leave the coadjoint vector \( B \) invariant. In terms of component fields this becomes the two coupled equations

\[
\begin{align*}
- \xi \partial D - \frac{1}{2} \epsilon \partial u - \frac{3}{2} \partial \epsilon u + b^* \partial^3 \xi - 2 \partial \xi D &= 0 , \\
- \xi \partial u - \frac{1}{2} \epsilon D - \frac{3}{2} \partial \xi u + q \partial^2 \epsilon &= 0 .
\end{align*}
\]

\(^8\)We can also refer to this as the NSR 0-brane Grassmann coordinate.
with \( \partial = \partial_z \). These equations are part of the field equations one extracts from the affirmative action. They have a direct analog to the Gauss’ Law constraints found in Yang-Mills.

In order to embed this into two dimensions, we replace the 1D Grassmann variable \( \zeta \) with 2-dimensional chiral Majorana spinors \( \zeta^\alpha \). Then the supersymmetric covariant derivative operator becomes

\[
D_\mu = \partial_\mu - \frac{i}{2} \gamma^N_{\mu\nu} \zeta^\nu \partial_N .
\]

With this

\[
\{D_\mu, D_\nu\} = -i \gamma^{N\mu} \frac{\partial}{\partial z^N} ,
\]

where \( \gamma^{\mu}_{\nu} \) is the associated Gamma matrix. We will use capital Latin indices for space-time indices and small Greek for spinor indices. With these \( \gamma^{\mu}_{\nu} \)'s we also introduce \( \gamma^{\alpha\beta} \) such that

\[
\gamma^A_{\alpha\beta} \gamma^{\beta\gamma}_B = \frac{1}{2} \delta^A_B + \frac{1}{2} \Sigma^{AB}_\alpha ,
\]

where \( \Sigma^{AB}_\alpha \) is anti-symmetric in its space-time indices.

Then, for example, an adjoint element is promoted to a vector superfield, \( F^M \), and has a \( \zeta \) expansion \( F^M = (\xi^M + \zeta^\alpha \gamma^M_{\alpha\beta} \epsilon^\beta + \sigma \zeta^\mu \zeta^\alpha \gamma^N_{\mu\alpha} ) \) while a two dimensional coadjoint element is promoted to the \( \frac{3}{2} \) spin superfield \( B_{\mu M} = (\Upsilon_{\mu M} + \zeta^\alpha \gamma^N_{\alpha\beta} D_{MN} + \zeta^\alpha \zeta^\beta \gamma^N_{[\alpha} A_{\beta]\mu N} ) \). The transformation of \( B_{\mu M} \) with respect to \( F^N \) is

\[
\delta_F B_{\mu M} = F^N \partial_N B_{\mu M} + \partial_M F^N B_{\mu N} + \frac{1}{2} (\partial_N F^N) B_{\mu M} + i (D_\lambda F^N \gamma^{\lambda}_{\mu N}) D_\nu B_{\mu M} .
\]

This is seen as the Lie derivative with respect to \( F^M \) on the space-time index in the first three summands followed by a supersymmetry transformation on \( B_{\mu M} \) with \( \epsilon^\nu \equiv (D_\lambda F^{\lambda N} \gamma^N_{\mu \nu} ) \). This combination of a Lie derivative and supersymmetry transformation is a natural extension of the Lie derivative. The supersymmetric isotropy equation in higher dimensions can now be thought of as setting \( \delta_F B_{\mu M} \) to zero. Then those fields \( F \) that satisfy this condition make up a subspace of vector fields that form the isotropy algebra for \( B_{\mu M} \). In order to be consistent with conformal field theory, every spinor index will carry a density weight of \( \frac{1}{2} \). Thus the superfield \( B_{\mu N} \) is a tensor density with weight \( \frac{1}{2} \).

3 Prepotentials for Superdiffeomorphisms and SO(N)

3.1 The Components of the Algebra and Its Dual

There have been many \( N \)-Extended Super Conformal algebras posed in the literature \[8\]. However we will focus on the \( N \)-extended Super Virasoro algebra proposed in \[11\]. There
one has

\[ G_A^I \equiv i \tau^{A+\frac{3}{2}} \left[ \partial^I - i 2 \zeta^I \partial_r \right] + 2(\mathcal{A} + \frac{1}{2}) \tau^A \frac{3}{2} \zeta^I \zeta^K \partial_K , \]

\[ L_A \equiv - \left[ \tau^{A+1} \partial_r + \frac{1}{2}(\mathcal{A} + 1) \tau^A \zeta^I \partial_I \right] , \]

\[ T_{A}^{11} \equiv \tau^A \left[ \zeta^I \partial^I - \zeta^J \partial^J \right] , \]

\[ U_A^{1\cdots i_q} \equiv (i)^{\frac{q}{2}} \tau^{(A-(\frac{q}{2})-\frac{1}{2})} \zeta^{1\cdots i_q} \zeta^{n-1} \partial^{i_q} , \quad q = 3, \ldots, N + 1 \ , \]

\[ R_{A}^{1\cdots i_p} \equiv (i)^{\frac{p}{2}} \tau^{(A-(\frac{p}{2})-\frac{1}{2})} \zeta^{1\cdots i_p} \partial_r , \quad p = 2, \ldots, N \ , \]

where \( N \) is the number of supersymmetries. Here we have used the notational conventions of [11]. In [11] one uses the dual representation of the algebra in order to develop a field theory. This idea is very different from that used in so called conformal field theories as in this approach there are no fields external to the algebra (actually its dual) introduced, that is no model or Lagrangian is introduced that is exterior to the algebra. The impetus behind this approach there are no fields external to the algebra (act ually its dual) introduced, that is no model or Lagrangian is introduced that is exterior to the algebra. The impetus behind the construction of the Lagrangian is for its field equations to correspond to constraints that arise on the coadjoint orbits.

The elements of the algebra can be realized as fields whose tensor properties can be determined by they way they transform under one dimensional coordinate transformations. How each field transforms under a Lie derivative with respect to \( \xi \) are summarized below. In conformal field theory these transformation laws characterize the fields by weight and spin. Here we treat the algebraic elements as one dimensional tensors so that higher dimensional realizations can be achieved.

### Table 1: Tensors Associated with the Algebra

| Element of algebra | Transformation Rule | Tensor Structure |
|--------------------|---------------------|-----------------|
| \( L_A \rightarrow \eta \) | \( \eta \rightarrow -\xi^a \eta + \xi^a \) | \( \eta^a \) |
| \( G_A^I \rightarrow \chi^I \) | \( \chi^I \rightarrow -\xi(\chi^I)^\tau + \frac{1}{2}\xi^\tau \chi^I \) | \( \chi^I \alpha \) |
| \( T^{RS} \rightarrow i^{RS} \) | \( i^{RS} \rightarrow -\xi(i^{RS})^\tau \) | \( i^{RS} \) |
| \( U^{V_1\cdots V_n} \rightarrow w^{V_1\cdots V_n} \) | \( w^{V_1\cdots V_n} \rightarrow -\xi(w^{V_1\cdots V_n})^\tau - \frac{1}{2}(n-2)\xi^\tau w^{V_1\cdots V_n} \) | \( w^{V_1\cdots V_n; a_{\alpha_1\cdots\alpha_n}} \) |
| \( R_{A}^{1\cdots T_n} \rightarrow r^{T_1\cdots T_n} \) | \( r^{T_1\cdots T_n} \rightarrow -\xi(r^{T_1\cdots T_n})^\tau + \frac{1}{2}(n-2)\xi^\tau r^{T_1\cdots T_n} \) | \( r^{T_1\cdots T_n; a_{\alpha_1\cdots\alpha_n}} \) |

In the above table we have used capital Latin letters, such as \( I, J, K \) to represent \( SO(N) \) indices, small Latin letters to represent tensor indices, and small Greek letters for spinor indices. Spinors with their indices up transform as scalar tensor densities of weight one (-1) while those with their indices down transform as tensor indices, and small Greek letters for spinor indices. Spinors with their indices up transform as scalar tensor densities of weight one (-1). For example the generator \( U^{V_1\cdots V_n} \) has a tensor density realization of contravariant tensor with rank one and weight \(-\frac{3}{2}\) living in the \( N \times N \times N \) representation of \( SO(N) \), i.e. \( \omega^{V_1\cdots V_n; a_{\alpha_1\cdots\alpha_n}} \). This is the first step at identifying a spectrum of physical fields that have
a natural connection to the algebra. The completion of this identification comes from identifying the tensors related to the coadjoint representation. The transformation laws for the coadjoint elements and tensor representation are tabulated below. We will raise and lower the SO($N$) indices using the $1$ matrix, while the spinor indices are raised and lowered with $C^{\alpha\beta}$ and $C_{\alpha\beta}$.

**Table 2: Tensors Associated with the Dual of the Algebra**

| Dual element of algebra | Transformation Rule | Tensor Structure |
|-------------------------|---------------------|-----------------|
| $L^*_{A} \rightarrow D$ | $D \rightarrow -2\xi^{\prime}D - \xi D'$ | $D_{ab}$ |
| $G^*_{A} \rightarrow \psi^1$ | $\psi^1 \rightarrow -\xi (\psi^1)' - \frac{3}{2} \xi' \psi^1$ | $\psi^1_{a\alpha}$ |
| $T^{RS} \rightarrow A^{RS}$ | $A^{RS} \rightarrow -(\xi)' A^{RS} - \xi (A^{RS})'$ | $A^{RS}_{a\alpha}$ |
| $U^{V_1...V_n} \rightarrow \omega^{V_1...V_n}$ | $\omega^{V_1...V_n} \rightarrow -\xi (\omega^{V_1...V_n})' - (2 - \frac{3}{2}) \xi' \omega^{V_1...V_n}$ | $\omega^{V_1...V_n}_{ab}, \alpha_1...\alpha_n$ |
| $R^*_{A}^{T_1...T_r} \rightarrow \rho^{T_1...T_r}$ | $\rho^{T_1...T_r} \rightarrow -(\rho^{T_1...T_r})' \xi - (2 - \frac{3}{2}) \xi' \rho^{T_1...T_r}$ | $\rho^{T_1...T_r}_{ab}, \alpha_1...\alpha_r$ |

Thus for $N$ supersymmetries there is one rank two tensor $D_{ab}$, $N$ spin-$\frac{3}{2}$ fields $\psi^1$, a spin-1 covariant tensor $A^{RS}$ that serves as the $N(N-1)/2$ SO($N$) gauge potentials associated with the supersymmetries, and $N (2^N - N - 1)$ fields for both the $\omega^{V_1...V_p}$ fields and $\rho^{T_1...T_p}$ sectors.

We would like to capture these component fields into superfields and write an action in terms of these superfields. One can see from the above table that at least two distinct superfields will be needed to absorb the field content. These two superfields will constitute a diffeomorphism sector and a gauge sector for the SO($N$) gauge symmetry.

### 3.2 The Chiral Diffeomorphism Superfield

Instead of using component fields we would like a superfield formulation of the algebra that can be used to construct an $N$-extended version of the affirmative action found in [3]. Just as in the above reference we will write the action using tensor notation so that future extension to higher dimensions and non-chirality can follow easily. Our focus will be on the two dimensional models, so we will assume in what follows that the Grassmann variables are Majorana and chiral. Therefore in this section we will display a fermionic index, "$a$" say, with the understanding that it is a one dimensional index.

Recall that up to the central extension the Virasoro algebra (or Witt algebra) may be realized as the one dimensional reduction of the Lie algebra of vector fields. Consider the vector fields $\xi^a$ and $\eta^a$. We know that the Lie derivative of $\eta^a$ with respect to $\xi^a$ is given by

$$\mathcal{L}_\xi \eta^a = -\xi^b \partial_b \eta^a + \eta^b \partial_b \xi^a = (\xi \circ \eta)^a,$$  \hspace{1cm} (12)

and further that

$$[\mathcal{L}_\xi, \mathcal{L}_\eta] = \mathcal{L}_{\xi \circ \eta}.$$  \hspace{1cm} (13)
Now consider the superfields $F = (\xi^n, \chi^{J; \beta})$ and $G = (\eta^n, \psi^{K; \alpha})$. We are assuming that we have Majorana fermions. Define the derivative operator $\mathcal{D}_{I; \mu}$ through

$$\{\mathcal{D}_{I; \mu}, \mathcal{D}_{J; \nu}\} = -i \delta_{IJ} \gamma_{\mu \nu}^n \frac{\partial}{\partial z^n}. \quad (14)$$

This implies that

$$\mathcal{D}_{I; \mu} = \frac{\partial}{\partial z_n} - \delta_{IJ} \frac{i}{2} \chi^{J; \nu} \gamma_{\mu \nu}^m \frac{\partial}{\partial z^n}. \quad (15)$$

The super Virasoro algebra contains both a diffeomorphism as well as a supersymmetry transformation with $\mathcal{D}_{I; \mu}$. We can construct vector fields

$$F = \xi^n \frac{\partial}{\partial z^n} + \frac{1}{2} \chi^{J; \beta} \mathcal{D}_{J; \beta}, \quad (16)$$

and

$$G = \eta^n \frac{\partial}{\partial z^n} + \frac{1}{2} \psi^{J; \beta} \mathcal{D}_{J; \beta}. \quad (17)$$

The commutator of $F$ and $G$ is

$$[F, G] = [\xi^n \frac{\partial}{\partial z^n} + \frac{1}{2} \chi^{J; \beta} \mathcal{D}_{J; \beta} , \eta^n \frac{\partial}{\partial z^n} + \frac{1}{2} \psi^{K; \alpha} \mathcal{D}_{I; \alpha}]$$

$$= \{\xi^n \frac{\partial}{\partial z^n} \eta^n - \eta^n \frac{\partial}{\partial z^n} \xi^n + \frac{1}{2} \chi^{J; \beta} (\mathcal{D}_{J; \beta} \eta^n) - \frac{1}{2} \psi^{K; \alpha} (\mathcal{D}_{K; \alpha} \xi^n)$$

$$+ i (\frac{1}{2} \chi^{J; \beta}) (\frac{1}{2} \psi^{K; \alpha}) \delta^{JK} \gamma^{n}_{\alpha \beta} \frac{\partial}{\partial z^n}$$

$$+ \{ - \frac{1}{4} (\chi^{J; \beta} (\mathcal{D}_{J; \beta} \psi^{K; \alpha}) + \psi^{J; \beta} (\mathcal{D}_{J; \beta} \chi^{K; \alpha})$$

$$+ \frac{1}{2} (\xi^n \frac{\partial}{\partial z^n} \psi^{K; \alpha} - \eta^n \frac{\partial}{\partial z^n} \chi^{K; \alpha}) \} \mathcal{D}_{K; \alpha} \} \mathcal{D}_{K; \alpha}. \quad (18)$$

With this in place we can now naturally extend the vector fields $\xi^n$ to $N$ supersymmetries. Let

$$\xi^{I_1 \ldots I_m; \alpha_1 \ldots \alpha_m} \equiv \xi^{I_1; \alpha_1} \ldots \xi^{I_m; \alpha_m}, \quad (19)$$

then an $N$-extension of the vector field $\xi^n$ is given by the vector superfield $F^n$, where

$$F^n = \xi^n + \lambda^n_{I_1; \alpha_1} \xi^{I_1; \alpha_1} + \rho^n_{I_1 I_2; \alpha_1 \alpha_2} \xi^{I_1 I_2; \alpha_1 \alpha_2} + \ldots + \gamma^n_{I_1 \ldots I_N; \alpha_1 \ldots \alpha_N} \xi^{I_1 \ldots I_N; \alpha_1 \ldots \alpha_N} \quad (20)$$

This superfield contains the superpartner that is used to perform the supersymmetric translation. With this we can write the superdiffeomorphism vector field that is analogous to the one used in Eq. (18) as

$$F = F^n \frac{\partial}{\partial z^n} + \delta^{AB} (\mathcal{D}_{A; \alpha} F^n) \gamma^{\alpha \beta}_n \mathcal{D}_{B; \beta}. \quad (21)$$

The $F^n$ superfield contains the field content of the $L_A, G^I_A$, and $R^T_{A \ldots A}$ generators seen in Table 1. Note that $\chi^{I; \alpha} = \delta^{I} I_1 \gamma^{\alpha \beta}_n \lambda^n_{I_1; \alpha_1}$.
However the centrally extended contributions demand that we also consider the prepotential $F^n_{I_1 \cdots I_N; \alpha_1 \cdots \alpha_N}$ given by
\[
F^n_{I_1 \cdots I_N; \alpha_1 \cdots \alpha_N} = \xi^n_{I_1 \cdots I_N; \alpha_1 \cdots \alpha_N} + \chi^n_{I_1 \cdots I_N; \alpha_1 \cdots \alpha_N} + r^n_{I_1 \cdots I_N; \alpha_1 \cdots \alpha_N} + \cdots + r^n_{I_1 \cdots I_N; \alpha_1 \cdots \alpha_N},
\]
where the “0” subscripted fields are the prepotentials defining the elements of $F^n$ through
\[
F^n = D_{I_1; \alpha_1} \cdots D_{I_N; \alpha_N} F^n_{I_1 \cdots I_N; \alpha_1 \cdots \alpha_N}.
\]
Then one may write down the commutation relations for centrally extended elements as
\[
[[ (F, a), (G, b)] = ([F, G], [G, F])
\]
where the two cocycle $[[F, G], [G, F]]$ is defined as
\[
[[F, G], [G, F]] = \frac{\delta \alpha}{2 \pi} \int d \zeta d \zeta \xi^{B_1 \cdots I_N; \alpha_1 \cdots \alpha_N}((F^n)^m G^m_{I_1 \cdots I_N; \alpha_1 \cdots \alpha_N} - (G^n)^m F^m_{I_1 \cdots I_N; \alpha_1 \cdots \alpha_N}).
\]
In one dimension the tensor indices $m$ and $n$ are not relevant but simply let us keep track of the transformation properties of the fields which will be useful in higher dimensions. The integrand transforms as a scalar density in one dimension. This preserves the $N = 0$ form of the two cocycle used for the central extension of the Virasoro algebra.

To continue we need the dual elements of these vector fields. This implies that there exists a bilinear two form $<\ast | \ast >$ that is invariant under diffeomorphisms. Consider the dual of $F^n$ for $N$ supersymmetries. In one dimension one has a pairing that can be represented tensorially as the integral
\[
< (F, a) | (B, \hat{b}) > = \int d \zeta d \zeta \xi_{\alpha_1; I_1} \cdots d \zeta_{\alpha_N; I_N} F^n B^n_{I_1 \cdots I_N; \alpha_1 \cdots \alpha_N} + \hat{a} \hat{b}.
\]
In one dimension $F^n$ is considered a rank one contravariant vector field, while $B^n_{I_1 \cdots I_N; \alpha_1 \cdots \alpha_N}$ is a quadratic differential with density $-\frac{N}{2}$. In higher dimensions we will take advantage of the fact that $\sqrt{g} F^{a b}$ transforms like $F^a$ in one coordinate. The invariant integral in $k$ dimensions will be written as
\[
< B | F > = \int d \zeta d \zeta \xi_{\alpha_1; I_1} \cdots d \zeta_{\alpha_N; I_N} \sqrt{g} F^{a b} B^n_{I_1 \cdots I_N; \alpha_1 \cdots \alpha_N}.
\]
For now, we use the one dimensional commutators and pairing to write the transformation law for the coadjoint elements as
\[
\delta_F B^n_{I_1 \cdots I_N; \alpha_1 \cdots \alpha_N} = -F^m \partial_m B^n_{I_1 \cdots I_N; \alpha_1 \cdots \alpha_N} - B^n_{I_1 \cdots I_N; \alpha_1 \cdots \alpha_N} \partial_m F^m + \hat{b} \nabla_m B^n_{I_1 \cdots I_N; \alpha_1 \cdots \alpha_N}.
\]
The transformation shows that $B^n_{I_1 \cdots I_N; \alpha_1 \cdots \alpha_N}$ transforms as a rank two tensor due to its $n$ and $p$ indices and for each contravariant fermion index we have assigned a density of weight $-\frac{1}{2}$. 


In terms of the fields in Table 2, the field $B_{n p}^{I_1 \cdots I_N; \alpha_1 \cdots \alpha_N}$ has a decomposition of

$$B_{n p}^{I_1 \cdots I_N; \alpha_1 \cdots \alpha_N} = D_{n p} \zeta^{I_1 \cdots I_N; \alpha_1 \cdots \alpha_N} + \psi_{[n p}^{[I_1; \alpha_1} \zeta^{I_2 \cdots I_N; \alpha_2 \cdots \alpha_N]} + \rho_{[n p}^{I_1 I_2; \alpha_1 \alpha_2} \zeta^{I_3 \cdots I_N; \alpha_3 \cdots \alpha_N]} + \cdots + \rho_{n p}^{I_1 \cdots I_N; \alpha_1 \cdots \alpha_N},$$

with the $\frac{3}{2}$ spin field $\psi_{[a \alpha}^{I}$ in Table 2 satisfying,

$$\psi_{a \alpha}^{I} = \psi_{n p}^{I; \alpha_1} \gamma_{a \alpha_1}^p .$$

In the construction of the actions that follow we will need to define $F^n$ in terms of the superfield $B_{n p}^{I_1 \cdots I_N; \alpha_1 \cdots \alpha_N}$. For the moment let us ignore the SO($N$) gauge symmetry that has been induced by the $T^{IJ}_M$ generators. We will use a tilde as a reminder of this. Then we may write

$$\tilde{B}_{n p} = \mathcal{D}_{I_1; \alpha_1} \cdots \mathcal{D}_{I_N; \alpha_N} B_{n p}^{I_1 \cdots I_N; \alpha_1 \cdots \alpha_N}.$$  

Since $F^n$ will generate time-independent coordinate transformations, we assign $F^n = \tilde{B}_0^n$. Before we put these ingredients into an action we must deal with the SO($N$) symmetry.

### 3.3 The SO($N$) Gauge Superfield

In the previous section we ignored the SO($N$) gauge field that has become manifest due to the fact that the $T^{IJ}$ generators transform as scalar fields while their dual elements transform as vector fields under diffeomorphisms. Again we would like to cast the algebra into the tidy language of superfields. Consider the superfields $\Lambda^{J; \alpha}$ that enjoy the $\zeta$ expansion in terms of the fields found in Table 1,

$$\Lambda^{J; \alpha} = w^{I; \alpha} + t^{I; \alpha} \zeta^{J; \alpha} + w_{I_1; \alpha_1} r^{I_1; \alpha_1 \alpha} + \cdots + w_{I_1 \cdots I_{N-1}; \alpha_1 \cdots \alpha_{N-1}} \zeta^{I_1 \cdots I_{N-1}; J; \alpha_1 \cdots \alpha_{N-1}, \alpha}. $$

Here $w^{I; \alpha} = w_\beta^{I; \alpha} \gamma_{\alpha}^\beta$, while $t^{I; \alpha}$ is the anti-symmetric field found in Table 1. A generic element is then

$$\Lambda = \Lambda^{J; \alpha} \frac{\partial}{\partial \zeta^{J; \alpha}}.$$  

We introduce the algebraic prepotential $\Lambda^{I_1 \cdots I_N; \alpha_1 \cdots \alpha_N}$ through

$$\Lambda^{J; \alpha} = \mathcal{D}_{I_1; \alpha_1} \cdots \mathcal{D}_{I_N; \alpha_N} \Lambda^{I; I_1 \cdots I_N; \alpha_1 \cdots \alpha_N}.$$  

Then the centrally extended commutation relations can be written as

$$[[[\Lambda, \lambda], (\Omega, \mu)], \Lambda, \Omega]] = [[\Lambda, \Omega], \Lambda, {}],$$

where the one dimensional two cocycle for the algebra is

$$<< \Lambda, \Omega >> = k \int d^2 \zeta_{I_1 \cdots I_N; \alpha_1 \cdots \alpha_N} \left( \mathcal{D}_{J; \alpha} \Lambda^{I; \alpha} \mathcal{D}_{N}; \beta \Omega_{M[1 \cdots I_N]; \beta [\alpha_1 \cdots \alpha_N]} - \mathcal{D}_{J; \alpha} \Omega^{I; \alpha} \mathcal{D}_{N; \beta} \Lambda^{M[1 \cdots I_N]; \beta [\alpha_1 \cdots \alpha_N]} \right) \delta^{I N}_{J M}.$$
The $\delta_{I,M}^{J,N}$ allows us to show that the SO($N$) invariant trace and is defined as

$$\delta_{I,M}^{J,N} = \delta_{I,M} \delta^{J,N} - \delta_{J,N} \delta^{I,M}.$$  \hspace{1cm} (37)

Again notice that the derivative in one dimension has the affect of changing a scalar into a scalar density. Thus the integral is invariant under one dimensional diffeomorphisms.

Again we would like to extract a dimension independent description of the dual elements of the SO($N$) adjoint. Here we use the fact that in one dimension $\Lambda^{I;\alpha}$ transforms the same way as $\sqrt{g} \Lambda^{I;\alpha p}$. This device allows us to find a dimension independent description of the dual of the algebra. The pairing between a centrally extended element, say $(\Lambda^{I;\alpha}, \alpha)$, and a coadjoint element is then

$$< \langle \Lambda, \alpha | (A, \beta) \rangle = \int dz^k d\zeta^{I_1 \cdots I_N;\alpha_1 \cdots \alpha_N} \sqrt{g} \Lambda^{I;\alpha p} A^{I_1 \cdots I_N;\alpha_1 \cdots \alpha_N} + \alpha \beta. \hspace{1cm} (38)$$

The dual element has the $\zeta$ decomposition

$$A^{I_1 \cdots I_N;\alpha_1 \cdots \alpha_N} = \Lambda^{I;\alpha p} \zeta^{I_1 \cdots I_N;\alpha_1 \cdots \alpha_N} + \omega^{I_1 \cdots I_N;\alpha_1 \cdots \alpha_N} + \cdots + \omega^{I_1 \cdots I_N;\alpha_1 \cdots \alpha_N}. \hspace{1cm} (39)$$

From the algebra and two cocycle we find that the transformation of the dual element with respect to $\Omega^{I;\alpha}$ is

$$\delta_{\Omega} A^{I_1 \cdots I_N;\alpha_1 \cdots \alpha_N} = -\Omega^{J;\beta} \mathcal{D}_{J;\beta} A^{I_1 \cdots I_N;\alpha_1 \cdots \alpha_N} + (\mathcal{D}_{I;\alpha} \Omega^{J;\beta}) A^{I_1 \cdots I_N;\alpha_1 \cdots \alpha_N}$$

$$- (\mathcal{D}_{J;\beta} \Omega^{I;\alpha}) A^{I_1 \cdots I_N;\alpha_1 \cdots \alpha_N}$$

$$+ \frac{\partial k \mathcal{D}_{J;\beta} \mathcal{D}_{N;\alpha} \nabla_M \Omega^{I_1 \cdots I_N;\alpha_1 \cdots \alpha_N} \delta^{J,N} \delta_{I,M}}{\delta_{I,M}^{J,N}}. \hspace{1cm} (40)$$

This transformation law is the Lie derivative on the SO($N$) group manifold where only the “T” index is recognized, since the other SO($N$) indices were contracted with the SO($N$) volume form. Notice that the coadjoint element transforms as a tensor density of weight 1 due to the penultimate term in the transformation law.

The gauge field can be extracted from the superpotential by writing

$$A^{I;\alpha} = \mathcal{D}_{I_2;\alpha_2} \cdots \mathcal{D}_{N;\alpha_N} A^{I_1 \cdots I_N;\alpha_1 \cdots \alpha_N}. \hspace{1cm} (41)$$

We can then build an SO($N$) covariant derivative operator so that the covariant derivative on a vector living in the left regular representation, say $B^L_q$, is

$$\nabla_p B^M_q = \partial_p B^M_q - \Gamma^r_{pq} B^M_r + A^{I;\alpha} B^L_q (\delta^I_L \delta_j^M - \delta_{JL} \delta_j^M) \hspace{1cm} (42)$$

Then Eq.(32) may be written as

$$B_{np} = \nabla_{I_1;\alpha_1} \cdots \nabla_{I_N;\alpha_N} B^I_{np;\alpha_1 \cdots \alpha_N}. \hspace{1cm} (43)$$
4 The $N$-Extended Affirmative Action

We are now in a position to construct the affirmative action. In [3] the principles used to construct the $N = 1$ action were distilled into a short procedure. In the $N = 0$ case, the action is written as

$$
S_{N=0} = - \int d^m x \sqrt{g} \left( X^{lmr} D^a_r X_{mla} + 2X^{lmr} D_{la} X^a_{rm} \right)
$$

(44)

$$
- \int d^m x \sqrt{g} \left( \frac{1}{4} X^{ab}_b \nabla_l \nabla_m X^{lm}_a + \frac{\beta}{2} X^{bga} X_{bga} \right)
$$

where $D_{mn}$ is the $N = 0$ remnant of the superfield $B^{I_1 \cdots I_N; \alpha_1 \cdots \alpha_N}$ and $X^{lmr} = \nabla^r D_{mn}$.

To the extent possible, we will use the $N = 0$ action as a prototype to building the $N$-extended affirmative action. The construction goes in the following stages:

1. Contract the conjugate momentum of the field with the variation of the field:

(a) $N = 0$

$$
\mathcal{L}_0 = X^{ij0} (\xi^i \partial_i D_{ij} + D_{ij} \partial_i \xi^j + D_{ij} \partial_j \xi^i)
$$

(45)

(b) $N$-Extended

The physical fields are contained in the superfield $B^{I_1 \cdots I_N; \alpha_1 \cdots \alpha_N}$ and the variation is with respect to the superfield $F^m$. We use the derivative operator $\nabla_p$ from Eq.(42) which is covariant with respect to the SO($N$) symmetry and general coordinate transformations. The $B^{I_1 \cdots I_N; \alpha_1 \cdots \alpha_N}$ field lives in the direct product space of $N$ left regular representations. In order to recover the $N = 0$ results we write

$$
\mathcal{L}_{0;h} = \nabla_0 B^{(f)}_{pq} \delta B^{pq}_{\{f\}}
$$

(46)

where

$$
B^{(f)}_{pq} \equiv B^{I_1 \cdots I_N; \alpha_1 \cdots \alpha_N}_{pq}
$$

(47)

and

$$
\delta B^{I_1 \cdots I_N; \alpha_1 \cdots \alpha_N}_{np} = - F^m \nabla_m B^{I_1 \cdots I_N; \alpha_1 \cdots \alpha_N}_{np} - B^{I_1 \cdots I_N; \alpha_1 \cdots \alpha_N}_{np} \nabla_n F^m

- B^{I_1 \cdots I_N; \alpha_1 \cdots \alpha_N}_{nm} \nabla_p F^m + \frac{\beta}{2} (\nabla_m F^m) B^{I_1 \cdots I_N; \alpha_1 \cdots \alpha_N}_{np} + \hat{bc} \nabla_n \nabla_p \nabla_m F^{mI_1 \cdots I_N; \alpha_1 \cdots \alpha_N}_{np}.
$$

(48)

2. Replace the fields $F^m (\xi^i)$ with a space-time component of the field, $B_0^m (D^0_a)$:

(a) $N = 0$

$$
\mathcal{L}_{1st} = X^{LM0}(D^A_0 \nabla_A D_{LM} + D_{AM} \nabla_L D^A_0 + D_{LA} \nabla_M D^A_0)
$$

(49)
where the vector potential is defined through Eqs.(39, 41). The SO(N) may be written as

\[ B_{np}^{\perp} = -B_{np}^{m} \nabla_{m} B_{qp}^{m} + \frac{\hat{b}}{2} (\nabla_{m} B_{pq}^{m}) B_{np}^{m} \] (50)

3. Extend the time directions to covariant directions:

(a) \( N = 0 \)

\[ \mathcal{L}_{\text{int}} = X^{LMR} (D_{R}^{A} \nabla_{A} D_{LM} + D_{AM} \nabla_{L} D_{R}^{A} + D_{LA} \nabla_{M} D_{R}^{A}) \] . (51)

(b) \( N \)-Extended

\[ \mathcal{L}_{\text{int}} = \nabla^{q} B_{\{q\}}^{np} ( - B_{q}^{m} \nabla_{m} B_{np}^{I_{N} \cdots I_{N}; \alpha_{1} \cdots \alpha_{N}} - B_{np}^{m} \nabla_{m} B_{q}^{m} \nabla_{p} B_{0}^{m} \nabla_{n} B_{0}^{m} \nabla_{I_{N} \cdots I_{N}; \alpha_{1} \cdots \alpha_{N}} + \hat{b} c \nabla_{m} \nabla_{n} B_{np}^{m} B_{q}^{m} B_{0}^{m} \nabla_{p} B_{0}^{m} \nabla_{n} B_{0}^{m} \nabla_{I_{N} \cdots I_{N}; \alpha_{1} \cdots \alpha_{N}} ) \]

(52)

At this point one may add the symplectic part of the Lagrangian to the above and write the \( N \)-extended superdiffeomorphism contribution to the action as

\[ S_{N-\text{diff}} = \int d^{2} x d^{a} \zeta \sqrt{g} \left\{ \frac{1}{4} \nabla_{a} B_{\{q\}}^{I_{I}} B_{\{q\}}^{pqa} + \frac{\beta}{2} \nabla_{a} B_{\{q\}}^{I_{I}} \nabla_{a} B_{\{q\}}^{pqa} \right\} \] (53)

where \( q = \hat{b} c \), the “charge” of the superdiffeomorphism field times the central extension.

In a similar way the SO(N) gauge field has a contribution to the action given by

\[ S_{\text{SO(N)}} = k \beta \int d^{2} x d^{a} \zeta \sqrt{g} (F_{\{q\}}^{LM}) \, (F_{\{I\}}^{CD}) (\delta_{LC} \delta_{MD} - \delta_{LD} \delta_{MC}) \] (54)

Here

\[ F_{\{q\}}^{LM} = \partial_{[q} A_{p]}^{LM} - \frac{1}{k \beta} A_{[q}^{BA} A_{p]}^{I N} f_{AB, I N}^{LM} \] (55)

where the vector potential is defined through Eqs.\([3, 11]\). The SO(N) structure constants \( f_{AB, I N, LM} \) may be written as

\[ f_{AB, I J, LM} = \delta^{AI} \delta^{BL} \delta^{JM} + \delta^{AJ} \delta^{BL} \delta^{IM} - \delta^{JM} \delta^{A} \delta^{B} \delta^{L} \delta^{M} \], (56)

There is one more contribution to this action that has to be accounted for and that is the interaction between the superdiffeomorphism field and the gauge field. In \([4]\) it was shown that the coadjoint elements also transform under gauge transformation. In terms of the \( \mathcal{GR} \) generators one has, for example, that

\[ \delta_{t_{IJ}} D = \frac{1}{2} (t_{IJ})^{t} A^{LM} (\delta_{I L} \delta_{J M} - \delta_{I M} \delta_{L J}) \] (57)
as well as the corresponding relationship that

\[ \delta_\xi A^{IJ} = -\xi (A^{IJ})' - (\xi)' A^{IJ}. \]  

We have already seen this second transformation from Table 2, which was used to show that \( A^{IJ} \) transforms as a vector under time-independent coordinate transformations. The first relationship is dual to this coordinate transformation. Such dual relationships have been discussed at length in [3] and [10] and constitute new interaction Lagrangians between the gauge fields and the diffeomorphism fields. The vanishing of the transformation laws determines the condition for the isotropy algebra. This condition is realized as a constraint equation when the adjoint elements are replaced with the conjugate momenta. This follows since the conjugate variables transform the same way as the adjoint elements under time-independent transformations. Therefore a new interaction term between the gauge sector and the superdiffeomorphism sector must be present.

In order to illustrate this we recall the result of [3]. There one considers the interactions due to an affine Lie algebra interacting with the Virasoro algebra. The coadjoint elements transform under simultaneous coordinate and gauge transformations as

\[ \delta \mathcal{F} B = (\xi (\theta), \Lambda (\theta), a) \ast (D (\theta), A (\theta), \mu) = (\delta D (\theta), \delta A (\theta), 0). \]  

Here \( \mathcal{F} = (\xi (\theta), \Lambda (\theta), a) \) is an arbitrary adjoint element, while \( B = (D (\theta), A (\theta), \mu) \) is the coadjoint element. The components of the variation are

\[ \delta D = 2\xi' D + D' \xi + \frac{c}{24\pi} \xi''' - \text{Tr} (AA'), \]  

and

\[ \delta A = A' \xi + \xi' A - [\Lambda A - A \Lambda] + k \mu \Lambda'. \]  

This leads to the interaction Lagrangian,

\[ \sqrt{g} F^{al}(D^a, \partial \tilde{A}_l + \tilde{A}_a \partial D^a - \partial_i (D^a, \tilde{A}_a)), \]  

where

\[ \tilde{A}_m = A_m - V^{-1} \partial_m V. \]

The \( V \) field corresponds to the WZNW field that maintains gauge invariance in the interaction of the gauge field with the diffeomorphism field. In fact the vacuum expectation value of \( V \) in the two dimensional case in [11] is the identity. We then also require a symplectic contribution to the action for this field and write it as

\[ S_V = m_A^2 \int \sqrt{g} (V^{-1} \partial_m V - A_m)(V^{-1} \partial_n V - A_n)(g^{mn} + D^{mn})d^n x. \]  

The coupling constant \( m_A^2 \) is to suggest a dynamical origin of a mass for the gauge field about the vacuum expectation value for \( V \).
Using the results of [4] we may write the corresponding interaction term between the superdiffeomorphism superfield and the SO($N$) super gauge field as

$$S_{int} = \frac{kb}{q^3} \int d^2 x d^a \zeta \sqrt{g} \left( F^{ab}_{(1)} \right)^{LM} \left( -B^m_a \partial_n \tilde{A}^{AB(1)} - \Gamma^p_{am} \tilde{A}^{AB(1)} \right) \left( \delta_{LA} \delta_{MB} - \delta_{LB} \delta_{MA} \right)$$

$$- \frac{kb}{q^3} \int d^2 x d^a \zeta \sqrt{g} \left( F^{ab}_{(1)} \right)^{LM} \left( \tilde{A}^{AB(1)} \nabla_b B^m_a \right) \left( \delta_{LA} \delta_{MB} - \delta_{LB} \delta_{MA} \right)$$

$$+ \frac{kb}{q^3} \int d^2 x d^a \zeta \sqrt{g} \left( F^{ab}_{(1)} \right)^{LM} \left( \frac{N}{2} \left( \nabla_a B^m_a \right) \tilde{A}^{AB(1)} \right)$$

$$+ m^2_A \int d^2 x d^a \zeta \sqrt{g} \left( \tilde{A}^{AB(1)} \right)^a \left( \delta^{AC} \delta^{BD} - \delta^{AD} \delta^{BC} \right).$$

The superfield $\tilde{A}^{IJ}_{p} = A^{IJ}_{p} - V^{IJ}_{p}$. It is the analog of $\tilde{A}^{I}_{p}$ in the $N = 0$ case mentioned above. The corresponding WZNW field is defined so that $V^{IJ}_{p} = \mathcal{D}_{\alpha_1} \cdots \mathcal{D}_{\alpha_N} V^{IJ_{\alpha_1 \cdots \alpha_N}}_p$ is pure gauge. The full chiral $N$-extended affirmative action is then

$$S_{N\chi AA} = \int d^2 x d^a \zeta \sqrt{g} \left( \frac{1}{q} \nabla_a B^a_{pq} \right) \left( B^{pq}_{(1)} \right) + \frac{kb}{q^3} \int d^2 x d^a \zeta \sqrt{g} \left( F^{ab}_{(1)} \right)^{LM} \left( -B^m_a \partial_n \tilde{A}^{AB(1)} - \Gamma^p_{am} \tilde{A}^{AB(1)} \right) \left( \delta_{LA} \delta_{MB} - \delta_{LB} \delta_{MA} \right)$$

$$+ \frac{kb}{q^3} \int d^2 x d^a \zeta \sqrt{g} \left( F^{ab}_{(1)} \right)^{LM} \left( \tilde{A}^{AB(1)} \nabla_b B^m_a \right) \left( \delta_{LA} \delta_{MB} - \delta_{LB} \delta_{MA} \right)$$

$$+ \frac{kb}{q^3} \int d^2 x d^a \zeta \sqrt{g} \left( F^{ab}_{(1)} \right)^{LM} \left( \frac{N}{2} \left( \nabla_a B^m_a \right) \tilde{A}^{AB(1)} \right)$$

$$+ m^2_A \int d^2 x d^a \zeta \sqrt{g} \left( \tilde{A}^{AB(1)} \right)^a \left( \delta^{AC} \delta^{BD} - \delta^{AD} \delta^{BC} \right).$$

5 Conclusions

We have constructed an action for two dimensional chiral supergravity interacting with the $N$-extended Virasoro group using methods developed in Refs.[1, 2]. The $N$-extended theory requires two superfields to carry the physical fields and auxiliary fields through a supersymmetric explicit construction. These two superfields correspond to the multiplet containing the gravitational fields related to the dual of the $N$-extended super Virasoro algebra [11] and another superfield to support the gauged SO($N$) vector potential that arises from gauging the $N$ supersymmetry generators. One finds that a new superfield, $V^{IJ}_{p}$, becomes manifest in order to maintain the SO($N$) gauge symmetry. In later work we will examine whether supersymmetry can be spontaneously broken when this field takes a vacuum expectation value.

The appearance of the superfields $B^{I_{\alpha_1 \cdots \alpha_N}}_{\alpha p}$ and $A^{I_{\alpha_2 \cdots \alpha N}}_{I_{\alpha_1 \cdots \alpha_N}}$ are very suggestive that in a completely geometrical approach to the description of 2D, $(N, 0)$ supergravity, these quantities could play the role of supergravity prepotentials. Should such an interpretation be viable, then the representation theory of the supergravity prepotentials in the context of 2D, $(N, 0)$ supergravity will have been shown to be a direct consequence of the geometrical arbitrary $N$ and model-independent realization of the super-Virasoro algebra.
Thus a future challenge of this line of investigation is to show that this algebra approach leads to the unconstrained supergravity variables required for the curved superspace. If we anticipate success in these efforts, then a pressing problem becomes their extension first to non-chiral 2D, \((N, N)\) supergravity theories and beyond to even more complicated higher dimensional theories. If these off-shell theories can be interpreted as the toric reduction of higher dimensional theories, then we will have a proof that all unconstrained supergravity theories are representations of the super-Virasoro algebra.
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