Optimizing Fund Allocation for Game-based Verifiable Computation Outsourcing

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Abstract—This paper considers the setting where a cloud server services a static set or a dynamic sequence of tasks submitted by multiple clients. Every client wishes to assure honest execution of tasks by additionally employing a trusted third party (TTP) to re-compute the tasks with a certain probability. The cloud server makes a deposit for each task it takes, each client allocates a budget (including the wage for the server and the cost for possibly hiring TTP) for each task submitted, and every party has its limited fund for either deposits or task budgets. We study how to allocate the funds optimally to achieve the three-fold goals: a rational cloud server honestly computes each task; the server’s wage is maximized; the overall delay for task verification is minimized. We apply game theory to formulate the optimization problems, and develop the optimal or heuristic solutions for three application scenarios. For each of the solutions, we analyze it through either rigorous proofs or extensive simulations. To the best of our knowledge, this is the first work on optimizing fund allocation for verifiable outsourcing of computation in the setting of one server and multiple clients, based on game theory.

Index Terms—Verifiable Computation Outsourcing, Game theory, Cloud Computing, Fund Allocation, Optimization.

1 INTRODUCTION

The popularity of cloud services and decentralized platforms promote the development and prosperity of computation outsourcing. Clients export heavy computational tasks (such as data mining and machine learning) to executors who have available and intensive computational resources to handle them. The executors could be cloud service providers or nodes in the decentralized network. Executors aim to efficiently utilize the available computational resources and achieve optimal benefits from the computation. Clients desire to outsource tasks without paying more than a certain predefined budget and get correct computation results with as short delay as possible.

In this paper, we focus on computation outsourcing in the cloud computing environment. Specifically, we consider a system composed of a cloud service provider (abbreviated as server hereafter) and multiple cloud clients (abbreviated as clients hereafter). Each client submits to the server either a static set or a dynamic sequence of tasks. The former corresponds to the application scenarios where the client processes tasks in batch while the latter the scenarios where tasks are generated and processed in real time.

To assure that the server returns correct computation results, certain verifiable outsourced computation mechanisms should be in place. A large variety of schemes have been proposed in the literature to verify outsourced computation. They can rely on cryptography [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], trusted hardware [13], [14], [15], [16], redundant system (that includes at least one trusted server) [17], [18], game theory [19], [20], [21], [22], [23], [24], [25], [26], or combinations of the above. As briefly surveyed in Section 2, the approaches purely relying on cryptography or trusted hardware usually have high costs and/or low performance/scalability, while the game-based approaches have gained more popularity for their lower costs due to the practical assumption of economically-rational participants. Hence, we also apply game theory in our study.

We adopt the basic model that each client outsources her tasks to only one server (without redundancy) but with probabilistic auditing. Additionally, for each task, the server is required to make a deposit, which can be taken by the client when the server is found misbehaving; each client should prepare a budget that includes the wage paid to the server (if the server is not found dishonest) and the cost for hiring a trusted third party (TTP) to check the result returned by the server (i.e., auditing). A relation among the deposit, wage and the auditing probability can be found such that, the server’s most beneficial strategy is to act honestly as long as the condition is satisfied.

It is natural to assume the cloud has a certain fund to spend as deposits for the tasks it takes. However, the fund is limited at a time and should be spent smartly so that the server can maximize its benefit, which we measure as the wage it can earn; it becomes more challenging when the server services a dynamic sequence of tasks, as it is unknown when new tasks will arrive and the sizes of the future tasks. For a client, it is also natural to assume she has some fund to spend on the tasks she outsources. The client’s fund is limited too, and thus should be smartly spent as well to maximize her benefit, which we measure as the overall delay that she has to experience when waiting for her tasks to complete. Here, the client’s spending strategy includes: first, how to distribute a given amount of fund to the tasks that are submitted simultaneously or within the same time window; second, for each of the tasks, how to further divide the assigned budget for paying the server’s wage and for hiring a TTP respectively. How can we smartly allocate the server and the clients’ funds to maximize their profits?
To the best of our knowledge, this is a question that has not been raised or answered in the literature. The focus of this paper is to formulate and solve this problem.

We formulate the problem in two steps. First, we formulate a per-task game-based outsourcing model. Specifically, the model enforces a security relation among three components, the server’s deposit, the server’s wage and the client’s auditing probability, where the latter two determines the client’s budget, to ensure the server’s best choice is to compute the task honestly. In addition, the model has the attractive property that, the wage and the auditing probability are not fixed but functions of the server’s deposit and the client’s budget; the larger is the deposit and/or the budget, the larger is the wage and the smaller is the auditing probability. Note that, larger wage and smaller auditing probability (and thus shorter delay) are desired by the server and the client, respectively. In the second step, we formulate the interactions between the server and the clients into an infinite extensive game with perfect information. Within this game, the server and the clients are the parties; the different ways to dividing the server’s fund into the tasks’ deposits and to dividing the clients’ funds into the tasks’ budgets are the parties’ actions; and the parties’ utilities are defined as functions of the actions.

We solve the problem in three steps. First, we develop an algorithm that finds the Nash equilibria of the game, which is also the optimal solution that maximizes the server’s wage meanwhile minimizes the client’s delay, for the special setting where there is only one client who submits a static set of tasks. Second, we develop an algorithm that finds the Nash equilibria and also the optimal solution for the more general setting where there are multiple clients each submitting a static set of tasks. Finally, we develop heuristic algorithms, which call the solution developed in the second step, to solve the problem when there are multiple clients each submitting a dynamic sequence of tasks. Rigorous proofs have been developed to show the optimality of the solutions developed in the first two steps. Extensive simulations have been conducted to evaluate the performance of the solutions developed in the third step.

In the rest of the paper, Section 2 surveys the related works. Section 3 introduces the system model and the per-task game-based outsourcing model. Section 4 defines the game between the server and the clients. Section 5, 6, and 7 develop the solutions in three steps. Finally, Section 8 concludes the paper and discusses the future work.

2 Related Works

There has been extensive research on verifying outsourced computation. We briefly summarize these efforts as follows.

Many schemes [11, 2, 3, 4, 5, 6, 7, 8, 9, 10] have been designed based on cryptographic primitives/algorithms. For example, Gennaro et al. [11] formalize the notion of verifiable computation; they utilize Yao’s garbled circuits to represent an outsourced function and homomorphic encryption to hide the circuits. Parno et al. [4] exploit quadratic programs to encode computation and generate a fixed-length proof independent of the input/output size. Geppetto [2] adapts multi-quadratic arithmetic programs to encode the computation, and constructs a commit-and-prove scheme to share data and prove execution of function. Various interactive proofs [8, 9, 10, 11, 12] have also been proposed. In general, the computational cost incurred by these schemes is high, which hampers their application in practice. As pointed out by Dong et al. [23] as well as Walfish and Blumberg [27], their computational overhead could be $10^3 - 10^5$ higher than the cost to compute the task. Hence, we do not adopt this approach in this paper.

Trusted Execution Environment (TEE) [13, 14, 15, 16] could be applied for verifiable computation. However, currently prevalent TEE such as Intel SGX is inappropriate for multi-threading and tasks requiring high demand of memory [16], which could make it inappropriate for enormous computation and validation for applications such as machine learning tasks [28] or heavy duty smart contracts [26] on blockchain. In this paper, we assume a TTP may use TEE; but we aim to minimize the employment of TTP and thus makes TEE an infrequently-used deterrence.

Alternatively, verifiable computation may be implemented based on redundancy and the assumption of at least one server being honest [17, 18]. Hence, when the servers return the same final result, the result can be immediately accepted; otherwise, the servers may be asked to provide intermediate computation results that can be compared to identify the correct computation. However, the assumption of existing at least one honest server could be impractical especially when only a small number of servers are employed.

Without the guarantee of trusted server employed, game theoretic approaches [19, 20, 21, 22, 23, 24, 25, 26] have been proposed to prevent rational servers from misbehaving. For example, Nix and Kantarcioglu [19] design two contracts for employing two servers. The games induced from the contracts have a Nash Equilibrium, where the servers behave honestly, as long as they cannot share information or collude with each other. Pham et al. [20] consider two settings, employing a single server or employing two servers that do not share information or collude with each other. They study how to coordinate the rewards, punishments, and auditing to make honest behavior the optimal strategy of the server(s).

Collusion among the servers has also been studied [21, 22, 23, 24]. In particular, Belenkiy et al. [21] study the proper fine-to-reward ratio to sabotage rational or malicious misbehavior and the impact of the probability that a hired server is misbehaving. Kupcu [23] generalizes the work by systematically studying the settings of hiring multiple servers with multiple types. Dong et al. [24] study the collusion and betrayal resulting from the interactions among the client-server and server-server contracts, and propose the design of smart contracts that make use of the blockchain for the participants to escort and distribute funds. In their recent works [25, 26], Liu and Zhang point out the necessity of auditing in hiring multiple possibly-colluding servers, even when an additional contract exists to encourage betrayal; they also consider that all hired servers may fully collude with each other, and design smart contracts for efficient execution of heavy-duty smart contracts.

In this paper, we also adopts game theory for verifiable computation. While the afore-discussed works consider the setting of one client outsourcing tasks to one server or multiple servers, we study the setting of multiple (cloud) clients outsourcing a static set or a dynamic sequence of
tasks to one (cloud) server. Similar to most of the afore-discussed works, the client needs to hire TTPs for probabilistic auditing, and we develop the security condition regarding the required relations among deposit, wage and auditing probability for each outsourced task. Significantly different from the state of the art, our work further studies the optimal distribution of the limited funds held by the server and the clients to maximize the system performance while meeting the security condition.

Game theory has also been applied in allocating cloud computation resources [29, 30, 31, 32, 33, 34]. For example, Wei et al. [31] propose to use the evolutionary game for cloud resource allocation. Kaewpuang et al. [32] formalize an optimization problem for allocating radio and computing resources for mobile devices and applies the core and Shapley value to coordinate the revenue allocation in the cooperative game. Pillai and Rao [33] study the problem of on-demand virtual machines allocation. Xu and Yu [34] propose a fairness-utilization game theoretical allocation for solving the problems of optimally allocating resources. While the cooperative game. Pillai and Rao [33] study the problem of on-demand virtual machines allocation. Xu and Yu [34] propose a fairness-utilization game theoretical allocation for solving the problems of optimally allocating resources. While aforementioned research aims to optimize the allocation of physical resources, our scheme is the first to allocate limited funds held by the cloud server and every client, respectively, to the tasks as deposits or task budgets, with the three-fold goals of (i) assuring honest computation of the tasks, (ii) maximizing the server’s overall wages, and (iii) minimizing the delay for verification.

3 System Architecture

In this section, we propose an architecture for game-based computation outsourcing to cloud server, which is facilitated by an underlying blockchain.

3.1 System Model

We consider a system consisting of a cloud service provider (called cloud server or server hereafter), m clients that need to outsource computation tasks to the server, and some trusted third parties (called TTPs hereafter) which the clients can resort to for verifying outsourced computation. Figure 1 illustrates the system architecture.

The server, denoted as S, is not completely trusted and its execution of the tasks outsourced by the clients may not always be correct. However, we assume the server is economically rational; that is, it always aims to maximize its profit and will not misbehave if that would cause penalty. As to be elaborated in Section 3.2 we introduce a game-based approach to guarantee that the server honestly executes the outsourced tasks. We assume that the server is willing to use a certain amount of fund as deposit to assure its client of its honest behavior.

We denote the m clients as C1,...,Cm. The tasks outsourced by each client Cj are denoted as tij, for j = 1,...,nj, where nj is the number of such tasks. Each task tij is associated with two costs denoted as cij and ̂cij, where cij is the server’s cost to execute the task and ̂cij is each TTP’s cost to execute the task. To simplify the presentation, we assume the execution time is proportional to the costs; that is, assuming k is a certain constant, the server’s execution time of the task is k · cij and each TTP’s execution time of the task is k · ̂cij. Each client Cj allocates a budget bij for each task tij, where bij ≥ cij so that the server is willing to take the task.

Each TTP can be hired at the price of ̂cij by a client to check if the server’s execution is correct via re-execution. A TTP can also be a cloud server that has a trusted execution environment (TEE) such as Intel SGX enclave.

Finally, we assume that the server, the clients and the TTPs can access a blockchain system so that no any centralized trusted authority is required.

3.2 Per-task Game-based Outsourcing Model

To ensure that the server honestly executes tasks, we adopt a game theoretic approach as follows. For each task tij, the server should make a deposit of dij and client Cj should promise a budget with a certain expected value of bij.

After the client outsources tij to the server, with a probability denoted as pij it also hires a TTP to execute the task. After the client has received a result of computation task from the server and/or the TTP, funds are distributed between the client and the server as follows: If no TTP is hired, or the results returned by the server and the hired TTP are the same, the client should pay a wage denoted as wij, where wij ≥ cij, to the server, and the server should also be returned with its deposit dij. If the results returned by the server and the TTP are different, deposit dij should be given to the client. Hence,

\[ b_{ij} = w_{ij} + p_{ij} \cdot \hat{c}_{ij}. \]  

(1)

Also, as stated in the following theorem, pij ≥ \( \frac{c_{ij}}{w_{ij} + d_{ij}} \) is the sufficient condition to deter the server from misbehaving and ensure it honestly executes task tij.

Theorem 1. As long as \( w_{ij} \geq c_{ij} \) and \( p_{ij} \geq \frac{c_{ij}}{w_{ij} + d_{ij}} \), an economically rational server must execute task \( t_{ij} \) honestly and submit a correct result to the client.
Proof. (sketch). If the server behaves dishonestly, it should lose its deposit with probability \( p_{i,j} \) while still receive a wage of \( w_{i,j} \) with probability \( 1 - p_{i,j} \); hence, its expected payoff is \( -p_{i,j} \cdot d_{i,j} + (1 - p_{i,j}) \cdot w_{i,j} \). If the server behaves honestly, it should receive a wage of \( w_{i,j} \) while pay the honest execution cost of \( c_{i,j} \); hence, its expected payoff is \( w_{i,j} - c_{i,j} \). For a rational server to behave honestly, it must hold that \( -p_{i,j} \cdot d_{i,j} + (1 - p_{i,j}) \cdot w_{i,j} \leq w_{i,j} - c_{i,j} \). Therefore, \( p_{i,j} \geq \frac{c_{i,j}}{w_{i,j} + d_{i,j}} \). □

4 Optimization Problem

To efficiently implement the proposed architecture, it is desired to optimize the allocation of the cloud server’s fund for deposits and the clients’ funds for tasks, to achieve the following dual goals: the server can maximize its wages earned from the clients; each clients can minimize the total time to verify the results of its tasks outsourced to the server.

4.1 Game between The Server and The Clients

We model the interactions between the server and the clients as an infinite extensive game with perfect information, denoted as \( G = (P, A, U) \).

- \( P = \{S, C_1, \ldots, C_m\} \): the set of players.
- \( A \): the set of actions taken by the players, including (i) all possibilities that each \( C_i \) can split its budget \( b_i \) to \( n_i \) tasks and (ii) all possibilities that \( S \) can split its deposit fund \( d \) to the \( n = \sum_{i=1}^{m} n_i \) tasks. Hence, the action set each \( C_i \) can take is denoted as \( A_{i} = \{(b_{1,i}, \ldots, b_{n_i,i}) \mid \sum_{j=1}^{n_i} b_{i,j} = b_i \} \), where \( b_i \) is \( C_i \)'s total budget for its tasks, and each action \((b_{1,i}, \ldots, b_{n_i,i})\) is one possible division of \( b_i \) to \( n_i \) tasks; the action set \( S \) can take is denoted as \( A_s = \{(d_{1,i}, \ldots, d_{m,n_m}) \mid \sum_{i=1}^{m} \sum_{j=1}^{n_i} d_{i,j} = d \} \), where \( d \) is the server’s fund for deposits, and each action \((d_{1,i}, \ldots, d_{m,n_m})\) is one possible division of \( d \) to \( n \) tasks.
- \( U = \{U_s, U_{c,1}, \ldots, U_{c,m}\} \): the players’ utility functions.

4.2 Constraints on Budgets and Deposit

According to the above definitions of the clients’ and the server’s actions, the following constraints are obvious:

\[
\sum_{j=1}^{n_i} b_{i,j} = b_i, \quad \forall i \in \{1, \ldots, m\}, \tag{2}
\]

and

\[
\sum_{i=1}^{m} \sum_{j=1}^{n_i} d_{i,j} = d. \tag{3}
\]

For each task \( t_{i,j} \), the server’s deposit for it should be at least \( \hat{c}_{i,j} \), to compensate client \( C_i \)'s cost for hiring a TTP if the server is found dishonest. Hence, we have the following constraint:

\[
d_{i,j} \geq \hat{c}_{i,j}. \tag{4}
\]

Regarding budget \( b_{i,j} \) for \( t_{i,j} \), according to Eq. \( 1 \), it includes wage \( w_{i,j} \) paid to the server for honest computation and the expected cost to hire TTP. First, based on Theorem 1 and that TTP should be hired as infrequently as possible, we set

\[
p_{i,j} = \frac{c_{i,j}}{w_{i,j} + d_{i,j}}. \tag{5}
\]

Second, \( w_{i,j} \geq c_{i,j} \) must hold to incentive the server. Because \( b_{i,j} = w_{i,j} + \frac{c_{i,j} \cdot \hat{c}_{i,j}}{w_{i,j} + d_{i,j}} \), which is from Equations \( 1 \) and \( 5 \), is an increasing function of \( w_{i,j} \), it holds that \( w_{i,j} \geq c_{i,j} \) is equivalent to \( b_{i,j} \geq c_{i,j} + \frac{c_{i,j} \cdot \hat{c}_{i,j}}{w_{i,j} + d_{i,j}} \). Further due to \( d_{i,j} \geq \hat{c}_{i,j} \), we set

\[
b_{i,j} \geq c_{i,j} + \frac{c_{i,j} \cdot \hat{c}_{i,j}}{c_{i,j} + \hat{c}_{i,j}}, \tag{6}
\]

which implies \( b_{i,j} \geq c_{i,j} + \frac{c_{i,j} \cdot \hat{c}_{i,j}}{c_{i,j} + \hat{c}_{i,j}} \) and \( w_{i,j} \geq c_{i,j} \).

4.3 Utility Functions

In the game, server \( S \) aims to maximize its total wage \( \sum_{i=1}^{m} \sum_{j=1}^{n_i} w_{i,j} \) under the constraints of \( 1 \), \( 2 \), \( 3 \), \( 4 \), and \( 6 \). From \( 1 \) and \( 5 \), it holds that \( b_{i,j} = w_{i,j} + \frac{c_{i,j} \cdot \hat{c}_{i,j}}{w_{i,j} + d_{i,j}} \), which can be written as a quadratic equation for variable \( w_{i,j} \) as \( w_{i,j}^2 + w_{i,j} \cdot (d_{i,j} - b_{i,j}) + c_{i,j} \cdot \hat{c}_{i,j} - b_{i,j} \cdot d_{i,j} = 0 \). Then, we have

\[
w(b_{i,j}, d_{i,j}) = \frac{b_{i,j} - d_{i,j} + \sqrt{(b_{i,j} + d_{i,j})^2 - 4c_{i,j} \cdot \hat{c}_{i,j}}}{2}. \tag{7}
\]

Therefore, the utility function of server \( S \) is

\[
U_s(A_s, A_{c,1}, \ldots, A_{c,m}) = \sum_{i=1}^{m} \sum_{j=1}^{n_i} w(b_{i,j}, d_{i,j}). \tag{8}
\]

Each \( C_i \) aims to minimize the expected time for verifying its \( n_i \) tasks. For each task \( t_{i,j} \), the expected verification time, denoted as \( T_{i,j} \), is

\[
T_{i,j}(b_{i,j}, d_{i,j}) = k \cdot (b_{i,j} - w(b_{i,j}, d_{i,j})) = k \cdot \frac{b_{i,j} + d_{i,j} - \sqrt{(b_{i,j} + d_{i,j})^2 - 4c_{i,j} \cdot \hat{c}_{i,j}}}{2}. \tag{9}
\]

Then, the utility function of client \( C_i \) is defined as

\[
U_{c,i}(A_s, A_{c,i}) = \sum_{j=1}^{n_i} [T_{i,j}(b_{i,j}, d_{i,j})]. \tag{10}
\]

4.4 Nash Equilibrium of the Game

A Nash equilibrium of the game is a combination of action, denoted as \( (A_{s,s}, A_{c,1}, \ldots, A_{c,m}) \), taken by the server and the clients respectively, such that: for the server and any \( A_s \neq A_{s,s} \), \( U_s(A_s, A_{c,1}, \ldots, A_{c,m}) \leq U_s(A_{s,s}, A_{c,1}, \ldots, A_{c,m}) \) for each client \( i \in \{1, \ldots, m\} \) and any \( A_{c,i} \neq A_{c,i,s} \), \( U_{c,i}(A_{s,s}, A_{c,i}) \leq U_{c,i}(A_{s,s}, A_{c,i}) \).
5.1 Client’s Optimization Problem

The client’s purpose is to minimize her utility, i.e., the expected time for verifying her tasks. Hence, the client’s optimization problem is as follows. (Note: parameter $k$ is ignored for the simplicity of exposition.)

$$\min \sum_{j=1}^{n_i} b_{i,j} + d_{i,j} - \sqrt{(b_{i,j} + d_{i,j})^2 - 4c_{i,j}\hat{c}_{i,j}}$$ \hspace{1cm} (11)

s.t. \hspace{1cm} \sum_{j=1}^{n_i} d_{i,j} = d_i; \hspace{1cm} b_{i,j} = b_i; \hspace{1cm} d_{i,j} \geq \hat{c}_{i,j}; \hspace{1cm} b_{i,j} \geq c_{i,j} + \frac{c_{i,j}\hat{c}_{i,j}}{c_{i,j} + \hat{c}_{i,j}}. \hspace{1cm} (12) \hspace{1cm} (13)

Note that, the sum of the objective functions of the above two optimization problems is

$$(11) + (14) = \sum_{j=1}^{n_i} b_{i,j} = b_i. \hspace{1cm} \tag{15}$$

Hence, the objective function of the server’s optimization problem can be re-written to

$$\max b_i - \sum_{j=1}^{n_i} b_{i,j} + d_{i,j} - \sqrt{(b_{i,j} + d_{i,j})^2 - 4c_{i,j}\hat{c}_{i,j}},$$

which is further equivalent to

$$\min \sum_{j=1}^{n_i} b_{i,j} + d_{i,j} - \sqrt{(b_{i,j} + d_{i,j})^2 - 4c_{i,j}\hat{c}_{i,j}}. \hspace{1cm} \tag{16}$$

Therefore, the above two optimization problems are equivalent. That is, a solution to the client’s optimization problem is also a solution to the server’s optimization problem, and thus is also the Nash equilibrium of the game between the server and the client.

5.2 Server’s Optimization Problem

The server’s purpose is also to maximize its utility, i.e., the total wage earned from the client. Hence, its optimization problem is as follows.

$$\max \sum_{j=1}^{n_i} b_{i,j} - d_{i,j} + \sqrt{(b_{i,j} + d_{i,j})^2 - 4c_{i,j}\hat{c}_{i,j}}$$ \hspace{1cm} (14)

s.t. \hspace{1cm} constraints \hspace{1cm} (12), \hspace{1cm} (13).

5.3 Proposed Algorithm

Due to the equivalence of the above two optimization problems, we only need to solve one of them. Next, we develop the algorithm, formally presented in Algorithm 1 to find the solution to the client’s optimization problem.

The core of the algorithm is to solve the following optimization problem, which is re-written from the afore-presented client’s optimization problem.

$$\min \sum_{j=1}^{n_i} f(s_{i,j}, i, j)$$ \hspace{1cm} (17)

where $f(x, i, j) = \frac{x - \sqrt{x^2 - 4c_{i,j}\hat{c}_{i,j}}}{2}$

Note that, $f(x, i, j)$ is the client’s utility associated with each task $t_{i,j}$, when the task is assigned with $x$ as the sum of $b_{i,j}$ and $d_{i,j}$. In the algorithm, we also use a partial derivative of $f(x, i, j)$, which is defined as

$$f'(x, i, j) = \frac{\partial f(x, i, j)}{\partial x}. \hspace{1cm} (16)$$

After the client and server exchange with each other their budget and deposit (i.e., $b_i$ and $d_i$), they each run Algorithm 1 to optimally allocate $b_i + d_i$ to the $n_i$ tasks, i.e., each task $t_{i,j}$ is assigned with budget $b_{i,j}$ and deposit $d_{i,j}$ where $\sum_{j=1}^{n_i} b_{i,j} = b_i$ and $\sum_{j=1}^{n_i} d_{i,j} = d_i$ with the goal of maximizing the client’s utility. Intuitively, the algorithm runs in the following three phases:

In the first phase, each task $t_{i,j}$ is assigned an initial value for $s_{i,j}$, which denotes the sum of $b_{i,j}$ and $d_{i,j}$. Here, the initial value is set to $\hat{c}_{i,j} + c_{i,j} + \frac{c_{i,j}\hat{c}_{i,j}}{c_{i,j} + \hat{c}_{i,j}}$ in order to satisfy constraints (13). After this phase completes, $s = b_i + d_i - \sum_{j=1}^{n_i}(\hat{c}_{i,j} + c_{i,j} + \frac{c_{i,j}\hat{c}_{i,j}}{c_{i,j} + \hat{c}_{i,j}})$ remains to be allocated in the second phase.

In the second phase, $s$ is split into units each of size $\delta$ and the units are further assigned to the tasks step by step. Specifically, with each step, one remaining unit is assigned to task $t_{i,j}$ whose $f'(s_{i,j}, i, j)$ is the minimal among all the tasks; this way, the units are assigned in a greedy manner to maximize the total utility of all the $n_i$ tasks.

After the $b_i + d_i$ have been greedily assigned to all the tasks, in the third phase, $s_{i,j}$ is further split into $b_{i,j}$ and $d_{i,j}$ such that, the shorter verification time a task has, the larger deposit is assigned to it. This way, the server’s deposit can be reclaimed as soon as possible from the tasks.

5.4 Analysis

We now prove that Algorithm 1 finds an optimal solution for the client’s optimization problem (which is also a solution for the server’s optimization problem), and the solution is a Nash equilibrium of the game between the client and the server. We develop the proof in the following steps. First, we introduce an optimization problem, as follows, which is relaxed from (15):

$$\min \sum_{j=1}^{n_i} f(s_{i,j}, i, j)$$ \hspace{1cm} (17)

where $f(x, i, j) = \frac{x - \sqrt{x^2 - 4c_{i,j}\hat{c}_{i,j}}}{2}$

s.t. \hspace{1cm} $-s_{i,j} + c_{i,j} + \frac{c_{i,j}\hat{c}_{i,j}}{c_{i,j} + \hat{c}_{i,j}} + \hat{c}_{i,j} \leq 0. \hspace{1cm} (18)$

Second, we derive the following Lemmas:

Lemma 1. The optimization problem defined in (17) has a unique solution.

Proof. As elaborated in Appendix 1, we show that the objective function is strictly convex and all the constraints are convex. \qed


Algorithm 1 Optimizing Resource Allocation (Server $S$ v.s. Client $C_i$ with Static Task Set)

Input:
- $b_i$: total budget of client $C_i$;
- $d_i$: total deposit of server $S$;
- $n_i$: total number of tasks;
- task set $\{t_{i,1}, \ldots, t_{i,n_i}\}$ and associated costs $\{c_{i,1}, \ldots, c_{i,n_i}\}$ and $\{\hat{c}_{i,1}, \ldots, \hat{c}_{i,n_i}\}$.

Output: $\{b_{i,1}, \ldots, b_{i,n_i}\}$ and $\{d_{i,1}, \ldots, d_{i,n_i}\}$.

Phase I: Initialization.
1: for $j \in \{1, \ldots, n_i\}$ do
2: $s_{i,j} \leftarrow (\hat{c}_{i,j} + c_{i,j} + \dfrac{\hat{c}_{i,j}}{c_{i,j} + \hat{c}_{i,j}}) \triangleright$ meet constraints \[13\]

Phase II: Greedy Allocation of the Remaining Fund.
1: $s \leftarrow [b_i + d - \sum_{j=1}^{n_i}(\hat{c}_{i,j} + c_{i,j} + \dfrac{\hat{c}_{i,j}}{c_{i,j} + \hat{c}_{i,j}})] \triangleright$ remaining fund to distribute
2: while $s \geq \delta$ do $\triangleright$ distribute remaining fund in unit $\delta$
3: $j^* \leftarrow \arg\min_{j \in \{1, \ldots, n_i\}} f(s_{i,j}, i, j)$
4: $s_{i,j^*} \leftarrow (s_{i,j} + \delta); s \leftarrow (s - \delta)$

Phase III: Splitting Sum to Budget/Deposit.
1: $d’ \leftarrow d - \sum_{j=1}^{n_i} \hat{c}_{i,j}$
2: $\text{tempSet} = \{1, \ldots, n_i\}$
3: while $\text{tempSet} \neq \emptyset$ do
4: $j^* = \arg\min_{j \in \{1, \ldots, n_i\}} f(s_{i,j}, i, j) \triangleright$ find the task with the shortest verification time
5: $x \leftarrow \min\{d’ - s_{i,j^*} - \hat{c}_{i,j^*}, - (c_{i,j^*} + \dfrac{\hat{c}_{i,j^*}}{c_{i,j^*} + \hat{c}_{i,j^*}})\}$
6: $d_{i,j^*} \leftarrow (\hat{c}_{i,j^*} + x) \triangleright$ assign as much deposit to task with the shortest verification time
7: $b_{i,j^*} \leftarrow (s_{i,j^*} - d_{i,j^*})$
8: $d’ \leftarrow (d’ - x)$
9: $\text{tempSet} \leftarrow (\text{tempSet} - \{j^*\})$

Lemma 2. Phases I and II of Algorithm 1 finds the unique solution to the optimization problem defined in [17].

Proof. See Appendix 2.

Lemma 3. Phase III of Algorithm 1 converts a solution of the optimization problem defined in [17] into a solution of the optimization problem defined in [15].

Proof. See Appendix 3.

Based on the above lemmas, we therefore have the following theorem:

Theorem 2. Algorithm 1 finds a solution of the optimization problem defined in [15].

Finally, we also prove in Appendix 4 the following theorem:

Theorem 3. Algorithm 1 finds a Nash equilibrium of the game between server $S$ and client $C_i$.

6 MULTIPLE CLIENTS WITH STATIC TASK SETS

In this section, we study the optimization problem in the context that server $S$ services $m$ clients $C_1, \ldots, C_m$. Different from the previous context of single client, optimizing for the server’s utility and for each client’s utility are not equivalent. So we cannot solve it in one step. Instead, we tackle the problem in two steps: we first optimize for the server’s utility, which produces an allocation of the server’s deposits to the clients; then, we optimize for each client’s utility based on the client’s budget and the deposit allocated by the server.

6.1 Algorithm

We propose an algorithm, formally presented in Algorithm 3, which runs in the following two steps.

First, we solve the server’s optimization problem, which produces the optimal allocation of the server’s deposits to the clients that maximizes the server’s wages. Thus, the optimization problem can be defined as follows:

$$\max \sum_{i=1}^{m} \sum_{j=1}^{n_i} w(b_{i,j}, d_{i,j})$$ \hspace{1cm} (19)

where $w(x, y)$ is defined as in [7] s.t. constraints [12] and [13].

Because

$$\sum_{i=1}^{m} \sum_{j=1}^{n_i} [b_i - \sum_{j=1}^{n_i} w(b_{i,j}, d_{i,j})]$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n_i} [b_i - w(b_{i,j}, d_{i,j})]$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n_i} b_i - d_{i,j} - \sqrt{(b_i + d_{i,j})^2 - 4c_{i,j}d_{i,j}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n_i} f(b_{i,j} + d_{i,j}, i, j),$$

the objective function of the above optimization problem, i.e., \[17\], is equivalent to $\min \sum_{i=1}^{m} \sum_{j=1}^{n_i} f(b_{i,j} + d_{i,j}, i, j)$. Furthermore, let $s_{i,j} = b_{i,j} + d_{i,j}$, and then the above optimization problem can be converted to:

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n_i} f(s_{i,j}, i, j)$$ \hspace{1cm} (20)

where $f(x, i, j)$ is defined as in [15] s.t.

$$s_{i,j} \geq c_{i,j} + \dfrac{c_{i,j} \hat{c}_{i,j}}{c_{i,j} + \hat{c}_{i,j}} + \hat{c}_{i,j},$$ \hspace{1cm} (21)

$$\sum_{j=1}^{n_i} s_{i,j} \geq b_i + \sum_{j=1}^{n_i} \hat{c}_{i,j},$$ \hspace{1cm} (22)

$$\sum_{i=1}^{m} \sum_{j=1}^{n_i} s_{i,j} = m b_i + d.$$ \hspace{1cm} (23)

Here, constraints [21], [22] and [23] are derived from constraints [12] and [13]. This optimization problem can be solved in three phases, as formally presented in Algorithm 2 and intuitively explained as follows:

Phase I: Initial allocation of clients’ budgets and the server’s deposits. The same as in Phase I of Algorithm 1, each task $t_{i,j}$ is initially allocated with a deposit of $\hat{c}_{i,j}$ and a budget of $c_{i,j} + \dfrac{\hat{c}_{i,j}}{c_{i,j} + \hat{c}_{i,j}}$, in order to satisfy constraints [13].
Phase IV: Greedy allocation of each client’s remaining budget. After Phase I, each client $C_i$ may have remaining budget, denoted as $b'_i = b_i - \sum_{j=1}^{n_i} (c_{i,j} + \frac{c_{i,j} \hat{c}_{i,j}}{c_{i,j} + \hat{c}_{i,j}})$. This remaining budget should be allocated step-by-step to the tasks of this client greedily. That is, similar to Phase II of Algorithm 1, $b'_i$ is divided into small units. With each step, the client whose utility function (a monotonically-decreasing function) has the minimal first-derivative value (i.e., the utility will drop the fastest), is allocated with one more unit from $b'_i$. This procedure continues until all units are allocated.

Phase III: Greedy allocation of the server’s remaining deposits. In this phase, the remaining deposits at the server is allocated to all the clients’ tasks step-by-step in the greedy manner, similar to Phase II.

Note that, Algorithm 2 also includes Phase IV, which computes each $d_i$, the total deposit allocated to the tasks of $C_i$. Each $d_i$ should be shared with $C_i$ for the client to find out the optimal allocation of its budget in the next step.

Algorithm 2 Optimal Splitting of Deposit (Server $S$ vs. Client $C_i$, $i = 1, \ldots, m$, with Static Task Set)

Input:
- $b_i$: total budget of each client $C_i$;
- $d$: total deposit of server $S$;
- $n_i$: total number of tasks from each $C_i$;
- task set $\{t_{1,1}, \ldots, t_{1,n_1}, \ldots, t_{m,1}, \ldots, t_{m,n_m}\}$ and associated costs $\{c_{1,1}, \ldots, c_{1,n_1}, \ldots, c_{m,1}, \ldots, c_{m,n_m}\}$ and $\{\hat{c}_{1,1}, \ldots, \hat{c}_{1,n_1}, \ldots, \hat{c}_{m,1}, \ldots, \hat{c}_{m,n_m}\}$.

Output: deposit $d_i$ allocated to each client $C_i$.

Phase I: Initialization.
1: for $i \in \{1, \ldots, m\}$ do
2: for $j \in \{1, \ldots, n_i\}$ do
3: \hfill $s_{i,j} \leftarrow (\hat{c}_{i,j} + c_{i,j} + \frac{c_{i,j} \hat{c}_{i,j}}{c_{i,j} + \hat{c}_{i,j}})$
4: end do
5: end do

Phase II: Greedy Allocation of Clients’ Remaining Budgets. 
1: for $i \in \{1, \ldots, m\}$ do
2: \hfill $b'_i \leftarrow b_i - \sum_{j=1}^{n_i} (c_{i,j} + \frac{c_{i,j} \hat{c}_{i,j}}{c_{i,j} + \hat{c}_{i,j}})$
3: while $b'_i \geq \delta$ do
4: \hfill $j^* \leftarrow \arg \min_{j \in \{1, \ldots, n_i\}} f'(s_{i,j^*}, i, j)$
5: \hfill $s_{i,j^*} \leftarrow (s_{i,j^*} + \delta); b'_i \leftarrow (b'_i - \delta)$
6: end while
7: end do

Phase III: Greedy Allocation of Remaining Deposit to tasks.
1: $d' \leftarrow d - \sum_{i=1}^{m} \sum_{j=1}^{n_i} d_{i,j}$
2: $T = \{(1, 1), \ldots, (1, n_1), \ldots, (m, 1), \ldots, (m, n_m)\}$
3: while $d' \geq \delta$ do
4: \hfill $i^* = \arg \min_{i \in T} f'(s_{i^*, j}, i, j)$
5: \hfill $s_{i^*, j} \leftarrow (s_{i^*, j} + \delta); d' \leftarrow (d' - \delta)$
6: end while

Phase IV: Preparing the Output.
1: for $i \in \{1, \ldots, m\}$ do
2: \hfill $d_i \leftarrow \sum_{j=1}^{n_i} s_{i,j^*} - b_i$
3: end do

In the second step, it is already known the server’s deposits allocated to the clients. Because the budget of each client is also known, each server-client pair can run Algorithm 2 presented in the previous section, to find out the optimal allocation of budget/deposit to the client’s tasks to minimize the client’s utility.

Algorithm 3 Optimal Resource Allocation (Server $S$ vs. Client $C_i$, $i = 1, \ldots, m$, with Static Task Set)

Input:
- $b_i$: total budget of each client $C_i$;
- $d$: total deposit of server $S$;
- $n_i$: total number of tasks from each $C_i$ for $i = 1, \ldots, m$;
- task set $\{t_{1,1}, \ldots, t_{1,n_1}, \ldots, t_{m,1}, \ldots, t_{m,n_m}\}$ and associated costs $\{c_{1,1}, \ldots, c_{1,n_1}, \ldots, c_{m,1}, \ldots, c_{m,n_m}\}$ and $\{\hat{c}_{1,1}, \ldots, \hat{c}_{1,n_1}, \ldots, \hat{c}_{m,1}, \ldots, \hat{c}_{m,n_m}\}$.

Output:
- $\{b_{1,1}, \ldots, b_{1,n_1}, \ldots, b_{m,1}, \ldots, b_{m,n_m}\}$ and $\{d_{1,1}, \ldots, d_{1,n_1}, \ldots, d_{m,1}, \ldots, d_{m,n_m}\}$.

6.2 Analysis
To analyze our proposed solution, we prove the following:

First, the optimization problem defined in (20) has only one unique solution. Second, Algorithm 2 solves the optimization problem defined in (20). Third, the optimization problem defined in (20) is equivalent to the one defined in (19). Finally, the budget and deposit allocation strategy produced by Algorithm 3 is a Nash equilibrium.

Lemma 4. The optimization problem defined in (20) has only one unique solution.

Proof. (sketch). Similar to the proof of Lemma 1 we can show that the objective function is strictly convex and every constraint is convex.

Theorem 4. Phases I, II and III of Algorithm 2 solves the optimization problem defined in (20).

Proof. As elaborated in Appendix 5, we prove by induction on the amount of fund to be allocated in Phase III.

Lemma 5. Optimization problem defined in (19) is equivalent to that defined in (20).

Proof. (sketch) On one hand, the constraints of the optimization problem defined in (20) are derived from (and necessary conditions) of those of (19), each optimal solution to (19) is a feasible solution to (20). On the other hand, each optimal solution to (20) can be converted to a feasible solution to (19), with Part IV of Algorithm 2 and steps 2-3 in Algorithm 3.

Theorem 5. Algorithm 3 finds a Nash Equilibrium for the game between server $S$ and $m$ clients $C_1, \ldots, C_m$.

Proof. (sketch). For server $S$, as the algorithm produces a unique optimal solution that maximizes its wage, there is no incentive to deviate from the solution. For each client, as proved in Theorem 5, there is no incentive, either, to deviate from the solution.
7 Multiple Clients with Dynamic Tasks

Tasks can be submitted to the cloud server dynamically. We assume that, each task is submitted with a budget promised by a client, and the server can start executing the task immediately due to the typically rich resource available at the cloud server.

Though the server can also immediately commit a deposit for a task at its arrival, this is not desired: First, the arrival of tasks is not predictable, making it difficult to optimize the distribution of the fund for deposit. Second, there could be a large number of tasks executed in parallel, which could divide the fund into very small pieces; the smaller are deposits, as we learn from the previous sections, the more frequently TTPs would be hired, which would decrease the amount of wage for the server and increase the latency for computation verification.

To address this issue, the server should distribute its fund to only a small number of tasks at a time. Also, the deposit made to tasks should be reclaimed as soon as possible and thus can be reused quickly; hence, it is more beneficial to make deposit to a task when it is completed and ready to be released than when it is just submitted. Moreover, as there could be a large number of tasks completing during a short period of time, the server should control the pace at which the completed tasks are released, and thus it can make deposits to only a selected subset of completed tasks at a time.

Based on the above ideas, we propose two algorithms. We first propose a baseline scheme named sequential releasing, with which the server releases only one completed task at a time and uses all of its fund as deposit for the task; this way, it can earn the most from each individual task, however, at the expense of slowed pace to release completed tasks. To address the limitation, we further propose a more generic scheme, parallel releasing, with which the server releases a subset of completed tasks at a time, based on certain criteria adjustable with some system parameters, to balance the trade off between the wage earned from each task and the pace of releasing completed tasks.

7.1 Sequential Releasing

As formally presented in Algorithm 4, the sequential releasing algorithm works as follows. When a task arrives, the server immediately start executing it. When the computation of a task finishes, it is released if currently the server has full fund available; otherwise, it is put into the queue waiting for its turn to be released. When a released task has been finally accepted by the client who submitted it, the server reclaims the deposit assigned to the task. Then, the waiting queue is checked; if there is one or more tasks there, one of them is picked to be released with the server’s fund as deposit.

7.2 Parallel Releasing

As formally presented in Algorithm 5 at the core of the parallel releasing algorithm is a function named AssignDeposit. Every time when the function is called, it works in the following three steps.

First, the server checks the following condition to determine if it is time to find another set of tasks to release: The server’s fund that has been currently locked due to being assigned to the tasks that are released but yet finalized, should be less than percentage $\alpha$ of the server’s whole fund for deposit.

Second, if it is time to start a new round of task releasing, the server selects a number of tasks waiting in the front of the queue of completed yet unreleased tasks, such that the sum of the required minimal deposits for these tasks is no more than percentage $\beta$ of the currently available fund. Note that, parameters $\alpha$ and $\beta$ are used to control the trade off between the pace of task releasing and the amount of deposit that a task can be assigned: the smaller are $\alpha$ and $\beta$, the slower is the pace of task releasing and the larger is the deposit that a task can be assigned (thus the larger is the wage that the server can earn from the task); and vice versa.

Third, once the set of tasks to be released has been selected, Algorithm 3 is called to find out the optimal strategy to deposit the currently available fund to the tasks, to attain the dual goals of maximizing the server’s wage and minimizing the delay for verification.

With the AssignDeposit function in place, the parallel releasing algorithm runs as follows: When a new task arrives, the server immediately starts executing the task; when a task is completed or finalized, the server calls the AssignDeposit function to make deposit assignments as long as the above-discussed conditions are satisfied.

7.3 Simulations

Due to dynamic and unpredictable nature of the incoming tasks, it is hard to analyze our proposed algorithms in theory. Hence, we evaluate the algorithms through simulations.

7.3.1 Settings

As inputs, we simulate 50 groups of task and each group has 100 tasks. Each task can be regular or heavy, different in the cost. The arrivals of tasks follows a Poisson distribution with average interval $\tau$. Specifically, the tasks are characterized...
Algorithm 5 Parallel Releasing

Variables:
- \( d \): server’s total fund for deposit;
- \( d_{\text{locked}} \): fund for deposit that have been locked by clients, initialized to 0;
- \( \{t_{i,j}\} \): a dynamic sequence of tasks;
- \( Q \): queue buffering completed but unreleased tasks.

Upon receiving a new task: start executing the task.
Upon completing execution of task \( t_{i,j} \):
1. \( Q . \text{append}(t_{i,j}) \)
2. call AssignDeposit
Upon task \( t_{i,j} \) being accepted by its owner (a client):
1. \( d_{\text{locked}} \leftarrow d_{\text{locked}} - d_{i,j} \)
2. call AssignDeposit

Function AssignDeposit:
1. \( S_{\text{task}} \leftarrow \emptyset \) \( \triangleright \) temporary task set
2. \( c_{\text{temp}} \leftarrow 0 \)
3. if \( d_{\text{locked}} \leq \alpha \cdot d \) then
4. while \( Q \neq \emptyset \) do
5. \( t_{i,j} \leftarrow Q . \text{front} \)
6. \( \hat{c}_{\text{temp}} \leftarrow \hat{c}_{\text{temp}} + \hat{c}_{i,j} \)
7. if \( \hat{c}_{\text{temp}} \geq \beta \cdot (d - d_{\text{locked}}) \) then
8. break
9. else
10. \( S_{\text{task}} \leftarrow S_{\text{task}} \cup \{t_{i,j}\} \)
11. \( Q . \text{dequeue} \)
12. call Algorithm 3 to allocate remaining fund for deposit, i.e., \( d - d_{\text{locked}} \), to tasks in \( S_{\text{task}} \)
13. \( d_{\text{locked}} \leftarrow d \)

by following parameters: (i) \( c \) is the average cost of each regular task, quantified by the required execution time; we let \( c \) range from 8τ to 128τ, with 32τ as the default value. (ii) \( T_{\text{confirm}} \) is the average time needed for the funds (including the client’s budget and the server’s deposit) to be distributed and confirmed on blockchain; we let \( T_{\text{confirm}} \) range from 0.25τ to 16τ, with 2τ by default. (iii) \( b \) is the server’s average budget for a task; we let \( b \in \{2c, 2.5c, 3c, 3.5c, 4c\} \), with 2.5c by default. (iv) \( p_{\text{heavy}} \) is the probability that a task is heavy (rather than regular), with the purpose of studying the impact of task heterogeneity; we let \( p_{\text{heavy}} \) take the default value of 0.1, and each heavy task has an average cost being 3 times of a regular task. (v) the default cost for hiring a TTP for verifying a task with cost \( c \) is \( \hat{c} = 3c \). (vi) \( d \) is the server’s total fund for deposit; we let \( d \) take the default value of \( d_0 = 768\tau \) and vary over the set of \{d_0, 2d_0, 4d_0, 8d_0, 16d_0, 32d_0\}.

We measure the following metrics: (i) \( \frac{\tau}{\tau} \) is the average percentage of the client’s budgets for tasks that are earned by the server as wages; (ii) \( \frac{\tau}{\tau} \) is the average amount of wage that the server can earn per time interval; (iii) \( \frac{\tau}{\tau} \) is the average amount of wage that the server can earn for each unit of computational cost it pays; (iv) average delay for a task is the time elapse (in the unit of τ) from the task is submitted until it is finalized (i.e., it is completed and the funds associated with it have been distributed and confirmed on blockchain). Specifically, the delay includes four parts: the computation delay \( c \), the release delay that is the time elapse from the task being completed till its result being released, the TTP delay which is time for a hired TTP to verify a task (if the client hires a TTP), and the confirmation delay which is the time for the funds associated with the task to be distributed and confirmed on blockchain. For each task, the computation and confirmation delays are fixed based on the task and system property. The release and the TTP delays, however, are affected by the algorithm used and the parameters of the algorithm.

In the simulation, we first evaluate how parameters \( \alpha \) and \( \beta \) affect the performance of the parallel releasing algorithm, which is followed by the comparison between the sequential releasing and the parallel releasing algorithms.

7.3.2 Impacts of \( \alpha \) and \( \beta \) on Parallel Releasing

In parallel releasing, \( \alpha \) controls when a new round of fund allocation starts; the larger is \( \alpha \), the shorter interval between the rounds. \( \beta \) controls how many tasks are released in parallel; the larger is \( \beta \), the more tasks released at the same time, and the less deposit each of the tasks is assigned.

Figure 30 shows the impact of \( \alpha \) and \( \beta \) on the average delay of task. When other parameters are the same, the delay decreases as \( \alpha \) increases; this is obvious because increasing \( \alpha \) shortens the interval between the rounds of fund allocation and task releasing. Given a fixed \( \alpha \), it is interesting to observe that \( \beta \) has different impacts on the delay when the value of \( \alpha \) is different. With a small \( \alpha \) (e.g., 0 and 0.3), the delay generally increases along with \( \beta \). This is because, as \( \beta \) increases, the deposit assigned to a task decreases while the probability of hiring TTP increases for verifying the task; when a task is checked with TTP, this not only increases the delay of the task, but also slows down the returning of the deposit and thus postpones the starting of the next round of allocation. The latter impact becomes even more significant with smaller \( \alpha \), where a new round of allocation can start only after all or a large percentage of the fund has been returned. On the other hand, with a large \( \alpha \) (e.g., 1 and 0.6), a new round of allocation can start even when a large percentage of fund is locked; thus, increasing \( \beta \) generally allows faster releasing of tasks and thus decreases the average delay of task.

From Figure 30 and Figure 31, as expected, the greater is \( \beta \), the smaller are \( \frac{\tau}{\tau} \) and \( \frac{\tau}{\tau} \); and this is nearly independent of \( \alpha \). This is because, when \( \beta \) increases, more tasks can be released at one round, which in turn leads to lower deposits allocated to each task. As we learn from the above sections, the server’s wage is an increasing function of the deposit; hence, lower deposits lead to lower wages earned by the server.

Figure 32 shows the impacts of \( \alpha \) and \( \beta \) on metric \( \frac{\tau}{\tau} \), which takes into account both the server’s concern (i.e., high wage) and the client’s concern (i.e., low delay); intuitively, higher \( \frac{\tau}{\tau} \) is desired. As we can observe, a larger \( \alpha \) generally leads to higher \( \frac{\tau}{\tau} \) when other parameters are the same. When \( \alpha \) is small (e.g., 0 and 0.3), \( \frac{\tau}{\tau} \) generally decreases as \( \beta \) increases, as it causes the wage to decrease and the delay to increase. When \( \alpha \) is large (e.g., 1 and 0.6), increasing \( \beta \) causes both the wage and the delay to drop; as a result, \( \frac{\tau}{\tau} \) increases with \( \beta \) as long as \( \beta \) is not too large (e.g., \( \beta \leq 0.7 \) for \( \alpha = 0.6 \) and \( \beta \geq 0.8 \) for \( \alpha = 1 \)).
The above simulation results reveal that, parameter $\alpha$ only affects the average delay of task experienced by the client and $\tau$, the wage per time unit (concern of the server); when other parameters are fixed, $\alpha = 1$ leads the smallest delay and the largest $\tau$. Hence, $\alpha = 1$ is desired and we will use this setting for the rest simulation.

### 7.3.3 Sequential vs. Parallel Releasing

Next, we compare the performance of sequential releasing and parallel releasing (with $\alpha = 1$ and $\beta = 0.6$) under various conditions.

Figure 2: The performance under different $\alpha$ and $\beta$

Figure 3: Impact of confirmation time

Figure 4: Impact of clients’ budgets

Figure 5: Impact of the server’s budget for deposit, $d$ varies from $d_0 = 768\tau$ to $32d_0$, while the other parameters take their default values. As demonstrated by Figure 5a, the average delay of task decreases for both algorithms. However, as the confirmation time increases, the returning of deposit funds gets slower for both, while the parallel one can still release tasks when funds are only partly returned; hence, the gap between two algorithms gets wider and wider though their delays both increase.

Without surprise, $w_b$ and $w_c$ have nearly no change when confirm time increases, for both the sequential and the parallel releasing algorithms. But as the two algorithms have different speeds in increasing the delay for task, they also decrease their $\tau$ at different rates. $\tau$ of the sequential releasing algorithm drops faster, and quickly becomes lower than that of its parallel counterpart.

Figure 4 shows the comparisons as $b$, the clients’ average budget per task, varies while other parameters take their default values. As we can see from Figure 4a, the average delay of task drops for both algorithms when the budget rises, and the parallel releasing algorithm has faster decrease. This is because: First, the delay as function $f(x, i, j)$ for each task $t_{i,j}$ as well as its totally assigned deposit and budget $x$, has the property that $f'(x, i, j) < 0$ and $f''(x, i, j) > 0$; hence, it decreases as each task is assigned with more budget while other conditions remain the same. Second, the sequential releasing algorithm assigns more deposit to each released task than the parallel one; hence, due to $f''(x, i, j) < 0$, a task released by the parallel releasing algorithm has smaller sum of deposit and budget, and thus its delay decreases faster as its budget increases. Similarly, the wage from a task as function $w(x, i, j)$ has the property that $w'(x, i, j) > 0$ and $w''(x, i, j) < 0$. Hence, it explains that, as shown in Figure 4b and 4c, both $w_b$ and $w_c$ increase with $b$ for both algorithms while the parallel releasing algorithm has faster increase. Resulting from the above trends, as shown in Figure 4d, $\tau$ increases with budget and the parallel releasing algorithm has faster increase.

Figure 5 shows the comparisons as the server’s budget for deposit, i.e., $d$ varies from $d_0 = 768\tau$ to $32d_0$, while the other parameters take their default values. As demonstrated by Figure 5a, the average delay of task decreases for both
the sequential and parallel algorithms as the server’s fund for budget increases. This is because, as the fund increases, every released task is assigned with larger deposit, which results in lower probability of hiring TTP and thus lower delay. The larger deposits also lead to larger wage that the server can earn from each released task, as shown in Figures 5b and 5c. These figures also demonstrate that, the parallel algorithm increases the wage at higher speed. This is because, as we know from the analysis in the previous sections, the wage is an increasing function of deposit and its second derivative is negative; hence, the increase of wages earned from multiple concurrently-released tasks is higher than the increase of wage earned from only one single task, when the wage increase is due to the same amount of increased deposit. Resulting from the above, as shown in Figure 5d, \( \frac{\tau}{s} \) is also increased with the fund for deposit, and the parallel algorithm is shown to have faster increase in \( \frac{\tau}{s} \) than the sequential one.

To summarize, we find that the following from the simulations:

- The sequential releasing algorithm earns higher wage per task than the parallel algorithm.
- In generally, the parallel releasing algorithm incurs lower average delay of task than the sequential algorithm.
- When the server’s deposit is much larger than the cost of each individual task (which is common in practical cloud computing systems), the parallel releasing algorithm can earn a similar level of wage per task as the parallel algorithm while incurring much lower delay of task; hence, the parallel algorithm is preferred in practice.

8 Conclusion and Future Works

In this paper, we study the verifiable computation outsourcing problem in the setting where a cloud server services a static set or a dynamic sequence of tasks submitted by multiple clients. We adopt a game-based model, where the cloud server should make a deposit for each task it takes, each client should allocate a budget that includes the wage paid to the server and the possible cost for hiring TTP for each task it submits, and every party (i.e., each of the server and the clients) has its limited fund that can be used for either deposits or task budgets. We study how the funds should be optimally allocated to achieve the three-fold goals: a rational cloud server should honestly compute each task it takes; the server’s wages earned from computing the tasks are maximized; and the overall delay experienced by each task for verifying her tasks is minimized. Specifically, we apply game theory to formulate the optimization problems, and develop the optimal or heuristic solutions for three application scenarios: one client outsources a static set of tasks to the server; multiple clients outsource a static set of tasks to the server; multiple clients outsource a dynamic sequence of tasks to the server. For each of the solutions, we analyze the solutions through either rigorous proofs or extensive simulations.

In the future, we will study in more depth the setting where there are multiple clients submitting dynamic sequences of tasks to the server. As it is challenging to develop optimal solution for the currently-defined general setting, we will explore to refine the problem with reasonable constraints and then develop an optimal solution for it.

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**APPENDIX 1 PROOF OF LEMMA 1**

For the optimization problem defined in (17), it is obvious that every constraint is convex. Next, we only need to show that the objective function is strictly convex. To simplify presentation, we introduce vector $\vec{s}_i$ to denote $<s_{1i}, \ldots, s_{ni}, i>$ and $f(\vec{s}_i)$ to denote the objective function. We further use $K_{ij}$ to denote $c_{ij}g_{ij}$, which is a constant. The Hessian matrix of $f(\vec{s}_i)$ has only $n_i$ non-zero diagonal elements, i.e., $\frac{\partial^2 f(\vec{s}_i)}{\partial s_{ij}^2} = \frac{2K_{ij}}{(s_{ij} - 4K_{ij})^2} > 0$, where $j \in \{1,2,\ldots,n_i\}$. Hence, the Hessian matrix of $f(\vec{s}_i)$, denoted as $H_i$, has the form of

\[
\begin{pmatrix}
2K_{i1} & \cdots & 0 \\
0 & \ddots & \vdots \\
0 & \cdots & 2K_{in_i}
\end{pmatrix}
\]

Letting $z = (z_1, \ldots, z_n)^T$ be any non-zero column vector, where $T$ denotes transpose, we have

\[
H_i z^T = \left(\sum_{j=1}^{n_i} z_j \frac{2K_{ij}}{(s_{ij} - 4K_{ij})^2}\right) z > 0,
\]

where $H_i$ is positive deﬁnite and $f(\vec{x})$ is strictly convex.

**APPENDIX 2 PROOF OF LEMMA 2**

Phase I of Algorithm 1 is simply to satisfy constraint (18). Hence we focus on studying Phase II, which allocate the remaining fund greedily. In the following, we prove (by contradiction) that the strategy of greedy allocation in Phase II leads to a solution of the optimization problem (15), which is unique due to Lemma 1.

Assume the optimal allocation strategy for the remaining fund is $\vec{\delta} = <(o_{i1}, o_{i2}, \ldots, o_{in_i}), i>$, where each $o_{ij}$ is the sum of budget and deposit assigned to task $t_{ij}$ from the remaining fund; but Phase II leads to a different allocation strategy $\vec{g} = <(g_{i1}, g_{i2}, \ldots, g_{in_i}), i>$. Then, there are some tasks (at least one) which receive more or less allocation than the optimal solution; let $M$ and $L$ denote the task sets respectively. Let $\vec{c} = <c_{i1}, c_{i2}, \ldots, c_{in_i}, i>$ denote the common parts of $\vec{\delta}$ and $\vec{g}$; that is, $c_j = \min(c_{ij}, o_{ij})$ for $j \in \{1, \ldots, n_i\}$.

Let $v_o, v_g$ and $v_o$ be outcomes of the objective function of (15) given by allocation $\vec{c}$, $\vec{g}$ and $\vec{\delta}$ respectively. As $\vec{\delta}$ leads to the unique optimal solution, we have: $v_g > v_o$. We deﬁne $diff_{f,c}$ and $diff_{f,g}$ as follows: $\text{diff}_{f,c} = v_o -$
which means

\[ f(x, i, j) \] 

increases. Therefore, for each task

\[ t \] 

increases (as shown by Fig. 6).

\[ \Delta = \sum_{i,j} f'(x, i, j) \] 

be the partial derivative function of \( f(x, i, j) \) which has

\[
f'(x, i, j) = \frac{1}{2} \left(1 - \frac{x}{\sqrt{x^2 - 4c_{i,j}}}ight) < 0;
\]

(24)

that is, \( f(x, i, j) \) is decreasing monotonously. Let \( f''(x, i, j) \) be the partial derivative function of \( f'(x, i, j) \) and it holds

\[
f''(x, i, j) = \frac{2c_{i,j}}{\sqrt{x^2 - 4c_{i,j}^2}} > 0;
\]

(25)

i.e., \( f(x, i, j) \) decreases monotonically but gets slower as \( x \) increases (as shown by Fig. 6).

According to (24) and (25), \( f'(x, i, j) \) increases as \( x_{i,j} \) increases. Therefore, for each task \( t_{i,j} \in L \),

\[
f'(x, i, j) > f'(c_{i,j}, i, j) \quad \forall c_{i,j} < x \leq a_{i,j}.
\]

Let

\[
f'_{\text{omin}} = \min(f'(c_{i,j}, i, j))
\]

for all tasks \( t_{i,j} \in L \), then we have

\[
diff_{c,o} = \sum_{t_{i,j} \in L} \int_{c_{i,j}}^{a_{i,j}} f'(x, i, j) dx
\]

> \sum_{t_{i,j} \in L} \int_{c_{i,j}}^{a_{i,j}} f'(c_{i,j}, i, j) dx

= \sum_{t_{i,j} \in L} ((a_{i,j} - c_{i,j}) \times f'(c_{i,j}, i, j))

\geq \sum_{t_{i,j} \in L} (a_{i,j} - c_{i,j}) \times f'_{\text{omin}} = \Delta \times f'_{\text{omin}},
\]

(26)

where \( \Delta = \sum_{t_{i,j} \in L} ((a_{i,j} - c_{i,j}) \sum_{t_{i,j} \in L} (a_{i,j} - c_{i,j}) = \sum_{t_{i,j} \in L} (a_{i,j} - c_{i,j}) = \max(f'(x, i, j)) \) for all tasks \( t_{i,j} \in M \) and \( c_{i,j} \leq x \leq a_{i,j} \). Since our greedy algorithm picks the task with the minimal first-derivative at each step and it chooses tasks in \( M \) over tasks in \( L \), we have

\[
f'_{\text{gmax}} \leq f'_{\text{omin}}.
\]

Therefore we have

\[
diff_{c,g} = \sum_{t_{i,j} \in M} \int_{c_{i,j}}^{a_{i,j}} f'(x, i, j) dx
\]

< \sum_{t_{i,j} \in M} \int_{c_{i,j}}^{a_{i,j}} f'_{\text{gmax}} dx = \sum_{t_{i,j} \in L} ((a_{i,j} - c_{i,j}) \times f'_{\text{gmax}}

\]

\[
= \Delta \times f'_{\text{gmax}} \leq \Delta \times f'_{\text{omin}} < \text{diff}_{c,o},
\]

(27)

which means \( v_g < v_o \), since \( v_g = v_c + \text{diff}_{c,g} \) and \( v_o = v_c + \text{diff}_{c,o} \). This contradicts with \( v_g > v_o \).

**APPENDIX 3 PROOF OF LEMMA 3**

By applying Phase III of Algorithm 1 it is obvious that, any allocation strategy \( \langle s_1, \cdots, s_{n_i} \rangle \) produced by Phase I and II, which is a solution to the optimization problem defined in (17), can always be converted into a detailed allocation strategy

\[
\langle b_{1,1}, d_{1,1}, \cdots, b_{n_i, n_i}, d_{n_i, n_i} \rangle,
\]

as the execution of Phase III will not abort before it is completed.

Because the above conversion assures \( b_{i,j} + d_{i,j} = s_{i,j} \) for each \( j \in \{1, \cdots, n_i\} \) and the objective functions in both (15) and (17) are determined only by the sums of every task’s budget and deposit, the value of the objective function in (15) by the detailed allocation strategy should be the same as the minimal value of the objective function in (17).

Moreover, as the optimization problems defined in (15) and (17) are the same except for that the former has more constraints than the latter, the solution to the former cannot be better than that for the latter; i.e., the minimal value of objective function in (15) must be no less than the minimal value of objective function in (17).

Based on the above reasoning, the detailed allocation strategy obtained from applying Phase III of Algorithm 1 to a solution to (17) must also be a solution to (15).

**APPENDIX 4 PROOF OF THEOREM 3**

Based on the definition of Nash equilibrium, we need to prove that neither the client nor the server has incentive to change strategy \( A_{s,i} = (b_{1,i}, \cdots, b_{n_i, i}) \) or \( A_{s,i} = (d_{1,i}, \cdots, d_{n_i, i}) \), which are produced by Algorithm 1 if the other player does not change its strategy.

Let \( s = \langle s_1, \cdots, s_{n_i} \rangle \) denote the unique solution to the optimization problem defined by (17), that is \( s_{i,j} = b_{i,j} + d_{i,j} \) for each \( j \in \{1, \cdots, n_i\} \).

First, let us consider the scenario that the server keeps strategy \( A_{s,i} = (d_{1,i}, \cdots, d_{n_i, i}) \), while the client changes its strategy from \( A_{c,i} = (b_{1,i}, \cdots, b_{n_i, i}) \) to a different strategy \( A'_{c,i} = (b'_{1,i}, \cdots, b'_{n_i, i}) \). Let \( s' = \langle s'_1, \cdots, s'_{n_i} \rangle \) denote every task’s sum of budget and deposit according to strategy set \( A_{s,i}, A'_{c,i} \); i.e., \( s'_{i,j} = b'_{i,j} + d_{i,j} \) for \( j \in \{1, \cdots, n_i\} \). Because \( \hat{s} \neq s' \) and \( \hat{s} \) is the unique solution to (17), it holds that the client’s utility under strategy set \( A_{s,i}, A'_{c,i} \) is greater than its utility under strategy set \( A_{s,i}, A_{c,i} \), i.e.,

\[
U_{c,i} = \sum_{j=1}^{n_i} s_{i,j} - 2\sqrt{s_{i,j}^2 - 4c_{i,j}^2},
\]

\[
U'_{c,i} = \sum_{j=1}^{n_i} s'_{i,j} - 2\sqrt{s'_{i,j}^2 - 4c_{i,j}^2}.
\]

So, the client should not have incentive to deviate from \( A_{c,i} \).

Similarly, let us consider the scenario that the client keeps strategy \( A_{c,i} \) while the server changes from \( A_{s,i} \) to a different strategy \( A'_{s,i} = (d_{1,i}, \cdots, d'_{n_i, i}) \). Let \( s'' = \langle s''_1, \cdots, s''_{n_i} \rangle \), where \( s''_{i,j} = b_{i,j} + d'_{i,j} \) for \( j \in \{1, \cdots, n_i\} \). The server’s total wage on strategy set \( A_{s,i}, A'_{c,i} \) is

\[
U_s = \sum_{j=1}^{n_i} s''_{i,j} - 2\sqrt{s''_{i,j}^2 - 4c_{i,j}^2},
\]

\[
U''_s = \sum_{j=1}^{n_i} s''_{i,j} - 2\sqrt{s''_{i,j}^2 - 4c_{i,j}^2}.
\]

Since \( \hat{s} = \langle s_1, \cdots, s_{n_i} \rangle \) is the unique solution for the optimization problem (17), \( U_s < U_s \). Therefore, the server has no incentive to deviate from \( A_{s,i} \) either.
OPT(t_0, \hat{b}_1', \cdots, \hat{b}_m') share the same optimal solution.

Finally, based on the induction assumption, \( \mathcal{P}(t_0, \hat{b}_1', \cdots, \hat{b}_m') \equiv \text{TRUE} \). That is, Algorithm 2 solves \( \text{OPT}(t_0, \hat{b}_1', \cdots, \hat{b}_m') \); thus, it also solves \( \text{OPT}(t_0 + 1, 1, \cdots, b_m) \), except that \( \{\hat{b}_i\} \) and \( d = \sum_{i=1}^{m} \sum_{j=1}^{n_i} \hat{c}_{i,j} + t_0 \cdot \delta \) (not \( \{\hat{b}_i\} \) and \( d = \sum_{i=1}^{m} \sum_{j=1}^{n_i} \hat{c}_{i,j} + (t_0 + 1) \cdot \delta \)) are the inputs to the algorithm. Hence, in the execution of the algorithm, let us move the last assignment to \( s_{i_*,j_*} \) in Phase II to be the first step in Phase III; this way, the algorithm works exactly as it takes \( \{\hat{b}_i\} \) and \( d = \sum_{i=1}^{m} \sum_{j=1}^{n_i} \hat{c}_{i,j} + (t_0 + 1) \cdot \delta \) as inputs to solve \( \text{OPT}(t_0 + 1, 1, \cdots, b_m) \). That is, \( \mathcal{P}(t_0 + 1, 1, \cdots, b_m) \equiv \text{TRUE} \).

APPENDIX 5 PROOF OF THEOREM 4

Let \( \text{OPT}(t, b_1, \cdots, b_m) \) denote the optimization problem defined in [20] where \( t \) is an integer, \( d = \sum_{i=1}^{m} \sum_{j=1}^{n_i} \hat{c}_{i,j} + t \cdot \delta \) and \( b_i \geq \sum_{j=1}^{n_i} (\hat{c}_{i,j} + \frac{c_{i,j} + \hat{c}_{i,j}}{\alpha_{i,j} + \gamma_{i,j}}) \) for each \( i \in \{1, \cdots, m\} \). Note that, \( c_{i,j} \) and \( \hat{c}_{i,j} \) are constants. Let \( \mathcal{P}(t, b_1, \cdots, b_m) \) denote the following predicate: Phases I, II and III of Algorithm 2 solves \( \text{OPT}(t, b_1, \cdots, b_m) \); i.e., Phases I-III of Algorithm 2 solves \( \text{OPT}(t, b_1, \cdots, b_m) \), because: Phase I simply initializes every \( s_{i,j} \) to satisfy constraint (21); Phase II minimizes \( \sum_{j=1}^{n_i} f(s_{i,j}, i, j) \) with the remaining budget of each client \( C_i \) independently, based on the arguments similar to the proof of Lemma 2; Phase III does nothing as \( d' = 0 \).

Induction step: Assuming \( \mathcal{P}(t, b_1, \cdots, b_m) \equiv \text{TRUE} \) for every integer \( 0 \leq t \leq t_0 \) and every \( \{b_i|i = 1, \cdots, m\} \) satisfying the above relevant constraints, next we prove \( \mathcal{P}(t_0 + 1, b_1, \cdots, b_m) \equiv \text{TRUE} \) for every \( \{b_i|i = 1, \cdots, m\} \) satisfying the relevant constraints.

First, let us consider optimization problem \( \text{OPT}(0, b_1, \cdots, b_m) \). According to the base case, \( \mathcal{P}(0, b_1, \cdots, b_m) \equiv \text{TRUE} \), and each \( s_{i,j}(0, b_1, \cdots, b_m) \) denotes the assignment of \( s_{i,j} \) in the optimal solution.

Second, let \( I = \{(i*, j*)\} = \arg \min_{i \in \{1, \cdots, m\} \forall j \in \{1, \cdots, n_i\}} f(s_{i,j}, i, j) \). Then, there must exist at least one \( (i*, j*) \in I \) such that, in the optimal solution to \( \text{OPT}(t_0 + 1, b_1, \cdots, b_m) \), the assignment of \( s_{i*,j*} \) is greater than the assignment of \( s_{i',j'} \) in the optimal solution to \( \text{OPT}(0, b_1, \cdots, b_m) \); i.e., \( s_{i*,j*}(0, b_1, \cdots, b_m) > s_{i',j'}(0, b_1, \cdots, b_m) \). This can be proved by contradiction. If this is not the case, as \( t_0 + 1 \geq 1 \), there should be at least \( (i', j') \not\in I \) such that \( s_{i',j'}(t_0 + 1, b_1, \cdots, b_m) > s_{i*,j*}(0, b_1, \cdots, b_m) \); then, if one unit assigned to \( s_{i',j'} \) instead is assigned to \( s_{i*,j*} \), the server can earn higher wage.

Third, let us consider optimization problem \( \text{OPT}(t_0, \hat{b}_1', \cdots, \hat{b}_m') \), where \( \hat{b}_i = b_i + \delta \) while \( \hat{b}_i' = b_i \) for every \( i \neq i^* \). We can prove the following: First, the optimal solution to \( \text{OPT}(t_0, \hat{b}_1', \cdots, \hat{b}_m') \) is a feasible solution to \( \text{OPT}(t_0 + 1, \hat{b}_1', \cdots, \hat{b}_m') \). This is because: \( \hat{b}_i' \geq \hat{b}_i \) for every \( i \), thus satisfying constraint (22) in \( \text{OPT}(t_0, \hat{b}_1', \cdots, \hat{b}_m') \) implies satisfying constraint (22) in \( \text{OPT}(t_0 + 1, \hat{b}_1', \cdots, \hat{b}_m) \); \( \sum_{i=1}^{m} \hat{b}_i' + \delta = \sum_{i=1}^{m} \hat{b}_i' \) and the value of \( d \) in \( \text{OPT}(t_0 + 1, \hat{b}_1', \cdots, \hat{b}_m) \) is greater than the \( d \) in \( \text{OPT}(t_0, \hat{b}_1', \cdots, \hat{b}_m') \) by \( \delta \), thus satisfying constraint (23) in \( \text{OPT}(t_0, \hat{b}_1', \cdots, \hat{b}_m') \) implies satisfying constraint (23) in \( \text{OPT}(t_0 + 1, \hat{b}_1', \cdots, \hat{b}_m) \). Similarly, the optimal solution to \( \text{OPT}(t_0 + 1, \hat{b}_1', \cdots, \hat{b}_m') \) is a feasible solution to \( \text{OPT}(t_0 + 1, \hat{b}_1', \cdots, \hat{b}_m) \).

Further due to the uniqueness of optimal solution to these optimization problems (based on Lemma 4).