Dependent measures in independent theories

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Abstract

We introduce the notion of dependence, as a property of a Keisler measure, and generalize several results of [HPS13] on generically stable measures (in NIP theories) to arbitrary theories. Among other things, we show that this notion is very natural and fundamental for several reasons: (i) all measures in NIP theories are dependent, (ii) all types and all fim measures in any theory are dependent, and (iii) as a crucial result in measure theory, the Glivenko-Cantelli class of functions (formulas) is characterized by dependent measures.

1 Introduction

In [HPS13], Hrushovski, Pillay, and Simon proved the following crucial theorem:

Theorem*: (Assuming NIP) Let $\mu$ be a global Keisler measure which is invariant over a small set. The following are equivalent:
(i) $\mu$ is both definable over and finitely satisfiable in a small model.
(ii) $\mu$ commutes with itself: $\mu_x \otimes \mu_y = \mu_y \otimes \mu_x$.
(iii) $\mu$ is finitely approximated.
(iv) $\mu$ is fim.

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1In this article, when we refer to the Hrushovski-Pillay-Simon theorem, we mean this theorem.
It is known [CGH21] that, without assuming NIP, (iv) is strictly stronger than (iii), and (iii) is strictly stronger than (i), and (ii) need not have any special properties.\footnote{See [CGH21] Propositions 8.12, 7.12, and Example 5.3, respectively.} This highlights the fact that to generalize the Hrushovski-Pillay-Simon theorem (to arbitrary theories) requires a weakened version of the NIP.

This paper aims to generalize this theorem to arbitrary theories. Indeed, we introduce the notion dependent measure and prove that, in any theory, every dependent measure for which (i) above holds is symmetric, i.e. (ii) holds. Then we show that, assuming a local version of NIP, for any symmetric measure the conditions (i)—(iv) are equivalent. We will show that the notion ‘dependent measure’ and the local version of NIP mentioned above are much weaker than NIP, in several ways. We will argue that, from several perspectives, these conditions are the minimum assumptions that lead to these results. It is worth noting that some of the arguments presented in this article are inherently similar to [HPS13], although the key notion ‘dependent measure’ allows us to use the facts in measure theory and make a connection between them and model theory. \textit{Surprisingly, in this case, model theory and measure theory have organic relationships, and the results of one domain corresponds to the results of another domain.} We believe that the new notion is valuable in itself and this approach can have more applications in future work.

Let us give our motivation and point of view on the importance of this work. It is very natural to generalize the results in the NIP context to arbitrary theories as this approach has already been pursued by generalizations of stable to simple and NIP theories. On the other hand, this study will clarify why the arguments work in NIP theories and how they can be generalized to the outside this context. Finally, this study identifies the deep links between two different areas of mathematics, namely model theory and measure theory, and their importance in applications as the results of [HPS13] are evidence of the usefulness of links between different domains, in the latter case probability theory and model theory. Apart from these, studying ‘measures’ as mathematical objects, which are the natural generalization of types, is interesting in itself and important in applications.

To simplify to read through the paper, we list some more important observations and results: Theorems 4.1, 4.8, 5.2, 5.4 and Proposition 5.3 and Corollary 4.11.
This paper is organized as follows: In the next section we review some basic notions from measure theory. In Section 3 we introduce the notion dependent measure and give some basic properties of dependent measures. In Section 4 we generalize some results of [HPS13] on the Morley products of measures and symmetric measures to arbitrary theories. In Section 5 we study fim measures and generalize the Hrushovski-Pillay-Simon theorem using a local version of NIP. We also refine a result of [CGH21] and give some new ideas for the future work.

2 Preliminaries from measure theory

In this section we give definitions of measure theory with which we shall concerned, especially the notion of stable set of functions (or µ-stability) and its properties.

Let \( X \) be a compact Hausdorff space. The space of continuous real-valued functions on \( X \) is denoted here by \( C(X) \). The smallest \( \sigma \)-algebra containing the open sets is called the class of Borel sets. By a Borel measure on \( X \) we mean a finite measure defined for all Borel sets. A Radon measure on \( X \) is a Borel measure which is regular. Recall that a measure is complete if for any null measurable set \( E \subseteq X \) with \( \mu(E) > 0 \) and \( s < r \) such that for each \( k = \{1, \ldots, k\} \)

\[
\mu^k \{ \pi \in E^k : \forall I \subseteq k \exists f \in A \bigwedge_{i \in I} f(w_i) \leq s \land \bigwedge_{i \notin I} f(w_i) \geq r \} = (\mu E)^k.
\]

In this paper, we always assume that every Radon measure is complete.

In the following, given a measure \( \mu \) and \( k \geq 1 \), the symbol \( \mu^k \) stands for \( k \)-fold product of \( \mu \) and \( \mu^* \) stands for the outer measure of \( \mu \).

The following fundamental notion has been invented by David H. Fremlin, namely \( \mu \)-stability, which in the model theory context we will call dependent measure. From a logical point of view, this notion was first studied in [Kha16] in the framework of integral logic.

**Definition 2.1 (µ-stability).** Let \( A \subseteq C(X) \) be a pointwise bounded family of real-valued continuous functions on \( X \). Suppose that \( \mu \) is a Borel probability measure on \( X \). We say that \( A \) is \( \mu \)-stable, if there is no measurable subset \( E \subseteq X \) with \( \mu(E) > 0 \) and \( s < r \) such that for each \( k = \{1, \ldots, k\} \)

\[
\mu^k \{ \pi \in E^k : \forall I \subseteq k \exists f \in A \bigwedge_{i \in I} f(w_i) \leq s \land \bigwedge_{i \notin I} f(w_i) \geq r \} = (\mu E)^k.
\]
Remark 2.2. (i) The notion \( \mu \)-stable is an adaptation of \([\text{Fre06}, 465B]\). Indeed, by Proposition 4 of \([T87]\), it is easy to check the equivalence. For this, notice that every function \( f \in A \) is continuous on \( X \) and so the left set in the equation above is measurable. This means that \((M)\) property of Proposition 4 of \([T87]\) holds.

(ii) A measure \( \mu \) is dependent iff its completion \( \bar{\mu} \) is dependent. Indeed, recall that the product measures of \( \mu, \bar{\mu} \) are the same. (See Proposition 465C(i) of \([\text{Fre06}]\) – Version of 26.8.13.)

The following are important results connecting the notion of ‘stable’ set of functions.

Fact 2.3 (\([\text{Fre06}],\) Pro. 465D(b)). Let \( X \) be a compact Hausdorff space, \( \mu \) a Radon probability measure on \( X \), and \( A \subseteq C(X) \). If \( A \) is \( \mu \)-stable, then every function in the pointwise closure of \( A \) is \( \mu \)-measurable.

As mentioned above, in this paper, we assume that all Radon measures are complete. Note that the completeness of \( \mu \) is absolutely necessary in Fact 2.3.

Recall that the convex hull of \( A \subseteq C(X) \), denoted by \( \Gamma(A) \) or \( \text{conv}(A) \), is the set of all convex combinations of functions in \( A \), that is, the set of functions of the form \( \sum_{i=1}^{k} r_i \cdot f_i \) where \( k \in \mathbb{N}, f_i \in A, r_i \in \mathbb{R}^+ \) and \( \sum_{i=1}^{k} r_i = 1 \).

The following theorem is the fundamental result on stable sets of functions. In fact, it asserts that a set of function is stable iff it is a Glivenko–Cantelli class iff its convex hull is a Glivenko–Cantelli class (cf. \([\text{Kha21a}]\)).

Fact 2.4 (\([\text{Fre06}]\)). Let \( X \) be a compact Hausdorff space, \( \mu \) a Radon probability measure on \( X \), and \( A \subseteq C(X) \) uniformly bounded. Then the following are equivalent:

(i) \( A \) is \( \mu \)-stable.

(ii) The convex hull of \( A \) is \( \mu \)-stable.

(iii) \( \lim_{k \to \infty} \sup_{f \in A} \left| \frac{1}{k} \sum_{i=1}^{k} f(w_i) - \int f \right| = 0 \) for almost all \( w \in X^\mathbb{N} \).

(Here, \( w = (w_1, w_2, \ldots) \in X^\mathbb{N} \) and the measure on \( X^\mathbb{N} \) is the usual product measure.)

Explanation. The direction (i) \( \implies \) (ii) is Theorem 465N(a) of \([\text{Fre06}]\). The converse is evident. (See also Proposition 465C(a)(i) of \([\text{Fre06}]\).) The equivalence (i) \( \iff \) (iii) is the equivalence (i) \( \iff \) (ii) of Theorem 465M of \([\text{Fre06}]\). Again, we emphasize that the completeness of \( \mu \) is necessary in the direction (i) \( \implies \) (iii).
The last fact shows that, on stable set of functions, the topology of pointwise convergence is stronger than the topology of convergence in measure.

**Fact 2.5** ([Fre06], Thm. 465G). Let $X$ be a compact Hausdorff space, $\mu$ a Radon probability measure on $X$, and $A \subseteq C(X)$ a $\mu$-stable set. Let $(f_i)$ be a net in $A$ such that $f_i \to f$ in the topology of pointwise convergence. Then $\int |f_i - f| \to 0$.

### 3 Dependent Keisler measures

In this section we introduce the notion of dependent measure (Definition 3.2), and give some of its principal properties and examples.

The model theory notation is standard, and a text such as [S15] will be sufficient background. We fix a first order language $L$, a complete $L$-theory $T$ (not necessarily NIP), an $L$-formula $\phi(x,y)$, and a subset $A$ of the monster model of $T$. The monster model is denoted by $U$. We let $\phi^*(y,x) = \phi(x,y)$. We define $p = \text{tp}_\phi(a/A)$ as the function $\phi(p,y) : A \to \{0,1\}$ by $b \mapsto \phi(a,b)$. This function is called a complete $\phi$-types over $A$. The set of all complete $\phi$-types over $A$ is denoted by $S_\phi(A)$. We equip $S_\phi(A)$ with the least topology in which all functions $p \mapsto \phi(p,b)$ (for $b \in A$) are continuous. It is compact and Hausdorff, and is totally disconnected. Let $X = S_\phi^*(A)$ be the space of complete $\phi^*$-types over $A$. Note that the functions $q \mapsto \phi(a,q)$ (for $a \in A$) are continuous, and as $\phi$ is fixed we can identify this set of functions with $A$. So, $A$ is a subset of all bounded continuous functions on $X$, denoted by $A \subseteq C(X)$.

A Keisler measure over $A$ in the variable $x$ is a finitely additive probability measure on the Boolean algebra of $A$-definable sets in the variable $x$, denoted by $L_x(A)$. Every Keisler measure over $A$ can be represented by a regular Borel probability measure on $S_x(A)$, the space of types over $A$ in the variable $x$. A measure over $U$ is called a global Keisler measure. The set of all measures over $A$ in the variable $x$ is denoted by $\mathcal{M}_x(A)$ or $\mathcal{M}(A)$. We will sometimes write $\mu$ as $\mu_x$ or $\mu(x)$ to emphasize that $\mu$ is a measure on the variable $x$.

For a formula $\phi(x,y)$, a Keisler $\phi$-measure over $A$ in the variable $x$ is a finitely additive probability measure on the Boolean algebra of $\phi$-definable sets over $A$ in the variable $x$, denoted by $L_\phi(A)$. Recall that a $\phi$-definable sets over $A$ is a Boolean combination of the instances $\phi(x,b), b \in A$. The set of all $\phi$-measures over $A$ in the variable $x$ is denoted by $\mathcal{M}_\phi(A)$.
Given a model $M$, a $L(M)$-formula $\theta(x)$, and types $p_1(x), \ldots, p_n(x)$ over $M$, the average measure of them (for $\theta(x)$), denoted by $\text{Av}(p_1, \ldots, p_n)$, is defined as follows:

$$\text{Av}(p_1, \ldots, p_n; \theta(x)) := \frac{|\{i : \theta(x) \in p_i, i \leq n\}|}{n}.$$ 

We first revisit a useful dictionary of [Kha21a] that is used in the rest of the paper and can also be used in future work. For the definition of finitely satisfiable (definable, Borel definable) measures see Definition 7.16 of [S15].

**Fact 3.1.** Let $T$ be a complete theory, $M$ a model of $T$ and $\phi(x, y)$ a formula.

(i) (Pillay) There is a correspondence between global $M$-finitely satisfiable $\phi$-types $p(x)$ and the functions in the pointwise closure of all functions $\phi(a, y) : S_{\phi^*}(M) \to \{0, 1\}$ for $a \in M$, where $\phi(a, q) = 1$ if and only if $\phi(a, y) \in q$.

(ii) The map $p \mapsto \delta_p$ is a correspondence between global $\phi$-types $p(x)$ and Dirac measures $\delta_p(x)$ on $S_{\phi}(U)$, where $\delta_p(A) = 1$ if $p \in A$, and $= 0$ in otherwise. Moreover, $p(x)$ is finitely satisfiable in $M$ iff $\delta_p(x)$ is finitely satisfiable in $M$.

(iii) There is a correspondence between global $\phi$-measures $\mu(x)$ and regular Borel probability measures on $S_{\phi}(U)$. Moreover, a global $\phi$-measures is finitely satisfiable in $M$ iff its corresponding regular Borel probability measure is finitely satisfiable in $M$.

(iv) The closed convex hull of Dirac measures $\delta(x)$ on $S_{\phi}(U)$ is exactly all regular Borel probability measures $\mu(x)$ on $S_{\phi}(U)$. Moreover, the closed convex hull of Dirac measures on $S_{\phi}(U)$ which are finitely satisfiable in $M$ is exactly all regular Borel probability measures $\mu(x)$ on $S_{\phi}(U)$ which are finitely satisfiable in $M$.

(v) There is a correspondence between global $M$-finitely satisfiable $\phi$-measures $\mu(x)$ and the functions in the pointwise closure of all functions of the form $\frac{1}{n} \sum^n \theta(a_i, y)$ on $S_{\phi^*}(M)$, where $\theta$ is a $\phi^*$-formula, $a_i \in M$, and $\theta(a_i, q) = 1$ if and only if $\theta(a_i, y) \in q$.

The above fact actually shows the ideology that we follow in this article. That is, the finitely satisfied (and invariant) measures in this paper are functions on specific topological spaces. The reader of this article can become more familiar with our approach as well as its historical trend by reading [Kha20], [Kha17], [KP18] and [Kha21a].

\^3Recall that a $\phi$-formula is a Boolean combination of instances of $\phi(x, b)$, $b \in M$. 

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Convention. Recall that every regular Borel probability measure \( \mu \) has a unique completion \( \bar{\mu} \). In this paper, without loose of generality we can assume that every measure is complete. That is, \( \mu = \bar{\mu} \). (The crucial notion of this paper (i.e. Definition 3.2) is neutral to completion. Cf. Remark 3.3(ii) below.)

Let \( \mu \in \mathcal{M}_x(U) \) and \( \mu_\phi \) its restriction to \( S_\phi(M) \) (equivalently, its restriction to Boolean algebra of \( \phi \)-formulas). As the restriction map \( r_\phi : S_x(M) \to S_\phi(M) \) is a quotient map, it is easy to verify that \( \mu_\phi(X) = \mu(r_\phi^{-1}(X)) \) for any Borel subset \( X \subseteq S_\phi(M) \). (See Remark 1.2 in [CGH21].) For \( A \subseteq B \) and \( \mu \in \mathcal{M}_x(B) \), \( \mu|_A \in \mathcal{M}_x(A) \) is the restriction of \( \mu \) by the quotient map \( r : S_x(B) \to S_x(A) \). The restriction of \( \mu|_A \) to \( S_\phi(A) \) is denoted by \( \mu_{\phi,A} \).

The crucial notion of the paper is as follows. (Compare Definition 2.1).

**Definition 3.2 (Dependent Measures).** Let \( T \) be a complete theory, \( M \) a model, and \( \mu_x \in \mathcal{M}(M) \).

(i) Suppose that \( A \subseteq B \subseteq M \). We say that \( \mu \) is dependent over \( B \) and in \( A \), if for any formula \( \phi(x,y) \) there is no \( E \subseteq S_\phi(B) \) measurable, \( \mu_{\phi,B}(E) > 0 \) (where \( \mu_{\phi,B} = (\mu|_B)_\phi \)) such that for each \( k \), \( (\mu_{\phi,B}^k)D_k(A, B, E, \phi) = (\mu_{\phi,B} E)^k \) where

\[
D_k(A, B, E, \phi) = \{ \bar{p} \in E^k : \forall I \subseteq k \exists b \in A \bigwedge_{i \in I} \phi(p_i, b) = 0 \land \bigwedge_{i \notin I} \phi(p_i, b) = 1 \}.
\]

(Recall that \( \phi(p_i, b) = 1 \) if \( \phi(x, b) \in p_i \) and \( \phi(p_i, b) = 0 \) in otherwise.)

(ii) We say that \( \mu \) is dependent, if \( \mu \) is dependent over \( M \) and in \( M \).

The above parameters \( A \) and \( B \) have complicated the definition. The reason for using parameters is that it gives us the possibility to expand or restrict functions/formulas and spaces, which will be used in some places. (In Remark 3.3(iii) below, we study monotonicity properties of “over \( B \) and in \( A \)” as one varies \( B \) and \( A \).)

On the other hand, it is possible to provide a definition that can be explained by model-theoretic intuition. In fact, using the argument of Theorem 4.1 below (or Fact 2.4), it is easy to show that \( \mu_x \in \mathcal{M}(M) \) is dependent if and only if

\[
\mu^\omega \left( \left\{ \bar{p} \in \prod_{i<\omega} S_{x_i}(M) : \lim_{k \to \infty} \sup_{x \in M} \left| \mu(\phi(x, b)) - \frac{1}{k} \sum_{i=1}^k \phi(p_i, b) \right| = 0 \right\} \right) = 1.
\]

(Here \( \mu^\omega \) is the usual product measure.)
Remark 3.3. (i) The notion dependent measure is an adaptation of Definition 2.1 above to the model theory context. Indeed, note that every function \( \phi(x, b) \) is continuous on \( S_x(B) \) and so \( D_k(A, B, E, \phi) \) is \( \mu_{\phi, B}^k \)-measurable. This means that \( (\mu_{\phi, B}^k)^*(D_k(A, B, E, \phi)) = \mu_{\phi, B}^k(D_k(A, B, E, \phi)) \) and (M) property of Proposition 4 of [T87] holds.

(ii) Recall from Remark 2.2(ii) that a Keisler measure \( \mu \) is dependent iff its completion \( \bar{\mu} \) is dependent.

(iii) (1) Let \( A \subseteq B \). As the restriction map \( r : S_x(B) \to S_x(A) \) is a quotient map, it is easy to verify that \( \mu|_A(X) = \mu|_B(r^{-1}(X)) \) for any Borel subset \( X \subseteq S_x(A) \). (See Remark 1.2 in [CGH21].) Now, by Proposition 465C(d) of [Fre06] version of 26.8.13, if \( \mu \) is dependent over \( A \) and in \( A \), then it is dependent over \( B \) and in \( A \). The converse is clear. (2) Let \( A \subseteq B \subseteq M \). It is easy to verify that: if \( \mu \) is dependent over \( M \) and in \( B \), then it is dependent over \( M \) and in \( A \). (Cf. [Fre06] version of 26.8.13, Proposition 465C(a)(i).)

By (1) and (2), \( \mu \in \mathcal{M}(M) \) is dependent if and only if for any \( A \subseteq B \subseteq M \), \( \mu \) is dependent over \( B \) and in \( A \).

(iv) As the restriction map \( r_\phi : S_x(B) \to S_\phi(B) \) is a quotient map, it is easy to verify that \( \mu_\phi(E) = \mu(r_\phi^{-1}(E)) \) for any Borel subset \( E \subseteq S_\phi(B) \). (See Remark 1.2 in [CGH21].) Therefore, by Proposition 465C(d) of [Fre06] again, if \( \mu \) is dependent over \( B \) and in \( A \), then whenever \( E \subseteq S_x(B) \) is measurable, \( \mu|_B(E) > 0 \), there is some \( k \geq 1 \) such that \( (\mu|_B)^k D_k(A, B, E) < (\mu|_B)^k \), where \( D_k(A, B, E) = \{ \overline{p} \in E^k : \forall I \subseteq k \exists b \in A \left( \bigwedge_{i \in I} \phi(p_i, b) = 0 \land \bigwedge_{i \not\in I} \neg \phi(p_i, b) = 1 \right) \} \).

The following is an important property of the notion dependent measure.

Proposition 3.4. Let \( T \) be a complete theory, \( M \) a model, and \( \mu \in \mathcal{M}(M) \). If \( \mu \) is dependent, then for any \( A \subseteq M \) and any \( L(A) \)-formula \( \phi(x, y) \), every function in the pointwise closure of the convex hull of \( \{ \phi(x, b) : S_\phi(A) \to \{0, 1\} \mid b \in A \} \) is \( \mu \)-measurable.

Proof. This is a consequence of Facts 2.3 and 2.4. Indeed, recall that \( \mu \) is dependent iff its completion \( \bar{\mu} \) is. Now, by the above convention, \( \mu_{\phi, A} \) (is complete and) satisfies in the assumptions of Facts 2.3 and 2.4. \( \Box \)

Recall from [KPI8] Def. 1.1 that a formula \( \phi(x, y) \) has NIP in a model \( M \) if there is no countably infinite sequence \( (a_i) \in M \) such that for all finite disjoint subsets \( I, J \subseteq \mathbb{N} \), \( M \models \exists y (\bigwedge_{i \in I} \phi(a_i, y) \land \bigwedge_{i \in J} \neg \phi(a_i, y)) \). The formula \( \phi \) has NIP for the theory \( T \) iff it has NIP in every model \( M \) of \( T \).
iff it has NIP in some model $M$ of $T$ in which all types over the empty set in countably many variables are realised.

**Proposition 3.5.** (i) For any theory, every type is dependent.  
(ii) A theory $T$ is NIP iff in any model $M$ of $T$, every Keisler measure over $M$ is dependent iff every Keisler measure over some model $M$ of $T$ in which all types over the empty set in countably many variables are realised.  
(iii) For any theory $T$ and any model $M$ of $T$, if every Keisler measure over $M$ is dependent, then every formula has NIP in $M$.

**Proof.** (i) is evident, by definition. (See also Theorem 5 of [T87].)  
(ii): It is easy to see that if $T$ is NIP, then for some $k$, $D_k(A, B, E, \phi) = \emptyset$. (Cf. Definition 3.2.) For the converse, by Proposition 3.4, for any model $M$ and any formula $\phi(x,y)$, every function in the pointwise closure of $\{\phi(x,b) : S_\phi(M) \to \{0,1\} \mid b \in M\}$ is measurable with respect to any Keisler measure over $M$ (in the variable $x$). This means that every function in the pointwise closure is universally Radon measurable. (Recall that every measure is complete and regular and therefore Radon.) By the equivalence $(iv) \iff (vi)$ of Theorem 2F of [BFT78], it is easy to see that $\phi$ has NIP in any model of $T$. By Remark 2.1 of [KP18], this means that $\phi$ has NIP for the theory $T$.  
(iii): By Proposition 3.4, for any formula $\phi(x,y)$, every function in the pointwise closure of $\{\phi(x,b) : S_\phi(M) \to \{0,1\} \mid b \in M\}$ is measurable with respect to any Keisler measure over $M$. By the equivalence $(iv) \iff (vi)$ of Theorem 2F of [BFT78], $\phi(x,y)$ has NIP in $M$.  

**Example 3.6.** (i) By Proposition 3.5 above, all types in any theory, and all measures in NIP theories are dependent.  
(ii) We say that a measure $\mu$ is purely atomic if there are Dirac measures $(\delta_n : n < \omega)$ such that $\mu = \sum_1^\infty r_n \delta_n$ where $r_n \in [0,1]$ and $\sum_1^\infty r_n = 1$. By definition, it is easy to verify that any purely atomic measure is dependent (cf. also Theorem 5 of [T87]). In [CG20], a measure $\mu$ is called trivial, if (1) it is purely atomic (i.e. $\mu = \sum_1^\infty r_n \delta_n$), and (2) any $\delta_n$ is realized in $U$, i.e. $\delta_n = \text{tp}(a_n/U)$ for some $a_n \in U$. It is shown [CG20, Theorem 4.9] that, in the theories of the random graph and the random bipartite graph, every definable and finitely satisfiable measure is trivial. This means that such measures are dependent. (Similarly, all definable and finitely satisfiable measures in the theories in [CG20, Corollary 4.10] are dependent.)  
(iii) Furthermore, in Proposition 5.3 below, we show that any finitely independent measure (in any theory) is dependent.  
In Example 4.9 below we will present a measure that is not dependent.
The following result allows us to create new dependent measures from the previous ones.

**Proposition 3.7.** The set \( M^d_x(U) \) of all global dependent measures (in the variable \( x \)) is convex.

**Proof.** Let \( \mu, \nu \in M^d_x(U) \) and \( r \in [0,1] \). Set \( \lambda = r\mu + (1-r)\nu \). Then, for formula \( \phi(x,y) \), \( b \in U \) and \( \bar{p} = (p_1, \ldots, p_k) \), we have

\[
|Av(\bar{p}; \phi(x,b)) - \lambda(\phi(x,b))| \leq r|Av(\bar{p}; \phi(x,b)) - \mu(\phi(x,b))| + (1-r)|Av(\bar{p}; \phi(x,b)) - \nu(\phi(x,b))|.
\]

We apply Fact 2.4 for \( \mu, \nu \) and conclude that \( \lim_{k \to \infty} \sup_{b \in U} |Av(\bar{p}; \phi(x,b)) - \lambda(\phi(x,b))| = 0 \) for almost all \( (p_i) \in (S_\phi(U))^N \). This means that \( \lambda \) is dependent. (Note that this argument works for all \( A \subseteq B \subseteq U \) in Definition 3.2. See also Remark 3.3(iii) above.) \( \square \)

## 4 Dependence and symmetry

In this section we generalize some results of [HPS13] on the Morley products of measures and symmetric measures. The following is the fundamental property of the notion of dependent measure.

**Theorem 4.1.** Let \( T \) be a complete theory, \( M \) a model, and \( \mu_x \in M(U) \). If \( \mu|_M \) is dependent, then for any \( \mu|_M \)-measurable subsets \( X_1, \ldots, X_m \subseteq S_x(M) \) and \( \epsilon > 0 \), there are \( n \in \mathbb{N} \) and \( E \subseteq (S_x(U))^n \), with \( (\mu^n)^*E \geq 1 - \epsilon \), such that for every \( b \in M \) and \( k \leq m \),

\[
|\mu(\phi(x,b) \cap X_k) - Av(p_1|_M, \ldots, p_n|_M; \phi(x,b) \cap X_k)| \leq \epsilon, \quad (*)
\]

for all \( (p_1, \ldots, p_n) \in E \). (Here \( p_i|_M \) is the restriction of \( p_i \) to \( S_x(M) \).)

**Proof.** First note that, by the definition and Remark 3.3(iv), we can use \( \mu|_M \) instead of \( \mu_{\phi,M} \) (in Definition 3.2). As \( \mu|_M \) is dependent (equivalently the set \( \{ \phi(x,b) : S_x(M) \to \{0,1\} \mid b \in M \} \) is \( \mu|_M \)-stable in the sense of Definition 2.1), by Proposition 465C(a)(i) and (b)(ii) of [Fre06]—Version of 26.8.13, the set \( \bigcup_{k=1}^m \{ \phi(x,b) \times \chi_{X_k} : b \in M \} \) is \( \mu|_M \)-stable, where \( \chi_{X_k} \) is the characteristic function of \( X_k \). Therefore, by Fact 2.4, we have

\[
\sup_{b \in M} \left| \frac{1}{n} \sum_{i=1}^n \phi(p_i, b) \times \chi_{X_k} - \mu(\phi(x,b) \times \chi_{X_k}) \right| \to 0
\]
as $n \to \infty$ for all $k \leq m$ and for almost every $(p_i) \in S_x(M)^N$ with the product measure $(\mu|_M)^N$. Now, it is easy to verify that the claim holds. Indeed, one can see directly (or using Theorem 11-1-1(c) of [184]) that there are $n \in \mathbb{N}$ and $F \subseteq (S_x(M))^n$, with $(\mu|_M)^*F \geq 1 - \epsilon$, such that for every $b \in M$ and $k \leq m$, $|\mu(\phi(x,b) \cap X_k) - Av(p'_1, \ldots, p'_n; \phi(x,b) \cap X_k)| \leq \epsilon$ for all $(p'_i) \in F$. Finally, use Remark 3.3(iii) above and find the desired set $E \subseteq (S_x(U))^n$ such that $(\ast)$ holds. (Here, $p'_i = p_i|_M$ for some $p_i \in S_x(U).$)

In the following, for $\mu \in \mathcal{M}(U)$, the support of $\mu$ is denoted by $S(\mu)$. (Cf. [15], p. 99.)

**Corollary 4.2.** Let $T$ be a complete theory, $M$ a model, and $\mu \in \mathcal{M}_x(U)$. If $\mu|_M$ is dependent, then for any $\mu|_M$-measurable subsets $X_1, \ldots, X_m \subseteq S_x(M)$ and $\epsilon > 0$, there are $p_1|_M, \ldots, p_n|_M \in S_x(M)$ such that for every $b \in M$ and $k \leq m$,

$$|\mu(\phi(x,b) \cap X_k) - Av(p_1|_M, \ldots, p_n|_M; \phi(x,b) \cap X_k)| \leq \epsilon.$$  

Furthermore, we can assume that $p_i \in S(\mu)$ for all $i$.

**Proof.** Immediate, by Theorem 4.1. (Recall from [G20, Pro. 2.10] that $\mu(S(\mu)) = 1$. This assures us that we can assume that $p_i \in S(\mu)$ for all $i$.)

The following result allows us to define the Morley product of a finitely satisfiable measure and a dependent measure.

**Proposition 4.3.** Let $\mu_x$ be global $M$-finitely satisfied measures, $\lambda_y$ a global dependent measure and $\phi(x,y;b)$ an $L(U)$-formula. Let $N \supseteq Mb$ be a model and define the function $f : S_{\phi^*}(N) \to [0,1]$ by $q \mapsto \mu(\phi(x,d;b))$ for some (any) $d \models q$. Then $f$ is $\lambda_y|_N$-measurable.

**Proof.** As $\mu$ is $M$-finitely satisfied, by Fact 3.1(v), $f$ is in the closure of the convex hull of $\{\phi(a,y;b) : S_{\phi^*}(N) \to \{0,1\} | a \in M\}$. Now, as $\lambda_y$ is dependent, by Proposition 3.3 above, $f$ is $\lambda_y|_N$-measurable. (Indeed, recall that $\lambda_y$ is a complete measure, with the above convention.)

\[\text{4\textsuperscript{Recall that }\phi^*(y,x;b) = \phi(x,y;b)}\]
Definition 4.4. By the assumptions of Proposition 4.3 we define the Morley product measure $\mu(x) \otimes \lambda(y)$ as follows:

$$\mu(x) \otimes \lambda(y)(\phi(x, y; b)) = \int_{S_\phi^*(N)} f d\lambda|_N.$$  

It is easy to verify that the definition does not depend on the choice of $N$. We will sometimes write $f$ as $f^\phi \mu$ (or $f^\phi \mu_N$) to emphasize that it is related to $\mu$, $\phi$ (and $N$) as above.

The following is a generalization of Lemma 7.1 of [S15], although the proof is essentially the same, using the previous observations.

Lemma 4.5. Let $\mu_x, \lambda_y$ be global dependent measures such that $\mu_x$ is $M$-finitely satisfied (or Borel definable over $M$) and $\lambda_y$ is $M$-finitely satisfied. If $\mu_x \otimes q_y = q_y \otimes \mu_x$ for any $q_y \in S_y(U)$ in the support of $\lambda_y$, then $\mu_x \otimes \lambda_y = \lambda_y \otimes \mu_x$.

Proof. Let $\phi(x, y; b) \in L(U)$ and $N \supseteq Mb$ a model. Let $f = f^\phi \mu$ be as above. As $\mu_x$ is $M$-finitely satisfied and $\lambda_y$ is dependent (or just $\mu_x$ is Borel definable over $M$), the Morley product $\mu \otimes \lambda$ is well-defined. (Cf. Definition 4.4 and Proposition 4.3) Fix $\epsilon > 0$. Let $\sum^n_{i=1} r_i \cdot \chi_{X_i}$ be a simple $\lambda|_N$-measurable functions such that $|f(q) - \sum^n_{i=1} r_i \cdot \chi_{X_i}(q)| < \epsilon$ for all $q \in S_\phi^*(N)$. (That is, $X_1, \ldots, X_n \in S_\phi^*(N)$ are $\lambda|_N$-measurable, $\chi_{X_i}$ is the characteristic function of $X_i$, and $r_i \in [0, 1]$ for $i \leq n$.) By Corollary 4.2 there are $q_1, \ldots, q_m \in S(\lambda)$ such that if $\tilde{\lambda} = \frac{1}{m} \sum q_i$ then

1. $|\tilde{\lambda}(X_i) - \lambda(X_i)| < \epsilon$ for all $i \leq n$, and
2. $|\tilde{\lambda}(\phi(a, y; b)) - \lambda(\phi(a, y; b))| < \epsilon$ for all $a \in U$.

(Here, we let $r^{-1}(X_i) := X_i$ again, where $r : S_\phi^*(U) \to S_\phi^*(N)$ is the restriction map.) Note that, as the $q_i$’s are types, the product measure $\tilde{\lambda} \otimes \mu$ is well-defined. The product measure $\tilde{\lambda} \otimes \mu$ is well-defined since $\mu$ is dependent and the $q_i$’s are $M$-finitely satisfied. As $\mu$ commutes with $\tilde{\lambda}$ and $\epsilon$ is arbitrary, by the conditions (1),(2), it is easy to see that $\mu_x \otimes \lambda_y(\phi(x, y; b)) = \lambda_y \otimes \mu_x(\phi(x, y; b))$. \qed

The above argument can be further visualized in the language of analysis. Given $\epsilon > 0$, and $r, s \in \mathbb{R}$, we write $r \approx \epsilon s$ to denote $|r - s| < \epsilon$. With the
above assumptions, the argument of Lemma 4.5 is as follows:

\[ \mu \otimes \lambda(\phi(x, y; b)) = \int f_\phi^* d\lambda \approx \epsilon \int \left( \sum r_i \cdot \chi_{X_i} \right) d\lambda = \sum r_i \cdot \lambda(X_i) \]

\[ \approx \epsilon \sum r_i \cdot \bar{\lambda}(X_i) = \int \left( \sum r_i \cdot \chi_{X_i} \right) d\tilde{\lambda} \approx \epsilon \int f_\phi^* d\tilde{\lambda} \]

\[ = \mu \otimes \bar{\lambda}(\phi(x, y; b)) = \tilde{\lambda} \otimes \mu(\phi(x, y; b)) = \int f_\phi^* d\mu \]

\[ \approx \epsilon \int f_\phi^* d\mu = \lambda \otimes \mu(\phi(x, y; b)). \]

Remark 4.6. Assuming that \( \mu \) and \( \lambda \) are dependent, and using Lemma 4.5, one can give a generalization of [HPS13, Lemma 3.1]. That is, if \( \mu \in \mathcal{M}_x(U) \) is definable over \( M \) and dependent, and \( \lambda \in \mathcal{M}_y(U) \) is \( M \)-finitely satisfied and dependent, then \( \mu_x \otimes \lambda_y = \lambda_y \otimes \mu_x \). (See also [S15, Prop. 7.22].) Nevertheless, we prove something stronger (cf. Theorem 4.8 below). In fact, the dependence of \( \lambda \) is unnecessary. Although, for the sake of completeness we give a model theoretic proof of the weaker result in Appendix.

First we need a lemma that is interesting in itself and is a generalization of [ChG21, Pro. 6.3].

**Lemma 4.7.** Let \( \mu \in \mathcal{M}_x(U) \), \( \lambda \in \mathcal{M}_y(U) \) such that \( \mu \) is dependent and \( \lambda \) is finitely satisfiable in a small model \( M \). Suppose that \( (\bar{a}_i)_i \) is a net in \( (M^y)^{<\omega} \) such that \( \text{Av}(\bar{a}_i) \to \lambda \), then:

\[ \text{Av}(\bar{a}_i) \otimes \lambda \to \lambda \otimes \mu. \]

**Proof.** Let \( \phi(x, y) \) be a formula, and set \( \lambda_i = \text{Av}(\bar{a}_i) \). By Fact 2.4 as \( \mu \) is dependent, the convex hull of the set \( \{ f_{\phi^*}(a) : a \in M \} \) is \( \mu \)-stable. Therefore,

\[ \lim_i [\lambda_i \otimes \mu](\phi(x, y)) = \lim_i \int f_{\phi^*} d\lambda_i = \lim_i \int f_{\phi^*} d\mu = \int f_{\phi^*} d\mu = \lambda \otimes \mu. \]

The second equality \( = \) holds by Fact 2.5. \( \square \)

The following is a generalization of [HPS13, Lemma 3.1] and [CGH21, Proposition 5.1].

**Theorem 4.8.** Let \( \mu_x \in \mathcal{M}(U) \) be a dependent measure and definable over \( M \), and \( \lambda_y \in \mathcal{M}(U) \) be \( M \)-finitely satisfied. Then \( \mu_x \otimes \lambda_y = \lambda_y \otimes \mu_x \).
Proof. By Fact 3.1(iv), there is \((\bar{a}_i) \in (M^y)^{<\omega}\) such that \(Av(\bar{a}_i) \to \lambda\). Set \(\lambda_i = Av(\bar{a}_i)\).

\[
\lambda \otimes \mu(\phi(x, y; d)) \overset{(1)}{=} \left[\lim_i (\lambda_i \otimes \mu)\right](\phi(x, y)) \\
\overset{(2)}{=} \left[\lim_i (\mu \otimes \lambda_i)\right](\phi(x, y)) \\
\overset{(3)}{=} [\mu \otimes (\lim_i \lambda_i)](\phi(x, y)) = \mu \otimes \lambda(\phi(x, y; d)).
\]

As the \(\lambda_i\)'s are smooth over \(M\), (2) follows. (1) and (3) follows from Lemma 4.7 and definability of \(\mu\), respectively. (Cf. also [CGH21, Lemma 5.4] for a proof of (3).) \(\Box\)

Example 4.9 (Non-example). (i) In Proposition 7.14 of [CGH21], it is shown that there is a complete theory \(T\), a global definable measure \(\mu_x\) and a finitely satisfiable (and definable) type \(q_y\) such that \(\mu \otimes q \neq q \otimes \mu\). Therefore, by Theorem 4.8, \(\mu\) is not dependent. (A direct examination of the fact that \(\mu\) is not dependent can be instructive and will be done elsewhere.)

(ii) By Proposition 3.5, in any non-NIP theory, there is a measure that is not dependent.

Definition 4.10. Let \(M\) be a small model and \(\mu \in \mathcal{M}(U)\).

(i) We say that \(\mu\) is dfs over \(M\) if it is both definable over and finitely satisfiable in \(M\).

(ii) We say that \(\mu\) is ddfs (over \(M\)) if it is both dfs (over \(M\)) and dependent.

Corollary 4.11. Let \(\mu \in \mathcal{M}(U)\) be ddfs. Then \(\mu\) is symmetric, that is, for any \(n \in \mathbb{N}\) and any permutation \(\sigma\) of \(\{1, \ldots, n\}\), \(\mu_{x_1} \otimes \cdots \otimes \mu_{x_n} = \mu_{\sigma x_1} \otimes \cdots \otimes \mu_{\sigma x_n}\).

Proof. This follows from Theorem 4.8 and associativity of \(\otimes\) for definable measures. (See [CGH21, Thm 2.18] for a proof of associativity of \(\otimes\) for definable measures.) \(\Box\)

Remark 4.12. An obvious question is whether the argument of Theorem 4.8 works with a weaker condition than dependence of measures. The answer is negative from one perspective: As measurability is necessary for the definition of \(\mu \otimes \lambda\), such a condition must require measurability. On the other hand, there is a weaker notion of ‘dependent measure’ which is equivalent to measurability, namely \(R\)-stable. (See [Fre06, 465S] or [T84].) The only
difference is in the definition of product of measures. Therefore, the above arguments work if and only if we use the notion $R$-stable (or $R$-dependent) measure instead of dependent measure. In the other word, the notion is optimal.

5 \textit{fim} and local NIP

In this section we show that, assuming a local version of NIP, any ddfs measure is \textit{fim}. Recall from [HPS13] that: a global measure $\mu$ is \textit{fim} (over $M$) if (i) for every $\phi(x,y) \in L$, and $\epsilon > 0$, for arbitrary sufficiently large $n$, there is an $L(M)$-formula $\theta(x_1, \ldots, x_n)$ such that $\mu^{(n)}(\theta_\epsilon) \geq 1 - \epsilon$, and (ii) for all $b$, $|\mu(\phi(x,b)) - Av(a_1, \ldots, a_n; \phi(x,b))| \leq \epsilon$ for all $(a_1, \ldots, a_n) \in \theta_\epsilon(U)$.

\textbf{Definition 5.1.} Let $M$ be a model and $\phi(x,y)$ a formula. We say that $\phi(x,y)$ is uniformly NIP in $M$ if there is a natural number $n = n_{\phi,M}$ such that there is no $a_1, \ldots, a_n \in M$ such that for any $I \subseteq \{1, \ldots, n\}$, $M \models \exists y \bigwedge_{i \in I} \phi(a_i, y) \land \bigwedge_{i \notin I} \neg \phi(a_i, y)$.

It is easy to see that $\phi$ has NIP for the theory $T$ iff it is uniformly NIP in the monster model of $T$ iff it is uniformly NIP in some model of $T$ in which all types over the empty set in countably many variables are realised.

Recall that $\mu$ is ddfs (over $M$) if it is both dfs (over $M$) and dependent.

\textbf{Theorem 5.2.} Let $\mu \in M(U)$ be ddfs over $M$. Suppose that every formula is uniformly NIP in $M$. Then $\mu$ is \textit{fim} (over $M$).

\textbf{Proof.} As $\mu$ is ddfs, by Corollary 4.11 $\mu$ is symmetric. Now, the argument of Lemma 3.4 of [HPS13] works well by using $\mu|M$ instead of $\mu$. Therefore, similar to [HPS13, Corollary 3.5], for every $\phi(x,y) \in L$, and $\epsilon > 0$, there is an $L(M)$-formula $\theta(x_1, \ldots, x_n)$ such that:

(i) $\mu^{(n)}(\theta_\epsilon) \geq 1 - \epsilon$, AND,

(ii) for all $b' \in M$, $|\mu(\phi(x,b')) - Av(a_1', \ldots, a_n'; \phi(x,b'))| \leq \epsilon$ for all $(a_1', \ldots, a_n') \in \theta_\epsilon(M)$.

Suppose that $(a_1, \ldots, a_n) \in \theta_\epsilon(U)$ and $b \in U$. Without loss of generality we can assume that $|M| = \aleph_0$. (See also Proposition 4.14 in [G21].) As

\footnote{In the notion, ‘uniformly’ emphasized that, in contrast to ‘NIP in a model’ in [KP18], there is a natural number $n_{\phi,M}$ for any formula $\phi$.}

\footnote{Although this assumption is not necessary in our argument, we considered it to simplify the discussion.
$M$ is countable, there is a sequence $c_i = (a_1^i, \ldots, a_n^i, b^i) \in M^{n+1}$ such that $tp(c_i/M) \rightarrow tp(a_1, \ldots, a_n, b/M)$ in the logic topology.\footnote{Notice that we can consider a countable fragment of $T$. Therefore, we can assume that $S_n(M)$ is separable, i.e. all types that were realized in $M$ are dense in $S_n(M)$. As $S_n(M)$ is separable and compact, it is metrizable.}

Now,

$$|\mu(\phi(x,b)) - Av(a_1, \ldots, a_n; \phi(x,b))| = \lim_i |\mu(\phi(x,b^i)) - Av(a_1^i, \ldots, a_n^i; \phi(x,b^i))|.$$ (Notic that definability of $\mu$ is used for $\lim_i \mu(\phi(x,b_i)) = \mu(\phi(x,b))$. See also Proposition 4.4 of [CGH21].) Also, as $tp(c_i/M) \rightarrow tp(a_1, \ldots, a_n, b/M)$ and $(a_1, \ldots, a_n) \in \theta_c(U)$, there is a natural number $k$ such that for all $i \geq k$, $(a_1^i, \ldots, a_n^i) \in \theta_c(M)$. Putting everything together, (ii) for all $b \in U$, $|\mu(\phi(x,b)) - Av(a_1, \ldots, a_n; \phi(x,b))| \leq \epsilon$ for all $(a_1, \ldots, a_n) \in \theta_c(U)$. This means that $\mu$ is fim.

An obvious question is whether each ‘fim’ measure is dependent. A positive answer indicates that the notion ‘dependent measure’ is necessary and the least possible one expects.

**Proposition 5.3.** Every fim measure is dependent.

**Proof.** Let $\mu_x \in \mathfrak{M}(U)$ be fim over a small model $M$. For any formula $\phi(x,y)$, there are formulas $\theta_{\epsilon_n}(x_1, \ldots, x_n) \in L(M)$ such that $\mu^{(n)}(\theta_{\epsilon_n}) \rightarrow 1$, $\epsilon_n \rightarrow 0$ and for all $b$, $|\mu(\phi(x,b)) - \frac{1}{n} \sum_1^n \phi(a_i, b)| \leq \epsilon_n$ for all $(a_1, \ldots, a_n) \in \theta_{\epsilon_n}(U)$. Therefore, it is easy to verify that for any $\epsilon > 0$,

$$(\mu^n)^* \{ (p_1, \ldots, p_n) \in (S_{x_1}(U))^n : \sup_{b \in U} |\mu(\phi(x,b)) - \frac{1}{n} \sum_1^n \phi(p_i, b)| \leq \epsilon \} \rightarrow 1.$$ Indeed, note that if $p \in S_{x_1, \ldots, x_n}(U)$ and $\theta_{\epsilon_n} \in p$ then $(p|_{x_1}, \ldots p|_{x_n})$ is in the measured set on the left. This is enough, by the equivalence (i) $\iff$ (iii) of Fact 2.4 (Recall also Definition 3.2 and Remark 3.3(i).)

From one perspective, the following result generalizes Theorem 5.16 of [CGH21]. It uses Proposition 5.3 to ensure that products of fim and finitely satisfied measures make sense.

**Theorem 5.4.** Let $\mu_x \in \mathfrak{M}(U)$ be fim (over $M$), and $\lambda_y$ be $M$-finitely satisfied. Then $\mu$ commutes with $\lambda$. 

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Proof. This follows from Theorem 4.8 and Proposition 5.3.

One can give a proof similar to [CGH21 Pro. 5.15]. The point here is that, as $\lambda$ is $M$-finitely satisfied and $\mu$ is dependent (by Proposition 5.3), every fiber function $f^{\varphi}_{\lambda|N}$ is $\mu|N$-measurable. (Cf. Definition 4.4.) We just have to check everything still works well.

Concluding remarks/questions

(1): (Terminology) On the question of whether it is a good idea to use the word “dependent” we give explanations. The author knows that the notion of “dependent type” already exists in the literature. That is, no Morley sequence has $IP$ (as in [HP11]); equivalently, some/any Morley sequence is convergent (as in [Kha22]). The same idea is used in [Kha21] for measures by randomization. We believe that the notion of “dependent measure” in the present paper is weaker than that in [Kha21]. In fact, the latter is a uniform version of the notion of we presented here. This allows finer layers to be studied using the notion of ‘dependent measure’. Recall that our motivation comes from the fact that all measures in $NIP$ theories and all $fim$ measures in arbitrary theories are dependent. We emphasize that the notion of ‘dependent measure’ wants to fill the gap between types and measures. We strongly believe that there is no need for a strong/uniform version of $NIP$ for this.

(2): (Right notion/presentation) We note that two properties have made the results known of model theory in this article generalizable: If $\lambda \in M(U)$ is dependent then:

(I) For any model $M$, $\lambda|M$ is approximated by types in its support (Corollary 4.2).

(II) The Morley product of $\mu \otimes \lambda$ is well-defined for any global measure $\mu$ that is finitely satisfiable in some small model (Proposition 4.3).

On the question of whether or not it is better to consider something along the lines (I)/(II) to generalize “dependence” outside the setting of $NIP$ we give

The notion of generically stable type is also defined in a similar way: $p$ is called generically stable type if no Morley sequence has $OP$ (cf. [PT11]). This suggests that it might be better to call the notion of “not having a divergent Morley sequence” generically dependence.
explanations. First, recall from Remark 4.12 that the notion of “dependence” is optimal for (I) and (II). Second, we agree that with (I) and (II) all the known results of model theory (in NIP) can be generalized to arbitrary theories, but results like Theorem 4.8, Lemma 4.7 (and even Theorem 4.1) are new and do not seem to be obtained by that approach, or at least not easily obtained. Thirdly and more importantly, finding links between different fields (model theory/measure theory) is welcome, and one field of mathematics is not supposed to be introspective.

(3): As the notion of $\mu$-stable is defined for real-valued functions, all the results of this article can be easily generalized to continuous logic [BBHU08].

At the end paper let us ask the following questions:

Question 5.5. (i) Is the product of two global ddfs measures ddfs? (ii) Is every ddfs measure dependent? If so, by Theorem 4.8 the answer to Question 5.10 of [CGH21] is positive (i.e., any two ddfs global measures commute.) As the product of two ddfs measures is ddfs, a positive answer to (ii) also automatically gives a positive answer to (i).

Although we strongly believe that the answer to (i) is negative. Indeed, suppose that $f : S_\phi(U) \times S_{\phi^*}(U) \to [0,1]$ is a function and for all $p \in S_\phi(U)$ and $q \in S_{\phi^*}(U)$, $x$-sections $f_p : S_{\phi^*}(U) \to [0,1]$ and $y$-sections $f^q : S_\phi(U) \to [0,1]$ are measurable. There is no guarantee that $f$ will be measurable. A similar idea may lead to the rejection of a claim in the initial version of [CGH21] that fim measures are closed under Morley product, i.e. the products of fim measures are fim. Finally, we believe that the answer to Question 5.10 of [CGH21] is negative, however, we should wait for such counterexamples in future work. In NIP theories, the answer to these questions is clearly positive. So we can make the questions more accurate.

Question 5.6. (i) In which theories is the product of two global ddfs measures ddfs? (ii) In which theories is every ddfs measure dependent?

Question 5.7. In Theorem 5.2, can the assumption “dependence” be removed? Is it possible to replace ‘uniformly NIP’ with ‘eventual NIP’ as in [Kha22]?

A Appendix

In this appendix, for the sake of completeness, we give a model theoretic proof of a weaker version of Theorem 4.8 using the previous observations.
Proposition A.1. Let $\mu_x \in \mathfrak{M}(\mathcal{U})$ be a global dependent measure and definable over $M$, and $\lambda_y \in \mathfrak{M}(\mathcal{U})$ be a global dependent measure and $M$-finitely satisfied. Then $\mu_x \otimes \lambda_y = \lambda_y \otimes \mu_x$.

**Proof.** By Lemma 4.5, we can assume that $\lambda_y = q_y$ is a type. Assume, for a contradiction, that $\mu_x \otimes q_y(\phi(x,y)) < r - 2\epsilon$ and $q_y \otimes \mu_x(\phi(x,y)) > r + 2\epsilon$, for some $r \in [0,1]$ and $\epsilon > 0$. Let $N \supseteq Md$ be a model where $\phi(x,y) = \phi(x,y;d)$. (Cf. Definition 4.4.) Recall that, as $q$ is $M$-finitely satisfied and $\mu$ is dependent, $f_{q,N}^{\phi}$ is $\mu|_N$-measurable and so $q_y \otimes \mu_x$ is well-defined. By definability of $\mu$, there is a formula $\psi(y) \in L(U)$ such that: (1) $q \models \psi(y)$ and for all $b \in \psi(M)$, $\mu(\phi(x,b)) < r - \epsilon$. (See also Fact 1.1(a) of [CGH21].)

As $f_{q,N}^{\phi}$ is $\mu|_N$-measurable, there is a $\mu|_N$-measurable set $X \subseteq S_{\phi}(N)$ such that: (2) $q \models \phi(a,y)$ iff $tp(a/N) \in X$.

By Corollary 4.2, pick $p_1, \ldots, p_n \in S_{\phi}(N)$ such that: (3) for all $b \in N$, $|Av(p_1, \ldots, p_n; \phi(x,b)) - \mu(\phi(x,b))| < \epsilon$ and, (4) $|Av(p_1, \ldots, p_n; X) - \mu(X)| < \epsilon$.

Let $a_i \models p_i$ ($i \leq n$). By finite satisfiability of $q$ there is $b_0 \in \psi(M)$ such that: (5) $\models \phi(a_i, b_0)$ iff $q \models \phi(a_i,y)$ ($i \leq n$).

It is easy to see, by (1) and (3), that $Av(p_1, \ldots, p_n; \phi(x, b_0)) < r$. Also, by (5) and (2), $Av(p_1, \ldots, p_n; \phi(x,b_0)) = \frac{1}{n} \sum q(\phi(a_i, y)) = \frac{1}{n} \sum \{i : a_i \in X\} = Av(p_1, \ldots, p_n; X)$ which is within $\epsilon$ of $\mu(\phi(x,y)) = q_y \otimes \mu_x(\phi(x,y)) > r + 2\epsilon$, by (4). This is a contradiction.

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