A DISCRETE WEIGHTED MARKOV–BERNSTEIN INEQUALITY
FOR POLYNOMIALS AND SEQUENCES

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Abstract. For parameters $c \in (0, 1)$ and $\beta > 0$, let $\ell_2(c, \beta)$ be the Hilbert space of real functions defined on $\mathbb{N}$ (i.e., real sequences), for which
\[
\|f\|_{c, \beta}^2 := \sum_{k=0}^{\infty} \frac{\beta^k}{k!} c^k [f(k)]^2 < \infty.
\]
We study the best (i.e., the smallest possible) constant $\gamma_n(c, \beta)$ in the discrete Markov-Bernstein inequality
\[
\|\Delta P\|_{c, \beta} \leq \gamma_n(c, \beta) \|P\|_{c, \beta}, \quad P \in \mathcal{P}_n,
\]
where $\mathcal{P}_n$ is the set of real algebraic polynomials of degree at most $n$ and $\Delta f(x) := f(x+1) - f(x)$.

We prove that
\begin{enumerate}
  \item $\gamma_n(c, 1) \leq 1 + \frac{1}{\sqrt{c}}$ for every $n \in \mathbb{N}$ and $\lim_{n \to \infty} \gamma_n(c, 1) = 1 + \frac{1}{\sqrt{c}}$;
  \item For every fixed $c \in (0, 1)$, $\gamma_n(c, \beta)$ is a monotonically decreasing function of $\beta$ in $(0, \infty)$;
  \item For every fixed $c \in (0, 1)$ and $\beta > 0$, the best Markov-Bernstein constants $\gamma_n(c, \beta)$ are bounded uniformly with respect to $n$.
\end{enumerate}
A similar Markov-Bernstein inequality is proved for sequences in $\ell_2(c, \beta)$. We also establish a relation between the best Markov-Bernstein constants $\gamma_n(c, \beta)$ and the smallest eigenvalues of certain explicitly given Jacobi matrices.

1. Introduction and statement of the results

Throughout this paper $\mathcal{P}_n$ and $\mathcal{P}_n^C$ stands for the set of real and complex algebraic polynomials of degree not exceeding $n$ and $\mathcal{P}$ for all real polynomials. The inequalities of the form
\[
\|p'\| \leq c_n\|p\|, \quad p \in \mathcal{P}_n \quad \text{or} \quad p \in \mathcal{P}_n^C,
\]
which hold for various norms are called Markov–Bernstein–type inequalities. Andrey Markov [26] settled the classical case of real polynomials and the uniform norm in $[-1, 1]$. Precisely, he showed that in this case the Chebyshev polynomial of the first kind, $T_n(x) = \cos n \arccos x$, $x \in [-1, 1]$, is the only (up to a constant factor) extremal polynomial and the best, that is, the smallest possible constant $c_n$ is equal to $T_n'(1) = n^2$. Later E. Hille, G. Szegő and J. D. Tamarkin [16] proved inequality (1.1) for the norm in $L^p[-1, 1]$, $1 \leq p < \infty$. The inequality
\[
\|p'\|_{L^p([-1, 1])} \leq n \|p\|_{L^p([-1, 1])}, \quad p \in \mathcal{P}_n^C,
\]

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where, for $0 \leq p \leq \infty$,
\begin{equation}
\|f\|_{L^p(\partial D)} = \left( \int_{-\pi}^{\pi} |f(e^{i\theta})|^p d\theta \right)^{1/p},
\end{equation}
holds for every $p > 0$. It is usually called the Bernstein inequality though the first proof for the case $p = \infty$ is due to M. Riesz [36] (see [29]). It was then established for $1 \leq p < \infty$ and V. Arestov [3] settled the case $p \in (0, 1)$. Similar inequalities hold for entire functions. Indeed, if $f$ is an entire function of exponential type $\sigma$, such that $f \in L^p(\mathbb{R})$, then
\begin{equation}
(1.3) \quad \|f'\|_{L^p(\mathbb{R})} \leq \sigma \|f\|_{L^p(\mathbb{R})}.
\end{equation}

We refer to [4, Theorem 11.3.3] and [34] for the cases $1 \leq p \leq \infty$ and $p \in (0, 1)$, respectively.

Various weighted versions of the above inequalities have been established. The challenging problem is to find the sharp constant in (1.1),
\begin{align*}
    c_n &= \sup \{ \|p'\|/\|p\| : p \in P_n, \ p \neq 0 \}
\end{align*}
and in some cases it has been determined explicitly.

When one considers the norm in a Hilbert space the sharp constant $c_n$ in Markov’s inequality for polynomials is the largest eigenvalue of a certain matrix. Despite this fact, even in the $L^2$ spaces induced by the classical weight functions of Jacobi ($w_{\alpha, \beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}$, $x \in [-1, 1]$, $\alpha, \beta > -1$), Laguerre ($w_\alpha(x) = x^\alpha e^{-x}$, $x \in (0, \infty)$) and Hermite $w_H(x) = e^{-x^2}$, $x \in (-\infty, \infty)$), the sharp Markov constants are known only in few cases (here we do not discuss inequalities relating different norms of a polynomial and its derivative). For the Laguerre case $\alpha = 0$ P. Turán [41] proved that
\begin{equation}
    c_n = \left(2 \sin \frac{\pi}{4n+2}\right)^{-1}.
\end{equation}

while in the Hermite case, which is a straightforward one,
\begin{equation}
    c_n = \sqrt{2n}
\end{equation}
and the Hermite polynomial $H_n(x)$ is the unique (up to a constant factor) extremal polynomial. In the case of a constant weight function $w(x) \equiv 1$, $x \in [-1, 1]$ (the Legendre case), E. Schmidt [38] proved that, with some $R \in (-6, 13)$,
\begin{equation}
    c_n = \frac{(2n+3)^2}{4\pi} \left(1 - \frac{\pi^2 - 3}{3(2n+3)^2} + \frac{16R}{(2n+3)^4} \right).
\end{equation}

Without any claim for completeness, we mention that bounds for the best constants in the $L^2$ norms induced by the Laguerre or the Gegenbauer weight functions are obtained in [1] [2] [7] [8] [9] [30] [31] [32]. Regarding the asymptotic behaviour of the best Markov constant $c_n$, we point out that $c_n$ is $O(n^{1/2})$, $O(n)$ and $O(n^2)$ as $n \to \infty$ in the cases of the $L^2$-norms induced by the Hermite, Laguerre, and Gegenbauer weight functions, respectively.

Weighted versions of (1.2) for the so-called weights with doubling properties were established by G. Mastroianni and V. Totik [27] when $1 \leq p \leq \infty$ and by T. Erdelyi [11] when $0 < p < 1$. Recently D. Lubinsky [25] proved the weighted analog of (1.3) for entire functions of exponential type, for all $p > 0$ and for the same type of doubling weights which, among others, contain those of the form $(1+x^2)^\alpha$, $\alpha \in \mathbb{R}$.
In this paper we study a discrete weighted Markov-Bernstein inequality for real algebraic polynomials. For any pair of parameters \((c, \beta)\) such that \(c \in (0, 1)\) and \(\beta > 0\), the Meixner inner product and Meixner norm in \(\mathcal{P}\) are defined by

\[
\langle f, g \rangle_{c, \beta} := \sum_{k=0}^{\infty} \frac{(\beta)_k}{k!} c^k f(k) g(k),
\]

where \((\beta)_k\) is the Poshhammer function, \((\beta)_k = \beta(\beta + 1) \cdots (\beta + k - 1)\), \(k \in \mathbb{N}\), with \((\beta)_0 = 1\), and

\[
\|f\|_{c, \beta} = \langle f, f \rangle_{c, \beta}^{1/2}.
\]

The forward difference (shift) operator \(\Delta\) is defined by

\[
\Delta f(x) := f(x + 1) - f(x), \quad x \in \mathbb{N}_0.
\]

The Markov-Bernstein inequality for \(\mathcal{P}_n\) associated with this norm is

\[
\|\Delta f\|_{c, \beta} \leq \gamma_n \|f\|_{c, \beta}, \quad f \in \mathcal{P}_n.
\]

We are interested in the best (the smallest possible) constant in (1.6),

\[
\gamma_n = \gamma_n(c, \beta) := \sup\{\|\Delta f\|_{c, \beta} : f \in \mathcal{P}_n, \|f\|_{c, \beta} = 1\}.
\]

Our main result is the following theorem:

**Theorem 1.1.** Let \(\gamma_n(c, \beta)\) be the best constant in Markov-Bernstein inequality (1.6). Then:

(i) For every \(n \in \mathbb{N}\), \(\gamma_n(c, 1)\) satisfies the inequality

\[
\gamma_n(c, 1) < 1 + \frac{1}{\sqrt{c}}.
\]

Moreover,

\[
\lim_{n \to \infty} \gamma_n(c, 1) = 1 + \frac{1}{\sqrt{c}}.
\]

(ii) For every fixed \(n \in \mathbb{N}\) and \(c \in (0, 1)\), \(\gamma_n(c, \beta)\) is a decreasing function of \(\beta \in (0, \infty)\).

(iii) For every fixed \(c \in (0, 1)\) and \(\beta > 0\) there exists a constant \(C(c, \beta) > 0\) such that \(\gamma_n(c, \beta) \leq C(c, \beta)\) for every \(n \in \mathbb{N}\).

**Remark 1.2.** Inequality (1.6) is discrete for two reasons: the Meixner norm \(\|\cdot\|_{c, \beta}\) is a “discrete” one, and the derivative is replaced by the forward difference operator. Theorem (1.7)(iii) reveals somewhat unusual phenomenon: while typically the sharp constants in the Markov-Bernstein inequalities tend to infinity as \(n\) grows, here the sequence \(\{\gamma_n(c, \beta)\}_{n \in \mathbb{N}}\) is bounded.

Theorem (1.7)(iii) follows from a Markov-Bernstein inequality for a wider set of functions. Let \(\ell_2(c, \beta)\) be the Hilbert space of real valued functions \(f\) defined on \(\mathbb{N}_0\) (i.e., sequences \(f = (f(0), f(1), \ldots))\) for which \(\|f\|_{c, \beta} < \infty\). We prove the following Markov-Bernstein inequality for sequences in \(\ell_2(c, \beta)\):

**Theorem 1.3.** Let \(c \in (0, 1)\).

(i) If \(\beta \geq 1\), then

\[
\|\Delta f\|_{c, \beta} \leq \left(1 + \frac{1}{\sqrt{c}}\right) \|f\|_{c, \beta}, \quad f \in \ell_2(c, \beta)
\]

and for \(\beta = 1\) the constant \(1 + \frac{1}{\sqrt{c}}\) cannot be replaced by a smaller one.
(ii) If $0 < \beta \leq 1$, then
\begin{equation}
\|\Delta f\|_{c,\beta} \leq \left(1 + \frac{1}{\sqrt{\beta c}}\right) \|f\|_{c,\beta}, \quad f \in \ell_2(c, \beta).
\end{equation}
Set
\begin{equation}
\tilde{\gamma}(c, \beta) := \sup_{f \in \ell_2(c, \beta), f \neq 0} \frac{\|\Delta f\|_{c,\beta}}{\|f\|_{c,\beta}},
\end{equation}
then Theorem 1.3 implies
\begin{equation}
\tilde{\gamma}(c, \beta) \leq \begin{cases} 
1 + \frac{1}{\sqrt{c}}, & \beta \geq 1, \\
1 + \frac{1}{\sqrt{\beta c}}, & 0 < \beta < 1.
\end{cases}
\end{equation}
Since $P_n \subset \ell_2(c, \beta)$, we have $\gamma_n(c, \beta) \leq \tilde{\gamma}(c, \beta), \quad n \in \mathbb{N}$, hence inequality (1.8) in Theorem 1.1(i) and Theorem 1.1(iii) are a consequence of (1.12).

The rest of the paper is structured as follows. Theorem 1.3 is proven in Section 2. In Section 3 we give some properties of the Meixner polynomials, the orthogonal polynomials with respect to the Meixner inner product. A relation between the best Markov constants $\gamma_n, \quad n \in \mathbb{N}$, and the smallest eigenvalues of some Jacobi matrices is established in Section 4. It is worth noticing that this result applies to more general situations, concerning the sharp constants in a wide class of polynomial inequalities in $L^2$-norms (see [1, 33]). In Section 5 we obtain two-sided estimates for $\gamma_n(c, 1)$ which complete the proof of Theorem 1.1(i). In Section 6 we apply the Hellmann-Feynman theorem to prove Theorem 1.1(ii). Section 7 contains some comments.

2. Proof of Theorem 1.3

Set $\tilde{f}(\cdot) := f(\cdot + 1)$, then by the triangle inequality
\begin{equation}
\|\Delta f\|_{c,\beta} = \|\tilde{f} - f\|_{c,\beta} \leq \|\tilde{f}\|_{c,\beta} + \|f\|_{c,\beta}.
\end{equation}
We have
\begin{equation}
\|\tilde{f}\|_{c,\beta}^2 = \sum_{k=0}^{\infty} c^k (\beta)_k [f(k + 1)]^2 = \frac{1}{c} \sum_{k=1}^{\infty} c^k (\beta)_k [f(k)]^2 \frac{k}{k - 1 + \beta}.
\end{equation}
Since
\begin{equation}
\frac{k}{k - 1 + \beta} \leq \begin{cases} 
1, & \beta \geq 1, \\
\frac{1}{\beta}, & 0 < \beta \leq 1
\end{cases}
\end{equation}
for every $k \in \mathbb{N}$, we conclude that
\begin{equation}
\|\tilde{f}\|_{c,\beta} \leq \begin{cases} 
\|f\|_{c,\beta} \sqrt{c}, & \beta \geq 1, \\
\|f\|_{c,\beta} \sqrt{\beta c}, & 0 < \beta \leq 1.
\end{cases}
\end{equation}
By substituting these upper bounds for $\|\tilde{f}\|_{c,\beta}$ in the right-hand side of (2.1) we obtain inequalities (1.10) and (1.11).
It remains to prove the sharpness of the constant $1 + 1/\sqrt{c}$ in the case $\beta = 1$. For an arbitrary fixed $n \in \mathbb{N}$ we consider the sequence

$$f(k) = \begin{cases} 
\frac{(-1)^k}{c^{k/2}}, & 0 \leq k \leq n \\
0, & k > n.
\end{cases}$$

We have $\|f\|_{c,1}^2 = n + 1$ and

$$\Delta f(k) = \begin{cases} 
\frac{(-1)^{k+1}}{c^{(k+1)/2}} \left( 1 + \frac{1}{\sqrt{c}} \right), & 0 \leq k \leq n - 1 \\
\frac{(-1)^{n+1}}{c^{n/2}}, & k = n \\
0, & k > n.
\end{cases}$$

Consequently,

$$\|\Delta f\|_{c,1}^2 = n + 1 > n \left( 1 + \frac{1}{\sqrt{c}} \right)^2.$$

Hence,

$$\|\Delta f\|_{c,1} > \left( \frac{n}{n+1} \right)^{1/2} \left( 1 + \frac{1}{\sqrt{c}} \right) \|f\|_{c,1}$$

and

$$\tilde{\gamma}(c, 1) \geq \lim_{n \to \infty} \left( \frac{n}{n+1} \right)^{1/2} \left( 1 + \frac{1}{\sqrt{c}} \right) = 1 + \frac{1}{\sqrt{c}}.$$

This inequality and (1.12) with $\beta = 1$ imply $\tilde{\gamma}(c, 1) = 1 + 1/\sqrt{c}$.

3. MEIXNER POLYNOMIALS

For any pair of parameters $(\beta, c)$ such that $\beta > 0$ and $c \in (0, 1)$, the Meixner inner product and norm are defined by

$$\langle f, g \rangle = \langle f, g \rangle_{c, \beta} := \sum_{x=0}^{\infty} \left( \frac{\beta}{x!} \right)^x c^x f(x; \beta, c) g(x; \beta, c), \quad \|f\|_{c, \beta} = \langle f, f \rangle_{c, \beta}^{1/2}.$$

The induced Hilbert space $\ell_2(c, \beta) = \{ f : \|f\|_{c, \beta} < \infty \}$ contains $P$ and the corresponding orthogonal polynomials are the Meixner polynomials $\{M_n(\cdot; \beta, c)\}_{n \in \mathbb{N}_0}$, defined by

$$M_n(x; \beta, c) := 2F_1 \left( \begin{array}{c} -n, -x \\ \beta \end{array} \middle| 1 - \frac{1}{c} \right).$$

Here, $2F_1$ is the hypergeometric function,

$$2F_1 \left( \begin{array}{c} p, q \\ r \end{array} \middle| t \right) = \sum_{k=0}^{\infty} \frac{(p)_k(q)_k}{(r)_k} \frac{t^k}{k!}.$$

In the following lemma we collect some properties of Meixner polynomials.

**Lemma 3.1.** The following are properties of Meixner polynomials:

(i) Orthogonality:

$$\langle M_m, M_n \rangle := \sum_{x=0}^{\infty} \left( \frac{\beta}{x!} \right)^x c^x M_m(x; \beta, c) M_n(x; \beta, c) = \frac{c^{-n} n!}{(\beta)_n (1 - \beta)^n} \delta_{m,n}, \quad m, n \in \mathbb{N}_0;$$
(ii) Forward shift operator identity:
\[ \Delta M_n(x; \beta, c) := M_n(x+1; \beta, c) - M_n(x; \beta, c) = \frac{n}{\beta} \frac{c-1}{c} M_{n-1}(x; \beta + 1, c); \]

(iii) Recurrence relation:
\[ (n + \beta)M_n(x; \beta + 1, c) = \beta M_n(x; \beta, c) + n M_{n-1}(x; \beta + 1, c); \]

(iv) Expansion formula:
\[ M_n(x+1; \beta, c) = \frac{n!}{(\beta + 1)^n} \sum_{k=0}^{n} \frac{(\beta)_k}{k!} M_k(x; \beta, c); \ n \in \mathbb{N}_0. \]

Proof. Properties (i) and (ii) are well known, see, e.g., [22, (1.9.2), (1.9.6)]. For the proof of property (iii), we write, with \( z = \frac{1}{1 - \frac{1}{c}} \), the formulae for \( M_n(x; \beta + 1, c) \) and \( M_n(x; \beta, c) \):

\[ M_n(x; \beta + 1, c) = 1 + \sum_{k=1}^{n} \frac{n!}{k!} \frac{x(x-1) \cdots (x-k+1)}{(\beta+1)(\beta+2) \cdots (\beta+k)} z^k, \]
\[ M_n(x; \beta, c) = 1 + \sum_{k=1}^{n} \frac{n!}{k!} \frac{x(x-1) \cdots (x-k+1)}{\beta(\beta+1) \cdots (\beta+k)} z^k. \]

Subtracting the second equality multiplied by \( \beta \) from the first one multiplied by \( n + \beta \), we obtain the result.

The proof of property (iv) is by induction with respect to \( n \). Obviously, the equality holds for \( n = 0 \), and we assume it is true for some \( n \in \mathbb{N}_0 \). Property (iii) and the inductional hypothesis then imply

\[ M_{n+1}(x; \beta + 1, c) = \frac{\beta}{n + 1 + \beta} M_{n+1}(x; \beta, c) + \frac{n + 1}{n + 1 + \beta} M_n(x; \beta + 1, c) \]
\[ = \frac{\beta}{n + 1 + \beta} M_{n+1}(x; \beta, c) + \frac{(n + 1)!}{(\beta + 1)n+1} \sum_{k=0}^{n} \frac{(\beta)_k}{k!} M_k(x; \beta, c) \]
\[ = \frac{(n + 1)!}{(\beta + 1)n+1} \sum_{k=0}^{n} \frac{(\beta)_k}{k!} M_k(x; \beta, c), \]

which accomplishes the induction step. \( \square \)

In view of Lemma 3.1 (i), the orthonormal Meixner polynomials \( \{p_m\}_{m \in \mathbb{N}_0} \) are given by

\[ p_m(x; \beta, c) := (1 - c)^{\frac{1}{2}} c^{\frac{m}{2}} \sqrt{\frac{(\beta)_m}{m!}} M_m(x; \beta, c). \]

The forward shift operator of the orthonormal Meixner polynomials obeys the following representation:

**Lemma 3.2.** For any \( m \in \mathbb{N} \),

\[ \Delta p_m(x; \beta, c) = \frac{c-1}{c} \sum_{k=0}^{m-1} \alpha_k p_k(x; \beta, c), \]

where

\[ \alpha_k := c^{-\frac{k}{2}} \sqrt{\frac{(\beta)_k}{k!}}. \]
the largest eigenvalue of the positive definite matrix $A$

or, equivalently,

$$\frac{(\beta)_m}{m!} \Delta M_m(x; \beta, c) = \frac{c - 1}{c} \sum_{k=0}^{m-1} \frac{(\beta)_k}{k!} M_k(x; \beta, c).$$

In this identity we replace $M_k(x; \beta, c)$ by

$$M_k(x; \beta, c) = c^{-\frac{k}{2}}(1 - c)^{-\frac{k}{2}} \sqrt{\frac{k!}{(\beta)_k}} p_k(x; \beta, c), \quad k = 0, \ldots, m,$$

and deduce the desired representation. \hfill \square

4. **Best Markov constants and extreme eigenvalues of Jacobi matrices**

In seeking for the best Markov constant

$$\gamma_n(c, \beta) := \sup \{ ||\Delta f||_{c, \beta} : f \in P_n, \ ||f||_{c, \beta} = 1 \},$$

we may assume without loss of generality that

$$f = t_1p_1 + t_2p_2 + \cdots + t_np_n = t^T p_1, \quad ||f|| = 1 = ||t|| = (t_1^2 + \cdots + t_n^2)^{1/2} = 1,$$

with $t^T = (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n$ and $p_1^T = (p_1, p_2, \ldots, p_n)$. Indeed, since $\Delta p_0 = 0$, $p_0$ cannot yield an increase of $||\Delta f||_{c, \beta}$.

According to Lemma 3.2 we have

$$\Delta p_1 = \frac{c - 1}{c} A_n p_0$$

where $p_0^T := (p_0, p_1, \ldots, p_{n-1})$ and

$$A_n := \begin{pmatrix}
\frac{\alpha_0}{\alpha_n} & 0 & 0 & \cdots & 0 \\
\frac{\alpha_1}{\alpha_n} & \frac{\alpha_1}{\alpha_n} & 0 & \cdots & 0 \\
\frac{\alpha_1}{\alpha_n} & \frac{\alpha_2}{\alpha_n} & \frac{\alpha_1}{\alpha_n} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{\alpha_1}{\alpha_n} & \frac{\alpha_2}{\alpha_n} & \frac{\alpha_3}{\alpha_n} & \cdots & \frac{\alpha_{n-1}}{\alpha_n}
\end{pmatrix}.$$

Hence, $\Delta f = (1 - 1/c) t^T \Delta p_1 = (1 - 1/c) t^T A_n p_0$, and

$$||\Delta f||^2_{c, \beta} = (1 - 1/c)^2 ||t^T A_n||^2 = (1 - 1/c)^2 (A_n^T t, A_n^T t) = (1 - 1/c)^2 (A_n A_n^T t, t).$$

Therefore,

$$\gamma_n^2(c, \beta) = (1 - 1/c)^2 \sup_{||t|| = 1} (A_n A_n^T t, t) = (1 - 1/c)^2 \mu_{\max}(c, \beta),$$

where $\mu_{\max} = \mu_{\max}(c, \beta)$ is the largest eigenvalue of the positive definite matrix $A_n A_n^T$.

Since $A_n^T A_n = A_n^{-1}(A_n A_n^T) A_n$, $A_n^T A_n \sim A_n A_n^T$, and therefore $\mu_{\max}$ is also the largest eigenvalue of the positive definite matrix $A_n^T A_n$. 

**Proof.** From Lemma 3.1(ii), (iv) we have

$$\Delta M_m(x; \beta, c) = \frac{m}{\beta} \frac{c - 1}{c} \frac{(m - 1)!}{(\beta + 1)_{m-1}} \sum_{k=0}^{m-1} \frac{(\beta)_k}{k!} M_k(x; \beta, c)$$

or, equivalently,

$$\frac{(\beta)_m}{m!} \Delta M_m(x; \beta, c) = \frac{c - 1}{c} \sum_{k=0}^{m-1} \frac{(\beta)_k}{k!} M_k(x; \beta, c).$$

With $M_k(x; \beta, c)$ by

$$M_k(x; \beta, c) = c^{-\frac{k}{2}}(1 - c)^{-\frac{k}{2}} \sqrt{\frac{k!}{(\beta)_k}} p_k(x; \beta, c), \quad k = 0, \ldots, m,$$

and deduce the desired representation. \hfill \square
It turns out that it is advantageous to work with the inverse matrices

\[ B_n = (A_n A_n^T)^{-1}, \quad C_n = (A_n^T A_n)^{-1}, \]

as we shall show that they are Jacobi matrices.

Let us find the explicit form of \( B_n \) and \( C_n \). The matrix \( A_n \) in (4.1) can be represented in the form

\[ A_n = \text{diag}\{\alpha_k^{-1}\} T_n \text{diag}\{\alpha_{k-1}\}, \tag{4.4} \]

where \( \text{diag}\{\alpha_k^{-1}\} \) and \( \text{diag}\{\alpha_{k-1}\} \) are diagonal \( n \times n \) matrices with entries on the main diagonal \((1/\alpha_1, \ldots, 1/\alpha_n)\) and \((\alpha_0, \alpha_1, \ldots, \alpha_{n-1})\), respectively, and \( T_n \) is an \( n \times n \) triangular matrix with entries \( t_{i,j} = 1 \), if \( i \geq j \), and \( t_{i,j} = 0 \), otherwise.

The matrices \( T_n^{-1} \) and \( (T_n^T)^{-1} \) are two-diagonal, namely the only nonzero entries of \( T_n^{-1} \) are \( t_{k,k}^{-1} = 1, k = 1, \ldots, n \), and \( t_{k+1,k}^{-1} = -1, k = 1, \ldots, n - 1 \). It follows from (4.4) that

\[ B_n = (A_n A_n^T)^{-1} = \text{diag}\{\alpha_k\} (T_n^T)^{-1} \text{diag}\{\alpha_{k-1}^{-2}\} T_n^{-1} \text{diag}\{\alpha_k\}, \]

and using the explicit form of \( T_n^{-1} \) and \( (T_n^T)^{-1} \), we conclude that \( B_n \) is a tridiagonal matrix whose diagonal entries are \( b_{k,k} = 1 + \alpha_k^2/\alpha_{k-1}^2 \), \( k = 1, \ldots, n - 1 \), and \( b_{n,n} = \alpha_n^2/\alpha_{n-1}^2 \), while the off-diagonal ones are \( b_{k,k+1} = b_{k+1,k} = -\alpha_{k+1}/\alpha_k \), \( k = 1, \ldots, n - 1 \).

In a similar manner, (4.4) implies

\[ C_n = (A_n^T A_n)^{-1} = \text{diag}\{\alpha_{k-1}^{-1}\} T_n^{-1} \text{diag}\{\alpha_k^2\} (T_n^T)^{-1} \text{diag}\{\alpha_{k-1}^{-1}\} \]

so that \( C_n \) is a tridiagonal matrix whose diagonal entries are \( c_{1,1} = \alpha_1^2/\alpha_0^2 \) and \( c_{k,k} = 1 + \alpha_k^2/\alpha_{k-1}^2 \), \( k = 2, \ldots, n \), and the off-diagonal ones are \( c_{k,k+1} = c_{k+1,k} = -\alpha_k/\alpha_{k-1} \), \( k = 1, \ldots, n - 1 \). Replacement of the explicit values of \( \alpha_k \) from (5.2) yields

\[ B_n = \begin{pmatrix}
\frac{\beta}{c} + 1 & -\sqrt{\frac{\beta+1}{2c}} & 0 & \cdots & 0 & 0 \\
-\sqrt{\frac{\beta+1}{2c}} & \frac{\beta+1}{2c} + 1 & -\sqrt{\frac{\beta+2}{3c}} & \cdots & 0 & 0 \\
0 & -\sqrt{\frac{\beta+2}{3c}} & \frac{\beta+2}{3c} + 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \frac{\beta+n-2}{(n-1)c} + 1 & -\sqrt{\frac{\beta+n-1}{nc}} \\
0 & 0 & 0 & \cdots & -\sqrt{\frac{\beta+n-1}{nc}} & \frac{\beta+n-1}{nc}
\end{pmatrix}, \tag{4.5} \]
Let $\tilde{B}_n$ and $\tilde{C}_n$ be the corresponding Jacobi matrices whose diagonal entries coincide with those of $B_n$ and $C_n$ but the off-diagonal ones are opposite to those of $B_n$ and $C_n$. Then obviously the eigenvalues of $B_n$ and $\tilde{B}_n$ coincide and those of $C_n$ and $\tilde{C}_n$ also do.

Since $\mu_{\max} = 1/\lambda_{\min}$, where $\lambda_{\min}$ is the smallest eigenvalue of either of the matrices $B_n$, $\tilde{B}_n$, $C_n$ and $\tilde{C}_n$, (4.3) yields the following

**Theorem 4.1.** The best constant $\gamma_n(c, \beta)$ in the Markov-Bernstein inequality
\[
\|\Delta p\|_{c, \beta} \leq \gamma_n \|p\|_{c, \beta}, \quad p \in \mathcal{P}_n
\]

admits the representation
\[
(4.7) \quad \gamma_n(c, \beta) = \frac{1/c - 1}{\sqrt{\lambda_{\min}(c, \beta)}},
\]

where $\lambda_{\min}(c, \beta) > 0$ is the smallest eigenvalue of either of the matrices $B_n$, $\tilde{B}_n$, $C_n$ and $\tilde{C}_n$.

As is well-known, every $n \times n$ Jacobi matrix $J_n$ defines through a three term recurrence relation a sequence of orthonormal polynomials $\{P_m\}_{m=0}^{n}$, and the zeros of $P_n$ are the eigenvalues of $J_n$. We therefore may reformulate Theorem 4.1 as

**Theorem 4.1’** The best Markov constant $\gamma_n(c, \beta)$ admits the representation (4.7), where $\lambda_{\min}(c, \beta)$ is the smallest zero of the $n$-th polynomial $P_n = P_n(c, \beta; \cdot)$ in the sequence of polynomials defined recursively by
\[
P_0(x) = 1, \quad P_1(x) = x - \frac{\beta}{c},
\]
\[
P_k(x) = \left( x - \frac{\beta + k - 1}{kc} - 1 \right) P_{k-1}(x) - \frac{\beta + k - 2}{(k-1)c} P_{k-2}(x), \quad k \geq 2.
\]

Since $\beta > 0$ and $c \in (0, 1)$, it follows from Favard’s theorem that $\{P_k\}_{k \in \mathbb{N}_0}$ form a system of orthogonal polynomials.
5. Two-sided estimates for \( \gamma(c, 1) \)

Matrices \( B = B_n(c, \beta) \) and \( C = C_n(c, \beta) \) have particularly simple form in the case \( \beta = 1 \), for instance,

\[
D_n := C_n(c, 1) = \begin{pmatrix}
\frac{1}{c} & -\frac{1}{\sqrt{c}} & 0 & \cdots & 0 & 0 \\
-\frac{1}{\sqrt{c}} & \frac{1}{c} + 1 & -\frac{1}{\sqrt{c}} & \cdots & 0 & 0 \\
0 & -\frac{1}{\sqrt{c}} & \frac{1}{c} + 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \frac{1}{c} + 1 & -\frac{1}{\sqrt{c}} \\
0 & 0 & 0 & \cdots & -\frac{1}{\sqrt{c}} & \frac{1}{c} + 1
\end{pmatrix}.
\]

We shall find estimates for \( \lambda_{\min}(c, 1) \), the smallest zero of \( |\lambda E_n - D_n| = 0 \). By change of variable

\[
\lambda = 1 + \frac{1}{c} + \frac{2z}{\sqrt{c}}
\]

this equation simplifies to \( \varphi_n(z) = 0 \), where

\[
\varphi_n(z) = \begin{vmatrix}
z + \frac{\sqrt{c}}{c} & \frac{1}{c} & 0 & \cdots & 0 & 0 \\
\frac{1}{c} & z & \frac{1}{c} & \cdots & 0 & 0 \\
0 & \frac{1}{c} & z & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & z & \frac{1}{c} \\
0 & 0 & 0 & \cdots & \frac{1}{c} & z
\end{vmatrix}.
\]

It is easy to see that

\[
\varphi_n(z) = \frac{1}{2^n} \left( U_n(z) + \sqrt{c} U_{n-1}(z) \right),
\]

where \( U_m(z) \) is the \( m \)-th Chebyshev polynomial of second kind,

\[
U_m(z) = \frac{\cos(m+1) \arccos(z)}{\sqrt{1 - z^2}}, \quad z \in [-1, 1].
\]

Indeed, (5.1) is readily verified to be true for \( n = 1, 2 \), and \( \{\varphi_m\} \) satisfy the recurrence relation

\[
\varphi_m(z) = z \varphi_{m-1}(z) - \frac{1}{4} \varphi_{m-2}(z), \quad m \geq 3,
\]

which is also satisfied by \( \{2^{-m} U_m\} \).

**Lemma 5.1.** The zeros of \( \varphi_n \), \( n \geq 2 \), are located in \((-1, 1)\) and interlace with the zeros of \( U_{n-1} \). Moreover, if \( \tau \) is the smallest zero of \( \varphi_n \), then

\[
\tau = -1 + \varepsilon_n, \quad \frac{1}{n(n+1)} < \varepsilon_n < 2 \sin^2 \frac{\pi}{2n}.
\]

**Proof.** Clearly, \( \varphi_n \) is a monic polynomial of degree \( n \). Let \( \eta_k = \cos \frac{k\pi}{n} \), \( k = 1, \ldots, n - 1 \), be the zeros of \( U_{n-1} \), then \(-1 < \eta_{n-1} < \eta_{n-2} < \cdots < \eta_1 < 1 \), and

\[
\text{sign } \varphi(\eta_k) = \text{sign } U_n(\eta_k) = (-1)^k, \quad k = 1, \ldots, n - 1.
\]

This and \( \varphi_n(-1) = (-1)^n 2^{-n} (n + 1 - n \sqrt{c}) \), \( \varphi_n(1) = 2^{-n} (n + 1 + n \sqrt{c}) \) imply that the zeros of \( \varphi_n \) lie in \((-1, 1)\) and interlace with the zeros of \( U_{n-1} \).
The upper bound for $\varepsilon_n$ follows from 
$$\tau < \eta_n - 1 = -\cos \frac{\pi}{n} = -1 + 2 \sin^2 \frac{\pi}{2n}.$$ 
To obtain the lower bound for $\varepsilon_n$, we apply one step of Newton’s method for finding $\tau$ as the smallest zero of $\varphi_n(z)$ with initial value $\tau(0) = -1$. We have 
$$\tau > \tau(1) = -1 + \frac{3(n + 1 - n\sqrt{c})}{n(n + 1)(n + 2 - (n - 1)\sqrt{c})} > -1 + \frac{1}{n(n + 1)},$$
where for the last inequality we have used that $g(x) = \frac{n + 1 - nx}{n + 2 - (n - 1)x}$ is a decreasing function in $(0, 1)$.

Going back to variable $\lambda$, we find 
$$\lambda_{\min}(c, 1) = 1 + \frac{1}{c} + \frac{2\tau}{\sqrt{c}} = 1 + \frac{1}{c} + \frac{2(-1 + \varepsilon_n)}{\sqrt{c}} = \left(\frac{1}{\sqrt{c}} - 1\right)^2 + \frac{2\varepsilon_n}{\sqrt{c}},$$

hence

\begin{equation}
\lambda_{\min}(c, 1) = \left(\frac{1}{\sqrt{c}} - 1\right)^2 \left(1 + \frac{2\varepsilon_n}{1 - \sqrt{c}}\right).
\end{equation}

Now (4.7), (5.2) and the estimates for $\varepsilon_n$ from Lemma 5.1 imply 

**Theorem 5.2.** For any $n \geq 2$, the best constant $\gamma_n(c, 1)$ in the Markov-Bernstein inequality 
$$\|\Delta p\|_{c, 1} \leq \gamma_n(c, 1) \|p\|_{c, 1}, \quad p \in P_n,$$

admits the estimates

\begin{equation}
1 + \frac{1}{\sqrt{c}} \left(1 + \frac{4\sqrt{c}}{(1 - \sqrt{c})^2} \sin^2 \frac{\pi}{2n}\right)^{1/2} \leq \gamma_n(c, 1) \leq \left(1 + \frac{2\sqrt{c}}{(1 - \sqrt{c})^2 n(n + 1)}\right)^{1/2}.
\end{equation}

Theorem 1.1(ii) now follows from the two-sided estimates (5.3). Note that the upper estimate for $\gamma_n(c, 1)$ in (5.3) sharpens the one in Theorem 1.1(i).

## 6. Monotone Dependence of Eigenvalues on $\beta$

The statement of Theorem 1.1(ii) is a consequence of the following

**Proposition 6.1.** For a fixed $c \in (0, 1)$, each eigenvalue $\lambda$ of the matrix $B_n(\beta, c)$, defined by (4.5), is a strictly monotone increasing function of $\beta$ in the interval $(0, \infty)$.

We apply the elegant method to establish monotonicity of zeros of orthogonal polynomials, or equivalently of eigenvalues of Jacobi matrices based on Hellmann-Feynman’s theorem [15] [12] and Wall-Wetzel’s criterion [43] for positive definiteness of Jacobi matrices. We describe it briefly and refer to Chapter 7.3 in Ismail’s book [18] as well as to [17, 19, 20] for more details. Consider the parametric sequence $\{p_k(x; \tau)\}_{k=0}^{\infty}$ of orthonormal polynomials which is generated by the three term recurrence relation

\begin{align*}
p_{-1}(x; \tau) &= 0, \\
p_0(x; \tau) &= 1, \\
x p_k(x; \tau) &= a_k(\tau) p_{k+1}(x; \tau) + b_k(\tau) p_k(x; \tau) + a_{k-1}(\tau) p_{k-1}(x; \tau), \quad k \geq 0,
\end{align*}
where \( a_{k-1}(\tau) > 0 \). The zeros of the polynomial \( p_n(x; \tau) \) coincide with the eigenvalues of the Jacobi matrix \( \mathbf{J}_n = \mathbf{J}_n(\tau) \), whose diagonal entries are \( b_k(\tau) \), \( k = 0, \ldots, n - 1 \), and the off-diagonal ones are \( a_k(\tau) \), \( k = 0, \ldots, n - 2 \). Moreover, if \( \lambda_j = \lambda_j(\tau) \) is a zero of \( p_n(x; \tau) \) and

\[
p_j = (p_0(\lambda_j; \tau), p_1(\lambda_j; \tau), \ldots, p_{n-1}(\lambda_j; \tau))^\top,
\]

then

\[
\mathbf{J}_n p_j = \lambda_j p_j.
\]

Let us denote by \( \tilde{\mathbf{J}}_n = \tilde{\mathbf{J}}_n(\tau) \) the tridiagonal matrix whose entries are the derivatives of the corresponding entries of \( \mathbf{J}_n(\tau) \). Then the Hellmann–Feynman theorem, in the particular case which is convenient for our objectives, reads as follows:

**Theorem 6.2.** For every zero \( \lambda_j(\tau) \) of \( p_n(x; \tau) \) we have

\[
\lambda_j'(\tau) = \frac{p_j^\top \tilde{\mathbf{J}}_n' p_j}{p_j^\top p_j}.
\]

Furthermore, if the numerator of the latter expression is positive, then the zeros \( \lambda_j(\tau) \) of \( p_n(x; \tau) \) are increasing functions of \( \tau \). In particular, the latter statement holds if \( \tilde{\mathbf{J}}_n \) is a positive definite matrix.

Let us recall that a sequence \( \{c_n\}_{n=1}^\infty \) of non-negative numbers is called a chain sequence if there exists another sequence \( \{\nu_n\}_{n=0}^\infty \), called a parametric one, such that \( 0 \leq \nu_0 < 1, 0 < \nu_n < 1 \) for all \( n \in \mathbb{N} \), and \( c_n = (1 - \nu_{n-1})\nu_n \) for every \( n \in \mathbb{N} \) (see [18]). A criterion for positive definiteness of Jacobi matrices, due to Wall and Wetzel [43], applied to \( \tilde{\mathbf{J}}_n \) yields:

**Proposition 6.3.** Let \( \tilde{\mathbf{J}}_n \) be a Jacobi matrix with positive diagonal entries. If \( b'_i > 0 \) for \( i = 0, \ldots, n - 1 \), and there is a chain sequence \( \{\kappa_i\} \) such that

\[
\frac{|a_i'|^2}{b'_i b'_{i+1}} < \kappa_i, \quad \text{for} \quad i = 0, 1, \ldots, n - 2,
\]

then \( \tilde{\mathbf{J}}_n \) is positive definite.

**Proof of Proposition 6.3.** All we need is to show that the matrix \( \tilde{\mathbf{B}}_n' = (\tilde{b}'_{i,j})_{n \times n} \) obtained from \( \tilde{\mathbf{B}}_n \) by partial differentiation of its entries with respect to \( \beta \) is positive definite. Straightforward calculations show that \( \tilde{b}'_{k,k} = 1/(k c) \), \( k = 1, \ldots, n \), and

\[
\tilde{b}'_{k,k+1} = \frac{1}{2\sqrt{(k+1)(k+\beta)c}}, \quad k = 1, \ldots, n - 1.
\]

It follows by Proposition 6.3 and the fact that that \( \{\kappa_i\} = \{1/4, 1/4, \ldots\} \) is a chain sequence that a sufficient condition for \( \tilde{\mathbf{B}}_n' \) to be positive definite is that the following inequalities are satisfied:

\[
\frac{[\tilde{b}'_{k,k+1}]}{b'_k b'_{k+1,k+1}} < \frac{1}{4}, \quad k = 1, \ldots, n - 1.
\]

They are equivalent to inequalities

\[
\beta + k(1-c) > 0, \quad k = 1, \ldots, n - 1,
\]

which are obviously true, as \( \beta > 0 \) and \( c \in (0,1) \). Hence, \( \tilde{\mathbf{B}}_n' \) is a positive definite matrix. \( \square \)
In particular, the smallest eigenvalue of $\tilde{B}$, $\lambda_{\min}(c, \beta)$, is a monotone increasing function of $\beta$. By Theorem 4.1, $\gamma_n(c, \beta) = \frac{1}{\sqrt{\lambda_{\min}(c, \beta)}}$ is a monotone decreasing function of $\beta$, which proves Theorem 1.1 (ii).

7. Comments

The Markov-Bernstein inequality for sequences, Theorem 1.3, was rather easy to prove, and then was used in the proof of parts (i) and (iii) of Theorem 1.1, the Markov-Bernstein inequality for polynomials. Since we observe, at least for $\beta = 1$, coincidence of the best Markov constant in $\ell_2(c, \beta)$ with the limit of $\gamma_n$ in the polynomial case, a natural question is whether, on the contrary, Markov-Bernstein inequality for sequences can be deduced from Markov-Bernstein inequality for polynomials. Such an approach would be possible if the corresponding general Bernstein’s problem had a solution in this particular case. Indeed, let us suppose that the following question has an affirmative answer: Is it true that, given $c \in (0, 1)$, $\beta \in [1, \infty)$, a sequence $f \in \ell_2(c, \beta)$ and $\epsilon > 0$, there is an algebraic polynomial $p$, such that

$$\sum_{k=0}^{\infty} \frac{(\beta)k}{k!} e^{\beta k} |f(k) - p(k)|^2 < \epsilon?$$

Then the Markov-Bernstein inequality for sequences would be an immediate consequence of the Markov-Bernstein inequality for polynomials.

A general version of Bernstein’s approximation problem reads as follows: given a weight function $W : \mathbb{R} \to [0, 1]$ and the corresponding weighted norm is $\| \cdot \|_W$, say $L^p_W(\mathbb{R})$, such that the corresponding moments are finite in the norm, i.e. $\|P\|_W < \infty$ for every $P \in \mathcal{P}$, is it true that for every function $f$ with $\|f\|_W < \infty$ and each $\epsilon > 0$, there is an algebraic polynomial $P$ such that $\|f - P\|_W < \epsilon$? The weights $W$ for which this problem has an affirmative answer are sometimes called admissible ones. The first results concerning characterisation of the admissible kernels for the uniform norm were obtained by S. N. Mergelyan, N. I. Akhiezer, H. Pollard, S. Izumi and T. Kawata, M. Dzrbasjan and L. Carleson. We refer to Lubinsky’s survey [24] and P. Koosis’s [23] and M. Ganzburg’s [14] books for details, further contributions and references.

The result in this direction which is the most relevant in our situation is due to G. Freud [13] Theorem 3.3. on p. 73 (see also [24 Theorem 1.7]). His contribution is an extension of M. Riesz’ one [37] which was obtained even before Bernstein posed his problem. G. Freud’s result applies to our problem but only for sequences in $\ell_2, c, \beta$ of at most polynomial growth, despite that it implies that such sequence would posses one-sided polynomial approximations.

We raise also a problem which is the “continuous” counterpart of the one stated above. More precisely, it would be of interest to know if $W(x) = e^{-ax}\Gamma(x+\beta)/\Gamma(x+1)$, where $a > 0$ and $\beta > 1$, is an admissible weight for Bernstein’s approximation problem in $L^p_W(0, \infty)$, for $p \geq 1$. The above mentioned results of S. Izumi and T. Kawata [21], M. Dzrbasjan [10] and L. Carleson [6] imply that this is true for $\beta = 1$.

Finally, it would be interest to study the eventual extensions of the results in Theorems 1.3 and 1.1 for the relevant weighted $\ell_p$ norms.
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