Extension of the Lagrange multiplier test for error
cross-section independence to large panels with non
normal errors

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Abstract: This paper reexamines the seminal Lagrange multiplier test for cross-section independence in a large panel model where both the number of cross-sectional units $n$ and the number of time series observations $T$ can be large. The first contribution of the paper is an enlargement of the test with two extensions: firstly the new asymptotic normality is derived in a simultaneous limiting scheme where the two dimensions $(n, T)$ tend to infinity with comparable magnitudes; second, the result is valid for general error distribution (not necessarily normal). The second contribution of the paper is a new test statistic based on the sum of the fourth powers of cross-section correlations from OLS residuals, instead of their squares used in the Lagrange multiplier statistic. This new test is generally more powerful, and the improvement is particularly visible against alternatives with weak or sparse cross-section dependence. Both simulation study and real data analysis are proposed to demonstrate the advantages of the enlarged Lagrange multiplier test and the power enhanced test in comparison with the existing procedures.

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1. Introduction

Consider the fixed effects panel data model

$$y_{it} = \alpha + x_{it}'\beta + \mu_i + \nu_{it}, \quad \text{for } i = 1, \ldots, n; \ t = 1, \ldots, T,$$

(1.1)

where $i$ indexes the cross-sectional (individual) units, and $t$ the time series observations. The dependent variable is $y_{it}$ and $x_{it}$ denotes the exogenous regressors of dimension $k \times 1$, $\beta$ is the corresponding $k \times 1$ vector of parameters, $\mu_i$ denotes the time-invariant individual effect which could be correlated with the regressors $x_{it}$. Throughout the paper the number of covariates $k$ is fixed while the two dimensions $n$ and $T$ may grow to infinity. A more general model is the following heterogeneous panel model

$$y_{it} = x_{it}'\beta_i + \nu_{it}, \quad \text{for } i = 1, \ldots, n; \ t = 1, \ldots, T,$$

(1.2)

where the slope parameters, $\beta_i$, are allowed to vary across $i$.

Our focus is to test the following cross-section independence hypothesis, that is

$$H_0: \text{the model errors } \{\nu_{it}\} \text{ are independent across the units } 1 \leq i \leq n. \quad (1.3)$$

Testing such cross-section independence is important because it is a preliminary step for many existing inference procedures for the panel model. These procedures become biased or even inconsistent when cross-sectional units are correlated, see for example (Chudik and Pesaran, 2013) for the case of commonly used panel unit root tests.

Several procedures exist in the literature for this independence test. A popular procedure is the Lagrange multiplier (LM) test proposed by Breusch and Pagan (1980) based on the sum of squared pair-wise sample correlation coefficients of the OLS residuals. Recent efforts have concentrated in large panel models where the panel size $n$ is large compared to the sample size $T$. It has been observed that the LM test is particularly biased in such large panels. Pesaran et al. (2008) found an approximation for the mean and variance of the LM statistic and established its asymptotic normality for their modified $LM_{adj}$ test under the following sequential limit scheme

$$\text{SEQ-L: } T \to \infty \text{ first, followed by } n \to \infty.$$
Pesaran (2004) also proposed a cross-sectional dependence (CD) test using sum of (non-squared) sample correlations of the OLS residuals. Pesaran (2015) showed that the null of the CD test is rather weak cross-sectional dependence as defined in Chudik et al. (2011). Again the asymptotic normality for the test statistics is derived under the sequential limit scheme.

When the panel size $n$ and the sample size $T$ are large while having comparable magnitude, it has been argued in recent high-dimensional statistic literature that more reliable asymptotic results can be found by considering a simultaneous limit scheme (Yao et al., 2015)

$$\text{SIM-L: } T \to \infty, n = n(T) \text{ such that } \lim_{T \to \infty} \frac{n}{T} = c > 0.$$  

Technically, the derivation of asymptotic normality for LM-type statistics under the SIM-L scheme is challenging. Actually, Baltagi et al. (2012) succeeded for the fixed effects homogeneous panel data model (1.1) by precisely identifying a mean shift in the asymptotic distribution of the LM statistic due to the SIM-L scheme. This bias-corrected test, $LM_{bc}$, however needs to assume the normality of the errors $(\nu_{it})$ as its development relies on a previous result established in Schott (2005) on sample correlations of normal-distributed errors. As for the heterogeneous model (1.2), a recent work Bailey et al. (2020) establishes the asymptotic normality of the LM test under the SIM-L scheme while also requiring normally distributed errors $(\nu_{it})$ as well as normally distributed regressors $(x_{it})$. This test, $LM_{RMT}$, is extensively compared to the CD and $LM_{adj}$ tests in various dimension and panel size combinations via simulation. The $LM_{RMT}$ has been shown to be comparable to $LM_{adj}$ in terms of size and power, with a slight preference for $LM_{RMT}$ when the sample size $T$ is not very large (or when the panel size $n$ is relatively large). The CD test is universally correctly sized but generally lacks power under local alternatives.

An important aim of this paper to extend the LM test to large panel models under the SIM-L scheme and without the normality assumption on the errors $(\nu_{it})$. Because the normality assumption is fundamental to the analytic methods used in both the $LM_{bc}$ and $LM_{RMT}$ tests, we achieve our goal by employing complete different tools and derive a new asymptotic normal distribution for the LM statistic in Theorem 3.1. A corresponding test $LM_e$ is thus derived for cross-section independence for the panel model (1.1). We also establish a surprising fact that this asymptotic distribution for the $LM_e$ coincide with those of both the $LM_{bc}$ and $LM_{RMT}$ tests although the technical derivations of the three tests are all different. This universality of
the asymptotic distribution for the LM statistic under the SIM-L scheme ensures a welcomed robustness of the LM test, thus extending its application scope to various large panel models.

A second contribution from the paper is to go beyond the LM statistic by considering fourth powers of residual correlations (instead of their squares in the LM statistic). The idea here is that fourth powers weight more heavily large correlations than smaller ones. This magnification effect enhances the power of the test, particularly under an alternative where the cross-section dependence is weak or sparse. Following the same setting and tools as for the LM statistic, we establish the asymptotic normality for this power-enhanced test (PET).

Next we designed several simulation experiments following commonly used settings in the large panel model literature. These experiments confirm the excellent empirical performance of the new \(LM_e\) test at a level comparable to the tests \(LM_{RMT}\) and \(LM_{adj}\). When it comes to compare the powers of the tests, the power-enhanced test PET indeed dominates all the existing tests discussed above. As expected, the advantage of the PET test is particularly significant under weak or sparse dependence alternatives.

We have also conducted a real data analysis to assess the properties of these tests in real-life large panels. This findings basically confirm the comparison results found through simulation experiments.

The rest of the paper is organised as follows. Section 2 presents OLS regression residuals in the panel model (1.1) and the existing tests discussed above. Sections 3 and 4 introduce, respectively, the extended LM test \(LM_e\) and the power-enhanced test PET, and establish their asymptotic normality under the null and the SIM-L scheme without assuming normality of the errors. Section 5 presents a detailed simulation study for comparison of finite-sample performance of the various tests. A real data analysis is carried out in Section 6. Some discussions are offered in the last conclusion section. All technical proofs are grouped to the Appendix.

2. OLS regression residuals and existing tests

Consider the population correlation matrix \(R = \left[\text{diag}(\Sigma)\right]^{-1/2}\Sigma\left[\text{diag}(\Sigma)\right]^{-1/2}\) of the error vectors \(\nu_t\). Clearly, under the independence hypothesis (1.3), one has \(R = I\). It is thus natural
to design test statistics based on estimates for these error correlations. Natural estimates for these correlations are obtained using the residuals from some fitted panel regression model. Precisely, consider the centralised variables
\[ \tilde{y}_{it} = y_{it} - \frac{1}{T} \sum_{t=1}^{T} y_{it}, \quad \tilde{x}_{it} = x_{it} - \frac{1}{T} \sum_{t=1}^{T} x_{it}, \quad \text{and} \quad \tilde{\nu}_{it} = \nu_{it} - \frac{1}{T} \sum_{t=1}^{T} \nu_{it}. \]

The model (1.1) takes a simpler form with these centralised variables,
\[ \tilde{y}_{it} = \tilde{x}_{it}' \beta + \tilde{\nu}_{it}, \quad \text{for } 1 \leq i \leq n; \ 1 \leq t \leq T. \]

The OLS estimator of the regression parameter \( \beta \) in model (1.1) is
\[ \hat{\beta} = \left( \sum_{t=1}^{T} \sum_{i=1}^{n} \tilde{x}_{it} \tilde{x}_{it}' \right)^{-1} \left( \sum_{t=1}^{T} \sum_{i=1}^{n} \tilde{x}_{it} \tilde{y}_{it} \right). \]

Introduce also the design matrices
\[ X_i = \left( \tilde{x}_{i1}, ..., \tilde{x}_{it}, ..., \tilde{x}_{iT} \right)_{k \times T}, \quad X = \left( X_1, X_2, ..., X_n \right)_{k \times nT}, \]
and the stacked observation vectors,
\[ Y_i = \left( \tilde{y}_{i1}, ..., \tilde{y}_{iT} \right)', \quad Y = \left( Y_1', ..., Y_n' \right)'. \]

Then \( \hat{\beta} = (XX')^{-1} XY \). It is well known that under fairly general assumptions on the panel model and the independence hypothesis \( H_0 \), \( \hat{\beta} \) is a consistent estimator of \( \beta \) no matter whether \( n \) is fixed or tends to infinity jointly with \( T \) (Baltagi et al., 2011, 2017). The regression residuals \( \hat{\nu}_{it} \) are
\[ \hat{\nu}_{it} = \tilde{y}_{it} - \tilde{x}_{it}' \hat{\beta} = \tilde{\nu}_{it} - \tilde{x}_{it}' \left( \hat{\beta} - \beta \right). \quad (2.1) \]

Let \( \hat{\nu}_t = (\hat{\nu}_{1t}, ..., \hat{\nu}_{nt})' \) for \( t = 1, ..., T \). The sample residual covariance matrix \( \hat{S}_T \) and the sample residual correlation matrix \( \hat{R}_T \) are respectively,
\[ \hat{S}_T = \frac{1}{T} \sum_{t=1}^{T} \hat{\nu}_t \hat{\nu}_t', \quad \text{and} \quad \hat{R}_T = \hat{D}_T^{-1/2} \hat{S}_T \hat{D}_T^{-1/2}, \quad \text{with} \quad \hat{D}_T = \text{diag}(\hat{S}_T). \quad (2.2) \]

The entries of \( \hat{R}_T \) are denoted as \( \hat{R}_t = \{ \hat{\rho}_{rs}, \ 1 \leq r, s \leq n \} \).

Introduce also the sample error covariance and correlation matrix defined analogously but with the centralised errors \( \{ \tilde{\nu}_t \} \), that is
\[ S_T = \frac{1}{T} \sum_{t=1}^{T} \tilde{\nu}_t \tilde{\nu}_t', \quad \text{and} \quad R_T = D_T^{-1/2} S_T D_T^{-1/2}, \quad \text{with} \quad D_T = \text{diag}(S_T). \quad (2.3) \]
2.1. The Lagrange Multiplier test (LM)

This very first test for cross-section independence is proposed by Breusch and Pagan (1980) and uses the statistic

$$LM = \frac{T}{2} \left\{ \text{tr}(\hat{R}_T^2) - n \right\} = \frac{T}{2} \sum_{r \neq s} \hat{\rho}_{rs}^2. \tag{2.4}$$

Note that $\sum_{r \neq s} \hat{\rho}_{rs}^2 = ||\hat{R}_T - I||_F^2$ is the squared Frobenius distance between $\hat{R}_T$ and $I$, the LM statistic is indeed a scaled estimator for the population distance $||R - I||_F^2$ which is zero under the null hypothesis.

Consider the following assumptions:

(A1) For each $i$, the disturbances, $\nu_{it}$, are serially independent with mean 0 and variance,

$$0 < \sigma_i^2 < \infty;$$

(A2) Under the null hypothesis defined by $H_0$ in (1.3): $\nu_{it} = \sigma_i \varepsilon_{it}$, with $\varepsilon_{it} \overset{i.i.d.}{\sim} N(0, 1)$ for all $i$ and $t$.

Under a large sample limit scheme where $n$ is fixed while $T \to \infty$ and Assumptions (A1) and (A2), Breusch and Pagan (1980) established that under the null, the LM statistic has an asymptotic $\chi^2_{n(n-1)/2}$ distribution. It has been well discussed in the literature that this LM test based on its large sample limiting chi-squared distribution suffers from severe size distortions for large panels where $n$ is large compared to time series size $T$.

2.2. The cross-section dependence test (CD)

To accommodate large panels, Pesaran (2004) proposed the following CD statistic

$$CD = \sqrt{\frac{T}{2n(n - 1)}} \sum_{r \neq s} \hat{\rho}_{rs}. \tag{2.5}$$

Consider the following additional assumptions:

(A3) The disturbances are $\nu_{it} = \sigma_i \varepsilon_{it}$, with $\varepsilon_{it} \overset{i.i.d.}{\sim} (0, 1)$ for all $i$ and $t$, the disturbances, $\varepsilon_{it}$, are symmetrically distributed around 0;

(A4) The regressors, $x_{it}$, are strictly exogenous such that $E(\nu_{it}|X_i) = 0$ for all $i$ and $t$, and $X'_iX_i$ is a positive definite matrix;
(A5) The OLS residuals, $\hat{\nu}_{it}$, defined by (2.1), are not all zero.

Under the Assumptions (A3)-(A4)-(A5), Pesaran (2004) established that the CD statistic is asymptotically standard normal under the sequential limit scheme. It is widely reported that the CD test enjoys a very accurate size in general while suffering from certain loss of power when the panel-wise correlations have varying signs. This loss of power can be understood by the fact that the CD statistic is averaging the sample residual correlations $\{\hat{\rho}_{rs}\}$; so if they carry varying signs across panel units, this averaging may lead to certain cancellation of correlations of opposite signs, thus neutralising the statistic and its power. This fact is confirmed in one setting of the simulation experiments where the cross-sectional dependence comes from a factor model with mean zero factor loadings, see Table 5.2.

2.3. The bias-adjusted Lagrange Multiplier test ($LM_{adj}$)

In order to adapt the LM test to large panels, Pesaran et al. (2008) proposed the following bias-adjusted version of the LM statistic:

$$LM_{adj} = \sqrt{\frac{1}{2n(n-1)}} \sum_{r \neq s} (T-k)\hat{\rho}_{rs}^2 - \mu_{Trs} \sigma_{Trs},$$

(2.6)

where

$$\mu_{Trs} = \frac{1}{T-k} \text{tr}(M_r M_s),$$

$$\sigma_{Trs}^2 = [\text{tr}(M_r M_s)]^2 a_{1T} + 2\text{tr}((M_r M_s)^2) a_{2T},$$

$$M_r = I_T - X_r'(X_r X_r')^{-1} X_r, \quad X_r = (x_{r1}, \ldots, x_{rT}),$$

$$a_{1T} = a_{2T} = \frac{1}{(T-k)^2}, \quad a_{2T} = \frac{3}{(T-k+2)^2}.$$

Under the assumptions (A1)-(A2)-(A4) (which imply the null hypothesis), the authors proved that under the SEQ-L scheme, $LM_{adj}$ is asymptotically standard normal. This bias-adjusted test indeed perform much better in large panels than the original LM test. The only known issue on this test is that because of the employed sequential limiting scheme, the test may be over-conservative in “micro-panels” where $T$ is quite limited compared to panel size $n$. 
2.4. A bias-corrected Lagrange Multiplier test \((LM_{bc})\)

Pesaran (2004) also suggested the following scaled version of the LM test for large panels:

\[
LM_P = CD_{lm} = \sqrt{\frac{1}{4n(n-1)}} \sum_{r \neq s} \left( T \hat{\rho}^2_{rs} - 1 \right).
\]

Under the Assumptions (A1)-(A2) and the SEQ-L scheme, \(LM_P\) is shown to have a standard normal limiting distribution. An issue with the test statistic as mentioned in Pesaran (2004) is that when \(T\) is not large, the scale adjustment made in \(LM_P\) may not be accurate, thus exhibiting substantial size distortions. Consequently, Baltagi et al. (2012) considered the SEQ-L scheme where a bias-corrected version was found for the fixed effects model (1.1). Precisely, the authors established that under assumptions (A1)-(A2)-(A4) and the SEQ-L scheme,

\[
LM_{bc} := LM_P - \frac{n}{2(T-1)} \overset{D}{\to} N(0,1).
\] (2.7)

It is remarkable that the asymptotic bias in \(LM_P\) is exactly identified as \(n/2(T-1)\), and the authors demonstrated that this bias is caused by the fact that the sample correlations are calculated using OLS residuals, instead of the (unobserved) model errors. Note that this bias disappears in the traditional large sample scheme where the ratio \(n/T \to 0\), thus emphasising the particular large panel effect. The derivation of (2.7) uses the asymptotic results on sample error correlation matrix in Schott (2005), thus requiring the normality assumption of the errors, see Assumption (A2).

Simulation experiments in Baltagi et al. (2012) show that the bias-corrected \(LM_{bc}\) test has an excellent finite sample performance in comparison to the CD test and the \(LM_{adj}\) test. It is particularly recommendable for micro-panels where \(T\) is relatively small.

2.5. A Gaussian Lagrange-Multiplier test for large panels \((LM_{RMT})\)

Based on asymptotic results from random matrix theory literature, Bailey et al. (2020) introduced another limiting distribution for the LM statistic under the SIM-L scheme and for the heterogeneous panel data (1.2). Precisely, they considered the statistic,

\[
LM_{RMT} = \frac{tr(\hat{R}_T^2) - \mu_{rmt}}{\sigma_{rmt}},
\] (2.8)
where, setting $c_T = n/T$ and $\kappa = \frac{3(T-k+2)}{(T+2)(T-k)}$,

$$
\mu_{rmt} = n(1 + c_T) + c_T^2 - c_T,
$$

$$
\sigma^2_{rmt} = 4c_T(1 + 2c_T)(c_T + 2) - 4(\kappa - 1)c_T(1 + c_T)^2 + (\kappa - 3)c_T(c_T - 4)^2(c_T + 1)^2.
$$

Consider the following assumptions.

(A6) Within each unit $i$, the regression vectors $\{x_{it}, 1 \leq t \leq T\}$ are i.i.d. $k$-variate normal $N(0, \Lambda_i)$ (centred with covariance matrix $\Lambda_i$).

(A7) The covariates $\{x_{it}\}$ and the errors $\{\nu_{it}\}$ are independent.

Under the assumptions (A1)-(A2) and (A6)-(A7), the null and the SIM-L scheme, Bailey et al. (2020) established

$$
LM_{RMT} \xrightarrow{D} N(0, 1).
$$

(2.9)

The simulation experiments in this reference showed that $LM_{RMT}$ is generally comparable to $LM_{adj}$ in terms of size and power, with however a slight preference for $LM_{RMT}$ when the sample size $T$ is relatively small. This result requires normality for both the model errors and the regressors, see Assumption (A2) and (A6).

3. An extended Lagrange multiplier test for large panels without normality assumption ($LM_e$)

As a main result of the present paper, we extend the test $LM_{RMT}$ to large panels where the errors are not necessarily normal distributed. Because in large panels, most of the previous results on the Lagrange multiplier rely on the normality assumption, the derivation of the asymptotic distribution of the LM statistic here requires new tools. This is achieved via recent results on sample correlation matrix from random matrix theory literature.

To introduce the conditions on the centralised design matrices $X_i$, we need to consider $E_k$, the subspace of $R^k$ which is orthogonal to the constant vector $1_k$ (with all coordinates equal to 1), that is,

$$
E_k = \{u = (u_1, \ldots, u_k) \in R^k : u_1 + \ldots + u_k = 0\}.
$$

We will use the following assumptions on the panel model.
(B1) The panel-wise error vectors \( \nu_1, \ldots, \nu_T \) are i.i.d. with mean zero and uniformly bounded sixth moments, that is
\[
\sup_{i,t} E|\nu_{it}|^6 \leq C_2, \quad \text{for some positive constant } C_2.
\]

(B2) (i) The regressors \( \{x_{it}\} \) are independent of the idiosyncratic disturbances \( \{\nu_{it}\} \).

(ii) There are positive constants \( a_1 \) and \( a_2 \) such that for all \( i, T \) and non zero \( u \in E_k \),
\[
a_1 \|u\|^2 \leq \frac{1}{T} u'X_iX_i' u \leq a_2 \|u\|^2.
\]

(iii) For any non zero vector \( u \in E_k \) and \( i \),
\[
\max_{1 \leq t \leq T} \left\langle u, \tilde{x}_{it} \right\rangle^2 \frac{u'X_iX_i' u}{u'X_iX_i' u} \to 0, \quad T \to \infty.
\]

The Assumption (B1) is a standard moment condition on the errors \( \{\nu_{it}\} \) which are serially uncorrelated over time with a constant cross-section covariance matrix \( \Sigma = \text{cov}(\nu_t) \). The Assumption (B2) ensures the regularity of the design matrices with the centralised regression variables \( \{\tilde{x}_{it}\} \). The independence between regressors and errors required in Assumption (B2) is slightly stronger than the strict exogeneity in Assumption (A4).

**Theorem 3.1.** Suppose Assumptions (B1) and (B2) hold for the panel data model (1.1). Then under the SIM-L scheme and the null hypothesis,
\[
\text{LM}_e := \frac{\text{tr}(\hat{R}_T^2) - \mu_{\text{LM}_e}}{\sigma_{\text{LM}_e}} \overset{D}{\to} N(0, 1), \quad (3.1)
\]

with
\[
\mu_{\text{LM}_e} = n(1 + c_T) + c_T^2 - c_T, \quad (3.2)
\]
\[
\sigma^2_{\text{LM}_e} = 4c_T^2. \quad (3.3)
\]

This result is established in two steps. Using recent results from random matrix theory, we first find that
\[
\frac{\text{tr}(R_T^2) - \mu_{\text{LM}_e}}{\sigma_{\text{LM}_e}} \overset{D}{\to} N(0, 1). \quad (3.4)
\]

Next, we show that
\[
\text{tr}(\hat{R}_T^2) - \text{tr}(R_T^2) = o_p(1). \quad (3.5)
\]
The first step result (3.4) is justified in Appendix A. The justification of the second step result (3.5) requires more calculations; they are detailed in Appendix C, see Proposition C.1.

We now compare the new test $LM_e$ to two existing tests also developed in the SIM-L scheme. With respect to the bias-corrected test $LM_{bc}$, note that the $LM_p$ statistic (Section 2.4) can be rewritten as

$$LM_p = \frac{\text{tr}(\hat{R}_T^2) - \mu_{LM_p}}{\sigma_{LM_p}}$$

with

$$\mu_{LM_p} = n(1 + c_T) - c_T + o(1), \quad \text{and} \quad \sigma^2_{LM_p} = 4c_T^2 = \sigma^2_{LM_e}.$$  

We have

$$\frac{\text{tr}(\hat{R}_T^2) - \mu_{LM_e}}{2c_T} = \frac{\text{tr}(\hat{R}_T^2) - \mu_{LM_p} - c_T^2}{2c_T} = LM_P - \frac{c_T}{2}.$$  

Thus by Theorem 3.1,

$$LM_P - \frac{c_T}{2} \xrightarrow{D} N(0,1),$$

under the SIM-L scheme. This coincide with the asymptotic normality of $LM_{bc}$ given in (2.7). Therefore, our $LM_e$ test can be seen as an extension of the $LM_{bc}$ test to panels with non-normal-distributed errors.

Next we compare the $LM_e$ test to the result (2.9) for the Gaussian $LM_{RMT}$ test. We find that $\mu_{rmt} = \mu_{LM_e}$, and $\sigma^2_{rmt} \sim \sigma^2_{LM_e} \sim 4c_T^2$ for large $T$ (and $n$) since as $\kappa \to 3$ when $k$ is fixed and $T \to \infty$. This means that under the SIM-L scheme, the asymptotic distribution of $\text{tr}(\hat{R}_T^2)$ derived in Theorem 3.1 for the extended LM test $LM_e$ is also valid for the heterogeneous model (1.2) assuming normality of the errors (and the regression variables).

In conclusion of these comparisons, the asymptotic distribution of $\text{tr}(\hat{R}_T^2)$ derived in Theorem 3.1 has a welcomed universality, being valid for the three tests $LM_e$, $LM_{adj}$ and $LM_{RMT}$ which cover quite different large panel models. Such distributional robustness enlarges the application scope of the LM statistic to various large panel models.

4. A power enhanced test for large panels

Anticipating the simulation results shown in Section 5, the various large-panel versions of the LM test, namely $LM_{adj}$, $LM_{bc}$, $LM_{RMT}$ as well as the new test $LM_e$ may suffer from the
problem of low power against large panels where the units are weakly dependent. Such weak dependence arises for example when the cross-sectional correlation matrix $R$ is sparse. Recent literature in high-dimensional statistics indicates that an efficient way for detection of sparse correlations is to weight those relatively significant sample correlations $\hat{\rho}_{rs}$ more heavily than those small sample correlations. An extreme method in this regard is for example to take the overall maximum $\max_{r \neq s} |\hat{\rho}_{rs}|$ as a test statistic. It is however unclear in the current SIM-L large panel setting how to derive a limiting distribution for such maximum type statistic. Here we propose a manageable compromise by considering the sum of fourth powers of the sample residuals correlations, that is, to consider

$$tr(\hat{R}_T^4) = \sum_{r \neq s} \hat{\rho}_{rs}^4,$$

as the new test statistic. The rational here is that compared to the sum of the squares $\{\hat{\rho}_{rs}^4\}$ used in the LM statistic, $tr(\hat{R}_T^4)$ is weighting more heavily larger sample correlations than smaller ones. This will enhance the power of the test when either very few sample correlations are significantly non zero, or they are many but with relatively small amplitudes. Such situations arise under an alternative with sparse cross-section dependence, or with globally weak cross-dependence.

We derive the asymptotic normality of $tr(\hat{R}_T^4)$ in the following theorem.

**Theorem 4.1.** Suppose Assumptions (B1) and (B2) hold for the panel data model (1.1). Then under the SIM-L scheme and under the null hypothesis $H_0$ in (1.3),

$$\frac{tr(\hat{R}_T^4) - \mu_{PET}}{\sigma_{PET}} \xrightarrow{D} N(0, 1).$$

with

$$\mu_{PET} = n \left(1 + \frac{6n}{T - 1} + \frac{6n^2}{(T - 1)^2} + \frac{n^3}{(T - 1)^3}\right) - 6c_T(1 + c_T)^2 - 2c_T^2,$$  

$$\sigma_{PET}^2 = 8c_T^4 + 96c_T^3(1 + c_T)^2 + 16c_T^2(3c_T^2 + 8c_T + 3)^2.$$  

The limiting normality of $tr(\hat{R}_T^4)$ in (4.1) under the null allows us to perform a level-$\alpha$ test for the null hypothesis $H_0$. This test is hereafter referred as the *power enhanced test (PET)* for cross-section independence in large panels.
The proof of Theorem 4.1 follows the strategy for that of Theorem 3.1 and also proceeds in two steps. In the first step, using recent results from random matrix theory, we find that

\[
\frac{\text{tr}(R^4_T) - \mu_{\text{PET}}}{\sigma_{\text{PET}}} \overset{D}{\to} N(0,1).
\] (4.4)

Next, we show that

\[
\text{tr}(\hat{R}^4_T) - \text{tr}(R^4_T) = o_p(1).
\] (4.5)

The first step result (4.4) is justified in Appendix B. The justification of the second step result (4.5) also requires more calculations; they are detailed in Appendix C, see Proposition C.2.

5. Simulation studies

We conduct a simulation study to investigate the finite sample performance of the proposed tests \(LM_e\) and \(PET\). Comparisons are made with the bias-adjusted LM test \(LM_{\text{adj}}\) in (2.6) and the cross-section dependence test \(CD\) in (2.5). The original Lagrange Multiplier test \(LM\) in (2.4) is excluded due to its well-known non applicability to large panels. The Gaussian high-dimensional Lagrange Multiplier test \(LM_{RMT}\) is also excluded in comparison as it is equivalent to our proposed general Lagrange multiplier test \(LM_e\), see comments after Theorem 3.1.

5.1. Empirical sizes of the tests

We consider the following data generating process proposed in Pesaran et al. (2008):

\[
y_{it} = a + \sum_{l=2}^{k} x_{lit} \beta_l + \mu_i + \nu_{it}, \quad 1 \leq i \leq n; \ 1 \leq t \leq T,
\] (5.1)

where \(a\) and \(\beta_l\) are set arbitrarily to 1 and \(l\), respectively, \(\mu_i \overset{\text{i.i.d.}}{\sim} N(1,1)\) and here \(k\) is the number of regressors including the intercept \(a\). The regressors are generated as

\[
x_{lit} = 0.6x_{i,t-1} + \sigma_{li} u_{lit}, \quad i = 1,2,\ldots,n; \ t = -49,\ldots,0,\ldots,T; \ l = 2,\ldots,k,
\] (5.2)

with \(x_{i,-50} = 0\), \(\sigma^2_{li} = \tau^2_{li}/(1 - 0.6^2)\), \(\tau^2_{li} \overset{\text{i.i.d.}}{\sim} \chi^2(6)/6\) and \(u_{lit} \overset{\text{i.i.d.}}{\sim} N(0,1)\). The first 50 observations in (5.2) are disregarded. Under the null, the errors \(\nu_{it}\) are assumed to be i.i.d. across individuals and over time, that take the form

\[
\nu_{it} = \sigma_i \varepsilon_{it}, \quad 1 \leq i \leq n; \ 1 \leq t \leq T,
\]
where $\sigma^2_i \sim \chi^2(2)/2$ and the $\varepsilon_{it}$'s are also i.i.d. and generated from different populations: (i) normal, $N(0,1)$, (ii) Student-t, $t_7/\sqrt{7/5}$, and (iii) chi-square, $(\chi^2(5) - 5)/\sqrt{10}$. The normalisation's in (ii) and (iii) are such that these variable have mean zero and unit variance.

We explore the performance of different tests using various combination of $(n,T)$ with $n \in \{50,100,200\}$ and $T \in \{50,100\}$. Empirical sizes and powers of the tests are evaluated from 2,000 independent replications. The nominal test level is 5%.

Table 5.1 presents the empirical sizes of $LMe$, PET, $LM_{adj}$ and CD tests (values close to 5% are better) under three different error distributions. The proposed PET test and the CD test have a similar performance a little better than $LMe$, which is in turn slightly better than $LM_{adj}$. However, the $LM_{adj}$ test is noticeably computationally more demanding than the other tests. It also has a large downside size distortion under the cases with $T = 50$, $n = 200$ and $k = 4$. 

Table 5.1  

|         | $k$ | 2   | 4   |
|---------|-----|-----|-----|
|         |     | 50  | 100 | 200 | 50  | 100 | 200 |
| T       |     |     |     |     |     |     |     |
| Normal  |     |     |     |     |     |     |     |
| $LMe$   |     |     |     |     |     |     |     |
| PET     | 5.00| 5.05| 4.70| 4.80| 6.05| 5.55|     |
| 50      |     |     |     |     |     |     |     |
| $LM_{adj}$ | 5.20| 5.15| 4.55| 4.20| 4.55| 2.80|     |
| CD      | 5.45| 4.95| 5.10| 4.95| 5.15| 5.30|     |
| $LMe$   |     |     |     |     |     |     |     |
| PET     | 5.45| 5.00| 5.30| 4.65| 5.25| 5.40|     |
| 100     |     |     |     |     |     |     |     |
| $LM_{adj}$ | 5.75| 5.05| 5.35| 4.70| 5.05| 4.30|     |
| CD      | 4.45| 5.50| 5.40| 5.20| 4.85| 5.15|     |
| Student-t |     |     |     |     |     |     |     |
| $LMe$   |     |     |     |     |     |     |     |
| PET     | 5.20| 4.90| 4.70| 5.15| 5.55| 5.35|     |
| 50      |     |     |     |     |     |     |     |
| $LM_{adj}$ | 5.55| 5.05| 4.20| 4.80| 4.10| 2.65|     |
| CD      | 5.55| 5.35| 4.45| 5.10| 5.05| 5.55|     |
| $LMe$   |     |     |     |     |     |     |     |
| PET     | 5.00| 4.45| 5.50| 4.80| 4.90| 5.60|     |
| 100     |     |     |     |     |     |     |     |
| $LM_{adj}$ | 5.30| 4.65| 5.50| 4.90| 4.75| 5.00|     |
| CD      | 4.55| 5.30| 5.30| 4.80| 5.25| 4.60|     |
| Chi-square |     |     |     |     |     |     |     |
| $LMe$   |     |     |     |     |     |     |     |
| PET     | 4.85| 6.45| 5.65| 5.75| 5.35| 5.10|     |
| 50      |     |     |     |     |     |     |     |
| $LM_{adj}$ | 5.45| 6.45| 5.30| 5.40| 4.15| 2.35|     |
| CD      | 5.20| 4.85| 4.90| 5.05| 5.00| 4.85|     |
| $LMe$   |     |     |     |     |     |     |     |
| PET     | 4.85| 4.90| 5.90| 5.05| 5.05| 5.80|     |
| 100     |     |     |     |     |     |     |     |
| $LM_{adj}$ | 5.15| 5.40| 6.35| 6.40| 4.80| 5.25|     |
| CD      | 4.70| 4.75| 4.95| 6.10| 4.90| 4.90|     |
5.2. Empirical powers of the tests

To evaluate the power of the tests considered, the disturbances are generated according to the following single-factor model,

$$\nu_{it} = \lambda_i f_t + \epsilon_{it}, \quad 1 \leq i \leq n; \quad 1 \leq t \leq T,$$

where $\lambda_i$ is the factor loading of individual $i$ for the common factor $f_t$ in period $t$, with $f_t \overset{i.i.d.}{\sim} N(0, 1)$. The factor loadings are constructed under three different scenarios:

- Dense case: here the strength of cross-sectional correlation is measured by a positive parameter $h > 0$. Given $h$, $\lambda_i \overset{i.i.d.}{\sim} U[-b, b]$ for $i = 1, \ldots, n$, where $b = \sqrt{3h/n}$ (the average of the squared length $\lambda_1^2 + \cdots + \lambda_n^2$ is thus $h$);

- Sparse case: $\lambda_i \overset{i.i.d.}{\sim} U(0.5, 1.5)$ for $i = 1, 2, \ldots, \lceil n^{0.3} \rceil$ and $\lambda_i = 0$ for $i = \lceil n^{0.3} \rceil + 1, \ldots, n$, where $\lceil n^{0.3} \rceil$ is the integer part of $n^{0.3}$. We have $\lceil n^{0.3} \rceil = 3, 3, 4$ for $n = 50, 100, 200$, respectively.

- Less-sparse case: $\lambda_i \overset{i.i.d.}{\sim} U(0.5, 1.5)$ for $i = 1, 2, \ldots, \lceil n^{0.5} \rceil$ and $\lambda_i = 0$ for $i = \lceil n^{0.5} \rceil + 1, \ldots, n$. We have $\lceil n^{0.5} \rceil = 7, 10, 14$ for $n = 50, 100, 200$, respectively.

In the dense case, all cross-sectional units are correlated. The correlation between units $i$ and $i'$ is $\lambda_i \lambda_{i'}$ and the overall strength of correlation is controlled by $h$. We study the empirical powers of the tests when $h$ varies while remaining bounded, that is, $h/n \to 0$. This thus corresponds to the setting of weak factor alternative used in Baltagi et al. (2017). In the sparse case, only a few, about $n^{0.3}$, cross-sectional units are correlated while other units are uncorrelated. When this number of correlated cross-sectional units increases to $n^{0.5}$, we call it a less-sparse case.

Table 5.2 shows the empirical powers of the tests in the dense case. The dimensions $n$, $T$ and $k$ are set to be $(50, 100)$, 100 and 2, respectively. The cross-sectional correlation strength $h$ varies from 1 to 7. As expected, the empirical powers of all tests increase with the strength $h$. The PET test largely outperforms the others. Compared to $LM_e$ and $LM_{adj}$, PET indeed boosts the power by up to 36% for cases with small value of $h$. $LM_{adj}$ performs a little better than $LM_e$. The CD test has very low powers, confirming the fact that its implicit null is rather weak cross-sectional dependence (Pesaran, 2004). Plots at the bottom of Table 5.2 illustrate
the evolution of these powers in function of the varying strength \( h \) and for the cases with chi-square distributed errors.

Tables 5.3 and 5.4 show the empirical powers for the sparse and the less-sparse cases, respectively. As expected, all the tests have higher power in the less-sparse case than in the sparse cases. The proposed PET test again performs best. It boosts the power by up to 16\% under these sparse cases as compared to \( LM_e \) and \( LM_{adj} \). The proposed \( LM_e \) test performs better than \( LM_{adj} \). The latter has low powers in the cases with \( T = 50 \) and \( k = 4 \), which is consistent with the conservative sizes observed in Table 5.1. The CD test has no powers for the sparse case, but has some powers for the less-sparse case; it has however an overall poor performance compared to the other tests.

### 5.3. Additional simulation experiments

The two proposed tests \( LM_e \) and \( PET \) are established for the fixed effects panel data model (1.1). Additional simulations are conducted to show the finite sample performance of the proposed tests for the heterogeneous panel data model (1.2) although this model is not covered by the developed theory. We consider the following data generating process of Pesaran et al. (2008):

\[
y_{it} = a + \sum_{l=2}^{k} x_{lit} \beta_{li} + \mu_i + \nu_{it}, \quad 1 \leq i \leq n; \quad 1 \leq t \leq T,
\]

with \( \beta_{li} \overset{i.i.d.}{\sim} N(1, 0.04) \). All other settings are kept the same as for the fixed effect model. It is striking and satisfactory to observe that for the heterogeneous panel model, conclusions from the simulation experiments are in general very similar to those reported in the previous section for the fixed-effect panel model. The following tables report some empirical results for the case of \( k = 2 \) (the other results with \( k = 4 \) are very similar and thus omitted).

Table 5.5 presents the empirical sizes of all tests under three different error distributions. The empirical sizes of the considered tests are all close to the nominal level. The proposed tests \( LM_e \) and \( PET \) perform slightly better than \( LM_{adj} \). Tables 5.6, 5.7 and 5.8 show the empirical powers of the tests for the dense case, the sparse case and the less-sparse case, respectively. As expected, \( LM_e \), \( PET \) and \( LM_{adj} \) all show higher powers when the correlation matrix of errors becomes denser. Again the proposed PET has consistently the highest power. In conclusion,
the proposed two tests seem also valid for the heterogeneous panel data model (1.2) according to these experiments.

6. A real data analysis

We apply our two new tests $LM_e$ and PET to the public health data sets in Lin et al. (2020) for the investigation of association between air pollution, hypertension and blood pressure of people. They used a unique sample of elderly people in Nanjing (China) containing 21 individuals with 441 medical records in total with precise examination dates, that is, $n = 21, T = 21$. Fixed effects panel data models are constructed to evaluate the effect of the special particulate $PM_{2.5}$ (diameter $< 2.5 \mu m$) on the hypertension (HY), systolic blood pressure (SBP) and diastolic blood pressure (DBP), respectively, by controlling the individual fixed effects $\delta_i, i = 1, \ldots, n$. Temperature (temp) is also considered as another control variable due to its positive effect on blood pressure. Therefore, three panel models are considered

- Model 1: $HY = \beta_1 \log(PM_{2.5}) + \beta_2 \log(temp) + \delta_i + \nu_1$;
- Model 2: $SBP = \beta_1 \log(PM_{2.5}) + \beta_2 \log(temp) + \delta_i + \nu_2$;
- Model 3: $DBP = \beta_1 \log(PM_{2.5}) + \beta_2 \log(temp) + \delta_i + \nu_3$.

To investigate whether the cross-sectional uncorrelation assumption in three models is justified, we applied the $LM_{adj}$ test, the $LM_e$ test, the PET test and the CD test to each regression model. The values of the corresponding test statistics are reported in Table 6.1.

Among the other four tests and considering a 5% nominal level, none of them can reject the null hypothesis of cross-sectional uncorrelation under Model 1. But they all reject the null under both Model 2 and Model 3. That means, individuals are uncorrelated in terms of hypertension, but correlated in terms of SBP and DBP. Meanwhile, the values of PET are much larger than others in Models 2 and 3 which confirm its power-enhancement property. Overall, the statistical results drawn from Model 1 seem reliable and consistent with each other, while more control variables need to be investigated to study the association of air pollution with SBP or DBP.

As illustrated by this example, the results of this paper seem to have potential application in the important area of panel data modelling.
7. Conclusion

For large fixed effects panel data model, we propose two new and efficient tests to detect the existence of cross-sectional correlation (dependence). The asymptotic normalities of test statistics are constructed under a simultaneous limit scheme (SIM-L) where the cross-sectional units dimension $n$ and the time series dimension $T$ are both large with comparable magnitude. Meanwhile, these results do not need the normality assumption on the errors and/or on the random design, while such normality assumptions are essential for the theoretical justification of most of the existing tests for large panels. Extensive Monte-Carlo experiments demonstrates the superiority of our proposed tests over some popular existing methods in terms of size and power. Especially, the power enhanced high-dimensional test PET consistently outperforms all the other methods considered.

There are still several avenues for future research. The tests proposed here are based on the fixed effects panel data model. It is highly valuable to investigate their validity in other large panel models. For example, our simulation experiments have shown the applicability of these proposed tests for the heterogeneous panel data model. A thorough theoretical investigation of such observation is missing though.

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### Appendix A: Proof of the asymptotic normality (3.4)

From the Theorem 3.1 of Yin et al. (2020), we have

$$
tr(R_T^2) − μ_c \overset{D}{\longrightarrow} N(μ_{\text{limit}}, σ_{\text{limit}}^2).
$$
From the result for $g_l = x^2$ in Example 3.2 of Yin et al. (2020), we get the centring term 
\[ \mu_c = n \left( 1 + \frac{n}{T-1} \right) = n \left( 1 + \frac{n}{T} \left( 1 + \frac{1}{T-1} \right) \right) = n(1 + c_T) + c_T^2. \]

From the results for $R = I_n$ and $g_l = x^2$ in Example 3.3 of Yin et al. (2020), we get the limiting terms 
\[ \mu_{\text{limit}} = -c, \quad \text{and} \quad \sigma_{\text{limit}}^2 = 4c^2. \]

The proof of Lemma 3.4 is complete.

**Appendix B: Proof of the asymptotic normality (4.4)**

From the Theorem 3.2 of Yin et al. (2020), we have 
\[ \text{tr}(R^T) - \mu_c' \xrightarrow{D} N(\mu_l', \sigma_l'^2). \]

From the result for $g_l = x^4$ in Example 3.2 of Yin et al. (2020), we get the centring term 
\[ \mu_c' = n \left( 1 + \frac{6n}{T-1} + \frac{6n^2}{(T-1)^2} + \frac{n^3}{(T-1)^3} \right). \]

From the results for $R = I_n$ and $g_l = x^4$ in Example 3.3 of Yin et al. (2020), we get the limiting terms 
\[ \mu_l' = -6c(1 + c)^2, \]
and 
\[ \sigma_l'^2 = 8c^4 + 96c^3(1 + c)^2 + 16c^2(3c^2 + 8c + 3)^2. \]

The proof of Lemma 4.4 is complete.

**Appendix C: Proof of the key estimates (3.5) and (4.5)**

The proof for the two key estimates is the main technical difficulty of the paper. They are given in Propositions C.1 and C.2 at the end of this section after a series of preliminary lemmas and calculations.

Recall first some useful notations related to the OLS residuals as given at the beginning of Section 2. The OLS estimator for $\beta$ is $\hat{\beta} = (XX')^{-1} XY$ ($k \times 1$ vector). For the errors $\tilde{\nu}_{it}$, let
$\hat{V}_i = \left( \hat{v}_{i1}, \ldots, \hat{v}_{iT} \right)'$ be a $T \times 1$ vector and then $\hat{V} = \left( \hat{V}_1, \ldots, \hat{V}_n \right)$ is a $T \times n$ matrix. Similarly for the residuals, define $\hat{v}_{it} = \hat{v}_{it} - \hat{\varepsilon}_{it} \left( \hat{\beta} - \beta \right)$, $\hat{V}_i = \hat{V}_i - X_i' \left( \hat{\beta} - \beta \right)$ and set $\hat{V} = \left( \hat{V}_1, \ldots, \hat{V}_n \right)$ ($T \times n$ matrix). Define also $W_i = X_i' \left( \hat{\beta} - \beta \right)$ ($T \times 1$ vector), $W = \left( W_1, \ldots, W_n \right)$ ($T \times n$ matrix). We can easily get that $\hat{V}_i = \hat{V}_i - W_i$, $\hat{V} = \hat{V} - W$.

We have for the sample covariance matrices: $S_T = \frac{1}{T} \hat{V}' \hat{V}$, $\hat{S}_T = \frac{1}{T} \hat{V}' \hat{V}$ with respective elements,

$$S_{T,i,j} = \frac{1}{T} \langle \hat{V}_i, \hat{V}_j \rangle = \frac{1}{T} \sum_{t=1}^{T} \hat{v}_{it} \hat{v}_{jt},$$

$$\hat{S}_{T,i,j} = \frac{1}{T} \langle \hat{V}_i, \hat{V}_j \rangle = \frac{1}{T} \sum_{t=1}^{T} \hat{v}_{it} \hat{v}_{jt}.$$

**Lemma C.1.** (Petrov, 1975, Theorem 13 of Chapter 13) Let $Y_1, \ldots, Y_n$ be independent and identically distributed random variables, such that $E(Y_1) = 0$, $E(Y_1)^2 = 1$ and $E|Y_1|^r < \infty$ for some $r \geq 3$. Then

$$\left| \mathbb{P} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i < y \right) - \Phi(y) \right| \leq \frac{C(r)}{(1 + |y|)^r} \left[ \frac{E|Y_1|^3}{\sqrt{n}} + \frac{E|Y_1|^r}{n^{\frac{r-2}{2}}} \right]$$

for all $y$, where $\Phi(\cdot)$ is the cumulative distribution function of standard normal and $C(r)$ is a positive constant depending only on $r$.

**Lemma C.2.** Let $\{Y_{ij}\}_{i \geq 1, j \geq 1}$ be an array of independent and identically distributed random variables such that $E(Y_{11}) = 0$, $E(Y_{11})^2 = 1$ and $E|Y_{11}|^r < \infty$ for some $r \geq 3$. Let $X_{in} = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} Y_{ij}$, then for any $\epsilon > 0$, we have

$$\max_{1 \leq i \leq n} |X_{in}| = O_p(n^{\frac{1}{2r} + \epsilon}).$$

**Proof.** For some $\alpha > 0$, let $c = \frac{1}{2r} + \epsilon$, we have

$$\mathbb{P}(\max_{1 \leq i \leq n} |X_{in}| \geq \alpha n^c) = 1 - [\mathbb{P}(|X_{in}| < \alpha n^c)]^n$$

$$\leq 1 - \left\{ \Phi(\alpha n^c) - \frac{C(r)}{(1 + \alpha n^c)^r} \left[ \frac{E|Y_{11}|^3}{\sqrt{n}} + \frac{E|Y_{11}|^r}{n^{\frac{r-2}{2}}} \right] \right\}^n$$

$$\sim 1 - \frac{1}{\sqrt{2\pi \alpha}} n^{-c} e^{-\frac{\alpha^2 n^{2c}}{2}} - \frac{C(r)}{(1 + \alpha n^c)^r} \left[ \frac{E|Y_{11}|^3}{\sqrt{n}} + \frac{E|Y_{11}|^r}{n^{\frac{r-2}{2}}} \right] \left[ \frac{E|Y_{11}|^3}{\sqrt{n}} + \frac{E|Y_{11}|^r}{n^{\frac{r-2}{2}}} \right] \sim \frac{1}{\sqrt{2\pi \alpha}} n^{-c} e^{-\frac{\alpha^2 n^{2c}}{2}} + \frac{C(r)E|Y_{11}|^3}{(1 + \alpha n^c)^r n^{\frac{r-2}{2}}},$$

where the first inequality follows by Lemma C.1, and the first approximation follows by the fact that $\Phi(x) \sim 1 - \frac{1}{\sqrt{2\pi x}} e^{-\frac{x^2}{2}}$ for large $x$. Therefore, $\max_{1 \leq i \leq n} |X_{in}| = O_p(n^c)$ holds. \hfill \Box
Remark C.1. For the panel data model, if the errors \( \{ \tilde{v}_{it} \}_{i \geq 1, t \geq 1} \) satisfy the conditions in Lemma C.2 and \( X_{iT} = \frac{1}{T} \sum_{t=1}^{T} \tilde{v}_{it} \), then for any \( \epsilon > 0 \), the estimate \( \max_{1 \leq i \leq n} |X_{iT}| = O_p(n^{\frac{1}{2} + \epsilon}) \) still holds once we further assume that \( K_1 \leq \frac{n}{T} \leq K_2 \) for some positive constants \( K_1 \) and \( K_2 \). The latter holds in the SIM-L scheme where indeed \( \frac{n}{T} \rightarrow c > 0 \).

Lemma C.3. Under Assumptions (B1) and (B2), for any \( \epsilon > 0 \) and some integer \( r_1 \geq 6 \) (such that \( E \nu_{it}^6 < \infty \)), we have the following results,

\[
\begin{align*}
(a) & \quad \max_{1 \leq i, j \leq n} |\tilde{V}_i, W_j| = O_p(n^{-1} + \epsilon - 1/2), \\
(b) & \quad \max_{1 \leq i, j \leq n} |W_i, W_j| = O_p(n^{-1}). \\
(c) & \quad \max_{1 \leq i, j \leq n} |S_{T,i,i} - \sigma^2| = O_p(n^{-1} + \epsilon - 1/2). \\
(d) & \quad \max_{1 \leq i, j \leq n} |\hat{S}_{T,i,i} - S_{T,i,i}| = O_p\left(n^{\frac{1}{2} + \epsilon - 3/2}\right). \\
(e) & \quad \max_{1 \leq i, j \leq n} |\hat{S}_{T,i,i} - \sigma^2| = O_p\left(n^{\frac{1}{2} + \epsilon - 3/2}\right). \\
(f) & \quad \max_{1 \leq i, j \leq n} |\hat{S}_{T,i,i}^2 - S_{T,i,i}^2| = O_p\left(n^{\frac{1}{2} + \epsilon - 3/2}\right).
\end{align*}
\]

Proof. Throughout the proof, the linear operators \( X_iX_i' \) is restricted to the subspace \( E_k \) (orthogonal to the constant vectors in \( R^k \)) where it is invertible by Assumption (B2). Note that \( \epsilon_i \) for \( i = 1, 2 \) are small positive numbers which may vary in different equations.

(a). Firstly, we consider the case \( i = j \). By CLT, we have \( \sqrt{T} \left( \frac{1}{T} \tilde{V}'_i \tilde{V}_i - 1 \right) \overset{D}{\rightarrow} N(0, \tau^2) \), where \( \tau^2 = \text{var}(\tilde{v}_{it}^2) \). By Assumption (B2), we have for any non null \( u \in E_k \),

\[
\frac{\langle u, \sum_t \tilde{x}_{it} \tilde{v}_{it} \rangle}{(u'X_iX_i'u)^{1/2}} \overset{D}{\rightarrow} N(0, 1).
\]

That is

\[
\xi_i = (X_iX_i')^{-1/2} \sum_t \tilde{x}_{it} \tilde{v}_{it} \overset{D}{\rightarrow} N_k(0, I_k).
\]

Therefore, for some \( r_1 \geq 3 \) and for any \( \epsilon_1 > 0 \), from Lemma C.2 we have

\[
\max_{1 \leq i \leq n} \| \xi_i \| = O_p(n^{\frac{1}{2r_1} + \epsilon_1}).
\]

Then

\[
\max_{1 \leq i \leq n} \left| (W_i, \tilde{V}_i) \right| = \max_{1 \leq i \leq n} \left| \sum_t \tilde{v}_{it} \tilde{x}_{it}' (\hat{\beta} - \beta) \right| = \max_{1 \leq i \leq n} \sqrt{T} \xi_i' \left( \frac{1}{T} X_iX_i' \right)^{1/2} (\hat{\beta} - \beta) \leq \sqrt{T} \left\| \left( \frac{1}{T} X_iX_i' \right)^{1/2} \right\| \max_{1 \leq i \leq n} \| \xi_i \| \| \hat{\beta} - \beta \|
\]
\[ = O_p \left( \sqrt{T} \cdot \frac{n^{\frac{1}{2T}+\epsilon_1}}{\sqrt{nT}} \right) = O_p(n^{\frac{1}{2T}+\epsilon_1-1/2}). \]

The calculations for \( i \neq j \) case is similar.

(b). By Assumption (B2), we have

\[ |(W_i, W_j)| \leq \|W_i\|\|W_j\| = \|X'_i(\hat{\beta} - \beta)\|\|X'_j(\hat{\beta} - \beta)\| \leq \|X_i\|\|X_j\|\|\hat{\beta} - \beta\|^2 \]
\[ \leq \|X'_iX_i\|^{1/2}\|X'_jX_j\|^{1/2}\|\hat{\beta} - \beta\|^2 \]
\[ = O(T^{1/2})O(T^{1/2})O_p(1/(nT)) = O_p(n^{-1}). \]

(c). Note for \( r_2 = r_1/2 \), we have \( E[\hat{\nu}_{it}^2 - \sigma^2]r_2 < \infty \). By CLT, we have \( Z_i := \sqrt{T}(S_{T,i,i} - \sigma^2) \xrightarrow{d} N(0, \tau^2) \) where \( \sigma^2 = E(\hat{\nu}_{it}^2) \) and \( \tau^2 = \text{var}(\hat{\nu}_{it}^2) \), then we have \( \max_{1 \leq i \leq n} |Z_i| = O_p(n^{\frac{1}{2r_2}+\epsilon_2-1/2}) \) for any \( \epsilon_2 > 0 \) by Lemma C.2. Therefore,
\[ \max_{1 \leq i \leq n} |S_{T,i,i} - \sigma^2| = O_p(n^{\frac{1}{2r_2}+\epsilon_2-1/2}) = O_p(n^{\frac{1}{r_1}+\epsilon_2-1/2}). \]

(d). By (a) and (b), we have
\[ \max_{1 \leq i \leq n} |S_{T,i,i} - S_{\tilde{T},i,i}| = \max_{1 \leq i \leq n} \left| -2T^{-1}(W_i, \tilde{V}_i) + T^{-1}\|W_i\|^2 \right| \]
\[ = \frac{1}{T}O_p(n^{\frac{1}{2T}+\epsilon_1-1/2}) + \frac{1}{T}O_p(n^{-1}) = O_p(T^{-1}n^{\frac{1}{2T}+\epsilon_1-1/2}). \]

(e). The conclusion holds from (c) and (d).

(f). By (a), (b) and (c), we have
\[ \max_{1 \leq i \leq n} |S_{T,i,i}^2 - S_{\tilde{T},i,i}^2| \]
\[ = \max_{1 \leq i \leq n} \frac{1}{T^2} \left| 4\langle W_i, \tilde{V}_i \rangle^2 + \|W_i\|^4 - 4\langle \tilde{V}_i, \tilde{V}_i \rangle\langle W_i, \tilde{V}_i \rangle + 2\langle \tilde{V}_i, \tilde{V}_i \rangle\|W_i\|^2 + 4\langle W_i, \tilde{V}_i \rangle\|W_i\|^2 \right| \]
\[ = \frac{1}{T^2} \left( O_p\left(n^{\frac{1}{2T}+2\epsilon_1-1}\right) + O_p\left(\frac{1}{n^2}\right) + O_p\left(n^{\frac{1}{2T}+\epsilon_1+1/2}\right) + O_p\left(1\right) + O_p\left(n^{\frac{1}{2T}+\epsilon_1-3/2}\right) \right) \]
\[ = O_p\left(T^{-2}n^{\frac{1}{2T}+\epsilon_1+1/2}\right). \]
Lemma C.4. Suppose Assumptions (B1)-(B2) hold. We have

(a) \( \sum_{i \neq j}^{n} \left\{ \hat{S}_{T,i,j}^2 - S_{T,i,j}^2 \right\} \xrightarrow{p} 0. \)

(b) \( \sum_{i \neq j \neq l}^{n} \left\{ \hat{S}_{T,i,j}^2 \hat{S}_{T,j,l}^2 - S_{T,i,j}^2 S_{T,j,l}^2 \right\} \xrightarrow{p} 0. \)

(c) \( \sum_{i \neq j \neq l \neq s}^{n} \left\{ \hat{S}_{T,i,j} \hat{S}_{T,j,l} \hat{S}_{T,i,s} \hat{S}_{T,l,s} - S_{T,i,j} S_{T,j,l} S_{T,i,s} S_{T,l,s} \right\} \xrightarrow{p} 0. \)

Proof. (a) We have

\[
\sum_{i \neq j}^{n} \left\{ \hat{S}_{T,i,j}^2 - S_{T,i,j}^2 \right\} = \frac{1}{T^2} \sum_{i \neq j}^{n} \left\{ \left( \bar{V}_i' - W_i' \right) \left( \bar{V}_j - W_j \right)^2 - \left( \bar{V}_i' \bar{V}_j \right)^2 \right\} 
\]

\[
= \frac{1}{T^2} \sum_{i \neq j}^{n} \left\{ -2\bar{V}_i'\bar{V}_j \left( W_i'\bar{V}_j + \bar{V}_i' W_j - W_i' W_j \right) + \left( W_i'\bar{V}_j + \bar{V}_i' W_j - W_i' W_j \right)^2 \right\} .
\]

Furthermore,

\[
\frac{1}{T^2} \sum_{i \neq j}^{n} |\bar{V}_i'\bar{V}_j W_i' W_j| = \frac{1}{T^2} \sum_{i \neq j}^{n} |\sqrt{T} \xi_{ij} W_i' W_j| \quad \text{set} \quad \bar{V}_i' \bar{V}_j = \sqrt{T} \cdot \xi_{ij} \quad (\xi_{ij} \text{ tends to } N(0, 1))
\]

\[
\leq \frac{1}{T^2} \left( \sum_{i \neq j}^{n} |\sqrt{T}||W_i' W_j| \right) \cdot \max_{i,j} |\xi_{ij}| 
\]

\[
= O_p \left( \frac{n^{1/2 + \epsilon}}{T^2 n} \right) = O_p \left( \frac{n^{1/2 + \epsilon - 1/2}}{T^2} \right) .
\]

After similar calculations, we conclude that

\[
\frac{1}{T^2} \sum_{i \neq j}^{n} |W_i'\bar{V}_j W_i' \bar{V}_j| = O_p \left( n^{1/2 + \epsilon - 1} \right),
\]

\[
\frac{1}{T^2} \sum_{i \neq j}^{n} |W_i'\bar{V}_j W_i' W_j| = O_p \left( n^{1/2 + \epsilon - 3/2} \right),
\]

\[
\frac{1}{T^2} \sum_{i \neq j}^{n} |W_i' W_j W_i' W_j| = O_p \left( n^{-2} \right).
\]

The last result we need to show is

\[
M_n := \sum_{i \neq j}^{n} \frac{\bar{V}_i' \bar{V}_j W_i' \bar{V}_j}{T^2} = o_p(1).
\]
Consider the $U$-statistic

$$U_n = \frac{1}{2} \sum_{i,j} h_T(\tilde{V}_i, \tilde{V}_j), \quad h_T(\tilde{V}_i, \tilde{V}_j) = \frac{\tilde{V}_i'\tilde{V}_j'X_i' + \tilde{V}_i'\tilde{V}_j'X_j'}{T}.$$  

We have

$$M_n = \sum_{i \neq j} \frac{\tilde{V}_i'\tilde{V}_j'X_i' + \tilde{V}_i'\tilde{V}_j'X_j'}{T^2} \cdot (\hat{\beta} - \beta) = 2 \frac{n}{T} U_n \cdot (\hat{\beta} - \beta). \quad (C.1)$$

Because the dimension $k$ is fixed, we suppose in the following that $k = 1$ to simplify the presentation. Thus $X_j'$ is a $T \times 1$ vector. By direct calculations, one can show that

$$E(\tilde{V}_i'\tilde{V}_j\tilde{V}_i'X_j')^2 = E(\tilde{\nu}_1^2) \{ E(\tilde{\nu}_1^4) + (T-1)[E(\tilde{\nu}_1^2)]^2 \} \sum_{t=1}^T x_{jt}^2.$$ 

Therefore there exist positive constants $a_2 > a_1 > 0$, such that

$$a_1 T^2 \leq E(\tilde{V}_i'\tilde{V}_j\tilde{V}_i'X_j')^2 \leq a_2 T^2.$$ 

It follows that exist positive constants $b_2 > b_1 > 0$, such that

$$b_1 \leq E\{ h_T(\tilde{V}_i, \tilde{V}_j) \} \leq b_2. \quad (C.2)$$

Similarly, one can show that

$$E\{ \sqrt{T} h_T(\tilde{V}_i, \tilde{V}_j) \} = o(1). \quad (C.3)$$

By CLT for $U$-statistic (e.g. Theorem 12.3 in Van der Vaart (1998)), we find that $\sqrt{n}U_n$ is asymptotic Gaussian. Finally, we have by (C.1)-(C.2)-(C.3)

$$M_n = \frac{2}{T} \left( \frac{n}{2} \right) U_n \cdot (\hat{\beta} - \beta) = \frac{2}{T} \left( \frac{n}{2} \right) \frac{1}{\sqrt{n}} O_p(1) \cdot O_p(n^{-1}) = O_p(1/\sqrt{n}).$$

The proof of the (a) is complete.

(b) We have

$$LHS = \frac{1}{T^2} \sum_{i \neq j \neq l} \left\{ \left( (\tilde{V}_i' - W_i') (\tilde{V}_j - W_j) \right)^2 \left( (\tilde{V}_j' - W_j') (\tilde{V}_i - W_i) \right)^2 - \left( \tilde{V}_i' \tilde{V}_j \right)^2 \left( \tilde{V}_j' \tilde{V}_i \right)^2 \right\}$$

$$= \frac{1}{T^2} \sum_{i \neq j \neq l} \left\{ (\tilde{V}_i' \tilde{V}_j)^2 (\tilde{V}_j' W_l + W_j' \tilde{V}_l - W_l' W_l)^2 + (\tilde{V}_j' \tilde{V}_i)^2 (\tilde{V}_i' W_j + W_i' \tilde{V}_j - W_j' W_j)^2 \right\}$$

$$+ \frac{1}{T^2} \sum_{i \neq j \neq l} \left( \tilde{V}_j' W_l + W_j' \tilde{V}_l - W_l' W_l \right)^2 \left( \tilde{V}_i' W_j + W_i' \tilde{V}_j - W_j' W_j \right)^2.$$
\[-2 \frac{1}{T^4} \sum_{i \neq j \neq l}^{n} \left( \tilde{V}_i' \tilde{V}_j \right) \left( \tilde{V}_j' W_i + W_j' \tilde{V}_i - W_i' W_j \right)^2 \left( \tilde{V}_i' W_j + W_i' \tilde{V}_j - W_j' W_i \right) \]

\[-2 \frac{1}{T^4} \sum_{i \neq j \neq l}^{n} \left( \tilde{V}_j' \tilde{V}_i \right) \left( \tilde{V}_i' W_j + W_i' \tilde{V}_j - W_j' W_i \right)^2 \left( \tilde{V}_j' W_i + W_j' \tilde{V}_i - W_i' W_j \right) \]

\[+ 4 \frac{1}{T^4} \sum_{i \neq j \neq l}^{n} \left( \tilde{V}_i' \tilde{V}_j \right) \left( \tilde{V}_i' W_j + W_i' \tilde{V}_j - W_j' W_i \right) \left( \tilde{V}_j' \tilde{V}_i \right) \left( \tilde{V}_i' W_j + W_i' \tilde{V}_j - W_j' W_i \right) \]

\[-2 \frac{1}{T^4} \sum_{i \neq j \neq l}^{n} \left( \tilde{V}_i' \tilde{V}_j \right)^2 \left( \tilde{V}_j' \tilde{V}_i \right) \left( \tilde{V}_i' W_j + W_i' \tilde{V}_j - W_j' W_i \right) \]

\[-2 \frac{1}{T^4} \sum_{i \neq j \neq l}^{n} \left( \tilde{V}_j' \tilde{V}_i \right)^2 \left( \tilde{V}_i' \tilde{V}_j \right) \left( \tilde{V}_i' W_j + W_i' \tilde{V}_j - W_j' W_i \right) . \]

It is easy to conclude that

\[\frac{1}{T^4} \sum_{i \neq j \neq l}^{n} \left| \tilde{V}_i' \tilde{V}_j \tilde{V}_i' \tilde{V}_j W_i \tilde{V}_j' W_l \right| = O_p \left( n^\frac{2}{r_1} + \epsilon - 1 \right) , \]

\[\frac{1}{T^4} \sum_{i \neq j \neq l}^{n} \left| \tilde{V}_i' \tilde{V}_j \tilde{V}_i' \tilde{V}_j W_i W_j \tilde{V}_i \tilde{V}_j W_l \right| = O_p \left( n^\frac{1}{r_1} + \epsilon - 2 \right) , \]

\[\frac{1}{T^4} \sum_{i \neq j \neq l}^{n} \left| \tilde{V}_i' \tilde{V}_j \tilde{V}_i' \tilde{V}_j \tilde{V}_j W_i W_l \right| = O_p \left( n^\frac{3}{4} + \epsilon - 3/2 \right) , \]

\[\frac{1}{T^4} \sum_{i \neq j \neq l}^{n} \left| \tilde{V}_i' \tilde{V}_j \tilde{V}_i' \tilde{V}_j W_i \tilde{V}_j W_i \tilde{V}_j \tilde{V}_i W_l \right| = O_p \left( n^\frac{1}{r_1} + \epsilon - 3 \right) , \]

\[\frac{1}{T^4} \sum_{i \neq j \neq l}^{n} \left| \tilde{V}_j' W_i \tilde{V}_i' W_i \tilde{V}_i \tilde{V}_j W_j \tilde{V}_j W_i \tilde{V}_j \tilde{V}_i W_l \right| = O_p \left( n^\frac{1}{r_1} + \epsilon - 4 \right) , \]

\[\frac{1}{T^4} \sum_{i \neq j \neq l}^{n} \left| W_j' W_i \tilde{V}_i' W_i \tilde{V}_i W_i \tilde{V}_i W_i \tilde{V}_i W_l \right| = O_p \left( n^{-5} \right) , \]

\[\frac{1}{T^4} \sum_{i \neq j \neq l}^{n} \left| \tilde{V}_i' \tilde{V}_j \tilde{V}_i' \tilde{V}_j W_i \tilde{V}_i W_i \tilde{V}_i \tilde{V}_j W_l \right| = O_p \left( n^\frac{2}{r_1} + \epsilon - 2 \right) , \]

\[\frac{1}{T^4} \sum_{i \neq j \neq l}^{n} \left| \tilde{V}_i' \tilde{V}_j \tilde{V}_i' \tilde{V}_j W_i \tilde{V}_i W_i \tilde{V}_i W_i \tilde{V}_i W_l \right| = O_p \left( n^\frac{3}{r_1} + \epsilon - 5/2 \right) , \]

\[\frac{1}{T^4} \sum_{i \neq j \neq l}^{n} \left| \tilde{V}_i' \tilde{V}_j \tilde{V}_i' \tilde{V}_j W_i \tilde{V}_i W_i \tilde{V}_i W_i \tilde{V}_i W_l \right| = O_p \left( n^\frac{1}{r_1} + \epsilon - 3 \right) , \]

\[\frac{1}{T^4} \sum_{i \neq j \neq l}^{n} \left| \tilde{V}_i' \tilde{V}_j \tilde{V}_i' \tilde{V}_j \tilde{V}_i W_i \tilde{V}_j W_i \tilde{V}_j W_i \tilde{V}_i W_l \right| = O_p \left( n^\frac{1}{r_1} + \epsilon - 7/2 \right) , \]

\[\frac{1}{T^4} \sum_{i \neq j \neq l}^{n} \left| \tilde{V}_i' \tilde{V}_j \tilde{V}_i' \tilde{V}_j \tilde{V}_i W_i \tilde{V}_j W_i \tilde{V}_i W_l \right| = O_p \left( n^\frac{3}{r_1} + \epsilon - 1/2 \right) . \]
Therefore there exist positive constants $a$.

It follows that exist positive constants.

One can show that

We have

The proof of (b) is completed.

Consider the $U$-statistic

We have

One can show that

Therefore there exist positive constants $a > a > 0$, such that

It follows that exist positive constants $b > b > 0$, such that

One can show that

By CLT for $U$-statistic, we find that $\sqrt{n}U_n$ is asymptotic Gaussian. Finally, we have

The proof of (b) is completed.

(c) We have

\[ \text{LHS} = \sum_{i \neq j \neq k} \{ \hat{S}_{T,i,j} \hat{S}_{T,j,k} \hat{S}_{T,i,k} \} \]
\[
\frac{1}{T^2} \sum_{i \neq j \neq l \neq s} \left( \tilde{V}_i' - W_i' \right) \left( \tilde{V}_j - W_j \right) \left( \tilde{V}_j' - W_j' \right) \left( \tilde{V}_l - W_l \right) \left( \tilde{V}_l' - W_l' \right) \left( \tilde{V}_s - W_s \right) \left( \tilde{V}_s' - W_s' \right) \left( \tilde{V}_l - W_l \right)
\]
\[
- \frac{1}{T^2} \sum_{i \neq j \neq l \neq s} \tilde{V}_i' \tilde{V}_j' \tilde{V}_l' \tilde{V}_s \tilde{V}_s' \tilde{V}_l
\]
\[
= \frac{1}{T^2} \sum_{i \neq j \neq l \neq s} \left( \tilde{V}_i' \tilde{V}_j' \tilde{V}_l' \tilde{V}_s \tilde{V}_s' \tilde{V}_l \right) \left\{ - \tilde{V}_i' \tilde{V}_s W_s' \tilde{V}_l - \tilde{V}_i' \tilde{V}_s \tilde{V}_s' W_l + \tilde{V}_i' \tilde{V}_s W_s' \tilde{V}_l + \tilde{V}_i' \tilde{V}_s W_s' \tilde{V}_l - W_i' \tilde{V}_s W_l + W_i' \tilde{V}_s W_l - W_i' \tilde{V}_s W_l - W_i' \tilde{V}_s W_l + W_i' \tilde{V}_s W_l \right\}
\]
\[
+ \left\{ - \tilde{V}_i' \tilde{V}_j W_j' \tilde{V}_l - \tilde{V}_i' \tilde{V}_j W_j' \tilde{V}_l + \tilde{V}_i' \tilde{V}_j W_j' \tilde{V}_l + \tilde{V}_i' \tilde{V}_j W_j' \tilde{V}_l - W_i' \tilde{V}_j W_l + W_i' \tilde{V}_j W_l - W_i' \tilde{V}_j W_l - W_i' \tilde{V}_j W_l + W_i' \tilde{V}_j W_l \right\} \cdot \left\{ - \tilde{V}_i' \tilde{V}_s W_s' \tilde{V}_l - \tilde{V}_i' \tilde{V}_s \tilde{V}_s' W_l + \tilde{V}_i' \tilde{V}_s W_s' \tilde{V}_l + \tilde{V}_i' \tilde{V}_s W_s' \tilde{V}_l - W_i' \tilde{V}_s W_l + W_i' \tilde{V}_s W_l - W_i' \tilde{V}_s W_l - W_i' \tilde{V}_s W_l + W_i' \tilde{V}_s W_l \right\}
\]
\[
+ W_i' \tilde{V}_s \tilde{V}_s' \tilde{V}_l - W_i' \tilde{V}_s W_s' \tilde{V}_l - \tilde{V}_i' \tilde{V}_s \tilde{V}_s' \tilde{V}_l + W_i' \tilde{V}_s W_s' \tilde{V}_l + W_i' \tilde{V}_s \tilde{V}_s' \tilde{V}_l - W_i' \tilde{V}_s W_l + W_i' \tilde{V}_s W_l - W_i' \tilde{V}_s W_l + W_i' \tilde{V}_s W_l \right\}
\]
It is easy to conclude that
\[
\frac{1}{T^4} \sum_{i \neq j \neq l \neq s}^{n} \left| \tilde{V}_i' \tilde{V}_j W_j' \tilde{V}_l \tilde{V}_l' W_i' \tilde{V}_s \tilde{V}_s' W_l' \right| = O_p \left( n^{-\frac{3}{2} \epsilon + \frac{1}{2}} \right),
\]
\[
\frac{1}{T^4} \sum_{i \neq j \neq l \neq s}^{n} \left| \tilde{V}_i' \tilde{V}_j W_j' \tilde{V}_l W_i' \tilde{V}_l' W_i' \tilde{V}_s \tilde{V}_s' \right| = O_p \left( n^{-\frac{3}{2} \epsilon - 1} \right),
\]
\[
\frac{1}{T^4} \sum_{i \neq j \neq l \neq s}^{n} \left| \tilde{V}_i' \tilde{V}_j W_j' \tilde{V}_l W_i' W_s W_i' \tilde{V}_s \tilde{V}_s' \right| = O_p \left( n^{-\frac{3}{2} \epsilon + \frac{3}{2}} \right),
\]
\[
\frac{1}{T^4} \sum_{i \neq j \neq l \neq s}^{n} \left| \tilde{V}_i' \tilde{V}_j W_j' \tilde{V}_l W_i' W_s W_l' \tilde{V}_s \tilde{V}_s' \right| = O_p \left( n^{-\frac{3}{2} \epsilon + \frac{3}{2}} \right),
\]
\[
\frac{1}{T^4} \sum_{i \neq j \neq l \neq s}^{n} \left| \tilde{V}_i' \tilde{V}_j W_j' \tilde{V}_l W_i' W_s W_l' \tilde{V}_s \tilde{V}_s' \right| = O_p \left( n^{-\frac{3}{2} \epsilon - 1} \right),
\]
\[
\frac{1}{T^4} \sum_{i \neq j \neq l \neq s}^{n} \left| \tilde{V}_i' \tilde{V}_j W_j' \tilde{V}_l W_i' W_s W_l' \tilde{V}_s \tilde{V}_s' \right| = O_p \left( n^{-\frac{3}{2} \epsilon - 1} \right),
\]
\[
\frac{1}{T^4} \sum_{i \neq j \neq l \neq s}^{n} \left| \tilde{V}_i' \tilde{V}_j W_j' \tilde{V}_l W_i' W_s W_l' \tilde{V}_s \tilde{V}_s' \right| = O_p \left( n^{-\frac{3}{2} \epsilon - 1} \right),
\]
\[
\frac{1}{T^4} \sum_{i \neq j \neq l \neq s}^{n} \left| \tilde{V}_i' \tilde{V}_j W_j' \tilde{V}_l W_i' W_s W_l' \tilde{V}_s \tilde{V}_s' \right| = O_p \left( n^{-\frac{3}{2} \epsilon - 1} \right).
\]

We need to show that
\[
O_n := \frac{1}{T^4} \sum_{i \neq j \neq l \neq s}^{n} \tilde{V}_i' \tilde{V}_j W_j' \tilde{V}_l \tilde{V}_l' W_i' \tilde{V}_s \tilde{V}_s' W_l = o_p(1),
\]
\[
Q_n := \frac{1}{T^4} \sum_{i \neq j \neq l \neq s}^{n} \tilde{V}_i' \tilde{V}_j W_j' \tilde{V}_l \tilde{V}_l' \tilde{V}_s \tilde{V}_s' W_l = o_p(1).
\]

Consider the $U$-statistic
\[
U_n = \frac{1}{\binom{n}{4}} \sum_{\{i,j,l,s\}} h_T(\tilde{V}_i, \tilde{V}_j, \tilde{V}_l, \tilde{V}_s), \quad h_T(\tilde{V}_i, \tilde{V}_j, \tilde{V}_l, \tilde{V}_s) = \frac{\tilde{V}_i' \tilde{V}_j \tilde{V}_j' \tilde{V}_l' \tilde{V}_l' \tilde{V}_s' \tilde{V}_s' \tilde{X}_s' \tilde{X}_s}{T^2},
\]
so we have
\[
O_n = \frac{24}{T^2} \binom{n}{4} U_n \cdot (\hat{\beta} - \beta)^2.
\]
One can show that
\[ E \left( \tilde{V}_i' \tilde{V}_i X_j' \tilde{V}_i \tilde{V}_i' X_i' \right)^2 \]
\[ = \sum_{t=1}^{T} x_{jt}^2 a_t \cdot \left( E^2(\tilde{\nu}_{11}^2) E^2(\tilde{\nu}_{11}^4) + (T-1) E^2(\tilde{\nu}_{11}^2) E^2(\tilde{\nu}_{11}^4) + 2(T-1) E^4(\tilde{\nu}_{11}^2) E(\tilde{\nu}_{11}^4) + 2(T-1)(T-2) E^4(\tilde{\nu}_{11}^2) E(\tilde{\nu}_{11}^4) + (T-1)(T-2)(T-3) E^6(\tilde{\nu}_{11}^2) \right). \]
Therefore there exist positive constants \( a_2 > a_1 > 0 \), such that
\[ a_1 T^4 \leq E \left( \tilde{V}_i' \tilde{V}_i X_j' \tilde{V}_i \tilde{V}_i' X_i' \right)^2 \leq a_2 T^4. \]
It follows that exist positive constants \( b_2 > b_2 > 0 \), such that
\[ b_1 \leq E\{h_T(\tilde{V}_i, \tilde{V}_j, \tilde{V}_l, \tilde{V}_s)\} \leq b_2. \]

Similarly, one can show that
\[ E\{\sqrt{T} h_T(\tilde{V}_i, \tilde{V}_j, \tilde{V}_l, \tilde{V}_s)\} = o(1). \]

By CLT for U-statistic, we find that \( \sqrt{n}U_n \) is asymptotic Gaussian. Finally, we have
\[ O_n = \frac{24}{T^2} \left( \frac{n}{4} \right) U_n \cdot (\beta - \beta)^2 = \frac{24}{T^2} \left( \frac{n}{4} \right) \cdot \frac{1}{\sqrt{n}} O_p(1) \cdot O_p \left( \frac{1}{n^2} \right) = O_p \left( \frac{1}{\sqrt{n}} \right). \]

Similarly, one can prove that \( Q_n = O_p \left( \frac{1}{\sqrt{n}} \right) \). The proof of (c) is completed.

**Lemma C.5.** Suppose Assumptions (B1)-(B2) hold. We have

(a) \( \sum_{i=1}^{n} \left\{ \tilde{S}_{T,i,i}^2 - S_{T,i,i}^2 \right\} \to_p 0. \)

(b) \( \sum_{i=1}^{n} \left\{ \tilde{S}_{T,i,i}^4 - S_{T,i,i}^4 \right\} \to_p 0. \)

**Proof.** (a) Indeed, we have
\[
\left| \sum_{i=1}^{n} \left\{ \tilde{S}_{T,i,i}^2 - S_{T,i,i}^2 \right\} \right| \leq \sum_{i=1}^{n} \left| \tilde{S}_{T,i,i} + S_{T,i,i} \right| \left| \tilde{S}_{T,i,i} - S_{T,i,i} \right|
\]
\[ \leq \left( \sum_{i=1}^{n} \left| \tilde{S}_{T,i,i} + S_{T,i,i} \right|^2 \right)^{1/2} \left( \sum_{i=1}^{n} \left| \tilde{S}_{T,i,i} - S_{T,i,i} \right|^2 \right)^{1/2}
\]
\[ = \left\{ O_p(n) \right\}^{1/2} \cdot \left\{ O_p \left( \frac{1}{T^2} n^{1+\epsilon-1} \right) \right\}^{1/2} = O_p \left( n^{1/2} \right). \]
where the last equality comes from Lemma C.3. The proof is completed.

(b) We have
\[
\left| \sum_{i=1}^{n} \left\{ \hat{S}_{T,i,i}^4 - S_{T,i,i}^4 \right\} \right| \leq \sum_{i=1}^{n} \left| \hat{S}_{T,i,i}^2 + S_{T,i,i}^2 \right| \left| \hat{S}_{T,i,i}^2 - S_{T,i,i}^2 \right|
\leq \left( \sum_{i=1}^{n} \left| \hat{S}_{T,i,i}^2 + S_{T,i,i}^2 \right|^2 \right)^{1/2} \left( \sum_{i=1}^{n} \left| \hat{S}_{T,i,i}^2 - S_{T,i,i}^2 \right|^2 \right)^{1/2}
\leq O_p \left( n^{1/2} + \epsilon^{-1/2} \right),
\]
where the last equality comes from Lemma C.3. The proof is completed.

\[\square\]

**Corollary 1.** Suppose Assumption 1-2 hold. We have

(a) \[ \sum_{i,j} \left\{ \hat{S}_{T,i,j}^2 - S_{T,i,j}^2 \right\} \xrightarrow{p} 0. \]

(b) \[ \sum_{i,j,l} \left\{ \hat{S}_{T,i,j}^2 \hat{S}_{T,j,l}^2 - S_{T,i,j}^2 S_{T,j,l}^2 \right\} \xrightarrow{p} 0. \]

(c) \[ \sum_{i,j,l,s} \left\{ \hat{S}_{T,i,j} \hat{S}_{T,j,l} \hat{S}_{T,i,s} \hat{S}_{T,l,s} - S_{T,i,j} S_{T,j,l} S_{T,i,s} S_{T,l,s} \right\} \xrightarrow{p} 0. \]

This corollary immediately holds with Lemma C.4 and Lemma C.5.

**C.1. Main calculations**

**Proposition C.1.** Under Assumption (B1)-(B2), we have
\[
\text{tr} \left( \hat{R}_{T}^2 - R_{T}^2 \right) = o_p(1).
\]

**Proof.** It is easy to verify that
\[
\text{tr} \left( \hat{R}_{T}^2 - R_{T}^2 \right) = \sum_{i \neq j}^{n} (A_{ij} + B_{ij}), \quad \text{(C.4)}
\]
where
\[
A_{ij} = \frac{\hat{S}_{T,i,j}^2 - \hat{S}_{T,i,j}^2}{\hat{S}_{T,i,j} \hat{S}_{T,j,i}}, \quad B_{ij} = \frac{\hat{S}_{T,i,j}^2 - S_{T,i,j}^2}{S_{T,i,j} S_{T,j,i}}.
\]
We calculate \[ \sum_{i \neq j}^{n} A_{ij} \] first. Note that
\[
A_{ij} = \hat{S}_{T,i,j}^2 \left( \frac{S_{T,j,i} - \hat{S}_{T,j,i}}{S_{T,j,i} \hat{S}_{T,i,j}} + \frac{S_{T,i,j} - \hat{S}_{T,i,j}}{\hat{S}_{T,i,j} \hat{S}_{T,j,i}} \right).
\]
For the first term, using Lemma C.3 we have

$$\left| \sum_{i \neq j}^n \hat{S}_{T,j,j}^2 \frac{S_{T,j,j} - \hat{S}_{T,j,j}}{S_{T,j,j} S_{T,i,i} S_{T,j,j}} \right| = \left| \sum_{i \neq j}^n \frac{\hat{S}_{T,i,i}^2}{S_{T,j,j} S_{T,i,i} S_{T,j,j}} \left( \frac{2}{T} \hat{V}_i'W_i - \frac{1}{T} W_i'W_i \right) \right| \leq \sum_{i \neq j}^n \frac{\hat{S}_{T,i,i}^2}{S_{T,j,j} S_{T,i,i} S_{T,j,j}} \left( \frac{2}{T} \hat{V}_i'W_i - \frac{1}{T} W_i'W_i \right) = O_p(n^{\frac{1}{N} + \epsilon - 1/2}) \cdot \frac{1}{T} \sum_{i \neq j}^n \hat{S}_{T,i,i}^2.$$  

By Lemma C.4, $\frac{1}{T} \sum_{i \neq j}^n \hat{S}_{T,i,i}^2 = O_p(1)$. Therefore, the first term has the order of $O_p(n^{\frac{1}{N} + \epsilon - 1/2})$.

The second term is estimated similarly with a same order, so that we conclude that $\sum_{i \neq j}^n A_{ij} = O_p(n^{\frac{1}{N} + \epsilon - 1/2}) \xrightarrow{p} 0$.

Next, $\sum_{i \neq j}^n B_{ij} = o_p(1)$ holds with Lemma C.4.

\[\square\]

**Proposition C.2.** Under Assumption (B1)-(B2), we have

$$tr \left( \hat{R}_T^4 - R_T^4 \right) = o_p(1).$$

**Proof.** It is easy to verify that

$$tr \left( \hat{R}_T^4 - R_T^4 \right) = \sum_{i,j,l}^n \left( \hat{S}_{T,i,i}^2 \hat{S}_{T,j,j}^2 - \hat{S}_{T,i,i} \hat{S}_{T,j,j} \right) \left( \frac{2}{T} \hat{V}_i'W_i - \frac{1}{T} W_i'W_i \right) + 2 \sum_{ijl}^n \sum_{s \geq l}^n \left( \hat{S}_{T,i,i} \hat{S}_{T,j,j} \hat{S}_{T,s,s} \hat{S}_{T,l,l} - \hat{S}_{T,i,i} \hat{S}_{T,j,j} \hat{S}_{T,s,s} \hat{S}_{T,l,l} \right)$$

$$= \sum_{ijl}^n (A_{ijl} + A_{ijl}^*) + 2 \sum_{ijl}^n \sum_{s \geq l}^n (B_{ijls} + B_{ijls}^*)$$

where

$$A_{ijl} = \frac{\hat{S}_{T,i,i} \hat{S}_{T,j,j} \hat{S}_{T,l,l} - \hat{S}_{T,i,i} \hat{S}_{T,j,j} \hat{S}_{T,l,l}}{\hat{S}_{T,j,j} \hat{S}_{T,i,i} \hat{S}_{T,l,l}} , \quad A_{ijl}^* = \frac{\hat{S}_{T,i,i} \hat{S}_{T,j,j} \hat{S}_{T,l,l} - \hat{S}_{T,i,i} \hat{S}_{T,j,j} \hat{S}_{T,l,l}}{\hat{S}_{T,j,j} \hat{S}_{T,i,i} \hat{S}_{T,l,l}} ,$$

$$B_{ijls} = \frac{\hat{S}_{T,i,i} \hat{S}_{T,j,j} \hat{S}_{T,s,s} \hat{S}_{T,l,l} - \hat{S}_{T,i,i} \hat{S}_{T,j,j} \hat{S}_{T,s,s} \hat{S}_{T,l,l}}{\hat{S}_{T,i,i} \hat{S}_{T,j,j} \hat{S}_{T,l,l} \hat{S}_{T,s,s}} , \quad B_{ijls}^* = \frac{\hat{S}_{T,i,i} \hat{S}_{T,j,j} \hat{S}_{T,s,s} \hat{S}_{T,l,l} - \hat{S}_{T,i,i} \hat{S}_{T,j,j} \hat{S}_{T,s,s} \hat{S}_{T,l,l}}{\hat{S}_{T,i,i} \hat{S}_{T,j,j} \hat{S}_{T,l,l} \hat{S}_{T,s,s}}.$$
We calculate \( \sum_{i,j,l} A_{ijl} \) first. Note that

\[
A_{ijl} = \hat{S}_{T,i,j} \hat{S}_{T,j,l} \left( \frac{S_{T,i,i} - \hat{S}_{T,i,i}}{\hat{S}_{T,i,i} \hat{S}_{T,j,j} \hat{S}_{T,l,l}} + \frac{S_{T,j,j} - \hat{S}_{T,j,j}}{\hat{S}_{T,i,i} \hat{S}_{T,j,j} \hat{S}_{T,l,l}} + \frac{S_{T,l,l} - \hat{S}_{T,l,l}}{\hat{S}_{T,i,i} \hat{S}_{T,j,j} \hat{S}_{T,l,l}} \right).
\]

For the first term, using Lemma C.3

\[
\left| \sum_{i,j,l} S_{T,i,i}^2 \frac{S_{T,i,i} - \hat{S}_{T,i,i}}{\hat{S}_{T,i,i} \hat{S}_{T,j,j} \hat{S}_{T,l,l} \hat{S}_{T,i,i}} \right| = \left| \sum_{i,j,l} S_{T,i,i}^2 \frac{S_{T,j,j} - \hat{S}_{T,j,j}}{\hat{S}_{T,i,i} \hat{S}_{T,j,j} \hat{S}_{T,l,l} \hat{T,i,i}} \right| \left( \frac{2}{T} \hat{V}_i W_i - \frac{1}{T} W_i W_i \right) 
\leq \sum_{i,j,l} S_{T,i,i}^2 \frac{S_{T,j,j} - \hat{S}_{T,j,j}}{\hat{S}_{T,i,i} \hat{S}_{T,j,j} \hat{S}_{T,l,l} \hat{T,i,i}} \left| \frac{2}{T} \hat{V}_i W_i - \frac{1}{T} W_i W_i \right| 
= O_p(n^{\frac{1}{2\alpha} + \epsilon - \frac{1}{2}}) \cdot \frac{1}{T} \sum_{i,j,l} S_{T,i,i}^2 \frac{S_{T,j,j} - \hat{S}_{T,j,j}}{\hat{S}_{T,i,i} \hat{S}_{T,j,j} \hat{S}_{T,l,l} \hat{T,i,i}}.
\]

By Corollary 1, \( \frac{1}{T} \sum_{i,j,l} S_{T,i,i}^2 \frac{S_{T,j,j} - \hat{S}_{T,j,j}}{\hat{S}_{T,i,i} \hat{S}_{T,j,j} \hat{S}_{T,l,l} \hat{T,i,i}} = O_p(1) \). Therefore, the first term has the order of \( O_p \left( n^{\frac{1}{2\alpha} + \epsilon - \frac{1}{2}} \right) \).

For the second term,

\[
\left| \sum_{i,j,l} S_{T,i,i}^2 \frac{S_{T,j,j} - \hat{S}_{T,j,j}}{\hat{S}_{T,i,i} \hat{S}_{T,j,j} \hat{S}_{T,l,l} \hat{T,i,i}} \right| \leq \sum_{i,j,l} S_{T,i,i}^2 \frac{S_{T,j,j} - \hat{S}_{T,j,j}}{\hat{S}_{T,i,i} \hat{S}_{T,j,j} \hat{S}_{T,l,l} \hat{T,i,i}} \left| \frac{2}{T} \hat{V}_i W_i - \frac{1}{T} W_i W_i \right| 
= O_p \left( n^{\frac{1}{2\alpha} + \epsilon - \frac{3}{2}} \right) \cdot \frac{1}{T} \sum_{i,j,l} S_{T,i,i}^2 \frac{S_{T,j,j} - \hat{S}_{T,j,j}}{\hat{S}_{T,i,i} \hat{S}_{T,j,j} \hat{S}_{T,l,l} \hat{T,i,i}}.
\]

The third term is estimated similarly with the first term with a same order, so that we conclude that \( \sum_{i,j,l} A_{ijl} = O_p \left( n^{\frac{1}{2\alpha} + \epsilon - \frac{1}{2}} \right) + O_p \left( n^{\frac{1}{2\alpha} + \epsilon - \frac{3}{2}} \right) \to 0 \).

Next, \( \sum_{i,j,l} A_{ijl} = o_p(1) \) holds with Corollary 1.

Then we calculate \( \sum_{i,j,l} B_{ijl} \). Note that

\[
B_{ijl} = S_{T,i,j} S_{T,j,l} S_{T,i,i} S_{\hat{T},i,i} \left( \frac{S_{T,i,i} - \hat{S}_{T,i,i}}{S_{T,i,i} S_{T,j,j} S_{T,l,l} S_{T,i,i}} + \frac{S_{T,j,j} - \hat{S}_{T,j,j}}{S_{T,i,i} S_{T,j,j} S_{T,l,l} S_{T,i,i}} + \frac{S_{T,l,l} - \hat{S}_{T,l,l}}{S_{T,i,i} S_{T,j,j} S_{T,l,l} S_{T,i,i}} \right).
\]
For the first term, we have
\[
\sum_{i,j,l} \sum_{s > l} \left| \frac{S_{T,i,i} - \hat{S}_{T,i,i}}{S_{T,i,i} \hat{S}_{T,j,j} \hat{S}_{T,l,l} \hat{S}_{T,s,s} S_{T,i,i}} \right|
\leq \sum_{i,j,l} \sum_{s > l} \left| \frac{\hat{S}_{T,i,j} \hat{S}_{T,j,l} \hat{S}_{T,i,s} \hat{S}_{T,l,s}}{\hat{S}_{T,i,i} \hat{S}_{T,j,j} \hat{S}_{T,l,l} \hat{S}_{T,s,s} S_{T,i,i}} \right| \cdot \left| \frac{2}{T} \hat{V} W_i - \frac{1}{T} W_i W_i \right|
= O_p \left( n^{\frac{1}{2} + \epsilon - \frac{1}{2}} \right) \cdot \frac{1}{T} \sum_{i,j,l} \sum_{s > l} |\hat{S}_{T,i,j} \hat{S}_{T,j,l} \hat{S}_{T,i,s} \hat{S}_{T,l,s}|
\]

By Corollary 1, \( \frac{1}{T} \sum_{i,j,l} \sum_{s > l} |\hat{S}_{T,i,j} \hat{S}_{T,j,l} \hat{S}_{T,i,s} \hat{S}_{T,l,s}| = O_p(1) \), so the first term has the order of \( O_p \left( n^{\frac{1}{2} + \epsilon - \frac{1}{2}} \right) \). The remaining terms are estimated similarly with same order, so that we conclude that \( \sum_{i,j,l} \sum_{s > l} B_{ijls} \xrightarrow{p} 0 \).

Finally, \( \sum_{i,j,l} \sum_{s > l} B_{ijls}^* = o_p(1) \) holds with Corollary 1. The proof is complete. \( \square \)
Table 5.2
Empirical powers of tests under dense case with $n = 50, 100$, $T = 100$, $k = 2$ and varying $h$.

| n   | $h$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 50  |     |     |     |     |     |     |     |     |
|     | $L_{Me}$ | 0.3610 | 0.8445 | 0.9930 | 1 | 1 | 1 | 1 |
|     | PET | 0.4800 | 0.9375 | 0.9990 | 1 | 1 | 1 | 1 |
|     | $L_{Madj}$ | 0.3685 | 0.8475 | 0.9935 | 1 | 1 | 1 | 1 |
|     | CD | 0.0580 | 0.0505 | 0.0480 | 0.0485 | 0.0465 | 0.0450 | 0.0455 |
|     |     |     |     |     |     |     |     |     |
| 100 |     |     |     |     |     |     |     |     |
|     | $L_{Me}$ | 0.3060 | 0.8245 | 0.9910 | 1 | 1 | 1 | 1 |
|     | PET | 0.4095 | 0.9195 | 0.9990 | 1 | 1 | 1 | 1 |
|     | $L_{Madj}$ | 0.3155 | 0.8315 | 0.9915 | 1 | 1 | 1 | 1 |
|     | CD | 0.0490 | 0.0475 | 0.0500 | 0.0595 | 0.0725 | 0.0850 | 0.0960 |
|     |     |     |     |     |     |     |     |     |

Plots for the Chi-square cases
Table 5.3

Empirical powers of tests under sparse case.

| T | n    | Normal | Student-t | Chi-square |
|---|------|--------|-----------|------------|
|   | 50   | 100    | 200       | 50         | 100        | 200       |
|   | 100  | 50     | 100       | 200        | 50          | 100        | 200       |
| 50 | $LM_e$ | 0.1990 | 0.1195    | 0.1005     | 0.2010      | 0.1005     | 0.0955     | 0.1415     | 0.1315     | 0.1125     |
|    | PET  | 0.2195 | 0.1240    | 0.1035     | 0.2175      | 0.1065     | 0.0920     | 0.1370     | 0.1355     | 0.1160     |
|    | $LM_{adj}$ | 0.2085 | 0.1230    | 0.0955     | 0.2130      | 0.1010     | 0.0915     | 0.1480     | 0.1345     | 0.1075     |
|    | CD   | 0.0830 | 0.0620    | 0.0660     | 0.0790      | 0.0605     | 0.0640     | 0.0690     | 0.0620     | 0.0580     |
| 100| $LM_e$ | 0.5735 | 0.2045    | 0.1805     | 0.4320      | 0.1970     | 0.2095     | 0.1695     | 0.1205     | 0.1165     |
|    | PET  | 0.6625 | 0.2295    | 0.1960     | 0.5050      | 0.2225     | 0.2360     | 0.1810     | 0.1315     | 0.1255     |
|    | $LM_{adj}$ | 0.5815 | 0.2095    | 0.1815     | 0.4455      | 0.2025     | 0.2110     | 0.1790     | 0.1255     | 0.1175     |
|    | CD   | 0.0865 | 0.0650    | 0.0660     | 0.0785      | 0.0620     | 0.0660     | 0.0655     | 0.0525     | 0.0510     |
| 50 | $LM_e$ | 0.1405 | 0.0755    | 0.0840     | 0.1345      | 0.1370     | 0.1095     | 0.2300     | 0.1160     | 0.1095     |
|    | PET  | 0.1475 | 0.0835    | 0.0970     | 0.1445      | 0.1475     | 0.1125     | 0.2425     | 0.1145     | 0.1125     |
|    | $LM_{adj}$ | 0.1320 | 0.0570    | 0.0495     | 0.1270      | 0.1040     | 0.0570     | 0.2240     | 0.0915     | 0.0580     |
|    | CD   | 0.0660 | 0.0660    | 0.0580     | 0.0690      | 0.0685     | 0.0580     | 0.0755     | 0.0580     | 0.0580     |
| 100| $LM_e$ | 0.2890 | 0.1210    | 0.1345     | 0.4655      | 0.1460     | 0.1030     | 0.1785     | 0.2955     | 0.1545     |
|    | PET  | 0.3350 | 0.1225    | 0.1395     | 0.5410      | 0.1640     | 0.1070     | 0.1920     | 0.3540     | 0.1835     |
|    | $LM_{adj}$ | 0.2910 | 0.1175    | 0.1155     | 0.4705      | 0.1400     | 0.0865     | 0.1805     | 0.2870     | 0.1395     |
|    | CD   | 0.0830 | 0.0705    | 0.0590     | 0.0915      | 0.0740     | 0.0560     | 0.0760     | 0.0715     | 0.0695     |

Plots for the Chi-square cases
Table 5.4
Empirical powers of tests under less-sparse case.

|       | Normal          | Student-t        | Chi-square       |
|-------|-----------------|------------------|------------------|
|       | n 50 100 200    | n 50 100 200     | n 50 100 200     |
| T     |                 |                  |                  |
| k=2   |                 |                  |                  |
| 50 LMe | 0.9780 0.9870 0.9910 | 0.9805 0.9610 0.9750 | 0.9850 0.9985 0.9135 |
| PET   | 0.9965 0.9980 0.9990 | 0.9960 0.9920 0.9935 | 0.9965 1.0000 0.9745 |
| LMadj | 0.9800 0.9870 0.9910 | 0.9820 0.9610 0.9735 | 0.9865 0.9985 0.9100 |
| CD    | 0.5135 0.5330 0.5535 | 0.5110 0.4815 0.4990 | 0.5190 0.6380 0.4120 |
| 100 LMe | 1.0000 1.0000 1.0000 | 1.0000 1.0000 1.0000 | 1.0000 1.0000 1.0000 |
| PET   | 1.0000 1.0000 1.0000 | 1.0000 1.0000 1.0000 | 1.0000 1.0000 1.0000 |
| LMadj | 1.0000 0.9995 0.9995 | 1.0000 1.0000 1.0000 | 1.0000 1.0000 1.0000 |
| CD    | 0.7750 0.7755 0.8240 | 0.8630 0.8665 0.7355 | 0.5685 0.7940 0.8120 |

Plots for the Chi-square cases

Table 5.5
Empirical sizes of tests (in %) under heterogeneous panel data model (1.2) with k = 2.

|       | Normal          | Student-t        | Chi-square       |
|-------|-----------------|------------------|------------------|
|       | n 50 100 200    | n 50 100 200     | n 50 100 200     |
| T     |                 |                  |                  |
| 50 LMe | 5.05 5.10 5.40  | 4.85 5.40 5.20  | 5.50 5.80 6.35   |
| PET   | 4.45 4.95 5.35  | 5.30 5.40 5.35  | 5.50 5.50 5.50   |
| LMadj | 5.45 5.25 5.05  | 5.35 5.50 5.10  | 5.80 6.05 5.90   |
| CD    | 5.25 5.00 4.30  | 5.10 4.85 4.55  | 5.40 4.05 5.25   |
| 100 LMe | 4.65 4.50 5.45  | 3.30 4.20 5.95  | 5.15 5.15 5.30   |
| PET   | 4.45 4.85 5.55  | 5.45 4.90 6.25  | 5.70 5.90 5.00   |
| LMadj | 4.80 4.75 5.55  | 5.60 4.40 5.95  | 5.40 5.25 5.35   |
| CD    | 4.25 5.20 4.85  | 4.25 5.10 4.60  | 4.45 4.85 4.75   |
Table 5.6
Empirical powers of tests under dense case with $n = 50, 100, T = 100, k = 2$ and varying $h$ for heterogeneous panel data model (1.2).

| $n$ | $h$ | 1     | 2     | 3     | 4     | 5     | 6     | 7     |
|-----|-----|-------|-------|-------|-------|-------|-------|-------|
| 50  |     |       |       |       |       |       |       |       |
|     | $LM_e$ | 0.2075 | 0.8490 | 0.9910 | 1     | 1     | 1     | 1     |
|     | PET    | 0.2615 | 0.9345 | 0.9995 | 1     | 1     | 1     | 1     |
|     | $LM_{adj}$ | 0.2160 | 0.8535 | 0.9910 | 1     | 1     | 1     | 1     |
|     | CD     | 0.0565 | 0.0505 | 0.0505 | 0.0490 | 0.0485 | 0.0505 | 0.0490 |
|     |        |       |       |       |       |       |       |       |
|     | $LM_e$ | 0.3725 | 0.8530 | 0.9915 | 1     | 1     | 1     | 1     |
|     | PET    | 0.4900 | 0.9385 | 0.9975 | 1     | 1     | 1     | 1     |
|     | $LM_{adj}$ | 0.3840 | 0.8575 | 0.9915 | 1     | 1     | 1     | 1     |
|     | CD     | 0.0645 | 0.0505 | 0.0495 | 0.0525 | 0.0530 | 0.0560 | 0.0650 |
|     |        |       |       |       |       |       |       |       |
|     | $LM_e$ | 0.2140 | 0.8590 | 0.9945 | 0.9990 | 1     | 1     | 1     |
|     | PET    | 0.2650 | 0.9470 | 0.9985 | 1     | 1     | 1     | 1     |
|     | $LM_{adj}$ | 0.2190 | 0.8575 | 0.9945 | 0.9990 | 1     | 1     | 1     |
|     | CD     | 0.0460 | 0.0560 | 0.0460 | 0.0460 | 0.0470 | 0.0475 | 0.0515 |

Table 5.7
Empirical powers of tests under sparse case for heterogeneous panel data model (1.2) with $k = 2$.

| $T$ | $n$ | 50  |   | 100 |   | 200 |   |     | 50  |   | 100 |   | 200 |   |     | 50  |   | 100 |   | 200 |
|-----|-----|-----|---|-----|---|-----|---|-----|-----|---|-----|---|-----|---|-----|-----|---|-----|---|-----|
|     | $LM_e$ | 0.1405 | 0.1045 | 0.1020 | 0.1240 | 0.0715 | 0.1040 | 0.1870 | 0.1220 | 0.1290 |
|     | PET    | 0.1445 | 0.1080 | 0.1035 | 0.1240 | 0.0700 | 0.1045 | 0.1925 | 0.1340 | 0.1320 |
|     | $LM_{adj}$ | 0.1480 | 0.1055 | 0.0975 | 0.1320 | 0.0720 | 0.0990 | 0.1945 | 0.1225 | 0.1200 |
|     | CD     | 0.0720 | 0.0665 | 0.0645 | 0.0780 | 0.0660 | 0.0605 | 0.0735 | 0.0655 | 0.0570 |
|     |        |       |       |       |       |       |       |       |       |       |
|     | $LM_e$ | 0.2985 | 0.1895 | 0.1840 | 0.1805 | 0.1675 | 0.1695 | 0.3555 | 0.1360 | 0.1185 |
|     | PET    | 0.3205 | 0.2050 | 0.2035 | 0.1890 | 0.1855 | 0.1885 | 0.3875 | 0.1380 | 0.1250 |
|     | $LM_{adj}$ | 0.3080 | 0.1945 | 0.1850 | 0.1885 | 0.1700 | 0.1720 | 0.3625 | 0.1375 | 0.1190 |
|     | CD     | 0.0800 | 0.0600 | 0.0590 | 0.0755 | 0.0545 | 0.0690 | 0.0730 | 0.0600 | 0.0545 |

Table 5.8
Empirical powers of tests under less-sparse case for heterogeneous panel data model (1.2) with $k = 2$.

| $l$ | $n$ | 50  |   | 100 |   | 200 |   |     | 50  |   | 100 |   | 200 |   |     | 50  |   | 100 |   | 200 |
|-----|-----|-----|---|-----|---|-----|---|-----|-----|---|-----|---|-----|---|-----|-----|---|-----|---|-----|
|     | $LM_e$ | 0.9030 | 0.9795 | 0.9925 | 0.9980 | 0.9620 | 0.9380 | 0.9995 | 0.9980 | 0.9820 |
|     | PET    | 0.9630 | 0.9965 | 0.9985 | 0.9960 | 0.9930 | 0.9850 | 0.9995 | 0.9980 | 0.9960 |
|     | $LM_{adj}$ | 0.9085 | 0.9800 | 0.9915 | 0.9980 | 0.9620 | 0.9345 | 0.9995 | 0.9980 | 0.9820 |
|     | CD     | 0.4085 | 0.5100 | 0.5665 | 0.5940 | 0.4695 | 0.4475 | 0.5865 | 0.6085 | 0.5355 |
|     |        |       |       |       |       |       |       |       |       |       |
|     | $LM_e$ | 0.9990 | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     |
|     | PET    | 0.9990 | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     |
|     | $LM_{adj}$ | 0.6405 | 0.7685 | 0.8020 | 0.6955 | 0.8475 | 0.7805 | 0.6490 | 0.6830 | 0.7915 |
Table 6.1
Values of test statistics.

| Tests | Model 1 | Model 2 | Model 3 |
|-------|---------|---------|---------|
| PET   | -0.2035 | 31.56   | 158.3   |
| $LM_e$| 0.0694  | 12.75   | 34.65   |
| $LM_{adj}$ | 0.9143 | 15.66   | 40.94   |
| CD    | 0.2049  | 16.71   | 28.35   |