The large-N phase transition of lattice SU($N$) gauge theories

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We investigate the large-$N$ phase transition of lattice SU($N$) gauge theories in the Wilson formulation, by performing a Monte Carlo simulation of the twisted Eguchi-Kawai model. A variant of the multicanonical algorithm allows a detailed exploration of the phase transition and a precise determination of the transition temperature.

1. Introduction

The large-$N$ limit of SU($N$) gauge theories is of considerable interest from the phenomenological point of view, and is one of our sources of understanding of non-perturbative QCD. The theoretical side is no less interesting, allowing e.g. to establish precise relationships between Yang-Mills theories and string theories.

A crucial property of large-$N$ field theories is factorization: connected Green’s functions of invariant quantities are suppressed with respect to the corresponding disconnected parts by powers of $1/N$. One of the most notable consequences of factorization is the possibility of constructing reduced models, i.e. single-site models which reproduce a lattice gauge theory in the $N \to \infty$ limit.

Despite the considerable simplifications occurring for large $N$, many interesting models have not been solved analytically; therefore we must resort to approximate techniques, such as numerical simulations.

2. The TEK model

The most promising reduced version of the SU($N$) lattice gauge theory (in the Wilson formulation) is the twisted Eguchi-Kawai model (TEK). Let us define a set of 4 traceless SU($N$) matrices $\Gamma_\mu$ obeying ‘t Hooft algebra

$$\Gamma_\nu \Gamma_\mu = Z_{\mu\nu} \Gamma_\mu \Gamma_\nu, \quad Z_{\mu\nu} = \exp\left(\frac{2i\pi}{N} n_{\mu\nu}\right),$$

where $n_{\mu\nu}$ is an antisymmetric integer-valued tensor. $\Gamma_\mu$ are the matrices implementing the translations by one lattice spacing in the $\mu$ direction.

The observables of the reduced model are obtained by applying the reduction prescription

$$U_\mu(x) \to T(x) U_\mu T(x)^\dagger, \quad T(x) = \prod_\mu (\Gamma_\mu)^{x_\mu}$$

to the corresponding quantity of the full model. The action of the TEK model is obtained by reduction of the Wilson action:

$$\beta N^2 E(U) \equiv S_{\text{TEK}}(U) = N \beta \sum_{\mu > \nu} \text{Tr} \left[ Z_{\mu\nu} U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger + \text{h.c.} \right].$$

For a proper choice of $\Gamma_\mu$, the Schwinger-Dyson equations of the TEK model reproduce in the large-$N$ limit the Schwinger-Dyson equations of the Wilson lattice gauge theory. We adopt the choice of Ref. [4]: $N$ is constrained to be a perfect square, $N = L^2$, and $n_{\nu\mu} = L$ for all $\nu > \mu$. $L$ takes the rôle of the lattice size, and the number of degrees of freedom is proportional to $L^4$.

The TEK model develops a first-order phase transition for $N \to \infty$. It corresponds to the large-$N$ limit of the first-order phase transition of SU($N$) lattice gauge theories. We will study the phase transition of the TEK model by Monte Carlo simulation.

3. The multicanonical algorithm

The choice of updating algorithm is extremely important in the neighborhood of a phase transition. Most numerical work on the TEK model adopted the heat-bath updating algorithm (HB) devised in Ref. [5]. The HB algorithm is generally quite efficient, but near the phase transition it is plagued by the familiar supercritical slowing
down; according to the experience accumulated in the last decade, this problem can be solved by devising a suitable multicanonical algorithm [2].

Since we are studying a temperature-driven phase transition, it is natural to choose a reweighting function depending only on the energy: we generate configurations according to the probability distribution
\[ P(\beta, U) \propto w(E(U)) \exp(-\beta N^2 E(U)), \]
and compute the canonical expectation value of an observable as
\[ \langle \mathcal{O} \rangle = \sum_c w^{-1}(E(c)) \mathcal{O}(c) / \sum_c w^{-1}(E(c)). \]

If we can find a reasonably accurate ansatz \( \bar{\rho} \) to the energy distribution \( \rho(E) \), we can construct the reweighting function
\[
\begin{align*}
w(E) &= 1/\bar{\rho}(E_-), \quad E \leq E_- , \\
w(E) &= 1/\bar{\rho}(E), \quad E_- \leq E \leq E_+, \\
w(E) &= 1/\bar{\rho}(E_+), \quad E_+ \leq E,
\end{align*}
\]
which flattens the probability distribution between the two peaks, and we can obtain a reasonable tunneling rate between the two vacua. The resulting multicanonical distribution can be simulated using an efficient Metropolis procedure.

Our first ansatz was
\[ \rho_{BL}(E) \equiv a_+ g_+(E) + a_- g_-(E) + \gamma \theta(E - E_+) \times \theta(E_+ - E) \left( 1 - g_+(E) \right) \left( 1 - g_-(E) \right), \]
\[ g_\pm(E) = \exp \left[ -\frac{(E - E_\pm)^2}{2\sigma_\pm^2} \right], \]
where all the parameters \( E_\pm, \sigma_\pm, a_\pm, \) and \( \gamma \) are \( N \)-dependent; the factors of \( (1 - g_\pm(E)) \) ensure the smoothness of the distribution for \( E = E_\pm \).

This is essentially a Binder-Landau ansatz [7] corrected for mixed-phase contributions [8]; the resulting algorithm (M1) works quite well up to \( N = 25 \), but it becomes very inefficient as \( N \) increases further.

It turns out that \( \rho \) does not follow the Gaussian behavior of Eq. (2) in the region \( E_- \ll E \ll E_+ \); if we start from one peak and move towards the other, the distribution will eventually follow an exponential law. In order to reproduce this behavior, we introduce two new parameters \( \bar{E}_\pm, \)
\[ E_- \leq \bar{E}_- < \bar{E}_+ < E_+, \]
and replace \( g_\pm \) in Eq. (2) with
\[ g_\pm(E) = \exp \left[ -\frac{(E - \bar{E}_\pm)^2}{2\sigma_\pm^2} \right], \]
where \( r_\pm \) and \( s_\pm \) are determined by the condition that the two branches of \( g_\pm \) join smoothly (up to the first derivative) for \( E = \bar{E}_\pm \). The new algorithm (M2) works remarkably well up to \( N = 64 \), as shown in Fig. [1].

The main drawback of algorithms M1 and M2 is that the multicanonical parameters must be tuned to an ever finer degree with increasing \( N \). Thanks to the absence of tunneling in a canonical simulation, \( E_\pm, \sigma_\pm, \) \( a_\pm, \) and \( \gamma \) can be estimated accurately by performing a canonical simulation starting from a disordered and an ordered configuration respectively, using the HB algorithm. The other parameters can be estimated roughly from the simulations at lower \( N \); this estimate needs to be refined by performing successive multicanonical simulations and looking at the resulting energy distribution. For \( N = 64 \) this required more than 10 simulations at moderate statistics (about 200k sweeps), with a computational effort comparable to the high-statistics simulation with the optimized parameters.
Table 1
Summary of high-statistics simulations.

| $N$ | $\beta$   | alg | stat | $\gamma$              |
|-----|-----------|-----|------|-----------------------|
| 25  | 0.3580    | HB  | 5M   | $5 \times 10^{-3}$    |
| 25  | 0.3574    | M1  | 5M   | $1 \times 10^{-5}$    |
| 36  | 0.3585    | M2  | 5M   | $1 \times 10^{-10}$   |
| 49  | 0.3588    | M2  | 4M   | $1 \times 10^{-10}$   |
| 64  | 0.3595    | M2  | 2M   | $5 \times 10^{-17}$   |

4. Results

A summary of our high-statistics simulations is presented in Table 1.

The quality of our ansatz (3) can be judged from Fig. 2, where the worst case $N = 64$ is presented. The ansatz is not consistent with the data within the statistical errors, but it is more than accurate enough for the purpose of optimizing the multicanonical algorithm.

It is interesting to notice the exponential fall-off of the energy distribution between the two peaks, which for $N = 64$ is followed to great accuracy for several orders of magnitude. On the other side of the two peaks, the energy distribution falls much faster, even faster than the Gaussian behavior assumed in Eqs. (2) and (3).

A single multicanonical simulation can be used to obtain results for a (small) range of $\beta$, using the reweighting technique. Our estimator of the inverse transition temperature $\beta_t$ is the value of $\beta$ which maximizes the specific heat. Fig. 2 shows that $\beta_t$ is in perfect agreement with the linear behavior in $1/N^2$ expected for a first order phase transition. Extrapolating to $N = \infty$ by a linear fit, we obtain

$\beta_t = 0.3596(2)$.

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