Parameter free calculation of hadronic masses from instantons.

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We propose a non-perturbative calculation scheme which is based on the semi-classical approximation of QCD and can be used to evaluate quantities of interest in hadronic physics. As a first application, we evaluate the mass of the pion and of the nucleon. Such masses are related to a particular combination of Green functions which, in some limit, is dominated by the contribution of very small-sized instantons. The size distribution of these pseudo-particles is determined by the 't Hooft tunneling amplitude formula and therefore our calculation is free from any model parameters. We prove that instanton forces generate a light pion and a nucleon with realistic mass ($M_n \sim 970\ MeV$). In connection with sum-rules approaches, we discuss the overlap of instantons with pion and nucleon resonances.

I. INTRODUCTION

After the discovery of the BPST instanton solution in QCD, two important results by 't Hooft originated a large interest in the semi-classical description of the QCD vacuum: the calculation of the tunneling amplitude and the discovery of the existence of quark zero-modes in the field of an instanton (which allowed to solve the U(1) problem). Unfortunately, it was immediately realized that in QCD (unlike in the electroweak case) the tunneling amplitude at one or two-loops accuracy is divergent in the infrared and therefore that instantons “tend to swell”. This fact led some to the conclusion that the QCD vacuum should be populated by large sized, topologically non trivial field configurations, which cannot be described semi-classically.

On the other hand, in Shuryak argued that the divergence of the single instanton density is non-physical and that the size distribution of instantons should eventually start to decay above some average size $\bar{\rho}$, either due to non-perturbative effects (which are not included in the 't Hooft formula) or due to interaction with other instantons. Along this line, he concluded that a semi-classical approach can still justified if a small fraction of the Euclidean space time is occupied by instantons, i.e. if $\rho^1 \bar{n} \ll 1$, where $\bar{n}$ is the average instanton density. From an analysis of the global properties of the vacuum (quark and gluon condensate) he estimated these parameters to be $\bar{n} \sim 1\ fm^{-4}$, $\rho \sim 1/3\ fm$. Based on such arguments, the Instanton Liquid Model (ILM) was developed in the 80’s and 90’s. The model was proven to be phenomenologically successful in describing many important low energy properties of QCD, such as the breaking of chiral symmetry, several hadronic masses and correlators and, recently, electro-magnetic formfactors. Moreover, studies of the classical content of the QCD vacuum, based on lattice simulations via cooling procedure, confirmed the presence of topologically non trivial “bumps” of gauge fields, with typical density and size very close to that assumed by the ILM.

Despite its phenomenological success, the ILM remains a model, because none has been able to derive from QCD why large instantons are not present in the vacuum. However, the fact that there are small-sized instantons in the vacuum, whose distribution is described (at two-loop level) by the $\rho \Lambda_{QCD} \ll 1$ limit of the ‘t Hooft tunneling amplitude, is a result of QCD.

In this work, we try to set up a calculation scheme for hadronic observables which emphasizes the contribution from such small-sized instantons, and we apply it to evaluate the mass of the pion and of the nucleon. In practice, in order to keep our calculation finite at each intermediate stage, we will need to regularize the size distribution by cutting-off large instantons. However we shall show that, in an appropriate limit to be defined below, at the end of the calculation our results will not depend on the particular choice of the cut-off and the only parameter left is $\Lambda_{QCD}$.

This way, we derive the existence of a light ($I = 1, J^P = 0^-$) meson state (pion) and of a ($I = \frac{1}{2}, J^P = \frac{1}{2}^+$) baryon state (nucleon). In the case of the nucleon, we are able to extract its mass and find $(3.9 \pm 0.2)\ \Lambda_{PV}$. However, this framework did not allow us to determine with sufficient accuracy the very small mass of the pion.

The paper is organized as follows. In section II we introduce our particular combination of two-point functions and show how this is related to the masses of the corresponding hadrons. In section III such a combination is evaluated.

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II. DETERMINATION OF THE HADRONIC MASSES FROM POINT-TO-WALL CORRELATORS AT LARGE MOMENTUM

Point-to-wall hadronic correlators are defined as the spatial Fourier transform of the point-to-point Euclidean correlation functions,

\[ G^H_2(t,p) = \int d^3x \, e^{i \cdot P \cdot x} \langle 0 | J_H(t, x) J'_H(0) | 0 \rangle, \]

where \( J_H \) is an interpolating operator, which excites states with the quantum numbers of the hadron \( H \). These correlators are often used to extract hadronic masses from lattice simulations. In fact, by computing their spectral decomposition, one finds that, in the large \( t \) limit, the contribution from all excitations is exponentially suppressed.

The energy of the ground state can then be determined from the logarithmic derivative

\[ -\frac{1}{dt} \log \left[ \frac{G^H_2(t + dt, p)}{G^H_2(t, p)} \right] \bigg|_{t \to \infty} \equiv E(p). \]

By setting \( p = 0 \) in (2), one can directly extract the mass of the lowest lying particle, with the quantum number of \( H \). In lattice simulations, \( p \) is typically chosen to be the smallest possible momentum compatible with the box size and \( dt \) to be the lattice spacing in the time direction.

In this work, however, we will need to consider ratio (2), for \( |p| \gtrsim 1 \, \text{GeV} \). In fact, we shall show that only in such kinematic range the pion and nucleon correlators are dominated by the effects of single small-sized instantons, which are calculable theoretically.

As we increase the momentum, the determination of the masses gets more and more difficult, and eventually becomes impossible if \( |p| >> m_H \). This is due to the fact that, in the ultra-relativistic regime where \( E(|p|) \sim |p| \), all states in the spectral decomposition have the same exponential decay \( \rho \). As a consequence, as we increase the momentum, larger and larger Euclidean time intervals are needed, in order to isolate the ground-state. Therefore, for a given value of the time interval \( t \), the LHS of eq.(2) will be dominated by the ground-state only for a limited range of momenta. Generally speaking, we expect the LHS of eq.(2) to become bigger than the RHS, at some large momenta.

The importance of the contribution form resonances depends on the details of the spectral function. In the case of the pion, we expect that they should become relevant only at very large momenta, because the first resonance (\( \pi(1330) \)) is much heavier than the ground-state. To estimate such contribution, let us consider a very simple two-poles model, in which only \( \pi \) and \( \pi(1330) \) states are included. Than the spectral decomposition of \( G^2_p(t) \) would read:

\[ G^2_p(t) \simeq \frac{a_\pi}{2\sqrt{p^2 + m_\pi^2}} e^{-\sqrt{p^2 + m_\pi^2} t} + \frac{a_{\pi(1330)}}{2\sqrt{p^2 + m_{\pi(1330)}^2}} e^{-\sqrt{p^2 + m_{\pi(1330)}^2} t}, \]

where \( a_\pi \) and \( a_{\pi(1330)} \) are the coupling of the states with the interpolating operator (which we shall assume to be of the same order of magnitude). For \( t = 7 \, \text{GeV}^{-1} \) and \( |p| \simeq 2 \, \text{GeV} \) we have that the contribution of \( \pi(1330) \) is suppressed by one order of magnitude.

The nucleon channel is more problematic because the first resonance is only about 500 MeV heavier than the ground state. In this case, by repeating the above analysis, we find that the desired suppression should be achieved for \( t \sim 8 \, \text{GeV}^{-1} \) and \( |p| \simeq 1.5 \, \text{GeV} \). Above such thresholds, a theoretical analysis of (2) can be used to obtain qualitative information about the strength of the resonances and the continuum.

III. THEORETICAL CALCULATION

In this section, we evaluate the LHS of (2) in a semi-classical approximation. Our starting point is the tunneling amplitude for one instanton, which is derived by integrating over small quantum fluctuations about the instanton solution. At two-loop accuracy it reads:

\[ d_{\text{Hooft}}(\rho) = \text{const} \cdot \rho^{-5} \rho^{N_c} \beta_1(\rho)^2 N_c \exp \left[ -\beta_2(\rho) + \left( 2 N_c - \frac{b'}{2 b} \right) \frac{b'}{2 b} \ln \beta_1(\rho) \right], \]
where $\beta_1(\rho)$ and $\beta_2(\rho)$ are the one and two-loop beta functions:

$$\beta_1(\rho) = -b \ln(\rho \Lambda_{PV}),$$

(5)

$$\beta_2(\rho) = \beta_1(\rho) + \frac{b'}{2b} \ln \left( \frac{2}{b} \beta_1(\rho) \right),$$

(6)

$$b = \frac{11}{3} N_c - \frac{2}{3} N_f, \quad b' = \frac{34}{3} N_c^2 - \frac{13}{3} N_c N_f + \frac{N_f}{N_c}. \quad (7)$$

This result was carried out in the Pauli-Villars regularization scheme and $\Lambda_{PV}$ is the corresponding scale parameter.

This formula is derived perturbatively and therefore is valid only for small instantons, for which $\rho \Lambda_{QCD} << 1$.

Its infrared divergence indicates that a fully perturbative description of a single tunneling amplitude is inconsistent. This is the so called 'infrared catastrophe' of the instanton calculus in QCD.

In the ILM, this problem is overcome by assuming that the vacuum can be described as an ensemble of interacting instantons, and that the single instanton size distribution is cut from above by an average instanton-instanton repulsion. The model depends on one dimensionless parameter, the instanton packing fraction $\bar{\rho}^4 \bar{n}$, which can not be derived and has to be estimated phenomenologically.

Such an approach necessarily introduces some additional approximations. First of all, in the presence of non negligible overlapping, the sum of many instantons is no longer an exact solution of the classical equation of motion. Moreover, the tunneling events can non be treated as completely independent and therefore it is not guaranteed \textit{a priori} that the contributions of such classical configurations is not overwhelmed by quantum fluctuations. However, one should mention that the ILM gives indeed a very good phenomenology, therefore one can argue, \textit{a posteriori}, that the assumptions made are legitimate.

Here, we try to go one step further and set up a calculation scheme which is sensitive only to the small-size tail of the instanton density and contains no phenomenological parameters. We shall only assume that the instanton packing fraction is small enough for the pseudo-particles not to loose completely their individuality. In other words, the QCD vacuum will still be described as some instanton ensemble. However, we will not need to choose a particular value for $\bar{\rho}$ and $\bar{n}$, nor we will have to model the instanton-instanton interaction. As a matter of fact, we shall also not require that the semi-classical approach is justified for \textit{all} pseudo-particle in the ensemble, because our results will be sensitive only to the contribution from the small ones.

The basic simplification introduced by the semi-classical approximation is that the infinite dimensional integration over all gauge configurations is replaced by a finite number of integrals over the collective coordinates of each pseudo-particle.

Let us consider, for sake of definiteness [21], the pseudo-scalar two-point correlation function, interpolated by the current:

$$J_5(x) = \bar{u}(x) \gamma_5 d(x).$$

(8)

In our approach, the two-point correlator is approximated with (for two flavors):

$$G(x,y) := \langle 0 | J_5(x) J_5^\dagger(y) | 0 \rangle \approx \frac{1}{Z} \int d\Omega f(\Omega) \left[ \prod_{f=u,d} \text{det}(D_\Omega + m_f) \right] Tr[\gamma_5 S(x,0;\Omega) \gamma_5 S(0,x;\Omega)],$$

(9)

$$Z = \int d\Omega f(\Omega) \left[ \prod_{f=u,d} \text{det}(D_\Omega + m_f) \right],$$

(10)

where $\Omega = (\Omega_1, ..., \Omega_{N^+_1}, \Omega_{N^+_2}, ..., \Omega_{N^+_1+N^+_2})$ is the set of all collective coordinate identifying the configurations of the instanton ensemble [22], $f(\Omega)$ is the weight associated to a particular configuration $\Omega$ in the pure Yang-Mills theory [23] and $S(x,y;\Omega)$ is the quark propagator in the configuration $\Omega$. The quark propagator can be formally written as:

$$S(x,y;\Omega) = \sum_\lambda \frac{\psi_\Lambda^\dagger(x) \psi_\Lambda^\dagger(y)}{-i \lambda + m},$$

(11)
where $\psi^\Omega_0(x)$ are eigenvalues of the Dirac operator, $D_\Omega \psi^\Omega_0 = \lambda \psi^\Omega_0$. In the field of a single instanton, the propagator is known exactly [4] and consists of a non zero-mode part and a zero-mode part:

$$S(x, y; \Omega) = S_{\text{zm}}(x, y; \Omega) + S_{\text{zm}}(x, y; \Omega),$$

(12)

where $\Omega_i$ denotes the set of collective coordinate relative to the particular instanton $i$. It can be verified that, when correlators receive contribution from more that one zero-mode propagator (like in the case we are considering), the non zero-mode part can be very well approximated with the free propagator:

$$S(x, y; \Omega) = S_0(x, y) + S_{\text{zm}}(x, y; \Omega).$$

(13)

The propagator in the field of many instantons was evaluated in [13] and consists of a non zero-mode part (which can again be approximated with a free propagator) and a quasi zero-mode part, which can be expanded in the functional space of zero-modes of individual instantons [24]:

$$S_{\text{zm}}(x, y; \Omega) = \sum_{I,J} C^\Omega_{I J} \psi^\Omega_0(x) \psi^\Omega_0\dagger(y),$$

(14)

where $\psi^\Omega_0(x)$ are the well known zero-mode wave functions [2, 3].

Inserting (14) in (13) we find:

$$G_2(x, y) - G_2_{\text{free}}(x, y) = \frac{1}{Z} \left( \sum_{I,J=1}^{N^+ + N^-} \sum_{I',J'=1}^{N^+ + N^-} \int d\Omega f(\Omega) \left[ \prod_{f=u,d} \det(D_\Omega + m_f) \right] C^\Omega_{I J} C^\Omega_{I' J'} \cdot \cdot \cdot \right)$$

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and of the anti-instanton is completely trivial. Moreover, total momentum conservation requires that the distribution of such instanton in the vacuum is homogeneous, that is:

\[ \tilde{f}(z; \rho) = \tilde{n} \delta(\rho - \bar{\rho}) \]  

which emphasizes the global properties of the instanton vacuum (i.e. average instanton size and density).

In our calculation, however, we want to focus on the effect of small instantons. At this purpose, we choose a parameterization that, in the limit \( \rho \to 0 \), reproduces the tunneling amplitude (4). For example we can choose the following regularization:

\[ d(\rho) = \frac{1}{1 + \exp((\rho - \bar{\rho})/\sigma)} \]  

We can now rewrite (17) as:

\[ G_{zm}^{2m}(x, y) = G_{2}(x, y) - G_{2}^{free}(x, y) = \tilde{n} \left( \int d^4z \int_0^{\infty} d\rho d(\rho) Tr[\gamma_5 S_{zm}^I(x, 0; z, \rho) \gamma_5 S_{zm}^A(0; x, z, \rho)] + \int d^4z \int_0^{\infty} d\rho d(\rho) Tr[\gamma_5 S_{zm}^A(x, 0; z, \rho) \gamma_5 S_{zm}^A(0; x, z, \rho)] \right) \]  

where we have taken equal density of instantons and anti-instantons, \( n_I = n_A =: \tilde{n} \), and \( S_{zm}^{I(A)}(x, y; z) \) is the zero-mode propagator of a particle of mass \( m^* \) in the field of an instanton (anti-instanton) of size \( \rho \) at \( z \). Hence, we can identify the parameter \( m^* \) as the quark effective mass originally introduced in [5]. In [17] such parameter was estimated from numerical simulations and form phenomenology to be \( m^* \approx 85 \text{ MeV} \).

Summarizing, what we have done so far is to provide a formal derivation of the Single Instanton Approximation (SIA), starting from the partition function (10). The validity of such approximation, in the case of the ILM, was studied in detail in [17]. It that analysis, we considered several point-to-point correlators that receive contribution from two zero-mode propagators and compared SIA calculations (with the ansatz (19)) with numerical simulations in the Random and Interacting Instanton Liquid. We found complete agreement up to distances of the order of \( 6 - 7 \text{ GeV}^{-1} \), depending on the particular correlator. Since this value was obtained in a particular model, it should be considered as a rough estimate. We may argue that, if a particular observable is mainly sensitive to the contribution of small instantons, the SIA should somehow increase its range of validity, because the packing fraction of such pseudo-particles is smaller. Also, we do not expect such SIA calculation for the point-to-wall propagator to be reliable for all values of the momentum \( \mathbf{p} \). In fact, if such momentum is small, the Fourier transform (1) receives non-negligible
contribution from point-to-point correlators which connect the origin with points on the wall that are very far away from the time axis (see fig. 2). From the discussion above it follows that such point-to-point correlators have size larger than the maximal compatible with the SIA. However, for \(|p|\) of the order of a GeV, only points at the distance smaller than roughly one inverse GeV from the time axis will contribute to the Fourier transform, and the SIA is applicable.

Summarizing, the feasibility of our approach relies on the existence of a window in the time and momentum variables, such that the SIA is still holding and the ground-state is not overwhelmed by resonances. For example, for \(t \gtrsim 9 \text{ GeV}^{-1}\) and \(p \lesssim 1 \text{ GeV}\), we do not trust our approximation, while for \(t < 6 \text{ GeV}^{-1}\) and \(p \lesssim 1 \text{ GeV}\), the SIA would be reliable, but the correlation functions would start to become very sensitive to the contribution of the resonances. Based on such arguments, we shall choose \(t\) to be large enough but the correlation functions would start to become very sensitive to the contribution of the resonances. For example, for \(|p| \gtrsim 1 \text{ GeV}\).

Once the general calculation scheme has been established, we can go ahead and evaluate the LHS of (2) for pion and nucleon. We begin with the pion pseudo-scalar point-to-wall correlation function. The leading instanton contribution to \(G_2^{\gamma m}(t,p)\) is given by:

\[
G_2^{\gamma m}(t,p) = \frac{n}{4m^2} \int_0^\infty dp \frac{d}{d|p|} \left| \left[\sqrt{\xi(z_4-t)} + \sqrt{\xi(z_4)}\right] \left(1 + |p| \frac{\xi(z_4-t)}{[\xi(z_4)]^{3/2}}\right) \right| \left(1 + |p| \frac{\xi(z_4)}{[\xi(z_4)]^{3/2}}\right),
\]

where \(\xi(x) = x^2 + \rho^2\).

The stationary phase method can also be applied to determine what instantons contribute the most, as \(|p|\) increases. By expanding the exponent around its minimum \((\rho = 0, z_4 = t/2)\), it is straightforward to verify that, at large momenta, \(G_2(t, |p|)\) will be dominated by small-sized instantons. Hence we expect that, in this regime, the prediction should be fairly insensitive to the detail of the cut-off function in (21) and depend essentially on \(n_{\text{Hooft}}\), only.

The result that a single, small instanton dominates the point-to-wall correlation function at large momentum has an intuitive interpretation [2]. The current at origin emits a quark and an anti-quark with undetermined momenta (see fig. 2). In order for such particles to form a bound-state at some later time \(t\), their momenta has to be focused and made almost collinear. The nearby instanton works as a “lens” and provides the required bending of the “quark beams”. At high momentum, the focusing is optimal if the “curvature” of the instanton lens is small and sits at a distance \(z_4 = 1/2 t\) from the source.

The same calculation can be repeated for the nucleon point-to-wall correlation function. In such case, we have more freedom in the choice of the point-to-point correlator. Previous studies [8] have shown that the single-instanton contribution is largest in the correlator:

\[
G_n^{\gamma m}(x,y) = \langle 0 | Tr [j_{sc}(x) j_{sc}(0)] \gamma_4 | 0 \rangle,
\]

where \(j_{sc}(x) = [u_a(x) (C \gamma_5) d_b(x)] u_c(x) \epsilon_{abc}\) is the so called ‘scalar current’. The leading instanton contribution to the nucleon point-to-wall correlator reads:

\[
G_2^{\gamma n}(t,p) = \frac{192\tilde{n}}{|p| m^2 \pi^2} \int_0^\infty dr \frac{r}{[r^2 + t^2]^{3/2}} \int_0^\infty d\rho d(p) \frac{\rho^4}{3} \int_{-\infty}^{\infty} dz_4 \int_0^\infty dz \frac{|z|^2 (r^2 + |z|^2 + \xi(z_4 - t))}{[|z|^2 + \xi(z_4)]^2 ([r^2 + |z|^2 + \xi(z_4 - t)]^2 - 4|z|^2 r^2)^2}
\]

(25)
Again, we expect this result to be reliable only for $|p| \gtrsim 1\, GeV$ and that small instantons should dominate, in the large momentum regime.

One can check that, at the scale we are working at, the free contribution to the pion and nucleon point-to-wall correlators is much smaller than the corresponding leading instanton term, for any reasonable value of $m^*$. Hence, in the ratio (3), the free contribution can be safely neglected and the model parameters $\bar{n}$ and $m^*$ cancel out. As a result, the only parameters that remain are $\Lambda_{QCD}$, appearing in $d_{tHooft}(\bar{\rho})$ and the position of the instanton size cut-off $\bar{\rho}$. However, from the above discussion, we expect the results to become independent on $\bar{\rho}$ for sufficiently large values of $p$.

In this section we have focused on the evaluation of the logarithmic derivative (2), which allows to extract hadronic masses. In principle, the same calculation scheme can be applied to other hadronic observables such as, e.g., electromagnetic form factors [12]. Following this prescription, one needs to relate the observable being investigated to some ratio of point-to-wall correlation functions and choose $t$ and $p$ according to the above discussion. One should keep in mind, however, that the SIA approximation is much more accurate if the relevant Green functions receive contribution from more than one quark zero-mode propagator [17].

IV. RESULTS AND DISCUSSION

Let us begin by discussing the results obtained in the case of the pion [28]. In fig. 4, several theoretical evaluations of the pion dispersion curve $E(|p|)$, are compared with the physical one. The different SIA predictions were obtained from (2) using different values of the time interval and of the size cut-off [29] see fig. 3. Notice that these values correspond to extremely different pictures of the vacuum, because the corresponding packing fraction is proportional to $\bar{\rho}^4$ and changes by one order of magnitude.

First of all we can verify that the pion pole has been isolated and that the results are fairly independent on the value of cut-off used. From the comparison in the region $1\, GeV < |p| < 2\, GeV$, we conclude that our semi-classical approach provides with the right amount of dynamics and is completely consistent with the existence of a light pion.

Following the discussion of section II, we would expect the excited states to contribute to the relevant green-functions, in the region above $2\, GeV$. This implies that the logarithmic derivative (2) must deviate from the pion dispersion curve, at some high momentum. However, we observe that the complete agreement between our calculation and the pion dispersion curve continues up to high momenta. This fact can be interpreted in terms of quark-hadron duality [18], which assumes that the contribution from the continuum is dominated by perturbative effects, while the ground-state pole can be described in terms of non-perturbative effects. Indeed our purely non-perturbative calculation does not reproduce the presence of a continuum.

The results obtained in the case of the nucleon, using different Euclidean time intervals and instanton size cut-offs, are reported in fig. 5. The numerical solution of the integral (25) is quite challenging and the signal becomes very noisy above $2\, GeV$. Again, we verify that the ground state pole has been isolated and that the results are insensitive to the choice of the cut-off. We observe agreement between our theoretical calculation and the experimental curve in the expected region, $1\, GeV \lesssim |p| \lesssim 1.6\, GeV^2$. Moreover, unlike in the case of the pion, the mass of the nucleon is
FIG. 4: Dispersion curve for the pion ($\Lambda_{PV} = 250 \text{ MeV}$). The dotted line is obtained from the physical pion mass, the points are SIA results.

FIG. 5: Dispersion curve of the nucleon ($\Lambda_{PV} = 250 \text{ MeV}$). The solid line is obtained from the physical nucleon mass, the points are SIA results.

sufficiently large do be extracted from the best fit curve. We found $M_n = 3.9 \pm 0.2 \Lambda_{PV} \sim 970 \text{ MeV}$, consistent with the experimental value.

In this case, by extrapolating the trend of the calculations above $2 \text{ GeV}$, we may say that some little deviation from the nucleon dispersion curve seems to occur, at high momenta. This fact may suggest that, unlike in the case of the pion, instanton non-perturbative effects have some overlap with nucleon resonances.

V. CONCLUSIONS AND OUTLOOK

In this paper we have presented a new calculation scheme which can be used to study several hadronic observables. The method is based on the idea that, in the appropriate limit, it is possible to isolate the contribution from small-sized instantons, upon which we have theoretical control.

The prescription was applied to evaluate the mass of the pion and of the proton. We found agreement with phenomenology and verify that our results are insensitive to the particular choice of cut-off $\bar{\rho}$ and therefore are model independent.

By comparing the results obtained for the pion and nucleon, we discussed the importance of instanton effects for the physics of resonances, in these channels. We found that, in the case of the pion, a purely non-perturbative instanton calculation does not reproduce the existence any resonance or of a continuum up to extremely high momenta. In the case of the nucleon, some deviation from the ground-state dispersion curve seems to occur at high momenta (although our results are not accurate enough to make a definitive statement). This would imply that instanton effects could
have some overlap with nucleon resonances. We intend to develop a more quantitative analysis of this issue in a future work.

We think that the method could be useful in investigating the contribution of instantons to a variety of hadronic observables. An application of the same ideas to the analysis of pion and proton electro-magnetic formfactors is in preparation.

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[20] We acknowledge a useful discussion with A. Schwenk on this point.
[21] The derivation can be straightforwardly generalized to all correlation functions receiving contribution from at least two zero-mode propagators.
[22] We shall assume that the first $N^+$ components of $\Omega$ will span the moduli space of instantons, while the next $N^-$ components that of anti-instantons.
[23] $f(\Omega)$ incorporates the effects of small quantum fluctuations around the classical solution.
[24] The coefficients $C_{IJ}$ correspond to the matrix elements of the inverse of the “overlapping matrix” (see [3] for details).
[25] The definition of the single anti-instanton is, of course, completely equivalent.
[26] The implication of total momentum conservation in SIA will be discussed in detail in [12].
[27] We would like to thank E.V. Shuryak for suggesting such interpretation.
[28] In performing the calculation, we have expressed all dimensional quantities in terms of $\Lambda_{PV}$. The results presented in this section have been converted to natural units by choosing $\Lambda_{PV} = 250 \text{MeV}$.
[29] These results were obtained using $\sigma = 0.1 \text{GeV}^{-1}$ in [21]. Different choices were tried and gave equivalent results.
[30] According to the discussion in (II), we have fitted only the points in the region $1 \text{GeV} < |p| < 1.6 \text{GeV}$.