Implementation with Uncertain Evidence*

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Abstract

We study a full implementation problem with hard evidence where the state is common knowledge but agents face uncertainty about the evidence endowments of other agents. We identify a necessary and sufficient condition for implementation in mixed-strategy Bayesian Nash equilibria called No Perfect Deceptions. The implementing mechanism requires only two agents and a finite message space, imposes transfers only off the equilibrium, and invoke no device with "...questionable features..." such as integer or modulo games. Requiring only implementation in pure-strategy equilibria weakens the necessary and sufficient condition to No Pure-Perfect Deceptions. In general type spaces where the state is not common knowledge, a condition called higher-order measurability is necessary and sufficient for rationalizable implementation with arbitrarily small transfers alongside.

1 Introduction

Consider a government funding a research grant for a vaccine for a new disease. Several firms have prior experience of developing vaccines for closely related diseases and can apply for the grant. Being mature firms in the same sector, firms are familiar (owing to prior experience) with the quality of research processes of the other firms. However, candidate vaccines for

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the new disease (which serve as evidence of the quality of a firm’s research) are products of
the firm’s internal research processes, and are, therefore, private information. Firms want
the grant to be awarded to them irrespective of the true quality of their research processes
while the government wants to maximize the probability of obtaining a functional vaccine.

Alternately, consider a personal injury trial where a judge faces the problem of de-
termining the level of compensation that a defendant should pay a plaintiff. Suppose that
the judge wants to set a level of compensation commensurate to the damage that has been
caused, but does not know what that damage is. Each agent knows the extent of the dam-
age, and irrespective of the level of damage, the plaintiff wants more compensation and the
defendant less. In such a setting, evidence is likely to be the result of investigation and
therefore uncertain. However, both the defendant and the plaintiff may have some notion of
the possible evidence which may arise.

The common threads that underlie the two situations described above are the following
- there is a (decision relevant) state commonly known to the agents but not to the designer,
agents have state independent incentives which are not aligned with those of the designer, and
while there is evidence in the setting, it is uncertain, so that agents do not know precisely
what evidence other agents may possess. This characterizes a wide variety of practical
situations and thus it is important to study how to obtain credible information for decision
making in these settings.

In this paper, we study a setup inspired by Ben-Porath and Lipman (2012), but different
in one important aspect - the evidence is uncertain. This means that articles of evidence
are drawn (independently) from a distribution, and the actual realization of evidence is
private information for the agents. The setup is Bayesian in the evidence dimension, i.e.
while agents do not know what evidence other agents have, the distribution from which this
evidence is drawn is common knowledge. This is more realistic than complete information
along all dimensions, evidence often being the product of research, which is fundamentally
uncertain in nature. It is however, conceivable, that even though the exact outcome of the
search for evidence may be unknown, prior experience may allow agents to roughly predict
what evidence may arise in a certain situation.

Evidence variation is used in two distinct ways in the mechanisms we construct - first,
in presenting evidence, an agent directly \textit{refutes} all the states where she may not possess
such evidence with positive probability; second, if an agent cannot present the evidence
for a state claim (perhaps because some evidence is not available with enough probability),
the distribution induced on her evidence report varies from the expected distribution under
truth-telling - a difference which we will show can be used to flag incorrect reporting using a
system of bets. Our result relies on a novel classification of lies (incorrect state claims) based
on their refutability with evidence, and for some lies which cannot be directly refuted with
evidence, the latter technique involving bets is used.

When constructing mechanisms, agents may attempt to deceive the designer by either
misreporting the state, or withholding some part of their evidence endowment, since their
incentives are not necessarily aligned with those of the designer. Such strategies, which
deviate from truth-telling, are called deceptions. Sometimes, deceptions might exist which
perfectly mimic another state in both the state and evidence dimensions. In such cases, we
will show that the designer cannot separate these states by using a mechanism, and this
yields our implementing condition - No Perfect Deceptions (NPD). More precisely, NPD can
be summarized as follows: if states $s$ and $s'$ are such that $s'$ is a lie in state $s$ which cannot
be directly refuted by evidence, and at state $s$, it is possible for all agents to induce the same
distribution on their evidence as they would under truth-telling in state $s'$, then the social
choice function must induce the same outcome at $s$ and $s'$.

To establish the sufficiency of NPD for mixed-strategy implementation, we present a
finite mechanism which fully implements any social choice function (SCF) which satisfies
NPD in mixed-strategy Bayesian Nash equilibria. The mechanism works with two or more
agents and uses off-equilibrium transfers. A striking feature of this mechanism is that while
most Bayesian implementation results (see for example Jackson (1991) and Serrano and
Vohra (2010)) require three or more agents and involve "...questionable features..." such
as integer or modulo games\(^1\), we obtain exact implementation with only two agents and
without the use of such "...questionable features...". We note however, that unlike in Jackson
(1991), where each agent knows only her own type, the domain of the social choice function
is common knowledge in our setup, and agents only contend with uncertainty along the
evidence dimension.

A strictly weaker condition, which we call No Pure-Perfect Deceptions (NPPD) is
necessary and sufficient for pure-strategy implementation. It requires that the strategies the
agents use to mimic the true distribution at another state be pure, and is therefore strictly
\(^1\text{See Jackson (1992) for a description of the issues which arise from the use of these questionable features.}\)
weaker than NPD. This result spells out a clear distinction in this setup between pure and mixed-strategy implementation, with the set of mixed-strategy implementable SCFs being strictly smaller than the set of pure-strategy implementable SCFs.\(^2\) This condition is still stronger than a stochastic version of the measurability condition in Ben-Porath and Lipman (2012) (which we term stochastic measurability) however, elucidating an important point - when uncertain evidence is considered, conditions in the flavour of measurability (which only require that the evidence distribution vary between states with different desired outcomes) no longer suffice, and stronger conditions are required even when limiting attention to pure strategies.

At a methodological level, we introduce two significant innovations. First, in line with our treatment of deterministic evidence in Banerjee, Chen, and Sun (2021), we use a novel classification of lies (incorrect state claims) in terms of their refutability with evidence and eliminate these lies successively in a mechanism. This eschews the canonical approach of first obtaining consistent reports using an integer game and then using the implementing condition to argue that consistent reports may only obtain on the truth. The versatility of this approach is evidenced by its utility in both deterministic and uncertain evidence settings. To motivate the second innovation, we first draw attention to the fact that our setting allows for the existence of lies which cannot be eliminated with positive probability by evidence possessed by any agent.\(^3\) In these cases, the necessary condition (NPD) yields that some agents presenting these lies will not be able to present every article of evidence with the same probability as they would be expected to under truthful reporting if the lie were indeed the truth. In this case, it becomes possible for agents to profitably signal to the designer that a lie is being presented by placing monetary bets on the evidence realizations of other agents. This is a form of arbitrage over probability distributions and is critical to the treatment of mixed strategies alongside Bayesian uncertainty. The technique is somewhat close in spirit to the approach in Crémer and McLean (1988), where it finds use in efficient

\(^2\)This is consistent with Bayesian implementation with preference variation, where Mixed Bayesian Monotonicity is strictly stronger than Bayesian Monotonicity. It is, however, different from implementation with complete information, where Maskin monotonicity characterizes implementation with preference variation, and measurability characterizes implementation with evidence irrespective of pure or mixed strategies.

\(^3\)This is possible even though the evidence distribution varies between two states. In particular, the support of the evidence distribution could be the same, while the probabilities of particular collections of evidence vary amongst the states.
partial implementation from agents about correlated valuations. This paper marks, to our knowledge, the first use of such an approach in the field of full implementation of general SCFs.\(^4\)

We then turn to a more general setup where agents’ private information is represented by a general type, and evidence is contained within this type. Further, arbitrary correlations are allowed in these types, so that evidence draws may now be correlated. In this context, we present a mechanism which implements a given SCF in rationalizable strategies but with arbitrarily small transfers alongside. This requires that at least one of the agents be able to separate between states (type profiles) at which different outcomes are prescribed in terms of their beliefs at some finite order, a condition we term higher-order measurability. While this condition bears a resemblance to the measurability condition in Abreu and Matsushima (1992), we spell out the difference here. In an environment without preference variation, only constant SCFs satisfy the measurability condition in Abreu and Matsushima (1992). However, in such a scenario, with sufficient evidence variation, we can build belief hierarchies which allow for implementing non-constant SCFs. In a setting with independent evidence endowments, such as the one we study for Bayesian Nash implementation, higher order measurability is identical to stochastic measurability, and therefore weaker than both NPD and NPPD. We show that higher-order measurability is necessary and sufficient for implementation in rationalizable strategies with small transfers, alongside the usual incentive compatibility condition. This significantly expands the scope of obtaining implementation with evidence, albeit at the cost of small transfers alongside the desired outcome and a significantly more complicated mechanism.

The rest of the paper is organized as follows - In Section 2, we describe a illustrating example more formally to provide context for the analysis in later sections. Section 3 defines the setup formally and introduces some properties of the evidence structure. Section 4 deals with the Bayes-Nash implementation result. Section 5 discusses pure-strategy Bayes-Nash implementation. Section 6 details rationalizable implementation in the setup. Finally, Section 7 reviews and compares our results with the literature.

\(^4\)Note that Crémér and McLean (1988) focuses on efficient allocation with cheap talk messages while we focus on implementation of SCFs.
2 Illustrating Example - Research Grant

For the aforementioned problem of research grants for a vaccine, suppose there are two firms, A and B, which apply for the grant. Firm A is a startup whose research process quality may either be low (L), medium (M) or high (H). A low quality research process is characterized either by having a low quality candidate vaccine (e.g. with major side effects). A firm with a medium quality research process has a probability 0.4 of having a high quality candidate (one with only minor side effects). A firm with a high quality research process produces a high quality candidate with a probability 0.6. Firm B is an incumbent with (commonly known) medium research quality (M). The government is not aware of Firm A’s research process quality, so that the states of the world are given by L, M, or H.

In state L, the government wants to allot the grant to firm B, in state H to firm A, and in state M to firm B. As in the introduction, we argue that the research process quality of firm A is common knowledge amongst the firms, but unknown to the government. With the social choice function so defined, we turn our attention to the following formal representation of the evidence structure.

| State | Evidence (Firm A) |
|-------|-------------------|
| H     | 0.6 × \{M, H\}, {L, M, H}\} + 0.4 × \{L, M, H\}\} |
| M     | 0.4 × \{M, H\}, {L, M, H}\} + 0.6 × \{L, M, H\}\} |
| L     | \{L, M, H\}\} |

Note that the articles of evidence in the above example have been associated with subsets of the state space. In general, we could have abstract articles of evidence, such as e₁ and e₂ instead of \{L, M, H\}, and \{M, H\} respectively. These ”names” are obtained by associating an article of evidence with the states in which they appear with positive probability. This form of nomenclature is used in Ben-Porath and Lipman (2012), albeit in a deterministic setting. We use this nomenclature to simplify the exposition (especially within examples) in several places in the paper. The articles of evidence above are interpreted thus - \{L, M, H\} is interpreted as possessing a low quality candidate, and \{M, H\} as possessing a high quality candidate. Thus, firm A, in presenting the article \{M, H\} proves that its research quality is not L. The above table essentially captures the intuition presented in the opening of this section - as we move upwards in research quality, we see an increasing probability of having a high quality candidate. Notice that firm A of type H maybe unable
to separate itself from a firm A type $M$, since either the type $M$ may have the high quality candidate or the type $H$ may not possess the high quality candidate. This yields that an unraveling result of the flavour of Milgrom (1981) or Grossman (1981) may have no bite in this setup. This is a property possessed by the setup in Ben-Porath and Lipman (2012), and shared by the more general setup we study here.

From the above intuition of evidence, it is clear that no agent can eliminate state $M$ at state $H$. This provides a major challenge in achieving implementation. To see this, suppose the mechanism were direct, requiring only a state and an evidentiary claim. Further, consider the following strategies for firm A in state $H$ possessed with $\{{{M, H}, {LMH}}\}$: Report $\{{{M, H}, {LMH}}\}$ with probability $\frac{2}{3}$ and report $\{{{LMH}}\}$ with probability $\frac{1}{3}$; In the state dimension, always claim that the state is $M$. With this strategy, the agents would induce the same distribution on their reports as if the true state were $M$, and they were reporting truthfully. This form of strategy is later dubbed a perfect deception. If perfect deceptions exist, it is not possible to separate states in any mechanism (direct or indirect) which relies on Bayesian Nash Implementation. This result is formally proved in Section 4.1 and followed by an intuitive description. Note that there is also a perfect deception for type $M$ to type $L$, but $L$ is refutable at $M$ with positive probability, so that types $M$ and $L$ are separable by a mechanism.

With the above evidence structure then, it is not possible for the government to allot the grant to firm $A$ in state $H$ and firm $B$ in state $M$. Consider though the following mild perturbation of the evidence structure: An additional article of evidence $\{{{H}}\}$ is available to firm $A$ with a small probability when its research quality is $H$. For instance, the distribution in state $H$ may be $0.1 \times \{{{H}}\} + 0.5 \times \{{{M, H}, {LMH}}\} + 0.4 \times \{{{L, M, H}}\}$. In this case, the government can actually implement different outcomes in states $M$ and $H$ - while in state $M$, $H$ is nonrefutable, the article $\{{{H}}\}$ is not available, so that a perfect deception is impossible; in state $H$, while it is possible to perfectly match the evidence distribution for state $M$, $M$ is actually refutable with positive probability.
3 Model and Preliminaries

3.1 Setup

There is a set of agents $\mathcal{I} = \{1, ..., I\} : I \geq 2$, a set of outcomes $A$ and a finite set of states $S$. Agents have bounded utility over the outcomes given by $\bar{u}_i : A \times S \to \mathbb{R}$. Their preferences are quasilinear, so that $u_i(a, \bar{\tau}_i, s) = \bar{u}_i(a, s) + \bar{\tau}_i$, where $\bar{\tau}_i$ is the transfer of money to agent $i$. The bound of $\bar{u}_i$ can be normalized so that it is strictly less than 1, that is, an agent can be persuaded to accept any outcome if the the alternative were any other outcome with a penalty of 1 dollar.

Each agent $i$ is endowed with a collection (mathematically, a set) of articles of evidence $E_i$ which can vary from state to state. A profile of collections (one for each agent) is denoted by $E$. The set of all possible collections of evidence for agent $i$ is denoted by $\mathbb{E}_i$ and we denote by $\mathbb{E} (= \Pi_{i \in \mathcal{I}} \mathbb{E}_i)$, the set of all possible profiles of evidence collections. As in the illustrating example, each collection of evidence can be associated with a set of subsets of the state space - namely those subsets in which the article occurs with positive probability.

We also assume that it is possible for an agent to submit the entire collection of evidence she is endowed with. This condition, which is commonly called normality, is present in Ben-Porath and Lipman (2012) as well, and is, generally speaking, a common feature of single stage mechanisms in such settings. In case of uncertain evidence though, there emerges a subtle difference. In Ben-Porath and Lipman (2012), normality is formulated by requiring that a single "most informative" evidence message be available in any state to any agent.\footnote{In context of the example for instance, the article $\{M, H\}$ is more informative than the article $\{L, M, H\}$. In general, the most informative article of evidence is given by the intersection of all the available articles of evidence, when they are named with subsets of the state space (as in the example).}

In this setting however, it is essential that agents be allowed to submit an entire collection of evidence, rather than just the most informative message. Intuitively, this is because several different collections of evidence may yield the same "most informative" evidence message. However, it may be possible to distinguish between these collections when each member is observed by the designer. We discuss this point formally in Appendix A.1.

Evidence at state $s$ is distributed according to the commonly known prior $p : S \to \Pi_{i \in \mathcal{I}} \Delta(\mathbb{E}_i)$. Denote the marginals of $p(s)$ on $\mathbb{E}_i$ and $\Pi_{j \neq i} \mathbb{E}_j$ by $p_i(s)$ and $p_{-i}(s)$, respectively. Deterministic evidence is a special case of this (the above distribution is degenerate). Note
that \( p \) is defined as a product distribution, eschewing the possibility of the evidence draws being correlated. This assumption is maintained in most of the paper, but relaxed in Section 6.

Let \( P_i \) denote the set of all possible evidence distributions for agent \( i \), i.e. \( P_i = \{ p_i(s) : s \in S \} \). We denote by \( P \), the set \( \prod_{i \in I} P_i \).

### 3.2 Mechanisms and Implementation

Any mechanism \( \mathcal{M} \) is represented by the tuple \((M, g, \tau)\) where \( M \) is the message space, \( g : M \to A \) is the outcome function, and \( \tau = (\tau_i)_{i \in I} \) is the profile of transfer rules with \( \tau_i : M \to \mathbb{R} \). At state \( s \), and with a prior \( p \), a mechanism \( \mathcal{M} \) induces a Bayesian game \( G(\mathcal{M}, u, s, p) = \langle I, s, p, \langle M_i, u_i, S \times E_i \rangle_{i \in I} \rangle \) with the following properties:

- The type space of a typical agent \( i \) is \( S \times E_i \).
- Every type of every agent forms beliefs on \( E_{-i} \) according to \( p_{-i} \).
- The action space of a type \((s, E_i)\) is given by \( M_i(E_i) \) (as some messages may be contingent on evidence availability). We only consider finite mechanisms, so that \( M_i(E_i) \) is finite for any \( E_i \).
- A typical (mixed) strategy for an agent \( i \) at the state \( s \) is a function: \( \sigma_i : S \times E_i \to \Delta(M_i) \) such that \( \sigma_i(s, E_i) \in M_i(E_i) \).

The common prior type space is specified by \( p \). Given a game \( G(\mathcal{M}, u, s, p) \), say \( \sigma_i(s, \cdot) \) is a best response to \( \sigma_{-i}(s, \cdot) \) if

\[
\sigma_i(s, E_i)(m_i) > 0 \Rightarrow m_i \in \arg \max_{m_i \in M_i} \sum_{E_{i,i} \in E_{-i}} p_{-i}(s)(E_{-i}) \sum_{m_{-i} \in M_{-i}} \sigma_{-i}(s, E_{-i})(m_{-i})[u_i(g(m_i', m_{-i}), \tau_i(m_i', m_{-i}), s)].
\]

Next, we define the notion of a Bayesian Nash equilibrium in this particular setting.

**Definition 1** A Bayesian Nash equilibrium (henceforth BNE) of the game \( G(\mathcal{M}, u, s, p) \) is defined as a profile of strategies \( \sigma \) such that \( \forall i, s, \sigma_i(s, \cdot) \) is a best response to \( \sigma_{-i}(s, \cdot) \).

A pure-strategy BNE is a BNE \( \sigma \) such that for any \( i \) and \( s, \sigma_i(s, E_i)(m_i) = 1 \) for some \( m_i \). We then define the main notion of implementation we work with in this paper.
Definition 2  An SCF $f$ is implementable in mixed-strategy BNE by a finite mechanism if there exists a finite mechanism $\mathcal{M} = (M, g, \tau)$ such that for any profile of bounded utility functions $\bar{u}$, at any state $s$, we have $g(m) = f(s)$ and $\tau(m) = 0$ for every message $m \in M$ in the support of $\sigma(s)$ of any BNE $\sigma$ of the Bayesian game $G(\mathcal{M}, u, s, p)$.

That is, for any BNE of the Bayesian game induced by the mechanism, the outcome is correct, and there are no transfers. We stress here that implementation should obtain regardless of the realized profile of evidence collections. Since we wish to implement by depending solely on evidence, rather than depending on preference variation, the notion of implementation above requires that implementation obtain regardless of the profile of bounded utility functions.

3.3 Lies and their classification

At the heart of the Bayesian implementation result is a classification of lies (states not identical to the truth), which we illustrate below.

Definition 3 A collection of evidence $E_i$ refutes a state $s$ (denoted $E_i \downarrow s$) if agent $i$ does not have $E_i$ or a superset of $E_i$ with positive probability at $s$. Otherwise, $E_i \not\downarrow s$.

That is, $E_i \downarrow s$ if $p_i(s)(E'_i) = 0 \forall E'_i \supseteq E_i$. In a mechanism, agents will always have the opportunity to withhold some articles from a collection. Therefore, even if a collection does not occur at $s$ with positive probability, being presented with this collection does not directly imply that $s$ is false. Indeed the evidence presented may be part of a larger collection which can occur. Therefore, only when a collection or any superset does not occur at $s$, can we claim that it refutes $s$.

In the context of the leading example, in state $M$ or $H$, the article $\{M, H\}$ refutes the state $L$. A difference from the deterministic evidence setting is that it is not available with probability 1. For instance, if some firm claims that another firm’s research process is of low quality (when it is of medium quality), the latter firm cannot necessarily prove this claim to be false by presenting the article $\{M, H\}$ because it is not available with probability 1. This is different from a setting of deterministic evidence, where the necessary condition (measurability) entails that the realization of the evidence vary between states, so that the
article \{M, H\} would have to be available with probability 1.\(^6\)

Also, if firm \(A\), when possessed with the article \{\(H\)\} claims a medium quality process, then it is the only agent who can prove this claim to be false.\(^7\) This scenario, referred to as a self-refutable lie, also exists under deterministic evidence, but under deterministic evidence, the firm would be able to refute a lie about itself with probability 1, rather than being able to do so only with some probability (as in the above example).

The following conditions arise as a result of the definition of refutation used in our setup:

1. (se1) For any agent \(i\), \(p_i(s)(E_i) = 0\) if \(E_i \downarrow s\). That is, at any state, the probability that collections of evidence which refute the truth are available is zero. In short, proof is true.

2. (se2) If \(E_i \not\subseteq s'\) then \(p_i(s')(E'_i) > 0\) for some \(E'_i \supseteq E_i\). That is, if a collection of evidence \(E_i\) does not refute \(s'\), then it (or a superset) is available in state \(s'\) as well.

In Appendix A.1, we study the connections between this setup and the deterministic evidence setup from Ben-Porath and Lipman (2012).

**Definition 4** A lie \(s' \in S\) is said to be refutable by \(i\) at \(s\) if \(\exists E_i \in E \text{ s.t. } p_i(s)(E_i) > 0\) and \(E_i \downarrow s'\).

That is, at \(s\), \(s'\) is refutable by \(i\) if \(i\) may have a collection of evidence \(E_i\) such that \(E_i \downarrow s'\). Clearly from (se1), \(E_i\) is not available at \(s'\) (if it were available, then \(E_i\) cannot refute \(s'\)). Then, the support of the evidence distribution for agent \(i\) contains a collection under the truth which is missing under a refutable lie.

**Definition 5** A lie \(s' \in S\) is said to be refutable at \(s\) if there is an agent \(i\) so that \(s'\) is refutable by \(i\) at \(s\). Otherwise, it is nonrefutable.

\(^6\)In a setting of complete information (e.g. Ben-Porath and Lipman (2012)), an SCF \(f\) satisfies measurability if whenever \(f(s) \neq f(s')\), there is an agent \(i\) whose evidence endowment varies between \(s\) and \(s'\).

\(^7\)Note that in context of this example, it would not be to the advantage of this firm to make such a claim, but since we do not rely on preference variation, in the general case, such issues may easily arise.
Returning to the leading example, $H$ is nonrefutable at $M$. This brings up another difference from the deterministic evidence setting, which is that it is not possible for a firm to prove that its research process is of high quality rather than being of medium quality. This is because there is no additional article of evidence available at state $H$ which is not available at state $M$. This scenario, called a nonrefutable lie (e.g. $H$ is nonrefutable at $M$), also occurs under deterministic evidence, but under deterministic evidence, it is necessary that some agent has additional evidence under a non-refutable lie than under the truth with probability 1. Indeed the treatment of non-refutable lies is one of the main differences between the deterministic and uncertain evidence setups.

We note that if $s'$ is a nonrefutable lie at $s$, then no agent may possess (with positive probability) any articles of evidence in state $s$ which they may not possess at state $s'$, since these articles would refute $s'$. Therefore, if $s'$ is nonrefutable at $s$, there are two possibilities with respect to the support of the evidence distribution at $s'$. Either it strictly includes the support under $s$, so that $s$ is refutable (with positive probability) at $s'$, or the supports are identical, so that only the probabilities with which various collections of evidence are available has changed. The latter scenario poses a major challenge in the implementation result we pursue.

4 Mixed-Strategy Bayesian Nash Implementation

4.1 A Necessary Condition: No Perfect Deceptions

Given that under deterministic evidence, the implementing condition (called measurability) entails only that at least some agent is endowed with a different set of evidence, it would be natural to expect that a stochastic version of this condition, which entails requiring $p(s) \neq p(s')$ whenever $f(s) \neq f(s')$ would suffice as the implementing condition for Bayesian implementation. We term this condition stochastic measurability. It turns out however, that not every non-refutable lie can be eliminated by a mechanism, necessitating a stronger implementing condition. We establish the condition and prove its necessity after some preliminaries below.

First, we define a deception for an agent of type $(s, E_i)$.

**Definition 6** A deception for an agent $i$ is a function $\alpha_i : S \times E_i \rightarrow \Delta(S \times E_i)$ such that if
We interpret $\alpha_i$ as being a function from the agent’s true type to a distribution over types she can mimic (which requires that true type has a weakly larger collection of evidence than the mimicked type). A profile of deceptions is denoted by $\alpha = (\alpha_i)_{i \in I}$. This notion is similar to the one with the same name in Jackson (1991).

Note that a deception can also be interpreted as a strategy for an agent in a direct mechanism (a mechanism where agents only report their types, i.e. $M_i = S \times E_i$ for each agent).

Definition 7 A deception $\alpha_i$ is perfect for state $s'$ at state $s$ for agent $i$ if,

$$\forall E'_i, \sum_{E_i \in E_i} p_i(s)(E_i)\alpha_i(s, E_i)(s', E'_i) = p_i(s')(E'_i).$$

Likewise, a profile of deceptions $\alpha = (\alpha_i)_{i \in I}$ is perfect for state $s'$ at state $s$ if $\alpha_i$ is perfect for state $s'$ at state $s$ for each agent $i$.

That is, a deception is perfect for state $s'$ at state $s$ if every agent induces the same distribution over her types as she would if the true state were $s'$ and if she were reporting truthfully.

We denote by $s \rightarrow_i s'$ the notion that agent $i$ has a perfect deception from $s$ to $s'$. Therefore, if there is a perfect deception from $s$ to $s'$, then, $s \rightarrow_i s' \forall i \in I$. Further, $s \not\rightarrow_i s'$ denotes the notion that $i$ does not have a perfect deception from $s$ to $s'$ and $s \not\rightarrow s'$ denotes that there is no perfect deception from $s$ to $s'$.

As mentioned earlier, it is not possible to eliminate non-refutable lies such that there is a perfect deception for the non-refutable lie at the true state. We now define the necessary condition for implementation.

Definition 8 (Condition NPD - No Perfect Deceptions) We say that a social choice function $f$ satisfies NPD whenever for two states $s$ and $s'$, if at $s$, state $s'$ is nonrefutable, and there exists a perfect deception for $s'$, then $f(s) = f(s')$.

That is, to have different desirable outcomes between $s$ and $s'$, either more evidence must be available at $s$ than is available at $s'$, or it must be impossible to have a perfect deception for state $s'$ at state $s$. We note that one way for a perfect deception to be
impossible is if there are articles of evidence at $s'$ which are not available at $s$. We can therefore also interpret the above condition as follows: we are able to eliminate all kinds of lies other than nonrefutable lies so that agents can mimic the lie both in the state and the evidence dimensions perfectly.

We also wish to stress the fact that the existence of a perfect deception from $s$ to $s'$ does not by itself preclude our ability to separate the states by a mechanism. Indeed it must be the case that $s'$ is nonrefutable at $s$. To see this, consider a scenario where the evidence structure is identical between $s$ and $s'$ except that each collection at $s$ is appended with an additional article of evidence $e_i$ for each agent $i$. Clearly, a perfect deception involves each agent claiming the state is $s'$ and withholding the article $e_i$. However, it is clear that these states can be separated by incentivizing refutation of a state claim by presenting any $e_i$.

For an example of a social choice function and an evidence structure under which the evidence distribution varies between two states, we refer the reader to the leading example in Section 2. We reproduce the evidence structure below:

| State | Evidence (Firm A) |
|-------|-----------------|
| $H$   | $0.6 \times \{\{M, H\}, \{L, M, H\}\} + 0.4 \times \{\{L, M, H\}\}$ |
| $M$   | $0.4 \times \{\{M, H\}, \{L, M, H\}\} + 0.6 \times \{\{L, M, H\}\}$ |
| $L$   | $\{\{L, M, H\}\}$ |

It is clear that the distribution of evidence does change between every pair of states, so that every SCF is stochastically measurable. However, it can be seen that it is possible to mimic the distribution at $M$ when the true state is $H$, as the article $\{M, H\}$ is available with extra probability at $H$. For instance, the deception $\alpha_A(H, \{\{M, H\}, \{L, M, H\}\}) \rightarrow \frac{1}{3}(M, \{\{L, M, H\}\}) + \frac{2}{3}(M, \{\{M, H\}, \{L, M, H\}\})$ suffices to do so. However, it is not possible to mimic the distribution at $H$ when the state is $M$, as the article $\{M, H\}$ isn’t available often enough. This establishes an important property of the NPD condition, that is, it is not bidirectional. Therefore, the structure induced by the social choice function on the state space is not partitional. We stress here that this is a stark point of differentiation between deterministic and uncertain evidence, since the necessary condition under deterministic evidence - measurability, induces a partitional structure on the state space.

We present the main result of the paper below.
Theorem 1 A social choice function is implementable in BNE if and only if it satisfies NPD.

We present the proof of necessity below, while the implementing mechanism which constitutes the sufficiency proof is presented in Section 4.4.

Proof. (Necessity of NPD) Suppose a mechanism $\mathcal{M} = (M, g, \tau)$ implements $f$ and consider a pair of states $s$ and $s'$ so that $s'$ is nonrefutable at $s$ and there is a perfect deception from $s$ to $s'$.

First, we claim that

For any collection $E_i$ in the support of the evidence distribution at $s, E_i$, or a (*) superset thereof must be in the support of the evidence distribution at $s'$.

This follows from the nonrefutability of $s'$ at $s$.

Now, consider any equilibrium $\sigma$ of the mechanism and the following strategy for agent $i$ at state $s$ when endowed with the collection $E_i$. First, the agent plays according to the deception $\alpha_i(s, E_i)$. This yields another type realization, which we denote by $(s', E'_i)$. Now, the agent plays $\sigma_i(s', E'_i)$. Since there is no additional collection of evidence at $s$, this object is well defined. With some abuse of notation, we denote the profile of such strategies by $\sigma \circ \alpha$.

We claim that $\sigma \circ \alpha$ forms an equilibrium at $s$ with outcome $f(s')$. To see this, consider an arbitrary agent $i$. First, given the strategies of other agents $j \neq i$, the belief induced over the actions of other agents $j$ is the same under $\sigma \circ \alpha$ at $s$ as that under $\sigma$ at the state $s'$. Given these beliefs, at state $s'$, $\sigma$ induces the outcome $f(s')$ with no transfers. Each type of agent $i$ can obtain this outcome in state $s$ by playing according to $\sigma_i \circ \alpha_i$. If any type $E_i$ of agent $i$ at $s$ has a profitable deviation, then there is another type of agent $i$ at $s'$ which has weakly more evidence than $E_i$ (from the claim (1) above) and can also deviate profitably. This contradicts the optimality of $\sigma_i$ against $\sigma_{-i}$ at $s'$. Therefore, playing according to $\sigma_i \circ \alpha_i$ continues to be a best response for the type. Thus, if there is a perfect deception $\alpha$ from $s$ to $s'$, then any implementable social choice function must have $f(s) = f(s')$. ■

Informally, under $\alpha$, agents pretend to be other types and play the equilibrium strategy under $\sigma$ of the mimiced type. The reason a type is best responding even though she may be playing the equilibrium strategy of another type is that the outcome is the same and the
preferences of agents do not change from state to state. That is, even though the concerned type may have extra evidence, and therefore be able to play other messages, there was another type at $s'$ who also had that extra evidence and that type’s best response still led to the outcome $f(s')$ which this agent is also getting from her strategy under $\sigma(\alpha(s))$, so that she continues to best respond. Therefore, since an equilibrium outcome at $s$ is $f(s')$, then $f$ is implementable only if $f(s) = f(s')$.

We note here that whereas the discussion above has centered around deceptions where an agent pretends to mimic herself at another state, this applies to mimicing a profile of distributions as well. More precisely, suppose that in the true state $s^*$, which is characterized by the profile of distributions $(p_i^*)_{i \in I}$, agents pretend to mimic themselves at another state $s'$, which is characterized by a profile of distributions $(p'_i)_{i \in I}$. If they can perfectly mimic this other state (in terms of playing their equilibrium strategies for state $s'$), then by Theorem 1, $f$ is implementable only if $f(s^*) = f(s')$.

This yields an interesting insight, which is that since it is always possible to mimic any cheap talk messages in the message space, non-refutable lies $s'$ for which agents do not have perfect deceptions from the truth must require presentation of evidence which is impossible in the true state. Then, if an agent $i$ does not have a perfect deception at state $s$ for $s'$, then she cannot mimic the distribution $p'_i$, as otherwise it would be possible for her to play the equilibrium strategies for the state $s'$. This highlights the role of evidence in this setup, which is twofold. Either agents can refute distributional claims using evidence, or agents are unable to mimic incorrect states perfectly owing to their inability to mimic the distribution of evidence.

### 4.2 A Challenge Scheme

The above discussion established that when agents report certain nonrefutable lies, they are unable to mimic the distribution of evidence for these lies. This brings up the question of how we can exploit this fact in the construction of mechanisms to facilitate the elimination of nonrefutable lies. To elucidate the underlying idea, we first consider an analogous situation.

Consider a setting with two agents. The first agent has a deck of cards, from which she draws a card and presents it. The second agent knows that the firsts’ deck of cards is missing one black card. How does she convey this information credibly to the designer if it is not possible to check the entire deck? If she simply says there is a card missing, this claim
may not be credible. Rather, the second agent can place a bet of the following form - if you
draw a black card, take a dollar from me. If you draw a red card, give me 99 cents. This
bet clearly loses the second agent money if there is no card missing. If she believes there is a
black card missing though, then this wins her money, as \( \frac{26}{51} \times 0.99 + \frac{25}{51} \times (-1) > 0 \). The fact
that the second agent is willing to take this bet credibly signals to the designer that there is
something wrong with the deck.

Note that the deck of cards is analogous to agent 1’s mixed-strategy. Therefore, it is
impossible to check the deck of cards and the second agent must find this alternate solution
to inform the designer that the first agent is not playing the truth-telling strategy. Individ-
ual cards are analogous to collections of evidence, and the inability to induce the correct
distribution of evidence is analogous to not having a black card. For the implementation
problem then, agents can blow the whistle on an incorrect (nonrefutable) state report by
identifying which agent does not have a perfect deception from the truth to the nonrefutable
lie, and then betting sums of money on that agent’s plausible collections of evidence so that
- (i) If the nonrefutable lie were to be the truth, then the bet loses money, and (ii) Since this
is not the case, the bet actually wins money. As we will discuss later, this can be thought
of as analogous to a monotonicity like condition, yielding a “reversal” between true and
false states. Returning to the illustrating example, if firm A tried to claim \( H \) when the true
state was \( M \), it would, at best be able to induce probability 0.4 on the article \{\( M, H \)\} as
that is the maximum probability with which it is available in state \( M \). Then, consider the
following bet: 1 dollar on \{\{L, M, H\}\} and 1 dollar against \{\{L, M, H\}, \{M, H\}\}. In state
\( H \) under truthful reporting, it yields an expected loss of 20 cents, while in state \( M \), it yields
a minimum expected profit of 20 cents. We will show below that this is possible whenever
there is no perfect deception. Before that, we will provide a simple way for an agent to place
such a bet.

Asking agents to place bets on (say) agent \( i \)’s evidence would mean that agents would
have to provide the designer a vector of numbers (one for each collection in \( \mathcal{E}_i \)). This yields
a complex message space. In fact, the designer can make the appropriate bets for agents.
All that she needs to know is what state (say \( s' \)) is being claimed, and what is the true state.
With this information, the designer can deduce which agent has no perfect deception for the
claimed state and place an appropriate bet against the evidence of that agent on behalf of
the agent who wishes to raise the challenge. We will denote these bets by \( b_i : S \times S \to \mathbb{R}^{|\mathcal{E}_i|} \).
Remark 1 The above can also be viewed as the building block of a monotonicity like condition. That is, for a pair of states $s$ and $s'$ such that $s \not\rightarrow_i s'$, if all agents claim that the state is $s'$, then there is an agent $j$ for whom $(f(s') + b_i(s', s)(E_i))$ is strictly preferred to $f(s')$ in state $s$ while $f(s')$ is weakly preferred to $(f(s') + b_i(s', s)(E_i))$ in state $s'$. The allocation $f(s') + b_i(s', s)(E_i)$ therefore, forms a test allocation which yields a reversal for an agent $j$ with $f(s')$ between the states $s$ and $s'$.

We formalize this notion below. First, denote by $A_i(s, E_i)$ the set of all deceptions $\alpha_i(s, E_i)$ for agent $i$ when endowed with evidence $E_i$ at state $s$. Let $\alpha_i(s)$ be defined as $\alpha_i(s) = (\alpha_i(s, E_i))_{E_i \in E_i}$, and analogously, let $A_i(s)$ be the set of all possible profiles $\alpha_i(s)$. Further, denote by $p_{\alpha_i, s}(E'_i)$ the probability induced over evidence collection $E'_i$ induced by agent $i$ under $\alpha_i(s)$. That is,

$$p_{\alpha_i, s}(E'_i) = \sum_{E_i \in E_i} p_i(s)(E_i)\alpha_i(s, E_i)((s', E'_i)).$$

Defining probability distributions over evidence, i.e. $(p_{\alpha_i, s}(E'_i))_{E_i \in E_i}$, by $p_{\alpha_i, s}$, we then denote by $P_i(s)$ the set $\{p_{\alpha_i, s} : \alpha_i \in A_i\}$, which is the set of all possible such probability distributions which agent $i$ can induce by playing any deception in state $s$. Among these distributions is also $p_{\alpha_i^*, s}$, the distribution over agent $i$'s evidence at $s$, which corresponds to the “no deception” scenario. At state $s'$ let $p_{\alpha_i^*, s'}$ denote the induced distribution over evidence by the “no deception” scenario.

With these preliminaries established, we now present the challenge scheme in the following lemma.

Lemma 1 For every agent $i$, there is a finite set $B_i = \{b_i(s, s') : s \not\rightarrow_i s'\}$ such that for any pair of states $s$ and $s'$ for which $s \not\rightarrow_i s'$, we have $b_i(s, s') \cdot p_{\alpha_i^*, s'} < 0$ and $b_i(s, s') \cdot p > 0$ $\forall p \in P_i(s)$.

Proof. Since $s \not\rightarrow_i s'$, $p_{\alpha_i^*, s'} \notin P_i(s)$. Further, $P_i(s)$ is closed and convex. Then, from the separating hyperplane theorem, there is a hyperplane $b_i$ such that $b_i \cdot p_{\alpha_i^*, s'} < 0$ and $b_i \cdot p > 0$ $\forall p \in P_i(s)$.

For a pair of states $s$ and $s'$ one such $b_i$ is sufficient. Iterating over all possible pairs of states which satisfy the premise of no perfect deception yields the set $B_i$. Therefore, each $B_i$ is finite. ■
Remark 2 The above lemma is interpreted as follows: there is a vector of real numbers (one for each possible collection of evidence for i) so that under truthful revelation in state $s'$, the expectation of this vector is negative whereas for any possible strategy of agent $i$ in state $s$, the expectation is positive. Accordingly, if an agent $j$ were to receive/give sums of money based on $b_i$ according to agent $i$’s evidence presentation in the direct mechanism, then in state $s'$ under truthful revelation by agent $i$, agent $j$ loses money while under any strategy for agent $i$ in state $s$, agent $j$ gains money. This yields a monotonicity like interpretation - if $s'$ is a lie, the agent $j$ can "blow the whistle” and indeed chooses to do so since it is profitable.

We remark here that the set $B_i$ is a set of possible bets, one for each pair of states such that there is no prefect deception for one to another. For the set to be usable in a mechanism, we need a way for an agent to utilize members of this set to challenge unanimous but incorrect state claims. This operationalization is achieved as follows. First, define a function $\hat{i} : S \times S \rightarrow I$ such that if $s \not\to s'$, $s \not\to_{i(s,s')}$ $s'$. That is, $\hat{i}$ identifies the agent who does not have a perfect deception from state $s$ to $s'$. Then, $b_{\hat{i}} : S \times S \rightarrow B_{\hat{i}}$ is defined such that $b_{\hat{i}}(s,s') \cdot p_{\sigma_{\hat{i},s'}^*} < 0$ and $b_{\hat{i}}(s,s') \cdot p > 0 \forall p \in P_{\hat{i}}(s)$. The existence of such functions follows immediately from Lemma 1.

That is, when an agent $j$ wants to bet against a report of $s'$ in state $s$, then merely telling the designer that the true state is $s$ suffices. This is because the designer can use the function $\hat{i}$ as defined above to infer which agent has no perfect deception from $s$ to $s'$ and can appropriately place a bet against $\hat{i}$’s evidence on behalf of $j$ using the function $b_{\hat{i}}$ so that the bet loses money for $j$ if $s'$ were indeed true and $\hat{i}$ were reporting truthfully, and wins money for $j$ if $s$ were true irrespective of what strategy $\hat{i}$ employs.

4.3 A Sketch of the Proof

In Section 4.1, we have established that NPD is necessary for implementation. In what follows, we will construct an implementing mechanism to establish that it is also sufficient. To fix ideas, we begin with a sketch of the proof first.

The mechanism asks each agent for the following reports: A distribution report about herself, a distribution report for the agent to her right, an evidence report, and a state report (used for placing bets). The proof utilizes the following steps. The first step is an
intermediate result that is used frequently, and thus forms one of the main building blocks of the proof. In essence, the result states that if each type of a particular agent (say $i$) submits all the evidence it is endowed with, then every agent ($i$ herself, and all others) present truthful reports about $i$’s evidence distribution. This result is obtained as follows. First, we use a proper scoring rule ($\tau^2$ in the proof) to incentivize other agents to predict the evidence report by agent $i$. Since agent $i$ presents maximal evidence in all her types, the optimal predictor for other agents is the true distribution for agent $i$. This truthful report is used (by means of a crosschecking penalty) to discipline agent $i$’s report about herself. This is denoted by $\tau^3$ in the proof. To reiterate, agent $i$’s maximal evidence presentation incentivizes other agents to make truthful predictions about her, whereupon a crosscheck incentivizes agent $i$ to make truthful reports about her own distribution. A direct corollary of this result is that if all agents present maximal evidence, then the entire profile of reports is truthful.

A key novelty of this argument is in the solution to the central tension when using scoring rules - on the one hand, the scoring rule must be active on all realizations of evidence to elicit the true distribution (otherwise it elicits the distribution conditional on being active, which may not yield truthful elicitation), but on the other hand, we require zero transfers on equilibrium. This is solved by breaking the scoring rule transfer into two parts - a direct part, which incentivizes predictions, and an offset part, which is used to cancel the direct part in equilibrium. When an agent makes a prediction, she only controls the direct part, so that from her point of view, the scoring rule is always active. This incentivizes her to make truthful predictions about others. However, her truthful report is used to discipline other agent’s reports about themselves, and this disciplined report is used to offset the scoring rule, so that it does not result in a transfer on equilibrium.

Armed with this intermediate result, the second step argues that the report profile must be pure (recall that agents are allowed to mix) and must be consistent with some state $s$. This step depends on an incentive for evidence presentation which is triggered if agents disagree with each other. This is denoted by $\tau^1$ in the proof. Broadly, if agents mix, then agents expect disagreement in the distribution reports, and if the message profile is inconsistent (due to such disagreements), then each agent is incentivized to present their maximal evidence. Then, from the above intermediate result, we can argue that the message profile cannot be inconsistent, since agents must make truthful reports if all evidence presentation is maximal.
and truthful reports must, by definition, be consistent with the true state.

The third step establishes that there cannot be a consensus on a refutable lie. We institute a penalty ($\tau^4$ in the proof) which penalizes agents if this commonly agreed upon state $s$ is refuted by another agent. There is also a small reward for refuting this consensus (this is implicit within $\tau^1$, which is also activated upon refutation). This penalty ($\tau^4$) is chosen to be large enough to deter agents from agreeing on a refutable lie.

The fourth and final step establishes that there cannot be a consensus on a nonrefutable lie which has a different outcome than the truth. If this were to be the case, then from Condition NPD, there is an agent (say $i$) who does not have a perfect deception from the true state to $s$. Then, the mechanism provides a way for agents to make a bet, as discussed in Section 4.2. Further $\tau^1$ is activated when there is an active bet as well. This leads to the submission of maximal evidence by all agents, which by the first step means that the profile of reports must have been true, so that there could not have been a consensus on a nonrefutable lie in the first place.

Thus, we are left only the possibility of a joint pure report of the true state (or a state which is outcome equivalent to the truth), so that implementation obtains. In Section 4.5.5, we also show that there are no transfers in equilibrium.

4.4 Mechanism

4.4.1 Message Space

Suppose the agents are seated around a circular table, so that for each agent $i$, there is an agent to her right (this is agent $i + 1$). For agent $I$, the agent considered as “being to the right of” her is agent 1. With this, the message space is defined as follows,

$$M_i = P_i \times P_{i+1} \times E_i \times S,$$

and a typical message ($m_i$) is given by $m_i = (p_{i,i}, p_{i,i+1}, E_i, s_i)$.

That is, each agent makes a claim about her own evidence distribution ($p_{i,i}$), a claim about the evidence distribution of the agent to her right ($p_{i,i+1}$), submits a collection of evidence and makes a state claim for (possibly) betting against other’s evidence reports.
4.4.2 Outcome

A message profile \( m \) is said to be consistent with state \( s \) if each distributional claim matches the true distribution in state \( s \). That is, \( m = (p_{i,j}, p_{i,i+1}, E_i, s_i)_{i \in \mathcal{I}} \) is consistent with \( s \) if \( \forall i, j \), \( p_{i,j} = p_{j,i}^{(s)} = p_{j,i} \). Otherwise, it is inconsistent.

The outcome is given by \( f(s) \) if \( m \) is consistent with a state \( s \), and an arbitrary outcome \( a \in A \) otherwise.

4.4.3 Transfers

The mechanism uses five transfers. There are scaling parameters with most of the transfers, and we will define the appropriate scaling in Section 4.4.4.

The first transfer provides an agent the incentive to submit evidence when there is inconsistency, or an active bet, or if a consistent profile has been refuted by an agent. That is,

\[
\tau_1^i(m) = \begin{cases} 
\varepsilon \times |E_i|, & \text{if } m \text{ involves inconsistency, or an active bet,} \\
0, & \text{or if } m \text{ is consistent with } s \text{ and } E_i \downarrow s;
\end{cases}
\]

where \( \varepsilon > 0 \) is a small positive number.

The second transfer is a modified proper scoring rule as defined below.

First, define \( Q_j(p_{i,j}, E_j) = [2p_{i,j}(E_j) - p_{i,j} \cdot p_{i,j}] \). That is, \( Q_j(p_{i,j}, E_j) \) gives the value of the scoring rule if \( j \) presents the collection \( E_j \) and agent \( i \)'s report about \( j \) is \( p_{i,j} \). With this, define

\[
\tau_2^{i,i+1}(m) = \{-\tau, \text{ if } p_{i,i} \neq p_{i-1,i}, 0 \text{ otherwise} \}
\]

Every agent’s report about herself is cross-checked with the report about her by the agent to her left. If the reports are not in agreement, then the agent is fined \( \tau \).
The fourth transfer is related to lies which other agents can refute.

\[ \tau^4_i (m) = \begin{cases} -\bar{\tau}, & \text{if } p_{i,i+1} = p_{i+1}(s) \forall i, \text{ and } E_j \downarrow s \text{ for some } j \neq i \\ 0 & \text{otherwise.} \end{cases} \]

That is, an agent \( i \) is fined \( \bar{\tau} \) if another agent has refuted state \( s \) obtained from the vector of reports by agents about the agents to their right. Note that this transfer does not apply to an agent’s report about herself. Further, this transfer is inactive when the profile \( (p_{i,i+1})_{i \in I} \) is not consistent.

\( \tau^5 \) is as defined below.

\[ \tau^5_i (m) = \begin{cases} \varepsilon \times b_j(s',s_i)(E_j), & \text{where } j = \hat{i}(s',s_i) \text{ if } m \text{ is consistent with } s' \\ 0 & \text{otherwise.} \end{cases} \]

where \( \varepsilon > 0 \) is a small positive number as defined earlier. Note that \( \tau^1 \) and \( \tau^5 \) are scaled using the same number \( \varepsilon \).

That is, if agent \( i \) bets against a unanimous report consistent with state \( s' \) with a bet claiming state \( s_i \), the designer evaluates the challenge in accordance with the set \( B_j \), where \( j = \hat{i}(s',s_i) \) and the function \( b_j \) defined in Section 4.2 and pays out the revenue from the bet to agent \( i \).

4.4.4 Scaling

In this section, we will define some parameters used for appropriately scaling the transfers, and then establish that there are values for \( \bar{\tau}, \varepsilon, \) and \( \varepsilon \) so that the appropriate scaling is possible.

First, we provide an intuitive overview of the scaling based on the proof sketch in Section 4.3. Recall that we are working in a normalized setup in which the utility of the outcome has been normalized to a number less than 1. Since an arbitrarily small \( \varepsilon \) provides sufficient incentive for agents to submit evidence and make bets, we can choose \( \varepsilon \) small enough so that a penalty of 1 dollar is enough to dominate not only any change in outcome, but also any gains or losses from \( \tau^1 \) or \( \tau^5 \). The scoring rule \( \tau^2 \) is at the medium scale (\( \bar{\tau} \)) and dominates any effect from a change of outcome.\(^8\) The crosscheck \( \tau^3 \) is also at this

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\(^8\)This is, strictly speaking, larger than it needs to be, since \( \tau^2 \) needs only to dominate the transfers at
scale since similar considerations apply to the agent’s report about herself. Since we wish to prioritize the elimination of lies which others can refute above other incentive considerations, \( \tau^4 \) dominates all other transfers, so that it belongs to the highest transfer scale (\( \tau \)). We define the transfer scales formally below.

Since \( \tau^2 \) is a proper scoring rule, if an agent \( i + 1 \) presents all her evidence, agent \( i \) maximizes her revenues from \( \tau^2 \) by truthfully reporting agent \( i + 1 \)’s evidence distribution. We require that the minimum expected loss for an agent \( i \) from lying about agent \( i + 1 \)’s distribution dominates any effect from a change in the outcome. That is,

\[
\min_{s \in S} \min_{p_{i,j} \neq p_j(s)} \sum_{E_j \in E_j} p_j(s)(E_j) \tau[Q_j(p_j(s), E_j) - Q_j(p_{i,j}, E_j)] > 1 \quad (1)
\]

Turning to the crosscheck, it must dominate any incentive from the outcome. Since the minimum loss from lying about agent \( i + 1 \)’s distribution scales linearly in \( \tau \), we can choose \( \tau > 1 \) such that inequality (1) is satisfied.

Further, define as \( \bar{\tau}^2 \) the maximum possible change in utility from \( \tau^2 \). That is, i.e.

\[
\bar{\tau}^2 = \max_{i \in I} \max_{E_{i+1} \in E_{i+1}} \max_{p_{i,i+1}, p'_{i,i+1} \in P_{i+1}} \tau[Q_{i+1}(p_{i,i+1}, E_{i+1}) - Q_{i+1}(p'_{i,i+1}, E_{i+1})]
\]

Now we define the fourth scaling inequality as follows:

\[
\tau \geq \frac{1 + \bar{\tau}^2}{\min_{s' \in RL_i(s)} \{p_i(s)(E_i) : E_i \downarrow s'\}}, \forall s \in S, \forall i \in I. \quad (2)
\]

That is, \( \tau^4 \) provides sufficient incentive to avoid being caught lying even for the smallest probability with which refutable lies may be refuted by the evidence of other agents.

It is then clear that the parameters \( \varepsilon \), \( \tau \) and \( \bar{\tau} \) can be chosen to satisfy the above inequalities simultaneously.

### 4.5 Proof of Implementation

In what follows, we assume that the true state is \( s^* \) and that it induces the profile of evidence distributions \( (p^*_i)_{i \in I} \).

\( \varepsilon \) scale (as in setting her report about the agent to her right, an agent may cause (or break) consistency). However, we scale it at the level of \( \tau \) for notational parsimony.
4.5.1 Preliminary Results

**Lemma 2** If agent $i$ presents all her evidence in $E_i$, then each agent $j \in \mathcal{I}$ presents the truth in each message $p_{j,i}$.

**Proof.** If agent $i$ presents all her evidence in $E_i$, then for agent $i-1$, setting $p_{i-1,i} = p_i^*$ maximizes her payoff from $\tau^2$ (since truthful reporting is optimal under a scoring rule). However, this could also cause losses from $\tau^1$ (by creating consistency) and $\tau^5$ (by activating a bet via creating consistency). However, since $\tau > 1$ (by choice), $p_{i-1,i} = p_i^*$ is the best response for agent $i-1$ in equilibrium. Then, since $\tau > 1$, $p_{i,i} = p_i^*$ is the best response for agent $i$. ■

4.5.2 Consistency

**Claim 1** In any equilibrium, there is a state $s$ such that the message profile is consistent with $s$.

**Proof.** If all agents present all their evidence, then Lemma 2 yields that all agents present true distributional claims in their reports and thus the profile of reports must be consistent with the truth, so that $s = s^*$.

Now, suppose that there is inconsistency but not all agents present all their evidence. We claim that this is only possible if precisely one agent $i$ randomizes in her reports $p_{i,i}$ or $p_{i,i+1}$. To see this, notice that if two or more agents randomize in their distribution reports, then each type of each agent expects inconsistency with positive probability and presents all their evidence. So suppose that only agent $i$ mixes. Then in each message, each agent $j \neq i$ expects inconsistency with positive probability and presents all her evidence ($\varepsilon > 0$). Then, from Lemma 2, $p_{k,j}$ is the truth for all $k \in \mathcal{I}$ and $j \neq i$. Therefore, $i$ can only mix in $p_{i,i}$. However, by hypothesis, each agent $j \neq i$ presents a pure report in $p_{j,i}$. Therefore, in any message $m_i$ where $p_{i,i} \neq p_{j,i}$, she incurs a loss from $\tau^3$. Note that there are no losses from $\tau^4$ since agent $i$ is only changing $p_{i,i}$ and $\tau^4$ does not apply to an agent’s report about herself. Consider a deviation to match $p_{j,i}$. This avoids the loss from $\tau^4$, but may cause losses from $\tau^2$ (by creating consistency) and $\tau^5$ (by activating a bet via creating consistency). Since $\tau > 1$, this is a profitable deviation. Therefore, we have a contradiction. ■
4.5.3 Eliminating Refutable Lies

**Claim 2** There is no equilibrium where the distribution reports are consistent with a refutable lie.

**Proof.** Suppose instead that $m$ is consistent with a refutable lie $s$. If it is refutable by two or more agents, then $\tau^1$ provides the incentive to refute the lie, and so all agents expect the lie to be refuted with positive probability so that they present maximal evidence. In turn, Lemma 2 implies the profile must be consistent with the truth, a contradiction.

So suppose $s$ can only be refuted by some agent $i$. Then, all agents other than $i$ expect agent $i$ to refute $s$ with positive probability and therefore present maximal evidence. Thus, from Lemma 2, $p_{i,j} = p_{j}^* \forall j \neq i$. Then, $p_{i-1,i} \neq p_{i}^*$ (otherwise the profile would be consistent with the truth). Therefore, in presenting $p_{i-1,i}$, agent $i-1$ knows that this claim will be refuted with probability at least $\min_{E_i \in E_i} \{p_i(s^*)(E_i) : E_i \downarrow s\}$. However, if she deviates to $p_{i-1,i} = p_{i}^*$, the profile has $p_{i,i+1} = p_{i+1}(s^*) \forall i$ and therefore cannot be refuted. So, the agent gains $\tau$ from $\tau^4$. Also, the agent is deviating away from consistency, so that $\tau^1$ yields no losses. Since $\tau$ dominates all the other incentives (Inequality (2)), deviating to $p_{i}^*$ instead of presenting $p_{i-1,i}$ is a profitable deviation for agent $i$. □

4.5.4 No Bets in Equilibrium

**Claim 3** In any equilibrium, the distribution reports are not consistent with a non-refutable lie.

**Proof.** From Claim 1, the agents must all present pure distribution reports which are consistent with a state $s$. From Claim 2, $s$ is not refutable by any agent, since this would mean that an agent is presenting a lie which another agent can refute. If $f$ prescribes a different outcome at $s$ than at the truth, then there must be an agent $i$ such that $s^* \not\rightarrow_i s$, that is, agent $i$ does not have a perfect deception from the truth to $s$. Then, any agent $j \neq i$ has the incentive to bet with probability 1, since $\varepsilon > 0$, whereupon all evidence presentation is maximal (due to $\tau^1$) and Lemma 2 presents a contradiction. □

4.5.5 Implementation

**Claim 4** In any equilibrium, the outcome is $f(s^*)$ and there are no transfers.
Proof. From Claims 1-3, the report profile is consistent with the true state. Therefore, the outcome is \( f(s^*) \). Clearly, such a consensus is not refutable, since any collection available under the truth cannot (by definition) refute it.

Now, we claim that there are no bets in equilibrium. To see this, notice that if an agent \( i \) bets in equilibrium, then it must be that agents have not presented all their evidence in \( E \), since from Lemma 1, a bet yields a loss if placed against a truthful reporting strategy. If some agent \( i \) finds it profitable to place a bet in any of her messages, then she must find it profitable to bet in all of her messages. Then, each agent finds it optimal to present all their evidence in \( E \), a contradiction. Since the message profile is consistent, and features no bets in equilibrium, \( \tau^1, \tau^3 \) and \( \tau^5 \) are inactive as well.

Finally, notice that the absence of any inconsistency yields that each agent’s claims about herself must be the same as those of the agent to her left. That is, \( p_{i,i+1} = p_{i+1,i+1} \forall i \). Then, \( \tau^2 \) is inactive as well. Thus, there are no transfers. □

5 Pure-Strategy Bayesian Nash Implementation

In this section, we take a brief digression to discuss pure-strategy implementation. While NPD was shown to be necessary for mixed-strategy implementation, a weaker version, called No Pure-Perfect Deceptions (NPPD) suffices for pure-strategy implementation. It is essentially identical to NPD except for the additional requirement that any deception \( \alpha \) be degenerate, since playing a non-degenerate deception requires mixing by the agents. A pure-perfect deception is defined below.

**Definition 9** A deception \( \alpha = (\alpha_i)_{i \in I} \) is pure-perfect for state \( s' \) at state \( s \) if it is a perfect deception for \( s \) at \( s' \) and \( \alpha \) is degenerate.

We now define the No Pure-perfect deceptions condition.

**Definition 10** (*Condition NPPD - No Pure-Perfect Deceptions*) We say that a social choice function \( f \) satisfies NPPD whenever for two states \( s \) and \( s' \), if at \( s \), state \( s' \) is nonrefutable, and there exists a pure-perfect deception for \( s' \), then \( f(s) = f(s') \).

We now state the pure-strategy implementation result.
Theorem 2 A social choice function is implementable in pure-strategy BNE if and only if it satisfies NPPD.

The proof of this result is derived as a simplification of our mixed-strategy implementation result, and is relegated to Appendix A.3.

No Pure-perfect deceptions is shown to be strictly weaker than NPD and strictly stronger than Stochastic Measurability in Appendix A.2. Therefore, in our setup, mixed-strategy implementation is a strictly harder problem than pure-strategy implementation. We note that this distinguishes this setup from the complete-information setup in Ben-Porath and Lipman (2012) where mixed-strategy Nash Implementation does not require a stronger condition on the social choice function. However, it is consistent with the rest of the Bayesian implementation literature though, in that mixed-strategy Nash implementation in Serrano and Vohra (2010) requires Mixed Bayesian Monotonicity, which is stronger than Bayesian Monotonicity which is required for pure-strategy implementation (Jackson (1991)).

6 Implementation with a General Type Space

In this section, we study a more general model than the one considered above. We allow agents to have a type $t_i \in T_i$ (where $T_i$ is finite) and allow agents’ types to be correlated. A type profile $t$ determines agents’ preferences over the outcomes in a set $A$, so that $\bar{u}_i(a, t)$ is agent $i$’s bounded utility when the outcome is $a$ and the type profile is given by $t$. Further, $u_i(a, \bar{\tau}_i, t) = \bar{u}_i(a, t) + \bar{\tau}_i$ is the quasilinear extension of $\bar{u}_i$. The type profile $t$ also determines agents’ evidence, which is represented by $\hat{E}_i(t_i) \in \mathbb{E}_i$. States of the world are represented by type profiles $t = (t_i)_{i \in I}$. Each agent has a belief $q_i : T_i \rightarrow \Delta(T_{-i})$ where $T_{-i} = \Pi_{j \neq i} T_j$. Such a model is represented by $\mathcal{T} = (T_i, \hat{E}_i, q_i)_{\in I}$. A social choice function maps a state to the desirable outcome in that state, so that $f : T \rightarrow A$, where $T = \Pi_i T_i$. Once again, we are interested in implementing $f$ without relying on preference variation, so that we consider constant preferences as a possibility.

In this section, we will adapt the solution concept of Interim Correlated Rationalizability from Dekel, Fudenberg, and Morris (2007) to our setting. Consider a mechanism $\mathcal{M} = (M, g, \tau)$, which induces a static Bayesian game denoted by $G(\mathcal{M}, \mathcal{T}, u)$. Define the set of messages feasible for an agent $i$ of type $t_i$ by $M_{i,t_i}$. Denote by $R_i^G(t_i) \in$ the set of
rationalizable messages for agent $i$ of type $t_i$. Then, we have $R_{G,0}^i (t_i) = M_{i,t_i}$.

$$R_{G,i,k+1}^i (t_i) = \begin{cases} 
    m_i \in M_{i,t_i} & \text{there exists } \pi_i \in \Delta (T_{-i} \times M_{-i}) \text{ such that} \\
    (1) \pi_i (t_{-i}, m_{-i}) > 0 \Rightarrow m_{-i} \in \Pi_{j \neq i} R_{G,j,k}^j (t_j) \\
    (2) m_i \in \arg \max_{m'_i \in M_{i,t_i}} \sum_{E_{-i},m_{-i}} u_i (g (m'_i, m_{-i}), \tau_i (m'_i, m_{-i}), t) \\
    (3) \sum_{m_{-i}} \pi_i (t_{-i}, m_{-i}) = q_i (t_i) (t_{-i}) 
\end{cases}$$

and $R_i^G (t_i) = \cap_{k=1}^\infty R_{i,k}^G (t_i)$. Then, denote by $R^G (t) = \Pi_{i \in I} R_i^G (t_i)$ as the profile of all rationalizable messages.

With this solution concept, we formalize the implementation notion below.

**Definition 11** A social choice function $f$ is rationalizably implementable with arbitrarily small transfers if for any $\varepsilon > 0$, there is a mechanism $\mathcal{M} = (M, g, \tau)$ such that for any profile of bounded utility functions $\bar{u} = (\bar{u}_i)_{i \in I}$, any type profile $t$, and any profile of rationalizable messages $m \in R^G (t)$ of the game $G (\mathcal{M}, \mathcal{T}, u)$, we have $g (m) = f (t)$. Further, $|\tau_i (m)| \leq \varepsilon$ for every message profile $m$.

That is, given an arbitrarily small (strictly) positive number $\varepsilon$, it should be possible to obtain the socially optimal outcome at any profile of rationalizable messages, and limit the transfers for any message profile (rationalizable or otherwise) to being no higher than $\varepsilon$. This represents a relaxation of the prior requirement that the transfers should be zero in equilibrium, but yields a relaxation of the solution concept to rationalizability, allows correlation among types, and (we will show) yields a weaker implementing condition than either NPD or NPPD.

We now describe the agent’s higher order beliefs. Since we wish to implement without relying on preference variation, so that constant preferences are a possibility, we restrict agents to form beliefs only on evidence to obtain a necessary condition which is strong enough to allow implementation when preferences are constant. Note that the implementing mechanism we will present later will allow for preference variation, but will not rely on it for implementation. To this end, define $Z^i_0 = E_{-i}$, and for $k \geq 1$, define $Z^i_k = Z^{k-1}_i \times \Delta (Z^{k-1}_{-i})$. For each $i$, let $T^*_i$ denote the set of $i$’s collectively coherent hierarchies (Mertens and Zamir (1985)). For each $t_i \in T_i$, let $\hat{\pi}_i (t_i) = \hat{E}_i (t_i)$; construct mappings $\hat{\pi}_i^k : T_i \rightarrow \Delta (Z^{k-1}_i)$ recursively for all $i \in I$ and $k \geq 1$, such that $\hat{\pi}_i^1$ is the push-forward of $q_i (t_i)$ given by the map from $T_{-i}$ to $Z^0_{-i}$, such that
and $\hat{\pi}^k_i(t_i)$ is the push-forward of $q_i(t_i)$ given by the map from $T_{-i}$ to $Z_{-i}^{k-1}$ such that

$$t_{-i} \mapsto (\hat{\pi}_0^i(t_{-i}), \hat{\pi}_1^i(t_{-i}), \ldots, \hat{\pi}_k^i(t_{-i})),$$

where for any $k = 0, 1, 2, \ldots$, $\hat{\pi}_k^i(t_{-i}) = \Pi_{j \neq i} \hat{\pi}_k^j(t_j)$.

The mappings thus constructed, i.e. $\hat{\pi}^*_i(t_i) = (\hat{\pi}_0^i(t_i), \hat{\pi}_1^i(t_i), \ldots)$ assign to each type a hierarchy of beliefs. We now propose a new implementing condition which we term higher-order measurability.

**Definition 12** A social choice function $f$ satisfies higher-order measurability if for any pair of states $t$ and $t'$ such that $f(t) \neq f(t')$, there is $k \in \mathbb{N}$ such that there is an agent $i$ whose $k^{th}$ order belief $\hat{\pi}_k^i(t_i) \neq \hat{\pi}_k^i(t'_i)$.

The necessity of higher-order measurability is demonstrated in Appendix A.4. Intuitively, higher-order measurability requires that the social choice function $f$ be measurable on a partition of $T$ where each cell of the partition is such that if $t$ and $t'$ belong to the same cell, then each agent has the same belief hierarchy in both $t$ and $t'$.

We now consider the setup we work with for Bayesian Nash implementation. Suppose an SCF $f$ satisfies stochastic measurability, so that if $f(s) \neq f(s')$, then there is an agent $i$ such that $p_i(s) \neq p_i(s')$. Then, the first order belief of any agent $j \neq i$ is different between $s$ and $s'$, so that $f$ must satisfy higher-order measurability. Conversely, if there are two states $s$ and $s'$ so that for each agent $i$, $p_i(s) = p_i(s')$, and evidence distributions are independent, then $f(s)$ must be identical to $f(s')$ if $f$ is to satisfy higher-order measurability. Therefore, in such a setup, higher-order measurability is identical to stochastic measurability, and therefore weaker than either NPD or NPPD.

Towards establishing that higher-order measurability is also sufficient for implementation with the above implementation notion, we will define the relevant version of incentive compatibility in this setting. To do so, consider the direct mechanism with respect to $\mathcal{T} = (T_i, \tilde{E}_i, q_i)_{i \in \mathcal{I}}$, where each agent $i$ makes a report $t_i \in T_i$ and the outcome chosen is $f((t_i)_{i \in \mathcal{I}})$. No transfers are induced.
Definition 13 A social choice function $f$ satisfies evidence incentive compatibility if for each agent $i$, and each type $t_i \in T_i$,

$$t_i \in \arg \max_{t_i' \in \hat{T}_i(t_i)} \sum_{t_{-i} \in T_{-i}} q_i(t_i)(t_{-i}) \bar{u}_i(f(t_i', t_{-i}), (t_i, t_{-i}))$$

where $\hat{T}_i(t_i) = \{t_i' : \hat{E}_i(t_i') \subseteq \hat{E}_i(t_i)\}$.

That is, if agent $i$ is endowed with type $t_i$, and anticipates every other agent to report their type truthfully, then it is among her best responses to also report truthfully. This is consistent with Deneckere and Severinov (2008), who show that if agents can send every combination of their available messages, then the above condition is necessary and sufficient for truthtelling to be a Bayesian Nash equilibrium. Theorems 1 and 2, however, do not require this condition since the use of large transfers causes it to be automatically satisfied. Now, we present the main result of this section.

Theorem 3 A social choice function is rationalizable implementable with arbitrarily small transfers iff it satisfies higher-order measurability and evidence incentive compatibility.

The sufficiency of higher-order measurability is demonstrated through the construction of an implementing mechanism which is presented in Appendix A.4. Here, we describe the intuition behind the construction, which follows ideas from Abreu and Matsushima (1992). The first step is to use a small incentive to extract the evidence from each agent. Irrespective of the other reports they make, it will be a strictly dominant strategy to present all their evidence. In the second step, we ask each agent to predict the evidence presentation of the other agents and score their predictions using a quadratic scoring rule. The input to the scoring rule is the agent’s own type, since common knowledge of the type allows the designer to compute the posterior distribution $q_i(t_i)$ and score the prediction appropriately. Since each agent expects every other agent will have presented all their evidence, she presents her true belief about the evidence presentations of the other agent in order to maximize the returns from this scoring rule. This elicits the other agent’s true first order belief. In the third step, we extract the agents true second order belief by asking them to predict the first order beliefs extracted in the previous step and scoring their predictions using a quadratic scoring rule. Higher-order measurability assures us that if we repeat this process enough times, we will have extracted the true types of the agents. This mechanism performs the
role of the *dictatorial choice function* in Abreu and Matsushima (1992). Each transfer here can be made small enough so that the overall transfer can be bounded below any preselected $\varepsilon$. In what follows, we refer to this extracted profile of types as $t^0$ and to agent $i$’s component of $t^0$ as $t^0_i$.

After achieving extraction, we proceed towards the goal of implementing the social choice function. The steps taken are essentially identical to those in Abreu and Matsushima (1992). Agents are asked for $J$ additional reports of their type, denoted $(t^j_i)_{j=1,2,\ldots,J}$ in what follows. Each profile of reports determines the outcome with a probability of $\frac{1}{J}$, but the following penalties are applied. To the first agent $i$ who deviates in their report $t^j_i$ from $t^0_i$, a penalty is applied which is large enough to dominate the incentive from affecting the outcome with a probability of $\frac{1}{J}$ but not so large as to incentivize lying in $t^0_i$. That such a balance can be achieved can be deduced from the fact that the designer can choose $J$ to be as large as required to reduce the incentive to affect the outcome. Then, no agent wants to be the first to deviate from the truthful report in $t^0$ and implementation obtains while the penalties remain switched off.

This mechanism however inherits some of the well known critiques of the Abreu-Matsushima type of mechanisms, most notably that it requires a high depth of reasoning, involves a fairly complex message space, and is more vulnerable to renegotiation (than say the mechanism proposed in the proof of Theorem 1) owing to the use of small transfers.

7 Related Literature

In this section, we discuss our paper’s results alongside some other related papers, to provide further insight into our contributions.

An early reference in the field of mechanism design with evidence is Green and Laffont (1986) who study the principal-agent problem when the agent cannot manipulate the truth arbitrarily. This corresponds to a notion of evidence - an agent, by presenting messages which are only available to her in a certain set of states, can prove that the true state is within that set. Early contributions in the field of game theory and evidence include Milgrom (1981), Grossman (1981) and Dye (1985). More recent papers involving mechanism design include Glazer and Rubinstein (2004), Glazer and Rubinstein (2006), Bull and Watson (2007), Deneckere and Severinov (2008), Hart, Kremer, and Perry (2017), and Ben-Porath,
Ben-Porath, Dekel, and Lipman (2019). Ben-Porath, Dekel, and Lipman (2021) study a setup involving stochastic evidence. While we assume that agents are endowed with evidence to begin with, Ben-Porath, Dekel, and Lipman (2021) study the process of evidence acquisition and the value of commitment. In addition, they focus on partial implementation whereas our focus is on full implementation.

Ben-Porath and Lipman (2012), Kartik and Tercieux (2012), and Banerjee, Chen, and Sun (2021) study full implementation with hard evidence as well. While Ben-Porath and Lipman (2012) and Banerjee, Chen, and Sun (2021) explicitly study hard evidence, Kartik and Tercieux (2012) model hard evidence within their more general costly evidence framework when articles of evidence are constrained to be either costless or infinitely costly. Specifically, in Banerjee, Chen, and Sun (2021) we provide a direct mechanism to fully implement SCFs in mixed strategies with two or more agents. The chief addition to this line of inquiry in Ben-Porath and Lipman (2012) and Banerjee, Chen, and Sun (2021) that the present paper contributes is uncertainty in the dimension of evidence. We are able to provide a finite mechanism that fully implements in mixed-strategy equilibria, but need an indirect mechanism to achieve this goal.\(^9\) Restricting attention to direct mechanisms, Peralta (2019) establishes necessary and sufficient conditions for full Bayesian implementation with uncertain evidence; however, he also points out that the restriction to direct mechanisms entails loss of implementability.

Turning away from evidence, Jackson (1991) is a seminal paper in the theory of Bayesian implementation, focusing on exploiting preference variation under incomplete information to implement SCFs. While focusing on pure strategies, it suggests that looking into mixed strategies is important. Serrano and Vohra (2010) indeed extend the analysis in Jackson (1991) to mixed strategies. The mechanisms in Jackson (1991) use modulo games and those in Serrano and Vohra (2010) use integer games. In our setup, preference variation does not necessarily exist and yet our implementation notion is robust to preference variation. We present in this paper, to our knowledge, the first mechanism in the literature that achieves implementation in a finite mechanism in a Bayesian setting which explicitly accounts for mixed-strategy equilibrium.

\(^9\)In a classical setting with preference variation and no evidence, Chen, Kunimoto, Sun, and Xiong (2022) also provides a finite (indirect) mechanism for mixed-strategy implementation. It focuses on a complete-information setting however.
A Appendix

A.1 Connection to Ben-Porath and Lipman (2012)

Ben-Porath and Lipman (2012) use a notion of nomenclature to associate articles of evidence with subsets of the state space. In essence, an article of evidence is associated with the set of states in which it occurs. A natural generalization to the setting we study here would be to associate an article of evidence with the set of states in which it occurs with positive probability. When probability distributions are degenerate (so that we have a deterministic evidence setting), it is clear that only one collection may occur with positive (and indeed unit) probability. We denote this collection by $E_i(s)$. With this notion of nomenclature, we state the analogous conditions from Ben-Porath and Lipman (2012) below.

1. (e1) $\forall e_i \in E_i(s), s \in e_i$
2. (e2) $e_i \in E_i(s) \implies e_i \in E_i(s')$ for every $s' \in e_i$.

The following proposition proves that (se1) and (se2) are indeed generalizations of (e1) and (e2).

**Proposition 1** If $p$ is degenerate, then (se1) and (se2) are identical to (e1) and (e2) respectively.

**Proof.** First, consider some $e_i$ such that $s \notin e_i$. Then, any collection $E_i$ containing $e_i$ refutes $s$. From se1 therefore, $p_i(s)(E_i) = 0$ for any $E_i \ni e_i$. Then, $E_i \neq E_i(s)$, whereupon we have obtained that if $s \notin e_i$, then $e_i \notin E_i(s)$, which is (e1). Thus, (se1) implies (e1).

Next, suppose $E_i \downarrow s$. Then, $\exists e_i \in E_i$ such that $s \notin e_i$. From (e1), $e_i \notin E_i(s)$, so that $p_i(s)(E_i) = 0$, which establishes (se1). Therefore, (e1) implies (se1) as well.

Now, we turn to (se2). Consider a pair of states $s$ and a collection $E_i$ such that $E_i \not\subseteq s'$. Notice that under degenerate probabilities, $p_i(s')(E_i') > 0 \implies E_i' = E_i(s')$ and $p_i(s)(E_i) > 0 \implies E_i = E_i(s)$, so that (se2) translates to the following: If $\forall e_i \in E_i(s)$, if $s' \in e_i$, then $E_i(s') \supseteq E_i(s)$. Now, consider any $e_i \in E_i(s)$ such that $s' \in e_i$. Then, $E_i(s') \supseteq E_i(s) \implies e_i \in E_i(s')$, which establishes (e2).

Finally, suppose $E_i \not\subseteq s'$ and $p_i(s)(E_i) > 0$. Under degenerate probabilities, this yields $E_i(s) \not\subseteq s'$, so that $\forall e_i \in E_i(s), s' \in e_i$. From (e2), $e_i \in E_i(s')$. Since this is true for all
we have $E_i(s') \supseteq E_i(s)$, which is identical to $(se2)$ under degenerate probabilities. We have, therefore established that $(se1)$ and $(se2)$ correspond to $(e1)$ and $(e2)$ under degenerate probabilities.

Next, we consider the following question: Since articles can be named according to the subset of states in which they occur with positive probability, does it suffice to allow an agent to reveal a “most informative” article of evidence rather than presenting an entire collection? More precisely, consider the following example evidence structure:

| State/Agent | A                        | B                        |
|-------------|--------------------------|--------------------------|
| U           | $\{\{L, M, H, U\}\}$    | $\{\{L, M, H, U\}\}$    |
| H           | $0.3 \times \{\{L, M, H, U\}, \{M, H\}\} + 0.7 \times \{\{L, M, H\}, \{M, H\}\}$ | $\{\{L, M, H, U\}\}$    |
| M           | $0.5 \times \{\{L, M, H, U\}, \{M, H\}\} + 0.5 \times \{\{L, M, H\}, \{M, H\}\}$ | $\{\{L, M, H, U\}\}$    |
| L           | $\{\{L, M, H\}, \{L, M, H, U\}\}$ | $\{\{L, M, H\}\}$    |

In both states $M$ and $H$, agent $A$, the only agent with evidence variation always has the article $\{M, H\}$ and this is the most informative article which is available to her. If we were to only allow her to present one article of evidence, then there would be a perfect deception in both directions between the states $M$ and $H$ so that we would not be able to distinguish them using any mechanism. However, if we were to allow the submission of entire collections of evidence, then there is no perfect deception from state $M$ to $H$ (the article $\{L, M, H\}$ is not available often enough) and no perfect deception from state $H$ to $M$ (the article $\{L, M, H, U\}$ is not available often enough). Therefore, we can distinguish between these states. Thus, there is a loss of generality when restricting attention to the most informative article of evidence. In a deterministic evidence setting though, this restriction is without loss of generality, as can be seen in both Ben-Porath and Lipman (2012) and Banerjee, Chen, and Sun (2021).

### A.2 NPPD vs NPD and Stochastic Measurability

In this section, we demonstrate that NPPD is strictly stronger than Stochastic Measurability and strictly weaker than NPD. To see that NPPD is strictly stronger than Stochastic Measurability, consider the example below.
Clearly, $U$ is nonrefutable at $H$. Consider the deception at state $H$ given by

$$\alpha_A(H, \{\{L, M, H, U\}, \{M, H, U\}\}) = (U, \{\{L, M, H, U\}\});$$

$$\alpha_A(H, \{\{L, M, H, U\}, \{M, H, U\}, \{H, U\}\}) = (U, \{\{L, M, H, U\}, \{M, H, U\}, \{H, U\}\}).$$

While the SCF satisfies stochastic measurability, this is a pure-perfect deception.

To see that NPPD is strictly weaker than NPD, we refer back to the example in Section 2. We reproduce the evidence structure below:

| State | Evidence (Firm A) |
|-------|-------------------|
| $H$   | $0.6 \times \{\{M, H\}, \{L, M, H\}\} + 0.4 \times \{\{L, M, H\}\}$ |
| $M$   | $0.4 \times \{\{M, H\}, \{L, M, H\}\} + 0.6 \times \{\{L, M, H\}\}$ |
| $L$   | $\{\{L, M, H\}\}$ |

It is easy to see that there is no pure-perfect deception in the above example - at $H$, the type $\{\{M, H\}, \{L, M, H\}\}$ must present all their evidence, as type $\{\{L, M, H\}\}$ does not have enough evidence to mimic $\{\{M, H\}, \{L, M, H\}\}$ at $M$, but then we induce too much probability (0.6 instead of 0.4) on $\{\{M, H\}, \{L, M, H\}\}$. This demonstrates that NPPD is strictly weaker than NPD.

### A.3 Proof of Theorem 2

The necessity of NPPD stems from arguments similar to those for the proof of necessity of NPD for mixed implementation. Sufficiency is established by referring to the mechanism used in the proof of Theorem 1. With the exception of the bets, the rest of the mechanism remains identical. Note that Claims 1 and 2 make no use of either the condition NPD or nontrivial...
randomization by agents. Further, Claims 1 and 2 imply that there is no equilibrium where agents present refutable lies. This leaves us with profiles consistent with nonrefutable lies.

So suppose agents present a profile consistent with a nonrefutable lie $s'$. In contrast to the arguments in Section 4.2, where it was sufficient for an agent to inform the designer of the true state, in this paradigm, depending on the strategy profile, it is possible for different evidence collections to be low or high in probability relative to the expected probabilities in state $s'$. Therefore, it is impossible for the designer to place an appropriate bet if she is informed only of the true state. Since $S$ and $E$ are finite, the number of pure deceptions is finite, since any pure deception profile is a map $\alpha : S \times E \rightarrow S \times E$. Therefore, they can be identified with numbers from 1 through $Z$ where $Z$ is the total number of possible pure deceptions.

We augment the message space of the base mechanism by adding the strategy identifier $z \in (1, 2, \ldots, Z)$. Once informed of the deception identifier, the designer can identify an agent $i$ and a pair of collections $E_i'$ and $E_i''$ such that $p_{\alpha_i(s)}(E_i') < p_i(s')(E_i')$ and $p_{\alpha_i(s)}(E_i'') > p_i(s')(E_i'')$ where $\alpha_i$ is the pure deception being played by agent $i$. Then, there are numbers $\gamma < 0$ and $\delta > 0$ such that (i) $\gamma \times p_{\alpha_i(s)}(E_i') + \delta \times p_{\alpha_i(s)}(E_i'') > 0$ and (ii) $\gamma \times p(s')(E_i') + \delta \times p(s')(E_i'') > 0$. These numbers $\gamma$ and $\delta$ form the bets in this setting. It is profitable to place a bet because of inequality (i). The rest of the analysis remains the same as earlier, in that the bet induces evidence submission and triggers a realignment through Lemma 2. Therefore, agents must present truthful reports and we obtain implementation. Note that there are no bets in equilibrium as betting against the truth yields losses (from inequality (ii)).

### A.4 Proof of Theorem 3

The reader will recall that higher-order measurability requires that the social choice function $f$ be measurable on a partition of $T$ where each cell of the partition is such that if $t$ and $t'$ belong to the same cell, then each agent has the same belief hierarchy in both $t$ and $t'$. We will show that higher-order measurability is necessary for rationalizable implementation with arbitrarily small transfers regardless of the mechanism in use. Accordingly, fix a mechanism $\mathcal{M} = (M, g, \tau)$.

A general type space model (see for instance Penta (2012)) is defined as $\mathcal{T} = (T_i, \hat{\theta}_i, q_i)_{i \in I}$. The function $\hat{\theta}_i$ maps each type to a payoff type. A payoff type consists of any constituent
component of an agent’s type which affects how she evaluates an allocation. In general, such models assume that all the actions are available to the agent in each state. In our case though, evidence is involved, which may vary in its availability with the state. Recalling that we work with utilities which are bounded by 1 dollar, we associate the following cost structure with evidence in order to model evidence availability. Suppose the mechanism \( M \) induces a maximum transfer of \( \bar{U} \). If an agent \( i \) of type \( t_i \) presents only such articles as are available to her in order to obtain an allocation \( a \) and transfer \( \tau_i \), then her utility is given by \( u_i(a, \tau_i, t_i) \). If an agent “presents” an unavailable article of evidence to do the same, her utility is given by \( u_i(a, \tau_i, t_i) - 2\bar{U} - 1 \). Since it is never worthwhile to obtain an outcome if it involves a \( 2\bar{U} - 1 \) dollar cost in order to do so, an agent never does so, so that we have effectively modelled evidence availability. Further, we avoid depending on preference variation for implementation, so that \( u_i(a, \tau_i, t_i) = u_i(a, \tau_i, t_i') \forall t, t' \in T \), i.e. preferences are constant. With these two restrictions then, only evidence is payoff relevant, so that in our case, \( \hat{\theta}_i : T_i \rightarrow E_i \), so that \( \hat{\theta}_i(t_i) = \hat{E}_i(t_i) \). Then, a general type space model yields our model \( \mathcal{T} = (T_i, \hat{E}_i, q_i)_{i \in I} \).

That higher-order measurability is necessary for implementation in rationalizable strategies in then immediate from Proposition 2 in Penta (2012) - if the set of rationalizable strategies is identical for each type of each agent, then different outcomes cannot be achieved in different states. Alternately, any profile of strategies that is rationalizable at \( t \) is also rationalizable at \( t' \) if \( \hat{\pi}_i^k(t_i) = \hat{\pi}_i^k(t_i') \) for all \( k \).

Now, we establish that the setup we worked with earlier can be obtained as a (further) special case of the model in Section 6. First, we set \( T_i = S \times E_i \) so that the type of each agent is defined. \( \hat{E}_i(s, E_i) \) is therefore simply given by \( E_i \). Note that \( p \) was defined as a prior in the previous setting, whereas here \( q_i \) represents a posterior belief once an agent is informed of her type. Since evidence is uncorrelated, we have

\[
q_i(s_i, E_i)((s_j, E_j))_{j \neq i} = \begin{cases} 
\Pi_{j \neq i} p_j(s)(E_j), & \text{if } s = s_i = s_j \forall j \neq i \\
0 & \text{otherwise.}
\end{cases}
\]

The map \( \hat{\theta}_i \) now maps \( S \times E_i \) to \( E_i \) as we continue to assert constant preferences and with the aforementioned cost structure, only evidence is payoff relevant. The social choice
function is defined as follows.

\[
f((s_i,E_i)_{i \in I}) = \begin{cases} 
  f(s), & \text{if } s = s_i \forall i \in I \\
  f(s_1) & \text{otherwise.}
\end{cases}
\]

This definition does not lead to difficulties since only such profiles occur with positive probability where \( s_i = s \) for all \( i \). Therefore, we have derived the previous setup as a simplification of this one.

Now we will prove that evidence incentive compatibility is necessary for rationalizable implementation with arbitrarily small transfers. If \( f \) is rationalizable implementable with arbitrarily small transfers, there is a mechanism \( \mathcal{M} = (M, g, \tau) \) which implements \( f \) such that for any profile of rationalizable messages \( m \in R^G(t) \) of the game \( G(\mathcal{M}, \mathcal{T}, u) \), we have \( g(m) = f(t) \). Further, \( \mathcal{M} \) can be chosen so that \( |\tau_i(m)| \leq \frac{\varepsilon}{2} \) for every message profile \( m \). The game \( G(\mathcal{M}, \mathcal{T}, u) \) must also have a Bayesian Nash equilibrium. Such an equilibrium also must have outcome \( f(t) \) (transfers notwithstanding), since every Nash Equilibrium survives the iterated elimination of strictly dominated strategies and thus constitutes a rationalizable message profile. Suppose this equilibrium is given by \( \sigma \). Consider an alternate mechanism \( \mathcal{M}' \) where agents merely report their type \( t_i \) including the evidence and the designer plays their strategy from \( \sigma \) on their behalf. The outcome and transfers are chosen to be identical to those induced by \( \mathcal{M} \). Such a mechanism must have a truth-telling equilibrium, say \( \sigma' \). That is, for each agent \( i \), and each type \( t_i \), this type must have the incentive to truthfully report her type \( t_i \) if she knows others are reporting their type accurately. That is,

\[
\sum_{t_{-i} \in T_{-i}} q_i(t_i)(t_{-i})u_i(f(t_i, t_{-i}), \tau_i(t_i, t_{-i}), (t_i, t_{-i})) \geq \sum_{t_{-i} \in T_{-i}} q_i(t_i)(t_{-i})u_i(f(t_i', t_{-i}), \tau_i(t_i', t_{-i}), (t_i, t_{-i})).
\]

The above must be satisfied with for all \( \varepsilon > 0 \). Then, taking limits with \( \varepsilon \to 0 \), we obtain

\[
\sum_{t_{-i} \in T_{-i}} q_i(t_i)(t_{-i})\bar{u}_i(f(t_i, t_{-i}), (t_i, t_{-i})) \geq \sum_{t_{-i} \in T_{-i}} q_i(t_i)(t_{-i})\bar{u}_i(f(t_i', t_{-i}), (t_i', t_{-i})),
\]

which shows that \( f \) satisfies evidence incentive compatibility.

The sufficiency proof is by construction of an implementing mechanism. Consider the following mechanism:
**Message Space:** $M_i = \mathbb{E}_i \times T_i \times T_i \times ... \times T_i$ (\(k + J + 1\)) times, where \(k\) is the minimum order of belief at which any pair of states with different \(f\)-optimal outcomes can be separated by some agent. A typical message for agent \(i\) is denoted by \(m_i = (E_i, (t_i^{0,k})_{k=1}^{\bar{k}+1}, (t_i^j)_{j=1}^J)\). Accordingly, \(m = (m_i)_{i \in I}\) is a profile of messages.

**Outcome:** The outcome is given by \(\frac{1}{J} \sum_{j=1}^J f(t^j)\).

**Transfers:**

The mechanism uses the following transfers, which we document below. Let \(\beta\) is a small positive number. Define

\[
\tau_i^1(m) = \beta \times |E_i|.
\]

This transfer incentivizes agents to submit all their evidence. Further, define

\[
\tau_i^{2,1}(m) = \beta \times [2 \hat{\pi}^1_i(t_i)(E_{-i}) - \hat{\pi}^1_i(t_i) \cdot \hat{\pi}^1_i(t_i)];
\]

\[
\tau_i^{2,k}(m) = \beta \times [2 \hat{\pi}^k_i(t_i)(E_{-i}) - \hat{\pi}^{k-1}_i(t_{-i}) \cdot \hat{\pi}^{k-2}_i(t_{-i}) \cdot \ldots \cdot \hat{\pi}^0_i(t_{-i}) - \hat{\pi}^k_i(t_i) \cdot \hat{\pi}^k_i(t_i) \cdot \hat{\pi}^k_i(t_i)], \forall k = 2, ..., \bar{k}+1.
\]

This transfer is a set of proper scoring rules which incentivizes agents to predict other agent’s lower order beliefs and their types using the report \(t_i^k\). Given the above transfer definitions, \(\beta\) can be chosen to be as small as required to meet any prescribed transfer bound \(\varepsilon\). This is critical, since these transfers are active for any profile of rationalizable strategies.

From the above transfers, it is easy to see that the strategy that maximizes agent \(i\)’s utility from \(\tau^1\) and \(\tau^2\) is to present all her evidence and report truthfully in each \(t_i^{0,k}\).

Given \(\beta\), define as \(\bar{\beta}_i\) the minimum loss in utility over any type profile that agent \(i\) endowed with any evidence can induce owing to \(\tau^1\) and \(\tau^2\) by changing either her evidence report or her first \(\bar{k} + 1\) type reports to be different from the truth.

Now, we define the transfers used to discipline the last \(J\) reports, which are used to set the outcome. The following transfer penalizes an agent for being among the first deviants from \(t_i^{0,k+1}\).

\[
\tau_i^3(m) = \begin{cases} 
-\bar{\tau}^3, & \text{if } \exists j \in \{1, ..., J\} \text{ s.t. } t_i^j \neq t_i^{k+1} \text{ and } t_i^h = t_i^{0,k+1} \forall h < j; \\
0 & \text{otherwise.}
\end{cases}
\]

Finally, we define the following check between an agent’s report \(t_i^{0,k+1}\) and \(t_i^j\).
\[ \tau_{ij}^4(m) = \begin{cases} -\bar{\tau}_4, & \text{if } t_i^{0,k+1} \neq t_i^j; \\ 0, & \text{otherwise}. \end{cases} \]

where \( \bar{\tau}_4 > 0 \). This transfer is repeated for every round.

We scale the transfers as follows. First, notice that for any given \( \varepsilon, \beta > 0 \) can be chosen so that the overall value of the transfers \( \tau^1 \) and \( \tau^2 \) does not exceed \( \varepsilon \). This fixes each \( \bar{\beta}_i \), and thereby \( \min_i \bar{\beta}_i \). We now choose a large enough number of rounds \( (J) \) such that \( \min_i \bar{\beta}_i > \frac{1}{J} \). This allows us to choose \( \bar{\tau}_3 \) such that \( \min_i \bar{\beta}_i > \bar{\tau}_3 > \frac{1}{J} \). Now, since \( \bar{\tau}_4 \) is only required to be positive, a sufficiently small value can be chosen so that \( \min_i \bar{\beta}_i > \bar{\tau}_3 + J\bar{\tau}_4 > \frac{1}{J} \).

Now, we will prove that this mechanism implements \( f \) in rationalizable strategies with arbitrarily small transfers. In the following proof, we denote the true state by \( t^* = (t^*_i)_{i \in I} \).

**Claim 5** Each agent presents all her evidence, i.e. \( \hat{E}_i(t^*_i) \).

**Proof.** If not, the agent can improve her payoff by providing additional articles of evidence due to the reward from \( \tau^1 \). There are no other incentives which are affected by the provision of additional evidence. ■

**Claim 6** Each agent \( i \) only presents such reports in \( (t_i^{0,k+1})_{k=1}^{k+1} \) which induce the same \( k^{th} \) order beliefs as \( t_i \) for any \( k \leq \bar{k} + 1 \). Further, \( f(t_i^{0,k+1}) = f(t) \).

**Proof.** From Claim 5, each agent presents all her evidence. Any report \( t_i^{0,k} \) for \( k < \bar{k} + 1 \), only controls the agent’s payoff from \( \tau^2 \) and therefore the only rationalizable report for any agent is to present a report \( t_i^{0,k} \) which induces the same belief as the agent’s true type \( t_i \). Any other report leads to a strictly lower payoff because \( \tau^2 \) is a quadratic scoring rule and yields a unique maximum.

Consider the agent’s report \( t_i^{0,k+1} \). It controls the agent’s payoff from \( \tau^2, \tau^3, \) and \( \tau^4 \). Changing \( t_i^{0,k+1} \) leads to a loss of at least \( \min_i \bar{\beta}_i \), which by construction dominates gains from \( \tau^3 \) and \( \tau^4 \) (recall that \( \min_i \bar{\beta}_i > J\bar{\tau}_4 + \bar{\tau}_3 \)), so that it is not profitable to lie in \( t_i^{0,k+1} \). Higher-order measurability then yields that the reports in \( t^{0,k+1} \) yield the correct cell of the partition of \( T \) to which \( t \) belongs. ■

**Claim 7** For any \( j \), \( t_i^j \) is the truth. Further, the mechanism implements and for any profile of rationalizable messages \( m \), \( \tau_i^3(m) = 0 \) and \( \tau_i^4(m) = 0 \) for all \( i \).
Proof. The proof is by induction. Consider any report \( t^j_i, j \geq 1 \) and suppose agents report truthfully all the way up to the \((j - 1)\)th message. If all agents \( k \neq i \) report truthfully in \( t^j_i \), then evidence incentive compatibility ensures that truth-telling is among the best responses for agent \( i \). The small penalty from \( \tau^4 \) makes it the unique best response since from Claim 6, \( t^{0,k+1} \) is true. If other agents \( k \neq i \) also lie in \( t^j_i \), then the truth yields the agent a profit of \( \bar{\tau}^3 \) from \( \tau^3 \), which dominates any gains from affecting the outcome with a probability \( \frac{1}{J} \), since \( \bar{\tau}^3 > \frac{1}{J} \). Therefore, such a message is not rationalizable. Therefore, it is rationalizable to present the truth in \( t^j_i \). Then, the transfers \( \tau^3 \) and \( \tau^4 \) are clearly inactive. Further, if every \( t^j_i \) is true, then the outcome is chosen correctly. ■

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