Statistics of delta peaks in the spectral density of large random trees

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Abstract

We present an analysis of the spectral density of the adjacency matrix of large random trees. We show that there is an infinity of delta peaks at all real numbers which are eigenvalues of finite trees. By exact enumerations and Monte-Carlo simulations, we have numerical estimations of the heights of peaks. In the large tree limit, the sum of their heights is \(0.19173 \pm 0.00005\). Moreover all associated eigenvectors are strictly localized on a finite number of nodes. The rest of the spectral density is a function which vanishes at all positions of peaks, which are a dense subset of real numbers: so this function is almost everywhere discontinuous.

Keywords: random tree, spectral density, density of states, adjacency matrix, localization, delta peak.

1 Introduction

Many models of graphs have been investigated in the last decades. This interest is mainly motivated by their numerous areas of applications in physics (propagation and percolation in disordered media, quantum gravity, etc.) and in other branches of science: combinatorial optimization, queuing theory, computers networks, interactions between biological molecules, quantum chemistry and many others. But their study by methods and concepts of statistical mechanics is more recent. The point of view of statistical physicists is generally different from the one of mathematicians, in particular by studying averaged quantities in the “thermodynamic” limit of infinite graphs rather than specific quantities to a given finite graph. Moreover some results have been obtained by Monte-Carlo methods, which are often regarded as “heretical” by mathematicians who prefer exact and proved results.

We can describe a graph as a set of \(n\) nodes (or vertices) with interactions between pairs of nodes, represented by a \(n \times n\) matrix. Then we are naturally interested by the spectrum of this adjacency matrix. See for example the book [Cvetkovic and al., 1995] for the theory and applications of these spectra. Many works have been done about the singularities of the spectral density and their connections with the localization of eigenvectors, also called quantum percolation [Evangelou, 1983; Mirlin, 2000; Bauer and Golinelli, 2001; id. 2001a]. The spectrum of the Laplacian matrix has also been studied [Biroli and Monasson, 1999].

In particular, it is known [Kirkpatrick and Eggarter, 1972] that strictly localized eigenvectors (i.e. with a finite number of non-zero coordinates) can appear on finite parts of large connected graph. Moreover the associated eigenvalues contribute to delta peaks in the spectral density. But the strict localization can occur only on parts of graph with special patterns. The motivation of our work is to study quantitatively this phenomenon for large (i.e in the infinite size limit) connected graphs.

As we have no reasons to choose a particular graph, we turn to models of random graphs. Of course, quantitative results depend on the particular choice of the model. Our first idea is to study the famous Erdős-Rényi model of random graph [Erdős and Rényi, 1960] but it presents a default: a graph on \(n\) nodes consists (in the high connectivity phase) of a giant connected component with \(O(n)\) nodes on which strict localization can occur, plus a number \(O(n)\) of finite connected components on which eigenvectors are always strictly localized. As we are not interested by these ones, we must consider either a modified Erdős-Rényi model by keeping only the giant component, or an other model.

We prefer the random labeled tree model described in Sect. 2, motivated by the following considerations. Firstly large Erdős-Rényi graphs on \(n\) nodes have circuits (or loops) with length \(O(\ln n)\), so they have locally a tree...
shape. As the strict localization occurs statistically on small number of nodes, it is little sensitive to circuits. Furthermore by definition a tree is always connected and we dismiss the problems due to small components. And finally a random tree is simpler to generate than other models of connected graphs. In this work, the interaction matrix is the adjacency matrix of the tree: it is the Hamiltonian of a particle which hops at each time step from one node to a connected node. Remark that it is different to the Laplacian matrix which describes the diffusion of a particle with continuous time.

As analytical methods give only partial results, we use two numerical methods: exact enumeration of small trees, described in Sect. 3.1 and Monte-Carlo simulations for large trees, described in Sect. 3.2. Qualitative description of the spectral density is done in Sect. 4: existence of an infinity of delta peaks is proved in Sect. 5.1 and their statistics is given in Sect. 5.2. Appendix A contains details about the use of symmetry for the numerical computation of the spectra. In Appendix B, we show that the set of eigenvalues of finite trees are a dense subset of the real numbers. Finally Appendix C explains the difficulties encountered to extend analytical results obtained in a previous work [Bauer and Golinelli, 2000].

2 Definitions and generalities

In this article, unless otherwise stated, the term tree refers to a labeled tree. A (labeled) tree on $n$ nodes is a connected graph with nodes (or vertex) labeled $\{1, 2, \ldots, n\}$ linked by $n-1$ simple (i.e., undirected, loopless and not multiple) edges. Consequently a tree is without circuit (or polygon). Two nodes are called adjacent or neighboring if they are connected by an edge. A leaf is a node with only one neighbor. Note that a tree is a bipartite graph: the set of nodes can be partitioned into two subsets so that adjacent nodes are in different subsets.

A theorem due to Cayley says that the number of different labeled trees on $n$ nodes is

$$T(n) = n^{n-2}. \quad (1)$$

A simple proof uses the Prüfer coding explained in Sect. 3.2. See the book [Van Lint and Wilson, 1992] for a general presentation.

The adjacency matrix of any tree on $n$ nodes is the $n \times n$ square matrix $A$ such that $A_{i,j} = 1$ if nodes $i$ and $j$ are adjacent and 0 otherwise. Then $A$ is symmetric, with zeroes on the diagonal. The walks on a tree are counted by its adjacency matrix: the number of walks of length $k$ starting at node $i$ and finishing at node $j$ is $(A^k)_{i,j}$. Similarly, $\text{Tr}A^k$ is the number of closed walks of length $k$.

The spectrum (set of eigenvalues) of the adjacency matrix $A$ of a tree $T$ is more simply called the spectrum of $T$. By definition, $\lambda$ is an eigenvalue associated with the eigenvector $V = (V_1, V_2, \ldots, V_n)$ if $AV = \lambda V$. So the eigenvalue equation on the node $i$ is

$$\lambda V_i = \sum_{j \text{ adj } i} V_j, \quad (2)$$

where the sum runs over the nodes $j$ adjacent to $i$. As $A$ is symmetric, its spectrum consists of real eigenvalues with a complete orthogonal basis of real eigenvectors. Furthermore, the spectrum is symmetric with respect to zero because a tree is bipartite: each eigenmode $(\lambda, V)$ has a partner $(-\lambda, V')$, where the vectors $V$ and $V'$ are equal on the nodes of one subset and opposite on the nodes of the other subset. See Appendix A for an application of this property to the numerical computation of the spectrum.

The symmetry factor of any tree is the number of permutations of the nodes that leave invariant this tree. Two trees are isomorphic if they differ only by a permutation of their nodes, or equivalently if their adjacency matrices differ only by a permutation of rows and columns. The number of trees isomorphic to any tree on $n$ nodes with symmetry factor $S$ is $n!/S$. The term labeled emphasizes that we are not identifying isomorphic trees. Therefore an unlabeled tree is an isomorphism class of labeled trees.

Clearly, isomorphic trees are cospectral (i.e. they have the same spectrum), but the reciprocal is wrong: the proportion of trees on $n$ nodes which have a cospectral but non isomorphic partner goes to 1 when $n$ becomes large [Cvetkovic and al., 1995, ch. 6].

The trees on $n \leq 5$ nodes (and their spectra) are listed in Table 1. A larger table for $n \leq 10$ is in [Cvetkovic and al., 1995].

In order to speak about average properties of trees (for example their spectral density), we turn the set of $T(n)$ trees on $n$ nodes into probability space with equiprobable elements, called random (labeled) trees. In other words, a random tree on $n$ nodes is randomly chosen among the $T(n)$ trees.

Note that the models of random labeled trees and of random unlabeled trees are not equivalent: trees with a high symmetry factor are less probable with the labeled model. For example, we have numerically computed that the average fraction of the spectrum occupied by the eigenvalue 0 in tree on $n = 23$ nodes is 0.1437... for a labeled tree and 0.2329... for an unlabeled tree. By considering the convergence, the size $n = 23$ is sufficiently large to so that a gap exists in the large $n$ limit. It can be explained because most of eigenvectors with $\lambda = 0$ have only two non-zero coordinates which are opposite and localized on two leaves adjacent to a same node. As this pattern gives a symmetry factor 2, eigenvalues zero are less frequent in the labeled model.

3 Numerical methods

We use two numerical methods: full enumeration for small trees and Monte-Carlo simulations for large trees.
3.1 Enumeration

If we want to enumerate all the \( T(n) = n^{n-2} \) trees on \( n \) nodes with a computer (for example with the Prüfer coding), the problem becomes very hard as soon as \( n \approx 10 \). It is faster to enumerate all the unlabeled (i.e non-isomorphic) trees. For each unlabeled tree \( T \) on \( n \) nodes, the spectrum (independent of labels on the nodes) is numerically computed. As \( T \) represents a class of isomorphic labeled trees, the spectrum is counted with a multiplicity equal to the number of ways of labeling \( T \). i.e. \( n! \) divided by its symmetry factor.

Let us call \( t(n) \) the number of unlabeled trees on \( n \) nodes. The sequence \( t(1), t(2), \ldots \), starts with 1, 1, 1, 2, 3, 6, 11, 23, 47, 106, \ldots Since works of Jordan, Cayley, Polya and Otter, we known [Knuth, 1997, p. 388] that \( t(n) \) grows exponentially like

\[
t(n) \sim \beta \frac{\alpha^n}{n^{3/2}},
\]

with \( \alpha = 2.955765285652 \ldots \) and \( \beta = 0.5349496061 \ldots \)

To enumerate the \( t(n) \) unlabeled trees, we used a simplified version of the WROM algorithm [Wright et al., 1986] with computation time \( \mathcal{O}(n^2 t(n)) \), and not \( \mathcal{O}(t(n)) \) as the original version. As the computation time for the spectrum of one \( n \times n \) matrix is \( \mathcal{O}(n^3) \), and \( \mathcal{O}(n^2) \) or less for the symmetry factor in the worst case, we see that the slow part is not the enumeration work, but the computation of the \( t(n) \) spectra. For the same reason, we did not find useful to work with algorithms like the recursive one of [Li and Ruskey, 1999]

Thus, in this work, we have enumerated trees up to \( n = 23 \), where \( t(23) = 14 \, 828 \, 074 \). As the computational time grows exponentially, it is difficult to enumerate spectra of trees with much more nodes.

3.2 Prüfer Monte-Carlo method

To study large trees, the full enumeration is impossible. The Monte-Carlo method consists by randomly generating a set of \( S \) independent trees, then measuring some properties and averaging them. Of course, the results have random noise, which can be estimated with usual formulae of statistics. In good cases, the noise decreases as \( 1/\sqrt{S} \).

Main points are to generate trees with the right probability law and with an efficient algorithm. For example, let us consider the naive method which generates a random graph by choosing \( n-1 \) edges between \( n \) nodes, then rejects this graph if not a tree. As trees are very rare among the set of graphs, this method is not efficient.

If we modify the procedure by rejecting edges which close a cycle, we generate at each time a tree but now with a non uniform probability law. For example, for trees on 4 nodes, \( \text{Prob}(T_{4,1}) = 1/15 \) and \( \text{Prob}(T_{4,2}) = 11/180 \) instead of \( 1/16 \). For \( n = 5 \), \( \text{Prob}(T_{5,1}) = 1/105 \), \( \text{Prob}(T_{5,2}) = 127/15120 \approx 1/119 \) and \( \text{Prob}(T_{5,3}) = 113/15120 \approx 1/134 \) instead of \( 1/125 \). We see that this procedure has a bias which favors trees with highly connected nodes. More seriously the bias grows exponentially with the size: the “star” tree (made of one central node connected to \( n-1 \) peripheral nodes) has a probability \( 1/(2n-3)! \) instead of \( 1/n^{n-2} \).

So we use the Prüfer Monte-Carlo method, which is efficient and not biased. We first describe the Prüfer coding, but without giving a proof. Let \( T \) be a labeled tree on \( n \) nodes. The Prüfer coding consists in removing successively one leaf at each step. We start with \( T_1 = T \). For \( 1 \leq i \leq n-1 \), the step \( i \) is the following: let \( b_i \) be the leaf of \( T_i \) with the smallest label, let \( a_i \) be the neighbor of \( b_i \) and let \( T_{i+1} \) be the tree obtained by deleting from \( T_i \) the leaf \( b_i \) and the edge \( \{b_i, a_i\} \). The Prüfer code of \( T \) is the sequence \( (a_1, a_2, \ldots, a_{n-2}) \). As \( a_{n-1} = n \) necessarily, it is
not included in the code.

Remark that the sequence \((b_1, b_2, \ldots, b_{n-1})\) is always a permutation of \((1, 2, \ldots, n-1)\). In contrast, repetitions can occur among the code. More precisely, the number of neighbors of the node \(j\) is the number of \(j\) in the code, plus one. Consequently the leaves are the nodes which never appear in the code.

To reverse this procedure, start with an arbitrary code \((a_1, a_2, \ldots, a_{n-2})\) with \(1 \leq a_i \leq n\). By convenience, we complete with \(a_{n-1} = n\). We built a tree \(T\) with the following iterative procedure: for \(1 \leq i \leq n-1\), at step \(i\), let \(b_i\) be the least number in \([1, n]\) not in \((b_1, b_2, \ldots, b_{i-1})\) and not in \((a_1, \ldots, a_{n-1})\). The tree \(T\) is made of the \(n-1\) edges \(\{b_i, a_i\}\).

Then the Prüfer coding is an one-to-one correspondence between the labeled trees on \(n\) nodes and the \(n^{n-2}\) codes \((a_1, a_2, \ldots, a_{n-2})\) with \(1 \leq a_i \leq n\). This proves in particular that \(T(n) = n^{n-2}\).

This coding gives an easy Monte-Carlo method: a random labeled tree on \(n\) nodes is built by choosing \(n-2\) independent random integer numbers in \([1, n]\) and by applying the reverse procedure. Then the spectrum is computed. As usual with Monte-Carlo methods, the averages are done with as many independent trees as possible, in order to reduce the statistical noise. Of course the eigenvalues are not independent: for example, each \(\lambda \neq 0\) is generated simultaneously with \(-\lambda\). So the estimators of variance must be calculated by considering that the independent events are the generated trees, then their whole spectra, but not the eigenvalues separately.

As the computation time to generate a random code and then the corresponding tree is \(O(n)\), it is possible to obtain trees on several millions nodes. But the slow part is always the computation of the spectrum which needs \(O(n^3)\) per tree (see details in Appendix A). This limits to several thousands nodes.

We simulate different sizes to evaluate finite size effects. For each size \(n = 30, 50, 100, 200, 500, 1000, 2000, 5000, 10000\), we obtained \(m = 30,000,000\) eigenvalues, i.e we simulated \(m/n\) random labeled trees of size \(n\). The main cpu time consumption was for the larger size. We judged that it is not interesting to simulate larger trees (with the same global cpu time): \(m\) would be smaller and the fluctuations bigger.

In principal we can simulate random unlabeled trees with the Prüfer Monte-Carlo method. This procedure generates trees with uniform probability if they are labeled, but non uniform if they are considered as unlabeled. So we can correct this bias by giving a weight on each generated tree equal to its symmetry factor, inversely proportional to its probability. We have not simulated unlabeled trees, but it is probable that new problems appear when \(n\) is large: as very few trees with very large weights dominate the averages, the fluctuations are large and non-gaussian. This is similar to the simulation of the low temperature phase of an Ising model by generating random independent up or down spins, i.e. at infinite temperature.

## 4 Spectral density of large random trees

By definition, the spectral density (or density of states) of any tree \(T\) on \(n\) nodes with spectrum \(\{\lambda_1, \lambda_2, \ldots, \lambda_n\}\) is the distribution

\[
\rho_T(x) = \frac{1}{n} \sum_{i=1}^{n} \delta(x - \lambda_i)
\]

and the average spectral density of random trees on \(n\) nodes is

\[
\rho_n(x) = \frac{1}{n^{n-2}} \sum_{T_n} \rho_T(x)
\]

where the sum runs over the \(n^{n-2}\) trees on \(n\) nodes. The bounds of the support of \(\rho_n(x)\) are \(\max\{\lambda_i\} = \sqrt{n-1}\) and \(\min\{\lambda_i\} = -\sqrt{n-1}\), which are eigenvalues of the “star” tree, made of one central node and \(n-1\) peripheral nodes. We are mainly interested by the asymptotic distribution \(\rho(x)\) when \(n\) is large, in the spirit of the statistical physics.

### 4.1 Moments of the spectral density

We consider \(\mu_k = \int dx x^k \rho_n(x)\), the \(k\)'th moment of the average spectral density \(\rho_n\). It is related to the number of closed walks of length \(k\) on the trees on \(n\) nodes by

\[
n^{n-1} \mu_{k,n} = \sum_{T_n} \text{Tr} T_n^k
\]

where the symbol \(T_n\) represents at the same time a tree on \(n\) nodes and its adjacency matrix.

As trees are bipartite, their spectra are symmetric with respect to zero and the odd moments \(\mu_{2k+1,n} = 0\). Furthermore, the first even moments \(\mu_{2k,n}\) can be calculated by enumeration of walks. For \(k = 2\), a closed walk of length 2 is just a “round trip” on an edge: as a tree has \(n-1\) edges, \(\text{Tr} T_n^2 = 2(n-1)\) for each tree, then \(\mu_{2,n} = 2 - 2/n\). The edges visited by a closed walk of length 2 on a tree form a subtree with \(l \leq k\) edges. By erasing the \(l\) visited edges, the rest (i.e the spectator edges) splits up into a “rooted forest” with \(n\) nodes and \(m = l+1\) rooted trees. The number of such rooted forests is

\[
F_{m,n} = n^{n-m} \frac{(n-1)!}{(n-m)!(m-1)!}
\]

The number of closed walks of length 2 visiting all nodes of trees with \(l\) edges is \((l+1)! J_{k,l}\) where the integer numbers \(J_{k,l}\) can be computed with a recursion relation: see Eq. 8 and Table 1 of [2001a, p 307]. Consequently, [Bauer, 2000]

\[
\mu_{2k,n} = \sum_{l=1}^{k} (l+1) J_{k,l} \frac{(n-1)(n-2)\ldots(n-l)}{n^l}
\]
valid for any \( n \geq 1 \) and the first ones are
\[
\begin{align*}
\mu_{2,n} &= 2 - \frac{2}{n}, \\
\mu_{4,n} &= 8 - \frac{20}{n} + \frac{12}{n^2}, \\
\mu_{6,n} &= 40 - \frac{176}{n} + \frac{256}{n^2} - \frac{120}{n^3}.
\end{align*}
\]

The large-\( n \) limits of the \( \mu_{k,n} \) are
\[
\mu_{2k} = \sum_{l=1}^{k} (l+1) \mathcal{J}_{k,l} \quad \text{and} \quad \mu_{2k+1} = 0.
\]

The \( \mu_k \) are the moments of the asymptotic distribution \( \rho(x) \) of the spectral density for large random trees. Note that \( \rho(x) \) is defined without the need to scale it with a power of \( n \). It is an “universal” distribution in the sense that it depends on no parameter, but only on the definition of the labeled trees.

The sequence \( \mu_{2k} \) starts \([Bauer, 2000]\) with \(2, 8, 40, 226, 1384, 8992, 61212, 433136\), etc. and we are not aware of any other problem involving the same sequence. It can be proved \([Bauer and Golinelli, 2001a]\) that the growth of the coefficients \( \mathcal{J}_{k,l} \), and then of the \( \mu_{2k} \), is sufficiently slow to characterize \( \rho(x) \), and sufficiently fast to assure that the support of \( \rho(x) \) is unbounded.

### 4.2 Delta peaks and continuous distribution

It is not easy to extract accurate local information on the asymptotic spectral density \( \rho(x) \) from the knowledge of a finite number of moments. So we have recourse to numerical computations as described in Sect. 3. A histogram of Monte-Carlo eigenvalues for \( n = 10000 \) is shown on Fig. 1. As the spectra of trees are always symmetric with respect to zero, the histogram is folded up around zero for convenience sake.

Only one choice of size \( n \) is represented on Fig. 1 because the dependence on \( n \) is small. In order to give an idea of these finite size effects, we show on Fig. 2 the (folded) cumulative density \( R_n(x) = \int_x^\infty \rho_n(y) \, dy \) for several choices on \( n \). We have observed that \( R_n(x) \) converges as \( 1/n \); for example, the values of the maximal distance \( D_n = \max_x |R_n(x) - R_{10000}(x)| \) are: \( D_{23} \approx 0.021, D_{50} \approx 0.0077, D_{100} \approx 0.0037 \) and \( D_{1000} \approx 0.00042 \). So we can consider that \( n = 10000 \) is sufficiently large to observe on Fig. 1 a distribution very close to the asymptotic distribution \( \rho(x) \).

The main observation is that \( \rho(x) \) has several peaks with holes around them. These peaks correspond with exact degeneracies of eigenvalues: they are delta peak in the sense of distribution theory. Unfortunately, their heights on Fig. 1 are not representative because they depend on the width of the histogram bin, chosen arbitrarily. On the other hand, they are regularly represented by the vertical steps on Fig. 2 but now only the main ones are visible. Numerical estimations of their heights are given in Sect. 5.2.

The largest delta peaks are, in order of importance, at \( x = 0, \pm 1, \pm \sqrt{2}, \text{etc} \). which are precisely the eigenvalues of the small trees listed in Table 1: \( T_1, T_2, T_3, \text{etc} \). This
has been previously observed in several models of random graphs [Kirkpatrick and Eggarter, 1972; Evangelou, 1983; Farkas and al., 2001]. In Sect. 5.1 we show that the spectral density, in the large $n$ limit, has an infinity of delta peaks at all eigenvalues of finite trees and that the corresponding eigenvectors are strictly localized, in the sense that their number of non-vanishing coordinates (i.e. the number of nodes on which the vector does not vanish) is finite.

Moreover we have noticed numerically that the reciprocal is true: all the degeneracies (i.e. two or more equal values) among the complete set of Monte-Carlo eigenvalues appear only at eigenvalues of finite trees and that the corresponding eigenvectors are strictly localized on a small number of nodes. In other words, the delta peaks appear at these special values and nowhere else. However the heights of peaks are exponentially small with the size of the corresponding finite subtree. Thus on Fig. 1 and 2, only the largest ones are visible because the smallest ones are drowning into Monte-Carlo fluctuations.

To summarize, our simulations indicate that the asymptotic distribution $\rho(x)$ has two components in the large $n$ limit: a discrete component $\rho_d(x)$ made of an infinity (but countable) of delta peaks at all eigenvalues of finite trees associated with strictly localized eigenvectors, and a continuous component $\rho_c(x)$ built with all eigenvalues which are not eigenvalues of finite trees which form a continuous support. As explained in Sect. 5.2, the total weight of $\rho_c(x)$ is $0.80827 \pm 0.00005$. Consequently, the total weight of $\rho_d(x)$ is $0.19173 \pm 0.00005$.

Note that the spectral density of the trees on $n$ nodes, for any $n$, is purely discrete because it is defined by a finite list of $n^{n-1}$ eigenvalues. So the continuous component $\rho_c(x)$ appears only in the large $n$ limit. However it is possible to study $\rho_c(x)$ with Monte-Carlo finite trees with the following procedure. Firstly we enumerate all “small” trees on $n \leq 23$ nodes and their spectra are listed. Next, during Monte-Carlo simulations of much larger trees, we research if the generated eigenvalues are among the spectra of small trees. If they are, we regard them as part of delta peaks; otherwise, we regard them as part of $\rho_c(x)$.

Of course, the threshold $n = 23$ is arbitrary and dictated by the power of our computer. But this threshold can be reduced up to $n \approx 15$ without significant changes because the delta peaks corresponding to trees on $n > 15$ nodes are so small that they are not visible in our simulations. As the heights of peaks decrease exponentially with $n$, the effective threshold increases as the logarithm of the computational time: then the value $n = 23$ is greatly sufficient.

With the previous criterion, we show on Fig. 3 the discrete spectral density $\rho_d(x)$ represented by the histogram of eigenvalues of Monte-Carlo trees on $n = 10000$ nodes which are also eigenvalues of small trees, drawn as delta peaks with now a correct height. As $30 \times 10^6$ eigenvalues are generated, only peaks with height bigger than $10^{-7}$ are observed with sufficient statistics. Because of this numerical cutoff, the reader could believe that the support of $\rho_d(x)$ has some gaps, but it is wrong because the eigenvalues of finite trees are a dense (but countable) subset of real numbers. A proof is given in Appendix B.
The histogram of the other eigenvalues (i.e. which are not eigenvalues of small trees) is drawn on Fig. 4: therefore it represents the continuous component $\rho_c(x)$ of the spectral density. As expected, it looks like Fig. 1 but without peaks. In particular, $\rho_c(x)$ has holes at $x = 0, \pm 1, \pm \sqrt{2}$, etc. which are positions of the delta peaks. By varying the width of the histogram bins, $\rho_c(x)$ appears to be zero and continuous at these special values, even if it is not really visible on Fig. 4. Around each special value, the spectral density concentrates to built a delta peak with a finite height and this concentration leaves a hole. This effect is proportional to the height of the peak. Those holes remember that in quantum mechanics, an eigenvalue with a non localized eigenvector is repelled by the others.

By generalization, we conjecture that any delta peak makes a hole in $\rho_c(x)$ proportional to its height. As delta peak positions are a dense (but countable) subset of the real numbers, $\rho_c(x)$ is a density function, non-continuous and non-zero almost everywhere except at these special values where it is continuous and zero. From a mathematical point of view, note that such “pathological” functions exist, for example $f(x) = \exp(-x^2) \min_i (2^i (x - \lambda_i)^2)$ where the $(\lambda_i)_{i \geq 1}$ are the special points. Remark that there is no contradiction with the word continuous: in the sense of the measure theory, $\rho_c(x)$ is a continuous distribution because its support is continuous or equivalently its integral $\int_{-\infty}^{x} \rho_c(y) \, dy$ is a continuous function. On the contrary, the discrete distribution $\rho_d$ has a discrete support and its integral has discontinuities (or jumps) for all delta peaks.

A sceptical reader could object that $\rho_c(x)$ could have a singular continuous component, i.e. a measure everywhere zero excepts at a non-countable set with Lebesgue measure zero. A famous example on $[0, 1]$ is the Cantor measure on the Cantor set. In this measure, the sum of the width of the gaps is 1 and all the measure is concentrated on the Cantor set. As this set has Lebesgue measure zero, the density is infinite on its points. But it is not a discrete distribution (with delta peaks) because this set is non-countable then the measure of any point remains always strictly zero. Equivalently, the integral is a continuous function.

Our numerical analysis can not determine if $\rho_c(x)$ contains or not a singular part because Monte-Carlo fluctuations prevent from distinguishing between a singular measure and a measure with a pathological density function. But we think that $\rho_c(x)$ vanishes only on the delta peaks positions, which are a set with Lebesgue measure zero, unlike the Cantor measure. Then the density remains finite everywhere, even if it is a pathological function. As we do not succeed to imagine a mechanism which concentrates the spectral density on a non-countable set with Lebesgue measure zero, we believe that $\rho_c(x)$ is not singular.

To summarize this section, the spectral density of large random trees have an infinity of delta peaks at all eigenvalues of finite trees, and a pathological density function vanishing at all peak positions.

5 Delta peaks in the spectral density

5.1 Existence of delta peaks

To show that a delta peak appears in the spectral density of large random trees at any eigenvalue of finite tree, we will adapt arguments of [Bauer and Golinelli, 2001a, p 322] used for the same property in random graphs. First, we show that in a large tree on $n$ nodes, the number of branches isomorphic to any finite tree $T$ is proportional to $n$. Consequently delta peaks appear in the spectral density because the multiplicity of eigenvalues of $T$ with eigenvectors strictly localized on branches is also $O(n)$ in large trees.

We now introduce two definitions. A rooted tree $(T, r)$ is a tree $T$ where the node $r$ is marked (the root). A branch $(B, r)$ at a node $r$ in a tree $T$ is a rooted subtree of $T$ with no edges between nodes of $B \setminus \{r\}$ and nodes of $T \setminus B$. Remark that it is not the usual definition.

Let us consider any rooted tree $(T, r)$ on $m$ nodes. The number of branches isomorphic to $(T, r)$ among all labeled trees on $n$ nodes is

$$B(n) = \frac{n!}{S \, (n-m)!} \, (n-m+1)^{(n-m-1)},$$

(13)

because the first factor counts the number of ways of labeling the branch ($S$ is the symmetry factor of $(T, r)$), and the second factor counts the choices of the rest of the tree, i.e. $(n-m)$ nodes, plus $r$. Consequently, a tree on $n$ nodes has in average $b(n) = B(n) / n^{n-2}$ branches isomorphic to $(T, r)$; in the large n limit,

$$b(n) \sim n \, \frac{e^{-(m-1)}}{S}.$$  

(14)

For example, $m = 2$ gives $n/e$ as the number of leaves in average in a large tree. Moreover, by considering more complex patterns with two branches, it can be proved that the variance of this number of branches per tree is $O(n)$ because contributions in $O(n^2)$ are canceled. Then this number is “self-averaging”: in the large $n$ limit, for almost all trees, the number of branches isomorphic to $(T, r)$ is $n e^{-(m-1)} / S$ plus $O(\sqrt{n})$ fluctuations. The amplitude of theses fluctuations has no simple formula because it depends not only on the size $m$ but also on the shape of $T$.

Let $U$ be a tree with a branch $(T, r)$ at a node $r$; let $(\lambda, V)$ be an eigenmode of $T$: if $V_r = 0$, then $\lambda$ is eigenvalue of $U$. Indeed the vector $W$ on $U$ which extends $V$ with coordinates zero outside $T$ is also eigenvector with same eigenvalue $\lambda$, independently of the size and the shape of
Moreover, the transformation can involve

$$T'' \rightarrow r''$$

and choices of roots, without forgetting symmetry factors. Secondly, in the case where the construction of the above paragraphs is done with $k \geq 3$ trees, the multiplicity of $\lambda$ is $k - 1$ but the tree $D$ is counted $k(k - 1)/2$ times in Eq. (13). So we must use inclusion-exclusion formula: this gives a $\mathcal{O}(e^{-3m})$ correction.

Because of these remarks, we do not know analytical formula for the heights of the delta peaks except at $\lambda = 0$ [Bauer and Golinelli, 2000] which corresponds to the tree on $m = 1$ node. However we keep the main result: in the large random tree limit, the multiplicity of any eigenvalue of finite tree is $\mathcal{O}(n)$ and a delta peaks appears with a self-averaging height.

### 5.2 Statistics of delta peaks

In this section, we give numerical estimations for the heights of delta peak in the spectral density of large random trees. We have explained in previous sections that these peaks correspond to eigenvalues of finite trees. As their number increases exponentially with their size, it would be cumbersome to give a list of heights for each eigenvalue. So we prefer to group eigenvalues by order.

Let $\lambda$ be an eigenvalue of a tree on $n$ nodes. We define the order of $\lambda$ as the size $d$ of the smallest tree(s) with eigenvalue $\lambda$. Of course, $d \leq n$. We define $L_d$ as the set of eigenvalues of order $d$: it is the finite set of eigenvalues of trees on $d$ nodes but not in $L_{d'}$ with $d' < d$.

In principle, $L_d$ can be explicitly written by enumerating all spectra of trees on $d' \leq d$ nodes, as done on Table 1. As the spectrum of the trivial tree $T_1$ on one node is $\{0\}$, then $L_1 = \{0\}$. For $d = 2$, the tree $T_2$ gives $L_2 = \{-1, 1\}$. For $d = 3$, the spectrum of $T_3$ is $\{-\sqrt{2}, 0, \sqrt{2}\}$. As 0 is already of order 1, then $L_3 = \{-\sqrt{2}, \sqrt{2}\}$. The following sets are $L_4 = \pm\{\sqrt{5} \pm 1\}/2, \sqrt{3}\}$, $L_5 = \pm\{\sqrt{2} \pm \sqrt{2}\}$, etc. Generally, as all spectra are symmetric with respect to zero, the $L_d$ are also symmetric.

An eigenvalue of order $d$ is by definition a root of a characteristic polynomial, which is an integer polynomial of degree $d$. But this polynomial can often be factorized, for example if $d$ is odd (because 0 is eigenvalue by symmetry) or if the tree has some symmetries [Cvetkovic et al., 1995, ch 4]. So an eigenvalue of order $d$ is an algebraic number of degree $d'$ with $d' \leq d$. It would be interesting to be able to determine the order of a given eigenvalue of a tree $T$ without have to enumerate all the trees smaller than $T$. However, we have only a partial answer expounded in Appendix C.

We define $F_d$ as the sum of the heights of delta peaks corresponding to eigenvalues of order $d$, in the large random tree limit. In other words, $F_d$ is the fraction of the spectrum of a large random tree occupied by eigenvalues
of order $d$.

For $d = 1$ (i.e. $\lambda = 0$) we know [Bauer and Golinelli, 2000] that $\mathcal{F}_1$ for a finite size $n$ has an asymptotic expansion in powers of $1/n$. We observe numerically the same behavior for $d \geq 2$. So we extrapolate for $n = \infty$ with least-square fits with linear or quadratic function of $1/n$. For these fits, we include Monte-Carlo results for $n = 20, 40, 60, 80, 100, 200, 500, 1000, 2000, 5000$ and 10000 and the standard errors for the extrapolations are smaller than the ones of the measures, as shown on Table 2. By simulating trees on $n = 10001$ nodes, we have also verified that the results do not show perceptible parity effects.

We also observe numerically that, for a given $d$, the variance of $\mathcal{F}_d$ for random trees on $n$ nodes is proportional to $1/n$: in other words, the $\mathcal{F}_d$ are self-averaging for large trees. The opposite would be very surprising because the eigenvectors corresponding to eigenvalue of finite order are strictly localized on finite patterns: the appearances of these patterns look like independent events on a large random tree. A consequence is that the standard error of $\mathcal{F}_d$ is $O(1/\sqrt{m})$ (for a given $d$) where $m$ is the total number of generated eigenvalues, independently of the size $n$.

We are also interested by the localization of eigenvectors. For an eigenvector $V$ corresponding to an eigenvalue of order $d$, let $l$ be the number of coordinates (or nodes) where $V_i \neq 0$. We observed that most of them are localized on $l = 2d$ nodes: they correspond to the construction shown on Fig. 5 with $T'$ and $T''$ as smallest as possible.

As explained in Sect. 5.1 and Appendix C, eigenvectors with arbitrary large $l$ can be built. But the probability of appearance of large patterns decreases exponentially with their size. So in our Monte-Carlo simulations, a few of eigenvectors have an extension $l > 2d$ without ever exceeding $4d$. In principle, with more important simulations, it would be possible to observe larger (and rarer) localized eigenvectors.

In particular we do not remark a delocalization phenomenon, as observed [Bauer and Golinelli 2001] in the Erdős-Rényi random graph model for eigenvectors with $\lambda = 0$, when the average effective connectivity is between 2.093 . . . and 3.312 . . . Note that in the random tree model, the average connectivity is by definition fixed to 2.

## 6 Conclusion

In this work we have studied the spectral density of the adjacency matrix of random labeled trees, as a model of a hopping particle on a connected graph. In the limit where the number $n$ of nodes is large, our main results are:

- The averaged spectral density converges to an asymptotic distribution $\rho(x)$ with a $O(1/n)$ behavior. Moreover the spectral density is self-averaging, i.e. the spectral density of a given tree is $\rho(x)$ almost surely.
- At all eigenvalues of finite trees, $\rho(x)$ has a delta peak. Its height decreases exponentially with the size of the corresponding finite tree. Except for the $\lambda = 0$ peak, we do not know how to calculate analytically these
heights, but numerical estimations are given. The total weight of these peaks is $0.19173 \pm 0.00005$.

- The rest of $\rho(x)$ is a distribution given by a density function which vanishes at each position of delta peak. As these positions form a dense set among real numbers, this function is almost everywhere discontinuous.

- The eigenvectors corresponding to eigenvalues of delta peaks are strictly localized, i.e. they vanish everywhere except on a finite number of nodes. On the other hand, the other eigenvectors are not strictly localized.

It seems difficult to extend the analytical calculation [Bauer and Golinelli, 2000] of the height of the $\lambda = 0$ peak to all peaks because it is specific to this particular eigenvalue: there is only one “pure” tree for $\lambda = 0$ and moreover this tree is the trivial tree on one node. Then the enumeration of composite trees with $\lambda = 0$ is possible. In contrast, each other peak corresponds to an infinite family of pure trees, which are not trivial.

We think that these results are shared by many models of graphs. For any eigenvalue of finite graph $g$, a delta peak appears in the spectral density of a large graph $G$ on $n$ nodes if the number of induced subgraphs of $G$ isomorphic to $g$ (and connected to the rest of $G$ only by nodes on which $V$ vanishes) is $O(n)$. Of course the height of the delta peak depends on the details of the model. Generally, the highest ones correspond to the smallest $g$ and their contribution gives an important fraction of the total weight of delta peaks.

Furthermore the existence of a pathological density function for the continuous part of the spectral density is not specific to the random trees. As it corresponds to eigenvectors which are not strictly localized and orthogonal to eigenvectors of the delta peaks, it is expected that the density function has a depression around each delta peak. Moreover we think that these conclusions are also valid for other kind of Hamiltonian, for example the Laplacian, or graphs with weighted links.

### A Symmetry of the spectrum

In this appendix, we show that the bipartition of any tree induces that its spectrum is symmetric with respect to zero. Moreover it allows to reduce the computation time of the spectrum by a factor 8.

Let $T$ be a tree on $n$ nodes. The set of nodes can be partitioned into two subsets, $P$ and $Q$, of sizes $p$ and $q$ (with $p + q = n$), so that all edges link a $P$ node with a $Q$ node. We consider the spectrum $(\lambda_i)_{i=1,n}$ of the symmetric adjacency matrix $A$ of $T$. In this notation, degenerate eigenvalues correspond to several indices $i$. Note that all eigenvalues are real. First, without changing the spectrum, we permute the labels of the nodes of $T$ so that $P$ nodes are labeled by $(1, 2, \ldots, p)$ and $Q$ nodes are labeled by $(p+1, p+2, \ldots, n)$. In this basis, the rows and columns of $A$ are permuted and $A$ has now a $(p+q) \times (p+q)$ block shape

$$A = \begin{pmatrix} 0 & R^T \\ R & 0 \end{pmatrix}$$

where $R$ is a rectangular $q \times p$ block and $R^T$ is the transpose of $R$. Then,

$$A^2 = \begin{pmatrix} R^T R & 0 \\ 0 & RR^T \end{pmatrix}$$

is block diagonal: its spectrum $(\lambda^2_i)_{i=1,n}$ is the union of the spectra of the square $p \times p$ block $R^T R$ and of the square $q \times q$ block $RR^T$. We will show that it is sufficient to diagonalize only one block, $R^T R$ for example.

Let $V$ be an eigenvector of $R^T R$ ($V$ is $p$-dimensional) with eigenvalue $\lambda^2$, then $R^T RV = \lambda^2 V$. We assume that $\lambda \neq 0$. Then $RV$ is a $(q$-dimensional) eigenvector of the other block $RR^T$ with the same eigenvalue $\lambda^2$, because $RR^T RV = R(\lambda^2 V) = \lambda^2 RV$. The two $n$-dimensional vectors

$$U_{\pm} = \begin{pmatrix} \pm \lambda V \\ RV \end{pmatrix}$$

are eigenvectors of $A$ with eigenvalues $\pm \lambda$ because

$$AU_{\pm} = \begin{pmatrix} R^T RV \\ \pm \lambda RV \end{pmatrix} = \begin{pmatrix} \lambda^2 V \\ \pm \lambda RV \end{pmatrix} = \pm \lambda U_{\pm}$$

If $\lambda^2$ is a degenerate eigenvalue of $R^T R$, this work can be done with any corresponding eigenvector.

To resume, any eigenvalue $\lambda^2 > 0$ of one block of $A^2$ is associated to an eigenvalue $\lambda^2$ of the other block and a couple $\pm \lambda$ of $A$. The rest is made of eigenvalues 0, with $m(A) = m(R^T R) + m(RR^T) = 2m(R^T R) + q - p$ by noting $m(\cdot)$ its multiplicity. Then the spectrum of $A$ is symmetric with respect to zero. Moreover eigenvalues (and eigenvectors) of $A$ can be deduced from the diagonalization of only one block.

From a numerical point of view, the complete diagonalization of a $n \times n$ full matrix needs a time $O(n^3)$. As $\min(p,q) \leq n/2$, it is at least 8 times faster to diagonalize the smallest block, $RR^T$ or $R^T R$, instead of $A$. In the large $n$ limit, $p \sim q \sim n/2$ for almost all trees, so the asymptotic factor is 8.

Remark that the characteristic polynomial of $A$ can by computed in $O(n^2)$ by using a recursion procedure with the characteristic polynomials of the subtrees obtained by deleting one node. However the computation of eigenvalues from the knowledge of the characteristic polynomial is numerically very instable, so we prefer diagonalize $A$ with Lapack routines for symmetric full matrix.
B Density of the eigenvalues of finite trees

In this appendix, we show that the set \( \mathcal{L} \) of eigenvalues of finite trees is a dense subset of real numbers. Remark that by definition \( \mathcal{L} \) is countable because finite trees are countable. We proceed in two steps: firstly the eigenvalues that by definition \( \mathcal{V} \) of the eigenvector on the respective nodes. The eigenvalue is everywhere dense.

The union of all these intervals covers the real numbers, \( \mathcal{L} \) is everywhere dense.

For a linear tree of length \( n \), pictured on Fig. 6, \( \lambda \) is eigenvalue with eigenvector \( V = \{ v_i \}_{i=1,n} \)

\[
\lambda v_i = v_{i-1} + v_{i+1} \quad (i = 1, 2, \ldots, n), \tag{19}
\]

with the conventions \( v_0 = v_{n+1} = 0 \). The \( n \) solutions are

\[
\lambda_p = 2 \cos \left( \frac{p \pi}{n+1} \right), \quad v_{i,p} = \sin \left( \frac{ip \pi}{n+1} \right) \tag{20}
\]

for \( p = 1, 2, \ldots, n \). Then the union of spectra of finite linear trees is dense in \([-2,2]\).

Now we consider a linear tree of length \( n \) decorated with bunches of \( k \) leaves: it is made of a linear backbone \( \{ a_i \}_{i=1,n} \), and for each \( i \), \( k \) nodes \( \{ b_{i,j} \}_{j=1,k} \) are connected to \( a_i \). See Fig. 6. Its size is \((k+1)n\).

For an eigenvalue \( \lambda \), we note \( a_i \) and \( b_{i,j} \) the coordinates of the eigenvector on the respective nodes. The eigenvalue equations are

\[
\lambda b_{i,j} = a_i \quad (i = 1, 2, \ldots, n; \quad j = 1, 2, \ldots, k) \tag{21}
\]

\[
\lambda a_i = a_{i-1} + a_{i+1} + \sum_{j=1}^{k} b_{i,j} \quad (i = 1, 2, \ldots, n) \tag{22}
\]

with the conventions \( a_0 = a_{n+1} = 0 \). If \( \lambda = 0 \), then

\[
\sum_{j=1}^{k} b_{i,j} = a_i = 0 \quad (i = 1, 2, \ldots, n), \tag{23}
\]

and the multiplicity of the eigenvalue 0 is \((k-1)n\). For \( \lambda \neq 0 \),

\[
b_{i,j} = a_i / \lambda, \quad (\lambda - k/\lambda) a_i = a_{i-1} + a_{i+1}. \tag{24}
\]

Then the equation for \( \mu = \lambda - k/\lambda \) is similar to Eq. (19). There are \( 2n \) non-zero eigenvalues given by

\[
\lambda_p^{(\pm)} = \frac{1}{2} \left( \mu_p \pm \sqrt{\mu_p^2 + 4k} \right), \tag{25}
\]

\[
\mu_p = 2 \cos \left( \frac{p \pi}{n+1} \right) \quad (p = 1, 2, \ldots, n). \tag{26}
\]

As the set of \( \mu_p \) for \( n \geq 1 \) is dense in \([-2,2]\), then the set of \( \lambda_p^{(\pm)} \) is dense in \([\sqrt{k+1} - 1, \sqrt{k+1} + 1]\) and the set of \( \lambda_p^{(-)} \) is dense in \([-\sqrt{k+1} - 1, -\sqrt{k+1} + 1]\).

The union of these intervals for \( k \geq 1 \), plus the interval \([-2,2]\) given by linear trees, covers all the real numbers. So the trees described above are sufficient to prove that eigenvalues of finite trees are a dense subset of real numbers.

We remark that the same calculation can be done with other decoration patterns. Let us consider now a tree with a linear backbone \( \{ a_i \}_{i=1,n} \), and for each \( i \), a rooted tree \((U,r)\) is grafted by merging \( r \) and \( a_i \), as pictured on Fig. 7. Then, Eq. (24) is generalized by

\[
f(\lambda) a_i = a_{i-1} + a_{i+1}, \tag{27}
\]

where the function \( f(x) \) is defined by

\[
f(x) = \frac{\chi_U(x)}{\prod_{j=1}^{s} \chi_{U_j}(x)} \tag{28}
\]

by noting \( \chi_T(x) \) the characteristic polynomial of the adjacency matrix of a tree \( T \), and \( U_1, U_2, \ldots, U_s \) the subtrees of \( U \) obtained by removing its root \( r \) (see Fig 7). The eigenvalues are the set of \( f^{-1}(\mu_p) \), solutions of \( f(x) = \mu_p \) and they are dense in \([f^{-1}([-2,2])]\).

That remembers the band theory in solid state physics. We look at \( U \) as an atom: an eigenvalue \( f^{-1}(0) \), solution of \( \chi_U(x) = 0 \), is an energy level. The linear tree decorated
with $U$ is like a crystal, and the terms $a_{i+1}$ in Eq. (27) mean that electrons can hop from an atom to its neighbors. Then each atomic level $f^{-1}(0)$ is stretched and forms a band $f^{-1}(2\cos(k))$ where $k$ is the wave vector.

C Large pure trees

For any finite tree $T$ with eigenvalue $\lambda$, we have seen in Sect. 5.1 that it is possible to build arbitrarily large trees with same eigenvalue $\lambda$ by connecting many copies of $T$ with additional nodes with coordinates zero. In this Appendix, we will show that arbitrarily large trees with eigenvectors without coordinates zero can be obtained as well.

Let $T$ be any tree with eigenvalue $\lambda$ and eigenvector $V$. If all coordinates $V_i \neq 0$, we say that $T$ is $\lambda$-pure. If $T$ is $\lambda$-pure, it can be proved that $\lambda$ is not degenerate in the spectrum of $T$. Then the $\lambda$-purity is really a property of the tree, and it does not depend on a particular choice of eigenvector. By noting $\lambda_m$ the largest eigenvalue of $T$, remark that $T$ is $\lambda_m$-pure because the Perron-Frobenius theorem assures that all coordinates of the corresponding eigenvector are all positive.

If $T$ is not $\lambda$-pure, it is composite. By deleting any node $i$ with $V_i = 0$, $T$ is split into a forest of one or more subtrees; on each subtree, the restriction of $V$ is again an eigenvector with eigenvalue $\lambda$. Then by deleting all nodes with coordinates 0, the rest is a forest of $\lambda$-pure subtrees. A consequence is that the order of $\lambda$ is less or equal to the size of the smallest pure subtree.

Naturally we wish enumerate $\lambda$-pure trees. The case $\lambda = 0$ can be easily solve: for any leaf $i$, its neighbor $j$ has $V_j = \lambda V_i = 0$. So there is only one 0-pure tree: the trivial tree on one node. On the contrary, if $\lambda \neq 0$, we will show that there is an infinity of $\lambda$-pure trees.

Many processes can build an arbitrary large $\lambda$-pure tree $U$ by using several $\lambda$-pure trees $T_i$ (not necessarily isomorphic) and additional nodes. Some processes are general: they apply to all trees $T_i$. But other processes are specialized to trees with special patterns or special $\lambda$ (for example integer $\lambda$). There is an infinity of kind of processes and we have not found a unified way to present them. So we content ourself to give some examples.

As it is cumbersome to explain the details with words, we show graphical representations of the resulting tree $U$ with its eigenvector $W$ by using the following rules. A triangle represents a $\lambda$-pure tree $T_i$ and the restriction of $W$ on $T_i$ is the eigenvector of $T_i$ with eigenvalue $\lambda$. When a node $r$ is labeled by $x$, it means that $W_r = x$. If $r$ is a vertex of a triangle $T_i$, the eigenvector of $T_i$ is scaled in order that $W_r = x$. Remark that it is always possible because $T_i$ is pure, i.e. without coordinate zero. When a node $r$ is a common vertex of triangles $T_i$ and $T_j$, it means that a node $r_i$ of $T_i$ is merged with a node $r_j$ of $T_j$. The neighbors of $r$ are the ones of $r_i$, plus the ones of $r_j$. Of course, all these processes are done in order that Eq. (2) is satisfied for $W$ on every nodes.

The simplest general process, called “2+1”, is shown on Fig. 8. It uses three $\lambda$-pure trees $T_{(1,2,3)}$ with one merge and one additional node. So $\text{Size}(U) = \sum_i \text{Size}(T_i)$. Remark that $U$ is pure because $\lambda \neq 0$. The reader can verify that Eq. (2) is satisfied on every nodes of $U$. For example, let us consider the common node $r$ between $T_1$ and $T_2$ with $W_r = 1$: the neighbors $j_1$ of $r$ inside the tree $T_1$ give $\sum_j W_{j_1} = \lambda W_r$, because the restriction of $W$ on $T_1$ satisfies Eq. (2). And the same for the neighbors $j_2$ inside the tree $T_2$. As $r$ is also linked with a node of $T_3$ with coordinate $-\lambda$, then all neighbors of $r$ give $\sum_j W_j = \lambda W_r$. This process can be iterated by using $U$ in place of $T_i$ to obtain arbitrary large $\lambda$-pure trees.

A generalization is the “$b+1,b$” process described on Fig. 8, with $b+1$ pure trees with a common node linked to $b$ patterns made of one pure tree and an additional node. Here again $\text{Size}(U) = \sum_i \text{Size}(T_i)$.

It is easy to imagine other processes of this kind with more complex “bridges” between pure trees. But we prefer to describe on Fig. 9 a process which exploits a specific pattern. It uses two pure trees with a couple of leaves linked to the same node: $U$ is built by merging these four leaves into a single node with a doubled coordinate. Then $\text{Size}(U) = \text{Size}(T_1) + \text{Size}(T_2) - 3$. This process can be generalized with $k$ trees with bunches of $k$ leaves, by merging the $k^2$ leaves into a single node with a coordinate multiplied by $k$. Once again, more complex processes can be imagined.

To calculate analytically the height of delta peaks in the spectral density, a method would be to enumerate the trees with eigenvalue $\lambda$ by first enumerating pure trees, and then by counting all combinations of pure trees into...
composite trees. Unfortunately, these few examples above show that there is a great variety of pure trees, and we have no method to do a full listing. The only case [Bauer and Golinelli, 2000] for which analytical results are known is for $\lambda = 0$ thanks to two facts: this is only one 0-pure tree which has one node, and then the count of combinations is done by a method specific to this trivial tree.

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