Two-photon statistics of nonclassical radiation in the dissipative finite-size Dicke model

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Abstract
The nonclassical feature of photons in the open finite-size Dicke model is investigated via the two-photon correlation function. The quantum dressed master equation combined with the extended coherent photonic states is applied to analyze the dissipative dynamics of both the photons and qubits. The antibunching to bunching transition of photons is clearly observed by tuning the qubit–photon coupling strength. The optimal qubit number is unraveled to enhance the two-photon correlation function. Moreover, the temperature bias of thermal baths induces significant two-photon bunching signature with deep strong qubit–photon interaction.

Keywords: photon statistics and coherence theory, cavity quantum electrodynamics, decoherence, open systems, light–matter interaction

1. Introduction
The light–matter interaction plays a fundamental role in understanding the optical coherence of quantum theory, which was originally characterized by Glauber [1]. It has been extensively investigated in quantum optics [2, 3], quantum information processing [4] and quantum dissipation [5]. The coupling between the radiation field and quantum matter induces the attractive nonclassical feature of photons, exhibiting the effective photon–photon correlation [6–9]. Based on the theory of quantum photon detection, the statistics of photon nonclassicality can be measured via the intensity correlation function [7, 10].

One prototype system to describe the quantum light–matter interaction is the quantum Rabi model, which is composed by a two-level qubit interacting with a single mode radiation field [11–13]. It has been theoretically studied ranging from the quantum optics [14], quantum entanglement [15] to quantum phase transition [16–18]. In particular, the integrability of the Rabi model was recently explored by Braak [19] and Chen [20] with the Bargmann space and extended coherent state approaches, respectively. The quantum Rabi model was experimentally realized in the cavity-QED platform, with the interaction between photon and qubit reaching the ultrastrong coupling regime (i.e. \( \lambda/\omega \geq 0.1 \), \( \lambda \) is the coupling strength and \( \omega \) is the bare frequency of photons) [3, 21]. Accordingly, the traditional rotating-wave-approximation becomes inapplicable.

Another seminal system is the quantum Dicke model, which constitutes many two-level qubits coupled to a single cavity mode [22, 23]. In sharp contrast to the quantum Rabi model, the Dicke model in thermodynamic limit undergoes ground state quantum phase transition, i.e. from the normal phase to the superradiant phase by crossing the critical qubit–photon coupling strength [24, 25]. Based on the large-\( N \) expansion with \( N \) as the qubit number, the system will exhibit universal scaling behaviors [25–27]. However, for the finite-\( N \) Dicke model, the traditional numerical methods is limited...
to the small qubit number regime, e.g. \( N \leq 32 \) in reference [25]. Interestingly, Chen et al proposed an extended bosonic coherent state approach, which is able to accurately obtain the ground state property with significant large qubit number, e.g. \( N \leq 1000 \) [28]. Moreover, Hirsch et al gave a similar treatment with an optimal bosonic dressed basis to study the excited-state quantum phase transition and the connection with chaos [29–33].

The representative phenomenon to exhibit the nonclassical character of the radiation field in the light–matter interacting systems is the photon-blockade effect, which is detected via the two-photon correlation function. The photon-blockade effect describes how the existence of one photon in the cavity strongly suppresses the simultaneous excitation of another photon [6], characterized by the dramatic photon antibunching signature. Such a blockade effect has been extensively investigated in various devices, e.g. optomechanical systems [34, 35], cavity-QED [36–38] and superconducting circuits [39]. Particularly for the open quantum Rabi model, it is interesting to find that via the two-photon correlation function the standard photon-blockade breaks down in strong qubit–photon coupling regime [8, 9].

The multi-qubits analogy of two-photon statistics in super-radiant spontaneous emission of the Dicke model was originally proposed by Eberly et al [40–42]. Consequently, the bunching and antibunching effects of the dissipative Dicke model have been extensively investigated at steady state [43–48]. However, most works on the dissipative Dicke model in the study of two-photon correlation function are conducted in the weak qubit–photon coupling regime, which are limited to two main conditions: (i) rotating-wave approximation for the qubit–photon interaction; (ii) phenomenological Lindblad dissipation. Hence, considering the importance of the strong qubit–photon coupling in the Dicke model [3, 25], it is intriguing to explore photon statistics in a strong qubit–photon interaction regime.

In this paper, we combine the quantum master equation with the extended bosonic coherent state approach to study two-photon statistics in the dissipative finite-size Dicke model, which enables us to handle the strong qubit–photon coupling and microscopical system-bath interaction. The influence of the finite qubit number on the two-photon correlation is investigated and the transition from photon antibunching to bunching is clearly exhibited. Moreover, the optimal enhancement effect is discovered. The effect of the temperature bias on the two-photon correlation is also analyzed. It is found that the large temperature bias significantly enhances the photon correlation in the strong qubit–photon coupling regime. The paper is organized as follows: in section 2.1, we describe the Dicke model; in sections 2.2 and 2.3 we apply the quantum master equation combined with the extended coherent photon state to obtain the dynamical equation of the qubit–photon hybrid system. In section 3 we introduce the two-photon correlation function. In section 4, we study the effects of finite qubit number and finite bath temperatures on the two-photon correlation. Finally, we give a conclusion in section 5.

2. Model and method

2.1. Dicke model

The Dicke model, composed by \( N \) identical two-level qubits interacting with a single photonic field, is described as (\( h = 1 \) and \( k_B = 1 \)) [22, 23]

\[
H_D = \omega \hat{a}^\dagger \hat{a} + \Delta \hat{J}_x + \frac{2\lambda}{\sqrt{N}} (\hat{a}^\dagger + \hat{a}) \hat{J}_y, \tag{1}
\]

where \( \hat{J}_x = \frac{1}{2} (\hat{J}_+ + \hat{J}_-) \) and \( \hat{J}_y \) are the pseudospin operators, composed by \( \hat{J}_x = \sum_n \hat{\sigma}_x \), \( \hat{J}_y = \sum_n \hat{\sigma}_y \), with \( \hat{\sigma}_\alpha \) the Pauli operators and \( \hat{\sigma}_\pm = \hat{\sigma}_x \pm i \hat{\sigma}_y \). They have the commutating relation \( [\hat{J}_+, \hat{J}_-] = 2\hat{J}_y \), \( [\hat{J}_x, \hat{J}_-] = \pm \hat{J}_+, \hat{a}^\dagger \) and \( \hat{a} \) are the photonic creating and annihilating operators, \( \Delta \) and \( \omega \) are the frequencies of the qubits and single photonic mode, and \( \lambda \) is the qubit–photon coupling strength. In the following, we set the frequency of photons \( \omega = 1 \) as the energy unit for simplicity without losing any generality. For the finite \( N \) Dicke model, the parity is conserved, i.e. \( \exp(i\pi \hat{N}_{\text{tot}}) H_D = 0 \) with the total number operator of excitations \( \hat{N}_{\text{tot}} = \hat{a}^\dagger \hat{a} + \hat{J}_x + \frac{N}{2} \). In the large \( N \) limit, the Dicke model undergoes a quantum phase transition [24, 25, 28], where the system transits from the normal phase to the superradiant phase, with the critical qubit–photon coupling strength \( \lambda_c = \sqrt{\omega \Delta / 2} \). While for \( N = 1 \), the Dicke model is reduced to the seminal quantum Rabi model \( \hat{H}_R = \omega \hat{a}^\dagger \hat{a} + \frac{\lambda}{2} \hat{J}_y + \lambda (\hat{a}^\dagger + \hat{a}) \hat{J}_x \) [11, 12].

2.2. Extended coherent bosonic state approach

The extended coherent bosonic state approach is considered as an efficient method to numerically solve the Dicke model with finite number of qubits [28, 30, 31, 33]. Before including the extended coherent bosonic state method, we first rotate the angular momentum operators with \( \pi / 2 \) along the \( y \)-axis \( \hat{J}_0 = \exp(i\hat{J}_y/2) \hat{H}_0 \exp(-i\hat{J}_y/2) \), resulting in

\[
\hat{H}_0 = \omega \hat{a}^\dagger \hat{a} - \Delta \hat{J}_x + \frac{2\lambda}{\sqrt{N}} (\hat{a}^\dagger + \hat{a}) \hat{J}_x. \tag{2}
\]

Under the qubits basis \( \{ | j, m \rangle, m = -j, -j + 1, \ldots, j - 1, j \} \) with \( j = N/2 \), the Hilbert space of the total system can be expressed in terms of the direct product basis \( \{ | \varphi_m \rangle \otimes | j, m \rangle \} \). In the Dicke model, the excitation number \( \hat{N}_{\text{tot}} = (\hat{a}^\dagger \hat{a}) + (\hat{J}_x + \frac{N}{2}) \) is not conserved. Therefore, the truncation of the excitation number of photons should be included in this system, especially in the strong qubit–photon coupling regime. Specifically, by considering the displacement transformation \( \hat{A}_m = \hat{a} + g_m \) with \( g_m = 2m/\omega \sqrt{N} \) and taking the total system basis into the Schrodinger equation, we obtain

\[
- \Delta \hat{J}_m | \varphi_m \rangle_{| j, m = 1 \rangle} - \Delta \hat{J}_m | \varphi_m \rangle_{| j, m = 1 \rangle} + \omega (\hat{A}_m^\dagger \hat{A}_m - g_m^2) | \varphi_m \rangle_{| j, m = 1 \rangle} = E | \varphi_m \rangle_{| j, m = 1 \rangle}. \tag{3}
\]

where \( \hat{J}_m | j, m \rangle = \hat{j}_m^\dagger | j, m \rangle \), with \( \hat{j}_m^\dagger = \sqrt{(j + 1)} - m(m \pm 1) \). Then, we left multiply \( \{ | n, j \rangle \} \) to equation (3), which results in
where $n = -j, -j + 1, \ldots, j$. Furthermore, the photonic state can be expanded as

$$|\varphi_m\rangle_b = \sum_{k=0}^{N_p} \frac{1}{\sqrt{k!}} c_{mk}(\hat{A}_m^\dagger)^k |0\rangle_A,$$

with $N_p$ the truncation number of photon excitations. Finally, we obtain the eigen-equation

$$\omega_l - g_n^2 c_{nl} - \Delta_j^m \sum_{k=0}^{N_p} c_{n+1,k} A_n \langle l|k\rangle A_{n+1} = E c_{nl},$$

where the coefficients are $A_n\langle l|k\rangle A_{n+1} = (-1)^l D_{lk}$ and $A_n\langle l|l\rangle A_{n+1} = (-1)^l D_{ll}$, with

$$D_{lk} = e^{-\gamma l^2/2} \sum_{r=0}^{\min\{l|k\rangle \langle k|l\rangle}, \quad G = \frac{2\lambda}{\omega \sqrt{N}}.$$

It should be noted that by applying the extended bosonic coherent state, we are solving the eigen-problem of the Dicke model in the full space of the total excitation parity ($\exp(\pi N_{ex}) = 1$ for $N \in$ even and $\exp(\pi N_{ex}) = -1$ for $N \in$ odd). Once we numerically solve the eigensolution $\hat{H}_0|\phi_k\rangle = E_k|\phi_k\rangle$, the original solution can be straightforwardly obtained as

$$\hat{H}_D|\phi_k\rangle = E_k|\phi_k\rangle,$$

with the eigenstate $|\phi_k\rangle = \exp(-i\pi \hat{\mathcal{J}}/2)|\phi_k\rangle$.

2.3. Quantum dressed master equation

For practical light–matter coupled systems, it is inevitable to interact with the dissipative environment, which leads to the Hamiltonian system

$$\hat{H} = \hat{H}_D + \hat{H}_B + \hat{V}.$$

Here, $\hat{H}_D$ is given by equation (2) and the bosonic thermal baths are expressed as,

$$\hat{H}_B = \sum_{u,q} \sum_k \omega_k b_{uk}^\dagger b_{uk},$$

where $b_{uk}^\dagger$ creates (annihilates) one boson in the $u$th bath with the frequency $\omega_k$. And the interactions between the Dicke system with bosonic thermal baths are specified as

$$\hat{V} = \hat{V}_q + \hat{V}_c,$$

with

$$\hat{V}_q = \sum_k (\lambda_{uk} b_{uk}^\dagger + \lambda_{uk}^\dagger b_{uk})(\hat{J}_+ + \hat{J}_-) / \sqrt{N},$$

and

$$\hat{V}_c = \sum_k (\lambda_{ck} b_{ck}^\dagger + \lambda_{ck}^\dagger b_{ck})(\hat{a}^\dagger + \hat{a}),$$

where $\lambda_{uk}^\dagger(\lambda_{uk})$ the coupling strength between the qubits (photon) and the corresponding bosonic bath. The $n$th thermal bath is characterized by the spectral function $\gamma_n(\omega) = 2\pi \Gamma \delta(\omega - \omega_n)$ in this paper, we specify $\gamma_n(\omega)$ the Ohmic case $\gamma_n(\omega) = \pi\alpha\omega \exp(-|\omega|/\omega_c) [5]$, where $\alpha$ is the coupling strength and $\omega_c$ is the cutoff frequency of thermal baths.

We assume the weak interaction between the Dicke system and thermal baths. Under the Born–Markov approximation, we obtain the quantum dressed master equation to investigate the dissipative dynamics of the Dicke system as [38, 49]

$$\frac{d}{dt}\rho = -i[H_B, \rho] + \sum_{u,k=1} \{\Gamma^u_{nk}(\Delta_u)|\phi_k\rangle\langle \phi_k|, \rho \} + \{\Gamma^u_{nu}(\Delta_u)|\phi_k\rangle\langle \phi_k|, \rho \}$$

where $|\phi_k\rangle$ is the eigenfunction of the Dicke model $\hat{H}_D$ as $\hat{H}_D|\phi_k\rangle = E_k|\phi_k\rangle$, the dissipator is $\mathcal{D}[\hat{O}, \rho] = \frac{1}{2} [2\hat{O}\rho^{\dagger} - \rho^{\dagger}\hat{O} - \hat{O}^{\dagger}\rho]$, the rate is $\Gamma_{nk}^u = \gamma_u(\Delta_u)S_{nk}^u$ [2], $S_{nk}^u = \langle \phi_k|\hat{O}^{\dagger} + \hat{O}|\phi_k\rangle$ and $\hat{S}_{nk}^u = \langle \phi_k|\hat{O}^{\dagger} + \hat{O}|\phi_k\rangle$, the Bose–Einstein distribution function is $\psi_{nk}(\Delta_u) = 1/[\exp(\Delta_u/k_B T_{q(u)}) - 1]$ with $T_{q(u)}$ the temperature of the $q(u)$th thermal bath, and the energy gap is $\Delta_u = E_c - E_k$.

In the eigen-basis, the population dynamics is given by

$$\frac{d}{dt}\rho_{nn} = \sum_{u,k=1} \Gamma^u_{nk}(\Delta_u)\rho_{nk} - \sum_{u,k\neq n} \Gamma^u_{nk}(1 + n_n(\Delta_u))\rho_{nn},$$

with $\Gamma^u_{nk} = -\Gamma^u_{nk}$. It should be noted that both the Dicke Hamiltonian in equation (1) and the total Hamiltonian in equation (9) commute with the collective angular momentum operator $\hat{J}^2 = \sum_{l,m=1} \hat{J}_l^2$. Hence, we can individually study the dissipative dynamics in separate subspaces of the collective angular momentum $\{\langle l,m\rangle \}$ having different good quantum number $l$, with $\hat{J}_l^2 = \hat{J}_l^2 | l,m \rangle \langle l,m |$ ($l = 0, 1, \ldots, N/2$) for $N \in$ even or ($l = 1/2, 3/2, \ldots, N/2$) for $N \in$ odd. In the present work, we select the initial state of qubits in the totally symmetric subspace, i.e. $|N/2, -N/2\rangle$ which makes both the transient state and steady state in the totally symmetric subspace. The generalization of the initial state to the mixed subspaces is straightforward, and the calculation procedure is quite similar.

In particular, as $T_q = T_c = T$ the Dicke system at steady state is in thermal equilibrium, such that the equilibrium density matrix operator is [9]

$$\hat{\rho}_s = \sum_k e^{-E_k/(k_B T)} |\phi_k\rangle\langle \phi_k|,$$

with the partition function $Z = \sum_k e^{-E_k/(k_B T)}$. And the steady state population is specified as

$$P_k = e^{-E_k/(k_B T)} / Z.$$
For the open quantum Rabi and Dicke models, the traditional way to investigate the dynamics of quantum systems is to apply the Lindblad master equation by phenomenologically considering the system-bath interaction, where the qubits and photons separately participate into the dissipative processes with the corresponding thermal baths [43–45, 48]. However, as the qubit–photon coupling becomes strong, the qubits and photons are strongly hybridized [38, 49]. Then, the quantum dressed master equation should be included to properly study the system dynamics, where the transitions assisted by thermal baths occur among the eigenstates, e.g. $|\phi_i\rangle$ in equation (8).

3. Zero-time delay two-photon correlation function

In quantum optics, the traditional definition of steady state two-photon correlation function, which was initially proposed by Glauber, is expressed as [1]

$$G^{(2)}(0) = \frac{\langle \hat{a}^\dagger \hat{a} \hat{a} \hat{a} \rangle}{\langle \hat{a} \hat{a} \rangle^2},$$

where $\langle \cdot \cdot \cdot \rangle$ means the expectation value at steady state. In particular, the bunching with super-Poisson distribution and antibunching with sub-Poisson distribution are two representative features of photon statistics. Quantitatively, the photon–photon correlation with the bunching is characterized as [10]

$$G^{(2)}(0) > 1.$$ (19)

In contrast, the photon antibunching is defined as

$$G^{(2)}(0) < 1.$$ (20)

Moreover, for the thermal state, the correlation function is $G^{(2)}(0) = 2$ [10, 50]. Such definition of the two-photon correlation function may be properly applied to investigate photon statistics in Lindblad form of open quantum systems with weak light–matter interaction.

However, as the light–matter interaction becomes strong, the two-photon correlation function should be measured in eigenspace. The generalized two-photon correlation function of the finite size Dicke model is given by [7, 8]

$$G_N^{(2)}(0) = \frac{\langle \hat{X}^+ \hat{X}^+ \rangle}{\langle \hat{X}^{-} \hat{X}^{-} \rangle},$$ (21)

where $N$ is the qubit number, $\hat{O} = \text{Tr}\{\hat{O}\rho_s(t \to \infty)\}$ and

$$\hat{X}^+ = -i \sum_{k>j} \Delta_{kj} X_k |\phi_j\rangle \langle \phi_k|,$$ (22)

with $\hat{X}^+ = (\hat{X}^+)\dagger$, $\Delta_{kj} = E_k - E_j$, and $X_k = \langle \phi_j | (\hat{a}^\dagger + \hat{a}) | \phi_k \rangle$. $X^+_{kj}$ describes the transition from the higher eigenstate $|\phi_k\rangle$ to the lower one $|\phi_j\rangle$. In particular, the two eigenstates $|\phi_k\rangle$ and $|\phi_j\rangle$ involved in the transition coefficient $X^+_{kj}$ should belong to the different space of the total excitation number parity $\exp(i\pi N_{\text{tot}})$, to make $X^+_{kj} \neq 0$. Then, we can correctly apply the two photon correlation function in equation (21) to investigate the behavior of photon statistics in the following section.

Hence, it is crucial to obtain the eigenstates and eigenvalues with both branches of the parity, as done in section 2.2. It should be noted that $\hat{X}^+ |\phi_0\rangle = 0$ for the ground state of $\hat{H} = \hat{H}_s + \hat{H}_B$, in contrast to $|\phi_i\rangle \neq 0$. Moreover, in absence of the qubit–photon interaction (i.e. $\lambda/\omega = 0$), the operator $\hat{X}^+$ is reduced to $\hat{X}^+ = -i\omega \hat{a}$. Then, the one and two photon correlation terms are simplified as

$$\langle \hat{X}^\dagger \hat{X}^+ \rangle = \omega^2 n_s(\omega),$$ (23)

$$\langle \hat{X}^{-} \hat{X}^{-} \rangle = 2\omega^2 n_s^2(\omega),$$ (24)

with the Bose–Einstein distribution function

$$n_s(\omega) = 1/[\exp(\omega/k_B T_c) - 1].$$ (25)

Consequently, the two-photon correlation function in equation (21) becomes $G_N^{(2)}(0) = 2$ and the expression returns back to the counterpart in equation (18). The expression of correlation function in equation (21) has been extensively analyzed in the dissipative quantum Rabi model and optomechanical systems [7–9]. In the following, we apply $G_N^{(2)}(0)$ to study the steady state two-photon statistics in the dissipative Dicke model with finite qubit number.

From the previous studies of the ground state phase transition with extended coherent bosonic states, it is surprising to find that $N_{\text{tr}} = 6$ is accurate enough to obtain the ground state energy with large qubit number $N = 32$ [28]. In this work, we select the truncation number $N_{\text{tr}} = 50$ up to $N = 160$, which is sufficient to guarantee the convergence of the two-photon correlation function as shown in figure 1.

4. Results and discussions

4.1. Effect of qubit–photon coupling strength

We first investigate the effect of qubit–photon interaction on the two-photon correlation function $G_N^{(2)}(0)$ with low temperatures of thermal baths (e.g. $T_c = T_q = 0.05\omega$), shown in figure 2(a). In the qubit–photon coupling regime $\lambda \in (0,0.3\omega)$, the finite eigenenergy difference (see figure 3(a))
results in the dramatic suppression of the steady state populations, i.e., $P_1 \gg P_2 \gg P_3 \gg P_4$, according to the steady state population distribution in equation (17). Hence, one and two photon correlation terms of $G_N^{(2)}(0)$ in equation (21) may be approximated as

$$\langle X^- X^+ \rangle \approx \omega^2 n_c(\omega)(P_1 A_1 + P_2 A_2 + P_3 A_3),$$

(26)

$$\langle (X^-)^2 (X^+)^2 \rangle \approx \omega^4 n_c^2(\omega)(P_2 B_2 + P_3 B_3 + P_4 B_4),$$

(27)

where $n_c(\omega)$ is the Bose–Einstein distribution function in equation (25), the coefficients are

$$A_k = \frac{1}{\omega^2 n_c(\omega)} \sum_{i<k} (\Delta_{ik} X_{ik})^2,$$

(28)

and

$$B_k = \frac{1}{\omega^2 n_c^2(\omega)} \sum_{p<k} (\Delta_{ik} X_{ik} \Delta_{ip} X_{ip})^2,$$

(29)

with the energy gap $\Delta_{ik} = E_i - E_k$ and transition coefficient $X_{ik}$ given in equation (22). Such approximation is numerically verified in figures 2(b) and (c).

From figure 3(b), the detection induced transition between eigenstates $|\phi_2\rangle$ and $|\phi_1\rangle$ assisted by the $c$th thermal bath is blocked ($X_{21} = 0$), which results from the same odd parity $(e^{\pm i(\theta + J_c + N/2)}) = -1$. And the detection coefficient $X_{32}$ is much smaller than $X_{31}$. Hence, one and two photon correlation terms in equations (26) and (27) are simplified as $\langle X^- X^+ \rangle \approx P_1(\Delta_{10}X_{10})^2$ and $\langle (X^-)^2 (X^+)^2 \rangle \approx P_3(\Delta_{31}X_{31}X_{10})^2$. Then, the two-photon correlation function is approximated as

$$G_s^{(2)}(0) \approx P_3(\Delta_{31}X_{31})^2 / [P_1^2(\Delta_{10}X_{10})^2].$$

(30)

It is found that by enhancing the interaction strength $\lambda$ the two-photon correlation function shows subthermal behavior (i.e. $G_s^{(2)}(0) < 1$).

In the regime $\lambda \in (0.3\omega, 0.6\omega)$, due to the avoided crossing of the energy levels $E_2$ and $E_3$, the corresponding parity is exchanged (see the yellow solid line with up-triangle and purple solid line with down-triangle in figure 3(a)). The two-photon correlation function is changed into

$$G_s^{(2)}(0) \approx P_2(\Delta_{21}X_{21})^2 / [P_1^2(\Delta_{10}X_{10})^2].$$

(31)

From figure 2(b), the fast increase of the output power $\langle X^- X^+ \rangle$ dominates the photon distribution, resulting in the two-photon blockade. Therefore, it clearly demonstrates the antibunching feature (i.e. $G_s^{(2)}(0) < 1$).

Then, by further increasing the coupling strength to the regime $\lambda \in (0.6\omega, 0.85\omega)$, the second and third excited energy levels become nearly degenerate, which both contribute to the correlation function $\langle (X^-)^2 (X^+)^2 \rangle$. Moreover, the detection induced transition efficiency $X_{20} = 0$ due to the same parity of $|\phi_2\rangle$ and $|\phi_0\rangle$. Hence, the two-photon correlation function is approximately expressed as

$$G_s^{(2)}(0) \approx P_2(\Delta_{21}X_{21}X_{10})^2 / P_1(\Delta_{10}X_{10})^4.$$

(32)
which can also be verified by the coefficients’ magnitudes in figures 2(b) and (c). It is interesting to observe an antibunching to bunching transition of photons. And the pronounced two photon enhancement signature is clearly exhibited (i.e. $G_8^{(2)}(0) \gg 2$). The fast decay of the two-photon correlation function in figure 4(a) is dramatically reduced to 2 due to formation of the thermal state of the Dicke system (see the appendix for details).

$$\hat{\rho}_s = \sum_m |m\rangle_s \langle m| e^{-\frac{1}{2} \left( \alpha_m \hat{a}_m - \left( \frac{2}{\sqrt{\omega N}} \right)^2 \right)} / (\hbar \omega T),$$

with the eigenstate of $\hat{J}_z$ as $|J_z, m\rangle_s = |m\rangle_s$, the displaced bosonic operator $\hat{a}_m = \hat{a} + 2m/(\omega \sqrt{N})$ and the partition function $Z = \frac{1}{1 - e^{-\omega \sqrt{N} / \hbar \omega T}} \sum_m \exp(\frac{1}{2} \frac{\hbar \omega T}{\sqrt{\omega \sqrt{N}}})$. Finally, the photons are inclined to be classically distributed. Therefore, we conclude that ultrastrong qubit–photon interaction may exhibit the antibunching to bunching transition. For the open Rabi model ($N = 1$), the photons are monotonically suppressed by increasing qubit–photon coupling strength [9], which is quite distinct from the counterpart in the dissipative finite-size Dicke model. It should be noted that the ultrastrong coupling condition is not necessary for the antibunching to bunching transition. It can also be realized by optimally tuning the photon loss rate [44], incoherent pumping rate [43, 45] and laser pumping frequency [47].

Moreover, the above analysis of the two-photon correlation function is carried out at resonance ($\Delta = 0$). It needs to be stressed that the extension to the biased regime (i.e. $\Delta \neq 0$) is straightforward based on the quantum dressed master equation (equation (14)). Then, we study the influence of the qubit–photon coupling strength on $G_8^{(2)}(0)$ at off-resonance ($\delta = \omega - \Delta$), shown in figure 4(a). It is found that the general profiles of $G_8^{(2)}(0)$ are similar for different $\delta$. However, by increasing the energy bias $\delta$ the antibunching effect of photons becomes comparatively weak. The main reason is that the peak magnitude of one photon correlation term $\langle X^- \rangle$ in figure 4(b) is suppressed monotonically.

Considering the general similarity of the profiles of the two-photon correlation function in figures 2(a) and 4(a), we focus on the resonant case as the representative for analysis in the following.

### 4.2. Effect of finite qubit number

We analyze the influence of the finite qubit number on the two-photon correlation function in figure 5(a). By increasing the qubit number, it is interesting to find that the minimum of $G_8^{(2)}(0)$ shows monotonic enhancement with the increase of qubit number. However, the peak of $G_N^{(2)}(0)$ of the finite size Dicke model is firstly enhanced and then suppressed. Such optimization can be clearly observed in figure 5(b). Hence,
we conclude that the antibunching to bunching transition is a finite-size effect, which vanishes in the thermodynamic limit. Such an optimal feature with finite qubit number is distinct from the divergent behavior of the quantum entanglement and stable value of the scaled concurrence in the thermodynamic limit to characterize superradiant phase transition [25–28].

Next, we analyze the scaling behavior of the qubit–photon coupling strength at the extreme values of the $G_k^{(2)}(0)$ with the finite qubit number $N$ in figure 5(c). It is found that they scale as

$$\lambda_{\text{max(min)}} - \lambda_c \propto N^{-0.8\pm0.6}, \quad (34)$$

where $\lambda_{\text{max(min)}}$ corresponds to the peak(valley) of the two-photon correlation function, and $\lambda_c = \frac{\omega_0^2}{4k_B T_{q,\text{th}}} \coth(\frac{\Delta}{2k_B T_{q,\text{th}}})$ is the critical coupling strength at finite temperature [51]. It implies that by increasing the qubit number, it is easier to observe the antibunching to bunching transition in a comparative weak qubit–photon coupling regime. Moreover, in the previous study of the quantum phase transition in the Dicke model, qubit–photon coupling strength $\lambda_{\text{max}}$ and $\lambda_{\text{max}}$, corresponding to the peaks of the entanglement entropy and scaled concurrence, are considered to be crucial indicators to quantify the phase transition [25]. Their behavior as $(\lambda_{\text{max}} - \lambda_0^0) \propto N^{-0.75\pm0.1}$ and $(\lambda_{\text{max}} - \lambda_0^0) \propto N^{-0.68\pm0.1}$, with $\lambda_0^0 = \frac{\omega_0^2}{4k_B T_{q,\text{th}}}$. However, such scaling exponents are apparently different from the counterpart of the two-photon correlation function in equation (34). Therefore, this demonstrates that the scaling behavior of the optimal qubit–photon coupling strength in equation (34) may be another potential indicator to detect the criticality of the finite qubit number Dicke model.

4.3. Effect of finite temperatures of thermal baths

We investigate the influence of the bath temperatures on the two-photon correlation function in figure 6(a) with finite qubit number (e.g., $N = 8$). In the ultrastrong coupling regime (e.g., $\lambda = 0.1\omega$, $0.4\omega$), by increasing the temperature the two-photon correlation function is enhanced from the antibunching to bunching feature, and approaches thermal distribution ($G_k^{(2)}(0) = 2$) in a comparatively high temperature regime (e.g., $T = 0.35\omega$). While in the qubit–photon coupling regime (e.g., $\lambda = 0.7\omega$), a giant two-photon bunching signature is clearly observed in the low temperature regime (e.g., $T = 0.1\omega$), whereas the two-photon correlation function is dramatically reduced as the temperature increases. Furthermore, in the deep strong coupling regime (e.g., $\lambda = \omega$), the photons are nearly thermally distributed with $G_k^{(2)}(0)$ slightly above 2 in the wide temperature zone. Then, we give a comprehensive picture of $G_k^{(2)}(0)$ by both tuning temperature and coupling strength in figure 6(b). It is found that the photon blockade and two-photon enhancement become significant in the low temperature regime with ultrastrong coupling strength. While as the temperature increases, the fluctuation of two-photon correlation function is suppressed monotonically, finally resulting in the thermal state of photons ($G_k^{(2)}(0) \approx 2$). Therefore, we conclude that the low temperature of thermal baths can enhance both the bunching and antibunching signatures of photons.

Next, we investigate the effect of the temperature bias ($T_e \neq T_q$) on the two-photon correlation function in figure 7. With ultrastrong qubit–photon interaction (e.g., $\lambda = 0.1\omega$ in figure 7(a) and $\lambda = 0.4\omega$ in figure 7(b)), the super-thermal behavior of photons (i.e., $G_k^{(2)}(0) > 2$) is exhibited at high $T_q$ and low $T_e$, while $T_e$ with large temperature bias. As the coupling strength increases (e.g., $\lambda = 0.7\omega$ in figure 7(c)), the giant photon bunching is exhibited with both low $T_q$ and $T_e$, compared to the counterpart with both high $T_e$ and $T_q$. Then, if we further increase the coupling strength to the deep regime (e.g., $\lambda = \omega$ in figure 7(d)), high $T_e$ and low $T_q$ jointly contribute to the

Figure 7. A 3D view of the two-photon correlation function $G_k^{(2)}(0)$ with (a) $\lambda = 0.1\omega$, (b) $\lambda = 0.4\omega$, (c) $\lambda = 0.7\omega$ and (d) $\lambda = \omega$. The other system parameters are the same as in figure 2.
significantly large two-photon bunching. Hence, we conclude that the two-photon correlation can be dramatically enhanced with deep strong qubit–photon interaction at large temperature bias.

5. Conclusion

To summarize, we study the two-photon correlation function in the dissipative Dicke model, where the qubits and the photons are individually coupled to thermal baths, respectively. The quantum dressed master equation combined with extended bosonic coherent state approach is applied to analyze the steady state behavior of the dissipative Dicke system, which is able to handle strong qubit–photon interaction. We investigate the influence of the qubit–photon coupling strength in the two-photon correlation function. An antibunching to bunching transition is clearly exhibited in the ultrastrong coupling regime. We also analyze the effect of the finite qubit number on the two-photon correlation function. It is found that the maximal two photon bunching feature is observed with the optimal qubit number. Moreover, the coupling strengths at the extreme values of two-photon correlation function scale as $\lambda_{\text{max(min)}} \approx 1/N$ with $\lambda$, the superradiant phase transition of the Dicke model at finite temperature. Then, we analyze the effect of the finite temperature on the two-photon correlation. The low bath temperature is crucial to exhibit the two-photon blockade and bunching behaviors. Moreover, we study the two-photon correlation function with temperature bias of thermal baths. It is found that strong qubit–photon interaction and large temperature bias cooperatively contribute to the giant two-photon bunching.

In the current work we mainly investigate the zero-time delay correlation function of photons. We should note that the finite-time delay two-photon correlation function is also a powerful tool to analyze the photon distribution, e.g. photon blockade in optomechanics [7]. We may apply the finite-time delay correlation function in further to analyze the photon behavior of the dissipative Dicke model.

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Appendix A. Two-photon correlation function at strong qubit–photon coupling

In the strong qubit–photon coupling limit, the qubit tunneling is strongly dressed and the Hamiltonian in equation (1) is simplified $\hat{H}_{\text{strong}} \approx \omega \hat{a} \hat{a} + \frac{2 \lambda}{\sqrt{N}} (\hat{a}^\dagger + \hat{a}) \hat{J}_x$, which can be re-expressed as

$$\hat{H}_{\text{strong}} \approx \sum_{m} \langle m | \hat{J}_x | m \rangle \left( \omega \hat{a} \hat{a} + \frac{2 \lambda}{\sqrt{N}} (\hat{a}^\dagger + \hat{a}) \right),$$

(A.1)

where $\hat{J}_x | m \rangle = m | m \rangle$. If we define the displaced bosonic operator associated with the angular momentum as $\hat{A}_m = \hat{a} + \frac{\sqrt{\lambda \omega}}{\sqrt{N}}$, the Hamiltonian is given by

$$\hat{H}_{\text{strong}} \approx \sum_{m} \langle m | \hat{J}_x | m \rangle \left( \omega \hat{A}_m^\dagger \hat{A}_m - \frac{2 \lambda}{\sqrt{N}} \right).$$

(A.2)

Hence, the steady state thermal state is given by

$$\hat{\rho}_s = \frac{1}{Z} \sum_{m} \langle m | \hat{J}_x | m \rangle e^{-\omega \hat{A}_m^\dagger \hat{A}_m / (k_B T)},$$

(A.3)

where the temperature $T_q = T_c = T$, $d_m = \exp(\frac{2 \lambda \omega}{\sqrt{N}} / k_B T)$ and $Z$ is the partition function to normalize $\hat{\rho}_s$. Moreover, the photon detection operator is specified as

$$\hat{X}^- = -i \omega \sum_{m} \langle m | \hat{J}_x | m \rangle \hat{A}_m^\dagger.$$ (A.4)

Therefore, it is easy to calculate the correlation functions at thermal state as

$$\langle \hat{X}^- \hat{X}^+ \rangle = \omega [e^{\omega / (k_B T)} - 1],$$

(A.5)

$$\langle (\hat{X}^-)^2 (\hat{X}^+)^2 \rangle = 2 \omega^2 [e^{\omega / (k_B T)} - 1]^2.$$ (A.6)

Finally, we obtain the two-photon correlation function as $G_2^{\text{th}}(0) = 2$.

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