Error analyses of Sinc-Nyström methods for initial value problems

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Abstract: Nurmuhammad et al. proposed two Sinc-Nyström methods for initial value problems. These two methods use different variable transformations; one method uses a single-exponential transformation and the other uses a double-exponential transformation. In the previous study, we improved their methods by replacing those variable transformations to achieve better performance. However, no error analysis of the improved methods has been conducted yet. Therefore, in this study, we perform error analyses of the improved methods.

Key Words: Sinc indefinite integration, Sinc-Nyström methods, SE transformation, DE transformation

1. Introduction

We are concerned with numerical methods for initial value problems of the following form:

\[
\begin{align*}
  y'(t) &= K(t)y(t) + g(t), \\
  y(0) &= r,
\end{align*}
\]

where \( K(t) \) is an \( m \times m \) matrix whose \((i, j)\) component is \( k_{ij}(t) \), and \( y(t), g(t), r \) are \( m \)-dimensional vectors. In this paper, the solution \( y(t) \) is supposed to decay exponentially as \( t \to \infty \). Most of the methods for the problems achieve polynomial convergence with respect to the sampling number \( l \), e.g., \( O(l^{-p}) \) for some \( p > 0 \). In contrast, Nurmuhammad et al. \cite{1} proposed exponentially converging numerical methods, which were developed based on the Sinc indefinite integration and a variable transformation. They considered two different variable transformations. A single-exponential (SE) transformation is employed in one method, and we call this method the SE-Sinc-Nyström method. The other method employs a double-exponential (DE) transformation, and we call this method the DE-Sinc-Nyström method. Based on their numerical investigation, Nurmuhammad et al. \cite{1} reported that the SE-Sinc-Nyström method achieved \( O(e^{-c_1\sqrt{l}}) \) for some \( c_1 > 0 \) and the DE-Sinc-Nyström method achieved \( O(e^{-c_2l/\log l}) \) for some \( c_2 > 0 \), even for stiff problems.

In the previous study \cite{2}, we improved the SE-Sinc-Nyström method and DE-Sinc-Nyström method by replacing the SE and DE transformation, respectively. By replacing the SE transformation, we observed faster convergence \( O(e^{-c'_1\sqrt{l}}) \) for some \( c'_1 > c_1 \), and by replacing the DE transformation,
we reduced the computational cost without compromising the performance (see [2] for the detailed discussion).

However, the error analyses of these methods have not been conducted yet. Therefore, in this study, we perform the error analyses of the improved SE- and DE-Sinc-Nyström method.

The remainder of this paper is organized as follows. The Sinc indefinite integration combined with a variable transformation is described in Section 2, the Sinc-Nyström methods are described in Section 3, the error analyses of the Sinc-Nyström methods (the main result of this paper) are described in Section 4, numerical examples are shown in Section 5, the proof of the new theorem in Section 2 is given in Section 6, the proofs of the new theorems in Section 4 are given in Section 7, and conclusions are given in Section 8.

2. Sinc indefinite integration

The Sinc indefinite integration over the whole real axis is expressed as

\[ \int_{-\infty}^{\xi} F(x) \, dx \approx \sum_{j=-N}^{N} F(jh)J(j, h)(\xi), \quad \xi \in \mathbb{R}, \] (2)

where \( M, N, \) and \( h \) are selected appropriately depending on given positive integer \( n, \) and \( J(j, h)(x) \) is defined by using the sine integral

\[ \text{Si}(x) = \int_{0}^{x} \frac{\sin t}{t} \, dt \]

as

\[ J(j, h)(x) = h \left\{ \frac{1}{2} + \frac{1}{\pi} \text{Si} \left( \frac{\pi(x - jh)}{h} \right) \right\}. \]

In what follows, we consider the application of the Sinc indefinite integration to

\[ I(t) := \int_{0}^{t} f(s) \, ds, \quad t \in (0, \infty). \]

For this purpose, we should employ a proper variable transformation to transform \( I(t) \) to the left-hand side in Eq. (2).

2.1 SE transformation

Muhammad–Mori [3] considered the SE transformation

\[ s = \psi(x) = \log(1 + e^{x}), \]

and put \( F(x) = f(\psi(x)) \psi'(x) \) in Eq. (2) to derive the following approximation formula:

\[ \int_{0}^{1} f(s) \, ds = \int_{-\infty}^{\psi^{-1}(t)} f(\psi(x)) \psi'(x) \, dx \approx \sum_{j=-M}^{N} f(\psi(jh)) \psi'(jh)J(j, h)(\psi^{-1}(t)). \] (3)

To describe its error analysis, we introduce the following function space.

**Definition 1.** Let \( D \) be a simply connected complex domain, let \( \beta \) be a positive constant, and let \( \alpha \) be a constant with \( 0 < \alpha \leq 1. \) Then, \( L_{\alpha, \beta}(D) \) denotes a family of functions \( f \) that are analytic on \( D, \) and such that for some constant \( C_1 > 0, \)

\[ |f(z)| \leq C_1 \left| \frac{z}{1 + z} \right|^{\alpha - 1} |e^{-z}|^{\beta} \] (4)

holds for all \( z \in D. \)

The error analysis of the formula Eq. (3) has already been given as follows. Here, let \( D_d \) denote a strip complex domain defined as \( D_d = \{ \zeta \in \mathbb{C} : |\text{Im}\, \zeta| < d \} \) for \( d > 0. \)
Theorem 1 (Hara–Okayama [4, Theorem 2]). Let \( \beta \) be a positive constant, let \( \alpha \) and \( d \) be constants with \( 0 < \alpha \leq 1 \) and \( 0 < d < \pi \), and let \( f \in L_{\alpha,\beta}(\varphi(D_d)) \). Let \( \mu = \min\{\alpha, \beta\} \), let \( M \) and \( N \) be defined as

\[
\begin{align*}
M &= n, \quad N = \left\lfloor \frac{\alpha}{\beta} \right\rfloor (if \ \mu = \alpha), \\
N &= n, \quad M = \left\lfloor \frac{\beta}{\alpha} \right\rfloor (if \ \mu = \beta),
\end{align*}
\]

and let \( h \) be defined as

\[
h = \sqrt{\frac{\pi d}{\mu n}}.
\]

Then, there exists a positive constant \( C \) independent of \( n \) such that

\[
\sup_{t \in (0, \infty)} \left| I(t) - \sum_{j=-M}^{N} f(\varphi(jh))\varphi'(jh)J(j,h)(\varphi^{-1}(t)) \right| \leq C e^{-\sqrt{\pi d \mu n}}.
\]

2.2 DE transformation

Okayama [5] proposed the DE transformation

\[
s = \phi(x) = \log(1 + e^{\pi \sinh(x)}),
\]

and put \( F(x) = f(\phi(x))\phi'(x) \) in Eq. (2) to derive the following approximation formula:

\[
\int_{0}^{t} f(s) \, ds = \int_{-\infty}^{\varphi^{-1}(t)} f(\phi(x))\phi'(x) \, dx \approx \sum_{j=-M}^{N} f(\varphi(jh))\phi'(jh)J(j,h)(\varphi^{-1}(t)).
\]

In this paper, we give its error analysis as below. The proof is given in Section 6.

Theorem 2. Let \( \beta \) be a positive constant, let \( \alpha \) and \( d \) be constants with \( 0 < \alpha \leq 1 \) and \( 0 < d < \pi/2 \), and let \( f \in L_{\alpha,\beta}(\varphi(D_d)) \). Let \( \mu = \min\{\alpha, \beta\} \), let \( M \) and \( N \) be defined as

\[
\begin{align*}
M &= n, \quad N = \left\lfloor \frac{1}{h \arcsinh \left( \frac{\alpha}{\beta} \sinh(nh) \right)} \right\rfloor (if \ \mu = \alpha), \\
N &= n, \quad M = \left\lfloor \frac{1}{h \arcsinh \left( \frac{\beta}{\alpha} \sinh(nh) \right)} \right\rfloor (if \ \mu = \beta),
\end{align*}
\]

and let \( h \) be defined as

\[
h = \frac{\arcsinh(\frac{dn}{\mu})}{n}.
\]

Then, there exists a positive constant \( C \) independent of \( n \) such that

\[
\sup_{t \in (0, \infty)} \left| I(t) - \sum_{j=-M}^{N} f(\varphi(jh))\varphi'(jh)J(j,h)(\varphi^{-1}(t)) \right| \leq C \frac{\arcsinh(\frac{dn}{\mu})}{n} e^{-\pi dn/\arcsinh(\frac{dn}{\mu})}.
\]

3. Sinc-Nyström methods

3.1 SE-Sinc-Nyström method

Integrating both sides of Eq. (1), we derive the following:

\[
y(t) = r + \int_{0}^{t} \{ K(s)y(s) + g(s) \} \, ds.
\]
Let $l = M + N + 1$ and let $y^{(l)}(t)$ be an approximate solution of $y(t)$. Based on Theorem 1, approximating the integral in Eq. (10), we derive

$$y^{(l)}(t) = r + \sum_{j=-M}^{N} \left\{ K(\psi(\phi))y^{(l)}(\phi(\phi)) + g(\phi(\phi)) \right\} \phi'(\phi(\phi))J(\phi(\phi))(\phi^{-1}(t)).$$

(11)

To determine the unknown coefficients $y^{(l)}(\phi(\phi))$, we set sampling points at $t = \phi(ih)$ ($i = -M, -M + 1, \ldots, N$). Then, we obtain a system of linear equations given by

$$(I_m \otimes I_l - (I_m \otimes \{ hI_l^{(-1)}D_l^{(\phi)} \})[K_{ij}^{(\phi)}])Y^{(\phi)} = R + (I_m \otimes \{ hI_l^{(-1)}D_l^{(\phi)} \})G^{(\phi)},$$

(12)

where $I_l$ and $I_m$ are identity matrices of order $m$ and $n$, respectively, $\otimes$ denotes the Kronecker product, and $I_l^{(-1)}$ is an $l \times l$ matrix whose $(i, j)$ components are defined as

$$(I_l^{(-1)})_{ij} = \frac{1}{2} + \frac{1}{\pi} \text{Si}(\pi(i-j)) \quad (i, j = -M, -M + 1, \ldots, N).$$

Moreover, $D_l^{(\phi)}$ and $K_{ij}^{(\phi)}$ are $l \times l$ diagonal matrices defined as

$$D_l^{(\phi)} = \text{diag}[\psi'(-Mh), \ldots, \psi'(Nh)],$$

$$K_{ij}^{(\phi)} = \text{diag}[\psi(ih), \ldots, \psi(ih)],$$

and $[K_{ij}^{(\phi)}]$ is a block matrix whose $(i, j)$ component is $K_{ij}^{(\phi)}$ ($i, j = 1, \ldots, m$). Furthermore, $R, Y^{(\phi)}$, and $G^{(\phi)}$ are $lm$-dimensional vectors defined as follows:

$$R = [r_1, \ldots, r_1, r_2, \ldots, r_2, \ldots, r_m, \ldots, r_m]^T,$$

$$Y^{(\phi)} = [y_1^{(l)}(\psi(-Mh)), \ldots, y_1^{(l)}(\psi(Nh)), \ldots, y_m^{(l)}(\psi(-Mh)), \ldots, y_m^{(l)}(\psi(Nh))]^T,$$

$$G^{(\phi)} = [g_1(\psi(-Mh)), \ldots, g_1(\psi(Nh)), \ldots, g_m(\psi(-Mh)), \ldots, g_m(\psi(Nh))]^T.$$

By solving Eq. (12), we can get the value of $y^{(l)}(\psi(\phi(\phi)))$, from which $y^{(l)}(t)$ is determined through Eq. (11). This procedure is the SE-Sinc-Nyström method proposed by the present authors [2].

### 3.2 DE-Sinc-Nyström method

Comparing Theorem 1 with Theorem 2, we can expect that the convergence rate should be accelerated by replacing $\psi$ with $\phi$ in the SE-Sinc-Nyström method. Using the replacement in Eq. (11), we derive

$$y^{(l)}(t) = r + \sum_{j=-M}^{N} \left\{ K(\phi(\phi))y^{(l)}(\phi(\phi)) + g(\phi(\phi)) \right\} \phi'(\phi(\phi))J(\phi(\phi))(\phi^{-1}(t)).$$

(13)

Setting sampling points at $t = \phi(ih)$ ($i = -M, -M + 1, \ldots, N$), we obtain

$$(I_m \otimes I_l - (I_m \otimes \{ hI_l^{(-1)}D_l^{(\phi)} \})[K_{ij}^{(\phi)}])Y^{(\phi)} = R + (I_m \otimes \{ hI_l^{(-1)}D_l^{(\phi)} \})G^{(\phi)},$$

(14)

for which $\phi$ is used instead of $\psi$. By solving Eq. (14), we can get the value of $y^{(l)}(\phi(\phi(\phi)))$, from which $y^{(l)}(t)$ is determined through Eq. (13). This procedure is the DE-Sinc-Nyström method proposed by the present authors [2].

### 4. Error analyses of the Sinc-Nyström methods

In this section, we state the error analyses of the SE-Sinc-Nyström method and the DE-Sinc-Nyström method described in Section 3. The proofs are given in Section 7.
4.1 Error analysis of the SE-Sinc-Nyström method
We analyzed the error of the SE-Sinc-Nyström method as follows.

**Theorem 3.** Let $\beta$ be a positive constant, and let $\alpha$ and $\mu$ be constants with $0 < \alpha \leq 1$ and $0 < \mu < \pi$. Assume that the function $k_{ij}$ ($i, j = 1, \ldots, m$) is analytic and bounded on $\psi(\mathcal{D})$, and $y_i$ and $g_i$ ($i = 1, \ldots, m$) belong to the function space $L_{\alpha, \beta}(\psi(\mathcal{D}))$. Let $h$ be set as Eq. (6), and let $M$ and $N$ be set as Eq. (5). Let $A_n$ be a coefficient matrix of the system of linear equation Eq. (12), i.e.,

$$A_n = I_m \otimes I_l - (I_m \otimes \{hI_l^{(1)} D_l^{(\psi)}\})[K_{ij}^{(\psi)}],$$

and let the inverse matrix of $A_n$ exist. Then, the error of the approximate solution $y^{(l)}(t)$ in Eq. (11) is estimated as

$$\max_{1 \leq i \leq m} \left\{ \sup_{t \in (0, \infty)} \left| y_i(t) - y_i^{(l)}(t) \right| \right\} \leq (C + \hat{C}||A_n^{-1}||_{\infty}) \sqrt{n} e^{-\sqrt{\pi \mu \nu}}, \quad (15)$$

where $C$ and $\hat{C}$ are positive constants independent of $n$.

4.2 Error analysis of the DE-Sinc-Nyström method
We analyzed the error of the DE-Sinc-Nyström method as follows.

**Theorem 4.** Let $\beta$ be a positive constant, and let $\alpha$ and $\mu$ be constants with $0 < \alpha \leq 1$ and $0 < \mu < \pi/2$. Assume that the function $k_{ij}$ ($i, j = 1, \ldots, m$) is analytic and bounded on $\phi(\mathcal{D})$, and $y_i$ and $g_i$ ($i = 1, \ldots, m$) belong to the function space $L_{\alpha, \beta}(\phi(\mathcal{D}))$. Let $h$ be set as Eq. (9), and let $M$ and $N$ be set as Eq. (8). Let $B_n$ be a coefficient matrix of the system of linear equation Eq. (14), i.e.,

$$B_n = I_m \otimes I_l - (I_m \otimes \{hI_l^{(1)} D_l^{(\phi)}\})[K_{ij}^{(\phi)}],$$

and let the inverse matrix of $B_n$ exist. Then, the error of the approximate solution $y^{(l)}(t)$ in Eq. (13) is estimated as

$$\max_{1 \leq i \leq m} \left\{ \sup_{t \in (0, \infty)} \left| y_i(t) - y_i^{(l)}(t) \right| \right\} \leq (C + \hat{C}||B_n^{-1}||_{\infty}) \arcsinh(\mu/\mu) e^{-\pi \mu \nu}/\arcsinh(\mu/\mu), \quad (16)$$

where $C$ and $\hat{C}$ are positive constants independent of $n$.

**Remark 1.** Both error estimates in Eq. (15) and Eq. (16) include the norm of the inverse of the coefficient matrix, which is not easy to give an a priori theoretical estimate. Therefore, in the next section, we investigate the norm numerically to observe whether it grows rapidly or not.

5. Numerical examples
In this section, we consider three initial value problems as derived below.

**Example 1.** Consider the following initial value problem:

$$y' = -\frac{2 + t}{1 + t} y + \frac{e^{-t}}{1 + t}, \quad y(0) = 0, \quad (17)$$

whose solution is $y(t) = \frac{t}{1 + t} e^{-t}$.

**Example 2** ([2, Example 1]). Consider the following initial value problem:

$$\begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} -2 & e^{-t} \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}, \quad \begin{pmatrix} y(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (18)$$

whose solution is $y(t) = t e^{-2t}$, $z(t) = e^{-t}$.
Fig. 1. Maximum error of the SE-Sinc-Nyström method and the DE-Sinc-Nyström method for the problem Eq. (17).

Fig. 2. $||A_n^{-1}||\infty$ (SE-Sinc) and $||B_n^{-1}||\infty$ (DE-Sinc) for the problem Eq. (17).

Fig. 3. Maximum error of the SE-Sinc-Nyström method and the DE-Sinc-Nyström method for the problem Eq. (18).

**Example 3** ([1, Example 3]). *Consider the following (stiff) initial value problem:*

$$
\begin{pmatrix}
y' \\
z'
\end{pmatrix} = \begin{pmatrix} 998 & 1998 \\ -999 & -1999 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}, \quad \begin{pmatrix} y(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},
$$

(19)

*whose solution is $y(t) = 2e^{-t} - e^{-1000t}$, $z(t) = -e^{-t} + e^{-1000t}$.*

The programs are performed in MATLAB R2016a on Windows 8 with Intel Xeon 2.50 GHz, 128 GB memory. The errors of the approximate solutions are investigated on the following 101 points:

$$
t = 2^{-50}, 2^{-49}, \ldots, 2^{-1}, 2^0, 2^1, \ldots, 2^{49}, 2^{50},
$$

and the maximum error among these points was plotted in Figs. 1, 3, 5. From the figures, we can observe exponential convergence suggested by Theorems 3 and 4: $O(e^{-\varepsilon_1/\sqrt{l}})$ in the case of the SE-Sinc-Nyström method, and $O(e^{-\varepsilon_2l/\log l})$ in the case of the DE-Sinc-Nyström method. This is because $||A_n^{-1}||\infty$ and $||B_n^{-1}||\infty$ do not grow rapidly, which can be observed from Figs. 2, 4, 6.
6. Proof of Theorem 2

In this section, Theorem 2 is proved. This is done in the following two steps: (i) estimate the error of the approximation Eq. (2) in Section 6.1, and (ii) estimate the error of the approximation Eq. (7) in Section 6.2.

6.1 Error estimate of the Sinc indefinite integration over the infinite interval

First, the error of the approximation Eq. (2) is estimated as

\[
\left| \int_{-\infty}^{x} F(\xi) \, d\xi - \sum_{j = -M}^{N} F(jh)J(j, h)(x) \right|
\]

\[
= \left| \int_{-\infty}^{x} F(\xi) \, d\xi - \sum_{j = -\infty}^{\infty} F(jh)J(j, h)(x) + \sum_{j = -\infty}^{\infty} F(jh)J(j, h)(x) - \sum_{j = -M}^{N} F(jh)J(j, h)(x) \right|
\]

\[
\leq \left| \int_{-\infty}^{x} F(\xi) \, d\xi - \sum_{j = -\infty}^{\infty} F(jh)J(j, h)(x) \right| + \sum_{j = -\infty}^{\infty} F(jh)J(j, h)(x) - \sum_{j = -M}^{N} F(jh)J(j, h)(x)
\]
\begin{align*}
&= \left| \int_{-\infty}^{x} F(\xi) \, d\xi - \sum_{j=-\infty}^{\infty} F(jh)J(j,h)(x) \right| + \sum_{j=-\infty}^{-M-1} F(jh)J(j,h)(x) + \sum_{j=N+1}^{\infty} F(jh)J(j,h)(x) \right|. \quad (20)
\end{align*}

The first term of Eq. (20) (called discretization error) is estimated as follows. The proof is omitted here because it is quite similar to that of the existing lemma \cite[Lemma 4]{7}.

**Lemma 1.** Let \( L, \alpha \) and \( \beta \) be positive constants, let \( \mu = \min\{\alpha, \beta\} \), and let \( d \) be a constant with \( 0 < d < \pi/2 \). Let \( F \) be analytic on \( \mathbb{D}_d \) and satisfy

\begin{equation}
|F(\zeta)| \leq \frac{L |\cosh \zeta|}{1 + e^{-\pi \sinh |\alpha|}|1 + e^{\pi \sinh \zeta}|^{\beta}}
\end{equation}

for all \( \zeta \in \mathbb{D}_d \). Then, it holds that

\begin{align*}
\sup_{x \in \mathbb{R}} \left| \int_{-\infty}^{x} F(\zeta) \, d\zeta - \sum_{j=-\infty}^{N} F(jh)J(j,h)(x) \right| &\leq \frac{2L}{\pi \mu (1 - e^{-2\pi d/h}) \cos^{\alpha+\beta}((\pi/2) \sin d) \cos d} h e^{-\pi d/h}.
\end{align*}

Next, we estimate the second term of Eq. (20) (called truncation error) as follows. Here, we define \( x_\gamma \) for \( \gamma > 0 \) as

\begin{equation}
x_\gamma = \begin{cases} \text{arcsinh} \left( \frac{1 + \sqrt{1 - (2\pi \gamma)^2}}{2\pi \gamma} \right) & (0 < \gamma < 1/(2\pi)), \\ \text{arcsinh}(1) & (\gamma \geq 1/(2\pi)). \end{cases}
\end{equation}

**Lemma 2.** Let \( R, \alpha \) and \( \beta \) be positive constants, let \( \mu = \min\{\alpha, \beta\} \), and let \( M \) and \( N \) be defined as Eq. (8). Let \( F \) satisfy

\begin{equation}
|F(x)| \leq \frac{R \cosh x}{(1 + e^{-\pi \sinh x})\alpha(1 + e^{\pi \sinh x})^{\beta}}
\end{equation}

for all \( x \in \mathbb{R} \), and \( Mh \geq x_\alpha \) and \( Nh \geq x_\beta \) hold. Then, it holds that

\begin{align*}
\left| \sum_{j=-\infty}^{-M-1} F(jh)J(j,h)(x) + \sum_{j=N+1}^{\infty} F(jh)J(j,h)(x) \right| &\leq \frac{2.2R}{\pi \mu} e^{-\pi \mu \sinh(nh)}.
\end{align*}

To prove this lemma, we use the following proposition and lemma.

**Proposition 1** (Okayama et al. \cite[Proposition 4.17]{7}). Let us define \( x_\gamma \) by Eq. (22). Then, the function \( G(x) = \cosh(x) e^{\pi \sinh(x)} \) is monotonically increasing for \( x \leq -x_\gamma \), and the function \( \tilde{G}(x) = \cosh(x) e^{-\pi \gamma \sinh(x)} \) is monotonically decreasing for \( x \geq x_\gamma \).

**Lemma 3** (Stenger \cite[Lemma 3.6.5]{6}). It holds that

\begin{equation}
\sup_{x \in \mathbb{R}} |J(j,h)(x)| \leq 1.1h.
\end{equation}

Using these results, we can prove Lemma 2 as follows.

**Proof.** By using Lemma 3, we have

\begin{align*}
\left| \sum_{j=-\infty}^{-M-1} F(jh)J(j,h)(x) + \sum_{j=N+1}^{\infty} F(jh)J(j,h)(x) \right| &\leq \sum_{j=-\infty}^{-M-1} |F(jh)||J(j,h)(x)| + \sum_{j=N+1}^{\infty} |F(jh)||J(j,h)(x)| \\
&\leq 1.1h \sum_{j=-\infty}^{-M-1} |F(jh)| + 1.1h \sum_{j=N+1}^{\infty} |F(jh)|.
\end{align*}

By using Eq. (23), we can estimate the second term of Eq. (24) as
Let \( \mu = \min\{\alpha, \beta\} \). Furthermore, using the relation Eq. (8) and \( \mu = \min\{\alpha, \beta\} \), we have

\[
\frac{1.1R}{\pi\alpha} e^{-\pi\alpha \sinh(Mh)} + \frac{1.1R}{\pi\beta} e^{-\pi\beta \sinh(Nh)} \leq \frac{1.1R}{\pi\alpha} e^{-\pi\mu \sinh(nh)} + \frac{1.1R}{\pi\beta} e^{-\pi\beta \sinh(nh)}
\]

\[
\leq \frac{1.1R}{\pi\mu} e^{-\pi\mu \sinh(nh)} + \frac{1.1R}{\pi\mu} e^{-\pi\mu \sinh(nh)}
\]

\[
= 2.2R \frac{1}{\pi\mu} e^{-\pi\mu \sinh(nh)}
\]

This completes the proof. \( \square \)

From the estimates of the discretization error (Lemma 1) and the truncation error (Lemma 2), we get the following theorem.

**Theorem 5.** Let \( L, R, \alpha \) and \( \beta \) be positive constants, and let \( d \) be a constant with \( 0 < d < \pi/2 \). Let \( F \) be analytic on \( D_d \) and satisfy Eq. (21) for all \( \zeta \in D_d \), and satisfy Eq. (23) for all \( x \in \mathbb{R} \). Let \( \mu = \min\{\alpha, \beta\} \), let \( M \) and \( N \) be defined as Eq. (8) and let \( h \) be defined as Eq. (9). Furthermore, let \( n \) be taken sufficiently large so that \( Mh \geq x_\alpha \) and \( Nh \geq x_\beta \) hold. Then, it holds that

\[
\sup_{x \in \mathbb{R}} \left| \int_{-\infty}^{x} F(\xi) d\xi - \sum_{j=-M}^{N} F(jh) J(j,h)(x) \right| \leq \frac{\arcsinh(dn/\mu)}{n} e^{-\pi d/\arcsinh(dn/\mu)} C, \quad (25)
\]

where \( C \) is a constant expressed as

\[
C = \frac{2}{\pi d\mu} \left\{ \frac{L}{(1 - e^{-2\pi d/\arcsinh(d/\mu)}) \cos \alpha + \beta ((\pi/2) \sin d) \cos d} + \frac{7.7R}{5\pi} e^{\pi^2 - 1} \right\}.
\]

We prepare the next lemma to prove Theorem 5.
Lemma 4. For \( x > 0 \), it holds that
\[
\frac{x}{\arcsinh(x)} \leq \frac{7}{5} \left( x + \frac{1}{x} - \frac{x}{\arcsinh(x)} \right).
\]

**Proof.** Notice that
\[
\frac{x}{\arcsinh(x)} \leq \frac{7}{5} \left( x + \frac{1}{x} - \frac{x}{\arcsinh(x)} \right) \iff 1 + \frac{1}{x^2} \geq \frac{12}{7} \frac{1}{\arcsinh(x)}.
\]

Furthermore, if we put \( t = \arcsinh(x) \), we have
\[
1 + \frac{1}{\sinh^2(t)} \geq \frac{12}{7} \frac{1}{t} \iff t \geq \frac{12}{7} \tanh^2(t).
\]

Therefore, to prove this lemma, we put \( f(t) = t - \frac{12}{7} \tanh^2(t) \), and we prove \( f(t) \geq 0 \) for \( t \geq 0 \). The derivative of \( f(t) \) is expressed as
\[
f'(t) = \frac{7 \cosh^3(t) - 24 \sinh(t)}{7 \cosh^3(t)} = \frac{1}{7 \cosh^3(t)} \left\{ \frac{7}{2} \left( \frac{e^t + e^{-t}}{2} \right)^3 - 24 \left( \frac{e^t - e^{-t}}{2} \right) \right\}.
\]

Here, if we put \( e^{2t} = z \), we have
\[
f'(t) = \frac{1}{7 \cosh^3(t)} \frac{7z^3 - 75z^2 + 117z + 7}{8z\sqrt{z}}.
\]

To determine \( t \) such that \( f'(t) = 0 \), we should solve the following cubic equation
\[
7z^3 - 75z^2 + 117z + 7 = 0.
\]

If we put \( g(z) \) as \( g(z) = 7z^3 - 75z^2 + 117z + 7 \), we can get
\[
\begin{align*}
g(-1) &= -192 < 0, \\
g(0) &= 7 > 0, \\
g(2) &= -3 < 0, \\
g(9) &= 88 > 0.
\end{align*}
\]

From the intermediate value theorem, there exist \( \alpha, \beta \) and \( \gamma \) such that
\[-1 < \alpha < 0, \quad 0 < \beta < 2, \quad 2 < \gamma < 9,\]

which satisfy \( g(\alpha) = g(\beta) = g(\gamma) = 0 \). In particular, \( \gamma \) is expressed as
\[
\gamma = \frac{1}{7} \left\{ 25 + 8\sqrt{22} \cos \left( \frac{1}{3} \arctan \left( \frac{7\sqrt{327}}{163} \right) \right) \right\}.
\]

Furthermore, noting \( z > 0 \) and \( t = \log \sqrt{z} \), we have \( f'(\log \sqrt{\beta}) = f'(\log \sqrt{\gamma}) = 0 \), and \( f(\log \sqrt{\gamma}) > 0.0013 > 0 \). Therefore, \( f(t) \geq f(0) = 0 \) was proved for \( t \geq 0 \).

From Lemma 4, we can prove Theorem 5 as follows.

**Proof.** By using Lemmas 1 and 2, and \( h \) defined as Eq. (9), we can get
\[
\left| \int_{-\infty}^{x} F(\xi) \, d\xi - \sum_{j=-M}^{N} F(jh)J(j,h)(x) \right| \\
\leq \frac{2L}{\pi d \mu (1 - e^{-2\pi d/h}) \cos \alpha + \beta ((\pi/2) \sin \delta) \cos \delta} \cdot \frac{h e^{-\pi d/h} + \frac{2.2 R}{\pi \mu} e^{-\pi \mu \sinh(nh)}}{1 - e^{-2\pi d/h} \cos \alpha + \beta ((\pi/2) \sin \delta) \cos \delta}.
\]

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The following lemma is important to prove Theorem 2.

6.2 Error estimate of the Sinc indefinite integration over the semi-infinite interval

Furthermore, from Lemma 4, we have

\[
\frac{dn/\mu}{\arcsinh(dn/\mu)} e^{-\pi dn/\arcsinh(dn/\mu)} = \frac{dn/\mu}{\arcsinh(dn/\mu)} e^{-\pi d n/\arcsinh(dn/\mu)} \leq \frac{7}{5} \left( \frac{d n}{\mu} + \frac{\mu}{dn/\mu} - \frac{n}{\arcsinh(dn/\mu)} \right) e^{-\pi d n/\arcsinh(dn/\mu)} \leq \frac{7}{5} e^{\pi d n/\mu} \left( \frac{d n}{\mu} + \frac{\mu}{dn/\mu} - \frac{n}{\arcsinh(dn/\mu)} \right) e^{-\frac{\pi}{\arcsinh(dn/\mu)}} \leq \frac{7}{5} e^{\frac{\pi}{\arcsinh(dn/\mu)}} \left( \frac{d n}{\mu} + \frac{\mu}{dn/\mu} - \frac{n}{\arcsinh(dn/\mu)} \right) e^{-\frac{\pi}{\arcsinh(dn/\mu)}},
\]

since \( x e^{-\mu x} \) has its maximum at \( x = 1/\pi \). This completes the proof. \( \square \)

6.2 Error estimate of the Sinc indefinite integration over the semi-infinite interval

The following lemma is important to prove Theorem 2.

Lemma 5 (Olayana [5, Lemma 5.11]). Let \( \beta \) be a positive constant, let \( \alpha \) and \( d \) be constants with \( 0 < \alpha \leq 1 \) and \( 0 < d < \pi/2 \), and let \( f \in L_{\alpha,\beta}(\phi(\mathcal{D})) \). Then, if we put \( F(\zeta) = f(\phi(\zeta))\phi'(\zeta), F \) is analytic on \( \mathcal{D} \), and Eq. (21) and Eq. (23) hold with

\[
L = \pi C_1 \hat{c}_d^{1-\alpha}, \quad R = \pi C_1 e^{\pi(1-\alpha)/12},
\]

where \( C_1 \) is a constant appearing in Eq. (4), and \( \hat{c}_d \) is a constant defined as

\[
\hat{c}_d = \frac{1 + \log \left( \frac{2 + \frac{1}{\cos((\pi/2) \sin d)}}{\log \left( \frac{2 + \frac{1}{\cos((\pi/2) \sin d)}}{1 + \frac{1}{\cos((\pi/2) \sin d)}} \right)} \right)}{1 + \frac{1}{\cos((\pi/2) \sin d)}}.
\]

Using this lemma, we can show the following theorem.

Theorem 6. Let \( \beta \) be a positive constant, let \( \alpha \) and \( d \) be constants with \( 0 < \alpha \leq 1 \) and \( 0 < d < \pi/2 \), and let \( f \in L_{\alpha,\beta}(\phi(\mathcal{D})) \). Let \( \mu = \min\{\alpha, \beta\} \), let \( M \) and \( N \) be defined as Eq. (8) and let \( h \) be defined as Eq. (9). Furthermore, let \( n \) be taken sufficiently large so that \( Mh \geq x_\alpha \) and \( Nh \geq x_\beta \) hold. Then, it holds that

\[
\sup_{t \in (0, \infty)} \left| I(t) - \sum_{j=-M}^{N} f(\phi(jh))\phi'(jh)J(j, h)(\phi^{-1}(t)) \right| \leq C_1 C_{\arcsinh(dn/\mu)} e^{-\pi d n/\arcsinh(dn/\mu)} \leq C_1 C_{\arcsinh(dn/\mu)} e^{-\pi d n/\arcsinh(dn/\mu)} e^{-\pi/\arcsinh(dn/\mu)},
\]

where \( C_1 \) is a constant appearing in Eq. (4), \( C \) is a constant expressed as

\[
C = \frac{2}{\pi d \mu} \left[ \frac{\hat{c}_d^{1-\alpha}}{(1 - e^{-2\pi d/\arcsinh(d/\mu)}) \cos^\alpha + \beta((\pi/2) \sin d) \cos d + \frac{7 \pi e^{(1-\alpha)/12}}{5\pi} e^{\pi^2/\hat{c}_d - 1} \right],
\]

and \( \hat{c}_d \) is a constant defined in Eq. (26).
Proof. According to Lemma 5, the function $F(\zeta) = f(\phi(\zeta))\phi'(\zeta)$ satisfies the assumptions of Theorem 5. Therefore, by using Eq. (25), we can get

$$
\int_0^t f(s) \, ds - \sum_{j=-M}^N f(\phi(jh))\phi'(jh)J(j, h)(\phi^{-1}(t)) = \int_{-\infty}^{\phi^{-1}(t)} F(\xi) \, d\xi - \sum_{j=-M}^N F(jh)J(j, h)(\phi^{-1}(t)) \leq \frac{2}{\pi d\mu} \frac{C_1}{n} e^{-2/\pi d\mu} \cos^{\alpha+\beta}((\pi/2) \sin d) \cos d \left[ \frac{\pi C_1 e^{1-\alpha}}{(1 - e^{-2\pi d/\arcsinh(d/\mu)}) \cos^{\alpha+\beta}((\pi/2) \sin d)} + \frac{7.7 \pi C_1 e^{(1-\alpha)/12}}{5\pi} e^{\frac{\pi \mu^2}{\beta^2} - 1} \right] \cdot \frac{\pi d\mu}{\arcsinh(d/\mu)} \cdot \frac{\pi d\mu}{\arcsinh(d/\mu)} \cdot \frac{\pi d\mu}{\arcsinh(d/\mu)} \cdot \frac{\pi d\mu}{\arcsinh(d/\mu)}.
$$

This completes the proof. \(\square\)

Theorem 2 immediately follows from Theorem 6.

7. Proofs of Theorem 3 and Theorem 4

In this section, Theorem 3 and Theorem 4 are proved in Section 7.1 and Section 7.2, respectively.

7.1 Proof of Theorem 3

To prove Theorem 3, we prepare the following two lemmas.

Lemma 6. Assume that the assumptions of Theorem 3 are fulfilled. Let $Y$ be an $lm$-dimensional vector defined as

$$
Y = [y_1(\psi(-Mh)), \ldots, y_1(\psi(Nh)), \ldots, y_m(\psi(-Mh)), \ldots, y_m(\psi(Nh))]^T.
$$

Then, there exists a positive constant $\overline{C}$ independent of $n$ such that

$$
||Y - Y^{(\psi)}||_\infty \leq \overline{C} ||A_n^{-1}||_\infty e^{-\sqrt{\pi d\mu} n}.
$$

Proof. Let us introduce $\tilde{y}(t)$ defined as

$$
\tilde{y}(t) = r + \sum_{j=-M}^N \{K(\psi(jh))y(\psi(jh)) + g(\psi(jh))\} \psi'(jh)J(j, h)(\psi^{-1}(t)),
$$

(27)

which shows the approximation of the right-hand side of Eq. (10) by the Sinc indefinite integration combined with the SE transformation. Furthermore, let $\hat{Y}$ be an $lm$-dimensional vector defined as

$$
\hat{Y} = [\hat{y}_1(\psi(-Mh)), \ldots, \hat{y}_1(\psi(Nh)), \ldots, \hat{y}_m(\psi(-Mh)), \ldots, \hat{y}_m(\psi(Nh))]^T.
$$

By taking samples at $t = \psi(-Mh), \psi((-M+1)h), \ldots, \psi(Nh)$ in Eq. (27), we have

$$
\hat{Y} = R + (I_m \otimes \{hI_l^{(-1)}D_l^{(\psi)}\}) \{K^{(\psi)} Y + G^{(\psi)}\}.
$$

Then, since $Y^{(\psi)} = A_n^{-1} (R + (I_m \otimes \{hI_l^{(-1)}D_l^{(\psi)}\}) G^{(\psi)})$, we can get

$$
Y - Y^{(\psi)} = Y - (I_m \otimes \{I_l \otimes \{hI_l^{(-1)}D_l^{(\psi)}\}\}) [K^{(\psi)}]^{-1} (R + (I_m \otimes \{hI_l^{(-1)}D_l^{(\psi)}\} G^{(\psi)})
$$

$$
= A_n^{-1} ((I_m \otimes \{I_l \otimes \{hI_l^{(-1)}D_l^{(\psi)}\}\}) [K^{(\psi)}] Y - (R + (I_m \otimes \{hI_l^{(-1)}D_l^{(\psi)}\} G^{(\psi)}))
$$

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Lemma 7. Let $h$ be defined as Eq. (6), and let $M$ and $N$ be defined as Eq. (5). Then, it holds that

$$h \sum_{j=-M}^{N} \psi'(jh) \leq \left\{ \log \left(1 + e^{2\sqrt{\pi d/\mu}} \right) \right\} \sqrt{n}.$$

Proof. $M \leq n$ and $N \leq n$ hold because $M$ and $N$ are defined as Eq. (5). Therefore, we have

$$h \sum_{j=-M}^{N} \psi'(jh) \leq h \sum_{j=-n}^{n} \psi'(jh) \leq \int_{-nh}^{nh} \psi'(x) \, dx \leq \int_{-\infty}^{(n+1)h} \psi'(x) \, dx = \log(1 + e^{(n+1)h}).$$

Furthermore, using $h$ defined as Eq. (6), we have

$$\log(1 + e^{(n+1)h}) = \log e^{(n+1)h} + \log \left(1 + e^{-(n+1)h}\right).$$
Proof. From Eq. (11),
\[ y_i(t) = r_i + \int_0^t f_i(s) \, ds. \] (30)

From Eq. (11), \( y_i^{(0)}(t) \) is expressed as
\[ y_i^{(0)}(t) = r_i + \sum_{j=-M}^N \{ k_i(\psi(jh)) \cdot y^{(0)}(\psi(jh)) + g_i(\psi(jh)) \} \psi'(jh)J(j, h)(\psi^{-1}(t)). \] (31)

We estimate the error by using
\[ \tilde{y}_i(t) = r_i + \sum_{j=-M}^N f_i(\psi(jh))\psi'(jh)J(j, h)(\psi^{-1}(t)), \] (32)

which is defined in Eq. (27). By using the triangle inequality, it holds that
\[ |y_i(t) - y_i^{(0)}(t)| = |y_i(t) - \tilde{y}_i(t) + \tilde{y}_i(t) - y_i^{(0)}(t)| \leq |y_i(t) - \tilde{y}_i(t)| + |\tilde{y}_i(t) - y_i^{(0)}(t)|. \] (33)

The first term of the right-hand side in Eq. (33) shows the difference between Eq. (30) and Eq. (32), and we have
\[ |y_i(t) - \tilde{y}_i(t)| = \left| \int_0^t f_i(s) \, ds - \sum_{j=-M}^N f_i(\psi(jh))\psi'(jh)J(j, h)(\psi^{-1}(t)) \right|. \]

By assumption, \( f_i \in L_{\alpha, \beta}(\mathcal{D}_d) \). Therefore, by using Theorem 1, we can obtain
\[ \left| \int_0^t f_i(s) \, ds - \sum_{j=-M}^N f_i(\psi(jh))\psi'(jh)J(j, h)(\psi^{-1}(t)) \right| \leq C_t e^{-\sqrt{\pi d/m}}. \]

where \( C_t \) is a positive constant.

Next, the second term of Eq. (33) shows the difference between Eq. (31) and Eq. (32), and we have
\[ |\tilde{y}_i(t) - y_i^{(0)}(t)| = \left| \sum_{j=-M}^N k_i(\psi(jh)) \cdot \{ y(\psi(jh)) - y^{(0)}(\psi(jh)) \} \psi'(jh)J(j, h)(\psi^{-1}(t)) \right| \]
\[ \leq \sum_{j=-M}^N \left| k_i(\psi(jh)) \cdot \{ y(\psi(jh)) - y^{(0)}(\psi(jh)) \} \right| \psi'(jh) \left| J(j, h)(\psi^{-1}(t)) \right|. \] (34)

Since \( \psi(x) = \log(1 + e^x) \), it follows for all \( x \in \mathbb{R} \) that
\[ \psi'(x) = \frac{1}{1 + e^{-x}} > 0, \]
and by using Lemma 3, we can further estimate the right-hand side in Eq. (34) as

\[
\sum_{j=-M}^{N} |k_i(\psi(jh)) \cdot \{y(\psi(jh)) - y^{(l)}(\psi(jh))\}| |\psi'(jh)| |J(j,h)(\psi^{-1}(t))| \\
\leq 1.1h \sum_{j=-M}^{N} |k_i(\psi(jh)) \cdot \{y(\psi(jh)) - y^{(l)}(\psi(jh))\}| \psi'(jh).
\]  

(35)

Here, in the summation of Eq. (35), we can get

\[
|k_i(\psi(jh)) \cdot \{y(\psi(jh)) - y^{(l)}(\psi(jh))\}| \\
= |k_{i1}(\psi(jh)) y_1(\psi(jh)) - y_1^{(l)}(\psi(jh))| + \cdots + |k_{im}(\psi(jh)) y_m(\psi(jh)) - y_m^{(l)}(\psi(jh))| \\
\leq |k_{i1}(\psi(jh))| |y_1(\psi(jh)) - y_1^{(l)}(\psi(jh))| + \cdots + |k_{im}(\psi(jh))| |y_m(\psi(jh)) - y_m^{(l)}(\psi(jh))|.
\]

By assumption, the function \(k_{ij}(t)\) is bounded. Therefore, setting

\[
\sup_{t \in (0,\infty)} |k_{ij}(t)| = \kappa_{ij},
\]  

(36)

we have

\[
|k_{i1}(\psi(jh))| |y_1(\psi(jh)) - y_1^{(l)}(\psi(jh))| + \cdots + |k_{im}(\psi(jh))| |y_m(\psi(jh)) - y_m^{(l)}(\psi(jh))| \\
\leq \kappa_{i1}||Y - Y^{(\psi)}||_{\infty} + \kappa_{i2}||Y - Y^{(\psi)}||_{\infty} + \cdots + \kappa_{im}||Y - Y^{(\psi)}||_{\infty}.
\]

Furthermore, setting \(\overline{\kappa}_i\) as \(\overline{\kappa}_i = \max \{\kappa_{i1}, \kappa_{i2}, \ldots, \kappa_{im}\}\), we have

\[
k_{i1}||Y - Y^{(\psi)}||_{\infty} + k_{i2}||Y - Y^{(\psi)}||_{\infty} + \cdots + k_{im}||Y - Y^{(\psi)}||_{\infty} \\
\leq |\overline{\kappa}_i||Y - Y^{(\psi)}||_{\infty} + |\overline{\kappa}_i||Y - Y^{(\psi)}||_{\infty} + \cdots + |\overline{\kappa}_i||Y - Y^{(\psi)}||_{\infty} \\
= \overline{\kappa}_i||Y - Y^{(\psi)}||_{\infty}.
\]

Therefore, from Lemmas 6 and 7, we can estimate Eq. (35) as

\[
1.1h \sum_{j=-M}^{N} |k_i(\psi(jh)) \cdot \{y(\psi(jh)) - y^{(l)}(\psi(jh))\}| \psi'(jh) \\
\leq 1.1h \sum_{j=-M}^{N} m\overline{\kappa}_i||Y - Y^{(\psi)}||_{\infty} \psi'(jh) \\
= 1.1hm\overline{\kappa}_i||Y - Y^{(\psi)}||_{\infty} \sum_{j=-M}^{N} \psi'(jh) \\
\leq 1.1hm\overline{\kappa}_i \left\{ \log \left(1 + e^{2\sqrt{\pi/d/\mu}} \right) \right\} \sqrt{n}||Y - Y^{(\psi)}||_{\infty} \\
\leq 1.1hm\overline{\kappa}_i C_i \left\{ \log \left(1 + e^{2\sqrt{\pi/d/\mu}} \right) \right\} ||A_n^{-1}||_{\infty} \sqrt{n} e^{-\sqrt{\pi/d/\mu}},
\]

where \(C_i\) is a positive constant.

Summing up the above results, we can estimate Eq. (33) as

\[
|y_i(t) - y_i^{(l)}(t)| \leq C_i e^{-\sqrt{\pi/d/\mu}} + 1.1hm\overline{\kappa}_i C_i \left\{ \log \left(1 + e^{2\sqrt{\pi/d/\mu}} \right) \right\} ||A_n^{-1}||_{\infty} \sqrt{n} e^{-\sqrt{\pi/d/\mu}} \\
\leq \left( C_i + 1.1hm\overline{\kappa}_i C_i \left\{ \log \left(1 + e^{2\sqrt{\pi/d/\mu}} \right) \right\} ||A_n^{-1}||_{\infty} \right) \sqrt{n} e^{-\sqrt{\pi/d/\mu}}.
\]

By setting \(C\) and \(\hat{C}\) as

\[
C = \max_{1 \leq i \leq m} C_i, \quad \hat{C} = \max_{1 \leq i \leq m} \left( 1.1hm\overline{\kappa}_i C_i \left\{ \log \left(1 + e^{2\sqrt{\pi/d/\mu}} \right) \right\} \right),
\]

we obtain the desired conclusion. 

\(\square\)
Lemma 8. Assume that the assumptions of Theorem 4 are fulfilled. Let \( Y \) be an \( lm \)-dimensional vector defined as

\[
Y = [y_1(\phi(-Mh)), \ldots, y_1(\phi(Nh)), \ldots, y_m(\phi(-Mh)), \ldots, y_m(\phi(Nh))]^T.
\]

Then, there exists a positive constant \( \overline{C} \) independent of \( n \) such that

\[
||Y - Y^{(\phi)}||_\infty \leq \overline{C}||B_n^{-1}||_\infty \frac{\text{arcsinh}(dn/\mu)}{n} e^{-\pi dn/\text{arcsinh}(dn/\mu)}.
\]

**Proof.** Let us introduce \( \tilde{y}(t) \) defined as

\[
\tilde{y}(t) = r + \sum_{j=-M}^{N} \{K(\phi(jh))y(\phi(jh)) + g(\phi(jh))\} \phi'(jh)J(j, h)(\phi^{-1}(t)),
\]

which shows the approximation of the right-hand side of Eq. (10) by the Sinc indefinite integration combined with the DE transformation. Furthermore, let \( \tilde{Y} \) be an \( lm \)-dimensional vector defined as

\[
\tilde{Y} = [\tilde{y}_1(\phi(-Mh)), \ldots, \tilde{y}_1(\phi(Nh)), \ldots, \tilde{y}_m(\phi(-Mh)), \ldots, \tilde{y}_m(\phi(Nh))]^T.
\]

By taking samples at \( t = \phi(-Mh), \phi(-M+1)h, \ldots, \phi(Nh) \) in Eq. (37), we have

\[
\tilde{Y} = R + (I_m \otimes \{hI_t^{(-1)}D_l^{(\phi)}\})\{[K_{ij}^{(\phi)}Y + G^{(\phi)}]\}.
\]

Then, since \( Y^{(\phi)} = B_n^{-1}(R + (I_m \otimes \{hI_t^{(-1)}D_l^{(\phi)}\})G^{(\phi)}) \), we can get

\[
Y - Y^{(\phi)} = Y - (I_m \otimes I_t - (I_m \otimes \{hI_t^{(-1)}D_l^{(\phi)}\})[K_{ij}^{(\phi)}])^{-1}(R + (I_m \otimes \{hI_t^{(-1)}D_l^{(\phi)}\})G^{(\phi)})
\]

\[
= B_n^{-1}((I_m \otimes I_t - (I_m \otimes \{hI_t^{(-1)}D_l^{(\phi)}\})[K_{ij}^{(\phi)}])Y - (R + (I_m \otimes \{hI_t^{(-1)}D_l^{(\phi)}\})G^{(\phi)}))
\]

\[
= B_n^{-1}(Y - (R + (I_m \otimes \{hI_t^{(-1)}D_l^{(\phi)}\})[K_{ij}^{(\phi)}][Y + G^{(\phi)}]))
\]

\[
= B_n^{-1}(Y - \tilde{Y}).
\]

Therefore, we have

\[
||Y - Y^{(\phi)}||_\infty = ||B_n^{-1}(Y - \tilde{Y})||_\infty \leq ||B_n^{-1}||_\infty ||Y - \tilde{Y}||_\infty.
\]

Let the matrix \( K(t) \) be expressed as Eq. (28) and let us define \( f_i \) as Eq. (29) for \( i = 1, 2, \ldots, m \). By assumption, we can apply Theorem 2 because \( f_i \) belongs to \( L_{a,b}(\phi(\mathcal{D})) \). Then, there exists a positive constant \( C_i \) independent of \( n \) such that

\[
||Y - \tilde{Y}||_\infty = \max_{-M \leq j \leq N, 1 \leq l \leq m} |y_l(\phi(jh)) - \tilde{y}_l(\phi(jh))|
\]

\[
\leq \sup_{0 < t < \infty, 1 \leq l \leq m} |y_l(t) - \tilde{y}_l(t)|
\]

\[
= \sup_{0 < t < \infty, 1 \leq l \leq m} \left| \int_0^t f_i(s) \, ds - \sum_{j=-M}^{N} f_i(\phi(jh))\phi'(jh)J(j, h)(\phi^{-1}(t)) \right|
\]

\[
\leq \max_{1 \leq l \leq m} \left\{ C_i \frac{\text{arcsinh}(dn/\mu)}{n} e^{-\pi dn/\text{arcsinh}(dn/\mu)} \right\}.
\]

If we put \( \overline{C} \) as \( \overline{C} = \max \{C_1, C_2, \ldots, C_m\} \), then we can get the following estimate:
\[ \|Y - \tilde{Y}\|_\infty \leq C \frac{\text{arcsinh}(dn/\mu)}{n} e^{-\pi dn/\text{arcsinh}(dn/\mu)}. \]

Therefore, it follows that
\[ \|Y - Y^{(\psi)}\|_\infty \leq C \|B_n^{-1}\|_\infty \frac{\text{arcsinh}(dn/\mu)}{n} e^{-\pi dn/\text{arcsinh}(dn/\mu)}. \]

This completes the proof. \(\square\)

**Lemma 9.** Let \( h \) be defined as Eq. (9), and let \( M \) and \( N \) be defined as Eq. (8). Then, it holds that
\[ h \sum_{j=-M}^{N} \phi'(jh) \leq \left[ \log(1 + e^{\pi d/\mu}) + \pi \frac{\text{arcsinh}(d/\mu)\sqrt{1 + (d/\mu)^2}}{1 + e^{-\pi d/\mu}} \right] n. \]

**Proof.** \( M \leq n \) and \( N \leq n \) hold because \( M \) and \( N \) are defined as Eq. (8). Therefore, we have
\[ h \sum_{j=-M}^{N} \phi'(jh) \leq h \sum_{j=-n}^{n} \phi'(jh) \]
\[ = h \sum_{j=-n}^{n-1} \phi'(jh) + h\phi'(nh) \]
\[ \leq \int_{-nh}^{nh} \phi'(x) \, dx + h\phi'(nh) \]
\[ \leq \int_{-\infty}^{\infty} \phi'(x) \, dx + h\phi'(nh) \]
\[ = \log(1 + e^{\pi \sinh(nh)}) + h \frac{\pi \cosh(nh)}{1 + e^{-\pi \sinh(nh)}} \]
\[ = \pi \sinh(nh) + \log(1 + e^{-\pi \sinh(nh)}) + h \frac{\pi \cosh(nh)}{1 + e^{-\pi \sinh(nh)}}. \]

Here, \( \cosh(\text{arcsinh}(x)) = \sqrt{1 + x^2} \) holds for \( x \in \mathbb{R} \). Using this and \( h \) defined as Eq. (9), we can get
\[ \pi \sinh(nh) + \log(1 + e^{-\pi \sinh(nh)}) + h \frac{\pi \cosh(nh)}{1 + e^{-\pi \sinh(nh)}} \]
\[ = \frac{\pi dn}{\mu} + \log(1 + e^{-\pi dn/\mu}) + \frac{d}{\mu} \cdot \frac{\text{arcsinh}(dn/\mu)}{dn/\mu} \cdot \frac{\pi \sqrt{1 + (dn/\mu)^2}}{1 + e^{-\pi dn/\mu}} \]
\[ \leq \frac{\pi dn}{\mu} + \log(1 + e^{-\pi d/\mu}) + \frac{d}{\mu} \cdot \frac{\text{arcsinh}(d/\mu)}{d/\mu} \cdot \frac{\pi \sqrt{1 + (d/\mu)^2}}{1 + e^{-\pi d/\mu}} \]
\[ \leq n \left[ \frac{\pi d}{\mu} + \log(1 + e^{-\pi d/\mu}) + \text{arcsinh}(d/\mu) \cdot \frac{\pi \sqrt{1 + (d/\mu)^2}}{1 + e^{-\pi d/\mu}} \right] \]
\[ = n \left[ \log(1 + e^{\pi d/\mu}) + \frac{\pi \text{arcsinh}(d/\mu)\sqrt{1 + (d/\mu)^2}}{1 + e^{-\pi d/\mu}} \right]. \]

This completes the proof. \(\square\)

We are in a position to prove Theorem 4.

**Proof.** We use \( f_i \) in Eq. (29). From Eq. (10), \( y_i \) is expressed as Eq. (30). From Eq. (13), \( y_i^{(l)}(t) \) is expressed as
\[ y_i^{(l)}(t) = r_i + \sum_{j=-M}^{N} \{ k_i(\phi(jh)) \cdot y^{(l)}(\phi(jh)) + g_i(\phi(jh)) \} \phi'(jh) J(j, h)(\phi^{-1}(t)). \] (38)

We estimate the error by using
\[ \hat{y}_i(t) = r_i + \sum_{j=-M}^{N} f_i(\phi(jh))\phi'(jh)J(j,h)(\phi^{-1}(t)), \] (39)

which is defined in Eq. (37). By the triangle inequality, it holds that
\[ |y_i(t) - y_i^{(1)}(t)| = |y_i(t) - \hat{y}_i(t) + \hat{y}_i(t) - y_i^{(1)}(t)| \leq |y_i(t) - \hat{y}_i(t)| + |\hat{y}_i(t) - y_i^{(1)}(t)|. \] (40)

The first term of the right-hand side in Eq. (40) shows the difference of Eq. (30) and Eq. (39), and we have
\[ |y_i(t) - \hat{y}_i(t)| = \left| \int_0^t f_i(s) \, ds - \sum_{j=-M}^{N} f_i(\phi(jh))\phi'(jh)J(j,h)(\phi^{-1}(t)) \right|. \]

By assumption, \( f_i \in \mathbf{L}_{a,\varphi}(\mathcal{P}_d) \). Therefore, by using Theorem 2, we obtain
\[ \left| \int_0^t f_i(s) \, ds - \sum_{j=-M}^{N} f_i(\phi(jh))\phi'(jh)J(j,h)(\phi^{-1}(t)) \right| \leq C_i \frac{\arcsinh(dn/\mu)}{n} e^{-\pi d/\arcsinh(dn/\mu)}, \]

where \( C_i \) is a positive constant.

Next, the second term of Eq. (40) shows the difference of Eq. (38) and Eq. (39), and we have
\[ |\hat{y}_i(t) - y_i^{(1)}(t)| = \left| \sum_{j=-M}^{N} k_i(\phi(jh)) \cdot \{ y(\phi(jh)) - y^{(1)}(\phi(jh)) \} \phi'(jh)J(j,h)(\phi^{-1}(t)) \right| \leq \sum_{j=-M}^{N} \left| k_i(\phi(jh)) \cdot \{ y(\phi(jh)) - y^{(1)}(\phi(jh)) \} \right| \left| \phi'(jh) \right| \left| J(j,h)(\phi^{-1}(t)) \right|. \] (41)

Since \( \phi(x) = \log(1 + e^{\pi \sinh(x)}) \), it follows for all \( x \in \mathbb{R} \) that
\[ \phi'(x) = \frac{\pi \cosh(x)}{1 + \pi \sinh(x)} > 0, \]

and by using Lemma 3, we can further estimate the right-hand side in Eq. (41) as
\[ \sum_{j=-M}^{N} \left| k_i(\phi(jh)) \cdot \{ y(\phi(jh)) - y^{(1)}(\phi(jh)) \} \right| \left| \phi'(jh) \right| \left| J(j,h)(\phi^{-1}(t)) \right| \leq 1.1h \sum_{j=-M}^{N} \left| k_i(\phi(jh)) \cdot \{ y(\phi(jh)) - y^{(1)}(\phi(jh)) \} \right| \phi'(jh). \] (42)

Here, in the summation of Eq. (42), we can get
\[ \left| k_i(\phi(jh)) \cdot \{ y(\phi(jh)) - y^{(1)}(\phi(jh)) \} \right| = \left| k_{i1}(\phi(jh)) \{ y_1(\phi(jh)) - y_1^{(1)}(\phi(jh)) \} + \cdots + k_{im}(\phi(jh)) \{ y_m(\phi(jh)) - y_m^{(1)}(\phi(jh)) \} \right| \leq \left| k_{i1}(\phi(jh)) \right| \left| y_1(\phi(jh)) - y_1^{(1)}(\phi(jh)) \right| + \cdots + \left| k_{im}(\phi(jh)) \right| \left| y_m(\phi(jh)) - y_m^{(1)}(\phi(jh)) \right|. \]

By assumption, the function \( k_{ij}(t) \) is bounded. Therefore, setting \( k_{ij} \) as Eq. (36), we have
\[ \left| k_{i1}(\phi(jh)) \right| \left| y_1(\phi(jh)) - y_1^{(1)}(\phi(jh)) \right| + \cdots + \left| k_{im}(\phi(jh)) \right| \left| y_m(\phi(jh)) - y_m^{(1)}(\phi(jh)) \right| \leq k_{i1}||Y - Y(\phi)||_\infty + k_{i2}||Y - Y(\phi)||_\infty + \cdots + k_{im}||Y - Y(\phi)||_\infty. \]

Furthermore, setting \( k_i \) as \( k_i = \max \{ k_{i1}, k_{i2}, \ldots, k_{im} \} \), we have
\[ k_{i1}||Y - Y(\phi)||_\infty + k_{i2}||Y - Y(\phi)||_\infty + \cdots + k_{im}||Y - Y(\phi)||_\infty. \]

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\[
\leq \kappa_i \| Y - Y^{(\phi)} \|_{\infty} + \kappa_i \| Y - Y^{(\phi)} \|_{\infty} + \cdots + \kappa_i \| Y - Y^{(\phi)} \|_{\infty}
= m\kappa_i \| Y - Y^{(\phi)} \|_{\infty}.
\]

Therefore, from Lemmas 8 and 9, we can estimate Eq. (42) as

\[
1.1h \sum_{j=-M}^{N} \left| k_i(\phi(jh)) \cdot \{ y(\phi(jh)) - y^{(l)}(\phi(jh)) \} \right| \phi'(jh)
\leq 1.1h \sum_{j=-M}^{N} m\kappa_i \| Y - Y^{(\phi)} \|_{\infty} \phi'(jh)
= 1.1m\kappa_i \| Y - Y^{(\phi)} \|_{\infty} h \sum_{j=-M}^{N} \phi'(jh)
\leq 1.1m\kappa_i \| Y - Y^{(\phi)} \|_{\infty} n \left[ \log(1 + e^{\pi d/\mu}) + \frac{\pi \arcsinh(d/\mu) \sqrt{1 + (d/\mu)^2}}{1 + e^{-\pi d/\mu}} \right]
\leq 1.1m\kappa_i \left[ \log(1 + e^{\pi d/\mu}) + \frac{\pi \arcsinh(d/\mu) \sqrt{1 + (d/\mu)^2}}{1 + e^{-\pi d/\mu}} \right] C_i \| B_n^{-1} \|_{\infty} n \arcsinh(d/\mu) e^{-\frac{\pi d n}{\arcsinh(d/\mu)}}
= \hat{C}_i \| B_n^{-1} \|_{\infty} \arcsinh(d/\mu) e^{-\frac{\pi d n}{\arcsinh(d/\mu)}}
\]

where \( C_i \) is a positive constant, and \( \hat{C}_i \) is a constant expressed as

\[
\hat{C}_i = 1.1m\kappa_i \left[ \log(1 + e^{\pi d/\mu}) + \frac{\pi \arcsinh(d/\mu) \sqrt{1 + (d/\mu)^2}}{1 + e^{-\pi d/\mu}} \right] C_i.
\]

Summing up the above results, we can estimate Eq. (40) as

\[
| y_i(t) - y_i^{(l)}(t) | \leq C_i \frac{\arcsinh(d/\mu)}{n} e^{-\frac{\pi d n}{\arcsinh(d/\mu)}} + \hat{C}_i \| B_n^{-1} \|_{\infty} \arcsinh(d/\mu) e^{-\frac{\pi d n}{\arcsinh(d/\mu)}}
\leq \left( C_i + \hat{C}_i \| B_n^{-1} \|_{\infty} \right) \arcsinh(d/\mu) e^{-\frac{\pi d n}{\arcsinh(d/\mu)}}.
\]

By setting \( C \) and \( \hat{C} \) as

\[
C = \max_{1 \leq i \leq m} C_i, \quad \hat{C} = \max_{1 \leq i \leq m} \hat{C}_i,
\]
we obtain the desired conclusion.

8. Conclusions

In this study, as numerical methods for initial value problems Eq. (1), we developed the SE-Sinc-Nyström method and DE-Sinc-Nyström method derived by improving the methods by Nurmuhammad et al. [1]. Numerical investigation suggested exponential convergence; however, no error analysis has been conducted yet. Therefore, in this study, we conducted error analyses of the methods. According to the results, we can theoretically prove that Sinc-Nyström methods can attain exponential convergence if the norm of the inverse of the coefficient matrix does not grow rapidly. That was in fact observed in numerical experiments in Section 5.

One of the disadvantages of the methods is that a special function \( \text{Si}(x) \) is included in the function \( J(j, h)(x) \), which has a higher computational cost compared to elementary functions. From this point of view, the basis function of the approximate solution should be written in elementary functions. We are working on this issue, and the result will be reported elsewhere.

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