Chapter 9  
Input-Output Model Based on Ordered Fuzzy Numbers

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Abstract The chapter presents the application of Ordered Fuzzy Numbers (OFNs) to the economic model. These numbers are used for input-output analysis (the Leontief model), which is a basic method of quantitative economics that presents macro-economic activity as a system of interrelated goods and services. OFNs allow us not only to apply mathematical modeling of imprecise or ambiguous data but also simultaneously portray more information than could be presented by real numbers. It is shown based on the Leontief model, where at the same time the current level, the forecast level, and the level of change of the final demand or the production level can be determined. The example shows that use of OFNs in economic modeling can simplify and deepen the economic analyses.

9.1 Introduction

Economics is a social science that studies and analyzes the production, distribution, and consumption of goods and services. One of the main tools used in economics is model. A model is a theoretical construction representing economic processes by a set of variables and a set of logical and quantitative relationships between them. Such model often take the form of systems of linear equations, due to their simplicity and ease of interpretation of their parameters and solution. One of the most famous and popular models of this type is the Leontief model, often called input-output analysis.

Input-output analysis is a quantitative economic technique that represents interdependencies between different sectors (industries, branches) and a way of describing the allocation of resources of a national economy or different regional economies. It is a particularly effective tool for the optimization of production processes, the improvement of economic conditions, and cost allocation analysis. The input-output
analysis also shows the combination of resources (raw materials, labor, capital), called inputs, that are required to achieve desired production goals, called outputs. It makes the Leontief model play a central role in planning and forecasting in economics.

The model proposed by Leontief does not take into account the natural uncertainty of variables of such complicated mathematical descriptions as the real-world economy models. The application of these variables in models involves knowing their numerical values. However, in reality many economic variables are difficult to be precisely measured. The elements of the Leontief model, such as a technical coefficients matrix, an output matrix, or a final demand matrix, are usually known as some intervals or fuzzy numbers. The use of only mean values in these matrices can lead to the loss of valuable information. In other words, in practical applications parameters are described using not only single values but intervals and fuzzy sets and numbers, in particular Ordered Fuzzy Numbers (OFNs). Properties of operations and the possibility of different interpretations of OFNs make these numbers widely used in economics.

There are papers in the literature in which the OFNs model is used in economic models. This model was applied, among others, to support decision making [9, 10, 14], for presentation of revenues and costs of a company [2, 3, 7], in the Leontief model [1, 2], for the presentation of stock prices [4, 8, 13], for the presentation of prices and the dynamics of their changes [5], and to determine the economic size of the delivery [6, 11]. These applications are based on the orientation of the OFNs as additional information and the arithmetics of the OFNs similar to the arithmetics of real numbers.

The chapter is organized as follows. Section 9.2 describes the fundamental concepts of input-output analysis. Next, in Sect. 9.3 a numerical example of input-output analysis based on OFNs is presented. Finally, concluding remarks are provided in Sect. 9.4.

9.2 Input-Output Analysis

The American economist Wassily W. Leontief developed the economic input-output model (the term “intersectoral” analysis is also used), which was first published in 1936. The beginnings of input-output analysis in economics are most often credited to Leontief and others who wrote the paper, “Studies in the Structure of the American Economy” [12]. In 1973 Leontief was awarded a Nobel prize for his great achievements in economics. The prize motivation was: “For the development of the input-output method and for its application to important economic problems.” His model is a basis for more models currently being used in many parts of the world. It can be applied to an economy of any size, from a small business or a region, to a whole country or the whole world. The Leontief input-output model leads to a better understanding of modeling economic systems, because it describes how the input and output of different sectors affect each other. The main goal of the Leontief
input-output model is to balance (equilibrium) the total amount of goods produced (total output) to the total demand (total input and final demand) for that production, in other words, consumption equals production.

Because the input-output model normally encompasses a large number of sectors, its framework is quite complicated. To simplify the analysis (model), the following assumptions are adopted:

- each sector produces only one homogeneous product,
- each sector uses a fixed input ratio (or factor combination) for the production of its output,
- production in every sector is subject to constant returns to scale, so that a $k$-fold change in every input will result in an exactly $k$-fold change in output.

In general, let us suppose an economy has $n$ sectors. The output of any sector, say the $i$th sector, is needed as an input in many other sectors, or even for that sector itself. It means that the level of the output of the $i$th sector will depend on the input requirements of all the $n$ sectors. On the other hand, the output of many other sectors will enter an input into the $i$th sector, and consequently the levels of other sectors products will depend partly on the input requirements of the $i$th sector. In view of these intersectoral dependences, it is clear that the input-output analysis should be of great use in production planning. This leads to the fact that the Leontief model allows answering the question: “What output level should reach each of the $n$ sectors in an economy in order to satisfy the total demand for that product?” To present the input-output model the following designations are introduced (they are expressed in financial terms, for example, in millions of dollars):

- $X = (X_1 \ X_2 \ ... \ X_n)^T$: A production vector, where $X_i$ ($i = 1, ..., n$) denotes the total value of the output of the $i$th sector for a year
- $d = (d_1 \ d_2 \ ... \ d_n)^T$: A final demand vector, where $d_i$ ($i = 1, ..., n$) denotes the total value of goods and services demanded from the $i$th sector by a nonproductive part of the economy (an open sector)
- $x_{ij}$ ($i, j = 1, ..., n$): Flow of production from $i$th sector to the $j$th sector; part of the output of the $i$th sector used in the $j$th sector for the production of its output

A starting point for an input-output analysis is an input-output table. An input-output table is a description of the flows (relationships) of goods and services among different sectors of a particular economic system. In the table, each horizontal row describes how one sector’s total product is divided among various sectors and final consumption. In turn, each vertical column denotes the combination of productive resources used within one sector. These flows concern a particular period, usually a one-year period. Table 9.1 shows a representation of an input-output table.

The structure of the table is a matrix that lists economic sectors, in the same sequence, both vertically and horizontally. If the rows of Table 9.1 are considered, for each $i$th sector ($i = 1, ..., n$), a linear equation describing how the sector distributes an output to other sectors can be written as
Table 9.1 An input-output table

| Sector | 1   | 2   | ... | n   | Total output | Final demand |
|--------|-----|-----|-----|-----|--------------|--------------|
| 1      | $x_{11}$ | $x_{12}$ | ... | $x_{1n}$ | $X_1$         | $d_1$         |
| 2      | $x_{21}$ | $x_{22}$ | ... | $x_{2n}$ | $X_2$         | $d_2$         |
| ...    | ... | ... | ... | ... | ...          | ...          |
| n      | $x_{n1}$ | $x_{n2}$ | ... | $x_{nn}$ | $X_n$         | $d_n$         |

\[
\begin{align*}
X_1 &= x_{11} + x_{12} + \ldots + x_{1n} + d_1 \\
X_2 &= x_{21} + x_{22} + \ldots + x_{2n} + d_2 \\
\vdots
X_n &= x_{n1} + x_{n2} + \ldots + x_{nn} + d_n
\end{align*}
\]

(9.1)

From the assumptions of the model: in order to produce each unit of the product of the $j$th sector, the input needed for the product of the $i$th sector must be a fixed amount, which should be denoted by $a_{ij}$ belonging to $[0, 1)$. Each element $a_{ij}$ is named the input coefficient or the technological coefficient and is calculated as

\[
a_{ij} = \frac{x_{ij}}{X_j} \quad (i, j = 1, \ldots n).
\]

(9.2)

The coefficients $a_{ij}$ are fixed by the current technology (when technology changes, coefficients change as well). From the formula (9.2)

\[
x_{ij} = a_{ij} X_j \quad (i, j = 1, \ldots n)
\]

(9.3)

where elements $a_{ij} X_j$ for $j = 1, \ldots, n$ are the inputs demand of the $j$th sector. If Eq. (9.3) are put into (9.1), a linear system of Eq. (9.1) takes the form

\[
\begin{align*}
X_1 &= a_{11} X_1 + a_{12} X_2 + \ldots + a_{1n} X_n + d_1 \\
X_2 &= a_{21} X_1 + a_{22} X_2 + \ldots + a_{2n} X_n + d_2 \\
\vdots
X_n &= a_{n1} X_1 + a_{n2} X_2 + \ldots + a_{nn} X_n + d_n
\end{align*}
\]

(9.4)

or, following application of a matrix algebra, it can be written as

\[
\begin{pmatrix}
X_1 \\
X_2 \\
\vdots \\
X_n
\end{pmatrix} =
\begin{pmatrix}
a_{11} & a_{12} & \ldots & a_{1n} \\
a_{21} & a_{22} & \ldots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \ldots & a_{nn}
\end{pmatrix}
\begin{pmatrix}
X_1 \\
X_2 \\
\vdots \\
X_n
\end{pmatrix} +
\begin{pmatrix}
d_1 \\
d_2 \\
\vdots \\
d_n
\end{pmatrix}
\]

(9.5)

or in short

\[
X = AX + d
\]

(9.6)
where $X$ denotes the output matrix, $A$ is the matrix of technological coefficients, and $d$ is the final demand matrix.

Very interesting from an economic point of view is matrix $A$, in which each column determines the input requirements for the production of one unit of a particular sector. This means that the production of each unit of the product of the $j$th sector requires $a_{1j}$ unit of the product of the first sector, $a_{2j}$ unit of the product of the second sector, ..., and $a_{nj}$ unit of the product of the $n$th sector. Because of the presence of the open sector, the sum of the elements in each column of the input coefficient matrix $A$ must be less than 1. If this sum is greater than or equal to 1, production is not economically justifiable.

Equation (9.6) can be written in the form of the Leontief model ($I$ denotes a unit matrix of size $n$)

$$X - AX = d \iff (I - A)X = d. \quad (9.7)$$

The matrix $(I - A)$ is called the Leontief matrix and converts a total production vector $X$ into a final product vector $d$. The question appears immediately: “If the final product vector $d$ is given, can the situation be reversed and can the total production vector $X$ be determined?” To answer the question the concept of a productive matrix should be introduced. A matrix of input coefficients $A$ is productive if there is a nonnegative vector of total production $X$ (i.e., all its entries are nonnegative, for each $i$ element $x_i \geq 0$), such that $X > AX$. The condition $X > AX$ has a simple economic interpretation. If $A$ is a productive matrix, the $i$th element of the vector $AX$ is the total production value of all sectors used by the $i$th sector in a year. Hence, the condition $X > AX$ means that, for each $i$, the value of the product produced by the $i$th sector exceeds the value of the products used by the $i$th sector. In other words, each sector runs at a profit.

**Theorem 1** Let $A$ be a square and a nonnegative matrix. Then $(I - A)$ is invertible and $(I - A)^{-1}$ nonnegative, if and only if the matrix $A$ is productive.

One can notice from the above theorem that in the real economy the final product $d$ determines the total production $X$, according to the rule

$$X = (I - A)^{-1}d. \quad (9.8)$$

If the symbol $\Delta$ is used, for example, $\Delta X$, to indicate a change in the value of a variable $X$, the change in the size of one variable due to changes in a second one using the Leontief model can be determined. The effect in the value of the final product, determined by changes in the total production value can be calculated using the equation in the following form

$$(I - A)\Delta X = \Delta d \quad (9.9)$$

or the change in total production due to the changes in the final product, using the equation in the form
\[ \Delta X = (I - A)^{-1} \Delta d. \] (9.10)

The Leontief model is used for forecasting. Three types of forecasts can be distinguished:

- first type forecast: When \( X \) or \( \Delta X \) are given, using formula (9.7) or (9.9) vector \( d \) or \( \Delta d \) can be determined.
- second type forecast: When \( d \) or \( \Delta d \) are given, using formula (9.8) or (9.10) vector \( X \) or \( \Delta X \) can be determined.
- mixed forecast: When complementary elements of \( X \) and \( d \) or \( \Delta X \) and \( \Delta d \) are given, using the Leontief model in the form (9.7) or (9.9) the remaining elements of \( X \) and \( d \) or \( \Delta X \) and \( \Delta d \) can be determined.

### 9.3 Example of Application of OFNs in the Leontief Model

The nature of the input-output analysis makes it possible to analyze an economy as an interconnected system of sectors that directly and indirectly affect one another. The Leontief model can be used, among others, to analyze how changes in the production level of different sectors affect changes in the final demand. More precisely, this model can be used to determine the final demand when production levels of different sectors are known, using the formula (9.7) or to analyze how changes in the production level of different sectors affect changes in the final demand, using the formula (9.9).

The use of OFNs allows us simultaneously to take into account the formulas (9.7) and (9.9). For this purpose, the triangular OFNs of the form \( X = (f_X(0), f_X(1), g_X(1), g_X(0)) \), where \( f_X(1) = g_X(1) \), is used with the following interpretation. The real number \( f_X(0) \) denotes the current production level, whereas the real number \( g_X(0) \) denotes the forecasted production level. Then the orientation of the OFN informs us about the direction of a change, that is, it describes the economic situation of the sector. More precisely, if \( f_X(0) < g_X(0) \) then the expected economic situation of the sector is good and the production level of the sector increases. Otherwise, that is, when \( f_X(0) > g_X(0) \) then the economic situation of the sector deteriorates and the production level is reduced. When \( f_X(0) = g_X(0) \) then there are no economic reasons to change the production level.

Let us consider a hypothetical economy consisting of only three sectors. The input-output table of this economy is presented in Table 9.2.

Using formula (9.2), the matrix \( A \) of the technological coefficients can be calculated and takes the form

\[
A = \begin{pmatrix}
0.1 & 0.2 & 0.1 \\
0.2 & 0.4 & 0.2 \\
0.3 & 0.1 & 0.4
\end{pmatrix}.
\]
Table 9.2: The input-output table of a hypothetical economy

| sector | 1  | 2  | 3  | \(X_i\) | \(d_i\) |
|--------|----|----|----|--------|--------|
| 1      | 140| 320| 180| 1400   | 760    |
| 2      | 280| 640| 360| 1600   | 320    |
| 3      | 420| 160| 720| 1800   | 500    |

The Leontief matrix \((I - A)\) that allows calculating the final demand of the economy takes the form

\[
I - A = \begin{pmatrix}
0.9 & -0.2 & -0.1 \\
-0.2 & 0.6 & -0.2 \\
-0.3 & -0.1 & 0.6
\end{pmatrix}.
\]

We consider how changes in the production level of different sectors affect changes in the final demand of each sector. For the particular sector, depending on the economic situation, three cases are analyzed: no change, and increase or decrease of the production level. Obtained results are presented in Tables 9.3 and 9.4, where \(X\) is the production vector, \(d\) is the final demand vector, \(\Delta X\) is the change of production level vector \((i.e., f_X(0) - g_X(0))\) where the sign determines the direction of change \((+\text{ increase and } -\text{ decrease})\), and \(\Delta d\) is the change of the final demand vector \((i.e., f_d(0) - g_d(0))\) where the sign determines the direction of change \((+\text{ increase and } -\text{ decrease})\).

Let us consider line 1 of Table 9.3. It shows the economy is stable and there are no causes influencing changes in the production level. The production level of each sector in the current \(X_c\) and forecasted \(X_f\) periods are the same and equal to \(X_c = X_f = (1400\ 1600\ 1800)^T\). It means that there are no causes to change the final demand level and in the current \(d_c\) and forecasted \(d_f\) periods it is equal to \(d_c = d_f = (760\ 320\ 500)^T\).

Line 2 of Table 9.3 shows the situation in which the first sector has a good economic situation and intends to increase the production level from 1400 to 1500, whereas the other sectors have a stable situation and their production levels do not change. The use of OFNs in the Leontief model allows us to compute, for the current production level equal to \(X_c = (1400\ 1600\ 1800)^T\), the final demand equal to \(d_c = (760\ 320\ 500)^T\). These numbers also allow us to compute the final demand equal to \(d_f = (850\ 300\ 470)^T\) after increasing the production level of the first sector equal to \(X_f = (1500\ 1600\ 1800)^T\). This means the increase of the production level of the first sector of 100 units and the unchanged production level of other sectors (described by the vector \(\Delta X = (100\ 0\ 0)^T\) cause changes in the final demand from \(d_c = (760\ 320\ 500)^T\) to \(d_f = (850\ 300\ 470)^T\) (described by the vector \(\Delta d = (90\ -20\ -30)^T\)). Changes in the final demands of the sectors result from the following facts.

- Increasing the production level of the first sector and the unchanged production level of other sectors causes the increase of the final demand of the first sector.
| No. | \(X\)                         | \(d\)                         | \(\Delta X\)        | \(\Delta d\)  |
|-----|--------------------------------|--------------------------------|---------------------|----------------|
| 1.  | \((1400, 1400, 1400, 1400)\)  | \((1600, 1600, 1600, 1600)\)  | \((500, 500, 500, 500)\) | \((0, 0)\)  |
| 2.  | \((1400, 1450, 1450, 1500)\) | \((1600, 1600, 1600, 1600)\)  | \((500, 485, 485, 470)\) | \((100, 0)\) \((-90, -20)\) |
| 3.  | \((1400, 1350, 1350, 1300)\) | \((1600, 1600, 1600, 1600)\)  | \((500, 515, 515, 530)\) | \((-100, 0)\) \((-90, 20)\) |
| 4.  | \((1400, 1400, 1400, 1400)\) | \((1600, 1650, 1650, 1700)\)  | \((500, 495, 495, 490)\) | \((0, 100)\) \((-20, 60)\) |
| 5.  | \((1400, 1400, 1400, 1400)\) | \((1600, 1550, 1550, 1500)\)  | \((500, 505, 505, 510)\) | \((0, -100)\) \((-60, 10)\) |
| 6.  | \((1400, 1400, 1400, 1400)\) | \((1600, 1600, 1600, 1600)\)  | \((500, 530, 530, 560)\) | \((0, 100)\) \((-10, -20)\) |
| 7.  | \((1400, 1400, 1400, 1400)\) | \((1600, 1600, 1600, 1600)\)  | \((500, 470, 470, 440)\) | \((0, -100)\) \((-60, 20)\) |
| 8.  | \((1400, 1400, 1400, 1400)\) | \((1600, 1650, 1650, 1700)\)  | \((500, 480, 480, 460)\) | \((100, 100)\) \((-70, 40)\) |
| 9.  | \((1400, 1400, 1400, 1400)\) | \((1600, 1800, 1800, 1800)\)  | \((500, 490, 490, 480)\) | \((100, -100)\) \((-80, 20)\) |
| 10. | \((1400, 1400, 1400, 1400)\) | \((1600, 1800, 1800, 1800)\)  | \((500, 510, 510, 520)\) | \((-100, 100)\) \((-110, 80)\) |
| 11. | \((1400, 1450, 1450, 1500)\) | \((1600, 1800, 1800, 1800)\)  | \((500, 520, 520, 540)\) | \((-100, 100)\) \((-70, 40)\) |
| 12. | \((1400, 1400, 1450, 1500)\) | \((1600, 1800, 1800, 1800)\)  | \((500, 515, 515, 530)\) | \((100, -100)\) \((-80, 40)\) |
| 13. | \((1400, 1450, 1450, 1500)\) | \((1600, 1800, 1800, 1800)\)  | \((500, 455, 455, 410)\) | \((100, 0)\) \((-90, -40)\) |
| 14. | \((1400, 1450, 1450, 1500)\) | \((1600, 1800, 1800, 1800)\)  | \((500, 485, 485, 470)\) | \((-100, 0)\) \((-80, -40)\) |

Table 9.3 Production vector \(X\) described by OFNs and the calculated final demand vector \(d\) and the change vectors \(\Delta X\) and \(\Delta d\).
Table 9.4 Production vector $X$ described by OFNs and the calculated final demand vector $d$ and the change vectors $\Delta X$ and $\Delta d$

| No. | $X$                               | $d$                          | $\Delta X$ | $\Delta d$ |
|-----|-----------------------------------|------------------------------|------------|------------|
| 1.  | $((1400, 1350, 1350, 1300))$      | $(760, 710, 710, 660)$       | $(-100, 0)$| $(-100, 0)$|
|     | $(1600, 1600, 1600, 1600)$        | $(320, 320, 320, 320)$       | $0$        | $0$        |
|     | $(1800, 1850, 1850, 1900)$        | $(500, 545, 545, 590)$       | $100$      | $90$       |
| 2.  | $((1400, 1400, 1400, 1400))$      | $(760, 745, 745, 730)$       | $0$        | $-30$      |
|     | $(1600, 1650, 1650, 1700)$        | $(320, 340, 340, 360)$       | $100$      | $40$       |
|     | $(1800, 1850, 1850, 1900)$        | $(500, 525, 525, 550)$       | $100$      | $50$       |
| 3.  | $((1400, 1400, 1400, 1400))$      | $(760, 755, 755, 750)$       | $0$        | $-10$      |
|     | $(1600, 1650, 1650, 1700)$        | $(320, 360, 360, 400)$       | $100$      | $80$       |
|     | $(1800, 1750, 1750, 1700)$        | $(500, 465, 465, 430)$       | $100$      | $70$       |
| 4.  | $((1400, 1400, 1400, 1400))$      | $(760, 765, 765, 770)$       | $0$        | $10$       |
|     | $(1600, 1550, 1550, 1500)$        | $(320, 280, 280, 240)$       | $-100$     | $-80$      |
|     | $(1800, 1850, 1850, 1900)$        | $(500, 535, 535, 570)$       | $100$      | $70$       |
| 5.  | $((1400, 1400, 1400, 1400))$      | $(760, 775, 775, 790)$       | $0$        | $30$       |
|     | $(1600, 1550, 1550, 1500)$        | $(320, 300, 300, 280)$       | $-100$     | $-40$      |
|     | $(1800, 1750, 1750, 1700)$        | $(500, 475, 475, 450)$       | $-100$     | $-50$      |
| 6.  | $((1400, 1450, 1450, 1500))$      | $(760, 790, 790, 820)$       | $100$      | $60$       |
|     | $(1600, 1650, 1650, 1700)$        | $(320, 330, 330, 340)$       | $100$      | $20$       |
|     | $(1800, 1850, 1850, 1900)$        | $(500, 510, 510, 520)$       | $100$      | $20$       |
| 7.  | $((1400, 1450, 1450, 1500))$      | $(760, 800, 800, 840)$       | $100$      | $80$       |
|     | $(1600, 1650, 1650, 1700)$        | $(320, 350, 350, 380)$       | $100$      | $60$       |
|     | $(1800, 1750, 1750, 1700)$        | $(500, 450, 450, 400)$       | $100$      | $60$       |
| 8.  | $((1400, 1450, 1450, 1500))$      | $(760, 810, 810, 860)$       | $100$      | $100$      |
|     | $(1600, 1550, 1550, 1500)$        | $(320, 270, 270, 220)$       | $100$      | $100$      |
|     | $(1800, 1850, 1850, 1900)$        | $(500, 520, 520, 540)$       | $100$      | $40$       |
| 9.  | $((1400, 1450, 1450, 1500))$      | $(760, 820, 820, 880)$       | $100$      | $120$      |
|     | $(1600, 1550, 1550, 1500)$        | $(320, 290, 290, 260)$       | $100$      | $60$       |
|     | $(1800, 1750, 1750, 1700)$        | $(500, 460, 460, 420)$       | $100$      | $80$       |
| 10. | $((1400, 1350, 1350, 1300))$      | $(760, 700, 700, 640)$       | $-100$     | $-120$     |
|     | $(1600, 1550, 1550, 1500)$        | $(320, 350, 350, 380)$       | $100$      | $60$       |
|     | $(1800, 1850, 1850, 1900)$        | $(500, 540, 540, 580)$       | $100$      | $80$       |
| 11. | $((1400, 1350, 1350, 1300))$      | $(760, 710, 710, 660)$       | $-100$     | $-100$     |
|     | $(1600, 1550, 1550, 1500)$        | $(320, 370, 370, 420)$       | $100$      | $100$      |
|     | $(1800, 1750, 1750, 1700)$        | $(500, 480, 480, 460)$       | $100$      | $-40$      |
| 12. | $((1400, 1350, 1350, 1300))$      | $(760, 720, 720, 680)$       | $-100$     | $-80$      |
|     | $(1600, 1550, 1550, 1500)$        | $(320, 290, 290, 260)$       | $-100$     | $-60$      |
|     | $(1800, 1850, 1850, 1900)$        | $(500, 550, 550, 600)$       | $100$      | $100$      |
| 13. | $((1400, 1350, 1350, 1300))$      | $(760, 730, 730, 700)$       | $-100$     | $-60$      |
|     | $(1600, 1550, 1550, 1500)$        | $(320, 310, 310, 300)$       | $-100$     | $-20$      |
|     | $(1800, 1750, 1750, 1700)$        | $(500, 490, 490, 480)$       | $-100$     | $-20$      |
• On the other hand, other sectors, in response to the increasing demand of the first sector, transfer to the first sector a greater amount of their production, which causes a decrease in their final demand.

Line 3 of Table 9.3 shows the situation in which the first sector has a bad economic situation and intends to decrease the production level from 1400 to 1300, whereas other sectors have a stable situation and their production levels do not change. In this situation, the results are symmetrical (opposite) to those shown in line 2 of Table 9.3.

Lines 4–7 of Table 9.3 show the situation in which any sector (second or third one), under the influence of the economic situation, changes the production level, whereas the production levels of other sectors are not changed. Economic interpretation of the situation of the hypothetical economy is analogous to that presented above.

Now, let us compare the state of the economy presented in lines 2 and 8 of Table 9.3. In line 8, in comparison to line 2, the second sector also has a good economic situation and intends to increase the production level from 1600 to 1700. In this situation, for the current production level equal to \(X_c = (1400 \ 1600 \ 1800)^T\) the final demand, equal to \(d_c = (760 \ 320 \ 500)^T\) can be computed. Moreover the final demand, equal to \(d_f = (830 \ 360 \ 460)^T\) after increasing the production level of the first and second sectors of 100 units is determined. These results mean that the increase of the production level of the first and second sectors of 100 units and the unchanged production of the third sector (described by the vector \(\Delta X = (100 \ 100 \ 0)^T\)) cause changes in the final demand from \(d_c = (760 \ 320 \ 500)^T\) to \(d_f = (830 \ 360 \ 460)^T\) (described by the vector \(\Delta d = (70 \ 40 \ -40)^T\)). Changes in the final demands of the sectors result from the following facts (in comparison to line 2).

• Increasing the production level in the first and second sectors and the unchanged production level of the third sector causes the increase of the final demand in the first and second sectors.
• The final demand in the first sector is smaller (in comparison to line 2) because part of the production level of the first sector, instead of the final demand, is transferred for the increase of the production level of the second sector.
• The final demand of the second sector, which decreased in line 2, increases because of the increase of the production level of the second sector.
• The final demand in the third sector is smaller (in comparison to line 2) because a greater part of the production level of the third sector, instead of the final demand, is transferred to the increasing production level of the first and second sectors.

Lines 9–14 of Table 9.3 and lines 1–13 of Table 9.4 present situations in which the production level of sectors of the economy increases, decreases, or remains unchanged, and how it affects the final demand of these sectors. These lines show and allow us to observe the current and future final demands of the sectors and how the final demands change as a result of changes in the production levels of the sectors.

A similar analysis can be carried out knowing the final demand vector \(d\) (or \(\Delta d\)) and calculating the production levels of the sectors \(X\) (or \(\Delta X\)) of the economy using matrix
\[(I - A)^{-1} = \begin{pmatrix} 1.36 & 0.52 & 0.40 \\ 0.72 & 2.04 & 0.80 \\ 0.80 & 0.60 & 2.00 \end{pmatrix}\]

and formulas (9.8) and (9.10).

### 9.4 Conclusions

The chapter presented the concept of an application of Ordered Fuzzy Numbers in an economical model, called the Leontief model. These numbers also allow us to present more information than real numbers. It is shown, based on the Leontief model, where OFNs simultaneously described the current levels, forecast levels, and the levels of change, both with regard to the final demand and the production level. It makes the OFNs model a very useful tool for mathematical modeling in economics.

### References

1. Kacprzak, D.: Leontief’s Model and Ordered Fuzzy Numbers (in Polish: Model Leontiewa i skierowane liczby rozmyte). VII Konferencja naukowo-praktyczna: Energia w nauce i technice 2, 797–815 (2008)
2. Kacprzak, D.: Ordered Fuzzy Numbers in Economic Modeling (in Polish: Skierowane liczby rozmyte w modelowaniu ekonomicznym). Optimum—Studia Ekonomiczne 3, 263–281 (2010)
3. Kacprzak, D.: Total revenue and total cost modeling by Ordered Fuzzy Numbers (in Polish: Przychód i koszt całkowity przedsiębiorstwa wyrażony przy użyciu skierowanych liczb rozmytych). J. Manag. Financ. 10(2), 139–149 (2012)
4. Kacprzak, D.: Application of Ordered Fuzzy Numbers to Present Stock Prices (in Polish: Zastosowanie skierowanych liczb rozmytych do prezentacji cen akcji). Optimum—Studia Ekonomiczne 6, 20–34 (2012)
5. Kacprzak, D.: The Presentation of Consumer Goods Prices and Their Dynamics with the Use of the Ordered Fuzzy Numbers (in Polish: Przetacacja cen dóbr konsumpcyjnych oraz dynamiki ich zmian za pomocą skierowanych liczb rozmytych). Optimum—Studia Ekonomiczne 1, 184–196 (2014)
6. Kacprzak, D., Kosinski, W.: Optimizing firm inventory costs as a fuzzy problem. Stud. Log. Gramm. Rhetor. 37, 89–105 (2014)
7. Kacprzak, D., Kosinski, W., Prokopowicz, P.: Fuzziness—representation of dynamic changes by ordered fuzzy numbers. Stud. Fuzziness Soft Comput. 243, 485–508 (2009)
8. Kacprzak, D., Kosinski, W., Kosinski, W.K.: Financial stock data and ordered fuzzy numbers. In: 12th International Conference Artificial Intelligence and Soft Computing, P1: 259–270 (2013)
9. Kacprzak, M., Starosta, B., Węgryn-Wolska, K.: Metasets and opinion mining in new decision support system. Artif. Intell. Soft Comput. Part II 9120, 625–636 (2015)
10. Kacprzak, M., Starosta, B., Węgryn-Wolska, K.: New approach to decision making. Proc. Second Int. Afro-Eur. Conf. Ind. Adv. AECIA 427, 397–407 (2015)
11. Sobol, I., Kacprzak, D., Kosinski, W.: Optimizing of a company’s costs under fuzzy data and optimal orders under dynamic conditions. Optimum—Studia Ekonomiczne 5, 172–187 (2014)
12. Leontief, W. et al.: Studies in the Structure of the American Economy. Oxford University Press (1953)

13. Marszałek, A., Burczyński, T.: Financial fuzzy time series models based on ordered fuzzy numbers. In: Pedrycz, W., Chen, S.M. (eds.) Time Series Analysis, Modelling and Applications ISRL 47. Springer, Berlin, Heidelberg (2013)

14. Roszkowska, E., Kacprzak, D.: The fuzzy SAW and fuzzy TOPSIS procedures based on ordered fuzzy numbers. Inf. Sci. 369, 564–584 (2016)