Asymptotics of Quantum Relative Entropy
From Representation Theoretical Viewpoint

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Abstract

In this paper it was proved that the quantum relative entropy $D(\sigma \parallel \rho)$ can be asymptotically attained by Kullback Leibler divergences of probabilities given by a certain sequence of POVMs. The sequence of POVMs depends on $\rho$, but is independent of the choice of $\sigma$.

1 Introduction

In classical statistical theory the relative entropy $D(p \parallel q)$ is an information quantity which means the statistical efficiency in order to distinguish a probability measure $p$ of a measurable space from another probability measure $q$ of the same measurable space. The states correspond to measures on measurable space. When $p, q$ are discrete probabilities, the relative entropy (called also information divergence) introduced by Kullback and Leibler is defined by [1]:

$$D(p \parallel q) := \sum_i p_i \log \frac{p_i}{q_i}.$$ 

In this paper, we consider the quantum mechanical case. Let $\mathcal{H}$ be a separable Hilbert space which corresponds to the physical system of interest. In quantum theory the states of a system correspond to positive operators of trace one on $\mathcal{H}$. (These operators are called densities.) The quantum relative entropy of a state $\rho$ with respect to another state $\sigma$ is defined by [2]:

$$D(\sigma \parallel \rho) := \text{Tr}[\sigma \log \frac{\sigma}{\rho}].$$

States are distinguished through the result of a quantum measurement on the system. The most general description of a quantum measurement probability is given by the mathematical concept of a positive operator valued measure (POVM) $M = \{M_i\}_{i=1}^{N(M)}$ which is a partition of the unit $\text{Id}_\mathcal{H}$ such that any $M_i$ is nonnegative operator. A POVM $M = \{M_i\}$ on $\mathcal{H}$ is called Projection Valued Measure (PVM), if any $M_i$ is projection. In quantum mechanics, $P^M(\rho) = \text{Tr}[M_i \rho]$ describes the probability distribution given by a POVM $M$ with respect to a state $\rho$. Then we define the quantity $D_M(\sigma \parallel \rho)$ as [2]:

$$D_M(\sigma \parallel \rho) := D(P^M_\sigma \parallel P^M_\rho).$$

Thus an information quantity we can directly access by a measurement $M$ is not $D(\sigma \parallel \rho)$ but $D(P^M_\sigma \parallel P^M_\rho)$. The map $\rho \mapsto P^M_\rho$ is the dual of a unitpreserving completely positive map. Therefore, we have the following by Uhlmann inequality [3]:

$$D_M(\sigma \parallel \rho) \leq D(\sigma \parallel \rho).$$

(1)
The equality is attained by a certain POVM $M$ when and only when $\rho\sigma = \sigma\rho$.

In this paper, we consider asymptotic attainment of the equality of the inequality (1). In order to answer the question we define the quantum i.i.d.-condition which is the quantum analogue of the independent and identically distribution condition. If there exist $n$ samples of the state $\rho$, the quantum state is described by $\rho^{\otimes n}$ defined by:

$$\rho^{\otimes n} := \rho \otimes \cdots \otimes \rho$$ on $\mathcal{H}^{\otimes n}$,

where the composite system $\mathcal{H}^{\otimes n}$ is defined as:

$$\mathcal{H}^{\otimes n} := \mathcal{H} \otimes \cdots \otimes \mathcal{H}.$$

In this paper, we call this condition the quantum i.i.d.-condition. Related to the inequality (1), it is well-known that $D(\sigma^{\otimes n} \parallel \rho^{\otimes n}) = nD(\sigma \parallel \rho)$.

Let $M_n$ be a POVM on $\mathcal{H}^{\otimes n}$, then we have

$$\frac{1}{n} D_{M_n}(\rho^{\otimes n} \parallel \sigma^{\otimes n}) \leq D(\sigma \parallel \rho). \quad (2)$$

Therefore, we consider the attainment of the equality of (2) in taking the limit of $n \to \infty$. Hiai and Petz proved the following theorem with respect to this problem.

**Theorem 1** Assume that the dimension of $\mathcal{H}$ is finite. Let $\sigma_n$ be a state on $\mathcal{H}^{\otimes n}$. If the sequence $\left\{\frac{1}{n} D(\sigma_n \parallel \rho^{\otimes n})\right\}$ convergence as $n \to \infty$, then we have

$$\lim_{n \to \infty} \frac{1}{n} D(\mathcal{E}_{\rho^{\otimes n}}(\sigma_n) \parallel \rho^{\otimes n}) = \lim_{n \to \infty} \frac{1}{n} D(\sigma_n \parallel \rho^{\otimes n}), \quad (3)$$

where $\mathcal{E}_{\rho^{\otimes n}}$ denotes the conditional expectation defined in (4) in the following section.

The preceding theorem implies that

$$\lim_{n \to \infty} \frac{1}{n} D(E(\mathcal{E}_{\rho^{\otimes n}}(\sigma_n)) \times E(\rho^{\otimes n}))(\sigma_n \parallel \rho^{\otimes n}) = \lim_{n \to \infty} \frac{1}{n} D(\sigma_n \parallel \rho^{\otimes n}),$$

where the PVM $E(\mathcal{E}_{\rho^{\otimes n}}(\sigma_n)) \times E(\rho^{\otimes n})$ is defined in the following section. In this paper, we consider whether a sequence of PVMs satisfying (3) depends on $\sigma_n$ in the case of that the state $\sigma_n$ satisfies the quantum i.i.d.-condition i.e. $\sigma_n = \sigma^{\otimes n}$:

$$\frac{D_{M_n}(\sigma^{\otimes n} \parallel \rho^{\otimes n})}{n} \to D(\sigma \parallel \rho) \text{ as } n \to \infty \forall \sigma. \quad (4)$$

We will consider this problem from a representation theoretical viewpoint. The main theorem of this paper is the following theorem.

**Theorem 2** Let $\rho$ be a state on $\mathcal{H}$, then there exists a sequence $\{(l_n, M_n)\}$ of pairs of an integer and a measurement on $\mathcal{H}^{\otimes l_n}$ such that

$$\frac{D_{M_n}(\sigma^{\otimes l_n} \parallel \rho^{\otimes l_n})}{l_n} \to D(\sigma \parallel \rho) \text{ as } n \to \infty \forall \sigma. \quad (5)$$

In the finite-dimensional case, the convergence of (5) is uniform for all $\sigma$. 2
2 Preliminary

Next, we consider the relation between a PVM and a quantum relative entropy. We put some definitions for this purpose. A state $\rho$ is called commutative with a PVM $E(=\{E_i\})$ on $\mathcal{H}$ if $\rho E_i = E_i \rho$ for any $i$. For PVMs $E(=\{E_i\}), F(=\{F_j\})$, we denote $E \leq F$ if for any $i$ there exist subsets $\{F/E_i\}$ such that $E_i = \sum_{j \in (F/E_i)} F_j$. For a state $\rho$, $E(\rho)$ denotes the spectral measure of $\rho$ which can be regarded a PVM. The conditional expectation $\mathcal{E}_E$ with respect to a PVM $E$ is defined as:

$$\mathcal{E}_E : \rho \mapsto \sum_i E_i \rho E_i.$$  \hspace{1cm} (6)

Therefore, the conditional expectation $\mathcal{E}_E$ is an affine map from the set of states to themselves. Then, the state $\mathcal{E}_E(\rho)$ is commutative with a PVM $E$. For simplicity, we denote the conditional expectation $\mathcal{E}_E(\rho)$ by $\mathcal{E}_\rho$.

**Theorem 3** Let $E$ be a PVM such that $w(E) := \sup_i \dim E_i < \infty$. If states $\rho, \sigma$ are commutative with a PVM $E$ and a PVM $F$ satisfies that $E, E(\rho) \leq F$, then we have

$$D_F(\sigma\|\rho) \leq D(\sigma\|\rho) \leq D_F(\sigma\|\rho) + \log w(E).$$

Note that there exists a PVM $F$ such that $E, E(\rho) \leq F$.

**Proof** It is proved by Lemma [1] and Lemma [2].

**Lemma 1** Let $\sigma, \rho$ be states. If a PVM $F$ satisfies that $E(\rho) \leq F$, then

$$D(\sigma\|\rho) = D_F(\sigma\|\rho) + D(\sigma\|\mathcal{E}_F(\sigma)).$$  \hspace{1cm} (7)

**Proof** Since $E(\rho) \leq F$, $F$ is commutative with $\rho$, we have $\text{Tr} \mathcal{E}_F(\sigma) \log \rho = \text{Tr} \sigma \log \rho$. Remark that $\text{Tr} \mathcal{E}_F(\sigma) \log \sigma = \text{Tr} \sigma \log \sigma$. Therefore, we get the following:

$$D_F(\sigma\|\rho) - D(\sigma\|\rho) = \text{Tr} \mathcal{E}_F(\sigma)(\log \mathcal{E}_F(\sigma) - \log \rho) - \text{Tr} \sigma(\log \sigma - \log \rho)
= \text{Tr} \mathcal{E}_F(\sigma)(\log \mathcal{E}_F(\sigma) - \log \sigma).$$

We get \hspace{1cm} (7).

**Lemma 2** Let $E, F$ be PVMs such that $E \leq F$. If a state $\sigma$ is commutative with $E$, then we have

$$D(\sigma\|\mathcal{E}_F(\sigma)) \leq \log w(E).$$  \hspace{1cm} (8)

**Proof** Let $a_i := \text{Tr} E_i \sigma E_i, \sigma_i := \frac{1}{a_i} E_i \sigma E_i$, then $\sigma = \sum_i a_i \sigma_i, \mathcal{E}_F(\sigma) = \sum_i a_i \mathcal{E}_F(\sigma_i)$. Therefore,

$$D(\sigma\|\mathcal{E}_F(\sigma)) = \sum_i a_i D(\sigma_i\|\mathcal{E}_F(\sigma_i)) \leq \sup_i D(\sigma_i\|\mathcal{E}_F(\sigma_i))
= \sup_i (\text{Tr} \sigma_i \log \sigma_i - \text{Tr} \mathcal{E}_F(\sigma_i) \log \mathcal{E}_F(\sigma_i))
\leq -\sup_i \text{Tr} \mathcal{E}_F(\sigma_i) \log \mathcal{E}_F(\sigma_i) \leq \sup_i \log \dim E_i = \log w(E).$$

Thus, we get \hspace{1cm} (8).

If a PVM $F = \{F_j\}$ is commutative with a PVM $E = \{E_i\}$, then we can define the PVM $F \times E = \{F_j E_i\}$. Then we have $F \times E \geq E, F$. If $E'$ is commutative with $E, F$ and $F \geq E'$, then we have $E' \times F \geq E' \times E$. If $F \geq E$ and $\frac{\text{Tr}[F_j \rho]}{\text{Tr}[E_i \rho]} = \frac{\text{Tr}[F_j \sigma]}{\text{Tr}[E_i \sigma]}$ for $j \in (F/E)_i$, then we have $D_F(\sigma\|\rho) = D_E(\sigma\|\rho)$. 

3
3 Quantum i.i.d. condition from group theoretical viewpoint

From the orthogonal direct sum decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_k$, we can naturally constitute the PVM $\{P_{\mathcal{H}_i}\}$, where $P_{\mathcal{H}_i}$ denotes the projection of $\mathcal{H}_i$. In the following, we consider the relation between irreducible representations and PVMs.

3.1 group representation and its irreducible decomposition

Let $V$ be a finite dimensional vector space over the complex numbers $\mathbb{C}$. A map $\pi$ from a group $G$ to the generalized linear group of a vector space $V$ is called a representation if the map $\pi$ is homomorphism i.e. $\pi(g_1)\pi(g_2) = \pi(g_1g_2)$, $\forall g_1, g_2 \in G$. For a subspace $W$ of $V$, it is invariant with respect to a representation $\pi$ if $\pi|_W(g_1)\pi|_W(g_2) = \pi|_W(g_1g_2)$, $\forall g_1, g_2 \in G$, where $\pi|_W$ denotes the restriction of $\pi$ to $W$. In this case, $\pi|_W$ is called a subrepresentation of $\pi$. Let $\pi$ be a representation to $V$, then $\pi$ is called irreducible if there no proper nonzero invariant subspace of $V$. Let $\pi_1(\pi_2)$ be representations of a group $G$ on $V_1(V_2)$ respectively. The tensored representation $\pi_1 \otimes \pi_2$ of $G$ on $V_1 \otimes V_2$ is defined as: $(\pi_1 \otimes \pi_2)(g) = \pi_1(g) \otimes \pi_2(g)$, and the direct sum representation $\pi_1 \oplus \pi_2$ of $G$ on $V_1 \oplus V_2$ is also defined as: $(\pi_1 \oplus \pi_2)(g) = \pi_1(g) \oplus \pi_2(g)$. If there is a invertible linear map $f$ from $V_1$ to $V_2$ such that $f\pi_1(g) = \pi_2(g)f$, $\pi_1$ is equivalent with $\pi_2$. If $W$ is an invariant subspace for a representation $\pi$ on $V$, then there is a complementary invariant subspace $W'$ for a representation $\pi$, so that $V = W \oplus W'$ and $\pi = \pi|_W \oplus \pi|_{W'}$. Therefore, any representation is a direct sum representation of irreducible representations.

Let $\pi_1$ ($\pi_2$) be a representation of $W_1$ ($W_2$) respectively. $W_1 \oplus W_2$ gives an irreducible decomposition of the direct sum representation $\pi := \pi_1 \oplus \pi_2$. If $\pi_1$ is equivalent with $\pi_2$, there is another irreducible decomposition. For example, there is an irreducible decomposition $\{v \oplus f(v)\mid v \in W_1\} \oplus \{v \oplus -f(v)\mid v \in W_1\}$, where $f$ is a map which gives the equivalence with $\pi_1$ and $\pi_2$. If $\pi_1$ isn’t equivalent with $\pi_2$, there is no irreducible decomposition except $W_1 \oplus W_2$. A direct sum decomposition $W = W_1 \oplus \cdots \oplus W_i$ is called isotypic with respect to a representation $\pi$ if it satisfies the following conditions: every irreducible component of $W_i$ with respect to a representation $\pi|_{W_i}$ is equivalent with each other. If $i \neq j$, then any irreducible component of $W_i$ with respect to a representation $\pi|_{W_i}$ isn’t equivalent with any irreducible component of $W_j$ with respect to a representation $\pi|_{W_j}$.

For a representation $\pi$ of $G$, we can define the subrepresentation $\pi|_{G_1}$ of a subgroup $G_1$ of $G$ by restricting a representation $\pi$ to $G_1$. If the subrepresentation $\pi|_{G_1}$ is irreducible, then the representation $\pi$ is irreducible. But, the converse isn’t true. In this paper, we call a subgroup $G_1$ of $G$ unramified if any subrepresentation $\pi|_{G}$ is irreducible when the representation $\pi$ of $G$ is irreducible.

3.2 Relation between a unitary representation and a PVM

Let $\mathcal{H}$ be a finite-dimensional Hilbert space. A representation $\pi$ to a Hilbert space $\mathcal{H}$ is called unitary if $\pi(g)$ is a unitary matrix for any $g \in G$. If $\mathcal{H}_1$ is an invariant subspace of $\mathcal{H}$ with respect to a unitary representation $\pi$, the orthogonal space $\mathcal{H}_1$ of $\mathcal{H}_2$ is invariant with respect to a unitary representation $\pi$. Therefore, we have $\pi = \pi|_{\mathcal{H}_1} \oplus \pi|_{\mathcal{H}_2}$. A unitary representation $\pi$ can be described by the orthogonal direct sum representation of irreducible representations.
which are orthogonal with one another. We can regarded the direct sum decomposition as a PVM. Remark that without unitarity we cannot deduce the orthogonality. If there is a pair of irreducible component whose elements are equivalent with one another. Therefore, a corresponding PVM is not unique. In this paper, we denote the set of PVMs corresponding to an orthogonal irreducible decomposition by \( \mathcal{M}(\pi) \).

Elements of the isotypic decomposition of a unitary representation \( \pi \) are orthogonal with one another. Thus, we can define a PVM \( N(\pi) \) as the isotypic decomposition. We call a representation \( \pi \) of a group \( G \) quasi-unitary if there exist an unramified subgroup \( G_1 \) such that the subrepresentation \( \pi|_{G_1}^{G} \) is unitary. For a quasi-unitary representation \( \pi \), we define \( N(\pi)(\mathcal{M}(\pi)) \) by \( N(\pi|_{G_1}^{G_1})(\mathcal{M}(\pi|_{G_1}^{G_1})) \) respectively. We can show the uniqueness of them. For a unitary representation \( \pi \) and \( g \in G \), \( \pi(g) \) is commutative with a PVM \( M \in \mathcal{M}(\pi) \) and a PVM \( N(\pi) \). Concerning a quasi-unitary representation \( \pi \), we can prove the same fact.

### 3.3 Relation between the tensored representation and PVMs

Let the dimension of the Hilbert space \( \mathcal{H} \) is \( k \). Irreducible representations of the special linear group \( \text{SL}(\mathcal{H}) \) and the special unitary group \( \text{SU}(\mathcal{H}) \) are classified by the highest weight. Thus, any irreducible representation of the special linear group \( \text{SL}(\mathcal{H}) \) is irreducible under restricting to the special unitary group \( \text{SU}(\mathcal{H}) \). The special unitary group \( \text{SU}(\mathcal{H}) \) is unramified subgroup of the special linear group \( \text{SL}(\mathcal{H}) \). Also, the special linear group \( \text{SL}(\mathcal{H}) \) is unramified subgroup of the general linear group \( \text{GL}(\mathcal{H}) \) since the general linear group \( \text{GL}(\mathcal{H}) \) is described as the direct sum group \( \text{SL}(\mathcal{H}) \times U(1) \).

We denote the natural representation of the general linear group \( \text{GL}(\mathcal{H}) \), the special linear group \( \text{SL}(\mathcal{H}) \), the special unitary group \( \text{SU}(\mathcal{H}) \) to \( \mathcal{H} \) by \( \pi_{\text{GL}(\mathcal{H})} \), \( \pi_{\text{SL}(\mathcal{H})} \), \( \pi_{\text{SU}(\mathcal{H})} \), respectively. We consider representations \( \pi_{\text{GL}(\mathcal{H})}^{\otimes n} := (\cdots (\pi_{\text{GL}(\mathcal{H})} \otimes \pi_{\text{GL}(\mathcal{H})}) \cdots ) \otimes \pi_{\text{GL}(\mathcal{H})} \), \( \pi_{\text{SU}(\mathcal{H})}^{\otimes n} := (\cdots (\pi_{\text{SU}(\mathcal{H})} \otimes \pi_{\text{SU}(\mathcal{H})}) \cdots ) \otimes \pi_{\text{SU}(\mathcal{H})} \) to the tensored \( \mathcal{H}^{\otimes n} \). Remark that \( \pi_{\text{GL}(\mathcal{H})}^{\otimes n}|_{\text{SU}(\mathcal{H})} = \pi_{\text{SL}(\mathcal{H})}^{\otimes n} \). From the unitarity of the representation \( \pi_{\text{SU}(\mathcal{H})}^{\otimes n} \), representations \( \pi_{\text{SU}(\mathcal{H})}^{\otimes n} \) and \( \pi_{\text{SL}(\mathcal{H})}^{\otimes n} \) are quasi-unitary. Therefore, the set \( \mathcal{M}(\pi_{\text{SU}(\mathcal{H})}^{\otimes n}) \) (the PVM \( N(\pi_{\text{SU}(\mathcal{H})}^{\otimes n}) \)) is consistent with the sets \( \mathcal{M}(\pi_{\text{SU}(\mathcal{H})}^{\otimes n} \cdot \mathcal{M}(\pi_{\text{GL}(\mathcal{H})}^{\otimes n}) \) (the PVMs \( N(\pi_{\text{SL}(\mathcal{H})}^{\otimes n}) \cdot N(\pi_{\text{GL}(\mathcal{H})}^{\otimes n}) \)) and we denote it by \( I_{\mathcal{H}^{\otimes n}} \otimes (IR_{\mathcal{H}^{\otimes n}}) \) respectively.

From the Weyl’s dimension formula ((7.1.8) or (7.1.17) in Goodman-Wallach[10]), The \( n \)-th symmetric space is the irreducible subspace in the representation \( \pi_{\text{GL}(\mathcal{H})}^{\otimes n} \) whose dimension is maximum. Its dimension is the repeated combination \( k \mathcal{H}_n \) evaluated by \( \mathcal{H}_n = \binom{n+k-1}{k-1} = \binom{n}{n} H_{k-1} \leq (n+1)^k-1 \). For \( M \in I_{\mathcal{H}^{\otimes n}} \), we have the following:

\[
 w(M) \leq (n+1)^k-1.
\]

**Lemma 3** Let \( \sigma \) be a state on \( \mathcal{H} \). Then a PVM \( M \in I_{\mathcal{H}^{\otimes n}} \) and the PVM \( IR_{\mathcal{H}^{\otimes n}} \) is commutative with tensored state \( \sigma_{\mathcal{H}^{\otimes n}} \).

**Proof** If \( \sigma \in \text{GL}(\mathcal{H}) \), then this lemma is trivial. If \( \sigma \notin \text{GL}(\mathcal{H}) \), there exists a sequence \( \{\sigma_i\}_{i=1}^{\infty} \) of elements of \( \text{GL}(\mathcal{H}) \) such that \( \sigma_i \to \sigma \) as \( i \to \infty \). Therefore we have \( \sigma_{\mathcal{H}^{\otimes n}} \to \sigma_{\mathcal{H}^{\otimes n}} \) as \( i \to \infty \). Because a PVM \( M \) is commutative with \( \sigma_{\mathcal{H}^{\otimes n}} \), the PVM \( M \) is commutative with \( \sigma_{\mathcal{H}^{\otimes n}} \). Similarly, we can prove that the PVM \( IR_{\mathcal{H}^{\otimes n}} \) is commutative with \( \sigma_{\mathcal{H}^{\otimes n}} \). \( \square \)
From the definition of $I_{\mathcal{H}}^{\otimes n}$ and $IR^{\otimes n}$, if $j \in (M/IR^{\otimes n})_i$, we have

$$\#(M/IR^{\otimes n})_i \Tr M_j E(\rho^{\otimes n})_k \sigma^{\otimes n} = \Tr IR^{\otimes n}_i E(\rho^{\otimes n})_k \sigma^{\otimes n}.$$  

for states $\rho, \sigma$ and a PVM $M \in I_{\mathcal{H}}^{\otimes n}$. The number $\#(M/IR^{\otimes n})_i$ corresponds to the number of equivalent irreducible representations in the representation $\pi^{\otimes n}_{GL(\mathcal{H})}$. Therefore we obtain

$$D_{IR^{\otimes n} \times E(\rho^{\otimes n})}(\sigma^{\otimes n} \parallel \rho^{\otimes n}) = D_{M \times E(\rho^{\otimes n})}(\sigma^{\otimes n} \parallel \rho^{\otimes n}). \quad (10)$$

### 4 Proof of Main Theorem

First we will prove Theorem 2 in the case of that the dimension of $\mathcal{H}$ is finite number $k$. Let $\rho$ be a state on $\mathcal{H}$. From Theorem 4, Lemma 5 and the preceding discussion, we obtain the following fact. For a PVM $E^n \in I_{\mathcal{H}}^{\otimes n}$, the PVM $M^n := E^n \times E(\rho^{\otimes n})$ satisfies:

$$\frac{D_{M_n}(\sigma^{\otimes n} \parallel \rho^{\otimes n})}{n} \leq D(\sigma \parallel \rho) \leq \frac{D_{M_n}(\sigma^{\otimes n} \parallel \rho^{\otimes n})}{n} + (k - 1) \frac{\log(n + 1)}{n} \forall \sigma. \quad (11)$$

Therefore we obtain

$$\frac{D_{M_n}(\sigma^{\otimes n} \parallel \rho^{\otimes n})}{n} \rightarrow D(\sigma \parallel \rho) \text{ as } n \rightarrow \infty \text{ uniformly for } \sigma. \quad (12)$$

From (10), the PVM $IR^{\otimes n} \times E(\rho^{\otimes n})$ satisfies (11) and (12). We get (3) in the finite-dimensional case. In spin 1/2 system, the PVM $IR^{\otimes n}$ corresponds to the measurement of the total momentum, and $E(\rho^{\otimes n})$ does to the one of the momentum of the direction specified by $\rho$. These observables are commutative with one another. Next, we consider the infinite-dimensional case. Let $\mathcal{B}(\mathcal{H})$ be the set of bounded operators on $\mathcal{H}$, and $\mathcal{B}(\mathcal{H})^{\otimes n}$ be $\mathcal{B}(\mathcal{H}) \otimes \cdots \otimes \mathcal{B}(\mathcal{H})$. According to (3), from the separability of $\mathcal{H}$, there exists a finite-dimensional approximation of $\mathcal{H}$, i.e. a sequence $\{\alpha_n : \mathcal{B}(\mathcal{H}_n) \rightarrow \mathcal{B}(\mathcal{H})\}$ of unit-preserving completely positive maps such that $\mathcal{H}_n$ is finite-dimensional and

$$\lim_{n \rightarrow \infty} D(\alpha_n^*(\sigma) \parallel \alpha_n^*(\rho)) = D(\sigma \parallel \rho) \quad (13)$$

for any states $\sigma, \rho$ on $\mathcal{H}$ such that $\mu \rho \leq \sigma \leq \lambda \rho$ for some positive real numbers $\mu, \lambda$. From (12), for any positive integer $n$ there exists a pair $(l_n, M_n')$ of an integer and a PVM on $\mathcal{H}_n^{\otimes l_n}$ such that

$$D(\alpha_n^*(\sigma) \parallel \alpha_n^*(\rho)) - \frac{D_{M_n'}\left((\alpha_n^*(\sigma))^{\otimes l_n} \parallel (\alpha_n^*(\rho))^{\otimes l_n}\right)}{l_n} < \frac{1}{n}. \quad (14)$$

The completely positive map $\alpha_n^{\otimes l}$ from $\mathcal{B}(\mathcal{H}_n)^{\otimes l}$ to $\mathcal{B}(\mathcal{H})^{\otimes l}$ is defined as $\alpha_n^{\otimes l}(A_1 \otimes A_2 \otimes \cdots \otimes A_l) = \alpha_n(A_1) \otimes \alpha_n(A_2) \otimes \cdots \otimes \alpha_n(A_l)$ for $\forall A_i \in \mathcal{B}(\mathcal{H})$. Therefore we have $(\alpha_n^{\otimes l})^* (\sigma^{\otimes l}) = \alpha_n^*(\sigma)^{\otimes l}$. 

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Let $M_n$ be $\alpha_n \otimes l_n(M'_n)$, then from (13),(14) we get

$$D_{M_n}(\sigma \otimes l_n \| \rho \otimes l_n) \leq D_{M'_n}(\alpha_n \otimes l_n \| \alpha_n \otimes l_n)$$

$$\leq D(\alpha_n^*(\sigma) \otimes l_n \| \alpha_n^*(\rho) \otimes l_n)$$

$$\geq D(\alpha_n^*(\sigma) \| \alpha_n^*(\rho)) + \frac{1}{n}$$

$$\rightarrow D(\sigma \| \rho) \text{ as } n \rightarrow \infty.$$ 

Therefore, we obtain Theorem 4. Note that such a POVM $M_n$ is independent of $\sigma$.

**Conclusions**

It was proved that the quantum relative entropy $D(\sigma \| \rho)$ is attained by the sequence of Kullback-Leibler divergences given by a certain sequence of POVMs which is independent of $\sigma$. This formula is thought to be important for the quantum asymptotic detection and the quantum asymptotic estimation. About the quantum asymptotic estimation, see [3]. The realization of the sequence of measurements are left for future study. In spin 1/2 system, it is enough to simultaneously measure the total momentum and the momentum of the direction specified by $\rho$.

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