A New Criterion for Crack Formation in Disordered Materials

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I. INTRODUCTION

Cracks are one of the most important sources for the failure of solids. Despite continuous efforts for more than a century a full understanding of fracture has not yet been reached. A simple but very appealing picture for the occurrence of cracks goes back to Griffith. Griffith describes the emergence of cracks as a nucleation phenomenon: To open a crack in a thin plate, atomic bonds have to be broken and two new surfaces have to be created. For a crack of linear size $a$, this costs an energy of order $a$. Simultaneously, the potential energy of the plate under external load is reduced due to the stress relaxation around the crack. This decreases the energy by an amount of order $a^2$. Thus, the total crack energy as a function of $a$ increases for small $a$ linearly and reaches a maximum at $a = a_c$ before it decreases quadratically. Cracks of length $a < a_c$ are stable whereas those with $a > a_c$ are unstable. However, contrary to conventional nucleation phenomena the typical energy barriers for crack propagation in a perfect solid under realistic stresses are much too high to be overcome by thermal fluctuations. Instead, the pre-existence of micro-cracks (or pre-weakened bonds) on scales $a \lesssim a_c$ is usually tacitly assumed. These will then grow under external load. It seems to be reasonable to consider micro-cracks as well as other heterogeneities in the material as a kind of frozen disorder only amenable to a statistical treatment.

The propagation of supercritical cracks in an inhomogeneous material has been the subject of a number of articles, which have attempted to explain the roughness of crack fronts found experimentally. Unfortunately a convincing explanation of the experimental data is still lacking. The other aspect, the occurrence of a critical crack in the first place has been, to the best of our knowledge, not yet considered. It is the aim of the present paper to address this point by calculating the probability of the occurrence of a critical crack in systems which includes various types of disorder. In particular, we will consider randomness in the atomic bond strength as well as randomly distributed impurities (or other kinds of heterogeneities) and frozen dislocations. It should however be mentioned, that our considerations are not restricted to crystalline material. The main ingredients of our theory is isotropic elasticity, which also applies to amorphous materials, supplemented by randomly distributed disorder. The latter can also include mesoscopic heterogeneities and microcracks which occur during fabrication. Although the various sources of disorder conceivable may differ considerably in their local properties, the most important aspect from the statistical point of view (which is adopted in this paper) is the spatial decay of the stress fields they create. It is in this sense that the three types of disorder considered below are generic.

The main body of the paper is related to crack creation in a thin plate of infinite extension. Some results can however be easily extended to bulk materials. The detailed investigation of cracks in slabs of finite width as well as those in three dimensional systems will be presented in forthcoming publications.

II. CRACKS IN AN INFINITE TWO-DIMENSIONAL SAMPLE

In this paper we consider a single planar crack extended parallel to the $x$-direction in a 2-dimensional elastically isotropic system of infinite extension. The 2-dimensional situation can be realized e.g. by a thin plate of width $h$ in the so-called plane stress configuration. The Lamé coefficients $\lambda$ and $\mu$ of the 2-dimensional system are then related to the Lamé coefficients $\lambda$ and $\mu$ of the bulk by $\lambda = 2\lambda_0 h/(\lambda + 2\mu)$, $\mu = \mu_0 h$. The coordinates of the crack are given by

$$-a \leq x \leq a, \quad y = 0. \quad (2.1)$$

Such planar crack appears for instance in experiments with preweakened bonds.

In two dimensions only mode I and mode II cracks occur characterized by the only non-zero components $\sigma_{yy}^{(e)} = \sigma_{yy}^{(e)} = \sigma_{xy}^{(e)} = \sigma^{(e)}$ respectively, of the applied stress $\delta^{(e)}$. In the further treatment we will apply the dislocation theory of cracks. The crack will be
described by virtual lattice planes filling its interior such that there is no free crack surface. The lattice planes terminate in crack dislocations with Burgers vector \( b^{(c)} \). The whole crack is then given by a collection of dislocations (and antidislocations) \( b^{(c)}_a \) at positions \( \mathbf{r}_a = (x_a, 0) \). Crack dislocations interact with the external stress \( \sigma \) and the disorder made up of impurities and frozen dislocations. For the further discussion, it turns out to be convenient to introduce a 2-dimensional dislocation density \( b(r) = \sum_a b_a \delta(\mathbf{r} - \mathbf{r}_a) \). The actual distribution of the crack dislocation will be determined later from a minimum condition for the free energy for given external stresses and crystal imperfections. It should be mentioned however, that the crack description by dislocations is not essential for the final results. We could also have used more traditional elasticity theory combined with the appropriate boundary conditions on the crack surface. In this sense also amorphous materials are included (but there are no frozen dislocations in this case).

The interaction between the external stress \( \sigma^{(e)} \) and a dislocation with Burgers vector \( b \) is given by the Peach-Köhler force \( f_i = -\epsilon_{ij} b_{m} \partial_m \sigma^{(e)} \), where \( \epsilon_{ij} \) denotes the total antisymmetric unit tensor. With the help of this relation one obtains for the total contribution of \( \sigma^{(e)} \) on the crack dislocation energy

\[
E^{(c)} = -\epsilon_{ij} \sigma^{(e)}_{lm} \sum_a x_a b^{(c)}_{a,m} = -\sigma^{(e)}_{ym} \int_{-a}^{a} dx b^{(c)}_{m}(x). \tag{2.2}
\]

Thus, in mode I (II) only the \( y(x) \) component of \( b^{(c)}(x) \) contributes to \( E^{(c)} \). Since \( E^{(c)} \) is the only macroscopic term which favors the existence of crack dislocations, it is clear that in mode I (II) only those with \( b^{(c)}(x) \) parallel to the \( y(x) \)-axis will occur. This will be used in the following.

The stress field \( \sigma_{ij} \) generated by dislocations is related to the Airy stress function \( \chi(\mathbf{r}) \) by \( \sqrt{2} \sigma_{ij} = \epsilon_{ik} \epsilon_{jl} \partial_k \partial_l \chi(\mathbf{r}) \), where \( \chi \) is a solution of

\[
(\nabla^2)^2 \chi(\mathbf{r}) = \bar{Y} \epsilon_{ij} \partial_j b_i(\mathbf{r}). \tag{2.3}
\]

Here \( \bar{Y} = 4\bar{\mu}(\bar{\lambda} + \bar{\mu})/(2\bar{\mu} + \bar{\lambda}) \) denotes the Young modulus in two dimensions. In an infinite system the solution of \(2.3 \) is given by \(2.4 \)

\[
\chi(\mathbf{r}) = \bar{Y} \int d^2 r' g(\mathbf{r} - \mathbf{r}') \epsilon_{ij} \partial_j b_i(\mathbf{r}'). \tag{2.4}
\]

Here \( g(\mathbf{r}) = r^2/(8\pi) \), \( r^2 = (\log |r| + C)/(8\pi) \), \( C \) is an arbitrary constant. The elastic energy of the dislocations is then given by

\[
E^{(c)} = \frac{1}{2} \int d^2 r \sigma_{ij} u_{ij} \tag{2.5}
\]

\[
= -\frac{\bar{Y}}{2} \int d^2 r \int d^2 r' \epsilon_{ij} \epsilon_{kl} b_j(\mathbf{r}) b_l(\mathbf{r}') \partial_k \partial_l g(\mathbf{r} - \mathbf{r}')
\]

where we used the relation \( u_{ik} = \frac{1}{2\mu} \sigma_{ik} - \frac{\lambda}{4\mu(\lambda + \mu)} \delta_{ik} \sigma_{ll} \). \( \tag{2.6} \)

The elastic energy of the crack dislocations is then given by

\[
E^{(c)} = -\frac{\bar{Y}}{8\pi} \int_{-a}^{a} dx \int_{-a}^{a} dx' b^{(c)}(x) b^{(c)}(x') \log \left| \frac{x - x'}{a_0} \right|. \tag{2.7}
\]

\( b^{(c)}(x) \) denotes the crack dislocation density along the \( x \)-axis and \( a_0 \) denotes a microscopic cut-off (of the order of the lattice spacing and in general different for mode I and mode II). Eq. \(2.7 \) is valid both for mode I and mode II cracks. Note that \( E^{(c)} \) is always positive.

The crack dislocation energy is given by

\[
E^{(c)} = 2 \int_{-a}^{a} dx \bar{\gamma}(x) = 4\bar{\gamma}_0 a + E^{(c)}(a) \tag{2.8}
\]

\( \bar{\gamma}_0(x) \) reflects the randomness in the strength of the bonds broken. For simplicity we assume Gaussian disorder with \( \langle \bar{\gamma}_0(x) \rangle = 0 \) and

\[
\langle \bar{\gamma}_1(x) \rangle = \delta \gamma^2 a_0 \delta_{a_0}(x - x') \tag{2.9}
\]

\( \langle \ldots \rangle \) denotes the average over the disorder and \( \delta_{a_0}(x - x') \) a delta function of width \( a_0 \). In general the correlation length \( a_0 \) of the disorder appearing in \(2.9 \) is different from the cut-off appearing in \(2.7 \). Similarly, further correlation lengths could be introduced for the distribution of impurities and frozen dislocations to be considered below. To avoid a too clumsy notation we will however use everywhere the length \( a_0 \) as a small scale cut-off but keeping in mind this complication. Depending on the type of material under consideration, \( a_0 \) may vary between the

![FIG. 1. Lattice planes terminating in crack dislocations in mode I (a) and mode II (b) cracks. The dislocation vectors are perpendicular to the lattice planes. The arrows denote the direction of the external forces \( f^{(c)} \). Changing \( f^{(c)} \) crack dislocations in mode I and II climb and glide, respectively.](image)
size of an atom in crystalline and the size of a grain in granular materials, respectively. The precise value of $a_0$ will be of course important if comparison with experiments is made. Then $\left\langle E_1^{(s)}(a) \right\rangle = 0$ and we find for the variance of $E_1^{(s)}(a)$

$$\left\langle \left[ E_1^{(s)}(a) - E_1^{(s)}(a') \right]^2 \right\rangle = \Delta_{(s)} |a - a'| \quad (2.10)$$

where $\Delta_{(s)} = 8\delta \gamma^2 a_0$. Clearly Eq. (2.3) is only valid for $|a - a'| \gtrsim a_0$.

In the following we add the contributions $E^{(d)} = E^{(fi)} + E^{(i)}$ from randomly frozen dislocations and impurities to the energy. $E^{(d)}$ is given by $\int d^2 r \tilde{\sigma}_{ij}^{(d)} u_{ij}^{(c)}$ where $\tilde{\sigma}_{ij}^{(d)}$ denotes the stress generated by the disorder and $u_{ij}^{(c)}$ the strain field generated from cracks, respectively.

Using Eq. (2.3) $E^{(d)}$ can be written in the form

$$E^{(d)} = -\frac{\bar{Y}}{4\pi} \int_{-a}^{a} dx b^{(c)}(x) V(x), \quad (2.11)$$

$$V(x) = V^{(fi)}(x) + V^{(i)}(x) \quad (2.12)$$

From (2.3), (2.4) and (2.5) one obtains for the potential created from dislocations

$$V^{(fi)}(x) = 4\pi \int d^2 r' \epsilon_{ij} (\delta_k \partial_j g(r-r'))_{y=0} b_{ij}^{(d)}(r') \quad (2.13)$$

where $k = x, y$ for mode I, II cracks, respectively. The frozen dislocations are assumed to have both random positions $r_a$ and directions of their Burgers vectors such that $\left\langle b^{(d)}(r) \right\rangle = 0$ and

$$\left\langle b_i^{(d)}(r) b_j^{(d)}(r') \right\rangle = b_i^{(d)} c_i^{(d)} \delta_{ab}(r-r') \delta_{ij} \quad (2.14)$$

Here $c_i^{(d)}$ and $b_i^{(d)}$ denote the concentration and the strength of the dislocation.

Impurities (or more macroscopic inclusions) also generate a long-range elastic displacement field $u^{(i)}(r)$. Repeating the calculation of Eshelby for $d = 2$ dimensions one finds for the strain tensor of an impurity located at the origin

$$u_{ij}^{(i)}(r) = \frac{\Omega}{2\pi} \frac{\lambda + \mu}{2\mu + \lambda} \partial_i \partial_j \log \left| r \right| \quad (2.15)$$

Here $\Omega$ denotes the 2-dimensional volume change due to the impurity which can be of either sign. The interaction energy between the crack dislocations and the impurities of density $\tilde{c}_{ij}(r)$ takes the form

$$E^{(i)} = \frac{1}{2} \Omega \int d^2 r \tilde{\sigma}_{ij}^{(c)}(r) \tilde{c}_{ij}(r) \quad (2.16)$$

Here we used (2.15) and

$$\tilde{c}_{ij}(r) = \sum_a \delta_{ab}(r-r_a) - \tilde{c}_{ij}(r) \quad (2.17)$$

where the summation is over all impurity sites $r_a$ and $\tilde{c}_{ij}(r)$ denotes the impurity concentration. With (2.3) - (2.6) we find

$$V^{(i)}(x) = \Omega \int d^2 r' c(r') \left( \partial_k \log \left| \frac{r-r'}{a_0} \right| \right)_{y=0} \quad (2.18)$$

again $k = x, y$ for mode I, II cracks, respectively.

The total energy $E = E^{(c)} + E^{(i)} + E^{(s)} + E^{(d)}$ is a functional of the crack dislocation density $b^{(c)}(x)$. Differentiating the saddle point equation $\delta E/\delta b^{(c)}(x) = 0$ with respect to $x$, we find

$$\frac{4\pi}{\bar{Y}} \sigma^{(c)} + \int_{-a}^{a} dx' b^{(c)}(x') \left( \frac{1}{x-x'} + V'(x) \right) = 0 \quad (2.19)$$

Eq. (2.19) has the solution

$$b^{(c)}(x) = \int_{-a}^{a} dx' f(x, x'; a) \left( \frac{4\pi}{\bar{Y}} \sigma^{(c)} + V'(x') \right)$$

$$= b^{(c)}_0(x) + b^{(c)}_1(x) \quad (2.20)$$

where

$$f(x, x'; a) = -\frac{1}{\pi a} \left( \frac{a^2 - x'^2}{a^2 - x^2} \right)^{1/2} \frac{1}{x' - x} \quad (2.21)$$

The total energy as a function of the crack length follows with the help of (2.19)

$$E(a) = \frac{\bar{Y}}{8\pi} \int_{-a}^{a} dx \int_{-a}^{a} dx' b^{(c)}(x) b^{(c)}(x') \log \left| \frac{x-x'}{a_0} \right|$$

$$+ 2 \int_{-a}^{a} dx \gamma(x) \quad (2.22)$$

where $b^{(c)}(x)$ given by (2.20).

For vanishing disorder $b^{(c)}(x) \rightarrow b^{(c)}_0(x, a)$ for which we obtain from (2.20), (2.21) and (2.1)

$$b^{(c)}_0(x, a) = \frac{4\sigma^{(c)}}{\bar{Y}} \frac{x}{(a^2 - x^2)^{1/2}} \quad (2.23)$$

which yields in mode I an elliptic crack of maximal height $2\sigma^{(c)}a/\bar{Y}$.

As follows from (2.20) the total energy (2.22) can be divided into contributions $E_n$ which are proportional to $(\bar{\sigma}_{ij})^{2-n}$ with $n = 0, 1, 2$, respectively. Here $E_1$ and $E_2$ depend on the disorder. The disorder-independent contributions to the energy are given by the Griffith expression

$$E_0(a) = 4\sigma_0 a - \frac{\pi a^2 \sigma_0^2}{\bar{Y}} = 4\sigma_0 a \left( 1 - \frac{a}{2a_c} \right) \quad (2.24)$$
which shows a maximum at \( a = a_c = 2\tilde{\gamma}_0 \tilde{Y} / (\pi \bar{\sigma}^2 c) \) corresponding to an energy barrier \( E_0(a_c) = 2\tilde{\gamma}_0 a_c \).

The contributions \( E_1 \) and \( E_2 \) depend on the frozen disorder and can be characterized by their mean value and variance. \( E_1 \) can be rewritten using partial integration and (A1) and (A2) as

\[
E_1 = \frac{\tilde{Y}}{4\pi} \int_{-a}^a \int_{-a}^a dx' dx \frac{b_0^c(x)b_1^c(x') \log \left| \frac{x-x'}{a_0} \right|}{a_0}
= -\frac{\tilde{Y}}{4\pi} \int_{-a}^a dx V(x)b_0^c(x) \tag{2.25}
\]

For impurities we obtain from (2.17) and (2.18)

\[
\left\langle V^{(i)}(x)V^{(i)}(x') \right\rangle = \Omega^2 \bar{e}^{(i)} \log \frac{R}{|x-x'|} \tag{2.26}
\]

where \( R \) is a cut-off of the order of the system size which has to be send to infinity. This gives with (A1), (A2) for the impurity contribution \( E^{(i)}_1 \) to the variance of \( E_1 \)

\[
\left\langle \left[ E^{(i)}_1(a) - E^{(i)}_1(a') \right]^2 \right\rangle = \Delta^{(i)} |a^2 - a'^2| \tag{2.27}
\]

Here \( \Delta^{(i)} = (\pi/2) \bar{e}^{(i)}(\Omega \bar{\sigma} c)^2 = \bar{e}^{(i)} \Omega^2 \tilde{Y} / a_c \).

For frozen dislocations we get from (2.13) and (2.14)

\[
\left\langle V^{(d)}(x)V^{(d)}(x') \right\rangle = \frac{6}{\pi} c^{(d)} b^{(d)} \left( \frac{\pi R^2}{3} - (x-x')^2 \right) \tag{2.28}
\]

which gives with (2.23) for the dislocation contribution \( E^{(d)}_1 \) to the variance of \( E_1 \)

\[
\left\langle \left[ E^{(d)}_1(a) - E^{(d)}_1(a') \right]^2 \right\rangle = \Delta^{(d)} (|a^2 - a'^2|) \tag{2.29}
\]

with \( \Delta^{(d)} = (3/\pi)c^{(d)}(\bar{b}^{(d)} \bar{\sigma}^{(c)})^2 = 6c^{(d)} b^{(d)} \tilde{Y}^2 / (\pi^2 a_c) \). It is easy to see from (2.11) and the condition that the first crack is closed, i.e. \( \int_{-a}^a dx b^{(c)}(x) = 0 \), that there dependence on \( R \) vanishes in (2.27), (2.28). These equations are clearly valid only for \(|a-a'| \) larger than \( a_0 \). For \(|a-a'| \) smaller than the mean distance between the impurities or dislocations, respectively, the statistics of \( E_1(a) \) is no longer Gaussian, but Eqs. (2.27), (2.29) still give the correct order of magnitude of the fluctuations of \( E_1(a) \).

The mean values as well as the fluctuations of \( E_2 \) are proportional to \( \bar{c}^{(i)} \) and \( \bar{c}^{(d)} \) and hence small if the disorder is weak as we will assume in the following. Then the average energy of the crack is given by the Griffith expression \( E_0(a) \). The energy barrier \( E_0(a_c) = 2\tilde{\gamma}_0 a_c \) is typically large and cannot be overcome by thermal fluctuations. Indeed, for crystalline solids with \( \tilde{\gamma}_0 \lesssim \tilde{Y} a_0 \) one finds

\[
E_0(a_c) \lesssim k_B T_m (\bar{Y} / \bar{\sigma})^2, \quad T_m = \tilde{\gamma}_0 a_0 / k_B \tag{2.30}
\]

where \( T_m \) is a characteristic temperature comparable to, but typically bigger than, the solid’s melting temperature \( \tilde{Y} \). For relatively large strain \( \bar{Y} / \bar{\sigma} \) is of the order 10 such that the nucleation rate for a supercritical crack is of the order \( \omega_0 \exp(-100 T_m / T) \). \( \omega_0 \) is a microscopic attempt frequency of the order \( 10^{18} \) s\(^{-1} \). In the further discussion we will therefore mostly neglect thermal fluctuations.

Let us denote the probability that a crack of length \( a \) has a negative energy by \( W_{E<0}(a) \). The smallest crack one can think of has a length of the order \( a_0 \). Thus a crack can only appear if this smallest crack has a negative energy, \( E(a_0) < 0 \). This occurs with the probability \( W_{E<0}(a_0) \). On this smallest scale even thermal fluctuations may help to create a crack as we will see below.

The crack can only then propagate further, if for a given disorder configuration the force on the crack tip \( f(a) = -\partial E / \partial a \) is positive for all \( a \geq a_0 \) (we neglect effects of inertia).

Because of its mathematical simplicity we consider here first a necessary condition to be fulfilled which is \( E(a) < 0 \) for all \( a \). Clearly, if \( E(a) > 0 \) an (essentially) macroscopic energy barrier exist and the crack cannot propagate. The probability \( W_{E<0}(a) \) is given by

\[
W_{E<0}(a) = \frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{-\phi(a)} dx e^{-x^2/2} \tag{2.31}
\]

with

\[
\phi(a) = \frac{4\tilde{\gamma}_0 a[1 - a/(2a_c)]}{(\Delta^{(i)} a + \Delta^{(d)} a^2 + \Delta^{(d)} a^4)^{1/2}} \tag{2.32}
\]

Here we used the fact that \( E^{(s)}_1 + E^{(i)}_1 + E^{(d)}_1 \) is Gaussian distributed with variance \( \Delta^{(s)} a + \Delta^{(i)} a^2 + \Delta^{(d)} a^4 \). Note that this expression for the variance makes sense only for \( a \) much larger than \( a_0 \). Among the cracks of various lengths \( a \) there is one length \( \bar{a}_c \) for which the probability \( W_{E<0} \) is minimal. This minimum appears at the maximum of \( \phi(a) \). One finds \( \bar{a}_c \) given by the solution of

\[
E^{(s)}_1(\bar{a}_c) + E^{(i)}_1(\bar{a}_c) + E^{(d)}_1(\bar{a}_c) = 0 \tag{2.33}
\]
\[ 0 = (1 - \frac{3\alpha}{2a_c})\Delta_{(s)} - \frac{a^2}{a_c}\Delta_{(i)} - 2a^3\Delta_{(id)} \]  
\hfill (2.33)

It is tempting to consider \( W_{E<0}(\tilde{a}_c) \equiv \bar{W}_{E<0} \) (compare (2.31), (2.33)) as the probability for the occurrence of the crack. This conclusion will be elucidated further below.

It is instructive to consider the three different sources of disorder separately:

(i) If only the surface energy is random, then \( \phi_{(s)}(a) \) vanishes at \( a = 0 \) and \( a = 2a_c \) and has a maximum at \( \tilde{a}_c = (2/3)a_c \). For weak disorder \( \phi^{(s)}(\tilde{a}_c) \equiv \tilde{\phi}^{(s)} \) is large and since for \( \phi(a) \gg 1 \), \( W_{E<0}(a) \approx -\exp(-\phi^2/2)/(\sqrt{2\pi}\phi) \) we get

\[ \bar{W}_{E<0}^{(s)} \approx \frac{1}{\sqrt{2\pi}\tilde{\phi}^{(s)}(\tilde{a}_c)} \exp\left( -\frac{16\gamma_0^2 a_c}{27\tilde{\phi}^{(s)}(\tilde{a}_c)a_0} \right) \]  
\hfill (2.34)

(ii) In the case of randomly distributed impurities \( \phi_{(i)}(a) \) has a maximum for vanishing \( a \). For weak disorder \( a_c \gg a_0 \) we get

\[ \bar{W}_{E<0}^{(i)} \approx \frac{1}{\sqrt{2\pi}\tilde{\phi}^{(i)}(\tilde{a}_c)} \exp\left( \frac{-8\gamma_0^2 a_c}{11\tilde{\phi}^{(i)}(\tilde{a}_c)a_0} \right) \]  
\hfill (2.35)

(iii) Finally, for frozen dislocations \( \phi_{(id)}(a) = 4\gamma_0(1/a - 1/(2a_c))/\sqrt{\Delta_{(id)}} \). Hence \( \phi_{(id)}(a) \) takes its maximum again at \( a \to 0 \) where \( \phi(a) \) diverges. In this case \( a \) has to be replaced by the minimal lengthscale \( a_0 \). This gives

\[ \bar{W}_{E<0}^{(id)} \approx \frac{1}{\sqrt{2\pi}\tilde{\phi}^{(id)}(\tilde{a}_c)} \exp\left( -\frac{4\pi^2\gamma_0^2 a_c}{3\tilde{\phi}^{(id)}(\tilde{a}_c)a_0^2} \right) \]  
\hfill (2.36)

It may be worthwhile to mention, that thermal fluctuations could be at least partially incorporated into the present treatment by relaxing the condition \( E(a) < 0 \) to \( E(a) < k_B T \log(\omega_0 t) \) since barriers of order \( E \) are overcome on time scales of the order \( t \approx \omega_0 \exp(E/T) \). This replacement changes the numerator of \( \phi(a) \) in eq. (2.32) from \( 4\gamma_0(1/a - (1/2a_c)) \) to \( 4\gamma_0(1/a - (1/a_c)) - k_B T \log(\omega_0 t) \). Since barriers on scales \( a \lesssim a_T \) with

\[ a_T \approx a_0 \frac{T}{4T_m} \log(\omega_0 t) \]  
\hfill (2.37)

disappear, \( a_0 \) has to be replaced by \( \max(a_0, a_T) \) in the expressions for \( W_{E<0} \) (we still assume \( a_c \gg a_T \)). This replacement effects essentially only the result for the frozen dislocations, (2.36). We emphasize however, that in general the three types of disorder will work in parallel.

Next we consider the probability to fulfill the sufficient condition \( \partial E/\partial a < 0 \) for \( 0 < a < \infty \), i.e. that the force \( f(a) \) acting on the crack tip is always positive and the crack can propagate forever. We decompose \( f \) into a deterministic and a stochastic contribution:

\[ f(a) = -\frac{\partial E(a)}{\partial a} = f_0(a) + f_1(a) \]  
\hfill (2.38)

\[ f_0(a) = 4\gamma_0 \left( 1 - \frac{a}{a_c} \right) \]  
\hfill (2.39)

\[ f_1(a) = f_1^{(s)}(a) + f_1^{(i)}(a) + f_1^{(id)}(a) \]  
\hfill (2.40)

Since \( f_1(a) = -\partial E/\partial a \) is Gaussian distributed, the joint probability distribution of \( f_1(a) \) for \( 0 < a < \infty \) is also Gaussian but in general non-local. Its form can be reconstructed from the second moments

\[ \langle f_1^{(s)}(a) f_1^{(s)}(a') \rangle = \Delta_{(s)} \delta_{aa}(a - a') \]
\[ \langle f_1^{(i)}(a) f_1^{(i)}(a') \rangle = 2a \Delta_{(i)} \delta_{aa}(a - a') \]
\[ \langle f_1^{(id)}(a) f_1^{(id)}(a') \rangle = 4a a' \Delta_{(id)} \]

In the case of random surface tension and randomly distributed impurities correlations are local and the joint probability distribution of the \( f \)'s factorizes. In this case the probability that the force \( f(a) \) on the tip of a crack of length \( a \) is positive is given by

\[ W_{f>0}(a) = \int_{-\infty}^{-f_0(a)/(f_1^2(a))^{1/2}} dx \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \]  
\hfill (2.41)

The total probability \( \bar{W}_{f>0} \) for \( f(a) > 0 \) for \( 0 < a < \infty \) is given by the product of all \( W_{f>0}(a) \) factors. Here we take into account, that the forces are correlated over a distance of order \( a_0 \) such that we can decompose the crack in pieces of the order \( a_0 \). It is more convenient to consider the logarithm of \( W_{f>0} \)

\[ \log \bar{W}_{f>0} = \sum_a \log W_{f>0}(a) \]  
\hfill (2.42)

The sum over \( a = n a_0 \) is here meant over integer numbers \( n \). Since for \( a \gg a_c \) the integrals in (2.41) are essentially equal to unity, it is sufficient to restrict the summation in (2.42) to the region \( 0 < a \lesssim a_c \). Moreover, the sum is dominated by the smallest values of \( W_{f>0}(a) \) for which we can replace the Gaussian integral (2.41) by the approximate expression used above. This gives

\[ \log \bar{W}_{f>0} \approx -\int_0^{a_c} da \frac{f_1^2(a)}{2a_0} \left( \frac{f_1^2(a)}{f_1^2(a)} + \log \left( \frac{f_1^2(a)}{2\pi f_0^2(a)} \right) \right) \]  
\hfill (2.43)

In the case of a random surface tension only we obtain

\[ \log \bar{W}_{f>0}^{(s)} \approx -a_c \frac{f_1^2(a)}{a_0(3a_0^2) \log \delta_{s}} \]  
\hfill (2.44)

Thus \( \bar{W}_{f>0}^{(s)} \) is essentially of the same form as \( W_{E<0} \) apart from a replacement of the numerical factor in the exponent \( (\frac{a_c}{a_0} \) is replaced by \( \frac{a}{a_0} \). A similar calculation for the case of random impurities gives
\[
\log \tilde{W}_{\ell>0}^{(i)} \approx -\frac{a_c}{a_0} \left( \frac{4\gamma_0 a_c}{c(i)\Omega^2 \gamma} (\log \frac{a_c}{a_0} - \frac{2}{3}) + 1 + \frac{1}{2} \log \frac{c(i)\Omega^2 \gamma}{32\pi a_0 \gamma_0} \right) \tag{2.45}
\]

which is again the same result for the exponent as for \(\tilde{W}_{E<0}\), apart from the logarithmic factor which replaces \(2\).

The third case of randomly distributed dislocations is more involved. Here we should indeed take into account the long range correlations of the forces \(f_3^{(fd)}\). This requires a more detailed mathematical investigation which we leave for further studies. However our experience with the two other cases makes it tempting to assume that \(\tilde{W}_{E<0}^{(fd)}\) gives essentially the correct expression of the probability for the occurrence of a crack.

### III. DISCUSSION AND CONCLUSIONS

It is interesting to remark that in all cases considered above \(\tilde{W}\) can be written in the form of an Arrhenius law for thermal nucleation with an effective temperature determined by the strength of the disorder:

\[
\tilde{W} \approx \tilde{W}_0 \exp \left( -\frac{\gamma_0 a_c}{k_B T_{\text{eff}}} \right) \tag{3.1}
\]

Here

\[
T_{\text{eff}}^{(s)} = \frac{6\gamma \gamma_0 a_0}{k_B \gamma_0} \tag{3.2}
\]

for random surface tension,

\[
T_{\text{eff}}^{(i)} = \frac{\Omega^2 c(i) \gamma}{4k_B} \tag{3.3}
\]

for randomly distributed impurities and

\[
T_{\text{eff}}^{(fd)} \approx \frac{3.4 h^3 b_{(fd)}^2 c_{(fd)} Y}{2\pi^2 k_B} \tag{3.4}
\]

for frozen dislocations. The present calculation is not accurate enough to determine the pre-exponential term \(\tilde{W}_0\), which we assume to be of the order one. Relation (3.1) can be given a very simple meaning: the probability that a crack of minimal length \(a_0\) occurs is given by \(\exp(-2\gamma_0 a_0 / (k_B T_{\text{eff}}))\). Here \(2\gamma_0 a_0\) denotes the energy of such a crack. The total probability is then the \((a_c/a_0)\)-th power of this elementary probability.

Below we want to estimate \(\tilde{W}\) for two different materials. We have to keep in mind, that our calculation was strictly two-dimensional. To compare the results with real experiments on thin plates we have to consider their dependence on the width \(h\) of the plate. A necessary condition for the application of the two-dimensional theory is, that the critical crack length \(a_c\) is much larger than \(h\). In the following we will assume, that this condition is always fulfilled.

Since \(\gamma_0 = \gamma_0 h, \tilde{Y} = Y h\) and \(\sigma_{(e)} = \sigma_{(e)} h\) are proportional to \(h\), the nucleation energy \(2\gamma_0 a_c = 4\gamma_0^2 Y / (\pi \sigma_{(e)}^2)\) is also proportional to \(h\).

Estimating the \(h\)-dependence of \(T_{\text{eff}}\) we have to make sure, that all relevant length scales in the \(xy\)-plane (like \(a_c\)) are much larger than the \(h\). For \(T_{\text{eff}}^{(i)}\) and \(T_{\text{eff}}^{(fd)}\), which were determined by the small scale cut-off \(a_0\), this leads to the severe restriction \(h < a_0\). We note however, that this cut-off will be in general larger than the lattice spacing. It is therefore appropriate to use in these two cases \(h\) as the small scale cut-off. The behavior on even smaller scales is described by three-dimensional physics which we will discuss at the end of the section.

With \(\delta \gamma = \delta \gamma h a_0\), where \(\delta \gamma\) denotes the fluctuation of the three-dimensional surface tension \(\gamma_0\). \(T_{\text{eff}}^{(s)}\) does not depend on \(h\).

Next, \(c(i) = c(i) h\) is a two-dimensional density proportional to \(h\) (the three-dimensional density \(c(i)\) is independent of \(h\)). \(\Omega\) as a two-dimensional cross section of the impurity should be replaced by \(\Omega^{1/2}/h\), i.e. we assume essentially spherical impurities. Thus

\[
T_{\text{eff}}^{(i)} \to \frac{\Omega^2 c(i) \gamma}{4k_B} \left( 1 - \frac{h}{2a_c} \right) \tag{3.5}
\]

is essentially independent of \(h(<< a_c)\).

For randomly distributed dislocations \(c_{(fd)}\) (as a line density) and \(b_{(fd)}\) are unchanged and hence

\[
T_{\text{eff}}^{(fd)} \to \frac{h^3 b_{(fd)}^2 c_{(fd)} Y}{4\pi^2 k_B} \tag{3.6}
\]

In (3.5) and (3.6) we replaced \(a_0\) by \(h\).

On scales smaller than \(h\) the cracks are three-dimensional. The results of this paper can however easily be extended to penny cracks in \(d\) dimensions. The Griffith energy then takes the form

\[
E_0(a) = \gamma_0 a^{d-1} - Y^{-1} \sigma_{(e)}^2 a^d \tag{3.7}
\]

where we neglect here and in the following all numerical factors which in general depend on the precise crack geometry. \(E_0(a)\) has a maximum at \(a_c \sim \gamma_0 Y / \sigma_{(e)}^2\), i.e. \(a_c\) is essentially unchanged.

If the disorder is taken into account, additional contributions to the energy appear. A randomness in the surface tension leads to a fluctuation of the order \(\langle (E_1^{(fd)}(a))^2 \rangle \approx \delta \gamma^2 (a_0 a_0)^{d-1}\). Following the discussion below Eq. (2.31), the minimum of \(W_{E<0}(a)\) follows again for \(a_c \sim a_c\). Rewriting the minimum \(W_{E<0,d}\) of \(W_{E<0}(a)\) in the form

\[
\tilde{W}_d \approx \tilde{W}_{d,0} \exp \left( -\frac{\gamma_0 a_c^{d-1}}{k_B T_{\text{eff}}} \right) \tag{3.8}
\]
we obtain for the effective disorder temperature $T_{\text{eff}}^{(a)}(d) \approx \delta \gamma/a^{(d-1)}/(k_B \gamma_0)$.

Randomly distributed impurities create an additional contribution $\sigma_{(i)}$ to the stress where

$$|\sigma_{(i)}| = Y \langle \Omega^d c_{(i)} / a^d \rangle^{1/2} \tag{3.9}$$

This expression can be understood as follows: An isolated impurity creates in a volume $a^d$ an average stress of the order $Y \Omega^d / a^d$. With $c_{(i)} a^d$ the number of impurities in this volume (the average stress created by the impurities is assumed to be already incorporated into $E_0(a)$) the fluctuations of the stress created by the impurities is given by (3.9). Thus we obtain

$$\langle (E_{(i)}^d(a))^2 \rangle \approx \Omega^d Y \gamma_0 a^d c_{(i)}/a_c.$$  

The minimal probability follows for $d > 2$ (contrary to the 2-dimensional case) for $a_c \sim a_c$. The disorder temperature is now of the order $T_{\text{eff}}^{(i)}(d) = \Omega^d c_{(i)} Y / (4k_B)$. Clearly, for $a_c \approx h$, $T_{\text{eff}}^{(i)}(d)$ agrees with the result for the plate, as it should be.

Similarly, randomly distributed dislocation lines will give a fluctuation contribution $\sigma_{(d)}$ to the stress of the order

$$|\sigma_{(d)}| = Y b_{(d)} h_{(d)}^{1/2} \tag{3.10}$$

Indeed, a single dislocation (line) creates a stress of the order $Y b_{(d)}/a$. With $c_{(d)} a^2$ for the number of dislocation lines in the volume $a^d$ which are assumed to have random orientations one obtains (3.10). The corresponding fluctuation of the energy due to dislocations is therefore given

$$\langle (E_{(d)}^d(a))^2 \rangle \approx a^2 \gamma_0 h_{(d)}^2 c_{(d)}/a_c.$$  

The minimal probability follows as for $d = 2$ from small scales $a \approx a_0$, which results in

$$T_{\text{eff}}^{(d)}(d) \approx a_0^2 a_c^{d-2} Y h_{(d)}^2 c_{(d)}/a_c.$$  

(3.11)

Clearly, also the case of multiple disorder can be considered as it was done in (2.3).

Next we have to compare the different probabilities in order to decide, which process dominates the formation of cracks in thin plates. For systems with a random surface tension, and under the condition $a_c \gg h$, the minimal probability $W$ arises both in two and in three dimensions from cracks of length $a_c \sim a_c$. In this case the probability for crack formation is given by (3.1).

For randomly distributed impurities $a_c(d = 2) \approx a_0 \approx h$ in two dimensions and $a_c(d = 3) \approx a_0$ in three dimensions. Thus crack formation is dominated here by two-dimensional cracks of minimal length $a_0 \approx h$. The corresponding probability is now given by (3.1), (3.2).

Finally, for randomly distributed dislocation, crack formation in both two and three dimensions is controlled by the formation of small cracks of size $a_c \approx a_0$. Comparing the corresponding probabilities in two and in three dimension (for $d = 2$ we have to use $a_0 \approx h$) we find that crack formation is dominated by penny cracks of size $a_0$. Its probability is given by (3.8) and (3.11) for $d = 3$.

### Table I. Estimates of the effective temperatures $T_{\text{eff}}$ for Glass and SiC using $\delta \gamma/\gamma_0 = 0.1$, $a_0 = 5 \times 10^{-10} \text{m}$. For frozen dislocations

| Method | Glass | SiC |
|--------|-------|-----|
| Weak disorder: $\delta \gamma/\gamma_0 = 0.1$, $a_0 = 5 \times 10^{-10} \text{m}$ | $T_{\text{eff}}^{(i)}(2) = 1087$ | $T_{\text{eff}}^{(i)}(2) = 4348$ |
| $a_0 = 5 \times 10^{-10} \text{m}$ | $A^{(i)} = 5.9\times10^{-10}$ | $A^{(i)} = 1.36\times10^{-10}$ |
| Strong disorder: $\delta \gamma/\gamma_0 = 0.3$, $a_0 = 10^{-9} \text{m}$ | $T_{\text{eff}}^{(i)}(2) = 3.9\times10^{-10}$ | $T_{\text{eff}}^{(i)}(2) = 1.57\times10^{-11}$ |
| $a_0 = 10^{-9} \text{m}$ | $A^{(i)} = 1.65\times10^{-23}$ | $A^{(i)} = 3.77\times10^{-24}$ |

The corresponding factors $A^{(i)}$ and $A^{(d)}$ are also given in Table I. The material constants are taken from Ref.

| Method | Glass | SiC |
|--------|-------|-----|
| Random surface energy | $\gamma_0 = 70$ | $\gamma_0 = 400$ |
| $\Omega_{\text{eff}}$ | $\text{random frozen dislocations}$ | $\text{random impurities}$ |
| Weak disorder: $\Omega = 2.5 \times 10^{-10} \text{m}^2, \gamma_0 = 8 \times 10^{-2} \text{m}^3$ | $T_{\text{eff}}^{(i)}(2) = 158.5$ | $T_{\text{eff}}^{(i)}(2) = 905.8$ |
| $a_0 = 5 \times 10^{-10} \text{m}$ | $A^{(i)} = 4.67\times10^{-11}$ | $A^{(i)} = 6.5\times10^{-12}$ |
| Strong disorder: $\Omega = 10^{-10} \text{m}^2, \gamma_0 = 10^{-11} \text{m}^3$ | $T_{\text{eff}}^{(i)}(2) = 1.26\times10^{-5}$ | $T_{\text{eff}}^{(i)}(2) = 7.24\times10^{-5}$ |
| $a_0 = 10^{-10} \text{m}$ | $A^{(i)} = 5.9\times10^{-12}$ | $A^{(i)} = 4.15\times10^{-9}$ |

In a macroscopic sample of linear size $L$ regions of distance greater than $a_c$ can be considered to be essentially independent. The total probability for crack formation is given by $W_d(L/a_c)^d$ where $d = 2$ for surface and impurity disorder and $d = 3$ for frozen dislocations.

Next we consider the probability for crack formation in two different materials, one is amorphous (glass) and one crystalline (SiC). The corresponding parameters are summarized in Table I. In the case of random surface tension and randomly distributed impurities we express the probability for crack formation as a function of $h/\sigma_{(c)}^2$ since the probability can be written in the form

$$\log(W/\bar{W}_0) = -\frac{4 \delta \gamma_0^2 Y / k_B T_{\text{eff}}}{\sigma_{(c)}^2} \equiv -A \frac{h}{\sigma_{(c)}^2} \tag{3.12}$$

For frozen dislocations we have

$$\log(W/\bar{W}_0) \approx -\frac{\gamma_0^2}{a_0^2 \delta \gamma_0^2 c_{(d)} / \sigma_{(c)}^2} \equiv -A^{(d)} \frac{a_0}{\sigma_{(c)}^2} \tag{3.13}$$

The corresponding values for $A^{(i)}$, $A^{(d)}$ and $A^{(id)}$ are also given in Table I.

A number of comments are in order:
(i) The results so far are based on the assumption of short range correlations of the disorder (surface energy, impurities, frozen dislocations). Experimentally this may not be the most important situation. Long range correlations of the disorder described by a power law decay on the r.h.s. of (2.9), (2.14) and (2.16) would in general lead to an increase of the probability for the occurrence of a crack.

(ii) In the case of impurities and frozen dislocations the probability $\tilde{W}$ is dominated by the energetics on small length scales. Since the stress on the crack tip diverges in the linear elasticity theory used throughout the paper, non-linear effects may be of particular importance for small cracks. This could diminish the numerical coefficients in the exponent of $\tilde{W}$.

(iii) Defects were assumed everywhere to be frozen. In real fatigue experiments often alternating stress is applied which leads to an accumulation of dislocations close to the crack tip, which makes crack propagation easier. This mechanism could also help in a situation, were due to disorder fluctuations, the stress is considerably higher than in the average (or the surface energy is lower).

(iv) An interesting question is the relation of our results to crack propagation in quasicrystals which take an intermediate position between periodic and random media.

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APPENDIX A: RELEVANT INTEGRALS

\begin{equation}
I_1 = \int_{-1}^{1} dx' \frac{1}{(1-x'^2)^{1/2}} \frac{1}{x'-x} = -\pi x \quad \text{for } |x| < 1
\end{equation}

\begin{equation}
I_2 = \int_{-1}^{1} dx' \frac{1}{(1-x'^2)^{-1/2}} \frac{x'}{x'-x} = \begin{cases} 
\pi & \text{for } |x| < 1 \\
\pi - \pi|x|(x^2 - 1)^{-1/2} & \text{for } |x| > 1
\end{cases}
\end{equation}