Global approximants with renormalization scale invariance in pQCD

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Truncated perturbative series (TPS’s) of any observable have the unphysical dependence on the choice of the renormalization scale (RScl). The diagonal Padé approximants (dPA’s) to any TPS of an observable possess the favorable property of being invariant in the large-\(\bar{\alpha}_0\) limit. This means that they are invariant under the change of the RScl \(\mu^2\) when the “running” coupling parameter \(\alpha_s(\mu^2)\) evolves according to the one-loop renormalization group equation. We present a method which generalizes this result – the resulting new approximants are fully RScl-invariant in the perturbative QCD (pQCD). Further, we present some numerical examples. Both the dPA’s and the new approximants are global, i.e., their structure goes beyond the usual (polynomial) TPS form and thus they could reveal some non-perturbative effects.

The contribution is based partly on [1]. It contains additionally some numerical examples. Since the construction of our approximants is related with the diagonal Padé approximants (dPA’s), we first explain what PA’s are.

1. What is Padé approximant?

Suppose we have a physical quantity \(F(z)\) which depends on the parameter \(z\). In the expansion of \(F(z)\) in powers of \(z\)

\[
F(z) = f_0 + f_1 z + \cdots f_n z^n + \cdots ,
\]

(1)

the \(f_j\)’s are in principle calculable. Suppose that \(f_0, \ldots, f_{L+M}\) have been calculated, i.e., the truncated perturbation series (TPS) \(F_{[L+M]}(z)\) is known. Then, the PA \([L/M] F(z)\) of order \(L/M\) to \(F(z)\) is defined as the ratio of two polynomials, of degree \(L\) (nominator) and \(M\) (denominator), such that, when expanded back in powers of \(z\), it reproduces the first \(L+M+1\) terms of (1). In general, this condition determines uniquely the PA. It is the minimal condition that any approximant to the TPS \(F_{[L+M]}(z)\) has to fulfill. PA has, in addition, the favorable property that it goes beyond the analytic form, showing structures (pole singularities in the complex \(z\) plane) not explicitly contained in series (1). We call such approximants global, since they can give us clues to some nonperturbative properties of \(F(z)\) at large \(|z|\). The diagonal PA’s (dPA’s) are those with \(L=M\).

2. Construction of new approximants

A generic observable \(S\) in pQCD is

\[
S \equiv a(q^2)f(q^2) = a(q^2)[1 + \sum_{r} r(q^2)a^r(q^2)], \tag{2}
\]

where \(q^2\) is a chosen renormalization scale (RScl), \(a(q^2) \equiv \alpha_s(q^2)/\pi\). \(S\) is RScl-independent. However, available is only a TPS \(S_{[n]}\)

\[
S_{[n]}(q^2) \equiv a(q^2)f^{[n]}(q^2) = a(q^2)[1 + \sum_{r} r(q^2)a^r(q^2)]. \tag{3}
\]

It has unphysical RScl-dependence (truncation).

How to extract as much information as possible from the available TPS \(S_{[n]}\)? Firstly, any approximant should fulfill the minimal condition (cf. previous Sec.). Secondly, the full observable \(S\) contains, in general, non-perturbative effects not explicitly manifested in power series (2) – this leads us to consider global approximants (cf. previous Sec.). Thirdly, \(S\) is RScl-independent, so it is natural to expect that RScl-invariant approximants bring us closer to \(S\). So, the question here is: How to construct global approximants which are based on the TPS \(S_{[n]}(q^2)\) and are RScl-independent?

It turns out that a partial answer to this question is the diagonal Padé approximant (dPA) \([M/M] S(a)\). To see this, we recall that the
evolution of $a(p^2)$ with the change of RScl $p^2$ is governed in pQCD by the RGE
\begin{equation}
\frac{da}{d \ln(-p^2)} = -\beta_0 a^2(1 + c_1 a + c_2 a^2 + \cdots). \tag{4}
\end{equation}
In the large-$\beta_0$ limit (i.e., $c_1 = c_2 = \cdots = 0$)
\begin{equation}
a^{(\text{I})}(p^2) = a(q^2) \left[1 + \ln(p^2/q^2) \beta_0 a(q^2)\right]^{-1}. \tag{5}
\end{equation}
Thus, the RScl change $q^2 \rightarrow p^2$ results in the change of
\begin{equation}
z \equiv a(q^2): \quad z \rightarrow z/(1 + bz). \tag{6}
\end{equation}

The dPA approach has to be extended. We rearrange power series (2) for where higher orders in (7) can be calculated by the large-$\beta_0$ decomposition into a sum of simple fractions $a(q^2) \equiv \sum_j a_j(q^2)$. The question was raised first by the authors of [3]. The answer: it is.

We now denote $p_j^2 = q^2 \exp[\tilde{u}_j(q^2)]$, i.e. $\tilde{u}_j(q^2) = \ln(p_j^2/q^2)$, (11) and therefore, in view of (3), rewrite (10) as
\begin{equation}
[M/M]_{a^2} = \sum_{j=1}^M \tilde{\alpha}_j a^{(11)}(p_j^2). \tag{12}
\end{equation}
Replace here $a^{(11)}(p_j^2) \rightarrow a(p_j^2)$, where $a(p_j^2)$ is evolved from $a(q^2)$ via the full RGE (8)
\begin{equation}
A_S^{[M/M]} = \sum_{j=1}^M \tilde{\alpha}_j a(p_j^2). \tag{13}
\end{equation}
This approximation satisfies all the conditions that we set forth at the outset: a) when expanded back in powers of $a(q^2)$ ($q^2$ is the original RScl), it reproduces all the terms of the TPS $S_{[2M-1]}(q^2)$ (8); b) it is global since it is a modification of the dPA approach (the latter is global); c) it is fully RScl-invariant in the pQCD-sense, i.e., independent of the initial choice of the RScl $q^2$, where $a(q^2)$ evolves according to the full available perturbative RGE (8). Explicit proof is given in [3] (first entry). In fact, coefficients $\tilde{\alpha}_j$ and squared momenta $p_j^2$ are all RScl-invariant.

Approximants (13) can be applied directly only to TPS’s $S_{[n]}$ (3) with an odd number $n = 2M - 1$. In pQCD, $S_{[1]}$ are available for many observables, $S_{[2]}$ for few, $S_{[3]}$ for none. Thus, (13) is applicable only to TPS’s $S_{[1]}$. In this case $M = 1$, (13) reduces to the effective charge (cf. [3]). To apply (13) to $S_{[2]}(q^2)$, a modification is needed. Introduce a new observable $\tilde{S} \equiv S + S$
\begin{equation}
\tilde{S} = (S)^2 = a(q^2) F(q^2) = a(q^2)[0 + 1a(q^2) + R_3 a^3(q^2) + \cdots], \tag{14}
\end{equation}
where $R_1 = 1$, $R_2 = 2r_1$, $R_3 = r_2 + 2r_3$, etc. Having $S_{[2]}(q^2)$ (3), we have $\tilde{S}_{[2]}(q^2)$ (i.e., up to and including the $R_3$-term). We apply (13) to $\tilde{S}_{[3]}$ $(M = 2)$. Although the leading order term in (13) is $0 \cdot a(q^2)$, the algorithm survives, results in
\begin{equation}
\tilde{A}_{\tilde{S}}^{[2/2]} = O(a^5), \quad S = \sqrt{\tilde{A}_{\tilde{S}}^{[2/2]}} + O(a^4). \tag{15}
\end{equation}
\begin{equation}
\sqrt{\tilde{A}_{\tilde{S}}^{[2/2]}} = \sqrt{\tilde{\alpha}_1 [a(p_1^2) - a(p_j^2)]}, \tag{16}
\end{equation}
where $\tilde{\alpha}_1$ and $p_j^2$ are determined by $r_1, r_2$ and $\beta_0, c_1, c_2$ (cf. [3], last entry). Approximant (13)
is fully RScl-invariant. The scales $p_j^2$ may be complex in some cases, but the result is real.

3. Specific numerical examples

3.1. A case of a Euclidean observable

Consider the Bjorken polarized sum rule

$$
\int_0^1 dx \left[ g_1^{(p)}(x, Q^2) - g_1^{(n)}(x, Q^2) \right] = \frac{[g_A/(3|g_V|)][1 - S(-Q^2)]}{1 S}, \quad (17)
$$

$p^2 = -Q^2 < 0$ is a momentum transfer. TPS $S_{[2]}(-Q^2)$ is now known: $r_1 = 3.584 \times (3); r_2 = 20.212 \times (4)$. These $r_j$'s are in the MS scheme, $n_f = 3$, and the chosen RScl $Q_{RScl}^2 = -Q^2$. We take $\sqrt{Q^2} = 1.76$ GeV [$a(-Q^2) = 0.083$].

First we note that any approximant can be used also to predict the next coefficient $r_3$, by reexpanding the approximant back in powers of $a$. We give in Table 1 results of various approximants to $S$ of (5): PA's, approximant (18), and the effective charge method (ECH (5), we set $c_3^{ECH} = 0$). In brackets, results are given for another scheme ($c_2 = 3, c_3 = 0$). We used the known $c_3$ parameter of the MS scheme (6) wherever possible, i.e., MS is characterized by $c_2 = 4.471$ and $c_3 = 20.99$. The results of method (18) differ somewhat from those of other methods. Further, values of $S(-Q^2)$ (cf. second column) predicted by global methods show up some scheme ($c_2$-)dependence ($\sim 1\%$). The RScl-dependence is another source of ambiguity in the results of the PA's. For example, if changing RScl $Q \to 2Q$, the result of the [1/2] PA changes by 3.5%, that of [2/2] 1/2 by 2%, and the original TPS $S_{[2]}$ by 11%. There is no RScl-dependence for approximants (19). The result $S(-Q^2)$ of the ECH method is scheme- and RScl-independent, since this method is local (i.e., a specific choice of RScl and scheme).

If changing the scheme more drastically, e.g. $\overrightarrow{MS} \to \hat{t}$ Hooft scheme ($c_2 = 4.471 \to 0; c_3 = 20.99 \to 0$), predictions for $S(-Q^2)$ of the global methods [1/2], [2/2] 1/2 and (18) change by 1.2% (0.1273 $\to 0.1260$), 1.2% (0.1326 $\to 0.1342$) and 4.4% (0.1378 $\to 0.1439$), respectively. Scheme dependence (i.e., $c_2$-dependence) of (18) does present a problem in this case.

3.2. A case of a Minkowskian observable

$$
R(p^2 = s) = \frac{\sigma(e^+e^- \to \gamma^* \to \text{hadr})}{\sigma(e^+e^- \to \mu^+\mu^-)}. \quad (18)
$$

Here, $p^2 = (p_e + p_e)^2 = s > 0$ is Minkowskian. Approximants generally lose predictability when applied to such observables (5). The associated Adler function $D(p^2 = -Q^2)$ ($Q^2 > 0$), RScl- and scheme-invariant, is Euclidean

$$
D(-Q^2) = Q^2 \int_0^\infty ds R(s)(s + Q^2)^{-2}. \quad (19)
$$

We apply approximants to it. The TPS up to NNLO has been calculated for $D(\overrightarrow{MS})$ For $n_f = 3, Q_{RScl}^2 = -Q^2$, in MS, it is

$$
D = \frac{11}{3}[1 + d^{(full)}], \quad d^{(full)} = d + d^{(t, l)}, \quad (20)
$$

$$
R(s) = a(-Q^2)[1 + 1.4902a(-Q^2)] - 0.6812a^2(-Q^2) + \cdots, \quad (21)
$$

$$
d^{(t, l)}(-Q^2) = a^3(-Q^2)((-0.3756) + \cdots. \quad (22)
$$

We take $\sqrt{Q^2} = 34$ GeV, $a(-Q^2) = 0.0452$. The $d^{(t, l)}$ is from light-to-light diagrams, should not be included in the approximants since the resummations cannot “see” separately diagrams of fundamentally different topologies. Reexpanding the approximants to $d(-Q^2)$ (21) in powers of $a(-Q^2)$, we predict $d_3$ in series (22) (note: $d_1 = 1.4902, d_2 = -0.6812$). Then relation (21) allows us to obtain the coefficients $r_j$ of the expansion of $r(s)$ in powers of $a(-Q^2)$, $|R| = (11/3)(1 + r^{(full)}), r^{(full)} = r + r^{(t, l)}, r^{(t, l)} = d^{(t, l)}, (-Q^2 = -s)]$

$$
r(s) = a(-s)[1 + r_1 a(-s) + \cdots], \quad (23)
$$

with $r_1 = d_1 = 1.4902, r_2 = d_2 - \pi^2/3 = -12.767, r_3 = d_3 - 89.190$, etc. Thus, we can predict also the coefficient $r_3$ of $r(s)$. Results are given in Table 2. In brackets are results in the $\hat{t}$ Hooft scheme ($c_j = 0$ for $j \geq 2$). Predictions of the PA methods and the ECH are clustered. Those of our method are slightly, but significantly, detached from them. Scheme ($c_2$-)dependence of the global approximants is weak (cf. last two digits in 2nd and 4th columns). Further, if the RScl is changed $Q_{RScl}^2 = -Q^2 \to -Q^2/4$, the $d(-Q^2)$'s of the PA's change about twice as strong as when the scheme is changed $c_2 = 1.475 \to 0$. Approximant (18) does not change when RScl changes.
### Table 1

Predictions of various approximants for the Bjorken polarized sum rule (with $n_f = 3$) in the $\overline{MS}$ ($c_2 = 4.471$) and the $c_2 = 3$ scheme (in brackets). $\sqrt{Q^2} = 1.76$ GeV; RScI chosen: $q_{RSCI}^2 = -Q^2$; $a(-Q^2) = 0.083$.

| Approx.     | $S(-Q^2)^{TPS}$ | $r_3^{TPS}$ | $S_{||3}(-Q^2; q_{RSCI})^{TPS}$ |
|-------------|------------------|-------------|----------------------------------|
| $S_{[2]}$ (TPS) | 0.1192 (0.1175)  | –           | 0.1192 (0.1175)                  |
| $[\frac{1}{2}]_{S}$ | 0.1273 (0.1261)  | 98.8 (109.4) | 0.1239 (0.1223)                  |
| $\sqrt{\frac{2}{2}]_{S}^{TPS}$ | 0.1326 (0.1322)  | 125.2 (141.1) | 0.1252 (0.1238)                  |
| $\sqrt{A_{S}^{[2/2]}}$ | 0.1378 (0.1370)  | 138.3 (151.4) | 0.1258 (0.1242)                  |
| $S_{[2]}$ (ECH) | 0.1320 (0.1320)  | 129.9 (140.4) | 0.1254 (0.1237)                  |

### Table 2

Predictions of various approximants for the ratio $R(s)$ (with $n_f = 5$) in the $\overline{MS}$ ($c_2 = 1.475$) and the ’t Hooft ($c_2 = 0$) scheme (in brackets). $\sqrt{Q^2} = 34$ GeV; RScI chosen: $q_{RSCI}^2 = -Q^2$; $a(-Q^2) = 0.0452$.

| Approx.     | $d(-Q^2)^{TPS}$ | $d_3^{TPS}$ | $d_{||3}(-Q^2; q_{RSCI})^{TPS}$ | $r_3^{TPS}$ |
|-------------|-----------------|-------------|---------------------------------|-------------|
| $[\frac{1}{2}]_4$ | 0.047996 (0.047974) | $-4.72 (-0.56)$ | 0.047996 (0.047974) | $-93.91 (-89.75)$ |
| $\frac{2}{3}d_{[d]}^{[1/2]}$ | 0.047978 (0.047956) | $-8.48 (-2.24)$ | 0.047981 (0.047967) | $-97.67 (-91.43)$ |
| $\sqrt{A_{d}^{[2/2]}}$ | 0.047931 (0.047939) | $-15.42 (-6.89)$ | 0.047952 (0.047948) | $-104.61 (-96.08)$ |
| $d_{[2]}$ (ECH) | 0.047959 (0.047959) | $-7.65 (-3.50)$ | 0.047984 (0.047962) | $-96.84 (-92.69)$ |

### 4. Summary

For a given $S_{[n]}$ (TPS) of an observable $S$, we can construct an RScI-independent approximant which reproduces that TPS to the given order $\sim a^{-n+1}_\beta$. It is global, i.e., it goes beyond the (polynomial) form of the TPS and could thus give us some clues to the nonperturbative effects – possibly in contrast to the local methods (ECH, PMS). It is, in principle, an improvement over another global approximant – the diagonal Padé approximant (dPA), the latter being RScI-invariant only in the large-$\beta_0$ limit. The question of how to eliminate the second major source of unphysical dependence, the scheme ($c_2$)-dependence, remains open. One possibility would be to choose an “optimal” $c_2$ (open problem). Another would be to extend the method so as to give us, in a global manner, approximants that are simultaneously RScI- and $c_2$-independent (an open problem).  

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