The derived algebra of a stabilizer, families of coadjoint orbits, and sheets

Anton Izosimov*

Abstract

Let $g$ be a finite-dimensional real or complex Lie algebra, and let $\mu \in g^*$. In the first part of the paper, the relation is discussed between the derived algebra of the stabilizer of $\mu$ and the set of coadjoint orbits which have the same dimension as the orbit of $\mu$. In the second part, semisimple Lie algebras are considered, and the relation is discussed between the derived algebra of a centralizer and sheets.

1 Introduction

Let $g$ be a finite-dimensional real or complex Lie algebra. The group $G$ acts on the dual space $g^*$ via the coadjoint action, and $g^*$ is foliated into the orbits of this action. Consider the union of all orbits which have the same codimension $k$. Denote this union by $g^*_k$. Each of the sets $g^*_k$ is a quasi-affine algebraic variety. The study of the varieties $g^*_k$ was initiated by A.Kirillov in connection with the orbit method [6], which relates the unitary dual of $G$ to the set of coadjoint orbits $g^*/G$. The sets $g^*_k/G$ appear in this picture as natural strata of $g^*/G$, therefore it is important to understand the geometry of $g^*_k$ for each $k$.

So, consider a finite-dimensional real or complex Lie algebra $g$. Let $g_\mu = \{x \in g \mid \text{ad}^* x(\mu) = 0\}$ be the stabilizer of an element $\mu \in g^*$ with respect to the coadjoint representation of $g$. In the present note, the following simple geometric fact is proved: any element $\xi \in g^*$ which is tangent to the variety $g^*_\mu$ at a point $\mu \in g^*$ vanishes on the derived algebra of $g_\mu$. As a corollary, the codimension of the set $\{\mu \in g^* \mid \dim [g_\mu, g_\mu] \geq k\}$ is at least $k$, which generalizes the well-known fact that the stabilizer of a generic element $\mu \in g^*$ is Abelian.

In the second part of the note, a semisimple Lie algebra $g$ is considered. In this case, the set $g^*_k$ can be identified with the variety of adjoint orbits of codimension $k$. The irreducible components of this latter variety are called sheets. Let $a \in g$, and let $g^a = \{x \in g \mid [x, a] = 0\}$ be the centralizer of $a$. Then the above-formulated statement becomes the following: the derived algebra of the centralizer of $a$ is orthogonal to any sheet passing through $a$. It was conjectured in the earlier version of the present paper [5] that if $g$ is a classical simple Lie algebra, and there is a unique sheet $S$ passing through $a \in g$, then $[g^a, g^a]$ is exactly the orthogonal complement to $S$. As it has been recently shown by A.Premet and L.Topley [12], this conjecture is true for any algebraically closed ground field of characteristic zero. The second conjecture, stating that if $S_1, \ldots, S_k$ are sheets passing through $a$, then $[g^a, g^a]$ is the orthogonal complement to $\sum T_{a,S_i}$, remains open.

*Moscow State University and Higher School of Economics. E-mail: a.m.izosimov@gmail.com
2 The derived algebra of a stabilizer and families of coadjoint orbits

Families of coadjoint orbits of the same dimension Let \( g \) be a real or complex Lie algebra. Let \( g_\mu = \{ x \in g \mid \text{ad}^*x(\mu) = 0 \} \) be the stabilizer of an element \( \mu \in g^* \) with respect to the coadjoint action. Let \( g^*_k = \{ \mu \in g^* \mid \dim g_\mu = k \} \). It is clear that \( g^*_k \) is a quasi-affine algebraic variety for each \( k \), and \( g^* \) is a disjoint union of all \( g^*_k \). The variety \( g^*_k \) can be also defined as the union of all coadjoint orbits of codimension \( k \).

For any \( \mu \in g^* \), denote \( g^*_\mu = g^*_\dim g_\mu \); in other words, \( g^*_\mu \) is \( g^*_k \) passing through \( \mu \).

Main statement

**Proposition 2.1.** Let \( \gamma \) be a smooth curve in \( g^* \) such that \( \gamma(0) = \mu \) and \( \gamma(t) \in g^*_\mu \) for all \( t \). Then the tangent vector \( \dot{\gamma}(0) \) vanishes on the derived algebra of \( g_\mu \):

\[
\langle \dot{\gamma}(0), [g_\mu, g_\mu] \rangle = 0.
\]

**Proof.** Since \( \dim g_\gamma(t) = \dim g_\mu \) for all \( t \), the bundle of stabilizers is locally trivial over the curve \( \gamma \), and it is possible to choose a basis \( e_1(t), \ldots, e_k(t) \) in \( g_\gamma(t) \) such that \( e_i(t) \) depends smoothly on \( t \). Since \( e_i(t) \in g_\gamma(t) \), the following equality holds:

\[
\langle \gamma(t), [e_i(t), e_j(t)] \rangle = 0.
\]

Differentiating with respect to \( t \) at \( t = 0 \), obtain

\[
\langle \dot{\gamma}(0), [e_i(0), e_j(0)] \rangle + \langle \mu, [\dot{e}_i(0), e_j(0)] \rangle + \langle \mu, [e_i(0), \dot{e}_j(0)] \rangle = 0.
\]

Since \( e_i(t) \) are elements of the stabilizer, the last two terms vanish, and

\[
\langle \dot{\gamma}(0), [e_i(0), e_j(0)] \rangle = 0,
\]

which implies that \( \dot{\gamma}(0) \) vanishes on the derived algebra of \( g_\mu \). q.e.d.

**Remark 2.1.** The proposition remains true if \( \gamma(t) \) is only defined for \( t \geq 0 \) and the right derivative \( \dot{\gamma}(0) \) exists. This may happen if \( g^*_\mu \) has a singularity at \( \mu \).

**Corollary 2.1.** Consider the case when \( g^*_\mu \) is smooth at the point \( \mu \). Then

1. Each element of the tangent space \( T_\mu g^*_\mu \) vanishes on the derived algebra of \( g_\mu \):

\[
\langle T_\mu g^*_\mu, [g_\mu, g_\mu] \rangle = 0.
\]

2. The following inequality is satisfied:

\[
\dim [g_\mu, g_\mu] \leq \text{codim}_\mu g^*_\mu.
\]

3. The equality

\[
\dim [g_\mu, g_\mu] = \text{codim}_\mu g^*_\mu
\]

is satisfied if and only if \( T_\mu g^*_\mu \) is exactly the annihilator of \( [g_\mu, g_\mu] \).
Remark 2.2. Inequality (1) shows that the derived algebra of a stabilizer cannot be too big. It resembles the following inequality for the index of a stabilizer: \( \text{ind} \mathfrak{g}_\mu \geq \text{ind} \mathfrak{g} \) (Vinberg, see [10]).

**Corollary 2.2.** The codimension of the set of elements \( \mu \in \mathfrak{g}^* \) such that

\[
\dim [\mathfrak{g}_\mu, \mathfrak{g}_\mu] \geq k
\]

is at least \( k \).

**Example 2.1.** For regular \( \mu \), obtain a well-known fact: \( \mathfrak{g}_\mu \) is abelian. Corollary 2.2 can be viewed as a natural generalization of this fact. It says that for a “not too singular” \( \mu \), its stabilizer is almost Abelian.

**Example 2.2.** Let \( \mathfrak{g} \) be complex semisimple. Then, for a generic singular element \( \mu \), the dimension of \([\mathfrak{g}_\mu, \mathfrak{g}_\mu]\) equals three. So, by Corollary 2.2, the codimension of the singular set for a complex semisimple Lie algebra is at least three. On the other hand, it is well known that this codimension is exactly three.

**Example 2.3.** Suppose that the set of singular elements in \( \mathfrak{g}^* \) is a hypersurface. Then the stabilizer of a generic singular element is one of the following types:

1. Abelian;
2. \( \text{aff}(1) \oplus \text{Abelian} \), where \( \text{aff}(1) \) is the Lie algebra of affine transformations of the line;
3. \( \mathfrak{h}_{2n+1} \oplus \text{Abelian} \), where \( \mathfrak{h}_{2n+1} \) is the \( 2n + 1 \)-dimensional Heisenberg algebra.

The class of Lie algebras for which the set of singular elements in \( \mathfrak{g}^* \) is a hypersurface is particularly important for integrable systems [1, 2].

**Remark 2.3.** Rewrite (1) as

\[
\dim \mathfrak{g}^*(\mu) - \dim O(\mu) \leq \dim \mathfrak{g}_\mu - \dim [\mathfrak{g}_\mu, \mathfrak{g}_\mu] \tag{2}
\]

where \( O(\mu) \) is the coadjoint orbit of \( \mu \). Coadjoint orbits form families and the difference

\[
\dim \mathfrak{g}^*(\mu) - \dim O(\mu)
\]

is exactly the local dimension of such a family. Inequality (2) estimates this dimension.

**Example 2.4.** Let \( \mathfrak{g} = \mathfrak{gl}(n) \),

\[
\mu = \text{diag}(\lambda_1, \ldots, \lambda_1, \lambda_2, \ldots, \lambda_2, \ldots, \lambda_s, \ldots, \lambda_s).
\]

Then \( \mathfrak{g}_\mu \simeq \mathfrak{gl}(k_1) \oplus \cdots \oplus \mathfrak{gl}(k_s) \), so \( \dim \mathfrak{g}_\mu - \dim [\mathfrak{g}_\mu, \mathfrak{g}_\mu] = s \). On the other hand, the set of orbits close to \( O(\mu) \) which have the same dimension as \( O(\mu) \) is parameterized by the eigenvalues \( \lambda_1, \ldots, \lambda_s \), so \( \dim \mathfrak{g}^*(\mu) - \dim O(\mu) \) is also equal to \( s \), and inequality (2) turns into equality.

**What happens if the transverse Poisson structure is linearizable.** Let \( M \) be a Poisson manifold, and \( \mu \in M \). Recall that, by the Weinstein splitting theorem [14], \( M \) can be locally decomposed into the direct product of a symplectic manifold and a manifold with a Poisson structure vanishing at \( \mu \). This latter Poisson structure is unique up to a diffeomorphism and is called the transverse Poisson structure at the point \( \mu \). More details can be found in [3].
In the case when $M$ is the dual $\mathfrak{g}^\ast$ of a Lie algebra $\mathfrak{g}$, the linear part of the transverse Poisson structure at a point $\mu$ is the Lie-Poisson structure of the stabilizer $\mathfrak{g}_\mu$. Consequently, if the transverse Poisson structure at a point $\mu$ is linearizable, then the Lie-Poisson structure on $\mathfrak{g}^\ast$ can be locally decomposed into the direct product of a symplectic structure and the Lie-Poisson structure of the stabilizer $\mathfrak{g}_\mu$, which allows to prove the following.

**Proposition 2.2.** Assume that the transverse Poisson structure at a point $\mu$ is linearizable. Then $\mathfrak{g}^\ast(\mu)$ is smooth at the point $\mu$, and

$$\dim_\mu \mathfrak{g}^\ast(\mu) - \dim O(\mu) = \dim \mathfrak{g}_\mu - \dim [\mathfrak{g}_\mu, \mathfrak{g}_\mu].$$

**Example 2.5** (M. Duflo, see [13, 3]). Let $\mathfrak{g}$ be a Lie algebra given by the following linear Poisson structure

$$\frac{\partial}{\partial x_1} \wedge (x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} + 2x_4 \frac{\partial}{\partial x_4}) + x_4 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3}.$$

Consider $\mu \in \mathfrak{g}^\ast$ with $x_4 = 0$ and $x_2^2 + x_3^2 > 0$. Then the stabilizer of $\mu$ is Abelian: $\dim [\mathfrak{g}_\mu, \mathfrak{g}_\mu] = 0$. On the other hand, $\text{codim} \mathfrak{g}^\ast(\mu) = 1$. Consequently, the transverse Poisson structure at $\mu$ is not linearizable.

### 3 Semisimple case: the derived algebra of a centralizer and sheets

**Sheets** In the semisimple case, the coadjoint and the adjoint actions can be identified by the means of the Killing form. This identification maps the variety $\mathfrak{g}_k^\ast$ to the variety

$$\mathfrak{g}^{(k)} = \{ a \in \mathfrak{g} \mid \dim g^a = k \}$$

where $g^a = \{ x \in g \mid [a, x] = 0 \}$ is the centralizer of $a$.

Irreducible components of the varieties $\mathfrak{g}^{(k)}$ are called sheets of $\mathfrak{g}$. Recall some facts about the topology of sheets.

1. Sheets are not necessarily smooth. However, if $\mathfrak{g}$ is a classical simple Lie algebra, then sheets are smooth (Im Hof [4]).

2. Sheets are not necessarily disjoint even in the classical case. However, they are for $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ (Kraft and Luna [7], Peterson [11]).

Study the relation between sheets and the derived algebra of a centralizer.

**The main statement in the semisimple case** Using Corollary [2.1] obtain the following.

**Proposition 3.1.** Let $\mathfrak{g}$ be a real or complex semisimple Lie algebra. Suppose that $a \in \mathfrak{g}$ belongs to a sheet $S$, and $S$ is smooth at the point $a$. Then the derived algebra of the centralizer of $a$ is orthogonal to $S$ at the point $a$: $$\langle [g^a, g^a], T_a S \rangle = 0.$$
Corollary 3.1. Let \( g \) be a real or complex semisimple Lie algebra. Suppose that \( a \in g \) belongs to a sheet \( S \), and \( S \) is smooth at the point \( a \). Then the following three statements are equivalent.

1. The derived algebra of the centralizer of \( a \) is exactly the orthogonal complement to \( S \) at the point \( a \):
   \[
   [g^a, g^a] = (T_\mu S)^\perp.
   \] (3)

2. The dimension of the derived algebra of the centralizer of \( \mu \) is equal to the codimension of \( S \):
   \[
   \dim [g^a, g^a] = \operatorname{codim} S.
   \]

3. The codimension of \( [g^a, g^a] \) in \( g^a \) is equal to the codimension of \( O(a) \) in \( S \):
   \[
   \dim g^a - \dim [g^a, g^a] = \dim S - \dim O(a).
   \]

If one of these three conditions is satisfied, the picture is the following: the centralizer is the orthogonal complement to the orbit while its derived algebra it is the orthogonal complement to the sheet (Figure 1).

The case of a semisimple element

Proposition 3.2. If \( g \) is a real or complex semisimple Lie algebra, and \( a \in g \) is semisimple, then there is only one sheet \( S \) passing through \( a \), \( S \) is smooth at \( a \), and equality (3) holds.

Proof. Since the transverse Poisson structure at a semisimple point is linearizable \cite{8}, the proof follows from Proposition 2.2. \( \square \)

Corollary 3.2. Let \( g \) be a compact real Lie algebra. Then all sheets of \( g \) are smooth, disjoint, and the equality (3) holds for every \( a \in g \).

Proof. The proof follows from the fact that all elements of a compact algebra are semisimple. \( \square \)
The case of \( sl(n, \mathbb{C}) \)

**Proposition 3.3.** The equality (3) holds for every \( a \in sl(n, \mathbb{C}) \).

**Proof.** Denote by \( h(\lambda) \) the size of the largest Jordan block of \( a \) with the eigenvalue \( \lambda \). Prove that

\[
\dim g^a - \dim [g^a, g^a] = \dim S - \dim D(a) = \left( \sum h(\lambda) \right) - 1.
\]

1. \( \dim g^a - \dim [g^a, g^a] = (\sum h(\lambda)) - 1 \).
   It suffices to prove this equality for the case when \( a \) is nilpotent. This can be easily done by studying the commutation relations for \( g^a \) found by O.Yakimova [15].

2. \( \dim S - \dim D(a) = (\sum h(\lambda)) - 1 \).
   This fact is known in the case when \( a \) is nilpotent (A.Moreau [9]). The idea of the proof for an arbitrary element is as follows. For each eigenvalue \( \lambda \), take a sequence of complex numbers \( \varepsilon_1(\lambda), \ldots, \varepsilon_k(\lambda) \) where \( k \) is the size of the block. This gives a family \( a_\varepsilon \in sl(n, \mathbb{C}) \) of dimension \( (\sum h(\lambda)) - 1 \). It is easy to check that the dimension of the centralizer of each \( x \in a_\varepsilon \) is equal to the dimension of \( g^a \), so \( a_\varepsilon \subset S \) where \( S \) is the sheet passing through \( a \). At the same time, the family \( a_\varepsilon \) is transversal to the orbit \( D(a) \), so

\[
\dim S \geq \dim D(a) + \left( \sum h(\lambda) \right) - 1.
\]

On the other hand, by Proposition 3.1

\[
\dim S - \dim D(a) \leq \dim g^a - \dim [g^a, g^a] = \left( \sum h(\lambda) \right) - 1,
\]

so \( \dim S - \dim D(a) = (\sum h(\lambda)) - 1 \), q.e.d.

The case of an arbitrary classical simple Lie algebra

In the previous version of the present paper [5], the following conjecture was formulated.

**Conjecture 3.1.** Let \( g \) be a complex classical simple Lie algebra.

1. If there is only one sheet \( S \) passing through \( a \in g \), then equality (3) holds, i.e.

\[
[g^a, g^a] = (T_a S)^\perp.
\]

2. If \( a \in g \) belongs to several sheets \( S_1, \ldots, S_k \), then

\[
[g^a, g^a] = \left( \sum_{i=1}^k T_a S_i \right)^\perp.
\]

Recently, A.Premet and L.Topley [12] have proved the first part of this conjecture for any algebraically closed ground field of characteristic 0. They have also provided a combinatorial description of those elements \( a \in g \) for which there is only one sheet passing through \( a \).

The second part of the conjecture remains open.

Also note that the conjecture is false for the exceptional Lie algebra \( G_2 \), as it follows from Remark 3 of [15].
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