Maximal $L^1$-regularity of generators for bounded analytic semigroups in Banach spaces

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Abstract

In this paper, we prove that the generator of any bounded analytic semigroup in $(\theta, 1)$-type real interpolation of its domain and underlying Banach space has maximal $L^1$-regularity, using a duality argument combined with the result of maximal continuous regularity. As an application, we consider maximal $L^1$-regularity of the Dirichlet-Laplacian and the Stokes operator in inhomogeneous $B^1_{q,1}$-type Besov spaces on domains of $\mathbb{R}^n$, $n \geq 2$.

2000 Mathematical Subject Classification: 35K90; 46B70; 47D06

Keywords: maximal $L^1$-regularity; sectorial operator; Stokes operator

1 Introduction

A central question for parabolic evolution equations in Banach spaces is whether a related linear operator will have maximal regularity. Maximal regularity of a closed linear operator can efficiently be applied to study related quasilinear and semilinear evolution equations; for relevant background material we refer e.g. to [2], [14] and [25]. In this paper we study maximal $L^1$-regularity of sectorial operators in Banach spaces. An advantage of maximal $L^1$-regularity of sectorial operators is that for the Cauchy problem (1.1) with nonzero initial value $u_0$ the existence of strong solutions is guaranteed without requiring higher regularity on $u_0$ than required for the mild solution; this is not the case with maximal $L^p$-regularity for $p \in (1, \infty)$. This property is efficient, for example, when considering density-dependent Navier-Stokes equations, cf. [12], [13].

Given a Cauchy problem

$$u'(t) + Au(t) = f(t) \quad \text{in } X, \quad u(0) = 0,$$

(1.1)
with a closed linear operator $A$ in a complex Banach space $X$, the operator $A$ is said to have maximal $L^p$-regularity in $X$ for $p \in [1, \infty]$, if for every $f \in L^p(\mathbb{R}_+; X)$, where $\mathbb{R}_+ = [0, \infty)$, the problem (1.1) has a unique solution satisfying
\[
\|u_t\|_{L^p(\mathbb{R}_+; X)} + \|Au\|_{L^p(\mathbb{R}_+; X)} \leq C\|f\|_{L^p(\mathbb{R}_+; X)} \quad (\exists C > 0).
\] (1.2)

Let $J = [0, T]$ for $T < \infty$ or $J = [0, \infty)$. The operator $A$ in $X$ is said to have maximal continuous regularity on $J$ if for every $f$ belonging to $BC(J, X)$ the problem (1.1) has a unique solution satisfying
\[
\|u_t\|_{BC(J, X)} + \|Au\|_{BC(J, X)} \leq C\|f\|_{BC(J, X)} \quad (\exists C > 0),
\] (1.3)
where $BC(J, X)$ is the space of all $X$-valued bounded and continuous functions on $J$.

Let us recall some known results for maximal regularity. If a sectorial operator in a Banach space has maximal $L^p$-regularity for some $p \in (1, \infty)$, then so does it for all $p \in (1, \infty)$ (see [8], [10], [29]); furthermore, (1.2) holds with temporally weighted space $L^p_t((\mathbb{R}_+, X)$ with norm $\|t^{1-\gamma}u\|_{L^p(\mathbb{R}_+, X)}$ for all $p \in (1, \infty)$, $\gamma \in (1/p, 1)$, i.e. maximal $L^p$-regularity, see [26]. If a densely defined closed linear operator $A$ has maximal $L^p$-regularity for $p \in (1, \infty)$ or maximal continuous regularity, then $-A$ must generate an analytic semigroup in the underlying Banach space ([15], [16], [23]).

By a classical result ([28]) generators of bounded analytic semigroups in Hilbert spaces have maximal $L^p$-regularity for $p \in (1, \infty)$. It was shown in [17] Theorem 3.2] that, if a sectorial operator $A$ in a UMD space $X$ admits bounded imaginary powers (BIP) in $L^p(X)$ with power angle less than $\frac{\pi}{2}$, i.e., $\|A^\tau\|_{L^p(X)} \leq ce^{\theta|\tau|}$, $\tau \in \mathbb{R}$, with some $\theta \in (0, \frac{\pi}{2})$, then $A$ has maximal $L^p$-regularity in $X$ for $p \in (1, \infty)$, see also [10], [14 Theorem 4.4], [2 Theorem III.4.10.7]. Note that UMD spaces are reflexive.

The operator $A$ with $-A$ being the generator of a bounded analytic semigroup in a Banach space $E$ has maximal continuous and $L^\infty_\gamma$($\gamma \in (0, 1]$)-regularity in the continuous interpolation space $X = (E, E_1)_{\theta, \infty}$, $\theta \in (0, 1)$, where $E_1 = D(A)$ is endowed with the graph norm of $A$, see [11 Théorème 3.1], [6 Theorem 2.14], [2 Theorem III.3.4.1] and [27 Theorem 3.5]. We note that if $A$ has maximal continuous regularity in a Banach space $X$, then $X$ can be neither reflexive nor weakly sequentially complete ([9 Remark 2.4 (d)], [2 Remarks III.3.1.3 (b)]). Moreover, if $A$ is unbounded and has maximal $L^\infty$-regularity, then $X$ must contain a complemented subspace isomorphic to $c_0$ ([21]).

Concerning maximal $L^1$-regularity, in [20], the property was proved for the negative Laplacian in $FM(\mathbb{R}^n)$, which is the Fourier image of the space of Radon measures on $\mathbb{R}^n$ with finite total variation. It is shown that the Stokes operator has maximal $L^1$-regularity in homogeneous Besov spaces $\dot{B}^s_p(\Omega), p \in (1, \infty), s \in (1/p - 1, 1/p)$, where $\Omega$ is the whole or half space, a smooth bounded or exterior domain, see [12], [13]. It is worth mentioning that in these papers the proof is essentially based on properties of the heat kernel on $\mathbb{R}^n$; moreover, in [12], [13] the Littlewood-Paley characterization of Besov spaces is crucial.

On the other hand, for generator $-A$ of an analytic semigroup, $A$ does not have maximal $L^1$-regularity property if it is unbounded and the underlying Banach space.
does not contain a complemented subspace isomorphic to $l_1$, see [21, Theorem 5]. Thus, it follows by [24, Theorem 2.e.7] that maximal $L^1$-regularity does not hold in reflexive Banach spaces. It is known by [22, Theorem 3.6] that when $-A$ is the generator of a bounded analytic semigroup in a Banach space $X$, maximal $L^1$-regularity of $A$ is equivalent to

$$\int_0^\infty \| Ae^{-tA} u \|_X dt \leq C \| u \|_X, \forall u \in X.$$  \hspace{1cm} (1.4)

One can check that (1.4) holds when $A$ is the generator of a bounded analytic semigroup in a Banach space $E$ with 0 in its resolvent set $\rho(A)$ and $X = (E, E_1)_{\theta, 1}$, $\theta \in (0, 1)$, where $E_1 = \mathcal{D}(A)$ is endowed with graph norm of $A$. However, it is not clear whether (1.4) can still be easily proved without the invertibility condition $0 \in \rho(A)$. We note that maximal regularity property without assuming $0 \in \rho(A)$ allows the estimate constant to be independent of time interval, which is very important in existence and stability theory for corresponding nonlinear problems.

In this paper we show that maximal $L^1$-regularity for the generator of any bounded analytic semigroup in $(\theta, 1)$-type real interpolation of Banach spaces can follow directly from the known maximal continuous regularity result by a duality argument without assuming $0 \in \rho(A)$. We remark here that (1.4) can still be easily proved without the invertibility condition $0 \in \rho(A)$ allows the estimate constant to be independent of time interval, which is very important in existence and stability theory for corresponding nonlinear problems.

Below, for Banach spaces $E$ and $E_1$, the notation $E_1 \stackrel{\mathcal{d}}{\hookrightarrow} E$ means that $E_1$ is continuously and densely embedded in $E$. The dual space of $E$ is denoted by $E'$. Let $\mathcal{L}(E_1, E)$ stand for the set of all linear bounded operators from $E_1$ to $E$, and let $\mathcal{L}(E) = \mathcal{L}(E, E)$. Furthermore, $\text{Isom}(E_1, E)$ denotes the set of all norm isomorphisms from $E_1$ to $E$. The notation $A \in \mathcal{H}(E_1, E)$ means that $A \in \mathcal{L}(E_1, E)$ and, considered as an unbounded linear operator in $E$ with $\mathcal{D}(A) = E_1$, $-A$ generates an analytic $C_0$-semigroup $\{e^{-tA}\}_{t \geq 0}$ in $E$.

The main result of the paper is as follows.

**Theorem 1.1 (Maximal $L^1$-regularity)** Let $E$ and $E_1$ be complex reflexive Banach spaces with $E_1 \stackrel{\mathcal{d}}{\hookrightarrow} E$ and let $A \in \mathcal{H}(E_1, E)$ generate a bounded analytic semigroup in $E$. Then, for $\theta \in (0, 1)$ the realization of $A$ in $E_{\theta, 1} := (E, E_1)_{\theta, 1}$ has maximal $L^1$-regularity in $E_{\theta, 1}$. More precisely, for $f \in L^1(\mathbb{R}_+; E_{\theta, 1})$, the Cauchy problem (1.1) with $X = E_{\theta, 1}$ has a unique solution $u$ such that $u_t, Au \in L^1(\mathbb{R}_+; E_{\theta, 1})$ and

$$\| u_t \|_{L^1(\mathbb{R}_+; E_{\theta, 1})} + \| Au \|_{L^1(\mathbb{R}_+; E_{\theta, 1})} \leq c \| f \|_{L^1(\mathbb{R}_+; E_{\theta, 1})},$$  \hspace{1cm} (1.5)

where $c = c(\theta)$ is independent of $f$.

The main idea for the proof of Theorem 1.1 is to use a duality argument based on maximal continuous regularity for a backward-in-time problem in continuous interpolations spaces which are obtained by a method of extrapolation of $E_1$ and $E$ introduced e.g. in [2], [3] (see also Subsection 2.1).

The following corollary is a direct consequence of Theorem 1.1.
Corollary 1.2 Under the same assumptions on the spaces $E, E_1$ and the sectorial operator $A$ as in Theorem [7], let $\{(E_\alpha, A_\alpha): \alpha \in \mathbb{R}\}$ be the interpolation and extrapolation scale generated by the real interpolation functor $\{(\cdot, \cdot)_{\theta_1}: \theta \in (0,1)\}$ and $(E, A)$, see Subsection 2.1. Moreover, if $\alpha \in \mathbb{Z}$, let

$$E_\alpha^* := (E_{\alpha-1/2}, E_{\alpha+1/2})_{1/2,1}.$$ 

(i) If $\alpha \in (k, k+1)$ with $k \in \mathbb{Z}$, then $A_\alpha$ has maximal $L^1$-regularity in $E_{\alpha-1}$.

(ii) If $\alpha \in \mathbb{Z}$, then $A_\alpha^*$, the $E_\alpha^*$-realization of $A_{\alpha-1/2}$, has maximal $L^1$-regularity in $E_\alpha^*$.

This paper is organized as follows. In Section 2, preliminaries on interpolation and extrapolation scales and vector measures with bounded variation are given. Section 3 is devoted to the proof of main results. As an application of the abstract theory, in Section 4, maximal $L^1$-regularity of the Dirichlet-Laplacian and the Stokes operator in $B_{q,1}$-type Besov spaces on domains is considered.

2 Preliminaries

2.1 Interpolation and extrapolation scales

Given an interpolation couple $(E_0, E_1)$ of Banach spaces and $\theta \in (0,1)$, we denote by $[\cdot, \cdot]_{\theta}$, $(\cdot, \cdot)_{\theta,r}$, $1 \leq r \leq \infty$, the complex and real interpolation functors, respectively, cf. [7], [30]. The continuous interpolation space $(E_0, E_1)^0_{\theta,\infty}$, $\theta \in (0,1)$, of $E_0$ and $E_1$ is defined as the closure of $E_0 \cap E_1$ in $(E_0, E_1)_{\theta,\infty}$.

To be more specific, in this subsection we assume that $E_1 \doteqdot E$ and $E$ is reflexive. Furthermore, let an interpolation functor $(\cdot, \cdot)_{\theta}$, $\theta \in (0,1)$, be given by

$$\langle \cdot, \cdot \rangle_{\theta} \in \{[\cdot, \cdot]_{\theta}, (\cdot, \cdot)_{\theta,r}, (\cdot, \cdot)_{\theta,\infty} : 1 \leq p < \infty\}.$$  \hfill (2.1)

Let $A$ be a linear, closed and sectorial operator (with angle $\vartheta \in (0, \pi)$) in $E_0$ with $\mathcal{D}(A) = E_1$ and $\mathcal{R}(A) = E$, i.e., $S_{\vartheta} := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \vartheta\} \subseteq \rho(-A)$ and

$$\|\lambda(\lambda + A)^{-1}\|_{L(E_0)} \leq K \quad \text{for all } \lambda \in S_{\vartheta}$$  \hfill (2.2)

with a constant $K > 0$ depending on $\vartheta$. Suppose that $A \in \mathcal{H}(E_1, E)$ and

$$\|e^{-tA}\|_{L(E)} \leq M \quad (\exists M > 0, \forall t \geq 0).$$  \hfill (2.3)

For $k \in \mathbb{N}$ let $E_k := \mathcal{D}(A^k)$ be the domain of $A^k$ in $E_0 := E$ endowed with its graph norm. Let $A^{\sharp} := A^*$ denote the dual (adjoint) operator of $A$ in the dual space $E^{\sharp} = E'$ which is a closed, densely defined unbounded operator with domain $\mathcal{D}(A^{\sharp}) \subset E^{\sharp}$. Then, for negative integers $k$, the space $E_k$ can be introduced as the dual space of $\mathcal{D}((A^{\sharp})^{-k})$ which is the domain of $(A^{\sharp})^{-k}$ in $E^{\sharp}$ endowed with its graph norm.

Finally, for $\alpha \in (k, k+1)$, $k \in \mathbb{Z}$, the space $E_\alpha$ is defined by

$$E_\alpha := (E_k, E_{k+1})_{\alpha-k},$$  \hfill (2.4)
and the operator $A_\alpha$ in $E_\alpha$ by the realization of $A$ in $E_\alpha$ if $\alpha \geq 0$ and as the closure of $A$ in $E_\alpha$ if $\alpha < 0$. Thus, the interpolation and extrapolation scale $\{(E_\alpha, A_\alpha) : \alpha \in \mathbb{R}\}$ is generated by $\langle \cdot, \cdot \rangle_\theta$ and $(E, A)$. It is well-known that for $-\infty < \beta < \alpha < \infty$

$$E_\alpha \overset{\text{d}_\beta}{\rightarrow} E_\beta, \quad A_\alpha \in \mathcal{H}(E_{\alpha+1}, E_\alpha), \quad e^{-tA_\alpha} = e^{-tA_\beta}|_{E_\alpha}, \quad \|e^{-tA_\alpha}\|_{\mathcal{L}(E_\alpha)} \leq M$$

(2.5)

with the same constant $M > 0$ as in (2.3), see [1], [2] Chapter V.1, V.2, [3] Section 1.

From now on, suppose that $\langle \cdot, \cdot \rangle_\theta$ for every $\theta \in (0, 1)$ does not coincide with $\langle \cdot, \cdot \rangle_\theta^0, \infty$ and let $\{(E_\alpha^\circ, A_\alpha^\circ) : \alpha \in \mathbb{R}\}$ be the interpolation and extrapolation scale generated by $\{E^\circ, A^\circ\}$ and the interpolation functor

$$\langle \cdot, \cdot \rangle^\circ_\theta = \begin{cases} \langle \cdot, \cdot \rangle_\theta & \text{for } \langle \cdot, \cdot \rangle_\theta = \langle \cdot, \cdot \rangle_\theta, \\ \langle \cdot, \cdot \rangle_{\theta,r} & \text{for } \langle \cdot, \cdot \rangle_\theta = \langle \cdot, \cdot \rangle_{\theta,r}, 1 < r < \infty, \\ \langle \cdot, \cdot \rangle^0_{\theta,\infty} & \text{for } \langle \cdot, \cdot \rangle_\theta = \langle \cdot, \cdot \rangle_{\theta,1}. \end{cases} \quad (2.6)$$

Note that compared to the literature, see e.g. [2, p. 282], [3, (1.9)], the role of $\langle \cdot, \cdot \rangle_\theta$ and $\langle \cdot, \cdot \rangle^\circ_\theta$ is interchanged since the third case in (2.6) starts with $\langle \cdot, \cdot \rangle_\theta = \langle \cdot, \cdot \rangle_{\theta,1}$ and defines the continuous interpolation functor $\langle \cdot, \cdot \rangle^0_{\theta,\infty}$. It is this case which is used in this article. Hence, in view of reflexivity of $E$ and $(A^\circ)' = A$ due to closedness of the operator $A$ in $E$, results from the literature can be used by formally interchanging $E$ and $E^\circ$, i.e., interchanging $(E_\alpha, A_\alpha)$ with $(E_\alpha^\circ, A_\alpha^\circ)$.

As an example we will frequently use that

$$E_\alpha = (E_\alpha^\circ)' = (E_\alpha)' = \mathbb{R}, \quad (2.7)$$

cf. [2] Theorem V.1.5.12], [3] (1.10)], whereas $(E_\alpha)'$ will be different from $E^\circ_{\alpha}$ by the third case of (2.6). Moreover, let us mention the smoothness properties of $e^{-tA_\alpha}$, see [1] Theorem 6.1 (vi)], [2] Theorem V.2.1.3, Corollary V.2.1.4]): For $-\infty < \beta < \alpha < \infty$ and $f \in E_\beta$ there holds $t^{\alpha-\beta}e^{-tA_\alpha}f \in BC((0,T), E_\alpha)$ and

$$\|e^{-tA_\alpha}f\|_{E_\alpha} \leq c(\alpha, \beta, M)t^{\beta-\alpha}\|f\|_{E_\beta}, \quad t \in (0,T).$$

(2.8)

Let $(A^\circ_{\alpha,-1})'$ be the dual operator of $A^\circ_{\alpha,-1} \in \mathcal{L}(E^\circ_{\alpha}, E^\circ_{\alpha-1},)$, $\alpha \in \mathbb{R}$, in the sense of bounded linear operators. In view of (2.7), $(A^\circ_{\alpha,-1})' \in \mathcal{L}((E^\circ_{\alpha-1})', (E^\circ_{\alpha})') = \mathcal{L}(E_{\alpha+1}, E_\alpha)$. Furthermore, the above argument of interchanging $(E_\alpha, A_\alpha)$ by $(E_\alpha^\circ, A_\alpha^\circ)$ and [2] Theorem V.2.3.2] imply that

$$(A^\circ_{\alpha,-1})' = A_\alpha \in \mathcal{L}(E_{\alpha+1}, E_\alpha), \quad \alpha \in \mathbb{R}.$$ 

(2.9)

### 2.2 Vector measures with bounded variation

Let $(E, \| \cdot \|_E)$ be a Banach space, $J$ be a $\sigma$-compact metrizable space, and let $\mathcal{B} = \mathcal{B}(J)$ be the Borel $\sigma$-algebra of $J$. A $\sigma$-additive map $\mu : \mathcal{B} \mapsto E$ is said to be an $E$-valued
vector measure if \( \mu(\emptyset) = 0 \) (cf. [5, Subsection 2.2]. For a vector measure \( \mu \) the total variation \( |\mu| : B \mapsto \mathbb{R}_+ \cup \{\infty\} \) is defined by

\[
|\mu|(G) := \sup_{\pi(G)} \sum_{F \in \pi(G)} \|\mu(F)\|_E, \quad G \in B,
\]

where the supremum is taken over all partitions \( \pi(G) \) of \( G \) into a finite number of pairwise disjoint Borel subsets. Then, \( \mu \) is said to be of bounded variation on \( J \) if

\[
\|\mu\|_{BV} := |\mu|(J) < \infty.
\]

We denote by \( \mathcal{M}(J, E) := (\mathcal{M}(J, E), \|\cdot\|_{BV}) \) the space of all \( E \)-valued vector measures on \( J \) with bounded total variation.

Next we replace \( E \) in \( \mathcal{M}(J, E) \) by \( E' \). Then, through the duality pairing

\[
\langle \mu, u \rangle_{\mathcal{M}(J, E'), BC(J, E)} = \int_J \langle u, d\mu \rangle_{E, E'}, \quad \mu \in \mathcal{M}(J, E'), \, u \in BC(J, E),
\]

it holds by [5, Theorem 2.2.4]

\[
(BC(J, E))' = \mathcal{M}(J, E'). \quad (2.11)
\]

If \( h \in L^1(J, E') \), then the \( E' \)-valued measure \( \mu_h \) on \( J \) defined by

\[
\mu_h(B) := \int_B h(t) \, dt, \quad B \in B,
\]

has bounded total variation, and the map \( h \mapsto \mu_h \) is a linear isometry from \( L^1(J, E') \) into \( \mathcal{M}(J, E') \). Hence \( L^1(J, E') \) can be identified with a closed subspace of \( \mathcal{M}(J, E') \), cf. [5, Remark 2.2.1].

3 Proof of the main result

Proof of Theorem 1.1. Let \( \{(E_\alpha, A_\alpha) : \alpha \in \mathbb{R}\} \) be the interpolation and extrapolation scale generated by the interpolation functor \( \{(\cdot, \cdot)_{\theta,1} : \theta \in (0,1)\} \) and \( (E, A) \), and let \( \{(E_\alpha, A_\alpha) : \alpha \in \mathbb{R}\} \) be the interpolation and extrapolation scale generated by the interpolation functor \( \{(\cdot, \cdot)_{0,\infty} : \theta \in (0,1)\} \) and \( (E^\natural, A^\natural) \equiv (E', A') \), see Subsection 2.1. Then, by (2.4) and (2.7), it holds that

\[
E_{\alpha+k} = (E_k, E_{k+1})_{\alpha,1}, \quad \alpha \in (0,1), \, k \in \mathbb{N} \cup \{0\},
\]

\[
E^\natural_{-\alpha-k} = (E^\natural_{-k}, E^\natural_{-k-1})_{\alpha,\infty}, \quad \alpha \in (0,1), \, k \in \mathbb{N} \cup \{0\},
\]

and

\[
E_\alpha = (E^\natural_{-\alpha})', \quad \alpha \in \mathbb{R}. \quad (3.2)
\]

Now, let \( 0 < T < \infty \) and consider the backward-in-time Cauchy problem

\[-v_t + A^\natural_{-1-\theta} v = A^\natural_{-1-\theta} g \quad \text{for} \quad 0 \leq t < T, \quad v(T) = 0,\]

\[
\quad (3.3)
\]

[5]
where $g \in C([0, T], E^2_{-\theta})$. Note that (3.3) is reduced by the change of variable $t \mapsto T - t$ to a parabolic Cauchy problem in $E^2_{-1-\theta}$ with initial time $t = 0$ and has a unique mild solution $v \in C([0, T], E^2_{-1-\theta})$ expressed by

$$v(T - t) = \int_0^t e^{-(t-\tau)A^2_{-1-\theta}} A^2_{-1-\theta} g(T - \tau) \, d\tau, \ t \in (0, T).$$

By the well-known property of analytic semigroups and the fact that $A^2_{-1-\theta}$ is an extension of $A^2_{-\theta}$, we have

$$v(T - t) = A^2_{-1-\theta} \int_0^t e^{-(t-\tau)A^2_{-1-\theta}} A^2_{-1-\theta} g(T - \tau) \, d\tau = A^2_{-\theta} \int_0^t e^{-(t-\tau)A^2_{-1-\theta}} A^2_{-1-\theta} g(T - \tau) \, d\tau = A^2_{-\theta} w(T - t), \ t \in (0, T),$$

where $w$ is the (unique) mild solution to the backward problem

$$-w_t + A^2_{-\theta} w = g \quad \text{in} \ (0, T), \ w(T) = 0.$$ 

Since the operator $A^2_{-\theta}$ is the $E^2_{-\theta}$-realization of $A^2_{-1}$ in $H(E^2_{0}, E^2_{-1})$ and $E^2_{-\theta} = (E^2_{0}, E^2_{-1})^{\theta, \infty}$ by (3.1), we get by the result of maximal continuous regularity (see Theorem IV.3.1.2, (Littlewood-Sobolev inequality) we get that $v \in C([0, T], E^2_{-1-\theta})$ expressed by

$$\|A^2_{-\theta} w\|_{C([0, T], E^2_{-\theta})} \leq C \|g\|_{C([0, T], E^2_{-\theta})}$$

with constant $C > 0$ independent of $T$. Consequently, we have

$$\|v\|_{C([0, T], E^2_{-\theta})} \leq C \|g\|_{C([0, T], E^2_{-\theta})}. \quad (3.4)$$

Now, let $f \in L^p(0, T; E_1)$, $1 < p < 1/(1-\theta)$. Then the mild solution to (1.1) with $X = E_{\theta}$ $(\equiv E_{\theta,1})$ is given by

$$u(t) \equiv u_f(t) := \int_0^t e^{-(t-s)A} f(s) \, ds, \quad (3.5)$$

and by (2.8) it follows that $\|Ae^{-tA}\|_{L(E_1, E_\theta)} \leq c \|e^{-tA}\|_{L(E_1, E_{1+\theta})} \leq c/t^\theta$. Therefore,

$$\|Au(t)\|_{E_\theta} \leq \int_0^t \|Ae^{-(t-s)A} f(s)\|_{E_\theta} \, ds \leq c \int_{\mathbb{R}^1} \frac{1}{(t-s)^\theta} \|f(s)\|_{E_1} \, ds, \ t \in (0, T),$$

where $\tilde{f}$ denotes the extension of $f$ by 0 from $[0, T]$ to $\mathbb{R}^1$. Hence, by the Hardy-Littlewood-Sobolev inequality we get that

$$\|Au\|_{L^p(0, T; E_\theta)} \leq C_p \|\tilde{f}\|_{L^p(\mathbb{R}; E_1)} = C_p \|f\|_{L^p(0, T; E_1)}$$
for \( p^* = p/(1 - (1 - \theta)p) \). Moreover, since \( \|e^{-tA}\|_{\mathcal{L}(E_{\theta})} \leq M \), see \((2.5)\), it follows from \((3.5)\) that \( \|u\|_{L^\infty([0,T];E_{\theta})} \leq M\|f\|_{L^1([0,T];E_{\theta})} \) and hence \( u \in L^1(0,T;E_{\theta}) \) in view of \( T < \infty \).

Thus we have

\[
Au \in L^1(0,T;E_{\theta}), \quad u \in L^1(0,T;E_{1+\theta})
\]  

(3.6)

for \( T < \infty \). Here, recall that \((E^2_{-\theta})' = E_{\theta}\) but \((E_{\theta})' \neq E^2_{-\theta}\). Hence \( L^1(0,T;E_{\theta}) \) is not the dual space of \( L^\infty(0,T;E^{-2}_{-\theta}) \) and vice versa.

By the properties of \( v, v_t, u, u_t \) in \((3.4), (3.5)\) and \((3.6)\) it follows by an approximation argument that for almost all \( t \in (0,T) \)

\[
\frac{d}{dt}\langle u(t), v(t) \rangle_{E_{\theta}, E^{-2}_{-\theta}} = \langle u_t(t), v(t) \rangle_{E_{\theta}, E^{-2}_{-\theta}} + \langle u(t), v_t(t) \rangle_{E_{1+\theta}, E^{-2}_{-1-\theta}} \in L^1(0,T);
\]

dence the map \( t \mapsto \langle u(t), v(t) \rangle_{E_{\theta}, E^{-2}_{-\theta}} \) is absolutely continuous in \([0,T]\). Therefore, in view of \( u(0) = v(T) = 0 \), we have

\[
0 = \int_0^T \frac{d}{dt}\langle u(t), v(t) \rangle_{E_{\theta}, E^{-2}_{-\theta}} \, dt
\]

(3.7)

Since, by \((2.9)\), the dual of \( A^2_{-1-\theta} \in \mathcal{L}(E^2_{-\theta}, E^{-2}_{-1-\theta}) \) equals \( A_{\theta} \in \mathcal{L}(E_{1+\theta}, E_{\theta}) \), we get from \((3.3)\) and \((3.7)\) that

\[
\int_0^T \langle Au, g \rangle_{E_{\theta}, E^{-2}_{-\theta}} \, dt = \int_0^T \langle u, A^2_{-1-\theta} g \rangle_{E_{1+\theta}, E^{-2}_{-1-\theta}} \, dt
\]

\[
= \int_0^T \langle u, -v_t + A^2_{-1-\theta} v \rangle_{E_{1+\theta}, E^{-2}_{-1-\theta}} \, dt
\]

\[
= \int_0^T \langle u_t + A_{\theta} u, v \rangle_{E_{\theta}, E^{-2}_{-\theta}} \, dt
\]

\[
= \int_0^T \langle f, v \rangle_{E_{\theta}, E^{-2}_{-\theta}} \, dt.
\]

In view of \((2.10), (2.11), (2.12)\) it follows that

\[
\langle \mu Au, g \rangle_{\mathcal{M}([0,T], E_{\theta}), C([0,T], E^2_{-\theta})} = \langle \mu f, v \rangle_{\mathcal{M}([0,T], E_{\theta}), C([0,T], E^2_{-\theta})},
\]

and hence, by \((3.4)\),

\[
|\langle \mu Au, g \rangle_{\mathcal{M}([0,T], E_{\theta}), C([0,T], E^2_{-\theta})}| \leq \|\mu f\|_{\mathcal{M}([0,T], E_{\theta})} \|v\|_{C([0,T], E^2_{-\theta})} \leq C(\theta) \|\mu f\|_{\mathcal{M}([0,T], E_{\theta})} \|g\|_{C([0,T], E^2_{-\theta})}.
\]

8
Moreover, since \( g \) is an arbitrary element of \( C([0,T], E_{\alpha,1}^s) \), we have

\[
\|\mu Au\|_{\mathcal{M}([0,T],E_{\alpha})} \leq C(\theta) \|\mu f\|_{\mathcal{M}([0,T],E_{\alpha})} \quad \forall f \in L^p(0,T; E_1),
\]

where \( C(\theta) \) is independent of \( T \). Then by the isometry property of the map \( h \mapsto \mu_h \) we conclude from (3.8) that

\[
\|Au\|_{L^1(0,T;E_{\alpha})} \leq C(\theta) \|f\|_{L^1(0,T;E_{\alpha})} \quad \forall f \in L^p(0,T; E_1).
\]

Now, due to the density of \( L^p(0,T; E_1) \) in \( L^1(0,T; E_{\alpha}) \) and linearity of the problem (1.1), it follows that (3.9) holds for all \( f \in L^1(0,T; E_{\alpha}) \).

Since \( f \in L^1(\mathbb{R}_+; E_{\alpha}) \), we get by the above conclusion for \( T < \infty \) and the representation formula (3.5) for the mild solution \( u \) that the unique solution \( u \) to (1.1) satisfies \( u_t, Au \in L^1_{\text{loc}}([0,\infty); E_{\alpha}) \). Moreover, since \( \|u_t, Au\|_{L^1(0,T;E_{\alpha})} \) can be estimated independently of \( T \), it follows that \( u_t, Au \in L^1(\mathbb{R}_+; E_{\alpha}) \) and the estimate (1.5) holds true.

**Proof of Corollary 1.2**

(i) For \( \alpha \in (k, k+1), k \in \mathbb{Z} \), we have \( E_\alpha = (E_k, E_{k+1})_{\alpha-k,1} \) and \( A_\alpha \in \mathcal{H}(E_{k+1}, E_k), \|e^{-tA_\alpha}\|_{\mathcal{L}(E_k)} \leq M \) by (2.5). Thus the assertion follows from Theorem 1.1 by considering the problem in the underlying Banach space \( E_k \).

(ii) If \( \alpha \in \mathbb{Z} \), then \( A_{\alpha-1/2} \in \mathcal{H}(E_{\alpha+1/2}, E_{\alpha-1/2}) \) and \( \|e^{-tA_{\alpha-1/2}}\|_{\mathcal{L}(E_{\alpha-1/2})} \leq M \) by (2.5). Hence, \( A_\alpha^* \), the \( E_\alpha^* \)-realization of \( A_{\alpha-1/2} \) where \( E_\alpha^* := (E_{\alpha-1/2}, E_{\alpha+1/2})_{1/2,1} \), has maximal \( L^1 \)-regularity in \( E_\alpha^* \) by Theorem 1.1.

### 4 Maximal \( L^1 \)-regularity of Dirichlet-Laplacian and Stokes operator in Besov spaces \( B_{q,1}^s \)

Important applications of the abstract theory derived in this paper concern, in particular, maximal \( L^1 \)-regularity of the Dirichlet-Laplacian and the Stokes operator in inhomogeneous Besov spaces. Let \( L^q(\Omega), W^{1,q}(\Omega) \) and \( B_{q,r}^s(\Omega), 1 \leq q, r \leq \infty, s \in \mathbb{R} \), denote the usual Lebesgue, Sobolev and Besov spaces, respectively, on a domain \( \Omega \subset \mathbb{R}^n \). If \( \Omega \neq \mathbb{R}^n \), the Besov space \( B_{q,r}^s(\Omega) \) is defined by restriction of tempered distributions in \( B_{q,r}^s(\mathbb{R}^n) \) to \( \Omega \); its norm is defined by the quotient norm, see [3, 30].

#### 4.1 Dirichlet-Laplacian

Let \( \Omega \subset \mathbb{R}^n, n \in \mathbb{N} \), be a domain with uniform \( C^2 \)-boundary \( \partial \Omega \). The Dirichlet-Laplacian \( -\Delta_\Omega \) is defined by \( -\Delta_\Omega u := -\Delta u \) for \( u \) in

\[
\mathcal{D}(-\Delta_\Omega) := \{u \in W^{2,q}(\Omega) : u|_{\partial \Omega} = 0\} = W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega), \quad 1 < q < \infty.
\]

It is well known that \( -\Delta_\Omega \) generates an analytic semigroup in \( L^q(\Omega) \). Hence, the \( q \)-dependent interpolation and extrapolation scale \( \{(E_{\alpha,1}, A_{\alpha,1}) : \alpha \in \mathbb{R}\} \) generated by \( (E, A) := (L^q(\Omega), -\Delta_\Omega) \) and the real interpolation functor \( \{(\cdot, \cdot)_{\theta,1} : \theta \in (0,1)\} \) is
well-defined. It is known ([3, Theorem 2.2, Proposition 2.4]) that \( E_{\alpha,1} = B_{q,1,0}^{2\alpha}(\Omega) \) for \( \alpha \in (-1 + 1/2q, 0) \cup (0, 1) \), where

\[
\begin{align*}
B_{q,1,0}^s(\Omega) = \begin{cases} 
\{ u \in B_{q,1}^s(\Omega) : u|_{\partial \Omega} = 0 \}, & s \in (1/q, 2) \\
\{ u \in B_{q,1}^s(\mathbb{R}^n) : \text{supp} \, u \subset \Omega \}, & s = 1/q \\
B_{q,1}^s(\Omega), & s \in (-2 + 1/q, 1/q) \setminus \{0\},
\end{cases}
\end{align*}
\]

and, of course, \( B_{q,1,0}^s(\Omega) = B_{q,1}^s(\mathbb{R}^n) \) if \( \Omega = \mathbb{R}^n \). In general, \( B_{q,1,0}^s(\Omega) \) for \( -2 < s < 0 \) is defined by

\[
B_{q,1,0}^s(\Omega) = ((W^{2,q'}(\Omega) \cap W_0^{1,q'}(\Omega))^s, L_0^q(\Omega))_{1+s/2,1}, -2 < s < 0, q' = q/(q-1); \quad (4.2)
\]

however, if \( \Omega \neq \mathbb{R}^n \), the third characterization in (4.1) holds for \( s < 0 \) only when \( s \in (-2 + 1/q, 0) \).

By the reiteration theorem yielding the identity \((E_{-1/2,1}, E_{1/2,1})_{1/2+\alpha,1} = E_{\pm \alpha,1} \) for sufficiently small \( \alpha > 0 \) (cf. [3, Lemma 1.1]), it also follows that

\[
\begin{align*}
E_{0,1}^s := (E_{-1/2,1}, E_{1/2,1})_{1/2,1} \\
= ((E_{-1/2,1}, E_{1/2,1})_{1/2-\alpha,1}, (E_{-1/2,1}, E_{1/2,1})_{1/2+\alpha,1})_{1/2,1} \\
= (E_{-\alpha,1}, E_{\alpha,1})_{1/2,1} \quad (0 < \alpha < 1/2q) \\
= (B_{q,1}^{-2\alpha}(\Omega), B_{q,1}^{2\alpha}(\Omega))_{1/2,1} = B_{q,1}^{0}(\Omega) =: B_{q,1,0}^0(\Omega).
\end{align*}
\]

Thus, Theorem 1.1, Corollary 1.2 result in the following proposition where \( -\Delta \Omega \) denotes the Dirichlet-Laplacian extended or restricted to \( B_{q,1,0}^s(\Omega) \), \( s \in (-2, 2) \), see (4.1), (4.2) and (4.3).

**Proposition 4.1** Let \( \Omega \in \mathbb{R}^n \), \( n \in \mathbb{N} \), be a domain with uniform \( C^2 \)-boundary such that semigroup \( e^{t \Delta \Omega} \) is bounded. Then the Dirichlet-Laplacian \( -\Delta \Omega \) has maximal \( L^1 \)-regularity in \( B_{q,1,0}^s(\Omega) \), \( q \in (1, \infty) \), \( s \in (-2, 2) \).

**Remark 4.2** If \( \Omega \neq \mathbb{R}^n \), the space \( B_{q,1,0}^s(\Omega) \) for \( s \in (-2, -2 + 1/q) \) in (4.2) can not be identified with a subspace of \( D(\Omega) \), the space of Schwartz distributions on \( \Omega \). In fact, with the notation of [3], in this case \( B_{q,1,0}^s(\Omega) = (\hat{B}_{q',\infty,0}^{-s}(\Omega))' \) where

\[
\hat{B}_{q',\infty,0}^{-s}(\Omega) := (W^{2,q'}(\Omega) \cap W_0^{1,q'}(\Omega), L_0^{q'}(\Omega))_{1+s/2,\infty}
\]

\[
= \{ u \in \hat{B}_{q',\infty}^{-s}(\Omega) : u|_{\partial \Omega} = 0 \},
\]

and

\[
\hat{B}_{q',\infty}^{-s}(\Omega) := (W^{2,q'}(\Omega), L_0^{q'}(\Omega))_{1+s/2,\infty}, \quad s \in (-2, -2 + 1/q],
\]

is the closure of \( W^{-s,q'}(\Omega) \) in the Besov space \( B_{q',\infty}^{-s}(\Omega) \) (cf. [3] (2.19), (2.20)) with \( \hat{B}_{q',\infty}^{-s} := n_{q',\infty}^{-s} \) yielding \((\hat{B}_{q',\infty}^{-s}(\Omega))' = B_{q,1}^s(\Omega)\). Moreover, for \( s \in (-2, -2 + 1/q] \) the set \( D(\Omega) \) is dense in neither \( B_{q,\infty,0}^{-s}(\Omega) \) nor \( \hat{B}_{q',\infty}^{-s}(\Omega) \), and neither \( B_{q,1}^s(\Omega) \) nor \( B_{q,1,0}^s(\Omega) \) are subspaces of distribution spaces.
4.2 Stokes operator

Maximal regularity of the Stokes operator is a crucial tool in the study of the Navier-Stokes equations. Below, we briefly mention how maximal $L^1$-regularity of the Stokes operator in solenoidal subspaces of inhomogeneous Besov spaces of $B^s_{q,1}$-type is implied by the abstract theory of this paper.

Let $\Omega \in \mathbb{R}^n$, $n \geq 2$, be a domain with uniform $C^2$-boundary $\partial \Omega$. Let $1 < q < \infty$ and $L^q_\sigma(\Omega)$ and $W^{1,q}_0(\Omega)$ be the closure of the set $C_0^\infty(\Omega) = \{ u \in C_0^\infty(\Omega)^n : \text{div} \, u = 0 \}$ in $L^q(\Omega)^n$ and $W^{1,q}(\Omega)^n$, respectively. Assume that

$$L^q_\sigma(\Omega) := \{ u \in L^q(\Omega)^n : \text{div} \, u = 0, \ u \cdot \mathbf{n}|_{\partial \Omega} = 0 \},$$

where $\mathbf{n}$ is the outward normal vector at $\partial \Omega$, and the Helmholtz decomposition

$$L^q(\Omega)^n = L^q_\sigma(\Omega) \oplus G_q(\Omega), \quad G_q(\Omega) = \{ v \in L^q(\Omega)^n : \exists \pi \in L^1_{\text{loc}}(\Omega) : v = \nabla \pi \}$$

holds algebraically and topologically. Let $\mathbb{P} = P_q$ be the Helmholtz projection from $L^q(\Omega)^n$ onto $L^q_\sigma(\Omega)$. Then the Stokes operator $A = A_q$ may be defined by

$$Au := -\mathbb{P} \Delta u \quad \text{for} \quad u \in \mathcal{D}(A) := W^{2,q}(\Omega)^n \cap W^{1,q}_0(\Omega), \ 1 < q < \infty. \quad (4.7)$$

We assume that

$$A \in \mathcal{H}(E_1, E), \quad \|e^{-tA}\|_{\mathcal{L}(E)} \leq M e^{\omega t} \quad (\exists M > 0, \exists \omega > 0, \forall t \geq 0), \quad (4.8)$$

where $E_1 := \mathcal{D}(A)$ is endowed with its graph norm and $E := L^q_\sigma(\Omega)$.

It is well known that the assumptions (1.5), (1.6) and (1.8) are satisfied for many kinds of standard domains, such as $\Omega = \mathbb{R}^n, \mathbb{R}^n_+,$ bounded and exterior domains, infinite layers and cylinders (with $\omega = 0$ in (1.8)) and aperture domains, and compact perturbations thereof (with $\omega \neq 0$, in general), see Introduction of [19] and the references therein. However, there are smooth unbounded domains for which the Helmholtz projection does not exist, see [IS].

Starting from $(E, A)$ and the real interpolation functor $\{(\cdot, \cdot)_{0,1} : \theta \in (0, 1)\}$, the interpolation and extrapolation scale $\{(E_{\alpha,1}, A_{\alpha,1}) : \alpha \in \mathbb{R}\}$ is generated. There exists an explicit representation of $E_{\alpha,1}$ for $|\alpha| \leq 1$, see [3] Theorem 3.4, Remark 3.7. More precisely, using the notation from [17], we have $E_{\alpha,1} = B^{2\alpha}_{q,1}(\Omega)$ with

$$E^{s}_{q,1}(\Omega) := \begin{cases} \{ u \in B^{s}_{q,1}(\Omega)^n : \text{div} \, u = 0, \ u|_{\partial \Omega} = 0 \}, & s \in (1/q, 2) \\ \{ u \in B^{s}_{q,1}(\mathbb{R}^n)^n : \text{div} \, u = 0, \ \text{supp} \, u \subset \bar{\Omega} \}, & s = 1/q \\ \{ u \in B^{s}_{q,1}(\Omega)^n : \text{div} \, u = 0, \ u \cdot \mathbf{n}|_{\partial \Omega} = 0 \}, & s \in (0, 1/q) \\ \mathcal{D}(A_q)' \cap L^q_\sigma(\Omega), & s \in (-2, 0). \end{cases} \quad (4.9)$$

Moreover, for $E^{\bullet}_{0,1} := (E_{-1/2,1}, E_{1/2,1})_{1/2,1}$ we have

$$E^{\bullet}_{0,1} = \{ u \in B^{0}_{q,1}(\Omega)^n : \text{div} \, u = 0, \ u \cdot \mathbf{n}|_{\partial \Omega} = 0 \} =: E^{0}_{q,1}(\Omega), \quad (4.10)$$

which can be proved exactly in the same way as [27] Lemma 4.3 just replacing $E_{s/2,\infty}(q)$, $B^{s}_{q,\infty}$ by $E_{s/2,1}$, $B^{s}_{q,1}$, respectively.

Finally, applying Theorem [1.1] and Corollary [1.2] we obtain the following result.
Proposition 4.3 Let \( \Omega \in \mathbb{R}^n, n \geq 2 \), be \( \mathbb{R}^n, \mathbb{R}^n_+, \) infinite layers, or bounded, exterior domains, or cylinders with uniform \( C^2 \)-boundaries. Moreover, let the Stokes operator \( \mathcal{A} \) be given as in (4.7). Then \( \mathcal{A} \) has maximal \( L^1 \)-regularity in \( B^{s,q}_{q,1}(\Omega), q \in (1, \infty), |s| < 2 \).

Remark 4.4 (1) If \( \Omega = \mathbb{R}^n \), it follows by [3, Remark 3.7 (a)] that for \( 1 < q < \infty \) and \( 0 < |s| < 2 \) there holds
\[
B^{s,q}_{q,1}(\mathbb{R}^n) = \{ u \in B^{s,q}_{q,1}(\mathbb{R}^n)^n : \text{div} \ u = 0 \}. \tag{4.11}
\]

(2) If \( \Omega \neq \mathbb{R}^n \) and \( s < 0 \), an explicit characterization of \( B^{s,q}_{q,1}(\Omega) \) is, in general, difficult to find, see [3]. If \( s \in (-2, -2 + 1/q) \), then \( (E_{s/2,1})^n \) is not a subspace of distributions on \( \Omega \).

For \( s \in (-2 + 1/q, 0) \) a characterization of \( B^{s,q}_{q,1}(\Omega) \) similar to (4.11) is possible provided there exists a Helmholtz decomposition of \( (E_{s/2,1})^n = B^{s,q}_{q,1}(\Omega)^n \) in the form
\[
B^{s,q}_{q,1}(\Omega)^n = M \oplus N \tag{4.12}
\]
with \( M := \{ u \in B^{s,q}_{q,1}(\Omega)^n : \text{div} \ u = 0 \} \) and \( N := \{ \nabla p \in (B^{s,q}_{q,1}(\Omega)^n : \exists p \in \mathcal{D}'(\Omega) \} \).

From the duality property (3.2) one infers that \( B^{s,q}_{q,1}(\Omega) \) is the dual of \( \mathbb{H}^{s,-q}_{q',\infty}(\Omega) \), where \( \mathbb{H}^{s,-q}_{q',\infty}(\Omega) \) is the completion of \( \mathcal{D}(\mathbb{A}_{q'}) \) in \( B^{s,-q}_{q',\infty}(\Omega)^n \). Note that \( \mathbb{H}^{s,-q}_{q',\infty}(\Omega) \subset B^{s,-q}_{q',\infty}(\Omega)^n \) and \( (\mathbb{B}^{s,-q}_{q',\infty}(\Omega), \mathbb{H}^{s,-q}_{q',\infty}(\Omega))' = B^{s,q}_{q,1}(\Omega) \), see §4.1. Then, by Hahn-Banach’s theorem,
\[
B^{s,q}_{q,1}(\Omega) \cong B^{s,q}_{q,1}(\Omega)^n/\tilde{N}, \ s \in (-2 + 1/q, 0), \tag{4.13}
\]
where “ \( \cong \) ” means isometric isomorphism, and
\[
\tilde{N} = \{ u \in B^{s,q}_{q,1}(\Omega)^n : \langle u, \varphi \rangle_{B^{s,q}_{q,1}, B^{s,-q}_{q',\infty}} = 0 \ \forall \varphi \in \mathbb{H}^{s,-q}_{q',\infty}(\Omega) \}.
\]

By de Rham’s lemma we have \( \tilde{N} = N \). Then, thanks to (4.12) and (4.13),
\[
B^{s,q}_{q,1}(\Omega) \cong M = \{ u \in B^{s,q}_{q,1}(\Omega)^n : \text{div} \ u = 0 \}.
\]

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