Gröbner-Shirshov bases for categories

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Abstract: In this paper we establish Composition-Diamond lemma for small categories. We give Gröbner-Shirshov bases for simplicial category and cyclic category.

Key words: Gröbner-Shirshov basis, simplicial category, cyclic category.

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1 Introduction

This paper devotes to Gröbner-Shirshov bases for small categories (all categories below are supposed to be small) presented by a graph (=quiver) and defining relations (see, MacLane [57]). As important examples, we use the simplicial and the cyclic categories (see, for example, MacLane [58], Gelfand, Manin [43]). In an above presentation, a category is viewed as a “monoid with several objects”. A free category $C(X)$, generated by a graph $X$, is just “free partial monoid of partial words” $u = x_{i_1} \ldots x_{i_n}, \ n \geq 0, \ x_{i_j} \in X$ and all product defined in $C(X)$. A relation is an expression $u = v, \ u,v \in C(X)$, where sources and targets of $u,v$ are coincident respectively. The same as for semigroups, we may use two equivalent languages: Gröbner-Shirshov bases language and rewriting systems language. Since we are using the former, we need a Composition-Diamond lemma (CD-lemma for short) for a free associative partial algebra $kC(X)$ over a field $k$, where $kC(X)$ is just a linear combination of uniform (with the same sources and targets) partial

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words. Then it is a routing matter to establish CD-lemma for $kC(X)$. It is a free category ("semigroup") partial algebra of a free category. Remark that in the literature one is usually used a language of rewriting system, see, for example, Malbos [53]. Let us stress that the partial associative algebras presented by graphs and defining relations are closely related to the well known quotients of the path algebras from representation theory of finitely dimensional algebras, see, for example, Assem, Simson, Skowroński [1]. In this respect Gröbner-Shirshov bases for categories are closely related to non-commutative Gröbner bases for quotients of path algebras, see Farkas, Feustel, Green [42]. Rewriting system language for non-commutative Gröbner bases of quotients of path algebras was used by Kobayashi [48]. Main new results of this paper are Gröbner-Shirshov bases for the simplicial and cyclic categories.

All algebras are assumed to be over a field.

2 A short survey on Gröbner-Shirshov bases

What is now called Gröbner and Gröbner-Shirshov bases theory was initiated by A. I. Shirshov (1921-1981) [66, 67], 1962 for non-associative and Lie algebras, by H. Hironaka [45, 46], 1964 for quotients of commutative infinite series algebras (both formal and convergent), and by B. Buchberger [32, 33], 1965, 1970 for commutative algebras.

English translation of selected works of A. I. Shirshov, including [66, 67], is recently published [68].

Remark that Shirshov’s approach was a most universal as we understand now since Lie algebra case becomes a model for many classes of non-commutative and non-associative algebras (with multiple operations), starting with associative algebras (see below). Hironaka’s papers on resolution of singularities of algebraic varieties become famous very soon and Hironaka got Fields Medal due to them few years latter. B. Buchberger’s thesis influenced very much many specialists in computer sciences, as well as in commutative algebras and algebraic geometry, for huge important applications of his bases, named him under his supervisor W. Gröbner (1898-1980).

Original Shirshov’s approach for Lie algebras [67], 1962, based on a notion of composition $[f, g]_w$ of two monic Lie polynomials $f, g$ relative to associative word $w$, i.e., $f, g$ are elements of a free Lie algebra $\text{Lie}(X)$ regarded as the subspace of Lie polynomials of the free associative algebra $k\langle X \rangle$, with $w \in X^*$, the free monoid generated by $X$. The definition of Lie composition relies on a definition of associative composition $(f, g)_w$ as (monic) associative polynomials (after worked out into $f, g$ all Lie brackets $[x, y] = xy - yx$) relative to degree-lexicographical order on $X^*$. Namely, $(f, g)_w = fb - ag$, where $w = acb$, $f = ac$, $g = cb$, $a, b, c \in X^*$, $c \neq 1$. Here $f$ means the leading (maximal) associative word of $f$. Then $(f, g)_w$ belongs to associative ideal $I_d(f, g)$ of $k\langle X \rangle$ generated by $f, g$, and the leading word of $(f, g)_w$ is less than $w$. Now we need to put some Lie brackets $[fb] - [ag]$ on $fb - ag$ in such a way that the result would belong to Lie ideal generated by $f, g$ (so we can not trouble bracketing into $f, g$) and the leading associative monomial of $[fb] - [ag]$ must be less than $w$. To overcome these obstacles Shirshov used his previous paper [64], 1958 with a new linear basis of free Lie algebra $\text{Lie}(X)$. As it happened the same linear basis of $\text{Lie}(X)$ was discovered in the paper Chen, Fox, Lyndon [34], 1958. Now this basis is called Lyndon-Shirshov basis, or, by a mistake,
Lyndon basis. It consists of non-associative Lyndon-Shirshov words (NLSW) \([u]\) in \(X\), that are in one-one correspondence with associative Lyndon-Shirshov words (ALSW) \(u\) in \(X\). The latter is defined as by a property \(u = vw > vw\) for any \(v, w \neq 1\). Shirshov [64], 1958 introduced and used the following properties of both associative and non-associative Lyndon-Shirshov words:

1) For any ALSW \(u\) there is a unique bracketing \([u]\) such that \([u]\) is a NLSW.

There are two algorithms for bracketing an ALSW. He mostly used “down-to-up algorithm” to rewrite an ALSW \(u\) on a new alphabet \(X_u = \{x_i(x_\beta)^j, \ i > \beta, \ j \geq 0, \ x_\beta\) is the minimal letter in \(u\}\); the result \(u'\) is again ALSW on \(X_u\) with the lex-order \(x_i > x_\beta x_\beta > (((x_i x_\beta) x_\beta) > . . . .)

It is Shirshov’s rewriting or elimination algorithm from his famous paper Shirshov [63], 1953, on what is now called Shirshov-Witt theorem (any subalgebra of a free Lie algebra is free). This rewriting was rediscovered by Lazard [49], 1960 and now called as Lazard elimination (it is better to call Lazard-Shirshov elimination).

There is “up-to-down algorithm” (see Shirshov [65], 1958, Chen, Fox, Lyndon [34], 1958): \([u] = [[[v]][w]]\), where \(w\) is the longest proper end of \(u\) that is ALSW, in this case \(v\) is also an ALSW.

2) Leading associative word of NLSW \([u]\) is just \(u\) (with the coefficient 1).

3) Leading associative word of any Lie polynomial is an associative Lyndon-Shirshov word.

4) A non-commutative polynomial \(f\) is a Lie polynomial if and only if

\[ f = f_0 \to \cdots \to f_i \to f_{i+1} \to \cdots \to f_n = 0, \]

where \(f_i \to f_{i+1} = f_i - \alpha_i[u_i], \ f_i = u_i\) is an ALSW, \(\alpha_i\) is the leading coefficient of \(f_i, \ i = 0, 1, \ldots \).

5) Any associative word \(c \neq 1\) is the unique product of (not strictly) increasing sequence of associative Lyndon-Shirshov words: \(c = c_1 c_2 \cdots c_n, \ c_1 \leq \cdots \leq c_n, \ c_i\) are ALSW’s.

6) If \(u = ab\), where \(u, v\) are ALSW, \(a, b \in X^*\), then there is a relative bracketing \([u]_v = [avb]\) of \(u\) relative to \(v\), such that the leading associative word of \([u]_v\) is just \(u\). Namely, \([u] = [avc][d], \ cd = b, \ [u]_v = [av[c][\cdots [c]\cdot n]]d, \ c = c_1 c_2 \cdots c_n\) as above.

7) If \(ac\) and \(cb\) are ASLW’s and \(c \neq 1\) then \(acb\) is an ALSW as well. If \(a, b\) are ALSW’s and \(a > b\), then \(ab\) is an ALSW as well.

Property 5) was known to Chen, Fox, Lyndon [34], 1958 as well. Lyndon [52], 1954, was actually the first for definition of associative “Lyndon-Shirshov” words. To the best of our knowledge, for many years, until PhD thesis by Viennot [69], 1978, no one mentioned the Lyndon’s discovery in 1954. On the other hand, there were dozens of papers and some books on Lie algebras that mentioned both associative and non-associative “Lyndon-Shirshov” words as Shirshov’s regular words, see, for example, P. M. Cohn [38], 1965, Bahturin [2], 1978.

Now a Lie composition \([f, g]_w\) of monic Lie polynomials \(f, g\) relative to a word \(w = fb = ag = acb, \ c \neq 1\) is defined by Shirshov [67], 1962, as follows

\[[f, g]_w = [fb]_f - [ag]_g,\]

where \([fb]_f\) means the result of substitution \(f\) for \([f]\) into the relative bracketing \([w]_ac\) of \(w\) with respect to \(ac\), the same for \([ag]_g\).
According to the definition and properties above, any Lie composition \([f, g]_w\) is an element of the Lie ideal generated by \(f, g\), and the leading associative word of the composition is less than \(w\).

The composition above is now called composition of intersection. Shirshov avoided what is now called composition of inclusion

\[
[f, g]_w = f - [a g b]_g, \quad w = \bar{f} = a \bar{g} b,
\]

assuming that any system \(S\) of Lie polynomials is reduced (irreducible) in a sense that leading associative word of any polynomial from \(S\) does not contain leading associative words of another polynomials from \(S\). This assumption relies on his algorithm of elimination of leading words for Lie polynomials below.

For associative polynomials the elimination algorithm is just non-commutative version of the Euclidean elimination algorithm. For Lie polynomial case Shirshov [67], 1962, defined the elimination of a leading word as follows:

If \(w = avb\), where \(w, v\) are ALSW’s, and \(v\) is the leading word of some monic Lie polynomial \(f\), then the transformation \([w] \mapsto [w] - [afb]_v\) is called an elimination of leading word of \(f\) into \([w]\). The result of Lie elimination is a Lie polynomial with a leading associative word less than \(w\).

Then Shirshov [67], 1962, formulated an algorithm to add to an initial reduced system of Lie polynomials \(S\) a “non-trivial” composition \([f, g]_w\), where \(f, g\) belong to \(S\). Non-triviality of a Lie polynomial \(h\) relative to \(S\) means that \(h\) is not going to zero using “elimination of leading words of \(S\)”. Actually, he defines to add to \(S\) not just a composition but rather the result of elimination of leading words of \(S\) into the composition in order to have a reduced system as well.

Then Shirshov proved the following

**Composition Lemma.** Let \(S\) be a reduced subset of \(\text{Lie}(X)\). If \(f\) belongs to the Lie ideal generated by \(S\), then the leading associative word \(\bar{f}\) contains, as a subword, some leading associative word of a reduced multi-composition of elements of \(S\).

He constantly used the following clear

**Corollary.** The set of all irreducible NLSW’s \([u]\) such that \(u\) does not contain any leading associative word of a reduced multi-composition of elements of \(S\) is a linear basis of the quotient algebra \(\text{Lie}(X)/Id(S)\).

Some later (see Bokut [8], 1972) the Shirshov Composition lemma was reformulated in the following form: Let \(S\) be a closed under composition set of monic Lie polynomials (it means that any composition \([f, g]_w\) of intersection and inclusion of elements of \(S\) is trivial, i.e., \([f, g]_w = \sum \alpha_i[a_i s_i b_i], \quad [a_i s_i b_i] = a_i \bar{s} b_i < w, \quad a_i, b_i \in X^*, \quad s_i \in S, \quad \alpha_i \in k\). If \(f \in Id(S)\), then \(\bar{f} = a \bar{s} b\) for some \(s \in S\). And \(S\)-irreducible NLSW’s is a linear basis of the quotient algebra \(\text{Lie}(X)/Id(S)\).

The modern form of Shirshov’s lemma is the following (see, for example, Bokut, Chen [12]).

**Shirshov’s Composition-Diamond Lemma for Lie algebras.** Let \(\text{Lie}(X)\) be a free Lie algebra over a field, \(S\) monic subset of \(\text{Lie}(X)\) relative to some monomial order on \(X^*\). Then the following conditions are equivalent:

1) \(S\) is a Gröbner-Shirshov basis (i.e., any composition of intersection and inclusion of elements of \(S\) is trivial).
2) If \( f \in Id(S) \), then \( \bar{f} = a\bar{s}b \) for some \( s \in S \), \( a, b \in X^* \).

3) \( Irr(S) = \{ [u][u] \text{ is an NLSW and } u \text{ does not contain any } \bar{s}, s \in S \} \) is a linear basis of the Lie algebra \( Lie(X|S) \) with defining relations \( S \).

The proof of the Shirshov’s Composition-Diamond lemma for Lie algebras becomes a model for proofs of number of Composition-Diamond lemmas for many classes of algebras. An idea of his proof is to rewrite any element of Lie ideal, generated by \( S \) in a form

\[
\sum \alpha_i[a_is_ib_i],
\]

where each \( s_i \in S \), \( a_i, b_i \in X^* \), \( \alpha_i \in k \) such that

i) leading words of each \( [a_is_ib_i] \) is equal to \( a_i\bar{s}b_i \) (in this case an expression \( [asb] \) is called normal Lie \( S \)-word in \( X \)) and

ii) \( a_1\bar{s}_1b_1 > a_2\bar{s}_2b_2 > \ldots \).

Now let \( F(X) \) be a free algebra of a variety (or category) of algebras. Following the idea of Shirshov’s proof, one needs

1) to define appropriate linear basis (normal words) of \( F(X) \),

2) to define monomial order of normal words,

3) to define compositions of element of \( S \) (they may be compositions of intersection, inclusion and left (right) multiplication, or may be else),

4) prove two key lemmas:

Key Lemma 1. Let \( S \) be a Gröbner-Shirshov basis (any composition of polynomials from \( S \) is trivial). Then any \( S \)-word is a linear combination of normal \( S \)-words.

Key Lemma 2. Let \( S \) be a Gröbner-Shirshov basis, \( [a_1s_1b_1] \) and \( [a_2s_2b_2] \) normal \( S \)-words, \( s_1, s_2 \in S \). If \( a_1\bar{s}_1b_1 = a_2\bar{s}_2b_2 \), then \( [a_1s_1b_1] - [a_2s_2b_2] \) is going to zero by elimination of leading words of \( S \) (elimination means composition of inclusion). There are number of CD-lemmas that realized Shirshov’s approach to them.

Shirshov [67], 1962, assumed implicitly that his approach, based on the definition of composition of any (not necessary Lie) polynomials, is equally valid for associative algebras as well (the first author is a witness that Shirshov understood it very clearly and explicitly; only lack of non-trivial applications prevents him from publication this approach for associative algebras). Explicitly it was done by Bokut [9] and Bergman [5].

CD-lemma for associative algebras is formulated and proved in the same way as for Lie algebras.

**Composition-Diamond Lemma for associative algebras.** Let \( k\langle X \rangle \) be a free associative algebra over a field \( k \) and a set \( X \). Let us fix some monomial order on \( X^* \). Then the following conditions are equivalent for any monic subset \( S \) of \( k\langle X \rangle \):

1) \( S \) is a Gröbner-Shirshov basis (that is any composition of intersection and inclusion is trivial).

2) If \( f \in Id(S) \), then \( \bar{f} = a\bar{s}b \) for some \( s \in S \), \( a, b \in X^* \).

3) \( Irr(S) = \{ u \in X^* | u \neq a\bar{s}b, s \in S, a, b \in X^* \} \) is a linear basis of the factor algebra \( k\langle X \rangle / Id(X) \).

There are a lot of applications of Shirshov’s CD-lemmas for Lie and associative algebras. Let us mention some connected to the Malcev embedding problem for semigroup algebras.
(Bokut [6, 7], 1969, there is a semigroup $S$ such that the multiplication semigroup of the semigroup algebra $k(S)$, where $k$ is a field, is embeddable into a group, but $k(S)$ is not embeddable into any division algebra), the unsolvability of the word problem for Lie algebras (Bokut [8]), Gröbner-Shirshov bases for semisimple Lie algebras (Bokut, Klein [25, 26, 27, 28]), Kac-Moody algebras (Poroshenko [59, 60, 61]), finite Coxeter groups (Bokut, Shiao [30]), braid groups in different set of generators (Bokut, Chainikov, Shum [23], Bokut [10], Bokut [11]), quantum algebra of type $A_n$ (Bokut, Malcolmson [29]), Chinese monoids (Chen, Qiu [35]).

There are applications of Shirshov’s CD-lemma [66], 1962 for free anti-commutative non-associative algebras: there are two anti-commutative Gröbner-Shirshov bases of a free Lie algebra, one gives the Hall basis (Bokut, Chen, Li [17]), another the Lyndon-Shirshov basis (Bokut, Chen, Li [18]).

Bokut, Chen, Mo [20] proved and reproved some embedding theorems for associative algebras, Lie algebras, groups, semigroups, differential algebras, using Shirshov’s CD-lemmas for associative and Lie algebras.

Bahturin, Olshanskii [3] found embeddings without distortion of associative algebras and Lie algebras into 2-generated simple algebras. They also used Shirshov’s CD-lemmas for associative and Lie algebras.

Mikhalev [54] used Shirshov’s approach and CD-lemma for associative algebras to prove CD-lemma for colored Lie super-algebras.

Mikhalev, Zolotykh [56] proved CD-lemma for free associative algebra over a commutative algebra. A free object in this category is $k[Y] \otimes k(X)$, tensor product of a polynomial algebra and a free associative algebra. Here one needs to use several compositions of intersection and inclusion.

Bokut, Fong, Ke [24] proved CD-lemma for free associative conformal (in a sense of V. Kac [47]) algebra $C(X,\langle n \rangle, n = 0, 1, \ldots, D, N(a, b), a, b \in X)$ of a fixed locality $N(a, b)$. Any normal basis of free associative conformal algebra was constructed by M. Roitman [62]. A linear basis of free associative conformal algebra was constructed by M. Roitman [62].

A more general case, the associative $H$-conformal algebra (or $H$-pseudo-algebra in a sense of Bakalov, D’Andrea, Kac [4]), where $H$ is any Hopf algebra, is still open.

Mikhalev, Vasilieva [55] proved CD-lemma for free supercommutative polynomial algebra.
bras. Here they use compositions of multiplication as well.

Bokut, Chen, Li [16] proved CD-lemma for free pre-Lie algebras (also known as Vinberg-Koszul-Gerstenhaber right-symmetric algebras).

Bokut, Chen, Liu [19] proved CD-lemma for free dialgebras in a sense of Loday [50]. Here conditions 1) and 2) are not equivalent but from 1) follows 2).

The cases of associative conformal algebras and dialgebras show that definition of Gröbner-Shirshov bases by condition 1) is in general preferable than the one using 2).

Bokut, Shum [31] proved CD-lemma for free Γ-associative algebras, where Γ is a group. It has applications to the Malcev problem above and to Bruhat normal forms for algebraic groups.

Eisenbud, Peeva, Sturmfels [41] found non-commutative Gröbner basis of any commutative algebra (extending any commutative Gröbner basis to a non-commutative one).

Bokut, Chen, Chen [14] proved CD-lemma for Lie algebras over commutative algebras. Here one needs to establish Key Lemma 1 in a more strong form – any Lie S-word is a linear combination of S-words of the form \([a s b]_2\) in the sense of Shirshov’s special Lie bracketing. As an application they proved Cohn’s conjecture [37] for the case of characteristics 2, 3 and 5 (that some Cohn’s examples of Lie algebras over commutative algebras are not embeddable into associative algebras over the same commutative algebras).

Bokut, Chen, Deng [15] proved CD-lemma for free associative Rota-Baxter algebras. As an application, Chen and Mo [36] proved that any dendriform algebra is embeddable into universal enveloping Rota-Baxter algebra. It was Li Guo’s conjecture, [44].

Bokut, Chen, Chen [13] proved CD-lemma for tensor product of two free associative algebras. As an application they extended any Mikhailov-Zolotykh commutative-non-commutative Gröbner-Shirshov basis laying into tensor product \(k[Y] \otimes k\langle X \rangle\) to non-commutative-non-commutative Gröbner-Shirshov basis laying into \(k\langle Y \rangle \otimes k\langle X \rangle\) (a la Eisenbud-Peeva-Sturmfels above). They also gave another proof of the Eisenbud-Peeva-Sturmfels theorem above.

As we mentioned in introduction, Farkas, Feustel, Green [42] proved CD-lemma for path algebras.

Drensky, Holtkamp [40] proved CD-lemma for nonassociative algebras with multiple linear operators.

Bokut, Chen, Qiu [21] proved CD-lemma for associative algebras with multiple linear operators.

Dotsenko, Khoroshkin [39] proved CD-lemma for operads.

3 Free categories and free category partial algebras

Let \(X = (V(X), E(X))\) be an oriented (multi) graph. Then the free category on \(X\) is \(C(X) = (Ob(X), Arr(X))\), where \(Ob(X) = V(X)\), and \(Arr(X)\) is the set of all paths (“words”) of \(X\) including the empty paths \(1_v, v \in V(X)\). It is easy to check \(C(X)\) has the following universal property. Let \(\mathcal{C}\) be a category and \(\Gamma_{\mathcal{C}}\) the graph relative to \(\mathcal{C}\) i.e., \(V(\Gamma_{\mathcal{C}}) = Ob(\mathcal{C})\) and \(E(\Gamma_{\mathcal{C}}) = mor(\mathcal{C})\). Let \(e : X \rightarrow C(X)\) be a monograph morphism of the graph \(X\) to the graph \(\Gamma_{C(X)}\), where \(e = (e_1, e_2)\), and \(e_1\) is a mapping on \(V(X), e_2\) on \(E(X)\), both \(e_1\) and \(e_2\) are mono. For any graph morphism
Let $C$ be a category and $k$ a field. Let

$$kC = \{ f = \sum_{i=1}^{n} \alpha_i \mu_i | \alpha_i \in k, \mu_i \in \text{mor}(C), n \geq 0, \mu_i (0 \leq i \leq n) \text{ have the same domains and the same codomains} \}.$$ 

Note that in $kC$, for $f, g \in kC$, $f + g$ is defined only if $f, g$ have the same domain and the same codomain.

A multiplication $\bullet$ in $kC$ is defined by linearly extending the usual compositions of morphisms of the category $C$. Then $(kC, \bullet)$ is called the category partial algebra over $k$ relative to $C$ and $kC(X)$ the free category partial algebra generated by the graph $X$.

### 4 Composition-Diamond lemma for categories

Let $X$ be a oriented (multi) graph, $C(X)$ the free category generated by $X$ and $kC(X)$ the free category partial algebra. Since we only consider the morphisms of the free category $C(X)$, we write $C(X)$ just for $\text{Arr}(X)$.

Note that for $f, g \in kC(X)$ if we write $gf$, it means $gf$ is defined.

A well ordering $>$ on $C(X)$ is called monomial if it satisfies the following conditions: $u > v \Rightarrow uw > vw$ and $wu > vw$, for any $u, v, w \in C(X)$. In fact, there are many monomial orders on $C(X)$. For example, let $E(X)$ be a well ordered set. Then the deg-lex order $>$ on $C(X)$ is defined by the following way: for any words $u = x_1 \cdots x_m, v = y_1 \cdots y_n \in C(X)$, $m = |u|, n = |v|$, $u > v \iff |u| > |v|$ or $|u| = |v|$ and $x_1 = y_1, x_2 = y_2, \ldots, x_t = y_t, x_{t+1} > y_{t+1}$ for some $0 \leq t < n$.

It is easy to check that $>$ is a monomial order on $C(X)$. In the following sections, we will see other monomial orders. Now, we suppose that $>$ is a fixed monomial order on $C(X)$. Given a nonzero polynomial $f \in kC(X)$, it has a word $\bar{f} \in C(X)$ such that $f = \alpha \bar{f} + \sum \alpha_i u_i$, where $\bar{f} > u_i, 0 \neq \alpha, \alpha_i \in k, u_i \in C(X)$. We call $\bar{f}$ the leading term of $f$ and $f$ is monic if $\alpha = 1$.

Let $S \subset kC(X)$ be a set of monic polynomials, $s \in S$ and $u \in C(X)$. We define $S$-word $u_s$ by induction:

(i) $u_s = s$ is an $S$-word of $s$-length 1.
(ii) Suppose that \( u_s \) is an \( S \)-word of \( s \)-length \( m \) and \( v \) is a word of length \( n \), i.e., the number of edges in \( v \) is \( n \). Then \( u_s v \) and \( vu_s \) are \( S \)-words of \( s \) length \( m + n \).

Note that for any \( S \)-word \( u_s = asb \), where \( a, b \in C(X) \), we have \( asb = a\overline{sb} \).
Let \( f, g \) be monic polynomials in \( kC(X) \). Suppose that there exist \( w, a, b \in C(X) \) such that \( w = \overline{f} = agb \). Then we define the composition of inclusion

\[
(f, g)_w = f - agb.
\]

For the case that \( w = \overline{fb} = a\overline{gb} \), \( w, a, b \in C(X) \), the composition of intersection is defined as follows:

\[
(f, g)_w = fb - ag.
\]

It is clear that

\[
(f, g)_w \in Id(f, g) \quad \text{and} \quad (f, g)_w < w,
\]
where \( Id(f, g) \) is the ideal of \( kC(X) \) generated by \( f, g \).

The composition \( (f, g)_w \) is trivial modulo \((S, w)\), if

\[
(f, g)_w = \sum_i \alpha_i a_i s_i b_i
\]
where each \( \alpha_i \in k \), \( a_i, b_i \in C(X) \), \( s_i \in S \), \( a_i s_i b_i \) an \( S \)-word and \( a_i\overline{s_i}b_i < \overline{f} \). If this is the case, then we write \( (f, g)_w \equiv 0 \ mod(S, w) \). In general, for \( p, q \in kC(X) \), we write

\[
p \equiv q \ mod(S, w)
\]
which means that \( p - q = \sum \alpha_i a_i s_i b_i \), where each \( \alpha_i \in k \), \( a_i, b_i \in C(X) \), \( s_i \in S \), \( a_i s_i b_i \) an \( S \)-word and \( a_i\overline{s_i}b_i < w \).

**Definition 4.1** Let \( S \subset kC(X) \) be a nonempty set of monic polynomials. Then \( S \) is called a Gröbner-Shirshov basis in \( kC(X) \) if any composition \( (f, g)_w \) with \( f, g \in S \) is trivial modulo \((S, w)\), i.e., \( (f, g)_w \equiv 0 \ mod(S, w) \).

**Lemma 4.2** Let \( a_1s_1b_1 \), \( a_2s_2b_2 \) be monic \( S \)-words. If \( S \) is a Gröbner-Shirshov basis in \( kC(X) \) and \( w = a_1\overline{s_1}b_1 = a_2\overline{s_2}b_2 \), then

\[
a_1s_1b_1 \equiv a_2s_2b_2 \ mod(S, w).
\]

**Proof.** There are three cases to consider.

Case 1. Suppose that subwords \( \overline{s}_1 \) and \( \overline{s}_2 \) of \( w \) are disjoint, say, \( |a_2| \geq |a_1| + |\overline{s}_1| \). Then, we can assume that \( a_2 = a_1\overline{s}_1c \) and \( b_1 = cs_2b_2 \) for some \( c \in C(X) \), and so, \( w = a_1\overline{s}_1c\overline{s}_2b_2 \). Now,

\[
a_1s_1b_1 - a_2s_2b_2 = a_1s_1c\overline{s}_2b_2 - a_1\overline{s}_1cs_2b_2 = a_1s_1c(\overline{s}_2 - s_2)b_2 + a_1(s_1 - \overline{s}_1)cs_2b_2.
\]

Since \( \overline{s}_2 - s_2 < \overline{s}_2 \) and \( \overline{s}_1 - \overline{s}_1 < \overline{s}_1 \), we conclude that

\[
a_1s_1b_1 - a_2s_2b_2 = \sum \alpha_i a_is_1v_i + \sum \beta_j a_js_2v_j
\]

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for some \( \alpha, \beta_j \in k \), \( S \)-words \( u_s v_i \) and \( u_j v_j \) such that \( u_s v_i, u_j v_j < w \).

This shows that \( a_1 s_1 b_1 \equiv a_2 s_2 b_2 \mod (S,w) \).

Case 2. Suppose that the subword \( s_1 \) of \( w \) contains \( s_2 \) as a subword. We may assume that \( s_1 = a s_2 b \), \( a = a_1 a \) and \( b = b b_1 \), that is, \( w = a_1 a s_2 bb_1 \) for some \( S \)-word \( a s_2 b \). We have

\[
\begin{align*}
a_1 s_1 b_1 - a_2 s_2 b_2 &= a_1 s_1 b_1 - a_1 a s_2 bb_1 \\
&= a_1 s_1 - a s_2 b b_1 \\
&= a_1 (s_1, s_2)_{\pi_1} b_1 \\
&\equiv 0 \mod (S,w)
\end{align*}
\]

since \( S \) is a Gröbner-Shirshov basis.

Case 3. \( s_1 \) and \( s_2 \) have a nonempty intersection as a subword of \( w \). We may assume that \( a_2 = a_1 a \), \( b_1 = b b_2 \), \( w_1 = s_1 b = a s_2 \). Then, we have

\[
\begin{align*}
a_1 s_1 b_1 - a_2 s_2 b_2 &= a_1 s_1 b b_2 - a_1 a s_2 b_2 \\
&= a_1 (s_1 b - a s_2) b_2 \\
&= a_1 (s_1, s_2)_{\pi_1} b_2 \\
&\equiv 0 \mod (S,w)
\end{align*}
\]

This completes the proof. \( \square \)

**Lemma 4.3** Let \( S \subset kC(X) \) be a subset of monic polynomials and \( \text{Irr}(S) = \{ u \in C(X) | u \neq a s b, a, b \in C(X), s \in S \} \). Then for any \( f \in kC(X) \),

\[
f = \sum_{u_i \leq f} \alpha_i u_i + \sum_{a_j s_j b_j \leq f} \beta_j a_j s_j b_j
\]

where each \( \alpha, \beta_j \in k \), \( u_i \in \text{Irr}(S) \) and \( a_j s_j b_j \) an \( S \)-word.

**Proof.** Let \( f = \sum_{i} \alpha_i u_i \in kC(X) \), where \( 0 \neq \alpha_i \in k \) and \( u_1 > u_2 > \cdots \). If \( u_i \in \text{Irr}(S) \), then let \( f_1 = f - \alpha_1 u_1 \). If \( u_i \not\in \text{Irr}(S) \), then there exist some \( s \in S \) and \( a_1, b_1 \in C(X) \), such that \( f = u_1 = a_1 s_1 b_1 \). Let \( f_1 = f - \alpha_1 a_1 s_1 b_1 \). In both cases, we have \( f_1 < f \). Then the result follows from the induction on \( f \). \( \square \)

**Theorem 4.4** (Composition-Diamond lemma for categories) Let \( S \subset kC(X) \) be a nonempty set of monic polynomials and \( < \) a monomial order on \( C(X) \). Let \( \text{Id}(S) \) be the ideal of \( kC(X) \) generated by \( S \). Then the following statements are equivalent:

(i) \( S \) is a Gröbner-Shirshov basis in \( kC(X) \).

(ii) \( f \in \text{Id}(S) \Rightarrow \tilde{f} = a s b \) for some \( s \in S \) and \( a, b \in C(X) \).

(\( \text{ii}' \)) \( f \in \text{Id}(S) \Rightarrow f = \alpha_1 a_1 s_1 b_1 + \alpha_2 a_2 s_2 b_2 + \cdots \), where each \( \alpha_i \in k \), \( a_is_i b_i \) is an \( S \)-word and \( a_1 s_1 b_1 > a_2 s_2 b_2 > \cdots \).
(iii) \( \text{Irr}(S) = \{ u \in C(X) | u \neq a\overline{s}b, a, b \in C(X), \ s \in S \} \) is a linear basis of the partial algebra \( kC(X)/\text{Id}(S) = kC(X|S) \).

**Proof.** (i) \( \Rightarrow \) (ii). Let \( S \) be a Gröbner-Shirshov basis and \( 0 \neq f \in \text{Id}(S) \). Then, we have

\[
f = \sum_{i=1}^{n} \alpha_{i}a_{i}s_{i}b_{i},
\]

where each \( \alpha_{i} \in k, \ a_{i}, b_{i} \in C(X), \ s_{i} \in S \) and \( a_{i}s_{i}b_{i} \) an \( S \)-word. Let

\[
w_{i} = a_{i}\overline{s_{i}}b_{i}, \ w_{1} = w_{2} = \cdots = w_{l} > w_{l+1} \geq \cdots, \ l \geq 1.
\]

We will use the induction on \( l \) and \( w_{1} \) to prove that \( f = a\overline{s}b \) for some \( s \in S \) and \( a, b \in C(X) \). If \( l = 1 \), then \( f = a_{1}\overline{s_{1}}b_{1} = a_{1}s_{1}b_{1} \) and hence the result holds. Assume that \( l \geq 2 \). Then, by Lemma 4.2, we have

\[
a_{1}s_{1}b_{1} \equiv a_{2}s_{2}b_{2} \mod(S, w_{1}).
\]

Thus, if \( \alpha_{1} + \alpha_{2} \neq 0 \) or \( l > 2 \), then the result holds by induction on \( l \). For the case \( \alpha_{1} + \alpha_{2} = 0 \) and \( l = 2 \), we use the induction on \( w_{1} \). Now, the result follows.

(ii) \( \Rightarrow \) (ii)'. Assume (ii) and \( 0 \neq f \in \text{Id}(S) \). Let \( f = \alpha_{1}\overline{f} + \cdots \). Then, by (ii), \( \overline{f} = a_{1}\overline{s_{1}}b_{1} \). Therefore,

\[
f_{1} = f - \alpha_{1}a_{1}s_{1}b_{1}, \ \overline{f_{1}} < \overline{f}, \ f_{1} \in \text{Id}(S).
\]

Now, by using induction on \( \overline{f} \), we have (ii)'.

(ii)' \( \Rightarrow \) (ii). This part is clear.

(ii) \( \Rightarrow \) (iii). Suppose that \( \sum_{i} \alpha_{i}u_{i} = 0 \) in \( kC(X|S) \), where \( \alpha_{i} \in k, \ u_{i} \in \text{Irr}(S) \). It means that \( \sum_{i} \alpha_{i}u_{i} \in \text{Id}(S) \) in \( kC(X) \). Then all \( \alpha_{i} \) must be equal to zero. Otherwise,

\[
\sum_{i} \alpha_{i}u_{i} = u_{j} \in \text{Irr}(S) \text{ for some } j \text{ which contradicts (ii)}.
\]

Now, by Lemma 4.3, (iii) follows.

(iii) \( \Rightarrow \) (i). For any \( f, g \in S \), by Lemma 4.3 and (iii), we have \( (f, g)_{w} \equiv 0 \mod(S, w) \). Therefore, \( S \) is a Gröbner-Shirshov basis.

**Remark.** If the category in Theorem 4.4 has only one object, then Theorem 4.4 is exact Composition-Diamond lemma for free associative algebras.

## 5 Gröbner-Shirshov bases for the simplicial category and the cyclic category

In this section, we give Gröbner-Shirshov bases for the simplicial category and the cyclic category respectively.
For each non-negative integer $p$, let $[p]$ denote the set $\{0, 1, 2, \ldots, p\}$ of integers in their usual order. A (weakly) monotonic map $\mu : [q] \to [p]$ is a function on $[q]$ to $[p]$ such that $i \leq j$ implies $\mu(i) \leq \mu(j)$. The objects $[p]$ with morphisms all weakly monotonic maps $\mu$ constitute a category $\mathcal{L}$ called simplicial category. It is convenient to use two special families of monotonic maps

$$
\varepsilon^i_q : [q-1] \to [q], \quad \eta^i_q : [q+1] \to [q]
$$

defined for $i = 0, 1, \ldots, q$ (and for $q > 0$ in the case of $\varepsilon^i$) by

$$
\varepsilon^i_q(j) = \begin{cases} 
  j, & \text{if } i > j, \\
  j+1, & \text{if } i \leq j,
\end{cases}
$$

$$
\eta^i_q(j) = \begin{cases} 
  j, & \text{if } i \geq j, \\
  j-1, & \text{if } j > i.
\end{cases}
$$

Let $X = (V(X), E(X))$ be an oriented (multi) graph, where $V(X) = \{|p| \in \mathbb{Z}^+ \cup \{0\}\}$ and $E(X) = \{\varepsilon^i_q : [p-1] \to [p], \eta^i_q : [q+1] \to [q] \mid p > 0, 0 \leq i \leq p, 0 \leq j \leq q\}$. Let $S \subseteq C(X) \times C(X)$ be the relation set consisting of the following:

$$
f_{q+1,i} : \varepsilon^i_q \varepsilon^j_q = \varepsilon^j_q \varepsilon^i_q, \quad j > i,
$$

$$
g_{q+1,i} : \eta^i_q \eta^j_q = \eta^j_q \eta^i_q, \quad j \geq i,
$$

$$
h_{q-1,i} : \eta^j_q \varepsilon^i_q = \begin{cases} 
  \varepsilon^j_q \varepsilon^i_q, & j \geq i, \\
  1_{q-1} \varepsilon^i_q, & i = j, \quad i = j+1, \\
  \varepsilon^i_q \eta^j_q, & i > j+1.
\end{cases}
$$

Then the simplicial category $\mathcal{L}$ is just the category $C(X|S)$ generated by $X$ with defining relation $S$, see MacLane [58], Theorem VIII. 5.2. We will give another proof in what follows.

We order $C(X)$ by the following way.

Firstly, for any $\eta^i_p, \eta^j_q \in \{\eta^i_p|p \geq 0, 0 \leq i \leq p\}$, $\eta^i_p > \eta^j_q$ iff $p > q$ or $(p = q$ and $i < j)$.

Secondly, for each $u = \eta^i_{p_1} \eta^i_{p_2} \cdots \eta^i_{p_n} \in \{\eta^i_p|p \geq 0, 0 \leq i \leq p\}^\ast$ (all possible words on $\{\eta^i_p|p \geq 0, 0 \leq i \leq p\}$, including the empty word $1_v$, $v \in \text{Ob}(X)$), let $wt(u) = (n, \eta^i_{p_n}, \eta^i_{p_{n-1}}, \cdots, \eta^i_{p_1})$. Then for any $u, v \in \{\eta^i_p|p \geq 0, 0 \leq i \leq p\}^\ast$, $u > v$ iff $wt(u) > wt(v)$ lexicographically.

Thirdly, for any $\varepsilon^i_p, \varepsilon^j_q \in \{\varepsilon^i_p|p \in \mathbb{Z}^+, 0 \leq i \leq p\}$, $\varepsilon^i_p > \varepsilon^j_q$ iff $p > q$ or $(p = q$ and $i < j)$.

Finally, for each $u = v_0 \varepsilon^i_{p_1} v_1 \varepsilon^i_{p_2} \cdots \varepsilon^i_{p_n} v_n \in C(X)$, $n \geq 0$, $v_j \in \{\eta^i_p|p \geq 0, 0 \leq i \leq p\}^\ast$, let $wt(u) = (n, v_0, v_1, \cdots, v_n, \varepsilon^i_{p_1}, \cdots, \varepsilon^i_{p_n})$. Then for any $u, v \in C(X)$,

$$
u >_1 v \iff wt(u) > wt(v) \text{ lexicographically}.
$$

It is easy to check that the $>_1$ is a monomial order on $C(X)$. Then we have the following theorem.

**Theorem 5.1** Let $X, S$ be defined as the above, the generating set and the relation set of the quotient category $C(X|S)$ respectively. Then with the order $>_1$ on $C(X)$, $S$ is a Gröbner-Shirshov basis for the category partial algebra $kC(X|S)$. 

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Proof. According to the order $\pi$, $\tilde{f}_{q+1,i} = \varepsilon_{q+1}^{i} \varepsilon_{q}^{j-1}$, $\tilde{g}_{q+1,j} = \eta_{q+1}^{j}$ and $\tilde{h}_{q+1,j} = \eta_{q}^{j} \varepsilon_{q}^{i}$. So, all the possible compositions of $S$ are the following:

(a) $(f_{q+2,q+1}, f_{q+1,q}) \eta_{q+1}^{j-1} \eta_{q+1}^{j-1}, \ k \leq i \leq j - 1$;

(b) $(g_{q-1,q}, g_{q+1,q}) \eta_{q-1}^{j-1} \eta_{q+1}^{j-1}, \ i \leq j \leq k$;

(c) $(h_{q+1,q}, f_{q+1,q}) \eta_{q}^{j-1} \eta_{q}^{j-1}, \ i \leq j - 1$;

(d) $(g_{q-2,q-1}, h_{q-1,q}) \eta_{q-2}^{j-1} \eta_{q-1}^{j-1}, \ j \leq k$.

We will prove that all possible compositions are trivial. Here, we only give the proof of the (b). For others cases, the proofs are similar.

Let us consider the following subcases of the case (b): (I) $i < j < k$; (II) $i < j, j = k, or j = k + 1$; (III) $i < k, k + 1 < j$; (IV) $j > k + 1, i = k, k + 1$; (V) $j > i > k + 1$.

For subcase (I),

$$(h_{q+1,q}, f_{q+1,q}) \eta_{q}^{j-1} \eta_{q}^{j-1} = \varepsilon_{q}^{j-1} \varepsilon_{q}^{j-1} - \eta_{q}^{j-1} \varepsilon_{q}^{j-1} \varepsilon_{q}^{j-1}$$

\[= \varepsilon_{q}^{j-1} \varepsilon_{q}^{j-1} - \eta_{q}^{j-1} \varepsilon_{q}^{j-1} \varepsilon_{q}^{j-1} \equiv 0 \mod(S, \eta_{q}^{j-1} \varepsilon_{q}^{j-1} \varepsilon_{q}^{j-1}).\]

For subcase (II),

$$(h_{q+1,q}, f_{q+1,q}) \eta_{q}^{j-1} \eta_{q}^{j-1} = \varepsilon_{q}^{j-1} \varepsilon_{q}^{j-1} \eta_{q}^{j-1} \varepsilon_{q}^{j-1} \eta_{q}^{j-1} \varepsilon_{q}^{j-1}$$

\[= \varepsilon_{q}^{j-1} \varepsilon_{q}^{j-1} \eta_{q}^{j-1} \varepsilon_{q}^{j-1} \eta_{q}^{j-1} \varepsilon_{q}^{j-1} \equiv 0 \mod(S, \eta_{q}^{j-1} \varepsilon_{q}^{j-1} \varepsilon_{q}^{j-1}).\]

For subcase (III),

$$(h_{q+1,q}, f_{q+1,q}) \eta_{q}^{j-1} \eta_{q}^{j-1} = \varepsilon_{q}^{j-1} \varepsilon_{q}^{j-1} \eta_{q}^{j-1} \varepsilon_{q}^{j-1} \eta_{q}^{j-1} \varepsilon_{q}^{j-1}$$

\[= \varepsilon_{q}^{j-1} \varepsilon_{q}^{j-1} \eta_{q}^{j-1} \varepsilon_{q}^{j-1} \eta_{q}^{j-1} \varepsilon_{q}^{j-1} \equiv 0 \mod(S, \eta_{q}^{j-1} \varepsilon_{q}^{j-1} \varepsilon_{q}^{j-1}).\]

For subcase (IV),

$$(h_{q+1,q}, f_{q+1,q}) \eta_{q}^{j-1} \eta_{q}^{j-1} = \varepsilon_{q}^{j-1} - \eta_{q}^{j-1} \varepsilon_{q}^{j-1} \varepsilon_{q}^{j-1}$$

\[= \varepsilon_{q}^{j-1} - \varepsilon_{q}^{j-1} \eta_{q-1} \varepsilon_{q}^{j-1} \varepsilon_{q}^{j-1} \equiv 0 \mod(S, \eta_{q}^{j-1} \varepsilon_{q}^{j-1} \varepsilon_{q}^{j-1}).\]

For subcase (V),

$$(h_{q+1,q}, f_{q+1,q}) \eta_{q}^{j-1} \eta_{q}^{j-1} = \varepsilon_{q}^{j-1} \varepsilon_{q}^{j-1} \eta_{q-1} \varepsilon_{q}^{j-1} \eta_{q-1} \varepsilon_{q}^{j-1}$$

\[= \varepsilon_{q}^{j-1} \varepsilon_{q}^{j-1} \eta_{q-1} \varepsilon_{q}^{j-1} \eta_{q-1} \varepsilon_{q}^{j-1} \equiv 0 \mod(S, \eta_{q}^{j-1} \varepsilon_{q}^{j-1} \varepsilon_{q}^{j-1}).\]
Therefore $S$ is a Gröbner-Shirshov basis of the category partial algebra $kC(X|S)$. □

By Theorem 4.4, $Irr(S) = \{\epsilon^i_{p} \cdots \epsilon^m_{p-m+1} p \geq i_1 > \ldots > i_m \geq 0, \ 0 \leq j_1 < \ldots < j_n < q, \text{ and } q-n+m=p\}$ is a linear basis of the category partial algebra $kC(X|S)$. Therefore, we have the following corollaries.

**Corollary 5.2** (Maclane [58], Lemma VIII. 5.1) In the category $C(X|S)$, each morphism $\mu : [q] \to [p]$ can be uniquely represented as

$$\epsilon^{i_1}_{p} \cdots \epsilon^{i_m}_{p-m+1} \eta^{j_1}_{q} \cdots \eta^{j_n}_{q-1},$$

where $p \geq i_1 > \ldots > i_m \geq 0, \ 0 \leq j_1 < \ldots < j_n < q, \text{ and } q-n+m=p$.

**Corollary 5.3** (Maclane [58], Theorem VIII. 5.2) $\mathcal{L} = C(X|S)$.

The cyclic category is defined by generators and defining relations as follows, see [43]. Let $Y = (V(Y), E(Y))$ be an oriented (multi) graph, where $V(Y) = \{p \mid p \in Z^\ast \cup \{0\}\}$, and $E(Y) = \{\epsilon^i_p : [p-1] \to [p], \eta^j_p : [q+1] \to [q], t_q : [q] \to [q] \mid p > 0, 0 \leq i \leq p, 0 \leq j \leq q\}$. Let $S \subseteq C(Y) \times C(Y)$ be the set consisting of the following relations:

- $f_{q+1,i} : \epsilon^{i}_{q+1} \epsilon^{j}_{q+1} = \epsilon^{j}_{q+1} \epsilon^{i}_{q}, \ j > i$,
- $g_{q+1,i} : \eta^{j}_{q+1} \eta^{j_{1}}_{q+1} = \eta^{j_{1}}_{q+1} \eta^{j}_{q+1}, \ j > i$,
- $h_{q-1,i} : \eta^{j}_{q-1} \eta^{i}_{q} = \begin{cases} \epsilon^{i-1}_{q-2} \epsilon^{j}_{q-2}, & j > i, \\ \epsilon^{i-1}_{q-2}, & i = j, \ i = j + 1, \\ \epsilon^{i-1}_{q-2}, & i > j + 1, \end{cases}$
- $\rho_1 : t_q \epsilon^{i}_{q} = \epsilon^{i-1}_{q} t_{q-1}, \ i = 1, \ldots, q$,
- $\rho_2 : t_q \eta^{i}_{q} = \eta^{i-1}_{q} t_{q+1}, \ i = 1, \ldots, q$,
- $\rho_3 : \eta^{i+1}_{q} = 1_q$.

The category $C(Y|S)$ is called cyclic category, denoted by $\Lambda$.

An order on $C(Y)$ is defined by the following way.

Firstly, for any $t_q^i, t_q^j \in \{t_q|q \geq 0\}^*$, $(t_q^i)^i > (t_q^j)^j$ if $i > j$ or $(i = j$ and $p > q)$.

Secondly, for any $\eta^i_p, \eta^j_q \in \{\eta^i_{p} \mid p \geq 0, 0 \leq i \leq p\}, \eta^i_p > \eta^j_q$ if $p > q$ or $(p = q$ and $i < j)$.

Thirdly, for each $u = w_0 \eta^{i_0}_{p_0} w_1 \eta^{i_1}_{p_1} \cdots w_{n-1} \eta^{i_{n-1}}_{p_{n-1}} w_n \in \{t_q, \eta^i_q, p \geq 0, 0 \leq i \leq p\}^*$, where $w_i \in \{t_q|q \geq 0\}^*$, let $wt(u) = (n, w_0, w_1, \ldots, w_n, \eta^i_{p_0}, \eta^i_{p_1}, \ldots, \eta^i_{p_n})$. Then for any $u, v \in \{t_q, \eta^i_q, p \geq 0, 0 \leq i \leq p\}^*$, $u > v$ if $wt(u) > wt(v)$ lexicographically.

Fourthly, for any $\epsilon^i_p, \epsilon^j_p \in \{\epsilon^i_p \mid p \in Z^+, 0 \leq i \leq p\}, \epsilon^i_p > \epsilon^j_p$ if $p > q$ or $(p = q$ and $i < j)$.

Finally, for each $u = v_0 \epsilon^{i_0}_{p_0} v_1 \epsilon^{i_1}_{p_1} \cdots \epsilon^{i_n}_{p_n} v_n \in C(Y), \ n \geq 0, \ v_j \in \{t_q, \eta^i_q, p \geq 0, 0 \leq i \leq p\}^*$, let $wt(u) = (n, v_0, v_1, \ldots, v_n, \epsilon^{i_0}_{p_0}, \ldots, \epsilon^{i_n}_{p_n})$.

Then for any $u, v \in C(Y),$

$$u \succ_2 v \iff wt(u) > wt(v) \text{ lexicographically}.$$

It is also easy to check the order $\succ_2$ is a monomial order on $C(Y)$, which is an extension of $\succ_1$. Then we have the following theorem.
**Theorem 5.4** Let $Y$, $S$ be defined as the above, the generating set and the relation set of cyclic category $C(Y|S)$ respectively. Let $S^C = S \cup \{ \rho_4, \rho_5 \}$, where

$$
\rho_4: \quad t_q \varepsilon^0_q = \varepsilon^q_q,
$$

$$
\rho_5: \quad t_q \eta^0_q = \eta^q_q t_{q+1}.
$$

Then

1. With the order $\succ_2$ on $C(Y)$, $S^C$ is a Gröbner-Shirshov basis for the cyclic category partial algebra $kC(Y|S)$.

2. For each morphism $\mu: [q] \rightarrow [p]$ in the cyclic category $\Lambda = C(Y|S)$, $\mu$ can be uniquely represented as

$$
\varepsilon_{i_1}^q \ldots \varepsilon_{i_m}^q \eta_{j_1}^1 \ldots \eta_{j_n}^n \eta_{k}^1 \ldots \eta_{k}^n q^k t_q,
$$

where $p \geq i_1 > \ldots > i_m \geq 0$, $0 \leq j_1 < \ldots < j_n < q$, $0 \leq k \leq q$ and $q - n + m = p$.

**Proof.** It is easy to check that $\bar{f}_{q+1} = \varepsilon^{q+1} \varepsilon^q \varepsilon^{q-1}, \bar{g}_{q,q+1} = \eta^1_{q+1} \eta^1_q, \bar{h}_{q-1} = \eta^1_{q-1} \varepsilon^1_q, \bar{p}_1 = t_q \varepsilon^1_q, \bar{p}_2 = t_q \eta^1_q, \bar{p}_3 = t_q \varepsilon^0_q, \bar{p}_4 = t_q \eta^0_q$, and $\bar{p}_5 = t_q \eta^0_q$.

First of all, we prove $Id(S) = Id(S^C)$. It suffices to show $\rho_4, \rho_5 \in Id(S)$. Since $(\rho_4, \rho_1) t_q^{i+1} \varepsilon^q = t_q^{i+1} \varepsilon^q, \bar{g}_{q,q+1} = \eta^1_{q+1} \eta^1_q, \bar{h}_{q-1} = \eta^1_{q-1} \varepsilon^1_q, \bar{p}_1 = t_q \varepsilon^1_q, \bar{p}_2 = t_q \eta^1_q, \bar{p}_3 = t_q \varepsilon^0_q, \bar{p}_4 = t_q \eta^0_q$, and $\bar{p}_5 = t_q \eta^0_q$. Clearly, the leading term of the polynomial $t_q \eta^0 q^q t_{q+1} - \eta^q_q$ is $t_q \eta^0 q^q t_{q+1}$. Therefore $(t_q \eta^0 q^q t_{q+1} - \eta^q_q, \rho_3) t_q \eta^0 q^q t_{q+1} = -t_q \eta^0 q^q t_{q+1}$ and thus $\rho_5 \in Id(S)$.

Secondly, we prove that all possible compositions of $S^C$ are trivial which are the following:

(a) $(f_{q+1}, f_{q+1}) t_q^{i+1} \varepsilon^q \varepsilon^{q-1}$, $k \leq i \leq j - 1$;

(b) $(g_{q-1}, g_{q+1}) t_q^{i+1} \eta^1 q^q t_{q+1}$, $i \leq j \leq k$;

(c) $(h_{q,q+1}, f_{q+1}) t_q^{i+1} \varepsilon^1 q^q t_{q+1}$, $i \leq j - 1$;

(d) $(g_{q-2}, h_{q-1}) t_q^{i+1} \eta^1 q^q t_{q+1}$, $j \leq k$;

(e) $(p_1, f_{q+1}) t_q^{i+1} \varepsilon^1 q^q t_{q+1}$, $j > i$ and $i = 1, 2, \ldots, q$;

(f) $(\rho_3, \rho_1) t_q^{i+1} \varepsilon^q i$, $i = 1, 2, \ldots, q$;

(g) $(\rho_2, g_{q+1}) t_q \eta^1 q^q t_{q+1}$, $j \geq i$ and $j = 1, 2, \ldots, q$;

(h) $(\rho_2, h_{q+1}) t_q \eta^1 q^q t_{q+1}$, $j > i$ and $j = 1, 2, \ldots, q$;

(i) $(\rho_3, \rho_2) t_q^{i+1} \varepsilon^1 q^q t_{q+1}$, $i = 1, 2, \ldots, q$;

(j) $(\rho_3, \rho_4) t_q^{i+1} \eta^1 q^q t_{q+1}$;

(k) $(\rho_3, \rho_5) t_q^{i+1} \eta^1 q^q t_{q+1}$.
(l) \((\rho_4, f_{q+1, q})_{q+1, q}^{0, 0, \ldots, 1, j} \), \(j > 0\);

(m) \((\rho_5, g_{q+1, q})_{q+1, q}^{0, 0, \ldots, 1, j} \);

(n) \((\rho_5, h_{q+1, q})_{q+1, q}^{0, 0, \ldots, 1, j} \), \(i \geq 0\).

Here, we only give the proof of the case (n) \((\rho_6, h_{q+1, q})_{q+1, q}^{0, 0, \ldots, 1, j} \). The others can be similarly proved. Let us consider the following subcases of the case (n): (I) \(i = 0\); (II) \(i = 1\); (III) \(i > 1\).

For subcase (I),
\[
(\rho_6, h_{q+1, q})_{q+1, q}^{0, 0, \ldots, 1, j} = \eta_q t_{q+1}^2 \varepsilon_{q+1}^0 - t_q \\
\equiv \eta_q t_{q+1}^2 \varepsilon_{q+1}^1 - t_q \\
\equiv \eta_q \varepsilon_{q+1}^1 t_q - t_q \\
\equiv 0 \mod(S, t_q \eta_q^{0, 0, \ldots, 1, j}).
\]

For subcase (II),
\[
(\rho_6, h_{q+1, q})_{q+1, q}^{0, 0, \ldots, 1, j} = \eta_q t_{q+1}^2 \varepsilon_{q+1}^1 - t_q \\
\equiv \eta_q t_{q+1}^2 \varepsilon_{q+1}^0 t_q - t_q \\
\equiv \eta_q \varepsilon_{q+1}^1 t_q - t_q \\
\equiv 0 \mod(S, t_q \eta_q^{0, 0, \ldots, 1, j}).
\]

For subcase (III),
\[
(\rho_6, h_{q+1, q})_{q+1, q}^{0, 0, \ldots, 1, j} = \eta_q t_{q+1}^2 \varepsilon_{q+1}^i - t_q \varepsilon_{q+1}^{i-1} \eta_q^{0, 0, \ldots, 1, j} \\
\equiv \eta_q \varepsilon_{q+1}^i t_q - \varepsilon_{q+1}^{i-1} t_q \eta_q^{0, 0, \ldots, 1, j} \\
\equiv 0 \mod(S, t_q \eta_q^{0, 0, \ldots, 1, j}).
\]

Thus \(S^C\) is a Gröbner-Shirshov basis of the category partial algebra \(kC(Y|S)\).

Now, by Theorem 4.4, for each morphism \(\mu : [q] \to [p]\) in \(\Lambda = C(Y|S)\) can be uniquely represented as
\[
\varepsilon_{i_1} \cdots \varepsilon_{i_m} p^{m+1} \eta_{q-n}^{j_1} \cdots \eta_{q-n}^{j_k} t_q,
\]
where \(p \geq i_1 > \ldots > i_m \geq 0\), \(0 \leq j_1 < \ldots < j_n < q\), \(0 \leq k \leq q\) and \(q - n + m = p\). □

**Remark.** According to Loday [51], the uniqueness property in Theorem 5.4 (2) was known.

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