ON ZERO MASS SOLUTIONS
OF VISCOSOUS CONSERVATION LAWS

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Abstract. In the paper, we consider the large time behavior of solutions to the convection-diffusion equation $u_t - \Delta u + \nabla \cdot f(u) = 0$ in $\mathbb{R}^n \times [0, \infty)$, where $f(u) \sim u^q$ as $u \to 0$. Under the assumption that $q \geq 1 + 1/(n + \beta)$ and the initial condition $u_0$ satisfies: $u_0 \in L^1(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} u_0(x) \, dx = 0$, and $\|e^{t\Delta} u_0\|_{L^1(\mathbb{R}^n)} \leq Ct^{-\beta/2}$ for fixed $\beta \in (0, 1)$, all $t > 0$, and a constant $C$, we show that the $L^1$-norm of the solution to the convection-diffusion equation decays with the rate $t^{-\beta/2}$ as $t \to \infty$. Moreover, we prove that, for small initial conditions, the exponent $q^* = 1 + 1/(n + \beta)$ is critical in the following sense. For $q > q^*$ the large time behavior in $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, of solutions is described by self-similar solutions to the linear heat equation. For $q = q^*$, we prove that the convection-diffusion equation with $f(u) = u|u|^{q^* - 1}$ has a family of self-similar solutions which play an important role in the large time asymptotics of general solutions.

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1 Introduction

In this paper, we study the large time behavior of solutions \( u = u(x,t) \) \((x \in \mathbb{R}^n, \ t > 0)\) to the Cauchy problem for the nonlinear convection-diffusion equation

\[
\begin{align*}
    u_t - \Delta u + a \cdot \nabla (u|u|^{q-1}) &= 0, \\
    u(x,0) &= u_0(x),
\end{align*}
\]

where \( q > 1 \) and the vector \( a \in \mathbb{R}^n \) are fixed. The assumptions \( u_0 \in L^1(\mathbb{R}^n) \) and \( \int_{\mathbb{R}^n} u_0(x) \, dx = 0 \) will also be required.

The typical nonlinear term occurring in hydrodynamics in the one-dimensional case has the form \( uu_x = (u^2/2)_x \) (as in the case of the viscous Burgers equation). The most obvious generalization of this nonlinearity consists in replacing the square by a power \( u^q \) where \( q \) is a positive integer. Here, however, we intend to observe a more subtle interaction of the nonlinearity with dissipation, consequently, we need to consider a continuous range of parameters \( q \). The problem then appears with the definition of \( u^q \) for negative \( u \) and for non-integer \( q \). In order to avoid this difficulty, we chose the nonlinear term of the from \( a \cdot \nabla (u|u|^{q-1}) \). This was done only to shorten notation in this report.

Note that, in fact, the following property of the nonlinearity will be essential throughout this work:

- the nonlinear term in (1.1) has the form \( \nabla \cdot f(u) \) where the \( C^1 \)-vector function \( f \) satisfies \( |f(u)| \leq C|u|^q \), \( |f'(u)| \leq C|u|^{q-1} \) for every \( u \in \mathbb{R} \), \( q > 1 \), and a constant \( C \). Moreover, if the balanced case is considered (i.e. \( q = 1 + 1/(n + \beta) \)), the limits

\[
\lim_{u \to 0^-} f(u)/|u|^q, \quad \text{and} \quad \lim_{u \to 0^+} f(u)/|u|^q
\]

should exist and the both should be different from 0.

Recent publications developed versatile functional analytic tools to study the long time behavior of solutions of this initial value problem. Concerning the decay of solutions of (1.1)-(1.2) and, more generally, of scalar parabolic conservation laws of the form \( u_t - \Delta u + \nabla \cdot f(u) = 0 \) with integrable initial conditions, Schonbek [29] was the first who proved that the \( L^2 \)-norm tends to 0 as \( t \to \infty \) with the rate \( t^{-n/4} \). To deal with this problem, she introduced the so-called Fourier splitting method. The results from [29] were extended in the later work [30], where the decay of solutions in \( L^p(\mathbb{R}^n) \), \( 1 \leq p \leq \infty \) was obtained, again, by a method based on the Fourier splitting
Convection-diffusion equation

Next, Escobedo and Zuazua \cite{13} proved decay estimates of the $L^p$-norms of solutions by a different method under more general assumptions on nonlinearity and under less restrictive assumptions on initial data. Finally, by the use of the logarithmic Sobolev inequality, Carlen and Loss \cite{8} showed that solutions of viscous conservation laws satisfy

$$\|u(\cdot, t)\|_p \leq Ct^{-(n/2)(1/r-1/p)}\|u_0\|_r$$

for each $1 \leq r \leq p \leq \infty$, all $t > 0$, and a numerical constant $C > 0$ depending on $p$ and $q$, only. Here, we would also like to recall results on algebraic decay rates of solution to systems of parabolic conservation laws, obtained by Kawashima \cite{24}, Hopf and Zumbrun \cite{18}, Jeffrey and Zhao \cite{20}, and Schonbek and S"{u}li \cite{32}. Smallness assumptions on initial conditions were often imposed in those papers.

The first term of the asymptotic expansion was studied as the next step in analysis of the long time behavior of solutions to (1.1)-(1.2). Assuming that $u_0 \in L^1(\mathbb{R}^n)$, roughly speaking, these results, cf. e.g. \cite{1, 24, 13, 14, 15, 2, 3, 4, 22, 23}, fall into three cases:

- **Case I:** $q > 1 + 1/n$, when the asymptotics is linear, i.e.

  $$(1.3) \quad t^{(n/2)(1-1/p)}\|u(\cdot, t) - MG(\cdot, t)\|_p \to 0 \text{ as } t \to \infty,$$

  where $M = \int_{\mathbb{R}^n} u_0(x) dx$, $G(x, t) = (4\pi t)^{-n/2} \exp(-|x|^2/(4t))$ is the fundamental solution of the heat equation. Hence, this case can be classified as *weakly nonlinear*, since in this situation the linear diffusion prevails and the nonlinearity is asymptotically negligible.

- **Case II:** $q = 1 + 1/n$, when

  $$(1.4) \quad t^{(n/2)(1-1/p)}\|u(\cdot, t) - U_M(\cdot, t)\|_p \to 0 \text{ as } t \to \infty,$$

  where $U_M(x, t) = t^{-n/2}U_M(xt^{-1/2}, 1)$ is the self-similar solution of (1.1) with $u_0(x) = M\delta_0$. Here, diffusion and the convection are balanced, and the asymptotics is determined by a special solution of a nonlinear equation.

- **Case III:** $1 < q < 1 + 1/n$, when the convection points in the $x_n$-direction (i.e. $a = (0, \ldots, 0, 1)$). Here

  $$(1.5) \quad t^{(n+1)(1-1/p)/(2q)}\|u(\cdot, t) - U_M(\cdot, t)\|_p \to 0 \text{ as } t \to \infty,$$
holds, where $U_M$ is a particular self-similar solution of the partly viscous conservation law $U_t - \Delta_y U + \frac{\partial}{\partial x_n} (|U|^q - 1) = 0$ such that $u_0(x) = M \delta_0$ in the sense of measures. Here $x = (y, x_n), y = (x_1, \ldots, x_{n-1})$, and $\Delta_y = \sum_{j=1}^{n-1} \frac{\partial^2}{\partial x_j^2}$. Hence, the asymptotics of solutions is determined by solutions of an equation with strong convection and partial dissipation.

Finally, we recall that, in the weakly nonlinear case, Zuazua [37] found, for solutions to (1.1)-(1.2), the second order term in the asymptotic expansion as $t \to \infty$. He observed that asymptotic behavior of the solution differs depending if $q$ satisfies $1 + 1/n < q < 1 + 2/n$, $q = 1 + 2/n$, or $q > 1 + 2/n$. Analogous results for Lévy conservation laws were obtained in [2, 3], and for convection-diffusion equations with dispersive effects in [22, 23]. Related results on the stability in $L^1(\mathbb{R}^n)$ of traveling waves (or shock waves) in scalar viscous conservation laws can be found in the papers by Serre [33] and Freistühler and Serre [17]. Some results on the $L^1$-stability of the zero solution of degenerate convection-diffusion equations can be found in the article by Feireisl and Laurençot [16].

Here, we assume that $M = \int_{\mathbb{R}^n} u_0(x) \, dx = \int_{\mathbb{R}^n} u(x, t) \, dx = 0$, thus the corresponding self-similar intermediate asymptotics in (1.3)-(1.5) are equal to 0 for every $q > 1$. Moreover, for $p = 1$ the asymptotic formulae in (1.3)-(1.5) say nothing else but $\|u(\cdot, t)\|_1 \to 0$ as $t \to \infty$.

The goal of this paper is to find self-similar asymptotics in $L^p(\mathbb{R}^n)$ of solutions to (1.1)-(1.2) with $M = 0$ imposing additional conditions on the initial data. We assume that $u_0$ satisfies $\|e^{t\Delta} u_0\|_1 \leq C t^{-\beta/2}$ for some $\beta \in (0, 1)$, all $t > 0$, and $C$ independent of $t$. Such a decay estimate of solutions to the linear heat equation is optimal for a large class of initial conditions (cf. Propositions 2.1 and 3.1 below). Under these assumptions, we improve the known algebraic decay rates of the solutions to (1.1)-(1.2) in the $L^p$-norms for every $1 \leq p \leq \infty$. In addition, if the initial data are sufficiently small, we discover the new critical exponent $q^* = 1 + 1/(n + \beta)$ such that

- for $q > q^*$ the asymptotics of solutions to (1.1)-(1.2) is linear and described by self-similar solutions to the heat equation (cf. Corollaries 2.2 and 3.1 below);

- $q = q^*$ corresponds to the balanced case, and the asymptotics of solutions corresponding to suitable small initial conditions is described by a new class of self-similar solutions to the nonlinear equation (1.1) (cf. Theorem 2.3 and the discussion in Section 5.).
In the next section of this paper, we briefly present the main results. The
results corresponding to the case $q > q^*$ are contained in Section 3. Section 4
considers the case when the exponent $q = q^*$ is critical. In Section 5, we explain
how to derive, from our general theorems, self-similar solutions to the nonlinear
equation (1.1) and how to study large time asymptotics of general solutions. In
the last section, we discuss possible applications of our ideas to other equations
such as the Navier-Stokes equations, the KdV-Burgers equation, and the BBM-
Burgers equation.

Notation. The notation to be used is mostly standard. For $1 \leq p \leq \infty$,
the $L^p$-norm of a Lebesgue measurable real-valued function defined on $\mathbb{R}^n$ is
denoted by $\|v\|_p$. We will always denote by $\|\cdot\|_\mathcal{X}$ the norm of any other Banach
space $\mathcal{X}$ used in this paper.

If $k$ is a nonnegative integer, $W^{k,p}(\mathbb{R}^n)$ will be the Sobolev space consisting
of functions in $L^p(\mathbb{R}^n)$ whose generalized derivatives up to order $k$ belong to
$L^p(\mathbb{R}^n)$.

The Fourier transform of $v$ is defined as
$$ \hat{v}(\xi) \equiv (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} v(x) \, dx. $$

Given a multi-index $\gamma = (\gamma_1, \ldots, \gamma_n)$, we denote $\partial^\gamma = \partial^{\gamma_1}/\partial x_1 \cdots \partial^{\gamma_n}/\partial x_n$. On the other hand, for $\beta > 0$, the operator $D^\beta$ is defined via the Fourier transform as
$$ (D^\beta w)(\xi) = |\xi|^\beta |\hat{w}(\xi)|. $$

The letter $C$ will denote generic positive constants, which do not depend on $t$
and may vary from line to line during computations.

2 Main results and comments

We recall that for every $u_0 \in L^1(\mathbb{R}^n)$, the Cauchy problem (1.1)-(1.2) has a
unique solution in $C([0, \infty); L^1(\mathbb{R}^n))$ satisfying
$$ u \in C((0, \infty); W^{2,p}(\mathbb{R}^n)) \cap C^1((0, \infty), L^p(\mathbb{R}^n)) $$

for all $p \in (1, \infty)$. The proof is based on a standard iteration procedure
involving the integral representation of solutions of (1.1)-(1.2)

$$ u(t) = e^{t\Delta} u_0 - \int_0^t a \cdot \nabla e^{(t-\tau)\Delta} (u |u|^{q-1})(\tau) \, d\tau. $$

(see, e.g., [13] for details). Here, $e^{t\Delta} u_0$ is the solution to the linear heat equation
given by the convolution of the initial datum $u_0$ with the Gauss-Weierstrass
kernel $G(x, t) = (4\pi t)^{-n/2} \exp(-|x|^2/(4t))$. Formula (2.1) will be one of the
main tools used in the analysis of the long time behavior of solutions.
Let us also recall that sufficiently regular solutions of \((1.1)-(1.2)\) satisfy the estimate
\[
\|u(\cdot, t)\|_p \leq C(p, r) t^{-(n/2)(1/r-1/p)} \|u_0\|_r
\]
for all \(1 \leq r \leq p \leq \infty\), all \(t > 0\), and a constant \(C(p, r)\) depending on \(p\) and \(r\), only. Inequalities (2.2) are due to Carlen and Loss [8, Theorem 1]. We also refer the reader to [2, 3] where counterparts of (2.2) were proved for more general equations: so-called Lévy conservation laws.

Section 3 contains the analysis of the large time asymptotics of solutions to the linear heat equation. Easy calculations show that for every \(u_0 \in L^1(\mathbb{R}^n)\) such that \(\int_{\mathbb{R}^n} u_0(x) \, dx = 0\) we have \(\|e^{t\Delta} u_0\|_1 \to 0\) as \(t \to \infty\). The following proposition asserts the existence of a large class of initial conditions for which the large time behavior of \(e^{t\Delta} u_0\) is self-similar. Here, we need the notion of the Riesz potential \(I_\beta\) and the fractional derivative \(D_\beta\) defined in the Fourier variables as
\[
(I_\beta w)(\xi) = \frac{\hat{w}(\xi)}{|\xi|^\beta} \quad \text{and} \quad (D_\beta w)(\xi) = |\xi|^\beta \hat{w}(\xi).
\]

**Proposition 2.1** Let \(\beta > 0\) and \(\gamma = (\gamma_1, \ldots, \gamma_n)\) be a multi-index with \(\gamma_i \geq 0\). Assume that \(I_\beta u_0 \in L^1(\mathbb{R}^n)\). Denote
\[
A = \lim_{|\xi| \to 0} \frac{\hat{u}_0(\xi)}{|\xi|^\beta} = \int_{\mathbb{R}^n} (I_\beta u_0)(x) \, dx.
\]
Then
\[
\| \partial^\gamma e^{t\Delta} u_0 \|_1 \leq C t^{-\beta/2-|\gamma|/2} \|I_\beta u_0\|_1
\]
for all \(t > 0\) and \(C = C(\beta, \gamma)\) independent of \(t\) and \(u_0\); moreover,
\[
t^{\beta/2+|\gamma|/2} \| \partial^\gamma e^{t\Delta} u_0(\cdot) - A \partial^\gamma D_\beta G(\cdot, t) \|_1 \to 0
\]
as \(t \to \infty\).

Let us emphasize that we do not assume that \(\beta < 1\) in Proposition 2.1. This condition only becomes necessary when the convection term is present. Here, we also refer the reader to Proposition 3.1 where self-similar asymptotics of the heat semigroup is studied under more general assumptions on initial data.

In our first theorem on the large time behavior of solutions to the nonlinear problem \((1.1)-(1.2)\), we assume the decay of \(\|e^{t\Delta} u_0\|_1\) with a given rate and we prove that the same decay estimate holds true for solutions to \((1.1)-(1.2)\).
Theorem 2.1  Fix $0 < \beta < 1$. Assume that $u_0 \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ satisfies the inequality
\begin{equation}
\|e^{t\Delta}u_0\|_1 \leq Ct^{-\beta/2}
\end{equation}
for all $t > 0$ and a constant $C$ independent of $t$. Let $u$ be the solution to (1.1)-(1.2) with $u_0$ as the initial datum. If $q > 1 + 1/n$, then there exists a constant $C$ such that
\begin{equation}
\|u(\cdot, t)\|_1 \leq C(1 + t)^{-\beta/2}
\end{equation}
for all $t > 0$. The estimate (2.8) holds also true for $1 + 1/(n + \beta) \leq q \leq 1 + 1/n$, with $0 < \beta < 1$, provided $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $\sup_{t>0} t^{\beta/2}\|e^{t\Delta}u_0\|_1$ is sufficiently small.

Remark 2.1. The assumption (2.7) means that $u_0$ belongs to the homogeneous Besov space $B^{-\beta, \infty}_1$ (cf. (2.11), below) which will play an important role in the analysis of the balanced case $q = 1 + 1/(n + \beta)$. □

The approach formulated in Theorem 2.1, saying that the decay estimates imposed on the heat semigroup lead to the analogous estimates of solutions to a nonlinear problem, appears in several recent papers. Here, we would like only to recall (the list is by no mean exhaustive) the works on the Navier-Stokes system by Schonbek [31] and Wiegner [36] where the $L^2$-decay of solutions was studied as well as by Miyakawa [26] where decay of the $L^1$-norm and $H^p$-norms (the Hardy spaces) of weak solutions was shown. Moreover, our results extend essentially the recent paper by Schonbek and Süli [32] where general conservation laws were considered.

If we combine the decay from (2.8) with inequalities (2.2), we obtain the improved $L^p$-decay of solutions to (1.1)-(1.2). Moreover, applying such estimates to (2.1) we find the asymptotics of solutions for $q > 1 + 1/(n + \beta)$. The following corollary contains these results.

Corollary 2.1 Under the assumptions of Theorem 2.1, for every $p \in [1, \infty]$ and $\beta \in (0,1)$, there exists $C = C(u_0, p)$ independent of $t$ such that
\begin{equation}
\|u(\cdot, t)\|_p \leq C(1 + t)^{-(n/2)(1-1/p) - \beta/2}
\end{equation}
for all $t > 0$. Moreover, for $q > 1 + 1/(n + \beta)$ and for every $p \in [1, \infty]$ it follows
\begin{equation}
t^{(n/2)(1-1/p)+\beta/2}\|u(\cdot, t) - e^{t\Delta}u_0(\cdot)\|_p \to 0 \quad \text{as} \quad t \to \infty.
\end{equation}
A slightly stronger version of this corollary is formulated and proved in the next section (cf. Corollary 3.1, below). Here, we only emphasize that combining (2.10) with Proposition 2.1 we obtain that the large time behavior of solutions to (1.1)-(1.2) with
\[ q > \frac{1 + 1}{n} \quad \text{or, if the data are sufficiently small, for} \quad q > \frac{1 + 1}{n + \beta} \]
is described by special self-similar solutions to the heat equation. This is worth stating more precisely.

**Corollary 2.2** Under the assumptions of Theorem 2.1 and Proposition 2.1 (or Proposition 3.1 with \( \ell(\xi) = |\xi|^\beta \), see Section 3) the solution to (1.1)-(1.2) with \( q > \frac{1 + 1}{n + \beta} \) satisfies
\[ t^{(n/2)(1 - 1/p) + \beta/2} \| u(\cdot, t) - AD^\beta G(\cdot, t) \|_p \to 0 \quad \text{as} \quad t \to \infty. \]

Our next results, studied in Section 4, correspond to the balanced case
\[ q = q^* = 1 + \frac{1}{n + \beta} \]
for some fixed \( 0 < \beta < 1 \). We will work in the homogeneous Besov space \( B_1^{-\beta, \infty} \) defined by
\[ B_1^{-\beta, \infty} = \{ v \in S'(\mathbb{R}^n) : \| v \|_{B_1^{-\beta, \infty}} < \infty \}, \]
where \( S'(\mathbb{R}^n) \) is the space of tempered distributions and the norm is given by
\[ (2.11) \quad \| v \|_{B_1^{-\beta, \infty}} \equiv \sup_{s > 0} s^{\beta/2} \| e^{s\Delta} v \|_1. \]

The standard way of defining norms in Besov spaces is based on the Paley-Littlewood dyadic decomposition. The choice of the equivalent norm (2.11) allows us to simplify several calculations. Recall here that Proposition 2.1 describes a large subset in \( B_1^{-\beta, \infty} \) of initial conditions \( u_0 \).

Section 4 contains the proofs of two main theorems. The first one provides a construction of global-in-time solutions to (1.1)-(1.2) with \( q = 1 + 1/(n + \beta) \) and suitably small initial data in the space \( B_1^{-\beta, \infty} \). The second theorem gives asymptotic stability of solutions in the balanced case. The precise statement of the theorems is the following.

**Theorem 2.2** Fix \( \beta \in (0, 1) \) and put \( q = 1 + 1/(n + \beta) \). There is \( \varepsilon > 0 \) such that for each \( u_0 \in B_1^{-\beta, \infty} \) satisfying \( \| u_0 \|_{B_1^{-\beta, \infty}} < \varepsilon \) there exists a solution of (1.1)-(1.2) for all \( t \geq 0 \) in the space
\[ \mathcal{X} \equiv \mathcal{C}([0, \infty) : B_1^{-\beta, \infty}) \cap \{ u : (0, \infty) \to L^q(\mathbb{R}^n) : \sup_{t > 0} t^{(n/2)(1-1/q) + \beta/2} \| u(t) \|_q < \infty \}. \]
This is the unique solution satisfying the condition
\[ \sup_{t>0} t^{(n/2)(1-1/q)+\beta/2} \|u(t)\|_q \leq 2\varepsilon. \]

**Theorem 2.3** Let the assumptions from Theorem 2.2 hold true. Assume that \( u \) and \( v \) are two solutions of (1.1)-(1.2) constructed in Theorem 2.2 corresponding to the initial data \( u_0, v_0 \in B_{1,\infty}^{-\beta} \), respectively. Suppose that
\[ \lim_{t \to \infty} t^{\beta/2} \|e^{t\Delta}(u_0 - v_0)\|_1 = 0. \]

Choosing \( \varepsilon > 0 \) in Theorem 2.2 sufficiently small, we have
\[ \lim_{t \to \infty} t^{(n/2)(1-1/p)+\beta/2} \|u(\cdot, t) - v(\cdot, t)\|_p = 0 \]
for every \( p \in [1, \infty] \).

In Section 5, we show how to use Theorem 2.2 in order to obtain self-similar solutions to equation (1.1) with the critical exponent \( q = 1 + 1/(n + \beta) \). Moreover, we explain the role of self-similar solutions in the large time asymptotics of other solutions to (1.1)-(1.2).

### 3 Asymptotics of solutions for \( q > 1 + 1/(n + \beta) \)

As noted in Section 2, the first problem is to find a class of data that will insure the decay of solutions to the heat equation in \( L^1(\mathbb{R}^n) \). This is obtained in Proposition 2.1, where this class of data is shown to be constituted by functions such that their convolutions with Riesz potentials lie in \( L^1(\mathbb{R}^n) \).

**Proof of Proposition 2.1.** Let us note that the limit in (2.4) exists, since \( \hat{u}_0(\xi)/|\xi|^\beta \) is continuous as the Fourier transform of an integrable function \( I_\beta u_0 \). First, we prove that \( \partial^\gamma D^\beta G(\cdot, 1) \in L^1(\mathbb{R}^n) \). Obviously, \( \partial^\gamma D^\beta G(\cdot, 1) \) is bounded and continuous because its Fourier transform \( (i\xi)^\gamma |\xi|^\beta e^{-|\xi|^2} \) is integrable. Moreover, it follows from [33, Ch. 5, Lemma 2] that for every \( \beta > 0 \) there exists a finite measure \( \mu_\beta \) on \( \mathbb{R}^n \) given by
\[ \hat{\mu}_\beta(\xi) = \frac{|\xi|^\beta}{(1 + |\xi|^2)^{\beta/2}}. \]
Hence, $\partial^\gamma D^\beta G(\cdot, 1) = \mu_\beta * K_{\beta, \gamma}$ where the function $K_{\beta, \gamma}$ is defined via the Fourier transform as $\hat{K}_{\beta, \gamma}(\xi) = (i\xi)^\gamma (1 + |\xi|^2)^{\beta/2} e^{-|\xi|^2}$. It is easy to prove that $K_{\beta, \gamma} \in S(\mathbb{R}^n)$ (the Schwartz class of rapidly decreasing smooth function), and this implies the integrability of $\partial^\gamma D^\beta G(\cdot, 1)$ for every multi-index $\gamma$.

Now, the change of variables yields that $\partial^\gamma D^\beta G(x, t)$ has the self-similar form:

$$\partial^\gamma D^\beta G(x, t) = t^{-n/2-\beta/2-|\gamma|/2} (\partial^\gamma D^\beta G)(x/\sqrt{t}, 1)$$

for all $x \in \mathbb{R}^n$ and $t > 0$.

To prove (2.5), use the Young inequality for the convolution, and thus by (3.1) it follows

$$\|\partial^\gamma e^{t\Delta} u_0\|_1 = \|\partial^\gamma D^\beta G(t) * I_\beta u_0\|_1 \leq \|\partial^\gamma D^\beta G(\cdot, t)\|_1 \|I_\beta u_0\|_1 \leq t^{-\beta/2-|\gamma|/2} \|\partial^\gamma D^\beta G(\cdot, 1)\|_1 \|I_\beta u_0\|_1$$

for all $t > 0$.

For the proof of (2.6), observe that the change of variables $z = x/\sqrt{t}$ combined with (3.1) leads to the following expression

$$t^{\beta/2+|\gamma|/2} \|\partial^\gamma e^{t\Delta} u_0(\cdot) - A \partial^\gamma D^\beta G(\cdot, t)\|_1$$

$$= t^{\beta/2+|\gamma|/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| \partial^\gamma D^\beta G(x - y, t) - \partial^\gamma D^\beta G(x, t) \right| I_\beta u_0(y) dy \, dx \leq \int \int_{\mathbb{R}^n \times \mathbb{R}^n} |I_\beta u_0(y)| \left| (\partial^\gamma D^\beta G)(z - y/\sqrt{t}, 1) - (\partial^\gamma D^\beta G)(z, 1) \right| dy dz$$

From the first part of this proof, the function $\partial^\gamma D^\beta G(z, 1)$ is continuous, hence the integrand on the right hand side of (3.2) tends to 0 as $t \to \infty$ for all $y, z \in \mathbb{R}^n$. Denote

$$A(z, y, t) \equiv (\partial^\gamma D^\beta G)(z - y/\sqrt{t}, 1) - (\partial^\gamma D^\beta G)(z, 1).$$

To apply the Lebesgue Dominated Convergence Theorem to the integral on the right hand side of (3.2), it is necessary to show that there exists $F \in L^1(\mathbb{R}^n)$ independent of $y \in \mathbb{R}^n$ and $t \geq 1$, such that

$$|A(z, y, t)| \leq F(z)$$

for all $z, y \in \mathbb{R}^n$ and $t \geq 1$. Note that

$$A(z, y, t) = \int_{\mathbb{R}^n} |\xi|^{\beta} e^{-iy/\sqrt{t}} \left[ e^{-|\xi|^2} e^{iz\xi} d\xi - 1 \right].$$
Moreover, the symbol $b(\xi, y, t) \equiv (1 + |\xi|^2/2(\imath \xi))\gamma \left[ e^{-\imath y/\sqrt{t}} - 1 \right] e^{-|\xi|^2}$ is a $C^\infty$ function of $(\xi, y) \in \mathbb{R}^n \times \mathbb{R}^n$, and satisfies the differential inequalities
\[
|\partial^\alpha_x \partial^\beta_y b(\xi, y, t)| \leq C(\alpha, \gamma, N)(1 + |\xi|)^{-N-\alpha}
\]
for all multi-indices $\alpha$ and $\gamma$, all $N \in \mathbb{N}$, and $C(\alpha, \gamma, N)$ independent of $\xi, y \in \mathbb{R}^n$ and $t \geq 1$. By [33, Ch. VI, Sec. 4, Prop. 1], the (inverse) Fourier transform with respect to $\xi$ of $b(\xi, y, t)$ satisfies the estimate
\[
|\mathcal{F}^{-1}_\xi b(\cdot, y, t)(z)| \leq C(1 + |z|)^{-N}
\]
for all $N \in \mathbb{N}$, and a constant $C = C(N)$ independent of $z, y \in \mathbb{R}^n$ and $t \geq 1$. Finally, the use of the measure $\mu_\beta$ from the first part of this proof combined with standard properties of the Fourier transform and the convolution lead to the representation $A(\cdot, y, t) = \mu_\beta \ast \mathcal{F}^{-1}_\xi b(\cdot, y, t)$. Hence, (3.3) holds true for the function $F(z) = C[\mu_\beta \ast (1 + |.|^{-N})](z)$ with any $N > n$. This completes the proof of Proposition 2.1. \hfill \Box

We recall that, in [27], Miyakawa obtained the $L^1$-decay of $e^{\imath \Delta} u_0$ provided the $|x|^\beta$-momentum of the data is bounded. Below, we will show that our assumptions is weaker than the one assumed by Miyakawa.

**Remark 3.2.** The $L^1$-decay of solutions to the linear heat equation formulated in (2.5) was proved by Miyakawa [27] under the assumptions
\[
(3.4) \quad u_0 \in L^1(\mathbb{R}^n), \quad \int_{\mathbb{R}^n} u_0(x) \, dx = 0, \quad \int_{\mathbb{R}^n} |x|^\beta |u_0(x)| \, dx < \infty
\]
for some $0 < \beta < 1$. To show that our assumption $I_\beta u_0 \in L^1(\mathbb{R}^n)$ is weaker than (3.4) it suffices to establish the inequality
\[
(3.5) \quad \|I_\beta u_0\|_1 \leq C \int_{\mathbb{R}^n} |x|^\beta |u_0(x)| \, dx
\]
valid for every $u_0$ satisfying (3.4) with $\beta \in (0, 1)$. Let us sketch the proof of (3.5), however, it does not play any role in our considerations, below. It is well known that $(I_\beta u_0)(x) = C(\beta, n) \int_{\mathbb{R}^n} |x - y|^{\beta - n} u_0(y) \, dy$ (in fact, this representation holds true for every $\beta \in (0, n)$). Hence, using the assumption $\int_{\mathbb{R}^n} u_0(y) \, dy = 0$ and changing the order of integration yield
\[
\|I_\beta u_0\|_1 \leq C(\beta, n) \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \frac{1}{|x - y|^{\beta - n}} - \frac{1}{|x|^{\beta - n}} \right) |u_0(y)| \, dy.
\]
Next, note that the integral with respect to $x$ in the inequality above is finite for every $y \in \mathbb{R}^n$, because its integrand $||x - y|^{\beta - n} - |x|^{\beta - n}|$ is locally integrable.
and behaves like \(|x|^{\beta-1-n}\) as \(|x| \to \infty\) (here, the assumption \(\beta \in (0,1)\) is crucial). Hence, by the change of variables, it follows that

\[
\int _{\mathbb {R}^n} \left| \frac{1}{|x-y|^{n-\beta}} - \frac{1}{|x|^{n-\beta}} \right| \, dx = |y|^{\beta} \int _{\mathbb {R}^n} \left| \frac{1}{|\omega-y|^{n-\beta}} - \frac{1}{|\omega|^{n-\beta}} \right| \, d\omega.
\]

Since \(\sup_{y \in \mathbb {R}^n \setminus \{0\}} \int _{\mathbb {R}^n} |\omega-y|/|y|^{\beta-n} - |\omega|^{\beta-n} | \, d\omega < \infty\) (the proof of this elementary fact is omitted), we obtain (3.7).

The self-similar asymptotics of \(e^{t\Delta}u_0\) in \(L^p(\mathbb {R}^n)\) with \(p \in [2, \infty]\) can be derived under weaker assumptions on \(u_0\). This is stated in the following proposition.

**Proposition 3.1** Let \(\ell = \ell(\xi)\) denote a function homogeneous of degree \(\beta > 0\). Assume that \(u_0\) satisfies

\[
(3.6) \quad \sup_{\xi \in \mathbb {R}^n \setminus \{0\}} \frac{\hat{u}_0(\xi)}{\ell(\xi)} < \infty \quad \text{and} \quad \lim_{|\xi| \to 0} \frac{\hat{u}_0(\xi)}{\ell(\xi)} = A
\]

for some \(A \in \mathbb {R}\). Denote by \(\mathcal {L}\) the Fourier multiplier operator defined via the formula \(\hat{\mathcal {L}}v(\xi) = \ell(\xi)\hat{v}(\xi)\). Under these assumptions, for every \(p \in [2, \infty]\) and for every multi-index \(\gamma\), it follows

\[
\ell^{n(1/p-1/2)+\beta/2+|\gamma|/2} \|\partial^\gamma e^{t\Delta}u_0 - A\partial^\gamma \mathcal {L}G(t)\|_p \to 0 \quad \text{as} \quad t \to \infty.
\]

**Proof.** The main tool here is the Hausdorff–Young inequality

\[
(3.7) \quad \|\hat{v}\|_p \leq C\|v\|_q,
\]

valid for every \(1 \leq q \leq 2 \leq p \leq \infty\) such that \(1/p + 1/q = 1\). Hence, (3.7), the change of variables \(\xi t^{1/2} = \omega\), and the homogeneity of \(\ell\) yield

\[
\|\partial^\gamma e^{t\Delta}u_0 - A\partial^\gamma \mathcal {L}G(t)\|_p^q \\
\quad \leq C \int_{\mathbb {R}^n} |(i\xi)^\gamma e^{-t|\xi|^2/2}\ell(\xi)\left(\frac{\hat{u}_0(\xi) - A\ell(\xi)}{\ell(\xi)}\right)|^q \, d\xi \\
\quad = Ct^{-n/2-(\beta/2+|\gamma|/2)q} \int_{\mathbb {R}^n} |(i\omega)^\gamma e^{-|\omega|^2/2}\ell(\omega)\left(\frac{\hat{u}_0(\omega/t^{1/2}) - A}{\ell(\omega/t^{1/2})}\right)|^q \, d\omega.
\]

Now, the assumptions on \(u_0\) in (3.6) allow us to apply the Lebesgue Dominated Convergence Theorem in order to prove that the integral on the right hand side tends to \(0\) as \(t \to \infty\). \(\square\)

**Remark 3.3.** The conditions formulated in (3.6) appear in a natural way if Hardy spaces are considered. Let us recall that a tempered distribution
$v$ belongs to the Hardy space $\mathcal{H}^p$ on $\mathbb{R}^n$ for some $0 < p < \infty$ whenever $v^+ = \sup_{t>0} |(\phi_t * v)| \in L^p(\mathbb{R}^n)$, where $\phi_t(x) = t^{-n} \phi(x/t)$ with $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \phi(x) \, dx = 1$. We refer the reader to [33] where several properties of Hardy spaces are derived. We recall that $\mathcal{H}^1$ is a Banach space strictly contained in $L^1(\mathbb{R}^n)$ and that $L^p(\mathbb{R}^n) = \mathcal{H}^p$ for $p > 1$ with equivalent norms. Suppose now that $p \leq 1$ and $u_0 \in \mathcal{H}^p$. It is known (cf. [35, Chapter III, §5.4]) that the Fourier transform $\hat{u}_0$ is continuous on $\mathbb{R}^n$ and $|\hat{u}_0(\xi)| \leq C|\xi|^{n(1/p-1)}\|u_0\|_{\mathcal{H}^p}$ for all $\xi \in \mathbb{R}^n$. Moreover, near the origin, this can be refined to $\lim_{t \to 0} |\hat{u}_0(\xi)||\xi|^{-n(1/p-1)} = 0$. Hence, assumptions (3.9) are satisfied with $\ell(\xi) = |\xi|^{\beta}$, $\beta \in (0,1)$, and $A = 0$, if e.g. $u_0 \in \mathcal{H}^{n/(n+\beta)}$.

Theorem 2.1 is the main decay theorem proved in this section. It ensures that the $L^1$-norm of solutions to the convection-diffusion equation decay at the rate $t^{-\beta/2}$ provided their initial data are such that the corresponding solutions to the heat equation decay at the same rate.

**Proof of Theorem 2.1.** The proof of this theorem relays on a systematic combination of the integral equation (2.1) with inequality (2.2). Note that since $u_0 \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, by (2.2), it follows that

$$
(3.8) \quad \|u(\cdot, t)\|^q_q \leq C(\|u_0\|_1, \|u_0\|_q)(1 + t)^{-(n/2)(q-1)}
$$

for all $t \geq 0$. Hence, computing the $L^1$-norm of (2.1), using the assumption on $u_0$, and (3.8) yield

$$
\begin{align*}
\|u(\cdot, t)\|_1 & \leq \|e^{t\Delta}u_0\|_1 + \int_0^t \|a \cdot \nabla G(\cdot, t - \tau)\|_1 \|u(\cdot, t)\|^q_q \, d\tau \\
& \leq Ct^{-\beta/2} + C \int_0^t (t - \tau)^{-1/2}(1 + \tau)^{-n/2(q-1)} \, d\tau \\
& \leq Ct^{-\beta/2} + C \begin{cases} 
\frac{t^{1/2-(n/2)(q-1)}}{q}, & \text{for } q \in \left(1 + \frac{1}{n}, 1 + \frac{2}{n}\right); \\
\frac{t^{-1/2} \log(e + t)}{q}, & \text{for } q = 1 + \frac{2}{n}; \\
\frac{t^{-1/2}}{q}, & \text{for } q > 1 + \frac{2}{n}.
\end{cases}
\end{align*}
$$

(3.9)

For $q \geq 1 + (\beta + 1)/n$, estimate (2.8) follows immediately from (3.9), since $-1/2 < -\beta/2$ and $1/2 - (n/2)(q-1) \leq -\beta/2$ for this range of $q$.

Next, consider $1 + 1/n < q < 1 + (\beta + 1)/n$. A simple calculation shows that $\alpha = -\left(1/2 - (n/2)(q-1)\right)$ satisfies $0 < \alpha < \beta/2$. Moreover, it follows from (3.9) that

$$
(3.10) \quad \|u(\cdot, t)\|_1 \leq C(1 + t)^{-\alpha}.
$$
Combining inequality (2.2) with (3.10) yields the improved decay of the $L^q$-norm

\begin{equation}
\|u(\cdot, t)\|_q \leq C(1 + t/2)^{-\alpha(q+1)/2}\|u(\cdot, t/2)\|_1 \\
\leq C(1 + t)^{-\alpha(q+1)/2}.
\end{equation}

Hence, repeating the calculations from (3.9), using (3.11) instead of (3.8), gives

\begin{equation}
\|u(\cdot, t)\|_1 \leq Ct^{-\beta/2} + \int_0^t (t - \tau)^{-\beta/2} (1 + \tau)^{-\alpha(q+1)/2} d\tau.
\end{equation}

If $-(n/2)(q-1) - q\alpha \leq -1$, the integral on the right hand side of (3.12) tends to 0 as $t \to \infty$ faster than $t^{-\beta/2}$ and this ends the proof. On the other hand, if $-(n/2)(q-1) - q\alpha > -1$, by the definition of $\alpha$, it follows from (3.12) that

\[ \|u(\cdot, t)\|_1 \leq Ct^{-\beta/2} + C t^{-\alpha(q+1)} \]

Hence, if $-\alpha(q+1) \leq -\beta/2$, the proof is complete. If, on the contrary $-\alpha(q+1) > -\beta/2$, we have the new estimate

\[ \|u(\cdot, t)\|_1 \leq C(1 + t)^{-\alpha(q+1)} , \]

which we use as in (3.11) (with $\alpha$ replaced by $\alpha(q+1)$) to get an improved decay of the $L^q$-norm: $\|u(\cdot, t)\|_q \leq C(1 + t)^{-\alpha(q+1)}$. Consequently, a finite number of repetitions of the above steps yields (2.8).

Finally, let us prove (2.8) for $1 + 1/(n+\beta) \leq q \leq 1 + 1/n$ under the assumption that $\sup_{t>0} \|e^{t\Delta} u_0\|_1$ is sufficiently small. For simplicity of notation, we put

\[ q = q^* = 1 + \frac{1}{n+\beta} , \]

and we use systematically the following inequality (obtained from the Hölder inequality and from (2.2))

\begin{equation}
\|u(\cdot, t)\|_q \leq \|u(\cdot, t)\|_q^\star \frac{q}{q-\alpha} \leq C(\|u_0\|_\infty)\|u(\cdot, t)\|_q^\star .
\end{equation}

for all $t > 0$. To proceed, we also define the auxiliary nonnegative continuous function

\[ g(t) \equiv \sup_{0 \leq \tau \leq t} \left( \tau^\beta/2 \|u(\cdot, \tau)\|_1 \right) + \sup_{0 \leq \tau \leq t} \left( \tau^{(1/2+\beta)/2} \|u(\cdot, \tau)\|_q^\star \right) . \]

Now, computing the $L^1$-norm of the integral equation (2.1) and using (3.13) yield

\begin{equation}
\begin{aligned}
t^\beta/2\|u(\cdot, t)\|_1 & \leq t^\beta/2\|e^{t\Delta} u_0\|_1 + C t^\beta/2 \int_0^t (t - \tau)^{-\beta/2} \|u(\cdot, \tau)\|_q^\star d\tau \\
& \leq t^\beta/2\|e^{t\Delta} u_0\|_1 \\
& \quad + g^\star(t) \ C t^\beta/2 \int_0^t (t - \tau)^{-1/2} \tau^{-1/2-\beta/2} d\tau
\end{aligned}
\end{equation}
for all $t > 0$. An elementary calculation shows that the quantity

$$t^{\beta/2} \int_0^t (t - \tau)^{-1/2} \tau^{-1/2 - \beta/2} \, d\tau$$

is finite for every $t > 0$ (since $0 < \beta < 1$) and independent of $t$. A similar reasoning gives

$$(3.15) \quad \|u(\cdot, t)\|_{q^*} \leq (t/2)^{(n/2)(1-1/q^*)} \|e^{(t/2)\Delta} u_0\|_1$$

$$+ g^{q^*}(t) C \int_0^t (t - \tau)^{-(n/2)(1-1/q^*)-1/2} \tau^{-1/2 - \beta/2} \, d\tau.$$  

Note that $-(n/2)(1 - 1/q^*) - \beta/2 = -(1/2 + \beta/2)/q^*$. Moreover, the quantity

$$t^{(1/2 + \beta/2)/q^*} \int_0^t (t - \tau)^{-(n/2)(1-1/q^*)-1/2} \tau^{-1/2 - \beta/2} \, d\tau$$

is finite (since $-(n/2)(1 - 1/q^*) - 1/2 > -1$) and independent of $t$ (by the change of variables).

Combining inequalities (3.14) and (3.15) yields

$$(3.16) \quad g(t) \leq C_1 \sup_{0 \leq \tau} \tau^{\beta/2} \|e^{t\Delta} u_0\|_1 + C_2 g^{q^*}(t)$$

for all $t \geq 0$ and constants $C_1$ and $C_2$ independent of $t$.

Finally, let

$$F(y) = A + C_2 y^{q^*} - y \quad \text{where} \quad A = C_1 \sup_{0 \leq \tau} \tau^{\beta/2} \|e^{t\Delta} u_0\|_1$$

and where $q^* > 1$. If $A > 0$ is sufficiently small, there exists $y_0 > 0$ such that $F(y_0) = 0$ and $F(y) > 0$ if $y \in [0, y_0)$. Moreover, it follows from (3.16) that $F(g(t)) \geq 0$. Since $g(t)$ is a nonnegative, continuous function such that $g(0) = 0$, we deduce that $g(t) \in [0, y_0)$ for all $t \geq 0$. This completes the proof of Theorem 2.1.$\blacksquare$

As a consequence of Theorem 2.1 we get Corollary 2.1. Actually, here we prove its slightly stronger version.

**Corollary 3.1** Under the assumptions of Theorem 2.1, for every $p \in [1, \infty]$ and $\beta \in (0, 1)$ there exists $C = C(u_0, p)$ independent of $t$ such that

$$\|u(\cdot, t)\|_p \leq C(1 + t)^{-(n/2)(1-1/p)-\beta/2}$$  

(3.17)
for all \( t > 0 \), and

\[
\|u(\cdot, t) - e^{t\Delta}u_0(\cdot)\|_p
\]

(3.18)

\[
\leq C \begin{cases}
  t^{-(n/2)(q-1/p)-(\beta q-1)/2} & \text{for } q \in \left(1 + \frac{1}{n+\beta}, \frac{n+2}{n+\beta}\right), \\
  t^{-(n/2)(1-1/p)-1/2} \log(e + t) & \text{for } q = \frac{n+2}{n+\beta}, \\
  t^{-(n/2)(1-1/p)-1/2} & \text{for } q > \frac{n+2}{n+\beta}
\end{cases}
\]

for all \( t \geq 1 \).

**Proof.** Inequality (3.17) is obtained combining (2.2) with (2.8) as in (3.11) where \( q \) is replaced by \( p \) and \( \alpha \) by \( \beta/2 \).

In view of the integral equation (2.1), to prove (3.18) it suffices to estimate the \( L^p \)-norm of the second term on the right hand side of (2.1). Here, split the integration range with respect to \( \tau \) into \([0, t/2] \cup [t/2, t]\) and study each term separately as follows. Using the Young inequality for the convolution and (3.17) yields

\[
\int_{t/2}^{t/2} \|a \cdot \nabla e^{(t-\tau)\Delta} (u|u|^{q-1})(\tau)\|_p d\tau
\]

(3.19)

\[
\leq \int_{0}^{t/2} \|a \cdot \nabla G(\cdot, t - \tau)\|_p \|u(\cdot, \tau)\|_q d\tau
\]

\[
\leq C \int_{0}^{t/2} (t - \tau)^{-(n/2)(1-1/p)-1/2} (1 + \tau)^{-(n/2)(q-1)-\beta q/2} d\tau
\]

\[
\leq C \begin{cases}
  t^{-(n/2)(q-1/p)-(\beta q-1)/2} & \text{for } q \in \left(1 + \frac{1}{n+\beta}, \frac{n+2}{n+\beta}\right), \\
  t^{-(n/2)(1-1/p)-1/2} \log(e + t) & \text{for } q = \frac{n+2}{n+\beta}, \\
  t^{-(n/2)(1-1/p)-1/2} & \text{for } q > \frac{n+2}{n+\beta}
\end{cases}
\]

for all \( t > 0 \).

A similar calculation gives

\[
\int_{t/2}^{t} \|a \cdot \nabla e^{(t-\tau)\Delta} (u|u|^{q-1})(\tau)\|_p d\tau
\]

(3.20)

\[
\leq \int_{t/2}^{t} (t - \tau)^{-1/2} \|u(\cdot, \tau)\|_{pq} d\tau
\]

\[
\leq C \int_{t/2}^{t} (t - \tau)^{-1/2} (1 + \tau)^{-(n/2)(q-1/p)-\beta q/2} d\tau
\]

\[
\leq Ct^{-(n/2)(q-1/p)-(\beta q-1)/2}
\]

for all \( t > 0 \).
Finally, to obtain (3.18), combine (3.19) and (3.20) (note that

\[-(n/2(q - 1/p) - (\beta q - 1)/2 \leq -(n/2)(1 - 1/p) - 1/2\]

for \(q \geq (n + 2)/(n + \beta)\). \qed

**Proof of Corollary 2.2.** As pointed out in Section 2, it follows from Corollary 3.1 that

\[t^{(n/2)(1 - 1/p)} \|\partial_t u(\cdot, t) - e^{t\Delta} u_0(\cdot)\|_p \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty,\]

for each \(q > 1 + 1/(n + \beta)\) and every \(p \in [1, \infty)\). Thus (3.21) combined with Propositions 2.1 and 3.1 yields Corollary 2.2. \qed

A few remarks are in order.

**Remark 3.4.** If the nonlinear term in (1.1) has the form \(\nabla \cdot f(u)\) and the function \(f\) is sufficiently regular at zero, it is possible to improve the conclusion of Corollary 2.2 to

\[t^{(n/2)(1 - 1/p) + \beta/2} \|\partial^{\gamma} u(\cdot, t) - A\partial^{\gamma} D^\beta G(\cdot, t)\|_p \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty\]

for the multi-index \(\gamma\) depending on the regularity of \(f\). \qed

**Remark 3.5.** Consider Theorem 2.1 and Corollary 2.2 in the context of the viscous Burgers equation

\[u_t - u_{xx} + (u^2/2)_x = 0, \quad u(x, 0) = u_0(x), \quad x \in \mathbb{R}.\]

This is problem (1.1)-(1.2) with \(n = 1, q = 2,\) and \(a = 1/2\). It is well-known (cf. e.g. [13, 18, 19, 12]) that the large time behavior of solutions to this equation supplemented with the integrable initial condition is described by so-called *nonlinear diffusion waves* (cf. (1.4), above). If, however, it is assumed that \(u_0\) satisfies the conditions from Proposition 2.1 with some \(0 < \beta < 1\), and if, moreover, \(\sup_{t > 0} \|e^{t\Delta} u_0\|_1\) is sufficiently small, the asymptotics for large \(t\) of solutions to the Burgers equation is given by the self-similar solutions \(A D^\beta G(x, t)\) to the heat equation.

For completeness of the exposition, we analyze problem (3.22) in more detail. Using the Hopf-Cole transformation one obtains the solution of (3.22) of the following form

\[u(x, t) = -\frac{(e^{t\Delta} u_0)_x(x)}{(e^{t\Delta} u_0)(x)}\]
where as usual \( w_0(x) = \exp\left(-\int_{-\infty}^{x} u_0(y) \, dy \right) \). Supposing that \( u_0 \in L^1(I\mathbb{R}) \), \( \int_{I\mathbb{R}} u_0(x) \, dx = 0 \), and \( u_0 \) satisfies the Miyakawa moment condition

\[
(3.24) \quad \int_{I\mathbb{R}^n} |x|^\beta |u_0(x)| \, dx < \infty,
\]

it is easy to show directly from the explicit formula (3.23) that the \( L^1 \)-norm of \( u(\cdot, t) \) decays at the rate \( t^{-\beta/2} \). Indeed, first note that the denominator is the solution to the heat equation with the datum \( w_0 \) and is bounded from below by \( \exp(-\|u_0\|_1) \). Thus, it is only necessary to bound the numerator

\[
\left( e^{t\Delta} w_0 \right)_x (x) = \int_{I\mathbb{R}} G(x - y, t)(w_0)(y) \, dy = - \int_{I\mathbb{R}} G(x - y, t)u_0(y) \exp\left(-\int_{-\infty}^{y} u_0(z) \, dz \right) \, dy.
\]

Obviously, \( (w_0)_y \in L^1(I\mathbb{R}) \) and \( \int_{I\mathbb{R}} |y|^\beta |(w_0)_y(y)| \, dy < \infty \), since \( u_0 \) has these properties and \( \exp\left(-\int_{-\infty}^{y} u_0(z) \, dz \right) \) is a bounded function. Let us skip an easy proof that \( \int_{I\mathbb{R}} (w_0)_y(y) \, dy = 0 \). Consequently, \( (w_0)_y \) satisfies the Miyakawa conditions, so the \( L^1 \)-norms of solutions to (3.22) decay with the rate \( t^{-\beta/2} \). Finally, repeating the calculations from the proof of Corollary 3.1 yields that the large time behavior of solutions to (3.22) is described by \( AD^\beta G(\cdot, t) \). Note that here non-smallness assumptions on \( u_0 \) have been imposed unlike it was done in Theorem 2.1 in the case \( 1 + 1/(n + \beta) \leq q \leq 1 + 1/n \). This example suggests that such an assumption in Theorem 2.1 is not necessary, however, the proof of a stronger version requires new ideas.

\[\square\]

**Remark 3.6.** In this paper, we limit ourselves to the case \( \beta \in (0, 1) \). We expect a completely different large time behavior of solutions to (1.1)-(1.2) in \( \beta \geq 1 \) for the following reason. Suppose that

\[
(3.25) \quad u_0 \in L^1(I\mathbb{R}^n, (1 + |x|) \, dx) \quad \text{and} \quad \int_{I\mathbb{R}^n} u_0(x) \, dx = 0.
\]

It is proved in [10] that \( \|e^{t\Delta} u_0\|_1 \leq Ct^{-1/2}\|u_0\|_{L^1(I\mathbb{R}^n, |x|) \, dx} \) for all \( t > 0 \) and a constant \( C \); moreover,

\[
t^{1/2}\left\|e^{t\Delta} u_0 - \int_{I\mathbb{R}^n} xu_0(x) \, dx \cdot \nabla G(x, t) \right\|_1 \to 0 \quad \text{as} \quad t \to \infty.
\]

Now, using the second order asymptotic expansion by Zuazua [37] (cf. also [3] for analogous results with more general diffusion operators and less regular initial conditions) of solutions to (1.1)-(1.2) with \( q > 1 + 2/n \), we obtain that the quantity

\[
t^{1/2}\left\|u(\cdot, t) - \left(\int_{I\mathbb{R}^n} xu_0(x) \, dx - a \int_{0}^{\infty} \int_{I\mathbb{R}^n} (u|u|^{q-1})(x, \tau) \, dx \, d\tau \right) \cdot \nabla G(x, t) \right\|_1
\]
tends to 0 as \( t \to \infty \). This asymptotic result shows that the large time behavior of solutions with the initial data satisfying (3.25) can be classified as weakly nonlinear in the sense of Zuazua [37]. Here, however, the first term of the asymptotics comes linearly from the heat kernel, but has a nonlinear dependence on the solution through a multiplicative factor (as noted by Zuazua in [37], it is an open question if this factor is different from zero). Hence, assuming that \( \| e^{t\Delta} u_0 \|_{1} \leq C t^{-(\beta/2)} \) for some \( \beta \geq 1 \) one should expect asymptotic expansions of solutions completely different from that in Corollary 2.2, specifically of the form just described.

\[ \Box \]

4 Nonlinear asymptotics

The following two lemmata give the crucial steps to yield the necessary estimates of the integral equation (2.1):

**Lemma 4.1** Let \( a \in \mathbb{R}^n \) be a fixed constant vector. There exists a constant \( C > 0 \) such that for every \( w \in L^1(\mathbb{R}^n) \) we have

\[
\| a \cdot \nabla e^{t\Delta} w \|_{B^{-\beta,\infty}_1} \leq C t^{(\beta-1)/2} \| w \|_1
\]

for all \( t > 0 \).

**Proof.** Using the definition of the norm in \( B^{-\beta,\infty}_1 \) and properties of the heat semigroup yields

\[
\| a \cdot \nabla e^{t\Delta} w \|_{B^{-\beta,\infty}_1} = \sup_{s>0} s^{\beta/2} \| e^{s\Delta} a \cdot \nabla w \|_1
\]

\[
= \sup_{s>0} s^{\beta/2} \| a \cdot \nabla e^{(t+s)\Delta} w \|_1
\]

\[
\leq C \| w \|_1 \sup_{s>0} s^{\beta/2} (t+s)^{-1/2}
\]

for all \( t > 0 \). Now, a direct calculation shows that \( \sup_{s>0} s^{\beta/2} (t+s)^{-1/2} = C(\beta) t^{(\beta-1)/2} \) with \( C(\beta) \) independent of \( t \). \( \Box \)

**Lemma 4.2** Assume that \( v \in B^{-\beta,\infty}_1 \). Then for each \( p \in [1, \infty] \) there exists a constant \( C > 0 \) such that

\[
\| e^{t\Delta} v \|_p \leq C t^{-(n/2)(1-1/p)-\beta/2} \| v \|_{B^{-\beta,\infty}_1}
\]

for all \( t > 0 \).
Proof. Standard properties of the heat semigroup $e^{t\Delta}$ and the definition of the norm in $B_{1}^{-\beta,\infty}$ give
\[
\|e^{t\Delta}v\|_p \leq C(t/2)^{-(n/2)(1-1/p)}\|e^{(t/2)\Delta}v\|_1 \leq Ct^{-(n/2)(1-1/p)-\beta/2}\|v\|_{B_{1}^{-\beta,\infty}}.
\]
for all $t > 0$ and a constant $C$. 

We are ready to prove the existence Theorem 2.2 in the critical case $q^*$. 

Proof of Theorem 2.2. Our reasoning is similar to that in [5, 6, 7, 21]. Moreover, the calculations below resemble those in the proof of Theorem 2.1 with $1 + 1/(n + \beta) \leq q \leq 1 + 1/n$, thus we shall be brief in details. Recall that in this section we consider
\[ q = q^* = 1 + \frac{1}{n + \beta} \]
which is equivalent to
\[ \frac{n}{2} \left( 1 - \frac{1}{q} \right) + \frac{\beta}{2} = \frac{1}{q} \left( \frac{1}{2} + \frac{\beta}{2} \right). \]

Equip the space $\mathcal{X}$ with the norm
\[
\|u\|_{\mathcal{X}} = \max\{\sup_{t>0} \|u(t)\|_{B_{1}^{-\beta,\infty}}, \sup_{t>0} t^{(n/2)(1-1/q)+\beta/2}\|u(t)\|_q\}.
\]
We will show that the nonlinear operator
\[ \mathcal{N}(u)(t) \equiv e^{t\Delta}u_0 - \int_0^t a \cdot \nabla e^{(t-\tau)\Delta} (u|u|^{q-1})(\tau) \, d\tau \]
is a contraction on the box
\[ B_{R,\varepsilon} = \{u \in \mathcal{X} : \|u(t)\|_{B_{1}^{-\beta,\infty}} \leq R \text{ and } \sup_{t>0} t^{(n/2)(1-1/q)+\beta/2}\|u(t)\|_q \leq 2\varepsilon\} \]
for sufficiently large $R > 0$ and a suitably small $\varepsilon > 0$. This will be guaranteed provided the following estimates can be shown to hold
\[ t^{(n/2)(1-1/q)+\beta/2}\|\mathcal{N}(u)(t)\|_q \leq C\|u_0\|_{B_{1}^{-\beta,\infty}} + C\varepsilon^q, \]
and
\[ \|\mathcal{N}(u)(t) - \mathcal{N}(v)(t)\|_{B_{1}^{-\beta,\infty}} \leq C\varepsilon^{q-1} \sup_{t>0} t^{(n/2)(1-1/q)+\beta/2}\|u(\cdot, t) - v(\cdot, t)\|_q \]
\[ t^{(n/2)(1-1/q)+\beta/2}\|\mathcal{N}(u)(t) - \mathcal{N}(v)(t)\|_q \leq C\varepsilon^{q-1} \sup_{t>0} t^{(n/2)(1-1/q)+\beta/2}\|u(\cdot, t) - v(\cdot, t)\|_q. \]
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with constants $C$ independent of $u$ and $t$.

For the proof of (4.3) observe that $\|e^{t\Delta}u_0\|_{B^{1,\beta}_t,\infty} \leq \|u_0\|_{B^{1,\beta}_t,\infty}$. Hence computing the $B^{1,\beta}_t,\infty$-norm of (1.2) for $u \in B_{R,\varepsilon}$ and applying Lemma 4.1 we obtain

\[
\|N(u)(t)\|_{B^{1,\beta}_t,\infty} \leq \|e^{t\Delta}u_0\|_{B^{1,\beta}_t,\infty} + \int_0^t \|a \cdot \nabla e^{(t-\tau)\Delta} (u|u|^{q-1})(\tau)\|_{B^{1,\beta}_t,\infty} d\tau
\]

\[
\leq \|u_0\|_{B^{1,\beta}_t,\infty} + C \int_0^t (t-\tau)^{(\beta-1)/2} \|u(\tau)\|^q d\tau
\]

\[
\leq \|u_0\|_{B^{1,\beta}_t,\infty} + C\varepsilon^q \int_0^t (t-\tau)^{(\beta-1)/2} \tau^{-\beta q/2} d\tau.
\]

Note now that the assumptions $\beta \in (0, 1)$ and $q = 1 + 1/(n + \beta)$ guarantee that the integral on the right hand side is finite for any $t > 0$. Moreover, since $(\beta-1)/2 - n(q-1)/2 - \beta q/2 + 1 = 0$, it follows that this integral is independent of $t$. Hence, estimate (4.3) holds true.

The proof of (4.4) is similar. It involves Lemma 4.2 as follows

\[
\|N(u)(t)\|_q \leq \|e^{t\Delta}u_0\|_q + \int_0^t \|a \cdot \nabla e^{(t-\tau)\Delta} (u|u|^{q-1})(\tau)\|_q d\tau
\]

\[
\leq Ct^{-n/(2(1-1/q)-\beta/2)} \|u_0\|_{B^{1,\beta}_t,\infty}
\]

\[
+ C\varepsilon^q \int_0^t (t-\tau)^{-n/(2(1-1/q)-1/2) - \beta q/2} d\tau.
\]

In this case, the conditions on $\beta, q$ imply again that the integral on the right hand side is finite for every $t > 0$. In fact, by a change of variables, it equals $Ct^{-n/(2(1-1/q)-\beta/2)}$ for a constant $C > 0$. Hence (4.4) is proved.

The proofs of (4.5) and (4.6) are completely analogous. The only difference consists in using elementary inequality

\[
\|u|u|^{q-1} - v|v|^{q-1}\|_1 \leq C\|u - v\|_q \left(\|u\|_q^{q-1} + \|v\|_q^{q-1}\right)
\]

valid for all $u, v \in L^q(\mathbb{R}^n)$.

Finally, it follows from (4.3)–(4.6) that $N : B_{R,\varepsilon} \rightarrow B_{R,\varepsilon}$ is a contraction for $R > 2\|u_0\|_{B^{1,\beta}_t,\infty}$ and a suitably small $\varepsilon > 0$. Hence the sequence defined as $u_0(t) = e^{t\Delta}u_0$ and $u_{n+1}(t) = N(u_n(t))$ converges to a unique (in $B_{R,\varepsilon}$) global-in-time solution to (1.1)–(1.2) provided $u_0(t) \in B_{R,\varepsilon}$, i.e. $\|u_0\|_{B^{1,\beta}_t,\infty}$ is sufficiently small (cf. Lemma 4.2).

The proof of Theorem 2.3 requires the following result from [21, Lemma 6.1].

**Lemma 4.3** Let $w \in L^1(0, 1)$, $w \geq 0$, and $\int_0^1 w(x) \, dx < 1$. Assume that $f$ and $g$ are two nonnegative, bounded functions such that

\[
f(t) \leq g(t) + \int_0^1 w(\tau)f(\tau t) \, d\tau.
\]
Then \( \lim_{t \to \infty} g(t) = 0 \) implies \( \lim_{t \to \infty} f(t) = 0 \). 

The next task is to prove the stability Theorem 2.3.

**Proof of Theorem 2.3.** The subtraction of equation (2.3) for \( v \) from the analogous expression for \( u \) leads to the following identity

\[
(4.10) \quad u(t) - v(t) = e^{t}\Delta(u_0 - v_0) - \int_0^t a \cdot \nabla e^{(t-\tau)\Delta}(u|u|^{q-1} - v|v|^{q-1})(\tau) \, d\tau.
\]

Repeating the reasoning from the proof of (4.4) involving inequality (4.8) gives

\[
(4.11) \quad \|u(\cdot, t) - v(\cdot, t)\|_q \leq Ct^{-(n/2)(1-1/q)-\beta/2} \left(\left(\frac{t}{2}\right)^{\beta/2}\|e^{(t/2)\Delta}(u_0 - v_0)\|_1\right)
\]

\[
\quad + C\int_0^t (t - \tau)^{-(n/2)(1-1/q)-1/2}\|u(\cdot, \tau) - v(\cdot, \tau)\|_q
\]

\[
\quad \times \left(\|u(\cdot, \tau)\|^{q-1}_q + \|v(\cdot, \tau)\|^{q-1}_q\right) \, d\tau.
\]

By Theorem 2.2, the both quantities

\[
\sup_{t > 0} t^{(n/2)(1-1/q)+\beta/2}\|u(\cdot, t)\|_q \quad \text{and} \quad \sup_{t > 0} t^{(n/2)(1-1/q)+\beta/2}\|v(\cdot, t)\|_q
\]

are bounded by \( 2\varepsilon \). Hence, multiplying (4.11) by \( t^{(n/2)(1-1/q)+\beta/2}, \) putting

\[
(4.12) \quad f(t) = t^{(n/2)(1-1/q)+\beta/2}\|u(\cdot, t) - v(\cdot, t)\|_q,
\]

and changing variable \( \tau = ts \), we get

\[
(4.13) \quad f(t) \leq C\left(\frac{t}{2}\right)^{\beta/2}\|e^{(t/2)\Delta}(u_0 - v_0)\|_1
\]

\[
\quad + 2C\varepsilon^{q-1} \int_0^1 (1 - s)^{-(n/2)(1-1/q)-1/2}s^{-(n/2)(q-1)-\beta q/2} f(ts) \, ds.
\]

Since \((1 - s)^{-(n/2)(1-1/q)-1/2}s^{-(n/2)(q-1)-\beta q/2} \in L^1(0, 1)\) (cf. comments following inequalities (4.7)), we may apply Lemma 4.3 obtaining \( f(t) \to 0 \) as \( t \to \infty \) for sufficiently small \( \varepsilon > 0 \). This proves (2.13) for \( p = q \).

Next, we prove (2.13) for \( p = 1 \). Computing the \( L^1 \)-norm of (4.10) and repeating the calculations from (11) and (111) yield

\[
(4.14) \quad t^{\beta/2}\|u(\cdot, t) - v(\cdot, t)\|_1 \leq t^{\beta/2}\|e^{t\Delta}(u_0 - v_0)\|_1
\]

\[
\quad + C\int_0^1 (1 - s)^{-1/2}s^{-(n/2)(q-1)-\beta q/2} f(ts) \, ds,
\]

where (4.11) leads to the following identity

\[
(4.15) \quad \|u(\cdot, t) - v(\cdot, t)\|_1 \leq C t^{\beta/2}\|e^{t\Delta}(u_0 - v_0)\|_1
\]

\[
\quad + C\int_0^1 (1 - s)^{-1/2}s^{-(n/2)(q-1)-\beta q/2} f(ts) \, ds.
\]
where \( f \), defined in (4.12), is a bounded function satisfying \( \lim_{t \to \infty} f(t) = 0 \), by the first part of this proof. Hence (2.12) and the Lebesgue Dominated Convergence Theorem give

\[
\lim_{t \to \infty} t^{\beta/2} \|u(\cdot, t) - v(\cdot, t)\|_1 = 0.
\]

The next stage of the proof deals with (2.13) for all \( p \in (1, \infty) \). The calculations from (3.11) show that \( \|u(\cdot, t)\|_\infty \) and \( \|v(\cdot, t)\|_\infty \) can be both bounded by \( Ct^{-n/2-\beta/2} \) for all \( t > 0 \) and a constant \( C \) independent of \( t \). Hence, by the Hölder inequality and (4.14) it follows that

\[
\|u(\cdot, t) - v(\cdot, t)\|_p \leq C\|u(\cdot, t) - v(\cdot, t)\|_1^{1/p} \\
\times \left( \|u(\cdot, t)\|_\infty^{1-1/p} + \|v(\cdot, t)\|_\infty^{1-1/p} \right) \\
= \alpha \left(t^{-n/2}(1-1/p-\beta/2) \right) \text{ as } t \to \infty,
\]

where we used the following inequality

\[
|g|q^{-1} - h|h|q^{-1} | \leq \frac{q}{2} |g - h| \left( |g|q^{-1} + |h|q^{-1} \right)
\]

valid for all \( g, h \in \mathbb{R} \) and \( q > 1 \).

Finally, the proof of (2.13) for \( p = \infty \) involves equation (4.10) and (2.13) proved already for all \( p \in [1, \infty) \). Standard \( L^p - L^q \) estimates of the of the heat semigroup imply that

\[
t^{n/2+\beta/2} \|e^{t\Delta}(u_0 - v_0)\|_\infty \leq C t^{n/2+\beta/2}(t/2)^{-n/2} \|e^{(t/2)\Delta}(u_0 - v_0)\|_1 \\
= C(t/2)^{\beta/2} \|e^{(t/2)\Delta}(u_0 - v_0)\|_1 \to 0
\]
as \( t \to \infty \) by assumption (2.12).

To study the second term on the right hand side of (4.10), the integration range with respect to \( \tau \) is decomposed into \([0, t] = [0, t/2] \cup [t/2, t] \).

Combining inequality (4.13) with estimates of the heat semi-group and the H"older inequality yields

\[
\|a \cdot \nabla e^{(t-\tau)\Delta} (u|u|^{q-1} - v|v|^{q-1}) (\tau)\|_\infty \leq C(t - \tau)^{-n/2-1/2} \|u(\tau) - v(\tau)\|_1 \left( \|u(\tau)\|_\infty^{q-1} + \|v(\tau)\|_\infty^{q-1} \right) \\
\leq C(t - \tau)^{-n/2-1/2} \tau^{-\beta/2-(n+\beta)(q-1)/2} f_1(\tau),
\]

where \( C \) is independent of \( t \) and \( \tau \), and \( f_1(\tau) = \tau^{\beta/2} \|u(\tau) - v(\tau)\|_1 \) is the bounded function which tends to 0 as \( t \to \infty \) by (2.13) for \( p = 1 \).
Moreover, choosing $1/r + 1/z = 1$, similar calculations lead to

$$
\left\| a \cdot \nabla e^{(t-\tau)\Delta} \left( u|u|^{q-1} - v|v|^{q-1} \right) (\tau) \right\|_\infty \leq C(t - \tau)^{-\alpha} \left( \frac{n}{2} \right)^{-\frac{1}{2}} \beta^{\frac{1}{2}} \left( n + \beta \right) \left( q - 1 \right) \left( q - 1 \right) \left( q - 1 \right) f_r(\tau)
$$

(4.17)

where $f_r(\tau) = \tau^{(n/2)\left(1-1/r\right)+\beta/2} \left\| u(\tau) - v(\tau) \right\|_r$ also tends to 0 as $t \to \infty$ by (2.13). Hence, by the change of variables $\tau = ts$, it follows from (4.16) that

$$
\int_0^{t/2} \left\| a \cdot \nabla e^{(t-\tau)\Delta} \left( u|u|^{q-1} - v|v|^{q-1} \right) (\tau) \right\|_\infty d\tau \leq C t^{-\alpha} \int_0^{1/2} (1 - s)^{-\alpha} f_1(\tau) ds.
$$

The integral on the right hand side is finite (recall that $q = 1 + 1/(n + \beta)$), because

$$
-\frac{\beta}{2} - \frac{(n + \beta)(q - 1)}{2} = -\frac{\beta + 1}{2} > -1 \quad \text{for } \beta \in (0, 1).
$$

This integral tends to 0 as $t \to \infty$ by the Lebesgue Dominated Convergence Theorem.

The case of the integral $\int_{t/2}^t \ldots d\tau$ involves inequality (4.17) with $z > 1$ chosen such that $-(n/2)(1-1/z) - 1/2 > -1$. The proof here is analogous as in the last case and as such will be omitted. This completes the proof of Theorem 2.3.

\[\square\]

5 Balance case: self-similar solutions

In this section, we continue our analysis on the asymptotic behaviour of solutions of (1.1) when $q$ is the critical exponent $q = q^* = 1 + 1/(n + \beta)$. Here, we would like to explain how Theorem 2.2 ensures the existence of a new class of self-similar solutions to (1.1) and how Theorem 2.3 shows that there is a large class of solutions whose asymptotic behaviour in $L^p(\mathbb{R}^n)$ corresponds to self-similar solutions.

Elementary calculations show that if $u(x,t)$ is a solution to the equation

$$
u_t - \Delta u + a \cdot \nabla (u|u|^{1/(n+\beta)}) = 0,$$

(5.1)

then so is $\lambda^{n+\beta} u(\lambda x, \lambda^2 t)$ for every $\lambda > 0$. Self-similar solutions should satisfy the equality $u(x,t) = \lambda^{n+\beta} u(\lambda x, \lambda^2 t)$, hence choosing $\lambda = \lambda(t) = 1/\sqrt{t}$ yields a self-similar form

$$
u(x,t) = t^{-\frac{n+\beta}{2}} U \left( \frac{x}{\sqrt{t}} \right),
$$

(5.2)
Convection-diffusion equation

where $U(x) = u(x, 1)$, $x \in \mathbb{R}^n$, and $t > 0$. Substituting $u(x, t)$ defined in (5.2) to equation (5.3), we show the function $U = U(x)$ satisfies the elliptic equation

$$-\Delta U - \frac{1}{2} x \cdot \nabla U = \frac{n + \beta}{2} U + a \cdot \nabla (U|U|^{1/(n+\beta)}) = 0.$$  

(5.3)

We believe that one can obtain solutions to (5.3) using ideas similar to those developed in [1]. In that paper, Aguirre, Escobedo and Zuazua establish a priori estimates and existence of solutions to the system

$$-\Delta f - \frac{1}{2} x \cdot \nabla f = \frac{n}{2} f + a \cdot \nabla \Psi(f) = 0.$$  

(5.4)

The main difference between our case and (5.4) is that their coefficient for $f$ is exactly $n/2$ which is the first eigenvalue of $L = -\Delta f - \frac{1}{2} x \cdot \nabla f$.

In our paper, however, we propose a completely different construction of self-similar solutions, based on the Cannone method [6]. Let us formulate this result.

**Theorem 5.1** Assume that $u_0 \in \mathcal{B}_{1}^{-\beta, \infty}$ is a homogeneous distribution of degree $-n - \beta$. Under the assumptions of Theorem 2.2, the constructed solution to (1.1)-(1.2) is self-similar; hence, of the form (5.2).

The proof of this theorem follows the standard reasoning (cf. e.g. [3], Section 3) and is based on the uniqueness result from Theorem 2.2. Let us skip other details. Here, we only mention that the fractional derivative of order $\beta$ of the Dirac delta $D^\beta \delta_0$ belongs to $\mathcal{B}_{1}^{-\beta, \infty}$. Indeed, this follows from the definitions of $e^{t\Delta}$ and $\delta_0$, since $e^{t\Delta} D^\beta \delta_0 = D^\beta G(x, t)$ (cf. the proof of Proposition 2.1). Hence, the self-similar form of $D^\beta G(x, t)$ (see (3.1)) yields $\|e^{t\Delta} D^\beta \delta_0\|_1 = t^{-\beta/2} \|D^\beta G(\cdot, 1)\|_1$. Finally, note that the tempered distribution $D^\beta \delta_0$ is homogeneous of degree $-n - \beta$. Consequently, Theorem 5.1 implies that every solution to (1.1)-(1.2) corresponding to $A D^\beta \delta_0$ with sufficiently small $|A|$ is self-similar.

Now, let $t^{-(n+\beta)/2} U_A(x/\sqrt{t})$ denote the self-similar solution corresponding to the initial datum $u_0 = A D^\beta \delta_0$ for some $A \in \mathbb{R}$. In the following theorem, we show that $U_A$ describes the asymptotic behavior of a large class of solutions to (1.1)-(1.2).

**Theorem 5.2** Let the assumptions from Theorem 2.2 hold true. Assume that $v$ is the solution of (1.1)-(1.2) constructed in Theorem 2.2 corresponding to
the initial data $v_0 \in B_{1,\infty}^{-\beta}$. Let $t^{-(n+\beta)/2}U_A(x/\sqrt{t})$ be the self-similar solution corresponding to the initial datum $u_0 = AD^\beta \delta_0$ for sufficiently small $|A|$. Suppose that

$$\lim_{t \to \infty} t^{\beta/2} \| e^{t\Delta} v_0 - AD^\beta G(\cdot, t) \|_1 = 0. \quad (5.5)$$

Choosing $\varepsilon > 0$ in Theorem 2.2 sufficiently small, we have

$$\lim_{t \to \infty} t^{(n/2)(1-1/p)+\beta/2} \| v(\cdot, t) - t^{-(n+\beta)/2}U_A(\cdot/\sqrt{t}) \|_p = 0$$

for every $p \in [1, \infty]$.

This theorem is a direct corollary of Theorem 2.3. Recall only that, by Proposition 2.1, the limit relation in (5.5) holds true if, in particular, $I^\beta v_0 \in L^1(\mathbb{R}^n)$. In this case, $A = \int_{\mathbb{R}^n} I^\beta v_0(x) \, dx$.

Let us compare Theorem 5.2 with its counterpart proved by Escobedo and Zuazua in [13], and recalled already in Introduction, formula (1.4). When $\int u_0 = M \neq 0$ and $q = 1 + 1/n$, equation (1.1) has a one-parameter family of self-similar solutions parameterized by $M$. Moreover, $U_M$ describes the large time asymptotics of all solutions with mass $M$. Note that for every $u_0 \in L^1(\mathbb{R}^n)$, the condition $\int u_0 = M$ is equivalent to

$$\| e^{t\Delta} u_0 - MG(\cdot, t) \|_1 \to 0 \quad \text{as} \quad t \to \infty.$$  

In our case, when $M = 0$, the set of self-similar solutions to (1.1) with $q = 1 + 1/(n + \beta)$ is more complicated, however, relation (5.5) (or, more generally, (2.12)) still allows us to identify solutions to (1.1)-(1.2) with the given self-similar large time behavior.

### 6 Conclusions

The ideas developed in this paper can be applied to other types of equations. As the first example, let us look at the Navier-Stokes equations for the incompressible fluid

$$u_t - \Delta u + u \cdot \nabla u + \nabla p = 0,$$

$$\text{div} \, u = 0,$$

$$u(\cdot, 0) = u_0.$$  

It well-known (see e.g. [27]) that any integrable solenoidal smooth vector field $u_0$ (i.e. $\text{div} \, u_0 \equiv 0$) satisfies $\int u_0 = 0$. This fact motivated Miyakawa to study
in [27] the $L^2$-decay of solutions to the Navier-Stokes system endowed with integrable initial conditions satisfying $\|e^{t\Delta}u_0\|_1 \leq C t^{-\beta/2}$ for some $0 < \beta < 1$, a constant $C$, and all $t > 0$. We believe that our methods will offer some improvements to the Miyakawa results.

We also expect that it will be possible to improve asymptotic expansions of solutions to the Korteweg-de Vries-Burgers equation as well as to the Benjamin-Bona-Mahony-Burgers equation obtained recently in [22, 23]. Some preliminary progress in this direction was already done by M. Mei in [25].

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