Critical phenomena in superlattices: Reentrant dimensional crossover and anomalous critical amplitudes

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Abstract

A crossover from $d$ to $d-1$, and then back to $d$-dimensional critical behavior is argued to be a generic feature characterizing ordering in a $d$-dimensional superlattice composed of atomically thick films of two ferromagnets. The crossover leads to anomalous changes in the amplitudes of critical singularities. In $d = 3$ Heisenberg and $XY$ superlattices large scale critical fluctuations persist over a wide temperature range.

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I. INTRODUCTION AND THE STATEMENT OF THE PROBLEM

Recent progress in creating artificial magnetic heterostructures [1] gives an impetus to the theory of phase transitions in multilayers. While the related experimental and theoretical efforts have been so far mostly concentrated at microscopic effects, this paper addresses the problem of universal features associated with large scale ordering in superlattices. Suppose the superlattice is built of atomically thick layers of two magnets, which taken separately both order through second order phase transitions at two respective bulk critical points $T_{c1} > T_{c2}$. Suppose also for simplicity that both components are ferro-magnets and that the critical behavior at their bulk Curie points is similar, i.e. both components belong to the same universality class. The question then is: at what temperature $T_c$ will long range ferromagnetic order appear in the superlattice and what critical behavior will be observed at that temperature? One can reasonably expect that in the limit of thick layers the answers to these questions will be largely universal, depending only on the bulk critical properties of the two components and the geometry of the superlattice; these universal aspects of the problem represent the subject of this paper.

We will consider the simplest superlattice geometry (Fig. 1) constructed of two elementary building blocks: slabs, or layers, of two ferromagnets, 1 and 2, of finite thickness $L_1$ and $L_2$, respectively. The slabs are stacked periodically in the $z$-direction, so that $L = L_1 + L_2$ is the period. The system is homogeneous in the remaining $d' = d - 1$ dimensions. While the dimensionality $d = 3$ (Fig. 1a) is naturally the most interesting one in view of experimental applications, the planar, $d = 2$, superlattice geometry can be quite conceivably realized by cutting a thin film out of a three-dimensional multilayered sample (Fig. 1b) or by deposition of a magnetic film on a substrate, which is itself cut out of a superlattice. Extension of our consideration to $d = 2$ is important because of analytical tractability of two-dimensional models. Being interested in the long wave length aspects of the problem, we will consider the limit of atomically thick slabs: $L_1 \gg a_1$, $L_2 \gg a_2$, where $a_{1,2}$ are the thicknesses of elementary, molecular layers of the two components, respectively; correspondingly, the in-
terfaces separating two subsequent slabs do not have to be atomically smooth. Naturally, universality extends applicability of the results obtained below beyond this particular magnetic model; among other interesting applications are a superfluid in a periodic matrix and a superfluid film deposited on a periodic substrate.

Universal critical phenomena related to three different geometrical elements present in a superlattice have been extensively studied theoretically and, to a variable extent, experimentally. First, at a single interface separating half-infinite near-critical, say \( |t_1| \equiv |T - T_{c1}|/T_{c1} \ll 1 \) and disordered, \( |t_2| \equiv |T - T_{c2}|/T_{c2} = O(1) \) magnets, one expects ordinary surface critical phenomena to take place at the near-critical side of the interface. It has been established theoretically \([2,3]\) that the critical exponents and amplitudes characterizing surface values of various observables are quite different from those in the bulk, as well as from the \( d' \)-dimensional ones. However, experimental confirmation of those theoretical findings has turned out to be difficult. Note that near \( T_{c2} \), where the other half of the single interface model goes through the criticality, the first one is already ordered and imposes a magnetic field acting at the surface of the second component. Therefore, one expects normal surface critical behavior \([4,5]\) to take place at the interface near \( T_{c2} \).

Second, near the bulk critical temperature \( T_{cj} \) of the \( j \)-th, \( j = 1, 2 \), component a thick, \( L_j \gg a_j \), single slab of that component exhibits a dimensional crossover from the critical behavior characteristic of bulk, \( d \)-dimensional, samples to the \( d' = d - 1 \)-dimensional critical behavior, as observed in thin films of the same material \([6]\). We will refer to this crossover as bulk-to-film below; it has been observed experimentally \([7]\). Finally, in a system of thin layers connected to each other by very weak interlayer bonds, near the critical temperature \( T'_c \) of a single layer one expects a crossover from \( d' \)-dimensional to the bulk, \( d \)-dimensional behavior, which we will call film-to-bulk crossover below. This type of dimensional crossover has been well experimentally documented in a variety of magnetic \([8]\) and non-magnetic layered systems \([9]\).

One source of motivation for studying critical phenomena in multilayers is that the significantly increased surface-to-volume ratio may improve chances of observing features
characteristic of the $d = 2$-dimensional and, especially, the elusive $d = 3$-surface critical behavior with respect to experiments involving single thin film or semi-bulk samples. However, finite separation between interfaces in a superlattice leads to correlations between fluctuations in different layers and at different interfaces. One therefore expects surface and two-dimensional scaling to be limited to a certain range of length scales, while eventually cooperative phenomena involving fluctuations on scales larger than the period of the superlattice will dominate the criticality. A crucial insight comes from the exact solution available for a ferromagnetic $d = 2$-Ising superlattice, as modelled by a planar Ising model composed of alternating strips of two components \[10\]. Recent analysis \[11\] demonstrated that the anomalous changes in the critical amplitudes and other interesting features of that model can be simply explained by reentrant dimensional crossover: as the temperature approaches the critical temperature of the superlattice the long wave length properties of the system are successively dominated by fluctuations characteristic of the the $d = 2$-Ising, then $d = 1$-Ising, and then again $d = 2$-Ising critical behavior.

Here we generalize and explore this picture by constructing and analyzing a renormalization group (RG) flow which provides a scaling description of criticality in experimentally interesting three dimensional superlattices as well as in the Heizenberg (isotropic) and XY (easy plane) universality classes. In the proposed scenario as the scaled temperature $t = (T - T_c)/T_c$, or $t_1 = (T - T_{c1})/T_{c1}$, decreases, the critical behavior of the superlattice displays the features of the three simpler systems listed above (Fig. 2): first, as $T_{c1}$ is approached, the original superlattice (i) in Fig. 2) can be approximated by sequence of thick layers of the first component weakly coupled through the 2-layers (ii) in Fig. 2). At larger values of $t$ this coupling can be neglected and the layers display regular bulk critical behavior with the ordinary surface critical behavior at the interfaces. Then, as the correlation length in the first component becomes comparable to the thickness $L_1$, each slab of the first component exhibits direct bulk-to-film, $d \to d'$-dimensional crossover. At still smaller values of $t$, the thickness of these slabs is irrelevant and the system becomes equivalent to the third paradigm mentioned above: a weakly coupled layered system (iii) in Fig. 2).
3). The reverse, film-to-bulk crossover finally takes the system back to the uniform, bulk $d$-dimensional behavior, characterized however by high degree of anisotropy ((iv) in Fig. 2). This reentrant crossover behavior leads to dramatic changes in the critical amplitudes, as shown in Fig. 3 for the $d = 3$-Ising class: the critical exponent characterizing divergence of a physical quantity, for which the specific heat per unit volume $C$ has been chosen in Fig. 3, changes two times, as $t$ goes to zero, between the values $\alpha$ and $\alpha'$ characteristic of the $d = 3$ and $d' = 2$-dimensional critical behavior, respectively. The relative widths $\tau_D$ (subscript $D$ denotes direct, bulk-to-film crossover here) and $\tau_R$ (subscript $R$ stands for the reverse, film-to-bulk crossover) of the temperature domains where the planar Ising and the reentrant bulk critical behaviors are observed will be calculated below; they provide convenient scaling combinations in terms of which a proper description of criticality in superlattices is achieved.

Two further points have to be added to this preview of the paper. First, a crucial element of our qualitative picture is that different layers of the first component remain effectively decoupled while the bulk-to-film crossover (from (ii) to (iii) in Fig. 2) happens within them. This turns out to be guarantied by the fact that the subsequent 1-slabs are coupled through the surface spins whose correlations are much weaker than of those in the bulk. Thus the surface critical behavior, which is not directly seen in temperature dependencies of bulk quantities, like that shown in Fig. 3, appears crucial to the nature of criticality in the superlattice; we will show that the values of surface critical exponents can thus be extracted from analysis of the dependence of the reentrant width $\tau_R$ on the thickness of the 2-slabs, $L_2$.

Second, in the case of the $d = 3$ Ising superlattice the $d' = d - 1$-dimensional model, describing a single film of the first component, has a regular finite temperature phase transition. Consequently, in the thick layer limit the critical temperature $T_c$ of the superlattice is only slightly depressed from $T_{c1}$ and the critical fluctuations are essentially localized within the 1-layers. However, in all other cases, i.e. $d = 3$ Heizenberg and XY-models and all models in $d = 2$, the corresponding $d'$-dimensional universality classes lack true ferromagnetic long range order at finite temperatures. The interlayer coupling through the layers of the
weaker, second component thus are crucial to existence of long range order. As a result, the critical temperature in these models is strongly shifted towards $T_{c2}$ and the critical domain, in which large scale fluctuations are active, is extended to the temperature range of the order of $T_{c1} - T_{c2}$.

Finally, a few remarks about the method of this paper. We focus here on the $n$-component vector spin systems with short range interactions. As usual, Ising, $XY$, and Heizenberg universality classes refer to the $n = 1$, or easy axis, $n = 2$, or easy plane, and $n = 3$, or isotropic, magnets respectively. The most important omission of this model, apart from restriction to the simplest type of the order parameter, is the absence of long-range dipolar forces; it seems however natural to make the first step within the simpler realm of local models. The actual method employed here is to combine the well known linear RG flows at the fixed points representing the three paradigms discussed above (Fig. 2), into the simplest global RG flow consistent with this linear behavior (Fig. 4). The resulting scaling theory then passes the test of comparison to the exact results available for the planar Ising model.

The outline of the paper is the following: in the next section we start by taking the conceptually simplest case of the three-dimensional Ising superlattice. The renormalization group flow will be proposed and scaling forms derived. In section III we consider the planar Ising case which requires extension of the previous consideration to the case of the zero-temperature critical point describing the $d' = 1$ dimensional Ising model. The results will then be shown to agree with the exact forms available for this case [11]. The extension of the method developed in section III will be used in the following section, where the Heizenberg and the $XY$ universality classes are considered. The results are summarized in section V.

II. THREE-DIMENSIONAL ISING SUPERLATTICE

We start now by focusing on the $d = 3$-Ising universality class. Its most important feature is that a single thick slab of the first component is fully capable of ordering at a finite temperature $T'_{c1}$, which continuously approaches the bulk transition temperature $T_{c1}$,
as the thickness $L_1 \to \infty$. We thus expect that the criticality in a three-dimensional Ising superlattice happens close to $T_{c1}$ and that the relevant fluctuations are localized within the 1-slabs. Much of what will be said below will apply however to other systems, so we will keep using general $d$ and $d' = d - 1$, as well as the general standard notation for the critical exponents in this section.

The renormalization group procedure which we use here has been extensively discussed in [3] for the half-infinite models bounded by one free surface, and applied to the superlattice geometry in [11]. It is based on subsequent reduction, $\Lambda' \to \Lambda' e^{-l}$, of the upper cut-off $\Lambda'$ imposed on the $d'$-dimensional wave numbers $k'$ characterizing the spatial variation of fluctuations in the directions parallel to the layers. Since the superlattice geometry is uniform in this plane, expansion in plane waves $\exp(ik'x')$ can be used as a basis for a perturbative RG of the type discussed in [3,12]. The details of the procedure will not be important to us here: all we need is the existence of an exact RG transformation; then according to the general principles [12,13] the linearized RG flow in the vicinity of the fixed points should be independent of the specific implementation. Note that there is no need to do anything about the cut-off in the $z$-direction (in fact one does not need such cut-off at all): elimination of fast modes in the directions parallel to the layers automatically induces coarse-graining in the $z$-direction [3,11,14].

We will now follow the RG flow probing fluctuations on larger and larger length scales parallel to the layers. The two crucial length scales, to which the current (running) length scale has to be compared, are the bulk correlation lengths $\xi_1(T)$ and $\xi_2(T)$, which can be (at least in principle) measured at the given temperature $T$ in independent bulk samples of the two components. Near $T_{c1}$ fluctuations in the second component are confined to scales smaller than the bulk correlation length $\xi_2(T) \approx \xi_2(T_{c1})$, which remains finite while $\xi_1$ diverges. Decreasing the upper momentum cutoff to $\xi_2^{-1}$ essentially eliminates fluctuations in the 2-layers. On larger scales the 2-layers are adequately described by the Ornstein-Zernike spin density functional corresponding to the Gaussian RG fixed point describing the low-temperature phase of the second component. As usual, Gaussian degrees of freedom can
be integrated out, the integration being equivalent to minimization of the Ornstein-Zernike functional. Such minimization is performed in the Appendix. It results in an effective interaction \[ J_0 e^{-L_2/\xi_2} \int s^T_n(x)s^B_{n+1}(x)d^{d-1}x' \] between the top surface spins \( s^T_n(x) \) of the \( n \)-th layer and the bottom surface spins \( s^B_{n+1}(x) \) of the \( n+1 \)-th layer of the first component. At this point the thickness \( L_2 \) gets absorbed into the bare coupling constant \( J_i = J_0 \exp(-L_2/\xi_2) \). A major simplification has occurred after this initial crossover: the RG flow has taken the original superlattice system to the first fixed point, \( FP_0 \), equivalent to a sample of the first component containing a periodic sequence of defect (hyper-) planes, characterized by very weak vertical bonds \( J_i \) ((ii) in Fig. 2). Note that all physics of the second component, as well as all details of the microscopic implementation of the interface between the subsequent layers, have been absorbed into two parameters: the directly measurable correlation length \( \xi_2 \) and the nonuniversal amplitude \( J_0 \).

A necessary condition for the reentrant dimensional crossover, described in the Introduction, is the smallness of the initial, bare value of the coupling in the temperature range around \( T_c \approx T_{c1} \):

\[ J_i = J_0 \exp[-L_2/\xi_2(T_{c1})] \ll k_B T. \] (2)

Under this condition, we can consider the coupling (1) as a weak perturbation with respect to the reference system consisting of noninteracting layers of the first component. Further, for the values of the RG parameter \( l \) such that \( \xi_2 \ll \Lambda' \ll L_1 \), typical fluctuations are correlated only over lengths much shorter than \( L_1 \), implying that the top and the bottom surfaces of each 1-layer are uncoupled. Provided the correlation length \( \xi_1 \) of the first component has not been encountered yet, the system appears to be at the critical fixed point \( CFP_s \) describing independent bulk critical behavior within the 1-layers coupled to the ordinary surface critical behavior at the noninteracting interfaces. Note that this scenario implies another condition,
Both conditions, (2,3), require the layers to be thick. In the case of two different materials,
\( T_{c1} - T_{c2} = O(1) \), so that \( \xi_2(T_{c1}) = O(a_2) \) and both inequalities are satisfied as soon as
\( L_1, L_2 \gg a_2 \). The situation becomes less trivial in the case of a superlattice created by a
weak periodic modulation of the properties of an originally uniform sample, then one has to
be sure that the period of modulation is large enough to offset the smallness of the difference
between the bulk Curie temperatures implying a relatively large \( \xi_2(T_{c1}) \propto (T_{c1} - T_{c2})^{-\nu} \),
where \( \nu \) is the standard correlation length exponent in \( d \) dimensions.

The crossover is now described as an RG-flow between three critical fixed points (Fig. 4):
\( CFP_s \) describing a sequence of uncorrelated thick layers of the first component ((ii) in Fig.
2), with extensive properties dominated by the \( d \)-dimensional bulk critical behavior taking
place inside the layers, while the associated ordinary surface scaling describes the observables
localized at the surface. From \( CFP \) the system flows to \( CFP' \) describing the \( d' = (d - 1) \)-
dimensional criticality in a system of uncorrelated thin films of the first component ((iii)
in Fig. 2), and, finally, to \( CFP_b \) describing uniform \( d \)-dimensional bulk behavior ((iv) in
Fig. 2). The flow between the fixed points is driven by two scaling fields (cf. Fig. 4),
the inverse thickness of the 1-layers, \( L_1^{-1} \), playing the role of a long wave length (infrared)
cutoff in the \( z \)-direction (cf. [16]), and the interlayer coupling strength \( J_i \). Yet another
one, scaled temperature field \( t_1 = (T - T_{c1})/T_{c1} \) controls the departure from the critical
manifold, \( t_{1c}(L_1, J_i) \), containing the flow attracted by \( CFP_b \), towards the massive, high-
and low-temperature fixed points. Instead of the scaled temperature field \( t_1 \) characterizing
the bulk criticality of the first component, one can use \( t = (T - T_c)/T_c \) defined relative to the
observed critical temperature \( T_c(L_1, J_i) \) of the superlattice. As we will see below, the
scaling forms look simpler when expressed via \( t \), but lack any information about the shift in
the critical temperature with respect to \( T_{c1} \). In the RG approach the scaling fields become
functions of the logarithmic length scale \( l \). While the explicit calculation of the crossover
scaling functions requires application of non-perturbative methods such as the one used in
the leading singularities are determined by the RG flow in close vicinities of the fixed points.

Specifically, at $CFP_s$ the linearized RG flow equations are

\[
\frac{dt_1}{dl} = \frac{1}{\nu} t_1, \quad (4) \\
\frac{d(1/L_1)}{dl} = 1/L_1, \quad (5) \\
\frac{dJ_i}{dl} = (d' - 2\omega_1)J_i = (\gamma_{11}/\nu)J_i. \quad (6)
\]

Here $\nu$ is the standard correlation length exponent of the bulk $d$-dimensional universality class. The second equation (5) reflects decrease in $L_1$ expressed in units of the running inverse length scale $\Lambda'$; the equality of the RG eigenvalue in (4) to one reflects asymptotic isotropy of the bulk critical behavior described by $CFP_s$. The RG eigenvalue for $J_i$ is read from (1). Because the spins entering (1) are located at the surfaces, the scaling dimension of the spin density $\omega_1$ determining the RG eigenvalue of $J_i$ is that characteristic of the surface spin density. Correspondingly $\gamma_{11}$ is the exponent characterizing the susceptibility of the surface spins to a perturbation by a surface field $\mathbf{3}$. The crucial point now is that since the surface spins are correlated more weakly than the bulk ones, the surface susceptibility usually does not diverge (see the estimates of surface critical exponents in $\mathbf{3}$): $\gamma_{11} \leq 0$. Consequently $J_i$ is irrelevant or marginal at the first encounter with $CFP$. Thus starting with a small coupling $J_i$, we are guarantied that it remains small until the system arrives at $CFP'$. Note that this picture implies a seemingly paradoxical prediction: a single layer of weak bonds cutting through a bulk sample effectively decouples the two halves, with two independent ordinary surface fluctuations developing at the two sides of it.

The two other scaling fields, $t_1$ and $L_1^{-1}$ are relevant at $CFP_s$. As $L_1(l)$ is positive definite, it is convenient to exclude $l$ from the equations (4,5), rewriting them as

\[
\frac{d \ln(t_1)}{d \ln(L_1)} = -\nu^{-1}, \quad (7) \\
\frac{d \ln(J_i)}{d \ln(L_1)} = -\gamma_{11}\nu^{-1}. \quad (8)
\]

The crossover to the $d'$-dimensional fixed point $CFP'$ occurs when $L_1(l)$ decreases to the
order of the microscopic length scale $a_1$. As a result of this first crossover (indicated by subscript $D$, for “direct,” below) the scaling field $1/L_1$ becomes irrelevant and disappears from the consideration, while the other two are renormalized to

$$t_{1D} = t_1/\tau_D$$  \hspace{1cm} (9)
$$J_{iD} = J_i \tau_D^{-\gamma_1}.$$  \hspace{1cm} (10)

Here the temperature rescaling factor

$$\tau_D \approx (a_1/L_1)^{1/\nu}$$  \hspace{1cm} (11)

conveniently characterizes the first crossover. We will reduce the number of nonuniversal parameters by defining $\tau_D$ not through $a_1$, but rather through the critical amplitude $X_1 = O(a_1)$ characterizing the divergence of the bulk correlation length $\xi_1 = X_1(-t_1)^{-\nu}$ at the low-temperature side of the bulk critical point $T_{c1}$:

$$\tau_D = (X_1/L_1)^{1/\nu}.$$  \hspace{1cm} (12)

In the RG formalism the correlation length $\xi_1$ characterizes the crossover from the critical to noncritical RG fixed points at nonzero values of $t_1$. Therefore $\tau_D$ determines the width of the temperature domain around $T_{c1}$ in which the dimensional crossover $CFP_s \rightarrow CFP'$ actually occurs: for $t_1 > \tau_D$ the $t_1$-field becomes large and drives the system away from the critical manifold before $1/L_1$ grows large; as a result the system never reaches $CFP'$.

If, on the other hand, $t_1 \lesssim \tau_D$ the system arrives at $CFP'$ with the values of the two relevant fields estimated by $t_1 \approx t_{1D}$, $J_i \approx J_{iD}$. In the absence of $J_i$ (which is small at this stage) $t_1$ completely determines the flow. The critical separatrix going into $CFP'$ corresponds to a certain initial value $t_{1D} = t'_c = O(1)$. One expects $t'_c < 0$, as finite thickness suppresses ordering. In fact, due to our definition of $\tau_D$ via the bulk correlation length $\xi_1$, the parameter $t'_c$ is universal: criticality in a free film of the first component of thickness $L_1$ happens at a temperature $T'_c(L_1)$ at which the ratio $L_1/\xi_1$ takes a universal value

$$L_1/\xi_1(T'_c) = (-t'_c)^\nu \approx 2.89,$$  \hspace{1cm} (13)
where the numerical estimate has been obtained by series expansion methods \[17\]. Expanding around the separatrix one obtains

\[
dt_1/dl = (1/\nu')(t_1 - t'_c)
\]  

\[
dJ_i/dl = (d' - 2\omega')J_i = (\gamma'/\nu')J_i,
\]

where the prime marks exponents related to the \(d'\)-dimensional criticality. These equations are to be solved with the initial data given by \(t_1 = t_{1D}, J_i = J_{iD}\). As a result of the first crossover the surface spins in (1) are strongly correlated with the rest of the corresponding layer, thus becoming \(d'\)-dimensional in nature. Correspondingly, the scaling dimension of the spin density \(\omega_1\) in the RG flow equation for \(J_i\) is changed from the ordinary surface value \(\omega_1\) to the \(d'\)-dimensional \(\omega'\). Correspondingly nonpositive \(\gamma_{11}\) is changed to positive \(\gamma'\), so that \(J_i\) is a relevant perturbation at \(CFP'\) driving the second crossover back to \(CFP\). The following analysis of the reverse crossover (as indicated by subscript \(R\) below) essentially repeats the one performed for \(CFP_s \rightarrow CFP'\) above: we divide one of the two linear RG equations by another to obtain (the coupling strength \(J_i\) is positive definite)

\[
d\ln(t_1 - t_{ac})/d\ln J_i = 1/\gamma'.
\]

A nonuniversal amplitude \(J_0^* = O(k_BT D^{-d'})\) is defined so that at \(J_i(l) \gtrsim J_0^*\) the \(d'\)-dimensional hyperplanes become strongly coupled and the system crosses over to the uniform bulk behavior at \(CFP_b\). As this happens, \(t_1\) is rescaled by the factor \(\tau_R = (J_{iD}/J_0^*)^{1/\gamma'}\), i. e.

\[
t_1(l) = t_{1R} \equiv (t_{1D} - t'_c)/\tau_R,
\]

where (recall that \(\gamma_{11} \leq 0\))

\[
\tau_R = (J_0/J_0^*)^{1/\gamma'}(L_1/X_1)^{-|\gamma_{11}|/\gamma'_\nu}\exp(-L_2/\gamma'\xi_2).
\]

Just as previously, for \(|t_{1R}| \gg 1\), i. e. \(|t_{1D} - t'_c| \gg \tau_R\), the system flows away from the critical separatrix before the reentrant crossover to \(CFP\) can take place. Hence \(\tau_R\) measures in units of \(t_{1D}\) the width of the temperature domain in which the reentrant scaling can be
observed; in the original units of $t_1$ the width is given then by the product $\tau_D \tau_R$. Within
the domain $t_{1R} \lesssim 1$ the system flows into the close vicinity of $CFP_b$. The critical separatrix
is again defined by a certain (this time positive: interlayer bonds enhance ordering) value of
$t_{1R} = t_{cR} = O(1)$. In fact, as at this point we have no other scale to measure the interlayer
coupling amplitude $J_0$, we can fix the arbitrary constant $J_0^*$ by setting $t_{cR} = 1$. Having
grown to the order of $J_0^*$ the interlayer coupling amplitude $J_i$ is absorbed into the rescaled
value of the temperature field $t_1(l) = t_{1R}$ and becomes irrelevant. The reentrant $CFP_b$ is
a regular critical fixed point with temperature (in the absence of magnetic field) being the
only relevant perturbation measuring the deviation from the critical separatrix (Fig. 4).
The evolution of the latter is again described by the RG flow equation (19) which has to be
solved with the initial condition
\[ t_1(l) = t_{1R} - 1 = (t_1/\tau_D - t'_c)/\tau_R - 1. \]

Having established the principle features of the RG flow we are in a position now to de-
velop a scaling description of the observable quantities. The two temperature scales $\tau_D$ and
$\tau_R$ defined by (2) and (3), together with the original value of the scaled temperature of the
first component, $t_1$, conveniently parametrize the scaling functions. If the bulk properties
of the two components are known, then the only new nonuniversal parameter appearing in
our description of the superlattice is the dimensionless ratio $J_0/J_0^*$, essentially characterizing
the strength of coupling between the spins across the interface between the two components
(see the Appendix for more precise definition). The standard two-scale factor universality
of the bulk critical points is thus extended to what may be called 2 + 2-factor univer-
sality: all scaling functions of a superlattice are universal apart from the two independent
bulk critical amplitudes of the first component plus the amplitude $J_0/J_0^*$ characterizing the
interface and the bulk correlation length of the second component $\xi_2(T_{c1})$. In fact, the two
additional parameters enter the description via the initial value of the effective coupling
$J_i(l = 0) = J_0 \exp[L_2/\xi_2(T_{c1})]$. Thus, if one were not interested in the (singular) dependence
of the critical behavior of the superlattice on the thickness $L_2$, only one extra amplitude, $J_i$,
would have to be added to complete the description of critical behavior in a superlattice. However, since for large \( L_2 \) the amplitude \( J_i \) is going to be anomalously small, and since \( L_2 \) is easily measured, we prefer to split \( J_i \) into the nonsingular amplitude \( J_0 \) and the singular exponential factor \( \exp(-L_2/\xi_2) \), adding the bulk correlation length \( \xi_2(T_{c1}) \) to the list of empirical parameters.

We start by analyzing the shift in the critical temperature. Unfolding back the two renormalizations of the temperature field, one obtains from (4,19)

\[
T_c = T_{c1}[1 + \tau_D(t'_c + \tau_R)] = T'_c(L_1) + T_{c1}\tau_D\tau_R. \tag{20}
\]

The shift consists of a larger, \( O(L_{1}^{-1/\nu}) \), shift to lower temperatures, slightly corrected by a much smaller, \( O[\exp(-L_2/\xi_2)] \), shift in the opposite direction. The last expression represents the latter as a shift with respect to \( T'_c(L_1) = T_{c1}[1 + \tau Dt'_c] \), the critical temperature in a film of the first component of thickness \( L_1 \) with free boundaries. If this latter temperature is known, then the observed shift towards the higher temperatures, \( T_c - T'_c(L_1) \), can be used to estimate the unknown amplitude \( J_0/J_0^* \) entering the definition of \( \tau_R \) (18).

Let us consider now an extensive observable, say the specific heat per unit volume \( C \) characterized by critical exponents \( \alpha \) and \( \alpha' \) in dimensions \( d \) and \( d' \) correspondingly. Since the surface-to-volume ratio in a superlattice vanishes as \( L_{1,2} \to \infty \), the observed signal at the first stage of the crossover is dominated by the interior of the 1-layers: as far as extensive properties are concerned \( CFP_s \) is equivalent to \( CFP_b \) and the RG flow is indeed reentrant. This flow can be represented by the scaling form

\[
C(t_1) = (L_1/L)A_{1-}|t_1|^{-\alpha}A_1(t_{1D}, t_{1R}), \tag{21}
\]

where \( t_{1D}, t_{1R} \) are given in (9,17), and \( A_{1-} \) is the bulk critical amplitude of the first component on the low-temperature side of the criticality. The factor \( (L_1/L)A_{1-} \) represents the amplitude of the contribution to \( C \) from the interior of the 1-layers at the first stage of the RG flow (governed by \( CFP_s \)) below \( T_{c1} \); the choice of \( A_{1-} \) instead of \( A_{1+} \) seems natural in view of \( T_c < T_{c1} \). The scaling function \( A_1 \) is completely universal. It has the following limits:
it approaches 1 at $t_{1D} \to -\infty$ and the universal amplitude ratio $A_{1+}/A_{1-}$ at $t_{1D} \to +\infty$; at $t_{1D} \to 0$ it develops a singularity $A_1 \propto |t_{1D}|^\alpha$ which cancels the one generated by the $|t|^{-\alpha}$ factor in (21). At $|t_{1D} - t_c'| \ll 1$ one has $A_1 \approx |t_{1D} - t_c'|^{-\alpha'} C_1(t_{1R})$. The asymptotic behavior of the new scaling function $C_1$ follows the same logic as the one employed above for $A_1$: it takes finite limits at $t_{1R} \to \pm \infty$, develops a singularity, $C_1 \propto |t_{1R}|^{\alpha'}$ at $t_{1R} \to 0$ to compensate for the singular prefactor, and, finally, diverges as $C_1 \propto |t_{1R} - 1|^{-\alpha}$, when $t_{1R} \to 1$.

As already mentioned above, because in the Ising universality class the critical temperature shifts satisfy $\tau_D \tau_R \ll \tau_D t_c' \ll 1$ a simpler scaling form is achieved in terms of the scaled temperature $t = (T - T_c)/T_c = t_1 - \tau_D (t_c' + \tau_R)$ shifted to the observed critical temperature $T_c(L_1, J_i)$. Defining $t_D = t/\tau_D$, $t_R = t_D/\tau_R$ one can write

$$C = (L_1/L)A_{1-}|t|^{-\alpha}A(t_D, t_R),$$

where the universal scaling function $A$ takes the following limits:

$$A \approx A_{1+}/A_{1-}, \text{ at } 1 \ll t_D \ll t_R,$$

$$A \approx 1, \text{ at } t_R \ll t_D \ll -1,$$

$$A \approx C_{D\pm} t^{-\alpha} t_D^{\alpha' - \alpha}, \text{ at } |t_D| \ll 1 \ll |t_R|,$$

$$A \approx C_{R\pm} t^{-\alpha} t_D^{\alpha' - \alpha} t_R^{\alpha' - \alpha}, \text{ at } |t_D| \ll |t_R| \ll 1.$$  

This form has a simple graphical interpretation given by the double logarithmic plot of Fig. 3. The continuity of $C(t)$ requires that both universal amplitude ratios $C_{D\pm}$, $C_{R\pm}$ (remember that the scaling function has been normalized by the bulk amplitude $A_{1-}$) are numbers of the order of unity. The last asymptotic expression in (26) shows that while the ultimate divergence $A = A_{1\pm}|t|^{-\alpha}$ is characterized by the bulk exponent $\alpha$, the critical amplitudes

$$A_{1\pm} = (L_1/L)A_{1\pm} C_{R\pm} \tau_R^{\alpha' - \alpha'}$$

are shifted on the logarithmic scale from the bulk amplitudes $A_{1\pm}$ of the first component by $(\alpha' - \alpha) \ln(1/\tau_R)$ (see Fig.3), i.e.
\[
\frac{\ln(A_{\pm}/A_{1\pm})}{\alpha'-\alpha} = \frac{1}{\gamma'}[L_2/\xi_2(T_c) + (-\gamma_{11}/\nu)\ln(L_1)] + O(1). \tag{28}
\]

More information about the critical state of a superlattice governed by \(CFP_b\) can be obtained by studying the anisotropy of correlation functions. The basic ratio \(X = \xi_\perp(T)/\xi_\parallel(T)\), of the critical amplitudes of correlation lengths across and along the layers, is given by expressions identical to (27), (28), where one has to use anisotropic correlation length exponents: \(\nu_\perp = \nu_\parallel = \nu\) at \(CFP_s\), \(CFP_b\), but \(\nu_\perp = 0, \nu_\parallel = \nu'\) at \(CFP'\), since at \(CFP'\) the correlations grow only in the direction along the layers. The result is:

\[
X \sim \tau^\nu R'. \tag{29}
\]

Similarly, at \(T_c\), the amplitude of the spin-spin correlation function, \(G_\perp(z) \propto z^{-2\omega}\) at \(z \gg L\) is suppressed compared to that of \(G_\parallel(x) \propto x^{-2\omega}\) by an amplitude ratio

\[
\Omega \equiv G_\perp/G_\parallel \sim X^{2\omega} \sim \tau^{2\beta' R}. \tag{30}
\]

One thus arrives at the large-scale description of the critical state of the superlattice as a highly anisotropic realization of the bulk universality class of the first component ((iv) in Fig. 2). In fact, following the standard ideas of scaling [19] one can relate all anomalous changes in the critical amplitudes (27), (28) to the anisotropy \(X\) of the basic length scales.

### III. PLANAR ISING SUPERLATTICE: EXTENSION TO THE CASE OF ZERO-TEMPERATURE CRITICALITY IN \(D' = 1\) DIMENSIONS AND COMPARISON TO EXACT RESULTS

Despite the generality of the above consideration, of all the \(O(n)\) spin lattice models in three and two dimensions only the \(d = 3, n = 1\)-superlattice behaves strictly according to this scenario, as the Ising model has regular finite-temperature critical fixed points without marginal operators in both \(d = 3\) and \(d' = 2\). The exactly solvable problem of a \(d = 2\)-Ising superlattice requires certain modifications because \(CFP'\) describing the one-dimensional Ising model is a zero-temperature fixed point. This results [11] in a large shift in the
critical temperature, \( T_{c1} - T_c = O(1) \). More precisely, in the interesting case of a \textit{thick layer} superlattice \([11]\), criticality happens when the correlation length \( \xi_1 \), characterizing the low-temperature phase of the first component, is much smaller than the width of the 1-layers, \( L_1 \). This means that the RG flow in this case comes to \( CFP' \) not from \( CFP_s \), but from the low-temperature fixed point \( LTFP \) describing the ordered phases of the first component. Nevertheless, many results can be formulated in terms of just one nonuniversal parameter characterizing \( LTFP \): the linear energy \( \Sigma_1 \) of a domain wall separating spin-up and spin-down phases of the first component.

Let us start our consideration from a close vicinity of \( T_{c1} \), where \( \xi_1 \gg L_1 \) and the crossover \( CFP_s \to CFP' \) proceeds directly, without encountering \( LTFP \). After leaving \( CFP_s \) the scaled temperature \( t_1 \) is renormalized to \( t_{1D} = t_1(L_1/X_{1-})^{1/\nu} \). Since \( \nu = 1 \) in the planar Ising universality class, below but close to \( T_{c1} \) one can write \( t_{1D} = -L_1/\xi_1 \) (note that the normalization \( |t_1| = 1/\xi_1 \), making \( X_{1-} = X_{1+} = 1 \) has been adopted in \([11]\)). Further, \( \gamma_{11} = 0 \) in this universality class \([4]\), so that the coupling \( J_i \propto \exp(-L_2/\xi_2) \) is not renormalized near \( CFP_s \) to the linear order in \( J_i \). While one should expect logarithmic corrections to \( J_i \) for this marginal case, those do not seem to affect the essential features of the exact solution. As discussed in \([11]\), at \( t_1 \ll 1 \)

\[
|t_{1D}| = L_1/\xi_1 = 2L_1\Sigma_1/k_BT \equiv 2J'.
\]  

(31)

The last identity, in which \( \Sigma_1(T) \) is the linear free energy of a domain wall between the two low-temperature phases of the first component, represents the dimensionless coarse-grained spin-spin coupling \( J' \) of the \( d = 1 \)-Ising model onto which a strip of the first component is mapped after the direct, bulk-to-film crossover. At \( CFP' \) this spin-spin coupling transforms according to \( dJ'/dl = -1/2 \), or

\[
dt_{1}/dl = 1
\]

(32)

with the critical separatrix corresponding to the extreme value \( t'_c = -\infty \), as implied by \( T'_c = 0 \) in the \( d' = 1 \)-dimensional Ising class. Note that the fugacity \( \zeta = \exp(-2J') \), taking
the initial value \( \exp(t_{1D}) \), behaves at \( CFP' \) as a regular thermal scaling field with the RG eigenvalue equal to one. Consequently, the criterion for \( CFP' \)-dominated behavior is \( \zeta \ll 1 \), or, equivalently, \( t_{1D} \ll -1 \).

Meanwhile, magnetization scales trivially, \( \omega' = 0 \), at \( CFP' \). Thus the eigenvalue of \( J_i \) is \( d' - 2\omega' = d = 1 \) and the corresponding RG equation at \( CFP' \) can be rewritten as \( d \ln J_i/dl = 1 \). Dividing one RG equation by another we obtain

\[
\frac{dt_1}{d \ln J_i} = 1,
\]

from which one can construct a renormalized temperature

\[
t_{1R} = t_{1D} - \ln(J_i/J_0^*) = L_2/\xi_2 - L_1/\xi_1 + O(1).
\]

It is convenient to define \( J_0^* \) here in such way that the criticality occurs at \( t_{1R} = 0 \). Then, in full agreement with [11], up to corrections vanishing as \( O(L_{1,2}^{-1}) \), the critical temperature \( T_c \) of the superlattice can be obtained from the equation

\[
L_1/\xi_1(T_c) = L_2/\xi_2(T_c) \equiv g_c,
\]

where the last identity defines what may be called the scaled thickness of the layers.

If this equation is satisfied at \( g_c \ll 1 \), the case called the thin layer limit in [11], then the flow indeed never encounters \( LTFP \). However, in this case the initial value of the coupling \( J_i \propto e^{-g_c} \) is not small, so that the criticality is reached before \( t_{1D} = -g_c \) becomes large and negative so as to display any features of the \( d' = 1 \) critical behavior. The flow in fact never leaves \( CFP \). The reentrant crossover does happen in the opposite, thick layer limit [11], \( g_c \gg 1 \), however in this case the behavior of the system on length scales between \( \xi_1 \) and \( L_1 \) is governed by the low-temperature, non-critical fixed point \( LTFP \).

To describe criticality in this limit we start at scales larger than \( \xi_1(T) \) well below \( T_c \). If the RG is implemented with rescaling the block spin by the factor of the area of the block (rather than the square root of the area leading to the Gaussian fixed point described by the Ornstein-Zernike functional [20]), then at these scales fluctuations in the amplitude
of magnetization are completely suppressed. The only allowed fluctuations are domain walls separating domains of two different spin orientations. At these scales the walls look geometrically sharp and are characterized (neglecting lattice effects) by a single parameter: free energy per unit length $\Sigma_1$. As discussed in [11], a single strip of the first component becomes identical to a $d = 1$-Ising model: spins are fully correlated across the strip; a domain wall cutting across the strip has finite free energy $\Sigma_1 L_1$ playing a role of an effective ferromagnetic coupling between neighboring blocks of size $\xi_1 \times L_1$. Characterization of the effective $d = 1$-model is completed by specifying the value of a block spin, $m_1(T)L_1\xi_1$, where $m_1(T)$ is the bulk magnetization density in the first component.

The crossover from $CFP'$ to $CFP_b$ proceeds exactly as described in the beginning of this section, because the representation (31) of $t_{1D}$ via the linear surface energy $\Sigma_1$ is valid independently of whether the system has arrived at $CFP'$ from $CFP_s$ or from $LTFP$. After the reverse crossover the thermal field is renormalized to

$$t_{1R} = t_{1D} - \ln(J_i / J^*_i) = L_2 / \xi_2 - L_1 \Sigma_1 / k_B T + O(1), \quad (36)$$

leading to the universal relation for the critical temperature

$$\frac{\Sigma_1(T_c) \xi_2(T_c)}{k_B T_c} = \frac{L_2}{L_1} \quad (37)$$

holding up to corrections of order $O(a/L)$. Note that the relation $\Sigma_1 = k_B T / 2 \xi_1$ used in (31) is valid in the planar Ising universality class up to corrections of order $O(a/\xi_1)$, which were neglected in the field-theoretical description of Ref. [11]. The expression (37) however has a much wider range of validity: it requires only that the strips are thick, with no condition on the values of the correlation lengths in the two components.

For an extensive observable like the specific heat per unit area we can write now

$$C = \frac{1}{L} A'(T) \exp(-\tilde{\alpha}' t_{1D}) A(t_{1R}). \quad (38)$$

Here the exponent $\tilde{\alpha}'$, equal to $-1$ in the case of the specific heat, and the critical amplitude $A'(T)$ characterize singularity $C = A' \zeta^{-\tilde{\alpha}'}$ in the one-dimensional regime. The scaling
function $\mathcal{A}$ has the following limits: it approaches 1 at $t_{1R} \gg 1$; diverges as $\mathcal{A} \approx C_{\pm} t_{1R}^{-\alpha}$ at $t_{1R} \ll 1$ with the amplitude ratios $C_{\pm} = O(1)$ being universal; an exponential singularity $\mathcal{A} \propto \exp(\tilde{\alpha}' t_{1R})$ cancels the singular prefactor in (38) at $t_{1R} \to -\infty$. The critical divergence of the specific heat,

$$C \approx \frac{1}{L} A'(T_c) \exp(-\tilde{\alpha}' g_c) C_{\pm} t_{1R}^{-\alpha} = A t^{-\alpha},$$

is characterized by the amplitude

$$A \sim A'(T_c) \exp(-\tilde{\alpha}' g_c),$$

where we have taken into account that $dt_{1R}/dt = O(L)$ cancels the $L^{-1}$ prefactor in (39).

In order to complete the estimate of $A$ we have to know the amplitude $A'(T_c)$. The one-dimensional amplitudes of extensive observables (per unit length of a layer) can be generated from the well-known expression for the free energy density $f'$ of a one-dimensional chain at temperature $T$ and magnetic field $H$ (see [21] and references there)

$$f'/k_B T \sim \left(H m_1 L_1 \right)^2 + \left(\zeta/\xi_1 \right)^2 \right)^{-1/2},$$

where we have taken into account that $\xi_1$ plays the role of the effective cutoff for the one-dimensional behavior along the layers, and both $\xi_1$ and $m_1$ are taken at $T = T_c$. Differentiating with respect to the temperature $T$ we obtain (at $T = T_c$ and $H = 0$) $A'(T) = g_c^2 k_B / \xi_1$. Combined with (40) this gives

$$A \sim k_B g_c e^{-g_c \xi_1^2}.$$  

This agrees fully with the result of [21]. Other amplitudes can be similarly estimated and agree with the exact solution.

An interesting feature of the form (38) is that below $T_c$, at $t_{1R} \ll -1$ the exponential part of the scaling function $\mathcal{A}$ compensating for the (now spurious) singular prefactor, leads to

$$C \propto \exp(\tilde{\alpha}' L_2 / \xi_2).$$
This can now be recognized as pointing to the dominance of the fluctuations in the second component as $T_{c2}$ is approached: in fact, due to the self duality of the planar Ising model, a completely similar construction can be developed starting from low-temperatures and focusing on the dual variables in the 2-layers. Both scenarios lead to essentially the same results.

IV. HEIZENBERG AND XY SUPERLATTICES

The other $O(n)$ models can be analyzed similarly; only the most interesting results will be briefly discussed below.

The case of the three-dimensional Heizenberg ($n = 3$) model is very close to the planar Ising one, as the $(d' = 2, n = 3)$-class is believed to be characterized by a zero-temperature critical point with the effective coupling being marginally relevant [22]. This effective coupling in a single Heizenberg film of the first component is the dimensionless helicity modulus $\Gamma_1$. In the low-temperature phase of the $d = 3$-Heizenberg model the helicity modulus $\Gamma_1$ has the dimension of inverse length, effectively defining the correlation length $\xi_1 \equiv \Gamma_1^{-1}$. In the thick layer limit the critical temperature $T_c$ of the superlattice is again shifted well below $T_{c1}$ of the first component. Thus on scales smaller than $L_1$ the first component is well ordered and the additive approximation $\Gamma' = \Gamma L_1 \gg 1$ is well justified [23]. Here $\Gamma_1$ can be (at least in principle) measured independently in a bulk sample of the first component. Once the direct crossover to $CFP'$ has occurred, one can form $t_{1D} = k_B T / \Gamma'$ and use the well known $d = 2$-Heizenberg model RG equation [22]

$$dt_{1D}/dl = 1/2\pi,$$  \hspace{1cm} (44)

while the coupling $J_i$ is trivially additive

$$dJ_i/dl = d' J_i,$$  \hspace{1cm} (45)

with $d' = 2$ here. Combining these two equations we obtain, in complete analogy with (37), a universal expression satisfied at the critical point of a Heizenberg superlattice:
Quite analogously to the consideration given above for the planar Ising case one arrives at the picture of the critical temperature shift $T_{c1} - T_c = O(T_{c1} - T_{c2})$, large scale fluctuations between in the wide temperature range between $T_{c1}$ and $T_{c2}$, and exponential dependence of critical amplitudes on the thickness of the layers.

The case of a $XY$ ($n = 2$) superlattice in $d = 3$ is special because, on one hand, the $d' = 2$-class is characterized by a finite-temperature Kosterlitz-Thouless transition, so that we expect the shift in the critical temperature to vanish in the limit of (atomically) thick layers. On the other hand, the spontaneous magnetization is zero in $d' = 2$ at any finite temperature. Thus the coupling $J_i$ through the 2-layers is crucial in supporting long-range order in a three-dimensional superlattice and we expect the observed spontaneous magnetization to be anomalously suppressed when $J_i \propto \exp(-L_2/\xi_2)$ is small, even well below $T_c \approx T_{c1}$. The suppression is easily estimated by combining the RG equation for the order parameter $dm/dl = -\omega'm$, where the scaling dimension [22]

$$\omega' = k_B T/4\pi \Gamma'_1 = k_B T/4\pi \Gamma_1 L_1,$$  

(47)

with the additive growth of $J_i$ given by (45); note that in the case of a superfluid the helicity modulus $\Gamma$ is proportional to the superfluid density $\rho_s$. The result is

$$\ln \frac{m(T)}{m_1(T)} = -\frac{k_B T}{8\pi \Gamma_1(T)\xi_2(T) L_1} + O(L^{-1}).$$

(48)

Note that $k_B T/\Gamma_1$ essentially plays the role of $\xi_1$ in this universality class, as well. One can thus see that in the case of $L_1/L_2 = O(1)$, spontaneous magnetization of a superlattice decreases exponentially as soon as $\xi_1$ exceeds $\xi_2$, long before $\xi_1$ becomes comparable to the thickness $L_1$. It would be interesting to see if this behavior could be observed at a superfluid transition of liquid helium-4 in some matrix with periodically modulated properties.

Similarly, a $d = 2 - XY$ superlattice may be realized in a helium film deposited on a periodically modulated substrate (the latter could be itself cut out of a solid three-dimensional
superlattice). In this case the $d' = 1$ $XY$-model orders only at $T = 0$. Thus we expect $T_c$ of the superlattice to be shifted well below $T_{c1}$. At such temperatures the 1-layers are equivalent to a sequence of one-dimensional (classical) $XY$-chains, described by the Gaussian Hamiltonians

$$\mathcal{H}_j = \int dx \frac{1}{2} \Gamma' (\nabla \phi_j)^2,$$  

(49)

where $\phi$ is the phase of the $XY$ order parameter, $j$ labels the layers. The effective one-dimensional helicity modulus $\Gamma'$ is given by the product $\Gamma_1 L_1$, as in all other cases considered in this section. Each pair of neighboring layers is coupled through the 2-layers separating them via

$$\mathcal{H}_{j,j+1} = J_i \int dx \cos(\phi_j - \phi_{j+1}).$$  

(50)

Combining the flow of the one-dimensional Gaussian coupling $d\Gamma'/dl = -\Gamma'$ with the ubiquitous (45) ($d' = 1$ here) we arrive at the condition of criticality $\Gamma_1 L_1 J_i / (k_B T)^2 = O(1)$, which with logarithmic accuracy can be rewritten as

$$L_2 / \xi_2(T_c) = \ln(L_1) + O(1).$$  

(51)

A little thinking shows that for $L_1/L_2 = O(1)$ the critical temperature of the superlattice is in fact shifted closer to the lower of the critical temperatures of the two components; it stays however sufficiently above $T_{c2}$ so that $\xi_2$ remains much smaller than $L_2$ and our focusing on fluctuations in the 1-layers remains well justified.

V. SUMMARY

In summary, this paper addresses the following, as yet hypothetical, experimental project: A sequence of superlattices is built of the same two ferromagnetic components, but with different basic thicknesses $L_1$ and $L_2$ (Fig. 1). The bulk critical properties of the components are assumed to be well known, the thicknesses $L_1, L_2 \gg a$, i.e. large compared to any important microscopic length of the two materials, and no long-range dipolar interaction
is present. Given the well developed theory of critical phenomena in the Ising, XY, and Heizenberg universality classes, can one predict the variation of the critical properties of the superlattices with the changes in \( L_1 \) and \( L_2 \)?

Naively one may expect that in the limit of \( L_1, L_2 \to \infty \) all extensive properties of the superlattice can be expressed as a weighted average of the bulk densities of the components. Indeed, if the limit of infinite thicknesses is taken at fixed temperature \( T \neq T_c \), surface corrections to the bulk density average vanish as \((\xi_1 + \xi_2)/L\), where the correlation lengths \( \xi_1, \xi_2 \) play the role of penetration depths for the perturbations induced by an interface in the two adjacent layers. However, experiment on a given sample studies the \( T \to T_c \) limit at fixed thicknesses \( L_1, L_2 \), so that the correlation length(s) are bound to exceed the period of the superlattice making the surface contributions crucial. The thick layer limit for the critical properties of a superlattice is singular; the thicknesses \( L_1, L_2 \) have to be combined with certain functions of the temperature to form proper scaling fields in terms of which a consistent scaling description is achieved.

Such scaling description has been constructed above based on the qualitative picture of reentrant dimensional crossover [11]. In this picture (Fig. 2) a superlattice, at its critical temperature \( T_c \), displays three different asymptotic types of critical behavior, when probed on different length scales. These three asymptotic models are, in the order of increasing wave length, (ii) a sequence of noninteracting semi-bulk samples, exhibiting ordinary surface critical behavior at the surfaces; (iii) a sequence of noninteracting thin layers, exhibiting \( d' = d - 1 \)-dimensional behavior; (iv) a highly anisotropic, but uniform critical system of the same universality class as that of the components. Away from \( T_c \) the finite correlation length falls into one of the three regimes, so that instead of probing different length scales at \( T_c \) one can study the crossover by observing the asymptotic behavior at different temperatures approaching \( T_c \).

Each of these limiting types of behavior corresponds to a previously studied renormalization group fixed point. Combining those into a global RG flow (Fig. 4) we have constructed a rather complete scaling picture of critical phenomena in superlattices composed of thick lay-
ers of two components. The simplest case is that of a $d = 3$-Ising superlattice, characterized by the possibility of a finite-temperature phase transition in a single layer of the first component. As a result, in the limit of thick layers the critical temperature $T_c$ of the superlattice approaches that of the stronger of the two ferromagnets, $T_{c1}$. The two crossovers then happen in a (relatively) narrow vicinity of $T_c$. They are conveniently characterized in terms of the reduced temperature scales $\tau_D, \tau_R$ given by (11), (18). These scales represent (Fig. 3) the relative widths of the temperature domains in which the superlattice behaves, respectively, as models (ii) and (iii) above. The simplest scaling description (26) is then achieved in terms of the reduced temperatures $t_D = (T - T_c)/\tau_D T_c$ and $t_R = (T - T_c)/\tau_R \tau_D T_c$, shifted to the observed critical temperature $T_c$. As represented graphically in the double logarithmic plot of Fig. 3, the ultimate divergence is characterized by the critical exponents characteristic of the bulk, $d$-dimensional behavior. However the presence of the intermediate domain of the $d'$-dimensional behavior (model (ii)) leads to anomalous changes in the critical amplitudes, as seen in (Fig. 3) and expressed analytically through (11,18) in (28).

The situation is somewhat more interesting in the other three- and two-dimensional spin-vector universality classes, where the weak bonds between subsequent 1-layers play a crucial role in maintaining the magnetic long-range order in the system. This leads to a large shift in the critical temperature of a superlattice, as implicitly given for various classes by the universal expressions (37,46, 49), as well as the anomalous, exponential suppression in the amplitude of spontaneous magnetization (48) in a three-dimensional $XY$-superlattice. Where applicable, the results obtained here by the method of this paper agree with the analysis of the exact solution for the two-dimensional Ising superlattice [10] given in [11]. The three-peaked form of the temperature dependence of the specific heat, characteristic of that model generalizes to the three-dimensional Heizenberg class. It has a transparent physical interpretation: most of the spin degrees of freedom in such a superlattice get ordered at one of the bulk critical temperatures of the two components, leading to finite, but large-amplitude peaks at both. However, a small, $O(L^{-1})$, fraction of the degrees of freedom participate in strong large-scale fluctuations at all $T_{c2} \lesssim T \lesssim T_{c1}$, giving small-amplitude,
but diverging contribution to the specific heat at $T_c$.

The abundance of interfaces in a three dimensional magnetic superlattice may greatly improve chances of experimental observation of surface and two-dimensional scaling. In particular, the elusive surface scaling exponents enter the definition of the parameter $\tau_R$ in (18) and can be obtained, for instance, by analyzing the dependence of the critical amplitudes, (28) and Fig. 3, on the thickness of 1-layers $L_1$.

Finally, the main restriction of the present analysis seems to be the neglect of dipolar interactions. When present, these interactions will lead to demagnetization effects making Heisenberg universality class asymptotically unstable. Otherwise, most of the results presented above can be easily modified in the case when the interlayer interaction dependence on the thickness of the separating layers, $J_i(L_2)$, is a power-law rather than an exponential as was always assumed above. However, the assumption of local interactions enters our consideration in much deeper ways than just through the exponential form of $J_i(L_2)$, so that the required revision could be much more serious. Anti-ferromagnets, of course, are free from the complications arising from the dipolar interactions. However, pinning of the spin density waves by the interfaces between the layers in a superlattice gives rise to a different source of the thickness dependent effects beyond the scope of this paper. In any case, the above consideration seems to represent a reasonable first step towards a complete analysis of scaling in superlattices. Note that recently critical phenomena in superlattices have been analyzed in the frames of the mean-field Landau-Ginzburg approximation [24]. A direct comparison of our results to theirs is hardly feasible in view of the failure of the mean-filed approximation $d = 2$ and 3 considered here. However a combination of the results of Ref. [24] with a renormalization group expansion in $d = 4 - \epsilon$ and the scaling framework developed above looks promising, although computationally very hard, direction for future work.

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Here we derive the effective spin-spin coupling (1) from the Ornstein-Zernike spin-density functional

\[ F_b[r] = \frac{k_B T}{2\chi^2} \int_0^{L_2} dz \, d^{d'}x' \{ s^2 + (\xi_2)^2(\nabla s)^2 \} \]  

(A1)

describing a layer of the second component on the length scales exceeding the correlation length \( \xi_2 \); the other parameter entering (A1) is the bulk spin susceptibility \( \chi_2 \). This functional corresponds to the Gaussian RG fixed point describing the bulk paramagnetic phase of the second magnet. In a layer of finite thickness \( L_2 \), the bulk part of the functional has to be complemented by the surface contributions, which at the paramagnetic fixed point take the form

\[ F_s = \sum_{j=1,2} F_{sj} = \sum_{j=1,2} \int d^{d'}x' \left\{ \frac{1}{2} b s^2(x', z_j) - h_j(x')s(x', z_j) \right\} . \]  

(A2)

In the above equation \( z_1 = 0, z_2 = L_2 \), the phenomenological parameter \( b \) accounts for the change in the number of nearest neighbors for the surface spins, while the surface fields \( h_j \) are induced by the neighboring spins in the layers of the first component:

\[ bh_j(x') = J_{12} s_j(x'), \]  

(A3)

where \( s_1 = s_n^T \) and \( s_2 = s_{n+1}^B \), as defined after (1), while \( J_{12} \) is the effective coupling across the interface. The minimization of the total free energy \( F_b + F_{s1} + F_{s2} \) proceeds via Fourier expansion in the \( x' \)-plane. For each Fourier harmonic of the spin density one obtains an independent one-dimensional functional. Minimizing the latter and summing all contributions one obtains in the limit \( \exp(-L_2/\xi_2) \ll 1 \) an effective interaction between the surface spins of the first component

\[ -k_B T \sum_{k'} G(k') h_1(k')h_2(k'); \]  

(A4)

\[ G(k') = \frac{(1 + (k'\xi_2)^2)^{1/2}\xi_2/\chi_2}{|b + (1 + (k'\xi_2)^2)^{1/2}\xi_2/\chi_2|^2}. \]  

(A5)
Since the Fourier components of the spin density with wave numbers $k' \gtrsim \xi_2^{-1}$ has been already eliminated by the RG transformation, the factors $(1 + (k'\xi_2)^2)^{1/2} \approx 1$. Using (A3) one finally obtains the interaction of the form (I) with the amplitude

$$J_0 = \frac{2\xi_2/\chi_2}{[b + \xi_2/\chi_2]^2} J_{12}^2.$$  \hspace{1cm} (A6)
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FIGURES

FIG. 1: Schematic view of a superlattice in, a), $d = 3$ and, b), $d = 2$ spatial dimensions.

FIG. 2: Graphical representation of the reentrant dimensional crossover. The observed behavior of the system changes from a noncritical superlattice, (i), to the sequence of thick layers of the first component, (ii), connected by exponentially weak links mediated by non-critical fluctuations in the second component. From this limit, the direct crossover takes the system to a sequence of weakly coupled thin layers, (iii), while the reentrant crossover takes it back to the bulk, although highly anisotropic, critical behavior, (iv).

FIG. 3: Schematic dependence of the specific heat per unit volume $C$ on the scaled temperature $t = (T - T_c)/T_c$ in a three-dimensional Ising superlattice.

FIG. 4: Reentrant renormalization group flow in the critical manifold of the parameter space representing the $d = 3$ Ising superlattices. The three fixed points, defined in the text, are connected by critical separatrices (bold lines). They correspond, in the order in which they are encountered by the flow, to the limits depicted by (ii), (iii), and (iv) in Fig. 2. The direct crossover, from $CFP_s$ to $CFP'$, and the reentrant one, from $CFP'$ to $CFP_b$, are driven by two scaling fields, measured by the inverse thickness $L_i^{-1}$ and by the interlayer coupling strength $J_i$, respectively. Note that yet another relevant scaling field, $t$, directed roughly perpendicular to the plane of the figure, drives the system away from this two-dimensional critical manifold to the high- and low-temperature fixed points.
\[ \log C = (\alpha - \alpha') \log \tau_R - \alpha \log t \]

\[ -\alpha' \log t \]

\[ -\alpha \log t \]

\[ -\log \tau_D \]

\[ -\log \tau_R \]

\[ -\log t \]
