CENTRALIZERS AND INVERSES TO INDUCTION AS EQUIVALENCE OF CATEGORIES

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Abstract. Given a ring homomorphism $B \to A$, consider its centralizer $R = A^B$, bimodule endomorphism ring $S = \text{End}_B A_B$ and sub-tensor-square ring $T = (A \otimes_B A)^B$. Nonassociative tensoring by the cyclic modules $R_T$ or $S_R$ leads to an equivalence of categories inverse to the functors of induction of restricted $A$-modules or restricted coinduction of $B$-modules in case $A | B$ is separable, H-separable, split or left depth two (D2). If $R_T$ or $S_R$ are projective, this property characterizes separability or splitness for a ring extension. Only in the case of H-separability is $R_T$ a progenerator, which replaces the key module $A e$ for an Azumaya algebra $A$. After establishing these characterizations, we characterize left D2 extensions in terms of the module $T_R$, and ask whether a weak generator condition on $R_T$ might characterize left D2 extensions as well, possibly a problem in $\sigma(M)$-categories or its generalizations. We also show that the centralizer of a depth two extension is a normal subring in the sense of Rieffel and pre-braided commutative. For example, its normality yields a Hopf subalgebra analogue of a factoid for subgroups and their centralizers, and a special case of a conjecture that D2 Hopf subalgebras are normal.

1. Introduction and Preliminaries

Given a ring homomorphism $B \to A$, we pass to its induced bimodule $B A_B$ and define its centralizer $R = A^B$, bimodule endomorphism ring $S = \text{End}_B A_B$ and sub-tensor-square ring $T = (A \otimes_B A)^B$. The ring $T$ (defined in eq. (4) below) replaces the enveloping ring $A^e$ which plays a major role in the study of separability and Azumaya algebras \cite{4}. We will see that it plays a similar role in the study of separability and H-separability below, perhaps in a troubling way from the point of view of Morita equivalence and its generalizations. When we tensor in a nonassociative way by the cyclic module $R_T$, an induced $A$-module $A \otimes_B M$ of a separable extension becomes naturally $A$-isomorphic to $M$ itself (Lemma \ref{2.1}). This shows that induction of $A$-modules is an equivalence of categories which characterizes separability if $R_T$ is just projective (Theorem \ref{2.2}) and characterizes H-separability if $R_T$ is a generator (Theorem \ref{3.4}). Thus only in the case of H-separability is this equivalence part of a Morita equivalence, in this case tensoring by the progenerator module $R_T$ with endomorphism ring isomorphic to the center of $A$.

A split extension is the dual of a separable extension in a unusual way involving the endomorphism ring \cite{4}; if we now pass from $T$ to its $R$-dual $S$ in depth two theory \cite{10}, we might experiment with the cyclic module $S_R$ (which does not generally form a bimodule with the right $T$-module action). It turns out that tensoring the restricted coinduction of a $B$-module $N$ by $S_R$ is naturally isomorphic to $N$
Thus restricted coinduction is an equivalence from the category of modules \(M_B\) to its image subcategory, a characterization of split extension if \(sR\) is projective (Theorem 2.6).

Left depth two extensions have a Galois theory and duality theorems for actions and coactions explored in [10, 9, 7, 8, 3]. What we do instead in this paper is show that a left D2 extension \(A \mid B\) has properties similar to a split extension as well as a separable extension. In Theorems 3.3 and 3.6 we show that a necessary condition for \(A \mid B\) to be left D2 is that restricted coinducted restriction and inducted restriction are equivalences as functors from the category of \(A\)-modules with inverse from the image subcategory defined by \(- \otimes_S R\) and \(R \otimes_T -\) once again. Given the known examples of D2 and separable extensions independently scattered on a Venn diagram on the one hand, and the characterization of H-separability (a strong form of both D2 extension and separable extension) on the other, we ask whether a weak generator condition on \(R\) together with the equivalence data might characterize left D2 extensions, perhaps with a flatness condition in addition. This might have an answer in the theory of \(\sigma(M)\)-categories or its generalizations by Wisbauer and Dress a subject we bring to bear in section 5 on a related question.

In section 4 of this paper we revisit a notion of normal subring in the sense of Rieffel [10]. We work with the simplest possible definition of normality in [10] since this already coincides with the notion of normality for Hopf subalgebras (Proposition 4.3). It is also a good notion for stating that the centralizer of a depth two extension is a normal subring (Proposition 4.2 and the closely related fact that it is pre-braided commutative in section 5). For example, this yields a Hopf subalgebra analogue of a factoid for subgroups and their centralizers (Proposition 4.4), and a special case of a conjecture that D2 Hopf subalgebras are normal (Corollary 4.5).

In a final section we discuss further categorical considerations and results such as which ring extensions generalizing separable extensions as well as D2 extensions possess the property that the inverse of induction-restriction of modules is tensoring by the centralizer module \(R_T\). We also discuss a depth two condition for bimodules using bicategories and endomorphism rings, and a functorial characterization of left D2 extension \(A \mid B\) in terms of induction from \(B\) to \(A\) on the one hand and induction from \(R\) to \(T\), or coinduction from \(R\) to \(S\), on the other hand.

1.1. Preliminaries for the general reader. The starting point in this paper is a ring homomorphism \(g : B \to A\) of associative unital rings preserving unity and admitting the possibility of noncommutative rings. (We may equally well work with \(K\)-algebras over a commutative ring \(K\), \(K\)-symmetric bimodules and \(K\)-linear morphisms throughout. In this paper then unadorned tensors between rings are over the integers.) The homomorphism induces a natural bimodule \(B A_B\) via \(b \cdot a \cdot b' = g(b) a g(b')\) if \(g : B \to A\), as well as bimodules \(_A A_B\) and \(_B A_A\) defined via \(g\) on one side only. The same holds for an \(A\)-module \(_A M\); it is a \(B\)-module \(_B M\) via \(g\). This defines the usual functor \(R\) (of “restriction”) from the category of left \(A\)-module denoted by \(_A M\) with \(A\)-linear morphism into the category \(_B M\). The data we focus on is contained in the subring \(g(B) \subseteq A\), leading us to suppress \(g\) and view \(B \to A\) as a ring extension \(A \mid B\), a proper ring extension if \(B \to A\) is monic. Sometimes a condition on \(_A B\), such as being a generator module, entails that \(A \mid B\) is a proper extension. If \(B\) is commutative and \(g\) maps \(B\) into the center \(Z(A)\) of \(A\), then \(A \mid B\) is the \(B\)-algebra \(A\), where a proper ring extension is a faithful algebra.
If \( A_M \) denotes an \( A\)-\( A \)-bimodule, we let
\[
M^B := \{ m \in M \mid bm = mb, \forall b \in B \}.
\]
Note that this is isomorphic to the group of \( A\)-\( A \)-bimodule homomorphisms from the tensor-square to \( M \),
\[
\text{Hom} \left( A_A \otimes_B A_A, A_M \right) \overset{\cong}{\longrightarrow} M^B \quad F \mapsto F(1_A \otimes 1_A)
\]
with inverse \( m \mapsto (x \otimes y \mapsto xmy) \) for \( m \in M^B, x, y \in A \).

In particular, we let \( M = A \) and note the centralizer subring of \( A \),
\[
R := A^B \cong \text{Hom} \left( A_A \otimes_B A_A, A_A \right),
\]
via \( r \mapsto (x \otimes y \mapsto xry) \).

As another particular case, we let \( M = A \otimes_B A \), the tensor-square with its usual endpoint \( A\)-\( A \)-bimodule structure, so that the endomorphism ring (under composition)
\[
\text{End} A_A \otimes_B A_A \cong (A \otimes_B A)^B := T
\]
duces the following ring structure on the construction \( T \) for a ring extension \( A \mid B \), where we suppress a possible summation over simple tensors and use a Sweedler-like notation to write \( t = t^1 \otimes t^2 \in T \subseteq A \otimes_B A \).
\[
tu = u^1 t^1 \otimes t^2 u^2, \quad 1_T = 1_A \otimes 1_A
\]
We will fix the notation \( T \) throughout the paper for this ring construction while not making explicit its dependence on the extension \( A \mid B \). Note the left ideal \((A \otimes_B A)^A\) of Casimir tensor elements.

From the End-representation of \( T \) we note that \( A \otimes_B A \) is the left \( T \)-module:
\[
T(A \otimes_B A) : \quad t \cdot (x \otimes y) = xt^1 \otimes t^2 y
\]
for all \( t \in T, x, y \in A \).

Thus any module \( A_M \) (restricted then) induced to \( A \otimes_B M \) becomes a left \( T \)-module via the canonical isomorphism \( A \otimes_B M \cong A \otimes_B A \otimes_A M \):
\[
T(A \otimes_B M) : \quad t \cdot (a \otimes m) = at^1 \otimes t^2 m
\]
for \( m \in M, a \in A, t \in T \).

Given the bimodule \( A_M \), the End-representation of \( T \) and eq. 11 leads via composition from the right to the module \((M^B)_T\) given by
\[
(M^B)_T : \quad m \cdot t = t^1 m t^2 \quad (m \in M^B, t \in T).
\]

In particular, we obtain the module \( R_T \), important in this paper, defined by
\[
R_T : \quad r \cdot t = r^1 t^2 r^2 \quad (r \in R, t \in T).
\]
This module was studied in [9] as a generalized Miyashita-Ulbrich module with roots in Hopf algebra and group representations.

The bimodule \( R_T \) is important in the depth two theory in e.g. [10 6 7 8] where \( T \) acts as a right bialgebroid over \( R \), and plays a minor role in this paper:
\[
R_T : \quad r \cdot t \cdot r' := r^1 t^2 r'
\]
for \( t \in T, r, r' \in R \), a restriction of scalars to a submodule of the tensor-square \( A\)-\( A \)-bimodule.

We also need the construction \( S := \text{End}_B A_B \), dual to \( T \) as a left \( R \)-bialgebroid in the depth two theory [10 6 7]. In the ring \( S \) under the usual composition we note
the right ideal \( \text{Hom} (B_A, B_B) \). For any \( B \)-module \( N_B \), we note that the *coinduced* right \( A \)-module \( \text{Hom} (A_B, N_B) \) (given by \((ha)(x) = h(ax)\)) is also a right \( S \otimes B \)-module; i.e., with commuting right \( S \)- and \( B \)-action since \((hb)(\alpha(a)) = (h \circ \alpha)(ba)\) for \( \alpha \in S, b \in B \). In other words, the \( A \)-module action restricted to a \( B \)-module action commutes with the \( S \)-module action to give a module \( \text{Hom} (B_A, B_N)_{S \otimes B} \) (tensor over the integers).

Since each \( r \in R \) defines an endomorphism in \( S \) in two ways - by left multiplication or by right multiplication,

\[
\lambda_r(x) = rx, \quad \rho_r(x) = xr \quad (x \in A, r \in R)
\]

which commute \( \lambda_r \rho_s = \rho_s \lambda_r \), we work with an \( R \)-\( R \)-bimodule structure on \( S \) given by

\[
R S R : \quad r \cdot \alpha \cdot s = \lambda_r \rho_s \alpha = r \alpha (-) s
\]

which is the appropriate bimodule for the left bialgebroid theory in [10, 6, 7, 8].

Finally, there is the natural left \( S \)-action on \( A \) by evaluation, which restricted to the centralizer becomes

\[
S R : \quad \alpha \cdot r = \alpha (r)
\]

which is also in \( R \) (for \( \alpha \in S, r \in R \)). The module \( S R \), studied to an extent in [9, 8], will be important to our study of split extensions in the next section and left \( D_2 \) extensions in the following section.

2. Separable and Split Extensions

Recall that a ring extension \( A \mid B \) is *separable* if the mapping \( \mu : A \otimes_B A \rightarrow A \) given by multiplying components, \( \mu(x \otimes y) = xy \), is a split \( A \)-\( A \)-epimorphism; an element \( e \in \mu^{-1}(1_A) \) is called a separability element and characterizes separable extensions with its two properties, \( ae \otimes e^2 = e \otimes e^2 a \) for all \( a \in A \) and \( e^1 e^2 = \mu(e) = 1 \).

**Lemma 2.1.** *Given a separable extension \( A \mid B \) and a module \( A M \), the mapping \( r \otimes_T (a \otimes_B m) \rightarrow \alpha m \) is an isomorphism.*

\[
R \otimes_T (A \otimes_B M) \xrightarrow{\cong} M
\]

of left \( A \)-modules.

**Proof.** Note that the left \( A \)-module structure on \( R \otimes_T A \otimes_B M \) is given by \( a \cdot r \otimes_T x \otimes_B m = r \otimes_B ax \otimes m \) \((a, x \in A)\), which is well-defined since the left \( T \)-module structure on the induced module \( A \otimes_B M \) is given by \( t \cdot a \otimes m = at \otimes t^2 m \). An inverse mapping \( M \rightarrow R \otimes_T (A \otimes_B M) \) is defined in terms of a separability element by \( m \mapsto 1_A \otimes e^1 \otimes e^2 m \) for each \( m \in M \), since on the one hand we have \( e^1 e^2 m = m \) and on the other hand, given \( a \in A, r \in R \),

\[
1 \otimes (e^1 \otimes e^2 a m) = 1 \otimes_T (e^1 \otimes e^2 r) \cdot (a \otimes m) = r \otimes (a \otimes m).
\]

The lemma recovers the isomorphism in [9, Theorem 4.1] when \( M = A \). Let \( F \) denote the induction functor on \( A \)-modules, so that \( F(M) = A \otimes_B M \) for an \( A \)-module \( A M \) and \( F(f) = id_A \otimes f \) for an arrow \( f \in \_A M \). In the theorem below, \( F \) is shown to be a fully faithful functor between \( A \_M \) and its image \( F(A M) \) (perhaps more traditionally denoted by \( F(A M) = \text{IndRes}_A M \) [12, p. 95]). In addition, the
lemma together with projectivity of $R_T$ is clearly seen from the proof to characterize separability.

**Theorem 2.2.** A ring extension $A|B$ is separable if and only if the module $R_T$ is projective and the induction functor $F : A\mathcal{M} \to F(A\mathcal{M})$ is an equivalence of categories with inverse functor given by $G(A \otimes_B M) = R \otimes_T (A \otimes_B M)$.

**Proof.** ($\Rightarrow$) That the module $R_T$ (given by $r \cdot t = t^1 r t^2$) is cyclic projective follows from [9 4.1], or more simply by noting that the right $T$-epi

$$
\varepsilon_T : T_T \to R_T : \varepsilon_T(t) = t^1 t^2
$$
is split by $r \mapsto e^1 \otimes e^2 r$ using a separability element $e$.

By lemma, we see that there is a natural isomorphism $\gamma : GF \to I$ given by $\gamma_M(r \otimes a \otimes_B m) := am$ on an object $M$, a natural transformation since for each arrow $f : N \to M$ in $A\mathcal{M}$, the equation $\gamma_M \circ GF(f) = f \circ \gamma_N$ just follows from $f$ being $A$-linear. Similarly, we see that there is a natural isomorphism $\phi : FG \to I$ given by $\phi_M : A \otimes_B (R \otimes_T (A \otimes_B M)) \cong A \otimes_B M, \phi_M(a \otimes_B r \otimes_T x \otimes_B m) = a \otimes_B x r m$ on an object $M$ in $A\mathcal{M}$, where for each arrow $id_A \otimes g : A \otimes_B N \to A \otimes_B M$ in $F(A\mathcal{M})$ we have $(id_A \otimes g) \phi_N = \phi_M(id_A \otimes id_R \otimes id_A g)$ since $g$ is $A$-linear. Thus, $F$ is an equivalence of the category $A\mathcal{M}$ with its image subcategory.

($\Leftarrow$) Let $M$ be the natural left module $A A$. Applying the hypothesis on the ring extension $A|B$, we have $R_T$ is projective and $R \otimes_T (A \otimes_B A) \cong A$ as left $A$-modules, and by naturality, as $A$-$A$-bimodules. By [9 4.1] this characterizes a separable extension (via an application of the functor $- \otimes_T (A \otimes_B A)$ to $R_T \otimes^* \cong T_T$ and the isomorphism $GF(A) \cong A$). \hspace{1cm} $\square$

For example, if $A|B$ is a separable algebra, the centralizer $R = A$ and $T = A^e = A^{op} \otimes A$. The module $A^e$ is projective, a characterization of separability [1]. The epimorphism $\varepsilon_T : T \to R$ corresponds to the well-known epimorphism $\mu : A^e \to A$ with interesting kernel, the module of universal differentials $\Omega^1 = \{x(dy)| x, y \in A\}$ where $dx = x \otimes 1 - 1 \otimes x$ (cf. [2]).

The following corollary is the well-known relative projectivity characterization of separable extension; e.g. [14 10.8]. From the commutative diagram we see that tensoring by $R_T$ is the ingredient that changes split epis to isomorphisms in a study of separability.

**Corollary 2.3.** A ring extension $A|B$ is separable if and only if the mapping $\mu_M : A \otimes_B M \to M$ given by $\mu_M(a \otimes m) = am$ is a split $A$-$C$-epimorphism for each $A$-$C$-bimodule $M$ where $C$ is a ring.

**Proof.** Let $A|B$ be any ring extension and $M$ any $A$-$C$-bimodule. The homomorphism $\psi_M : A \otimes_B M \to R \otimes_T A \otimes_B M, \psi_M(a \otimes m) = 1 \otimes a \otimes m$, which is induced by the epi $\varepsilon_T : T_T \to R_T$, leads us to a commutative diagram:

$$
\begin{array}{ccc}
A \otimes_B M & \xrightarrow{\psi_M} & R \otimes_T (A \otimes_B M) \\
\downarrow{\mu_M} & & \downarrow{\gamma_M} \\
M & & \\
\end{array}
$$

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If $A | B$ is separable, then $\gamma_M$ is an isomorphism of $A$-$C$-bimodules and $\varepsilon_T$ is a split epi, whence $\psi_M$ is split. It follows that $\mu_M$ is a split $A$-$C$-epimorphism as stated.

Of course, if $\mu_A$ is split as an $A$-$A$-epi, then $A | B$ is separable by definition. □

Next we recall that a ring extension $A | B$ is split, if $B \to A$ has a left inverse as $B$-$B$-homomorphisms; thus $A | B$ is a proper extension, $A$ is a right and left $B$-generator and a left inverse $E$, called a conditional expectation, satisfies $E(1_A) = 1_B$.

Recall that for a ring homomorphism $B \to A$, the coinduced module $(\text{CoInd}N)_A$ of a $B$-module $N_B$ is an $A$-module $\text{Hom}(A_B, N_B)$, which is of course a right $S$-module via composition with elements of $S = \text{End}_BA_B$. The module action of $S$ however only commutes with the restricted action of $A$ to $B$.

**Lemma 2.4.** Given a split extension $A | B$ and a $B$-module $N_B$, the mapping $f \otimes r \mapsto f(r)$ defines an isomorphism

\[
\text{Hom}(A_B, N_B) \otimes_S R \overset{\sim}{\longrightarrow} N_B
\]

of right $B$-modules.

*Proof.* The mapping is $S$-balanced since $S R$ is given by evaluation, $\alpha \cdot r = \alpha(r)$. The mapping is right $B$-linear since the $B$-module structure is given by $(f \otimes r) \cdot b = fb \otimes r$, which is $S$-balanced as well, and $f(br) = f(rb) = f(r)b$ for $f \in \text{Hom}(A_B, N_B), b \in B, r \in R = A^B$.

An inverse is given $n \mapsto nE(-) \otimes 1_A$ where $n \in N$ and $nE(-)$ denotes the mapping in $\text{Hom}(A_B, N_B)$ given by $x \mapsto nE(x)$. This is indeed an inverse since $nE(1) = n$ on the one hand and

\[
f(r)E(-) \otimes 1_A = f \circ \rho_r \circ E \otimes_S 1_A = f \otimes r,
\]

on the other hand. □

Now consider the functor corresponding to the restriction of coinduction, $F : \mathcal{M}_B \to F(\mathcal{M}_B)$ given by $F(N_B) = \text{Hom}(A_B, N_B)_B$. (The image of $F$ is the category we might denote by $\text{ResCoInd} \mathcal{M}_B$.) For a split extension this functor is an equivalence as we see next.

**Theorem 2.5.** A ring extension $A | B$ is split if and only if the module $S R$ is projective and the induction functor $F : \mathcal{M}_B \to F(\mathcal{M}_B)$ is an equivalence of categories with inverse functor given by $G(\text{Hom}(A_B, N_B)) = \text{Hom}(A_B, N_B) \otimes_S R$.

*Proof.* ($\Rightarrow$) That $S R$ is cyclic projective follows from [9, Lemma 1.1], or by other means, noting that the left $S$-homomorphism $\varepsilon_S : S \to R$ given by $\alpha \mapsto \alpha(1)$ splits the monomorphism $S R \to S S$ given by $r \mapsto r E(-)$.

By lemma $GF \cong I$ on objects in $\mathcal{M}_B$, and is natural w.r.t. arrows $f : N_B \to U_B$ in $\mathcal{M}_B$ since the evaluation mappings in the lemma clearly commute with $\text{Hom}(A, f) \otimes \text{id}_R$ (where $\text{Hom}(A, f) = \lambda_f$) and $f$. Also $I \cong FG$ on objects $\text{Hom}(A_B, N_B)_B$ in $F(\mathcal{M}_B)$, since

\[
\text{Hom}(A_B, N_B) \overset{\cong}{\longrightarrow} \text{Hom}(A_B, \text{Hom}(A_B, N_B) \otimes_S R), \quad g \mapsto (a \mapsto g(a)E(-) \otimes 1_A)
\]

and is natural w.r.t. an arrow $f : N_B \to U_B$, since $f(g(a)E(-)) = f(g(a))E(-)$.

($\Leftarrow$) Let $N$ be the natural module $B_B$, whence $\text{Hom}(A_B, B_B) \otimes_S R \cong B$ as right $B$-modules, and also as left $B$-modules by naturality. Apply the functor
Hom\((A_B, B_B) \otimes S\) — (from left \(S\)-modules to \(B-B\)-bimodules) to \(sR \oplus * \cong sS\).

Whence

\[ B_B \oplus * \cong B \text{Hom}(A_B, B_B)_B. \]

It follows that there is \(f \in \text{Hom}(B \text{Hom}(A_B, B_B)_B, B_B)\) and

\[ g \in \text{Hom}(B_B, B \text{Hom}(A_B, B_B)_B) \cong \text{Hom}(B_A B, B_B) \]

via \(g \mapsto g(1_B)\), such that \(f \circ g = \text{id}_B\). If \(F := g(1)\), this becomes \(f(F) = 1_B\). Define \(E(a) = f(\lambda_a \circ F)\), an arrow in \(\text{Hom}(B_A B, B_B)\) satisfying \(E(1_A) = 1_B\). □

Note that the lemma and the proof of the theorem show that

**Corollary 2.6.** A ring extension \(A|B\) is split if and only if the module \(sR\) is finite projective and \(\text{Hom}(A_B, B_B) \otimes s R \cong B\) as \(B-B\)-bimodules.

### 3. Left D2 and H-separable Extensions

A ring extension \(A|B\) is said to be left D2 (or left depth two) if its tensor-square \(A \otimes_B A\) is centrally projective w.r.t. \(A\) as natural \(B-A\)-bimodules; i.e.,

\[ BA \otimes_B A_A \oplus * \cong \oplus^n_B A_A \]

(16)

Under some natural identifications of \(\text{Hom}(B_A B, B_A A_A) \cong T\) and \(\text{Hom}(B_A \otimes_B A_A, B_A A_A) \cong S\) valid for any ring extension, we arrive at the equivalent condition of left D2: there are matched elements \(\beta_i \in S\) and \(t_i \in T\) (called left D2 quasibases) such that the following equation in \(A \otimes_B A\) holds,

\[ x \otimes y = \sum_{i=1}^n t_i^1 \otimes t_i^2 \beta_i(x) y \]

(17)

for all \(x, y \in A\).

By letting \(y = 1\), this last equation is quite analogous to the equation \(a(1) \otimes \tau(a(2))a(3) = a \otimes 1\) for \(a\) in some Hopf algebra with antipode \(\tau\). Let’s see this idea in action in the next lemma, which is going to be used in our last section on normality.

**Lemma 3.1.** Let \(A|B\) be a left D2 extension and \(A \otimes_B A\) a bimodule. Then \(A \otimes_B A \subseteq M_B A\).

**Proof.** For each \(m \in M_B = \{m \in M : mb = bm, \ \forall b \in B\}\), define a mapping \(\varepsilon_m : A \otimes_B A \to M\) by \(\varepsilon_m(x \otimes y) = xmy\), which is well-defined. Extending the notion of the module \(R_T\), one defines a right \(T\)-module structure on \(M_B\) in eq. (17) by \(m \cdot t = t^1 m t^2\). Apply \(\varepsilon_m\) to eq. (17) with \(y = 1\) to obtain \(xm = \sum_i (m \cdot t_i) \beta_i(x) \in M_B A\). □

Naturally we would get equality in the lemma if we also had a corresponding right D2 condition, something we will do in the next section (and such extensions are said to be D2). For the next lemma we recall that we have \(T \otimes_R A \cong A \otimes_B A\) via a mapping \(\pi_A\) sending \(t \otimes_R a \mapsto t^1 \otimes_B t^2 a\) with inverse given by \(x \otimes_B y \mapsto \sum_t t_i \otimes_R \beta_i(x) y\) (an application of eq. (17) [9] 2.1(4)).

**Lemma 3.2.** If \(A|B\) is left D2 and \(A \otimes_B A\) is a module, then \(\gamma_M(r \otimes a \otimes m) = \text{arm}\) defines an isomorphism

\[ R \otimes_T (A \otimes_B M) \cong M \]

(18)

of right \(A\)-modules, which is natural w.r.t. \(A\)-module homomorphisms \(A M \to A N\).
Proof. Define a mapping \( \pi_M = \pi_A \otimes A \text{id}_A \) via the canonical identification \( A \otimes_A M \cong M \) as a component of the top horizontal arrow below. Since \( A \mid B \) is left D2, it follows that this arrow is an isomorphism. The left vertical and bottom horizontal arrows are again canonical isomorphisms.

The diagram is commutative by following an element \( r \otimes t \otimes m \) in the upper lefthand corner around from below to \( t^1 r^2 t^2 m \) and around from above to \( r \otimes t^1 \otimes t^2 m \), then to \( t^1 r^2 m \), the same element. Thus, \( \gamma_M \) is an isomorphism. It is clearly natural w.r.t. arrows in \( A \text{M} \) as in the proof of Theorem 2.2. \( \square \)

Let \( F \) again denote the induction functor on \( A \)-modules, so that \( F(M) = A \otimes_B M \) for an \( A \)-module \( M \).

**Theorem 3.3.** If a ring extension \( A \mid B \) is left D2, then the induction functor \( F : A \text{M} \to F_A \text{M} \) is an equivalence of categories with inverse functor given by \( G(A \otimes_B M) = R \otimes_T (A \otimes_B M) \).

Proof. By lemma, \( GF \cong I \) via the natural transformation \( \gamma_M : GF(M) \cong M \). Also \( FG \cong I \) on \( F(A \text{M}) \) by applying \( F \) to the natural transformation \( \gamma \). \( \square \)

In contrast to Theorem 2.2 for separable extensions, I am not aware of any characterization of left D2 extensions in terms of \( F \) and \( G \) with some condition on \( R_T \) alone. There are examples that suggest we avoid weakening or strengthening the projectivity condition on \( R_T \), while the theorem below suggests weakening a generator condition on \( R_T \). There are of course separable extensions that are not left D2 such as the complex group algebras corresponding to a non-normal subgroup pair \( H < G \) in a finite group \( G \) \([9, 3.2]\). There are also left D2 extensions that are not separable such as the group algebras corresponding to a proper normal subgroup \( H \triangleleft G \) of a \( p \)-group over a characteristic \( p \) field; or more mundanely, any finite projective algebra that is not a separable algebra. The H-separable extensions (named after Hirata) are both separable, left (and right) D2 \([10, \text{section 3}]\) for which we have the following theorem. Recall that \( A \mid B \) is H-separable if there are matched elements (called an H-separability system) \( e_i \in (A \otimes_B A)^n \) and \( r_i \in R \) \((i = 1, \ldots, n)\) such that in \( A \otimes_B A \) we have \( 1 \otimes 1 = \sum_i e_i r_i \). For example, from an H-separability system we see \( A \mid B \) is left D2 with \( \beta_i = \rho r_i \) and \( t_i = e_i \).

**Theorem 3.4.** A ring extension \( A \mid B \) is H-separable if and only if \( R_T \) is a pro-generator and \( F, G \) defined in Theorems 2.2 and 3.3 are inverse equivalences.

Proof. (\( \Rightarrow \)) This follows from Theorems 2.2 and 3.3 and the observation that \( R_T \) is a generator \([9, 4.2]\), which we also see as follows. Given an H-separability system, the epi \( \oplus^n R_T \to T_T \) given by \((x_1, \ldots, x_n) \mapsto \sum_{i=1}^n x_i e_i \) is split by the right \( T \)-monomorphism \( t \mapsto (t^1 r_1 t^2, \ldots, t^1 r_n t^2) \).
(⇐) From the generator condition \( T_{T} \oplus * \cong \oplus^{n} R_{T} \) we arrive at
\[
A A \otimes_{B} A A \oplus * \cong \oplus^{n} A A,
\]
which is another characterization of H-separable extension \( A \mid B \) \[1\] ch. 2], by tensoring with \(- \otimes_{T} (A \otimes_{B} A)\) and using naturality together with \( GF \cong I \).

In this theorem, \( F \) and \( G \) are not Morita equivalences, although if we expand the category \( F(A,M) \) to the category \( T_{M} \), the functor \( R \otimes_{T} - \) is a Morita equivalence to the category \( Z(A)_{M} \), where \( Z(A) \) is the center of \( A \) \[9\] 4.3]. For example, if \( A \mid B \) is an Azumaya algebra (a separable algebra over its center \( B \)), then we recover the well-known fact that \( A^{c} \) and \( B \) are Morita equivalent via the progenator \( A_{A^{-1}} \).

We now turn to coinduction of modules in relation to left \( D_{2} \) extension. The lemma below extends an aspect of the characterization of the endomorphism ring as a smash product with the bialgebroid \( S \) in \[10\] 3.8].

**Lemma 3.5.** If the extension \( A \mid B \) is left \( D_{2} \) and \( M_{A} \) is a module, then there is an isomorphism \( \text{Hom}(A_{B}, M_{B}) \cong M \otimes_{R} S \) of right \( B \)-modules and right \( S \)-modules, which is natural with respect to arrows in \( M_{A} \).

**Proof.** The isomorphism is given by
\[
\chi_{M} : M \otimes_{R} S \to \text{Hom}(A_{B}, M_{B}), \quad \chi_{M}(m \otimes \alpha)(a) := m \alpha(a).
\]
This is \( B \)-linear w.r.t. the right \( B \)-module \( M \otimes_{R} S \) given by \( m \otimes_{R} \alpha \cdot b = mb \otimes_{R} \alpha \) since \( \chi_{M}(mb \otimes \alpha)(a) = \chi_{M}(m \otimes \alpha)(ba) \). It is also \( S \)-linear since \( \chi_{M}(m \otimes \alpha \cdot \beta)(a) = \chi_{M}(m \otimes \alpha)(\beta(a)) \) for \( \alpha, \beta \in S \).

The transformation \( \phi \) is natural w.r.t. \( g : M_{A} \to N_{A} \) since \( g(m \alpha(-)) = g(m) \alpha(-) \) for each \( \alpha \in \text{End}_{B}A_{B} \).

An inverse mapping is given by
\[
\text{Hom}(A_{B}, M_{B}) \to M \otimes_{R} S : \quad f \mapsto \sum_{i} f(t_{i}^{1})t_{i}^{2} \otimes_{R} \beta_{i},
\]
inverse by short computations using \( \alpha(t^{1})t^{2} \in R \) and \( \mu_{M}(f \otimes \text{id}_{A}) \) applied to eq. \( \text{[17]} \). □

Let \( R : M_{A} \to M_{B} \) denote the usual restriction (pullback or forgetful) functor from \( A \)-modules to \( B \)-modules for a ring homomorphism \( B \to A \). The next theorem is like Theorem \[2\] for split extensions in that restricted coinduction is shown to be a fully faithful functor but restricted this time to the subcategory \( R(M_{A}) \) of \( M_{B} \).

Again let \( F \) be the functor of restricted coinduction from \( M_{A} \) into \( M_{B} \) given by \( F(N) = \text{Hom}(A_{B}, N_{B})_{B} \) for any module \( N_{B} \).

**Theorem 3.6.** If \( A \mid B \) is left \( D_{2} \), then \( F \) is a category equivalence from \( R(M_{A}) \) into \( FR(M_{A}) \) with inverse functor \( G \) given on a module \( M_{A} \) by
\[
G(\text{Hom}(A_{B}, M_{B})) = \text{Hom}(A_{B}, M_{B}) \otimes_{S} R.
\]

**Proof.** Note that applying the lemma and associativity of tensor product gives us
\[
(19) \quad (\text{Hom}(A_{B}, M_{B}) \otimes_{S} R) \cong M \otimes_{R} S \otimes_{S} R \cong M \otimes_{R} R \cong M
\]
a composition of isomorphisms from left to right giving the mapping \( \rho_{M} : f \otimes_{S} r \mapsto f(r) \) for \( f \in \text{Hom}(A_{B}, M_{B}), r \in R \). Since \( \rho \) is natural w.r.t. to morphism \( R(f) \) for arrows \( f \) in \( M_{A} \), it follows that \( GF \cong I \) on \( R(M_{A}) \). From this it follows that also \( FG \cong I \) on \( FR(M_{A}) \).

□
In contrast to Theorem 2.6 characterizing split extensions, which will also have $F$ as an equivalence on the subcategory $R(M_A)$ of $M_B$, I am not aware of any characterization of left D2 extensions in terms of $F$ and $G$ with some condition on $sR$ alone. There are left D2 extensions that are not split such as the endomorphism algebra over the $2 \times 2$ upper triangular algebra [7 5.5]. There are of course split extensions that are not left D2 such as the complex group algebras corresponding to a non-normal subgroup $H < G$ of a finite group [7 3.2], or certain exterior algebras over their centers. If $A \mid B$ is split and centrally projective (so $B_0 A_B \oplus \ast \cong \oplus^n B B_B$, thus it is left D2), we may characterize these by placing the progenator condition on $sR [4 1.1]$ in a theorem formulated like Theorems 3.6 and 2.6.

We end this section with an endomorphism ring characterization of left D2 for right f.g. projective extensions. Let $E = \text{End}_B$ denote the right endomorphism ring of a ring extension $A \mid B$. This has $A$-$A$-bimodule structure given by $(xfy)(a) = xfy(a)$, equivalently $x \cdot f \cdot y = \lambda_x \circ f \circ \lambda_y$.

**Proposition 3.7.** Suppose $A \mid B$ is a ring extension where $A_B$ is finite projective. Then $A \mid B$ is left D2 if and only if

$$AE_B \oplus \ast \cong \oplus^n A_A B_B.$$

**Proof.** We note that for any ring extension, there is an isomorphism $\Phi$,

$$\text{End}_B \xrightarrow{\sim} \text{Hom}(A \otimes_B A_A, A_A), \quad \Phi(f)(x \otimes y) = f(x)y,$$

with inverse given by $F \mapsto F(\cdot \otimes 1_A)$. Note that $\Phi$ is an $A$-$A$-isomorphism since $\Phi(xfy)(a \otimes c) = xfy(a)c = x\Phi(f)(ya \otimes c)$.

($\Rightarrow$) If we are given that the tensor-square is centrally projective w.r.t. $B A_B$,

$$BA \otimes_B A_A \oplus \ast \cong \oplus^n B A_A,$$

we apply the functor $\text{Hom}(-, A_A)$ from $B$-$A$-bimodules into $A$-$B$-bimodules to this, obtaining eq. (20). This only uses eq. (21) and does not use the hypothesis on $A_B$.

($\Leftarrow$) Since $A_B$ is finite projective, it follows that $A \otimes_B A_A$ and $AE$ are finite projective by applying $- \otimes_B A$ and $A \text{Hom}(-, A_B)$ to $A_B \oplus \ast \cong \oplus^n B B_B$. Hence the module $A \otimes_B A_A$ is reflexive, so $\text{Hom}(A_E, A_A) \cong A \otimes_B A$. Then we apply $B \text{Hom}(A_A, - A_A)_A$ to eq. (20) and obtain eq. (22) as a consequence. \hfill \Box

4. **Normality for Subrings**

A ring extension $A \mid B$ is right D2 if there are $n$ matched elements $u_j \in T$ and $\gamma_j \in S$ such that the following equation in the tensor-square $A \otimes_B A$ holds,

$$x \otimes y = \sum_{j=1}^n x\gamma_j(y)u_j^1 \otimes u_j^2$$

for all $x, y \in A$.

Right D2 extensions are dual to left D2 extensions in a certain category of ring extensions, via the notion of opposite ring [7]. While there is no theoretical reason given yet for right D2 extensions to be left D2, there has never been observed a one-sided D2 ring extension that is not two-sided. We say a ring extension is D2 if it is both left and right D2. For example, finite right Galois extensions with Hopf algebroid actions are automatically right D2, but they can be shown to be left D2 as well via the antipode. Finding a Galois extension without antipode might give us a
one-sided D2 extension; however, there is Schauenburg’s theorem that an antipode
is definable from a Galois isomorphism for a bialgebra action.

In this section we study normality for subrings in relation to D2 subrings (or
proper extensions via inclusion). The reason for this study is that normal subgroups
of finite groups correspond precisely to D2 complex finite dimensional subgroup
algebras [9], which led to Nikshych asking the interesting generalized question of
whether a D2 Hopf subalgebra is normal; we answer this question affirmatively in
case the Hopf subalgebra enjoys equality with its double centralizer in corollary 4.5.

We use the simplest possible definition of normal subring in [16] while dropping
the requirement that the ring and subring be semisimple artinian. (The requirements
of semisimplicity and the ideals be maximal are not needed in our study; however, the
reader is urged to read [16] before making any further use of this notion.) First, let
B be a subring of A; we say an (two-sided) ideal I in B is A-invariant if IA = AI
as subsets of A. An ideal J in A may be contracted to an ideal J ∩ B in B.

Definition 4.1. For the purposes of this section, a subring B ⊆ A is said to be
normal if all contracted ideals are A-invariant; i.e., for every ideal J in A, we have
A(J ∩ B) = (J ∩ B)A.

For example, the center of a ring is a normal subring. Normal subgroups corre-
spond to normal subrings via group algebra by the proposition on Hopf subalgebras
below.

Proposition 4.2. If A | B is a D2 extension, then the centralizer R is a normal
subring in A.

Proof. By lemma 3.1 and its right D2 dual, we have the following equality of subsets
in an A-A-bimodule AM
(24) AMB = MB A

Let J be an ideal in A. Then J B = J ∩ R, so J contracted to R is A-invariant by
the equation.

For the same reasons as in eq. (24), each submodule J ⊆ RT , given a D2 extension
A | B, satisfies A-invariance:

(25) AJ = JA

Recall that a Hopf subalgebra K ⊂ H (with antipode τ) is normal if a(1)Kτ(a(2)) ⊆ K
and τ(a(1))Ka(2) ⊆ K for all a ∈ H. The two terminologies do not conflict:

Proposition 4.3. Suppose K ⊆ H is a Hopf subalgebra. Then K is normal subring
of H if and only if K is a normal Hopf subalgebra of H.

Proof. (⇒) Suppose K is a normal Hopf subalgebra in H. Let J be an ideal in H
and x ∈ J ∩ K. Given a ∈ H, we note that

(26) ax = a(1)xτ(a(2))a(3) ∈ (J ∩ K)H

by the normality condition. Hence H(J ∩ K) ⊆ (J ∩ K)H, and similarly (J ∩ K)H ⊆
H(J ∩ K).

(⇒) Here we use the well-known characterization of normality for a Hopf sub-
algebra K in H by HK+ = K+ H [3] ch. 3]where K+ = ker ε|K and H+ = ker ε
are the augmentation ideals of the counit (and its restriction to K). We note that
K+ = H+ ∩ K, whence by hypothesis HK+ = K+ H. □
It is well-known in textbooks on group theory that given a subgroup $H < G$, the centralizer subgroup $C_G(H)$ is normal in the normalizer subgroup $N_G(H)$; consequently, if $H$ is normal in $G$, then $C_G(H)$ is normal in $G$. The converse though is not true by some easy examples; e.g., within the dihedral group of transformations of a square. Here is a Hopf subalgebra analogue.

**Proposition 4.4.** Suppose $K$ is a Hopf subalgebra of a finite dimensional Hopf algebra $H$, in which the centralizer $C_H(K)$ is a Hopf subalgebra as well. If $K \subseteq H$ is a normal Hopf subalgebra, then $C_H(K) \subseteq H$ is a normal Hopf subalgebra.

**Proof.** Since $K \subseteq H$ is a normal Hopf subalgebra, it is Hopf-Galois (with coaction by the quotient Hopf algebra $H/HK^+$) and therefore D2 [9]. By proposition 4.2, the centralizer $C_H(K)$ is a normal subring in $H$, therefore a normal Hopf subalgebra by proposition 4.3. \[\square\]

Notice that the proof shows the following.

**Corollary 4.5.** Suppose $K$ is a Hopf subalgebra of a finite dimensional Hopf algebra $H$, in which the centralizer $C_H(K)$ and the double centralizer $C_H(C_H(K))$ are Hopf subalgebras. If $K \subseteq H$ is a D2 Hopf subalgebra, then the double centralizer $C_H(C_H(K)) \subseteq H$ is a normal Hopf subalgebra.

While $K \subseteq C_H(C_H(K))$ is always the case, this will be a proper subset even for certain normal subgroups, such as the index 2 subgroups in the quaternion 8-element group.

5. **Discussion**

In this section, we append some additional thoughts about the previous four sections. First, we pose in terms of modules the problem of when a ring extension $A \mid B$ satisfies $R \otimes_T (A \otimes_B A) \cong A$ via the mapping $\gamma_A$ defined in section 2. For any ring $C$ and modules $M_C, N_C$, this question becomes simply when is evaluation $\text{Hom}(M_C, N_C) \otimes_E M_C \xrightarrow{\cong} N_C$ an isomorphism where $E := \text{End}_M C$? We view this question in terms of Morita theory and its generalizations, thereby extending Theorem 2.2.

Secondly, two characterizations of the left D2 condition on a ring extension $A \mid B$ are obtained via naturally isomorphic functors of induction or coinduction from the category of $A$-modules into the category of $B$-modules. Third, the centralizer of a D2 extension is noted to be a pre-braided commutative ring. Finally, a depth two arrow in a bicategory of bimodules is considered in comparison with the endomorphism ring extension of a bimodule, and two candidates for a definition of D2 bimodule are shown to be the same if the bimodule is finite projective.

5.1. **The Morita viewpoint on $\gamma_A$.** Let $A \mid B$ be a ring extension. An interesting question is when $A \mid B$ has isomorphic $\gamma_A$ (which recall is given by $\gamma_A(r \otimes a \otimes c) = arc$ for $r \in R$, $a, c \in A$). Such an extension is a generalization of separable extension, right D2 extension and left D2 extension, which is clear from lemmas 2.1 and 3.2 (as well as an adaptation of the latter to right D2 extensions).

Thinking in terms of the canonical isomorphisms introduced in the preliminaries of section 1, there are $R \cong \text{Hom}_{A^e}(A \otimes_B A, A)$, $T \cong \text{End}_{A^e}(A \otimes_B A)$, where
$A^e = A^{op} \otimes A$ and $\gamma_A$ is identical with evaluation,

$$\text{Hom}_{A^e}(A \otimes B, A) \otimes_{\text{End}_{A^e}(A \otimes B, A)} (A \otimes B, A) \rightarrow A,$$

where $R_T$ is identical with the right $\text{End}_{A^e}(A \otimes B, A)$-module $\text{Hom}_{A^e}(A \otimes B, A)$
given by right composition. This leads to the extended question of when given a ring $C$, there are modules $M_C$ and $N_C$ such that the evaluation mapping

$$\gamma_{M,N} : \text{Hom}(M_{C}, N_{C}) \otimes_{\text{End}_{M_C}} M_C \rightarrow N_C$$

given by $\gamma_{M,N}(f \otimes m) = f(m)$, is an isomorphism?

A solution to this when $N = C$ and $M_C$ is a progenerator comes from Morita theory, since $\text{Hom}(M_C, C_C) \otimes E M_C \cong C$ as $C$-$C$-bimodules where $E := \text{End} M_C$
in addition to $M \otimes_C \text{Hom}(M_C, C_C) \cong \text{End} M_C$ via rank one projection.

More generally, if given a $C$-module $X$, let $\sigma[X]$ denote the full subcategory of right $C$-modules that are submodules of quotients of direct sums $X^{(I)}$ for an arbitrary indexing set $I$.

**Proposition 5.1.** Suppose $M_C$ is $\sigma[X]$-projective and $\sigma[X]$-generator and $N_C \in \sigma[X]$ w.r.t. a third $C$-module $X$. Then the evaluation mapping $\gamma_{M,N}$ above is a right $C$-isomorphism.

The proof follows directly from the proof in [11, p. 499].

If we pass to the Dress subcategory $D[M]$ of $\sigma[M]$, whose objects are direct summands of direct sums of copies of $M$, there is a solution given by the following.

**Theorem 5.2.** Suppose $M_C, N_C$ are two $C$-modules such that $N_C \in D[M]$. Then $\gamma_{M,N}$ is an isomorphism.

**Proof.** Let $\pi_i : M_C \rightarrow N_C$ and $\iota_i : N_C \rightarrow M_C$ be $2n$ mappings such that $\sum_{i=1}^{n} \pi_i \circ \iota_i = \text{id}_N$. Then an inverse to $\gamma_{M,N}(f \otimes m) = f(m)$ is given by $n \mapsto \sum_{i} \pi_i \otimes \iota_i(n)$. □

Lemma 2.1 is a corollary of this theorem, since if $A | B$ is separable, we have $N_C \oplus \ast \cong M_C$ via the multiplication mapping, where $C = A^e$, $N = A$ and $M = A \otimes_B A$.

As a last remark note that there is a version of the commutative triangle in Figure 1 if $D M_C$ is a bimodule w.r.t. rings $C$ and $D$. This triangle consists of the evaluation mapping $\gamma_{M,N}$ and the counit of adjunction $\varepsilon$ to the standard Hom-Tensor adjunction between the functors $\text{Hom}(M_C, -)$ and $- \otimes_D M$ between $M_C$ and $M_D$, which is another evaluation mapping. Between them is a canonical quotient mapping $\psi_M$ induced from left module structure $D \rightarrow \text{End} M_C$. The commutative diagram is the following.

$$\begin{array}{ccc}
\text{Hom}(M_C, N_C) \otimes_D M & \xrightarrow{\psi_{M,N}} & \text{Hom}(M_C, N_C) \otimes_{\text{End}_{M_C}} M \\
\varepsilon & & \gamma_{M,N} \\
N & \xrightarrow{\varepsilon} & \text{End}_{M_C} M
\end{array}$$
5.2. Two Functorial Characterizations of Left D2 Extension. Given a ring extension \( A \mid B \), we consider two functors of induction from the category of modules \( A M \) into \( B M \). First we have \( I_B^A := \text{Res}_B^A \text{Ind}_B^A \text{Res}_B^A \) which is defined as the notation suggests, viz. \( I_B^A(AM) = BA \otimes_B M \). Second, we make use of constructed ring \( T \) and its right module structure over the centralizer \( R \) given by \( t \cdot r = t^1 \otimes t^2 r \) (recalled from the preliminaries of section 1). From this we obtain a second functor \( I^T_R : A M \rightarrow B M \) defined by \( I_R^T(AM) = T \otimes_R M \) where the left \( B \)-module structure makes use of elements of \( R \) and \( B \) commuting in \( A \): \( b \cdot (t \otimes m) = t \otimes bm \). Left D2 extensions are characterized by these two functors being naturally isomorphic and having a special property.

**Theorem 5.3.** A ring extension \( A \mid B \) is left D2 if and only if \( T_R \) if f.g. projective and the two functors \( I_B^A \) and \( I^T_R \) are naturally isomorphic.

**Proof.** The theorem is a restatement of \([7, \text{Prop. 2.2}]\) in terms of functors. If \( A \mid B \) is left D2, then \( A \otimes_B M \cong T \otimes_R M \) via \( a \otimes_B m \mapsto \sum t_i \otimes_R \beta_i(a)m \) with inverse \( t \otimes_R m \mapsto t^1 \otimes_B t^2 m \) in terms of a left D2 quasibase. Either of these maps is clearly natural w.r.t. an arrow \( AM \rightarrow AN \) in \( A M \). That \( T_R \) is finite projective follows from applying eq. \((17)\) to elements in \( T = (A \otimes_B A)^B \subseteq A \otimes_B A \).

Conversely, given \( T_R \ast \ast \cong \otimes^\ast B_R \), apply the functor \( - \otimes_R A \) into \( B M A_1 \), one obtains from naturality and \( I_B^A(AM) \cong I^T_R(AM) \) that \( B A \otimes_B A \ast \ast \ast \cong \otimes^\ast_B B A \), the left D2 condition. \( \square \)

Note that if \( A \mid B \) is left D2 then the two functors are also naturally isomorphic as functors to the category of left bimodules over \( T \) and \( B \):

\[
I_B^A \cong I^T_R : A M \rightarrow T \otimes_B M
\]

where \( T \otimes_B T \otimes_R M \) for a module \( AM \) is naturally given by \((t \otimes b) \cdot (u \otimes m) = tu \otimes bm \), and \( T \otimes_B A \otimes_B M \) is given by \((t \otimes b) \cdot (a \otimes_B m) = bat \otimes_B t^2 m \).

Now recall the endomorphism ring \( S \equiv \text{End}_B A_B \) and introduce a third functor of coinduction \( CoI^S_R : A M \rightarrow B M \) defined by \( CoI^S_R(AM) = B \text{Hom} \left( (R_S, R M) \right) \) where one again makes use elements of \( B \) and \( R \) commuting in \( A \) in order to define the left \( B \)-module \( \text{Hom} \left( (R_S, R M) \right) \). In analogy with the previous results, the two functors below are naturally isomorphic if \( A \mid B \) is left D2, also as functors into \( S \otimes_B A_M \).

**Theorem 5.4.** A ring extension \( A \mid B \) is left D2 if and only if the module \( R S \) is f.g. projective and \( I_B^A \) is naturally isomorphic to \( CoI^S_R \).

**Proof.** This is a restatement of \([7, \text{Prop. 2.2(iii)}] \) in functorial terms. If \( A \mid B \) is left D2, then \( A \otimes_B M \cong \text{Hom} \left( (R_S, R M) \right) \) via

\[
(29) \quad \chi_M(a \otimes_B m)(\alpha) := \alpha(a)m
\]

for \( \alpha \in S \), with inverse \( F \mapsto \sum t_i \otimes_R t_i^2 F(B_i) \). Either mapping is natural w.r.t. an arrow \( g : AM \rightarrow AN \) (where \( \chi_M \) and \( \chi_N \) fit into a commutative square with \( \text{Hom} \left( (S, g) \right) \) and \( A \otimes g \)).

Conversely, from \( R S \ast \ast \ast \cong \otimes^\ast_R R \), apply \( B \text{Hom} \left( (\cdot, R A) \right) \) to obtain via naturality and \( B I_B^A(AM) \cong B CoI^S_R(AM) \) the left D2 condition, eq. \((16)\). \( \square \)

Note that \( \chi_M \) is also a left \( S \)-homomorphism where \( S A \otimes_B M \) is naturally given by \( \alpha \cdot a = \alpha(a) \) and \( S \text{Hom} \left( (R S, R M) \right) \) is induced from the natural module \( S_M \). This follows from \( \chi_M(\beta(a) \otimes m)(\alpha) = \chi_M(a \otimes m)(\alpha \circ \beta) \) for \( \alpha, \beta \in S \).
5.3. The centralizer of a D2 extension is a pre-braided commutative ring. In [3] a natural coaction $\delta : A \to A \otimes R T$ on a (right) D2 extension $A | B$ is introduced, and it is noted that $A | B$ is a right $T$-Galois extension if $A_B$ is faithfully flat (although this condition is not needed for the discussion to follow). The coaction $\delta$ is given in Sweedler notation by

$$a_{(0)} \otimes a_{(1)} := \sum_j \gamma_j(a) \otimes_R u_j$$

using the notation in eq. (23). The coaction $\delta$ restricts to the centralizer $R$ of $A | B$ to give the mapping $s_R : R \to T$, where $s_R(r) = 1 \otimes_B r$, after the simplification $R \otimes_R T \cong T$:

$$r_{(0)} \otimes r_{(1)} = \sum_j 1 \otimes \gamma_j(r)u_j = 1 \otimes_R 1 \otimes_B r.$$  

Recall the module $R_T$ with action temporarily denoted by $r \cdot t = t^1 r t^2$. Then the ring $R$ satisfies the following pre-braided commutativity condition (cf. [1]):

$$sr = r_{(0)}(s \cdot r_{(1)}).$$

Another equivalent way to look at this type of commutativity condition is for a ring $R$ with module structures $sR$ and $RT$ which do not combine to give a bimodule, but instead there are elements $\gamma_j \in S$, $u_j \in T$ such that

$$sr = \sum_j (\gamma_j \cdot r)(s \cdot u_j)$$

The Miyashita-Ulbrich module $R_T$, the right $R$-bialgebroid $T$ and the right $T$-comodule $R$ formally satisfy the Yetter-Drinfeld condition.

5.4. Depth Two Bimodules. We consider possible definitions of a depth two bimodule from two viewpoints. First, there is the viewpoint of a depth two arrow in a bicategory with a recipe for right D2 in [17]. There is a bicategory of rings where arrows or 1-cells are bimodules and 2-cells are bimodule homomorphisms [12], p. 283. (Horizontal) composition of arrows is tensoring bimodules, which is only associative up to an isomorphism, while (vertical) composition of 2-cells is composition of bimodule homomorphisms. If $\alpha : R M_S \to R N_S$, $\beta : R N_S \to R Q_S$, $\gamma : sU_T \to sV_T$, and $\delta : sV_T \to sW_T$ are bimodule homomorphisms, then the horizontal and vertical compositions satisfy a middle four exchange law as an equality of homomorphisms between $R M \otimes_S U_T \to R Q \otimes_S W_T$

$$\beta \otimes_S \delta \circ (\alpha \otimes_S \gamma) = (\beta \circ \alpha) \otimes_S (\delta \circ \gamma)$$

To this we make some changes. Replace an object ring $R$ by its category of modules $M_R$, then the arrows above become induction functors, to which we add coinduction functors $\text{Hom}(sM_R, -) : M_R \to M_S$. The coinduction functor just given has left adjoint the induction functor $- \otimes S M_R : M_S \to M_R$. The recipe for bimodule $sM_R$ to be a right D2 arrow in [17] is functorial with value on $M$ given by

$$\text{Hom}(M_R, \text{End}(M_R) \otimes S M_R) \oplus \ast \cong \oplus^n \text{End}(M_R)$$

as right $S$-modules. For example, if $B \to A$ is a ring homomorphism and $M = B A_A$ the induced bimodule, then the induction functor is $\text{Ind}^A_B$, the coinduction functor
is $\text{Res}_B^A$, and by naturality one arrives at the right D2 condition,

$$AA \otimes BA_B \oplus * \cong \oplus^n A_B.$$  

On the other hand, one might make use of the endomorphism ring of a ring extension to define a bimodule $SM_R$ to be right D2 if the left module structure mapping $S \to \text{End} M_R$ is right D2. For example, given a ring homomorphism $B \to A$ and its induced bimodule $BA_A$, this map simplifies $B \to \text{End} A_A \cong A$ to $B \to A$. Now let $E$ denote $\text{End} M_R$, then $S \to E$ is right D2 if $E \otimes S E_S$ is centrally projective w.r.t. $E E_S$. What relation does the central projectivity condition have to the condition in eq. (34)? They are the same if $M_R$ is finite projective:

**Lemma 5.5.** Suppose $SM_R$ is a bimodule which is f.g. projective as a right module. Then there is an $E$-$S$-bimodule isomorphism,

$$\text{End} M_R \otimes S \text{End} M_R \cong \hom(M_R, \text{End} M_R),$$

given by $\phi(f \otimes g)(m) = f \otimes g(m)$ for every $f, g \in E, m \in M$.

The proof of the lemma is left as an exercise. Note that for any ring extension $A | B$ the canonical bimodule $BA_A$ is right rank one free. These considerations lead us then to define when a right finite projective bimodule is right D2.

**Definition 5.6.** A right f.g. projective bimodule $SM_R$ is said to be right D2 if its endomorphism ring extension $S \to \text{End} M_R$ is right D2.

The definition admits an extension of the definition to left D2 left finite projective bimodules by opposite categorical considerations as in [7]: a left finite projective bimodule $SM_R$ is left D2 if its left endomorphism ring extension $R \to (\text{End} S M)^{op}$ is left D2. Then a left D2 extension $A | B$ yields the left D2 bimodule $A_A B$. However, left and right D2 bimodules do not generalize properly D2 ring extension unless we restrict ourselves to Frobenius extensions and Frobenius bimodules, since a Frobenius extension is left D2 iff it is right D2. Recall that a bimodule $SM_R$ is Frobenius if $SM$ and $M_R$ are finite projective modules, and the left and right duals of $M$ are $R$-$S$-bimodule isomorphic:

$$\hom(S M, S S) \cong \hom(M_R, R_R).$$

The main property of a Frobenius bimodule is that its right and left endomorphism ring extensions are Frobenius (Morita, cf. [3] ch. 2)).

**Definition 5.7.** A Frobenius bimodule $SM_R$ is said to be D2 if its endomorphism ring extensions $R \to (\text{End} S M)^{op}$ and $S \to \text{End} M_R$ are D2.

Note that a Frobenius extension $A | B$ is D2 iff its bimodule $BA_A$ (or $A_A B$) is Frobenius and D2, which follows from the endomorphism ring theorem for D2 extensions [7]. For example, the Frobenius bimodule in [3, 2.7] is D2 since its endomorphism ring is H-separable, therefore left and right D2. D2 bimodules would in principle provide a wealth of examples of D2 extensions via the endomorphism ring.

**References**

[1] I. Bálint and K. Szlachányi, Finitary Galois extensions over non-commutative bases, KFKI preprint (2004), RA/0412122.

[2] G. Böhm and T. Brzeziński, Strong connections and the relative Chern-Galois character for corings, preprint, 2005. arXiv:math.RA/0503469
[3] S. Caenepeel, D. Quinn and S. Raianu, Duality for finite Hopf algebra explained by corings, preprint, 2005. RA/0504427.

[4] F. DeMeyer and E. Ingraham, Separable Algebras over Commutative Rings, LNM 181, Springer, New York, 1971.

[5] L. Kadison, New Examples of Frobenius Extensions, University Lecture Series 14, AMS, Providence, 1999. Update, (2004) 6 pp.: www.ams.org/bookpages.

[6] L. Kadison, Depth two and the Galois coring, in: Noncommutative Geometry and Representation Theory in Mathematical Physics, ed. J. Fuchs, AMS Contemporary Math, to appear. math.RA/0408155.

[7] L. Kadison, The endomorphism ring theorem for Galois and D2 extensions, preprint, 2005, QA/0503194.

[8] L. Kadison, Galois theory for bialgebroids, depth two and normal Hopf subalgebras, in: Proc. Conf. Swansea and Ferrara (2004), eds. T. Brzezinski and C. Menini, Annali dell’Università di Ferrara, Sez. VII, Scienze Matematiche, to appear. arXiv:math.QA/0402188

[9] L. Kadison and B. Külshammer, Depth two, normality and a trace ideal condition for Frobenius extensions, Commun. Algebra, to appear. GR/0409346.

[10] L. Kadison and K. Szlachányi, Bialgebroid actions on depth two extensions and duality, Adv. in Math. 179 (2003), 75–121.

[11] T.Y. Lam, Lectures on Modules and Rings, GTM 189, Springer, New York, 1999.

[12] S. Mac Lane, Categories for the Working Mathematician, GTM 5, Springer, New York, 2nd edition, 1998.

[13] S. Montgomery, Hopf Algebras and Their Actions on Rings, CBMS Regional Conf. Series in Math. Vol. 82, AMS, Providence, 1993.

[14] R. Pierce, Associative Algebras, GTM 88, Springer, New York, 1982.

[15] F. Van Oystaeyen et al, Separable functors revisited, Commun. Algebra 18 (1990), 1445-1459.

[16] M.A. Rieffel, Normal subrings and induced representations, J. Algebra 59 (1979), 364–386.

[17] K. Szlachányi, Finite quantum groupoids and inclusions of finite type, in: Mathematical physics in mathematics and physics (Siena, 2000), 393–407, Fields Inst. Commun., 30, A.M.S., Providence, RI, 2001.

[18] R. Wisbauer, Foundations of Module and Ring Theory, Gordon and Breach, Philadelphia, 1991.

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