THE ISOMORPHISM PROBLEM FOR UNIVERSAL ENVELOPING ALGEBRAS OF FOUR-DIMENSIONAL SOLVABLE LIE ALGEBRAS

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Abstract. This paper is a contribution to the isomorphism problem for universal enveloping algebras of finite-dimensional Lie algebras. We focus on solvable Lie algebras of small dimensions over fields of arbitrary characteristic. We prove, over an arbitrary field, that the isomorphism type of a metabelian Lie algebra whose derived subalgebra has codimension one is determined by its universal enveloping algebra. As an application of the results in this paper, we solve the isomorphism problem for solvable Lie algebras of dimension four over fields of characteristic zero and also point out the problems that occur in prime characteristic.

1. Introduction

The main results of this paper make a contribution to the study of the isomorphism problem for universal enveloping algebras of Lie algebras which asks if an isomorphism $U(L) \cong U(H)$ between the universal enveloping algebras of two Lie algebras $L$ and $H$ implies the isomorphism $L \cong H$. It is known that the answer to this question is, in general, "no" and explicit examples of non-isomorphic finite-dimensional Lie algebras with isomorphic universal enveloping algebras were presented in [RU07, SU11]. On the other hand, all such known examples are nilpotent and are defined over fields of characteristic $p$ with small $p$ (in the sense that $p < \dim L$). Moreover, the known results show that the isomorphism problem does have a positive solution in several natural and important classes of finite-dimensional Lie algebras. Indeed, over an arbitrary field of characteristic different from 2, the isomorphism problem was solved for the class of 3-dimensional Lie algebras by Malcolmson [Mal92] and by Chun, Kajiwara and Lee [CKL04].

The recent paper [CPR-NW] by Campos, Petersen, Robert-Nicoud, and Wierstra announced the positive solution of the isomorphism problem for a large class of finite-dimensional Lie algebras including finite-dimensional nilpotent Lie algebras over fields of characteristic zero. The proof of this result relies on the deep theory of quasi-isomorphism
and Koszul duality. The isomorphism problem for small-dimensional nilpotent Lie algebras over fields of arbitrary characteristic was also addressed using computational techniques by the last two authors [SU11]. A detailed treatment of the isomorphism problem can be found in the article by Riley and the third author [RU07] and in the survey [Use15].

It might be worth noting the analogies with a similar isomorphism problem for integral group rings where a positive solution for the class of all nilpotent groups was given independently in [RS87] and [Wei88]. There exists, however, a pair of non-isomorphic finite solvable groups of derived length 4 whose integral group rings are isomorphic (see [Her01]). In view of Cartier-Kostant-Milnor-Moore theorem, enveloping algebras and group algebras are building blocks of cocommutative Hopf algebras, so one might be able to view these problems in this broader context.

In analogy with group rings and in light of a recent positive solution in the class of finite-dimensional nilpotent Lie algebras, one may ask whether there is a counterexample of solvable Lie algebras for the isomorphism problem. So, in this paper we focus on solvable Lie algebras. We study the isomorphism problem for finite-dimensional but not necessarily nilpotent Lie algebras and we mostly focus on techniques that are valid over fields of arbitrary characteristic. After the preliminary Section 2 in which we summarize the required tools for the rest of the paper, in Section 3, we prove the following result concerning a class of metabelian Lie algebras over arbitrary fields. This can be viewed as an adaptation of the corresponding theorem on group rings by Whitcomb [Whi68] (see also [PMS02, Theorem 9.3.13]).

**Theorem 1.1.** Let $L = M \rtimes \langle x \rangle$ and $H = N \rtimes \langle y \rangle$ be finite-dimensional Lie algebras over an arbitrary field, where $M$ and $N$ are ideals of $L$ and $H$, respectively. Suppose that $\alpha : U(L) \to U(H)$ is an algebra isomorphism such that $\alpha(MU(L)) = NU(H)$. Then, $L/M' \cong H/N'$.

The other main result of Section 3 is the following theorem which is obtained using the theory developed by Elashvili and Ooms [EO03, Oom04] concerning the relationship between the center $Z(U(L))$ and the Frobenius semiradical of $L$.

**Theorem 1.2.** Suppose that $L$ and $H$ are finite-dimensional metabelian Lie algebras over a field $\mathbb{F}$ of characteristic zero, and that $L = (L' + Z(L)) \rtimes \langle x \rangle$ and $H = (H' + Z(H)) \rtimes \langle y \rangle$. If $U(L) \cong U(H)$, then $L \cong H$.

Given a Lie algebra $L$, an abelian ideal $M$ of $L$ and a certain maximal ideal $m$ of $U(L)$, we construct in Section 4 a Lie algebra $\mathcal{L}_{M,m}$ and an associative algebra $\mathcal{A}_{M,m}$ using sections of $U(L)$. Choosing $M$ and $m$ appropriately, $\mathcal{L}_{M,m}$ and $\mathcal{A}_{M,m}$ will be invariant under certain automorphisms of $U(L)$ (Theorem 4.2 and Corollary 4.3). The algebras $\mathcal{L}_{M,m}$ and $\mathcal{A}_{M,m}$ are explicitly computable for finite-dimensional Lie algebras (Section 5) and $\mathcal{L}_{M,m}$ often turns out to be isomorphic to $L$ (Corollary 4.4). Hence, for given Lie algebras $L$ and $H$ with abelian ideals $M$ and $N$ and adequate maximal ideals of $U(L)$ and $U(H)$, respectively, computing $\mathcal{L}_{M,m}$ and $\mathcal{L}_{N,n}$ or $\mathcal{A}_{M,m}$ and $\mathcal{A}_{N,n}$ can lead
to verifying that \( U(L) \not\cong U(H) \). The Lie algebra \( \mathcal{L}_{M,m} \) and the associative algebra \( A_{M,m} \) are explicitly calculated for several 4-dimensional solvable Lie algebras in Section 5.

Our results allow us to prove the following theorem which gives a positive solution for the isomorphism problem for universal enveloping algebras of 4-dimensional solvable Lie algebras over fields of characteristic zero.

**Theorem 1.3.** Let \( L \) and \( H \) be solvable Lie algebras of dimension at most 4 over a field of characteristic zero. If the universal enveloping algebras \( U(L) \) and \( U(H) \) are isomorphic, then \( L \) and \( H \) must also be isomorphic.

The proof of Theorem 1.3 is presented in Section 6 and relies on de Graaf’s classification of 4-dimensional solvable Lie algebras [dG05] (see also Section 6). First, we divide the class of 4-dimensional solvable Lie algebras into 6 groups in such a way that two Lie algebras in distinct groups cannot have isomorphic universal enveloping algebras by the results of the aforementioned article [RU07].

The use of the Frobenius semiradical is the main reason why we could not state Theorem 1.3 in an arbitrary characteristic. Indeed, while Theorem 1.1 and the constructions of \( \mathcal{L}_{M,m} \) and \( A_{M,m} \) do not require conditions on the characteristic, the results that rely on the Frobenius semiradical are only valid in characteristic zero. We conjecture that Theorem 1.3 holds also in positive characteristic, and in Conjecture 6.2, we state three statements whose validity would imply this claim, but the verification of these statements is beyond the means of the current paper.

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## 2. Preliminaries

Let \( L \) be a Lie algebra over a field \( \mathbb{F} \) and denote by \( U(L) \) the universal enveloping algebra of \( L \). Suppose that \( L \) is finite-dimensional and let \( \{x_1, \ldots, x_d\} \) be a basis of \( L \). By the Poincaré–Birkhoff–Witt Theorem (see [Hum78, Theorem, p. 92]), the monomials of the form \( x_1^{a_1} \cdots x_d^{a_d} \), with \( a_i \geq 0 \) for all \( i \), form a basis for \( U(L) \). In particular, the linear subspace generated by the \( x_i \) is a Lie subalgebra of \( U(L) \) which can be identified with the original Lie algebra \( L \). Hence the associative algebra \( U(L) \) is also generated by \( L \). The augmentation map \( \varepsilon_L : U(L) \to \mathbb{F} \) is the unique algebra homomorphism that
extends the map \( x_i \mapsto 0 \) for all \( i \). The kernel of \( \varepsilon_L \) is denoted by \( \omega(L) \) and is referred to as the **augmentation ideal** of \( U(L) \). The augmentation ideal \( \omega(L) \) is the unique maximal ideal of \( U(L) \) that contains \( x_i \) for all \( i \). An easy application of the Poincaré–Birkhoff–Witt Theorem shows that \( \omega(L) = LU(L) = U(L)\,L \). In \( U(L) \), the \( n \)-th power \( \omega(L)^n \) is an ideal which is denoted by \( \omega^n(L) \). It is clear that the ideals \( \omega^1(L) = \omega(L) \), \( \omega^2(L) \), \( \omega^3(L) \), \ldots form a descending chain.

If \( S \) is a subalgebra of \( L \), then a basis of \( S \) can be extended to a basis of \( L \), and so, in virtue of the Poincaré–Birkhoff–Witt Theorem, \( U(S) \) can be considered as a subalgebra of \( U(L) \) and \( \omega(S) \) can be considered as a subalgebra of \( \omega(L) \).

The following lemma collects some important properties of \( U(L) \) and \( \omega(L) \) that will be used in the paper. For the proof of parts (1) and (2) see [Dix96, Proposition 2.2.14], for part (3) see [RU07, Proposition 6.1(1)]. The \( i \)-th term of the lower central series of a Lie algebra \( L \) is denoted by \( \gamma_i(L) \) starting with \( \gamma_1(L) = L \) and \( \gamma_2(L) = L' \).

**Lemma 2.1.** Let \( L \) be a Lie algebra, \( M \) an ideal of \( L \) and \( S \) a subalgebra of \( L \). Then

1. The right ideal \( MU(L) \) of \( U(L) \) generated by \( M \) coincides with the left ideal \( U(L)\,M \) of \( U(L) \) generated by \( M \). Hence \( MU(L) = U(L)\,M \) is a two-sided ideal.
2. The homomorphism \( U(L) \to U(L/M) \) induced by the projection \( \pi : L \to L/M \) is surjective with kernel \( U(L)\,M \).
3. \( \omega(S) \cap \omega^n(S)\omega(L) = \omega^{n+1}(S) \); hence, \( L \cap \omega^n(S)\omega(L) = \gamma_{n+1}(S) \). In particular \( L \cap \omega(M)\omega(L) = L \cap M\omega(L) = M' \).

The following elementary result that we shall frequently use can be quickly derived from Lemma 2.1(1).

**Lemma 2.2.** The ideal \( L'U(L) \) coincides with the two-sided ideal of \( U(L) \) generated by \( \{[a, b] := ab - ba \mid a, b \in U(L)\} \). Consequently, \( L'U(L) \) is invariant under \( \text{Aut}(U(L)) \).

It is known that two non-isomorphic Lie algebras may have isomorphic universal enveloping algebras; examples are presented in [RU07, SU11]. Nevertheless, several properties of a Lie algebra are determined by the isomorphism type of its enveloping algebra. For instance, it is a well known that if \( L \) is a finite-dimensional Lie algebra, then the (linear) dimension of \( L \) coincides with the Gelfand–Kirillov dimension of \( U(L) \) (see [MR01, 8.1.15(iii)]); thus \( \dim L \) is determined only by the isomorphism type of \( U(L) \). A detailed study of some properties of Lie algebras that are determined by the isomorphism type of its universal enveloping algebra is given in [RU07]; in the following lemma we give a summary of the ones that we use in this paper.

**Lemma 2.3.** Let \( L \) and \( H \) be finite-dimensional Lie algebras and \( \alpha : U(L) \to U(H) \) an algebra isomorphism. Then the following hold.

1. \( \dim L = \dim H \) and \( \dim L/L' = \dim H/H' \).
2. If \( L \) is nilpotent, then so is \( H \). Moreover, in this case the nilpotency classes of \( L \) and \( H \) coincide.
(3) $L'/L'' \cong H'/H''$; in particular, $\dim L'/L'' = \dim H'/H''$.
(4) If $L$ is metabelian, then so is $H$.
(5) If $L$ is solvable, then so is $H$.
(6) If $M$ and $N$ are ideals of $L$ and $H$, respectively, such that $\alpha(MU(L)) = NU(H)$, then $M/M' \cong N/N'$.

The next lemma is a variation of [RU07, Lemma 2.1].

**Lemma 2.4.** Let $L$ and $H$ be Lie algebras and suppose that $\alpha : U(L) \to U(H)$ is an isomorphism. Then there exists an isomorphism $\overline{\alpha} : U(L) \to U(H)$ such that

1. $\overline{\alpha}(\omega(L)) = \omega(H)$; and
2. if $x \in L$ such that $\alpha(x) \in \omega(H)$, then $\overline{\alpha}(x) = \alpha(x)$.

At the end of this preliminary section, let us review some facts concerning the Frobenius semiradical of a Lie algebra in characteristic zero. This concept will be used in the proof of Theorem 1.3. A more complete treatment can be found in [EO03, Oom04]. We denote the dual of a vector space $V$ by $V^*$. Let $L$ be a Lie algebra over a field $\mathbb{F}$ of characteristic zero and let $f \in L^*$. Then we define

$$L(f) = \{ x \in L \mid f([x, y]) = 0 \text{ for all } y \in L \}.$$ 

It is easy to see that $L(f)$ is a Lie subalgebra of $L$ containing the center $Z(L)$ of $L$. Set

$$i(L) = \min_{f \in L^*} \dim L(f);$$

the number $i(L)$ is referred to as the index of $L$. An element $f \in L^*$ is called regular if $\dim L(f) = i(L)$, the set of regular elements is denoted by $L^*_\text{reg}$. The set

$$F(L) = \sum_{f \in L^*_\text{reg}} L(f)$$

is called the Frobenius semiradical of $L$. The Frobenius semiradical is a subalgebra of $L$, containing $Z(L)$, which is invariant under $\text{Aut}(L)$.

The following result links the Frobenius semiradical of a Lie algebra defined over a field of characteristic zero to the center $Z(U(L))$ of its enveloping algebra. If $X$ is a subset or an element of a Lie algebra $L$, then $C_L(X)$ denotes the centralizer of $X$ in $L$.

**Lemma 2.5.** Let $L$ be a Lie algebra over a field of characteristic zero. Then the following statements hold.

1. Let $M$ be a Lie subalgebra of $L$ of codimension one. Then $i(M) = i(L) + 1$ if and only if $F(L) \subseteq M$.
2. If $x \in L$ is such that $C_L(x)$ has codimension one in $L$, then $i(C_L(x)) = i(L) + 1$.
3. $Z(U(L)) \subseteq U(F(L))$.

**Proof.** See [EO03, Proposition 1.6(4)] for part (1), [EO03, Proposition 1.9(1)] for part (2) and [Oom04, Theorem 2.5(1)] for part (3). $\square$
Part (3) of Lemma 2.5 gives a necessary condition for an element to be central in $U(L)$ and this will be used in the proof of Theorem 1.3. The containment stated in Lemma 2.5(3) is not valid in characteristic $p$, since, in this case, there exists a non-zero polynomial expression $p(x)$, for each $x \in L$, such that $p(x) \in Z(U(L))$; see [Jac79, Chapter VI, Lemma 5].

The following consequence of Lemma 2.5 will be used in the proof of Theorem 1.2.

**Lemma 2.6.** Suppose that $L$ is a metabelian and non-abelian Lie algebra over a field of characteristic zero such that $L' + Z(L)$ has codimension one in $L$. Then $F(L) \subseteq L' + Z(L)$.

**Proof.** Set $M = L' + Z(L)$. Clearly, $M \leq C_L(L')$ and, in fact, as $L$ is non-abelian, $M = C_L(L')$. Since $M$ has codimension one in $L$, there is $x \in L$ such that $L = M \rtimes \langle x \rangle$.

Moreover, there exists $y \in L'$ such that $[y, x] \neq 0$, and it follows that $M = C_L(y)$. Now, Lemma 2.5 implies that $F(L) \subseteq M$. $\square$

3. Universal enveloping algebras of metabelian Lie algebras

In this section we prove Theorem 1.1 which shows that the isomorphism problem for universal enveloping algebras has a positive solution for a rather large class of metabelian Lie algebras. This theorem was inspired by a corresponding result for group algebras. In the proof we use a well-known fact that if $\varphi$ is an automorphism of a polynomial ring $\mathbb{F}[x]$, then $\varphi(x) = ax + b$ with $a, b \in \mathbb{F}$ and $a \neq 0$; see [Nag72, Proposition 3.1].

**The proof of Theorem 1.1.** Invoking Lemma 2.4, we assume without loss of generality that $\alpha(MU(L)) = NU(H)$ and $\alpha(\omega(L)) = \omega(H)$. We claim that

(1) for every $z \in L$ there exists $y_z \in H$ such that $\alpha(z) \equiv y_z \pmod{N\omega(H)}$.

Let us first verify claim (1) in the case when $z = x$. The map $\alpha$ induces an isomorphism between $U(L)/MU(L)$ and $U(H)/NU(H)$. On other hand, by Lemma 2.1(2),

$$U(L)/MU(L) \cong U(L/M) \quad \text{and} \quad U(H)/NU(H) \cong U(H/N),$$

and so we can view $U(L)/MU(L)$ and $U(H)/NU(H)$ as polynomial rings in the variables $x + MU(L)$ and $y + NU(H)$, respectively. Thus, using the remark preceding the proposition, we have $\alpha(x) + NU(H) = ay + b + NU(H)$ where $a, b \in \mathbb{F}$ and $a \neq 0$. Further, as we assume $\alpha(\omega(L)) = \omega(H)$, it follows that $b = 0$. Hence, $\alpha(x) + NU(H) = ay + NU(H)$, and it follows that there exist $z_1, \ldots, z_n \in N$ and $u_1, \ldots, u_n \in U(H)$ such that

$$\alpha(x) = ay + \sum_{i=1}^{n} z_i u_i.$$
Denoting the augmentation map of $U(H)$ by $\varepsilon_H$, we have
\[
\alpha(x) = ay + \sum_{i=1}^{n} z_i u_i = ay + \sum_{i=1}^{n} z_i (u_i - \varepsilon_H(u_i) + \varepsilon_H(u_i))
\]
\[
= ay + \sum_{i=1}^{n} \varepsilon_H(u_i) z_i + \sum_{i=1}^{n} z_i (u_i - \varepsilon_H(u_i)).
\]
Since $u_i - \varepsilon_H(u_i) \in \omega(H)$, it follows that
\[
(2) \quad \alpha(x) \equiv ay + \sum_{i=1}^{n} \varepsilon_H(u_i) z_i \pmod{N\omega(H)}.
\]
Note that the element on the right-hand side of equation (2) lies in $H$. Denoting this element by $y_x$, we obtain that
\[
(3) \quad \alpha(x) \equiv y_x \pmod{N\omega(H)}.
\]
Hence (1) is valid for the element $x \in L$. If $z \in M$, then $\alpha(z) \in NU(H) = N + N\omega(H)$, and so there is some $y_z \in N$ such that $\alpha(z) \equiv y_z \pmod{N\omega(H)}$. Therefore property (1) holds also for elements of $M$. As $L = M + \langle x \rangle$, claim (1) must hold for all elements $z \in L$.

Now, using Lemma 2.1(3) we have that
\[
\alpha(M\omega(L)) = \alpha(MU(L)\omega(L)) = \alpha(MU(L))\alpha(\omega(L)) = NU(H)\omega(H) = N\omega(H).
\]
Therefore the assignment $z + M\omega(L) \mapsto y_z + N\omega(H)$ for all $z \in L$ defines an injective Lie algebra homomorphism
\[
L/(L \cap M\omega(L)) \cong (L + M\omega(L))/M\omega(L) \to (H + N\omega(H))/N\omega(H) \cong H/(H \cap N\omega(H)).
\]
Applying Lemma 2.1(3), we obtain that $L/(L \cap M\omega(L)) \cong L/M'$ and, similarly, that $H/(H \cap N\omega(H)) \cong H/N'$. Thus there exists an injective Lie algebra homomorphism $L/M' \to H/N'$. On the other hand, Lemma 2.3(6) shows that $\dim M/M' = \dim N/N'$, and since $\dim L/M = \dim H/N = 1$ we can conclude that $\dim L/M' = \dim H/N'$. Now it follows that the injective homomorphism $L/M' \to H/N'$ is an isomorphism. □

**Corollary 3.1.** Suppose that $L$ and $H$ are as Theorem 1.1 and that $\alpha : U(L) \to U(H)$ is an algebra isomorphism. Then the following are valid.

1. If either $M$ or $N$ is abelian, then $L \cong H$.
2. If $M = L'$ and $N = H'$, then $L/L'' \cong H/H''$. If, in addition, either $L$ or $H$ is metabelian, then $L \cong H$.

**Proof.** (1) By Lemma 2.3(1), $\dim L = \dim H$. Suppose that $M$ is abelian. By Theorem 1.1, we have $L \cong H/N'$, and hence $N' = 0$. Thus $L \cong H$, as claimed.

(2) Lemma 2.2 implies that $\alpha(L'^{U(L)}) = H'^{U(H)}$. Thus Corollary 3.1(1), with $M = L'$ and $N = H'$, gives us $L/L'' \cong H/H''$. If, in addition, $L$, say, is metabelian, then so is $H$, by Lemma 2.3(4). In particular, $L'' = 0$ and $H'' = 0$, which implies $L \cong H$. □
It is worth noting that in [SU11, Theorem 3.1(ii)] we can find examples of non-isomorphic metabelian Lie algebras with isomorphic universal enveloping algebras, in which the derived subalgebras have codimension different from one. These algebras are nilpotent and are defined over fields of positive characteristic.

We end this section with the proof of Theorem 1.2. Recall that the Frobenius semiradical \( F(L) \) of a Lie algebra \( L \) was introduced in Section 2.

**The proof of Theorem 1.2.** First, we note that the theorem is true if either \( L \) or \( H \) is abelian. Therefore, we assume that \( L \) and \( H \) are non-abelian. Set \( M = L' + Z(L) \) and \( N = H' + Z(H) \). By Lemma 2.6, \( F(L) \subseteq M \) and \( F(H) \subseteq N \). Applying Lemma 2.5(3), we have the following chains of inclusions:

\[
(4) \quad Z(U(L)) \subseteq U(F(L)) \subseteq U(M) \quad \text{and} \quad Z(U(H)) \subseteq U(F(H)) \subseteq U(N).
\]

Let \( \alpha : U(L) \to U(H) \) be an isomorphism and suppose using Lemma 2.4 that \( \alpha(\omega(L)) = \omega(H) \). We claim that \( \alpha(MU(L)) = NU(H) \). By Lemma 2.2,

\[
\alpha(L'U(L)) = H'U(H) \subseteq NU(H),
\]

and since \( Z(L) \subseteq Z(U(L)) \), it follows that

\[
\alpha(Z(L)) \subseteq Z(U(H)).
\]

Further, \( Z(L) \subseteq \omega(L) \), and hence we obtain from equation (4) that

\[
\alpha(Z(L)) \subseteq Z(U(H)) \cap \omega(H) \subseteq U(N) \cap \omega(H) = \omega(N) \subseteq NU(H).
\]

Thus \( \alpha(Z(L)U(L)) \subseteq NU(H) \), and

\[
\alpha(MU(L)) = \alpha(L'U(L)) + \alpha(Z(L)U(L)) \subseteq NU(H).
\]

Applying the same argument to the isomorphism \( \alpha^{-1} : U(H) \to U(L) \), we have that \( \alpha^{-1}(NU(H)) \subseteq MU(L) \). Therefore \( \alpha(MU(L)) = NU(H) \), and since \( M \) and \( N \) are abelian ideals of \( L \) and \( H \), respectively, Corollary 3.1(1) implies that \( L \cong H \). \( \square \)

4. **Finite-dimensional automorphism invariants of universal enveloping algebras**

The aim of this section is to present constructions that, for a Lie algebra \( L \), for an abelian ideal \( M \) of \( L \), and for a certain maximal ideal \( m \) of \( U(L) \), outputs another Lie algebra \( L_{M,m} \) and an associative algebra \( A_{M,m} \) that are constructed using sections of \( U(L) \); the algebras \( L_{M,m} \) and \( A_{M,m} \) are invariant under certain automorphisms of \( U(L) \). We will show that \( L_{M,m} \) can be isomorphic to \( L \), and in such cases the isomorphism type of \( L \) can be determined by the isomorphism type of \( U(L) \). The constructions are based on arguments presented by [CKL04].

Throughout this section, \( L \) denotes a finite-dimensional Lie algebra over a field \( \mathbb{F} \) and \( M \) an abelian ideal of \( L \). Let \( K \) denote the quotient \( L/M \). Lemma 2.1 implies that \( MU(L) \) is a two-sided ideal of \( U(L) \) and \( U(K) \cong U(L)/MU(L) \). Since \( M \) is abelian, \( K \) has a well-defined action \( \text{ad}_{K,M} : K \to \mathfrak{gl}(M) \) given by \( \text{ad}_{K,M}(k + M)(a) = [k, a] \) for all
We can take, for instance, $m_k$ of $U \varpi$ (5), the map and consider $\beta$ (6).

**Proposition 4.1.** Let $\beta : U(L)/MU(L) \to \mathfrak{gl}(\widetilde{I})$ by setting

$$\beta(w + MU(L))(a + Mm) = [w, a] + Mm = (wa - aw) + Mm$$

for all $w \in U(L)$ and $a \in MU(L)$.

**Proposition 4.1.** Let $L$, $M$, $m$, $\widetilde{I}$ and $\beta$ be as above. Then the following are valid.

1. The map $\beta : U(L)/MU(L) \to \mathfrak{gl}(\widetilde{I})$ is a well-defined homomorphism of Lie algebras.
2. Let $J$ be the unique Lie ideal of $U(L)$ that contains $MU(L)$ and such that $\ker \beta = J/MU(L)$. Considering $\widetilde{I}$ and $\widetilde{K} = U(L)/J$ as Lie algebras and setting

$$\mathcal{L}_{M,m} = \widetilde{I} \rtimes_\beta \widetilde{K},$$

we have that $\mathcal{L}_{M,m}$ is a finite-dimensional Lie algebra. Moreover, if $K$ is abelian, then $\mathcal{L}_{M,m}$ is metabelian.
3. Let $\mathfrak{J} = J \cap m$. Then $\mathfrak{J}$ is a two-sided associative ideal of $U(L)$ and $\mathfrak{J}$ has codimension one in $J$. In particular, $\dim U(L)/\mathfrak{J} = \dim U(L)/J + 1$.
4. If $m = \omega(L)$, then $L \cap J = C_L(M)$ and $C_L(M)U(L) \subseteq \mathfrak{J}$.
5. If $m = \omega(L)$, then there is an injective homomorphism $\widetilde{\chi}$ from $M \rtimes L/C_L(M)$ to $\mathcal{L}_{M,m}$ which is an isomorphism if and only if $\dim L/C_L(M) = \dim \widetilde{K}$.

**Proof.** (1) We claim that $\beta$ above is well-defined. Suppose that $w \in U(L)$, $a, v \in MU(L)$, $u \in Mm$ and recall that $MU(L) \subseteq m$. Then

$$[w + v, a + u] = [w, a] + [w, u] + [v, a] + [v, u].$$

Now $[w, u] = uw - uw \in Mm$, $[v, a] = va - av \in Mm$, and $[v, u] = vu - vw \in Mm$. Thus $[w + v, a + u] + Mm = [w, a] + Mm$. Since $\beta$ is induced by the adjoint action of $U(L)$ on the ideal $MU(L)$, we have that $\beta : U(L)/MU(L) \to \mathfrak{gl}(\widetilde{I})$ is a Lie algebra homomorphism.
(2) Since $\beta$ is a Lie algebra homomorphism, it follows that $\mathcal{L}_{M,m}$ is a Lie algebra. Note that $\beta$ induces an injective homomorphism $\tilde{K} \to \mathfrak{gl}(\tilde{I})$, and so $\tilde{K}$ is finite-dimensional. Hence $\mathcal{L}_{M,m}$ is finite-dimensional. Now, if $K = L/M$ is abelian then so is $U(K) \cong U(L)/MU(L)$, and consequently $\tilde{K}$ is also abelian. In this case $\mathcal{L}_{M,m}$ is the semidirect sum of two abelian Lie algebras, and so $\mathcal{L}_{M,m}$ is metabelian.

(3) Since $\mathfrak{J}$ is a Lie ideal of $U(L)$, it is enough show that $\mathfrak{J}$ is a right ideal. Suppose that $w \in \mathfrak{J}$ and let $u \in U(L)$. Then we have, for $a \in MU(L)$, that

$$[wu, a] = [w, a]u + w[u, a] = [w, a]u + [w, [u, a]] + [u, a]w.$$ 

Since $w \in J$, the first two summands lie in $Mm$, and since $w \in m$ the last summand is in $Mm$. Thus $wu \in \mathfrak{J} = J \cap m$, and so $\mathfrak{J}$ is a two-sided ideal of $\omega(L)$. Since $1 \in J$, we have that $J + m = U(L)$, and so

$$J/\mathfrak{J} = J/(J \cap m) = (J + m)/m = U(L)/m \cong \mathbb{F}.$$ 

Therefore $\mathfrak{J}$ has codimension one in $J$.

(4) Suppose that $m = \omega(L)$ and let $x \in L \cap J$. Then, by the definition of $J$, we have, for all $y \in M$, that

$$[x, y] \in L \cap M\omega(L) = M' = 0$$

(see Lemma 2.1(3) for the second equality). Thus $x \in C_L(M)$, and so $L \cap J \subseteq C_L(M)$. For the converse, let $x \in C_L(M)$ and let $a = \sum_{i=1}^{n} x_iu_i$, where $x_i \in M$ and $u_i \in U(L)$, be an arbitrary element of $MU(L)$. Since $[x, x_i] = 0$ for all $i$, it follows that

$$[x, a] = x \sum_{i=1}^{n} x_iu_i - \sum_{i=1}^{n} x_iu_ix = \sum_{i=1}^{n} x_iu_i - \sum_{i=1}^{n} x_iu_ix.$$ 

Since every summand of the right-hand side of the last equation lies in $M\omega(L)$, we have that $x \in L \cap J$. Thus $C_L(M) \subseteq L \cap J$ and the equality $C_L(M) \subseteq L \cap J$ follows.

For the second statement, note that $C_L(M)$ is an ideal of $L$, and so $C_L(M)U(L)$ is a two-sided ideal of $U(L)$, by Lemma 2.1. As $\mathfrak{J}$ is a two-sided ideal of $U(L)$ which contains $C_L(M)$, it follows directly that $C_L(M)U(L) \subseteq \mathfrak{J}$.

(5) Note that $M$ is a $K$-module with the adjoint action $\text{ad}_{K,M}$. Hence $M$ is also a module for the associative algebra $U(K)$. Now $U(K) \cong U(L)/MU(L)$, and so $U(K)$ acts on the vector space $\tilde{I}$ by the composition of the isomorphism $U(K) \to U(L)/MU(L)$ and the homomorphism defined in (6). Hence the isomorphic vector spaces $M$ and $\tilde{I}$ are both $U(K)$-modules considering $U(K)$ as a Lie algebra. Let $\psi : U(K) \to \mathfrak{gl}(M)$ and $\tilde{\psi} : U(K) \to \mathfrak{gl}(\tilde{I})$ denote the corresponding homomorphisms. Recall that $\vartheta : M \to \tilde{I}$ as defined in equation (5) is a linear isomorphism. If $\alpha \in \mathfrak{gl}(M)$ then $\tilde{\alpha} = \vartheta \alpha \vartheta^{-1}$ is a corresponding endomorphism of $\tilde{I}$ and the map $\alpha \mapsto \tilde{\alpha}$ is an isomorphism $\mathfrak{gl}(M) \to \mathfrak{gl}(\tilde{I})$.

We claim that $\tilde{\psi}(k) = \tilde{\psi}(k)$ for all $k \in K$. In other words, under the bijection $\vartheta : M \to \tilde{I}$, an element $k \in K$ induces the same transformation on $M$ and on $\tilde{I}$. Suppose that $k \in K$ and $a + Mm \in \tilde{I}$. As noted above, the elements $x_i + Mm$ form a
basis for $\tilde{I}$, where the set $\{x_i\}_i$ is a basis for $M$. Hence we may assume without loss of
genularity that $a \in M$. Then
\[
\tilde{\psi}(k)(a + Mm) = [k, a] + Mm = \psi(k)(a) + Mm = \vartheta \psi(k)\vartheta^{-1}(a + Mm)
\]
\[
= \tilde{\psi}(k)(a + Mm).
\]
Thus the claim holds.

Define the map $\chi : M \times K \to \mathcal{L}_M = \tilde{I} \rtimes \tilde{K}$, by $\chi(a, k + M) = (\vartheta(a), k + J)$. Clearly, $\vartheta : M \to \tilde{I}$ is a linear isomorphism and the map $k + M \mapsto k + J$ is a well-defined linear homomorphism $K \to U(L)/J$. It follows from the argument in the previous paragraph that $\chi$ is a homomorphism of Lie algebras. It is immediate verify that the kernel of the map $k + M \mapsto k + J$ is equal to $C_L(M)$. Further, $\chi$ induces an injective homomorphism $\tilde{\chi} : M \rtimes L/C_L(M) \to \tilde{I} \rtimes \tilde{K}$ which is an isomorphism if and only if $\dim L/C_L(M) = \dim \tilde{K}$. $\square$

In Proposition 4.1(2) we defined the Lie algebra
\[
\mathcal{L}_{M,m} = \tilde{I} \rtimes \tilde{K}
\]
and we can also define the associative algebra
\[
(7) \quad \mathcal{A}_{M,m} = U(L)/\mathfrak{J}
\]
where $\mathfrak{J}$ is the two-sided ideal of $U(L)$ defined in Proposition 4.1(3). If $m = \omega(L)$ then we simply write $\mathcal{L}_M$ and $\mathcal{A}_M$ instead of $\mathcal{L}_{M,m}$ and $\mathcal{A}_{M,m}$, respectively. If $L$ is a finite-dimensional Lie algebra, then both $\mathcal{L}_{M,m}$ and $\mathcal{A}_{M,m}$ are finite-dimensional and they will be explicitly computed for some 4-dimensional solvable Lie algebras in Section 5. Now the main result of this section states that the Lie algebra $\mathcal{L}_{M,m}$ and the associative algebra $\mathcal{A}_{M,m}$ are preserved by certain automorphisms of $U(L)$.

**Theorem 4.2.** Let $L$ and $H$ be finite-dimensional Lie algebras over a field $\mathbb{F}$ and let $M$ and $N$ be abelian ideals of $L$ and $H$, respectively. Suppose that $\alpha : U(L) \to U(H)$ is an algebra isomorphism such that $\alpha(MU(L)) = NU(H)$. Let $m$ be a maximal ideal of $U(L)$ such that $U(L)/m \cong \mathbb{F}$ and $M \subseteq m$ and set $n = \alpha(m)$. Then $\alpha$ induces an isomorphism between the Lie algebras $\mathcal{L}_{M,m}$ and $\mathcal{L}_{N,n}$ defined in Proposition 4.1(2) and between the associative algebras $\mathcal{A}_{M,m}$ and $\mathcal{A}_{N,n}$ defined in (7).

**Proof.** Denote by $\tilde{I}_1 = MU(L)/Mm$ and $\tilde{I}_2 = NU(H)/Nn$. Since $\alpha(MU(L)) = NU(H)$ we have that
\[
\alpha(Mm) = \alpha(MU(L)m) = \alpha(MU(L))\alpha(m) = NU(H)n = Nn.
\]
Thus $\alpha$ induces an isomorphism $\overline{\alpha}$ between $\tilde{I}_1$ and $\tilde{I}_2$.

Let $J_1/MU(L)$ and $J_2/NU(H)$ be the kernels of the actions of $U(L)/MU(L)$ and $U(H)/NU(H)$ on $\tilde{I}_1$ and $\tilde{I}_2$, respectively, defined by equation (6). We will show that
\( \alpha(J_1) = J_2 \). By definition
\[
J_1 = \{ w \in U(L) \mid [w,a] \in Mm \text{ for all } a \in MU(L) \}.
\]
Suppose that \( w \in J_1 \). Then \([w,a] \in Mm \text{ for all } a \in MU(L)\). As \( \alpha(Mm) = Nn \), we have \([\alpha(w),\alpha(a)] = \alpha([w,a]) \in Nn \) for all \( a \in MU(L) \). As \( a \) runs through all elements of \( MU(L) \), \( \alpha(a) \) runs through all elements of \( NU(H) \), and so we have that \([\alpha(w),a] \in Nn \) for all \( a \in NU(H) \). Therefore \( \alpha(w) \in J_2 \), and so \( \alpha(J_1) \subseteq J_2 \). The inclusion \( \alpha(J_2) \subseteq J_1 \) can be proved similarly. Therefore
\[
\alpha(J_1) = J_2.
\]
Setting \( \mathcal{J}_1 = J_1 \cap m \) and \( \mathcal{J}_2 = J_2 \cap n \), this implies that
\[
\alpha(\mathcal{J}_1) = \alpha(J_1 \cap m) = J_2 \cap n = \mathcal{J}_2,
\]
and so \( \alpha \) induces an isomorphism between \( \mathcal{A}_{M,m} = U(L)/\mathcal{J}_1 \) and \( \mathcal{A}_{N,n} = U(L)/\mathcal{J}_2 \). In addition, equation (8) also implies that the map \( \tilde{\alpha} : U(L)/J_1 \rightarrow U(H)/J_2 \) defined by \( \tilde{\alpha}(w + J_1) = \alpha(w) + J_2 \), \( w \in U(L) \), is an isomorphism.

Let us complete the proof of the claim concerning the isomorphism between \( \mathcal{L}_{M,m} \) and \( \mathcal{L}_{N,n} \). Define the map \( (\overline{\alpha}, \tilde{\alpha}) : \mathcal{L}_{M,m} \rightarrow \mathcal{L}_{N,n} \), by
\[
(\overline{\alpha}, \tilde{\alpha})(a + Mm, w + J_1) = (\overline{\alpha}(a + Mm), \tilde{\alpha}(w + J_2)).
\]
We will show that \( (\overline{\alpha}, \tilde{\alpha}) \) is an isomorphism. Since \( \overline{\alpha} \) and \( \tilde{\alpha} \) are isomorphisms, we are only required to show that
\[
\tilde{\alpha}([a + Mm, w + J_1]) = [\alpha(a) + Nn, \alpha(w) + J_2]
\]
for all \( a \in MU(L) \) and \( w \in U(L) \). Let us compute
\[
[\alpha(a) + Nn, \alpha(w) + J_2] = [\alpha(a), \alpha(w)] + Nn
= \alpha([a, w]) + Nn = \tilde{\alpha}([a + Mm, w + J_1]).
\]
Hence \( (\overline{\alpha}, \tilde{\alpha}) \) is an isomorphism, as claimed. \( \square \)

Using Lemma 2.2, Lemma 2.4 and Theorem 4.2, and taking \( M = L' \), \( N = H' \), \( m = \omega(L) \), and \( n = \omega(H) \), we obtain the following corollary.

**Corollary 4.3.** Let \( L \) and \( H \) be finite-dimensional Lie algebras over a field \( \mathbb{F} \) such that \( L' \) and \( H' \) are abelian. If \( U(L) \cong U(H) \), then \( \mathcal{L}_{L'} \cong \mathcal{L}_{H'} \) and \( \mathcal{A}_{L'} \cong \mathcal{A}_{H'} \).

The following results will be useful in the determination of \( \mathcal{L}_M \) and \( \mathcal{A}_M \) for some 4-dimensional solvable Lie algebras in Section 5.

**Corollary 4.4.** Let \( L \) be a Lie algebra and let \( M \) be an abelian ideal of \( L \) containing \( L' \). Suppose that \( L = M \times K \), where \( \dim L/C_L(M) \geq [(\dim M)^2/4] + 1 \). Then, the following hold.

1. \( \mathcal{L}_M \cong M \times L/C_L(M) \).
2. If \( C_L(M) = M \), then \( U(L) = J \oplus K \) (as vector spaces).
Proof. (1) By Proposition 4.1(5), it suffices to show that \( \dim L/C_L(M) = \dim \widetilde{K} \). Since \( L' \subseteq M \), the Lie algebras \( K = L/M \) and \( \widetilde{K} \) are abelian. By Proposition 4.1(4), \( J \cap L = C_L(M) \), and hence
\[
L/C_L(M) = L/(J \cap L) \cong (L + J)/J \leq U(L)/J \cong \widetilde{K}.
\]
Thus \( \dim \widetilde{K} \geq \dim L/C_L(M) \geq \lfloor (\dim M)^2/4 \rfloor + 1 \). On the other hand, \( \widetilde{K} \) is an abelian subalgebra of \( \mathfrak{gl}(\widetilde{I}) \) and the maximal dimension of an abelian subalgebra of \( \mathfrak{gl}(\widetilde{I}) \) is \( \lfloor (\dim M)^2/4 \rfloor + 1 \) (see [Jac44, Theorem 1]). Thus \( \widetilde{K} \leq \lfloor (\dim M)^2/4 \rfloor + 1 \), and so \( \dim \widetilde{K} = \dim L/C_L(M) \) must hold.

(2) Since \( C_L(M) = M \), we obtain \( J \cap L = M \) by Proposition 4.1(4). This implies that
\[
J \cap K = J \cap L \cap K = M \cap K = 0.
\]

On the other hand, \( \dim U(L)/J = \dim \widetilde{K} \) and we have seen in the proof of statement (1) that \( \dim K = \dim \widetilde{K} \). Thus \( J \) and \( K \) are linear subspaces of \( U(L) \) such that \( J \cap K = 0 \) and \( \dim U(L)/J = \dim K \). As \( K \) is finite-dimensional, the vector space direct decomposition \( U(L) = J \oplus K \) follows. \( \square \)

The next lemma can be used to explicitly determine \( \mathcal{A}_M \) for some Lie algebras.

**Lemma 4.5.** Let \( L \) be a finite-dimensional Lie algebra, suppose that \( M \) is an abelian ideal of \( L \) and let \( x \in L \) be such that \( L = C_L(M) \times \langle x \rangle \). We have that
\[
\mathcal{A}_M = \mathbb{F}[x]/(f(x))
\]
where \( f(x) \) is the smallest degree polynomial such that \( f((\text{ad } x)|_M) = 0 \) and \( f(0) = 0 \).

*Proof.* Let \( J \) and \( \mathfrak{J} \) be as in Proposition 4.1. By Proposition 4.1(4), \( C_L(M) \subseteq J \) and, clearly, \( C_L(M) \subseteq \omega(L) \). Thus \( C_L(M) U(L) \subseteq \mathfrak{J} \). Therefore
\[
\mathcal{A}_M = U(L)/\mathfrak{J} \cong (U(L)/(C_L(M) U(L)))/(\mathfrak{J}/C_L(M) U(L)).
\]
Furthermore, by Lemma 2.1(2),
\[
U(L)/(C_L(M) U(L)) \cong U(L/C_L(M))
\]
and, since \( L/C_L(M) \cong \langle x \rangle \), we have that \( U(L/C_L(M)) \cong \mathbb{F}[x] \). Therefore, the composition
\[
\mathbb{F}[x] \to U(L/C_L(M)) \to (U(L)/(C_L(M) U(L)))/(\mathfrak{J}/C_L(M) U(L)) \to \mathcal{A}_M
\]
is a surjective homomorphism from \( \mathbb{F}[x] \) to \( \mathcal{A}_M \) with kernel \( \mathbb{F}[x] \cap \mathfrak{J} \). In particular \( \mathcal{A}_M \) is a commutative algebra and \( \mathcal{A}_M = \mathbb{F}[x]/(\mathbb{F}[x] \cap \mathfrak{J}) \). As \( \mathbb{F}[x] \) is a principal ideal domain, the ideal \( \mathbb{F}[x] \cap \mathfrak{J} \) is generated by a smallest degree polynomial in \( \mathbb{F}[x] \cap \mathfrak{J} \). Given a polynomial \( f(x) \in \mathbb{F}[x] \) we have that \( f(x) \in \mathfrak{J} = J \cap \omega(L) \) if and only if \( f(x) \in J \) and \( f(0) = 0 \). Since \( f(0) = 0 \), \( f(x) = a_1 x + a_2 x^2 + \cdots + a_k x^k \), and so \( f(x) \in J \) if and only if
\[
[f(x), y] \in M \omega(L)
\]
for every \( y \in M \).
We claim that
\[(ad x^n)y \equiv (ad x)^n y \pmod{M \omega(L)} \text{ for all } y \in M \text{ and } n \geq 1.\]

In fact, the claim is clear for \(n = 1\). Assume that (9) is valid for \(n \geq 1\) and note that
\[[x^{n+1}, y] = x[x^n, y] + [x, y]x^n \equiv x(ad x^n)y \equiv x(ad x)^n y \equiv (ad x)^{n+1}y \pmod{M \omega(L)}.

Now claim (9) follows by induction. This also implies that
\[\{f(x), y\} \equiv f(ad x)y \pmod{M \omega(L)}.

Thus, since \(f(ad x)y \in L\) for every \(y \in M\), we have that \(f(x) \in J\) if and only if
\[f(ad x)y \in M \omega(L) \cap L = M' = 0;

see Lemma 2.1(3). That is \(f(x) \in J\) if and only if \(f(\text{ad } x|_M) = 0\).

**Corollary 4.6.** Let \(L\) be a finite-dimensional Lie algebra. Suppose that \(M\) is an abelian ideal of \(L\) and \(x \in L\) such that \(L = C_L(M) \rtimes \langle x \rangle\). Then \(L_M \cong M \rtimes L/C_L(M)\) if and only if \((\text{ad } x|_M)^2 \equiv \lambda (\text{ad } x)|_M\) for some \(\lambda \in \mathbb{F}\).

**Proof.** By Proposition 4.1(5), \(L_M \cong M \rtimes L/C_L(M)\) is equivalent to the condition \(\dim \tilde{K} = \dim L/C_L(M) = 1\), which (by Proposition 4.1(3)) amounts to \(\dim A_M = 2\). By Lemma 4.5 \(\dim A_M = 2\) if and only if the degree of the smallest degree polynomial such that \(f(\text{ad } x|_M) = 0\) and \(f(0) = 0\) is 2. This condition is equivalent to the condition that \(f(x) = x^2 - \lambda x\) with some \(\lambda \in \mathbb{F}\), which, in turn, amounts to \((\text{ad } x|_M)^2 = \lambda \text{ad } x|_M\).

5. \(L_M\) and \(A_M\) for some 4-dimensional solvable Lie algebras

In this section, we determine the Lie algebra \(L_M\) (defined in Proposition 4.1) and the associative algebra \(A_M\) (defined in (7)) for several of the isomorphism classes of solvable Lie algebras \(L\) of dimension 4. A classification of these Lie algebras was given by de Graaf [dG05], we reproduce this classification in Section 6. In each of the following calculations we determine \(L_M\) and \(A_M\) taking \(M = L'\).

5.1. \(L = M_0^3\). First we calculate \(L_M\) for the Lie algebra \(L = M_0^3\) defined, over an arbitrary field \(\mathbb{F}\), by the multiplication table
\[L = M_0^3 = \langle x_1, x_2, x_3, x_4 \mid [x_4, x_1] = x_1, [x_4, x_2] = x_3, [x_4, x_3] = x_3 \rangle.

Note that only the non-zero products are displayed in the presentation of \(L\); for example \([x_1, x_2] = 0\), but it is not explicitly stated. We have that \(M = \langle x_1, x_3 \rangle\) and \(C_L(M) = \langle x_1, x_2, x_3 \rangle\), thus \(L = C_L(M) \rtimes \langle x_4 \rangle\). Note that
\[(\text{ad } x_4)|_M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.

Hence, applying Lemma 4.5 and Corollary 4.6, we have that
\[L_M \cong \langle x_1, x_3 \rangle \rtimes \langle x_4 \rangle\quad \text{and} \quad A_M = \mathbb{F}[x_4]/(x_4^2 - x_4),\]
since \( f(x) = x^2 - x \) is the smallest degree polynomial such that \( f(\text{ad} x_4|_M) = 0 \) and \( f(0) = 0 \).

5.2. \( L = M^6_{0,b} \). Let \( \mathbb{F} \) be a field and let us now calculate \( \mathcal{L}_M \) and \( \mathcal{A}_M \) for the following family of isomorphism classes of Lie algebras

\[
L = M^6_{0,b} = \langle x_1, x_2, x_3, x_4 \mid [x_4, x_1] = x_2, [x_4, x_2] = x_3, [x_4, x_3] = bx_2 + x_3 \rangle,
\]

where \( b \in \mathbb{F} \). In this family, \( M^6_{0,b} \cong M^6_{0,c} \) if and only if \( b = c \); see Group 5 in Section 6. We have that \( M = \langle x_2, x_3 \rangle \) and \( C_L(M) = \langle x_1, x_2, x_3 \rangle \), thus \( L = C_L(M) \rtimes \langle x_4 \rangle \). Note that

\[
\text{ad} x_4|_M = \begin{pmatrix} 0 & b \\ 1 & 1 \end{pmatrix}.
\]

First suppose that \( b = 0 \). Then we have that \( (\text{ad} x_4)|_M^2 = \text{ad} x_4|_M \). Hence, applying Lemma 4.5 and Corollary 4.6, we obtain

\[
\mathcal{A}_M = \mathbb{F}[x_4]/(x_4^2 - x_4) \quad \text{and} \quad \mathcal{L}_M \cong \langle x_2, x_3 \rangle \rtimes \langle x_4 \rangle.
\]

Now we suppose that \( b \neq 0 \). The smallest degree polynomial \( f(x) \) such that \( f((\text{ad} x_4)|_M) = 0 \) and \( f(0) = 0 \) is \( f(x) = x^3 - x^2 - bx \). Thus by Lemma 4.5,

\[
\mathcal{A}_M = \mathbb{F}[x_4]/(x_4^2 - x_4 - bx_4).
\]

Choosing the basis \( \{1, x_4, x_4^2 \} \) of \( \mathcal{A}_M \), we have that a basis for \( \tilde{K} \) is \( \{x_4, x_4^2 - x_4\} \).

Now, as \( \{x_4, x_3\} \) is a basis of \( \tilde{I} \) and the action of \( \tilde{K} \) on \( \tilde{I} \) is induced by the map \( \beta \) in (6), we can calculate the brackets in \( \mathcal{L}_M = \tilde{I} \rtimes \tilde{K} \). First we note that

\[
[x_4^2, x_2] = [x_4, x_2]x_4 + x_4[x_4, x_2] = x_3x_4 + x_4x_3 \equiv x_3x_4 + [x_4, x_3] \equiv bx_2 + x_3 \pmod{M\omega(L)}.
\]

and that

\[
[x_4^2, x_3] = [x_4, x_3]x_4 + x_4[x_4, x_3] = (bx_2 + x_3)x_4 + x_4(bx_2 + x_3) = bx_4x_2 + x_4x_3 \equiv bx_3 + bx_2 + x_3 = (b+1)x_3 + bx_2 \pmod{M\omega(L)}.
\]

Thus \( [x_4^2, x_2] \equiv bx_2 \pmod{M\omega(L)} \) and \( [x_4^2, x_3] \equiv bx_3 \pmod{M\omega(L)} \). Therefore, \( \mathcal{L}_M \) is the 4-dimensional Lie algebra spanned by \( \{x_2, x_3, x_4, x_4^2 - x_4\} \) and non-zero brackets

\[
[x_4, x_2] = x_3, \quad [x_4, x_3] = bx_2 + x_3, \quad [x_4^2 - x_4, x_2] = bx_2, \quad [x_4^2 - x_4, x_3] = bx_3.
\]

5.3. \( L = M^7_{0,b} \). Let \( \mathbb{F} \) be a field and let us now calculate \( \mathcal{L}_M \) and \( \mathcal{A}_M \) for the following family of isomorphism classes of Lie algebras

\[
L = M^7_{0,b} = \langle x_1, x_2, x_3, x_4 \mid [x_4, x_1] = x_2, [x_4, x_2] = x_3, [x_4, x_3] = bx_2 \rangle,
\]

where \( b \in \mathbb{F} \). In this family \( M^7_{0,b} \cong M^7_{0,c} \) if and only if there exists \( \lambda \in \mathbb{F}^{\times} \) such that \( b = \lambda^2 c \), see Section 6.
We have that $M = \langle x_2, x_3 \rangle$ and $C_L(M) = \langle x_1, x_2, x_3 \rangle$, and thus $L = C_L(M) \rtimes \langle x_4 \rangle$. Note that

$$(\text{ad} x_4)|_M = \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}.$$  

Suppose that $b = 0$; then $(\text{ad} x_4|_M)^2 = 0$. Applying Corollary 4.6 and Lemma 4.5, we have that

$$L_M \cong \langle x_2, x_3 \rangle \rtimes \langle x_4 \rangle \quad \text{and} \quad A_M = \mathbb{F}[x_4]/(x_4^2).$$

Now suppose that $b \neq 0$. Then the smallest degree monic polynomial $f(x)$ such that $f((\text{ad} x_4)|_M) = 0$ and $f(0) = 0$ is $f(x) = x^3 - bx$. Thus, by Lemma 4.5,

$$(10) \quad A_M = \mathbb{F}[x_4]/(x_4^3 - bx).$$

Choosing the basis $\{1, x_4, x_4^2\}$ of $A_M$, we have that $\tilde{K} = \{\overline{x_4}, \overline{x_4^2}\}$. Since $\{\overline{x_2}, \overline{x_3}\}$ is a basis of $\overline{I}$ and the action of $\tilde{K}$ over $\overline{I}$ is induced by the map $\beta$ in (6), we can calculate the brackets in $L_M = I \rtimes_{\beta} \tilde{K}$. For this, note that

$$[x_2^2, x_2] = [x_1, x_2]x_4 + x_4[x_1, x_2] = x_3x_4 + x_4x_3 \equiv x_3x_4 + [x_4, x_3] \equiv bx_2 \pmod{M\omega(L)},$$

and

$$[x_2^2, x_3] = [x_1, x_2]x_4 + x_4[x_1, x_3] = bx_2x_4 + bx_4x_2 \equiv bx_2x_4 + b[x_4, x_2] \equiv bx_3 \pmod{M\omega(L)}.$$  

Therefore $L_M$ is a 4-dimensional Lie algebra spanned by $\{\overline{x_2}, \overline{x_3}, \overline{x_4}, \overline{x_4^2}\}$ and non-zero brackets

$$[\overline{x_2}, \overline{x_2}] = \overline{x_3}, \quad [\overline{x_4}, \overline{x_3}] = \overline{bx_2}, \quad [\overline{x_4}, \overline{x_2}] = \overline{bx_4}, \quad [\overline{x_4^2}, \overline{x_3}] = \overline{bx_3}.$$  

It is interesting to notice that in the family $M_{0,b}^f$, the isomorphism type of the original Lie algebra $L$ is not determined by the Lie algebra $L_M$; it is, however, determined by the commutative algebra $A_M$ in (10) as implied by the following lemma.

**Lemma 5.1.** Let $\mathbb{F}$ be a field and if char $\mathbb{F} = 2$ then also assume that $\mathbb{F}$ is perfect. Let $b, c \in \mathbb{F}^\times$. Then $\mathbb{F}[x]/(x^3 - bx) \cong \mathbb{F}[x]/(x^3 - cx)$ if and only if there exists $\lambda \in \mathbb{F}^\times$ such that $c = \lambda^2 b$.

**Proof.** First, if $\lambda \in \mathbb{F}^\times$ such that $c = \lambda^2 b$, then the automorphism of $\mathbb{F}[x]$ induced by $x \mapsto \lambda^{-1}x$ takes $x^3 - bx$ to $\lambda^{-3} x^3 - \lambda^{-1} bx$ and hence the image of the ideal $(x^3 - bx)$ is $(x^3 - \lambda^2 bx) = (x^3 - cx)$. Therefore the quotients $\mathbb{F}[x]/(x^3 - bx)$ and $\mathbb{F}[x]/(x^3 - cx)$ are isomorphic. This argument implies the claim of the lemma for perfect fields of characteristic two.

Assume now char $\mathbb{F} \neq 2$ and also that $\mathbb{F}[x]/(x^3 - bx)$ and $\mathbb{F}[x]/(x^3 - cx)$ are isomorphic. Suppose first that $b \in \mathbb{F}^2$; let $b = \beta^2$ for some $\beta \in \mathbb{F}^\times$. Then $x^3 - bx = x(x - \beta)(x + \beta)$ and so, by the Chinese Remainder Theorem,

$$\mathbb{F}[x]/(x^3 - bx) \cong \mathbb{F}[x]/(x) \oplus \mathbb{F}[x]/(x - \beta) \oplus \mathbb{F}[x]/(x + \beta) \cong \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}.$$  

Since $\mathbb{F}[x]/(x^3 - bx) \cong \mathbb{F}[x]/(x^3 - cx)$, the decomposition $\mathbb{F}[x]/(x^3 - cx) \cong \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}$ must hold which forces that $x^2 - c$ is reducible, which implies that $c \in \mathbb{F}^2$. This argument shows
that if one of $c$ or $b$ is an element of $\mathbb{F}^2$, then both are, and in this case the conclusion of the lemma holds with $\lambda = 1$.

Thus, only remains to analyze the case $b, c \notin \mathbb{F}^2$. Then

$$\mathbb{F}[x]/(x^3 - bx) = \mathbb{F}[x]/(x) \oplus \mathbb{F}[x]/(x^2 - b)$$

and

$$\mathbb{F}[x]/(x^3 - cx) = \mathbb{F}[x]/(x) \oplus \mathbb{F}[x]/(x^2 - c)$$

are the decompositions of the respective algebras into direct sums of simple algebras. Since $\mathbb{F}[x]/(x^3 - bx) \cong \mathbb{F}[x]/(x^3 - cx)$, it follows that

$$\mathbb{F}[x]/(x^2 - b) \cong \mathbb{F}[x]/(x^2 - c).$$

Suppose that $\delta + \lambda \overline{\alpha} \in \mathbb{F}[x]/(x^2 - c)$ is the image of $\overline{\alpha} \in \mathbb{F}[x]/(x^2 - b)$ under an isomorphism $\alpha : \mathbb{F}[x]/(x^2 - b) \rightarrow \mathbb{F}[x]/(x^2 - c)$, where $\delta, \lambda \in \mathbb{F}$ and necessarily $\lambda \neq 0$. Then

$$0 = \alpha(x^2 - b) = (\delta + \lambda \overline{\alpha})^2 - b = \delta^2 + 2\delta \lambda \overline{\alpha} + \lambda^2 c - b.$$ 

This implies that $\delta = 0$ and that $\lambda^2 c = b$ as required. \hfill \qedsymbol

6. The Proof of Theorem 1.3

For the proof of Theorem 1.3, we use the classification of the 4-dimensional solvable Lie algebras given by de Graaf [dG05]. Since $\dim L$ is determined by the isomorphism type of $U(L)$ (Lemma 2.3(1)), and the isomorphism problem for enveloping algebras of solvable Lie algebras of dimension less than or equal to three was solved in [CKL04] over an arbitrary field, we concentrate on the case of solvable Lie algebras of dimension 4. The first step is to divide the algebras into six groups in such a way that two algebras in different groups cannot have isomorphic enveloping algebras. The second step will be to verify that two non-isomorphic Lie algebras inside the same group cannot have isomorphic enveloping algebras.

First we state the classification of 3-dimensional solvable Lie algebras. Recall that only the non-zero brackets are displayed in the multiplication tables of Lie algebras.

**Theorem 6.1.** Let $L$ be a 3-dimensional solvable Lie algebra over a field $\mathbb{F}$. Then $L$ is isomorphic to one of the following algebras:

1. $L^1 = \langle x_1, x_2, x_3 \rangle$ (the abelian Lie algebra);
2. $L^2 = \langle x_1, x_2, x_3 \mid [x_3, x_1] = x_1, [x_3, x_2] = x_2 \rangle$;
3. $L^3_a = \langle x_1, x_2, x_3 \mid [x_3, x_1] = x_2, [x_3, x_2] = ax_1 + x_2 \rangle$ with some $a \in \mathbb{F}$;
4. $L^4_a = \langle x_1, x_2, x_3 \mid [x_3, x_1] = x_2, [x_3, x_2] = ax_1 \rangle$ with some $a \in \mathbb{F}$.

If $L_i^j \cong L_n^u$, then $i = u$. Moreover $L^3_a \cong L^3_b$ if and only if $a = b$ and $L^4_a \cong L^4_b$ if and only if there is an $\alpha \in \mathbb{F}^*$ with $a = \alpha^2 b$.

In dimension four, we have 14 families of isomorphism classes of solvable Lie algebras. We group these Lie algebras into six groups in such a way that two Lie algebras in distinct groups cannot have isomorphic enveloping algebras. We follow the notation introduced
by de Graaf [dG05]. Several of de Graaf’s families, such as \( M^3_a, M^{13}_a, M^6_{a,b} \) and \( M^7_{a,b} \), are split between two groups. Although Theorem 1.3 is only stated in characteristic zero, we present the isomorphism types of 4-dimensional solvable Lie algebras over an arbitrary field and we consider the isomorphism problem of enveloping algebras of certain classes of such Lie algebras also over a field of prime characteristic.

The 4-dimensional solvable Lie algebras are presented in several tables, each table corresponding to one group. Each table contains five columns and the information contained in the columns is as follows.

Name: The name of the algebra in the notation of de Graaf [dG05].
relations: The non-trivial Lie brackets in the multiplication table of the algebra with respect to a basis \( \{x_1, x_2, x_3, x_4\} \).
char: Restrictions on the characteristic of the field over which the algebra is defined. If this entry is blank, then the algebra is defined over an arbitrary field.
parameter: Restrictions on the parameters \( a, b \) that may appear in the multiplication table of the algebra.
isomorphism: The conditions under which isomorphism occurs between two algebras in the same family. If the family is described by a single parameter \( a \), then the condition given in this entry is equivalent to the isomorphism \( M^i_a \cong M^i_b \). If the family is described by two parameters \( a \) and \( b \), then the condition given in this entry is equivalent to the isomorphism \( M^i_{a,b} \cong M^i_{c,d} \).

By the main result of [dG05], there is no isomorphism between two Lie algebras in different families, while two Lie algebras belonging to the same family are isomorphic if and only if the condition in the “isomorphism” column of the corresponding line of the table holds.

**Group 1: The abelian Lie algebra.** The first group contains only the abelian Lie algebra.

| Name  | relations | char | parameter | isomorphism |
|-------|-----------|------|-----------|-------------|
| \( M^4 \) |           |      |           |             |

**Group 2: The non-metabelian Lie algebras.** A 4-dimensional non-metabelian solvable Lie algebra over a field \( \mathbb{F} \) is isomorphic to one of the following Lie algebras.
Name | relations | char | parameter | isomorphism
--- | --- | --- | --- | ---
$M^{12}$ | $[x_4, x_1] = x_1, [x_4, x_2] = 2x_2, [x_4, x_3] = x_3, [x_3, x_1] = x_2$ |  |  |  

$M^{13}_a$ | $[x_4, x_1] = x_1 + ax_3, [x_4, x_2] = x_2, [x_4, x_3] = x_1, [x_3, x_1] = x_2$ | $a \in \mathbb{F}^*$ |  | $a = b$ 

$M^{14}_a$ | $[x_4, x_1] = ax_3, [x_4, x_3] = x_1, [x_3, x_1] = x_2$ | $a \in \mathbb{F}^*$ |  | $a/b \in \mathbb{F}^2$ 

$M^{11}_{a,b}$ | $[x_4, x_1] = x_1, [x_4, x_2] = bx_2, [x_4, x_3] = (1 + b)x_3, [x_3, x_1] = x_2, [x_3, x_2] = ax_1$ | 2 | $a \in \mathbb{F} \setminus \{0\}, b \in \mathbb{F} \setminus \{1\}$ | $a/c \in \mathbb{F}^2, (\delta^2 + (b + 1)\delta + b)/c \in \mathbb{F}^2$ where $\delta = (b + 1)/(d + 1)$ 

**Group 3:** Metabelian Lie algebras with one-dimensional derived subalgebra.
Let $L$ be a 4-dimensional metabelian Lie algebra such that $\dim L' = 1$. Then $L$ is isomorphic to one of the following algebras.

Name | relations | char | parameter | isomorphism
--- | --- | --- | --- | ---
$M^4$ | $[x_4, x_2] = x_3, [x_4, x_3] = x_3$ |  |  |  

$M^5$ | $[x_4, x_2] = x_3$ |  |  |  

**Group 4:** Metabelian Lie algebras with three-dimensional derived subalgebra.
Suppose that $L$ is a 4-dimensional metabelian Lie algebra with $\dim L' = 3$. Then $L$ is isomorphic to one of the following Lie algebras.

Name | relations | char | parameter | isomorphism
--- | --- | --- | --- | ---
$M^2$ | $[x_4, x_1] = x_1, [x_4, x_2] = x_2, [x_4, x_3] = x_3$ |  |  |  

$M^{3}_a$ | $[x_4, x_1] = x_1, [x_4, x_2] = x_3, [x_4, x_3] = -ax_2 + (a + 1)x_3$ | $a \in \mathbb{F}^*$ |  | $a = b$ 

$M^{6}_{a,b}$ | $[x_4, x_1] = x_2, [x_4, x_2] = x_3, [x_4, x_3] = ax_1 + bx_2 + x_3$ | $a \in \mathbb{F}^*, b \in \mathbb{F}$ |  | $a = c, b = d$ 

$M^{7}_{a,b}$ | $[x_4, x_1] = x_2, [x_4, x_2] = x_3, [x_4, x_3] = ax_1 + bx_2$ | $a \in \mathbb{F}^*, b \in \mathbb{F}$ |  | $a = \alpha^2c, b = \alpha^2d$ with $\alpha \in \mathbb{F}^*$ 

**Group 5:** Metabelian Lie algebras with two-dimensional derived subalgebra and non-trivial center.
Let $L$ be a 4-dimensional metabelian Lie algebra with
dim $L' = 2$ and dim $Z(L) \neq 0$. Then $L$ is isomorphic to one of the following algebras.

| Name   | relations | char parameter | isomorphism     |
|--------|-----------|----------------|-----------------|
| $M_0^{a,b}$ | $[x_4, x_1] = x_1, [x_4, x_2] = x_2, [x_4, x_3] = x_3, [x_4, x_3] = x_3$ | $b \in \mathbb{F}$ | $b = c$         |
| $M_{0,b}^b$ | $[x_4, x_1] = x_2, [x_4, x_2] = x_3, [x_4, x_3] = bx_2 + x_3$ | $b \in \mathbb{F}$ | $b = \alpha^2 c$ with $\alpha \in \mathbb{F}^*$ |

**Group 6: Metabelian Lie algebras with two-dimensional derived subalgebra and trivial center.** Let $L$ be a Lie algebra over a field $\mathbb{F}$ such that dim $L' = 2$ and $Z(L) = 0$. Then $L$ is isomorphic to one of the following Lie algebras.

| Name   | relations | char parameter | isomorphism                  |
|--------|-----------|----------------|------------------------------|
| $M^3$ | $x_1, x_2 = x_2, [x_3, x_4] = x_4$ |               |                              |
| $M_0^a$ | $[x_4, x_1] = x_1 + ax_2, [x_4, x_2] = x_1, [x_3, x_1] = x_1, [x_3, x_2] = x_2$ | $T^2 - T - a \in \mathbb{F}[T]$ is irreducible | if char $\mathbb{F} \neq 2$
|       |           |                | $a + 1/4 = \alpha^2 (b + 1/4)$ with $\alpha \in \mathbb{F}^*$ if char $\mathbb{F} = 2$
| $M_{0}^{13}$ | $[x_4, x_1] = x_1, [x_4, x_2] = x_2, [x_4, x_3] = x_1, [x_3, x_1] = x_2$ | $\neq 2$ | $T^2 + T + a + b \in \mathbb{F}[T]$ is reducible |
| $M_{a}^{10}$ | $[x_4, x_1] = x_2, [x_4, x_2] = ax_1, [x_3, x_1] = x_1, [x_3, x_2] = x_2$ | 2 | $a \notin \mathbb{F}^2$
|       |           |                | or $a = 0$ $Y^2 + bx^2 + a$
|       |           |                | has solution $(X, Y) \in \mathbb{F} \times \mathbb{F}$
|       |           |                | with $X \neq 0$ |

In [dG05, Section 5.5], it is shown that if char $\mathbb{F} = 2$ and $a \in \mathbb{F}^2 \setminus \{0\}$ then $M_{a}^{10} \cong M_{0}^{13}$. For this reason, the Lie algebras $M_{a}^{10}$ are considered only for $a \notin \mathbb{F}^2$ or for $a = 0$. Similarly, we will consider the Lie algebra $M_{0}^{13}$ only in characteristic different from two and in the family $M_{a}^{10}$ we allow the parameter $a = 0$.

**The proof of Theorem 1.3.** We prove Theorem 1.3 by verifying a sequence of claims. Before the first claim, recall that the Frobenius semiradical $F(L)$ of a Lie algebra $L$ was defined in Section 2.

**Claim 1.** Suppose that $L$ is a Lie algebra in Group 6 over a field $\mathbb{F}$ of characteristic zero. Then $F(L) = 0$ and $Z(U(L)) = \mathbb{F}$.

**The proof of Claim 1.** Suppose that $L$ is a Lie algebra in Group 6 given by the basis $\{x_1, x_2, x_3, x_4\}$ and let $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ be the dual basis of $L^*$. Consider the following
linear forms on $L$:

\[
\begin{align*}
L &= M^8 : \quad f = \phi_2 + \phi_4; \\
L &= M^9_a : \quad f = \phi_1 + \phi_2; \\
L &= M_{13}^a : \quad f = \phi_2; \\
L &= M_{10}^a : \quad f = \phi_2. \\
\end{align*}
\]

Then, in each of the cases, the alternating bilinear form

\[ B_f : L \times L \to \mathbb{F}, \quad (x, y) \mapsto f([x, y]) \]

is non-degenerate. This implies that the index $i(L)$, introduced in Section 2, is equal to zero, and hence $F(L) = 0$. By Lemma 2.5, this shows that $Z(U(L)) = \mathbb{F}$. □

Claim 2. Suppose that $L$ and $H$ are 4-dimensional solvable Lie algebras over a field of characteristic zero such that $L$ and $H$ belong to distinct groups in the group division given above. Then $U(L) \not\cong U(H)$.

The proof of Claim 2. Lemma 2.3 implies that the isomorphism type of $U(L)$ determines whether $L$ is abelian, whether $L$ is metabelian, and, if $L$ is metabelian, it determines $\dim L'$. Hence we may assume without loss of generality that $L$ is in Group 5 and $H$ is in Group 6. In this case, $Z(L) \neq 0$, and so $Z(U(L)) \neq \mathbb{F}$. On the other hand, Claim 1 implies that $Z(U(H)) = \mathbb{F}$. Therefore $U(L) \not\cong U(H)$. □

Claim 3. If $L$ and $H$ are two Lie algebras in Group 2 over a field $\mathbb{F}$ of characteristic different from 2 such that $U(L) \cong U(H)$, then $L \cong H$.

The proof of Claim 3. First note that the Lie algebras $M_{11}^{a,b}$ only exist in characteristic two and hence we can disregard them in this argument. Suppose that $L$ and $H$ are two Lie algebras in Group 2 such that $U(L) \cong U(H)$ and assume that $\alpha : U(L) \to U(H)$ is an isomorphism. Note that $L'$ and $H'$ are ideals of codimension one in $L$ and in $H$, respectively. Therefore, there are elements $x \in L$ and $y \in H$ such that $L = L' \rtimes \langle x \rangle$ and $H = H' \rtimes \langle y \rangle$. Thus Corollary 3.1(2) implies that $L/L'' \cong H/H''$. We have that

\[
\begin{align*}
M^{12}/(M^{12})'' &\cong L^2; \\
M^{13}_a/(M^{13}_a)'' &\cong L^3_a; \\
M^{14}_a/(M^{14}_a)'' &\cong L^4_a;
\end{align*}
\]

where $L^2, L^3_a$ and $L^4_a$ are the Lie algebras that appear in Theorem 6.1. Hence, by Theorem 6.1, if $L \not\cong H$, then $L/L'' \not\cong H/H''$. Thus it follows that $L \cong H$. □

The following claim addresses Group 3.

Claim 4. $U(M^4) \not\cong U(M^5)$ holds over an arbitrary field $\mathbb{F}$. 

The proof of Claim 4. This follows at once from Lemma 2.3(2), as $M^4$ is non-nilpotent, but $M^5$ is nilpotent.

Claim 5. If $L$ and $H$ are Lie algebras in Group 4 such that $U(L) \cong U(H)$, then $L \cong H$ holds over an arbitrary field $\mathbb{F}$.

The Proof of Claim 5. We have that $L = L' \rtimes \langle x_4 \rangle$ and $H = H' \rtimes \langle x_4 \rangle$ and that $L'$ and $H'$ are abelian. Hence if $U(L) \cong U(H)$, then Corollary 3.1 implies that $L \cong H$.

Claim 6. If $\text{char} \mathbb{F} = 0$ and $L$ and $H$ are Lie algebras of Group 5 such that $U(L) \cong U(H)$, then $L \cong H$.

The proof of Claim 6. For a Lie algebra $L$ in Group 5, let $\mathcal{L}_L'$ denote the Lie algebra constructed in Proposition 4.1 with respect to the abelian ideal $M = L' \leq L$ and the maximal ideal $m = \omega(L) \leq U(L)$. If $L$ and $H$ are two Lie algebras in Group 5 and $U(L) \cong U(H)$, then by Corollary 4.3 $\mathcal{L}_L' \cong \mathcal{L}_H'$.

In Section 5, we determined that $\mathcal{L}_L'$ is 3-dimensional if $L = M_3$ while it is 4-dimensional, if $L = M_0^6$ or $L = M_0^7$ with $b \neq 0$. Moreover, the algebras $\mathcal{L}_L'$ constructed for $M_0^3$, $M_0^6$, and $M_0^7$ are pairwise non-isomorphic. Thus, the possible non-isomorphic algebras with isomorphic enveloping algebras must be isomorphic to $M_0^6$ or $M_0^7$ with some $b, c \in \mathbb{F}$. Let us hence assume that $L$ and $H$ are Lie algebras that are isomorphic to either $M_0^6$ or $M_0^7$ with some $b, c \in \mathbb{F}$ such that $U(L) \cong U(H)$. Note that $L = M \rtimes \langle x_4 \rangle$ and $H = N \rtimes \langle x_4 \rangle$, where $M = L' + Z(L)$ and $N = H' + Z(H)$. Now Theorem 1.2 implies that $L \cong H$.

Claim 7. If $L$ and $H$ are Lie algebras in Group 6 over an arbitrary field, such that $U(L) \cong U(H)$, then $L \cong H$.

The proof of Claim 7. Suppose that $L$ and $H$ are as in the claim, and let $\alpha : U(L) \to U(H)$ be an isomorphism. We may assume by Lemma 2.2 that $\alpha(\omega(L)) = \omega(H)$. Let us compute $\mathcal{L}_L'$ and $\mathcal{L}_H'$ as in Proposition 4.1 using the abelian ideals $L'$ and $H'$, respectively. Note that both $L$ and $H$ can be written as $\langle x_1, x_2 \rangle \rtimes \langle x_3, x_4 \rangle$ where $\langle x_1, x_2 \rangle$ is the derived subalgebra. Further, the second component of this semidirect sum decomposition acts faithfully on the first component, and hence Corollary 4.4(1) shows that $\mathcal{L}_L' \cong L$ and $\mathcal{L}_H' \cong H$. On the other hand, Theorem 4.2 implies that $\mathcal{L}_L' \cong \mathcal{L}_H'$. Thus $L \cong H$, as claimed.

Our arguments in the proof of Theorem 1.3 work over arbitrary characteristic, except for some parts of the proofs of Claims 2, 3, and 6. In Claim 2, the argument in prime characteristic fails if $L$ is in Group 5 and $H$ is in Group 6. Hence in order to extend Theorem 1.3 to fields of arbitrary characteristic, one would only need to prove the following statements.

Conjecture 6.2. (1) If $L$ and $H$ are Lie algebras over a field of positive characteristic such that $L$ belongs to Group 5, $H$ belongs to Group 6 and $U(L) \cong U(H)$, then $L \cong H$.
(2) If $L$ and $H$ are Lie algebras over a field of characteristic two that belong to Group 2 such that $U(L) \cong U(H)$, then $L \cong H$.

(3) If $L$ and $H$ are Lie algebras over a field of positive characteristic that belong to Group 5 such that $U(L) \cong U(H)$, then $L \cong H$.

The calculations in Section 5.3 and Lemma 5.1 imply that assertion (3) of Conjecture 6.2 is true if both $L$ and $H$ are of the form $M_{0,b}^7$ for some $b \in F$ and $F$ is either of characteristic different from two or perfect.

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