Degenerations of Pascal lines

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Abstract
Let $K$ denote a nonsingular conic in the projective plane. Pascal’s theorem says that, given six distinct points $A$, $B$, $C$, $D$, $E$, $F$ on $K$, the three intersection points $AE \cap BF$, $AD \cap CF$, $BD \cap CE$ are collinear. The line containing them is called the Pascal line of the sextuple. However, this construction may fail when some of the six points come together. In this paper, we find the indeterminacy locus where the Pascal line is not well-defined and then use blow-ups along polydiagonals to define it. We analyse the geometry of Pascals in these degenerate cases. Finally we offer some remarks about the indeterminacy of other geometric elements in Pascal’s \textit{hexagrammum mysticum}.

Keywords Pascal’s theorem · Pascal lines · Hexagrammum Mysticum

Mathematics Subject Classification 14N05 · 51N35

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1 Introduction

Pascal’s theorem is one of the most elegant results in classical projective geometry. Given a collection of six distinct points on a conic, it allows us to define a highly symmetrical configuration called the *hexagrammum mysticum*. We begin with an elementary introduction to this subject; the main results will be described in Sect. 1.4 after the required notation is available.

1.1. Let $k$ denote an algebraically closed field of characteristic $\neq 2$, and let $\mathbb{P}^2$ be the projective plane over $k$. Fix a nonsingular conic $K$ in $\mathbb{P}^2$. Given six distinct points $A, B, C, D, E, F$ on $K$, one can arrange them into an array $\begin{bmatrix} A & B & C \\ F & E & D \end{bmatrix}$. Then Pascal’s theorem says that the three cross-hair intersection points

$$AE \cap BF, \quad AD \cap CF, \quad BD \cap CE,$$

(corresponding to the $2 \times 2$ minors of the array) are collinear (see Diagram 1).

The line containing them is called the Pascal line (or just the Pascal) of the array; we will denote it by $\{ABC\}_{\mathbf{FED}}$.

The Pascal remains unchanged if we shuffle the rows or columns of the array; thus we have 12 different ways $\{ABC\}_{\mathbf{FED}} = \{FED\}_{\mathbf{ABC}} = \{FDE\}_{\mathbf{ACB}}$ etc.

of denoting the same Pascal. Any essentially different arrangement of the same points, such as $\{DAB\}_{\mathbf{FCE}}$, will generally correspond to a different Pascal. Thus there are
6!/12 = 60 notionally different Pascals. It is a theorem due to Pedoe (1941) that these sixty lines are pairwise distinct if the initial six points A, . . . , F are chosen in general position.

1.2. If exactly two of the points amongst A, . . . , F coincide, then all the Pascals remain well-defined as long as we follow a natural convention. If, say \( P = Q \), then we should interpret \( PQ \) as the tangent line to the conic at \( P \). (Henceforth, we denote this tangent by \( \mathbb{T}_P \).) For instance, in the case of the Pascal \( \{A\ B\ C\ F\ E\ D\} \), up to relabelling there are three possibilities for two of the points to coincide; namely \( A = B \), \( A = F \) or \( A = E \). In the first case, the Pascal is simply the line \( AD \), and in the second case it is the line joining \( A \) to \( BD \cap CE \). The third case is shown in Diagram 2.

1.3. However, things start breaking down if three points coincide, say \( A = B = C \). Then it is no longer obvious how to define the Pascal \( \{A\ A\ A\ F\ E\ D\} \), since all the three cross-hair intersections are at \( A \). Similarly, if two pairs of points coincide, say \( A = B \) and \( E = F \), then it is not obvious how to define \( \{A\ A\ C\ E\ E\ D\} \).

1.4 Results

In the next section, we will introduce a projective variety \( \mathcal{H} \) which acts as a parameter space for labelled sextuples of points on the conic. Given a formal arrangement of letters such as \( s = \{A\ B\ C\ F\ E\ D\} \), the definition of the Pascal line will correspond to a rational map

\[
p_s : \mathcal{H} \rightarrow (\mathbb{P}^2)^* \]
from the space of sextuples to the space of lines in the projective plane.

- In Proposition 2.1 below, we will characterise the indeterminacy scheme of this map. It will turn out to be a union of polydiagonals in $\mathcal{H}$.
- We will remove a codimension three subvariety from $\mathcal{H}$, and blow-up the resulting space $\mathcal{H}^o$ along certain polydiagonals. The result is a diagram

\[
\begin{array}{c}
\mathbb{X} \\
\downarrow \beta \\
\mathcal{H}^o \\
\xrightarrow{p_*} (\mathbb{P}^2)^* \\
\end{array}
\]

where $\beta$ is a sequence blow-ups, and $q_*$ is a regular map. In other words, $q_*$ resolves the indeterminacy in the rational map $p_*$ (restricted to $\mathcal{H}^o$). This is the content of Theorem 3.3.
- It is of geometric interest to see how the Pascals behave on the exceptional loci of the blow-ups. This analysis is carried out in Sect. 4. There are three cases to consider, amongst which the partition $(2, 2, 2)$ is the most symmetric and leads to the most elegant results.
- The sixty Pascals are part of a larger geometric configuration called the hexagrammum mysticum. It is comprised of 35 more lines (apart from the Pascals) and 95 more points. The questions of definability can also be raised for these elements. Although we do not venture into this analysis deeply, we offer some remarks about it in Sect. 5.

1.5 References

The literature on Pascal’s theorem is enormous. The standard classical reference is by George Salmon (see Salmon 2005, Notes). For later development of this material, it is convenient to introduce the so-called ‘dual notation’ which makes crucial use of the outer automorphism of the symmetric group $S_6$ (see Howard et al. 2008). This notation, together with a host of results discovered by Cremona and Richmond are explained by H. F. Baker in his note ‘On the Hexagrammum Mysticum of Pascal’ in (Baker (1923), Note II, pp. 219–236). One of the best modern surveys of this material is by Conway and Ryba (2012). We refer the reader to Coxeter (1987), Kadison and Kromann (1996), Seidenberg (1962) for foundational notions in projective geometry, and to Eisenbud and Harris (2000), Harris (1992), Hartshorne (1992) for those in algebraic geometry. In particular, we will use the notion of a blow-up which is extensively discussed in (Eisenbud and Harris (2000), Ch. IV.2) and (Hartshorne (1992), Ch. II.7).

2 Partitions and polydiagonals

2.1. In this section we will introduce the necessary geometric set-up. Let $ltr$ denote the set of letters \{A, B, C, D, E, F\}, and let $\mathcal{P}$ denote the set of partitions of $ltr$. For
the sake of readability, we will write the partition
\[ \pi = \{ \{ A, C, D \}, \{ B \}, \{ E, F \} \} \in \mathfrak{P} \]
as \( ACD \cdot B \cdot EF \), and we will say that it is of type \((3, 2, 1)\). Define \( n(\pi) \) to be the cardinality of \( \pi \) (which is 3 in this case).

Now let \( \mathcal{H} = \mathcal{K}^{ltr} \) denote the set of maps \( ltr \to \mathcal{K} \). Of course, \( \mathcal{H} \) is a projective variety isomorphic to \((\mathbb{P}^1)^6\). Given \( h \in \mathcal{H} \), we will usually write
\[ h(A) = A, \quad h(B) = B, \quad \text{etc} \]
to denote the corresponding points on \( \mathcal{K} \).

Every \( h \in \mathcal{H} \) determines a partition \( \theta(h) \in \mathfrak{P} \) by the rule that \( x, y \in ltr \) belong to the same element of \( \theta(h) \), if and only if \( h(x) = h(y) \). In particular, \( h \) is injective exactly when \( \theta(h) = \{ \text{A, B, C, D, E, F} \} \).

2.2. For every partition \( \pi \in \mathfrak{P} \), we have a polydiagonal inside \( \mathcal{H} \), defined as follows:
\[ \Delta_\pi := \{ h \in \mathcal{H} : h(x) = h(y) \text{ if } x \text{ and } y \text{ belong to the same element of } \pi \}. \]
Note that the defining condition says ‘if’, and not ‘iff’. For instance, if \( \pi = \{ A, C, D \} \cdot B \cdot EF \), then \( \Delta_\pi \) is the set of maps \( ltr \to \mathcal{K} \) which satisfy
\[ h(A) = h(C) = h(D) \quad \text{and} \quad h(E) = h(F). \]
Thus \( \Delta_\pi \) is a nonsingular closed subvariety in \( \mathcal{H} \). It is isomorphic to \((\mathbb{P}^1)^{n(\pi)}\).

2.3. Partitions are partially ordered by refinement. We will write \( \pi_1 \leq \pi_2 \) if \( \pi_2 \) is a refinement of \( \pi_1 \), in which case \( \Delta_\pi_1 \subseteq \Delta_\pi_2 \). In particular, the smallest polydiagonal \( \Delta_\pi \) is the one for which \( \pi = \{ A, B, C, D, E, F \} \). At the other end, if \( \pi = \{ A, B, C, D, E, F \} \) is the trivial partition, then \( \Delta_\pi = \mathcal{H} \).

If \( h \in \Delta_\pi \), then \( \theta(h) \leq \pi \) and hence \( \Delta_{\theta(h)} \subseteq \Delta_\pi \). For a general point \( h \in \Delta_\pi \), we have \( \theta(h) = \pi \).

2.4. Let \([z_0, z_1, z_2]\) be the homogeneous coordinates in \( \mathbb{P}^2 \). Then, for instance, the coordinates of the line \( 3z_0 + 5z_1 + 7z_2 = 0 \) will be written as \( (3, 5, 7) \). Identify \( \mathcal{K} \) with the conic \( z_0z_2 = z_1^2 \), and fix an isomorphism \( \tau : \mathbb{P}^1 \to \mathcal{K} \) by the formula \( \tau(a) = [1, a, a^2] \) for \( a \in \mathbb{K} \), and \( \tau(\infty) = [0, 0, 1] \).

Choose indeterminates \( a, \ldots, f \), and write
\[ A = \tau(a), \ldots, F = \tau(f). \quad (2.1) \]
Define a Pascal symbol to be an array such as \( s = \{ E \ C \ A \} \), determined up to row and column shuffles. There are sixty such symbols.
2.5. For the moment, we fix the symbol $s = \left\{ \begin{array}{c} A \ B \ C \\ F \ E \ D \end{array} \right\}$. Given the points $A, \ldots, F$ as in (2.1), it is easy to calculate the coordinates of the lines $AE, BF$ etc, and eventually those of the Pascal $\left\{ \begin{array}{c} A \ B \ C \\ F \ E \ D \end{array} \right\}$. They turn out to be $\langle u_0, u_1, u_2 \rangle$, where

$$
\begin{align*}
    u_0 &= abde - abdf - acde + acef + bcdf - bcef, \\
    u_1 &= -abe + abf + acd - acf + adf - aef \\
          &\quad - bcd + bce - bde + bef + cde - cdf, \\
    u_2 &= -ad + ae + bd - bf - ce + cf.
\end{align*}
$$

(2.2)

(This computation was programmed by us in MAPLE.)

Given the polynomial ring $R = k[a, b, c, d, e, f]$, we have a rational map

$$
\begin{align*}
    \text{Spec } R &\rightarrow (\mathbb{P}^2)^*, \\
    (a, \ldots, f) &\rightarrow (u_0, u_1, u_2).
\end{align*}
$$

The indeterminacy scheme of this map is defined by the ideal $I_s = \langle u_0, u_1, u_2 \rangle \subseteq R$. Now it is straightforward to calculate the minimal primary decomposition of $I_s$; in fact all of its primary components turn out to be prime ideals. (This computation was done in MACAULAY2.) We have

$$
I_s = \bigcap_{i=1}^{6} p_i,
$$

(2.3)

where $p_1, \ldots, p_6$ are the following prime ideals:

$$
\begin{align*}
    p_1 &= (b - c, a - c), & p_2 &= (e - f, d - f), \\
    p_3 &= (e - f, a - b), & p_4 &= (d - e, b - c), & p_5 &= (d - f, a - c), \\
    p_6 &= (c - d, b - e, a - f).
\end{align*}
$$

The construction above globally corresponds to a rational map

$$
\begin{align*}
p_s : \mathcal{H} &\rightarrow (\mathbb{P}^2)^*.
\end{align*}
$$

Let $\Omega_s \subseteq \mathcal{H}$ denote its indeterminacy scheme. The preceding calculation proves the following:

**Proposition 2.1** The scheme $\Omega_s$ is equal to the union $\bigcup \Delta_\pi$, where $\pi$ ranges over the following six partitions:

$$
\begin{align*}
    \vartheta_1 &= A.B.C.D.E.F, & \vartheta_2 &= A.B.C.D.E.F, \\
    \vartheta_3 &= A.B.C.D.E.F, & \vartheta_4 &= A.B.C.D.E.F, & \vartheta_5 &= A.C.B.D.E.F, \\
    \vartheta_6 &= A.F.B.E.C.D.
\end{align*}
$$

(2.4)

In particular, this is a reduced scheme with six irreducible components.
These partitions encode a simple geometric pattern, which says that the Pascal
\[
\begin{bmatrix}
A & B & C \\
F & E & D \\
\end{bmatrix}
\]
can be undefined in the following three ways:

- all three points in either of the two rows become equal, or
- any two of the columns become equal, or
- the two rows become equal.

The first situation is captured by \( \vartheta_1 \) and \( \vartheta_2 \) which are of type \((3, 1, 1, 1)\), the second by \( \vartheta_3, \vartheta_4, \vartheta_5 \) which are of type \((2, 2, 1, 1)\), and the last by \( \vartheta_6 \) which is of type \((2, 2, 2)\).

For later use, observe that if \( i, j \) are distinct indices between 1 and 6, then \( \Delta_{\vartheta_i} \cap \Delta_{\vartheta_j} \) is a polydiagonal of type \((6), (3, 3), (3, 2, 1)\) or \((4, 2)\). For instance, \( \Delta_{\vartheta_1} \cap \Delta_{\vartheta_3} = \Delta_{ABC.DEF} \) is of type \((3, 2, 1)\).

2.6. Assume that \( h \in H \) belongs to exactly one of the polydiagonals from (2.4), say \( \Delta_{\vartheta_1} \). Then the corresponding maximal ideal \( m_h \) in \( R \) contains \( p_1 \), but not \( p_2, \ldots, p_6 \). For \( 2 \leq i \leq 6 \), choose an element \( g_i \in p_i \setminus m_h \), and let \( g = \prod g_i \). Localising at \( g \), we have an equality of ideals \( (I)_{g} = (p_1)_{g} \) in \( R_g \). Thus, inside the open set \( \operatorname{Spec} R_g \subseteq \operatorname{Spec} R \) containing \( h \), the indeterminacy scheme coincides with \( (p_1)_{g} \).

Of course, we have a similar rational map
\[ p_s : H \rightarrow (\mathbb{P}^2)^* , \]
for any Pascal symbol \( s \). Its indeterminacy scheme \( \Omega_s \) is obtained by appropriately permuting the letters \( A, \ldots, F \).

2.7. As an example, consider the symbols
\[ s_1 = \begin{bmatrix} A & E & D \\ F & B & C \end{bmatrix} \quad \text{and} \quad s_2 = \begin{bmatrix} A & F & C \\ D & E & B \end{bmatrix} . \]

Now let \( h \) be a general point of \( \Delta_{AF,BE,CD} \); that is to say, a map \( ltr_h \rightarrow K \) such that
\[ h(A) = h(F) = A, \quad h(B) = h(E) = B, \quad h(C) = h(D) = C, \]
where \( A, B, C \) are distinct points on the conic. Then \( s_1 \) leads to the undefined Pascal \( \begin{bmatrix} A & B & C \\ A & B & C \end{bmatrix} \) since the two rows become equal. However, \( s_2 \) becomes the Pascal \( \begin{bmatrix} A & A & C \\ C & B & B \end{bmatrix} \) which remains well-defined (and in fact equals the line \( AB \)). Thus \( \Delta_{AF,BE,CD} \) is contained in \( \Omega_{s_1} \), but not in \( \Omega_{s_2} \).

On the other hand, the smaller polydiagonal \( \Delta_{ACDF,BE} \) is contained in \( \Omega_{s_2} \), since its generic point leads to the undefined Pascal \( \begin{bmatrix} A & A & A \\ A & B & B \end{bmatrix} \) whose last two columns are equal.
2.8. It is a little unsatisfactory that the proof of Proposition 2.1 should rely upon a machine computation. In this section, we show how to prove a weaker version of the result using only elementary algebra. It will not be needed in the rest of the paper.

Define the expressions

\[ P_0 = bf - be + ce, \quad Q_0 = (e - f) (b - c) (ae + bd - bf - ce), \]
\[ P_1 = -(c + f), \quad Q_1 = (f - e) (b - c) (a - c + d - f), \]

and

\[ \delta = \det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ f & e & d \end{bmatrix}. \]

Then it is straightforward to check that

\[ u_0 = P_0 \delta + Q_0, \quad u_1 = P_1 \delta + Q_1, \quad u_2 = \delta. \]

Define the conditions

\[ C_1 : a = b = c, \quad C_2 : d = e = f, \]
\[ C_3 : a = b, e = f, \quad C_4 : b = c, d = e, \quad C_5 : a = c, d = f, \]
\[ C_6 : a = f, b = e, c = d. \]

These conditions are exactly parallel to the definitions of \( p_i \) and \( \vartheta_i \) in the previous section. The following is a set-theoretic (and hence weaker) version of Proposition 2.1.

**Proposition 2.2** Given elements \( a, b, c, d, e, f \) in \( k \), we have \( u_0 = u_1 = u_2 = 0 \) if and only if at least one of the conditions \( C_i \) is satisfied.

**Proof** The ‘if’ part follows by a simple computation. Any of the \( C_i \) makes the matrix \( M \) have rank \( \leq 2 \), forcing \( \delta = 0 \). Furthermore, we have \( Q_0 = Q_1 = 0 \) implying \( u_i = 0 \) for all \( i \).

Now assume that \( u_0 = u_1 = u_2 = 0 \), and hence \( Q_0 = Q_1 = \delta = 0 \). This implies that \( \text{rank}(M) \leq 2 \). Now assume that none of the conditions \( C_i \) for \( 1 \leq i \leq 5 \) are satisfied. Since \( C_1 \) is false, the second row of \( M \) is not a multiple of the first. Hence the third row must be a linear combination of the first two; that is to say,

\[ f = ra + s, \quad e = rb + s, \quad d = rc + s, \]

for some \( r, s \in k \). Substituting this into the \( Q_i \), we get

\[ Q_0 = rs (a - b) (b - c) (c - a) = 0 \quad \text{and} \quad Q_1 = r (r - 1) (a - b) (b - c) (c - a) = 0. \]

Now \( a \neq b \), since otherwise \( f = e \) and then \( C_3 \) would hold. A similar argument for \( C_4 \) and \( C_5 \) shows that \( a, b, c \) are pairwise distinct. Hence \( rs = 0 \), and \( r (r - 1) = 0 \).
If \( r = 0 \), then \( d = e = f \) which is disallowed. Hence we must have \( r = 1, s = 0 \), which implies that \( C_6 \) must hold. \( \square \)

This implies the set-theoretic equality

\[
V(I_s) = \bigcup_{i=1}^{6} V(p_i),
\]

but we will need the stronger version in (2.3).

2.9. Given a Pascal symbol \( s \), we should like to resolve the indeterminacy in the definition of the rational map

\[
p_s : \mathcal{H} \rightarrow (\mathbb{P}^2)^{\ast}.
\]

That is to say, we want to construct a proper birational morphism \( X \rightarrow \mathcal{H} \) such that there is a well-defined map \( X \rightarrow (\mathbb{P}^2)^{\ast} \) which factors through \( p_s \). There are possibly many ways to do this, but we mention two which do not turn out to be geometrically feasible.

- According to the standard formalism of (Hartshorne (1992), Ch. II.7), we can take \( X \) to be the blow-up of \( \mathcal{H} \) along the subscheme \( \Omega_s \). However, since \( \Omega_s \) has several components with multiple intersections between them, the resulting space would be highly singular and unwieldy to work with.

- Another possibility is to construct \( X \) in stages, first by blowing up the smallest polydiagonal \( \Delta_{ABCDEF} \), followed by blowing up the proper transforms of the next smallest polydiagonals and so on. This is the central idea behind the configuration space constructed by Ulyanov (2002). However, after trying out this approach we encountered several obstacles. Already at the first stage, there is a ‘problematic’ locus inside the exceptional hypersurface on which the Pascal is not well-defined. (In a nutshell, it arises due to the appearance of \( \delta \) in the formulae for \( u_i \).) This problem recurs at several intermediate stages; which indicates that the total number of blow-ups needed would be very large, and their combinatorics would be difficult to control.

The approach we have chosen is to remove certain polydiagonals in \( \mathcal{H} \), and then to blow-up the open sublocus \( \mathcal{H}^{\circ} \subseteq \mathcal{H} \) along the remaining ones. In other words, we resolve the indeterminacies of the restricted map

\[
p_s : \mathcal{H}^{\circ} \rightarrow (\mathbb{P}^2)^{\ast}.
\]

As mentioned above, this turns out to be geometrically the most pragmatic solution. Our construction is uniform in the sense that it simultaneously works for all Pascal symbols \( s \).

3 Resolution of indeterminacy

Recall the following connection between blow-ups and the extension of rational maps:
Proposition 3.1 Let \( V \) be a variety with a rational map \( f : V \to \mathbb{P}^n \). Let \( \Sigma \) be the indeterminacy scheme (also known as the scheme of base points) of \( f \). Let \( \text{Bl}_{\Sigma}(V) \) denote the blow-up of \( V \) along \( \Sigma \). Then \( f \) extends to a regular morphism
\[
\tilde{f} : \text{Bl}_{\Sigma}(V) \to \mathbb{P}^n.
\]

Proof See (Hartshorne 1992, Ch. 2.II, Example 7.17.3).

This result will be used in the proof of Theorem 3.3 below. In summary, given an affine chart \( \text{Spec } R \subseteq V \), the map \( f \) is given by a projective \((n + 1)\)-tuple of functions \([v_0, \ldots, v_n]\) such that the \( v_i \) generate the ideal of \( \Sigma \cap \text{Spec } R \). By construction, the pullback of \( \Sigma \) to \( \text{Bl}_{\Sigma}(V) \) is Cartier (i.e., locally defined by a single nonzero divisor). This divisor can be ‘cancelled out’ from the tuple, which gives a well-defined expression for \( \tilde{f} \). The reader will find many such examples in (Eisenbud and Harris 2000, Ch. IV.2). A thematically similar discussion is given in (Hartshorne 1992, Ch. 2.II) preceding the proof of the proposition.

3.1. We now proceed in the following steps:

- Define the open set
\[
\mathcal{H}^o = \mathcal{H} - \bigcup_{\mu} \Delta_\mu,
\]
where the union is over all polydiagonals of type \((3, 2, 1)\) or \((4, 1, 1)\). Notice that the union automatically includes all polydiagonals of type \((5, 1)\), \((3, 3)\), \((4, 2)\) and \((6)\).

- If \( \pi \) is of type \((2, 2, 1, 1), (3, 1, 1, 1)\) or \((2, 2, 2)\), then define \( \Delta^o_\pi = \mathcal{H}^o \cap \Delta_\pi \), which we call an open polydiagonal of type \( \pi \).

- Now let \( Z \subseteq \mathcal{H}^o \) denote the union of all open polydiagonals of type \((2, 2, 2)\), and let \( Y \subseteq \mathcal{H}^o \) denote the union of all open polydiagonals of type \((2, 2, 1, 1)\) or \((3, 1, 1, 1)\). Then we have
\[
Z \subseteq Y \subseteq \mathcal{H}^o.
\]

A schematic picture is shown in Diagram 3. The thick green line represents a typical open polydiagonal of type \((2, 2, 2)\), such as \( \Delta^o_{\text{AF.BE.CD}} \). It is the transverse intersection of three open polydiagonals of type \((2, 2, 1, 1)\), shown as blue rectangles. In this example,
\[
\Delta^o_{\text{AF.BE.CD}} = \Delta^o_{\text{AF.BE.CD}} \cap \Delta^o_{\text{AF.BE.CD}} \cap \Delta^o_{\text{AF.BE.CD}} \tag{3.1}
\]

The red rectangle represents a typical open polydiagonal of type \((3, 1, 1, 1)\). Thus \( Z \) is the union of all green lines, and \( Y \) is the union of all blue and red rectangles.

In general, a point \( h \in Y \) will be in the indeterminacy locus of \( p_s \) for some Pascal symbols \( s \) and not for others. The issue is decided by how the combinatorial structure of \( \theta(h) \) interacts with that of \( s \) (cf. Sect. 2.7).
Diagram 3  Polydiagonals in $H^\circ$. The green line has type $(2, 2, 2)$ and each of the three blue rectangles has type $(2, 2, 1, 1)$. The red rectangle has type $(3, 1, 1, 1)$

**Lemma 3.2**  Let $X'$ be the blow-up of $H^\circ$ along $\Delta^\circ_{\Delta, FE, CE}$. Then the proper transforms of the three polydiagonals appearing on the right-hand side of (3.1) are pairwise disjoint in $X'$.

**Proof**  The result will follow from a local calculation. We should like to blow up the ideal $J = (f - a, e - b, d - c)$ inside the ring $R = k[a, b, c, d, e, f]$. The resulting space is covered by three affine charts corresponding to the generators of $J$. For instance, choose a parameter $w = f - a$, and let $e - b = q_1 w$, $d - c = q_2 w$. The blow-up locally corresponds to the ring map

$$R \longrightarrow k[a, b, c, w, q_1, q_2],$$

which sends $a, b, c$ to themselves, together with

$$f \rightarrow a + w, \quad e \rightarrow b + q_1 w, \quad d \rightarrow c + q_2 w.$$  

The three polydiagonals respectively correspond to the ideals $(f - a, e - b), (e - b, d - c)$ and $(f - a, d - c)$ in $R$. Since their proper transforms are the $S$-ideals $(1), (q_1, q_2)$ and $(1)$ respectively, only the middle one has nonempty support in the affine chart $\text{Spec } S$. The calculation for the other two charts is essentially identical, which proves the lemma.  

Pictorially, the lemma says that when the green line is blown up, the proper transforms of the three blue rectangles get separated.

3.2. Now let $Y' = \text{Bl}_Z(H^\circ)$ denote the blow-up of $H^\circ$ along $Z$, with structure morphism

$$Y' \xrightarrow{\beta_1} H^\circ.$$  

Let $Y' \subseteq X'$ denote the proper transform of $Y$. Notice that, before the blow-up, any of the red rectangles was already disjoint from any of the blue ones, since we have
removed their intersections from $\mathcal{H}$ while arriving at $\mathcal{H}^0$. Hence $Y'$ is a disjoint union of nonsingular varieties which are individually easy to handle.

If $h$ is a point in $\Delta_{AF, BE, CD}$, then $\beta_1^{-1}(h)$ is isomorphic to $\mathbb{P}^2$. This plane intersects the proper transform of each of the blue rectangles in a point. In order to represent these three points, it will be convenient to have a more symmetric affine chart inside the blow-up. In the notation of Lemma 3.2, choose the parameter $t = (f + e + d) - (a + b + c)$ and let $f - a = p_1 t$ and $e - b = p_2 t$. The blow-up locally corresponds to the ring map

$$R \longrightarrow \mathbb{k}[a, b, c, t, p_1, p_2],$$

which sends $a, b, c$ to themselves, together with

$$f \rightarrow a + p_1 t, \quad e \rightarrow b + p_2 t, \quad d \rightarrow c + t - p_1 t - p_2 t.$$

The proper transforms of the three polydiagonals correspond to the ideals $(p_1, p_2), (p_2, 1 - p_1 - p_2)$ and $(p_1, 1 - p_1 - p_2)$. Hence the three marked points are given by

$$W_{AF, BE} : p_1 = p_2 = 0, \quad W_{BE, CD} : p_1 = 1, \quad p_2 = 0, \quad W_{AF, CD} : p_1 = 0, \quad p_2 = 1. \quad (3.2)$$

Of course, similar statements hold for any partition of type $(2, 2, 2)$.

Finally, let $X = \text{Bl}_{Y'}(X')$ denote the blow-up along $Y'$, with structure morphism

$$X \xrightarrow{\beta_2} X'.$$

Now let $\beta = \beta_2 \circ \beta_1$, so that we have a morphism

$$X \xrightarrow{\beta} \mathcal{H}^0.$$

The next theorem says that the rational map $p_s$ extends to a morphism $q_s$ on $X$.

**Theorem 3.3** Let $s$ be any Pascal symbol. Then we have a commutative diagram

$$\begin{array}{ccc}
X & \xrightarrow{q_s} & \mathcal{H}^0 \\
\downarrow{\beta} & & \downarrow{\beta_1} \\
\mathcal{H}^0 & \xrightarrow{p_s} & (\mathbb{P}^2)^* \\
\end{array}$$

The commutativity has the following meaning: if $\tilde{h} \in X$ is a point such that $p_s$ is already defined at $h = \beta(\tilde{h})$, then $q_s(\tilde{h}) = p_s(h)$.

**Proof** Let $\tilde{h} \in X$, and write $h = \beta(\tilde{h})$. If $h \in \mathcal{H}^0 \setminus Y$, then $\beta$ is an isomorphism in an open neighbourhood of $\tilde{h}$ and hence $q_s$ is well-defined at $\tilde{h}$. 

\[ \nabla \text{ Springer} \]
Now assume that \( h \in Y \). If \( p_s \) is already defined at \( h \), then we set \( q_s(\tilde{h}) = p_s(h) \). Hence we may further assume that \( p_s \) is undefined at \( h \). If \( h \not\in Z \), then in a sufficiently small open neighbourhood \( U \) of \( h \) we have \( \beta^{-1}(U) \simeq Bl_{Y \cap U}(U) \). By the argument of Sect. 2.6, the indeterminacy scheme of \( p_s \) coincides with \( Y \cap U \) in \( U \). Then Proposition 3.1 implies that \( p_s \) extends to a regular map over \( \beta^{-1}(U) \).

Now assume \( h \in Z \). Then the same argument implies that \( p_s \) extends to a regular morphism \( q_s' \) on \( \beta_1^{-1}(U) \) for an open neighbourhood \( U \) of \( h \). Now define \( q_s(h) = q_s'(\beta_2(\tilde{h})) \). This proves the theorem.

**Proposition 3.4** The schemes \( X' \) and \( X \) are irreducible and nonsingular quasiprojective varieties.

**Proof** By construction, \( Z \) is a nonsingular variety lying inside \( H^o \) which is itself nonsingular and quasiprojective. Hence the result for \( X' \) follows from the local description of blow-ups in (Eisenbud and Harris 2000, Ch. IV.2). Since \( Y' \) is also nonsingular, the same argument gives the result for \( X \). \( \square \)

### 3.3.

We can calculate the hitherto undefined Pascals using the recipe described in the theorem. For instance, let \( h \in H^o \) be such that

\[
\begin{align*}
  h(\mathbb{A}) &= h(\mathbb{B}) = h(\mathbb{C}) = \tau(3), \\
  h(\mathbb{D}) &= \tau(1), \\
  h(\mathbb{E}) &= \tau(7), \\
  h(\mathbb{F}) &= \tau(4),
\end{align*}
\]

in the notation of Sect. 2.4. Then the Pascal is undefined for \( s = \{ \mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{F}, \mathbb{E}, \mathbb{D} \} \), since all points in the top row become equal.

In order to define the Pascal over the fibre \( \beta^{-1}(h) \), we need to blow up the ideal \( I = (b-a, c-a) \) inside the ring \( R = k[a, b, c, e, d, f] \). Choose a parameter \( b-a = t \) and let \( c-a = pt \). Locally above the point \( h \), the morphism \( \beta \) is given by the ring map

\[
R \longrightarrow \frac{k[a, d, e, f, p, t]}{s}
\]

which acts by sending \( a, d, e, f \) to themselves, together with \( b \to a+t, c \to a+pt \). Now make the substitutions

\[
\begin{align*}
  a &\to 3, \\
  b &\to 3+t, \\
  c &\to 3+pt, \\
  d &\to 1, \\
  e &\to 7, \\
  f &\to 4,
\end{align*}
\]

into the expressions \( u_i \) from Section 2.4; then we get

\[
\begin{align*}
  u_0 &= -3t(8pt+3p+21), \\
  u_1 &= 6t(pt+2p+5), \\
  u_2 &= -3t(p+1).
\end{align*}
\]

Since \( \langle u_0, u_1, u_2 \rangle \) is a projective triple, we may cancel the \( t \). After substituting \( t = 0 \) (which corresponds to the exceptional locus of the blow-up), we get the formula

\[
\langle u_0, u_1, u_2 \rangle = ( -3(3p+21), 6(2p+5), -3(p+1) ) = (3p+21, -4p - 10, p+1)
\]
for the line coordinates of the Pascal, as \( p \) describes a variable point in \( \beta^{-1}(h) \simeq \mathbb{P}^1 \).

(It is understood that the point \( p = \infty \) will be captured in a different affine chart.) Notice the identity \( u_0 + 3u_1 + 3^2u_2 = 0 \), which implies that the Pascal always passes through the triple point \( \tau(3) \). In the next section, we will carry out this analysis more generally.

4 The Pascal morphism on the exceptional loci

If \( h \in H^\circ \) is a sextuple which lies in one of the polydiagonals which have been blown up, then it is of interest to see how the Pascals behave on the fibre \( \beta^{-1}(h) = X_h \).

4.1. In this section we will describe the maps

\[ q_s : X_h \longrightarrow (\mathbb{P}^2)^*, \]

as \( s \) runs through all Pascal symbols. There are three cases to consider; namely when \( \theta(h) \) is of type \((3, 1, 1, 1), (2, 2, 1, 1) \) or \((2, 2, 2) \). The last case is the most symmetric, and as such geometrically the most elegant.

The first two of these partitions are of codimension two in \( H^\circ \), and \( X_h \simeq \mathbb{P}^1 \) in these cases. In the \((2, 2, 2) \) case, \( X_h \) is isomorphic to \( \mathbb{P}^2 \) blown up in the three marked points described in Section 3.2. The exceptional locus of this blow-up consists of three disjoint copies of \( \mathbb{P}^1 \). We label them as \( L_{AF}, L_{BE}, L_{CD} \) in the context of the example given there, and similarly for other partitions.

4.2 The \((3, 1, 1, 1) \) case

Let \( \theta(h) = ABC, D.E.F \). Fix distinct fix points \( M, P, Q, R \) on the conic, and assume that \( h \) acts as follows:

\[ A, B, C \rightarrow M, \quad D \rightarrow P, \quad E \rightarrow Q, \quad F \rightarrow R. \]

Let \( s \) be a Pascal symbol. First assume that \( A, B, C \) are not in the same row of \( s \). Then, up to the permutation of these three letters, \( s \) has either of the two forms:

\[ \begin{cases} A \ B \ x \\ C \ y \ z \end{cases} \text{ or } \begin{cases} A \ B \ x \\ y \ z \ C \end{cases}, \]

where \( \{x, y, z\} = \{D, E, F\} \). In either case, it follows from the definition of the Pascal that it is already defined at \( h \). For the first class of symbols, it is the line \( Mz \) (where \( z \) stands for \( P, Q \) or \( R \) as the case may be). For the second, it is the tangent line \( T_M \) to the conic at \( M \).
Now assume that \( s = \{ ABC \ \ x \ y \ z \} \), where \( \{ x, y, z \} = \{ D, E, F \} \). Then \( h \) lies in the indeterminacy locus of \( p_s \).

**Proposition 4.1** With notation as above, \( q_s \) induces an isomorphism of the fibre \( X_h \) with the pencil of lines through \( M \).

**Proof** This will follow by an explicit calculation as in Section 3.3. Assume \( s = \{ ABC \ \ FED \ xy \} \). Using an automorphism of \( \mathbb{P}^1 \), we may assume that \( M, P, Q, R \) respectively correspond to the points \( m, -1, 0, 1 \) on \( \mathbb{P}^1 \). Now substitute

\[
\begin{align*}
 a & \to m, \quad b \to m + t, \quad c \to m + p \ t, \\
 d & \to -1, \quad e \to 0, \quad f \to 1,
\end{align*}
\]

into the formulae for \( u_i \), factor out \( t \) from the projective triple \( \langle u_0, u_1, u_2 \rangle \) and substitute \( t = 0 \). Then the formula for the Pascal turns out to be

\[
\lambda_p = (-m \ p, 2 \ m - p \ m + p, p - 2).
\]

Its dot product with the vector \( \tau(m) = [1, m, m^2] \) is zero, which implies that the line passes through \( M \). Since its coordinates are linear in the parameter \( p \), we get the desired isomorphism. The other cases of \( s \) follow by symmetry. \( \square \)

### 4.3 The \((2, 2, 1, 1)\) case

Now let \( \theta(h) = AB.EF.C.D \). Fix distinct fix points \( M, N, P, Q \) on the conic, and assume that \( h \) acts as follows:

\[
A, B \to M, \quad E, F \to N, \quad C \to P, \quad D \to Q.
\]

There are four Pascal symbols \( s \) for which \( h \) lies in the indeterminacy locus, namely

\[
\begin{align*}
\{ A \ B \ x \} \quad \text{and} \quad \{ A \ B \ x \},
\end{align*}
\]

where \( \{ x, y \} = \{ C, D \} \). We will give the result for one of these patterns, and the others will follow by symmetry.

**Proposition 4.2** Assume \( s = \{ ABC \ \ FED \} \). Then \( q_s \) induces an isomorphism of the fibre \( X_h \) with the pencil of lines through the point \( M Q \cap NP \).

**Proof** As before, this follows by an explicit calculation. We may assume that \( M, N, P, Q \) respectively correspond to \( m, n, 1, -1 \) on \( \mathbb{P}^1 \). The recipe involves blowing up the ideal \(( b - a, f - e)\) inside the ring \( R = \mathbb{k}[a, b, c, d, e, f] \). Hence, make
substitutions \( b = a + t, \quad f = e + pt \) into the \( u_i \) and calculate the Pascal as above. Its line coordinates turn out to be

\[
(m^2 p - mp - n^2 - n, m^2 p - 2mp + n^2 + 2n + p + 1, -mp - n + p - 1),
\]

whose dot product with \( MQ \cap NP = [n - m + 2, m + n, 2mn + m - n] \) is zero. Then the linearity in parameter \( p \) establishes the isomorphism. \( \square \)

Now let us consider those \( s \) for which the Pascal is already defined at \( h \). These sets \( \{A, B\}, \{E, F\} \) are in symmetric positions, and so are the letters within each set. Hence, up to these shuffles, the essentially distinct cases are as follows:

\[
\{ A \ B \ x \} \quad \{ A \ E \ F \} \quad \{ A \ E \ x \} \quad \{ A \ E \ F \} \quad \{ A \ E \ x \} \quad \{ A \ E \ x \} \quad \{ A \ E \ x \} \quad \{ A \ E \ x \},
\]

where \( \{x, y\} = \{C, D\} \). The corresponding Pascals can be found by tracking the cross-hair intersections. In the first three cases it is the line \( MN \). In the fourth case, it is the line joining \( My \cap Nx \) and \( Ny \cap Mx \), where \( x, y \) stand for either \( P \) or \( Q \) depending on the choice of the bijection \( \{x, y\} \rightarrow \{C, D\} \). In the fifth case, it is the line joining \( T_M \cap yN \) and \( T_N \cap xM \). This completes the discussion of the \((2, 1, 1)\) case.

### 4.4 The \((2, 2, 2)\) case

Let \( \theta(h) = AF.BE.CD \). Fix three distinct points \( P, Q, R \) on the conic, and assume that \( h \) acts as follows:

\[
A, F \rightarrow P, \quad B, E \rightarrow Q, \quad C, D \rightarrow R.
\]

We should like to describe the morphisms \( q_s : X_h \rightarrow (\mathbb{P}^2)^* \) for varying symbols \( s \).

Recall that each point in the projective plane has a polar line with respect to the conic \( K \), and conversely each line has a pole (see Seidenberg 1962, Ch. VI). In Diagram 4, the points \( P', Q', R' \) are respectively the poles of lines \( QR, PR, PQ \). Then \( P'Q'R' \) is called the polar triangle of \( PQR \). It is a theorem due to Chasles that these two triangles are in perspective (see Salmon 2005, §99); that is to say,

- lines \( P', Q', R' \) are concurrent in a point \( CH \), and
- the points \( PQ \cap P'Q', PR \cap P'R' \) and \( QR \cap Q'R' \) are collinear on a line \( ch \).

(The intersection \( PQ \cap P'Q' \) is not shown in the diagram.) Now the behaviour of \( q_s \) (restricted to fibre \( X_h \)) is described in the following theorem.

**Theorem 4.3**  
(1) The map \( q_s \) is constant for 44 values of the symbol \( s \). Amongst these, there are 8 values each for which the image is the line \( PQ, PR \) or \( QR \). There are 4 values each for which the image is \( PP', QQ' \) or \( RR' \). Finally, there are 8 values of \( s \) for which the image is \( ch \).

(2) The map \( q_s \) is non-constant for 16 values of \( s \). There are 4 values for which the image is the pencil of lines through \( P \), and similarly 4 each for \( Q \) and \( R \). For the remaining 4 values, the image is all of \((\mathbb{P}^2)^*\).
In order to explain the patterns precisely, let the letters $x, y, z$ stand for elements of the three subsets $\{A, F\}, \{B, E\}, \{C, D\}$ in some order. Moreover, if for instance $x$ stands for one of the letters $\{B, E\}$, then it will also serve as a placeholder for the corresponding point on the conic, namely $Q$.

For the moment, let us allot the letters arbitrarily in the following way:

$$\begin{align*}
x &\rightarrow \{A, F\}, \\
y &\rightarrow \{C, D\}, \\
z &\rightarrow \{B, E\}. \\
\end{align*}$$

Now $q_s$ is a constant map in these cases:

- If $s = \begin{bmatrix} x & x & y \\ x & y & z \\ y & z & z \end{bmatrix}$, then image $(q_s) = xz$. For instance, $s = \begin{bmatrix} F & A & D \\ C & B & E \end{bmatrix}$ obeys this pattern, and hence the map $q_s$ is constant with image $PQ$.
- If $s = \begin{bmatrix} x & y & z \\ y & z & x \\ x & z & y \end{bmatrix}$, then the image is the line $xx'$. For instance, if $s = \begin{bmatrix} A & C & E \\ F & B & D \end{bmatrix}$ then it is the line $PP'$.
- For $s = \begin{bmatrix} x & y & z \\ y & z & x \end{bmatrix}$, it is the line $ch$.

As for the variable lines,

- For $s = \begin{bmatrix} x & y & y \\ z & x & z \\ z & y & z \end{bmatrix}$, the image of $q_s$ is the pencil of lines through $x$.
- For $s = \begin{bmatrix} x & y & z \\ x & y & z \end{bmatrix}$, the image is all of $(\mathbb{P}^2)^*$. 
The same is true of any of the six possible allotments. This covers all the sixty Pascal symbols.

**Proof** All the proofs follow by straightforward computations. We will prove three cases for the sake of illustration.

For example, suppose that $s = \{A B C, F E D\}$ which fits the pattern $\{x y z, x y z\}$. Using an automorphism of $\mathbb{P}^1$, we may assume that $P = \tau(1), Q = \tau(0), R = \tau(-1)$. Following the generators of the ideal $I = (f - a, e - b, d - c)$ in $R = \mathbb{k}[a, b, c, d, e, f]$, make substitutions

$$f \rightarrow a + t = 1 + t, \quad e \rightarrow b + p_1 t = p_1 t, \quad d \rightarrow c + p_2 t = -1 + p_2 t$$

into the formulae for the $u_i$. Cancel the factor $t$ and substitute $t = 0$. After substituting $p_1 \rightarrow p_1/p_0$, $p_2 \rightarrow p_2/p_0$ and homogenizing, we see that the Pascal morphism is given by the composite

$$X_h \rightarrow \mathbb{P}^2 \rightarrow (\mathbb{P}^2)^*,$$

where the first map blows down the three $L$-lines (see Section 4.1), and the second map is the isomorphism

$$\begin{bmatrix} p_0 & p_1 & p_2 \end{bmatrix} \rightarrow \begin{bmatrix} p_0 & p_1 & p_2 \end{bmatrix} \begin{bmatrix} 0 & -1 & -1 \\ -2 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}.$$

As a second example, suppose that $s = \{A C D, F B E\}$ which fits the pattern $\{x y y, x z z\}$. The Pascal is not defined at $h$, since the second and third column are equal. According to the recipe, we first need to blow up the open polydiagonal $\Delta^0_{A F, B E, C D}$, and then the proper transform of $\Delta^0_{A F, B E, C D}$. We will follow the notation of Section 3.2, where the first blow-up is already done. Since the proper transform corresponds to the ideal $(p_2, 1 - p_1 - p_2)$ in $S$, we let $1 - p_1 - p_2 = p_2 r$. Then the second blow-up corresponds to the map

$$\mathbb{k}[a, b, c, t, p_1, p_2] \rightarrow \mathbb{k}[a, b, c, t, p_2, r],$$

where $a, b, c, t, p_2$ map to themselves and $p_1 \rightarrow 1 - p_2 r - p_2$. Now make these substitutions in the $u_i$, cancel the factor $t p_2$ and set $t = 0$. Using the same $P, Q, R$ as above, we get

$$\langle u_0, u_1, u_2 \rangle = (-2, r, 2 - r).$$

Its dot product with $\tau(1) = [1, 1, 1]$ is zero, hence this represents a pencil of lines through $P$. 

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As a third example, suppose that \( s = \{ \text{ACB DEF} \} \), which fits the pattern \( \{ \text{x y zyz x} \} \). Then the Pascal is already defined at \( h \). By definition, it is the line passing through 

\[
\mathbb{T}_P \cap QR , \quad \mathbb{T}_Q \cap PR \quad \text{and} \quad \mathbb{T}_R \cap PQ ,
\]

which is \( \text{ch} \).

The other cases follow by similar calculations. We leave the details to the industrious reader. \( \square \)

5 The hexagrammum mysticum

We have seen that six points on a conic lead to a collection of sixty Pascal lines in the plane. These lines satisfy some incidence relations, leading to several other geometric elements such as the Kirkman and Steiner points, and the Cayley and Salmon lines. This entire structure is sometimes called the Hexagrammum Mysticum; it is described in detail in the article by Conway and Ryba (2012). The question of definability which we have considered for Pascals can also be raised for these geometric elements. We offer some brief remarks in this direction.

5.1 Kirkman points

It is a theorem due to Kirkman that the Pascals

\[
\left\{ \begin{array}{ccc}
A & E & C \\
D & B & F \\
\end{array} \right\} , \quad \left\{ \begin{array}{ccc}
B & D & A \\
F & C & E \\
\end{array} \right\} , \quad \left\{ \begin{array}{ccc}
C & F & B \\
E & A & D \\
\end{array} \right\}
\]

(5.1)

are concurrent; their common point is called a Kirkman point. This pattern is obtained by starting from the array

\[
\begin{bmatrix}
A & B & C \\
F & E & D \\
\end{bmatrix},
\]

and arranging the red (respectively blue) letters in a \( \lor \) shape (respectively a \( \land \) shape) as shown. As we scan the arrays in (5.1) from left to right, the red letters undergo a cyclic shift \( ABC - BCA - CAB \), whereas the blue letters undergo an anti-cyclic shift \( DEF - FDE - EFD \).

By shuffling the letters \( \{A, \ldots, F\} \) in all possible ways, we get altogether sixty Kirkman points in the plane. As in the case of the Pascal line (cf. Section 2.5), it is straightforward to calculate the coordinates of the Kirkman point described above and find the indeterminacy locus. We have carried out this computation in MACAULAY2. It turns out that this locus is scheme-theoretically defined by the ideal \( \bigcap J_w \), where \( w \) ranges over the symbols

\[
\text{def, cef, ab.ef, bdf, ac.df, bc.de, ade, bcf, bd.cf, ae.cf, ad.cf, ce.bf, ce.bd, ace, ad.ce, ae.bf, ad.bf, ae.bd, abd, abc.}
\]
This is to be interpreted as follows: if \( w \) is a letter triple such as \( def \), then \( J_w \) stands for the ideal \((d - e, d - f)\). If \( w \) is a dot-separated pair such as \( ab.e\ f \), then \( J_w \) stands for \((a - b, e - f)\). Hence the indeterminacy scheme is a union of polydiagonals of types \((3, 1, 1, 1)\) and \((2, 2, 1, 1)\). In particular, all the Kirkman points remain well-defined as long as the initial six points are pairwise distinct or if there is a single double point.

### 5.2 Steiner points

The Pascal lines satisfy another incidence theorem due to Steiner. The lines

\[
\begin{align*}
&\begin{vmatrix}
A & B & C \\
F & E & D
\end{vmatrix}, \quad \\
&\begin{vmatrix}
A & B & C \\
D & F & E
\end{vmatrix}, \quad \\
&\begin{vmatrix}
A & B & C \\
E & D & F
\end{vmatrix},
\end{align*}
\]

are also concurrent, and their common point is called a Steiner point. In this case, the top row is fixed at \((A, B, C)\) and the bottom row goes through cyclic permutations of \((F, E, D)\). We get altogether 20 Steiner points by shuffling the labels \(\{A, \ldots, F\}\).

Now two of these Pascals may become simultaneously undefined on certain polydiagonals, and then the corresponding Steiner point is also undefined. However, in contradistinction to Pascal lines and Kirkman points, a Steiner point may become undefined even if \(A, \ldots, F\) are pairwise distinct. In summary, the situation is as follows (see Chipalkatti 2018). We will say that a sextuple of points \(A, \ldots, F\) on \(K \simeq \mathbb{P}^1\) is tri-symmetric, \(^1\) if it is projectively equivalent to the set

\[
\left\{0, 1, \infty, \alpha, \frac{\alpha - 1}{\alpha}, \frac{1}{1 - \alpha}\right\}
\]

for some \(\alpha \in k\). In that case, at least one of the Steiner points becomes undefined.

### 5.3 Cayley lines, Plücker lines and Salmon points

The Kirkman points satisfy a collinearity theorem, leading to the so-called 20 Cayley lines. Similarly, incidences of Steiner points lead to 15 Plücker lines, and those of Cayley lines lead to 15 Salmon points. We will not describe the combinatorics of these incidences here since this is best done via the ‘dual notation’ (see Baker 1923; Conway and Ryba 2012).

One can carry out a similar analysis for these geometric elements, and deduce the following (see Chipalkatti 2018):

- If the sextuple is tri-symmetric, then at least one of the Cayley lines becomes undefined.
- If it is tri-symmetric with \(\alpha = \sqrt{-1}\), then at least one of the Plücker lines and one of the Salmon points becomes undefined.

Thus, the indeterminacy loci for Cayley and Plücker lines, as well as those for Steiner and Salmon points are intricate subvarieties of \(\mathcal{H}\) which are not confined to

\(^1\) The rationale behind this term is explained in Chipalkatti (2018).
the polydiagonals. One would need to carry out a detailed analysis of their geometry in order to resolve the indeterminacies. We hope to take this up in a possible sequel to this paper.

5.4. There is an octagonal variant of Pascal’s theorem which goes as follows (see Baralić and Spasojević 2015, §5) or (Evans and Rigby 2002, Theorem 2.1). Let $A_1, \ldots, A_8$ be eight distinct points on $\mathcal{K}$. Consider the reducible quartic curves

$$Q = A_1 A_2 \cup A_3 A_4 \cup A_5 A_6 \cup A_7 A_8, \quad Q' = A_2 A_3 \cup A_4 A_5 \cup A_6 A_7 \cup A_8 A_1.$$

Then $Q$ and $Q'$ intersect in a collection of sixteen points which may be decomposed into two octads

$$\{A_1, \ldots, A_8\} \cup \{B_1, \ldots, B_8\}.$$

Then we have the theorem that all the $B_i$ lie on a conic, which may be called the Pascal conic of the original points $A_i$. This construction will eventually break down as some of the $A_i$ come together. It would be of interest to know the indeterminacy locus inside $\mathcal{K}^8$ where the Pascal conic fails to be well-defined and whether one can use blow-ups as above to define it in those cases. This has every promise to be a rich area of geometric investigation.

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