Covering functors without groups

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Abstract

Coverings in the representation theory of algebras were introduced for the Auslander-Reiten quiver of a representation finite algebra in [15] and later for finite dimensional algebras in [2, 7, 11]. The best understood class of covering functors is that of Galois covering functors $F : A \to B$ determined by the action of a group of automorphisms of $A$. In this work we introduce the balanced covering functors which include the Galois class and for which classical Galois covering-type results still hold. For instance, if $F : A \to B$ is a balanced covering functor, where $A$ and $B$ are linear categories over an algebraically closed field, and $B$ is tame, then $A$ is tame.

Introduction and notation

Let $k$ be a field and $A$ be a finite dimensional (associative with 1) $k$-algebra. One of the main goals of the representation theory of algebras is the description of the category of finite dimensional left modules $A\text{mod}$. For that purpose it is important to determine the representation type of $A$. The finite representation type (that is, when $A$ accepts only finitely many indecomposable objects in $A\text{mod}$, up to isomorphism) is well understood. In that context, an important tool is the construction of Galois coverings $F : \hat{A} \to A$ of $A$ since $\hat{A}$ is a locally representation-finite category if and only if $A$ is representation-finite [7, 12]. For a tame algebra $A$ and a Galois covering $F : \hat{A} \to A$, the category $\hat{A}$ is also tame, but the converse does not hold [9, 14].

Coverings were introduced in [15] for the Auslander-Reiten quiver of a representation-finite algebra. For algebras of the form $A = kQ/I$, where $Q$ is a quiver and $I$ an admissible ideal of the path algebra $kQ$, the notion of covering was introduced in [2, 7, 11]. Following [2], a functor $F : A \to B$, between two locally bounded $k$-categories $A$ and $B$, is a covering functor if the following conditions are satisfied:

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(a) $F$ is a $k$-linear functor which is onto on objects;

(b) the induced morphisms

$$\bigoplus_{Fb = j} A(a, b') \to B(Fa, j) \quad \text{and} \quad \bigoplus_{Fa = i} A(a', b) \to B(i, Fb)$$

are bijective for all $i, j$ in $B$ and $a, b$ in $A$.

We denote by $(\nu f^*_a)_{\nu} \mapsto f$ and $(\mu f_a')_{\mu} \mapsto f$ the corresponding bijections. We shall consider $F_\lambda: A_{\text{mod}} \to B_{\text{mod}}$ the left adjoint to the pull-up functor $F^*: B_{\text{mod}} \to A_{\text{mod}}$, $M \mapsto MF$, where $C_{\text{mod}}$ denotes the category of left modules over the $k$-category $C$, consisting of covariant $k$-linear functors.

The best understood examples of covering functors are the Galois covering functors $A \to B$ given by the action of a group of automorphisms $G$ of $A$ acting freely on objects and where $F: A \to B = A/G$ is the quotient defined by the action. See [2, 5, 7, 11, 12] for results on Galois coverings. Examples of coverings which are not of Galois type will be exhibited in Section 1.

In this work we introduce balanced coverings as those coverings $F: A \to B$ where $b f^*_a = b f_a$ for every $f \in B(Fa, Fb)$. Among many other examples, Galois coverings are balanced, see Section 2. We shall prove the following:

**THEOREM 0.1** Let $F: A \to B$ be a balanced covering. Then every finitely generated $A$-module $X$ is a direct summand of $F F_\lambda X$.

In fact, according to the notation in [1], we show that a balanced covering functor is a cleaving functor, see Section 3. This is essential for extending Galois covering-type results to more general situations. For instance we show the following result.

**THEOREM 0.2** Assume that $k$ is an algebraically closed field and let $F: A \to B$ be a covering functor. Then the following hold:

(a) If $F$ is induced from a map $f: (Q, I) \to (Q', I')$ of quivers with relations, where $A = kQ/I$ and $B = kQ'/I'$, then $B$ is locally representation-finite if and only if so is $A$;

(b) If $F$ is balanced and $B$ is tame, then $A$ is tame.

More precise statements are shown in Section 4. For a discussion on the representation type of algebras we refer to [11, 13, 9, 6, 14].
1 Coverings: examples and basic properties

1.1 The pull-up and push-down functors

Following [2, 7], consider a locally bounded $k$-category $A$, that is, $A$ has a (possibly infinite) set of non-isomorphic objects $A_0$ such that

(a) $A(a, b)$ is a $k$-vector space and the composition corresponds to linear maps $A(a, b) \otimes_k A(b, c) \to A(a, c)$ for every $a, b, c$ objects in $A_0$;

(b) $A(a, a)$ is a local ring for every $a$ in $A_0$;

(c) $\sum_b A(a, b)$ and $\sum_b A(b, a)$ are finite dimensional for every $a$ in $A_0$.

For a locally bounded $k$-category $A$, we denote by $A_{\text{Mod}}$ (resp. $\text{Mod}_A$) the category of covariant (resp. contravariant) functors $A \to \text{Mod}_k$; by $A_{\text{mod}}$ (resp. $\text{mod}_A$) we denote the full subcategory of locally finite-dimensional functors $A \to \text{mod}_k$ of the category $A_{\text{Mod}}$ (resp. $\text{Mod}_A$). In case $A_0$ is finite, $A$ can be identified with the finite-dimensional $k$-algebra $\bigoplus_{a,b \in A_0} A(a, b)$; in this case the category $A_{\text{Mod}}$ (resp. $A_{\text{mod}}$) is equivalent to the category of left $A$-modules (resp. finitely generated left $A$-modules).

According to [6], in case $k$ is algebraically closed, there exist a quiver $Q$ and an ideal $I$ of the path category $kQ$, such that $A$ is equivalent to the quotient $kQ/I$. Then any module $M \in A_{\text{Mod}}$ can be identified with a representation of the quiver with relations $(Q, I)$. Usually our examples will be presented by means of quivers with relations.

Let $F: A \to B$ be a $k$-linear functor between two locally bounded $k$-categories. The pull-up functor $F^*: B_{\text{Mod}} \to A_{\text{Mod}}, M \mapsto MF$ admits a left adjoint $F_\lambda: A_{\text{Mod}} \to B_{\text{Mod}}$, called the push-down functor, which is uniquely defined (up to isomorphism) by the following requirements:

(i) $F_\lambda A(a, -) = B(Fa, -)$;

(ii) $F_\lambda$ commutes with direct limits.

In particular, $F_\lambda$ preserves projective modules. Denote by $F_\rho: A_{\text{Mod}} \to B_{\text{Mod}}$ the right adjoint to $F_\lambda$.

For covering functors $F: A \to B$ we get an explicit description of $F_\lambda$ and $F_\rho$ as follows:

Lemma 1.1 [2]. Let $F: A \to B$ be a covering functor. Then
(a) For any $X \in A_{\text{mod}}$ and $f \in B(i, j)$,

$$F_\lambda X(f) = (X(b^*f_a)): \bigoplus_{F_a=i} X(a) \to \bigoplus_{F_b=j} X(b), \text{ with } \sum F(b^*f_a) = f.$$ 

In particular, $F_\lambda(a,-): F_\lambda A(a,-) \to B(Fa,-)$ is the natural isomorphism given by $(b^*f_a)_b \mapsto f$.

(b) For any $X \in A_{\text{mod}}$ and $f \in B(i, j)$

$$F_\rho X(f) = (X(f^*a)_b): \prod_{F_a=i} X(a) \to \prod_{F_b=j} X(b), \text{ with } \sum F(f^*a)_b = f.$$ 

In particular, $F_\rho D(-, b): F_\rho DA(-, b) \to DB(-, Fb)$ is the natural isomorphism induced by $(f^*a)_a \mapsto f$. 

\[\square\]

### 1.2 The order of a covering

The following lemma allows us to introduce the notion of order of a covering.

**Lemma 1.2** Let $F: A \to B$ be a covering functor. Assume that $B$ is connected and a fiber $F^{-1}(i)$ is finite, for some $i \in B_0$. Then the fibers have constant cardinality.

**Proof.** Let $i \in B_0$ and $0 \neq f \in B(i, j)$. For $a \in F^{-1}(i)$, $\sum_{F_b=j} \dim_k A(a, b) = \dim_k B(i, j)$. Hence $|F^{-1}(i)| \dim_k B(i, j) = \sum_{F_a=i} \sum_{F_b=j} \dim_k A(a, b) = \sum_{F_b=j} \sum_{F_a=i} \dim_k A(a, b) = |F^{-1}(j)| \dim_k B(i, j)$ and $|F^{-1}(i)| = |F^{-1}(j)|$. Since $B$ is connected, the claim follows. \[\square\]

In case $F: A \to B$ is a covering functor with $B$ connected and $A_0$ is finite, we define the **order** of $F$ as $\text{ord}(F) = |F^{-1}(i)|$ for any $i \in B_0$. Thus $\text{ord}(F)|B_0| = |A_0|$.

We recall from the Introduction that a covering functor $F: A \to B$ is **balanced** if $b^*f_a = b^*f_a$ for every couple of objects $a, b$ in $A$.

**Lemma 1.3** Let $F: A \to B$ be a balanced covering functor, then $F_\lambda = F_\rho$ as functors $A_{\text{mod}} \to B_{\text{mod}}$. \[\square\]
1.3 Examples

(a) Let \( A \) be a locally bounded \( k \)-category and let \( G \) be a group of \( k \)-linear automorphisms acting freely on \( A \) (that is, for \( a \in A_0 \) and \( g \in G \) if \( ga = a \), then \( g = 1 \)). The quotient category \( A/G \) has as objects the \( G \)-orbits in the objects of \( A \); a morphism \( f : i \to j \) in \( A/G \) is a family \( f : (a, b) \in \prod_{a,b} A(a, b) \), where \( a \) (resp. \( b \)) ranges in \( i \) (resp. \( j \)) and \( g \cdot b = g \cdot f \cdot a \) for all \( g \in G \). The canonical projection \( F : A \to A/G \) is called a Galois covering defined by the action of \( G \).

A particular situation is illustrated by the following algebras (given as quivers with relations):

\[
A: \begin{array}{ccc}
\bullet & \rho_0 & \bullet \\
\gamma_0 & \gamma_1 & \bullet \\
\sigma_0 & \sigma_1 & \bullet
\end{array}
\begin{array}{c}
\begin{cases}
\rho_1 \rho_0 = \nu_1 \gamma_0 \\
\sigma_1 \sigma_0 = \gamma_1 \nu_0 \\
\rho_1 \nu_0 = \nu_1 \sigma_0 \\
\sigma_1 \gamma_0 = \gamma_1 \rho_0
\end{cases}
\end{array}
\begin{array}{c}
\begin{cases}
\alpha_1 \alpha_0 = \beta_1 \beta_0 \\
\beta_1 \alpha_0 = \alpha_1 \beta_0
\end{cases}
\end{array}
B: \begin{array}{ccc}
\bullet & \alpha_0 & \bullet \\
\beta_0 & \alpha_1 & \bullet \\
\beta_1 & \beta_2 & \bullet
\end{array}
\begin{array}{c}
\begin{cases}
\alpha_1 \alpha_0 = \beta_1 \beta_0 \\
\beta_1 \alpha_0 = \alpha_1 \beta_0
\end{cases}
\end{array}
\]

The algebra \( A \) is tame, but \( B \) is wild when \( \text{char } k = 2 \] [1]. The cyclic group \( C_2 \) acts freely on \( A \) and \( A/C_2 \) is isomorphic to \( B \).

(b) Consider the algebras given by quivers with relations and the functor \( F \) as follows:

\[
\begin{array}{c}
a_2 \quad a_1 \\
\beta_2 \quad \beta_1
\end{array}
\begin{array}{c}
b_2 \quad b_1 \\
\rho_2 \quad \rho_1
\end{array}
\begin{array}{c}
\alpha_2 \quad \alpha_1 \\
\beta_2 \quad \beta_1
\end{array}
\xrightarrow{F}
\begin{array}{c}
a \quad b \\
\alpha \quad \beta
\end{array}
\]
both algebras with \( \text{rad }^2 = 0 \) and \( F\alpha_1 = \alpha, F\alpha_2 = \alpha + \beta, F\beta_1 = \beta, F\rho_1 = \rho, i = 1, 2 \). It is a simple exercise to check that \( F \) is a balanced covering, but obviously it is not of Galois type.

(c) Consider the functor

\[
A: \begin{array}{ccc}
a_1 & a_2 \\
\beta_1 & \alpha_1 \\
\beta_2 & \alpha_2
\end{array}
\xrightarrow{F} B: \begin{array}{cc}
a & b \\
\alpha & \beta
\end{array}
\]
where \( F\alpha_i = \alpha_i, i = 1, 2, \) \( F\beta_1 = \beta, F\beta_2 = \alpha + \beta. \) Since \( F(\beta_2 - \alpha_2) = \beta \) and \( F(\beta_1) = \beta, \) then \( b_2\beta_2 = -\alpha_2 \) and \( b_2\beta_2 = 0. \) Hence \( F \) is a non-balanced covering functor.

For the two dimensional indecomposable \( A \)-module \( X \) given by \( X(a_2) = k, X(b_2) = k, X(\alpha_2) = id \) and zero otherwise, it follows that \( F\lambda X \) is indecomposable and hence \( X \) is not a direct summand of \( F\lambda X. \)

(d) As a further example, consider the infinite category \( A \) and the balanced covering functor defined in the obvious way:

\[
\begin{array}{c}
\alpha_1 \\
\beta_1 \\
\circ \\
\beta_2 \\
\circ \\
\vdots \\
\end{array} \quad \xrightarrow{F} \quad \begin{array}{c}
\beta \\
\alpha \\
\end{array}
\]

where both categories \( A \) and \( B \) have \( \text{rad}^2 = 0. \)

1.4 Coverings of schurian categories

We say that a locally bounded \( k \)-category \( B \) is schurian if for every \( i, j \in B_0, \) \( \dim_k B(i, j) \leq 1. \)

**Lemma 1.4** Let \( F: A \to B \) be a covering functor and assume that \( B \) is schurian, then \( F \) is balanced.

**Proof.** Let \( 0 \neq f \in B(i, j) \) and \( Fa = i, Fb = j. \) Since \( B \) is schurian, there is a unique \( 0 \neq \nu f_a^* \in A(a, b') \) with \( Fb' = j \) and a unique \( 0 \neq \nu f_a' \in A(a', b) \) with \( Fa' = i \) satisfying \( Fb'f_a^* = f = Fb'f_a'. \) In case \( b = b' \), then \( a = a' \) and \( b f_a^* = \nu f_a. \) Else \( b \neq b' \) and \( b f_a^* = 0. \) In this situation \( a \neq a' \) and \( b f_a^* = 0. \) \( \Box \)

**Proposition 1.5** Let \( F: A \to B \) be a covering functor with finite order and \( B \) schurian. Then for every \( M \in B_{\text{mod}}, F\lambda F\lambda M \cong M^{\text{ord}(F)}. \)

**Proof.** For any \( 0 \neq f \in B(i, j) \) we get

\[
\begin{array}{c}
\nu f_a^* \\
(M(Fb f_a^*)) \\
\downarrow \quad \downarrow \\
\nu f_a' \\
(M(f)) \\
\end{array} \quad \xrightarrow{F\lambda F\lambda M(i)} \quad \xrightarrow{M^{\text{ord}(F)(i)}} \quad \xrightarrow{(M(f), \ldots, M(f))} \quad \xrightarrow{\text{diag}(M(f), \ldots, M(f))} \quad \xrightarrow{F\lambda F\lambda M(j)} \quad \xrightarrow{M^{\text{ord}(F)(j)}}
\]

Since for each \( a \) there is a unique \( b \) with \( b f_a^* \neq 0 \) such that \( Fb f_a^* = f, \) then the square commutes. \( \Box \)

**Remark:** If \( B \) is not schurian the result may not hold as shown in \([9, (3.1)]\) for a Galois covering \( F: B \to C \) with \( B \) as in Example (1.3a).
1.5 Coverings induced from a map of quivers

Let \( q: Q' \to Q \) be a covering map of quivers, that is, \( q \) is an onto morphism of oriented graphs inducing bijections \( i^+ \to q(i)^+ \) and \( i^- \to q(i)^- \) for every vertex \( i \) in \( Q' \), where \( x^+ \) (resp. \( x^- \)) denotes those arrows \( x \to y \) (resp. \( y \to x \)). For the concept of covering and equitable partitions in graphs, see [10].

Assume that \( Q \) is a finite quiver. Let \( I \) be an admissible ideal of the path algebra \( kQ \), that is, \( J^n \subset I \subset J^2 \) for \( J \) the ideal of \( kQ \) generated by the arrows of \( Q \). We say that \( I \) is admissible with respect to \( q \) if there is an ideal \( I' \) of the path category \( kQ' \) such that the induced map \( kq: kQ \to kQ' \) restricts to isomorphisms \( \bigoplus_{q(a)=i} kQ'(a, b) \to I(i, j) \) for \( q(b) = j \) and \( \bigoplus_{q(b)=j} kQ'(a, b) \to I(i, j) \) for \( q(a) = i \).

Observe that most examples in (1.3) (not Example (c)) are built according to the following proposition:

**Proposition 1.6** Let \( q: Q' \to Q \) be a covering map of quivers, \( I \) an admissible ideal of \( kQ \) and \( I' \) an admissible ideal of \( kQ' \) making \( I \) admissible with respect to \( q \) as in the above definition. Then the induced functor \( F: kQ'/I' \to kQ/I \) is a balanced covering functor.

**Proof.** Since \( q \) is a covering of quivers, it has the unique lifting property of paths. Hence for any pair of vertices \( i \) in \( Q \) and \( a \) in \( Q' \) with \( q(a) = i \), we have that

\[
\begin{array}{ccc}
\bigoplus_{q(b)=j} kQ'(a, b) & \xrightarrow{kq} & kQ(i, j) \\
\downarrow & & \downarrow \\
\bigoplus_{F(b)=j} kQ'/I'(a, b) & \xrightarrow{F} & kQ/I(i, j)
\end{array}
\]

is a commutative diagram with \( F \) an isomorphism. This shows that \( F \) is a covering functor.

For any arrow \( i \xrightarrow{\alpha} j \) in \( Q \) and \( q(a) = i \), there is a unique \( b \) in \( Q' \) and an arrow \( a \xrightarrow{\alpha'} b \) with \( q(\alpha') = \alpha \). Hence the class \( \epsilon f^*_{\alpha} \) of \( \alpha' \) in \( kQ'/I'(a, b) \) satisfies that \( F(\epsilon f^*_{\alpha}) \) is the class \( f = \bar{\alpha} \) of \( \alpha \) in \( kQ/I(i, j) \). By symmetry, \( \epsilon f^*_{\alpha} = \bar{\rho} f_{\alpha} \). For arbitrary \( f \in kQ/I(i, j) \), \( f \) is the linear combination \( \sum \lambda_i f_i \), where \( f_i \) is the product of classes of arrows in \( Q \). Observe that for arrows \( i \xrightarrow{\alpha} j \xrightarrow{\beta} m \) we have \( c(\bar{\beta}\bar{\alpha})^* = (c\bar{\beta}^*)(\bar{\alpha}a^*) = (c\bar{\beta}^*)(\bar{\alpha}a^*) \). It follows that \( F \) is balanced. \( \square \)

In the above situation we shall say that the functor \( F \) is induced from a map \( q: (Q', I') \to (Q, I) \) of quivers with relations.
2 On Galois coverings

2.1 Galois coverings are balanced

Proposition 2.1 Let $F: A \to B$ be a Galois covering, then $F$ is balanced.

Proof. Assume $F$ is determined by the action of a group $G$ of automorphisms of $A$, acting freely on the objects $A_0$. Let $a, b$ in $A$ with $Fa = i$, $Fb = j$ and $(\nu f^*_a)_\nu \in \bigoplus_{F\nu = j} A(a, b)$ with $\sum_{F\nu = j} F(\nu f^*_a) = f$.

For each object $b'$ with $Fb' = j$, there is a unique $g_{b'} \in G$ with $g_{b'}(b') = b$. Then $(g_{b'}(\nu f^*_a))_{\nu} \in \bigoplus_{\nu} A(g_{b'}(a), b) = \bigoplus_{F\alpha' = i} A(a', b)$ with $\sum_{\nu} F(g_{b'}(\nu f^*_a)) = \sum_{\nu} F(\nu f^*_a) = f$. Hence $g_{b'}(\nu f^*_a) = b f_{g_{b'}(a)}$ for every $Fb' = j$. In particular, for $g_b = 1$ we get $b f^*_a = b f_a$.

□

2.2 The smash-product

We say that a $k$-category $B$ is $G$-graded with respect to the group $G$ if for each pair of objects $i, j$ there is a vector space decomposition $B(i, j) = \bigoplus_{g \in G} B^g(i, j)$ such that the composition induces linear maps

$$B^g(i, j) \otimes B^h(j, m) \to B^{gh}(i, m).$$

Then the smash product $B \# G$ is the $k$-category with objects $B_0 \times G$, and for pairs $(i, g), (j, h) \in B_0 \times G$, the set of morphisms is

$$(B \# G)((i, g), (j, h)) = B^{g^{-1}h}(i, j)$$

with compositions induced in natural way.

In [4] it was shown that $B \# G$ accepts a free action of $G$ such that

$$(B \# G)/G \xrightarrow{\sim} B.$$

Moreover, if $B = A/G$ is a quotient, then $B$ is a $G$-graded $k$-category and

$$(A/G) \# G \xrightarrow{\sim} A.$$

Proposition 2.2 Let $F: A \to B$ be a covering functor and assume that $B$ is a $G$-graded $k$-category. Then
(i) Assume $A$ accepts a $G$-grading compatible with $F$, that is, $F^g(a, b) \subseteq B^g(Fa, Fb)$, for every pair $a, b \in A_0$ and $g \in G$. Then there is a covering functor $F \# G: A \rightarrow B \# G$ completing a commutative square

\[ A \# G \xrightarrow{F\#G} B \# G \]

\[ A \rightarrow F \rightarrow B \]

where the vertical functors are the natural quotients. Moreover $F$ is balanced if and only if $F \# G$ is balanced.

(ii) In case $B$ is a schurian algebra, then $A$ accepts a $G$-grading compatible with $F$.

**Proof.** (i): For each $a, b \in A_0$, consider the decomposition $A(a, b) = \bigoplus_{g \in G} A^g(a, b)$ and $B(Fa, Fb) = \bigoplus_{g \in G} B^g(Fa, Fb)$. Since these decompositions are compatible with $F$, then $A^g(a, b) = F^{-1}(B^g(Fa, Fb))$, for every $g \in G$.

For $\alpha \in (A \# G)((a, g), (b, h)) = A^{g^{-1}h}(a, b) = F^{-1}(B^{g^{-1}h}(Fa, Fb))$, we have

\[ (F \# G)(\alpha) = F\alpha \in B^{g^{-1}h}(Fa, Fb) = (B \# G)((Fa, g), (Fb, h)). \]

(ii): Assume $B$ is schurian and take $a, b \in A_0$ and $g \in G$. Either $B^g(Fa, Fb) = B(Fa, Fb) \neq 0$, if $A(a, b) \neq 0$ or $B^g(Fa, Fb) = 0$, correspondingly we set $A^g(a, b) = A(a, b)$ or $A^g(a, b) = 0$. Observe that the composition induces linear maps $A^g(a, b) \otimes A^h(b, c) \rightarrow A^{gh}(a, c)$, hence $A$ accepts a $G$-grading compatible with $F$. \qed

**Remark:** In the situation above, the fact that $A$ and $B \# G$ are connected categories does not guaranty that $A \# G$ is connected. For instance, if $B = A/G$, then $A \# G = A \times G$.

The following result is a generalization of Proposition 2.2(ii).

**Proposition 2.3** Let $F: A \rightarrow B$ be a (balanced) covering functor induced from a map of quivers with relations. Let $F': B' \rightarrow B$ be a Galois covering functor induced from a map of quivers with relations defined by the action of a group $G$. Assume moreover that $B'$ is schurian. Then $A$ accepts a $G$-grading compatible with $F$ making the following diagram commutative

\[ A \# G \xrightarrow{F\#G} B' \]

\[ A \rightarrow F \rightarrow B \]

\[ F' \]
Proof. Let $A = k\Delta/J$, $B = kQ/I$ and $B' = kQ'/I'$ be the corresponding presentations as quivers with relations, $F$ induced from the map $\delta: \Delta \to Q$, while $F'$ induced from the map $q: Q' \to Q$. For each vertex $a$ in $\Delta$ fix a vertex $a'$ in $Q'$ such that $F'a' = Fa$.

Consider an arrow $a \xrightarrow{\alpha} b$ in $\Delta$ and $\pi$ the corresponding element of $A$. We claim that there exists an element $g_{\alpha} \in G$ such that $F(\pi) \in B^{g_{\alpha}}(Fa, Fb)$. Indeed, we get $F(\pi) = \overline{\beta} = F'(\overline{\beta'})$ for arrows $Fa \xrightarrow{\beta} Fb$ and $a' \xrightarrow{\beta'} g_{\alpha}b'$ for a unique $g_{\alpha} \in G$. Therefore $F(\pi) \in B^{g_{\alpha}}(Fa, Fb)$. We shall define $A^{g_{\alpha}}(a, b)$ as containing the space $k\alpha$.

For this purpose, consider $g \in G$ and any vertices $a, b$ in $\Delta$, then $A^{g}(a, b)$ is the space generated by the classes $u$ of the paths $u: a \to b$ such that $F(u) \in B^{g}(Fa, Fb)$. Since the classes of the arrows in $\Delta$ generate $A$, then $A(a, b) = \bigoplus_{g \in G} A^{g}(a, b)$. We shall prove that there are linear maps

$$A^{g}(a, b) \otimes A^{h}(b, c) \to A^{gh}(a, c).$$

Indeed, if $\overline{u} \in A^{g}(a, b)$ and $\overline{v} \in A^{h}(b, c)$ for paths $u: a \to b$ and $v: b \to c$ in $\Delta$, let $F(\overline{u}) = F'(\overline{u'})$ and $F(\overline{v}) = F'(\overline{v'})$ for paths $u': a' \to gb'$ and $v': b' \to hc'$ in $Q'$. Since $B'$ is schurian then the class of the lifting of $F(\overline{uv})$ to $B'$ is $(gv'\alpha')u'$. Therefore $F(\pi)F(\overline{u}) = F'(\overline{(gv')u'}) \in B^{gh}(Fa, Fb)$.

By definition, the $G$-grading of $A$ is compatible with $F$. We get the commutativity of the diagram from Proposition 2.2. \hfill \Box

2.3 Universal Galois covering

Let $B = kQ/I$ be a finite dimensional $k$-algebra. According to [11] there is a $k$-category $\tilde{B} = k\tilde{Q}/\tilde{I}$ and a Galois covering functor $\tilde{F}: \tilde{B} \to B$ defined by the action of the fundamental group $\pi_1(Q, I)$ which is universal among all the Galois coverings of $B$, that is, for any Galois covering $F: A \to B$ there is a covering functor $F': \tilde{B} \to A$ such that $\tilde{F} = FF'$. In fact, the following more general result is implicitly shown in [11]:

Proposition 2.4 [11]. The universal Galois covering $\tilde{F}: \tilde{B} \to B$ is universal among all (balanced) covering functors $F: A \to B$ induced from a map $q: (Q', I') \to (Q, I)$ of quivers with relations, where $A = kQ'/I'$. \hfill \Box
3 Cleaving functors

3.1 Balanced coverings are cleaving functors

Consider the $k$-linear functor $F: A \to B$ and the natural transformation $F(a, b): A(a, b) \to B(Fa, Fb)$ in two variables. The following is the main observation of this work.

**Theorem 3.1** Assume $F: A \to B$ is a balanced covering, then the natural transformation $F(a, b): A(a, b) \to B(Fa, Fb)$ admits a retraction $E(a, b): B(Fa, Fb) \to A(a, b)$ of functors in two variables $a, b$ such that $E(a, b)F(a, b) = 1_{A(a, b)}$ for all $a, b \in A_0$.

**Proof.** Set $E(a, b): B(Fa, Fb) \to A(a, b), f \mapsto \ast f_a$ which is a well defined map. For any $\alpha \in A(a, a'), \beta \in A(b, b')$, we shall prove the commutativity of the diagrams:

$$
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
B(Fa, Fb) \xrightarrow{E(a, b)} A(a, b) \\
\downarrow \quad A(a, b)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
B(Fa, Fb) \xrightarrow{E(a', b)} A(a', b)
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
B(Fa, Fb) \xrightarrow{E(a, b)} A(a, b)
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}

$$

For the sake of clarity, let us denote by $\circ$ the composition of maps. Indeed, let $f \in B(Fa, Fb)$ and calculate $\sum_{a' = Fa} F(\beta \circ \ast f_{a'}) = F\beta \circ f$, hence

$$
A(a, \beta) \circ E(a, b)(f) = \beta \circ \ast f_a = \ast(F\beta \circ f)_a = E(a, b') \circ B(Fa, F\beta)(f),
$$

and the first square commutes. Moreover, let $h \in B(Fa', Fb)$ and calculate $\sum_{b' = Fb} F(\gamma \circ \ast f_{b'}) = h \circ F\alpha$ and therefore $\ast(h \circ F\alpha)_a = \ast(h \circ F\alpha)$. Using that $F$ is balanced we get that $E(a, b) \circ B(Fa, Fb)(h) = \ast(h \circ F\alpha)_a = \ast(h \circ F\alpha) = A(\alpha, b) \circ E(a', b')(h)$. \hfill \Box

Given a $k$-linear functor $F: A \to B$ the composition $F_\ast F_\lambda: A_{\text{Mod}} \to A_{\text{Mod}}$ is connected to the identity $1$ of $A_{\text{Mod}}$ by a canonical transformation $\varphi: F_\ast F_\lambda \to 1$ determined by $F_\ast F_\lambda A(a, -)(b) = \bigoplus_{Fb' = Fb} A(a, b') \to A(a, b), (f_{b'}) \mapsto f_b$, see [1, page 234]. Following [1], $F$ is a cleaving functor if the canonical transformation $\varphi$ admits a natural section $\varepsilon: 1 \to F_\ast F_\lambda$ such that $\varphi(X)\varepsilon(X) = 1_X$ for each $X \in A_{\text{Mod}}$. The following statement, essentially from [1], yields Theorem 0.1 in the Introduction.

**Corollary 3.2** Let $F: A \to B$ be a balanced covering, then $F$ is a cleaving functor.
Proof. Observe that $F \circ F$ is exact, preserves direct sums and projectives (the last property holds since $F \circ B(i, -) = \oplus_{F a = i} A(a, -)$). Hence to define $\varepsilon : 1 \rightarrow F \circ F$ it is enough to define $\varepsilon(A(a, -)) : A(a, -) \rightarrow F \circ F A(a, -)$ with the desired properties. For $b \in A_0$, consider $\varepsilon_b : A(a, b) \rightarrow \bigoplus_{F a' = F b} A(a, b') = F \circ F A(a, -)(b)$ the canonical inclusion. For $h \in A(b, c)$ we shall prove the commutativity of the following diagram:

\[
\begin{array}{ccc}
A(a, b) & \xrightarrow{\varepsilon_b} & \bigoplus_{F a' = F b} A(a, b') \\
A(a, c) & \xrightarrow{\varepsilon_c} & \bigoplus_{F a' = F c} A(a, c')
\end{array}
\]

Let $f \in A(a, b)$, since $F$ is balanced $A(\cdot, c F h_b) \circ \varepsilon_b(f) = \varepsilon_c F h_b \circ f = \varepsilon_b \circ A(a, h) f)$, since $c F h_b = h$ if $c' = c$ and it is 0 otherwise. This is what we wanted to show. \[\square\]

4 On the representation type of categories

4.1 Representation-finite case

Recall that a $k$-category $A$ is said to be locally representation-finite if for each object $a$ of $A$ there are only finitely many indecomposable $A$-modules $X$, up to isomorphism, such that $X(a) \neq 0$. For a cleaving functor $F : A \rightarrow B$ it was observed in \cite{1} that in case $B$ is of locally representation-finite then so is $A$. In particular this holds when $F$ is a Galois covering by \cite{7}. We shall generalize this result for covering functors.

Part (a) of Theorem 0.2 in the Introduction is the following:

THEOREM 4.1 Assume that $k$ is algebraically closed and let $F : A \rightarrow B$ be a covering induced from a map of quivers with relations. Then $B$ is locally representation-finite if and only if so is $A$. Moreover in this case the functor $F \circ A_{\text{mod}} \rightarrow B_{\text{mod}}$ preserves indecomposable modules and Auslander-Reiten sequences.

Proof. Let $F : A \rightarrow B$ be induced from $q : (Q', I') \rightarrow (Q, I)$ where $A = kQ'/I'$ and $B = kQ/I$. Let $\tilde{B} = k\tilde{Q}/\tilde{I}$ be the universal cover of $B$ and $\tilde{F} : \tilde{B} \rightarrow B$ the universal covering functor. By Proposition 2.4 there is a covering functor $F' : \tilde{B} \rightarrow A$ such that $\tilde{F} = F F'$.

(1) Assume that $B$ is a connected locally representation-finite category. Since $F$ is induced by a map of quivers with relations, then Proposition 1.6 implies that $F$ is balanced. Hence Corollary 3.2 implies that $F$ is a cleaving functor. By \cite{1} (3.1), $A$ is locally representation-finite; for the sake of completeness, recall the simple argument: each indecomposable $A$-module $X \in A_{\text{mod}}$ is a direct summand of
\[ F_{\lambda}X = \bigoplus_{i=1}^{n} F_{\lambda}N_{i}^{n_i} \] for a finite family \( N_1, \ldots, N_n \) of representatives of the isoclasses of the indecomposable \( B \)-modules with \( N(i) \neq 0 \) for some \( i = F(a) \) with \( X(a) \neq 0 \).

(2) Assume that \( A \) is a locally representation-finite category. First we show that \( B \) is representation-finite. Indeed, by case (1), since \( F' : \tilde{B} \to A \) is a covering induced by a map of a quiver with relations, then \( \tilde{B} \) is locally representation-finite. By [12], \( \tilde{B} \) is representation-finite. In particular, [2] implies that \( \tilde{F}_\lambda \) preserves indecomposable modules, hence \( F_\lambda \) and \( F'_\lambda \) also preserve indecomposable modules.

Let \( X \) be an indecomposable \( A \)-module. We shall prove that \( X \) is isomorphic to \( F'_\lambda N \) for some indecomposable \( \tilde{B} \)-module \( N \). Since indecomposable projective \( A \)-modules are of the form \( A(a,-) = F'_\lambda \tilde{B}(x,-) \) for some \( x \) in \( \tilde{B} \), using the connectedness of \( \Gamma_A \), we may assume that there is an irreducible morphism \( Y \overset{f}{\to} X \) such that \( Y = F'_\lambda N \) for some indecomposable \( \tilde{B} \)-module \( N \). If \( N \) is injective, say \( N = D\tilde{B}(-,j) \), there is a surjective irreducible map \((h_i) : N \to \oplus_i N_i \) such that all \( N_i \) are indecomposable modules and \( 0 \to S_j \to N \overset{(h_i)}\to \oplus_i N_i \to 0 \) is an exact sequence. Then \( Y = DA(-,F'j) \) and the exact sequence

\[ 0 \to S_{F'j} \to Y \overset{(F'_\lambda(h_i))}\to \oplus_i F'_\lambda(N_i) \to 0 \]

yields the irreducible maps starting at \( Y \) (ending at the indecomposable modules \( F'_\lambda(N_i) \)). Therefore \( X = F'_\lambda(N_r) \) for some \( r \), as desired. Next, assume that \( N \) is not injective and consider the Auslander-Reiten sequence \( \xi : 0 \to N \overset{g}{\to} N' \overset{g'}{\to} N'' \to 0 \) in \( \tilde{B} \text{-mod} \). We shall prove that the push-down \( F'_\lambda \xi : 0 \to F'_\lambda N \overset{F'_\lambda g}{\to} F'_\lambda N' \overset{F'_\lambda g'}{\to} F'_\lambda N'' \to 0 \) is an Auslander-Reiten sequence in \( A \text{-mod} \). This implies that there exists a direct summand \( \tilde{N} \) of \( N' \) such that \( X \overset{\sim}{\to} F'_\lambda \tilde{N} \) which completes the proof of the claim.

To verify that \( F'_\lambda \xi \) is an Auslander-Reiten sequence, let \( h : F'_\lambda \tilde{N} \to Z \) be non-split mono in \( A \text{-mod} \). Consider \( \text{Hom}_A(F'_\lambda N, Z) \overset{\sim}{\to} \text{Hom}_A(N, F'_\lambda Z) \), \( h \mapsto h' \) which is not a split mono (otherwise, then \( \text{Hom}_A(F'_\lambda Z, N) \overset{\sim}{\to} \text{Hom}_A(Z, F'_\lambda N) \), \( \nu \mapsto \nu' \) with \( \nu h' = 1_{F'_\lambda Z} \). By Lemma [13], \( F'_\lambda = F'_\rho \) and \( F'_\rho h = F'_\rho Z \). Then there is a lifting \( \bar{h} : N' \to F'_\lambda Z \) with \( \bar{h} g = h' \). Hence \( \text{Hom}_B(N', F'_\lambda Z) \overset{\sim}{\to} \text{Hom}_A(F'_\lambda N', Z) \), \( \bar{h} \mapsto \bar{h}' \) with \( \bar{h}' F'_\lambda g = h \).

We show that \( F_\lambda \) preserves Auslander-Reiten sequences. Let \( X \) be an indecomposable \( A \)-module of the form \( X = F'_\lambda N \) for an indecomposable \( \tilde{B} \)-module \( N \). Then \( F_\lambda X = F_\lambda F'_\lambda N = \tilde{F}_\lambda N \). Since by [12], \( \tilde{F}_\lambda \) preserves indecomposable modules, then \( F_\lambda X \) is indecomposable. Finally, as above, we conclude that \( F_\lambda \) preserves Auslander-Reiten sequences.\[ \square \]
4.2 Tame representation case

Let $k$ be an algebraically closed field. We recall that $A$ is said to be of tame representation type if for each dimension $d \in \mathbb{N}$ and each object $a \in A_0$, there are finitely many $A - k[t]$-bimodules $M_1, \ldots, M_s$ which satisfy:

(a) $M_i$ is finitely generated free as right $k[t]$-module $i = 1, \ldots, s$;
(b) each indecomposable $X \in A\text{mod}$ with $X(a) \neq 0$ and $\dim_k X = d$ is isomorphic to some module of the form $M_i \otimes_{k[t]} (k[t]/(t - \lambda))$ for some $i \in \{1, \ldots, s\}$ and $\lambda \in k$.

In fact, it is shown in [13] that $A$ is tame if (a) and (b) are substituted by the weaker conditions:

(a') $M_i$ is finitely generated as right $k[t]$-module $i = 1, \ldots, s$;
(b') each indecomposable $X \in A\text{mod}$ with $X(a) \neq 0$ and $\dim_k X = d$ is a direct summand of a module of the form $M_i \otimes_{k[t]} (k[t]/(t - \lambda))$ for some $i \in \{1, \ldots, s\}$ and $\lambda \in k$.

The following statement covers claim (b) of Theorem 0.2 in the Introduction.

THEOREM 4.2 Let $F: A \to B$ be a balanced covering functor. If $B$ is tame, then $A$ is tame.

Proof. Let $a \in A_0$ and $d \in \mathbb{N}$. Let $M_1, \ldots, M_s$ be the $B - k[t]$-bimodules satisfying (a) and (b): each indecomposable $M \in B\text{mod}$ with $M(Fa) \neq 0$ and $\dim_k M \leq d$ is isomorphic to some $M_i \otimes_{k[t]} (k[t]/(t - \lambda))$ for some $i \in \{1, \ldots, s\}$ and $\lambda \in k$.

By Corollary 3.2 each indecomposable $X \in A\text{mod}$ with $X(a) \neq 0$ and $\dim_k X = d$ is a direct summand of some $F_* (M_i \otimes_{k[t]} (k[t]/(t - \lambda)))$, which is isomorphic to $F_* M_i \otimes_{k[t]} (k[t]/(t-\lambda))$, for some $i \in \{1, \ldots, s\}$ and $\lambda \in k$. Hence $A$ satisfies conditions (a') and (b'). □

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