A Markovian and Roe-algebraic approach to asymptotic expansion in measure

Kang Li¹ · Federico Vigolo² · Jiawen Zhang³

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Abstract
In this paper, we conduct further studies on geometric and analytic properties of asymptotic expansion in measure. More precisely, we develop a machinery of Markov expansion and obtain an associated structure theorem for asymptotically expanding actions. Based on this, we establish an analytic characterisation for asymptotic expansion in terms of the Drutu–Nowak projection and the Roe algebra of the associated warped cones. As an application, we provide new counterexamples to the coarse Baum–Connes conjecture.

Keywords Asymptotic expansion in measure · Coarse Baum–Connes conjecture · Markov expansion · Spectral gap · Strong ergodicity · Warped cones

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Jiawen Zhang
jiawenzhang@fudan.edu.cn

Kang Li
kang.li@fau.de

Federico Vigolo
federico.vigolo@uni-goettingen.de

¹ Department of Mathematics, Friedrich-Alexander-Universität Erlangen-Nürnberg, Cauerstraße 11, 91058 Erlangen, Germany
² Mathematical Institute, Georg-August-Universität Göttingen, Bunsenstraße 2-3, 37073 Göttingen, Germany
³ School of Mathematical Sciences, Fudan University, 220 Handan Road, Shanghai 200433, China

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1 Introduction

This paper is the second part of a broader study of the notion of asymptotic expansion in measure for measurable actions of countable groups on probability spaces. We introduced this notion in [28], as a dynamical analogue of a previously defined notion of asymptotic expansion for metric spaces [27].

Asymptotic expansion in measure is a weakening of expansion in measure as defined in [57] and it turns out that—for measure-class-preserving actions—it is also equivalent to the classical notion of strong ergodicity introduced by Schmidt [50] and Connes and Weiss [9] (see [28] for more details).

More precisely, a measurable action $\rho : \Gamma \curvearrowright (X, \nu)$ of a countable group on a probability space is asymptotically expanding in measure if for each $\alpha \in (0, \frac{1}{2}]$ there exist $c_\alpha > 0$ and a finite symmetric set $S_\alpha \subseteq \Gamma$ such that for every measurable subset $A \subseteq X$ with $\alpha \leq \nu(A) \leq \frac{1}{2}$ we have

$$\nu\left(\bigcup_{s \in S_\alpha} s \cdot A\right) > (1 + c_\alpha)\nu(A). \quad (1.1)$$

The action $\rho$ is called expanding in measure if we let $c_\alpha \equiv c$ and $S_\alpha \equiv S$ for some $c > 0$ and a finite subset $S$ in $\Gamma$.

In [28] we studied the general structure theory of asymptotically expanding actions. Most notably, we showed that an action is asymptotically expanding in measure if and only if it admits an exhaustion by domains of expansion (see Sect. 2.5 for a more detailed account). This fact allowed us to reprove a few recent—and–old results for strongly ergodic actions and it is also a key technical tool for the present paper. In addition, we also made explicit the connection between the notion of asymptotic expansion for measurable actions and that of asymptotic expansion for metric spaces. This allowed us to provide a rich source of concrete examples of asymptotic expander graphs (see [28] for details).

In this paper, we will further study the notion of asymptotic expansion in measure in the context of measure-class-preserving actions (in particular, all the results here described hold for strongly ergodic actions). Adapting the techniques developed in [23] to the dynamical setting, we are able to prove some rather striking analytic and geometric properties of asymptotic expansion in measure. More precisely, we obtain an analytic characterisation of asymptotic expansion in measure in terms of quasi-locality of averaging projections and Roe algebras of the associated warped cones. As a consequence, we will provide a new source of counterexamples to the coarse Baum–Connes conjecture, which is a central problem in higher index theory (see, e.g., [40, 61]).

To obtain the above results, we develop a spectral characterisation of (asymptotic) expansion in measure which we find of independent interest. This is obtained by associating measure-class-preserving actions with some reversible Markov kernels and by studying the resulting Laplacians and averaging operators. The spectral characterisation is obtained by extending some classical results for Markov processes on finite state-spaces to general Markov kernels. We find that this theory provides
a solid framework to study spectral properties of actions that are not necessarily measure-preserving.

1.1 Spectral gaps and Markov expansion

A probability measure-preserving action $\rho : \Gamma \curvearrowright (X, \nu)$ always induces a unitary representation $\pi : \Gamma \curvearrowright L^2(X, \nu)$. If $\Gamma$ is generated by a finite symmetric subset $S$, the action $\rho$ has a spectral gap if there exists some positive constant $\kappa > 0$ such that every $f \in L^2(X, \nu)$ with $\int_X f \, d\nu = 0$ satisfies

$$\|f\|_2 \leq \kappa \sum_{s \in S} \|\pi(s)f - f\|_2.$$  \hspace{1cm} (1.2)

This can be seen as an extremely strong version of ergodicity, and it is not very hard to show that $\rho$ has a spectral gap if and only if it is expanding in measure (this was shown more or less independently in [6, 15, 19, 55], and was already implicit in earlier works of K. Schmidt and Connes–Feldmann–Weiss).

With the action $\rho$ is associated a Markov operator $\mathcal{P} \in \mathcal{B}(L^2(X, \nu))$ defined by $\mathcal{P} := \frac{1}{|S|} \sum_{s \in S} \pi(s)$ and a Laplacian $\Delta := 1 - \mathcal{P} \in \mathcal{B}(L^2(X, \nu))$. These operators are self-adjoint, and $\rho$ has a spectral gap if and only if $0$ is a simple (i.e., with multiplicity one) isolated point in the spectrum of $\Delta$ (equivalently, $1$ is a simple isolated point in the spectrum of $\mathcal{P}$). This characterisation in terms of self-adjoint operators is crucial to provide explicit examples of actions with spectral gap, as it opens a door to algebraic and representation theoretical tools. In fact, this point of view leads to very deep connections between dynamical systems, analysis and number theory. These connections make the study of the spectral gap property for measure-preserving actions into a very active and important field of research ([4–6, 14, 29, 31]).

As an intermediate step toward an analytic study of asymptotic expansion in measure, we set a framework to extend the above connections to the setting of measure-class-preserving actions. It follows from the work of Houdayer–Marrakchi–Verraedt [19, Theorem 3.2] that expansion in measure is equivalent to (1.2), whenever $\rho(s)$ has bounded Radon–Nikodym derivative for every $s \in S$. In turn, (1.2) holds if and only if $0$ is a simple isolated point in the spectrum of the self-adjoint operator $T := \sum_{s \in S} |1 - \pi(s)|$. However, the spectrum of the operator $T$ remains difficult to control. It is therefore desirable to produce some spectral condition which can more adequately describe the notion of expansion.

In this paper, we provide a rather satisfactory answer to the above need using Markov kernels and Markov expansion. Our approach is based on a shift in paradigm, and can be justified by some analogies between finite graphs and dynamical systems. A systematic study of these analogies by means of the approximation procedure can be found in [55] and further developed in [28]. According to this procedure, expansion in measure corresponds to vertex-expansion for finite graphs [55] and, if the action is measure-preserving, the spectral gap condition (1.2) can be seen as an analogue of spectral expansion for graphs (see also [46]). It is a classical result that spectral-expansion is equivalent to edge-expansion ([2, 3, 10]), and it is easy to verify that the latter
is equivalent to vertex-expansion. This can be seen as the graph-theoretic analogue of the equivalence between (1.2) and expansion in measure for measure-preserving actions.

To be more precise, spectral expansion for a regular finite graph $G$ is defined in terms of the spectral gap of the discrete Laplacian $\Delta \in \mathcal{B}(L^2(G, \nu))$. Here $\nu$ is the counting measure on the set of vertices of $G$ and $\Delta$ is defined as $1 - P$, where $P$ is the averaging operator (a.k.a. Markov operator) defined by $Pf(v) := \sum_{v \sim w} f(w)/\text{degree}(v)$ for $f \in L^2(G, \nu)$. Importantly, if the graph $G$ is not regular then the discrete Laplacian is no longer self-adjoint in $\mathcal{B}(L^2(G, \nu))$, Here $\nu$ is the counting measure on the set of vertices of $G$ and $\Delta$ is defined as $1 - P$, where $P$ is the averaging operator (a.k.a. Markov operator) defined by $Pf(v) := \sum_{v \sim w} f(w)/\text{degree}(v)$ for $f \in L^2(G, \nu)$. Instead, it is self-adjoint in $\mathcal{B}(L^2(G, \tilde{\nu}))$, where $\tilde{\nu}$ is a different measure which takes into account the degree of each vertex. Spectral expansion is then defined in terms of the spectrum of $\Delta$ seen as an operator on $L^2(G, \tilde{\nu})$. A more sophisticated way of rephrasing this is that the (lazy) simple random walk on a finite connected graph $G$ has a unique stationary probability measure $\tilde{\nu}$. The probability distribution of the $n$-th step of such a random walk converges exponentially fast to $\tilde{\nu}$ (in the $L^2$-norm), and the spectral expansion measures the rate of exponential convergence.

The above discussion can be used as heuristics in the dynamical setting. We remark that graphs corresponding to a measure-preserving action are “regular on a large scale”. It is therefore natural to expect a correspondence between spectral expansion and expansion in measure. On the other hand, actions that are not measure-preserving correspond to irregular graphs (the “large scale degrees” are governed by the Radon–Nikodym derivatives). This suggests us to search for a spectral characterisation of expansion in measure in terms of some operator in $\mathcal{B}(L^2(X, \tilde{\nu}))$—where $\tilde{\nu}$ is some stationary measure depending on the Radon–Nikodym derivatives. This is precisely the approach that we take in this paper.

Let $\Gamma$ be a finitely generated group and $S \subseteq \Gamma$ a finite symmetric generating set containing the identity element, and let $\rho : \Gamma \curvearrowright (X, \nu)$ be a measure-class-preserving action with the Radon–Nikodym derivatives $r(\gamma, x) := \frac{d\nu_{\gamma^{-1}}}{d\nu}(x)$. It turns out that the measure $\tilde{\nu}$ defined by

$$
d\tilde{\nu}(x) := \sum_{s \in S} r(s, x) \frac{1}{2} d\nu(x)
$$

is a stationary measure for the reversible Markov kernel

$$
\Pi(x, -) := \frac{1}{\sum_{s \in S} r(s, x)^{\frac{1}{2}}} \sum_{s \in S} r(s, x)^{\frac{1}{2}} \delta_{s \cdot x}.
$$

Naturally associated to $\Pi$, there are a Markov operator $P$ and a Laplacian $\Delta = 1 - P$. Both of these are self-adjoint operators in $\mathcal{B}(L^2(X, \tilde{\nu}))$—we defer to Sect. 3 for

1 Assuming that the graphs have uniformly bounded degree.

2 To give a somewhat precise meaning to the notion of “regular on a large scale” it is necessary to use the terminology of [28, 55]: given any measurable subset $A \subseteq X$ and a sufficiently fine approximation $[A]$, the ratio $\frac{\nu([A])}{|A|}$ will be roughly equal to $\nu(S \cdot A)/\nu(A)$. If $\rho$ is measure-preserving and $A$ is disjoint from $s \cdot A$ for every $s \in S$, then the latter ratio is equal to $|S|$. That is, the approximating graphs are “$|S|$-regular on a large scale.”
preliminaries and definitions regarding Markov kernels. Every measurable subset $A \subseteq X$ has a natural notion of “measure of the boundary” $|\partial_{\Pi}(A)| \in \mathbb{R}_{\geq 0}$ (Definition 3.2 or [21]), and we say that $\rho$ is Markov expanding if there is a $c > 0$ such that

$$|\partial_{\Pi}(A)| > c \tilde{v}(A)$$

for every $A \subseteq X$ with $0 < \tilde{v}(A) \leq \frac{1}{2} \tilde{v}(X)$. This should be thought of as a dynamical analogue of edge-expansion for graphs. Importantly, the equivalence between edge-expansion and spectral expansion can be extended from the context of random walks on graphs to that of general reversible Markov kernels:

**Theorem A** ([24, Theorem 2.1], see also the appendix to this paper) *Let $\Pi$ be a reversible Markov kernel on $X$ with finite reversing measure $m$. Let $\lambda_2$ be the infimum of the spectrum of the restriction of $\Delta$ to the space of functions with zero average, and let $\kappa := \inf |\partial_{\Pi}(A)| / m(A)$ for $A \subseteq X$ with $0 < m(A) \leq \frac{1}{2} m(X)$. Then

$$\frac{\kappa^2}{2} \leq 1 - \lambda_2 \leq 2\kappa.$$*

As a consequence, we obtain a characterisation for Markov expansion in terms of the spectrum of $\Delta \in \mathcal{B}(L^2(X, \tilde{\nu}))$. Furthermore, it is relatively easy to show that, when the Radon–Nikodym derivatives are bounded, Markov expansion is equivalent to the original notion of expansion in measure (this is analogous to the equivalence between edge-expansion and vertex-expansion for graphs of uniformly bounded degrees). This leads us to the following:

**Proposition B** (Corollary 3.16, Remark 3.17) *Let $\Theta \geq 1$ be a constant. A measure-class-preserving action $\rho: \Gamma \curvearrowright (X, \nu)$ with $1/\Theta \leq r(s, x) \leq \Theta$ for every $s \in S$ and $x \in X$ is expanding in measure if and only if $0$ is a simple isolated point in the spectrum of $\Delta \in \mathcal{B}(L^2(X, \tilde{\nu}))$.*

**Remark 1.1** It is not hard to show that Proposition B and [19, Theorem 3.2] are in fact equivalent. However, we find that our approach has various advantages:

1. We find that the Laplacian operator $\Delta$ is more natural than $T$. It should be easier to handle (e.g., to control spectral gap), and it allows us to borrow several calculations and results from the classical setting of random walks on graphs.
2. The spectral gap condition can be rephrased by saying that the restriction of the Markov operator $\mathcal{P}$ to the space functions with zero-average has operator norm strictly less than 1. It can be useful to know that $\mathcal{P}^n$ converges in the operator norm to the projection onto constant functions (see also Sect. 4.3).
3. It allows for a finer control of the expansion constants.
4. The spectral characterisation of Markov expansion holds true also for actions with unbounded Radon–Nikodym derivatives (this should be of independent interest).

We restricted the previous discussion to the case of actions of finitely generated groups for the sake of simplicity. However, the machinery of Markov kernels is very flexible, and all the results mentioned above will actually be proved for actions of
arbitrary discrete countable groups. Furthermore, we will also study restrictions of actions to subsets of $X$ which are not necessarily invariant.\(^3\) As a sample application, we note that Proposition B implies the following (see Sect. 2 for the relevant definitions and Corollary 3.16):

**Corollary C** [15] A measure-preserving action $\Gamma \curvearrowright (X, \nu)$ has local spectral gap with respect to $Y \subseteq X$ if and only if $Y$ is a domain of expansion.

Being able to work with subsets of $X$ is a necessary requirement to use the structure theorems established in [28], which characterise asymptotic expansion in terms of exhaustions (see Sect. 2.5). Combining those results with the Markov machinery developed above, we are able to prove an additional structure result which will play a key role in the rest of the paper:

**Theorem D** (Theorem 3.20) A measure-class-preserving action $\Gamma \curvearrowright (X, \nu)$ on a probability space is asymptotically expanding in measure if and only if every subset $Y \subseteq X$ admits an exhaustion by domains of Markov expansion.

**Remark 1.2** The above theorem remains true when replacing “probability” by “$\sigma$-finite” and “asymptotically expanding in measure” by “strongly ergodic”.

### 1.2 Warped cones and finite propagation approximations

Our next aim is to study asymptotically expanding actions via analytic properties of certain projection operators. This is done by using the warped cone construction as a bridge between the metric and dynamical setting, and then utilizing Markov expansion. The end result is a dynamical analogue of the theory developed in [23] to characterise asymptotic expanders using averaging projections.

The notion of warped cone was firstly introduced by Roe [42] to explore more examples with/without Yu’s property A and coarse embeddings into Hilbert spaces. The geometry of warped cones was subsequently studied by a number of people, e.g., [11, 13, 37, 44–47, 55, 56, 58]. Roughly speaking, given a continuous action $\Gamma \curvearrowright (X, d)$ on a compact metric space with diameter at most 2, the associated unified warped cone is the metric space $(\mathcal{O}_\Gamma X, d_{\Gamma})$, where $\mathcal{O}_\Gamma X = X \times [1, \infty)$ as a set and $d_{\Gamma}$ is a metric on $\mathcal{O}_\Gamma X$ defined in terms of the group action (see Sect. 4.1 for details).

Given a probability measure $\nu$ on $(X, d)$, we consider the averaging projection $P_X$ on $L^2(X, \nu)$, which is the rank-one orthogonal projection onto the space of constant functions on $X$. Denoting by $\lambda$ the Lebesgue measure on $[1, \infty)$, the Drutu–Nowak projection is defined as $\mathcal{G} = P_X \otimes \text{Id}_{L^2([1, \infty))} \in \mathcal{B}(L^2(\mathcal{O}_\Gamma X, \nu \times \lambda))$, which is the orthogonal projection onto $\mathbb{C} \otimes L^2([1, \infty), \lambda)$.

The Drutu–Nowak projection $\mathcal{G}$ was first introduced in [11, Section 6.e.] in their study on the coarse Baum–Connes conjecture (more details will be provided later). They showed that if an action is measure-preserving and has a spectral gap, then the projection $\mathcal{G}$ is a norm limit of finite propagation operators in $\mathcal{B}(L^2(\mathcal{O}_\Gamma X, \nu \times \lambda))$. Recall that an operator $T \in \mathcal{B}(L^2(\mathcal{O}_\Gamma X, \nu \times \lambda))$ has finite propagation if there exists

\[^3\] It would be also possible to extend this theory to include general countable measurable equivalence relations.
$R > 0$ such that for any $f, g \in C_0(\mathcal{O}_\Gamma X)$ with $d_\Gamma(\text{supp}(f), \text{supp}(g)) > R$ we have $f^T g = 0$, where $f$ and $g$ are regarded as diagonal operators on $L^2(\mathcal{O}_\Gamma X, \nu \times \lambda)$ via the multiplication representation.

In this paper, we study the converse of Druţu–Nowak’s result and prove the following analytic characterisation for asymptotically expanding actions:

**Theorem E** (Theorems 4.8 and 4.16) Let $(X, d)$ be a metric space with diameter at most $2$ equipped with a Radon probability measure $\nu$, and $\rho : \Gamma \curvearrowright X$ be a continuous measure-class-preserving action. The following are equivalent:

1. $\rho$ is asymptotically expanding;
2. the Druţu–Nowak projection $\mathcal{G}$ is quasi-local;
3. the Druţu–Nowak projection $\mathcal{G}$ is a norm limit of operators with finite propagation.

**Remark 1.3** The notion of quasi-locality was introduced by Roe [41]. It is weaker than the property of admitting an approximation by finite propagation operators, and it is relatively easy to verify. For more details on quasi-locality, we refer readers to [12, 26, 27, 53, 54].

Theorem E is a dynamical analogue of [23, Theorem 6.1], and there are two main ingredients in its proof. Firstly, we introduce a dynamical notion of finite propagation approximation and quasi-locality (see Sects. 4.2 and 4.4) as an intermediate bridge to connect asymptotic expansion and analytic properties of the Druţu–Nowak projection. Secondly, we apply the tool of Markov expansion to approximate dynamical quasi-local operators with finite dynamical propagation ones. Due to some correspondence results (Propositions 4.7 and 4.15), we can then pass from the dynamical notions to their analytic analogues for unified warped cones and obtain Theorem E.

As a byproduct, we construct numerous projections which can be approximated by operators with finite propagation (see Corollary 4.17). These projections will be important in the next section, where we deal with the coarse Baum–Connes conjecture.

### 1.3 Roe algebras and the coarse Baum–Connes conjecture

Roe algebras are $C^*$-algebras associated with metric spaces. These $C^*$-algebras encode coarse geometric information of the metric spaces and play key roles in higher index theory (see, e.g., [40, 41, 61] for more details). We conclude this paper by studying Roe algebras of warped cones associated to asymptotically expanding actions and provide an application to the so-called coarse Baum–Connes conjecture.

Given a continuous action $\Gamma \curvearrowright (X, d)$ on a compact metric space with diameter at most $2$ and a non-atomic probability measure $\nu$ on $(X, d)$ with full support, we consider the multiplication representation $C_0(\mathcal{O}_\Gamma X) \to \mathbf{B}(L^2(\mathcal{O}_\Gamma X, \nu \times \lambda))$. The **Roe algebra** of the unified warped cone, denoted by $C^*(\mathcal{O}_\Gamma X)$, is the norm closure of all finite propagation locally compact operators in $\mathbf{B}(L^2(\mathcal{O}_\Gamma X, \nu \times \lambda))$ (see Sect. 5.1 for more details).

Although the Druţu–Nowak projection $\mathcal{G}$ can be approximated by finite propagation operators, it is **not** locally compact because its restriction on $L^2([1, \infty), \lambda)$

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4. It is conjectured that quasi-locality should be strictly weaker than admitting such approximations.
is the identity operator. To obtain non-trivial projections in the Roe algebra, Sawicki [45] suggested to consider the integral warped cone and the associated integral Druţu–Nowak projection (see Sect. 5.1). Since the integral warped cone is coarsely equivalent to the original warped cone, they have *-isomorphic Roe algebras. Based on [11], Sawicki [45, Proposition 1.3] showed that for a measure-preserving action with spectral gap, the integral Druţu–Nowak projection belongs to the associated Roe algebra.

Theorem E allows us to both extend and provide a converse to Sawicki’s result:

**Theorem F** (Theorem 5.2, Corollary 5.6) Let $(X, d)$ be a compact metric space with diameter at most 2, $\nu$ a non-atomic Radon probability measure on $X$ of full support, and $\rho : \Gamma \curvearrowright (X, d, \nu)$ a continuous measure-class-preserving action. Then $\rho$ is asymptotically expanding if and only if the integral Druţu–Nowak projection belongs to the Roe algebra $C^*(\mathcal{O}_\Gamma X)$. Moreover, the integral Druţu–Nowak projection is non-compact and ghost.

The study of projections in Roe algebras is motivated by the computation of their K-theories. The coarse Baum–Connes conjecture asserts that K-theories of Roe algebras can be computed in terms of homology information of underlying metric spaces. When true, this establishes a connection between geometry, topology and analysis. One ground-breaking result on the subject is due to Yu [63], as he showed that the coarse Baum–Connes conjecture holds for all metric spaces with bounded geometry that are coarsely embeddable into Hilbert spaces. On the other hand, counterexamples to the conjecture were subsequently discovered by Higson [16] (see also [18]) using expander graphs. In a recent joint work with Khukhro, we found more counterexamples using asymptotic expanders [23].

Understanding which spaces satisfy the coarse Baum–Connes conjecture is still one of the major questions in higher index theory, as it has significant applications to other areas of mathematics, such as topology and geometry (see [17, 48, 52, 62] for more details).

It is an open question whether warped cones arising from actions with spectral gap are counterexamples to the coarse Baum–Connes conjecture. This question was the motivation behind the introduction of the Druţu–Nowak projection in [11]. Recently, Sawicki [45, Theorem 3.5] proved that sparse warped cones (see Sect. 5.2) do provide counterexamples to the coarse Baum–Connes conjecture. His proof follows a similar outline of Higson’s original proof for expander graphs. Using our work on asymptotically expanding actions, we can generalise Sawicki’s result as follows:

**Theorem G** (Corollary 5.15) Let $(X, d)$ be a compact metric space of diameter at most 2 equipped with a non-atomic probability measure $\nu$ of full support, and $\rho : \Gamma \curvearrowright (X, d, \nu)$ be a free Lipschitz measure-class-preserving asymptotically expanding action. Under either of the following conditions:

1. if $\Gamma$ has property A and $X$ is a manifold;
2. if the asymptotic dimension of $\Gamma$ is finite and $X$ is an ultrametric space;

the coarse Baum–Connes conjecture for the sparse warped cone fails.
Remark 1.4 We can produce examples whose violation of the coarse Baum–Connes conjecture can be deduced from Theorem G, but not from any previously known results (see Example 5.16).

Under some extra conditions (ONL and bounded geometry), it follows by combining Theorem G with Yu’s result [63] that warped cones arising from asymptotically expanding actions cannot coarsely embed into Hilbert spaces. Our last result shows that these extra conditions are in fact unnecessary (this partially generalises [37, Theorem 3.1]):

Proposition H (Proposition 5.18) Let \((X, d)\) be a compact metric space of diameter at most 2 equipped with a non-atomic probability measure \(\nu\), and \(\rho : \Gamma \curvearrowright (X, d, \nu)\) be a continuous measure-class-preserving and asymptotically expanding action. Then the warped cone \(O_{\Gamma} X\) does not admit a coarse embedding into any Hilbert space.

1.4 Structure of the paper

Section 2 covers some preliminaries and further illustrates the connections between this paper and other works. The first half of Sect. 3 can be read independently from the rest of the paper and is devoted to introducing reversible Markov kernels/expansion and the statement of Theorem A. A self-contained proof of Theorem A is given in the appendix. The second part of Sect. 3 connects this theory to the study of measure-class-preserving actions. Here we prove Proposition B, Corollary C and Theorem D. These results will be important to both of the following sections. In Sect. 4 we recall the warped cone construction and study asymptotic expansion from the point of view of warped cones. Here we prove Theorem E. Section 5 is mostly devoted to the study of Roe algebras of warped cones. In the first part, we prove Theorem F, and in the second part we provide new counterexamples to the coarse Baum–Connes conjecture by proving Theorem G. Finally, we conclude this section by proving Proposition H.

2 Preliminaries

2.1 Standing conventions

Throughout the paper, \(\Gamma\) will always be a countable discrete group. The group \(\Gamma\) will be made into a metric space by fixing a proper length function (see below). The letter \(S\) will always denote a finite subset in \(\Gamma\). Such a set will often—but not always—be symmetric (i.e., \(\gamma \in S\) implies that \(\gamma^{-1} \in S\)) and containing the identity element \(1 \in \Gamma\). We will not generally assume that \(S\) generates \(\Gamma\).

All the measure spaces will be \(\sigma\)-finite and all the actions will be measurable. More precisely, we say that \(\Gamma \curvearrowright (X, \nu)\) is an action as shorthand for saying that \(\Gamma\) is a countable discrete group acting measurably on a \(\sigma\)-finite measure space \((X, \nu)\). When we equip a metric space \((X, d)\) with a measure \(\nu\), we will always assume that \(\nu\) is defined on the Borel \(\sigma\)-algebra.
2.2 Actions on measure spaces

Let \((X, \nu)\) be a measure space. A measurable subset \(A \subseteq X\) of positive finite measure is called a domain. An exhaustion of \((X, \nu)\) is a sequence of nested measurable subsets \(Y_1 \subseteq Y_2 \subseteq \cdots\) such that \(\bigcup_{n \in \mathbb{N}} Y_n = X\) up to measure zero. We denote exhaustions by \(Y_n \nearrow (X, \nu)\), or simply \(Y_n \nearrow X\) if the measure is clear from the context.

A proper length function on \(\Gamma\) is a function \(\ell : \Gamma \to \{0\} \cup \mathbb{N}\) which satisfies the following:

- \(\ell(\gamma) = 0\) if and only if \(\gamma = 1\) (the identity element in \(\Gamma\));
- \(\ell(\gamma) = \ell(\gamma^{-1})\) for every \(\gamma \in \Gamma\);
- \(\ell(\gamma_1 \gamma_2) \leq \ell(\gamma_1) + \ell(\gamma_2)\) for every \(\gamma_1, \gamma_2 \in \Gamma\);
- the number of \(\gamma \in \Gamma\) with \(\ell(\gamma) \leq k\) is finite for every \(k \in \mathbb{N}\).

It is easy to show that every countable discrete group \(\Gamma\) admits a proper length function (see e.g. [36, Proposition 1.2.2]). For example, if \(\Gamma\) is a finitely generated group then we can simply take the word length with respect to an arbitrary finite symmetric generating set. Any proper length function \(\ell\) induces a left-invariant metric \(d_\ell\) on \(\Gamma\) by \(d_\ell(\gamma_1, \gamma_2) := \ell(\gamma_1^{-1} \gamma_2)\). This makes \(\Gamma\) into a proper discrete metric space.

Choosing a different length function \(\ell'\) will yield a coarsely equivalent metric on \(\Gamma\) (we will not need this fact).

For each \(k \in \mathbb{N}\), we denote by \(B_k\) the closed ball in \((\Gamma, \ell)\) with radius \(k\) and centred at the identity:

\[
B_k := \{\gamma \in \Gamma | \ell(\gamma) \leq k\}.
\]

It follows from the definition of length function that each \(B_k\) is finite and symmetric, \(1 \in B_k\) and \(B_k \cdot B_l \subseteq B_{k+l}\) for every \(k, l \in \mathbb{N}\).

We will be concerned with actions of \(\Gamma\) on \((X, \nu)\). Given \(A \subseteq X\) and \(K \subseteq \Gamma\), let

\[
K \cdot A := \bigcup_{\gamma \in K} \gamma \cdot A.
\]

Since \(1 \in B_k\), we note that \(A \subseteq B_k \cdot A\) for every \(A \subseteq X\) and every \(k \in \mathbb{N}\).

Recall that an action \(\Gamma \curvearrowright (X, \nu)\) is measure-class-preserving if it sends measure-zero sets to measure-zero sets. In this case, for every \(\gamma \in \Gamma\) there is an associated Radon–Nikodym derivative \(d\gamma_*^{-1} \nu/d\nu\) that is well-defined up to measure-zero sets.

2.3 Expansion in measure

Let \(\Gamma \curvearrowright X\) be an action and \(S \subseteq \Gamma\) a finite symmetric set. For any measurable subset \(A \subseteq X\) we denote \(\partial^\Gamma_S A := S \cdot A \setminus A\), which should be regarded as the “boundary of \(A\) with respect to the action by \(S\)”.

**Definition 2.1** [57] An action \(\rho : \Gamma \curvearrowright (X, \nu)\) on a probability measure space \((X, \nu)\) is called expanding (in measure) if there exist a constant \(c > 0\) and a finite \(S \subseteq \Gamma\) such...
that for any measurable subset $A \subseteq X$ with $0 < \nu(A) \leq \frac{1}{2}$, we have $\nu(\partial_S A) > c\nu(A)$. In this case, we say that $\rho$ is $(c, S)$-expanding or simply $S$-expanding.

If an action is $(c, B_k)$-expanding for some $k \in \mathbb{N}$, we may also say that it is $(c, k)$-expanding. Note that every expanding action is $(c, k)$-expanding for some $c > 0$ and $k \in \mathbb{N}$.

In an independent work, Grabowski–Máthé–Pikhurko defined a “local” version of expansion under the name of domain of expansion:

**Definition 2.2** [15] Let $\rho : \Gamma \curvearrowright (X, \nu)$ be an action. A domain $Y \subseteq X$ is called a domain of expansion for $\rho$ if there exist a constant $c > 0$ and a finite $S \subseteq \Gamma$ such that for every measurable subset $A \subseteq Y$ with $0 < \nu(A) \leq \frac{\nu(Y)}{2}$, we have

$$\nu((S \cdot A) \cap Y) > (1 + c)\nu(A).$$

In this case, we say that $Y$ is a domain of $(c, S)$-expansion or simply of $S$-expansion.

As before, if $S = B_k$ we may say that $Y \subseteq X$ is a domain of $(c, k)$-expansion.

We note that when $\nu$ is finite, $\rho : \Gamma \curvearrowright (X, \nu)$ is expanding if and only if $X$ is a domain of expansion for $\rho$. We end this subsection by recalling the following elementary fact, which will be used in the proof of Proposition 3.18:

**Lemma 2.3** [28, Lemma 3.14] Let $\rho : \Gamma \curvearrowright (X, \nu)$ be an action and $Y \subseteq X$ a domain. Assume that $Y_1, Y_2 \subseteq Y$ are domains of $S$-expansion. If $\nu(Y_1) > \frac{3}{4}\nu(Y)$ and $\nu(Y_2) > \frac{3}{4}\nu(Y)$ then the union $Y_1 \cup Y_2$ is a domain of $S$-expansion as well.

### 2.4 (Local) spectral gap

We will work with complex $L^p$-spaces for $p \in [1, \infty)$. If we wish to stress that the $L^p$-norm of a function on $X$ is computed with respect to the measure $\nu$, we denote it by $\|f\|_{\nu, p}$. Similarly, we will denote the inner product on the Hilbert space $L^2(X, \nu)$ by $\langle f, g \rangle_\nu$.

Given a measurable subset $Y$ in a measure space $(X, \nu)$, we denote the restriction of $\nu$ to $Y$ by $\nu|_Y$. With a slight abuse of notation, we also use the symbol $\nu|_Y$ to denote the measure on $X$ which gives measure 0 to $X \setminus Y$ and coincides with $\nu$ on all measurable subsets of $Y$ (i.e., $\nu|_Y = \chi_Y \cdot \nu$ where $\chi_Y$ is the indicator function of $Y$). This will not cause confusion, as the meaning will be clear from the context.

A measure-preserving action $\rho : \Gamma \curvearrowright (X, \nu)$ on a probability measure space $(X, \nu)$ has a spectral gap if there exist a constant $\kappa > 0$ and a finite $S \subseteq \Gamma$ such that for every function $f \in L^2(X, \nu)$ with $\int_X f \, d\nu = 0$ we have

$$\|f\|_2 \leq \kappa \sum_{\gamma \in S} \|\gamma \cdot f - f\|_2, \quad (2.1)$$

---

[5] The authors of [15] only consider measure-preserving actions, but their definition makes sense for general measurable actions as well.
where $\gamma \cdot f(x) := f(\gamma^{-1} \cdot x)$. It can be shown that the action $\rho$ is expanding in measure if and only if it has a spectral gap (see, e.g., [57, Section 7]).

In [6], Boutonnet–Ioana–Golsefidy introduced the following localised version of spectral gap:

**Definition 2.4** [6, Definition 1.2] Let $\rho : \Gamma \curvearrowright (X, \nu)$ be a measure-preserving action and $Y \subseteq X$ be a domain. The action $\rho$ has **local spectral gap** with respect to $Y$ if there exist a constant $\kappa > 0$ and a finite $S \subseteq \Gamma$ such that

$$\|f\|_{Y,2} \leq \kappa \sum_{\gamma \in S} \|\gamma \cdot f - f\|_{Y,2}$$

for every $f \in L^2(X, \nu)$ with $\int_Y f \, d\nu = 0$.

It is clear that when $\nu$ is a probability measure, $\rho$ has spectral gap if and only if it has local spectral gap with respect to the whole $X$.

It is shown in [15, Lemma 5.2] that a measure-preserving action $\rho : \Gamma \curvearrowright (X, \nu)$ has local spectral gap with respect to a domain $Y \subseteq X$ if and only if $Y$ is a domain of expansion for $\rho$. This fact can also be deduced from [19, Theorem 3.2] (or by adapting the arguments of [57, Section 7]). Later on, we will provide an alternative proof based on our study of Markov kernels (see Corollary 3.16).

**Remark 2.5** Equations (2.1) and (2.2) make sense also if the action is not measure-preserving (although in this case it would be perhaps more appropriate to refer to them as Poincaré inequalities, rather than spectral gaps). It follows from [19, Theorem 3.2] that—as long as the Radon–Nikodym derivatives are bounded—the characterisation of (domains of) expansion in measure in terms of (local) spectral gaps also holds for actions that do not necessarily preserve measures.

### 2.5 Asymptotic expansion in measure and structure theorems

The following weakening of expansion in measure was defined in [28] in analogy with [23, 27]:

**Definition 2.6** [28, Definition 3.1] Let $\rho : \Gamma \curvearrowright (X, \nu)$ be an action on a space $(X, \nu)$ of finite measure. The action $\rho$ is called **asymptotically expanding (in measure)** if there exist functions $c : (0, \frac{1}{2}] \to \mathbb{R}_{>0}$ and $k : (0, \frac{1}{2}] \to \mathbb{N}$ such that for every $\alpha \in (0, \frac{1}{2}]$ we have

$$\nu\left(B_{\frac{1}{k}(\alpha)} \cdot A\right) > (1 + c(\alpha))\nu(A)$$

for every measurable subset $A \subseteq X$ with $\alpha \nu(X) \leq \nu(A) \leq \frac{\nu(X)}{2}$.

For a finite $S \subseteq \Gamma$, we say that $\rho$ is $(c, S)$-asymptotically expanding (in measure) (or simply $S$-asymptotically expanding) if for every $\alpha \in (0, \frac{1}{2}]$ and measurable subset $A \subseteq X$ with $\alpha \nu(X) \leq \nu(A) \leq \frac{\nu(X)}{2}$, we have $\nu(S \cdot A) > (1 + c(\alpha))\nu(A)$. 

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**Remark 2.7** When the acting group $\Gamma$ is finitely generated by a finite symmetric set $S$, a measure-class-preserving action $\rho : \Gamma \curvearrowright (X, \nu)$ on a probability space is asymptotically expanding if and only if it is $S$-asymptotically expanding (see [28, Lemma 3.16]). We will not need this fact in this paper.

This notion turns out to be naturally related to strong ergodicity. Recall that a measure-class-preserving action $\rho : \Gamma \curvearrowright (X, \nu)$ on a probability space is asymptotically expanding if and only if it is $S$-asymptotically expanding (see [28, Lemma 3.16]).

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We will not need this fact in this paper.
Lemma 2.11 [28, Lemma 3.8] Let $Y \subseteq X$ be a domain of asymptotic expansion for an action $\Gamma \curvearrowright (X, \nu)$. Then there exist functions $b: [\frac{1}{2}, 1) \rightarrow \mathbb{R}_{>0}$ and $h: [\frac{1}{2}, 1) \rightarrow \mathbb{N}$ such that for every $\beta \in [\frac{1}{2}, 1)$, we have

$$
\nu\left( (B_{h(\beta)} \cdot A) \cap Y \right) > (1 + b(\beta)) \nu(A)
$$

for every measurable subset $A \subseteq Y$ with $\frac{1}{2} \nu(Y) \leq \nu(A) \leq \beta \nu(Y)$.

For later use, we end this subsection by recalling some structure results established in [28, Section 4].

Proposition 2.12 [28, Proposition 4.5 and 4.11] Let $\Gamma \curvearrowright (X, \nu)$ be an action, $Y \subseteq X$ be a domain of asymptotic expansion and $(Z_n)_{n \in \mathbb{N}}$ be a sequence of nested subsets of $Y$ with $\nu(Z_n) \to 0$. Then there exist $N_0 \in \mathbb{N}$, a sequence of finite subsets $S_n \subseteq \Gamma$ and an exhaustion $Y_n \nearrow Y$ by domains of $S_n$-expansion such that $Y_n \subseteq Y \setminus Z_n$ for every $n > N_0$.

Theorem 2.13 [28, Theorem 4.9] Let $\rho : \Gamma \curvearrowright (X, \nu)$ be a measure-class-preserving action. Then the following are equivalent:

1. $\rho$ is strongly ergodic;
2. every finite measure subset is a domain of asymptotic expansion;
3. $\rho$ is ergodic and $X$ admits a domain of expansion.

Remark 2.14 For measure-preserving actions, the equivalence “(1) $\iff$ (3)” of Theorem 2.13 had been previously proved in [32, Theorem A].

3 Expansion and reversible Markov kernels

In the first part of this section, we introduce the language of Markov kernels and review a general estimate for the Cheeger constant of a reversible Markov kernel in terms of the spectrum of the associated Laplacian operator. In the second part, we show that measure-class preserving actions give rise to reversible Markov kernels. This allows us to define the notion of (domain of) Markov expansion and to characterise asymptotic expansion in terms of exhaustions by domains of Markov expansion. This result will be pivotal in the subsequent sections.

3.1 Preliminaries on Markov kernels

We begin by recalling a few elementary properties of reversible Markov kernels. We refer to the first chapters of [39] for more background and details.

Definition 3.1 Let $\mathcal{E}$ be a $\sigma$-algebra on a set $X$. A Markov kernel on the measurable space $(X, \mathcal{E})$ is a function $\Pi: X \times \mathcal{E} \to [0, 1]$ such that:

1. for every $x \in X$, the function $\Pi(x, \cdot): \mathcal{E} \to [0, 1]$ is a probability measure;
2. for every \( A \in \mathcal{E} \), the function \( \Pi(-, A): X \to [0, 1] \) is \( \mathcal{E} \)-measurable.

If \( f: X \to \mathbb{R} \) is integrable with respect to the probability measure \( \Pi(x, -) \), we denote its integral by

\[
\int_X f(y) \Pi(x, dy) := \int_X f(y) d\Pi(x, -)(y)
\]

(the integral is then naturally extended to complex-valued functions). The associated \textit{Markov operator} \( \mathcal{P} \) is a linear operator on the space of bounded \( \mathcal{E} \)-measurable functions, defined by

\[
\mathcal{P} f(x) := \int_X f(y) \Pi(x, dy).
\]

Since \( \Pi(-, A) \) is measurable for every \( A \in \mathcal{E} \), we can define an operator \( \hat{\mathcal{P}} \) on the space of measures on \((X, \mathcal{E})\) by letting

\[
\hat{\mathcal{P}} \nu(A) := \int_X \Pi(x, A) d\nu(x)
\]

for every measure \( \nu \) on \((X, \mathcal{E})\). The operators \( \mathcal{P} \) and \( \hat{\mathcal{P}} \) are dual to one another in the sense that

\[
\int_X \mathcal{P} f(x) d\nu(x) = \int_X f(x) d\hat{\mathcal{P}} \nu(x), \tag{3.1}
\]

whenever the integrals are defined.

\textbf{Definition 3.2} [21] Given a measure \( \nu \) on \((X, \mathcal{E})\) and an \( A \in \mathcal{E} \), the \textit{(\( \nu \)-)size of the boundary} of \( A \) (with respect to \( \Pi \)) is defined as

\[
|\partial_\Pi A|_\nu := \int_A \Pi(x, X \setminus A) d\nu(x).
\]

\textbf{Remark 3.3} Heuristically, a Markov kernel can be described as “moving mass across \( X \)” without creating nor destroying it: the value \( \Pi(x, A) \) is the proportion of the mass that is moved from the point \( x \) into the set \( A \). The measure \( \hat{\mathcal{P}} \nu \) is the distribution of mass on \( X \) that is obtained after moving the initial distribution \( \nu \) according to the kernel \( \Pi \). The function \( \mathcal{P} f \) assigns to a point \( x \in X \) the expected value of \( f \) when spreading \( x \) across \( X \) according to the kernel \( \Pi \).

The duality formula (3.1) on the indicator function \( f = \chi_A \) can be understood as saying that the total \( \nu \)-mass that is moved into a set \( A \) by the kernel \( \Pi \) is equal to \( \nu \)-integral of the likelihood that \( \Pi \) will take \( x \) into \( A \). The size of the boundary of \( A \in \mathcal{E} \) is the amount of \( \nu \)-mass that is carried outside \( A \) by \( \Pi \).

We will only be concerned with some special Markov kernels:
Definition 3.4 A Markov kernel $\Pi$ is called reversible if there exists a measure $m$ on $(X, \mathcal{E})$ such that

$$\int_X f(x) \Pi g(x) dm(x) = \int_X \Pi f(x) g(x) dm(x)$$

for every pair of measurable bounded functions $f, g : X \to \mathbb{R}$. The measure $m$ is said to be a reversing measure for $\Pi$ (note that $m$ need not be unique in general). To specify which reversing measure is being considered, we say that $\Pi$ is a reversible Markov kernel on $(X, m)$.

Let $m$ be a measure on $X$. We define the measure $\mu$ on $X \times X$ by letting

$$\mu(A \times B) := \int_A \Pi(x, B) dm(x) = \int_X \chi_A(x) \Pi \chi_B(x) dm(x)$$

(3.2)

for every $A, B \in \mathcal{E}$. Then $m$ is a reversing measure if and only if $\mu$ is symmetric, i.e., $\mu(A \times B) = \mu(B \times A)$ for every $A, B \in \mathcal{E}$. In this case, we have

$$|\partial \Pi(A)|_m = \mu\left(A \times (X \setminus A)\right) = \mu\left((X \setminus A) \times A\right) = |\partial \Pi(X \setminus A)|_m.$$  (3.3)

In other words, the $m$-size of the boundary of any measurable set is equal to the $m$-size of the boundary of its complement.

For the rest of this section, let us fix a reversible Markov kernel $\Pi$ on $(X, m)$. We note that

$$\tilde{\Pi} m(A) = \mu(X \times A) = \mu(A \times X) = \int_A \Pi(x, X) dm(x) = m(A),$$

i.e., $m$ is invariant under $\tilde{\Pi}$. Hence, the Jensen inequality yields:

$$\int_X |\Pi f(x)|^2 dm(x) \leq \int_X \Pi |f|^2(x) dm(x) = \int_X |f|^2(x) d\tilde{\Pi} m(x)$$

$$= \int_X |f|^2(x) dm(x) = \|f\|_{m,2}^2.$$  

Therefore, the Markov operator $\Pi$ can be regarded as a bounded operator on $L^2(X, m)$ with norm $\|\Pi\| \leq 1$. Since $m$ is reversing, the operator $\Pi$ is self-adjoint.

Now, for any $p \in [1, \infty)$ and any $f \in L^p(X, m)$, we define its $p$-Dirichlet energy as

$$\mathcal{E}_p(f) := \frac{1}{2} \int_{X \times X} |f(x) - f(y)|^p d\mu(x, y).$$
Since $\mu$ is symmetric, we note that
\[
\int_{X \times X} |\chi_A(x) - \chi_A(y)|^p \, d\mu(x, y) = \mu\left( A \times (X \setminus A) \right) + \mu\left( (X \setminus A) \times A \right) \\
= 2\mu\left( A \times (X \setminus A) \right).
\]

Hence for every $p \in [1, \infty)$, we have
\[
E_p(\chi_A) = |\partial_\Pi (A)|_m.
\] (3.4)

Finally, we observe that for any $f \in L^2(X, m)$ we have
\[
E_2(f) = \frac{1}{2} \int_{X \times X} |f|^2(x) + |f|^2(y) - 2\Re(f(x)\overline{f(y)}) \, d\mu(x, y) \\
= \|f\|_{m,2}^2 - \langle f, \mathcal{P}f \rangle_m,
\]
where the last equality uses the reversibility.

We define the Laplacian of $\Pi$ as $\Delta := 1 - \mathcal{P}$, then we have $E_2(f) = \langle f, \Delta f \rangle_m$ for every $f \in L^2(X, m)$. In particular, the Laplacian $\Delta$ is a positive self-adjoint operator whose spectrum is contained in $[0, 2]$.

### 3.2 Isoperimetric inequalities and spectra of Markov kernels

It is a well-known result that a sequence of finite graphs is a family of expanders if and only if the Markov operators associated with the simple random walks have a uniform spectral gap [2, 3, 10]. A similar result is true—albeit not as widely known—in the context of Markov kernels.

Let $\Pi$ be a reversible Markov kernel on $(X, m)$, where $m$ is a finite measure. Then all constant functions on $X$ belong to $L^2(X, m)$ and are fixed by $\mathcal{P}$. It follows that $\|\mathcal{P}\| = 1$ and 1 belongs to the spectrum of $\mathcal{P}$. Denote the orthogonal complement of the constant functions in $L^2(X, m)$ by $L^2_0(X, m)$, i.e.,
\[
L^2_0(X, m) := \left\{ f \in L^2(X, m) \mid \int_X f(x) \, dm(x) = 0 \right\}.
\]

Note that $L^2_0(X, m)$ is $\mathcal{P}$-invariant and that the spectrum of the restriction of $\mathcal{P}$ on $L^2_0(X, m)$ is contained in $[-1, 1]$. We denote the supremum of this spectrum by $\lambda_2 \in \mathbb{R}$.

We make the following definition:

**Definition 3.5** A reversible Markov kernel on a finite measure space $(X, m)$ is said to have a spectral gap if $\lambda_2 < 1$.

It is clear from the definition that the reversible kernel $\Pi$ has a spectral gap if and only if 1 is isolated in the spectrum of $\mathcal{P}$ and the 1-eigenspace consists of constant...
functions on $X$. Equivalently, this happens if and only if $0$ is isolated in the spectrum of $\Delta = 1 - \Psi$ and the $0$-eigenspace consists of constant functions. Obviously, we have that

$$1 - \lambda_2 = \inf \left\{ \frac{\mathcal{E}_2(f)}{\| f \|_{m,2}^2} \left| f \in L^2_0(X, m) \right. \right\}.$$  (3.5)

In analogy with the notion of Cheeger constants for finite graphs, we define:

**Definition 3.6** The Cheeger constant for a reversible Markov kernel $\Pi$ on $(X, m)$ is

$$\kappa := \inf \left\{ \frac{|\partial \Pi(A)|_m}{m(A)} \left| A \in \mathcal{E}, \ 0 < m(A) \leq \frac{1}{2}m(X) \right. \right\}.$$  

We can now state the following theorem relating Cheeger constants and spectral gaps in the context of Markov kernels:

**Theorem 3.7** [24, Theorem 2.1] Let $\Pi$ be a reversible Markov kernel on $(X, m)$ where $m$ is finite. Then

$$\frac{\kappa^2}{2} \leq 1 - \lambda_2 \leq 2\kappa.$$  

**Remark 3.8** We are grateful to the anonymous referee for pointing out [24] to us. We should remark that the authors of [24] use a slightly different notion of Cheeger constant for Markov kernels. Moreover, the inequality they prove has slightly different constants and it is not sharp (see also the remark below [24, Proposition 2.2]). For completeness, we provide a self-contained proof of Theorem 3.7 in the appendix.

### 3.3 Markov kernels from actions

Let $\Gamma \rhd (X, \nu)$ be a measure-class-preserving action. For every $\gamma \in \Gamma$ and $x \in X$, let $r(\gamma, x) := \frac{d\gamma^{-1}\nu}{d\nu}(x)$ be the Radon–Nikodym derivative. Note that $r(\gamma, x) = r(\gamma^{-1}, \gamma(x))^{-1}$ and for any measurable function $f$ on $X$ we have

$$\int_X f(\gamma \cdot x) d\nu(x) = \int_X f(x) r(\gamma^{-1}, x) d\nu(x)$$

when the integrals exist. In particular, for every measurable $Y \subseteq X$ we have

$$\int_Y f(\gamma \cdot x) r(\gamma, x)^{\frac{1}{2}} d\nu(x) = \int_{\gamma(Y)} f(x) r(\gamma^{-1}, x)^{\frac{1}{2}} d\nu(x)$$  (3.6)

when the integrals exist.

Now fix a finite symmetric subset $S \subseteq \Gamma$ containing the identity $1$, and a measurable subset $Y \subseteq X$ (which might have infinite measure). For every $x \in Y$, let $S_{Y,x} := \{ s \in S | s \cdot x \in Y \}$ and

$$\sigma_{Y,S}(x) := \sum_{s \in S_{Y,x}} r(s, x)^{\frac{1}{2}}.$$  (3.7)
Definition 3.9 Let $\Gamma \curvearrowright (X, \nu)$ be a measure-class-preserving action. The *normalised local Markov kernel* associated with $Y$ and $S$ is the Markov kernel on $Y$ defined by

$$\Pi_{Y,S}(x, -) = \frac{1}{\sigma_{Y,S}(x)} \sum_{s \in SY, x} r(s, x)^{1/2} \delta_{S,x}$$

where $\delta_y$ is the Dirac delta measure on the point $y$, and we denote the associated Markov operator by $\mathcal{P}_{Y,S}$. We say that $\Pi_{S}:=\Pi_{X,S}$ is the *normalised Markov kernel* associated with $S$.

For later use, we record here an elementary but convenient integration formula: for every measurable function $G : S \times Y \to \mathbb{C}$ we have

$$\int_Y \sum_{s \in SY, x} G(s, x) \nu(x) = \int_Y \sum_{s \in S} \chi_{\{s^{-1}(Y)\}}(x) G(s, x) \nu(x) = \sum_{s \in S} \int_{Y \cap s^{-1}(Y)} G(s, x) \nu(x)$$

(3.8)

(when the integrals are defined).

One of the key properties of normalised local Markov kernels is that they are reversible. In fact, consider the measure $\tilde{\nu}_{Y,S}$ on $Y$ defined by

$$d\tilde{\nu}_{Y,S} = \sigma_{Y,S} \cdot d(\nu|_Y).$$

In other words, $\tilde{\nu}_{Y,S}$ is obtained by rescaling the restriction of $\nu$ to $Y$ by the density function $\sigma_{Y,S}$. Then the following holds true:

**Proposition 3.10** Let $\Gamma \curvearrowright (X, \nu)$ be a measure-class-preserving action and $S$ be a finite symmetric subset of $\Gamma$ containing the identity 1. Then:

1. The measure $\tilde{\nu}_{Y,S}$ is equivalent to the restriction $\nu|_Y$.
2. If $\nu(Y)$ is finite, then $\nu(A) \leq \tilde{\nu}_{Y,S}(A) \leq |S|^{1/2} \nu(A) \nu(Y)$ for any measurable $A \subseteq Y$. In particular, in this case $\tilde{\nu}_{Y,S}(Y)$ is also finite.
3. The measure $\tilde{\nu}_{Y,S}$ is reversing for the normalised local Markov kernel $\Pi_{Y,S}$. The associated measure $\mu$ on $Y \times Y$—defined by (3.2)—is determined by the formula:

$$\int_{Y \times Y} F(x, y) d\mu(x, y) = \sum_{s \in S} \int_{Y \cap s^{-1}(Y)} r(s, x)^{1/2} F(x, s \cdot x) \nu(x)$$

(3.9)

for every integrable function $F$ on $Y \times Y$.

**Proof** (1). Note that $0 < r(s, x) < \infty$ for $\nu$-almost every $x \in X$ because the action is measure-class-preserving. Since $S$ contains the identity 1, we know that $SY, x$ is non-empty for every $x \in Y$. It follows immediately that a measurable subset of $Y$ is $\nu$-null if and only if it is $\tilde{\nu}_{Y,S}$-null.
(2). Since $1 \in S_{Y,x}$, we have $\nu(A) \leq \tilde{\nu}_{Y,S}(A)$ for any measurable $A \subseteq Y$. On the other hand, by (3.8) and the Cauchy–Schwarz inequality we have

$$\tilde{\nu}_{Y,S}(A) = \sum_{s \in S} \int_{A \cap s^{-1}(Y)} r(s, x)^{\frac{1}{2}} \, d\nu(x)$$

$$\leq \sum_{s \in S} \nu(A \cap s^{-1}(Y))^\frac{1}{2} \left( \int_{A \cap s^{-1}(Y)} r(s, x) \, d\nu(x) \right)^\frac{1}{2}$$

$$\leq \left( \sum_{s \in S} \nu(A) \right)^{\frac{1}{2}} \cdot \left( \sum_{s \in S} \nu((s \cdot A) \cap Y) \right)^{\frac{1}{2}} \leq |S| \sqrt{\nu(A) \nu(Y)}.$$

(3). Let us first verify the formula for $\mu$. By definition, for any measurable function $F$ on $Y \times Y$ we have that

$$\int_{Y \times Y} F(x, y) \, d\mu(x, y) = \int_{Y} \int_{Y} F(x, y) \Pi_{Y,S}(x, dy) \, d\tilde{\nu}_{Y,S}(x)$$

$$= \int_{Y} \frac{1}{\sigma_{Y,S}(x)} \sum_{s \in S_{Y,x}} r(s, x)^{\frac{1}{2}} F(x, s \cdot x) \, d\tilde{\nu}_{Y,S}(x)$$

$$= \int_{Y} \sum_{s \in S_{Y,x}} r(s, x)^{\frac{1}{2}} F(x, s \cdot x) \, d\nu(x)$$

$$= \sum_{s \in S} \int_{Y \cap s^{-1}(Y)} r(s, x)^{\frac{1}{2}} F(x, s \cdot x) \, d\nu(x),$$

where the last step follows from (3.8).

To show that $\tilde{\nu}_{Y,S}$ is reversing for $\Pi_{Y,S}$, it suffices to prove:

$$\int_{Y \times Y} F(x, y) \, d\mu(x, y) = \int_{Y \times Y} F(y, x) \, d\mu(x, y)$$

for every measurable function $F$ on $Y \times Y$. From (3.9), we have that

$$\int_{Y \times Y} F(y, x) \, d\mu(x, y) = \sum_{s \in S} \int_{Y \cap s^{-1}(Y)} r(s, x)^{\frac{1}{2}} F(s \cdot x, x) \, d\nu(x)$$

$$= \sum_{s \in S} \int_{s \cap Y} r(s^{-1}, x)^{\frac{1}{2}} F(x, s^{-1} \cdot x) \, d\nu(x)$$

$$= \sum_{s \in S} \int_{Y \cap s^{-1}(Y)} r(s, x)^{\frac{1}{2}} F(x, s \cdot x) \, d\nu(x)$$

$$= \int_{Y \times Y} F(x, y) \, d\mu(x, y),$$

where we use (3.6) for the second equation, and use $S = S^{-1}$ for the third one. \qed

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Remark 3.11 The assumption that \( 1 \in S \) is only used to ensure that \( S_{Y,x} \) is always non-empty for every \( x \in Y \). This assumption can be dropped if one already knows, \( a \text{ priori} \), that \( S_{Y,x} \) is non-empty (e.g., \( Y \) is \( \Gamma \)-invariant). On the contrary, the condition that \( S = S^{-1} \) is essential for the proof of reversibility in Proposition 3.10.

Having introduced the reversible normalised local Markov kernel \( \Pi_{Y,S} \), we would like to apply the techniques developed in previous subsections to asymptotically expanding actions. Firstly, let us give the following definition:

**Definition 3.12** Let \( \Gamma \curvearrowright (X, \nu) \) be a measure-class-preserving action and \( Y \subseteq X \) be a domain. Let \( S \subseteq \Gamma \) be a finite symmetric subset with \( 1 \in S \), then \( Y \) is called a *domain of Markov \( S \)-expansion (for the action)* if the associated normalised local Markov kernel \( \Pi_{Y,S} \) has strictly positive Cheeger constant (see Definition 3.6). \( Y \) is called a *domain of Markov expansion* if it is a domain of Markov \( S \)-expansion for some finite symmetric \( S \subseteq \Gamma \) with \( 1 \in S \).

By Theorem 3.7, \( Y \) is a domain of Markov \( S \)-expansion if and only if the normalised local Markov kernel \( \Pi_{Y,S} \) has spectral gap. In other words, \( 1 \) is isolated in the spectrum of the Markov operator \( \mathcal{P}_{Y,S} \) and the 1-eigenspace consists of constant functions on \( X \). When this is the case, restriction of the Markov operator \( \mathcal{P}_{Y,S} \) on \( L^2_0(Y, \tilde{\nu}_{Y,S}) \) has spectrum contained in \([-1, \lambda_2] \subset [-1, 1) \). The restriction of the operator \( \frac{1}{2} + \frac{1}{2} \mathcal{P}_{Y,S} \) to \( L^2_0(Y, \tilde{\nu}_{Y,S}) \) has spectrum contained in \([-\frac{3}{4}, \lambda_2+\frac{1}{2}] \) (this is the Markov operator obtained by lazyfying the Markov process). For future reference, we record this observation as a lemma.

**Lemma 3.13** The Markov kernel \( \Pi_{Y,S} \) has spectral gap if and only if the restriction of the lazy Markov operator \( \frac{1}{2} + \frac{1}{2} \mathcal{P}_{Y,S} \) to \( L^2_0(Y, \tilde{\nu}_{Y,S}) \) has spectrum contained in \([-\frac{3}{4}, 1-\epsilon] \) with \( \epsilon > 0 \) (and hence has norm strictly less than 1).

The following result provides the connection between expansion in measure (Definition 2.2) and Markov expansion (Definition 3.12) under an assumption of bounded Radon–Nikodym derivatives:

**Lemma 3.14** Let \( \Gamma \curvearrowright (X, \nu) \) be a measure-class-preserving action, \( Y \subseteq X \) be a domain and \( S \) be a finite symmetric subset of \( \Gamma \) containing the identity. If there is a constant \( \Theta \geq 1 \) such that \( 1/\Theta \leq r(s, x) \leq \Theta \) for every \( x \in Y \) and \( s \in S_{Y,x} \), then \( Y \) is a domain of \( S \)-expansion if and only if it is a domain of Markov \( S \)-expansion.

**Proof** By definition and (3.9), for any measurable subset \( A \subseteq Y \) we have that

\[
\left| \partial_{\Pi_{Y,S}}(A) \right|_{\tilde{\nu}_{Y,S}} = \int_{Y \times Y} \chi_A(x) \chi_{Y \setminus A}(y) \, d\mu(x, y) \\
= \sum_{s \in S} \int_{Y \cap s^{-1}(Y)} r(s, x)^\frac{1}{2} \chi_A(x) \chi_{Y \setminus A}(s \cdot x) \, d\nu(x) \\
= \sum_{s \in S} \int_{(A \setminus s^{-1}(A)) \cap s^{-1}(Y)} r(s, x)^\frac{1}{2} \, d\nu(x) \\
= \sum_{s \in S} \int_{(s \cdot A \setminus A) \cap Y} r(s^{-1}, x)^\frac{1}{2} \, d\nu(x),
\]

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where the last equality uses (3.6).

Hence, it follows from the assumption on \( r \) that

\[
\frac{1}{\sqrt{\Theta}} |\partial_{\Pi, S}(A)|_{\tilde{\nu}_Y, S} \leq \sum_{s \in S} \nu((s \cdot A \setminus A) \cap Y) \leq \sqrt{\Theta} |\partial_{\Pi, S}(A)|_{\tilde{\nu}_Y, S}.
\] (3.10)

Moreover, it follows by the definition of \( \tilde{\nu}_Y, S \) that

\[
\nu(A) \leq \tilde{\nu}_Y, S(A) \leq |S| \sqrt{\Theta} \cdot \nu(A)
\] (3.11)

for any measurable subset \( A \subseteq Y \).

Now we assume that \( Y \) is a domain of \((c, S)\)-expansion for some constant \( c > 0 \). Fix a measurable subset \( A \subseteq Y \) with \( 0 < \tilde{\nu}_Y, S(A) \leq \frac{1}{2} \tilde{\nu}_Y, S(Y) \). In particular, both \( \nu(A) > 0 \) and \( \nu(Y \setminus A) > 0 \) by Proposition 3.10(1).

If \( \nu(A) \leq \frac{1}{2} \nu(Y) \), it follows from Definition 2.2 and (3.10) that

\[
c \nu(A) < \nu((S \cdot A \setminus A) \cap Y) \leq \sum_{s \in S} \nu((s \cdot A \setminus A) \cap Y) \leq \sqrt{\Theta} |\partial_{\Pi, S}(A)|_{\tilde{\nu}_Y, S}.
\]

Together with (3.11), we conclude that

\[
|\partial_{\Pi, S}(A)|_{\tilde{\nu}_Y, S} > \frac{c}{|S| \Theta} \tilde{\nu}_Y, S(A).
\]

If \( \nu(A) > \frac{1}{2} \nu(Y) \), we can apply the same argument to \( Y \setminus A \) and deduce from (3.3) that

\[
|\partial_{\Pi, S}(A)|_{\tilde{\nu}_Y, S} = |\partial_{\Pi, S}(Y \setminus A)|_{\tilde{\nu}_Y, S} > \frac{c}{|S| \Theta} \tilde{\nu}_Y, S(Y \setminus A) \geq \frac{c}{|S| \Theta} \tilde{\nu}_Y, S(A),
\]

where the last inequality follows from the assumption that \( \tilde{\nu}_Y, S(A) \leq \frac{1}{2} \tilde{\nu}_Y, S(Y) \). Thus, \( Y \) is a domain of Markov \( S \)-expansion as desired.

The proof of the converse implication is similar. Let \( \kappa > 0 \) be the Cheeger constant for the normalised local Markov kernel \( \Pi_{Y, S} \) and fix any measurable subset \( A \subseteq Y \) with \( 0 < \nu(A) \leq \frac{1}{2} \nu(Y) \).

If \( \tilde{\nu}_Y, S(A) \leq \frac{1}{2} \tilde{\nu}_Y, S(Y) \), then (3.10) implies that

\[
\nu(\partial S \cap Y) \geq \frac{1}{|S|} \sum_{s \in S} \nu((s \cdot A \setminus A) \cap Y) \\
\geq \frac{1}{|S| \sqrt{\Theta}} |\partial_{\Pi, S}(A)|_{\tilde{\nu}_Y, S} \\
\geq \frac{\kappa}{|S| \sqrt{\Theta}} \tilde{\nu}_Y, S(A).
\]
where $\partial^\Gamma_S A = S \cdot A \setminus A$. Together with (3.11) we obtain that

$$\nu(\partial^\Gamma_S A \cap Y) \geq \frac{\kappa}{|S| \sqrt{\Theta}} \nu(A).$$

If $\tilde{v}_{Y,S}(A) > \frac{1}{2} \tilde{v}_{Y,S}(Y)$, then $\partial^\Gamma_S A \cap Y \supseteq \partial^\Gamma_S (Y \setminus S \cdot A) \cap Y$ implies that

$$\nu(\partial^\Gamma_S A \cap Y) \geq \nu\left(\partial^\Gamma_S (Y \setminus S \cdot A) \cap Y\right) \geq \frac{\kappa}{|S| \sqrt{\Theta}} \nu(Y \setminus S \cdot A).$$

Moreover, using $1 \in S$ we note that

$$\nu(Y \setminus S \cdot A) = \nu(Y) - \nu(A) - \nu(\partial^\Gamma_S A \cap Y) \geq \nu(A) - \nu(\partial^\Gamma_S A \cap Y).$$

So it is easy to conclude that

$$\nu(\partial^\Gamma_S A \cap Y) \geq \frac{\kappa}{|S| \sqrt{\Theta} + \kappa} \cdot \nu(A).$$

This shows that $Y$ is a domain of $S$-expansion for the action. \hfill \Box

**Remark 3.15** The statement of Lemma 3.14 is an analogue of the fact that for graphs with bounded degrees, there are bounds between edge-expansion and vertex-expansion. More precisely, the Cheeger constant of the normalised (local) Markov kernel should be regarded as the “measured” Cheeger constant of the edge-expansion, while the notion of expansion in measure is clearly an analogue of the (exterior) vertex-expansion for graphs. The assumption that the Radon-Nikodym derivatives are bounded corresponds to that the graphs have bounded degree.

Consequently, we obtain an alternative and direct proof for [15, Lemma 5.2]:

**Corollary 3.16** [15, Lemma 5.2] Let $\Gamma \curvearrowright (X, \nu)$ be a measure-preserving action and $Y \subseteq X$ a domain. Then $Y$ is a domain of expansion if and only if the action has local spectral gap with respect to $Y$.

**Proof** Using indicator functions, it is easy to see that the existence of a local spectral gap implies that $Y$ is a domain of expansion. Hence, we only focus on the converse implication.

Let $Y$ be a domain of $(c, k)$-expansion and let $S := B_k$. Then the normalised local Markov kernel $\Pi_{Y,S}$ has a spectral gap by Theorem 3.7 and Lemma 3.14. Using (3.9) in Proposition 3.10, we obtain that for every $g \in \mathcal{L}^2_{\tilde{\nu}}(Y, \tilde{\nu}_{Y,S})$:

$$(1 - \lambda_2) \cdot \|g\|^2_{\tilde{\nu}_{Y,S}, 2} \leq \mathcal{E}_2(g) = \frac{1}{2} \sum_{s \in S} \int_{Y \cap \mu^{-1}(Y)} |g(x) - g(s \cdot x)|^2 \, d\nu(x). \tag{3.12}$$

where $1 - \lambda_2$ is bounded away from zero (Definition 3.5).
Now we fix an $f \in L^2(X, \nu)$ with $\int_Y f \, d\nu = 0$. Then $f|_Y \in L^2(Y, \tilde{\nu}_Y, S)$ and $g := f|_Y - \int_Y f|_Y \, d\tilde{\nu}_Y \in L^2_0(Y, \tilde{\nu}_Y, S)$ by construction and Proposition 3.10(2). Thus, it follows from $S = S^{-1}$ and (3.12) that

$$\sum_{s \in S} \| s \cdot f - f \|_{\tilde{\nu}_Y, 2} \geq \left( \sum_{s \in S} \int_Y |f(x) - f(s \cdot x)|^2 \, d\nu(x) \right)^{\frac{1}{2}} \geq (2\varepsilon_2(g))^{\frac{1}{2}} \geq \sqrt{2(1 - \lambda_2)} \cdot \| g \|_{\tilde{\nu}_Y, 2}.$$  

Moreover, since $\int_Y f \, d\nu = 0$ we see that

$$\| g \|_{\tilde{\nu}_Y, 2}^2 = \| g \|_{\tilde{\nu}_Y, 2}^2 = \| f \|_{\tilde{\nu}_Y, 2}^2 + \nu(Y) \left( \int_Y f|_Y \, d\tilde{\nu}_Y, S \right)^2 \geq \| f \|_{\tilde{\nu}_Y, 2}^2.$$  

Combining the above inequalities we conclude that

$$\sum_{s \in S} \| s \cdot f - f \|_{\tilde{\nu}_Y, 2} \geq \sqrt{2(1 - \lambda_2)} \cdot \| f \|_{\tilde{\nu}_Y, 2},$$  

as required. \hfill \Box

**Remark 3.17** Note that the proof of Corollary 3.16 holds also for non-measure-preserving actions as long as the action is measure-class-preserving and there is a uniform upper bound $\Theta \geq 1$ on the Radon–Nikodym derivatives $r(s, x)$. This can be used to provide an alternative proof for [19, Theorem 3.2].

### 3.4 Markov expansion and the structure of strongly ergodic actions

Now we are in the position to prove the Markovian analogue of the structure theorem for strongly ergodic actions (Theorem 2.13). Let us start with the following local version (compare with Proposition 2.12):

**Proposition 3.18** Let $\sigma: \Gamma \acts (X, \nu)$ be a measure-class-preserving action. If $Y \subseteq X$ is a domain of asymptotic expansion, then $Y$ admits an exhaustion by domains $Y_n$ of Markov $S^{(n)}$-expansion such that for each $n \in \mathbb{N}$ there is a constant $\Theta_n \geq 1$ such that $1/\Theta_n \leq r(s, y) \leq \Theta_n$ for every $y \in Y_n$ and $s \in S^{(n)}_{Y_n, y}$.

Moreover, if $Y \subseteq X$ is a domain of $S$-asymptotic expansion, then $Y$ admits an exhaustion by domains $Y_n$ of Markov $S$-expansion.

**Proof** It follows from Proposition 2.12 that there exists an exhaustion $Y^{(k)} \acts Y$ by domains of $S^{(k)}$-expansion in measure. Without loss of generality, we can assume that $S^{(k)}$ is symmetric, $1 \in S^{(k)}$ and $S^{(k)} \subseteq S^{(k+1)}$ for every $k \in \mathbb{N}$. Let

$$Z^{(k)}_m := \left\{ y \in Y^{(k)} \mid r(s, y) < \frac{1}{m} \text{ or } r(s, y) > m \text{ for some } s \in S^{(k)}_{Y^{(k)}, y} \right\}.$$  

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Since each $Y^{(k)}$ has finite measure and the action is measure-class-preserving, for every $k \in \mathbb{N}$ we have $\nu(Z_{m}^{(k)}) \to 0$ as $m \to \infty$. Hence, we can choose for every $n \in \mathbb{N}$ a sequence of integers $(m_{k}^{(n)})_{k \in \mathbb{N}}$ such that

$$\sum_{k \in \mathbb{N}} \nu\left(Z^{(k)}_{m_{k}^{(n)}}\right) \leq \frac{1}{n}.$$  

Let

$$\tilde{Z}_{n} := \bigcup_{k \in \mathbb{N}} Z^{(k)}_{m_{k}^{(n)}},$$

then we have $\nu(\tilde{Z}_{n}) \to 0$ as $n \to \infty$. We can further assume that $m_{k}^{(n+1)} \geq m_{k}^{(n)}$ for every $n \in \mathbb{N}$ so that $\tilde{Z}_{n+1} \subseteq \tilde{Z}_{n}$.  

Now for every $k$, it follows from Proposition 2.12 that $Y^{(k)}$ admits an exhaustion $Y_{l}^{(k)} \nearrow Y^{(k)}$ by domains of $S^{(k)}$-expansion such that $Y_{l}^{(k)} \cap \tilde{Z}_{l} = \emptyset$. By a diagonal argument, there exists a sequence $(l_{k})_{k \in \mathbb{N}}$ such that $Y_{l_{k}}^{(k)}$ converges in measure to $Y$. Let

$$Y_{n} := \bigcup_{k=0}^{n} Y_{l_{k}}^{(k)}.$$  

Ignoring finitely many $k$ if necessary, we can assume that $\nu(Y_{l_{0}}^{(0)}) \geq \frac{3}{4} \nu(Y)$. Then we conclude from Lemma 2.3 that each $Y_{n}$ is a domain of $S^{(n)}$-expansion. Since $Y_{n} \cap \tilde{Z}_{n} = \emptyset$, it follows from Lemma 3.14 that it is also a domain of Markov $S^{(n)}$-expansion. The second statement is obtained by the special case where $S^{(k)} = S$ for all $k \in \mathbb{N}$.  

\begin{remark}
The main technical difficulty in the previous proof is to obtain increasing sequences of domains of Markov expansion. This is largely due to the fact that it is hard to control Markov $S$-expansion as the set $S$ varies. More precisely, choosing different $S$ could yield widely different measures $\nu_{Y,S}$ and this would in turn influence the Cheeger constant of $\Pi_{Y,S}$.

An alternative approach to Proposition 3.18 would be to go through the proof of Proposition 2.12 and reprove it using the language of Markov expansion.
\end{remark}

It is now simple to prove a structure result for strongly ergodic actions in terms of Markovian expansion:

\begin{theorem}
Let $\rho : \Gamma \curvearrowright (X, \nu)$ be a measure-class-preserving action. Then $\rho$ is strongly ergodic if and only if every domain $Y \subseteq X$ admits an exhaustion by domains of Markov expansion.
\end{theorem}

\begin{proof}
Necessity: This follows from Theorem 2.13 “(1)⇒(2)” and Proposition 3.18.
\end{proof}
Sufficiency: It follows from the same argument as in the proof of [28, Theorem 4.9 “(5)⇒(6)”] that \( \rho \) must be ergodic. By Theorem 2.13 “(3)⇒(1)”, it is hence enough to show that \( X \) admits a domain of expansion.

We can choose a domain \( Y \subseteq X \) for which there exist constants \( C(\gamma) \geq 1 \) depending on \( \gamma \in \Gamma \) such that \( C(\gamma)^{-1} \leq r(\gamma, y) \leq C(\gamma) \) for every \( \gamma \in \Gamma \) (such a domain can be constructed using an argument similar to that in the proof of Proposition 3.18). By the hypothesis, there is an exhaustion \( Y_n \nearrow Y \) by domains of Markov expansion. Finally, Lemma 3.14 implies that any such \( Y_n \) produces the desired domain of expansion in measure.

\[ \square \]

4 Warped cones and finite (dynamical) propagation approximations

The aim of this section is to introduce warped cones associated with group actions on metric measure spaces and to study the effects of asymptotic expansion on the analytic properties of said warped cones. More precisely, adapting the techniques in [23] to the context of group actions and using the structure results in Sect. 3.4, we can characterise asymptotic expansion in terms of finite propagation approximations of the Druţu–Nowak projections. As an intermediate bridge, we introduce dynamical versions of quasi-locality and finite propagation approximation to connect actions and projections on warped cones. In turn, this allows us to construct a multitude of non-compact ghost projections which will be used in Sect. 5 to construct counterexamples to the coarse Baum–Connes conjecture.

4.1 Preliminaries on warped cones

Recall that the countable group \( \Gamma \) is equipped with a proper length function \( \ell \). Let \((X, d)\) be a metric space and \( \rho: \Gamma \curvearrowright X \) be a continuous action. For every \( t \geq 1 \) let \( d^t \) be the rescaling of \( d \) by \( t \), i.e., \( d^t(x, y) := td(x, y) \).

**Definition 4.1** The **warped cone** associated with the action \( \Gamma \curvearrowright X \) is the family of metric spaces \( \mathcal{WC}(\Gamma \curvearrowright X) := \{(X, d^t\Gamma) | t \in [1, \infty)\} \), where \( d^t\Gamma \) is the largest metric such that

\[ d^t\Gamma \leq d^t \quad \text{and} \quad d^t\Gamma(x, \gamma \cdot x) \leq \ell(\gamma) \]

for every \( x \in X \) and \( \gamma \in \Gamma \).

If the diameter of \((X, d)\) is at most 2, we can also define the **unified warped cone** as the metric space \((O\Gamma X, d\Gamma)\), where \( O\Gamma X = X \times [1, \infty) \) as a set and

\[ d\Gamma((x_1, t_1), (x_2, t_2)) := d^{t_1 \wedge t_2}(x_1, x_2) + |t_1 - t_2| \]

where \( t_1 \wedge t_2 = \min \{t_1, t_2\} \). The requirement on the diameter is necessary to ensure that \( d\Gamma \) is a metric.

We will also need the following:
Lemma 4.2 Let $\Gamma \curvearrowright (X, d)$ be a continuous action and $R > 0$ fixed. Given $A \subseteq X$, let $N_R(A; d^\Gamma_1) \subseteq X$ be the closed $R$-neighbourhood of $A$ with respect to the metric $d^\Gamma_1$. Then

$$\bigcap_{t \geq 1} N_R(A; d^\Gamma_1) = B_R \cdot A.$$  

**Proof** It is clear that $B_R \cdot A$ is contained in $N_R(A; d^\Gamma_1)$ for every $t \geq 1$. For the converse, it suffices to prove it for $A$ closed. If $R < 1$, we see that $N_R(A; d^\Gamma_1) = N_R(A; d') = N_{R/t}(A; d)$, because $\ell$ only takes integer values. So the result holds trivially.

By induction on $n \in \mathbb{N}$, we will prove that the claim holds for every $R < n$ and every closed $A \subseteq X$. First note that for every fixed $\gamma \in \Gamma$ we have

$$\bigcap_{t \geq 1} \gamma \cdot N_R(A; d') = \gamma(A). \quad (4.1)$$

Also note that the warped distance can be computed as

$$d^\Gamma_1(x, y) = \inf_{\xi} \left( \sum_{i=0}^{k} d'(x_i, y_i) + \sum_{i=1}^{k} |\gamma_i| \right) \quad (4.2)$$

where the infimum is taken over $k \in \mathbb{N}$ and sequences $\xi$ of points $x_0, \ldots, x_k, y_0, \ldots, y_k \in X$ and elements $\gamma_1, \ldots, \gamma_k \in \Gamma$ so that $x = x_0, y = y_k$ and $x_i = \gamma_{i-1}(y_{i-1})$ for every $1 \leq i \leq k$ (this expression is obtained by imposing that $d^\Gamma_1$ satisfies the triangle inequality).

Fix now some $0 < R < n$. For every $y \in N_R(A; d^\Gamma) \setminus N_R(A; d')$ we can take a sequence of sequences $\xi_l$ converging to the infimum in (4.2). Since $y \notin N_R(A; d')$ we can assume that each sequence $\xi_l$ has length $k \geq 1$. Since $B_R$ is finite, we can also pass to a subsequence $\xi_{l_m}$ so that each sequence $\xi_{l_m}$ has the same $\gamma_1$. Denote this element by $y_{1, y} \in B_R$. Since $A$ is closed, it follows that

$$y \in N_{R - |\gamma_{1, y}|} \left( y_{1, y} \cdot N_{R - |\gamma_{1, y}|}(A; d') ; d^\Gamma_1 \right).$$

As a consequence, we deduce that

$$N_R(A; d^\Gamma) \subseteq N_R(A; d') \cup \left( \bigcup_{m=1}^{n-1} N_{R-m} \left( B_m \cdot N_{R-m}(A; d') ; d^\Gamma_1 \right) \right). \quad (4.3)$$

For every $1 \leq m \leq n-1$ and $t > 0$, the set $C_{m,t} := B_m \cdot N_{R-m}(A; d')$ is closed. If we fix $t_0 > 1$ we can apply the induction hypothesis on the neighbourhoods of $C_{m,t_0}$
to deduce that

$$
\bigcap_{t \geq 1} N_{R-m}(C_{m,t}; d^{l}_\Gamma) \subseteq \bigcap_{t > t_0} N_{R-1}(C_{m,t_0}; d^{l}_\Gamma) \\
= B_{R-m} \cdot C_{m,t_0} \\
\subseteq B_R \cdot N_{R-m}(A; d^{l}_0).
$$

Therefore, for each $1 \leq m \leq n - 1$ we have

$$
\bigcap N_{R-m}(C_{m,t}; d^{l}_\Gamma) \subseteq \bigcap_{t_0 > 1} B_R \cdot N_{R-m}(A; d^{l}_0) = B_R \cdot A,
$$

where we used (4.1) on the finitely many elements $\gamma \in B_R$ to obtain the last equality. This shows the the right hand side of (4.3) shrinks down to $B_R \cdot A$ as $t$ goes to infinity, thus proving the lemma. \hfill \Box

For more details and elementary facts on the geometry of warped cones, we refer to [42, 44, 55, 59].

### 4.2 (Dynamical) quasi-local characterisations for asymptotic expansion

In this subsection, we will introduce a notion of dynamical quasi-locality and explain its relation with the ordinary quasi-locality for operators on warped cones. Using the dynamical quasi-locality, we will study the Drutu–Nowak projection associated to a warped cone, and show that the ordinary quasi-locality of this projection characterises asymptotic expansion in measure.

Let $\rho: \Gamma \curvearrowright X$ be a continuous action on a metric space $(X, d)$ of diameter at most 2. Let $\nu$ be a probability measure on $(X, d)$ and $\lambda$ be the Lebesgue measure on $[1, \infty)$. Equip the unified warped cone $O/\Gamma \times X$ with the product measure $\nu \times \lambda$.

For any measurable non-null $Y \subseteq X$, denote by $P_Y \in B(L^2(X, \nu))$ the averaging projection on $Y$, which is the orthogonal projection onto the one-dimensional subspace in $L^2(X, \nu)$ spanned by $\chi_Y$. In other words,

$$
P_Y f := \langle f, \frac{1}{\nu(Y)} \cdot \chi_Y \rangle \chi_Y,
$$

where $f \in L^2(X, \nu)$. The Drutu–Nowak projection (see [11, Section 6.c.]) is defined as $G = P_X \otimes \text{Id}_{L^2([1, \infty))} \in B(L^2(O/\Gamma \times X, \nu \times \lambda))$. In other words, it is the orthogonal projection onto $\mathbb{C} \otimes L^2([1, \infty), \lambda)$.

Recall from [40, 41] that an operator $T \in B(L^2(O/\Gamma \times X, \nu \times \lambda))$ is quasi-local if for every $\epsilon > 0$, there exists an $R > 0$ such that for any two measurable subsets $A, C \subseteq O/\Gamma \times X$ with $d_{\Gamma}(A, C) > R$ we have $\| \chi_A T \chi_C \| < \epsilon$. Analogously, a family of operators $\{T_t\}_{t \in [1, \infty)}$ in $B(L^2(X, \nu))$ is uniformly quasi-local on $\mathcal{W}(O/\Gamma \times X)$ if for every $\epsilon > 0$ there exists an $R > 0$ such that for every $t \in [1, \infty)$ and every pair of measurable subsets $A, C \subseteq X$ with $d^{l}_\Gamma(A, C) > R$, we have $\| \chi_A T_t \chi_C \| < \epsilon$. 
Now we introduce the following dynamical analogue of quasi-locality for operators in $\mathcal{B}(L^2(X, \nu))$ where $(X, \nu)$ is a probability space with a $\Gamma$-action:

**Definition 4.3** Let $\rho : \Gamma \curvearrowright (X, \nu)$ be an action on a probability space $(X, \nu)$. An operator $T \in \mathcal{B}(L^2(X, \nu))$ is called $\rho$-quasi-local if for every $\epsilon > 0$ there exists a $k \in \mathbb{N}$ such that for any measurable subsets $A, C \subseteq X$ with $\nu((B_k \cdot A) \cap C) = 0$, we have $\|\chi_A T \chi_C\| < \epsilon$ (recall that $B_k = \{\gamma \in \Gamma | \ell(\gamma) \leq k\}$).

Similarly to [27, Lemma 3.8], quasi-locality of the averaging projection $P_X$ can be detected by the following calculation:

**Lemma 4.4** For every measurable subsets $A, C$ in $X$, we have that

$$\|\chi_A P_X \chi_C\|_{\mathcal{B}(L^2(X, \nu))} = \sqrt{\nu(A) \nu(C)}.$$

**Proof** By direct calculations, we have that

$$\|\chi_A P_X \chi_C\| = \sup_{\|v\| = \|w\| = 1} |\langle \chi_A P_X \chi_C v, w \rangle|$$

$$= \sup_{\|v\| = \|w\| = 1} |\langle P_X \chi_C v, P_X \chi_A w \rangle|$$

$$= \sup_{\|v\| = \|w\| = 1} |\langle \chi_C v, 1 \rangle \langle \chi_A w, 1 \rangle|$$

$$= \sup_{\|v\| = \|w\| = 1} |\langle v, \chi_C \rangle \langle w, \chi_A \rangle |1, 1\rangle|$$

$$\leq \sqrt{\nu(A) \nu(C)},$$

where the last inequality follows from the Cauchy–Schwarz inequality. On the other hand, if we let $v$ and $w$ be the normalised characteristic functions of $C, A$ respectively then we have that $\langle P_X \chi_C v, P_X \chi_A w \rangle = \sqrt{\nu(A) \nu(C)}$. \qed

The following corollary is a dynamical analogue of [27, Proposition 3.9] and it is an immediate consequence of Lemma 4.4:

**Corollary 4.5** Let $\rho : \Gamma \curvearrowright (X, \nu)$ be an action on a probability space $(X, \nu)$ and $P_X$ be the associated averaging projection on $X$. Then $P_X$ is $\rho$-quasi-local if and only if

$$\lim_{k \to +\infty} \sup \{\nu(A) \nu(C) | A, C \subseteq X \text{ measurable, } \nu((B_k \cdot A) \cap C) = 0\} = 0.$$

We are now ready to show that asymptotic expansion in measure can be characterised by $\rho$-quasi-locality of the associated averaging projections. This is an analogue of [27, Theorem 3.11].

**Proposition 4.6** Let $\rho : \Gamma \curvearrowright (X, \nu)$ be an action on a probability space $(X, \nu)$ and $P_X$ be the associated averaging projection on $X$. Then $\rho$ is asymptotically expanding if and only if $P_X$ is $\rho$-quasi-local.
Proof Necessity: Suppose $P_X$ is not $\rho$-quasi-local, then by Corollary 4.5 we have that

$$\alpha := \frac{1}{2} \lim_{k \to +\infty} \sup \{ \nu(A) \nu(C) \mid A, C \subseteq X \text{ measurable}, \nu((B_k \cdot A) \cap C) = 0 \}.$$ 

is strictly positive. In particular, $\frac{1}{2} \leq 1 - \alpha < 1$. Thus, we can choose a sequence $(A_n, C_n)_{n \in \mathbb{N}}$ where $A_n, C_n \subseteq X$ are measurable subsets with $\nu((B_n \cdot A_n) \cap C_n) = 0$ such that $\nu(A_n) \nu(C_n) \geq \alpha$. Since $\nu(A_n) \leq 1$ and $\nu(C_n) \leq 1$, both $\nu(A_n)$ and $\nu(C_n)$ are at least $\alpha$. Furthermore, $\nu((B_n \cdot A_n) \cap C_n) = 0$ implies that $\nu(A_n \cap C_n) = 0$. In particular, both $\nu(A_n)$ and $\nu(C_n)$ are not greater than $1 - \alpha$ for each $n \in \mathbb{N}$.

If the action was asymptotically expanding, then Definition 2.6 and Lemma 2.11 would imply that there exist constants $b > 0$ and $h \in \mathbb{N}$ such that for every measurable subset $A \subseteq X$ with $\alpha \leq \nu(A) \leq 1 - \alpha$, we have $\nu(B_h \cdot A) > (1 + b)\nu(A)$. Let $k := mh$, where $m := \lceil \log_2 \left( \frac{1}{\alpha} - 1 \right) \rceil$. Then either

$$\nu(B_k \cdot A) > 1 - \alpha$$

or we deduce by induction on $m$ that

$$\nu(B_k \cdot A) > (1 + b)^m \nu(A) \geq 1 - \alpha.$$ 

Note that $A_n$ satisfies $\alpha \leq \nu(A_n) \leq 1 - \alpha$ for all $n \in \mathbb{N}$. Hence for $n \geq k$, we have $\nu(B_n \cdot A_n) > 1 - \alpha$. This is a contradiction to $\nu(C_n) \geq \alpha$ and $\nu((B_n \cdot A_n) \cap C_n) = 0$.

Sufficiency: Assume that $\rho$ is not asymptotically expanding. Then there exists $\alpha_0 \in (0, \frac{1}{2}]$ such that for every $n \in \mathbb{N}$ there exists a measurable subset $A_n \subseteq X$ with $\alpha_0 \leq \nu(A_n) \leq \frac{1}{2}$ and $\nu(B_n \cdot A_n) \leq \frac{3}{2} \nu(A_n)$. For every $n$ we have

$$\nu(X \setminus (B_n \cdot A_n)) = 1 - \nu(B_n \cdot A_n) \geq 1 - \frac{3}{2} \nu(A_n) \geq \frac{1}{4}.$$ 

Hence, we have that

$$\nu(A_n) \cdot \nu(X \setminus (B_n \cdot A_n)) \geq \frac{\alpha_0}{4} > 0,$$

which implies that the limit

$$\lim_{n \to +\infty} \sup \{ \nu(A) \nu(C) \mid A, C \subseteq X \text{ measurable}, \nu((B_n \cdot A) \cap C) = 0 \} \geq \frac{\alpha_0}{4}.$$ 

Hence, $P_X$ is not $\rho$-quasi-local by Corollary 4.5. \hfill \Box

We will now show that the dynamical quasi-locality completely determines the ordinary quasi-locality for those operators of the (unified) warped cone that arise as transformations of the base space. To be precise, consider the following $*$-homomorphism:

$$\Phi : \mathcal{B}(L^2(X, \nu)) \to \mathcal{B}(L^2(O_\Gamma X, \nu \times \lambda)), \quad T \mapsto T \otimes \text{Id}_{L^2([1,\infty))} \quad (4.4)$$

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(note that the Druţu–Nowak projection $\mathcal{G}$ equals to $\Phi(P_X)$). We can then prove the following:

**Proposition 4.7** Let $(X, d)$ be a metric space with diameter at most 2 equipped with a probability measure $\nu$, and $\rho : \Gamma \curvearrowright X$ be a continuous action. For any $T \in \mathcal{B}(L^2(X, \nu))$, we consider the following conditions:

1. $T$ is $\rho$-quasi-local;
2. $\Phi(T)$ is quasi-local;
3. the family of operators $T_t \equiv T$ for $t \in [1, \infty)$ is uniformly quasi-local on $\mathcal{W}_C(\Gamma \curvearrowright X)$.

Then we have $(1) \Rightarrow (2) \Rightarrow (3)$. Furthermore, if $\nu$ is Radon then they are all equivalent.

**Proof** $(1) \Rightarrow (2)$: Fix an $\epsilon > 0$. Since $T$ is $\rho$-quasi-local, there exists $k \in \mathbb{N}$ such that for any measurable subsets $A^i, C^i \subseteq X$ with $\nu((B_k \cdot A^i) \cap C^i) = 0$, we have $\|\chi_A^i T^i \chi_{C^i}\| < \epsilon$.

Given a pair of measurable subsets $A, C \subseteq \mathcal{O}_\Gamma X = X \times [1, \infty)$ with $d_{\Gamma}(A, C) > k$, we can write $A = \bigcup_{t \in [1, \infty)} A_t \times \{t\}$ and $C = \bigcup_{t \in [1, \infty)} C_t \times \{t\}$, where $A_t, C_t$ are measurable subsets in $X$.

For every $(x, t) \in \mathcal{O}_\Gamma X = X \times [1, \infty)$ and every $\gamma \in \Gamma$, we have $d_{\Gamma}((\gamma \cdot x, t), (x, t)) \leq \ell(\gamma)$. Since $d_{\Gamma}(A, C) > k$, it follows that $\nu((B_k \cdot A_t) \cap C_t) = 0$ for every $t \in [1, \infty)$. Hence, we conclude that $\|\chi_{A_t} T \chi_{C_t}\| < \epsilon$ for every $t \in [1, \infty)$.

For every $\xi \in L^2(\mathcal{O}_\Gamma X, \nu \times \lambda)$, we set $\xi_t(x) = \xi(x, t)$ so that $\xi_t \in L^2(X, \nu)$ for almost every $t \in [1, \infty)$. Using Fubini’s Theorem, we obtain that

$$
\|\chi_A \Phi(T) \chi_C \xi\|^2 = \int_{\mathcal{O}_\Gamma X} \left|\frac{(\chi_A(T \otimes \text{Id}_{L^2([1, \infty])}) \chi_C \xi)}{2d(\nu \times \lambda)(x, t)} \right|^2 dt
$$

$$
= \int_1^\infty \int_X \left|\chi_{A_t} T \chi_{C_t} \xi_t \right|^2 d\nu(x)dt
$$

$$
= \int_1^\infty \left\|\chi_{A_t} T \chi_{C_t} \xi_t \right\|^2 dt
$$

$$
\leq \int_1^\infty \epsilon^2 \|\xi_t\|^2 dt = \epsilon^2 \|\xi\|^2.
$$

It follows that $\Phi(T)$ is quasi-local.

$(2) \Rightarrow (3)$: For any measurable subsets $A, C \subseteq X$, we note that $d_{\Gamma}^t(A, C)$ equals $d_{\Gamma}(A \times [t, t+1], C \times [t, t+1])$. For every $f \in L^2(X, \nu)$ and $t \in [1, \infty)$, we construct a $F_t \in L^2(\mathcal{O}_\Gamma X, \nu \times \lambda)$ by letting $F_t(x, s) = f(x)$ if $t \leq s \leq t + 1$ and zero otherwise. Note that $\|f\|_\nu = \|F_t\|_{\nu \times \lambda}$ and $\|\chi_A T \chi_{C} f\| = \|\chi_{A} T \chi_{C} f\| = \|\chi_{A \times [t, t+1]} \Phi(T) \chi_{C \times [t, t+1]} F_t\|$. Now the rest of the proof is obvious.

$(3) \Rightarrow (1)$: Fix an $\epsilon > 0$. Then by the assumption, there exists an $R > 0$ such that for every $t \in [1, \infty)$ and measurable subsets $A, C \subseteq X$ with $d_{\Gamma}^t(A, C) > R$ we have $\|\chi_A T \chi_C\| < \epsilon$. We will verify that $\|\chi_A T \chi_C\| < \epsilon$ for measurable subsets $A, C \subseteq X$ with $\nu((B_R \cdot A) \cap C) = 0$. 

\[ \text{Birkhäuser} \]
Assume first that $A, C \subseteq X$ are compact subsets such that $(B_R \cdot A) \cap C = \emptyset$. It follows from Lemma 4.2 that

$$\bigcap_{t \geq 1} N_R(A; d_1^t) \cap C = (B_R \cdot A) \cap C = \emptyset.$$  

Since $C$ is compact and all $N_R(A; d_1^t)$ are closed, we deduce that $N_R(A; d_1^{t_0}) \cap C = \emptyset$ for some $t_0$ large enough. This means that $d_1^{t_0}(A, C) > R$ and hence $\|\chi_A T \chi_C\| < \epsilon$ by the hypothesis.

For general measurable subsets $A, C \subseteq X$ with $v((B_R \cdot A) \cap C) = 0$, replacing $C$ by $C \setminus (B_R \cdot A)$ if necessary (which only differ by a null set) we may assume that $(B_R \cdot A) \cap C = \emptyset$. Since the measure $v$ is Radon and finite, there exist increasing sequences of compact subsets $\{A_n \subseteq A\}_{n \in \mathbb{N}}$ and $\{C_n \subseteq C\}_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} v(A \setminus A_n) = 0$ and $\lim_{n \to \infty} v(C \setminus C_n) = 0$. A fortiori, we have $(B_R \cdot A_n) \cap C_n = \emptyset$ and it follows from the discussion in the second paragraph that $\|\chi_{A_n} T \chi_{C_n}\| < \epsilon$ for all $n \in \mathbb{N}$. Thus, $\|\chi_A T \chi_C\| = \sup_n \|\chi_{A_n} T \chi_{C_n}\| < \epsilon$.  

Combining Proposition 4.6 with Proposition 4.7 implies the desired characterisation of asymptotic expansion in measure in terms of quasi-locality:

**Theorem 4.8** Let $(X, d)$ be a metric space with diameter at most 2 equipped with a Radon probability measure $v$, and $\rho : \Gamma \rightharpoonup (X, d)$ be a continuous action. If $P_X$ is the associated averaging projection on $X$ and $\mathcal{G} = P_X \otimes \text{Id}_{L^2([1, \infty))}$ is the Druţu–Nowak projection, then the following are equivalent:

1. $\rho$ is asymptotically expanding;
2. $P_X$ is $\rho$-quasi-local;
3. $\mathcal{G}$ is quasi-local;
4. the family of operators $(P_X)_t \equiv P_X$ for $t \in [1, \infty)$ is uniformly quasi-local on $\mathcal{WC}(\Gamma \rightharpoonup X)$.

### 4.3 Projections approximated by finite dynamical propagation operators

In the previous subsection, we showed that asymptotic expansion in measure can be characterised by (dynamical) quasi-locality of certain projections (Theorem 4.8). In the same spirit of [23, Section 6], we would like to connect (dynamical) quasi-locality with finite (dynamical) propagation operators.

In doing so, we will show that unified warped cones arising from asymptotically expanding actions admit plenty of projections which can be approximated by finite propagation operators. This greatly generalise [11, Theorem 6.6]. It will turn out that all of these projections lie outside the image of the coarse Baum–Connes assembly map. We will return to these aspects in Sect. 5.

We once again introduce a dynamical analogue of an analytic property of operators, namely, the dynamical propagation (see Sect. 4.4 for the notion of ordinary finite propagation operators):

$$\text{Birkhäuser}$$
Definition 4.9 Let $\rho : \Gamma \actson (X, \nu)$ be an action on a probability space $(X, \nu)$. We say that an operator $T \in \mathcal{B}(L^2(X, \nu))$ has finite $\rho$-propagation if there is a $k \in \mathbb{N}$ such that $\chi_A T \chi_C = 0$ for any measurable subsets $A, C \subseteq X$ with $\nu((B_k \cdot A) \cap C) = 0$. The smallest $k$ satisfying the above condition is called the $\rho$-propagation of $T$.

Throughout the rest of this subsection, let $\Gamma \actson (X, \nu)$ be a measure-class-preserving action, $Y \subseteq X$ be a domain and $S \subseteq \Gamma$ be a finite symmetric set containing the identity. Recall from Proposition 3.10 that such an action induces a normalised local Markov kernel $\Pi_{Y,S}$ on $Y$ (Definition 3.9). This kernel is reversible with a reversing measure $\tilde{\nu}_{Y,S}$, where $\tilde{d}\tilde{\nu}_{Y,S} = \sigma_{Y,S} \tilde{d}(\nu|Y)$ for the function $\sigma_{Y,S}$ defined in (3.7). We denote by $\mathcal{P}_{Y,S} \in \mathcal{B}(L^2(Y, \tilde{\nu}_{Y,S}))$ and $\Delta_{Y,S} = 1 - \mathcal{P}_{Y,S} \in \mathcal{B}(L^2(Y, \tilde{\nu}_{Y,S}))$ the Markov and Laplacian operators associated with $\Pi_{Y,S}$, respectively.

We will present two different ways to produce projections from operators of finite $\rho$-propagation using Markov $S$-expansion. One is normalised to better accommodate the associated Markov kernel, while the other is non-normalised and more related to the original averaging projection $P_X$. In either case, our construction relies heavily on the techniques developed in Sect. 3.

4.3.1 Normalised projections

Let $\tilde{P}_{Y,S} \in \mathcal{B}(L^2(Y, \tilde{\nu}_{Y,S}))$ be the orthogonal projection onto constant functions on $Y$ (this need not coincide with $P_Y$, as the projection is taken with respect to the inner product $\langle \cdot, \cdot \rangle_{\tilde{\nu}_{Y,S}}$). Let us consider the isometric embedding

$$\tilde{T}_{Y,S} : L^2(Y, \tilde{\nu}_{Y,S}) \hookrightarrow L^2(X, \nu)$$

defined by pointwise multiplication by the function $\sqrt{\sigma_{Y,S}}$ on $Y$ and then extending by 0 on $X \setminus Y$. This induces the following adjoint $*$-homomorphism:

$$\tilde{\text{Ad}} : \mathcal{B}(L^2(Y, \tilde{\nu}_{Y,S})) \rightarrow \mathcal{B}(L^2(X, \nu)), \quad T \mapsto \tilde{T}_{Y,S} \circ T \circ (\tilde{T}_{Y,S})^*.$$ 

(4.5)

Note that $\tilde{T}_{Y,S}(1) = \sqrt{\sigma_{Y,S}}$, where 1 is the constant function 1 in $L^2(Y, \tilde{\nu}_{Y,S})$ and $\sigma_{Y,S}$ is defined to be 0 on every $x \in X \setminus Y$. It follows that

$$\tilde{P}_{Y,S} := \tilde{\text{Ad}}(\tilde{P}_{Y,S}) \in \mathcal{B}(L^2(X, \nu))$$

is the orthogonal projection onto the 1-dimensional subspace of $L^2(X, \nu)$ spanned by the vector $\sqrt{\sigma_{Y,S}}$. We also transfer the lazy Markov operator $\mathcal{P}_{Y,S} \in \mathcal{B}(L^2(Y, \tilde{\nu}_{Y,S}))$ to $\tilde{\mathcal{P}}_{Y,S} := \tilde{\text{Ad}}(\mathcal{P}_{Y,S}) \in \mathcal{B}(L^2(X, \nu))$. Similarly, $\tilde{\text{Ad}}$ sends the lazy Markov operator $\frac{1}{2} + \frac{1}{2} \mathcal{P}_{Y,S}$ to $\frac{1}{2} \chi_Y + \frac{1}{2} \tilde{\mathcal{P}}_{Y,S} \in \mathcal{B}(L^2(X, \nu))$.

Now the techniques developed in Sect. 3 can be used to prove the following:

Proposition 4.10 Let $\rho : \Gamma \actson (X, \nu)$ be a measure-class-preserving action and $Y \subseteq X$ be a domain of Markov $S$-expansion (Definition 3.12). Then the associated projection $\tilde{P}_{Y,S} \in \mathcal{B}(L^2(X, \nu))$ is a norm limit of operators $\left(\frac{1}{2} \chi_Y + \frac{1}{2} \tilde{\mathcal{P}}_{Y,S}\right)^n$, which all have finite $\rho$-propagation.
Proof Since the operator $\frac{1}{2} \mathcal{X}_Y + \frac{1}{2} \hat{\mathcal{P}}_Y, S$ has $\rho$-propagation at most $\max \{ \ell(s) | s \in S \}$, all of its powers $(\frac{1}{2} \mathcal{X}_Y + \frac{1}{2} \hat{\mathcal{P}}_Y, S)^n$ have finite $\rho$-propagation as well. By Theorem 3.7 and Lemma 3.13, the lazy Markov operator $\frac{1}{2} \mathcal{X}_Y + \frac{1}{2} \hat{\mathcal{P}}_Y, S$ on $L^2(Y, \tilde{\nu}_Y, S)$ has spectrum contained in $[-\frac{3}{4}, 1 - \varepsilon] \cup \{1\}$ for some $\varepsilon > 0$. It follows that the sequence $(\frac{1}{2} \mathcal{X}_Y + \frac{1}{2} \hat{\mathcal{P}}_Y, S)^n$ converges (as $n \to \infty$) in the operator norm to the projection onto the 1-eigenspace, which is exactly the projection $\hat{\mathcal{P}}_Y, S$. Since $\hat{\text{Ad}}$ is a $\ast$-homomorphism, $\| (\frac{1}{2} \mathcal{X}_Y + \frac{1}{2} \hat{\mathcal{P}}_Y, S)^n - \hat{\mathcal{P}}_Y, S \| \to 0$ for $n \to \infty$. This finishes the proof. \hfill $\Box$

Theorem 3.20 shows that strongly ergodic actions provide plenty of domains of Markov expansion. We can hence use Proposition 4.10 as an abundant source of projections which can be approximated by finite $\rho$-propagation operators.

As a corollary to Proposition 4.10 we also recover the following result by Druţu and Nowak:

Corollary 4.11 [11, Theorem 6.6] Let $\rho: \Gamma \curvearrowright (X, \nu)$ be a measure-preserving action on a probability space $(X, \nu)$. Suppose that $\rho$ has spectral gap, then the averaging projection $P_X$ is a norm limit of operators with finite $\rho$-propagation.

Proof Since the action is measure preserving, we have $\tilde{\nu}_{X, S} = |S| \cdot \nu$. It follows that $\hat{P}_{X, S} = P_X$ for any choice of $S \subseteq \Gamma$. If $\rho$ has spectral gap, then $X$ is a domain of expansion (see e.g. Corollary 3.16). Hence, $X$ is a domain of Markov expansion by Lemma 3.14 and we can apply Proposition 4.10 to conclude the proof. \hfill $\Box$

4.3.2 Non-normalised projections

Now we move on to the second construction, where we show that the averaging projections $P_Y$ are norm limits of operators with finite $\rho$-propagation as well. Unlike the previous construction, these projections will not be limits of powers of a fixed Markov operator. Instead, we will apply our structure theory for asymptotically expanding actions to produce appropriate sequences of operators.

We define a different embedding

$I_{Y, S}: L^2(Y, \tilde{\nu}_Y, S) \hookrightarrow L^2(X, \nu)$

simply by extending each function in $L^2(Y, \tilde{\nu}_Y, S)$ by 0 on $X \setminus Y$. In general, $I_{Y, S}$ is not isometric and may even be unbounded.

Assume now that there exists $\Theta \geq 1$ such that $1/\Theta \leq r(s, y) \leq \Theta$ for every $y \in Y$ and $s \in S_{Y, y}$. Under this assumption, it is clear that $I_{Y, S}$ is bounded. So it induces the following adjoint map:

$\text{Ad}: \mathfrak{B}(L^2(Y, \tilde{\nu}_Y, S)) \to \mathfrak{B}(L^2(X, \nu))$, by $T \mapsto I_{Y, S} \circ T \circ (I_{Y, S})^\ast$.

Note that— while being a bounded linear map preserving $\ast$-operations—the adjoint map $\text{Ad}$ might not be multiplicative.

6 Strictly speaking, [11, Theorem 6.6] concerns the operator $\mathfrak{G} = P_X \otimes \text{Id}_{L^2([1, \infty))}$ and also shows that it is “ghost”. We will recover these facts in Sect. 5.
As before, let \( \tilde{P}_{Y,S} \in \mathcal{B}(L^2(Y, \tilde{v}_{Y,S})) \) be the orthogonal projection onto constant functions, while \( P_Y \in \mathcal{B}(L^2(X, \nu)) \) is the orthogonal projection onto the one-dimensional subspace in \( L^2(X, \nu) \) spanned by \( \chi_Y \). Since \( (I_{Y,S})^*(g) = \frac{1}{\sigma_{Y,S}} g|_Y \) for \( g \in L^2(X, \nu) \), we have that

\[
\text{Ad}(\tilde{P}_{Y,S}) = \frac{v(Y)}{\tilde{v}_{Y,S}(Y)} \, P_Y.
\] (4.6)

We prove the following:

**Lemma 4.12** Let \( \rho : \Gamma \curvearrowright (X, \nu) \) be a measure-class-preserving action and \( Y \subseteq X \) be a domain of Markov S-expansion. Assume further that there exists \( \Theta \geq 1 \) such that \( \frac{1}{\Theta} \leq r(s, y) \leq \Theta \) for every \( y \in Y \) and \( s \in S_{Y,Y} \). Then the averaging projection \( P_Y \in \mathcal{B}(L^2(X, \nu)) \) is a norm limit of operators with finite \( \rho \)-propagation.

**Proof** By Theorem 3.7 and Lemma 3.13, the lazy Markov operator \( \frac{1}{2} + \frac{1}{2} P_{Y,S} \) on \( L^2(Y, \tilde{v}_{Y,S}) \) has spectrum contained in \( [-\frac{3}{4}, 1 - \varepsilon] \cup \{1\} \) for some \( \varepsilon > 0 \). Hence, \( (\frac{1}{2} + \frac{1}{2} P_{Y,S})^n \) converges in the operator norm to the projection \( \tilde{P}_{Y,S} \) in \( \mathcal{B}(L^2(Y, \tilde{v}_{Y,S})) \) as \( n \to \infty \).

Since the embedding \( I_{Y,S} \) is bounded, we obtain that

\[
I_{Y,S} \circ \left( \frac{1}{2} + \frac{1}{2} P_{Y,S} \right)^n \circ (I_{Y,S})^* = \text{Ad} \left( \left( \frac{1}{2} + \frac{1}{2} P_{Y,S} \right)^n \right) \xrightarrow{n \to \infty} \text{Ad}(\tilde{P}_{Y,S}).
\]

By (4.6), the latter is a multiple of \( P_Y \). The conclusion then holds because each \( I_{Y,S} \circ (\frac{1}{2} + \frac{1}{2} P_{Y,S})^n \circ (I_{Y,S})^* \) has \( \rho \)-propagation bounded by \( n \cdot \max \{ \ell(s) \mid s \in S \} \). \( \square \)

Unlike Proposition 4.10, Lemma 4.12 concerns projections that do not depend on the finite symmetric set \( S \). This allows us to prove a result for domains of asymptotic expansion as well:

**Proposition 4.13** Let \( \rho : \Gamma \curvearrowright (X, \nu) \) be a measure-class-preserving action. Then for any domain \( Y \subseteq X \) of asymptotic expansion, the averaging projection \( P_Y \) is a norm limit of operators with finite \( \rho \)-propagation.

**Proof** From Proposition 3.18, it follows that there is an exhaustion \( Y_n \not\supset Y \) by domains of Markov \( S^{(n)} \)-expansion such that for every \( n \in \mathbb{N} \) there is a \( \Theta_n \geq 1 \) such that \( \frac{1}{\Theta_n} \leq r(s, y) \leq \Theta_n \) for every \( y \in Y_n \) and \( s \in S_{Y_n,Y}^{(n)} \).

Now it follows from Lemma 4.12 that each \( P_{Y_n} \) is a norm limit of operators with finite \( \rho \)-propagation. Since \( Y_n \) increasingly converges to \( Y \) in measure and \( Y \) has finite measure, then \( P_{Y_n} \) converges to \( P_Y \) in the operator norm. Hence, a diagonal argument will conclude the proof. \( \square \)

It follows easily from the definitions that norm limits of operators with finite \( \rho \)-propagation are \( \rho \)-quasi-local. Hence, combining Proposition 4.6 with Proposition 4.13 we immediately obtain the following:

**Corollary 4.14** Let \( \rho : \Gamma \curvearrowright (X, \nu) \) be a measure-class-preserving action on a probability space \( (X, \nu) \). Then \( \rho \) is asymptotically expanding if and only if \( P_X \) is a norm limit of operators with finite \( \rho \)-propagation.
4.4 Characterising asymptotic expansion by finite propagation approximations

Finally, we conclude this section by combining results in Subsections 4.2 and 4.3 to prove that an action is asymptotically expanding if and only if the Druţu–Nowak projection can be approximated by operators with finite propagation.

Let \((X, d)\) be a metric space of diameter at most 2, \(\rho : \Gamma \curvearrowright X\) be a continuous action and \(O_\Gamma X\) the associated unified warped cone. If \(X\) is equipped with a probability measure \(\nu\), we give \(O_\Gamma X\) the product measure \(\nu \times \lambda\) and say that an operator \(T \in \mathcal{B}(L^2(O_\Gamma X, \nu \times \lambda))\) has finite propagation if there exists an \(R > 0\) such that for any two measurable subsets \(A, C \subseteq O_\Gamma X\) with \(d_\Gamma(A, C) > R\), we have \(\chi_A T \chi_C = 0\).

**Proposition 4.15** Let \((X, d)\) be a metric space with diameter at most 2 equipped with a probability measure \(\nu\), and \(\rho : \Gamma \curvearrowright X\) be a continuous action. If \(T \in \mathcal{B}(L^2(X, \nu))\) has finite \(\rho\)-propagation, then \(\Phi(T)\) has finite propagation. If in addition \(\nu\) is Radon, the converse implication holds as well.

**Proof** The argument is identical to that of Proposition 4.7 with \(\epsilon = 0\). \(\square\)

Since norm limits of operators with finite propagation are quasi-local, we can combine Proposition 4.15 and Corollary 4.14 with Theorem 4.8 (3) \(\Rightarrow\) (1) to obtain a dynamical counterpart of [23, Theorem C]:

**Theorem 4.16** Let \((X, d)\) be a metric space with diameter at most 2 equipped with a probability measure \(\nu\), and \(\rho : \Gamma \curvearrowright X\) be a continuous measure-class-preserving action. The following are equivalent:

1. \(\rho\) is asymptotically expanding;
2. the averaging projection \(P_X\) is a norm limit of operators with finite \(\rho\)-propagation;
3. the Druţu–Nowak projection \(G\) is a norm limit of operators with finite propagation.

For later use, we record that we can apply Proposition 4.15 to the projections constructed in Propositions 4.10 and 4.13 and obtain the following:

**Corollary 4.17** Let \((X, d)\) be a metric space with diameter at most 2 equipped with a probability measure \(\nu\), and \(\rho : \Gamma \curvearrowright X\) be a continuous and measure-class-preserving action. Let \(P \in \mathcal{B}(L^2(X, \nu))\) be one of the following rank-one projection:

1. \(P = \hat{P}_{Y, S}\) for a domain \(Y \subseteq X\) of Markov S-expansion;
2. \(P = P_Y\) for a domain \(Y \subseteq X\) of asymptotic expansion.

Then the projection \(\Phi(P) = P \otimes \text{Id}_{L^2([1, \infty))}\) is a norm limit of operators with finite propagation.

5 The coarse Baum–Connes conjecture

In this section, we will use the projections constructed in Sect. 4.3 to provide new counterexamples to the coarse Baum–Connes conjecture. These arise from certain warped cones associated with asymptotically expanding actions. We will follow the outline of [45, Section 3] (the origin of this method goes back to [16, 60]).
Throughout this section, \((X, d)\) will be a compact metric space with diameter at most 2 endowed with a non-atomic probability measure \(\nu\) of full support (i.e., every singleton has measure zero and every open set has positive measure). As usual, \(\Gamma\) is a countable discrete group with a proper length function \(\ell\). Furthermore, \(\Gamma \bowtie (X, d, \nu)\) will be a continuous measure-class-preserving action.

### 5.1 Roe algebras and projections

Let us begin by recalling some basic notions concerning Roe algebras.

Let \((Y, d)\) be any proper metric space. In particular, \(Y\) is locally compact and \(\sigma\)-compact. Let \(C_0(Y)\) be the \(C^*\)-algebra of continuous functions on \(Y\) vanishing at infinity. A non-degenerate \(*\)-representation \(C_0(Y) \to \mathcal{B}(\mathcal{H})\) on some separable Hilbert space \(\mathcal{H}\) is called \(\textit{ample}\) if no non-zero element of \(C_0(Y)\) acts as a compact operator on \(\mathcal{H}\). An operator \(a \in \mathcal{B}(\mathcal{H})\) has \(\textit{finite propagation}\) if there is \(r > 0\) such that \(f a g = 0\) whenever \(f, g \in C_0(Y)\) satisfy \(d(\text{supp}(f), \text{supp}(g)) > r\).\(^7\) Moreover, an operator \(a \in \mathcal{B}(\mathcal{H})\) is called \(\textit{locally compact}\) if \(fa\) and \(af\) are compact for all \(f \in C_0(Y)\).

The \textit{algebraic Roe algebra} \(\mathbb{C}[Y]\) of \(Y\) is the \(*\)-algebra of locally compact finite propagation operators in \(\mathcal{B}(\mathcal{H})\), and the \textit{Roe algebra} \(C^*(Y)\) of \(Y\) is the norm-closure of \(\mathbb{C}[Y]\) in \(\mathcal{B}(\mathcal{H})\). Note that the Roe algebra \(C^*(Y)\) does not depend on the choice of the non-degenerate ample \(*\)-representation of \(C_0(Y)\), but only up to non-canonical \(*\)-isomorphism (see e.g. [61, Remark 5.1.13]). On the other hand, the \(K\)-theory groups \(K_\bullet(C^*(Y))\) do not depend on the choice of such representations up to \(\textit{canonical} \ (*\)-isomorphism (see e.g. [61, Theorem 5.1.15]). It is well-known that the isomorphism class of \(C^*(Y)\) is a coarse invariant for the metric space \(Y\).

Let now \((X, d, \nu)\) be a metric measure space as outlined at the beginning of Sect. 5. Since \(\nu\) has full support and is non-atomic, the multiplication representation of \(C(X)\) on \(L^2(X, \nu)\) is non-degenerate and ample. Hence the multiplication representation of \(C_0(\Gamma \bowtie X)\) on \(L^2(X \times [1, \infty), \nu \times \lambda)\) is also non-degenerate and ample. We can thus use it to form the Roe algebra \(C^*(\Gamma \bowtie X)\).

As explained by Sawicki in [45, Proposition 1.1], the original Drutu–Nowak projection \(\mathcal{G} \in \mathcal{B}(L^2(\Gamma \bowtie X, \nu \times \lambda))\) is \(\textit{not} \) locally compact because its image contains a copy of \(L^2([1, \infty), \lambda)\). In particular, \(\mathcal{G}\) cannot belong to the Roe algebra. One way to overcome this issue is to consider the subspace \((X \times \mathbb{N}, d_\Gamma)\) of the unified warped cone \(\Gamma \bowtie X\) instead. We will call this the \textit{integral warped cone}. Since the embedding \((X \times \mathbb{N}, d_\Gamma) \hookrightarrow (\Gamma \bowtie X, d_\Gamma)\) is a quasi-isometry, their Roe algebras are isomorphic. We will hence abuse the notation and denote also the integral warped cone by \(\Gamma \bowtie X\).

Similarly, we also define the following analogue of the \(*\)-homomorphism \(\Phi\) defined in (4.4) (still denoted by \(\Phi\)):

\[
\Phi : \mathcal{B}(L^2(X, \nu)) \to \mathcal{B}(L^2(\Gamma \bowtie X, \nu \times \lambda_{\mathbb{N}})), \quad T \mapsto T \otimes \text{Id}_{L^2(\mathbb{N})},
\]

\(^7\) It is easy to check that for a proper metric space \((X, d)\), \(T \in \mathcal{B}(L^2(X, \nu))\) has finite propagation \(\textit{if and only if}\) there exists an \(R > 0\) such that \(\chi_A T \chi_C = 0\) whenever \(A, C \subseteq X\) are measurable subsets with \(d(A, C) > R\). In particular, this definition is equivalent to the one given in Sect. 4.4.
where \( \lambda_\mathbb{N} \) denotes the counting measure on \( \mathbb{N} \). It is elementary to check that Theorem 4.8 and Proposition 4.15 still hold in the integral setting. It follows that the integral analogues of Theorem 4.16 and Corollary 4.17 hold true as well. We will henceforth use their integral versions without further notice.

Let us now focus on the projections considered in Corollary 4.17. More precisely, we denote by \( \mathcal{P} \) the set of rank one projections in \( \mathcal{B}(L^2(X, \nu)) \) as follows:

\[
P \in \mathcal{P} \iff \text{either } P = \hat{P}_{Y, S} \text{ for a domain } Y \subseteq X \text{ of Markov } S\text{-expansion}
\]

or

\[
P = P_Y \text{ for a domain } Y \subseteq X \text{ of asymptotic expansion}.
\]

For the averaging projection \( P_X \), the associated projection \( \Phi(P_X) = P_X \otimes \text{Id}_{\ell^2(\mathbb{N})} \) (still denoted by \( \Phi \)) is called the integral Druţu–Nowak projection (see [45, Proposition 1.3]). It follows from Corollary 4.17 that the projection \( \Phi(P) \) can be approximated by finite propagation operators for every \( P \in \mathcal{P} \). Actually, we can even show the following stronger statement:

**Proposition 5.1** For every \( P \in \mathcal{P} \), the projection \( \Phi(P) \) is non-compact and belongs to the Roe algebra \( C^*(\mathcal{O}_\Gamma X) \) of the integral warped cone \( \mathcal{O}_\Gamma X \). In particular, when the action is asymptotically expanding the integral Druţu–Nowak projection \( \Phi \) belongs to \( C^*(\mathcal{O}_\Gamma X) \).

**Proof** Clearly, each \( \Phi(P) = P \otimes \text{Id}_{\ell^2(\mathbb{N})} \) is non-compact for \( P \in \mathcal{P} \). We only show that \( \Phi(P) \) belongs to \( C^*(\mathcal{O}_\Gamma X) \) when \( P = \hat{P}_{Y, S} \) for a domain \( Y \subseteq X \) of Markov \( S\)-expansion, as the other case is similar and almost identical to the proof of [45, Proposition 1.3]. Recall that \( \hat{P}_{Y, S} \) is the orthogonal projection onto the one-dimensional subspace of \( L^2(X, \nu) \) spanned by the vector \( \sqrt{\sigma_{Y, S}} \) defined in (3.7).

Since \( X \) is compact, there exists a Borel partition \( \mathcal{V} = \{ V_i \mid i \in I \} \) of \( \mathcal{O}_\Gamma X \) such that each \( V_i \) has diameter at most 1 and is contained in some level set \( X \times \{ n \} \), and for each \( n \in \mathbb{N} \) only finitely many \( V_i \) are contained in \( X \times \{ n \} \). For each \( i \in I \), we write \( V_i = U_i \times \{ n(i) \} \) for Borel \( U_i \subseteq X \) and \( n(i) \in \mathbb{N} \). We consider the closed subspace \( W \subseteq L^2(\mathcal{O}_\Gamma X, \nu \times \lambda_\mathbb{N}) \) spanned by

\[
\left\{ (\chi_{U_i} \cdot \sqrt{\sigma_{Y, S}}) \otimes \chi_{\{ n(i) \}} \mid i \in I \right\}.
\]

Let \( R \in \mathcal{B}(L^2(\mathcal{O}_\Gamma X, \nu \times \lambda_\mathbb{N})) \) be the orthogonal projection onto \( W \). It is clear that \( \Phi(P) \) is a subprojection of \( R \), so \( \Phi(P) = R \circ \Phi(P) \circ R \). Moreover, the projection \( R \) has propagation at most one.

By Corollary 4.17, \( \Phi(P) \) is a norm limit of finite propagation operators \( T_n \in \mathcal{B}(L^2(\mathcal{O}_\Gamma X, \nu \times \lambda_\mathbb{N})) \). In particular, we have \( \Phi(P) = \lim_{n \to \infty} RT_n R \). Since each \( RT_n R \) has finite propagation, it suffices to show that it is also locally compact. If \( \phi \in C_0(\mathcal{O}_\Gamma X) \) is a function of compact support, then its range is contained in \( L^2(X \times \{ 1, 2, \ldots, N_0 \}) \) for some \( N_0 \in \mathbb{N} \). This implies that \( R \phi \) is of finite rank. Since the set of compact operators is norm-closed, we have that \( R \psi \) is compact for every \( \psi \in C_0(\mathcal{O}_\Gamma X) \). Since \( R \) is self-adjoint, \( \psi R \) is compact as well. Hence, we conclude that both \( RT_n R \psi \) and \( \psi RT_n R \) are compact for every \( \psi \in C_0(\mathcal{O}_\Gamma X) \), as desired. \( \square \)

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Proposition 5.1 allows us to use Theorem 4.8 to deduce the main theorem of this subsection. Namely, the following dynamical version of [23, Theorem C]:

**Theorem 5.2** Let \((X, d)\) be a compact metric space with diameter at most 2 equipped with a non-atomic Radon probability measure \(\nu\) of full support, and \(\rho : \Gamma \curvearrowright (X, d, \nu)\) be a continuous and measure-class-preserving action.

If \(P_X\) is the associated averaging projection on \(X\) and \(\mathcal{G} = P_X \otimes \text{Id}_{\ell^2(\mathbb{N})}\) is the integral Druţu–Nowak projection, then the following are equivalent:

1. \(\rho\) is asymptotically expanding;
2. \(P_X\) is \(\rho\)-quasi-local;
3. \(\mathcal{G}\) is quasi-local;
4. \(\mathcal{G}\) belongs to the Roe algebra \(C^*(\mathcal{O}_\Gamma X)\) of the integral warped cone \(\mathcal{O}_\Gamma X\).

Another important feature of the projections \(\Phi(P)\) for \(P \in \mathcal{P}\) is that they are ghost operators. This notion was originally introduced by Yu (unpublished) in his study of the coarse Baum–Connes conjecture. We will use the following:

**Definition 5.3** [11, Definition 6.5] Given a metric measure space \((Z, d, \nu)\), an operator \(T \in \mathcal{B}(L^2(Z, \nu))\) is called ghost if for every \(R, \epsilon > 0\), there exists a bounded subset \(C \subseteq Z\) such that for any \(\phi \in L^2(Z, \nu)\) with \(\|\phi\| = 1\) and \(\text{supp}(\phi) \subseteq B_R(x; d)\) for some \(x \in Z \setminus C\) we have \(\|T\phi\| \leq \epsilon\).

Firstly, we observe the following easy fact:

**Lemma 5.4** A non-atomic probability measure \(\nu\) on a metric space \((Z, d)\) is necessarily upper uniform ([11, Definition 6.1]) in the sense that

\[
\lim_{r \to 0} \sup_{z \in Z} \nu(B_r(z; d)) = 0.
\]

**Proof** If there exist an \(\epsilon > 0\) and a sequence \(z_n \in Z\) such that \(\nu(B_{1/n}(z_n; d)) \geq \epsilon > 0\) for every \(n \in \mathbb{N}\), then there must be some point \(z \in Z\) that belongs to \(B_{1/n}(z_n; d)\) for infinitely many \(n\). To see this, it is sufficient to note that \(\nu(\bigcap_{N \in \mathbb{N}} \bigcup_{n > N} B_{1/n}(z_n; d)) = \lim_{N \to \infty} \nu(\bigcup_{n > N} B_{1/n}(z_n; d)) = \epsilon\) as the probability measure \(\nu\) is continuous from above. On the other hand, such a \(z\) must be an atom for \(\nu\) so that \(\nu\) cannot be non-atomic.

The following lemma is a generalisation of [11, Theorem 6.6]:

**Lemma 5.5** If \(T \in \mathcal{B}(L^2(X, \nu))\) is any orthogonal rank one projection, then \(\Phi(T) \in \mathcal{B}(L^2(\mathcal{O}_\Gamma X, \nu \times \lambda_{\mathbb{N}}))\) is ghost. In particular, \(\Phi(P)\) is a ghost projection for every \(P \in \mathcal{P}\).

**Proof** Since \(X\) is compact, the action \(\Gamma \curvearrowright X\) is uniformly continuous. Then the proof of Lemma 4.2 can be adapted to show that the balls \(B_R(x; d^\text{fil}_\Gamma)\) are contained in \(N_{\delta_n}(B_{[R]} \cdot x; d) \subseteq X\) for some positive \(\delta_n\) independent of \(x\) and such that \(\delta_n \to 0\), where \(B_{[R]}\) denotes the ball in \(\Gamma\). Since \(N_{\delta_n}(B_{[R]} \cdot y; d) \subseteq \bigcup_{y \in B_{[R]}} B_{\delta_n}(y \cdot x; d)\) and \(B_{[R]}\) is finite, we easily deduce from the upper uniformity of \(\nu\) (Lemma 5.4) that \(\lim_{n \to \infty} \sup_{x \in X} \nu(B_R(x; d^\text{fil}_\Gamma)) = 0\) (see also [11, Lemma 6.3]).
Let \( \epsilon, R > 0 \) be fixed and let \( C_N := \mathcal{O}_\Gamma X \cap (X \times \{1, N\}) \) for \( N \in \mathbb{N} \). We note that \( C_N \) is a bounded subset of \( \mathcal{O}_\Gamma X \) and any point in \( \mathcal{O}_\Gamma X \setminus C_N \) is of the form \((x, n)\) for some \( n \geq N \) and \( x \in X \). It is well-known that every rank one projection \( T \in \mathcal{B}(L^2(X, \nu)) \) is of the form \( T \eta = \langle \eta, \xi \rangle \xi \) for some unit vector \( \xi \in L^2(X, \nu) \). To show that \( \Phi(T) \in \mathcal{B}(L^2(\mathcal{O}_\Gamma X, \nu \times \lambda_{[N]})) \) is ghost, we fix any \( \phi \in L^2(\mathcal{O}_\Gamma X, \nu \times \lambda_{[N]}) \) with \( \|\phi\| = 1 \) and \( \text{supp}(\phi) \subseteq B_R((x, n); d_\Gamma) \) for some \((x, n) \in \mathcal{O}_\Gamma X \setminus C_N \). So we have that

\[
\|\Phi(T)(\phi)\|^2 = \sum_{m \in \mathbb{N}} \|\xi\|^2 \cdot \left| \int_X \phi(y, m)\bar{\xi}(y) d\nu(y) \right|^2
\]

\[
\leq \sum_{m=n-R}^{n+R} \left( \int_{B_R(x; d_\Gamma^n)} |\phi(y, m)\bar{\xi}(y)| d\nu(y) \right)^2
\]

\[
\leq \sum_{m=n-R}^{n+R} \left( \int_X |\phi(y, m)|^2 d\nu(y) \cdot \int_{B_R(x; d_\Gamma^n)} |\bar{\xi}(y)|^2 d\nu(y) \right)
\]

\[
\leq \sum_{m=n-R}^{n+R} \int_{B_R(x; d_\Gamma^n)} |\bar{\xi}(y)|^2 d\nu(y),
\]

where the last inequality uses the fact that \( \int_X |\phi(y, m)|^2 d\nu(y) \leq \|\phi\|^2 = 1 \) for every \( m \in \mathbb{N} \). Since \( \xi \in L^2(X, \nu) \) and \( \sup_{x \in X} \nu(B_R(x; d_\Gamma^n)) \to 0 \) as \( n \to \infty \), it follows that \( \|\Phi(T)(\phi)\|^2 \to 0 \) for \( n \to \infty \). We can hence choose \( N \) large enough so that \( \|\Phi(T)(\phi)\| \leq \epsilon \) for every \( \phi \) with \( \text{supp}(\phi) \subseteq B_R((x, n); d_\Gamma) \) for some \((x, n) \in \mathcal{O}_\Gamma X \setminus C_N \), as desired. \( \square \)

Combining Proposition 5.1 with Lemma 5.5, we obtain the following:

**Corollary 5.6** Let \((X, d)\) be a compact metric space with diameter at most 2 endowed with a non-atomic probability measure \(\nu\) of full support, and \(\Gamma \curvearrowright (X, d, \nu)\) a measure-class-preserving continuous action. Then each \(\Phi(P) \in \mathcal{B}(L^2(\mathcal{O}_\Gamma X, \nu \times \lambda_{[N]}))\) for \(P \in \mathcal{P}\) is a non-compact ghost projection in the Roe algebra \(C^*(\mathcal{O}_\Gamma X)\) of the integral warped cone \(\mathcal{O}_\Gamma X\).

### 5.2 Counterexamples to the coarse Baum–Connes conjecture

In this subsection, we will consider the subset \(Q_X := X \times \{2^n | n \in \mathbb{N}\}\) of \(X \times \{1, \infty\}\) and the associated subspace \(\mathcal{Q}_\Gamma X\) (which we will call *sparse warped cone*) of the unified warped cone \(\mathcal{O}_\Gamma X\). The main goal is to show that under certain mild assumptions all non-compact ghost projections in the Roe algebra \(C^*(\mathcal{Q}_\Gamma X)\) lie outside the image of the coarse Baum–Connes assembly map. In particular, they all violate the coarse Baum–Connes conjecture.

As before, we define a \(\ast\)-homomorphism \(\Phi_Q\) as follows:

\[
\Phi_Q: \mathcal{B}(L^2(X, \nu)) \to \mathcal{B}(L^2(\mathcal{Q}_\Gamma X, \nu \times \lambda_{[N]})), \quad T \mapsto T \otimes \text{Id}_{\ell^2([2^n | n \in \mathbb{N}])}.
\]
It is easy to see that Corollary 5.6 still holds in this setting: under the same assumption, each $\Phi_Q(P)$ with $P \in P$ is a non-compact ghost projection in the Roe algebra $C^*(Q\Gamma X)$. We call $\mathfrak{S}_Q = \Phi_Q(P_X)$ the sparse Drutu–Nowak projection.

The idea of the proof is to construct two “trace” maps $\tau_d$ and $\tau_u$ on $K_0(C^*(Q\Gamma X))$, whose restrictions to the image of the coarse assembly map coincide and yet take different values on every non-compact ghost projection in $C^*(Q\Gamma X)$. The following argument is a combination of those in [16, 45, 60]. We have decided to provide here a fair amount of details, because it also requires a few (minor) adaptations and extensions.

**Remark 5.7** The choice of $2^n$ in the definition of $Q\Gamma X$ is rather arbitrary and made for the sake of concreteness. We could equally set $Q X = X \times \{a_n|n \in \mathbb{N}\}$ for any other sequence $\{a_n\}_{n \in \mathbb{N}} \subseteq [1, \infty)$ as long as $\lim_{n,m \to \infty} |a_n - a_m| = \infty$.

### 5.2.1 The trace $\tau_d$

For each $n \in \mathbb{N}$, we denote by $Q_n \in \mathfrak{B}(L^2(Q\Gamma X, \nu \times \lambda_n))$ the orthogonal projection onto $L^2(X \times \{2^n\}, \nu)$. For $T \in \mathfrak{B}(L^2(Q\Gamma X, \nu \times \lambda_n))$ with propagation at most $2^n-1$, we have $Q_n T = T Q_n$ and define $T_n := Q_n T Q_n \in C^*(X \times \{2^n\})$. Hence, the map

$$\mathbb{C}[Q\Gamma X] \ni T \mapsto (T_n)_{n \in \mathbb{N}} \in \prod_n C^*(X \times \{2^n\})$$

is multiplicative, contractive and $*$-preserving on the algebraic Roe algebra $\mathbb{C}[Q\Gamma X]$. Thus, it yields a $*$-homomorphism on the entire Roe algebra $C^*(Q\Gamma X)$.

As each $X \times \{2^n\}$ is compact, $C^*(X \times \{2^n\})$ is $*$-isomorphic to the $C^*$-algebra of compact operators $\mathfrak{R}(L^2(X \times \{2^n\}))$. Hence, the canonical trace map $\text{Tr}$ on $\mathfrak{R}(L^2(X \times \{2^n\}))$ induces $\text{Tr}_*: K_0(C^*(X \times \{2^n\})) \to \mathbb{Z}$. As in [60, Section 6] and [45, Section 3], we define the trace map

$$\tau_d : K_0(C^*(Q\Gamma X)) \to \prod \mathbb{R}$$

as the composition of the trace $\text{Tr}_* : K_0(C^*(X \times \{2^n\})) \to \mathbb{Z}$ with $\mathbb{Z} \subseteq \mathbb{R}$ with the map

$$K_0(C^*(Q\Gamma X)) \to K_0\left(\prod_n C^*(X \times \{2^n\})\right)$$

induced by $T \mapsto (T_n)_{n \in \mathbb{N}}$ under the identification.
\[
K_0\left( \frac{\prod_n C^*(X \times [2^n])}{\bigoplus_n C^*(X \times [2^n])} \right) \cong \frac{K_0(\prod_n C^*(X \times [2^n]))}{K_0\left( \bigoplus_n C^*(X \times [2^n]) \right)} \cong \prod_n \frac{K_0(C^*(X \times [2^n]))}{K_0(\bigoplus_n C^*(X \times [2^n]))}.
\]

The proof of the following lemma is almost identical to the proof of [60, Theorem 6.1], we include here a short proof for the convenience of the reader.

**Lemma 5.8** Let \( p \in C^*(Q_\Gamma X) \) be any projection, then \( \tau_d([p]) = 0 \) if and only if \( p \) is compact. In particular, we have \( \tau_d([\Phi_Q(P)]) \neq 0 \) for every \( P \in \mathcal{P} \).

**Proof** Firstly, we note that for every \( T \in C^*(Q_\Gamma X) \), we have \( [T, Q_n] \to 0 \) as \( n \to \infty \).

In particular, for every projection \( p \in C^*(Q_\Gamma X) \) we have that \( Q_n p Q_n \) gets arbitrarily close to some honest projections \( q_n \) in \( C^*(X \times [2^n]) \) as \( n \to \infty \). In other words, \( [(Q_n p Q_n)_{n \in \mathbb{N}}] = [(q_n)_{n \in \mathbb{N}}] \) in \( \prod_n C^*(X \times [2^n]) / \bigoplus_n C^*(X \times [2^n]) \).

By the definition of \( \tau_d \), we have that
\[
\tau_d([p]) = [(\operatorname{Tr}(q_1), \operatorname{Tr}(q_2), \ldots)] = [\dim(q_1), \dim(q_2), \ldots],
\]

where \( \dim(q_n) \) denotes the dimension of the range of \( q_n \). On the other hand, as \( \|Q_n p Q_n - q_n\| \to 0 \) it follows that the projection \( p \) is compact if and only if \( \dim(q_n) = 0 \) for all but finitely many \( n \). So we conclude that \( \tau_d([p]) = 0 \) if and only if \( p \) is compact. \( \square \)

### 5.2.2 The trace \( \tau^U \)

To construct the other trace map \( \tau^U \), we need some extra assumptions and preliminaries. Following [47], we equip \( QX = X \times [2^n]_{n \in \mathbb{N}} \) with the open cone metric
\[
d_Q((x_1, t_1), (x_2, t_2)) := (t_1 \wedge t_2) \cdot d(x_1, x_2) + |t_1 - t_2|
\]
so that \( QX \) and \( Q_\Gamma X \) coincide as sets but are equipped with different metrics. We can then define a metric \( d_{\Gamma \times Q} \) on the product \( \Gamma \times QX \) as the largest metric such that

- \( d_{\Gamma \times Q}((\gamma, (x_1, t_1)), (\gamma, (x_2, t_2))) \leq d_Q((x_1, t_1), (x_2, t_2)) \);
- \( d_{\Gamma \times Q}((\eta \gamma, (x, t)), (\eta \gamma, x, t)) \leq \ell(\eta) \)

for every \( \gamma, \eta \in \Gamma \) and \((x_1, t_1), (x_2, t_2) \in QX \).

The projection to the second coordinate gives a natural quotient map \( \pi : \Gamma \times QX \to Q_\Gamma X \) and the metric \( d_{\Gamma \times Q} \) is defined so that the quotient metric on \( Q_\Gamma X \) coincides with the warped metric \( d_\Gamma \). Since \( X \) is compact, it is shown in [47, Proposition 3.10] that the action on \( X \) is free if and only if \( \pi \) is asymptotically faithful. Recall that a surjective map between metric spaces \( \pi : (Y, d_Y) \to (Z, d_Z) \) is called asymptotically faithful if for every \( R > 0 \) there is a bounded subset \( C_R \subseteq Z \) such that the restriction

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8 We remark that the metric \( d_{\Gamma \times Q} \) is denoted by \( d'_1 \) in [45] and it is isometric—but not equal—to the metric \( d^1 \) used in [47, Definition 3.6] The latter is also denoted by \( d_1 \) in [45].
of \( \pi \) to every \( R \)-ball centred at a point outside of \( \pi^{-1}(C_R) \) is an isometry \[47, 60\]. Asymptotic faithfulness will play an important role later on, we thus need to restrict our attention to free actions.

To estimate operator norms of finite propagation operators in \( \mathcal{B}(L^2(\Gamma \times QX)) \), we assume that the metric space \( (\Gamma \times QX, d) \) has the operator norm localisation property (ONL) (see \[7\]). Namely, if we equip \( \Gamma \times QX \) with the product measure \( \lambda \times \nu \times \lambda \mathbb{N} \) (here \( \lambda \) is the counting measure on the discrete group \( \Gamma \)), we say that \( (\Gamma \times QX, d) \) has ONL if for every \( c \in (0, 1) \) and \( r > 0 \) there exists an \( R > 0 \) so that for any operator \( T \in \mathcal{B}(L^2(\Gamma \times QX)) \) of propagation at most \( r \) there exists a unit vector \( \xi \in L^2(\Gamma \times QX) \) with \( \text{diam}(\text{supp}\xi) \leq R \) satisfying \( \|T\xi\| \geq c\|T\| \).

**Remark 5.9** It follows from \[7, Proposition 2.4\] that the above definition of ONL is equivalent to the original definition in \[7, Definition 2.3\]. It follows from the work of Sako \[43\] that—for metric spaces that are proper and have bounded geometry—ONL is also equivalent to property A in the sense of \[42, Definition 2.1\] (see \[45, Corollary 2.5\] for a proof).

**Remark 5.10** For a Lipschitz action \( \Gamma \actson X \) on a compact space \( X \), the metric space \( (\Gamma \times QX, d) \) has ONL under either of the following conditions:

1. \( \Gamma \) has property A and \( X \) is a manifold;
2. \( \Gamma \) is locally compact and has ONL.

We refer to \[45, Corollary 2.11\] for a more general statement.

As in \[45, Section 3.2\], let \( \rho : \Gamma \actson X \) be a free action so that \( \pi : \Gamma \times QX \to Q \Gamma X \) is asymptotically faithful. Let \( T \in C^*(Q \Gamma X) \) be an operator with propagation at most \( r \) and let \( n_0 \) be large enough so that for every \( n > n_0 \) the quotient map \( \pi \) restricts to an isometry on every ball of radius \( 3r \) in \( \Gamma \times X \times \{2^n\} \subseteq \Gamma \times QX \). This allows us to define, for every \( n > n_0 \), a canonical \( \Gamma \)-equivariant lift \( T_n' \in \mathbb{C}[\Gamma \times X \times \{2^n\}]^\Gamma \) of the operator \( T_n = Q_n T_n Q_n \in C^*(X \times \{2^n\}) \). Specifically, given \( \xi, \eta \in L^2(\Gamma \times X \times \{2^n\}) \) with support of diameter at most \( r \) we define

\[
\langle T_n' \xi, \eta \rangle := \begin{cases} 
\langle T_n(\xi \circ \sigma), \eta \circ \sigma \rangle, & \text{if } d_{\Gamma \times Q}(\text{supp}\xi, \text{supp}\eta) \leq r, \\
0, & \text{otherwise},
\end{cases}
\]

where \( \sigma \) is the inverse of the restriction of \( \pi \) to \( \text{supp}(\xi) \cup \text{supp}(\eta) \). Note that the subspace spanned by vectors with diameter of supports at most \( r \) is dense in \( L^2(\Gamma \times X \times \{2^n\}) \), hence \( T_n' \) is well-defined. It is verified in \[45, Lemma 3.1\] that each \( T_n' \) is bounded, and it is clear that each \( T_n' \) has propagation at most \( r \) and is locally compact and invariant under conjugations.

Moreover, \[45, Lemma 3.2\] shows that if \( (\Gamma \times QX, d) \) has ONL, then \( \|T_n'\| \leq c\|T\| \) for every \( n \in \mathbb{N} \) and some uniform constant \( c > 0 \) coming from ONL. It follows that the map \( T \mapsto \{(T_n')_{n \in \mathbb{N}}\} \) induces an algebraic \(*\)-homomorphism:

\[
\Psi : \mathbb{C}[Q \Gamma X] \longrightarrow \prod_n C^*(\Gamma \times \{2^n\} \times X)^\Gamma \oplus \prod_n C^*(\Gamma \times \{2^n\} \times X)^\Gamma
\]
which can be extended to a $C^*$-homomorphism on the whole $C^*(\Gamma \times X)$. As a matter of fact, it is possible to obtain a slightly improved control on the norm of $\Psi(T)$ (the proof is omitted as it is equal to the proof of [61, Lemma 13.3.11]):

**Lemma 5.11** Let $\Gamma \curvearrowright X$ be a free action and assume that $(\Gamma \times QX, d_{\Gamma \times Q})$ has ONL. Then for every $T \in \mathbb{C}[\mathcal{Q}_\Gamma X]$ we have that

$$
\|\Psi(T)\| = \sup_{R \geq 0} \lim_{n \to \infty} \sup \left\{ \|T_n\| \left| \xi \in L^2(X \times \{2^n\}), \|\xi\| = 1 \text{ and diam}(\text{supp}\xi) \leq R \right. \right\}.
$$

Lemma 5.11 allows us to identify the kernel of $\Psi$ with the closed ideal consisting of all ghost operators in $C^*(\mathcal{Q}_\Gamma X)$ (compare with [61, Corollary 13.3.14]):

**Corollary 5.12** Let $\Gamma \curvearrowright X$ be a free action and assume that $(\Gamma \times QX, d_{\Gamma \times Q})$ has ONL. Given $T \in C^*(\mathcal{Q}_\Gamma X)$, then $\Psi(T) = 0$ if and only if $T$ is a ghost operator.

**Proof** By continuity of $\Psi$, it follows from Lemma 5.11 that the formula

$$
\|\Psi(T)\| = \sup_{R \geq 0} \lim_{n \to \infty} \sup \left\{ \|T_n\| \left| \xi \in L^2(X \times \{2^n\}), \|\xi\| = 1 \text{ and diam}(\text{supp}\xi) \leq R \right. \right\}.
$$

also holds for every $T \in C^*(\mathcal{Q}_\Gamma X)$. So $\Psi(T) = 0$ if and only if for every $R, \epsilon > 0$, there exists an $N_0 \in \mathbb{N}$ such that for every $n > N_0$ and every unit vector $\xi \in L^2(X \times \{2^n\})$ with $\text{diam}(\text{supp}\xi) \leq R$, we have $\|T_n\xi\| \leq \epsilon.$ The latter condition holds if and only if $T$ is a ghost operator. $\square$

We resume the construction of the trace $\tau^u$ following [45, Section 3.2]. It can be shown that for every $n \in \mathbb{N}$ we have a $*$-isomorphism

$$
C^*((\Gamma \times X \times \{2^n\}), d_{\Gamma \times QX})^\Gamma \cong C_r^*(\Gamma) \otimes \mathfrak{K}(L^2(X \times \{2^n\})),
$$

where $\mathfrak{K}(L^2(X \times \{2^n\}))$ denotes the compact operators. The latter admits a trace $\tau$ coming from the canonical traces on both tensor factors. More precisely, we let

$$
\tau(p) := \text{Tr}(\chi_1 p \chi_1),
$$

where $\chi_1$ is the characteristic function of $\{1\} \times X \times \{2^n\}$, and $\text{Tr}$ is the canonical trace on $\mathfrak{K}(L^2(X \times \{2^n\}))$. Finally, we define the trace $\tau^u$ on $K_0(C^*(\mathcal{Q}_\Gamma X))$ as the following composition:

$$
K_0(C^*(\mathcal{Q}_\Gamma X)) \xrightarrow{\Psi_*} K_0 \left( \bigoplus_n C^*((\Gamma \times X \times \{2^n\}), d_{\Gamma \times Q})^\Gamma \right) \cong \bigoplus_n K_0(C^*((\Gamma \times X \times \{2^n\}), d_{\Gamma \times Q})^\Gamma) \xrightarrow{\tau} \prod_{\mathbb{R}} \bigoplus_{\mathbb{R}}.
$$

Consequently, Corollary 5.12 together with Corollary 5.6 prove the following:
Proposition 5.13 Let $(X, d)$ be a compact metric space of diameter at most 2 equipped with a non-atomic probability measure $\nu$ of full support, and $\Gamma \curvearrowright (X, d, \nu)$ be a free measure-class-preserving continuous action. Assume that $(\Gamma \times QX, d_{\Gamma \times Q})$ has ONL. Then for every ghost projection $p \in C^*(Q\Gamma X)$, we have $\tau^u([p]) = 0$. In particular, for any projection $P \in \mathcal{P}$ we have $\tau^u([\Phi^Q(P)]) = 0$.

5.2.3 Comparing the two traces

The concluding argument goes exactly as in [16, 45, 60]. The key idea is to use Atiyah $\Gamma$-index Theorem [33] to show that whenever $p$ is a projection in the Roe algebra such that $[p]$ belongs to the range of the coarse assembly map, then

$$\tau_d([p]) = \tau^u([p]) \in \prod \mathbb{R} \bigoplus \mathbb{R}.$$ 

This argument first appeared in [16, Proposition 5.6]. The detailed proof (in the case of graphs) can be found in [60, Lemma 6.5] (see also [45, Theorem 3.3] for the case of compact metric spaces).

Together with Lemma 5.8 and Proposition 5.13, we deduce that every $K$-theory class of a non-compact ghost projection in the Roe algebra is not in the image of the coarse assembly map (see [60, Theorem 6.1] for the case of graphs). In particular this applies to any projection $\Phi^Q(P)$ with $P \in \mathcal{P}$. We record this fact as a theorem, as it is the main result of Sect. 5 and it generalises [45, Theorem 3.5] from measure-preserving actions with a spectral gap to measure-class-preserving asymptotically expanding actions:

Theorem 5.14 Let $(X, d)$ be a compact metric space of diameter at most 2 equipped with a non-atomic probability measure $\nu$ of full support, and $\rho: \Gamma \curvearrowright (X, d, \nu)$ be a free continuous measure-class-preserving action. Further assume that $(\Gamma \times QX, d_{\Gamma \times Q})$ has ONL.

If $p = \Phi^Q(P_Y)$ for any domain $Y \subseteq X$ of asymptotic expansion or $p = \Phi^Q(\hat{P}_Y, S)$ for any domain $Y \subseteq X$ of Markov $S$-expansion, then $[p]$ does not belong to the image of the coarse assembly map.

In particular, if the action $\rho$ is asymptotically expanding then the class of the sparse Drutu–Nowak projection $\mathcal{G}_Q$ violates the coarse Baum–Connes conjecture for the sparse warped cone $Q\Gamma X$.

The following corollary follows immediately from Theorem 5.14 and Remark 5.10:

Corollary 5.15 Let $(X, d)$ be a compact metric space of diameter at most 2 equipped with a non-atomic probability measure $\nu$ of full support, and let $\rho: \Gamma \curvearrowright (X, d, \nu)$ be a free Lipschitz measure-class-preserving asymptotically expanding action under either of the following conditions:

(1) if $\Gamma$ has property A and $X$ is a manifold; 
(2) if the asymptotic dimension of $\Gamma$ is finite and $X$ is an ultrametric space.

Then the coarse Baum–Connes conjecture for the sparse warped cone $Q\Gamma X$ fails.
Example 5.16 Given a chain of finite index subgroups $\Gamma > \Gamma_1 > \Gamma_2 > \cdots$, we consider the inverse limit $X = \lim \leftarrow_{\Gamma / \Gamma_i}$. This space is homeomorphic to a Cantor set, and the uniform measures on $\Gamma / \Gamma_i$ induce a natural probability measure $\nu$ on $X$ (it is obviously non-atomic and with full-support). Further, $X$ can also be given an ultrametric by letting $d((\gamma_i \Gamma_i)_{i \in \mathbb{N}}, (\gamma'_i \Gamma_i)_{i \in \mathbb{N}}) = 2^{-n}$ where $n$ is the smallest index such that $\gamma_n \Gamma_n \neq \gamma'_n \Gamma_n$. Clearly, $\Gamma$ acts $X$ by left multiplication and the action is isometric and measure-preserving. Further, if $\bigcap_{i \in \mathbb{N}} \Gamma_i = \{1\}$ then the action is free. Such an action is called a profinite action.

Abért–Elek constructed in [1, Theorem 5] a free profinite action $F_k \curvearrowright (X, d, \nu)$ of any finitely generated non-abelian free group $F_k$ that is strongly ergodic (and hence asymptotically expanding) but does not have a spectral gap.

Since the free group has asymptotic dimension 1, we can hence apply Theorem 5.14 and Corollary 5.15 to deduce that the sparse Drutu–Nowak projection over $Q_{F_k}X$ violates the coarse Baum–Connes conjecture. This fact does not directly follow from [45, Theorem 3.5], as the action does not have spectral gap. As pointed out by the anonymous referee, it is also possible to deduce that the sparse warped cone $Q_{F_k}X$ violates the coarse Baum–Connes conjecture by combining the approximating space construction from [28] with the results in [23]. However, the latter argument is somewhat more opaque. For example, it is not clear to us whether this approach implies that sparse Drutu–Nowak projection violates the coarse Baum–Connes. On the contrary, the approach developed in this paper implies all non-compact ghost projections—including the sparse Drutu–Nowak projection—violate the conjecture.

Remark 5.17 It is not hard to check that the sparse and unified warped cones arising from a free profinite action have bounded geometry if and only if there is a uniform upper bound on the indices $[\Gamma_i : \Gamma_{i+1}]$ for $i \in \mathbb{N}$.

It follows that the sparse warped cone in Example 5.16 does not have bounded geometry in general: the construction of Abért–Elek requires chains of subgroups with indices growing very quickly (this is important in the proof of [1, Lemma 6.2]). It would be interesting to know if it is possible to find a chain $\Gamma > \Gamma_1 > \cdots$ with uniformly bounded indices $[\Gamma_i : \Gamma_{i+1}]$ such that the induced profinite action is strongly ergodic but has no spectral gap.

5.3 Non-coarse embeddability

In this subsection we prove that warped cones arising from asymptotically expanding actions do not coarsely embed into any Hilbert space. One of the ground-breaking results by Yu was to verify the coarse Baum–Connes conjecture for every proper bounded geometry metric space which coarsely embeds into some Hilbert space [63]. It follows from Theorem 5.14 that the sparse warped cone $Q_{\Gamma}X$ coming from an asymptotically expanding action of $\Gamma$ on a compact metric space $X$ cannot coarsely embed into Hilbert spaces provided that $(\Gamma \times QX, d_{\Gamma \times Q})$ has ONL and $Q_{\Gamma}X$ has bounded geometry (it is not hard to show that if $(X, d)$ is proper, so is $(O_{\Gamma}X, d_{\Gamma})$).

---

9 A metric space $(X, d)$ has bounded geometry if for every $\epsilon, R > 0$ there exists an $N \in \mathbb{N}$ such that any $\epsilon$-separated subset of an $R$-ball of $X$ has at most $N$ elements.
Proposition 5.18

Let $\rho, \Gamma \times QX, d_{\Gamma} \times Q$ and bounded geometry of $QX$ are redundant.

Recall that a map $F : (X, d_X) \rightarrow (Z, d_Z)$ is a coarse embedding between metric spaces if there exist non-decreasing unbounded functions $\rho_{\pm} : [0, \infty) \rightarrow [0, \infty)$ such that

$$\rho_{-}(d_X(x, x')) \leq d_Z(F(x), F(x')) \leq \rho_{+}(d_X(x, x')),$$

for all $x, x' \in X$.

The following proposition is a (partial) extension of [37, Theorem 3.1] from the setting of measure-preserving actions with a spectral gap to asymptotically expanding measure-class-preserving actions. The proof combines the idea in [37, Theorem 3.1] with Proposition 3.18:

**Proposition 5.18** Let $(X, d)$ be a compact metric space of diameter at most 2 equipped with a non-atomic probability measure $\nu$, and $\rho : \Gamma \curvearrowright (X, d, \nu)$ be a continuous measure-class-preserving and asymptotically expanding action.

If $A \subseteq [1, \infty)$ is any unbounded subset and $d_{\Gamma}$ is the warped cone metric on $O_{\Gamma}X$, then $(X \times A, d_{\Gamma})$ does not admit a coarse embedding into any Hilbert space.

**Proof** From Proposition 3.18, there exist a finite symmetric subset $1 \subseteq S \subseteq \Gamma$ and a domain $Y \subseteq X$ of Markov $S$-expansion such that there is a constant $\Theta \geq 1$ such that $\frac{1}{\Theta} \leq r(s, x) \leq \Theta$ for every $x \in Y$ and $s \in S_{Y, x} = \{s \in S | s \cdot x \in Y\}$. Hence, we have $1 - \lambda_2 > 0$ by Theorem 3.7 (see also (3.5)) and we let $\kappa := \frac{1}{\Sigma(1 - \lambda_2)} > 0$.

By Proposition 3.10(3) (see also (3.12)) we have that for every $g \in L^2_0(Y, \tilde{\nu}_{Y, S})$

$$\|g\|_{\tilde{\nu}_{Y, S}, 2}^2 \leq \kappa \sum_{s \in S} \int_{Y \cap s^{-1}(Y)} r(s, x)^{1/2} |g(x) - g(s \cdot x)|^2 \, d\nu(x) \leq \kappa \sqrt{\Theta} \sum_{s \in S} \int_{Y \cap s^{-1}(Y)} |g(x) - g(s \cdot x)|^2 \, d\nu(x).$$

Assume now that $(X \times A, d_{\Gamma})$ admits a coarse embedding into Hilbert space $\ell^2(\mathbb{N})$. If $F : (Y \times A, d_{\Gamma}) \rightarrow \ell^2(\mathbb{N})$ denotes the coarse embedding, then we let $F_t : Y \rightarrow \ell^2(\mathbb{N})$ be the restriction of $F$ to the level set $Y \times \{t\}$ for $t \in A$. For each $t \in A$ and $n \in \mathbb{N}$, denote by $F_t^{(n)}$ the associated coefficient of the function of $F_t$. Every $F_t^{(n)}$ is a bounded function (which is hence in $L^2(Y, \nu)$), and we have

$$\sum_{n \in \mathbb{N}} \sum_{s \in S} \int_{Y \cap s^{-1}(Y)} \left| F_t^{(n)}(x) - F_t^{(n)}(s \cdot x) \right|^2 \, d\nu(x) = \sum_{s \in S} \int_{Y \cap s^{-1}(Y)} \|F_t(x) - F_t(s \cdot x)\|_{\ell^2(\mathbb{N})}^2 \, d\nu(x) \leq \sum_{s \in S} \int_{Y \cap s^{-1}(Y)} \rho_{+}(d_{\Gamma}(x, s \cdot x))^2 \, d\nu(x) \leq \rho_{+}(M)^2 |S|,$$
where \( M := \max_{s \in S} \ell(s) \).

After translating \( F_t \) if necessary, we may assume that \( F_t^{(n)} \in L_0^2(Y, \tilde{\nu}_Y, S) \). This implies that for each \( t \in A \) we have

\[
\| F_t \|_{\tilde{\nu}_Y, 2}^2 = \sum_{n \in \mathbb{N}} \| F_t^{(n)} \|_{\tilde{\nu}_Y, 2}^2 \leq \kappa \sqrt{\Theta \rho_+(M)}^2 |S| < \infty.
\]

On the other hand, since \( F_t^{(n)} \in L_0^2(Y, \tilde{\nu}_Y, S) \) we also have

\[
\int_{Y \times Y} \| F_t(x) - F_t(y) \|_{\ell^2(\mathbb{N})}^2 \tilde{\nu}_Y, S(x) \tilde{\nu}_Y, S(y) = 2 \| F_t \|_{\tilde{\nu}_Y, 2}^2.
\]

We will thus reach a contradiction by showing that

\[
\int_{Y \times Y} \| F_t(x) - F_t(y) \|_{\ell^2(\mathbb{N})}^2 \tilde{\nu}_Y, S(x) \tilde{\nu}_Y, S(y) \to \infty, \quad \text{as } t \to \infty. \tag{5.1}
\]

Since \( \tilde{\nu}_Y, S \) is non-atomic and \( \Gamma \) is countable, the set

\[
N := \{(x, y) \in Y \times Y \mid \Gamma \cdot x = \Gamma \cdot y\}
\]

is measurable and has measure zero by Fubini’s theorem. On the other hand, for any \( x, y \in Y \) lying in different \( \Gamma \)-orbits the distance \( d_\Gamma((x, t), (y, t)) = d_\Gamma(x, y) \to \infty \) as \( t \to \infty \). Since \( F \) is a coarse embedding, it follows that \( \| F_t(x) - F_t(y) \|_{\ell^2(\mathbb{N})}^2 \to \infty \).

Hence, we deduce (5.1) and obtain the desired contradiction. \( \square \)

**Remark 5.19** Let \( 1 < p < \infty \). By interpolation, if \( Y \) is a domain of Markov \( S \)-expansion then the lazy Markov operator \( \frac{1}{2} + \frac{1}{2} \Phi_{Y, S} \) has norm strictly less than one also when regarded as an operator on \( L_p^0(Y, \tilde{\nu}_Y, S) \). An easy modification of the proof of Proposition 5.18 shows that the warped cone does not coarsely embed into \( L^p \) for any \( 1 < p < \infty \). Moreover, the warped cone cannot coarsely embed into \( L^1 \)-spaces as well because it is shown in [35, Proposition 4.1] that every \( L^1 \)-space coarsely embeds into a Hilbert space.

**Example 5.20** [37, Theorem 3.1] does not apply to the warped cones arising from the profinite actions \( F_k \acts (X, d, \nu) \) of Abért–Elek (Example 5.16). However, we may use Proposition 5.18 to conclude that the sparse warped cone \( Q_{F_k} X \) as well as the unified warped \( O_{F_k} X \) cannot be coarsely embedded into any Hilbert space.

Note also that the non-embeddability of \( Q_{F_k} X \) does not immediately follow from the fact that it violates the coarse Baum–Connes conjecture. In fact, Yu’s argument only applies to *bounded geometry* proper metric spaces, while the warped cones in Example 5.16 have unbounded geometry (Remark 5.17).

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Appendix A: Proof of the spectral characterisation of expansion for Markov kernels

In this appendix, we will provide a proof of Theorem 3.7. As mentioned before, some special cases of this result are especially well-known. Two such instances are given by simple random walks on finite or countably infinite graphs: the former gives a spectral characterisation of expansion [2, 3, 10], while the latter characterises non-amenability [10, 22, 34]. Lawler–Sokal [24] proved the general result already in the 1980s. However, their work seems to have been overlooked by a part of the mathematical community. In [21] Kaimanovich proved a version of Theorem 3.7 for reversible Markov kernels on infinite measure spaces (he actually proved much more refined results concerning $p$-capacities and Dirichlet norms). Lyon–Nazarov proved it for Markov kernels arising from measure-preserving actions on probability spaces [30, Theorem 3.1].

Before finding out about [24], we managed to prove Theorem 3.7 by extending the standard argument used for Markov processes on finite state spaces [25, Theorem 13.10] (see also [38]). Most of its key points generalise without difficulties (Lemma A.1 and Lemma A.2), but the concluding argument is considerably more involved. We decided to include the proof in this appendix for the convenience of the reader and also because it provides a slightly better lower bound on the spectral gap.

**Lemma A.1** If $g \in L^1(X, m)$ is a function that takes value in $[0, \infty)$ and such that $m(\{g > 0\}) \leq \frac{1}{2}m(X)$, then $\mathcal{E}_1(g) \geq \kappa \|g\|_{m,1}$.

**Proof** By hypothesis, we have that

$$\int_X g(x)dm = \int_0^\infty m(\{g \geq t\})dt \leq \frac{1}{\kappa} \int_0^\infty |\partial \Pi(\{g \geq t\})|_m dt,$$

where $\kappa$ denotes the Cheeger constant. Using (3.4) we deduce:

$$\int_X g(x)dm \leq \frac{1}{\kappa} \int_0^\infty \mathcal{E}_1(\chi_{\{g \geq t\}}) dt.$$

$$= \frac{1}{2\kappa} \int_0^\infty \int_{X \times X} |\chi_{\{g \geq t\}}(x) - \chi_{\{g \geq t\}}(y)| \ d\mu(x, y) dt$$

---

10 [30, Theorem 3.1] also claims that the spectrum of the Markov operator is bounded away from $-1$, but this is not correct: there is a small mistake at the very end of their proof. It is also worth pointing out that their proof is based on an inequality which they claim holds true by “checking cases”. We are unable to verify such inequality.
thus proving the lemma. \hfill \Box

In turn, this is used to prove the estimate that lies at the heart of the proof of Theorem 3.7:

**Lemma A.2** If \( g \in L^2(X, m) \) is a function that takes value in \([0, \infty)\) and such that 

\[
\mu(\{ g > 0 \}) \leq \frac{1}{2} \mu(X),
\]

then

\[
\mathcal{E}_2(g) \geq \frac{\kappa^2}{2} \| g \|_{m,2}^2.
\]

**Proof** Firstly, we note that 

\[
\| g \|_{m,2}^2 = \| g^2 \|_{m,1}.
\]

We can hence apply Lemma A.1 to the function \( g^2 \) to obtain

\[
\kappa \| g \|_{m,2}^2 \leq \mathcal{E}_1(g^2).
\]  \quad (A.1)

Using the Cauchy–Schwarz inequality, we can estimate the value \( \mathcal{E}_1(g^2) \) as follows:

\[
\mathcal{E}_1(g^2) = \frac{1}{2} \int_{X \times X} \left( g^2(x) - g^2(y) \right)^2 \, d\mu(x, y)
= \frac{1}{2} \int_{X \times X} |g(x) - g(y)| \cdot |g(x) + g(y)| \, d\mu(x, y)
\leq \frac{1}{2} \left( \int_{X \times X} |g(x) - g(y)|^2 \, d\mu(x, y) \right)^{\frac{1}{2}} \left( \int_{X \times X} |g(x) + g(y)|^2 \, d\mu(x, y) \right)^{\frac{1}{2}}
= \frac{1}{2} \left( 2 \mathcal{E}_2(g) \right)^{\frac{1}{2}} \left( 2 \| g \|_{m,2}^2 + 2 \langle g, \mathfrak{P}g \rangle_m \right)^{\frac{1}{2}}
\leq \sqrt{2} \mathcal{E}_2(g)^{\frac{1}{2}} \| g \|_{m,2}.
\]

The proof is complete once we combine the above estimate with (A.1). \hfill \Box

Finally, we are ready to prove the main theorem of this subsection:

**Proof of Theorem 3.7** Given a measurable \( A \subseteq X \) with \( 0 < m(A) \leq \frac{1}{2} m(X) \), let

\[
f_A := \chi_A - \frac{m(A)}{m(X)}
\]

be the projection of \( \chi_A \) to \( L_0^2(X, m) \). Then

\[
\| f_A \|_{m,2}^2 = (m(X) - m(A)) \frac{m(A)}{m(X)} \geq \frac{1}{2} m(A).
\]
Using (3.4) we deduce that

\[ 1 - \lambda_2 \leq \frac{\mathcal{E}_2(\mathbf{f}_A)}{\|\mathbf{f}_A\|_{m,2}^2} = \frac{|\partial \mathbf{\Pi}(A)|_m}{\|\mathbf{f}_A\|_{m,2}^2} \leq 2 \frac{|\partial \mathbf{\Pi}(A)|_m}{m(A)}, \]

and hence \( 1 - \lambda_2 \leq 2\kappa. \)

For the other direction, we need to show

\[ \frac{\kappa^2}{2} \leq \inf_{f \in L^2_0(X,m)} \frac{\langle f, \Delta f \rangle_m}{\|f\|_{m,2}^2} = \inf_{f \in L^2_0(X,m)} \frac{\mathcal{E}_2(f)}{\|f\|_{m,2}^2} = 1 - \lambda_2. \]

Since \( \mathcal{P} \) is self-adjoint, the spectral theorem implies that there exists a sequence of real-valued functions \( f_n \in L^2_0(X,m) \) with \( \|f_n\|_{m,2} = 1 \) such that \( \|\mathcal{P} f_n - \lambda_2 f_n\|_{m,2} \to 0 \). In particular, \( \langle f_n, \Delta f_n \rangle \to 1 - \lambda_2 \). Write \( f_n = f_n^+ - f_n^− \), where \( f_n^+(x) := \max\{0, f_n(x)\} \) and \( f_n^−(x):= \max\{0, -f_n(x)\} \). Replacing \( f_n \) with \( -f_n \) if necessary, we can assume that \( m(\{f_n(x) > 0\}) \leq \frac{1}{2}m(X) \).

If each \( f_n \) was an eigenfunction for \( \Delta \), we would immediately have

\[ \frac{\langle f_n^+, \Delta f_n^+ \rangle_m}{\|f_n^+\|_{m,2}^2} = \frac{\langle f_n, \Delta f_n \rangle_m}{\|f_n\|_{m,2}^2}. \tag{A.2} \]

In this case, the proof of the theorem would easily follow from Lemma A.2. Yet, this need not be the case for general Markov kernels. This is the place where our argument differs from the classical proof for finite-state processes.

On the way to overcome this difficulty, we will first need to modify \( f_n \) to ensure that \( \|f_n^+\|_{m,2} \) is bounded away from 0. If \( \|f_n^+\|_{m,2} \) does not tend to 0, we simply pass to a subsequence \( h_n := f_{k_n} \) so that \( \|h_n^+\|_{m,2} \) is bounded away from 0. Otherwise, we have \( \|f_n^+\|_{m,2} \to 0 \). Since \( m \) is finite, we also have \( \|f_n^+\|_{m,1} \to 0 \). On the other hand, \( \|f_n^−\|_{m,1} = \|f_n^−\|_{m,1} \) because \( f_n \in L^2_0(X,m) \). It follows that there exists a sequence \( c_n > 0 \) such that \( c_n \to 0 \) and \( m(\{f_n^−(x) \geq c_n\}) \to 0 \). We then define \( h_n := -f_n^+ + c_n \) and also note that

\[ \|\mathcal{P} h_n - \lambda_2 h_n\|_{m,2} \leq \|\mathcal{P} h_n - \lambda_2 f_n\|_{m,2} + \|\mathcal{P} - \lambda_2 \| \cdot \|c_n\| \to 0 ~\text{as} ~n \to \infty. \]

For \( n \) large enough we have \( m(\{h_n^+(x) > 0\}) \leq m(X)/2 \) and \( \|h_n^+\|_{m,2} \geq \|f_n^−\|_{m,2} - \|c_n\|_{m,2} \) tends to 1, as \( 1 = \|f_n^+\|_{m,2}^2 + \|f_n^−\|_{m,2}^2 \) and \( \|f_n^+\|_{m,2} \to 0 \).

We are now ready to complete the proof. Note that

\[ \langle h_n^+, (\mathcal{P} h_n)^+ \rangle_m - \lambda_2 \|h_n^+\|_{m,2}^2 = \langle h_n^+, (\mathcal{P} h_n)^+ - \lambda_2 h_n^+ \rangle_m \]

and by the Cauchy–Schwarz inequality, we have that

\[ \frac{\langle h_n^+, (\mathcal{P} h_n)^+ - \lambda_2 h_n^+ \rangle}{\|h_n^+\|_{m,2}^2} \leq \frac{\|\mathcal{P} h_n - \lambda_2 h_n\|_{m,2}}{\|h_n^+\|_{m,2}} \leq \frac{\|\mathcal{P} h_n - \lambda_2 h_n\|_{m,2}}{\|h_n^+\|_{m,2}}. \]
Since $\|h_n^+\|_{m,2}$ is bounded away from 0, the right hand side in the above inequality tends to 0 and therefore

$$1 - \lambda_2 = \lim_{n \to \infty} \frac{\|h_n^+\|_{m,2}^2 - \langle h_n^+, (\mathcal{P} h_n^+)\rangle_m}{\|h_n^+\|_{m,2}^2}.$$ 

Finally, since $\langle h_n^+, \mathcal{P}(h_n^+)\rangle_m \geq \langle h_n^+, (\mathcal{P} h_n^+)\rangle_m$, we deduce that

$$1 - \lambda_2 \geq \lim_{n \to \infty} \frac{\|h_n^+\|_{m,2}^2 - \langle h_n^+, \mathcal{P}(h_n^+)\rangle_m}{\|h_n^+\|_{m,2}^2} = \lim_{n \to \infty} \frac{\langle h_n^+, \Delta(h_n^+)\rangle_m}{\|h_n^+\|_{m,2}^2} = \lim_{n \to \infty} \frac{\mathcal{E}_2(h_n^+)}{\|h_n^+\|_{m,2}^2},$$

and the latter is greater or equal to $\frac{\kappa^2}{2}$ by Lemma A.2, as desired. \hfill \Box

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