ERGODICITY OF THE NUMBER OF INFINITE GEODESICS ORIGINATING FROM ZERO

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Abstract. First-passage percolation is a random growth model which has a metric structure. An infinite geodesic is an infinite sequence whose all sub-sequences are shortest paths. One of the important quantity is the number of infinite geodesics originating from the origin. When $d = 2$ and an edge distribution is continuous, it is proved to be almost surely constant [D. Ahlberg, C. Hoffman. Random coalescing geodesics in first-passage percolation]. In this paper, we will prove the same result for higher dimensions and general distributions.

1. Introduction

First-passage percolation was first introduced by Hammesley and Welsh in 1965, as a model of fluid flow in random medium. In this model, we consider the first passage time on $\mathbb{Z}^d$-lattice equipped with random weights. A path is said to be optimal if it attains the first passage time. Under weak conditions on distributions, the first passage times between two points define a metric structure. Therefore, optimal paths can be seen as geodesics and are central objects of this model. An infinite geodesic is an infinite path of $\mathbb{Z}^d$ whose all sub-sequences are optimal paths. One of the important quantity is the number of infinite geodesics originating from the origin. It is expected to be infinity and proved rigorously under un-proven limiting shape assumption when the dimension is greater than or equal to 2 in [8]. However, it is currently best known to be at least 4, which is shown in [7]. See [1, 2] for more background and related works on infinite geodesics.

The important property is that two infinite geodesics tend to coalesce, which is called ”coalescing property”. It is established in the case $d = 2$ case for continuous distributions [4, 2]. This property allows us to use ergodic theory and Ahlberg and Hoffman showed that the number of infinite geodesics originating from the origin is almost surely constant [2]. Note that their methods rely on the uniqueness of optimal paths between any two points, which follows from the continuity of the distribution, and special geometric properties of $\mathbb{Z}^2$-lattice, which for example allows one to define the counter-clockwise labeling of infinite geodesics. Our aim of this paper is to develop new techniques to establish the coalescing property for more general frameworks. And we will prove the above result both for general dimensions and distributions.

1.1. Setting. We consider the first-passage percolation on the lattice $\mathbb{Z}^d$ with $d \geq 2$. The model is defined as follows. The vertices are the elements of $\mathbb{Z}^d$. Let us denote by $E^d$ the set of edges:

$$E^d = \{\{v, w\} | v, w \in \mathbb{Z}^d, |v - w|_1 = 1\},$$

where we set $|v - w|_1 = \sum_{i=1}^d |v_i - w_i|$ for $v = (v_1, \ldots, v_d)$, $w = (w_1, \ldots, w_d)$. Note that we consider non-oriented edge in this paper and we sometimes regard $\{v, w\}$ as a subset of $\mathbb{Z}^d$ with a slight abuse of notation. We assign a non-negative random variable $\tau_e$ on each edge $e \in E^d$ as the passage time of $e$. The collection $\tau = \{\tau_e\}_{e \in E^d}$ is assumed to be independent and identically distributed with common distribution $F$. A path $\gamma$ is a finite sequence of vertices $(x_1, \ldots, x_l) \subset \mathbb{Z}^d$ such that for any $i \in \{1, \ldots, l - 1\}$, $\{x_i, x_{i+1}\} \in E^d$. It is useful to regard a path as a subset of edges:

$$\gamma = \{(x_i, x_{i+1})\}_{i=1}^{l-1}. \tag{1.1}$$
Without otherwise noted, we use this convention. Let us define the length of a path $\gamma$ as $l(\gamma) = l - 1$. Given a path $\gamma$, we define the passage time of $\gamma$ as
\[
\tau(\gamma) = \sum_{v \in \gamma} \tau_v.
\]
Given two vertices $v, w \in \mathbb{R}^d$, we define the first passage time between vertices $v$ and $w$ as
\[
t(v, w) = \inf_{\gamma: v \to w} t(v, w),
\]
where the infimum was taken over all finite paths $\gamma$ starting at $v$ and ending at $w$. A path from $v$ to $w$ is said to be optimal if it attains the first passage time, i.e., $t(\gamma) = t(v, w)$. We denote by $\mathcal{O}(v, w)$ the set of all optimal paths from $v$ to $w$. If $F$ is continuous, i.e., $\mathbb{P}(\tau_e = a) = 0$ for any $a \in \mathbb{R}$, when we fix starting and ending point, then an optimal path is uniquely determined. Then we still denote by $\mathcal{O}(v, w)$ this optimal path with a slight abuse of notation.

We say that an infinite sequence $(x_1, x_2 \cdots) \subset \mathbb{Z}^d$ is an infinite geodesic if for any $1 \leq i < j$, $(x_i, \cdots, x_j)$ is an optimal path from $x_i$ to $x_j$. Denote by $\mathcal{I}$ the set of all infinite geodesics and $\mathcal{I}(v)$ the set of all infinite geodesics originating from $v$. Given two infinite geodesics $\Gamma_1$ and $\Gamma_2$, we say that they are distinct if $\sharp \{x \in \mathbb{Z}^d | x \in \Gamma_1 \cap \Gamma_2\} < \infty$, where we regard $\Gamma_1$ and $\Gamma_2$ as subsets of vertices in this definition. Otherwise, we say that $\Gamma_1$ and $\Gamma_2$ coalesce and write $\Gamma_1 \sim \Gamma_2$. Let $\mathcal{N} = \mathcal{N}(\tau) \in \mathbb{N} \cup \{\infty\}$ be the number of distinct infinite geodesics:
\[
\mathcal{N} = \max\{k \in \mathbb{N} \cup \{\infty\} | \exists \Gamma_1, \cdots, \Gamma_k \in \mathcal{I} \text{ such that } \Gamma_i \not\sim \Gamma_j \text{ for any } i \neq j\}.
\]
We define the number of distinct infinite geodesics originating from $v \in \mathbb{Z}^d$ as
\[
\mathcal{N}_v = \max\{k \in \mathbb{N} \cup \{\infty\} | \exists \Gamma_1, \cdots, \Gamma_k \in \mathcal{I}(v) \text{ such that } \Gamma_i \not\sim \Gamma_j \text{ for any } i \neq j\}.
\]
Since $\mathcal{N}$ is invariant under lattice shift, by ergodicity, it is almost surely constant [1]: there exists $N \in \mathbb{N} \cup \{\infty\}$ such that
\[
(1.2) \quad \mathbb{P}(\mathcal{N} = N) = 1.
\]
If $F$ is continuous, then since an optimal path is uniquely determined between any two vertices, it is easy to check that $\sim$ is an equivalence relation, $\mathcal{N} = \sharp[\mathcal{I}/ \sim]$ and $\mathcal{N}_v = \sharp[\mathcal{I}_v]$.

1.2. Main results.

**Definition 1.** A distribution $F$ is said to be useful if
\[
(1.3) \quad \mathbb{P}(\tau_e = F^-) < \begin{cases} p_e(d) & \text{if } F^- = 0 \\ \bar{p}_e(d) & \text{otherwise,} \end{cases}
\]
where $p_e(d)$ and $\bar{p}_e(d)$ stand for the critical probabilities for $d$-dimensional percolation and oriented percolation model, respectively and $F^-$ is the infimum of the support of $F$.

Note that if $F$ is continuous, then $F$ is useful.

**Theorem 1.** Suppose that $F$ is useful and there exists $\alpha > 0$ such that $\mathbb{E}\exp(\alpha \tau_e) < \infty$. Then the following holds almost surely: for any $v \in \mathbb{Z}^d$,
\[
(1.4) \quad \mathcal{N}_v = \mathcal{N}.
\]
In particular, by (1.2),
\[
\mathcal{N}_v \text{ is almost surely constant.}
\]

**Remark 1.** In the case $d = 2$ with a continuous distribution, the above result was shown in [2].
1.3. Notation and terminology. This subsection collects some notations and terminologies for the proof.

- Given a path \( \gamma = (x_i)_{i=1}^l \), we set \( \gamma[i] = x_i \).
- Given two paths \( \gamma_1 = (\gamma_1[i])_{i=1}^l \) and \( \gamma_2 = (\gamma_2[i])_{i=1}^l \) with \( \gamma_1[i] = \gamma_2[i] \), we denote the concatenated path by \( \gamma_1 \oplus \gamma_2 \), i.e. \( \gamma_1 \oplus \gamma_2 = (\gamma_1[1], \ldots, \gamma_1[l], \ldots, \gamma_2[l]) \).
- Given two paths \( \gamma = (y_i)_{i=1}^l \) and \( \Gamma = (x_i)_{i=1}^l \), we write \( \gamma \subset \Gamma \) if there exists \( k \) such that \( y_i = x_{k+i} \) for any \( i \in \{1, \ldots, l\} \). Then we say that \( \gamma \) is a sub-path of \( \Gamma \).
- Given \( x, y \in \mathbb{R}^d \), we define \( d_\infty(x, y) = \max\{|x_i - y_i| \mid i = 1, \ldots, d\} \). It is useful to extend the definition as

\[
d_\infty(A, B) = \inf\{d_\infty(x, y) \mid x \in A, y \in B\} \text{ for } A, B \subset \mathbb{R}^d.
\]

When \( A = \{x\} \), we write \( d_\infty(x, B) \).

- Given \( x \in \mathbb{R} \), we denote by \( \lceil x \rceil \) the greatest integer less than or equal to \( x \).
- Given a set \( D \subset \mathbb{Z}^d \), let us define the outer boundary of \( D \) as

\[
\partial^+ D = \{v \notin D \mid \exists w \in D \text{ such that } |v - w|_1 = 1\}.
\]

- Let \( F^- \) and \( F^+ \) be the infimum and supremum of the support of \( F \), respectively:

\[
F^- = \inf\{\delta \geq 0 \mid \mathbb{P}(\tau_e < \delta) > 0\}, \quad F^+ = \sup\{\delta \geq 0 \mid \mathbb{P}(\tau_e > \delta) > 0\},
\]

where if \( F \) is unbounded distribution, then we set \( F^+ = \infty \).

- Given a finite path \( \gamma_0 \) starting at \( v \), we define \( \mathcal{I}(\gamma_0) = \{\Gamma \in \mathcal{I}(v) \mid \gamma_0 \subset \Gamma\} \).

- Given \( M \in \mathbb{N} \), let \( \mathcal{T}_M \) be the set of all paths whose length is \( M \).

2. Proof

2.1. Heuristic. We will explain the heuristic behind the proof of \( \mathcal{N} = \mathcal{N}_0 \) in this subsection. Let \( \Gamma \) be an infinite geodesic originating from some vertex with which all infinite geodesics originating from the origin do not coalesce. Then one can construct infinitely many optimal paths from the origin intersecting \( \Gamma \) at only one point. We call the intersecting points bad points and the sub-path from the origin do not coalesce. Then one can construct infinitely many optimal paths from the origin. This implies that there exists \( \Gamma \in \mathcal{N}_0 \) such that \( \Gamma \sim \mathcal{N} \) almost surely. We obtain \( \mathcal{N}_0 = \mathcal{N} \).

![Figure 1](image)

Figure 1.

Left: We resample the configurations on the third sub-path.

Right: After resampling, 4-th and the subsequent bad points vanish.
There are mainly two obstacles to put the above argument into practice. First, to lower the passage times, each $k$-th paths needs to have sufficiently large passage time before resampling. Second, we need to take $\Gamma = \Gamma(\tau)$ depending on configurations. Then when we resample them, $\Gamma$ might change, i.e., $\Gamma(\tilde{\tau}) \neq \Gamma(\tau)$ where $\tilde{\tau} = \{\tilde{\tau}_k\}_{k \in E^d}$ is resampled configurations. Therefore, the above heuristics does not work straightforwardly.

2.2. **Proof for continuous distributions with unbounded support.** In this subsection, suppose that $F$ is continuous and $F^+ = \infty$. Recall that we denote by $\mathcal{O}(v, w)$ the unique optimal path between $v$ and $w$. It suffices to show that $\mathbb{P}(\mathcal{N}_0 = N) = 1$.

**Definition 2.** In this definition, we consider a path as a subset of vertices. Given an infinite path $\Gamma$ and a vertex $x \in \Gamma$, we say that $x$ is bad for $\Gamma$ if

$$\mathcal{O}(0, x) \cap \Gamma = \{x\}.$$ 

Otherwise, we say that $x$ is good for $\Gamma$.

**Definition 3.** Given an infinite path $\Gamma$, we say that $\Gamma$ is bad if

$$\sharp\{i \in \mathbb{N} | \Gamma[i] \text{ is bad for } \Gamma\} = \infty.$$ 

Otherwise, we say that $\Gamma$ is good.

**Lemma 1.** If $\Gamma$ is good, then there exists an infinite geodesic $\Gamma_0 \in \mathcal{I}(0)$ such that $\Gamma \sim \Gamma_0$.

**Proof.** Let $m = \max\{i \in \mathbb{N} | \Gamma[i] \text{ is bad for } \Gamma\}$. It suffices to prove that $\mathcal{O}(0, \Gamma[m]) \oplus (\Gamma[i]_{i=m}^\infty)$ is an infinite geodesic. We take $l \geq m$. Let $k = \min\{i \in \mathbb{N} | \Gamma[i] \in \mathcal{O}(0, \Gamma[l]) \cap \Gamma\}$. By the definition of $m$, we have $k \leq m$. Since $\mathcal{O}(\Gamma[k], \Gamma[l]) = (\Gamma[i]_{i=k}^l)$, we have $\Gamma[m] \in \mathcal{O}(0, \Gamma[l])$ and $\mathcal{O}(0, \Gamma[m]) \oplus (\Gamma[i]_{i=m}^\infty) = \mathcal{O}(0, \Gamma[l])$. Since any sub-path of an optimal path is also an optimal path, we have that $\mathcal{O}(0, \Gamma[m]) \oplus (\Gamma[i]_{i=m}^\infty)$ is an infinite geodesic.

**Lemma 2.** If $\mathcal{N}_0 < N$, then there exists a finite path $\gamma_0 = (\gamma_0[i]_{i=1}^l)$ such that $\mathcal{I}(\gamma_0)$ is non-empty and for any $\Gamma' \in \mathcal{I}(\gamma_0)$, $\Gamma'$ is bad.

**Proof.** Since $\mathcal{N}_0 < N$, there exists a bad infinite geodesic $\Gamma \in \mathcal{I}$. Note that $\sharp\{\Gamma' \in \mathcal{I}(\Gamma[1]) | \Gamma' \text{ is good }\} \leq \mathcal{N}_0 < N$.

Therefore, there exists $l \in \mathbb{N}$ such that for any $\Gamma' \in \mathcal{I}(\Gamma[i]_{i=1}^l)$, $\Gamma'$ is bad. □

This lemma yields

$$\mathbb{P}(\mathcal{I}(0) < N) \leq \sum_{\gamma_0} \mathbb{P}(\mathcal{I}(\gamma_0) \text{ is non-empty and } \forall \Gamma \in \mathcal{I}(\gamma_0), \Gamma \text{ is bad}),$$

where the summation is taken over all finite path. We fix a finite path $\gamma_0$ and set $\nu = \gamma_0[1]$. Let us define the event $\mathcal{A}$ as

$$\mathcal{A} = \{\mathcal{I}(\gamma_0) \text{ is non-empty and } \forall \Gamma \in \mathcal{I}(\gamma_0), \Gamma \text{ is bad}\}.$$

We will prove that for any finite path $\gamma_0$, $\mathbb{P}(\mathcal{A}) = 0$.

Let $\epsilon, M, L, \delta > 0$. We define the event $\mathcal{B}$ as

$$\mathcal{B} = \{t(0, \nu) \leq M\}.$$ (2.1)

Then if we take $M > 0$ sufficiently large depending on $\epsilon$, we get

$$\mathbb{P}(\mathcal{B}) \geq 1 - \epsilon/4.$$ (2.2)

**Definition 4.** Given $a, b \in \mathbb{Z}^d$, $(a, b)$ is said to be black if

$$\begin{align*}
\begin{cases}
|a - b|_1 &\geq \delta \sharp \mathcal{O}(a, b), \\
t(a, b) &\geq \delta \sharp \mathcal{O}(a, b), \\
\sharp \{e \in \mathcal{O}(a, b) | \tau_e \geq 3M\} &\geq \delta \sharp \mathcal{O}(a, b).
\end{cases}
\end{align*}$$ (2.3)
Lemma 3. For any $M > 0$, there exist $c_1, c_2 > 0$ and $\delta > 0$ such that for any $k \in \mathbb{N}$,
\[
\mathbb{P}
\left(\forall a, b \in [-k, k]^d, \begin{cases} (a, b) \text{ is black} & \text{if } |a - b|_1 \geq \sqrt{k} \\ \mathbb{P}(a, b) \leq k/2 & \text{otherwise} \end{cases} \right) \leq c_1 \exp\left(-c_2 \sqrt{k}\right).
\]

We postpone the proof until Appendix. The condition for $|a - b|_1 < \sqrt{k}$ is necessary to restrict our attention to optimal paths whose length is sufficiently large, in order to use the condition that $(a, b)$ is black. We define the event $\mathcal{C}$ as
\[
\mathcal{C} = \left\{ \forall k \geq L, \forall a, b \in [-k, k]^d, \begin{cases} (a, b) \text{ is black} & \text{if } |a - b|_1 \geq \sqrt{k} \\ \mathbb{P}(a, b) \leq \delta & \text{otherwise} \end{cases} \right\}.
\]

By Lemma 3, we have the following lemma:

Lemma 4. For any $\epsilon > 0$ and $M > 0$, there exist $\delta, L > 0$ such that
\[
(2.4) \quad \mathbb{P}(\mathcal{C}) \geq 1 - \epsilon/4.
\]

On the event $\mathcal{A}$, we take $\bar{\Gamma} \in \mathcal{I}(\gamma_0)$ such that $\bar{\Gamma}$ is bad with a deterministic rule. We define the event $\mathcal{D}(a_1, \ldots, a_k)$ as
\[
\mathcal{D}(a_1, \ldots, a_k) = \{ \forall 1 \leq j \leq k, \exists i \in (a_{j-1}, a_j) \text{ such that } \bar{\Gamma}[i] \text{ is bad for } \bar{\Gamma} \cap \mathcal{A} \}
\]
with the convention that $a_0 = 1$. Note that $\mathcal{D}(a_1, \ldots, a_{k+1}) \subset \mathcal{D}(a_1, \ldots, a_k)$ and
\[
\lim_{a_{k+1} \to \infty} \mathbb{P}(\mathcal{D}(a_1, \ldots, a_{k+1})) = \mathbb{P}(\mathcal{D}(a_1, \ldots, a_k)).
\]
Then we define the sequence $\{a_k\} \subset \mathbb{N}$ inductively as follows: Let $a_1 > 2(\delta^{-1} M + L + |v|_1 + 1)$ be $\mathbb{P}(\mathcal{A} \setminus \mathcal{D}(a_1)) < \epsilon/4$. Suppose that we have defined $\{a_{j-1}\}_{j=1}^k$. We set $a_{k+1}$ such that $a_{k+1} > a_k$ and
\[
\mathbb{P}(\mathcal{D}(a_1, \ldots, a_k) \setminus \mathcal{D}(a_1, \ldots, a_{k+1})) < \epsilon/2^{k+2}.
\]
We define
\[
\mathcal{D} = \{ \exists \Gamma \in \mathcal{I}(\gamma_0) \text{ s.t. } \forall j \in \mathbb{N}, \exists i \in (a_j, a_{j+1}) \text{ s.t. } \Gamma[i] \text{ is bad for } \Gamma \}.
\]
Note that
\[
\mathcal{D} \supset \bigcap_{k \in \mathbb{N}} \mathcal{D}(a_1, \ldots, a_k).
\]
Thus, we have
\[
(2.5) \quad \mathbb{P}(\mathcal{A} \setminus \mathcal{D}) < \epsilon/4.
\]
We define $\mathcal{P} = \mathcal{A} \cap \mathcal{B} \cap \mathcal{D} \cap \mathcal{C}$. By (2.2), (2.4) and (2.5), for any $\epsilon > 0$, there exist $M, L, \delta > 0$ such that
\[
\mathbb{P}(\mathcal{A}) \leq \mathbb{P}(\mathcal{P}) + \epsilon.
\]

Proposition 1. For any $M, L, \delta > 0$
\[
\mathbb{P}(\mathcal{P}) = 0.
\]

Since $\epsilon > 0$ is arbitrary, this proposition leads to $\mathbb{P}(\mathcal{A}) = 0$ and we conclude the proof. Before going into the proof of Proposition 1, we prepare some definitions.

Definition 5. We say that $\Gamma \in \mathcal{I}(\nu)$ has $k$-step if there exists $i \in (a_k, a_{k+1})$ such that $\Gamma[i]$ is bad and for any $i > a_{k+1}$, $\Gamma[i]$ is good.

Definition 6. An edge $e \in E^d$ is said to be $k$-pivotal if there exist $\Gamma \in \mathcal{I}(\gamma_0)$ and $i \in (a_k, a_{k+1})$ such that $e = \{\Gamma[i-1], \Gamma[i]\}$ and for any $\Gamma' \in \mathcal{I}(\gamma_0)$ satisfying that $e \notin \Gamma'$ and $\Gamma'$ is good, there exists $j \in \mathbb{N}$ such that for any $m \geq j$, $e \in \mathcal{O}(0, \Gamma'[m])$.

Definition 7. $\Gamma \in \mathcal{I}(\nu)$ is said to be very bad if for any $k \in \mathbb{N}$, there exists $i \in (a_k, a_{k+1})$ such that $\Gamma[i]$ is bad.

Definition 8. Given $\Gamma \in \mathcal{I}(\nu)$, let $S(\Gamma) = \sup\{i \in \mathbb{N} \mid \Gamma[i] \text{ is bad}\}$. If $\Gamma$ is bad, then we set $S(\Gamma) = \infty$. Let $\mathcal{R} = \inf\{S(\Gamma) \mid \Gamma \in \mathcal{I}(\gamma_0)\}$ and $\mathcal{K} = \inf\{|t(0, x_R)| \mid \Gamma \in \mathcal{I}(\gamma_0) \text{ with } R = S(\Gamma)\}$. 
Proposition 2. For any $k \in \mathbb{N}$,
\[ \mathbb{P}(\delta a_{k-1}/2 \leq K < \infty) \geq \frac{\delta^2}{4} \mathbb{P}(\mathcal{P}) \mathbb{P}(\tau_e < M). \]

Proof of Proposition 2. Since $\lim_{l \to \infty} \mathbb{P}(l \leq K < \infty) = 0$, letting $k \to \infty$, that is $a_{k-1} \to \infty$, we have $\mathbb{P}(\mathcal{P}) = 0$.

To prove Proposition 2, we will use the following lemma.

Lemma 5. For any $k \geq 2$ and $\eta \in E^d$,
\[ \mathbb{P}\left( \{\eta \text{ is } k\text{-pivotal} \} \cap \left\{ \delta a_{k-1}/2 \leq K < \infty \right\} \cap \left\{ \sharp\{e \in E^d| e \text{ is } k\text{-pivotal}\} \leq 2\delta^{-1}a_{k+1} \right\} \right) \]
\[ \geq \mathbb{P}\left( \left\{ \exists \Gamma \in \mathcal{I}(\gamma_0) \text{ s.t. } \Gamma \text{ is } \text{very } \text{bad}, \tau_\eta \geq 3M, \exists j \in (a_k,a_{k+1}] \text{ s.t. } \eta = \{\Gamma[j-1],\Gamma[j]\} \right\} \cap \mathcal{P} \right) \mathbb{P}(\tau_\eta < M) \]

Proof. Let $\{\tau^*_e\}_{e \in E^d}$ be independent copy of $\{\tau_e\}_{e \in E^d}$. Define $\{\tau_e^{(n)}\}_{e \in E^d}$ as
\[ \tau_e^{(n)} = \begin{cases} \tau^*_e & \text{if } e = \eta, \\ \tau_e & \text{if } e \neq \eta. \end{cases} \]

We write that $\Gamma$ ia bad$^{(n)}$ if $\Gamma$ is bad with respect to $\tau^{(n)}$. We will use this convention for other properties. We have that the right hand side of (2.6) equals to
\[ \mathbb{P}\left( \left\{ \exists \Gamma \in \mathcal{I}(\gamma_0) \text{ s.t. } \Gamma \text{ is } \text{very } \text{bad}, \tau_\eta \geq 3M, \exists j \in (a_k,a_{k+1}] \text{ s.t. } \eta = \{\Gamma[j-1],\Gamma[j]\} \right\} \cap \mathcal{P} \cap \{\tau^{(n)}_\eta < M\} \right). \]

We suppose the event inside of (2.6) and take such a path $\Gamma$ and $j$. It suffices to show that
\[ \{\eta \text{ is } k\text{-pivotal}^{(n)}\} \cap \{\delta a_{k-1}/2 \leq K^{(n)} < \infty\} \cap \{\sharp\{e \in E^d| e \text{ is } k\text{-pivotal}^{(n)}\} \leq 2\delta^{-1}a_{k+1}\}. \]

The proof is divided into five steps.

Step 1: $\Gamma \in \mathcal{I}^{(n)}(\gamma_0)$.

Proof. Note that for any $l$ with $l > j$, $(\Gamma[i])_{i=1}^l$ is an optimal path with respect to $\tau^{(n)}$. Since any sub–path of an optimal path is also optimal, we have $\Gamma \in \mathcal{I}^{(n)}(\gamma_0)$.

Step 2: $\Gamma$ has $k$–step or $k - 1$–step with respect to $\tau^{(n)}$. In particular, $K^{(n)} < \infty$.

Proof. Let $l > j$. Since
\[ M + t(0,\Gamma[l]) \geq t(v,\Gamma[l]) > t^{(n)}(v,\Gamma[l]) + 2M, \]
we have
\[ t(0,\Gamma[l]) > t^{(n)}(v,\Gamma[l]) + M > t^{(n)}(0,\Gamma[l]). \]
Thus $\eta \in \mathcal{O}^{(n)}(0,\Gamma[l])$ and $\Gamma[l]$ is good$^{(n)}$.

Next we take $l \leq a_k$ such that $\Gamma[l]$ is bad for $\Gamma$. Then we will show that $\Gamma[l]$ is also bad$^{(n)}$ for $\Gamma$. If $\mathcal{O}^{(n)}(0,\Gamma[l]) \cap \Gamma \neq \{\Gamma[l]\}$, then $\eta \in \mathcal{O}^{(n)}(0,\Gamma[l])$. Since $\Gamma$ is an infinite geodesic for $\tau^{(n)}$, there exists $l_1 \geq j$ such that $\Gamma[l_1]$ is bad$^{(n)}$, which contradicts the above conclusion.

Step 3: For any good$^{(n)}$ $\Gamma_1 \in \mathcal{I}^{(n)}(\gamma_0)$ with $\eta \notin \Gamma$ and for any sufficiently large $i \in \mathbb{N}$, we have
\[ \eta \in \mathcal{O}^{(n)}(0,\Gamma_1[i]) \text{ and } t^{(n)}(0,\Gamma_1[S^{(n)}(\Gamma_1)]) \geq \delta a_{k-1}/2. \]
In particular, $\eta$ is $k$-pivotal$^{(n)}$.

Proof. By the same argument of Step 1, we get $\Gamma_1 \in \mathcal{I}^{(n)}(\gamma_0)$. Thus by the condition of $A$, $\Gamma_1$ is bad. We take $k_1 \in \mathbb{N}$ so that $\Gamma_1$ has $k_1$-step for $\tau^{(n)}$. Let $l > a_{k_1+1}$ be such that $\Gamma_1[l]$ is bad for $\Gamma_1$. Then for any $l_1 > l$, then since $\mathcal{O}(0,\Gamma_1[l_1]) \neq \mathcal{O}^{(n)}(0,\Gamma_1[l_1])$, we have $\eta \in \mathcal{O}^{(n)}(0,\Gamma_1[l_1])$. Since $\eta \notin \mathcal{O}^{(n)}(\Gamma_1[S^{(n)}(\Gamma_1)],\Gamma_1[l_1])$, we obtain $\eta \in \mathcal{O}^{(n)}(0,\Gamma_1[S^{(n)}(\Gamma_1)])$. 


Recall that \( \eta = \{ \Gamma[j - 1], \Gamma[j] \} \). Since \( \mathcal{O}(v, \Gamma[j - 1]) = j - 2 \geq \sqrt{j + |v|} \), using the condition \( C \) with \( k = j + |v| \), we have

\[
\begin{align*}
& t^{(n)}(0, \Gamma[1]|S^{(n)}(\Gamma[1])) \geq t^{(n)}(0, \Gamma[j - 1]) \\
& \quad \geq t^{(n)}(v, \Gamma[j - 1]) - t^{(n)}(0, v) \\
& \quad = t(v, \Gamma[j - 1]) - t^{(n)}(0, v) \\
& \quad \geq \delta a_k - M \geq \delta a_{k - 1}/2.
\end{align*}
\]

(2.7)

\[ \Box \]

Step 4: For any \( \Gamma_1 \in \mathcal{I}^{(n)}(\gamma_0) \) with \( \eta \in \Gamma_1 \) and \( S(\Gamma_1) < \infty \),

\[
t^{(n)}(0, \Gamma_1|S^{(n)}(\Gamma_1)) \geq \delta a_{k - 1}/2.
\]

\[ \Box \]

Proof. Since \( \mathcal{I}^{(n)}(\gamma_0) \) with \( \eta \in \Gamma_1 \) and \( S(\Gamma_1) < \infty \),

\[
t^{(n)}(0, \Gamma_1|S^{(n)}(\Gamma_1)) \geq \delta a_{k - 1}/2.
\]

(2.8)

Combing Step 2-4, \( \delta a_{k - 1}/2 \leq K < \infty \) holds.

Step 5: If \( e \in E^d \) is \( k \)-pivotal, then \( e \in \mathcal{O}^{(n)}(0, \Gamma[j]) \) or \( e \in \{ \Gamma[1], \ldots, \Gamma[a_k + 1] \} \). In particular, for \( \mathcal{O}^{(n)}(0, \Gamma[j]) \) or \( e \in \{ \Gamma[1], \ldots, \Gamma[a_k + 1] \} \), we have

\[
\sharp\{ e \in E^d | e \text{ is } k\text{-pivotal} \} \leq 2\delta^{-1}a_{k + 1}.
\]

Proof. If \( e \notin \Gamma \), then there exists \(\eta \in \Gamma \cap \Gamma \), \( e \in \mathcal{O}^{(n)}(0, \Gamma[m]) \). On the other hand, by Step 2, for any \( m \geq l \), \( \mathcal{O}^{(n)}(\Gamma[j], \Gamma[m]) \subset \mathcal{O}^{(n)}(0, \Gamma[m]) \), which leads to \( e \in \mathcal{O}^{(n)}(0, \Gamma[j]) \). If \( e \in \Gamma \), then there exists \( \Gamma_1 \in \mathcal{I}^{(n)}(\gamma_0) \) and \( \eta \in \{ \Gamma[1], \ldots, \Gamma[a_k + 1] \} \).

Note that we have proved in Step 2 that \( \mathcal{O}^{(n)}(0, \Gamma[j - 1]) = \mathcal{O}(0, \Gamma[j - 1]) \) and \( \mathcal{O}^{(n)}(0, \Gamma[j]) = \mathcal{O}(0, \Gamma[j - 1]) \). Thus, by the condition \( C \), we obtain

\[
\sharp\{ e \in E^d | e \in \mathcal{O}^{(n)}(0, \Gamma[j]) \} \leq \delta^{-1}|\Gamma[j]|_1
\]

(2.9)

Since \( \sharp\{ e \in E^d | e \in \{ \Gamma[i] \}_{i=1}^{a_{k+1}} \} \leq a_{k+1} \), we have the conclusion.

We turn to the proof of Lemma \( \sharp \) By Step 1-5, we have

\[
\mathbb{P} \left( \{ \eta \text{ is k-pivotal} \} \cap \{ \sharp\{ e \in E^d | e \text{ is } k\text{-pivotal} \} \leq 2\delta^{-1}a_{k + 1} \} \cap \{ \delta a_{k - 1}/2 \leq K < \infty \} \right)
\]

\[ = \mathbb{P} \left( \{ \eta \text{ is k-pivotal} \} \cap \{ \sharp\{ e \in E^d | e \text{ is } k\text{-pivotal} \} \leq 2\delta^{-1}a_{k + 1} \} \cap \{ \delta a_{k - 1}/2 \leq K < \infty \} \right)
\]

\[ \geq \mathbb{P} \left( \{ \exists \Gamma_1 \in \mathcal{I}^{(n)}(\gamma_0) \text{ s.t. } \Gamma_1 \text{ is very bad, } \tau_\eta \geq 3M, \eta = \{ \Gamma[j - 1], \Gamma[j] \} \}, \mathcal{P}, \tau^*_e < M \right),
\]

as desired.

\[ \Box \]

Proof of Proposition \( \sharp \) Note that if \( C \) holds and there exists \( \Gamma \in \mathcal{I}^{(n)}(\gamma_0) \) such that \( \Gamma \) is very bad,

\[
\sharp\{ e \in E^d | \tau_e \geq 3M, e \in \{ x_i \}_{i=a_k}^{a_{k+1}} \} \geq \delta(a_{k+1} - a_k).
\]
Therefore,
\[
25^{-1}a_{k+1}\mathbb{P}(\delta a_{k-1}/2 \leq K < \infty) \\
\geq \mathbb{E}\left[\mathbb{P}\left(\{e \in E^d \mid e \text{ is } k\text{-pivotal}\} \cap \{\delta a_{k-1}/2 \leq K < \infty\}\right)\right] \\
= \sum_{e \in E^d} \mathbb{P}\left(\{\delta a_{k-1}/2 \leq K < \infty\} \cap \{e \text{ is } k\text{-pivotal}\} \cap \{\delta e \leq (a_{k+1} - a_{k})\}\right) \\
\geq \sum_{e \in E^d} \mathbb{P}\left(\left\{\exists \Gamma \in \mathcal{I}(\gamma_0), \Gamma \text{ is very bad}, \tau_e \geq 3M, e \in (\Gamma[i])_{i=a_k}^{a_{k+1}} \right\}, \mathcal{P}\right) \mathbb{P}(\tau_e < M), \\
= \mathbb{E}\left[\mathbb{P}\left(\exists \Gamma \in \mathcal{I}(\gamma_0) \text{ s.t. } \Gamma \text{ is very bad}, e \in (\Gamma[i])_{i=a_k}^{a_{k+1}} \geq 3M; \mathcal{P}\right) \mathbb{P}(\tau_e < M), \\
\geq \delta(a_{k+1} - a_k)\mathbb{P}(\mathcal{P})\mathbb{P}(\tau_e < M) \geq \delta(a_{k+1} - a_k)\mathbb{E}\left[\sum_{e \in E^d} \mathbb{P}(\{e \in E^d \mid e \text{ is } k\text{-pivotal}\} \cap \{\delta a_{k-1}/2 \leq K < \infty\})\right] \mathbb{P}(\tau_e < M).
\]

\[\square\]

2.3. Proof for continuous distributions with bounded support. Suppose that \(F\) is continuous and \(F^+ < \infty\). The proof is similar as before, so we sketch the difference of them. We take positive constants \(\alpha_1, \alpha_2\) such that \(F^- < \alpha_1 < \alpha_2 < F^+\). We replace the definitions of \(\mathcal{B}\) and \(\mathcal{C}\) as follows. Let us define the event \(\mathcal{B}_2\) as
\[
\mathcal{B}_2 = \left\{\left(t(0, e) \leq \frac{M}{3(\alpha_2 - \alpha_1)}\right)\right\}.
\]

**Definition 9.** \((a, b) \in \mathbb{Z}^d \times \mathbb{Z}^d\) is said to be black\(^2\) if
\[
|a - b| \geq \delta\Omega(a, b), \\
t(a, b) \geq \delta\Omega(a, b), \\
\exists \gamma \in \mathcal{T}_M \mid \gamma \subset \Omega(a, b), \forall e \in \gamma, \tau_e \geq \delta \Omega(a, b).
\]

**Lemma 6.** There exist \(c_1, c_2 > 0\) such that for any \(k \in \mathbb{N}\),
\[
\mathbb{P}
\left[
\forall a, b \in [-k, k]^d, \left\{(a, b) \text{ is black}^{2}, t(\Omega(a, b)) \leq \frac{k}{2} \text{ if } |a - b| \geq \sqrt{k}\right\}
\right] \leq c_1 \exp(-c_2 \sqrt{k}).
\]

We postpone the proof until Appendix. Then if we take \(L\) sufficiently large, we have the following:
\[
(2.11) \quad \mathbb{P}
\left[
\forall k \geq L, \forall a, b \in [-k, k]^d, \left\{(a, b) \text{ is black}^{2}, t(\Omega(a, b)) \leq \delta k \text{ if } |a - b| \geq \sqrt{k}\right\}
\right] \geq 1 - \epsilon/4.
\]

Let \(C_2\) be the event inside \(2.11\). We define \(\mathcal{P}_2 = \mathcal{A} \cap \mathcal{B}_2 \cap \mathcal{D} \cap C_2\). Then as in subsection 2.2 we have that for any \(\epsilon > 0\), there exist \(M, L, \delta > 0\) such that
\[
\mathbb{P}(\mathcal{A}) \leq \mathbb{P}(\mathcal{P}_2) + \epsilon.
\]

**Definition 10.** Given \(\gamma = (\gamma_i)_{i=1}^{\infty} \in \mathcal{T}_M\), \(\gamma\) is said to be k-pivotal if there exists \(\Gamma \in \mathcal{I}(\gamma_0)\) such that \(\gamma \subset (\Gamma[i])_{i=a_k}^{a_{k+1}}\) and for any \(\Gamma' \in \mathcal{I}(\gamma_0)\) satisfying that \(\gamma \cap \Gamma' = \emptyset\) and \(\Gamma'\) is good, there exists \(j \in \mathbb{N}\) such that for any \(m \geq j\), \(\gamma \cap \Omega(0, \Gamma'[m]) \neq \emptyset\).

Then Lemma \([\text{3}]) will be replaced as follows:

**Lemma 7.** For any \(k \geq 2\) and \(\gamma_1 \in \mathcal{T}_M\),
\[
\mathbb{P}\left(\{\gamma_1 \text{ is k-pivotal}\} \cap \{\exists \Gamma \in \mathcal{I}(\gamma_0) \text{ such that } \Gamma \text{ is very bad}, \forall e \in \gamma_1, \tau_e \geq \alpha_2, \gamma_1 \subset (\Gamma[i])_{i=a_k}^{a_{k+1}}\} \cap \mathcal{P}_2\right) \mathbb{P}(\tau_e < a_1) \leq 25^{-1}(2d)^M a_{k-1} \cap \{\delta a_{k-1}/2 \leq K < \infty\})
\]

**Proof.** Let \(\{\tau^*_e\}_{e \in E^d}\) be independent copy of \(\{\tau_e\}_{e \in E^d}\). Define \(\{\tau^*_{e_{\gamma_1}}\}_{e \in E^d}\) as
\[
\tau_{e_{\gamma_1}}^* = \begin{cases} 
\tau_e^* & \text{if } e \in \gamma_1, \\
\tau_e & \text{otherwise.}
\end{cases}
\]
Then Step 1, 2, 4 can be proved in the same way as before. We replace Step 3 and Step 5 by

Step 3’: For any good \( \Gamma_1 \in \mathcal{I}(\gamma_0) \) with \( \gamma \cap \Gamma = \emptyset \) and for any sufficiently large \( i \in \mathbb{N} \), we have
\[
\gamma \cap O(\gamma)(0, \Gamma_1[i]) \neq \emptyset \quad \text{and} \quad t(\gamma)(0, \Gamma_1[S(\gamma)(\Gamma_1)]) \geq \delta a_{k-1}/2.
\]

In particular, \( \gamma \) is \( k \)-pivotal \( \gamma_0 \).

Step 5’: If \( \gamma \in \mathcal{T}_M \) is \( k \)-pivotal \( \gamma_0 \), \( \gamma \cap O(\gamma)(0, \Gamma[a_{k+1}]) \neq \emptyset \) or \( \gamma \cap \{ \Gamma[0] \cdots , \Gamma[a_{k+1}] \} \neq \emptyset \). In particular,
\[
\sharp \{ \gamma \in \mathcal{T}_M \mid \gamma \text{ is } k \text{-pivotal}(\gamma_0) \} \leq 2\delta^{-1}(2d)^Ma_{k+1}.
\]
They can be proved in the same way as in Lemma 3.

\[\square\]

**Proposition 3.** For any \( k \in \mathbb{N} \),
\[
\mathbb{P}(\delta a_{k-1}/2 \leq K < \infty) \geq \frac{\delta^2}{4(2d)^M}\mathbb{P}(P_2)\mathbb{P}(\tau_e < M).
\]

**Proof.**
\[
2\delta^{-1}(2d)^Ma_{k+1}\mathbb{P}(\delta a_{k-1}/2 \leq K < \infty)
\]
\[
\geq \mathbb{E} \left[ \sharp \{ \gamma \in \mathcal{T}_M \mid \gamma \text{ is } k \text{-pivotal} \} \mid \{ \delta a_{k-1}/2 \leq K < \infty \} \right] \cap \mathbb{E} \{ \sharp \{ \gamma \in \mathcal{T}_M \mid \gamma \text{ is } k \text{-pivotal} \} \leq 2\delta^{-1}(2d)^Ma_{k+1} \}
\]
\[
= \sum_{\gamma \in \mathcal{T}_d} \mathbb{P} \left( \{ \gamma \text{ is } k \text{-pivotal} \} \cap \{ \delta a_{k-1}/2 \leq K < \infty \} \cap \{ \sharp \{ \gamma' \in \mathcal{T}_M \mid \gamma' \text{ is } k \text{-pivotal} \} \leq 2\delta^{-1}(2d)^Ma_{k+1} \} \right)
\]
\[
\geq \sum_{\gamma \in \mathcal{T}_d} \mathbb{P} \left( \left\{ \Gamma \text{ is very bad, } \forall \gamma \in \gamma_1, \tau_e \geq \alpha_2, \gamma \subset \{ \Gamma[i] \}_{i=\alpha_1}^{a_{k+1}} \right\} \cap P_2 \right) \mathbb{P}(\tau_e < \alpha_1)^M
\]
\[
\geq \delta(a_{k+1} - a_k)\mathbb{P}(P_2)\mathbb{P}(\tau_e < \alpha_1)^M \geq \frac{\delta a_{k+1}}{2} \cdot \mathbb{P}(P_2)\mathbb{P}(\tau_e < \alpha_1)^M.
\]
Rearranging it, we conclude the proof. \[\square\]

Letting \( k \to \infty \), we have \( \mathbb{P}(P_2) = 0 \). Finally, letting \( \epsilon \to 0 \), we have \( \mathbb{P}(A) = 0 \) as desired.

2.4. **Proof for general distributions.** We only consider the case \( F^+ = \infty \). For the case \( F^+ < \infty \) the proof is similar, combining the argument in subsection 2.3. Let \( K \in \mathbb{N} \). We replace Definition 2 as follows:

**Definition 11.** Given an infinite path \( \Gamma \) and a vertex \( x \in \Gamma \), we say that \( x \) is bad for \( \Gamma \) from \( v \in \mathbb{Z}^d \) if there exists \( \gamma \in O(v, x) \) such that
\[
\gamma \cap \Gamma = \{ x \}.
\]
Otherwise, we say that \( x \) is good for \( \Gamma \) from \( v \).

**Definition 12.** Given an infinite path \( \Gamma \), we say that \( \Gamma \) is bad from \( v \) if
\[
\sharp \{ i \in \mathbb{N} \mid \Gamma[i] \text{ is bad for } \Gamma \text{ from } v \} = \infty.
\]
Otherwise, we say that \( \Gamma \) is good from \( v \).

We simply say that \( \Gamma \) is bad if \( v = 0 \). As in Lemma 3 we have the following lemma.

**Lemma 8.** If \( \Gamma \) is good, then there exists \( \Gamma_0 \in \mathcal{I}(0) \) such that \( \sharp \Gamma \triangle \Gamma_0 < \infty \), where \( \triangle \) is symmetric difference and we regard \( \Gamma \) and \( \Gamma_0 \) as subsets of vertices.

Note that \( \sharp \Gamma \triangle \Gamma_0 < \infty \) is a stronger property than \( \sharp \Gamma \sim \Gamma_0 \).

**Definition 13.** Given \( a, b \in \mathbb{Z}^d \), \( (a, b) \) is said to be black3 if
\[
\begin{cases}
|a-b| \geq \delta \max_{\gamma \in O(a,b)} \sharp \gamma, \\
t(a,b) \geq \delta \max_{\gamma \in O(a,b)} \sharp \gamma, \\
\min_{\gamma \in O(a,b)} \sharp \{ e \in \gamma \mid \tau_e \geq 3M \} \geq \delta \max_{a \in O(a,b)} \sharp O(a,b).
\end{cases}
\]

(2.12)
Lemma 9. For any \( v \in \mathbb{Z}^d \) and \( K \in \mathbb{N} \),
\[
\mathbb{P}(\{N_v \leq K\} \cap \{\exists \Gamma \in \mathcal{I}(v) \text{ such that } \Gamma \text{ is bad}\}) = 0.
\]

Proof. We follow the argument of subsection 2.2. Let \( \mathcal{A}_3 = \{N_v \leq K\} \cap \{\exists \Gamma \in \mathcal{I}(v) \text{ such that } \Gamma \text{ is bad}\} \) and \( \mathcal{P}_3 = \mathcal{A}_3 \cap \mathcal{B} \cap \mathcal{C}_3 \cap \mathcal{D} \). An edge \( e \in E^\tau \) is said to be \( k \)-pivotal if for any distinct infinite geodesics \( \Gamma_1, \ldots, \Gamma_{K \wedge N_v} \in \mathcal{I}(v) \),
\[
e \in \bigcup_{i=1}^{K \wedge N_v} (\Gamma_i[i])_{i=1}^{\delta^{-1} a_k + 1},
\]
Given an infinite geodesic \( \Gamma \in \mathcal{I}(v) \), let \( \hat{S}(\Gamma) = \sup \{S(\Gamma') | \Gamma' \in \mathcal{I}(v), \Gamma \sim \Gamma'\} \). Let
\[
\mathcal{M}_k = \min_{\Gamma_1,\ldots,\Gamma_k} \max_{1 \leq i \leq k} \hat{S}(\Gamma_i),
\]
where \( \Gamma_1, \ldots, \Gamma_k \) run over all distinct \( k \) infinite geodesics in \( \mathcal{I}(v) \). We define the event \( \mathcal{E}_3(k) \) as
\[
\mathcal{E}_3(k) = \{\{i \in \mathbb{N} | a_{k-1} \leq i \leq a_{k+1}\} \cap \{\mathcal{M}_1, \ldots, \mathcal{M}_{K \wedge N_v}\} \neq \emptyset\}.
\]
Note that \( \lim_{k \to \infty} \mathbb{P}(\mathcal{E}_3(k)) = 0 \). By definition, we have
\[
\sharp\{e \in E^\tau | e \text{ is } k\text{-pivotal}\} \leq K \delta^{-1} a_{k+1}.
\]
Let \( \{\tau^*_\gamma\} \) be independent copy of \( \tau \) and we define \( \tau^{(n)} \) as before. By the same argument as before, it suffices to prove the following: for any \( \eta \in E^\tau \),
\[
\mathbb{P}(\{\eta \text{ is } k\text{-pivotal}^{(n)}\} \cap \mathcal{E}_3^{(n)}(k)) \geq \mathbb{P}\left(\left\{\exists \Gamma \in \mathcal{I}(v) \text{ s.t. } \Gamma \text{ is very bad, } \tau_\eta \geq 3M, \exists j \in (a_k,a_{k+1}] \text{ s.t. } \eta = \{\Gamma[j-1] \in \mathcal{I}(v), \Gamma[j]\} \right\} \cap \mathcal{P}_3 \cap \{\tau^*_\gamma < M\}\right).
\]
To this end, suppose that the event inside of the right hand side holds.

Lemma 10. The following hold:
(i) for any \( \Gamma' \in \mathcal{I}^{(n)}(v) \) with \( \eta \notin \Gamma' \), \( \Gamma' \in \mathcal{I}(v) \),
(ii) for any \( \Gamma' \in \mathcal{I}^{(n)}(v) \) with \( \Gamma' \sim \Gamma, \eta \in \Gamma' \),
(iii) for any \( \Gamma' \in \mathcal{I}^{(n)}(v) \) with \( \eta \in \Gamma', \hat{S}(\Gamma') \geq a_{k-1} \),
(iv) \( \hat{S}(\Gamma^{(n)}(\eta)) \leq a_{k+1} \),
(v) \( \max\{i \in \mathbb{N} | \Gamma_i, \ldots, \Gamma_i \in \mathcal{I}^{(n)}(v), \text{ s.t. } \eta \notin \Gamma_i \text{ and } \Gamma_i \not\sim \Gamma \text{ if } i \neq j\} \leq (K \wedge N_v^{(n)}) - 1 \).

Proof. (i)-(iv) can be proved in a similar way as in Step 2 of Lemma 5. If \( \eta \notin \Gamma_i \in \mathcal{I}^{(n)}(v) \), then we have that \( \Gamma_i \in \mathcal{I}(v) \) and \( \Gamma \not\sim \Gamma_i \). Therefore, we obtain \( \eta \).

By (v), for any distinct infinite geodesics \( \Gamma_1, \ldots, \Gamma_{K \wedge N_v^{(n)}} \in \mathcal{I}^{(n)}(v), \eta \in \bigcup_{i=1}^{K \wedge N_v^{(n)}} \Gamma_i \), Since \( |\Gamma[j] - v| \leq a_{k+1} \), by the condition \( C_3 \),
\[
\max_{\gamma \in [0(v,\Gamma[j])] } \sharp\gamma \leq \delta^{-1} a_{k+1}.
\]
Therefore \( \eta \) is \( k\)-pivotal\(^{(n)}\). Next we prove that \( \mathcal{E}_3(k) \) holds. Let \( j \in \mathbb{N} \) be such that for any \( i \leq j, \mathcal{M}_i < a_{k-1} \) and for any \( i > j, \mathcal{M}_i \geq a_{k-1} \). By (ii) and (v) in Lemma 10, we get \( j \leq (K \wedge N_v^{(n)}) - 1 \). Thus, it suffices to show \( \mathcal{M}_{j+1} \leq a_{k+1} \). Take distinct infinite geodesics \( \Gamma_1, \ldots, \Gamma_j \) such that \( \hat{S}(\Gamma_i) < a_{k-1} \). Then since \( \eta \notin \Gamma_i \) and \( \Gamma_j \not\sim \Gamma \) for any \( i \) by (iii) in Lemma 10 defining \( \Gamma_{i+1} = \Gamma, (\Gamma_1, \ldots, \Gamma_{i+1}) \) are distinct infinite geodesics. Thus, by (iv) in Lemma 10 we have \( \mathcal{M}_{j+1} \leq a_{k+1} \).

The rest of the proof is the same as before. \( \square \)
Letting $K$ goes to infinity, we have
\[ P(\{N_0 < \infty \} \cap \{ \exists \Gamma \in I(v) \text{ such that } \Gamma \text{ is bad} \}) = 0. \]

Exchanging the roles of $0$ and $v$, we have that
\[ P(\{ N_0 < \infty \} \cap \{ \exists v \in \mathbb{Z}^d \text{ and } \Gamma \in I(0) \text{ such that } \Gamma \text{ is bad from } v \}) \]
\[ \leq \sum_{v \in \mathbb{Z}^d} P(\{ N_0 < \infty \} \cap \{ \exists \Gamma \in I(0) \text{ such that } \Gamma \text{ is bad from } v \}) = 0. \tag{2.14} \]

Next lemma corresponds to Lemma \[2\].

**Lemma 11.**
\[ P(N_0 < N) \leq P(\exists \text{ finite path } \gamma_0 \text{ such that } I(\gamma_0) \neq \emptyset \text{ and } \forall \Gamma' \in I(\gamma_0), \Gamma' \text{ is bad}). \tag{2.15} \]

**Proof.** By \[2.14\], it suffices to show that if $N_0 < \infty$ and for any $\Gamma' \in I(0)$ and $v \in \mathbb{Z}^d$, then $\Gamma'$ is good from $v$, then the event of the right hand side \[2.15\] holds. Let $v \in \mathbb{Z}^d$ and $\Gamma \in I(v)$ such that for any $\Gamma' \in I(0)$, $\Gamma' \not\sim \Gamma$. Then by Lemma \[8\], $\Gamma$ is bad. We take distinct infinite geodesics $\{ \Gamma_i \}_{i=1}^{n_0} \subset I(0)$. For any $i \in \{1, \ldots, N_0\}$, since $\Gamma_i$ is good from $v$, there exists $\ell_i \in \mathbb{N}$ such that for any $\Gamma' \in I(v)$ with $(\Gamma[i])_{\ell_i} \subset \Gamma'$, $\Gamma' \not\sim \Gamma_i$. Let $\ell = \max_{1 \leq i \leq N_0} \ell_i$ and $\gamma_0 = (\Gamma[i])_{\ell_i=1}$. Note that for any $\Gamma' \in I(v)$, if $\Gamma' \not\sim \Gamma_i$ for any $i$, then since $N_0$ is the maximum number of distinct infinite geodesics, $\Gamma'$ is bad. Therefore, for any $\Gamma' \in I(v)$ with $\gamma_0 \subset \Gamma'$, $\Gamma'$ is bad. \[\square\]

**Definition 14.** An edge $e \in E^d$ is said to be $k$-pivotal if there exists $\Gamma \in I(\gamma_0)$ with $e \in (\Gamma[i])_{i=a_k+1}^{a_k+1}$ and for any $\Gamma' \in I(\gamma_0)$, if $e \notin \Gamma'$ and $\Gamma'$ is good, then for any sufficiently large $m$,
\[ e \in \bigcap_{\gamma \in O(0,\Gamma'[m])} \gamma. \]

**Definition 15.**
\[ K_4 = \min_{\Gamma \in I(\gamma_0)} \min\{ t(0, \Gamma[S(\Gamma)]) \mid \Gamma' \in I(\gamma_0), \Gamma \sim \Gamma', S(\Gamma') = \hat{S}(\Gamma) \}. \]

Fix $v \in \mathbb{Z}^d$ and a finite path $\gamma_0$ starting at $v$. Let us define
\[ A_4 = \{ I(\gamma_0) \text{ is non-empty and } \forall \Gamma \in I(\gamma_0), \Gamma \text{ is bad} \}, \]
and $P_4 = A_4 \cap B \cap C_3 \cap D$.

**Lemma 12.** Given $\eta \in E^d$, we define $\tau^{(n)}$ and the term 'very bad' as before. Then,
\[ \mathbb{P} \left( \left\{ \eta \text{ is } k \text{-pivotal}^{(n)} \cap \{ e \in E^d \mid e \text{ is } k \text{-pivotal}^{(n)} \} \leq 2\delta^{-1} a_{k+1} \cap \{ \delta a_{k-1}/2 \leq K_4 < \infty \} \right\} \right) \]
\[ \geq \mathbb{P} \left( \left\{ \exists \Gamma \in I(\gamma_0), \Gamma \text{ is very bad}, \tau_\eta \geq 3M, \right. \right. \]
\[ \left. \left. \exists j \in (a_k, a_{k+1}) \text{ s.t. } \eta = \{ \Gamma[j-1], \Gamma[j] \} \right\}, \ P_4, \ \tau^*_e < M \right). \]

**Proof.** Step 2–5 will be replace by:

**Step 2':** $S^{(n)}(\Gamma) < \infty$. In particular, $K_4 < \infty$.

**Step 3':** For any good $\Gamma_1 \in I^{(n)}(\gamma_0)$ with $\eta \notin \Gamma_1$, for any sufficiently large $i \in \mathbb{N}$,
\[ \eta \in \bigcap_{\gamma \in O^{(n)}(0,\Gamma_1[i])} \gamma \text{ and } t^{(n)}(0, \Gamma_1[S^{(n)}(\Gamma_1)]) \geq \delta a_{k-1}/2. \]

In particular, $\eta$ is $k$-pivotal $^{(n)}$.

**Step 4':** The following hold:

(i) for any $\Gamma_1 \in I^{(n)}(\gamma_0)$ with $\eta \notin \Gamma_1$, there exists $\Gamma_2 \in I^{(n)}(\gamma_0)$ such that $(\Gamma[i])_{i=j-1}^{\infty} \subset \Gamma_2$ and $S^{(n)}(\Gamma_2) \geq a_{k-1}$, in particular, $S^{(n)}(\Gamma_1) \geq a_{k-1}$.

(ii) if $\Gamma_2 \in I^{(n)}(\gamma_0)$ satisfying that $S^{(n)}(\Gamma_2) \geq a_{k-1}$, then
\[ t^{(n)}(0, \Gamma_2[S^{(n)}(\Gamma_2)]) \geq \delta a_{k-1}/2. \]
Step 5*: If \( e \in E^d \) is \( k \)-pivotal\(^{(n)} \), then \( e \in \cap_{\gamma \in O^{(n)}(0,x)} \gamma \) or \( e \in \{ \Gamma[1], \ldots, \Gamma[a_{k+1}] \} \). In particular,
\[
\sharp \{ e \in E^d \mid e \text{ is } k \text{-pivotal } \}
\leq 25^{-1} a_{k+1}.
\]
Except for Step 2" and Step 3", the proofs are the same as in Lemma 3.

Proof of Step 2". Let \( \Gamma_1 \in I^{(n)}(\gamma_0) \) be such that \( \Gamma_1 \sim \Gamma \). Then by Lemma 10-(i) and (ii), we obtain \( \eta \in \Gamma_1 \) and \( \Gamma_1 \in I(\gamma_0) \). Let \( l \in \mathbb{N} \) be such that \( \Gamma_1[l] = \Gamma[j] \). By the same argument as in Step 2, we have that for any \( i \geq l \), \( \Gamma_1[i] \) is good\(^{(n)} \) for \( \Gamma_1 \). It follows that \( S^{(n)}(\Gamma) \leq \infty \). \( \square \)

Proof of Step 3". By the same argument of Step 1, we get \( \Gamma_1 \in I(\gamma_0) \). Thus by the condition of \( \mathcal{A}_3 \), \( \Gamma_1 \) is bad. Let \( i \in \mathbb{N} \) be such that \( i > S^{(n)}(\Gamma_1) \) and \( \Gamma_1[i] \) is bad. Since \( t^{(n)}(0, \Gamma_1[i]) < t(0, \Gamma[i]) \), we have \( \eta \in \cap_{\gamma \in O^{(n)}(0, \Gamma[i])} \gamma \). Let \( l > i \) and \( \gamma \in O^{(n)}(0, \Gamma_1[l]) \). We define \( l_1 = \min \{ l_2 \in \mathbb{N} \mid \gamma[l_2] \in \Gamma_1 \} \) and let \( l_1' \) be \( \Gamma_1[l_1'] = \gamma[l_1] \). Note that since \( l_1' \leq S^{(n)}(\Gamma_1) \), \( (\gamma[1], \ldots, \gamma[l_1]) \oplus (\Gamma_1[l_1'], \ldots, \Gamma_1[i]) \in O^{(n)}(0, \Gamma_1[l]) \), we get \( (\gamma[1], \ldots, \gamma[l_1]) \oplus (\Gamma_1[l_1'], \ldots, \Gamma_1[i]) \in O^{(n)}(0, \Gamma_1[i]) \). Together with \( \eta \notin (\Gamma_1[l_1'], \ldots, \Gamma_1[i]) \), we have \( \eta \notin \gamma \).

By the same argument as before, for any \( \gamma \in O^{(n)}(0, \Gamma_1[S^{(n)}(\Gamma_1)]) \), we have
\[
\gamma \oplus (\Gamma_1[S^{(n)}(\Gamma_1)], \ldots, \Gamma_1[i]) \in O^{(n)}(0, \Gamma_1[i]) \quad \text{and} \quad \eta \notin \bigcap_{\gamma \in O^{(n)}(0, \Gamma_1[S^{(n)}(\Gamma_1)])} \gamma.
\]
With a similar argument to (2.7), we obtain
\[
t^{(n)}(0, \Gamma_1[S^{(n)}(\Gamma_1)]) \geq \delta a_{k-1}/2.
\]
\( \square \)

We now turn to the proof of Lemma 12. Let \( \Gamma_1, \Gamma_2 \in I^{(n)}(\gamma_0) \) be such that \( \Gamma_1 \sim \Gamma_2 \) and \( S^{(n)}(\Gamma_1) = S^{(n)}(\Gamma_2) \). If \( \eta \notin \Gamma_2 \), then Step 3" yields \( t^{(n)}(0, \Gamma_2[S^{(n)}(\Gamma_2)]) \geq \delta a_{k-1}/2 \). If \( \eta \in \Gamma_2 \), then by Step 4"-(i), there exists \( \Gamma_3 \in I^{(n)}(\gamma_0) \) such that \( (\Gamma_3[l])_{l=1}^{\infty} \subset \Gamma_3 \) and \( S^{(n)}(\Gamma_3) \geq a_{k-1} \). Then since \( \Gamma_3 \sim \Gamma_1 \), we have \( S^{(n)}(\Gamma_1) \geq a_{k-1} \) and \( t(0, \Gamma_2[S^{(n)}(\Gamma_2)]) \geq \delta a_{k-1}/2 \) by using Step 4"-(ii). This yields \( \delta a_{k-1}/2 \leq \mathcal{K}_1^{(n)} \).

The rest of the proof is the same as before and we skip the details. \( \square \)

3. Appendix

3.1. Proof of Lemma 3

Lemma 13. For any \( M > 0 \), there exists \( c_1, c_2 > 0 \) such that for any \( x \in \mathbb{Z}^d \),
\[
\mathbb{P} \left( \min_{\Gamma \in O(0,x)} \sharp \{ e \in \Gamma \mid \tau_e \geq M \} \leq c|x|_1 \right) \leq c_1 \exp \{ -c_2|x|_1 \}.
\]

Proof. We take \( \tilde{\tau}_e \) such that if \( \tau_e < M \), \( \tilde{\tau}_e = \tau_e \) and otherwise, \( \tilde{\tau}_e = \tau_e + 1 \). The results of [3] imply that there exists \( c > 0 \) such that for any \( x \in \mathbb{Z}^d \),
\[
\mathbb{E}[\tilde{\tau}(0,x)] \geq \mathbb{E}[t(0,x)] + c|x|_1.
\]
Although they only discuss the first passage time from 0 to \( N \mathbf{x}_1 \), the same proof works. By Theorem 3.11 in [1], we have that there exists \( c_1, c_2 > 0 \) such that
\[
\mathbb{P}(t(0,x) - \mathbb{E}[t(0,x)] \geq c|x|_1/4) \leq c_1 \exp (-c_2|x|_1),
\]
\[
\mathbb{P}(\tilde{\tau}(0,x) - \mathbb{E}[\tilde{\tau}(0,x)] \geq c|x|_1/4) \leq c_1 \exp (-c_2|x|_1).
\]
The yields that
\[
\mathbb{P}(\tilde{\tau}(0,x) - t(0,x) \geq c|x|_1/2) \leq 2c_1 \exp (-c_2|x|_1).
\]
Note that \( \min_{\Gamma \in O(0,x)} \sharp \{ e \in \Gamma \mid \tau_e \geq M \} \leq c|x|_1/2 \) implies \( \tilde{\tau}(0,x) - t(0,x) \leq c|x|_1/2 \). Therefore the proof is completed. \( \square \)
Lemma 14. There exists $C, c_1, c_2 > 0$ such that

$$\mathbb{P} \left( \max_{\Gamma \in \Omega(0,z)} \sharp \Gamma \geq C|x|_1 \right) \leq c_1 \exp(-c_2|x|_1). \tag{3.3}$$

There exists $c_1, c_2 > 0$ such that for any $k \in \mathbb{N}$ and $a, b \in \mathbb{Z}^d$ with $|a - b| \leq \sqrt{k}$,

$$\mathbb{P} \left( \max_{\Gamma \in \Omega(a,b)} \sharp \Gamma \geq k/2 \right) \leq c_1 \exp(-c_2|x|_1). \tag{3.4}$$

Proof. From Proposition 5.8 in [6], there exist $A, B, C > 0$ such that for any $r > 0$

$$\mathbb{P} (\exists \text{ self-avoiding path } \Gamma \text{ from } 0 \text{ with } |\Gamma| \geq r \text{ and } t(\Gamma) < Ar) < B \exp(-Cr). \tag{3.5}$$

We take a positive constant $C$ sufficiently large. We use Lemma 3.13 in [1] and (3.5) with $r = C|x|_1$

to obtain,

$$\mathbb{P} \left( \max_{\Gamma \in \Omega(0,z)} \sharp \Gamma \geq C|x|_1 \right) \leq \mathbb{P} (\exists \text{ self-avoiding path } \Gamma \text{ from } 0 \text{ with } |\Gamma| \geq C|x|_1 \text{ and } t(\Gamma) < AC|x|_1) + \mathbb{P}(t(0,x) \geq AC|x|_1) \leq c_1 \exp(-c_2|x|_1),$$

with some constant $c_1, c_2 > 0$. This yields (3.3).

Note that

$$\mathbb{P} \left( \max_{\Gamma \in \Omega(a,b)} \sharp \Gamma \geq k/2 \right) \leq \mathbb{P} \left( \max_{\Gamma \in \Omega(a,b)} \sharp \Gamma \geq k/2, t(a, b) < Ak/2 \right) + \mathbb{P}(t(a, b) \geq Ak/2).$$

The first term can be bounded by (3.5). By exponential Markov inequality, the second term also can be bounded from above by $c_1 e^{-c_2k}$ with some $c_1, c_2 > 0$. \qed

Proof of Lemma 3 If we take $\delta > 0$ sufficiently small and $C, L > 0$ sufficiently large,

$$\mathbb{P}(C^c) \leq \sum_{a, b \in [-k, k]^d, |a - b| \geq \sqrt{k}} \mathbb{P}(a, b \text{ is not black}) + \sum_{a, b \in [-k, k]^d, |a - b| < \sqrt{k}} \mathbb{P}(\max_{\Gamma \in \Omega(a,b)} \sharp \Gamma \geq k/2) \leq 3(2k + 1)^d c_1 e^{-c_2 \sqrt{k}} \leq c_1 e^{-c_2 \sqrt{k}/2}. \tag{3.7}$$

\qed

3.2. Proof of Lemma 6

Lemma 15. Suppose that $F^- < \infty$ and $F$ is useful. Let $F^- < \alpha_2 < F^+$. For any $M \in \mathbb{N}$ there exists $c, c_1, c_2 > 0$ such that for any $x \in \mathbb{Z}^d$,

$$\mathbb{P} \left( \min_{\Gamma \in \Omega(0,x)} \sharp \{ \gamma \in T_M, \gamma \cap \Gamma, \forall e \in \gamma, \tau_e \geq \alpha_2 \} \geq c|x|_1 \right) \leq c_1 \exp(-c_2|x|_1).$$

Proof. Given a path $\Gamma$, we define the new passage time as

$$t^+(\Gamma) = \sum_{e \in \Gamma} \tau_e + \beta \sharp \{ \Gamma \in T_M, \gamma \cap \Gamma, \forall e \in \gamma, \tau_e \geq \alpha_2 \},$$

where $\beta$ is a positive constant chosen to be later. Let us denote the corresponding first passage time from 0 to $x$ by $t^+(0,x)$.

Lemma 16. There exists $c > 0$ such that for any $x \in \mathbb{Z}^d$

$$\mathbb{E}[t^+(0,x)] - \mathbb{E}[t(0,x)] \geq c|x|_1.$$
First we prove Lemma 15. By Theorem 3.13 in [1], we have that there exist \(c_1, c_2\) such that
\[
\Pr(t(0, x) - \mathbb{E}[t(0, x)]) \geq c_3|x|/4 \leq c_1 \exp(-c_2|x|).
\]
The same argument of [5] leads to that
\[
\Pr(t^+(0, x) - \mathbb{E}[t^+(0, x)]) \leq -c_3|x|/4 \leq c_1 \exp(-c_2|x|).
\]
Therefore \(\Pr(t^+(0, x) - t(0, x) \leq c_3|x|/2 \leq c_1 \exp(-c_2|x|).
\]
Note that
\[
\beta \min_{\Gamma \in O(0, x)} \#\{\gamma \in T_M \mid \gamma \subseteq \Gamma, \forall e \in \gamma, \tau_e \geq \alpha_2\} \geq t^+(0, x) - t(0, x).
\]
Thus, we complete the proof. \(\square\)

The proof of Lemma 6 is the same as before. The rest will be devoted to Lemma 16. Since
\[
\beta \min_{\Gamma \in O(0, x)} \#\{\gamma \in T_M \mid \gamma \subseteq \Gamma, \forall e \in \gamma, \tau_e \geq \alpha_2\} \leq t^+(0, x) - t(0, x),
\]
it suffices to show that
\[
(3.8) \quad \mathbb{E} \left[ \min_{\Gamma \in O(0, x)} \#\{\gamma \in T_M \mid \gamma \subseteq \Gamma, \forall e \in \gamma, \tau_e \geq \alpha_2\} \right] \geq c|x|.
\]

\textbf{Proof of (3.8).} The proof is very similar to that of Lemma 5 in [9]. We only touch with the difference of them. Let \(n \in \mathbb{N}\). We consider the following boxes:

- \(S(l, n) = \{v \in \mathbb{Z}^d : nl \leq v_i < n(l + 1)\}\) for any \(i\).
- \(T(l, n) = \{v \in \mathbb{Z}^d : nl - n \leq v_i \leq n(l + 2)\}\) for any \(i\).
- \(B^1(l, n) = T(l, n) \cap T(l + 2 \text{sgn}(j) e_{ij}: n)\).

\textbf{Lemma 17.} If \(F\) is useful, then there exists \(\delta > 0\) and \(D > 0\) such that for any \(v, w \in \mathbb{Z}^d\),
\[
\Pr(t(v, w) < \delta|v - w|_1) \leq e^{-D|v - w|_1}.
\]

For the proof of this lemma, see Lemma 5.5 in [3].

For simplicity, we set \(B = B^1(l, n)\). We take sufficiently large \(R > 0\) to be chosen later.

\textbf{Definition 16.} We define following conditions;

1. for any \(v, w \in B^1(l, n)\) with \(|v - w|_1 \geq n^{1/3}\),
   \[
t(v, w) \geq (F^- + \delta)|v - w|_1,
   \]
   where \(\delta > 0\) is in Lemma 14. (Note that \(t^+(v, w) \geq t(v, w)\).)

2. for any \(e \cap B \neq \emptyset, \tau_e \leq F^+ - R^{-1}\).

An \(n\)-box \(B\) is said to be black if \(\begin{cases} (1) \text{ and (2) hold} & \text{if } \Pr(\tau_e = F^+) = 0 \\ (1) \text{ holds} & \text{if } \Pr(\tau_e = F^+) > 0 \end{cases}\)

Hereafter “crossing an \(n\)-box” means crossing in the short direction. See Figure 2.

![Figure 2](image-url)

**Figure 2.**
Left: Boxes: \(S, T, B\).
Right: \(\bigcup(0, x)\) crosses an \(n\)-box in the short direction.
Definition 17.
An n-box $B$ is said to be white if there exists $\Gamma \in \mathcal{G}(0, x)$ such that $\Gamma$ cross $B$.

An n-box $B$ is said to be gray if $B$ is black and white.

As in (2.4) of [9], we obtain that there exists $\epsilon > 0$ such that for any $x \in \mathbb{Z}^d$,

$$E[\mathbb{I}\{\text{distinct gray n-box } B\}] \geq \epsilon|x|/2$$

Definition 18. Define

$$F_R^+ = \begin{cases} 
F^+ - R^{-2} & \text{if } F^+ < \infty \text{ and } \mathbb{P}(\tau_c = F^+) = 0, \\
F^+ & \text{if } F^+ < \infty \text{ and } F(\{F^+\}) > 0,
\end{cases}$$

and

$$F_R^- = \begin{cases} 
F^- + R^{-2} & \text{if } \mathbb{P}(\tau_c = F^-) = 0, \\
F^- & \text{if } \mathbb{P}(\tau_c = F^-) > 0.
\end{cases}$$

Note that if $R$ is sufficiently large,

$$F_R^- < F^- + \delta/2 < F_R^+ \text{ and } F_R^- \leq \alpha_2 \leq F_R^+$$.  

Denote by $\partial^+ B$ the outer boundary of an n-box $B$. Let $n_1 = [d \sqrt{n}] + d$. If we take $n$ sufficiently large, for any $a, b \in \partial^+ B$ with

$$|a - b|_1 \geq \delta n/(2F^+)$$

there exists a self-avoiding path $\gamma_{a, b} = (x_0, \ldots, x_l)$ from $a$ to $b$ satisfying

$$\{x_i\}_{i=1}^{l+1} \subseteq B$$

such that the following hold:

1. $d_{\infty}(x_{n_1}, B^c), d_{\infty}(x_{l-n_1}, B^c) \geq \sqrt{n}$,
2. $|x_{n_1} - x_{l-n_1}|_1 = l - 2n_1$,
3. $d_{\infty}(x_i, B^c) \geq n_1$ for any $i \in \{n_1, \ldots, l, n_1\}$,
4. $(x_{i-M}, \ldots, x_{i+M})$ is a straight line for any $i \in I_{a,b}$,

where $I_{a,b} = \{n_1, \ldots, l - n_1\} \cap n_1\mathbb{Z}$. The reason why we use $\sqrt{n}$ is just $\sqrt{n} \ll n$ and not important. We take such a path to each $a, b \in \partial^+ B$ with $|a - b|_1 \geq \delta n/(2F^+)$. For $a, b \in \partial^+ B$ with $|a - b|_1 \geq \delta n/(2F^+)$, we take arbitrary self-avoiding path from $a$ to $b$.

Let $a, b \in \partial^+ B$ with $|a - b|_1 \geq \delta n/(2F^+)$ and $\gamma_{a, b} = (x_i)_{i=1}^{l}$. Given a path $\gamma = \gamma_{a,b} = (x_0, \cdots, x_l)$ and n-Box $B$, $\tau$ is said to be satisfied ($\gamma, B$)-condition if (1) $\tau(x_{i-1}, x_i) \in (\alpha_2, F^+_R]$ if there exists $j \in I_{a,b}$ such that $|i - j| \leq M$, (2) $\tau(x_{i-1}, x_i) \leq F_R^-$ otherwise, (3) $\tau_c \geq F^+_R$ if $e \notin \gamma$ and $e \cap B \neq \emptyset$. Denote the independent copy of $\tau$ by $\tau^c$ and set $\tau^c_e = \tau_c^e$ if $e \cap B \neq \emptyset$, $\tau^c_e = \tau_c$ otherwise. Let $(\tilde{a}, \tilde{b})$ be random variable on $\partial^+ B \times \partial^+ B$ with uniform distribution and its probability measure $P$. Given a path $\Gamma = (x_0, \cdots, x_l)$ and an n-box $B$, we set

$$\text{st}(\Gamma, B) = x_{\min\{i\, |\, x_i \in \partial^+ B\}}$$,

$$\text{fin}(\Gamma, B) = x_{\max\{i\, |\, x_i \in \partial^+ B\}}$$.

Note that if $\Gamma$ cross $B$ and $B$ is black, then since $t^+(\text{st}(\Gamma, B), \text{fin}(\Gamma, B)) \geq (F^- + \delta)n$,

$$|\text{st}(\Gamma, B) - \text{fin}(\Gamma, B)|_1 \geq \frac{(F^- + \delta)n}{2F^+} + 1.$$

Definition 19. An n-box $B$ is called $\mathcal{C}$-good if for any $\Gamma \in \mathcal{G}^+(0, x)$, there exists $\gamma \in \mathcal{T}_M$ such that $\gamma \subset B$, $\gamma \subset \Gamma$ and for any $e \in \gamma$, $\tau_e \geq \alpha_2$. 

ERGODICITY OF THE NUMBER OF INFINITE GEODESICS ORIGINATING FROM ZERO 15
Lemma 18. We take $\beta = R^{-2}$. If $R \geq n^{2d}$ and $n$ is sufficiently large, then there exists $c > 0$ such that for any $N \in \mathbb{N}$, unless $0 \in B$ or $Ne_1 \in B$,

\[
P(B \text{ is Good for } \tau) = P \otimes P(B \text{ is Good for } \tau^B)
\]

\[
\geq P \otimes P \left( (\tilde{a}, \tilde{b}) = (st(\Gamma, B), \text{fin}(\Gamma, B))) \text{ and } \tau^* \text{ satisfies}\ (\gamma_{\tilde{a}, \tilde{b}}, B)\text{-condition}\right)
\]

(3.12)

\[
= \frac{1}{\partial+B^2} \sum_{(a,b)} P \left( (a,b) = (st(\Gamma, B), \text{fin}(\Gamma, B))), \tau^* \text{ satisfies}\ (\gamma_{a,b}, B)\text{-cond.}\left)\right.
\]

\[
\geq cP(B \text{ is gray}).
\]

Proof. The proof is the same as in Lemma 5 of [9] and we skip the details. \(\Box\)

From (3.9) and (3.12), we have that there exists $c > 0$ such that

\[
E \left[ \min_{\Gamma \in \mathcal{O}^+(0,x)} \sharp \{ \gamma \in \mathcal{T}_M \mid \gamma \sqsupseteq \Gamma, \forall e \in \gamma, \tau_e \geq \alpha_2 \} \right] \geq \frac{1}{2d} \sum_{B^i(l; n); n-box} P(B^i(l; n) \text{ is Good})
\]

\[
\geq \frac{c}{2d} \sum_{B^i(l; n); n-box} P(B^i(l; n) \text{ is gray})
\]

\[
\geq \frac{c}{2d} \mathbb{E} \left[ \sharp \{ \text{distinct gray n-box } B \} \right]
\]

\[
\geq \frac{c}{4d} \left| \partial \right|_1,
\]

(3.13)

where $2d$ appears because of the overlap of $n$-boxes. Thus the proof is completed.

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