Exceptional band touching for strongly correlated systems in equilibrium

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Quasi-particles described by Green’s functions of equilibrium systems exhibit non-Hermitian topological phenomena because of their finite lifetime. This non-Hermitian perspective on equilibrium systems provides new insights into correlated systems and attracts much interest because of its potential to solve open questions in correlated compounds. We provide a concise review of the non-Hermitian topological band structures for quantum many-body systems in equilibrium, as well as their classification.

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1. Introduction

After the discovery of topological insulators/superconductors, the topological perspective of condensed matter is of growing importance [1–7]. While the notion of topology was originally utilized to understand the band structure of a gapped quadratic Hamiltonian (i.e. free-fermion systems), it has been extended to gapless systems; it has been shown that topologically protected band touching exists for Weyl semi-metals [8–11] or nodal line superconductors [12]. The notion of topological phases has been further extended to correlated systems where correlations and topology induce a variety of exotic phenomena [13–23], such as topologically ordered systems [24–36], topological Mott insulators [37–41], and the reduction of topological classifications [42–56].

Intriguingly, recent studies have shown that correlations even induce non-Hermitian topological phenomena [57–64], which have been extensively analyzed in various contexts [65–103] (e.g. photonic systems [104–114], open quantum systems [115–128], etc.). In particular, Ref. [57] has pointed out that the finite lifetime of quasi-particles induces an exceptional point (EP) in the Brillouin zone (BZ) which is a representative example of the non-Hermitian topological band structure. Correspondingly, topologically protected band touching occurs for both the real and imaginary parts, which we call exceptional band touching in this paper. The above EPs in many-body systems in equilibrium are connected by Fermi arcs, meaning that correlation induces gapless excitations even for band insulators. The emergence of EPs accompanied by Fermi arcs is numerically demonstrated by applying the dynamical mean-field theory (DMFT) to heavy fermions [59]. The above non-Hermitian perspective of Green’s functions for equilibrium systems has been further developed with symmetry of many-body Hamiltonians [60,94–97]; the interplay between symmetry and non-Hermiticity results in symmetry-protected exceptional rings (SPERs) in two dimensions [60] and symmetry-protected exceptional surfaces (SPESs) in three dimensions [60,62]. This recently
developed non-Hermitian perspective in equilibrium systems has attracted much interest because it provides new insights into quasi-particle spectrums which potentially solve open questions in condensed matter physics [129–132].

The aim of this article is to provide a concise review of these advances in the non-Hermitian perspective in correlated systems in equilibrium. As a $2 \times 2$ Hamiltonian describes the essential properties, we start with this simplest case and review numerical results demonstrating the emergence of exceptional band touching.

The rest of this paper is organized as follows. In Sect. 2 we demonstrate the emergence of EPs for a heavy-fermion system by applying the DMFT to a heavy-fermion system. In Sect. 3 we show that SPERs and SPESs can emerge for correlated systems with chiral symmetry. In Sect. 4 we address the ten-fold classification of exceptional band touching for the single-particle spectrum by taking into account $PT$ ($CP$) and chiral symmetry, where $PT$ ($CP$) symmetry denotes symmetry under the product of time reversal and inversion (charge conjugation and inversion), respectively. A short summary and remaining open questions appear at the end of this paper.

2. Exceptional points for strongly correlated systems

In this section we show that EPs emerge due to finite lifetimes of quasi-particles for strongly correlated systems [59]. Specifically, the origin of the above non-Hermitian topological phenomena is the imaginary part of the self-energy [see Eqs. (11) and (12)] which describes the lifetime of quasi-particles. The emergence of EPs results in a significant difference in the single-particle spectrum.

In the following, after a brief explanation of EPs (Sect. 2.1) and the single-particle Green’s function (Sect. 2.2), we demonstrate the emergence of EPs for heavy-fermion systems and see that EPs significantly change the single-particle spectrum.

2.1. Topological properties of EPs

2.1.1. Case of a $2 \times 2$ Hamiltonian

Let us first analyze a non-Hermitian $2 \times 2$ Hamiltonian, which shows the essential properties of EPs.

It is well known that a generic $2 \times 2$ matrix can be expanded by the Pauli matrices $\tau$ and the identity matrix $\tau_0$,

$$H(k) = \sum_{\mu} [b_\mu(k) + id_\mu(k)] \tau_\mu, \quad (1)$$

where $b_\mu$ and $d_\mu$ ($\mu = 0, 1, 2, 3$) are continuous functions taking real values.

One can numerically and analytically confirm that the above non-Hermitian matrix may show EPs. In Fig. 1, energy eigenvalues taking complex numbers are plotted for a specific choice of the $b_\mu$ and $d_\mu$. At the EPs, the Hamiltonian becomes non-diagonalizable. Correspondingly, as one can see in Fig. 1, band touching occurs for both the real and imaginary parts of the energy eigenvalues.

In order to see the details we diagonalize the Hamiltonian in Eq. (1), which yields

$$E_{\pm} = b_0 + id_0 \pm \sqrt{b^2 - d^2 + 2ib \cdot d}, \quad (2)$$

with $b \cdot d := \sum_{j=1,2,3} b_j d_j$, $b^2 = b \cdot b$, and $d^2 = d \cdot d$. The above equation indicates that band touching occurs for both the real and imaginary parts when the following conditions are satisfied:

$$b^2 - d^2 = 0, \quad (3a)$$

$$b \cdot d = 0, \quad (3b)$$

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In other words, these are necessary conditions for the emergence of EPs. One can see that the above conditions are also sufficient conditions; supposing that Eq. (3) is satisfied, we can see that the Hamiltonian can be rewritten as

$$H(k) = (b_0 + id_0)\tau_0 + 2d \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

with a proper choice of the basis. In this basis, one can see that the Hamiltonian is generically non-diagonalizable for $d \neq 0$.

In the above, we have seen the following facts. At the EP, the $2 \times 2$ Hamiltonian in Eq. (1) becomes non-diagonalizable, resulting in exceptional band touching. For the $2 \times 2$ Hamiltonian, the EP emerges if and only if Eqs. (3a) and (3b) are satisfied. We note that the band touching is protected by non-trivial topology whose topological invariant is discussed in the next subsection.

### 2.1.2. Topological invariant characterizing EPs

As shown in Fig. 1, band touching occurs at the EPs. Such band touching for two-dimensional systems can be topologically characterized by the vorticity, akin to the winding number,

$$\nu = \frac{1}{4\pi i} \oint d^2 k \cdot \nabla_k \log \det[H(k) - E_0]$$

Here, we have considered that the band touching occurs at energy $E_0$. $H(k)$ ($\dim H \geq 2$) denotes a generic non-Hermitian matrix, and $\nabla_k := (\partial_{k_x}, \partial_{k_y})$. The path of the integral is chosen so that it encloses the EP. For $\dim H = 2$, the vorticity can be written as

$$\nu = \frac{1}{2\pi} \oint d^2 k \cdot \nabla_k \arg[E_+(k) - E_-(k)]$$

where $E_\pm$ are the energy eigenvalues [see Eq. (2)].

---

1 This can be seen as follows. First, by diagonalizing the Hamiltonian, we rewrite the vorticity as

$$\nu = \frac{1}{4\pi i} \oint dk \cdot \nabla_k \sum_n \log[E_n(k) - E_0],$$

where $E_n (n = 1, \ldots, \dim H)$ denotes the energy eigenvalues of $H$. Substituting Eq. (1) into the above equation, we obtain

$$\nu = \frac{1}{4\pi i} \oint dk \cdot \nabla_k \sum_{n=\pm} \log[E_n(k) - E_0],$$
In the following, we see how the vorticity defined in Eq. (5) characterizes the EPs. Consider a generic non-Hermitian matrix $H(k)$ with $\text{dim } H \geq 2$ which shows band touching at energy $E_0$. The band touching point can be formulated as

$$\det[H(k_0) - E_0 \mathbb{1}] = 0, \quad (7)$$

where $k_0$ denotes the EP in momentum space. Mapping the non-Hermitian Hamiltonian to the Hermitian matrix $\tilde{H}$, we can rewrite the above condition as

$$\det[\tilde{H}(k_0)] = 0, \quad (8a)$$

with

$$\tilde{H} = \begin{pmatrix} 0 & H(k) - E_0 \mathbb{1} \\ H^\dagger(k) - E_0^* \mathbb{1} & 0 \end{pmatrix}. \quad (8b)$$

Here, we have extended the Hilbert space on which the Pauli matrices $\rho$ act. This can be easily confirmed by noticing that Eq. (8a) can be written as $|\det[H(k_0) - E_0]|^2 = 0$. The above fact means that the exceptional band touching can be described by the zero modes of the Hermitian matrix $\tilde{H}$, which is chiral symmetric: $[\tilde{H}, \tilde{\Sigma}] = 0$ with $\tilde{\Sigma} := \mathbb{1} \otimes \rho_3$. Therefore, remembering that the zero modes of the chiral symmetric Hermitian Hamiltonian are characterized by the winding number,

$$\nu_W = \frac{1}{4\pi i} \oint d^2 k \cdot \text{tr} [\tilde{\Sigma} \tilde{H}^{-1} \nabla_k \tilde{H}], \quad (9)$$

we can see that the EPs can be characterized by the vorticity in Eq. (5); substituting $\tilde{\Sigma} = \mathbb{1} \otimes \rho_3$ into Eq. (9) yields Eq. (5).\(^2\) We note that the vorticity is half-quantized due to the extra prefactor 1/2, which is just a convention.

where $E_\pm$ are the energy eigenvalues [see Eq. (2)]. This can be rewritten as

$$\nu = \frac{1}{4\pi i} \oint d^2 k \cdot \nabla_k [\log \Delta(k) + \log(-\Delta(k))]$$

$$= \frac{1}{4\pi i} \oint d^2 k \cdot \nabla_k [2 \log \Delta(k)]$$

$$= \frac{1}{2\pi} \oint d^2 k \cdot \nabla_k \arg(E_+ - E_-),$$

with $\Delta := (E_+ - E_-)/2 = \sqrt{b^2 - d^2 + 2ib \cdot d}$. Here, we have omitted the term proportional to $b_0 + id_0$ by assuming that it cancels with $E_0$. The last line of the above equation corresponds to the right-hand side of Eq. (6).

\(^2\) Equation (5) is obtained as follows. With $\Sigma = \mathbb{1} \otimes \rho_3$, Eq. (9) is rewritten as

$$\nu_W = \frac{1}{4\pi i} \oint d^2 k \cdot \text{tr} \left[ \begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ H^\dagger(k) & H(k) \end{pmatrix}_{\rho}^{-1} \nabla_k \begin{pmatrix} 0 & H(k) \\ H^\dagger(k) & 0 \end{pmatrix}_{\rho} \right]$$

$$= \frac{1}{4\pi i} \oint d^2 k \cdot \text{tr} [\nabla_k \log H^\dagger(k) - \nabla_k \log H(k)]$$

$$= -\frac{1}{2\pi} \oint d^2 k \cdot \nabla_k \log \det H(k).$$

The last line corresponds to Eq. (5) up to the prefactor: $\nu = -\nu_W/2$. 

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In this section we have considered two-dimensional systems. We note, however, that the vorticity is well defined along a one-dimensional path in the three-dimensional BZ. In this case, the vorticity characterizes exceptional loops in the BZ (see also Table 1). For a $2 \times 2$ Hamiltonian, there is a complementary understanding. The EPs appear when both Eqs. (3a) and (3b) are satisfied, meaning that one degree of freedom is left in the three dimensions. This remaining degree of freedom forms a loop which is nothing but the exceptional loop in three dimensions.

2.2. **EPs appearing in the single-particle spectrum**

In the above, we have seen that a non-Hermitian matrix may show EPs which are characterized by the vorticity in Eq. (5). In this section we see that a non-Hermitian matrix governs the single-particle excitation spectrum of correlated systems in equilibrium (i.e. the energy is conserved).

First, we define the retarded single-particle Green’s function $G_{\alpha\beta}^R(t, k)$ whose imaginary part corresponds to the single-particle spectrum:

$$G_{\alpha\beta}^R(t, k) = -i \langle \hat{c}_{k\alpha}(t) \hat{c}_{k\beta}^\dagger(0) + \hat{c}_{k\beta}^\dagger(0) \hat{c}_{k\alpha}(t) \rangle \theta(t),$$

(10)

where $\hat{c}_{k\alpha}$ creates a fermion with momentum $k$ in the state $\alpha$ (spin, orbital, etc.). $\langle \cdot \rangle$ denotes the expectation value for temperature $\beta^{-1}$ [{\cdot} := \text{tr}(\cdot e^{-\beta \hat{H}})/Z$ with the partition function $Z := \text{tr}(e^{-\beta \hat{H}})]$. $\hat{c}_{k\alpha}^\dagger(t) := e^{i H t} \hat{c}_{k\alpha}^\dagger e^{-i H t}$, with the second quantized Hamiltonian $H$ describing the correlated system in equilibrium (i.e. $H$ is a Hermitian operator). $\theta(t)$ takes 0, 1/2, and 1 for $t < 0$, $t = 0$, and $t > 0$, respectively. Applying the Fourier transformation, we obtain Dyson’s equation [133]:

$$G^{-1}(\omega + i\delta, k) = g^{-1}(\omega + i\delta, k) - \Sigma(\omega + i\delta, k),$$

(11)

which defines the self-energy $\Sigma(\omega + i\delta, k)$. Here, $g(\omega + i\delta, k)$ denotes the retarded Green’s function for free fermions: $g^{-1}(\omega + i\delta, k) := (\omega + i\delta) \mathbb{1} - \hat{h}(k)$ with the identity matrix $\mathbb{1}$ and the Bloch Hamiltonian $\hat{h}(k)$ for the non-interacting case; $\delta$ is an infinitesimal constant ($\delta > 0$). With the Green’s function, the single-particle spectral function is defined as $A(\omega, k) = -\text{Im} \sum_{\alpha} G_{\alpha\alpha}(\omega + i\delta, k)/\pi$, which can be rewritten as

$$A(\omega, k) = -\frac{1}{\pi} \text{Im} \text{tr}[(\omega + i\delta) \mathbb{1} - H_{\text{eff}}(\omega, k)]^{-1},$$

(12a)

$$H_{\text{eff}}(\omega, k) = \hat{h}(k) + \Sigma(\omega + i\delta, k).$$

(12b)

We note that the self-energy $\Sigma(\omega + i\delta, k)$ is a non-Hermitian matrix describing the lifetimes of quasi-particles. Therefore, Eq. (12a) indicates that the single-particle excitations of energy $\omega$ are governed by the non-Hermitian matrix $H_{\text{eff}}(\omega, k)$.

In addition to EPs, the non-Hermiticity of the effective Hamiltonian yields low-energy excitations. The energy gap can be pure imaginary because of the non-Hermiticity of $H_{\text{eff}}$. In this case, even when the Bloch Hamiltonian is gapped, the system may show Fermi arcs connecting EPs.

We finish this section by commenting on an additional condition for EPs in the single-particle spectrum. The effective Hamiltonian appears in the denominator of the spectral weight in Eq. (12a), meaning that the EPs are seriously smeared when the denominator is large. Therefore, in order for EPs to emerge as a peak in the single-particle spectral function, the frequency $\omega$ should satisfy an additional condition—see, for instance, Eq. (18a).
2.3. EPs for two-dimensional heavy-fermion systems

In this section we demonstrate that EPs emerge in the single-particle spectrum of a heavy-fermion system, by employing the DMFT. In particular, we analyze the Kondo lattice in two dimensions. The Hamiltonian reads

\[ \hat{H} = \sum_{i} t_{i\alpha,i\beta} \hat{c}_{\alpha}^\dagger_i \hat{c}_{\beta}^\prime_i + J \sum_{i} \hat{S}_{ib} \cdot \hat{S}_i, \]

(13)

where \( \hat{c}_{\alpha,i}^\dagger \) creates an electron with spin \( s = \uparrow, \downarrow \) in orbital \( \alpha = a, b \) of site \( i \). \( \hat{S}_{ib} := \frac{1}{2} \hat{c}_{ibs}^\dagger \sigma_{ss'} \hat{c}_{ibs'} \) with the Pauli matrices \( \sigma \) acting on the spin space. \( \hat{S} \) is the spin-1/2 operator for the localized spins. Here, the Kondo coupling of electrons in orbital \( \alpha \) is neglected for simplicity. The hopping \( t_{i\alpha,i\beta} \) is defined so that the Bloch Hamiltonian is written as

\[ h(k) = -2t' \sin k_{y} \tau_{1} + [-\epsilon_{0} - 2t(\cos k_{x} + \cos k_{y})] \tau_{3}, \]

(14)

where \( \epsilon_{0}, t, \) and \( t' \) take real values. The Pauli matrices \( \tau \) act on the orbital space. In the non-interacting case, this model shows two Dirac cones for \( t = 1 \) and \( 0 < \epsilon_{0} < 4 \).

In order to analyze the above correlated electron system we employ the DMFT [134–137], which treats local correlation exactly. In the DMFT framework, the lattice model is mapped to an effective impurity model where the self-energy of spin \( s \), \( \{ \Sigma_{s}(\omega + i\delta) := \text{diag} (0, \Sigma_{bs}(\omega + i\delta)) \} \) is computed self-consistently. Here, \( \Sigma_{bs}(\omega + i\delta) \) denotes the self-energy for orbital \( b \) and spin \( s \). In order to compute the self-energy for the effective impurity model, we employ the numerical renormalization group method (NRG) [138–140]. This method directly provides the single-particle spectral function, while other methods based on Monte Carlo calculations [141–143] require the analytic continuation.

Once the self-energy is obtained, the single-particle spectrum is obtained as

\[ A(\omega, k) = -\frac{1}{\pi} \text{Im} \, \text{tr} \left[ \Sigma_{s}(\omega + i\delta) \right]^{-1}, \]

(16a)

\[ H_{\text{eff}}(\omega, k) = h(k) + \Sigma_{s}(\omega + i\delta). \]

(16b)

Here, we have omitted the subscript \( s \) \( \{ \Sigma(\omega + i\delta) := \text{diag} (0, \Sigma_{bs}(\omega + i\delta)) \} \) by assuming that the system is in the paramagnetic phase. We note that the effective Hamiltonian is a \( 2 \times 2 \) matrix.

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3 The action of the effective impurity model is written as

\[ Z_{\text{imp}} = \int \prod_{s} \mathcal{D} \hat{c}_{hs} \mathcal{D} \hat{c}_{hi} \mathcal{W}_{S} \exp(-S_{\text{imp}}), \]

\[ -S_{\text{imp}} = \int d\tau d\tau' \sum_{s} \hat{c}_{hs} G_{s}^{-1}(\tau - \tau') \hat{c}_{hs} - H_{\text{int}} \delta(\tau - \tau'), \]

where \( H_{\text{int}} \) denotes the local interaction term \( H_{\text{int}} := Js_{h0} \cdot S. \delta(\tau - \tau') \) is the delta function. \( \mathcal{W}_{S} \) denotes taking trace for the localized spin. \( G_{s}(\tau - \tau') \) denotes the Green’s function of the effective bath. \( \hat{c}_{hs} \) (\( \hat{c}_{hi} \)) is a Grassmannian variable which corresponds to the creation operator \( \hat{c}_{ih} \) (annihilation operator \( \hat{c}_{hi} \)) at site \( i = 0 \). Solving the above model with an impurity solver, we obtain the self-energy \( \Sigma_{hs} \), which allows us to compute the Green’s function as

\[ G_{s}^{-1}(\omega_{n}) = \left[ \frac{1}{N} \sum_{k} \frac{1}{\omega_{n} - h(k) - \Sigma_{s}(\omega_{n})} \right] + \Sigma_{s}(\omega_{n}), \]

with the Matsubara frequency \( \omega_{n} = (2n + 1)\pi T \). Here, \( N \) denotes the number of unit cells. Computing the effective Green’s function \( G_{s}^{-1} \) yields the self-energy \( \Sigma_{s}(\omega_{n}) \).
Expanding it with the Pauli matrices as Eq. (1), we obtain the following coefficients:

\[
(b_0, b_1, b_2, b_3) = \left( \frac{\text{Re}\Sigma_b(\omega + i\delta)}{2}, 2t' \sin k_x, 0, -\epsilon_0 - 2t(\cos k_x + \cos k_y) - \frac{\text{Re}\Sigma_b(\omega + i\delta)}{2} \right),
\]  

(17a)

\[
(d_0, d_1, d_2, d_3) = \left( \frac{\text{Im}\Sigma_b(\omega + i\delta)}{2}, 0, 0, -\frac{\text{Im}\Sigma_b(\omega + i\delta)}{2} \right).
\]  

(17b)

Therefore, the conditions for EPs appearing as the peak of the single-particle spectral function are written as

\[
2\omega_0 - \text{Re}\Sigma_b(\omega_0 + i\delta) = 0,
\]  

(18a)

\[-\epsilon_0 - 2t(\cos k_{0x} + \cos k_{0y}) - \frac{\text{Re}\Sigma_b(\omega_0 + i\delta)}{2} = 0,
\]  

(18b)

\[-[\text{Im}\Sigma_b(\omega_0 + i\delta)]^2 + 16t'^2 \sin^2 k_{0y} = 0.
\]  

(18c)

Here, the second and third equations are obtained from Eq. (3), specifying the position of the EP \(k_0\) in the BZ. The first equation specifies the energy \(\omega_0\) where the EPs emerge as peaks of the spectral function. We note that in the DMFT framework, the momentum dependence of the self-energy is neglected. However, the EPs should emerge even in calculations beyond the DMFT framework because they are topologically protected.

Let us now analyze the Kondo lattice model of Eq. (13). In the rest of this section, we set the parameters to \((t, t', \epsilon_0) = (1, 0.667, 0.667)\). The phase diagram obtained is shown in Fig. 2. When the Kondo coupling is small, an anti-ferromagnetic phase emerges because the Ruderman–Kittel–Kasuya–Yosida interaction [144–146] becomes dominant. Increasing the interaction \(J\), itinerant electrons and localized spins form singlets due to the Kondo effect. As a result, the anti-ferromagnetic phase is suppressed in the region of strong \(J\).
Fig. 3. Single-particle spectral function $A(\omega_0, k)$ with $\omega_0 = 0.06t$ for $J = 1.8t$ and $T = 0.048t$. The data are plotted around the boundary of the BZ ($k_y = \pi$). Panel (a) is plotted by setting the imaginary part of the self-energy to zero $\text{Im} \Sigma_b(\omega_0) = 0$. We can see that the peak for $k_y = \pi$, indicating the emergence of the Dirac cone with chiral symmetry. Panel (b) shows that the Dirac cone splits into two EPs (green dots) because of the imaginary part of the self-energy. These EPs are connected with Fermi arcs. These figures are adapted with permission from Ref. [59]. Copyright 2018 American Physical Society.

We numerically observe the EPs in the paramagnetic phase. The Kondo effect plays an important role in the emergence of EPs. The self-energy is shown in Figs. 2(b) and 2(c) for $T = 0.048t$, which corresponds to the horizontal line in Fig. 2(a). For small $J$ ($J = t$), the real and imaginary parts of the self-energy take small values because the electrons are almost decoupled from the localized spins. Increasing the coupling $J$ enhances the Kondo effect, which results in a dip structure of $\text{Im} \Sigma_b(\omega + i\delta)$ in the low-energy region (i.e. around $\omega \sim 0$). This dip structure of the self-energy induces the EPs. The single-particle spectral function for $J = 1.8t$ is plotted in Fig. 3. This shows the data obtained by assuming that the imaginary part of the self-energy is zero [see Fig. 3(a)] in order to show that the imaginary part of the self-energy is essential for the EPs. In this figure, we can see a single peak due to the existence of a Dirac cone. Fig. 3(b) shows the spectral function obtained by the DMFT. We can see that the dip structure of the imaginary part splits the Dirac cone into two EPs, as represented by green dots. Furthermore, we can see that the EPs are connected by the Fermi arc where the bulk gap $\Delta_c = E_+ - E_-$ becomes pure imaginary. The emergence of the Fermi arc enhances the local density of states (LDOS) around $\omega \sim 0$ [see Fig. 4(a)]. In the above, we have seen that the imaginary part of the self-energy splits each of two Dirac cones into a pair of two EPs connected with the bulk Fermi arc. As we see below, these bulk Fermi arcs are robust because the EPs are topologically protected.

Here, we address the characterization of the above EPs. Because the vorticity is written as in Eq. (6) for the $2 \times 2$ Hamiltonian, we can compute its value by plotting the argument of $\Delta_c^2$ [see Fig. 4(b)]. In this figure, the branch cut of $\Delta_c$ is represented by white dashed lines which end at EPs. Therefore, taking the integral along the green line illustrated in Fig. 4(b), we can see that the vorticity takes $\nu = -1/2$. We note that the vorticity takes $\nu = 1/2$ for the EP around $k_x = -\pi/2$.

Changing the Kondo coupling results in pair annihilation of EPs. Here, we note that there are two scenarios: (i) a pair of EPs originating at a Dirac point are annihilated by themselves; (ii) two pairs of EPs exchange the pairs and are annihilated. The former scenario can be observed by decreasing the Kondo coupling $J$. In Fig. 5(a), we can see that two EPs approach and are annihilated. Correspondingly, the Fermi arc vanishes. The latter scenario can be observed by increasing the interaction $J$. When the Kondo effect is enhanced, the EPs approach the boundary of the BZ specified by $k_x = \pi$. On this boundary, the pair of EPs arising from two distinct Dirac cones annihilate each other [see
3. Symmetry-protected exceptional rings and surfaces in correlated systems

In the previous section we saw that electron correlations induce EPs in the absence of symmetry. In addition, it is well known that symmetry enriches the topological structures for Hermitian systems [147–150]. Therefore, it should be valuable to analyze the effects of symmetry on EPs, which is the main subject of this section.

Reference [60] has revealed that many-body chiral symmetry results in novel types of exceptional band touching: SPERs in two dimensions and SPESs in three dimensions. In the following, after elucidating the topological properties of SPERs and SPESs, we demonstrate their emergence in correlated systems.

Fig. 4. (a) The local density of states $\rho(\omega) = \sum_k A(\omega, k)/N$ for $J = 2$ and $T = 0.048t$. Here, $N$ denotes the number of unit cells. The red line indicates the data computed with the obtained self-energy. For comparison, we also plot the data obtained by setting $\text{Im} \Sigma(\omega) = 0$ (see blue line). (b) Color map of $\text{Arg}[\Delta^2(\omega_0, k)]$ with $\omega_0 = 0.06t$. On the white dashed lines the value $\text{Arg}[\Delta^2(k_x, k_y)]$ jumps from $-\pi$ to $\pi$, which corresponds to the branch cut of $\Delta_s$. These figures are adapted with permission from Ref. [59]. Copyright 2018 American Physical Society.

Fig. 5. The single-particle spectral function $A(\omega_0, k)$ for $T = 0.048t$. Data for $J = t$ and $J = 2t$ are plotted in panels (a) and (b), respectively. Panel (a) indicates that the Fermi arc shrinks, corresponding to the fusion of two EPs. Panel (b) indicates that the Fermi loop emerges because two EPs merge at the boundary of the BZ. These figures are adapted with permission from Ref. [59]. Copyright 2018 American Physical Society.

Fig. 5(b)]. The qualitative difference from the previous case is that a Fermi loop emerges after the pair annihilation of EPs, enhancing the LDOS in the low-energy region [see Fig. 6(a)]. The emergence of the Fermi loop is again due to the energy gap taking a pure imaginary value [see Fig. 6(b)].
Fig. 6. (a) The local density of states $\rho(\omega) = \sum_k A(\omega, k)/N$ for $J = 2t$ and $T = 0.048t$. Here, $N$ denotes the number of unit cells. The red line indicates the data computed with the obtained self-energy. For comparison, we plot the data obtained by setting $\text{Im}\Sigma_\omega = 0$ (see blue line). (b) Color map of $\text{Arg}[\Delta^\Gamma_\omega(k_x, k_y)]$. On the white dashed lines, the value $\text{Arg}[\Delta^\Gamma_\omega(k_x, k_y)]$ jumps from $-\pi$ to $\pi$. For this parameter set, the white dashed line forms a closed loop. These figures are adapted with permission from Ref. [59]. Copyright 2018 American Physical Society.

3.1. Symmetry protection of exceptional band touching

3.1.1. Case of a $2 \times 2$ Hamiltonian

First, we analyze the case of the $2 \times 2$ Hamiltonian [see Eq. (1)] which captures the essential properties. Here, let us suppose that the Hamiltonian for a two-dimensional system satisfies the relation

$$\tau_3 H^\dagger(k) \tau_3 = -H(k),$$

which indicates that the system is chiral symmetric [see Eq. (30a)]. The above condition imposes the following symmetry condition on the coefficients $b_\mu$ and $d_\mu$:

$$b_0 = b_3 = d_1 = d_2 = 0.$$  (20)

Now let us consider the effects of the symmetry constraint on the EPs. As we saw in Sect. 2.1.1, EPs emerge when the two conditions in Eqs. (3a) and (3b) are satisfied. We note, however, that one of the conditions, Eq. (3b), is always satisfied by the symmetry constraint, meaning that the number of conditions for EPs is reduced. This indicates that for the two-dimensional BZ, fixing one degree of freedom is sufficient to obtain the EPs. Therefore, the remaining degree of freedom forms a ring of EPs which is denoted as an SPER [60]. On an arbitrary point of the SPERs, the band touching occurs both for the real and imaginary parts.

We can apply the same argument to a three-dimensional system where SPEESs emerge [60]. In this case, the two degrees of freedom are left in the BZ.

3.1.2. Topological invariant characterizing SPERs and SPEESs with chiral symmetry

In the above we have seen that the symmetry constraint results in SPERs or SPEESs where exceptional band touching occurs. In this section, we show that the band touching is topologically characterized by the zeroth Chern number, a zero-dimensional topological invariant.

Let us suppose that the $2n \times 2n$ Hamiltonian satisfies the relation

$$U_1 H^\dagger(k) U_1^\dagger = -H(k),$$

(21)
where $U_\Gamma$ is a unitary matrix satisfying $U_\Gamma^2 = \mathbb{1}$. The above equation is a generic form of the symmetry constraint in Eq. (19). We now consider the following Hermitian Hamiltonian composed of $H(k)$:

$$
\tilde{H}(k) = \begin{pmatrix}
0 & H(k) - E_0 \\
H^*(k) - E_0^* & 0
\end{pmatrix}_{\rho},
$$

(22)

where we have assumed that the exceptional band touching occurs at energy $E_0 \in i\mathbb{R}$. In a similar way to the case of Sect. 2.1.2, we can define the topological invariant characterizing the SPERs and SPESs by addressing topological characterization of zero-energy excitations described by the Hermitian Hamiltonian $\tilde{H}$. The essential difference from the previous case (Sect. 2.1.2) is that the Hermitian Hamiltonian preserves the two distinct constraints of chiral symmetry:

$$
\tilde{\Sigma} \tilde{H}(k) \tilde{\Sigma}^{-1} = -\tilde{H}(k),
$$

(23a)

$$
\tilde{U}_\Gamma \tilde{H}(k) \tilde{U}_\Gamma^{-1} = -\tilde{H}(k),
$$

(23b)

with

$$
\tilde{\Sigma} = \mathbb{1} \otimes \rho_3,
$$

(23c)

$$
\tilde{U}_\Gamma = U_\Gamma \otimes \rho_1.
$$

(23d)

The additional chiral symmetry allows us to define the zeroth Chern number. Due to two distinct constraints of chiral symmetry, the Hamiltonian can be block diagonalized with a unitary operator $\tilde{U} = i\tilde{\Sigma} \tilde{U}_\Gamma (\tilde{U}^2 = \mathbb{1})$,

$$
\tilde{H} = \begin{pmatrix}
H_+ & 0 \\
0 & H_-
\end{pmatrix}.
$$

(24)

Here, $H_+$ ($H_-$) denotes the Hamiltonian acting on the subspace where the operator $\tilde{U}$ is reduced to $\mathbb{1}$ ($-\mathbb{1}$). We denote these subspaces by plus and minus sectors. We note that applying either $\tilde{\Sigma}$ or $\tilde{U}_\Gamma$ exchanges the plus and minus sectors because of the anti-commutation relation $\{\tilde{U}_\Gamma, \tilde{\Sigma}\} = \{\tilde{\Sigma}, \tilde{U}\} = 0$. Namely, letting $|+\rangle$ be a state of the plus sector ($\tilde{U}|+\rangle = |+\rangle$), we obtain $\tilde{U} \tilde{\Sigma} |+\rangle = -\tilde{\Sigma} |+\rangle$, which means that $\tilde{\Sigma} |+\rangle$ belongs to the minus sector. The above facts indicate that the block-diagonalized Hamiltonians $H_+$ and $H_-$ are related to each other and belong to symmetry class A. Therefore, the characterization of the zero-energy excitations of $\tilde{H}(k)$ can be done with the zeroth Chern number for the plus sector, which corresponds to the number of eigenstates with negative eigenvalues of $H_+$. This fact suggests $\mathbb{Z}$ classification of zero-dimensional Hermitian systems belonging to class A.

We note that the block-diagonalized Hamiltonian can be rewritten as $H_+ = iU_\Gamma H$, which can be seen as follows. Noticing that the unitary matrix $V$ block diagonalizes the unitary operator $\tilde{U} = U_\Gamma \otimes \rho_2$,

$$
\tilde{V}^* \tilde{U} \tilde{V} = \begin{pmatrix}
\mathbb{1} & 0 \\
0 & -\mathbb{1}
\end{pmatrix}_{\rho},
$$

(25a)

with

$$
\tilde{V} = \frac{1}{\sqrt{2}} \begin{pmatrix}
\mathbb{1} & -i \mathbb{1} \\
iU_\Gamma & -U_\Gamma
\end{pmatrix}_{\rho},
$$

(25b)
Table 1. Objects formed by EPs in the BZ for each case of spatial dimensions. In the presence of chiral symmetry, exceptional band touching forms objects which are one dimension higher than in the absence of symmetry.

| Dimension | 1 | 2 | 3 |
|-----------|---|---|---|
| No symmetry | — | point | loop |
| With chiral symmetry | point | ring | surface |

we can block diagonalize the Hermitian Hamiltonian \( \tilde{V}^\dagger \tilde{H} \tilde{V} \):

\[
\tilde{V}^\dagger \tilde{H} \tilde{V} = \begin{pmatrix}
iHU_\Gamma & 0 \\
0 & -iHU_\Gamma
\end{pmatrix}.
\]

(26)

Here, we have used the relation \( U_\Gamma^2 = \mathbb{I} \).

Therefore, the SPERs and the SPESs are characterized by the zeroth Chern number, which is the number of negative eigenvalues of the Hermitian Hamiltonian \( H_+(k) = iH(k)U_\Gamma \) at each point in the BZ.

The above result indicates that the dimension of an object composed of the exceptional band touching becomes one dimension higher by chiral symmetry [Eq. (21)] compared to a system without the symmetry (see Table 1). Table 1 also indicates that the EPs emerging in one-dimensional systems are either unstable or symmetry protected.

We finish this section with the complementary understanding for the \( 2 \times 2 \) Hamiltonian. Namely, exceptional band touching appears at points where both Eqs. (3a) and (3b) are satisfied. Thus, in the absence of symmetry, exceptional band touching forms \( (d-2) \)-dimensional objects in the \( d \)-dimensional BZ. On the other hand, in the presence of chiral symmetry, exceptional band touching forms \( (d-1) \)-dimensional objects because Eq. (3b) is always satisfied by symmetry.

3.2. SPERs for a correlated honeycomb lattice

The SPERs can emerge for strongly correlated systems in equilibrium. In order to demonstrate the emergence of SPERs, we apply the DMFT and NRG to a Hubbard model of a honeycomb lattice. The Hamiltonian reads

\[
\hat{H} = \sum_{\langle ij \rangle \alpha \beta} t_{i\alpha,j\beta} \hat{c}_{i\alpha s}^\dagger \hat{c}_{j\beta s'} + \sum_{i\alpha} U_\alpha \left( \hat{n}_{i\alpha \uparrow} - \frac{1}{2} \right) \left( \hat{n}_{i\alpha \downarrow} - \frac{1}{2} \right),
\]

(27)

where \( \hat{c}_{i\alpha s}^\dagger \) creates a fermion with spin \( s = \uparrow, \downarrow \) at site \( i \) and sublattice \( \alpha = A, B \); \( \hat{n}_{i\alpha s} = \hat{c}_{i\alpha s}^\dagger \hat{c}_{i\alpha s} \). The first term describes nearest-neighbor hopping with \( t_{i\alpha,j\beta} \in \mathbb{R} \). The second term describes on-site repulsion \( (U_\alpha \geq 0) \). Applying the Fourier transformation for \( U_\alpha = 0 \), we obtain the Bloch Hamiltonian which is written as

\[
h(k) = b_1(k) \tau_1 + b_2(k) \tau_2,
\]

(28a)

where \( b_1(k) \) and \( b_2(k) \) \([b_1(k), b_2(k) \in \mathbb{R}]\) are defined as

\[
b_1(k) - ib_2(k) = te^{ik \cdot a_1} + te^{ik \cdot a_2} + t'e^{ik \cdot a_3}.
\]

(28b)

Here, we have assumed hopping \( t \) \((t' := rt)\) between sites connected with gray (brown) lines, respectively. The vectors \( a_1, a_2, \) and \( a_3 \) are illustrated in Fig. 7. We consider that this model can be
Fig. 7. Sketch of the honeycomb Hubbard model. The $A$ ($B$) sublattice is illustrated with blue (red) spheres. The vectors $a_1$, $a_2$, and $a_3$ specify the neighboring sites; $a_1 := (\sqrt{3}, 1)/2$, $a_2 := (−\sqrt{3}, 1)/2$, and $a_3 := (0, −1)$. Nearest-neighbor hopping with $t$ ($rt$) is represented by gray (brown) bonds. This figure is adapted with permission from Ref. [60]. Copyright 2019 American Physical Society.

fabricated for cold atoms because the inhomogeneous Hubbard interaction is implemented with the optical Feshbach resonance [151,152].

The above model preserves the chiral symmetry for an arbitrary value of the interaction $U_\alpha$:

\[
\hat{\Gamma} \hat{H} \hat{\Gamma}^{-1} = \hat{H}, \quad (29a)
\]

\[
\hat{\Gamma} = \prod_{j, \alpha} \left( \hat{c}_{j, \alpha \uparrow} + \text{sgn} (\alpha) \hat{c}_{j, \alpha \downarrow} \right) \left( \hat{c}_{j, \alpha \downarrow} + \text{sgn} (\alpha) \hat{c}_{j, \alpha \uparrow} \right), \quad (29b)
\]

with sgn($\alpha$) taking 1 ($−1$) for $\alpha = A$ ($\alpha = B$). This symmetry imposes the following constraint on the Green’s function:

\[
\tau_3 G(−\omega + i\delta, \mathbf{k}) \tau_3 = −G(\omega + i\delta, \mathbf{k}). \quad (30a)
\]

In particular, for $\omega = 0$ the above condition can be rewritten as

\[
\tau_3 \hat{H}_{\text{eff}}^\dagger (0, \mathbf{k}) \tau_3 = −\hat{H}_{\text{eff}} (0, \mathbf{k}) \quad (30b)
\]

in terms of the effective Hamiltonian $H_{\text{eff}} (\omega, \mathbf{k})$. This constraint is nothing but the symmetry discussed in the previous section [see Eq. (19)]. Therefore, the chiral symmetry of the correlated systems in Eq. (29) protects the SPERs emerging in the single-particle spectrum.

The DMFT results show the emergence of the SPERs. In Fig. 8, the spectrum at $\omega = 0$ is plotted for several values of the temperature. In the non-interacting case, it is well known that the Dirac cones appear at the corners of the BZ illustrated with the white hexagon. In the presence of the correlations, the Dirac cones split into rings [see the green rings in Fig. 8(a)]. Increasing the temperature suppresses the lifetimes of quasi-particles. Correspondingly, the SPERs become large [Fig. 8(b)]. In this figure we can also see the effect of symmetry on the Fermi arcs shown in Fig. 3(b). Because of the chiral symmetry, the Fermi arcs change into Fermi planes. This is because the energy eigenvalues $E_n$ appear in a pair $(E_n, −E_n^*)$ or become pure imaginary in the presence of the chiral symmetry of Eq. (30b).\(^4\)

\(^4\) This fact can be understood as follows. Suppose that the Hamiltonian is chiral symmetric [see Eq. (21)]. Then, with the eigenvalue $E_n$ and the right eigenvector $|\psi^R_n\rangle$ ($H|\psi^R_n\rangle = |\psi^R_n\rangle E_n, n \in \mathbb{Z}$), we obtain the relation

\[
H^\dagger U^\dagger_1 |\psi^R_n\rangle = −U^\dagger_1 |\psi^R_n\rangle E_n.
\]

Here, we have used Eq. (21). Noticing that the eigenvalues problem of the left eigenvector $|\psi^L_n\rangle$ ($n \in \mathbb{Z}$) is written as $H^\dagger |\psi^L_n\rangle = |\psi^L_n\rangle E_n^*$, we can see that the vector $U^\dagger_1 |\psi^R_n\rangle$ is the left eigenvector with the eigenvalue $E_n$.\(^4\)
For higher temperatures, the SPERs, arising from distinct Dirac cones, merge into a single loop [see Fig. 8(c)]. We also note that the presence of the Dirac cones is not a necessary condition for SPERs. Introducing the anisotropy of the hopping, the Dirac cones disappear because of the pair annihilation. Even in the absence of the Dirac cones SPERs emerge [see Fig. 8(d)]. We note that the SPERs are topologically stable; Fig. 9(a) shows that numerical characterization of the SPERs with the zeroth Chern number can be done.

Finally, we show that the emergence of the Fermi plane accompanying a SPER enhances the specific heat $C = d \langle H \rangle / dT$ because the Fermi plane induces additional low-energy excitations. In Fig. 9(b), the specific heat is shown with a red line. For comparison, we also plot data with a blue line by assuming that the imaginary part of the self-energy for the $A$ and $B$ sublattices takes the average value $\text{Im} [\Sigma_A(\omega + i\delta) + \Sigma_B(\omega + i\delta)]/2$. We note that the system does not show SPERs when the imaginary part for the $A$ sublattice is identical to that for the $B$ sublattice. We can see that the specific heat is enhanced because of the Fermi planes accompanying SPERs.

### 3.3. SPESs for a correlated diamond lattice

The emergence of SPESs can also be demonstrated by applying the DMFT to a Hubbard model of a diamond lattice, which is a three-dimensional extension of the honeycomb Hubbard model of Eq. (27). The lattice structure and the BZ are shown in Fig. 10(a) and 10(b), respectively. In a similar way to the previous section, we introduce an inhomogeneity of the interaction.

In the following, we see the details. For $U_A = 8t$, $U_B = 0$, and $T = 0.8t$, SPESs emerge as shown in Fig. 10(c). Here, we have employed the iterative perturbation method [153,154] as the impurity...
Fig. 9. (a) Colormap of the zeroth Chern number for $r = 1$, $T = 0.0325t$, $U_A = 10t$, and $U_B = 5t$. The black lines represent the SPERs separating domains where the zeroth Chern number takes distinct values. The black hexagon illustrates the BZ. The numbers enclosed with black squares denote the zeroth Chern number, which is defined so that it takes zero for the Hermitian case $H_{\text{eff}} = b_1 \tau_1 + b_2 \tau_2$. (b) Temperature dependence of the specific heat for $r = 1$, $U_A = 10t$, and $U_B = 0$. Data plotted with the blue line are for comparison; the data are obtained by assuming that the imaginary part of the self-energy takes the same value $\text{Im}[\Sigma_A(\omega) + \Sigma_B(\omega)]/2$. Namely, the data on the blue line do not show SPERs. Fermi planes emerge for $T \gtrsim 0.1$. These figures are adapted with permission from Ref. [60]. Copyright 2019 American Physical Society.

Fig. 10. (Color Online). (a) Sketch of the diamond lattice. This lattice is composed of two sublattices, $A$ and $B$, which are illustrated with red and blue spheres, respectively. We assume that interaction for the $A$ sublattice is stronger than that of the $B$ sublattice. (b) The BZ and the high symmetric points for the diamond lattice. The BZ is illustrated with orange lines. (c) Exceptional surface for $U_A = 8t$, $U_B = 0$, and $T = 0.8t$. These figures are adapted with permission from Ref. [62]. Copyright 2019 American Physical Society.

The solver of the DMFT. In Fig. 11(a), the single-particle spectral function at zero energy $A(\omega = 0, \mathbf{k})$ is plotted for the $k_{xy} - k_z$ plane [i.e. the blue plane in Fig. 10(b)]. The green dots plotted in Fig. 11(a) correspond to the sections of SPESs. We note that in the region enclosed with the SPESs the energy gap becomes pure imaginary, meaning that zero-energy excitations appear in this region. Thus, the Fermi volume appears instead of the Fermi arc discussed in Sect. 2.3. Figure 11(b) shows the single-particle spectral function $A(\omega, \mathbf{k})$ along the lines connecting the high symmetry points in the BZ. In this figure, we can confirm the emergence of the Fermi volume by the presence of zero-energy excitations between X and K points. Outside of the SPESs, the zero-energy excitations disappear.

We finish this section with a comment concerning the effect of SPESs on the magnetic response. As shown in Fig. 11(c), the LDOS of the $B$ sublattice is enhanced by the Fermi volume accompanying the SPESs. We note that the LDOS of the $A$ sublattice is just renormalized. This imbalance of the LDOS can induce a counterintuitive behavior of the local magnetic susceptibility. In Fig. 11(b), the
Fig. 11. (Color Online). Spectral properties and the magnetic response. (a) The single-particle spectral function $A(\omega = 0, \mathbf{k})$ for the $k_{xy} - k_z$ plane shown in Fig. 10(b). (b) The single-particle spectral function $A(\omega, \mathbf{k})$ along the lines connecting high symmetry points in the BZ. The blue lines in this panel illustrate the dispersion relation for the non-interacting case. (c) The local density of states $\rho_\alpha(\omega) = -\sum_k \text{Im} G_{\alpha\alpha}(\omega, \mathbf{k})/(\pi N)$, with $N$ denoting the number of unit cells. Data shown in panels (a), (b), and (c) are obtained for $U_A = 8t$, $U_B = 0$, and $T = 0.8t$. (d) Magnetic susceptibility $\chi_s$ against interaction $U_A$ and temperature $T$. Here, $U_B$ is set to $U_B = U_A/2$. These figures are adapted with permission from Ref. [62]. Copyright 2019 American Physical Society.

Local magnetic susceptibility computed with the random-phase approximation (RPA)\(^5\) is plotted. As shown in Fig. 11(d), due to the imbalance of the LDOS, the magnetic susceptibility of the $B$ sublattice becomes larger than that of the $A$ sublattice, although the interaction of the $B$ sublattice is weaker than that of the $A$ sublattice.

\(^5\) The local magnetic susceptibility $\chi_s$ is computed as follows. With the RPA, the matrix of the susceptibility is written as

$$\chi_{\text{RPA}}^{\alpha\beta}(i\epsilon_n, \mathbf{q}) = (\mathbb{1} - \chi_0 U)\chi_0^{\alpha\beta},$$

with $U = \text{diag}(U_A, U_B)$ and $\chi_0^{\alpha\beta}$ defined as

$$\chi_0^{\alpha\beta}(i\epsilon_n, \mathbf{q}) = -\frac{T}{N} \sum_{k,m} G_{\alpha\beta}^{0}(i\omega_n + i\epsilon_n, \mathbf{q} + \mathbf{k}) G_{\beta\alpha}^{0}(i\omega_n, \mathbf{k}).$$

Here, $\omega_n$ and $\epsilon_n$ denote the Matsubara frequency [$\omega_n = (2n + 1)\pi T$ and $\epsilon_n = 2n\pi T$ with $n \in \mathbb{Z}$]; $N$ denotes the number of unit cells. The local magnetic susceptibility $\chi_s^\alpha$ is obtained as

$$\chi_s^A = (\chi_{AA}^{\text{RPA}} + \chi_{AB}^{\text{RPA}})/2,$$

$$\chi_s^B = (\chi_{BB}^{\text{RPA}} + \chi_{BA}^{\text{RPA}})/2,$$

with $q = 0$. We set $\epsilon_n \rightarrow 0$ instead of doing analytic continuation.
4. Ten-fold classification of the exceptional band touching in equilibrium systems

In Sect. 3 we saw that the correlated systems with chiral symmetry may show SPERs and SPESs in two and three dimensions, respectively. These SPERs and SPESs are characterized by the zeroth Chern number, a zero-dimensional topological invariant taking an arbitrary integer (see Sect. 3.1.2). In other words, the topological classification of the exceptional band touching is \( \mathbb{Z} \) for a system with chiral symmetry.

In this section, by generalizing the argument in Sect. 3, we address the topological classification of the exceptional band touching. Specifically, we carry out the ten-fold classification of exceptional band touching in the presence/absence of \( \text{PT}, \text{CP} \), and chiral symmetry for correlated systems. This is because \( \text{PT} (\text{CP}) \) symmetry is closed at each point in the BZ, as is chiral symmetry (i.e. the corresponding symmetry transformation does not flip the momentum). We note that 38-fold classification for exceptional band touching is carried out in Ref. [97] for a generic Bloch Hamiltonian. However, our analysis clarifies which symmetry classes are relevant for correlated systems. Our ten-fold classification is consistent with the corresponding classification results for 38 symmetry classes.

In what follows, we address the classification of exceptional band touching after a brief description of the relevant symmetry.

4.1. Symmetry constraints

4.1.1. PT symmetry

For the correlated systems preserving \( \text{PT} \) symmetry (i.e. symmetry under the product of time-reversal and spatial inversion), the second quantized Hamiltonian \( \hat{H} \) satisfies

\[
\hat{P}\hat{T}\hat{H}\hat{P}\hat{T}^{-1} = \hat{H}.
\]

(31)

Here, the anti-unitary operator \( \hat{P}\hat{T} \) is written as

\[
\hat{P}\hat{T} = \hat{U}_{PT}\mathcal{K},
\]

(32a)

\[
\hat{U}_{PT}c_{i\alpha}^\dagger \hat{U}_{PT}^\dagger = \sum_{\beta} c_{-i\beta}^\dagger U_{PT,\beta\alpha},
\]

(32b)

where \( c_{i\alpha}^\dagger \) creates a fermion with state \( \alpha \) at site \( i \). \( \hat{U}_{PT} \) is a unitary operator. \( \mathcal{K} \) is an operator taking complex conjugation. \( U_{PT} \) is a matrix satisfying \( U_{PT}U_{PT}^* = \pm \mathbb{I} \). Here we have supposed that under inversion, site \( j \) is mapped to \( -j \).

For \( \text{PT} \)-symmetric systems, the Green’s function satisfies [20]

\[
G(\omega + i\delta, \mathbf{k}) = U_{PT}G^T(\omega + i\delta, \mathbf{k})U_{PT}^\dagger,
\]

(33a)

which can be rewritten as

\[
H_{\text{eff}}(\omega + i\delta, \mathbf{k}) = U_{PT}H_{\text{eff}}^T(\omega + i\delta, \mathbf{k})U_{PT}^\dagger.
\]

(33b)
Equation (33a) can be seen by following a straightforward calculation.  

4.1.2. CP symmetry

For correlated systems preserving CP symmetry (i.e. symmetry under the product of charge conjugation and inversion), the second quantized Hamiltonian \( \hat{H} \) satisfies

\[
\overline{CP} \hat{H} \overline{CP}^{-1} = \hat{H},
\]

with \( \overline{CP} \) corresponding to the unitary operator \( \overline{CP} = \hat{U}_{CP} \) which transforms \( \hat{c}_{ia} \) as

\[
\hat{U}_{CP} \hat{c}_{ia} \hat{U}_{CP}^\dagger = \sum_{\beta} \hat{c}_{-i\beta} U_{CP,\beta a}.
\]

Here, \( U_{CP} \) is a unitary matrix satisfying \( U_{CP} U_{CP}^\dagger = \pm \mathbb{1} \).

---

6 Equation (33a) can be obtained as follows. First, we note that the following relations hold:

\[
\begin{align*}
    it\hat{H} &= it\hat{U}_{PT}\hat{H}^*\hat{U}_{PT}^\dagger, \\
    \hat{U}_{PT}\hat{c}_{ia}\hat{U}_{PT}^\dagger &= \sum_{a'} \hat{c}_{a'a} U_{a'a}, \\
    \langle n^*|\hat{A}|m^*\rangle &= \sum_{\{\alpha,\beta\}} \langle n^*|\{\alpha\}\{\alpha\}|\{\beta\}\{\beta\}|m^*\rangle \\
    &= \sum_{\{\beta\}} \langle m|\{\beta\}\{\beta\}|\{\alpha\}\{\alpha\}|n\rangle \\
    &= \langle m|\hat{A}^\dagger|n\rangle,
\end{align*}
\]

where \( |\{\alpha\}\rangle \) and \( |\{\beta\}\rangle \) denote the states generated by applying the creation operators \( \hat{c}_{i\alpha} \) on the vacuum \( |0\rangle \); \( |n\rangle \) and \( |m\rangle \) are arbitrary states, and \( |n^*\rangle := \sum_{\{\alpha\}} |\{\alpha\}\rangle \langle \{\alpha\}| \). By using the above relations, the correlation function \( \langle \hat{c}_{ia}(t)\hat{c}_{j\gamma}^\dagger \rangle \) is rewritten as

\[
\begin{align*}
    \langle \hat{c}_{ia}(t)\hat{c}_{j\gamma}^\dagger \rangle &= Z^{-1} \text{tr}[e^{-\beta H^*} \hat{c}_{ia} e^{-\beta H} \hat{c}_{j\gamma}^\dagger] \\
    &= Z^{-1} \text{tr}[e^{-\beta H^*} e^{\beta \hat{H}} \hat{U}_{PT}\hat{c}_{ia}\hat{U}_{PT}^\dagger e^{-\beta \hat{H}} \hat{U}_{PT}\hat{c}_{j\gamma}^\dagger \hat{U}_{PT}^\dagger] \\
    &= Z^{-1} \sum_{a'\gamma'} \text{tr}[e^{-\beta \hat{H}} e^{\beta \hat{H}} U_{PT,a'a} e^{-\beta \hat{H}} e^{\beta \hat{H}} \hat{c}_{a'\alpha} e^{\beta \hat{H}} \hat{c}_{j\gamma'}^\dagger U_{PT,y'\gamma'}^\dagger] \\
    &= Z^{-1} \sum_{a'\gamma'} U_{PT,a'a} U_{PT,y'\gamma'}^\dagger \text{tr}[\hat{c}_{a'\alpha} e^{-\beta \hat{H}} e^{\beta \hat{H}} \hat{c}_{j\gamma'}^\dagger e^{-\beta \hat{H}} e^{\beta \hat{H}}] \\
    &= \sum_{a'\gamma'} U_{PT,a'a} U_{PT,y'\gamma'}^\dagger \langle \hat{c}_{a'\alpha}(t)\hat{c}_{j\gamma'}^\dagger \rangle,
\end{align*}
\]

which is equivalent to

\[
\langle \hat{c}_{ia}(t)\hat{c}_{j\gamma}^\dagger \rangle = \sum_{a'\gamma'} U_{PT,a'a} U_{PT,y'\gamma'}^\dagger \langle \hat{c}_{a'\alpha}(t)\hat{c}_{j\gamma'}^\dagger \rangle.
\]

In a similar way, we obtain

\[
\langle \hat{c}_{k\alpha}(t)\hat{c}_{ia}(t) \rangle = \sum_{\gamma'\alpha'} U_{PT,a\alpha'} U_{PT,y'\gamma'}^\dagger \langle \hat{c}_{\alpha'\gamma'}(t)\hat{c}_{k\alpha}(t) \rangle.
\]

Remembering the definition of the retarded Green’s function [see Eq. (10)], we end up with Eq. (33a).
For CP-symmetric systems, the Green’s function satisfies

\[ G(\omega + i\delta, k) = -U_{CP} G^*(\omega + i\delta, k) U_{CP}^\dagger, \]  

(36a)

which can be rewritten as

\[ H_{\text{eff}}(\omega + i\delta, k) = -U_{CP} H_{\text{eff}}^*(\omega + i\delta, k) U_{CP}^\dagger. \]  

(36b)

Equation (36a) can be obtained by using the following relations:

\[ G(\omega + i\delta, k) = -U_{CP} G^T(\omega - i\delta, k) U_{CP}^\dagger, \]  

(37)

\[ G^*_{\alpha\beta}(\omega - i\delta, k) = G_{\beta\alpha}(\omega + i\delta, k). \]  

(38)

We note that, applying the Fourier transformation, \( G(\omega - i\delta, k) \) is rewritten as \( G^A(-t, k) \), which is defined as

\[ G^A_{\alpha\beta}(t, k) := i<\hat{c}_{\alpha\beta}(t)\hat{c}_{\alpha\beta}^\dagger + \hat{c}_{k\beta}^\dagger\hat{c}_{k\alpha}(t)>\theta(-t). \]  

(39)

Equation (37) is obtained by a straightforward calculation.\(^7\) Equation (38) holds when the correlated

---

\(^7\) Equation (37) can be obtained by the following calculations. First, we note that the following relation holds:

\[ \hat{U}_{CP}^\dagger \hat{c}_{\alpha\beta}^\dagger \hat{U}_{CP} = \sum_{\alpha'} \hat{c}_{\alpha\alpha'} U_{\alpha\alpha'}^\dagger. \]

By using the above relations, \( \langle \hat{c}_{\alpha\alpha'}(t)\hat{c}_{\alpha\alpha'}^\dagger \rangle \) is rewritten as

\[ \langle \hat{c}_{\alpha\alpha'}(t)\hat{c}_{\alpha\alpha'}^\dagger \rangle = Z^{-1} \text{tr} \left[ e^{-\beta\hat{H}} e^{i\hat{U}_{CP}} e^{-\alpha\hat{H}} e^{i\hat{U}_{CP}^\dagger} \hat{c}_{\alpha\alpha'} e^{-\alpha\hat{H}} e^{i\hat{U}_{CP}} e^{-\alpha\hat{H}} e^{i\hat{U}_{CP}^\dagger} \hat{c}_{\alpha\alpha'}^\dagger \right] \]

\[ = Z^{-1} \text{tr} \left[ e^{-\beta\hat{H}} e^{i\hat{U}_{CP}} e^{-\alpha\hat{H}} e^{i\hat{U}_{CP}^\dagger} \hat{c}_{\alpha\alpha'} e^{\alpha\hat{H}} e^{-i\hat{U}_{CP}^\dagger} \hat{c}_{\alpha\alpha'}^\dagger \hat{U}_{CP} \right] \]

\[ = Z^{-1} \sum_{\alpha',\gamma'} \text{tr} \left[ e^{-\beta\hat{H}} e^{i\hat{U}_{CP}} e^{-\alpha\hat{H}} e^{i\hat{U}_{CP}^\dagger} \hat{c}_{\alpha\alpha'} e^{\alpha\hat{H}} e^{-i\hat{U}_{CP}^\dagger} \hat{c}_{\gamma\gamma'} e^{\alpha\hat{H}} e^{-i\hat{U}_{CP}^\dagger} \hat{c}_{\gamma\gamma'}^\dagger \right] \]

\[ = \sum_{\alpha',\gamma'} U_{CP,\alpha\alpha'} U_{CP,\gamma\gamma'} \langle \hat{c}_{\alpha\alpha'}(t)\hat{c}_{\gamma\gamma'}^\dagger \rangle, \]

which is equivalent to

\[ \langle \hat{c}_{\alpha\alpha'}(t)\hat{c}_{\alpha\alpha'}^\dagger \rangle = \sum_{\alpha',\gamma'} U_{CP,\alpha\alpha'} U_{CP,\gamma\gamma'} \langle \hat{c}_{\alpha\alpha'}^\dagger \hat{c}_{\gamma\gamma'}(t) \rangle \theta(-t). \]

In a similar way, we obtain

\[ \langle \hat{c}_{\gamma\gamma'}^\dagger \hat{c}_{\alpha\alpha'}(t) \rangle = \sum_{\alpha',\gamma'} U_{CP,\alpha\alpha'} U_{CP,\gamma\gamma'} \langle \hat{c}_{\gamma\gamma'}^\dagger \hat{c}_{\alpha\alpha'}(t) \rangle \theta(-t). \]

Namely, the above calculations yield the following relation between the retarded and the advanced Green’s function:

\[ \langle \hat{c}_{\alpha\alpha'}(t)\hat{c}_{\gamma\gamma'}^\dagger + \hat{c}_{\gamma\gamma'}^\dagger\hat{c}_{\alpha\alpha'}(t) \rangle \theta(t) = \sum_{\alpha',\gamma'} U_{CP,\alpha\alpha'} U_{CP,\gamma\gamma'} \langle \hat{c}_{\gamma\gamma'}(t')\hat{c}_{\alpha\alpha'}^\dagger(t') \rangle \theta(-t'), \]

with \( t' = -t \). We note that the right-hand (left-hand) side of the above equation corresponds to \( iG^R(-iG^A) \). Applying the Fourier transformation, we obtain Eq. (37).
system is described by a Hermitian Hamiltonian.\(^8\)

### 4.1.3. Chiral symmetry

For correlated systems preserving chiral symmetry, the second quantized Hamiltonian \( \hat{H} \) satisfies

\[
\hat{\Gamma} \hat{H} \hat{\Gamma}^{-1} = \hat{H},
\]

with

\[
\hat{\Gamma} = \hat{U}_\Gamma \mathcal{K}.
\]

Here, \( \hat{U}_\Gamma \) is a unitary operator transforming the annihilation operator as

\[
\hat{U}_\Gamma \hat{c}_{i\alpha} \hat{U}_\Gamma^\dagger = \sum_\beta \hat{c}_{i\beta} U_{\Gamma,\beta\alpha},
\]

where \( U_{\Gamma} \) is a matrix satisfying \( U_{\Gamma}^2 = \mathbb{1} \).

For chiral symmetric systems, the Green’s function satisfies

\[
G(\omega + i\delta, k) = -U_{\Gamma} G^\dagger(\omega - i\delta, k) U_{\Gamma}^\dagger,
\]

which can be rewritten as

\[
H_{\text{eff}}(\omega + i\delta, k) = -U_{\Gamma} H_{\text{eff}}^\dagger(\omega - i\delta, k) U_{\Gamma}^\dagger.
\]

Equation (41a) can be obtained by a straightforward calculation.\(^9\) This equation can also be obtained from Eqs. (33a) and (36a) by noticing that applying the operator \( \hat{\Gamma} \) is equivalent to applying the product of the operators \( \hat{P} \hat{T} \) and \( \hat{C} \hat{P} \).

---

\(^8\) Equation (38) can be obtained by making use of the Hermiticity of the Hamiltonian. With the Lehmann representation, the Green’s function can be written as

\[
G_{\alpha\beta}(\omega + i\delta, k) = Z^{-1} \sum nm e^{-\beta E_n} \frac{e^{i(\omega + i\delta + E_n - E_m)}}{(\omega + i\delta + E_n - E_m)} \langle n | \hat{c}_{k\beta} | m \rangle \langle m | \hat{c}_{k\alpha}^\dagger | n \rangle,
\]

where the \( |n\rangle \) are eigenstates of the Hamiltonian \( \hat{H} \). With this representation, we can see that the following relation holds:

\[
G_{\alpha\beta}^*(\omega + i\delta, k) = Z^{-1} \sum nm e^{-\beta E_n} \frac{e^{i(\omega - i\delta + E_n - E_m)}}{(\omega - i\delta + E_n - E_m)} \langle n | \hat{c}_{k\beta}^\dagger | m \rangle \langle m | \hat{c}_{k\alpha} | n \rangle^* = G_{\alpha\beta}(\omega - i\delta, k),
\]

which is nothing but the relation shown in Eq. (38).

\(^9\) Equation (41a) can be obtained as follows. First, we note that the following relations hold:

\[
\hat{H} = i \hat{U}_\Gamma \hat{H}^* \hat{U}_\Gamma^\dagger,
\]

\[
\hat{U}_\Gamma \hat{c}_{i\alpha} \hat{U}_\Gamma^\dagger = \sum_\beta U_{\Gamma,\beta\alpha} \hat{c}_{i\beta}^\dagger,
\]

\[
\langle n^* | \hat{A}^\dagger | m^* \rangle = \langle m | \hat{A} | n \rangle,
\]

\[
\langle n^* \rangle = \langle m | \hat{A} | n \rangle,
\]

\[
\langle n^* \rangle = \langle m | \hat{A}^\dagger | n \rangle,
\]

\[
\langle n^* \rangle = \langle m | \hat{A} | n \rangle,
\]

\[
\langle n^* \rangle = \langle m | \hat{A}^\dagger | n \rangle,
\]

\[
\langle n^* \rangle = \langle m | \hat{A} | n \rangle,
\]
4.2. Ten-fold classification

Prior to the topological classification of exceptional band touching, we note the following two facts.
(i) Exceptional band touching of the non-Hermitian Hamiltonian $H_{\text{eff}}(\omega = 0, k)$ can be described by a Hermitian Hamiltonian satisfying $\{\hat{H}, \hat{\Sigma}\} = 0$ with $\hat{\Sigma} = \mathbb{I} \otimes \rho_3$ [see, e.g., Eq. (8b)] [90,92,93,97,155,156]. (ii) For Hermitian systems, the classification of $d_{\text{EP}}$-dimensional gapless excitations in $d$ spatial dimensions is accomplished by classifying the $(\delta-1)$-dimensional gapped Hermitian Hamiltonian with $\delta = d - d_{\text{EP}}$ [157,158].

Thus, the problem is reduced to classifying gapless excitations of the Hermitian Hamiltonian $\hat{H}$ in the presence/absence of the following symmetry constraints:

\[
\begin{align*}
\hat{U}_{\text{PT}}\hat{H}^*(k)\hat{U}_{\text{PT}}^\dagger &= \hat{H}(k), \\
\hat{U}_{\text{CP}}\hat{H}^*(k)\hat{U}_{\text{CP}}^\dagger &= -\hat{H}(k), \\
\hat{U}_{\Gamma}\hat{H}^*(k)\hat{U}_{\Gamma}^\dagger &= -\hat{H}(k),
\end{align*}
\]

where $|n\rangle$ and $|m\rangle$ are arbitrary states, $|n^*\rangle$ is defined as $|n^*\rangle := \sum_{|i\omega\rangle}|(i\omega)\rangle\langle n| (i\omega)\rangle$, with the states $|(i\omega)\rangle$ obtained by applying the operators $\hat{c}_{i\omega}^\dagger$ to the vacuum. By using the above relations, $\langle \hat{c}_{i\omega}(t)\hat{c}_{j\nu}^\dagger \rangle$ is rewritten as

\[
\langle \hat{c}_{i\omega}(t)\hat{c}_{j\nu}^\dagger \rangle = Z^{-1}\text{tr}[e^{-\beta\hat{H}} e^{it\hat{H}} \hat{c}_{i\omega} e^{-it\hat{H}} \hat{c}_{j\nu}^\dagger]
\]

\[
= Z^{-1}\text{tr}[e^{-\beta\hat{H}} e^{it\hat{H}} \hat{U}_{\Gamma}^\dagger \hat{c}_{i\omega} \hat{U}_{\Gamma} e^{-it\hat{H}} \hat{U}_{\Gamma}^\dagger \hat{c}_{j\nu}^\dagger \hat{U}_{\Gamma}]
\]

\[
= Z^{-1}\sum_{\alpha\gamma'} U_{\Gamma,\alpha\nu'} U_{\Gamma,\gamma'\gamma} \text{tr}[e^{-\beta\hat{H}} e^{it\hat{H}} \hat{c}_{i\omega} e^{-it\hat{H}} \hat{c}_{j\nu}^\dagger e^{-it\hat{H}} \hat{c}_{i\omega}]
\]

\[
= Z^{-1}\sum_{\alpha\gamma'} U_{\Gamma,\alpha\nu'} U_{\Gamma,\gamma'\gamma} \langle \hat{c}_{i\omega}(t)\hat{c}_{j\nu}^\dagger (-t) \rangle,
\]

which is equivalent to

\[
\langle \hat{c}_{k\alpha}(t)\hat{c}_{k\alpha}^\dagger \rangle = \sum_{\alpha'\gamma'} U_{\Gamma,\alpha'\nu'} U_{\Gamma,\gamma'\gamma} \langle \hat{c}_{k\gamma'}^\dagger \hat{c}_{k\alpha'} (-t) \rangle.
\]

In a similar way, we obtain

\[
\langle \hat{c}_{k\gamma'}^\dagger \hat{c}_{k\alpha}(t) \rangle = \sum_{\alpha'\gamma'} U_{\Gamma,\alpha'\nu'} U_{\Gamma,\gamma'\gamma} \langle \hat{c}_{k\alpha'} (-t)\hat{c}_{k\gamma'}^\dagger \rangle.
\]

Namely, the above calculation yields the following relation between the retarded and advanced Green’s functions:

\[
\langle \hat{c}_{k\alpha}(t)\hat{c}_{k\gamma'}^\dagger + \hat{c}_{k\gamma'}^\dagger \hat{c}_{k\alpha}(t) \rangle \theta(t) = \sum_{\alpha'\gamma'} U_{\Gamma,\alpha'\nu'} U_{\Gamma,\gamma'\gamma} \langle \hat{c}_{k\alpha'} (-t)\hat{c}_{k\gamma'}^\dagger + \hat{c}_{k\gamma'}^\dagger \hat{c}_{k\alpha'} (t) \rangle \theta(-t'),
\]

with $t' = -t$. We note that the right-hand (left-hand) side of the above equation corresponds to $iG^R (-iG^A)$, respectively. With the Fourier transformation and Eq. (38), we obtain Eq. (41a).
with

$$\tilde{H} = \begin{pmatrix} 0 & H_{\text{eff}}(0, \mathbf{k}) \\ H^\dagger_{\text{eff}}(0, \mathbf{k}) & 0 \end{pmatrix},$$  \hspace{1cm} (42d)

$$\tilde{U}_{PT} = \begin{pmatrix} 0 & U_{PT} \\ U^\dagger_{PT} & 0 \end{pmatrix},$$  \hspace{1cm} (42e)

$$\tilde{U}_{CP} = U_{CP} \rho_0,$$  \hspace{1cm} (42f)

$$\tilde{U}_\Gamma = \begin{pmatrix} 0 & U_\Gamma \\ U^\dagger_\Gamma & 0 \end{pmatrix},$$  \hspace{1cm} (42g)

and $\tilde{U}_{PT} \tilde{U}^\dagger_{PT} = \pm \mathbb{1}$, $\tilde{U}_{CP} \tilde{U}^\dagger_{CP} = \pm \mathbb{1}$, and $\tilde{U}_\Gamma \tilde{U}_\Gamma = \mathbb{1}$.

The above relation can also be written with the two anti-unitary operators ($\tilde{P}_T = \tilde{U}_{PT} K$, $\tilde{C}_P = \tilde{U}_{CP} K$) and a unitary operator ($\tilde{\Gamma} := \tilde{U}_\Gamma$). We note that the unitary matrices $\tilde{U}_{PT}$, $\tilde{U}_{CP}$, and $\tilde{U}_\Gamma$ satisfy the following commutation/anti-commutation relations:

$$\{ \tilde{U}_{PT}, \tilde{\Sigma} \} = 0,$$  \hspace{1cm} (43a)

$$[ \tilde{U}_{CP}, \tilde{\Sigma} ] = 0,$$  \hspace{1cm} (43b)

$$\{ \tilde{U}_\Gamma, \tilde{\Sigma} \} = 0.$$  \hspace{1cm} (43c)

Therefore, exceptional band touching can be classified by addressing the classification of gapless excitations in Hermitian systems with additional chiral symmetry whose operator $\tilde{\Sigma}$ satisfies Eq. (43). We address the classification based on Clifford algebra [148,159]. The specific procedure of the classification is summarized in Sect. 4.2.2. In the next section, we discuss the classification results.

### 4.2.1. Classification results

The classification results of $d_{\text{EP}}$-dimensional exceptional band touching for $H_{\text{eff}}(\omega = 0, \mathbf{k})$ are summarized in Table 2. Here, we consider the $d$-dimensional BZ.

For each case of $d = d_{\text{EP}}$ and symmetry class, this table elucidates the presence/absence of the $(\delta - 1)$-dimensional topological invariant in the BZ: “Z” (“$\mathbb{Z}_2$”) indicates the presence of a topological invariant taking an arbitrary integer (0 or 1), respectively; “0” appearing as the classification result (i.e. from the 6th to the 13th columns) indicates the absence of such topological invariants.

These classification results explain the exceptional band touching reported so far. For instance, this table indicates the $\mathbb{Z}$ classification for class A with $\delta = 2$, meaning that there exists exceptional band touching characterized by a one-dimensional topological invariant. This classification result explains the presence of EPs observed in Fig. 3(b) ($d = 2$ and $d_{\text{EP}} = 0$). We note that the emergence of EPs for class A is also reported for systems with disorder [58,61] or electron–phonon coupling [57]. With $d = 3$ and $d_{\text{EP}} = 1$ we obtain the same $\delta$, resulting in the $\mathbb{Z}$ classification for class A. This fact also explains the emergence of exceptional loops in three-dimensional systems [64]. The classification results for symmetry classes AI, AII, D, and C show the stability of these band touching points in the presence/absence of $PT$ or $CP$ symmetry.

The emergence of SPERs observed in Fig. 8 is also consistent with Table 2 ($d = 2$ and $d_{\text{EP}} = 1$). For class AIII with $\delta = 1$ we obtain the $\mathbb{Z}$ classification, implying the presence of the zeroth Chern number. The $\mathbb{Z}$ classification for class AIII with $\delta = 1$ is also consistent with the emergence of SPEs.
observed in Fig. 10(c) \((d = 3\) and \(d_{EP} = 2\)). We note that the classification results for symmetry classes BDI, DIII, CII, and CI show the stability of the exceptional band touching in the presence of \(PT\) or \(CP\) symmetry.

While we have mainly analyzed exceptional band touching for symmetry class A or AIII in the previous sections, the classification results summarized in Table 2 imply the existence of novel exceptional band touching. The verification of exceptional band touching for other cases of symmetry is still missing, along with the material realization.

It is also worth noting that the above table may explain the exceptional band touching away from \(\omega = 0\) for class A by recognizing the frequency as an additional momentum, although we have restricted ourselves to \(\omega = 0\) so far. Indeed, the emergence of exceptional rings in the \(\omega\)--\(k\) space has been demonstrated for two-dimensional heavy fermions [63] \((d = 3\) and \(d_{EP} = 1\)), which is consistent with the \(Z\) classification for symmetry class A with \(\delta = 2\). The above fact allows us to interpret the \(Z\) classification for class A with \(\delta = 4\); it implies the presence of novel EPs in the \(\omega\)--\(k\) space for three spatial dimensions. Further analysis in this direction is required.

### 4.2.2. Details of the classification for the Hermitian Hamiltonian

As discussed at the beginning of this section, classification of the \(d_{EP}\)-dimensional exceptional band touching in \(d\) spatial dimensions is accomplished by classifying the \((\delta - 1)\)-dimensional gapped Hermitian Hamiltonian with additional chiral symmetry satisfying Eq. (43). Here, \(\delta\) denotes the codimension \((\delta = d - d_{EP})\). In this section, we address the classification of the gapped Hermitian Hamiltonian based on Clifford algebra [148,159].

In what follows we describe the technical details of the derivation of Table 2. Thus, readers who are interested in the physical interpretation of the classification results rather than the technical details can skip this section.
Table 3. Classifying space \((C_q \text{ and } R_q)\) and the corresponding homotopy group \([\pi_0(C_q) \text{ and } \pi_0(R_q)]\).

| Classifying space | \(C_0\) | \(C_1\) | \(R_0\) | \(R_1\) | \(R_2\) | \(R_3\) | \(R_4\) | \(R_5\) | \(R_6\) | \(R_7\) |
|------------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| \(\pi_0(C_q)\) or \(\pi_0(R_q)\) | \(\mathbb{Z}\) | 0 | \(\mathbb{Z}\) | \(\mathbb{Z}_2\) | \(\mathbb{Z}_2\) | \(\mathbb{Z}\) | 0 | 0 | 0 | 0 |

Specifically, the topological classification based on the Clifford algebra can be carried out by the following steps [148,159]:

(i) Deform the Hermitian Hamiltonian \(\tilde{H}\) to the Hermitian Dirac Hamiltonian \(H_0\),

\[
H_0(k) = \sum_{j=1,\ldots,\delta-1}^{} k_j \gamma_j + m \gamma_0,
\]

where the \(\gamma\) satisfy \(\{\gamma_i, \gamma_j\} = 2\delta_{ij}\) for \(i, j = 0, \ldots, \delta - 1\). Because such deformation is possible for an arbitrary gapped Hamiltonian, the problem is reduced to classifying the possible mass term \(\gamma_0\).

(ii) Consider a Clifford algebra \(Cl_q\) or \(Cl_{p,q}\) with the \(\gamma\) matrices and the symmetry operators. \(Cl_q\) denotes the Clifford algebra composed of \(q\) generators,

\[
\{e_1, e_2, \ldots, e_q\},
\]

where the generator \(e_i\) satisfies \(e_i^2 = 1\) for \(i = 1, \ldots, q\). \(Cl_{p,q}\) represents the Clifford algebra composed of \(p + q\) generators,

\[
\{e_1, e_2, \ldots, e_p, e_{p+1}, \ldots, e_{p+q}\},
\]

where the generator \(e_i\) satisfies \(e_i^2 = -1\) (\(e_i^2 = 1\) for \(i = 1, \ldots, p\) \((i = p + 1, \ldots, p + q)\). We note that an operator \(J\) satisfying \(\{\tilde{P}T, J\} = \{\tilde{CP}, J\} = [H_0(k), J] = 0\) needs to be introduced in the presence of \(PT\) or \(CP\) symmetry because \(\tilde{P}T\) and \(\tilde{CP}\) are anti-unitary operators.

(iii) By adding the mass term \(\gamma_0\), consider the extension problem to obtain the corresponding classifying space which turns out to be \(C_q\) \((R_{q-p})\) when the extension problem is \(Cl_q \rightarrow Cl_{q+1}\) \((Cl_{p,q} \rightarrow Cl_{p,q+1}\)). Here, we note that the corresponding classifying space of the extension problem \(Cl_{p,q} \rightarrow Cl_{p+1,q}\) is \(R_{2+p-q}\) [159].

(iv) By making use of the relation summarized in Table 3, obtain the classification result \(\pi_0(C_q)\) \([\pi_0(R_q)]\). We note that the relations \(\pi_0(C_{q+2}) = \pi_0(C_q)\) and \(\pi_0(R_{q+8}) = \pi_0(R_q)\) hold, which are known as the Bott periodicity.

With steps (i)–(iv) above we can obtain the classification results shown in Table 2.

Table 4 shows the Clifford algebra for each symmetry class generated by the mass term, the kinetic terms, and symmetry operators. Although one can reproduce the classification results from Table 4, we explicitly apply the above procedure for class A and AII as examples.

**Class A:** Remembering that the Hamiltonian \(\tilde{H}\) in \(\delta - 1\) dimensions is chiral symmetric, \(\{\tilde{H}, \tilde{S}\} = 0\), we obtain the Clifford algebra \(C_{\delta}\) generated by

\[
\{\gamma_1, \ldots, \gamma_{\delta-1}, \tilde{S}\}.
\]
Introducing the mass term $\gamma_0$ results in the extension problem which is written as $Cl_\delta \to Cl_{\delta+1}$. Here, the Clifford algebra $Cl_{\delta+1}$ is generated by
\begin{equation}
\{\gamma_0, \gamma_1, \ldots, \gamma_{\delta-1}, \tilde{\Sigma}\},
\end{equation}
which is shown in the last column of Table 2. Therefore, the corresponding classifying space is $C_\delta$, which indicates that the classification result is computed with $\pi_0(C_\delta)$. By making use of the Bott periodicity and the relation summarized in Table 3, we obtain the classification results for $\delta = 1, \ldots, 8$.

Class AII: First, note that $\tilde{\mathcal{PT}}^2 = -1$ holds. Remembering that the Hamiltonian $\tilde{H}$ in $\delta - 1$ dimensions is chiral symmetric, $\{\tilde{H}, \tilde{\Sigma}\} = 0$, we obtain the Clifford algebra $C_{\delta}$ generated by
\begin{equation}
\{J\gamma_1, \ldots, J\gamma_{\delta-1}, \tilde{\mathcal{PT}}, J\tilde{\mathcal{PT}}; \tilde{\Sigma}\}.
\end{equation}

Introducing the mass term $\gamma_0$ results in the extension problem which is written as $Cl_{\delta+1,1} \to Cl_{\delta+2,1}$. Here, the Clifford algebra $Cl_{\delta+2,1}$ is generated by
\begin{equation}
\{J\gamma_0, J\gamma_1, \ldots, J\gamma_{\delta-1}, \tilde{\mathcal{PT}}, J\tilde{\mathcal{PT}}; \tilde{\Sigma}\},
\end{equation}
which is shown in the last column of Table 2. Therefore, the corresponding classifying space is $R_{2+\delta}$, which indicates that the classification result is computed with $\pi_0(R_{2+\delta})$. By making use of the Bott periodicity and the relation summarized in Table 3, we obtain the classification results for $\delta = 1, \ldots, 8$.

We note that for $\delta = d + 1$, Table 2 indicates the classification results for the $d$-dimensional gapped Hamiltonian with additional chiral symmetry satisfying Eq. (43). In this case, the classification results are given by the homotopy group $\pi_0(C_{q-1+d})$ or $\pi_0(C_{q-1+d})$ with integer $q$, while the original tenfold classification for topological insulators/superconductors is given by $\pi_0(C_{q-d})$ or $\pi_0(C_{q-d})$. This is due to the fact that applying $\tilde{\mathcal{PT}}$ or $\tilde{\mathcal{CP}}$ does not flip the momentum $k$ [155,160,161], while applying the time-reversal or particle–hole operator does ($k \to -k$).
5. Summary and outlook

We have briefly reviewed the recently developed non-Hermitian perspective of the band structure in equilibrium systems. We have seen that the finite lifetime of quasi-particles induces EPs. In addition, we have seen that the symmetry of the many-body Hamiltonian results in SPERs (SPESs) in two (three) dimensions. While the above non-Hermitian perspective has been developed recently, there are several open questions to be addressed.

For instance, the effects of EPs on transport properties should be further analyzed. As seen in this paper, the exceptional band touching induces low-energy excitations such as Fermi arcs. The emergence of these low-energy excitations may change the conductivity or other electromagnetic responses.

The experimental observation of EPs in electronic systems is also a crucial issue in this direction. Topological Kondo insulators such as SmB$_6$ [162–168] and YbB$_{12}$ [169,170] might serve as platforms for EPs because they are strongly correlated materials and show Dirac cones at surfaces. Prior to experimental observation, quantitative analysis such as LDA+DMFT calculations should be carried out as well as theoretical proposals for how to experimentally observe the EPs.

While this paper focuses on exceptional band touching, non-Hermiticity induces richer topological physics. The non-Hermitian skin effect is another representative unique phenomenon [78,85–89,171–174]; the energy spectrum of a non-Hermitian matrix significantly depends on the boundary condition when the skin effect occurs. Determining whether the non-Hermiticity from finite lifetimes induces the skin effect is an intriguing theoretical open question to be addressed.

Finally, we comment on another significant issue of non-Hermiticity and correlations. Recently, a fractional quantum Hall phase, a topologically ordered phase, has been extended to non-Hermitian systems [118]. The extension of topologically ordered phases to non-Hermitian systems has been further addressed for a non-Hermitian toric code [175,176]. Developing the effective field theory to describe these non-Hermitian topologically ordered phases should be addressed, as well as extending them to systems with symmetry (e.g. time-reversal symmetry).

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