QUALIFICATION CONDITIONS-FREE CHARACTERIZATIONS OF THE $\varepsilon$-SUBDIFFERENTIAL OF CONVEX INTEGRAL FUNCTIONS

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Abstract. We provide formulae for the $\varepsilon$-subdifferential of the integral function $I_f(x) := \int_T f(t, x) d\mu(t)$, for an integrand $f : T \times X \to \mathbb{R}$ being measurable in $(t, x)$ and convex in $x$ with respect to a complete $\sigma$-finite measure space $(T, A, \mu)$. The state variable lies in a possibly non-separable and locally convex space. The resulting characterizations are given in terms of the $\varepsilon$-subdifferential of the data functions involved in the integrand, but without any qualification conditions. We also derive new formulas when some usual continuity-type conditions are in force. These results are new even for the finite sum of convex functions and for the finite-dimensional setting.

1. Introduction

Several problems in applied mathematics such as calculus of variations, optimal control theory and stochastic programming among others, rely on the study of integral functions and functionals given by the following expression

$$x(\cdot) \in \mathfrak{X} \rightarrow \hat{I}_f(x(\cdot)) := \int_T f(t, x(t)) d\mu(t),$$

for a functional space $\mathfrak{X}$ and an integrand function $f$ that are given by the data of the underlying system. Problems which consider this class of functionals represent a tremendous territory for developing variational analysis, and indeed it is especially under this class of problems where the theory has traditionally been organized. Models which consider integrals with respect to time are common in the study of dynamical systems and problems of optimal control. Also, when the problem involves uncertainty, as in stochastic programming problems, the design of such mathematical models is represented by a probability space, so that the problem is modeled using the integral sign. Applications to stochastic programming problems often concern the study of density distributions, which can also be presented under integration with respect to the Lebesgue measure.

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In classical studies, such as in the calculus of variations, the integrand $f(t, x)$ is usually supposed to be continuous in $t$ and $x$, jointly, or even with some order of differentiability. Later, integrands with finite values and satisfying the Caratheodory condition, that is, continuity in $x$ and measurability in $t$, have been considered. It can be easily noticed in all of these cases that for every measurable function $x(\cdot)$, the function $t \to f(t, x(t))$ is at least measurable and, hence, can be well-defined using the convention adopted for the extended-real line. However, new mathematical models, especially the emergence of the modern control theory, leads to the consideration of general integrands with possibly infinite values. In this way, important kinds of constraints can most efficiently be represented. Such integrands require a distinctly new theoretical approach, where questions of measurability, meaning of the integral and the existence of measurable selections are prominent and are reflected in the concept of normal integrands.

As is traditional in optimization and generally in variational analysis, one could replace the continuity of $f_t := f(t, \cdot)$ by the weaker property of lower semicontinuity, but maintaining the measurability of $f(t, x)$ with respect to $t$. Nevertheless, it is not enough to ensure the measurability of $t \to f(t, x(t))$ for any measurable function $x(\cdot)$. Indeed, consider $T = [0, 1]$ and let $\mathcal{A}$ be the $\sigma$-Algebra of Lebesgue measurable sets in $[0, 1]$. If $D$ is a non-measurable set in $[0, 1]$, and $f(t, x) := 0$ if $t = x \in D$ and $f(t, x) := 1$ otherwise, then the measurability and the lower semicontinuity of $f$ hold trivially. However, for $x(t) = t$ we lack the measurability of the function $t \to f(t, x(t))$. This example shows that although the lower semicontinuity assumption in $x$ is certainly right, the assumption of measurability in $t$ for each fixed $x$ is not adequate. The way out of this impasse was found by Rockafellar [36], using the concept of normal convex integrands, which is an equivalent definition to the one presented in Section 3, such that $f(t, \cdot)$ is proper and lower semi-continuous for each $t$, and there exists a countable collection $U$ of measurable functions $u$ from $T$ to $\mathbb{R}^n$ having the following properties: (a) for each $u \in U$, $f(t, u(t))$ is measurable in $t$; (b) for each $t$, $U_t \cap \text{dom } f_t$ is dense in $\text{dom } f_t$, where $U_t = \{u(t) : u \in U\}$.

The notion of convex normal integrands provides the link that allowed to connect the theories of measurable multifunctions and subdifferentials. The preservation of the measurability of multifunctions under a broad variety of operations including countable intersections, countable unions, sums, Painlevé-Kuratowski limits, and so on, as well as the validity of Castaing’s representations (12, 11), made this theory very popular in various problems of applied mathematics during the last four decades. Castaing’s representation theorem is intrinsically related with the possibility of extending the definition of the classical integration to the one of set-valued mappings considering, using measurable and integrable selections. In our case, we shall deal with the subdifferential mapping to get similar results as in the Leibniz integral rule.

Some of the classical studies about this class of integral functions and functionals can be found in Castaing-Valadier [14], Ioffe-Levin [26], Ioffe-Tikhomirov [27], Levin [30] and Rockafellar [36, 37, 40]. Ather recent works are Borwein-Yao [7], Ioffe [25], Lopez-Thibault [31], Mordukhovich-Sagara [32] among others. A summary of the elementary theory of measurability and integral functionals in finite-dimension can be found in [29, 11], and in [11, 24, 41, 45] for infinite-dimensional spaces.
The aim of this research is to give formulae for the $\varepsilon$-subdifferential of the convex integral function $I_f$, given by

$$x \in X \rightarrow I_f(x) := \int_T f(t, x) d\mu(t),$$

that is, when the space $X$ in (1.1) is the space of constant functions. This particular case is also known as the continuous sum. A well-known formula, given by Ioffe-Levin \[26\] for the finite-dimensional setting, shows that under certain continuity assumptions the following formula holds for the subdifferential of $I_f$

$$\partial I_f(x) = \int_T \partial f_t(x) d\mu(t) + N_{\text{dom}} f(x), \text{ for all } x \in \mathbb{R}^n,$$

where the set $\int_T \partial f_t(x) d\mu(t)$ is understood in the sense of Aumann’s integral (see Definition \[3.1\]), that is to say, as the set of points of the form $\int_T x(t) d\mu(t)$ where $x$ is an integrable function such that $x(t) \in \partial f_t(x)$ for almost all $t \in T$ (ae for short). One can compare (1.3) with its discrete counterpart, which declares that for every two convex lsc functions $f_1, f_2$ such that $f_1$ is continuous at some point of the domain of $f_2$ one gets $\partial(f_1 + f_2)(x) = \partial f_1(x) + \partial f_2(x)$ for all $x \in \mathbb{R}^n$. So, a reasonable idea is to give similar formulae as those given by Hiriart-Urruty and Phelps without the use of qualification conditions in the discrete case (see e.g. \[22\]). Whence, it feels natural to think about a generalization of (1.3) as

$$\partial I_f(x) = \bigcap_{\eta > 0} \text{cl} \left\{ \int_T \partial_\eta f_t(x) d\mu(t) + N_{\text{dom}} f(x) \right\} \text{ for all } x \in \mathbb{R}^n.$$  

Such an expression does not hold without any qualification conditions, since that one can find counterexamples where the set $\int_T \partial_\eta f_t(x) d\mu(t)$ is empty and the integrand $f_t$ is smooth at the point of interest (see Example \[4.5\]).

The above impediment leads us to use enlargements like $\int_T \partial_\eta f_t(x) d\mu(t)$ to generalize (1.3). With this idea in mind we provide general formulae for the $\varepsilon$-subdifferential of the convex integral functional $I_f$ defined in an arbitrary locally convex space. For sake of brevity, we have divided the investigation of the subdifferential of the convex integral functional $I_f$ into two papers (see \[16\] for the second part). We also have investigated the nonconvex integral functionals given by the form of (1.2) (see \[17\] for more details).

The rest of the paper is organized as follow: In Section 2 we summarize the notation which is classical in convex analysis and agrees with many monographs (see e.g. \[3, 6, 21, 29, 33, 38, 46\]). In Section 3 we give some definitions and preliminary results of the vector integration, measurable multifunctions, measurable selections and integral of multifunctions, which are used to study the subdifferential of the integral functional $I_f$. In Section 4 we present our main formulae, which characterize the $\varepsilon$-subdifferential of the integral functional without any qualification conditions in an arbitrary locally convex space (see Theorem \[4.1\]). In this result we explore the idea of considering enlargements of $\int_T \partial_\varepsilon f_t(x) d\mu(t)$ using the family of all finite-dimensional subspaces of $X$. It is important to mention that this technique of intersecting over the family of all finite-dimensional subspaces has been used to study the subdifferential of the supremum function (see, e.g., \[14, 20, 34, 35\] and the references therein). Later, we provide corollaries and simplifications of our main formulae under some qualification conditions of the data (see Corollary \[4.7\].
Also, general formulae for the discrete sum are derived from Theorem 4.1. Finally, in Section 5 we use calculus rules for the \( \varepsilon \)-subdifferential to get tighter formulae in two different frameworks: the first one corresponds to the case of a dual pair of Suslin spaces, being the framework where the most important results in the theory of measurable multifunctions and measurable selections have been developed. The second one corresponds to the case when the measure space is a countably discrete space, that is, when \( T \) is the set of natural numbers and the \( \sigma \)-algebra is given by its power set. This context is principally motivated by the studies of the subdifferential of series of convex functions (see e.g. [43]).

2. Notation

In this section we give the main notations and definitions that will be used in the sequel. We denote by \((X, \tau_X)\) and \((X^*, \tau_{X^*})\) two Hausdorff (separated) locally convex spaces (lcs, for short), which are in duality by the bilinear form \( \langle \cdot, \cdot \rangle : X^* \times X \to \mathbb{R} \), \( \langle x^*, x \rangle = x^*(x) \). For a point \( x \in X \) (resp. \( x^* \in X^* \)) \( \mathcal{N}(\tau_X) \) (resp. \( \mathcal{N}^*(\tau_{X^*}) \)) represents the (convex, balanced and symmetric) neighborhood system of \( x \) (resp. \( x^* \)); we omit the reference to the topology when there is no confusion. Examples of \( \tau_X \) are the weak* topology \( w(X^*, X) \) (\( w^* \), for short) where the convergence is denoted by \( \rightharpoonup \), the Mackey topology denoted by \( \tau(X^*, X) \), and the strong topology denoted by \( \beta(X^*, X) \). We will write \( \mathbb{R} := \mathbb{R} \cup \{-\infty, \infty\} \) and adopt the conventions that \( 0 \cdot \infty = 0 = 0 \cdot (-\infty) \) and \( \infty + (-\infty) = (-\infty) + \infty = \infty \). We denote \( B_{\rho}(x,r) := \{ y \in X : \rho(x - z) \leq r \} \) if \( \rho : X \to \mathbb{R} \) is a seminorm, \( x \in X \), and \( r > 0 \).

For a given function \( f : X \to \mathbb{R} \), the (effective) domain of \( f \) is \( \text{dom} f := \{ x \in X \mid f(x) < +\infty \} \). We say that \( f \) is proper if \( \text{dom} f \neq \emptyset \) and \( f > -\infty \), and inf-compact if for every \( \lambda \in \mathbb{R} \) the sublevel set \( [f \leq \lambda] := \{ x \in X \mid f(x) \leq \lambda \} \) is compact. We denote by \( \Gamma_0(X) \) the class of proper lower semicontinuous (lsc) convex functions on \( X \). The conjugate of \( f \) is the function \( f^* : X^* \to \mathbb{R} \) defined by

\[
 f^*(x^*) := \sup_{x \in X} \{ \langle x^*, x \rangle - f(x) \},
\]

and the biconjugate of \( f \) is \( f^{**} := (f^*)^* : X \to \mathbb{R} \). For \( \varepsilon \geq 0 \) the \( \varepsilon \)-subdifferential of \( f \) at a point \( x \in X \) where it is finite is the set

\[
 \partial_{\varepsilon} f(x) := \{ x^* \in X^* \mid \langle x^*, y - x \rangle \leq f(y) - f(x) + \varepsilon, \forall y \in X \};
\]

if \( f(x) \) is not finite, we set \( \partial_{\varepsilon} f(x) := \emptyset \). For a subspace \( F \) of \( X \), \( \overline{\text{co}} f : F \to \mathbb{R} \) is the function such that \( \text{epi}(\overline{\text{co}} f) = \overline{\text{co}}(\text{epi} f \cap (F \times \mathbb{R})) \).

The indicator and the support functions of a set \( A \subseteq X, X^* \) are, respectively,

\[
 \delta_A(x) := \begin{cases} 
 0 & x \in A \\
 +\infty & x \notin A,
\end{cases}
\]

\[
 \sigma_A := \delta_A^*.
\]

The inf-convolution of \( f, g : X \to \mathbb{R} \) is the function \( f \square g := \inf_{z \in X} \{ f(z) + g(\cdot - z) \} \); it is said to be exact at \( x \) if there exists \( z \) such that \( f \square g(x) = f(z) + g(x - z) \).

For a set \( A \subseteq X \), we denote by \( \text{int}(A), \overline{A} \) (or \( \text{cl}A \), \( \text{co}(A), \overline{\text{co}}(A) \), \( \text{lin}(A) \) and \( \text{aff}(A) \), the interior, the closure, the convex hull, the closed convex hull, the linear subspace and the affine subspace of \( A \). The relative interior of \( A \) with respect to an affine subspace \( F \), denoted by \( \text{ri}_F(A) \), is the interior of \( A \) with respect to \( F \). By the
symbol $\text{ri}(A)$ we denote $\text{ri}_{\text{aff}(A)}(A)$ if $\text{aff}(A)$ is closed, and the emptyset otherwise. The **polar** of $A$ is the set
\[
A^\circ := \{ x^* \in X^\ast \mid \langle x^*, x \rangle \leq 1, \forall x \in A \},
\]
and the **recession cone** of $A$ (when $A$ is convex) is the set
\[
A_\infty := \{ x \in X \mid \lambda x + y \in A \text{ for some } y \in A \text{ and all } \lambda \geq 0 \}.
\]
The $\varepsilon$-normal set of $A$ at $x$ is $N^\varepsilon_A(x) := \partial_\varepsilon \delta_A(x)$.

### 3. Preliminary results

In what follows $(X, \tau_X)$ and $(X^\ast, \tau_{X^\ast})$ are two lcs, as in Section 2. We give the main definitions and results which are used in the sequel.

A Hausdorff topological space $S$ is said to be a Suslin space if there exist a Polish space $P$ (complete, metrizable and separable) and a continuous surjection from $P$ to $S$ (see [8, 11, 42]). For example, if $X$ is a separable Banach spaces, then $(X, \|\cdot\|)$ and $(X^\ast, w^\ast)$ are Suslin.

Let $(T, \Sigma, \mu)$ be a complete $\sigma$-finite measure space. Given a function $f : T \to \overline{\mathbb{R}}$, we denote
\[
D_f := \{ g \in L^1(T, \mathbb{R}) : f(t) \leq g(t) \text{ } \mu\text{-almost everywhere} \},
\]
and define the upper integral of $f$ by
\[
\int_T f(t) d\mu(t) := \inf_{g \in D_f} \int_T g(t) d\mu(t)
\]
whenever $D_f \neq \emptyset$. If $D_f = \emptyset$, we set $\int_T f(t) d\mu(t) := +\infty$. A function $f : T \to U$, with $U$ being a topological space, is called simple if there are $k \in \mathbb{N}$, a partition $T_1 \in \Sigma$ and elements $x_i \in U$, $i = 0, \ldots, k$, such that $f = \sum_{i=0}^k x_i 1_{T_i}$ (here, $1_{T_i}$ denotes the characteristic function of $T_i$, equaling to 1 in $T_i$ and 0 outside). Function $f$ is called strongly measurable (measurable, for short) if there exists a countable family $(f_n)_n$ of simple functions such that $f(t) = \lim_{n \to \infty} f_n(t)$ for almost every (ae, for short) $t \in T$.

A strongly measurable function $f : T \to X$ is said to be strongly integrable (integrable for short), and we write $f \in L^1(T, X)$, if $\int_T \sigma_B(f(t)) d\mu(t) < \infty$ for every bounded balanced subset $B \subset X^\ast$. Observe that in the Banach spaces setting, $L^1(T, X)$ is the set of Bochner integrable functions (see, e.g., [18 §II]).

A function $f : T \to X$ is called (weakly or scalarly integrable) weakly or scalarly measurable if for every $x^* \in X^\ast$, $t \to \langle x^*, f(t) \rangle$ is (integrable, resp.) measurable. We denote $L^w_\mu(T, X)$ the space of all weakly integrable functions $f$ such that
\[
\int_T \sigma_B(f(t)) d\mu(t) < \infty
\]
for every bounded balanced subset $B \subset X^\ast$. Similarly, for functions taking values in $X^\ast$, we say that $f : T \to X^\ast$ is ($w^\ast$-integrable, resp.) $w^\ast$-measurable if for every $x \in X$, the mapping $t \to \langle x, f(t) \rangle$ is (integrable, resp.) measurable. Also, we denote $L^w_{\mu^*}(T, X^\ast)$ the space of all $w^\ast$-integrable functions $f$ such that $\int_T \sigma_B(f(t)) d\mu(t) < \infty$ for every bounded balanced subset $B \subset X$.

It is clear that every strongly integrable function is weakly integrable. However, the weak measurability of a function $f$ does not necessarily imply the measurability
of the function $\sigma_B(f(\cdot))$, and so the corresponding integral of this last function must be understood in the same sense as $\int f \, d\lambda$.

Also, observe that if in addition $X$ is a Suslin, then every $(\Sigma, \mathcal{B}(X))$-measurable function $f : T \to X$ (that is, $f^{-1}(B) \in \Sigma$ for all $B \in \mathcal{B}(X)$) is weakly measurable, where $\mathcal{B}(X)$ is the Borel $\sigma$-Algebra of the open (equivalently, weak) set of $X$ (see, e.g., [11, Theorem III.36]).

The quotient spaces $L^1(T, X)$ and $L^1_w(T, X)$ of $\mathcal{L}^1(T, X)$ and $\mathcal{L}^1_w(T, X)$, respectively, are those given with respect to the equivalence relations $f = g$ ae, and $\langle f, x^* \rangle = \langle g, x^* \rangle$ ae for all $x^* \in X^*$, respectively (see, for example, [28]).

It is worth observing that when $X$ is a separable Banach space, both notions of (strong and weak) measurability and integrability coincide; hence, if, in addition, $(X^*, \| \cdot \|)$ is separable, then $\mathcal{L}^1(T, X^*) = \mathcal{L}^1_w(T, X^*)$ (see [18, II, Theorem 2]).

It is worth recalling that when the space $X$ is separable, but the dual $X^*$ is not $\| \cdot \|$-separable, $\mathcal{L}^1(T, X^*)$ and $\mathcal{L}^1_w(T, X^*)$ may not coincide (see [18, II Example 6]). For every $w^*$-integrable function $f : T \to X^*$ and every $E \in \Sigma$, the function $x_E^f$ defined on $X$ as $x_E^f(x) := \int_T f(t, x) \, d\mu$ is a linear mapping (not necessarily continuous), which we call the weak integral of $f$ over $E$, and we write $\int_E f \, d\mu := x_E^f$. Moreover, if $f$ is strongly integrable, this element $\int_E f \, d\mu$ also refers to the strong integral of $f$ over $E$. Observe that, in general, $\int_E f \, d\mu$ may not be in $X^*$. However, when the space $X$ is Banach, and the function $f : T \to X^*$ is $w^*$-integrable, the linear function $\int_E f \, d\mu$ belongs to $X^*$, and it is called the Gelfand integral of $f$ over $E$ (see [18, II, Lemma 3.1] and details therein).

When $X$ is Banach, $L^\infty(T, X)$ is the normed space of (equivalence classes with respect to the relation $f = g$ ae) strongly measurable functions $f : T \to X$, which are essentially bounded; that is, $\|f\|_\infty := \text{ess sup}\{|f(t)| : t \in T\} < \infty$. A functional $\lambda^* \in L^\infty(T, X)^*$ is called singular if there exists a sequence of measurable sets $T_n$ such that $T_{n+1} \subseteq T_n$, $\mu(T_n) \to 0$ as $n \to \infty$ and $\lambda^*(g1_{T_n}) = 0$ for every $g \in L^\infty(T, X)$. We will denote $L^{\text{sing}}(T, X)$ the set of all singular functionals. It is well-known that each functional $\lambda^* \in L^\infty(T, X)^*$ can be uniquely written as the sum $\lambda^*(\cdot) = \int_T (\lambda_1^*(t) \cdot) \, d\mu(t) + \lambda_2^*(\cdot)$, where $\lambda_1^* \in L^w(T, X^*)$ and $\lambda_2^* \in L^{\text{sing}}(T, X)$ (see, for example, [11, 30]).

For a vector space $L$ of function $x : T \to X$, where $X$ is endowed with a locally convex topology $\tau$, by an integral functional on $L$ we mean an extended-real-valued functional $I_f$ of the form

$$x(\cdot) \in L \to I_f(x(\cdot)) := \int_T f(t, x(t)) \, d\mu(t),$$

where $f : T \times X \to [\overline{\mathbb{R}}]$ is any function. A function $f : T \times X \to [\overline{\mathbb{R}}]$ is called a $\tau$-normal integrand (or, simply, normal integral when no confusion occurs), if $f$ is $\Sigma \otimes \mathcal{B}(X, \tau)$-measurable and the functions $f(t, \cdot)$ are lsc for ae $t \in T$. In addition, if $f(t, \cdot) \in \Gamma_0(X)$ for ae $t \in T$, then $f$ is called convex normal integrand. For simplicity, we denote $f_t := f(t, \cdot)$. When $L$ is the linear space of constant functions, we also consider the integral function $I_f$ defined on $X$ as

$$x \in X \to I_f(x) := \int_T f(t, x) \, d\mu(t).$$

A multifunction $G : T \rightrightarrows X$ is called $\Sigma \otimes \mathcal{B}(X)$-measurable (measurable, for simplicity) if its graph, $\text{gph} G := \{(t, x) \in T \times X : x \in G(t)\}$, is an element of $\Sigma \otimes \mathcal{B}(X)$. 
We say that \( G \) is weakly measurable if for every \( x^* \in X^* \), \( t \to \sigma_{G(t)}(x^*) \) is a measurable function.

**Definition 3.1.** The strong and the weak integrals of a (non-necessarily measurable) multifunction \( G : T \rightrightarrows X^* \) are given respectively by

\[
\int_T G(t)d\mu(t) := \left\{ \int_T m(t)d\mu(t) \in X^* : m \text{ is integrable and } m(t) \in G(t) \text{ a.e.} \right\},
\]

\[
(w)\int_T G(t)d\mu(t) := \left\{ \int_T m(t)d\mu(t) \in X^* : m^* \text{-integrable and } m(t) \in G(t) \text{ a.e.} \right\}.
\]

It is important to recall that the original definition of integral of set-valued mappings is due to R. J. Aumann and it was given for multifunctions defined on a closed interval \([0, T]\) in \( \mathbb{R} \); see for example [2]. For this reason many authors give the name of Aumann Integral to Definition 3.1.

The next definition corresponds to the notion of decomposability in locally convex Suslin spaces [11, Definition 3, §7].

**Definition 3.2.** (i) Assume that \((T, \Sigma) = (\mathbb{N}, \mathcal{P}(\mathbb{N}))\). A vector space \( L \subset X^T \) is said to be decomposable if

\[ c_{00}(X) := \{ (x_n) : \exists k_0 \in \mathbb{N} \text{ such that } x_k = 0, \forall k \geq k_0 \} \subset L. \]

(ii) Assume that \((T, \Sigma) \neq (\mathbb{N}, \mathcal{P}(\mathbb{N}))\). A vector space \( L \) of weakly integrable functions in \( X^T \) is said to be decomposable if for every \( u \in L \), every weakly integrable function \( f \in X^T \) such that \( f(T) \) is relatively compact, and every set \( A \in \Sigma \) with finite measure, we have that \( f1_A + u1_{A^c} \in L \).

The specification of the decomposability above with the underlying \( \sigma \)-Algebra \((T, \Sigma)\) makes sense, since the two definitions may not coincide. For instance, if \( X = \mathbb{R} \) and \( \mu \) is a finite measure over \((\mathbb{N}, \mathcal{P}(\mathbb{N}))\), then the space \( L = c_{00}(X) \) is obviously decomposable in the sense of Definition 3.2(i), but not with respect to Definition 3.2(ii). Indeed, the decomposability of \( L \) in the sense of Definition 3.2(ii) would imply that \( \ell^\infty \subseteq L \).

We shall use the following result extensively, which characterizes the Fenchel conjugate of \( I_f \). The first part of it, corresponding to the case when \((X, X^*)\) is a dual pair of Suslin spaces, can be found in [11] Theorem VII-7. In the second part, we obtain a similar representation of the conjugate of \( I_f \) when \((T, \Sigma) = (\mathbb{N}, \mathcal{P}(\mathbb{N}))\).

**Proposition 3.3.** Let \( L(T, X) \) and \( L(T, X^*) \) be two vector spaces of weakly integrable functions from \( T \) to \( X \) and \( X^* \), resp., such that \( L(T, X) \) is decomposable and the function \( t \to \langle v(t), u(t) \rangle \) is integrable for every \((u,v) \in L(T,X) \times L(T, X^*) \). If \( f : T \times X \to \mathbb{R} \) is a normal integrand such that \( \bar{I}_f(u_0) < \infty \) for some \( u_0 \in L(T, X) \), and either \((X, X^*)\) is a dual pair of Suslin spaces, or \((T, \Sigma) = (\mathbb{N}, \mathcal{P}(\mathbb{N}))\), then for all \( v \in L(T, X^*) \)

\[
\bar{I}_f(v) = \sup_{v \in L(T, X)} \int_T (\langle u(t), v(t) \rangle - f(t, u(t)))d\mu(t),
\]

**Proof.** First, we may suppose w.l.o.g. that \( \bar{I}_f(u_0) \in \mathbb{R} \); since otherwise, \( \bar{I}_f(u_0) = -\infty \) and the conclusion holds trivially. So, the proof in the first case of Suslin spaces
Theorem 3.4. \( X \) proof because the converse inequality
\( \alpha \)
\( I \) \( \cup \{ {R} \) where
\( w \)
\( x \) and consider the sequence \((x_k) \in L(N, X)\) defined for \( n > k \) by \( x_k(n) = u_0(n) \), and for \( n \leq k \) by
\( x_k(n) := w_n \),
where \( w_n \in X \) is any vector such that \( \langle w_n, v(n) \rangle - f(n, w_n) \geq \max\{f^*(n, v(n)) - \frac{k}{n}, \delta(n) \} \) when \( f^*(n, v(n)) < +\infty \), and \( \langle w, v(n) \rangle - f(n, w) \geq \max\{k, \delta(n) \} \), otherwise. Then, for every \( k, k_0 \in \mathbb{N} \), with \( k_0 < k \),
\[
\alpha \geq \int_{n \leq k_0} \langle x_k(n), v(n) \rangle - f(n, x_k(n))d\mu(n) + \int_{n > k_0} \langle x_k(n), v(n) \rangle - f(n, x_k(n))d\mu(n)
\]
\[
\geq \int_{n \leq k_0} \langle x_k(n), v(n) \rangle - f(n, x_k(n))d\mu(n) + \int_{n > k_0} \delta(n)d\mu(n).
\]
Then, taking the limit on \( k \) we get \( \alpha \geq \int \frac{f^*(n, v(n))d\mu(n)}{k_0} + \int_{n > k_0} \delta(n)d\mu(n) \), and the inequality \( \alpha \geq \int f^*(n, v(n))d\mu(n) \) follows as \( k_0 \) goes to \( +\infty \). This finishes the proof because the converse inequality \( \alpha \leq \int f^*(n, v(n))d\mu(n) \) holds trivially. \( \Box \)

The second well-known result gives a representation of the Fenchel conjugate of \( \hat{I}_f \) in \( L^\infty(T, X) \). This result was first proved in [37] Theorem 1\] for the case \( X = \mathbb{R}^n \), and in [30] Theorem 4\] when \( X \) is an arbitrary separable reflexive Banach space.

**Theorem 3.4.** Let \( X \) be a separable reflexive Banach space, and \( f : T \times X \to \mathbb{R} \cup \{ +\infty \} \) be a normal convex integrand. Assume that the integral functional \( I_f \) defined on \( L^\infty(T, X) \) is finite at some point in \( L^\infty(T, X) \), and that \( I_f^\ast \) is finite at some point in \( L^1(T, X) \). Then the Fenchel conjugate of \( I_f^\ast \) on \( L^\infty(T, X) \) is given by, for every \( u^\ast = \ell^\ast + s^\ast \) with \( \ell^\ast \in L^\infty_w(T, X^\ast) \) and \( s^\ast \in L^{sing}(T, X) \),
\[
(\hat{I}_f)^\ast(u^\ast) = \int_{T} f^\ast(t, \ell^\ast(t))d\mu(t) + \sigma_{\text{dom}I_f}(s^\ast).
\]

A straightforward application of the above theorem gives us a representation of the subdifferential of integrand functionals. The proof can be found (for \( \varepsilon = 0 \)) in [37] Corollary 1\] for the finite-dimensional case, and in [31] Proposition 1.4.1\] for arbitrary separable reflexive Banach spaces. The proof of the general case \( \varepsilon \geq 0 \) is similar, and is given here for completeness.

**Proposition 3.5.** With the assumptions of Theorem 3.4\] for every \( u \in L^\infty(T, X) \) and \( \varepsilon \geq 0 \), one has that \( u^\ast = \ell^\ast + s^\ast \in \partial_\varepsilon I_f(u) \) (with \( \ell^\ast \in L^\infty_w(T, X^\ast) \) and \( s^\ast \in L^{sing}(T, X) \)) if and only if there exists an integrable function \( \varepsilon_1 : T \to [0, +\infty) \) and a constant \( \varepsilon_2 \geq 0 \) such that
\[
\ell^\ast(t) \in \partial_{\varepsilon_1(t)} f(t, u(t)) \text{ ae}, \ s^\ast \in N_{\text{dom}I_f}(u), \text{ and } \int_{1}^{\varepsilon_1(t)}d\mu(t) + \varepsilon_2 \leq \varepsilon.
\]
Proof. Take \( u^* = \ell^* + a^* \) in \( \partial \hat{I}_f(u) \); hence, \( u \in \text{dom} \hat{I}_f \). Then by Theorem 3.3 and the definition of \( \varepsilon \)-subdifferentials we have
\[
\int_T \left( f(t, u(t)) + f^*(t, \ell^*(t)) - \langle \ell^*(t), u(t) \rangle \right) \, d\mu(t) + \left( \sigma_{\text{dom} I_f}(s^*) - \langle s^*, u \rangle \right) \leq \varepsilon.
\]
Hence, we conclude by setting \( \varepsilon_1(t) := f(t, u(t)) + f^*(t, \ell^*(t)) - \langle \ell^*(t), u(t) \rangle \geq 0 \) and \( \varepsilon_2 := \sigma_{\text{dom} I_f}(s^*) - \langle s^*, u \rangle \geq 0 \).

The next result, also given in [37, Theorem 2], will be used in the proof of Theorem 4.1.

**Theorem 3.6.** Let \( f : T \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) be a normal convex integrand. Assume that \( \hat{u} \in L^\infty(T, \mathbb{R}^n) \), and that for some \( r > 0 \) the function \( f(\cdot, \hat{u}(\cdot) + x) \) is integrable for every \( x \) such that \( \|x\| < r \). Then there is some \( u^* \) in \( L^1(T, \mathbb{R}^n) \) such that \( u^* \in \text{dom} \hat{I}_f \). Moreover, \( \hat{I}_f \) is continuous (in the \( L^\infty(T, \mathbb{R}^n) \)-norm) at \( u \) wherever \( \|u - \hat{u}\|_\infty < r \).

The next two results deal with measurable selections in both Suslin spaces and non-separable Banach spaces.

**Proposition 3.7.** [11, Theorem III.22] Let \( S \) be a Suslin space and \( G : T \to S \) be a measurable multifunction with non-empty values. Then there exists a sequence \( (g_n) \) of \( (\Sigma, \mathcal{B}(S)) \)-measurable selections of \( G(t) \) such that \( \{g_n(t)\}_{n \geq 1} \) is dense in \( G(t) \) for every \( t \in T \).

**Proposition 3.8.** [10, Corollary 3.11] Assume that \( (T, \Sigma, \mu) \) is finite (complete measure), and assume that \( X \) is Asplund. Then every \( w^*-\)measurable multifunction \( C : T \to X^* \) with nonempty and weak*-compact values admits a \( w^* \)-measurable selection.

In the following two lemmas we consider a finite-dimensional Banach subspace \( F \) of \( X \) and its dual \( F^* \), both endowed with norms \( \| \cdot \|_F \) and \( \| \cdot \|_{F^*} \), respectively, together with a continuous linear projection \( P : X \to F \), whose adjoint mapping is denoted by \( P^* \).

**Lemma 3.9.** There exist constant \( M \geq 0 \) and neighborhood \( W \in \mathcal{N}_0 \) (depending only on \( P \) and \( F \)) such that for every integrable function \( u^*(\cdot) : T \to F^* \), the composite function \( P^* \circ u^*(\cdot) \) is integrable and satisfies \( \sigma_{W}(u^*(t) \circ P) \leq M\|u^*(t)\|_{F^*} \).

**Proof.** Since \( P : X \to F \) is a continuous linear mapping, there exist \( M \geq 0 \) and \( W \in \mathcal{N}_0 \) such that \( \|P(x)\|_F \leq M_1\sigma_{W_0}(x) \) for all \( x \in X \) and, hence,
\[
\sigma_{W}(u^*(t) \circ P) = \sup_{x \in W} \langle u^*(t), P(x) \rangle \leq M_1 \sup_{y \in B_F(0,1)} \langle u^*(t), y \rangle = M_1\|u^*(t)\|_{F^*}.
\]
We are done since the function \( P^* \circ u^*(\cdot) \) inherits the measurability from \( u^* \).

**Lemma 3.10.** Assume that both \( X \) and \( X^* \) are Suslin and let \( u^* : T \to X^* \) be a weak*-measurable function. Then the set
\[
\mathcal{S} := \{(t, x^*, y^*, v^*) \in T \times X^* \times X^* \times F^* \mid x^* + y^* + P^*(v^*) = u^*(t) \}
\]
belongs to \( \Sigma \otimes \mathcal{B}(X^* \times X^* \times F^*) \). Consequently, given measurable multifunctions \( C_1, C_2 : T \Rightarrow X^* \), \( C_3 : T \Rightarrow F^* \), the multifunction \( C : T \Rightarrow X^* \times X^* \times F^* \) defined as
\[
(x^*, y^*, z^*) \in C(t) \Leftrightarrow (x^*, y^*, z^*) \in C_1(t) \times C_2(t) \times C_3(t) \text{ and } u^*(t) = x^* + y^* + P^*(z^*),
\]
is measurable.

Proof. Consider the functions \( g(t, x^*, y^*, v^*) = x^* + y^* + P^*(v^*) - u^*(t) \) and \( h(x^*, y^*, v^*) = x^* + y^* + P^*(v^*) \), we claim that \( h \) is \((\Sigma \otimes \mathcal{B}(X^* \times X^* \times F^*), \mathcal{B}(X))\)-measurable. First assume that \( u^* \) is a simple function, that is, there exists a measurable partition of \( T, \{T_i\}_{i=1}^n \) and there are elements \( x_i^* \in X \) such that \( u(t) = \sum x_i^* 1_{T_i}(t) \), then it is easy to see that for every open set \( U \) on \( X^*, h^{-1}(U) = \bigcup_{i=1}^n 1_{T_i} \varphi^{-1}(B + x_i^*) \in \Sigma \otimes \mathcal{B}(X^* \times X^* \times F^*) \) so \( h \). Now if \( u^* \) is measurable, then by Theorem III.36 \( u^* \) is the limit of a sequence of simple functions \( u_n^* \), so considering a countable dense set \( D \) on \( X \) we can write

\[
\mathcal{G} = \bigcap_{i \in D} \bigcap_{i \geq j} \bigcap_{j \geq j} \{ (t, x^*, y^*, v^*) \mid \| x^* + y^* + P^*(v^*) - u_n^*(t), v^* \| < 1/i \},
\]

therefore \( \mathcal{G} \in \Sigma \otimes \mathcal{B}(X^* \times X^* \times F^*) \). \( \square \)

4. Characterizations via \( \varepsilon \)-subdifferentials

In this section we characterize the sudifferential of the convex function \( I_f \), by means of the \( \varepsilon \)-subdifferentials of the functions \( f(t, \cdot) \), \( t \in T \). As before, \( X \) and \( X^* \) are two lcs paired in duality, and \( f : T \times X \to \overline{\mathbb{R}} \) is a convex normal integrand with respect to \( \Sigma \otimes \mathcal{B}(X, \tau_X) \).

We start with the main result of this section.

**Theorem 4.1.** For every \( x \in X \) and \( \varepsilon \geq 0 \) we have

\[
(4.1) \quad \partial \varepsilon I_f(x) = \bigcap_{L \in \mathcal{F}(x)} \bigcup_{\ell \in \mathcal{I}(\varepsilon)} \left\{ \int_T \partial \ell(t)(f_t + \delta_{L \cap \text{dom} I_f})(x) d\mu(t) + N_{\text{dom} I_f \cap L}(x) \right\}
\]

\[
(4.2) \quad = \bigcap_{L \in \mathcal{F}(x)} \bigcup_{\ell \in \mathcal{I}(\varepsilon)} \left\{ \int_T \partial \ell(t)(f_t + \delta_{L \cap \text{dom} I_f})(x) d\mu(t) \right\},
\]

where \( \mathcal{F}(x) := \{ V \subseteq X : V \text{ is a finite-dimensional linear space and } x \in V \} \) and \( \mathcal{I}(\eta) := \{ \ell \in L^1(T, \mathbb{R}^+) : \int \ell \leq \eta \} \).

**Proof.** First suppose \( x = 0 \). Take \( x^* \in \partial \varepsilon I_f(0) \) and choose \( L \in \mathcal{L}(0) \) and define \( F = \text{span} \{ L \cap \text{dom} I_f \} = \text{span} \{ e_i \}_{i=1}^n \) (where \( e_i \in L \cap \text{dom} I_f \) is a basis of \( F \)).

Consider a continuous projection \( P : X \to F \) and \( \hat{f} : T \times F \to \overline{\mathbb{R}} \) the restriction of \( f \) to \( F \), then \( x^* \in \partial \varepsilon I_f(0) \), hence define \( y^* = x^* \circ P \) and \( N^* = x^* - y^* \).

Now \( y^* \in \partial \varepsilon (I_f + \delta_{\mathcal{F}})(0) \), \( y^* \in \partial \varepsilon (I_f + \delta_{\mathcal{F}})(0) \), and \( N^* = F_{\perp} \).

Because \( \text{dom} I_f = F \cap \text{dom} I_f \neq \emptyset \) and \( \text{span} \{ \text{dom} I_f \cap F \} = \text{span} \{ \text{dom} I_f \cap L \} = F \) is a finite-dimensional subspace we have that \( I_f \) is continuous on \( \text{ri} \text{ dom} I_f \), that is to say, there exist \( \eta > 0 \) and \( x_0 \in \text{dom} I_f \cap L \) such that \( x_0 + \eta \text{ co} \{ \pm e_i \} \subseteq \text{dom} I_f \). Hence if \( h \in F \) belongs to \( \eta \text{ co} \{ \pm e_i \} \) we have that \( f(\cdot, x_0 + h) \) is integrable, so applying Theorem 3.6 we have \( \hat{I}_f \) is continuous in a neighborhood of \( x_0 \) in \( L^\infty(T, F) \) and the hypotheses of Theorem 4.4 are satisfied. Then we can apply the composition rule to \( I_f \), so \( \partial \varepsilon I_f(0) = A^*(\partial \varepsilon I_f(0)) \), where \( A : X \to L^\infty(T, F) \) is given by \( A(h) = h1_T \) and \( A^*(u^* + v^*)(h) = \int_T (u^*(t), h) + v(h1_T) \), where \( u^* \in L^1(T, F^*) \)
and $v^* \in L^\text{sing}(T,F)$. Then there are $\alpha^* \in L^1(T,F^*)$ and $\beta^* \in L^\text{sing}(T,F)$ such that $y^*_f(h) = y^*(h) = \int_T (\alpha^*(t), h) d\mu(t) + \beta^*(h 1_T)$ for all $h \in F$. Moreover, by Proposition \ref{proposition:coincide} there exist $\varepsilon_1, \varepsilon_2 \geq 0$ and $\ell \in \mathcal{I}(\varepsilon_1)$ such that $\alpha(t) \in \partial \ell(t) f(t)(0)$ $\mu$-ae and $\sigma_{\text{dom} I_f}(\beta) \leq \varepsilon_2$, so we define $z^* \in L^1(T,X^*)$ and $\nu^* \in X^*$ by $z^*(t) = P^*(\alpha^*(t)) = \alpha^*(t) \circ P$ and $\nu^*(x) = \beta(P(x) 1_T)$ respectively.

From Lemma \ref{lemma:coincide} we get that $z^* \in L^1(T,X^*)$. Now from the fact that $\alpha(t) \in \partial \ell(t) f(t)(0)$ $\mu$-ae and the definition of $z^*$ we have $z^*(t) \in \partial \ell(t) (f(t) + \delta F)(0)$ $\mu$-ae. Since $A(\text{dom}I_f) \subseteq \text{dom} \hat{I}_f$ we conclude that $\nu^* \in N_{\text{dom}I_f \cap L}(0)$. So we get

$$x^* \in \bigcup_{\varepsilon_1 \varepsilon_2 \geq 0 \ell \in \mathcal{I}(\varepsilon_1)} \left\{ \int_T \partial \ell(t) (f(t) + \delta \text{aff}(L \cap \text{dom} I_f))(0) d\mu(t) + N^\varepsilon_{\text{dom} I_f \cap L}(0) \right\}.$$ 

Finally, using the fact that $\bigcap_{L \in \mathcal{F}(x)} \partial \ell(I_f + \delta I_f)(0) = \partial \ell(I_f)(0)$, we obtain the first equality of \ref{equation:main} for $x = 0$.

In the general case consider $g(t,y) = f(t,y + x)$ is easy to verify $\text{epi} g_t = \text{epi} f_t - (x,0)$. Then $\partial I_f(x) = \partial I_g(0)$ and

$$\partial I_g(0) = \bigcap_{L \in \mathcal{F}(0)} \bigcup_{\varepsilon_1 \varepsilon_2 \geq 0 \ell \in \mathcal{I}(\varepsilon_1)} \left\{ \int_T \partial \ell(t) (g_t + \delta \text{aff}(L \cap \text{dom} I_f))(0) d\mu(t) + N^\varepsilon_{\text{dom} I_f \cap L}(0) \right\}.$$ 

Then if we suppose that $L \in \mathcal{F}(x)$, we have $\partial \ell(t) (g_t + \delta \text{aff}(L \cap \text{dom} I_f))(0) = \partial \ell(t) (f_t + \delta \text{aff}(L \cap \text{dom} I_f))(x)$ and $N^\varepsilon_{\text{dom} I_f \cap L}(0) = N^\varepsilon_{\text{dom} I_f \cap L}(x)$.

To prove \ref{equation:general} consider $x^* \in \partial \ell f(x)$ and $L \in \mathcal{F}(x)$, then there exists $\varepsilon_1, \varepsilon_2 \geq 0$ such that $\varepsilon = \varepsilon_1 + \varepsilon_2$, integrable functions $\ell \in \mathcal{I}(\varepsilon_2)$, $y(t) \in \partial \ell(t) (f_t + \delta \text{aff}(L \cap \text{dom} I_f))(x)$ and $\nu^* \in N^\varepsilon_{\text{dom} I_f \cap L}$ such that $x^* = \int_T y(t) d\mu(t) + \nu^*$, so taking $\nu^*(t) = \mu(T)^{-1} \nu^* 1_T$ and $\ell_2(t) = \mu(T)^{-1} \varepsilon_2$ we get:

$$y^*(t) + \nu^*(t) \in \partial \ell(t) + \ell_2(t) (f_t + \delta \text{aff}(L \cap \text{dom} I_f))(x)$$

and $\int_T \ell(t) + \ell_2(t) = \varepsilon$.

Finally the right side of \ref{equation:general} is trivially included in $\partial \ell I_f(x)$. \hfill \qed

\begin{remark}
As it can be easily seen from the proof of Theorem \ref{theorem:main}, instead of assuming that $f : T \times X \to \mathbb{R} \cup \{+\infty\}$ is a normal convex integrand, it is sufficient to suppose that for every finite-dimensional subspace $F$ of $X$, the function $f|_{F} : T \times F \to \mathbb{R} \cup \{+\infty\}$ is a convex normal integrand; of course, both assumptions coincide in finite-dimensional setting, but they are not equivalent in general.

\end{remark}

\begin{remark}
It is worth mentioning that the theorem above also holds if instead of the set $\mathcal{F}(x)$ we take some subfamily of finite-dimensional $\mathcal{L} \subseteq \mathcal{F}(x)$ such that $\bigcap_{n \in \mathbb{N}} \partial \ell(I_f + \delta_{L_n})(x) = \partial \ell I_f(x)$; for example if the space $X$ is separable, or more generally, if $\text{epi} I_f$ is separable, we can take $(x_i, \alpha_i)_{i \geq 1}$ dense in $\text{epi} I_f$, so we define $L_n := \text{span}\{x, x_i\}_{i \geq 1}$ and it is easy to see that $\bigcap_{n \in \mathbb{N}} \partial \ell(I_f + \delta_{L_n})(x) = \partial \ell I_f(x)$.
\end{remark}
Remark 4.4. In Theorem 4.1 one can weaken the convexity hypothesis by assuming that, for every finite-dimensional subspace \(F \subset X\), \(f_{|F}\) is a normal integrand and \(\partial^+ I_f = \partial^+ \mu_fJ_f\).

In this case, formulae (4.1) and (4.2) change to

\[
\partial_x I_f(x) \subseteq \bigcap_{L \in \mathcal{F}(x)} \bigcup_{t \in \mathcal{I}(x)} \left\{ \int_T \partial_{t(t)} + m_{x,L}(t) \left(f_t + \delta_{\text{aff}(L \cap \text{dom}I_f)} \right)(x) d\mu(t) \right\},
\]

\[
\partial_x I_f(x) \subseteq \bigcap_{L \in \mathcal{F}(x)} \bigcup_{t \in \mathcal{I}(x)} \left\{ \int_T \partial_{t(t)} + m_{x,L}(t) \left(f_t + \delta_{\text{aff}(L \cap \text{dom}I_f)} \right) d\mu(t) \right\},
\]

respectively, where \(m_{x,L}(\cdot) := f(\cdot, x) - \overline{\omega}_L f(\cdot, x)\) is the modulus of convexity over \(F\). Moreover, we observe that if \(\partial_x I_f(x) \neq \emptyset\), then \(\int_T m_{x,L}(t) d\mu(t) \leq \varepsilon\) for all \(L \in \mathcal{F}(x)\). Indeed, take \(L \in \mathcal{F}(x)\). On the one hand, for every \(t \in T\), \(f(t, x) \geq \overline{\omega}_L f(t, \cdot, \varepsilon, \eta, \varepsilon) \geq \overline{\omega}_L f(t, x)\). On the other hand, the nonemptiness of \(\partial_x I_f(x)\) ensures that \(I_f(x) \leq \overline{\omega}_L I_f(x) + \varepsilon\), so that the hypothesis \(I_{\overline{\omega}_L} f(x) = \overline{\omega}_L I_f(x)\) leads to \(I_f(x) \leq I_{\overline{\omega}_L} f(x) + \varepsilon\). This implies that

\[
\int_T f(t, x) - \overline{\omega}_L f(t, \cdot, \varepsilon, \eta, \varepsilon) d\mu(t) \leq \varepsilon.
\]

In particular, if \(f\) is a convex normal integrand, or if \(\partial I_f(x) \neq \emptyset\), then we have \(m_{x,L}(\cdot) = 0\).

The next example justifies the use of an indicator function inside the integral symbol, even if the data function \(f_t\) is smooth and the space is finite dimensional.

Example 4.5. Consider the function \(f(x) := \frac{b}{a}x + b + \delta_{[-\eta, \eta]}(x)\) \(a, b, \eta > 0\); then we have

\[
\partial_0 f(x) = \left[ -\frac{\varepsilon}{\eta}, \frac{b}{a}, \frac{\varepsilon}{\eta}, \frac{b}{a} \right].
\]

We consider Lebesgue measure on \([0, 1]\) and the convex normal integrand \(f : [0, 1] \times \mathbb{R} \to [0, +\infty]\) given by \(f(t, x) := \frac{b(t)}{a(t)} x + b(t) + \delta_{[-\eta(t), \eta(t)]}(x)\), where \(a(t) = \eta(t) = t\) and \(b(t) = \frac{1}{\sqrt{t}} + 1\). Hence,

\[
I_f(x) = \left\{ \int_0^1 \left(1 + \frac{1}{\sqrt{t}}\right) dt \quad \text{if} \quad x = 0, \quad +\infty \right\} \text{if} \quad x \neq 0,
\]

and we obtain \(\partial_0 I_f(0) = \mathbb{R}\), while

\[
\partial_0 f_t(0) = \left[ -\frac{\varepsilon}{t}, \frac{1}{\sqrt{t}} + 1, \frac{\varepsilon}{t}, \frac{1}{\sqrt{t}} + 1 \right] = \left[ 1 - \frac{\varepsilon}{t}, \frac{1}{t^{3/2}}, \frac{1 + \varepsilon}{t}, \frac{1}{t^{3/2}} \right].
\]

Consequently the set \(\int_0^1 \partial_0 f_t(0) dt\) is empty for every \(\varepsilon < 1\).
Remark 4.6. Observe that formulae of Theorem 4.1 can be simplified if some qualification conditions (QC, for short) are in force. For instance, each one of the following conditions in [46, Theorem 2.8.3] ensures the validity of the exact sum rule \( \partial f_t(x) = \partial f_t(x) + \partial \delta_{\text{aff}(L \cap \text{dom} I_f)}(x), \ t \in T, \) giving rise to characterizations of \( \partial I_f(x) \) by means of only the \( \varepsilon \)-subdifferentials of the \( f_t \)'s:

QC(i) \( X = \mathbb{R}^n \) and \( \text{ri}(\text{dom} f_t) \cap \text{aff}(L \cap \text{dom} I_f) \neq \emptyset. \)

QC(ii) \text{Attouch-Brézis' condition; that is,} \( X \) Banach and \( \mathbb{R}_+ (\text{dom} f_t - \text{aff}(\text{dom} I_f \cap L)) \) is a closed subspace for every \( L \in F(x). \)

QC(iii) \text{Fenchel-Moreau-Rockafellar's condition; that is,} \( f_t \) is continuous at some point of \( \text{dom} I_f. \)

QC(iv) for every \( L \in F(x) \) and every \( U \in N_0 \) there exist \( \lambda > 0 \) and \( V \in N_0 \) such that \( V \cap \text{span} \{ \text{dom} f_t - \text{aff}(\text{dom} I_f \cap L) \} \subseteq \{ f_t \leq r \} \cap B - \text{aff}(\text{dom} I_f \cap L). \)

These conditions all imply the following property (see e.g. [3,4,9,13–15]), which also ensures the above sum rule:

QC(v) \( \text{For every} \ x^* \in X^* \text{and every} \ L \in F(x) \)

\[
(f + \delta_{\text{aff}(I_f \cap L)})(x^*) = \min \{ f^*(y^*) : x^* - y^* \in \text{aff}(\text{dom} I_f \cap L)^* \}
\]

Corollary 4.7. In the setting of Theorem 4.1 suppose that one of conditions [QC(i)] to [QC(v)] holds in a measurable set \( T_0 \subset T. \) Then for all \( x \in X \)

\[
\partial I_f(x) = \bigcap_{L \in F(x)} \left\{ \int_{T_0} \left( \partial f_t(x) + (\text{aff}(\text{dom} I_f \cap L - x))^\perp \right) d\mu(t) \right\}
\]

(4.4)

\[
+ \int_{T_0} \partial f_t + \delta_{\text{aff}(L \cap \text{dom} I_f)}(x)d\mu(t) + N_{L \cap \text{dom} I_f}(x).
\]

In particular, if \( T \) is finite, then

(4.5) \( \partial \left( \sum_{i \in T} f_i \right)(x) = \bigcap_{\varepsilon > 0} \left\{ \sum_{t \in T_0} \partial f_t(x) + \sum_{i \in T_0} \partial f_i(x) \right\}. \)

Proof. Equation (4.4) is direct from Theorem 4.1 and Remark 4.6 and so, we only need to prove (4.5). Fix \( x \in X, \ V \in N_0 \) and choose \( L \in F(x) \) such that \( L^\perp \subseteq V. \)

We may assume that \( \partial I_f(x) \neq \emptyset. \) By (4.1), and taking into account [22, Theorem 3.1], we have for every \( \varepsilon > 0, \)

\[
\partial \left( \sum_{i \in T} f_i \right)(x) \subseteq \sum_{t \in T_0} \partial f_t(x) + \left( \text{aff}(\text{dom} I_f \cap L - x))^\perp \right)
\]

\[
+ \sum_{i \in T_0} \partial f_i(x) + \delta_{\text{aff}(L \cap \text{dom} I_f)}(x) + N_{L \cap \text{dom} I_f}(x)
\]

\[
\subseteq \sum_{i \in T_0} \partial f_i(x) + \left( \text{aff}(\text{dom} I_f \cap L - x))^\perp \right) + \sum_{i \in T_0} \partial f_i(x)
\]

\[
+ \left( \text{aff}(\text{dom} I_f \cap L - x))^\perp + N_{L \cap \text{dom} I_f}(x) + V; \right.
\]

hence, \( \partial f_i(x) \neq \emptyset \) for all \( i \in T_0. \) But

\[
(\text{aff}(\text{dom} I_f \cap L - x))^\perp + (\text{aff}(\text{dom} I_f \cap L - x))^\perp + N_{L \cap \text{dom} I_f}(x) \subseteq N_{L \cap \text{dom} I_f}(x),
\]
and so we conclude that
\[ \partial \left( \sum_{i \in T} f_i \right)(x) \subseteq \sum_{i \in T_0} \partial f_i(x) + \sum_{i \in T_0} \partial_x f_i(x) + N_{\text{dom} I_f}(x) + V. \]

Moreover, since (see [20, Lemma 11]),
\[ N_{\text{dom} I_f \cap L}(x) = \left[ \text{cl} \left( \sum_{i \in T_0} \partial f_i(x) + \sum_{i \in T_0} \partial_x f_i(x) + L^+ \right) \right], \]
it follows that
\[ \partial \left( \sum_{i \in T} f_i \right)(x) \subseteq \sum_{i \in T_0} \partial f_i(x) + \sum_{i \in T_0} \partial_x f_i(x) + V + V, \]
which in turn implies
\[ \partial \left( \sum_{i \in T} f_i \right)(x) \subseteq \bigcap_{\varepsilon > 0, i \in T_0} \bigcap_{V \in N_{\text{dom} I_f}} \left[ \sum_{i \in T_0} \partial f_i(x) + \sum_{i \in T_0} \partial_x f_i(x) + V + V \right] \]
\[ = \bigcap_{\varepsilon > 0, i \in T_0} \text{cl} \left\{ \sum_{i \in T_0} \partial f_i(x) + \sum_{i \in T_0} \partial_x f_i(x) \right\}. \]
This yields the direct inclusion "\( \subseteq \)" and then completes the proof, since the opposite inclusion is easy.

For the importance of the finite-dimensional applications we give an explicit formulation of the finite-dimensional case.

**Corollary 4.8.** Let \( f : T \times \mathbb{R}^n \to \mathbb{R} \) be a convex normal integrand. Assume that \( \text{ri}(\text{dom} f_t) \cap \text{aff}(\text{dom} I_f) \neq \emptyset \) for almost all \( t \in T \). Then for every \( x \in \mathbb{R}^n \)

\[ (4.6) \quad \partial I_f(x) = \int_T \left( \partial f_i(x) + N_{\text{dom} I_f}(x) \right) d\mu(t). \]

**Proof.** Consider \( x \in \mathbb{R}^n \). Using Corollary 4.7 we have that
\[ \partial I_f(x) = \int_T \left( \partial f_i(x) + (\text{aff}(\text{dom} I_f - x))^{\perp} \right) d\mu(t) + N_{\text{dom} I_f}(x) \]
\[ = \int_T \left( \partial f_i(x) + (\text{aff}(\text{dom} I_f - x))^{\perp} + N_{\text{dom} I_f}(x) \right) d\mu(t) \]
\[ = \int_T \left( \partial f_i(x) + N_{\text{dom} I_f}(x) \right) d\mu(t). \]

\[ \square \]

**Remark 4.9.** It is worth to compare [25, Theorem 1] with Corollary 4.8. In [25, Theorem 1] the author assumes that \( \text{ri}(\text{dom} f_t) \cap \text{dom} I_f \neq \emptyset \) for almost every \( t \in T \) (which is equivalent to \( \text{ri}(\text{dom} f_t) \cap \text{aff}(\text{dom} I_f) \neq \emptyset \) for almost all \( t \in T \)). He claims that (4.6) can be replace by
\[ \partial I_f(x) = \int_T \partial f_i(x) d\mu(t) + N_{\text{dom} I_f}(x). \]
However, Example 4.5 shows that the above equality does not hold without any qualification condition over $I_f$.

**Example 4.10.** The main feature of the finite parameterized case given in (4.5) is that the characterization of $\partial I_f(x)$ does not involve the normal cone $N_{\text{dom } I_f \cap L}(x)$. This fact is specific to this finite case and cannot be true in general, even for smooth data functions $f_t$ with $\int_T \partial f_t(x) \, d\mu(t) \neq 0$. For example, consider the Lebesgue measure on $[0, 1]$ and the integrand $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ given by $f(t, x) = x^2/t$. Then we obtain $I_f = \delta_{[0]}$ and, so, $\partial f_t(0) = \{0\}$, while $\partial I_f(0) = \mathbb{R}$. The same example can be adapted to construct a counterexample for a countable measure over the measurable space $(\mathbb{N}, P(\mathbb{N}))$.

Next, we give another formula for the subdifferential of finite sums of convex functions, where a qualification condition involving the relative interiors is satisfied by only a part of the family $\{f_t, t \in T\}$. We need the following technical Lemma, which is an adaptation of classical techniques in finite-dimension setting. We include the proof for completeness.

**Lemma 4.11.** Let $g \in \Gamma_0(X)$ and let $L \subset X$ be a finite-dimensional affine subspace. If $g$ is continuous relative to $\text{aff}(\text{dom } g)$ at some point in $\text{dom } g \cap L$, then for every $x^* \in X^*$,

$$ (g + \delta_L)^*(x^*) = \min \{g^*(y^*) + \delta_L^*(x^* - y^*) : y^* \in X^* \}. $$

**Proof.** Because the inequality $\leq$ always holds, we only prove the opposite inequality for every given $x^* \in X^*$ with $(g + \delta_L)^*(x^*) < +\infty$. Let $x \in \text{dom } g \cap L$ be a continuity point of $g$ as in the assumption of the corollary that, up to a translation, we may suppose equal to 0. First, since $0 \in \text{dom } g \cap L$, we observe that the function $g + \delta_L$ is proper, and so $(g + \delta_L)^* = \text{cl}^{\text{w}^*}\{g^* \Gamma \delta_L^*\}$ (see [40 Corollary 2.3.5]). Then there are net $(x_{1,i})_{i \in I}$ and $(x_{2,i})_{i \in I}$ such that $w_i^* := x_{1,i}^* + x_{2,i}^* \to x^*$ and $g^*(x_{1,i}^*) + \delta_L^*(x_{2,i}) = g^*(x_{1,i}) + \delta_L(x_{2,i}) \to (g + \delta_L)^*(x^*)$; hence, $(x_{2,i})_{i \in I} \subset L^\perp$ and

$$ g^*(x_{1,i}^*) \to (g + \delta_L)^*(x^*). $$

Let $\tilde{L} \subset L$ be a finite-dimensional subspace such that $Z = W \oplus \tilde{L}$, where $W = \text{lin}(\text{dom } g)$ and $Z := \text{lin}(\text{dom } g - L)$, and denote by $P_W$ and $P_{\tilde{L}}$ some continuous projections from $Z$ to $W$ and $L$, respectively.

From the fact that $\text{dom } g \subset W$ one easily see that for every $z^* \in W^*$ and every continuous linear extension $\tilde{z}^* \in X^*$ of $z^*$,

$$ g^*(\tilde{z}) = (g|_W)^*(\tilde{z}|_W) = (f|_W)^*(z^*). $$

Because $g$ is continuous relative to $W$ at 0 we can find some $r \geq \sup\{g(0), g^*(x_{1,i}) : i \in I\}$ such that $(g \leq r)$ is a neighborhood of 0 in $W$. Then $U := P_{\tilde{L}}^{-1}(\{g \leq r\}) \cap P_{\tilde{L}}^{-1}(\mathbb{B}_{\tilde{L}})$, where $\mathbb{B}_{\tilde{L}}$ is the closed unit ball in $\tilde{L}$, is also a neighborhood of 0 relative to $Z$. Observe that for every $i \in I$ and every $u \in U$ it holds

$$ \langle x_{1,i}^*, u \rangle = \langle x_{1,i}^*, P_W(u) \rangle + \langle x_{1,i}^*, u - P_W(u) \rangle $$

$$ = \langle x_{1,i}^*, P_W(u) \rangle - g(P_W(u)) + g(P_W(u)) + \langle x_{1,i}^*, P_{\tilde{L}}(u) \rangle $$

$$ \leq g^*(x_{1,i}^*) + g(P_W(u)) + \langle x_{1,i}^*, x_{2,i}^* \rangle, P_{\tilde{L}}(u) \rangle \quad (\text{as } x_{2,i}^* \in L^\perp) $$

$$ = g^*(x_{1,i}^*) + g(P_W(u)) + \langle w_i^*, P_{\tilde{L}}(u) \rangle \leq 2r + M, $$

where $M := \sup\{g(0), g^*(x_{1,i}) : i \in I\}$.
where $M := \sup \{w_i^*: v \in B_L, i \in I\} < +\infty$. By Banach-Alaoglu-Bourbaki’s theorem [19, Theorem 3.37] there exists a subnet of $(x^*_i)_i$ such that $(x^*_i)_i \to x^*_i$ and, consequently, $(x^*_i f_j)_i \to x^*_i$, for some $x^*_1, x^*_2 \in Z^*$; hence, since $(x^*_i)_i \subset L^*$ we also have that

\begin{equation}
(x^*_i h) = 0 \text{ for all } h \in L.
\end{equation}

Moreover, since $\text{dom } g \subset Z$ we get (recall (4.7))

\begin{equation}
(g|_Z)^*(x^*_1) \leq \lim \inf_j (g|_Z)^*((x^*_2)_j) = \lim \inf_j g^*(x^*_2)_j = \lim_j g^*(x^*_2, x^*_1) = (g + \delta_L)^*(x^*_1).
\end{equation}

Finally, if $\tilde{x}_1, \tilde{x}_2 \in X^*$ are extensions of $x^*_1$ and $x^*_2$, respectively, and $\tilde{x}^* := \tilde{x}_1 + \tilde{x}_2$, then using (4.9) we get $g^*(\tilde{x}^*) \leq (g + \delta_L)^*(x^*)$.

Let $\bar{\delta}_L(x^*_2) \leq (g|_Z)^*(x^*_1) \leq (g + \delta_L)^*(x^*)$.

which finishes the proof.

**Corollary 4.12.** Assume that $T$ is finite and let $\{f_t\}_{t \in T} \subseteq \Gamma_0(X)$ and $T_0 \subseteq T$ be given. Then the following statements hold true:

(i) If $\bigcap_{t \in T_0} \text{r}_{\text{aff}(\text{dom } f_t)}(\text{dom } f_t) \cap \bigcap_{t \in T_0} \text{dom } f_t \neq \emptyset$ and each $f_t$ with $t \in T_0$ is continuous on $\text{r}_{\text{aff}(\text{dom } f_t)}(\text{dom } f_t)$. Then for every $x \in X$

\[
\partial \left(\sum_{t \in T} f_t\right)(x) = \bigcap_{\epsilon > 0} \left(\sum_{t \in T_0} \partial f_t(x) + \sum_{t \in T_0} \partial \epsilon f_t(x)\right).
\]

(ii) If $\bigcap_{t \in T_0} \text{int}(\text{dom } f_t) \cap \bigcap_{t \in T_0} \text{dom } f_t \neq \emptyset$ and each $f_t$ with $t \in T_0$ is continuous on $\text{int}(\text{dom } f_t)$. Then for every $x \in X$

\[
\partial \left(\sum_{t \in T} f_t\right)(x) = \sum_{t \in T_0} \partial f_t(x) + \bigcap_{\epsilon > 0} \left(\sum_{t \in T_0} \partial \epsilon f_t(x)\right).
\]

**Proof.** Let $f := \sum_{t \in T} f_t$. Fix $t \in T_0$ and consider $L \in F(x)$. Then, applying Lemma 4.11 with $g = f_t$ and $L = \text{aff}(\text{dom } f \cap L)$ we ensure the validity of condition 4.3 Therefore statement (i) follows by applying Corollary 4.7 Statement (ii) follows from (i) by arguing as in Lemma 4.11.

5. Suslin spaces or discrete measure space

In this section we give more sharp characterizations of the $\epsilon$-subdifferential of $I_f$ under the cases where either $X, X^*$ are Suslin spaces, or $(T, \Sigma) = (N, \mathcal{P}(N))$. These settings indeed permit the use of measurable selection theorems, which give us more control over the integration of the multifunctions $\partial_{\epsilon(t)} f_t(x)$ and $N_{\text{dom } f \cap L}(x)$. We recall that $f : T \times X \to \mathbb{R} \cup \{+\infty\}$ is a given normal convex integrand, and $(T, \Sigma, \mu)$ is a complete $\sigma$-finite measure space. The function $I_f : X \to \mathbb{R} \cup \{+\infty\}$ is defined as

\[
I_f(x) = \int_T f(t, x) d\mu(t).
\]
Then the following corollary makes sharper the characterization given in Theorem 4.1 by using only the \( \varepsilon \)-subdifferential of the \( f_i \)’s.

**Theorem 5.1.** We suppose that either \( X \) and \( X^* \) are Suslin spaces or \( (T, \Sigma) = (\mathbb{N}, \mathcal{P}(\mathbb{N})) \). Then for every \( x \in X \) and \( \varepsilon \geq 0 \) we have

\[
\partial_{\varepsilon} I_f(x) = \bigcap_{L \in \mathcal{F}(x)} \bigcup_{\varepsilon_1, \varepsilon_2, \eta \geq 0, \eta \in L^1(T, (0, +\infty))} \bigcap_{\ell \in I(\varepsilon_1)} \text{cl} \left\{ \int_T \left( \partial_{\ell(t)} + \eta(t) f_t(x) + N_{\text{dom} I_f}^2(x) \right) d\mu(t) \right\},
\]

where the closure is taking in the strong topology \( \beta(X^*, X) \).

**Proof.** We only need to prove the inclusion “\( \subseteq \)” in which we suppose that \( x = 0 \). Take \( F \in \mathcal{F}(0) \), \( \varepsilon > 0 \) and \( L := \text{span}\{F \cap \text{dom} I_f\} = \text{aff}\{F \cap \text{dom} I_f\} \), say \( L = \text{span}\{e_i\}_{i=1}^p \) with \( \{e_i\}_{i=1}^p \) being linearly independent so that \( \text{co}\{\pm e_i\}_{i=1}^p \) is the united closed ball in \( L \) with respect to a norm \( \| \cdot \|_L \) (on \( L \)). Let \( P : X \to L \) be a continuous projection, \( M \geq 0 \), and \( W \in \mathcal{N}_0 \) as in Lemma 3.9. Given \( \delta > 0 \), we pick an integrable function \( \gamma : T \to (0, +\infty) \) such that

\[
M \int_T \gamma(t) d\mu \leq \delta,
\]

and define the measurable multifunctions \( U, V : T \rightrightarrows L^* \) as

\[
U(t) := \{ x^* \in X^* : |\langle x^*, e_i \rangle | \leq \gamma(t), i = 1, \ldots, p \},
\]

\[
V(t) := \{ x^* \in L^* : |\langle x^*, e_i \rangle | \leq \gamma(t), i = 1, \ldots, p \}.
\]

Now take \( x^* \in \partial_{\varepsilon} I_f(0) \) and fix a positive measurable function \( \eta \). By formula (4.1) in Theorem 4.1 there exist \( \varepsilon_1, \varepsilon_2 \geq 0 \) with \( \varepsilon_1 + \varepsilon_2 = \varepsilon \), \( \ell \in I(\varepsilon_1) \), an integrable selection \( x_{L, \varepsilon}^* \ell(t) \) of the multifunction \( t \to \partial_{\ell(t)}(f_t + \delta_L)(0) \), and \( \lambda^* \in N_{\text{dom} I_f \cap L(0)}^2 \) such that \( x^* = \int_T x_{L, \varepsilon}^* \ell(t) d\mu(t) + \lambda^* \). Since \( U(t) \in \mathcal{N}_0 \), by [22, Theorems 3.1 and 3.2] we have that, for \( \text{ae} \ t \in T \),

\[
x_{L, \varepsilon}^* \ell(t) \in \partial_{\ell(t)}(f_t + \delta_L)(0) \subseteq \partial_{\ell(t) + \eta(t)} f_t(0) + L^\perp + U(t)
\]

\[
\subseteq \partial_{\ell(t) + \eta(t)} f_t(0) + N_{\text{dom} I_f \cap L}(0) + P^*(V(t)).
\]

We define the multifunction \( G : T \rightrightarrows X^* \times X^* \times L^* \) as

\[
(y^*, w^*, v^*) \in G(t) \Leftrightarrow \begin{cases} y^* \in \partial_{\ell(t) + \eta(t)} f(t, 0), \ w^* \in N_{\text{dom} I_f \cap L}(0), \text{ and } v^* \in V(t), \\ x_{L, \varepsilon}^* t = y^* + w^* + P^*(v^*). \end{cases}
\]

If \( X, X^* \) are Suslin spaces, then by Lemma 5.11 \( G \) is measurable, and so, by Proposition 6.7 it admits a measurable selection \( (y^* (\cdot), w^* (\cdot), v^* (\cdot)) \). This also obviously holds when \( (T, \Sigma) = (\mathbb{N}, \mathcal{P}(\mathbb{N})) \). Thus, by Lemma 3.9 the function \( u^*(t) := v^*(t) \circ P \) is integrable and we get

\[
\sigma_W(u^*(t)) \leq M \max_{i=1, \ldots, p} \langle v^*(t), e_i \rangle \leq M \gamma(t) \text{ for } \text{ae } t \in T.
\]
Consequently, the function $y^* + w^* = x^*_L,ε(\cdot) - u^*(\cdot)$ is strongly integrable and we have (recall (5.1))

$$\sigma_W(x^* - \int_T (y^*(t) + w^*(t))d\mu(t) - \lambda^*) = \sigma_W(\int_T x^*_L,ε(t)d\mu(t) - \int_T (y^*(t) + w^*(t))d\mu(t))$$

$$= \sigma_W(\int_T u^*(t)d\mu(t))$$

$$\leq \int_T \sigma_W(u^*(t))d\mu(t)$$

$$\leq M \int_T T(\cdot)d\mu \leq \delta;$$

that is,

$$x^* - \int_T (y^*(t) + w^*(t))d\mu(t) - \lambda^* \in \delta W^*,$$

and due to the arbitrariness of $\delta$ we obtain

$$x^* \in \text{cl}^{\beta(X^*,X)} \int_T \left(\partial_{t(t)+\eta(t)}f(t)\big|_{0} + N_{\text{dom}I_f \cap L(0)} \right) d\mu(t) + N_{\text{dom}I_f \cap L(0)}.$$

Finally, since $N_{\text{dom}I_f \cap L(0)} \subset \int_T N_{\text{dom}I_f \cap L(0)}d\mu(t)$ we conclude that

$$x^* \in \text{cl}^{\beta(X^*,X)} \int_T \left(\partial_{t(t)+\eta(t)}f(t)\big|_{0} + N_{\text{dom}I_f \cap L(0)} \right) d\mu(t).$$

$\square$

The next result is a finite-dimensional-like characterization of the subdifferential of $I_f$. Recall that a closed affine subspace $A \subset X$ is said to have a continuous projection if there exists an affine continuous projection from $X$ to $A$, or equivalent if there exits a continuous linear projection from $X$ to $A - x_0$, where $x_0 \in A$.

**Theorem 5.2.** Let $X$, $X^*$ and $T$ be as in Theorem 5.1. If $I_f$ is continuous on $\text{ri} (\text{dom}I_f) \neq \emptyset$ and $\text{aff} (\text{dom}I_f)$ has a continuous projection, then

$$\partial I_f(x) = \bigcap_{\eta \in L^1(T,(0,\infty))} \text{cl}^{w^*} \left\{ (w) - \int_T \left(\partial_{\eta(t)}f(t)\big|_{x} + N_{\text{dom}I_f(x)} \right) d\mu(t) \right\}.$$

**Proof.** Because the inclusion “$\subseteq$” is immediate we only need to prove the other inclusion “$\supseteq$” when $x = 0$; hence, $F : = \overline{\text{aff}(\text{dom}I_f)}$ is a closed subspace of $X$. Let $x^* \in \partial I_f(0), \eta \in L^1(T,(0,\infty))$, and $V := \{ h^* \in X^* : ||h^*, e_i|| \leq 1, \; i = 1, \ldots, p \}$ for some $\{e_i\}_{i=1}^p \subset X$. By the current assumption, we take $x_0 \in \text{ri}(\text{dom}I_f)$ and a continuous projection $P : X \to F$. Define $L = \text{span}\{e_i, P(e_i), x_0\}_{i=1}^p$ and $W(t) := \{h^* \in X^* : \max\{||h^*, e_i||, ||h^*, P(e_i)||, ||h^*, x_0||\} \leq \varepsilon(t), \; i = 1, \ldots, p\}$, where $\varepsilon(\cdot)$ is any positive integrable function with values on $(0,1)$ and $\int_T \varepsilon d\mu \leq 1/2$. Then $L^\perp + W(t) \subseteq W(t) \subseteq V$. Because $L \cap \text{ri}(\text{dom}I_f) \neq \emptyset$ we have (see, e.g., Corollary 4.12)

$$N_{\text{dom}I_f \cap L(0)} = \text{cl}^{w^*} (L^\perp + N_{\text{dom}I_f(0)}).$$

(5.3)
By Theorem 5.1 there exists a (strong) integrable selection \( y^*(t) \in \partial_{\eta(t)} f_t(0) + N_{\text{dom} f_t(0)} \) and the weak integrability of \( (5.5) \)

\[
x^* - \int_T y^* d\mu \in V.
\]

Also, by the measurability of multifunctions \( \partial_{\eta(t)} f_t(0), N_{\text{dom} f_t(0)}, \) and \( W(\cdot) \) (see, e.g., [26]), there exists a (weakly) measurable selection \( z^*(\cdot) \) of \( \partial_{\eta(t)} f_t(0) + N_{\text{dom} f_t(0)} \) such that \( y^*(t) - z^*(t) \in W(t) \) for all \( t \) (the existence of such a selection is guaranteed for Suslin spaces by the representation theorem of Castaing, while it is straightforward in the discrete case). Let us verify that the function \( z^*(\cdot) \circ P \) is weakly integrable: Given \( U \in \mathcal{N}_0 \) such that \( x_0 + P(U) \subset \text{ri}(\text{dom} f_t) \) (using the the continuity of \( P \)) we have, for every \( y \in U, \)

\[
\langle z^*(t) \circ P, y \rangle = (z^*(t), P y) \\
= (z^*(t), x_0 + P y) - \langle z^*(t), x_0 \rangle \\
\leq f(t, x_0 + P(y)) - f(t, 0) + \eta(t) - \langle z^*(t), x_0 \rangle + \sigma_{N_{\text{dom} f_t(0)}}(x_0 + P(y)) \\
\leq f(t, x_0 + P(y)) - f(t, 0) + \eta(t) + \langle z^*(t) - y^*(t), x_0 \rangle + \langle y^*(t), x_0 \rangle \\
(5.5) \\
\leq f(t, x_0 + P(y)) - f(t, 0) + \eta(t) + \varepsilon(t) + \langle y^*(t), x_0 \rangle,
\]

and the weak integrability of \( z^*(\cdot) \circ P \) follows as

\[
\int_T |\langle z^*(t) \circ P, y \rangle| d\mu(t) \leq \int_T |f(t, x_0 + P(y))| d\mu(t) \\
+ \int_T |f(t, x_0 - P(y))| d\mu(t) - I_f(0) \\
+ \int_T (\eta(t) + \varepsilon(t) + |\langle y^*(t), x_0 \rangle|) d\mu(t) < +\infty.
\]

Moreover, \( (5.5) \) implies that \( \int_T z^* \circ P d\mu \) is uniformly bounded on a neighborhood of zero, so that \( \int_T z^* \circ P d\mu \in X^* \). Finally, we have \( z^*(t) \circ P = z^*(t) + z^*(t) \circ P - z^*(t), \)

\[
z^*(t) \in \partial_{\eta(t)} f_t(0) + N_{\text{dom} f_t(0)} \) and \( z^*(t) \circ P - z^*(t) \in F^+, \)

consequently

\[
z^*(t) \circ P \in \partial_{\eta(t)} f_t(0) + N_{\text{dom} f_t(0)} + F^+ = \partial_{\eta(t)} f_t(0) + N_{\text{dom} f_t(0)}
\]

and (recall \( (5.4) \))

\[
x^* - \int_T z^* \circ P d\mu = x^* - \int_T y^* d\mu + \int_T (y^* - z^*) d\mu - \int_T (z^* \circ P - z^*) d\mu \in V + V + F^+,
\]

so that (observing that \( F^+ \subset \int_T N_{\text{dom} f_t(0)} d\mu(t) \))

\[
x^* \in (w) - \int_T \left( \partial_{\eta(t)} f_t(0) + N_{\text{dom} f_t(0)} \right) d\mu(t) + V + V + F^+ \\
\subset (w) - \int_T \left( \partial_{\eta(t)} f_t(0) + N_{\text{dom} f_t(0)} \right) d\mu(t) + V + V.
\]
Hence, by intersecting over \( V \) we get

\[
x^* \in \text{cl} w^* \left\{ (w) - \int_T (\partial_{\eta(t)} f_t(x) + N_{\text{dom} I_f}(x)) \, d\mu(t) \right\},
\]

which gives the desired inclusion due to the arbitrariness of the function \( \eta \).

\[ \square \]

**Remark 5.3.** It has not escaped our notice that under the setting of Theorem 5.2 If we assume that \( X \) is a Banach space and for a given point \( \bar{x} \) we have that there exists some sequence \( \eta_n \in L^1(T, (0, +\infty)) \) such that \( \int T \eta(t) d\mu(t) \to 0 \) and \( (w) - \int T \partial_{\eta(t)} f_t(\bar{x}) d\mu(t) \neq \emptyset \) is nonempty for every \( n \in \mathbb{N} \). Then

\[ (5.6) \quad \partial I_f(\bar{x}) = \bigcap_{n \in \mathbb{N}} \text{cl} w^* \left\{ (w) - \int T \partial_{\eta_n(t)} f_t(\bar{x}) d\mu(t) + N_{\text{dom} I_f}(\bar{x}) \right\}. \]

Indeed, it is straightforward the inclusion \( \supseteq \). Moreover, for each \( n \in \mathbb{N} \) (due to the fact that the integral of \( \partial_{\eta_n(t)} f_t(\bar{x}) \) and \( N_{\text{dom} I_f}(\bar{x}) \) are nonempty) we have that

\[ (w) - \int T \partial_{\eta_n(t)} f_t(\bar{x}) d\mu(t) \subseteq \text{cl} w^* \left\{ (w) - \int T \partial_{\eta_n(t)} f_t(\bar{x}) d\mu(t) + N_{\text{dom} I_f}(\bar{x}) \right\} \]

(see e.g. [23, Proposition 5.6], [1, Proposition 8.6.2]). Then, by (5.2) the left-hand side is included in the right-hand side.

The final result correspond to the explicit formula when the space \( X \) is finite dimensional.

**Corollary 5.4.** Let \((T, \Sigma, \mu)\) be a finite-measure space and let \( f : T \times \mathbb{R}^n \to \mathbb{R} \) be a convex normal integrand. Then, for every \( x \in \mathbb{R}^n \)

\[ (5.7) \quad \partial I_f(x) = \bigcap_{\eta > 0} \text{cl} w^* \left\{ \int T (\partial_{\eta f_t(x)} + N_{\text{dom} I_f}(x)) \, d\mu(t) \right\}. \]

**Proof.** Since the measure is finite we have that the right-hand side of (5.7) is included in \( \partial I_f(x) \). Moreover, due to the fact that the measure is finite the constant positive functions belongs to \( L^1(T, (0, +\infty)) \), then by Theorem 5.2 the subdifferential of \( I_f \) at \( x \) is included in the right-hand side of (5.7).

\[ \square \]

**Remark 5.5.** It is worth mentioning that (5.7) is very similar to the expected formulation (1.4), but with the difference that in this final formula the normal cone of the domain must be inside of the sign of integral. It is explained due to the fact that the set \( \int T \partial_{\eta f_t(x)} d\mu(t) \) could be empty, as it was shown in Example 4.5. Nonetheless, under the additional assumption that \( \int T \partial_{\eta_n f_t(x)} d\mu(t) \) is non-empty for some sequence \( \eta_n \to 0 \) we have that (1.4) holds at \( x \).
6. Conclusions

In this work, we presented general characterizations of the $\varepsilon$-subdifferential of the integral functional $I_f$, without any qualification conditions, when defined over a locally convex space (see Theorem 4.1 for the main result). However, as far as we know, when the space is a non-separable locally convex space, there is no theory about measurable selections and integration of multifunctions, but we bypass this inconvenient using intersection over the family of finite-dimensional subspaces. The use of indicator functions required in our techniques is also justified in Example 4.5. We provided simplifications under qualification conditions of the nominal data (see Corollary 4.7). Our approach allows us to give formulae for subdifferential of the sum of convex functions (see Corollaries 4.7 and 4.12). Moreover, using calculus rules for the $\varepsilon$-subdifferential and measurable selections theorems, we presented formulae, which do not involve the use of indicator functions (see Theorems 5.1 and 5.2). Finally, we included simplification of the main results in finite-dimensional setting, discussed and compared our results with the previous results from the literature (see Corollaries 5.4 and 4.8 and Remarks 4.9 and 5.5).

References

1. J.-P. Aubin and H. Frankowska, *Set-valued analysis*, Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2009, Reprint of the 1990 edition [MR1048347].
2. Robert J. Aumann, *Integrals of set-valued functions*, J. Math. Anal. Appl. 12 (1965), 1–12.
3. R. I. Boţ, S. M. Grad, and G. Wanka, *New constraint qualification and conjugate duality for composed convex optimization problems*, J. Optim. Theory Appl. 135 (2007), no. 2, 241–255.
4. Radu Ioan Boţ, Sorin-Mihai Grad, and Gert Wanka, *A new constraint qualification for the formula of the subdifferential of composed convex functions in infinite dimensional spaces*, Math. Nachr. 281 (2008), no. 8, 1088–1107.
5. J. M. Borwein and A. S. Lewis, *Convex analysis and nonlinear optimization*, second ed., CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, vol. 3, Springer, New York, 2006, Theory and examples.
6. J. M. Borwein and J. D. Vanderwerff, *Convex functions: constructions, characterizations and counterexamples*, Encyclopedia of Mathematics and its Applications, vol. 109, Cambridge University Press, Cambridge, 2010.
7. Jonathan M. Borwein and Liangjin Yao, *Legendre-type integrands and convex integral functions*, J. Convex Anal. 21 (2014), no. 1, 261–288.
8. N. Bourbaki, *Éléments de mathématique. VIII. Première partie: Les structures fondamentales de l’analyse. Livre III: Topologie générale. Chapitre IX: Utilisation des nombres réels en topologie générale*, Actualités Sci. Ind., no 1045, Hermann et Cie., Paris, 1948.
9. R. S. Burachik and V. Joyakumar, *A dual condition for the convex subdifferential sum formula with applications*, J. Convex Anal. 12 (2005), no. 2, 279–290.
10. B. Cascales, V. Kadets, and J. Rodríguez, *The Gelfand integral for multi-valued functions*, J. Convex Anal. 18 (2011), no. 3, 873–895.
11. C. Castaing and M. Valadier, *Convex analysis and measurable functions*, Lecture Notes in Mathematics, Vol. 580, Springer-Verlag, Berlin-New York, 1977.
12. Ch. Castaing, *Sur les multi-applications mesurables*, Rev. Française Informat. Recherche Opérationnel 1 (1967), no. 1, 91–126.
13. C. Corneli, M. L. Lázaro, and L. Thibault, *On subdifferential calculus for convex functions defined on locally convex spaces*, Ann. Sci. Math. Québec 23 (1999), no. 1, 23–36.
14. R. Correa, A. Hantoute, and M. A. López, *Towards supremum-sum subdifferential calculus free of qualification conditions*, SIAM J. Optim. 26 (2016), no. 4, 2219–2234.
15. R. Correa, A. Hantoute, and P. Pérez-Aros, *Complete characterization of the subdifferential of convex integral functionals II: Qualification conditions, conjugate and sequential formulæ*, (2018), submitted.
17. R. Correa, A. Hantoute, and P. Pérez-Aros, *Sequential and exact formulae for the subdifferential of nonconvex integral functionals*, ArXiv e-prints (2018).

18. J. Diestel and J. J. Uhl, Jr., *Vector measures*, American Mathematical Society, Providence, R.I., 1977, With a foreword by B. J. Pettis, Mathematical Surveys, No. 15.

19. M. Fabian, P. Habala, P. Hájek, V. Montesinos, and V. Zizler, *Banach space theory*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, Springer, New York, 2011, The basis for linear and nonlinear analysis.

20. A. Hantoute, M. A. López, and C. Zălinescu, *Subdifferential calculus rules in convex analysis: a unifying approach via pointwise supremum functions*, SIAM J. Optim. 19 (2008), no. 2, 863–882.

21. J.-B. Hiriart-Urruty and C. Lemaréchal, *Fundamentals of convex analysis*, Grundlehren Text Editions, Springer-Verlag, Berlin, 2001, Abridged version of it Convex analysis and minimization algorithms. I [Springer, Berlin, 1993; MR1261420 (95m:90001)] and it II [ibid.; MR1295240 (95m:90002)].

22. J.-B. Hiriart-Urruty, M. Moussaoui, A. Seeger, and M. Volle, *Subdifferential calculus without qualification conditions, using approximate subdifferentials: a survey*, J. Convex Anal. 25 (2018), no. 2, 643–673.

23. S. Hu and N. S. Papageorgiou, *Handbook of multivalued analysis. Vol. I*, Mathematics and its Applications, vol. 419, Kluwer Academic Publishers, Dordrecht, 1997, Theory.

24. A. D. Ioffe, *Survey of measurable selection theorems: Russian literature supplement*, SIAM J. Control Optim. 16 (1978), no. 5, 728–732.

25. A. D. Ioffe and V. M. Tikhomirov, *Duality of convex functions and extremum problems*, Russian Mathematical Surveys 23 (1968), no. 6, 53.

26. A. Ionescu Tulcea and C. Ionescu Tulcea, *Topics in the theory of lifting*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 48, Springer-Verlag New York Inc., New York, 1969.

27. P.-J. Laurent, *Approximation et optimisation*, Hermann, Paris, 1972.

28. V. L. Levin, *Convex integral functionals and the theory of lifting*, Russian Mathematical Surveys 30 (1975), no. 2, 119.

29. B. S. Mordukhovich and N. Sagara, *Subdifferentials of nonconvex integral functionals in banach spaces with applications to stochastic dynamic programming*, J. Convex Anal. 25 (2018), no. 2, 643–673.

30. P. P. Mörtz, *Subdifferential formulae for the supremum of an arbitrary family of functions*, ArXiv e-prints (2017).

31. R. T. Rockafellar, *Convex integral functionals and duality*, Contributions to nonlinear functional analysis (Proc. Sympos., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1971), Academic Press, New York, 1971, pp. 215–236.

32. L. Schwartz, *Radon measures on arbitrary topological spaces and cylindrical measures*, Published for the Tata Institute of Fundamental Research, Bombay by Oxford University Press, London, 1973, Tata Institute of Fundamental Research Studies in Mathematics, No. 6.
43. C. Vallée and C. Zălinescu, *Series of convex functions: subdifferential, conjugate and applications to entropy minimization*, J. Convex Anal. 23 (2016), no. 4, 1137–1160.

44. D. H. Wagner, *Survey of measurable selection theorems*, SIAM J. Control Optimization 15 (1977), no. 5, 859–903.

45. ——, *Survey of measurable selection theorems: an update*, Measure theory, Oberwolfach 1979 (Proc. Conf., Oberwolfach, 1979), Lecture Notes in Math., vol. 794, Springer, Berlin-New York, 1980, pp. 176–219.

46. C. Zălinescu, *Convex analysis in general vector spaces*, World Scientific Publishing Co., Inc., River Edge, NJ, 2002.

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