Comparing symbolic powers of edge ideals of weighted oriented graphs

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Abstract
Let \( D \) be a weighted oriented graph and \( I(D) \) be its edge ideal. If \( D \) contains an induced odd cycle of length \( 2n+1 \), under certain condition, we show that \( I(D)^{n+1} \neq I(D)^n \). We give necessary and sufficient condition for the equality of ordinary and symbolic powers of edge ideals of weighted oriented graphs having each edge in some induced odd cycle of it. We characterize the weighted naturally oriented unicyclic graphs with unique odd cycles and weighted naturally oriented even cycles for the equality of ordinary and symbolic powers of their edge ideals. Let \( D' \) be the weighted oriented graph obtained from \( D \) after replacing the weights of vertices with non-trivial weights which are sinks, by trivial weights. We show that the symbolic powers of \( I(D) \) and \( I(D') \) behave in a similar way. Finally, if \( D \) is any weighted oriented star graph, we prove that \( I(D)^{(s)} = I(D)^s \) for all \( s \geq 2 \).

Keywords Weighted oriented graph · Sink vertex · Edge ideal · Symbolic power · Induced odd cycle · Even cycle · Star graph

Mathematics Subject Classification Primary: 05C22 · 05C25 · 05C38 · Secondary: 05E40

1 Introduction
Let \( k \) be a field and \( R = k[x_1, \ldots, x_n] \) be a polynomial ring in \( n \) variables. Let \( I \) be a homogeneous ideal of \( R \). Then for \( s \geq 1 \), the \( s \)-th symbolic power of \( I \) is defined as
$I^{(s)} = \bigcap_{P \in \text{Ass } I} (I^s R_P \cap R)$. Geometrically, the symbolic powers are important since they capture, all the polynomials that vanish with a given multiplicity. We refer [4] to the reader to analyze the background results of symbolic powers of ideals. By definition it is clear that $I^s \subseteq I^{(s)}$ for all $s \geq 1$ but the reverse containment may fail. It is always an interesting problem to find the necessary and sufficient condition for holding the reverse containment. There is no such criteria to be known when the equality $I^s = I^{(s)}$ holds for any arbitrary ideal. But for certain classes of ideals such as prime or radical ideals, there are equivalent conditions given by Hochster in [6] and by Li and Swanson in [7]. In this paper we compare the ordinary and symbolic powers of edge ideals of weighted oriented graphs.

Let $D = (V(D), E(D), w)$ be a weighted oriented graph with the vertex set $V(D) = \{x_1, \ldots, x_n\}$, the edge set $E(D)$ consists of ordered pairs of the form $(x_i, x_j)$ which represents a directed edge from the vertex $x_i$ to the vertex $x_j$ and the weight function $w : V(D) \to \mathbb{N}$. The weight of a vertex $x_i \in V(D)$ is $w(x_i)$ denoted by $w_i$ or $w_{x_i}$. If a vertex $x_i$ of $D$ is a source (i.e., has only arrows leaving $x_i$), we set $w_i = 1$. This edge ideal of $D$ is denoted by $I(D)$ and is defined as $I(D) = (x_i x_j \mid (x_i, x_j) \in E(D))$. This ideal was introduced in [5, 9]. Let $G = (V(G), E(G))$ be the underlying graph of $D$ whose vertex set is $V(G) = V(D)$ and edge set is $E(G) = \{(x_i, x_j) \mid (x_i, x_j) \in E(D)\}$. The edge ideal of $G$ is $I(G) = (x_i x_j \mid \{x_i, x_j\} \in E(G))$.

In general, even for monomial ideals comparison of ordinary and symbolic powers is a difficult problem. For simple graphs, by [10, Theorem 5.9], we know that all the ordinary and symbolic powers of edge ideal coincide if and only if the graph is bipartite. But there is no such result for weighted oriented graphs. As the edge ideal of a weighted oriented complete graphs and weighted oriented complete bipartite graphs for the equality of ordinary and symbolic powers of their edge ideals (see Theorem 3.11, Corollary 3.19 and Proposition 3.20, respectively).

Let $D'$ be the weighted oriented graph obtained from $D$ after replacing the weights of vertices with non-trivial weights which are sinks, by trivial weights. If we assume all vertices of $D$ with non-trivial weights are sinks, then $D' = G$ and in [8], we proved that the symbolic powers of $I(D)$ and $I(G)$ behave in a similar way. In this paper, even if we do not assume all vertices of $D$ with non-trivial weights are sinks, we show that
the symbolic powers of \( I(D) \) and \( I(D') \) behave in a similar way (see Theorem 4.4). We prove that the symbolic defects of edge ideals of \( D \) and \( D' \) are the same in Proposition 4.5 and so we have \( I(D')^{(s)} = I(D)^{s} \) if and only if \( I(D)^{(s)} = I(D)^{s} \) for each \( s \geq 1 \). As an instant application of this result, we characterize the weighted oriented even cycles of length 4 which are not naturally oriented for the equality of all the ordinary and symbolic powers of their edge ideals (see Proposition 4.10). Finally in Theorem 4.12, we prove the equality of ordinary and symbolic powers of edge ideal of any weighted oriented star graph.

2 Preliminaries

In this section, we recall some definitions and results for the weighted oriented graphs.

Definition 2.1 A vertex cover \( C \) of \( D \) is a subset of \( V(D) \) such that if \( (x, y) \in E(D) \), then \( x \in C \) or \( y \in C \). A vertex cover \( C \) of \( D \) is minimal if each proper subset of \( C \) is not a vertex cover of \( D \). We set \((C)\) to be the ideal generated by the vertices of \( C \).

Definition 2.2 Let \( x \) be a vertex of a weighted oriented graph \( D \), then the sets \( N^{+}_{D}(x) = \{y \mid (x, y) \in E(D)\} \) and \( N^{-}_{D}(x) = \{y \mid (y, x) \in E(D)\} \) are called the out-neighbourhood and the in-neighbourhood of \( x \), respectively. Moreover, the neighbourhood of \( x \) is the set \( N_{D}(x) = N^{+}_{D}(x) \cup N^{-}_{D}(x) \). For a subset of the vertices \( W \subseteq V(D) \), we define \( N^{+}_{D}(W), N^{-}_{D}(W) \) and \( N_{D}(W) \) similarly. Define \( \deg_{D}(x) = |N_{D}(x)| \) for \( x \in V(D) \). A vertex \( x \in V(D) \) is called a source vertex if \( N_{D}(x) = N^{+}_{D}(x) \). A vertex \( x \in V(D) \) is called a sink vertex if \( N_{D}(x) = N^{-}_{D}(x) \). We set \( V^{+}(D) \) as the set of vertices of \( D \) with non-trivial weights.

Definition 2.3 For \( T \subseteq V(D) \), we define the induced digraph \( D_{T} = (V(D_{T}), E(D_{T}), w) \) of \( D \) on \( T \) to be the weighted oriented graph such that \( V(D_{T}) = T \) and for any \( u, v \in V(D_{T}) \), \( (u, v) \in E(D_{T}) \) if and only if \( (u, v) \in E(D) \). Here \( D_{T} = (V(D_{T}), E(D_{T}), w) \) is a weighted oriented graph with the same orientation as in \( D \) and for any \( u \in V(D_{T}) \), if \( u \) is not a source in \( D_{T} \), then its weight equals to the weight of \( u \) in \( D \), otherwise, its weight in \( D_{T} \) is 1. For a subset \( W \subseteq V(D) \) of the vertices in \( D \), define \( D \setminus W \) to be the induced digraph of \( D \) with the vertices in \( W \) (and their incident edges) deleted.

Definition 2.4 [9, Definition 4] Let \( C \) be a vertex cover of a weighted oriented graph \( D \). We define

\[
\begin{align*}
L_{1}^{D}(C) &= \{x \in C \mid N^{+}_{D}(x) \cap C^{c} \neq \phi\}, \\
L_{2}^{D}(C) &= \{x \in C \mid x \notin L_{1}^{D}(C) \text{ and } N^{-}_{D}(x) \cap C^{c} \neq \phi\} \text{ and} \\
L_{3}^{D}(C) &= C \setminus (L_{1}^{D}(C) \cup L_{2}^{D}(C))
\end{align*}
\]

where \( C^{c} \) is the complement of \( C \), i.e., \( C^{c} = V(D) \setminus C \).

Lemma 2.5 [9, Proposition 6] Let \( C \) be a vertex cover of \( D \). Then \( L_{3}^{D}(C) = \phi \) if and only if \( C \) is a minimal vertex cover of \( D \).
Lemma 2.6 [9, Proposition 5] If $C$ is a vertex cover of $D$, then $L^D_3(C) = \{x \in C \mid N_D(x) \subset C\}$.

Definition 2.7 [9, Definition 7] A vertex cover $C$ of $D$ is strong if for each $x \in L^D_3(C)$ there is $(y, x) \in E(D)$ such that $y \in L^D_2(C) \cup L^D_3(C)$ with $y \in V^+(D)$ (i.e., $w(y) \neq 1$).

Remark 2.8 [9, Remark 8, Proposition 5] A vertex cover $C$ of $D$ is strong if and only if for each $x \in L^D_3(C)$, we have $N_D^-(x) \cap V^+(D) \cap [C \setminus L^D_1(C)] \neq \phi$.

Lemma 2.9 [9, Corollary 9] If $C$ is a minimal vertex cover of $D$, then $C$ is strong.

Definition 2.10 A strong vertex cover $C$ of $D$ is said to be a maximal strong vertex cover of $D$ if it is not contained in any other strong vertex cover of $D$.

Definition 2.11 [9, Definition 32] A weighted oriented graph $D$ has the minimal-strong property if each strong vertex cover is a minimal vertex cover.

Lemma 2.12 [9, Lemma 47] Let $D$ be a weighted oriented graph. Then $I(D) \subseteq I_C$, for each vertex cover $C$ of $D$.

The next lemma describes the irreducible decomposition of the edge ideal of a weighted oriented graph $D$.

Lemma 2.15 [9, Theorem 25, Remark 26] Let $D$ be a weighted oriented graph and $C_1, \ldots, C_s$ are the strong vertex covers of $D$, then the irredundant irreducible decomposition of $I(D)$ is

$$I(D) = I_{C_1} \cap \cdots \cap I_{C_s}$$

where each $I_{C_i} = (L^D_1(C_i) \cup \{x^{w(x_j)}_j \mid x_j \in L^D_2(C_i) \cup L^D_3(C_i)\})$, $\text{rad}(I_{C_i}) = P_i = (C_i)$.

Corollary 2.16 [9, Remark 26] Let $D$ be a weighted oriented graph. Then $P$ is an associated prime of $I(D)$ if and only if $P = (C)$ for some strong vertex cover $C$ of $D$.

Let $I \subseteq R$ and $I = Q_1 \cap \cdots \cap Q_m$ be a primary decomposition of ideal $I$. For $P \in \text{Ass}(R/I)$, we denote $Q_{\leq P}$ to be the intersection of all $Q_i$ with $\sqrt{Q_i} \subseteq P$. If $C$ is a strong vertex cover of a weighted oriented graph $D$, then $(C) \in \text{Ass}(R/I(D))$. We denote $I_{\leq C}$ as $I_{(C)}$. In the following lemma, we write the [3, Theorem 3.7] for edge ideals of weighted oriented graphs.
Lemma 2.17 [3, Theorem 3.7] Let $I$ be the edge ideal of a weighted oriented graph $D$ and $C_1, \ldots, C_r$ are the maximal strong vertex covers of $D$. Then

$$I^{(s)} = (I \subseteq C_1)^s \cap \cdots \cap (I \subseteq C_r)^s.$$ 

Lemma 2.18 [8, Lemma 3.1] Let $D$ be a weighted oriented graph. If $V(D)$ is a strong vertex cover of $D$, then $I(D)^{(s)} = I(D)^s$ for all $s \geq 2$.

Lemma 2.19 [2, Lemma 3.1] Let $D$ be a weighted oriented graph. Then $V(D)$ is a strong vertex cover if and only if $N^+_D(V^+(D)) = V(D)$.

Lemma 2.20 [8, Corollary 3.8] Let $D$ be a weighted oriented bipartite graph where the vertices of $V^+(D)$ are sinks. Then $I(D)^{(s)} = I(D)^s$ for all $s \geq 2$.

Definition 2.21 Let $I \subset R$ be a monomial ideal. Let $G(I)$ be the set of minimal generators of the ideal $I$. Let $J$ be the ideal we need to add to $I^s$ to achieve $I^{(s)}$, i.e., $I^{(s)} = I^s + J$. We set $\text{sdefect}(I, s)$ equal to the number of elements of $G(J)$.

The following lemma is based on the extension of ideals.

Lemma 2.22 [1, Exercise 1.18] Let $f : A \longrightarrow B$ be a ring homomorphism. Let $I$ be an ideal of $A$. The extension $I^e$ of $I$ is the ideal $Bf(I)$ generated by $f(I)$ in $B$. If $I_1$ and $I_2$ are ideals of $A$, then

(a) $(I_1 \cap I_2)^e \subseteq (I_1)^e \cap (I_2)^e$,

(b) $(I_1 I_2)^e = (I_1)^e (I_2)^e$.

Notation 2.23 Let $g \in k[x_1, \ldots, x_n]$ be a monomial. We define support of $g = \{x_i : x_i | g\}$ and we denote it by $\text{supp}(g)$.

3 Comparing ordinary and symbolic powers of weighted oriented graphs

In this section, we compare the ordinary and symbolic powers of edge ideals of weighted oriented graphs containing induced odd cycles and weighted naturally oriented even cycles.

In [4, Proposition 4.10], if a simple graph contains an induced odd cycle $C_{2n+1} = (x_1, \ldots, x_{2n+1})$, the authors have shown that the $(n+1)$-th ordinary and symbolic power of its edge ideal are different. In this paper we extend this result for weighted oriented graphs under certain condition.

Proposition 3.1 Let $D$ be a weighted oriented graph. Let $D'$ be an induced odd cycle with underlying graph $C_{2n+1} = (x_1, \ldots, x_{2n+1})$ where $V(C_{2n+1}) \not\subseteq N^+_D(V^+(D))$ and it satisfies the condition “$V(C_{2n+1}) \setminus N^+_D(V^+(D))$ contains one vertex which is not source in $D'$, otherwise, it contains a vertex which is source in $D'$ with trivial weight in $D'$”. Then $I(D)^{(n+1)} \neq I(D)^{n+1}$. 

Proof Let \( w_i = w(x_i) \) for \( x_i \in V(C_{2n+1}) \). Let \( f = x_1^{a_1} \cdots x_{2n+1}^{a_{2n+1}} \) where each \( a_i = w_i \) if \( N_D^+(x_i) \neq \emptyset \) (i.e., \( x_i \) is not source in \( D' \)) and \( a_i = 1 \) if \( N_D^-(x_i) = \emptyset \) (i.e., \( x_i \) is source in \( D' \)). We claim that \( f \in I(D)^{(n+1)} \setminus I(D)^{n+1} \). We set \( m_{[x_i, x_j]} \) as the minimal generator of \( I(D') \) corresponding to the edge \( \{x_i, x_j\} \in E(C_{2n+1}) \). Note that \( m_{[x_i, x_j]} \) can be \( x_i x_j \) or \( x_j x_i^{a_i} \) and \( x_i x_j \) is multiple of \( m_{[x_i, x_j]} \). Let \( u \in V(C_{2n+1}) \) be the vertex which is not source in \( D' \) and \( u \) is source in \( D' \), then its weight is 1 in \( D' \). Without loss of generality we can assume that \( u = x_1 \). Here \( N_D'(x_1) = \{x_2, x_{2n+1}\} \). Let \( C \) be a maximal strong vertex cover of \( D \). Suppose \( x_1 \notin C \). Then for any strong vertex cover \( C' \subsetneq C \), \( x_1 \notin C' \) and so \( x_2 \) and \( x_{2n+1} \) \( \in C' \). If \( (x_2, x_1) \in E(D'), \) then for each strong vertex cover \( C' \subsetneq C, x_1 \in N_D'^+(x_2) \cap C'^c \) and hence \( x_2 \in L_D'(C') \). This implies \( x_2 \in I_{\leq C} \). If \( (x_1, x_2) \in E(D'), \) then \( x_2^{a_2} = x_2^{u_2} \in I_{\leq C} \). In both cases \( x_2^{a_2} \in I_{\leq C} \). By the same argument we can show \( x_{2n+1}^{a_{2n+1}} \in I_{\leq C} \). By Lemma 2.14, \( m_{[x_3, x_4]}, \ldots, m_{[x_{2n-1}, x_{2n}]} \in I_{\leq C} \). So \( x_2^{a_2} \cdot x_{2n+1}^{a_{2n+1}} \cdot m_{[x_3, x_4]} \cdot \ldots \cdot m_{[x_{2n-1}, x_{2n}]} \in (I_{\leq C})^{n+1} \). Hence \( f = x_1^{a_1} \cdots x_{2n+1}^{a_{2n+1}} \in (I_{\leq C})^{n+1} \). Suppose \( u_1 \in C \). Since \( x_1 \notin N_D^+(V(D)) \), by Remark 2.8, we have \( x_1 \notin L_D'(C) \). Thus by Lemma 2.6, at least one element of \( N_D(x_1) \) does not belong to \( C \). Then for any strong vertex cover \( C' \subsetneq C \), at least one element of \( N_D(x_1) \) does not belong to \( C' \) and \( x_1 \) \( \in C' \). If \( x_1 \) is not source in \( D' \), \( x_1^{a_1} = x_1^{u_1} \in I_{\leq C} \). If \( x_1 \) is source in \( D' \), by our assumption \( u_1 = 1 \) in \( D' \) and \( x_1 \in I_{\leq C} \). In both cases \( x_1^{a_1} \in I_{\leq C} \). By Lemma 2.14, \( m_{[x_2, x_3]}, \ldots, m_{[x_{2n-1}, x_{2n}]} \in I_{\leq C} \). So \( x_1^{a_1} \cdot m_{[x_2, x_3]} \cdot \ldots \cdot m_{[x_{2n-1}, x_{2n}]} \in (I_{\leq C})^{n+1} \). Hence \( f = x_1^{a_1} \cdots x_{2n+1}^{a_{2n+1}} \in (I_{\leq C})^{n+1} \). Similarly for any maximal strong vertex cover \( C \) of \( D \), we can show \( f = x_1^{a_1} \cdots x_{2n+1}^{a_{2n+1}} \in (I_{\leq C})^{n+1} \). Hence \( f \in I(D)^{(n+1)} \). It remains to show that \( f \notin I(D)^{n+1} \). Since \( \text{supp}(f) = V(D') \) and \( D' \) is an induced digraph of \( D \), it is enough to show \( f \notin I(D')^{n+1} \). Here \( \text{supp}(f) = 2n + 1 \). Thus if we want to express \( f \) as a multiple of product of some \( n + 1 \) minimal generators of \( I(D') \), then one \( x_i^{a_i} \) of \( f \) must involve in two minimal generators of \( I(D') \). But by definition of \( a_i \), any \( x_i^{a_i} \) of \( f \) cannot involve in two minimal generators of \( I(D') \). Therefore \( f = x_1^{a_1} \cdots x_{2n+1}^{a_{2n+1}} \notin I(D')^{n+1} \). Thus \( f \in I(D)^{(n+1)} \setminus I(D)^{n+1} \). Hence the proof follows.

Remark 3.2 The above proposition may not be true if we remove the given condition.

For example consider the weighted oriented graph \( D \) as in Fig. 1. Then \( I(D) = (x_1x_2, x_2x_3^3, x_1x_3^3, x_4x_1^3) \). Here \( D \) contains an induced cycle \( D' \) of length 3 where \( x_3 \in N_D^+(V(D)), x_3 \in N_D^+(V(D)) \) and \( x_1 \in V(D') \setminus N_D^+(V(D)) \). Note that \( x_1 \) is source in \( D' \) but \( w(x_1) \neq 1 \) in \( D \). Using Macaulay 2, we see that \( I(D)^{(2)} = I(D)^2 \).
Now we see some applications of Proposition 3.1 to weighted oriented graphs containing induced odd cycles.

**Theorem 3.3** Let $D$ be a weighted oriented graph such that each edge of $D$ lies in some induced odd cycle of it. Then $V(D)$ is a strong vertex cover of $D$ if and only if $I(D)^{(s)} = I(D)^s$ for all $s \geq 2$.

**Proof** If $V(D)$ is a strong vertex cover of $D$, then by Lemma 2.18, $I(D)^{(s)} = I(D)^s$ for all $s \geq 2$.

Assume that $I(D)^{(s)} = I(D)^s$ for all $s \geq 2$. Suppose $V(D)$ is not a strong vertex cover. By Lemma 2.19, we have $N_D^+(V(D)) \neq V(D)$. Let $u \in V(D) \setminus N_D^+(V(D))$.

**Case (1)** Suppose $u$ is not source in $D$.
Since $u$ is not source in $D$, there exists some induced odd cycle $D'$ of $D$ such that $u$ is not source in $D'$. Here $u \in V(D') \setminus N_D^+(V(D))$. If $|V(D')| = 2m + 1$ for some $m$, then by Proposition 3.1, we get $I(D)^{(m+1)} \neq I(D)^{m+1}$, which is a contradiction.

**Case (2)** Suppose $u$ is source in $D$.
Then consider any induced odd cycle $D''$ of $D$ containing the vertex $u$. Here $u$ is source in $D''$ and $w(u) = 1$ in $D$. Here $u \in V(D'') \setminus N_D^+(V(D))$. If $|V(D'')| = 2k + 1$ for some $k$, then by Proposition 3.1, we get $I(D)^{(k+1)} \neq I(D)^{k+1}$, which is a contradiction.

The following result is an immediate consequence of the above result.

**Corollary 3.4** Let $D$ be a weighted oriented odd cycle. Then $V(D)$ is a strong vertex cover of $D$ if and only if $I(D)^{(s)} = I(D)^s$ for all $s \geq 2$.

**Corollary 3.5** Let $D$ be a weighted oriented graph whose underlying graph $G$ is a clique sum of finite number of odd cycles and complete graphs. Then $V(D)$ is a strong vertex cover of $D$ if and only if $I(D)^{(s)} = I(D)^s$ for all $s \geq 2$.

**Proof** Since $G$ is a clique sum of finite number of odd cycles and complete graphs, each edge of $G$ lies in some complete graph or odd cycle. In a complete graph, any three vertices form a triangle. Thus each edge of complete graph lies in some induced odd cycle of length 3. This implies each edge of $D$ lies in some induced odd cycle of it. Hence the proof follows from Theorem 3.3.

**Corollary 3.6** Let $D$ be a weighted oriented graph whose underlying graph $G$ is a complete $m$–partite graph for some $m \geq 3$. Then $V(D)$ is a strong vertex cover of $D$ if and only if $I(D)^{(s)} = I(D)^s$ for all $s \geq 2$.

**Proof** Note that each edge of $G$ lies in some induced odd cycle of length 3. Therefore each edge of $D$ lies in some induced odd cycle of it. Hence the proof follows from Theorem 3.3.

In the next result we see that presence of certain induced weighted naturally oriented path guarantees the failure in the equality of 3rd ordinary and symbolic power of edge ideal of weighted oriented graph.
Definition 3.7 A path is naturally oriented if all edges of path are oriented in one direction.

Lemma 3.8 Let $D$ be a weighted oriented graph such that at most one edge is oriented into each vertex. Let $D'$ be an induced weighted naturally oriented path of length 3 of $D$ with $V(D') = \{x_{i-1}, x_i, x_{i+1}, x_{i+2}\}$, $E(D') = \{(x_i, x_{j+1}) | i - 1 \leq j \leq i + 1\}$, $w(x_i) \geq 2$ and $w(x_{i+1}) = 1$. Then $I(D)^3 \neq I(D)^3$.

Proof Let $w_j = w(x_j)$ for $x_j \in V(D')$. We claim $g = x_{i-1}x_i^{w_i}x_{i+1}^{w_{i+1}}x_{i+2}^{w_{i+2}} \in I(D)^3$. Let $C$ be a maximal strong vertex cover of $D$. Suppose $x_{i+2} \notin C$. Then for any strong vertex cover $C' \subseteq C$, $x_{i+2} \notin C'$ and so $x_{i+1} \in C'$. Since $(x_{i+1}, x_{i+2}) \in E(D)$, for each strong vertex cover $C' \subseteq C$, $x_{i+2} \in N_D^+(x_{i+1}) \cap C'^c$ and hence $x_{i+1} \in L_D^+(C')$. This implies $x_{i+1} \in I_{\subseteq C}$. By Lemma 2.14, $x_{i-1}x_i^{w_i}I_{\subseteq C}$. So $x_{i-1}x_i^{w_i}(x_{i+1})^2 \in (I_{\subseteq C})^3$. Hence $g \in (I_{\subseteq C})^3$. Suppose $x_{i+2} \in C$. By definition of $D$, $|N_D^-(x_{i+2})| = 1$. Since $N_D^-(x_{i+2}) = \{x_{i+1}\} \notin V^+(D)$, by Remark 2.8, we have $x_{i+2} \notin L_D^+(C)$. Thus by Lemma 2.6, at least one element of $N_D^-(x_{i+2})$ does not belong to $C$. Then for any strong vertex cover $C' \subseteq C$, at least one element of $N_D(x_{i+2})$ does not belong to $C'$ and so $x_{i+2} \notin C'$. This implies $x_{i+2}^{w_{i+2}} \in I_{\subseteq C}$. By Lemma 2.14, $x_{i+1}x_{i+2} \in I_{\subseteq C}$. So $(x_{i+1})^2x_{i+2}^{w_{i+2}} \in (I_{\subseteq C})^3$. Hence $g \in (I_{\subseteq C})^3$. Similarly for any maximal strong vertex cover $C$ of $D$, we can prove that $g \notin (I_{\subseteq C})^3$. Therefore $g \in I(D)^3$. Notice that $g \notin I(D)^3$. Since $\text{supp}(g) = V(D')$ and $D'$ is an induced digraph of $D$, $g \notin I(D)^3$. Hence the result follows. □

Next we see some applications of Lemma 3.8 to some weighted oriented graphs.

Definition 3.9 A cycle is naturally oriented if all edges of cycle are oriented in clockwise direction. In a naturally oriented unicyclic graph, the cycle is naturally oriented and each edge of the tree connected with the cycle is oriented away from the cycle.

Remark 3.10 Let $D$ be a weighted naturally oriented unicyclic graph. By [9, Proposition 15], $V(D)$ is a strong vertex cover of $D$ if and only if $D$ is naturally oriented and $w(x) \geq 2$ when $\deg_D(x) \geq 2$ for all $x \in V(D)$.

In the next result, we characterize the weighted naturally oriented unicyclic graphs with a unique odd cycles for the equality of ordinary and symbolic powers of their edge ideals.

Theorem 3.11 Let $D$ be a weighted naturally oriented unicyclic graph with a unique odd cycle $C_{2n+1} = (x_1, \ldots, x_{2n+1})$. Then $I(D)^{(s)} = I(D)^{s}$ for all $s \geq 2$ if and only if $w(x) \geq 2$ when $\deg_D(x) \geq 2$ for all $x \in V(D)$.

Proof If $w(x) \geq 2$ when $\deg_D(x) \geq 2$ for all $x \in V(D)$, then by Remark 3.10 and Lemma 2.18, we have $I(D)^{(s)} = I(D)^{s}$ for all $s \geq 2$.

Now we assume that $I(D)^{(s)} = I(D)^{s}$ for all $s \geq 2$. First we claim $w(x) \neq 1$ for all $x \in V(C_{2n+1})$. Suppose it’s not true. Without loss of generality we can assume that $w(x_1) = 1$. Here $N_D^-(x_2) = \{x_1\} \nsubseteq V^+(D)$. Thus $x_2 \in V(C_{2n+1}) \setminus N_D^+(V^+(D))$. Hence by Proposition 3.1, $I(D)^{(n+1)} \neq I(D)^{n+1}$, which is a contradiction. So our claim follows.
Now we claim \( w(x) \geq 2 \) when \( \deg_D(x) \geq 2 \) for all \( x \in V(D) \setminus V(C_{2n+1}) \). Suppose it’s not true. Then there exists some \( x_i \in V(D) \setminus V(C_{2n+1}) \) such that \( \deg_D(x_i) \geq 2 \) and \( w(x_i) = 1 \). Without loss of generality we can assume that \( x_i \) is in some tree \( T \) connected with \( x_1 \). As \( T \) is a tree, there is only one path \( P \) from \( x_1 \) to \( x_i \). Let \( P = x_1, x_i_1, x_i_2, \ldots, x_i \) be that path whose length is \( t \) and since \( \deg_D(x_i) \geq 2 \), there exists some \( y_j \in N^+_D(x_i) \). Without loss of generality we can assume that \( t \) be the least integer such that \( \deg_D(x_i) \geq 2 \) and \( w(x_i) = 1 \). That means \( w(x_i) \geq 2, \ldots, w(x_i_{t-1}) \geq 2 \) and \( w(x_i_t) = 1 \). Note that \( t \geq 1 \).

**Case (1)** Suppose \( t = 1 \).

There exists an induced weighted naturally oriented path \( D' \) of \( D \) with \( V(D') = \{x_{2n+1}, x_1, x_i, y_j\} \), \( E(D') = \{(x_{2n+1}, x_1), (x_1, x_i), (x_i, y_j)\} \), \( w(x_1) \geq 2 \) and \( w(x_i) = 1 \).

**Case (2)** Suppose \( t = 2 \).

There exists an induced weighted naturally oriented path \( D' \) of \( D \) with \( V(D') = \{x_1, x_i, x_i, y_j\} \), \( E(D') = \{(x_1, x_i), (x_i, x_i), (x_i, y_j)\} \), \( w(x_i) \geq 2 \) and \( w(x_i) = 1 \).

**Case (3)** Suppose \( t \geq 3 \).

There exists an induced weighted naturally oriented path \( D' \) of \( D \) with \( V(D') = \{x_i, x_i, x_i, y_j\} \), \( E(D') = \{(x_i, x_i), (x_i, x_i), (x_i, y_j)\} \), \( w(x_i) \geq 2 \) and \( w(x_i) = 1 \).

Thus by Lemma 3.8, for any \( t \geq 1 \), we have \( I(D)^{(1)} \neq I(D)^3 \), which is a contradiction. Hence the claim follows. \( \square \)

Now we compare the ordinary and symbolic powers of edge ideals of weighted oriented even cycles.

We observe that [9, Lemma 48] is true for weighted oriented even cycle of length \( 4 \). Hence we get the following result.

**Lemma 3.12** [9, Lemma 48] Let \( D \) be a weighted oriented cycle whose underlying graph is \( C_n = (x_1, \ldots, x_n) \) where \( n \geq 4 \) and \( n \neq 5 \). Then \( D \) has the minimal-strong property if and only if the vertices of \( V^+(D) \) are sinks.

**Theorem 3.13** Let \( D \) be a weighted oriented even cycle. If \( V(D) \) is a strong vertex cover of \( D \) or \( D \) has the minimal-strong property, then \( I(D)^{(s)} = I(D)^s \) for all \( s \geq 2 \).

**Proof** If \( V(D) \) is a strong vertex cover of \( D \), then by Lemma 2.19, we get \( I(D)^{(s)} = I(D)^s \) for all \( s \geq 2 \).

If \( D \) has the minimal-strong property, then by Lemmas 3.12 and 2.20, we have \( I(D)^{(s)} = I(D)^s \) for all \( s \geq 2 \). \( \square \)

**Remark 3.14** Converse of the Theorem 3.13 need not be true.

For example consider the weighted oriented even cycle \( D \) as in Fig. 2. Let \( I = I(D) \). Then \( I = (x_1x_2^{w_2}, x_2x_3, x_3x_4, x_4x_1) \). Since \( N_D^+(x_2) \cap V^+(D) = \phi \), by Remark 2.8, \( V(D) \) is not a strong vertex cover of \( D \). The vertex covers of \( D \) except \( V(D) \) are \( C_1 = \{x_1, x_3\}, C_2 = \{x_2, x_4\}, C_3 = \{x_2, x_3, x_4\}, C_4 = \{x_1, x_2, x_4\}, C_5 = \{x_1, x_3, x_4\} \) and \( C_6 = \{x_1, x_2, x_3\} \). Consider \( C_1 = \{x_1, x_3\} \). Here \( x_2 \in N_D^+(x_1) \cap C_1^c \) and \( x_4 \in N_D^+(x_3) \cap C_1^c \). So \( L_1^D(C_1) = \{x_1, x_3\} \). Note that \( C_1 \) is minimal and by Lemma 2.9,
By the above remark and Corollary 3.4, we see that, if non-trivial weights.

Let \( C_1 = (x_1, x_3) \). By the same argument we can prove that \( C_2 \) is strong and \( I_{C_2} = (x_2, x_4) \). Consider \( C_3 = \{x_2, x_3, x_4\} \). Here \( x_1 \in N^+_D(x_4) \cap C^c_3 \), \( N^+_D(x_2) \cap C^c_3 = \emptyset \) and \( N^+_D(x_3) \cap C^c_3 = \emptyset \). So \( L^D_1(C_3) = \{x_4\} \). Here \( x_1 \in N^+_D(x_2) \cap C^c_3 \) and \( N^+_D(x_3) \cap C^c_3 = \emptyset \). Thus \( L^D_2(C_3) = \{x_2\} \) and \( L^D_3(C_3) = \{x_3\} \). Since \( x_2 \in N^+_D(x_3) \cap V^+(D) \cap L^D_2(C_3) \), \( C_3 \) is strong. Hence we have \( I_{C_3} = (x_2^{w_2}, x_3^{w_3}, x_4) = (x_2^{w_2}, x_3, x_4) \). Consider \( C_4 = \{x_1, x_2, x_4\} \). By Lemma 2.6, \( L^D_1(C_4) = \{x_1\} \). Since \( N^+_D(x_1) = \{x_4\} \notin V^+(D) \), by Remark 2.8, \( C_4 \) is not strong. By the same argument we can prove that \( C_5 \) and \( C_6 \) are not strong. Thus \( C_1, C_2, \) and \( C_3 \) are the only strong vertex covers of \( D \). Note that \( C_3 \) is not a minimal vertex cover of \( D \). So \( V(D) \) is neither a strong vertex cover of \( D \) nor \( D \) has the minimal-strong property. We claim that \( I^{(s)} = I^s \) for all \( s \geq 2 \).

By Lemma 2.17, we have \( I^{(s)} = ((x_2^{w_2}, x_3, x_4) \cap (x_2, x_4))^s \cap (x_1, x_3)^s = (x_2^{w_2}, x_2 x_3, x_4)^s \cap (x_1, x_3)^s \). Let \( m = \text{lcm}(m_1, m_2) \) for some \( m_1 \in G((x_2^{w_2}, x_2 x_3, x_4)^s) \) and \( m_2 \in G((x_1, x_3)^s) \). Thus \( m_1 = (x_2^{w_2})^{a_1}(x_2 x_3)^{a_2}(x_4)^{a_3} \) and \( m_2 = (x_1)^{b_1}(x_3)^{b_2} \) for some \( a_i, b_i \geq 0 \) with \( a_1 + a_2 + a_3 = s \) and \( b_1 + b_2 = s \).

**Case (1)** Assume that \( b_2 \leq a_2 \). Then \( b_1 \geq a_1 + a_3 \).

Thus \( m = \text{lcm}(m_1, m_2) = (x_2^{w_2})^{a_1}(x_2 x_3)^{a_2}(x_4)^{a_3}(x_1)^{b_1} = (x_1 x_2^{w_2})^{a_1}(x_2 x_3)^{a_2}(x_4 x_1)^{a_3}(x_1)^{b_1-(a_1+a_3)} \). Hence \( m \in I^{a_1-a_2+a_3} = I^s \).

**Case (2)** Assume that \( b_2 > a_2 \).

Thus \( m = \text{lcm}(m_1, m_2) = (x_2^{w_2})^{a_1}(x_2 x_3)^{a_2}(x_4)^{a_3}(x_3)^{b_2-a_2}(x_1)^{b_1} \). Here \( b_2-a_2+b_1 = s-a_2 = a_1 + a_3 \). Note that \( x_2^{w_2} \) can pair up with \( x_1 \) to get some element of \( \mathcal{G}(I) \). Also \( x_2^{w_2} \) can pair up with \( x_3 \) to get a multiple of some element of \( \mathcal{G}(I) \). Similarly \( x_4 \) can pair up with \( x_3 \) or \( x_1 \) to get some element of \( \mathcal{G}(I) \). Thus \( (x_2^{w_2})^{a_1}(x_4)^{a_3}(x_3)^{b_2-a_2}(x_1)^{b_1} \) can be expressed as a multiple of product of \( a_1 + a_3 \) elements of \( \mathcal{G}(I) \).

Therefore \( m \in I^{(a_1+a_3)+a_2} = I^s \). Hence the claim follows.

In this paper we characterize the weighted naturally oriented even cycles for the equality of all the ordinary and symbolic powers of their edge ideals.

**Remark 3.15** Let \( D \) be a weighted oriented cycle. By Remark 3.10, \( V(D) \) is a strong vertex cover of \( D \) if and only if \( D \) is naturally oriented and all vertices of \( D \) have non-trivial weights.

By the above remark and Corollary 3.4, we see that, if \( D \) is a weighted naturally oriented odd cycle, then all vertices of \( D \) have non-trivial weights if and only if \( I^{(s)} = I^s \) for all \( s \geq 2 \).
Lemma 3.16 Let $D$ be a weighted naturally oriented cycle. If all vertices of $D$ have non-trivial weights, then $I(D)^{(s)} = I(D)^3$ for all $s \geq 2$.

Proof It follows from Remark 3.15 and Lemma 2.18.

In the next result, if $D$ is a weighted naturally oriented cycle of length $n \neq 4$ where at least one vertex has non-trivial weight, we see that only the equality of 3rd ordinary and symbolic powers ensures the non-trivial weight of each vertex.

Theorem 3.17 Let $D$ be a weighted naturally oriented cycle whose underlying graph $G$ is $C_n = (x_1, x_2, \ldots, x_n)$, where $n \neq 4$ and at least one vertex of $D$ has non-trivial weight. Then all vertices of $D$ have non-trivial weights if and only if $I(D)^{(3)} = I(D)^3$.

Proof Here $V(D) = \{x_1, \ldots, x_n\}$. Let $w_i = w(x_i)$ for $x_i \in V(D)$. If all vertices of $D$ have non-trivial weights, then by Lemma 3.16, we have $I(D)^{(3)} = I(D)^3$. Now we assume that $I(D)^{(3)} = I(D)^3$. Suppose all vertices of $D$ do not have non-trivial weights. We know that at least one vertex of $D$ has non-trivial weight. Without loss of generality we can assume that $w(x_2) \geq 2$ and $w(x_3) = 1$.

Case (1) Assume that $n \geq 5$.

Then there exists an induced weighted naturally oriented path $D'$ of $D$ with $V(D') = \{x_1, x_2, x_3, x_4\}$, $E(D') = \{(x_1, x_2), (x_2, x_3), (x_3, x_4)\}$, $w(x_2) \geq 2$ and $w(x_3) = 1$. By Lemma 3.8, we have $x_1^w x_2^2 x_3^w x_4^w \in I(D)^{(3)} \setminus I(D)^3$, which is a contradiction.

Case (2) Assume that $n = 3$.

Note that $x_1 \in V(C_3) \setminus N^+_D(V^+(D))$. By Proposition 3.1, $x_1^w x_2^2 x_3^w \in I(D)^{(2)} \setminus I(D)^2$. Then by Lemmas 2.14 and 2.17, we have $(x_1^w x_2^2 x_3^w)(x_1 x_2^w) \in I(D)^{(3)} \setminus I(D)^3$, which is a contradiction.

Remark 3.18 Let $D$ be a weighted naturally oriented cycle of length $n$ whose underlying graph $G$ is $C_n = (x_1, x_2, \ldots, x_n)$, where $n \neq 4, 6$ and at least one vertex of $D$ has non-trivial weight. If we assume $w(x_2) \geq 2$ and $w(x_3) = 1$, then by the similar argument as in Theorem 3.17, we find that $x_1^w x_2^w x_3 x_6^w \in I(D)^{(2)} \setminus I(D)^2$ for $n \geq 7$ and $x_1^w x_2^w x_3 \in I(D)^{(2)} \setminus I(D)^2$ for $n = 3$ and $5$. Hence $I(D)^{(2)} = I(D)^2$ implies all vertices of $D$ have non-trivial weights and it ensures the equality of all the ordinary and symbolic powers. But it is not true for weighted naturally oriented even cycles of length 6. For example consider $D$ to be a weighted naturally oriented cycle $D$ where the underlying graph is $C_6 = (x_1, x_2, \ldots, x_6)$ and only $x_2$ has non-trivial weight 2. Then $I(D) = (x_1 x_2^2, x_2 x_3, x_3 x_4, x_4 x_5, x_5 x_6, x_6 x_1)$. Using Macaulay 2, we observe $I(D)^{(2)} = I(D)^2$ but $I(D)^{(3)} \neq I(D)^3$ and in this case all the vertices except $x_2$ have trivial weights. In Theorem 3.17, $I(D)^{(3)} = I(D)^3$ guarantees that all vertices of $D$ have non-trivial weights and it ensures the equality of all the ordinary and symbolic powers. But if $D$ is a weighted naturally oriented cycle with underlying graph is $C_4 = (x_1, x_2, x_3, x_4)$ and at least one vertex of $D$ has non-trivial weight, we do not even need each vertex to be of non-trivial weight to ensure the equality of all the ordinary and symbolic powers (see Proposition 3.20).

In the next two results, we give necessary and sufficient condition for the equality of ordinary and symbolic powers of edge ideals of weighted naturally oriented even cycles.
Corollary 3.19 Let $D$ be a weighted naturally oriented even cycle whose underlying graph is $C_n = (x_1, \ldots, x_n)$, where $n \neq 4$ and at least one vertex of $D$ has non-trivial weight. Then $I(D)(s) = I(D)^s$ for all $s \geq 2$ if and only if all vertices of $D$ have non-trivial weights.

Proof If $I(D)(s) = I(D)^s$ for all $s \geq 2$, then by Theorem 3.17, all vertices of $D$ have non-trivial weights. If all vertices of $D$ have non-trivial weights, then by Lemma 3.16, we have $I(D)(s) = I(D)^s$ for all $s \geq 2$. \qed

In the next result, we characterize the weighted naturally oriented even cycles of length 4 for the equality of ordinary and symbolic powers of their edge ideals.

Proposition 3.20 Let $D$ be a weighted naturally oriented even cycle whose underlying graph is $C_4 = (x_1, x_2, x_3, x_4)$ and at least one vertex of $D$ has non-trivial weight. Then $I(D)(s) = I(D)^s$ for all $s \geq 2$ if and only if $D$ satisfies one of the following conditions:

1. all vertices of $D$ have non-trivial weights,
2. one vertex of $D$ has non-trivial weight,
3. only two non-consecutive vertices of $D$ have non-trivial weights.

Proof Let $I = I(D)$. If $D$ satisfies (1), then by Lemma 3.16, we have $I(s) = I^s$ for all $s \geq 2$. If $D$ satisfies (2), then by Remark 3.14, we get $I(s) = I^s$ for all $s \geq 2$. Now assume $D$ satisfies (3). Here two non-consecutive vertices of $D$ have non-trivial weights. Without loss of generality we can assume that $w(x_2) \neq 1$ and $w(x_4) \neq 1$. Then $I = (x_1 x_2 w_2, x_2 x_3, x_3 x_4 w_4, x_4 x_1)$ and by the similar argument as in Remark 3.14, we find $I(s) = (x_1, x_3)^s \cap (x_2, x_4)^s \cap \left( (x_1, x_2, x_4 w_1) \cap (x_2, x_4) \right)^s = (x_1, x_3)^s \cap (x_2 w_2, x_2 x_3, x_4)^s \cap (x_2, x_1 x_4, x_4 w_4)^s$. We claim that $I(s) \subseteq I^s$.

We prove this by induction on $s$. The case for $s = 1$ is trivial. Let $m = G(I(s))$. Then $m = \text{lcm}(m_1, m_2, m_3)$ for some $m_1 \in G((x_1 x_3)^s)$, $m_2 \in G((x_2 w_2, x_2 x_3, x_4)^s)$ and $m_3 \in G((x_2, x_1 x_4, x_4 w_4)^s)$. Thus $m_1 = (x_1)^{a_1} (x_3)^{w_2}, m_2 = (x_2 w_2)^{b_1} (x_2 x_3)^{b_2} (x_4)^{b_3}$ and $m_3 = (x_2)^{c_1} (x_1 x_4)^{c_2} (x_4 w_4)^{c_3}$ for some $a_i, b_i, c_i \geq 0$ with $a_1 + a_2 = s, b_1 + b_2 + b_3 = s$ and $c_1 + c_2 + c_3 = s$.

Case (1) Assume that $a_1 \neq 0$. If $b_3 \neq 0$, then $\tilde{m}$ is divisible by $x_1 x_4$ and observe that $\tilde{m}_{x_1 x_4} \in (x_1, x_3)^{s-1} \cap (x_2 w_2, x_2 x_3, x_4)^{s-1} \cap (x_2, x_1 x_4, x_4 w_4)^{s-1} = I^{s-1}$. Hence by induction hypothesis $\tilde{m}_{x_1 x_4} \in I^{s-1}$ and so $\tilde{m} \in I^s$.

If $b_2 \neq 0$, then $\tilde{m}$ is divisible by $x_2 x_3$ and notice that $\tilde{m}_{x_2 x_3} \in (x_1, x_3)^{s-1} \cap (x_2 w_2, x_2 x_3, x_4)^{s-1} \cap (x_2, x_1 x_4, x_4 w_4)^{s-1} = I^{s-1}$. Hence by induction hypothesis $\tilde{m}_{x_2 x_3} \in I^{s-1}$ and so $\tilde{m} \in I^s$.

Now we assume $b_3 = b_2 = 0$. Then $b_1 = s$ and $\text{lcm}(m_1, m_2) \in I^s$. So $\tilde{m} \in I^s$.

Case (2) Assume that $a_1 = 0$. Then $a_2 = s$ and $\text{lcm}(m_1, m_3) \in I^s$. Hence $\tilde{m} \in I^s$.

Next we prove the converse part. Let us assume that $I(s) = I^s$ for all $s \geq 2$. Suppose none of (1), (2) and (3) is true. Then $D$ must satisfy one of the following conditions:

(a) only two consecutive vertices of $D$ have non-trivial weights,
(b) only three vertices of $D$ have non-trivial weights.

(a) Assume that $D$ has only two consecutive vertices with non-trivial weights. Without loss of generality we can assume that $w(x_2) \neq 1$ and $w(x_3) \neq 1$. Then $I = (x_1x_2^{w_2}, x_2x_3^{w_3}, x_3x_4, x_4x_1)$ and by the similar argument as in Remark 3.14, we find that $I^{(3)} = ((x_1, x_3^{w_3}, x_4) \cap (x_1, x_3))^3 \cap ((x_2^{w_2}, x_3^{w_3}, x_4) \cap (x_2, x_4))^3 = (x_1, x_3^{w_3}, x_3x_4)^3 \cap (x_2^{w_2}, x_2x_3^{w_3}, x_4)^3$. Observe that $x_1x_2x_3^{w_3}x_4^2 \in I^{(3)} \setminus I^3$, which is a contradiction.

(b) Assume that $D$ has only three vertices with non-trivial weights. Without loss of generality we can assume that $w(x_2) \neq 1$, $w(x_3) \neq 1$ and $w(x_4) \neq 1$. Then $I = (x_1x_2^{w_2}, x_2x_3^{w_3}, x_3x_4^{w_4}, x_4x_1)$ and by the similar argument as in Remark 3.14, we find that $I^{(2)} = ((x_1, x_3^{w_3}, x_4^{w_4}) \cap (x_1, x_3))^2 \cap ((x_2, x_4^{w_4}) \cap (x_2, x_4))^2 \cap (x_2^{w_2}, x_3^{w_3}, x_4)^2 \cap (x_2^{w_2}, x_2x_3^{w_3}, x_4)^2$. Notice that $x_1x_2x_3^{w_3}x_4^{w_4} \in I^{(2)} \setminus I^2$, which is a contradiction. Hence the proof follows.

\[ \square \]

4 Comparing symbolic powers of weighted oriented graphs

In this section, we show that the symbolic powers of edge ideals of a weighted oriented graph $D$ and the new weighted oriented graph $D’$ obtained from $D$ after replacing the weights of vertices with non-trivial weights which are sink, by trivial weights, behave in a similar way. We see that using the symbolic powers of edge ideals of one class of weighted oriented graphs, we can compute the symbolic powers of edge ideals of another class of weighted oriented graphs.

**Notation 4.1** Let $D$ be a weighted oriented graph, where $U \subseteq V^+(D)$ be the set of vertices which are sinks and $w_j = w(x_j)$ if $x_j \in V^+(D)$. Let $D’$ be the weighted oriented graph obtained from $D$ after replacing $w_j$ by $w_j = 1$ if $x_j \in U$. Let $V(D) = V(D’) = V = \{x_1, \ldots, x_n\}$. Let $R = k[x_1, \ldots, x_n] = \bigoplus_{d=0}^{\infty} R_d$ be the standard graded polynomial ring. Consider the map

$$\Phi : R \longrightarrow R \text{ where } x_j \longrightarrow x_j \text{ if } x_j \notin U \text{ and } x_j \longrightarrow x_j^{w_j} \text{ if } x_j \in U.$$  

Here $\Phi$ is an injective homomorphism of $k$–algebras. By [5, Corollary 5], $I(D)$ is Cohen–Macaulay if and only if $I(D’)$ is Cohen–Macaulay. We want to investigate the relationship between the symbolic powers of $I(D)$ and $I(D’).$ The next two lemma’s are very important to prove our result in Theorem 4.4.

**Lemma 4.2** Let $D$, $D’$ and $\Phi$ are same as defined in Notation 4.1. Then $C$ is a strong vertex cover of $D$ if and only if $C$ is a strong vertex cover of $D’$.

**Proof** Since $D$ and $D’$ have the same underlying graph, $C$ is a vertex cover of $D$ if and only if $C$ is a vertex cover of $D’$. Now consider $C$ to be a vertex cover of both $D$ and $D’$. Here $D$ and $D’$ have the same orientation on edges. Thus $L_i^D(C) = L_i^{D’}(C)$.
for $1 \leq i \leq 3$. This implies $C \setminus L^D_1(C) = C \setminus L^D_1(C')$. Since each element of $U$ is sink, $N_D^−(x) \cap U = \phi$ for $x \in V \setminus U$. Since two adjacent vertices cannot be sink vertices, $N_D^−(x) \cap U = \phi$ for $x \in U$. Hence for each $x \in C$, $N_D^−(x) \cap U = \phi$.

Note that $V^+(D) \setminus U = V^+(D')$. Since $D$ and $D'$ have the same orientation on edges, $N_D^−(x) = N_D^−(x)$ for each $x \in C$. Thus for each $x \in C$, $N_D^−(x) \cap V^+(D) = N_D^−(x) \cap [V^+(D) \setminus U] = N_D^−(x) \cap [V^+(D) \setminus U] = N_D^−(x) \cap V^+(D')$. Therefore for each $x \in L^D_3(C) = L^D_3(C')$, we have $N_D^−(x) \cap V^+(D) \cap [C \setminus L^D_1(C)] = N_D^−(x) \cap V^+(D') \cap [C \setminus L^D_1(C')]$ and $N_D^−(x) \cap V^+(D) \cap [C \setminus L^D_1(C)] \neq \phi$ if and only if $N_D^−(x) \cap V^+(D') \cap [C \setminus L^D_1(C')] \neq \phi$. Hence by Remark 2.8, $C$ is a strong vertex cover of $D$ if and only if $C$ is a strong vertex cover of $D'$.

**Lemma 4.3** Let $D$, $D'$ and $\Phi$ are same as defined in Notation 4.1. Let $I$ and $\tilde{I}$ be the edge ideals of $D$ and $D'$, respectively. Let $C_{1i}, \ldots, C_{rt_i}$ are the maximal strong vertex covers of both $D$ and $D'$. Let $C_{1i}, \ldots, C_{rt_i}$ are the strong vertex covers of both $D$ and $D'$ such that $C_{ij} \subset C_{ii}$ for $2 \leq j \leq t_i$ and $1 \leq i \leq r$. Let $IC_{1i}$ and $IC_{ij}$ are the irreducible ideals associated to $C_{ij}$ and $C_{ji}$, respectively. Then

$$\Phi(\tilde{I}C_{1i} \cap \tilde{I}C_{12} \cap \cdots \cap \tilde{I}C_{rt_i}) = IC_{1i} \cap IC_{12} \cap \cdots \cap IC_{rt_i} \text{ for } 1 \leq i \leq r.$$

Moreover, every element of $G((IC_{1i} \cap IC_{12} \cap \cdots \cap IC_{rt_i})^\times)$ is of the form

$$\left(\prod_{x_j \in V \setminus U} x_j^{d_j} | x_j \in V \setminus U\right)\left(\prod_{x_k \in U} (x_k^{w_k})^{e_k} | x_k \in U\right) \text{ for some } d_j, e_k \geq 0.$$

**Proof** First we claim that $\Phi(\tilde{I}C_{1i} \cap \tilde{I}C_{12} \cap \cdots \cap \tilde{I}C_{rt_i}) = IC_{1i} \cap IC_{12} \cap \cdots \cap IC_{rt_i}$. Consider any strong vertex cover $C_{1i}$ of both $D$ and $D'$. From the proof of Lemma 4.2, $L^D_p(C_{1i}) = L^D_p(C_{1i})$ for $1 \leq p \leq 3$. Notice that $N_D^+(x) = N_D^+(x) = \phi$ for each $x \in U$. So $L^D_1(C_{1i}) \cup L^D_3(C_{1i}) = L^D_1(C_{1i}) \cup L^D_3(C_{1i})$. By definition $\tilde{I}C_{1i} = (L^D_2(C_{1i}) \cup \{x_j^{w_j} | x_j \in [L^D_2(C_{1i}) \cup L^D_3(C_{1i})] \setminus U\}) \cup \{x_k | x_k \in U\})$ and $IC_{1i} = (L^D_2(C_{1i}) \cup \{x_j^{w_j} | x_j \in [L^D_2(C_{1i}) \cup L^D_3(C_{1i})] \setminus U\}) \cup \{x_k^{w_k} | x_k \in U\})$. Note that $\Phi(\tilde{I}C_{1i}) = IC_{1i}$ for $1 \leq i \leq t_i$. By Lemma 2.22, we have $\Phi(\tilde{I}C_{1i} \cap \tilde{I}C_{12} \cap \cdots \cap \tilde{I}C_{rt_i}) \subseteq \Phi(\tilde{I}C_{1i}) \cap \Phi(\tilde{I}C_{12}) \cap \cdots \cap \Phi(\tilde{I}C_{rt_i}) = IC_{1i} \cap IC_{12} \cap \cdots \cap IC_{rt_i}$.

Let $f \in G(IC_{1i} \cap IC_{12} \cap \cdots \cap IC_{rt_i})$. Then $f = \text{lcm}(f_{i1}, f_{i2}, \ldots, f_{in})$ for some $f_{ii} \in G(IC_{1i})$ where $1 \leq i \leq t_i$. Here each $f_{ii}$ involves only one variable. Fix any $i \in [t_i]$. If $f_{ii}$ involves the variable $x_l$, then

$$f_{ii} = \begin{cases} x_l & \text{if } x_l \in L^D_2(C_{1i}) \\ x_l^{w_j} & \text{if } x_l \in [L^D_2(C_{1i}) \cup L^D_3(C_{1i})] \setminus U \\ x_l^{w_j} & \text{if } x_l \in U \end{cases}$$

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Thus we can express $f$ as

$$f = \left( \prod_{x_j} (x_j^{a_j} b_j | x_j \in V \setminus U) \right) \left( \prod_{x_k} (x_k^{u_k} c_k | x_k \in U) \right)$$

for some $a_j = 1$ or $w_j$, $b_j = 0$ or 1 and $c_k = 0$ or 1. We want to find some $g \in \tilde{I}_{C_{i_1}} \cap \tilde{I}_{C_{i_2}} \cap \ldots \cap \tilde{I}_{C_{i_{t_1}}}$ such that $\Phi(g) = f$. We set $g_i = f_i$ if $f_i$ involves variable from $V \setminus U$ and $g_i = x_l^{w_l}$ where $x_l \in U$ for $1 \leq i \leq t_1$. Therefore

$$g_i = \begin{cases} x_l & \text{if } x_l \in L^D(C_{i_1}) \\ x_l^{w_l} & \text{if } x_l \in [L^D(C_{i_1}) \cup L^D(C_{i_1})] \setminus U \\ x_l & \text{if } x_l \in U \end{cases}$$

Here each $g_i \in \mathcal{G}(\tilde{I}_{C_{i_1}})$ and lcm$(g_1, g_2, \ldots, g_{t_1}) = \left( \prod_{x_j} (x_j^{a_j} b_j | x_j \in V \setminus U) \right) \left( \prod_{x_k} (x_k^{u_k} c_k | x_k \in U) \right)$. Let $g = \text{lcm}(g_1, g_2, \ldots, g_{t_1})$. Notice that $\Phi(g) = f$. Thus $f = \Phi(g) = \Phi(\text{lcm}(g_1, g_2, \ldots, g_{t_1})) = \Phi(\tilde{I}_{C_{i_1}} \cap \tilde{I}_{C_{i_2}} \cap \ldots \cap \tilde{I}_{C_{i_{t_1}}})$. Hence $\Phi(\tilde{I}_{C_{i_1}} \cap \tilde{I}_{C_{i_2}} \cap \ldots \cap \tilde{I}_{C_{i_{t_1}}}) = I_{C_{i_1}} \cap I_{C_{i_2}} \cap \ldots \cap I_{C_{i_{t_1}}}$.

By the similar argument, for $2 \leq i \leq r$, we can prove that

$$\Phi(\tilde{I}_{C_{i_1}} \cap \tilde{I}_{C_{i_2}} \cap \ldots \cap \tilde{I}_{C_{i_{t_1}}}) = I_{C_{i_1}} \cap I_{C_{i_2}} \cap \ldots \cap I_{C_{i_{t_1}}}.$$ 

Let $h \in \mathcal{G}(I_{C_{i_1}} \cap I_{C_{i_2}} \cap \ldots \cap I_{C_{i_{t_1}}})$. Then $h = h_1 \cdot h_s$ for some $h_i$’s $\in \mathcal{G}(I_{C_{i_1}} \cap I_{C_{i_2}} \cap \ldots \cap I_{C_{i_{t_1}}})$. By (1), for $1 \leq i \leq s$, $h_i = \left( \prod_{x_j} (x_j^{a_{j_i}} b_{j_i} | x_j \in V \setminus U) \right) \left( \prod_{x_k} (x_k^{u_k} c_{k_i} | x_k \in U) \right)$ for some $a_{j_i} = 1$ or $w_{j_i}$, $b_{j_i} = 0$ or 1 and $c_{k_i} = 0$ or 1. Hence $h = \left( \prod_{x_j} (x_j^{d_{j_i}} | x_j \in V \setminus U) \right) \left( \prod_{x_k} (x_k^{e_k} | x_k \in U) \right)$ where $d_{j_i} = a_{j_i} b_{j_i} + \ldots + a_{j_s} b_{j_s}$ and $e_{k_i} = c_{k_1} + \ldots + c_{k_r}$.

In the following theorem, we show that the symbolic powers of edge ideals of $D$ and $D'$ behave in a similar way.

**Theorem 4.4** Let $I$ and $\tilde{I}$ be the edge ideals of $D$ and $D'$, respectively. Then $\Phi(\tilde{I}^s) = I^s$ and $\Phi(\tilde{I}^{(s)}) = I^{(s)}$ for all $s \geq 1$.

**Proof** By the definitions of $I$ and $\tilde{I}$, $\Phi(\tilde{I}) = I$. Thus by Lemma 2.22, we have $\Phi(\tilde{I}^s) = (\Phi(\tilde{I}))^s = I^s$ for all $s \geq 1$. Now we claim that $\Phi(\tilde{I}^{(s)}) = I^{(s)}$. Let $C_{i_1}, \ldots, C_{i_t}$ be the maximal strong vertex covers of both $D$ and $D'$. Let $C_{i_2}, \ldots, C_{i_t}$
Hence Proposition 4.5

Let $D, D'$ be same as defined in Notation 4.1. Let $I$ and $\tilde{I}$ be the edge ideals of $D$ and $D'$, respectively. Then for each $s \geq 1$,

$$s\text{defect}(\tilde{I}, s) = s\text{defect}(I, s).$$
Proof Fix any \( s \geq 1 \). Let \( X = G(I^{(s)}) \setminus I^s \) and \( X' = \tilde{G}(\tilde{I}^{(s)}) \setminus \tilde{I}^s \). First we claim that \( \Phi(X') \subseteq X \).

Suppose \( p \in G(\tilde{I}^{(s)}) \setminus \tilde{I}^s \). By the similar argument used to get the general form of any element of \( G(I^{(s)}) \) in Theorem 4.4, we can write \( p = (\prod x_j^{u_j} | x_j \in V \setminus U) (\prod x_k^{v_k} | x_k \in U) \) for some \( u_j, v_k \geq 0 \). By Theorem 4.4, \( \Phi(p) = (\prod x_j^{u_j} | x_j \in V \setminus U) (\prod x_k^{v_k} | x_k \in U) \). Let \( q = \Phi(p) \). Suppose \( q \notin G(I^{(s)}) \). Then \( q \) must be multiple of some element of \( G(I^{(s)}) \) (say \( q' \)). By Theorem 4.4, we can write \( q' = (\prod x_j^{u_j} | x_j \in V \setminus U) (\prod x_k^{v_k} | x_k \in U) \) where each \( u_j' \leq u_j, v_k' \leq v_k \) and at least one \( u_j' < u_j \) or \( v_k' < v_k \). Let \( p' = (\prod x_j^{u_j} | x_j \in V \setminus U) (\prod x_k^{v_k} | x_k \in U) \). By Theorem 4.4, we have \( I^{(s)} = \Phi(\tilde{I}^{(s)}) \). Since \( \Phi(p') = q' \in I^{(s)} = \Phi(\tilde{I}^{(s)}) \) and \( \Phi \) is an injective, \( p' \in \tilde{I}^s \). Thus \( p \) is multiple of \( p' \in \tilde{I}^{(s)} \), which is a contradiction because \( p \notin G(I^{(s)}) \). Therefore \( q \notin G(I^{(s)}) \). Since \( p \notin \tilde{I}^s \), it is easy to see that \( q \notin \tilde{I}^s \). So \( q \notin G(I^{(s)}) \setminus I^s \) and hence \( \Phi(X') \subseteq X \).

Now consider the map \( \Phi|_{X'} : X' \rightarrow X \). It is enough to show \( \Phi|_{X'} \) is bijective. We know \( \Phi|_{X'} \) is injective. Suppose \( g \in G(I^{(s)}) \setminus I^s \). Then by Theorem 4.4, there exists \( f \in G(\tilde{I}^{(s)}) \setminus \tilde{I}^s \) such that \( \Phi|_{X'}(f) = g \). So \( \Phi|_{X'} \) is surjective and hence \( \Phi|_{X'} \) is bijective.

Corollary 4.6 Let \( D, D' \) and \( \Phi \) be same as defined in Notation 4.1. Let \( I \) and \( \tilde{I} \) be the edge ideals of \( D \) and \( D' \), respectively. Then \( \tilde{I}^{(s)} = \tilde{I}^s \) if and only if \( I^{(s)} = I^s \) for each \( s \geq 1 \).

Proof By Proposition 4.5, sdefect(\( \tilde{I}, s \)) = 0 if and only if sdefect(\( I, s \)) = 0 for each \( s \geq 1 \). Hence the proof follows.

If all the vertices of \( V^+(D) \) are sinks, we get the following two results.

Corollary 4.7 Let \( D \) be a weighted oriented graph \( D \) where the vertices of \( V^+(D) \) are sinks and its underlying graph is \( G \). Then \( I(G)^{(s)} = I(G)^s \) if and only if \( I(D)^{(s)} = I(D)^s \) for each \( s \geq 1 \).

Proof Let \( D' \) is same as defined in Notation 4.1. Then \( D' = G \) and the proof follows from Corollary 4.6.

Corollary 4.8 Let \( D \) be a weighted oriented graph where the vertices of \( V^+(D) \) are sinks and its underlying graph is \( G \). Then \( G \) is bipartite if and only if \( I(D)^{(s)} = I(D)^s \) for all \( s \geq 2 \).

Proof By [10, Theorem 5.9], \( G \) is bipartite if and only if \( I(G)^{(s)} = I(G)^s \) for all \( s \geq 2 \). Then the proof follows from Corollary 4.7.
Fig. 3 All classes of weighted oriented even cycles of length 4 which are not naturally oriented

Remark 4.9 As an application of Theorem 4.4 and Corollary 4.6, by studying the symbolic powers of edge ideals of one class of weighted oriented graphs, we can get information about the symbolic powers of edge ideals of another class of weighted oriented graphs. When we try to find the necessary and sufficient condition for the equality of ordinary and symbolic powers of edge ideals of a certain class of weighted oriented graphs, as an application of Corollary 4.6, we can omit the checking of equality of ordinary and symbolic powers of the edge ideals of those weighted oriented graphs where some vertex with non-trivial weight is sink. Hence we need to check the equality only for a smaller class of graphs.

In the next result, as an application of Corollary 4.6, we give necessary and sufficient condition for the equality of ordinary and symbolic powers of edge ideals of weighted oriented even cycles of length 4 which are not naturally oriented.

Proposition 4.10 Let $D$ be a weighted oriented even cycle which is not naturally oriented, with underlying graph is $C_4 = (x_1, x_2, x_3, x_4)$ and at least one vertex of $D$ has non-trivial weight. Then $I(D)^{(s)} = I(D)^s$ for all $s \geq 2$ if and only if $D$ is not of class (7) (See Fig. 3).

Proof If all vertices of $D$ have non-trivial weights, then $D$ is naturally oriented. So $D$ has at most three vertices with non-trivial weights. Let $D'$ is same as defined in Notation 4.1. Let $I = I(D)$ and $\tilde{I} = I(D')$. We check the equality of ordinary and symbolic powers by considering different cases depending upon the number of vertices with non-trivial weights. We do not consider any weighted oriented even cycle which can be regarded as some weighted naturally oriented even cycle by changing the orientation of edges.

Case (1) $D$ has only one vertex with non-trivial weight. Then $D$ is of class (1) (see Fig. 3). Here we can think $D'$ as a simple bipartite graph. Hence by Corollary 4.8, we get $I^{(s)} = I^s$ for all $s \geq 2$.

Case (2) $D$ has only two vertices with non-trivial weights.
Then $D$ is one of the classes (2), (3), (4) and (5) (see Fig. 3). If $D$ is of class (2), then we can think $D'$ as a weighted naturally oriented even cycle where only one vertex has non-trivial weight. Using Proposition 3.20, we get $\tilde{I}^{(s)} = I^{s}$ for all $s \geq 2$. Hence by Corollary 4.6, we have $I^{(s)} = I^{s}$ for all $s \geq 2$. If $D$ is of class (3), then by the similar argument as in class (2), $I^{(s)} = I^{s}$ for all $s \geq 2$. If $D$ is of class (4), then by the similar argument as in class (1), $I^{(s)} = I^{s}$ for all $s \geq 2$. Now assume $D$ is of class (5). There is no vertex with non-trivial weight which is sink. Without loss of generality we can assume that $w(x_2) \neq 1$ and $w(x_4) \neq 1$. Then $I = (x_1x_2^{w_2}, x_2x_3, x_3x_4, x_4x_4^{w_4})$. By the similar argument as in Remark 3.14, we find $I^{(s)} = (x_1, x_3)^s \cap ((x_2^{w_2}, x_3, x_4^{w_4}) \cap (x_2, x_4))^s = (x_1, x_3)^s \cap (x_2^{w_2}, x_2x_3, x_3x_4, x_4^{w_4})^s$. Let $\tilde{m} \in G(I^{(s)})$. Then $\tilde{m} = \text{lcm}(m_1, m_2)$ for some $m_1 \in G((x_1, x_3)^s)$ and $m_2 \in G((x_2^{w_2}, x_2x_3, x_3x_4, x_4^{w_4})^s)$. Thus $m_1 = x_1^{a_1}x_3^{a_2}$ and $m_2 = (x_2^{w_2})^{b_1}(x_2x_3)^{b_2}(x_3x_4)^{b_3}(x_4^{w_4})^{b_4}$ for some $a_1, b_i \geq 0$ with $a_1 + a_2 = s$ and $b_1 + b_2 + b_3 + b_4 = s$.

Assume that $a_1 \geq b_1 + b_4$. Then $x_1^{a_1}(x_2^{w_2})^{b_1}(x_2x_3)^{b_2}(x_3x_4)^{b_3}(x_4^{w_4})^{b_4} | \text{lcm}(m_1, m_2) = \tilde{m}$. Since $x_1x_2^{w_2}$ and $x_1x_4^{w_4} \in G(I)$, $x_1^{a_1}(x_2^{w_2})^{b_1}(x_3x_4)^{b_3}(x_4^{w_4})^{b_4}$ can be expressed as a multiple of product of $b_1 + b_4$ elements of $G(I)$. So $\tilde{m} \in I^{(b_2 + b_3) + (b_1 + b_4)} = I^{s}$.

Now assume that $a_1 < b_1 + b_4$. Then $a_2 > b_2 + b_3$.

Thus $\tilde{m} = \text{lcm}(m_1, m_2) = x_1^{a_1}x_3^{a_2 - (b_2 + b_3)}(x_2^{w_2})^{b_1}(x_2x_3)^{b_2}(x_3x_4)^{b_3}(x_4^{w_4})^{b_4}$.

Here $a_1 + a_2 - (b_2 + b_3) = s - (b_2 + b_3) = b_1 + b_4$. Thus $x_1^{a_1}x_3^{a_2 - (b_2 + b_3)}(x_2^{w_2})^{b_1}(x_4^{w_4})^{b_4}$ can be expressed as a multiple of product of $b_1 + b_4$ elements of $G(I)$. So $\tilde{m} \in I^{(b_2 + b_3) + (b_1 + b_4)} = I^{s}$.

Hence $I^{(s)} = I^{s}$ for all $s \geq 2$.

Case (3) $D$ has only three vertices with non-trivial weights.

Then $D$ is one of the classes (6) and (7) (see Fig. 3). If $D$ is of class (6), then $D'$ is of class (5). We know $I^{(s)} = I^{s}$ for all $s \geq 2$. Hence by Corollary 4.6, we have $I^{(s)} = I^{s}$ for all $s \geq 2$. If $D$ is of class (7), then we can think $D'$ as a weighted naturally oriented even cycle where only two consecutive vertices have non-trivial weights. By Proposition 3.20, we have $I^{(s)} \neq I^{s}$ for some $s \geq 2$. Then by Corollary 4.6, we get $I^{(s)} \neq I^{s}$. Hence the proof follows.\]

Next we see another application of Corollary 4.6 to weighted oriented star graphs.

**Definition 4.11** A star graph $S_n$ of order $n$ is a tree on $n + 1$ vertices with one vertex having degree $n$ and the other $n$ vertices having degree 1.

In the next theorem, we show that the ordinary and symbolic powers of edge ideal of any weighted oriented star graph are equal.

**Theorem 4.12** Let $D$ be a weighted oriented star graph whose underlying graph is $S_n$ for some $n \geq 2$. Then $I(D)^{(s)} = I(D)^{s}$ for all $s \geq 2$.

**Proof** Let $V(D) = \{x_0, x_1, \ldots, x_n\}$ with $\text{deg}_D(x_0) = n$ and $\text{deg}_D(x_i) = 1$ if $i \neq 0$. Here $E(S_n) = \{(x_0, x_1), (x_0, x_2), \ldots, (x_0, x_n)\}$.

Case (1) Assume that $w(x_0) = 1$.\]
If \( w(x_i) \neq 1 \) for some \( i \neq 0 \), then \((x_0, x_i) \in E(D)\). This implies \( x_i \) is a sink vertex. So all vertices of \( V^+(D) \) are sinks. Thus by Corollary 4.8, we have \( I(D)^{(s)} = I(D)^s \) for all \( s \geq 2 \).

**Case (2)** Assume that \( w(x_0) \neq 1 \), i.e., \( x_0 \in V^+(D) \).

Then \( x_0 \) is not a source vertex, i.e., \( N_D^+(x_0) \neq \emptyset \).

**Case (2.a)** Suppose \( N_D^+(x_0) = \emptyset \).

Then \( x_0 \) is a sink vertex. This implies that each \( x_i \) for \( i \neq 0 \) is a source vertex. So \( w_i = 1 \) for each \( i \neq 0 \). Hence the only vertex of \( V^+(D) \) is \( x_0 \) and it is sink. Then by Corollary 4.8, we have \( I(D)^{(s)} = I(D)^s \) for all \( s \geq 2 \).

**Case (2.b)** Suppose \( N_D^+(x_0) \neq \emptyset \).

Without loss of generality we can assume that \( N_D^+(x_0) = \{x_1, x_2, \ldots, x_r\} \) and \( N_D^-(x_0) = \{x_{r+1}, x_{r+2}, \ldots, x_n\} \) for some \( r \geq 1 \). If \( x_i \in V^+(D) \) for some \( i \in [r] \), then \( x_i \) is sink. Let \( D' \) is same as defined in Notation 4.1. Then \( w_i = 1 \) for \( 1 \leq i \leq n \) in \( D' \). Let \( \tilde{I} = I(D') \). By Corollary 4.6, it is enough to show that \( \tilde{I}^{(s)} = \tilde{I}^s \) for all \( s \geq 2 \).

Note that the two minimal vertex covers of \( D' \) are \( \{x_0\} \) and \( \{x_1, \ldots, x_n\} \). By Lemma 2.9, these are strong vertex covers of \( D' \).

Consider a vertex cover \( C = \{x_0, x_1, x_2, \ldots, x_r\} \). Then \( C^c = \{x_{r+1}, x_{r+2}, \ldots, x_n\} \). Here \( N_D^+(x_0) \cap C^c = \emptyset \) and \( N_D^-(x_0) \cap C^c \neq \emptyset \). So \( x_0 \in L^D_2(C) \). By Lemma 2.6, \( L^D_3(C) = \{x_1, x_2, \ldots, x_r\} \). Since \( x_0 \in N_D^-(x_i) \cap V^+(D') \cap L^D_2(C) \) for \( 1 \leq i \leq r \), \( C \) is a strong vertex cover of \( D' \).

Consider \( C' \subseteq C \) as a vertex cover of \( D' \). Since \( x_{r+1} \notin C' \), \( x_0 \in C' \). Thus there exist \( j \) and \( k \) such that \( x_j \in C' \) and \( x_k \notin C' \). Without loss of generality \( x_j = x_1 \) and \( x_k = x_2 \). By Lemma 2.6, \( x_1 \in L^D_3(C') \). Here \( N_D^-(x_1) = \{x_0\} \subseteq L^D_1(C') \) because \( N_D^+(x_0) \cap C^c \neq \emptyset \). Hence by Remark 2.8, \( C' \) is not strong.

Let \( C_1 = \{x_0\}, C_2 = \{x_1, \ldots, x_n\} \) and \( C_3 = \{x_0, x_1, \ldots, x_r\} \). Suppose there exists a strong vertex cover \( C_4 \) of \( D' \) other than \( C_1, C_2 \) and \( C_3 \). We know that any vertex cover which is a proper subset of \( C_3 \) cannot be strong. Thus \( C_4 \) must contain \( x_0 \) and some vertex \( x_i \in \{x_{r+1}, x_{r+2}, \ldots, x_n\} \). Without loss of generality we can assume that \( C_4 \) contains \( x_0 \) and \( x_{r+1} \). By Lemma 2.6, \( x_{r+1} \in L^D_3(C_4) \). Since \( N_D^-(x_{r+1}) = \emptyset \), by Remark 2.8, \( C_4 \) is not a strong vertex cover of \( D' \). Hence \( C_1, C_2 \) and \( C_3 \) are the only strong vertex covers of \( D' \). Consider \( C_1 = \{x_0\} \). Here \( x_1 \in N_D^-(x_0) \cap C_1^c \).

So \( L^D_1(C_1) = \{x_0\} \). Hence \( \tilde{I}C_1 = (x_0) \). Consider \( C_2 = \{x_1, \ldots, x_n\} \). Then \( \tilde{I}C_2 = (x_1, \ldots, x_r, x_{r+1}, \ldots, x_n) \). Consider \( C_3 = \{x_0, x_1, \ldots, x_r\} \). We know \( L^D_2(C_3) = \{x_0\} \). This implies \( \tilde{I}C_3 = (x_0^{w_0}, x_1, \ldots, x_r) \). Hence by Lemma 2.17, we have

\[
\tilde{I}^{(s)} = ((x_0^{w_0}, x_1, \ldots, x_r) \cap (x_0))^{s} \cap (x_1, \ldots, x_r, x_{r+1}, \ldots, x_n)^{s} = (x_0^{w_0}, x_0x_1, \ldots, x_0x_r)^{s} \cap (x_1, \ldots, x_r, x_{r+1}, \ldots, x_n)^{s}.
\]
Let $\tilde{m} \in G(\tilde{I}^s)$. Then $\tilde{m} = \text{lcm}(m_1, m_2)$ for some $m_1 \in G((x_0^{w_0}, x_0x_1, \ldots, x_0x_r)^s)$ and $m_2 \in G((x_1, x_r, x_{r+1}, \ldots, x_n)^s)$. Thus $m_1 = (x_0^{w_0})^{a_0} (x_0x_1)^{a_1} \cdots (x_0x_r)^{a_r}$ and $m_2 = x_1^{b_1} \cdots x_r^{b_r} x_{r+1}^{b_{r+1}} \cdots x_n^{b_n}$ for some $a_i, b_i \geq 0$ with $\sum_{i=0}^r a_i = s$ and $\sum_{i=1}^n b_i = s$.

If $a_0 = 0$, then $m_1 \in \tilde{I}^s$ and so $\tilde{m} \in \tilde{I}^s$.

Now we assume that $a_0 \neq 0$ and $b_{r+1} + \cdots + b_n \geq a_0$. Here $m_1 x_{r+1}^{b_{r+1}} \cdots x_n^{b_n} \mid \text{lcm} (m_1, m_2) = \tilde{m}$. Then we can express $m_1 x_{r+1}^{b_{r+1}} \cdots x_n^{b_n} = (x_0x_1)^{a_1} \cdots (x_0x_r)^{a_r} \cdot [((x_0^{w_0})^{a_0} x_{r+1}^{b_{r+1}} \cdots x_n^{b_n}].$ Since $x_i x_0^{w_0} \in G(\tilde{I})$ for $r+1 \leq i \leq n$, $(x_0^{w_0})^{a_0} x_{r+1}^{b_{r+1}} \cdots x_n^{b_n}$ can be expressed as a multiple of $a_0$ elements of $G(\tilde{I})$. So $m_1 x_{r+1}^{b_{r+1}} \cdots x_n^{b_n} \in I^{(a_1 + a_2 + \cdots + a_r) + a_0} = \tilde{I}^s$. Hence $\tilde{m} \in \tilde{I}^s$.

Finally, we assume that $a_0 \neq 0$ and $b_{r+1} + \cdots + b_n < a_0$. Here $(x_0^{w_0})^{a_0} x_0^{a_1 + \cdots + a_r} m_2 | (\text{lcm}(m_1, m_2) = \tilde{m})$. Then we can express

\[
(x_0^{w_0})^{a_0} x_0^{a_1 + \cdots + a_r} m_2 = (x_0^{w_0})^{a_0} x_0^{a_1 + \cdots + a_r} x_1^{b_1} \cdots x_r^{b_r} x_{r+1}^{b_{r+1}} \cdots x_n^{b_n} \\
= (x_0^{w_0})^{a_0 -(b_{r+1} + \cdots + b_n)} x_0^{a_0 + b_{r+1} + \cdots + b_n} \cdot (x_0x_1)^{b_1} \cdots (x_0x_r)^{b_r} [x_{r+1}^{b_{r+1}} \cdots x_n^{b_n}] \\
= (x_0^{w_0})^{a_0 -(b_{r+1} + \cdots + b_n)} [x_0^{b_1} \cdots x_r^{b_r} x_{r+1}^{b_{r+1}} \cdots x_n^{b_n}] \\
= (x_0^{w_0})^{a_0 -(b_{r+1} + \cdots + b_n)} [x_0 x_1]^{b_1} \cdots [x_0 x_r]^{b_r} [x_{r+1}^{b_{r+1}} \cdots x_n^{b_n}] \\
= (x_0^{w_0})^{a_0 -(b_{r+1} + \cdots + b_n)} [x_0 x_1 x_2 x_3 \cdots x_n]^{b_1} \cdots [x_0 x_1 x_2 x_3 \cdots x_n]^{b_n}.
\]

So $(x_0^{w_0})^{a_0} x_0^{a_1 + \cdots + a_r} m_2 \in \tilde{I}^{(b_{r+1} + \cdots + b_r) + (b_{r+1} + \cdots + b_n)} = \tilde{I}^s$. Therefore $\tilde{m} \in \tilde{I}^s$. Hence the proof follows. □

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Data Availability Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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