Artificial market model based on deterministic agents and derivation of limit of GARCH type process

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abstract

We investigate the dealer model – an artificial market model based on deterministic agents both numerically and theoretically. The agents refer to the past market prices and changes their ask/bid price. The temporal development of the market price fluctuations is calculated numerically. A probability density function of the market price changes has power law tails. Autocorrelation coefficient of the changes has an anti-correlation, and autocorrelation coefficient of squared changes (volatility correlation function) has a long time correlation. A probability density function of intervals between two successive transactions follows a geometric distribution. The GARCH type stochastic process is approximately derived from the market changes of the model in a limit case. We discuss two factors of the market price fluctuations and display a relation between the volatility of the market prices and a demand-supply curve. We conclude that the

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power law tails and the long time volatility result from both positive and negative heterogenous feedbacks in the agents.

**keywords:** artificial market model, GARCH type stochastic process, power-law distribution, volatility clustering, econophysics

## 1 Introduction

Economically motivated problems are attracting the attention of physicists, economists and engineers. Recently physicists have become interested in these problems and established a new field, “econophysics”, which has been made tremendous progress since 1997. The price fluctuation is one of the most exciting interests in econophysics [1, 2, 3, 4]. Statistical properties of market price fluctuations were clarified from analyzing financial time series empirically. Specifically we focus on two statistical properties of price fluctuations; (1) the probability distribution (pdf) of price changes has fat tails [?], and (2) the squared or absolute price changes have long time correlation, which is well-known as the volatility clustering [6]. The volatility clustering is measured by autocorrelation function of squared or absolute price changes, the volatility autocorrelation function, which has a long tail when the volatility is clustered.

On one hand stochastic models have been proposed in order to describe these phenomena. For example, the truncated Lévy flights [5, 7], the random multiplicative processes [8, 9, 10] and the GARCH models [11, 12]. On the other hand, various agent-based models have been proposed in order to explain the basic properties of price fluctuations from the viewpoint of complex dynamical systems. These studies are called a agent-based approach [13, 14, 15, 16, 17].

The first agent-based model of a market was introduced by one of the au-
thors (H.T.) et. al. in order to explain why market price apparently fluctuate randomly [18]. It was shown that trading involve a kind of likely nonlinear interaction among agents in general. A pseudo-random walk fluctuation of market price results from the effect of chaos in an artificial market. That is verified in the market that includes only 3 agents with very simple deterministic strategies. Also, it was pointed out that when the agents have the tendency of following the latest market trend, large fluctuations such as crashes or bubbles occur spontaneously. The market model is a variant of the deterministic agent-based model.

Bak et. al. investigated a stock market model with "rational traders" and "noise traders". Noise traders’ actions depend on their current volatility in the market and imitate ways to buy or sell. Rational traders optimize their own utility functions. They emphasize that it is important for noise traders to exist in order to have fat tails of the pdf [13]. Lux et. al. investigated a stochastic multi-agent model with the pool of traders divided into fundamentalists and noise traders [15]. They reported that a pdf of returns in their model has fat tails. These models assume that a market has a balance of demand and supply at a unit time to decide the next market price, and the market price changes depending on an unbalance of the demand and supply. Unlike the stochastic model of Lux stimulative extensions to the basic Minority Game (MG), multi-agent market models without noise trader, have been developed and are successively investigated [16, 17]. In fact these studies demonstrate the stylized facts (the fat tails of the pdf of market price changes and the volatility clustering) and the numerical results show that the model contains the GARCH properties. However the GARCH process was analytically approximated in these studies.

Specifically GARCH model are useful tools in econometrics. However there are few studies that give the theoretical basis of the GARCH from microscopic
viewpoint. The aim of the present article is that we develop an artificial model characterized by a few parameters and directly derive the GARCH type stochastic process from a price change of the model. In our preceding study we proposed a dynamical model of a market having mean field interactions of the dealers [19]. We derived a Langevin equation with both positive and negative feedbacks in a stochastic manner for market prices and, the pdf of time interval between successive trading follows exponential decay. But autocorrelation function of the squared change does not have a long time correlation. In the present article we introduce heterogeneity of the dealers instead of the mean-field interaction in our preceding study.

The present article is organized as follows. In Sec. 2 we introduce an artificial model of a market based on deterministic agents who modify their ask/bid price depending on past price changes. In Sec. 3 the statistical properties of artificial market price fluctuations from numerical simulations are shown. In Sec. 4 we show how the GARCH type stochastic process for the market price changes is theoretically derived from the market model in a limit manner. The GARCH type stochastic process is a familiar model for time series analysts but its theoretical basis in microscopic viewpoint has not been clarified yet. In Sec. 5 we discuss the volatility clustering demonstrated from numerical simulation in the model comparing with the demand and supply curves of the market. Sec. 6 is devoted to concluding remarks.

2 Market model

We show a brief explanation of the dealer model – an artificial market model with many simple deterministic dealers [20]. The fundamental idea of the model is an interaction between agents (dealers) through a common board and histor-
ical market prices. We consider a market in which $N$ dealers exchange a single commodity, for example, in foreign currency market. We assume that the dealers have limited information due to a short time trading. It takes a finite time for them to decide to buy or sell. Therefore they have each simple strategy to predict the next market price from the limited information. By the same token we do not assume that they have utility function since they cannot optimize their action in short time. Dealers put a sell/buy order on the market. By a competitive mechanism in the market adequate orders are selected, and the next market price is determined from the ordered price.

As shown in fig. 1 we explain the dealer model for two parts: (1) a market rule, which describes how to determine a market price from orders, (2) a strategy of the agent, which governs how to order a sell/buy into the market and how to modify their ask/bid prices. In the following subsections we explain a market rule, a strategy of the agent, initial conditions and parameters.

2.1 Market rule

A market has a competitive mechanism for orders. We assume that $N$ dealers put their order (sell/buy) to a common board. Here $p_i(t)$ represents an ask/bid price of the $i$th dealer at time $t$. The ask prices and the bid prices individually compete in the market. Namely the maximum buying price and the minimum selling price are effective in the market. Thus the condition for a trading to occur is given by the following inequality,

$$\max_{\text{for all buyers}} \{p(t)\} \geq \min_{\text{for all sellers}} \{p(t)\},$$

(1)

where the right hand side represents the maximum bid price for all the buyers, and the left hand side represents the minimum ask price for all the sellers. Assume that the market price $P(t)$ is determined as an arithmetic mean of the buy
price and the sell price when trading occurs. Otherwise the last market price is maintained. Namely,

\[
P(t) = \begin{cases} 
\frac{1}{2} (\max\{p(t)\} + \min\{p(t)\}) & (\max\{p(t)\} \geq \min\{p(t)\}) \\
P(t - 1) & (\max\{p(t)\} < \min\{p(t)\})
\end{cases}.
\]  \tag{2}

\section{2.2 Strategy of the dealer}

In general a dealer determines his/her action from several causes. The causes are separated into “endogenous” factors and “exogenous” ones. The endogenous ones mean what has happened in the market and contains the historical market prices and a rumor in the market. The exogenous ones mean what has happened outside market, a balance of domestic commerce or international commerce among countries and so on. Here we consider that the dealer has to make a decision based on the market price fluctuations since they regards only prices for a short period and determine their action and ask/bid price.

Here we consider a trade between a seller and a buyer. Suppose that the both must make a trade. Then the seller decreases his/her sell price a little bit unless he/she can make a trade. On the other hand the buyer increases his/her buy price a little bit. Repeating this they find a satisfying exchange price.

The same is consistent in the market. Sellers go on decreasing their sell price a little bit until they can sell and buyers go on increasing their buy price until they can buy. From the assumption a modification of the ask/bid price is negative/positive for sellers/buyers. The temporal development of the ask/bid price can be described by,

\[
p_i(t + 1) = p_i(t) + \alpha_i(t) D_i(t),
\]  \tag{3}

\(\alpha_i(t)\) represents the \(i\)’th dealer’s modification per a unit time step, and \(D_i(t) = -1\) when the \(i\)th dealer is a seller, and \(D_i(t) = 1\) a buyer.
Moreover we assume that a value of modification depends on the past market prices due to limited information. The assumption means that the next action of dealers is only affected by the historical data of market prices.

\[ \alpha_i(t) = \alpha(P(t), P(t-1), \ldots) \] (4)

We assume that the dealers are sensitive to price changes rather than exact prices. One of the simplest modification algorithms is a linear inner product of dealer-dependent coefficients and past price changes. Namely the modification \( \alpha_i(t) \) is described by

\[ \alpha_i(t) = |1 + \sum_{s'=1}^{T} c_{i,s'} \Delta P_{\text{prev}}(s')|a_i, \] (5)

where \( \Delta P_{\text{prev}}(s') \) denotes the \( s' \)th change of the past market price. It is obvious that \( \Delta P_{\text{prev}}(1) \) is the latest market price change. \( c_{i,s'} \) represents coefficients of the \( i \)th dealer for the last \( s' \)th market price change, \( a_i \) is a positive coefficient.

For simplicity we consider the case of \( T = 1 \). Then a rule to modify dealers’ expectation price is written by

\[ p_i(t+1) = p_i(t) + |1 + c_i \Delta P_{\text{prev}}(1)|a_i D_i(t), \] (6)

where \( c_i \) is a coefficient, which corresponds to accelerating and deaccelerating his/her modification of expectation price depending on the latest price. After the large price change each dealer goes up or goes down quickly his/her expectation price in order to make his/her portfolio balanced as soon as possible.

We assume that the dealers open a sell/buy position till they exchange. They determine whether they open the sell/buy position after they have made a trade. We assume that the sellers want to go on selling when the market price goes up and that the buyers want to go on buying when the market price goes down.

Therefore after trading the commodity the two dealers who have exchange open either a sell position or a buy position depending on the past price change.
\[ \Delta P_{\text{prev}}. \] If the market price goes up then the dealers may expect to make a profit from selling the commodity. Hence we assume that they open the buy position, i.e., \( D_i(t + 1) = -1 \), when \( \Delta P_{\text{prev}} > 0 \). In striking contrary we assume that they open the buy position, i.e., \( D_i(t + 1) = 1 \) when \( \Delta P_{\text{prev}} < 0 \) in order to make a profit from future selling the commodity. Furthermore they open either a sell position or a buy position at the same probability \( p = 1/2 \) when \( \Delta P_{\text{prev}} = 0 \).

Moreover we assume that dealers have difference between the market price and the retried ask/bid price, which denotes \( \Lambda_i(\equiv P(t) - p_i(t)) \). In a mind of a dealer it is clear that the retried ask price is greater than the market price. On the other hand the retried bid price is less than the market price. Therefore the ask/bid price of the \( i \)th dealer at \( t \) is written by,

\[ p_i(t) = P(t) - D(t + 1)\Lambda_i \quad (7) \]

For simplicity \( \Lambda_i = \Lambda \) for the all dealers.

### 2.3 Initial conditions and parameters

Each dealer has two coefficients: \( a_i \) and \( c_i \). \( a_i \) and \( c_i \) are initially given by uniform random numbers distributed in interval \([0, a^*]\) and \([-c^*, c^*]\), respectively. These coefficients exhibit dealer’s personality and are fixed throughout a numerical simulation. Each dealer has two variables: \( D_i(t) \) and \( p_i(t) \). Initial condition \( D_i(0) \) is given by either +1 or −1 randomly, i.e. a probability for \( D_i(0) \) to be 1 is 1/2, and a probability for \( D_i(0) \) to be −1 is 1/2. Then \( p_i(0) \) is given by \( P_0 - \Lambda \) when \( D_i(0) = 1 \) and \( P_0 + \Lambda \) when \( D_i(0) = -1 \), where \( P_0 \) is an initial market price fixed as \( P_0 = 120.0 \) throughout simulations.

We assume that the latest price change \( \Delta P_{\text{prev}} \) is initially zero. The dealer’s rule is deterministic except initial conditions. This model has four parameters; amount of the dealers \( N \), \( a^* \) for \( a_i \), difference between the ask/bid price and the
market price $\Lambda$ and $c^*$ for $c_i$. Finally we summarize the model parameters in Tab. 1.

3 Numerical simulation

Fixing parameters $N = 100$, $\Lambda = 1.0$ and changing $c^*$ and $a^*$, we numerically simulate the dealer model introduced in the above section. Figs. 2 and 3 show a typical example of time series of market prices $P(t)$ and of price changes $\Delta P(t) = P(t) - P(t-1)$. The market prices and their changes apparently fluctuate although the model is completely deterministic. The reason is high-dimensionality of the system as the freedom of the system is proportional to the agent-number.

Let $\tau_s$ denote the time when the $s$th trade occurs. Fig. 4 exhibits a conceptual illustration of market price fluctuations. The time series may be characterized by a price change and a time difference between successive trading. $P(\tau_s)$ represents a market price at $\tau_s$. $\Delta p_s$ denotes a price change at $\tau_s$, namely, $\Delta p_s \equiv P(\tau_s) - P(\tau_{s-1})$, and $n_s$ a time difference between successive trading, $n_s \equiv \tau_s - \tau_{s-1}$.

Fig. 5 displays semi-log plots of pdfs of $\Delta p_s$ for various values of $a^*$ and $c^*$. The pdfs seem to have fat tails and their tails depend on the values of both $a^*$ and $c^*$. The corresponding cumulative distribution function (cdf) is defined as,

$$F(\geq |x|) = \int_{-\infty}^{-|x|} f(x')dx' + \int_{|x|}^{\infty} f(x')dx',$$

(8)

where $f(x)$ is a pdf. If the pdf follows power-law distribution the corresponding cdf is given by

$$F(\geq |x|) \propto |x|^{-\beta},$$

(9)

where $\beta$ is a power law exponent ($\beta > 0$), which is estimated from a slope of the cdf in the double-log scale. As shown in fig. 6 the power law exponent depends on the value of both $c^*$ and $a^*$. For large $a^*$ and $c^*$ the power law exponent becomes
small. Moreover let be $\kappa \equiv a^* c^*$. Then the power law exponent $\beta$ is a function of $\kappa$ as shown in fig. 7. We discuss why the power law exponent depends on $\kappa$ in the next section.

Typical example of time series of $n_s$ and its pdf are shown in fig. 8, respectively. The pdf can be approximated by a geometric distribution,

$$W(n) = p(1 - p)^n \quad (10)$$

This means that occurrence of trading is fully random. By numerical fitting in fig. 8 (right) we obtain $p = 0.36$. From analysis of high-frequency financial data it is clarified that the pdf trading intervals at the same time is fitted by exponential distribution [21].

4 Stochastic approximation as the GARCH process

Here we consider why the pdf of the price changes has fat tails and that its tail index depends on the parameter both $a^*$ and $c^*$. We will show that the market price is approximately dominated by the GARCH process through this section.

As shown by the numerical simulations the market price fluctuates discontinuously. Here we consider $\Delta p_s$, a change of the market price on the $s$th transaction as shown in fig. 4. Let $M_s$ denote the buying price at the $s$th transaction, and $m_s$ the selling price as shown in fig. 9. From the definition $P(\tau_s) = \frac{1}{2}(M_s + m_s)$ and $\Delta p_s = P(\tau_s) - P(\tau_{s-1})$, we get

$$\Delta p_s = \frac{1}{2}(M_s + m_s) - \frac{1}{2}(M_{s-1} + m_{s-1})$$

$$= \frac{1}{2}(M_s - M_{s-1}) + \frac{1}{2}(m_s - m_{s-1}). \quad (11)$$

From eq. (6) the next buyer at $\tau_{s-1}$ adds $|1 + c_j \Delta p_{s-1}|a_j$ into his/her bid price $n_s$ times until he/she can trade, and the next seller also subtracts $|1 + c_i \Delta p_{s-1}|a_i$
from his/her ask price $n_s$ times. Hence the first term and the second term in eq. (11) are given by
\begin{align*}
M_s - M_{s-1} &= |1 + c_j \Delta p_{s-1}| a_j n_{s-1} - K_s, \tag{12} \\
m_s - m_{s-1} &= k_s - |1 + c_i \Delta p_{s-1}| a_i n_{s-1}, \tag{13}
\end{align*}
where the subscript $j$ represents the dealer who gives the highest bid price at the time $\tau_s$, and $i$ represents the dealer who gives the lowest ask one. $K_s$ represents a difference of the bid prices between the $(s - 1)$th buyer and the $s$th buyer at $\tau_{s-1}$, and $k_s$ of the ask prices between the $(s - 1)$th seller and the $s$th seller.

Substituting eqs. (13) and (12) into eq. (11) yields
\begin{equation}
\Delta p_s = \frac{1}{2} |1 + c_j \Delta p_{s-1}| a_j n_{s-1} - \frac{1}{2} |1 + c_i \Delta p_{s-1}| a_i n_{s-1} + \frac{1}{2} (k_s - K_s). \tag{14}
\end{equation}

Since the dealer number $N$ is large we assume that \{c\} and \{a\} are mutually independent stochastic variable, which is uniformly distributed in intervals $[-c^*,c^*]$ and $[0,a^*]$. Therefore we have $\langle c \rangle = 0$, $\langle c^2 \rangle = c^*^2/3$, $\langle a \rangle = a^*/2$ and $\langle a^2 \rangle = a^*^2/3$. Moreover since the sellers and the buyers are symmetric $k_s - K_s$ is also symmetric. Actually it is easily confirmed by a numerical simulation. Fig. 11 (left) is a typical example of time series of $k_s - K_s$. It is obvious that $k_s - K_s$ is symmetric. Therefore $\langle k_s - K_s \rangle = 0$.

By taking conditionally averaging eq. (14) over dealers’ indices $i$ and $j$ we have,
\begin{equation}
\langle \Delta p_s \rangle = 0. \tag{15}
\end{equation}

By taking square of eq. (14) and averaging over dealers’ indices $i$ and $j$ under the condition that $\Delta p_{s-1}$ is realized, we get
\begin{equation}
\langle \Delta p_s^2 \rangle = \frac{1}{4} \langle (k_s - K_s)^2 \rangle + \frac{1}{4} \langle |1 + c \Delta p_{s-1}|^2 \rangle \langle a^2 \rangle \langle n_s^2 \rangle
\end{equation}
\[ + \frac{1}{4} \langle (1 + c\Delta p_{s-1})^2 \rangle \langle a^2 \rangle \langle n_s^2 \rangle - \frac{1}{2} \langle (1 + c\Delta p_{s-1}) \rangle \langle (1 + c\Delta p_{s-1}) \rangle \langle a \rangle \langle n_s^2 \rangle \]
\[ = \frac{1}{4} \langle (k_s - K_s)^2 \rangle + \frac{1}{2} \langle (1 + c\Delta p_{s-1})^2 \rangle \langle a^2 \rangle \langle n_s^2 \rangle - \frac{1}{2} \langle (1 + c\Delta p_{s-1}) \rangle^2 \langle a \rangle^2 \langle n_s^2 \rangle \quad (16) \]

The first term in the left hand side of eq. (16) is not constant. \( k_s - K_s \) has large fluctuations. Fig. 11 (right) is the pdf of \( k_s - K_s \). The pdf is the same distribution as \( \Delta p \). Hence we assumed that \( \langle (k_s - K_s)^2 \rangle \) is proportional to a conditional variance of \( \Delta p_{s-1} \), namely

\[ \frac{1}{4} \langle (k_s - K_s)^2 \rangle = \eta \langle \Delta p_{s-1}^2 \rangle, \quad (17) \]

where \( \eta \) is a positive coefficient. The second term in the left hand side of eq. (16) is calculated as,

\[ \frac{1}{2} \langle (1 + c\Delta p_{s-1})^2 \rangle \langle a^2 \rangle \langle n_s^2 \rangle = \frac{a^2}{6} \langle n_s^2 \rangle + \frac{a^2 c^*}{18} \Delta p_{s-1}^2 \langle n_s^2 \rangle. \quad (18) \]

The third term in the left hand side of eq. (16) is calculated as,

\[ \frac{1}{2} \langle (1 + c\Delta p_{s-1}) \rangle^2 \langle a \rangle^2 \langle n_s^2 \rangle = \left\{ \begin{array}{ll}
\frac{a^2}{32} \left( c^* \Delta p_{s-1} \right) + c^* \Delta p_{s-1}^2 \langle n_s^2 \rangle & (|\Delta p_{s-1}| \geq 1/c^*) \\
\frac{a^2}{32} \langle n_s^2 \rangle & (|\Delta p_{s-1}| < 1/c^*)
\end{array} \right. \quad (19) \]

Therefore by using \( \sigma_s^2 = \langle \Delta p_s^2 \rangle - \langle \Delta p_s \rangle^2 \) (16) is described as,

\[ \sigma_s^2 = \left\{ \begin{array}{ll}
\omega_1 + \eta \sigma_{s-1}^2 + \rho_1 \Delta p_{s-1}^2 + \frac{1}{\Delta p_{s-1}^2} & (|\Delta p_{s-1}| \geq 1/c^*) \\
\omega_2 + \eta \sigma_{s-1}^2 + \rho_2 \Delta p_{s-1}^2 & (|\Delta p_{s-1}| < 1/c^*)
\end{array} \right., \quad (20) \]

where \( \omega_1 \equiv \frac{5}{18} a^2 \langle n_s^2 \rangle, \rho_1 \equiv \frac{7}{288} \kappa^2 \langle n_s^2 \rangle, \gamma \equiv \frac{a^2}{32a^2} \langle n_s^2 \rangle, \omega_2 \equiv \frac{1}{24} a^2 \langle n_s^2 \rangle \) and \( \rho_2 \equiv \frac{1}{18} \kappa^2 \langle n_s^2 \rangle \). From eq. (10) we obtain \( \langle n_s^2 \rangle = \frac{(2-p)(1-p)}{p^2} \approx 2.92 \)

For large \( |\Delta p_{s-1}| \) the term \( \frac{1}{\Delta p_{s-1}^2} \) of eq. (20) is almost zero. Then eq. (20) is approximated as a GARCH(1,1) process. The probability density function of the GARCH(1,1) process has fat tail [22, 23]. The power law exponent is a function of \( \sigma_1 \) and \( \eta \). Namely the power law exponent depends on \( \kappa \). Furthermore if \( \kappa \) is small then the second term of eq. (20) vanishes, and we obtain \( \sigma_s^2 = \omega_2 + \eta \sigma_{s-1}^2 \).
The result of the iteration is given by,

\[
\sigma_s^2 \to \frac{\sigma_0^2}{1 - \eta} + \frac{\omega_2 \eta}{(1 - \eta)^2} \quad (s \to \infty).
\] (21)

Namely the variance of \(\Delta p_s\) is nearly constant. Then \(p_s\) behaves similarly to a Brownian motion.

5 Discussion

5.1 Correlation function

We calculate an autocorrelation coefficient of the market price changes and one of squared changes. The volatility clustering is the well-known fact that the squared changes of the market price are clustering in financial data. It is indicated that the volatility clustering is related to a long time correlation of the market \([6]\).

These two autocorrelation coefficients are defined by

\[
R^{(1)}_s = \frac{\langle \Delta p_{t+s} \Delta p_t \rangle - \langle \Delta p_{t+s} \rangle \langle \Delta p_t \rangle}{\langle \Delta p_t^2 \rangle - \langle \Delta p_t \rangle^2},
\] (22)

\[
R^{(2)}_s = \frac{\langle \Delta p_{t+s}^2 \Delta p_t^2 \rangle - \langle \Delta p_{t+s}^2 \rangle \langle \Delta p_t^2 \rangle}{\langle \Delta p_t^4 \rangle - \langle \Delta p_t^2 \rangle^2}.
\] (23)

As shown in fig. 10 (left) the autocorrelation coefficient is negative at small \(s\). It means that the price change tends to move to an opposite direction of the last price change. On the other hand autocorrelation coefficient of the squared change in fig. 10 (right) has a long time correlation. It is interesting that the volatility of price changes has long time correlation although each dealer just depends on the latest price change. Actually it is reported that the volatility oocorrelation function of the GARCH(1,1) process decreases exponentially \([24]\). However that is only true when a ARCH(1) parameter plus a GARCH(1) parameter is less than or equal to 1. When the ARCH(1) parameter plus the GARCH(1) parameter is greater than 1 it is impossible to derive the autocorrelation function in the same
analytical way. From numerical simulations of the pure GARCH(1,1) process we confirm that the autocorrelation function has long tail when the ARCH(1) parameter plus the GARCH(1) parameter is greater than 1. Specifically its long tail is significant when the GARCH(1) parameter is near 1. Hence we can explain from this property that the market price fluctuations of the dealer model exhibits the volatility clustering when $\eta + \rho_1 > 1$ and $\eta$ is near 1.

5.2 Demand and Supply curve of the model

In the model introduced in the article we treat the case that demand and supply are automatically balancing, so that the total number of both sellers and buyers is conserved. We introduce time-dependent density of buyers and sellers for a market price $P$ at time $t$, $d(P,t)$ and $s(P,t)$, respectively [25]. The cumulative frequency distributions of sellers and buyers $D(P,t)$ and $S(P,t)$ are respectively defined by

$$D(P,t) = \int_0^P d(P',t)dP'$$

$$S(P,t) = \int_P^\infty s(P',t)dP'.$$

Of course in practical markets numbers of sellers and buyers can be observed partially. However in the numerical simulation one can calculate them from all the dealers’ variable. Here we consider relation between volatility and a demand-supply curve. Fig. 12 shows $D(P,t) - S(P,t)$ in high volatility regime (A) and in low volatility one (B). It is found that a slope of $D(P,t) - S(P,t)$ in the high volatility regime is gradual. On the other hand one in the low is rapid. This means that a slope of $D(P,t) - S(P,t)$ is related to the volatility. Hence the volatility clustering is attributed to gradually changing of the demand-supply curve.
5.3 Amount of sellers and amount of buyers

In the model the seller and buyer who have exchanged commodity open his/her position depending on the latest price change after they make a trade. We assume that the sellers go on selling when the market price goes down and that the buyers go on buying when the market price goes up. Let be \( N_S \) and \( N_B \) represent an amount of the sellers and one of buyers. Then \( N_S - N_B \) fluctuates around zero.

The reason is as follows: When \( N_S - N_B > 0 \) the market price tends to go down. When \( N_S - N_B < 0 \) it goes up. However when the market goes down \( N_S \) decreases and \( N_B \) increases. \( N_S - N_B \) becomes less than zero. When the market price goes up \( N_S \) increases and \( N_B \) decreases. \( N_S - N_B \) becomes greater than zero. If the sellers/buyers go on selling/buying when the market price goes up/down then a crash/bubble occurs. Throughout both the GARCH process and the demand and supply we think that there are two reasons why the market price fluctuates. One results from rules of a dealers’ prediction depending on the past market price. Another results from unbalancing of demand and supply.

6 Conclusion

We introduced the dealer model in which a commodity is exchanged such as a foreign exchange market. The model generates the time series statistically similar to real financial one although the model has a few parameters. This means that the model contains primitive factor for the market price fluctuations. The probability density function of the artificial price changes has power law tails. The autocorrelation function of the changes has anti-correlation for a few ticks. This implies that the market price changes tend to move in opposite. A squared autocorrelation function (volatility correlation function) has a long tail.
although the dealers depend on the latest price change. We derived the GARCH process for price changes under the assumption that the dealer homogeneously trades irrespective of their coefficients i.e. we can take average of $\Delta p_i^2$ over all the dealers’ indices. From the GARCH(1,1) limit we explained the power-law of the probability density function and the long time correlation of volatility correlation function. The long time correlation of volatility can be also seen in a demand-supply curve. The slope of the demand-supply curve is gradual in high volatility regime. In contrary the slope of demand-supply curve is rapid in a low volatility case. We guess that the market price fluctuations result from both the dealers’ prediction depending on the past market prices and unbalancing of demand and supply.

The relation between microscopic behavior of dealers and macroscopic feature of a market will be bridged by statistical mechanics. Then the intriguing problem is to clarify what kind of microscopic interaction between dealers generate statistical feature of macroscopic variables.

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References

[1] H.E. Stanley, L.A.N. Amaral, P. Gopikrishnan and V. Plerou, Physica A, 283 (2000) 31–41.

[2] M. Ausloos, Physica A, 285 (2000) 48–65.
[3] X. Gabaix, P. Gopikrishnan, V. Plerou and H.E. Stanley, Nature, 423 (2003) 267–270.

[4] X. Gabaix, P. Gopikrishnan, V. Plerou and H.E. Stanley, Physica A, 324 (2003) 1–5.

[5] R.N. Mantegna and H.E. Stanley, An Introduction to Econophysics: Correlations and Complexity in Finance, Cambridge University Press, Cambridge (1999).

[6] M. Pasquini and M. Serva, Physica A, 269 (1999) 140–147.

[7] H. Nakao, Physics Letters A, 266 282–289.

[8] H. Takayasu, A.-H. Sato and M. Takayasu, Physical Review Letters, 79 (1997) 966–969.

[9] D. Sornette, Physica A, 250 (1998) 295–314.

[10] A.-H. Sato, H. Takayasu and Y. Sawada, 61 (2000) 1081–1087.

[11] R.F. Engle, Econometrica, 50 (1982) 987–1007.

[12] T. Bollerslev, R.Y. Chou and K.F. Kroner, Journal of Econometrics, 52 (1992) 5–59.

[13] P. Bak, M. Paczuski and M. Shubik, Physica A, 246 (1997) 430–453.

[14] N.F. Johnson, S. Jarvis, R. Jonson, P. Cheung, Y.R. Kwong and P.M. Hui, Physica A, 258 (1998) 230–236.

[15] T. Lux and M. Marchesi, Nature, 397 (1999) 498–500.

[16] D. Challet, M. Marsili and Yi-Cheng Zhang, Physica A, 294 (2001) 514–524.
[17] P. Jefferies, M.L. Hart, P.M. Hui and N.F. Johnson, The European Physical Journal B, 20 (2001) 493–501.

[18] H. Takayasu, H. Miura, T. Hirabayashi and K. Hamada, Physica A, 184 (1992) 127–134.

[19] A.-H. Sato and H. Takayasu, Physica A, 250 (1998) 231–252.

[20] A.-H. Sato and H. Takayasu, Empirical science of financial fluctuations – The Advent of Econophysics (Ed. by H. Takayasu), Springer-Verlag, Tokyo, (2002) 171–178.

[21] M. Takayasu and H. Takayasu, Physica A, 324 (2003) 101-107.

[22] D. Ghose and K.F. Kroner, Journal of Empirical Finance, 2 (1995) 225–251.

[23] P.A. Groenendijk, A. Lucas and C.G. de Vries, Journal of Empirical Finance, 2 (1995) 253–264.

[24] Z. Ding and C.W.J. Granger, Journal of Econometrics, 73 (1996) 185–215.

[25] H. Takayasu and M. Takayasu, Physica A, 269 (1999) 24–29.

| parameter | detail |
|-----------|--------|
| N         | amount of the dealers |
| $a^*$     | a prediction coefficient $a_i$ |
| $\Lambda$ | difference between retried ask/bid price and the market price. |
| $c^*$     | for a prediction coefficient $c_i$ |
Figure 1: The conceptual illustration of the dealer model. The inputs of the market are orders from dealers. The output of the market is a market price. The input of an agent is a sequence of past price changes. The output of an agent is a sell price or a buy price.

Figure 2: A typical example of time series of market prices at $N = 100$, $a^* = 0.01$ and $\Lambda = 1.0$ for $c^* = 0.0$ (right) and 375.0 (left).
Figure 3: A typical example of time series of price changes at \( N = 100, a^* = 0.01 \) and \( \Lambda = 1.0 \) for \( c^* = 0.0 \) (left) and \( 375.0 \) (right).

Figure 4: A conceptual illustration of a market price fluctuation. A trading occurs at random. \( \tau_s \) represents the time when the \( s \)th trading occurs. \( P(\tau_s) \) exhibits a market price at \( \tau_s \), \( \Delta p_s \) a market price change at \( \tau_s \), namely \( \Delta p_s = P(\tau_s) - P(\tau_{s-1}) \), and \( n_s \) a difference between \( \tau_{s-1} \) and \( \tau_s \), \( n_s \equiv \tau_s - \tau_{s-1} \).
Figure 5: Semi-log plots of the probability density functions of the price changes $\Delta p_s$. We fix $N = 100$, $\Lambda = 1.0$ and $a^* = 0.01$, and change $a^*$ (left). We fix $N = 100$, $\Lambda = 1.0$ and $c^* = 100.0$, and change $a^*$ (right).

Figure 6: Log-log plots of the cumulative distribution function of the price changes $\Delta p_s$. We fix $N = 100$, $\Lambda = 1.0$ and $a^* = 0.01$, and change $a^*$ (left). We fix $N = 100$, $\Lambda = 1.0$ and $c^* = 100.0$, and change $a^*$ (right).
Figure 7: Log-log plots of the cumulative distribution functions of the price changes $\Delta p_s$ at $\kappa = 3.8$ and $\kappa = 3.7$. It shows the cumulative distribution functions at three different parameter sets for each $\kappa$.

Figure 8: A typical example of time series of an interval between successive trading $n_s$ (left) and its probability distribution function in semi-log scale (right). Parameters are fixed at $N = 100$, $a^* = 0.01$, $\Lambda = 1.0$ and $c^* = 375.0$. 
Figure 9: A conceptual illustration for explaining dealers interaction between \( \tau_{s-1} \) and \( \tau_s \). \( M_s \) represents a buy price at \( \tau_s \), and \( m_s \) a sell price. The Next buyer at \( \tau_{s-1} \) adds \( |1 + c_j \Delta p_{s-1}| \) into his/her bid price \( n_s \) times, and the next seller subtracts \( |1 + c_i \Delta p_{s-1}| \) from his/her bid price \( n_s \) times.
Figure 10: Autocorrelation coefficient of price changes (left) and Autocorrelation coefficient of squared price changes (right) are calculated from temporal development of price changes. Parameters are fixed at \( N = 100, a^* = 0.01, \Lambda = 1.0 \) and \( c^* = 375.0 \).

Figure 11: A typical example of time series of \( k_s - K_s \) at \( N = 100, a^* = 0.01, \Lambda = 1.0 \) and \( c^* = 375.0 \) (left). The pdf of \( k_s - K_s \) and \( \Delta p_s \) at the same parameters (right).
Figure 12: A difference between cumulative frequency distribution $D(P,t)$ and $S(P,t)$. Capital A exhibits a high volatility regime, and B a low volatility one. Parameters are fixed at $N = 100$, $a^* = 0.01$, $\Lambda = 1.0$ and $c^* = 375.0$. 