Stability and bifurcations of a discrete-time prey-predator system with constant prey refuge

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Abstract. In ecology, by refuge an organism attains protection from predation by hiding in an area where it is unreachable or cannot simply be found. In population dynamics, once refuges are available, both prey-predator populations are expressively greater and meaningfully extra species can be sustained in the region. This examine the stability of a discrete predator prey model incorporating with constant prey refuge. Existence results and the stability conditions of the system are analyzed by obtaining fixed points and Jacobian matrix. The chaotic behavior of the system is discussed with bifurcation diagrams. Numerical experiments are simulated for the better understanding of the qualitative behavior of the considered model.

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1. Introduction

In applied mathematics and theoretical biology, the Lotka-Volterra and logistic systems have been reflected as samples of modeling plus dynamics. But, this one was assumed newly how different methods of regulatory mechanisms, like competition, consumption, birth and death and the like, result in modifications of the stability and dynamics of biological systems. Modeling in mathematics may be observed as the utmost capable tool used for calculating the response of natural ecology to external impacts. Conversely, researchers explore complex behavior and difficulties that is fundamental to any effort to apply mathematics to biology. However, in such efforts, the biological properties appear, possibly in their cleanest form with all their exact features.

The study of dynamical nature involving two or three interacting populations extracted from an ecology system looks to be biologically defensible along with practical attention in a numeral significant cases. Structures, as “agricultural crop – pest”, “forest-pest”, “valued profitable animal types its leading resource-its main predator”, and the rest, aid as models. The first mathematical discussion that is truthfully intended at relating interacting populations seemed as before as the 1920’s \cite{14}. Far ahead, Gause et al. \cite{3} operated on the investigation confirmation of the outcomes attained
by Volterra – Lotka and some mathematical ideologies established to authenticate their studies. Later, Kolmogorov [13] started Volterra’s studies who recommended a theoretically new methodology to the mathematical ecosystem.

Dynamical populations contain two considerations: one is reproduction and another one death of individuals. Three main categories of interactions among classes are familiar in ecosystem and are specified by $[++]$, $[-[-]$ & $[++]$ [19]. Here $+$ positions for a positive or constructive effect of one species upon another, and $-$ for an unfavourable consequence. The corresponding types of interspecies relations are identified as $[++]$ proto collaboration, symbiosis or mutualism; $[-[-]$ common competitive conquest, or competition for a mutual source; and $[++]$ is either parasite-host or predatory-prey population interactions.

Many works considered the dynamical significances of a two species prey predator system along with the effect of prey refuges [4, 8, 9, 16, 5, 7]. Furthermost combined the hiding behaviour of prey in predator prey structure as a novel component of them and its main result on the stability property has been investigated [18, 21, 22, 6]. Though, many of the research attentive on the predator-prey structures influence of prey refuges along by Holling nature functional response [4, 11, 12]. For reliability with the earlier research work in this arena, the following prey-predator structure studied including prey refuges [17] as an ecological place where predation amount is lesser;

$$\frac{du}{dt} = \gamma u (1-u) - \alpha (1-\mu)uv$$
$$\frac{dv}{dt} = \beta (1-\mu)uv - \sigma v,$$

where $u(t)$ and $v(t)$ are prey and predator population density at time $t$ individually, hence both are non-zero. The additional factors have the following biological significances: $\gamma$ is the inherent per capita growing amount of $u(t)$, $\alpha$ is the maximal per capita intake amount of $v(t)$, $\beta$ is the change aspect representing the quantity of a new born $v(t)$’s for each seized prey, $\sigma$ is the per capita death amount of $v(t)$ and parameter $\mu \in (0,1)$ signifies the constant measure of $u(t)$ by refuges. The factors are all non-zero values. Next, the effects of prey refuges over investigation of the following structures are estimated: (I) Fixed point density of two species populations, (II) Stability property of the positive interior fixed point.

Applying the argument as in [1, 20], seeing that the functions as well as variables switch only at ordered intervals, the following resulting formula is constructed over the translation from (1) by applying the main integer function $[t]$:

$$\frac{1}{u(t)} \frac{du}{dt} = [1-u(t)]\gamma - (1-\mu)\alpha v(t)$$
$$\frac{1}{v(t)} \frac{dv}{dt} = (1-\mu)\beta u(t) - \sigma,$$

using the piecewise constant arguments technique for differential equations $[t]$, thus integrating the model (2) on interval $[\ell, \ell + 1)$,

$$\ln \frac{u_{\ell + 1}}{u_{\ell}} = \gamma [1-u(\ell)] - (1-\mu)\alpha v(\ell)$$
$$\ln \frac{v_{\ell + 1}}{v_{\ell}} = (1-\mu)\beta u(\ell) - \sigma,$$

For $\ell = 0, 1, 2, \cdots$, the system of discrete-time 2-D incorporating prey refuges attain as follows:

$$u_{\ell + 1} = u_{\ell} \exp\left[1-u(\ell)\gamma - (1-\mu)\alpha v(\ell)\right]$$
$$v_{\ell + 1} = v_{\ell} \exp\left[(1-\mu)\beta u(\ell) - \sigma\right],$$
The paper is structured with solution of existence and uniqueness of system (1.1) in section 2 followed with analysis of stability condition in Section 3 with the help of fixed points. Section 4 provides some numerical examples with simulations. Section 5 discusses the chaos control of the system and the paper concludes with brief conclusion in the last section.

2. Boundedness, existence and uniqueness of solutions
The following subsequent lemma establishes the boundedness of solutions of (4).

2.1. Lemma [23, 24] Suppose that \( u(t) \) satisfies \( u(0) > 0 \) and \( u(t + 1) \leq u(t) \exp \left[ A_0 \left[ 1 - B u(t) \right] \right] \) for \( t \in [1, \infty) \), where \( B > 0 \) is a constant then,

\[
\limsup_{t \to \infty} u(t) \leq \frac{1}{A_0 B} \exp(1 - 1).
\]

The theorem given below establishes that each positive solution of system (4) is uniformly bounded.

2.2. Theorem Each positive solution \( \{(u(t), v(t))\} \) of the system (4) is uniformly bounded.
Proof. Let \( \{(u(t), v(t))\} \) be any positive solution of (4). Then from the first part of the system (4),

\[
u_{(t+1)} \leq u(t) \exp\left[ \gamma(1-u(t)) \right], \forall t = \{0,1,2,\ldots\}.
\]

Consider \( u_{(t)} > 0 \), using Lemma (2.1),

\[
\limsup_{t \to \infty} u_{(t)} \leq u \exp(\gamma - 1) = M_1. 
\]

From the 2nd part of the system (4),

\[
v_{(t+1)} = v_{(t)} \exp\left[ \beta(1-\mu)u - \sigma \right] \leq v_{(t)} \exp\left[ \beta u - \beta \mu u - \sigma \right],
\]

\[
v_{(t+1)} \leq v_{(t)} \exp\left[ \beta u(1 + \mu) \left( 1 - \frac{\sigma}{\beta u(1 + \mu)} \right) \right].
\]

By the assumption that \( v_{(t)} > 0 \), and Lemma 2.1,

\[
\limsup_{t \to \infty} v_{(t)} \leq \frac{1}{\sigma} \exp[\beta(1 + \mu) - 1] = M_2.
\]

This shows that \( \lim_{t \to \infty} \sup\{u_{(t)}, v_{(t)}\} \leq M \), where \( M = \max\{M_1, M_2\} \).

The sufficient condition for the solution of existence and uniqueness of the system (1) is:

2.3. Theorem For each initial condition which is positive, there exists a unique solution for the system (1).
Proof. A sufficient condition for the solution of existence and uniqueness of the system (1) in the area \( \Psi \times (0, \tau) \) is determined, where

\[
\Psi = \{(u(t), v(t)) \in \mathbb{R}^2 : \max \{u(t), v(t)\} \leq \mathbb{H}\}.
\]

The technique applied in [10] is implemented here. Consider \( L(\Lambda) = (L_1(\Lambda), L_2(\Lambda)) \), a mapping is defined by

\[
L_1(\Lambda) = (1-u)u - (1 - \mu)\alpha uv,
L_2(\Lambda) = (1 - \mu)\beta uv - \sigma v.
\]

(7)

For any \( \Lambda, \bar{\Lambda} \in \Psi \), it follows from (7) that
3. Fixed points and its stability

First, the existence of fixed points for the system (4) is discussed in this section, and then the stability are investigated from the fixed points by using the characteristic polynomial or the eigenvalues of the matrix evaluated.

3.1. Existence of fixed points of the model (4)

In this section, fixed points of the equation (4) are calculated through solving the system of algebraic equations:

\[ \begin{align*}
\exp \left[ \gamma (1 - u) - \alpha (1 - \mu) v \right] &= 0, \\
\exp \left[ \beta (1 - \mu) u - \sigma \right] &= 0.
\end{align*} \tag{8} \]

Solving (8), the authors arrive at three positive fixed points; hence the following theorem concluded.

3.2. Theorem. The existence of fixed points of the system (4) satisfies:

1) The trivial and axial fixed points \( FP_t = (0,0) \) and \( FP_\lambda = (1,0) \) always exist.

2) If \( \mu < \frac{\sigma}{\beta} \), then the interior fixed point \( FP_1 = \left( u_*, \frac{\gamma}{\alpha (1 - \mu)} (1 - u_*) \right) \) exists, where \( u_* = \frac{\sigma}{\beta (1 - \mu)} \).

3.3. Stability of the model (3)

Now the stability are investigated for fixed points \( FP_t, FP_\lambda \) and \( FP_1 \).

3.4. Theorem. [15, 25] The following situations are valid for the fixed point \( (u, v) \) of any system

1) If \( |\theta_1| < 1 \) and \( |\theta_2| < 1 \), it is a sink point and stable;

2) If \( |\theta_1| > 1 \) and \( |\theta_2| > 1 \), it is a source point and locally unstable;

3) If \( |\theta_1| < 1 \) and \( |\theta_2| > 1 \) (or \( |\theta_1| > 1 \) and \( |\theta_2| < 1 \)), it is a saddle point;

4) If \( |\theta_1| = 1 \) or \( |\theta_2| = 1 \), it is non-hyperbolic.
Consider the system (4),
\[ f(u, v) = u \exp[\gamma(1-u) - \alpha(1-\mu)v] \]
\[ g(u, v) = v \exp[\beta(1-\mu)u - \sigma]. \]

The Jacobian matrix obtained from the above system is
\[ J_{(u,v)} = \begin{bmatrix} (1-\gamma)u \exp[\gamma(1-u) - \alpha(1-\mu)v] & -\alpha(1-\mu) \exp[\gamma(1-u) - \alpha(1-\mu)v] \\ v \beta(1-\mu) \exp[\beta(1-\mu)u - \sigma] & \exp[\beta(1-\mu)u - \sigma] \end{bmatrix}. \] (9)

3.5. **Theorem** If \( \gamma > 0 \) and \( \sigma > 0 \), then the trivial fixed point \( FP_t \) is a saddle point.

**Proof.** The Jacobian matrix of the fixed point \( FP_t \) is obtained in the following form
\[ J(FP_t) = \begin{bmatrix} \exp(\gamma) & 0 \\ 0 & \exp(-\sigma) \end{bmatrix}. \]

Now, \( \exp(\gamma) + \exp(-\sigma) \) is the trace and \( \exp(\gamma - \sigma) \) is determinant of the Jacobian matrix. If \( |J(FP_t) - \theta I| = 0 \), the eigenvalues of \( J(FP_t) \) are \( \theta_1 = \exp(\gamma) \) and \( \theta_2 = \exp(-\sigma) \). The results proved by using Theorem 3.4. This completes the proof.

3.6. **Theorem** The axial fixed point \( FP_\lambda \) is

1) stable if \( 0 < \gamma < 2 \) and \( \mu > 1 - \frac{\sigma}{\beta} \) and unstable if \( \mu < 1 - \frac{\sigma}{\beta} \) and \( \gamma > 2 \).

2) saddle if \( \mu < 1 - \frac{\sigma}{\beta} \) and \( 0 < \gamma < 2 \) and non-hyperbolic if \( \mu = 1 - \frac{\sigma}{\beta} \) and \( \gamma = 2 \).

**Proof.** For \( FP_\lambda \), Jacobian matrix is
\[ J(FP_\lambda) = \begin{bmatrix} 1-\gamma & -\alpha(1-\mu) \\ 0 & \exp(\beta(1-\mu) - \sigma) \end{bmatrix}. \] (10)

Now, \( 1 - \gamma + \exp(\beta(1-\mu) - \sigma) \) is the trace and \( (1-\gamma)\exp(\beta(1-\mu) - \sigma) \) is determinant of the matrix (10). If \( |J(FP_\lambda) - \theta I| = 0 \), the eigenvalues of \( J(FP_\lambda) \) are \( \theta_1 = 1 - \gamma \) and \( \theta_2 = \exp(\beta(1-\mu) - \sigma) \). The results (1)–(2) are proved by using Theorem 3.4. This completes the proof.

3.7. **Theorem** Let \( \mu < 1 - \frac{\sigma}{\beta} \) and

1) if \( 1 - \frac{\sigma+1}{\beta} < \mu < \frac{4\beta - \gamma\sigma(\sigma+1)}{4\beta + \gamma\sigma} \), then \( FP_1 \) is a sink point.

2) if \( \mu < \min\left\{ \frac{4\beta - \gamma\sigma(\sigma+1)}{4\beta + \gamma\sigma}, 1 - \frac{\sigma+1}{\beta} \right\} \), then \( FP_1 \) is a source point.

3) if \( \mu > \frac{4\beta - \gamma\sigma(\sigma+1)}{4\beta + \gamma\sigma} \), then \( FP_1 \) is a saddle point.

4) if \( \mu > \frac{4\beta - \gamma\sigma(\sigma+1)}{4\beta + \gamma\sigma} \) and \( trace \neq 2 \), then \( FP_1 \) is a Flip bifurcation point.

5) if \( T^2 - 4D < 0 \) and \( \mu = 1 - \frac{\sigma+1}{\beta} \), then \( FP_1 \) is a Neimark-Saker bifurcation point.
Proof. At $FP_1$, the matrix as

$$J(FP_1) = \begin{bmatrix}
1 - \frac{\gamma \sigma}{\beta(1 - \mu)} & -\frac{\alpha \sigma}{\beta} \\
\frac{\gamma}{\alpha} & \frac{\beta - \sigma}{1 - \mu}
\end{bmatrix}.$$ (11)

From the matrix (11), the trace and determinant can be written as

$$B = \text{Trace}[J(FP_1)] = 2 - \frac{\gamma\sigma}{\beta(1 - \mu)},$$

$$C = \text{Det}[J(FP_1)] = 1 - \frac{\gamma\sigma(1 - \mu)}{\beta(1 + \sigma - (1 - \mu)\beta)}.$$ The characteristic equation of $J(FP_1)$ is $\Phi(\theta) = \theta^2 - B\theta + C = 0$. The results (1)–(5) proved by using Theorem 3.4. This completes the proof.

4. Numerical simulation

Theoretical discussion is confirmed in this section and sustained through suitable examples via considering certain distinctive cases of model (4). Numerical results obviously motivating rich dynamical behaviors. Furthermore, the trajectories of the solutions with time series and phase portraits for the model (4) are exhibited. Dynamical behavior of the model (4) about the interior fixed point in different sets of values is offered.

4.1. Example Let $\gamma = 0.85, \alpha = 0.99, \mu = 0.1, \beta = 2.07$ and $\sigma = 0.9$ be the parameter values of the system (4) with the initial points $u(0) = 0.65$ and $v(0) = 0.35$. Computation yields $(u, v) = (0.48, 0.49)$. The Jacobian matrix is $\begin{bmatrix} 0.5894 & -0.4304 \\ 0.9187 & 1 \end{bmatrix}$. Here $T = 1.5894$, $D = 0.9848$ and the eigenvalue are $\theta_{1,2} = 0.7944 \pm 0.5944i$ such that $|\theta_{1,2}| = 0.9924 < 1$. Since the stability criteria are fulfilled, the system (4) is stable, see Figure 1. The phase plane diagram in 1(b) displays a sink point and spiraling of the path towards the fixed point $(u, v)$.

![Figure 1. Stability Time Line and Phase Plane of FP1](image-url)
4.2. Example Let $\gamma = 0.85, \alpha = 0.99, \mu = 0.1, \beta = 1.99$ and $\sigma = 0.75$ be the values of the model (4) with the initial points $u(0) = 0.65$ and $v(0) = 0.35$. The Jacobian matrix is
\[
\begin{pmatrix}
0.6441 & -0.3731 \\
0.9931 & 1
\end{pmatrix}.
\]
Here $T = 1.6441$, $D = 1.0146$ and the eigenvalues are $\theta_{1,2} = 0.8220 \pm 0.5821i$ such that $|\theta_{1,2}| = 1.0073 > 1$. The criteria for stability are not satisfied. Hence the system (4) is unstable, see Figure 2. The phase plane diagram 2(b) displays a source and the path spirals inwards but settles as a limit cycle and does not approach the fixed point $(u_*, v_*)$.

![Figure 2. Unstability Time Line and Phase Plane of FP](image)

4.3. Example Existence of limit cycle is established for a selective set of parameters in phase portrait, see Figure 3. Consider two sets of parameters as $\gamma = 0.85, \alpha = 0.99, \mu = 0.05, \beta = 2.07$ and $\sigma = 0.5$ and $\gamma = 0.75, \alpha = 0.99, \mu = 0.075, \beta = 2.01$ and $\sigma = 0.85$ with the initial points $u(0) = 0.65$ and $v(0) = 0.35$. It is clearly seen that from Figure 3(a) the path spirals outwards however do not move towards a fixed point and the path spirals inwards but not tending to a fixed point, see Figure 3(b). Thus in both the figures, the path settle down as a limit cycle.

![Figure 3. Limit Cycles of FP of the System (4)](image)

4.4. Bifurcation analysis
The existence of bifurcations for the system (4) are analyzed in this section.

4.4.1 Bifurcation at FP$_1(1,0)$
The parameter for analyzing the existence of bifurcation is chosen as $\gamma$. The fixed point $FP_\lambda$ is said to undergo periodic doubling bifurcation if one of the eigenvalue is $-1$ and and other shall not be $1$ (or) $-1$, see [2].

The quadratic equation obtained from (10) is

$$\Phi(\theta) = \theta^2 - (1 - \gamma + \exp(\beta(1 - \mu - \sigma)))\theta + (1 - \gamma)\exp(\beta(1 - \mu - \sigma)).$$

By Theorem 3.6, $\gamma = 2$, the eigenvalues are

$$\theta_1 = -1, \theta_2 = \exp(\beta(1 - \mu - \sigma)).$$

Now, the following theorem stated for the periodic doubling bifurcation of (4).

4.4.2. Theorem The axial fixed point $FP_\lambda$ loses its stability at $\gamma = 2$ and

$$\theta_1 = -1, \theta_2 = \exp(\beta(1 - \mu - \sigma)) \neq \pm 1,$$

and become a periodic doubling bifurcation.

Consider $\mu = 1.1, \alpha = 0.65, \beta = 0.15, \sigma = 0.01$ and $1.9 \leq \gamma \leq 4$ in system (4) with initial points $u(0) = 0.55; v(0) = 0.35$. Periodic doubling bifurcation is exhibited for these parameter values. Fixed point is estimated to be $FP_\lambda = (1,0)$ by using simple calculations. Also, the conditions of Theorem 4.4.2 are proved as follows: $\gamma = 2$. Eignevalues are $\theta_1 = -1$ and $\theta_2 = 0.9753 \neq \pm 1$. The conditions for periodic doubling bifurcation are attained nearby the fixed point $FP_\lambda$ at $\gamma$ (bifurcation point). Periodic doubling bifurcation diagram of $(\gamma, u)$ plus lyapunov exponent are displayed in Figure 4(a) and 4(b).

4.4.3. Bifurcation at $FP_t(u, v)$

Let $\beta$ be the bifurcation parameter considered to analyze Neimark-Sacker (NS) bifurcation. The occurrence of this bifurcation is ensured when the eigenvalues at endemic equilibrium states are complex conjugate with modulus equal to 1 [2]. Utilizing Theorem 3.7, $\beta = \frac{\sigma + 1}{1 - \mu}$, the eigenvalues are

$$\theta_{1,2} = 1 - \frac{\sigma \gamma}{2(1 + \sigma)} \pm i \frac{1}{2} \sqrt{\frac{4\sigma \gamma}{1 + \sigma} + \frac{4\sigma}{\alpha(1 + \sigma)} - \left(\frac{\sigma \gamma}{1 + \sigma}\right)^2};$$

Now, the following theorem concluded for the NS bifurcation of (4).
4.4.4. Theorem The NS bifurcation of system (4) occurs when \( \beta = \frac{\sigma + 1}{1 - \mu} \) and

\[
|\theta_{1,2}| = \left| 1 - \frac{\sigma \gamma}{2(1 + \sigma)} \pm \frac{i}{2} \left( \frac{4\sigma \gamma + 4\sigma}{\alpha(1 + \sigma)} - \left( \frac{\sigma \gamma}{1 + \sigma} \right)^2 \right) \right| = 1.
\]

Taking \( \gamma = 1.99, \mu = 0.5, \alpha = 0.05, \sigma = 0.65 \) and \( 3.1 \leq \beta \leq 5 \) in the system (4) with the initial points \( u(0) = 0.65 \) and \( v(0) = 0.35 \). These parameter values are considered for NS bifurcation. Unique interior positive fixed point is calculated to be \( (u, v) = (0.3939, 48.2456) \) by simple calculation. Furthermore, from Theorem 4.4.4, \( \beta = 3.3 \) and the eigenvalues are \( \theta_{1,2} = 0.6080 \pm 0.7939i \) with \( |\theta_{1,2}| = 1 \). The NS bifurcation conditions are achieved near the interior fixed point \( FP_1 \) at \( \beta \) (bifurcation point). NS bifurcation diagrams of the interior fixed point \( FP_1 \) of the system (4) in \((\beta, u)\) and \((\beta, v)\). From Figure 5, it is perceived that interior positive fixed value \( FP_1 \) of the model (4) is stable for \( \beta = 3.3 \), lose its stability at \( \beta = 3.3 \) and for \( \beta > 3.3 \), the quasi-periodic orbits on the invariant cycle arise and period orbits appear in the period-windows, the paths change to near chaos by the growing of \( \beta \) value.

![Figure 5 The Bifurcation Diagram for Interior Fixed Point Corresponding to \( \beta \)](image)

![Figure 6 Local Amplifications of the Bifurcation Diagram for Interior Fixed Point.](image)
5. Chaos control

Now, the feedback chaos control strategy introduced for the system (4). Hence, the controlled discrete system of (4) is

\[
\begin{align*}
  u_{t+1} &= u_t \exp \left[ \gamma (1-u) - \alpha (1-\mu)v \right] + N(u_t - u^*), \\
  v_{t+1} &= v_t \exp \left[ \beta (1-\mu)u - \sigma \right] + N(v_t - v^*),
\end{align*}
\]

(12)

where \( N \) is the control parameter and \((u^*, v^*)\) is the interior fixed point for system (4). With appropriate selection of control parameter, the influence of bifurcation in system (4) can be vanished, delayed, or advanced.

The Jacobian matrix of (12) at \( FP_t \) is

\[
J(FP_t) = \begin{bmatrix}
  1 - \frac{\gamma \alpha}{\beta (1-\mu)} + N & -\frac{\alpha \sigma}{\beta} \\
  \frac{\gamma}{\alpha} \left( \beta - \frac{\sigma}{1-\mu} \right) & 1 + N
\end{bmatrix}
\]

The characteristic polynomial is

\[
\mathbb{P}(\nu) = \nu^2 - p_t(u^*, v^*)\nu + q_t(u^*, v^*),
\]

(13)

Where
\[ p_i(u^*, v^*) = 2 - \frac{\gamma \sigma}{\beta (1 - \mu)} + 2N, \]
\[ q_i(u^*, v^*) = \left( 1 - \frac{\gamma \sigma}{\beta (1 - \mu)} + N \right) (1 + N) + \left( \frac{r}{\alpha} \beta - \frac{\sigma}{1 - \mu} \right) \left( \frac{\alpha \sigma}{\beta} \right). \]

Then both eigenvalues of (13) lies in the unit disk if
\[ |p_i(u^*, v^*)| < 1 + q_i(u^*, v^*) < 2. \]

6. Conclusion
The dynamical nature of a discrete time two species system with including constant prey refuge is carried out. The stability conditions and bifurcation analysis is studied. Numerical simulations for periodic doubling bifurcation and Neimark-Sacker bifurcations are performed as well in accordance with the theoretical work. The time plots and different phase plane diagrams are presented to understand the dynamics exhibited by the system.

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