Modal-Hamiltonian interpretation of quantum mechanics and Casimir operators: the road toward quantum field theory

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Abstract

The general aim of this paper is to extend the Modal-Hamiltonian interpretation of quantum mechanics to the case of relativistic quantum mechanics with gauge \( U(1) \) fields. In this case we propose that the actual-valued observables are the Casimir operators of the Poincaré group and of the group \( U(1) \) of the internal symmetry of the theory. Moreover, we also show that the magnitudes that acquire actual values in the relativistic and in the non-relativistic cases are correctly related through the adequate limit.

1 Introduction

In spite of the impressive success of quantum theory, its interpretation is still an open problem. In previous works ([1], [2]) we have presented the Modal-Hamiltonian Interpretation (MHI) of non-relativistic quantum mechanics: a realist, non-collapse interpretation, which defines the preferred context of the system (the set of the actual-valued observables) in terms of its Hamiltonian. In subsequent works ([3], [4], [5]), we have shown that the Modal-Hamiltonian interpretative rule of actual-value ascription can be formulated in a Galilei-invariant form when expressed in terms of the Casimir operators of the Galilei group. In this way, the preferred context selected by the MHI turns out to be Galilei-invariant, a reasonable result from a realist viewpoint.

Although these interpretative conclusions were obtained for non-relativistic quantum mechanics, the idea of extending the interpretation to quantum field
theory by replacing the relevant symmetry group sounds rather natural. The
general aim of this paper is, precisely, to open the road toward that exten-
sion by beginning with the case of relativistic quantum mechanics with gauge
$U(1)$ fields, whose symmetry groups are the Poincaré group and the internal
symmetry group $U(1)$. In this context, we will propose an interpretative rule
according to which the preferred context is defined by the Casimir operators of
the Poincaré group and of the group $U(1)$ of the internal symmetry. We will ar-
gue that this rule leads to physically reasonable results in the relativistic realm,
since the resulting actual-valued observables can be considered objective mag-
nitudes because invariant under the relevant groups. However, one should also
expect the adequate relationship between the results obtained in the relativistic
case and those obtained in non-relativistic quantum mechanics. In fact, we will
also show that the magnitudes that acquire actual values in the relativistic and
in the non-relativistic cases are correctly related through the adequate limit.

For this purpose, the paper is organized as follows. In Section 2 we will briefly
introduce the central tenets of the MHI, in particular, the Modal-Hamiltonian
actualization rule of actual-value ascription. On the basis of the main features
of the Galilei group and of its central extension, presented in Section 3, in
Section 4 we will show how the Modal-Hamiltonian actualization rule can be
formulated under a Galilei-invariant form in terms of the Casimir operators
of the group. Section 5 will be devoted to the study of the Poincaré group,
its central extension and the two limits leading to the central-extended Galilei
group: the standard non-relativistic limit and the Inönü-Wigner contraction. In
Section 6, the limits of the Casimir operators of the trivially extended Poincaré
group will be obtained in order to show that those limits lead precisely to the
Casimir operators of the central-extended Galilei group; this result will count
in favor of the extrapolation of MHI to relativistic quantum mechanics. In
Section 7 we will focus on the internal symmetry $U(1)$, in order to propose
that the Casimir operator of the corresponding symmetry group also acquires
an actual-value; in particular, we will show that such a Casimir operator is
the charge which, as a consequence, may be legitimately considered and actual-
valued physical magnitude. Finally, in Section 8 we will draw our conclusions.

2 The Modal-Hamiltonian Interpretation

The MHI belongs to the modal family of interpretations of quantum mechanics
(see [6]); as a consequence, it is a realist interpretation according to which the
quantum state describes the possible properties of a system but not its actual
properties. Here we will only recall the interpretative postulates relevant to our
discussion.

The first step is to identify the systems that populate the quantum ontol-
ogy. By adopting an algebraic perspective, a quantum system is defined in the
following terms:

\textbf{Systems postulate (SP):} A \textit{quantum system} $\mathcal{S}$ is represented
by a pair $(\mathcal{O}, H)$ such that (i) $\mathcal{O}$ is a space of self-adjoint operators
on a Hilbert space $\mathcal{H}$, representing the observables of the system,
(ii) $H \in \mathcal{O}$ is the time-independent Hamiltonian of the system $\mathcal{S}$,
and (iii) if $\rho_0 \in \mathcal{O}^\prime$ (where $\mathcal{O}^\prime$ is the dual space of $\mathcal{O}$) is the initial
state of $\mathcal{S}$, it evolves according to the Schrödinger equation in its
von Neumann version.

Of course, any quantum system can be partitioned in many ways; however,
not any partition will lead to parts which are, in turn, quantum systems (see
\cite{7}, \cite{8}). On this basis, a composite system is defined as

**Composite systems postulate (CSP):** A quantum system
represented by $\mathcal{S} : (\mathcal{O}, H)$, with initial state $\rho_0 \in \mathcal{O}$, is composite
when it can be partitioned into two quantum systems $\mathcal{S}^1 : (\mathcal{O}^1, H^1)$
and $\mathcal{S}^2 : (\mathcal{O}^2, H^2)$ such that (i) $\mathcal{O} = \mathcal{O}^1 \otimes \mathcal{O}^2$, and (ii) $H = H^1 \otimes
I^2 + I^1 \otimes H^2$, (where $I^1$ and $I^2$ are the identity operators in the
 corresponding tensor product spaces). In this case, the initial states
of $\mathcal{S}^1$ and $\mathcal{S}^2$ are obtained as the partial traces
$\rho^1_0 = \text{Tr}(2)\rho_0$ and
$\rho^2_0 = \text{Tr}(1)\rho_0$; we say that $\mathcal{S}^1$ and $\mathcal{S}^2$ are subsystems of the composite
system, $\mathcal{S} = \mathcal{S}^1 \cup \mathcal{S}^2$. If the system is not composite, it is elemental.

Since the contextuality of quantum mechanics, implied by the Kochen-
Specker theorem (\cite{9}), prevents us from consistently assigning actual values to
all the observables of a quantum system in a given state, the second step is to
identify the preferred context, that is, the set of the actual-valued observables
of the system. For this purpose, we formulate a rule of actual-value assignment:

**Actualization rule (AR):** Given an elemental quantum system
represented by $\mathcal{S} : (\mathcal{O}, H)$, the actual-valued observables of $\mathcal{S}$ are $H$
and all the observables commuting with $H$ and having, at least, the
same symmetries as $H$.

This preferred context where actualization occurs is independent of time,
since it depends on the Hamiltonian: the actual-valued observables always
commute with the Hamiltonian and, therefore, they are constants of motion of the
system. In other words, the observables that receive actual values are the same
during all the “life” of the quantum system as such—precisely, as a closed
system—: there is no need of accounting for the dynamics of the actual properties
of the quantum system as in other modal interpretations (see \cite{10}).

The fact that the Hamiltonian always belongs to the preferred context agrees
with the many physical cases where the energy has definite value. The Modal-
Hamiltonian actualization rule has been applied to several well-known physical
situations (hydrogen atom, Zeeman effect, fine structure, etc.), leading to results
consistent with experimental evidence (see \cite{1}, Section 5). Moreover, it has
proved to be effective for solving the measurement problem, both in its ideal
and its non-ideal versions (see \cite{1}, Section 6).
3 The Galilei Group in quantum mechanics

The space-time symmetry group of non-relativistic—classical or quantum—
mechanics is the Galilei group \( G \), a Lie group with its associated Galilei algebra of
generators \( g \). This algebra is defined by ten symmetry generators \( G_\alpha \), with \( \alpha = 1 \) to 10: one time-displacement \( G_\tau \), three space-displacements \( G_r \), three spatial rotations \( G_\theta \), and three Galilei-boost-velocity components \( G_u \), with \( i = x, y, z \).

The group is defined by the commutation relations between its generators,

\[
\begin{align}
[G_{r_i}, G_{r_j}] &= 0 \quad (1a) \\
[G_{u_i}, G_{u_j}] &= 0 \quad (1b) \\
[G_{\theta_i}, G_{\theta_j}] &= i\varepsilon_{ijk} G_{\theta_k} \quad (1c) \\
[G_{\theta_i}, G_{r_j}] &= i\varepsilon_{ijk} G_{r_k} \quad (1d) \\
[G_{\theta_i}, G_{u_j}] &= i\varepsilon_{ijk} G_{u_k} \quad (1e) \\
[G_{r_i}, G_{\tau_j}] &= 0 \quad (1f) \\
[G_{r_i}, G_{\tau_j}] &= 0 \quad (1g) \\
[G_{r_i}, G_{\tau_j}] &= 0 \quad (1h) \\
[G_{r_i}, G_{\tau_j}] &= 0 \quad (1i)
\end{align}
\]

where \( i, j, k = x, y, z \), and \( \varepsilon_{ijk} \) is the Levi-Civita tensor, such that \( i \neq k \), \( j \neq k \), \( \varepsilon_{xyz} = \varepsilon_{yxz} = \varepsilon_{zyx} = 1 \), \( \varepsilon_{xzy} = \varepsilon_{yxz} = -1 \), and \( \varepsilon_{ijk} = 0 \) if \( i = j \).

Each Galilei transformation \( T_\alpha \in G \) acts on observables and states as

\[
O \rightarrow O' = U_{s_\alpha} O U_{s_\alpha}^{-1} \quad |\varphi\rangle \rightarrow |\varphi'\rangle = U_{s_\alpha} |\varphi\rangle \quad (2)
\]

where \( s_\alpha \) is the parameter corresponding to the transformation \( T_\alpha \), and \( U_{s_\alpha} \) is the family of unitary operators describing \( T_\alpha \). Since in any case \( s_\alpha \) is a continuous parameter, each \( U_{s_\alpha} \) can be expressed in terms of the corresponding symmetry generator \( G_\alpha \) as

\[
U_{s_\alpha} = e^{iG_\alpha s_\alpha} \quad (3)
\]

The combined action of all the transformations is given by

\[
U_s = \prod_{\alpha=1}^{10} e^{iG_\alpha s_\alpha} \quad (4)
\]

The Galilei group admits a nontrivial central extension by a central charge
that commutes with all its generators. Such an extension is obtained as a
semi-direct product between the Galilei algebra \( G \) and the algebra generated
by the central charge, which in this case denotes the mass operator \( M = mI \),
where \( I \) is the identity operator and \( m \) is the mass, \( G \times \langle M \rangle \) (see \[11], \[12\]).

The commutators corresponding to the extension are those of eqs.\[11], with the
exception of eq.(1f), which is replaced by

\[
[G_{u_i}, G_{r_j}] = i\delta_{ij} M \quad (5)
\]

While for an ordinary representation (or at the classical level) this extension
is unnecessary, for quantum representations with an arbitrary phase (i.e., such
that \( |\phi\rangle \sim \exp(i\omega) |\phi\rangle \) ) it is unavoidable (\[13], Chapter 3). In this central
extension, the symmetry generators represent the basic magnitudes of the
theory: the energy \( H = \hbar G_\tau \), the three momentum components \( P_i = \hbar G_{r_i} \),

\[
H = \hbar G_\tau, \quad P_x = \hbar G_{r_x}, \quad P_y = \hbar G_{r_y}, \quad P_z = \hbar G_{r_z}
\]
the three angular momentum components $J_i = \hbar G \theta_i$, and the three Galilei-boost components $K_i^{(G)} = \hbar G u_i$. The rest of the physical magnitudes can be defined in terms of these basic ones: for instance, the three position components are $Q_i = K_i^{(G)}/m$, the three orbital angular momentum components are $L_i = \varepsilon_{ijk}Q^j P^k$, the three spin components are $S_i = J_i - L_i$. Then, by taking $\hbar = 1$, the commutation relations result

\[
\begin{align*}
[P_i, P_j] &= 0 \quad (6_a) \\
[K_i^{(G)}, P_j] &= 0 \quad (6_b) \\
[J_i, J_j] &= i\delta_{ij} M \quad (6_c) \\
[J_i, P_j] &= i\varepsilon_{ijk} P_k \quad (6_d) \\
[J_i, K_j^{(G)}] &= i\varepsilon_{ijk} K_k^{(G)} \quad (6_e)
\end{align*}
\]

Let us recall that a Casimir operator of a Lie group is an operator that commutes with all the generators of the group and, therefore, is invariant under all the transformations of the group. In the case of the mass central-extended Galilei group, the Casimir operators are

\[
\begin{align*}
C_1^G &= M = mI \quad (7) \\
C_2^G &= MH - P^2/2 = M(H - P^2/2m) = mW \quad (8) \\
C_4^G &= M^2 J_i J^i + (P_i P^i)(K_i^{(G)} K^{(G)}) - (P_i K^{(G)i})^2 - 2M J_i J^i \varepsilon_{ijk} P^j K^{(G)k} \quad (9)
\end{align*}
\]

where $W$ is the internal energy operator. In the reference frame at rest with respect to the center of mass, these operators have the form

\[
\begin{align*}
C_1^G &= M = mI \quad C_2^G = mW \quad C_4^G = m^2 J_i J^i = m^2 S^2 \quad (10)
\end{align*}
\]

The eigenvalues of the Casimir operators label the irreducible representations of the group ([15], [16], [17]). So, in each irreducible representation, the Casimir operators are multiples of the identity since $M = mI$, $W = wI$ (where $w$ is the internal energy), and $S^2 = s(s + 1)I$ (where $s$ is the eigenvalue of the spin).

### 4 Interpretation and Galilei group

A continuous transformation, as in the case of the Galilei group, admits two interpretations. Under the active interpretation, the transformation corresponds to a change from one system to another—transformed—system; under the passive interpretation, the transformation consists in a change of the viewpoint—reference frame—from which the system is described (see [18]). Nevertheless, in both cases the validity of a group of symmetry transformations expresses the fact that the identity and the behavior of the system are not altered by the application of the transformations: in the active interpretation language, the original and the transformed systems are equivalent; in the passive interpretation language, the original and the transformed reference frames are equivalent. Then, any realist interpretation should agree with that physical fact: the rule
of actual-value ascription should select a set of actual-valued observables that remains unaltered under the transformations. Since the Casimir operators of the central-extended Galilei group are invariant under all the transformations of the group, one can reasonably expect that those Casimir operators belong to the set of the actual-valued observables.

As we have seen, the preferred context selected by the Modal-Hamiltonian actualization rule only depends on the Hamiltonian of the system. Then, the requirement of invariance of the preferred context under the Galilei transformations is directly fulfilled when the Hamiltonian is invariant, that is, in the case of time-displacement, space-displacement and space-rotation:

\[
H' = e^{iH\tau} H e^{-iH\tau} = H \quad \text{(since } [H, H] = 0) \\
H' = e^{iP_i \tau} H e^{-iP_i \tau} = H \quad \text{(since } [P_i, H] = 0, \text{ see eq.}(6_g)) \\
H' = e^{iJ_i \theta_i} H e^{-iJ_i \theta_i} = H \quad \text{(since } [J_i, H] = 0, \text{ see eq.}(6_h)) \quad (11)
\]

However, it is not clear that the requirement of invariance of the preferred context completely holds, since the Hamiltonian is not invariant under Galilei-boosts. In fact, under a Galilei-boost transformation corresponding to a velocity \(u_x\), \(H\) changes as

\[
H' = e^{iK_x'(G)_{ux}} H e^{-iK_x'(G)_{ux}} \neq H \quad \text{(since } [K_x'(G), H] = iP_x \neq 0, \text{ see eq.}(6_i)) \quad (12)
\]

Nevertheless, as we have shown in a previous work ([3]), when space is homogeneous and isotropic – when there are no external fields applied to the system –, a Galilei-boost transformation only introduces a change in the subsystem that carries the kinetic energy of translation: the internal energy \(W\) remains unaltered under the transformation. This should not sound surprising to the extent that the internal energy – multiplied by \(m\) – is a Casimir operator of the central-extended Galilei group (see eq.([10])).

On this basis, we can reformulate the actualization rule in an explicit Galilei-invariant form in terms of the Casimir operators of the central-extended group:

**Actualization rule’ (AR’):** Given a quantum system free from external fields and represented by \(\mathcal{S} = (\mathcal{O}, H)\), its actual-valued observables are the observables \(C_G^i\) represented by the Casimir operators of the central-extended Galilei group in the corresponding irreducible representation, and all the observables commuting with the \(C_G^i\) and having, at least, the same symmetries as the \(C_G^i\).

Since the Casimir operators of the central-extended Galilei group – in the reference frame of the center of mass – are \(M, mW\) and \(m^2S^2\), this reformulation of the rule is in agreement with the original AR when applied to a system free from external fields:

- The actual-valuedness of \(M\) and \(S^2\), postulated by AR’, follows from AR: these observables commute with \(H\) and do not break its symmetries because, in non-relativistic quantum mechanics, both are multiples of the
identity in any irreducible representation. The fact that $M$ and $S^2$ always acquire actual values is completely natural from a physical viewpoint, since mass and spin are properties supposed to be always possessed by any quantum system and measurable in any physical situation.

- The actual-valuedness of $W$ might seem to be in conflict with AR because $W$ is not the Hamiltonian: whereas $W$ is Galilei-invariant, $H$ changes under the action of a Galilei-boost. However, this is not a real obstacle because a Galilei-boost transformation only introduces a change in the subsystem that carries the kinetic energy of translation, which can be considered a mere shift in an energy defined up to a constant (see [3]).

Summing up, the application of the modal-Hamiltonian actualization rule leads to reasonable results, since the actual-valued observables turn out to be invariant and, therefore, objective magnitudes. The assumption of a strong link between invariance and objectivity is rooted in a natural idea: what is objective should not depend on the particular perspective used for the description; or, in group-theoretical terms, what is objective according to a theory is what is invariant under the symmetry group of the theory. This idea is not new. It was widely discussed in the context of special and general relativity with respect to the ontological status of space and time (see [19]). The claim that objectivity means invariance is also a central thesis of Weyl’s book Symmetry ([20]). In recent times, the idea has strongly reappeared in several works ([21], [22], [23], [24], [25]). From this perspective, the Modal-Hamiltonian actualization rule says that the observables that acquire actual values are those representing objective magnitudes. When expressed in so simple terms, we can expect that the rule can be extrapolated to any quantum theory endowed with a symmetry group. In particular, the actual-valued observables of a system in relativistic quantum mechanics would be those represented by the Casimir operators of the Poincaré group and of the internal symmetry group. In the following sections we will develop this idea in detail.

5 The Poincaré group and its limits

In the case of the Poincaré group, the generators are $H, P_i, J, K_i^{(P)}$, where the $K_i^{(P)}$ are the Lorentz-boost components. The commutation relations of the Poincaré group can be formulated in the 4-dimension Lorentz space-time as

$$
\begin{align*}
[P_\mu, P_\nu] & = 0, \\
[M_{\mu\nu}, P_\rho] & = \eta_{\mu\rho} P^\nu - \eta_{\nu\rho} P^\mu, \\
[M_{\mu\nu}, M_{\rho\sigma}] & = \eta_{\mu\rho} M^{\nu\sigma} - \eta_{\nu\rho} M^{\mu\sigma} - \eta_{\nu\sigma} M^{\mu\rho} + \eta_{\mu\sigma} M^{\nu\rho}
\end{align*}
$$

(13)

where $\mu, \nu, ... = 0, 1, 2, 3$, $\eta_{\mu\nu}$ is the metric tensor of space-time, and

$$
P_\mu = (H, P_i) \quad M_{\mu\nu} = \begin{pmatrix} 0 & K_i^{(P)} \\ -K_i^{(P)} & J_{ij} \end{pmatrix} \quad J_k = \varepsilon_{kij} J^{ij}
$$

(14)
Then, eqs. (13) can be rewritten in a form that permit them to be compared with the Galilei case:

\[
\begin{align*}
[P_i, P_j] &= 0 \quad (15a) \\
[K_i^{(P)}, K_j^{(P)}] &= -i\varepsilon_{ijk} J^k \quad (15b) \\
[J_i, J_j] &= i\varepsilon_{ijk} J^k \quad (15c) \\
[J_i, P_j] &= i\varepsilon_{ijk} P^k \quad (15d) \\
[J_i, K_j^{(P)}] &= i\varepsilon_{ijk} K^{(P)k} \quad (15e) \\
[K_i^{(P)}, P_j] &= i\delta_{ij} H \quad (15f) \\
[P_i, H] &= 0 \quad (15g) \\
[J_i, H] &= 0 \quad (15h) \\
[K_i^{(P)}, H] &= iP_i \quad (15i) \\
[J_i, K_j^{(P)}] &= i\varepsilon_{ijk} K^{(P)k} \quad (15e)
\end{align*}
\]

In turn, the Casimir operators of the Poincaré group are (see [26])

\[
\begin{align*}
C_2^P &= H^2 - P_i P^i \\
C_4^P &= H^2 J_i J^i + (P_i P^i) (K_i^{(P)} K^{(P)i}) - (J_i P^i)^2 - (P_i K^{(P)i})^2 - 2H J^k \varepsilon_{ijk} P^i K^{(P)j} \\
C_2^P &= m_0^2 I \\
C_4^P &= m_0^2 J_i J^i = m_0^2 S^2
\end{align*}
\]

(16)

In the reference frame at rest with respect to the center of mass, where \( P_i = 0 \) and \( H = E = m_0 \), these operators result

\[
\begin{align*}
C_2^P &= m_0^2 I \\
C_4^P &= m_0^2 J_i J^i = m_0^2 S^2
\end{align*}
\]

(17)

If we now extrapolate the invariant Modal-Hamiltonian actualization rule AR' to the relativistic case, we have to conclude that the Casimir operators \( C_2^P \) and \( C_4^P \) are the operators that define the actual-valued observables of the quantum system. This result is physically reasonable because mass and spin are properties supposed to be always possessed by any elemental particle (see [27]); moreover, mass and spin are two of the properties that contribute to the classification of elemental particles. However, the adequacy of the interpretation in the relativistic realm is not guaranteed yet, since it is still necessary to prove that the actual-valued observables in the relativistic and in the non-relativistic theories are correctly related through an adequate limit. This task leads us to analyze the relationship between the Galilei group and the Poincaré group.

As it is well known, the Galilei group can be recovered from the Poincaré group by means of an Inönü-Wigner contraction (see [26]). However, as we have seen, the physically meaningful group of quantum mechanics is not the Galilei group, but its central extension. Therefore, the question is whether the central-extended Galilei group can be obtained from a central extension of the Poincaré group. But the answer to this question is not straightforward, because the Poincaré group does not admit non-trivial central extensions ([28]). For this reason, in the following subsections we will consider two limiting procedures. First, we will review the traditional non-relativistic limit, which has a clear physical meaning but does not admit a direct representation in group terms. Then, we will introduce a generalized Inönü-Wigner contraction of a trivial extension of the Poincaré group, which, as it will be shown, leads to the central-extended Galilei group.

\footnote{A trivial extension of a Lie algebra \( g \) is a direct sum \( g \oplus M \), where \( M \) is an additional commuting generator.}
5.1 The traditional non-relativistic limit

Let us recall the relativistic transformations of coordinates:

\[
\begin{align*}
\vec{x'} &= R \vec{x} + \frac{(\gamma - 1)}{\beta^2} (\vec{\beta} \cdot \vec{x}) \vec{\beta} - \gamma ct \vec{\beta} + \vec{r} \\
c't &= \gamma (ct - \vec{\beta} \cdot \vec{x}) + c\tau
\end{align*}
\]

(18)

(19)

where \( \vec{r} \) is the space-displacement vector, \( \tau \) is the time-displacement scalar, \( R \) is the space-rotation matrix, and \( \gamma = (1 - \beta^2)^{-1/2} \) with \( \beta = \vec{v}/c \). These are the transformations that lead to the Poincaré group given by eqs. (15), where the parameters corresponding to each generator are: \( \tau \) for \( H \), \( \vec{r} \) for the \( P_i \), \( R \) for the \( J_i \), and \( \vec{\beta} \) for the \( K_i^{(P)} \).

The traditional relativistic limit is the limit \( \beta \to 0 \) \( (\gamma \to 1) \). This means that the limit affects only the boost-transformation, and not the remaining transformations. This fact can also be noted by comparing the central-extended Galilei group in eqs. (6) with the Poincaré group in eqs. (15): the two groups share a splittable seven dimensional subgroup \( ISO(3) \times \langle H \rangle \), defined by the commutation relations (15a), (15c), (15d), (15e) and (15f). In particular, \( \langle H \rangle \) is the time-displacement group generated by \( H \), and \( ISO(3) = \langle T_i \rangle \times SO(3) \) is the inhomogeneous rotation group in three dimensions, where \( \langle T_i \rangle \) is the space-displacement group generated by the \( P_i \) and \( SO(3) \) is the space-rotation group generated by the \( J_i \). Therefore, the difference between the Galilei and the Poincaré groups is confined to the commutation relations that involve the boost generators: the relativistic limit should turn the Poincaré boost-generators \( K_i^{(P)} \) into the Galilei boost-generators \( K_i^{(G)} \), and the commutation relations (15b), (15e), (15f) and (15i) of the Poincaré group into the commutation relations (6b), (6e), (6f) and (6i) of the Galilei group respectively.

Since in this case we are interested only in boosts, we can simplify the transformations of coordinates of eqs. (18) and (19) by making \( \tau = 0 \), \( \vec{r} = \vec{0} \) and \( R = I \):

\[
\begin{align*}
\vec{x'} &= \vec{x} + \frac{(\gamma - 1)}{\beta^2} (\vec{\beta} \cdot \vec{x}) \vec{\beta} - \gamma ct \vec{\beta} \\
c't &= \gamma (ct - \vec{\beta} \cdot \vec{x})
\end{align*}
\]

(20)

(21)

Let us also consider that energy, mass and momentum are

\[
E = \gamma m_0 c^2 \quad m = \gamma m_0 \quad p_i = \gamma m_0 v_i
\]

(22)

As it is well known, in the traditional relativistic limit \( \beta \to 0 \) \( (\gamma \to 1) \) we obtain

\[
\begin{align*}
\vec{x'} &= \vec{x} \\
t' &= t \\
E &= m_0 c^2 \\
m &= m_0 \quad p_i = m_0 v_i
\end{align*}
\]

(23)

(24)

On the other hand, the Poincaré-boost generators \( K_i^{(P)} \) can be expressed as

\[
K_i^{(P)} = X_i H - X_0 P_i
\]

(25)
where the $X_i$ are the position operators corresponding to the $x_i$, $H$ is the Hamiltonian operator corresponding to the energy $E$, $X_0$ is the operator conjugate to $H$ and, then, it corresponds to $ct$, and the $P_i$ are the momentum operators corresponding to the $p_i$. Therefore, by considering eqs. (23) and (24), the relativistic limit of the $K_i^{(P)}$ is

$$\lim_{\beta \to 0} K_i^{(P)} = X_i m_0 c^2 - c t P_i = K_i^{(\beta \to 0)}$$ (26)

Now, we can compute the commutation relations (15b), (15e), (15f) and (15i) with the just obtained $K_i^{(\beta \to 0)}$:

$$\lim_{\beta \to 0} \left[ K_i^{(P)}, K_j^{(P)} \right] = \left[ K_i^{(\beta \to 0)}, K_j^{(\beta \to 0)} \right] = 0$$ (27)

$$\lim_{\beta \to 0} \left[ J_i^{(P)}, K_j^{(P)} \right] = \left[ J_i^{(\beta \to 0)}, K_j^{(\beta \to 0)} \right] = i \varepsilon_{ijk} K_k^{(\beta \to 0)}$$ (28)

$$\lim_{\beta \to 0} \left[ K_i^{(P)}, P_j \right] = \left[ K_i^{(\beta \to 0)}, P_j \right] = i \delta_{ij} M_0 c^2$$ (29)

$$\lim_{\beta \to 0} \left[ K_i^{(P)}, H \right] = \left[ K_i^{(\beta \to 0)}, H \right] = i P_i$$ (30)

In turn, by making $c = 1$, eq. (26) becomes

$$\lim_{\beta \to 0} \left[ K_i^{(P)}, P_j \right] = \left[ K_i^{(\beta \to 0)}, P_j \right] = i \delta_{ij} M_0$$ (31)

Finally, let us compare eqs. (27), (28), (31), and (30) with the corresponding commutation relations (6b), (6e), (6f) and (6i) of the Galilei group: if the limit $K_i^{(\beta \to 0)}$ of the Poincaré-boost is identified with the Galilei-boost $K_i^{(G)}$, and the Poincaré operator $M_0$ is identified with the Galilei mass operator $M$, the central-extended Galilei group can be considered the non-relativistic limit of the Poincaré group.

### 5.2 A generalized Inönü-Wigner contraction

The traditional non-relativistic limit has a clear physical meaning and, then, it is desirable to express it in group terms. We know that the traditional Inönü-Wigner contraction maps the Poincaré group onto the Galilei group. But, since the mass generator has been added to the Galilei group, an analogous map between the Poincaré and the central-extended Galilei groups is not possible, to the extent that both groups have different numbers of generators. Therefore, a natural way of obtaining the desired map is by extending the Poincaré group. This is the strategy that we will follow below.

Let us consider the Poincaré group $ISO(1,3)$, with generators $\{H, P_i, J_i, K_i^{(P)}\}$, and its corresponding commutation relations given by eqs. (15). Let us recall that the Poincaré group does not admit non-trivial extensions. Therefore, we extend the group trivially, in such a way that all the generators commute with
a central charge \( M \). In this case, we obtain a new group \( ISO(1,3) \times \langle M \rangle \), with generators \( \{ H, P_i, J_i, K_i^{(P)}, M \} \), corresponding to a \textit{trivially extended Poincaré group}. Now, we can introduce the following change in the basis of generators:

\[
\mathcal{H} = H - M
\]

In the new basis \( \{ \mathcal{H}, P_i, J_i, K_i^{(P)} \} \), the commutation relations given by eqs. (15) preserve their form, with the only exception of eq.(15f), which becomes:

\[
\left[ K_i^{(P)}, P_j \right] = i\delta_{ij} H = i\delta_{ij}(\mathcal{H} + M)
\]

Now the task is to show that this trivially extended Poincaré group represented by \( ISO(1,3) \times \langle M \rangle \) contracts to the centrally extended Galilei group \( \mathcal{G} \times \langle M \rangle \):

\[
ISO(1,3) \times \langle M \rangle \longrightarrow \mathcal{G} \times \langle M \rangle
\]

The contraction is obtained by rescaling the generators as

\[
J_i' = J_i, \quad P_i' = \varepsilon P_i, \quad K_i^{(P)'} = \varepsilon K_i^{(P)}, \quad \mathcal{H}' = \mathcal{H}, \quad M' = \varepsilon^2 M
\]

The commutation relations given by eqs. (15) are left unchanged by the rescaling, with the exception of eq.(15b), and of eq.(15f) now replaced by eq.(33):

\[
\left[ K_i^{(P)'}, K_j^{(P)'} \right] = i\varepsilon_{ijk} \varepsilon^2 J^k
\]

As in the original Inönü-Wigner contraction, the operation is completed by introducing the limit \( \varepsilon \to 0 \), which turns eqs.(36) and (37) into

\[
\left[ K_i^{(P)'}, K_j^{(P)'} \right] = 0, \quad \left[ K_i^{(P)'}, P_j' \right] = 0
\]

The Inönü-Wigner contraction admits a physical interpretation (see [26]). The factor \( \varepsilon \) affects the boost generators \( K_i^{(P)'} \) and the momentum generators \( P_i' \). As a consequence, \( \varepsilon \) also affects the boost-velocities and the space-displacements resulting from the exponentiation of those generators. Then, by introducing the limit \( \varepsilon \to 0 \), we describe a situation where boost-velocities and space-displacements are “small”. Boost-velocities are small with respect to the velocity of light \( c \), which here was taken as \( c = 1 \). Space-displacements are small with respect to \( c \tau \), where \( \tau \) is the time-displacement associated with the Hamiltonian \( H \), which is not affected by \( \varepsilon \). For these reasons, this kind of contraction is known as “speed-space contraction” ([26]).

Summing up, the result of the application of this generalized Inönü-Wigner
contraction to the trivially extended Poincaré group is

\[ [P'_i, P'_j] = 0 \]  \hspace{0.5cm} (39a)
\[ [K'_i, K'_j] = 0 \]  \hspace{0.5cm} (39b)
\[ [J'_i, J'_j] = i\varepsilon_{ijk} J'^k \]  \hspace{0.5cm} (39c)
\[ [J'_i, P'_j] = i\varepsilon_{ijk} P'^k \]  \hspace{0.5cm} (39d)
\[ [J'_i, K'^j] = i\varepsilon_{ijk} K'^k \]  \hspace{0.5cm} (39e)
\[ [K'_i, P'_j] = i\delta_{ij} M' \]  \hspace{0.5cm} (39f)
\[ [K'_i, H'] = 0 \]  \hspace{0.5cm} (39g)
\[ [J'_i, K'^j] = i\varepsilon_{ijk} K'^k \]  \hspace{0.5cm} (39h)
\[ [J'_i, P'^j] = i\varepsilon_{ijk} P'^k \]  \hspace{0.5cm} (39i)
\[ [K'_i, H'] = iP'_i \]  \hspace{0.5cm} (39j)

Let us compare these eqs. (39) with the commutation relations given by eqs. (6), which define the central-extended Galilei group. If the mass \( M' \) of relation (39f) is identified with the mass \( M \) of relation (6f), and the Poincaré-boost \( K(P) \) of eqs. (39) is identified with the Galilei-boost \( K(G) \) of eqs. (6), then it can be said that the generalized İnönü-Wigner contraction of the trivially extended Poincaré group leads to the central-extended Galilei group, as originally expected (see [29]).

6 The limits of the Casimir operators

Let us recall that the physically meaningful group of non-relativistic quantum mechanics is not the Galilei group, but its central extension, whose Casimir operators, expressed in the reference frame of the center of mass, are (see eqs. (10))

\[ C^G_1 = M = mI \]  \hspace{0.5cm} (40)
\[ C^G_2 = mW = mwI \]  \hspace{0.5cm} (41)
\[ C^G_4 = m^2 J^i J^i = m^2 S^2 = m^2 s(s + 1)I \]  \hspace{0.5cm} (42)

In turn, the Casimir operators of the Poincaré group, expressed in the reference frame of the center of mass, are (see eqs. (17))

\[ C^P_2 = m_0^2 I \]  \hspace{0.5cm} (43)
\[ C^P_4 = m_0^2 J^i J^i = m_0^2 S^2 = m_0^2 s(s + 1)I \]  \hspace{0.5cm} (44)

It is quite clear that there is no limit that can introduce a map between the two \( C^G_i \) and the three \( C^G_j \). Nevertheless, in the traditional relativistic limit \( \beta \to 0 \) \((\gamma \to 1)\), \( m = \gamma m_0 \) becomes \( m_0 \) and \( E = \gamma m_0 c^2 \) becomes \( m_0 c^2 \). Therefore, by making \( c = 1 \), in the non-relativistic limit, both the mass \( m \) and the internal energy \( E = w \) are \( m_0 \). This means that, conceptually, the limit of \( C^P_i \) is \( C^G_i \), but the limit of \( C^P_2 \) leads to the two remaining Casimir operators \( C^G_1 \) and \( C^G_2 \); since in this limit \( m = w = m_0 \) and, thus, \( C^G_2 = (C^G_1)^2 \):

\[ C^P_4 \to C^G_4 \] \hspace{0.5cm} \[ C^P_2 \to C^G_2 = (C^G_1)^2 \]  \hspace{0.5cm} (45)

Of course, this is a conceptual argument that cannot be expressed in group language, to the extent that the limit relates a non-extended group with an
extended group. Then, we may expect that, by following the strategy developed in the previous section, the generalized Inönü-Wigner contraction of the Casimir operators of the trivially extended Poincaré group leads to the Casimir operators of the central-extended Galilei group.

The Casimir operators of the trivially extended Poincaré group represented by \( \text{ISO}(1, 3) \times (M) \) in the basis \( \{ \overline{H}, P_i, J_i, K^{(P)}_i, M \} \) are

\[
\begin{align*}
C^{PE}_1 &= M = mI \\
C^{PE}_2 &= -(P_iP^i) + \overline{H}^2 + M^2 + 2\overline{H}M \\
C^{PE}_4 &= (\overline{H} + M)^2 J_iJ^i - (J_iP^i)^2 + (P_iP^i) \left( K^{(P)}_iK^{(P)}_i \right) - (P_iK^{(P)}_i)^2 - 2(\overline{H} + M) J^k\epsilon_{ijk}P^jK^{(P)}_j 
\end{align*}
\]

By means of the rescaled basis introduced in eqs. (46), the Casimir operators are

\[
\begin{align*}
\tilde{C}^{PE}_1 &= \varepsilon^{-2}M' \\
\tilde{C}^{PE}_2 &= -\varepsilon^{-2}(P'_iP'^i) + \overline{H}'^2 + \varepsilon^{-4}M'^2 + 2\varepsilon^{-2}\overline{H}'M' \\
\tilde{C}^{PE}_4 &= (\overline{H}' + \varepsilon^{-2}M')^2 J'_iJ'^i - \varepsilon^{-2}(J'_iP'^i)^2 + \varepsilon^{-4}(P'_iP'^i)(K^{(P')}_iK^{(P')}_i) \\
&\quad - \varepsilon^{-4}(P'_iK^{(P')}_i)^2 - 2\varepsilon^{-2}(\overline{H}' + \varepsilon^{-2}M') J'^k\epsilon_{ijk}P'^jK^{(P')}_j 
\end{align*}
\]

As usual, the contracted Casimir operators are obtained by applying the limit \( \varepsilon \to 0 \) to the adequately rescaled operators:

\[
\begin{align*}
\tilde{C}^{PE}_1 &= \lim_{\varepsilon \to 0} \varepsilon^2\tilde{C}^{PE}_1 = M' \\
\tilde{C}^{PE}_2 &= \lim_{\varepsilon \to 0} \varepsilon^4\tilde{C}^{PE}_2 = M'^2 \\
\tilde{C}^{PE}_4 &= \lim_{\varepsilon \to 0} \varepsilon^4\tilde{C}^{PE}_4 = M'^2 J'_iJ'^i + (P'_iP'^i)(K^{(P')}_iK^{(P')}_i) - (P'_iK^{(P')}_i)^2 - 2M' J'^k\epsilon_{ijk}P'^jK^{(P')}_j 
\end{align*}
\]

Let us compare these eqs. (52), (53) and (54) with eqs. (7), (8) and (9), which express the Casimir operators of the mass central-extended Galilei group in the reference frame at rest with respect to the center of mass. As in the case of the commutation relations, if the mass \( M' \) of the first group of equations is identified with the mass \( M \) of the second group, and the Poincaré-boost \( K^{(P)\hat{n}} \) is identified with the Galilei-boost \( K^{(G)\hat{n}} \), it can be said that the generalized Inönü-Wigner contraction of the Casimir operators of the trivially extended Poincaré group leads to the Casimir operators of the central-extended Galilei group.

Summing up, when the Modal-Hamiltonian actualization rule is expressed in an explicit Galilei-invariant form, it leads to a physically reasonable result: the actual-valued observables are those represented by the Casimir operators of the mass central-extended Galilei group, \( M, W \) and \( S^2 \), which acquire their actual values \( m, w \) and \( s(s + 1) \). The natural strategy is to extrapolate the interpretation to the relativistic realm by replacing the Galilei group with the
Poincaré group. But when one takes into account that the relevant group of non-relativistic quantum mechanics is not the Galilei group but its central extension, the mere replacement of the relevant group is not sufficient: one has to show also that the actual-valued observables in the relativistic and the non-relativistic cases are related through the adequate limit. As a consequence, the Poincaré group has to be trivially extended, in order to show that the limit between the corresponding Casimir operators holds, and this result counts in favor of the proposed extrapolation of our MHI to relativistic quantum mechanics.

7 Relativistic quantum mechanics

Since the spirit of the MHI is to consider the observables representing invariances as the actual-valued observables of the system, when this interpretation is extrapolated to the relativistic domain, all the symmetries have to be considered. In particular, in relativistic quantum theories, besides the space-time symmetries represented by the Poincaré group, quantum systems have internal gauge-symmetries. Therefore, according to the MHI, the invariant magnitudes corresponding to those gauge-symmetries should also be actual-valued. As an illustration of this claim, in this section we will analyze the case of relativistic quantum mechanics with gauge $U(1)$ fields.

7.1 Internal symmetry

Let us consider a free Dirac field $\Psi$ whose Lagrangian has the following form:

$$L_D = \overline{\Psi} \left( i\hbar \frac{\partial}{\partial t} - c \overrightarrow{\alpha} \cdot \overrightarrow{p} - \beta \gamma^0 \right) \Psi$$

where $\overrightarrow{\alpha}$ and $\beta$ are the Dirac matrices, and $\Psi$ is a four component spinor that is composed of two spinors. This means that the field is $\Psi = (\phi, \chi)$ (and the conjugate transposed is $\Psi^\dagger = (\phi^\dagger, \chi^\dagger)$). So, we can write the Lagrangian of eq.(55) explicitly in terms of this spinors as

$$L_D(\phi, \chi) = i\hbar \frac{\partial \phi}{\partial t} - c \overrightarrow{\sigma} \cdot \overrightarrow{p} \phi - \beta m_o c^2 \phi$$

$$L_D(\phi, \chi) = i\hbar \frac{\partial \chi}{\partial t} - c \overrightarrow{\sigma} \cdot \overrightarrow{p} \chi - \beta m_o c^2 \chi$$

This Lagrangian is invariant under a global gauge-symmetry represented by the Abelian Lie group $U(1)$, such that the field transforms as

$$\Psi \rightarrow e^{-iQ\alpha} \Psi$$

$$\Psi \rightarrow \overline{\Psi} e^{iQ\alpha}$$

where $Q$ is the generator of the transformation and $\alpha$ is a constant real number.
that transforms the field as

\[ \Psi \rightarrow e^{-iQ\alpha(x)}\Psi \quad \overline{\Psi} \rightarrow \overline{\Psi}e^{iQ\alpha(x)} \]  

(59)

where \( \alpha(x) \) is now a real function of the space-time position \( x \). In order to recover invariance, a field \( A_\mu \) has to be included, such that it is transformed as

\[ A_\mu \rightarrow A_\mu + \partial_\mu \alpha \]  

(60)

and

\[ L_M = \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \]  

(61)

Then, the invariant Lagrangian is

\[ L_{Df} = \overline{\Psi} \left( i\hbar \left( \partial_t - ieA_o \right) - e\overrightarrow{\alpha} \cdot (\overrightarrow{p} - e\overrightarrow{A}) - \beta m_o c^2 \right) \Psi + L_M \]  

(62)

In this case, \( Q \) is trivially the only Casimir operator \( C_U^1 \) of the internal group \( U(1) \).

Since the internal gauge-symmetry \( U(1) \) is a symmetry of the theory, according to the MHI extrapolated to this case, the only Casimir operator \( C_U^1 = Q \) of this symmetry group—invariant under the corresponding transformations—is an actual valued observable of the system. Again, this leads to a physically reasonable result since the operator \( Q \) of the internal gauge-symmetry \( U(1) \) is the charge operator, \( Q = eI \).

### 7.2 The limit of the internal gauge-symmetry

In the literature, it is usual to find the non-relativistic limit of the Euler-Lagrange equations, but not of the Lagrangian. In order to obtain this limit, we can introduce the following ansatz:

\[ \begin{pmatrix} \phi \\ \chi \end{pmatrix} = e^{-\frac{im_o c^2}{\hbar} t} \begin{pmatrix} \phi_o \\ \chi_o \end{pmatrix} \]  

(63)

where \( \phi_o \) and \( \chi_o \) still depend on space and time coordinates. Eq.(63) expresses the spinors in terms of two separate time-dependent factors: one unknown, given by \( \phi_o \) and \( \chi_o \), and the other an oscillating factor with frequency \( \omega_o = \frac{m_o c^2}{\hbar} \). By introducing eq.(63) into eq.(57), we obtain

\[ L_D(\phi_o, \chi_o) = i\hbar \phi_o^\dagger \partial_t \phi_o + i\hbar \chi_o^\dagger \partial_t \chi_o - \phi_o^\dagger c \overrightarrow{\sigma} \cdot \overrightarrow{p} \chi_o - \chi_o^\dagger c \overrightarrow{\sigma} \cdot \overrightarrow{p} \phi_o + 2m_o c^2 \chi_o^\dagger \chi_o \]  

(64)

which can be rearranged as

\[ L_D(\phi_o, \chi_o) = i\hbar \phi_o^\dagger \partial_t \phi_o - \phi_o^\dagger c \overrightarrow{\sigma} \cdot \overrightarrow{p} \chi_o - \chi_o^\dagger c \overrightarrow{\sigma} \cdot \overrightarrow{p} \phi_o + i\hbar \chi_o^\dagger (\partial_t + 2m_o c^2) \chi_o \]  

(65)

The time-derivatives of the spinors \( \phi_o \) and \( \chi_o \) are related with the time-oscillation with frequency \( \omega = E_o/\hbar \).

Up to this point, no non-relativistic limit has been introduced yet. In order to perform such a limit, we have to notice that the total energy for the spinor
\( \phi_o \) is \( E_{\phi_o} = E_o \) and for the spinor \( \chi_o \) is \( E_{\chi_o} = E_o + 2m_o c^2 \). So, if we consider that \( E_o \ll m_o c^2 \), then

\[
E_{\phi_o} = E_o \quad E_{\chi_o} = E_o + 2m_o c^2 \sim 2m_o c^2 \tag{66}
\]

These two relations imply that the last two terms of eq.(65) can be written as

\[
\text{i} \hbar \chi_o^\dagger (\partial_t + 2m_o c^2) \chi_o = \text{i} \hbar \chi_o^\dagger (E + 2m_o c^2) \chi_o \sim \text{i} \hbar \chi_o^\dagger 2m_o c^2 \chi_o 
\]

and the non-relativistic limit of the Lagrangian reads

\[
\tilde{L}_D(\phi_o, \chi_o) = \text{i} \hbar \phi_o^\dagger \partial_t \phi_o - \phi_o^\dagger c \partial_t \phi_o - \chi_o^\dagger c \partial_t \chi_o - \chi_o^\dagger c \partial_t \phi_o + \text{i} \hbar 2m_o c^2 \chi_o^\dagger \chi_o \tag{68}
\]

In order to write the Lagrangian of eq.(68) in terms of only one of the spinors, say \( \phi_o \), we have to begin by computing the Euler-Lagrange equation for \( \chi_o \),

\[
\partial_t \left( \frac{\partial \tilde{L}_D}{\partial (\partial_t \chi_o)} \right) + \partial_t \left( \frac{\partial \tilde{L}_D}{\partial (\partial_t \phi_o)} \right) - \frac{\partial \tilde{L}_D}{\partial \chi_o} = 0 \tag{69}
\]

which results

\[
c \vec{\sigma} \cdot \vec{p} \phi_o - 2m_o c^2 \chi_o = 0 \tag{70}
\]

or, equivalently,

\[\chi_o = \frac{c \vec{\sigma} \cdot \vec{p} \phi_o}{2m_o c} \tag{71}\]

If we now replace eq.(71) into eq.(68), we obtain

\[
\tilde{L}_D(\phi_o, \chi_o) = \text{i} \hbar \phi_o^\dagger \partial_t \phi_o - \phi_o^\dagger c \vec{\sigma} \cdot \vec{p} \frac{\vec{\sigma} \cdot \vec{p} \phi_o}{2m_o c} - \frac{1}{2m_o c} \phi_o^\dagger \phi_o \frac{\vec{\sigma} \cdot \vec{p}^2 \phi_o}{2m_o c} + \text{i} \hbar 2m_o c^2 \chi_o^\dagger \chi_o \tag{72}
\]

Since \( \frac{\vec{\sigma} \cdot \vec{p} \phi_o}{2m_o c} = \frac{1}{2m_o c} \phi_o^\dagger (\vec{\sigma} \cdot \vec{p}) \dagger \), eq.(72) reads

\[
\tilde{L}_D(\phi_o) = \text{i} \hbar \phi_o^\dagger \partial_t \phi_o - \frac{1}{2m_o} \phi_o^\dagger \phi_o \frac{\vec{\sigma} \cdot \vec{p}^2 \phi_o}{2m_o c} \tag{73}\]

where we have used \( (\vec{\sigma} \cdot \vec{p})(\vec{\sigma} \cdot \vec{p}) = \vec{\sigma}^2 \). In particular, the last term of eq.(73) can be written as

\[
\phi_o^\dagger \phi_o \frac{\vec{\sigma} \cdot \vec{p}^2 \phi_o}{2m_o c} = \hbar^2 \phi_o^\dagger \nabla^2 \phi_o = \hbar^2 (\nabla \phi_o \nabla \phi_o - \nabla \phi_o \nabla \phi_o) \tag{74}\]

The first term of the r.h.s. of eq.(74) is a divergent term, which only contributes with a surface term that becomes zero when the Lagrangian density is integrated. Then, eq.(73) finally results

\[
L_{NR} = \tilde{L}_D(\phi_o) = \text{i} \hbar \phi_o^\dagger \partial_t \phi_o - \frac{\hbar^2}{2m_o} \nabla \phi_o \nabla \phi_o \tag{75}\]
which is the desired non-relativistic Schrödinger Lagrangian. In turn, the Euler-Lagrange non-relativistic equation reads

\[ i\hbar \partial_t \phi_o = \frac{\hbar^2}{2m_o} \nabla^2 \phi_o \]  

(76)

### 7.3 The full group of relativistic quantum mechanics with $U(1)$ fields

With respect to the space-time symmetries, we can see that:

- The relativistic Lagrangian $L_D$ of eq.(55) is invariant under the Poincaré group $ISO(1,3)$, and also under its trivial extension $ISO(1,3) \times \langle M \rangle$.

- The non-relativistic Schrödinger Lagrangian $L_{NR}$ of eq.(75) is invariant under the Galilei group $G$, and also under its trivial extension $G \times \langle M \rangle$, which, as we have seen, can be obtained from the Inönü-Wigner contraction of $ISO(1,3) \times \langle M \rangle$.

With respect to the internal symmetries, in turn:

- The relativistic Lagrangian $L_D$ of eq.(55) is invariant under a global gauge-transformation $U(1)$ acting on the spinors. If we want to preserve invariance under a local gauge-transformation, we have to introduce gauge-fields, which turn out to be the electromagnetic potentials obeying the Maxwell equations (see [30]).

- The non-relativistic Schrödinger Lagrangian $L_{NR}$ of eq.(75) is also invariant under a global gauge-transformation $U(1)$, and it is also invariant under a local gauge-transformation by means of the introduction of electromagnetic fields (see [31]).

The fact that electromagnetism can be obtained from a non-relativistic theory might sound weird. However, nowadays it is clear that the structure of the Maxwell equations is not determined by the symmetry properties of space-time but by the properties of gauge-symmetries. The only difference between the non-relativistic and the relativistic cases is that, whereas the gauge-potentials in the relativistic Lagrangian are strictly electromagnetic potentials, the gauge-potentials in the non-relativistic Lagrangian are components of a Galilean vector field, and this means that they transform as irreducible representations of the central extension of the Galilei group. As a consequence, in the non-relativistic case the potentials are the “magnetic limit” or the “electric limit” of the electromagnetic potential (see [32]).

Now the question is how the kinematical Poincaré group and the internal gauge-group combine together to lead to a new group, whose Casimir operators would represent the actual-valued observables of the relativistic quantum system according to our MHI. The point is relevant because the Casimir operators of that new group might be different than the Casimir operators of the component
groups \((M\) and \(S^2\) coming from the Poincaré group, and \(Q\) coming from the gauge group). Fortunately, this is not the case: according to the Coleman-Mandula theorem \([33]\), for a simpler presentation, see \([34, 35]\), there is no non-trivial union of the Poincaré group and the internal group. In other words, the only possible combination between the two groups is the direct product.

Summing up, in this case the full group is

\[
ISO(1,3) \times \langle M \rangle \times U(1)
\]

(77)

whose Casimir operators are those of the trivially extended Poincaré group —given by the mass \(M\) and the spin \(S^2\) and of the internal gauge-group \(U(1)\) —given by the charge \(Q\). In turn, the full group of non-relativistic quantum mechanics is

\[
\mathcal{G} \times \langle M \rangle \times U(1)
\]

(78)

whose Casimir operators are those of the central extended Galilei group —given by the mass \(M\), the internal energy \(W\), and the spin \(S^2\) and of the internal gauge-group \(U(1)\) —given by the charge \(Q\). The Inönü-Wigner contraction applied to the full group of eq.\((77)\) leads to the full non-relativistic group:

\[
ISO(1,3) \times \langle M \rangle \times U(1) \to \mathcal{G} \times \langle M \rangle \times U(1)
\]

(79)

As a consequence, according to our MHI, the actual-valued observables in the relativistic case are \(M, S^2\) and \(Q\), with their actual values: mass \(m_0\), spin \(s\) and charge \(\epsilon\). This is the result that one expects from a physical viewpoint, since mass, spin and charge are properties supposed to be always possessed by any quantum system and measurable in any physical situation, and their values are precisely what define the different kinds of the elemental particles of the theory.

### 7.4 The many-particle case

As it is well known, the many-particle case cannot rigorously treated by relativistic quantum mechanics, and this fact leads us to the realm of quantum field theory. We also know that, in general, the particle number is not a conserved quantity in quantum field theory, nor a Casimir operator of the relevant symmetry group of the theory. Nevertheless, in the particular cases of the “in” and the “out” stages of the scattering process, it is assumed that the system can be modelled as a collection of \(N\) non-interacting particles (the experimentally detected particles). Therefore, at those stages the relevant group is the tensor product of \(N\) copies of the full group \(ISO(1,3) \times \langle M \rangle \times U(1)\) (for simplicity we will only consider collections consisting of a single kind of elementary particles). Since the representations of this \(N\)-tensor product can be expressed as products of the representations of the factor groups, they are labelled by \(N\) and the Casimir operators of the group \(ISO(1,3) \times \langle M \rangle \times U(1)\). This means that, in the “in” and the “out” stages, the particle number operator \(N\) becomes an extra Casimir operator to be taken into account.

Let us consider a particle labelled by the Casimir operators \(C_1^{PE} = M\), \(C_2^{PE} = M^2\) and \(C_4^{PE} = S^2\) of the trivially extended Poincaré group —in the
reference frame of the center of mass— and the Casimir operator \( C_{11}^U = Q \) of the gauge-group. A non-interacting \( n \)-particle state is given by

\[
|n\rangle = |1\rangle \otimes |1\rangle \otimes ... |1\rangle = (a^\dagger a^\dagger ... a^\dagger)|0\rangle = (a^\dagger)^n|0\rangle,
\]

where \( a^\dagger \) is the creation operator of each particle, and

\[ N|n\rangle = n|n\rangle, \]

is the particle number, which can be easily seen to be additive. Now we can combine the already known Casimir operators, and define the new operators for the collection of \( N \) particles with the same mass and spin:

\[
\text{Mass}=(C_{2P}^E)^{1/2}N \quad \text{Spin}=(C_{4P}^E)^{1/2} \quad \text{Charge}=Q \quad \text{Particle Number}=N.
\]

According to our MHI, all these operators represent actual-valued observables of the system of \( N \) non–interacting particles. Let us stress again that this result is not general in quantum field theory, but it is only valid in the “in” and “out” stages of the scattering process. Nevertheless, the fact that \( N \) turns out to be a definite-valued observable is a reasonable result in those stages, where the number of particles is always considered a definite magnitude of the system. In turn, since in the interacting stage \( N \) is not a Casimir operator, according to our MHI it is not a definite-valued observable; and this is also reasonable because the particle number is not expected to be definite in the presence of interaction.

8 Conclusions

The interpretation of quantum mechanics is still one of the most debated problems in the foundations of physics. Although many new formal results were obtained during the last decades, the links with physical models have lost their strength in the discussions. With our MHI, we have tried to revert this situation by taking into account physical observables and their physical meaning as generators of symmetries. In particular, we have expressed our rule of actual-value ascription in a Galilei-invariant form, in terms of the Casimir operators of the Galilei group.

On the other hand, the interpretation of quantum relativistic theories has been a much less discussed topic. In this paper we have argued that, since group considerations play a central role in our interpretation of non-relativistic quantum mechanics from the very beginning, the extrapolation of the strategy to the relativistic case—by replacing the relevant group— is straightforward: the actual-valued observables of the system turn out to be the Casimir operators of the Poincaré group and of the internal gauge-group. In particular, we have shown that this extrapolation leads to physically reasonable results, since the actual-valued observables so selected are magnitudes supposed to be always possessed by the systems, and they are also the properties that contribute to the classification of elemental particles. Moreover, we have also proved that
the actual-valued observables in the relativistic and the non-relativistic theories are correctly related through an adequate limit, which can also be expressed in group terms. On the basis of these results, we consider that the further extension of the MHI to quantum field theory is an issue that deserves to be studied.

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