Dynamics of the universal area-preserving map associated with period doubling: hyperbolic sets

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Abstract

It is known that the famous Feigenbaum–Coullet–Tresser period doubling universality has a counterpart for area-preserving maps of $\mathbb{R}^2$. A renormalization approach has been used in Eckmann et al (1982 Phys. Rev. A. 26 720–2) and Eckmann et al (1984 Mem. Am. Math. Soc. 47 1–121) in a computer-assisted proof of existence of a ‘universal’ area-preserving map $F_*$—a map with orbits of all binary periods $2^k, k \in \mathbb{N}$. In this paper, we consider maps in some neighbourhood of $F_*$ and study their dynamics.

We first demonstrate that the map $F_*$ admits a ‘bi-infinite heteroclinic tangle’: a sequence of periodic points $\{z_k\}, k \in \mathbb{Z}$,

\[ |z_k| \xrightarrow{k \to \infty} 0, \quad |z_k| \xrightarrow{k \to -\infty} \infty, \]

(1)

whose stable and unstable manifolds intersect transversally; and, for any $N \in \mathbb{N}$, a compact invariant set on which $F_*$ is homeomorphic to a topological Markov chain on the space of all two-sided sequences composed of $N$ symbols. A corollary of these results is the existence of unbounded and oscillating orbits.

We also show that the third iterate for all maps close to $F_*$ admits a horseshoe. We use distortion tools to provide rigorous bounds on the Hausdorff dimension of the associated locally maximal invariant hyperbolic set:

\[ 0.7673 \geq \dim_H(C_F) \geq \epsilon \approx 0.00013 e^{-7499}. \]

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(Some figures in this article are in colour only in the electronic version)
1. Introduction

Following the pioneering discovery of the Feigenbaum–Coullet–Tresser period doubling universality in unimodal maps (Feigenbaum 1978, 1979, Tresser and Coullet 1978) universality has been demonstrated to be a rather generic phenomenon in dynamics.

To prove universality one usually introduces a renormalization operator on a functional space, and demonstrates that this operator has a hyperbolic fixed point.

This renormalization approach to universality has been very successful in one-dimensional dynamics, and has led to the explanation of universality in unimodal maps (Epstein 1986, 1989, Lyubich 1999), critical circle maps (de Faria 1992, 1999, Yampolsky 2002, 2003) and holomorphic maps with a Siegel disc (McMullen 1998, Gaidashev and Yampolsky 2007, Yampolsky 2007).

Universality has been abundantly observed in higher dimensions, in particular, in two and more dimensional dissipative systems (cf. Collet et al (1980), Sparrow (1982)), in area-preserving maps, both as the period doubling universality (Benettin et al 1980, Derrida and Pomeau 1980, Helleman 1980, Collet et al 1981, Eckmann et al 1982, 1984, Gaidashev and Koch 2008) and as the universality associated with the break-up of invariant surfaces (MacKay 1982, Shenker and Kadanoff 1982, MacKay 1983, Mehr and Escande 1984) and in Hamiltonian flows (Escande and Doveil 1981, Abad et al 1998, 2000, Koch 2002, 2004a,b, Gaidashev and Koch 2004, Gaidashev 2005, Kocić 2005).

It has been established that the universal behaviour in dissipative and conservative higher dimensional systems is fundamentally different. The case of the dissipative systems is often reducible to the one-dimensional Feigenbaum–Coullet–Tresser universality (Collet et al 1980, de Carvalho et al 2005, Lyubich and Martens 2008). Universality for highly dissipative Hénon like maps

\[ F(x, u) = (f(x) - \epsilon(x, u), x), \]

where \( f \) is a unimodal map, \( \epsilon \) is sufficiently small together with its derivatives, has been demonstrated in de Carvalho et al (2005). Specifically, it has been shown that these maps are in the Feigenbaum–Coullet–Tresser universality class, and that the degenerate map

\[ F^*(x, u) = (f^*(x), x), \]

where \( f^* \) is the Feigenbaum–Coullet–Tresser fixed point, is a renormalization fixed point in that class. The authors of de Carvalho et al (2005) have also constructed a Cantor set (an attractor) for infinitely renormalizable maps, and demonstrated that for such Hénon-like maps universality coexists with non-rigidity: the Hölder exponent of the conjugacy between the actions of any two infinitely renormalizable maps with nonequal average Jacobians on their Cantor sets has an upper bound less than 1.

The case of area-preserving maps seems to be very different, and at present there is no deep understanding of universality in conservative systems, other than in the ’trivial’ case of the universality for systems ’near integrability’ (Koch 2002, 2004a, Gaidashev 2005, Kocić 2005, Khanin et al 2007).

An infinite period doubling cascade in families of area-preserving maps was observed by several authors in the early 1980s (Benettin et al 1980, Derrida and Pomeau 1980, Helleman 1980, Bountis 1981, Collet et al 1981). The period-doubling phenomenon can be illustrated with the area-preserving Hénon family (cf Bountis (1981)):

\[ H_a(x, u) = (-u + 1 - ax^2, x). \]

Maps \( H_a \) have a fixed point \(((1 + \sqrt{1 + a})/a, (-1 + \sqrt{1 + a})/a)\) which is stable for \(-1 < a < 3\). When \( a_1 = 3 \) this fixed point becomes unstable, at the same time an orbit of
period two is born with \( H_a(x_\pm) = (x_\mp, x_\mp) \), \( x_\mp = (1 \pm \sqrt{a - 3})/a \). This orbit, in turn, becomes unstable at \( a_2 = 4 \), giving birth to a period 4 stable orbit. Generally, there exists a sequence of parameter values \( a_k \), at which the orbit of period 2\(^{k-1} \) turns unstable, while at the same time a stable orbit of period 2\(^k \) is born. The parameter values \( a_k \) accumulate on some \( a_\infty \). The crucial observation is that the accumulation rate

\[
\lim_{k \to \infty} \frac{a_k - a_{k-1}}{a_{k+1} - a_k} = 8.721... \tag{2}
\]

is universal for a large class of families, not necessarily Hénon.

Furthermore, the 2\(^k \) periodic orbits scale asymptotically with two scaling parameters

\[
\lambda = -0.249... \quad \mu = 0.061... \tag{3}
\]

To explain how orbits scale with \( \lambda \) and \( \mu \) we will follow Bountis (1981). Consider an interval \((a_k, a_{k+1})\) of parameter values in a ‘typical’ family \( F_\alpha \). For any value \( \alpha \in (a_k, a_{k+1}) \) the map \( F_\alpha \) possesses a stable periodic orbit of period 2\(^k \). We fix some \( \alpha_k \) within the interval \((a_k, a_{k+1})\) in some consistent way; for instance, by requiring that the restriction of \( F_\alpha \) to a neighbourhood of a stable periodic point in the 2\(^k \)-periodic orbit is conjugate, via a diffeomorphism \( H_k \), to a rotation with some fixed rotation number \( r \). Let \( p_k \) be some unstable periodic point in the 2\(^{k-1} \)-periodic orbit, and let \( p_k' \) be the further of the two stable 2\(^k \)-periodic points that bifurcated from \( p_k' \). Denote with \( d_k = |p_k' - p_k| \), the distance between \( p_k \) and \( p_k' \). The new elliptic point \( p_k \) is surrounded by invariant ellipses; let \( c_k \) be the distance between \( p_k \) and \( p_k' \) in the direction of the minor semi-axis of an invariant ellipse surrounding \( p_k \), see figure 1. Then,

\[
\frac{1}{\lambda} = - \lim_{k \to \infty} \frac{d_k}{d_{k+1}}, \quad \frac{1}{\mu} = - \lim_{k \to \infty} \frac{\rho_k}{\rho_{k+1}}, \quad \frac{1}{\lambda^2} = \lim_{k \to \infty} \frac{c_k}{c_{k+1}},
\]

where \( \rho_k \) is the ratio of the smaller and larger eigenvalues of \( D_{H_k}(p_k) \).

This universality can be explained rigorously if one shows that the renormalization operator

\[
R[F] = \Lambda_F^{-1} \circ F \circ F \circ \Lambda_F, \tag{4}
\]

where \( \Lambda_F \) is some \( F \)-dependent coordinate transformation, has a fixed point and the derivative of this operator is hyperbolic at this fixed point.

It has been argued in Collet et al (1981) that \( \Lambda_F \) is a diagonal linear transformation. Furthermore, such \( \Lambda_F \) has been used in Eckmann et al (1982) and Eckmann et al (1984) in a computer-assisted proof of existence of a reversible renormalization fixed point \( F_\star \) and hyperbolicity of the operator \( R \).
An exploration of a possible analytic machinery has been undertaken in Gaidashev and Koch (2008) where it has been demonstrated that the fixed point $F^*$ is very close, in some appropriate sense, to an area-preserving Hénon-like map

$$H^*(x, u) = (\phi(x) - u, x - \phi(\phi(x) - u)), \quad (5)$$

where $\phi$ solves the following one-dimensional problem of the non-Feigenbaum type:

$$\phi(y) = \frac{2}{\lambda} \phi(\phi(\lambda y)) - y. \quad (6)$$

In this paper we will study the dynamics of the renormalization fixed point $F^*$ and maps in some neighbourhood of $F^*$. We would like to emphasize that the preservation of area for such maps leads to one important difference between the case at hand and the highly dissipative case of Hénon-like maps: the former maps have no obvious invariant subsets in their domain. In particular, this means, that the construction of invariant Cantor sets carried out in de Carvalho et al (2005), which is using existence of invariant subsets in a crucial way, is not readily applicable to the case of area-preserving maps.

To construct hyperbolic sets, we will use the idea of covering relations (see, e.g. Zgliczyński and Gidea (2004), Zgliczyński (2009)) in rigorous computations. The Hausdorff dimension of the hyperbolic sets will be estimated with the help of the Duarte distortion theorem (see, for example, Duarte (2000)) which enables one to use the distortion of a Cantor set to ultimately find bounds on the dimension.

The structure of the paper is as follows: we begin by recalling the basic properties of area-preserving reversible maps in section 2. In section 3 we introduce some notation and recall some standard definitions from hyperbolic dynamics. In section 4 we study the domain of analyticity of maps in a neighbourhood of the renormalization fixed point, and analytic continuation of the renormalization fixed point. Section 5 consists of the statements of our main theorems. In section 6, we recall the definition and the main properties of the covering relations. In section 7 we prove that any area-preserving reversible map in a neighbourhood of the renormalization fixed point has a transversal homoclinic orbit in its domain of analyticity. In section 8 we construct a heteroclinic tangle for the renormalization fixed point. From its existence the existence of unbounded and oscillating trajectories follow. In section 9 we recall the Duarte distortion theorem. In sections 10 and 11 we prove that the third iterate of any area-preserving reversible map in a neighbourhood of the renormalization fixed point has a horseshoe, and compute bounds on its Hausdorff dimension using the Duarte distortion theorem.

In a satellite paper (Gaidashev and Johnson 2009), we prove that infinitely renormalizable maps in the neighbourhood of existence of the hyperbolic set for the third iterate also admit a 'stable' set. This set is a bounded invariant set, such that the maximal Lyapunov exponent for the third iterate is zero. In Gaidashev and Johnson (2009) we provide an upper bound on the Hausdorff dimension of the stable set, and prove that the Hausdorff dimension is constant for all maps in some subset of infinitely renormalizable maps.

### 2. Renormalization for area-preserving reversible maps

An ‘area-preserving map’ will mean an exact symplectic diffeomorphism of a subset of $\mathbb{R}^2$ onto its image.

Recall, that an area-preserving map can be uniquely specified by its generating function $S$:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ S_2(x, y) \end{pmatrix}, \quad S_i \equiv \partial_i S. \quad (7)$$
Furthermore, we will assume that $F$ is reversible, that is
\[ T \circ F \circ T = F^{-1}, \quad \text{where} \quad T(x, u) = (x, -u). \tag{8} \]

For such maps it follows from (7) that
\[ S_1(y, x) = S_2(x, y) = s(x, y), \tag{9} \]
and
\[ \left( \begin{array}{c} x \\ -s(y, x) \end{array} \right) \mapsto \left( \begin{array}{c} y \\ s(x, y) \end{array} \right). \tag{10} \]

It is this ‘little’ $s$ that will be referred to below as ‘the generating function’. It follows from (9) that $s_1$ is symmetric. If the equation $-s(y, x) = u$ has a unique differentiable solution $y = y(x, u)$, then the derivative of such a map $F$ is given by the following formula:
\[ DF(x, u) = \begin{bmatrix} -\frac{s_2(y(x, u), x)}{s_1(y(x, u), x)} & \frac{1}{s_1(y(x, u), x)} \\ \frac{s_1(x, y(x, u)) - s_2(x, y(x, u))}{s_1(y(x, u), x)} & -\frac{s_2(x, y(x, u))}{s_1(y(x, u), x)} \end{bmatrix}. \tag{11} \]

We will now derive an equation for the generating function of the renormalized map $\Lambda F \circ F \circ F^{-1}$.

Applying a reversible $F$ twice we get
\[ \left( \begin{array}{c} x' \\ -s(z', x') \end{array} \right) \mapsto \left( \begin{array}{c} z' \\ s(x', z') \end{array} \right) \mapsto \left( \begin{array}{c} y' \\ -s(y', z') \end{array} \right). \]

It has been argued in Collet et al (1981) that
\[ \Lambda F(x, u) = (\lambda x, \mu u). \]

We therefore set $(x', y') = (\lambda x, \lambda y), z'(\lambda x, \lambda y) = z(x, y)$ to obtain
\[ \left( \begin{array}{c} x \\ -\frac{1}{s(z, \lambda x)} \end{array} \right) \mapsto \left( \begin{array}{c} \frac{\lambda}{s(z, \lambda x)} \\ -s(z, \lambda x) \end{array} \right) \mapsto \left( \begin{array}{c} \lambda \frac{y}{s(z, \lambda y)} \\ s(z, \lambda y) \end{array} \right) \mapsto \left( \begin{array}{c} \frac{1}{\mu} s(z, \lambda y) \\ \lambda \frac{y}{s(z, \lambda y)} \end{array} \right), \tag{12} \]
where $z(x, y)$ solves
\[ s(\lambda x, z(x, y)) + s(\lambda y, z(x, y)) = 0. \tag{13} \]

If the solution of (13) is unique, then $z(x, y) = z(y, x)$, and it follows from (12) that the generating function of the renormalized $F$ is given by
\[ \tilde{s}(x, y) = \mu^{-1} s(z(x, y), \lambda y). \tag{14} \]

One can fix a set of normalization conditions for $\tilde{s}$ and $z$ which serve to determine scalings $\lambda$ and $\mu$ as functions of $s$. For example, the normalization
\[ s(1, 0) = 0 \]
is reproduced for $\tilde{s}$ as long as
\[ z(1, 0) = z(0, 1) = 1. \]

In particular, this implies that
\[ s(\lambda, 1) + s(0, 1) = 0. \]

Furthermore, the condition
\[ \partial_1 s(1, 0) = 1 \tag{15} \]
is reproduced as long as
\[ \mu = \partial_1 z(1, 0). \]

We will now summarize the above discussion in the following definition of the renormalization operator acting on generating functions originally due to the authors of Eckmann et al (1982) and Eckmann et al (1984).

**Definition 2.1.**

\[ R_{EKW}[s](x, y) = \mu^{-1}s(z(x, y), \lambda y), \]  

where
\[ 0 = s(\lambda x, z(x, y)) + s(\lambda y, z(x, y)), \]  
\[ 0 = s(\lambda, 1) + s(0, 1) \quad \text{and} \quad \mu = \partial_1 z(1, 0). \]  

**Definition 2.2.** The Banach space of functions \( s(x, y) = \sum_{i,j=0}^{\infty} c_{ij} x^i y^j \), analytic on a bi-disc \( |x - 0.5| < \rho, |y - 0.5| < \rho \), for which the norm \( \|s\|_\rho = \sum_{i,j=0}^{\infty} |c_{ij}| \rho^i \) is finite, will be referred to as \( \mathcal{A}(\rho) \). \( A_s(\rho) \) will denote its symmetric subspace \( \{ s \in \mathcal{A}(\rho) : s_1(x, y) = s_1(y, x) \} \).

As we have already mentioned, the following has been proved with the help of a computer in Eckmann et al (1982) and Eckmann et al (1984).

**Theorem 2.3.** There exist a polynomial \( s_a \in A_s(\rho) \) and a ball \( B_r(s_a) \subset A_s(\rho) \), \( r = 6.0 \times 10^{-7}, \rho = 1.6 \), such that the operator \( R_{EKW} \) is well defined, analytic and compact on \( B_r \).

Furthermore, its derivative \( DR_{EKW}|_{B_r} \) has exactly two eigenvalues \( \delta_1 \) and \( \delta_2 \) of modulus larger than 1, while
\[ \text{spec}(DR_{EKW}|_{B_r}) \setminus \{ \delta_1, \delta_2 \} \subset \{ z \in \mathbb{C} : |z| \leqslant \nu < 1 \}. \]

Finally, there is an \( s^* \in B_r \), such that
\[ R_{EKW}[s^*] = s^*. \]

The scalings \( \lambda_s \) and \( \mu_s \) corresponding to the fixed point \( s^* \) satisfy
\[ \lambda_s \in [-0.248 876 81, -0.248 873 76], \]  
\[ \mu_s \in [0.061 107 811, 0.061 112 465]. \]

**Remark 2.4.** The radius of the contracting part of the spectrum \( \text{spec}(DR_{EKW}(s_a)) \setminus \{ \delta_1, \delta_2 \} \) has been estimated in Eckmann et al (1984) to be \( \nu = 0.8 \).

It follows from the above theorem that there exists a codimension 2 local stable manifold \( W_{\text{loc}}(s^*) \subset B_r \).

**Definition 2.5.** A reversible map \( F \) of the form (10) such that \( s \in W_{\text{loc}}(s^*) \) is called infinitely renormalizable. The set of all reversible infinitely renormalizable maps is denoted by \( \mathcal{W} \).
3. Some notation and definitions

We will use the following notation for the sup norm of a function \( h \) and a transformation \( H \) defined on some set \( S \subseteq \mathbb{R}^2 \) or \( \mathbb{C}^2 \):

\[
|h|_S \equiv \sup_{(x, u) \in S} |h|, \\
|H|_S \equiv \max \left\{ \sup_{(x, u) \in S} |\mathcal{P}_x H|, \sup_{(x, u) \in S} |\mathcal{P}_u H| \right\},
\]

where \( \mathcal{P}_x \) and \( \mathcal{P}_u \) are projections on the corresponding components.

We will also use the notation \( |\cdot| \) for the \( l_2 \) norm for vectors in \( \mathbb{R}^2 \).

The interval enclosures of \( \lambda_a \) and \( \mu_a \) will be denoted

\[
\lambda_a = [\lambda_-, \lambda_+]; \quad \lambda_- = -0.248 \, 876 \, 81, \quad \lambda_+ = -0.248 \, 873 \, 76, \\
\mu_a = [\mu_-, \mu_+]; \quad \mu_- = 0.061 \, 107 \, 811, \quad \mu_+ = 0.061 \, 112 \, 465.
\]

The corresponding interval enclosure for the linear map \( \Lambda_a \) will be denoted \( \Lambda_a^* \); if \( (x, u) \in \mathbb{C}^2 \), then

\[
\Lambda_a(x, u) \equiv \left\{ (\lambda x, \mu u) \in \mathbb{C}^2 : \lambda \in \lambda_a, \mu \in \mu_a \right\}.
\]

The bound on the renormalization fixed point \( F_s \) will be referred to as \( F_s \):

\[
F_s \equiv \left\{ F : (x, -s(y, x)) \mapsto (y, s(x, y)) : s \in s^* \right\},
\]

where \( s_a \) is as in theorem 2.3; the third iterate of this bound will be referred to as \( G_s \). With \( DG_s : p \mapsto [DG_s(p)] \) we denote an interval matrix valued function such that

\[
[DG_s(p)]_{ij} \in [DG_s(p)]_{ij}, \quad \text{for all } G \in G_s, \quad p \in D_3,
\]

where \( D_3 \) is the domain of \( G_s \), and the bound on the operator norm of \( DG \) for \( G = F \circ F \circ F, \quad F \in F_s \) on a set \( S \) will be denoted

\[
\|DG_s\|_S \equiv \sup_{F \in F_s} \|D(F \circ F \circ F)\|_S.
\]

Given a non-empty open set \( D \subseteq \mathbb{C}^n \) we will denote by \( \mathcal{O}_R(D) \) the set of reversible area-preserving maps \( F : D \mapsto \mathbb{C}^n \), analytic on \( D \).

We will proceed with a collection of classical notations (see, for example, Katok and Hasselblatt (1995)) relevant to our following discussion.

**Definition 3.1 (Hyperbolic set).** Let \( \mathcal{M} \) be a smooth manifold, and let \( F \) be a diffeomorphism of an open subset \( \mathcal{U} \subseteq \mathcal{M} \) onto its image.

A set \( \mathcal{C} \) is called hyperbolic for the map \( F \) if there is a Riemannian metric on a neighbourhood \( \mathcal{U} \) of \( \mathcal{C} \), and \( \beta < 1 < \delta \), such that for any \( p \in \mathcal{C} \) and \( n \in \mathbb{N} \) the tangent space \( T_{F^n(p)} \mathcal{M} \) admits a decomposition in two invariant subspaces:

\[
T_{F^n(p)} \mathcal{M} = E^s_n \oplus E^u_n, \quad DF(F^n(p))E^s_n = E^{u}_{n+1},
\]

on which the sequence of differentials is hyperbolic:

\[
\|DF(F^n(p))\|_{E^s_n} < \beta, \quad \|DF^{-1}(F^n(p))\|_{E^u_n} < \delta^{-1}.
\]

**Definition 3.2 (Locally maximal hyperbolic set).** Let \( \mathcal{C} \) be a hyperbolic set for \( F : \mathcal{U} \mapsto \mathcal{M} \). If there is a neighbourhood \( \mathcal{V} \) of \( \mathcal{C} \) such that \( \mathcal{C} = \cap_{n \in \mathbb{Z}} F^n(\mathcal{V}) \), then \( \mathcal{C} \) is called locally maximal.
Definition 3.3 (Bernoulli shift). Let \( \{0, 1, \ldots, N - 1\}^\mathbb{Z} \) be the space of all two-sided sequences of \( N \) symbols:

\[
\{0, 1, \ldots, N - 1\}^\mathbb{Z} = \{\omega = (\omega_{-1}, \omega_0, \omega_1, \ldots) : \omega_i \in \{0, 1, \ldots, N - 1\}, i \in \mathbb{Z}\}.
\]

Define the Bernoulli shift on \( \{0, 1, \ldots, N - 1\}^\mathbb{Z} \) as

\[
\sigma_N(\omega) = \omega', \quad \omega'_i = \omega_{i+1}.
\]

Definition 3.4 (Topological Markov chain). Let \( A = (a_{ij})_{i,j=0}^{N-1} \) be an \( N \times N \) matrix whose entries are either 0 or 1. Let

\[
\{0, 1, \ldots, N - 1\}^\mathbb{Z}_A = \{\omega \in \{0, 1, \ldots, N - 1\}^\mathbb{Z} : a_{\omega_i, \omega_{i+1}} = 1, n \in \mathbb{Z}\}.
\]

The restriction

\[
\sigma_N\big|_{\{0,1,\ldots,N-1\}^\mathbb{Z}_A} \equiv \sigma_A
\]

is called a topological Markov chain determined by \( A \).

Definition 3.5 (Homoclinic and heteroclinic points). Let \( F : X \mapsto X \) be a homeomorphism on a metric space \( (X, d) \). A point \( x \in X \) is said to be homoclinic to the point \( y \in X \) if

\[
\lim_{|n| \to \infty} d(F^n(x), F^n(y)) = 0.
\]

A point \( x \) is said to be heteroclinic to points \( y_1 \) and \( y_2 \) if

\[
\lim_{n \to -\infty} d(F^n(x), F^n(y_1)) = \lim_{n \to +\infty} d(F^n(x), F^n(y_2)) = 0.
\]

In this case we will also say that there exists a heteroclinic orbit between points \( y_1 \) and \( y_2 \).

If \( M \) is a differentiable manifold, \( F \in \text{Diff}^1(M) \) and \( x \in M \) is a hyperbolic fixed point of \( F \), then we say that \( q \in M \) is a transversal homoclinic point to \( x \) if it is a point of transversal intersection of the stable and unstable manifolds of \( x \).

Definition 3.6. Let \( X \) be a metric space. If \( S \subset X \), and \( d \in [0, \infty) \), the \( d \)-dimensional Hausdorff content of \( S \) is defined as

\[
C^H_d(S) = \inf \left\{ \sum_i \text{diam}(S_i)^d : \{S_i\} \text{ is a cover of } S \right\}.
\]

The Hausdorff dimension of \( S \) is defined as

\[
\dim_H(S) = \inf \left\{ d \geq 0 : C^H_d(S) = 0 \right\}.
\]

4. Domain of analyticity of \( F_* \) and \( G_* \)

\( F \) is defined implicitly by the generating function \( s \), its domain is given as in Eckmann et al (1984):

\[
\mathcal{D}_s = \{(x, y) \in \mathbb{C}^2 : |x - 0.5| < 1.6, |y - 0.5| < 1.6\}.
\]

To find the domain of \( F \in F_* \), we note that its second argument is equal to \(-s(y, x)\), for some \( s \in s^* \) (see (10)). Thus, the domain of \( F, \mathcal{D} \), is given by

\[
\mathcal{D} = \{(x, u) \in \mathbb{C}^2 : u = -s(y, x), s \in s^*(y, x), |x - 0.5| < 1.6, |y - 0.5| < 1.6\}.
\]

We denote by

\[
\mathcal{D}_r = \{(x, u) \in \mathcal{D} : \exists x = \exists u = 0\},
\]

the real slice of \( \mathcal{D} \).

To solve non-linear equations on the computer we use the interval Newton operator, see, for example, Neumaier (1990).
Definition 4.1 (Interval Newton Operator). Let $F : \mathcal{D} \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$, and let $DF$ be an interval matrix valued function such that $[DF(x)]_{ij} \in [D(F)(x)]_{ij}$, for all $x \in \mathcal{D}$. Let $X \subset \mathcal{D}$ be a Cartesian product of finite intervals, $\hat{x} \in X$, and assume that if $A \in DF(X)$, then $A$ is non-singular. We define the interval Newton operator as

$$N(F, X, \hat{x}) = \hat{x} - (DF)^{-1}(X)F(\hat{x}).$$

The main properties of $N$ are that if $N(F, X, \hat{x}) \subset X$, then there exists a unique solution to $F(x) = 0$ in $X$, which is contained in $N(F, X, \hat{x})$, and if $N(F, X, \hat{x}) \cap X = \emptyset$, then there is no solution to $F(x) = 0$ in $X$.

Lemma 4.2. There exists a non-empty open set $\hat{\mathcal{D}}$,

$$\hat{\mathcal{D}} \subset \{ x \in \mathbb{R} : |x - 0.5| < 1.6 \},$$

such that for every $(x, u) \in \hat{\mathcal{D}}$ there exists a unique solution of the interval Newton operator of the function $h_{(x,u)}(y) = u + s(y, x)$, $s \in s^*(y, x)$, that satisfies $|y - 0.5| < 1.6$.

Proof. Set $Y = \{ y \in \mathbb{R} : |y - 0.5| < 1.6 \}$. Given $(x, u)$, let $N(h_{(x,u)}, Y, \hat{y})$ be the interval Newton operator (for some appropriately chosen $\hat{y} \in Y$). We have verified that there exists a non-empty set $\hat{\mathcal{D}}$ such that for all $(x, u) \in \hat{\mathcal{D}}$

$$N(h_{(x,u)}, Y, \hat{y}) \subset Y.$$  

This verification is implemented in the program findDomain of Progs1 (2009). It follows, see for example, Neumaier (1990), that there is a unique $y$, such that $(y, x)$ is in the real slice of $\hat{\mathcal{D}}$, and $h_{(x,u)}(y) = 0$. Thus, $y$ is defined as a function of $(x, u)$, and by (31) $(x, u) \in \hat{\mathcal{D}}$.

Clearly, $\hat{\mathcal{D}} \subset \hat{\mathcal{D}}.$

The generating function $s$ is analytic on a bi-disc, preferably $F$ should have a similar property. From the above construction, we can at least show that $F$ is defined on a complex neighbourhood of $\hat{\mathcal{D}}$ from the above lemma 4.2.

Lemma 4.3. $\mathcal{D}$ contains an open complex neighbourhood of the set $\hat{\mathcal{D}}$.

Proof. $N(h_{(x,u)}, y)$ is only well defined if $s^*_1(y, x) \neq 0$, so that the condition of the implicit function theorem is automatically satisfied on the solution set of $N(h_{(x,u)}, Y, \hat{y})$. Since $s^*_1$ is analytic on the bi-disc $\{ |x - 0.5| < 1.6 \} \times \{ |y - 0.5| < 1.6 \}$, there is an open neighbourhood in $\mathbb{C}^2$ of the solution set where $s^*_1(y, x) \neq 0$. It follows from the implicit function theorem that $F_w$ is analytic on this neighbourhood.

Generally, we will denote the third iterate of a map $F \in \mathcal{O}_R(\mathcal{D})$ as $G$. The domain of $G$, $\mathcal{D}_3$, is given by

$$\mathcal{D}_3 = F^{-2}(\mathcal{D}'' \cap \mathcal{D}),$$

where

$$\mathcal{D}'' = F(\mathcal{D}' \cap \mathcal{D}), \quad \mathcal{D}' = F(\mathcal{D}).$$

Using the program findDomain, we have verified that the real slice $\hat{\mathcal{D}}_3$ of $\mathcal{D}_3$ is an open non-empty set. Approximations of $\hat{\mathcal{D}}$ and $\hat{\mathcal{D}}_3$ are shown in figure 2. We note that by using the renormalization equation $F_w = R[F_w]$, $F_w$ has, for any $k$, an analytic continuation to domains

$$\mathcal{D}^k = \Lambda w^{-1}(F_w^{-1}(\mathcal{D}^{k-1} \cap \mathcal{D}^{k-1})), \quad \mathcal{D}^0 = \mathcal{D},$$
Figure 2. The real slices of the domains of $F$ (blue/outer contour) $G_*$ (red).

Figure 3. The real slices of the domains $D_0^3$ (red/grey in print) and $D_1^3$ (blue/black in print).

while $G_*$ has an analytic continuation to

$$D_3^k \equiv \Lambda_*^{-1}(G_*^{-1}(D_3^{k-1} \cap D_3^{k-1})), \quad D_3^0 \equiv D_3.$$ 

The real slices of the domains $D_3^0$ and $D_3^1$ are given in figure 3.

We will conclude this section with the following.

**Lemma 4.4.** All maps $F \in F_*$ possess a hyperbolic fixed point $p_0 = p_0(F)$, such that

1. $P_x p_0 \in (0.577 618 43, 0.577 619 89)$, and $P_x p_0 = 0$;
2. $DF(p_0)$ has two real eigenvalues:

$$e_+ \in (-2.057 635 59, -2.057 599 28),$$
$$e_- \in (-0.486 017 15, -0.485 980 84).$$
Proof. The bound on the fixed point has been obtained with the help of the interval Newton method. Hyperbolicity has been demonstrated by computing a bound on $DF$ using formula (11).

5. Statement of the main results

We will now summarize our main findings.

Our first theorem describes the two-sided heteroclinic tangle for the fixed point map $F_*$.

Main theorem 1. The renormalization fixed point $F_*$ has the following properties:

1. $F_*$ possesses a point $p_\star$ which is transversally homoclinic to the fixed point $p_0$.
2. There exists a positive integer $n$ such that for any negative integer $k$ the map $F_n^k$ has a heteroclinic orbit $O_k$ between the periodic points $\Lambda_n^k(p_0)$ and $\Lambda_n^{k+1}(p_0)$, and for any positive integer $k$ the map $F_n^{-2k}$ has a heteroclinic orbit $O_k$ between the periodic points $\Lambda_n^k(p_0)$ and $\Lambda_n^{k-1}(p_0)$.
3. For any $N \in \mathbb{N}$ and $M \in \mathbb{N}$ there exists an integer $n$ and an invariant set $Z_{F_2}$, such that $F_n^1 | Z_{F_2} \approx \text{homeo} \sigma_{A(|\{-N,-N+1,\ldots,M-1,M\}|^2}$, where $\sigma_A$ is the topological Markov chain defined by a $(N+M+1) \times (N+M+1)$ tridiagonal matrix

\[
A = \begin{bmatrix}
1 & 1 & 0 & \ldots & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & \ldots & 0 & 0 \\
0 & 1 & 1 & 1 & \ldots & 0 & 0 \\
& & & & & & \\
0 & 0 & 0 & \ldots & 1 & 1 & 0 \\
0 & 0 & 0 & \ldots & 0 & 1 & 1 \\
0 & 0 & 0 & \ldots & 0 & 0 & 1 & 1
\end{bmatrix}
\]

4. For any $R > \varepsilon > 0$ there exists a point $p \in D$ and an $n \in \mathbb{N}$ such that $|F_n^k(p)| > R$ (an unbounded orbit) and a point $q \in D$ and $m \in \mathbb{N}$ such that $|F_m^k(q)| > R$ and $|F_m^k(q)| < \varepsilon$ (an oscillating orbit).

Our second result demonstrates that all locally infinitely renormalizable maps admit a hyperbolic set in their domain of analyticity.

Main theorem 2. Any $F \in F_*$ admits a hyperbolic set $C_G \subset D_3$ for $G \equiv F \circ F \circ F$;

\[G |_{\tilde{C}_2} \approx \text{homeo} \sigma_2([0,1]^2),\]

whose Hausdorff dimension satisfies

\[0.7673 \geq \dim_H(C_G) \geq \varepsilon,\]

where $\varepsilon \approx 0.00013 e^{-7.499}$ is strictly positive.

6. Topological tools

The main tools of our proofs are covering relations (Zgliczyński 1997, Zgliczyński and Gidea 2004) and cone conditions (Kokubu et al 2007, Zgliczyński 2009), see also Galias and Zgliczyński (2001) for proving the existence of homoclinic and heteroclinic orbits. To make this paper reasonably self-contained we include a brief introduction to the necessary concepts.
6.1. H-sets and covering relations

The notion of an h-set and a covering relation first appeared in Zgliczyński (1997), the most thorough treatment is Zgliczyński and Gidea (2004). The basic idea is to construct computable conditions for the existence of a semi-conjugacy to symbolic dynamics. This is done by constructing h-sets, i.e. hyperbolic-like sets, that cross each other in a (topologically) non-trivial way. We denote by $B_n(c, r)$, the open ball in $\mathbb{R}^n$ with centre $c$ and radius $r$, and $S^n(c, r) = \partial B_{n+1}(c, r)$.

**Definition 6.1.** An h-set is a quadruple consisting of

- a compact subset $|N|$ of $\mathbb{R}^n$,
- a pair of numbers $u(N), s(N) \in \{0, 1, 2, \ldots\}$, with $u(N) + s(N) = n$,
- a homeomorphism $c_N : \mathbb{R}^n \rightarrow \mathbb{R}^n = \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)}$, such that $c_N(|N|) = \overline{B_{u(N)}(0, 1)} \times \overline{B_{s(N)}(0, 1)}$.

We denote such a quadruple by $N$. We usually drop the bars on the support and refer to it as $N$. Furthermore, $N_c = \partial B_{u(N)}(0, 1) \times \overline{B_{s(N)}(0, 1)}$, $N^+ = \partial \overline{B_{u(N)}(0, 1)} \times \overline{B_{s(N)}(0, 1)}$, $N^+_c = \partial B_{u(N)}(0, 1) \times B_{s(N)}(0, 1)$, $N^- = c_N^{-1}(N^+)$.

The maps that we study in this paper are reversible, which we use to reduce the amount of computation. For such maps the following definition is useful, see also the discussion in Zgliczyński and Gidea (2004).

**Definition 6.2.** Assume $N, M$ are h-sets, such that $u(N) = u(M) = u$ and $s(N) = s(M) = s$. Let $f : N \rightarrow \mathbb{R}^n$ be a continuous map. Let $f_c = c_M \circ f \circ c_N^{-1} : N_c \rightarrow \mathbb{R}^u \times \mathbb{R}^s$. We say that $N \overset{f}{\Rightarrow} M$ (N f-covers M) iff the following conditions are satisfied.

1. There exists a continuous homotopy $h : [0, 1] \times N_c \rightarrow \mathbb{R}^u \times \mathbb{R}^s$, such that the following conditions hold true
   
   \[ h_0 = f_c, \]
   \[ h([0, 1], N_c^-) \cap M_c = \emptyset, \]
   \[ h([0, 1], N_c) \cap M^+_c = \emptyset. \]

2. There exists a linear map $A : \mathbb{R}^u \rightarrow \mathbb{R}^u$, such that
   
   \[ h_1(p, q) = (Ap, 0) \quad \text{for} \quad p \in \overline{B_{u}(0, 1)} \quad \text{and} \quad q \in \overline{B_{s}(0, 1)}, \]
   \[ A(\partial B_{u}(0, 1)) \subset \mathbb{R}^u \setminus \overline{B_{u}(0, 1)}. \]

$h_1$ is called a model map for the relation $N \overset{f}{\Rightarrow} M$.

The maps that we study in this paper are reversible, which we use to reduce the amount of computation. For such maps the following definition is useful, see also the discussion in Zgliczyński and Gidea (2004).
Definition 6.3. Let \( N \) be an \( h \)-set. We define the \( h \)-set \( N^T \) as follows:

- the compact subset of the quadruple \( N^T \) is the compact subset of the quadruple \( N \), also denoted by \( N \),
- \( u(N^T) = s(N) \), \( s(N^T) = u(N) \).

The homeomorphism \( c_{N^T} : \mathbb{R}^n \to \mathbb{R}^n = \mathbb{R}^{u(N^T)} \times \mathbb{R}^{s(N^T)} \) is defined by

\[
c_{N^T}(x) = j(c_N(x)),
\]

where \( j : \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)} \to \mathbb{R}^{s(N)} \times \mathbb{R}^{u(N)} \) is given by \( j(p, q) = (q, p) \).

Definition 6.4. Assume \( N, M \) are \( h \)-sets, such that \( u(N) = u(M) = u \) and \( s(N) = s(M) = s \).

Let \( g : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n \). Assume that \( g^{-1} : M \to \mathbb{R}^n \) is well defined and continuous. We say that \( N g \prec N \) if \( f_1 \cdots f_{k-1} f_k(x) \in \text{int} N_k \).

Moreover, if \( N_k = N_0 \), then \( x \) can be chosen so that

\[
f_k \circ f_{k-1} \circ \cdots \circ f_1(x) = x.
\]

In our proofs the hypothesis of theorem 6.6 is verified using the routine checkCoveringRelations of Progs1 (2009).

6.2. Cone conditions for \( h \)-sets

Theorem 6.6 gives a computational tool to prove the existence of orbits with prescribed symbolic dynamics. To prove that such orbits are unique one would ideally require hyperbolicity of the map in a neighbourhood of the orbit. Typically, this is proved by constructing invariant cone fields. An alternative method to prove uniqueness is provided by covering relations with cone conditions, first described in Kokubu et al (2007), the method is studied in further detail in Zgliczyński (2009), which we follow below.

Definition 6.7. Let \( N \subset \mathbb{R}^n \) be an \( h \)-set and \( Q : \mathbb{R}^n \to \mathbb{R}^n \) be a quadratic form

\[
Q((x, y)) = \alpha(x) - \beta(y), \quad (x, y) \in \mathbb{R}^u \times \mathbb{R}^s,
\]

where \( \alpha : \mathbb{R}^{u(N)} \to \mathbb{R} \) and \( \beta : \mathbb{R}^{s(N)} \to \mathbb{R} \) are positive definite quadratic forms.

The pair \((N, Q)\) is called an \( h \)-set with cones.

Definition 6.8. Assume that \((N, Q_N)\) and \((M, Q_M)\) are \( h \)-sets with cones, such that \( u(N) = u(M) = u \), and let \( f : N \to \mathbb{R}^{\dim(M)} \) be continuous. Assume that \( N \to M \). We say that \( f \) satisfies the cone condition (with respect to the pair \((N, M)\)) iff for any \( p_1, p_2 \in N \), \( p_1 \neq p_2 \) holds

\[
Q_M(f_c(p_1) - f_c(p_2)) > Q_N(p_1 - p_2).
\]

(35)
Definition 6.9. Assume that \((N, Q_N)\) and \((M, Q_M)\) are \(h\)-sets with cones, such that \(u(N) = u(M) = u\) and \(s(N) = s(M) = s\), and let \(f : N \to \mathbb{R}^{\mathbb{N}}\) be continuous. Assume that \(N \leftrightarrow^f M\). We say that \(f\) satisfies the cone condition (with respect to the pair \((N, M)\)) iff for any \(q_1, q_2 \in M\), \(q_1 \neq q_2\) holds

\[
Q_M(q_1 - q_2) > Q_N(f^{-1}_c(q_1) - f^{-1}_c(q_2)).
\]

(36)

Remark 6.10. Note, for details see the discussion in Zgliczyński (2009), that stable manifolds are propagated backwards through a chain of covering relations with cone conditions. Similarly, unstable manifolds are propagated forwards through a chain of covering relations. In the planar case, used in this paper, the Lipschitz constant of the stable manifold is given by \(\sqrt{\beta/\alpha}\), where \(\alpha\) and \(\beta\) are the constant coefficients of the corresponding quadratic forms.

To verify that the cone conditions hold, we use the following lemma (Zgliczyński 2009, lemma 8).

Lemma 6.11. Assume that for any \(B \in DF_c(N_C)\), the quadratic form

\[
V(x) = Q_M(Bx) - Q_N(x)
\]

is positive definite, then for any \(p_1, p_2 \in N_c\) such that \(p_1 \neq p_2\)

\[
Q_M(f_c(p_1) - f_c(p_2)) > Q_N(p_1 - p_2).
\]

(38)

In our proofs the hypothesis of lemma 6.11 is verified using the routine checkConeConditions of Progs1 (2009).

Remark 6.12. The proof of lemma 6.11 also yields a uniform lower bound, \(\epsilon_{MN}\) on the difference \(Q_M(f_c(p_1) - f_c(p_2)) - Q_N(p_1 - p_2)\) between each pair of \(h\)-sets.

Theorem 6.13. Let \(A = (a_{ij})_{i,j=0}^{k-1}\) be a \(k \times k\) matrix whose entries are either 0 or 1. Assume that the following set of covering relations with cone conditions holds

\[
N_i \leftrightarrow^f N_j \quad \text{if} \quad a_{ij} = 1.
\]

Then, each sequence in \([0, 1, \ldots, k-1]^\mathbb{Z}_A\) (see definition 3.4) is realized by a unique orbit.

Proof. Assume that \(x \in N_{\omega x}\) and \(y \in N_{\omega y}\) are such that \(f^i(x) \in N_{\omega x}\) and \(f^i(y) \in N_{\omega y}\), for all \(i \in \mathbb{Z}\), where \(\omega = (\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots) \in [0, 1, \ldots, k-1]^\mathbb{Z}_A\). Let \(\epsilon = \min \{(i,j): a_{ij} = 1\} \epsilon_{N_j, N_i}\), where \(\epsilon_{N_j, N_i}\) is as in remark 6.12. We have

\[
Q_{N_{\omega x}}(f^i_c(p_1) - f^i_c(p_2)) > Q_{N_{\omega y}}(f^{i-1}_c(p_1) - f^{i-1}_c(p_2)) + \epsilon
\]

\[
> \cdots
\]

\[
> Q_{N_{\omega}}((p_1) - (p_2)) + i\epsilon.
\]

(39)

This is a contradiction, since by assumption \(f^i(p_1) - f^i(p_2)\) stays inside a compact neighbourhood of the origin, for all \(i \in \mathbb{Z}\), where \(Q_{N_{\omega}}\) is bounded from above. \(\square\)
7. A transversal homoclinic orbit for \( F_* \)

The proofs of the theorems in this and the next sections were implemented in a computer program Progs1 (2009) using the software package CAPD (2009) for interval arithmetic and covering relations. All computations were performed on an Intel Xeon 2.3 GHz, 64bit processor with 16 Gb of RAM. The program was compiled with gcc, version 4.1.2.

Numerical experiments indicate that for all \( F \in F_* \), there exists a transversal homoclinic point \( p_\bowtie \), approximately located at \((1.067 707, 0)\), to the fixed point \( p_0 \), approximately located at \((0.577 619, 0)\), see figure 4. We construct a sequence of covering relations and quadratic forms, and verify that the hypotheses of theorem 6.6 and lemma 6.11 are satisfied. Here, and in the rest of the paper, we use formula (11) to provide bounds on the derivative of \( F \).

**Theorem 7.1.** Any map \( F \in F_* \) possesses a point \( p_\bowtie \), transversally homoclinic to \( p_0 \). The map \( F^6 \) admits a locally maximal invariant set \( C^F_* \ni p_\bowtie, p_0 \):

\[
F^6|_{C^F_*} \approx \text{homeo} \sigma^2 \mathbb{Z}^{[0,1]}.
\]

**Proof.** To avoid the ‘flip’ of the system we study the second iterate of \( F \), following the stable manifold from the homoclinic point to the fixed point. We construct four \( h \)-sets, \( B_i \), \( i = 1, 2, 3, 4 \), as in table 1: box \( B_4 \) around \( p_0 \), box \( B_1 \) around \( p_\bowtie \) and two intermediate boxes \( B_2 \) and \( B_3 \). \( B_4 \) is spanned by the unstable and stable eigenvectors of \( DF(p_0) \). The other boxes are experimentally rotated until two sides are roughly parallel with the stable manifold. We prove (using the routines checkCoveringRelations and checkConeConditions in Progs1 (2009)) that the following covering sequence with cone conditions holds

\[
B_1 \Rightarrow B_2 \Rightarrow B_3 \Rightarrow B_4 \Rightarrow B_4.
\]

The map \( F \) is reversible, by reversing time and reflecting the \( h \)-sets in \([\mu = 0]\) our cover is also a back-cover from \( p_\bowtie \) to \( p_0 \) along the unstable manifold of \( p_0 \). Therefore we also have the following covering relations with cone conditions

\[
B_4 \Leftarrow T(B_3) \Leftarrow T(B_2) \Leftarrow B_1.
\]
Existence of a homoclinic orbit follows from theorem 6.13. The transversality of the intersection follows from the fact that we use cones with Lipschitz constant less than one, \( \text{Lip} = 0.95 \).

Note that both \( p_0 \) and \( p_0^* \) depend analytically on \( F \). Part 1 of the first main theorem follows immediately, since \( F_* \in F_\ast \).

Remark 7.2. The numbers describing boxes \( B_i \) in table 1 are non-representable on the computer. We have used these numbers as the input for the routine checkConeConditions. During its execution, the numbers are rounded up to the nearest representable number; therefore, strictly speaking, theorem 7.1 has been proved for boxes different from (although very close to) those reported in the table. Since the specific values of the true parameters are largely irrelevant, we will not go into pains of computing and reporting bounds on these parameters. This will be the situation throughout the paper: all specific numerical data appearing in various tables and statements of theorems are (very accurate) approximations of the true data for which the results have been proved on the computer.

8. A bi-infinite heteroclinic tangle for \( F_* \)

In this section we will demonstrate that the fixed point \( F_* \) has a sequence of hyperbolic periodic points \( \{z_k\} \), \( k \in \mathbb{Z} \),

\[
|z_k| \xrightarrow{k \to \infty} 0 \quad \text{and} \quad |z_k| \xrightarrow{k \to -\infty} \infty,
\]

such that their stable and unstable manifolds form a heteroclinic tangle.

We first demonstrate that there exists a heteroclinic orbit between the fixed point \( z_0 \equiv p_0 \) and the period two point \( z_1 \equiv \Lambda_+(p_0) \).

We proceed as in the previous section: we construct a covering sequence of \( h \)-sets, as in table 2, consisting of one box \( B_0 \) around \( p_0 \), one box \( B_1 \) around \( \Lambda_+(p_0) \) and 11 intermediate boxes on the orbit. In this sequence, \( B_0 \) is defined as \( \Lambda_{-1}^\ast(B_1) \).

Since the orbit is on the tangle between the unstable and stable manifolds, there are parts of it where it is hard to construct expanding and contracting directions directly for the second iterate of the map. The fourth iterate, however, exhibits nice hyperbolic-like behaviour along a pseudoorbit. We use the programs checkCoveringRelations and checkConeConditions to verify that the hypotheses of theorem 6.6 and lemma 6.11 are satisfied for \( F_*^4 \) on the sequence of \( h \)-sets. In addition, we prove that they are satisfied for \( F_*^2 \) for the covering \( B_1 \Rightarrow B_1 \) (which implies that the covering relations and cone conditions are also satisfied for \( B_0 \Rightarrow B_0 \)). We use cones with Lipschitz constant 0.9 in the proof.

The reversibility of \( F_* \) implies that the existence of a heteroclinic orbit from \( z_1 \) to \( z_0 \) entails the existence of a heteroclinic orbit from \( z_0 \) to \( z_1 \). It also implies existence of homoclinic
The data used to prove the existence of a heteroclinic orbit for $F_2^*$ between $z_0$ and $z_1$. Vectors $e_i^{u,v}$ are the ‘stable’ and ‘unstable’ spanning vectors of the rectangles. The length of the sides of the rectangles $B_i$ is $2 \cdot |e_i^{u,v}|$. The covering sequence is $B_1 \Rightarrow B_1 \Rightarrow B_3 \Rightarrow B_4 \Rightarrow B_5 \Rightarrow B_3 \Rightarrow B_6 \Rightarrow B_9 \Rightarrow B_{10} \Rightarrow B_{11} \Rightarrow B_3 = \Lambda_{-1}(B_3) \Rightarrow B_0 = \Lambda_{-1}(B_1)$. The first and the last two covering relations are for $F_2^*$, the other ones are for $F_1^*$.

| Box # | Centre $l_i^u$ | Centre $l_i^u$ | $e_i^u$ | $e_i^v$ |
|-------|----------------|----------------|--------|--------|
| $B_1$ | $(-1.143, 0)$  | 0.00072        | 0.00072 | (0.982114, 0.188285) | (0.982114, -0.188285) |
| $B_2$ | $(-1.141, 0)$  | 0.00054        | 0.00054 | (0.982114, 0.188285) | (0.982114, -0.188285) |
| $B_3$ | $(-1.133, 0)$  | 0.00045        | 0.00045 | (0.982114, 0.188285) | (0.982114, -0.188285) |
| $B_4$ | $(-0.998, 0)$  | 0.00047        | 0.00047 | (0.987, 0.16072)     | (0.997, -0.077416)    |
| $B_5$ | $(0.058, 0)$   | 0.00055        | 0.00055 | (0.994, -0.10938)    | (0.995, 0.0997849)    |
| $B_6$ | $(0.106, 0)$   | 0.00044        | 0.00044 | (0.975, -0.222205)   | (0.982114, 0.188285) |
| $B_7$ | $(-0.2842, 0)$ | 0.00044        | 0.00044 | (0.84, 0.542856)     | (0.866, 0.025, -0.5)  |
| $B_8$ | $(0.533, 0)$   | 0.00003        | 0.00003 | (0.8, -0.6)          | (0.788579, 0.614934)  |
| $B_9$ | $(0.575, 0)$   | 0.00003        | 0.00003 | (0.788579, -0.614934)| (0.788579, 0.614934)  |
| $B_{10}$ | $(0.577, 0)$ | 0.00005        | 0.00005 | (0.788579, -0.614934)| (0.788579, 0.614934)  |
| $B_{11}$ | $(0.577, 0)$ | 0.0001         | 0.0001  | (0.788579, -0.614934)| (0.788579, 0.614934)  |

Orbits for both points, that enters a small neighbourhood of the other one. The sizes of these neighbourhoods are given by the sizes of the boxes $B_0$ and $B_1$ in table 2.

To construct the pseudorobbit we non-uniformly distribute points in the unstable eigendirection of $z_1$, and locate an initial point that passes as close as possible to $z_1$ and $z_0$. The finest discretization of the approximate unstable vector for the period two point is done on the scale $10^{-14}$. The first point on the pseudorobbit is in the unstable eigendirection of $z_1$ at a distance of $0.00058120854283$, and the last point is at a distance of $0.000167835$ from $z_0$.

We will now demonstrate the following eight covering relations:

$B_1 \Rightarrow B_0$, $B_1 \Rightarrow B_1$, $B_0 \Leftarrow B_1$, $B_1 \Leftarrow B_1$;

$B_0 \Rightarrow B_0$, $B_0 \Leftarrow B_0$, $B_0 \Leftarrow B_1$, $B_1 \Leftarrow B_0$.

Step 1. Set

$B_0^0 = F^{-4}(F_0^*(B_1) \cap F^{*4}(F_0^*(B_0) \cap F^{*2}(B_{10} \cap B_{11})))$;

$B_1^0 = F^{-4}(F_0^*(B_1) \cap F^{*4}(F_0^*(B_0) \cap F^{*2}(B_{10} \cap B_{11})))$.

(40)

Clearly, the iterate $\hat{F}_u F_0^t$ is well defined and analytic on $B_0^0 \cup B_1^0$, and

$B_0^0 \xrightarrow{\hat{F}_u} B_0$, $B_1^0 \xrightarrow{\hat{F}_u} B_1$.

(42)

The set $B_1$ is a parallelogram. Let $\pi_i^u$ be a projection on the span of $e_i^u$ ‘along’ the vector $e_i^u$ and $\pi_i^v$ be a projection on the span of $e_i^v$ ‘along’ the vector $e_i^v$. Then the sets $B_i^0$, $i = 0, 1$, are ‘vertical’ in the sense that $\pi_i^u(B_i^0) = \pi_i^u(B_1)$, while the sets $\hat{F}_u(B_i^0)$, $i = 0, 1$, are ‘horizontal’ in the sense that $\pi_i^v(\hat{F}_u B_i^0) = \pi_i^v(B_0)$ and $\pi_i^v(\hat{F}_u B_i^0) = \pi_i^v(B_1)$.

Step 2. Since the set $B_1$ is symmetric: $T(B_1) = B_1$, we have that

$T(B_0^0) \xrightarrow{\hat{F}_u^{-1}} B_0$, $T(B_1^0) \xrightarrow{\hat{F}_u^{-1}} B_1$.

(43)
Step 3. We get from the period doubling equation:
\[ \hat{F}_s(\Lambda^{-1}_s(B^1_i)) = \Lambda^{-1}_s(\hat{F}^2_s(B^1_i)). \]

Clearly, since \( \hat{F}_s \) is well defined and analytic on \( 1 B^1_i \), and \( 1 B^1_i \xrightarrow{\hat{F}_s} 1 B^1_i \) there exists a vertical subset \( 2 B^1_i \subset 1 B^1_i \) on which \( \hat{F}^2_s \) is analytic. Set \( 1 B^0_0 = \Lambda^{-1}_s(2 B^1_i) \). Then, since \( B_0 = \Lambda^{-1}_s(B^1_i) \), \( 1 B^0_0 \) covers \( B_0 \):
\[ 1 B^0_0 \xrightarrow{\hat{F}_s} B_0, \quad T(1 B^0_0) \xrightarrow{\hat{F}^{-1}_s} B_0. \] (44)

Step 4. Let \( 1 B^0_0 \supset 1 B^1_i \) be any set such that \( \varepsilon \leq \text{dist}_{p \in \tilde{B}_i} p, 1 B^0_0 \leq 2 \varepsilon \) for some sufficiently small \( \varepsilon > 0 \), chosen so that \( \hat{F}_s \) is analytic on \( 1 B^1_i \). The set \( 1 B^1_i = \hat{F}^{-1}_s(\bigwedge_{T(B^1_i)}) \cap B_0 \) satisfies
\[ 1 B^1_i \xrightarrow{\hat{F}_s} B_1, \quad T(1 B^1_i) \xrightarrow{\hat{F}^{-1}_s} B_1. \] (45)

Set
\[ C^\infty_{\hat{F}_s} = \lim_{k \to \infty} C^k_{\hat{F}_s}, \]
where
\[ C^k_{\hat{F}_s} = \hat{F}_s(\hat{F}^{-1}_s C^{k+1}_{\hat{F}_s}) \quad C^1_{\hat{F}_s} = \hat{F}_s(\bigwedge_{T(B^1_i)}) \cap \hat{F}^{-1}_s(\bigwedge_{T(B^1_i)}).
\]

We have proved the following:

Lemma 8.1. The map \( \hat{F}_s = F_s^{2^k} \) admits a locally maximal invariant set \( C^\infty_{\hat{F}_s} \) on which its action is homeomorphic to the full Bernoulli shift on \([0, 1]^Z\).

Proof of main theorem 1 (2–4). The covering relations (42) imply that, for any \( m \), there exist vertical sets \( m B^0_0 \subset 1 B^1_i \) and \( m B^1_1 \subset 1 B^1_i \) such that \( \hat{F}_s^m \) is analytic on them, and \( m B^0_0 \xrightarrow{\hat{F}_s^m} B_0 \) and \( m B^1_1 \xrightarrow{\hat{F}_s^m} B_1 \).

Consider the map \( \hat{F}_s^{2^m} \) for some integer \( k > 1 \). The fixed point equation
\[ \Lambda^{-k}_s \circ \hat{F}_s^{2^m} \circ \Lambda^{-k}_s = \hat{F}_s \] implies that
\[ \Lambda^{-k}_s(m B^0_0) \xrightarrow{\hat{F}_s^{2^m}} \Lambda^{-k}_s(B_0) = \Lambda^{-k+1}_s(B_1), \]
\[ \Lambda^{-k}_s(T(m B^0_0)) \xrightarrow{\hat{F}_s^{2^m}} \Lambda^{-k}_s(T(B_0)) = \Lambda^{-k+1}_s(T(B_1)), \]
\[ \Lambda^{-k}_s(m B^1_1) \xrightarrow{\hat{F}_s^{2^m}} \Lambda^{-k}_s(B_1), \]
\[ \Lambda^{-k}_s(T(m B^1_1)) \xrightarrow{\hat{F}_s^{2^m}} \Lambda^{-k}_s(B_1). \]

Below we will use the notation that \( A \xrightarrow{\hat{F}_s} B \) if there exists some vertical set \( A \subset A \) (depending on \( n \) and \( A \)) on which \( \hat{F}_s^n \) is analytic, and \( \hat{F}_s \xrightarrow{\hat{F}_s} B \). A \( \xrightarrow{\hat{F}_s} B \) and \( A \xrightarrow{\hat{F}_s} B \) are used similarly. The covering relations (42) together with the fixed point equation (46) imply that, in this notation, we get for any integer \( k > 0 \)
\[ \Lambda^{-k}_s(B_1) \xrightarrow{\hat{F}_s^{2^m}} \Lambda^{-k-1}_s(B_1) \xrightarrow{\hat{F}_s^{2^m-1}} \ldots \xrightarrow{\hat{F}_s^2} B_1, \]
\[ \Lambda^{-k}_s(T(B_1)) \xrightarrow{\hat{F}_s^{2^m}} \Lambda^{-k-1}_s(T(B_1)) \xrightarrow{\hat{F}_s^{2^m-1}} \ldots \xrightarrow{\hat{F}_s^2} T(B_1). \]
We have, therefore, shown that for any two natural $N$ and $M$, there exists an integer $n = 2^N$, such that

\[
\begin{align*}
\Lambda^N_\sigma(B_1) &\leftrightarrow \Lambda^{N-1}_\sigma(B_1) &\leftrightarrow \ldots &\leftrightarrow B_1 &\leftrightarrow \Lambda^{-1}_\sigma(B_1) &\leftrightarrow \Lambda^N_\sigma(B_1) \\
\ldots &\leftrightarrow \Lambda^{-M+1}_\sigma(B_1) &\leftrightarrow \Lambda^{-M}_\sigma(B_1) &\leftrightarrow \Lambda^{-1}_\sigma(B_1) &\leftrightarrow \Lambda^M_\sigma(B_1) \\
\ldots &\leftrightarrow \Lambda^{-1}_\sigma(B_1) &\leftrightarrow B_1 &\leftrightarrow \Lambda^0_\sigma(B_1) &\leftrightarrow \Lambda^{-1}_\sigma(B_1)
\end{align*}
\]

Set

\[\Delta_{N,M} = \bigcup_{k=-M}^{N} \Lambda^k_\sigma(\phi(k)B_1^0) \cup \Lambda^k_\sigma(\phi(k)B_1^1),\]

where $\phi(k) = 1$ if $k \geq 0$ and $\phi(k) = 2^{N-k}$ if $k < 0$, and define recursively:

\[Z^1_{F_1} \equiv \hat{F}^n_\sigma(\Delta_{N,M}) \cap \hat{F}^n_\sigma(T(\Delta_{N,M}))\quad \text{and} \quad Z^1_{F_2} \equiv \hat{F}^n_\sigma(\mathring{Z}^{k-1}) \cap \hat{F}^n_\sigma(\mathring{Z}^{k-1}).\]

Clearly, the set $Z_{F_2}^1 \equiv Z_{F_2}^0$ is a locally maximal invariant set for $F_2^x$, appearing in part (3) of main theorem 1.

If $\mathcal{O}_0$ is a heteroclinic orbit between points $z_1$ and $z_0$ for the iterate $\bar{F}_1 = F_1^x$ (or vice versa), then the fixed point equation (46) implies that, on the one hand, $\mathcal{O}_k \equiv \Lambda^k_\sigma(\mathcal{O}_0)$ is a heteroclinic orbit for iterate $\bar{F}_2^x$ between periodic points $z_{k+1}$ and $z_k$.

Therefore, $\mathcal{O}_{-k}$ contains a heteroclinic orbit between fixed points $z_{-k+1}$ and $z_{-k}$ for iterate $\bar{F}_x$. Stable and unstable manifolds of points $z_k$ and $z_{k+1}$ intersect in a two-sided tangle, see figure 5. This proves part (2) of the main theorem 1.

9. Distortion tools

We will say that a pair $(\mathcal{K}, \psi)$ is a dynamically defined Cantor set if $\mathcal{K} \subseteq R$ is a Cantor set and $\psi : \mathcal{K} \mapsto \mathcal{K}$ is a locally Lipschitz expanding map, topologically conjugate to the full Bernoulli shift $\sigma : [0, 1]^\mathbb{N} \mapsto [0, 1]^\mathbb{N}$.

A Markov partition $\mathcal{P} = \{\mathcal{K}_0, \mathcal{K}_1\}$ of $(\mathcal{K}, \psi)$ is a partition of $\mathcal{K}$ in two disjoint subsets, $\mathcal{K} = \mathcal{K}_0 \cup \mathcal{K}_1$, $\mathcal{K}_0 \cap \mathcal{K}_1 = \emptyset$, such that $\psi|_{\mathcal{K}_i} : \mathcal{K}_i \mapsto \mathcal{K}$ is a strictly monotone Lipschitz expanding homeomorphism.

Given a sequence $(a_0, \ldots, a_{n-1}) \in \{0, 1\}^n$, denote

\[\mathcal{K}(a_0, \ldots, a_{n-1}) = \bigcap_{i=0}^{n-1} \psi^{-1}(\mathcal{K}_{a_i}).\]

A bounded component of $\mathbb{R} \setminus \mathcal{K}$ is called a gap of $\mathcal{K}$. Gaps can be ordered as follows. The unique gap of order 0 is the interval $U_0 = \hat{\mathcal{K}} \setminus (\hat{\mathcal{K}}_0 \cup \hat{\mathcal{K}}_1)$, where $\hat{\mathcal{K}}$ denotes the convex hull of a set $A \subseteq \mathbb{R}$. A connected component of

\[\hat{\mathcal{K}} \setminus \bigcup_{(a_0, \ldots, a_{n-1}) \in \{0, 1\}^n} \mathcal{K}(a_0, \ldots, a_{n-1})\]
is called a gap of order $n$ if it is not a gap of order $k < n$. Any gap of $\mathcal{K}$ is a gap of some finite order.

**Definition 9.1.** Given a gap $U$ of $\mathcal{K}$ of order $n$, we denote $L_U$ (respectively, $R_U$) the unique left (respectively, right) adjacent to the $U$ interval of the form $\mathcal{K}(a_0, \ldots, a_{n-1})$, 

Figure 5. Intersections of (a) the stable manifolds of $z_1$ and $F_*(z_1)$ (blue) with the unstable ones for $z_2$, $F_*(z_2)$, $F^2_*(z_2)$ and $F^3_*(z_2)$ (red and magenta); (b) the stable manifold of $z_0$ (blue) with the unstable ones for $z_1$ and $F_*(z_1)$ (red); (c) the stable manifold of $z_{-1}$ (blue) with the unstable one for $z_0$ (red).
Definition 9.2. The distortion of the dynamically defined Cantor set
 distortion theorem (see Duarte (2000), Gorodetsky and Kaloshin (2008)).

Definition 9.5. Given positive constants $(a, \ldots, a_{n-1}) \in [0, 1]^n$.

The numbers

$$\tau_L(K) \equiv \inf \left\{ \frac{|L_U|}{|U|} : U \text{ is a gap of } K \right\},$$

$$\tau_R(K) \equiv \inf \left\{ \frac{|R_U|}{|U|} : U \text{ is a gap of } K \right\}$$

will be called the left and right thicknesses of $K$. The numbers

$$\tau_L(\mathcal{P}) \equiv \frac{|L_{U_0}|}{|U_0|} \quad \text{and} \quad \tau_R(\mathcal{P}) \equiv \frac{|R_{U_0}|}{|U_0|}$$

are called the left and right thicknesses of the Markov partition $\mathcal{P}$. Given a Lipschitz expanding
map $\psi : I \mapsto \mathbb{R}$, $I \subset \mathbb{R}$, define the distortion of $\psi$ on $I$:

$$\text{Dist}(\psi, I) \equiv \sup_{\{x,y,z \in I : z \neq x \neq y \neq z\}} \log \left\{ \frac{|\psi(y) - \psi(x)|}{|\psi(z) - \psi(x)|} \frac{|z - x|}{|y - x|} \right\} \in [0, \infty]$$

( note, $\psi(z) \neq \psi(x)$, $\psi(y) \neq \psi(x)$).

Definition 9.2. The distortion of the dynamically defined Cantor set $(L, \psi)$ is

$$\text{Dist}_\psi(K) \equiv \sup_{(a_0, \ldots, a_{n-1}) \in [0, 1]^n} \text{Dist}(\psi, K(a_0, \ldots, a_{n-1})).$$

Lemma 9.3 (see Palis and Takens (1993), Duarte (2000)). Let $(K, \psi)$ be a dynamically
defined Cantor set with a Markov partition $\mathcal{P}$ and distortion $\text{Dist}_\psi(K) = c$. Then

$$e^{-\tau_L(\mathcal{P})} \leq c_{\tau_L(K)} \leq e^{\tau_L(\mathcal{P})}, \quad e^{-\tau_R(\mathcal{P})} \leq c_{\tau_R(K)} \leq e^{\tau_R(\mathcal{P})}.$$ 

This lemma is important since it allows one to estimate the thicknesses of a Cantor set in terms
of thicknesses of its Markov partition.

Definition 9.4 (see Duarte (2000), Gorodetsky and Kaloshin (2008)). Define $\mathcal{F}$ to be the
set of all maps $F : \Delta_0 \cup \Delta_1 \mapsto \mathbb{R}^2$ such that:

1. $\Delta_0$ and $\Delta_1$ are compact sets, diffeomorphic to rectangles, with non-empty interior;
2. $F$ is $C^2$ on a neighbourhood of $\Delta_0 \cup \Delta_1$, that maps $\Delta_0 \cup \Delta_1$ diffeomorphically onto its
   image;
3. the locally maximal invariant set $\mathcal{C}_F = \bigcap_{n \in \mathbb{Z}} F^{-n}(\Delta_0 \cup \Delta_1)$ is a hyperbolic set, and the
   action of $F$ on $\mathcal{C}_F$ is conjugated to the Bernoulli shift $\sigma : [0, 1]^{\mathbb{Z}} \mapsto [0, 1]^{\mathbb{Z}}$.
4. $\mathcal{P} = (\Delta_0, \Delta_1)$ is a Markov partition for $F : \mathcal{C}_F \mapsto \mathcal{C}_F$, in particular, $F$ has two fixed
   points $p_0 \in \Delta_0$ and $p_1 \in \Delta_1$, whose stable and unstable manifolds contain the boundaries
   of $\Delta_0$ and $\Delta_1$.

We will now introduce our main tool for computing the Hausdorff dimension—the Duarte
distortion theorem (see Duarte (2000), Gorodetsky and Kaloshin (2008)).

Definition 9.5. Given positive constants $C$, $\epsilon$ and $\gamma$ define $\mathcal{F}(C, \epsilon, \gamma)$ to be the class of maps
$F : \Delta_0 \cup \Delta_1 \mapsto \mathbb{R}^2$, $F \in \mathcal{F}$, such that

1. $diam(\Delta_0 \cup \Delta_1) \leq 1$, $diam(F(\Delta_0) \cup F(\Delta_1)) \leq 1$;
2. the derivative of $F$, $DF(x, u)$ is $\begin{bmatrix} a(x, u) & b(x, u) \\ c(x, u) & d(x, u) \end{bmatrix}$, where $a, b, c$ and $d$ are $C^1$, satisfies on
   $\Delta_0 \cup \Delta_1$
   (a) $\det DF(x, u) = 1$.
   (b) $|d| < 1 < |a| \leq \frac{C}{\epsilon}$,
   (c) $|b|, |c| \leq \epsilon |a| - 1$;
Remark 9.6. Unlike in Duarte (2000) and Gorodetsky and Kaloshin (2008) we do not require the eigenvalues of the maps $\psi_s \equiv c \circ F^{-1}, \tilde{c} \equiv c \circ F^{-1} \circ \pi_1(p)$, respectively unstable leaves of $\mathcal{F}^u$, and $\mathcal{F}^u$ can be identified with the set of stable leaves of $\mathcal{F}^s$ on $\mathcal{F}(\Delta_0 \cup \Delta_1)$. The positivity of the eigenvalues is important only in the construction of the boundaries of the two components $\Delta_0$ and $\Delta_1$ of the Markov partition $\mathcal{P}$ (see lemma 10.5 for the construction of $\mathcal{P}$ in our case), but does not enter the proof of theorem 9.7.

We will now continue with the notion of stable and unstable Cantor sets. Denote the stable and unstable foliations of $\mathcal{C}_F$ as $\mathcal{F}^s$ and $\mathcal{F}^u$:

$$\mathcal{F}^s \equiv \{ \text{connected comp. of } \mathcal{W}^s(\mathcal{C}_F) \cap (\Delta'_0 \cup \Delta'_1) \}$$

$$\mathcal{F}^u \equiv \{ \text{connected comp. of } \mathcal{W}^u(\mathcal{C}_F) \cap (\Delta'_0 \cup \Delta'_1) \}$$

Also define

$$\mathcal{I}^s \equiv \mathcal{W}^s_{bc}(p_0) \cap \Delta'_0$$

and

$$\mathcal{I}^u \equiv \mathcal{W}^u_{bc}(p_0) \cap \Delta_0.$$
Theorem 9.7 (Duarte distortion theorem, Duarte (2000)). If \( F \in \mathcal{F}(C, \epsilon, \gamma) \), then the distortion of the dynamically defined Cantor sets \( (\mathcal{K}^{u,s}, \psi_{u,s}) \), \( \psi_{u,s} = \pi_{u,s} \circ F \), is bounded:
\[
\text{Dist}_{\psi_{u,s}}(\mathcal{K}^{u,s}) \leq D(C, \epsilon, \gamma) \equiv 4(C + 3)\gamma + 2\epsilon.
\]
(59)

In particular,
\[
e^{-D(C, \epsilon, \gamma) \tau_L(P_{u,s})} \leq \tau_L(K_{u,s}) \leq e^{D(C, \epsilon, \gamma) \tau_L(P_{u,s})},
\]
\[
e^{-D(C, \epsilon, \gamma) \tau_R(P_{u,s})} \leq \tau_R(K_{u,s}) \leq e^{D(C, \epsilon, \gamma) \tau_R(P_{u,s})}.
\]

The Duarte distortion theorem has been used as a powerful tool in conservative dynamics in several instances. It has been proved, and applied in Duarte (2000) to demonstrate that accumulation of a locally maximal invariant set by periodic elliptic points is generic in two parameter families. Furthermore, Gorodetsky and Kaloshin (2008) use it to show that generic unfoldings of homoclinic tangencies of two-dimensional area-preserving diffeomorphisms give rise to hyperbolic sets of dimension arbitrarily close to 2. To show this, the authors of Gorodetsky and Kaloshin (2008) complement the Duarte distortion theorem with the following results, which will be useful to us.

Lemma 9.8. The Hausdorff dimension of a Cantor set \( K \) satisfies
\[
\dim_H K > d,
\]
where \( d \) is the solution of
\[
\tau_L(K)^d + \tau_R(K)^d = (1 + \tau_L(K) + \tau_R(K))^d.
\]

Lemma 9.9. Suppose that for given \( t_L > 0, t_R > 0 \) the solution \( d \) of the equation
\[
t_L^d + t_R^d = (1 + t_L + t_R)^d
\]
is in \((0, 1)\), then
\[
d > \max \left\{ \frac{\log \left( 1 + \frac{t_R}{1+t_L} \right)}{\log \left( 1 + \frac{t_L}{1+t_R} \right)}, \frac{\log \left( 1 + \frac{t_L}{1+t_R} \right)}{\log \left( 1 + \frac{t_R}{1+t_L} \right)} \right\}.
\]
(60)

10. A horseshoe for \( G^* \)

In this section we demonstrate the existence of a horseshoe for the third iterate of \( G \in G^* \). We start with definitions.

Definition 10.1 (Full component). Suppose \( \Delta \subset \mathcal{U} \subset \mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^l \) is homeomorphic to a rectangle (a set of the form \( \mathcal{D}_1 \times \mathcal{D}_2 \subset \mathbb{R}^k \oplus \mathbb{R}^l \)), and let \( F : \mathcal{U} \mapsto \mathbb{R}^n \) be a diffeomorphism. A connected component \( \Delta_0 = F(\Delta_0) \) of \( \Delta \cap F(\Delta) \) is called full, if
\[
(1) \quad \tilde{\mathcal{P}}_2(\Delta_0) = D_2,
\]
\[
(2) \quad \text{for any } z \in \tilde{\Delta}_0, \tilde{\mathcal{P}}_1|_{\tilde{F}(\tilde{\mathcal{U}}_0 \times \tilde{\mathcal{D}}_2 \times \tilde{\mathcal{P}}_2(z))} \text{ is a bijection onto } D_1.
\]
Here, \( \tilde{\mathcal{P}}_{1,2} \equiv \mathcal{P}_{1,2} \circ h \), \( h \) the homeomorphism \( \Delta_0 \cong \mathcal{D}_1 \times \mathcal{D}_2 \), \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) are the canonical projections on \( \mathbb{R}^k \) and \( \mathbb{R}^l \).

Definition 10.2 (Two-component horseshoe). Let \( \mathcal{U} \subset \mathbb{R}^n \) be an open set, then the set \( \Delta \subset \mathcal{U} \) homeomorphic to a rectangle \( D_1 \times D_2 \) is called a two-component horseshoe for the diffeomorphism \( F : \mathcal{U} \mapsto \mathbb{R}^n \) if \( \Delta \cap F(\Delta) \) contains at least two full components \( \Delta_0 \) and
The rectangles that approximate the Markov partition for the horseshoe of $G_*$.

| Component | Centre   | ‘Stable’ scale | ‘Unstable’ scale |
|-----------|----------|----------------|-----------------|
| $\Delta'_0$ | (0.670 198.0, 0.0) | 0.083 | 0.083 |
| $\Delta'_1$ | (−0.441 811.0, 0.0) | 0.0655 | 0.0655 |

$\Delta_1$ such that,

1. $\tilde{T}_2(\Delta_0 \cup \Delta_1) \subset \text{int } D_2$, $\tilde{T}_1(F^{-1}(\Delta_0 \cup \Delta_1)) \subset \text{int } D_1$;
2. $DF|_{F^{-1}(\Delta_0 \cup \Delta_1)}$ preserves and expands an unstable (‘horizontal’) cone family on $F^{-1}(\Delta_0 \cup \Delta_1)$;
3. $DF^{-1}|_{\Delta_0 \cup \Delta_1}$ preserves and expands a stable (‘vertical’) cone family on $\Delta_0 \cup \Delta_1$.

By a standard result of the theory of dynamical systems, if $\Delta$ is a two-component horseshoe for $F$, then the action of $F$ on the locally maximal hyperbolic set $C_F \equiv \bigcap_{i=-\infty}^{\infty} F^i(\Delta)$ is homeomorphic to the Bernoulli shift $\sigma_2$ on $[0,1]^Z$.

To demonstrate the existence of a horseshoe for the third iterate of $G \in G_*$, we construct two rectangles: $\Delta'_0$ which contains the fixed point $p_0$ and $\Delta'_1$ which contains the period three point $p_1$. We will prove that these two rectangles constitute two full components of a horseshoe, and are such that $G(\Delta'_0) \cap W^u_{\text{loc}}(p_0)$ and $G(\Delta'_1) \cap W^s_{\text{loc}}(p_0)$ contain the components of the Markov partition of the locally maximal hyperbolic set $K_\sigma^*$.

The centres of the rectangles and sizes of the spanning vectors are given in Table 3. Note that these boxes are symmetric with respect to the involution $T: (x, u) \mapsto (-x, -u)$.

**Theorem 10.3.** For any $G \in G_*$, the inverse map $G^{-1}$ admits a horseshoe whose two full components are the sets $\Delta'_0$ and $\Delta'_1$, defined in Table 3. $\Delta'_0$ and $\Delta'_1$ are independent of $G$.

**Proof.** To demonstrate the existence of the horseshoe we first use the routine `checkCoveringRelations` to show that

$$\Delta'_0 \overset{G}{\Rightarrow} \Delta'_0 \overset{G}{\Rightarrow} \Delta'_1 \overset{G}{\Rightarrow} \Delta'_1 \overset{G}{\Rightarrow} \Delta'_0,$$

where the $h$-sets are as in Table 3. This implies that (1) in definition 10.2 holds.

The remainder of the proof of the existence of a horseshoe is done for the map $\tilde{G}_\sigma^{-1} \equiv T \circ G_\sigma^{-1} \circ T^{-1}$, where $T(x, u)$ is some coordinate transformation that approximately diagonalizes the derivative $DG_\sigma$ on $\Delta'_0 \cup \Delta'_1$. (See `Progs1 (2009)` for programs used in this part of the proof). We have used the following coordinate change:

$$T(x, u) = 0.55(x - 0.5, u + 0.026 546 163 211 697 7382)
- \begin{align*}
-0.658 & 388 496 175 704 694 (x - 0.577 619) \\
0.611 & 583 723 237 529 069 (x - 0.577 619)^2 \\
+ 0.102 & 408 008 658 008 658 (x - 0.577 619)^3.
\end{align*}\ \ (61)$$
Denote $\tilde{\Delta}'_{0,1} = T(\Delta'_{0,1})$. We show that there exists an invariant unstable cone field $\tilde{C}_u$ on $\tilde{\Delta}'_0 \cup \tilde{\Delta}'_1$.

$$\tilde{C}_u(p) = \left\{ v = (v_1, v_2) \in \mathbb{R}^2 : -0.15 \leq \frac{v_2}{v_1} \leq 0.2 \right\}, \quad p \in \tilde{\Delta}'_0,$$  

(62)

$$\tilde{C}_u(p) = \left\{ v = (v_1, v_2) \in \mathbb{R}^2 : -0.1 \leq \frac{v_2}{v_1} \leq 0.05 \right\}, \quad p \in \tilde{\Delta}'_1. \quad (63)$$

We have verified that

$$D\tilde{G}_s(p) \cdot v \in \text{int} \tilde{C}_u(\tilde{G}_s(p)), \quad \text{for all} \quad v \in \tilde{C}_u(p), \quad p \in \tilde{\Delta}'_0 \cup \tilde{\Delta}'_1,$$  

(64)

and that vectors inside the cone field are expanded:

$$|D\tilde{G}_s(p) \cdot v| = A|v|, \quad v \in \text{int} \tilde{C}_u(\tilde{G}_s(p)),$$

where the expansion rate $A$ is in $\mathbb{A} = [A_-, A_+]$

$$A_- = 6.091, \quad A_+ = 26.892. \quad (65)$$

Verification of invariance and expansion of the cone fields is carried out (among other things) in the routine Distortion of Progs1 (2009).

Reversibility and symmetry of the partition imply that there exists a stable cone field

$$\tilde{C}_u(p) \equiv DT(T^{-1}(p))(T(DT^{-1}(p))\tilde{C}_u(p))$$

(66)

on $\tilde{G}_s(\tilde{\Delta}'_0 \cup \tilde{\Delta}'_1) \cap (\tilde{\Delta}'_0 \cup \tilde{\Delta}'_1)$, which is mapped into its interior and expanded with the rate $A$ by $D\tilde{G}^{-1}$. $\square$

**Lemma 10.4.** The cone fields

$$C_u(p) = DT^{-1}(T(p))\tilde{C}_u(T(p)), \quad p \in \Delta'_0 \cup \Delta'_1, \quad (67)$$

$$C_u(q) = T(C_u(q)), \quad q \in G_s(\Delta'_0 \cup \Delta'_1) \cap (\Delta'_0 \cup \Delta'_1), \quad (68)$$

are transversal in the sense that the angle between any $u \in C_u(q)$ and $v \in \tilde{C}_u(q)$, $q \in G_s(\Delta'_0 \cup \Delta'_1) \cap (\Delta'_0 \cup \Delta'_1)$ is bounded from below.

Furthermore, the leaves of the foliations $\mathcal{F}^s$ and $\mathcal{F}^u$, defined in (56)–(57), are graphs over the $x$-axis.

**Proof.** We verify on the computer (this is done in the routine Distortion in Progs1 (2009)) that any $v \in DT^{-1}(T(p))\tilde{C}_u(T(p))$, $p \in \Delta'_0$, satisfies

$$\frac{v_2}{v_1} \geq 0.461,$$

and that any $v \in DT^{-1}(T(p))\tilde{C}_u(T(p))$, $p \in \Delta'_1$, satisfies

$$\frac{v_2}{v_1} \leq -0.675.$$

This implies that the angle between any $u \in C_u(q)$ and $v \in \tilde{C}_u(q)$, $q \in G_s(\Delta'_0 \cup \Delta'_1) \cap (\Delta'_0 \cup \Delta'_1)$ is bounded from below. $\square$

The horseshoe for $G$ is illustrated in figure 6, for both original $(a)$ and $T (b)$ coordinates.

Note that the full components $\Delta'_0$ and $\Delta'_1$ are chosen uniformly for all $G \in G_*$. This is not compatible with the definition of full components in definition 9.4. We can, however, prove that any $G \in G_*$ has full components, as in definition 9.4, contained in the components constructed in theorem 10.3.
Lemma 10.5. For any $G \in G_\ast$ there exist full components $\tilde{\Delta}_0 \subset \tilde{\Delta}_0'$ and $\tilde{\Delta}_1 \subset \tilde{\Delta}_1'$ as in definition 9.4.

Proof. (see Progs1 (2009) for programs) Let $G \in G_\ast$, by theorem 10.3

$$\tilde{p}_1 \equiv T(p_1) \in \tilde{\Delta}_1 \cap \tilde{G}(\tilde{\Delta})$$

hence there exists $\epsilon > 0$ such that $B_\epsilon(\tilde{p}_1) \subset \tilde{\Delta}^\ast$. Therefore, $\tilde{W}_u^\epsilon \equiv \tilde{W}_u^\epsilon(\tilde{p}_1) \cap B_\epsilon(\tilde{p}_1) \subset \tilde{\Delta}^\ast$. Let $t \mapsto \tilde{W}_u^\epsilon(t)$ be some parametrization of $\tilde{W}_u^\epsilon$. For sufficiently small $\epsilon$ the tangent vector $\tau(t)$ to $\tilde{W}_u^\epsilon(t)$ is close to $e^1_\epsilon(\tilde{p}_1)$, that is, there exists $\epsilon > 0$ such that $\tau(t) \in C_\epsilon(\tilde{W}_u^\epsilon(t))$.

The cone conditions imply that $\tilde{W}_u^\epsilon(t)$ is expanded in the forward (unstable) cone into $\text{cvx} \tilde{G}(\tilde{\Delta})$: there exists a positive integer $N$ such that $\tilde{G}^N(\tilde{W}_u^\epsilon) \cap (\tilde{\Delta}_0 \cap \tilde{\Delta}_1') \neq \emptyset$ and
The right thicknesses of the Markov partition for the unstable Cantor set according to Lemma 9.8 and the Duarte distortion theorem 9.7, the knowledge of the left and right thicknesses of the Markov partition for the unstable Cantor set together with the distortion of this Cantor set is sufficient to compute a lower bound on its Hausdorff dimension. For any $G \in G_*$ the intersections of the local stable manifold $\tilde{W}_s(p_0)$ with $\partial G(\tilde{\Delta}_0)$ and $\partial G(\tilde{\Delta}_1)$ are at the points $k_0^b, k_0', k_1^b, k_1'$, respectively, where

$$k_0^b = [0.041476215, 0.04509142] \times [0.01162711, 0.015242315],$$

$$k_0' = [0.034578335, 0.03698849] \times [0.023247785, 0.02565794]$$

and

$$k_1^b = [0.099814, 0.1022241] \times [-0.07865165, -0.07624155],$$

$$k_1' = [0.09487115, 0.09660585] \times [-0.06980765, -0.06710275],$$

respectively.

**Proof.** (see Progs1 (2009) for programs) We prove that there exist two points $q_1, q_2 \in W_{loc}(p_1) \cap G(\Delta_1) \cap W^s(p_0)$, such that

$$G(q_1) \cap W_{loc}(p_0) \neq \emptyset,$$

$$G(q_2) \cap \Delta_0 \neq \emptyset,$$

$$G^2(q_2) \cap \Delta_1 \neq \emptyset,$$

$$G^3(q_2) \cap W_{loc}(p_0) \neq \emptyset.$$
consisting of one box around $p_0$, one box around $p_1$ and five intermediate boxes on the orbit; two of them close to the unstable vector of $p_1$ and three of them close to the stable vector of $p_0$. We use the programs checkCoveringRelations and checkConeConditions to verify that the hypotheses of theorem 6.6 and lemma 6.11 are satisfied on the sequence of $h$-sets.

We emphasize that the choice of boxes is very delicate. To construct the sequence we first construct relatively large boxes around $p_0$ and $p_1$, spanned by their unstable and stable eigenvectors. Second, we reduce the size of these boxes until we are able to prove that the hypothesis of lemma 6.11 is satisfied. Third, we non-uniformly distribute points in the unstable eigendirection inside of the box containing the period three point; on the smallest scale the points are only separated by $10^{-11}$. Fourth, we iterate, non-rigorously, these points to locate a heteroclinic pseudoorbit, which starts within the box containing $p_1$ and ends in the box containing $p_0$. The first point on the orbit is in the unstable eigendirection of the period three point, at a distance of 0.000 158 885 from our approximation of the period three point, and the last point is at a distance of 0.001 774 01 from the fixed point. Finally, we use the points on this pseudoorbit as centres of our covering sequence. The first two intermediate points are on the local unstable manifold of the period three point, and we use its unstable and stable eigenvectors to span the corresponding $h$-sets. The other three intermediate points are on the local stable manifold of the fixed point, and we use its unstable and stable eigenvectors to span the corresponding $h$-sets. Note that if we change the location of the original point on the stable vector by as little as $10^{-3}$, the proof fails. The existence of $q_1$ follows from the set of covering relations constructed in table 4.

To find $q_2$ we do a similar non-rigorous computation to locate a pseudoorbit with the properties above. The corresponding sequence of $h$-sets and covering relations is given in table 5. Finally, we put $k_0^1 = TG(q_1)$, $k_0^0 = TG^3(q_1)$, $k_0^0 = TG^3(q_1)$ and $k_1^1 = TG^4(q_2)$. By the construction of the components of the Markov partition $\tilde{\Delta}_0$ and $\Delta_1$, it is immediate that $\{k_0^0, k_0^0\} = \tilde{\mathcal{V}}_{\text{loc}}^u(\tilde{p}_0) \cap \partial\tilde{G}(\tilde{\Delta}_0)$ and $\{k_1^1, k_1^1\} = \tilde{\mathcal{V}}_{\text{loc}}^s(\tilde{p}_0) \cap \partial\tilde{G}(\Delta_1)$. Evaluating $k_0^1$ at $q_0/1$ with $G_*$ yields the enclosures given in the statement of the lemma.

Consider a $G \in G_*$, by lemma 10.5 the two components of the Markov partition for $k_0^u$ are contained in the intersections

$$\tilde{G}(\tilde{\Delta}_0) \cap \left(\tilde{\mathcal{V}}_{\text{loc}}^u(\tilde{p}_0) \cap \tilde{\Delta}_0\right), \quad \tilde{G}(\Delta_1) \cap \left(\tilde{\mathcal{V}}_{\text{loc}}^s(\tilde{p}_0) \cap \tilde{\Delta}_0\right).$$

| Box # | Centre            | Eigenvector | Scale |
|-------|-------------------|-------------|-------|
| 0     | $(0.577\,619,0)$  | Fixed       | 0.004 |
| 1     | $(-0.527\,156,0)$ | Period three| 0.0005|
| 2     | $(-0.524\,339,-0.003\,025\,53)$ | Period three| 0.001 |
| 3     | $(-0.458\,451,-0.069\,5428)$ | Period three| 0.002 |
| 4     | $(0.683\,671,-0.090\,7786)$ | Fixed       | 0.002 |
| 5     | $(0.565\,061,0.009\,741\,95)$ | Fixed       | 0.002 |
| 6     | $(0.578\,698,-0.001\,4086)$ | Fixed       | 0.003 |
are the approximate unstable and stable eigenvectors of $D$. This property in the program to propagate stable manifolds backwards through a sequence of covering relations. We use $W$ to enclose the stable manifold at the fixed point, we work in the coordinates (61). First, we choose 200 points $q_i$, distributed uniformly on the interval $(0, p)$. These 2200 points $z_{i,n} \equiv F_n^{-1}(q_i)$, $1 \leq i \leq 200$, $0 \leq n \leq 10$, provide an approximation of the stable manifold. We next construct rectangles

\[ B_{i,n} = \{ (x, u) \in \mathbb{R}^2 : \| (x - P_x T(z_{i,n}), u - P_u T(z_{i,n})) \cdot e^u \| < 0.001, \| (x - P_x T(z_{i,n}), u - P_u T(z_{i,n})) \cdot e^s \| < 0.001 \}, \]

where

\[ e^u = (0.570 868 623 900 820 281, -0.821 041 420 541 974 730), \]

\[ e^s = (0.992 704 972 028 756 258, 0.120 568 812 341 277 774) \]

are the approximate unstable and stable eigenvectors of $D\tilde{F}_z$ at $p_0$, and verify that

\[ B_{i,0} \xrightarrow{\tilde{F}} B_{i,9} \xrightarrow{\tilde{F}} \cdots \xrightarrow{\tilde{F}} B_{i,0} \xrightarrow{\tilde{F}} B_0, \]

for all $F \in F_*$, where

\[ B_0 = \{ (x, u) \in \mathbb{R}^2 : \| (x - P_x T(p_0), u - P_u T(p_0)) \cdot e^u \| < 0.001, \| (x - P_x T(p_0), u - P_u T(p_0)) \cdot e^s \| < 0.001 \}. \]

Therefore, our next task will be to find an enclosure of the local stable manifold at the fixed point $p_0$. By the property of $h$-sets with cone conditions mentioned in remark 6.10, it is possible to propagate stable manifolds backwards through a sequence of covering relations. We use this property in the program $\text{encloseStableManifold}$ to enclose the local stable manifold $W_{\text{loc}}(\tilde{p}_0) \cap \Delta_0$; estimates of the length of its intersection with $G_*(\Delta)$ are needed to estimate the left and right thicknesses of the Markov partition for $\mathcal{K}^\infty$, and ultimately the Hausdorff dimension.

To enclose the stable manifold of the fixed point, we work in the coordinates (61). First, we choose 200 points $q_i$, distributed uniformly on the interval

\[ (p_0, p_0 + 0.000 11(0.788 578 889 012 330, -0.614 933 602 760 558)) \subset \mathbb{R}^2, \]

and iterate each of them 10 times with $F_n^{-1}$. These 2200 points $z_{i,n} \equiv F_n^{-1}(q_i)$, $1 \leq i \leq 200$, $0 \leq n \leq 10$, provide an approximation of the stable manifold.

Table 5. The boxes used to prove the existence of a heteroclinic orbit between the period three point and the fixed point, passing through $q_2$. Note that the numbers in this table are in the original coordinates. $k_i'$ corresponds to box #7 and $q_2$ to box #4. The spanning vectors are the stable and unstable vectors for either the fixed point or the period three point. Note, the stable vector is constructed by reflecting the unstable vector in $[u = 0]$. The approximate unstable eigenvectors for the fixed and period three points are $(0.788 578 889 012 330, 0.614 933 602 760 558)$ and $(0.682 259 082 166 558, -0.731 107 710 380 897)$, respectively. To construct the corresponding parallelogram the spanning vectors are scaled as indicated in the table. The covering sequence is $1 \Rightarrow 1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 6 \Rightarrow 7 \Rightarrow 8 \Rightarrow 0 \Rightarrow 0$.

| Box # | Centre | Eigenvector | Scale |
|-------|--------|-------------|-------|
| 0     | (0.577 619, 0) | Fixed        | 0.002 |
| 1     | (−0.527 156, 0) | Period three | 0.00015 |
| 2     | (−0.527 035, −0.000 143 356) | Period three | 0.0001(0.9) |
| 3     | (−0.523 879, −0.003 516 78) | Period three | 0.0003 |
| 4     | (−0.447 416, −0.079 9474) | Period three | 0.0003 |
| 5     | (0.787 595, −0.008 880 05) | Fixed        | 0.0005 |
| 6     | (−0.366 87, 0.010 9682) | Period Three | 0.0003 |
| 7     | (0.674 07, −0.081 9091) | Fixed        | 0.001 |
| 8     | (0.566 215, 0.008 822 67) | Fixed        | 0.001 |
To prove that the cone conditions are satisfied we verify the hypothesis of lemma 6.11 using the routine checkConeConditions. This proves that \( \bigcup_{i,n} B_{i,n} \cup B_0 \) covers the local stable manifold.

The sets \( K_0^u, K_0^l, K_1^u, K_1^l \), from lemma 11.1, serve as bounds on the end points of the sets \( K_{0,1}^u \) in the Markov partition \( \mathcal{P} = \{ K_0^u, K_1^u \} \). Therefore,

\[
\tau_L(\mathcal{P}) \geq \frac{\text{dist}(k_1^b, k_1^s)}{\sqrt{1 + \text{Lip}^2 \sup_{q \in k_1^b, q' \in k_1^s}|q - q'|}} \geq 0.0650166, \quad (69)
\]

\[
\tau_R(\mathcal{P}) \geq \frac{\text{dist}(k_0^b, k_0^s)}{\sqrt{1 + \text{Lip}^2 \sup_{q \in k_0^b, q' \in k_0^s}|q - q'|}} \geq 0.0514139, \quad (70)
\]

(these quantities are computed in the routine Gaps in Progs1 (2009)).

Finally, we compute (in routine Distortion of Progs1 (2009)) the quantities \( C, \gamma \) and \( \epsilon \) appearing in the definition 9.5, and verify parts (3), (4) and (5) of this definition. In particular, we have set

\[
\epsilon = \sup_{\Delta_0 \cup \Delta_1} \left\{ \frac{\max(|b|, |c|)}{|a| - 1} \right\}, \quad C = \sup_{\Delta_0 \cup \Delta_1} \left\{ |a| \frac{\max(|b|, |c|)}{|a| - 1} \right\},
\]

while \( \gamma \) has been chosen as the supremum of all possible values that verify conditions (52)–(53) over \( \Delta_0 \cup \Delta_1 \). To verify part (4) we have bound the total variation of \( \eta(x, u) = \log |a(x, u)| \) on \( \Delta_0 \cup \Delta_1 \) from above by \( \text{Area}(\Delta_0) \cdot \sup_{(x,u) \in \Delta_0} |\nabla \eta(x, u)| \).

**Lemma 11.2.** Any function \( G \in G_s : \Delta_0 \cup \Delta_1 \mapsto \mathbb{R}^2 \) is of class \( \mathcal{C}(C, \gamma, \epsilon) \) with

\[
C = 16.6, \quad \gamma = 47.8, \quad \epsilon = 0.88.
\]

A straightforward implementation of the Duarte distortion theorem and lemma 9.8 gives us a bound on the Hausdorff dimension of the unstable Cantor set:

**Corollary 11.3.** For all \( G \in G_s \), the distortion of the unstable Cantor set \( K_G^u \) is less than or equal to

\[
\text{dim}_H(K_G^u) \geq 3749.3
\]

while its Hausdorff dimension satisfies:

\[
\dim_H(K_G^u) \geq \frac{\log \left( 1 + \frac{e^{-\tau_L(\mathcal{P})}}{\tau_L(\mathcal{P})} \right)}{\log \left( 1 + \frac{e^{-\tau_R(\mathcal{P})}}{\tau_R(\mathcal{P})} \right)} \approx \frac{1}{2D} e^{-2D} \frac{\tau_R(\mathcal{P})}{\tau_L(\mathcal{P})},
\]

where \( \tau_L(\mathcal{P}) \) and \( \tau_R \) are as in (69)–(70).

The Hausdorff dimension of the set \( C_G \) satisfies

\[
\dim_H(C_G) \geq 2\dim_H(K_G^u) \approx 0.00013e^{-7499}.
\]

**Proof.** The fact that the stable and unstable foliations are transversal (see lemma 10.4) implies that \( C_G = K_G^s \times K_G^u \). Reversibility and the symmetry of the Markov partition \( \Delta_0 \cap \Delta_1 \) implies that \( K_G^s = T(K_G^u) \), and \( \dim_H(K_G^u) = \dim_H(K_G^s) \). which in turn implies the last claim.

Corollary 11.3 together with theorem 10.3 proves the first part of the second main theorem.

Let us now construct a convergent sequence of approximations of the Cantor sets \( C_G \). As before, we denote \( \Delta_0 \) and \( \Delta_1 \) the two full components of the Markov partition as in definition 9.5.
and lemma 10.5, \( \Delta \equiv \Delta_0 \cup \Delta_1 \). Define recursively
\[
U_G^0 = G(\Delta) \cap G^{-1}(\Delta) \quad \text{and} \quad U_G^k = G(U_G^{k-1}) \cap G^{-1}(U_G^{k-1}).
\]

Each of the sets \( U_G^k \) contains \( 2 \cdot 4^k \) components \( U_G^{k,n} \), \( n = 1, 2, 4^k \).

Recall that \( \partial \Delta = b_j \cup t_j \cup l_j \cup r_j \), \( j = 0, 1 \). By definition of \( U_G^{k,n} \), the boundary \( \partial U_G^{k,n} \) consists of two images, \( t_j \), \( b_j \), \( l_j \), \( r_j \), \( j = 0, 1 \), under \( G \), and two images, \( t^{k,n} \), \( b^{k,n} \), \( l^{k,n} \), \( r^{k,n} \), of subsets of \( t_j \), \( b_j \), \( l_j \), \( r_j \), \( j = 0, 1 \), under \( G^{-k} \). We will refer to \( t^{k,n} \), \( b^{k,n} \), \( l^{k,n} \), \( r^{k,n} \) as ‘edges’ or ‘sides’, and to their intersections as ‘corners’ of \( U_G^k \).

**Lemma 11.4.** Let
\[
\rho_{k,n} = \sup_{B \subseteq \partial U_G^{k,n}}(\rho)
\]

and set \( \rho_k = \min_{n}(\rho_{k,n}) \). There exist constants \( C > 0 \) and \( c > 0 \) such that
\[
\text{diam}(U_G^{k,n}) \leq C A^{-k} \equiv CK^k,
\]
\[
\rho_k \geq c A^{-k} \equiv CK^k
\]

where \( A_k \) are as in (65).

**Proof.** The length of an edge of \( U_G^{k,n} \) is bounded by \( cA^{-k} \) for some \( A \in A \). This follows from the fact that \( \Delta_j \), \( j = 0, 1 \), are constructed so that the tangent vectors to \( t_j \) and \( b_j \), \( j = 0, 1 \), are contained in \( C_\alpha \), and those to \( l_j \) and \( r_j \), \( j = 0, 1 \), — in \( C_t \) (see (67) and (68)) and from the fact that an edge of \( U_G^{k,n} \) is contained in the image of an edge of \( U_G^{k-1,j} \) for some \( j \).

Consider an edge of \( U_G^k \), say \( t^{k,n} \). Set
\[
\alpha^+ = \sup_{v \in \partial t^{k,n}}(\angle v : v \in C_\alpha(p)), \quad \alpha^- = \inf_{v \in \partial t^{k,n}}(\angle v : v \in C_\alpha(p))
\]

where \( \angle v \) signifies the angle between vector \( v \) and the \( x \)-axis, measured, say, counterclockwise. Similarly for \( b^{k,n} \), \( l^{k,n} \), \( r^{k,n} \). Straightforward geometric considerations demonstrate that the angles \( \alpha^+ \), \( \alpha^- \) specify two quadrilaterals, containing and contained in \( U_G^{k,n} \), respectively.

Since the angle between any \( v \in C_\alpha(p) \) and any \( u \in C_\alpha(p) \) is bounded from below (see lemma 10.4), the angles of the quadrilaterals are bounded away from \( 0 \), and their diameters satisfy
\[
c A^+ \leq \text{diam (inner quadrilateral)},
\]
\[
C A^- \geq \text{diam (outer quadrilateral)}
\]

The claim follows. \( \square \)

**Corollary 11.5.** The Hausdorff dimension of the Cantor set \( C_G \) satisfies
\[
\text{dim}_H(C_G) \leq \frac{\log(4)}{-\log \kappa} \leq 0.7673.
\]

**Proof.** Clearly \( C_G \subset U_G^k \), for all \( k \). As above, \( U_G^k = \bigcup_{n=1}^{2 \cdot 4^k} U_G^{k,n} \), where
\[
\text{diam}(U_G^{k,n}) \leq \text{const} \kappa^k, \quad \text{for all } 1 \leq n \leq 2 \cdot 4^k.
\]

Hence, for any \( k > 1 \), \( C^H_G(C_G) \leq \text{const} 4^k \kappa^k \), and if \( 4 \kappa^k < 1 \), then \( C^H_G(C_G) = 0 \). \( \square \)
12. Overview of the programs

We will now give a brief overview of the programs used in the proofs.

A subset of the domain of a map $F \in F_*$ is determined using the program findDomain of Progs1 (2009), used in lemma 4.2. findDomain discretizes the domain of the generating function in the $x$-domain, and computes rigorous bounds on the maximum and minimum of the generating function, $s(y, x)$, on this slice of the domain. During the process of computing the maximum and minimum, the program also verifies that the conditions of the implicit function theorem are verified on the slice.

To verify that a pair of $h$-sets satisfies the hypothesis of theorem 6.6, we use the routine checkCoveringRelations of Progs1 (2009). It discretizes the boundary of an $h$-set, and verifies that each piece of the boundary is mapped so that the covering relations hold.

To verify that a pair of $h$-sets with cones satisfies the hypothesis of lemma 6.11, we use the routine checkConeConditions of Progs1 (2009). It computes the derivative of the map on the initial $h$-set in appropriate coordinates, and checks the positive definite condition of the quadratic form $V$ in the statement of the lemma using the Sylvester criterion.

The sequences of $h$-sets used to enclose the stable manifold in section 11, are constructed in the program encloseStableManifold of Progs1 (2009). This program uses the routines checkCoveringRelations and checkConeConditions, to verify that on each sequence, the hypotheses of theorem 6.6 and lemma 6.11, respectively, are satisfied.

As we have already mentioned, parts of theorem 10.3, lemmas 10.4 and 11.2 are proved with the help of the routine Distortion of Progs1 (2009). The input of the routine is a set of parameters that define the components $\Delta_0'$ and $\Delta_1'$ (see table 3) and the cone fields (see (62)--(63)). The routine verifies that the cone fields are invariant and computes the expansion rates (65). It also computes the quantities $C, \gamma$ and $\epsilon$ appearing in lemma 11.2.

The thicknesses (69) and (70) of the Markov partition for the horseshoe of $G_*$ are computed in the routine Gaps of Progs1 (2009).

13. Concluding remarks

An obvious question is whether the Hausdorff dimension of the set $C_G$ for all $G$ in some subset of $G_*$ is independent of $G$.

In the satellite work (Gaidashev and Johnson 2009) we were able to use a renormalization approach to show that the Hausdorff dimension of the ‘stable’ set, that is the set on which the maximal Lyapunov exponent is zero, is indeed invariant for a subset of infinitely renormalizable maps. An essential ingredient of that proof is the fact that the renormalizations of all infinitely renormalizable maps in a neighbourhood of the fixed point converge to that fixed point on their domain of analyticity $D$.

In a somewhat similar fashion, the invariance of the Hausdorff dimension of the set $C_G$ could be demonstrated if one can show that renormalizations of the infinitely renormalizable maps converge to $G_*$ on neighbourhoods $V_k$ of the rescalings $\Lambda^{-1}_{k, G}(C_G)$ of the hyperbolic sets. However, $D \cup_{k=1}^{\infty} V_k$ is not a connected domain, and convergence on $D$ does not imply that on $\cup_{k=1}^{\infty} V_k$.

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