MANIFOLDS, STRUCTURES CATEGORICALLY

G.V. Kondratiev

Abstract. A notion of general manifolds is introduced. It covers all usual manifolds in mathematics. Essentially, it is a way how to get a bigger 'fibration' over a site which locally coincides with a given one. An enrichment with generalized elements is regarded which allows to see hom-sets of a given category as (almost) objects and to transfer some technics from objects onto hom-sets. Lifting problem for a group action and actions of group objects are also included.

1. Fibrations and cofibrations

(Co)fibrations play role of structures over objects in a given category which can be transported along morphisms. Transport system is called (co)cartesian morphisms. The situation is very similar to fibrations with connection as in Differential Geometry.

Definition 1.1. For a functor

\[
\begin{array}{ccc}
E & \xrightarrow{\pi} & B \\
\downarrow & & \downarrow \\
E' & \xrightarrow{f} & E
\end{array}
\]

- morphism \( f : B' \to p(E) \) has a \textbf{cartesian lifting} \( \tilde{f} : E' \to E \in Ar\ E \) if \( \forall f' : E'' \to E \) such that \( p(f') \) factors through \( f \) (i.e., \( \exists g : p(E'') \to B' \) such that \( p(f') = f \circ g \)) \( f' \) itself uniquely factors through \( \tilde{f} \) over the base factorization (i.e., \( \exists! \tilde{g} : E'' \to E' \) such that \( f' = \tilde{f} \circ \tilde{g} \) and \( p(\tilde{g}) = g \)) [Jac]

\[
\begin{array}{ccc}
E'' & \xrightarrow{\exists! \tilde{g}} & E' \\
\downarrow \forall f' & & \downarrow \forall f \\
E & \xrightarrow{\tilde{f}} & E \\
\downarrow p(f') & & \downarrow p(f) \\
B' & \xrightarrow{f} & p(E)
\end{array}
\]

- morphism \( f : p(E) \to B' \) has a \textbf{cocartesian lifting} \( \tilde{f} : E \to E' \in Ar\ E \) if \( \forall f' : E \to E'' \) such that \( p(f') \) factors through \( f \) (i.e., \( \exists g : B' \to p(E'') \) such that \( p(f') = g \circ f \)) \( f' \) itself uniquely factors through \( \tilde{f} \) over the base factorization (i.e., \( \exists! \tilde{g} : E' \to E'' \) such that \( f' = \tilde{g} \circ \tilde{f} \) and

1991 Mathematics Subject Classification. CT Category Theory.
Key words and phrases. (co)fibration, (almost) structures, enrichment, manifolds, stacks, group object, actions.
\[ p(\tilde{g}) = g \] [Jac]

\[ \begin{array}{ccc}
E & \xrightarrow{f} & E' \\
\downarrow & & \downarrow \\
p(E) & \xrightarrow{f} & B'
\end{array} \]

\[ \exists \tilde{g} \]

Remark. (Co)cartesian morphisms if exist are unique up to vertical isomorphism \((v : E \to E' \in Ar E)\) is vertical if \(p(v) = 1_B\) for some \(B \in Ob B\).

**Definition 2.1.2.** A functor \( \xrightarrow{p} \) is called [Jac, Str]

- **fibration** if for each \( f : B' \to p(E) \in Ar B \) there exists cartesian lifting \( \tilde{f} : E' \to E \in Ar E \)
- **cofibration** if for each \( f : p(E) \to B' \in Ar B \) there exists cocartesian lifting \( \tilde{f} : E \to E' \in Ar E \)
- **bifibration** if it is both fibration and cofibration

Subcategory \( E_B := p^{-1}(B, 1_B) \hookrightarrow E \) is called fiber over \( B \). \( E_B \) consists of all vertical morphisms over \( B \).

**Examples**

1. For a category \( C \) with pullbacks **codomain fibration** is

\[ \xrightarrow{\text{cod}} \]

\[ \begin{array}{ccc}
C & \xrightarrow{f} & \bullet \\
\downarrow & & \downarrow \\
\bullet & \xrightarrow{\phi} & \bullet
\end{array} \]

Cartesian lifting is a pullback square

2. For a category \( C' \) with pushouts **domain cofibration** is

\[ \xrightarrow{\text{dom}} \]

\[ \begin{array}{ccc}
C' & \xrightarrow{f} & \bullet \\
\downarrow & & \downarrow \\
\bullet & \xrightarrow{\phi} & \bullet
\end{array} \]

Cocartesian lifting is a pushout square

3. [Kom] Denote \( \text{Rng–Mod} \), category of left modules with variable rings.

\( Ob(\text{Rng–Mod}) \) are pairs \((R, M)\) where: \( R \) is a ring, \( M \) is an \( R \)-module. \( Ar(\text{Rng–Mod}) \) are pairs \((\varphi : R_1 \to R_2, f : M_1 \to M_2)\) such that \( \forall r \in R_1, m \in M_1 \quad f(r \cdot m) = \varphi(r) \cdot f(m) \).

Then 'projection on first component' \( \text{Rng–Mod} \xrightarrow{p_1} \text{Rng} \) is a bifibration. If \( \varphi : R_1 \to R_2 \in Ar \text{Rng} \) then **direct image** of \((R_1, M_1)\) over \( \varphi \) is \((R_2, R_2 \otimes R_1 M_1)\). Cocartesian lifting of \( \varphi \) is \((\varphi, \tilde{\varphi})\) with \( \tilde{\varphi} : M_1 \to R_2 \otimes R_1 M_1 : m \mapsto 1 \otimes m \).
where: $\psi \cdot f := \mu \circ (\psi \otimes f) : R_2 \otimes_{R_1} M_1 \xrightarrow{\otimes \varphi} R_3 \otimes_{R_1} M_3 \xrightarrow{\psi} M_3$, $\mu$ is module multiplication, $\psi(r_1 \cdot r_2) \cdot f(m) = (\psi \cdot f)(r_1 \cdot r_2 \otimes m) = \psi(r_1) \cdot (\psi(f)(r_2 \otimes m)) = \psi(r_1) \cdot (\psi(r_2) \cdot f(m))$.

**Inverse image** of $(R_1, M_1)$ over $\alpha : R \rightarrow R_1$ is $(R, M_1)$ with action $(r, m) \mapsto \alpha(r) \cdot m$.

**Cartesian lifting** of $\alpha$ is $(\alpha, 1_{M_1})$.

### 4. (Differential Equations)

$\text{Set} \xleftarrow{p_1} \text{Top} \xleftarrow{p_2} \text{Diff} \xleftarrow{p_3} \text{Diff} \xleftarrow{p_4} \text{Jet}^\infty(\text{Diff}^\rightarrow) \xleftarrow{p_5} \text{Sub}(\text{Jet}^\infty(\text{Diff}^\rightarrow))$

where $\text{Diff}^\rightarrow$ is a full subcategory of $\text{Diff}^\rightarrow$ consisting of surjective submersions (surjective maps with surjective differential at each point), $\text{Jet}^\infty(\text{Diff}^\rightarrow)$ is the corresponding category of $\infty$-jet-bundles, $\text{Sub}(\text{Jet}^\infty(\text{Diff}^\rightarrow))$ is the category of subobjects of $\infty$-jet-bundles (differential relations),

$p_1 : \text{Top} \rightarrow \text{Set}$ is a fibration ($\forall$ arrow $S' \xrightarrow{u} p_1(T) \exists$ Cartesian 'completion'),

$T' \xrightarrow{v} T$

$T'$ has initial topology w.r.t. $u$,

$p_2 : \text{Diff} \rightarrow \text{Top}$ is not a fibration if differentiable manifolds are regarded in usual sense (as locally Euclidean), but it is a fibration if differentiable manifolds are topological spaces endowed with a subsheaf of continuous functions closed under smooth operations,

$p_3 : \text{Diff}^\rightarrow \rightarrow \text{Diff}$ is a codomain fibration since pullback of a surjective submersion is a surjective submersion,

$p_4 : \text{Jet}^\infty(\text{Diff}^\rightarrow) \rightarrow \text{Diff}^\rightarrow$ is not a fibration (if we admit arbitrary fibre bundle arrows between objects in $\text{Jet}^\infty(\text{Diff}^\rightarrow)$), but it is a structure over $\text{Diff}^\rightarrow$ (see 2.1.1),
Lemma 1.1. Every functor \( F : C \to B \) factors through a free fibration \( C \xrightarrow{i} 1/F \) where

\[ C \xrightarrow{i} 1/F \]

\[ \downarrow \]

\[ \text{dom} \]

\[ \downarrow \]

\[ B \]

\[ i : C \to 1/F : \begin{cases} C \mapsto 1_{F(C)} & C \in \text{Ob} C \\ (f : C \to C') \mapsto (F(f), f) & f \in \text{Ar} C \end{cases} \]

Proof is straightforward. See [Jac].

Another equivalent description of (co)fibrations is via (co)contravariant pseudofunctors \( B \to \text{CAT} \).

Proposition 1.1.
Proof is straightforward. See [Jac].

Two processes above are (weakly) inverse to each other.

For each choice of (co)cartesian liftings for all arrows \( f : B' \to p(E) \) (or respectively, \( f : p(E) \to B' \)) in the base category of a (co)fibration \( p \)

\[
F_p : B \to \text{CAT} : \begin{cases} B \mapsto E_B & B \in \text{Ob} \ B \\ f \mapsto \text{Cart}_f & f \in \text{Ar} \ B \end{cases}
\]

Conversely, for a given (co)contravariant pseudofunctor \( F : B \to \text{CAT} \) there is a (co)fibration \( E_F \)

\[
p_F : (\text{Grothendieck construction}).
\]

\[
\left\{ \begin{array}{ll}
\text{Ob } E_F & \text{are pairs } \left( \begin{array}{c} E \\ B \end{array} \right), & E \in \text{Ob } F(B), \ B \in \text{Ob } B \\
\text{Ar } E_F & \text{are pairs } \left( \begin{array}{c} h \\ f \end{array} \right) \in E_F(\left( \begin{array}{c} E \\ B \end{array} \right), \left( \begin{array}{c} E' \\ B' \end{array} \right)) & h \in F(B)(E, F(f)(E')), \ f \in B(B, B')
\end{array} \right.
\]

\[
1_{\left( \begin{array}{c} E \\ B \end{array} \right)} := \left( \begin{array}{c} E \xrightarrow{\sim} F(1_B)(E) \\ 1_B \end{array} \right),
\]

for \( \left( \begin{array}{c} u \\ f \end{array} \right) : \left( \begin{array}{c} E \\ B \end{array} \right) \to \left( \begin{array}{c} E' \\ B' \end{array} \right) \) and \( \left( \begin{array}{c} v \\ g \end{array} \right) : \left( \begin{array}{c} E' \\ B' \end{array} \right) \to \left( \begin{array}{c} E'' \\ B'' \end{array} \right) \) the composite

\[
\left( \begin{array}{c} v \\ g \end{array} \right) \circ \left( \begin{array}{c} u \\ f \end{array} \right) := \left( \begin{array}{c} w \\ g \circ f \end{array} \right)
\]

where \( w : E \xrightarrow{\sim} F(f)(E') \xrightarrow{F(g)(\cdot)} F(g)(F(f)(E')) \xrightarrow{\sim} F(g \circ f)(E'') \)

or respectively,

\[
\left\{ \begin{array}{ll}
\text{Ob } E_F & \text{are pairs } \left( \begin{array}{c} E \\ B \end{array} \right), & E \in \text{Ob } F(B), \ B \in \text{Ob } B \\
\text{Ar } E_F & \text{are pairs } \left( \begin{array}{c} h \\ f \end{array} \right) \in E_F((E, B), (E', B')) & h \in F(B')(F(f)(E), E'), \ f \in B(B, B')
\end{array} \right.
\]

\[
1_{\left( \begin{array}{c} E \\ B \end{array} \right)} := \left( \begin{array}{c} F(1_B)(E) \xrightarrow{\sim} E \\ 1_B \end{array} \right),
\]

for \( \left( \begin{array}{c} u \\ f \end{array} \right) : \left( \begin{array}{c} E \\ B \end{array} \right) \to \left( \begin{array}{c} E' \\ B' \end{array} \right) \) and \( \left( \begin{array}{c} v \\ g \end{array} \right) : \left( \begin{array}{c} E' \\ B' \end{array} \right) \to \left( \begin{array}{c} E'' \\ B'' \end{array} \right) \) the composite

\[
\left( \begin{array}{c} v \\ g \end{array} \right) \circ \left( \begin{array}{c} u \\ f \end{array} \right) := \left( \begin{array}{c} w \\ g \circ f \end{array} \right)
\]

where \( w : F(g \circ f)(E) \xrightarrow{\sim} F(g)(F(f)(E)) \xrightarrow{F(g)(\cdot)} F(g)(E') \xrightarrow{\sim} E'' \),

\( p_F \) is the projection onto the bottom component in both cases.

Two processes above are (weakly) inverse to each other.

Proof is straightforward. See [Jac].

(It is essential for Grothendieck construction that every (co)fibration \( p \)

is equivalent to
(co)fibration $p_2$ where

$$
egin{align*}
\text{Ob}_p := & \left\{ \frac{E}{p(E)} \left| E \in \text{Ob} \mathcal{E} \right. \right\}, \\
\text{Ar}_p := & \left\{ \frac{f}{p(f)} \left| f \in \text{Ar} \mathcal{E} \right. \right\}, \\
p_2 \text{ is the projection onto the bottom component, and that every morphism in } \mathcal{E} \text{ factors through (co)cartesian one)
\end{align*}
$$

2. Almost structures

Definition 2.1. Structure of type $\mathcal{E}$ on (objects of) category $\mathcal{B}$ is a functor $p \downarrow \mathcal{E} \downarrow \mathcal{B}$ which is

- faithful
- admits lifting of iso’s of type $f : B' \sim \rightarrow p(E)$ (or, the same, $f : p(E) \sim \rightarrow B'$)
- each fiber $\mathcal{E}_B$ is skeletal

Lemma 2.1. Let $p \downarrow \mathcal{E} \downarrow \mathcal{B}$ be a structure on $\mathcal{B}$, $E', E'' \in \text{Ob} (\mathcal{E}_B)$ for some $B \in \text{Ob} (\mathcal{B})$.

The following are equivalent

a) $E' = E''$

b) $\forall E \in \text{Ob} \mathcal{E} \ p(E(E, E')) = p(E(E, E''))$

c) $\forall E \in \text{Ob} \mathcal{E} \ p(E(E', E)) = p(E(E'', E))$

d) $\forall E \in \text{Ob} \mathcal{E}_B \ E_B (E, E') = \emptyset$ iff $E_B (E, E'') = \emptyset$

e) $\forall E \in \text{Ob} \mathcal{E}_B \ E_B (E', E) = \emptyset$ iff $E_B (E'', E) = \emptyset$

Proof. a) $\Rightarrow$ b), c), d), e) is obvious.

b) $\Rightarrow$ a) Take $E = E'$ and $E = E''$ then $\exists f : E' \rightarrow E''$ and $\exists g : E'' \rightarrow E'$ such that $p(f) = 1_B = p(g)$. So, $p(f \circ g) = p(g \circ f) = 1_B$. By faithfulness of $p$ and skeletal condition on $\mathcal{E}_B$, $f$ and $g$ are trivial iso’s.

c) $\Rightarrow$ a) is the same as b) $\Rightarrow$ a).

d) $\Rightarrow$ a) Take $E = E'$ and $E = E''$ then $\exists f : E' \rightarrow E''$, $f$ is vertical, and $\exists g : E'' \rightarrow E'$, $g$ is vertical. So, $f \circ g = 1_{E''}$, $g \circ f = 1_{E'} \Rightarrow E' \simeq E'' \Rightarrow E' = E''$.

e) $\Rightarrow$ a) is the same as d) $\Rightarrow$ a).

Proposition 2.1. For structure $p \downarrow \mathcal{E} \downarrow \mathcal{B}$ each fiber $\mathcal{E}_B$ is a poset category with (vertical) cartesian morphisms, just identities.

Proof. $\mathcal{E}_B$ is a preorder since $\exists$ at most one morphism between objects ($p$ is faithful). $\mathcal{E}_B$ is a partial order since all isomorphic objects are the same (skeletal condition). Cartesian lift of iso is iso, so cartesian lift $\widetilde{1}_B$ is an identity (since $\mathcal{E}_B$ is a partial order).
Proposition 2.2.

- **Pullback of a fibration is a fibration,**

- Pullback of a structure (on $B$) is a structure (on $A$) $\xrightarrow{\text{Pullback}}$

\[ F^*E \xrightarrow{p'} E \]

\[ A \xrightarrow{F} B \]

Proof.

- Pullbacks in $\mathbf{CAT}$ exist and they are certain subcategories of direct products. Cartesian morphisms are preserved under pullbacks which is seen from the following diagram

\[
\begin{array}{ccc}
(A'',E'') & \xrightarrow{(w,\beta)} & (A',E') \\
\downarrow{(w,\alpha)} & & \downarrow{(u,v)} \\
(A,E) & \xrightarrow{(u,v)} & (A,E)
\end{array}
\]

\[
\begin{array}{ccc}
E'' & \xrightarrow{\alpha} & E' \\
\downarrow{\beta} & & \downarrow{v} \\
E & \xrightarrow{v} & E
\end{array}
\]

\[
\begin{array}{ccc}
A'' & \xrightarrow{v} & A' \\
\downarrow{u} & & \downarrow{u} \\
A & \xrightarrow{u} & A
\end{array}
\]

\[
\begin{array}{ccc}
E & \xrightarrow{F} & E \\
\downarrow{p} & & \downarrow{p} \\
B & \xrightarrow{B} & B
\end{array}
\]

\[
\begin{array}{ccc}
FA'' & \xrightarrow{Fv} & FA' \\
\downarrow{Fw} & & \downarrow{Fu} \\
FA & \xrightarrow{FA} & FA
\end{array}
\]

(where: $p(E) = F(A)$, $p(\alpha) = F(u \circ w)$, $v$ is cartesian over $F(u)$).

- If $\xrightarrow{p}$ admits lifting of iso’s of type $B' \xrightarrow{\sim} p(E)$ then $\xrightarrow{p'}$ admits lifting of iso’s of the same type (obvious). If $p$ is faithful then $p'$ is faithful (assume, $p'(u, v) = p(u, v_1) = u$ then $F(u) = p(v) = p(v_1)$, so, $v = v_1$). Fibres of $\xrightarrow{p'}$ are skeletal (assume, $\xrightarrow{(1_A, v)} : (A, E') \xrightarrow{\sim} (A, E)$ is an iso in $(F^*E)_A$ then $v : E' \xrightarrow{\sim} E$ is an isomorphism in $E_{F(A)}$, so, $E' = E$, and $(A, E') = (A, E)$).

Partial cases of a pullback are ‘fiber’ and ‘intersection of structures’:

\[
\begin{array}{ccc}
E_B & \xrightarrow{p} & E \\
\downarrow{1} & & \downarrow{p} \\
B & \xrightarrow{B} & B
\end{array}
\]

\[
\begin{array}{ccc}
E_1 \land E & \xrightarrow{\pi} & E \\
\downarrow{p_1} & & \downarrow{p} \\
E_1 & \xrightarrow{E_1} & B
\end{array}
\]

Remark. The notion of ‘structure on objects’ of a category was introduced in [Kom] in order to deal with usual structures in Differential Geometry like smooth structures or fibre bundles on topological spaces. However, it turned out too weak (no inverse and direct images) and too strict (skeletal fibres) simultaneously. Appropriate framework was created with theory of (co)fibrations. Nevertheless, a weaker notion of almost structure emphasizes direct connection with the main structure, which is the case of importance especially when almost structure is introduced on hom-sets.
Proposition 2.3. For each structure $\mathbf{E}$ of type $\mathbf{E}$ on objects of category $\mathbf{B}$

- there is an embedding $i_p : \mathbf{E} \hookrightarrow \mathbf{Set}^\mathbf{E}$:
  \[
  \begin{cases}
  (E, p(E)) \mapsto p(E(-, E)) & \text{on objects} \\
  (v, p(v)) \mapsto p(E(-, v)) & \text{on arrows}
  \end{cases}
  \]

- $p(E(-, E)) \hookrightarrow \mathbf{B}(p(-), p(E)) : \mathbf{E} \hookrightarrow \mathbf{Set}$ (hom-subfunctor)

Proof. Functoriality is obvious. Injectivity follows from Lemma 2.1.1.1. \qed

Remark. It means that every structure $\mathbf{E}$ on objects in $\mathbf{B}$ is faithfully representable by a specific subcategory of $\mathbf{B}$-hom-subfunctors (in which sufficient to take arrows of only simple type $f \circ -$).

The reasonable question is if can an object $E \in \text{Ob}\mathbf{E}$ be recovered from a functor $F \hookrightarrow \mathbf{B}(p(-), p(E))$? If can it is unique but the answer is no, in general. Even when it is impossible subfunctor $F \hookrightarrow \mathbf{B}(p(-), p(E))$ behaves like an object in $\mathbf{E}$.

Definition 2.2.
- Arbitrary subfunctor $F \hookrightarrow \mathbf{B}(p(-), B) : \mathbf{E} \hookrightarrow \mathbf{Set}$ is called almost-$\mathbf{E}$ structure over object $B \in \text{Ob}\mathbf{B}$.
- Category with objects $\begin{pmatrix} F \\ B \end{pmatrix}$, $B \in \text{Ob}\mathbf{B}$, $F \hookrightarrow \mathbf{B}(p(-), B)$ and morphisms \[
  \begin{pmatrix} f \circ - \\ f \end{pmatrix} \equiv \begin{pmatrix} B(p(-), f) \\ f \end{pmatrix}, f : B \to B' \text{ is called a category of almost-$\mathbf{E}$ structures over $\mathbf{B}$.}
  \]
- Almost-$\mathbf{E}$ costructure over $B \in \text{Ob}\mathbf{B}$ is a subfunctor $F' \hookrightarrow \mathbf{B}(B, p(-)) : \mathbf{E} \to \mathbf{Set}$ [almost costructures are not dual to almost structures, they behave all together in a covariant way].
- Category with objects $\begin{pmatrix} F' \\ B \end{pmatrix}$, $B \in \text{Ob}\mathbf{B}$, $F' \hookrightarrow \mathbf{B}(B, p(-))$ and morphisms $f : \begin{pmatrix} F' \\ B \end{pmatrix} \to \begin{pmatrix} F' \\ B_1 \end{pmatrix}$, if $f : B \to B_1 \in \text{Ar}\mathbf{B}$ and $\forall E_1 \in \text{Ob}\mathbf{E}$, $\forall g \in \mathbf{B}(B_1, p(E_1)), g \circ f \in \mathbf{B}(B, p(E_1))$, is called a category of almost-$\mathbf{E}$ costructures over $\mathbf{B}$. \qed

Example

Take $\text{Poly}(E_1, E_2, ..., E_n; -) \hookrightarrow \mathbf{Set}(p(E_1 \times \cdots \times E_n, p(-))) : \mathbf{Vect} \to \mathbf{Set}$, subfunctor of polylinear maps. Then $\text{Poly}(+, +, ..., +; -) : \mathbf{Vect}^n \to \mathbf{Set}^\mathbf{Vect}$ determines a subcategory of almost-$\mathbf{Vect}$ costructures over $\mathbf{Vect}$ (note $\mathbf{Vect}$ is forgetful).
Proposition 2.4. For a structure of type \( \mathbf{E} \) on \( \mathbf{B} \),

\[
\begin{array}{ccc}
\mathbf{AE} & \longrightarrow & \mathbf{E} \\
\downarrow & & \downarrow \mathbf{p} \\
\mathbf{B} & & \mathbf{B}
\end{array}
\]

- is a fibration;
- \( \mathbf{p} \) is a subcategory

Proof.

- If \( (F_B) \in \text{Ob} \mathbf{B} \) and \( f : B' \rightarrow B \) take

\[
f^* F := \{ g : p(X) \rightarrow B' \mid X \in \text{Ob} \mathbf{E} \}, \quad f \circ g \in F(X) \subset \mathbf{B}(p(X), B) \}. \]

Then \( f^* F \hookrightarrow \mathbf{A} \mathbf{E} \) is a subfunctor, and

\[
\left( f^* F \right) \left( f' \right) \left( f \circ - \right) \rightarrow \left( f \right) \left( f' \right) \left( f \circ - \right) \text{ is cartesian over } f
\]

- Assignment

\[
\begin{array}{l}
\left( p(E) \right) \mapsto p(E(-, E)) \hookrightarrow \mathbf{B}(p(-), p(E)) \quad \text{on objects} \\
\left( p(v) \right) \mapsto p(E(-, v)) = \mathbf{B}(p(-), p(v)) \quad \text{on arrows}
\end{array}
\]

gives the required embedding.

3. Enrichment with generalized elements in hom-sets

Definition 3.1. 1-category \( \mathbf{C} \) is enriched in tensor category \( (\mathcal{V}, I, \otimes) \) \cite{Kel, Bor2} if

- \( \forall x, y \in \text{Ob} \mathbf{C} \quad \mathbf{C}(x, y) \in \mathcal{V} \)
- \( \forall x, y, z \in \text{Ob} \mathbf{C} \quad \mu_{x, y, z} : \mathbf{C}(y, z) \otimes \mathbf{C}(x, y) \rightarrow \mathbf{C}(x, z) \in \text{Ar} \mathcal{V} \)
- \( \forall x, y, z, w \in \text{Ob} \mathbf{C} \)

\[
\begin{array}{ccc}
\mathbf{C}(z, w) \otimes \mathbf{C}(y, z) & \otimes \mathbf{C}(x, y) & \mathbf{C}(z, w) \otimes \mathbf{C}(x, z) \\
\downarrow \mu \otimes 1 & & \mu \\
\mathbf{C}(w, y) \otimes \mathbf{C}(x, y) & \mathbf{C}(x, w)
\end{array}
\]

- \( \forall x \in \text{Ob} \mathbf{C} \exists u_x : I \rightarrow \mathbf{C}(x, x) \in \text{Ar} \mathcal{V} \) such that

\[
\begin{array}{ccc}
\mathbf{C}(x, y) & \mathbf{C}(x, y) & \mathbf{C}(z, x) \\
\mu & u_x \otimes 1 & 1 \otimes u_x
\end{array}
\]
Generalized elements in hom-sets are just parametrized families of arrows in the same way as continuous or smooth families of maps. They form almost-C structure on hom-sets of category C [Kom].

**Definition 3.2.** Assume, C has binary products and |−| : C → Set is a faithful functor. Then Set-map \( f : |Z| \rightarrow C(X,Y) \), \( Z \in \text{Ob} \ C \) is called a generalized (or acceptable) element of \( C(X,Y) \) with domain \(|Z|\) if arrow \( f \circ \gamma_Z \) can be lifted to C

\[
\begin{array}{c}
\text{C}(X,Y) \times |X| \xrightarrow{ev} |Y| \\
\downarrow f \times 1 \\
|Z| \times |X| \\
\downarrow \gamma_Z \\
|Z \times X|
\end{array}
\]

where: \( \gamma_Z \) is the mediating arrow to product, \( ev(g,x) := |g|(x) \)

Denote \( G(|Z|, C(X,Y)) \rightarrow \text{Set}(|Z|, C(X,Y)) \) subset of generalized elements of \( C(X,Y) \) with domain \(|Z|\).

**Proposition 3.1.** Assignment \( Z \mapsto G(|Z|, C(X,Y)) \) is extendable to a functor 
\( G(|−|, C(X,Y)) : C^{op} \rightarrow \text{Set} \).

**Proof.** Assume, \( \alpha : Z' \rightarrow Z \) is an arrow, \( f \in G(|Z|, C(X,Y)) \) is a generalized element. We need to show that \( f \circ \alpha \in G(|Z'|, C(X,Y)) \) is a generalized element as well, i.e., that \( \exists h : Z' \times X \rightarrow Y \) such that \( |h| = ev \circ (f \times 1) \circ (\alpha \times 1) \circ \gamma_{Z'} = f \circ (\alpha \times 1) \circ \gamma_{Z'} \).

Since \( \gamma_Z : |Z \times X| \rightarrow |Z| \times |X| \) is natural in \( Z \) \( |Z' \times X| \xrightarrow{\gamma_{Z'}} |Z'| \times |X| \)
\[
\begin{array}{c}
|Z' \times X| \xrightarrow{\gamma_{Z'}} |Z'| \times |X| \\
\downarrow |\alpha \times 1| \\
|Z \times X| \xrightarrow{\gamma_Z} |Z| \times |X| \xrightarrow{f} |Y|
\end{array}
\]

the requirement will be \( \exists h : Z' \times X \rightarrow Y \) such that \( |h| = f \circ \gamma_Z \circ |\alpha \times 1| \). By assumption on \( f \), \( \exists g : Z \times X \rightarrow Y \) such that \( |g| = f \circ \gamma_Z \). So, take \( h := g \circ (\alpha \times 1) \).

**Proposition 3.2.** If |−| : C → Set is a faithful functor which preserves binary products, then category C is enriched with generalized elements in presheaves category Set^{C^{op}}.

**Proof.**
- \( \forall X,Y \in \text{Ob} \ C \) \( (G(|−|, C(X,Y)) : C^{op} \rightarrow \text{Set}) \in \text{Ob}(\text{Set}^{C^{op}}) \)
- \( \forall X,Y,Z \in \text{Ob} C \) take \( \mu_{X,Y,Z} : G(|−|, C(Y,Z)) \otimes G(|−|, C(X,Y)) \rightarrow G(|−|, C(X,Z)) \) such that \( \forall W \in \text{Ob} C \) \( \mu_{X,Y,Z} : \text{Ob} C \times C(Y,Z) \times C(X,Y) \rightarrow C(X,Z) \) is the composite in C, \( < f,g > : |W| \rightarrow C(Y,Z) \times C(X,Y) \) is the mediating arrow to product. \( \mu_{X,Y,Z} \) is natural in \( W \) since \( (\mu_{X,Y,Z} \circ < f,g >) \circ |h| = \mu_{X,Y,Z} \circ < f \circ |h|, g \circ |h| > \) for \( h : W' \rightarrow W \).

Why \( ev \circ ((\mu_{X,Y,Z} \circ < f,g >) \times 1) \circ \gamma \) can be lifted to C? By condition,
\[ \text{C}(Y, Z) \times |Y| \xrightarrow{ev} |Z| \]

\[ \text{C}(X, Y) \times |X| \xrightarrow{ev} |Y| \]

\[ \text{C}(X, Z) \times |X| \xrightarrow{ev} |Z| \]

\[ \text{C}(Y, Z) \times \text{C}(X, Y) \times |X| \xrightarrow{\text{1}_{\text{C}(Y, Z)} \times ev} \text{C}(Y, Z) \times |Y| \]

\[ \text{C}(X, Z) \times |X| \xrightarrow{ev} |Z| \]

\[ (f \times g) \times 1_{|X|} \]

\[ \gamma = 1 \]

\[ |W| \times |X| \]

\[ |W| \times |X| \]

The required dotted way \( ev \circ (\mu^C \circ <f, g>) \times 1_{|X|}) \circ \gamma \) can be lifted to \( C \) since the most right way \( f \circ (1_{|W|} \times g) \circ (<1_{|W|}, 1_{|W|}) \times 1_{|X|}) \circ 1_{|W| \times X|} \) can be lifted (for take liftings for \( f, g \) identities and identities for identities).

- (associativity) \( \forall f, g, h \) such that \( f : |W| \to \text{C}(Z, Z') \), \( g : |W| \to \text{C}(Y, Z) \), \( h : |W| \to \text{C}(X, Y) \)

\[ \mu^C_{X, Z, Z'} \circ <f, \mu^C_{Y, Z, Z} \circ <g, h > = \mu^C_{X, Y, Z} \circ <\mu^C_{Y, Z, Z}, \circ <f, g >, h > \] because the equality holds at each point \( w \in |W| \).

- (identities) \( \forall X \in Ob \text{C} \) take \( u_{X; W} : 1 \to \text{Set}(|W|, \text{C}(X, X)) : * \mapsto (w \mapsto 1_X) \). It is natural in \( W \) and \( \forall f, g \) such that \( f : |W| \to \text{C}(X, Y) \), \( g : |W| \to \text{C}(Z, X) \) the equalities hold

\[ \mu^C \circ <f, u_{X; W} > (w) = \mu^C(f(w), 1_X) = f(w), \mu^C \circ <u_{X; W}, g > (w) = \mu^C(1_X, g(w)) = g(w) \]

\[ w \in |W| \].

**Corollary** (refinement of Proposition 3.2). Under the above assumptions (\( C \) has binary products, \(|-| : C \to \text{Set} \) is a faithful functor preserving binary products) hom-sets of \( C \) enriched with almost-\( C \) structure of generalized elements.

**Proof** is immediate (because presheaves of generalized elements are of specific form \( \mathcal{G}(|-|, \text{C}(X, Y)) \mapsto \text{Set}(|-|, \text{C}(X, Y)) \) and \( \mu \) is actually postcomposite \( \mu^C \circ - \)).

We mean further that \( C \) is AC-category if this specific enrichment with generalized elements is given. Moreover, we call \( D \) is AC-category if it is enriched with presheaves of generalized elements with domains in \( C \).
Remark. All usual concrete categories, like Top, Grp, Rng, etc. carry corresponding almost structures (which in some cases can be strict).

Example

**Proposition 3.3.** If $X$ is a locally compact topological space (so that, family $\mathcal{T}$ of topologies on $\text{Top}(X,Y)$ for which evaluation map $ev : \text{Top}(X,Y) \times |X| \to |Y|$ is continuous is not empty and contains minimal element, compact-open topology on $\text{Top}(X,Y)$) then $\tau \in \mathcal{T}$ is compact-open iff $\forall Z \in \text{Ob Top}$ each generalized element $f : |Z| \to \text{Top}(X,Y)$ is continuous.

**Proof.**$\Rightarrow$ Regard the diagram

\[
\begin{array}{c}
\text{Top}(X,Y) \times |X| \\
\downarrow \text{ev} \\
|Y|
\end{array}
\]

We want to show that $f : |Z| \to \text{Top}(X,Y)$ is continuous (with compact-open topology in $\text{Top}(X,Y)$) if $\tilde{f} : |Z| \times |X| \to |Y|$ is continuous, i.e., that $\forall z \in |Z| \forall (\text{subbase})\text{ compact-open set } U^K \ni \tilde{f}(z) \exists \text{nbhd } V \ni z \text{ such that } \tilde{f}(V) \subset U^K$. It is equivalent that $\forall z \in |Z| \forall U^K \ni \tilde{f}(z) \exists V \ni z \text{ such that } \tilde{f}(V \times K) \subset U$. Since $\tilde{f}$ is continuous $\forall (z,x) \in \{z\} \times K$ and $\forall \text{ open } U \ni \tilde{f}(z,x) \exists \text{ open nbhd } V_z \times W_x \ni (z,x) \text{ such that } \tilde{f}(V_z \times W_x) \subset U$. $\bigcup_{x \in K} W_x \ni K$ (open cover). So, by compactness of $K$, $\exists W_{x_1},...,W_{x_n}$, such that $\bigcup_{i=1}^n W_{x_i} \ni K$. Take $V := \bigcap_{i=1}^n (V_i)_z$, where $(V_i)_z$ corresponds to $W_{x_i}$ (i.e., $(V_i)_z$ is open, $(V_i)_z \ni z, \tilde{f}((V_i)_z \times W_{x_i}) \subset U$). Then $\tilde{f}(V \times K) \subset U$.

$\Leftarrow$ Take $Z = \text{Top}(X,Y)$ with compact-open topology. Take $\text{Top}(X,Y)$ itself (on the top of the diagram) with non minimal $\tau \in \mathcal{T}$, $f := 1 \in \text{Ar Set}$. Then $1 : \text{Top}(X,Y) \to \text{Top}(X,Y)$ is an admissible generalized element, since $ev$ is continuous, but $1$ is not continuous. $\square$

Remark. Therefore, for locally compact space $X$ almost-$\text{Top}$ structure $\mathcal{G}(|Z|, \text{Top}(X,Y))$ coincides with compact-open topology, i.e., is actually $\text{Top}$ structure.

If we agree that functor $\mathcal{G}(|-|, \text{C}(X,Y)) : \text{C}^{\text{op}} \to \text{Set}$ reflects essential properties of $\text{C}$-hom-sets we immediately get a unique (up to isomorphism) extension of each functor $F : \text{C} \to \text{D}$, i.e., deal with $\text{C}$-hom-sets as with $\text{C}$-objects. In this way, for example, tangent or jet functor can be introduced directly on $\text{Aut}(X)$, $X \in \text{Ob Diff}$ to give rise a calculus on $\text{Aut}(X)$. Possibility of such an extension follows from the fact that each presheaf is a certain (canonical) colimit of representables [Mac, M-M].

**Proposition 3.4.**

- Yoneda embedding $y : \text{C} \to \text{Set}^{\text{C}^{\text{op}}}$ is a universal cocompletion of $\text{C}$, i.e., $\forall F : \text{C} \to \text{E}$, where $\text{E}$ is cocomplete, $\exists ! (\text{up to iso})$ cocontinuous $\tilde{F} : \text{Set}^{\text{C}^{\text{op}}} \to \text{E}$ such that

\[
\begin{array}{ccc}
\text{Set}^{\text{C}^{\text{op}}} & \xrightarrow{\tilde{F}} & \text{E} \\
\downarrow \text{y} & & \downarrow F \\
\text{C} & \xrightarrow{\pi} & \text{E}
\end{array}
\]

- $\tilde{F}(P) = \text{Colim}(\int_{\text{C}} P \xrightarrow{\pi_{\text{C}}} \text{C} \xrightarrow{F} \text{E})$, where $P \in \text{Ob Set}^{\text{C}^{\text{op}}}$, $\int_{\text{C}} P$ is a category of elements of $P$, $\pi$ is the natural projection.
• \( \text{Cat} \xrightarrow{\text{forgetful}} \text{Ccomp} \), adjunction between \( \text{Cat} \) and full subcategory of cocomplete categories \( \text{Ccomp} \) with Yoneda embedding \( \gamma_C : C \to \text{Set}^{\text{Ccomp}} \) as a unit.

• Each functor \( F : C \to D \) admits a unique (up to iso) cocontinuous extension \( F : \text{Set}^{\text{Ccomp}} \to \text{Set}^{D^{op}} \) such that \[
\begin{array}{ccc}
\text{Set}^{\text{Ccomp}} & \xrightarrow{F} & \text{Set}^{D^{op}} \\
\gamma_C & \downarrow & \downarrow \gamma_D \\
C & \xrightarrow{F} & D
\end{array}
\]

Proof. See [M-M]. \( \square \)

For example, if \( T : \text{Diff} \to \text{Diff} \) is a tangent functor, \( \text{Diff}(X, Y) \) is a presheaf on \( \text{Diff} \) (hom-set enriched as above) then \( T(\text{Diff}(X, Y)) = \int_{\text{Diff}} \text{Diff}(X, Y) \xrightarrow{T} \text{Diff} \xrightarrow{y} \text{Set}^{\text{Diff}^{op}} \).

3.1. Tangent functor for smooth algebras.

It is an example of dual (and invariant) construction for the main functor of Differential Geometry (which gives suggestion how it can be extended over spectra of commutative algebras).

Let \( T : \text{Diff} \to \text{Diff} \) be tangent functor on the category of real \( \infty \)-smooth manifolds. In local coordinates it looks like \[
\begin{align*}
X & \to TX : (x^i) \to (x^i, \xi^i) \quad X \in \text{Ob} \text{Diff} \\
f & \to Tf : (f^i(x)) \to (f^i(x), \frac{\partial f^i}{\partial x^j} \xi^j) \quad f \in \text{Ar} \text{Diff}
\end{align*}
\]

\( \text{Diff} \xrightarrow{i} \mathbb{R}\text{-Alg}^{op} \) is a subcategory of the opposite of real commutative algebras. Working in \( \text{Diff} \) it is hard (if possible at all) to give coordinate-free characterization of \( T \). The question is how it looks like in \( \mathbb{R}\text{-Alg} \)?

Definition 3.1.1. Let \( \mathcal{A} \in \text{Ob} \mathbb{R}\text{-Alg} \).

• \( \rho : \mathcal{A} \to \text{Top}(\text{Spec}_\mathbb{R}(\mathcal{A}), \mathbb{R}) \) is called functional representation homomorphism of \( \mathcal{A} \), where \( \text{Spec}_\mathbb{R}(\mathcal{A}) = \mathbb{R}\text{-Alg}(\mathcal{A}, \mathbb{R}) \) with initial topology w.r.t. all functions \( \rho(a), a \in \mathcal{A}, \rho(a)(f) := \text{ev}(f, a) := |f|(a) \).

• \( \mathcal{A} \) is called smooth if \( \forall a_1, a_2, \ldots, a_n \in \mathcal{A} \) and \( \forall f : \mathbb{R}^n \to \mathbb{R} \in C^\infty(\mathbb{R}^n) \) the composite \( f \circ \rho(a_1), \rho(a_2), \ldots, \rho(a_n) \in \text{Im}(\rho) \).

Denote by \( \text{R-Sm-Alg} \xhookrightarrow{\mathbb{R}} \mathbb{R}\text{-Alg} \) full subcategory of smooth algebras.

Lemma 3.1.1. \( \text{R-Sm-Alg} \to \mathbb{R}\text{-Alg} \) is a reflective subcategory, i.e., the inclusion has a left adjoint \( \text{Sm} : \mathbb{R}\text{-Alg} \to \text{R-Sm-Alg} \), smooth completion of \( \mathbb{R}\)-algebras.

Proof. Just take for each \( \mathbb{R}\)-algebra \( \mathcal{A} \mathbb{R}\)-algebra \( \text{Sm}(\mathcal{A}) \) of all terms \( \{f(a_1, \ldots, a_n) \mid f : \mathbb{R}^n \to \mathbb{R}, a_1, \ldots, a_n \in \mathcal{A} \} \) (all smooth operations are admitted). Each morphism \( f \) from an \( \mathbb{R}\)-algebra \( \mathcal{A} \) to a smooth algebra \( \mathcal{B} \) is uniquely extendable to \( f : \text{Sm}(\mathcal{A}) \to \mathcal{B} \). \( \square \)

Let \( \text{Sym-Alg} \) be a category of symmetric partial differential algebras. \( \text{Ob}(\text{Sym-Alg}) \) are graded commutative algebras over commutative \( \mathbb{R}\)-algebras with a differential \( d : \mathcal{A}^0 \to \mathcal{A}^1 \) of degree 1 determined only on elements of degree 0 (\( d \) is \( \mathbb{R}\)-linear and satisfies Leibniz rule). \( \text{Ar}(\text{Sym-Alg}) \) are graded degree 0 algebra homomorphisms which respect \( d \).

Lemma 3.1.2. There is an adjunction \( \mathbb{R}\text{-Alg} \xrightarrow{p_0} \text{Sym-Alg} \).

where: \( p_0 \) is the projection onto 0-component \[
\begin{cases}
  p_0(\mathcal{A}) := \mathcal{A}^0 \\
  p_0(\mathcal{A} \xrightarrow{f} \mathcal{B}) := (\mathcal{A}^0 \xrightarrow{f^0} \mathcal{B}^0)
\end{cases}
\]

Sym is taking symmetric algebra over module of differentials of the given algebra

\[\text{Sym}(\mathcal{C}) := \text{Sym}(\Lambda^1(\mathcal{C}))\]

\[\text{Sym}(\mathcal{C} \xrightarrow{h} \mathcal{D}) := (\text{Sym}(\mathcal{C}) \xrightarrow{h} \text{Sym}(\mathcal{D}))\]

\[h(\sum c_{1}^i \cdots c_{k}^i \cdot (dc_{1})^{i_1} \cdots (dc_{k})^{i_k}) := \sum h(c_{1}^i \cdots c_{k}^i)(dh(c_{1}))^{i_1} \cdots (dh(c_{k}))^{i_k}\]

**Proposition 3.1.1.**

- C weaker topology. For

**Lemma 3.1.3.**

**Proof.** \(\forall\) conversely, each such \(\tilde{\alpha}\) is uniquely restricted to \(\alpha\). Initial topology on \(\mathbb{R}\text{-Alg}((\text{Sm})(\mathcal{A}), \mathbb{R})\) does not change because new functions are functionally dependent on old ones. \(\square\)

**Remark.** With Zarisski topology in spectra smooth completion yields the same set with a weaker topology. For \(C^\infty(X), X \in \text{ObDiff}\) Zarisski and initial topologies coincide.

**Proposition 3.1.1.**

- Tangent functor \(T : \mathbb{R}\text{-Sm-Alg} \rightarrow \mathbb{R}\text{-Sm-Alg}\) is equal to the composite \(\mathbb{R}\text{-Alg} \xrightarrow{\text{Sym}} \text{Sym-Alg} \xrightarrow{U} \mathbb{R}\text{-Alg} \xrightarrow{\text{Sm}} \mathbb{R}\text{-Sm-Alg}\), where \(U\) forgets differential and grading.

\[TX \xrightarrow{T(C^\infty(X))} \]

- To canonical projection \(p_X\) there corresponds canonical embedding \(i_{C^\infty(X)}(X) \rightarrow C^\infty(X)\).

**Proof.**

- If \(X \in \text{ObDiff}\) \(TX \sim \text{Spec}_\mathbb{R}(U \circ \text{Sym}(C^\infty(X))) \sim \text{Spec}_\mathbb{R}(\text{Sm} \circ U \circ \text{Sym}(C^\infty(X)))\).

- immediate. \(\square\)

**Remark.** It is reasonable to define \(T\) on \(\mathbb{R}\text{-Alg}\) as \(T := U \circ \text{Sym}\) and transfer it to spectra via duality \(\mathbb{R}\text{-Alg}^{op} \xrightarrow{F} \text{Spec}_\mathbb{R}\) as \(F \circ T^{op} \circ G\).

### 4. General manifolds

**Definition 4.1.** Functor \(F\) is called a fibration with respect to class of arrows \(\mathcal{C} \subset Ar\ B\)

\[\text{if } \forall f : B' \rightarrow F(E) \in \mathcal{C} \exists \tilde{f} : E' \rightarrow E \in Ar\ E\ \text{such that } \tilde{f} \text{ is over } f \text{ and } \tilde{f} \text{ is cartesian}.\]

**Definition 4.2.** [M-M]

- Grothendieck pretopology \(\tau_0\) on a category \(\mathcal{B}\) with pullbacks is a family of coverings \(\tau_{0B}\) for each object \(B \in \text{Ob\ }\mathcal{B}\) (elements of a covering are just arrows with codomain \(B\)) such that

  - if \(f : B' \rightarrow B\) is an is then \(\{f\} \in \tau_{0B}\) is an one-element covering
  - if \(g : B'' \rightarrow B\) is an arrow and \(c \in \tau_{0B}\) then pullback family \(\text{plbk}_g(c) \in \tau_{0B''}\)
  - (coverings are composable) if \(c \in \tau_{0B}\) and \(\forall B' \in d(c)\) there is given a covering \(c_{B'} \in \tau_{0B'}\) then \(c \circ \bigcup\ c_{B'} \in \tau_{0B}\)
• Grothendieck **topology** \( \tau \) on a category \( \mathcal{B} \) (not necessarily with pullbacks) is a family of hom-subfunctors \( \tau_B \) for each object \( B \in \text{Ob} \mathcal{B} \) such that

- \( \mathcal{B}(-, B) \in \tau_B \)
- if \( f : B' \to B \) and \( t \in \tau_B \) then the **inverse image** \( (f^*(t) : X \to (f^*t)(X, B') \subset \mathcal{B}(X, B')) \in \tau_{B'} \) \( (h \in f^*(t)(X, B') \iff f \circ h \in t(X, B)) \)
- if \( s \hookrightarrow \mathcal{B}(-, B) \) is any hom-subfunctor such that \( \forall f : B' \to B \ f^*(s) \in \tau_{B'} \) then \( s \in \tau_B \)

Every topology is a pretopology if \( \mathcal{B} \) has pullbacks, and every pretopology generates a topology [M-M]. Category \( \mathcal{B} \) with (pre)topology is called **site** \( (\mathcal{B}, \tau_0) \).

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{F} & \mathcal{E} \\
\downarrow & & \downarrow \circ \tau_0 \\
(B, \tau_0) & \xleftarrow{\mathcal{E}} & \text{E-Man}
\end{array}
\]

**Definition 4.3.** [Kom] Functor \( F \) is called **local** if it is a fibration with respect to all elements of all coverings \( \bigcup_{B \in \text{Ob} \mathcal{B}} \tau_0_B \).

**Definition 4.4.** For a given functor to a site \( F \) smallest local functor \( p_F \) is called **\( E \)-manifold structure** over \( \mathcal{B} \). It means

- \( \forall X \in \text{Ob} \text{E-Man} \) \( \exists \) covering \( \mathcal{C}_{p_F(X)} = \{ i : B_i \to p_F(X) \}_{i \in I} \in \tau_{0_{p_F(X)}} \) such that there are inverse images \( i^*(X) \in \text{Ob} \mathcal{E} \) (i.e., \( \mathcal{E} \) contains isomorphic representatives of the inverse images)

- \( \forall f : X' \to X \in \text{Ar} \text{E-Man} \) \( \exists \) coverings \( \mathcal{C}_{p_F(X')} = \{ i' : B'_i \to p_F(X') \}_{i \in I} \in \tau_{0_{p_F(X')}} \) and \( \mathcal{C}_{p_F(X)} = \{ i : B_i \to p_F(X) \}_{i \in I} \) such that \( \forall i \in \mathcal{C}_{p_F(X)} \exists i' \in \mathcal{C}_{p_F(X')} \) such that

\[
\begin{array}{ccc}
B'_i & \xrightarrow{\exists \varphi} & B_i \\
\downarrow & & \downarrow \\
p_F(X') & \xrightarrow{p_F(f)} & p_F(X) \\
\end{array}
\]

and over it

\[
\begin{array}{ccc}
i' & \xrightarrow{i} \quad & \exists \Phi & \xrightarrow{\exists \Phi} & i^*(X) \\
\downarrow & & \downarrow & & \downarrow \\
i' & \xrightarrow{i} & \tilde{i} & \xrightarrow{\tilde{i}} & i^*(X) \\
\end{array}
\]

\( \tilde{i}, \tilde{i} \) are cartesian, \( p_F(\Phi) = \varphi, \quad \Phi \in \text{Ar} \mathcal{E} \) (arrows are locally in \( \mathcal{E} \))

- **E-Man** is maximal with respect to two above properties

**Examples**

1. **Set** as a manifold structure \( 1 \)

\[
\begin{array}{ccc}
\text{Set} & \xrightarrow{\text{Set}_{inj}} & \text{Set}_{inj} \\
\downarrow & & \downarrow \\
(\text{Set}, \tau_0) & \xleftarrow{\text{Set}_{inj}} & \text{Set}_{inj}
\end{array}
\]

where \( \text{Set}_{inj} \) is a category of sets with injective maps only, \( \tau_0 \) is a pretopology consisting of all families of injective maps with common codomains.

\[
\begin{array}{ccc}
\mathcal{C}^r - \text{Man}_k & \xleftarrow{\text{Try}} \mathcal{C}^r - \text{Man}_k \\
\downarrow & & \downarrow \\
(\text{Top}, \tau_0) & \xrightarrow{\text{Top}} & (\text{Top}, \tau_0)
\end{array}
\]

2. **Differentiable** manifolds \( k = \mathbb{R} \) or \( \mathbb{C} \),

\( \tau_0 \) consists of all open coverings, \( r = 0, 1, ..., \infty \) (or \( \omega \) for complex manifolds),
\[
\begin{align*}
\{ \text{Ob} (\text{Triv}C^r\text{-}\text{Man}_k) &= \{ k^0, k^1, \ldots, k^n, \ldots \} \\
\text{Ar} (\text{Triv}C^r\text{-}\text{Man}_k) &= C^r\text{-}\text{maps} 
\end{align*}
\]

3. Locally trivial fibre bundles

\[
\xymatrix{ 
\text{B}_n(E, p) \ar[r] & \text{B}_0(E, p) \\
(\text{Man}^{-1}, \tau_0) \ar[u] & 
}
\]

(see 5)

\[
\xymatrix{ 
\text{Fol}(E, p) \ar[r] & \text{Fol}_0(E, p) \\
(\text{Man}, \tau_0) \ar[u] & 
}
\]

associated to \(\text{AMan}\)-functor sequence

4. Foliations over \(\text{Man}\)

\[
\xymatrix{ 
\mathcal{E} \ar[r] & \text{Man} \ar[r] & \text{Top} \\
(\text{Man}^{-1}, \tau_0) \ar[u] & 
}
\]

(see 5), where: \(\text{Man}\) is a category of manifolds (of type \(E'\)) over \(\text{Top}\), \(\tau_0\) are all 'open coverings' of objects in \(\text{Man}\), \(\text{Fol}_0(E, p) = \text{B}_0(E, p)\) is a category of trivial foliations ('direct products') with leaves in \(E\), projection functor is the first (top) component of projection \(\tau_0\) are different in these two cases and corresponding categories of manifolds are glued differently).

5. \(E\)-manifolds over \(\text{Top}\).

Let \(\xymatrix{\mathcal{E} \ar[r] & \text{Top}}\) be a local structure on \(\text{Top}\) with \(\tau_0\), all open coverings.

- **Local \(E\)-map** on a topological space \(X\) is a pair \(\xymatrix{\mathcal{E} \ar[r] & \text{Top} \quad \left( \begin{array}{c} E \\ U \end{array} \right) \in \text{Ob} \quad \xymatrix{\text{Top} \ar[r] & \mathcal{E}} \quad \text{U is open.} \)

- Family \(\left\{ \left( \begin{array}{c} E_i \\ U_i \end{array} \right) \right\}_{i \in I}\) is compatible iff \(\forall (i, j) \in I^2\) \(E_i|_{U_i \cap U_j} \sim_{\varphi} U_j|_{U_i \cap U_j}, \varphi\) is a vertical iso.

- **\(E\)-atlas** \(\mathcal{A}\) on \(X\) is a compatible family \(\left\{ \left( \begin{array}{c} E_i \\ U_i \end{array} \right) \right\}_{i \in I}\) such that \(\bigcup_{i \in I} U_i = X\).

- Two \(E\)-atlases \(\mathcal{A}\) and \(\mathcal{A}'\) are equivalent iff \(\mathcal{A} \cup \mathcal{A}'\) is still an \(E\)-atlas on \(X\) (so, there exist maximal atlases, call them \(A_{\text{max}}, B_{\text{max}}, \text{etc.}\)).

- The above 'equivalence' on atlases is not transitive in general. So, there can be different maximal atlases containing a given one. But, it is transitive if \(\forall \left( \begin{array}{c} E \\ U \end{array} \right), \left( \begin{array}{c} E' \\ U \end{array} \right) \in \text{Ob} \quad \xymatrix{\text{Top} \ar[r] & \mathcal{E} \quad \xymatrix{\text{Top} \ar[r] & \mathcal{E}}}

and \(\forall\) open covering \(\bigcup_{i \in I} U_i \supset U\) \(E|_{U_i} \sim_{\text{vert}} E'|_{U_i}\) (for all \(i \in I\)) implies \(E \sim_{\text{vert}} E'\).

- Topological space \(X\) together with an \(E\)-atlas \(\mathcal{A}\) on it is called \(E\)-manifold, i.e., \((X, \mathcal{A}) \in \)
$\text{Ob } \text{E-Man}$.  
- An arrow in $\text{E-Man}$ is $f : (X, A) \to (Y, B)$ such that $f : X \to Y$ is continuous and 
\[ \forall \left( \begin{array}{c} E' \\ V \\ U \end{array} \right) \in A, \left( \begin{array}{c} E' \\ V' \\ U' \end{array} \right) \in B \text{ if } U \cap f^{-1}(V) \neq \emptyset \text{ then } f|_{U \cap f^{-1}(V)} : U \cap f^{-1}(V) \to V \text{ admits (unique) lifting } f|_{U \cap f^{-1}(V)} : E|_{U \cap f^{-1}(V)} \to E' \in \text{Ar E}. \]

5. Fibre bundles

Locally trivial fibre bundles give an important example of general manifolds over $\text{Man} \to [\text{Kom}]$.

**Definition 5.1.** Category of trivial fibre bundles $\text{Bn}_0(\mathcal{E}, p)$ over $\text{Man} \to$ with typical fibres in a category $\mathcal{E}$ consists of the following data

- $\mathcal{E} \xrightarrow{\pi} \text{Top}$, where $\mathcal{E}$ and $\text{Man}$ are $\text{AMan}$-categories, $p$ is $\text{AMan}$-functor, $\pi$ preserves binary products [i.e., $\text{Man}$ is enriched in $\text{Set}^{\text{Man}^{op}}$ with presheaves of generalized elements $G(|-|, \text{Man}(A, A'))$ for each hom-set $\text{Man}(A, A')$, $\mathcal{E}$ is enriched in $\text{Set}^{\text{Man}^{op}}$ with subfunctors $\mathcal{H}(|-|, \mathcal{E}(E, E')) \to \text{Set}^{\text{Man}^{op}}(|-|, \mathcal{E}(E, E'))$ for each hom-set $\mathcal{E}(E, E')$, $p_{E, E', X} : \mathcal{H}(|X|, \mathcal{E}(E, E')) \to G(|X|, \text{Man}(p(E), p(E'))): f \mapsto p_{E, E', E} \circ f$ is natural in $X \in \text{Ob Man}$, $p_{E, E', E} : \mathcal{E}(E, E') \to \text{Man}(p(E), p(E'))$ is the restriction of functor $p$ on the hom-set]

- $\text{Ob } \text{Bn}_0(\mathcal{E}, p) := \{(X, E) | X \in \text{Ob Man}, E \in \text{Ob } \mathcal{E}\}$; 
- $\text{Ar } \text{Bn}_0(\mathcal{E}, p) := \{(X, E) \xrightarrow{(f, \Phi)} (X', E') | f : X \to X', \Phi \in \mathcal{H}(|X|, \mathcal{E}(E, E'))\}$

\[
\begin{array}{c}
\text{Bn}_0(\mathcal{E}, p) \\
\downarrow p_0
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
(X, E) \mapsto X \times p(E) \xrightarrow{p_1} X \\
(f, \Phi) \mapsto (f \circ \Phi) \xrightarrow{p_1} f \downarrow X \xrightarrow{p} X' \end{array} \\
\end{array}
\end{array}
\]

where $\phi := e \circ (p_{E, E'} \circ \Phi) \times 1_{p(E)}$, $p_{E, E'} \circ \Phi \in G(|X|, \text{Man}(p(E), p(E')))$

**Definition 5.2.** Category of locally trivial fibre bundles $\text{Bn}(\mathcal{E}, p)$ over site $(\text{Man} \to, \tau_0)$, where $\tau_0$ is pullbacks of all open coverings of codomains (i.e., if $q : Y \to X \in \text{Ob Man} \to$ and $\{U_i\}_{i \in I}$ is an open covering of $X$ then $\{q^{-1}(U_i) \to U_i \}_{i \in I}$ is a covering of $q$), is a manifold structure of type $\text{Bn}_0(\mathcal{E}, p)$ over $\text{Man} \to$.

A usual way of construction of new fibre bundles from old ones is by fibrewise operations. Let $\mathcal{E} \xrightarrow{\phi} \text{Top}$ and $\mathcal{E}' \xrightarrow{\phi'} \text{Top}$ be two sequences generating categories of fibre bundles $\text{Bn}(\mathcal{E}, p)$ and $\text{Bn}(\mathcal{E}', p')$ of types $\mathcal{E}$ and $\mathcal{E}'$, respectively, $F : \mathcal{E} \to \mathcal{E}'$ be an $\text{AMan}$-functor. Then there exists a corresponding functor $\text{Bn}(F) : \text{Bn}(\mathcal{E}, p) \to \text{Bn}(\mathcal{E}', p')$.

Denote by $\text{AMan-CAT}$ an 1-category such that

\[
\left\{ \begin{array}{l}
(\text{Ob } \text{AMan-CAT}) \ni (\mathcal{E}, p), \text{ if } \mathcal{E} \text{ is } \text{AMan}-\text{category, } p : \mathcal{E} \to \text{Man} \text{ is } \text{AMan}-\text{functor}
\end{array} \right.
\]

and by $\text{Bn}_0$ and $\text{Bn}$ subcategories of $\text{1-CAT}$ consisting of categories of trivial and locally trivial fibre bundles with fibres of a fixed type (i.e., of categories like $\text{Bn}_0(\mathcal{E}, p)$ and $\text{Bn}(\mathcal{E}, p))$ and functors preserving atlases as arrows (see 2.5, remarks). Of course, $\text{Bn}_0(\mathcal{E}, p) \to \text{Bn}(\mathcal{E}, p)$.

**Proposition 5.1.** There are functors $\text{Bn}_0(-) : \text{AMan-CAT} \to \text{Bn}_0 \to \text{1-CAT}$.
Proposition 6.1. categories of prestacks and stacks respectively.

\[ \begin{cases} (\mathcal{E}, p) \mapsto \text{Bn}_0(\mathcal{E}, p) \quad & \text{on objects} \\ (F : \mathcal{E} \to \mathcal{E}') \mapsto \text{Bn}_0(F) : \begin{cases} (X, E) \mapsto (X, F(E)) & (X, E) \in \text{Ob}(\text{Bn}_0(\mathcal{E}, p)) \\ (f, \Phi) \mapsto (f, F_{E,E'} \circ \Phi) & (f, \Phi) \in \text{Ar}(\text{Bn}_0(\mathcal{E}, p)) \end{cases} \quad & \text{on arrows} \end{cases} \]

and \( \text{Bn}(-) : \text{AMan-CAT} \to \text{Bn} \hookrightarrow \text{1-CAT} \), such that \( \text{Bn}(-) = \text{Man}(\text{Bn}_0(-)) \) (see 6, remarks) (i.e., to each fibrewise functor there corresponds an actual functor on fibre bundles).

Proof. The given assignment for \( \text{Bn}_0(-) \) is obviously functorial. If \( \mathcal{A} := \left\{ \begin{array}{c} \downarrow \\ \bigcup_{i \in I} \end{array} \right\} \) is an \( \mathcal{E} \)-atlas for \( \bigcup_{i \in I} U_i \times p(E_i) \) then \( \mathcal{A}' := \left\{ \begin{array}{c} \downarrow \\ \bigcup_{i \in I} \end{array} \right\} \) is an \( \mathcal{E}' \)-atlas for \( \bigcup_{i \in I} U_i \times p'(F(E_i)) \), \( \mathcal{A}' \) is a compatible family of arrows, if \( \mathcal{A} \) is compatible, and can be glued to an arrow \( \bigcup_{i \in I} \).

So, \( \text{Bn}_0(F) \) and \( \text{Bn}(F) \) preserve atlases.

Remark. Similarly, there can be defined fibrewise functors of more than one variable (e.g., \( \text{Bn}(\mathcal{E}, p) \times \text{Bn}(\mathcal{E}', p') \to \text{Bn}(\mathcal{E}'', p'') \)) induced by \( F : \mathcal{E} \times \mathcal{E}' \to \mathcal{E}'' \), an \( \text{AMan} \)-functor. In this way usual fibrewise operations like \( \times, \oplus, \otimes, \) etc., are introduced.

6. Stacks and construction of general manifolds

Stacks give an example of relative higher order Category Theory. \( (n+1) \)-categories form an \((n+1)\)-category, so that (forgetting set-theoretical difficulties) \( \text{Hom}_{(n+1)-\text{CAT}}(\mathcal{C}, n-\text{CAT}) \) is an \( n \)-category.

Definition 6.1. Let \( (B, \tau) \) be a site (\( B \) is 1-category), \( F : B^{op} \to n-\text{CAT} \) be a (weak) functor.

- For a sieve \( i_s : s \to B(-, B) \) \((n-1)\)-category \( \text{Desc}(s, F) := \text{Hom}_{(n+1)-\text{CAT}}(s, F) \) is called descent data for functor \( F \) and sieve \( s \).
- For all \( B \in \text{Ob}(B) \) and all sieves \( i_s : s \to B(-, B) \) there is an induced (restriction) functor \( i_s^* : \text{Hom}_{(n+1)-\text{CAT}}(B(-, B), F) \to \text{Hom}_{(n+1)-\text{CAT}}(s, F) \). If \( i_s^* \) is full and faithful \((\forall B \text{ and } \forall s)\) then \( F \) is called prestack. If, moreover, it is an equivalence \( F \) is called stack, i.e., \( F \) is prestack iff \( \forall B, s \text{ Hom}_{(n+1)-\text{CAT}}(s, F) \xleftarrow{i_s^*} \quad \text{full, faith.} \quad \text{Hom}_{(n+1)-\text{CAT}}(B(-, B), F) \xrightarrow{\sim} F(B) \text{ Yoneda} \) and stack iff \( i_s^* \) is an equivalence.

For \( n = 1 \) there is another definition of stack via matching families [Moe, Vis].

Denote by \( \text{PreSt}(B^{op}, n-\text{CAT}), \text{St}(B^{op}, n-\text{CAT}) \mapsto \text{Hom}_{(n+1)-\text{CAT}}(B^{op}, n-\text{CAT}) \) full subcategories of prestacks and stacks respectively.

Proposition 6.1. Both inclusions
have left adjoints.

Proof. See [Moe, Vis]. □

Construction of manifolds of type $\mathbf{E}$ over site $(\mathbf{B}, \tau)$

1. Factor $(1)$-functor $F : \mathbf{E} \to (\mathbf{B}, \tau)$ through a free fibration (see Lemma 1.1)

2. For fibration \( F \) form a corresponding (weak) functor $\hat{F} : \mathbf{B}^{op} \to \mathbf{1}$ and complete it to a stack $(\Phi \circ \Psi) \hat{F} : \mathbf{B}^{op} \to \mathbf{1}$ with respect to topology $\tau$.

3. Get back (by Grothendieck construction) from stack $(\Phi \circ \Psi) \hat{F} : \mathbf{B}^{op} \to \mathbf{1}$ to a fibration $\tilde{\mathbf{E}} \to 1/F$

4. Choose a (correct) class of arrows $\mathcal{M}$ in $\mathbf{B}$ representing 'embeddings of simple pieces into manifolds'.

5. Take a full subcategory $\mathbf{E-Man} \hookrightarrow \tilde{\mathbf{E}}$ consisting of all objects $(B, \mathcal{E}) \in \text{Ob}(\tilde{\mathbf{E}})$ such that $\exists$ a sieve $s \hookrightarrow \mathbf{B}(-, B)$ (depending on $B$) and $\forall f \in s \text{ Cart}_{f}(\mathcal{E}) = ((\Phi \circ \Psi) \hat{F}(f)) (\mathcal{E}) : (df \to F(\mathcal{E})) \in \mathcal{M}$ for some $E \in \text{Ob}(\mathbf{E})$. Then is the required category of manifolds of type $\mathbf{E}$ over base site $(\mathbf{B}, \tau)$. □

Remarks.

- Depending on the choice of class $\mathcal{M}$ categories $\mathbf{E-Man}$ will be different (so, $\mathcal{M}$ is an additional parameter). For cases of usual manifolds (smooth real or complex) $\mathcal{M}$ is always class of topological embeddings of open subspaces.

- An object $(B, \mathcal{E})$ in $\mathbf{E-Man}$ consists of a base object $B$ and an 'atlas' $\mathcal{E}$, where $\mathcal{E}$ is a class of compatible charts $(U \to F(\mathcal{E})) \in \mathcal{M}, U \in \text{Ob}(\text{Im}F), E \in \mathbf{E}$. All arrows are represented by vertical arrows for the stack completion of $1/F$.

- $\text{Im}(p_{F}) \supset \text{Im}(F)$.

- The resulting category of manifolds $\mathbf{E-Man} \to 1$ is not usually a fibration.

- Denote by $\mathbf{Man}_{0} \hookrightarrow \mathbf{1}$ a category consisting of subcategory of $\mathbf{E-Man}$ of trivial manifolds of type $\mathbf{E}$ for each type $\mathbf{E}$ and functors 'mapping $\mathbf{E}$-atlases to $\mathbf{E}'$-atlases'. Respectively, by $\mathbf{Man} \hookrightarrow \mathbf{1}$ a category consisting of $\mathbf{E-Man}$ for each type $\mathbf{E}$ and functors 'mapping...
 Proposition 7.1. If \( \sigma \) is an obvious way) to each category without group action a category with groups actions, namely, on \( X \) prestack on \( X \) \( M \). If \((\sigma,\mathcal{F})\) is an (equivariant) arrow in \( \text{Grp} \), and arrows from \( M \) with codomains in \( \mathcal{E} \), \( F : \mathcal{E} \to \mathcal{B} \) be the inclusion functor. Assignment \( X \subset U \mapsto \{ f : U \to k^n \mid n = 0, 1, ..., f \in \mathcal{M} \} \) gives a prestack on \( X \). It is a nontrivial stack iff \( \mathcal{X} \) is a manifold. \((X,E) \in \text{Ob}(\mathbb{E} \text{-Man})\) iff \( E \) is an atlas on \( X \).

### 7. Lifting problem for a group action

Let \( \text{Grp} \) be a category of groups, \( (-)\text{Grp} : 1\text{-CAT} \to 1\text{-CAT} \) be a functor which assigns (in an obvious way) to each category without group action a category with groups actions, namely,

- \( \mathcal{C} \mapsto \text{Grp-}\mathcal{C} \)
- \( (F : \mathcal{C} \to \mathcal{C}') \mapsto (\text{Grp-}F : \text{Grp-}\mathcal{C} \to \text{Grp-}\mathcal{C}') \) \( F \in \text{Ar}(1\text{-CAT}) \)

\( \text{Grp-}\mathcal{C} \) consists of triples \((G,C,\rho)\) \((G \in \text{Ob}(\text{Grp}), C \in \text{Ob}(\mathcal{C}), \rho : G \to \text{Aut}(C) \) is a group homomorphism) as objects, and pairs \((\sigma : G \to G', f : C \to C') : (G,C,\rho) \to (G',C',\rho') \) as arrows,

\[ \text{Grp-}F : \begin{cases} (G,C,\rho) \to (G,F(C),F(C,\circ \rho)) & (G,C,\rho) \in \text{Ob}(\text{Grp-}\mathcal{C}) \\ (\sigma,f) \to (\sigma,F(f)) & (\sigma,f) \in \text{Ar}(\text{Grp-}\mathcal{C}) \end{cases} \]

\[ [(\sigma,F(f))] \text{ is an (equivariant) arrow in } \text{Grp-}\mathcal{C} \text{ because } F'(\rho'(\sigma(g)))) \circ F(f) = F(f) \circ \rho(g) \forall g \in G \]

**Proposition 7.1.** If \( p \) is a structure over \( \mathcal{B} \) \( (i.e., \text{all isomorphisms of type } (B' \xrightarrow{\sim} p(E)) \text{ in } \mathcal{B} \) \( \text{Ar } \mathcal{B} \text{ can be lifted to isomorphisms } (E' \xrightarrow{\sim} E) \in \text{Ar } \mathcal{E}) \) then \( \text{Grp-}p \) is a structure over \( \text{Grp-}\mathcal{B} \).

**Proof.** If \((\varphi,f) : (G',B',\rho') \xrightarrow{\sim} (G,p(E),p \circ \rho) \) is an iso then \( \left( \frac{E'}{B'} \xrightarrow{f} \frac{E}{B} \right) \) is an iso, because \( p \)

is a structure over \( \mathcal{B} \). Regard the diagram (of group homomorphisms)

\[
\begin{array}{ccc}
\text{Aut}_{\mathcal{E}}(E') & \xleftarrow{f^{-1} \circ_{\mathcal{E}} \circ f} & \text{Aut}_{\mathcal{E}}(E) \\
\downarrow p & \swarrow \rho' \circ \varphi & \downarrow p \\
\text{Aut}_{\mathcal{B}}(B') & \xleftarrow{f^{-1} \circ_{\mathcal{B}} \circ f} & \text{Aut}_{\mathcal{B}}(p(E)) \\
\end{array}
\]

\[ \rho''(g') := f^{-1} \circ_{\mathcal{E}} \rho(\varphi(g')) \circ_{\mathcal{E}} f \]
\[(*)\] commutes because \( f \circ_B \rho'(g') = (p \circ \rho)(\varphi(g')) \circ_B f \) by equivariance condition.

\[(**\)] commutes because \( p(\rho''(g')) = f^{-1} \circ_B p(\rho(\varphi(g'))) \circ_B f = \rho'(g'). \)

So, \( \exists \, \text{iso} \left( (G', E', \rho'') \rightarrow (G', B', \rho') \right) \) i.e. \( \text{Grp}_p \rightarrow \) is a structure over \( \text{Grp}_B. \)

There is a commutative diagram in \( 1\text{-CAT} \)

\[
\begin{array}{ccc}
E & \longrightarrow & \text{Grp}_E \\
p & & \downarrow \text{Grp}_p \\
B & \longleftarrow & \text{Grp}_B
\end{array}
\]

(where horizontal arrows forget group actions). So, there exists a forgetful fiber functor \( E_B \leftarrow \text{Grp}_E(G, B, \rho). \)

**Definition 7.1.** For a given \( G \)-action \((G, B, \rho) \in \text{Ob} \,( \text{Grp}_B)\), an object \( E \in \text{Ob} \,( \text{E}_B(G, B, \rho)) \) admits lifting of \( G \)-action if \( \exists \, \text{iso} \left( (G, E, \rho') \rightarrow (G, B, \rho) \right) \) (essentially, \( \rho = p \circ \rho' \)).

**Lifting problem** for a \( G \)-action \( \rho : G \rightarrow \text{Aut}_B(B) \) is equivalent to completion of the diagram of group homomorphisms with exact row

\[
\begin{array}{ccc}
1 & \longrightarrow & \text{Aut}_B(E) \\
p & & \downarrow \text{Grp}_p \\
B & \longrightarrow & \text{Aut}_B(B)
\end{array}
\]

where \( \text{Aut}_B(E) \) are vertical automorphisms of \( E \) over \( B \).

For single element \( g \in \text{Aut}_B(B) \) there is a simple criterion of existance of \( g' \in \text{Aut}_B(E) \) such that \( p(g') = g \) (see [Kom]).

**Proposition 7.2.** For a fibration \( p \downarrow \text{Grp}_p \) (or structure over \( B \)) \( g \in \text{Aut}_B(B) \) can be lifted to \( g' \in \text{Aut}_B(E) \) iff \( \text{Cart}_g(E) \rightarrow E \) (vertical iso).

**Proof.** \( \iff \)

\[
\begin{array}{ccc}
E & \rightarrow & \text{Cart}_g(E) \\
\sim & & \sim \\
g' & \rightarrow & E
\end{array}
\]

\[
\begin{array}{ccc}
B & \longrightarrow & B
\end{array}
\]

**Proposition 7.3.** If \( \text{Grp}_p \rightarrow \) is a structure on \( B \) (or (co)fibration with unique (co)Cartesian lifting),
and \((G, B, \rho) \in \text{Ob}(\text{Grp-B})\), then \(\exists\) a representation

\[
G \xrightarrow{\rho} \text{Aut}_{\text{CAT}}(E_B)
\]

where \(R(g) : \)

\[
\begin{aligned}
E \xrightarrow{\rho(g)} \text{Cart}_{\rho(g)}(E) & \quad E \in \text{Ob}(E_B) \\
E \xrightarrow{\rho(g)} \text{Cart}_{\rho(g)}(E) & \quad f \xrightarrow{\rho(g)} (\alpha(g)) f \in \text{Ar}(E_B) \\
E' \xrightarrow{\rho(g)} \text{Cart}_{\rho(g)}(E') &
\end{aligned}
\]

Proof is straightforward. \(\square\)

Corollary. If \(E \in \text{Ob}(E_B)\) is such that \(\forall g \in G \text{ Cart}_{\rho(g)}(E) = E\) then Cartesian lifting \(\rho(g) \mapsto \rho(g)\) lifts action \((G, B, \rho) \in \text{Ob}(\text{Grp-B})\) to the action \((G, E, \tilde{\rho}) \in \text{Ob}(\text{Grp}(G, B, \rho))\). \(\square\)

Example (Covering Space)

A covering space is a (co)fibration \(p \xrightarrow{} B\) over groupoid \(B\) with unique (co)cartesian lifting in which all morphisms are (co)cartesian. Moreover, representation \(\text{Aut}(b) \to \text{Aut}(E_b)\), \(b \in \text{Ob}(B)\) (induced by (co)cartesian lifting) is transitive on objects of \(E_b\).

Proposition 7.4. For a covering space \(p \xrightarrow{} B\) over connected groupoid \(B\)

- \(\text{Aut}(\rho) \simeq \text{Aut}(E_b)\) (where \(g \in \text{Aut}(E_b)\) iff \(g \circ f^* = f^* \circ g\), \(f^* \equiv \text{coCart}_f\), \(\forall f \in \text{Aut}(b)\))
- \(\text{Aut}(E_b) \simeq \text{W}(\text{Stab}(e)) \simeq \text{N}(\text{Stab}(e))/\text{Stab}(e)\) (where \(\text{Stab}(e) \hookrightarrow \text{Aut}(b)\) is the stabilizer of an object \(e \in \text{Ob}(E_b)\), \(\text{N}(\text{Stab}(e))\), \(\text{W}(\text{Stab}(e))\) are its normalizer and Weil group respectively).

Proof.

- An automorphism \(g\) of covering space \(p\) is given by family of fiberwise functors \(g_b, b \in \text{Ob}(B)\), such that \(f^* \circ g_b = g_{b'} \circ f^*, f^* \equiv \text{coCart}_f, \forall (f : b \to b') \in \text{Ar}(B)\). Take \(g_b \in \text{Aut}(E_b)\) and define \(g_{b'} := h^* \circ g_b \circ (h^*)^{-1}\) for some \(h : b \to b'\). Then \(g_{b'}\) is well-defined (if \(h_1 : b \to b'\) is another morphism then \(h^* \circ g_b \circ (h^*)^{-1} = h_1^* \circ g_b \circ (h_1^*)^{-1}\) since \((h_1^{-1} \circ h^*)^* \circ g_b = (h_1^*)^{-1} \circ h^* \circ g_b = g_b \circ (h_1^*)^{-1} \circ h^* = g_b \circ (h_1^{-1} \circ h^*), h_1^{-1} \circ h \in \text{Aut}(b)\), and it is an automorphism of covering space \(p\) (if \(f : b' \to b''\) then \(f^* \circ g_{b''} = g_{b'} \circ f^*\) since \(f^* \circ h^* \circ g_b = g_b \circ f^* \circ h^*, f \circ h : b \to b''\)).
- See [May]. \(\square\)
7.1. Lifting of a groupoid action for a sheaf.

**Definition 7.1.1.** Let \((\mathbf{Top}, \tau_0)\) be a site for all open coverings on topological spaces.

- **Set-valued presheaf** \(P : \mathbf{Top}^{op} \to \mathbf{Set}\) is a **sheaf** iff \(\forall\) sieve \(S \hookrightarrow B(\_ , B)\) and \(\forall\) natural transformation \(f : S \to P \ni \tilde{f} : B(\_ , B) \to P\) such that \(S \xrightarrow{\tilde{f}} B(\_ , B)\).

- **Cat-valued presheaf** \(P : \mathbf{Top}^{op} \to \mathbf{Cat}\) is a **sheaf** iff its object and morphism parts are sheaves, i.e., \(\mathbf{Top}^{op} P \to \mathbf{Ob} \mathbf{Set}\) and \(\mathbf{Top}^{op} P \to \mathbf{Mor} \mathbf{Set}\) are \(\mathbf{Set}\)-valued sheaves.

- For presheaf \(P : \mathbf{Top}^{op} \to \mathbf{Cat}\), space \(X \in \mathbf{Ob}(\mathbf{Top})\) and sieve \(S \hookrightarrow \mathbf{Top}(\_ , X)\) matching familiar of objects \(\tilde{E} : S \to \mathbf{Ob} P\) (nat. trans.) (or matching family of arrows \(\tilde{f} : S \to \mathbf{Mor} P\) (nat. trans.)) **has a germ** \(\text{germ}_x(\tilde{E})\) (respectively, \(\text{germ}_x(\tilde{f})\)) at point \(x \in X\) iff \(\exists \text{Colim}_{s \in S, \text{Im}(s) \ni x} (\tilde{E}(s)) =: \text{germ}_x(\tilde{E})\) (respectively, \(\text{Colim}_{s \in S, \text{Im}(s) \ni x} (\tilde{f}(s)) =: \text{germ}_x(\tilde{f})\)) (if germ exists it is unique up to iso and does not depend on the choice of sieve \(S\))

- **Etale space** is \(E := \bigsqcup_{x \in X} \text{germ}_x(\tilde{E})\) (respectively, \(f := \bigsqcup_{x \in X} \text{germ}_x(\tilde{f})\)) (depending on two variables: 'point' \(x \in X\) and 'matching family' \(\tilde{E}\) or \(\tilde{f}\)) with topology generated by basic open sets \((U, \{\text{germ}_x(\tilde{E})\mid x \in U\})\) (or, \((U, \{\text{germ}_x(\tilde{f})\mid x \in U\})\)), \(U\) is open in \(X\). There is a natural continuous projection \(p : E \to X : (x, \text{germ}_x(\tilde{E})) \mapsto x\) (respectively, \(p : f \to X : (x, \text{germ}_x(\tilde{f})) \mapsto x\)) which is a local homeomorphism.

**Lemma 7.1.1.** Every fibration is a cofibration with respect to iso's (every cofibration is a fibration with respect to iso's).

**Proof.** Let \(\xymatrix{E \ar[d]_p \ar[r]_{\tilde{f}} & B'}\) be a fibration, and \(p(E) \xrightarrow{\tilde{f}} B'\) be an iso in \(B\). Then \(\tilde{f} := (\tilde{f}^{-1})^{-1} : E \to E'\) (where \(\sim\) on the right is a cartesian lifting) is a cocartesian lifting of \(f\) (obvious). \(\square\)

**Corollary.** For a (co)fibration \(\xymatrix{E \ar[d]_p \ar[r] & B}\) for each iso \((f : B \xrightarrow{\sim} B') \in \mathbf{Ar} B\) inverse image \(\mathbf{Cart}_f : E_{B'} \to E_B : s_{B'} \mapsto f^*(s_{B'})\) and direct image \(\mathbf{Cart}_f : E_B \to E_{B'} : s_B \mapsto f_*(s_B)\) (where \(s_B\) is a 'section' (object or morphism) over \(B\)) exist. \(\square\)

**Definition 7.1.2.**

- For a space \(X \in \mathbf{Ob}(\mathbf{Top})\) groupoid of local homeomorphisms of \(X\) is a subcategory \(\mathbf{Gr}_X \hookrightarrow \mathbf{Top}\) such that \(\big\{\mathbf{Ob}(\mathbf{Gr}_X)\mid\) open subsets in \(X\) \big\}

- (Nonfull) subcategory \(\mathbf{Gr}_{X,x} \hookrightarrow \mathbf{Gr}_X\) with objects \(U \ni x\) and morphisms \(f : U \to V, f(x) = x\), is called groupoid of local homeomorphisms of \(X\) with fixed point \(x \in X\).

- \(X\) is **transitive** with respect to \(\mathbf{Gr}_X\) if \(\forall x, y \in X \exists U, V \in \mathbf{Ob}(\mathbf{Gr}_X), (f : U \to V) \in \mathbf{Ar}(\mathbf{Gr}_X)\) such that \(U \ni x, V \ni y, f(x) = y\).

- For a (co)fibration \(\xymatrix{E \ar[d]_p \ar[r] & \mathbf{Top}}\) with unique (co)cartesian lifting and space \(X \in \mathbf{Ob}(\mathbf{Top})\) two actions
Lemma 7.1.2.

Let \( f \in \text{Gr}_X(U,V) \) and \( f^* \equiv \text{Cart}_f : E_V \rightarrow E_U : s_V \mapsto f^*s_V \).

**left action** \( \forall f \in \text{Gr}_X(U,V) \) \( f^* \equiv \text{Cart}_f : E_V \rightarrow E_U : s_V \mapsto f^*s_V \),

**right action** \( \forall f \in \text{Gr}_X(U,V) \) \( f_* \equiv \text{coCart}_f : E_U \rightarrow E_V : s_U \mapsto f_*s_U \)

(where \( s_V, s_U \) are objects or morphisms).

- To each of actions \( f^*, f_* \) (on local sections of \( E_X \)) there correspond respectively left and right actions of \( \text{Gr}_{X,x} \) on \( \{ \text{germ}_x(\tilde{s}) \mid \tilde{s} \text{ is a matching family of local sections of } E_X \} \). If \( s = \text{germ}_x(s_U) \) is a germ at point \( x \) presented by a local section \( s_U \) (i.e., \( s = \text{Colim}_{U \supset V \ni x} (s_U |_V) \)), then if \( (f : W \rightarrow V) \in \text{Gr}_{X,w}(W,V) \)

  **left action** \( f^*s := \text{germ}_x((f \mid_{f^{-1}(U \cap V)})^*(s_U |_U)) \)

  **right action** \( f_*s := \text{germ}_x((f \mid_{U \cap W})_*(s_U |_W)) \).

- For a subgroupoid \( G \hookrightarrow \text{Gr}_X \), \( \left( \begin{array}{c} s \\ X \end{array} \right) \in \left( \begin{array}{c} E \\ \text{Top} \end{array} \right) \) is \( G \)-invariant if \( \forall (f : U \rightarrow V) \in \text{Ar}(G) \)

  \[
  f^*(s_V) = s_U \text{ (or } f_*(s_U) = s_V) \text{ (where: } s_U := s \mid_U := (i : U \rightarrow X)^*s, \quad \begin{array}{c} E \\ \text{Top} \end{array} \text{ is a fibration with unique cartesian lifting (or a local structure with respect to inclusions of open sets), } s \text{ is a section (object or morphism) over } X \text{). In other words, } G \text{-invariant actions admit lifting of groupoid } G. \]

- **germ** \( \text{germ} \) \((f) \) of a map \( (f : U \rightarrow V) \in \text{Ar}(\text{Gr}_X) \), such that \( f(x) = y \), is an equivalence class of maps \( \{ g \in \text{Ar}(\text{Gr}_X) \mid g(x) = y, \exists \text{ opens } W_x \ni x, W_y \ni y, \text{ such that } \exists \text{ the same restrictions } f \mid_{W_x,W_y} = g \mid_{W_x,W_y} \in \text{Ar}(\text{Gr}_X) \} \). Assume, \( s_x = \text{germ}_x(s_{U_1}), s'_y = \text{germ}_y(s'_{V_1}) \). Then

  **left action** \( (\text{germ}_{x,y}(f))^*s'_y := \text{germ}_x((f \mid_{f^{-1}(V \cap U_1)})^*(s'_y |_{V \cap U_1})) \)

  **right action** \( (\text{germ}_{x,y}(f))_*s_x := \text{germ}_y((f \mid_{U \cap U_1})_*(s_x |_{U \cap U_1})) \).

\( \square \)

**Lemma 7.1.2.**

Let \( X \) be a topological space, \( G \hookrightarrow \text{Gr}_X \) be a subgroupoid, \( G_x \hookrightarrow G \) be a subgroupoid of pointed maps with fixed point \( x \in X \). Then \( \forall x, y \in X \) and \( \forall f \in \text{Ar}(G) \), \( s.t. f(x) = y \),

\[
\text{germ}_{x,y}(G_x) = \text{germ}_{y,x}(f^{-1}) \cdot \text{germ}_{y,y}(G_y) \cdot \text{germ}_{x,y}(f) \quad \text{(for certain unique composite · of germs of maps)}.}
\]

- If \( s_x = \text{germ}_x(s_U) \in S_x \subset S \) is a point of etale space \( S \quad \begin{array}{c} S \\ X \end{array} \) (corresponding to objects or morphisms over \( X \) for a fibration \( \begin{array}{c} E \\ \text{Top} \end{array} \) ) then \( s_x \) is \( G_x \)-invariant iff it is \( \text{germ}_{x,x}(G_x) \)-invariant.

- If \( G \) is transitive on \( X \) and \( s_x \) is \( G_x \)-invariant then \( \forall f, g \in \text{Ar}(G) \), \( s.t. f(x) = y, g(x) = y \), there is a unique induced germ at point \( y \) \( (\text{germ}_{x,y}(f))_*s_x = (\text{germ}_{x,y}(g))_*s_x \) and this germ
is $G_y$-invariant (or, respectively, $(\text{germ}_{y,x}(f^{-1}))^*s_x = (\text{germ}_{y,x}(g^{-1}))^*s_x$ is a unique $G_y$-invariant germ at point $y$), i.e. $s_x$ can be distributed in a unique way over all $X$ (to give rise to a section $s : X \to S$ of etale space $S$ consisting of invariant germs at each point). □

**Proposition 7.1.1.** For sheaf $P : \text{Top}^{op} \to \text{CAT}$, space $X \in \text{Ob}(\text{Top})$, and transitive groupoid $G \hookrightarrow \text{Gr}_X$ $G$-invariant sections over $X$ are in bijective correspondence with a subset of $G_x$-invariant germs (of local sections) for a fixed point $x \in X$.

**Proof.** To each $G$-invariant section over $X$ there corresponds $G_x$-invariant germ of this section at point $x$. Conversely, by lemma 7.1.2, each $G_x$-invariant germ generates a section of the corresponding etale space. When this section is continuous there is a global section over $X$ of sheaf $P$ (which is locally invariant). □

**Remark.** B.P. Komrakov [Kom] asserts (without a proof) that the above bijection is with the whole set of $G_x$-invariant germs. But, without additional assumptions it is not clear why the corresponding section of invariant germs is continuous and the sheaf section is invariant. □

**8. Equivalence, groups, actions**

Let $\mathcal{R}$ be a category of sets with a given equivalence relation for each set. There are following functors:

- **forgetful** $p : \mathcal{R} \to \text{Set} : (A, R) \mapsto A$
- **quotient** $Q : \mathcal{R} \to \text{Set} : (A, R) \mapsto A/R$
- **inclusion** $\Delta : \text{Set} \to \mathcal{R} : A \mapsto (A, \Delta_A)\), $\Delta_A := \{(a, a) \mid a \in A\}$

such that $\mathcal{R} \xrightarrow{\mathcal{T}} \text{Set}$, i.e. $\text{Set}(Q(A, R), B) \xrightarrow{\sim} \mathcal{R}((A, R), \Delta(B)) : f \mapsto f \circ \pi$, where $A \xrightarrow{\pi} A/R$ is the canonical projection (so, quotient object $Q(A, R)$ represents functor $\mathcal{R}((A, R), \Delta(-)) : \text{Set} \to \text{Set}$).

For arbitrary category $\mathcal{C}$ equivalence relation on objects is introduced as usual via hom-sets.

**Definition 8.1.**

- A functor $R : \mathcal{C}^{op} \to \mathcal{R}$ is called an equivalence relation on object $C \in \text{Ob}\mathcal{C}$ if $\mathcal{C}^{op} \xrightarrow{R} \mathcal{R}$ (i.e. usual equivalence relations are introduced on hom-sets $\mathcal{C}(C', C)$, $C' \in \text{Ob}\mathcal{C}$ and they are preserved under precomposition $- \circ f, f : C'' \to C'$).
- Let $\mathcal{C}_R$ be a category such that $\text{Ob}(\mathcal{C}_R)$ are pairs $(C, R), C \in \text{Ob}\mathcal{C}, R$ is an equivalence relation on $C$, $\text{Ar}(\mathcal{C}_R)$ are maps $(C, R) \xrightarrow{(f, F)} (C', R')$, where $(f : C \to C') \in \text{Ar}\mathcal{C}$ and $F : R \Rightarrow R'$ is a
natural transformation of equivalence relations such that

\[ p_F = C(-, f) \] (it means \( (C, R) \overset{(f,F)}{\longrightarrow} (C', R') \) is a morphism in \( C_R \) iff \( f : C \to C' \) is an

arrow in \( C \) and \( f \circ - \) preserves equivalence relation, i.e. if \( g_1 \sim_R g_2 \) then \( f \circ g_1 \sim_{R'} f \circ g_2 \) for

\( g_1, g_2 : X \to C \)).

\( C_R \) is an analogue of \( \mathcal{R} \) for arbitrary category \( C \). Again, there are the following functors

• **forgetful** \( p : C_R \to C : (C, R) \mapsto C \)

• **inclusion** \( \Delta : C \to C_R : C \mapsto (C, \Delta \circ C(-, -)) \), where \( \Delta : \text{Set} \to \mathcal{R} : A \mapsto (A, \Delta_A) \),

\( \Delta_A := \{(a, a) \mid a \in A\} \)

• **quotient** \( Q : C_R \to C : (C, R) \mapsto C/R \) which is a left adjoint to \( \Delta : C \to C_R \), i.e.

\[ C_R \xrightarrow{\Delta} C \quad \text{or} \quad C(Q(C, R), C') \xrightarrow{\sim_{\text{nat.iso}}} C_R((C, R), \Delta(C')) \] (quotient object \( C/R := Q(C, R) \))

represents functor \( C_R((C, R), \Delta(-)) : C \to \text{Set} \) which means that \( \exists \) an arrow \( \pi : (C, R) \to \Delta(Q(C, R)) \) such that \( \forall f : (C, R) \to \Delta(C') \exists \hat{f} : Q(C, R) \to C' \) with \( f = \Delta(\hat{f}) \circ \pi \), in other words, quotient map \( \pi : C \to Q(C, R) \) is a common coequalizer of all equivalent arrows \( f \sim_R g \) with arbitrary domain and codomain \( C \) and, in particular, is always an epimorphism.

Quotient functor may not exist for the whole category \( C_R \), but there always exists a (maximal)

full subcategory \( C_{R/Q} \hookrightarrow C_R \) for which \( C_{R/Q} \xrightarrow{\Delta} C \) (indeed, \( C_{R/Q} \) is always non empty

since \( Q \circ \Delta(C) = C \), i.e. \( \Delta(C) \in \text{Ob}(C_{R/Q}) \)).

If \( C \) is a concrete category with representable underlying functor \( U := C(I, -) \) then to each equivalence relation \( R : C^{op} \to \mathcal{R} \) on object \( C \) with quotient map \( \pi : (C, R) \to \Delta(Q(C, R)) \) there corresponds a usual equivalence relation on \( C(I, C) \) with quotient map \( \pi \circ - : C(I, C) \to C(I, Q(C, R)) \), and, conversely, to usual equivalence relation on \( C(I, C) \) with quotient map \( \pi \circ - : C(I, C) \to C(I, C') \) there corresponds a maximal 'saturated' equivalence relation \( R : C^{op} \to \mathcal{R} \)

on object \( C \) with quotient map \( \pi : C \to C' \equiv Q(C, R) \) such that \( f \sim_R g \) iff \( \pi \circ f = \pi \circ g \). In general, equivalence relation on hom-sets is weaker than usual one.

Let \( C \in \text{Ob} C, \sigma : G \to \text{Aut}_C(C) \), then \( G \) also acts on hom-sets \( C(C', C), \quad C' \in \text{Ob} C, \)

\[ G \times C(C', C) \to C(C', C) : \{(g, f) \mapsto \sigma(g) \circ f \} \quad \text{left action} \]

\[ \{(g, f) \mapsto \sigma(g^{-1}) \circ f \} \quad \text{right action} \]

, i.e. \( \exists \) a functor \( \Sigma : C^{op} \to G\text{-Set} \) such that

\[ \Sigma \downarrow p \] (it means that all hom-sets \( C(C', C), \quad C' \in \text{Ob} C, \) are

regarded with the given \( G \)-action).

There are functors

• \( r : G\text{-Set} \to \mathcal{R} : \{(X, G, \sigma) \mapsto (X, R_\sigma) \} \quad \text{on objects} \)

\[ \{(X, G, \sigma) \overset{f}{\mapsto} (X', G, \sigma') \mapsto ((X, R_\sigma) \overset{f}{\mapsto} (X', R_{\sigma'})) \} \quad \text{on arrows} \]

(where \( R_\sigma \) is an equivalence relation on \( X \) such that \( (x, y) \in R_\sigma \) iff \( \exists g \in G \ y = \sigma(g)x \)
$r$ is a functor over $\text{Set}$, i.e.

$$G\text{-}\text{Set} \xrightarrow{r} \mathcal{R} \xrightarrow{p} \text{Set}$$

- $r : G\text{-}C \to C_{\mathcal{R}}$:
  
  \[
  \begin{cases}
  (C, G, \sigma) \mapsto (C, R_{\sigma}) & \text{on objects} \\
  ((C, G, \sigma) \xrightarrow{\delta} (C', G, \sigma')) \mapsto ((C, R_{\sigma}) \xrightarrow{\delta} (C', R_{\sigma'})) & \text{on arrows}
  \end{cases}
  \]

  (where $R_{\sigma} := r \circ \Sigma$ is an equivalence relation on object $C$ corresponding to $\sigma$, $C_{op}$)

$r$ is a functor over $C$, i.e.

$$G\text{-}C \xrightarrow{r} C_{\mathcal{R}} \xrightarrow{p} C \xrightarrow{\pi} \text{Set}$$

Let $G\text{-}C_{Q} := r^{-1}(C_{\mathcal{R}Q})$. Then $\exists$ a quotient functor $G\text{-}C_{Q} \xrightarrow{r_{Q}} C_{\mathcal{R}Q} \xrightarrow{Q} C$. Denote it again by $Q$, and $Q \circ r (C, G, \sigma)$ by $C/G$.

For arbitrary functor $F : C \to D$ we have $G\text{-}F : G\text{-}C \to G\text{-}D$ such that

$$G\text{-}C \xrightarrow{r} C_{\mathcal{R}} \xrightarrow{p} C \xrightarrow{\pi} \text{Set}$$

needs not preserve quotients, i.e. the diagram

$$C \xrightarrow{F} D \xrightarrow{p} C \xrightarrow{\pi} \text{Set}$$

can be wrong (the dotted arrow may not exist and the natural isomorphism may not hold). If the above diagram holds (up to iso) then $F : C \to D$ preserves quotients (of category $G\text{-}C$). In this case $F(C/G) \cong C'/C$.

$F(C)/G$. Quotient $C/G$ is called universal [Kom] if $\forall C' \in \text{Ob} C$ $(C' \times C/G \xrightarrow{\pi} C')$ is with trivial $G$-action.

** Proposition 8.1.** Let $p$ be a structure on $B$, $p$ preserve quotients of category $G\text{-}E$, and $(E, G, \sigma) \in \text{Ob} (G\text{-}E)$ be an object such that $E/G$ exists with $\pi : E \to E/G$, canonical projection, then

$$\left(\frac{E}{p(E/G)}\right) = (p(\pi))_{*} \left(\frac{E}{p(E)}\right)$$

is a direct image of $\left(\frac{E}{p(E)}\right)$.

**Proof.** We need to prove that $\left(\frac{E}{p(E)}\right) \xrightarrow{(\pi/p(\pi))_{*}} \left(\frac{E}{p(E/G)}\right)$ is cocartesian. Take $u/v : \left(\frac{E}{p(E)}\right) \to \left(\frac{E'}{p(E')}\right)$ such that $v = k \circ p(\pi)$ for some $k : p(E/G) \to p(E')$, i.e. $\forall f \sim_{G} f' : B \to p(E) \ x \circ f$
v \circ f'.

Assume, \( h \sim E \colon E_1 \to E \) then \( p(h) \sim p(E_1) \to p(E) \) (because, if \( h' = \sigma(g) \circ h \) then \( p(h') = p(\sigma(g)) \circ p(h) \)). So, \( v \circ p(h) = v \circ p(h') \) and \( p(u) \circ p(h) = p(u) \circ p(h'), u \circ h = u \circ h' \) (\( p \) is faithful), i.e. \( u \) coequalizes all \( \sim \) equivalent arrows (in \( R_\alpha \)).

Therefore, \( u = \hat{u} \circ \pi \) for a unique \( \hat{u} : E/G \to E' \).

Finally, \( \exists! (\hat{u}/k) : (E/G \to p(E'/G)) \) such that \( (u/v) = (\hat{u}/k) \circ (p/\pi) \), i.e. \( (\pi/p) \) is cocartesian.

\[ \square \]

8.1. Group objects, subgroups, quotient objects.

Definition 8.1.1. Let \( \mathbf{C} \) be a category with binary products and terminal object 1.

- \( G \in \text{Ob} \mathbf{C} \) is called a **group object** if \( \exists \) maps \( m : G \times G \to G \), \( e : 1 \to G \), \( \text{inv} : G \to G \) such that the following group-like diagrams hold

\[
\begin{array}{cccc}
G \times G \times G & \overset{1 \times m}{\longrightarrow} & G \times G & \overset{m}{\longrightarrow} & G \\
G \times G & \overset{e \times 1}{\longrightarrow} & G \times G & \overset{1 \times e}{\longrightarrow} & G \\
G & \overset{\Delta}{\longrightarrow} & G \times G & \overset{\Delta}{\longrightarrow} & G \\
\end{array}
\]

- Subobject \( K \twoheadrightarrow G \) of group object \( G \) is called a **subgroup** (object) if \( \exists \) maps \( m_K : K \times K \to K \)

\[
\begin{array}{ccc}
K \times K & \overset{e_K}{\longrightarrow} & K \\
1 & \overset{e_K}{\longrightarrow} & K \\
K & \overset{\text{inv}_K}{\longrightarrow} & K \\
\end{array}
\]

- For two elements \( f, g : 1 \to G \) multiplication \( f \cdot g : 1 \to G \) is

\[
\begin{array}{cccc}
G \times G & \overset{m}{\longrightarrow} & G \\
G & \overset{\sim}{\longrightarrow} & G \times G \overset{m}{\longrightarrow} G \\
G & \overset{\sim}{\longrightarrow} & G \times G \overset{m}{\longrightarrow} G \\
\end{array}
\]

- **Right shift** \( R_g : G \to G \) (by element \( g : 1 \to G \)) is

\[
\begin{array}{cccc}
G & \overset{1 \times g}{\longrightarrow} & G \times G & \overset{m}{\longrightarrow} G \\
G & \overset{\sim}{\longrightarrow} & G \times G \overset{m}{\longrightarrow} G \\
G & \overset{\sim}{\longrightarrow} & G \times G \overset{m}{\longrightarrow} G \\
\end{array}
\]

- **Left shift** \( L_g : G \to G \) (by element \( g : 1 \to G \)) is

\[
\begin{array}{cccc}
G & \overset{g \times 1}{\longrightarrow} & G \times G & \overset{m}{\longrightarrow} G \\
G & \overset{\sim}{\longrightarrow} & G \times G \overset{m}{\longrightarrow} G \\
G & \overset{\sim}{\longrightarrow} & G \times G \overset{m}{\longrightarrow} G \\
\end{array}
\]

\[ \square \]

Proposition 8.1.1. For a group object \( G \in \text{Ob} \mathbf{C} \)

- \( \mathbf{C}(1, G) \) is a group,
- \( \exists \) (anti)representation \( \mathbf{C}(1, G) \to \text{Aut}_C(G) : g \mapsto R_g \) (by right shifts) and representation \( \mathbf{C}(1, G) \to \text{Aut}_C(G) : g \mapsto L_g \) (by left shifts).

Proof.

- It follows immediately from group object axioms \( (e : 1 \to G \) is the identity, \( \text{inv} \circ g : 1 \to G \) is the inverse of \( g : 1 \to G \)).
- \( R_e = 1_G : G \to G \) (obvious)
- \( R_f \circ R_g = R_{gof} \) follows from the diagram

\[
\begin{array}{cccc}
G & \overset{R_f \circ R_g}{\longrightarrow} & G & \overset{R_{gof}}{\longrightarrow} G \\
G & \overset{\sim}{\longrightarrow} & G \times G \overset{m}{\longrightarrow} G \\
G & \overset{\sim}{\longrightarrow} & G \times G \overset{m}{\longrightarrow} G \\
\end{array}
\]
Two dotted paths $G \times 1 \xrightarrow{1 \times g} G \times G \xrightarrow{m} G \sim G$, $G \times 1 \xrightarrow{1 \times g \cdot f} G \times G \xrightarrow{1 \times (g \cdot f)} G \times G \sim G$ and $G \times 1 \xrightarrow{1 \times g} G \times G \xrightarrow{m} G \sim G \times 1$ are equal since their composites with projections $p_1 : G \times 1 \xrightarrow{\sim} G$ and $p_2 \equiv ! : G \times 1 \rightarrow 1$ are equal, indeed,

$$p_1 \circ (p_1^{-1} \circ m \circ (1 \times g)) = m \circ (1 \times g)$$

and

$$p_2 \circ (p_1^{-1} \circ m \circ (1 \times g)) = !$$

Since $p = p_1 \circ ((m \circ (1 \times g)) \times 1) \circ (1 \times <1,1>) = !$ and $p_2 \circ ((m \circ (1 \times g)) \times 1) \circ (1 \times <1,1>) = !$, we have $p \circ (m \circ (1 \times g) \circ <1,1> \times 1) = m \circ (1 \times g) \circ <1,1> \circ 1 = m \circ (1 \times g) \circ ! \circ 1 = m \circ (1 \times g) \circ 1 \times 1 = *$

Proof for left shift $L_g$ is similar. □

**Corollary.** If $K \xrightarrow{\sim} G$ is a subgroup (object) of $G$ then $\mathbf{C}(1, K) \xrightarrow{\sim} \text{Aut}_C(G) : k \mapsto R_k$ is a (right) action $(G, K, \sigma)$ on $G$ (by right shifts from $K$). If quotient $Q(G, K, \sigma) \in \text{Ob C}$ exists it is called **quotient object** $G/K$ (under right action of $K$).

**Proposition 8.1.2.** Let $p : G \rightarrow G/K$ be a quotient map s.t. $\mathbf{C}(1, p) : \mathbf{C}(1, G) \rightarrow \mathbf{C}(1, G/K)$ is surjective. Then

- $L_g : G \rightarrow G$ induces a Set-map $\bar{L}_g' : \mathbf{C}(1, G/K) \xrightarrow{\sim} \mathbf{C}(1, G/K)$
- If $G/K$ is universal then $\exists \bar{L}_g : G/K \xrightarrow{\sim} G/K$ in $\mathbf{C}$ s.t. $\mathbf{C}(1, \bar{L}_g) = \bar{L}_g'$ and $p \xrightarrow{\sim} G \xrightarrow{\sim} G/K \xrightarrow{\bar{L}_g} G/K$

Proof.

- Claim 1: $\bar{L}_g' : p x \mapsto p L_g x$ is well-defined and iso

Proof of Claim 1: If $p x = p x'$ then $\exists k \in \mathbf{C}(1, K)$ such that $x' = R_k x = x \cdot k$. Then $L_g x' = L_g (x \cdot k) = (g \cdot x) \cdot k = (L_g x) \cdot k = R_k (L_g x)$, i.e. $p L_g x' = p L_g x$.

- Claim 2: $G \times G \xrightarrow{1 \times p} G \times G/K$ is a quotient map of $(G \times G, K, <1, \sigma>) \in \text{Ob C-K}$, where $\text{C-K}$ is a category of right actions of $\mathbf{C}(1, K)$ on objects of $\mathbf{C}$, i.e. $<1, \sigma> : \mathbf{C}(1, K) \rightarrow \text{Aut}_C(G \times G) : k \mapsto 1 \times R_k$. □
8.2. C-group actions.

Definition 8.2.1.
- Let $G$ be a group object in $\mathbf{C}$, $X \in \text{Ob}\, \mathbf{C}$, then $\mathbf{C}$-map $\rho : G \times X \to X$ is a (left) group
action on }X\text{ if }
\begin{align*}
G \times G \times X & \xrightarrow{1 \times \rho} G \times X \\
m \times 1 & \downarrow \\
G \times X & \xrightarrow{\rho} X
\end{align*}
\begin{align*}
1 \times X & \xrightarrow{\rho} G \times X \\
\xrightarrow{p_2} & \downarrow \\
X & \xrightarrow{\rho} X
\end{align*}

$\bullet$ **Left shift** $L^X_g : X \to X$ (by $g \in C(1,G)$) is the composite

$\bullet$ If $K \overset{i_1}{\to} G$ is a subgroup of $G$, $Y \overset{i_2}{\to} X$ is a subobject of $X$ then $K$ stabilizes $Y$ if

$\exists f : K \times Y \to Y$ such that

$$
\begin{array}{ccc}
G \times X & \xrightarrow{\rho} & X \\
\downarrow & & \downarrow \\
Z \times Y & \xrightarrow{\rho_x} & Y
\end{array}
$$

Lemma 8.2.1. Let $Y \overset{i}{\to} X$ be a subobject of object $X$ with $G$-action $\rho : G \times X \to X$. Assignment $\text{Stab}_Y : \text{Ob} \ C \to \text{Ob} \text{Set} : Z \mapsto \text{Stab}_Y(Z) \subset C(Z,G)$ such that $(x : Z \to G) \in \text{Stab}_Y(Z)$ iff $\exists \rho_x : Z \times Y \to Y$ such that

$$
\begin{array}{ccc}
G \times X & \xrightarrow{\rho} & X \\
\downarrow & & \downarrow \\
Z \times Y & \xrightarrow{\rho_x} & Y
\end{array}
$$
is functorial (hom-subfunctor).

Proof. For $(f : W \to Z) \in \text{Ar} \ C$ define $\text{Stab}_Y(f) : \text{Stab}_Y(Z) \to \text{Stab}_Y(W) : x \mapsto x \circ f$ (as precomposite with $f$). This is correct since if $x \in \text{Stab}_Y(Z)$ then $x \circ f \in \text{Stab}_Y(W)$ which can be seen from the diagram

$$
\begin{array}{ccc}
G \times X & \xrightarrow{\rho} & X \\
\downarrow & & \downarrow \\
Z \times Y & \xrightarrow{\rho_x} & Y
\end{array}
$$

Functorial properties of $\text{Stab}_Y$ are obvious. So, $\exists$ functor $\text{Stab}_Y : \text{C}^{\text{op}} \to \text{Set}$ and $\text{Stab}_Y \to C(-,G)$ is a hom-subfunctor.

Definition 8.2.2. If $\text{Stab}_Y : \text{C}^{\text{op}} \to \text{Set}$ is representable then denote its representing object by $\text{Stab}_Y \in \text{Ob} \ C$ and call it a stabilizer of $Y \overset{i}{\to} X$ (for group $G$ acting on $X$).

Proposition 8.2.1. Let $\text{Stab}_Y : \text{C}^{\text{op}} \to \text{Set}$ be represented by $\text{Stab}_Y \in \text{Ob} \ C$. Then

$\bullet$ $\text{Stab}_Y \overset{j}{\to} G$ is a subobject of group $G$ (but not necessarily a group object itself),

$\bullet$ $j$ is the universal element of functor $\text{Stab}_Y$,

$\bullet$ each element in $\text{Stab}_Y(Z)$ has form $j \circ x$ for a unique $x : Z \to \text{Stab}_Y$, and all elements of this form $\forall x : Z \to \text{Stab}_Y$ are in $\text{Stab}_Y(Z)$ (in other words, $(z \in Z \ G) \& (z \in \text{Stab}_Y(Z)) \iff (z \in Z \ G)$,}
Proposition 8.2.2.

Proof. First three points follow from Yoneda Lemma \((j \text{ is the universal element of representation } C(-, \text{Stab}_Y) \Rightarrow \text{Stab}_Y \text{ corresponding (under Yoneda embedding) to monic (natural transformation) } C(-, \text{Stab}_Y) \Rightarrow \text{Stab}_Y \Rightarrow C(-, G))\). Fourth point follows from the definition and above properties of functor \(\text{Stab}_Y\) and that \(Y \Rightarrow X\) is monic. \(\square\)

Lemma 8.2.2. Subobject \(H \rightarrow \ X G \) of a group object \(G \in \text{Ob C}\) is itself a group object iff \(\forall Z \in \text{Ob C}\) there are induced group operations in hom-set \(C(Z, H)\) in the following way

\[
\begin{array}{ccc}
G \times G & \xrightarrow{m} & G \\
\downarrow & & \downarrow \\
H \times H & \xrightarrow{\alpha} & H
\end{array}
\quad
\begin{array}{ccc}
G & \xrightarrow{\text{inv}} & G \\
\downarrow & & \downarrow \\
H & \xrightarrow{\alpha} & H
\end{array}
\quad
\begin{array}{ccc}
1 & \xrightarrow{e} & G \\
\downarrow & & \downarrow \\
Z & \xrightarrow{\exists e(x)} & H
\end{array}
\]

Proof is obvious in both directions. \(\square\)

Proposition 8.2.2.

- \(\text{Stab}_Y \rightarrow \ X G\) is always a submonoid of group object \(G \in \text{Ob C}\).
- \(\text{Stab}_Y \rightarrow \ X G\) is a subgroup of group object \(G \in \text{Ob C}\) if \(\forall Z \in \text{Ob C} \ \forall x \in \text{Stab}_Y(Z)\) the corresponding map \(\rho_x : Z \times Y \rightarrow Y\) as in the diagram \(\xrightarrow{z \times i} \) is surjective in \(Z \times Y \rightarrow Y\).

the second argument, i.e. \(\forall t : T \rightarrow Z\) the map \(C(T, Y) \ni s \mapsto \rho_x \xrightarrow{s} t\) is surjective \(\text{it holds in classical case in } \text{Set}\).

Proof.
In general, for $x \in Z \text{Stab}_Y \ inv(x) \in Z \ G$, but $inv(x) \notin Z \text{Stab}_Y$.
Proof of Lemma. \[ \begin{array}{ccc} Hom(X,A) \xrightarrow{f_X} Hom(X,B) & \xrightarrow{g'_X} \rightarrow & Hom(X,C) \\ \downarrow h_X \downarrow & \xrightarrow{Im(k)} \downarrow & \downarrow k_X \\ Hom(X,B) \xrightarrow{g_X} Hom(X,D) & \xrightarrow{g'_X} \rightarrow & \downarrow k_X \downarrow \\ \end{array} \]

\( k_X, g_X \) factor through \( Im(k) \) since \( k_X \) is injective and \( f_X \) is surjective. Define diagonal \( d_X := (k'_X)^{-1} \circ g'_X \). Arrows \( d_X \) (\( X \) is a parameter) form natural transformation which can be seen from the diagram

\[ \text{So, apply the above lemma to the square} \]

\[ \begin{array}{ccc} C(-,Z \times Y) & \xrightarrow{(1 \times \rho_x) \circ (1,1 \times 1)} & C(-,Z \times Y) \\ \downarrow \exists \rho_{inv(x)} \downarrow & \xrightarrow{\rho_0(\rho_{inv(x)} \times i)} \downarrow & \downarrow \rho_0(\rho_{inv(x)} \times i) \\ C(-,Y) & \xrightarrow{i} & C(-,X) \\ \end{array} \]

(The top arrow is componentwise surjective since \((1 \times \rho_x) \circ (1,1 > 1) \circ t, s > = (1 \times \rho_x) \circ < t, t, s > = < t, s >\), and, so that, \( \forall < t, s > : T \rightarrow Z \times Y \exists \) its preimage \( t, s : T \rightarrow Z \times Y \) with \( t = m \) and \( s \) is a solution of the equation \( \rho_x \circ < t, s > = \) \( l \) [which exists because \( \rho_x \) is surjective in the second argument]. The bottom arrow is componentwise injective since \( i \) is monic.)

Consequently, \( \exists(!) \rho_{inv(x)} : Z \times Y \rightarrow Y \) such that \( inv(x) \times i \)

\[ \begin{array}{ccc} G 	imes X & \xrightarrow{\rho} & X \\ \downarrow Z \times Y & \xrightarrow{\rho_{inv(x)}} & \downarrow Y \\ \end{array} \]

\[ \text{i.e. if } x \in Z \text{ Stab}_Y \]

then \( inv(x) \in Z \text{ Stab}_Y. \)

Lemma 8.2.4. If \( L_g : X \xrightarrow{\sim} X \) is a left shift, and \( Y,Z \xrightarrow{\sim} X \) are subobjects of \( X \) such that an induced isomorphism \( \overline{L_g} : Y \xrightarrow{\sim} Z \) exists, i.e. the diagram \( \xrightarrow{inv} \)

\[ \begin{array}{ccc} Y & \xrightarrow{\sim} & Z \\ \overline{L_g} \end{array} \]

commutes, then \( \exists \)

an induced map (iso) \( \overline{L_g} \circ R_g^{-1} : \text{Stab}_Y \xrightarrow{\sim} \text{Stab}_Z \), corresponding to \( \overline{L_g} : Y \xrightarrow{\sim} Z \), such that the
Proof. The only difficulty is that the left side square commutes, namely,

\[
\begin{array}{c}
G \times X \\
\downarrow_{iZ \times iZ} \quad \downarrow_{(L_g \circ R_{g^{-1}}) \times L_g} \quad \downarrow_{(L_g \circ R_{g^{-1}}) \times L_g} \\
Stab_Y \times Y \\
\end{array}
\]

\[
\begin{array}{c}
\downarrow_{iZ} \\
\downarrow_{(L_g \circ R_{g^{-1}}) \times L_g} \quad \downarrow_{(L_g \circ R_{g^{-1}}) \times L_g} \\
Stab_Y \times Y \\
\end{array}
\]

Sufficient to show that \((L_g \circ R_{g^{-1}}) \circ j_Y \in \text{Stab}_Y\), i.e. that \((L_g \circ R_{g^{-1}}) \circ j_Y \in \text{Stab}_Y(\text{Stab}_Y)\), or that \(\exists \rho': \text{Stab}_Y \times Z \to Z\) such that

\[
((L_g \circ R_{g^{-1}}) \circ j_Y) \times (L_g \circ i_Y) = ((L_g \circ R_{g^{-1}}) \circ j_Y) \times (i_Z \circ \bar{T}_g) = (((L_g \circ R_{g^{-1}}) \circ j_Y) \times i_Z) \circ (1 \times \bar{T}_g),
\]

then \(\rho \circ (((L_g \circ R_{g^{-1}}) \circ j_Y) \times i_Z) \circ (1 \times \bar{T}_g) = i_Z \circ \bar{T}_g \circ \rho_J Y \circ (1 \times \bar{T}_g)^{-1}, \) and so, \(\rho \circ (((L_g \circ R_{g^{-1}}) \circ j_Y) \times i_Z) = i_Z \circ \bar{T}_g \circ \rho_J Y \circ (1 \times \bar{T}_g)^{-1}.\) Therefore, \(\forall x \in T\) \(L_g \circ R_{g^{-1}}(x) \in T \text{Stab}_Y\), i.e. \(\exists\) the induced map \(L_g \circ R_{g^{-1}}: \text{Stab}_Y \to \text{Stab}_Y.\)

\[\Box\]

Proposition 8.2.3.
• Two objects \((G, C(1, \text{Stab}_Y), \sigma_1)\) and \((G, C(1, \text{Stab}_Y), \sigma_2)\) from \(\text{C-Grp}\) (a category of right group actions on objects from \(C\)) are (equivariantly) isomorphic if \(\exists g \in C(1, G)\) and an induced isomorphism (as in lemma 2.7.2.4) \(\bar{T}_g : Y \to Z.\) The required isomorphism has form

\[
(G, C(1, \text{Stab}_Y), \sigma_1) \xrightarrow{(L_g \circ R_{g^{-1}}, \bar{T}_g \circ R_{g^{-1}}(0))} (G, C(1, \text{Stab}_Y), \sigma_2).
\]

• \(G/\text{Stab}_Y \simeq G/\text{Stab}_Z\) (if these quotients exist).

Proof.
• It is necessary to prove that \(\forall g : 1 \to G\) and \(k : 1 \to \text{Stab}_Y\) \(L_g \circ R_{g^{-1}} \circ R_k = R_{g^{-1}} \circ R_k \circ L_g \circ R_{g^{-1}}.\)

It follows from two facts \(R_{g^{-1}} = R_{g^{-1}} \circ R_k \circ R_{g^{-1}}\) (antihomomorphism) and commutativity of left and right shifts \(L_{g1} \circ R_{g2} = R_{g2} \circ L_{g1}\) [the last fact follows from associativity axiom \(\forall t, r, s >: T \to G \times G \times G\) \(m(t, m(r, s)) = m(m(t, r), s)\), and so, \(L_{g1} \circ R_{g2} \circ t = m(g_1 \circ t, m(g_2 \circ t)) = m(g_1 \circ t, g_2 \circ t) = R_{g2} \circ L_{g1} \circ t.\)]

• Isomorphic objects in \(\text{C-Grp}\) have isomorphic quotients in \(C\) since \(Q : \text{C-Grp} \to C\) is a functor. So, \(Q(G, C(1, \text{Stab}_Y), \sigma_1) \simeq Q(G, C(1, \text{Stab}_Y), \sigma_2).\)

\[\Box\]

Definition 8.2.3. An object \(X \in \text{Ob} C\) with a group action \(\rho : G \times X \to X\) such that \(\forall x : 1 \to X\) both \(\text{Stab}_x\) and \(G/\text{Stab}_x\) exist, and \(G/\text{Stab}_x\) is universal, is called homogenous if \(\exists\)
Proof. 

\[ G \times (G/\text{Stab}_x) \xrightarrow{\rho'} G/\text{Stab}_x \]

an isomorphism \( f : G/\text{Stab}_x \xrightarrow{\sim} X \) such that 

\[ 1 \times f \sim f \quad \text{(for an } x : 1 \to X) \]

\[ G \times X \xrightarrow{\rho} X \]

where \( \rho' \) is defined from 

\[ G \times (G/\text{Stab}_x) \xrightarrow{\sim} G/\text{Stab}_x \]

where \( \rho' \) is defined from 

\[ 1 \times p \xrightarrow{\sim} \exists ! \rho' \to G/\text{Stab}_x \]

(1 \times p and \( p \) are quotient maps). □

Proposition 8.2.4. If \( X \) is a homogenious object (with \( G \)-action \( \rho : G \times X \to X \)) and \( C(1,p) : C(1,G) \to C(1,\text{G}/\text{Stab}_x) \) is surjective, where \( G \xrightarrow{p} G/\text{Stab}_x \) is a quotient map, then

- \( C(1,G) \) acts transitively on \( C(1,X) \), i.e. \( \forall x, y : 1 \to X \exists g : 1 \to G \) such that \( y = L_g \circ x \),
- definition of homogenious object \( X \xrightarrow{\sim} G/\text{Stab}_x \) does not depend on the choice of \( x : 1 \to X \).

Proof. 

\[ \forall a', b' : 1 \to G/\text{Stab}_x \exists a, b : 1 \to G \text{ s.t. } pa = a', pb = b', \text{ and } \exists g : 1 \to G \text{ s.t. } b = L_g a. \]

By proposition 8.1.2, \( L'_g(a') = L_g(pa) = pL_g(a) = pb = b' \) (where \( L'_g \) is the induced left shift on \( C(1,\text{G}/\text{Stab}_x) \)). So, \( C(1,G) \) acts transitively on set \( C(1,\text{G}/\text{Stab}_x) \), and consequently, on \( C(1,X) \).

\[ \begin{array}{c}
\xymatrix{ G \times G \ar[r]^m \ar[d]^{1 \times p_x} & G \ar[d]^{p_y} \\
G \times (\text{G}/\text{Stab}_x) \ar[r]^{\rho'} \ar[d]^{1 \times f} & \text{G}/\text{Stab}_x \ar[d]^{\sim} \\
G \times X & \text{X} \ar[l]_\rho \}
\end{array} \]

• Regard the diagram

\[ \alpha \text{ exists as a mediating arrow because } f \circ p_x \text{ is a quotient map of } (G, C(1,\text{Stab}_x), \sigma_1), \text{ and } p_y \circ (L_g \circ R_{g^{-1}}) \text{ coequalizes } \sim_{\sigma_1} \text{-equivalent arrows } (L_g \circ R_{g^{-1}} \text{ is equivariant, and } p_y \text{ is a quotient map of } (G, C(1,\text{Stab}_y), \sigma_2)), \text{ essentially, } \alpha = Q(L_g \circ R_{g^{-1}}). \text{ The bottom square commutes because } (1 \times f) \circ (1 \times p_x) \text{ is a quotient map, and so, epi} \]

\[ \begin{array}{c}
\xymatrix{ G \times (\text{G}/\text{Stab}_y) \ar[r]^{\rho''} \ar[d]^{(L_g \circ R_{g^{-1}}) \times \alpha} & \text{G}/\text{Stab}_y \ar[d]^\sim \\
G \times X \ar[r]^p \ar[d]^{(L_g \circ R_{g^{-1}}) \times L_{g^{-1}}} & X \ar[d]^{L_{g^{-1}}} \\
G \times X & X \ar[l]_\rho \}
\end{array} \]

Therefore, \( \sim \) \( 1 \times (\alpha \circ L_{g^{-1}}) \)

required isomorphism (by definition 8.2.3). □
Bibliography

[A-H-S] J. Adamek, H. Herrlich, G.E. Strecker, *Abstract and Concrete Categories. The Joy of Cats*, online edition, 2004.

[A-V-L] D.V. Alekseevskiy, A.M. Vinogradov, V.V. Lychagin, *Main Ideas and Concepts of Differential Geometry*, Moscow, 1988. (Russian)

[Bi-Cr] R.L. Bishop, R.J. Crittenden, *Geometry of Manifolds*, Academic Press, New York and London, 1964.

[Bor1] F. Borceux, *Handbook of Categorical Algebra 1. Basic Category Theory*, Cambridge University Press, 1994.

[Bor2] F. Borceux, *Handbook of Categorical Algebra 2. Categories and Structures*, Cambridge University Press, 1994.

[Bor3] F. Borceux, *Handbook of Categorical Algebra 3. Categories of Sheaves*, Cambridge University Press, 1994.

[Bru] U. Bruzzo, *Introduction to Algebraic Topology and Algebraic Geometry*, International School for Advanced Studies, Trieste, 2002.

[C-C-L] S.S. Chern, W.H. Chen, K.S. Lam, *Lectures on Differential Geometry*, World Scientific, 2000.

[D-N-F] B.A. Dubrovin, S.P. Novikov, A.T. Fomenko, *Modern Geometry*, Moscow, 1979. (Russian)

[Eng] R. Engelking, *General Topology*, 1977.

[ELOS] L.E. Evtushik, U.G. Lumiste, N.M. Ostianu, A.P. Shirokov, *Differential Geometric Structures on Manifolds*, Moscow, 1979. (Russian)

[Hir] F. Hirzebruch, *Topological Methods in Algebraic Geometry*, Springer-Verlag, 1966.

[Jac] B. Jacobs, *Categorical Logic and Type Theory*, Elsevier, North-Holland, 2001.

[Kel] G.M. Kelly, *Basic Concepts of Enriched Category Theory*, Reprints in TAC, No. 10, 2005.

[Kom] B.P. Komrakov, *Structures on Manifolds and Homogeneous Spaces*, Minsk, 1978. (Russian)

[Mac] S. MacLane, *Categories for the Working Mathematician*, Springer-Verlag, 1971.

[M-M] S. MacLane, I. Moerdijk, *Sheaves in Geometry and Logic*, Springer-Verlag, 1992.

[May] J.P. May, *A Concise Course in Algebraic Topology*, The University of Chicago Press, 1999.

[Moe] I. Moerdijk, *Introduction to the Language of Stacks and Gerbes*, University of Utrecht, arXiv:math.AT/0212266v1, 2002.

[Nar] R. Narasimhan, *Analysis on Real and Complex Manifolds*, North-Holland, 1968.

[Nes] J. Nestruev, *Smooth manifolds and observables*, Moscow, 2003. (Russian)

[Str] T. Streicher, *Fibred Categories a la Jean Benabou*, online lecture notes, 1999.

[Strt] R. Street, *Categorical and Combinatorial Aspects of Descent Theory* (2003), talk at ICIAM.

[Vis] A. Vistoli, *Notes on Grothendieck Topologies, Fibered Categories and Descent Theory*, Bologna, Italy, online lecture notes.