SPECTRAL CURVES FOR SUPER-YANG-MILLS WITH ADJOINT
HYPERMULTIPLET FOR GENERAL LIE ALGEBRAS *

Eric D’Hoker\textsuperscript{1} and D.H. Phong\textsuperscript{2}

\textsuperscript{1} Department of Physics
University of California, Los Angeles, CA 90024, USA;
Institute for Theoretical Physics
University of California, Santa Barbara, CA 93106, USA

\textsuperscript{2} Department of Mathematics
Columbia University, New York, NY 10027, USA

ABSTRACT

The Seiberg-Witten curves and differentials for $\mathcal{N} = 2$ supersymmetric Yang-Mills theories with one hypermultiplet of mass $m$ in the adjoint representation of the gauge algebra $\mathcal{G}$, are constructed for arbitrary classical or exceptional $\mathcal{G}$ (except $G_2$). The curves are obtained from the recently established Lax pairs with spectral parameter for the (twisted) elliptic Calogero-Moser integrable systems associated with the algebra $\mathcal{G}$. Curves and differentials are shown to have the proper group theoretic and complex analytic structure, and to behave as expected when $m$ tends either to 0 or to $\infty$. By way of example, the prepotential for $\mathcal{G} = D_n$, evaluated with these techniques, is shown to agree with standard perturbative results. A renormalization group type equation relating the prepotential to the Calogero-Moser Hamiltonian is obtained for arbitrary $\mathcal{G}$, generalizing a previous result for $\mathcal{G} = SU(N)$. Duality properties and decoupling to theories with other representations are briefly discussed.

* Research supported in part by the National Science Foundation under grants PHY-95-31023, PHY-94-07194 and DMS-95-05399.
I. INTRODUCTION

Substantial evidence has accumulated in support of the connection between the structure of the low energy effective action of $\mathcal{N} = 2$ super-Yang-Mills theory in four dimensions [1] and certain integrable systems [2,3,4,5,6,7]. For reviews, see [8]. Compelling arguments were given on general grounds that this connection should hold true [4]. In particular, the Hitchin system [9] was proposed for $\mathcal{N} = 2$ super-Yang-Mills theory with gauge algebra $SU(N)$ and one massive hypermultiplet in the adjoint representation of $SU(N)$, and the Seiberg-Witten curve and differential [4] were naturally obtained from it. A possible relation between the spectral curves arising from the Hitchin system and those associated with the elliptic Calogero-Moser systems [10,11,12] was suggested in [5] and established for the $SU(N)$ gauge algebra by Krichever in unpublished work.

In a recent paper [7], we showed that the Calogero-Moser integrable system indeed captures the physics of the low energy dynamics of $\mathcal{N} = 2$ supersymmetric Yang-Mills theory with gauge algebra $SU(N)$ and with one hypermultiplet of mass $m$ in the adjoint representation of $SU(N)$. We checked the perturbative contributions and evaluated 1- and 2-instanton corrections with the help of a renormalization group type equation which was also established in [7] (see also [13]). Decoupling the full hypermultiplet by letting $m \rightarrow \infty$ while keeping the vacuum expectation values of the gauge scalar fixed reproduced the gauge theory without hypermultiplet [14]. By letting the mass $m$, as well as some of the vacuum expectation values of the gauge scalar tend to $\infty$ while tuning their differences, we recovered the gauge theory with massive hypermultiplets in the fundamental representation of $SU(N)$ [15]. In special cases, it was possible to get product unitary gauge groups with hypermultiplets in fundamental and bi-fundamental representations, of the type solved by Witten using branes, string theory and M-theory [16].

In the present paper, we propose Seiberg-Witten curves and associated differentials in terms of elliptic Calogero-Moser systems for $\mathcal{N} = 2$ supersymmetric Yang-Mills theories with arbitrary gauge algebra $G$ (except $G_2$), and with one hypermultiplet of mass $m$ in the adjoint representation of the gauge algebra. The precise correspondence is with the ordinary elliptic Calogero-Moser system when $G$ is simply laced, and with the twisted Calogero-Moser system when $G$ is non-simply laced. The latter was introduced in a companion paper [17], and both will be reviewed below. The modulus $\tau$ of the elliptic curve $\Sigma$ that underlies the Calogero-Moser systems is given in terms of the super-Yang-Mills coupling $g$ and theta angle $\theta$ by

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}.$$  (1.1)
In view of the ultra-violet finiteness of these theories, this coupling is well-defined. In terms of the Lax operators $L(z)$ and $M(z)$ for the (twisted) elliptic Calogero-Moser system, with spectral parameter $z \in \Sigma$, the curve and the differential take the form

$$\Gamma : R(k, z) = \det(kI - L(z)) = 0,$$
$$d\lambda = kdz.$$  \hspace{1cm} (1.2)

Until now, the construction of the Seiberg-Witten curve and differential from the elliptic Calogero-Moser system could be carried out only for $G = SU(N)$, (as in [7]) since it is only for $G = SU(N)$ that the relevant Lax pair with spectral parameter was known [11]. For $G \neq SU(N)$, the situation was as follows. For classical Lie algebras $G$, Lax operators without spectral parameter for the elliptic Calogero-Moser systems were discovered long ago [12]. However, from the very proposal of the spectral curves in (1.2), it is clear that Lax pairs with spectral parameter are needed. Though a Lax pair with a free parameter was introduced by Inozemtsev [18], its dependence on this parameter appears unsuited for Seiberg-Witten theory. For exceptional Lie algebras, no Lax pairs (with or without spectral parameter) appear to be known at all. (See e.g. [28] for a recent discussion.)

In a first companion paper [17], we give an explicit and systematic construction of the Lax pairs with spectral parameter for the ordinary (and twisted) elliptic Calogero-Moser systems, associated with any simple Lie algebra $G$, including exceptional ones. In a second companion paper [19], we show that under certain scaling behaviors of $\tau$ and $m \to \infty$, the ordinary (resp. twisted) Calogero-Moser systems tend towards Toda systems associated with the untwisted affine Lie algebras $G^{(1)}$ (resp. dual affine Lie algebras $(G^{(1)})^{\vee}$). Using the results of [17] and of [19], we shall argue in the present paper that the spectral curves constructed from the elliptic Calogero-Moser systems indeed generate the Seiberg-Witten curves for super-Yang-Mills theory with one massive adjoint hypermultiplet. The crucial requirements on the curve and differential that we shall check here are as follows.

(a) The curve $R(k, z) = 0$ must be invariant under the action of the Weyl group of $G$ and have appropriate analytic behavior;
(b) The differential $d\lambda$ must be meromorphic on the curve $R(k, z) = 0$, with poles independent of moduli, and residues linear in the hypermultiplet mass $m$.
(c) In the $m = 0$ limit, the curve and differential must reduce to those of the $N = 4$ supersymmetric gauge theory which receives no quantum corrections;
(d) In the $m \to \infty$ limit, while keeping vacuum expectation values of the gauge scalars fixed, and running the gauge coupling according to the renormalization group, the
curve and differential for the super-Yang-Mills theory without hypermultiplet must be recovered.

The remainder of this paper is organized as follows. In §II, we present the Hamiltonians and Lax pairs of the ordinary and twisted Calogero-Moser systems, obtained in [17]. In §III, we justify the Seiberg-Witten curves and differential, by checking points (a), (b), (c), (d) and (e), discussed above, using the results of [19]. In §IV, we obtain by way of example, the spectral curve for $G = D_n$ in the weak coupling limit, and show that the effective prepotential obtained in this limit agrees with the standard result from quantum field theory. In §V, we present a number of applications. We generalize to the case of an arbitrary Lie algebra $G$ the renormalization group type equation established previously for $SU(N)$ in [7], in which the variation of the prepotential with the gauge coupling in terms of the Calogero-Moser Hamiltonian. We discuss the duality properties for various $G$. We indicate how curves for $N = 2$ super-Yang-Mills theories with hypermultiplets in certain other representations of the gauge algebra $G$ may be obtained from the adjoint hypermultiplet case by suitable decoupling limits.

Finally, we point out that Seiberg-Witten curves have also been derived by using string theory methods. One method is by exploiting the appearance of enhanced gauge symmetries at certain singular compactifications. (See for example [20].) A second method is by obtaining supersymmetric Yang-Mills theory as an effective theory on a configuration of branes in string theory or M-theory. This approach was pioneered in [16], (see also [21]) for $SU(N)$ gauge group (and products thereof). The extension to other classical groups is discussed in [22]. Relations between the string theory and M-theory approaches and integrable systems were proposed in [16,23].

II. CALOGERO-MOSER HAMILTONIANS AND LAX OPERATORS

The elliptic Calogero-Moser integrable systems associated with a finite-dimensional simple Lie algebra $G$ of rank $n$, involve $n$ complex degrees of freedom $x_i$ and their canonical momenta $p_i$, $i = 1, \cdots, n$. The Hamiltonian is defined by

$$H = \frac{1}{2} p \cdot p - \sum_{\alpha \in R(G)} \frac{1}{2} m_{|\alpha|}^2 \varphi_{\nu(\alpha)}(\alpha \cdot x).$$

(2.1)

Here, $R(G)$ is the set of all (non-zero) roots of $G$, and $m_{|\alpha|}$ are complex constants dependent only on the Weyl orbit $|\alpha|$. The $\varphi_{\nu}$ are Weierstrass functions defined by

$$\varphi_{\nu}(u) = \sum_{l=0}^{\nu-1} \varphi(u + 2 \omega_a \frac{l}{D}).$$

(2.2)
where \( \omega_a \) is one of the half periods \( \omega_1, \omega_2 \) or \( \omega_3 = \omega_1 + \omega_2 \); for definiteness, we choose \( \omega_1 \).

Finally, the function \( \nu(\alpha) \) enters as follows.

1. The ordinary Calogero-Moser system is defined by \( \nu(\alpha) = 1 \) for all roots of \( \mathcal{G} \).
2. The twisted Calogero-Moser system is defined for non-simply laced \( \mathcal{G} \) by
   - \( \nu(\alpha) = 1 \) for all long roots of \( \mathcal{G} \);
   - \( \nu(\alpha) = 2 \) for all short roots of \( B_n, C_n \) and \( F_4 \);
   - \( \nu(\alpha) = 3 \) for all short roots of \( G_2 \).
3. For simply laced \( \mathcal{G} \), the twisted and ordinary Calogero-Moser systems coincide.

The elliptic Calogero-Moser Hamiltonians are completely integrable in the sense that there exists a Lax pair of \( N \times N \) dimensional matrix valued functions of \( x \) and \( p \), denoted by \( L \) and \( M \), such that the Hamilton-Jacobi equations of the elliptic Calogero-Moser system on \( x \) and \( p \) are equivalent to the Lax equation, given by

\[
\dot{L} = [L, M].
\]  

Integrability follows from the existence of the Lax equation, since the latter automatically guarantees that the quantities \( \text{tr}L^{\gamma+1} \) are conserved integrals of motion for \( \gamma = 0, \cdots, \infty \).

In fact, in [17] a stronger result was shown to hold. There exists a pair of Lax operators \( L(z) \) and \( M(z) \) which depend upon a spectral parameter \( z \), and which are such that the Lax equation (2.3) is equivalent to the elliptic Calogero-Moser Hamiltonian-Jacobi equations for arbitrary value of \( z \). The Lax operators are not unique since the Lax equation is invariant under the following gauge transformations by an arbitrary \( N \times N \) matrix-valued function \( S \) of \( x \), \( p \) and \( z \),

\[
L \to L^S = SLS^{-1}
\]

\[
M \to M^S = SMS^{-1} - \dot{S}S^{-1}.
\]  

In particular, the action of the Weyl group \( W_\mathcal{G} \) of \( \mathcal{G} \) on the operators \( L \) and \( M \) is realized in terms of such a transformation. As a result, the spectral curve, defined in (1.1) is invariant under the Weyl group \( W_\mathcal{G} \) and under time evolution.

We shall now summarize the final expressions for the Lax operator \( L(z) \) obtained in [17].* The form of the operator \( M(z) \) will not be needed for our purposes, since it does not enter into the form of the curve or of the differential in (1.1). We choose a Cartan subalgebra \( \mathcal{H}_\mathcal{G} \) of \( \mathcal{G} \), with generators \( h_j, j = 1, \cdots, n \), conveniently assembled into a vector of generators, denoted by \( h \). The Lie algebra \( \mathcal{G} \) is embedded into \( GL(N, \mathbb{C}) \) by via an

---

* Some key facts about Lie algebra theory are collected in the Appendix §A of [17]; useful general sources are in [24].
$N$-dimensional representation $\Lambda$ with weight vectors $\lambda_I$, $I = 1, \cdots, N$. We denote the generators of $GL(N, \mathbb{C})$ by $E_{IJ}$ with $I, J = 1, \cdots, N$. The general form of $L(z)$ for both ordinary and twisted Calogero-Moser systems is then given by

$$L(z) = p \cdot h + \sum_{I,J=1;I\neq J}^{N} C_{I,J} \Phi_{IJ}((\lambda_I - \lambda_J) \cdot x, z) E_{IJ}. \quad (2.5)$$

We shall establish in §III below that the systems relevant to super-Yang-Mills dynamics for gauge algebra $\mathcal{G}$ are the ordinary (resp. twisted) Calogero-Moser system for simply laced $\mathcal{G}$ (resp. non-simply laced $\mathcal{G}$). We list the entries $C_{I,J}$ and $\Phi_{IJ}$ separately for each case.

(a) Simply Laced $\mathcal{G}$ : Ordinary Calogero-Moser Systems

The elliptic functions $\Phi_{IJ}$ are independent of $I$ and $J$ and given by [17], (see also [11])

$$\Phi_{IJ}(u, z) = \Phi(u, z) \equiv \sigma(z - u) / \sigma(z) \sigma(u) e^{u \zeta(z)}, \quad (2.6)$$

where $\sigma(u)$ and $\zeta(u)$ are the standard Weierstrass functions* satisfying $\sigma(u) = u + O(u^5)$, $\zeta(u) = \sigma'(u)/\sigma(u)$ and $\zeta'(u) = -\varphi(u)$. To define the constants $C_{I,J}$, we fix the representations of $\mathcal{G}$ to those of smallest (non-trivial) dimension. For $\mathcal{G} = A_n, D_n, E_6, E_7$ we have respectively $N = n+1, 2n, 27, 56$. Each of these representations only has non-zero weights $\lambda$, which belong to a single orbit of the Weyl group $W_{\mathcal{G}}$. It is very convenient to replace the labels $I$ and $J$ in (2.5) by the $N$ weight vectors of the representation. The precise mapping between the labels $I$ and $\lambda$ is immaterial, since they will be permuted under the action of the Weyl group. We have for $\mathcal{G} = A_n, D_n, E_6, E_7$ the following expressions [DP]

$$C_{I,J} = C_{\lambda,\mu} = \begin{cases} m & \lambda - \mu \in \mathcal{R}(\mathcal{G}) \\ 0 & \lambda - \mu \notin \mathcal{R}(\mathcal{G}) \end{cases} \quad (2.7)$$

For $\mathcal{G} = E_8$, we have $N = 248$. This representation contains 240 non-zero weights $\lambda$ (which are roots) and 8 zero weights. It is convenient to replace the 248 labels $I$ and $J$ by the 240 nonzero weights $\lambda$ and $\mu$ and by $a, b = 1, \cdots, 8$ which label the zero weights. We have then for $\mathcal{G} = E_8$

$$C_{a,b} = 0 \quad a, b = 1, \cdots, 8$$

$$C_{\lambda,\mu} = \begin{cases} m c(\lambda, \mu) & \lambda - \mu \in \mathcal{R}(E_8) \\ 0 & \lambda - \mu \notin \mathcal{R}(E_8) \end{cases} \quad (2.8)$$

$$C_{\lambda,a} = \frac{1}{2} m \sum_{b=1}^{8} c(\lambda, \beta_b) O_{\beta_b,a}.$$  

* A useful source for information on elliptic functions is [25].
Here, \( \beta_b, b = 1, \ldots, 8 \) is a choice of 8 (which is the maximum number) mutually orthogonal roots of \( E_8 \), corresponding to the maximal subalgebra \([A_1]^8\) of \( E_8 \). The \( 8 \times 8 \) matrix \( O_{\beta_b,a} \) is an arbitrary orthogonal matrix. The functions \( c(\lambda, \mu) \) take values \( \pm 1 \) only, and are defined by a set of equations discussed in [17].

(b) Non-Simply Laced \( G \) : Twisted Calogero-Moser Systems

To define the functions \( \Phi_{IJ} \) and the constants \( C_{I,J} \), we fix the representations to be of smallest dimension. For non-simply laced \( G = B_n, C_n, F_4 \), we have respectively \( N = 2n, 2n + 2, 24 \). For \( G = G_2 \), only partial results on the existence of a Lax pair could be obtained in [17]; as a result, we shall refrain from discussing this case here. Several different functions now enter into (2.5),

\[
B_n \quad \Phi_{IJ}(x, z) = \begin{cases} 
\Phi(x, z) & I - J \neq 0, \pm n \\
\Phi_2(x, z) & I - J = \pm n
\end{cases}
\]

\[
C_n \quad \Phi_{IJ}(x, z) = \Phi(x + \omega_{IJ}, z)
\]

\[
F_4 \quad \Phi_{\lambda\mu}(x, z) = \begin{cases} 
\Phi(x, z) & \lambda \cdot \mu = 0 \\
\Phi_1(x, z) & \lambda \cdot \mu = \frac{1}{2} \\
\Phi_2(x, z) & \lambda \cdot \mu = -1
\end{cases}
\]

The new functions are defined in terms of the fundamental function \( \Phi(x, z) \) of (2.6) by the following relations. For more details, see [17].

\[
\Phi_1(x, z) = \Phi(x, z) - \Phi(x + \omega_1, z)e^{\pi i \zeta(z) + z \zeta(\omega_1)}
\]

\[
\Phi_2(x, z) = \Phi(x, z)\Phi(x + \omega_1, z)\Phi(\omega_1, z)^{-1}.
\]

The constants \( C_{I,J} \) are given by

\[
B_n \quad C_{I,J}(x, z) = \begin{cases} 
m \sqrt{2m_1} & I - J \neq 0, \pm n \\
m & I - J = \pm n
\end{cases}
\]

\[
C_n \quad C_{I,J} = \begin{cases} 
m \sqrt{2m} & I, J = 1, \ldots, 2n; I - J \neq \pm n \\
m & I = 1, \ldots, 2n, J = 2n + 1, 2n + 2; I \leftrightarrow J \\
2m & I = 2n + 1, J = 2n + 2; I \leftrightarrow J
\end{cases}
\]

\[
F_4 \quad C_{\lambda\mu} = \begin{cases} 
m \sqrt{2m_1} & \lambda \cdot \mu = 0 \\
m & \lambda \cdot \mu = -\frac{1}{2} \\
1 \sqrt{2m_1} & \lambda \cdot \mu = \frac{1}{2} \\
2m_1 & \lambda \cdot \mu = -1
\end{cases}
\]

The cocycle shifts \( \omega_{IJ} \) are given by

\[
\omega_{IJ} = \begin{cases} 
+\omega_2 & I = 1, \ldots, 2n + 1; J = 2n + 2, \\
-\omega_2 & J = 1, \ldots, 2n + 1; I = 2n + 2, \\
0 & \text{otherwise}
\end{cases}
\]
We note here that the twisted Calogero-Moser systems for $B_n$ and $F_4$ involve two independent Calogero-Moser couplings $m$ and $m_1$. We shall discuss their physical significance at the end of §III. (b).

**III. CURVES, DIFFERENTIALS FROM CALOGERO-MOSER**

Given the Lax operators for the ordinary Calogero-Moser systems associated with simply laced Lie algebras $G$, and of the twisted Calogero-Moser systems associated with non-simply laced Lie algebras $G$, our proposal for the Seiberg-Witten curves $\Gamma$ and differentials is

$$\Gamma : R(k, z) = \det(kI - L(z)) = 0$$

$$d\lambda = kd\lambda.$$  

(3.1)

The quantum order parameters $a_i$, their dual $a_{Di}$, $i = 1, \cdots, n$ and the prepotential $F$ are then defined by

$$a_i = \frac{1}{2\pi i} \oint_{A_i} d\lambda, \quad a_{Di} = \frac{1}{2\pi i} \oint_{B_i} d\lambda, \quad \frac{\partial F}{\partial a_i} = a_{Di}. \quad (3.2)$$

Here, the cycles $A_i$ and $B_i$, $i = 1, \cdots, n$ are constructed below. We shall now carry out the consistency checks, discussed in the introduction.

(a) **Analytic behavior, Weyl Invariance, Counting Moduli**

- The function $R(k, z) = \det(kI - L(z))$ is polynomial in $k$ and meromorphic as a function of $z$, despite the fact that the entries $L_{IJ}(z)$ of the matrix $L(z)$ themselves have essential singularities, as can be seen from the very definition of $\Phi$ in (2.6). In fact, the expression for $L(z)$ in (2.5) shows that conjugation of $L(z)$ by the diagonal matrix $S$ with components

$$S_{IJ}(z) = \delta_{IJ} e^{\lambda_I x(z)}$$

(3.3)

leads to an operator $L^S(z)$, defined by (2.4) with meromorphic entries. For simply laced $G$, this follows from the fact that only $\Phi$ enters (see (2.6)), and from the fact that under the transformation (3.3), $p$ is unchanged and the function $\Phi(u, z)$ is effectively replaced by the meromorphic function

$$\tilde{\Phi}(u, z) = \frac{\sigma(z - u)}{\sigma(z)\sigma(u)}. \quad (3.4)$$

For non-simply laced $G$, it follows from their definition in (2.10) that the functions $\Phi_1(u, z)$, $\Phi_2(u, z)$ have the same essential singularity as $\Phi(u, z)$ itself. Thus, the argument presented for simply laced $G$ then also holds for non-simply laced $G$.  

8
• The Weyl group $W_G$ is generated by Weyl reflections $W_\alpha$, $\alpha \in \mathcal{R}(G)$, which act on $x$, and $p$ in the standard way

$$x \rightarrow W_\alpha(x) = x - 2\alpha \frac{x \cdot \alpha}{\alpha_2}, \quad p \rightarrow W_\alpha(p).$$  \hfill (3.5)

The action of $W_\alpha$ preserves the inner product, $W_\alpha(x) \cdot W_\alpha(y) = x \cdot y$. Thus, a Weyl reflection on $x$ and $p$ in $L(z)$ may be recast in terms of the action of the Weyl reflection on the weights $\lambda_I$ and on the Cartan generators $h$. This action is given by

$$W_\alpha(\lambda_I) = \sum_{J=1}^{N} (S_\alpha)_{IJ} \lambda_J$$

$$W_\alpha(h) = S_\alpha h S_\alpha^{-1}$$

$$W_\alpha(L(z)) = S_\alpha L(z) S_\alpha^{-1},$$  \hfill (3.6)

where $S_\alpha$ is a permutation matrix with entries $(S_\alpha)_{IJ}$ defined by the first line in (3.6).

Thus, the action of the Weyl group on $L(z)$ is simply by conjugation, and the spectral curve (3.1) is invariant under $W_G$.

• The curves $R(k, z) = 0$ are expected to depend upon precisely $n$ complex moduli, which are the independent integrals of motion of the Calogero-Moser system. Let us briefly explain why. Each Lax operator $L(z)$ depends upon all of the $2n$ degrees of freedom $x_i$ and $p_j$, $i, j = 1, \cdots, n$, with a non-degenerate Poisson bracket $\{x_i, p_j\} = \delta_{ij}$. The quantities $\text{tr} L(z)^{\gamma+1}$ are all integrals of motion for $\gamma = 0, \cdots, \infty$. On general grounds, at most $n$ of these can be functionally independent. By taking the $m \rightarrow 0$ limit, one establishes that precisely $n$ values of $\gamma = \gamma_i, i = 1, \cdots, n$ yield functionally independent integrals of motion, with the $\gamma_i$ corresponding to the exponents of the Lie algebra $\mathcal{G}$, as given in [1], Appendix A, Table 4. By continuity in $m$, these integrals of motion are also expected to be mutually independent for $m \neq 0$. Thus, we have precisely $n$ functionally independent integrals of motion for all $m$.

Using the fact that time evolution acts by conjugation on $L(z)$, we immediately derive from (3.1) that the function $R(k, z)$ is conserved under time evolution,

$$\frac{d}{dt} R(k, z) = \{H, R(k, z)\} = 0.$$  \hfill (3.7)

Thus, $R(k, z)$ must be a function of only the $n$ independent integrals of motion $\text{tr} L(z)^{\gamma_i+1}$, which in super-Yang-Mills theory play the role of moduli, parametrizing the supersymmetric vacua of the gauge theory. In the case of the elliptic Calogero-Moser system for
\( \mathcal{G} = SU(n + 1) \), this result was further confirmed in [7], where the explicit dependence upon the \( n \) integrals of motion was exhibited explicitly, as will be discussed also in \( \S \) IV below.

(b) Meromorphicity of \( d\lambda \), pole structure

- Meromorphicity of the Seiberg-Witten differential \( d\lambda = kdz \) is readily established, once it is realized that the spectral curve may be expressed as \( R(k, z) = \det(kI - L^S(z)) \), where \( S \) was defined in (3.3) and the entries of \( L^S(z) \) are meromorphic functions of \( z \).

- A simple pole in \( z \) arises in \( L^S(z) \) when \( z = 0 \), or more generally when \( z \) approaches \( z_P = 2\omega_1n_1/\nu + 2\omega_2n_2 \), for \( n_1, n_2 \in \mathbb{Z} \). The behavior at these poles is readily read off from the behavior of \( L^S(z) \), which may be derived from the structure of \( L(z) \) in (2.5) and from (3.3). We find

\[
L^S(z) = -\frac{C_{I,J}}{z - z_P} + \text{regular terms.} \tag{3.8}
\]

From the explicit expressions given in (2.8), (2.9) and (2.11) for the constants \( C_{I,J} \), it is clear that

(i) the position and the residues of the poles are independent of the moduli,

(ii) the residues of the poles are linear in the hypermultiplet mass \( m \).

This shows that the residues of \( k \) are linear functions of \( m \). Note however that it is more difficult to determine their exact values for general \( \mathcal{G} \) than it was for the \( SU(N) \) case, when they were all \(-m\), except for the last coefficient which is \((N - 1)m\).

There is one further important issue concerning the mass of the adjoint hypermultiplet that still needs to be addressed. The Calogero-Moser systems for simply laced \( \mathcal{G} \), as well the twisted Calogero-Moser system for \( \mathcal{G} = C_n \) involve only a single Calogero-Moser coupling \( m \), as in (2.7), (2.8) and (2.11b), and this parameter is identified with the mass of the hypermultiplet in the adjoint representation of \( \mathcal{G} \) since it arises as a residue of a pole of the Seiberg-Witten differential by (3.8). Remarkably, the twisted Calogero-Moser systems for \( B_n \) and \( F_4 \), (and as far as we know also \( G_2 \)) involve two Calogero-Moser couplings \( m \) and \( m_1 \).

Let us begin by discussing the case of \( \mathcal{G} = B_n \). From considering the subset of roots of \( B_n \) associated with the subalgebra \( D_n \), it follows immediately that the coupling \( m \) in (2.11a) is exactly the mass of the adjoint hypermultiplet. Understanding the role of \( m_1 \) is slightly more delicate. At the level of the integrable system, this coupling is à priori unrelated to \( m \). At the level of Seiberg-Witten theory however, there can be only a single mass parameter for the adjoint hypermultiplet, since the latter transforms under a single
irreducible representation of the gauge algebra $\mathcal{G}$. Given that the residue of the pole at the half period $z = \omega_1$ is linear in $C_{1,j}$, in view of (3.8), and takes the value $m_1$, we see that by the general arguments of Seiberg-Witten theory, $m_1$ must be linear in $m$, the hypermultiplet mass. The precise coefficient does not appear to be determined from Calogero-Moser dynamics.

The ratio $m_1/m$ may be fixed by comparing for example the one loop contribution to the prepotential, obtained from standard field theory methods, with the result derived from the Calogero-Moser approach. It is likely that the special value selected this way by Seiberg-Witten theory corresponds to a point of enhanced symmetry at the level of the classical Calogero-Moser system. At present, we do not know what that symmetry might be, and leave this issue open for further investigation.

(c) **Structure of Homology Cycles**

Here, we specify a set of homology cycles $A_i$ and $B_i$, $i = 1, \cdots, n$, for the spectral curve $\Gamma$, (where $n$ is the rank of $\mathcal{G}$), to be used in the evaluation of the quantum order parameters and effective prepotential in (3.2).

Let $A$ and $B$ be a canonical basis of homology cycles on the base elliptic curve $\Sigma$, and let $A$ be the cycle which shrinks to a point when $\tau \to +i\infty$. The spectral curve $\Gamma$ is obtained by gluing along certain cuts $N$ copies of $\Sigma$. In the limit $m \to 0$, $L(z) = p \cdot h$ admits $N$ constant eigenvalues of which $n$ are linearly independent. Select the $n$ copies of $\Sigma$ corresponding to such a maximal set of linearly independent eigenvalues. The desired $A_i$ and $B_i$ cycles, $i = 1, \cdots, n$ are obtained by lifting to these sheets the $A$ and $B$ cycles of the base torus $\Sigma$.

This prescription has been shown to reproduce the correct prepotential in the case of $SU(N)$ in [7]. We shall show below, explicitly, that it is also appropriate for $D_n$. See also [3] for a prescription of Prym varieties when the spectral curve arises from a group theoretic gluing of several copies of the sphere.

(d) **The limit $m \to 0$ to $\mathcal{N} = 4$ super-Yang-Mills**

In the $m \to 0$ limit, the effective prepotential of the Calogero-Moser system should reproduce the classical metric $ds^2 = (\text{Im} \, \tau) \sum_{i=1}^r da_i d\bar{a}_i$ on the space of vacua. This metric is known to receive no quantum corrections, since the $m = 0$ limit is the $\mathcal{N} = 4$ theory.

In the original work of Donagi-Witten [4] for the $SU(N)$ case, the verification of this requirement was carried out via a Hitchin system. In terms of Calogero-Moser systems,
the verification is even simpler. Indeed, as we saw in (c) above, at \( m = 0 \), the Lax operator \( L(z) \) reduces to \( L(z) = p \cdot h \) for all \( z \in \Sigma \), so that the spectral curve \( \Gamma \), given by \( \det(kI - L(z)) = 0 \), reduces to \( N \) unglued copies of \( \Sigma \), indexed by the constant eigenvalues of \( L(z) = p \cdot h \). Of these only \( n \) are linearly independent, say \( k_1, \cdots, k_n \). The \( A_i \) and \( B_i \) cycles, \( i = 1, \cdots, n \), are the lifts to the corresponding sheets of the \( A \) and \( B \) cycles on \( \Sigma \). Thus, both the order parameters \( a_i \) and their duals \( a_{Di} \) may be evaluated in the \( m \to 0 \) limit and we find

\[
a_i = \frac{1}{2\pi i} \oint_{A_i} d\lambda = \frac{1}{2\pi i} \oint_A dz = \frac{2\omega_1}{2\pi i} k_i
\]

\[
a_{Di} = \frac{1}{2\pi i} \oint_{B_i} d\lambda = \frac{1}{2\pi i} \oint_B dz = \frac{2\omega_1}{2\pi i} \tau k_i
\]

The prepotential \( F \), also defined in (3.2) is then easily read off in the \( m \to 0 \) limit and we have

\[
F = \frac{\tau}{2} \sum_{i=1}^r a_i^2.
\]

As a result, \( \text{Im} \partial_{a_i} \partial_{a_j} F = \text{Im} \tau \delta_{ij} \) correctly reproduces the classical metric.

(e) The limit \( m \to \infty \) to the theory without hypermultiplets

All the requirements analyzed above would be fulfilled by the ordinary as well as by the twisted Calogero-Moser systems associated with the gauge algebra \( G \). Thus, the final requirement on the behavior of the system as \( m \to \infty \) is crucial in distinguishing between these two possibilities, at least in the case of non-simply laced \( G \) where the twisted system differs from the ordinary one.

As \( m \to \infty \), standard renormalization group (and \( R \)-symmetry) arguments dictate the dependence of the gauge coupling \( g \) and the angle \( \theta \) of (1.1) on the mass \( m \), in terms of a renormalization scale \( M \), which is kept fixed in the limit,

\[
\tau = \frac{i}{2\pi} h_\text{dual} \ln \frac{m^2}{M^2}.
\]

Here, \( h_\text{dual} \) is the dual Coxeter number of the gauge algebra \( G \), which coincides with the quadratic Casimir of the algebra \( G \). Physically, when \( m \to \infty \) while \( \tau \) obeys (3.11), the \( \mathcal{N} = 2 \) super-Yang-Mills theory with gauge algebra \( G \) and one hypermultiplet in the adjoint representation of \( G \) converges to the \( \mathcal{N} = 2 \) super-Yang-Mills theory without hypermultiplets. In order to describe the \( \mathcal{N} = 2 \) super-Yang-Mills theory with one adjoint hypermultiplet, the associated Calogero-Moser system (1) must converge to a finite limit,
which must give an integrable system for the theory without hypermultiplets.

In a companion paper [19], we have systematically analyzed the limits of the ordinary and twisted Calogero-Moser systems according to general scaling behavior of the form

\[ m = M e^{\pi i \delta \tau}. \]  

(3.12)

(The variable \( x \) is shifted as well, but \( p \) is kept fixed upon taking the limit; we shall not need the precise form of this behavior here.) The results from [19] relevant to this analysis are as follows.

(i) The ordinary Calogero-Moser system has a finite limit for all \( 0 < \delta \leq \frac{1}{h_G} \), and diverges when \( \delta > \frac{1}{h_G} \). At the critical scaling with \( \delta = \frac{1}{h_G} \), the Calogero-Moser Hamiltonian and Lax pair tend to those of the (affine) Toda system associated with the untwisted affine Lie algebra \( G^{(1)} \).

(ii) The twisted Calogero-Moser system has a finite limit for all \( 0 < \delta \leq \frac{1}{h_G^\vee} \), and diverges for \( \delta > \frac{1}{h_G^\vee} \). At the critical scaling with \( \delta = \frac{1}{h_G^\vee} \), the Calogero-Moser Hamiltonian and Lax pair tend to those of the (affine) Toda system with the dual algebra \( (G^{(1)})^\vee \).

Now, the scaling (3.12) of the Calogero-Moser systems agrees with that required by the renormalization group when

\[ \delta = \frac{1}{h_G^\vee}. \]  

(3.13)

By comparing the scalings of (i) and (ii) above and the value of (3.13), we find the following requirements on the scaling behavior.

For \textit{simply laced} \( G \), \( h_G^\vee = h_G \) and the twisted and ordinary Calogero-Moser systems coincide. From (i), the limit is finite, confirming (1) above. Furthermore, the limit is the Toda system for \( G^{(1)} \), thus reproducing the result of [3], as required by (2). This confirms that for simply laced \( G \), the ordinary Calogero-Moser system indeed passes the last consistency test of (d).

For \textit{non-simply laced} \( G \), it is always true that \( h_G > h_G^\vee \). If \( \delta \) is given by (3.13), as required by the renormalization group arguments of (3.11), then the ordinary Calogero-Moser system will diverge as the limit \( m \to \infty \) is taken according to (3.12), violating the requirement (1) above. On the other hand, when \( \delta \) is given by (3.13), the twisted Calogero-Moser system converges to the (affine) Toda system for the affine Lie algebra \((G^{(1)})^\vee \). Thus, requirement (1) above is satisfied and the limit reproduces the result of [3], as required by (2). This confirms that for non-simply laced \( G \), the twisted Calogero-Moser system passes the last consistency test of (d).
IV. CURVES FOR LOW RANK; WEAK COUPLING $D_n$

In our treatment [7] of the case $\mathcal{G} = SU(N)$, we succeeded in reformulating equation (3.1) for the Seiberg-Witten curve in terms of a very simple expression involving the Jacobi theta function $\vartheta_1$,

$$\vartheta_1\left(\frac{1}{2\pi i}(z - m \frac{\partial}{\partial k})|\tau\right)H(k) = 0. \quad (4.1)$$

Here, $H(k)$ stands for a polynomial of degree $N$, whose overall normalization may be fixed to be $H(k) = k^N + O(k^{N-2})$. We may re-express $H(k) = \det(kI - \bar{k} \cdot h)$, where $h$ are the Cartan generators of $\mathcal{G}$ and the $N-1$ free parameters $\bar{k}_i$ play the role of the classical order parameters of the super-Yang-Mills theory. (The variable $k$ in (4.1) actually differs from that used in (3.1) via a shift by a function that depends upon $z$ and $\tau$ but is independent of the moduli, and is thus irrelevant for our considerations.)

In the case of general gauge algebra $\mathcal{G}$, we expect the Seiberg-Witten curves of (3.1) to admit simplified expressions analogous to those for the $SU(N)$ case given in (4.1), where the role of $H(k)$ is played by $\det(kI - \bar{k} \cdot h)$. We plan to address this problem in a subsequent publication.

We now present a considerable simplification in the evaluation of the spectral curve $\Gamma$ of (3.1), by making a judicious choice of classical order parameters. At $m = 0$, the curve $\Gamma$ is given exactly by $\det(kI - p \cdot h) = 0$, which depends only upon $k$ and the $n$ independent Casimir invariants $u_i^0$. The $u_i^0$ are polynomials in $p$, homogeneous of degree $\gamma_i + 1$, where the exponents $\gamma_i$ are given in [17], Appendix §A, Table 4. At $m \neq 0$, the curve $\Gamma$ depends upon both $p_i$ and $x_i$. However, the fact that the spectral curve is built on an integrable system guarantees that $\Gamma$ depends on $p$ and $x$ only through $n$ combinations $u_i = u_i(m)$, which are polynomial in $m$ and satisfy $u_i(0) = u_i^0$. Thus, $u_i(m)$ may be viewed as the deformation of $u_i^0$ away from $m = 0$, and may be identified by the leading $p$ behavior. Thus, to compute $\Gamma$ in terms of the Casimirs $u_i(m)$, it suffices to carry out the calculation of the determinant for any arbitrary convenient choice of the variables $x_i$, since the $p$-dependence alone will allow for the identification of the Casimirs $u_i(m)$. One very convenient choice for $x$ is in terms of the level vector $\rho^\vee$ of the Lie algebra $\mathcal{G}$, and the associated level function $l(\alpha)$

$$x = \xi \rho^\vee, \quad \alpha \cdot x = \xi l(\alpha). \quad (4.2)$$

Here, the parameter $\xi$ is arbitrary. Direct calculations of the curves $\Gamma$ are still cumbersome for large rank and for exceptional algebras. An indirect method in which the trigonometric
limit (i.e. zero gauge coupling) is evaluated first allows for further simplifications, as will be explained in (c) below.

(a) Curves for Low Rank Classical $G$

For low rank classical groups, we have the following explicit forms of the spectral curves. For $G = B_2 = C_2$, the curve reads

$$0 = k^4 - 2k^2(u_2 - 2m^2\varphi(z) - m_1^2\varphi_2(z)) + 4m_1m^2k\varphi'(z) + m_1^4\varphi_2(z)^2$$

$$- m_1^2m^2\varphi(z)^2 + 2m_1^2u_2\varphi_2(z) + 4m_1^2m^2\varphi_2(z)\varphi(\omega_1) + u_4. \quad (4.3)$$

Here, $u_2$ and $u_4$ are two independent classical order parameters, and $\varphi_2$ is the twisted Weierstrass function, defined in (2.2). Notice that it may be expressed in terms of $\varphi$ alone via the relation

$$\varphi_2(z) = \varphi(z) + \frac{(\varphi(\omega_1) - \varphi(\omega_2)(\varphi(\omega_1) - \varphi(\omega_3))}{\varphi(z) - \varphi(\omega_1)). \quad (4.4)$$

A discussion of the interpretation of the mass parameter $m_1$ was given at the end of §III (b).

For $G = D_n$, the curves are

$$0 = \sum_{j=0}^{n} Q_{2j}(k)u_{2n-2j}, \quad (4.5)$$

where $u_{2n-2j}$ are the Casimir invariants with $u_0 = 1$. The functions $Q_{2j}(k)$ are polynomials in $k$ of degree $2j$. To order $j \leq 5$, we have

$$Q_0 = 1$$

$$Q_2 = k^2$$

$$Q_4 = k^4 - 4k^2m^2\varphi$$

$$Q_6 = k^6 - 12k^4m^2\varphi - 8k^3m^3\varphi'$$

$$Q_8 = k^8 - 24k^6m^2\varphi - 32k^5m^3\varphi' - 48k^4m^4\varphi^2 + 64g_2k^2m^6\varphi$$

$$Q_{10} = k^{10} - 40k^8m^2\varphi - 80k^7m^3\varphi' - 240k^6m^4\varphi^2 - 64k^5m^5\varphi_2\varphi'$$

$$+ 704g_2k^4m^6\varphi + 512g_2k^3m^7\varphi' - 768k^2m^8g_3\varphi, \quad (4.6)$$

where we have used the abbreviations $\varphi = \varphi(z)$, $\varphi' = \varphi'(z)$ and where $g_2$ and $g_3$ stand for the modular forms of degrees 4 and 6 respectively, normalized by the equation $\varphi'^2 = 4\varphi^3 - g_2\varphi - g_3$. The combination of (4.5) and (4.6) yields the $D_n$ curves for $n = 2, 3, 4, 5$. It would not be easy to establish these low order curves by direct calculation of the determinants.
in (1.2), even using the simplifications explained in the preceding paragraph. Instead, indirect methods, developed in (c) below, were used to derive (4.5) and (4.6). We expect that a more general method can be found, analogous to the one used for $G = SU(N)$ to derive (4.1), from which (4.6) may be obtained for general $D_n$.

(b) Trigonometric Calogero-Moser and Perturbative Limit: $D_n$ Example

The perturbative limit of gauge theory corresponds to $g \to 0$ in (1.1), which implies that $\tau \to +i\infty$ and $q \to 0$. One further confirmation that we have indeed uncovered the correct curves for general Lie algebras $G$ is that the correct perturbative limit for the prepotential will be reproduced by these curves. For the sake of brevity, we discuss only the case $G = D_n$ here.

At the level of the elliptic Calogero-Moser system, the perturbative limit produces the trigonometric Calogero-Moser system in which

$$\varphi(z) \to \frac{1}{Z^2} \frac{1}{6} \frac{1}{4 \sinh^2 \frac{z}{2}} + \frac{1}{12},$$

$$\Phi(x, z) \to \frac{1}{2} \coth \frac{1}{2} x - \frac{1}{Z} \frac{1}{Z} = \frac{1}{2} \coth \frac{1}{2} z.$$  \hspace{1cm} (4.7)

Here, we have introduced a natural variable $Z$ which will prove to be convenient shortly. The curves in the trigonometric limit may be evaluated completely explicitly for $G = D_n$, and from this information, the low order curves of (4.5) and (4.6) may be inferred. We begin by deriving the curves in this limit. We make use of the additional simplification by choosing $x$ as in (4.3) and letting $\xi$ be real and $\xi \to +\infty$. The function $\Phi(\alpha \cdot x, z)$ which enters the Lax operator $L(z)$ has a particularly simple form in this limit, given by

$$\Phi(\alpha \cdot x, z) \to -\frac{1}{Z} + \left\{ \begin{array}{ll} +\frac{1}{2} & \alpha > 0 \\ -\frac{1}{2} & \alpha < 0 \end{array} \right. \hspace{1cm} (4.8)$$

Introducing the $n \times n$ matrices $\mu^\pm$ by

$$\mu^+_{ij} = \begin{cases} 1 & i < j \\ 0 & i \geq j \end{cases} \hspace{1cm} \mu^-_{ij} = \begin{cases} 1 & i > j \\ 0 & i \leq j \end{cases},$$  \hspace{1cm} (4.9)

the matrix $\mu = \mu^+ + \mu^-$, and $P = \text{diag}(p_1, \cdots, p_n)$, the equation (1.2) for the curve becomes $R(k, z) = 0$ with

$$R(k, z) = \det \begin{pmatrix} kI - P + \frac{m}{Z} \mu - \frac{m}{2} (\mu^+ - \mu^-) & (\frac{m}{Z} - \frac{m}{2}) \mu \\ (\frac{m}{Z} + \frac{m}{2}) \mu & kI + P + \frac{m}{Z} \mu + \frac{m}{2} (\mu^+ - \mu^-) \end{pmatrix}. \hspace{1cm} (4.10)$$
By taking suitable linear combinations of rows and columns, one easily shows that the evaluation of the above determinant can be reduced to the evaluation of a determinant of an $n \times n$ matrix, given as follows

$$R(k, z) = \det [(kI + P - m\mu^-)(kI - P - m\mu^+) + k(m + 2\frac{m}{Z})\mu]. \quad (4.11)$$

The determinants of the factors $kI \pm P - m\mu^\mp$ in the first term in the brackets are easy to compute since each factor is a triangular matrix. However, the second term in the brackets would seem to spoil this advantage. Actually, by rearranging the expansion of both terms in the bracket, the determinant can be expressed as follows

$$R(k, z) = \det [(AI + P - m\mu^-)(AI - P - m\mu^+) + (mA + 2k\frac{m}{Z})(\mu + I)], \quad (4.12)$$

where the new variable $A$ is defined by a quadratic relation in terms of $k$ and $Z$,

$$0 = A^2 + mA + 2k\frac{m}{Z} - k^2. \quad (4.13)$$

The definition is chosen in such a way that a matrix of rank 1 appears in the second term in the bracket in (4.12). (This remarkable relation is the analogue for $D_n$ of a linear change of variables made for the case of $A_n$ in (3.5) of [7]; in both cases, this change of variables is the key relation that allows for a completely explicit solution.) Any symmetric matrix of rank 1, such as $I + \mu$ may be written in terms of a column vector $u$ as $I + \mu = uu^T$. We use this fact and the following fundamental relation

$$\det[M + uu^T] = \det M(1 + u^T M^{-1} u),$$

where $M$ is any invertible matrix and $u$ is any column vector, to complete the evaluation of the determinant of (4.12). We find

$$R(k, z) = \prod_{j=1}^{n} (A^2 - p_j^2) + (mA + 2\frac{m}{Z}) \sum_{j=1}^{n-1} \prod_{i=1}^{j-1} [(A + m)^2 - p_i^2] \prod_{i=j+1}^{n} (A^2 - p_i^2). \quad (4.15)$$

Further algebraic manipulations permit us to recast the final result in the following form,

$$R(k, z) = \frac{m^2 + mA - 2k\frac{m}{Z}}{m^2 + 2mA} H(A) + \frac{mA + 2k\frac{m}{Z}}{m^2 + 2mA} H(A + m) \quad (4.16a)$$

$$H(A) = \prod_{j=1}^{n} (A^2 - p_j^2) = \sum_{j=0}^{n} (-1)^{n-j} A^{2j} u_{2n-2j} \quad (4.16b)$$
Here, we have identified the invariant polynomials in $p_j$ with the integral invariants of the system $u_{2n-2j}$, which are also the classical order parameters, as discussed in (b). The curve $\Gamma$ given by $R(k, z) = 0$, in the trigonometric limit, is now simply expressed in the variables $A$ and $Z$ by

$$
(m^2 + mA - 2k^2 \frac{m}{Z})H(A) + (mA + 2k^2 \frac{m}{Z})H(A + m) = 0.
$$

This equation is remarkably close to the equation found within the same approximation for $A_n$ in (4.5) of [7] with $q = 0$.

(c) Inferring Elliptic from Trigonometric Curves for $D_n$

Given a spectral curve $\Gamma$ for the elliptic Calogero-Moser system (say for $G = D_n$), the limit $q \to 0$ will produce the curves of the trigonometric Calogero-Moser system (up to redefinitions of the classical order parameters, which is physically irrelevant). By identifying the limit with the corresponding curve of (4.16), we can learn a great deal about the elliptic case. What will be missed are quantities that are proportional to variables that vanish in this limit. Clearly, $p_j$, $\wp$, $g_2$ and $g_3$ all have non-zero limits. However, the discriminant $\Delta$ of the underlying elliptic curve tends to zero,

$$
\Delta = g_2^3 - 27g_3^2 \to 0,
$$

since $g_2 \to 1/12$ and $g_3 \to -1/216$. Thus, the form of the curves in the trigonometric case determines the form of the curves in the elliptic case, up to functions that vanish with $\Delta$. Since $\Delta$ enters polynomially in $R(k, z)$, and the scaling degree of $\Delta$ is 12, curves will be uniquely determined by their trigonometric limits when $n \leq 5$.

To work out the trigonometric curves from (4.16) we proceed as follows. The combination $A$ is somewhat inconvenient, since it is not rational in $k$. Using (4.16a), we may however obtain a recursion relation for the coefficients $P_{2j}$ of the expansion

$$
R(k, z) = \sum_{j=0}^{n} (-1)^{n-j} P_{2j} u_{2n-2j}
$$

in terms of classical order parameters $u_{2n-2j}$ which is manifestly polynomial in $k$. The result is

$$
0 = P_{2(j+1)} - (2k^2 + m^2 - 4k^2 \frac{m}{Z})P_{2j} + k^2(k - 2 \frac{m}{Z})^2 P_{2(j-1)},
$$

(4.20a)
with the initial conditions $P_0 = 1$, $P_2 = k^2$. The lowest non-trivial orders are then

$$P_4 = k^4 - 4k^2 \frac{m^2}{Z^2} + m^2 k^2$$

$$P_6 = k^6 - 12k^4 \frac{m^2}{Z^2} + 16k^3 \frac{m^3}{Z} - 4k^3 \frac{m^3}{Z} - 4k^2 \frac{m^4}{Z^2} + 3k^4 m^2 + k^2 m^4$$

$$P_8 = k^8 - 24k^6 \frac{m^2}{Z^2} + 6k^6 m^2 + 64k^5 \frac{m^3}{Z} - 16k^5 \frac{m^3}{Z} - 48k^4 \frac{m^4}{Z^4}$$

$$- 8k^4 \frac{m^4}{Z^2} + 5k^4 m^4 + 32k^3 \frac{m^5}{Z^3} - 8k^3 \frac{m^5}{Z} - 4k^2 \frac{m^6}{Z^2} + k^2 m^6.$$

and

$$P_{10} = k^{10} - 40k^8 \frac{m^2}{Z^2} + 160k^7 \frac{m^3}{Z^3} - 240k^6 \frac{m^4}{Z^4} + 128k^5 \frac{m^5}{Z^5}$$

$$+ m^2 [10k^8 - 40k^7 \frac{m}{Z} + 160k^5 \frac{m^3}{Z^3} - 160k^4 \frac{m^4}{Z^4}]$$

$$+ m^4 [15k^6 - 48k^5 \frac{m}{Z} + 12k^4 \frac{m^2}{Z^2} + 48k^3 \frac{m^3}{Z^3}]$$

$$+ m^6 [7k^4 - 12k^3 \frac{m}{Z} - 4k^2 \frac{m^2}{Z^2} + m^8 k^2].$$

Now using the limit of $\varphi$ and its derivative, as in (4.7), we may uniquely identify which functional dependence in $\varphi$ gave rise to each of the terms in (4.20). Doing so, (and allowing for redefinitions of the classical order parameters $u_{2n-2j}$), we find the results of (4.5) and (4.6), with $Q_{2j} \rightarrow P_{2j}$.

(d) Agreement with Perturbation Theory : $D_n$ Example

Since we now possess the Calogero-Moser curve for $G = D_n$ in the trigonometric limit, we should be able to compute the contribution to the effective prepotential of the $D_n$ theory with a massive adjoint hypermultiplet to perturbative order. To do so, it is convenient to make use of the form (4.16a) of the curve : $R(k, z) = 0$ implies the following expression for the curve

$$e^u = \frac{H(A + m)}{H(A)},$$

where we define the complex variable $u$ by

$$e^u \equiv \frac{(k + A + m)(k - A - m)}{(k + A)(k - A)}.$$

Recall that $A$ was defined in (4.13) as a function of $k$ and $Z$. To evaluate the prepotential, we need the Seiberg-Witten differential $d\lambda = kdz$ in terms of the new variable $A$. This is achieved by first changing variables from $(k, z)$ to $(A, u)$, using (4.13) and (4.22), i.e.
without using the curve equation \( R(k, z) = 0 \). First, we obtain \( z \) as a function of \( Z \), by inverting the last line in (4.7), and then use (4.13) to express \( Z \) as a function of \( A \) and \( k \),

\[
e^z = \frac{+1 + \frac{2}{Z}}{-1 + \frac{2}{Z}} = \frac{(k - A)(k + A + m)}{(k + A)(k - A - m)}.
\]

Finally, \( k \) may be expressed in terms of \( A \) and \( u \) using (4.22). Now, it is easy to work out \( d\lambda = kdz \),

\[
d\lambda = \left\{ \frac{k}{k - A} - \frac{k}{k - A - m} \right\}(dk - dA) + \left\{ \frac{k}{k + A + m} - \frac{k}{k + A} \right\}(dk + dA)
\]

\[
= \left\{ \frac{A}{k - A} - \frac{A + m}{k - A - m} \right\}(dk - dA) - \left\{ \frac{A}{k + A + m} - \frac{A}{k + A} \right\}(dk + dA),
\]

which is readily re-expressed in terms of \( A \) and \( u \),

\[
d\lambda = -Adu - md\log(k^2 - (A + m)^2).
\]

The last term in (4.25), integrated around any closed curve, as is always the case in Seiberg-Witten theory, gives rise to moduli independent contributions only and is physically irrelevant. Remarkably, the curve (4.21) and the Seiberg-Witten differential (4.25) in terms of the variables \( k \) and \( z \) are identical to the ones for \( G = SU(N) \) in terms of the variables \( A \) and \( u \). (See [7], eq. (4.13).) Thus, the calculation of the effective prepotential for \( D_n \) to perturbative order follows directly from the our calculation for \( SU(N) \). We find

\[
F^{pert} = -\frac{1}{8\pi i} \sum_{\alpha \in \mathcal{R}(D_n)} \{(\alpha \cdot a)^2 \log(\alpha \cdot a)^2 - (\alpha \cdot a + m)^2 \log(\alpha \cdot a + m)^2\},
\]

which agrees with the standard perturbation theory result for the effective prepotential of a theory with an adjoint hypermultiplet with mass \( m \).

V. FURTHER RESULTS AND ISSUES

(a) The Effective Prepotential Equation

In the analysis of the \( N = 2 \) super-Yang-Mills theory with hypermultiplets in the fundamental representation of (classical) gauge algebras [13] or in that of the theory with one hypermultiplet in the adjoint representation of the \( SU(N) \) gauge algebra [7] powerful renormalization group type equation were obtained for the prepotential. We propose
that the same relation should hold between the prepotential of the gauge theory and the Hamiltonian of the integrable system,

\[ a_i = \frac{1}{2\pi i} \oint_{A_i} dz \ \partial F \frac{\partial F}{\partial \tau} = H = \frac{1}{4\pi i} \oint_A dz \text{tr} L^2. \tag{5.1a} \]

Here, \( H \) is the (twisted) elliptic Calogero-Moser Hamiltonian of (2.1). \( H \) may be expressed solely in terms of the quantum order parameters \( a_i \) and the modulus \( \tau \), by inverting the relation (5.1a) to obtain \( a_i \) as a function of the classical order parameters and \( \tau \).

\( \text{(f) Duality Properties} \)

We consider transformations of the (half) periods \( \omega_1 \) and \( \omega_2 \) of the following form

\[ \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} \rightarrow \begin{pmatrix} \omega'_2 \\ \omega'_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} \quad a, b, c, d \in \mathbb{Z}, \tag{5.2} \]

with \( \delta = ad - bc \neq 0 \). When \( \delta = 1 \), these transformations form the modular group, or a subgroup thereof. For \( \delta = 2 \) these are the Landen or Gauss transformations familiar from the theory of elliptic functions, and associated with mapping the period lattice into a lattice where one of the periods is reduced to half, while the other is left intact [25]. As we have defined it here, the spectral parameter \( z \) is unchanged under modular transformations. The Weierstrass functions \( \sigma(z) \), \( \zeta(z) \) and \( \wp(z) \), and thus the function \( \Phi(x, z) \) are similarly seen to be invariant.

We immediately conclude that the curves \( R(k, z) = 0 \) for simply laced \( \mathcal{G} \) are modular invariant. Physically, for these gauge algebras, the super-Yang-Mills theories are thus self-dual, namely they are invariant under the interchange of weak and strong gauge coupling \( \tau \), defined in (1.1), under the modular transformation \( S \) by \( \tau \rightarrow -1/\tau \). As such, these \( \mathcal{N} = 2 \) super-Yang-Mills theories provide explicit realizations of the Montonen-Olive duality conjecture [26], just as the massless \( \mathcal{N} = 4 \) theory does. (See also [1], [4] and [27].)

For non-simply laced \( \mathcal{G} \), functions other than \( \sigma(z) \), \( \zeta(z) \) and \( \wp(z) \) are involved in the expressions for the curves \( R(k, z) = 0 \). Specifically, non-simply laced \( \mathcal{G} \) corresponds to twisted elliptic Calogero-Moser systems in which the short roots are twisted with a preferred half period \( \omega_a, a = 1, 2, 3 \), taken to be \( \omega_1 \) here. (resp. third period in the case of \( \mathcal{G} = G_2 \)) Having singled out a preferred half (resp. third) period, the full modular
invariance is broken to a subgroup which leaves the preferred half (resp. third) period invariant. Defining the congruence subgroups in the usual manner,

\[ \Gamma_0(\nu) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1, \ c \equiv 0 \pmod{\nu} \right\}, \] (5.3)

we see that the remaining subgroup of the modular group is \( \Gamma_0(2) \) for \( G = B_n, C_n, F_4 \) and \( \Gamma_0(3) \) for \( G = G_2 \).

In [17], it was shown that under one of the Landen or Gauss transformations [25] with \( \delta = 2 \), the elliptic Calogero-Moser Hamiltonian for \( G = B_n, C_n, F_4 \) are mapped into Calogero-Moser Hamiltonians for the dual algebras \( G^\vee = C_n, B_n, F_4 \), and that for \( \delta = 3 \), \( G_2 \) is mapped into itself. We have not been able to show anything analogous for the Lax operators or for the spectral curves. We do not know at present what the precise role of these transformations with \( \delta \neq 1 \) is. The mapping between the Calogero-Moser Hamiltonians leads us to speculate that there may exist an underlying such symmetry of the spectral curve and perhaps of the gauge theory as well.

(c) Decoupling to smaller representations

Decoupling of all or part of the adjoint hypermultiplet by tuning the vacuum expectation values of the gauge scalar and of the hypermultiplet mass was used in [7] as a powerful tool to obtain \( \mathcal{N} = 2 \) supersymmetric Yang-Mills theories with different gauge groups and with hypermultiplets in different representations of the gauge group. In these decouplings, we showed that only the most asymptotically free part of the gauge group will survive in the decoupling limit (i.e. at energies low compared to the decoupling scale). In particular, any \( U(1) \) factors that may arise in the group theoretic decomposition of the gauge group, will not survive in the physical low energy theory, a fact also familiar from [16].

Specifically, starting with a hypermultiplet in the adjoint representation of the gauge group \( SU(N_c + N_f) \), we were able to reach, by decoupling, a theory with gauge group \( SU(N_c) \) and \( N_f \) hypermultiplets (with \( N_f < 2N_c - 1 \)) in the fundamental representation of \( SU(N_c) \) of arbitrary masses. For certain special arrangements of the groups and couplings, we could also achieve product gauge groups \( SU(N_1) \times \cdots \times SU(N_p) \) with hypermultiplets in fundamental and bi-fundamental representations, such as those solved in [16].

It should be clear that the same decoupling techniques may be applied to the \( \mathcal{N} = 2 \) super Yang-Mills theories with adjoint hypermultiplet for which we have derived curves for general gauge groups in this paper. We shall leave a detailed discussion for a later publication, and limit ourselves here to pointing out some interesting cases.
(1) Decoupling of $SO(2n)$ (resp. $SO(2n + 1)$) to a subgroup $SU(p)$ with $1 < p < n$, minimally embedded into the maximal $SU(n)$ subgroup of $SO(2n)$ (resp. $SO(2n + 1)$) should yield a theory with $SU(p)$ gauge group and hypermultiplets in the fundamental and rank 2 anti-symmetric reps of $SU(p)$. *

(2) Decoupling of $Sp(2n)$ to a subgroup $SU(p)$ with $1 < p < n$, minimally embedded into the maximal $SU(n)$ subgroup of $Sp(2n)$ should yield a theory with $SU(p)$ gauge group and hypermultiplets in the fundamental and rank 2 symmetric reps of $SU(p)$.

(3) Decoupling of $E_8$, $E_7$ or $E_6$ to one of its exceptional subgroups (say $E_7$ or $E_6$) is expected to yield a theory with exceptional gauge group and one or more hypermultiplets in the 56-dimensional rep of $E_7$ and the 27-dimensional representation of $E_6$.

(4) Decoupling of $E_8$, $E_7$ or $E_6$ to one of its $SO(p)$ subgroups is expected to yield a theory with $SO(p)$ gauge group and one or more hypermultiplets in fundamental and spinor representations of $SO(p)$.

ACKNOWLEDGEMENTS

We have benefited from useful conversations with Elena Caceres, Ron Donagi and Igor Krichever. The first author wishes to thank Edward Witten for a generous invitation to the Princeton Institute for Advanced Study, where this research was initiated, as well as the Aspen Center for Physics. He would also like to acknowledge David Gross and the members of the Institute for Theoretical Physics in Santa Barbara for the hospitality extended to him while most of this work was being carried out. Both authors would like to thank David Morrison, I.M. Singer and Edward Witten for inviting them to participate in the 1998 workshop on “Geometry and Duality”, at the Institute for Theoretical Physics.

REFERENCES

[1] Seiberg, N. and E. Witten, “Electro-magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory”, Nucl. Phys. B 426 (1994) 19, hep-th/9407087. Seiberg, N. and E. Witten, “Monopoles, duality, and chiral symmetry breaking in N=2 supersymmetric QCD”, Nucl. Phys. B431 (1994) 494, hep-th/9410167.

* Curves for $SU(p)$ gauge group and hypermultiplets in (anti-)symmetric representations of the gauge group were constructed using brane technology and M-theory by Landsteiner and Lopez in [22]. Explicit checks of perturbative contributions, and instanton corrections were carried out in [29].
[2] Gorski, A., I.M. Krichever, A. Marshakov, A. Mironov, A. Morozov, “Integrability and Seiberg-Witten Exact Solution”, Phys. Lett. B\textbf{355} (1995) 466, hep-th/9505035.

Matone, M., “Instantons and Recursion Relations in N=2 SUSY Gauge Theories”, Phys. Lett. B\textbf{357} (1996) 342, hep-th/9506102.

Nakatsu, T. and K. Takasaki, “Whitham-Toda Hierarchy and N=2 Supersymmetric Yang-Mills Theory”, Mod. Phys. Lett. A \textbf{11} (1996) 157-168, hep-th/9509162.

“Isomonodromic Deformations and Supersymmetric Gauge Theories”, Int. J. Mod. Phys. A\textbf{11} (1996) 5505, hep-th/9603069.

[3] Martinec, E. and Warner, N., “Integrable systems and supersymmetric gauge theories”, Nucl. Phys. B\textbf{459} (1996) 97-112, hep-th/9509161.

[4] Donagi, R. and E. Witten, “Supersymmetric Yang-Mills theory and integrable systems”, Nucl. Phys. B\textbf{460} (1996) 299-334, hep-th/9510101.

[5] Martinec, E., “Integrable structures in supersymmetric gauge and string theory”, Phys. Lett. B\textbf{367} (1996) 91, hep-th/9510204.

[6] Sonnenschein, J., S. Theisen, and S. Yankielowicz, “On the Relation between the Holomorphic Prepotential and the Quantum Moduli in SUSY Gauge Theories”, Phys. Lett. B\textbf{367} (1996) 145-150, hep-th/9510129.

Eguchi, T. and S.K. Yang, “Prepotentials of N=2 supersymmetric gauge theories and soliton equations”, hep-th/9510183.

Itoyama, H. and A. Morozov, “Prepotential and the Seiberg-Witten theory”, Nucl. Phys. B\textbf{491} (1997) 529, hep-th/9512161; “Integrability and Seiberg-Witten theory”, hep-th/9601168. “Integrability and Seiberg-Witten Theory: Curves and Periods”, Nucl. Phys. B\textbf{477} (1996) 855, hep-th/9511126.

Ahn, C. and S. Nam, “Integrable Structure in Supersymmetric Gauge Theories with Massive Hypermultiplets”, Phys. Lett. B\textbf{387} (1996) 304, hep-th/9603028.

Krichever, I.M. and D.H. Phong, “On the integrable geometry of soliton equations and N=2 supersymmetric gauge theories”, J. Differential Geometry \textbf{45} (1997) 349-389, hep-th/9604199.

Bonelli, G., M. Matone, “Nonperturbative Relations in N=2 SUSY Yang-Mills WDVV Equation”, Phys. Rev. Lett. \textbf{77} (1996) 4712, hep-th/9605090.

Marshakov, A., A. Mironov, and A. Morozov, “WDVV-like equations in N=2 SUSY Yang-Mills theory”, Phys. Lett. B\textbf{389} (1996) 43, hep-th/9607109.

Marshakov, A. “Non-perturbative quantum theories and integrable equations”, Int. J. Mod. Phys. A\textbf{12} (1997) 1607, hep-th/9610242.

Nam, S. “Integrable Models, Susy Gauge Theories and String Theory”, Int. J. Mod. Phys. A\textbf{12} (1997) 1243, hep-th/9612134.
Marshakov, A., A. Mironov, A. Morozov, "More Evidence for the WDVV Equations in N=2 SUSY Yang-Mills Theories", hep-th/9701123.

Marshakov, A., “On Integrable Systems and Supersymmetric Gauge Theories”, Theor. Math. Phys. 112 (1997) 791, hep-th/9702083.

Krichever, I.M. and D.H. Phong, “Symplectic forms in the theory of solitons”, hep-th/9708170, to appear in Surveys in Differential Geometry, Vol. III.

[7] D’Hoker, E. and D.H. Phong, “Calogero-Moser Systems in SU(N) Seiberg-Witten Theory”, Nucl. Phys. B513 (1998) 405, hep-th/9709053.

[8] Lerche, W., “Introduction to Seiberg-Witten theory and its stringy origin”, Proceedings of the Spring School and Workshop in String Theory, ICTP, Trieste (1996), hep-th/9611190; Nucl. Phys. Proc. Suppl. 55B (1997) 83, and references therein;

Donagi, R., “Seiberg-Witten Integrable Systems”, alg-geom/9705010;

Freed, D., “Special Kähler Manifolds”, hep-th/9712042;

Carroll, R., “Prepotentials and Riemann Surfaces”, hep-th/9802130.

[9] Hitchin, N., “Stable bundles and integrable systems”, Duke Math. J. 54 (1987) 91.

[10] Calogero, F., “Solution of the one-dimensional N-body problem with quadratic and/or inversely quadratic pair potentials”, J. Math. Physics 12 (1971) 419-436;

Moser, J., “Integrable systems of non-linear evolution equations”, in Dynamical Systems, Theory and Applications, J. Moser, ed., Lecture Notes in Physics 38 (1975) Springer-Verlag.

[11] Krichever, I.M., “Elliptic solutions of the Kadomtsev-Petviashvili equation and integrable systems of particles”, Funct. Anal. Appl. 14 (1980) 282-290.

[12] M.A. Olshanetsky and A.M. Perelomov, “Classical integrable finite-dimensional systems related to Lie algebras”, Phys. Rep. 71C (1981) 313-400.

Perelomov, A.M., “Integrable Systems of Classical Mechanics and Lie Algebras”, Vol. I, Birkhäuser (1990), Boston; and references therein.

Leznov, A.N. and M.V. Saveliev, Group Theoretic Methods for Integration of Non-Linear Dynamical Systems, Birkhauser 1992.

[13] D’Hoker, E., I.M. Krichever, and D.H. Phong, “The renormalization group equation for N=2 supersymmetric gauge theories”, Nucl. Phys. B 494 (1997), 89-104, hep-th/9610156.

[14] Argyres, P.C., and A. E. Faraggi, “The Vacuum Structure and Spectrum of N=2 Supersymmetric SU(N) Gauge Theories”, Phys. Rev. Lett. 74 (1995) 3931, hep-th/9411057.

Klemm, A., W. Lerche, S. Yankielowicz, S. Theissen, “Simple Singularities and N=2 Supersymmetric Yang-Mills Theory”, Phys. Lett. B344 (1995) 169, hep-th/9411048.
Klemm, A., W. Lerche, and S. Theisen, “Non-perturbative actions of N=2 supersymmetric gauge theories”, Int. J. Mod. Phys. A11 (1996) 1929-1974, hep-th/9505150.

[15] Hanany, A., “On the Quantum Moduli Space of Vacua of N=2 Supersymmetric SU(N) Gauge Theories”, Nucl. Phys. B452 (1995) 283, hep-th/9505075.

D’Hoker, E., I.M. Krichever, and D.H. Phong, “The effective prepotential for N=2 supersymmetric SU(Nc) gauge theories”, Nucl. Phys. B489 (1997) 179-210, hep-th/9609041.

D’Hoker, E. and D.H. Phong, “Strong coupling expansions in SU(N) Seiberg-Witten theory”, hep-th/9701151. Phys. Lett. B397 (1997) 94-103.

[16] Witten, E., “Solutions of four-dimensional field theories via M-theory”, Nucl. Phys. B500 (1997) 3, hep-th/9703166.

[17] D’Hoker, E., and D.H. Phong, “Calogero-Moser Lax Pairs with Spectral Parameter for General Lie Algebras”, April 1998, hep-th/9804124.

[18] V.I. Inozemtsev, “Lax representation with spectral parameter on a torus for integrable particle systems”, Lett. Math. Phys. 17 (1989) 11-17.

[19] D’Hoker, D.H. and D.H. Phong, “Calogero-Moser and Toda systems for twisted and untwisted affine Lie algebras”, April 1998 preprint, hep-th/9804123.

[20] Kachru, S., C. Vafa, “Exact Results for N=2 Compactifications of Heterotic Strings”, Nucl. Phys. B450 (1995) 69, hep-th/9505103.

Bershadsky, M., K. Intrilligator, S. Kachru, D.R. Morrison, V. Sadov and C. Vafa, “Geometric Singularities and Enhanced Gauge Symmetries”, Nucl. Phys. B481 (1996) 215, hep-th/9605200.

Katz, S., A. Klemm, C. Vafa, “Geometric Engineering of Quantum Field Theories”, Nucl. Phys. B497 (1997) 173, hep-th/9609239.

Katz, S., P. Mayr, and C. Vafa, “Mirror symmetry and exact solutions of 4D N=2 gauge theories”, Adv. Theor. Math. Phys. 1 (1998) 53, hep-th/9706110.

[21] Hanany, A., and E. Witten, “Type IIB Superstrings, BPS Monopoles, and Three-Dimensional Gauge Dynamics”, Nucl. Phys. B492 (1997) 152.

[22] Brandhuber, A., J. Sonnenschein, S. Theisen and S. Yankielowicz, “M-Theory and Seiberg-Witten Curves : Orthogonal and Symplectic Groups”, Nucl. Phys. B504 (1997) 175, hep-th/9705232.

Landsteiner, K., E. Lopez, “New Curves From Branes”, hep-th/9708118.

Landsteiner, K., E. Lopez, DA. Lowe, “N=2 Supersymmetric Gauge Theories, Branes and Orientifolds”, Nucl. Phys. B507 (1997) 197, hep-th/9705193.

Uranga, A.M., “Towards Mass Deformed N=4 SO(N) and Sp(K) Gauge Theories from
Brane Configurations", hep-th/9803054;
Yokono, T., “Orientifold four plane in brane configurations and N=4 USp(2N) and
SO(2N) theory”, hep-th/9803123.

[23] Gorskii, A., “Branes and Integrability in the N=2 SUSY YM Theory”, Int. J. Mod.
Phys. A12 (1997) 1243, hep-th/9612238;
Gorskii, A., S. Gukov and A. Mironov, “Susy Field Theories, Integrable Systems and
their Stringy/Brane Origin”, hep-th/9710239;
Cherkis, S.A., A. Kapustin, “Singular Monopoles and Supersymmetric Gauge Theories
in Three Dimensions”, hep-th/9711145.

[24] Kac, V., “Infinite-dimensional Lie algebras”, Birkhäuser (1983) Boston;
Goddard, P. and D. Olive, “Kac-Moody and Virasoro algebras in relation to quantum
physics”, International J. of Modern Physics A, Vol. I (1986) 303-414;
McKay, W.G., J. Patera and D.W. Rand, “Tables of Representations of Simple Lie
Algebras”, Vol. I : Exceptional Simple Lie algebras, Centre de Mathématiques, Uni-
versité de Montréal, 1990.

[25] Erdelyi, A., ed., “Higher Transcendental Functions”, Bateman Manuscript Project,
Vol. II, R.E. Krieger (1981) Florida.

[26] Montonen, C., and D. Olive, “Magnetic Monopoles as Gauge Particles ?”, Phys. Lett.
B72 (1977) 117.

[27] Vafa, C. and E. Witten, “Strong Coupling Test of S-Duality”, Nucl. Phys. B431
(1994) 3, hep-th/9408074;
Minahan, J.A., D. Nemeschansky, “N=2 Super-Yang-Mills and Subgroups of SL(2,Z)”,
Nucl. Phys. B468 (1996) 72;
J.A. Minahan, D. Nemeschansky, N.P. Warner, “Instanton Expansions for Mass De-
formed N=4 SuperYang-Mills Theories, hep-th/9710147.

[28] Braden, H.W. “R-Matrices, Generalized Inverses and Calogero Moser Sutherland
Models”, to appear in the Proceedings of the Workshop on Calogero-Moser-Sutherland
Models, in the CRM Series in Mathematical Physics, Springer-Verlag; available from
http://www.maths.ed.ac.uk/preprints/97-017.

[29] Naculich, N.G., H. Rhedin, H.J. Schnitzer, “One Instanton Predictions of a Seiberg-
Witten Curve from M-Theory : The Anti Symmetric Representation of SU(N)”, hep-

27