Gravity Assist as a Test of Relativistic Gravity

S. V. Bolokhov1*, K. A. Bronnikov1,2,3**, and M. V. Skvortsova1***

1Institute of Gravitation and Cosmology, Peoples’ Friendship University of Russia (RUDN University), ul. Miktukho-Maklaya 6, Moscow 117198, Russia
2Center for Gravitation and Fundamental Metrology, VNIIMS, Ozyornaya ul. 46, Moscow 119361, Russia
3National Research Nuclear University “MEPhI”, Kashirskoe sh. 31, Moscow 115409, Russia

Received July 6, 2022; revised August 7, 2022; accepted August 10, 2022

Abstract—We consider the gravity assist maneuver, that is, a correction of spacecraft motion at its passing near a planet, as a tool for evaluating the Eddington post-Newtonian parameters $\beta$ and $\gamma$, characterizing vacuum spherically symmetric gravitation fields in metric theories of gravity. We estimate the effect of variation in $\beta$ and $\gamma$ on a particular trajectory of a probe launched from the Earth’s orbit and passing closely near Venus, where relativistic corrections slightly change the impact parameter of probe scattering in Venus’s gravitational field. It is shown, in particular, that a change of $10^{-4}$ in $\beta$ or $\gamma$ leads to a shift of about 50 km in the probe’s aphelion position.

DOI: 10.1134/S0202289322040053

1. INTRODUCTION

By a gravity assist (GA) maneuver, we mean a way to adjust a spacecraft trajectory using the gravitational field of a massive celestial body (a planet) when the spacecraft enters the sphere of gravitational influence of this planet (the Hill sphere). The goal of this maneuver is to increase or decrease the spacecraft velocity in the Sun’s (quasi-inertial) reference frame (RF), saving fuel by borrowing kinetic energy from the planet’s orbital motion. In astronautics, GA has been successfully used for more than 50 years in various space missions in the Solar System. A good introduction to the GA problems including many technical and computational aspects can be found in [1], see also [2, 3].

Besides fuel economy and technical convenience for tasks of a particular mission, one more important aspect of GA is a high sensitivity of the spacecraft trajectory to variations of its initial parameters of motion such as the impact parameter [4]. Due to this, it was emphasized that GA can serve as one of possible tools for revealing the difference between Newtonian gravity and Einstein’s general relativity (GR) [5].

In this paper, we develop this idea and demonstrate that GA can be regarded as a tool for quite a precise test of metric theories of gravity, whose difference from GR can be formulated in a suitable approximation in terms of the parametrized post-Newtonian (PPN) formalism [6–8]. In the case of a spherically symmetric gravitational field of a central body, relevant are two well-known PPN parameters $\beta$ and $\gamma$, called the Eddington parameters [9], that slightly modify the Schwarzschild metric [10]. In GR we have $\beta = \gamma = 1$. The influence of these parameters is greatly amplified by GA that governs the probe’s subsequent motion, in particular, the values of $\beta$ and $\gamma$ affect such measurable quantities as the coordinates of probe’s destination points and travel times. Therefore, GA can in principle contribute to the current estimates and restrictions on $\beta$ and $\gamma$ from the astronomical observations in the Solar System. At present, deviations of these parameters from unity are estimated as $\sim 10^{-4}$ [8].

Remarkably, unlike astronomical events, a gravitational maneuver is an initiated and technically controlled event with a desired accuracy and calibration, which in principle opens the way for testing theories of gravity in an active manner rather than on the base of passive astronomical observations.

Needless to say that real calculations of spacecraft motion take into account the elliptic nature of planetary orbits and perturbations due to various factors acting in space. Therefore, such calculations are impossible in the sense of exact solutions to the equations of motion and require either direct numerical integration or approximate analytical methods. In our paper we follow the latter way and describe the geodesic motion up to some negligible terms using a realistic approximation from the viewpoint of the
currently achievable measurement accuracy. Also, for our demonstration purposes it is sufficient to adhere to spherically symmetric metrics and to assume circular planetary orbits.

As a particular problem to be considered, we take the probe flight model described in [4]. The probe starts from the Earth’s orbit and approaches Venus, its flyby near Venus is implemented as a gravity assist maneuver, and, as a result, the probe flies far beyond the Earth’s orbit. Eventually, the influence of relativistic corrections and the GA lead to a measurable shift of an arrival point, which is here taken to be the aphelion of the probe’s orbit.

A note on the notations: we mostly use small letters for parameters of motion in the Sun’s RF, capital letters for the same in Venus’s RF, and the indices 1 and 2 for probe motion before and after the assist, respectively.

In Section 2 we consider the general formalism of motion in spherically symmetric space-times and verify the correctness of the resulting formulas by reproducing the known expression for the post-Newtonian perihelion shift with arbitrary $\beta$ and $\gamma$. Section 3 is devoted to a particular kind of probe orbits with GA at Venus, and Section 4 is a brief conclusion.

2. ORBITAL MOTION AND PERIHELION SHIFT

2.1. Post-Newtonian Approximation

Consider the general static, spherically symmetric metric and its post-Newtonian (PN) representation,

$$ds^2 = f(r)dt^2 - g(r)dr^2 - r^2 d\Omega^2$$  \hspace{1cm} (1)

$$\equiv \left(1 - 2\frac{\mu}{r} + 2(\beta - \gamma)\frac{\mu^2}{r^2}\right)dt^2$$

$$- \left(1 + 2\gamma\frac{\mu}{r}\right)dr^2 - r^2 d\Omega^2,$$  \hspace{1cm} (2)

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2$, the constants $\beta$ and $\gamma$ are PN coefficients, both equal to 1 in the case of GR, and different for different metric theories of gravity; $\mu = GM/c^2$, $M$ being the mass of the gravitating center (so that $r_g = 2\mu$ is its Schwarzschild gravitational radius).

We will deal with geodesic motion in the equatorial plane, $\theta = \pi/2$, so that in the general metric (1) we have two integrals of the geodesic equations and the 4-velocity normalization condition $u_\mu u^\mu = 1$ (the dot denotes $d/ds$, and the displacement $ds$ coincides with that of proper time of a moving body, $d\tau$):

$$\dot{t} = E/f(r),$$  \hspace{1cm} (3)

$$\dot{\phi} = L/r^2,$$  \hspace{1cm} (4)

$$E^2/f(r) - g(r)i^2 - L^2/r^2 = 1.$$  \hspace{1cm} (5)

where the constants $E$ and $L$ are conserved energy and angular momentum, respectively, per unit mass of the moving body. If the geodesic motion is finite and takes place between a minimum radius $r_-$ and a maximum radius $r_+$, then these $r_{\pm}$ can be found from the condition $\dot{r} = 0$, which, being substituted to (5), leads to

$$\frac{L^2}{r_{\pm}^2} = \frac{E^2}{f_{\pm}} - 1, \quad \frac{L^2}{r_{\pm}^2} = \frac{E^2}{f_-} - 1,$$  \hspace{1cm} (6)

where $f_{\pm} = f(r_{\pm})$. This allows for expressing the constants $E$ and $L$ in terms of $r_{\pm}$:

$$E^2 = \frac{f_+ f_- (r_{+}^2 - r_{-}^2)}{f_{+} r_{+}^2 - f_{-} r_{-}^2}, \quad L^2 = \frac{(f_+ - f_-) r_{+}^2 r_{-}^2}{f_{-} r_{+}^2 - f_{+} r_{-}^2}.$$  \hspace{1cm} (7)

Comparing $\phi$ in (4) and $\dot{r}$ expressed from (5), we can exclude $ds$, which results in

$$\frac{d\phi}{dr} = \frac{L}{r^2 \sqrt{E^2 - f(r)g(r)(1 + L^2/r^2)}}$$  \hspace{1cm} (8)

Integration of (8) makes it possible to find the total azimuthal angle $\phi$ covered at test body motion between two specified values of $r$ (on segments where $r$ changes monotonically), provided that $E$ and $L$ are known.

Equations (3)–(8) are valid for an arbitrary metric (1). However, dealing with the post-Newtonian metric (2) applied to the Solar system, it makes sense to fix the necessary accuracy of further calculations. Specifically, if $M = M_\odot$ is the solar mass, then the Schwarzschild radius is $r_g = 2\mu \sim 1\ km$. The orbital radii are of the order $\sim 10^8\ km$, while the Earth’s orbital velocity $v \sim 30\ km/s \sim 10^{-4}\ m/s$ in units where $c = 1$ shows the magnitudes of all other relevant velocities. Thus,

$$\frac{\mu}{r} \sim 10^{-8}, \quad L \sim 10^4\ km,$$

$$\frac{L^2}{r^2} \sim v^2 \sim E - 1 \sim 10^{-8}.$$  \hspace{1cm} (9)

It proves to be reasonable to calculate angles up to $10^{-8}$, and in all expressions we will neglect any terms that make smaller contributions.

Substituting $f(r)$ and $g(r)$ from (2) to (8), we obtain

$$\frac{d\phi}{dr} = \frac{L(1 - \mu/r)(1 + \gamma\mu/r)}{\sqrt{2E + 2\mu/r - k + L^2/r^2 - 2\mu L^2/r^3}},$$  \hspace{1cm} (10)

where we have denoted

$$E_m = E - 1 = O(10^{-8}), \quad 2\dot{E} = 2E_m + E_m^2,$$

$$k = 2\mu^2(\beta - \gamma).$$  \hspace{1cm} (11)
The term with $1/r^3$ in (10) is eliminated by putting $r = R - \mu$, which results in

$$d\phi = \frac{L}{R^2 \sqrt{2\dot{E} + 2\frac{\mu}{R} + \frac{2\mu^2 - L^2 - k}{R^2}}} dr,$$

where we have neglected some contributions of the order $O(10^{-16})$. This expression is integrated in elementary functions by putting $R = 1/x$: the angle covered between two radius values $r_1$ and $r_2$ is

$$\phi = \left(1 + \frac{2\mu^2 - k}{2L^2}\right) \int_{r_1}^{r_2} \frac{-dx}{\sqrt{A_1 + B_1 x - x^2}} + (\gamma + 1)\mu \int_{r_1}^{r_2} \frac{-x dx}{\sqrt{A_1 + B_1 x - x^2}}$$

$$= \left[\left(1 + \frac{2\mu^2 - k}{2L^2} + \frac{(\gamma + 1)\mu B_1}{2}\right) \times \arccos \frac{2x - B_1}{\sqrt{4A_1 + B_1^2}} + (\gamma + 1)\mu \sqrt{A_1 + B_1 x - x^2}\right]_{r_1}^{r_2},$$

where, by our substitutions, $x = 1/(r + \mu)$, and

$$A_1 = \frac{2E_m + E_m^2}{L^2 + k - 2\mu^2}, \quad B_1 = \frac{2\mu}{L^2 + k - 2\mu^2}. \quad (14)$$

Now, with (13), we are ready to calculate the anomalous perihelion shift of a planetary orbit due to relativistic gravity. To obtain this shift per revolution, we must calculate the integral (13) between $r_1 = r_-$ (perihelion) and $r_2 = r_+$ (aphelion) and multiply it by two: $2\int_{r_-}^{r_+} d\phi$.

We first of all notice that $dr/d\phi = 0 \Rightarrow dx/d\phi = 0$ at $r = r_\pm$ (since these are extremal points of the function $r(\phi)$), therefore, according to (12), $A_1 + B_1 x - x^2 = 0$ at these points (with necessary accuracy), and thus only the arccos term in (13) contributes to $\phi$. Next, after some simple algebra, taking into account the notations (11), (14) and the expression (7) for $L^2$, and again neglecting terms of the order $\sim 10^{-16}$, we find that

$$\frac{2x(r_\pm) - B_1}{\sqrt{4A_1 + B_1^2}} = \mp 1$$

$$\Rightarrow \arccos \frac{2x - B_1}{\sqrt{4A_1 + B_1^2}} \Big|_{r_1}^{r_2} = \pi,$$

and we conclude that

$$\phi = 2\pi \left(1 + \frac{2\mu^2 - k}{2L^2} + \frac{(\gamma + 1)\mu B_1}{2}\right). \quad (16)$$

It is important that Eq. (15) is valid with appropriate accuracy for relativistic orbits with arbitrary PN coefficients $\beta$ and $\gamma$.

Since in Newtonian theory we have $\phi = 2\pi$, the anomalous shift per revolution is

$$\Delta \phi = \phi - 2\pi = \pi \left(\frac{2\mu^2 - k}{L^2} + (\gamma + 1)\mu B_1\right)$$

$$\approx \frac{\pi \mu (r_+ + r_-)}{r_+ r_-} (2 + 2\gamma - \beta), \quad (17)$$

where the last approximate equality is obtained by substitution of $B_1$ from (14) and $L^2$ from (7) and preserving only the main order of magnitude. Recalling that the major semiaxis $a$ of an ellipse and its eccentricity $e$ are related to $r_+$ and $r_-$ by

$$a(1 - e^2) = \frac{r_+ r_-}{r_+ + r_-},$$

we can finally write

$$\Delta \phi = \frac{2\pi \mu}{a(1 - e^2)} (2 + 2\gamma - \beta), \quad (18)$$

in agreement with [10] (see also references therein). In particular, in GR, where $\beta = \gamma = 1$, we obtain the well-known expression $\Delta \phi = 6\pi GM/a(1 - e^2)$, leading to the famous value of 43" per century for the planet Mercury.

### 2.2. Newtonian Orbital Motion

For an arbitrary point of the path we should use the full expression (13), separately for each monotonicity range of $r(\phi)$. To obtain the relativistic correction to $\phi$ at the same point, one should subtract the corresponding expression obtained in Newtonian gravity.

The Newtonian equations of motion of a test body in the gravitational field of an attracting center of mass $M$, moving in the equatorial plane $\theta = \pi/2$ in the spherical coordinates $(r, \theta, \phi)$, read

$$\ddot{r} - r\dot{\phi}^2 = -\frac{GM}{r^2}, \quad \ddot{\phi} - 2r\dot{r}\dot{\phi} = 0,$$

where the dot denotes a time derivative, $d/dt$. Equations (19) have the integrals

$$r^2 \dot{\phi} = L_N, \quad \frac{1}{2} \dot{r}^2 + \frac{L_N^2}{2r^2} - \frac{GM}{r} = E_N, \quad (20)$$

where $L_N$ and $E_N$ have the meaning of the Newtonian angular momentum and total energy, respectively, per unit mass of the test body. Excluding $dt$ from the two equations (20), we obtain

$$\left(\frac{d\phi}{dr}\right)^2 = \frac{L_N^2}{r^4(2E_N + 2GM/r - L_N^2/r^2)}, \quad (21)$$
whence it follows
\[ \pm \phi = \int \frac{dr}{r^2 \sqrt{A_N + B_N/r - 1/r^2}} = -\int \frac{d\xi}{\sqrt{A_N + B_N \xi - \xi^2}}, \]  \hspace{1cm} \text{(22)}

where
\[ \xi = \frac{1}{r}, \quad A_N = \frac{2E_N}{L_N^2}, \quad B_N = \frac{2GM}{L_N^2}. \]  \hspace{1cm} \text{(23)}

Integrating (22) between some radii \( r_1 \) and \( r_2 \), we obtain
\[ \pm \phi = \arccos \left( \frac{2/r - B_N}{\sqrt{4A_N + B_N^2}} \right) \bigg|_{r_1}^{r_2}. \]  \hspace{1cm} \text{(24)}

It is the expression (24) that should be subtracted from (13) to obtain the relativistic correction, and a specific example of such a calculation will be considered in the next section. Let us also present some useful formulas for such Kepler elliptic orbits (see, e.g., [11]):
\[ r(\phi) = \frac{p}{1 - e \cos \phi}, \quad r_+ = \frac{p}{1 \pm e} = a(1 \pm e), \]
\[ p = \frac{L_N^2}{GM}, \quad e = \frac{r_+ - r_-}{r_+ + r_-} = \sqrt{1 + \frac{2E_N L_N^2}{(GM)^2}}, \]  \hspace{1cm} \text{(25)}

where \( e \) is the orbit eccentricity, \( p \) is the focal parameter (which in our case coincides with Venus’s orbital radius, equal to \( OB = OC \) in Fig. 1), and \( a = (r_+ + r_-)/2 \) is the major semiaxis of the ellipse.

It is of interest that at the intersection points B and C of the two orbits, the horizontal \( x \) component \( v_x \) of the probe orbital velocity coincides with that of Venus:
\[ v_x = v_V = \pm \sqrt{\frac{GM}{p}}, \]  \hspace{1cm} \text{(26)}

and this circumstance makes easier our further calculations. This relation may be proved as follows. Since \( \mathbf{r} = (x, y) = (r \cos \phi, r \sin \phi) \), both \( v_x = \dot{x} \) and \( v_y = \dot{y} \) are easily found from the expression of \( r(\phi) \) in (25):
\[ v_x = \frac{d[r(\phi) \cos \phi]}{d\phi} \dot{\phi} = -\frac{L_N}{p} \sin \phi, \]
\[ v_y = \frac{d[r(\phi) \sin \phi]}{d\phi} \dot{\phi} = -\frac{L_N}{p} (\cos \phi - e), \]  \hspace{1cm} \text{(27)}

where we have used that \( \dot{\phi} = L_N/r^2 \). At points B and C where \( x = 0, \ y = \pm p \), and \( \phi = \pi/2 \) or \( 3\pi/2 \), respectively, we have
\[ v_x = \mp \frac{L_N}{p}, \quad v_y = -\frac{eL_N}{p} = -\dot{r}(C), \]  \hspace{1cm} \text{(28)}

where the last equality follows from \( r\dot{r} = x\dot{x} + y\dot{y} \), while at points B and C we have \( x = 0, \ y = \pm p = \pm r \). Then Eq. (26) follows from the standard expression for the focal parameter of Kepler elliptic motion, \( p = L_N^2/(GM) \). (The latter can be verified by comparing expressions for the conserved quantity \( E_N \) in (20) applied to \( r = r_+ = r_E \) and \( r = p \), with substitution of \( \dot{r} \) at \( r = p \) and the equality \( r_+ = p/(1 - e) \).)

Curiously, it happens that the specific angular momentum \( L_N \) is the same for the probe and the planet whose circular orbit is crossed by the probe at \( \phi = \pi/2 \) and \( \phi = 3\pi/2 \).

Let us note that the expression (24) cannot be obtained directly from (13) by using the Newtonian approximation of the metric (1)
\[ ds^2 = \left(1 - \frac{2\mu}{r}\right) dt^2 - dr^2 - r^2 d \Omega^2, \]  \hspace{1cm} \text{(29)}

which corresponds to (2) with \( \beta = \gamma = 0 \), leading to \( k = 0 \) in Eqs. (10)–(14). Indeed, comparing (13) with \( \beta = \gamma = k = 0 \) with (24), we see many differences: an addition to unity in the factor before the arccosine, the nonzero second term, and distinctions of \( A_1 \) and \( B_1 \) from \( A_N \) and \( B_N \) that play similar roles in the integration. A reason is, at least partly, in the different meaning of similar derivatives in relativistic and Newtonian equations (a particle’s proper time versus Newtonian absolute time), so that, for example, the Newtonian angular momentum \( L_N \) is different from \( L \). More generally, while using (29), we are still dealing with 4D Riemannian space-time, whose geometry requires non-Newtonian laws of motion. The correspondence principle, the requirement that relativistic gravity becomes Newtonian in the case of weak gravity and small velocities, works precisely only at zero velocity of a test particle at large \( r \) in the metric (29). As we see, at nonzero velocities there emerge corrections comparable to post-Newtonian ones.

3. GRAVITY ASSIST WITH VENUS

As an example of the effect of different metric theories of gravity on the results of a gravity assist, we will consider such an assist by Venus at point C according to Fig. 1. We assume that Earth and Venus are moving in circular orbits with radii \( r_E \) and \( r_V \) and orbital velocities \( v_E \) and \( v_V \), respectively; the probe under consideration is launched at point A of Earth’s orbit with a velocity \( v_0 < v_E \) so that its perihelion \( P \) is located inside the orbit of Venus, \( r_P < r_V \). The initial conditions are
\[ r = r_E, \quad \dot{r} = 0, \quad \phi = 0, \quad \dot{\phi} = v_0/r_E. \]  \hspace{1cm} \text{(30)}
The relevant numerical data to be used in the calculations and some calculated quantities are
\[
\begin{align*}
    r_E &= 1.495878159 \times 10^8 \text{ km}, \\
    r_V &= 1.082076791 \times 10^8 \text{ km}, \\
    r_P &= 0.847605588 \times 10^8 \text{ km}, \\
    v_0 &= 25.336 \text{ km/s} = 0.0000845118 \text{ (if } c = 1), \\
    v_V &= 35.02530368 \text{ km/s}. \quad (31)
\end{align*}
\]

We will study the probe motion in the framework of Newtonian gravity, while the small relativistic corrections affect only the gravity assist parameters. Their influence reduces to the fact that the probe’s impact parameter \( b \) at its scattering against Venus depends on the correction \( \Delta \phi \) of the azimuthal angle \( \phi \) at point \( C \), considered in Sun’s reference frame (Fig. 1).

### 3.1. The Relativistic Effect

Let us calculate the relativistic correction \( \Delta \phi \), the difference between the expressions (13) and (24) for a fixed value of \( r \). For the probe motion from point \( A \) to point \( C \) (or any other point between the perihelion \( P \) and a would-be return to \( A \)) we can write
\[
\begin{align*}
    \Delta \phi &= \frac{\pi \mu}{a(1 - e^2)} (2 + 2\gamma - \beta) \\
    &+ \left[ \left( 1 + \frac{2\mu^2 - k}{2L^2} + \frac{(\gamma + 1)\mu B_1}{2} \right) \arccos \eta_1 \\
    &+ (\gamma + 1)\mu \sqrt{A_1 + B_1 x - x^2} \right]_{r_P}^{r} - \arccos \eta_N \bigg|_{r_P}^{r}, \quad (32)
\end{align*}
\]
where \( x = 1/(r + \mu) \), and
\[
\begin{align*}
    \eta_1 &= \frac{2r - B_1}{\sqrt{4A_1 + B_1^2}}, \\
    \eta_N &= \frac{2/r - B_N}{\sqrt{4A_N + B_N^2}}, \quad (33)
\end{align*}
\]
the quantities \( A_1 \) and \( B_1 \) are defined in (14), \( A_N \) and \( B_N \) in (23). The first term in (32) is the relativistic shift of the angle \( \phi \) gained on the half-revolution from the aphelion \( A \) to the perihelion \( P \), and all the rest corresponds to the motion from \( P \) to a given value of \( r \).

Our next task is to single out the Newtonian part of the relativistic term in (32), bearing in mind that we need only first-order corrections to \( \phi \), expected to be of the order \( 10^{-8} \) according to (9). In particular, since the source root involved in (32) is a small quantity, it can be calculated in Newtonian terms, as \( \sqrt{A_N + B_N/r - 1/r^2} \).

More generally, to compare relativistic and Newtonian expressions, we should use directly measurable quantities which can be further used in the framework of any theory. These are the radius \( r \) (equal to the circumference around the Sun divided by \( 2\pi \)), the azimuthal angle \( \phi \) counted from point \( A \) in Fig. 1, and the time \( t \) whose Newtonian version is identified with the time measured by a distant \( (r \to \infty) \) observer and used in the metric (1). The initial velocity \( v_0 = r_E(d\phi/dt) \) at point \( A \) is also such a measurable quantity.

Then we obtain for the relativistic integrals of motion in terms of the Newtonian ones:
\[
2\tilde{E} := E^2 - 1 = 2E_m + E_m^2 \approx 2E_N + v_0^4 + \frac{k}{r_E^4},
\]
\[
L^2 \approx L_N^2 \left( 1 + v_0^2 + \frac{2\mu}{r_E^2} \right), \quad (34)
\]
whose orders of magnitude are \( \tilde{E} \sim 10^{-8}, \ L^2 \sim 10^8 \text{ km}^2 \). In obtaining this, we took into account that at probe motion near point \( A \) we have \( ds^2 = f(r_E)dt^2 - r_E^2d\phi^2 \). Furthermore, for the quantities appearing in (32), we get
\[
\begin{align*}
    A_1 &\approx \frac{2\tilde{E}}{L_N^2} (1 - \xi_1), \\
    B_1 &\approx \frac{2\mu}{L_N^2} (1 - \xi_1), \\
    \xi_1 &:= v_0^2 \frac{2\mu}{r_E^2} + \frac{2\mu^2(\beta - \gamma - 1)}{L_N^2}, \quad (35)
\end{align*}
\]
where \( A_1 \sim 10^{-16} \text{ km}^{-2}, \ B_1 \sim 10^{-8} \text{ km}^{-1}, \ \xi_1 \sim 10^{-8} \).

Let us evaluate \( \Delta \phi \) at point \( C \) where \( r = r_V \), and \( \phi \) is close to its Newtonian value \( \phi_N = 3\pi/2 \). According to the last, Newtonian term in (32) is equal to \(-\pi/2\). On the other hand, we can recall that at the lower limit, \( r = r_P \), according to (15), the relativistic quantity \( \eta_1 = 1 \Rightarrow \arccos \eta_1 = 0 \). At the upper limit...
Summarizing all that, we can rewrite (32) as follows with appropriate accuracy:

\[
\Delta \phi = \frac{3\pi}{2} \frac{\mu(2 + 2\gamma - \beta)}{a(1 - e^2)} - \eta_1(r_V) + (\gamma + 1)\mu \sqrt{A_N + B_N/r_V - 1/r_V^2},
\]

where \(A_N\) and \(B_N\) are given by (23), and

\[
\eta_1(r_V) = \frac{B_N(r_V\xi_1 - \mu)}{r_V \sqrt{4A_N + B_N^2}}.
\]

Thus we have an analytical expression for the relativistic azimuthal angle shift at point C.

### 3.2. Probe Motion Near Venus and Later

In the case where the assist takes place at point C with \(\phi = 3\pi/2\) in the Newtonian approximation, the calculated values of the probe velocity components at C are

\[
v_{1x} = v_V = 35.02530368 \text{ km/s},
\]

\[
v_{1y} = -9.68976496 \text{ km/s}
\]

in the Cartesian coordinates of Fig. 1, in the Sun’s RF. It is convenient to consider the probe motion in Venus’s gravitational field in its own RF, using the Cartesian coordinates parallel to those of Fig. 1, since Venus’s orbital velocity \(v_V\) at point C is \(x\)-directed. Thus in Venus’s RF the probe velocity has the components

\[
V_{1x} = 0, \quad V_{1y} = -9.68976496 \text{ km/s}.
\]

We consider the probe motion in Venus’s RF with the initial data (39) and

\[
X_1 = -b, \quad Y_1 = \infty,
\]

where \(b\) is the impact parameter (Fig. 2), a free parameter of the problem. The characteristic length scale at this motion is \(\sim 10^3 - 10^4\) km, much smaller than that of the interplanetary motion (\(\sim 10^8\) km), which allows us to put \(Y_1 = \infty\) at the beginning of the process. Our goal is to find the final velocity components \(v_{2x}\) and \(v_{2y}\) to be used as initial data for the probe’s further motion in the solar gravitational field. This approach, called the single instant hyperbola (SIH) method [4], corresponds to the approximation in which the sphere of a planet’s gravitational influence (the Hill sphere) is infinitesimally small from the viewpoint of the probe’s interplanetary motion, making it possible to consider the GA as an instantaneous event.

Under the conditions (39) and (40), in full similarity with Eqs. (19)–(23), it is easy to calculate the total angle \(\Gamma\) covered by the radius vector \(R\) of the probe (see Fig. 2) at its hyperbolic motion in Venus’s gravitational field:

\[
\Gamma = 2 \int_{R_{\text{min}}}^{\infty} \frac{dZ}{\sqrt{A_V + B_V Z - Z^2}} = 2 \arccos \frac{-2/R + B_V}{\sqrt{4A_V + B_V^2}} |_{R_{\text{min}}},
\]

where \(R_{\text{min}}\) is the minimum distance of the probe from the center of Venus, \(Z = 1/R\), and

\[
A_V = \frac{2E_V}{L_V^2} = \frac{1}{b^2}, \quad B_V = \frac{2\mu_V}{L_V^2} = \frac{2\mu_V}{b^2 V_1^2};
\]

here \(V_1 = -V_{1y}\) is the absolute value of the initial velocity of the probe, \(E_V = V_1^2/2\) and \(L_V = bV_1\) are the conserved energy and angular momentum per unit mass of the probe, respectively, and

\[
\mu_V = GM_V \approx 3.24872 \times 10^5 \text{ km}^3/\text{s}^2,
\]

\(M_V \approx 4.8675 \times 10^{24}\) kg being Venus’s mass. The value of \(R_{\text{min}}\) is obtained from the condition \(R = 0\) in the integral of the equations of motion similar to (20):

\[
R_{\text{min}} = \frac{1}{V_1^2} \left( -\mu_V + \sqrt{\mu_V^2 + b^2 V_1^4} \right).
\]

As a result, Eq. (41) leads to the expression confirming the general relations for hyperbolic motion in a
Newtonian gravitational field

\[ \Gamma = 2 \arccos \frac{1}{e_V}, \quad e_V = \sqrt{1 + \frac{v_1^2b^2}{\mu v_1^2}}. \]  

(45)

where \( e_V \) is the eccentricity of the probe’s hyperbolic orbit near Venus. The total deflection angle \( \Phi \) is related to \( \Gamma \) as

\[ \Phi = \pi - \Gamma = 2 \arcsin \frac{1}{e_V}. \]  

(46)

It is now straightforward to find the components of the probe final velocity \( \vec{V}_2 \) in Venus’s RF since, due to energy conservation, \( V_2 = |\vec{V}_2| = V_1 \). Thus,

\[ V_{2X} = V_1 \sin \Phi, \quad V_{2Y} = -V_1 \cos \Phi \]  

(47)

the minus sign is here due to the initial negative direction of the velocity along the \( Y \) axis).

In Sun’s RF the velocity components are

\[ v_{2x} = V_{2X} + v_Y, \quad v_{2y} = V_{2Y}. \]  

(48)

These components must be used as the initial data for calculating the probe motion in Sun’s gravitational field after gravity assist at point \( C \), to be considered in the framework of Newtonian gravity. The conserved quantities (integrals of motion) \( E_{N2} \) and \( L_{N2} \) are

\[ E_{N2} = \frac{1}{2} (v_{2x}^2 + v_{2y}^2) - \frac{GM_\odot}{r_v}, \]  

\[ L_{N2} = r_v v_{2x}, \]  

(49)

both being functions of the impact parameter \( b \). The corresponding geometric parameters (the eccentricity \( e_2 \), the orbital parameter \( p_2 \), the major semiaxis \( a_2 \) and the aphelion and perihelion radii \( r_{2\pm} \)) of the resulting probe orbit are

\[ e_2 = \sqrt{1 + 2E_{N2}L_{N2}^2/\mu^2}, \]  

\[ p_2 = \frac{L_{N2}^2}{\mu}, \quad a_2 = \frac{\mu}{2E_{N2}}, \quad r_{2\pm} = \frac{p_2}{1 \mp e_2}. \]  

(50)

where, as before, \( \mu = GM_\odot \approx 1.4777 \text{ km in units where } c = 1 \), or, in usual units more appropriate for calculations in Newtonian theory,

\[ \mu \approx 1.327461 \times 10^{11} \frac{\text{km}^3}{\text{s}^2}. \]  

(51)

Of interest for us is the sensitivity of the probe aphelion position at changes of the impact parameter \( b \). In addition to the radius \( r_{2\pm} \), we should determine the aphelion location in the tangential direction, which can be characterized by the angle

\[ \phi_2(b) = \pi - \arccos \frac{p_2 - r_v}{e_2r_v}, \]  

(52)

covered during probe motion from the GA position (crossing Venus’s orbit, point \( C \) in Fig. 1) to the

aphelion, calculated according to (24). The tangential shift \( \Delta \ell_{\text{tan}} \) of the aphelion due to a small change of \( b \) \((\Delta b) \) is then estimated as

\[ \Delta \ell_{\text{tan}} = r_{2+}(b) \Delta \phi_2(b), \]  

(53)

where \( \Delta \phi_2(b) \) is the increment of \( \phi_2(b) \) due to \( \Delta b \). Both radial and tangential shifts of the aphelion due to small \( \Delta b \) are illustrated in Fig. 1 as the position difference between the points \( A' \) and \( A'' \).

One more quantity of interest is the travel time \( t_{2+} \) from the assist position to the aphelion, which, according to Eq. (20), is given by

\[ t_{2+} = \int_{r_v}^{r_{2+}} \frac{r \, dr}{\sqrt{2r^2E_{N2} + 2\mu r - L_{N2}^2}}. \]  

(54)

3.3. Numerical Estimates

Let us estimate the relativistic correction (36) according to the data (31). A term by term calculation of this correction in Eq. (36) gives

\[ \Delta \phi \approx 6.432 \times 10^{-8} (2 + 2\gamma - \beta) + 5.084 \times 10^{-8} \]  

\[ + 9.871 \times 10^{-8} (\gamma - \beta) + 3.774 \times 10^{-9} (\gamma + 1) \]  

\[ \approx (1.83 - 1.63 \beta + 2.31 \gamma) \times 10^{-7}. \]  

(55)

At motion near point \( C \), this \( \Delta \phi \) shifts the probe trajectory to the right by the distance \( r_v \Delta \phi \), thus decreasing the impact parameter \( b \) (see Fig. 2) by

\[ \Delta b \approx -(19.83 - 17.64 \beta + 25.01 \gamma) \text{ km}. \]  

(56)

In particular, in the case of GR with \( \beta = \gamma = 1 \), we obtain \( \Delta b \approx 27.20 \text{ km} \).

This change in \( b \) results in a substantial change in the parameters of probe motion at stage 2, as follows from Table 1 that presents these parameters for a few values of \( b \).

In particular, the table shows that if \( b \) changes by 1 km from its value of 10 000 km, the aphelion radius changes by more than 16 000 km, and the aphelion itself “moves aside” by more than 15 000 km. Then the relativistic correction (56) shifts the aphelion point by more than 500 thousand kilometers. The parameters \( \beta \) and \( \gamma \) are known to be close to unity (their GR values) up to \( \sim 10^{-4} \) [8], hence to improve the knowledge of these quantities using the GA method as described here, it is necessary to determine the aphelion position.
Table 1. Probe motion parameters at and after the gravity assist at different values of the impact parameter $b$: the minimum distance from the center of Venus $R_{\text{min}}$, the deflection angle $\Phi$ in Venus’s RF, the velocity components $v_{2x}$ and $v_{2y}$ in Sun’s RF after the assist, the orbit eccentricity $e_2$, the aphelion radius $r_{2+}$ after the assist, the angle $\phi_2(b)$ given by (52), the travel time $t_{2+}$ from the orbit of Venus to the aphelion, and the corresponding sensitivities $dr_{2+}/db$, $\Delta t_{\text{fin}}/\Delta b$ and $dt_{2+}/db$ to variations of $b$

| $b$ (km) | 10000 | 12000 | 14000 | 16000 | 18000 |
|----------|--------|--------|--------|--------|--------|
| $R_{\text{min}}$ (km) | 7121.61 | 9028.81 | 10961.2 | 12909.8 | 14869.5 |
| $\Phi$ (rad) | 0.666227 | 0.56145 | 0.484585 | 0.42595 | 0.379819 |
| $v_{2x}$ (km/s) | 41.0138 | 40.1843 | 39.5392 | 39.029 | 38.6178 |
| $v_{2y}$ (km/s) | −7.6177 | −8.20223 | −8.57416 | −8.82395 | −8.99919 |
| $e_2$ | 0.450155 | 0.414992 | 0.389411 | 0.370431 | 0.356034 |
| $r_{2+}$ ($10^6$ km) | 269.84 | 243.47 | 225.84 | 213.415 | 204.271 |
| $dr_{2+}/db$ | −16308.1 | −10606.2 | −7303.74 | −5274.42 | −3958.22 |
| $\phi_2$ (rad) | 2.54024 | 2.4374 | 2.35258 | 2.28159 | 2.22147 |
| $\Delta t_{\text{fin}}/\Delta b$ | −15346.2 | −11319.4 | −8731.82 | −6950.56 | −5664.2 |
| $t_{2+}$ (days) | 236.271 | 205.297 | 184.508 | 169.698 | 158.65 |
| $dt_{2+}/db$ (s/km) | −1658.40 | −1076.59 | −747.845 | −546.146 | −416.512 |

The probe travel time is also highly sensitive to the values of $b$: as follows from the last line of the table, the whole relativistic correction to $t_{2+}$ is about 45 000 seconds for $b = 10 000$ km, therefore, to improve the knowledge of $\beta$ and $\gamma$ one should fix the time when the probe reaches its aphelion up to a few seconds.

4. CONCLUDING REMARKS

We have demonstrated that the gravity assist maneuver can be regarded as an efficient tool for measuring the parameters of metric theories of gravity. Using the Eddington parameters $\beta$ and $\gamma$, which characterize vacuum spherically symmetric gravitational fields in such theories, we have estimated the sensitivity of probe trajectories to variation in $\beta$ and $\gamma$ on the basis of the flight model considered in [4, 5]. We found that variations of the order of $10^{-4}$ in $\beta$ or $\gamma$ (corresponding to the accuracy of the present knowledge of their values) lead to shifts of about 50 km in the probe’s aphelion position. Thus the GA maneuver significantly amplifies small variations in the trajectory parameters thus making the method exceedingly sensitive.

As follows from Table 1, the method sensitivity rapidly grows as the impact parameter $b$ gets smaller, but we evidently cannot approach too close to the planet to avoid touching its atmosphere: in the case of Venus, $R_{\text{min}}$ should not be smaller than $\approx 6500$ km.

On the other hand, according to [4], an even larger sensitivity may be gained by observing the probe arrival point back on the Earth’s orbital radius; it seems, however, that such arrival is harder to precisely observationally fix than the aphelion point.

For our demonstration purposes it was sufficient to adhere to spherically symmetric metrics and circular planetary orbits, and to use some realistic approximations from the viewpoint of the currently achievable measurement accuracy. In particular, the probe motion at the GA stage and afterwards was studied in the framework of the Kepler problem in Newtonian gravity, while small relativistic corrections appreciably affect only the GA impact parameter $b$. However, in possible future practical experiments it will be evidently necessary to take into account all significant factors affecting the probe motion, such as ellipticity of planetary orbits, perturbations from the gravitational field of other planetary and the solar wind, and subtle details of the GA itself beyond the single instant hyperbola approximation.

ACKNOWLEDGMENTS

The authors are grateful to Alexander P. Yefremov for fruitful discussions.

FUNDING

This publication has been supported by the RUDN University Strategic Academic Leadership Program.
K.B. was also supported in part by the Ministry of Science and Higher Education of the Russian Federation, Project “Fundamental properties of elementary particles and cosmology” No. 0723-2020-0041.

CONFLICT OF INTEREST
The authors declare that they have no conflicts of interest.

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