ANALYSIS OF GENERALIZED PROBABILITY DISTRIBUTIONS ASSOCIATED WITH HIGHER LANDAU LEVELS

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Abstract. To a higher Landau Level corresponds a generalization of the Poisson distribution arising from generalized coherent states. In this paper, we write down the atomic decomposition of this probability measure and expressed its weights through $2F_2$ hypergeometric polynomials. Then, we prove that it is not infinitely divisible in opposite to the Poisson distribution corresponding to the lowest Landau level. We also derive the Lévy-Kintchine representation of its characteristic function when the latter does not vanish and deduce that the representative measure is signed. By considering the total variation of the last measure, we obtain the characteristic function of a new infinitely divisible discrete probability distribution for which we compute also the weights.

1. Introduction

The Poisson distribution is the cornerstone of the set of integer-valued infinitely divisible distributions ([12]). In the realm of quantum optics and precisely in photoemission, this probability distribution describes the random number of photons per unit of time when the intensity of the incident light is constant and the counts during distinct time intervals are statistically independent ([8], p.16-19). It also arises from the standard coherent states of the Bargmann-Fock space of entire functions in the $L^2$-space of the complex Gaussian distribution. In [2], this space was identified to the null eigenspace of a magnetic Laplacian which is unitarily equivalent to the Landau operator. The latter represents the Hamiltonian of a charged particle evolving in the plane subject to the action of a homogeneous normal magnetic field. More generally, the spectrum of the Landau operator, known as Euclidean Landau levels, was shown to be the set of nonnegative integers and the corresponding eigenspaces were called generalized Bargmann-Fock spaces (known also as true polyanalytic spaces, [II]).

In [9], generalized coherent states were attached to each eigenspace and it was proved in [10] that for any $m \geq 0$, the number $X_m$ of photons per unit of time associated with the $m$-th Landau level is governed by the following probability distribution

\[ \Pr(X_m = j) = \frac{(\min(m,j))! (\max(m,j))!}{\lambda^{m-j} e^{-\lambda} \left( L^{(m-j)}_{\min(m,j)}(\lambda) \right)^2}, j \geq 0, \]

where $L^{(\alpha)}_n$ is the $n$-th Laguerre polynomial of parameter $\alpha > -1$ ([13]) and $\lambda > 0$ is the intensity of the light. In particular, $X_0$ reduces to the Poisson distribution of parameter $\lambda$. When $m \geq 1$, the characteristic function of $X_m$ was computed in ([10]): it splits into
the product of the characteristic functions of $X_0$ and of a Dirac mass at $m$, together with an additional oscillating factor expressed as a Laguerre polynomial composed with a cosine function. Since this last factor is the Fourier transform of a finitely-supported measure, it does not correspond to an infinitely divisible distribution ([12]). However, this fact does not imply that the distribution of $X_m$, $m \geq 1$ is not infinitely divisible and it is therefore natural to wonder whether this property remains valid for higher Landau levels $m \geq 1$.

In this paper, we analyze the structure of the distribution of $X_m$, $m \geq 1$. More precisely, we write down its atomic decomposition and express its weights through $_2F_2$ hypergeometric polynomials of $\lambda$. We also prove that the infinite divisibility of $X_m$ breaks down as soon as $m \geq 1$: even when the characteristic function of $X_m$ does not vanish, the canonical sequence of the moment generating function of this random variable has a signed first element ([12]). In order to get more insight into the failure of this property, we derive a Lévy-Kintchine representation of the characteristic function of $X_m$ when it does not vanish, whence it is readily seen that the representative measure is signed. This fact not only proves again that $X_m$ is not infinitely divisible but also provides a new infinitely divisible discrete probability distribution whose Lévy measure is the total variation of the last signed measure. Using similar computations as the previous ones leading to the Lévy-Kintchine representation, we compute the characteristic function of the new distribution and write down its atomic decomposition.

The paper is organized as follows. In section 2, we recall the coherent states formalism used to construct the generalized Poisson distribution displayed in (1.1) and write down its atomic decomposition. In section 3, we prove the non infinite divisibility of $X_m$ and derive of the aforementioned Lévy-Kintchine-type representation. Section 4 is devoted to the analysis of the newly obtained infinitely divisible distribution.

### 2. Generalized Poisson distribution: atomic decomposition

In this paragraph, we briefly recall from [10] the construction of the distribution of $X_m$ which we refer to below as the generalized Poisson distribution. To this end, we introduce the Hamiltonian $H$ with constant homogeneous magnetic field which may be written, up to normalizations, as:

$$
(2.1) \quad H := -\frac{1}{4} \left( \left( \frac{\partial}{\partial x} + \sqrt{-1}y \right)^2 + \left( \frac{\partial}{\partial y} - \sqrt{-1}x \right)^2 \right) - \frac{1}{2}.
$$

Conjugating by the complex Gaussian function $\phi \mapsto e^{\frac{1}{2}|z|^2} \phi$, $\phi \in L^2(\mathbb{R}^2, dx dy)$ where $z = x + \sqrt{-1}y$ then $H$ is unitarily mapped into the magnetic Laplacian

$$
L := -\frac{\partial^2}{\partial z \partial \bar{z}} + \bar{z} \frac{\partial}{\partial \bar{z}}.
$$
This is a self-adjoint operator in the Hilbert space $L^2(\mathbb{C}, e^{-|z|^2} dz)$ where $dz$ is the Lebesgue measure in $\mathbb{C}$ and has purely discrete positive spectrum given by the set of nonnegative integers ($\mathbb{N}$). Actually, the direct sum decomposition holds

$$L^2(\mathbb{C}, e^{-|z|^2} dz) = \bigoplus_{m=0}^{\infty} A_m(\mathbb{C})$$

where

$$A_m(\mathbb{C}) := \{ Lf = mf \} \cap L^2(\mathbb{C}, e^{-|z|^2} dz)$$

is the eigenspace of $L$ corresponding to the $m$-th Landau level. In particular, $A_0(\mathbb{C})$ coincides with the Bargmann-Fock space of holomorphic functions in $L^2(\mathbb{C}, e^{-|z|^2} dz)$, for which an orthonormal basis is given by ($\psi_j(z) := z^j/\sqrt{\pi j!}, j \geq 0$) yielding the reproducing kernel

$$K_0(z, w) = \sum_{j \geq 0} \psi_j(z)\overline{\psi_j(w)} = \frac{1}{\pi} e^{(z,w)}, \ z,w \in \mathbb{C}.$$ 

It follows that the sequence

$$\frac{|\psi_j(z)|^2}{K_0(z, z)}, \ j \geq 0,$$

defines a discrete probability distribution which is nothing else but the Poisson distribution of parameter $\lambda = |z|^2$. When $m \geq 1$, the reproducing kernel of $A_m(\mathbb{C})$ was given closed formula in [2] and the basis elements coincide, up to a normalization, with the complex Hermite polynomials (see [10] where they were denoted $h_{m,j}$):

$$\psi_{m,j}(z) := \frac{1}{\sqrt{\pi m! j!}} \sum_{n=0}^{\min(m,j)} (-1)^n \binom{m}{n} \frac{j!}{(j-n)!} z^{j-n}.$$ 

Note in passing that these polynomials appears in relation with complex multiple Wiener integrals ([6]). Moreover, the generalized Poisson distribution ([11] was obtained in [10] in the same fashion the Poisson distribution was obtained from $A_0(\mathbb{C})$. Independently and at nearly the same time, this distribution appeared in [4] where few statistical properties were derived. In [10], its characteristic functions was computed and takes the form:

$$(2.2) \quad E[e^{iuX_m}] = \exp \left( \lambda (e^{iu} - 1) \right) e^{imuL_m^{(0)}(2\lambda (1 - \cos u))}, \ u \in \mathbb{R}.$$ 

With this expression in hands, we perform the analysis of the generalized Poisson distribution and write down in the subsequent paragraph its atomic decomposition.

2.1. Atomic decomposition. Clearly, we only need to write $L_m^{(0)}[2\lambda(1 - \cos u)]$ as a linear combination of $\cos ju, 1 \leq j \leq m$. This is the content of the following lemma.

Lemma 1. Let $m \geq 1$ and $\lambda > 0$, then

$$(2.3) \quad L_m^{(0)}[2\lambda (1 - \cos u)] = \gamma_0(\lambda, m) + 2 \sum_{j=1}^{m} \gamma_j(\lambda, m) \cos ju,$$
where
\[ \gamma_j (\lambda, m) = (-1)^j \sum_{k=j}^{m} \binom{m}{k} \binom{2k}{k-j} \frac{(-\lambda)^k}{k!}. \]

Moreover, \( \gamma_j (\lambda, m) \) are \( _2F_2 \) hypergeometric polynomials of \( \lambda \) (see \cite{3}, CH.IV for the definition of \( _2F_2 \)).

**Proof.** The expression of \( \gamma_j (\lambda, m) \) follows from the expansion \((13), p.101)\]
\[ L_m^{(0)} [2\lambda (1 - \cos u)] = \sum_{k=0}^{m} \binom{m}{k} \left( \frac{-2\lambda}{k!} \right)^k (1 - \cos u)^k = \sum_{k=0}^{m} \binom{m}{k} \frac{(-\lambda)^k}{k!} 4^k \sin^{2k}(u/2) \]
together with the linearization formula \((5), p.31)\]
\[ 4^k \sin^{2k}(u/2) = \binom{2k}{k} + 2 \sum_{j=1}^{k} (-1)^j \binom{2k}{k-j} \cos(ju). \]

As to the last claim of the lemma, it suffices to change the index summation \( k \mapsto m+k \) and to use the duplication formula \((3), p.5, eq.15)\]
\[ (2j+2k)! = \frac{4^{j+k}}{\sqrt{\pi}} (j+k)! \Gamma(j+k+1/2) = 4^{j+k}(1/2)_j(j+1/2)_k(j+k)! \]
where \((a)_k = a(a+1) \cdots (a+k-1)\) is the Pochhammer symbol. More precisely
\[
\gamma_j (\lambda, m) = \frac{m!}{(m-j)!} \sum_{k=0}^{m-j} \frac{(-1)^k(m-j)!}{(m-j-k)!(k+j)!} \binom{2k+2j}{k} \frac{\lambda^{k+j}}{(k+j)!} \\
= \frac{(1/2)_j(4\lambda)^j}{(2j)!} \binom{m}{j} \sum_{k=0}^{m-j} \frac{(j-m)_k(j+1/2)_k(4\lambda^k)}{(j+1)_k(2j+1)_k k!} \\
= \frac{\lambda^j}{j!} \binom{m}{j} _2F_2(j-m, j+1/2, j+1, 2j+1, 4\lambda).
\]

\( \square \)

From this lemma, it follows that:

**Proposition 1.** The distribution of \( X_m \) is decomposed as
\[
P(\lambda) * \delta_m * \left( \sum_{j=-m}^{m} \gamma_{|j|} (\lambda, m) \delta_j \right) = P(\lambda) * \left( \sum_{j=0}^{2m} \gamma_{|m-j|} (\lambda, m) \delta_j \right) := \sum_{j=0}^{\infty} q_j (\lambda, m) \delta_j
\]
where \( P(\lambda) \) stands for the Poisson distribution of parameter \( \lambda \) and
\[
q_j (\lambda, m) := e^{-\lambda} \sum_{k=0}^{j^2 2m} \frac{\lambda^{j-k}}{(j-k)!} \gamma_{|m-k|}.
\]
3. Infinite divisibility

This section is devoted to the proof of the non infinite divisibility of $X_m$ as well as to the Lévy-Khintchine representation for its characteristic function when the latter does not vanish. Recall that a random variable $Y$ is infinitely divisible if for any $n \geq 2$, there exist $n$ independent and identically distributed random variables $(Y_i^{(n)})_{i=1}^n$ such that

$$Y = \sum_{i=1}^n Y_i^{(n)}.$$ 

A necessary condition for this property to hold is the non vanishing of the characteristic function of the given random variable ([12], CH.IV, Proposition 2.4). Accordingly and with regard to (2.2), if $m \geq 1$ is fixed then the parameter $\lambda$ must satisfy

$$2 \lambda (1 - \cos u) \neq x_k^{(m)}, \quad 1 \leq k \leq m,$$

for all $u \in \mathbb{R}$, where $x_k^{(m)}, 1 \leq k \leq m$ are the increasingly-ordered zeros of Laguerre polynomial $L_m^{(0)}$, which are known to be simple and positive ([13]). But since

$$0 \leq 2 \lambda (1 - \cos u) \leq 4 \lambda,$$

then the distribution of $X_m$ cannot be infinitely divisible unless probably when $\lambda < x_1^{(m)}/4$. For those values of $\lambda$, this property does not hold neither:

**Proposition 2.** For any $m \geq 1$ and any $\lambda < x_1^{(m)}/4$, the distribution of $X_m$ is not infinitely divisible.

**Proof.** Since $X_m$ is a integer-valued random variable, we can use for instance the following criterion ([12], CH.II, Theorem 4.1) rather than other sophisticated criteria from the general theory: a discrete probability distribution $(p_k)_{k \in \mathbb{N}}$ with $p_0 > 0$ is infinitely divisible if and only if the logarithmic derivative of its moment generating function

$$P(z) := \sum_{k \geq 0} p_k z^k, \quad z \in [0, 1]$$

satisfies

$$\frac{P'(z)}{P(z)} = \sum_{k \geq 0} r_k z^k$$

for non negative reals $r_k, k \geq 0$. The sequence $(r_k)_{k \geq 0}$ is referred to as the canonical sequence of $P$ and we prove below that $r_1 < 0$. To proceed, we first deduce the moment generating function of $X_m$ from its characteristic function (2.2). Actually, the latter extends to an analytic function in the upper-half plane $\mathbb{C}^+$ so that

$$\mathbb{E} \left( e^{-uX_m} \right) = \exp \left( \lambda \left( e^{-u} - 1 \right) - mu \right) L_m^{(0)} \left( 2\lambda (1 - \cosh u) \right).$$
for any $u \geq 0$. Setting $z = e^{-u}$ then the moment generating function of $X_m$ reads

$$P_m(z) := \mathbb{E}(z^{X_m}) = \exp(\lambda(1 - z - 1)) z^m \left(-\frac{\lambda}{z}(1 - z)^2\right).$$

Next, since the leading coefficient of $L_m^{(0)}$ is $(-1)^m$ then

$$\frac{P'_m(z)}{P_m(z)} = \lambda + \frac{m}{z} + \frac{d}{dz} \ln L_m^{(0)} \left(-\frac{\lambda}{z}(1 - z)^2\right)$$

$$= \lambda + \frac{m}{z} - \frac{1}{z} \sum_{k=1}^{m} \frac{\lambda(1 - z^2)}{z x_k^{(m)} + \lambda(1 - z)^2}$$

$$= \lambda + \sum_{k=1}^{m} \frac{x_k^{(m)}}{z x_k^{(m)} + \lambda(1 - z)^2}$$

$$= \lambda + \sum_{k=1}^{m} \frac{x_k^{(m)}}{1 + (x_k^{(m)} - 2\lambda)z + z^2}.$$ 

By assumption $x_k^{(m)} - 2\lambda > 0$ therefore

$$r_1 = \frac{d}{dz} \left(\frac{P'_m}{P_m}\right)(0) = \sum_{k=1}^{m} x_k^{(m)}(2\lambda - x_k^{(m)}) < 0.$$ 

3.1. A Lévy-Khintchine representation. In order to get more insight into the failure of the non infinite divisibility property of $X_m$ under the assumption $4\lambda < x_1^{(m)}$, we shall derive a Lévy-Khintchine representation for its characteristic function. Recall from [11] (see p.37, Theorem 8.1) that if $Y$ is an infinitely divisible random variable without Gaussian component, then its Lévy-Khintchine representation may be written as

$$\ln \mathbb{E}[e^{iux}] = iu \lambda + \int (e^{iux} - 1 - iux1_{|x|<1}) \nu(dx)$$

where $b \in \mathbb{R}$ and $\nu$, the Lévy measure of $Y$, is a positive measure satisfying

$$\nu\{0\} = 0, \quad \int (1 \wedge x^2) \nu(dx) < \infty.$$ 

Coming back to $X_m$, it is obvious that

$$\lambda(e^{iu} - 1) +imu = imu + \lambda \int (e^{iux} - 1)\delta_1(dx),$$

so that we are only concerned with the representation of

$$\ln \left(L_m^{(0)}(2\lambda(1 - \cos u))\right) = \sum_{k=1}^{m} \ln \left(x_k^{(m)} - 4\lambda \sin^2(u/2)\right) = \sum_{k=1}^{m} x_k^{(m)} + \sum_{k=1}^{m} \ln \left(1 - a_k^{(m)} \sin^2(u/2)\right)$$
where
\[ a_k^{(m)} := \frac{4\lambda}{x_k^{(m)}} \in (0,1), \; 1 \leq k \leq m. \]

In this respect, we prove

**Proposition 3.** Let \( m \geq 1. \) Then, for every \( 1 \leq k \leq m, \) there exists a signed measure \( \mu_k^{(m)} \) satisfying (3.4) and such that
\[ \ln \left( L_m^{(0)}(2\lambda (1 - \cos u/2)) \right) = \int (e^{iux} - 1) \left[ \sum_{k=1}^{m} \mu_k^{(m)}(dx) \right]. \]

**Proof.** Since \( L_m^{(0)}(x) = (-1)^m x^m + \text{terms of lower degrees and } L_m^{(0)}(0) = 1 \) then
\[ \prod_{i=1}^{m} x_i^{(m)} = 1 \quad \text{whence} \]
\[ \ln \left( L_m^{(0)}(2\lambda (1 - \cos u/2)) \right) = \sum_{k=1}^{m} \ln \left( 1 - a_k^{(m)} \sin^2(u/2) \right). \]

Now, fix \( k \in \{1, \ldots, m\} \) and expand
\[ \ln \left( 1 - a_k^{(m)} \sin^2 \frac{u}{2} \right) = - \sum_{j=1}^{\infty} \frac{[a_k^{(m)}]^j}{j!} \sin^{2j}(u/2) \]
\[ = - \sum_{j=1}^{\infty} \frac{[a_k^{(m)}]^j}{j!} \left\{ \binom{2j}{j} + 2 \sum_{s=1}^{j} (-1)^s \binom{2j}{j-s} \cos(su) \right\} \]
(3.5)
where the second equality follows from (2.4). Using the duplication formula
\[ (2j)! = 4^j (1/2)_j j!, \]
we get
\[ \sum_{j=1}^{\infty} \frac{[a_k^{(m)}]^j}{j!} \binom{2j}{j} = \sum_{j=1}^{\infty} \frac{(1/2)_j [a_k^{(m)}]^j}{j!} \int_0^{a_k^{(m)}} \frac{(1/2)_j x^j dx}{x} \]
\[ = \int_0^{a_k^{(m)}} \frac{1}{\sqrt{1 - x(1 - x + 1)}} dx = 2 \ln \left( \frac{2}{\sqrt{1 - a_k^{(m)} + 1}} \right). \]
(3.6)
Now consider the series
\[ \sum_{j \geq s} \binom{2j}{j-s} \frac{[a_k^{(m)}]^j}{j!} = \int_0^{a_k^{(m)}} \sum_{j \geq s} \binom{2j}{j-s} \frac{x^j dx}{4^j x} \]
for fixed \( s \geq 1. \) Performing an index change \( j \mapsto j + s \) and using again the duplication formula
\[ (2j + 2s)! = \frac{4^{j+s}}{\sqrt{\pi}} \Gamma(j + s + 1/2)(j + s)!, \]
we get
\[
\sum_{j \geq s} \left( \frac{2j}{j - s} \right) x^{j} = \frac{\Gamma(s + 1/2)\Gamma(s + 1)}{\sqrt{\pi}\Gamma(2s + 1)} \sum_{j \geq 0} \frac{(s + 1/2)_j(s + 1)_j}{(2s + 1)j!} x^{j+s} = x^{s} 2F_1(s + 1/2, s + 1, 2s + 1; x)
\]
where \(2F_1\) is the Gauss hypergeometric function ([3], CH.II). Set
\[
\alpha(x) := \frac{x}{1 + \sqrt{1 - x}} = \frac{1 - \sqrt{1 - x}}{1 + \sqrt{1 - x}}, \quad x \in [0, 1],
\]
recall formula (6), p.101 in [3]
\[
2F_1(s + 1/2, s + 1, 2s + 1; x) = \frac{4^s}{\sqrt{1 - x}(1 + \sqrt{1 - x})^{2s}}.
\]
Then
\[
\sum_{j \geq s} \left( \frac{2j}{j - s} \right) x^{j} \frac{x^j}{4^j} = \frac{\alpha(x)^s}{\sqrt{1 - x}}.
\]
But
\[
\alpha'(x) = \frac{\alpha(x)}{x \sqrt{1 - x}},
\]
which implies
\[
\sum_{j \geq s} \left( \frac{2j}{j - s} \right) \frac{[a^{(m)}]_j}{j!} = \frac{1}{s} [\alpha^s(x)]_0^{a^{(m)}_k} = \frac{1}{s} \alpha^s(a_{(m)}^k).
\]
As a result
\[
(3.7) \quad 2 \sum_{j \geq 1} \frac{[a^{(m)}]_j}{j!} \sum_{s=1}^j (-1)^s \left( \frac{2j}{j - s} \right) \cos(su) = 2 \sum_{s \geq 1} \frac{(-1)^s}{s} \alpha^s(a_{(m)}^k) \cos(su).
\]
The RHS of (3.7) can be written as
\[
\int e^{iux} \left\{ \sum_{s \geq 1} \frac{(-1)^s}{s} \alpha^s(a_{(m)}^k) [\delta_s + \delta_{-s}](x) \right\} := - \int e^{iux} \mu^{(m)}_k(dx).
\]
The signed measure \(\mu^{(m)}_k\) is finite:
\[
\int \mu^{(m)}_k(dx) = 2 \sum_{s \geq 1} \frac{(-1)^s}{s} \alpha^s(a_{(m)}^k) = \ln[1 + \alpha(a_{(m)}^k)] = 2 \ln \left( \frac{2}{1 + \sqrt{1 - a_{(m)}^k}} \right),
\]
has finite moments of all orders and obviously satisfies
\[
\int x 1_{|x| < 1} \mu^{(m)}_k(dx) = 0.
\]
Combining (3.5), (3.6) and (3.7), the proposition is proved. \qed
4. AN INFINITELY DIVISIBLE DISTRIBUTION

The preceding computations show that
\[
\ln \mathbb{E}[e^{iuX_m}] = imu + \lambda \int (e^{iu} - 1) \left\{ \delta_1 + \sum_{k=1}^{m} \mu_k^{(m)} \right\} (dx)
\]
which implies again that \(X_m\) is not infinitely divisible since the representative measure is signed. Nonetheless, we have seen that apart from positivity, \(\mu_k^{(m)}\) share the same properties (3.4) of Lévy measures. Since its total variation
\[
|\mu_k^{(m)}| := \sum_{s \geq 1} \frac{1}{s} \alpha^s(a_k^{(m)}) [\delta_s + \delta_{-s}](x)
\]
does so and is positive, then
\[
\delta_1 + \sum_{k=1}^{m} |\mu_k^{(m)}|
\]
is a Lévy measure and the corresponding Lévy-Kintchine representation
\[
imu + \lambda \int (e^{iu} - 1 - iux1_{|x|<1}) \left\{ \delta_1 + \sum_{k=1}^{m} |\mu_k^{(m)}| \right\} (dx)
\]
gives rise to an infinitely divisible probability distribution. Using the linearization formula (4.1)
\[
\cos^{2j}(u/2) = \frac{1}{2^{2j}} \left\{ \binom{2j}{j} + 2 \sum_{s=1}^{j} \binom{2j}{j-s} \cos(su) \right\}.
\]
(see [5], p.31) and reading backward the proof of the previous proposition, we deduce that the characteristic function of this new distribution reads
\[
\exp(\lambda (e^{iu} - 1)) \frac{1}{L_m^{(0)} (2\lambda (1 + \cos u))}.
\]
Recalling \(a_k^{(m)} = 4\lambda / x_k^{(m)} \in (0, 1)\) then the expansion
\[
\frac{1}{L_m^{(0)} (2\lambda (1 + \cos u))} = \prod_{k=1}^{m} \frac{1}{1 - a_k^{(m)} \cos^2 u} = \sum_{\tau_1, \ldots, \tau_m \geq 0} \prod_{j=1}^{m} \left[ (x_j^{(m)})^{-\tau_j} [4\lambda \cos u]^{2(\tau_1 + \cdots + \tau_m)} \right]
\]
then with the linearization formula (4.1) show that \(u \mapsto \left[ L_m^{(0)} (2\lambda (1 + \cos u)) \right]^{-1}\) is the characteristic function of a discrete probability distribution, say \(\kappa_m\). Besides, if we denote by \(\tau = (\tau_1 \geq \tau_2 \geq \cdots \geq \tau_m \geq 0) \in \mathbb{N}^m\) a partition of length \(m\) and weight \(|\tau| := \tau_1 + \cdots + \tau_m\), then
\[
\frac{1}{L_m^{(0)} (2\lambda (1 + \cos u))} = m! \sum_{\tau} \prod_{j=1}^{m} \left[ (x_j^{(m)})^{-\tau_j} [4\lambda \cos u]^{2|\tau|} \right].
\]
Consequently, the weights of $\kappa_m$ are seen to be

$$\kappa_{m\{0\}} = m! \sum_{\tau} \left( \frac{2|\tau|}{|\tau|} \right)^{m} \prod_{j=1}^{m} \left( \frac{\lambda}{x_j^{(m)}} \right)^{\tau_j}$$

and for any integer $n \geq 1$,

$$\kappa_{m\{-n\}} = \kappa_{m\{n\}} = 2m! \sum_{\tau,|\tau|\geq n} \left( \frac{2|\tau|}{|\tau| - n} \right)^{m} \prod_{j=1}^{m} \left( \frac{\lambda}{x_j^{(m)}} \right)^{\tau_j}.$$

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