THE CUNTZ SEMIGROUP OF SOME SPACES OF DIMENSION AT MOST 2

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Abstract. The Cuntz semigroup is computed for spaces $X$ of dimension at most 2 such that $\tilde{H}^2(X,\mathbb{Z}) = 0$. This computation is then extended to spaces of dimension at most 2 such that $\tilde{H}^2(X\setminus\{x\},\mathbb{Z}) = 0$ for all $x \in X$ (e.g., any compact surface). It is also shown that for these two classes of spaces the Cuntz equivalence of countably generated Hilbert $C^*$-modules (over $C_0(X)$) amounts to their isomorphism.

1. Introduction

Let $X$ be a locally compact Hausdorff space of dimension at most 2. Let $\text{Cu}(X)$ denote the (stabilized) Cuntz semigroup of the $C^*$-algebra $C_0(X)$. This semigroup can be defined in terms of positive elements of $C_0(X) \otimes \mathcal{K}$ or countably generated Hilbert $C^*$-modules over $C_0(X)$ (see [3]). In this paper the latter definition is used to compute the Cuntz semigroup of all spaces of dimension 0 or 1, and of some spaces of dimension 2 (see Theorems 1 and 2 below). In doing this computation, we shall also see that the relations of Cuntz equivalence and of isomorphism coincide for the countably generated Hilbert $C^*$-modules over the spaces covered by our results. Thus, by computing the Cuntz semigroup we are giving a description of the isomorphism classes of countably generated Hilbert $C^*$-modules over such spaces.

Let $\mathbb{N}$ denote the set of natural numbers with 0 and $\infty$ adjoined. Let $\text{LSC}_\sigma(X,\mathbb{N})$ denote the ordered semigroup of lower semicontinuous functions $f: X \to \mathbb{N}$ such that the open set $f^{-1}((n, \infty])$ is $\sigma$-compact for all $n \geq 0$.

Theorem 1. Let $X$ be a locally compact Hausdorff of covering dimension at most 2 and such that $\tilde{H}^2(X,\mathbb{Z}) = 0$ (the Čech cohomology with integer coefficients).

(i) The ordered semigroup $\text{Cu}(X)$ is isomorphic to $\text{LSC}_\sigma(X,\mathbb{N})$.

(ii) Cuntz equivalent Hilbert $C^*$-modules over $C_0(X)$ are isomorphic.

The isomorphism from $\text{Cu}(X)$ to $\text{LSC}_\sigma(X,\mathbb{N})$ of the above theorem is given by the map $[H] \mapsto \dim([H])$, where $H$ is a countably generated Hilbert $C^*$-module and $\dim([H])(x) := \dim H_x$ for all $x \in X$. Here $H_x$ denotes the Hilbert space $H_x = H/HC_0(X\setminus\{x\})$. We call $H_x$ the fibre of $H$ over $x$.

Theorem 1 covers all spaces of dimension 0 or 1. The case $\dim(X) = 0$ was obtained previously by Perera in [10] and the case $X = [0, 1]$ is implicit.
in Ivanescu’s proof of [7, Theorem 2.3]. For spaces of dimension 0 and 1, Theorem 1 was conjectured by Kris Coward in his Ph.D. thesis (see [4]).

Let \( X \) be a connected compact Hausdorff space of dimension 2. Let \( V(X) \) denote the semigroup of isomorphism classes of finitely generated projective modules over \( C(X) \). Let \((LSC_\sigma(X,\mathbb{N}) \cup V(X))/\sim\) be the quotient of the disjoint union of \( LSC_\sigma(X,\mathbb{N}) \) and \( V(X) \) obtained by identifying the class \( n[C(X)] \) in \( V(X) \) with the constant function \( n \) in \( LSC_\sigma(X,\mathbb{N}) \). Let us define on this set an order and an addition operation. Inside the sets \( LSC_\sigma(X,\mathbb{N}) \) and \( V(X) \), we retain the order and addition with which these sets are endowed. Suppose \( f \in LSC_\sigma(X,\mathbb{N}) \) is a non-constant function and \([P] \in V(X)\). Let \( f + [P] \in LSC_\sigma(X,\mathbb{N}) \) be the function \( f + \text{Dim}([P]) \). Let the order relation be \( f \leq [P] \) if \( f \leq \text{Dim}([P]) \) and \([P] \leq f \) if \( \text{Dim}([P]) \leq f \).

**Theorem 2.** Let \( X \) be a connected compact Hausdorff space of dimension 2 such that \( H^2(X\setminus\{x\},\mathbb{Z}) = 0 \) for all \( x \in X \).

(i) The ordered semigroup \( \text{Cu}(X) \) is isomorphic to \((LSC_\sigma(X,\mathbb{N}) \cup V(X))/\sim\).

(ii) Cuntz equivalent Hilbert \( C^\ast \)-modules over \( C(X) \) are isomorphic.

From the classification of compact surfaces, any compact surface \( X \) satisfies that \( X\setminus\{x\} \) is homotopic to a 1-dimensional space for all \( x \in X \). Thus, all compact surfaces satisfy the hypotheses of Theorem 2.

Theorems 1 and 2 will be obtained as an application of two results on Hilbert \( C^\ast \)-modules, Corollary 1 and Proposition 1 below, which are interesting in their own rights. The next section is dedicated to proving these results. The section thereafter contains the proofs of the theorems on the computation of the Cuntz semigroup.

### 2. Results on Hilbert \( C^\ast \)-modules

All Hilbert \( C^\ast \)-modules are assumed to be right modules. For a Hilbert \( C^\ast \)-module \( H \) over a \( C^\ast \)-algebra \( A \), and a closed two-sided ideal \( I \) of \( A \), \( HI \) denotes the submodule of \( H \) of elements of the form \( h \cdot i \), with \( h \in H \) and \( i \in I \). The quotient \( H/HI \) is endowed with its natural structure of \( A/I \)-Hilbert \( C^\ast \)-module. If \( H \) is a Hilbert \( C^\ast \)-module over \( C_0(X) \), \( H_x \) denotes the Hilbert space \( H/HC_0(X\setminus\{x\}) \). The notation \( l_2(C_0(X)) \) is used for the \( C_0(X) \)-Hilbert \( C^\ast \)-module \( C_0(X) \otimes l_2(\mathbb{N}) \) (i.e., the infinite sequences \((f_i), f_i \in C_0(X)\), such that \( \sum_{i=1}^{\infty} f_i^* f_i \) is convergent in norm). Finally, \( \mathcal{K} \) denotes the \( C^\ast \)-algebra of compact operators over a separable Hilbert space.

**Lemma 1.** Let \( A \) be a \( C^\ast \)-algebra. Let \( a, b \in A^+ \) be of norm at most 1 and let \( \epsilon \) be such that \( \|a - b\| < \epsilon \). Then there is \( y \) such that \( y^* y = (a - \epsilon)_+, \ y y^* \leq b, \) and \( \|y y^* - b\| < 3\epsilon + 2\sqrt{\epsilon} \).

**Proof.** The proof of this lemma follows closely the first part of the proof of [9, Lemma 2.2]. Let \( \epsilon_1 \) be such that \( \|a - b\| < \epsilon_1 < \epsilon \). Let \( \epsilon \in C^\ast(a) \) be a positive contraction such that \( \epsilon(a - \epsilon_1) \epsilon = (a - \epsilon)_+ \). Set \( b^{1/2} \epsilon = x \) and let \( x = v|x| \) be the polar decomposition of \( x \), where \( v \in A^{**} \). From \( a - \epsilon_1 \leq b \) we
get \((a - \epsilon)_+ \leq ebe = x^*x\), and so the element \(v(a - \epsilon)_+^{1/2}\) belongs to \(A\). Set 
v(a - \epsilon)_+^{1/2} = y. We have \(y^*y = (a - \epsilon)_+\) and

\[yy^* = (a - \epsilon)_+v^*vxv^* = xx^* = b^{1/2}e^2b^{1/2} \leq b.\]

Also,

\[\|yy^* - b\| \leq \|v(a - \epsilon)_+ - vx^*vx\| + \|b^{1/2}e^2b^{1/2} - b\|.\]

With a few of simple computations it may be shown that the right-hand side is bounded by \(3\epsilon + 2\sqrt{\epsilon}\). This is done using that \(e, v, a\) and \(b\) are contractions, and that \(\|a^{1/2} - b^{1/2}\| < \sqrt{\epsilon}\) (this is implied by \(\|a - b\| < \epsilon\)). \(\Box\)

Given \(f \in (C_0(X) \otimes \mathcal{K})^+\) and \(x \in X\) we denote by \(\sigma_1(f)(x), \sigma_2(f)(x), \ldots\)
the eigenvalues of \(f(x)\) arranged in decreasing order.

**Theorem 3.** Let \(X\) be a locally compact Hausdorff space of dimension at most 2. Let \(B\) be a \(\sigma\)-unital hereditary subalgebra of \(C_0(X) \otimes \mathcal{K}\). Then the set of strictly positive elements \(f \in B^+\) such that all the non-zero eigenvalues of \(f(x)\) have multiplicity 1 for all \(x \in X\) is a dense subset of \(B^+\).

**Proof.** Let \(f_0 \in B^+\) be a strictly positive element. Let \(U_n = \{x \in X \mid \sigma_n(f_0)(x) > 0\}\). We remark that the sets \(U_n\) depend only on the algebra \(B\) and not on the choice of the strictly positive element \(f_0\).

Let \(K \subseteq U_n\) be compact and \(\epsilon > 0\). Let us denote by \(V(n, K, \delta)\) the subset of \(B^+\) of elements \(g \in B^+\) such that

(1) \(\sigma_n(g)(x) > \sigma_{n+1}(g)(x)\) for all \(x \in K\), and

(2) \(\|f_0 - gb\| < \delta\) for some \(b \in B\).

Let us show that \(V(n, K, \delta)\) is open and dense in \(B^+\). Neither of the properties (1) and (2) is destroyed by a sufficiently small perturbation of \(g\). Hence \(V(n, K, \delta)\) is open. Let us show that any \(f \in B^+\) may be approximated by elements of \(V(n, K, \delta)\). Since the set of strictly positive elements is dense in \(B^+\), we may assume that \(f\) is strictly positive. Then \(\sigma_n(f)(x)\) is non-zero for \(x \in K\) by the strict positivity of \(f\). Let \(\epsilon_0\) denote the minimum of \(\sigma_n(f)\) on \(K\).

By [3, Theorem 3.5] (see also [1, Theorem 1] for the metrizable case) we can approximate \(f\) by \(g_1 \in (C_0(X) \otimes \mathcal{K})^+\) such that the first \(n\) largest eigenvalues of \(g_1(x)\) are simple for all \(x \in X\). Let us choose \(g_1\) such that \(\|f - g_1\| < \epsilon\), where \(0 < \epsilon < \epsilon_0\). By Lemma[1], there is \(h \in C(X) \otimes \mathcal{K}\) such that \((g_1 - \epsilon)_+ = h^*h\) and \(hh^* \leq f\). Since the \(n\) largest nonzero eigenvalues of \(g_1\) have multiplicity 1, the same is true for \((g_1 - \epsilon)_+\), and hence also for \(hh^*\). Notice also that the \(n\)-th largest eigenvalue of \(hh^*\) is nonzero on \(K\), since we have chosen \(\epsilon < \epsilon_0\). Let us choose \(\epsilon_1 > 0\) small enough such that the \(n\)-th largest eigenvalue of \(\epsilon_1 f_0 + hh^*\) has multiplicity 1 for all \(x \in K\) and set \(\epsilon_1 f_0 + hh^* = g\). The function \(g\) satisfies that \(\epsilon_1 f_0 \leq g \leq (1 + \epsilon_1)f\), so \(g \in B^+\) and \(g\) satisfies (2) for any value of \(\delta > 0\). Notice also that by letting \(\epsilon\) and \(\epsilon_1\) be arbitrarily small the function \(g\) will get arbitrarily close to \(f\). Hence \(V(n, K, \delta)\) is dense in \(B^+\).

The open set \(U_n\) is \(\sigma\)-compact for all \(n\). So, there are compact sets \((K_{n,m})_{m=1}^\infty\) such that \(\bigcup_{m=1}^\infty K_{n,m} = U_n\). By the Baire category theorem the intersection \(\bigcap_{n,m,l=1}^\infty V(n, K_{n,m}, 1/2^l)\) is a dense subset of \(B^+\). It is easily verified that the elements of this intersection have the properties stated in the theorem. \(\Box\)
Corollary 1. Let $X$ be a locally compact Hausdorff space of dimension at most 2. Let $H$ be a countably generated Hilbert C*-module over $C_0(X)$. Then

$$H \cong \bigoplus_{i=1}^{\infty} P_i C_0(U_i),$$

where $(U_i)_{i=1}^{\infty}$ is a decreasing sequence of $\sigma$-compact open subsets of $X$, and $P_i$ are finitely generated projective modules over $C_0(U_i)$ of constant dimension 1 over their fibres.

Proof. Since $H$ is countably generated there is $f \in (C_0(X) \otimes K)^+$ such that $H \cong fl_2(C_0(X))$. By the previous theorem applied to the hereditary subalgebra generated by $f$, we may assume that the non-zero eigenvalues of $f(x)$ have multiplicity 1 for all $x \in X$. For $i = 1, 2, \ldots$ let $U_i = \{x \in X \mid \sigma_i(f)(x) > 0\}$. Then $f$ has the form $f = \sum_{i=1}^{\infty} p_i \sigma_i(f)$, where $p_i \in C_0(U_i) \otimes K$ is a rank 1 projection for all $i$ and the elements $(p_i \sigma_i(f))_{i=1}^{\infty}$ are mutually orthogonal. We have

$$fl_2(C_0(X)) \cong \bigoplus_{i=1}^{\infty} p_i \sigma_i(f) l_2(C_0(X)).$$

So, setting $p_i l_2(C_0(U_i)) = P_i$ we get the desired result. \hfill \Box

The second result used in the computation of the Cuntz semigroup applies to countably generated Hilbert C*-modules over an arbitrary C*-algebra.

Proposition 1. Let $I$ be a closed two-sided ideal of a C*-algebra $A$. Let $H$ and $E$ be countably generated Hilbert C*-modules over $A$ such that $H/HI$ is isomorphic to $E/EI$ as $A/I$-Hilbert C*-modules, and let $\phi: H/HI \to E/EI$ be an isomorphism between them. Then the Hilbert C*-modules $H \oplus EI$ and $E \oplus HI$ are isomorphic and there is an isomorphism $\Phi: H \oplus EI \to E \oplus HI$ that lifts $\phi$.

Proof. By the noncommutative Tietze extension Theorem (see [2, Theorem 3]), there is $\psi: H \to E$ that lifts $\phi$, and $\|\psi\| \leq 1$. Let $\Phi: H \oplus E \to E \oplus H$ be given by the matrix

$$\Phi = \begin{pmatrix} \psi & \sqrt{1 - \psi^* \psi} \\ \sqrt{1 - \psi \psi^*} & \psi^* \end{pmatrix}. $$

It is easily verified that $\Phi$ is an isomorphism of Hilbert C*-modules. Since $\psi$ lifts an isomorphism from $H/HI$ to $E/EI$, we have that the image of $1 - \psi \psi^*$ is contained in $EI$ and the image of $1 - \psi^* \psi$ is contained in $HI$. The same is true for the square roots of these operators. This implies that the restriction of $\Phi$ to $H \oplus EI$ is an isomorphism from $H \oplus EI$ to $E \oplus HI$. \hfill \Box

Remark 4. The previous theorem can be used in combination with Kasparov’s Stabilization Theorem to give an alternative proof of [2, Theorem 2].

Corollary 2. Let $I$ and $J$ be $\sigma$-unital closed two-sided ideals of $A$ and let $H$ be a countably generated Hilbert C*-module over $A$. Then

$$HI \oplus HJ \cong H(I + J) \oplus H(I \cap J).$$
Proof. This follows from Proposition 1 applied to the countably generated Hilbert C*-modules $H(I + J)$ and $HJ$, which are isomorphic after passing to the quotient by $I$.

3. Proof of Theorems 1 and 2

The following lemma clarifies the role of the cohomological condition assumed in Theorems 1 and 2.

Lemma 2. Let $X$ be a locally compact Hausdorff space such that $\dim X \leq 2$ and $\check{H}^2(X) = 0$. Then for any $\sigma$-compact open subset $U$ of $X$ and finitely generated projective module $P$ over $C_b(U)$ of dimension 1 on its fibres we have $P \cong C_0(U)$.

Proof. By the exact sequence in cohomology we deduce from $\dim X \leq 2$ and $\check{H}^2(X) \leq 0$ that $\check{H}^2(U) = 0$. The projective modules over $C_b(U)$ are in bijective correspondence with the vector bundles of finite type over $U$ (see [11]). The line bundles over $U$ are classified by their first Chern class, which takes values in $\check{H}^2(U)$ (see [6] and [11]). Therefore, every projective module over $C_b(U)$ of dimension 1 over its fibres is trivial, i.e., isomorphic to $C_0(U)$. □

Proof of Theorem 1. By Corollary 1 and the previous lemma, every countably generated Hilbert C*-module over $C_0(X)$ is isomorphic to a module of the form $\bigoplus_{i=1}^\infty C_0(U_i)$, where $(U_i)_{i=1}^\infty$ is a decreasing sequence of $\sigma$-compact open subsets of $X$.

Let $H = \bigoplus_{i=1}^\infty C_0(U_i)$ for some decreasing sequence of $\sigma$-compact open sets $(U_i)_{i=1}^\infty$, where $1_{U_i}$ denotes the characteristic function of $U_i$. Since every function $f$ in $\text{LSC}_\sigma(X, \overline{N})$ can be represented in the form $\sum_{i=1}^\infty 1_{U_i}$ (with $U_i = f^{-1}((i, \infty])$), the map $\text{Dim}: \text{Cu}(X) \to \text{LSC}_\sigma(X, \overline{N})$ is surjective. If the sequence of open sets $(U_i)_{i=1}^\infty$ is required to be decreasing then the representation of $f$ in this form is unique. Hence $\text{Dim}(\cdot)$ is also injective. It is easily verified that $\text{Dim}(\cdot)$ preserves order and addition.

Finally, notice that the isomorphism class of $H$ is uniquely determined by the function $\text{Dim}([H])$. Hence, Cuntz equivalent modules over $C_0(X)$ are isomorphic. □

Proof of Theorem 2. Let $P$ be a finitely generated projective module over $C(X)$ of constant dimension 1, and let $U$ be a nonempty proper subset of $X$. Since $\check{H}^2(U) = 0$, $PC_0(U) \cong C_0(U)$ as $C(X)$-Hilbert C*-modules. Also, $P$ and $C(X)$ become isomorphic (as $C_0(X \setminus U)$-modules) after passing to the quotient by $C_0(U)$, because $X \setminus U \subseteq X \setminus \{x'\}$ for some $x' \in X$ and $\check{H}^2(X \setminus \{x'\}) = 0$. Therefore, by Proposition 1 $P \oplus C_0(U) \cong C_0(X) \oplus C_0(U)$ as $C(X)$-Hilbert C*-modules.

Consider now an arbitrary countably generated Hilbert C*-module $H$ over $C(X)$. By Corollary 1 we may assume it has the form $\bigoplus_{i=1}^\infty P_iC_0(U_i)$. Suppose that there are indices $i$ such that the subset $U_i$ is proper and nonempty. Let $i_0$ be the smallest such index. Then $P_iC_0(U_j) \cong C_0(U_j)$ for $j \geq i_0$. Also, by
the remarks of the previous paragraph, we have

\[ P_1 \oplus P_2 \oplus \cdots \oplus P_{i_0-1} \oplus C_0(U_{i_0}) \cong C(X) \oplus C(X) \oplus \cdots \oplus C(X) \oplus C_0(U_{i_0}). \]

Hence, \( H \cong \bigoplus_{i=1}^{\infty} C_0(U_i) \). Suppose on the other hand that the subsets \( U_i \) are never proper and nonempty. If \( U_i = X \) for all \( i \) then \( H \cong l_2(C(X)) \) (by Theorem \( 5 \)). If \( U_i \) is eventually empty then \( H \) is a finitely generated projective module over \( C(X) \). Summarizing, every countably generated Hilbert \( \text{C}^* \)-module over \( C(X) \) is either a finitely generated projective module or has the form \( \bigoplus_{i=1}^{\infty} C_0(U_i) \), where \( (U_i)_{i=1}^{\infty} \) is a decreasing sequence of \( \sigma \)-compact open subsets of \( X \).

Let us map the elements of \( \text{Cu}(X) \) to elements of \( \text{LSC}_\sigma(X, \mathbb{N}) \sqcup V(X)/\sim \) by mapping

\[ \bigoplus_{i=1}^{\infty} C_0(U_i) \quad [Q] \quad \sum_{i=1}^{\infty} 1_{U_i}. \]

Let \( Q \) be a finitely generated projective module of constant dimension \( m \) over its fibres and let \( H = \bigoplus_{i=1}^{\infty} C_0(U_i) \), with at least one \( U_i \) proper and nonempty; say \( U_{i_0} \) is the first such set. Since the module \( Q \oplus H \) has non-constant dimension function (equal to \( m + \text{Dim}([H]) \)), we must have \( Q \oplus H \cong C(X)^m \oplus \bigoplus_{i=1}^{\infty} C_0(U_i) \). This shows that the map defined by (1) is linear.

Let \( Q \) and \( H \) be as before. Suppose that \( \text{Dim}([Q]) = m \) and \( \text{Dim}([H]) \geq m \). Notice that \( i_0 \geq m \). By Corollary [1], \( Q \cong P_1 \oplus P_2 \cdots \oplus P_m \), for some projective modules \( P_i, i = 1, 2, \ldots, m \), of constant dimension 1. For every such module \( P_i \), we have \( P_i \oplus C(U_{i_0}) \cong C(X) \oplus C(U_{i_0}) \), and so \( P_i \) is isomorphic to a submodule of \( C(X) \oplus C(U_{i_0}) \). Hence, \( P_1 \oplus P_2 \cdots \oplus P_m \) is isomorphic to a submodule of \( C(X)^m \oplus C(U_{i_0}) \). Since \( i_0 \geq m \), we have \( [Q] \leq [H] \).

Suppose on the other hand that \( \text{Dim}([Q]) = m \) and \( \text{Dim}([H]) \leq m \). Then \( H = C_0(U_1) \oplus \cdots \oplus C_k(U_k) \) for some \( k \leq m \). The modules \( P_1 C_0(U_1) \oplus \cdots \oplus P_k C_0(U_k) \) and \( H \) are isomorphic, since they have the same (non-constant) dimension function. On the other hand, \( P_1 C_0(U_1) \oplus \cdots \oplus P_k C_0(U_k) \) is clearly a submodule of \( Q \). It follows that \( [H] \leq [Q] \). This completes proving that the map defined by (1) is an isomorphism of ordered semigroups.

Let us show that the relations of Cuntz equivalence and isomorphism coincide for the Hilbert modules over \( C(X) \). It follows from the arguments given before that if the Hilbert module \( H \) has non-constant dimension function, or constant infinite dimension, then its isomorphism class is determined by \( \text{Dim}([H]) \), which only depends on its Cuntz equivalence class. Thus, the two relations coincide in this case. If \( H \) has constant finite rank then it is a projective module over \( C(X) \). It is known that in this case the Cuntz equivalence class of \( H \) coincides with its isomorphism class (see [4]). \( \square \)

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