Correlations in interference and diffraction

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Abstract

Quantum formalism of Fraunhofer diffraction is obtained. The state of the diffraction optical field is connected with the state of the incident optical field by a diffraction factor. Based on this formalism, correlations of the diffraction modes are calculated with different kinds of incident optical fields. Influence of correlations of the incident modes on the diffraction pattern is analyzed and an explanation of the "ghost" diffraction is proposed.

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1 Introduction

Correlations of states play an important role in quantum cryptography \[1,2\], teleportation \[3,4\], and computation \[5\textsuperscript{−}11\] theory. Correlated states are generated usually by nonlinear optical processes\[12\] or by the beam splitter\[13\]. In this paper, we consider correlations in interference and diffraction. On the one hand, the diffraction or interference modes have some interesting correlation properties. On the other hand, correlations of the incident modes has a notable influence on the interference or diffraction pattern, in particular, it is the key to the explanation of the ”ghost” diffraction\[14\], an interesting quantum effect. Interference can be regarded as a special case of diffraction. To analyze correlations in interference and diffraction, we need a quantum formalism of diffraction. In the early days of quantum electrodynamics (QED), it had been proved that the Maxwell equations which underpin diffraction remain true when the fields are quantized\[15\textsuperscript{−}17\]. In quantum optics the entire mode structure of the diffraction field is still determined by the Helmholtz part of the wave equation. The role played by quantum mechanics is in determining the states of the diffraction modes from the states of the incident modes. However, no systematic approach in determining the states of the diffraction modes has been proposed. In this paper, we first solve this problem. By introducing the quantum Kirchhoff boundary condition, we connect the states of the diffraction modes with the states of the incident modes by a diffraction factor. Then correlations of the diffraction modes with different kinds of incident optical fields are calculated. Influence of correlations of the incident modes on the diffraction pattern is analyzed. The ”ghost” diffraction is also explained based on this formalism.

We consider Fraunhofer diffraction. This kind of diffraction is most important. In Section 2, we introduce the equivalent scalar optical field and the quantum Kirchhoff boundary condition. The equivalent scalar optical field simplifies the problem of scalar diffraction, in which the variation of polarization through diffraction is not considered. The quantum Kirchhoff boundary condition is equivalent in physics to the Kirchhoff boundary condition in classical scalar diffraction yet overcomes the difficulty that the classical Kirchhoff boundary condition destroys the commutation relations of the field operators. In section 3, we obtain quantum formalism of Fraunhofer diffraction. The normal characteristic functions of the diffraction modes are connected with those of the incident modes by a diffraction factor. From the characteristic functions,
correlation properties of the diffraction modes are analyzed in Sec. 4. In this section the diffraction pattern is also calculated with entangled incident states. An explanation of the ”ghost” diffraction is proposed.

2 The equivalent optical field and the quantum Kirchhoff boundary condition

In the diffraction problem the incident and diffraction optical fields are free. The free quantized electromagnetic field can be expanded into plane wave modes:

\[
\vec{E} = \sum_{-\vec{k}, \mu=1,2} i \sqrt{\frac{\hbar \omega}{2V}} a_{-\vec{k}, \mu} \vec{e}^{-i (\vec{k} \cdot \vec{r} - \omega t)} + h.c.,
\]  

where \( \mu \) is polarization index and \( \vec{k} \cdot \vec{e}_{-\vec{k}, \mu} = 0 \). The annihilation and creation operators \( a_{-\vec{k}, \mu}, a^+_{-\vec{k}', \mu'} \) satisfy the commutation relation

\[
\left[ a_{-\vec{k}, \mu}, a^+_{-\vec{k}', \mu'} \right] = \delta_{\vec{k}, \vec{k}'} \delta_{\mu, \mu'}. \tag{2}
\]

The frequency of the optical field remains unchanged through diffraction. So we only need consider fields with a definite frequency \( \omega \). That is, in the expansion (1) only the terms with \( |\vec{k}| = \frac{\omega}{c} \) need be considered. Let \( \vec{k} = (k_x, k_y, \sqrt{\frac{\omega^2}{c^2} - k_x^2 - k_y^2}) \) and \( \vec{k} \) has only two degrees of freedom \( k_x, k_y \) (The symbols \( \vec{k} \) below all have this meaning). The incident and diffraction optical fields are in a half space. Suppose the plane \( z = 0 \) is the diffraction plane, then there may exist evanescent waves with a depression factor \( e^{-|k_z|z} \) at both sides of the diffraction plane. So the value domains of \( k_x, k_y \) are \((-\infty, +\infty)\), i.e., \( k_z \) can be imaginary. This is different from the plane wave expansions in the whole space.

In scalar diffraction theory, the boundary condition at the diffraction plane is independent of the orientation of the optical field, and the variation of polarization of the optical field through diffraction need not be considered. So we can introduce the following equivalent scalar optical field by neglecting the polarization index.

\[
\varepsilon \left( \vec{r} \right) = \frac{1}{\sqrt{S}} \sum_{k_x, k_y} a_{-\vec{k}} e^{i \vec{k} \cdot \vec{r}}, \tag{3}
\]

where the box-normality of space has been used and \( S \) is the cross-section area of the box. The commutator (2) yields the following commutation relation of
the equivalent optical field at the diffraction plane $z = 0$

$$\left[ \varepsilon (x, y, 0), \varepsilon^+ (x', y', 0) \right] = \delta (x - x') \delta (y - y'). \quad (4)$$

In scalar diffraction the equivalent scalar optical field can be in place of the real optical field. The diffraction problem is much simplified by introducing the equivalent scalar optical field.

In classical scalar diffraction theory the Kirchhoff boundary condition states: the optical field remains unchanged through the diffraction aperture $\Sigma$ and decays to zero through the diffraction screen\cite{18}. This boundary condition can not be used directly in the quantum case because the postulate that the optical field decays to zero through the diffraction screen destroys the commutation relations of the field operators. To keep consistent with quantum theory, we introduce the following quantum Kirchhoff boundary condition. The equivalent optical field $\varepsilon (x, y, z = 0)$ before diffraction is generally in a complicated entangled state and we use $\rho (z = 0^-)$ to represent its whole density operator. The quantum Kirchhoff boundary condition says: When passing the diffraction screen all modes of the field $\varepsilon (x, y, z = 0)$ ($x, y \in S - \Sigma$) at the screen undergo such a strong dissipation that after the screen they are all in the vacuum state. At the same time, the modes of the field $\varepsilon (x, y, z = 0)$ ($x, y \in \Sigma$) at the aperture undergo no dissipation at all. From quantum dissipation theory\cite{19,20}, the total density operator $\rho (z = 0^+)$ after diffraction is expressed as

$$\rho (z = 0^+) = tr_{S - \Sigma} \rho (z = 0^-) \otimes \prod_{(x, y) \in S - \Sigma} |0\rangle_{xy} \langle 0|, \quad (5)$$

where the notation $tr_{S - \Sigma}$ indicates trace of all modes at the screen. This boundary condition for scalar diffraction is equivalent in physics to the classical Kirchhoff boundary condition. Yet it is consistent with quantum mechanics as it results from the quantum dissipation theory. In next section we use this boundary condition to derive quantum formalism of diffraction.

### 3 Quantum formalism of Fraunhofer diffraction

In Fraunhofer diffraction the incident and diffraction optical fields are expanded into the plane wave modes and the role played by quantum mechanics is in determining the states of the diffraction modes from the states of the incident modes. Let $a_{\vec{k}'}$ and $b_{\vec{k}}$ represent the annihilation operators of the incident
mode \( \vec{k}' \) and the diffraction mode \( \vec{k} \), respectively. \( \rho \left( a_{\vec{k}_0} \right) \) is the density operator of the incident mode \( a_{\vec{k}_0} \) and other incident modes are supposed in the vacuum state. First we derive the reduced normal characteristic function \( \chi^{(n)} \left( b_{\vec{k}}; \xi \right) \) of the diffraction mode \( b_{\vec{k}} \). Using Eq. (5) and the inverse transformation of Eq. (3)

\[
a_{\vec{k}} = \frac{1}{\sqrt{S}} \int_S dx dy \varepsilon \left( \vec{r} \right) e^{-i \vec{k} \cdot \vec{r}},
\]
we get

\[
\chi^{(n)} \left( b_{\vec{k}}; \xi \right) = \langle e^{i \xi^* b_{\vec{k}}^+} \cdot e^{i \xi b_{\vec{k}}^+} \rangle
= Tr \left\{ tr_{S_0} \rho \left( 0 \right) \otimes \prod_{(x,y) \in \Sigma_{S}} \left| 0 \right>_{xy} \left< 0 \right| \cdot \exp \left[ i \xi^* \frac{1}{\sqrt{S}} \int_S dx dy \varepsilon^+ \left( x, y, 0 \right) e^{i(k_x x + k_y y)} \right] \cdot \exp \left[ i \xi \frac{1}{\sqrt{S}} \int_S dx dy \varepsilon \left( x, y, 0 \right) e^{-i(k_x x + k_y y)} \right] \right\}
= Tr \left\{ \rho \left( a_{\vec{k}_0} \right) \otimes \prod_{\vec{k}_0' \neq \vec{k}_0} \left| 0 \right>_{\vec{k}_0'} \left< 0 \right| \exp \left\{ i \xi^* \sum_{\vec{k}_0'} a^+_{\vec{k}_0'} \frac{1}{\sqrt{S}} \int_{\Sigma_{S}} dx dy e^{i \left[ \left( k_{x0} - k_{x0}' \right) x + \left( k_{y0} - k_{y0}' \right) y \right]} \right\}
\cdot \exp \left\{ i \xi \sum_{\vec{k}_0'} a_{\vec{k}_0'} \frac{1}{\sqrt{S}} \int_{\Sigma_{S}} dx dy e^{-i \left[ \left( k_{x0} - k_{x0}' \right) x + \left( k_{y0} - k_{y0}' \right) y \right]} \right\} \right\},
\]

where \( \Sigma \) and \( S \) represent area of the diffraction aperture and the whole diffraction plane, respectively, and the notation \( Tr \) indicates trace of all modes. We define the energy transmissivity \( \lambda \) as \( \lambda = \frac{\Sigma}{S} \). Its physical meaning is the ratio of the energy of the diffraction optical field to the energy of the incident optical field. The Fraunhofer diffraction factor \( f \left( \vec{k} \right) \) is defined as

\[
f \left( \vec{k} \right) = \frac{\sqrt{\lambda}}{\Sigma} \int_{\Sigma} e^{-i(k_x x + k_y y)} dx dy.
\]

\( f \left( \vec{k} \right) \) is normalized by

\[
\sum_{\vec{k}} f^* \left( \vec{k} \right) f \left( \vec{k} \right) = 1.
\]

Eq. (7) is therefore simplified to

\[
\chi^{(n)} \left( b_{\vec{k}}; \xi \right) = \chi^{(n)} \left[ a_{\vec{k}_0} \sqrt{\lambda} x f \left( \vec{k} - \vec{k}_0' \right) \right].
\]
Eq. (10) connects the reduced normal characteristic function of the diffraction mode \( b_k \) with that of the incident mode \( a_{k_0} \) by a simple diffraction factor.

Similar to the derivation of Eq. (10), the total normal characteristic function of all diffraction modes \( \{ b_k \} \) has the form

\[
\chi_T^{(n)} \left( \{ b_k \}, \{ \xi_k \} \right) = \left< e^{i \sum_k \xi_k^* b_k^+ e^{-ik \cdot \xi}} \right>.
\]

(11)

The above results are obtained with the supposition that only the incident mode \( k_0 \) is not in the vacuum state. If all the incident modes are in an entangled state, and we use \( \chi_T^{(n)} \left( \{ a_{k'} \}, \{ \xi_{k'} \} \right) \) to indicate its whole normal characteristic function. Eq. (11) can thus be generalized to

\[
\chi_T^{(n)} \left( \{ b_k \}, \{ \xi_k \} \right) = \chi_T^{(n)} \left[ \{ a_{k'} \}; \sqrt{\lambda} \sum_k \xi_k f \left( \hat{k} - \hat{k'} \right) \right].
\]

(12)

Eq. (12) determines the states of all diffraction modes from the states of the incident modes. It is a fundamental equation in the quantum formalism of Fraunhofer diffraction.

The final result (12) is similar to the quantum description of the beam splitter. For the beam splitter, the input and output modes are linked by a canonical transformation

\[
\left( \begin{array}{c} b_1 \\ b_2 \end{array} \right) = \left( \begin{array}{cc} r & t \\ -t & r \end{array} \right) \left( \begin{array}{c} a_1 \\ a_2 \end{array} \right),
\]

(13)

where \( a_1, a_2 \) are input operators and \( b_1, b_2 \) are output operators. The parameters \( r \) and \( t \) should satisfy \( r^2 + t^2 = 1 \). From Eq. (13), we obtain the relation of the normal characteristic function between the input and output modes

\[
\chi^{(n)}(b_1, b_2; \xi_1, \xi_2) = \chi^{(n)}(a_1, a_2; r\xi_1 - t\xi_2, t\xi_1 + r\xi_2).
\]

(14)

Eqs. (14) and (12) are very alike in the form. However, some important differences lie in their derivation. In diffraction the input and output modes cannot be put in a canonical transformation, which may be seen from the relation

\[
\sum_{\hat{k}} \sqrt{\lambda} f^* \left( \hat{k} - \hat{k'} \right) \sqrt{\lambda} f \left( \hat{k} - \hat{k'} \right) = \lambda < 1.
\]

(15)
Only when the energy transmissivity $\lambda = \frac{\Sigma}{S} = 1$, i.e., when there is no diffraction screen, the input and output modes can be linked by a trivial canonical transformation. So unlike Eq. (14), Eq. (12) is not a direct result of the input-output theory$^{[22,19]}$. In the derivation of Eq. (12), the quantum Kirchhoff boundary condition plays an essential role.

The general equation (12) can describe interference as well as diffraction. If there are two diffraction apertures $\Sigma_1, \Sigma_2$, the diffraction factor $f\left(\vec{\kappa}\right)$ simply becomes

$$f\left(\vec{\kappa}\right) = \sqrt{\lambda} \int_{\Sigma_1 + \Sigma_2} e^{-i(k_xx + k_yy)} dxdy,$$

(16)

where $\lambda = \frac{\Sigma_1 + \Sigma_2}{S}$. When $\Sigma_1, \Sigma_2$ tend to zero, Eq. (12) with this $f\left(\vec{\kappa}\right)$ gives quantum description of the double-slit interference.

4 Correlations in interference and diffraction

4.1 correlations of the diffraction (or interference) modes

In this subsection we consider correlations of the diffraction modes. Suppose all the incident modes except $\vec{k}_0'$ are in the vacuum state. First we show that the diffraction modes are not correlated only when the incident mode $\vec{k}_0'$ is in a coherent state. If the diffraction modes are independent, the decomposition $\chi_T^{(n)}\left(\{b_{\vec{k}}\}, \{\xi_{\vec{k}}\}\right) = \prod_{\vec{k}} \chi^{(n)}\left(b_{\vec{k}}, \xi_{\vec{k}}\right)$ should hold. From Eq. (11) this decomposition holds if and only if $\chi^{(n)}\left[a_{\vec{k}_0'}; \xi\right]$ has the following form

$$\chi^{(n)}\left[a_{\vec{k}_0'}; \xi\right] = e^{i(\xi^*\alpha + \xi\alpha)},$$

(17)

i.e., the incident mode is in a coherent state. Under this condition, the diffraction modes are not correlated and all in coherent states. With any other kinds of incident optical fields the diffraction modes are in an entangled state.

The above discussion shows that the diffraction modes are generally correlated. In experiments correlation of the photon number is widely used, so we first calculate the correlation coefficient of the photon number of two diffraction modes. The correlation coefficient is defined by

$$\eta = \frac{\left\langle \Delta n_{\vec{k}_1} \Delta n_{\vec{k}_2} \right\rangle}{\sqrt{\left\langle \left(\Delta n_{\vec{k}_1}\right)^2 \right\rangle \left\langle \left(\Delta n_{\vec{k}_2}\right)^2 \right\rangle}},$$

(18)
where \( n_{\vec{k}_i} \) \((i = 1, 2)\) denotes the number operator of the mode \( \vec{k}_i \). After some calculation, from Eq. (11) we obtain

\[
\eta = \frac{F_n - 1}{(F_n + h_1 - 1)(F_n + h_2 - 1)^{1/2}},
\]

where \( F_n \) is the Fano factor of the incident mode \( \vec{k}_0' \), i.e.,

\[
F_n = \frac{\left\langle \left( \Delta n_{\vec{k}_0'} \right)^2 \right\rangle}{\left\langle n_{\vec{k}_0'} \right\rangle},
\]

and \( h_i \) in Eq. (19) is defined by

\[
h_i = \frac{1}{\lambda |f(\vec{k}_1 - \vec{k}_0')|^2} \quad (i = 1, 2).
\]

The relation between \( \eta \) and \( F_n \) is illustrated in Fig. 1.

If the incident mode is in a thermal state, \( F_n = \left\langle n_{\vec{k}_0'} \right\rangle + 1 \) and \( \eta \) tends to its maximum value 1 with \( \left\langle n_{\vec{k}_0'} \right\rangle \gg 1 \). The correlation coefficient \( \eta \) gets its minimum value \(-[\left( h_1 - 1 \right) \left( h_2 - 1 \right)]^{-1/2}\) with the incident mode in a Fock state.

Though \( \eta \approx 1 \) if the incident mode is in a thermal state with \( \left\langle n_{\vec{k}_0'} \right\rangle \gg 1 \), the diffraction modes are not correlated perfectly in this case. That can be seen from residual variance of the variables \( n_{\vec{k}_1} \) and \( n_{\vec{k}_2} \) in the linear regression. The residual variance of the variable \( n_{\vec{k}_1} \) has the form[23]

\[
Var \left( n_{\vec{k}_1} - \beta_1 n_{\vec{k}_2} - \beta_2 \right) = \left\langle \left( \Delta n_{\vec{k}_1} \right)^2 \right\rangle (1 - \eta^2)
= \left\langle n_{\vec{k}_0'} \right\rangle h_1^{-1} \left( F_n + h_1 - 1 \right) (1 - \eta^2),
\]

where \( \beta_1 \) and \( \beta_2 \) are linear regression coefficients. Suppose \( h_1 = h_2 \) and \( \left\langle n_{\vec{k}_0'} \right\rangle \gg 1 \), then

\[
Var \left( n_{\vec{k}_1} - \beta_1 n_{\vec{k}_2} - \beta_2 \right) \approx \frac{2 \left\langle n_{\vec{k}_0'} \right\rangle}{h_1} = 2 \left\langle n_{\vec{k}_1} \right\rangle.
\]
So in this case the residual variance is very large. In fact, the equation \( \eta \to 1 \) under the condition \( \langle n_{\rightarrow k_0} \rangle \to \infty \) results from the infinite variance of \( n_{\rightarrow k_1} \) and \( n_{\rightarrow k_2} \). We can not conclude from \( \eta \to 1 \) that the diffraction modes are correlated perfectly.

For the beam splitter, the correlation of the number operator of the output modes has the same form as Eq.(19). However, there are still some differences. First, the equation \( \frac{1}{n_1} + \frac{1}{n_2} = 1 \) holds for the beam splitter whereas in diffraction we have \( \frac{1}{n_1} + \frac{1}{n_2} < 1 \). So for the beam splitter, the correlation coefficient of the output number operators can attain its minimum value -1 with the input mode in a Fock state. Second, in diffraction or interference correlations of many modes can be generated whereas the beam splitter is only used to prepare two-mode entangled states.

Correlation coefficients describe correlation properties of a pair of specialized operators. Several approaches to the description of quantum entanglement have been proposed. In particular, Schlienz and Mahler interpreted the difference between the entangled state and the product state as the entanglement\[24\]. Suppose \( \rho \) is the density operator of the whole system and \( \rho_a = tr_b \rho \), \( \rho_b = tr_a \rho \), where the subscripts \( a \) and \( b \) represent two subsystems. The Schlienz-Mahler measure is defined by\[24\]

\[
\gamma = \sqrt{\frac{N(\rho)}{N(\rho) - 1} tr \left[ (\rho - \rho_a \otimes \rho_b)^2 \right]},
\]

where \( N(\rho) \) indicates the dimension of the density operator \( \rho \) and \( \gamma \) defined above satisfies \( 0 \leq \gamma \leq 1 \). However, the more recent papers distinguish quantum entanglement from classical correlations\[25-30\]. The entanglement is interpreted as the degree of inseparability. The entangled state is said to be inseparable if it can not be expressed as a mixture of product states of two subsystems. In this interpretation, the Schlienz-Mahler quantity \( \gamma \) measures the total correlations rather than pure quantum entanglement. It is now believed that pure quantum entanglement can not be fully described by a single quantity\[29\]. Bennett et. al. defined two quantities:\[26,29\] ”entanglement of formation” defined as the least number of shared singlets asymptotically required to prepare \( \rho \) by local operations and classical communication, and ”distillable entanglement” defined as the greatest number of pure singlets that can asymptotically be prepared from \( \rho \) by local operations and classical communication. And recently, Vedral et. al. introduced a new measure of entanglement\[30\], which interprets the entangle-
ment as the minimum distance to all separable states. These measures have the desirable feature that their expectations can not be increased by local operations, but the disadvantage of being hard to evaluate because of the implied optimization. The question is still open in this direction.

Though the Schlienz-Mahler quantity \( \gamma \) in fact measures the total correlations, it is superior to the correlation coefficients, since it is not limited to specialized observables. In the following we use the Schlienz-Mahler quantity to analyze correlation properties of the diffraction modes. Before doing this, we first introduce the following lemma.

**Lemma.** Suppose \( \rho_1, \rho_2 \) are two density operators of boson fields, and \( \chi_1^{(n)}(\xi), \chi_2^{(n)}(\xi) \) are normal characteristic functions of \( \rho_1 \) and \( \rho_2 \), respectively, then we have

\[
\text{tr} (\rho_1 \rho_2) = \int \frac{d^2 \xi}{\pi} \chi_1^{(n)}(\xi) \chi_2^{(n)}(-\xi) e^{-|\xi|^2}.
\]

**(25)**

**Proof.** If generalized functions (such as derivatives of delta functions) are permitted, the existence proof of \( P \)-functions of Boson fields has been given by Klauder and Sudarshan\(^{[31,19]} \). So \( \text{tr} (\rho_1 \rho_2) \) can be expressed as

\[
\text{tr} (\rho_1 \rho_2) = \int P_1(\alpha) \langle \alpha | \rho_2 | \alpha \rangle \, d^2\alpha
\]

\[= \pi \int P_1(\alpha) Q_2(\alpha) \, d^2\alpha,
\]

where \( P_1(\alpha) \) and \( Q_2(\alpha) \) are \( P,Q\)-functions of the density operators \( \rho_1, \rho_2 \) respectively. The \( P,Q\)-functions are Fourier transformations of the normal and anti-normal characteristic functions. So Eq. (26) can be rewritten as

\[
\text{tr} (\rho_1 \rho_2) = \int \frac{d^2 \xi}{\pi} \chi_1^{(n)}(\xi) \chi_2^{(a)}(-\xi),
\]

**(27)**

where \( \chi^{(a)}(\xi) \) indicates the anti-normal characteristic function. Eq. (27) is equivalent to Eq. (25). This completes the proof.

We calculate the Schlienz-Mahler quantity \( \gamma \) with a thermal incident optical field. From Eq. (11) the normal characteristic function of the diffraction modes \( \overrightarrow{k}_1 \) and \( \overrightarrow{k}_2 \) has the form

\[
\chi_T^{(n)}(\overrightarrow{b}_{\overrightarrow{k}_1}, b_{\overrightarrow{k}_2}) = \frac{\langle N \rangle \sqrt{X}}{\pi} \left[ e^{\langle N \rangle} f(\overrightarrow{k}_1 - \overrightarrow{k}_0') \xi_{\overrightarrow{k}_1} f(\overrightarrow{k}_2 - \overrightarrow{k}_0') \xi_{\overrightarrow{k}_2} \right],
\]

**(28)**

where \( \langle N \rangle \) is the mean photon number of the incident mode. For thermal states, the dimension of the density operator \( N (\rho) \to \infty \). Eq. (24) together with Eq. (25) yields

\[
\gamma^2 = \frac{1}{x_1 x_2 - 4y^2} - \frac{2}{x_1 x_2 - y^2} + \frac{1}{x_1 x_2},
\]

**(29)**
where

\[ x_i = 2 \langle N \rangle \lambda \left| f \left( \vec{k}_i - \vec{k}_0 \right) \right|^2 + 1 \quad (i = 1, 2), \quad (30) \]

\[ y = \langle N \rangle \lambda \left| f \left( \vec{k}_1 - \vec{k}_0 \right) f \left( \vec{k}_2 - \vec{k}_0 \right) \right| \quad (31) \]

From Eq. (29) it is obvious that \( \gamma \) tends to zero if \( \langle N \rangle \rightarrow \infty \) or \( \langle N \rangle \rightarrow 0 \).

If the diffraction factor satisfies \( \left| f \left( \vec{k}_1 - \vec{k}_0 \right) \right| = \left| f \left( \vec{k}_2 - \vec{k}_0 \right) \right| \), we have

\[ x_1 = x_2 = 2y + 1 \quad \text{and} \quad \gamma \text{ is simplified to} \]

\[ \gamma = \left[ \frac{1}{4y + 1} - \frac{2}{(3y + 1)(y + 1)} + \frac{1}{(2y + 1)^2} \right]^{\frac{1}{2}}. \quad (32) \]

The relation between \( \gamma \) and \( y \) is illustrated in Fig. 2.

From the figure we see the Schlienz-Mahler quantity \( \gamma \) attains the maximum when \( y \approx 1.1 \). The maximum value is 0.25. With a larger or smaller mean photon number, the correlation of the diffraction modes decreases.

### 4.2 Influence of correlations of the incident modes on the diffraction (or interference) pattern

To show the influence of correlations on the diffraction pattern, we consider the circumstance with two incident modes being in an entangled state. The entangled state is prepared by a beam splitter with the input mode in a Fock state. From Eq. (14) the normal characteristic function of the two incident modes has the form

\[ \chi^{(n)} \left( a_{\vec{k}_1}, a_{\vec{k}_2}; \xi_{\vec{k}_1}, \xi_{\vec{k}_2} \right) = e^{i \left( r^* \xi_{\vec{k}_1} - t^* \xi_{\vec{k}_2} \right)} a^+ e^{i \left( r \xi_{\vec{k}_1} - t \xi_{\vec{k}_2} \right)} |n\rangle, \quad (33) \]

where \( a \) denotes the input mode of the beam splitter. The diffraction pattern is shown by the mean photon number distribution \( \langle n_{\vec{k}} \rangle \) of the diffraction modes. Eq. (12) gives

\[
\langle n_{\vec{k}} \rangle = \lambda \left\{ \left| f \left( \vec{k} - \vec{k}_1 \right) \right|^2 \langle n_{\vec{k}_1} \rangle + \left| f \left( \vec{k} - \vec{k}_2 \right) \right|^2 \langle n_{\vec{k}_2} \rangle \right. \\
+ \left. \left[ f^* \left( \vec{k} - \vec{k}_1 \right) f \left( \vec{k} - \vec{k}_2 \right) \langle a_{\vec{k}_1}^+, a_{\vec{k}_2} \rangle + h.c. \right] \right\} \\
= \lambda n \left| rf \left( \vec{k} - \vec{k}_1 \right) - tf \left( \vec{k} - \vec{k}_2 \right) \right|^2. \quad (34) \]
If the two incident modes are not correlated, i.e., if they are represented by the density operator $\rho_{k_1,k_2} \otimes \rho_{k',2}$, the mean photon number distribution of the diffraction modes becomes

$$\langle n_{\vec{k}} \rangle = \lambda n \left[ r f \left( \vec{k} - \vec{k}_1 \right) \right]^2 + \left[ t f \left( \vec{k} - \vec{k}'_2 \right) \right]^2. \quad (35)$$

So the two incident modes are superposed coherently if they are correlated and incoherently if they are not.

The influence of correlations on the diffraction (or interference) is dramatically illustrated by the "ghost" diffraction (or interference) effect. In the observation experiment of the "ghost" diffraction\,[14], a light beam, which is generated from spontaneous parametric down-conversion (SPDC) and consists of two orthogonal polarization components (usually called signal and idler), is split by a polarization beam splitter into two beams, and detected by two distinct pointlike photoncounting detectors for coincidence. A Young’s double-slit or single-slit aperture is inserted into the signal beam. Surprisingly, an interference or diffraction pattern is observed in the coincidence counts by scanning the detector in the idler beam. Here we give an exact explanation of the "ghost" diffraction. For the SPDC, the output light is in a superposition of the vacuum and two-photon states\,[12]

$$|\Psi\rangle = |0\rangle + F \sum_{\vec{k}'} a_{\vec{k}'}^+, c_{\vec{k}'}^+, |0\rangle, \quad (36)$$

where the operators $a_{\vec{k}'}^+$, and $c_{\vec{k}'}^+$ represent the signal and idler modes, respectively. The normal characteristic function of the signal and idler modes is indicated by $\chi^{(n)}_T \left( \left\{ a_{\vec{k}'} \right\} ; \left\{ c_{\vec{k}'} \right\} ; \left\{ \xi_{\vec{k}_1} \right\} ; \left\{ \xi_{\vec{k}_2} \right\} \right)$. Then the signal light meets a diffraction screen and the idler light remains unchanged. The second-order correlation coefficient between a fixed diffraction mode and arbitrary idler modes is to be measured. From Eq. (12), the normal characteristic function of the diffraction and idler modes has the form

$$\chi^{(n)}_T \left( \left\{ b_{\vec{k}} \right\} ; \left\{ c_{\vec{k}'}, f \right\} ; \left\{ \xi_{\vec{k}_1} \right\} ; \left\{ \xi_{\vec{k}_2} \right\} \right)$$

$$= \chi^{(n)}_T \left( \left\{ a_{\vec{k}'} \right\} ; \left\{ c_{\vec{k}'} \right\} ; \left\{ \sqrt{\lambda} \sum_{\vec{k}} \xi_{\vec{k}} f \left( \vec{k} - \vec{k}' \right) \right\} ; \left\{ \xi_{\vec{k}_2} \right\} \right). \quad (37)$$

With Eqs. (36) and (37), we obtain the second-order correlation coefficient between a fixed diffraction mode $\vec{k}$ and an arbitrary idler mode $\vec{k}'$

$$G^{(2)} \left( b_{\vec{k}}, c_{\vec{k}'}^+ \right) = \langle b_{\vec{k}}^+ b_{\vec{k}} c_{\vec{k}'}^+, c_{\vec{k}'}^+ \rangle = \lambda |F|^2 \left[ f \left( \vec{k} - \vec{k}' \right) \right]^2. \quad (38)$$
It is directly proportional to square of the diffraction factor. The diffraction pattern occurs by fixing the diffraction mode and scanning the idler modes. Therefore, Eq. (38) explains the observation in the "ghost" diffraction.
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**Caption 1:** The relation between the correlation coefficient $\eta$ of the diffraction modes and the Fano factor $F_n$ of the incident mode. We choose $h_1 = h_2 = 3$.

**Caption 2:** The relation between the Schlienz-Mahler quantity $\gamma$ and the mean photon number of the incident mode. $y$ is expressed by Eq.(31).