Single Loop Gaussian Homotopy Method for Non-convex Optimization

Hidenori Iwakiri∗
The University of Tokyo, RIKEN AIP
iwakiri-hidenori2020@g.ecc.u-tokyo.ac.jp

Yuhang Wang∗
The University of Tokyo
utyuuhikou@gmail.com

Shinji Ito
NEC Corporation, RIKEN AIP
i-shinji@nec.com

Akiko Takeda
The University of Tokyo, RIKEN AIP
takeda@mist.i.u-tokyo.ac.jp

Abstract

The Gaussian homotopy (GH) method is a popular approach to finding better stationary points for non-convex optimization problems by gradually reducing a parameter value $t$, which changes the problem to be solved from an almost convex one to the original target one. Existing GH-based methods repeatedly call an iterative optimization solver to find a stationary point every time $t$ is updated, which incurs high computational costs. We propose a novel single loop framework for GH methods (SLGH) that updates the parameter $t$ and the optimization decision variables at the same. Computational complexity analysis is performed on the SLGH algorithm under various situations: either a gradient or gradient-free oracle of a GH function can be obtained for both deterministic and stochastic settings. The convergence rate of SLGH with a tuned hyperparameter becomes consistent with the convergence rate of gradient descent, even though the problem to be solved is gradually changed due to $t$. In numerical experiments, our SLGH algorithms show faster convergence than an existing double loop GH method while outperforming gradient descent-based methods in terms of finding a better solution.

1 Introduction

Let us consider the following non-convex optimization problem:

$$\min_{x \in \mathbb{R}^d} f(x),$$ (1)

where $f : \mathbb{R}^d \to \mathbb{R}$ is a non-convex function. Let us also consider the following stochastic setting:

$$f(x) := \mathbb{E}_\xi [f(x; \xi)],$$ (2)

where $\xi$ is the random variable following a probability distribution $P$ from which i.i.d. samples can be generated. Such optimization problems attract significant attention in machine learning, and at the same time, the need for optimization algorithms that can find a stationary point with smaller objective value is growing. For example, though it is often said that simple gradient methods can find global minimizers for deep learning (parameter configurations with zero or near-zero training loss), such beneficial behavior is not universal, as noted in [19]; the trainability of neural nets is highly dependent on network architecture design choices, variable initialization, etc. There are also various other highly non-convex optimization problems in machine learning (see e.g., [16]).

∗The first two authors contributed equally.

36th Conference on Neural Information Processing Systems (NeurIPS 2022).
We propose two variants of the SLGH method: SLGH and SLGH, which have different update rules for $t$. SLGH updates $t$ using the derivative of $F(x, t)$ in terms of $t$, based on the idea of viewing $F(x, t)$ as the objective function with respect to the variable $(x, t)$. Though this approach is effective in finding good solutions (as demonstrated in Appendix D.4), it requires additional computational cost due to the calculation of $\frac{\partial F}{\partial t}$. To avoid this additional computational cost, we also consider...
SLGH, that uses fixed-rate update rule for $t$. We also show that both SLGH$_d$ and SLGH$_r$ have the same theoretical guarantee.

Table 1 summarizes the convergence rate of our SLGH method to reach an $\epsilon$-stationary point under a number of problem settings. Since the convergence rate depends on the decreasing speed of $t$, we list two kinds of complexity in the table; details are described in the caption.

We consider the three settings in which available oracles differ. In Case 1), the full (or stochastic) gradient of $F(x, t)$ in terms of $x$ is available for the deterministic problem (1) (or stochastic problem (2), respectively). However, in this setting, we have to calculate Gaussian convolution for deriving GH functions and their gradient vectors, which becomes expensive, especially for high-dimensional applications, unless closed-form expression of Gaussian convolution is possible. While [22] provides closed-form expression for some specific functions $f$, such as polynomials, Gaussian RBFs, and trigonometric functions, such problem examples are limited. As Case 2), we extend our deterministic and stochastic GH methods to the zeroth-order setting, for which the convolution computation is approximated using only the function values. Another zeroth-order setting, Case 3), is also considered in this paper: the inexact function values (more precisely, the function value with bounded error) can be queried similarly as in the setting in [17]. See Appendix C for more details.

Although no existing studies have analyzed the complexity of a double loop GH method to find an $\epsilon$-stationary point, we can see that its inner loop requires the same complexity as GD (gradient descent) method up to constants. Furthermore, as noted above, the complexity of the SLGH method with a tuned hyperparameter matches that of GD method. Thus, the SLGH method becomes faster than a double loop GH method by around the number of outer loops. The SLGH method is also superior to double loop GH methods from practical perspective, because in order to ensure convergence of their inner loops, we have to set the stepsize conservatively, and furthermore a sufficiently tuned terminate condition must be required.

**Contributions** We can summarize our contribution as follows:

1. We propose novel deterministic and stochastic single loop GH (SLGH) algorithms and analyze their convergence rates to an $\epsilon$-stationary point. As far as we know, this is the first analysis of convergence rates of GH methods for general non-convex problems (1) and (2). For non-convex optimization, the convergence rate of SLGH with a tuned hyperparameter becomes consistent with the convergence rate of gradient descent, even though the problem to be solved is gradually changed due to $t$. At this time, the SLGH algorithms become faster than a double loop one by around its number of outer loops.

2. We propose zeroth-order SLGH (ZOSLGH) algorithms based on zeroth-order estimators of gradient and Hessian values, which are useful when Gaussian smoothing convolution is difficult. We also consider the possibly non-smooth case in which the accessible function contains error, and we derive the upper bound of the error level for convergence guarantee.

3. We empirically compare our proposed algorithm and other algorithms in experiments, including artificial highly non-convex examples and black-box adversarial attacks. Results show that the proposed algorithm converges much faster than an existing double loop GH method, while it is yet able to find better solutions than are GD-based methods.

## 2 Standard Gaussian homotopy methods

**Notation:** For an integer $N$, let $[N] := \{1, \ldots, N\}$. We express $\chi_{[N]} := \{\chi_1, \ldots, \chi_N\}$ for a set of some vectors. We also express the range of the smoothing parameter $t$ as $T := [0, t_1]$, where $t_1$ is an initial value of the smoothing parameter. Let $\| \cdot \|$ denote the Euclidean norm and $\mathcal{N}(0, I_d)$ denote the $d$-dimensional standard normal distribution.

Let us first define Gaussian smoothed function.

**Definition 2.1.** Gaussian smoothed function $F(x, t)$ of $f(x)$ is defined as follows:

$$F(x, t) := \mathbb{E}_{u \sim \mathcal{N}(0, I_d)}[f(x + tu)] = \int f(x + ty)k(y)dy,$$

where $k(y) = (2\pi)^{-d/2} \exp (-\|y\|^2/2)$ is referred to as the Gaussian kernel.
The idea of Gaussian smoothing is to take an expectation over the function value with a Gaussian distributed random vector \( u \). For any \( t > 0 \), the smoothed function \( F(x, t) \) is a \( C^\infty \) function, and \( t \) plays the role of a smoothing parameter that controls the level of smoothing.

Here, let us show the link between Gaussian smoothing and the heat equation [34]. The Gaussian smoothing convolution is basically the solution of the heat equation [34].

\[
\frac{\partial}{\partial t} \hat{u} = \Delta_x \hat{u}, \quad \hat{u}(\cdot, 0) = f(\cdot),
\]

where \( \Delta_x \) denotes the Laplacian. The solution of the heat equation is \( \hat{u}(x, t) = \left(\frac{1}{4\pi t}\right)^{\frac{d}{2}} \int f(y) e^{-\frac{|x-y|^2}{4t}} dy. \) This can be made the same as the Gaussian smoothing function \( F(x, t) \) by scaling its coefficient, which only changes the speed of progression.

Corollary 9 in [25] shows a sufficient condition for ensuring that \( f \) has the asymptotic strict convexity in which the smoothed function \( F(x, t) \) becomes convex if a sufficiently large smoothing parameter \( t \) is chosen. On this basis, the standard GH method, Algorithm 1, starts with a (almost) convex optimization problem \( F(x, t) \) with large parameter value \( t \in \mathbb{R} \) and gradually changes the problem toward the target non-convex \( f(\cdot) = F(\cdot, 0) \) by decreasing \( t \) gradually. [14] reduces \( t \) by multiplying by a factor of \( 1/2 \) for each iteration \( k \). [24] focuses more on theoretical work w.r.t. the general setting and do not discuss the update rule for \( t \).

**Algorithm 1** Standard GH method ([24, 14])

**Require:** Objective function \( f \), iteration number \( T \), sequence \( \{t_1, \ldots, t_T\} \) satisfying \( t_1 > \cdots > t_T \).

- Find a solution \( x_1 \) for minimizing \( F(x, t_1) \).
- for \( k = 1 \) to \( T \) do
  - Find a stationary point \( x_{k+1} \) of \( F(x, t_{k+1}) \) with the initial solution \( x_k \).
- end for
- return \( x_T \)

### 3 Single loop Gaussian homotopy algorithm

A function \( h(x) \) is \( L_0 \)-Lipschitz with a constant \( L_0 \) if for any \( x, y \in \mathbb{R}^d \), \( |h(x) - h(y)| \leq L_0 \|x - y\| \) holds. In addition, \( h(x) \) is \( L_1 \)-smooth with a constant \( L_1 \) if for any \( x, y \in \mathbb{R}^d \), \( \|\nabla h(x) - \nabla h(y)\| \leq L_1 \|x - y\| \) holds. Let us here list assumptions for developing algorithms with convergence guarantee.

**Assumption A1.**

- (i) Objective function \( f \) satisfies \( \sup_{x \in \mathbb{R}^d, t} \mathbb{E}_u[|f(x + tu)|] < \infty \) (In the stochastic setting, \( f \) satisfies \( \sup_{x \in \mathbb{R}^d, \xi} \mathbb{E}_u[|\hat{f}(x + tu; \xi)|] < \infty \)).
- (ii) The optimization problem (1) has an optimal value \( f^* \).
- (iii) Objective function \( f(x) \) is \( L_0 \)-Lipschitz and \( L_1 \)-smooth on \( \mathbb{R}^d \) (In the stochastic setting, \( \hat{f}(x; \xi) \) is \( L_0 \)-Lipschitz and \( L_1 \)-smooth on \( \mathbb{R}^d \) in terms of \( x \) for any \( \xi \)).

Assumption (i) for making \( F(x, t) \) well-defined and enabling to exchange the order of differentiation and integration, as well as Assumption (ii), is mandatory for theoretical analysis with the GH method. Assumption (iii) is often imposed for gradient-based methods. This is a regular boundedness and smoothness assumption in recent non-convex optimization analyses (see e.g., [2, 21, 10]).

In the remainder of this section, we consider the nature of the GH method and propose a more efficient algorithm, a SLGH algorithm. We then provide theoretical analyses for our proposed SLGH algorithm.

#### 3.1 Motivation

The standard GH algorithm needs to solve an optimization problem for a given smoothing factor \( t \) in each iteration and manually reduce \( t \), e.g., by multiplying some decreasing factor. To simplify this process, we consider an alternative problem as follows:

\[
\minimize_{x \in \mathbb{R}^d, t \in \mathcal{T}} F(x, t),
\]
where $F(x, t)$ is the Gaussian smoothed function of $f(x)$. This single loop structure can reduce the number of iterations by optimizing $x$ and $t$ at the same time.

The following theorem is a (almost) special case of Theorem 6 in [12], which is studied in statistical physics but may not be well-known in machine learning and optimization communities. This theorem shows that the optimal solution of (5) $(x^*, t^*)$ satisfies $t^* = 0$, and thus $x^*$ is also a solution for (1). Therefore, we can regard $F(x, t)$ as an objective function in the SLGH method.

**Theorem 3.1.** Suppose that Assumptions A1 (i) and (ii) are satisfied. Unless $f$ is constant a.e., the minimum of the GH function $F(x, t)$ will be always found at $t = 0$, and the corresponding $x$ will be an optimal solution for (1).

We present a proof of this theorem in Appendix B.1. The proof becomes much easier than that in [12] due to its considering a specific case.

Let us next introduce an update rule for $t$ utilizing the derivative information. When we solve the problem (5) using a gradient descent method, the update rule for $t$ becomes $t_{k+1} = t_k - \eta \frac{\partial F}{\partial x}$, where $\eta$ is a step size. The formula (4) in the heat equation implies that the derivative $\frac{\partial F}{\partial t}$ is equal to the Laplacian $\Delta_x F$, i.e., $\frac{\partial F}{\partial t} = \text{tr}(H_F(x))$, where $H_F(x)$ is the Hessian of $F$ in terms of $x$. Since $\text{tr}(H_F(x))$ represents the sharpness of minima [11], this update rule can sometimes decrease $t$ quickly around a minimum and find a better solution. See Appendix D.4 for an example of such a problem.

### 3.2 SLGH algorithm

Let us next introduce our proposed SLGH algorithm, which has two variants with different update rules for $t$: SLGH with a fixed-ratio update rule (SLGH$_H$) and SLGH with a derivative update rule (SLGH$_d$). SLGH$_H$ updates $t$ by multiplying a decreasing factor $\gamma$ (e.g., 0.999) at each iteration. In contrast to this, SLGH$_d$ updates $t$ while using derivative information. Details are described in Algorithm 2. Algorithm 2 transforms a double loop Algorithm 1 into a single loop algorithm. This single loop structure can significantly reduce the number of iterations while ensuring the advantages of the GH method.

**Algorithm 2** Deterministic/Stochastic Single Loop GH algorithm (SLGH)

**Require:** Iteration number $T$, initial solution $x_1$, initial smoothing parameter $t_1$, step size $\beta$ for $x$, step size $\eta$ for $t$, decreasing factor $\gamma \in (0, 1)$, sufficient small positive value $\epsilon$

**for** $k = 1$ to $T$ **do**

\[
x_{k+1} = x_k - \beta \hat{G}_x, \quad \hat{G}_x = \begin{cases} 
\nabla_x F(x_k, t_k) & \text{(determ.)} \\
\nabla_x F(x_k, t_k; \xi_k) & \text{(stoc.)}
\end{cases},
\]

\[
t_{k+1} = \begin{cases} 
\gamma t_k & \text{(SLGH$_H$)} \\
\max\{\min\{t_k - \eta \hat{G}_t, \gamma t_k\}, \epsilon\} & \text{(SLGH$_d$)}
\end{cases}, \quad \hat{G}_t = \begin{cases} 
\frac{\partial F(x_k, t_k)}{\partial t} & \text{(determ.)} \\
\frac{\partial F(x_k, t_k; \xi_k)}{\partial t} & \text{(stoc.)}
\end{cases}
\]

**end for**

In the stochastic setting of (2), the gradient of $F(x, t)$ in terms of $x$ is approximated by $\nabla_x \hat{F}(x, t; \xi)$ with randomly chosen $\xi$, where $\hat{F}(x, t; \xi)$ is the GH function of $\hat{f}(x; \xi)$. Likewise, the derivative of $F(x, t)$ in terms of $t$ is approximated by $\frac{\partial F(x, t; \xi)}{\partial t}$. The stochastic algorithm in Algorithm 2 uses one sample $\xi_k$. We can extend the stochastic approach to a minibatch one by approximating $\nabla_x \hat{F}(x, t)$ by $\frac{1}{M} \sum_{i=1}^{M} \nabla_x \hat{F}(x, t; \xi_i)$ with samples $\{\xi_1, \ldots, \xi_M\}$ of some batch size $M$, but for the sake of simplicity, we here assume one sample in each iteration. In this setting, the gradient complexity matches the iteration complexity; thus, we also use the term “iteration complexity” in the stochastic setting. Other methods, such as momentum-accelerated method [33] and Adam [18] can also be applied here. According to Theorem 3.1, the final smoothing parameter needs to be zero. Thus, we

---

2 Although the assumptions in Theorem 3.1 are stronger than those in the theorem proved by Evans, the statement of ours is also stronger than that of his theorem, in a sense that our theorem guarantees that all optimal solutions satisfy $t = 0$. 

---
multiply $\gamma$ by $t$ even in SLGH when the decrease of $t$ is insufficient. We also assure that $t$ is larger than a sufficiently small positive value $\epsilon' > 0$ during an update to prevent $t$ from becoming negative.

### 3.3 Convergence analysis for SLGH

Let us next analyze the worst-case iteration complexity for both deterministic and stochastic SLGHs, but, before that, let us first show some properties for Gaussian smoothed function $F(x, t)$ under Assumption A1 for the original function $f(x)$. In the complexity analyses in this paper, we always assume that $\gamma$ is bounded from above by a universal constant $\bar{\gamma} < 1$, which implies $1/(1 - \gamma) = O(1)$.

**Lemma 3.2.** Let $f(x)$ be a $L_0$-Lipschitz function. Then, for any $t > 0$, its Gaussian smoothed function $F(x, t)$ will then also be $L_0$-Lipschitz in terms of $x$. Let $f(x)$ be a $L_1$-smooth function. Then, for any $t > 0$, $F(x, t)$ will also be $L_1$-smooth in terms of $x$.

Lemma 3.2 indicates that Assumption A1 given to the function $f(x)$ also guarantees the same properties for $F(x, t)$. Below, we give some bounds between the smoothed function $F(x, t)$ and the original function $f(x)$.

**Lemma 3.3.** Let $f$ be a $L_0$-Lipschitz function. Then, for any $x \in \mathbb{R}^d$, $F(x, t)$ is also $L_0\sqrt{d}$-Lipschitz in terms of $t$, i.e., for any $x$, smoothing parameter values $t_1, t_2 > 0$, we have $\|F(x, t_1) - F(x, t_2)\| \leq L_0\sqrt{d}|t_1 - t_2|$.

On the basis of Lemmas 3.2 and 3.3, the convergence results of our deterministic and stochastic SLGH algorithms can be given as in Theorems 3.4 and 3.5, respectively. Proofs of the following theorems are given in Appendix B.2. Let us first deal with the deterministic setting.

**Theorem 3.4 (Convergence of SLGH, Deterministic setting).** Suppose Assumption A1 holds, and let $\hat{x} := x_k \in \mathbb{R}^d$, $k' = \arg\min_{k \in [T]} \|\nabla f(x_k)\|$. Set the stepsize for $x$ as $\beta = 1/L_1$. Then, for any setting of the parameter $\gamma$, $\hat{x}$ satisfies $\|\nabla f(\hat{x})\| \leq \epsilon$ with the iteration complexity of $T = O\left(d^{3/2}/\epsilon^2\right)$.

Further, if we choose $\gamma \leq d^{-\Omega(\epsilon^2)}$, the iteration complexity can be bounded as $T = O(1/\epsilon^2)$.

This theorem indicates that if we choose $\gamma$ close to 1, then the iteration complexity can be $O\left(d^{3/2}/\epsilon^2\right)$, which is $O(d^{3/2})$ times larger than the $O(1/\epsilon^2)$-iteration complexity by the standard gradient descent methods [27]. However, we can remove this dependency on $d$ to obtain an iteration complexity matching that of the standard gradient descent, by choosing $\gamma \leq d^{-\Omega(\epsilon^2)}$, as shown in Theorem 3.4. Empirically, settings of $\gamma$ close to 1, e.g., $\gamma = 0.999$, seem to work well enough, as demonstrated in Section 5.

An inner loop of the double loop GH method using the standard GD requires the same complexity as the standard GD method up to constants since the objective smoothed function of inner optimization problem is $L_1$-smooth function. By considering the above results, we can see that the SLGH algorithm becomes faster than the double loop one by around the number of outer loops.

To provide theoretical analyses in the stochastic setting, we need additional standard assumptions.

**Assumption A2.**

(i) The stochastic function $\bar{f}(x; \xi)$ becomes an unbiased estimator of $f(x)$. That is, for any $x \in \mathbb{R}^d$, $f(x) = \mathbb{E}_\xi[\bar{f}(x; \xi)]$ holds.

(ii) For any $x \in \mathbb{R}^d$, the variance of the stochastic gradient oracle is bounded as $\mathbb{E}_\xi[\|\nabla_x \bar{f}(x; \xi) - \nabla f(x)\|^2] \leq \sigma^2$. Here, the expectation is taken w.r.t. random vectors $\xi_k$.

The following theorem shows the convergence rate in the stochastic setting.

**Theorem 3.5 (Convergence of SLGH, Stochastic setting).** Suppose Assumptions A1 and A2 hold. Take $k_1 := \Theta(1/\epsilon^4)$ and $k_2 := O\left(\log_2 \min\{d^{-1/2}, d^{-3/4} \epsilon^{-2}\}\right)$ and define $k_0 = \min\{k_1, k_2\}$. Let $\hat{x} := x_{k'}$, where $k'$ is chosen from a uniform distribution over $\{k_0 + 1, k_0 + 2, \ldots, T\}$. Set the stepsize for $x$ as $\beta = \min\{1/\sqrt{L_1}, 1/\sqrt{T - k_0}\}$. Then, for any setting of the parameter $\gamma$, $\hat{x}$ satisfies $\mathbb{E}[\|\nabla f(\hat{x})\|] \leq \epsilon$ with the iteration complexity of $T = O\left(d^2 + d^{3/2}/\epsilon^2\right)$ where the expectation is taken w.r.t. random vectors $\xi_k$. Further, if we choose $\gamma \leq \left(\min\{d^{1/2}, d^{3/2} \epsilon^2\\}\right)^{-\Omega(\epsilon^2)}$, the iteration complexity can be bounded as $T = O(1/\epsilon^4)$. 


We note that the iteration complexity of $T = O(1/\epsilon^4)$ for sufficiently small $\gamma$ matches that for the standard stochastic gradient descent (SGD) shown, e.g., by [13].

4 Zeroth-order single loop Gaussian homotopy algorithm

In this section, we introduce a zeroth-order version of the SLGH algorithms. This ZOSLGH algorithm is proposed for those optimization problems in which Gaussian smoothing convolution is difficult to compute, or in which only function values can be queried.

4.1 ZOSLGH algorithm

For cases in which only function values are accessible, approximations for the gradient in terms of $x$ and derivative in terms of $t$ are needed. [28] has shown that the gradient of the smoothed function $F(x, t)$ can be represented as

$$\nabla_x F(x, t) = \frac{1}{t} \mathbb{E}_u([f(x + tu) - f(x)]u), \ u \sim \mathcal{N}(0, I_d).$$

(6)

Thus, the gradient $\nabla_x F(x, t)$ can be approximated by an unbiased estimator $\tilde{g}_x(x, t; u)$ as

$$\tilde{g}_x(x, t; u) := \frac{1}{t} (f(x + tu) - f(x))u, \ u \sim \mathcal{N}(0, I_d).$$

(7)

The derivative $\frac{\partial F}{\partial t}$ is equal to the trace of the Hessian of $F(x, t)$ because the Gaussian smoothed function is the solution of the heat equation $\frac{\partial F}{\partial t} = \text{tr}(H_F(x))$. We can estimate $\text{tr}(H_F(x))$ on the basis of the second order Stein’s identity [32] as follows:

$$H_F(x) \approx \frac{(mv - I_d)}{t^2} (f(x + tv) - f(x)), \ v \sim \mathcal{N}(0, I_d).$$

(8)

Thus, the estimator for derivative can be written as:

$$\tilde{g}_t(x, t; v) := \frac{(v^Tv - d)(f(x + tv) - f(x))}{t^2}, \ v \sim \mathcal{N}(0, I_d).$$

(9)

As for the stochastic setting, $f(x)$ in (7) and (9) is replaced by the stochastic function $\tilde{f}(x; \xi)$ with some randomly chosen sample $\xi$. The gradient $\nabla_x \tilde{F}(x, t; \xi)$ of its GH function $\tilde{F}(x, t; \xi)$ can then be approximated by $\tilde{G}_x(x, t; \xi, u) := \frac{\tilde{f}(x + tu; \xi) - \tilde{f}(x; \xi)}{u}$, and the derivative $\frac{\partial F}{\partial \xi}$ can be approximated by $\tilde{G}_t(x, t; \xi, v) := \frac{(v^Tv - d)(\tilde{f}(x + tv; \xi) - \tilde{f}(x; \xi))}{t^2}$ (see Algorithm 3 for more details).

Algorithm 3 Deterministic/Stochastic Zeroth-Order Single Loop GH algorithm (ZOSLGH)

Require: Iteration number $T$, initial solution $x_1$, initial smoothing parameter $t_1$, step size $\beta$ for $x$, step size $\eta$ for $t$, decreasing factor $\gamma \in (0, 1)$, sufficient small positive value $\epsilon$

for $k = 1$ to $T$ do

Sample $u_k$ from $\mathcal{N}(0, I_d)$

$x_{k+1} = x_k - \beta \tilde{G}_{x, u}, \ \tilde{G}_{x, u} = \left\{ \begin{array}{ll}
\tilde{g}_x(x_k, t_k; u_k) \text{ (determin.)} \\
\tilde{G}_x(x_k, t_k; \xi_k, u_k), & \xi_k \sim P \text{ (stoc.)}
\end{array} \right.$

Sample $v_k$ from $\mathcal{N}(0, I_d)$

$t_{k+1} = \left\{ \begin{array}{ll}
\gamma t_k \text{ (SLGH)} \\
\max\{\min\{t_k - \eta \tilde{G}_{t, v}, \gamma t_k\}, \epsilon'\} \text{ (SLGHd)}
\end{array} \right.$

end for

4.2 Convergence analysis for ZOSLGH

We can analyze the convergence results using concepts similar to those used with the first-order SLGH algorithm. Below are the convergence results for ZOSLGH in both the deterministic and stochastic
settings. Proofs of the following theorems are given in Appendix B.3, and the definitions of \( \hat{x} \) are provided in the proofs. We start from the deterministic setting, which is aimed at the deterministic problem (1).

**Theorem 4.1 (Convergence of ZOSLGH, Deterministic setting).** Suppose Assumption A1 holds. Take \( k_1 := \Theta(d/\epsilon^2) \) and \( k_2 := O(\log_d d^{-1/2}) \), and define \( k_0 = \min\{k_1, k_2\} \). Let \( \bar{x} := x_{k'} \), where \( k' \) is chosen from a uniform distribution over \( \{k_0 + 1, k_0 + 2, \ldots, T\} \). Set the stepsize for \( x \) as \( \beta = 1/(2(d + 4)L_1) \). Then, for any setting of the parameter \( \gamma, \bar{x} \) satisfies \( \mathbb{E}[\|\nabla f(\bar{x})\|] \leq \epsilon \) with the iteration complexity of \( T = O(d^2/\epsilon^2) \), where the expectation is taken w.r.t. random vectors \( \{u_k\} \) and \( \{v_k\} \). Further, if we choose \( \gamma \leq d^{-\Omega(\epsilon^2/d)} \), the iteration complexity can be bounded as \( T = O(d/\epsilon^2) \).

This complexity of \( O(d/\epsilon^2) \) for \( \gamma \leq d^{-\Omega(\epsilon^2/d)} \) matches that of zeroth-order GD (ZOOGD) [28].

Let us next introduce the convergence result for the stochastic setting. As shown in [13], if we take the expectation for our stochastic zeroth-order gradient oracle with respect to both \( \xi \) and \( u \), we get

\[
\mathbb{E}_{\xi,u}[\hat{G}_x(x, t; \xi, u)] = \mathbb{E}_u[\mathbb{E}_\xi[\hat{G}_x(x, t; \xi, u)|u]] = -\nabla_x f(x, t).
\]

Therefore, \( \zeta_k := (\xi_k, u_k) \) behaves similarly to \( u_k \) in the deterministic setting.

**Theorem 4.2 (Convergence of ZOSLGH, Stochastic setting).** Suppose Assumptions A1 and A2 hold. Take \( k_1 := \Theta(d'/\epsilon^2) \) and \( k_2 := O(\log_d d^{-1/2}) \), and define \( k_0 = \min\{k_1, k_2\} \). Let \( \bar{x} := x_{k'} \), where \( k' \) is chosen from a uniform distribution over \( \{k_0 + 1, k_0 + 2, \ldots, T\} \). Set the stepsize for \( x \) as \( \beta = \min\{\frac{1}{2(d'/\epsilon^2)}, \frac{1}{\sqrt{(d'-k_0)(d'+4)}}\} \). Then, for any setting of the parameter \( \gamma, \bar{x} \) satisfies \( \mathbb{E}[\|\nabla f(\bar{x})\|] \leq \epsilon \) with the iteration complexity of \( T = O(d^2/\epsilon^4) \), where the expectation is taken w.r.t. random vectors \( \{u_k\}, \{v_k\}, \) and \( \{\xi_k\} \). Further, if we choose \( \gamma \leq d^{-\Omega(\epsilon^2/d)} \), the iteration complexity can be bounded as \( T = O(d/\epsilon^4) \).

This complexity of \( O(d/\epsilon^4) \) for \( \gamma \leq d^{-\Omega(\epsilon^2/d)} \) also matches that of ZOSGD [13].

## 5 Experiments

In this section, we present our experimental results. We conducted two experiments. The first was to compare the performance of several algorithms including the proposed ones, using test functions for optimization. We were able to confirm the effectiveness and versatility of our SLGH methods for highly non-convex functions. We also created a toy problem in which ZOSLGH for optimization. We were able to confirm the effectiveness and versatility of our SLGH methods.

**Generation of per-image black-box adversarial attack example.** Let us consider the unconstrained black-box attack optimization problem in [9], which is given by

\[
\min_{x \in \mathbb{R}^d} f(x) := \lambda \ell(0.5\tanh(\tanh^{-1}(2a + x)) + \|0.5\tanh(\tanh^{-1}(2a) + x) - a\|^2,
\]

where \( \lambda \) is a regularization parameter, \( a \) is the input image data, and \( \tanh \) is the element-wise operator which eliminates the constraint representing the range of adversarial examples. The first term \( \ell(\cdot) \) of \( f(x) \) is the loss function for the untargeted attack in [6], and the second term \( L_2 \) is the adversarial distortion (the lower the better). The goal of this problem is to find the perturbation that makes the loss \( \ell(\cdot) \) reach its minimum while keeping \( L_2 \) distortion as small as possible. The initial adversarial perturbation \( x_0 \) was set to 0. We say a successful attack example has been generated when the loss \( \ell(\cdot) \) is lower than the attack confidence (e.g., \( 1e-10 \)).

Let us here compare our algorithms, ZOSLGH, and ZOSLGH, to three zeroth-order algorithms: ZOSGD [13], ZOAdaMM [9], and ZOGradOpt [14]. ZOGradOpt is a homotopy method with a
double loop structure. In contrast to this, ZOSGD and ZOAdaMM are SGD-based zeroth-order methods and thus do not change the smoothing parameter during iterations.

Table 2 and Figure 1 show results for our experiment. We can see that SGD-based algorithms are able to succeed in the first attack with far fewer iterations than our GH algorithms (e.g., Figure 1(a), Figure 1(d)). Accordingly, the value of $L_2$ distortion decreases slightly more than GH methods. However, SGD-based algorithms have lower success rates than do our SLGH algorithms. This is because SGD-based algorithms remain around a local minimum $x = 0$ when it is difficult to attack, while GH methods can escape the local minima due to sufficient smoothing (e.g., Figure 1(b), Figure 1(e)). Thus, the SLGH algorithms are, on average, able to decrease total loss over that with SGD-based algorithms. In a comparison within GH methods, ZOGradOpt requires more than 6500 iterations to succeed in the first attack due to its double loop structure (e.g., Figure 1(c), Figure 1(f)). In contrast to this, our SLGH algorithms achieve a high success rate with far fewer iterations. Please note that SLGH$_d$ takes approximately twice the computational time per iteration than the other algorithms because it needs additional queries for the computation of the derivative in terms of $t$. See Appendix E for a more detailed presentation of the experimental setup and results.

Table 2: Performance of a per-image attack over 100 images of CIFAR-10 under $T = 10000$ iterations. “Succ. rate” indicates the ratio of success attack, “Avg. iters to 1st succ.” is the average number of iterations to reach the first successful attack, “Avg. $L_2$ (succ.)” is the average of $L_2$ distortion taken among successful attacks, and “Avg. total loss” is the average of total loss $f(x)$ over 100 samples. Please note that the standard deviations are large since the attack difficulty varies considerably from sample to sample.

| Methods     | Succ. rate | Avg. iters to 1st succ. | Avg. $L_2$ (succ.) | Avg. total loss |
|-------------|------------|------------------------|-------------------|-----------------|
| SGD algo.   |            |                        |                   |                 |
| ZOSGD       | 88%        | 835 ± 1238             | 0.076 ± 0.085     | 27.70 ± 74.80   |
| ZOAdaMM     | 85%        | 3335 ± 2634            | 0.050 ± 0.055     | 20.24 ± 62.48   |
| GH algo.    |            |                        |                   |                 |
| ZOGradOpt   | 65%        | 6789 ± 1901            | 0.249 ± 0.159     | 41.45 ± 76.04   |
| ZOSLGH$_r$ ($\gamma = 0.999$) | 93%        | 4979 ± 756             | 0.246 ± 0.178     | 14.26 ± 54.61   |
| ZOSLGH$_d$ ($\gamma = 0.999$) | 92%        | 4436 ± 805             | 0.150 ± 0.084     | 16.49 ± 58.69   |

Figure 1: Total loss for generating per-image black-box adversarial examples for different images of CIFAR-10 and MNIST (log scale).
6 Summary and future work

We have presented here the deterministic/stochastic SLGH and ZOSLGH algorithms as well as their convergence results. They have been designed for the purpose of finding better solutions with fewer iterations by simplifying the homotopy process into a single loop. We consider this work to be a first attempt to improve the standard GH method.

Although this study has considered the case in which the accessible function contains some error and is possibly non-smooth, we assume the underlying objective function to be smooth. Further work should be carried out to investigate the case in which the objective function itself is non-smooth.

Acknowledgements  This work was supported by JSPS KAKENHI Grant Number 19H04069, JST ACT-I Grant Number JPMJPR18U5, and JST ERATO Grant Number JPMJER1903.
References

[1] Neculai Andrei. An unconstrained optimization test functions collection. *Advanced Modeling and Optimization*, 10(1):147–161, 2008.

[2] S. Aydore, T. Zhu, and D. P. Foster. Dynamic local regret for non-convex online forecasting. In *Advances in Neural Information Processing Systems*, pages 7982–7991, 2019.

[3] A. Blake and A. Zisserman. *Visual reconstruction*. MIT press, 1987.

[4] T. Brox and J. Malik. Large displacement optical flow: descriptor matching in variational motion estimation. *IEEE transactions on pattern analysis and machine intelligence*, 33(3):500–513, 2010.

[5] G. Bourmaud C. Zach. Descending, lifting or smoothing: Secrets of robust cost optimization. In *Proc. ECCV*, volume 12, pages 558–574, 2018.

[6] N. Carlini and D. Wagner. Towards evaluating the robustness of neural networks. In *2017 IEEE Symposium on Security and Privacy*, pages 39–57, 2017.

[7] B. Chen and P. T. Harker. A non-interior-point continuation method for linear complementarity problems. *SIAM Journal on Matrix Analysis and Applications*, 14(4):1168–1190, 1993.

[8] X. Chen. Smoothing methods for nonsmooth, nonconvex minimization. *Mathematical programming*, 134(1):71–99, 2012.

[9] X. Chen, S. Liu, K. Xu, X. Li, X. Lin, M. Hong, and D. Cox. Zo-adamm: Zeroth-order adaptive momentum method for black-box optimization. In *Advances in Neural Information Processing Systems*, pages 7204–7215, 2019.

[10] A. Cutkosky and F. Orabona. Momentum-based variance reduction in non-convex sgd. In *Advances in Neural Information Processing Systems*, pages 15236–15245. Curran Associates, Inc., 2019.

[11] Laurent Dinh, Razvan Pascanu, Samy Bengio, and Yoshua Bengio. Sharp minima can generalize for deep nets. In *Proceedings of the 34th International Conference on Machine Learning*, pages 1019–1028. PMLR, 2017.

[12] L.C. Evans. *Partial Differential Equations*. American Mathematical Society, 2010.

[13] S. Ghadimi and G. Lan. Stochastic first-and zeroth-order methods for nonconvex stochastic programming. *SIAM Journal on Optimization*, 23(4):2341–2368, 2013.

[14] E. Hazan, K. Y. Levy, and S. Shalev-Shwartz. On graduated optimization for stochastic non-convex problems. In *Proceedings of the 33rd International Conference on Machine Learning*, pages 1833–1841, 2016.

[15] A. A. Hemeda. Homotopy perturbation method for solving systems of nonlinear coupled equations. *Applied Mathematical Sciences*, 6(93-96):4787–4800, 2012.

[16] P. Jain and P. Kar. Non-convex optimization for machine learning. *Foundations and Trends in Machine Learning*, 10(3–4):142–336, 2017.

[17] C. Jin, L. T. Liu, R. Ge, and M. I. Jordan. On the local minima of the empirical risk. In *Proceedings of the 32nd International Conference on Neural Information Processing Systems*, pages 4901–4910, 2018.

[18] D. P. Kingma and J. Ba. Adam: A method for stochastic optimization. In *3rd International Conference on Learning Representations*, 2015.

[19] H. Li, Z. Xu, G. Taylor, C. Studer, and T. Goldstein. Visualizing the loss landscape of neural nets. In *Advances in Neural Information Processing Systems*, pages 6389–6399, 2017.

[20] S. Liu, B. Kailkhura, P. Y. Chen, P. S. Ting, S. Y. Chang, and L. Amini. Zeroth-order stochastic variance reduction for nonconvex optimization. In *Advances in Neural Information Processing Systems*, page 3727–3737, 2018.
[21] P. Mertikopoulos, N. Hallak, A. Kavis, and V. Cevher. On the almost sure convergence of stochastic gradient descent in non-convex problems. In Advances in Neural Information Processing Systems, pages 1117–1128. Curran Associates, Inc., 2020.

[22] H. Mobahi. Closed form for some gaussian convolutions. arXiv preprint arXiv:1602.05610, 2016.

[23] H. Mobahi and J. W. Fisher III. On the link between gaussian homotopy continuation and convex envelopes. In Energy Minimization Methods in Computer Vision and Pattern Recognition, pages 43–56, 2015.

[24] H. Mobahi and J. W. Fisher III. A theoretical analysis of optimization by gaussian continuation. In Proceedings of the AAAI Conference on Artificial Intelligence, volume 29, 2015.

[25] H. Mobahi and Y. Ma. Gaussian smoothing and asymptotic convexity. Technical report, Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, 2012.

[26] Marcin Molga and Czeslaw Smutnicki. Test functions for optimization needs. 2005.

[27] Y. Nesterov. Introductory Lectures on Convex Optimization: a basic course. Kluwer Academic Publishers, 2004.

[28] Y. Nesterov and V. Spokoiny. Random gradient-free minimization of convex functions. Foundations of Computational Mathematics, 17(2):527–566, 2017.

[29] M. Nielsen. Graduated non-convexity by smoothness focusing. In Proceedings of the British Machine Vision Conference, pages 60.1—-60.10. BMVA Press, 1993.

[30] W. J. Shao, C. Geißler, and F. Sivrikaya. Graduated optimization of black-box functions. arXiv preprint arXiv:1906.01279, 2019.

[31] A. Sokolov, J. Kreutzer, S. Riezler, and C. Lo. Stochastic structured prediction under bandit feedback. In Advances in neural information processing systems, pages 1489–1497, 2016.

[32] C. Stein. A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. In Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability, Volume 2: Probability Theory. The Regents of the University of California, 1972.

[33] I. Sutskever, J. Martens, G. Dahl, and G. Hinton. On the importance of initialization and momentum in deep learning. In Proceedings of the 30th International Conference on Machine Learning, pages 1139–1147. PMLR, 2013.

[34] D. V. Widder. The heat equation. Academic Press, 1976.

[35] Z. Wu. The effective energy transformation scheme as a special continuation approach to global optimization with application to molecular conformation. SIAM Journal on Optimization, 6(3):748–768, 1996.

[36] Y. C. Xu, A. Joshi, A. Singh, and A. Dubrawski. Zeroth order non-convex optimization with dueling-choice bandits. In Proceedings of the 36th Conference on Uncertainty in Artificial Intelligence, pages 899–908. PMLR, 2020.
A Related work

Iteration complexity analysis for GH methods To the best of our knowledge, there are no existing works that give theoretical guarantee for the convergence rate except for [14]. It characterized a parameterized family of non-convex functions referred to as “σ-nice”, for which a GH algorithm converges to a global optimum. Moreover, it derived the convergence rate to an ϵ-optimal solution for the σ-nice function. The framework of σ-nice imposes the two conditions: (i) the solution obtained in each inner loop is located sufficiently close to an optimal solution of the optimization problem in the next inner loop; (ii) the optimization problem in each inner loop is strongly convex around its optimal solutions. Unfortunately, it is not obvious whether we can efficiently judge a function is “σ-nice”, and we cannot apply the analysis results to general non-convex functions. On the other hand, this work tackles a problem of different nature from [14] since it analyzes the convergence rate to an ϵ-stationary point for general non-convex functions.

Guarantee for the value of the objective function [24] provided an upper bound on the objective value attained by a homotopy method. The bound was characterized by a quantity that they referred to as “optimization complexity”, which can be analytically computed when the objective function is expressed in some suitable basis functions such as Gaussian RBFs.

Other smoothing methods Smoothing methods other than Gaussian smoothing include [7, 8]. The smoothing kernel in those works is simpler but restricted to specific problem settings. For example, [8] constructs smoothing approximations for optimization problems that can be reformulated by using the plus function \((t)_+ := \max\{0, t\}\).

Zeroth-order techniques In problem settings in which the explicit gradient of the objective function cannot be calculated but the exact function values can be queried, zeroth-order optimization has become increasingly popular due to its potential for wide application. Such a class of applications appears in black-box adversarial attacks on deep neural networks [9], structured prediction [31], and reinforcement learning [36]. Various zeroth-order methods (ZOSGD [13], ZOAdaMM [9], ZOSVRG [20]) have been proposed for such black-box situations. All of them have been developed from ZOGD in [28], which introduces random gradient-free oracles based on Gaussian smoothing with fixed \(t\). This trend also applies to research on the GH method. [14] developed a GH method in the zeroth-order setting for which the objective is only accessible through a noisy value oracle. [30] proposed a GH method for hyperparameter tuning based on [14] using two-point zeroth-order estimators [28].

B Proofs for theorems and lemmas in Sections 3 and 4

Notation: We sometimes denote the expectation with respect to random variables \(\chi_{S+1}, \ldots, \chi_T\) \((S, T \in \mathbb{N}, T > S)\) as \(\mathbb{E}_{\chi}[\cdot]\) for the sake of simplicity.

B.1 Theorem 3.1

Proof for Theorem 3.1: Since the optimization problem (1) has an optimal value \(f^*\) by Assumption A1 (ii), for any \(t \in \mathcal{T}\) and for any \(x \in \mathbb{R}^d\), we have

\[
F(x, t) - f^* = \mathbb{E}_u[f(x + tu) - f^*] \geq 0.
\]

Together with the relationship \(F(x, 0) = f(x)\), for any \(x \in \mathbb{R}^d\), for any \(t \in \mathcal{T}\) and for any optimal solution \(x^* \in \mathbb{R}^d\) of the optimization problem (1), we have \(F(x, t) - F(x^*, 0) \geq 0\). Furthermore, if we exclude cases where \(f(x)\) is constant (a.e.), for any \((x, t) \in \mathbb{R}^d \times \mathcal{T} \setminus \{(x, 0) \mid f(x) = f(x^*)\}\), we obtain

\[
F(x, t) - f^* = \mathbb{E}_u[f(x + tu) - f^*] > 0.
\]

\(^3\)Their method is not exactly a GH method because it smooths the objective function using random variables sampled from the unit ball (or the unit sphere in a zeroth-order setting) rather than Gaussian random variables. However, for the sake of simplicity, we treat it as a GH method in this paper.
Therefore, a minimum of the optimization problem of the GH function minimize $F(x, t)$ holds only at $t = 0$ and the corresponding $x$ becomes an optimal solution of the original optimization problem minimize $f(x)$.

\section*{B.2 First-order SLGH algorithm}

At the beginning of the subsection, we introduce a lemma that gives upper bounds for moments of Gaussian random variables, and then prove the two lemmas which appeared in the main paper.

\textbf{Lemma B.1 (Lemma 1 in [28])}. Let $u \in \mathbb{R}^d$ be a standard normal random variable. For $p \in [0, 2]$, we have $E_u[\|u\|^p] \leq dp^{p/2}$. If $p \geq 2$, $E_u[\|u\|^p] \leq (d + p)^{p/2}$ holds.

\textbf{Proof for Lemma 3.2}: According to the definition of Gaussian smoothing in the main paper, we have

\[|F(x, t) - F(y, t)| = \left| \int (f(x + tz)k(z) - f(y + tz)k(z))dz \right| \leq \int |f(x + tz) - f(y + tz)|k(z)dz \leq L_0\|x - y\|k(z)dz \leq L_0\|x - y\|.

The proof of $L_1$-smooth is similar to that of $L_0$-Lipschitz:

\[|\nabla_x F(x, t) - \nabla_x F(y, t)| \leq \int |\nabla f(x + tz) - \nabla f(y + tz)|k(z)dz \leq L_1\|x - y\|k(z)dz \leq L_1\|x - y\|.

The lemma has proved that the Lipschitz constants of $F(x, t)$ and $\nabla_x F(x, t)$ in terms of $x$ are smaller than those of $f(x)$ and $\nabla f(x)$, respectively. Therefore we can use the Lipschitz constants $L_0$ and $L_1$ of $f(x)$ and $\nabla f(x)$ for $F(x, t)$ and $\nabla_x F(x, t)$.

\textbf{Proof for Lemma 3.3}:

\[|F(x, t_1) - F(x, t_2)| = |E_u[f(x + t_1u) - f(x + t_2u)]| \leq E_u[L_0|t_1 - t_2|\|u\|] \leq L_0|t_1 - t_2|\sqrt{d},

where the last inequality holds due to Lemma B.1.

Before going to the convergence theorems, we introduce an additional useful lemma to estimate the gap between the gradient of the smoothed function and the true gradient.

\textbf{Lemma B.2}. Let $f$ be a $L_1$-smooth function.

\begin{enumerate}[(i)]
\item (Lemma 4 in [28]) For any $x \in \mathbb{R}^d$ and $t > 0$, we have

\[\|\nabla f(x)\|^2 \leq 2\|\nabla_x F(x, t)\|^2 + \frac{t^2}{2}L_1^2(d + 6)^3.
\]

\item Further, if $f$ is $L_0$-Lipschitz, for any $x \in \mathbb{R}^d$ and $t > 0$, we have

\[\|\nabla f(x)\|^2 \leq \|\nabla_x F(x, t)\|^2 + tL_0L_1(d + 3)^{3/2}.
\]
\end{enumerate}

\textbf{Proof for (i)}: We have

\[\|\nabla f(x)\|^2 - \|\nabla_x F(x, t)\|^2 = (\|\nabla f(x)\| + \|\nabla_x F(x, t)\|)(\|\nabla f(x)\| - \|\nabla_x F(x, t)\|) \leq 2L_0||\nabla f(x)|| - ||\nabla_x F(x, t)|| \leq 2L_0\|\nabla_x F(x, t) - \nabla f(x)\|.
\]
The term \( \| \nabla_x F(x, t) - \nabla f(x) \| \) can be upper bounded as follows:

\[
\| \nabla_x F(x, t) - \nabla f(x) \| \leq \left\| \mathbb{E}_u \left[ \left( \frac{f(x + tu) - f(x) - \langle \nabla f(x), u \rangle}{t} \right) u \right] \right\|
\leq \mathbb{E}_u \left[ \frac{1}{t} \left( f(x + tu) - f(x) - t\langle \nabla f(x), u \rangle \right) \| u \| \right]
\leq \mathbb{E}_u \left[ \frac{tL_1}{2} \| u \|^3 \right]
\leq \frac{tL_1}{2} (d + 3)^{3/2},
\]

where the last second inequality follows from a property of \( L_1 \)-smooth function \((\forall x, y \in \mathbb{R}^d, |f(y) - f(x) - \langle \nabla f(x), y - x \rangle| \leq \frac{L_1}{2} \| y - x \|^2)\), and the last inequality holds due to Lemma B.1. Therefore, we obtain

\[
\| \nabla f(x) \|^2 \leq \| \nabla_x F(x, t) \|^2 + tL_0L_1(d + 3)^{3/2}.
\]

Now, we are ready to prove Theorem 3.4.

**Proof for Theorem 3.4:** We follow the convergence analysis of gradient descent. According to Assumption A1 and Lemma 3.2, \( F(x, t) \) is \( L_0 \)-Lipschitz and \( L_1 \)-smooth in terms of \( x \). Therefore, we have

\[
F(x_{k+1}, t_k) \leq F(x_k, t_k) + \langle \nabla_x F(x_k, t_k), (x_{k+1} - x_k) \rangle + \frac{L_1}{2} \| x_{k+1} - x_k \|^2
= F(x_k, t_k) - \left( \beta - \frac{L_1 \beta^2}{2} \right) \| \nabla_x F(x_k, t_k) \|^2,
\]

where the last equation holds due to the updating rule of the gradient descent: \( x_{k+1} - x_k = -\beta \nabla_x F(x_k, t_k) \). Then, we can get the upper bound for \( \| \nabla_x F(x, t) \|^2 \):

\[
\left( \beta - \frac{L_1 \beta^2}{2} \right) \| \nabla_x F(x_k, t_k) \|^2 \leq F(x_k, t_k) - F(x_{k+1}, t_k)
= F(x_k, t_k) - F(x_{k+1}, t_{k+1}) + F(x_{k+1}, t_{k+1}) - F(x_{k+1}, t_k)
\leq F(x_k, t_k) - F(x_{k+1}, t_{k+1}) + L_0 |t_{k+1} - t_k| \sqrt{d},
\]

where the last inequality follows from Lemma 3.3.

Now, sum up the above inequality for all iterations \( k_0 + 1 \leq k \leq T \) \((T > k_0 \in \mathbb{N})\), and denote the minimum of \( f \) as \( f^* \), then we have

\[
\left( \beta - \frac{L_1 \beta^2}{2} \right) \sum_{k=k_0+1}^{T} \| \nabla_x F(x_k, t_k) \|^2 \leq F(x_{k_0+1}, t_{k_0+1}) - F(x_{T+1}, t_{T+1}) + L_0 \sqrt{d} \sum_{k=k_0+1}^{T} |t_{k+1} - t_k|
\leq F(x_{k_0+1}, t_{k_0+1}) - f^* + L_0 \sqrt{d} \sum_{k=k_0+1}^{T} |t_{k+1} - t_k|
\leq f(x_{k_0+1}) - f^* + L_0 \sqrt{d} \left( t_{k_0+1} + \sum_{k=k_0+1}^{T} |t_{k+1} - t_k| \right),
\]

(10)
where the last inequality holds due to Lemma 3.3. Then, we can get the upper bound for $\|\nabla f(\hat{x})\|^2$ as

\[
\|\nabla f(\hat{x})\|^2 = \min_{k \in [T]} \|\nabla f(x_k)\|^2 \\
\leq \min_{k=k_0+1, \ldots, T} \|\nabla f(x_k)\|^2 \\
\leq \frac{1}{T-k_0} \sum_{k=k_0+1}^T \|\nabla f(x_k)\|^2 \\
\leq \frac{1}{T-k_0} \sum_{k=k_0+1}^T \|\nabla F(x_k, t_k)\|^2 + \frac{1}{T-k_0} L_0 L_1 (d + 3)^{3/2} \sum_{k=k_0+1}^T t_k \\
\leq \frac{2 \left(f(x_{k_0+1}) - f^* + L_0 \sqrt{d} (t_{k_0+1} + \sum_{k=k_0+1}^T |t_{k+1} - t_k|)\right)}{(T-k_0)(2\beta - L_1 \beta^2)} + \frac{1}{T-k_0} L_0 L_1 (d + 3)^{3/2} \sum_{k=k_0+1}^T t_k,
\]

where the third inequality holds due to Lemma B.2 (ii) and the last inequality follows from (10).

If we choose the step size $\beta$ as $\frac{1}{L_1}$, we have

\[
\|\nabla f(\hat{x})\|^2 \\
\leq \frac{2 L_1 \left(f(x_{k_0+1}) - f^* + L_0 \sqrt{d} (t_{k_0+1} + \sum_{k=k_0+1}^T |t_{k+1} - t_k|)\right)}{T-k_0} + \frac{1}{T-k_0} L_0 L_1 (d + 3)^{3/2} \sum_{k=k_0+1}^T t_k \\
= O \left(\frac{1}{T-k_0} \left(1 + d^{3/2} \sum_{k=k_0+1}^T t_k\right)\right), \quad (11)
\]

where the last equality holds since $\sum_{k=k_0+1}^T t_{k+1} - t_k = O \left(\sum_{k=k_0+1}^T t_k\right)$ is satisfied. If we update $t_k$ as in Algorithm 2, we have $\sum_{k=k_0+1}^T t_k \leq \sum_{k=k_0+1}^T \max\{t_1 \gamma^{k-1}, \epsilon'\} \leq \sum_{k=k_0+1}^T \left(t_1 \gamma^{k-1} + \epsilon'(T-k_0)\right)$. By taking $\epsilon'$ sufficiently close to 0, together with the assumption of $1/(1-\gamma) = O(1)$, we have $\sum_{k=k_0+1}^T t_k = O(\gamma^{k_0})$. This implies that $\|\nabla f(\hat{x})\|^2 \leq O\left(\frac{1+\gamma^{k_0} d^{3/2}}{T-k_0}\right)$. Hence, we can obtain $\|\nabla f(\hat{x})\| \leq \epsilon$ in $T = k_0 + O\left(\frac{1+\gamma^{k_0} d^{3/2}}{\epsilon^2}\right)$ iterations.

Now, set $k_0 = O\left(\frac{1}{\epsilon^2}\right)$, then, the iteration complexity can be bounded as $T = O\left(\frac{d^{3/2}}{\epsilon^2}\right)$. Furthermore, when $\gamma$ is chosen as $\gamma \leq d^{-3/2}$, we can obtain $\gamma^{k_0} = O\left(d^{-3/2}\right)$ for some $k_0 = O\left(\frac{1}{\epsilon^2}\right)$. This yields the iteration complexity of $T = O\left(\frac{1}{\epsilon^2}\right)$.

Before going to the proof of Theorem 3.5 in the stochastic setting, we prove that the gradient of the smoothed stochastic function $\nabla F(x; t; \xi)$ is unbiased, and it has a finite variance.

\section*{Lemma B.3.} Suppose that $f$ satisfies Assumption A1 (i) and Assumption A2.

(i) The stochastic gradient of the smoothed function $\nabla_x F(x; t; \xi)$ becomes an unbiased estimator of $\nabla_x F(x; t)$. That is, for any $x \in \mathbb{R}^d$ and $t > 0$, $E_\xi[\nabla_x F(x; t; \xi)] = \nabla_x F(x; t)$ holds.

(ii) For any $x \in \mathbb{R}^d$ and $t > 0$, the variance of $\nabla_x F(x; t; \xi)$ is bounded as $E_\xi[\|\nabla_x F(x; t; \xi) - \nabla_x F(x; t)\|^2] \leq \sigma^2$. 

16
where the first equation holds due to the updating rule which yields that for simplicity. From (12), we obtain the upper bound for \( \| \nabla L \| \),

\[
\text{Proof for (i): From Assumption A1 (i), we can exchange the order of integration in terms of } \xi \text{ and } u, \text{ which yields that}
\]

\[
\mathbb{E}_\xi[\nabla_x \bar{F}(x, t; \xi)] = \mathbb{E}_u \left[ \mathbb{E}_\xi \left[ \frac{\bar{f}(x + tu; \xi) - \bar{f}(x; \xi)}{t} u \right] \right]
\]

\[
= \mathbb{E}_u \left[ \mathbb{E}_\xi \left[ \frac{\bar{f}(x + tu; \xi) - \bar{f}(x; \xi)}{t} \right] u \right]
\]

\[
= \mathbb{E}_u \left[ \frac{f(x + tu) - f(x)}{t} u \right]
\]

\[
= \nabla_x F(x, t).
\]

\[
\text{Proof for (ii): We have}
\]

\[
\mathbb{E}_\xi[\| \nabla_x \bar{F}(x, t; \xi) - \nabla_x F(x, t) \|^2] = \mathbb{E}_\xi[\| \nabla_x \mathbb{E}_u[f(x + tu; \xi)] - \nabla_x \mathbb{E}_u[f(x + tu)] \|^2]
\]

\[
= \mathbb{E}_\xi[\| \mathbb{E}_u[\nabla_x \bar{f}(x + tu; \xi) - \nabla \bar{f}(x + tu)] \| ^2]
\]

\[
\leq \mathbb{E}_\xi[\| \mathbb{E}_u[\nabla_x \bar{f}(x + tu; \xi) - \nabla \bar{f}(x + tu)] \| ^2]
\]

\[
= \mathbb{E}_u[\mathbb{E}_\xi[\| \nabla_x \bar{f}(x + tu; \xi) - \nabla \bar{f}(x + tu) \| ^2]]
\]

\[
\leq \sigma^2,
\]

where the second and third equalities hold due to Assumption A1 (i), and the last inequality follows from Assumption A2 (ii).

\[
\text{Proof for Theorem 3.5: Denote } \delta_k := \nabla_x \bar{F}(x_k, t_k; \xi_k) - \nabla_x F(x_k, t_k). \text{ We follow the convergence analysis of stochastic gradient descent. According to Lemma 3.2, since } f(x) \text{ is } L_0\text{-Lipschitz and } L_1\text{-smooth}, F(x, t) \text{ is also } L_0\text{-Lipschitz and } L_1\text{-smooth in terms of } x. \text{ Thus, we have}
\]

\[
F(x_{k+1}, t_k) \leq F(x_k, t_k) + \langle \nabla_x F(x_k, t_k), (x_{k+1} - x_k) \rangle + \frac{L_1}{2} \| x_{k+1} - x_k \|^2
\]

\[
= F(x_k, t_k) - \beta \langle \nabla_x F(x_k, t_k), \nabla_x \bar{F}(x_k, t_k; \xi_k) \rangle + \frac{L_1}{2} \beta^2 \| \nabla_x \bar{F}(x_k, t_k; \xi_k) \|^2
\]

\[
= F(x_k, t_k) - \left( \beta - \frac{L_1}{2} \beta^2 \right) \| \nabla_x F(x_k, t_k) \|^2 + L_1 \beta^2 \| \delta_k \|^2
\]

(12)

where the first equation holds due to the updating rule \( x_{k+1} - x_k = -\beta \nabla_x \bar{F}(x_k, t_k; \xi_k) \), and the last equation holds due to the definition of \( \delta_k \). Denote

\[
A_k := -(\beta - L_1 \beta^2) \langle \nabla_x F(x_k, t_k), \delta_k \rangle + \frac{L_1}{2} \beta^2 \| \delta_k \|^2
\]

for simplicity. From (12), we obtain the upper bound for \( \| \nabla_x F(x, t) \|^2 \) as follows:

\[
\left( \beta - \frac{L_1}{2} \beta^2 \right) \| \nabla_x F(x_k, t_k) \|^2 \leq F(x_k, t_k) - F(x_{k+1}, t_k) + A_k
\]

\[
= F(x_k, t_k) - F(x_{k+1}, t_{k+1}) + F(x_{k+1}, t_{k+1}) - F(x_{k+1}, t_k) + A_k
\]

\[
\leq F(x_k, t_k) - F(x_{k+1}, t_{k+1}) + L_0 | t_{k+1} - t_k | \sqrt{d} + A_k,
\]

17
where the last inequality follows from Lemma 3.3.

Now, sum up the above inequality for all iterations $k_0 + 1 \leq k \leq T$ ($k_0 < T$). Then we have

$$\left( \beta - \frac{L_1}{2} \beta^2 \right) \sum_{k=k_0+1}^{T} \| \nabla_x F(x_k, t_k) \|^2$$

$$\leq F(x_{k_0+1}, t_{k_0+1}) - F(x_{T+1}, t_{T+1}) + L_0 \sqrt{d} \sum_{k=k_0+1}^{T} |t_{k+1} - t_k| + \sum_{k=k_0+1}^{T} A_k$$

$$\leq F(x_{k_0+1}, t_{k_0+1}) - f^* + L_0 \sqrt{d} \sum_{k=k_0+1}^{T} |t_{k+1} - t_k| + \sum_{k=k_0+1}^{T} A_k.$$ 

$$\leq f(x_{k_0+1}) - f^* + L_0 \sqrt{d} \left( t_{k_0+1} + \sum_{k=k_0+1}^{T} |t_{k+1} - t_k| \right) + \sum_{k=k_0+1}^{T} A_k.$$

Take the expectation with respect to the random vectors $\{\xi_{k_0+1}, \ldots, \xi_T\}$, then we have

$$\left( \beta - \frac{L_1}{2} \beta^2 \right) \sum_{k=k_0+1}^{T} \mathbb{E}_{\xi} [\| \nabla_x F(x_k, t_k) \|^2]$$

$$\leq f(x_{k_0+1}) - f^* + L_0 \sqrt{d} \left( t_{k_0+1} + \sum_{k=k_0+1}^{T} \mathbb{E}_{\xi} [|t_{k+1} - t_k|] \right) + \sum_{k=k_0+1}^{T} \mathbb{E}_{\xi} [A_k]. \quad (13)$$

The expectation of $A_k$ is evaluated as

$$\sum_{k=k_0+1}^{T} \mathbb{E}_{\xi} [A_k] = - \sum_{k=k_0+1}^{T} (\beta - L_1 2 \beta^2) \mathbb{E}_{\xi} [\langle \nabla_x F(x_k, t_k), \delta_k \rangle] + \sum_{k=k_0+1}^{T} \frac{L_1}{2} \beta^2 \mathbb{E}_{\xi} [||\delta_k||^2]$$

$$\leq (T - k_0) \frac{L_1}{2} \beta^2 \sigma^2, \quad (14)$$

where the last equality holds due to Lemma B.3 (ii) ($\mathbb{E}_{\xi} [||\delta_k||^2] \leq \sigma^2$) and the fact that each point $x_k$ is a function of the history $\xi_{[k-1]}$ in the random process, thus $\mathbb{E}_{\xi_k} [\langle \nabla_x F(x_k, t_k), \delta_k \rangle | \xi_{[k-1]} = 0].$

Then, we can estimate the upper bound for $\mathbb{E}_{\xi, \delta} [||\nabla f(\hat{x})||^2]$ as

$$\mathbb{E}_{\xi, \delta} [||\nabla f(\hat{x})||^2] = \frac{1}{T - k_0} \sum_{k=k_0+1}^{T} \mathbb{E}_{\xi} [||\nabla f(x_k)||^2]$$

$$\leq \frac{1}{T - k_0} \sum_{k=k_0+1}^{T} \mathbb{E}_{\xi} [||\nabla_x F(x_k, t_k)||^2] + \frac{1}{T - k_0} L_0 L_1 (d + 3)^{3/2} \sum_{k=k_0+1}^{T} \mathbb{E}_{\xi} [t_k]$$

$$\leq 2 \left( f(x_{k_0+1}) - f^* + L_0 \sqrt{d} \left( t_{k_0+1} + \sum_{k=k_0+1}^{T} \mathbb{E}_{\xi} [|t_{k+1} - t_k|] \right) \right)$$

$$\frac{T - k_0}{(T - k_0)(2\beta - L_1 \beta^2)} + \frac{1}{T - k_0} L_0 L_1 (d + 3)^{3/2} \sum_{k=k_0+1}^{T} \mathbb{E}_{\xi} [t_k] + \frac{L_1 \beta^2 \sigma^2}{2\beta - L_1 \beta^2},$$

where the first inequality holds due to Lemma B.2 (ii) and the last inequality follows from (13) and (14).

If the step size $\beta$ is chosen as $\beta = \min \left\{ \frac{1}{L_1}, \frac{1}{\sqrt{T - k_0}} \right\}$, then we have

$$\frac{1}{2\beta - L_1 \beta^2} \leq \frac{1}{\beta},$$

$$\frac{1}{\beta} \leq L_1 + \sqrt{T - k_0}.$$
Hence, we can obtain
\[
2 \left( f(x_{k_0+1}) - f^* + L_0 \sqrt{d} \left( t_{k_0+1} + \sum_{k=k_0+1}^{T} \mathbb{E}_x [||t_{k+1} - t_k||] \right) \right)
= \frac{1}{T - k_0} L_0 L_1 (d + 3)^{3/2} \sum_{k=k_0+1}^{T} \mathbb{E}_x [t_k] + \frac{L_1 \beta^2 \sigma^2}{2 \beta - L_1 \beta^2}
= O \left( \frac{1 + \sqrt{d} \mathbb{E}_x \left[ \sum_{k=k_0+1}^{T} ||t_{k+1} - t_k|| \right]}{T - k_0} + \frac{d^{3/2} \mathbb{E}_x \left[ \sum_{k=k_0+1}^{T} t_k \right]}{T - k_0} \right)
\]

If \( t_k \) is updated as in Algorithm 2, we have \( \sum_{k=k_0+1}^{T} ||t_{k+1} - t_k|| \leq t_1 \gamma_{k_0} = O(\gamma k_0) \) and \( \sum_{k=k_0+1}^{T} t_k \leq t_1 \gamma_{k_0} + \epsilon T = O(\gamma k_0) \) in the same argument that showed Theorem 3.4. Combining the above inequalities, we obtain
\[
\mathbb{E}_{x,k} [||\nabla f(\hat{x})||^2] = \frac{1}{T - k_0} \sum_{k=k_0+1}^{T} \mathbb{E}_x [||\nabla f(x_k)||^2] = O \left( \frac{1 + \mathbb{E}_x \left[ ||\nabla f(x)||^2 \right]}{\sqrt{T - k_0}} + \frac{d^{3/2} \mathbb{E}_x \left[ ||\nabla f(x)||^2 \right]}{T - k_0} \right).
\] (15)

Here, we have \( k_0 = O \left( \frac{1}{\epsilon^2} \right) \) by the definition of \( k_0 \). Thus, by setting \( T = k_0 = O \left( \frac{d + d^{1/2}}{\epsilon^2} \right) \), we can obtain \( \mathbb{E}_{x,k} [||\nabla f(\hat{x})||^2] \leq \epsilon^2 \). This implies \( \mathbb{E}_{x,k} [||\nabla f(\hat{x})||^2] \) follows from Jensen’s inequality. Furthermore, when \( \gamma \) is chosen as \( \gamma = (\max \{d^{1/2},d^{-3/2}\epsilon^{-2}\})^{-1} \), we have \( \log_\gamma \min \{d^{-1/2},d^{-3/2}\epsilon^{-2}\} = O \left( \frac{1}{\epsilon^2} \right) \), which implies \( k_0 = \Omega \left( \log_\gamma \min \{d^{-1/2},d^{-3/2}\epsilon^{-2}\} \right) \). Therefore, we can obtain \( \gamma k_0 = O \left( \min \{d^{-1/2},d^{-3/2}\epsilon^{-2}\} \right) \), which yields the iteration complexity of \( T = O \left( \frac{1}{\epsilon^2} \right) \).

\( \square \)

### B.3 Zeroth-order SLGH algorithm

In the zeroth-order setting, we can evaluate the gap between the zeroth-order gradient estimator and the true gradient using the following lemma.

**Lemma B.4 (Theorem 4 in [28]).** Let \( f \) be a \( L_1 \)-smooth function, then for any \( x \in \mathbb{R}^d \) and for any \( t > 0 \), we have
\[
\mathbb{E}_u \left[ \frac{1}{t^2} (f(x + tu) - f(x))^2 \right] \leq \frac{t^2}{2} L_1^2 (d + 6)^3 + 2(d + 4) ||\nabla f(x)||^2.
\]

**Proof for Theorem 4.1:** Let \( w_k := (u_k,v_k) \), \( k \in [T] \), and denote \( \delta_k := \tilde{g}_x(x_k,t_k;u_k) - \nabla_x F(x_k,t_k) \), where \( \tilde{g}_x(x_k,t_k;u_k) \) is the zeroth-order estimator of gradient defined in the main paper. Utilize the updating rule of \( x \) and \( L_1 \)-smoothness of \( F(x,t) \) in terms of \( x \). Then we have
\[
F(x_{k+1},t_k) \leq F(x_k,t_k) + \langle \nabla_x F(x_k,t_k), (x_{k+1} - x_k) \rangle + \frac{L_1}{2} ||x_{k+1} - x_k||^2
= F(x_k,t_k) - \beta \langle \nabla_x F(x_k,t_k), \tilde{g}_x(x_k,t_k;u_k) \rangle + \frac{L_1}{2} \beta^2 ||\tilde{g}_x(x_k,t_k;u_k)||^2
= F(x_k,t_k) - \beta ||\nabla_x F(x_k,t_k)||^2 - \beta \langle \nabla_x F(x_k,t_k), \delta_k \rangle + \frac{L_1}{2} \beta^2 ||\tilde{g}_x(x_k,t_k;u_k)||^2,
\] (16)

where the first equation holds due to the updating rule \( x_{k+1} - x_k = -\beta \tilde{g}_x(x_k,t_k;u_k) \).

Denote
\[
B_k := -\beta \langle \nabla_x F(x_k,t_k), \delta_k \rangle + \frac{L_1}{2} \beta^2 ||\tilde{g}_x(x_k,t_k;u_k)||^2
\]
for simplicity. From Lemma 3.3 and (16), we get the upper bound for $\|\nabla_x F(x, t)\|^2$ as

$$
\beta\|\nabla_x F(x_k, t_k)\|^2 \leq F(x_k, t_k) - F(x_{k+1}, t_k) + B_k
$$

$$
= F(x_k, t_k) - F(x_{k+1}, t_{k+1}) + F(x_{k+1}, t_k) - F(x_{k+1}, t_k) + B_k
$$

$$
\leq F(x_k, t_k) - F(x_{k+1}, t_{k+1}) + L_0|t_{k+1} - t_k|\sqrt{d} + B_k.
$$

Now, sum up the above inequality for all iterations $k_0 + 1 \leq k \leq T$ ($k_0 < T$). Then we have

$$
\sum_{k=k_0+1}^{T} \beta\|\nabla_x F(x_k, t_k)\|^2 \leq F(x_{k_0+1}, t_{k_0+1}) - F(x_{T+1}, t_{T+1}) + L_0 \sum_{k=k_0+1}^{T} |t_{k+1} - t_k|\sqrt{d} + \sum_{k=k_0+1}^{T} B_k
$$

$$
\leq F(x_{k_0+1}, t_{k_0+1}) - f^* + L_0\sqrt{d} \sum_{k=k_0+1}^{T} |t_{k+1} - t_k| + \sum_{k=k_0+1}^{T} B_k
$$

$$
\leq f(x_{k_0+1}) - f^* + L_0\sqrt{d} \left( t_{k_0+1} + \sum_{k=k_0+1}^{T} |t_{k+1} - t_k| \right) + \sum_{k=k_0+1}^{T} B_k.
$$

Next, take the expectations with respect to random vectors $\{w_{k_0+1}, \ldots, w_T\}$ on both sides. Then we can get

$$
\sum_{k=k_0+1}^{T} \beta\mathbb{E}_w[\|\nabla_x F(x_k, t_k)\|^2] \leq f(x_{k_0+1}) - f^* + L_0\sqrt{d} \left( t_{k_0+1} + \sum_{k=k_0+1}^{T} \mathbb{E}_w[|t_{k+1} - t_k|] \right)
$$

$$
+ \sum_{k=k_0+1}^{T} \mathbb{E}_w[B_k].
$$

(17)

Observe by the definition of $\tilde{g}_x(x_k, t_k; u_k)$ in the main paper that $\mathbb{E}_{u_k}[\tilde{g}_x(x_k, t_k; u_k) | u_{|k-1]}] = \nabla_x F(x_k, t_k)$, thus $\mathbb{E}_{u_k}[\|\nabla_x F(x_k, t_k, \delta_k) | u_{|k-1]}] = 0$ holds. Then we have

$$
\mathbb{E}_{u_k}[B_k | u_{|k-1}]] = -\beta\mathbb{E}_{u_k}[\|\nabla_x F(x_k, t_k, \delta_k) | u_{|k-1}]] + \frac{L_1}{2} \beta^2 \mathbb{E}_{u_k}[\|\tilde{g}_x(x_k, t_k; u_k)\|^2 | u_{|k-1}]]
$$

$$
\leq \frac{L_1}{2} \beta^2 \left( \mathbb{E}_{u_k}[t_k^2 | u_{|k-1}]] L_1^3(d+6)^3 + 2(d+4)\mathbb{E}_{u_k}[\|\nabla f(x_k)\|^2 | u_{|k-1}]] \right)
$$

$$
= \frac{\mathbb{E}_{u_k}[t_k^2 | u_{|k-1}]] L_1^3\beta^2(d+6)^3 + L_1\beta^2(d+4)\mathbb{E}_{u_k}[\|\nabla f(x_k)\|^2 | u_{|k-1}]]}{4},
$$

(18)

where the inequality holds due to Lemma B.4.

Lemma B.2 (ii) together with the above inequalities yields that

$$
\sum_{k=k_0+1}^{T} \beta\mathbb{E}_w[\|\nabla f(x_k)\|^2]
$$

$$
\leq \sum_{k=k_0+1}^{T} \beta\mathbb{E}_w[\|\nabla x F(x_k, t_k)\|^2] + \sum_{k=k_0+1}^{T} \beta L_0 L_1 (d + 3)^{3/2} \mathbb{E}_w[t_k]
$$

$$
\leq f(x_{k_0+1}) - f^* + L_0\sqrt{d} \left( t_{k_0+1} + \sum_{k=k_0+1}^{T} \mathbb{E}_w[|t_{k+1} - t_k|] \right) + \sum_{k=k_0+1}^{T} \mathbb{E}_w[B_k]
$$

$$
+ \sum_{k=k_0+1}^{T} \beta L_0 L_1 (d + 3)^{3/2} \mathbb{E}_w[t_k]
$$

$$
\leq f(x_{k_0+1}) - f^* + L_0\sqrt{d} \left( t_{k_0+1} + \sum_{k=k_0+1}^{T} \mathbb{E}_w[|t_{k+1} - t_k|] \right) + \sum_{k=k_0+1}^{T} \frac{\mathbb{E}_w[t_k^2]}{4} L_1^3\beta^2(d+6)^3
$$

$$
+ \sum_{k=k_0+1}^{T} L_1\beta^2(d+4)\mathbb{E}_w[\|\nabla f(x_k)\|^2] + \sum_{k=k_0+1}^{T} \beta L_0 L_1 (d + 3)^{3/2} \mathbb{E}_w[t_k],
$$

(19)
where the second inequality holds due to (17), and the last inequality follows from (18). Rearrange the terms in the above inequality. Then we can get

\[
\begin{align*}
(\beta - (d + 4)L_1\beta^2) \sum_{k=k_0+1}^{T} E_w[\|\nabla f(x_k)\|^2] &\leq f(x_{k_0+1}) - f^* + L_0\sqrt{d} \left( t_{k_0+1} + \sum_{k=k_0+1}^{T} E_w[|t_{k+1} - t_k|] \right) \\
+ \frac{L_1^2\beta^2(d + 6)^3}{4} \sum_{k=k_0+1}^{T} E_w[t_k^2] &+ L_0L_1\beta(d + 3)^{3/2} \sum_{k=k_0+1}^{T} E_w[t_k].
\end{align*}
\]

(20)

Divide both sides of the above inequality by \((T - k_0)(\beta - (d + 4)L_1\beta^2)\) and set the step size \(\beta\) as \(\frac{1}{2(d+4)l_1}\). Since \(\frac{1}{\beta-(d+4)l_1\beta^2} \leq 4(d+4)L_1\) holds, we can obtain

\[
\begin{align*}
\frac{1}{T - k_0} \sum_{k=k_0+1}^{T} E_w[\|\nabla f(x_k)\|^2] &\leq \frac{4(d+4)L_1}{T - k_0} \left( f(x_{k_0+1}) - f^* + L_0\sqrt{d} \left( t_{k_0+1} + \sum_{k=k_0+1}^{T} E_w[|t_{k+1} - t_k|] \right) \right. \\
+ \frac{L_1^2(d + 6)^3}{16(d+4)^2} \sum_{k=k_0+1}^{T} E_w[t_k^2] &+ \frac{L_0(d + 3)^{3/2}}{2(d+4)} \sum_{k=k_0+1}^{T} E_w[t_k] \\
= O \left( \frac{d}{T - k_0} \right) &\left( 1 + dE_w \left[ \sum_{k=k_0+1}^{T} t_k^2 \right] + \sqrt{dE_w} \left[ \sum_{k=k_0+1}^{T} t_k \right] \right)
\end{align*}
\]

(21)

where the last equality follows from the update rule of \(t_k\), as shown in the proof of Theorem 3.4 as well.

Here, we have \(k_0 = O(\frac{d}{\epsilon})\) by the definition of \(k_0\). Thus, by setting \(T = k_0 + O\left( \frac{\epsilon^2}{d^2} \right) = O\left( \frac{\epsilon^2}{d^2} \right)\), we can obtain

\[
E_{w,k^*}[\|\nabla f(\hat{x})\|^2] = \frac{1}{T - k_0} \sum_{k=k_0+1}^{T} E_w[\|\nabla f(x_k)\|^2] \leq \epsilon^2.
\]

This implies \(E_{w,k^*}[\|\nabla f(\hat{x})\|^2] \leq \epsilon\) as \(E_{w,k^*}[\|\nabla f(\hat{x})\|^2] \leq E_{w,k^*}[\|\nabla f(x)\|^2]\) follows from Jensen’s inequality. Furthermore, when \(\gamma\) is chosen as \(\gamma \leq \frac{d}{d^2/2d} = O\left( \frac{d}{\sqrt{2d}} \right)\), we have \(\log \gamma d^{-1/2} = O\left( \frac{d}{\sqrt{2d}} \right)\), which implies \(k_0 = \Omega(\log \gamma d^{-1/2})\). Therefore, we can obtain \(\gamma^{k_0} = O(d^{-1/2})\), which yields the iteration complexity of \(T = O\left( \frac{\epsilon^2}{d^2} \right)\).

\[\square\]

**Proof for Theorem 4.2:** Let \(\xi_k := (\xi_k, u_k, v_k)\), \(k \in [T]\) and denote \(\delta_k := \tilde{G}_x(x_k, t_k; \xi_k, u_k) - \nabla_x \tilde{F}(x_k, t_k)\). As discussed in the main paper, we have

\[
E_{\xi,w}[\tilde{G}_x(x, t; \xi, u)] = E_w[E_{\xi}[\tilde{G}_x(x, t; \xi, u)|u]] = \nabla_x \tilde{F}(x, t).
\]

(22)

From the update rule for \(x\), we can obtain

\[
F(x_{k+1}, t_k) \leq F(x_k, t_k) + \langle \nabla_x F(x_k, t_k), (x_{k+1} - x_k) \rangle + \frac{L_1}{2} \|x_{k+1} - x_k\|^2
\]

\[
= F(x_k, t_k) - \beta \langle \nabla_x \tilde{F}(x_k, t_k), \tilde{G}_x(x_k, t_k; \xi_k, u_k) \rangle + \frac{L_1}{2} \beta^2 \|\tilde{G}_x(x_k, t_k; \xi_k, u_k)\|^2
\]

\[
= F(x_k, t_k) - \beta \|\nabla_x \tilde{F}(x_k, t_k)\|^2 - \beta \langle \nabla_x \tilde{F}(x_k, t_k), \delta_k \rangle + \frac{L_1}{2} \beta^2 \|\tilde{G}_x(x_k, t_k; \xi_k, u_k)\|^2.
\]

Now, denote

\[
D_k := -\beta \langle \nabla_x \tilde{F}(x_k, t_k), \delta_k \rangle + \frac{L_1}{2} \beta^2 \|\tilde{G}_x(x_k, t_k; \xi_k, u_k)\|^2
\]

for simplicity. Then we can get the upper bound for \(\|\nabla_x \tilde{F}(x, t)\|^2\) with \(D_k\):

\[
\beta \|\nabla_x \tilde{F}(x_k, t_k)\|^2 \leq F(x_k, t_k) - F(x_{k+1}, t_k) + D_k
\]

\[
= F(x_k, t_k) - F(x_{k+1}, t_{k+1}) + F(x_{k+1}, t_{k+1}) - F(x_{k+1}, t_k) + D_k
\]

\[
\leq F(x_k, t_k) - F(x_{k+1}, t_{k+1}) + L_0|t_{k+1} - t_k|\sqrt{d} + D_k.
\]
Sum up the above inequality for all iterations \( k_0 + 1 \leq k \leq T \) \((T > k_0)\). Then we have

\[
\sum_{k=k_0+1}^{T} \beta ||\nabla_x F(x_k, t_k)||^2 \\
\leq F(x_{k_0+1}, t_{k_0+1}) - F(x_{T+1}, t_{T+1}) + L_0 \sqrt{d} \sum_{k=k_0+1}^{T} |t_{k+1} - t_k| + \sum_{k=k_0+1}^{T} D_k \\
\leq F(x_{k_0+1}, t_{k_0+1}) - f^* + L_0 \sqrt{d} \sum_{k=k_0+1}^{T} |t_{k+1} - t_k| + \sum_{k=k_0+1}^{T} D_k \\
\leq f(x_{k_0+1}) - f^* + L_0 \sqrt{d} \left( t_{k_0+1} + \sum_{k=k_0+1}^{T} |t_{k+1} - t_k| \right) + \sum_{k=k_0+1}^{T} D_k, \quad (23)
\]

where the last inequality follows from Lemma 3.3. Observe from (22) that

\[
\mathbb{E}_{\zeta_k} [||\nabla_x F(x_k, t_k), \delta_k|| | \zeta_{k-1}] = 0.
\]

Thus, we have

\[
\mathbb{E}_{\zeta_k}[D_k | \zeta_{k-1}] = -\beta \mathbb{E}_{\zeta_k} [||\nabla_x F(x_k, t_k), \delta_k|| | \zeta_{k-1}] + \frac{L_1}{2} \beta^2 \mathbb{E}_{\zeta_k} [||\bar{G}_x(x_k, t_k; \xi_k, u_k)||^2 | \zeta_{k-1}] \\
= \frac{L_1}{2} \beta^2 \mathbb{E}_{\zeta_k} (||\bar{G}_x(x_k, t_k; \xi_k, u_k)||^2 | \zeta_{k-1}) \\
\leq \frac{L_1}{2} \beta^2 \left( \frac{\mathbb{E}_{\zeta_k}[t_k^2 | \zeta_{k-1}]}{2} L_1^2 (d + 6)^3 + 2(d + 4)(\mathbb{E}_{\zeta_k} [||\nabla_x F(x_k; \xi_k) | \zeta_{k-1}||^2]) \right) \\
\leq \frac{L_1}{2} \beta^2 \left( \frac{\mathbb{E}_{\zeta_k}[t_k^2 | \zeta_{k-1}]}{2} L_1^2 (d + 6)^3 + 2(d + 4)(\mathbb{E}_{\zeta_k} [||\nabla f(x_k) | \zeta_{k-1}||^2] + \sigma^2) \right), \quad (24)
\]

where the fist inequality follows from Lemma B.4 and the last inequality holds due to Assumption A2 (ii).

Take the expectation for (23) with respect to \( \zeta_{k_0+1}, \ldots, \zeta_T \). Together with Lemma B.2 (ii), we have

\[
\sum_{k=k_0+1}^{T} \beta \mathbb{E}_{\zeta} [||\nabla f(x_k)||^2] \\
\leq \sum_{k=k_0+1}^{T} \beta \mathbb{E}_{\zeta} [||\nabla_x F(x_k, t_k)||^2] + \sum_{k=k_0+1}^{T} \beta \mathbb{E}_{\zeta} [t_k] L_0 L_1 (d + 3)^{3/2} \\
\leq f(x_{k_0+1}) - f^* + L_0 \sqrt{d} \left( t_{k_0+1} + \sum_{k=k_0+1}^{T} \mathbb{E}_{\zeta} [t_{k+1} - t_k] \right) + \sum_{k=k_0+1}^{T} \mathbb{E}_{\zeta}[D_k] \\
+ \sum_{k=k_0+1}^{T} \mathbb{E}_{\zeta}[t_k] L_0 L_1 (d + 3)^{3/2} \\
\leq f(x_{k_0+1}) - f^* + L_0 \sqrt{d} \left( t_{k_0+1} + \sum_{k=k_0+1}^{T} \mathbb{E}_{\zeta} [t_{k+1} - t_k] \right) + \sum_{k=k_0+1}^{T} \mathbb{E}_{\zeta}[t_k] L_0 L_1 (d + 3)^{3/2} \\
+ \sum_{k=k_0+1}^{T} \frac{\mathbb{E}_{\zeta}[t_k^2]}{4} L_1^3 \beta^2 (d + 6)^3 + \sum_{k=k_0+1}^{T} L_1 \beta^2 (d + 4) \mathbb{E}_{\zeta} [||\nabla f(x_k)||^2] + L_1 \beta^2 (d + 4) \sigma^2 (T - k_0),
\]

22
where the last inequality holds due to (24). Rearrange the terms in the above inequality. Then we can get

$$
(\beta - (d + 4)L_1\beta^2) \sum_{k=k_0+1}^{T} \mathbb{E}_\zeta[||\nabla f(x_k)||^2] \leq f(x_{k_0+1}) - f^* + L_0 \sqrt{d} \left( t_{k_0+1} + \sum_{k=k_0+1}^{T} \mathbb{E}_\zeta[|t_{k+1} - t_k|] \right) \\
+ \frac{L_1^2\beta^2(d + 6)^3}{4} \sum_{k=k_0+1}^{T} \mathbb{E}_\zeta[t_k^2] + L_1\beta^2(d + 4)\sigma^2(T - k_0) \\
+ \sum_{k=k_0+1}^{T} \mathbb{E}_\zeta[t_k]L_1\beta(d + 3)^{3/2}, \quad (25)
$$

If the step size $\beta$ is chosen as min \( \frac{1}{2(d + 4)L_1}, \frac{1}{\sqrt{(T - k_0)(d + 4)}} \), then we have

$$
\frac{1}{\beta} \leq 2(\bar{d} + 4)L_1, \quad \frac{1}{\beta} \leq 2(d + 4)L_1 + \sqrt{(T - k_0)(d + 4)}.
$$

Hence, by dividing both sides of (25) by $T - k_0(\beta - 2(d + 4)L_1\beta^2)$, we can obtain

$$
\frac{1}{T - k_0} \sum_{k=1}^{T} \mathbb{E}_\zeta[||\nabla f(x_k)||^2] \\
\leq \frac{f(x_{k_0+1}) - f^* + L_0 \sqrt{d} \left( t_{k_0+1} + \sum_{k=k_0+1}^{T} \mathbb{E}_\zeta[|t_{k+1} - t_k|] \right)}{(T - k_0)(\beta - (d + 4)L_1\beta^2)} \\
+ \frac{L_1^2\beta^2(d + 6)^3}{4(T - k_0)} \sum_{k=k_0+1}^{T} \mathbb{E}_\zeta[t_k^2] + \frac{L_1\beta^2(d + 4)\sigma^2 T}{(T - k_0)} \\
\leq \frac{2}{T - k_0} \left( f(x_{k_0+1}) - f^* + L_0 \sqrt{d} \left( t_{k_0+1} + \sum_{k=k_0+1}^{T} \mathbb{E}_\zeta[|t_{k+1} - t_k|] \right) \right) \left( 2(d + 4)L_1 + \sqrt{(T - k_0)(d + 4)} \right) \\
+ \frac{2}{T - k_0} L_1\beta(d + 3)^{3/2} \sum_{k=k_0+1}^{T} \mathbb{E}_\zeta[t_k] + \frac{L_1^2\beta(d + 6)^3}{2(T - k_0)} \sum_{k=k_0+1}^{T} \mathbb{E}_\zeta[t_k^2] + 2L_1\beta(d + 4)\sigma^2 \\
= O \left( \sqrt{d} \left( 1 + \sqrt{d} \sum_{k=k_0+1}^{T} \mathbb{E}_\zeta[|t_{k+1} - t_k|] \right) \right) + d \left( d\mathbb{E}_\zeta \left( \sum_{k=k_0+1}^{T} t_k^2 + \sqrt{d}\mathbb{E}_\zeta \left( \sum_{k=k_0+1}^{T} t_k \right) + 1 \right) \right) \\
= O \left( \sqrt{d} \left( 1 + \sqrt{d}\gamma_{k_0} \right) \right),
$$

where the last equality follows from the update rule of $t_k$, as shown in the proof of Theorem 3.4 as well.

Here, we have $k_0 = O \left( \frac{d^2}{\beta} \right)$ by the definition of $k_0$. Thus, by setting $T = k_0 + O \left( \frac{d^2}{\gamma} \right) = O \left( \frac{d^2}{\beta^2} \right)$, we can obtain $\mathbb{E}_\zeta, k'[||\nabla f(\hat{x})||^2] = \frac{1}{T - k_0} \sum_{k=k_0+1}^{T} \mathbb{E}_\zeta[||\nabla f(x_k)||^2] \leq \epsilon^2$. This implies $\mathbb{E}_\zeta, k'[||\nabla f(\hat{x})||] \leq \epsilon$ as $\mathbb{E}_\zeta, k'[||\nabla f(\hat{x})||^2] \leq \mathbb{E}_\zeta, k'[||\nabla f(\hat{x})||^2]$ follows from Jensen’s inequality. Furthermore, when $\gamma$ is chosen as $\gamma \leq d^{-1/2d}$, we have $\log_{\gamma} d^{1/2} = O \left( \frac{d^{1/2}}{\gamma} \right)$, which implies that $k_0 = \Omega \left( \log_{d^{1/2}} d^{1/2} \right)$. Therefore, we can obtain $\gamma_{k_0} = O(d^{-1/2})$, which yields the iteration complexity of $T = O \left( \frac{d^2}{\gamma} \right)$.

\(\square\)
C ZOSLGH algorithm with error tolerance

In Sections 3 and 4, we assumed that we had access to the exact function value or a gradient oracle whose variance was finite. However, in some practical cases, we will have access only to the function values containing error, and it would be impossible to obtain accurate gradient oracles of an underlying objective function. Figure 2 illustrates such a case; although the objective function $f$ (Figure 2(a)) is smooth, the accessible function $f'$ (Figure 2(b)) contains some error, and thus many local minima arise. In this section, we consider optimizing a smooth objective function $f$ using only the information of $f'$. We assume that the following condition holds between $f$ and $f'$.

**Assumption A3.** The supremum norm of the difference between $f$ and $f'$ is uniformly bounded:

$$\sup_{x \in \mathbb{R}^d} |f(x) - f'(x)| \leq \nu.$$  

In the stochastic setting, we assume $\sup_{x \in \mathbb{R}^d} |f(x; \xi) - f'(x; \xi)| \leq \nu$ for any $\xi$.

![Figure 2](image)

**(a) Smooth objective function**  
**(b) Accessible function with error**

Figure 2: Illustration of a smooth objective function and the accessible function that contains error.

Please note that we do not impose any other assumptions on the accessible function $f'$. Thus, $f'$ can be non-Lipschitz or even discontinuous. Even in such cases, we can develop an algorithm with a convergence guarantee because its smoothed function $F'(\cdot, t)$ is smooth as far as $t$ is sufficiently large. In the following, we denote the Lipschitz and gradient Lipschitz constant of $F'(\cdot, t)$ as $L_0(t)$ and $L_1(t)$, respectively.

The ZOSLGH algorithm in this setting is almost the same as Algorithm 3. The only difference is $\sqrt{\nu}$ rather than $\epsilon$ in the update rule of $t_{k+1}$. See the Algorithm 4 for a more detailed description. Please note that $F', \tilde{g}'_x, \tilde{G}'_{x,u}, \tilde{g}'_t, \tilde{G}'_{t,v}$ are defined in the same way as the no-error setting using $f'$.

**Algorithm 4** Deterministic/Stochastic Zeroth-Order Single Loop GH algorithm (ZOSLGH) with error tolerance

**Require:** Iteration number $T$, initial solution $x_1$, initial smoothing parameter $t_1$, sequence of step sizes $\{\beta_k\}$ for $x$, step size $\eta$ for $t$, decreasing factor $\gamma \in (0, 1)$, error tolerance $\nu$

**for** $k = 1$ to $T$ **do**

- Sample $u_k$ from $\mathcal{N}(0, I_d)$
- Update $x_k$ by
  $$x_{k+1} = x_k - \beta_k \tilde{G}'_{x,u},$$
  where $\tilde{G}'_{x,u} = \begin{cases} 
  \tilde{g}'_x(x_k, t_k; u_k) & \text{(deterministic)} \\
  \tilde{G}'_{x}(x_k, t_k; u_k), \xi_k \sim P & \text{(stochastic)}
  \end{cases}$

- Sample $v_k$ from $\mathcal{N}(0, I_d)$
- Update $t_k$ by
  $$t_{k+1} = \begin{cases} 
  \max\{\gamma t_k, \sqrt{\nu}\} & \text{(SLGH)} \\
  \max\{\min\{t_k - \eta \tilde{G}'_{t,v}, \gamma t_k\}, \sqrt{\nu}\} & \text{(SLGH)}
  \end{cases},$$
  where $\tilde{G}'_{t,v} = \begin{cases} 
  \tilde{g}'_t(x_k, t_k; v_k) & \text{(deterministic)} \\
  \tilde{G}'_{t}(x_k, t_k; v_k), \xi_k \sim P & \text{(stochastic)}
  \end{cases}$

**end for**
We provide the convergence analyses in the following theorems. The definitions of \( \hat{x} \) in the deterministic and stochastic settings are given in Appendix C.2 and C.3, respectively.

**Theorem C.1 (Convergence of ZOSLGH with error tolerance, Deterministic setting).** Suppose Assumptions A1 and A3 hold.

Take \( k_1 := \Theta(d/e^2) \) and \( k_2 := O(\log_2 1/d) \) and define \( k_0 = \min\{k_1, k_2\} \). Let \( \hat{x} := x_{k'} \), where \( k' \) is chosen from a uniform distribution over \( \{k_0 + 1, k_0 + 2, \ldots, T\} \). Set the stepsize for \( x \) at iteration \( k \) as \( \beta_k = \frac{1}{16(d+4)L_x(t_k)} \), \( k \in [T] \). Then, for any setting of the parameter \( \gamma \), if the error level \( \nu \) satisfies \( \nu = O(e^2/d^3) \), \( \hat{x} \) satisfies \( E[\|\nabla f(\hat{x})\|] \leq \epsilon \) with the iteration complexity of \( T = O(d^3/e^2) \), where the expectation is taken w.r.t. random vectors \( \{u_k\} \) and \( \{v_k\} \). Further, if we choose \( \gamma \leq d^{-\Omega(e^3/d)} \), the iteration complexity can be bounded as \( T = O(d/e^4) \).

**Theorem C.2 (Convergence of ZOSLGH with error tolerance, Stochastic setting).** Suppose Assumptions A1, A2 and A3 hold. Take \( k_1 := \Theta(d/e^4) \) and \( k_2 := O(\log_2 1/d) \) and define \( k_0 = \min\{k_1, k_2\} \). Let \( \hat{x} := x_{k'} \), where \( k' \) is chosen from a uniform distribution over \( \{k_0 + 1, k_0 + 2, \ldots, T\} \). Set the stepsize for \( x \) at iteration \( k \) as \( \beta_k = \min \left\{ \frac{1}{16(d+4)L_x(t_k)}, \sqrt{\frac{1}{(T-k_0)(d+4)}} \right\} \).

Then, for any setting of the parameter \( \gamma \), if the error level \( \nu \) satisfies \( \nu = O(e^2/d^3) \), \( \hat{x} \) satisfies \( E[\|\nabla f(\hat{x})\|] \leq \epsilon \) with the iteration complexity of \( T = O(d^3/e^2 + d^3/e^2) \), where the expectation is taken w.r.t. random vectors \( \{u_k\}, \{v_k\} \) and \( \{\xi_k\} \). Further, if we choose \( \gamma \leq d^{-\Omega(e^3/d)} \), the iteration complexity can be bounded as \( T = O(d/e^4) \).

### C.1 Proofs for technical lemmas

We introduce several lemmas before going to the convergence analysis. All of them describe properties of the function with error \( f' \) and its Gaussian smoothing \( F' \). Throughout this subsection, we assume that \( f \) is \( L_0 \)-Lipschitz and \( L_1 \)-smooth function. We also suppose that the function pair \( (f, f') \) satisfies \( \sup_{x \in \mathbb{R}^d} |f(x) - f'(x)| \leq \nu \).

**Lemma C.3.** For any \( x \in \mathbb{R}^d \) and \( t > 0 \), we have

\[
E_u \left[ \frac{1}{t^2} (f'(x + tu) - f'(x))^2 \|u\|^2 \right] \leq 4(d + 4)\|\nabla f(x)\|^2 + t^2 L_x^2 (d + 6)^3 + 8d \frac{\nu^2}{t^2}.
\]

**Proof:**

\[
E_u \left[ \frac{1}{t^2} (f'(x + tu) - f'(x))^2 \|u\|^2 \right] = E_u \left[ \frac{1}{t^2} (f(x + tu) - f(x) + (f' - f)(x + tu) - (f' - f)(x))^2 \|u\|^2 \right] \\
\qquad \leq 2E_u \left[ \frac{1}{t^2} (f(x + tu) - f(x))^2 \|u\|^2 \right] + 2E_u \left[ \frac{1}{t^2} (2\nu)^2 \|u\|^2 \right] \\
\qquad \leq 4(d + 4)\|\nabla f(x)\|^2 + t^2 L_x^2 (d + 6)^3 + 8d \frac{\nu^2}{t^2},
\]

where the last inequality holds due to Lemma B.1 and Lemma B.4.

**Lemma C.4.** For any \( x \in \mathbb{R}^d \) and \( t > 0 \), we have

\[
E_x \left[ \frac{1}{t^2} (\hat{f}'(x + tu; \xi) - \hat{f}'(x; \xi))^2 \|u\|^2 \right] \leq 4(d + 4)(\|\nabla f(x)\|^2 + \sigma^2) + t^2 L_x^2 (d + 6)^3 + 8d \frac{\nu^2}{t^2}.
\]
Proof:

\[ \mathbb{E}_\xi \left[ \frac{1}{t^2} (\hat{f}'(x + tu; \xi) - \hat{f}'(x; \xi))^2 \|u\|^2 \right] \]
\[ = \mathbb{E}_\xi \left[ \mathbb{E}_u \left[ \frac{1}{t^2} (\hat{f}(x + tu; \xi) - \hat{f}(x; \xi) + (\hat{f}' - \hat{f})(x + tu; \xi) - (\hat{f}' - \hat{f})(x; \xi))^2 \|u\|^2 \right] \right] \]
\[ \leq 2\mathbb{E}_\xi \left[ \mathbb{E}_u \left[ \frac{1}{t^2} (\hat{f}(x + tu; \xi) - \hat{f}(x; \xi))^2 \|u\|^2 \right] \right] + \frac{2}{t^2} \mathbb{E}_\xi [\mathbb{E}_u ((2\nu)^2 \|u\|^2)] \]
\[ \leq 2\mathbb{E}_\xi \left[ \frac{t^2}{2} L_1^2 (d + 6)^3 + 2(d + 4) \|\nabla \hat{f}(x; \xi)\|^2 \right] + 8d \nu^2 \]
\[ \leq 4(d + 4)(\|\nabla f(x)\|^2 + \sigma^2) + t^2 L_1^2 (d + 6)^3 + 8d \nu^2, \]

where the second inequality follows from Lemma B.1 and Lemma B.4, and the last inequality holds due to Assumption A2 (ii).

**Lemma C.5.** For any \( x \in \mathbb{R}^d \) and for any \( t_1, t_2 \in \mathcal{T} \), we have

\[ |F'(x, t_1) - F'(x, t_2)| \leq L_0 |t_1 - t_2| \sqrt{d} + 2\nu. \]

Proof:

\[ |F'(x, t_1) - F'(x, t_2)| = |F(x, t_1) - F(x, t_2) + (F' - F)(x, t_1) - (F' - F)(x, t_2)| \]
\[ \leq |F(x, t_1) - F(x, t_2)| + |\mathbb{E}_u ((f' - f)(x + t_1 u))| + |\mathbb{E}_u ((f' - f)(x + t_2 u))| \]
\[ \leq |F(x, t_1) - F(x, t_2)| + \mathbb{E}_u [((f' - f)(x + t_1 u)) + \mathbb{E}_u [((f' - f)(x + t_2 u))] \]
\[ \leq |F(x, t_1) - F(x, t_2)| + 2\nu \]
\[ \leq L_0 |t_1 - t_2| \sqrt{d} + 2\nu, \]

where the last inequality holds due to Lemma 3.3.

**Lemma C.6 (Lemma 30 in [17]).** For any \( x \in \mathbb{R}^d \) and for any \( t_1, t_2 \in \mathcal{T} \), we have

\[ \|\nabla_x (F' - F)(x, t)\| \leq \sqrt{\frac{2 \nu}{\pi t}}. \]

**Lemma C.7.**

(i) \( F'(x, t) \) is \( L_0 + \sqrt{\frac{2 \nu}{\pi t}} \)-Lipschitz in terms of \( x \).

(ii) (Lemma 20 in [17]) \( F'(x, t) \) is \( L_1 + \frac{2\nu}{\sqrt{\pi t}} \)-smooth in terms of \( x \).

Proof for (i):

\[ \|\nabla_x F'(x, t)\| \leq \|\nabla_x F(x, t)\| + \|\nabla_x (F' - F)(x, t)\| \]
\[ \leq L_0 + \sqrt{\frac{2 \nu}{\pi t}}, \]

where the last inequality holds due to Lemma 3.2 and Lemma C.6.

**Lemma C.8.** For any \( x \in \mathbb{R}^d \) and \( t > 0 \), we have

\[ \|\nabla f(x)\|^2 \leq 4 \|\nabla_x F'(x, t)\|^2 + \frac{t^2}{2} L_1^2 (d + 6)^3 + \frac{8 \nu^2}{t^2}. \]
Proof: We have
\[
\|\nabla f(x)\| = \|\mathbb{E}_u[(\nabla f(x), u)]\|^2
\]
\[
= \left\| \frac{1}{t} \mathbb{E}_u[(f(x + tu) - f(x) - f(x) - t(\nabla f(x), u))u] \right\|^2
\]
\[
\leq \left\| \nabla_x F(x, t) - \frac{1}{t} \mathbb{E}_u[(f(x + tu) - f(x) - t(\nabla f(x), u))u] \right\|^2
\]
\[
= 2\|\nabla_x F(x, t)\| + \frac{2}{t^2} \|\mathbb{E}_u[(f(x + tu) - f(x) - t(\nabla f(x), u))u]\|^2
\]
\[
\leq 2\|\nabla_x F(x, t)\| + \frac{2}{t^2} \|\mathbb{E}_u[(f(x + tu) - f(x) - t(\nabla f(x), u))u]\|^2
\]
\[
\leq 2\|\nabla_x F(x, t)\| + \frac{2L_1^2}{t^2} \|\mathbb{E}_u[\|u\|^6]\|
\]
\[
\leq 2\|\nabla_x F(x, t)\| + \frac{2L_1^2}{2} (d + 6)^3
\]
\[
\leq 2\|\nabla_x F(x, t)\| + \frac{2L_1^2}{2} (d + 6)^3 + \left(\frac{8\nu^2}{\pi} t\right)^2,
\]
where the third last inequality holds due to Lemma B.1, and the last inequality holds due to Lemma C.6.

C.2 Proof for the deterministic setting

Proof for Theorem C.1: Let \( w_k := (u_k, v_k) \) and denote \( \delta_k := \tilde{g}_x(x_k, t_k; u_k) - \nabla_x F'(x_k, t_k) \).
Utilize the updating rule for \( x \) and \( L_1(t) \)-smoothness of \( F'(\cdot, t) \). Then we have

\[
F'(x_{k+1}, t_k) \leq F'(x_k, t_k) + \langle \nabla_x F'(x_k, t_k), (x_{k+1} - x_k) \rangle + \frac{L_1(t_k)}{2} \|x_{k+1} - x_k\|^2
\]
\[
= F'(x_k, t_k) - \beta_k \langle \nabla_x F'(x_k, t_k), \tilde{g}_x(x_k, t_k; u_k) \rangle + \frac{L_1(t_k)}{2} \beta_k^2 \|\tilde{g}_x(x_k, t_k; u_k)\|^2
\]
\[
= F'(x_k, t_k) - \beta_k \|\nabla_x F'(x_k, t_k)\|^2 - \beta_k \langle \nabla_x F'(x_k, t_k), \delta_k \rangle + \frac{L_1(t_k)}{2} \beta_k^2 \|\tilde{g}_x(x_k, t_k; u_k)\|^2.
\]
\[
(26)
\]
Denote
\[
E_k := -\beta_k \langle \nabla_x F'(x_k, t_k), \delta_k \rangle + \frac{L_1(t_k)}{2} \beta_k^2 \|\tilde{g}_x(x_k, t_k; u_k)\|^2
\]
for simplicity. From Lemma C.5 and (26), we get the upper bound for \( \|\nabla_x F'(x, t)\|^2 \) as

\[
\beta_k \|\nabla_x F'(x_k, t_k)\|^2 \leq F'(x_k, t_k) - F'(x_{k+1}, t_k) + E_k
\]
\[
= F'(x_k, t_k) - F'(x_{k+1}, t_{k+1}) + F'(x_{k+1}, t_{k+1}) - F'(x_k, t_k) + E_k
\]
\[
\leq F'(x_k, t_k) - F'(x_{k+1}, t_{k+1}) + L_0 \|t_{k+1} - t_k\| \sqrt{d} + 2\nu + E_k.
\]

27
Now, sum up the above inequality for all iterations $k_0 + 1 \leq k \leq T$ ($T > k_0$). Then we have

$$
\sum_{k=k_0+1}^{T} \beta_k \| \nabla_x F'(x_k, t_k) \|^2
\leq F'(x_{k_0+1}, t_{k_0+1}) - F'(x_{T+1}, t_{T+1}) + L_0 \sum_{k=k_0+1}^{T} \| t_{k+1} - t_k \| \sqrt{d} + 2\nu (T - k_0) + \sum_{k=k_0+1}^{T} E_k
\leq F'(x_{k_0+1}, t_{k_0+1}) - f^* + \nu + L_0 \sqrt{d} \sum_{k=k_0+1}^{T} \| t_{k+1} - t_k \| + 2\nu (T - k_0) + \sum_{k=k_0+1}^{T} E_k.
\leq f'(x_{k_0+1}) - f^* + 3\nu + L_0 \sqrt{d} \left( t_{k_0+1} + \sum_{k=k_0+1}^{T} |t_{k+1} - t_k| \right) + 2\nu (T - k_0) + \sum_{k=k_0+1}^{T} E_k
\leq f(x_{k_0+1}) - f^* + 4\nu + L_0 \sqrt{d} \left( t_{k_0+1} + \sum_{k=k_0+1}^{T} |t_{k+1} - t_k| \right) + 2\nu (T - k_0) + \sum_{k=k_0+1}^{T} E_k,
$$

(27)

where the third inequality holds due to Lemma C.5. We can bound the conditional expectation of $E_k$ as

$$
\mathbb{E}_{w_k} [E_k | w_{[k-1]}]
\leq -\beta_k \mathbb{E}_{w_k} [(\nabla_x F'(x_k, t_k), \delta_k) | w_{[k-1]}] + \frac{\mathbb{E}_{w_k} [L_1(t_k) | w_{[k-1]}]}{2} \beta_k^2 \mathbb{E}_{w_k} [|| g_k'(x_k, t_k; u_k) ||^2 | w_{[k-1]}]
\leq \frac{\mathbb{E}_{w_k} [L_1(t_k) | w_{[k-1]}]}{2} \beta_k^2 \mathbb{E}_{w_k} [|| g_k'(x_k, t_k; u_k) ||^2 | w_{[k-1]}]
\leq \frac{\mathbb{E}_{w_k} [L_1(t_k) | w_{[k-1]}]}{2} \beta_k^2 \left( 4(d + 4) \mathbb{E}_{w_k} [|| \nabla f(x_k) ||^2 | w_{[k-1]}] + L_1^2 (d + 6)^3 \mathbb{E}_{w_k} [t_k^2 | w_{[k-1]}]
+ 8d \mathbb{E}_{w_k} [\nu^2 / t_k^2 | w_{[k-1]}] \right),
$$

where the first inequality holds since we have $\mathbb{E}_{w_k} [\delta_k | w_{[k-1]}] = \mathbb{E}_{w_k} [\delta_k | u_{[k-1]}] = 0$, and the last inequality holds due to Lemma C.3. Take the expectations of (27) w.r.t. random vectors $\{w_{k_0+1}, ..., w_T\}$. Then we can get

$$
\sum_{k=k_0+1}^{T} \beta_k \mathbb{E}_{w_k} [|| \nabla_x F'(x_k, t_k) ||^2]
\leq f(x_{k_0+1}) - f^* + 4\nu + L_0 \sqrt{d} \left( t_{k_0+1} + \mathbb{E}_{w_c} [|| t_{k+1} - t_k ||] \right) + 2\nu (T - k_0)
+ \frac{1}{2} \left( 4(d + 4) \sum_{k=k_0+1}^{T} \beta_k^2 \mathbb{E}_{w_c} [L_1(t_k) || \nabla f(x_k) ||^2] + L_1^2 (d + 6)^3 \sum_{k=k_0+1}^{T} \beta_k^2 \mathbb{E}_{w_c} [L_1(t_k) t_k^2]
+ 8d \sum_{k=k_0+1}^{T} \beta_k^2 \mathbb{E}_{w_c} \left[ L_1(t_k) \frac{\nu^2}{t_k^2} \right] \right).
$$

(28)
Lemma C.8 together with (28) yields
\[
\sum_{k=k_0+1}^{T} \beta_k E_w[\|\nabla f(x_k)\|^2] 
\leq 4 \sum_{k=k_0+1}^{T} \beta_k E_w[\|\nabla f(x_k, t_k)\|^2] + \frac{1}{2} \sum_{k=k_0+1}^{T} \beta_k E_w[L_k^2(1) + L_k^2(2) + 1] \sum_{k=k_0+1}^{T} \beta_k E_w \left[ \frac{\nu^2}{t_k^2} \right] + 8d \sum_{k=k_0+1}^{T} \beta_k E_w \left[ L_1(t_k) \frac{\nu^2}{t_k^2} \right] + \frac{1}{2} \sum_{k=k_0+1}^{T} \beta_k E_w \left[ \frac{\nu^2}{t_k^2} \right].
\]

By rearranging the terms, we obtain
\[
\sum_{k=k_0+1}^{T} \left( \beta_k E_w[\|\nabla f(x_k)\|^2] - 8(d + 4) \beta_k^2 E_w[L_1(t_k)\|\nabla f(x_k)\|^2] \right) 
\leq 4 \left( f(x_{k_0+1}) - f^* + 4\nu + L_0 \sqrt{d} \left( t_{k_0+1} + \sum_{k=k_0+1}^{T} E_w[|t_{k+1} - t_k|] \right) + 2\nu(T - k_0) \right) 
+ 2 \left( L_1^2(1)(d + 6)^3 \sum_{k=k_0+1}^{T} \beta_k^2 E_w[L_1(t_k)t_k^2] + 8d \sum_{k=k_0+1}^{T} \beta_k^2 E_w \left[ L_1(t_k) \frac{\nu^2}{t_k^2} \right] \right) 
+ \frac{1}{2} \sum_{k=k_0+1}^{T} \beta_k E_w \left[ \frac{\nu^2}{t_k^2} \right].
\]

If we update \( t_k (k \in [T]) \) as in Algorithm 4, we have \( \nu = O(t_k^2) \), which yields \( L_1(t_k) = O(1) \) from Lemma C.7. Hence, by setting the step size \( \beta_k \) as \( \frac{1}{\sqrt{(d+4)\|\nabla f(x_k)\|^2}} \) (\( k \in [T] \)), we can obtain
\[
\frac{1}{T - k_0} \sum_{k=k_0+1}^{T} E_w[\|\nabla f(x_k)\|^2] = O \left( \frac{d}{T - k_0} \left( 1 + \sqrt{d} \sum_{k=k_0+1}^{T} E_w[|t_{k+1} - t_k|] + d^2 \sum_{k=k_0+1}^{T} E_w[t_k^2] \right) \right)
\]
in the same way as before. We can also get \( \sum_{k=k_0+1}^{T} |t_{k+1} - t_k| = \sum_{k=k_0+1}^{T} (t_k - t_{k+1}) = t_{k_0+1} - t_{T+1} = t_{k_0+1} = O(\gamma^{k_0}) \). Further, we have
\[
\sum_{k=k_0+1}^{T} t_k^2 \leq \sum_{k=k_0+1}^{T} \max \{ t_k^2 \gamma^{2(k-1)} \}, \nu \leq \sum_{k=k_0+1}^{T} \left( t_k^2 \gamma^{2(k-1)} \right) \leq \frac{t_1^2 \gamma^{2k_0}}{1 - \gamma^2} + \nu(T - k_0) = O(\gamma^{2k_0} + \nu(T - k_0))
\]
where the first inequality follows from the update rule of \( t_k \) in Algorithm 4. Hence, we obtain
\[
\frac{1}{T - k_0} \sum_{k=k_0+1}^{T} E_w[\|\nabla f(x_k)\|^2] = O \left( \frac{d}{T - k_0} \left( 1 + \sqrt{d} \gamma^{k_0} + d^2 (\gamma^{2k_0} + \nu(T - k_0)) \right) \right)
\]
\[
= O \left( \frac{d(1 + d^2 \gamma^{2k_0})}{T - k_0} + d^3 \nu \right) = O \left( \frac{d(1 + d^2 \gamma^{2k_0})}{T - k_0} + \epsilon^2 \right),
\]

29
where the last equality follows from the assumption of $\nu = O(\epsilon^2/d^3)$.

Here, we have $k_0 = O\left(\frac{d^4}{\epsilon^2}\right)$ by the definition of $k_0$. Thus, by setting $T = k_0 + O\left(\frac{d^4}{\epsilon^2}\right) = O\left(\frac{d^4}{\epsilon^2}\right)$, we can obtain $\mathbb{E}_{w,k'}[\|\nabla f(\hat{x})\|^2] = \frac{1}{T-k_0} \sum_{k=k_0+1}^{T} \mathbb{E}_w[\|\nabla f(x_k)\|^2] \leq \epsilon^2$. This implies $\mathbb{E}_{w,k'}[\|\nabla f(\hat{x})\|^2] \leq \epsilon$ as $\mathbb{E}_w[\|\nabla f(x_k)\|^2] \leq \mathbb{E}_{w,k'}[\|\nabla f(\hat{x})\|^2]$ follows from Jensen’s inequality. Furthermore, when $\gamma$ is chosen as $\gamma \leq d^{-\epsilon^2/d}$, we have $\log_\gamma d^{-1} = O\left(\frac{d^4}{\epsilon^2}\right)$, which implies $k_0 = \Omega\left(\log_\gamma d^{-1}\right)$. Therefore, we can obtain $\gamma^{k_0} = O(d^{-1})$, which yields the iteration complexity of $T = O\left(\frac{d^4}{\epsilon^2}\right)$.

\[ \square \]

### C.3 Proof for the stochastic setting

**Proof for Theorem C.2:**

Let $\zeta_k := (\xi_k, u_k, v_k), k \in [T]$ and denote $\delta_k := \hat{G}'_x(x_k, t_k; \xi_k, u_k) - \nabla_x F'(x_k, t_k)$. Since $\hat{G}'_x(x; t; \xi, u)$ is an unbiased estimator of $\nabla_x F'(x, t)$, we have

\[
F'(x_{k+1}, t_k) - F'(x_k, t_k) = \nabla_x F'(x_k, t_k) \cdot (x_{k+1} - x_k) + \frac{L_1(t_k)}{2} \|x_{k+1} - x_k\|^2
\]

Furthermore, when $T$ for simplicity. Then, we can get the upper bound for $\|\nabla_x F(x, t)\|^2$ with $I_k$:

\[
\beta_k \|\nabla_x F'(x_k, t_k)\|^2 \leq F'(x_k, t_k) - F'(x_{k+1}, t_k) + I_k
\]

Now, denote $I_k := -\beta_k \langle \nabla_x F'(x_k, t_k), \delta_k \rangle + \frac{L_1(t_k)}{2} \beta_k^2 \|\hat{G}'_x(x_k, t_k; \xi_k, u_k)\|^2$

for simplicity. Then, we can get the upper bound for $\|\nabla_x F(x, t)\|^2$ with $I_k$:

\[
\beta_k \|\nabla_x F'(x_k, t_k)\|^2 \leq F'(x_k, t_k) - F'(x_{k+1}, t_k) + I_k
\]

where the last inequality follows from Lemma C.5. Sum up the above inequality for all iterations $k_0 + 1 \leq k \leq T$. Then, we have

\[
\sum_{k=k_0+1}^{T} \beta_k \|\nabla_x F'(x_k, t_k)\|^2 \leq F'(x_{k_0+1}, t_{k_0+1}) - F'(x_{T+1}, t_{T+1}) + L_0 \sqrt{d} \sum_{k=k_0+1}^{T} |t_{k+1} - t_k| + 2\nu(T - k_0) + \sum_{k=k_0+1}^{T} I_k
\]

We can also obtain $\mathbb{E}_{\zeta_k}[I_k \mid \zeta_{[k-1]}]$

\[
\mathbb{E}_{\zeta_k}[I_k \mid \zeta_{[k-1]}] = -\beta_k \mathbb{E}_{\zeta_k}[\langle \nabla_x F'(x_k, t_k), \delta_k \rangle \mid \zeta_{[k-1]}] + \mathbb{E}_{\zeta_k}[L_1(t_k) \mid \zeta_{[k-1]}] \beta_k^2 \mathbb{E}_{\zeta_k}[\|\hat{G}'_x(x_k, t_k; \xi_k, u_k)\|^2 \mid \zeta_{[k-1]}]
\]

\[
\leq \mathbb{E}_{\zeta_k}[L_1(t_k) \mid \zeta_{[k-1]}] \beta_k^2 \left(4(d + 4) \mathbb{E}_{\zeta_k}[\|\nabla f(x_k)\|^2 \mid \zeta_{[k-1]}] + \sigma^2\right) + \mathbb{E}_{\zeta_k}[t_k^2 \mid \zeta_{[k-1]}] L_3^2 (d + 6)^3 + 8d \mathbb{E}_{\zeta_k}[\nu^2/t_k^2 \mid \zeta_{[k-1]}],
\]

(30)
where the last inequality holds due to Lemma C.4.

Take the expectation of (30) with respect to $k + 1$. Then we have

$$\sum_{k=k_0+1}^{T} \beta_k \mathbb{E}_\zeta[\|\nabla_x F'(x_k, t_k)\|^2]$$

$$\leq f(x_{k_0+1}) - f^* + 4\nu + L_0 \sqrt{d} \left( t_{k_0+1} + \sum_{k=k_0+1}^{T} \mathbb{E}_\zeta[|t_{k+1} - t_k|] \right) + 2\nu (T - k_0) + \sum_{k=k_0+1}^{T} \mathbb{E}_\zeta[I_k]$$

$$\leq f(x_{k_0+1}) - f^* + 4\nu + L_0 \sqrt{d} \left( t_{k_0+1} + \sum_{k=k_0+1}^{T} \mathbb{E}_\zeta[|t_{k+1} - t_k|] \right) + 2\nu (T - k_0) +$$

$$+ \frac{1}{2} \left( 4(d+4) \sum_{k=k_0+1}^{T} \beta_k^2 \mathbb{E}_\zeta[L_1(t_k)\|\nabla f(x_k)\|^2 + \sigma^2] + L_1^2(d+6)^3 \sum_{k=k_0+1}^{T} \beta_k^2 \mathbb{E}_\zeta[L_1(t_k)t_k^2] \right)$$

$$+ 8d \sum_{k=k_0+1}^{T} \beta_k^2 \mathbb{E}_\zeta \left[ L_1(t_k) \frac{\nu^2}{t_k^2} \right] ,$$

From Lemma C.8 (ii), we have

$$\sum_{k=k_0+1}^{T} \beta_k \mathbb{E}_\zeta[\|\nabla f(x_k)\|^2]$$

$$\leq 4 \sum_{k=k_0+1}^{T} \beta_k \mathbb{E}_\zeta[\|\nabla_x F'(x_k, t_k)\|^2] + \frac{L_1^2(d+6)^3}{2} \sum_{k=k_0+1}^{T} \beta_k \mathbb{E}_\zeta[t_k^2] + \frac{8}{\pi} \sum_{k=k_0+1}^{T} \beta_k \mathbb{E}_\zeta \left[ \frac{\nu^2}{t_k^2} \right]$$

$$\leq 4 \left( f(x_{k_0+1}) - f^* + 4\nu + L_0 \sqrt{d} \left( t_{k_0+1} + \sum_{k=k_0+1}^{T} \mathbb{E}_\zeta[|t_{k+1} - t_k|] \right) + 2\nu (T - k_0) \right)$$

$$+ 2 \left( 4(d+4) \sum_{k=k_0+1}^{T} \beta_k^2 \mathbb{E}_\zeta[L_1(t_k)(\|\nabla f(x_k)\|^2 + \sigma^2)] + L_1^2(d+6)^3 \sum_{k=k_0+1}^{T} \mathbb{E}_\zeta \beta_k^2 \mathbb{E}_\zeta[L_1(t_k)t_k^2] \right)$$

$$+ 8d \sum_{k=k_0+1}^{T} \beta_k^2 \mathbb{E}_\zeta \left[ L_1(t_k) \frac{\nu^2}{t_k^2} \right] + \frac{L_1^2(d+6)^3}{2} \sum_{k=k_0+1}^{T} \beta_k \mathbb{E}_\zeta[t_k^2] + \frac{8}{\pi} \sum_{k=k_0+1}^{T} \beta_k \mathbb{E}_\zeta \left[ \frac{\nu^2}{t_k^2} \right] .$$

By rearranging the terms, we obtain

$$\sum_{k=k_0+1}^{T} \left( \beta_k \mathbb{E}_\zeta[\|\nabla f(x_k)\|^2] - 8(d+4)\beta_k^2 \mathbb{E}_\zeta[L_1(t_k)\|\nabla f(x_k)\|^2] \right)$$

$$\leq 4 \left( f(x_{k_0+1}) - f^* + 4\nu + L_0 \sqrt{d} \left( t_{k_0+1} + \sum_{k=k_0+1}^{T} \mathbb{E}_\zeta[|t_{k+1} - t_k|] \right) + 2\nu (T - k_0) \right)$$

$$+ 2 \left( 4(d+4)\sigma^2 \sum_{k=k_0+1}^{T} \beta_k^2 \mathbb{E}_\zeta[L_1(t_k)] + L_1^2(d+6)^3 \sum_{k=k_0+1}^{T} \beta_k \mathbb{E}_\zeta[L_1(t_k)t_k^2] + 8d \sum_{k=k_0+1}^{T} \beta_k \mathbb{E}_\zeta \left[ L_1(t_k) \frac{\nu^2}{t_k^2} \right] \right)$$

$$+ \frac{L_1^2(d+6)^3}{2} \sum_{k=k_0+1}^{T} \beta_k \mathbb{E}_\zeta[t_k^2] + \frac{8}{\pi} \sum_{k=k_0+1}^{T} \beta_k \mathbb{E}_\zeta \left[ \frac{\nu^2}{t_k^2} \right] .$$

If we update $t_k (k \in [T])$ as in Algorithm 4, we have $\nu = O(t_k^2)$, which yields $L_1(t_k) = O(1)$ from Lemma C.7. Furthermore, if we set the step size $\beta_k$ as

$$\left\{ \frac{1}{16(d+4)L_1(t_k)}, \frac{1}{\sqrt{(T-k_0)(d+4)}} \right\} \quad (k \in [T])$$
[T]), then we have
\[
\frac{1}{\beta_k} - 8(d + 4)L_1(t_k)\beta_k^2 \leq \frac{2}{\beta_k},
\]
\[
\frac{1}{\beta_k} \leq 16(d + 4)L_1(t_k) + \sqrt{(T - k_0)(d + 4)}.
\]
for all \( k \in [T] \). Using the above inequalities, we can obtain
\[
\frac{1}{T - k_0} \sum_{k=k_0+1}^{T} \mathbb{E}_c[||\nabla f(x_k)||^2] = O \left( \frac{\sqrt{d} \left( 1 + \sqrt{d} \sum_{k=k_0+1}^{T} \mathbb{E}_c[|t_{k+1} - t_k|] \right)}{\sqrt{T - k_0}} + \frac{d \left( 1 + d^2 \sum_{k=k_0+1}^{T} \mathbb{E}_c[|t_k^2|] \right)}{T - k_0} \right)
\]
\[
= O \left( \frac{\sqrt{d} + d\gamma_{k_0}}{\sqrt{T - k_0}} + \frac{d + d^3\gamma_{2k_0}}{T - k_0} + d^3\nu \right)
\]
\[
= O \left( \frac{\sqrt{d} + d\gamma_{k_0}}{\sqrt{T - k_0}} + \frac{d + d^3\gamma_{2k_0}}{T - k_0} + \epsilon^2 \right),
\]
where the second and last equality can be shown via a similar way as in the proof of Theorem C.1.

Here, we have \( k_0 = O \left( \frac{d^2}{\gamma^2} \right) \) by the definition of \( k_0 \). Thus, by setting \( T = O \left( \frac{d^2}{\gamma^2} + \frac{d^2}{\epsilon^2} \right) = O \left( \frac{d^2}{\gamma^2} + \frac{d^2}{\epsilon^2} \right) \), we can obtain \( \mathbb{E}_c,\kappa'[||\nabla f(\hat{x})||^2] = \frac{1}{T-k_0} \sum_{k=k_0+1}^{T} \mathbb{E}_c[||\nabla f(x_k)||^2] \leq \epsilon^2 \). This implies \( \mathbb{E}_c,\kappa'[||\nabla f(\hat{x})||^2] \leq \epsilon \) as \( \mathbb{E}_c,\kappa'[||\nabla f(\hat{x})||^2] \leq \mathbb{E}_c,\kappa'[||\nabla f(\hat{x})||^2] \) follows from Jensen’s inequality. Furthermore, when \( \gamma \) is chosen as \( \gamma \leq d^{-\epsilon/\epsilon^2} \), we have \( \log_\gamma d^{-1} = O \left( \frac{d^2}{\epsilon^2} \right) \), which implies that \( k_0 = \Omega \left( \log_\gamma d^{-1} \right) \). Therefore, we can obtain \( \gamma_{k_0} = O(d^{-1}) \), which yields the iteration complexity of \( T = O \left( \frac{d^2}{\epsilon^2} \right) \).

\[
\square
\]

D Optimization of test functions

In the first three subsections, let us compare the performance of our SLGH algorithms with GD-based algorithms and double loop GH algorithms using highly-non-convex test functions for optimization: the Ackley function [26], Rosenbrock function, and Himmelblau function [1]. We implemented the following five types of algorithms: (ZOS)GD, (ZO)GradOpt, in which the factor for decreasing the stepsize \( \beta \) due to cosine functions which are included in the second term. We ran the aforementioned five types of zeroth-order algorithms with the stepsize \( \beta = 0.1 \) for \( T = 1000 \) iterations. The initial smoothing parameter for the GH algorithms (ZOGradOpt and ZOSLGH) was set to \( t_1 = 1 \), where local minima of the smoothed function almost disappeared (Figure 3(b)). The smoothing parameter for ZOSGD was chosen as \( t = 0.005 \). We set the initial point for the optimization as \( (x, y) = (5, 5) \).

D.1 Ackley Function

The Ackley function is defined as
\[
f(x, y) = -20 \exp \left[ -0.2 \sqrt{0.5 (x^2 + y^2)} \right] - \exp[0.5\cos 2\pi x + \cos 2\pi y] + e + 20,
\]
whose global optimum is \( f(0,0) = 0 \). As shown in Figure 3(a), it has numerous small local minima due to cosine functions which are included in the second term. We ran the aforementioned five types of zeroth-order algorithms with the stepsize \( \beta = 0.1 \) for \( T = 1000 \) iterations. The initial smoothing parameter for the GH algorithms (ZOGradOpt and ZOSLGH) was set to \( t_1 = 1 \), where local minima of the smoothed function almost disappeared (Figure 3(b)). The smoothing parameter for ZOSGD was chosen as \( t = 0.005 \). We set the initial point for the optimization as \( (x, y) = (5, 5) \).
We illustrate the optimization results in Table 3 and Figure 4. The GH methods successfully reach near the optimal solution \((0, 0)\) when the decreasing speed of \(t\) is not so fast, while ZOSGD is stuck in a local minimum in the immediate vicinity of the initial point \((5, 5)\). Please note that GradOpt succeeds in optimization without decreasing the smoothing parameter since the optimal solution of the smoothed function with \(t = 1\) almost matches that of the original target function.

Table 3: Optimization results of the Ackley function. The global optimum is \(f(0, 0) = 0\).

| Methods          | \((x, y)\)     | \(f(x, y)\)     |
|------------------|----------------|-----------------|
| SGD algo. ZOSGD  | (4.99, 4.99)   | 12.63           |
| GH algo. ZOGradOpt \((\gamma = 0.5)\) | \((4.2 \times 10^{-3}, 1.9 \times 10^{-3})\) | \(1.4 \times 10^{-2}\) |
| ZOGradOpt \((\gamma = 0.8)\) | \((-2.2 \times 10^{-3}, 6.7 \times 10^{-3})\) | \(8.1 \times 10^{-2}\) |
| ZOSLGH \((\gamma = 0.995)\) | \((1.97, 1.97)\) | 6.56            |
| ZOSLGH \((\gamma = 0.999)\) | \((-3.6 \times 10^{-3}, -4.6 \times 10^{-3})\) | \(1.7 \times 10^{-2}\) |

Figure 3: Visualization of the Ackley function and its Gaussian smoothed function.

Figure 4: Plots of the function value and the smoothing parameter during optimization of the Ackley function.
D.2 Rosenbrock Function

Let us define the Rosenbrock function in 2D as

\[ f(x, y) = 100 \left( y - x^2 \right)^2 + (1 - x)^2, \]

whose global optimum is \( f(1, 1) = 0 \). This function is difficult to optimize because the global optimum lies inside a flat parabolic shaped valley with low function value (Figure 5(a)). Since this function is polynomial, we can calculate the GH smoothed function analytically (see [25]):

\[
F(x, y, t) := \mathbb{E}_{u_x, u_y}[f(x + tu_x, y + tu_y)], \quad (u_x, u_y \sim \mathcal{N}(0, 1))
\]

\[
= 100x^4 + (-200y + 600t^2 + 1)x^2 - 2x + 100y^2 - 200t^2y + (300t^4 + 101t^2 + 1).
\]

Thus, we applied first-order methods to this function. The stepsize and iteration number were set to \( \beta = 1 \times 10^{-4} \) and \( T = 20000 \), respectively. The initial smoothing parameter for the GH algorithms (GradOpt and SLGH) was set to \( t_1 = 1.5 \), where the smoothed function became almost convex around the optimal solution (Figure 5(b)). We set the initial point for the optimization as \((x, y) = (-3, 2)\).

![Rosenbrock function](image1.png)

(a) Rosenbrock function  
(b) Smoothed function with parameter \( t = 1.5 \)

Figure 5: Visualization of the Rosenbrock function and its Gaussian smoothed function.

We illustrate the optimization results in Table 4, Figure 6 and Figure 7. The GH methods can decrease the function value much faster than GD. This is because the smoothed function is much easier to optimize than the original function while its optimal solution is close to that of the original one. In the early stage of optimization, the GH methods reach near a point \((0, 2)\), which is a good initial point for optimization, while GD falls into a point in the flat valley, which is far from the optimal solution. (Figure 7).

Table 4: Optimization results of the Rosenbrock function. The global optimum is \( f(1, 1) = 0 \).

| Methods    | \((x, y)\)   | \(f(x, y)\)  |
|------------|---------------|---------------|
| GD algo.   | (0.408, 0.210) | 0.284         |
| GH algo.   | GradOpt \((\gamma = 0.5)\) | (0.817, 0.667) | \(3.36 \times 10^{-2}\) |
|            | GradOpt \((\gamma = 0.8)\) | (0.808, 0.652) | \(3.70 \times 10^{-2}\) |
|            | SLGH \((\gamma = 0.995)\) | (0.819, 0.670) | \(3.27 \times 10^{-2}\) |
|            | SLGH \((\gamma = 0.999)\) | (0.795, 0.631) | \(4.19 \times 10^{-2}\) |
Figure 6: Plots of the function value and the smoothing parameter during optimization of the Rosenbrock function.

Figure 7: Comparison of output sequences between GD and SLGH, \( \gamma = 0.999 \) with contours of the Rosenbrock function.

**D.3 Himmelblau Function**

The Himmelblau function is defined as

\[
f(x, y) = (x^2 + y - 11)^2 + (x + y^2 - 7)^2.
\]

It has four minimum points in the vicinity of \( (x, y) = (3.000, 2.000), (-2.805, 3.131), (-3.779, -3.283), (3.584, -1.848) \) and one maximum point in the vicinity of \( (x, y) = (-0.271, -0.923) \). It takes the optimal value 0 at the four points. Since this function is also polynomial, we can calculate the GH smoothed function analytically:

\[
F(x, y, t) := \mathbb{E}_{u_x, u_y} [f(x + tu_x, y + tu_y)], \quad (u_x, u_y \sim \mathcal{N}(0, 1))
\]

\[
= x^4 + (2y + 6t^2 - 21)x^2 + (2y^2 + 2t^2 - 14)x + y^4 + (6t^2 - 13)y^2 + (2t^2 - 22)y + (6t^4 - 34t^2 + 170).
\]

Thus, we applied first-order methods to this function. The stepsize and iteration number were set to \( \beta = 1 \times 10^{-4} \) and \( T = 2000 \), respectively. The initial smoothing parameter for GH algorithms was set to \( t_1 = 2 \), where the smoothed function became almost convex around the optimal solution (Figure 8(b)). We set the initial point for the optimization as \( (x, y) = (5, 5) \).
Table 5, Figure 9, and Figure 10 show the optimization results. GD and our SLGH algorithms successfully reach near the global optimum, while GradOpt fails to decrease the function value. This is because the optimal solution of the smoothed function when $t = 2$ lies near the maximum point of the original Himmelblau function $(-0.271, -0.923)$. Figure 10 describes detailed optimization process. Our SLGH algorithm succeeds in returning to the optimal solution once it has passed by reducing $t$. In contrast, GradOpt reaches the vicinity of a minimum of the smoothed function without knowing the detailed shape of the original function; as a result, it is stuck around a local maximum of the original function.

Table 5: Results of optimization of the Himmelblau function. It has a global optimum $f(3, 2) = 0$.

| Methods  | $(x, y)$ | $f(x, y)$  |
|----------|----------|------------|
| GD alg.  | GD       | (2.998, 2.003) | $1.6 \times 10^{-4}$ |
| GH alg.  | GradOpt ($\gamma = 0.5$) | (2.575, 1.437) | 14.14 |
|         | GradOpt ($\gamma = 0.8$) | (1.573, 0.868) | 80.51 |
| SLGH ($\gamma = 0.995$) | (2.999, 2.002) | $6.9 \times 10^{-5}$ |
| SLGH ($\gamma = 0.999$) | (2.983, 1.897) | 0.21 |

Figure 8: Visualization of the Himmelblau function and its Gaussian smoothed function.

(a) Himmelblau function  
(b) Gaussian smoothed function with parameter $t = 2$

Figure 9: Plots of the function value and the smoothing parameter during optimization of the Himmelblau function.

(a) Function value $f(x, y)$ versus iterations.  
(b) Smoothing parameter $t$ versus iterations.

Figure 10: Visualization of the Gaussian smoothed function with parameter $t = 2$.

Table 5: Results of optimization of the Himmelblau function. It has a global optimum $f(3, 2) = 0$. 

| Methods  | $(x, y)$ | $f(x, y)$  |
|----------|----------|------------|
| GD alg.  | GD       | (2.998, 2.003) | $1.6 \times 10^{-4}$ |
| GH alg.  | GradOpt ($\gamma = 0.5$) | (2.575, 1.437) | 14.14 |
|         | GradOpt ($\gamma = 0.8$) | (1.573, 0.868) | 80.51 |
| SLGH ($\gamma = 0.995$) | (2.999, 2.002) | $6.9 \times 10^{-5}$ |
| SLGH ($\gamma = 0.999$) | (2.983, 1.897) | 0.21 |
Figure 10: Comparison of output sequences of GradOpt, in which the factor for decreasing
the smoothing parameter is 0.8, and SLGH_r (γ = 0.999) with contours of the smoothed
Himmelblau function. The blue and orange lines represent output sequences of GradOpt when
\( t = 2.0 \) and \( t = 1.6 \), respectively.

D.4 Additional Toy Example

At the end of this section, let us present a toy example problem in which SLGH_r, which utilizes
the derivative \( \frac{\partial F}{\partial t} \) for the update of \( t \), outperforms SLGH_d. Let us consider the following artificial
non-convex function:

\[
f(x, y) = \begin{cases} 
    x^2 - 150 \times 1.1^{-(x-10)^2+y^2} & (x \geq 0) \\
    x^2/50 - 150 \times 1.1^{-(x-10)^2+y^2} & (x < 0) 
\end{cases}
\]

The second term creates a hole around \((x, y) = (10, 0)\) (see Figure 12(a)), and this function has an
optimum in the vicinity of \( f(9.319, 0) \approx -56.670 \). This function is difficult to optimize for GH
methods since the hole around the optimum disappears when the smoothing parameter \( t \) is large
(Figure 12(b)).

We ran SLGH_r (\( \gamma = 0.995 \) or 0.999) and SLGH_d (\( \gamma = 0.999 \)) with the stepsize (for \( x \)) \( \beta = 0.01 \) for
\( T = 1000 \) iterations. The initial point and initial smoothing parameter were set to \((x, y) = (15, 0)\)
and \( t_1 = 5 \), respectively. We set the stepsize for \( t \) as 0.01.

Table 6 and Figure 11 show the optimization results. We can see that only SLGH_d can decrease \( t \)
around the hole adaptively, and thus successfully can find the optimal solution.

Table 6: Optimization results of the artificial non-convex function. It has a global optimum in the
vicinity of \( f(9.319, 0) \approx -56.670 \).

| Methods     | \((x, y)\)                                   | \(f(x, y)\)          |
|-------------|---------------------------------------------|----------------------|
| GH algo.    |                                             |                      |
| SLGH_r (\( \gamma = 0.995 \)) | \((-0.248, 2.38 \times 10^{-2}\)) | \(-5.52 \times 10^{-3}\) |
| SLGH_r (\( \gamma = 0.999 \)) | \((-2.959, -2.18 \times 10^{-3}\)) | 0.175                |
| SLGH_d (\( \gamma = 0.999 \)) | \((9.319, 8.33 \times 10^{-3}\)) | \(-56.670\)          |
Figure 11: Plots of the function value and the smoothing parameter during optimization of the artificial non-convex function.
Figure 12: Visualization of the artificial non-convex function and its Gaussian smoothed function.

(a) Artificial non-convex function

(b) Gaussian smoothed function with parameter $t = 5$
E Black-box adversarial attack

E.1 Experimental Setup

We used well-trained DNNs\(^4\) for CIFAR10 and MNIST classification tasks as target models, respectively. We adopted the implementation\(^5\) in [9] for ZOSGD and ZOAdaMM. GradOpt [14] in our implementation adopts the same random gradient-free oracles [28] as with our ZOSLGH methods, rather than their smoothed gradient oracle, where random variables are sampled from the unit sphere. Moreover, we set the stepsize in its inner loop as a constant instead of \(\Theta(1/k)\), where \(k\) denotes an iteration number in the inner loop, due to less efficiency of the original setting. Therefore, the essential difference between GradOpt and ZOSLGH is whether or not the structure of algorithms is single loop.

As recommended in their work, we set the parameter for ZOAdaMM as \(v_0 = 10^{-5}\), \(\beta_1 = 0.9\), and \(\beta_2 = 0.3\). The factor for decreasing the smoothing parameter in ZOGradOpt was set to \(0.5\). For all algorithms, we chose the regularization parameter \(\lambda\) as \(\lambda = 10\) and set attack confidence \(\kappa = 1 \times 10^{-4}\). We chose minibatch size as \(M = 10\) to stabilize estimation of values and gradients of the smoothed function. The initial adversarial perturbation was chosen as \(x_0 = 0\), and the initial smoothing parameter \(t_0\) was 10 for GH methods and 0.005 for the others. The decreasing factor for \(t\) in the ZOSLGH algorithm was set to \(\gamma = 0.999\) for both of ZOSLGH\(_r\) and ZOSLGH\(_d\), unless otherwise noted. Other parameter settings are described in Table 7. We used different step sizes for ZOAdaMM because it adaptively penalizes the step size using the information of past gradients [9].

!![](https://github.com/carlini/nn_robust_attacks)

!![](https://github.com/KaidiXu/ZO-AdaMM)

| T          | \(\beta\) (other than ZOAdaMM) | \(\beta\) (for ZOAdaMM) | \(\eta\) | \((N_0, \epsilon_0)\) |
|------------|---------------------------------|-------------------------|----------|----------------------|
| CIFAR-10   | 10000 0.01/3072                | 0.5/3072                 | 1 \times 10^{-4}/3072 | (100, 5 \times 10^{-3}) |
| MNIST      | 20000 1/784                    | 100/784                  | 0.1/784  | (100, 1 \times 10^{-3}) |

E.2 CIFAR-10

Additional plots Figures 13 and 14 show additional plots for total loss and \(L_2\) distortion, respectively. We can see that our ZOSLGH algorithms successfully decrease the total loss value except in cases where images are so difficult to attack that no algorithms succeed in attacking (Figure 13(i), 13(j)). Plots in Figure 14 imply that the algorithms are stuck around a local minimum \(x = 0\) when they are failed to decrease the loss value.

\(^4\)https://github.com/carlini/nn_robust_attacks
\(^5\)https://github.com/KaidiXu/ZO-AdaMM
Figure 13: Additional plots of total loss versus iterations on CIFAR-10 (log scale). (a)-(c) All algorithms can successfully decrease the loss value when images are easy to attack. In particular, in plot (c), SGD-based algorithms can find better solutions than GH-based algorithms. (d)-(f) Only GradOpt fails to attack due to its slow convergence. (g) Only ZOSGD is stuck around a local minimum $x = 0$. (h) Only our ZOSLGH$_H$ algorithm succeeds in escaping the local minimum, and thus it can decrease the loss value more than 200 than other algorithms. (i), (j): These images are so difficult to attack that no algorithms can succeed in attacking.
Figure 14: Plots of $L_2$ distortion versus iterations for images that are difficult to attack on CIFAR-10. Each plot of (a)-(d) corresponds to Figure 13(g)-Figure 13(j).
Effect of choice of the parameter $\gamma$ in the ZOSLGH algorithm  We also investigated the effect of choice of the decreasing parameter $\gamma$ in the ZOSLGH algorithm. We compared ZOSGD, ZOSLGH$_r$ with $\gamma = 0.995$, and ZOSLGH$_r$ with $\gamma = 0.999$. All other parameters were set to the same values as before. Figure 15 implies that the decreasing speed of $t$ is associated with a trade-off: a rapid decrease of $t$ yields fast convergence, but reduces the possibility to find better solutions.

![Comparison of total loss transition of ZOSGD, ZOSLGH$_r$ with $\gamma = 0.995$, and ZOSLGH$_r$ with $\gamma = 0.999$ (log scale).](image)

Figure 15: Comparison of total loss transition of ZOSGD, ZOSLGH$_r$ with $\gamma = 0.995$, and ZOSLGH$_r$ with $\gamma = 0.999$ (log scale).
Generated adversarial examples  Table 8 shows adversarial images generated by different algorithms and their original images.

Table 8: Comparison of adversarial images for CIFAR-10 with different algorithms.

| Image ID | 39 | 79 | 89 | 115 |
|----------|----|----|----|-----|
| Original | ![Image](image1.png) | ![Image](image2.png) | ![Image](image3.png) | ![Image](image4.png) |
| Classified as | dog | ship | truck | cat |
| $L_2$ distortion: | 0 | 0 | 0 | 0 |
| ZOSGD | ![Image](image5.png) | ![Image](image6.png) | ![Image](image7.png) | ![Image](image8.png) |
| Classified as | dog (fail.) | airplane | truck (fail.) | horse |
| $L_2$ distortion: | $6.7 \times 10^{-5}$ | 0.154 | $5.6 \times 10^{-5}$ | $4.5 \times 10^{-3}$ |
| ZOAdaMM | ![Image](image9.png) | ![Image](image10.png) | ![Image](image11.png) | ![Image](image12.png) |
| Classified as | dog (fail.) | airplane | truck (fail.) | horse |
| $L_2$ distortion: | 0.226 | 0.145 | 0.131 | $1.6 \times 10^{-3}$ |
| ZOGradOpt | ![Image](image13.png) | ![Image](image14.png) | ![Image](image15.png) | ![Image](image16.png) |
| Classified as | cat | airplane | truck (fail.) | horse |
| $L_2$ distortion: | 0.304 | 0.254 | $1.1 \times 10^{-30}$ | 0.192 |
| ZOSLGH \text{r} | ![Image](image17.png) | ![Image](image18.png) | ![Image](image19.png) | ![Image](image20.png) |
| Classified as | cat | airplane | automobile | horse |
| $L_2$ distortion: | 0.540 | 0.212 | 0.282 | 0.076 |
| ZOSLGH \text{d} | ![Image](image21.png) | ![Image](image22.png) | ![Image](image23.png) | ![Image](image24.png) |
| Classified as | cat | airplane | automobile | horse |
| $L_2$ distortion: | 0.359 | 0.174 | 0.241 | 0.075 |

E.3 MNIST

Finally, let us show the experimental results on the MNIST dataset. Our ZOSLGH algorithms attain higher success rates than other algorithms on this dataset as well as CIFAR-10 (Table 9). Moreover, the average number of iterations to achieve the first successful attack becomes comparable to ZOSGD. The main difference from the results on CIFAR-10 is that the average of $L_2$ distortion at successful time becomes far larger, from $0.050 \sim 0.250$ to $4.25 \sim 5.20$. This implies that attacks on MNIST are more difficult than those on CIFAR-10. See Figure 16 and Figure 17 for additional plots for total loss and $L_2$ distortion. Figure 10 shows adversarial images generated by different algorithms and their original images.
Table 9: Performance of a per-image attack over 100 images of MNIST under $T = 20000$ iterations. “Succ. rate” indicates the ratio of success attack, “Avg. iters to 1st succ.” is the average number of iterations to reach the first successful attack, “Avg. $L_2$ (succ.)” is the average of $L_2$ distortion taken among successful attacks, and “Avg. total loss” is the average of total loss $f(x)$ over 100 samples. Please note that the standard deviations are large since the attack difficulty varies considerably from sample to sample.

| Methods     | Succ. rate | Avg. iters to 1st succ. | Avg. $L_2$ (succ.) | Avg. total loss |
|-------------|------------|-------------------------|--------------------|-----------------|
| SGD algo.   |            |                         |                    |                 |
| ZOSGD       | 67%        | 1171 ± 1954             | 4.83 ± 4.13        | 73.60 ± 102.70  |
| ZOAdaMM     | 71%        | 261 ± 1068              | 4.25 ± 3.36        | 67.49 ± 100.25  |
| ZOGradOpt   |            |                         |                    |                 |
| ZOSLGH$_r$ ($\gamma = 0.999$) | 84%    | 6166 ± 4354            | 5.16 ± 2.28        | 28.25 ± 65.35   |
| ZOSLGH$_d$ ($\gamma = 0.999$)  | 96%        | 1537 ± 277             | 4.32 ± 2.44        | 11.83 ± 37.88   |
| GH algo.    |            |                         |                    |                 |
| ZOSLGH$_r$ ($\gamma = 0.999$) | 96%    | 1342 ± 242             | 4.37 ± 2.58        | 12.09 ± 38.56   |
| ZOSLGH$_d$ ($\gamma = 0.999$)  |           |                         |                    |                 |
Figure 16: Additional plots of total loss versus iterations on MNIST (log scale). (a)-(b) All algorithms can successfully decrease the loss value when images are easy to attack. (c)-(d) Only GradOpt fails to attack due to its slow convergence. (e) ZOSGD and ZOAdaMM are stuck around a local minimum $x = 0$. (f) Only our ZOSLGH algorithms succeed in escaping the local minimum, and thus they can decrease the loss value more than 200 than other algorithms. (g), (h): These images are so difficult to attack that no algorithms can succeed in attacking.
Figure 17: Plots of $L_2$ distortion versus iterations for images that are difficult to attack on MNIST. Each plot of (a)-(d) corresponds to Figure 16(e)-Figure 16(h).
Table 10: Comparison of the adversarial images for MNIST with different algorithms.

| Image ID | 10 | 21 | 48 | 83 |
|----------|----|----|----|----|
| Original | ![Image](image1.png) | ![Image](image2.png) | ![Image](image3.png) | ![Image](image4.png) |
| Classified as | 0 | 6 | 4 | 7 |
| $L_2$ distortion: | 0 | 0 | 0 | 0 |
| ZOSGD | ![Image](image5.png) | ![Image](image6.png) | ![Image](image7.png) | ![Image](image8.png) |
| Classified as | 0 (fail.) | 5 | 9 | 7 (fail.) |
| $L_2$ distortion: | $4.1 \times 10^{-7}$ | 1.194 | 1.183 | $1.8 \times 10^{-4}$ |
| ZOAdaMM | ![Image](image9.png) | ![Image](image10.png) | ![Image](image11.png) | ![Image](image12.png) |
| Classified as | 0 (fail.) | 5 | 9 | 7 (fail.) |
| $L_2$ distortion: | $4.9 \times 10^{-14}$ | 1.334 | 1.100 | $4.0 \times 10^{-14}$ |
| ZOGradOpt | ![Image](image13.png) | ![Image](image14.png) | ![Image](image15.png) | ![Image](image16.png) |
| Classified as | 2 | 5 | 9 | 9 |
| $L_2$ distortion: | 3.898 | 1.378 | 1.903 | 6.379 |
| ZOSLGH | ![Image](image17.png) | ![Image](image18.png) | ![Image](image19.png) | ![Image](image20.png) |
| Classified as | 2 | 5 | 9 | 9 |
| $L_2$ distortion: | 3.867 | 1.261 | 1.106 | 6.075 |
| ZOSLGHd | ![Image](image21.png) | ![Image](image22.png) | ![Image](image23.png) | ![Image](image24.png) |
| Classified as | 2 | 5 | 9 | 9 |
| $L_2$ distortion: | 4.048 | 1.222 | 1.059 | 5.722 |