NONLINEAR FORCE-FREE EXTRAPOLATION OF THE CORONAL MAGNETIC FIELD BASED ON THE MAGNETOHYDRODYNAMIC RELAXATION METHOD

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ABSTRACT

We develop a nonlinear force-free field (NLFFF) extrapolation code based on the magnetohydrodynamic (MHD) relaxation method. We extend the classical MHD relaxation method in two important ways. First, we introduce an algorithm initially proposed by Dedner et al. to effectively clean the numerical errors associated with $\nabla \cdot B$. Second, the multigrid type method is implemented in our NLFFF to perform direct analysis of the high-resolution magnetogram data. As a result of these two implementations, we successfully extrapolated the high resolution force-free field introduced by Low & Lou with better accuracy in a drastically shorter time. We also applied our extrapolation method to the MHD solution obtained from the flux-emergence simulation by Magara. We found that NLFFF extrapolation may be less effective for reproducing areas higher than a half-domain, where some magnetic loops are found in a state of continuous upward expansion. However, an inverse S-shaped structure consisting of the sheared and twisted loops formed in the lower region can be captured well through our NLFFF extrapolation method. We further discuss how well these sheared and twisted fields are reconstructed by estimating the magnetic topology and twist quantitatively.

Key words: Sun: corona – Sun: magnetic fields – Sun: photosphere

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1. INTRODUCTION

Solar active phenomena such as solar flares, coronal mass ejections (CMEs), and filament eruptions are widely attributed to the release of magnetic energy in the solar corona (Priest & Forbes 2002; Shibata & Magara 2011). Many theoretical and numerical models have been proposed for understanding their dynamics and triggering mechanisms (the details are summarized in Linton & Moldwin 2009; Chen 2011; Shibata & Magara 2011). However, a number of issues remain unanswered. For instance, we still do not have a proper view of the three-dimensional (3D) coronal magnetic field related to an active region. Earlier studies based on analytical or numerical models were categorized in terms of the “loss of stability” or “loss of equilibrium”; however, there are also a few cases that strongly depend on the magnetic configurations because the coronal magnetic field deduced from observational images is very complicated, and one cannot extract its physical essence in term of the simplified analytical models. Therefore, it becomes important to construct the coronal magnetic field on the basis of a numerical model by using the observational data and to investigate the physical condition of the equilibrium state before the flare.

Unfortunately, the 3D coronal magnetic field cannot be directly observed even with the state-of-art solar physics satellite, whose observations can currently provide only the vector field on the photosphere. For these reasons, force-free extrapolation has been performed based on a vector field. The force-free field is expressed as

$$\nabla \times B = \alpha(r)B,$$  \hspace{1cm} (1)

or

$$B \cdot \nabla \alpha(r) = 0,$$  \hspace{1cm} (2)

and it has been widely accepted as an approximation of the coronal magnetic field because the value of the plasma $\beta$ is very low ($10^{-1}$–$10^{-2}$) in the solar corona. The force-free state is classified into three energy levels. One of them, called the potential field, is the current free state, i.e., $\alpha = 0$; hence, this corresponds to the minimum energy state. The linear force-free field (LFFF) has a uniform distribution of $\alpha(r)$, and this energy level is higher than the potential field. However, the observed $\alpha$ is generally a function of space on the photosphere. Observations at various wavelengths often reveal a localized strong shear field close to the neutral line in the active regions before a flare (e.g., Hagyard et al. 1984; Su et al. 2007). From these observational images and results, we know that the potential and LFFF cannot adequately explain the coronal magnetic field before the flare; thus, the nonlinear force-free field (NLFFF) has been considered to model the active region’s magnetic field. Because the force-free equation is essentially nonlinear, it is not straightforward to solve it for the coronal magnetic field. The solution is obtained only numerically through an iteration process for a fixed vector field on the bottom boundary, whereas the potential field and LFFF are calculated easily from the normal component of the magnetic field on the solar surface (Sakurai 1989). Various methods of obtaining NLFFF solutions have been proposed and developed. For the sake of brevity, we do not review them thoroughly here; interested readers are urged to see the comprehensive reviews on this topic by Schrijver et al. (2006), Metcalf et al. (2008), or Wiegelmann & Sakurai (2012).

Representative methods for extrapolating the force-free coronal magnetic field include time-evolutionary methods as well as iterative methods (e.g., the boundary integral method of Yan & Sakurai 1997, 2000 and the Grad-Rubin methods of Sakurai 1981; Amari et al. 1997, 2006; Wheatland 2004), which
iterate an equation to find a solution (1). McClymont & Mikic (1994) and Mikic & McClymont (1994) developed the magnetohydrodynamic (MHD) relaxation method, which directly solves the MHD equations under the zero $\beta$ approximation (Mikic et al. 1988). These equations include the resistivity that allows the field lines to change their topology rapidly toward a force-free state. The calculation begins with the construction of a potential field from the normal component of the magnetic field on the photosphere, and a force-free state is obtained by controlling the transverse electric field and keeping the magnetic flux $B_t$ according to an induction equation toward the normal component of the current density deduced from the vector field. Jiang et al. (2011) and Jiang & Feng (2012) recently developed a force-free extrapolation code based on the MHD relaxation method that includes the gas pressure, as well as viscous and resistive terms. That code is implemented into the space-time conservation-element and solution-element method constructed using a full MHD system and a modern high-performance numerical method (Feng et al. 2007; Feng et al. 2010).

Roumeliotis (1996) developed a force-free extrapolation code that included an induction equation with a hypothetical velocity against the Lorentz force, which is a simplified formation from McClymont & Mikic (1994) and Mikic & McClymont (1994). This formula was originally introduced by Yang et al. (1986) to obtain a magneto-frictional method. In addition, this calculation is classified into two phases: stress and relaxation after a potential field is constructed as an initial state from the normal component of the magnetic field on the photosphere. In the stress phase, a Lorentz force is injected from the bottom boundary so that the transverse components of the vector potential approach the observed transverse field. In the relaxation phase, the upper coronal field relaxes toward a force-free state under the fixed bottom boundary. Because of the two combined effects in this method, it is called the stress and relaxation method. Some authors have already implemented this method into their own code (Valori et al. 2005; Jiang et al. 2011). The resistivity included in the induction equation permits the magnetic reconnection to accelerate the process of the force-free state; thus, it plays the same role as in McClymont & Mikic (1994) and Mikic & McClymont (1994). Valori et al. (2005) introduced the magnetic induction field vector to replace the vector potential. This makes it easier to implement the boundary condition than in the original stress and relaxation method of Roumeliotis (1996). They applied their extrapolation method to the twisted loops obtained from Török & Kliem (2003) and found that the NLFFF performs reasonably well for reconstructing the twisted loops in the localized area close to the neutral line. The improved code of Valori et al. (2007, 2010) is applied to the ideal force-free field introduced by Low & Lou (1990) as well as to a more complex situation by Titov & Démoulin (1999). van Ballegooijen (2004) inserts a twisted magnetic flux tube into a potential field and the magnetofriction (van Ballegooijen et al. 2000) drives a system toward a force-free state without the transverse component on the photosphere; its magnetic configuration is then compared with observational images. Another method is an optimization method originally proposed and developed by Wheatland et al. (2000) and an improved version of it presented by Wiegemann (2004), which minimizes a function consisting of divergence-free and force-free fields. Although the basic equation in the optimization method also includes a higher-order differential equation, which is difficult to solve even numerically, highly accurate reconstruction is recorded in some articles (e.g., Schrijver et al. 2006). Recently, the Solar Optical Telescope (SOT) on board Hinode (Kosugi et al. 2007; Tsuneta et al. 2008) can provide images of the vector field with a high spatial resolution (more than 1K pixels). Moreover, the Helioseismic and Magnetic Imager (HMI) on board the Solar Dynamics Observatory (SDO) can provide vector field data with a high temporal resolution (every 12s), which enables analysis of the NLFFF in unprecedented temporal resolution. Thus it would be interesting to see the performance of the NLFFF with these high-resolution data.

The purpose of this study is to develop an extrapolation code for the NLFFF that accelerates the calculation time even when these high-resolution data are used. We extended the original MHD relaxation method of McClymont & Mikic (1994) and Mikic & McClymont (1994) in two important ways. First, we implemented an algorithm to prevent the deviation from $\nabla \cdot B = 0$ introduced by Dedner et al. (2002), in which the time-dependent term corresponding to $\phi$ (see Equation (6)) is used to remove the numerical error of $\nabla \cdot B$. Second, we implemented a multigrid-type method (Brandt 1977) to more rapidly propagate information on the boundary condition at a larger scale inside the domain than in the smaller component, which accelerates the speed toward a force-free state. The accuracy and reliability are investigated by using the ideal force-free solution introduced by Low & Lou (1990). We further apply our extrapolation code to the MHD solution obtained from the flux emergence simulation by Magara (2012) to investigate the reliability of the NLFFF extrapolation in a real physical situation.

This article is constructed as follows. The extrapolation and numerical methods are described in Section 2. The result of the reconstruction using the Low & Lou solution is presented in Section 3, whereas the MHD solution from Magara (2012) is shown in Section 4. Finally, some important discussions and conclusions are summarized in Section 5.

2. NUMERICAL METHOD

We developed an NLFFF extrapolation code based on MHD relaxation by implementing the multigrid-type procedure and an algorithm for cleaning the errors related to $\nabla \cdot B$. We demonstrated the performance of this method in our previous studies, e.g., Inoue et al. (2011, 2012a, 2012b, 2013). Nevertheless, several issues were not covered extensively in the previous works and require more detailed explanations.

This method is formulated using the zero-beta MHD equations where the gas pressure and gravity are neglected (Mikic et al. 1988) to achieve a force-free state. In this study, we numerically solve the following equations:

$$\frac{\partial v}{\partial t} = -(v \cdot \nabla)v + \frac{1}{\rho} J \times B + v \nabla^2 v. \tag{3}$$

$$\frac{\partial B}{\partial t} = \nabla \times (v \times B - \eta J) - \nabla \phi, \tag{4}$$

$$J = \nabla \times B. \tag{5}$$

$$\frac{\partial \phi}{\partial t} + c_h^2 \nabla \cdot B = - \frac{c_h^2}{\epsilon_p^2} \phi, \tag{6}$$

where $B$ is the magnetic flux density, $v$ is the velocity, $J$ is the electric current density, $\rho$ is the pseudo density, and $\phi$ is the convenient potential. The pseudo density is assumed to be proportional to $|B|$ in order to ease the relaxation by equalizing...
the Alfvén speed in space. The last equation (6) introduced by Dedner et al. (2002) plays a crucial role in avoiding deviation from $\nabla \cdot \mathbf{B} = 0$. From Equations (4) and (6), we can obtain the following equation:

$$\frac{\partial^2 (\nabla \cdot \mathbf{B})}{\partial t^2} + \frac{c_s^2}{c_p^2} \frac{\partial (\nabla \cdot \mathbf{B})}{\partial t} = \epsilon_h \nabla^2 (\nabla \cdot \mathbf{B}),$$

(7)

which illustrates the propagating and diffusing nature of the numerical errors related to $\nabla \cdot \mathbf{B}$, where $c_h$ and $c_p$ correspond to the advection and diffusion coefficients, respectively. The main advantages of this method are that (1) it can be very easily implemented in our numerical code without the need for many improvements, and (2) it accelerates the process of removing errors and does not take as much time as that required to remove errors by solving the Poisson equation (see Tóth 2000 or Tanaka 1995).

The length, magnetic field, velocity, time, and electric current density are normalized by $L_0$, $B_0$, $V_A \equiv B_0/(\mu_0 \rho_0)^{1/2}$, $\tau_A \equiv L_0/V_A$, and $J_0 = B_0/\mu_0 L_0$, respectively. The nondimensional viscosity $\nu$ is set to a constant, $(1.0 \times 10^{-3})$, and the nondimensional resistivity $\eta$ is given by the functional

$$\eta = \eta_0 + \eta_1 \frac{|\mathbf{J} \times \mathbf{B}| |\mathbf{v}|^2}{|\mathbf{B}|^2},$$

(8)

where $\eta_0$ depends on each case, as shown in Table 1, and $\eta_1$ is fixed at $1.0 \times 10^{-3}$ in nondimensional units. The second term is introduced to accelerate the relaxation to the force-free state particularly in the weak field region. The parameters $c_s^2$ are fixed at constants 0.1, whereas $c_p^2$ varies according to Table 1.

The velocity field is adjusted in such a way that it does not correspond to a large value; otherwise, it would affect the Courant-Friedrichs-Lewy condition. We define $v^* = |\mathbf{v}|/|\mathbf{v}_A|$, and if the value of $v^*$ becomes larger than the value of $v_{\text{max}}$ as given in Table 1, the velocity is modified as

$$\mathbf{v} \Rightarrow \frac{v_{\text{max}}}{v^*} \mathbf{v}.$$  

(9)

Two different types of boundary conditions are applied in this study to extrapolate the 3D coronal magnetic field. The first is that all six boundaries are set to the exact solutions obtained from Low & Lou. We denote this boundary condition as EX(exact). In the second, only the bottom boundary is set to the exact solution from Low & Lou or Magara (2012), and the other boundaries are assumed to act like rigid walls; i.e., the normal component of the magnetic field is fixed at the original solutions, and the tangential component is determined by the induction equation as described in Equation (4). We denote this boundary condition as RW(Rigid Wall), which is less information than in EX. In all cases, the velocity field ($\mathbf{v}$) is set to zero on all the boundaries. A Neumann-type boundary condition ($\partial_n \phi = 0$) is applied for the potential $\phi$ at all the boundaries, where $\partial_n$ represents the derivative for the normal direction on the surface. The initial condition is given by a potential field calculated from the normal component on all the boundaries for all cases.

In the ideal force-free cases (the Low & Lou solution), we apply the exact solutions directly on each boundary surface. On the other hand, in the MHD solution obtained from Magara (2012), the handling of the bottom boundaries differs from that in the ideal force-free case except for the normal component. In this case, we introduce a procedure analogous to the stress and relaxation method. The transverse component $B_{BC}$ is defined as a linear combination of $B_{\text{obs}}$ and $B_{\text{pot}}$ on the bottom surface, as follows:

$$B_{BC} = \gamma B_{\text{obs}} + (1 - \gamma) B_{\text{pot}},$$

(10)

where $B_{\text{obs}}$ and $B_{\text{pot}}$ are the transverse components of the observational (MHD solution in this study) and the potential field, respectively. $\gamma$ is a coefficient ranging from 0 to 1. When $R = \int |\mathbf{J} \times \mathbf{B}|^2 dV$, which is introduced as an indication for the force-free state, drops below a critical value denoted by $R_{\text{min}}$ during an iteration, then $\gamma$ grows according to $\gamma = \gamma + d\gamma$, where $d\gamma$ is also given as a parameter. $\gamma$ becomes equal to 1, then, $B_{BC}$ can be completely consistent with the observational data.

As for the numerical method, the spatial derivative is approximated by the second-order finite difference and a time integration is conducted using the Runge–Kutta–Gill method to fourth-order accuracy. Furthermore, we adapt the multigrid-type method to accelerate the procedure for achieving a force-free state. This method contains several distinct numerical grids with different resolutions; the first calculation starts using the coarsest one to obtain a force-free field, and then we use this as an initial condition for the second high-resolution grid. Consequently, by repeating these procedures, the high-resolution force-free state can be obtained in a short time.

The simulation domain in the ideal force-free case is set to $(0, 0, 0) < (x, y, z) < (2, 2, 2)$ defined as nondimensional values, and this is divided into $64^3$ grids, $128^3$ grids, $256^3$ grids, or $512^3$ grids. Case 3M1, case 3M2, case 1M1, and case 1M2 shown in Table 1 are applied for the multigrid-type method, whereas direct calculation is applied for case 0-case 5, without the multigrid-type method. On the other hand, for the MHD solution, the entire numerical domain is set to $(0, 0, 0) < (x, y, z) < (43.2, 43.2, 32.4)$ (Mm$^3$) following Magara (2012), and it is extracted from the original data. Then the total grid number is assigned as $80 \times 80 \times 60$. All the parameters in each case are given in Table 1. All of the physical

| Parameters | $R_{\text{min}}$ | $v_{\text{max}}$ | $c_h$ | $\eta_0$ | $dy$ | $GN$ |
|------------|-----------------|-----------------|-------|---------|------|------|
| case 1     | EX              | $1.0 \times 10^{-4}$ | 1.0   | 5.0     | $3.75 \times 10^{-3}$ | ... | 64$^1$ |
| case 1M1   | EX              | $1.0 \times 10^{-4}$ | 1.0   | 5.0     | 0    | ... | 128$^2$ |
| case 1M2   | EX              | $1.0 \times 10^{-4}$ | 1.0   | 5.0     | 0    | ... | 256$^3$ |
| case 2     | RW              | $1.0 \times 10^{-4}$ | 1.0   | 5.0     | $3.75 \times 10^{-5}$ | ... | 64$^4$ |
| case 3     | EX              | $1.0 \times 10^{-4}$ | 1.0   | 5.0     | $3.75 \times 10^{-5}$ | ... | 128$^3$ |
| case 3M1   | EX              | $1.0 \times 10^{-4}$ | 1.0   | 5.0     | 0    | ... | 256$^3$ |
| case 3M2   | EX              | $1.0 \times 10^{-4}$ | 1.0   | 5.0     | 0    | ... | 512$^3$ |
| case 4     | EX              | $1.0 \times 10^{-4}$ | 1.0   | 5.0     | $3.75 \times 10^{-5}$ | ... | 256$^3$ |
| case 5     | RW              | $5.0 \times 10^{-3}$ | $5.0 \times 10^{-2}$ | 0.2     | $5.0 \times 10^{-5}$ | 0.02 | $80 \times 80 \times 60$ |
values are normalized using $L_0 = 43.2$ (Mm) and $B_0 = 262$ (G) (see Magara & Longcope 2003 for details). Consequently, the numerical domain is set to $(0, 0, 0) < (x, y, z) < (1, 1, 0.75)$ in nondimensional space.

3. RESULT OF THE NLFFF EXTRAPOLATION OF THE LOW & LOU SOLUTION

3.1. Role in the Cleaning of the Numerical Error Related to $\nabla \cdot B$

We first check the accuracy of the numerical code for the ideal force-free solution introduced by Low & Lou (1990). A 3D view of this solution is shown in Figure 1(a). The lines and background color indicate the magnetic field lines and distribution of the normal component of the magnetic field, respectively. Figure 1(b) shows the potential field extrapolated from the normal component of the magnetic field on all the boundaries, which is used as an initial condition in the NLFFF calculation. We calculated the three cases denoted as case $i$ where $i = 0–2$. Case 0 corresponds to the boundary condition EX, where we also do not use Equation (6). Case 1 and case 2 correspond to the boundary conditions Ex and RW, respectively. More detailed information on case 1 and case 2 is given in Table 1.

Figures 2(a) and (b) show iteration profiles of $R = \int |J \times B|^2 dV$ and $D = \int |\nabla \cdot B|^2 dV$ for different cases. We clearly found that case 1 and case 2 tend toward a force-free state because the $R$ and $D$ profiles decrease with each iteration. Iteration was stopped when $R$ reached a minimum value. Case 0 shows a much different profile from those of case 1 and case 2. This result indicates that an iteration profile approaching a force-free state is very sensitive to numerical errors in the deviation from $\nabla \cdot B = 0$. The difference between case 1 and case 2 is determined by the differences in the lateral and top boundary conditions between them. Even though incomplete lateral and top boundaries in RW are given in case 2, the values of $R$ and $D$ are found to be of equal orders of magnitude as found in case 1. However, case 2 takes about twice as long as case 1 to search for the force-free solution.

3.2. Topology Analysis of the 3D Magnetic Field Lines

The 3D NLFFF structures for case 1 and case 2 are shown in Figures 3(a) and (b), respectively. The color contours represent a connectivity error that is defined as

$$\Delta = |\Delta_{\text{Exact}} - \Delta_{\text{NLFFF}}|,$$

$\Delta_{\text{Exact}}$ ($\Delta_{\text{NLFFF}}$) is the distance from one magnetic field line footpoint to another measured on the bottom surface in the exact(NLFFF) solution. The NLFFF solutions in case 1 and case 2 seem to have almost the same configuration as that of the exact Low & Lou solution shown in Figure 1(a). The connectivity errors between these cases also have the same distributions, a random distribution in the entire domain.

We investigate the magnetic topology to clarify the cause of the connectivity error. We used the photospheric cross section of the quasi-separatrix layers (QSLs) introduced by Demoulin et al. (1996). We calculated the following quantity at each pixel on the vector field maps:

$$N(x, y) = \sqrt{\sum_{i=1,2} \left(\frac{\partial X_i}{\partial x}\right)^2 + \left(\frac{\partial X_i}{\partial y}\right)^2},$$

where $(X_1, X_2)$ is the relative distance corresponding to $(x' - x^*, y' - y^*)$. $(x', y')$ and $(x^*, y^*)$ are the positions of the end points of the field lines whose starting points are two adjacent grid points located at $(x_{i0}, y_{i0})$ and $(x_{i0}', y_{i0}')$ on the photospheric surface. This means that the locations of the end points of these
field lines, which are traced from these start points across a large $N(x, y)$ value, may differ greatly.

Figures 3(c) and (d) show the connectivity error in white contours whose magnitude corresponds to 0.05 over the distribution $\log(N)$ mapped on the bottom surface. We clearly see that the connectivity errors are almost on the enhancement layers at considerable distance from a polarity inversion line. From this analysis, we found that, remarkably, the error in the NLFFF appears in particular regions where the magnetic topology is changing dramatically.

We show these particular regions in detail. Figures 3(e) and (f) show the connectivity errors in the same format as Figures 3(c) and (d) over a map of the open-closed field lines in case 1 and case 3, respectively. Closed means that both footpoints of each field line are anchored in the bottom surface; for the open field, one footpoint goes through the side or top boundaries. These results clearly show that the connectivity errors appear along the boundaries between open and closed field lines. On the other hand, the values obtained from this study are $\sim 0.5$, but most of the regions are occupied by values less than 0.25, which is much smaller than the entire length of the numerical domain. Consequently, this is not due to a change in the topology from open to closed lines or vice versa; rather, each outer loop of the open or closed field lines deviates slightly from the reference field. Furthermore, Figure 3(f) shows a plot in the same format with a higher resolution than that of Figure 3(e), which can reduce the error distribution.

### 3.3. Quantitative Comparison of the Low & Lou Solution and NLFFF

We further performed a detailed quantitative analysis as introduced by Schrijver et al. (2006). When $B$ and $b$ represent the semi-analytical Low & Lou solution and the extrapolated solution, respectively, the accuracy of the NLFFF is estimated by the following sequential relations:

$$C_{\text{vec}} = \frac{\sum_i B_i \cdot b_i}{(\sum_i |B_i|^2 \sum_i |b_i|^2)^{1/2}},$$

$$C_{Cs} = \frac{1}{N} \sum_i \frac{B_i \cdot b_i}{|B_i||b_i|},$$

$$1 - E_N = 1 - \frac{\sum_i |b_i - B_i|}{\sum_i |B_i|},$$

$$1 - E_M = 1 - \frac{1}{N} \sum_i \frac{|b_i - B_i|}{|B_i|},$$

$$\epsilon = \frac{\sum_i |b_i|^2}{\sum_i |B_i|^2},$$

where $C_{\text{vec}}$ is the vector correlation, $C_{Cs}$ is the Cauchy-Schwarz inequality, $E_M$ is the mean vector error, $E_N$ is the normalized vector error, $\epsilon$ is the energy ratio, and $N$ is the number of vectors in the field. These results are summarized in Table 2. Figure 4(a) shows the iteration profiles of $1 - E_m$ for case 1 and case 2. The final values reach 0.95 in both cases. We clearly see that both case 1 and case 2 can reconstruct the original Low & Lou solution with good accuracy and no significant difference is found between them even though case 2 requires a longer calculation time than case 1.

Finally, we performed another quantitative analysis by evaluating the force-free $\alpha$ in both footpoints of each field line. Because the value of the force-free $\alpha$ should be constant along the field line (see Equation (2)), their values at the both footpoints of each field line should be equal in order to satisfy the force-free condition. The force-free $\alpha$ in both footpoints for case 1 and case 2, measured on the surface above the first grid above the bottom one, are mapped in Figures 4(b) and (c), respectively. The horizontal and vertical axes represent the values of the force-free $\alpha$ at each footpoint where the diagonal green line corresponds to $y = x$. If an extrapolated field completely satisfies the force-free state, the force-free $\alpha$ will be distributed along this line. Because most points in case 1 and case 2 are along the green line, this result clearly shows that these cases almost satisfy the force-free state well.

### 3.4. Multigrid Strategy

#### 3.4.1. Procedure of the Multigrid-type Method

We present a procedure for a multigrid-type method, which is needed to accelerate the calculation time for high-resolution
η conditions are maintained, and its location and other parameters & Feng 2012). First, we extrapolate an NLFFF with the coarsest calculation speed was reported (e.g., Metcalf et al. 2008; Jiang algorithms have already implemented it, and an accelerated magnetogram data obtained from, e.g., SOT / Hinode. Some algorithms have already implemented it, and an accelerated calculation speed was reported (e.g., Metcalf et al. 2008; Jiang & Feng 2012). First, we extrapolate an NLFFF with the coarsest grid number 128^3 grid points denoted as case 3. After attaining a force-free state with this number of grid points, a higher resolution of (256^3) was achieved in case 3M2. For comparison with case 3M2, we also calculated case 4, in which 256^3 grid points are assigned but without an implementation of the multigrid-type method.

### Table 2

| Star     | Method        | C_{ve} | C_d | 1 − E_N | 1 − E_M | ϵ  | Grid Number |
|----------|---------------|--------|-----|---------|---------|----|-------------|
| Low & Lou| -             | 1.00   | 1.00| 1.00    | 1.00    | 1.00| 64^3        |
| Wiegemann| Optimization  | 1.00   | 1.00| 0.98    | 0.98    | 1.02| 64^3        |
| case 1   | MHD relaxation| 1.00   | 1.00| 0.97    | 0.95    | 1.02| 256^3       |
| case 2   | MHD relaxation| 1.00   | 1.00| 0.97    | 0.95    | 1.04| 64^3        |
| case 3   | MHD relaxation| 1.00   | 1.00| 0.99    | 0.98    | 1.00| 128^3       |
| case 4   | MHD relaxation| 1.00   | 1.00| 0.98    | 0.96    | 0.99| 256^3       |
| case 1M1 | MHD relaxation| 1.00   | 1.00| 0.97    | 0.94    | 1.02| 128^3       |
| case 1M2 | MHD relaxation| 1.00   | 1.00| 0.97    | 0.93    | 1.02| 256^3       |

magnetogram data obtained from, e.g., SOT / Hinode. Some algorithms have already implemented it, and an accelerated calculation speed was reported (e.g., Metcalf et al. 2008; Jiang & Feng 2012). First, we extrapolate an NLFFF with the coarsest grid number 128^3 grid points denoted as case 3. After attaining a force-free state with this number of grid points, a higher resolution of (256^3) was obtained by using the force-free field realized in the previous step as an initial condition, which is referred to case 3M1. In the same way, the highest resolution (of 512^3) was achieved in case 3M2. For comparison with case 3M1, we also calculated case 4, in which 256^3 grid points are assigned but without an implementation of the multigrid-type method.

### 3.4.2. Accuracy and Calculation Time in Run 1

We performed our calculations using two different patterns, i.e., run 1 and run 2, using a multigrid-type method whose final results are obtained in three steps. First, we examined the performance related to run 1 with an initial assignment of the coarsest grid number 128^3 grid points denoted as case 3. After attaining a force-free state with this number of grid points, a higher resolution of (256^3) was obtained by using the force-free field realized in the previous step as an initial condition, which is referred to case 3M1. In the same way, the highest resolution (of 512^3) was achieved in case 3M2. For comparison with case 3M1, we also calculated case 4, in which 256^3 grid points are assigned but without an implementation of the multigrid-type method.

Figures 5(a) and (b) show the results related to the iteration profiles of $R = \int |J \times B|^2 dV$ and $D = \int |\nabla \cdot B|^2 dV$ corresponding to run 1. The coarsest grid level 128^3 gradually decreases by $1.0 \times 10^{-7}$ at about $2.0 \times 10^4$ iterations; the calculation takes 12h^5 in real time and $1 − E_m$ reaches about 0.98, as shown in Table 2. Although $R$ and $D$ suddenly increase as the grid number changes from 128^3 to 256^3, they immediately decrease by about $1.0 \times 10^{-7}$ again. This sudden increment in $R$ and $D$ due to the change in the grid is clearly the result of a numerical error arising from an interpolation. However, this error rapidly decreases within $10^2$ iterations, as this scale is small compared to the previous grid, so the diffusion may be effective in decreasing the numerical error associated with a higher mode. The green line corresponding to 256^3 grids can reach $1.0 \times 10^{-7}$ at about $3.0 \times 10^4$ iterations, which is marked by the green circle; the total calculation time takes about 45 hr in real time. On the other hand, in case 4, which is also assigned to the 256^3 grids but without the multigrid-type method, the value of $R$ marked after a total calculation time of 100 hr by a black circle is found to be one order larger than that for case 3M1. The multigrid-type method significantly reduces the calculation time. Hence, we clearly see that it is an effective method for analyzing high-resolution data. The final state, plotted by the purple lines, can achieve a high-resolution force-free field given by 512^3 grids.

Figure 5(c) shows a distribution map of the force-free $\alpha$ for case 3M1 at the green solid circle in Figure 5(a); the map is in the same format as Figures 4(b) or (c). We see that many red points appear along the green lines, although a slight deviation appears in the range of $0 < \alpha < 2.0$. However, the overall pattern of the extrapolated field satisfies a state close to the force-free state. On the other hand, Figure 5(d) shows the results for the case 4, marked by the dotted black circle in Figure 5(a), where the calculation time is the same as that for $3.0 \times 10^4$ iterations at

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8 The numerical code was parallelized by Message Passing Interface (MPI), and the calculation speed was measured by using a 3.06 GHz Xeon X5500 eight-core processor implemented in a DELL T7500.
the end of the calculation in case 3M1. As expected, many points deviate from the force-free state. Thus, an implementation of the multigrid-type method yields a force-free state in a dramatically short time.

### 3.4.3. Accuracy and Calculation Time in Run2

The procedure for run 2 is basically the same as that of run 1 except for the assigned grid numbers. In run 2, 64^3 grids points are initially assigned, corresponding to the coarsest grid and initial condition in case 1. Eventually, following the same procedure in run 1, we obtain a force-free state with 128^3 and 256^3 grid points, which are called case 1M1 and case 1M2, respectively.

Figures 6(a) and (b) show the results on the $R$ and $D$ profiles, respectively, for each case (case 1, case 1M1, and case 1M2). The black line corresponds to case 4, which is the same as in run 1. The green and black solid circles represent 2.0 \times 10^4 iterations, corresponding to the end of the calculation with 256^3 grid points, for case 1M2 and case 4, respectively. The total calculation times were approximately 30 hr and 67 hr, respectively. Furthermore, the black dotted circle represents 8625 iterations for case 4, corresponding to the same calculation time as that required for 2.0 \times 10^4 iterations of the multigrid-type case. Although the $R$ and $D$ profiles in multigrid cases reach to values of less than 1.0 \times 10^{-7}, as with the previous multigrid-type case, run 1, the quantitative value $1 - E_m$ shown in Table 2 does not increase from its initial value of 0.95 (obtained from the initial coarsest grid) with increasing grid numbers. We further checked the distribution map of the force-free $\alpha$.

Figures 6(c) and (d) show a distribution map of the force-free $\alpha$ for case 1M2 and case 4, marked by the solid green and dotted black circles, respectively, in Figure 6(a); the format is the same as in Figures 5(c) or (d). Case 1M2 in the region $\alpha < 0$ is found to yield a better reconstruction in a short time than case 4. However, for $0 < \alpha < 3.0$, this case, as well as case 3M1, seems to deviate slightly from the force-free state. In comparison, Figure 4(b), which shows a distribution map of the force-free $\alpha$ in the initial state, shows a better force-free state than case 1M2 even for $\alpha > 0$. Hence, this error is clearly derived from an interpolation through a change in the grid that critically affects the value of $1 - E_m$, as reported by Jiang & Feng (2012).

### 3.5. 2D Distribution of the Force-free $\alpha$

Figure 7(a) plots contours of the force-free $\alpha$ to clarify why the reconstructed field in the region of weak force-free $\alpha$ ($0 < |\alpha| < 3.0$) deviated from the force-free state when the multigrid-type method is used, such as shown in Figures 5 and 6. The red, green, and blue contours indicate strengths of the force-free $\alpha$ corresponding to 2.0, $-2.0$, and $-4.0$, respectively. From Figure 7(a), a region of strong force-free $\alpha$ appears in the central region, where the extrapolated field satisfies the force-free state well, as shown in the previous results. On the other hand, as regions of weak negative and positive force-free $\alpha$, R1...
Figure 6. (a) Iteration profiles of $R = \int |J \times B|^2 dV$ in run 2; the format is essentially the same as in Figure 5(a). The initial grid number differs from that of run 1 changing from $64^3$ (case 1, red) to $256^3$ (case 1M2, green) through $128^3$ (case 1M1, blue). The black line the shows result for case 4, which is the same as in Figure 5(a). The green and black solid circles represent $2 \times 10^4$ iterations for case 1M2 and case 4, corresponding to the end of calculation with $256^3$ grid points. The black dotted circle represents 8625 iterations, corresponding to the same calculation time as at the end of case 1M2. (b) Iteration profile of $D = \int |\nabla \cdot B|^2 dV$ in run 2. (c)–(d) Distributions of force-free $\alpha$ obtained from case 1M2 and case 4, marked by the green solid and black dotted circles in (a), respectively. These states are obtained in the same calculation time in each case.

(A color version of this figure is available in the online journal.)

Figure 7. (a) Selected contours of the force-free $\alpha$ plotted over the $B_z$ component in gray scale. The red, green, and blue lines represent the strengths $2.0$, $-2.0$, and $-4.0$, respectively. R1 and R2, enclosed by the red and green lines, fall within $2.0 < \alpha < 4.0$ and $-2.0 < \alpha < 0$, respectively. (b) Magnetic field lines (orange lines) plotted over (a).

(A color version of this figure is available in the online journal.)

and R2 lie at considerable distances from the central region in the numerical domain where the force-free $\alpha$ is distributed in a range of $2.0 < \alpha < 4.0$ and $-2.0 < \alpha < 0$, respectively. Figure 7(b) also shows the magnetic field lines, most of which in R1 and R2 are rooted in a region near the boundaries of the domain; i.e., the numerical errors remarkably appear near the boundaries through an interpolation accompanying a change in the grid number. However, as in Figure 6, the extrapolated field
in the strong force-free \( \alpha \) regions exhibits a better force-free state even when the multigrid process is used; thus, the core region in particular can be reconstructed with good accuracy in a dramatically short time by using it.

4. NLFFF Extrapolation in a Flux Emergence Region Produced by the MHD Simulation

We found that our NLFFF extrapolation method performed remarkably well in reproducing an ideal force-free state. Next, we applied it to a flux-emergence region obtained from an MHD simulation (Magara 2012). The idea was to check its performance for a region that is quite close to the real corona. The simulation results of the Magara (2012) provided a hypothetical state of the solar corona affected by the pressure, gravity field, and nonequilibrium state that differs greatly from the ideal force-free field introduced by Low & Lou. In this study, we focus on how well the sheared and twisted field lines in the lower corona are reconstructed by our NLFFF method; most of the free energy is accumulated in these lines, and they are treated as the most important parts for solar active phenomena such as solar flares and CMEs. In the following section, we quantitatively compare the differences in the 3D configurations yielded by the MHD solution and NLFFF extrapolation and finally estimate the degree of twist in them.

4.1. Overview of the Active Region from the MHD Simulation

Magara (2012) surveyed the dynamics of flux emergence with respect to a wide range of parameters set at the initial time of the twisted magnetic flux tube. In this study, we select one snapshot at the last moment of the medium twist (MT) case (see Table 1 in their article), in which a flux tube embedded in the convection zone has emerged into the solar corona and formed coronal magnetic loops. First, we introduce the basic components of the magnetic field obtained from the MHD simulation, which is used as the boundary condition in our NLFFF extrapolation method.

Figure 8(a) shows a height profile of the integrated plasma \( \beta (z) \). The black circles indicate case 5. (b) Distribution of \( B_z \) component at 2700 (km) above the photosphere plotted in gray scale, which is obtained from the flux-emergence simulation in the MT case in Magara (2012). This magnetic field is normalized by the maximum value \( B_0 = 262 \) (G); i.e., the maximum and minimum values correspond to 1 and 0. (c) Height profiles of integrated force-free \( \alpha (z) \) (solid line) and nonforce-free component \( \alpha' (z) \) (dashed line) according to Equations (18) and (19). (d) Velocity profile \( |J|/dy \) in white contours plotted over the distribution of the current density \( J \). Strength of contours is 8.0. The red dotted line indicates the half-height of the entire domain.

(A color version of this figure is available in the online journal.)
MHD solution (dashed line) and where $S$ represents a surface on the bottom boundary, and other parameters used in this NLFFF calculation are shown in Table 1 (see case 5). The NLFFF is selected at $1.0 \times 10^4$ iterations, approximately at which $R$ and $E$ begin to saturate.

Figure 9(b) shows the height profiles of $\langle \alpha \rangle$ in the MHD solution (dashed line) and $\langle \alpha_{nlfff} \rangle$ in NLFFF (solid line) for case 5. The values of $\langle \alpha_{nlfff} \rangle$ and its pattern deviate slightly from those of $\langle \alpha \rangle$, corresponding to those regimes where the value of $\langle \alpha \rangle$ is dominant over $\langle \alpha' \rangle$, as shown in Figure 8(b). On the other hand, these values and profiles of $\langle \alpha_{nlfff} \rangle$ are found to deviate greatly from those of $\langle \alpha \rangle$ in the upper area, above the half-height of the entire domain.

4.2.1. 1D Profiles from the NLFFF and MHD Solutions

First, we show the one-dimensional result from the NLFFF and compare them with the MHD solution. Figure 9(a) shows iteration profiles of the total Lorentz force $R$ (solid line) and magnetic energy $E$ (dashed line) corresponding to case 5. $\gamma$ as defined in the Equation (10) is equal to 1, and the other parameters used in this NLFFF calculation are shown in Table 1 (see case 5). The NLFFF is selected at $1.0 \times 10^4$ iterations, approximately at which $R$ and $E$ begin to saturate.

Figure 9(b) shows the height profiles of $\langle \alpha \rangle$ in the MHD solution (dashed line) and $\langle \alpha_{nlfff} \rangle$ in NLFFF (solid line) for case 5. The values of $\langle \alpha_{nlfff} \rangle$ and its pattern deviate slightly from those of $\langle \alpha \rangle$, corresponding to those regimes where the value of $\langle \alpha \rangle$ is dominant over $\langle \alpha' \rangle$, as shown in Figure 8(b). On the other hand, these values and profiles of $\langle \alpha_{nlfff} \rangle$ are found to deviate greatly from those of $\langle \alpha \rangle$ in the upper area, above the half-height of the entire domain.

4.2. Result of the NLFFF

4.2.1. 1D Profiles from the NLFFF and MHD Solutions

Next, we show a 3D view of the MHD simulation and NLFFF and present a detailed comparison in terms of the magnetic topology. Figure 10(a) shows a top view of the selected field lines obtained from the MHD solutions for case 5. These are traced from the surface, which is $2700$ km above the photosphere, as shown in Figure 8(a). Figure 10(b) also shows the selected 3D field lines in the NLFFF, which are also extrapolated from the same surface as those described above. From these results, we infer that although they are not exactly the same, the NLFFF seems to reproduce an inverse S-shaped structure lying above the polarity inversion line and is qualitatively similar to the MHD solution. It is important to capture these S- or inverse S-shaped structures because they are considered to be precursors for huge flares (Canfield et al. 1999) observed in solar active regions. We further investigate the magnetic topology in more detail to clarify the differences in between the NLFFF and MHD solutions.

Figures 10(c) and (d) show the distribution of the field line length mapped on the bottom surface for the MHD solution and the NLFFF, respectively. One footpoint of the field lines rooted in the white areas is not at the bottom surface; that is, their other footpoints are rooted in the lateral boundary surfaces, whereas the other field lines traced from colored areas are closed. Therefore, the boundaries between the colored and white areas represent the separatrix separating the closed and open field lines, in which the QSL values are enhanced. The inverse S-shaped structure can be formed by the NLFFF as well as by the MHD solution, and this structure is better captured in the NLFFF except in the regions marked by the dashed circle in which the closed loops are anchored.

To provide more clarification, we also show the field line structure within the dashed circle in Figure 10(d) in more detail. Figures 11(a) and (b) show the field lines (red) in the MHD and NLFFF, respectively, traced from the area marked by the

\[
\epsilon_{\text{force}} = \frac{\langle | \int B_x B_x | + | \int B_y B_y | + | \int B_z B_z \rangle \rangle}{\int | B_x^2 + B_y^2 + B_z^2 | dx dy},
\]

\[
\epsilon_{\text{torque}} = \frac{\langle | \int x \int B_x \rangle + | \int y \int B_y \rangle + | \int z \int B_z \rangle \rangle}{\int \sqrt{x^2 + y^2 + z^2} \int B_x^2 + B_y^2 + B_z^2 dx dy},
\]

where $S$ represents a surface on the bottom boundary, and $\epsilon_{\text{force}}$ and $\epsilon_{\text{torque}}$ correspond to the force balance and torque balance parameters, respectively. When $\epsilon_{\text{force}} \ll 1$ and $\epsilon_{\text{torque}} \ll 1$, the boundary surface approximately satisfies the force-free condition (Wiegelmann et al. 2006). As a result, in this case ($\epsilon_{\text{force}} = 0.275$ and $\epsilon_{\text{torque}} = 0.382$), they deviate from the force-free state; nevertheless, these values are close to the SP/Hinode data for 2006 December 12 according to Wiegelmann et al. (2012).

Figure 8(d) shows profiles of the integrated velocity with respect to the $y$ direction ($\int u(x, y, z) dy$) with a map of the current density in the same format as ($\int J(x, y, z) dy$). We clearly see that the color is strongly enhanced in the lower central area in which the core field is formed. The large velocity fields, plotted by the white contours, are concentrated on the area around the half-height of the entire box, marked by the red dashed line. Consequently, we infer that extrapolation of this region is difficult; however, one of our interests is to address how well the core field is reconstructed under this condition.
Figure 10. Selected field lines obtained using (a) an MHD solution and (b) an NLFFF in case 5, plotted over the $B_z$ component. Distribution of the field line lengths mapped on the bottom surface obtained from (c) an MHD solution and (d) an NLFFF. All the field lines are traced from the bottom surface. The colored areas are occupied by closed field lines, both footpoints of which are anchored in the bottom surface and whose maximum length is $L_{\text{max}} = 2.5$. The white areas are dominated by another type of field line with one footpoint rooted in the lateral boundaries. These are plotted in a range of $(0.1, 0.1) < (x, y) < (0.9, 0.9)$. The black solid lines represent a contour for $|B_z| = 0.1$. The dashed circle marks region where closed loops are anchored.

(A color version of this figure is available in the online journal.)

The field line profiles in MHD and NLFFF are remarkably different. One footpoint of the field line in the MHD solution touches the lateral surface: i.e., the line crosses the boundary surface, whereas both footpoints in the NLFFF touch the bottom surface. Figures 11(c) and (d) show the distribution of the height of one footpoint of the field line measured from the bottom surface in the MHD and NLFFF. All of the field lines are traced from the bottom surface, and most of their footpoints appear in the lower areas plotted in red. On the other hand, we see strong enhancement areas, which are marked by dashed circles, in the MHD solution, whereas these regions are not seen in the NLFFF. These enhanced areas indicate that the height of one footpoint of the field lines is above the half-height of the numerical domain; therefore, it might be difficult for the NLFFF to capture these field lines in this case.

4.2.3. Magnetic Twist in the NLFFF and MHD Simulations

Finally, we compare the magnetic twist obtained using the NLFFF with that in the MHD solution. This value represents the degree of twist of a magnetic field line as determined by the measurement of the magnetic helicity generated due to the current parallel to a field line (Berger & Field 1984; Berger & Prior 2006; Török et al. 2010; Inoue et al. 2011, 2012b, 2013). Because a large amount of magnetic twist can lead to an unstable condition (Kruskal & Kulsrud 1958; Hood & Priest 1979; Török et al. 2004; Török & Kliem 2005; Fan 2005; Inoue & Kusano 2006; Birn et al. 2006), an estimation of the magnetic twist is important for analyzing the stability of the solar coronal magnetic field. We are interested in addressing the extent to which the magnetic twist can be reconstructed. The magnetic twist is defined as

$$T_n = \frac{1}{4\pi} \int \alpha dl,$$

where the line integral $\int dl$ is taken along a magnetic field line, and the force-free $\alpha$ is calculated from $\alpha = J \cdot B / |B|^2$.

Figure 12 shows the distributions of the magnetic twist in each field line from the MHD solution, where the NLFFF is also mapped on the surfaces at the same height. Positive and negative values indicate right-handed and left-handed twists, respectively, depending on the value of the magnetic helicity accumulated in an initial flux tube embedded in the subsurface.
Inoue et al. -0.5 0.5 [Bz] (a) MHD(case5) (b) NLFFF(case5)

Figure 11. Selected field lines obtained using (a) an MHD solution and (b) an NLFFF for case 5 traced from the dashed circle in Figure 10(d). The $B_z$ component is plotted in gray scale in the same format as in Figure 8(b). Distribution maps of height of one footpoint in the field line measured from bottom surface for (c) an MHD solution and (d) an NLFFF. The solid lines represent contours for $|B_z| = 0.1$.

5. SUMMARY AND DISCUSSION

In this study, we developed an NLFFF extrapolation code based on the MHD relaxation method and applied it to the ideal force-free field introduced by Low & Lou (1990). Our NLFFF extrapolation code generally produces the original ideal force-free state, even though incomplete lateral and top boundary conditions are imposed. Moreover, although errors related to the connectivity of the field lines clearly appeared along the separatrix layers (QSL), their topology is not changed dramatically. We further implemented the multigrid-type method in our NLFFF extrapolation code, which increased calculation speed toward a force-free state compared to the code without it. We see that the inner region of the numerical domain in particular can be reconstructed with high accuracy. Thus, our code can be used as a possible method for extrapolating the NLFFF in a shorter time using a high-resolution vector field obtained from SOT/Hinode or HMI/SDO.

Next, we also applied our extrapolation method to the MHD solutions obtained from Magara (2012), which are influenced by the gas pressure, gravity, and nonequilibrium state. In principle, it does not seem appropriate to apply the NLFFF to MHD solutions; nevertheless, in the interest of readers, we also checked the effectiveness of the NLFFF extrapolation for a more realistic situation of the solar corona. As a result, we found that the S-shaped sheared structure formed in the lower corona can be captured well by our NLFFF despite the difference in the value of the twist and the profile compared to the original MHD solutions.

The black contours show the normal component of the magnetic field. We clearly see that the strongly twisted regions in the MHD solution are localized at positive and negative polarities, in which the strong twist values over one turn ($|T_n| > 1.0$) are found near the vicinity of the tip. In contrast, although the strongly twisted regions in the NLFFF are also localized in the same areas at the both polarities, their distributions and values are not the same as those of the MHD solution. The areas marked by the red circles are remarkably different in both cases; in the NLFFF, closed field lines exist in these areas, whereas that type of field line is not seen in the MHD solution. Although the reconstructed twisted lines in other areas tend to capture those in the MHD solution, their values are relatively weak in most regions near the tip at both polarities, whereas a high twist value, $|T_n| > 1.0$, is also observed in some areas. These results show that the shape of the sheared field lines is reconstructed well qualitatively; on the other hand, the value of the magnetic twist tends to be weaker than that of the reference field.

(Figure 12. Twist value of each field line in (a) an MHD solution and (b) an NLFFF for case 5 mapped on the bottom surface. $B_z$ contours are plotted by solid lines ($|B_z| = 0.1$) in the same format as in Figures 10 or 11. The domain is set in the range of $(0.2,0.2) < (x, y) < (0.8,0.8)$.

(A color version of this figure is available in the online journal.)
Thus, we conclude that our NLFFF extrapolation can work in the lower coronal region, in which a strong current is stored.

Our NLFFF extrapolation may be less effective for reproducing the upper region of the entire numerical domain. Note, however, that the numerical solutions of Magara (2012) describe an extreme situation compared to the real solar corona. For example, a preexisting coronal field is not assumed in this MHD simulation, and in such a situation the magnetic loops filling the upper area can expand continuously in all directions, as pointed out by Magara (2004) and Magara (2011, see Figure 8). Therefore, it might be difficult for the NLFFF to reconstruct this type of field line, as shown in our results (Figures 11 and 12). We believe the consideration of the preexisting coronal magnetic field may be important in suppressing the expanded loops during the process of flux emergence and achieving a steady state under which an S-shaped or inverse S-shaped field may be formed. We expect that our NLFFF extrapolation might be able to reproduce these regions with better accuracy than what is presented in the results here. These results are derived from one of the results of Magara (2012), and our interests are in further addressing the extrapolation of the coronal magnetic field using boundary conditions formed by stronger or weaker twisted flux tubes or with various plasma $\beta$ values. However, these interests remain as future works.

On the other hand, a non-force-free extrapolation method has been developed recently by other authors (e.g., Wiegelmann & Wheatland 2013). These methods are still being improved and are expected to be a solid tool for capturing the MHD solution more accurately in the near future.

From this study, we conclude that the extrapolated field can robustly reproduce an ideal force-free state, e.g., the Low & Lou solution. In contrast for the MHD solutions obtained from the flux-emergence simulation, this method captures the sheared field, such as elbow-like structures lying in the lower corona, in which the strongest energy is accumulated. Therefore, the extrapolated field may provide a better understanding of active phenomena in the solar corona. Because the SP/Hinode and HMI/SDO can observe the vector field with high spatial and temporal resolution, we will consider a comprehensive view of the flare dynamics and onset mechanism by using the 3D extrapolated magnetic field obtained from this data set in a future work.

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