Generalized q-deformed oscillators, $q$-Hermite polynomials, generalized coherent states

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Abstract. The aim of this paper is to study generalized $q$-analogs of the well-known $q$-deformed harmonic oscillators and to connect them with $q$-Hermite polynomials. We give a construction of the appropriate oscillator-like algebras and show that corresponding Hermite polynomials are generalization of the discrete $q$-Hermite I and the discrete $q$-Hermite II polynomials. We also construct generalized coherent states of Barut-Girardello type for oscillator-like systems connected with these polynomials.

1. Introduction

The simplest deformation of the canonical commutation relations has been emerged in context of the study of the dual resonance models of the strong interaction theories [1]. More general deformation of these relations was considered in connection with description of representations of quantum groups [2], [3]. It was introduced in order to extend the method of realization of generators of Lie algebras by creation and annihilation operators (the Jordan-Wigner construction) to the quantum case. Since then various generalized $q$-analogs of these deformations became the subject of mathematical and theoretical physics. Their relation to the noncommutative geometry, special functions of the $q$-analysis and other subjects of the mathematics became evident. A connection between harmonic oscillators and Hermite polynomials in the quantum mechanics is well-known. This connection was generalized to the $q$-deformed cases as well. The generalized $q$-deformed oscillators [4], [5], [6], [7] are related to the $q$-deformed Hermite polynomials in the same way as the standard quantum oscillator is connected with the classical Hermite polynomials. In the work [8] the spectra of the position $Q$ and the momentum $P$ operators for various $q$-deformed oscillators in the Fock representation has been described. There the spectral measures and the generalized eigenfunctions of these operators
has been found. They are expressed in terms of certain $q$-Hermitian polynomials. Various orthogonal deformed $q$-Hermite polynomials can appear depending on the type of a deformation of the oscillator algebra.

For example, in the case of Arik-Coon deformation with parameter $q > 1$ one gets Hermite polynomials for which orthogonal measure is known. The same problems has been studied for more general form of the operators $Q$ and the $P$ [9]. In this case the spectral measure and the eigenfunctions has been expressed in terms of the discrete $q$-Hermite polynomials. The spectral measure of the position operator of Biedenharn-McFarlane oscillator has been calculated in the case indetermine Hamburger moment problem [10]. Coherent states of the oscillator-like systems, connected with some $q$-Hermite polynomials have been constructed in [6], [15].

Naturally, a problem of generalization of these results to the case of other $q$-deformations of the harmonic oscillator algebra arises. In this paper we consider oscillator-like systems giving a description of the generalized $q$-deformed oscillators and connect them with the generalized discrete $q$-Hermite I and $q$-Hermite II polynomials. These systems involve, as particular cases, the known one-parameter deformations of oscillator algebras.

In Section 2, we give a structure function, defining relations and the position and momentum operators $Q$ and $P$ of the corresponding deformed oscillator algebra. In the Fock representation the operators $Q$ and $P$ have a Jacobi matrix form. We investigate the self-adjointness properties of these operators.

In Section 3 we give examples of the oscillator-like systems connected with discrete $q$-Hermite I and the generalized discrete $q$-Hermite I polynomials. We establish the orthogonality relations for these polynomials and, as a consequence, obtain a spectrum of the position operators of these systems.

The same problem is solved in Section 4 for discrete Hermite II and generalized discrete Hermite II polynomials of the corresponding oscillator algebras.

In Section 5 we study generalized coherent states of oscillator algebras corresponding to discrete $q$-Hermite I, generalize discrete $q$-Hermite I and II polynomials on the basis of the method suggested in [6], [15].

The generalized $q$-deformed oscillator and its Heisenberg-Weyl algebra is defined by the structure function $f(n) = f_n$ (a positive function satisfying $f(0) = 0$), which fix an associative algebra generated by the elements $\{1, a, a^+, N\}$, satisfying the defining relations

$$[N, a] = -a, \quad [N, a^+] = a^+, \quad (1)$$
\[ a^+ a = f(N), \quad a a^+ = f(N+1). \]  

The structure functions \( f(n) \) characterize the deformation scheme. The Fock realization of these relations and the number of particles operator have the form

\[ a|n\rangle = f_{n-1}|n-1\rangle, \quad a^+|n\rangle = f_n|n+1\rangle, \quad (3) \]

\[ N|n\rangle = n|n\rangle. \quad (4) \]

As an example we consider the special case \( f_n = (n+1)^{1/2} \) to get the Heisenberg-Weyl algebra of the quantum oscillator of quantum mechanics. It is generated by the generators \( a, a^+, N \) and its defining relations are

\[ [N, a] = -a, \quad [N, a^+] = a^+, \quad (5) \]

\[ [a, a^+] = 1, \quad [a, a] = [a^+, a^+] = 0. \quad (6) \]

Recall that the Fock realization of this Heisenberg-Weyl algebra is

\[ a|n\rangle = \sqrt{n}|n - 1\rangle, \quad a^+|n\rangle = \sqrt{n+1}|n + 1\rangle, \quad (7) \]

\[ N|n\rangle = a^+ a|n\rangle = n|n\rangle. \quad (8) \]

The position \( Q \) and the momentum \( P \) operators are unbounded operators defined on a dense domain in the Hilbert space \( \mathcal{H} \) and satisfy the famous commutation relation \([Q, P] = iI\). These operators are related to the creation and annihilation operators \( a^+ \) and \( a \) from (7) by the formulae

\[ Q = a^+ + a, \quad P = \frac{1}{i}(a^+ - a). \quad (9) \]

Each of these operators have a Jacobi matrix form and due to the condition

\[ \sum_{n=0}^{\infty} \frac{1}{\sqrt{n}} = \infty \]

is a self-adjoint operator (Carleman’s Lemma) \([11]\). There is a well-known the connection between the quantum-mechanical position and momentum operators \( Q \) and \( P \) and the Hermite polynomials \( H_n(x) \). The spectrum of the operator \( Q \) is continuous and in order to find its generalized eigenvectors we have to describe solutions of the equation \( Q|x\rangle = x|x\rangle \). To do
this, one considers the expansion $|x\rangle = \sum_{n=0}^{\infty} P_n(x)|n\rangle$ of the vector $|x\rangle$ in the Fock space $\mathcal{H}$. Using the equations (7) and $Q|x\rangle = x|x\rangle$ we obtain the following recurrence relation

$$(n + 1)^{1/2} P_{n+1}(x) + n^{1/2} P_{n-1}(x) = xP_n(x), \quad n = 0, 1, \ldots$$

(11)

for coefficients $P_n(x)$. This relation has the solutions

$$P_n(x) = \frac{1}{(2^n\pi n!)^{1/2}} H_n(x),$$

(12)

with the initial conditions $P_{-1}(x) = 0$, $P_0(x) = 1$. The polynomials $H_n(x)$ satisfy the equation

$$xH_n(x) = \frac{1}{2} H_{n+1}(x) + nH_{n-1}(x)$$

(13)

and can be written as

$$H_n(x) = \sum_{k=0}^{[n/2]} \frac{n!(-1)^k}{k!(n-2k)!} x^{n-2k}.$$ 

(14)

They can be represented by means of hypergeometric function as

$$H_n(x) = (2x)^n {}_2F_0\left( -n/2, \ -\frac{n-1}{2} \bigg| -\frac{1}{x^2} \right).$$

(15)

The polynomials

$$\psi_n(x) = \left( \frac{1}{\sqrt{\pi 2^n n!}} \right)^{1/2} H_n(x)$$

(16)

are orthonormal with respect to the measure $d\omega(x) = e^{-x^2} dx$ and give wave functions of the Hamiltonian $H = a^+a + aa^+$ of the harmonic oscillator corresponding to the eigenvalues $\lambda_n = 2n + 1, n \geq 0$.

2. Generalized oscillator algebras, position and momentum operators and spectrum of Hamiltonian

We consider oscillator-like systems defined by the structure function

$$f_n = q^{\alpha(n+1)+\beta/2} \left( \frac{1 - q^{(l-1)(n+1)}}{1 - q^{l-1}} \right)^{1/2}$$

(17)
of oscillator algebra (see (Sec. 3) and (Sec. 4)). For the special values of the parameters $\alpha, \beta, l, q$ this function reproduce known versions of deformations of the oscillator Heisenberg-Weyl algebra: the Arik-Coon, the Biedenharn-Macfarlane and the other ones. The defining relations for these algebras can be written as

$$aa^+ - q^{2\alpha} a^+ a = q^{2\alpha(N+1) + \beta} q^{(l-1)N},$$  \hspace{1cm} (18)

$$aa^+ - q^{2\alpha+l-1} a^+ a = q^{2\alpha(N+1) + \beta},$$  \hspace{1cm} (19)

or in a more compact form

$$aa^+ - q^{2\alpha} a^+ a = q^{2\alpha(N+1) + \beta} q'^N,$$  \hspace{1cm} (20)

$$aa^+ - q^{2\alpha} q' a^+ a = q^{2\alpha(N+1) + \beta},$$  \hspace{1cm} (21)

where $q' = q^{l-1}$. The Fock realization of the operators $a, a^+$ in the Fock space $H$ is

$$a|n\rangle = q^{\alpha n + \beta/2} \left( \frac{1 - q^{(l-1)n}}{1 - q^{l-1}} \right)^{1/2} |n - 1\rangle,$$  \hspace{1cm} (22)

$$a^+|n\rangle = q^{\alpha(n+1) + \beta/2} \left( \frac{1 - q^{(l-1)(n+1)}}{1 - q^{l-1}} \right)^{1/2} |n + 1\rangle.$$  \hspace{1cm} (23)

A spectrum of the Hamiltonian $H = a^+ a + aa^+$ of these oscillator-like systems is discrete and is given by the expression

$$\lambda_n = q^{2\alpha n + \beta} (1 - q^{l-1})^{-1} \left( 1 - q^{(l-1)n} \right) + q^{2\alpha(n+1) + \beta} (1 - q^{l-1})^{-1} \left( 1 - q^{(l-1)(n+1)} \right),$$  \hspace{1cm} (24)

where $n \geq 0$. However, wave functions corresponding to these eigenvalues are defined in a more complicated way. In the orthonormal basis $|n\rangle, n = 1, 2, \ldots$ of the Hilbert space $H$ the position and momentum operators $Q$ and $P$ of these oscillators systems are given by by the Jacobi matrices

$$Q|n\rangle = f_n|n + 1\rangle + f_{n-1}|n - 1\rangle,$$  \hspace{1cm} (25)

$$P|n\rangle = \frac{1}{l} (f_n|n + 1\rangle - f_{n-1}|n - 1\rangle).$$  \hspace{1cm} (26)

Unlike to the case of standard quantum oscillator, in this one the self-adjointness of the operator $Q$ depends on the values of parameters $q, \alpha, l$. Due to Theorem 1.1, Chapter VII in [11] the deficiency indices of the operator $Q$ are $(0,0)$ and then its closure $\bar{Q}$ is a self-adjoint operator,
or they are \((1, 1)\) and then the operator \(\tilde{Q}\) allows self-adjoint extensions. According to Theorem 1.5, Chapter VII in [11], if the function \(f(n)\) from (17) satisfy the conditions

\[
f_{n-1}f_{n+1} \leq f_n^2, \quad \sum_{n=0}^{\infty} \frac{1}{f_n} < \infty,
\]

then the deficiency indices of \(Q\) are \((1, 1)\). The first condition of (27) is reduced to the inequality \(q^{-(l-1)} - q^{-1} \geq 2\) what is satisfied for all positive \(q\). The convergence of the series (27) depends on the values of parameters of the deformation. Namely,

\[
q < 1, \quad \begin{cases} 
\alpha < 0, & \text{convergent}, \\
\alpha > 0, & \text{divergent}, \\
\alpha + l - 1 < 0, & \text{convergent}, \\
\alpha + l - 1 > 0, & \text{divergent},
\end{cases}
\]

and

\[
q > 1, \quad \begin{cases} 
\alpha < 0, & \text{divergent}, \\
\alpha > 0, & \text{convergent}, \\
\alpha + l - 1 < 0, & \text{divergent}, \\
\alpha + l - 1 > 0, & \text{convergent}.
\end{cases}
\]

In particular, this choice of the structure function unify the following cases of the \(q\)-deformed of the oscillator algebras: the Biedenharn-Macfarlane deformation \((\alpha = 1/2, \beta = -1, \quad l = -1, \quad q < 1)\) and \((\alpha = -1/2, \beta = 1, \quad l - 1 = 2, \quad q > 1)\)

\[
[N, a] = -a, \quad [N, a^+] = a^+,
\]

and its symmetric generalization \((\alpha = 1/2, \beta = -1, \quad l \in \mathbb{R})\),

\[
[N, a] = -a, \quad [N, a^+] = a^+,
\]

the deformation associated with the discrete \(q\)-Hermite I polynomials \((\alpha = 1/2, \quad \beta = -1, \quad l = 2, \quad q < 1)\)

\[
[N, a] = -a, \quad [N, a^+] = a^+,
\]
\[ a a^+ - q a^+ a = q^{2N}, \quad a a^+ - q^2 a^+ a = q^N \]  

(35)

and the deformation associated with the discrete \( q \)-Hermite II polynomials 
\( (\alpha = -1, \quad \beta = 2, \quad l = 2, \quad q < 1) \)

\[ [N, a] = -a, \quad [N, a^+] = a^+, \]  

(36)

\[ a a^+ - q^{-1} a^+ a = q^{-2N}, \quad a a^+ - q^{-2} a^+ a = q^{-N}. \]  

(37)

3. Generalized \( q \)-deformed oscillator-like systems and generalized discrete \( q \)-Hermite I polynomials

First of all we consider a deformed oscillator in the case when the parameters in (17) take the values  
\( \alpha = 1/2, \quad \beta = -1, \quad l = 2, \quad 0 < q < 1 \). The structure function (17) in this case is written as

\[ f_n = q^{(n+1)/2-1/2}(1-q)^{-1/2}(1-q^{n+1})^{1/2}. \]  

(38)

The Fock representation of the \( a \) and \( a^+ \) operators for (34), (35) are

\[ a|n\rangle = (\frac{1}{1-q})^{1/2}q^{(n-1)/2}(1-q^{n})^{1/2}|n-1\rangle, \]  

(39)

\[ a^+|n\rangle = (\frac{1}{1-q})^{1/2}q^{n/2}(1-q^{n+1})^{1/2}|n+1\rangle. \]  

(40)

The deformed canonical commutation relations take the form (34), (35) or

\[ [a, a^+] = q^N \frac{1-q^{N+1}}{1-q} - q^{-N} \frac{1-q^{N}}{1-q}. \]  

(41)

Recall that the Hamiltonian of this oscillator \( H = a a^+ + a^+ a \) has a discrete spectrum \( H|n\rangle = \lambda_n|n\rangle \), where

\[ \lambda_n = q^n(1-q)^{-1}(1-q^{n+1}) + q^{n-1}(1-q)^{-1}(1-q^n), \]  

(42)

To find the wave functions corresponding to these eigenvalues we proceed as in the case of the standard quantum oscillator (see Introduction).

The position operator \( Q \) and the momentum operator \( P \) are given in the basis \( |n\rangle, n = 0, 1, \ldots \) of the Hilbert space \( \mathcal{H} \) by Jacobi matrices. The generalized eigenvectors \( \{|x\rangle\} \) of the operator \( Q, \ Q|x\rangle = x|x\rangle \), form a continuous basis of the Hilbert space \( \mathcal{H} \) and coefficients \( P_n^{(0)}(x; q) \) of the
transition from the basis \{\ket{n}\} to the basis \{\ket{x}\},
\[ |x\rangle = \sum_{n=0}^{\infty} P_n^{(0)}(x; q) |n\rangle, \]
satisfy the recurrence relation
\[ xP_n^{(0)}(x; q) = \left(\frac{1}{1-q}\right)^{1/2} q^{n/2} \]
\[ \times (1 - q^{n+1})^{1/2} P_{n+1}^{(0)}(x; q) + \left(\frac{1}{1-q}\right)^{1/2} q^{(n-1)/2}(1 - q^n)^{1/2} P_{n-1}^{(0)}(x; q). \] (43)

If we do the rescaling of variables \( y = (1-q)^{1/2}x \) and denote \( \psi_n^{(0)}(x; q) = P_n^{(0)}((1-q)^{-1/2}x; q) \), then the previous relation is reduced to
\[ x\psi_n^{(0)}(x; q)\]
\[ = q^{n/2}(1 - q^{n+1})^{1/2}\psi_{n+1}(x; q) + q^{(n-1)/2}(1 - q^n)^{1/2}\psi_{n-1}(x; q). \] (44)

After replacement
\[ \psi_n^{(0)}(x; q) = \frac{q^{-n(n-1)/4}}{(q; q)_n^{1/2}} h_n^{(0)}(x; q), \] (45)
we obtain the recurrence relation for the discrete \( q \)-Hermite I polynomials
\[ x h_n^{(0)}(x; q) = h_{n+1}^{(0)}(x; q) + q^{n-1}(1 - q^n)h_{n-1}^{(0)}(x; q). \] (46)

Together with the initial condition \( h_0^{(0)}(x; q) = 1 \) it defines the discrete \( q \)-Hermitian I polynomials \cite{12}, \cite{13} represented as
\[ h_n^{(0)}(x; q) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(q; q)_n}{(q^2; q^2)_k(q; q)_{n-2k}}(-1)^k q^{k(k-1)}x^{n-2k}. \] (47)

They can be written by means of the basic hypergeometric function as
\[ h_n^{(0)}(x; q) = x^n 2\phi_0(\frac{q^{-n}}{q}, \frac{q^{-n+1}}{q} - \frac{q^2}{q}; \frac{2q^{n-1}}{x^2}). \] (48)

Now, the solution of the equation \cite{13} can be represented by the expression
\[ P_n^{(0)}(x; q) = \frac{q^{-n(n-1)/4}}{(q; q)_n^{1/2}} h_n^{(0)}(\sqrt{1-qx}; q). \] (49)
It follows from [12, 13] that these polynomials are orthogonal with respect to the discrete measure

\[
\begin{align*}
&d \omega^{(0)}(x) = \frac{1}{2}(q; q^2)_{\infty} \delta(x - \frac{q^0}{\sqrt{1-q}}) \, dx \\
&+ \sum_{k>0} \frac{\sqrt{1-q} |x|}{2} \frac{(q^2(1-q)x^2, q; q^2)_{\infty}}{(q; q)_{\infty}} \delta(x - \frac{q^k}{\sqrt{1-q}}) \, dx \\
&+ \sum_{k>0} \frac{\sqrt{1-q} |x|}{2} \frac{(q^2(1-q)x^2, q; q^2)_{\infty}}{(q; q)_{\infty}} \delta(x + \frac{q^k}{\sqrt{1-q}}) \, dx.
\end{align*}
\]  

(50)

and the orthogonality relation is

\[
\frac{\delta_{mn}}{(q; q)_n} = \frac{1}{2}(q; q^2)_{\infty} P_m^{(0)}(1; q) P_n^{(0)}(1; q)
\]

\[+
\sum_{k=0}^{\infty} \left\{ P_m^{(0)}(q^k; q) P_n^{(0)}(q^k; q) + P_m^{(0)}(-q^k; q) P_n^{(0)}(-q^k; q) \right\} \frac{q^k (q^{2k+2}; q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \frac{(q^{2k+2}; q^2)_{\infty}}{(q^2; q^2)_{\infty}}.
\]  

(51)

It follows that spectrum of the position operator \( Q \) is

\[
Sp Q = \left\{ \frac{\pm 1}{\sqrt{1-q}}, \frac{\pm q}{\sqrt{1-q}}, \ldots, \frac{\pm q^k}{\sqrt{1-q}} \ldots; k \geq 0 \right\}.
\]  

(52)

The extension of this method for the generalized oscillator (17), (20), (21), determined by the formulas (22) and (23) for \( q < 1 \) gives the recurrence relations

\[
x P_n(x; q) = \left( \frac{1}{1-q'} \right)^{1/2} q^{\alpha(n+1)+\beta/2} (1 - q^\beta(n+1))^{1/2} P_{n+1}(x; q)
\]

\[+
\left( \frac{1}{1-q'} \right)^{1/2} q^{\alpha n+\beta/2} (1 - q^\beta)^{1/2} P_{n-1}(x; q).
\]  

(53)

If we rescale the variables \( y = (1-q')^{1/2}x \), then \( P_n(x; q) = \psi_n((1-q')^{1/2}x; q) \) yields

\[
x \psi_n(x; q)
\]

\[=
q^{\alpha(n+1)+\beta/2} (1 - q^\beta)^{1/2} \psi_{n+1}(x; q) + q^{\alpha n+\beta/2} (1 - q^\beta)^{1/2} \psi_{n-1}(x; q).
\]  

(54)
Representing the function $\psi_n(x; q)$ as

$$\psi_n(x; q) = \frac{q^{-\alpha n^2/2}}{q^{(\alpha + \beta)n/2}(q'; q')_n^{1/2}} h_n(x; q)$$  \hspace{1cm} (55)$$

we obtain from (54) the recurrent relation for the generalized q-Hermite polynomials $h_n(x; q)$:

$$x h_n(x; q) = h_{n+1}(x; q) + q^{2\alpha + \beta} (1 - q^n) h_{n-1}(x; q).$$  \hspace{1cm} (56)$$

This equation can be solved by means of the anzatz

$$h_n(x; q) = \sum_{k=0}^{[n/2]} \frac{(q'; q')_n}{((a_n, c_n); (1, q^d))_k(q'; q')_{n-2k}} (-1)^k q^{k(k-1)} x^{n-2k},$$  \hspace{1cm} (57)$$

where we use the notation (14)

$$((a, c); (p, q))_k = \begin{cases} 1, & \text{if } k = 0; \\ (a - c)(ap - cq) \cdots (ap^{k-1} - cq^{k-1}), & \text{otherwise}, \end{cases}$$  \hspace{1cm} (58)$$

$q' = q^{l-1}$, and $a_n, c_n, d$ are unknown quantities. It is easy to see that this anzatz leads the relation (56) to the identity

$$1 - q^{n+1} - q^{2\alpha + \beta} (a_n - c_n q^{d(k-1)}) q^{k-2(k-1)} = 1 - q^{n-2k+1}.$$  \hspace{1cm} (59)$$

This identity admits the solutions

$$a_n = q^{-2\alpha - \beta} q^{n-1}, \hspace{0.5cm} c_n = q^{2\alpha - \beta} q^{n+1}, \hspace{0.5cm} d = 2$$  \hspace{1cm} (60)$$

and an easy calculation gives $((a_n, c_n); (1, q^d))_k = q^{-k(2\alpha + \beta)} q^{k(k-1)} (q^{2}; q^2)_k$.

The resulting expressions for the generalized q-Hermite polynomials can be written as the polynomial of degree $n$ in $x$

$$h_n(x; q) = \sum_{k=0}^{[n/2]} \frac{(q'; q')_n}{(q^2; q^2)_k(q'; q')_{n-2k}} (-1)^k q^{k(k+1) - 2(k-1)} x^{n-2k}.$$  \hspace{1cm} (61)$$

They can be represented in terms of the basic hypergeometric function as

$$h_n(x; q) = x^n \phi_0 \left( \begin{array}{c} q^{l-n} \qquad q^{l-n+1} \\ q^2 \qquad q^{2\alpha + \beta} q^m \end{array} \right).$$  \hspace{1cm} (62)$$

It is easy to see that for $\alpha = \frac{1}{2}$, $\beta = -1$, $l = 2$ the solution (61) of (56) reduces to the solution (47) of (46).
At last, the solutions $P_n(x; q)$ of the equations \[53\] with the initial conditions $P_{-1}(x; q) = 0, \ P_0(x; q) = 1$ can be written as polynomials of degree $n$ in $x$:

$$P_n(x; q) = \frac{q^{-\alpha n^2/2}}{q^\alpha \beta} n_0(q'; q')_{1/2} \sqrt{1 - q' x; q}. \quad (63)$$

Now we restrict ourselves by the condition $\alpha = (l - 1)/2$ in \[63\]. Then

$$P_n(x; q) = \frac{q^{-\alpha n(n-1)/2}}{(q'; q')_{1/2} \sqrt{1 - q' x; q'}} q^{-n(2\alpha + \beta)/2} \sqrt{1 - q' x; q'}. \quad (64)$$

These polynomials are orthogonal with respect to the discrete measure

$$d\omega(x) = \frac{q^{-2\alpha + \beta}/2}{2} \sqrt{1 - q'} (q'; q')_\infty \delta(x - \frac{q_0}{q^{-(2\alpha + \beta)/2} \sqrt{1 - q'}}) dx$$

$$+ \sum_{k>0} \frac{q^{-2\alpha + \beta}/2}{2} \sqrt{1 - q'} |x| (q^{-(2\alpha + \beta)}(q'^2 - 1 - q') x^2; q'; q')_\infty \delta(x - \frac{q^k}{q^{-(2\alpha + \beta)/2} \sqrt{1 - q'}}) dx$$

$$+ \sum_{k>0} \frac{q^{-2\alpha + \beta}/2}{2} \sqrt{1 - q'} |x| (q^{-(2\alpha + \beta)}(q'^2 - 1 - q') x^2; q'; q')_\infty \delta(x + \frac{q^k}{q^{-(2\alpha + \beta)/2} \sqrt{1 - q'}}) dx. \quad (65)$$

The orthogonality relation has the form

$$\frac{\delta_{nn}}{(q'; q')_n} = \frac{1}{2 (q'^2; q'^2)_\infty} P_n(1; q)) P_n(1; q)$$

$$+ \sum_{k>0} \{P_m(q; q) P_n(q; q) + P_m(-q; q) P_n(-q; q) \} \frac{q^{2k+2}}{2} \frac{q'^2}{(q'^2; q'^2)_\infty} \frac{(q'^2; q'^2)_\infty}{(q'^2; q'^2)_\infty} \quad (66)$$

From this it follows that spectrum of the position operator $Q$ is

$$Sp Q = \left\{ \pm \frac{q^{(2\alpha + \beta)/2}}{\sqrt{1 - q}}, \ pm \frac{q^{(2\alpha + \beta)/2} q'}{\sqrt{1 - q'}}, \ldots, \pm \frac{q^{(2\alpha + \beta)/2} q^k}{\sqrt{1 - q'}}, \ldots; k \geq 0 \right\}. \quad (67)$$

4. Generalized $q$-deformed oscillator-like systems and generalized discrete $q$-Hermite II polynomials
If we fix in (17) the values of the parameters \( \alpha = -1, \beta = 2, l = 2, 0 < q < 1 \), the structure function \( f_n \) is reduced to the form
\[
f_n = q^{-(n+1)+1}(1 - q)^{-1/2}(1 - q^{n+1})^{1/2}.
\] (68)

The Fock representation of the creation and the annihilation operators of the relations (36), (37) is given by
\[
a|n\rangle = \left( \frac{q}{1 - q} \right)^{1/2} q^{-n+1/2}(1 - q^n)^{1/2} |n - 1\rangle,
\]
\[
a^+|n\rangle = \left( \frac{q}{1 - q} \right)^{1/2} q^{-n-1/2}(1 - q^{n+1})^{1/2} |n + 1\rangle.
\] (69)

It follows that
\[
aa^+|n\rangle = q^{-2n} \frac{1 - q^{n+1}}{1 - q} |n\rangle, \quad a^+a|n\rangle = q^{-2n+2} \frac{1 - q^n}{1 - q} |n\rangle,
\] (70)

and commutation relation (37) can be written in the symbolic form
\[
[a, a^+] = q^{-2N} \frac{1 - q^{N+1}}{1 - q} - q^{-2N+2} \frac{1 - q^N}{1 - q}.
\] (71)

The Hamiltonian \( H \) of this oscillator-like system has the discrete spectrum
\[
H|n\rangle = \lambda_n |n\rangle, \quad \lambda_n = q^{-2n}(1 - q)^{-1}(1 - q^{n+1}) + q^{2-2n}(1 - q)^{-1}(1 - q^n), \quad n \geq 0.
\] (72)

As in the previous section the position and momentum operators \( Q \) and \( P \) in the basis \(|n\rangle\) of the Hilbert space \( \mathcal{H} \) are represented by Jacobi matrices. The coefficients \( \tilde{P}_n^0(x; q) \) of the transition \(|x\rangle = \sum_{n=0}^{\infty} \tilde{P}_n^0(x; q)|x\rangle \) from the basis \(|n\rangle\) to the basis \(|x\rangle\), \( Q|x\rangle = x|x\rangle \), satisfy the relations
\[
x \tilde{P}_n^0(x; q) = \left( \frac{q}{1 - q} \right)^{1/2} q^{-(n+1)+1/2}
\]
\[
\times (1 - q^{n+1})^{1/2} \tilde{P}_{n+1}^0(x; q) + \left( \frac{q}{1 - q} \right)^{1/2} q^{-n+1/2}(1 - q^n) \tilde{P}_{n-1}^0(x; q).
\] (73)

Introducing the rescaling \( y = q^{-1/2}(1 - q)^{1/2}x \) and the function \( \tilde{\psi}_n^{(0)}(x; q) = \tilde{P}_n^0(q^{1/2}(1 - q)^{-1/2}x; q) \) we obtain the equation
\[
x \tilde{\psi}_n^{(0)}(x; q) = q^{-(n+1)+1/2}
\]
\begin{equation}
\times (1 - q^{n+1})^{1/2} \tilde{\psi}_{n+1}(x; q) + q^{-n+1/2} (1 - q^n)^{1/2} \tilde{\psi}_n(x; q).
\end{equation}

After the replacement
\begin{equation}
\tilde{\psi}_n^{(0)}(x; q) = \frac{q^{n^2/2}}{(q^q)_{n/2}^{1/2}} \tilde{h}_n^{(0)}(x; q),
\end{equation}
we obtain the recurrence relation for the discrete $q$-Hermite II polynomials \[12\]
\begin{equation}
x \tilde{h}_n^{(0)}(x; q) = \tilde{h}_n^{(0)}(x; q) + q^{-2n+1} (1 - q^n) \tilde{h}_n^{(0)}(x; q)
\end{equation}
which together with the initial condition $\tilde{h}_0^{(0)}(x; q) = 1$ define the discrete $q$-Hermite II polynomials
\begin{equation}
\tilde{h}_n^{(0)}(x; q) = \sum_{k=0}^{[n/2]} \frac{(q^q)_n}{(q^q)_{k+1}^{1/2}(q^q)_{n-2k}^{1/2}} (-1)^k q^{2k(k-n)+k} x^{n-2k}.
\end{equation}

These polynomials can be represented in terms of the basic hypergeometric function:
\begin{equation}
\tilde{h}_n^{(0)}(x; q) = x^n 2\phi_1 \left( q^{-n} q^{-n+1} \begin{array}{c} 0 \\ q^2; -q^2 \end{array} x \right).
\end{equation}

The solution of the equations \[13\] with the initial conditions $\tilde{P}_0^{(0)}(x; q) = 0$, $\tilde{P}_1^{(0)}(x; q) = 1$ can be given in the form
\begin{equation}
\tilde{P}_n^{(0)}(x; q) = \frac{q^{n^2/2}}{(q^q)_{n/2}^{1/2}} \tilde{h}_n^{(0)}(q^{-2n} \sqrt{1 - qx}; q).
\end{equation}

It follows from \[13\] that these polynomials are orthogonal with respect to the discrete measure
\begin{equation}
d\tilde{\omega}^{(0)}(x)
\end{equation}
\begin{equation}
= \sum_{k=-\infty}^{\infty} c^{-1} q^{-1/2} \sqrt{1 - qw(q^{-1/2} \sqrt{1 - qx}; q)} x \delta(x - \frac{cq^k}{q^{-1/2} \sqrt{1 - q}}) dx
\end{equation}
\begin{equation}
- \sum_{k=-\infty}^{\infty} c^{-1} q^{-1/2} \sqrt{1 - qw(-q^{-1/2} \sqrt{1 - qx}; q)} x \delta(x + \frac{cq^k}{q^{-1/2} \sqrt{1 - q}}) dx.
\end{equation}
The orthogonality relation for these polynomials has the form
\begin{equation}
\sum_{k=-\infty}^{\infty} \{ \tilde{P}_m^{(0)}(cq^k; q) \tilde{P}_n^{(0)}(cq^k; q) + \tilde{P}_m^{(0)}(-cq^k; q) \tilde{P}_n^{(0)}(-cq^k; q) \} w(cq^k; q) q^k.
\end{equation}
\[ w(x; q) = \frac{1}{(-x^2; q^2)_\infty}. \]

where \( w(x; q) = 1/(-x^2; q^2)_\infty \). It follows that spectrum of the position operator \( Q \) is

\[ SpQ = \left\{ \frac{\pm c}{q^{-1/2} \sqrt{1 - q}}, \frac{\pm cq}{q^{-1/2} \sqrt{1 - q}}, \ldots, \frac{\pm cq^k}{q^{-1/2} \sqrt{1 - q}}, \ldots; k \geq 0 \right\}. \]

A connection of the discrete q-Hermite I polynomials and the discrete q-Hermite II polynomials are given by

\[ q \rightarrow \frac{1}{q}. \]

Indeed, we have the relations

\[ \frac{(1/q'; 1/q')_n q^{-(k-1)}}{(1/q'; 1/q')_n - 2k} = \frac{(q'; q')_n q^{2k^2 - 2kn}}{(q'; q')_{n-2k}}, \]

leading to the identity

\[ h_n^{(0)}(ix; q^{-1}) = i^n \tilde{h}_n^{(0)}(x; q). \]

This identity reflects the transition \( q \rightarrow q^{-1} \) from the oscillator \( 81 \), \( 82 \) to the oscillator \( 53 \). \( 54 \).

Now we consider the generalized oscillator \( 17 \), \( 20 \), \( 21 \) represented by operators \( 22 \), \( 23 \), \( 29 \) for \( q < 1 \). Then instead \( 53 \) we have the equality

\[ x \tilde{P}_n(x; q) = \left( \frac{q^{\beta/2}}{1 - q} \right)^{1/2} q^{\alpha(n+1) + \beta/4} \]

\[ \times (1 - q^{n+1})^{1/2} \tilde{P}_{n+1}(x; q) + \left( \frac{q^{\beta/2}}{1 - q} \right)^{1/2} q^{\alpha n + \beta/4} (1 - q^n)^{1/2} \tilde{P}_{n-1}(x; q), \]

or

\[ x \tilde{\varphi}_n \]

\[ = q^{\alpha(n+1) + \beta/4} (1 - q^{n+1})^{1/2} \tilde{\varphi}_{n+1}(x; q) + q^{\alpha n + \beta/4} (1 - q^n)^{1/2} \tilde{\varphi}_{n-1}(x; q), \]

where \( \tilde{\varphi}_n(x; q) = \tilde{P}_n((q^{\beta/4}(1-q'))^{-1/2}x; q). \) Representing the function \( \tilde{\varphi}_n(x; q) \) as

\[ \tilde{\varphi}_n(x; q) = \frac{q^{-\alpha n^2/2}}{q^{(2\alpha + \beta)n/4}(q'; q')^n_{1/2}} \tilde{h}_n(x; q) \]

\[ \tilde{\varphi}_n(x; q) = \frac{q^{-\alpha n^2/2}}{q^{(2\alpha + \beta)n/4}(q'; q')^n_{1/2}} \tilde{h}_n(x; q) \]

\[ \tilde{\varphi}_n(x; q) = \frac{q^{-\alpha n^2/2}}{q^{(2\alpha + \beta)n/4}(q'; q')^n_{1/2}} \tilde{h}_n(x; q) \]

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we obtain the recurrence relation

\[ \tilde{h}_{n+1}(x; q) + q^{2\alpha + \beta/2}(1 - q^n)\tilde{h}_{n-1}(x; q) = x\tilde{h}_n(x; q). \]  

(89)

The solution of this equation can be obtained by means of the anzatz

\[ \tilde{h}_n(x; q) = \sum_{k=0}^{[n/2]} \frac{(q'; q')_n}{((a_n, c_n); (1, q^d))_k(q'; q')_{n-2k}}(-1)^k q^{k(2k-2n+1)} x^{n-2k}, \]  

(90)

which generalizes (77). It is easy to see that it reduces the relations (89) to the identity

\[ (1 - q^{m-2k+1}) q^{2k} \]

\[ = 1 - q^{m+1} - q^{2\alpha + \beta/2}(a_n - c_n q^d(k-1)) q^{-(2k-2n-1)} q^2(n-1). \]  

(91)

We obtain the solution

\[ a_n = q^{-2\alpha - \beta/2} q^{-2n+1}, \quad c_n = q^{-2\alpha - \beta/2} q^{-2n+3}, \quad d = 2. \]  

(92)

An easy calculation gives \(((a_n, c_n); (1, q^d))_k = q^{-k(2\alpha + \beta/2)} q^{-(2k-2n-1)} (q^2; q^2)_k. \]

The resulting expression

\[ \tilde{h}_n(x; q) = \sum_{k=0}^{[n/2]} \frac{(q'; q')_n}{(q^2; q^2)_k(q'; q')_{n-2k}}(-1)^k q^{(2\alpha + \beta/2)k} q^{2k^2} x^{n-2k} \]  

(93)

defines a generalized of the q-Hermite polynomials which can be written in terms of the basic hypergeometric function.

\[ \tilde{h}_n(x; q) = x^n _2\phi_1 \left( q'^{-n}, q'^{-n+1} \begin{array}{c} 0 \\ q^2; -\frac{q^{2\alpha + \beta/2} q^{2n+1}}{x^2} \end{array} \right). \]  

(94)

It is easy to see that for the special values \( \alpha = -1, \quad \beta = 2, \quad l = 2 \) the solution (93) is reduced to the solution (77) of the relation (76).

Finally, the solution \( \tilde{P}_n(x; q) \) with the initial conditions \( \tilde{P}_{-1}(x; q) = 0, \quad \tilde{P}_0(x; q) = 1 \) of the equation (86) are given by the formula

\[ \tilde{P}_n(x; q) = \frac{q^{-\alpha n^2/2}}{q^{2\alpha + \beta n/4}(q'; q')_n^{1/2}} \tilde{h}_n(q^{-\beta/4} \sqrt{1 - q' x}; q). \]  

(95)

From now on we restrict ourselves by the condition \( \alpha = -(l - 1) \) in (95). Then

\[ \tilde{P}_n(x; q) = \frac{q^{-\alpha n^2/2}}{(q'; q')_n^{1/2}} \tilde{h}_n(q^{-(\alpha + \beta)/2} \sqrt{1 - q' x}; q). \]  

(96)
These polynomials are orthogonal with respect to the discrete measure

\[ d\tilde{\omega}(x) = \sum_{k=-\infty}^{\infty} c^{-1} q^{-(\alpha+\beta)/2} \sqrt{1 - q'} w(q^{-(\alpha+\beta)/2} \sqrt{1 - q'} x; q') x \delta(x - \frac{cq^k}{q^{-(\alpha+\beta)/2} \sqrt{1 - q'}}) \, dx \]

\[ - \sum_{k=-\infty}^{\infty} c^{-1} q^{-(\alpha+\beta)/2} \sqrt{1 - q'} w(q^{-(\alpha+\beta)/2} \sqrt{1 - q'} x; q') x \delta(x + \frac{cq^k}{q^{-(\alpha+\beta)/2} \sqrt{1 - q'}}) \, dx \]

and the orthogonality relation is

\[ \sum_{k=-\infty}^{\infty} \{ \tilde{P}_m(cq^k; q) \tilde{P}_n(cq^k; q) + \tilde{P}_m(-cq^k; q) \tilde{P}_n(-cq^k; q) \} w(cq^k; q) q^k \]

\[ = 2 \frac{(q^2, -c^2 q', -c^{-2} q'; q')_\infty (q'; q')_n q^{-\alpha n^2}}{(q', -c^{-2} q'; q')_\infty q^{n^2}} \delta_{mn}, \quad c > 0, \quad (98) \]

where \( w(x; q) = 1/(x^2; q^2)_\infty \). It follows that spectrum of the position operator \( Q \) is

\[ SpQ = \left\{ \frac{\pm c}{q^{-(\alpha+\beta)/2} \sqrt{1 - q'}}, \frac{\pm cq'}{q^{-(\alpha+\beta)/2} \sqrt{1 - q'}}, \ldots, \frac{\pm cq^k}{q^{-(\alpha+\beta)/2} \sqrt{1 - q'}}, \ldots ; k \geq 0 \right\}. \quad (99) \]

A connection of the generalized q-Hermitian I and the generalized q-Hermitian II polynomials is not evident at all as in non-generalized case. Unfortunately the change \( q \to 1/q \) does not lead to the analogous of (85) for the generalized q-Hermitian I and the generalized q-Hermitian II polynomials (61) and (93). It is evident from (22) and (23). Instead of this in this case we have

\[ h_n(x; 1/q) = \sum_{k=0}^{[n/2]} \frac{(q'; q')_n}{(q'^2; q'^2)_k (q'; q')_{n-2k}} (-1)^k q^{-k(2\alpha n+\beta)} q^k (2k-n) x^{n-2k}. \quad (100) \]

It is easy to see that for \( \alpha = \frac{1}{2} \), \( \beta = -1 \), \( l-1 = 1 \) this relation gives identity (85).

5. Barut-Girardello coherent states of oscillators associated with generalized discrete q-Hermite I and II polynomials
It is known that the coherent states in the ordinary Lie algebras are very useful for studying the representation theory. The generalized coherent states are very useful in the study of representation of quantum group and physics, in particular, in quantum optics. Barut-Girardello type coherent states of oscillator algebras have been studied for oscillator-like system connected with some orthogonal polynomials. The family coherent states associated with discrete q-Hermite polynomials of type II have been described in [6] and [15]. In this section we give solution the same problem for discrete q-Hermite I and generalized discrete q-Hermite I and II polynomials.

First of all we prove the formula for a generating function of q-Hermite I polynomials connected with the appropriate q-oscillator

\[ \sum_{n=0}^{\infty} \frac{(-1)^n q^{-n(n-1)/2}}{(q; q)_n} h_n^{(0)}(x; q) t^n = (qt; 1/q)_\infty \phi_1 \left( \frac{x}{tq} \left| \frac{1/q; -tq}{1/q; -tq} \right. \right). \] (101)

Let us denote the left hand side of this identity by

\[ \Phi(x, q, t) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{-n(n-1)/2}}{(q; q)_n} h_n^{(0)}(x; q) t^n. \] (102)

Then

\[ \Phi(ix, 1/q, t) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{-n(n-1)/2}}{(q; q)_n} h_n^{(0)}(x; q) \left( \frac{it}{q} \right)^n \] (103)

Using formula (3.29.12) of ref. [13]

\[ \sum_{n=0}^{\infty} \frac{(-1)^n q^{-n(n-1)/2}}{(q; q)_n} h_n^{(0)}(x; q) t^n = (-it; q)_\infty \phi_1 \left( \frac{ix}{-it} \left| \frac{q; it}{q; it} \right. \right) \] (104)

we obtain

\[ \Phi(ix, 1/q, t) = (-t/q; q)_\infty \phi_1 \left( \frac{ix}{t/q} \left| \frac{q; -t/q}{q; -t/q} \right. \right) \] (105)

from which easy follows (101).

The Barut-Girardello coherent states of a oscillator (1), (2) in the Fock representation space $H$ are defined as eigenvectors of annihilation operator $a$:

\[ a |z\rangle = z |z\rangle, \quad z \in \mathbb{C}, \] (106)

given by the formula

\[ |z\rangle = \mathcal{N}^{-1} \sum_{n=0}^{\infty} \frac{z^n}{\tilde{f}_{n-1}!} |n\rangle, \] (107)
where \( \mathcal{N} \) is normalized factor. By definition the basis vectors \( |n\rangle \) of \( \mathcal{H} \) are taken as polynomials \( \psi_n^{(0)}(x; q) \) of (45). In the case of the oscillator corresponding q-Hermite I polynomials we have 
\[
f_{n-1}! = \sqrt{1/(1-q)^n q^{n(n-1)/2} (q; q)_n}
\]
It follows 
\[
f_{n-1}! = \sqrt{1/(1-q)^n q^{n(n-1)/2} (q; q)_n}
\]
and coherent state (107) can be written as 
\[
|z\rangle = \mathcal{N}^{-1}(|z|^2) \sum_{n=0}^{\infty} \frac{(\sqrt{1-q}z)^n}{q^{n(n-1)/2} (q; q)_n^{1/2}} h_n^{(0)}(x; q)
\]
(109)
\[
= \mathcal{N}^{-1}(|z|^2) \sum_{n=0}^{\infty} \frac{(-1)^n q^{-n(n-1)/2} (q; q)_n^{-1/2}}{h_n^{(0)}(x, q)(-\sqrt{1-q}z)^n}
\]
(110)
(take into account of (101))
\[
= \mathcal{N}^{-1}(|z|^2)(q(-\sqrt{1-q}z); 1/q) \sum_{n=0}^{\infty} \frac{(-1)^n q^{-n(n-1)/2}}{q^{n(n-1)/2} (q; q)_n^{-1/2}} h_n^{(0)}(x, q)(-\sqrt{1-q}z)^n
\]
Easy calculation gives the normalized factor
\[
\mathcal{N}^2(|z|^2) = 2 \phi_0 \left( 0, 0 \Big| q; (1-q)z \right).
\]
(112)
The overlapping of two coherent states is
\[
\langle z_1 | z_2 \rangle = 2 \phi_0 \left( 0, 0 \Big| q; (1-q)z_1 z_2 \right).
\]
(113)
The terminal expression for coherent state \( |z\rangle \) of the q-oscillator corresponding discrete q-Hermite I polynomials has the form
\[
|z\rangle = \left\{ 2 \phi_0 \left( 0, 0 \Big| q; (1-q)z \right) \right\}^{1/2}
\]
\[
\times (-\sqrt{1-q}z; 1/q) \sum_{n=0}^{\infty} \frac{x}{q^{n(n-1)/2} (q; q)_n^{-1/2}} h_n^{(0)}(x, q)(-\sqrt{1-q}z)^n
\]
(114)
The family of coherent states associated with oscillator-like system corresponding to generalized discrete q-Hermite I polynomials (61) can be obtained the same method. In this case
\[
f_{n-1}! = \sqrt{1/(1-q')^n q'^{\alpha n^2} q'^{(\alpha+\beta)n} (q'; q')_n}
\]
(115)
and basis vectors $|n\rangle$ are taken as polynomials $\psi_n(x; q)$ of (66). This leads to

$$|z\rangle = \mathcal{N}^{-1}(|z|^2) \sum_{n=0}^{\infty} \frac{(\sqrt{1-q}z)^n}{\sqrt{q^{2n+1}}q^{(\alpha+\beta)n}(q'; q)_n} q^{-\alpha n^2/2} h_n(x; q)$$

or

$$|z\rangle = \mathcal{N}^{-1}(|z|^2) \sum_{n=0}^{\infty} \frac{(-1)^n q^{-\alpha n(n-1)}(q'; q)_n}{(q; q)_n} h_n(x; q) \left(-\frac{\sqrt{1-q}}{q^{2\alpha+\beta}} z\right)^n$$

Comparing the expressions (116) and (117) we obtain

$$|z\rangle = \mathcal{N}^{-1}(|z|^2) (-q^{-2\alpha+\beta} q' \sqrt{1-q} z; 1/q') \times \phi_1 \left( \frac{q^{-2\alpha+\beta}/2 (q'-1/2x)}{-q^{-2\alpha+\beta} q' \sqrt{1-q} z} \right) \left(1/q'; q^{-2\alpha+\beta} q' \sqrt{1-q} z\right).$$

The short calculation of a normalizing factor of the coherent state (118) gives

$$\mathcal{N}^2(|z|^2) = \sum_{n=0}^{\infty} \left(\frac{\sqrt{1-q}}{q^{2\alpha+\beta}} |z|^2\right)^n q^{-\alpha n(n-1)} \frac{1}{(q'; q)_n}.$$
\[ |z \rangle = \mathcal{N}^{-1}(|z|^2) \sum_{n=0}^\infty \frac{(-1)^n q^{-\alpha n^2}}{(q;q)_n} h_n(x; q) \left(-q^{-\beta/4} q^{-(2\alpha+\beta)/2} \sqrt{1-q'} z \right)^n. \tag{122} \]

The generation function for polynomials \( h_n(x; q) \) is defined by
\[
\sum_{n=0}^\infty \frac{(-1)^n q^{-\alpha n^2}}{(q;q)_n} h_n(x; q) t^n
= (-iq^{\beta/4} q'^{n/2} t; q')_\infty \Phi_1 \left( \begin{array}{c} \frac{i q^{-(2\alpha n+\beta)/2} q'^{(2n-1)/2} x}{-i q^{\beta/4} q'^{n/4} t} \\ -i q^{\beta/4} q'^{n/4} z \end{array} \right) q'; -i \sqrt{q'(1-q')} q'^{n/2}. \tag{123} \]

At last, the normalizing factor of the generalized coherent state \( |z \rangle \) can be written as
\[
\mathcal{N}^2(|z|^2) = \sum_{n=0}^\infty \frac{q^{-\alpha n(n+1)}}{(q'; q')_n} \left( \frac{1 - q'}{q^3} |z|^2 \right)^n. \tag{125} \]

We have not established completeness (over-completeness) of the given set of generalized coherent states. It will be done in a forthcoming paper.

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References

[1] Arik A., Coon D. D. and Lam, J. Math. Phys. 16 (1975), 1776.

[2] Biedenharn, L. C., The quantum group $SU_q(2)$ and a $q$ analog of the boson operator, J. Phys. A 22 (1989), L837.

[3] Macfarlane A. J., On $q$ analogs of the quantum harmonic oscillator and the quantum group $SU_q(2)$, J. Phys. A 22 (1989) L4581.

[4] Damaskinsky E. V., Kulish P. P., Deformed oscillators and their applications, Zap. Nauchn. Sem. LOMI 189, (1991) 37.

[5] Damaskinsky E. V., Kulish P. P., $q$-Hermite polynomials and $q$-oscillators, Zap. Nauch. Sem. POMI 199 1992, 81.

[6] Borzov V. V., Damaskinsky E. V., Generalized coherent states for $q$-oscillator connected with $q$-Hermite polynomials, arXiv:math.QA/0307356.

[7] Borzov V. V., Damaskinsky E. V., Yegorov S. B., Some remarks on the representations of the generalized deformed oscillator algebra, arXiv:q-alg/9509022.

[8] Burban I. M., Klimyk A. U., On spectral properties of $q$-oscillator operators, Lett. Math. Phys. 29 (1991), 13.

[9] Chung W.-Sang, Klimyk A. U., On position and momentum operators in the $q$-oscillator algebra, J. Math. Phys. 37(2) (1992) 917.

[10] Borzov V.V., Damaskinsky E.V., Kulish P.P., On position operator spectral measure for deformed in the case of indetermined Hamburger moment problem, Reviews in Math. Phys. 12 (2000), 691.

[11] Berezanskii, Ju. M., Expansion in eigenfunctions of self-adjoint operators, (American Mathematical Society, Providence, R. I. 1968).

[12] Gasper G., Rahman M., Basic Hypergeometric Series, (Cambridge University Press, Cambridge, 1996).

[13] Koekoek R., Swarttouw R. F., The Askey-scheme of hypergeometric orthogonal polynomials and its $q$-analogue, (Report no.94 - 05, Delft University of Technology, 1994), [arXiv: math. CA/9602214].
[14] Jagannathan, R., Rao K., *Two-parameter quantum algebras, twin-basic numbers, and associated generalized hypergeometric series*, arXiv: math. NT/0602613.

[15] Borzov V. V., Damaskinsky E.V., *Generalized coherent states for q-oscillator connected with discrete q-Hermite polynomials*, arXiv: math. quant-ph/0407252.