Influence of high-order QED correction on phase transition of the Euler-Heisenberg AdS black hole

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Two phase transition branches of the Euler-Heisenberg AdS (EHAdS) black hole (BH) was proposed by Magos et al. from the phase transition critical behavior [Magos et al., Phys. Rev. D. 102, 084011 (2020)]. We found that the thermodynamic of the first phase transition branch for the EHAdS BH is instable. We suspect that this branch may depends on the high-order QED correction. The corrected EHAdS BH solution by considering the high-order QED correction is derived and the phase transition of this scenario is investigated. It is found that there is only one phase transition branch, indicating that the phase transition instability disappears for the EHAdS BH. By calculating the critical thermodynamic quantities and critical exponents of the corrected EHAdS BH, we find that the universal constant is $\varepsilon_Q = 3/8$ for the zeroth-order term of $a$ and the 2nd-order term correction is $13/82994$ in comparison that without considering the high-order QED correction. The critical exponents are equivalent to that of the vdW system. Based on the Ruppeiner geometry, the normalized scalar curvature is drieved. It is found that there is one sunken surface for the high-order QED correction situation. The two sunken surfaces occur in the EHAdS BH without high-order QED correction, indicating that the high-order QED correction leads to the phase transition instability disappears for the EHAdS BH from the microscopic point of view. These results suggest that the high-order QED correction eliminates the metastable state phase transition branch of the EHAdS BH, and the phase transition of the corrected EHAdS BH satisfies Maxwell behavior.

I. INTRODUCTION

Black hole (BH) thermodynamics is considered as a bridge connecting general relativity, quantum mechanics, and classical thermodynamics. Regarding the BH in the anti-de Sitter (AdS) spacetime as a thermodynamic system, Hawking and Page found that its thermodynamic properties is similar to the classical thermodynamic system \(1\). Taking the negative cosmological constant as the thermodynamic pressure, Dolan et al. developed the AdS BH thermodynamics to the extended phase space \(2\,3\). By investigating $P - \nu$ critical behavior of the charged AdS BH in the extended phase space, Kubiznak et al. found that the critical exponents of the charged AdS BH are precisely same as the vdW system \(4\). Johnson proposed that the charged AdS BH can be built as a heat engine model as the vdW fluid, and the heat engine efficiency can be calculated similarly \(5\). Aydner et al. investigated the Joule-Thomson expansion of the charged AdS BH and obtained the phase transition heating-cooling regions in the $T - P$ plane \(6\). Thermodynamics properties of various charged AdS BHs in the extended phase space are also extensively discussed \(7\,16\).

Based on the Dirac’s positron theory, Euler and Heisenberg proposed a new approach to describe the electromagnetic field. They derived the effective Lagrangian density of the electromagnetic field by revising the Maxwell’s equations in the vacuum \(17\). This new effective Lagrangian density has the high-order terms of the nonlinear electromagnetic (NEM) field \(18\). Schwinger reformulated this non-perturbative one-loop effective Lagrangian density within the quantum electrodynamics (QED) framework. The one-loop effective Lagrangian density carries the main characteristics of the Euler-Heisenberg (EH) NEM field \(19\). When the electric field strength is higher than the critical value ($m^2 c^3 / e^2$), the QED effect leads to the emergence of particle pairs in the vacuum \(20\). Coupling the one-loop effective Lagrangian density with the Einstein field equation, Yajima et al. obtained the EH BH solution \(21\). Kruglov provided an approximation approach for the EH NEM field within the high-order QED framework and gave the corresponding static spherically symmetric BH solution \(22\). By applying the Newman-Janis algorithm and its Azreg-Aïnou formulation, Bretón et al. obtained the rotating BH solution in the Einstein EH theory and analyzed its event horizons, ergoregions and test particle circular orbits \(23\). Subsequently, they investigated the birefringence and the quasinormal modes of the spherically symmetric EH BH. It is found that the quasinormal modes are suppressed by the effect of EH NEM field, which makes the charged BH behave more Schwarzschild-like \(24\).

Magos et al. generalized the EH BH solution to the AdS spacetime by considering the cosmological constant \(\Lambda\). It is found that EHAdS BH is characterized by the BH mass $M$, electric charge $Q$, cosmological constant $\Lambda$, and EH parameter $a$. By deriving the equation of the state and critical behavior, they found that the critical volume represents two phase transition branches of the EHAdS BH. The second phase transition branch is sim-
ilar to the vdW system, and the first phase transition branch depends on the EH NEM field, which leads to the phase transition splitting in small BH region [25]. Note that some AdS BHs come from the Einstein field equation coupled with the NEM fields (Bardeen-AdS BH and Hayward-AdS BH). Their phase transition have only one branch and represent similar properties to the vdW system [7, 20]. It is different from the two phase transition branches of the EHAdS BH [23]. We suspect whether the first phase transition branch results from the effect of the high-order QED.

We derive the EHAdS BH correction solution by considering the high-order QED correction and investigate the phase transition of this scenario. We also analyze the critical behavior and equal area law in the cases of the EHAdS BH and the corrected EHAdS BH. It is found that this disturbance correction as a thermodynamic stable probe can be used to reveal the physical properties of the EHAdS BH. The paper is organized as following: In Sec. II we analyze the critical behavior and equal area law in the framework of the EHAdS BH thermodynamics. Our solution of the EHAdS BH with high-order QED correction and investigation of thermodynamic behaviors are presented in Sec. III. Our conclusions is presented in Sec. IV.

II. THE EHADS BH AND THERMODYNAMIC CHARACTERISTICS

The line element of the EHAdS BH is given by [28]

\[ ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2d\Omega^2, \quad (1) \]

where \( f(r) \) is the metric potential, which is written as

\[ f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{aQ^4}{20r^6} - \frac{\Lambda r^2}{3}, \quad (2) \]

in which \( M \) and \( Q \) are the mass and charge of the BH, \( a \) is the EH parameter, \( \Lambda \) can be defined as the thermodynamic pressure \( (P) \), satisfying \( \Lambda = -8\pi P \) [29, 30]. The event horizon radius \( r_+ \) of the BH is derived from the largest root of the equation \( f(r_+) = 0 \). Hence, the BH mass is given as

\[ M = \frac{r_+}{2} + \frac{Q^2}{2r_+} - \frac{aQ^4}{40r_+^5} - \frac{\Lambda r_+^3}{6}. \quad (3) \]

According to the first law of the BH thermodynamics, the BH temperature is

\[ T = \frac{1}{4\pi r_+} \left( 1 - \frac{Q^2}{r_+^2} + \frac{aQ^4}{4\pi r_+^8} - \frac{\Lambda r_+^3}{3} \right). \quad (4) \]

Using \( \Lambda = -8\pi P \) and Eq. (4), the equation of the state for the EHAdS BH can be expressed as

\[ P = \frac{T}{2r_+} - \frac{1}{8\pi r_+^2} + \frac{Q^2}{8\pi r_+^4} - \frac{aQ^4}{32\pi r_+^8}. \quad (5) \]

Based on Eq. (5) and the critical conditions, the critical thermodynamic quantities of the EHAdS BH are obtained, i.e.

\[ T_{gc} = \frac{1}{2\pi r_+^2} \left( 1 - \frac{2Q^2}{r_+^2} + \frac{aQ^4}{4\pi r_+^8} \right), \quad (6) \]

\[ P_{gc} = \frac{1}{8\pi r_+^2} \left( 1 - \frac{3Q^2}{r_+^2} + \frac{7aQ^4}{4\pi r_+^8} \right), \quad (7) \]

\[ r_{gc}^2 = 2Q^2 \left( \frac{1}{2} \arccos \left( 1 - \frac{7a}{16Q^4} \right) - \frac{2\pi k}{3} \right) + 1 \]

where \( g \) in the subscript stands for the general EHAdS BH case. In case of \( k = 2, r_{gc}^2 \) is negative. In cases of \( k = 0 \) and \( k = 1 \), they represent the two phase transition branches respectively. Note that the above critical thermodynamic quantities are derived from the condition of the EH parameter \( a \) satisfying 0 \( \leq a \leq 32Q^2/7 \) [29]. Utilizing the critical conditions and state equation, one can obtain

\[ g(r_+) \equiv r_+^6 - 6Q^2r_+^4 + 7aQ^4 = 0. \quad (9) \]

By setting EH parameter \( a = 1 \) and BH charge \( Q = 0.6 \), \( g(r_+) \) as a function of \( r_+ \) is plotted in Figure 1. \( g(r_+) \) function curve intersects the horizontal axis twice (blue point and red point), indicating that there are two phase transition branches for the EHAdS BH situation. The blue point corresponds to parameter \( k \) equal to 1, which is the “first critical point”, and red point corresponds to parameter \( k \) equal to 0, which is the “second critical point”.

![Figure 1](image-url)

**FIG. 1.** \( g(r_+) \) as a function of \( r_+ \). We take EH parameter \( a = 1 \) and BH charge \( Q = 0.6 \).

The universal constant is defined as \( \varepsilon \equiv P_{gc}v_{gc}/T_{gc} \), where \( v_{gc} = 2r_{gc} \) is the specific volume [31]. By using Eqs. (6), (7), and (9), the \( \varepsilon \) of the EHAdS BH can be written as

\[ \varepsilon = \frac{21c}{16 + 32c}. \quad (10) \]
where $c = 1 - 2Q^2/r_{g1}^2$. When $-1/2 < c < 0$, the value of $\varepsilon$ is negative. From Figure 2, one can observe that $c$ ranges in $(-\infty, 1/2)$ for the $\varepsilon_{g1}$, and $c$ ranges in $(1/2, 3/8)$ for the $\varepsilon_{g2}$, where $\varepsilon_{g1}$ corresponds to the universal constant of the first/second critical point. Hence, the universal constant at the first critical point could be negative if $8Q^2/7 < a < 16Q^2/7$, which corresponds to unstable configurations of $a$ and $Q$. We also found that $\varepsilon_{g2}$ is closer to $\varepsilon_{vdW}$ than the $\varepsilon_{g1}$.

Furthermore, the heat capacity of the EHAdS BH is

$$C_P = T(g_{12}/T) = \frac{2\pi r}{\gamma Q^4 - 4Q^2r_{g1}^4 + 4r_{g1}^6 + 32Q^4r_{g1}^8}{-7\gamma Q^4 + 12Q^2r_{g1}^4 - 4r_{g1}^6 + 32Q^4r_{g1}^8}.$$  \hspace{1cm} (11)

Figure 3 illustrates $C_P$ as a function of $r_+$. We can observe that $C_P$ curves are discontinuity. A sign change for both critical points in the $C_P$ diagram, showing the thermodynamic instability of the EHAdS BH.

The $P - v$ and $T - S$ type Maxwell’s equal area law can be given by [15]

$$P_1(v_2 - v_1) = \int_{v_1}^{v_2} Pdv,$$  \hspace{1cm} (12)

$$T_1(S_2 - S_1) = \int_{S_1}^{S_2} TdS,$$  \hspace{1cm} (13)

where $P_1$ (or $T_1$) represents the pressure (or temperature) of isobar (or isotherm), $v$ is the specific volume, $S$ is the entropy, the subscripts 1 (or 2) stands the start (or end) phase of isobaric (or isothermal) process. According to Eqs. (5) and (12), we have

$$P_{g1} = \frac{T_g}{2r_{g1}} - \frac{1}{8\pi r_{g1}^2} + \frac{Q^2}{8\pi r_{g1}^4} - \frac{aQ^4}{32\pi r_{g1}^8},$$ \hspace{1cm} (14)

$$P_{g2} = \frac{T_g}{2r_{g2}} - \frac{1}{8\pi r_{g2}^2} + \frac{Q^2}{8\pi r_{g2}^4} - \frac{aQ^4}{32\pi r_{g2}^8},$$ \hspace{1cm} (15)

$$2P_{g1} = \frac{3T_g(1 + x_g)}{2r_{g2}(\sum_{i=0}^{7}x_i^g)} - \frac{3}{4\pi r_{g2}^4(\sum_{i=0}^{7}x_i^g)} + \frac{3Q^2}{4\pi r_{g2}^4x_g(\sum_{i=0}^{7}x_i^g)} - \frac{3aQ^4(\sum_{i=0}^{7}x_i^g)}{80\pi r_{g2}^4x_g(\sum_{i=0}^{7}x_i^g)},$$ \hspace{1cm} (16)

where $x_g \equiv r_{g1}/r_{g2}$ (0 < $x_g$ < 1). Based on Eqs. (14) and (15), one can get

$$T_g = \frac{1 + x_g}{4\pi r_{g2}^2x_g} + \frac{aQ^4(\sum_{i=0}^{7}x_i^g)}{16\pi r_{g2}^4x_g^3} - \frac{Q^2(\sum_{i=0}^{7}x_i^g)}{4\pi r_{g2}^4x_g^3}$$ \hspace{1cm} (17)

$$2P_{g1} = \frac{T_g(1 + x_g)}{2r_{g2}x_g} - \frac{1 + x_g^2}{8\pi r_{g2}^4x_g} + \frac{Q^2(1 + x_g^2)}{8\pi r_{g2}^4x_g^2} - \frac{aQ^4(1 + x_g^2)}{32\pi r_{g2}^8x_g^2}.$$ \hspace{1cm} (18)

Using the Eqs. (16)-(18), $r_{g2}$ is,

$$r_{g2}^2 = \frac{1 + 4x_g + x_g^2}{6x_g} B,$$ \hspace{1cm} (19)

where $B = 2Q^2[1 + 2\cos(arccos(1 - (27a)/(40Q^2))]/3 - 2k\pi/3]$ and $C = (5 + 20x_g + 29x_g^2 + 32x_g^3 + 29x_g^4 + 20x_g^5 + 5x_g^6)/9 + 4x_g + x_g^2)^3$. The phase transition temperature $T_g$ can be derived as

$$T_g = \frac{\sqrt{6}(1 + x_g)}{4\pi B^{1/2}(1 + 4x_g + x_g^2)^{1/2}} - \frac{3\sqrt{6}Q^2(1 + x_g)(1 + x_g^2)}{2\pi B^{3/2}(1 + 4x_g + x_g^2)^{3/2}} + \frac{27\sqrt{6}aQ^4(1 + x_g)(1 + x_g^2)}{4\pi B^{3/2}(1 + 4x_g + x_g^2)^{3/2}}.$$ \hspace{1cm} (20)

We also construct the $T - S$ type equal area law for the EHAdS BH by utilizing Eqs. (4) and (13). The phase transition pressure $P_g$ can be written as

$$P_g = \frac{3x_g}{4\pi B(1 + 4x_g + x_g^2)} - \frac{9Q^2x_g(1 + x_g + x_g^2)}{2\pi B^2(1 + 4x_g + x_g^2)^2} + \frac{81aQ^4x_g(\sum_{i=0}^{6}x_i^g)}{32\pi B^2(1 + 4x_g + x_g^2)^2}.$$ \hspace{1cm} (21)

We define a parameter $\chi_{n} = \frac{z_{n+1} - z_{n-1}}{z_{n+1} + z_{n-1}}$ (0 < $\chi$ < 1) to measure the temperature (pressure) of different phase transition branches, where $Z$ is the phase transition temperature $T_g$ (pressure $P_g$). The subscripts $c1$ and $c2$ denote the two critical points. Using Eqs. (19)-(21), the isobaric (isothermal) curves of the EHAdS BH on the $P - v$ ($T - S$) plane are shown in Figure 4. We can see that the length of the isothermal (isobaric) horizontal segment increases gradually with the increase of temperature (pressure) for the first phase transition branch, while its different from the second phase transition branch. Furthermore, we also found that the instability appears after the phase transition (the blue dash curves bulge) when only the first phase transition occurring. When only the second phase transition occurring, the instability appears before the phase transition (the red dash curves bulge). The two phase transition branches co-exist in certain ranges of the temperature and pressure, suggesting that the phase transition instability disappears and the reentrant phase transition can be occurred in this scenario. Figure 5 shows the region of the reentrant phase transition occurs. In the coexistence region of blue and red curves, the thermodynamics is stable. The $\chi_{P_g}$ (and $\chi_{T_g}$) is in the range of 0.65 ~ 0.85 (and 0.51 ~ 0.79).

III. The EHAdS BH With High-Order QED Correction and Thermodynamic Characteristics

A. The solution of the corrected EHAdS BH

Classical electrodynamics is modified for strong electromagnetic fields because of self-interaction of photons [27]. QED becomes nonlinear electrodynamics due to loop corrections and the effect of vacuum birefringence takes place. In Ref. [22], the main feature of the QED correction is to ensure the electric field and electrostatic
The action of the EHAdS BH is given by \[ I = \int d^4 \sqrt{-g} \left( \frac{1}{2\kappa^2} (R - 2\Lambda) - \mathcal{L}(\mathcal{F}, \mathcal{G}) \right), \] (22)

where \( \kappa^2 \equiv 8\pi G \), \( R \) is the Ricci scalar, \( \mathcal{L}(\mathcal{F}, \mathcal{G}) \) is the one-loop effective Lagrangian density of the EH nonlinear electrodynamics, which is \( \mathcal{L}(\mathcal{F}, \mathcal{G}) = -\mathcal{F} + \frac{2}{7} \mathcal{F}^2 + \frac{2}{3} \mathcal{G}^2 \).

\[ \mathcal{F} = \frac{1}{2} F_{\mu\nu} F^{\mu\nu} \quad \text{and} \quad \mathcal{G} = \frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu}, \] where \( F_{\mu\nu} \) is the electromagnetic field strength tensor, and its dual \( \tilde{F}^{\mu\nu} = \frac{1}{2\sqrt{g}} \epsilon_{\mu\nu\sigma\rho} F^{\sigma\rho} \). The field equations read as \[ \nabla_{\mu} P^{\mu\nu} = 0, \quad G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa^2 T_{\mu\nu}, \] (23)

where \( P^{\mu\nu} = (1 - a\mathcal{F}) F_{\mu\nu} - \mathcal{F}^{\mu\nu} \frac{\pi G}{4} \mathcal{G} \). Similarly, when only the electric field is considered \( (\mathcal{B}=0) \), the above equations can be rewritten as

\[ \partial_t \left( r^2 E \left( \frac{aE^2}{2} + 1 \right) \right) = 0, \] (24)

\[ G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa^2 T_{\mu\nu}. \] (25)

Assuming that the EH nonlinear electric field near BH is generated by the charge carried by the EHAdS BH, the Eq. (24) should be replaced with the active form of the EH nonlinear electric field, i.e., \( \partial_t \left( r^2 E \left( \frac{aE^2}{2} + 1 \right) \right) = 4\pi \rho \), where \( \rho \) is charge density of the EHAdS BH. This effect can also be achieved by imposing Coulomb’s law constraints on the EH nonlinear electric field \( \frac{\partial E}{\partial t} = \frac{Q}{\pi r^2} \). Hence, \( E(r) \) is

\[ E(r) = \frac{\sqrt{3}}{\sqrt{3}a} \sinh \left( \frac{1}{3} \ln \left( \frac{\sqrt{27}aQ}{\sqrt{8r^2}} + \sqrt{\frac{27aQ^2}{8r^4}} + 1 \right) \right). \] (26)

Utilizing the condition \( r \to \infty \), \( E(r) \) expansion form is

\[ E(r) = Q \left( \frac{1}{r^2} - \frac{aQ^3}{2r^6} + \frac{3a^2Q^5}{4r^{10}} - \frac{12a^3Q^7}{8r^{14}} \right) + O(r^{-18}). \] (27)

The \( (0,0) \) component of the energy-momentum tensor \( T_{\mu\nu} \) is

\[ T_{00} = \frac{E^2}{2} \left( 1 + \frac{3aE^2}{4} \right) = \frac{Q^2}{2r^4} - \frac{aQ^4}{8r^8} + \frac{a^2Q^6}{8r^{12}} - \frac{3a^3Q^8}{16r^{16}} + O(r^{-20}). \] (28)
which approximately equal to 1. Using the same method, Eq. (28) can be written as

\begin{equation}
\frac{dm}{dr} = \frac{Q^2}{2r^2} - \frac{aQ^4}{8r^6} + \frac{\Lambda r^2}{2} + \frac{\beta a^2 Q^6}{8r^{10}}.
\end{equation}

(29)

The metric potential (Eq. 2) of the EHAdS BH is obtained by integrating the above equation. Treating the third-order term and above as the high-order QED correction, Eq. (28) can be rewritten as

\begin{equation}
T_{00} = \frac{Q^2}{2r^4} - \frac{aQ^4}{8r^8} + \frac{\beta a^2 Q^6}{8r^{12}},
\end{equation}

(30)

where the parameter \( \beta \) replaces higher-order terms, which approximately equal to 1. Using the same method, Eq. (29) can be rewritten as

\begin{equation}
\frac{dm}{dr} = \frac{Q^2}{2r^2} - \frac{aQ^4}{8r^6} + \frac{\Lambda r^2}{2} + \frac{\beta a^2 Q^6}{8r^{10}}.
\end{equation}

(31)

Hence, the metric potential \( f(r) \) of the corrected EHAdS BH is

\begin{equation}
f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{aQ^4}{20r^6} - \frac{\Lambda r^2}{3} + \frac{\beta a^2 Q^6}{36r^{10}}.
\end{equation}

(32)

B. Thermodynamics of the EHAdS BH within the high-order QED correction framework

In this scenario, the BH mass is

\begin{equation}
M = \frac{r_+}{2} + \frac{Q^2}{2r_+} - \frac{aQ^4}{40r_+^4} - \frac{\Lambda r_+^2}{6} + \frac{\beta a^2 Q^6}{72r_+^9}.
\end{equation}

(33)

The temperature and equation of the state can be written as

\begin{equation}
T = \frac{1}{4\pi r_+} \left(1 - \frac{Q^2}{r_+^2} + \frac{aQ^4}{4r_+^4} - \frac{\Lambda r_+^2}{2} - \frac{\beta a^2 Q^6}{4r_+^{10}}\right),
\end{equation}

(34)

\begin{equation}
P = \frac{T}{2r_+} - \frac{1}{8\pi r_+^2} + \frac{Q^2}{8\pi r_+^4} - \frac{aQ^4}{32\pi r_+^8} + \frac{\beta a^2 Q^6}{32r_+^{12}}.
\end{equation}

(35)

Utilizing Eq. (35) and critical conditions, one can get

\begin{equation}
Q(r_+) \equiv r_+^6 - 6Q^2r_+^4 + 7aQ^4 - \frac{33a^2 Q^6 \beta}{2r_+^4} = 0.
\end{equation}

(36)

\( Q(r_+) \) as a function of \( r_+ \) is plotted in Figure 6. \( Q(r_+) \) function curve intersects the horizontal axis once for different \( \beta \) values, indicating that there is one phase transition of the corrected EHAdS BH.

According to the critical conditions, the critical thermodynamic quantities of the corrected EHAdS BH are obtained, i.e.

\begin{equation}
T_{Qc} = -\frac{3a^2 Q^6 \beta}{4\pi r_{Qc}^{11}} + \frac{aQ^4}{2\pi r_{Qc}^{7}} - \frac{Q^2}{\pi r_{Qc}^{3}} - \frac{1}{2\pi r_{Qc}},
\end{equation}

(37)

\begin{equation}
P_{Qc} = -\frac{11a^2 Q^6 \beta}{32\pi r_{Qc}^{12}} + \frac{7aQ^4}{32\pi r_{Qc}^{8}} - \frac{3Q^2}{8\pi r_{Qc}^{4}} + \frac{1}{8\pi r_{Qc}^{2}},
\end{equation}

(38)

\begin{equation}
r_{Qc}^2 = r_{Qc}^2 - A + (-1)^k \sqrt{33a^2 Q^6 \beta + A^2},
\end{equation}

(39)
FIG. 6. $Q(r_+)$ as a function of $r_+$ under different values of parameter $β$ for the EH parameter $a = 1$ and BH charge $Q = 0.6$. 

where $Q$ in the subscript stands for the EHAdS BH with high-order QED correction situation, $A \equiv 14aQ^4r_{gc}^2 - 24Q^2r_{gc}^6 + 5r_{gc}^8$, and $B \equiv 14aQ^4 - 72Q^2r_{gc}^4 + 20r_{gc}^6$. If $β \to 0$, Eqs. (37)–(39) degenerate into the EHAdS BH case (Eqs. (6-8)). In case of $β \to 1$, the $κ$ value should be zero, and $r_{Qc}$ is positive. The critical radius can be approximated by the limit of $a \to 0$,

$$\lim_{a \to 0} r_{Qc}^2 = \lim_{a \to 0} r_{Qc}^2,$$

$$= 4Q^2\cos\left(\frac{1}{3} \arccos\left(1 - \frac{7a}{16Q^2}\right)\right) + 2Q^2 = 6Q^2. (40)$$

In this limit, we have

$$v_{Qc} \approx 2\sqrt{6}Q, (41)$$

$$T_{Qc} \approx \frac{1}{3\sqrt{6}\pi Q} + \frac{a}{432\sqrt{6}\pi Q^3} - \frac{a^2β}{10368\sqrt{6}πQ^5}, (42)$$

$$P_{Qc} \approx \frac{1}{9\pi Q^2} + \frac{\frac{7a}{41472\pi Q^4}}{11a^2β} - \frac{11a^2β}{1492992πQ^6}. (43)$$

Therefore, the universal constant $ε_Q$ of the corrected EHAdS BH at $a \to 0$ is

$$ε_Q \approx \frac{3}{8}a^0 + \frac{1}{288Q^2}a^{1 - (2 + 13β)/8}944Q^4a^2 + O(a^3). (44)$$

On can see that the $ε_3 = 3/8$ for the zeroth-order term of $a$, which is same as the vdW system, the 1st-order term correction is 1/288, and the 2nd-order term correction is 13/82994 in comparison that without the high-order QED correction. For simplicity, we have omitted the 3rd-order and subsequent correction.

Furthermore, we calculate the critical exponents of the EHAdS BH with high-order QED correction situation. In the reduced parameter space, the Eq. (45) can be given as

$$p_Q = \frac{τ_Q v_Q}{ε_Q P_{Qc}^2} + \frac{1}{πP_{Qc}^2ε_Q^2} - \frac{1}{2 ν^2} + \frac{1}{12ν^2}$$

$$- \frac{a}{1728Q^2ν^2} + \frac{β a^2}{62208Q^4ν^12}, (45)$$

where

$$p_Q = \frac{P}{P_{Qc}}, \quad τ_Q = \frac{T}{T_{Qc}}, \quad ν_Q = \frac{v}{v_{Qc}}. (46)$$

Utilizing Eqs. (42)–(43), the Eq. (45) can be rewritten as

$$p_Q = \frac{τ_Q}{ε_Q P_{Qc}^2} - \frac{2}{ν^2} + \frac{1}{3ν^2} + h(ν_Q), (47)$$

where $h(ν_Q)$ is

$$h(ν_Q) = \frac{7ν_1^2(3 + 7ν_1^2 - 42ν_1^4)}{5832ν_1^2} + 12ν_1(36 + 132ν_1^2 - 792ν_1^4) a^2ν_1^3 - \frac{3 + 7ν_1^2 - 42ν_1^4}{1250ν_1^2} a^2 + O[a^3]. (48)$$

By introducing $t_Q$ and $ω_Q$ defined as

$$t_Q = τ_Q - 1, \quad ω_Q = (ν_Q - 1)^{1/3}, (49)$$

one can get the expansion of the reduced state equation near the critical point ($t_Q \to 0, ω_Q \to 0$),

$$p_Q(t_Q, ω_Q) \approx 1 + \frac{1}{ε_Q}(t_Q - \frac{1}{ε_Q})ω_Q t_Q$$

$$+ \left(\frac{4}{3} - \frac{1}{ε_Q} + \frac{h(3)(1)}{6}\right)ω_Q^2 + O(ω_Q^3 t_Q, ω_Q^4), (50)$$

where $h(3)(1)$ is given as

$$h(3)(1) = \frac{a}{Q^2} \left(\frac{83}{324} - \frac{581 + 3144β a}{139968} Q^2\right). (51)$$

With the equal area law, the volume of small BH ($ω_{Q1}$) and large BH ($ω_{Q2}$) satisfy

$$\int_{ω_{Q1}}^{ω_{Q2}} ω_Q dP_Q = 0, (52)$$

where $dP_Q = \left(-\frac{ω_Q}{ε_Q} + (4 - \frac{3}{ε_Q} + \frac{h(3)(1)}{6}ω_Q^3) dω_Q$. Thus, the above equation has a unique nontrivial solution, that is

$$ω_{Q1} = -ω_{Q2} = \sqrt{\left(-\frac{4}{3ε_Q} + \frac{1}{ε_Q} - \frac{h(3)(1)}{6ε_Q}\right)} - t, (53)$$

We known that the critical exponents of the AdS BH can be given by

$$C_v = T\left(\frac{∂S}{∂T}\right)_V \propto |t|^{-α}, \quad η = V_2 - V_1 \propto |t|^{-β}, (54)$$

$$κ_T = -\frac{1}{V}\left(\frac{∂V}{∂T}\right) \propto |t|^{-γ}, \quad P = P_c \propto |V - V_c|^{δ}. (55)$$
For the corrected EHAdS BH, the heat capacity at constant volume is $C_v = 0$, inferring that the first critical exponent as $\alpha = 0$. According to Eq. (53), we have

$$\eta = v_{Qc}(\omega_{Q1} - \omega_{Q2}) = 2v_{Qc}\omega_{Q1} \propto |t|^{1/2}. \quad (56)$$

The second critical exponent is $\beta = 1/2$. The third and fourth critical exponents are $\gamma = 1$ and $\delta = 3$ since

$$\kappa_T = -\frac{1}{V} \left( \frac{\partial V}{\partial P} \right) = -\frac{1}{P_{Qc}(\omega_{Q1} + 1) \frac{\partial Q}{\partial t}} \propto \omega_{Q1}, \quad (57)$$

$$|p_Q - 1| = \left( \frac{4}{3} \frac{1}{\omega_{Q1}} + \frac{h^{(3)}(1)}{6} \right) \omega_{Q1} \propto |Q_{13}^3|. \quad (58)$$

In this situation, the critical exponents ($\alpha$, $\beta$, $\gamma$, $\delta$) are $(0, 1/2, 1, 3)$, which are equivalent to the critical exponents of the vdW system [4].

The heat capacity is

$$C_P = 2\pi r^2 \left( \frac{a Q^4 + 4a^4}{24\pi r^2 + 4a^4} + 287 \right) + \alpha^2 Q^6. \quad (59)$$

Figure 7 reports the $C_P$ as a function of $r_+$. It is found that the sign of the $C_P$ does not change, meaning that the thermodynamic instability of the BH disappears.

![FIG. 7. $C_P$ as a function of $r_+$ of the corrected EHAdS BH. The red point is the critical point. We take EH parameter $a = 1$, BH charge $Q = 0.6$ and parameter $\beta = 1$.](image)

According to Eqs. (12) and (55), the $P - v$ type Maxwells equal area law in this scenario is constructed. We have

$$P_{Q1} = \frac{T_Q}{2r_{Q1}} - \frac{1}{8\pi r_{Q1}^2} + \frac{Q^2}{32\pi r_{Q1}^4} + \frac{\alpha Q^4}{32\pi r_{Q1}^6} + \frac{\beta a^2 Q^6}{32\pi r_{Q1}^8} \quad (60)$$

$$P_{Q2} = \frac{T_Q}{2r_{Q2}} - \frac{1}{8\pi r_{Q2}^2} + \frac{Q^2}{32\pi r_{Q2}^4} + \frac{\alpha Q^4}{32\pi r_{Q2}^6} + \frac{\beta a^2 Q^6}{32\pi r_{Q2}^8} \quad (61)$$

$$2P_{Q1} = \frac{37Q}{2r_{Q1}(1 + x_Q + x_Q^2)} - \frac{3Q^2}{4\pi r_{Q1}^2(1 + x_Q + x_Q^2)} + \frac{3a Q^4}{80\pi r_{Q1}^8 x_Q^2(1 + x_Q + x_Q^2)}$$

$$+ \frac{a^2 Q^6}{856\pi r_{Q1}^6(1 + x_Q)} + \frac{a^2 Q^6}{856\pi r_{Q1}^6(1 + x_Q)} + \frac{a^2 Q^6}{856\pi r_{Q1}^6(1 + x_Q)} \quad (62)$$

where $x_Q = r_{Q1}/r_{Q2}$ ($0 < x_Q < 1$). Based on Eqs. (60) and (61), one can get

$$T_Q = \frac{1 + x_Q}{4\pi r_{Q2} x_Q} - \frac{Q^2(\sum_{i=0}^{\infty} x_Q^i)}{4\pi r_{Q2}^3 x_Q} + \frac{a Q^4(\sum_{i=0}^{\infty} x_Q^i)}{16\pi r_{Q2}^3 x_Q}$$

$$+ \frac{\alpha^2 Q^6(\sum_{i=0}^{\infty} x_Q^i)}{16\pi r_{Q2}^3 x_Q} \quad (63)$$

and

$$2P_{Q1} = \frac{T_Q(1 + x_Q) - 1}{2r_{Q2} x_Q} - \frac{Q^2}{8\pi r_{Q2}^3 x_Q} + \frac{Q^2}{8\pi r_{Q2}^3 x_Q}$$

$$- \frac{a Q^4(1 + x_Q^2)}{32\pi r_{Q2}^8 x_Q^3} + \frac{a^2 Q^6(1 + x_Q^2)}{32\pi r_{Q2}^8 x_Q^3} \quad (64)$$

Using Eqs. (62)-(64), $r_{Q2}$ is

$$r_{Q2} = \frac{r_2}{\omega_{Q1}} - \frac{2Dr_{e_2}^2 - 4\epsilon_{e_2} r_{e_2}^2 - 5\omega_{Q1}}{2Dr_{e_2}^2 - 4\epsilon_{e_2} r_{e_2}^2 + 2\omega_{Q1} r_{e_2}^2} \quad (65)$$

$$+ \frac{(-1)^{k}}{8\pi r_{Q2}^3 x_Q} \frac{Q^2(\sum_{i=0}^{\infty} x_Q^i)}{8\pi r_{Q2}^3 x_Q} + \frac{a Q^4(\sum_{i=0}^{\infty} x_Q^i)}{32\pi r_{Q2}^8 x_Q^3}$$

$$- \frac{\alpha^2 Q^6(\sum_{i=0}^{\infty} x_Q^i)}{32\pi r_{Q2}^8 x_Q^3} \quad (66)$$

Using Eq. (63), (65), and (66), the isobaric (and isothermal) curves of the corrected EHAdS BH on the $P - v$ and $T - S$ planes are shown in Figure 8. One can observe that the length of isothermal (or isobaric) horizontal segment decreases gradually with the increase of temperature (or pressure). It is same as the vdW system and charged AdS BH, while the instability features of the EHAdS BH disappear.

On the other hand, the Ruppeiner geometry of the BH system can reveal the microstructure of BH phase transition. We analyze the microscopic phase transition behavior of the EHAdS BHs by investigating the Ruppeiner geometry. The normalized scalar curvature $R_N$ is given by [9]

$$R_N = \frac{(\partial_t^2)^2 - T^2(\partial_t^2)^2 + 2T^2(\partial_t^2)(\partial_{t,t})}{2(\partial_t^2)^2}. \quad (67)$$

According to Eqs. (44) and (47), the normalized scalar curvature of the EHAdS BH with high-order QED co-
The numerical results of the coexistence curves between the BHs space, we can use the parametrization form to fit the numerical data of the coexistence curves between the BHs. Figure 10 shows that the correction leads to the phase transition instability disappears for the EHAdS BH without high-order QED correction, indicating that the correction leads to the phase transition instability disappears for the EHAdS BH from the microscopic point of view.

Because of the vdW type phase transition shows a charge independent property in the reduced parameter space, we can use the parametrization form to fit the numerical data of the coexistence curves between the BHs. The parametrization form is

$$\nu_Q = \sum_{i=0}^{10} a_{Q_i} \tau_Q^i, \quad \tau_Q \in (0, 1).$$

The numerical results of the $a_{Q_i}$ are listed in Table 1, where $\nu_{Q_{10}}$ and $\nu_{Q_{20}}$ represents the BH charge in the large and small BH region. Figure 11 shows that $\nu_Q$ as a function of $\tau_Q$. It is found that the volume of the small BH increases with the increasing of temperature, while it is opposite for the large BH.

Figure 12 reports $R_N$ as a function of $\tau_Q$. We can see that the behaviour of the normalized scalar curvature $R_N$ along the coexistence saturated large BH and small BH curves meets the relationship $R_N(1-T)^2 \sim a^{-1}$. In the small BH region, $R_N$ has a sign change from positive to negative, implying that the dominant micro interaction force transits from repulsion to attraction in the small BH region. These features also appear in charged AdS BHs system [9].

IV. CONCLUSIONS

The effect of the high-order QED correction on the phase transition of the EHAdS BH has been revealed in this analysis. Without considering the high-order QED correction, two phase transition branches and thermodynamic instability are found in the EHAdS BH. By establishing its equal area law, we show that the two phase transition branches co-exist in certain ranges of the temperature and pressure, suggesting that the instability disappears and the reentrant phase transition can be occurred in this scenario.

Considering the high-order QED correction, we derive the corrected EHAdS BH solution. Only one phase transition branch is found in this scenario by using the critical condition and equation of the state. Its universal constant is $\varepsilon_Q = 3/8$ for the zeroth-order term of $a$ and the 2nd-order term correction is $13/82994$ in comparison that without the high-order QED correction. Its critical exponents $(\alpha, \beta, \gamma, \delta)$ are $(0, 1/2, 1, 3)$, which is equivalent to the vdW system. This implies that the phase transition of the high-order QED corrected EHAdS BH satisfies the Maxwell behavior. Meanwhile, the sign of the $C_P$ in this scenario does not change, suggesting that the thermodynamic instability disappears for the BH. We also constructed the Maxwells equal area law in this situation and found that it is same as the vdW system and charged AdS BH.

We further investigated the phase transition microstructure of the corrected EHAdS BH through the Ruppeiner geometry. It is found that the surface of the
FIG. 9. Illustration of the normalized scalar curvature $R_N$ as a function of $\nu$ and $\tau$ by setting the EH parameter $a = 1$ and the BH charge $Q = 0.6$. The left panel is the corrected EHAdS BH ($\beta = 1$) and the right panel is the EHAdS BH.

TABLE I. Values of the coefficients $\alpha_Qi$ in the fitting formula of the coexistence curves for $a = 1$, $Q = 0.6$ and $\beta = 1$.

| $\nu_Qi$ | $\alpha_Q0$ | $\alpha_Q1$ | $\alpha_Q2$ | $\alpha_Q3$ | $\alpha_Q4$ | $\alpha_Q5$ | $\alpha_Q6$ | $\alpha_Q7$ | $\alpha_Q8$ | $\alpha_Q9$ | $\alpha_Q10$ |
|----------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| -        | 90.66      | -1131.11   | 7625.30     | -31455.36   | 83299.04    | -143438.45  | 157364.01   | -102674.48  | 30636.87    | 1373.82     | -2289.27    |

FIG. 10. The coexistence curves of $\nu_Q$ and $\tau_Q$. The discrete points denote the numerical values and the red/blue line is the fitting formula Eq. [69]. We take $a = 1$, $Q = 0.6$ and $\beta = 1$.

normalized scalar curvature is sunken where the scalar curvature diverges. There is one sunken surface for the high-order QED correction situation, which is same as the vdW system. Compared with this scenario, two sunken surfaces occur in the EHAdS BH without high-order QED correction, indicating that the correction leads to the phase transition instability disappears for the EHAdS BH from the microscopic point of view. From the critical behavior of the normalized scalar curvature, it can be further found that the micro characteristics of the corrected EHAdS BH are no different from that of AdS charged. These results suggest that the first phase transition branch of the EHAdS BH is in metastable state, and its thermodynamics is unstable. The high-order QED correction eliminates this instability, and the phase transition of the EHAdS BH under this correction satisfies Maxwell behavior.

FIG. 11. $R_N$ along the coexistence saturated large BH (red curve) and small BH (blue curve). The green curve represents the $R_N = -\frac{1}{2}(1 - \tau_Q)^2$ as standard property of vdW type phase transition. We take $a = 1$, $Q = 0.6$ and $\beta = 1$. 
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