ON ERDŐS CHAINS IN THE PLANE

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Abstract. Let $P$ be a finite point set in $\mathbb{R}^2$ with one defines the graph-distance set $\Delta_G(P)$ as $\Delta_G(P) = \{(|p_i - p_j|)_{(i,j) \in E(G)} : p_i, p_j \in P\}$, where $G$ is a simple connected graph on $k = O(|P|)$ vertices. Then if $G$ has a Hamiltonian path we have

$$|\Delta_G(P)| \geq \frac{|P|^{k-1}}{\log^{\frac{1}{1+\frac{1}{k}}}|P|}.$$ 

This builds on work of Ioseivich and the author [15] which used the rigidity of the underlying graph and Rudnev [21] on the 2-chain (or hinge). The above result follows from a generalisation of Rudnev’s result to any chain of constant length. To prove this generalisation we use the Elekes-Sharir-Guth-Katz Framework to prove an iterative reduction to the result of Rudnev.

1. Introduction

Given a set $P$ in $\mathbb{R}^d$, ine define the distance set of $P$ as

$$\Delta_d(P) = \{|x - y| : x, y \in P\} \subseteq \mathbb{R}^d.$$ 

The famous distance conjecture of Erdős [7] asked what is the minimal number of distinct distances determined by a finite point set $P$ in $\mathbb{R}^d$? This was resolved in the plane by Guth and Katz [14] building upon the work of Elekes and Sharir [6]. This followed decades of work by, among others, Moser [17], Chung [4], Chung-Szemerédi-Trotter [5], Székely [24], Solymosi-Tóth [23], Tardos [25]. See the book of Garibaldi, Iosevich and Senger [12] for a more complete introduction.

Recently progress has been made by Iosevish and the author [15] and Rudnev [21] on a configuration based variant of Erdős’ conjecture. Suppose one has a graph $G$ with $k$ vertices, what is the minimum number of distinct-distance realisations when one takes the vertices from a set of $n$ elements and considers distances only along edges. When the graph concerned is the complete graph on two vertices we see that this is exactly the distinct-distance problem of Erdős, when the graph is a triangle the question asks for distinct congruence classes of triangles. To give the

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precise formulation, for a finite point set \( P \), we define the graphical-distance set of \( P \) as

\[
\Delta_G(P) = \{(|p_i - p_j|)_{i,j} \in E(G) : p_i, p_j \in P\}.
\]

Then one asks for a lower bound on the size of \( \Delta_G(P) \), as \(|P|\) grows. Using the integer lattice as an upper bound one conjectures for a connected graph \( G \) on \( k \) vertices that for all \( \varepsilon > 0 \) one can find a constant \( C_\varepsilon \) such that

\[
|\Delta_G(P)| \geq C_\varepsilon |P|^{k-1-\varepsilon}.
\] (1.1)

Configurations in the Euclidean setting were studied by Fürstenberg, Katznelson and Weiss [11] in the context of positive upper density. They expanded distance results in positive density sets due to Bourgain [2] and Falconer-Marstrand [9] to show that one can find triangles. This result was then greatly expanded by Ziegler [26] who showed one could find any simplex. Lyall and Magyar [16] recently provided a sharp extension of the result of Bourgain. Bennett, Iosevich and Taylor [1] building on earlier work of Chan, Laba and Pramanik [3] answered a version building on the Falconer conjecture (see [8]), showing that if one takes a set of sufficiently high Hausdorff dimension then if the graph is a chain of any length the graphical-distance set contains an open set. Using the improvements to the Falconer threshold in the plane due to Guth, Iosevich, Ou and Wang [13], Ou and Taylor [18] recently improved the threshold for chains.

In the Erdős setting János Pach asked how many similar triangles are defined by a set of \( n \) points in the plane. Solymosi and Tardos [22] found the tight bound that points determine at most \( O(|P|^4 \log(|P|)) \) similar triangles using bounds on \( k \)-rich complex transformations. One can quickly adapt this to bound the set of similar triangles by \( \Omega(|P|^2 / \log(|P|)) \). This bound was reproved by Rudnev [20], who also improved the bound on classes of congruent triangles to \( \Omega(|P|^2) \) using the framework established by Elekes-Sharir-Guth-Katz.

Iosevich and the author provided the first class of graphs for which (1.1) holds. We established that if \( G \) is a minimally infinitesimally rigid connected graph on \( k \) vertices then \( |\Delta_G(P)| \geq |P|^{k-1} \). Where the lack of logarithm in the bound is expected as rigid graphs necessary contain loops.

Iosevich and the author [15] also note that (1.1) quickly follows from the pinned Erdős conjecture. One can see this by noting that if \( T_G \) is a spanning tree for \( G \) then \( |\Delta_G(P)| \geq |\Delta_{T_G}(P)| \) and thus to prove (1.1) in generality it suffices to prove the conjecture for trees. Using the pinned version of the Erdős distance result gives
many rich pins and one can use these construct a sufficient number of trees to verify (1.1).

With this idea of trees in mind Iosevich and the author posed the question of whether one could verify (1.1) for the 2-chain or hinge, the simplest non-rigid structure. This was recently verified by Rudnev [21] who used a clever partitioning setup and a generalisation of the Guth-Katz incidence result due to Sharir and Solomon. In this paper we extended Rudnev’s result to all chains establishing that

$$|\Delta_{k-\text{chain}}(P)| \gtrsim \frac{|P|^{k-1}}{\log^{3(k+2)} |P|}.$$  \hspace{1cm} (1.2)

By the spanning-tree reduction we note that (1.2) establishes (1.1) for all polygons; chains of triangles; and most generally any graph with a Hamiltonian path. We note also that (1.2) doesn’t apply to all rigid graphs, see Figure 3.
most cases. Neither of these results are strong enough to quickly establish a result as strong as (1.2).

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2. **Statement of Results**

We prove the following

**Theorem 2.1.** Let $P$ be a finite point set in $\mathbb{R}^2$ with the set of distance $k$-chains defined as

$$ \Delta_k(P) = \{ (|p_1 - p_2|, |p_2 - p_3|, \ldots, |p_k - p_{k+1}|) : p_i \in P \}.$$ 

We show that for $k = O(|P|)$ we have

$$ |\Delta_k(P)| \geq \frac{|P|^k}{\log^{\frac{3(k+2)}{2^k}} |P|}.$$ 

As noted above by the spanning tree argument of Iosevich and the author [15] we have teh following corollary.

**Corollary 2.2.** Let $P$ be a point set in $\mathbb{R}^2$. Let $G$ is a connected simple graph with $k = O(|P|)$ vertices. If $G$ has a Hamiltonian path we have

$$ |\Delta_G(P)| \geq \frac{|P|^{k-1}}{\log^{\frac{3(k+1)}{2^k}} |P|}.$$ 

We note that Corollary 2.2 is sharp up to log factors for any graph $G$. We expect that one can improve the log factor in the general case to $\log^{k-1} |P|$ and even further when the graph contains cycles. For example we expect further improvement in the case of the 4-cycle to

$$ |\Delta_{C_4}(P)| \geq \frac{|P|^3}{\log^2 |P|},$$

an improvement here would be of extreme interest as we believe this a hard problem.
We note that the $l^2$ approach will fail to give a sharp bound for the 3-star. Indeed if one preforms the energy calculation below, see (3.1), then to obtain the sharp bound of $\Delta_{3\text{-star}}(P) \gtrsim |P|^{3-\alpha(1)}$ one would need $\mathbb{E}_{3\text{-star}}(P) \lesssim |P|^5$. However, if one considers the example of a point set with $n$ points on each of three concentric circles and a point at their centre we have $n^3$ realisations of the three star rooted at the centre. Choosing a pair of such centre rooted 3-stars gives a member of the energy thus $|\mathbb{E}_{3\text{-star}}(P)| \gtrsim |P|^6$.

3. Proof of Theorem 2.1

To prove Theorem 2.1 use the Elekes-Sharir–Guth-Katz framework to reduce long chains to the result of Rudnev from [21].

Recall the setup of Elekes which allows us to count distances via counting the energy. Then we will use the ideas of Elekes-Sharir and Guth-Katz to count this energy via an incidence problem in $\mathbb{R}^3$.

First let $\nu(z) = \{(p_1, p_2, \ldots, p_{k+1}) \in P^{k+1} : |p_i - p_{i+1}| = z_i \text{ for } i = 1, \ldots, k\}$ be the number of times the $k$-chain with distances $t = (z_1, z_2, \ldots, z_k)$ arises. Then we can count the number of $k$-chains using

$$\left(|P|^{k+1}\right)^2 = \left(\sum_{z \in \Delta_k(P)} \nu(z)\right)^2 \leq |\Delta_k(P)| \sum_{z} \nu^2(z). \quad (3.1)$$

We note that this final sum gives the size of the following energy set,

$$\mathbb{E}_k(P) = \{(p_1, \ldots, p_{k+1}, p'_1, \ldots, p'_{k+1}) : |p_i - p_{i+1}| = |p'_i - p'_{i+1}| \text{ for } i = 1, \ldots, k\}.$$ 

So we aim to bound $|\mathbb{E}_k(P)|$ by $|P|^{k+2} \log \frac{3k+7}{2} |P|$.

These energies can be thought of as configurations of Guth-Katz lines in $\mathbb{R}^3$. Recall the for each pair of points the Guth-Katz line $l_{pq}$ represents all rotations taking $p$ to $q$. An intersection $l_{pq} \cap l_{p'q'}$ represents a rotation that takes $p$ to $q$ and $p'$ to $q'$ simultaneously and hence maps the distance $|p - p'|$ to $|q - q'|$.

So each entry in $\mathbb{E}_k(P)$ corresponds to the $k + 1$ lines $l_{p_1p'_1}, \ldots, l_{p_{k+1}p'_{k+1}}$ having the intersections
as in Figures 4 and Figure 5 for the 3-chain and 4-chain respectively.

We separate our approach depending on whether $k$ is odd or even. For $k$ odd we have an even number of lines in the configuration in $\mathbb{R}^3$ and thus we have an off central line, see $l$ in Figure 4. For $k$ even we have an odd number of lines and thus we have a central line, see $l$ in Figure 5.

We define iterative line weights on this critical line $l$ by

$$
\nu_1(l) = \sum_{v \in \mathcal{L}} \delta_{l,v} \quad \nu_{i+1}(l) = \sum_{v \in \mathcal{L}} \nu_i(l') \delta_{l,v}.
$$

So $\nu_i(l)$ counts the number of distinct $i$-chains of lines that originate from $l$. By picking a central line $l$ (see Figure 4 and 5) we can see that the energy $E_k$ is given by
We then proceed by Cauchy-Schwarz and induction. We note the following result of Rudnev [21].

**Theorem 3.1.** (Rudnev [21, Theorem 1]) For any point set $P$ in $\mathbb{R}^2$ we have that $|E_2(P)| \lesssim |P|^2 \log^3 |P|$.

Suppose that $k$ is even, then

$$|E_k(P)| = \sum_l \nu_{k/2}(l) \nu_{k/2}(l') \delta_{l,l'}$$

$$\leq \left( \sum_l \nu_{k/2}(l)^2 \right)^{1/2} \left( \sum_{l',l''} \nu_{k/2-1}(l') \nu_{k/2-1}(l'') \delta_{l,l'} \delta_{l''} \right)^{1/2}$$

$$\leq \left( \sum_l \nu_{k/2}(l)^2 \right)^{1/2} \left( \sum_{l',l''} \nu_{k/2-2}(l') \right)^{1/2} \left( \sum_{l,l',l''} \delta_{l,l'} \delta_{l,l''} \right)^{1/4}.$$ 

We observe that the final sum here is the number of pairs of lines that intersect a line $l$ and thus gives exactly $|E_2(P)|$. So rearranging and stating the above in terms of energies gives

$$|E_k(P)| \leq |E_{k-2}(P)||E_2(P)|^{1/2}$$

Thus by induction when $k$ is even we have

$$|E_k(P)| \lesssim |P|^{k+2} \log^{\frac{3k+6}{k}} |P|.$$ 

Suppose that $k$ is odd, we have
\[ |\mathbb{E}_k(P)| = \sum_l \nu_{(k-1)/2}(l) \sum_{l'} \nu_{(k-1)/2}(l') \delta_{l,l'} \]
\[ \leq \left( \sum_l \nu_{(k-1)/2}(l) \right)^{1/2} \left( \sum_l \sum_{l'',l'} \nu_{(k-1)/2}(l') \nu_{(k-1)/2}(l'') \delta_{l,l'} \delta_{l'',l''} \right)^{1/2} \]
\[ \leq \left( \sum_l \nu_{(k-1)/2}(l) \right)^{1/2} \left( \sum_{l'} \nu_{(k-1)/2}(l') \right)^{1/2} \left( \sum_{l,l',l''} \delta_{l,l'} \delta_{l'',l''} \right)^{1/4}. \]

Thus when \( k \) is odd we have

\[ |\mathbb{E}_k(P)| \leq |\mathbb{E}_{k-1}(P)||\mathbb{E}_2(P)|^{1/4}. \]

Thus by the above result for \( k \) even and Theorem 3.1 we have

\[ |\mathbb{E}_k(P)| \lesssim |P|^{k+2} \log^{\frac{4k+6}{4}} |P|. \]

Combining with the above we have \( |\mathbb{E}_k(P)| \lesssim |P|^{k+2} \log^{\frac{4k+6}{4}} |P|. \)

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