Spectral gaps for the linear surface wave model in periodic channels

F.L. Bakharev, K. Ruotsalainen, J. Taskinen

May 6, 2014

Abstract

We consider the linear water-wave problem in a periodic channel which consists of infinitely many identical containers connected with apertures of width $\epsilon$. Motivated by applications to surface wave propagation phenomena, we study the band-gap structure of the continuous spectrum. We show that for small apertures there exists a large number of gaps and also find asymptotic formulas for the position of the gaps as $\epsilon \to 0$: the endpoints are determined within corrections of order $\epsilon^{3/2}$. The width of the first bands is shown to be $O(\epsilon)$. Finally, we give a sufficient condition which guarantees that the spectral bands do not degenerate into eigenvalues of infinite multiplicity.

1 Introduction

1.1 Overview of the results

Research on wave propagation phenomena in periodic media has been very active during many decades. The topics and applications include for example photonic crystals, meta-materials, Bragg gratings of surface plasmon polariton waveguides, energy harvesting in piezoelectric materials as well as surface wave propagation in periodic channels, which is the subject of this paper. A standard...
mathematical approach consists of linearisation and posing a spectral problem for an elliptic, hopefully self-adjoint, equation or system.

Early on it was noticed that waves propagating in periodic media have spectra with allowed bands separated by forbidden frequency gaps. This phenomenon was first discussed by Lord Rayleigh [22]. It has also attracted some interest in coastal engineering because it provides a possible means of protection against wave damages [11, 14], for example by varying the bottom topography by periodic arrangements of sandbars. The existence of forbidden frequencies is conventionally related to Bragg reflection of water waves by periodic structures. Here, Bragg reflection is an enhanced reflection which occurs when the wavelength of an incident surface wave is approximately twice the wavelength of the periodic structure. This mechanism works, if the waves are relatively long so that the depth changes can effect them [14].

A similar phenomenon may also happen, when waves are propagating along a channel with periodically varying width. In [9], and later [13], the authors studied a channel, the wall of which had a periodic stepped structure. Using resonant interaction theory they were able to verify that significant wave reflection could occur. These results are based on the assumption of small wall irregularities.

Gaps in the continuous spectrum for equations or systems in unbounded waveguides have been studied in many papers, and we refer to [7] for an introduction to the topic. In [19] the authors studied the linear elasticity system and proved the existence of arbitrarily (though still finitely) many gaps, the number of them depending on a small geometric parameter; the approach is similar to Section 3.1, below, and the result is analogous to Corollary 3.2. In the setting of the linear water-wave problem, spectral gaps have been studied in [8], [12], [17] and [4], though the point of view is different from the present work.

In this paper we consider surface wave propagation using the linear water wave equation with spectral Steklov boundary condition on the free water surface, see the equations (1.8)–(1.10), which are called the original problem here. The water-filled domain \( \Pi^\varepsilon \) forms an unbounded periodic channel consisting of infinitely many identical bounded containers connected by apertures of width \( \varepsilon > 0 \), see Figure 1.1. The first results, Theorem 3.1 and Corollary 3.2, show that the essential spectrum \( \sigma \) of the original problem (which is expected to be nonempty due to the unboundedness of the domain) has gaps, and the number of them can be made arbitrarily large depending on the parameter \( \varepsilon \). An explanation of this phenomenon can be outlined rather simply using the Floquet-Bloch theory, though a lot of technicalities will eventually be involved. Namely, if \( \varepsilon = 0 \), the domain becomes a disjoint union of infinitely many bounded containers, and the water-wave problem reduces to a problem on a bounded domain (we call it the limit problem), hence it has a discrete spectrum consisting of an increasing sequence of eigenvalues \((\Lambda_k^0)_{k=1}^\infty\). On the other hand, for \( \varepsilon > 0 \), one can use the Gelfand transform to render the original problem into another bounded domain problem depending on the additional parameter \( \eta \in [0, 2\pi] \). For each fixed \( \eta \) this problem again has a sequence of eigenvalues \((\Lambda_k^\varepsilon(\eta))_{k=1}^\infty\). Moreover, by results of [15, 18, Theorem 3.4.6, and [16, Theorem 2.1, the

2
essential spectrum $\sigma$ of the problem (1.8)–(1.10) equals

$$\sigma = \bigcup_{k=1}^{\infty} \Upsilon'_k, \quad \Upsilon'_k = \{ \Lambda'_k(\eta) : \eta \in [0, 2\pi) \},$$  \hspace{1cm} (1.1)

where the sets $\Upsilon'_k$ are subintervals of the positive real axis, or bands of the spectrum. (For the use of this so called Bloch spectrum in other problems, see for example [?], or [2].) In general, those bands may overlap making $\sigma$ connected, but in Theorem 3.1 we obtain asymptotic estimates for the lower and upper endpoints of $\Upsilon'_k$: we show that $\Lambda'_0 \leq \Lambda'_k(\eta) \leq \Lambda'_0 + C_k \epsilon$ for all $k$ and $\eta$ and for some constants $C_k > 0$. In view of (1.1) this implies the existence of a spectral gap between $\Upsilon'_k$ and $\Upsilon'_{k+1}$ for small $\epsilon$ and $k$ such that $\Lambda'_k \neq \Lambda'_{k+1}$. However, since the estimates depend also on $k$, we can only open a gap for finitely many $k$, though the number of gaps tends to infinity as $\epsilon \to 0$.

The asymptotic position (as $\epsilon \to 0$) of the gaps is determined more accurately in Theorems 3.5 and 3.6: those main results state that

$$\Upsilon'_k = (\Lambda'_k + A_k \epsilon + O(\epsilon^{3/2}), \Lambda'_0 + B_k \epsilon + O(\epsilon^{3/2}))$$

where the numbers $A_k \leq B_k$ depend linearly on the three dimensional capacity of the set $\theta$. This result also ensures that in case $A_k \neq B_k$ the bands $\Upsilon'_k$ do not degenerate into single points, which means that the spectrum of the original problem indeed has a genuine band-gap structure. Facts concerning the numbers $A_k$, $B_k$ are discussed after Theorem 3.6.

As for the structure of this paper, we recall in Section 1.2 the exact formulation of the linear water-wave problem, its variational formulation as well as the parameter dependent problem arising from the Gelfand transform, and the limit problem. Section 2 contains the formal asymptotic analysis which relates the spectral properties of the original problem with the limit problem and which is rigorously justified in Section 3. The main results, Theorems 3.1, 3.5 and 3.6 as well as Corollary 3.2 are also given in Section 3. The proofs are based on the max-min principle and construction of suitable test functions adjusted to the geometric characteristics of the domains under study.

Acknowledgement. The authors want to thank Prof. Sergey A. Nazarov for many discussions on the topic of this work.

1.2 Formulation of the problem, operator theoretic tools

Let us proceed with the exact formulation of the problem. We consider an infinite periodic channel $\Pi'$ (see (1.7)), consisting of water containers connected by small apertures of diameter $O(\epsilon)$. The coordinates of the points in the channel are denoted by $x = (x_1, x_2, x_3) = (y_1, y_2, z) = (y, z)$, and $x' = (x_2, x_3)$ stands for the projection of $x$ to the plane $\{x_1 = 0\}$. We choose the coordinate system in such a way that the axis of the channel is in $x_1$-direction and the free surface is in the plane $\{x_3 = 0\}$.

**Definition 1.1.** We describe the geometric assumptions on the periodicity cell in detail, as well as some related technical tools including the cut-off functions.
Let us denote by $\omega_0 \subset \mathbb{R}^3$ a domain with a Lipschitz boundary and compact closure such that its intersections with $\{x_1 = 0\}$- and $\{x_1 = 1\}$-planes are simply connected planar domains with positive area and contain the points $P^0 = (0, P_2, P_3)$ and $P^1 = (1, P_2, P_3)$ with $P_3 < 0$, respectively; these points are fixed throughout the paper. Then the periodicity cell and its translates are defined by setting (see Figure 1.1)

$$\omega = \{x \in \omega_0 : x_3 < 0, x_1 \in (0, 1)\}, \quad \omega_j = \{x : (x_1 - j, x_2, x_3) \in \omega\}, \quad j \in \mathbb{Z}. \tag{1.2}$$

Furthermore, we assume that the set $\theta \subset \mathbb{R}^2$ is a bounded planar domain containing the origin $(0, 0)$ and that the boundary $\partial \theta$ is at least $C^2$-smooth. We assume that $\theta$ is so small that the set $\{0\} \times (2\theta + (P_2, P_3))$ is contained in $\partial \omega$ and $\sup_{(x_2, x_3) \in \theta} (x_3 + P_3) =: d_\theta < 0$. We define the apertures between the container walls as the sets

$$\theta_j^\epsilon = \{x = (j, x') : \epsilon^{-1}(x' - (P_2, P_3)) \in \theta\}, \quad j \in \mathbb{Z}. \tag{1.3}$$

It is plain that $x_3 < 0$ for $x \in \theta_j^\epsilon$ for all $0 < \epsilon \leq 1$, by the choice of $d_\theta$. We shall need at several places a cut-off function

$$\chi_\theta \in C_0^\infty(\mathbb{R}^3), \tag{1.4}$$

which is equal to one in a neighbourhood of the set $\{0\} \times \overline{\theta}$ and vanishes outside another compact neighbourhood of $\{0\} \times \overline{\theta}$. More precisely, we require that

$$\{\text{supp} (\chi_\theta) + (0, P_2, P_3)\} \cap \{x_1 = 0\} \subset \partial \omega,$n

$$\{\text{supp} (\chi_\theta) + (0, P_2, P_3)\} \cap \{x_1 > 0\} \subset \omega \tag{1.5}$$

(this is possible by the specifications made on $\theta$) and $\chi_\theta(x)$ vanishes, if $|x_1| \geq 1/4$ or $x_3 + P_3 \geq d_\theta/2$. We also assume that $\partial_{x_1} \chi_\theta = 0$, when $x_1 = 0$. Furthermore,
denoting \( \chi_j(x) = \chi_\theta(x - P^j) \), it follows from the above specifications that \( \chi_j(x) = 0 \) if \( x_3 \geq d_\theta/2 \); in particular \( \chi_j \) vanishes on the free water surface \( \gamma \).

Finally, we shall need the scaled cut-off functions

\[
X^\varepsilon_j = \chi_\theta(\varepsilon^{-1}(x - P^j)).
\]

(1.6)

It is plain that also \( X^\varepsilon_j \) vanishes on \( \gamma \) for \( 0 < \varepsilon \leq 1 \) and that \( X^\varepsilon_j(x) = 1 \) for \( x \in \theta_j^\varepsilon, j = 0, 1 \).

**Definition 1.2.** The periodic water channel is defined by

\[
\Pi^\varepsilon = \bigcup_{j \in \mathbb{Z}} (\sigma_j \cup \theta_j^\varepsilon),
\]

(1.7)

and it will be the main object of our investigation. The free surface of the channel is denoted by \( \Gamma^\varepsilon = \partial \Pi^\varepsilon \cap \{x_3 = 0\} \), and the wall and bottom part of the boundary is \( \Sigma^\varepsilon = \partial \Pi^\varepsilon \setminus \Gamma^\varepsilon \). The boundary of the isolated container \( \varpi \), the periodicity cell, consists of the free surface \( \gamma \) and the wall and bottom \( \sigma^\varepsilon \) with two apertures \( \theta^\varepsilon_0 \) and \( \theta^\varepsilon_1 \).

**Remark 1.3.** We shall use the following general notation. Given a domain \( \Xi \), the symbol \((\cdot,\cdot)_\Xi\) stands for the natural scalar product in \( L^2(\Xi) \), and \( H^k(\Xi) \), \( k \in \mathbb{N} \), for the standard Sobolev space of order \( k \) on \( \Xi \). The norm of a function \( f \) belonging to a Banach function space \( X \) is denoted by \( \|f;X\| \). For \( r > 0 \) and \( a \in \mathbb{R}^N \), \( B_r(a) \) (respectively, \( S_r(a) \)) stands for the Euclidean ball (resp. ball surface) with centre \( a \) and radius \( r \). By \( C, c \) (respectively, \( C_k, c_k, C(k) \) etc.) we mean positive constants (resp. constants depending on a parameter \( k \)) which do not depend on functions or variables appearing in the inequalities, but which may still vary from place to place. The gradient and Laplace operators \( \nabla \) and \( \Delta \) act in variable \( x \), unless otherwise indicated.

In the framework of the linear water-wave theory we consider the spectral Steklov problem in the channel \( \Pi^\varepsilon \),

\[
-\Delta u^\varepsilon(x) = 0 \quad \text{for all } x \in \Pi^\varepsilon, \quad (1.8)
\]

\[
\partial_n u^\varepsilon(x) = 0 \quad \text{for a.e. } x \in \Sigma^\varepsilon, \quad (1.9)
\]

\[
\partial_z u^\varepsilon(x) = \lambda^\varepsilon u^\varepsilon(x) \quad \text{for a.e. } x \in \Gamma^\varepsilon. \quad (1.10)
\]

Here \( u^\varepsilon \) is the velocity potential, \( \lambda^\varepsilon = g^{-1}\omega^2 \) is a spectral parameter related to the frequency of harmonic oscillations \( \omega > 0 \) and the acceleration of gravity \( g \).

By the geometric assumptions made above, the outward normal derivative \( \partial_n \) is defined almost everywhere on \( \Sigma^\varepsilon \). It coincides with \( \partial_z = \partial/\partial_z \) on the free surface \( \Gamma^\varepsilon \).

The rest of this section is devoted to presenting the operator theoretic tools which will be needed later to prove our results: Gelfand transform, variational formulation of the boundary value problems, and max-min-formulas for eigenvalues. The spectral problem (1.8)–(1.10) can be transformed into a family of
spectral problems in the periodicity cell using the Gelfand transform. We briefly recall its definition:

\[ v(y, z) \mapsto V(y, z, \eta) = \frac{1}{\sqrt{2\pi}} \sum_{j \in \mathbb{Z}} \exp(-i\eta(z + j))v(y, z + j), \tag{1.11} \]

where \((y, z) \in \Pi^c\) on the left while \(\eta \in [0, 2\pi)\) and \((y, z) \in \varpi\) on the right.

As is well known, the Gelfand transform establishes an isometric isomorphism between the Lebesgue spaces,

\[ L^2(\Pi^c) \cong L^2(0, 2\pi; L^2(\varpi)), \]

where \(L^2(0, 2\pi; B)\) is the Lebesgue space of functions with values in the Banach space \(B\) endowed with the norm

\[ \|V; L^2(0, 2\pi; B)\| = \left( \int_0^{2\pi} \|V(\eta); B\|^2 \, d\eta \right)^{1/2}. \]

The Gelfand transform is also an isomorphism from the Sobolev space \(H^l(\Pi^c)\) onto \(L^2(0, 2\pi; H^l_{c,\eta}(\varpi))\) for \(l = 1, 2\). The space \(H^{2}_{c,\eta}(\varpi)\) consists of Sobolev functions \(u\) which satisfy the quasi-periodicity conditions

\[ u(0, x') = e^{-i\eta}u(1, x'), \quad (0, x') \in \theta^c_0, \tag{1.12} \]

\[ \partial_x u(0, x') = e^{-i\eta}\partial_x u(1, x'), \quad (0, x') \in \theta^c_0, \tag{1.13} \]

whereas \(H^{1}_{c,\eta}(\varpi)\) is the Sobolev space with the condition (1.12) only.

Applying the Gelfand transform to the differential equation (1.8) and to the boundary conditions (1.9)–(1.10), we obtain a family of model problems in the periodicity cell \(\varpi\) parametrized by the dual variable \(\eta\),

\[ -\Delta U^c(x; \eta) = 0, \quad x \in \varpi, \tag{1.14} \]

\[ \partial_n U^c(x; \eta) = 0, \quad x \in \sigma^c, \tag{1.15} \]

\[ \partial_z U^c(x; \eta) = \Lambda^c(\eta)U^c(x; \eta), \quad x \in \gamma, \tag{1.16} \]

\[ U^c(0, x'; \eta) = e^{-i\eta}U^c(1, x'; \eta), \quad x \in \theta^c_0, \tag{1.17} \]

\[ \partial_x U^c(0, x'; \eta) = e^{-i\eta}\partial_x U^c(1, x'; \eta), \quad x \in \theta^c_0. \tag{1.18} \]

Here, \(\Lambda^c = \Lambda^c(\eta)\) is a new notation for the spectral parameter \(\lambda^c\). More details on the use of the Gelfand-transform can be found e.g. in [19], Section 2.

The apertures disappear at \(\epsilon = 0\) so in that case the also quasi-periodicity conditions cease to exist. Hence, we can consider the problem (1.14)–(1.18) as a singular perturbation of the limit spectral problem

\[ -\Delta U^0(x) = 0, \quad x \in \varpi, \tag{1.19} \]

\[ \partial_n U^0(x) = 0, \quad x \in \sigma, \tag{1.20} \]

\[ \partial_z U^0(x) = \Lambda^0 U^0(x), \quad x \in \gamma \tag{1.21} \]

with \(\Lambda^0\) as a spectral parameter.
Our approach to the spectral properties of model and limit problems is similar to [20], Sections 1.2, 1.3. We first write the variational form of the problem (1.14)–(1.18) for the unknown function $U \in H^1_{\varepsilon,\eta}(\varpi)$ as

$$
(\nabla U, \nabla V)_\varpi = \Lambda(\varpi) U, V \in H^1_{\varepsilon,\eta}(\varpi),
$$

(1.22)
and the corresponding variational formulation of the limit problem for $U \in H^1(\varpi)$ reads as

$$
(\nabla U, \nabla V)_\varpi = \Lambda(U, V) \gamma, V \in H^1(\varpi).
$$

(1.23)
We denote by $H^1_{\varepsilon}$ the space $H^1_{\varepsilon,\eta}(\varpi)$ endowed with the new scalar product

$$
(u, v)_\varepsilon = (\nabla u, \nabla v)_\varpi + (u, v)_{\gamma},
$$

(1.24)
and define a self-adjoint, positive and compact operator $B_{\varepsilon}(\eta) : H^1_{\varepsilon} \rightarrow H^1_{\varepsilon}$ using

$$
(B_{\varepsilon}(\eta)u, v)_\varepsilon = (u, v)_{\gamma}.
$$

(1.25)
The problem (1.22) is then equivalent to the standard spectral problem

$$
B_{\varepsilon}(\eta)u = M_{\varepsilon}u
$$
with another spectral parameter

$$
M_{\varepsilon} = (1 + \Lambda_{\varepsilon})^{-1}.
$$

(1.26)
Clearly, the spectrum of $B_{\varepsilon}(\eta)$ consist of 0 and a decreasing sequence $(M_{\varepsilon_k}(\eta))_{k=1}^\infty$ of eigenvalues, which moreover can be calculated from the usual min-max formula

$$
M_{\varepsilon_k}(\eta) = \min_{E_k} \max_{v \in E_k} \frac{(B_{\varepsilon}(\eta)v, v)_\varepsilon}{(v, v)_{\gamma}},
$$

(1.27)
where the minimum is taken over all subspaces $E_k \subset H^1_{\varepsilon}$ of co-dimension $k - 1$. Using (1.24) and (1.25), we can write a max-min formula for the eigenvalues of the problem (1.22):

$$
\Lambda_{\varepsilon_k}(\eta) = \frac{1}{M_{\varepsilon_k}(\eta)} - 1 = \max_{E_k} \min_{v \in E_k} \frac{(\nabla v, \nabla v)_\varpi + (v, v)_{\gamma}}{(v, v)_{\gamma}} - 1
$$

= \max_{E_k} \min_{v \in E_k} \frac{||\nabla v; L^2(\varpi)||^2}{||v; L^2(\gamma)||^2}
$$

(1.28)
On the other hand, the connection (1.27) and the properties of the sequence $(M_{\varepsilon_k}(\eta))_{k=1}^\infty$ mean that the eigenvalues (1.29) form an unbounded sequence

$$
0 \leq \Lambda_1(\eta) \leq \Lambda_2(\eta) \leq \ldots \leq \Lambda_k(\eta) \leq \ldots \rightarrow +\infty.
$$

(1.30)
The eigenfunctions can be assumed to form an orthonormal basis in the space $L^2(\varpi)$. The functions $\eta \mapsto \Lambda_{\varepsilon_k}(\eta)$ are continuous and $2\pi$-periodic (see for example [5], Ch. 9). Hence the sets

$$
\Upsilon_{\varepsilon_k} = \{\Lambda_{\varepsilon_k}(\eta) : \eta \in [0, 2\pi)\}
$$

(1.31)
are closed connected segments, which may degenerate into single points; their relation to the original problem was already mentioned in (1.1).

The spectral concepts of the limit problem (1.19)–(1.21) can be treated in the same way as in (1.24)–(1.30). Since the quasi-periodicity conditions vanish for \( \varepsilon = 0 \), the space \( \mathcal{H} \) is replaced by \( H^1(\varpi) \); the norm induced by (1.24) is now equivalent to the original Sobolev norm of \( H^1(\varpi) \). We denote by \( B : H^1(\varpi) \to H^1(\varpi) \) the operator defined as in (1.24)–(1.25). The limit problem has an eigenvalue sequence \( (\Lambda_k^0)_{k=1}^\infty \) like (1.30), however, neither the eigenvalues nor the operator \( B \) depend on \( \eta \) (cf. [19], Section 3). The first eigenvalue \( \Lambda_1^0 \) equals 0, and the first eigenfunction is the constant function. Analogously to (1.32) we can write

\[
\Lambda_k^0 = \max_{U \in F_k} \min_{v \in F} \frac{\|\nabla v; L^2(\varpi)\|^2}{\|v; L^2(\gamma)\|^2},
\]

where again \( F_k \subset H^1(\varpi) \) is running over all subspaces of codimension \( k - 1 \). We denote by

\[
(U_k^0)_{k=1}^\infty
\]

an \( L^2(\gamma) \)-orthonormal sequence of eigenfunctions corresponding to the eigenvalues (1.32).

**Lemma 1.4.** For all \( k \) there exists a constant \( C_k > 0 \) such that

\[
|U_k^0(x)| \leq C_k, \quad |\nabla U_k^0(x)| \leq C_k
\]

for all \( x \in \text{supp}(\chi_j) \cap \varpi, j = 0, 1 \) (and hence for all \( x \in \text{supp}(X_j') \cap \varpi, 0 < \varepsilon \leq 1 \)).

**Proof.** Let for example \( j = 0 \) (the other case is treated similarly), and define the domains \( G_1, G_2 \subset \mathbb{R}^3 \) with \( C^\infty \) boundary such that \( G_0 := \text{supp}(\chi_j) \subset G_1 \subset G_1 \subset G_2 \subset \{x_3 < 0\} \) and \( G_2 \) still so small that

\[
G_2 \cap \{x_1 = 0\} \subset \partial \varpi \quad \text{and} \quad \overline{G_2} \cap \{x_1 > 0\} \subset \varpi.
\]

As a consequence, these domains are smooth enough so that we can use the local elliptic estimates [H, Theorem 15.2], to the solutions \( U_k^0 \) of the equation (1.19): this yields for every \( l = 1, 2, \ldots, \) a constant \( C_{l,k} > 0 \) such that

\[
\|U_k^0; H^{l+1}(G_n \cap \varpi)\| \leq C_{l,k}(\|U_k^0; H^{l-1}(G_{n+1} \cap \varpi)\| + \|U_k^0; L^2(G_{n+1} \cap \varpi)\|)
\]

for \( n = 0, 1 \). Applying this first with \( n = 1 \) and \( l = 1 \) we get a bound for \( \|U_k^0; H^2(G_1 \cap \varpi)\| \) and then, with \( n = 0 \) and \( l = 2 \), for \( \|U_k^0; H^3(G_0 \cap \varpi)\| \). The standard embeddings \( H^2(G_1 \cap \varpi) \subset C_B(G_0 \cap \varpi) \) and \( H^3(G_0 \cap \varpi) \subset C_B^1(G_0 \cap \varpi) \) imply the result. \( \square \)

## 2 The formal asymptotic procedure

### 2.1 The case of a simple eigenvalue

To describe the asymptotic behaviour (as \( \varepsilon \to 0 \)) of the eigenvalues \( \Lambda_k^\varepsilon(\eta) \) of the problem (1.14)–(1.18) we consider first the case \( \Lambda_k^\varepsilon \) is a simple eigenvalue of the
problem (1.19)-(1.21) for some fixed $k$. Let us make the following ansatz:

$$\Lambda_k^e(\eta) = \Lambda_k^0 + \epsilon \Lambda'_k(\eta) + \tilde{\Lambda}_k^e(\eta),$$  \hfill (2.1)

where $\Lambda_k^e(\eta)$ is a correction term and $\tilde{\Lambda}_k^e(\eta)$ a small remainder to be evaluated and estimated. In this section we derive the expression (2.13) for $\Lambda'_k(\eta)$, cf. also (2.18) and (2.19), and the remainder will be treated in Section 3.2.

The corresponding asymptotic ansatz for the eigenfunction reads as follows:

$$U_k^e(x; \eta) = U_k^0(x) + \chi_0(x)w_{k0}(\epsilon^{-1}(x - P^0)) + \chi_1(x)w_{k1}(\epsilon^{-1}(x - P^1)) + \epsilon U'_k(x; \eta) + \tilde{U}_k^e(x; \eta),$$  \hfill (2.2)

where $(U_k^0)_{k=1}^\infty$ is as in (1.33). The functions $w_{k0}$ and $w_{k1}$ are of boundary layer type, and $\chi_j$ is given above (1.6).

The boundary layers $w_{kj}$ depend on the “fast” variables (“stretched” coordinates)

$$\xi^j = (\xi_1^j, \xi_2^j, \xi_3^j) = \epsilon^{-1}(x - P^j), \quad j = 0, 1.$$

They are needed to compensate the fact that the leading term $U_k^0$ in the expansion (2.2) does not satisfy the quasi-periodicity conditions (1.17)-(1.18). By Lemma 1.4 and the mean value theorem, the eigenfunction $U_k^0(x)$ has the representation

$$U_k^0(x) = U_k^0(P^j) + O(\epsilon), \quad x \in \theta^j,$$

near the points $P^j$, $j = 0, 1$. We look for $w_{k0}$ and $w_{k1}$ as the solutions of the problems

$$\Delta_{\xi} w_{k0}(\xi^0) = 0, \quad \xi^0_1 > 0,$$
$$\partial_{\xi_1} w_{k0}(\xi^0) = 0, \quad \xi^0_0 \in \{0\} \times (\mathbb{R}^2 \setminus \overline{\theta}),$$
$$w_{k0}(\xi^0) = a_{k0}, \quad \xi^0_0 \in \{0\} \times \theta,$$

and

$$\Delta_{\xi} w_{k1}(\xi^1) = 0, \quad \xi^1_1 < 0,$$
$$\partial_{\xi_1} w_{k1}(\xi^1) = 0, \quad \xi^1_0 \in \{0\} \times (\mathbb{R}^2 \setminus \overline{\theta}),$$
$$w_{k1}(\xi^1) = a_{k1}, \quad \xi^1_0 \in \{0\} \times \theta,$$

in the half spaces $\{\xi^0_1 > 0\}$ and $\{\xi^1_1 < 0\}$, respectively; the meaning of the numbers $a_{kj}$ will be explained below. Both of the functions $w_{kj}$, $j = 0, 1$, can be extended to even harmonic functions in the exterior of the set $\{0\} \times \theta$:

$$\Delta_{\xi} w_{kj}(\xi^j) = 0, \quad \xi^j_0 \in \mathbb{R}^3 \setminus (\{0\} \times \overline{\theta}),$$
$$w_{kj}(\xi^j) = a_{kj}, \quad \xi^j_0 \in \partial(\{0\} \times \overline{\theta}).$$  \hfill (2.3)

Furthermore, the problem (2.3) admits a solution (see (2.1))

$$w_{kj}(\xi^j) = a_{kj} \frac{\text{cap}_3 \theta}{|\xi^j_0|} + \bar{w}_{kj}(\xi^j),$$  \hfill (2.4)
\[ \tilde{w}_{kj}(\xi^j) = O(|\xi^j|^{-2}), \quad \nabla_{\xi^j} \tilde{w}_{kj}(\xi^j) = O(|\xi^j|^{-3}), \quad (2.5) \]

where \( \text{cap}_3(\theta) \) is the 3-dimensional capacity of the set \( \{0\} \times \theta \) and concerns large \( \xi^j \)-behaviour. Moreover, the solution has a finite Dirichlet integral:

\[
\int_{\mathbb{R}^3} |\nabla_{\xi^j} w_{kj}(\xi^j)|^2 d\xi^j \leq C \quad (2.6)
\]

for some constant \( C > 0 \).

We aim to choose the coefficients \( a_{kj} \) such that \( U'_k \) satisfies the quasi-periodicity conditions (1.17)–(1.18). Clearly, for each \( \epsilon > 0 \)

\[
U'_k(P^0; \eta) = e^{-i\eta} U'_k(P^1; \eta),
\]
\[
\partial_z U'_k(P^0; \eta) = e^{-i\eta} \partial_z U'_k(P^1; \eta),
\]

which together with the asymptotic expansion (2.5) yield the relations

\[
U_k^0(P^0) + a_{k0} = e^{-i\eta}(U_k^0(P^1) + a_{k1}) \quad \text{and} \quad a_{k0} = -e^{-i\eta}a_{k1}
\]

for the coefficients. Hence,

\[
a_{k1} = -e^{i\eta}a_{k0}, \quad a_{k0} = \frac{1}{2} \left( e^{-i\eta}U_k^0(P^1) - U_k^0(P^0) \right), \quad (2.7)
\]

Now we can write a model problem for the main asymptotic correction term \( U'_k \):

\[
- \Delta U'_k(x; \eta) = \Delta W_k(x) \quad x \in \varrho,
\]
\[
(\partial_z - \Lambda^0_k)U'_k(x; \eta) = \Lambda_k(\eta)U_k^0(x), \quad x \in \gamma,
\]
\[
\partial_n U'_k(x; \eta) = 0, \quad x \in \sigma, \quad (2.10)
\]

where we denote

\[
W_k(x) = \left( \sum_{j=0}^1 \chi_j(x) \frac{a_{kj} \text{cap}_3(\theta)}{|x - P_j|} \right), \quad x \in \varrho, \quad (2.11)
\]

In addition to \( U'_k \), the problem (2.8)–(2.10) will also determine the number \( \Lambda'_k(\eta) \) in a unique way for every \( k \) and \( \eta \). This will follow by requiring the solvability condition to hold in the Fredholm alternative, see Lemma 2.1 and its proof, below. Indeed, using the Green formula and the normalization in (1.33) we write (\( ds \) is the surface measure):

\[
\Lambda_k(\eta) = \Lambda'_k(\eta)\|U_k^0; L^2(\gamma)\|^2 = \int_\gamma \left( \partial_z U'_k(x; \eta) - \Lambda_k^0 U_k^0(x; \eta) \right) \overline{U_k^0(x)} \, ds(x) = \int_{\partial\varrho} \left( \frac{\overline{U_k^0(x)}}{U_k^0(x)} \partial_n U'_k(x; \eta) - U'_k(x; \eta) \partial_n \overline{U_k^0(x)} \right) \, ds(x) =
\]

10
\[ = \int_{\Omega} U^0_k(x) \Delta U'_k(x; \eta) - \int_{\Omega} \overline{U^0_k(x)} \Delta W_k(x) \, dx. \]  

(2.12)

Taking into account that the last integral converges absolutely and using the
Green formula again yield

\[ \Lambda'_k(\eta) = \lim_{t \to 0} \sum_{j=0}^{1} \int_{S_r(P_j) \cap \Omega} \frac{U^0_k(x) \partial_n}{|x - P_j|} \left( - \frac{a_{kj}}{|x - P_j|} \right) ds(x) \]

\[ = -2\pi \capp_3(\theta) \left( a_{k0} U^0_k(P^0) + a_{k1} U^0_k(P^1) \right); \]

see (2.11) and Remark [1.3] for notation. According to (2.7) we finally obtain

\[ \Lambda'_k(\eta) = \pi \capp_3(\theta) |U^0_k(P^0) - e^{-i\eta} U^0_k(P^1)|^2. \]  

(2.13)

**Lemma 2.1.** Choosing \( \Lambda'_k(\eta) \) as in (2.13), the problem (2.8)–(2.10) has a so-
lution \( U'_k \in H^1(\Omega) \).

**Proof.** The variational formulation of the problem (2.8)–(2.10) reads as

\[(\nabla U'_k, \nabla V)_{\Omega} - \Lambda'_k(U'_k, V) = (\nabla W_k, \nabla V)_{\Omega} - \Lambda'_k(U^0_k, V). \]

(2.14)

We remark that the function \( 1/|x - P^j| \) is harmonic in \( \Omega \), and since \( \chi_j \)
equals constant one in a neighbourhood of \( P^j \), the function \( \Delta W_k \) vanishes there, hence,
\( W_k \) and \( \nabla W_k \) are smooth as well as uniformly bounded everywhere in \( \Omega \). Moreover,
\( U^0_k \in L^2(\gamma) \).

Using the definition of the operator \( B : H^1(\Omega) \to H^1(\Omega) \) (cf. (1.24), (1.25)
and the remarks above (1.32)) we can rewrite (2.14) as follows:

\[(U'_k, V)_0 - (\Lambda^0_k + 1)(BU'_k, V)_0 = (W_k, V)_0 - \Lambda'_k(BU^0_k, V)_0 - (BW_k, V)_0 \]

(2.15)

which means that \( U'_k \) must be a solution of the equation

\[(B - M^0_k)U'_k = -M^0_k(W_k - BW_k - \Lambda'_k BU^0_k). \]

(2.16)

Notice that \( U^0_k \) is the solution of the homogeneous problem (2.10), so, by the
Fredholm alternative, (2.16) is solvable, if and only if the right hand side of it
is orthogonal to the function \( U^0_k \). This condition is satisfied by choosing \( \Lambda'_k(\eta) \)
as above, since

\[(W_k - BW_k - \Lambda'_k BU^0_k, U^0_k)_0 = (\nabla W_k, \nabla U^0_k)_{\Omega} - \Lambda'_k \|U^0_k; L^2(\gamma)\|^2 = 0 , \]

by (2.12) and \((\nabla W_k, \nabla U^0_k)_{\Omega} = -(\Delta W_k, U^0_k)_{\Omega} \). This last identity follows from
the first Green formula, because the normal derivative of \( W_k \) vanishes on \( \partial\Omega \)
due to the properties of the function \( \chi_j \), see below (1.5). \( \Box \)
2.2 The case of a multiple eigenvalue

In this section we complete the asymptotic analysis by studying the behaviour of eigenvalues $\Lambda_k^0(\eta)$ in the case some $\Lambda_k^0$ has multiplicity $m$ greater than one: we have

$$\Lambda_{k-1}^0 < \Lambda_k^0 = \ldots = \Lambda_{k+m-1}^0 < \Lambda_{k+m}^0.$$  

The ansatz (2.1) is used again. Furthermore, as in (1.33) we denote by $(U_{k+j}^0)_{0 \leq j \leq m-1} \subset L^2(\gamma)$ an orthonormal system of eigenfunctions associated with the eigenvalue $\Lambda_k^0$. Any eigenfunction $U_0^0$ corresponding to $\Lambda_k^0$ can be presented as a linear combination

$$U_0^0(x) = \sum_{j=0}^{m-1} \alpha_j U_{k+j}^0(x).$$

Analogously to (2.2) we introduce the asymptotic ansatz

$$U^\varepsilon(x; \eta) = U_0^0(x) + \chi_0(x) w_{k0}(\varepsilon^{-1}(x-P_0)) + \chi_1(x) w_{k1}(\varepsilon^{-1}(x-P_1)) + \varepsilon U'(x; \eta) + \tilde{U}^\varepsilon(x; \eta).$$

Using the same argumentation as in the previous section we construct the boundary layers $w_{kj}, j = 0, 1$, which satisfy the conditions

$$w_{kj}(\xi^j) = a_{kj} \frac{\text{cap}_3 \theta}{|\xi^j|} + O(|\xi^j|^{-2});$$

here the coefficients $a_{kj}$ come from the equations (2.7), where $U_0^0$ is replaced by $U_k^0$. The main asymptotic term $U'$ is also treated in the same way as in Section 2.1. To use the Fredholm alternative for finding $\Lambda_{k+j}^0(\eta)$, $j = 0, \ldots, m-1$, we write

$$\Lambda_{k+j}^0(\eta) \alpha_j = \Lambda_{k+j}^0(\eta)(U_0^0, U_{k+j}^0)_\gamma,$$

and making use of the Green formula as above we get

$$\Lambda_{k+j}^0(\eta) \alpha_j = \sum_{l=0}^{m-1} \beta_{lj} \alpha_j,$$

where

$$\beta_{lj} = \pi \text{cap}_3(\theta)(U_{k+l}^0(P_0) - e^{-i\eta}U_{k+l}^0(P_1))(U_{k+j}^0(P_0) - e^{-i\eta}U_{k+j}^0(P_1)).$$

Hence, $\Lambda_{k+j}^0(\eta)$ is an eigenvalue of the matrix $B = (\beta_{lj})_{l,j=0}^{m-1}$. This matrix has rank one, because it can be represented in the form $B = \bar{v} v^T$, where $v$ is a vector with components $v_{j+1} = U_{k+j}^0(P_0) - e^{-i\eta}U_{k+j}^0(P_1), j = 0, \ldots, m-1$. This means that

$$\Lambda_k^0(\eta) = \pi \text{cap}_3(\theta) \sum_{l=0}^{m-1} |U_{k+l}^0(P_0) - e^{-i\eta}U_{k+l}^0(P_1)|^2, \quad (2.18)$$

$$\Lambda_{k+j}^0(\eta) = 0, \quad 1 \leq j \leq m-1. \quad (2.19)$$
3 Existence and position of spectral gaps

3.1 Existence of gaps

The first estimate on the eigenvalues of the problem (1.22) can now be stated as follows.

**Theorem 3.1.** For any $k \in \mathbb{N}$ there are numbers $\epsilon_k > 0$ and $C_k > 0$ such that for every $\epsilon \in (0, \epsilon_k)$ and any dual variable $\eta \in [0, 2\pi)$, the eigenvalues of the problem (1.22) and the eigenvalues of the limit problem (1.23) are related as follows:

$$\Lambda_k^0 \leq \Lambda_k^\epsilon(\eta) \leq \Lambda_k^0 + C_k \epsilon.$$  \hfill (3.1)

As mentioned in the introduction (see the explanations around (1.1) and (1.31)), this result implies the existence of any prescribed number of gaps in the essential spectrum $\sigma$, since (3.1) also establishes an estimate for the endpoints of the intervals $\Upsilon_k^\epsilon$. To prove that result one needs to take enough many distinct eigenvalues $\Lambda_k^0$ and a small enough $\epsilon$.

**Corollary 3.2.** Given any number $N \in \mathbb{N}$, the essential spectrum $\sigma$ of the problem (1.8)–(1.10) on $\Pi^\epsilon$ has at least $N$ gaps, if $\epsilon$ is small enough.

**Proof of Theorem 3.1** We apply the max-min-principle described in Section 1.2 and first prove the estimate

$$\Lambda_k^\epsilon(\eta) \geq \Lambda_k^0.$$  \hfill (3.2)

Indeed, we recall that in the equations (1.32) and (1.29) both $F_k \subset H^1(\varpi)$ and $E_k \subset H' = H_{k,\eta}(\varpi)$ are arbitrary subspaces of co-dimension $k - 1$. Since $H_{k,\eta}(\varpi) \subset H^1(\varpi)$, each $E_k$ is contained in some $F_k$, and thus the infimum in (1.32) is smaller than that in (1.29).

So we turn to the upper estimate in (3.1) and fix a $k \in \mathbb{N}$. Let the eigenfunctions $U^0_j$ be as in (1.33) and let $H_k \subset H' = H_{\gamma}$ be a subspace spanned by the functions $Y^\epsilon U^0_j$, where $j = 1, \ldots, k$ and

$$Y^\epsilon = 1 - X^\epsilon_0 - X^\epsilon_1 \in C^\infty(\varpi)$$  \hfill (3.3)

and $X^\epsilon_1$ are as in (1.6). We remark that the functions $Y^\epsilon U^0_j$ satisfy the quasi-periodicity condition (1.12) in the definition of the space $H'$, since $Y^\epsilon$ vanishes in a neighbourhood of the apertures, see the remarks around (1.6). Moreover, the sequence $(Y^\epsilon U^0_1, Y^\epsilon U^0_2, \ldots, Y^\epsilon U^0_k)$ is still linearly independent, due to the $L^2(\gamma)$-orthogonality in (1.33) and the fact that $Y^\epsilon$ equals 1 in the set $\gamma$. Hence, the dimension of $H_k$ is $k$.

If $E_k$ is an arbitrary subspace of $H'$ of co-dimension $k - 1$ (cf. (1.29)), the intersection $E_k \cap H_k$ contains a non-trivial linear combination

$$U(x) = Y^\epsilon(x) \sum_{j=1}^k a_j U^0_j(x), \quad \sum_{j=1}^k |a_j|^2 = 1.$$  \hfill (3.4)
By the remarks just above we have \( \|U; L^2(\gamma)\| = 1 \). Hence, from (1.29) and (1.33) we infer that

\[
\Lambda_k(\eta) \leq \frac{\|\nabla U; L^2(\omega)\|^2}{\|U; L^2(\gamma)\|^2} = \|\nabla U; L^2(\omega)\|^2
\]

\[
= \left\| \nabla \left( (Y^\epsilon - 1) \sum_{j=1}^{k} a_j U_j^0 \right) + \nabla \left( \sum_{j=1}^{k} a_j U_j^0 \right) \right\|^2
\]

\[
= \left\| \nabla \left( \sum_{j=1}^{k} a_j U_j^0 \right) \right\|^2 + 2 \left( \nabla \left( -X_0^\epsilon \sum_{j=1}^{k} a_j U_j^0 \right), \nabla \left( \sum_{j=1}^{k} a_j U_j^0 \right) \right) \]

\[
+ \left\| \nabla \left( X_0^\epsilon + X_1^\epsilon \sum_{j=1}^{k} a_j U_j^0 \right) \right\|^2
\]

(3.5)

To evaluate the first term on the right hand side notice that the functions \( U_j^0 \) satisfy (1.29) so that the \( L^2(\gamma) \)-orthogonality of (1.33) implies

\[
\left\| \nabla \left( \sum_{j=1}^{k} a_j U_j^0 \right) \right\|^2 = \sum_{j,l=1}^{k} a_j a_l \langle \nabla U_j^0, \nabla U_l^0 \rangle \]

\[
= \sum_{j=1}^{k} a_j a_j \Lambda_j^0(\eta, U_j^0) \leq \sum_{j=1}^{k} a_j^2 \Lambda_j^0 \leq \Lambda_k^0,
\]

(3.6)

where the last inequality follows from (3.4) and the fact that the eigenvalues \( \Lambda_j^0 \) are indexed in increasing order. Furthermore, we use Lemma 1.4 as well as the facts that the supports of \( X_l^\epsilon, l = 0, 1 \), have measure of order \( \epsilon^3 \), and \( |\nabla X_l^\epsilon| \) are of order \( \epsilon^{-1} \), and \( |a_j| \leq 1 \) to estimate

\[
\left| \left( \nabla \left( X_0^\epsilon + X_1^\epsilon \sum_{j=1}^{k} a_j U_j^0 \right), \nabla \left( \sum_{j=1}^{k} a_j U_j^0 \right) \right) \right|
\]

\[
\leq k^2 \left( \sup_{x \in S} (1, |U_j^0(x)|, |\nabla U_j^0(x)|) \right)^2 \sup_{x \in S} (1, |\nabla X_l^\epsilon(x)|) \int_S dx \leq C_k \epsilon^2,
\]

\[
\left\| \nabla \left( X_0^\epsilon + X_1^\epsilon \sum_{j=1}^{k} a_j U_j^0 \right) \right\|^2
\]

\[
\leq k^2 \left( \sup_{x \in S} (1, |U_j^0(x)|, |\nabla U_j^0(x)|) \right)^2 \left( \sup_{x \in S} (1, |\nabla X_l^\epsilon(x)|) \right)^2 \int_S dx \leq C_k \epsilon \cdot (3.7)
\]

where \( S = \text{supp}(X_0^\epsilon + X_1^\epsilon) \). Combining this with (3.6) and (3.7) yields the result.

\[\square\]

3.2 Asymptotic position of spectral bands

In this section we shall prove the validity of the asymptotic ansatz (2.1), see Theorem 3.5. This yields our main result concerning the asymptotic position of
the spectral bands, Theorem 3.6.

We start the proof by recalling a classical lemma on near eigenvalues and eigenvectors (see [23] and also, e.g., [3]).

**Lemma 3.3.** Let $T$ be a selfadjoint, positive, and compact operator in a Hilbert space $H$. If a number $\mu > 0$ and an element $V \in H$ satisfy $\|V; H\| = 1$ and $\|T V - \mu V; H\| = \tau \in (0, \mu)$, then the segment $[\mu - \tau, \mu + \tau]$ contains at least one eigenvalue of $T$.

To apply Lemma 3.3 to the operator $B(\eta)$ of (1.23), we fix an arbitrary $k$ and, keeping in mind the formula (1.27), define the approximate $k$:th eigenvalue and eigenvector of $B(\eta)$ by

$$
\mu_k = (1 + \Lambda_k^0 + \epsilon \Lambda_k^1(\eta))^{-1},
\lambda_k(x) = \|\mathcal{U}_k; \mathcal{H}\|^{-1}\mathcal{U}_k(x),
$$

where

$$
\mathcal{U}_k(x) = (1 - X^0(x) - X^1(x))U^0_k(x)
+ X^0(x)U^0_k(P^0) + X^1(x)U^0_k(P^1)
+ \chi_0(x)w_{k0}(\epsilon^{-1}(x - P^0)) + \chi_1(x)w_{k1}(\epsilon^{-1}(x - P^1))
+ \epsilon(1 - X^0(x) - X^1(x))U^0_k(x, \eta),
$$

$U^0_k$ is as in (1.33), $X^j(x) = \chi_j(\epsilon^{-1}(x - P^j))$ and $\chi_j$ are as in (1.0).

We need a lower bound for the norm of $\mathcal{U}_k$.

**Lemma 3.4.** For all $k$ there exists a constant $C_k > 0$ such that

$$
\left|\|\mathcal{U}_k; \mathcal{H}\|^2 - 1 - \Lambda_k^0\right| \leq C_k \epsilon^{1/2}.
$$

**Proof.** Recall that the expression for $\mathcal{U}_k$, (3.9), contains the term $U^0_k$: let us denote $\overline{U}_k := \mathcal{U}_k - U^0_k$. By (1.24), (1.32) - (1.33) - (1.23), we have $\|U^0_k; \mathcal{H}\|^2 = \|U^0_k; L^2(\gamma)\|^2 + \|\nabla U^0_k; L^2(\pi)\|^2 = 1 + \Lambda_k^0$. Hence, by the Cauchy-Schwartz inequality,

$$
\left|\|\mathcal{U}_k; \mathcal{H}\|^2 - 1 - \Lambda_k^0\right| = \left|2(U^0_k, \overline{U}_k)\epsilon + \|\overline{U}_k; \mathcal{H}\|^2\right|
\leq 2\sqrt{1 + \Lambda_k^0}\|\overline{U}_k; \mathcal{H}\| + \|\overline{U}_k; \mathcal{H}\|^2.

$$

Taking into account the definition of the norm of $\mathcal{H}$, the formula (3.9) and the fact that the functions $\chi_j$ and $X^j$ vanish on $\gamma$ we find that $\|\mathcal{U}_k; \mathcal{H}\|$ is bounded by the sum of the expressions

$$
\begin{align*}
\|\nabla (X^0_j(U^0_k - U^0_k(P^j))): L^2(\pi)\|, & \quad j = 0, 1, \\
\|\nabla (\chi_jw_{kj}(\epsilon^{-1}(x - P^j))): L^2(\pi)\|, & \quad j = 0, 1, \\
\|\epsilon(1 - X^0_j - X^1_j)U^0_k; H^1(\pi)\|.
\end{align*}
$$

15
First we use the observation that the supports of the functions \(X_j^0\) and \(\nabla X_j^0\) are contained in balls of radius \(O(\epsilon)\) and that \(|U_k^0(x)|\) and \(|\nabla U_k^0(x)|\) are uniformly bounded in these balls (Lemma [43]), hence \(U_k^0(x) = O(\epsilon)\) there. So, (3.12) can be bounded by a constant times
\[
\int |U_k^0 - U_k^0(P_j)|^2 |\nabla X_j^0|^2 dx + \int |\nabla U_k^0|^2 |X_j^0|^2 dx
\]
\[
\leq C\left( \int_{\text{supp} X_j^0} c^2 \epsilon^{-2} dx + \int_{\text{supp} X_j^0} dx \right)^{1/2} \leq C\epsilon^{3/2}.
\]

We estimate the terms (3.13) using the fact that the support of the function \(\nabla \chi_j\) is contained in a set \(\{c \leq |x - P| \leq C\} =: S_j\) for some constants \(0 < c < C\) (see above (1.6)), hence, by the estimate (2.4)–(2.5),
\[
|w_{kj}(\epsilon^{-1}(x - P))| \leq C\epsilon |x - P|^{-1} \text{ for } x \in S_j.
\]

Applying (2.6) yields
\[
\int |\nabla (\chi_j(x)w_{kj}(\epsilon^{-1}(x - P)))|^2 dx
\]
\[
\leq \int_{S_j} |w_{kj}(\epsilon^{-1}(x - P))|^2 dx + \int_{S_j} |\nabla (w_{kj}(\epsilon^{-1}(x - P)))|^2 dx
\]
\[
\leq \int_{S_j} C\epsilon^2 |x - P|^{-2} dx + \epsilon^3 \epsilon^2 \int_{\mathbb{R}^3} |\nabla \chi_j w_{kj}(\xi)|^2 d\xi
\]
\[
\leq C_1 \epsilon.
\]

Finally, by Lemma 2.4, \(U_k^0\) belongs to the space \(H^1(\varpi)\). For the terms (3.14) we thus get the bound
\[
\|\epsilon(1 - X_0^0 - X_j^0)U_k^0; H^1(\varpi)\|
\]
\[
\leq C\epsilon \|U_k^0; H^1(\varpi)\| + \epsilon \max_{j=0,1} \|\nabla X_j^0; L^2(\varpi)\|^{1/2} \|U_k^0; L^2(\varpi)\|^{1/2} \leq C' \epsilon,
\]
since \(\|\nabla X_j^0; L^2(\varpi)\| \leq C \epsilon^{1/2}\), due to the measure of the support of \(\nabla X_j^0\). □ □

As a corollary of this lemma, if \(\epsilon \in (0, \epsilon_0]\), then the bounds
\[
0 < \mu_k \leq c_\mu \|U_k; \mathcal{H}_\theta\| \geq c_\mu > 0,
\]
hold true with some positive constants \(c_\mu\) and \(c_\mu\) depending on \(\varpi\) and \(\theta\) only.

The next theorem provides quite accurate asymptotic information on the eigenvalues of the model problem and in particular justifies the ansatz (2.1).

**Theorem 3.5.** For every \(k \geq 1\) there exists a constant \(C_k\) such that, for each \(\eta \in [0, 2\pi)\),
\[
|\Lambda_k^j(\eta) - \Lambda_k^0 - \epsilon \Lambda_k^j(\eta)| \leq C_k \epsilon^{3/2},
\]
where \(\Lambda_k^j(\eta) = \pi \text{cap}_j(\theta)|U_k^0(P) - e^{-\eta}U_k^0(P)|^2\) (cf. (2.13)) in the case the eigenvalue \(\Lambda_k^0\) is simple and \(\Lambda_k^j(\eta)\) is given by the formulas (2.18)–(2.19) in the case \(\Lambda_k^0\) is a multiple eigenvalue.
Proof. We apply Lemma 3.3 to the operator $B'(\eta)$ with $\mu = \mu_k$ and $V = V_k$ as in (3.8). Our aim is to show that $\tau$ of the lemma can be chosen as small as $C_k \epsilon^{3/2}$. The lemma then gives an eigenvalue $M(\epsilon, \eta)$ of $B'(\eta)$ with the estimate

$$|M(\epsilon, \eta) - \mu_k| \leq C_k \epsilon^{3/2}.$$ 

Using (3.8) and (1.27) this turns into an eigenvalue $\lambda(\epsilon, \eta)$ (of (1.14)–(1.18)) satisfying (3.18) in the place of $\Lambda_k'$. However, if $\epsilon$ is small enough, Theorem 3.1 guarantees that in a neighbourhood of $\Lambda_k'$ there is only one eigenvalue of the model problem, namely $\Lambda_k'$. So $\lambda(\epsilon, \eta)$ must coincide with it, and the estimate (3.18) follows.

We are thus left with the task of proving

$$\tau = \|B'(\eta)V_k - \mu_k V_k; H^c\| \leq C_k \epsilon^{3/2}.$$ 

To this end we write, using $V_k = c_{ul}^{-1} U_k$, (1.25), (1.24), (3.8), (3.17),

$$\tau = \sup_Z \left| (B'(\eta)V_k - \mu_k V_k, Z)_\gamma \right|$$ 

$$= c_{ul}^{-1} \sup_Z \left| (U_k, Z)_\gamma - \mu_k (U_k, Z)_\gamma - \mu_k (\nabla U_k, \nabla Z)_\gamma \right|$$ 

$$\leq c_{ul}^{-1} \sup_Z \left| (\Lambda_k^0 + \epsilon \Lambda_k')(U_k, Z)_\gamma - (\nabla U_k, \nabla Z)_\gamma \right| =: c_{ul}^{-1} \sup_Z |T(Z)| (3.19)$$

The supremum is calculated here over all functions $Z \in H^c$ with unit norm. The expression $T(Z)$ can be represented as a sum of the terms

$$S_1(Z) = - (\nabla U_k^0, \nabla Z)_\gamma + \Lambda_k^0 (U_k^0, Z)_\gamma,$$

$$S_2(Z) = - (\nabla (X_0^0 (U_k^0 (P^0) - U_k^0)), \nabla Z)_\gamma,$$

$$S_3(Z) = - \left( \nabla \left( \chi_0 w_0 \epsilon^{-1} (x - P^0) + \chi_1 w_1 \epsilon^{-1} (x - P^0) \right), \nabla Z \right)_\gamma$$ 

$$+ \epsilon (\nabla w_k, \nabla Z)_\gamma,$$

$$S_4(Z) = - \epsilon (\nabla U_k', \nabla Z)_\gamma - \epsilon (\nabla u_k, \nabla Z)_\gamma$$ 

$$+ \epsilon \Lambda_k^0 (U_k, Z)_\gamma + \epsilon \Lambda_k^0 (U_k', Z)_\gamma,$$

$$S_5(Z) = - \epsilon (\nabla ((X_0^0 + X_1^0) U_k'), \nabla Z)_\gamma + \epsilon^2 \Lambda_k^0 (U_k, Z)_\gamma.$$ 

where $w_k$ is given by

$$w_k(x) = \chi_0(x) \frac{a_{k0} \text{cap}_2(\theta)}{|x - P^0|} + \chi_1(x) \frac{a_{k1} \text{cap}_2(\theta)}{|x - P^1|}. \quad (3.20)$$

First we note that $S_1(Z) = S_4(Z) = 0$, because $U_k^0$ and $U_k'$ are the solutions of the problems (1.19)–(1.21) and (2.8)–(2.10), respectively; see also (2.14).

To estimate $S_2(Z)$ we use the Cauchy-Schwartz inequality:

$$\left| (\nabla (X_j^0 (U_k^0 (P^0) - U_k^0)), \nabla Z)_\gamma \right|$$ 

$$\leq \left\| \nabla (X_j^0 (U_k^0 (P^0) - U_k^0)) \right\| L^2(\gamma) \| Z; H^1(\gamma) \| \leq C_k \epsilon^{3/2}.$$
Hence, using \( \text{Lemma 2.1} \)

\[
\epsilon \left| \nabla \left( \nabla (X_0^j + X_1^j U') \right) \right| \leq \epsilon \left| \nabla \left( \nabla \left( X_0^j + X_1^j U'_j \right) \right) \right| \left| H^1(\omega) \right| \leq \epsilon \left| \nabla \left( X_0^j + X_1^j \right) \right| \left| U'_j \right| L^2(\omega) + \epsilon \left| X_0^j + X_1^j \right| \left| \nabla U'_j \right| L^2(\omega) \leq C_k \epsilon^{3/2}.
\]

The surface integral in \( S_3(Z) \) can be estimated simply by the trace inequality.

To provide an upper bound for \( S_3(Z) \) we notice that by (2.4),

\[
w_{kj}(\epsilon^{-1}(x - P')) = \frac{\alpha_{kj} \cap \rho_j(\theta)}{|x - P'|} + \bar{w}_{kj}(\epsilon^{-1}(x - P')), \quad j = 0, 1, \quad (3.21)
\]

hence, using (3.20) we can write

\[
S_3(Z) = -\sum_{j=0}^{1} \left( \nabla \left( \chi_j \bar{w}_{kj}(\epsilon^{-1}(x - P')) \right), \nabla Z \right)_\omega.
\]

After integrating by parts and taking into account that \( \bar{w}_{kj} \) are harmonic functions we obtain

\[
S_3(Z) = \sum_{j=0}^{1} \left( \bar{w}_{kj}(\epsilon^{-1}(x - P')) \Delta \chi_j, Z \right)_\omega + 2 \left( \nabla \chi_j \nabla \left( \bar{w}_{kj}(\epsilon^{-1}(x - P')) \right), Z \right)_\omega. \quad (3.22)
\]

In the second term, the support of the function \( \nabla \chi_j \) is contained in a set \( \{ \epsilon \leq |x - P'| \leq C \} =: S_j \) for some constants \( 0 < \epsilon < C \), hence, by the estimate (2.5),

\[
\left| \nabla \left( \bar{w}_{kj}(\epsilon^{-1}(x - P')) \right) \right| \leq C \epsilon^{-1} \left( \epsilon^{-1} |x - P'| \right)^{-3} = C \epsilon^2 |x - P'|^{-3} \quad \text{for } x \in S_j.
\]

Hence,

\[
\left| \left( \nabla \chi_j \nabla \left( \bar{w}_{kj}(\epsilon^{-1}(x - P')) \right), Z \right) \right|_\omega \leq \left( \int_{S_j} \left| \nabla \left( \bar{w}_{kj}(\epsilon^{-1}(x - P')) \right) \right|^2 dx \right)^{1/2} \left| Z \right| L^2(\omega) \leq C' \epsilon^2. \quad (3.24)
\]

The first term in (3.22) is treated with a similar argument, since \( S_j \) still contains the support of \( \Delta \chi_j \) and the estimate

\[
\left| \bar{w}_{kj}(\epsilon^{-1}(x - P')) \right| \leq C \epsilon^2 |x - P'|^{-3} \quad \text{for } x \in S_j.
\]

\[
\epsilon \left| \nabla ((X_0^j + X_1^j) U'_j), \nabla Z \right|_\omega \leq \epsilon \left| \nabla \left( X_0^j + X_1^j U'_j \right) \right| \left| H^1(\omega) \right| \leq \epsilon \left| \nabla \left( X_0^j + X_1^j \right) \right| \left| U'_j \right| L^2(\omega) + \epsilon \left| X_0^j + X_1^j \right| \left| \nabla U'_j \right| L^2(\omega) \leq C_k \epsilon^{3/2}.
\]
again holds, by (2.5).

We thus get the bound

$$\tau \leq C_k(\theta)\epsilon^{3/2}. \quad \square$$

□

As a consequence we can now provide the asymptotic widths and positions of the spectral bands.

**Theorem 3.6.** Let the index $k$ be such that the eigenvalue $\Lambda_k^0$ is simple. Then, the band $\Upsilon_k^\epsilon$ of the continuous spectrum of the problem (1.8)–(1.10) has the asymptotic form

$$\Upsilon_k^\epsilon = [\Lambda_k^0 + A_k \epsilon + O(\epsilon^{3/2}), \Lambda_k^0 + B_k \epsilon + O(\epsilon^{3/2})]$$

where

$$A_k = \pi \text{cap}_3(\theta) \min \{|U_k^0(P^0) - U_k^0(P^1)|^2, |U_k^0(P^0) + U_k^0(P^1)|^2\}, \quad (3.25)$$

$$B_k = \pi \text{cap}_3(\theta) \max \{|U_k^0(P^0) - U_k^0(P^1)|^2, |U_k^0(P^0) + U_k^0(P^1)|^2\}. \quad (3.26)$$

**Remark 3.7.** Returning to the band-gap structure of the Bloch spectrum (1.1), in general it may happen that a spectral band, closed interval, degenerates into a single point. In this case the band consists of a single eigenvalue of infinite multiplicity, and the band is thus contained in the essential but not in the continuous spectrum. However, by Theorem 3.6 if $k$ is such that both $U_k^0(P^0)$ and $U_k^0(P^1)$ are nonzero and $\epsilon$ is small enough, the numbers $A_k$ and $B_k$ are distinct, and in this case the spectral band $\Upsilon_k^\epsilon$ is indeed an interval with positive length. This obviously provides a way to construct examples where the spectrum of the linear water-wave problem has a genuine band-gap structure with proper intervals as bands and with at least a given number of spectral gaps, cf. Theorem 3.2.

**References**

[1] Agmon, S., Douglis, A., Nirenberg, L., Estimates near the boundary for solutions of elliptic differential equations satisfying general boundary conditions. I., Comm. Pure Appl. Math. 12, 623–727, 1959.

[2] Allaire, G., Conca, C., Bloch wave homogenization and spectral asymptotic analysis, J. Math. Pures Appl., 77, 153–208, 1998.

[3] Birman, M.S., Solomyak, M.Z., Spectral Theory of Self-Adjoint Operators in Hilbert Space, Reidel Publishing Company, Dordrecht, 1986.

[4] Carter, B.G., McIver, P., Water-wave propagation through an infinite array of floating structures. Journal of Engineering Mathematics, 2012
[5] Kato, T., Perturbation theory for linear operators. Die Grundlehren der
mathematischen Wissenschaften, Band 132 Springer-Verlag New York,
1966.

[6] Kozlov, V.A., Mazja, V.G., Rossmann, J., Elliptic boundary value
problems in domains with point singularities, American Mathematical Soc.,
Providence, 1997

[7] Kuchment, P., Floquet theory for partial differential equations. Operator
Theory: Advances and Applications, 60, Birkhuser Verlag, Basel, 1993.

[8] Linton, C. M., Water waves over arrays of horizontal cylinders: band gaps
and Bragg resonance. Journal of Fluid Mechanics, 670, 504-526, 2011.

[9] Liu, P.L.-F., Resonant reflection of water waves in a long channel with
corrugated boundaries, J. Fluid. Mech., 245, 371-381, 1987.

[10] Mazya V. G., Nazarov S. A. and Plamenevskii B. A., Asymptotic the-
tory of elliptic boundary value problems in singularly perturbed domains,
Birkh"auser Verlag, Basel, 2000.

[11] Mattioli, F., Resonant reflection of a series of submerged breakwaters, Il
Nuovo Cimento C 13 C, pp. 823-833, 1990.

[12] McIver, P., Water-wave propagation through an infinite array of cylindrical
structures. Journal of Fluid Mechanics, 424, 101-125, 2000.

[13] McKee, W.D., The propagation of water waves along a channel of variable
width, Applied Ocean Research, 21, 145-156, 1999.

[14] Mei, C.C., Resonant reflection of surface water waves by periodic sandbars,
J. Fluid Mech., 152, 315-335, 1985.

[15] Nazarov S.A., Elliptic boundary value problems with periodic coefficients in
a cylinder, Izv. Akad. Nauk SSSR. Ser. Mat. 45 (1) 101-112, 1981. (English
transl.: Math. USSR. Izvestija. 18 (1), 89-98, 1982)

[16] Nazarov, S.A., Properties of spectra of boundary value problems in cylin-
drical and quasicylindrical domains, Sobolev Spaces in Mathematics, vol.
II (Maz'ya V., Ed.) International Mathematical Series 9, 261–309, 2008.

[17] Nazarov S.A., Opening gaps in the spectrum of the water-wave problem
in a periodic channel, Zh. Vychisl. Mat. i Mat. Fiz., 50, 6, 1092–1108,
2010 (English transl.: Comput. Math. and Math. Physics 50, 6, 1038–1054,
2010).

[18] Nazarov, S.A, Plamenevskii, B.A, Elliptic problems in domains with piece-
wise smooth boundaries, Walter de Gruyter, Berlin, New York, 1994.
[19] Nazarov, S.A., Ruotsalainen, K., Taskinen, J., Essential spectrum of a periodic elastic waveguide may contain arbitrarily many gaps, Appl. Anal. 89, 1, 109-124, 2010.

[20] Nazarov, S.A., Taskinen, J., On essential and continuous spectra of the linearized water-wave problem in a finite pond, Math. Scand. 106, 1, 141-160, 2010.

[21] Pólya G. and Szegö G.; Isoperimetric inequalities in mathematical physics, Princeton University Press, N.J., 1951.

[22] Lord Rayleigh: On the maintenance of vibrations by forces of double frequency, and on the propagation of waves through a medium endowed with a periodic structure, Philos. Mag., 24, 145-159, 1887.

[23] Visik M. I. and Ljusternik L. A.; Regular degeneration and boundary layer of linear differential equations with small parameter, Amer. Math. Soc. Transl. 20, 239-364, 1962.
