Stress-energy of a quantized scalar field in static wormhole spacetimes

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Abstract

Static traversable wormhole solutions of the Einstein equations require “exotic” matter which violates the weak energy condition. The vacuum stress-energy of quantized fields has been proposed as the source for this matter. Using the Dewitt-Schwinger approximation, analytic expressions for the stress-energy of a quantized massive scalar field are calculated in five static spherically symmetric Lorentzian wormhole spacetimes. We find that in all cases, for both minimally and conformally coupled scalar fields, the stress-energy does not have the properties needed to support the wormhole geometry.
I. INTRODUCTION

The subject of Lorentzian wormholes has been of some interest in recent years. Since the publication of the general form for the static spherically symmetric Lorentzian wormhole by Morris and Thorne [1], a great deal of work has been done in an effort to understand whether such solutions to the Einstein equations are compatible with the known laws of physics. While the very existence of such solutions is clearly exotic, in allowing alternate “shortcut” paths connecting otherwise distant regions of the universe, even more disturbing properties of Lorentzian wormholes have been noted.

Following the initial publication of Morris and Thorne, a number of groups [2,3] showed that it was possible to utilize these solutions to construct a spacetime with closed timelike curves (CTC)—a “time machine”. This discovery spurred a large amount of work on whether and how one can make sense of various sorts of physical processes [4–6] that may take place in such spacetimes. Others [7–9] have attempted to determine if the known laws of physics will prevent the conversion of a initially chronal wormhole (or other) spacetime into one in which closed timelike curves are present—the “chronology protection conjecture”. Excluding the ill-understood possibility of quantum gravity acting as a chronology protection agent, it appears that the only possible source of chronology protection would be due to diverging vacuum energy of quantized fields on the chronology horizon of spacetimes where CTCs are formed [10], though the issue is not resolved. It is not clear whether the gravitational backreaction to the divergences is sufficiently strong to alter the spacetime causal structure, preserving chronology, before quantum gravity becomes important. It also appears that divergences may be avoided for at least some quantized fields [11].

A fundamental difficulty associated with the wormhole solutions, even in the absence of closed timelike curves, is the nature of the stress-energy tensor associated with such a geometry. Morris and Thorne [1] demonstrated that the stress-energy tensor of a static spherical wormhole must satisfy two conditions at the wormhole throat. First, it was shown that the radial stress must be negative, i.e., a tension $\tau$ rather than a pressure; second,
the tension must be larger in magnitude than the value of the energy density at the throat \((\tau > \rho)\). They termed matter obeying these two conditions “exotic”, since matter obeying the second condition will necessarily violate the weak energy condition. Such behavior is not observed in known forms of classical matter; however, it has been repeatedly speculated that the stress-energy of quantized fields might satisfy these conditions, supporting traversible wormholes.

The purpose of this paper is to examine whether the stress-energy of quantized fields in fact will have the appropriate form to support a traversible wormhole geometry. Calculating the stress-energy tensor of a quantized field in a curved spacetime is a difficult and arduous task. In this paper, the DeWitt-Schwinger approximation will be used to evaluate the stress-energy tensor of a quantized massive scalar field in five candidate wormhole geometries. Recent work by Anderson, Hiscock, and Samuel [12] has demonstrated that the DeWitt-Schwinger approximation is quite accurate so long as the radius of curvature of the spacetime is greater than the Compton wavelength of the massive field. In contrast, the various analytic approximations for massless fields are not as robust, and, in all but Ricci flat spacetimes, contain an arbitrary parameter whose value cannot be fixed except by experiment [12].

We find that in all five wormhole geometries, for the most physically plausible values of the curvature coupling – minimal \((\xi = 0)\) or conformal \((\xi = 1/6)\) coupling – the stress-energy tensor of the quantized massive scalar field is never of the correct form to support the wormhole. Three of the wormhole geometries contain adjustable parameters in the metric; our conclusion holds for all values of these parameters. In each case, either the radial stress is positive (pressure rather than tension) or the magnitude of the tension is less than that of the energy density (not “exotic”).

If one allows arbitrary values of the curvature coupling, \(\xi\), then in three of the metrics examined it is possible to find cases where the stress-energy does have the form to help support the wormhole. In the other two cases, the stress-energy fails to support the wormhole for all possible values of \(\xi\).

It thus appears questionable whether the vacuum stress-energy of quantized fields can in
fact supply the exotic matter required for a traversable wormhole. For physically plausible, minimally or conformally coupled fields in the cases examined, the vacuum stress-energy of a quantized massive field would actively oppose the formation or maintenance of such a wormhole. Since vacuum energies, unlike classical fields, cannot be “engineered away”, whatever form of matter present which would satisfy the exotic conditions must also overcome the vacuum energy contributions of fields such as these, to yield a total stress-energy for the sum of all fields present which obeys the exotic conditions.

The general static spherically symmetric Lorentzian wormhole spacetime is described in Sec. II. The conditions which the stress-energy of the quantized field must satisfy if it is to help hold the wormhole open are developed there in terms of the parameters describing the spacetime. The calculation of the stress-energy for a quantized scalar field in a general static spherically symmetric spacetime is discussed as well as the range of validity for these approximation methods. In Sec. III the five separate wormhole spacetimes for which the stress-energy tensor of the quantized scalar field was calculated are described and the DeWitt-Schwinger expressions for the stress-energy of quantized massive scalar fields for each spacetime are given. Throughout the paper we use units such that $\hbar = c = G = 1$. Our sign conventions are those of Misner, Thorne, and Wheeler [13].

II. GENERAL WORMHOLE SPACETIMES AND CALCULATIONS OF STRESS-ENERGY FOR STATIC SPHERICALLY SYMMETRIC SPACETIMES

In 1988, Morris and Thorne [1] published the general form for a static spherically symmetric Lorentzian wormhole. It is given by the line element

$$ds^2 = -e^{2\Phi(r)}dt^2 + \left(1 - \frac{b(r)}{r}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2.$$  \hspace{1cm} (1)

There are two arbitrary functions in the line element: $\Phi(r)$, which is called the redshift function as it describes the gravitational redshift in this spacetime; and $b(r)$, which is called the shape function as it describes the shape of the spatial geometry of the wormhole in an
embedding diagram. In addition the coordinate $r$ is constrained to run between $r_0 \leq r < \infty$, where $r_0$ is the throat radius. In a static orthonormal frame, the energy density and radial tension are found to be

$$\rho \equiv T_{\hat{t}\hat{t}} = \frac{b'}{8\pi r^2}, \quad (2)$$

$$\tau \equiv -T_{\hat{r}\hat{r}} = \frac{1}{8\pi} \left[ \frac{b}{r^3} - 2 \left( 1 - \frac{b}{r} \right) \frac{\Phi'}{r} \right], \quad (3)$$

where a prime represents a derivative with respect to $r$. Morris and Thorne also proved that the matter associated with the wormhole by the Einstein equations must satisfy two conditions. First, at the throat,

$$\tau_0 > 0, \quad (4)$$

the tension must be positive. Second, the matter in the neighborhood of the throat must be “exotic” in the sense that:

$$\frac{\tau_0 - \rho_0}{|\rho_0|} > 0 \quad (5)$$

where the subscript indicates that the quantities are evaluated at the throat. Such matter inevitably must violate the weak energy condition; that is, some timelike observers will measure local energy densities to be negative. Our primary goal in this paper is to examine the stress-energy tensor of a quantized field in several wormhole spacetimes, and to determine whether that stress-energy satisfies Eqs. (4,5). If so, then the quantized field is acting in such a way as to maintain the wormhole geometry. If not, then the vacuum state stress-energy of the quantized field would act in a fashion so as to close off the throat and (in a self-consistent treatment) presumably hinder the formation of macroscopic wormholes.

Methods for calculating the stress-energy tensor for quantized fields in static spherically symmetric spacetimes have been developed by a number of groups. Approximate analytic expressions for $\langle T_{\mu\nu} \rangle$ for conformally invariant massless scalar, spinor, and vector fields in Einstein spacetimes (for which $R_{\mu\nu} = \Lambda g_{\mu\nu}, \Lambda = constant$) have been found by Page,
Brown, and Ottewill \cite{14,15}. Frolov and Zel’nikov \cite{16} have developed a geometrically based analytic approximate expression for $\langle T_{\mu\nu} \rangle$ for conformally invariant massless fields in static spacetimes. The only direct input to their approximation from quantum field theory is in the form of the trace anomaly. Direct, rather than approximate, calculation of $\langle T_{\mu\nu} \rangle$ is generally a very difficult task in curved spacetimes. Howard and Candelas calculated $\langle T_{\mu\nu} \rangle$ for a conformal massless scalar field in the Schwarzschild spacetime \cite{17,18}. Such direct calculations are of course the only way the possible validity of the analytic approximations can be firmly established.

Recently Anderson, Hiscock, and Samuel \cite{19,12} have developed a method and numerical program able to calculate the stress-energy tensor of a quantized scalar field with arbitrary curvature coupling and mass in a general static spherically symmetric spacetime. In the course of developing this method, an analytic approximation schemes for calculating the stress-energy tensor for a massive scalar field with arbitrary curvature coupling was developed.

The approximation developed amounts to an alternate derivation of the DeWitt-Schwinger approximation utilizing a WKB expansion. In a static spherically symmetric spacetime, the large $m$ limit of the WKB approximation for $\langle T_{\mu\nu} \rangle$ for the massive scalar field is equivalent to the DeWitt-Schwinger expansion. In order to obtain the expansion to order $1/m^2$ it is necessary to carry the WKB expansion out to sixth order. This approximation, denoted by $\langle T_{\mu\nu} \rangle_{DS}$ was found to be exceedingly accurate in Reissner-Nordström black hole spacetimes as long as the product of the mass of the field, $m$, and the mass of the black hole, $M$, was greater than unity. For example, choosing $Mm = 2$ gave fractional accuracy in the values of $\langle T_{\mu\nu} \rangle_{DS}$ of order $10^{-2}$ everywhere outside the event horizon. The approximation is described in detail in Ref. \cite{12}. 

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III. STRESS-ENERGY FOR QUANTIZED SCALAR FIELDS IN WORMHOLE SPACETIMES

In this section analytic approximate expressions for the stress-energy tensor of a quantized massive scalar field are evaluated using the DeWitt-Schwinger method [12] in five different exemplar wormhole spacetimes. The DeWitt-Schwinger approximation is carried out to order $1/m^2$.

The five different spacetimes are characterized by particular choices of redshift function $\Phi(r)$, and shape function $b(r)$. The energy density and radial tension are defined as:

$$\rho = -\langle T^t_t \rangle$$

$$\tau = -\langle T^r_r \rangle.$$  

A. Zero tidal force Schwarzschild wormhole

A particularly simple set of wormhole geometries are those for which

$$\Phi(r) = 0.$$  

For these cases all stationary observers experience zero tidal forces. A simple particular example is where the spatial geometry is chosen to have the Schwarzschild form:

$$b(r) = constant = r_0,$$

where $r_0$ is the throat radius. Since $b$ is constant in this solution, the background energy density $\rho$ vanishes identically, by Eq.(2)

The DeWitt-Schwinger approximation provides the following expressions for the stress-energy tensor components of a quantized massive scalar field in this spacetime:

$$\langle T^t_t \rangle = \frac{r_0^2 \left(-405r + 448r_0 + 2520r_0\xi - 2772r_0\xi \right)}{53760\pi^2r^9m^2},$$

$$\langle T^r_r \rangle = \frac{7}{5} 

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\[ \langle T^r_r \rangle = \frac{r_0^2 (261r - 238r_0 - 1008r_\xi + 924r_0\xi)}{53760\pi^2 r^9m^2}, \]  \hspace{0.5cm} (11) \\
\[ \langle T^\theta_\theta \rangle = \frac{r_0^2 (-783r + 833r_0 + 3024r_\xi - 3234r_0\xi)}{53760\pi^2 r^9m^2}. \]  \hspace{0.5cm} (12)

From these equations it can be seen that the radial tension is positive only if \( \xi > \frac{23}{84} \), while the “exotic” condition of Eq.(5) is satisfied only if \( \xi < \frac{10}{84} \). It is thus not possible to simultaneously satisfy the two conditions of Eqs.(4,5) for any value of the curvature coupling. Therefore, the stress-energy of the quantized field will never have the form required to act in support of the wormhole in this case.

**B. The simple wormhole**

This wormhole metric was used on a final examination in an introductory relativity class at Caltech; it is discussed in Box 2 and on p. 400 of Ref. [1]. The metric functions are

\[ \Phi(r) = 0, \]  \hspace{0.5cm} (13) \\
\[ b(r) = \frac{r_0^2}{r}. \]  \hspace{0.5cm} (14)

We find the stress-energy tensor components for the massive scalar field to be:

\[ \langle T^t_t \rangle = \frac{r_0^2}{20160\pi^2 r^{12}m^2} \left[ 5940r^4 - 22932r^2r_0^2 + 18025r_0^4 \\
+ \xi \left( -60480r^4 + 238168r^2r_0^2 - 188930r_0^4 \right) \\
+ \xi^2 \left( 151200r^4 - 631680r^2r_0^2 + 513660r_0^4 \right) \\
+ \xi^3 \left( 141120r^2r_0^2 - 160440r_0^4 \right) \right], \]  \hspace{0.5cm} (15)

\[ \langle T^r_r \rangle = \frac{r_0^2}{20160\pi^2 r^{12}m^2} \left[ -2484r^4 + 7116r^2r_0^2 - 4445r_0^4 \\
+ \xi \left( 24192r^4 - 69048r^2r_0^2 + 43050r_0^4 \right) \\
+ \xi^2 \left( -60480r^4 + 181440r^2r_0^2 - 115500r_0^4 \right) \\
+ \xi^3 \left( -40320r^2r_0^2 + 36120r_0^4 \right) \right]. \]  \hspace{0.5cm} (16)
\[
\langle T_\theta^\theta \rangle = \frac{r_0^2}{20160\pi^2 r_0^2 m^2} \left[ 7452r^4 - 28464r^2 r_0^2 + 22225r_0^4 \right. \\
+ \xi \left( -72576r^4 + 276192r^2 r_0^2 - 215250r_0^4 \right) \\
+ \xi^2 \left( 181440r^4 - 725760r^2 r_0^2 + 577500r_0^4 \right) \\
+ \xi^3 \left( 161280r^2 r_0^2 - 180600r_0^4 \right) \] .
\]

(17)

In this case, the radial tension of the quantized massive scalar field is positive at the throat if \( \xi > 0.860358 \); the exotic condition is satisfied if either \( \xi < 0.151551 \) or \( 0.2596 < \xi < 1.42218 \). Both conditions are satisfied, and the wormhole is supported by the stress-energy of the quantized field, only if \( 1.42218 > \xi > 0.860358 \). The stress-energy of a quantized massive field with either conformal or minimal coupling does not have the form necessary to support the wormhole throat.

C. The “absurdly benign” wormhole

In this case,

\[
\Phi(r) = 0, \\
\]

(18)

\[
b(r) = \frac{r_0 (a + r_0 - r)^2}{a^2},
\]

(19)

where \( r_0 \) is again the throat radius and \( a \) is an adjustable length. This spacetime is called “absurdly benign” by Morris and Thorne, because classically all of the exotic material is contained in the region \( r_0 \leq r < r_0 + a \). The value of \( b(r) \) given is valid only within this range of values of \( r \). For \( r \geq r_0 + a \), the shape function \( b(r) = 0 \) so outside this radius the spacetime is just Minkowski space.

Due to the length of the algebraic expressions for the general components of the stress-energy tensor, we will give only the values of the components calculated at the throat, \( r = r_0 \), for the remaining three spacetimes beginning with this spacetime.

The components of the stress-energy tensor for the massive scalar field at the throat are found to be:
\[ \langle T^t_t \rangle_0 = \frac{1}{161280 \pi^2 a^5 r_0^6 m^2} \left[ 129a^5 + 1202a^4 r_0 + 5134a^3 r_0^2 + 7856a^2 r_0^3 + 3884ar_0^4 + 264r_0^5 \\
+ \xi \left( -756a^5 - 11536a^4 r_0 - 59080a^3 r_0^2 - 95200a^2 r_0^3 - 46368ar_0^4 - 2688r_0^5 \right) \right] \\
+ \xi^2 \left( 23520a^4 r_0 + 191520a^3 r_0^2 + 342720a^2 r_0^3 + 164640ar_0^4 + 6720r_0^5 \right) \\
+ \xi^3 \left( -161280a^3 r_0^2 - 349440a^2 r_0^3 - 161280ar_0^4 \right), \quad (20) \]

\[ \langle T^r_r \rangle_0 = \frac{1}{161280 \pi^2 a^4 r_0^6 m^2} \left[ 69a^4 + 450a^3 r_0 + 1878a^2 r_0^2 + 2272ar_0^3 + 552r_0^4 \\
+ \xi \left( -252a^4 - 3360a^3 r_0 - 19824a^2 r_0^2 - 25536ar_0^3 - 5376r_0^4 \right) \right] \\
+ \xi^2 \left( 6720a^3 r_0 + 70560a^2 r_0^2 + 94080ar_0^3 + 13440r_0^4 \right) \\
+ \xi^3 \left( -80640a^2 r_0^2 - 107520ar_0^3 \right), \quad (21) \]

\[ \langle T^\theta_\theta \rangle_0 = \frac{1}{80640 \pi^2 a^5 r_0^6 m^2} \left[ 75a^5 + 876a^4 r_0 + 3198a^3 r_0^2 + 5132a^2 r_0^3 + 2325ar_0^4 + 138r_0^5 \\
+ \xi \left( -315a^5 - 5838a^4 r_0 - 29484a^3 r_0^2 - 55776a^2 r_0^3 - 25200ar_0^4 - 1344r_0^5 \right) \right] \\
+ \xi^2 \left( 10080a^4 r_0 + 84840a^3 r_0^2 + 198240a^2 r_0^3 + 85680ar_0^4 + 3360r_0^5 \right) \\
+ \xi^3 \left( -60480a^3 r_0^2 - 215040a^2 r_0^3 - 80640ar_0^4 \right), \quad (22) \]

where the subscript zero denotes the value of a quantity at the throat.

The range of values of the curvature coupling constant which will satisfy the conditions specified by Eqs. (4,5) will depend on the value chosen for \( a \). The regions in the \((\xi, a)\) plane for which the stress-energy conditions necessary for wormhole support are satisfied are illustrated in Figure (1). For no value of \( a \) in this geometry will the tension be positive for either the minimally coupled or conformally coupled massive field. Thus, a massive minimally or conformally coupled field never contributes to the total stress-energy in a fashion so as to support the wormhole.

D. Wormhole with finite radial cutoff in background \( T^\mu_\nu \)

In this case a zero-tidal-force throat solution is joined at a finite radius to an exterior Schwarzschild solution. Since we are only concerned with the stress-energy of the quantized
fields in the neighborhood of the throat, and our analytic approximate expressions for \( \langle T_{\mu\nu} \rangle \) are sufficiently local, only the interior geometry in the neighborhood of the throat is needed. Therefore

\[
\Phi(r) = 0, \quad (23)
\]

\[
b(r) = r_0 \left( \frac{r}{r_0} \right)^{1-\eta}, \quad (24)
\]

where \( \eta \) is a constant bounded by \( 0 < \eta < 1 \).

For the massive scalar field, the values for the stress-energy components at the throat are given by:

\[
\langle T_{t\,t} \rangle_0 = \frac{1}{161280 \pi^2 m^2 r_0^6} \left[ 64 + 112\eta - 490\eta^2 - 92\eta^3 + 403\eta^4 + 132\eta^5 \right. \\
+ \xi \left( -672 - 1792\eta + 6860\eta^2 + 1148\eta^3 - 4956\eta^4 - 1344\eta^5 \right) \\
+ \xi^2 \left( 3360 + 10080\eta - 30240\eta^2 - 5040\eta^3 + 18480\eta^4 + 3360\eta^5 \right) \\
+ \xi^3 \left( -6720 - 20160\eta + 40320\eta^2 + 6720\eta^3 - 20160\eta^4 \right) \bigg], \quad (25)
\]

\[
\langle T_{r\,r} \rangle_0 = \frac{1}{161280 \pi^2 m^2 r_0^6} \left[ 64 - 210\eta^2 + 146\eta^3 + 69\eta^4 \\
+ \xi \left( -672 + 2940\eta^2 - 1848\eta^3 - 672\eta^4 \right) \\
+ \xi^2 \left( 3360 - 13440\eta^2 + 8400\eta^3 + 1680\eta^4 \right) \\
+ \xi^3 \left( 6720 + 20160\eta^2 - 13440\eta^3 \right) \bigg], \quad (26)
\]

\[
\langle T_{\theta\,\theta} \rangle_0 = \frac{1}{161280 \pi^2 m^2 r_0^6} \left[ -128 + 504\eta - 840\eta^2 - 19\eta^3 + 495\eta^4 + 138\eta^5 \\
+ \xi \left( 1344 - 7392\eta + 11760\eta^2 + 462\eta^3 - 5460\eta^4 - 1344\eta^5 \right) \\
+ \xi^2 \left( -6720 + 38640\eta - 53760\eta^2 - 840\eta^3 + 19320\eta^4 + 3360\eta^5 \right) \\
+ \xi^3 \left( 13440 - 70560\eta + 80640\eta^2 - 3360\eta^3 - 20160\eta^4 \right) \bigg]. \quad (27)
\]

Despite having two adjustable parameters, \( \eta \) and \( \xi \), there are no cases in which this stress-energy will satisfy both conditions, Eqs. (4, 5). It thus appears that the vacuum stress-energy of a massive scalar field will always oppose this sort of wormhole.
E. The Proximal Schwarzschild Wormhole

This metric is similar to the Schwarzschild metric except for an additional term in $g_{tt}$,

$$- g_{tt} = 1 - \frac{r_0}{r} + \frac{\epsilon}{r^2},$$

(28)

$$b(r) = r_0.$$  

(29)

The variable $\epsilon$ is a small positive constant. The addition of this term to the metric prevents the appearance of an event horizon in this spacetime, keeping the wormhole traversable by Morris and Thorne’s definition [1]. Due to its similarity to Schwarzschild, this spacetime is called proximal Schwarzschild. This metric is discussed in Ref. [20] in Sec. 13.4.3.

For a massive scalar field, the stress-energy components at the throat are:

$$\langle T^t_t \rangle_0 = \frac{1}{322560\pi^2 \epsilon^3 r_0^6 m^2} \left[ -442 \epsilon^3 - 491 \epsilon^2 r_0^2 - 1044 \epsilon r_0^4 + 358 r_0^6 \right. $$

$$+ \xi \left( 5600 \epsilon^3 + 3332 \epsilon^2 r_0^2 + 10332 \epsilon r_0^4 - 3402 r_0^6 \right) $$

$$+ \xi^2 \left( -14280 \epsilon^3 - 4620 \epsilon^2 r_0^2 - 31500 \epsilon r_0^4 + 9870 r_0^6 \right) $$

$$+ \xi^3 \left( -26880 \epsilon^3 - 10080 \epsilon^2 r_0^2 + 22680 \epsilon r_0^4 - 5460 r_0^6 \right] ,$$

(30)

$$\langle T^r_r \rangle_0 = \frac{1}{322560\pi^2 \epsilon^3 r_0^6 m^2} \left[ 58 \epsilon^3 - 107 \epsilon^2 r_0^2 - 20 \epsilon r_0^4 + 22 r_0^6 \right. $$

$$+ \xi \left( 112 \epsilon^3 + 1428 \epsilon^2 r_0^2 + 56 \epsilon r_0^4 - 210 r_0^6 \right) $$

$$+ \xi^2 \left( 840 \epsilon^3 - 5460 \epsilon^2 r_0^2 + 630 r_0^6 \right) $$

$$+ \xi^3 \left( -6720 \epsilon^3 + 5040 \epsilon^2 r_0^2 - 420 r_0^6 \right] ,$$

(31)

$$\langle T^\theta_\theta \rangle_0 = \frac{1}{322560\pi^2 \epsilon^3 r_0^6 m^2} \left[ -538 \epsilon^3 - 382 \epsilon^2 r_0^2 - 833 \epsilon r_0^4 + 283 r_0^6 \right. $$

$$+ \xi \left( 7112 \epsilon^3 + 3220 \epsilon^2 r_0^2 + 9296 \epsilon r_0^4 - 3087 r_0^6 \right) $$

$$+ \xi^2 \left( -14280 \epsilon^3 - 4620 \epsilon^2 r_0^2 - 29400 \epsilon r_0^4 + 9135 r_0^6 \right) $$

$$+ \xi^3 \left( -26880 \epsilon^3 - 7560 \epsilon^2 r_0^2 + 20160 \epsilon r_0^4 - 4830 r_0^6 \right] .$$

(32)
If one assumes either minimal or conformal values for the curvature coupling, \( \xi \), then it is easy to show that there is no value of \( \epsilon \) which will result in a positive tension and also satisfy the exotic condition of Eq.(5). Therefore, the vacuum stress-energy tensor of any conformal or minimally coupled massive scalar field will not help support the wormhole. There are three small regions in the \((\xi, \epsilon)\) plane in which the stress-energy conditions necessary to support the wormhole will be satisfied. These regions are illustrated in Figure (2). Therefore, the vacuum stress-energy tensor of any conformal or minimally coupled massive scalar field will not help support the wormhole.

**IV. DISCUSSION**

We have shown that, within the context of the DeWitt-Schwinger approximation for the vacuum stress-energy tensor of a quantized massive scalar field, such a field will never have the needed “exotic” properties to support a static wormhole in the five exemplar cases examined, if the field is either minimally or conformally coupled to the scalar curvature. In several of the cases examined, the stress-energy tensor will not satisfy the exotic conditions for any value of the curvature coupling, while for other cases there are ranges of metric parameters and curvature couplings which will result in a stress-energy tensor which is exotic in the sense of Morris and Thorne. While there is no experimental evidence concerning values of the curvature coupling (indeed, there is a general lack of evidence at the present for fundamental scalar fields), other theoretical arguments have been made to demonstrate that only small values of the curvature coupling, near zero, are physically plausible. For example, by requiring the contribution of a quantized massless scalar field to the entropy of an equilibrium black hole to be positive, it is possible to limit the curvature coupling to the range \(-3.431 < \xi < 0.84\) \(^{[2]}\).

Our results are dependent upon the use of the DeWitt-Schwinger approximation. This approximation has been shown to be quite accurate and robust by direct comparison with exact numerically calculated stress-energy tensors in black hole spacetimes \(^{[12]}\). So long
as the mass of the scalar field is large compared with the local radius of curvature of the spacetime, we expect the analytic approximation to be very close to the exact values which could be obtained numerically (with great effort). In none of the cases examined does it appear likely that the slight changes in going to exact values for $\langle T_{\mu \nu} \rangle$ would cause the scalar field to satisfy the exotic conditions and support the wormhole geometry.

In recent work, Ford and Roman [22] have used a bound they had previously developed [23] on negative energy densities in Minkowski space to argue that a traversable wormhole must either be microscopic—of order the Planck size—or the wormhole must have two very different length scales, those being the size of the region where the exotic matter is located and the throat radius. They show that for the absurdly benign wormhole that if one chooses a throat radius of about two meters, one finds that $a$ must be about $10^{14}$ Planck lengths $\approx 10^{-19} \text{cm}$. Using these values in our results for the absurdly benign case yields interesting results. For the massive scalar field, the field acts to support the geometry only if the curvature coupling is assigned obviously unphysical values, namely $\xi > 1.25 \times 10^{29}$.

Finally, while there may be other sorts of fields which will yield vacuum stress-energies which satisfy the exotic conditions and thus could aid in supporting a wormhole, it is important to note that the vacuum stress-energy of the fields studied here would also be present in such a case, opposing the action of the hypothetical wormhole-supporting fields. Thus, the present results not only indicate that massive quantized fields are unlikely to support wormhole geometries, but in addition show that the vacuum stress-energy of such fields will oppose the formation and maintenance of traversible wormholes.

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FIGURES

FIG. 1. Points within the shaded region represent values of $\xi$ and $a$ for which the tension is positive and the exotic condition is satisfied, so that the vacuum stress-energy is of the correct form to help support the absurdly benign wormhole.

FIG. 2. Points within the shaded regions represent values of $\xi$ and $\epsilon$ for which the tension is positive and the exotic condition is satisfied, so that the vacuum stress-energy is of the correct form to help support the proximal Schwarzschild wormhole.
