Growth of Turaev-Viro invariants and cabling

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Abstract

The Chen-Yang volume conjecture [4] states that the growth rate of the Turaev-Viro invariants of a compact oriented 3-manifold determines its simplicial volume. In this paper we prove that the Chen-Yang conjecture is stable under \((2n + 1, 2)\) cabling.

1 Introduction

For \(M\) a 3-manifold that is either closed or with boundary, the Turaev-Viro invariants \(TV_r(M)\) are real valued topological invariants that can be computed using state sums over a triangulation of \(M\). They depend of the choice of an \(2r\)-th root of unity \(q\) whose square is a primitive \(r\)-th root; in this paper we will always choose \(q = e^{2\pi i/r}\) and assume \(r\) odd. Moreover, when \(M\) is a manifold with empty or toroidal boundary, the invariants \(TV_r(M)\) are always non-negative.

The geometric meaning of the Turaev-Viro invariants is hard to understand from their state sum definition. However, a conjecture of Chen and Yang [4] states that the asymptotics of the \(TV_r\) invariants at root \(q = e^{2\pi i/r}\) is related to hyperbolic volume:

**Conjecture 1.1.** [4] For any hyperbolic manifold \(M\) (closed or with boundary), we have

\[
\lim_{r \to \infty, \ r \text{ odd}} \frac{2\pi}{r} \log |TV_r(M, q = e^{2\pi i/r})| = \text{Vol}(M)
\]

where \(\text{Vol}(M)\) is the hyperbolic volume of \(M\).

Conjecture 1.1 is reminiscent of the Volume Conjecture of Kashaev and Murakami-Murakami [7][9] where the \(TV_r\) invariants are replaced with evaluations \(J_n(K, e^{2\pi i/n})\) of the normalized colored Jones polynomials of an hyperbolic knot.

In [6], Yang, Kalfagianni and the author gave a formula relating the Turaev-Viro invariants of a link complement to colored Jones polynomials, establishing a connection between the two conjectures. In the same paper, Conjecture 1.1 was proved for the complements of the figure-eight knot, the borromean link and knots of Gromov norm zero.

Let us define the growth rate of Turaev-Viro invariants by:

**Definition 1.2.** Let \(M\) be a 3-manifold, closed or with boundary. Then the Turaev-Viro growth rate is

\[
LTV(M) = \limsup_{r \to \infty, \ r \text{ odd}} \frac{2\pi}{r} \log |TV_r(M, q = e^{2\pi i/r})|.
\]

A way to restate Conjecture 1.1 for general 3-manifold (not necessarily hyperbolic), is the following:

**Conjecture 1.3.** For any compact oriented 3-manifold \(M\), we have

\[
LTV(M) = \text{Vol}(M),
\]

where \(\text{Vol}(M)\) is the simplicial volume of \(M\).
We recall that the simplicial volume of $M$ can be thought either as the sum of the hyperbolic volumes of the hyperbolic pieces in the JSJ decomposition of $M$, or as $v_3 ||M||$ where $||M||$ is the Gromov norm of $M$. Note that the Turaev-Viro invariants sometimes vanish on lens spaces, thus replacing the limit by an upper limit is necessary.

In [5], Kalfagianni and the author investigated the growth rate of Turaev-Viro invariants. They showed that the growth rate of Turaev-Viro invariants satisfies properties similar to that of the simplicial volume:

**Theorem 1.4.** [5] Let $M$ be a compact oriented 3-manifold, with empty or toroidal boundary.

1. If $M$ is a Seifert manifold, then there exists constants $B > 0$ and $N$ such that for any odd $r \geq 3$, we have $TV_r(M) \leq Br^N$ and $LTV(M) \leq 0$.
2. If $M$ is a Dehn-filling of $M'$, then $TV_r(M) \leq TV_r(M')$ and $LTV(M) \leq LTV(M')$.
3. If $M = M_1 \cup M_2$ is obtained by gluing two 3-manifolds $M_1$ and $M_2$ along a torus boundary component, then $TV_r(M) \leq TV_r(M_1)TV_r(M_2)$ and $LTV(M) \leq LTV(M_1) + LTV(M_2)$.

These properties are parallel to the properties of the simplicial volume: the simplicial volume of Seifert manifolds is 0, the simplicial volume decreases under Dehn-filling and is subadditive under gluing along tori [12].

Let $p, q$ be coprime integers, the $(p, q)$-cabling space is the complement of a $(p, q)$-torus knot standardly embedded in a solid torus. A $(p, q)$-cabling of a manifold $M$ with toroidal boundary is a manifold $M'$ obtained by gluing a $(p, q)$-cabling space to a boundary component of $M$. In this paper, we will investigate the compatibility of Conjecture 1.1 with $(p, 2)$-cabling. We will show the following:

**Theorem 1.5.** Let $M$ be a manifold with toroidal boundary and $M'$ be obtained by gluing a $(p, 2)$-cabling space $C_{p,2}$ to a boundary component of $M$. Then there are constants $B > 0$ and $N > 0$ such that

$$
\frac{1}{Br^N} TV_r(M') \leq TV_r(M) \leq Br^N TV_r(M').
$$

In particular, this means that if Conjecture 1.1 is true for $M$ then it is true for $M'$.

## 2 Preliminaries

### 2.1 Reshetikhin-Turaev $SO_3$-TQFTs and TQFT basis

We briefly sketch the properties of Reshetikhin-Turaev $SO_3$-TQFTs, defined by Reshetikhin and Turaev in [10]. We will introduce them in the skein-theoretic framework of Blanchet, Habegger, Masbaum and Vogel [3]. We refer to [2][3] for the details of these constructions.

For any odd integer $r \geq 3$, and primitive $2r$-th root of unity $A$, there is an associated TQFT functor $RT_r$, with the following properties:

- For $\Sigma$ a closed compact oriented surface, $RT_r(\Sigma)$ is a finite dimensional $\mathbb{C}$-vector space, with a natural Hermitian form. Moreover for a disjoint union $\Sigma \bigsqcup \Sigma'$ one has $RT_r(\Sigma \bigsqcup \Sigma') = RT_r(\Sigma) \otimes RT_r(\Sigma')$.

- For $M$ a compact oriented closed 3-manifold, $RT_r(M)$ is the $SO_3$ Reshetikhin-Turaev invariant, a complex valued topological invariant of 3-manifolds, and for $M$ with boundary, $RT_r(M)$ is a vector in $RT_r(\partial M)$.
- If \((M, \Sigma_1, \Sigma_2)\) is a cobordism, \(RT_r(M) : RT_r(\Sigma_1) \rightarrow RT_r(\Sigma_2)\) is a linear map. Moreover, the composition of cobordisms is sent to the composition of linear maps, up to a power of \(A\).

Moreover, some basis of the TQFT spaces \(RT_r(\Sigma)\) of surfaces has been explicitly described in \([3]\). We recall that for \(\Sigma\) a surface, \(RT_r(\Sigma)\) is a quotient of the Kauffman module of a handlebody of the same genus.

In the case of a torus we get the following picture: the torus \(T^2\) is the boundary of a solid torus \(D^2 \times S^1\). One gets a family of elements of the Kauffman module of \(D^2 \times S^1\) by taking the core \(\{0\} \times S^1\) and coloring it by the \(i-1\)-th Jones-Wenzl idempotents, thus obtaining an element \(e_i \in RT_r(T^2)\). For a definition of Jones-Wenzl idempotents we refer to \([3]\). For \(r = 2m + 1\), and \(A\) a \(2r\)-th root of unity, only finitely many Jones-Wenzl idempotents can be defined, thus only the elements \(e_1, \ldots, e_{2m-1}\) are well defined (see \([3]\) [Lemma 3.2]).

As elements of the Kauffman module of the solid torus, the \(e_i\)'s can also be considered as elements of \(RT_r(T^2)\). One gets a basis of \(RT_r(T^2)\) consisting of elements \(e_i\):

**Theorem 2.1.** \([3]\) [Theorem 4.10] If \(r = 2m + 1 \geq 3\), then the family \(e_1, e_2, \ldots, e_m\) is an orthonormal basis of \(RT_r(T^2)\). Moreover one has \(e_{m-i} = e_{m+1+i}\) for \(0 \leq i \leq m-1\).

Note that the last part implies that the family \(e_1, e_3, \ldots, e_{2m-1}\) is the same basis of \(RT_r(T^2)\) as the family \(e_1, e_2, \ldots, e_m\), in a different order.

As a consequence of TQFTs axioms, the \(RT_r\) vectors associated to link complements can be tied to values of colored Jones polynomials. Indeed, if \(M = S^3 \setminus L\) is a link complement where \(L\) has \(n\) components, \(RT_r(M)\) will be a vector in \(RT_r(T^2)^{\otimes n}\) whose coefficient along \(e_{i_1} \otimes e_{i_2} \cdots \otimes e_{i_n}\) is obtained by gluing solid tori with cores colored by Jones-Wenzl idempotents to the boundary components of \(M\) and taking the Reshetikhin-Turaev invariant of that. Thus the coefficient we get is \(\eta_r J_i(L, A^4)\), where \(J_i(L, t)\) is the \(i\)-th unnormalized colored Jones polynomial of \(L\), \(i\) is a multi-index of colors, and

\[
\eta_r = RT_r(S^3) = \frac{A^2 - A^{-2}}{\sqrt{-r}}.
\]

### 2.2 Relationship with the Turaev-Viro invariants

While the Turaev-Viro invariants of compact oriented 3-manifolds \(M\) are defined as state sums over triangulations of \(M\) (see \([13]\)), we will only use a well-known identity relating the Turaev-Viro invariants and Reshetikhin-Turaev invariants. This property was first proved by Roberts \([11]\) in the case of closed 3-manifolds, and extended to manifolds with boundary by Benedetti and Petronio \([1]\). For simplicity we state it only in the case of manifolds with toroidal boundary:

**Theorem 2.2.** Let \(M\) be a compact oriented manifold with toroidal boundary, let \(r \geq 3\) be an odd integer and let \(A\) be a primitive \(2r\)-th root of unity. Then have

\[
TV_r(M, A^2) = ||RT_r(M, A)||^2
\]

where \(|| \cdot ||\) is the natural Hermitian norm on \(RT_r(\partial M)\).

For \(T^2\) a torus, the natural Hermitian form on \(RT_r(T^2)\) is definite positive for any \(A\), thus \(TV_r(M)\) is non-negative.
2.3 The cabling formula

For \( r = 2m + 1 \), it is convenient to extend the definition of vectors \( e_i \in RT_r(T^2) \) to all possible values of \( i \in \mathbb{Z} \) as follows: we formally set \( e_{-l} = -e_l \) for any \( l \geq 0 \) (and particular \( e_0 = 0 \)) and

\[
e_{l+kr} = (-1)^k e_l
\]

for any \( k \in \mathbb{Z} \). This is compatible with the above mentioned symmetry of the vectors \( e_i \)'s.

Recall that for \( p \) and \( q \) coprime integers with \( q > 0 \), the \((p, q)\)-cabling space \( C_{p,q} \) is the complement in a solid torus of a standardly embedded \((p, q)\)-torus knot. The cobordism \( C_{p,q} \) gives rise in TQFT to a linear map

\[
RT_r(C_{p,q}) : RT_r(T^2) \to RT_r(T^2).
\]

We now describe the action of this map on the basis \( \{e_i\}_{1 \leq i \leq m} \) of \( RT_r(T^2) \). By TQFT axioms, the map \( RT_r(C_{p,q}) \) sends the element \( e_i \) to the element of \( RT_r(T^2) \) corresponding to a \((p, q)\)-torus knot embedded in the solid torus and colored by the \( i - 1 \)-th Jones-Wenzl idempotent. Morton computed these elements using skein calculus, yielding the following formula:

**Theorem 2.3.** (Cabling formula)

For any odd \( r = 2m + 1 \geq 3 \), for any \( 1 \leq i \leq m \), for any coprime integers \((p, q)\), one has:

\[
RT_r(C_{p,2})(e_i) = A^{pq(i^2-1)/2} \sum_{k \in S_i} A^{-2pk(qk+1)} e_{2qk+1}.
\]

where \( S_i \) is the set

\[
S_i = \{-\frac{i-1}{2}, -\frac{i-3}{2}, \ldots, \frac{i-3}{2}, \frac{i-1}{2}\}.
\]

3 Stability under \((p, 2)\)-cabling

In this section, we will let \( r = 2m + 1 \geq 3 \) be an odd integer and let \( A = e^{\frac{i\pi}{r}} \), which is a primitive \( 2r \)-th root of unity.

Recall that a \((p, q)\)-cabling \( M' \) of a manifold \( M \) consist of gluing the exterior torus boundary component of \( C_{p,q} \) to a boundary component of \( M \). The JSJ decomposition of \( M' \) will consist of the pieces in the JSJ decomposition of \( M \), plus an extra piece that is the cabling space \( C_{p,q} \). As \( C_{p,q} \) is a Seifert manifold, \( M \) and \( M' \) have the same simplicial volume.

**Theorem 3.1.** Let \( M \) be a manifold with toroidal boundary, let \( p \) be an odd integer and let \( M' \) be a \((p, 2)\)-cabling of \( M \). Then there exists constants \( B \geq 0 \) and \( N \) such that

\[
\frac{r^{-N}}{B} TV_r(M) \leq TV_r(M') \leq Br^N TV_r(M).
\]

In particular, we have \( LT V(M) = LT V(M') \).

As \( M \) and \( M' \) in the theorem have the same volume, the conclusion implies that if Conjecture I.1 is true for \( M \) it is true for \( M' \).
Proof. First, $M'$ is obtained by gluing $C_{p,2}$, which is Seifert and thus has volume 0, to $M$ along a torus. We know by Theorem 1.4 that

$$TV_r(M') \leq TV_r(M)TV_r(C_{p,2}).$$

But as $C_{p,2}$ has volume 0, one has $TV_r(C_{p,2}) \leq Br^N$ for some constants $B > 0$ and $N$.

To prove the other inequality, we study the map induced by $C_{p,2}$ in the $RT_r$-TQFT. If $T$ is the boundary component coming from the $(p,2)$-torus knot and $T'$ is the exterior boundary component, then $C_{p,2}$ induces a map

$$RT_r(C_{p,2}) : RT_r(T) \to RT_r(T').$$

If $M$ has only one boundary component, as $RT_r$ is a TQFT, we have that $RT_r(M') = RT_r(C_{p,2})(RT_r(M))$. If $M$ has other boundary components than the one used to glue $C_{p,2}$, then for any coloring $i$ of the other boundary components of $M$, we have

$$RT_r(M', i) = RT_r(C_{p,2})(RT_r(M, i)).$$

In all cases, if $RT_r(C_{p,2})$ is invertible, we can write

$$||RT_r(M)|| \leq ||RT_r(C_{p,2})^{-1}|| : ||RT_r(M')||,$$

where $|| \cdot ||$ is the norm induced by the Hermitian form on $RT_r(\partial M)$ or $RT_r(\partial M')$ and $|| \cdot ||$ is the corresponding operator norm.

To conclude the proof of the theorem, we thus only need to prove that $RT_r(C_{p,2})$ is invertible, and that $||RT_r(C_{p,2})^{-1}||$ grows at most polynomially.

We can compute the matrix of $RT_r(C_{p,2})$ in the basis $e_1, \ldots, e_m$ of $RT_r(T)$ by the cabling formula recalled above as Theorem 2.3.

For $q = 2$, the formula states:

$$RT_r(C_{p,2})(e_i) = Ap^{(i^2-1)} \sum_{k \in S_i} A^{-2pk(2k+1)}e_{4k+1}$$

The cabling formula implies that the image lies in the vector space spanned by $e_1, e_3, \ldots, e_{2m-1}$, but by the Symmetry Principle recalled in 2.1 $e_{m-i} = e_{m+1+i}$ for all $0 \leq i \leq m-1$. Thus $\{e_1, e_3, \ldots, e_{2m-1}\}$ is actually the basis $\{e_1, \ldots, e_m\}$ in a different order.

From the cabling formula we get that $RT_r(C_{p,2})(e_i)$ lies in $\text{Span}(e_1, e_3, \ldots, e_{2i-1})$ and that the coefficient in $e_{2j-1}$ in $RT_r(C_{p,2})(e_i)$ is $Ap^{(i^2-1)}Ap^{(j^2-1)}$ if $j$ has same parity as $i$, and $-Ap^{(i^2-1)}Ap^{(j^2-1)}$ else.

We can write $RT_r(C_{p,2})$ in the basis $\{e_1, \ldots, e_m\}$ and $\{e_1, e_3, \ldots, e_{2m-1}\}$ as a product of two diagonal matrices and a triangular matrix:

$$RT_r(C_{p,2}) = \begin{pmatrix}
1 & 0 & \ldots & 0 \\
0 & Ap^{(2-2^2)} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & Ap^{(m-m^2)}
\end{pmatrix}
\begin{pmatrix}
1 & -1 & 1 & \ldots & 0 \\
0 & 1 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & \ldots & 0 \\
0 & Ap^{(2^2-1)} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & Ap^{(m^2-1)}
\end{pmatrix}$$

And thus we have:

$$RT_r(C_{p,2})^{-1} = \begin{pmatrix}
1 & 0 & \ldots & 0 \\
0 & Ap^{(1-2^2)} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & Ap^{(1-m^2)}
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 0 & \ldots & 0 \\
0 & 1 & 1 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & \ldots & 0 \\
0 & Ap^{(2^2-2)} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & Ap^{(m^2-m)}
\end{pmatrix}$$
The two diagonal matrices are isometries and the middle matrix clearly has norm bounded by a polynomial in $r$, which concludes the proof of Theorem 3.1.

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