Prüfer sheaves and generic sheaves over the weighted projective lines of genus one

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Abstract: In the present paper, we introduce the concepts of Prüfer sheaves and adic sheaves over a weighted projective line of genus one, show that Prüfer sheaves and adic sheaves can classify coherent sheaves over a weighted projective line. Moreover, we describe the relationship between Prüfer sheaves and generic sheaves, and provide two methods to construct generic sheaves by using coherent sheaves and Prüfer sheaves.

Keywords: weighted projective lines, quasi-coherent sheaves, pure injective objects

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1 Introduction

The notion of weighted projective lines was introduced by Geigle and Lenzing [6] to give a geometric treatment to canonical algebras which was studied by Ringel [13]. Let $k$ be an algebraically closed field, a weighted projective line over $k$ can be viewed as obtained from a projective line $\mathbb{P}^1_k$ by endowing with positive integral multiplicities $p_1, \ldots, p_t$ (which were called weights) on pairwise distinct points $\lambda_1, \ldots, \lambda_t$. Much interesting work has

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been done on the category of coherent sheaves over a weighted projective
line. But we still know little about the category of quasi-coherent sheaves
over a weighted projective line, even the properties of some special quasi-
coherent sheaves.

As we know that, quasi-coherent sheaves and coherent sheaves over a
scheme play the similar role with modules and finitely generated modules
over rings. And in the representation theory of algebras, some special mod-
ules, for example generic modules, Prüfer modules, adic modules, have been
widely studied ([1], [2], [12], [14], [15]). These infinitely dimensional mod-
ules play an important role in the category of modules. In 1997, Lenzing [9]
extended the concept of generic modules to generic sheaves over weighted
projective lines of genus one and determined all indecomposable generic
sheaves. Moreover, he proved that the left perpendicular category of the
generic sheaf $G_q$ with $q \in \mathbb{Q} \cup \{\infty\}$ intersecting the category of coherent
sheaves is exactly the subcategory $C_q$.

In view of the important role of generic modules, Prüfer modules and
adic modules in the representation theory of finite-dimensional algebras and
generic sheaves over weighted projective lines of genus one, we make further
study about two special quasi-coherent sheaves which we call Prüfer sheaves
and adic sheaves in this paper. The paper is organized as follows:

In section 2, we recall the structure of the category of coherent sheaves
over a weighted projective line of genus one. In section 3, we extend the
concepts of Prüfer modules and adic modules to Prüfer sheaves and adic
sheaves over a weighted projective line of genus one and show how to use
Prüfer sheaves and adic sheaves to classify the category of coherent sheaves.
In section 4, we prove that generic sheaves, Prüfer sheaves and adic sheaves
are all pure injective objects. Section 5 describes an important relationship
between Prüfer sheaves, adic sheaves and generic sheaves in Theorem 5.3
and Theorem 5.4. We provide two methods to construct generic sheaves by
using Prüfer sheaves and coherent sheaves in section 6.

Throughout the paper, $\text{Hom}(X, Y)$ means $\text{Hom}_{\text{Coh}(X)}(X, Y)$ and $\text{Ext}^1(X,$
2 The category of coherent sheaves on a weighted projective line of genus one

Let $k$ be an algebraically closed field, $\mathbb{X}$ be a weighted projective line of genus one over $k$. It is well-known that every quasi-coherent sheaf on $\mathbb{X}$ is a direct limit of coherent sheaves, and the category $\text{Qcoh}(\mathbb{X})$ of quasi-coherent sheaves on $\mathbb{X}$ is a locally noetherian Grothendieck category. Hence, the structure of a quasi-coherent sheaf on $\mathbb{X}$ much depend on that of coherent sheaves on $\mathbb{X}$. In this section, we recall the structure of the category of coherent sheaves on $\mathbb{X}$.

**Proposition 2.1** (see [6]) The category $\text{coh}(\mathbb{X})$ of coherent sheaves on $\mathbb{X}$ is an abelian, Ext-finite, noetherian, hereditary and Krull-Schmidt $k$-category. $\text{coh}(\mathbb{X})$ satisfies Serre duality, i.e. for any two coherent sheaves $F$ and $G$, there is an isomorphism $\text{Hom}(F,\tau G) = D\text{Ext}^1(G,F)$, where $D = \text{Hom}_k(-,k)$.

In addition, $\text{coh}(\mathbb{X}) = \text{coh}^+(\mathbb{X}) \vee \text{coh}_0(\mathbb{X})$, that is, each indecomposable object of $\text{coh}(\mathbb{X})$ lies either in $\text{coh}^+(\mathbb{X})$ or in $\text{coh}_0(\mathbb{X})$, and there are no non-zero morphisms from $\text{coh}_0(\mathbb{X})$ to $\text{coh}^+(\mathbb{X})$, where $\text{coh}^+(\mathbb{X})$ denotes the full subcategory of $\text{coh}(\mathbb{X})$ consisting of all objects which do not have a simple subobject, and $\text{coh}_0(\mathbb{X})$ denotes the full subcategory of $\text{coh}(\mathbb{X})$ consisting of all objects of finite length.

For more detail structure of $\text{coh}(\mathbb{X})$, we need introduce rank, degree and slope of coherent sheaves.

Let $F,G \in \text{coh}(\mathbb{X})$, the Euler form of $F$ and $G$ is defined by

$$\langle F, G \rangle = \dim_k \text{Hom}(F,G) - \dim_k \text{Ext}^1(F,G).$$

which can induce a non-degenerated bilinear form $\langle -,- \rangle$ on the Grothendieck group $K_0(\mathbb{X})$, also called Euler form.
Lemma 2.2 The radical of the Grothendieck group $K_0(X)$ has a $\mathbb{Z}$-basis $u, w$ such that $\langle u, w \rangle = p = \text{l.c.m.}(p_1, \ldots, p_t)$.

Definition 2.3 For each coherent sheaf $F$, define the rank of $F$ by $\text{rk}(F) = \langle [F], w \rangle$, and the degree of $F$ by $\text{deg}(F) = \langle u, [F] \rangle$, where $[F] \in K_0(X)$ is the corresponding class of $F$. Then the slope of a coherent sheaf $F$ is an element in $\mathbb{Q} \cup \{\infty\}$ defined as $\mu(F) = \text{deg}(F)/\text{rk}(F)$.

Proposition 2.4 (see [10]) For each $q \in \mathbb{Q} \cup \{\infty\}$, let $\mathcal{C}^{(q)}$ be the additive closure of the full subcategory of $\text{coh}(X)$ formed by all indecomposable coherent sheaves of slope $q$. Then the following holds:

(i) $\mathcal{C}^{(q)}$ is isomorphic to $\text{coh}_0(X)$ for each $q \in \mathbb{Q} \cup \{\infty\}$. In particular, $\mathcal{C}^{(\infty)}$ is just $\text{coh}_0(X)$, which is uniserial, i.e. each indecomposable object has a unique finite composition series, and admits a natural decomposition $\text{coh}_0(X) = \coprod_{x \in X} U_x$, where $U_x$ are connected uniserial categories indexed by $X$.

(ii) $\text{coh}(X)$ is the additive closure of $\bigcup_{q \in \mathbb{Q} \cup \{\infty\}} \mathcal{C}^{(q)}$.

(iii) $\text{Hom}_X(\mathcal{C}^{(q)}, \mathcal{C}^{(r)}) \neq 0$ if and only if $q \leq r$.

(iv) (Riemann-Roch formula) For each $F, G \in \text{coh}(X)$, there has $\sum_{i=0}^{p-1} \langle [\tau^i F], [G] \rangle = \text{rk}(F)\text{deg}(G) - \text{rk}(G)\text{deg}(F)$, where $p = \text{l.c.m.}(p_1, \ldots, p_t)$.

3 Prüfer sheaves and adic sheaves

In this section, we introduce the concepts of Prüfer sheaves and adic sheaves over $X$, and discuss the set of morphisms between these two classes of quasi-coherent sheaves and coherent sheaves.

By Proposition 2.4, we know that for each $q \in \mathbb{Q} \cup \{\infty\}$, the Auslander-Reiten quiver of $\mathcal{C}^{(q)}$ consists of stable tubes indexed by $X$.

Definition 3.1 Let $\mathcal{T}$ be a stable tube in $\mathcal{C}^{(q)}$ with the rank $d$. Let $S$ be a quasi-simple sheaf (i.e., simple object in $U_x$ which Auslander-Reiten quiver is $\mathcal{T}$) belonging to $\mathcal{T}$ and $S[i]$ be the indecomposable sheaf of length $i$ in $\mathcal{C}^{(q)}$ satisfies $\text{Hom}(S, S[i]) \neq 0$. Then there is a sequence of embeddings $S \to S[2] \to \ldots \to S[i] \to \ldots$. Denote by $S[\infty]$ the corresponding direct limit. Composing the irreducible morphisms between the sheaves belonging to $\mathcal{T}$
in the appropriate way we obtain a generalized tube $T' = (S[di + 1])_{i \in \mathbb{N}_0}$. Comparing to the definition of Prüfer modules in [8], we call $S[\infty]$ a Prüfer sheaf over the weighted projective line $\mathbb{X}$.

Similarly, there is also an indecomposable sheaf $S[-i]$ of length $i$ in $\mathcal{C}_q$ satisfies $\text{Hom}(S[-i], S) \neq 0$, and we can obtain a sequence of epimorphisms $\ldots \to S[-i] \to \ldots \to S[-2] \to S$. Denote by $S[-\infty]$ the corresponding inverse limit. We call $S[-\infty]$ an adic sheaf on $\mathbb{X}$.

Next we talk about the sets of morphisms between coherent sheaves and Prüfer sheaves, and then between coherent sheaves and adic sheaves. We need the following lemmas.

**Lemma 3.2**

(i) Let $X \in \text{coh}(\mathbb{X})$. If $\{Y_i \mid i \in I, Y_i \in \text{Qcoh}(\mathbb{X})\}$ is a direct system, then

$$\text{Hom}(X, \lim_{\to} Y_i) = \lim_{\to} \text{Hom}(X, Y_i) \quad \text{and} \quad \text{Ext}^1(X, \lim_{\to} Y_i) = \lim_{\to} \text{Ext}^1(X, Y_i).$$

If $\{Y_i \mid i \in I, Y_i \in \text{Qcoh}(\mathbb{X})\}$ is an inverse system, then

$$\text{Hom}(X, \lim_{\leftarrow} Y_i) = \lim_{\leftarrow} \text{Hom}(X, Y_i).$$

(ii) Let $X, Y \in \text{Qcoh}(\mathbb{X})$. If $Y = \lim_{\to} Y_i$ with $Y_i \in \text{Qcoh}(\mathbb{X})$ and $\text{Ext}^1(Y_i, X) = 0$ for every $i$, then $\text{Ext}^1(Y, X) = 0$. Dually, if $Y = \lim_{\leftarrow} Y_i$ with $Y_i \in \text{Qcoh}(\mathbb{X})$ and $\text{Ext}^1(X, Y_i) = 0$ for every $i$, then $\text{Ext}^1(X, Y) = 0$.

**Proof**

(i) We only proof $\text{Ext}^1(X, \lim_{\to} Y_i) = \lim_{\to} \text{Ext}^1(X, Y_i)$, the rest formulas is obvious. For Qcoh($\mathbb{X}$) have enough injective objects, we have an exact sequence

$$0 \to Y_i \to I_i \to H_i \to 0, \quad \text{for each } Y_i$$

where $I_i$ is injective. Applying $\text{Hom}(X, -)$ on these exact sequences, we have long exact sequences

$$\text{Hom}(X, I_i) \to \text{Hom}(X, H_i) \to \text{Ext}^1(X, Y_i) \to 0,$$

Taking direct limit, we have

$$\lim_{\to} \text{Hom}(X, I_i) \to \lim_{\to} \text{Hom}(X, H_i) \to \lim_{\to} \text{Ext}^1(X, Y_i) \to 0. \ (*)$$
On the other hand, applying direct limit on \(0 \to Y_i \to I_i \to H_i \to 0\), we obtain new exact sequences \(0 \to \lim Y_i \to \lim I_i \to \lim H_i \to 0\). Applying \(\text{Hom}(X, -)\) to it, we have long exact sequences

\[
\text{Hom}(X, \lim Y_i) \to \text{Hom}(X, \lim I_i) \to \text{Ext}^1(X, \lim Y_i) \to \text{Ext}^1(X, \lim I_i).
\]

\(\lim I_i\) is injective since \(\text{Qcoh}(X)\) is hereditary, so we obtain another long exact sequence

\[
\text{Hom}(X, \lim I_i) \to \text{Hom}(X, \lim H_i) \to \text{Ext}^1(X, \lim Y_i) \to 0. (**)
\]

Compare (*) with (**), by the five lemma, we have \(\text{Ext}^1(X, \lim Y_i) = \lim \text{Ext}^1(X, Y_i)\).

(ii) We prove the first result, the rest is duality. Let \(0 \to X \to Z \to Y \to 0\) be an exact sequence. Since \(Y = \lim Y_i\) and \(\text{Ext}^1(Y_i, X) = 0\), there exists an exact commutative diagram

\[
\begin{array}{ccc}
0 & \to & X \\
\pi_i & \to & Y_i \\
\phi_i & \downarrow & \\
0 & \to & \text{Z}_i
\end{array}
\]

\[
\begin{array}{ccc}
0 & \to & X \\
\psi_i & \to & \text{Y}_i \\
\phi_i & \downarrow & \\
0 & \to & \text{Z}_i
\end{array}
\]

for each \(i < j\), where \(\pi_i = (0, 1)\) and \(\psi_i = \begin{pmatrix} 1 & 0 \\ 0 & \phi_i \end{pmatrix}\). Moreover, there exists a morphism \(\eta_i = \begin{pmatrix} 0 \\ 1 \end{pmatrix}: Y_i \to Z_i\) satisfying \(\pi_i \eta_i = 1_{Y_i}\) and \(\psi_i \eta_i = \eta_j \phi_i\). Then by the university of direct limit, it induces a morphism \(\eta: Y \to Z\), we can prove that \(\pi \eta = 1_{Y}\). It implies \(\text{Ext}^1(Y, X) = 0\). \(\square\)

**Lemma 3.3** Let \(X \in \text{coh}(\mathcal{X}), Y \in \text{Qcoh}(\mathcal{X})\). There is an Auslander-Reiten formula

\[D\text{Ext}^1(X, Y) = \text{Hom}(Y, \tau X).\]

**Proof**: Noticing that each quasi-coherent sheaf on \(\mathcal{X}\) is a direct limit of its coherent subsheaves, we may assume that \(Y = \lim Y_i\), where \(\{Y_i \mid i \in I, Y_i \in \text{coh}(\mathcal{X})\}\) is a direct system. Using Lemma 3.2 and by Serre duality, we have
DExt^1(X, Y) = DExt^1(X, \lim_{i} Y_i) = \lim_{i} DExt^1(X, Y_i) = \lim_{i} \text{Hom}(Y_i, \tau X) = \text{Hom}(Y, \tau X). \hfill \Box

**Proposition 3.4** Let $S_q[\infty]$ be the Prüfer sheaf of slope $q$ and $E$ be an indecomposable coherent sheaf which lies in the mouth of a tube.

(i) If $\mu(E) < q$, then $\text{Ext}^1(E, S_q[\infty]) = 0$ and $\text{Hom}(E, S_q[\infty]) \neq 0$.

(ii) If $\mu(E) = q$, then $\text{Ext}^1(E, S_q[\infty]) = 0 = \text{Hom}(E, S_q[\infty])$ when $E \neq S_q$, otherwise $\text{Ext}^1(E, S_q[\infty]) = 0$ and $\text{Hom}(E, S_q[\infty]) \neq 0$. In particular, $\dim_k \text{Hom}(S_q, S_q[\infty]) = 1$.

(iii) If $\mu(E) > q$, then $\text{Ext}^1(E, S_q[\infty]) \neq 0$ and $\text{Hom}(E, S_q[\infty]) = 0$.

**Proof**: Assume that $S_q$ lies in a tube with rank $d$.

(i) If $\mu(E) < q$, we have $\text{Ext}^1(E, S_q[\infty]) = \lim_{i} \text{Ext}^1(E, S_q[i]) = 0$, and $\text{Hom}(E, S_q[\infty]) = \lim_{i} \text{Hom}(E, S_q[i])$. By Riemann-Roch formula, $\text{Hom}(E, S_q[id]) \neq 0$ for $i \in \mathbb{N}$. So $\text{Hom}(E, S_q[\infty]) \neq 0$.

(ii) If $\mu(E) = q$, the result is obvious by Auslander-Reiten formulas.

(iii) If $\mu(E) > q$, then $\text{Hom}(E, S_q[\infty]) = 0$. By Riemann-Roch formula, we know that $\text{Hom}(S_q[id], \tau E) \neq 0$ for $i \in \mathbb{N}$. Moreover, a non zero morphism from $S_q[id]$ to $\tau E$ can be extended to a non zero morphism form $S_q[\infty]$ to $\tau E$, so by Lemma 3.4, we have $\text{Ext}^1(E, S_q[\infty]) \neq 0$. \hfill \Box

**Proposition 3.5** Let $S_q[-\infty]$ be the adic sheaf of slope $q$ and $E$ be an indecomposable coherent sheaf lies in the mouth of a tube.

(i) If $\mu(E) < q$, then $\text{Hom}(E, S_q[-\infty]) \neq 0$ and $\text{Ext}^1(E, S_q[-\infty]) = 0$.

(ii) If $\mu(E) = q$, then $\text{Hom}(E, S_q[-\infty]) = 0 = \text{Ext}^1(E, S_q[-\infty])$ when $E \neq \tau^{-1} S_q$, otherwise $\text{Hom}(E, S_q[-\infty]) = 0$ and $\text{Ext}^1(E, S_q[-\infty]) \neq 0$. In particular, we have $\dim_k \text{Ext}^1(\tau^{-1} S_q, S_q[-\infty]) = 1$.

(iii) If $\mu(E) > q$, then $\text{Hom}(E, S_q[-\infty]) = 0$ and $\text{Ext}^1(E, S_q[-\infty]) \neq 0$.

**Proof**: Assume that $S_q$ lies in a tube with rank $d$.

(i) By Riemann-Roch Theorem, $\text{Hom}(E, S_q[-d]) \neq 0$, there has a non-zero morphism $f' : E \to S_q[-d]$. Since $\text{Ext}^1(E, C^q) = 0$, $f'$ can be extended to a non-zero morphism $f : E \to S_q[-\infty]$. Thus $\text{Hom}(E, S_q[-\infty]) \neq 0$. By Lemma 3.2(ii), $\text{Ext}^1(E, S_q[-\infty]) = 0$. 

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(iii) If $\mu E > q$, $\text{Hom}(E, S_q[-\infty]) = \lim_{\leftarrow} \text{Hom}(E, S_q[-i]) = 0$. By Riemman-Roch Theorem, $\text{Hom}(S_q[-d], \tau E) \neq 0$. So $\text{Hom}(S_q[-\infty], \tau E) \neq 0$, thus $\text{Ext}^1(E, S_q[-\infty]) \neq 0$.

(ii) If $E$ lies in a different tube with $S_q$, obviously $\text{Hom}(E, S_q[-\infty]) = \text{Ext}^1(E, S_q[-\infty]) = 0$. Otherwise $\text{Hom}(E, S_q[-\infty]) = \lim_{\leftarrow} \text{Hom}(E, S_q[-i]) = 0$. If $E \neq \tau^{-1}S_q$, we get $\text{Ext}^1(E, S_q[-\infty]) = 0$ since $\text{Ext}^1(E, S_q[-i]) = 0$ for each $i$. If $E = \tau^{-1}S_q$, applying $\text{Hom}(-, S_q)$ to the exact sequence $0 \to (\tau S_q)[-\infty] \to S_q[-\infty] \to S_q \to 0 (\ast)$. If $\tau S_q \neq S_q$, $\text{Hom}(S_q[-\infty], S_q) = \text{Hom}(S_q, S_q) \neq 0$ and $\dim_k \text{Hom}(S_q[-\infty], S_q) = 1$. If not, applying $\text{Hom}(S_q, -)$ to $(\ast)$, we have the exact sequence

$$
\text{Ext}^1(S_q, S_q[-\infty]) \rightarrow \text{Ext}^1(S_q, S_q[-\infty]) \rightarrow \text{Ext}^1(S_q, S_q) \rightarrow 0
$$

By the similar consideration as Lemma 3.2(ii), $g$ is a monomorphism. So there has $\dim_k \text{Ext}^1(S_q, S_q[-\infty]) = 1$. □

Combining the results of Proposition 3.4 and Proposition 3.5, we have

**Corollary 3.6** Let $q \in \mathbb{Q} \cup \{\infty\}$, then $(\perp S_q[-\infty] \cap \perp S_q[\infty]) \cap \text{coh}(X) = \mathcal{C}^{(q)}$.

Moreover, we obtain that

**Corollary 3.7** Prüfer sheaves and adic sheaves are indecomposable.

**Proof**: Assume $S[\infty]$ is decomposable, writes $S[\infty] = U \bigoplus V$ with $U, V \in \text{Qcoh}(X)$. By Proposition 3.4 we may assume $U$ satisfies $\text{Hom}(E, U) = 0$ for $E \in \text{coh}(X)$ with $\mu E = q$. But there exists a surjective morphism from $\bigoplus S[i]$ to $U$, so it is impossible. Therefore $S[\infty]$ is indecomposable. Dually, adic sheaves are also indecomposable . □

**Corollary 3.8** Let $q, r \in \mathbb{Q} \cup \{\infty\}$, $S_q$, $S'_r$ be quasi-simple sheaves of slope $q$ and $r$ respectively. Then

(i) If $q < r$, then $\text{Hom}(S_q[\infty], S'_r[\infty]) \neq 0$;

(ii) If $q = r$, then $\text{Hom}(S_q[\infty], S'_r[\infty]) \neq 0$ when $S_q$, $S'_r$ lie in the same tube, otherwise $\text{Hom}(S_q[\infty], S'_r[\infty]) = 0$;

(iii) If $q > r$, then $\text{Hom}(S_q[\infty], S'_r[\infty]) = 0$.

**Proof**: (ii), (iii) is obvious.
(i) Noticing that there are exact commutative diagrams
\[
\begin{array}{ccccccc}
0 & \rightarrow & S_q[i] & \rightarrow & S_q[i+1] & \rightarrow & E_i & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & S'_i[\infty] & \rightarrow & H & \rightarrow & E_i & \rightarrow & 0
\end{array}
\]
for \(i \in \mathbb{N}\). By Proposition 3.4, there is a non-zero morphism from \(S_q\) to \(S'_i[\infty]\). Since the second rows of commutative diagrams are split, so there are non-zero maps from \(S_q[i+1]\) to \(S'_i[\infty]\) which implies \(\text{Hom}(S_q[\infty], S'_i[\infty]) \neq 0\). □

**Remark 3.9** There has the dual property of the morphisms between adic sheaves as Corollary 3.8 which was not showed here.

### 4 The Purity of generic sheaves, Prüfer sheaves and adic sheaves

Recall that the pure injective object in a locally finitely presented category was defined as follows.

**Definition 4.1** (see [3]) Let \(\mathcal{A}\) be a locally finitely presented category, \(\text{fp}(\mathcal{A})\) be the subcategory of \(\mathcal{A}\) consists of all finitely presented objects.

(i) A sequence \(0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0\) in \(\mathcal{A}\) is called pure-exact if \(0 \rightarrow \text{Hom}_\mathcal{A}(X, A) \rightarrow \text{Hom}_\mathcal{A}(X, B) \rightarrow \text{Hom}_\mathcal{A}(X, C) \rightarrow 0\) is exact for all \(X \in \text{fp}(\mathcal{A})\).

(ii) An object \(A \in \mathcal{A}\) is called pure injective if every pure exact sequence \(0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0\) is split.

(iii) An object \(A \in \mathcal{A}\) is called \(\Sigma\)-pure injective if \(\bigoplus_I A\) is pure injective for any set \(I\).

Obviously, \(\Sigma\)-pure injective is pure injective. Noticing that for a locally finitely presented category \(\mathcal{A}\) with products, the subgroup of finite definition of \(\text{Hom}_\mathcal{A}(X, A)\) for any \(A \in \mathcal{A}\) and \(X \in \text{fp}(\mathcal{A})\) is defined as the image of the morphism \(\alpha^* : \text{Hom}_\mathcal{A}(Y, A) \rightarrow \text{Hom}_\mathcal{A}(X, A)\), arising from a morphism \(\alpha : X \rightarrow Y\) from \(X\) to any object \(Y \in \text{fp}(\mathcal{A})\). Notice that a subgroup of
finite definition of $\text{Hom}(X, A)$ is a sub $\text{End}(A)$-module of $\text{Hom}(X, A)$. There has the following property.

**Lemma 4.2** (see [3]) Let $A$ be an object in a locally finitely presented category $\mathcal{A}$ with products, then $A$ is $\Sigma$-pure injective if and only if $A$ satisfies the descending condition for the subgroup of finite definition of $\text{Hom}_\mathcal{A}(X, A)$ for any $X \in \text{fp}(\mathcal{A})$.

Since $\text{Qcoh}(\mathcal{X})$ is a locally finitely presented category with product, we can also consider purity of quasi-coherent sheaves, and we get

**Proposition 4.3** A pure exact sequence in $\text{Qcoh}(\mathcal{X})$ is an exact sequence.

**Proof:** Let $0 \to A \to B \to C \to 0$ be a pure exact sequence. By definition, there has an exact sequence $0 \to \text{Hom}(\bigoplus L, A) \to \text{Hom}(\bigoplus L, B) \to \text{Hom}(\bigoplus L, C) \to 0$ where $L$ runs through all line bundles in $\text{coh}(\mathcal{X})$. It implies $0 \to \Gamma(A) \to \Gamma(B) \to \Gamma(C) \to 0$ is an exact sequence where $\Gamma(-)$ is the global section functor. By sheafication, it finishes the proof. □

In this section, we discuss the purity of generic sheaves, Prüfer sheaves and adic sheaves. The notion of generic sheaves on $\mathcal{X}$ was introduced in [9]. Let $T$ be a tilting sheaf on $\mathcal{X}$, by definition, an indecomposable quasi-coherent sheaf $G$ is called generic if $G$ is not a coherent sheaf, and $\text{Hom}(T, G)$ and $\text{Ext}^1(T, G)$ have finite $\text{End}(G)$-length.

Lenzing [9] proved that, there exists a unique indecomposable generic sheaf $G_q$ of slope $q$ for each $q \in \mathbb{Q} \cup \{\infty\}$ under isomorphism. In particular, the sheaf $K$ of rational functions on $\mathcal{X}$ is the generic sheaf of slope $\infty$, and there exists an automorphism $\Phi_{q\infty}$ of the bounded derived category $\text{D}^b(\text{Qcoh}(\mathcal{X}))$ such that $G_q = \Phi_{q\infty}(K)$. Moreover, the morphisms between generic sheaf $G_q$ and coherent sheaves as follows.

**Lemma 4.4** (see [9]) Let $E \in \text{coh}(\mathcal{X})$ be indecomposable.

(i) If $\mu(E) < q$, then $\text{Ext}^1(E, G_q) = 0$ and $\text{Hom}(E, G_q) \neq 0$.

(ii) If $\mu(E) = q$, then $\text{Hom}(E, G_q) = 0 = \text{Ext}^1(E, G_q)$.

(iii) If $\mu(E) > q$, then $\text{Hom}(E, G_q) = 0$ and $\text{Ext}^1(E, G_q) \neq 0$.

**Theorem 4.5** Generic sheaves, Prüfer sheaves are $\Sigma$-pure injectives, adic sheaves are pure injectives.

**Proof:** Since each generic sheaf $G_q$ is of finite length over $\text{End}(G_q)$ by...
definition, generic sheaves are $\Sigma$-pure injective by Lemma 4.4.

Next, we prove Prüfer sheaves are $\Sigma$-pure injective sheaves. Let $E$ be an indecomposable coherent sheaf and $S_q[\infty]$ be a Prüfer sheaf with quasi-simple sheaf $S_q$ of slope $q$, it is suffices to show $\text{Hom}(E, S_q[\infty])$ are artinian over $\text{End}(S_q[\infty])$. By Proposition 3.4, we only need to consider the situation when $\mu(E) < q$ and $S_q$ lies in the tube of rank one.

In this case, there exist an exact commutative diagram

$$
\begin{array}{ccccccc}
0 & \to & S_q & \to & S_q[i] & \to & S_q[i - 1] & \to & 0 \\
\| & & \| & & \| & & \| & \downarrow & \\
0 & \to & S_q & \to & S_q[i + 1] & \to & S_q[i] & \to & 0
\end{array}
$$

where $i \in \mathbb{N}$ and $i > 1$. Taking direct limit, we obtain a new exact sequence (*)

$$
\begin{array}{ccccccc}
0 & \to & S_q & \to & S_q[\infty] & \to & \phi S_q[\infty] & \to & 0
\end{array}
$$

satisfies $\phi|_{S_q[i]} = \phi_i$. Since $\text{coh}(\mathcal{X})$ is a Krull-Schmidt category, we can choose a basis of $\text{Hom}(E, S_q)$, write $m_{11}, m_{12}, \ldots, m_{1n}$. Applying $\text{Hom}(E, -)$ to the exact sequence

$$
\begin{array}{ccccccc}
0 & \to & S_q & \to & S_q[2] & \to & S_q & \to & 0
\end{array}
$$

we obtain a set of elements $m_{21}, m_{22}, \ldots, m_{2n}$ in $\text{Hom}(E, S_q[2])$ satisfies $\phi m_{2j} = m_{1j}$. For $i > 2$, inductively, choose elements $m_{ij}, m_{i2}, \ldots, m_{in}$ in $\text{Hom}(E, S_q[i])$ such that $\phi m_{ij} = m_{i(i-1)j}$. Let $\bigoplus_{i \in \mathbb{N}} k x_i$ be a free $k$-module with basis $(x_i)_{i \in \mathbb{N}}$ where $x_i$ satisfies $\phi x_i = x_{i-1}$ for $i > 1$ and $\phi x_1 = 0$. Then it induces a $k[\phi]$-module structure on $\bigoplus_{i \in \mathbb{N}} k x_i$. Now set $N_j = \bigoplus_{i \in \mathbb{N}} k x_{ij}$ be a copy of the $k[\phi]$-module $\bigoplus_{i \in \mathbb{N}} k x_i$ for every $j \in \{1, \ldots, n\}$. Then the assignment $x_{ij} \mapsto m_{ij}$ induces an epimorphism $\bigoplus_{j=1}^n N_j \to M$ of $k[\phi]$-modules. Since $\bigoplus_{j=1}^n N_j$ is artinian over $k[\phi] [8]$, we have $\text{Hom}(E, S_q[\infty])$ is artinian over $k[\phi]$. So $\text{Hom}(E, S_q[\infty])$ is artinian over $\text{End}(S_q[\infty])$.

At last, we show that $S_q[-\infty]$ is pure injective object. Let

$$
\begin{array}{ccccccc}
0 & \to & S_q[-\infty] & \xrightarrow{f} & E & \to & F & \to & 0
\end{array}
$$

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be a pure exact sequence. The pushout of $f$ and the canonical morphism $S_q[-\infty] \to S_q[-i]$ induces a commutative diagram

$$
\begin{array}{ccccccc}
0 & \to & S_q[-\infty] & \xrightarrow{f} & E & \to & F & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & S_q[-j] & \xrightarrow{f_j} & E_j & \to & F & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & S_q[-i] & \xrightarrow{f_i} & E_j & \to & F & \to & 0 \\
\end{array}
$$

Obviously, the rows of commutative diagrams are all pure exact sequences. Since $S_q[-i]$ are pure injectives, the rows are split, i.e. there exist $h_i : E_i \to S_q[-i]$ for $i \in \mathbb{N}$ satisfying $h_if_i = 1_{S_q[-i]}$. By the universal property of inverse limit, we obtain a morphism $h : E \to S_q[-\infty]$ satisfying $hf = 1_{S_q[-\infty]}$. So the first row of the commutative diagram is split, i.e. $S_q[-\infty]$ is pure injective object.

\[\square\]

5 Relationship between Prüfer sheaves, adic sheaves and generic sheaves

In this section, we describe the relationship between Prüfer sheaves, adic sheaves and generic sheaves.

Recall in [9] that, for a quasi-coherent sheaf $X$, its torsion part $tX$ is defined as the sum of all subobjects of $X$ having finite length. If $tX = 0$, i.e. $\text{Hom}(S, X) = 0$ for each simple sheaf $S$, then $X$ is called torsion-free. $X$ is called divisible if $\text{Ext}^1(S, X) = 0$ for each simple sheaf $S$. We extend these definitions to the following.

**Definition 5.1** Let $G \in \text{Qcoh}(\mathcal{X})$, $G$ is called $q$-torsion-free if $\text{Hom}(E, G) = 0$ for $E \in \text{coh}(\mathcal{X})$ and $\mu E \geq q$. $G$ is called $q$-divisible if $\text{Ext}^1(E, G) = 0$ for $E \in \text{coh}(\mathcal{X})$ and $\mu(E) = q$, i.e. $\text{Ext}^1(S, G) = 0$ for each quasi-simple sheaf $S$ of slope $q$.

We having following theorem.
Theorem 5.2 Let $G$ be a $q$-torsion-free divisible sheaf and $G_q$ be the generic sheaf of slope $q$. Then $G = \oplus G_q$.

Proof: Since $G$ is $q$-torsion-free, we get $G = \lim\limits_\leftarrow X_i$ for $X_i \in \text{coh}(X)$ with $\mu(X_i) < q$. Noticing that there is an automorphism $\Phi_{q}\infty$ of $D^b(\text{Qcoh}(X))$ which sends $C(\infty)$ to $C(q)$ and $G_q = \Phi_{q}\infty(K)$ for $q \in \mathbb{Q}$, we have $\Phi_{q}\infty^{-1}(G) = \lim\limits_\leftarrow \Phi_{q}\infty^{-1}(X_i) \in \text{Qcoh}(X)$.

On the other hand, let $S$ be a simple sheaf, then $\Phi_{q}\infty^{-1}(S)$ is a quasi-simple sheaf of slope $q$. And we have $\text{Ext}^1(S, \Phi_{q}\infty^{-1}(G)) = \text{Ext}^1(\Phi_{q}\infty^{-1}(S), G) = 0$ and $\text{Hom}(S, \Phi_{q}\infty^{-1}(G)) = \text{Hom}(\Phi_{q}\infty^{-1}(S), G) = 0$. So $\Phi_{q}\infty^{-1}(G)$ is a torsion-free divisible sheaf of slope $\infty$. Using the similar method as [4], we have $\Phi_{q}\infty^{-1}(G) = \oplus K$, so $G = \oplus \Phi_{q}\infty^{-1}(K) = \oplus G_q$. □

Let $S_q$ be a quasi-simple sheaf of slope $q$ which lies in a tube of rank $d$ and $S_q[\infty]$ be the correspondence Prüfer sheaf. Now we will describe the relationship between Prüfer sheaves and generic sheaves as follows.

Theorem 5.3 There are two exact sequences as follows in $\text{Qcoh}(X)$:

$$0 \rightarrow S_q \rightarrow S_q[\infty] \rightarrow (\tau^{-1}S_q)[\infty] \rightarrow 0 \quad \text{and}$$

$$0 \rightarrow S_q[d] \rightarrow S_q[\infty] \rightarrow S_q[\infty] \rightarrow 0$$

which produce two inverse systems $\{(\tau^{-1}S_q)[\infty] \mid i \in \mathbb{N}\}$ and $\{S_q[\infty] \mid i \in \mathbb{N}\}$. Moreover, we have $\lim\limits_{\leftarrow i}(\tau^{-1}S_q)[\infty] = \oplus G_q$ and $\lim\limits_{\leftarrow i}S_q[\infty] = \oplus G_q$.

Proof: According the Auslander-Reiten quiver of $\mathcal{C}(q)$ we have an exact commutative diagram

$$\begin{array}{cccccc}
0 & \rightarrow & S_q & \rightarrow & S_q[i] & \rightarrow (\tau^{-1}S_q)[i - 1] & \rightarrow 0 \\
0 & \rightarrow & S_q & \rightarrow & S_q[i + 1] & \rightarrow (\tau^{-1}S_q)[i] & \rightarrow 0 \\
\end{array}$$

for $i \in \mathbb{N}$. Taking the direct limit, we obtain the required first exact se-
quence. We also have another exact commutative diagram

\[
\begin{array}{cccccccccc}
0 & \rightarrow & S_q[d] & \rightarrow & S_q[id] & \rightarrow & S_q[(i-1)d] & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & S_q[d] & \rightarrow & S_q[(i+1)d] & \rightarrow & S_q[id] & \rightarrow & 0
\end{array}
\]

for \( i \in \mathbb{N} \). Taking the direct limit, we obtain the required second exact sequence.

For the first sequence, we get a inverse system \( \{ (\tau^i S_q)[\infty] \mid i \in \mathbb{N} \} \), and another inverse system \( \{ S_q[\infty] \mid i \in \mathbb{N} \} \) from the second sequence. By Theorem 5.2, we only need to show that \( \lim_{\leftarrow} (\tau^i S_q)[\infty] \) and \( \lim S_q[\infty] \) are \( q \)-torsion-free divisible sheaves.

By Lemma 3.2 we know that \( \text{Ext}^1(E, \lim S_q[\infty]) = 0 \) and \( \text{Ext}^1(E, \lim (\tau^i S_q)[\infty]) = 0 \) for \( E \) is a coherent sheaf of slope \( q \), i.e. they are \( q \)-divisible.

Let \( E \in \text{coh}(X) \) and \( \mu(E) \geq q \), then \( \text{Hom}(E, S_q[\infty]) \neq 0 \) when \( E = S_q \). But \( S_q \) is a subobject of \( S_q[d] \), so \( \text{Hom}(S_q, \lim S_q[\infty]) = 0 \), and then \( \lim S_q[\infty] \) is \( q \)-torsion-free.

By Proposition 3.4, we have \( \text{Hom}(E, \lim (\tau^i S_q)[\infty]) = 0 \) when \( \mu(E) \geq q \). So \( \lim (\tau^i S_q)[\infty] \) is also \( q \)-torsion-free. \( \square \)

By duality, we obtain the relationship between adic sheaves and generic sheaves.

**Theorem 5.4** There are two exact sequences as follows in \( \text{Qcoh}(X) \):

\[
0 \rightarrow (\tau S_q)[-\infty] \rightarrow S_q[-\infty] \rightarrow S_q \rightarrow 0 \quad \text{and} \quad 0 \rightarrow S_q[-\infty] \rightarrow S_q[-d] \rightarrow S_q \rightarrow 0
\]

which produce two direct systems \( \{ (\tau^{-i} S_q)[-\infty] \mid i \in \mathbb{N} \} \) and \( \{ S_q[-\infty] \mid i \in \mathbb{N} \} \). Moreover, we have \( \lim (\tau^{-i} S_q)[-\infty] = \oplus G_q \) and \( \lim S_q[-\infty] = \oplus G_q \).

Moreover, there has

**Corollary 5.5** There is an exact sequence

\[
0 \rightarrow (\tau S_q)[-\infty] \rightarrow \oplus G_q \rightarrow S_q[\infty] \rightarrow 0.
\]
Proof: This proof can also be seen in [1]. Firstly, by Theorem 5.3 we have an exact sequence

\[ 0 \rightarrow S_q[d] \rightarrow S_q[\infty] \rightarrow \phi \rightarrow S_q[\infty] \rightarrow 0. \]

with \( \text{Ker} \phi^i = S_q[i]. \) Since \( S_q[i] = (\tau S_q)[-id], i \in \mathbb{N}, \) we have commutative diagrams

\[
\begin{array}{c}
0 \rightarrow (\tau S_q)[-(i+1)d] \rightarrow S_q[\infty] \downarrow \phi \rightarrow S_q[\infty] \rightarrow 0 \\
0 \rightarrow (\tau S_q)[-id] \rightarrow S_q[\infty] \rightarrow S_q[\infty] \rightarrow 0
\end{array}
\]

Since \( \{ (\tau S_q)[-id] \mid i \in \mathbb{N} \} \) satisfies Mittag-Leffler condition, taking inverse limit, by Corollary 4.3 in [7], we have the required exact sequence. □

Next, we consider the morphisms between Prüfer sheaves, adic sheaves and generic sheaves. We can obtain the following results.

**Corollary 5.6** Let \( q \in \mathbb{Q} \cup \{ \infty \}, S_q \) be a quasi-simple sheaf of slope \( q, \)

(i) If \( q < r, \) then \( \text{Hom}(S_q[\infty], G_r) \neq 0 \) and \( \text{Hom}(G_r, S_q[\infty]) = 0. \)

(ii) If \( q \geq r, \) then \( \text{Hom}(S_q[\infty], G_r) = 0 \) and \( \text{Hom}(G_r, S_q[\infty]) \neq 0. \)

Proof: (i) If \( q < r, \) there exist exact commutative diagrams

\[
\begin{array}{c}
0 \rightarrow S_q[i] \rightarrow S_q[i+1] \rightarrow E_i \rightarrow 0 \\
0 \rightarrow G_r \rightarrow H \rightarrow E_i \rightarrow 0
\end{array}
\]

for \( i \in \mathbb{N}. \) By Lemma 4.4, there is a non-zero morphism from \( S_q \) to \( G_r. \) Since the second rows of commutative diagrams are split, there are non-zero maps from \( S_q[i+1] \) to \( G_r \) which implies \( \text{Hom}(S_q[\infty], G_r) \neq 0. \) By [9], \( G_q \) can be written as a direct limit \( G_q = \lim_{\rightarrow} E_i \) where \( E_i \in \text{coh}(\mathcal{X}) \) with \( \lim \mu(E_i) = q. \) So \( \text{Hom}(G_r, S_q[\infty]) = \text{Hom}(\lim_{\rightarrow} E_i, S_q[\infty]) = \lim_{\rightarrow} \text{Hom}(E_i, S_q[\infty]) = 0. \)

(ii) If \( q = r, \) using Theorem 5.3, the result is obvious.

If \( q > r, \) by Lemma 4.4, we know that \( \text{Hom}(S_q[\infty], G_r) = 0. \) And \( \text{Hom}(G_r, S_q) = \text{DExt}^1(\tau^{-1} S_q, G_r) \neq 0, \) so \( \text{Hom}(G_r, S_q[\infty]) \neq 0. \) □

**Corollary 5.7** Let \( q \in \mathbb{Q} \cup \{ \infty \}, S_q \) be a quasi-simple sheaf of slope \( q, \)
(i) If $q \leq r$, then $\text{Hom}(S_q[-\infty], G_r) \neq 0$ and $\text{Hom}(G_r, S_q[-\infty]) = 0$.

(ii) If $q > r$, then $\text{Hom}(S_q[-\infty], G_r) = 0$ and $\text{Hom}(G_r, S_q[-\infty]) \neq 0$.

**Proof:** (i) If $q < r$, there exists a non-zero morphism from $S_q$ to $G_r$, and since there has a canonical surjective morphism from $S_q[-\infty]$ to $S_q$, so $\text{Hom}(S_q[-\infty], G_r) \neq 0$. By Lemma 4.4 and Lemma 3.2, $\text{Hom}(G_r, S_q[-\infty]) = 0$. If $q = r$, by Theorem 5.4 $S_q[-\infty]$ is a subobject of direct sum of $G_q$, i.e. $\text{Hom}(S_q[-\infty], G_q) \neq 0$. Obviously, $\text{Hom}(G_q, S_q[-\infty]) = 0$.

(ii) If $q > r$, there exists a non-zero morphism from $G_r$ to $S_q$. Since $\text{Ext}^1(G_r, C(q)) = 0$, it can be extended to a non-zero morphism from $G_r$ to $S_q[-\infty]$. Therefore, we obtain that $\text{Hom}(G_r, S_q[-\infty]) \neq 0$. By Proposition 3.5, $S_q[-\infty]$ can be written a direct limit of coherent sheaves of slope greater than $r$, so $\text{Hom}(S_q[-\infty], G_r) = 0$. □

### 6 The construction of generic sheaves

In this section, we always assume that $q \in \mathbb{Q} \cup \{\infty\}$. For each $q$, denote

$\mathcal{C}_q = \{F \in \text{Qcoh}(\mathcal{X}) | F \text{ is } q'-\text{torsion-free where } q' \in \mathbb{Q} \cup \{\infty\} \text{ and } q' > q\}$,

$\mathcal{Q}_q = \{F \in \text{Qcoh}(\mathcal{X}) | F \text{ is a factor of direct sum of sheaves in } \bigcup_{q' \in \mathbb{Q} \cup \{\infty\}, q' > q} \mathcal{C}(q')\}$,

$w_q = \{W \in \text{Qcoh}(\mathcal{X}) | W \in \mathcal{C}_q \text{ and } W \text{ is } q\text{-divisible}\}$.

And let $w'_q \subseteq w_q$ be the full subcategory of all direct sums of Prüfer sheaves of slope $q$.

In this section, we will show the $w_q$-approximation of each quasi-coherent sheaf, and then provide two methods to construct generic sheaves over a weighted projective line by using coherent sheaves and Prüfer sheaves. Firstly, we obtain the following important property.

**Proposition 6.1** ($\mathcal{Q}_q, \mathcal{C}_q$) is a split torsion pair.

**Proof:** Notice that when $q = \infty$, we have $\mathcal{C}_q = \text{coh}(\mathcal{X})$ and $\mathcal{Q}_q = 0$. By [3], ($\mathcal{Q}_q, \mathcal{C}_q$) is always a torsion pair. We only need consider the cases $q \in \mathbb{Q}$.

Let $\eta : 0 \to F \to G \to H \to 0$ be an exact sequence with $F \in \mathcal{Q}_q$ and $H \in \mathcal{C}_q$. We first show if $H$ is a coherent sheaf, then $\eta$ is split. Without loss of generality, we assume $H$ is indecomposable. If $\eta$ is not split, there exists
an commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & F & \rightarrow & G & \rightarrow H & \rightarrow 0 \\
\downarrow \alpha & & \downarrow & & \downarrow & & \downarrow 0 \\
0 & \rightarrow & \tau H & \rightarrow & E & \rightarrow H & \rightarrow 0
\end{array}
\]

where the second row is an Auslander-Reiten sequence. Since \(\tau H \in \mathcal{C}_q\), then \(\alpha = 0\), which is impossible. So \(\eta\) is split. Secondly, we assume that \(H\) is a quasi-coherent sheaf, we can write \(H = \varinjlim H_i\), where \(H_i\) is coherent subsheaf of \(H\). Certainly, the slope of each \(H_i\) is not greater than \(q\). So by Lemma 3.2, we obtain \(\text{Ext}^1(H, F) = \text{Ext}^1(\varinjlim H_i, F) = 0\). Therefore \((Q_q, \mathcal{C}_q)\) is a split torsion pair. □

**Remark 6.2** Similarly, let \(\mathcal{C}'_q = \{F \in \text{Qcoh}(X) | F\text{ is }q\text{-torsion-free}\}\) and \(Q'_q = \{F \in \text{Qcoh}(X) | F\text{ is a factor of direct sum of sheaves in } \bigcup_{q'\in \mathbb{Q} \cup \{\infty\}, q' \geq q} \mathcal{C}^{(q')}\}\), then \((Q'_q, \mathcal{C}'_q)\) is also a split torsion pair.

For any class \(Z\) of quasi-coherent sheaves, we denote by \(l(Z)\) the class of all quasi-coherent sheaves \(F\) with \(\text{Hom}(F, Z) = 0\) and \(r(Z)\) the class of all quasi-coherent sheaves \(F\) with \(\text{Hom}(Z, F) = 0\). Then we have \((l(r(Z)), r(Z))\) is a torsion pair in \(\text{Qcoh}(X)\). Moreover, denote by \(g(\mathcal{C}^{(q)})\) the class of all quasi-coherent sheaves \(F\) generated by \(\mathcal{C}^{(q)}\), that is, \(F\) is a factor of direct sums of objects in \(\mathcal{C}^{(q')}\). To study the structure of quasi-coherent sheaves in \(w_q\), we need the following lemma.

**Lemma 6.3** \(g(\mathcal{C}^{(q)}) = l(r(\mathcal{C}^{(q)})\) and then \((g(\mathcal{C}^{(q)}), r(\mathcal{C}^{(q)}))\) is a torsion pair in \(\text{Qcoh}(X)\).

**Proof:** We only need to show that \(g(\mathcal{C}^{(q)})\) is closed under extension. The left of the proof are similar to the proof of Lemma 1.3 in [15].

Let \(0 \rightarrow M \rightarrow N \rightarrow N/M \rightarrow 0\) be an exact sequence in \(\text{Qcoh}(X)\) with \(M\) and \(N/M\) in \(g(\mathcal{C}^{(q)})\). If \(N \in \text{coh}(X)\), then there exist surjective morphisms \(f : E \rightarrow M\) and \(g : F \rightarrow N\) where \(E\) and \(F\) lie in \(\mathcal{C}^{(q)}\). Since \(\text{Qcoh}(X)\) is a
hereditary category, we have the following commutative diagram

```
0 → E → H → F → 0
|    |    |    |    |
| f  | f' |    |    |
0 → M → H' → F → 0
|    |    |    |     |
g'    |    |     |   g    |
0 → M → N → N/M → 0
```

Thus $H \in \mathcal{C}^{(q)}$ and $g'f'$ is surjective which implies $N \in g(\mathcal{C}^{(q)})$. Now assume $N$ be a quasi-coherent sheaf. Since $N/M \in g(\mathcal{C}^{(q)})$, there exists a surjective morphism $h : \bigoplus E_i \to N/M$ with $E_i \in \mathcal{C}^{(q)}$. Then $N/M = \bigcup h(E_i)$, that is, we can write $N/M = \bigcup (N_i/M)$, where $M \subset N_i \subset N$ and $N_i/M$ is finitely generated by $\mathcal{C}^{(q)}$. Therefore, we have the following commutative diagram

```
0 → M → N_i → N_i/M → 0
|    |    |    |    |
|    |    |    |     |
0 → M → \bigcup N_i → (\bigcup N_i)/M → 0
|    |    |    |     |
|    |    |    |     |
0 → M → N → N/M → 0
```

Since $N/M = \bigcup (N_i/M)$, we have $\iota$ is an isomorphism which implies $N = \bigcup N_i$. Thus without loss of generality, we only need to show that if $N/M$ is finitely generated by $\mathcal{C}^{(q)}$ and $M \in g(\mathcal{C}^{(q)})$, then $N \in g(\mathcal{C}^{(q)})$. Write $N = \bigcup N_i$, where $\{N_i\}$ is a set of filtered subcoherent sheaves of $N$. Thus $N/M = (\bigcup N_i)/M = \bigcup (N_i/M) = \bigcup (N_i \bigcup M)/M$, where the second equality is according to Proposition 11.2 in [11]. Notice that $(N_i \bigcup M)/M \in \text{coh}(\mathbb{X})$ and $(N_i \bigcup M)/M \subset (N_{i'} \bigcup M)/M$ when $i < i'$. We get $\deg(N_i \bigcup M)/M > \deg(N_{i'} \bigcup M)/M$ or $\text{rk}(N_i \bigcup M)/M > \text{rk}(N_{i'} \bigcup M)/M$. Thus there exists $i$ such that $N/M = (N_i \bigcup M)/M$, which implies $N = N_i \bigcup M$. Now write $M = \bigcup M_j$, where $\{M_j\}$ is a set of filtered subcoherent sheaves of $M$. There exists $j$ such that $M_j \bigcap N_i = M \bigcap N_i$. So $(N_i \bigcup M_j)/N_i = N_i/(M_j \bigcap N_i) = N_i/(M \bigcap N_i) = (N_i \bigcup M)/M = N/M$ for $j' \geq j$. We obtain that $N_i \bigcup M_j \in g(\mathcal{C}^{(q)})$. Thus $N \in g(\mathcal{C}^{(q)})$. \qed
Theorem 6.4 Each $W \in w_q$ is a direct sum of Prüfer sheaves and the generic sheaf $G_q$.

Proof: Let $W \in w_q$, according to Lemma 6.3, there has an exact sequence $0 \to tW \to W \to W/tW \to 0$, where $tW \in g(C^{(q)})$ and $W/tW \in r(C^{(q)})$. Now, $W/tW$ is $q$-torsion-free divisible implies $W/tW = \oplus G_q$. Since $tW$ is a direct limit of its subsheaves which lie in $C^{(q)}$, there exist a quasi-simple sheaf $S_q$ and a non-zero morphism $\alpha' : S_q \to tW$. Obviously, $\alpha'$ must be a monomorphism and it can be extended to a monomorphism $\alpha : S_q[\infty] \to tW$ since $tW$ is $q$-divisible. For $tW/S_q[\infty]$ is also a direct limit of coherent sheaves in $C^{(q)}$, by Lemma 3.2, $S_q[\infty]$ is a direct summand of $tW$. Using induction, $tW$ is a direct sum of Prüfer sheaves with slope $q$. Since $\text{Ext}^1(G_q, S_q[\infty]) = 0$, we have $W = tW \oplus W/tW$, this finishes the proof. □

Proposition 6.5 Each $F \in Q_q$ is generated by $w_q'$, i.e. $F$ is a factor of an object in $w_q'$.

Proof: We only need to prove that if $F$ is a coherent sheaf of slope greater than $q$, $F$ is generated by $w_q'$. Denote by $tF$ be the union of all images of non-zero morphism from $C^{(q)}$ to $F$. If $tF \neq F$, then $F/tF$ is a non-zero coherent sheaf of slope greater than $q$. By Riemann-Roch formula, there is a non-zero morphism from $C_q$ to $F/tF$, it is a contradiction by Lemma 6.3. So $F$ is generated by $C^{(q)}$. Since $\text{Ext}^1(C^{(q)}, F) = 0$, any morphism from $C^{(q)}$ to $F$ can be extended to the morphism from Prüfer sheaves of slope $q$ to $F$, so $F$ is generated by $w_q'$. □

When considering the left $w_q$-approximation of $C_q$, we have the following theorem which can be immediately obtained from Theorem 4.1 in [15] by using similar method.

Theorem 6.6 For each $F \in C_q$, there exists an exact sequence

$$
\begin{array}{cccccc}
0 & F & \xrightarrow{\mu_q} F_{w_q} & \oplus S_q \oplus S_q[\infty] & \to 0 \\
\end{array}
$$

where $\mu_q$ is the left minimal $w_q$-approximation. Moreover, if $F$ is $q$-torsion-free, then $F_{w_q}$ is also $q$-torsion-free. □

We obtain the connection between Prüfer sheaf and generic sheaf by exact sequence as follows.
**Theorem 6.7** Let $F \in C_q$ and $F$ is $q$-torsion-free. Moreover, if that $\dim \text{End}(G_q) \text{Hom}(F, G_q) < \infty$, then there exists an exact sequence

$$0 \longrightarrow F \overset{\mu_q}{\longrightarrow} \oplus_n G_q \longrightarrow \oplus S_q \oplus e_{S,F} S_q[\infty] \longrightarrow 0$$

where $e_{S,F} = \dim \text{Ext}^1(S_q, F)_{\text{End}(S_q)}$ and $S_q$ runs through all quasi-simple sheaves of slope $q$.

**Proof:** By Theorem 6.6, if $F$ is $q$-torsion-free, $F_{w_q}$ is direct sums of $G_q$ and $\mu_q$ is naturally a minimal $\mathcal{G}$-approximation where $\mathcal{G} = \{F \in \text{Qcoh}(X) \mid F$ is direct sums of $G_q\}$. On the other hand, if $\dim \text{End}(G_q) \text{Hom}(F, G_q) = n$, write $\{e_1, e_2, \ldots, e_n\}$ be a basis of $\text{Hom}(F, G_q)$ over $\text{End}(G_q)$. Let $f = (e_1, e_2, \ldots, e_n)^T: F \rightarrow \oplus_n G_q$, obviously $f$ is also a minimal $\mathcal{G}$-approximation which implies $f = \mu_q$. This finishes our proof. $\square$

Denote $\lg$ be the length of modules, according to [9], there exists a linear form over $\text{coh}(X)$ as follows.

$$\langle [E], [G_q] \rangle = \lg_{\text{End}(G_q)}(\text{Hom}(E, G_q)) - \lg_{\text{End}(G_q)}(\text{Ext}^1(E, G_q)),$$

where $E \in \text{coh}(X)$, $q = d/r$, $d$ and $r$ are coprime integers, $r \geq 0$, $[G_q] = ru + dw$.

**Corollary 6.8** Let $F$ be a coherent sheaf with slope less than $q$. If $F$ satisfies the equation $\text{rk}(F) - r\deg(F) = 1$, then there has an exact sequence

$$0 \longrightarrow F \longrightarrow G_q \longrightarrow \oplus S_q \oplus e_{S_q} S_q[\infty] \longrightarrow 0,$$

where $e_{S,F} = \dim \text{Ext}^1(S_q, F)_{\text{End}(S_q)}$ and $S_q$ runs through all quasi-simple sheaves of slope $q$.

**Proof:** If $F$ is a coherent sheaf with slope less than $q$, $F$ is naturally $q$-torsion-free. Moreover, $\langle [F], [G_q] \rangle = \langle [F], ru + dw \rangle = \text{rk}(F) - r\deg(F)$. According to Theorem 6.7, we immediately get the required statement. $\square$

**Remark 6.9** Corollary 6.8 provide a method to construct generic sheaves. Notice that if $q = \infty$, then $G_\infty = K$ and a coherent sheaf $F$ satisfies the conditions in Corollary 6.8 must be a line bundle. Corollary 6.8 in fact popularizes the method of construction for $K$ over elliptic curves in [4].
Next we consider the minimal right $w_q$-approximation of $Q_q$.

**Theorem 6.10** Let $q \in \mathbb{Q}$. For each $F \in Q_q$, there exists an exact sequence

$$0 \rightarrow \oplus G_q \rightarrow \oplus S_q \oplus S_q[\infty] \rightarrow F \rightarrow 0$$

where $\beta_q$ is the minimal right $w_q$-approximation.

**Proof:** We only need to prove that there is an exact sequence $0 \rightarrow G \rightarrow \oplus S_q \oplus S_q[\infty] \rightarrow F \rightarrow 0$ satisfying $G$ is $q$-torsion-free, the rest is similar to the proof of Theorem 7.1 in [15].

According to Proposition 6.5, we obtain an exact sequence $0 \rightarrow G \rightarrow \oplus S_q \oplus S_q[\infty] \rightarrow F \rightarrow 0$ where $G$ lies in $C_q$. Moreover, $G$ can be chosen to be $q$-torsion-free. In fact, if it is not, let $tG$ be the union of all images of non-zero morphism from $C^{(q)}$ to $G$, then $G/tG$ is $q$-torsion-free. We have a new exact sequence $0 \rightarrow G/tG \rightarrow (\oplus S_q \oplus S_q[\infty])/tG \rightarrow F \rightarrow 0$. Since $tG$ lies in $C_q$ and $tG$ is generated by $C^{(q)}$, we can write $tG = \lim_{\rightarrow} E_i$ where $E_i$ lies in $C^{(q)}$, so $(\oplus S_q \oplus S_q[\infty])/tG$ is $q$-divisible and generated by $C^{(q)}$ which means $(\oplus S_q \oplus S_q[\infty])/tG$ is a direct sum of Prüfer sheaves. Therefore, we obtain an exact sequence $0 \rightarrow G \rightarrow \oplus S_q \oplus S_q[\infty] \rightarrow F \rightarrow 0$ where $G$ is $q$-torsion-free and lies in $C_q$. \[\square\]

**Remark 6.11** (i) Compared to Theorem 6.7, $q$ can not be $\infty$ in Theorem 6.10.

(ii) Let $E$ be a coherent sheaf of slope greater than $q$. If $\dim_{\text{End}(G_q)} \text{Ext}^1(E, G_q) = 1$, then the kernel of right $w_q$-approximation of $E$ is just $G_q$. So in this sense, we obtain another method to construct $G_q$ by using coherent sheaves and Prüfer sheaves.

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