Complexity of Acyclic Colorings of Graphs and Digraphs with
Degree and Girth Constraints

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Abstract

We consider acyclic $r$-colorings in graphs and digraphs: they color the vertices in $r$ colors, each of which induces an acyclic graph or digraph. (This includes the dichromatic number of a digraph, and the arboricity of a graph.) For any girth and sufficiently high degree, we prove the NP-completeness of acyclic $r$-colorings; our method also implies the known analogue for classical colorings. The proofs use high girth graphs with high arboricity and dichromatic numbers. High girth graphs and digraphs with high chromatic and dichromatic numbers have been well studied; we re-derive the results from a general result about relational systems, which also implies the similar fact about high girth and high arboricity used in the proofs. These facts concern graphs and digraphs of high girth and low degree; we contrast them with acyclic colorings of tournaments (which have low girth and high degree). We prove that even though acyclic two-colorability of tournaments is known to be NP-complete, random acyclically $r$-colorable tournaments allow recovering an acyclic $r$-coloring in deterministic linear time, with high probability.

1 Introduction

Let $G$ be either a graph or a digraph. An *acyclic $r$-coloring of $G$* is an assignment of $r$ colors to the vertices in $G$, so that the vertices of each color $i$ induce an acyclic subgraph $G_i$ of $G$. Note that the vertex sets $V(G_i)$ partition $V(G)$. An equivalent condition on the $r$-coloring is that no cycle $C$ in $G$ is monochromatic. (In the case of digraphs $C$ is a directed cycle.) The least $r \geq 1$ such that $G$ admits an acyclic $r$-coloring is called the *arboricity of $G$* if $G$ is a graph [16], and is called the *dichromatic number of $G$* if $G$ is a digraph [1]. In both cases, the problem of deciding if $G$ has an acyclic $r$-coloring is NP-complete, for any fixed $r \geq 2$ [12]. In this paper we study the effect of restrictions on the girth and the degrees in $G$. For digraphs, we define the *directed girth* to be the minimum length of a directed cycle, if one exists, and leave it undefined otherwise; and we take the degree of a vertex to be the sum of its in-degree and out-degree.
We focus on a combination of girth and degree constraints, and we look at the two opposite ends of the spectrum: small girth with high degree on the one hand, and large girth with small degree on the other. For the former problem, graphs of high degree and small girth are typified by complete graphs and the arboricity of complete graphs is trivial, but digraphs of high degree and small directed girth are typified by tournaments, and the dichromatic number of tournaments is already a hard problem: even just deciding acyclic two-colorability of a tournament is known to be NP-complete [3] (see also [11]). We prove that for random acyclically r-colorable tournaments we can recover the unknown acyclic r-coloring in deterministic linear time, with high probability over the choices of T. (Such a coloring is unique with high probability, as long as r is a constant.) This underscores the fact that the NP-completeness does not come from random instances. We placed this discussion in the last section, as it is quite technical.

For the latter problem, we consider graphs and digraphs of low degree and high girth. It is known that in the context of classical graph colorings, for each r and k there exists a d such that deciding r-colorability of graphs with girth at least k and all degrees at most d is NP-complete [5]). We offer a simple proof of this fact to illustrate our techniques, and then prove an analogous result for acyclic colorings. For both graphs and digraphs, we consider the special cases of r = 2 separately, as we can offer simpler proofs and/or better bounds. In any case, even at this opposite end of the scale, the arboricity and the dichromatic number remain mostly NP-complete.

Our NP-completeness constructions depend on gadgets constructed from graphs and digraphs with high girth and high arboricity or dichromatic number. There are well known constructions of graphs and digraphs with high girth and high chromatic and dichromatic numbers [6, 3]. As far as we were able to determine, there does not appear to be such a result for arboricity, so we provide a short proof. The proof depends on a general result for high girth relational systems from [9]; in fact the same result implies the corresponding results for the chromatic and dichromatic numbers as well.

To highlight the gap for digraphs between the largest acyclic induced subgraph and the largest acyclic induced subgraph that can be found in polynomial time, we prove that even for digraphs (without digons) that have an acyclic n^{1−ε}-coloring, and hence must have acyclic subgraphs of size n^{1−ε}, it is hard to find one of size greater than n^{1/2+ε} (Theorem 4.2).

## 2 Graphs and Digraphs with High Girth and Low Degrees

The following prototype result for ordinary r-coloring of undirected graphs is known [5]. We offer an easy proof to illustrate our technique.

**Theorem 2.1** There exists a function d = d(r, k) such that given r, k ≥ 3, the r-colorability of a graph G of girth at least k, and of maximum degree at most d = d(r, k), is NP-complete.

**Proof.** We shall reduce from the problem of r-colorability. Given a graph G, we shall construct a graph G’ with maximum degree at most d and girth at least k that is r-colorable if and only if G is r-colorable. The first step is to replace each vertex x with a binary tree T_x having deg(x) leaves, and place each edge xy of G between the y-th leaf of T_x and the x-th leaf of T_y. The resulting graph G* has maximum degree three. In the second step we shall replace each edge of every tree T_x by a gadget J designed to ensure that the girth of the resulting graph G’ is at least k and that all vertices that were leaves of any one T_x obtain the same colour in any r-colouring of G’. This ensures that G’ is r-colourable if and only if G is r-colourable. The maximum degree of G’ is then three times the maximum degree Δ(J) of the gadget J, so d(r, k) = 3Δ(J). It remains to construct J: it is well known that for any k and r there exists a graph K which is not r-colorable and has
girth at least $k$ [6] (for a constructive proof see [15]). We may assume that $K$ contains an edge $uv$ such that $K - uv$ is $r$-colorable. We let $J$ to be the graph $K - uv$, and replace each edge $st$ of every $T_x$ by a copy of $J$, identifying $s$ with $u$ and $t$ with $v$. Since the girth of $K$ was at least $k$, each path joining $u$ and $v$ in $J$ has at least $k$ vertices, and the girth of the entire graph $G'$ is at least $k$. Any $r$-colouring of $G'$ assigned the same color to $u$ and $v$ in each copy of $J$, since otherwise $K$ would have been $r$-colorable.

The same technique can be applied to acyclic coloring problems. We start with discussing the computational complexity of arboricity of graphs of high girth and low degree, and prove that it remains NP-complete. We begin with the special case of $r = 2$.

**Theorem 2.2** There is a function $d(k)$ such that given $k \geq 3$, the problem of acyclic two-coloring a graph $G$ of girth at least $k$, and of maximum degree at most $d(k)$, is NP-complete.

**Proof.** Fix $k \geq 3$. In this case we reduce from the not-all-equal $k$-satisfiability problem with three occurrences of each variable, which is NP-complete by Feder and Ford [8]. An instance of this problem is a set of variables with binary values, and a set of clauses each consisting of $k$ variables. The question is if values can be chosen so that no clause has all variables of the same value. (This is also known as the two-colorability problem for $k$-uniform hypergraphs [12].) Given such an instance, we replace each clause by a disjoint cycle of $k$ vertices, one corresponding to each variable. Clearly, in every acyclic two-colouring each of these cycles will receive both colours. We need to ensure that all three occurrences of a variable are given the same value. We shall add for every variable $x$ a simple claw $T_x = K_{1,3}$ with the three leaves identified with the three occurrences of $x$. The resulting graph $G'$ has maximum degree three. For this construction we shall similarly use a gadget $J$ to replace each edge of every claw $T_x$; the gadget $J$ will ensure that the final graph $G$ has girth at least $k$ and have the same color on the three occurrences of each variable, in every acyclic two-colouring of $G$. This guarantees that $G$ is acyclically two-colorable if and only if the original instance is satisfiable. We construct $J$ from a graph $K$ that has girth at least $k$ that does not admit an acyclic $r$-coloring, but contains an edge $uv$ such that $J = K - uv$ is acyclically $r$-colorable. We then replace each edge of each $T_x$ by a copy of $J$, identifying $u$ with $x$ and $v$ with $y$. The construction of such a graph $K$ is discussed in the next section.

The general result is the following.

**Theorem 2.3** There exists a function $d = d(r, k)$ such that given $r, k \geq 3$, the problem of acyclic $r$-coloring a graph $G$ of girth at least $k$, and maximum degree at most $d = d(r, k)$, is NP-complete.

**Proof.** Here we combine both of the previous tricks. We again reduce from the problem of graph $r$-colorability. Thus let $G$ be an instance. We again replace each vertex $x$ by a binary tree $T_x$ with $\deg(v)$ leaves. We will use two gadgets, $J_1$ and $J_2$ with the following properties:

- $J_1$ has vertices $u_1, v_1$ such that any acyclic $r$-coloring of $J_1$ assigns the same color to $u_1$ and $v_1$;
- $J_2$ has vertices $u_2, v_2$ such that any acyclic $r$-coloring of $J_2$ assigns different colors to $u_2$ and $v_2$;
- $J_1$ has girth at least $k$ and contains no path with fewer than $k$ vertices between $u_1$ and $v_1$; and
• \(J_2\) has girth at least \(k\).

Each edge \(st\) of each \(T_x\) will be replaced by \(J_1\) identifying \(s\) with \(u_1\) and \(t\) with \(v_1\), and each edge \(xy\) of \(G\) will similarly be replaced by a copy of \(J_2\) between the corresponding leaves of \(T_x\) and \(T_y\). The resulting graph \(G'\) has girth at least \(k\) because every cycle in \(G'\) is either inside a copy of some \(J_i\), or passes through some \(J_1\). Clearly, \(G'\) is acyclically \(r\)-colorable if and only if \(G\) is \(r\)-colorable in the usual sense. The degrees of \(G'\) are maximized by three times the maximum degree of any \(u_1, u_2, v_1, v_2\) in \(J_1, J_2\).

It remains to explain how to construct \(J_1, J_2\). Let \(K\) be a graph of girth \(k\) that is not acyclically \(r\)-colorable but has an edge \(uv\) such that \(K - uv\) is acyclically \(r\)-colorable. (These graphs are constructed in the next section.) Let \(J_1 = K - uv\) and let \(J_2\) be obtained from \(K\) by subdividing the edge \(uv\) by a new vertex \(w\). We also take \(u_1 = u, v_1 = v\) and \(u_2 = u, v_2 = w\). Then it is easy to verify that \(J_1, J_2\) satisfy the required properties. Indeed, in any acyclic \(r\)-coloring of \(J_1\), the vertices \(u_1 = u, v_1 = v\) must obtain the same colour, otherwise \(J_1 \cup uv = K\) would also be acyclically \(r\)-colorable. The same argument holds for \(J_2\) and \(u = u_2\) and \(v\), and therefore \(u_2 = u\) and \(v_2 = w\) must obtain different colors. Any path between \(u_1\) and \(v_1\) in \(J_1\) contains at least \(k\) vertices, otherwise \(K\) would contain a cycle shorter than \(k\).

We are ready to tackle the desired result for the dichromatic number.

**Theorem 2.4** There exists a function \(d = d(r, k)\) such that given \(r, k \geq 3\), the problem of acyclically \(r\)-coloring a digraph \(G\) of directed girth at least \(k\), and of in-degrees and out-degrees at most \(d = d(r, k)\), is NP-complete.

**Proof.** The proof is similar to the undirected case above. We again reduce from graph \(r\)-colorability. Let \(G\) be an instance, and replace each vertex \(x\) by an oriented binary tree \(T_x\) with \(\text{deg}(v)\) leaves. The tree is first rooted at a non-leaf vertex, then oriented away from the root. We will use two digraph gadgets, \(J_1\) with the following properties:

• \(J_1\) has vertices \(u_1, v_1\) such that any acyclic \(r\)-coloring of \(J_1\) assigns the same color to \(u_1\) and \(v_1\);

• \(J_2\) has vertices \(u_2, v_2\) such that any acyclic \(r\)-coloring of \(J_2\) assigns different colors to \(u_2\) and \(v_2\);

• \(J_1\) has directed girth at least \(k\) and contains no directed path with fewer than \(k\) vertices from \(u_1\) to \(v_1\); and

• \(J_2\) has directed girth at least \(k\).

Each directed edge \(st\) of each \(T_x\) will be replaced by \(J_1\) identifying \(s\) with \(u_1\) and \(t\) with \(v_1\), and each directed edge \(xy\) of \(G\) will similarly be replaced by a copy of \(J_2\) between the corresponding leaves of \(T_x\) and \(T_y\). The resulting graph \(G'\) has directed girth at least \(k\) because every directed cycle in \(G'\) is either inside a copy of some \(J_i\), or passes through some \(J_1\). Clearly, \(G'\) is acyclically \(r\)-colorable if and only if \(G\) is \(r\)-colorable in the usual sense. The in- and out-degrees of \(G'\) are maximized by three times the maximum in- and out-degree of any \(u_1, u_2, v_1, v_2\) in \(J_1, J_2\).

In this case there is again a simpler construction when \(r = 2\).

**Theorem 2.5** Given \(k \geq 3\), the problem of acyclic 2-coloring a digraph \(G\) of directed girth at least \(k\) and of in-degrees and out-degrees at most \(k + 1\), is NP-complete.
Proof. We proceed as in Theorem 2.2, reducing from the not-all-equal $k$-satisfiability problem with three occurrences of each variable $x$, replacing each clause with a disjoint directed $k$-cycle. To ensure that the three occurrences of a variable $x$ in clauses have the same value, we consider for each variable $x$ a separate digraph $H_k$ whose vertices are partitioned into $k$ sets $S_0, S_1, \ldots, S_{k-1}$ with $S_0$ of size one, $S_1$ an independent set of size three and each $S_i$ for $2 \leq i < k$ inducing a directed $k$-cycle. In addition, we include all edges from $S_i$ to $S_{i+1}$, and $S_{k-1}$ to $S_0$. In any acyclic two-coloring of $H_k$, each $S_i$ for $2 \leq i < k$ must have both colors, so the colors in $S_1$ must all be different from the color in $S_0$, and hence the same. The three elements of $S_1$ can thus be identified with the three occurrences of $x$.

3 High Girth Graphs and Digraphs

In this section we discuss the existence of high-girth graphs and digraphs without acyclic $r$-colorings. For ordinary graph $r$-colorings, we have the following well-known result of [6].

Theorem 3.1 For any $r, k \geq 3$, there exists a graph with girth at least $k$ which is not $r$-colorable.

For dichromatic number we have the following theorem [3].

Theorem 3.2 For any $r, k \geq 3$, there exists a digraph with directed girth at least $k$ which is not acyclically $r$-colorable.

A very general version of such results is proved in [9] (Theorem 5). The proof in [9] is probabilistic but there is a constructive proof in [14]. We refer the reader to [9] for the definition of a constraint satisfaction problem, the girth of an instance, and equivalence of problems. We explain below the special case sufficient for our applications here.

Theorem 3.3 For every constraint satisfaction problem $P$, any instance $I$ of $P$, and any integer $k \geq 3$, there exists an instance $I'$, equivalent to $I$, with girth at least $k$.

The $r$-valued not-all-equal $k$-satisfiability problem is an example of a constraint satisfaction problem. Here an instance is a set of variables $x_1, x_2, \ldots, x_n$ each taking one of $r$ possible values, and a set of constraints $C_1, C_2, \ldots, C_m$, each consisting of exactly $k$ variables. The solution of an instance is an assignment of values to the variables such that no constraint $C_i$ has all its variables assigned the same value. (This can also be viewed as the $r$-coloring problem of $k$-uniform hypergraphs; cf. the special case $r = 2$ in the proof of Theorem 2.2.) In this case, the girth of an instance is the smallest set of variables $y_0, y_1, \ldots, y_{k-1}$ such that any two consecutive $y_j, y_{j+1}$ (subscripts modulo $k$) occur together in some constraint $C_i$. We say that an instance $I$ is equivalent to an instance $I'$ if $I$ has a solution if and only if $I'$ has a solution. There obviously are instances without a solution, for example $n = (k - 1)r + 1$ variables and all $m = \binom{n}{k}$ constraints imposed on each subset of size $k$. (If each variable is assigned one of $r$ values, some $k$ variables will have the same value, so $I$ has no solution.) We obtain the following corollary of the theorem.

Theorem 3.4 For any $r, k \geq 3$, there exists an instance of $r$-valued not-all-equal $k$-satisfiability problem with girth at least $k$, which has no solution.
We can transform the instance into a digraph by taking a vertex for each variable \(x_i\) and form a directed \(k\)-cycle on any set of \(k\) variables occurring in a constraint \(C_i\). Clearly, this digraph has directed girth at least \(k\). We obtain a new proof of Theorem 3.2.

By replacing each constraint with an undirected \(k\)-cycle, we similarly conclude the following useful fact.

**Theorem 3.5** For any \(r, k \geq 3\), there exists a graph with girth at least \(k\) which is not acyclically \(r\)-colorable.

We close this section by noting that graph \(r\)-coloring is another example of a constraint satisfaction problem, and applying Theorem 3.3 to the graph \(K_{r+1}\) which is not \(r\)-colorable, we obtain Theorem 3.1.

The digraph \(K\) from Theorem 3.2 may be assumed to contain an edge \(uv\) such that \(K - uv\) is acyclically \(r\)-colorable, e.g., by assuming that \(K\) is minimal with respect to inclusion. A similar remark applies to the graph \(K\) from Theorem 3.5.

The obvious question is whether a more explicit construction for \(H\) and \(H'\) can be given, thus avoiding randomization [6, 9] or a more complex construction [14]. For example, in the case \(k = 3\), a random tournament as in Theorem 5.2 of size \(O(r \log r)\) suffices for \(H\), yet it remains hard to find \(H\) and \(H'\). Our construction below gives \(H = H'\) of size polynomial in \(k\) for \(r\) fixed, or polynomial in \(r\) for \(k\) fixed.

**Theorem 3.6** For every \(r \geq 1, k \geq 3\), there exists a digraph \(H_r^k\) with the following properties.

1. \(H_r^k\) has at most \(k^r\) vertices;
2. moreover, if \(k \leq r\), then \(H_r^k\) has at most
   \[
   k^{\left[k(1 + \log(\frac{r}{k}))\right]} \leq k^k \left(\frac{er}{k}\right)^{k \log k}
   \]
   vertices;
3. \(H_r^k\) can be constructed in time linear in the number of vertices;
4. \(H_r^k\) does not have an acyclic \(r\)-coloring;
5. for each edge \(uv\), the graph \(H_r^k - uv\) does have an acyclic \(r\)-coloring; and
6. \(H_r^k\) has directed girth \(k\).

This gives the bound \(d(r, k) \leq 3|V(H_r^k)|\) in Theorem 2.3.

**Proof.** We fix \(k\), and let \(H_r^k\) be a \(k\)-cycle, satisfying all conditions. For \(H_r^k\) with \(r \geq 2\), write \(r - 1 = a \left(\frac{r - 1}{k}\right) + b \left(\frac{r - 1}{k}\right)\) with \(a, b \geq 0\) and \(a + b = k\).

Let \(r' = r - 1 - \left\lfloor \frac{r - 1}{k} \right\rfloor\) and \(r'' = r - 1 - \left\lceil \frac{r - 1}{k} \right\rceil\). Define \(H_r^k\) as the disjoint union of \(a\) copies of \(H_r^{k'}\) and \(b\) copies of \(H_r^{k''}\), for a total of \(k\) copies, with all edges joining each such copy to the next, or the last one to first one.

We prove the last three conditions by induction on \(r\). The first \(a\) copies need at least \(r' + 1\) colors, avoiding at most only \(\left\lceil \frac{r - 1}{k} \right\rceil\) colors. The last \(b\) copies need at least \(r'' + 1\) colors, avoiding at most only \(\left\lfloor \frac{r - 1}{k} \right\rfloor\) colors. By the definition of \(a, b\), at most \(r - 1\) colors are avoided by at least one copy, so some color \(i\) appears in all the copies, and this gives a cycle of color \(i\) of length \(k\) across all the copies. This proves condition (4).
For condition (5), suppose the removed edge \(uv\) is inside the \(j\)th copy \(H_j\). Then \(H_j\) can be colored with only \(r'\) colors \((r''\) colors), giving one more color than in the definition of \(a, b\), for a total of \(r\) avoided colors across all the copies, so no color appears in all the copies, and there is no cycle across all the copies that gives the same color in all copies. This proves condition (5) in this case.

If the removed edge is \(uv\) joins say \(H_j, H_{j+1}\), then color \(H_j - u\) and \(H_{j+1} - v\) with only \(r'\) (or \(r''\)) colors by condition (5), avoiding one more color in each of \(H_j, H_{j+1}\), with only color \(i\) avoided in both cases. Assign color \(i\) to \(u, v\). This gives us again \(r - 1\) avoided colors, but including color \(i\) in all copies does not give a cycle of length \(k\) of color \(i\), since the cycle would have to go through edge \(uv\). This proves condition (5).

For condition (6), note that all cycles either go through only one copy and are thus inductively of length at least \(k\), or go through all the copies and must thus be of length at least \(k\).

For conditions (1, 2), note that each step of the induction has \(r', r'' \leq r(1 - \frac{1}{\min(r, k)})\) and \(k|V(H^k_r)| \geq |V(H^k_r)|\).

Note that this last result allows us to prove Theorems 2.5 and 2.3 without necessarily assuming that \(r, k\) are constants, but may depend on \(n\), for as long as the bound \(|V(H^k_r)| \leq n^{1-\epsilon}\) holds with \(\epsilon > 0\) constant.

4 Approximation

We now prove a hardness of approximation result.

**Lemma 4.1** Let \(0 < \epsilon < 1\) be a constant. Given a complete bipartite graph \(H = (U, V, E)\) with \(|U| = |V| = n\), let \(H'\) be the digraph obtained from \(H\) by orienting the edges in either direction independently with equal probability \(\frac{1}{2}\). Then with probability approaching 1 as \(n\) goes to infinity, for every two subsets \(U' \subseteq U, V' \subseteq V\) having \(|U'|, |V'| \geq n^{\epsilon}\), the digraph induced by \(U' \cup V'\) contains a cycle.

**Proof.** Say \(|U'| = |V'| = n^{\epsilon}\) and choose \(U', V'\) ordered in at most \(n^{2n^{\epsilon}}\) ways. If \(U' \cup V'\) induces an acyclic subgraph consistent with these ordering, then the order of the neighbors of a vertex \(u \in U'\), for some such order, will have first the edges incoming to \(u\) from \(V'\), then the outgoing edges from \(u\) to \(V'\). This can happen in \(n^{\epsilon} + 1\) ways out of \(2n^{\epsilon}\) possible choices for the edes joining \(u\). Multiplying resulting probability over all \(n^{\epsilon}\) choices of \(u\) from choices of subsets \(U', V'\) gives probability of there being \(U' \cup V'\) acyclic at most

\[n^{2n^{\epsilon}}(n^{\epsilon} + 1)n^{\epsilon - 2n^{2\epsilon}},\]

which tends to zero as \(n\) goes to infinity.

Noga Alon informed us that the known relatively recent explicit construction for bipartite Ramsey graphs [2] will give a derandomization of this lemma. Indeed, if \(U' \cup V'\) induces an acyclic digraph with a corresponding linear order \(L\), then the middle vertices of \(U', V'\) in \(L\) are \(u, v\), respectively. Say the edge joining \(u, v\) goes in the direction \(uv\). Then there are edges going from vertices below \(u\) in \(U'\) to vertices above \(v\) in \(V'\). (The other case is symmetric, from below \(v\) in \(V'\) to above \(u\) in \(U'\) if the direction is \(vu\)). We can define a bipartite graph from \(H'\) by including only edges from \(U\) to \(V\). Then we just saw that we would have either a bipartite clique or a bipartite independent set with \(k\) vertices in each side, \(k = n^{\epsilon}/2\). The bipartite Ramsey construction in [2] guarantees this does not happen even with \(k = n^{o(1)}\).
Feige and Kilian [10] proved that it is NP-hard to find an independent set of size greater than $n^\epsilon$ (thus hard to $n^{1-\epsilon}$ color) in a graph $G$ that is colorable with $n^\epsilon$ colors, for any $\epsilon > 0$. As a result, it is equally hard to find a large acyclic subgraph in a digraph, since we could replace the edges of $G$ with digons (girth 2), so that acyclic sugraphs in the resulting digraph correspond to independent sets in $G$.

**Theorem 4.2** Fix $0 < \epsilon < \frac{1}{4}$. It is NP-hard to find an acyclic induced subgraph of size greater that $N^{\frac{1}{2}+\epsilon}$ (thus hard to find an acyclic $N^{\frac{1}{2}-\epsilon}$ coloring) in an $N$-vertex digraph $G'$ without digons, i.e., of girth at least 3, that has an acyclic $r$-coloring with $r \leq N^\epsilon$.

**Proof.** Let $G$ be an instance of the NP-hard question of Feige and Kilian. For each vertex $v_i \in V(G)$, let $U_i$ be a set with $|U_i| = n$. For each edge $v_i v_j \in E(G)$, join $U_i, U_j$ with the random bipartite digraph as in the lemma (which can be derandomized). This gives a digraph $G'$ with $N = n^2$ vertices that has an acyclic $r$-coloring with $r \leq N^{\frac{1}{2}}$, by copying each color of a $v_i$ into the corresponding $U_i$.

However, an acyclic induced subgraph $S$ can only meet sets $U_i$ in at least $n^\epsilon$ vertices if the corresponding $v_i$ form an independent set, by the lemma. Therefore there will only be found $n^\epsilon$ such large intersections by the result of Feige and Kilian, giving us $n^{1+\epsilon}$ vertices of $S$, plus small intersections, of size at most $n^\epsilon$ for the remaining $U_i$, for a total $|S| \leq 2n^{1+\epsilon} = 2N^{\frac{1+\epsilon}{2}}$. \(\square\)

## 5 Random Tournaments

We begin with two simple observations to introduce random tournaments, as in [7].

**Theorem 5.1** Every tournament $T$ on $n$ vertices contains a transitive subtournament on $\lceil \log_2(n+1) \rceil$ vertices, and therefore has an acyclic $\frac{n}{(1-\epsilon) \log_2 n} + n^{1-\epsilon} = O\left(\frac{n}{\log n}\right)$ coloring.

**Proof.** Greedily select a vertex $v$ from $T$ of outdegree at least $\frac{n-1}{2}$, remove $v$ and its in neighbors from $T$ to obtain $T'$ of size at least $\frac{n+1}{2}$. This halving can be done $\left\lceil \log_2(n+1) \right\rceil - 1$ times. The chosen $\left\lceil \log_2(n+1) \right\rceil$ vertices $v$ will form a transitive tournament. For the acyclic coloring, select and remove greedily transitive tournaments from $T$, each of size at least $(\log_2 n)1 - \epsilon$, until we reach a tournament $T'$ of size at most $n^{1-\epsilon}$, and we use at most these many colors for $T'$. \(\square\)

Random tournaments essentially match the preceding bound.

**Theorem 5.2** With high probability, a tournament $T$ on $n$ vertices only contains a transitive subtournament on $O(\log n)$ vertices, and therefore only has an acyclic $\Omega(\frac{1}{\log n})$-coloring.

**Proof.** Selecting a sequence of $\lceil 3 \log n \rceil$ distinct vertices $v$ of $T$ can be done in at most $s = n^{3 \log n}$ ways. The probability that such a sequence will give the ordering of a transitive tournament is at most $\frac{1}{t}$ for $t = 2^{(3 \log n - 1)}$. The ratio $\frac{1}{t}$ tends to 0 as $n$ goes to infinity. \(\square\)

We now define a random model for acyclic $r$-colorable tournaments $T$ on $n$ vertices. Consider $r$ integers $s_1 \geq s_2 \geq \cdots \geq s_r \geq 1$ with $s_1 + s_2 + \cdots + s_r = n$. To define $T$, first consider $r$ disjoint sets of vertices $S_i$ with $|S_i| = s_i$, and impose on each $S_i$ a transitive (acyclic) tournament. Finally, orient each edge joining vertices in two different $S_i$ independently with probability $\frac{1}{2}$ in either direction.
The tournaments $T$ so generated have an acyclic $r$-coloring, obtained by assigning color $i$ to the vertices in $S_i$. We consider the problem of acyclic $r$-coloring such a $T$ when the vertices of $T$ are given in arbitrary order, and the $S_i$ and $s_i$ are not given. We give a deterministic algorithm that finds such a coloring with probability arbitrarily close to 1. If $r$ is fixed, the algorithm runs in time $O(n^2)$, linear in the size of the input $T$.

The algorithm runs in three phases, which we describe below.

The first phase starts with the tournament $T_{n_1,r_1} = T$ with $n_1 = n$ and $r_1 = r$, and operates in rounds. At the beginning of the $j$th round, we have $T_{n_j,r_j}$.

We define $d_j = c\sqrt{n_j \log n_j}$ for some constant $c$. Given a tournament $R$ and a vertex $v$ in $R$, we define

$$d^R_{\text{diff}}(v) = d^R_{\text{out}}(v) - d^R_{\text{in}}(v)$$

as the difference between the out-degree and the in-degree of $v$ in $R$. Note that $E(d^T_{\text{diff}}^{S_i,v}(v)) = d^T_{\text{diff}}^{S_i}(v)$ if $v \in S_i$. Consider the Chernoff bounds for $X$ equal to the sum of independent Bernoulli random variables, with $\mu = E(X)$.

$$Pr(X \leq (1 - \delta)\mu) \leq e^{-\frac{\delta^2 \mu}{2}}, 0 \leq \delta \leq 1,$$

$$Pr(X \geq (1 + \delta)\mu) \leq e^{-\frac{\delta^2 \mu}{3}}, 0 \leq \delta \leq 1.$$

Letting

$$X = d^{T_{n_j,r_j}}_{\text{out}}(v) - d^{S_i}_{\text{out}}(v)$$

we have that

$$Pr(|X - \frac{n_j - s_i}{2}| \geq d_j) \leq 2e^{-\frac{2d_j^2}{n(n_j - s_i)}}$$

$$\leq 2e^{-\frac{\frac{1}{4}n^2 \log^2 n_j}{}},$$

and therefore

$$Pr(|d^{T_{n_j,r_j}}_{\text{diff}}(v) - d^{S_i}_{\text{diff}}(v)| \geq 2d_j) \leq 2e^{-\frac{\frac{1}{4}n^2 \log^2 n_j}{}},$$

The probability that this holds for all $v$ in $T_{n_j,r_j}$ is at most $n_j$ times the bound. Let $u^*$ be the vertex that maximizes $|d^{T_{n_j,r_j}}_{\text{diff}}(v)|$, say the quantity within the absolute value is nonnegative. Then $u^* \in S^* = S_i^*$ with $s^* = |S^*|$. Let $S_1$ be the largest of the $S_i$ in $T_{n_j,r_j}$, and let $u_1$ be the starting vertex of the transitive tournament $S_1$. Then

$$|S_1| - |S^*| \leq d^{S_i}_{\text{diff}}(u_1) - d^{S^*}_{\text{diff}}(u^*)$$

$$\leq (d^{S_i}_{\text{diff}}(u_1) - d^{T_{n_j,r_j}}_{\text{diff}}(u_1)) + (d^{T_{n_j,r_j}}_{\text{diff}}(u_1) - d^{T_{n_j,r_j}}_{\text{diff}}(u^*)) + (d^{T_{n_j,r_j}}_{\text{diff}}(u^*) - d^{S^*}_{\text{diff}}(u^*))$$

$$\leq 2d_j + 0 + 2d_j = 4d_j$$

with probability at least $1 - 2(n_j + 1)e^{-\frac{\frac{1}{4}n^2 \log^2 n_j}{}},$

The algorithm seeks to determine $S^*$ given the vertex $u^* \in S^*$. For $v \neq u^*$, and $w \neq u^*, v$, the probability that $uwvw$ is a directed 3-cycle in either direction is $\frac{1}{4}$, unless $v, w \in S^*$, in which case the probability is zero. Let $X(v)$ be the number of $uw$ that form such a directed 3-cycle with $u^*$ and $v$. Then $E(X(v)) = \frac{1}{4}(n - 2)$ if $v \notin S^*$, and $E(X(v)) = \frac{1}{4}(n - s^*)$ if $v \in S^*$. Then

$$Pr(|X(v) - E(X(v))| \geq d_j) \leq 2n_j e^{-\frac{dd_j^2}{2n_j}}.$$
\[ \leq 2n_j e^{-\frac{3}{4}c^2 \log^2 n_j}, \]

where the factor of \( n_j \) is needed to account for the fact that \( u^* \) could be any vertex in \( T_{n_j, r_j} \). With probability \( n_j \) times this much this holds for all \( X(v) \).

Suppose \( s^* - 2 > 2d_j \). With probability at least \( 1 - 4n_j^2 e^{-\frac{3}{4}c^2 \log^2 n_j} \), vertices \( v \in S^* \) have \( X(v) \leq A' = \frac{1}{4} (n - s^*) + d_j \), and vertices \( v \not\in S^* \) have \( X(v) \geq B' = \frac{1}{4} (n - 2) - d_j \) and \( B' > A' \).

Thus for as long as \( s_1 > 6d_j + 2 \), we have \( s^* > 2d_j + 2 \), and the algorithm can determine \( u^* \) and test the \( A', B' \) bounds to determine \( S^* \). This works with probability at least

\[
1 - p_j = 1 - 2(n_j + 1) e^{-\frac{3}{4}c^2 \log^2 n_j} - 4n_j^2 e^{-\frac{3}{4}c^2 \log^2 n_j} = 1 - e^{-(1-\epsilon_j)\frac{3}{4}(c^2 \log^2 n_j)}
\]

where \( \epsilon_j \leq \frac{3}{c^2 \log^2 n_j} \).

**Lemma 5.3** Suppose the \( j \)th round starts with \( T_{n_j, r_j} \). Let \( d_j = c\sqrt{n_j} \log n_j \). With probability at least \( 1 - p_j \), if the largest \( S_1 \) has \( s_1 > 6d_j + 2 \), then we find \( S^* \) with \( s^* \geq s_1 - 4d_j > 2d_j + 2 \), and reduce the problem to \( T_{n_{j+1}, r_{j+1}} = T_{n_j - s^*, r_j - 1} \).

The last round of the first phase takes \( T_{n_j, r_j} \) to \( T_{n_{j+1}, r_{j+1}} \). We let \( n^- = n_j, r^- = r_j, n' = n_{j+1}, r' = r_{j+1} \), and \( j = j \) for this \( j \). Note that after phase one is over, we have \( s'_1 \leq 6d_j + 2 \leq 2 + 6c\sqrt{n'} \log n' \) with \( s'_1 \) and \( d' \) defined similarly with \( j = \hat{j} \). When phase one no longer applies, \( r' \geq \frac{n'}{s'_1} \geq \frac{n'}{c \log n} \).

In particular, if \( r = O(1) \), then \( r' = O(1) \) and \( n' = O(1) \) and the rest of the problem can be solved in \( O(1) \) time.

We may assume \( n^- > \max \left( \frac{\log n}{\log r}, \frac{\log n}{\log n'} \right) \) since otherwise the problem can be solved in linear time avoiding the \( \hat{j} \)th round.

**Theorem 5.4** Finding \( S^* \) takes \( O(n^2) \) time, linear in the size of the input \( T \). This yields a total running time of \( O(rn^2) \) over at most \( r \) rounds. The probability bound for the first phase is \( 1 - n_j p_j = 1 - e^{-\Omega(c^2 \log^2 n^-)} \) with \( j = \hat{j} \).

For \( r \) constant, we can find an acyclic \( r \)-coloring of \( T \) on \( n \) vertices in time \( O(n^2) \), linear in the size of the input, with probability as above.

After the first phase is over, we have \( T' = T_{n', r'} \). The second phase first identifies all sets \( U \) in \( T' \) with \( |U| \leq c \log n' \) that could be the least elements of an \( S_i \). The number of possible such sets \( U \) is at most \( n' c \log n' \).

First \( U \) must be a transitive tournament. Suppose \( U \) of size \( c \log n' \) is the bottom of an \( S_i \). Let \( V \) be \( U \) plus all the elements above all of \( U \).

We claim that \( |V \setminus S_i| < c \log n' \). Otherwise choose \( W \) with \( |W| = c \log n' \) contained in \( V \setminus S_i \). The sets \( U, W \) are joined by edges joining different colors, thus the probability that they will all be oriented upwards is at most \( 2^{-c^2 \log^2 n'} \). There are at most \( n'^{2c \log n'} \) possible choices of \( V, W \), so with probability at least \( 1 - e^{(2c - c^2 \log 2) \log^2 n'} \) the claim follows.

Suppose \( k_0 \geq c_0 \log_2 n' \) for some sufficiently large constant \( c_0 \). The algorithm repeatedly chooses sets \( Z = V \setminus W \) that define transitive (acyclic) tournaments within \( T_{n', r'} \). The sets \( Z \) are considered in nonincreasing order of \( z = |Z| \geq k_0 \). We claim that with high probability, we will have \( Z = S_i \) for one of the sets \( S_i \), so we choose such a \( Z \) and discard all later \( Z' \) that intersect \( Z \), with \( |Z'| \leq |Z| \). The algorithm ends when there are no more \( Z \) with \( z \geq k_0 \).
Theorem 5.5 The second phase correctly selects the remaining $S_i$ with $s_i \geq k_0$, with probability at least $1 - e^{(2c_r-\epsilon)\log 2}n^2 - 2^{-\frac{k_0}{\log 2}+1}$. for $k_0 \geq 24\log n'$ and $n'$ sufficiently large. The running time is bounded by $O(e^{3c\log 2 n'})$.

Proof. It only remains to show that all chosen $Z$ are $S_i$, with probability at least $1 - 2^{-\frac{k_0}{\log 2}+1}$. If not, $Z$ meets at least two $S_i$. Let $Z = \bigcup Z_i$, where $Z_i = Z \cap S_i$, with $z_i = |Z_i|$. Order the $Z_i$ in decreasing order of $z_i$, and let $i$ be such that $\sum_{i<\hat{i}} z_i < \frac{\hat{i}}{2}$ and $\sum_{i>\hat{i}} z_i < \frac{\hat{i}}{2}$.

Let $\ell = z - z_i$. The $z_i$ can be partitioned into two sets into one of the two cases $A = \{z_1, \ldots, z_i\}, B = \{z_{i+1}, \ldots, z_l\}$ or $A = \{z_1, \ldots, z_{\hat{i}-1}\}, B = \{z_{\hat{i}}, \ldots, z_l\}$, and one of these two partitions has corresponding sizes at least $\frac{\hat{i}}{2}, \frac{\hat{i}}{2}$.

Once the edges within $A$ with $|A| \geq \frac{\hat{i}}{2}$ have been oriented, the probability that a vertex $w$ in $B$ with $|B| \geq \frac{\hat{i}}{2}$ will fit in some order among $A$ is $(|A| + 1)2^{-|A|} = (\frac{\hat{i}}{2})2^{-\frac{\hat{i}}{2}}, \text{ or } ((\frac{\hat{i}}{2})2^{-\frac{\hat{i}}{2}})^\ell$ over $B$. The $\ell$ vertices in $Z - Z_1$ and at most $\ell$ vertices in $S_1 - Z_1$ (since $s_1 \leq z$) can be chosen in at most $n^{2\ell}$ ways, giving the probability bound

$$n^{2\ell} ((\frac{\hat{i}}{2})2^{-\frac{\hat{i}}{2}})^\ell$$

$$= 2^{-\ell(\sum_{i=0}^{\log 2 n - \log 2 \frac{\hat{i}}{2}, \frac{\hat{i}}{2}})}$$

$$= 2^{-\ell(1-(\log 2 \epsilon)(\log n - \frac{\log \frac{\hat{i}}{2}}{\log 2} + \frac{2\log 2}{\log 2}))}$$

$$\leq 2^{-\ell \frac{\epsilon n}{2}}$$

$$\leq 2^{-\ell \frac{k_0}{\log 2}}$$

for $z \geq k_0 \geq 24\log n'$ and $n'$ sufficiently large. Summing over all $\ell \geq 1$ gives the bound $\leq 2^{-\frac{k_0}{\log 2}+1}$. \[\Box\]

The third phase starts with a resultant $T_{n',r''}$ and possible sets $Z$ with $z \leq 24\log n'$, so $r'' \geq \frac{n'}{24\log n'}$. Each $S_i$ has $e^{2c\log 2 n'}$ choices of $U_i, W_i$, for a total of $e^{2c'\log 2 n'}$ choices for the $r''$ sets $S_i$ to be selected, giving running time $O(n''e^{2c'\log 2 n'})$.

The third phase seems the most expensive, since the running time is exponential in $r$ versus quasi-polynomial in the second phase, and polynomial in the first phase. We can avoid the third phase by approximating the bound $24\log n'$ on color classes with the bound $\log_2 n$ from Theorem 5.1, for an approximation factor of $24\log 2$.

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