On a Generalization of the van der Waerden Theorem

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Abstract

For a given length and a given degree and an arbitrary partition of the positive integers, there always is a cell containing a polynomial progression of that length and that degree; moreover, the coefficients of the generating polynomial can be chosen from a given subsemigroup and one can prescribe the occurring powers. A multidimensional version is included.

1 Introduction

A sequence in $\mathbb{R}$ will be called a polynomial progression if it is of the form $\{P(1), P(2), P(3), \ldots\}$ for some polynomial $P(x) = a_dx^d + a_{d-1}x^{d-1} + \cdots + a_1x + a_0$. This progression is said to be of degree $d$ if $P$ has degree equal to $d$ and not less.

Theorem 1. Given two positive integers $d$ and $l$, if the set of the positive integers is split up into finitely many non-overlapping parts, there exists a polynomial progression of length $l$ and of degree $d$ that belongs to precisely one of these parts.

For $d = 1$ the polynomials look like $P(x) = a + bx$ and the $l$-segment of the polynomial progression takes the form $\{a + b, a + 2b, a + 3b, \ldots, a + lb\}$: the theorem boils down to the well-known van der Waerden Theorem on monochromatic arithmetic progressions. It is fun to write down the $d = 2$ case.

Corollary 1. Given any $l \in \mathbb{N}$ and any finite coloring of $\mathbb{N}$, there exist three positive integers $a, b$ and $c$ for which all terms in $\{a + b + c, a + 2b + 4c, a + 3b + 9c, \ldots, a + lb + l^2c\}$ have the same color.

The 1927 proof of van der Waerden’s Theorem is quite complicated, involving a double induction argument. The 1927 issue of the journal, $[3]$, is difficult to access nowadays, but a very clear exposition is found in R.L. Graham, B.L. Rothschild and J.H. Spencer ($[1]$), pp 29 – 34. As B.L. van der Waerden once remarked, around 1927 he was not aware of the impact of his result as a prototypical Ramsey Theorem - after all, Ramsey’s famous paper stems from 1930 - and merely considered it as a clever exercise. A proof of the above
theorem by means of induction seems a Sisyphean task. We rather use some ideal theory in the semigroup $\beta \mathbb{N}$. As a matter of fact, the argument in the HINDMAN-strauss treatise $^{(2)}$ for the van der Waerden theorem (see 14.1 l.c.) is readily adapted to the present situation. By preferring the smooth $\beta \mathbb{N}$-argument to a complicated induction proof we ignore the calvinistic concern (see $^{(2)}$ p.280) that it “is enough to make someone raised on the work ethic feel guilty”.

The more restrictions one puts on the admissible polynomials, the fewer polynomials one has at his/her disposal and the more difficult it seems to force the ensuing polynomial progressions into one and the same cell. The polynomials $P(x) = \sum_{k=0}^{d} a_k x^k$ we admit here satisfy

- the admissible coefficients $a_k$ belong to one and the same subsemigroup $S$ of $(\omega, +)$, where $\omega = \mathbb{N} \cup 0$;
- the admissible exponents in the powers $x^k$ belong to a subset $D \subset \{0, 1, 2, \ldots, d\}$ containing $d$.

Such polynomials will be called $(S,D)$-polynomials.

The sharpened theorem reads

**Theorem 2.** Given two positive integers $d$ and $l$, if the set of the positive integers is split up into finitely many non-overlapping parts, there exists a polynomial progression of length $l$ and of degree $d$, generated by a $(S,D)$-polynomial, that belongs to precisely one of these parts.

## 2 Proof

Since Theorem 1 concerns the special case where $S = \omega$ and $D = \{0, 1, 2, \ldots, d\}$, we only need to prove Theorem 2.

Fix $d$ and $l$ in $\mathbb{N} = \{1, 2, 3\ldots\}$. Without loss of generality we may assume that $l > d$. In fact, once the theorem has been proved for “long” progressions (that is $l > d$), then the pertinent cell certainly contains shorter segments ($l \leq d$). We consider polynomials $P(x) = \sum_{i=0}^{d} a_i x^i$ in one indeterminate $x$ of degree $\leq d$ with coefficients in $\omega^{d+1}$. Consider the following sets $S_o$ and $I_o$ in $\omega^l$ consisting of $l$ consecutive polynomial values

$$S_o = \{\{P(1), P(2), \ldots, P(l)\} \in \omega^l : P(x) = \sum_{k \in D} a_k x^k, \text{ with } \{a_0, a_1, \ldots, a_d\} \in \mathbb{S}^{d+1}\}$$

$$I_o = \{\{P(1), P(2), \ldots, P(l)\} \in \mathbb{N}^l : P(x) = \sum_{k \in D} a_k x^k, \text{ with } \{a_0, a_1, \ldots, a_d\} \in (\mathbb{S} \cap \mathbb{N})^{d+1}\}$$

The impact of the assumption that $l > d$ is that each element in $S_o$ corresponds to a unique polynomial. In fact, if such an $l$-tuple would be generated by two different polynomials, the difference of these polynomials would have more zeros (viz. at the $l$ points $1, 2, \ldots, l$ in $\mathbb{C}$) than its degree $d < l$ permits.

$S_o$ is a subsemigroup of $\mathbb{S}^{d+1}$ under coordinatewise addition, the restrictions $k \in D$ meaning that only addition of coordinates $k$ from $D$ matters. In fact, the sum of two $l$-tuples
in $S_o$ corresponds to the sum of their unique polynomials and the latter is again a polynomial of degree $\leq d$ with coefficients in the semigroup $S$.

The progressions $\{P(1), P(2), \ldots, P(l)\}$ in $I_o$ all have degree $= d$, since $a_d \geq 1$. It follows that $I_o$ is a proper subset of $S_o$. Obviously, $I_o$ is also a semigroup. Moreover, $I_o$ is an ideal in $S_o$. In fact, upon adding any point in $S_o$ to an arbitrary element of $I_o$, all coefficients of the sum polynomial are again $\geq 1$ and this polynomial is of exact degree $d$. Although trivial, we notice that $S_o$ contains constant $\mathbb{N}$-valued polynomials, but $I_o$ contains none of these. This will be instrumental shortly.

Consider the Stone-Čech compactification $\beta \omega$. We are going to use a few facts about $\beta \omega$ that are found in N. Hindman and D. Strauss [2]. We find it convenient to ignore the slight differences in the ideal theory between the two semigroups (see [2], Chap. 4) $\beta \omega$ and $\beta \mathbb{N}$, writing $\beta \mathbb{N}$ where $\beta \omega$ would sometimes be more appropriate. From this point on we can follow the proof of the van der Waerden Theorem in [2], Theorem 14.1, almost verbatim.

Take the compact product space $Y = (\beta \mathbb{N})^I$ and the closures $S = cl_Y(S_o)$ and $I = cl_Y(I_o)$. The semigroup $\beta \mathbb{N}$ has a smallest ideal $K(\beta \mathbb{N}) \neq \emptyset$ (see [2], Chap 4), which will be our main tool.

Take any point $p \in K(\beta \mathbb{N})$ and consider the constant $l$-tuple $\vec{p} = \{p,p,\ldots,p\}$. The crucial step is to show that $\vec{p}$ belongs to $S$.

The closures $cl_{\beta \mathbb{N}} B$ of the members $B \in p$ form a neighborhood basis in $\beta \mathbb{N}$ around $p$. It follows that for the product topology in $Y$ there exist members $B_1, B_2, \ldots, B_r \in p$ for which the box $U = \prod^{1 \leq i \leq r}_{1 \leq i \leq r} cl_{\beta \mathbb{N}}(B_i)$ is a $Y$-neighborhood of $\vec{p}$. The intersection $\cap_{1 \leq i \leq r} cl_{\beta \mathbb{N}}(B_i)$ is a $\beta \mathbb{N}$-neighborhood of $p$. The set $N$ lying dense in $\beta \mathbb{N}$, it is intersected by this neighborhood. Select $a \in N \cap (\cap_{1 \leq i \leq r} cl_{\beta \mathbb{N}}(B_i))$. The constant $l$-string $\vec{a} = \{a, a, \ldots, a\}$ thus belongs to $U$. Also, $S_o$ containing all constant $l$-tuples, we have $\vec{a} \in S_o$. Consequently, we have $\vec{a} \in S_o \cap U$. This shows that $\vec{p}$ belongs to the closure of $S_o$ in $Y$, and so $\vec{p} \in S$, indeed.

Next we use the fact that by [2], Theorem 2.23, the $K$-functor preserves products. From $p \in K(\beta \mathbb{N})$ we infer $\vec{p} \in (K(\beta \mathbb{N}))^I = K((\beta \mathbb{N})^I) = K(Y)$. Conclusion: $\vec{p} \in S \cap K(Y)$.

Having shown that $S \cap K(Y) \neq \emptyset$, we can invoke [2], Theorem 1.65 to determine the smallest ideal of the semigroup $S$: it simply is $K(S) = S \cap K(Y)$. This leads to

$$\vec{p} \in K(S).$$

(2.1)

Obviously, $I$ is an ideal in $S$. The smallest ideal in $S$ is contained in $I$: $K(S) \subset I$. It follows from (2.1) that $\vec{p} \in I$.

Finally, let $N = \bigcup_i A_i$ be a finite partition. The closures $\vec{A}_i = cl_{\beta \mathbb{N}} A_i$ are open and form a partition of $\beta \mathbb{N}$. Hence, precisely one of them, $\vec{A}_j$ say, is a $\beta \mathbb{N}$-neighborhood of our point $p \in K(\beta \mathbb{N})$. Then $V = (\vec{A}_j)^I$ is a $Y$-neighborhood of $\vec{p}$. Because $\vec{p} \in I$, $V$ must meet the dense subset $I_o$ of $I$ and we can select a polynomial $P$ in in such a manner that $\{P(1), P(2), \ldots, P(l)\}$ belongs to $V$. But the $P(1), P(2), \ldots, P(l)$ still are integers in $\mathbb{N}$. For this reason

$$\{P(1), P(2), \ldots, P(l)\} \subset \vec{A}_j \cup \mathbb{N} = A_j$$

and the segment $\{P(1), P(2), \ldots, P(l)\}$ has the color of $A_j$. [2]
3 Free gifts

The essential property of the set $A_j$ used in the last part of the above proof is the fact that $A_j$ contains a point $p$ belonging to $K(\beta N)$, or $A_j \in p$. Sets $A \subset N$ belonging to some $p \in K(\beta N)$ are called piecewise syndetic sets. We recall that in terms of $N$ itself, $A$ is piecewise syndetic if and only if the gaps between its intervals of consecutive elements remain bounded in lengths, (see (2) Theorem 4.40). It follows that $A_j$ may be replaced by any infinite piecewise syndetic set $A$ and we get as a

**Bonus 3.** Given a piecewise syndetic set $A \subset N$, a length $l$ and a degree $d$, there exists a polynomial progression of degree $d$ for which the first $l$ terms belong to $A$.

Finally we consider a multidimensional version of the theorem, dealing with $m$ polynomial progressions of varying lengths and degrees simultaneously.

**Bonus 4.** Pick the following items in $N$: a dimension parameter $m$, degrees $d_1, d_2, \ldots, d_m$, and lengths $l_1, l_2, \ldots, l_m$. If the set $N$ is split up into finitely many non-overlapping parts, there exist $m$ polynomial progressions of length $l_i$ and of degree $d_i$ each, $1 \leq i \leq m$, that simultaneously belong to one of these parts. Also, any given piecewise syndetic set contains such a collection of polynomial progressions.

**Remark** There is an obvious $(S, D)$ version.

**Proof.** We introduce arrays

$$\mathcal{P} = \begin{pmatrix}
P_1(1) & P_1(2) \cdots P_1(l_1) \\
P_2(1) & P_2(2) \cdots P_2(l_2) \\
\vdots & \vdots \\
P_m(1) & P_m(2) \cdots P_m(l_m)
\end{pmatrix}$$

generated by polynomials $P_1, P_2, \ldots, P_m$ with coefficients from $\omega$.

These arrays $\mathcal{P}$ need not have the customary rectangular form, the $i^{th}$ row having $l_i$ entries. Extending these rows by putting zeros in the empty places until they all get max\{$l_i : i = 1, 2, \ldots, m$\} entries would unnecessarily complicate the definition of $I_o infra.$

We have avoided to call these $\mathcal{P}$ matrices since they are not intended to act as transformations in some vector space. In order to describe the set these arrays belong to we write $e_i = \{0, \ldots, 1, 0, \ldots, 0\} = \delta_{ij}$ for the usual unit vectors in $\mathbb{R}^m$. These unit vectors are customarily envisaged as rows; upon transposition we get the unit columns $e_i^T$. The direct sum decomposition

$$\omega^m = e_1^T \omega \oplus e_2^T \omega \oplus \cdots \oplus e_m^T \omega$$

divides $\omega^m$, and thereby $N^m$, into $m$ horizontal layers, each equal to $N$ and each row is an additive semigroup at its own.

$$e_i^T N = \begin{pmatrix}
0 & \cdots & 1 & 2 & 3 & 4 & 5 & \cdots \\
0 & \cdots \\
0
\end{pmatrix}$$
Picture: the $i^{th}$ row of an array $P$ is contained in the $i^{th}$ layer.

Upon replacing the $l$-tuples in the definitions of $S_o$ and $I_o$ by the arrays $P$ we get

$$S_o = \{ P \in \bigoplus_{i=1}^{m} e_i^T \mathbb{N} : P_i(x) = \sum_{k=0}^{d_i} a_{ki}x^k, \text{ with } \{a_0, a_1, \ldots, a_{d_i}\} \in \omega^{d_i+1} \text{ for } i = 1, 2, \ldots, m \},$$

$$I_o = \{ P \in \bigoplus_{i=1}^{m} e_i^T \mathbb{N} : P_i(x) = \sum_{k=0}^{d_i} a_{ki}x^k, \text{ with } \{a_0, a_1, \ldots, a_{d_i}\} \in \mathbb{N}^{d_i+1} \text{ for } i = 1, 2, \ldots, m \}.$$

These are subsemigroups of the $\bigoplus_{i=1}^{m} e_i^T \mathbb{N}$ and $I_o$ is a proper ideal in $S_o$.

We refrain from repeating all details the above proof for the $m = 1$ case.

For a start, we may assume without loss of generality that $l_1 > d_1, l_2 > d_2, \ldots, l_m > d_m$. Define $l = \max_{1 \leq i \leq m} l_i$. This time we have to deal with the compact space $Y^m$, one $Y = (\beta \mathbb{N})^l$ for each layer, so that $Y^m = (\beta \mathbb{N})^{lm}$. The closure $I = \text{cl}_{Y^m}(I_o)$ is an ideal in the semigroup $S = \text{cl}_{Y^m}(S_o)$

To every $p \in K(\beta \mathbb{N})$ we assign the constant $m \times l$ array

$$\bar{p} = \begin{pmatrix} p & p \cdots & p \\ p & p \cdots & p \\ \vdots \\ p & p \cdots & p \end{pmatrix}$$

After a little twist the above argument leads to $\bar{p} \in K(S) \subset I$. For a piecewise syndetic set $A \in p$ the product $V = (\bar{A})^l$ is a $Y^m$-neighborhood of $\bar{p}$ which intersects the dense subset $I_o$ of $I$ in at least one point. This point is an array $P$, say. It follows that the entries $P_i(j)$ of $P$ belong to $\text{cl}_{Y^m}A$ and thus to $\text{cl}_{\beta \mathbb{N}}A$. All $P_i(j)$ being positive integers, we may write

$$\bigcup \{P_i(j) : i = 1, \ldots, m; j = 1, \ldots, l_i\} \subset \bar{A} \cap \mathbb{N}.$$

Conclusion: these $m$ polynomial progressions do lie in $A$ itself.

References

[1] R.L. GRAHAM, B.L. ROTHSCHILD AND J.H. SPENCER (1990), Ramsey Theory; Second Edition; New York: John Wiley & Sons

[2] N. HINDMAN AND D. STRAUSS (1998), Algebra in the Stone-Čech compactification, Theory and Applications; Berlin: W. De Gruyter

[3] B.L. VAN DER WAERDEN (1927), Beweis einer Baudetschen Vermutung, Nieuw Archief voor Wiskunde 19, 212 – 216.