Non-reciprocal elastic wave propagation in spatiotemporal periodic structures

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Abstract

We study longitudinal and transverse wave propagation in beams with elastic properties that are periodically varying in space and time. Spatiotemporal modulation of the elastic properties breaks mechanical reciprocity and induces one-way propagation. We follow an analytic approach to characterize the non-reciprocal behavior of the structures by analyzing the symmetry breaking of the dispersion spectrum, which results in the formation of directional band gaps and produces shifts of the first Brillouin zone limits. This approach allows us to relate position and width of the directional band gaps to the modulation parameters. Moreover, we identify the critical values of the modulation speed to maximize the non-reciprocal effect. We numerically verify the theoretical predictions by using a finite element model of the modulated beams to compute the transient response of the structure. We compute the two-dimensional Fourier transform of the collected displacement fields to calculate numerical band diagrams, showing excellent agreement between theoretical and numerical dispersion diagrams.

1. Introduction

Reciprocity of wave propagation is a fundamental principle of many wave phenomena and applies in electromagnetism (Landau and Lifshitz 1960), optics (von Helmholtz 1924) and acoustics (Pierce 1981), to name a few. Loosely, it states that waves propagate symmetrically in space from one point to the other, no matter which one is the source or the receiver. A growing area of research is concerned with the possibility of breaking this form of symmetry in order to realize one-way propagation which is highly desirable in many technological applications. In acoustics, for instance, it might be expedient to protect a source from its echo or achieve full-duplex sound communication. Furthermore, non-reciprocal devices can be used to realize one-way filters and isolation (Fleury et al 2015, Cummer et al 2016). In mechanical systems, weak non-reciprocal wave propagation occurs in rotating rings, in which the rotation introduces a mechanical bias that leads to different wavenumber and phase velocity values for waves propagating in opposite directions (Huang et al 2013). The mechanical bias is due to Coriolis forces (Beli et al 2015), which are responsible for breaking reciprocity (Onsager 1931a). A similar non-reciprocal behavior is observed in systems subjected to a magnetic field (Onsager 1931b). Strong non-reciprocal effect are achieved by relaxing some of the assumptions of the Onsager–Casimir principle of microscopic reversibility (Casimir 1945), which states that reciprocity is not guaranteed when nonlinearity or time-dependent material properties are exploited (Fleury et al 2015). Therefore, considerable efforts have been devoted to achieve non-reciprocal behavior through nonlinear or time-modulated devices. Indeed, giant non-reciprocal transmission can be obtained by combining a nonlinear medium with a superlattice (Liang et al 2009, 2010) or with a gain/loss pair (Gu et al 2016), hence exploiting asymmetric frequency conversion of the nonlinear medium due to second-harmonic generation (SHG) and frequency selectivity of sonic crystals. Experimental evidence shows rectifying ratios up to $10^4$, although efficacy of such nonlinear devices depends on the amplitude of the input signal, which has to be large enough to trigger the SHG mechanism of the nonlinear medium (Liang et al 2010). Another way to exploit nonlinearity is to use a
compact active acoustic metamaterial coupled to a nonlinear electronic circuit (Popa and Cummer 2014), which produces an isolation factor $>10$ dB. Several recent studies have investigated the possibility of modifying in time the material properties of the system to achieve non-reciprocal behavior. For example, one-way acoustic isolation has been shown in graphene based nanoelectromechanical systems (Zanjani et al 2015), acoustic waveguides (Zanjani et al 2014), acoustic circulators (Fleury et al 2014) and time-dependent superlattices (Swinteck et al 2015). The common strategy in these studies is to extend to the time domain the spatial-only periodic variation of material properties that in general is associated with metamaterials. This strategy thus exploits spatiotemporal modulated systems to manipulate wave propagation (Yu and Fan 2009, Sounas et al 2013, Wang et al 2013, Hadad et al 2015). Space and time modulation of the medium supporting wave propagation already gained the attention of the scientific community many decades ago. For instance, Slater (1958) studied the scattering of an electronic wave by a sinusoidal perturbation. In this problem, a particle is moving in a region with potential energy having the form of a traveling plane wave $V = V_0 \cos(\omega t - \mathbf{k} \cdot \mathbf{r})$. The Schrödinger’s equation for this particle, in the one dimensional case, reduces to a Mathieu’s equation (Brillouin 1953). A perturbation approach is employed to obtain the first three harmonics $n = 0, \pm 1$ of the wave function, under the assumption that the amplitude of modulation $V_0$ of the potential energy is small. Moreover, an approximate relation between the energy $\hbar \omega_0$ and the wavevector $\mathbf{k}_0$ is derived, which is able to predict the first order stop-band or Bragg reflection, but fails at describing higher-order stop-bands. Simon (1960) and Hessel et al (Hessel and Oliner 1961) suggested a more general approach to study the wave propagation of an electromagnetic wave in a medium with sinusoidal disturbance, i.e. with dielectric constant $\varepsilon = \varepsilon_0 + \varepsilon_1 \cos(\omega t - \mathbf{k}_0 \cdot \mathbf{r})$. The wave equation for the electric field is solved by imposing a Floquet solution with space–time harmonics, which leads to a description of higher-order stop-bands by computing the dispersion relation given in the form of a rapidly convergent continued fraction. While many of these and others pioneering works focused mainly on issues related to traveling wave parametric amplification (Cullen 1958, Tien and Suhl 1958) and the condition for stability of the system associated to frequency conversion effects (Cassedy and Oliner 1963, Cassedy 1967), to the best of the authors’ knowledge, the possibility of achieving one-way propagation in spatiotemporal modulated system was not properly recognized and described.

In the present work, we describe the dispersion diagrams of elastic waves propagating in beams with space–time periodic material properties. We study both non-dispersive longitudinal motion and dispersive transverse motion in the low-frequency range by assuming, in our derivations, the Euler–Bernoulli beam theory assumptions. The dispersion diagrams are used to identify a new class of band gaps that allow wave propagation in one direction only, thus they are referred to as non-reciprocal or directional band gaps. In one-dimensional systems, the directional band gaps signal and quantify the non-reciprocal behavior of the structure. We show that reciprocity is broken when a spatiotemporal modulation of the material properties of the structure, i.e. the Young’s modulus $E$ and $\rho$, is imposed in the form of a traveling wave. In this case, we assume a solution in the Floquet form with space–time harmonics, following the approach of (Slater 1958). The Floquet solution is substituted into the general equations of motion of beams with space and time varying material properties and leads to a quadratic eigenvalues problem (Tisseur and Meerbergen 2001) (QEP) that can be solved for the frequency $\omega$ as a function of the wavenumber $\kappa$. This approach also allows us to obtain approximate analytical relationships between the modulation parameters and the position and width of the directional band gaps for the simple case of harmonic modulation. Moreover, we compute the minimum speed at which the modulating traveling wave has to travel in order to maximize the non-reciprocal effect. We also discuss the shift of the first Brillouin zone (FBZ) limits due to spatiotemporal modulation. We remark here that many recent works focused on the possibility to control wave propagation through active components, which can change dynamically the properties of the material they are coupled with. For instance, it is possible to obtain active acoustic metamaterials with varying density and bulk modulus by using water–filled cylinders coupled to piezoelectric elements (Alk and Baz 2013). By using the same piezoelectric effect, spatiotemporal piezoelectric beams could be obtained by properly switching the connection each piezoelectric patch has to an electric circuit (Yan et al 2016), effectively achieving a time-varying equivalent stiffness.

The paper is divided into 4 sections. The present introduction (section 1) is followed by section 2, which firstly gives a description of the systems under investigation, the relative equations of motion and the considered modulation strategies, secondly expounds the methodology employed for the analysis. Results are presented and discussed in section 3 for both longitudinal and transverse motion. Concluding remarks are presented in section 4.

2. Theoretical background

2.1. Time-spatial periodic beams

Consider a beam with Young’s modulus $E$ and density $\rho$ periodic functions of space and time. We define this periodic variation of the medium’s characteristic in space and time as modulation. We consider a periodic
variation along the axial direction, thus the modulated Young’s modulus $E$ and density $\rho$ are expressed as:

$$E(x, t) = E_0 + E_m \cos(\omega_m t - \kappa_m x) + E_m/5 \cos(3(\omega_m t - \kappa_m x)),$$

for any location $x$ and at any instant $t$, in which $\lambda_m$ and $T_m$ define the spatial and time periodicity, respectively, while $\kappa_m = 2\pi/\lambda_m$ and $\omega_m = 2\pi/T_m$ are the wavenumber and the angular frequency associated to the properties of modulation. The periodic modulation pattern can be understood as a traveling wave with velocity $v_m = \omega_m/k_m$. We limit our analysis to the case in which this pattern travels with constant velocity and we define $v_m$ as ‘modulation speed’. The behavior of spatial-only periodic structures can be fully characterized by studying a single unit cell, which represents the building block of the structure (Brillouin 1953, Kittel 2004). For systems periodic both in space and time, the concept of unit cell has to be extended to account for periodicity both in the spatial and temporal domains. Therefore, in our study we focus on the spatiotemporal unit cell in order to obtain the dispersion properties of the structure. The spatiotemporal unit cell is associated to $\lambda_m$ and $T_m$, it travels with velocity $v_m$ along the axial direction and coincides with the classically defined unit cell of spatial-only periodic structures for $v_m = 0$. An example of spatiotemporal unit cell is given in figure 1(a), in which we consider a material with Young’s modulus given by $E(x, t) = E_0 + E_m \cos(\omega_m t - \kappa_m x) + E_m/5 \cos(3(\omega_m t - \kappa_m x))$, with $E_0 = 70$ GPa and $E_m = 3$ GPa. At $t = 0$, the stiffness of the structure is defined by the periodic function shown in figure 1(b). We remark here that the unit cell can travel in both forward and backward directions. In the present study, we consider forward propagating modulation only, since analogous considerations can be made for backward propagating modulation.

2.2. Analysis of dispersion

We first derive the equations of motions of beams with material properties that depend upon space and time. We consider both longitudinal and transverse motion. The study is restricted to the case of slender beams, in which shear deformation and rotational inertia are neglected (Meirovitch 1997). Time-space periodic structures are then investigated by restricting the class of functions that describe the material properties to periodic functions in time and space only. As shown in figure 2, in the case of longitudinal motion we assume $u = u(x, t)$ to be the longitudinal displacement of the beam along the axial direction. For transverse motion, the displacement is described by $w = w(x, t)$. We impose the conservation of linear momentum along both axial and transverse directions and the constitutive laws for the cross-section of the beam to get the following equation of motion for longitudinal motion:

$$\frac{\partial}{\partial x} \left[ E(x, t) \frac{\partial u(x, t)}{\partial x} \right] - \frac{\partial}{\partial t} \left[ \rho(x, t) \frac{\partial u(x, t)}{\partial t} \right] = 0 \quad (2)$$

and transverse motion:

$$R^2 \frac{\partial^2}{\partial x^2} \left[ E(x, t) \frac{\partial^2 w(x, t)}{\partial x^2} \right] + \frac{\partial}{\partial t} \left[ \rho(x, t) \frac{\partial w(x, t)}{\partial t} \right] = 0 \quad (3)$$
in which \( A \) is the cross-sectional area, \( I \) is the area moment of inertia and \( R_g = \sqrt{I/A} \) is the radius of gyration of the beam. If \( E(x, t) = E \) and \( \rho(x, t) = \rho \) are constant in space and time, equations (2) and (3) reduce to the familiar equations of motion for uniform beams.

We study wave propagation properties in time-spatial periodic structures by computing and analyzing band diagrams for said structures. The approach followed in this work exploits the periodic nature of the considered modulation of the material properties functions \( E(x, t) \) and \( \rho(x, t) \). Both are periodic in the variables \( x \) and \( t \) with periodicity \( \lambda_m \) and \( T_m \), respectively. It is then possible to express them by using the following Fourier series representations:

\[
E(x, t) = \sum_{p=-\infty}^{+\infty} \hat{E}_p e^{i( \omega_p t - \kappa_px)} \quad \rho(x, t) = \sum_{p=-\infty}^{+\infty} \hat{\rho}_p e^{i( \omega_p t - \kappa_px)}
\]  

in which \( \hat{E}_p \) and \( \hat{\rho}_p \) are the Fourier coefficients of the series associated with the traveling harmonic terms \( e^{i( \omega_p t - \kappa_px)} \). An explicit expression for these coefficients can be obtained by performing a double integration over the unit cell’s spatial and time periods \( \lambda_m \) and \( T_m \), respectively:

\[
\hat{E}_p = \frac{1}{\lambda_m} \int_{-\lambda_m/2}^{+\lambda_m/2} \int_{-T_m/2}^{+T_m/2} E(x, t) e^{-i( \omega_p t - \kappa_px)} dx dt,
\]

\[
\hat{\rho}_p = \frac{1}{\lambda_m} \int_{-\lambda_m/2}^{+\lambda_m/2} \int_{-T_m/2}^{+T_m/2} \rho(x, t) e^{-i( \omega_p t - \kappa_px)} dx dt.
\]

The solutions to equations (2) and (3) are chosen in the generalized Floquet form (Slater 1958) and write:

\[
u(x, t) = e^{i( \omega t - \kappa x)} \sum_{n=-\infty}^{+\infty} \hat{u}_n e^{i( \omega_n t - \kappa_n x)},
\]

\[
w(x, t) = e^{i( \omega t - \kappa x)} \sum_{n=-\infty}^{+\infty} \hat{w}_n e^{i( \omega_n t - \kappa_n x)}.
\]

In a study on wave propagation in spatiotemporal periodic transmission lines (Cassedy 1967), allowing solutions in the generalized Floquet form, it is discussed the possibility of having time-growing waves for certain modulation parameters. These time-growing waves are associated to unstable interactions between the wave propagating in the system and the modulation wave. In the same study, the author discusses under which conditions the governing equations of the system are satisfied, for real wavenumbers, by complex frequencies only, leading to time-growing oscillations. Such instabilities occur when the modulation speed is greater than the wave speed in the uniform system. Although interesting, an analogous study of the stability conditions for wave propagation in modulated beams falls outside the scope of the present article. Nevertheless, knowing that non-periodic, time-growing solutions can arise for certain combinations of the modulation parameters \( \omega_m, \kappa_m \) and \( \hat{E}_p, \hat{\rho}_p \), we assume stability to hold by imposing a modulation speed smaller than the wave speed in the uniform structure. Substituting equations (7) and (8) into equations (2) and (3), together with the expressions for the material properties in equation (4), leads to expressions that can be simplified by exploiting orthogonality of the Fourier basis in equations (2) and (3). This gives:

\[
\sum_{n=-\infty}^{+\infty} \left[(\kappa + n\kappa_m)(\kappa + p\kappa_m)\right] \hat{E}_{p-n} \hat{u}_n = \sum_{n=-\infty}^{+\infty} \left[(\omega + n\omega_m)(\omega + p\omega_m)\right] \hat{\rho}_{p-n} \hat{u}_n,
\]

\[
\sum_{n=-\infty}^{+\infty} \left[(\kappa + n\kappa_m)(\kappa + p\kappa_m)\right] \hat{E}_{p-n} \hat{w}_n = \frac{1}{R_g} \sum_{n=-\infty}^{+\infty} \left[(\omega + n\omega_m)(\omega + p\omega_m)\right] \hat{\rho}_{p-n} \hat{w}_n
\]

which hold for the longitudinal and transverse motion of the beam, respectively. If a finite number \( N \) of terms is considered in equations (7) and (8), then equations (9) and (10) represent each a finite set of \( 2N + 1 \) equations that can be cast in the form of a quadratic eigenvalue problem (QEP):

\[
(\omega^2 \hat{L}_2 + \omega \hat{L}_1 + \hat{L}_0) \hat{u} = 0,
\]
where \( \hat{L}_j = \hat{L}_j(\kappa) \) and \( \hat{T}_j = \hat{T}_j(\kappa) \), with \( j = 0, 1, 2 \), are coefficient matrices whose entries depend on the modulation parameters \( \hat{\omega}_m \) and \( \hat{\kappa}_m \), the Fourier coefficients \( \hat{E}_p \) and \( \hat{\rho}_p \) and the wavenumber \( \kappa \). Moreover, \( \hat{u} \) and \( \hat{w} \) are the displacement coefficient vectors. The QEP in equations (11) and (12) is solved in terms of \( \omega \) by letting \( \kappa \) vary in a convenient range. For spatial-only periodic structure, this range is given by the irreducible Brillouin zone (IBZ) (Brillouin 1953). We show that spatiotemporal modulation challenges the classical definition of the IBZ, leading to a shift in the wavenumber range to be considered. The dispersion properties of the time–spatial periodic structure are then obtained by representing the relation \( \omega = \omega(\kappa) \) in the form of band diagrams.

3. Results

The results are presented in the form of band diagrams for both longitudinal and transverse wave. We also numerically verify the proposed dispersion analysis through a finite element study of the transient response to external excitation of beams with modulated properties. We consider two distinct modulation strategies of the Young’s modulus \( E \) only, i.e. harmonic and square traveling wave, while we assume the density \( \rho \) to be constant. In discussing the results, we refer to the dimensionless frequency \( \Omega \) and wavenumber \( \mu \):

\[
\Omega = \frac{\hat{\omega}_m}{c_0}, \quad \mu = \kappa \lambda_m, \tag{13}
\]

where \( \hat{\omega} \) is the frequency and \( c_0 = \sqrt{E_0/\rho_0} \) is the velocity of longitudinal waves in a non-modulated beam.

We first consider a harmonic modulation of the Young’s modulus only, thus assuming the following expressions for the material parameters:

\[
E(x, t) = E_0 + E_m \cos(\omega_m t - \kappa_m x), \quad \rho(x, t) = \rho_0, \tag{14}
\]

where \( E_m \) is the amplitude of the modulation. For \( E_m = 0 \), the material is not modulated and the beam is uniform, while for \( E_m \neq 0 \) and \( \omega_m \neq 0 \), the value of the Young’s modulus can be described as the harmonic traveling wave defined in equation (14). The wave travels along the positive direction of the x axis with speed \( v_m = \omega_m / \kappa_m \). We define the dimensionless modulation amplitude \( \alpha_m \) and the dimensionless modulation velocity parameter \( \nu_m \):

\[
\alpha_m = \frac{E_m}{E_0}, \quad \nu_m = \frac{v_m}{c_0}. \tag{15}
\]

An example of harmonic modulation is given in figure 3: spatial-only modulation is shown in figure 3(a) and corresponds to the case with \( T_m \rightarrow \infty \), thus \( \omega_m \rightarrow 0 \) and \( \nu_m \rightarrow 0 \), while spatiotemporal modulation with \( \alpha_m = 0.20 \) and \( \nu_m = 0 \) is shown in figure 3(b). The Fourier coefficients in equation (14) are given by equation (5) and write:

\[
\hat{E}_p = \begin{cases} 
E_0 & \text{for } p = 0 \\
E_m/2 & \text{for } p = \pm 1 
\end{cases} \tag{16}
\]

while \( \hat{\rho}_p = \rho_0 \delta_{p0} \), where \( \delta_{p0} \) is the Kronecker delta.

We also consider a square traveling wave of the Young’s modulus only given by the following expressions:

\[
E(x, t) = E_1 + (E_2 - E_1) H[\cos(\omega_m t - \kappa_m x)], \quad \rho(x, t) = \rho_0, \tag{17}
\]

in which \( H \) denotes the Heaviside function. The Young’s modulus takes the two values \( E_1 \) and \( E_2 \), with \( E_2 > E_1 \). In this case, \( E_m = E_2 - E_1 \). The spatial and time periods of the modulation are \( \lambda_m = 2\pi / \kappa_m \) and \( T_m = 2\pi / \omega_m \), respectively. The modulation travels with velocity \( v_m = \omega_m / \kappa_m \).

It can be shown that the Fourier coefficients for square wave modulation given by equation (5) write:

\[
\hat{E}_p = E_2 \delta_p + \frac{E_1 - E_2}{2} \sin \epsilon \left[ \frac{\pi p}{2} \right], \tag{18}
\]

in which \( \sin \epsilon(x) = \sin(x)/x \) and \( \hat{\rho}_p = \rho_0 \delta_{p0} \).

3.1. Band diagrams

3.1.1. Harmonic modulation

We obtain the band diagrams shown in figures 4 and 5 by solving the QEP in equations (11) and (12) for longitudinal and transverse motion, respectively. For conventional periodic system with spatial–only modulation, the dispersion diagrams are periodic in the wavenumber domain, therefore the calculation of the dispersion diagrams can be restricted to one period, known as FBZ, where the dimensionless wavenumber \( \mu \) ranges in \([ - \pi, \pi ] \). Moreover, it is customary to further restrict the domain of the computation by considering just half of the period, called IBZ, with \( \mu \) ranging in \([ 0, \pi ] \). This approach is justified by an important property of
conventional periodic structures, namely that their dispersion relation satisfies the relation $\Omega(\mu) = \Omega(-\mu)$. Such property is due to the fact that the structure can support both backward propagating waves and forward propagating waves, and the behavior of the waves is the same no matter which direction they are propagating to. It is also understood that the relation $\Omega(\mu) = \Omega(-\mu)$ implies symmetry of the dispersion diagrams with respect to the frequency axis, allowing the analysis to be restricted on the IBZ symmetric with respect to the frequency axis. For this reason, we need to consider a sufficiently broad range for the values of the wavenumber. In our analysis, we impose $\mu$ to range in $[-3/2\pi, 3/2\pi]$. In all the computations, we focus on the two lowest dispersion branches where a band gap opens for $\alpha_m \neq 0$, so we can truncate the series expressing the solution imposing $N = 3$. As discussed in appendix B, increasing the truncation order does not provide any additional advantage, while the CPU time needed to solve the QEP increases quickly. Therefore, the matrices $\hat{L}_j$ and $\hat{T}_j$, with $j = 0$, 1, 2, in the QEP are square matrices of order $2 \cdot 3 + 1 = 7$. Both for longitudinal and transverse motion in absence of modulation, thus for $\alpha_m = 0$ and $\nu_m = 0$, the beam does not display any band gap (figures 4(a) and 5(a)). The only dispersion branches having physical meaning are the ones departing from the origin, which are associated to the two 0th order harmonics, one associated to the backward-propagating wave, the other associated to the forward-propagating wave. On the contrary, when space-only modulation is applied, in our example for $\alpha_m = 0.40$ and $\nu_m = 0$, the system displays a complete band gap for $\Omega$ in the range $[0.43, 0.53]$ for longitudinal motion and $[0.0197, 0.0244]$ for transverse motion, in which both backward-propagating waves and forward-propagating waves are not supported by the structure, as shown in figures 4(b) and 5(b). Due to spatial periodic modulation of the medium, higher spatial-only harmonics in the solutions expressed by equations (7) and (8) have non-zero amplitude and coupling between the associated modes leads to dispersive behavior and band gaps of Bragg type at the edge of the FBZ. The band diagram is periodic with period $2\pi$, with FBZ ranging from $\mu = -\pi$ to $\mu = \pi$. When modulation in space and time is introduced, symmetry of the dispersion diagram with respect to $\Omega$ and $\mu$ is broken, as shown in figures 4(c) and 5(c), which are obtained respectively for $\alpha_m = 0.40$ and $\nu_m = 0.05$ for longitudinal motion, and for $\alpha_m = 0.40$ and $\nu_m = 0.002$ for transverse motion. The band diagrams for spatiotemporal periodic structures clearly show that, within certain frequency ranges, only modes propagating in one direction are allowed. For longitudinal motion, a closer look at figure 4(c) reveals that waves propagating with $\Omega$ in the range $[0.41, 0.46]$ have positive group velocity only; therefore forward propagating waves only are allowed to propagate, while backward-propagating longitudinal waves only are supported by the structure for $\Omega$ in the range $[0.51, 0.56]$. Such frequency ranges are then associated to one-way propagation, thus we can call them ‘directional band gaps’. By comparing the band diagrams for longitudinal motion in figures 4(c) and (d), we also observe that for $\alpha_m = 0.40$ and $\nu_m = 0.05$ the structure displays a complete band gap for $\Omega$ in the range $[0.46, 0.51]$ together with the two directional band gaps for $\Omega$ in the ranges $[0.41, 0.46]$ and $[0.51, 0.56]$, while for $\alpha_m = 0.40$ and $\nu_m = 0.20$ the structure exhibit just two wider directional band gaps. Analogous considerations can be done for transverse motion by analyzing the dispersion diagrams in figure 5. Furthermore, figures 4(c) and (d) reveal that when $\nu_m \neq 0$, the local maxima of the dispersion branches do not coincide with the limits of the classically
defined FBZ $\mu = \pm \pi$, instead they are shifted in the direction of positive wavenumbers $\kappa$ for forward propagating modulation. Similar considerations hold in the case of transverse motion, as shown in figures 5(c) and (d). In other words, spatiotemporal modulation of the material properties introduces shifts in the position of the band gaps, breaking symmetry of the dispersion diagrams and leading to directional band gaps. In addition, spatiotemporal modulation also challenges the classic definition of the FBZ and its limits, as already observed by Cassedy (Cassedy and Oliner 1963, Cassedy 1965).

3.1.2. Square modulation
For square modulation we also compute the band diagrams for both longitudinal and transverse motion and compare the cases of spatial-only modulation and spatiotemporal modulation. We consider $\alpha = 3$ and $\nu_m = 0.15$ for the beam in longitudinal motion, while for transverse motion we consider the case with $\alpha = 3$ and $\nu_m = 0.015$. We focus on the first two band gaps only. We assume $N = 5$ to be a good compromise between quality of the approximation and time required for the computation, as discussed in appendix B. As shown in figures 6(a) and (c), the spatial-only modulation, thus for $\nu_m = 0$, induces multiple band gaps in the structure. From figures 6(b) and (d), we can see that when spatiotemporal modulation is considered, in this case for

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**Figure 4.** Band diagrams for beam in longitudinal motion and harmonic modulation: (a) non-modulated beam with $\alpha_m = 0$ and $\nu_m = 0$, (b) space modulated only beam with $\alpha_m = 0.40$ and $\nu_m = 0$, (c) space–time modulated beam with $\alpha_m = 0.40$ and $\nu_m = 0.05$, (d) space–time modulated beam with $\alpha_m = 0.40$ and $\nu_m = 0.20$. For $\alpha_m \neq 0$ and $\nu_m \neq 0$ the mirror symmetry with respect to the frequency axis is relaxed, leading to directional band gaps which signal the possibility of one-way propagation.
the band gaps are shifted upwards or downwards depending on the direction of propagation, hence they become directional band gaps and inform on the non-reciprocal behavior of the structure.

3.2. Parametric analysis for harmonic modulation

We now analytically characterize the effect of the modulation parameters on the dispersion properties of the beam. We consider a reduced version of the QEP used to compute the band diagrams in figures 4 and 5 and obtain approximate analytical expressions for the dispersion branches of spatiotemporal modulated beams. Such expressions allow us to quantify the effect of the modulation parameters $\alpha_m$ and $n_m$ not only on the position and width of the directional band gaps, but also on the shift of the limits of the FBZ. We confine our analysis to the harmonic modulation only. We start with the longitudinal motion case by investigating the values of the FBZ limits induced by the modulation. Figure 7 shows a detail of the regions of the band diagram associated with the directional band gaps. The plot presents different cases of modulation, each corresponding to the same value $n_m = 0.10$ and increasingly smaller values of $\alpha_m$. In figures 7(b) and (c) we observe that for $\alpha_m \to 0$, the band gaps close and two dispersion branches intersect at the point $(\mu_p, \Omega_p)$ for forward propagating waves, with $\mu_p = \pi$, and at the point $(\mu_b, \Omega_b)$ for backward propagating waves, with $\mu_b = -\pi$. We remark here that, in the limit of zero modulation, the dispersion branches departing from the origin and

$\nu_n = 0$, the band gaps are shifted upwards or downwards depending on the direction of propagation, hence they become directional band gaps and inform on the non-reciprocal behavior of the structure.

**Figure 5.** Band diagrams for beam in transverse motion and harmonic modulation: (a) non-modulated beam with $\alpha_m = 0$ and $n_m = 0$, (b) space modulated only beam with $\alpha_m = 0.40$ and $n_m = 0$, (c) space–time modulated beam with $\alpha_m = 0.40$ and $n_m = 0.002$, (d) space–time modulated beam with $\alpha_m = 0.40$ and $n_m = 0.01$. For $\alpha_m \neq 0$ and $n_m \neq 0$ the mirror symmetry is lost and directional band gaps are obtained.
associated to the 0th order harmonic in equation (7) are the only having physical reality, as discussed by Cassedy (Cassedy and Oliner 1963). We seek for analytic expressions for $m_B$, $W_B$ and $m_F$, $W_F$ by truncating the series expressing the solution in equation (7) to the first order, then equation (11) reduces to:

$$
\begin{bmatrix}
L_{11} & L_{12} & L_{13} \\
L_{21} & L_{22} & L_{23} \\
L_{31} & L_{32} & L_{33}
\end{bmatrix}
\begin{bmatrix}
\hat{u}_{-1} \\
\hat{u}_{0} \\
\hat{u}_{+1}
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
$$

(19)

where the coefficients can be found in the appendix A. We first focus on the directional band gap for forward propagating waves. In this case, the band gap is generated by the interaction of the forward 0th order harmonic wave with frequency $\omega$ and the backward (or folded) first order harmonic wave with frequency $\omega - \omega_{00}$. Therefore, the first order harmonic wave with frequency $\omega + \omega_{00}$ can be neglected and we can further simplify equation (19) by solving the problem for the coefficients $\hat{u}_{-1}$ and $\hat{u}_{0}$ only to obtain:

$$
\begin{bmatrix}
L_{11} & L_{12} \\
L_{21} & L_{22}
\end{bmatrix}
\begin{bmatrix}
\hat{u}_{-1} \\
\hat{u}_{0}
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0
\end{bmatrix}
$$

(20)

Figure 6. Band diagrams for beam in longitudinal (a), (b) and transverse (c), (d) motion and square modulation. For longitudinal motion: (a) space modulated only beam with $n_m = 3$ and $v_m = 0$, (b) space–time modulated with $n_m = 3$ and $v_m = 0.13$. Similarly for transverse motion: (c) space modulated only beam with $n_m = 3$ and $v_m = 0$, (d) space–time modulated with $n_m = 3$ and $v_m = 0.015$. The structure is non-reciprocal for $n_m = 0$ and $v_m = 0$ in multiple frequency ranges.
A compact solution to this QEP is obtained by observing that for $a \to 0$, then $L_{22}, L_{21} \to 0$. The characteristic equation for equation (20) reduces to:

$$L_{11} = \left\{ \Omega^2 - 2 \nu_m \Omega - \left[ \left( \frac{\mu}{2\pi} - 1 \right)^2 - \nu_m^2 \right] \right\} \left\{ \Omega^2 - \left( \frac{\mu}{2\pi} \right)^2 \right\} = 0$$

whose solutions are:

$$\Omega_{2,2} = \nu_m \pm \frac{\mu}{2\pi} - 1,$$

$$\Omega_{3,4} = \pm \frac{\mu}{2\pi}.$$  

Each of these four roots is associated to a different dispersion branch. In general these roots are distinct, but at $\mu = \mu_F$, equation (20) allows double roots for $a_m \to 0$. Two of the dispersion branches intersect at the point $(\mu_F, \Omega_F)$, which can be obtained by imposing:

\[\text{Figure 7. Band diagrams for longitudinal motion with } \nu_m = 0.10 \text{ and different values of } a_m. \text{ For } a_m \to 0, \text{ the first two dispersion branches intersect at } (\mu_m, \Omega_m) \text{ and } (\mu_m, \Omega_m) \text{ for backward and forward propagating waves, respectively. Specifically, the point } (\mu_m, \Omega_m) \text{ is the intersection of the branch } \Omega_2 \text{ in equation (22) with the branch } \Omega_3 \text{ in equation (23). Comparison with the case in which } a_m \neq 0 \text{ is given in (b) for backward propagating waves and in (c) forward propagating waves.}\]
In this case, the terms \( \mu = \pi \) only if \( \nu_m = 0 \), otherwise the limit of the FBZ between the first and the second dispersion branches is shifted by the value \( \Delta \mu = \nu_m \pi \), which in first approximation depends upon the modulation speed only. As already recognized by Cassidy and Oliner [1963], when spatiotemporal modulation is introduced, the dispersion diagrams present a peculiar skew representation in the frequency-wavenumber plane, with repetition of the unit cell along a line of slope dependent on the modulation frequency, rather than repetition along the wavenumber axis, as customary in spatially modulated only systems. Therefore, in the context of spatiotemporally modulated systems, it is not possible to apply the concept of FBZ as defined for spatially modulated systems only. For this reason, Cassidy talks about generalized Brillouin diagrams. If the dimensionless modulation amplitude is small but different from zero, \( \alpha_m \ll 1 \), the characteristic equation for equation (20) is a quartic and does not allow double roots at \( \mu = \pi (1 + \nu_m) \). Instead, it allows four distinct solutions, two of which lie in a neighborhood of \( \Omega_d \). In a neighborhood of the point \((\mu_d, \Omega_d)\), we can write \( \mu = \mu_d + \delta \mu \) and \( \Omega = \Omega_d + \delta \Omega \), with \( \delta \mu, \delta \Omega \ll 1 \).

Substituting the assumed \( \mu \) and \( \Omega \) into \( L_{11} = L_{22} \), this quantity is expressed as a function of the small quantities \( \delta \mu, \delta \Omega \) only, hence we can neglect them and write:

\[
L_{11} - L_{22} \approx 0
\]

for \( \mu = \mu_d = \pi (1 + \nu_m) \), in which equality holds for \( \alpha_m \to 0 \). The latter key approximation allows us to study two simple problems:

\[
L_j^1 - L_j^2 = 0 \quad \Rightarrow \quad (L_{11} - L_{12})(L_{11} + L_{12}) = 0,
\]

(27)

\[
L_j^2 - L_j^3 = 0 \quad \Rightarrow \quad (L_{22} - L_{23})(L_{22} + L_{23}) = 0
\]

(28)

both representing an approximation of the original problem expressed by the characteristic equation given by solving equation (20) in a neighborhood of \((\mu_d, \Omega_d)\). The solutions of equation (27) write:

\[
\Omega_d^I = \nu_m \pm \frac{1 - \nu_m}{2} \sqrt{1 \pm \frac{\alpha_m (1 + \nu_m)}{2}}
\]

(29)

while the solutions of equation (28) write:

\[
\Omega_d^H = \nu_m \pm \frac{1 - \nu_m}{2} \sqrt{1 \pm \frac{\alpha_m (1 - \nu_m)}{2}}
\]

(30)

with \( r = 1, 2, 3, 4 \). For both of the approximate problems, we only consider the two positive values of frequency in a neighborhood of \( \Omega_d \), which approximate of the upper and lower limits of the directional band gap. By properly averaging such frequencies we define:

\[
\Omega_{d}^{\text{top}} = \frac{1}{2} \nu_m + \frac{1 - \nu_m}{2} \sqrt{1 + \frac{\alpha_m (1 + \nu_m)}{2}} + \frac{1 + \nu_m}{2} \sqrt{1 + \frac{\alpha_m (1 - \nu_m)}{2}}
\]

(31)

\[
\Omega_{d}^{\text{bot}} = \frac{1}{2} \nu_m + \frac{1 - \nu_m}{2} \sqrt{1 - \frac{\alpha_m (1 + \nu_m)}{2}} + \frac{1 + \nu_m}{2} \sqrt{1 - \frac{\alpha_m (1 - \nu_m)}{2}}
\]

(32)

where \( \Omega_{d}^{\text{top}} \) and \( \Omega_{d}^{\text{bot}} \) are the upper and the lower limits of the directional band gap for the forward propagating waves, respectively.

Since \( \alpha_m \ll 1 \), linearized expressions can be obtained by using the Taylor series of \( \sqrt{1 + x} = 1 + 1/2x + O(x^2) \), thus equation (31) rewrites as:

\[
\Omega_{d, \text{lin}}^{\text{top}} = \frac{\alpha_m}{8} + \frac{1}{2} (1 + \nu_m),
\]

(33)

\[
\Omega_{d, \text{lin}}^{\text{bot}} = -\frac{\alpha_m}{8} + \frac{1}{2} (1 + \nu_m).
\]

(34)

The dependency of the directional band gap width on the modulation parameters \( \alpha_m \) and \( \mu_m \) is given by the following expression:

\[
\Delta \Omega_{d, \text{lin}} = \Omega_{d, \text{lin}}^{\text{top}} - \Omega_{d, \text{lin}}^{\text{bot}} = \frac{\alpha_m}{4}.
\]

(35)

We can use a similar approach to study the directional band gap associated to the backward propagating waves. In this case, the terms \( \dot{u}_0 \) and \( \dot{u}_{1,1} \) are retained in equation (19) to give:
\[
\begin{bmatrix}
L_{22} & L_{23} \\
L_{32} & L_{33}
\end{bmatrix}
\begin{bmatrix}
\tilde{u}_0 \\
\tilde{u}_{i+1}
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]  
(36)

It can be shown that modulation induces a shift of the left edge of the FBZ, thus the following relations hold:

\[
\mu_B = -\pi (1 - \nu_m) \quad \Omega_B = \frac{1}{2} (1 - \nu_m).
\]  
(37)

Moreover the expression for the upper and lower limits of the directional band gaps for backward propagating waves are:

\[
\Omega_B^{\text{top}} = \frac{1}{2} \left[ -\nu_m + \frac{1 + \nu_m}{2} \sqrt{1 + \frac{\alpha_m(1 - \nu_m)}{2(1 + \nu_m)}} + \frac{1 - \nu_m}{2} \sqrt{1 - \frac{\alpha_m(1 + \nu_m)}{2(1 - \nu_m)}} \right]
\]  
(38)

\[
\Omega_B^{\text{bot}} = \frac{1}{2} \left[ -\nu_m + \frac{1 + \nu_m}{2} \sqrt{1 - \frac{\alpha_m(1 - \nu_m)}{2(1 + \nu_m)}} + \frac{1 - \nu_m}{2} \sqrt{1 + \frac{\alpha_m(1 + \nu_m)}{2(1 - \nu_m)}} \right]
\]  
(39)

while the corresponding linearized expressions write:

\[
\Omega_{B,\text{lim}}^{\text{top}} = +\frac{\alpha_m}{8} + \frac{1}{2} (1 - \nu_m),
\]  
(40)

\[
\Omega_{B,\text{lim}}^{\text{bot}} = -\frac{\alpha_m}{8} + \frac{1}{2} (1 - \nu_m).
\]  
(41)

The linearized expression for the backward directional band gap width is given by:

\[
\Delta \Omega_{B,\text{lim}}^{\text{top}} = \Omega_{B,\text{lim}}^{\text{top}} - \Omega_{B,\text{lim}}^{\text{bot}} = \frac{\alpha_m}{4}
\]  
(42)

hence:

\[
\Delta \Omega_{F,\text{lim}} = \Delta \Omega_{B,\text{lim}} = \frac{\alpha_m}{4}
\]  
(43)

both backward and forward directional band gap widths have the same value, which depends on the modulation amplitude \(\alpha_m\) only. We can also determine the minimum value of the modulation velocity parameter \(\nu_m^c\) such that, for \(\alpha_m \neq 0\), the beam shows two distinct directional band gaps, one associated to the forward propagating waves, the other one associated to the backward propagating waves. In order to obtain an expression for \(\nu_m^c\) we impose:

\[
\Omega_{B,\text{lim}}^{\text{top}} = \Omega_{B,\text{lim}}^{\text{bot}}
\]  
(44)

which, solved for \(\nu_m\) gives:

\[
\nu_m^c = \frac{\alpha_m}{4}.
\]  
(45)

This results shows that, for smalls values of \(\alpha_m\), the modulation velocity parameter \(\nu_m\) determines a shift \(\Omega_{\text{shift}}\) in the location of the band gaps that writes \(\Omega_{\text{shift}} = \nu_m\) for forward propagating waves and \(\Omega_{\text{shift}} = -\nu_m\) for backward propagating waves, thus breaking the mirror symmetry of the band diagram about the \(\Omega\) axis.

Following the same procedure outlined in the case of longitudinal motion, we characterize backward and forward wave propagation in the case of transverse motion through approximate analytic expressions for the directional band gaps. We study the following QEP:

\[
\begin{bmatrix}
T_{11} & T_{12} & T_{13} \\
T_{21} & T_{22} & T_{23} \\
T_{31} & T_{32} & T_{33}
\end{bmatrix}
\begin{bmatrix}
\tilde{w}_{-1} \\
\tilde{w}_0 \\
\tilde{w}_{+1}
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]  
(46)

which we obtain from equation (12) by retaining the 0th and first order harmonics in equation (8). The dimensionless expressions for the coefficients in equation (46) can be found in appendix A. For forward wave propagation, figure 8 shows that for \(\nu_m = 0.005\), the right edge of the FBZ between the first and the second dispersion branch is shifted due to modulation. Similarly to the case of longitudinal motion, in the limit of zero modulation, the only dispersion branches having physical reality are the ones departing from the origin and associated to the 0th order harmonic in equation (8). Also, as \(\alpha_m \to 0\), the band gap closes and the dispersion branches intersect at the point corresponding to the following dimensionless wavenumber and frequency:

\[
\mu_F = \pi \left(1 + \frac{\nu_m}{\chi}\right) \quad \Omega_F = \frac{1}{4} \left(1 + \frac{\nu_m}{\chi}\right)^2 \chi
\]  
(47)

in which we define \(\chi = R g \kappa_m\). Also we observe that \(\mu_F \to \pi\) and \(\Omega_F \to \chi/4\) as \(\nu_m \to 0\), hence when no temporal modulation is applied. The limits of the directional band gaps write:
The radical in the previous expression can be linearized to get:

$$\Omega_{\mu_F}^{\text{top}} = \frac{1}{2} \left[ \nu_m + \frac{1}{4} \left( 1 - \frac{\nu_m}{\chi} \right)^2 \sqrt{1 + \frac{\alpha_m}{2} \left( 1 + \frac{\nu_m}{\chi} \right)^2} + \frac{1}{4} \left( 1 + \frac{\nu_m}{\chi} \right)^2 \sqrt{1 + \frac{\alpha_m}{2} \left( 1 - \frac{\nu_m}{\chi} \right)^2} \right],$$

$$\Omega_{\mu_F}^{\text{bot}} = \frac{1}{2} \left[ \nu_m + \frac{1}{4} \left( 1 - \frac{\nu_m}{\chi} \right)^2 \sqrt{1 - \frac{\alpha_m}{2} \left( 1 + \frac{\nu_m}{\chi} \right)^2} + \frac{1}{4} \left( 1 + \frac{\nu_m}{\chi} \right)^2 \sqrt{1 - \frac{\alpha_m}{2} \left( 1 - \frac{\nu_m}{\chi} \right)^2} \right].$$

The radical in the previous expression can be linearized to get:

$$\Omega_{\mu_F, \text{lin}}^{\text{top}} = \frac{1}{2} \left[ \nu_m + \frac{1}{2} \left( 1 + \frac{\alpha_m}{4} \right) \chi \right],$$

$$\Omega_{\mu_F, \text{lin}}^{\text{bot}} = \frac{1}{2} \left[ \nu_m + \frac{1}{2} \left( 1 - \frac{\alpha_m}{4} \right) \chi \right].$$

Figure 8. Band diagrams for transverse motion with \( \nu_m = 0.005 \) and different values of \( \alpha_m \). For \( \alpha_m \to 0 \) (a), the first two dispersion branches intersect at \((\mu_{\text{top}}, \Omega_{\text{top}})\) and \((\mu_{\text{bot}}, \Omega_{\text{bot}})\) for backward and forward propagating waves, respectively. Comparison with the case in which \( \alpha_m = 0 \) is given in (b) for backward propagating waves and in (c) forward propagating waves.
in which we also assume that \((v_m/\chi)^2 \ll 1\). The linearized expression for the band gap width writes:

\[
\Delta \Omega_{\text{F,lim}} = \Omega_{\text{F,lim}}^{\text{top}} - \Omega_{\text{F,lin}}^{\text{bot}} = \frac{\alpha_m}{8} \chi.  \tag{52}
\]

For waves traveling backward, one can show that, for the first band gap with \(\alpha_m \rightarrow 0\), the modulation shifts the left edge of the FBZ leading to the following values:

\[
\mu_B = -\pi \left( 1 - \frac{v_m}{\chi} \right), \quad \Omega_B = \frac{1}{4} \left( 1 - \frac{v_m}{\chi} \right)^2 \chi.  \tag{53}
\]

For \(\alpha_m \ll 1\) but different from zero, directional band gaps open and their limiting frequencies can be obtained as functions of the modulation parameters and written as follows:

\[
\Omega_{B,\text{lin}}^{\text{top}} = \frac{1}{2} \left[ -v_m + \frac{1}{4} \left( 1 + \frac{v_m}{\chi} \right)^2 \sqrt{1 + \frac{\alpha_m}{2} \left( 1 - \frac{v_m}{\chi} \right)^2 + \frac{1}{4} \left( 1 - \frac{v_m}{\chi} \right)^2 \left( 1 + \frac{\alpha_m}{2} \left( 1 - \frac{v_m}{\chi} \right)^2 \right)} \right] \tag{54}
\]

\[
\Omega_{B,\text{lin}}^{\text{bot}} = \frac{1}{2} \left[ -v_m + \frac{1}{4} \left( 1 + \frac{v_m}{\chi} \right)^2 \sqrt{1 - \frac{\alpha_m}{2} \left( 1 - \frac{v_m}{\chi} \right)^2 + \frac{1}{4} \left( 1 - \frac{v_m}{\chi} \right)^2 \left( 1 + \frac{\alpha_m}{2} \left( 1 - \frac{v_m}{\chi} \right)^2 \right)} \right] \tag{55}
\]

Linearized expressions can be obtained and write:

\[
\Omega_{B,\text{lin}}^{\text{top}} = \frac{1}{2} \left[ -v_m + \frac{1}{2} \left( 1 + \frac{\alpha_m}{4} \right) \chi \right],  \tag{56}
\]

\[
\Omega_{B,\text{lin}}^{\text{bot}} = \frac{1}{2} \left[ -v_m + \frac{1}{2} \left( 1 - \frac{\alpha_m}{4} \right) \chi \right] \tag{57}
\]

therefore the band gap width has the following expression:

\[
\Delta \Omega_{B,\text{lin}} = \Omega_{B,\text{lim}}^{\text{top}} - \Omega_{B,\text{lim}}^{\text{bot}} = \frac{\alpha_m}{8} \chi = \Delta \Omega_{\text{F,lim}} \tag{58}
\]

hence also for beams in transverse motion, the directional band gaps width has the same expression both for backward and forward propagating waves. Moreover the gap width does not depend on the modulation velocity parameter \(v_m\), similarly to the case of longitudinal motion. Instead, the gap width depends on the modulation amplitude \(\alpha_m\) and on the parameter \(\chi\). We can compute the critical modulation velocity to get:

\[
v_m^\ast = \frac{\alpha_m}{8} \chi. \tag{59}
\]

We assess the validity of the approximate relations discussed above by comparing the band gaps given by the full solution of the QEP with \(N = 3\) and by the analytic expression valid for \(\alpha_m \ll 1\). Results are presented in figure 9 for both longitudinal and transverse motion. We track the gap amplitude as a function of the modulation parameters and written as follows:

\[
\frac{\alpha_m}{8} \chi = \Delta \Omega_{\text{F,lim}} \tag{58}
\]

hence also for beams in transverse motion, the directional band gaps width has the same expression both for backward and forward propagating waves. Moreover the gap width does not depend on the modulation velocity parameter \(v_m\), similarly to the case of longitudinal motion. Instead, the gap width depends on the modulation amplitude \(\alpha_m\) and on the parameter \(\chi\). We can compute the critical modulation velocity to get:

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\[
\frac{\alpha_m}{8} \chi = \Delta \Omega_{\text{F,lim}} \tag{58}
\]

hence also for beams in transverse motion, the directional band gaps width has the same expression both for backward and forward propagating waves. Moreover the gap width does not depend on the modulation velocity parameter \(v_m\), similarly to the case of longitudinal motion. Instead, the gap width depends on the modulation amplitude \(\alpha_m\) and on the parameter \(\chi\). We can compute the critical modulation velocity to get:

\[
v_m^\ast = \frac{\alpha_m}{8} \chi. \tag{59}
\]

3.3. Numerical simulations

We perform numerical simulations to verify the non-reciprocal behavior of the beams when an external excitation is imposed. Specifically, we compute the transient response of the structure to narrow-band excitation and analyze the displacement field. The computation are performed by using the commercial finite element code COMSOL Multiphysics. We consider a 2L long beam, with \(L = 70\lambda_m\) and a total of 140 unit cells. In our model, each unit cell is discretized by 20 quadrilateral elements, therefore we consider 2800 elements for the whole structure. In this section we assume \(\chi = 0.0144\). The excitation is imposed halfway through the length of the structures at \(x = 0\) as a point load, so that both backward and forward propagating waves are excited. The load acts along the \(x\) direction when we target longitudinal motion, while it acts normally to the \(x\) direction for transverse motion, as shown in figure 10. We introduce the dimensionless time \(\tau\) as:

\[
\tau = \frac{\varepsilon_0 t}{\lambda_m} \tag{60}
\]
which represents the number of unit cells of length \( l_m \) that a longitudinal wave, traveling at the speed \( c_0 \) in a uniform beam, covers in the time \( t \). Results are presented in figures 11 and 12 in the form of waterfall plots, that allow us to track how the waves propagate throughout the system. We compare a non-modulated beam, in which we consider \( a_m = 0, n_m = 0 \), with a spatiotemporal modulated beam with \( a_m = 0.40 \) and \( n_m = 0.20 \) for longitudinal motion and with \( a_m = 0.40 \) and \( n_m = 0.01 \) for transverse motion. The results can be better interpreted when compared to the band diagrams of the corresponding system in figures 4(a) and 5(a) for the non-modulated beam and figures 4(d) and 5(d) for the modulated beams, respectively. When no modulation is applied, the system is perfectly reciprocal and waves propagate symmetrically in the backward and forward directions, as shown in figures 11(a) and 12(a), with excitation centered at \( W = 0.485 \) and \( W = 0.0225 \) for longitudinal and transverse motion, respectively. When the same excitation is applied but the properties of the beam are modulated, the systems still behave in a reciprocal fashion since the excitation is centered in a pass band, as shown in band diagram in figure 4(d). The waterfall plots for this case are shown in figures 11(b) and 12(b) for longitudinal and transverse motion, respectively. Conversely, strong one-directional wave propagation is achieved when the excitation is centered in a directional band gap. We recognize forward-only propagation in figures 11(c) and 12(c), being the modulated structure excited with a narrow band signal centered at \( \Omega = 0.3844 \) and \( \Omega = 0.0175 \), for longitudinal and transverse motion, respectively. Instead, backward-only propagation can be observed in figures 11(d) and 12(d) for longitudinal and transverse motion when the excitation is centered at \( \Omega = 0.5844 \) and \( \Omega = 0.0275 \), respectively.

Another numerical approach to characterize the dispersion properties of a beam consists in exciting the structure and computing the two-dimensional Fourier transform (2DFT) of the recorded displacement fields \( u(x, t) \) and \( w(x, t) \):

\[
U(\kappa, \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x, t) e^{-i(\omega t - \kappa x)} dx dt,
\]

Figure 9. Band maps for beam in longitudinal (a) and transverse (b) motion. The gap amplitude is plotted as a function of the increasing modulation amplitude parameter \( \alpha_m \) for the two cases \( \kappa_m = 0 \) and \( \kappa_m \neq 0 \), i.e. with zero and non-zero modulation velocity parameter, respectively. The colored regions represent the band gap evolution as computed solving the full QEP, while the approximate analytic expression, valid for \( \alpha_m \ll 1 \), are represented by the dashed lines.

Figure 10. Schematic of a 2L long beam loaded in its mid span. Longitudinal and transverse motion are excited by the horizontal force \( F_{ext,L} \) and the transverse force \( F_{ext,T} \), respectively.
The transformation allows one to describe the response of the beam in the wavenumber/frequency domain, effectively obtaining a band diagram for the structure. This technique is successfully applied in experimental validations as well (Airoldi and Ruzzene 2011). We compare the band diagrams given by plotting the magnitude $|U(\kappa, \omega)|$ and $|W(\kappa, \omega)|$ of the complex quantities defined by equations (61) and (62), in the form of contour lines, with the band diagrams that are given by the method discussed in the previous sections.

The results are presented in figures 13(b) and (c) for longitudinal motion of respectively a uniform and modulated beam with $\alpha_m = 0.40$ and $\nu_m = 0.20$. Similarly, figures 14(b) and (c) show the results for transverse motion of a uniform and modulated beam with $\alpha_m = 0.40$ and $\nu_m = 0.005$, respectively. When longitudinal waves are concerned, the structure is excited with a load characterized by a broadband signal centered at $\Omega = 0.485$ and acting along the $x$ direction as shown in figure 13(a), while for transverse waves a vertical load centered at $\Omega = 0.0225$ is used. The broadband signal is now required in order to excite the structure within the

$$W(\kappa, \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w(x, t) e^{-i(\omega t - \kappa x)} \, dx \, dt.$$  (62)
frequency range of interest, which in our case has to be wide enough to obtain the band diagram. We recognize that the contour lines used to plot both $U (\kappa, \omega)$ and $W (\kappa, \omega)$ overlap with the dispersion branches given by the proposed method, thus verifying the validity of the methodology for the calculation of the dispersion diagrams of spatiotemporal structures. For comparison, figures 13 and 14 show also the case in which the modulation is spatial only, hence for $\kappa_m = 0$.

4. Conclusions

In summary, we characterize dispersion properties of spatiotemporal periodic beams in longitudinal and transverse motion. By employing a solution in the Floquet form with space and time harmonics, we are able to compute the dispersion diagrams for such structures by solving a quadratic eigenvalue problem. The analysis of the dispersion diagrams, limited by the 1D beam modeling assumptions to the low-frequency range, allows us to...
describe the unique features that spatiotemporal modulation induces on the wave propagation properties of the structure, such as symmetry breaking of the dispersion relation and the relative dispersion diagrams. Specifically, we identify the signature of one-way propagation as directional band gaps. Such band gaps clearly describe in which frequency ranges forward-propagating or backward-propagating waves only are supported by the structure. The key finding of the study allow us to analytically relate the modulation parameters to the position and width of the directional band diagrams, and also compute the minimum modulation speed required in order to have a fully non-reciprocal wave propagation for harmonic modulation. Finally, we verify our prediction with numerical simulations.

Nonreciprocal systems are the object of an emerging field of study and promise to deeply impact the way we control wave propagation, enriching the design space for technological applications in acoustics, phononics and photonics, to name a few. For this reason, when spatiotemporal periodic structures are used to achieve one-way propagation, it is important to be able to properly describe and characterize their non-reciprocal behavior, correctly predicting the influence of the modulation parameters. The proposed methodology was exposed in the context of elastic waves propagating in an elastic solid, but it could be applied to other one-dimensional wave propagation problems with a similar mathematical structure and involving a periodic variation of the medium characteristics in both space and time. Further investigation should be devoted in understanding the role of frequency conversion phenomena in spatiotemporal modulated structures in relation to their use in manipulating and controlling elastic waves, similarly to what has been done in the study of traveling-wave

Figure 13. For longitudinal motion of 2L long beam excited at its mid span, comparison between the band diagram obtained solving the QEP (dashed lines) and through the normalized magnitude $|U(\tau, \omega)|$ of the 2DFT of the displacement field (contour lines): (a) excitation signal and its frequency spectrum; (b) band diagram for uniform beam ($a_1=0$ and $b_1=0$); (c) band diagram for modulated beam ($a_1=0.40$ and $b_1=0.20$).
parametric circuits and parametric amplification of electromagnetic waves, possible applications being one-directional vibration insulators or directional spatial filters.

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Appendix A. Coefficients of the QEP with first order wave expansion

The coefficients of equation (19) for a modulated beam in longitudinal motion write as follows:

\[ L_{11} = \Omega^2 - 2\nu_m \Omega - \left( \frac{\mu}{2\pi} - 1 \right)^2 - \nu_m^2, \]

(A.1)

\[ L_{12} = -\frac{\mu}{2\pi} \left( \frac{\mu}{2\pi} - 1 \right) \frac{\Omega_m}{2}. \]

(A.2)

Figure 14. For transverse motion of 2L long beam excited at its mid span, comparison between the band diagram obtained solving the QEP (dashed lines) and through the normalized magnitude \(|W(\tau, \omega)|\) of the 2DFT of the displacement field (contour lines): (a) excitation signal and its frequency spectrum; (b) band diagram for uniform beam (\(\alpha_m = 0\) and \(\nu_m = 0\)); (c) band diagram for modulated beam (\(\alpha_m = 0.40\) and \(\nu_m = 0.005\)).
Similarly, the coefficients of equation (46) for a modulated beam in transverse motion write as follows:

\begin{align}
L_{13} &= 0, \\
L_{21} &= -\frac{\mu}{2\pi} \left( \frac{\mu}{2\pi} - 1 \right) \alpha_m \frac{\chi}{2}, \\
L_{22} &= \Omega^2 - \left( \frac{\mu}{2\pi} \right)^2, \\
L_{23} &= -\frac{\mu}{2\pi} \left( \frac{\mu}{2\pi} + 1 \right) \alpha_m \frac{\chi}{2}, \\
L_{31} &= 0, \\
L_{32} &= -\frac{\mu}{2\pi} \left( \frac{\mu}{2\pi} + 1 \right) \alpha_m \frac{\chi}{2}, \\
L_{33} &= \Omega^2 + 2\nu_m \Omega - \left( \frac{\mu}{2\pi} + 1 \right)^2 \chi^2 - \nu_m^2, \tag{A.9}
\end{align}

Figure B1. Analysis of the error introduced by the truncation of the QEP’s solution for harmonic modulation: ratio between the value of the band gap limits computed with truncation order \( N \) and the value \( \Omega \) computed with \( N = 1 \) as a function of \( N \) (a); CPU time needed to compute the QEP for increasing truncation order \( N \) as a function of \( N \), normalized by the time needed to solve the QEP for \( N = 1 \) (b). We assumed \( \alpha_m = 0.40, \nu_m = 0.05 \) for longitudinal motion and \( \alpha_m = 0.40, \nu_m = 0.002 \) for transverse motion.
Appendix B. Truncation order analysis

The truncation order was selected as a compromise between accuracy of the predictions and required computational time. A short convergence study was conducted to determine this compromise. The relatively low number of harmonics used in the expansion is due to the fact that the considered modulations and their effect on the wave properties are well represented with few harmonics. Figure B1 (a) shows the ratio $\Omega_N / \Omega_l$ of the frequency bounds of the the band gap limits when considering $N$ and a single term ($N = 1$) respectively, both for the harmonic and square modulations cases. For harmonic modulation of the beam in longitudinal motion, we considered $\alpha_m = 0.40$ and $\nu_m = 0.05$, while for harmonic modulation of beam in transverse motion we considered $\alpha_m = 0.40$ and $\nu_m = 0.002$. When other modulation parameters are considered, the results did not

$$T_{33} = \Omega^2 + 2\nu_m \Omega - \left( \frac{\mu}{2\pi} + 1 \right)^4 \chi^2 - \nu_m^2$$

(A.18)

in which we define $\chi = R_g \kappa_m$.
change significantly. As shown in figure B1(a), the ratio $\Omega_N / \Omega_1$ decreases from unity for $N > 1$, due to the better approximation obtained when increasing the terms retained in the series. Figure B1(b) shows the CPU time required for the computation as a function of the number $N$ of terms retained in the series, normalized with respect to the CPU time associated to the case $N = 1$. Since retaining more than $N = 3$ terms seems not to provide any advantage, while the CPU time needed increases quickly, we chose $N = 3$ to be a good compromise when harmonic modulation is considered. When we studied square modulation, we focused on the first two band gaps only. The second band gap is obtained when considering $N \geq 2$. Figure B2 shows that the ratios $\Omega_N / \Omega_1$ associated to the band gap limits decrease more slowly than when harmonic modulation is considered. Due to the fact that the matrices in the QEP are fully populated, since $\hat{B}_p \neq 0$ for $|p| \geq 2$, more time is required to compute the solution of the QEP. For this reason, we assumed $N = 5$ to be a good compromise between quality of the approximation and time required for the computation.

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