A MODEL TO EXPLAIN VARYING $\Lambda$, $G$ AND $\sigma^2$ SIMULTANEOUSLY

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Abstract

Models with varying cosmical parameters, which were earlier regarded constant, are getting attention. However, different models are usually invoked to explain the evolution of different parameters. We argue that whatever physical process is responsible for the evolution of one parameter, should also be responsible for the evolution of others. This means that the different parameters are coupled together somehow. Based on this guiding principle, we investigate a Bianchi type I model with variable $\Lambda$ and $G$, in which $\Lambda$, $G$ and the shear parameter $\sigma^2$, all are coupled. It is interesting that the resulting model reduces to the FLRW model for large $t$ with $G$ approaching a constant.

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1. Introduction
In the past few years, evidence has mounted indicating that some constants, which were earlier treated as true constants, are no longer constant in cosmology. The examples are – Einstein’s cosmological constant $\Lambda$, Newton’s gravitational constant $G$, the fine structure constant, etc. Different phenomenological models have been suggested to explain the evolutions of different constants (let us call them parameters). However, we believe that there should be only one model to explain all these parameters if the underlying theory is correct. Moreover, whatever physical process is responsible for the evolution of one parameter, should also be responsible for the evolution of others, implying that the different parameters are coupled together somehow. It should, therefore, be the evolution of the universe itself which should explain the dynamics of all the parameters. In this paper, we investigate such a model from the Einstein field equations which explains the variability of $\Lambda$, $G$ and the anisotropy parameter $\sigma^2$ simultaneously. The cosmological consequences of the model are also discussed.

Now we shall describe briefly the motivation for considering the different parameters and their variations. The one which comes first in the list is undoubtedly the Einstein’s cosmological parameter $\Lambda$, whose existence is favoured by the recent supernovae (SNe) Ia observations [1] and which is also consistent with the recent anisotropy measurements of the cosmic microwave background (CMB) made by the WMAP experiment [2]. However, there is a fundamental problem related with the existence of $\Lambda$, which has been extensively discussed in the literature. It’s value expected from the quantum field theory- calculations is about 120 orders of magnitude higher than that estimated from the observations. A phenomenological solution to this problem is suggested by considering $\Lambda$ as a function of time, so that it was large in the early universe and got reduced with the expansion of the universe [3].

Variation of Newton’s gravitational parameter $G$ was originally suggested by Dirac on the basis of his large numbers hypothesis [4]. As $G$ couples geometry to matter, it is reasonable to consider $G = G(t)$ in an evolving universe when one considers $\Lambda = \Lambda(t)$. Many extensions of general relativity with $G = G(t)$ have been made ever since Dirac first considered the possibility of a variable $G$, though none of these theories has gained wide acceptance. However a new approach, which has been widely investigated in the past few years [5], is appealing. It assumes the conservation of the energy-momentum tensor which consequently renders $G$ and $\Lambda$ as coupled fields, similar to the
case of \( G \) in original Brans-Dicke theory. This leaves Einstein’s fields equations formally unchanged. In this context, an approach is worth mentioning in which the scaling of \( G(t) \) and \( \Lambda(t) \) arise from an underlying renormalization group flow near an infrared attractive fixed point [6]. The resulting cosmology explains the high redshift SNe Ia and radio sources observations successfully [7]. It also describes the Planck era reliably and provides a resolution to the horizon and flatness problems of the standard cosmology without any unnatural fine tuning of the parameters [8]. Gravitational theories with variable \( G \) have also been discussed in the context of induced gravity model where \( G \) is generated by means of a non-vanishing vacuum expectation value of a scalar field [9]. Recently a constraint on the variation of \( G \) has been obtained by using WMAP and the big bang nucleosynthesis observations [10], which comes out as
\[
-3 \times 10^{-13} \text{ yr}^{-1} < (\dot{G}/G)_\text{today} < 4 \times 10^{-13} \text{ yr}^{-1}.
\]

Another important quantity which is supposed to be damped out in the course of cosmic evolution is the anisotropy of the cosmic expansion. It is believed that the early universe was characterized by a highly irregular expansion mechanism which isotropized later [11]. The level of anisotropy left out by the era of decoupling is only about \( 10^{-5} \), as is revealed by the CMB observations. It could be that whatever mechanism diminished \( \Lambda \) to its present value, could have also rendered the early highly anisotropic universe to the present smoothed out picture. This will be our guiding principle in investigating the model.

We shall keep ourselves limited to Einstein’s field equations and to the parameters which appear explicitly therein. It would be worthwhile to mention that models with varying speed of light are recently being promoted. These are supported by the claims, based on the measurements of distant quasar absorption spectra, that the fine structure constant may have been smaller in the past. However, the speed of light \( c \) has a complex character having six different facets which come from many laws of physics that are \textit{a priori} disconnected from the notion of light itself [12]. If it is the causal speed of which these theories are talking about, then one should not consider a varying \( c \) in general relativity unless the structure of the spacetime metric is changed and reinterpreted. We consider \( c = 1 \) throughout our calculations.

We consider the Bianchi type I metric, which is the simplest anisotropic generalization of the flat Robertson-Walker metric and allows for different expansion factors in three orthogonal directions. In the comoving coordinates
(u^i = \delta^i_0), the metric can be written as
\[ ds^2 = -dt^2 + X^2(t) \, dx^2 + Y^2(t) \, dy^2 + Z^2(t) \, dz^2. \]  

An average expansion scale factor can be defined by \( R(t) = (XYZ)^{1/3} \) implying that the Hubble parameter \( H = \dot{R}/R \).

2. Field Equations

We consider \( G \) and \( \Lambda \) as functions of the cosmic time \( t \). For the metric (1), the Einstein field equations, with perfect fluid, read

\[ \frac{\dot{X}Y}{XY} + \frac{\dot{Y}Z}{YZ} + \frac{\dot{Z}X}{ZX} = 8\pi G \rho + \Lambda \]  
\[ \frac{\ddot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\ddot{Y}}{Y} = -8\pi Gw \rho + \Lambda \]  
\[ \frac{\dot{Y}}{Y} + \frac{\ddot{Z}}{Z} + \frac{\ddot{X}}{X} = -8\pi Gw \rho + \Lambda \]  

(5)

Here we have assumed, as usual, an equation of state \( p = w \rho \), where \( 0 \leq w \leq 1 \) is a constant. The non-vanishing components of the shear tensor \( \sigma_{ij} \), defined by \( \sigma_{ij} = u_{i,j} + u_{j,i} - \frac{2}{3} \, g_{ij} \, u^k_k \), are obtained as

\[ \sigma^1 = \frac{4}{3} \frac{\dot{X}}{X} - \frac{2}{3} \left( \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z} \right), \]  
\[ \sigma^2 = \frac{4}{3} \frac{\dot{Y}}{Y} - \frac{2}{3} \left( \frac{\dot{Z}}{Z} + \frac{\dot{X}}{X} \right), \]  
\[ \sigma^3 = \frac{4}{3} \frac{\dot{Z}}{Z} - \frac{2}{3} \left( \frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} \right). \]

Thus the magnitude \( \sigma^2 \equiv \sigma_{ij} \sigma^{ij}/8 \) is obtained as

\[ \sigma^2 = \frac{1}{3} \left[ \frac{\dot{X}^2}{X^2} + \frac{\dot{Y}^2}{Y^2} + \frac{\dot{Z}^2}{Z^2} - \left( \frac{\dot{X} \dot{Y}}{XY} + \frac{\dot{Y} \dot{Z}}{YZ} + \frac{\dot{Z} \dot{X}}{ZX} \right) \right]. \]  

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It can be shown\(^2\) that \(\sigma^2\) is proportional to \(R^{-6}\), i.e., \(\sigma = \alpha R^{-3}\), where \(\alpha = \text{constant}\). This implies that

\[
\frac{\dot{\sigma}}{\sigma} = -\left(\frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z}\right) = -3H. \tag{10}
\]

Equations (2) and (9) allow to write the analogue of the Friedmann equation as

\[
3H^2 = 8\pi G \rho + \sigma^2 + \Lambda. \tag{11}
\]

So far, there has been no effect of the varying characters of \(G\) and \(\Lambda\) on the equations and they are formally the same as those with constant \(G\) and \(\Lambda\). However, the generalized conservation equation is different from the ordinary one. This can be obtained either from the Bianchi identities or by using equations (3–5) in the differentiated form of equation (2) and can be written, after doing some simple algebra, in the form

\[
8\pi G \left[ \dot{\rho} + 3(1 + w)H \rho \right] + 8\pi \rho \dot{G} + \dot{\Lambda} = 0. \tag{12}
\]

We assume, as is common in cosmology, that the conservation of energy-momentum tensor of matter holds \((T_{ij}^m = 0)\) leading to

\[
\dot{\rho} + 3(1 + w)H \rho = 0, \tag{13}
\]

leaving \(G\) and \(\Lambda\) as some kind of coupled fields:

\[
8\pi \rho \dot{G} + \dot{\Lambda} = 0. \tag{14}
\]

Equation (13 has a simple solution \(\rho = CR^{-3(1+w)}\), where \(C = \text{constant} > 0\). Equation (14) can be integrated as

\[
G(R) = G_0 - \frac{1}{8\pi C} \left[ \Lambda(R) R^{3(1+w)} - 3(1 + w) \int \Lambda(R) R^{2+3w} dR \right], \tag{15}
\]

where \(G_0\) is a constant of integration. Equations (10–15) supply only 4 independent equations in 5 unknowns \(\rho, R, G, \Lambda\) and \(\sigma\). In search of one more equation, we do some algebra in the following.

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\(^2\)By subtracting (4) from (3), and (5) from (4) and integrating the resulting equations, one can get \(\dot{X} - \dot{Y} \propto \frac{1}{XYZ^2}, \dot{Y} - \dot{Z} \propto \frac{1}{XYZ}, \dot{Z} - \dot{X} \propto \frac{1}{XYZ^2}\). By squaring and adding these equations one gets \(\sigma^2 \propto 1/(XYZ)^2\).
An elimination of $H$ between (11) and (13) gives
\[
\frac{\dot{\rho}^2}{\rho^3} = 3(1 + w)^2 \left(8\pi G + \frac{\sigma^2}{\rho} + \frac{\Lambda}{\rho}\right).
\]
(16)

Differentiating this and using (10), (13) and (14) therein, we obtain
\[
2\frac{\ddot{\rho}}{\rho} - 3\frac{\dot{\rho}^2}{\rho^2} = 3(1 + w)^2 \left[\left(\frac{1 - w}{1 + w}\right) \sigma^2 - \Lambda\right], \ w \neq -1, \ \dot{\rho} \neq 0,
\]
(17)

which is the central equation of our investigation whose solution will supply the required ansatz. Substituting (13) in (17), we obtain an equation for $R$ as
\[
\frac{2}{1 + w} \dot{H} + 3H^2 + \left(\frac{1 - w}{1 + w}\right) \sigma^2 - \Lambda = 0.
\]
(18)

3. Models

If the physical processes, responsible for reducing the early highly anisotropic universe to a smooth present universe, are also responsible for bringing down the large value of $\Lambda$ to its small present value, the two parameters $\sigma^2$ and $\Lambda$ must be related somehow. In view of this guiding principle, the simplest solution of equation (17) is
\[
\Lambda = \left(\frac{1 - w}{1 + w}\right) \sigma^2,
\]
(19)

which is our required ansatz, indicates a linear coupling between the cosmological constant and anisotropy. The parameters $G$ and $\Lambda$ are already coupled through equation (15). We find that the model in this case is described by
\[
R = a \ t^{2/3(1+w)}, \ a = \text{constant} > 0,
\]
(21)

\[
\rho = \left[\frac{C}{a^{3(1+w)}}\right] t^{-2},
\]
(22)

\[
\sigma = \left[\frac{\alpha}{a^3}\right] t^{-2/(1+w)},
\]
(23)
\[ \Lambda = \left[ \frac{1-w}{1+w} \right] \frac{\alpha^2}{a^6} t^{4/(1+w)}, \quad (24) \]

\[ G = G_0 - \left[ \frac{\alpha^2}{4\pi C(1+w)} a^{-3(1-w)} \right] t^{-2(1-w)/(1+w)}. \quad (25) \]

The model has a constant deceleration parameter \( q = (1+3w)/2 \) and evolves to isotropy as \( t \to \infty \), with \( \Lambda \to 0 \) and \( G \to G_0 \). Thus for large \( t \), the model approaches the flat FLRW model which is very encouraging. It may be noted that though the current observations of SNe Ia and CMB favour accelerating models \( (q < 0) \), but they do not altogether rule out the decelerating ones which are also consistent with these observations [13]. One can even fit the models with zero \( \Lambda \) if one takes into account the extinction of light by the metallic dust ejected from the supernovae explosions [13].

We note that for \( t < 3\alpha^2/16\pi CG_0a^2 \), \( G \) becomes negative unless \( w = 1 \) (with \( G_0 > \alpha^2/8\pi C \)). One can however choose the constants \( \alpha \) and \( a \) (which are arbitrary) appropriately so that \( G \) remains positive in the range of validity of general relativity. However, taken at the face value, the model predicts a repulsive gravity in the range \( 0 \leq t < 3\alpha^2/16\pi CG_0a^2 \). For \( w = 1 \), the model reduces to \( \Lambda = 0 \), \( G = \text{constant} \) and \( \sigma \propto H = 1/3t \).

The model can be generalized very easily by generalizing the ansatz (19) in the form:

\[ \Lambda = \gamma \sigma^2, \quad \gamma = \text{constant}, \quad (26) \]

which can allow a negative \( \Lambda \) as well (until we have precise enough SN Ia data to rule out certain models, we should keep all our options open). Now with the new ansatz (26), equation (15) reduces to

\[ G = G_0 - \frac{\gamma \alpha^2}{4\pi C(1-w)} R^{-3(1-w)}, \quad w \neq 1. \quad (27) \]

The model starts from a big bang (or a big bang-like state) with \( G, \quad |\Lambda| \) and \( \sigma^2 \) all infinite and evolves to isotropy with \( G \to G_0 \) and \( \Lambda \to 0 \) as \( t \to \infty \). The time-evolution of \( R \) is given by

\[ t + t_0 = \int \left[ \frac{8\pi CG_0}{3} R^{-(1+3w)} + \frac{\alpha^2}{3} \left( 1 - \frac{1+w}{1-w} \right) R^{-4} \right]^{-1/2} dR, \quad w \neq 1, \quad (28) \]
where $t_0$ is a constant of integration. It is hard to integrate r.h.s. of equation (28) for a general $w$ ($\neq 1$) unless $\gamma = (1 - w)/(1 + w)$ (which has already been investigated); or $G_0 = 0$ i.e., $\gamma < 0$. If $G_0 = 0$, equation (28) reduces to

$$R = \alpha^{1/3} \left[3 \left(1 - \frac{1+w}{1-w} \gamma\right)\right]^{1/6} t^{1/3}. \quad (29)$$

When $G_0 \neq 0$, $\Lambda$ can assume both—negative as well as positive values. In the case of a positive $\Lambda$, similar kind of argument, as above, can be given when $G$ becomes negative in the beginning of the universe. For a positive $G_0$, equation (28) can be integrated in different phases of evolution, as is shown in the following.

$w = 1/3 :$

$$t = \frac{1}{2\ell} \left[R\sqrt{\ell R^2 + m - \frac{m}{\sqrt{\ell}}} \sinh^{-1} \left(\frac{\sqrt{\ell} R}{m}\right)\right], \quad (30)$$

$w = 0 :$

$$R = \left[\frac{9}{4} \ell (t + t_0)^2 - \frac{n}{\ell}\right]^{1/3}, \quad (31)$$

where $\ell = 8\pi CG_0/3$, $m = (1 - 2\gamma)\alpha^2/3$, $n = (1 - \gamma)\alpha^2/3$, $t_0 = 2\sqrt{n}/3\ell$ and $\gamma < 1/2$.

5. Conclusion

Einstein’s field equations with time-dependent $G$ and $\Lambda$ have been considered in the context of Bianchi type-I spacetime in such a way which conserves the energy-momentum tensor of matter. We assume that the physical processes responsible for the evolution of one parameter, should also be responsible for the evolutions of others. This means that the different parameters are coupled. In this view, the field equations give a trivial ansatz implying a coupling between $\sigma^2$ (shear), $\Lambda$ and $G$. The resulting model, for the baryonic matter, approaches the standard FLRW model in the later epochs, with $G$ approaching a constant value. However, the earlier phases of the model are altogether different from that in the standard cosmology. For stiff matter,
the model reduces to the standard Bianchi type-I model with \( q = 2, G = \text{constant}, \Lambda = 0, \rho \sim t^{-2} \) and \( \sigma \sim H \sim t^{-1} \).

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