ON QUANTUM GROUP $GL_{p,q}(2)$

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Abstract. In the Hopf algebra $GL_{p,q}(2)$ the determinant is central iff $p = q$. In this case we put determinant to be equal to 1 to get $SL_q(2)$. In this paper we consider the case when $p/q$ is a root of unity; and, consequently, a power of the determinant is central.

Introduction.

In the paper [B-Kh] the Universal enveloping algebras for metaplectic quantum groups of $SL(2)$-type were constructed. As we look back at the history of $SL_q(2)$ we can notice that on the levels of formulas the algebra of functions on $SL_q(2)$ was always presented the same way ([M], [R-T-F]), while the Universal enveloping algebra had different presentations. This fact encouraged me and Joseph Bernstein [B-Kh] to consider the enveloping algebra as the secondary object in comparison with the Hopf algebra of functions. As a byproduct of our axiomatic search we constructed the metaplectic quantum groups of $SL(2)$-type. This construction was carried out on the level of universal algebras.

On my way to find nice formulas for algebra of functions on metaplectic quantum group, I found some nice formulas for other algebras. These formulas are presented in this paper.

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1. Algebra of regular functions on $GL_{p,q}(2)$.

1.1. For every $p, q \in \mathbb{C}^*$ we consider the bialgebra $Mat_{p,q}$ which is generated by four noncommuting elements $(a, b, c, d)$, satisfying the following relations (see, [M], [O-W] and references there):

\begin{align*}
ab &= p^{-1}ba \\
ac &= q^{-1}ca \\
bd &= q^{-1}db \\
bc &= q^{-1}pcb \\
ad - da &= (p^{-1} - q)bc.
\end{align*}

(*)
Introduce matrices
\[ Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Mat(2, Mat_{p,q}) \]
\[ P = \begin{pmatrix} 0 & -1 \\ p^{-1} & 0 \end{pmatrix} \quad Q = \begin{pmatrix} 0 & -1 \\ q^{-1} & 0 \end{pmatrix}. \]

Then we can rewrite the relations \((*)\) in a more compact form:
\[ YPY^t = DP \]
\[ Y^tQY = DQ, \]
where \(D\) is an element in \(Mat_{p,q}\) and has a meaning of quantum determinant.

**Comment.** The matrix \(P\) defines a quantum plane as an algebra generated by two generators \(x, y\) and the relation \(xy = p^{-1}yx\). Analogously the matrix \(Q\) defines a quantum plane. We consider \(Mat_{p,q}\) as an algebra of operators preserving quantum plane defined by \(P\) and dual quantum plane defined by \(Q\).

1.2. The comultiplication in the algebra \(Mat_{p,q}\) is defined as follows:
\[ \Delta a = a \otimes a + b \otimes c \]
\[ \Delta b = a \otimes b + b \otimes d \]
\[ \Delta c = c \otimes a + d \otimes c \]
\[ \Delta d = c \otimes b + d \otimes d. \]

Using the natural imbeddings \(i', i'' : Mat_{p,q} \rightarrow Mat_{p,q} \otimes Mat_{p,q}, \ (i'(x) = x \otimes 1, \ i''(x) = 1 \otimes x)\), we can rewrite comultiplication formulae \((**)\) as follows:
\[ \Delta(Y) = i'(Y) \cdot i''(Y), \]
which is an equality in \(Mat(2, Mat_{p,q} \otimes Mat_{p,q})\).

1.3. This algebra has a multiplicative quantum determinant \(D = det_{p,q}(Y)\) (see [M], [O-W], [K]):
\[ D = da - pcb = da - qbc = ad - p^{-1}bc = ad - q^{-1}cb. \]

Multiplicativity means that \(\Delta D = D \otimes D\), or, equivalently, \(det_{p,q}(Y_1Y_2) = det_{p,q}(Y_1) \cdot det_{p,q}(Y_2)\) whenever the entries of \(Y_1\) commute with the entries of \(Y_2\).

Localizing by \(D^{-1}\) we will get the algebra of functions on \(GL_{p,q}(2)\). This localization can be described easily due to the fact that \(D\) is normalizing (see [M]):
\[ Da = aD \quad Db = p^{-1}qbD \]
\[ Dc = pq^{-1}cD \quad Dd = dD. \]

1.4. The bialgebra Fun\((GL_{p,q})\) is the Hopf algebra; an antipode \(S\) is defined by: \(S(Y) = Y^{-1}\). Specifically:
\[ S(a) = dD^{-1} \quad S(b) = -pbD^{-1} \]
\[ S(c) = -p^{-1}cD^{-1} \quad S(d) = aD^{-1}. \]
In a more compact form:

\[ Y^{-1} = S(Y) = P Y^t P^{-1} D^{-1} = D^{-1} Q^{-1} Y^t Q. \]

1.5. If \( p = q \) then \( D \) is central; and we can take quotient of \( \text{Fun}(GL_{p,q}(2)) \) by the Hopf ideal generated by \( D - 1 \). We would get standard \( \text{Fun}(SL_q(2)) \) (see [Kas]).

1.6. Suppose now that \( p^{-1} q \) equals \( \xi \), where \( \xi \) is the \( n \)-th root of unity: \( \xi^n = 1 \). In this case \( D^n \) is central. An ideal generated by \( D^n - 1 \) is a Hopf ideal, so we can consider a quotient algebra, which we would denote by \( \text{Fun}(SL_{q,\xi}(2)) \):

\[ \text{Fun}(SL_{q,\xi}(2)) = \text{Fun}(GL_{p,q}(2))/(D^n - 1). \]

2. Axiomatic approach.

2.1. Analyzing the Hopf algebra \( A = \text{Fun}(SL_{q,\xi}(2)) \) we note that it has the following important property:

Let \( I \subset A \) be a two-sided ideal generated by \( b \) and \( c \). Then \( I \) is a Hopf ideal in \( A \), i.e. \( \Delta I \in A \otimes I + I \otimes A \) and \( S(I) \subset I \). The quotient Hopf algebra \( A/I \) is isomorphic to the algebra of functions on the algebraic group \( \mathbb{C}^* \otimes \mathbb{Z}_n \):

\[ \mathbb{C}[a, d, D, D^{-1}]/(ad - D, D^n - 1) \]

\[ \Delta a = a \otimes a \quad \Delta d = d \otimes d \quad \Delta D = D \otimes D \]

\[ S(a) = dD^{-1} \quad S(d) = aD^{-1} \quad S(D) = D^{-1}. \]

Or, equivalently:

\[ \mathbb{C}[a, a^{-1}, D, D^{-1}]/(D^n - 1) \]

\[ \Delta a = a \otimes a \quad \Delta D = D \otimes D \]

\[ S(a) = a^{-1} \quad S(D) = D^{-1}. \]

Informally, this means that our quantum group \( A = \text{Fun}(SL_{q,\xi}(2)) \) contains the group \( \mathbb{C}^* \otimes \mathbb{Z}_n \) as a subgroup.

3. Dual picture.

3.1. Let us denote by \( T^2 \) the two-dimensional torus. Denote by \( t \) a point of \( T^2 \). To each point \( t = (x_1, x_2) \) we can correspond a generator \( \hat{t} \) in the group algebra of torus. This correspondence is multiplicative — \( \hat{t}_1 \hat{t}_2 = \hat{t}_1 \hat{t}_2 \).

We can describe (we use notations similar to [B-Kh]) a universal enveloping algebra \( U_{p,q}(2) \) of \( GL_{p,q}(2) \) as a Hopf algebra generated by \( \hat{t} \) (\( t \in T^2 \)) and two elements \( E, F \), satisfying the relations:

\[ \hat{t} E \hat{t}^{-1} = \alpha(t) E \]

\[ \hat{t} F \hat{t}^{-1} = (-\alpha)(t) F \] (1)
\[ \Delta \hat{t} = \hat{t} \otimes \hat{t} \]
\[ \Delta E = E \otimes 1 + Q_1 \otimes E \]
\[ \Delta F = F \otimes Q_2^{-1} + 1 \otimes F \]

(2)

\[ [E, F] = \frac{Q_1 - Q_2^{-1}}{q - p^{-1}}. \]

(3)

Here \( Q_1, Q_2 \) are generators corresponding to points of the torus with coordinates \((q, p^{-1})\) and \((p, q^{-1})\) respectively; and \( \alpha \) denotes the weight \((1, -1)\) on torus: if \( t = (x, y) \), then \( \alpha(t) = xy^{-1} \).

**Comments.**

1. The commutator \([E, F]\) is the element \( X \) of an algebra, generated by torus, satisfying the equation: \( \Delta X = Q_1 \otimes X + X \otimes Q_2^{-1} \). In denominator (3) we can choose any constant. We chose our constant so that the evaluation of the right hand side element on the weight \((1, 0)\) would be equal to 1.

2. In this case and in the following cases, the antipode is uniquely defined and could be easily recovered:

\[ S(\hat{t}) = \hat{t}^{-1} \quad S(E) = -Q_1^{-1}E \quad S(F) = -Q_2F. \]

3.2. To get the universal algebra \( U_q(2) \) of \( SL_q(2) \) we have to put \( p = q \) and to take the group subalgebra of \( T^1 \subset T^2 \) generated by elements \( \hat{h} \) corresponding to \((h, h^{-1}) \in T^2 \). Denote by \( K \) an element corresponding to \((q, q^{-1})\), then we would have the following relations:

(1)
\[ \hat{h}E \hat{h}^{-1} = h^2E \]
\[ \hat{h}F \hat{h}^{-1} = h^{-2}F \]

(2)
\[ \Delta \hat{h} = \hat{h} \otimes \hat{h} \]
\[ \Delta E = E \otimes 1 + K \otimes E \]
\[ \Delta F = F \otimes K^{-1} + 1 \otimes F \]

(3)
\[ [E, F] = \frac{K - K^{-1}}{q - q^{-1}}. \]

3.3. Consider the case \( SL_q,\xi(2) \): namely, \( p^{-1}q = \xi \), where \( \xi^n = 1 \). We can describe the universal enveloping algebra \( U_q,\xi(2) \) as a subalgebra in \( U_{p,q}(2) \), generated by elements \( E, F, \hat{h} = (h, h^{-1}) \in T^2 \) for \( h \in T \) and \( W = (1, \xi) \in T^2 \) \((W^n = 1)\).

(1)
\[ WEW^{-1} = \xi^{-1}E \quad \hat{h}E \hat{h}^{-1} = h^2E \]
\[ WFW^{-1} = \xi F \quad \hat{h}F \hat{h}^{-1} = h^{-2}F \]

\[ \Delta W = W \otimes W \]

(2)
\[ \Delta \hat{h} = \hat{h} \otimes \hat{h} \]
\[ \Delta E = E \otimes 1 + W \hat{q} \otimes E \]
\[ \Delta F = F \otimes W \hat{\xi} \hat{\xi}^{-1} + 1 \otimes F \]
\[ [E, F] = \frac{\hat{q} - \xi \hat{q}^{-1}}{q - \xi q^{-1}} W. \]

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