Inverse scattering on non-compact manifolds with general metric

Dedicated for the memory of Yaroslav Kurylev

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Introduction and summary

The problem we address in this paper is the spectral theory and inverse problem associated with Laplacians on some class of non-compact Riemannian manifolds (or more general manifolds admitting conic singularities). By observing behaviors of solutions to the Helmholtz equation on the manifold, we introduce an analogue of Heisenberg’s scattering matrix in quantum mechanics. We then show that the knowledge of the scattering matrix determines the topology and the metric of the manifold. We begin with a brief overview of our results, leaving precise statements in the text.

0.1. Scattering of waves on non-compact manifolds. We consider a connected, non-compact $n \geq 2$-dimensional Riemannian manifold $\mathcal{M}$ of the form

$$\mathcal{M} = \mathcal{K} \cup \left( \bigcup_{j=1}^{N} M_j \right) \cup \left( \bigcup_{j=N+1}^{N+N'} M_j \right),$$

where $\mathcal{K}$ is an open relatively compact subset and $M_j$, $1 \leq j \leq N + N'$, is an open non-compact subset in $\mathcal{M}$, henceforth called an end. As will be discussed later, we allow conical singularities for our manifolds. For the sake of simplicity of explanation, however, we first consider $C^\infty$-manifolds.

We assume that each end $M_j$ is diffeomorphic to $(0, \infty) \times M_j$, where $M_j$ is a compact $(n - 1)$-dimensional Riemannian manifold endowed with metric $h_j(x, dx)$. Moreover, on each $M_j$, the Riemannian metric of $\mathcal{M}$ is written in the form

$$ds^2 = (dr)^2 + \rho_j(r)h_j(r, x, dx),$$

where $h_j(r, x, dx)$ is an $r$-dependent metric on $M_j$ satisfying\footnote{Until the end of subsection 0.6, we state only main parts of the assumptions. Precise assumptions are given in Subsection 0.11.}

$$h_j(r, x, dx) = h_j(x, dx) + O(r^{-\gamma_j}), \quad \text{as} \quad r \to \infty,$$
where \( \gamma_j > 1 \). As for the behavior of \( \rho_j(r) \), we assume that for \( 1 \leq j \leq N \), the section of \( M_j \) at \( r \) is either exponentially or polynomially growing,

\[
(0.2) \quad \rho_j(r) = O(e^{\gamma_j r}), \quad \text{or} \quad \rho_j(r) = O(r^{2\beta_j}), \quad c_j > 0, \quad \beta_j > 0,
\]

and for \( N + 1 \leq j \leq N + N' \), the section of \( M_j \) at \( r \) is either exponentially or polynomially decaying,

\[
(0.3) \quad \rho_j(r) = O(e^{-\gamma_j r}), \quad \text{or} \quad \rho_j(r) = O(r^{-2\beta_j}), \quad c_j > 0, \quad \beta_j > 0.
\]

We represent the case (0.2) by

\[
\rho_j(r) \geq O(r^{2\beta_j})
\]

and, similarly the case (0.3) by

\[
\rho_j(r) \leq O(r^{-2\beta_j}).
\]

We put

\[
E_j = \left( \frac{(n-1)c_j}{2} \right)^2,
\]

when \( \rho_j(r) \) is exponentially growing or decaying, and

\[
E_j = 0,
\]

when \( \rho_j(r) \) grows or decays polynomially, but not exponentially. We put

\[
E = \min_{1 \leq j \leq N + N'} E_j.
\]

Let \( H = -\Delta M \) be the Laplacian of \( M \). Then, the essential spectrum of \( H \) is

\[
\sigma_e(H) = [E, \infty).
\]

Let

\[
E = \{ E_1, \ldots, E_{N+N'} \} \cup \sigma_p(H),
\]

which is a discrete set with possible accumulation points in \( \{ E_1, \ldots, E_{N+N'} \} \).

We introduce a function space \( B^*(M) \): \( f \in B^*(M) \) if and only if \( f \in L^2(K) \) and

\[
\sup_{R>1} \frac{1}{R} \int_0^R \|f(r)\|_{L^2(M_j)}^2 dr < \infty
\]

for \( 1 \leq j \leq N + N' \). For two functions \( f, g \in L^2_{\text{loc}}(M) \), we denote \( f \simeq g \) if they satisfy

\[
\lim_{R \to \infty} \frac{1}{R} \int_{0<r<R} \|f(r) - g(r)\|_{L^2(M_j)}^2 dr = 0
\]

on each \( M_j, 1 \leq j \leq N + N' \).

Our results are most transparent when \( \rho_j(r) \geq r^{2\beta_j} \) with \( \beta_j > 1/2 \) for \( 1 \leq j \leq N \). Take \( \lambda \in \sigma_e(H) \setminus \mathcal{E} \) and let \( \mathcal{N}(\lambda) \) be defined by

\[
\mathcal{N}(\lambda) = \{ u \in B^*(M) : (-\Delta_M - \lambda) u = 0 \}.
\]

Let

\[
h_{\infty,j} = \begin{cases} L^2(M_j), & 1 \leq j \leq N, \\ \mathbb{C}, & N + 1 \leq j \leq N + N', \end{cases}
\]

and put

\[
h_\infty(\lambda) = \oplus_{j=1}^{N+N'} c_j(\lambda) h_{\infty,j},
\]
where \( e_j(\lambda) \) is the characteristic function of \([E_j, \infty)\). Note that for \( a = (a_1, \ldots, a_{N+N'}) \in h_\lambda \), \( a_j \) is an \( L^2 \)-function on \( M_j \) for \( 1 \leq j \leq N \), while \( a_j \in \mathbb{C} \) for \( N+1 \leq j \leq N+N' \), moreover \( a_j = 0 \) if \( \lambda < E_j \). We put

\[
\Phi_j(r, \lambda) = \int_0^r \phi_j(t, \lambda) \, dt, \\
\phi_j(r, \lambda) = \sqrt{\lambda - \frac{(n - 1)\rho_j'(r)^2}{2\rho_j(r)}}.
\]

**Theorem A.** For \( 1 \leq j \leq N \), assume \( \rho_j(r) \geq O(r^{2\beta_j}) \) with \( \beta_j > 1/2 \), and for \( N+1 \leq j \leq N+N' \), assume \( \rho_j(r) \leq O(r^{-2\beta_j}) \) with \( \beta_j > 0 \). Let \( \lambda \in (E_j, \infty) \setminus \mathcal{E} \). Then, for any \( a^{(in)} \in h_\infty(\lambda) \), there exist unique \( u \in N(\lambda) \) and \( a^{(out)} \in h_\infty(\lambda) \) such that, \( u \) behaves as follows on each end \( M_j \).

1. For \( 1 \leq j \leq N \),
   \[
   u \simeq \left( \frac{\pi}{\sqrt{\lambda - E_j}} \right)^{1/2} \rho_j(r)^{-(n-1)/2} \left( e^{-i\Phi_j(r, \lambda)} a_j^{(in)}(x) - e^{i\Phi_j(r, \lambda)} a_j^{(out)}(x) \right).
   \]

2. For \( N+1 \leq j \leq N+N' \),
   \[
   u \simeq \left( \frac{\pi}{\sqrt{\lambda - E_j}} \right)^{1/2} \rho_j(r)^{-(n-1)/2} \left( e^{-i\Phi_j(r, \lambda)} a_j^{(in)}(r) - e^{i\Phi_j(r, \lambda)} a_j^{(out)}(r) \right) e_{j,0},
   \]

where \( e_{j,0} \) is the normalized eigenvector of the Laplace-Beltrami operator on \( M_j \).

3. The operator
   \[
   S(\lambda) : a^{(in)} \rightarrow a^{(out)}
   \]
   is unitary on \( h_\infty(\lambda) \).

Note that in Theorem A \( a_j^{(in)} = a_j^{(out)} = 0 \) if \( \lambda < E_j \).

The meaning of the above expansion is as follows. For \( u \in N(\lambda) \), we put

\[
v = e^{-i\Phi j(r, \lambda)\sqrt{-\Delta}} u \quad \text{and} \quad v^{(in)}_j = \rho_j(r)^{-(n-1)/2} e^{i\Phi_j(r, \lambda)} a_j^{(in)}(r),
\]

\[
v^{(out)}_j = \rho_j(r)^{-(n-1)/2} e^{i\Phi_j(r, \lambda)} a_j^{(out)}(r),
\]

for \( 1 \leq j \leq N \). We also put

\[
v^{(in)}_j = \rho_j(r)^{-(n-1)/2} e^{i\Phi_j(r, \lambda)} a_{j,0} e_{j,0},
\]

\[
v^{(out)}_j = \rho_j(r)^{-(n-1)/2} e^{i\Phi_j(r, \lambda)} b_{j,0} e_{j,0},
\]

for \( j = N+1, \ldots, N+N' \). Then, \( v \) satisfies the wave equation \( \partial^2_t v + Hv = 0 \) on \( M_j \), and \( v^{(in)}_j, v^{(out)}_j \) satisfies \( \partial^2_t v_j + Hv_j = 0 \) asymptotically as \( r \to \infty \) in \( M_j \). Let \( \psi_j(t) = \pm \Phi_j(r, t) - \sqrt{\lambda t} \). Then, the phase \( \psi_j(t) \) of the wave \( v_j^{(in)}, v_j^{(out)} \) is constant. The expansions in \[1\] and \[2\] show that the wave front is diverging to infinity as \( t \to -\infty \) for \( v^{(in)} \) and \( t \to \infty \) for \( v^{(out)} \). By this reason, in the expansion in Theorem A, the part with factor \( -i\Phi_j(r, \lambda) \) is called *incoming* and that with factor \( i\Phi_j(r, \lambda) \) is called *outgoing*.

Theorem A thus has the following interpretation. Omitting the time factor \( e^{-i\Phi j(r, \lambda)} \), we send a wave \( \psi^{(in)}_j = e^{-i\Phi_j(r, \lambda)} a_j^{(in)} \) in the remote past at infinity of \( M_j \), and

\[2\] In this case, it means a surface on which the phase is constant.
observe the wave \( v^{(\text{out})} = e^{i\Phi^{(\text{out})}} a^{(\text{out})} \) coming back to infinity in the remote future. Then, the mapping, which is an analogue of Heisenberg’s \( S\)-matrix in physics,

\[
S(\lambda) : a^{(\text{in})} \rightarrow a^{(\text{out})}
\]

is unitary. It is generally believed, and has been proved in various cases, that the \( S\)-matrix determines the whole physical system. Our aim is to prove this belief for the case of non-compact manifolds with (at most) exponentially growing or decaying ends.

### 0.2. Inverse scattering from regular end

For \( j = 1, \cdots, N \), i.e. when \( \mathcal{M}_j \) is growing to infinity, we call \( \mathcal{M}_j \) regular end. For \( j = N + 1, \cdots, N + N' \), i.e. when \( \mathcal{M}_j \) is shrinking to a point, we call \( \mathcal{M}_j \) cusp. Letting

\[
S_{jk}(\lambda) : a_j^{(\text{in})} \rightarrow a_k^{(\text{out})},
\]

where \( a^{(\text{in})} = (a_1^{(\text{in})}, \cdots, a_N^{(\text{in})}) \), \( a^{(\text{out})} = (a_1^{(\text{out})}, \cdots, a_{N+N'}^{(\text{out})}) \), the \( S\)-matrix \( S(\lambda) \) is an \((N + N') \times (N + N')\)-matrix with operator entries \( S_{jk}(\lambda) \). Our first main theorem asserts that the manifold \( \mathcal{M} \) is determined by one entry \( S_{jj}(\lambda) \) of \( S(\lambda) \) associated with a regular end \( \mathcal{M}_j \) for all energies \( \lambda \in \sigma_e(H) \setminus \mathcal{E} \).

Assume that we are given two manifolds \( \mathcal{M}^{(1)}, \mathcal{M}^{(2)} \) of the form \( \mathcal{M}_j \). Let \( S^{(1)}(\lambda) = (S_{jk}^{(1)}(\lambda)) \) be the \( S\)-matrix for \( \mathcal{M}^{(1)} \) with size \((N^{(1)} + N^{(1)}) \times (N^{(1)} + N^{(1)})\).

Note that the number of ends of \( \mathcal{M}^{(1)} \) is not assumed to be equal for \( i = 1, 2 \) a-priori.

**Theorem B.** Assume that \( \rho_j(r) \geq O(r^{2\beta_j}) \) with \( \beta_j > 1/2 \) (\( 1 \leq j \leq N \)) on all regular ends and \( \rho_j(r) \leq O(r^{-2\beta_j}) \) with \( \beta_j > 0 \) (\( N + 1 \leq j \leq N + N' \)) on all cusps (Note that we are omitting the superscript \( (i) \)). Assume that a regular end \( \mathcal{M}_1^{(1)} \) and a regular end \( \mathcal{M}_1^{(2)} \) are isometric, and the associated \((1,1)\) components \( S_{11}^{(1)}(\lambda) \) and \( S_{11}^{(2)}(\lambda) \) coincide for all \( \lambda \in (\sigma_e(H^{(1)}) \setminus \mathcal{E}^{(1)}) \cap (\sigma_e(H^{(2)}) \setminus \mathcal{E}^{(2)}) \). Then, \( \mathcal{M}^{(1)} \) and \( \mathcal{M}^{(2)} \) are isometric.

### 0.3. Inverse scattering from slowly increasing end

The spectral analysis becomes harder when the growth rate of the volume of the end becomes slower. Assume that on an regular end \( \mathcal{M}_j \), \( \rho_j(r) \geq O(r^{2\beta_j}) \) with \( 0 < \beta_j \leq 1/2 \). We fix such an end \( \mathcal{M}_j \). Assume that \( \mathcal{M}_j \) is diffeomorphic to \((0, \infty) \times M_j \), let \( -\Delta_{M_j} \) be the Laplac-Beltrami operator of \( M_j \), \( \lambda_{0,j} < \lambda_{1,j} < \cdots \rightarrow \infty \) the eigenvalues of \(-\Delta_{M_j} \) and \( \lambda_{\ell,j} \) the eigenprojection associated with \( \lambda_{\ell,j} \). Take a large constant \( C \) which depends only on \( \rho(r) \) and put for \( \lambda, E > 0 \)

\[
r_0(\lambda, E) = \left( \frac{2C(1 + E)}{\lambda} \right)^{1/\epsilon}, \tag{0.4}
\]

\( \epsilon > 0 \) being a small constant. Take \( \chi(r) \in C^\infty(\mathbb{R}) \) such that \( \chi(r) = 0 \) for \( r < 1 \), \( \chi(r) = 1 \) for \( r > 2 \), and put

\[
\chi_{\ell,j}(\lambda, r) = \chi \left( \frac{r}{r_0(\lambda, \lambda_{\ell,j})} \right). \tag{0.6}
\]

---

\( ^3 \)It can be seen, for example, in the asymptotic expansion of solutions to the Helmholtz equation in \( \text{(3.4)} \) and \( \text{(3.6)} \). The latter is more complicated than the former.
We also put
\[ \varphi_j(\lambda, E, r) = \int_{r_0(\lambda, E)}^r \alpha_j(\lambda, E, s) ds, \]
\[ \alpha_j(\lambda, E, r) = \sqrt{\lambda - \left( \frac{(n-1)\rho_j'}{2\rho_j} \right)^2 - \frac{E}{\rho_j^2}}. \]

Theorem C. If \( 0 < \beta_j \leq 1/3 \), we assume that \( \mathcal{M}_j \), the metric is of the form
\[ ds^2 = (dr)^2 + \rho_j(r) h_j(x, dx). \]

For the other regular ends, assume \( \beta_j > 1/3 \), and for the cusp ends, assume \( \beta_j > 0 \).

Let \( \lambda \in (E, \infty) \setminus \mathcal{E} \). Then, we have the same conclusion as in Theorem A except that on the end where \( 0 < \beta_j \leq 1/2 \), we have the asymptotic expansion
\[ u \simeq \left( \frac{\pi}{\sqrt{\lambda - E_j}} \right)^{1/2} \rho_j(r)^{-(n-1)/2} \]
\[ \times \sum_{\ell=0}^{\infty} \chi_{\ell,j}(\lambda, r) \left( e^{-i\varphi_j(\lambda, \lambda_{\ell,j}, r) a_{\ell,j}^{(in)}} - e^{i\varphi_j(\lambda, \lambda_{\ell,j}, r) a_{\ell,j}^{(out)}} \right) e_{\ell,j}, \]
where \( a_{\ell,j}^{(in)} = \sum_{\ell=0}^{\infty} a_{\ell,j}^{(in)} e_{\ell,j}(x), a_{\ell,j}^{(out)} = \sum_{\ell=0}^{\infty} a_{\ell,j}^{(out)} e_{\ell,j}(x), \) and \( e_{\ell,j}(x) \) is a normalized eigenvector of \(-\Delta_{\mathcal{M}_j}\) associated with the eigenvalue \( \lambda_{\ell,j} \).

Theorem D. Under the same assumption as in Theorem C, assume that a regular end \( \mathcal{M}_1^{(1)} \) and a regular end \( \mathcal{M}_1^{(2)} \) are isometric, and the associated \((1,1)\) components \( S_{11}^{(1)}(\lambda) \) and \( S_{11}^{(2)}(\lambda) \) coincide for all \( \lambda \in (\sigma_e(H^{(1)}) \setminus \mathcal{E}^{(1)}) \cap (\sigma_e(H^{(2)}) \setminus \mathcal{E}^{(2)}) \). Then, \( \mathcal{M}^{(1)} \) and \( \mathcal{M}^{(2)} \) are isometric.

Comparing (0.1) with (0.6), we see that the asymptotic behavior of solutions to the Helmholtz equation changes at the threshold \( \beta_j = 1/2 \).

0.4. Inverse scattering from cylindrical end. An end \( \mathcal{M}_j \) is said to be asymptotically cylindrical if the metric has the behavior
\[ ds^2 = (dr)^2 + h_j(r, x, dx), \]
\[ h_j(r, x, dx) = h_j(x, dx) + O(r^{-\gamma_j}), \quad \gamma_j > 0. \]
In this case, the expansion (0.6) is modified as follows:
\[ u \simeq \left( \frac{\pi}{\sqrt{\lambda}} \right)^{1/2} \rho_j(r)^{-(n-1)/2} \]
\[ \times \sum_{\lambda_{\ell,j} < \lambda} \left( e^{-iy\sqrt{\lambda - \lambda_{\ell,j}}} a_{\ell,j}^{(in)} - e^{iy\sqrt{\lambda - \lambda_{\ell,j}}} a_{\ell,j}^{(out)} \right) e_{\ell,j}. \]

Namely, for a finite energy \( \lambda \), we have only a finite number of scattering waves (channels), and the S-matrix becomes a matrix of finite size. Theorem D holds also for this case, i.e. the manifold \( \mathcal{M} \) is determined from the S-matrix associated with the cylindrical end (52).
0.5. Inverse scattering from cusp. It is known that the information of the S-matrix for the cusp does not determine the manifold ([108]). To get more information, we generalize the notion of S-matrix by enlarging the solution space of the Helmholtz equation \((-\Delta_M - \lambda)u = 0\) on the cusp. For this purpose, we assume that the end in question is a pure cusp. Namely, we assume that the metric is of the form
\[ ds^2 = (dr)^2 + \rho_N + N'(r)h_N + N'(x, dx) \]
on \(M_{N+N'}\). We put \(N + N' = \kappa\) for the sake of simplicity. For the other cusp ends, we assume as before.

Let \(0 = \lambda_{0,\kappa} \leq \lambda_{1,\kappa} \leq \lambda_{2,\kappa} \leq \cdots\) be the eigenvalues of \(-\Delta_M\) with complete orthonormal system of eigenvectors \(e_{\ell,\kappa}(x), \ell = 0, 1, 2, \ldots\) We put
\[
\Phi_\kappa(r, B) = \int_{r_0}^{r} \sqrt{\frac{B}{\rho_\kappa}} - \lambda + \frac{((n^2 - 2n)j^2 + (n - 2)}{2}(\frac{\rho'_\kappa}{\rho_\kappa})' \ dr.
\]
Then, there exist solutions \(u_{\ell,\kappa,\pm}\) to the equation
\[-u'' - \frac{(n - 1)\rho'_\kappa}{\rho_\kappa}u' + \left(\frac{\lambda_{\ell,\kappa}}{\rho_\kappa} - \lambda\right)u = 0,
\]
which behave like
\[u_{\ell,\kappa,\pm} \sim \rho_\kappa(r)^{-n/2}e^{\pm \Phi_\kappa(r,\lambda_{\ell,\kappa})}, \quad r \to \infty.
\]
Take any solution \(u\) of the equation
\[(-\Delta_M - \lambda)u = 0, \quad \text{on} \quad M_\kappa.
\]
Expanding it by \(e_{\ell,\kappa}\), we have
\[(u(r, \cdot), e_{\ell,\kappa})_{L^2(M_j)} = a_{\ell,\kappa}u_{\ell,\kappa,\pm}(r) + b_{\ell,\kappa}u_{\ell,\kappa,\mp}(r).
\]
We introduce two spaces of sequences \(A_{\kappa,\pm}\):
\[
A_{\kappa,\pm} \ni \{c_{\ell,\pm}\}_{\ell=0}^{\infty} \iff \sum_{\ell=0}^{\infty} |c_{\ell,\pm}|^2 |u_{\ell,\kappa,\pm}(r)|^2 < \infty, \quad \forall r > 0.
\]
We take a partition of unity \(\{\chi_j\}\) on \(M\) such that \(\chi_j(r) = 1\) on \(M_j \cap \{r > 2\},\)
\(\chi_j(r) = 0\) on \(M_j \cap \{r < 1\}\) for \(1 \leq j \leq N + N'\). We define the generalized incoming solution on the cusp end \(M_\kappa\) by
\[
\Psi^{(in)}_\kappa = \chi_\kappa \sum_{\ell=0}^{\infty} a_{\ell,\kappa}u_{\ell,\kappa,\pm}(r) e_{\ell,\kappa}(x), \quad \{a_{\ell,\kappa}\}_{\ell=0}^{\infty} \in A_{\kappa,\pm},
\]
which is growing as \(r \to \infty\), and the generalized outgoing solution by
\[
\Psi^{(out)}_\kappa = \chi_\kappa \sum_{\ell=0}^{\infty} b_{\ell,\kappa}u_{\ell,\kappa,\mp}(r) e_{\ell,\kappa}(x), \quad \{b_{\ell,\kappa}\}_{\ell=0}^{\infty} \in A_{\kappa,\mp},
\]
which is decaying as \(r \to \infty\). We also define the spaces of generalized scattering data by
\[
h^{(in)}_\kappa(\lambda) = \bigoplus_{j=0}^{N+1} c_j(\lambda) L^2(M_j) \oplus \bigoplus_{j=N+1}^{N+N'-1} c_j(\lambda) C_j \oplus \{c_\kappa(\lambda) A_{\kappa,\pm}\},
\]
\[4\text{Here, f} \sim g \text{means that } f/g \to 1 \text{ as } r \to \infty.\]
\[ h_{\infty}^{(\text{out})}(\lambda) = \left( \sum_{j=1}^{N} c_j(\lambda)L^2(M_j) \right) \oplus \left( \sum_{j=N+1}^{N+N'} c_j(\lambda)C_j \right) \oplus \left( c_{\kappa}(\lambda)A_{\kappa,-} \right), \]

where \( c_j(\lambda) \) is the characteristic function of the interval \((E_j, \infty)\).

**Theorem E.** For any generalized incoming data \( a^{(\text{in})} \in h_{\infty}^{(\text{in})}(\lambda) \), there exist a unique solution \( u \) of the equation \(-\Delta_M - \lambda u = 0\), and the outgoing data \( a^{(\text{out})} \in h_{\infty}^{(\text{out})}(\lambda) \) such that

\[ u - \Psi^{(\text{in})}_\kappa \in B^*, \]

\[ u = \Psi^{(\text{in})}_\kappa - \Psi^{(\text{out})}_\kappa, \quad \text{on } M_{\kappa}, \]

and on the ends \( M_j \), \( 1 \leq j \leq N + N' - 1 \), \( u \) has the asymptotic form in Theorems A and C. Here, \( \Psi^{(\text{in})}_\kappa \) and \( \Psi^{(\text{out})}_\kappa \) are written by \([0, \ell], [0, \lambda] \) with \( a_{\ell, \kappa} \), \( b_{\ell, \kappa} \) replaced by the associated components of \( a^{(\text{in})}_\kappa \) and \( a^{(\text{out})}_\kappa \).

We call the mapping

\[ S(\lambda) : h_{\infty}^{(\text{in})}(\lambda) \ni a^{(\text{in})} \rightarrow a^{(\text{out})} \in h_{\infty}^{(\text{out})}(\lambda) \]

the generalized scattering matrix. Then, the inverse scattering theorem can be extended to the generalized S-matrix. Note that the \((\kappa, \kappa)\) component \( S_{\kappa, \kappa}(\lambda) \) is an infinite matrix whose \((0, 0)\) component is the usual S-matrix \( S_{N+N', N+N'}(\lambda) \), which is a complex number of modulus 1.

**Theorem F.** Under the same assumption as in Theorem E, assume that the cusp ends \( M^{(1)}_{\kappa} \) and \( M^{(2)}_{\kappa} \) are isometric, and the associated \((\kappa, \kappa)\) components of the generalized S-matrix coincide for all \( \lambda \in \left( \sigma_e(H^{(1)}) \setminus \mathcal{E}^{(1)} \right) \cap \left( \sigma_e(H^{(2)}) \setminus \mathcal{E}^{(2)} \right) \). Then, \( M^{(1)} \) and \( M^{(2)} \) are isometric.

As above, the number of ends of \( M^{(1)} \) and \( M^{(2)} \) are not assumed to be equal a-priori.

**0.6. Riemannian metric with continuous spectrum.** Properties of continuous spectrum of a manifold (i.e. that of the Laplacian) depend largely on its volume growth. Let \( M \) be a compact manifold of dimension \( n - 1 \). We consider the Riemannian metric on \( M = (0, \infty) \times M \) of the form

\[ ds^2 = (dr)^2 + g^{(M)}(r, x, dx), \]

where\[ g^{(M)}(r, x, dx) = g^{(M)}_{ij}(r, x)dx^i dx^j \]is a metric on \( M \) depending smoothly on \( r > 0 \). Identifying \( g^{(M)} \) with an \((n-1) \times (n-1)\) matrix \( (g^{(M)}_{ij}) \), we let

\[ g = g(r, x) = \det (g^{(M)}_{ij}). \]

For \( \kappa \in \mathbb{R} \), let \( S^\kappa \) be the set of \( C^\infty \) functions on \((0, \infty) \times M \) having the following property:

\[ (0.9) \quad S^\kappa \ni f \iff |\partial_x^\ell \partial_r^\alpha f(r, x)| \leq C_{\ell_\alpha}(1 + r)^{n - \kappa - \ell}, \quad \forall \ell, \alpha, \]

where \( C_{\ell_\alpha} \) is a constant. This definition is naturally extended for tensor fields\(^5\).

\(^5\)Using Einstein’s summation convention, \( a_{ij}b^{ij} = \sum_{i,j=1}^{n} a_{ij}b^{ij} \).

\(^6\)We define the \( S^\kappa \) norm for manifolds with conic singularities in Definition 2.9 which is actually used throughout this article.
Now let us consider \( g = g(r,x) \) such that

\[
g' = \frac{g'}{4g} - \frac{(n-1)c_0}{2} - c_1r^{-\alpha} \in S^{-1-\epsilon},
\]

where \( r = \partial_r \), and \( c_0, c_1, \alpha, \epsilon \) are real constants such that \( \alpha > 0, \epsilon > 0 \). This constant \( c_0 \) is important, since it determines the infimum of the continuous spectrum of the Laplacian. Integrating the equation

\[
g' = \frac{g'}{4g} - \frac{(n-1)c_0}{2} - c_1r^{-\alpha} = O(r^{-1-\epsilon}),
\]

we obtain \( g = \rho^{2(n-1)}O(1) \), where

\[
(0.10) \quad \rho(r) = \begin{cases} 
\exp(c_0r + c_1r^{1-\alpha}), & 0 < \alpha < 1, \\
\exp(c_0r)r^\beta, & \alpha = 1, \\
\exp(c_0r)(1 + O(r^{-\delta})), & \alpha > 1, \quad \delta = \min\{\alpha - 1, \epsilon\},
\end{cases}
\]

with \( c'_1 = \beta/(1-\alpha), \beta = 2c_1/(n-1) \). We put \( h(r, x, dx) = \rho(r)^{-2}g(r, x, dx) \), which is asymptotically equal to a metric independent of \( r > 0 \), i.e. \( h(r, x, dx) \rightarrow h^{(M)}(x, dx) \) as \( r \rightarrow \infty \). Thus, our typical example of the metric is written in terms of \( \rho(r) \) in \( (0.11) \) as

\[
ds^2 = (dr)^2 + \rho(r)^2h(r, x, dx).
\]

Let \( S(r) = \{r\} \times M \) be the section of \( M \) at \( r \). Then the volume of \( S(r) \) is growing as \( r \rightarrow \infty \) if either \( c_0 > 0 \) or \( c_0 = 0, \beta > 0 \), and is shrinking to 0 if either \( c_0 < 0 \), or \( c_0 = 0, \beta < 0 \). Let us call the metric of the form

\[
(0.12) \quad ds^2 = (dr)^2 + \rho(r)^2h(r, x, dx),
\]

a perturbed warped product metric, if it has the property \( h(r, x, dx) \rightarrow h^{(M)}(x, dx) \) as \( r \rightarrow \infty \). In Chapter 1, §3 we show that the metric with cross term

\[
(0.13) \quad ds^2 = a(t, z)(dt)^2 + 2w(t)b_i(t, z)dz^i + w(t)^2c_{ij}(t, z)dz^i dz^j,
\]

\[
w(t)^{-1} \in S^{-\kappa}, \quad a(t, z) - 1 \in S^{-\lambda}, \quad b_i(t, z) \in S^{-\mu}, \quad c_{ij}(t, z) - h_{ij}(z) \in S^{-\nu},
\]

is transformed into the form \( (0.12) \), where \( z \) denote local coordinates on \( M \).

### 0.7. Examples of manifolds.

Simple examples can be constructed by taking \( M \) to be the surface of revolution with the metric induced from the Euclidean metric in \( \mathbb{R}^{n+1} : x_{n+1} = (x_1^2 + \cdots + x_n^2)^{1/(2\beta)} \). Neglecting the singularity at \( x = 0 \), the case \( \beta = 1 \) corresponds to the conical surface, and the case \( \beta = 1/2 \) to the parabola.

In \( (0.11) \), we have restricted the growth order of \( |\log g| \) at most linearly. This is a natural restriction, since outside this range, the Laplacian may not have continuous spectrum. To see it, let us consider the warped product metric

\[
ds^2 = (dr)^2 + \rho(r)^2h_M(x, dx).
\]

The associated Laplacian is unitarily equivalent to (see §4.12)

\[
-\partial_r^2 - \rho^{-2}\Lambda + \left(\frac{\rho'}{4\rho}\right)^2 + \left(\frac{\rho'}{4\rho}\right)',
\]
where $\Lambda$ is the Laplace-Beltrami operator for $h_M(x, dx)$. Letting $\{\lambda_\ell\}_{\ell=0}^\infty$ be the eigenvalues for $-\Lambda$, it is unitarily equivalent to
\[\oplus_{\ell=0}^\infty \left( -\partial_r^2 + \frac{\lambda_\ell}{\rho^2} + \left(\frac{\rho'}{4\rho}\right)^2 + \left(\frac{\rho'}{4\rho}\right)' \right).\]
If $\rho'/\rho = \pm r^\epsilon$, it has only the discrete spectrum.

The metric of the warped product form includes many important examples. It is the hyperbolic metric when $\rho(r) = e^{\pm r}$, and the Euclidean metric when $\rho(r) = r^2$. The manifold with cylindrical end is the case $\rho(r) = \text{constant}$. Our assumption means that $\rho(r)$ is in between $e^{cr}$ and $e^{-cr}$ with $c > 0$. Therefore, as long as we start from the asymptotic expansion (0.10), the class of the metric we employ in this paper seems to be optimal for the study of forward and inverse scattering on Riemannian manifolds.

0.8. Conic singularities. Our another aim is to introduce a class of manifolds allowing cone-like singularities. A simple example of manifolds with cone-like singularities is the sector
\[S_\alpha = \{z \in \mathbb{C}; 0 \leq \arg(z) \leq \alpha\}, \quad 0 < \alpha < 2\pi\]
with two boundaries $\{z \in S_\alpha; \arg(z) = 0\} \cup \{z \in S_\alpha; \arg(z) = \alpha\}$ identified (see Figure 2 and Example 1.9). One can induce the differential structure of $\mathbb{R}^2$ to
\[S_\alpha \setminus \{0\}\]
to make it a $C^\infty$-manifold. One can also induce the Euclidean metric to $S_\alpha \setminus \{0\}$, which is not smooth at $z = 0$. If $2\pi/\alpha \in \mathbb{N}$, by the group action of rotation of angle $\alpha$, $S_\alpha$ becomes a $C^\infty$-manifold including 0, although the metric is singular at 0. This is a simple example of orbifold. Similarly, hyperbolic manifolds are orbifolds, and the singularity at the top of the cone does no harm for the spectral analysis (see e.g. [33]). This is no longer the case when $2\pi/\alpha \notin \mathbb{Q}$. However, we can develop a theory of conic manifold with group action (CMGA) in order to allow this sort of singularities in the spectral analysis and inverse problems on more general class of manifolds.

Manifolds with singularities have been objects of long issue. For example, the regularity of solutions to the Dirichlet problem for Laplacians around corners, the behavior of solutions to the wave equation near cracks or thin sets are significant problems of classical physics. In differential geometry and in global analysis, spaces having conic or more general singularities have been extensively studied, see e.g.
In particular, scattering problems have been studied from microlocal point of view e.g. in [3; 18; 19; 63; 82; 86; 85; 87; 38]. In this paper we use more classical functional analytical techniques and our focus is on the inverse scattering problems.

In inverse spectral problems for elliptic equations and in equivalent problems for heat and wave equations [65], singular spaces have appeared in the study of stability, see [2; 13; 35; 80]. Indeed, for the stability problem one has to consider a class of spaces that is compact in suitable topology, for instance in the Gromov-Hausdorff or Lipschitz sense. In [2], a stability result for the inverse boundary spectral problem is studied in the class of manifolds for which the curvature of the manifold, the diameter and, in addition, the injectivity radius are bounded. The Gromov-Hausdorff closure of these manifolds contain manifolds which metric tensor is not smooth. When the injectivity radius is not bounded from below, the theory changes radically as geometric collapses can appear. The study of collapsing manifolds has been an important trend in modern differential geometry. The celebrated papers of by Perelman [91] and Cheeger-Fukaya-Gromov [17] are examples of the progress of collapse and the metric geometry of the non-smooth spaces.

When the manifolds converge to a lower dimensional object we say that the geometry collapses. The limit space can be very non-smooth and not even a manifold, but generally an Alexandrov space [15]. In the case that manifolds of dimension $n$ converge to a space whose Hausdorff dimension is $n - 1$, the limiting objects are orbifolds. The inverse interior spectral problems for compact orbifolds is studied in [72]. A related inverse scattering problem for non-compact two-dimensional orbifolds is studied in [56]. In this paper we extend this research for higher dimensional manifolds with more general singularities and study manifolds, which are non-compact and may have certain type of conic singularities. In particular, the class of conic singularities studied in this paper contain all orbifold type singularities. This enables us to extend the boundary control method\(^7\) on manifolds with conic singularities, since it is based on the combination of spectral theory for Laplacian, wave equation and geodesics on it. We can then determine the manifold from the knowledge of S-matrix via the boundary control method. The details of the metric with conic singularities will be given in Definition 1.6 in §1.

0.9. Main subjects. We assume that $M$ is a regular conic manifold in the sense to be defined in Chapter 1, §1. We study the spectral and scattering theory for the Laplace operator $\Delta_M$ on $M$, in particular,

- Limiting absorption principle for the resolvent $R(z) = (\Delta_M - z)^{-1}$,
- Spectral representation of $\Delta_M$,
- Helmholtz equation for $\Delta_M$ and the S-matrix,
- Inverse scattering.

The first 3 subjects constitute the main parts of the forward problem. Once we have solved them, the inverse scattering problem can be solved in the same way as in [52; 55].

\(^7\)As will be explained in Subsection 0.10 this is the main tool for the inverse scattering procedure.
Our main results are Theorem 6.1 in Chapter 2, §6, which asserts that the whole manifold is determined from the knowledge of the (generalized) S-matrix for all energies. To achieve this final goal, we prepare the following theorems, which are of independent importance from the viewpoint of spectral theory:

(1) Chapter 1: Forward problem
   - Theorem 4.2 (Rellich-Vekua theorem)
   - Theorem 8.2 (Limiting absorption principle for the resolvent)
   - Theorem 11.4 (Resolvent asymptotics and spectral representation)
   - Theorem 12.8 (Characterization of the solution space of the Helmholtz equation)
   - Theorem 12.9 (Asymptotic expansion of solutions to the Helmholtz equation and the S-matrix)

(2) Chapter 2: Inverse problem
   - Theorem 1.1 (The S-matrix and the generalized S-matrix determine the source-to-solution map)
   - Theorem 1.2 (The generalized S-matrix determine the N-D map)

0.10. Related works. Analysis of Laplacians on non-compact manifolds and inverse spectral problems on compact or non-compact manifolds are our main concern. Since both of them are rather big subjects, we should better restrict ourselves to only recent developments. As for the latter subject, one should take notice of the evolution of multi-dimensional inverse problems of PDE which started in 1980s from the works of Sylvester-Uhlmann [101], Nachman [88], Khenkin-Novikov [69]. There is a close connection between the inverse boundary value problem and the inverse scattering problem. For the survey of the recent development of inverse boundary value problems, see [104], [105], [99] and the references therein. The other important stream is the boundary control method due to Belishev [4] and its development to Riemannian manifold by Belishev-Kurylev [6]. See [5] for a survey of this method. See also an expository work [64], and [51] for the extension of BC method to inverse scattering problems. As for the spectral analysis on non-compact manifolds, a general framework based on microlocal calculus is provided by Melrose in [84], [83]. Hyperbolic manifold is an interesting topic from analysis, geometry as well as number theory, and has been discussed in many articles ([34], [8], [9], [10], [92], [93], [39], [40], [44], [45]). In particular, inverse scattering on hyperbolic manifolds by the boundary control method was studied by [61], [62], [96], [41], [42]. For the case including more slowly (polynomially) growing manifolds, see [43], [70], [71], and recent works [57], [58], [59]. These works are based on the functional analytic technique utilized in spectral theory.

We have been working on important typical cases of such metrics as cylindrical ends in [52], asymptotically hyperbolic ends in [55], [53], [56], the asymptotically Euclidean case in [51]. In [56], a relation between the generalized S-matrix and arithmetic surfaces was discussed. As was stated in [54], we recognized that these typical metrics are embedded in a series of metrics, which can be understood in
Not only the behavior of the metric at infinity, but also the singularity of the manifold has been our concern.

In this paper, we employ the method of integration by parts, which originates from the work of Eidus \cite{32}. Since this technique is elementary, hence basic, it is flexible in applying to new problems. Moreover, although it is classical, its basic ideas are absorbed and transformed into new machineries of this field. We do not pursue full generalities, although we recognize that many parts of our arguments can be performed under weaker assumptions. Our intention is to try to simplify the whole theory and to make the assumptions and statements as clear and understandable as possible.

0.11. More precise assumptions of the metric. Our assumptions on the metric change slightly from section to section according to the subjects. Therefore, let us summarize here the final assumptions on the metric. After introducing CMGA in Definition 1.6 and $S^\infty$ in Definition 2.9, we assume that each end is of the form $(0, \infty) \times M_i$, where $M_i$ is CMGA of dimension $n-1$. We further assume as follows.

(A-1) The ends $M_1, \ldots, M_N$ are regular, and the ends $M_{N+1}, \ldots, M_{N+N'}$ are cusp.

(A-2) For each end $M_j$, $j = 1, \ldots, N + N'$, there exist constants $c_{0,j} \in \mathbb{R}$, $\alpha_{0,j} > 0$, $\gamma_{0,j} > 1$ and a metric $h_{M_j}(x, dx)$ on $M_j$ such that

\begin{equation}
\frac{\rho_j'(r)}{\rho_j(r)} = c_{0,j} \in S^{-\alpha_{0,j}},
\end{equation}

\begin{equation}
h_j(r, x, dx) - h_{M_j}(x, dx) \in S^{-\gamma_{0,j}}.
\end{equation}

(A-3) On regular ends $M_j$, there exist constants $\beta_{0,j} > 0, r_{0,j} > 0$ such that

\begin{equation}
\frac{\rho_j'(r)}{\rho_j(r)} \geq \frac{\beta_{0,j}}{r} \quad \text{for} \quad r > r_{0,j}.
\end{equation}

Moreover, the constants $\alpha_{0,j}, \gamma_{0,j}$ in (0.14), (0.15) satisfy

\begin{align*}
\alpha_{0,j} &> 0, \text{ for the regular ends with } \beta_{0,j} > \frac{1}{2}, \ c_{0,j} \geq 0, \\
\alpha_{0,j} &> 1, \text{ for the regular ends with } 0 < \beta_{0,j} \leq \frac{1}{2}, \ c_{0,j} \neq 0, \\
\alpha_{0,j} &> \frac{1}{2}, \text{ for the regular ends with } 0 < \beta_{0,j} \leq \frac{1}{2}, \ c_{0,j} = 0.
\end{align*}

On cusp ends $M_j$, there exists a constant $\beta_{0,j} < 0$ such that

\begin{equation}
\rho_j(r) \leq C(1 + r)^{\beta_{0,j}}
\end{equation}

holds. Furthermore, for all cusp ends

\begin{equation}
\alpha_{0,j} > 1/2, \quad \text{if} \quad c_{0,j} = 0.
\end{equation}

\begin{equation}
\alpha_{0,j} > 1/2, \quad \text{if} \quad c_{0,j} = 0.
\end{equation}

\textsuperscript{8}The main part of this paper, in particular for the case of smooth metric, has already been completed around 2014.
The results for the forward problem are proven under the assumptions (A-1), (A-2), (A-3). The inverse scattering from the regular end with $\beta_j > 1/3$ is proven also under these assumptions.

To consider the inverse scattering from the regular end with $0 < \beta_j \leq 1/3$ and the cusp end, we impose the following assumption (A-4).

(A-4) The metric on $M_j$ is of the warped product form
\[
 ds^2 \big|_{M_j} = (dr)^2 + \rho_j(r)^2 h_{M_j}(x, dx),
\]
for either of the following cases (A-4-1) or (A-4-2):

(A-4-1) There exists $1 \leq j \leq N$ such that for the regular end $M_j$, $\beta_j$ satisfies
\[
 0 < \beta_j \leq 1/3,
\]
and $\rho_j(r)$ satisfies (A-2) and (A-3).

(A-4-2) There exists $N + 1 \leq j \leq N + N'$ such that for the cusp end $M_j$, $\rho_j(r)$ satisfies the condition (13.6).

In addition to the assumptions (A-1) $\sim$ (A-4), we also impose the assumptions (C-1) $\sim$ (C-4) (to appear in Chapter 1, Subsection 1.3, (D) (to appear in Chapter 1, Subsection 2.1) and (L) (to appear in Chapter 2, Section 6).

0.12. Thresholds. The potential term of the Schrödinger operator $-\Delta + V(x)$ in $\mathbb{R}^n$ with $V(x) = O(|x|^{-\alpha})$ as $|x| \to \infty$ is said to be short-range if $\alpha > 1$ and long-range if $0 < \alpha \leq 1$. The border line $\alpha = 1$ appears in the case of the Coulomb potential $V(x) = -C/|x|$, across which solutions to the Schrödinger equation have different asymptotic behavior at infinity. In our case, this change of asymptotic behavior occurs across $\alpha = 1$ when $c_0 \neq 0$, and $\alpha = 1/2$ when $c_0 = 0$.

We now explain the thresholds we will encounter in the spectral analysis of $M$.

(i) In the first place, we have avoided the case in which $\rho(r)$ is constant. This corresponds to the case of asymptotically cylindrical end, and we have studied it in [52]. Of course, we can include it in this paper, however, the statements of results are different in many places. Therefore, to simplify the description, it is better to deal with it separately.

(ii) For the regular ends, we have thresholds.

- Transformation of the metric (0.13) into the perturbed warped product (3): $\beta = 1$, and $\beta = 1/2$.

Note that the case $\beta = 1$ corresponds to the asymptotically Euclidean metric.

- Rellich-Vekua theorem (4): $\beta = 1/3$.

We have proved the Rellich-Vekua type theorem for the case $\beta > 1/3$, however it is open for $\beta \leq 1/3$.

- Resolvent asymptotics (9, 11):
  \[
  \alpha = 1 \ (c_0 \neq 0), \quad \alpha = 1/2 \ (c_0 = 0), \quad \beta = 1/2, \quad \gamma = 1.
  \]

\footnote{As a matter of fact, we still do not know whether it is a real border-line or not.}
The asymptotic behavior at infinity of the resolvent is strongly affected by all constants $c_0, \alpha_0, \beta_0, \gamma_0$. For $\beta > 1/2$, we allow the metric of long-range behavior, however, for $0 < \beta \leq 1/2$, we deal with only metrics of short-range behavior. Across the threshold $\alpha = 1$ for the case $c_0 \neq 0$, and $\alpha = 1/2$ for the case $c_0 = 0$, the asymptotic expansion at infinity of the resolvent changes its form. As for $\gamma$, we consider only the case $\gamma > 1$ in this article. For $\gamma \leq 1$, the behavior will be different from the one we give here.

0.13. Notations. For Banach spaces $X$ and $Y$, $\mathcal{B}(X; Y)$ is the space of all bounded linear operators from $X$ to $Y$, and $\mathcal{B}(X) = \mathcal{B}(X; X)$. For a Hilbert space $\mathfrak{h}$, $L^2(I; \mathfrak{h}; m(r)dr)$ is the $L^2$-space of $\mathfrak{h}$-valued functions on an interval $I \subset \mathbb{R}$ with respect to the measure $m(r)dr$, whose inner product is defined by

$$(f, g) = \int_I (f(r), g(r))_\mathfrak{h} m(r)dr.$$ 

$H^s(\mathcal{M})$ denotes the Sobolev space of order $s$ (with respect to $L^2$ derivatives) on a manifold $\mathcal{M}$. For a self-adjoint operator $A$, $\rho(A)$ denotes its resolvent set, and $\sigma(A)$, $\sigma_d(A)$, $\sigma_p(A)$, $\sigma_c(A)$, $\sigma_e(A)$ and $\sigma_{ac}(A)$ denote its spectrum, discrete spectrum, point spectrum (the set of eigenvalues), essential spectrum, continuous spectrum, and absolutely continuous spectrum, respectively (see e.g. [67, 95]).

For the reader's convenience, we give here a brief list of symbols used frequently in the text.

| $S^\kappa$ | (3.4), (2.39) |
|-----------------|----------------|
| $D(k)$ | (5.15) |
| $w = \sqrt{B(r)}u$ | (5.17) |
| $E_{0,\text{tot}}$ | (7.7) |
| $\mathcal{E}$ | (13.1) |
| $f(r) \sim g(r)$ | (13.7) |

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We (H. I., M. L. and Yaroslav Kurylev) were working on the theme of this paper for more than 15 years, and know that there are still lots of room for improvements. However, H. I. and M. L think that now it is the time to publish and dedicate this paper to Y. Kurylev, who passed away in 2019.
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CHAPTER 1

Spectral and scattering theory

We start with introducing manifolds with conical singularities (CMGA) and study the domain of its Laplacian. The manifold \( M \) is assumed to be CMGA and its ends \( M_i \) are of the form \((0, \infty) \times M_i\), where \( M_i \) is CMGA of dimension \( n-1 \). Chapter 1 is devoted to the forward problem of scattering, and the main aim is to prove the limiting absorption principle of the resolvent and derive the spectral representation. Finally, the S-matrix is introduced.

1. Group action and conic manifolds

Our definition of conic manifold is rather involved. We begin with the notion of orbifolds, and then introduce conic manifolds admitting group action. In the following, \( B^k(x_0, R) \) is an open ball in \( \mathbb{R}^k \) of radius \( R \) centered at \( x_0 \in \mathbb{R}^k \).

1.1. Conic chart.

Definition 1.1. Given a topological space \( U \), we call a triple \((\tilde{U}, \Gamma, \pi)\) a conic chart of \( U \) if it has the following properties:

- \( \tilde{U} \) is an open set in \( \mathbb{R}^n \) having the form \( \tilde{U} = \tilde{W} \times \tilde{V} \), where \( \tilde{W} = B^k(0, R_0) \), \( \tilde{V} = B^{n-k}(0, R_1) \) for some \( R_0, R_1 > 0 \) and \( 0 \leq k \leq n \) satisfying \( k \neq n-1 \).
- \( \Gamma \) is a finite group acting on \( \tilde{U} \), leaving \( \tilde{W} \) invariant.
- \( \pi : \tilde{U} \to U \) is defined to be \( \pi = \Phi \circ \tilde{\pi} \), where \( \tilde{\pi} : \tilde{U} \to \Gamma \setminus \tilde{U} \) is the canonical projection to the set of orbits, and \( \Phi : \Gamma \setminus \tilde{U} \to U \) is a homeomorphism.

\[
\tilde{U} = \tilde{W} \times \tilde{V}
\]

\[
\begin{array}{ccc}
\tilde{U} & \xrightarrow{\pi} & \Gamma \setminus \tilde{U} \\
\Phi & \downarrow & \\
U & \xrightarrow{\tilde{\pi}} & \\
\end{array}
\]

Figure 1. Conical chart

Define \( Y : \tilde{U} \to \tilde{W} \) and \( Z : \tilde{U} \to \tilde{V} \) by

\[
\tilde{U} \ni x \to (y, z) = (Y(x), Z(x)).
\]
Then, by definition, $\Gamma$ leaves $\tilde{W}$ invariant if

$$Y(\gamma \cdot x) = Y(x), \quad \forall x \in \tilde{U}, \quad \forall \gamma \in \Gamma.$$ 

The isotropy group for $x \in \tilde{U}$ is defined by

$$\Gamma(x) = \{ \gamma \in \Gamma ; \gamma \cdot x = x \}.$$ 

For $k = 0$, we put

$$\tilde{U}_{\text{reg}} = \{ x \in \tilde{U} ; \Gamma(x) = \{ e \} \},$$

where $\{ e \}$ means the subgroup of $\Gamma$ consisting only of the unit. For $0 < k < n$, we put

$$\tilde{U}_{\text{reg}} = \left( \tilde{W} \times (B^{n-k}(0, R_1) \setminus \{ 0 \}) \right) \cap \{ x \in \tilde{U} ; \Gamma(x) = \{ e \} \}.$$ 

Finally, we put

$$\tilde{U}_{\text{sing}} = \tilde{U} \setminus \tilde{U}_{\text{reg}},$$

$$U_{\text{reg}} = \pi(\tilde{U}_{\text{reg}}).$$

Note that $\tilde{U} = \tilde{V}$ for $k = 0$, in which case $\tilde{U}_{\text{sing}}$ is either an empty set or $\{ 0 \}$ in $\tilde{U}$, and $\tilde{U} = \tilde{W}$ for $k = n$, in which case $\tilde{U}_{\text{sing}}$ is an empty set.

Later, we introduce a Riemannian metric $\tilde{g}$ on $\tilde{U}$. Then a natural choice of $\Gamma$ is a group of isometric transformations with respect to $\tilde{g}$. Typical examples are a finite subgroup of $SO(2)$ when $\tilde{g}$ is the Euclidean metric (Example 1.9), and a finite subgroup of $SL(2, \mathbb{R})$ when $\tilde{g}$ is the hyperbolic metric (Example 1.3).

It is not necessary to define the differential structure of $U$, since the calculus on $U$ is done by lifting to $\tilde{U}$. Let $F_{\tilde{U}}$ and $F_U$ be the set of all functions on $\tilde{U}$ and $U$, and $S_{\tilde{U}}$ the set of $\Gamma$-invariant functions on $\tilde{U}$, i.e.

$$v \in S_{\tilde{U}} \iff v(\gamma \cdot x) = v(x), \quad \forall x \in \tilde{U}, \quad \forall \gamma \in \Gamma.$$ 

For $v \in F_U$, define

$$(1.1) \quad \pi^* v(x) = v(\pi(x)).$$

Then, since $\pi(\gamma \cdot x) = \pi(x)$ for all $\gamma \in \Gamma$ and $x \in \tilde{U}$ by definition, we have $\pi^* v \in S_{\tilde{U}}$, and the map

$$(1.2) \quad F_U \ni v \to \pi^* v \in S_{\tilde{U}}$$

is a bijection. The operator $P$ defined by

$$(1.3) \quad F_{\tilde{U}} \ni u(x) \to Pu(x) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} u(\gamma \cdot x) \in S_{\tilde{U}}$$

is a surjection. Moreover, $P$ is an orthogonal projection on $L^2(\tilde{U})$, i.e.

$$P^2 = P, \quad P^* = P.$$ 

The regularity of functions on $U$ is defined through $P$. Namely, $f$ on $U$ is in $C^m$ if and only if there exists $\tilde{f} \in C^m(\tilde{U})$ such that $P \tilde{f} = f$, which is equivalent to $\pi^* f \in C^m(\tilde{U})$. The same remark applies to Sobolev spaces, which we explain in Subsection 1.7.
EXAMPLE 1.2. (An n-dimensional topological manifold). For an n-dimensional topological manifold \( M \), any \( p \in M \) has a coordinate neighborhood \((U, \varphi)\), where \( U \) is an open neighborhood of \( p \) in \( M \) and \( \varphi : U \to \bar{U} \) is a homeomorphism with \( \bar{U} \) an open set in \( \mathbb{R}^n \). Then, \((\bar{U}, \{e\}, \varphi^{-1})\) is the conic chart, where \( e \) is the unit in \( SO(n) \).

EXAMPLE 1.3. (Modular surface). Let \( H = \mathbb{C}_+ = \{x + iy; x \in \mathbb{R}, y > 0\} \) be the upper-half plane. By taking a suitable discrete subgroup \( \Gamma \) of \( SL(2, \mathbb{R}) \), we can define an action on \( H \) by

\[
\Gamma \times H \ni (\gamma, z) \to \gamma \cdot z = \frac{az + b}{cz + d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix},
\]

and obtain a 2-dimensional orbifold \( \Gamma \backslash H \) (see Definition [1.3]). A well-known example is the modular surface, which corresponds to \( \Gamma = SL(2, \mathbb{Z}) \). In this case, \( \Gamma \) is generated by two elements \( \gamma^{(T)} \) and \( \gamma^{(I)} \), where

\[
\gamma^{(T)} \cdot z = z + 1, \quad \gamma^{(I)} \cdot z = -\frac{1}{z}.
\]

Let \( M = SL(2, \mathbb{Z}) \backslash H \). Then, its fundamental domain \( M^f \) is written as

\[
M^f = \{z \in \mathbb{C}_+ : |z| \geq 1, |\text{Re} z| \leq 1/2\}
\]

with boundary

\[
\partial M^f = \partial M^f_1 \cup \partial M^f_2,
\]

\[
\partial M^f_1 = L_- \cup L_+, \quad L_\pm = \left\{ \pm \frac{1}{2} + iy ; \frac{\sqrt{3}}{2} \leq y < \infty \right\},
\]

\[
\partial M^f_2 = \left\{ e^{i\varphi} ; \frac{\pi}{3} \leq \varphi \leq \frac{2\pi}{3} \right\}.
\]

To obtain \( M \) from \( M^f \), we glue \( \partial M^f_1 \) by the action of \( \gamma^{(T)} : z \to z + 1 \), and \( \partial M^f_2 \) by the action of \( \gamma^{(I)} : e^{i\varphi} \to e^{i(\pi - \varphi)} \). For \( w \in H \), let \( \Gamma(w) \) be the isotropy group:

\[
\Gamma(w) = \{\gamma \in SL(2, \mathbb{Z}) : \gamma \cdot w = w\}.
\]

For \( \gamma \in SL(2, \mathbb{Z}) \), let \( \langle \gamma \rangle \) be the cyclic group generated by \( \gamma \). Then, for \( w \in M^f \), we have the following 4 cases:

- \( \Gamma(i) = \left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\} \),
- \( \Gamma(e^{\pi i/3}) = \left\{ \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \right\} \),
- \( \Gamma(e^{2\pi i/3}) = \left\{ \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \right\} \),
- \( \Gamma(w) = \{ \pm I_2 \}, \quad w \neq i, e^{\pi i/3}, e^{2\pi i/3} \),

where \( I_2 \) is the 2 \times 2 identity matrix. In the first case, \( \Gamma(i) \) is order 2, in the 2nd and the 3rd cases, \( \Gamma(e^{\pi i/3}) \) and \( \Gamma(e^{2\pi i/3}) \) are order 3, and in the last case, the action \( \gamma \cdot z \) is the identity. If \( w \neq i, e^{\pi i/3}, e^{2\pi i/3} \), the manifold structure of \( M \) is easily defined by taking an open set \( U = U \subset \mathbb{C}_+ \) such that \( w \in U, i, e^{\pi i/3}, e^{2\pi i/3} \not\in U \) and \( \varphi(z) = z \).

An element \( \gamma \in SL(2, \mathbb{R}) \) is said to be elliptic if it has only one fixed point in \( \mathbb{C}_+ \), which is equivalent to \( |\text{tr} \gamma| < 1/2 \). Let \( p \) be the fixed point of an elliptic \( \gamma \).
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Then its isotropy group is cyclic. Let $n$ be the order of this cyclic group. Then, the generator $\gamma_0$ of this isotropy group satisfies

$$\frac{w - p}{w - \overline{p}} = e^{2\pi i/n} \frac{z - p}{z - \overline{p}}, \quad w = \gamma_0 \cdot z$$

(see Lemma 2.4 of [56]). We can then take the local coordinates near $p$ by

$$\zeta = \left(\frac{z - p}{z - \overline{p}}\right)^n, \quad \zeta(p) = 0.$$ 

Then, we have

$$z = \frac{p - \overline{p}\zeta^{1/n}}{1 - \zeta^n} = p + (p - \overline{p})\zeta^{1/n} + \cdots.$$ 

(See Subsection 2.3 of [56]). Letting $\zeta = \rho e^{i\theta}$, the hyperbolic metric $((dx)^2 + (dy)^2)/y^2$ is rewritten as

$$\frac{(dx)^2 + (dy)^2}{y^2} = (dr)^2 + \frac{1}{n^2} \sinh^2(r)(d\theta)^2.$$ 

(See Subsection 2.4 of [56]). We have thus seen that around the elliptic fixed points, we can introduce a $C^\infty$-differentiable structure on $M$ and a Riemannian metric except for fixed points. Note that:

- Although the covering space $\mathbb{C}^+$ has a global $C^\infty$-Riemannian metric $\frac{(dx)^2 + (dy)^2}{y^2}$, the induced metric on $M$ is singular at elliptic fixed points.

**Example 1.4.** (3-dimensional orbifold). An example of the 3-dimensional orbifold is given by the upper-half space model of the hyperbolic space $\mathbb{H}^3 = \mathbb{R}^3_+$.
Let us represent points in $H^3$ by quaternions: $(x_1, x_2, y) \leftrightarrow x_1 \mathbf{i} + x_2 \mathbf{j} + y \mathbf{k}$.

We identify $x_1 \mathbf{i} + x_2 \mathbf{j}$ with $z = x_1 + ix_2 \in \mathbb{C}$. Then,

$$x_1 \mathbf{i} + x_2 \mathbf{j} + y \mathbf{k} = \left( \begin{array}{c} z \\ y/z \end{array} \right) =: \zeta = z + yj.$$

The action of $SL(2, \mathbb{C})$ is defined by

$$SL(2, \mathbb{C}) \times H^3 \ni (\gamma, \zeta) \rightarrow \gamma \cdot \zeta := \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \left( \begin{array}{c} z \\ y \end{array} \right) =: \zeta = (a\zeta + b)(c\zeta + d)^{-1}, \quad \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right).$$

Since the mapping $\gamma \rightarrow \gamma \cdot \zeta$ is 2 to 1, we consider $PSL(2, \mathbb{C}) = SL(2, \mathbb{C})/\{\pm I\}$. A counterpart of the modular group is the Picard group

$$\Gamma = PSL(2, \mathbb{Z}[i]) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) : a, b, c, d \in \mathbb{Z}[i], \ ad - bc = 1 \right\},$$

where $\mathbb{Z}[i] = \mathbb{Z} + i\mathbb{Z}$, the ring of Gaussian integers. As is well-known (see e.g. \cite{33}, \cite{55}), the fundamental domain of $\Gamma$ is

$$\mathcal{M} = \Gamma \backslash H^3 = \left\{ z + yj : \Re z \leq \frac{1}{2}, \ 0 \leq \Im z \leq \frac{1}{2}, \ |z|^2 + y^2 \geq 1 \right\}.$$

The vertices of $\mathcal{M}$ are

$$\infty, \ -\frac{1}{2} + \frac{\sqrt{3}}{2}j, \ \frac{1}{2} + \frac{\sqrt{3}}{2}j, \ \frac{1}{2} + \frac{1}{2}j + \frac{\sqrt{3}}{2}j, \ -\frac{1}{2} + \frac{1}{2}j + \frac{\sqrt{3}}{2}j.$$
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Letting $\mathcal{M}_\pm = \{z + iy \in \mathcal{M}; 0 \leq \pm z \leq 1/2\}$, the boundary of the fundamental domain is split into 8 parts:

- $S_1 = \mathcal{M}_- \cap \{\text{Re} z = -1/2\}$,
- $S_2 = \mathcal{M}_+ \cap \{\text{Re} z = 1/2\}$,
- $S_3 = \mathcal{M}_- \cap \{\text{Im} z = 0\}$,
- $S_4 = \mathcal{M}_+ \cap \{\text{Im} z = 0\}$,
- $S_5 = \mathcal{M}_- \cap \{\text{Im} z = 1/2\}$,
- $S_6 = \mathcal{M}_+ \cap \{\text{Im} z = 1/2\}$,
- $S_7 = \mathcal{M}_- \cap \{|z|^2 + y^2 = 1\}$,
- $S_8 = \mathcal{M}_+ \cap \{|z|^2 + y^2 = 1\}$.

Letting $L_{ij} = S_i \cap S_j$, the sets of singular points are given by

- $L_{13} = L_{24} = \{z = -1/2, y \geq \sqrt{3}/2\}$, $L_{34} = \{z = 0, y \geq 1\}$
- $L_{15} = L_{26} = \{z = -1/2 + i/2, y \geq 1/\sqrt{2}\}$, $L_{56} = \{z = i/2, y \geq \sqrt{3}/2\}$
- $L_{17} = L_{28} = \{|z|^2 + y^2 = 1, \text{Re} z = -1/2, 0 \leq \text{Im} z \leq 1/2\}$
- $L_{78} = \{|z|^2 + y^2 = 1, \text{Re} z = 0, 0 \leq \text{Im} z \leq 1/2\}$
- $L_{37} = L_{48} = \{|z|^2 + y^2 = 1, -1/2 \leq \text{Re} z \leq 0, \text{Im} z = 0\}$
- $L_{57} = L_{68} = \{|z|^2 + y^2 = 1, -1/2 \leq \text{Re} z \leq 0, \text{Im} z = 1/2\}$

We now let

- $L_1 = L_{13}$, $L_2 = L_{15}$, $L_3 = L_{34}$, $L_4 = L_{56}$
- $L_5 = L_{17}$, $L_6 = L_{78}$, $L_7 = L_{37}$, $L_8 = L_{57}$

The isotropy group for $L_n$ are finite groups of rotations. In fact, $\Gamma(L_i) = \mathbb{Z}_2$, $i \neq 5, 8$, $\Gamma(L_i) = \mathbb{Z}_3$, $i = 5, 8$.

The point is:

- The singular points form 1-dimensional curves.
- Some of the curves of singular points go to infinity.
- Some of the curves of singular points intersect.

Since the isotropy groups for $L_i (1 \leq i \leq 4)$ are rotation groups around them, for any $t > 1$, the horizontal slice $M_t = \mathcal{M} \cap \{y = t\}$ is a compact 2-dimensional orbifold with singular points $-1/2 + tj, -1/2 + i/2 + tj, yj, i/2 + tj$. Here, the covering is a disc in $\mathbb{R}^2$. For the details, see e.g. [33] and [55].

1.2. Orbifolds. The notion of orbifold was first introduced by Satake [100]. We recall its definition for the sake of comparison.

**Definition 1.5.** An $n$-dimensional orbifold is a paracompact Hausdorff space $X$ endowed with the set of local charts $\{U_i; i \in I\}$ having the following properties:

- $X = \bigcup_{i \in I} U_i$.
- For each $U_i$, there exists a triple $(\bar{U}_i, \Gamma_i, \pi_i)$, where $\bar{U}_i \subset \mathbb{R}^n$ is an open set, $\Gamma_i$ is a finite group acting effectively (injectively) on $\bar{U}_i$, and $\pi_i : \bar{U}_i \to U_i$ is a continuous map inducing a homeomorphism $U_i \simeq \Gamma_i \backslash \bar{U}_i$.
- For any $U_i \subset U_j$, there is a set $I_{ij} = \{(\phi, h)\}$, whose elements are called injections, where $\phi : \bar{U}_i \to \bar{U}_j$ is a smooth imbedding and $h : \Gamma_i \to \Gamma_j$ is an injective homeomorphism, such that $\phi$ is $h$-invariant and $\pi_i = \pi_j \circ \phi$. Moreover, $\Gamma_i \times \Gamma_j$ acts transitively on $I_{ij}$ by
  $$ (g, g') \circ (\phi, h) = (g' \circ \phi \circ g^{-1}, \text{Ad}(g') \circ h \circ \text{Ad}(g^{-1})), $$
  $$ \forall g \in \Gamma_i, g' \in \Gamma_j, (\phi, h) \in I_{ij}. $$
- For each $p \in U_i \cap U_j$, there exists $U_k$ such that $p \in U_k \subset U_i \cap U_j$, and the injections are closed under composition for any $U_k \subset U_i \subset U_j$. 


Therefore, \( (\tilde{U}, \Gamma, \pi_i) \) is the conic chart in the sense of Definition 1.1.

The examples 1.3 and 1.4 are 2 and 3 dimensional orbifolds arising in number theory. For more detailed exposition of orbifolds, see e.g. [23] and [103].

Let us repeat important points. Let \( M \) be an orbifold with \( C^\infty \)-smooth coordinates. Then, every point \( x_0 \in M \) has a neighborhood \( U \) where \( U \) is either an open smooth manifold with smooth metric, or, \( U \) has the structure \( U \equiv \Gamma \setminus \tilde{U} \), where \( (\tilde{U}, \Gamma, \pi) \) is a conic chart of \( U \). This means that (1) \( \tilde{U} \subset \mathbb{R}^n \) is a local covering neighborhood having the product structure \( \tilde{U} = \tilde{W} \times \tilde{V} \), where \( \tilde{W} \subset B^k(0, R_1) \subset \mathbb{R}^k \) and \( \tilde{V} = B^{n-k}(0, R_0) \subset \mathbb{R}^{n-k} \), and (2) \( \Gamma \) is a finite group acting on \( \tilde{U} \). Therefore, in \( U \), there are local coordinates

\[
X : U \to \Gamma \setminus \mathbb{R}^n, \quad X(U) = \Gamma \setminus \tilde{U}.
\]

1.3. Conic manifolds with group action. We now define the conic manifold which admits group action, which is an extension of orbifolds.

Definition 1.6. A conic manifold admitting group action, which is abbreviated to CMGA, is a topological space \( M \) equipped with the following structure: There exists a family of open covering \( \{ U_j : j \in J \} \) of \( M \) having the following properties (C-1) \~ (C-4):

(C-1) For any \( j \in J \), \( U_j \) has a conic chart \( (\tilde{U}_j, \Gamma_j, \pi_j) \), \( \tilde{U}_j = \tilde{W}_j \times \tilde{V}_j \), where for some \( 0 \leq k \leq n, k \neq n-1 \),

\[
\tilde{W}_j = B^k(0, R_0) \subset \mathbb{R}^k, \quad \tilde{V}_j = B^{n-k}(0, R_1) \subset \mathbb{R}^{n-k}.
\]

(C-2) Define \( Y_j : \tilde{U}_j \to \tilde{W}_j, Z_j : \tilde{U}_j \to \tilde{V}_j \) and \( \tilde{U}^\text{reg} \) by

\[
\tilde{U}_j \ni x \to x = (y, z) = (Y_j(x), Z_j(x)),
\]

Then:

(C-2-1) The action of \( \gamma \in \Gamma_j \) keeps the \( y \)-coordinates invariant, i.e.

\[
Y_j(\gamma \cdot x) = Y_j(x), \quad \forall x \in \tilde{U}_j, \quad \forall \gamma \in \Gamma_j.
\]

(C-2-2) There exists a \( \Gamma_j \)-invariant \( C^\infty \)-metric \( \bar{g}_j \) on \( \tilde{U}^\text{reg} \), i.e.

\[
\gamma_* \bar{g}_j = \bar{g}_j \quad \text{on} \quad \tilde{U}^\text{reg}, \quad \forall \gamma \in \Gamma_j.
\]

(C-2-3) In the spherical coordinates \( Z_j(x) = s\omega = z, s = s(x) = |z|, \omega = \omega(x) = \frac{z}{|z|} \) such that \( s \in (0, R_1) \) and \( \omega \in S^{n-k-1} \) on \( B^{n-k}(0, R_0) \setminus \{0\} \), \( \bar{g}_j \) has the form

\[
\bar{g}_j = \sum_{p, q = 1}^k a_{pq}^{(j)}(y, s, \omega)dy^pd\omega^q
\]

\[
+ ds^2 + s^2 \sum_{\ell, m = 1}^{n-k} b_{\ell m}^{(j)}(y, s, \omega)d\omega^\ell d\omega^m + s \sum_{p = 1}^k \sum_{\ell = 1}^{n-k} h_{p\ell}^{(j)}(y, s, \omega)dy^p d\omega^\ell.
\]
The coefficients satisfy
\[
\begin{cases}
\tilde{a}_{pq}^{(j)}(y, s, \omega) \to \tilde{a}_{pq}^{(j)}(y), \\
\tilde{b}_{\ell m}^{(j)}(y, s, \omega) \to \tilde{b}_{\ell m}^{(j)}(y, \omega), \\
\tilde{h}_{\ell m}^{(j)}(y, s, \omega) \to 0,
\end{cases}
\] (1.7)
uniformly in \((y, \omega)\) as \(s \to 0\). Moreover, there exist constants \(C_1 \geq C_0 > 0\) and a positive continuous function \(T_j(y)\) such that
\[
C_0 \ g_{S^{n-k-1}} \leq \sum_{\alpha, \beta = 1}^{n-k} \tilde{a}_{\ell m}^{(j)}(y, \omega) d\omega^\ell d\omega^m \leq T_j(y)^2 g_{S^{n-k-1}},
\] (1.8)
where \(g_{S^{n-k-1}}\) is the standard metric of \(S^{n-k-1}\) and \(C_0 \leq T_j(y) \leq C_1\).

(C-2) For \(e \neq \gamma \in \Gamma_j\), \(e\) being the unit of \(\Gamma_j\),
\[
\text{cap}_2(\{x \in \bar{U} : \gamma \cdot x = x\}) = 0,
\] (1.9)
where \(\text{cap}_2(E)\) denotes the 2-capacity of a subset \(E \subset \mathbb{R}^n\).

(C-3) If \(U_j \cap U_k \neq 0\), there exists \(\ell \in J\) such that \(U_\ell \subset U_j \cap U_k\).

(C-4) If \(U_\ell \subset U_k\), there exist an injective homomorphism \(I_{\ell k} : \Gamma_\ell \to \Gamma_k\), and a \(C^\infty\) injective map \(\tilde{I}_{\ell k} : \bar{U}_\ell \to \bar{U}_k\).

Recall that the 2-capacity of a subset \(E \subset \mathbb{R}^n\) is defined by
\[
\text{cap}_2(E) = \inf \int_{\mathbb{R}^n} (|u|^2 + |
abla u|^2) \ dx,
\]
where the infimum is taken over all \(u \in H^2(\mathbb{R}^n)\) such that \(u \geq 1\) almost everywhere on a neighborhood of \(E\).

To simplify the analysis below, we assume that the constants \(R_0, R_1 > 0\) in (1.4) are independent of \(j\) and so are \(C_0, C_1\) in (1.8).

We have another expression of (1.4), which does not contain local coordinates on \(S^{n-k-1}\). Omitting the subscript \(j\),
\[
\tilde{g} = \sum_{p,q=1}^{k} \tilde{a}_{pq}(y, z) dy^p dy^q + \sum_{\ell,m=1}^{n-k} \tilde{b}_{\ell m}(y, z) dz^\ell dz^m + \sum_{p=1}^{k} \sum_{m=1}^{n-k} \tilde{h}_{p\ell}(y, z) dy^p dz^\ell,
\] (1.10)
where \(z = (z^p)_{p=1}^{n-k} \in \mathbb{R}^{n-k}, |z| = (\sum_{p=1}^{n-k} |z^p|^2)^{1/2}\), and
\[
\begin{aligned}
\tilde{a}_{pq}(y, z) &= a_{pq}(y, |z|, \frac{z}{|z|}), \\
\tilde{b}_{\ell m}(y, z) &= \frac{z^\ell z^m}{|z|^2} + \sum_{u,v=1}^{n-k} (\delta_{u \ell} - \frac{z^u z^\ell}{|z|^2}) b_{uv}(y, |z|, \frac{z}{|z|}) (\delta_{vm} - \frac{z^v z^m}{|z|^2}), \\
\tilde{h}_{p\ell}(y, z) &= \sum_{u=1}^{n-k} h_{pu}(y, |z|, \frac{z}{|z|}) (\delta_{u \ell} - \frac{z^u z^\ell}{|z|^2}),
\end{aligned}
\]
By the conditions (1.7), (1.8), there exist constants \(R_0, C > 0\) such that
\[
C|\xi|^2 \leq \tilde{g}_\pm(\xi, \xi) \leq C^{-1}|\xi|^2, \ |z| \leq R_0.
\] (1.11)
for any \( \xi \), where \( \tilde{g}_s(\xi, \xi) \) is the metric \( \tilde{g} \) in \((1.10)\), and we identify \( \xi \in \mathbb{R}^n \) with the vector field \( \sum_{i=1}^{k} \xi_i \frac{\partial}{\partial y^i} + \sum_{i=k+1}^{n} \xi_i \frac{\partial}{\partial z^i} \).

Let us check the validity of \((1.9)\) for the case of linear action by \( \text{SO}(n) \).

**Lemma 1.7.** Let a finite group \( \Gamma \subset \text{SO}(n) \) act linearly on \( \mathbb{R}^n \), and for \( e \neq \gamma \in \Gamma \), put \( L_\gamma = \{ x \in \mathbb{R}^n : \gamma \cdot x = x \} \). Then, \( \dim(L_\gamma) \leq n - 2 \).

**Proof.** Assume that \( \dim(L_\gamma) = n - 1 \). Then, 1 is an eigenvalue with multiplicity at least \( n - 1 \). Since \( \gamma \in \text{SO}(n) \), the remaining eigenvalue must be 1. This is a contradiction. \( \square \)

**Lemma 1.8.** If \( \Gamma_j \) is a finite subgroup of \( \text{SO}(n) \) acting linearly on \( \tilde{U}_j \), the condition \((1.9)\) is satisfied.

**Proof.** Lemma \((1.7)\) implies that \((n - 2)\)-dimensional Hausdorff measure of \( \{ x \in \tilde{U} : \gamma \cdot x = x \} \) is finite. Then, its 2-capacity is 0. See \[47\], p. 18. \( \square \)

For Hausdorff measures and dimensions, see \[81\].

**1.4. Orbifold and CMGA.** The examples \((1.3)\) and \((1.4)\) are both orbifold and CMGA. The difference between CMGA and orbifold is the singularity of the metric. For the case of orbifold, instead of \((1.5)\), the metric \( \tilde{g} \) on the covering space is assumed to be smooth in a neighborhood of \( z = 0 \).

**Example 1.9.** A simple example of CMGA is constructed from a sector

\[ U = \{ se^{i\psi} : s \geq 0, \psi \in [0, 2\pi \kappa] \}, \quad 0 < \kappa < 1. \]

We regard \( U \) as a cone by identifying \( \psi = \psi_0 \) with \( \psi = 0 \). Letting \( \theta = \psi/\kappa \), and \( \omega = (\cos \theta, \sin \theta) \), which varies over whole \( S^1 \), we equip \( U \), which is homeomorphic to \( \mathbb{R}^2 \), with the standard polar coordinates \( (s, \theta) \in [0, \infty) \times S^1 \). Then, \( x = s \cos \theta, \ y = s \sin \theta \) in the rectangular coordinates. Define a metric \( g \) on \( U \) by

\[ g = (ds)^2 + \kappa^2 s^2 (d\theta)^2. \]

Then, \( U \) is a \( C^\infty \) manifold with the standard differential structure of \( \mathbb{R}^2 \), but its metric is singular at the origin. In fact, \( g \) is written as

\[ g = \frac{x^2 + \kappa^2 y^2}{x^2 + y^2} (dx)^2 - \frac{2(\kappa^2 - 1) xy}{x^2 + y^2} dxdy + \frac{\kappa^2 x^2 + y^2}{x^2 + y^2} (dy)^2 \]

\[ = (dx)^2 + (dy)^2 + (\kappa^2 - 1)(d\theta)^2. \]

If \( 1/\kappa \) is a natural number, then \( U \) is an orbifold. In fact, put \( \tilde{U} = \mathbb{R}^2 \), \( \Gamma = \) the group generated by the rotation of angle \( 2\pi/n \), where \( n = 1/\kappa \), and \( \pi : se^{i\psi} \to se^{in\psi} \). Then, \( (\tilde{U}, \Gamma, \pi) \) is a conic chart of \( U \). Endow \( \tilde{U} \) with the metric \( \tilde{g} = (dx)^2 + (dy)^2 \). Then, \( g \) is induced from \( \tilde{g} \).

If \( \kappa \notin \mathbb{Q} \), \( U \) is not an orbifold, since the group generated by a rotation of angle \( 2\pi \kappa \) is an infinite group. However, we can regard \( U \) as a CMGA, where we take \( \tilde{U} = \mathbb{R}^2 \) equipped with the metric \((1.12)\), \( \Gamma = \{ e \} \) (the unit group) and \( \pi \) to be identity.
Example 1.10. We consider a higher dimensional analogue of Example 1.9. Let
\[
\tilde{U} = \tilde{W} \times \tilde{V} = \mathbb{R}^k \times \mathbb{R}^{n-k},
\]
and \(\Gamma\) a finite subgroup of \(SO(n)\) acting on \(\tilde{U}\). As in Definition 1.6 assume that this action leaves \(y\)-coordinates invariant, i.e.
\[
Y(\gamma \cdot x) = Y(x), \quad \forall x \in \tilde{U}, \quad \forall \gamma \in \Gamma.
\]
We then have
\[
\Gamma \backslash \tilde{U} = \tilde{W} \times (\Gamma \backslash \tilde{V}).
\]
We define a \(\Gamma\)-invariant metric \(\tilde{g}\) by
\[
\tilde{g} = (dy)^2 + (ds)^2 + a(y)s^2g_{S^{n-k-1}},
\]
where \(a(y)\) is any positive \(C^\infty\)-function on \(\tilde{W}\). Then, \(\tilde{g}\) is a smooth metric on
\[
U \setminus (\mathbb{R}^k \times \{0\}) = \mathbb{R}^k \times \left((0, \infty) \times (\Gamma \backslash S^{n-k-1})\right),
\]
where we deal with \(\Gamma \backslash S^{n-k-1}\) as an orbifold. This space has the structure described in the left-hand side of Figure 1.4. As is seen in its right-hand side, this CMGA has screen type singularities.
1.5. Global distance on $M$. Assume that a CMGA, denoted by $M$, is covered by locally finite coordinate neighborhoods $U_j, j \in J = \{1, 2, \cdots \}$, (allowing the case $\# J < \infty$). We define

$$M^{reg} = \bigcup_{j=1}^{\infty} \pi_j \left( \left( \widetildetilde{W}_j \times (B^{n-k}(0, R_1) \setminus \{0\}) \right) \cap \{ x \in \widetildetilde{U}_j ; \; \Gamma_j(x) = \{ e \} \} \right),$$

$$M^{sing} = M \setminus M^{reg}.$$ We also denote

$$(1.13) \quad M_j^{reg} = \pi_j \left( \left( \widetildetilde{W}_j \times (B^{n-k}(0, R_0) \setminus \{0\}) \right) \cap \{ x \in \widetildetilde{U}_j ; \; \Gamma_j(x) = \{ e \} \} \right),$$

$$U_j^{reg} = \pi_j(\widetildetilde{U}_j^{reg}).$$

The metric tensors $g_j$ in (1.6) define a smooth metric on the regular part $M^{reg}$ of $M$ and we denote this metric by $g$. These coordinates also determine a topology on $M$. As $M$ is the topological closure of $M^{reg}$, we define

$$(1.14) \quad d_M(x, y) = \inf_{\gamma} \text{Length}_g(\gamma([0, 1]) \cap M^{reg}),$$

where the infimum is taken over paths $\gamma : [0, 1] \to M$ that are piecewise $C^1$-smooth on the lifted local coordinates, $\gamma(0) = x, \gamma(1) = y$, and $\gamma([0, 1]) \cap M^{sing}$ is a finite set. Note that $\gamma([0, 1]) \cap M_j^{reg}$ is rectifiable in all local coordinate charts $U_j^{reg}$. Also, when $\gamma((s_1, s_2)) \subset U_j^{reg} \subset M^{reg}$, the above length in (2.10) is written using local coordinates as

$$\text{Length}_g(\gamma((s_1, s_2))) = \int_{s_1}^{s_2} \left( g_{jk}(\gamma(s)) \gamma^j(s) \gamma^k(s) \right)^{1/2} ds.$$  

1.6. Integration on CMGA. Integration over $M$ is actually done on its covering as explained below. Let $\{U_j\}_{j \in J}$ be a covering of $M$, and $(\widetildetilde{U}_j, \Gamma_j, \pi_j)$ the conic chart of $U_j$. We can construct a partition of unity $\{\chi_j\}_{j \in J}$ on $M$ and $\widetildetilde{\chi}_j \in C^\infty(\widetildetilde{U}_j)$ satisfying $\text{supp} \; \chi_j \subset U_j$ and $\chi_j(\pi_j(x)) = \widetildetilde{\chi}_j(x)$). Then, assuming that integration over $M$ is defined, for any $v, w \in L^2(M),$ 

$$(v, w)_{L^2(M)} = \left( \sum_{j=1}^{\infty} \chi_j v, w \right)_{L^2(M)} = \sum_{j=1}^{\infty} (\chi_j v, w)_{L^2(M)},$$

since $\sum_{j=1}^{\infty} \chi_j = 1$. This reduces the computation on each patch $U_j$. We omit the subscript $j$ and assume that the supports of $v, w$ are contained in $U$. As was noticed before, the functions on $U$ are regarded as $\Gamma$-invariant functions on $\widetildetilde{U}$. Therefore, $L^2(U)$ should be regarded as the closed subspace of $L^2(\widetildetilde{U})$ consisting of $\Gamma$-invariant functions. With in mind, we define integration over $U$ by the one over $\widetildetilde{U}$, and put

$$(v, w)_{L^2(U)} = \frac{1}{|\Gamma|} (\pi^* v, \pi^* w)_{L^2(\widetildetilde{U})}.$$
1.7. **Sobolev spaces.** We fix a coordinate patch $U$ in $M$. Letting $	ilde{g} = \tilde{g}_{ab}dx^a dx^b$ be the $\Gamma$-invariant Riemannian metric on $\tilde{U}^r e g$, we define a quadratic form $\tilde{q}$ with form domain $D(\tilde{q}) = C_0^\infty(\tilde{U}) \cap S_0$ by

$$\tilde{q}(\tilde{v}, \tilde{w}) = \frac{1}{2}\int_{\tilde{U}} (\tilde{v}, \tilde{w})_{L^2(\tilde{U})} + \frac{1}{2\det g} (\tilde{g}^{ab} \partial_a \tilde{v}, \partial_b \tilde{w})_{L^2(\tilde{U})},$$

where $\partial_a = \partial / \partial x^a$, $\tilde{v}, \tilde{w} \in C_0^\infty(\tilde{U}) \cap S_0$. Letting $v(\pi(x)) = \tilde{v}(x)$, $w(\pi(x)) = \tilde{w}(x)$, we put

$$q(v, w) = \tilde{q}(\tilde{v}, \tilde{w}),$$

and call it a quadratic form on $C_0^\infty(U)$. Note the abuse of notation, since $C_0^\infty(U)$ is not defined. For convenience, we write the right-hand side of (1.15) as

$$\int_U v\sqrt{g} dx + \int_U g^{ab}(x)(\partial_a v(x))(\partial_b w(x)) \sqrt{g(x)} dx.$$

Since the metric $\tilde{g}$ is $\Gamma$-invariant, so is $\tilde{g}^{ab}(\partial_a \tilde{v})(\partial_b \tilde{v})$ for $\Gamma$-invariant $\tilde{v}$. Taking account of (1.15), the usual calculus for integral, e.g. integration by parts, can be applied also to this formal expression. Returning to the total manifold $M$, the form $q$ is rewritten formally as

$$q(v, w) = \sum_j \int_{U_j} x_j v\sqrt{\tilde{g}} dx + \sum_j \int_{U_j} g^{ab}(\partial_a (x_j v)) \partial_b w \sqrt{\tilde{g}} dx$$

for $v, w$ whose lifts $\tilde{v}, \tilde{w}$ are smooth and compactly supported. Actually, it is represented as

$$\tilde{q}(\tilde{v}, \tilde{w}) = \sum_j \frac{1}{\det g} \int_{\tilde{U}_j} \tilde{x}_j (\tilde{\pi}_j^* v)(\tilde{\pi}_j^* w) \sqrt{\tilde{g}_j} dx$$

$$+ \sum_j \frac{1}{\det g} \int_{\tilde{U}_j} \tilde{g}^{ab}(\partial_a (\tilde{x}_j \tilde{\pi}_j^* v)) (\partial_b \tilde{\pi}_j^* w) \sqrt{\tilde{g}_j} dx.$$

Since the computation is reduced to that on $\tilde{U}_j$, the usual theory of quadratic form is applied to $q$. Then, due to (1.14), $q$ is closable. The Sobolev space $H^1(M)$ is defined to be the domain of $\overline{q} = \text{the closure of }\tilde{q}$:

$$H^1(M) = D(\overline{q}).$$

This is more clearly stated in the following lemma.

**Lemma 1.11.** $H^1(M) = H^1(M^r e g)$.

**Proof.** Recall that $v \in H^1(M^r e g)$ means that $\pi^* v \in H^1(\tilde{U}_j \emptyset \tilde{U}_j^s i n g)$ for all $j$. Therefore $H^1(M) \subset H^1(M^r e g)$ is obvious. To show the converse inclusion, we use the following fact.

**Lemma 1.12.** Let $\Omega$ be a domain in $\mathbb{R}^n$, and assume that there exists a subset $S \subset \Omega$ whose $2$-capacity is equal to 0. Then, there exists a bounded operator $\text{Ext} : H^1(\Omega \emptyset S) \to H^1(\Omega)$ such that $\text{Ext} v = v$ on $\Omega \emptyset S$ for any $v \in H^1(\Omega \emptyset S)$. 
See [? Theorem 4.6 (and its proof). Therefore, if we show that the 2-capacity of \( U_j^{sing} \) is 0, the inclusion \( H^1(M) \supset H^1(M^{reg}) \) follows. Now, omitting the subscript \( j \), \( U^{sing} \) consists of two parts.

\[
U^{sing,1} = \mathbb{W} \times \{0\}, \quad U^{sing,2} = \{x \in \tilde{U} : \Gamma(x) \neq \{e\}\}.
\]

If \( k = n \), \( \tilde{U} \) has no singularities. If \( k \neq n \), then \( k \leq n - 2 \) by the assumption \( (C-1) \). Therefore \( U^{sing,1} \) has a finite \( (n-2) \)-dimensional Hausdorff measure, hence its 2-capacity is 0, as was seen in the proof of Lemma 1.8. Noting that \( \tilde{U}^{sing, \gamma} \). Therefore, there is no problem for the regularity, and we have

\[
\tilde{U}^{sing,2} \subset \bigcup_{\gamma \in \Gamma} \{x : \gamma \cdot x = x\},
\]

and using (1.9), we see that the 2-capacity of \( U^{sing,2} \) is 0.

Returning to the quadratic form (1.10), let \( L' \) be the associated Friedrichs extension, i.e. the unique self-adjoint operator \( L' \) satisfying \( D(\sqrt{\mathcal{L}}) = H^1(M) \) and

\[
\mathfrak{P}(u, v) = (L'u, v), \quad \forall u \in D(L'), \quad \forall v \in D(\mathfrak{P}).
\]

We put \( L = L' - 1 = -\Delta_M \). Then, we have

\[
H^1(M) = D(\sqrt{1-\Delta_M}).
\]

We define the Sobolev space \( H^s_x(M) \) by \( H^s_x(M) = D(L')^{s/2}, s \in \mathbb{R} \). In other words,

\[
H^s_x(M) = D((1-\Delta_M)^{s/2}), \quad s \in \mathbb{R}.
\]

We write \( H^s_x(M) \) as \( H^s(M) \) for the sake of simplicity. We need a uniformity for the bounds of the metric on each \( \tilde{U}_j \) to define \( L' \). However, we do not pursue this condition here, since in our later applications the number of charts are finite. The projection \( P \) defined on \( L^2(\tilde{U}) \) is naturally extended to \( H^m(\tilde{U}), m = 0, 1, 2, \ldots \).

In the case of orbifolds, the metric \( \tilde{g} \) on the covering space is assumed to be smooth even on \( \{z = 0\} \). Therefore, there is no problem for the regularity, and we have

\[
\bigcap_{m=0}^{\infty} H^m(M) \subset C^\infty(M).
\]

However, for the case of CMGA, although

\[
\bigcap_{m=0}^{\infty} H^m(M) \subset C^\infty(M^{reg})
\]

holds, \( u \in \cap_{m=0}^{\infty} H^m(M) \) may not be regular on \( M^{sing} \), since \( \tilde{g} \) is not assumed to be smooth around \( \{z = 0\} \).

2. Laplacian on conic manifold

2.1. Regularity of the domain of Laplacian. Let \( L = -\Delta \) be the Laplacian on \( M \) defined in the previous section. We study the regularity of the elements in \( D(L) \) near \( M^{sing} \). Recall that \( M \) is covered by \( \{U_j : j \in J\} \). By taking local coordinates \( \theta \in \mathbb{R}^{n-k-1} \) on \( S^{n-k-1} \), we introduce the following norm for functions defined on \( \tilde{U}_j \):

\[
(p^{(m)}_j(f)) = \sup_{|y| < R_0, 0 < s < R_1, \theta} \sum_{|\alpha| + |\beta| + |\gamma| \leq m} s^{-|\gamma|} |\partial_\beta \partial_\theta \partial_\alpha f(y, s, \theta)|.
\]

We make the condition (1.7) more precise.
(D) Letting \( c(y,s,\omega) \) be any of \( a_{pq}^{(j)}(y,s,\omega) \), \( b_{lm}^{(j)}(y,s,\omega) \) and \( h_{pq}^{(j)}(y,s,\omega) \), we assume that

\[
\sup_{|y|<R_0,0<s<R_1,\theta} p_j^{(2)}(c) < \infty,
\]

and \( c(y,0,\omega) = \lim_{s \to 0} c(y,s,\omega) \) satisfies \( c(y,0,\omega) = c(y) \) if \( c = a_{pq}^{(j)} \), \( c(y,0,\omega) = 0 \)
if \( c = h_{pq}^{(j)} \). Moreover, (1.8) is satisfied.

The collection of norms \( \{p_j^{(m)}\}_{j \in J} \), together with a partition of unity \( \{\chi_j\}_{j \in J} \)
on \( M \), endows a norm on \( M \), which we denote by \( | \cdot |_m \):

\[
|f|_m = \sup_{j \in J} p_j^{(m)}(\chi_j f).
\]

Take a conic chart \((\widetilde{U}_j, \Gamma_j, \pi_j)\). Omitting the subscript \( j \), we rewrite the metric \((1.0)\) in terms of local coordinates \( \theta \) on \( S^{n-k-1} \) as follows:

\[
(2.3)
\widetilde{g}_{ij} = \begin{pmatrix}
a & 0 & s^j h \\
0 & 1 & 0 \\
s h & 0 & s^j b
\end{pmatrix},
\]

where \( a = (a_{pq}), b = (b_{pq}), h = (h_{pq}) \), with \( 1 \leq p, q \leq k, 1 \leq \alpha, \beta \leq n-k-1 \).

Note that \( a \) is the same as in \((1.0)\), and \( b \) and \( h \) are different, since we have used
\( d\omega = \sum_{\ell=1}^{n-k-1} \frac{\partial \omega}{\partial \theta} d\theta^\ell \).
However, \( a, b, h \) still have the properties in (D).

We then have

\[
(2.4)
T(\widetilde{g}_{ij})T = \begin{pmatrix}
a & 0 & \frac{t}{s} h \\
0 & 1 & 0 \\
h & 0 & b
\end{pmatrix}, \quad T = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1/s
\end{pmatrix}.
\]

The inverse of this matrix is computed as follows:

\[
(2.5)
A = (a - \frac{t}{s} h b^{-1} h)^{-1}, \quad H = -b^{-1} h A, \quad B = (b - h a^{-1} t h)^{-1}.
\]

In fact, consider the equation

\[
\begin{pmatrix}
a & 0 & \frac{t}{s} h \\
0 & 1 & 0 \\
h & 0 & b
\end{pmatrix}
\begin{pmatrix}
A & 0 & \frac{t}{s} H \\
0 & 1 & 0 \\
H & 0 & B
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

that is

\[
a A + \frac{t}{s} H = 1, \quad a \frac{t}{s} H + \frac{t}{s} H B = 0, \quad
h A + b H = 0, \quad h \frac{t}{s} H + b B = 1.
\]

Since \( \det b \neq 0 \) by \((1.3)\), we have \( H = -b^{-1} h A \), which implies \( (a - \frac{t}{s} h b^{-1} h) A = 1 \).
Then, \( A \) is written as in \((2.5)\). \( B \) can be computed similarly.

Therefore, we have

\[
(2.6)
(\widetilde{g}_{ij})^{-1} = (\widetilde{g}^{ij}) = \begin{pmatrix}
A & 0 & \frac{t}{s} H/s \\
0 & 1 & 0 \\
H/s & 0 & B/s^2
\end{pmatrix}.
\]
We put
\[ g = \det (\tilde{g}_{ij}). \]
Then, by (2.6), the Laplacian is formally decomposed as
\[ L = L_A + L_S + L_B + L_H, \]
where
\[ L_A = -\frac{1}{\sqrt{g}} \partial_p (\sqrt{g} a^{pq} \partial_q), \quad L_B = -\frac{1}{\sqrt{g}} \partial_a (\sqrt{g} b^{a\beta} \partial_{\beta}), \]
\[ L_S = -\frac{1}{\sqrt{g}} \partial_s (\sqrt{g} \partial_s), \quad L_H = -\frac{1}{\sqrt{g}} \left( \frac{1}{s} \partial_p (\sqrt{g} h^{pq} \partial_a) + \frac{1}{s} \partial_a (\sqrt{g} h^{a\beta} \partial_p) \right), \]
where \( \partial_p = \partial/\partial y^p, \partial_s = \partial/\partial s \) and \( \partial_a = \partial/\partial y^a, \theta^a \) being the local coordinate on \( S^{n-k-1} \). Actually, we decompose the associated quadratic form as
\[ q = q_A + q_S + q_B + q_H, \]
where
\[ q_A(u, v) = (a^{pq} \partial_q u, \partial_p v), \quad q_B(u, v) = (b^{a\beta} \partial_q u, \partial_{\beta} v), \]
\[ q_S(u, v) = (\partial_s u, \partial_s v), \quad q_H(u, v) = (h^{pq} \partial_q u, \partial_p v) + (h^{a\beta} \partial_q u, \partial_{\beta} v). \]
Here, \( q \) is \( q(u, v) - (u, v) L^2(M) \) where \( q(u, v) \) is defined by (1.16), and we omit \( \sharp \Gamma \) for the sake of simplicity. Recall that in the computation below \( M^{sing} \) and its neighborhood are understood to be lifted to \( \tilde{U} \).

To study the regularity of \( u \in D(L) \) near \( M^{sing} \), it is sufficient to consider on a small coordinate patch intersecting \( M^{sing} \). We take local coordinates \( y, s, \theta \) near \( M^{sing} \), where \( \theta \) is a local coordinate on \( S^{n-k-1} \). Letting \( \delta \) be a small positive constant, we can thus identify the \( \delta \)-neighborhood of \( M^{sing} \) with the following set:
\[ M^{sing}_\delta = \{(y, s, \theta); y \in \mathcal{O}^k, 0 \leq s < \delta, \theta \in \mathcal{O}^{n-k-1}\}, \]
where \( \mathcal{O}^k \) and \( \mathcal{O}^{n-k-1} \) are bounded open sets in \( \mathbb{R}^k \) and \( \mathbb{R}^{n-k-1} \). Letting \( c(y, s, \theta) \) be any of \( a_{pq}, b_{lm} \) and \( h_{pd} \), we have by the assumption (D):
\[ \sup_{y, s, \theta} |s|^{-\gamma} |\partial_\gamma \partial_\beta \partial_{\alpha} c(y, s, \theta)| < \infty \]
for \( |\alpha| + |\beta| + |\gamma| \leq 2 \). Moreover, \( a_{pq}, b_{lm} \) and \( h_{pd} \) are extended to \( s \leq 0 \) as \( C^2 \)-functions. Taylor expansion and (1.17) yield
\[ s^{-|\gamma|} |\partial_\gamma \partial_\beta \partial_{\alpha} (a_{pq}(y, s, \theta) - \tilde{a}_{pq}(y))| \leq Cs, \]
\[ s^{-|\gamma|} |\partial_\gamma \partial_\beta \partial_{\alpha} (b_{lm}(y, s, \theta) - \tilde{b}_{lm}(y, \theta))| \leq Cs, \]
\[ s^{-|\gamma|} |\partial_\gamma \partial_\beta \partial_{\alpha} (h_{pd}(y, s, \theta))| \leq Cs, \]
for \( |\alpha| + |\beta| + |\gamma| \leq 1 \).

Recall that the inner product of \( L^2(\tilde{U}) \), hence \( L^2(M) \), is defined by
\[ (u, v) = \int u \sqrt{g} dy ds d\theta. \]
By (2.24),
\[ g = s^{2(n-k-1)} \{ \det a \det b + o(1) \}, \quad s \to 0. \]
Therefore, letting
\[ g(y, s, \theta) = s^{2(n-k-1)} g_0(y, s, \theta), \]
where \( g_0 \) satisfies
\[ g_0(y, s, \theta) = \left\{ \begin{array}{ll} 1 & \text{if } y = 0, \quad 0 < s < \delta, \quad \theta \in \mathcal{O}^{n-k-1} \text{ and } s \theta = 0, \\
\end{array} \right. \]
with respect to similarity, however, it requires careful computation. In order to make the singularity

Note

and (2.14), we rewrite

(2.12)  \[ \|u\|_{a,b,c} = \left( \sum_{|\alpha| \leq a,\beta \leq b,|\gamma| \leq c} \int_{M^{'\text{sing}}_{\delta}} s^{-2|\gamma|} |\partial_y^\alpha \partial_r^\beta \partial_{\delta}^\gamma u|^2 s^{n-k-1} dy ds d\theta \right)^{1/2}, \]

(2.13)  \[ |u|_{0,2,0} = \left( \int_{M^{'\text{sing}}_{\delta}} \left| \partial_s^2 u + \frac{n-k-1}{s} \partial_s u \right|^2 s^{n-k-1} dy ds d\theta \right)^{1/2}. \]

We put

\[ \|u\|_{H^1} = \|u\|_{0,0,0} \]

Note that for \( u \) supported in \( M^{'\text{sing}}_{\delta} \), \( |u|_{0,0,0} \) is equivalent to \( \|u\|_{L^2(M)} \). We also use the following inner product:

(2.14)  \[ (u,v)_0 = \int_{M^{'\text{sing}}_{\delta}} u \bar{\psi} s^{n-k-1} dy ds d\theta. \]

**Definition 2.1.** Let \( \tilde{H}^2(M^{'\text{sing}}_{\delta}) \) be the set of functions \( u \) such that

\[ \|u\|_{\tilde{H}^2(M^{'\text{sing}}_{\delta})} := |u|_{0,2,0} + \sum_{a+b+c \leq 2, b \leq 1} \|u\|_{a,b,c} < \infty. \]

The aim of this subsection is to show the following theorem.

**Theorem 2.2.** \( D(L) = H^2_{\text{reg}}(M^{'\text{reg}}_{\delta}) \cap \tilde{H}^2(M^{'\text{sing}}_{\delta}) \).

The proof of this theorem is done by the standard argument for elliptic regularity, however, it requires careful computation. In order to make the singularity with respect to \( s \) of the volume element of the quadratic form \( q \) more explicit, we make the gauge transformation \( v \to g_0^{1/4} v \), where \( g_0 \) is in (2.11). In view of (2.10) and (2.11), we rewrite \( q \) in (2.8) as

\[ q(g_0^{-1/4} u, g_0^{-1/4} v) = Q(u,v) = Q_A + Q_S + Q_B + Q_H, \]

\[ Q_A(u,v) = (a^{pq}(\partial_q + \partial_q \log g_0^{-1/4})u, (\partial_p + \partial_p \log g_0^{-1/4})v)_0, \]

\[ Q_B(u,v) = (\nu^{\alpha\beta}(\partial_\delta + \partial_\delta \log g_0^{-1/4})u, (\partial_\alpha + \partial_\alpha \log g_0^{-1/4})v)_0, \]

\[ Q_S(u,v) = ((\partial_s + \partial_s \log g_0^{-1/4})u, (\partial_s + \partial_s \log g_0^{-1/4})v)_0, \]

\[ Q_H(u,v) = (h^{\alpha\beta}(\partial_\delta + \partial_\delta \log g_0^{-1/4})u, (\partial_\alpha + \partial_\alpha \log g_0^{-1/4})v)_0 \]

\[ + (h^{\alpha\beta}(\partial_\delta + \partial_\delta \log g_0^{-1/4})u, (\partial_\alpha + \partial_\alpha \log g_0^{-1/4})v)_0. \]

Note \( g_0^{1/4}(\partial_p g_0^{-1/4})u = \partial_p u + (\partial_p \log g_0^{-1/4})u, \text{etc.} \). We put

\[ L_0 = g_0^{1/4} L g_0^{-1/4}, \quad D(L_0) = g_0^{1/4} D(L). \]
Then we have for \( u \in D(L_0) \) and \( v \in g_0^{1/4}D(\sqrt{L}) \)
\[
Q(u,v) = (L_0u,v)_0.
\]

Here, we recall a well-known lemma on Freidrich’s mollifier. Take \( \rho(x) \in C^\infty_0(\mathbb{R}^m) \) such that \( \rho(x) = 0 \) for \( |x| > 1 \) and \( \int_{\mathbb{R}^m} \rho(x)dx = 1 \). For a sufficiently small \( \delta > 0 \), we put \( \rho_\delta(x) = \delta^{-m}\rho(x/\delta) \) and let the operator \( \rho_\delta \) be defined by
\[
\rho_\delta : u \rightarrow \rho_\delta * u(x) = \int_{\mathbb{R}^m} \rho_\delta(x-y)u(y)dy.
\]

We also put
\[
[\rho_\delta *, a(x) \frac{\partial}{\partial x_j}]u = \rho_\delta * (a(x) \frac{\partial u}{\partial x_j}) - a(x) \frac{\partial}{\partial x_j} (\rho_\delta * u).
\]

**Lemma 2.3.** Let \( a(x) \in C^1(\mathbb{R}^m) \) be such that
\[
|a|_{G_1} := \sup_{x \in \mathbb{R}^m} \sum |\partial_x^\alpha a(x)| < \infty.
\]

Then, there exists a constant \( C > 0 \) independent of \( \delta > 0 \) such that
\[
\left\| [\rho_\delta *, a(x) \frac{\partial}{\partial x_j}]u \right\|_{L^2(\mathbb{R}^m)} \leq C|a|_{G_1} \|u\|_{L^2(\mathbb{R}^m)}, \tag{2.15}
\]
and as \( \delta \rightarrow 0 \)
\[
[\rho_\delta *, a(x) \frac{\partial}{\partial x_j}]u \rightarrow 0, \text{ in } L^2(\mathbb{R}^m) \tag{2.16}
\]
for any \( u \in L^2(\mathbb{R}^m) \).

**Proof.** We have
\[
[\rho_\delta *, a(x) \frac{\partial}{\partial x_j}]u = \int_{\mathbb{R}^m} u(y) \frac{\partial}{\partial y_j} \left\{ \rho_\delta(x-y)(a(x) - a(y)) \right\} dy.
\]

Noting that
\[
|a(x) - a(y)| \leq C|a|_{G_1} |x-y|, \quad \int_{|x-y| < \delta} |x-y| \left| \frac{\partial \rho_\delta}{\partial y_j}(x-y) \right| dy < C,
\]
we obtain (2.15). To prove (2.16), we only have to consider the case in which \( u \in C^\infty_0(\mathbb{R}^m) \). Then, we have
\[
a(x) \frac{\partial}{\partial x_j} \rho_\delta * u - \rho_\delta * (a(x) \frac{\partial}{\partial x_j} u) = \int_{\mathbb{R}^m} (a(x) - a(y)) \rho_\delta(x-y) \frac{\partial u}{\partial y_j}(y)dy.
\]

Noting that
\[
\int_{|x-y| \leq \delta} \left| \frac{\partial u}{\partial y_j}(y) \right| dy \leq C\delta \int \rho_\delta(x-y) \left| \frac{\partial u}{\partial y_j}(y) \right| dy,
\]
we obtain (2.16). \( \square \)

It is sufficient to prove Theorem 2.2 with \( D(L) \) replaced by \( D(L_0) \). Let \( u \in D(L_0) \). Since \( g_0^{1/4} u \in D(\sqrt{L}) \), we have by (1.17)
\[
\sum_{|\alpha|+|\beta|+|\gamma| \leq 1} \int s^{-2|\gamma|} |\partial_y^\alpha \partial_s^\beta \partial_y^\gamma u|^2 s^{n-k-1} dydsd\theta < \infty. \tag{2.17}
\]
We consider the 2nd order derivatives of $u \in D(L_0)$. Take $\rho(y, \theta) \in C_0^\infty(\mathbb{R}^{n-1})$ such that $\rho(y, \theta) = 0$ for $|y|^2 + |\theta|^2 > 1$ and $\int_{\mathbb{R}^{n-1}} \rho(y, \theta) dy d\theta = 1$. We put $\rho_\delta(y, \theta) = \delta^{-(n-1)} \rho(y/\rho, \theta/\rho)$. Note that the operator $\rho_\delta \ast$ has the property in Lemma 2.2 with $L^2(\mathbb{R}^m)$-norm replaced by $\| \cdot \| = \| \cdot \|_{0,0,0}.$

**Lemma 2.4.** If $u \in D(L_0)$, then $\rho_\delta \ast u \in D(L_0)$, and $L_0(\rho_\delta \ast u) \to f$ in $L^2(M)$ as $\delta \to 0$.

**Proof.** If $u \in D(L_0)$, letting $L_0u = f$, we have for $v \in g_0^{1/4}D(\sqrt{L})$, $Q(u, v) = (f, v)_0$. We replace $v$ by $\rho_\delta \ast v$. Then, since $\rho_\delta \ast$ is self-adjoint and commutes with $\partial_q$ and $\partial_\theta$, we have

$$Q_A(u, \rho_\delta \ast v) = Q_A(\rho_\delta \ast u, v) + (f_1\delta, v)_0,$$

$$f_1\delta = (\partial_p + \partial_p \log g_0^{-1/4} \ast \rho_\delta \ast, a p q (\partial_q + \partial_q \log g_0^{-1/4}) u) + (\partial_p + \partial_p \log g_0^{-1/4} \ast \rho_\delta \ast a p q (\partial_q + \partial_q \log g_0^{-1/4}) u),$$

$$Q_B(\rho_\delta \ast u, v) = Q_B(\rho_\delta \ast u, v) + (f_2\delta, v)_0,$

$$f_2\delta = (\partial_\alpha + \partial_\alpha \log g_0^{-1/4} \ast \rho_\delta \ast, b^{\alpha\beta} \partial_\beta (\partial_\beta + \partial_\beta \log g_0^{-1/4}) u) + (\partial_\alpha + \partial_\alpha \log g_0^{-1/4} \ast \rho_\delta \ast b^{\alpha\beta} \partial_\beta (\partial_\beta + \partial_\beta \log g_0^{-1/4}) u),$$

$$Q_H(\rho_\delta \ast u, v) = Q_H(\rho_\delta \ast u, v) + (f_3\delta, v)_0,$

$$f_3\delta = (\partial_p + \partial_p \log g_0^{-1/4} \ast \rho_\delta \ast, h p q (\partial_q + \partial_q \log g_0^{-1/4}) u) + (\partial_p + \partial_p \log g_0^{-1/4} \ast \rho_\delta \ast h p q (\partial_q + \partial_q \log g_0^{-1/4}) u) + (\partial_\alpha + \partial_\alpha \log g_0^{-1/4} \ast \rho_\delta \ast h p q (\partial_q + \partial_q \log g_0^{-1/4}) u),$$

$$Q_S(\rho_\delta \ast u, v) = Q_S(\rho_\delta \ast u, v) + (f_4\delta, v)_0,$$

$$f_4\delta = (\partial_\alpha \log g_0^{-1/4} \ast \rho_\delta \ast (\partial_\alpha + \partial_\alpha \log g_0^{-1/4}) u).$$

Summing up these 4 terms, we obtain

$$Q(u, \rho_\delta \ast v) = Q(\rho_\delta \ast u, v) + \left(\sum_{i=1}^{4} f_i\delta, v\right)_0.$$

On the other hand, we have

$$Q(u, \rho_\delta \ast v) = (L_0u, \rho_\delta \ast v)_0 = (f, \rho_\delta \ast v)_0 = (\rho_\delta \ast f, v)_0.$$

Therefore, we have

$$Q(\rho_\delta \ast u, v) = (\rho_\delta \ast f, v)_0 - \left(\sum_{i=1}^{4} f_i\delta, v\right)_0.$$

Since this holds for all $v \in g_0^{-1/4}D(\sqrt{L})$, we see that $\rho_\delta \ast u \in D(L_0)$ and $L_0(\rho_\delta \ast u) = \rho_\delta \ast f - \left(\sum_{i=1}^{4} f_i\delta\right).$
Since \( u \in H^1(M) \), by Lemma 2.3 \( L_0(\rho_\delta * u) \to f \) in \( \| \cdot \|_{0,0,0} \) norm as \( \delta \to 0 \). This proves the lemma. \( \square \)

As a formal differential operator, \( L_0 \) is rewritten as
\[
L_0 = L_A^{(0)} + L_S^{(0)} + L_B^{(0)} + L_H^{(0)} + L_R^{(0)},
\]
where \( L_R^{(0)} \) consists of first order and zeroth order terms of differential operators \( \partial_p, \frac{\partial}{\partial \rho}, \partial_s \) with bounded coefficients. We also decompose the quadratic form \( Q \) as
\[
Q(u,v) = Q_A^{(0)}(u,v) + Q_S^{(0)}(u,v) + Q_B^{(0)}(u,v) + Q_H^{(0)}(u,v) + Q_R^{(0)}(u,v),
\]
where
\[
Q_A^{(0)}(u,v) = (a_{\alpha p} \partial_\alpha u, \partial_p v)_0, \\
Q_S^{(0)}(u,v) = (\partial_s u, \partial_s v)_0, \\
Q_B^{(0)}(u,v) = (b_{\alpha \beta} \partial_\alpha \partial_\beta u, \partial_s v)_0, \\
Q_H^{(0)}(u,v) = (h^{\alpha \beta} \partial_\alpha u, \partial_\beta v)_0 + (h^{\alpha p} \partial_p u, \partial_\alpha v)_0, \\
Q_R^{(0)}(u,v) = (L_R^{(0)}(u,v))_0.
\]

We are going to estimate \( \text{Re} Q_E^{(0)}(u, L_F^{(0)} u), F \neq S \), from below. In the following, \( o(H^2) \) denotes a term which is estimated as follows: For any \( \epsilon > 0 \), there exists a constant \( C_\epsilon > 0 \) such that
\[
|o(H^2)| \leq \epsilon \sum_{a+b+c \leq 2, b \leq 1} \|u\|^2_{a,b,c} + C_\epsilon \|u\|^2_{H^1}.
\]

We put
\[
m(\delta) = \max_{p, \alpha \leq 0 \leq s < \delta, y, \theta} \left( |h^{\alpha \beta}(y, s, \theta)| + |h^{\alpha p}(y, s, \theta)| \right).
\]

By the assumption (D), \( m(\delta) \to 0 \) as \( \delta \to 0 \).

**Lemma 2.5.** There exists a constant \( C > 0 \) such that if \( u \) is in \( D(L_0) \) and three times differentiable with respect to \( y \) and \( \theta \), we have
\[
\|u\|^2_{0,0,0} \leq C \text{Re} Q_A^{(0)}(u, L_A^{(0)} u) + o(H^2),
\]
\[
\|u\|^2_{1,0,1} \leq C \text{Re} Q_B^{(0)}(u, L_A^{(0)} u) + o(H^2),
\]
\[
\|u\|^2_{1,1,0} \leq C \text{Re} Q_S^{(0)}(u, L_A^{(0)} u) + o(H^2),
\]
\[
|Q_H^{(0)}(u, L_A^{(0)} u)| \leq C m(\delta) \sum_{a+b+c \leq 2, b \leq 1} \|u\|^2_{a,b,c},
\]
\[
Q_R^{(0)}(u, L_A^{(0)} u) = o(H^2).
\]
PROOF. By integration by parts if necessary, \( \text{Re} \, Q^{(0)}_R(u, L^{(0)}_A u) \) consists of the terms like e.g.,

\[
(\cdots \partial_\alpha u, \cdots \partial_\alpha \frac{\partial_\beta}{s} u)_0, \quad (\cdots \partial_\beta u, \cdots \partial_\alpha \frac{\partial_\beta}{s} u)_0,
\]

which proves (2.23). Similar integration by parts yields (2.22), (2.21), (2.19). We prove (2.20). By integration by parts,

\[
Q^{(0)}_B(u, L^{(0)}_A u) = (b^{\alpha\beta}_p \partial_\beta \frac{\partial_\alpha}{s} u, a^{pq}\partial_q \partial_r \frac{\partial_\alpha}{s} u)_0 + o(H^2).
\]

Let \( C(\eta) \) be the cube in \((y, \theta)\) of size \( \eta \) with center at the origin. We take a partition of unity \( \{ \chi_j \} \) such that \( \sum_j \chi_j^2 = 1 \) and each \( \chi_j \) is a translation of fixed \( \chi \) which has support inside \( C(2\eta) \) and \( \chi = 1 \) on \( C(\eta) \). Let \( C_j(\eta) \) be the associated translation of \( C(\eta) \) and \( x_j \) the center of \( C_j(\eta) \). Inserting \( 1 = \sum_j \chi_j^2 \), we have

\[
Q^{(0)}_B(u, L^{(0)}_A u) = \sum j \left( b^{\alpha\beta}_p \partial_\beta \frac{\partial_\alpha}{s} (\chi_j u), a^{pq}\partial_q \partial_r \frac{\partial_\alpha}{s} (\chi_j u) \right)_0 + o(H^2).
\]

We put

\[
a^{pq}_j = a^{pq}(x_j), \quad b^{\alpha\beta}_j = b^{\alpha\beta}(x_j).
\]

Given \( \epsilon > 0 \), taking \( \eta \) small enough independently of \( j \), we have

\[
\text{Re} \left( b^{\alpha\beta}_p \partial_\beta \frac{\partial_\alpha}{s} (\chi_j u), a^{pq}\partial_q \partial_r \frac{\partial_\alpha}{s} (\chi_j u) \right)_0 \geq \text{Re} \left( b^{\alpha\beta}_j \partial_\beta \frac{\partial_\alpha}{s} (\chi_j u), a^{pq}\partial_q \partial_r \frac{\partial_\alpha}{s} (\chi_j u) \right)_0 - \epsilon \| \chi_j u \|_{1,0,1}^2.
\]

Since \((a^{pq}_j), (b^{\alpha\beta}_j)\) are positive definite matrices corresponding to independent variables \( y, \theta \), by passing to the Fourier transform with respect to \( y \) and \( \theta \), we have

\[
\text{Re} \left( b^{\alpha\beta}_p \partial_\beta \frac{\partial_\alpha}{s} (\chi_j u), a^{pq}\partial_q \partial_r \frac{\partial_\alpha}{s} (\chi_j u) \right)_0 \geq C_0 \left( \left( -\frac{\Delta_y}{s^2} \right) \chi_j u, \left( -\Delta_\theta \right) \chi_j u \right)_0,
\]

where \( \Delta_y \) and \( \Delta_\theta \) are the Laplacians with respect to \( y \) and \( \theta \), and \( C_0 \) is a positive constant independent of \( j \). It is easy to show that

\[
\| \chi_j \|^2_{0,1} \leq C \left( \left( -\frac{\Delta_y}{s^2} \right) \chi_j u, \left( -\Delta_\theta \right) \chi_j u \right)_0 + \| \chi_j u \|^2_{1,0}.
\]

The inequalities (2.24), (2.25) and (2.26) prove (2.20). \( \square \)

Similarly, one can prove the following two lemmas.

**Lemma 2.6.** There exists a constant \( C > 0 \) such that if \( u \) is in \( D(L_0) \) and three times differentiable with respect to \( y \) and \( \theta \), we have

\[
\| u \|_{1,0}^2 \leq C \text{Re} \, Q^{(0)}_A(u, L^{(0)}_B u) + o(H^2),
\]

\[
\| u \|_{0,2}^2 \leq C \text{Re} \, Q^{(0)}_B(u, L^{(0)}_B u) + o(H^2),
\]

\[
\| u \|_{1,0}^2 \leq C \text{Re} \, Q^{(0)}_B(u, L^{(0)}_B u) + o(H^2),
\]

\[
\| Q^{(0)}_H(u, L^{(0)}_B u) \| \leq Cm(\delta) \sum_{\alpha + \beta + \epsilon \leq 2, b \leq 1} \| u \|_{a,b,c}^2,
\]

\[
Q^{(0)}_R(u, L^{(0)}_B u) = o(H^2).
\]
2. LAPLACIAN ON CONIC MANIFOLD

Lemma 2.7. There exists a constant $C > 0$ such that if $u$ is in $D(L_0)$ and three times differentiable with respect to $y$ and $\theta$, we have for $E \neq H$

$$\left| Q_E^{(0)}(u, L_H^{(0)} u) \right| \leq C \sum_{a+b+c \leq 2, b \leq 1} \|u\|^2_{a,b,c}.$$

The following estimates are our main purpose.

Lemma 2.8. There exists a constant $C > 0$ such that for any $u \in D(L_0)$

$$\sum_{a+b+c \leq 2, b \leq 1} \|u\|_{a,b,c} \leq C(\|L_0 u\| + \|u\|_{H^1}).$$

Proof. By virtue of Lemma 2.4, we only have to prove this lemma when $0 = 0$ small enough, and put $L_0 = L_0^{(0)}$. We put $v = L_A^{(0)} u$ in the quadratic form $Q(u, v) = (f, v)_0$. Then,

$$Q_A^{(0)}(u, L_A^{(0)} u) + \sum_{a,b+1 \leq 2, b \leq 1} \|u\|^2_{a,b,c}.$$

By Lemma 2.5, we have

$$- \text{Re} Q_S^{(0)}(u, L_A^{(0)} u) \leq o(H^2), \quad - \text{Re} Q_B^{(0)}(u, L_A^{(0)} u) \leq o(H^2),$$

$$- \text{Re} Q_H^{(0)}(u, L_A^{(0)} u) \leq C \sum_{a+b+c \leq 2, b \leq 1} \|u\|^2_{a,b,c}. \quad - \text{Re} Q_R^{(0)}(u, L_A^{(0)} u) \leq o(H^2).$$

We take $\epsilon > 0$ small enough, and put

$$C_\epsilon(H^2) = \epsilon \sum_{a+b+c \leq 2, b \leq 1} \|u\|^2_{a,b,c} + C_\epsilon \|u\|^2_{H^1}.$$  

Noting that $Q_A^{(0)}(u, L_A^{(0)} u) = \|L_A^{(0)} u\|^2$, we have by (2.34), choosing $\delta$ small enough

$$\|L_A^{(0)} u\|^2 \leq C \|f\|^2 + o_\epsilon(H^2).$$

Using (2.19), we then have

$$\|u\|^2_{2,0,0} \leq C \|f\|^2 + o_\epsilon(H^2).$$

By the similar arguments, we have

$$\|L_B^{(0)} u\|^2 \leq C \|f\|^2 + o_\epsilon(H^2),$$

$$\|L_H^{(0)} u\|^2 \leq C \|f\|^2 + o_\epsilon(H^2),$$

$$\|L_R^{(0)} u\|^2 \leq C \|f\|^2 + o_\epsilon(H^2).$$

Since $L_0 = L_A^{(0)} + L_S^{(0)} + L_B^{(0)} + L_H^{(0)} + L_R^{(0)}$, we then have

$$\|L_0^{(0)} u\|^2 \leq C \|f\|^2 + o_\epsilon(H^2).$$
This yields
\begin{equation}
(2.38) \quad \| (\partial_s^2 + \frac{n - k - 1}{s}) u \| \leq C \| f \|^2 + o_{\epsilon}(H^2).
\end{equation}
Summing up (2.35), (2.35), (2.35) and (2.38), we get finally
\begin{equation}
\| u \|_{H^2} \leq C \| f \|_2 + \epsilon \| u \|_{H^2} + C_{\epsilon} \| u \|^2_{H^1}.
\end{equation}
This proves the lemma.

Theorem 2.2 now follows from Lemma 2.8. In particular, for any $u \in D(L)$,
\begin{equation}
Lu = L_A u + L_S u + L_B u + L_H u,
\end{equation}
where each term of the right-hand side makes sense in the sense of distribution and belongs to $L^2(M)$.

\subsection*{2.2. Limit metric and its perturbation.}
We return to our manifold $M$ of the form (0.1). We need to change the definition of $S^\kappa$ in (0.9) as follows. Let $\{\chi_j\}_{j \in J}$ be the set of partition of unity on $M$, and define $p_j^{(m)}(f)$ by (2.1). We put
\begin{equation}
p_M(f) = \sum_{j \in J} p_j^{(2)}(f) = \sum_{j \in J} \sup_{y,s,\theta} \sum_{|\alpha| + |\beta| + |\gamma| \leq 2} s^{-|\gamma|} |\partial_y^\alpha \partial_s^\beta \partial_\theta^\gamma (\chi_j f(y,s,\theta))|.
\end{equation}

**Definition 2.9.** We define $S^\kappa$ to be the set of $C^\infty((0,\infty); C^2(M_{reg}))$-functions $f$ on $M$ satisfying
\begin{equation}
p_M(\partial_\ell^rf(r)) \leq C(1 + r)^{\kappa - \ell}, \quad \forall \ell \geq 0.
\end{equation}

The assumptions on $M$ are as follows.
- $K$ is a relatively compact $n$-dimensional regular conic manifold.
- Each end $M_i$ ($i = 1, \cdots, N + N'$) is an $n$-dimensional CMGA $M_i$ with the following properties. There exist an $(n-1)$-dimensional CMGA $M_i$ and a family of metrics $h_i(r,x,dx)$ ($r > 0$) on $M_i$ such that $M_i$ is diffeomorphic to $(0,\infty) \times M_i$ and equipped with the metric
  \begin{equation}
ds_i^2 = (dr)^2 + \rho_i(r)^2 h_i(r,x,dx),
\end{equation}
where $\rho_i(r)$ satisfies (A-2), (A-3). Moreover, there exists a metric $h_{M_i}(x,dx)$ as a CMGA on $M_i$ such that
  \begin{equation}
h_i(r,x,dx) - h_{M_i}(x,dx) \in S^{-\gamma_0,i}, \quad \gamma_0,i > 1.
\end{equation}

**Remark 2.10.** The above assumption is stronger than actually needed. In Definition 2.9, we have only to assume that $f \in S^\kappa$ if and only if $C^2((0,\infty); C^2(M_{reg}))$-functions $f$ on $M$ satisfying
\begin{equation}
p_M(\partial_\ell^rf(r)) \leq C(1 + r)^{\kappa - \ell}, \quad 0 \leq \ell \leq 2.
\end{equation}
For the assumption for the metric $g_{ij}$, this is still stronger. However, in order not to make the assumption too complicated, we proceed under the above condition.
Our main concern for the conic singularities in the previous subsection is the local structure of the singular set $M^\text{sing}$ and the regularity of the domain of the Laplacian. They are obviously invariant by the multiplication by $C^1$-functions to the metric. Then we can assume without loss of generality that for each end $M_i$, $\mathcal{M}_i^\text{sing} = (0, \infty) \times M_i^\text{sing}$, and $M_i^\text{sing}$ has the structure described in the previous section. We omit the subscript $i$ for the sake of simplicity. Around $\mathcal{M}_i^\text{sing}$, letting $x = (y, z)$ be the local coordinate on $M$, $s = |z|$, $\omega = z/s$ and $\theta$ the local coordinate on $S^{n-k-1}$, the Riemannian metric of $M$ is rewritten as
\begin{equation}
g = (dr)^2 + \rho(r)^2 \left( \sum_{p, q = 1}^{k} a_{pq}(r, y, s, \theta) dy^p dy^q + ds^2 \right) + s^2 \sum_{\ell, m = 1}^{n-k} b_{\ell m}(r, y, s, \theta) d\omega^\ell d\omega^m + s \sum_{p = 1}^{k} n-1-k h_p(r, y, s, \theta) dy^p d\omega^t \right),
\end{equation}
where the coefficients $a_{pq}(r, y, s, \theta)$, $b_{\ell m}(r, y, s, \theta)$, $h_p(r, y, s, \theta)$ satisfy the assumptions (C-1), (C-2) and (C-3) uniformly with respect to $r > 0$. Moreover, there exist $a_{pq}^\infty(y, s, \theta)$, $b_{\ell m}^\infty(y, s, \theta)$, $h_p^\infty(y, s, \theta)$ such that
\begin{equation}
p_M(\partial_r^m \alpha - \partial_r^m \alpha^\infty) \leq C r^{-\gamma_0 - m},
\end{equation}
where $\alpha$ is any of $a_{pq}(r, y, s, \theta)$, $b_{\ell m}(r, y, s, \theta)$, $h_p(r, y, s, \theta)$, and $\alpha^\infty$ is any of $a_{pq}^\infty(y, s, \theta)$, $b_{\ell m}^\infty(y, s, \theta)$, $h_p^\infty(y, s, \theta)$. Using (2.40), we also see that $g$ satisfies
\begin{equation}
g = \rho(r)^{2(n-1)} s^{2(n-k-2)} g_1, \quad C < g_1 < C^{-1}
\end{equation}
for a constant $C > 0$.

2.3. Laplacians. The Laplacian $-\Delta_M$ on $M$ has a self-adjoint realization in $L^2(\mathcal{M})$ through the quadratic form $\sum_{i, j} (g^{ij} \partial_i u \partial_j v)$, which is denoted by $H$. By Lemma 1.11 and Theorem 2.2 it satisfies
\begin{equation}
D(\sqrt{H}) = H^1(\mathcal{M}),
\end{equation}
\begin{equation}
D(H) = H^2_{\text{loc}}(\mathcal{M}_{\text{reg}}) \cap \tilde{H}^2(\mathcal{M}^\text{sing}_\delta).
\end{equation}
We take $\chi_{\text{reg}}, \chi_{\text{sing}} \in C^\infty(\mathcal{M})$ such that $\chi_{\text{reg}} + \chi_{\text{sing}} = 1$ on $\mathcal{M}$, $\chi_{\text{reg}} = 0$ on a small neighborhood of $\mathcal{M}^\text{sing}_\delta$. Choose $\delta > 0$ so that $\text{supp} \chi_{\text{sing}} \subset \mathcal{M}^\text{sing}_\delta$. We put
\begin{equation}
P_{a, b, c, d}(u) = \left( \sum_{a + b + c + d \leq 2} s^{n-k-1-2|\gamma|} \rho(r)^{-2(|a| + |b| + |c| + |d|)} \left| \partial_r^a \partial_y^b \partial_\theta^c \partial_s^d u \right|^2 \right)^{1/2},
\end{equation}
and for $t \in \mathbb{R}$, define the $\tilde{H}^{2, t}(\mathcal{M}^\text{sing}_\delta)$ norm by
\begin{equation}
\|u\|_{\tilde{H}^{2, t}(\mathcal{M}^\text{sing}_\delta)}^2 = \sum_{a + b + c + d \leq 2} \int_{\mathcal{M}^\text{sing}_\delta} (1 + r)^{2t} P_{a, b, c, d}(u)^2 \rho^{n-1}(r) dr dy ds d\theta
\end{equation}
\begin{equation}
+ \int_{\mathcal{M}^\text{sing}_\delta} (1 + r)^{2t} \left| \partial_s^2 u + \frac{n-k-1}{s} \partial_s u \right|^2 s^{n-k-1} \rho^{n-1}(r) dr dy ds d\theta.
\end{equation}
We also put
\[
\|u\|_{\widetilde{H}^2(M^{\text{cusp}})} = \|u\|_{\widetilde{H}^2(\chi_{\text{reg}}(\chi_{\text{cusp}}))},
\]
(2.44) \[
\|u\|_{\widetilde{H}^2(M)} = \|\chi_{\text{reg}}u\|_{H^2(M^{\text{cusp}})} + \|\chi_{\text{sing}}u\|_{\widetilde{H}^2(M^{\text{cusp}})}.
\]
By Lemma 2.28 the following elliptic regularity theorem holds.

**Theorem 2.11.** There exists a constant \(C > 0\) such that for any \(u \in D(H)\)
\[
\|u\|_{\widetilde{H}^2(M)} \leq C(\|Hu\|_{L^2(M)} + \|u\|_{L^2(M)}).
\]

By our assumption, each end \(M_i\) is diffeomorphic to \((0, \infty) \times M_i\). Recalling that \(M_i\) has two metrics \(h_i(r, x, dx)\) and \(h_{M_i}(x, dx)\), let \(\Lambda_i(r)\) and \(\Lambda_i\) be the associated Laplace-Beltrami operators on \(M_i\), respectively. By the same arguments as above, 
\[-\rho_i(r)^{-2}\Lambda_i(r)\text{ and } -\rho_i(r)^{-2}\Lambda_i\text{ have self-adjoint realizations in }L^2(M_i)\text{ as Friedrichs extensions, which are denoted by }B_i(r)\text{ and }B_i.\]
Therefore,
\[-\Delta_M = -\frac{\partial^2}{\partial r^2} - \frac{\partial g_i}{2} \frac{\partial}{\partial r} + B_i(r) \quad \text{on } M_i,
\]
\[B_i(r) = \rho_i(r)^{-2}\Lambda_i(r) \quad \text{on } M_i.\]
They satisfy
\[D(\sqrt{B_i(r)}) = D(\sqrt{B_i}) = H^1(M_i).\]
We impose the Dirichlet boundary condition at \(r = 1\), and let \(H_i\) and \(H_{0,i}\) be the Laplacians on \(\bar{M}_i = (1, \infty) \times M_i\) associated with metrics \((dr)^2 + \rho_i(r)^2 h_i(r, x, dx)\) and \((dr)^2 + \rho_i(r)^2 h_i(x, dx)\), respectively. They are the Friedrichs extensions of the Laplace operators restricted to \(C_0^\infty(M_i)\). Hence [2.42] and [2.43] hold also for \(H_i\) and \(H_{0,i}\).

We derive resolvent equations for \(H_i\) and \(H\) restricted to \(M_i\). Let
\[R(z) = (H - z)^{-1}, \quad R_i(z) = (H_i - z)^{-1}.\]
Take \(\chi_i \in C^\infty(M)\) such that
\[
\chi_i = \begin{cases} 
1 & \text{on } (2, \infty) \times M_i \subset M_i, \\
0 & \text{on } ((0,1) \times \bar{M}_i) \cup (M \setminus M_i) .
\end{cases}
\]
Since the elements in the domains of \(H\) and \(H_i\) are twice differentiable with respect to \(r\), and \(\chi_i\) depends only on \(r\), we have
(2.45) \[
\chi_i R_i(z)f \in D(H), \quad \forall f \in L^2(\bar{M}_i),
\]
(2.46) \[
\chi_i R_i(z)f = R(z)\chi_i f + R(z)[H, \chi_i]R_i(z)f.
\]
Conversely, we have
(2.47) \[
\chi_i R(z)f \in D(H_i), \quad \forall f \in L^2(\bar{M}_i),
\]
(2.48) \[
\chi_i R(z)f = R_i(z)\chi_i f + R_i(z)[H_i, \chi_i]R(z)f.
\]
We have thus finished preliminary consideration for the Laplacian on CMGA. We add here two more facts which are useful to study the cusp end.
2.4. Functional calculus. We introduce a formula due to Helffer-Sjöstrand [46] on the representation of functions of self-adjoint operators in terms of their resolvents. We use the following notation.

\[ C \ni z = x + iy, \quad \overline{\partial_z} = \frac{1}{2}(\partial_x + i\partial_y), \quad dzd\overline{\sigma} = -2i\sigma xdy. \]

**Lemma 2.12.** If \( f(x) \in C^\infty(\mathbb{R}) \) satisfies for some \( s \in \mathbb{R} \)

\[ |f^{(k)}(x)| \leq C_k(1 + |x|)^{s-k}, \quad \forall k \geq 0, \]

there exists \( F(z) \in C^\infty(\mathbb{C}) \), called an almost analytic extension of \( f(x) \), satisfying

\[
\begin{cases}
F(x) = f(x), & x \in \mathbb{R}, \\
|F(z)| \leq C(1 + |z|^s), & z \in \mathbb{C}, \\
|\overline{\partial_z}F(z)| \leq C_N|\text{Im } z|^N (1 + |z|)^{s-1-N}, & \forall N \geq 0, \quad z \in \mathbb{C}, \\
\text{sup} F(z) \in \{z \in \mathbb{C} : |\text{Im } z| \leq 2 + 2|\text{Re } z|\}. 
\end{cases}
\]

One can take \( F(z) \in C_0^\infty(\mathbb{C}) \), if \( f(x) \in C_0^\infty(\mathbb{R}) \).

**Lemma 2.13.** If \( f(x) \) satisfies \( 2.49 \) for some \( s < 0 \),

\[ f(A) = \frac{1}{2\pi i} \int_C \overline{\partial_z}F(z)(z - A)^{-1}dzd\overline{\sigma} \]

holds for any self-adjoint operator \( A \), where \( F \) is an almost analytic extension of \( f \).

For the proof, see [31], p. 392.

This formula is suitable to deal with the perturbation \( f(B) - f(A) \). For two self-adjoint operators \( A \) and \( B \), assume that \( D(A) = D(B) \) and \( (A - B)(i + A)^{-1} \) and \( (A - B)(i + B)^{-1} \) are bounded operators. Then, we have for any \( \varphi \in C_0^\infty(\mathbb{R}) \)

\[ \varphi(B) - \varphi(A) = \frac{1}{2\pi i} \int_C \overline{\partial_z}\varphi(z)(z - B)^{-1}(B - A)(z - A)^{-1}dzd\overline{\sigma}, \]

where \( \varphi \in C_0^\infty(\mathbb{C}) \) is an almost analytic extension of \( \varphi \). This is formally obvious by the resolvent equation. We show the convergence of the integral of the right-hand side. In fact, we have

\[ \|(B - A)(z - A)^{-1}\| \leq \|(B - A)(i + A)^{-1}\| \|((i + A)(z - A)^{-1}\| \leq C\|(i + A)(z - A)^{-1}\|. \]

By using the spectral decomposition of \( A \), we have

\[ \|(i + A)(z - A)^{-1}\| \leq \sup_{\lambda \in \mathbb{R}} \left| \frac{i + \lambda}{z - \lambda} \right| \leq C|\text{Im } z|^{-1}(1 + |z|). \]

Using the estimate

\[ |\overline{\partial_z}\varphi(z)| \leq C|\text{Im } z|^2(1 + |z|)^{-4}, \]

we see that the right-hand side of \( 2.51 \) is a bounded operator.

Returning to our manifold \( \mathcal{M} \), we pick up one end \((0, \infty) \times M \) equipped with the metric \( ds^2 = (dr)^2 + \rho(r)^2h(r, x, dx) \). Let \( h_M(x, dx) \) be the limit metric on \( M \) satisfying

\[ h(r, x, dx) - h_M(x, dx) \in S^{-\gamma} \]

for \( \gamma > 0 \). Let \( \Lambda(r) \) be the Laplace-Beltrami operator on \( M \) associated with the metric \( h(r, x, dx) \).
Lemma 2.14. For any \( \varphi \in C_0^\infty(\mathbb{R}) \), \( \varphi(\Lambda(r)) \) is strongly differentiable on \( L^2(M) \) with respect to \( r > 0 \) and satisfies

\[
\left\| \frac{d}{dr} \ell \varphi(\Lambda(r)) \right\| \leq C_n (1 + r)^{-\ell - \gamma}, \quad \forall \ell \geq 1.
\]  

\[
\left\| \frac{d^k}{dr^k} \varphi(\Lambda(r)) \right\| \leq C (1 + r)^{-1 - \gamma},
\]

Proof. By (2.2), \( D(\Lambda(r)) \) is independent of \( r > 0 \). In view of (2.71), for any \( u \in D(\Lambda(r)) \), each term in \( \Lambda(r)u \) is differentiable with respect to \( r \). Let us check it for the most delicate term \( L_Su \). Letting \( h(r, x) = s^{2(n-k-2)}h_0(r, x) \), we have \( C \leq h_0 \leq C^{-1} \) and

\[
L_S = -\sqrt{h_0} \left( \partial_s^2 + \frac{n - k - 2}{s} \partial_s \right) - \frac{\sqrt{h_0}}{\sqrt{h_0}} \partial_s.
\]

Therefore, \( L_Su \) is differentiable with respect to \( r \). This proves that \( (z - \Lambda(r))^{-1} \) is strongly differentiable with respect to \( r > 0 \). Using (2.54) and arguing as above, we obtain (2.52).

By virtue of (2.51), we have

\[
\left[ \varphi(\Lambda(r)), \frac{g'}{g} \right] = \frac{1}{2\pi i} \int_{C} \partial_z \bar{\varphi}(z)(z - \Lambda(r))^{-1} \left[ \Lambda(r), \frac{g'}{g} \right](z - \Lambda(r))^{-1} dzd\bar{z}.
\]

Since \( g'/g \in S^{-\gamma} \), \( [\Lambda(r), g'/g] \) is a 1st order differential operator with coefficients decaying like \( r^{-1-\gamma} \). This proves (2.53). \( \square \)

2.5. 1-dimensional equation. We summarize here basic facts about the 1-dimensional Helmholtz equation, since they elucidate the role of radiation condition and are also utilized in the spectral analysis of cusp.

Lemma 2.15. Consider the equation on \( \mathbb{R}^1 \):

\[-u'' - k^2 u = f \quad \text{for} \quad -\infty < r < \infty.\]

Assume that \( k > 0 \), \( u = 0 \) for \( r < 1 \), and \( f \in L^1(0, \infty) \).

(1) If \( \lim_{r \to \infty} |u'(r) - iku(r)| = 0 \), we have

\[
u(r) = \frac{i}{2k} \int_{0}^{\infty} e^{ik|r-s|} f(s) ds,
\]

\[
|u'(r) - iku(r)| \leq \int_{r}^{\infty} |f(s)| ds.
\]

(2) If furthermore \( \lim_{r \to \infty} |u(r)| = 0 \), we have

\[
|u(r)| \leq \frac{1}{k} \int_{r}^{\infty} |f(s)| ds,
\]

\[
\int_{0}^{\infty} (1 + r)^{2(s-1)} |u|^2 dr \leq C_{k,s} \int_{0}^{\infty} (1 + r)^{2s} |f(r)|^2 dr,
\]

for any \( s > 1/2 \), where the constant \( C_{k,s} \) is independent of \( k \) when \( k \) varies over a compact interval in \( (0, \infty) \).
PROOF. Let \( g_0(k, r, s) = \frac{i}{2k} e^{ik|r-s|} \). For \( 0 < r < a \), by integration by parts
\[
- \int_0^a u''(s)g_0(k, r, s)ds = u(r) + \frac{e^{ik(a-r)}}{2ik} (u'(a) - iku(a)) + k^2 \int_0^a u(s)g_0(k, r, s)ds.
\]
Using the equation, we have
\[
(2.58) \quad u(r) + \frac{e^{ik(a-r)}}{2ik} (u'(a) - iku(a)) = \frac{i}{2k} \int_0^a e^{ik|r-s|} f(s)ds.
\]
Letting \( a \to \infty \), we obtain \((2.55)\). This and \((2.58)\) yield
\[
(2.59) \quad u'(a) - iku(a) = e^{ik(r-a)} \int_a^\infty e^{ik|r-s|} f(s)ds,
\]
which implies \((2.56)\). If \( \lim_{r \to -\infty} |u(r)| = 0 \), we have \( \int_0^\infty e^{-iks} f(s)ds = 0 \), hence
\[
(2.60) \quad u(r) = \int_r^\infty \frac{\sin k(r-s)}{k} f(s)ds,
\]
which yields \((2.56)\).

Let us recall well-known Hardy’s inequality: Let \( h(r) \in L^1((0, \infty); dr) \) and put \( w(r) = \int_r^\infty h(t)dt \). Then for \( s > 1/2 \),
\[
(2.60) \quad \int_0^\infty r^{2(s-1)}|w(r)|^2dr \leq \frac{4}{(2s-1)^2} \int_0^\infty r^{2s}|h(r)|^2dr.
\]
(See e.g. [51], p. 106). This and \((2.59)\) imply \((2.57)\).

3. Transformation of the metric

We pick up one end \( M_i \) of \( M \), omit the subscript \( i \) and consider the perturbed warped product metric
\[
(3.1) \quad ds^2 = (dr)^2 + \rho(r)^2 h(r, x, dx)
\]
on \((0, \infty) \times M \) having the property
\[
(3.2) \quad h(r, x, dx) - h_M(x, dx) \in S^{-\gamma}.
\]
The perturbation term \( h(r, x, dx) - h_M(x, dx) \) is said to be short-range if \( \gamma > 1 \) in \((3.2)\), and long-range if \( 0 < \gamma \leq 1 \). In the study of spectral properties of the associated Laplace operator, the former can be dealt with by the standard perturbation technique, however the latter requires involved analysis. In this section, given a metric
\[
(3.3) \quad ds^2 = a(t, z)(dt)^2 + 2w(t)b_i(t, z)dt dz^i + w(t)^2 c_{ij}(t, z)dz^i dz^j,
\]
where \( z = (z^1, \cdots, z^{n-1}) \) are local coordinates on \( M \), we seek the condition under which we can transform this metric into the perturbed warped product form \((3.1)\).

The decay order \( \gamma \) in \((3.2)\) will be affected by the growth order of the volume of the manifold. For the sake of simplicity, we consider the case in which \( M \) has no conic singularities hence the metric is \( C^\infty \). Let \( \{U_j\}_{j \in J} \) be a finite open covering of \( M \), each \( U_j \) being diffeomorphic to a bounded open subset \( V_j \subset \mathbb{R}^{n-1} \). We assume that there exists an \( r_0 > 0 \) such that for \( r > r_0, M \) is covered by \( \{(r_0, \infty) \times U_j\}_{j \in J} \). We take one of \( U_j \), assume that \( z \in V_j \) and omit the subscript \( j \).
Assume that
\begin{align}
(3.4) & \quad w(t)^{-1} \in S^{-\kappa}, \\
(3.5) & \quad a(t, z) - 1 \in S^{-\lambda}, \\
(3.6) & \quad b_i(t, z) \in S^{-\mu}, \\
(3.7) & \quad c_{ij}(t, z) - h_{ij}(z) \in S^{-\nu},
\end{align}
where \(\kappa, \lambda, \mu, \nu\) are constants such that
\begin{align}
(3.8) & \quad \kappa > 1/2, \quad \lambda > 1, \quad \mu > 0, \quad \nu > 0, \quad \kappa + \mu > 1,
\end{align}
\(h_{ij}(z)dz^idx^j\) is a \(C^\infty\) metric on \(M\), and \(S^\kappa\) is defined as in (0.9).
Letting \(y = (t, z)\), we rewrite (3.3) as
\begin{align}
ds^2 &= g_{ij}dy^i dy^j,
\end{align}
where \(g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix} \begin{pmatrix} a & \tilde{b} \\ \tilde{b} & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & w^{-1} \end{pmatrix},
\end{align}
where \(\tilde{a} - 1 \in S^{-\lambda}, \quad \tilde{b} \in S^{-\mu}, \quad \tilde{c} - \tilde{h} \in S^{-\nu}, \quad \tilde{h} = (h_{ij})^{-1}\).
Define the classical Hamiltonian \(H\) by
\begin{align}
H(t, z, \tau, \zeta) &= \tilde{a}\tau^2 + 2w^{-1}\tilde{b}\tau\zeta_i + w^{-2}\tilde{c}^{ij}\zeta_i\zeta_j.
\end{align}
We rewrite it as
\begin{align}
H(t, z, \tau, \zeta) &= \tau^2 + K(t, z, \tau, \zeta),
\end{align}
\begin{align}
K &= (\tilde{a} - 1)\tau^2 + 2w^{-1}\tilde{b}\tau\zeta_i + w^{-2}\tilde{c}^{ij}\zeta_i\zeta_j.
\end{align}
By (3.8), we have as a function of \(t, z\)
\begin{align}
(3.9) & \quad m = \min\{\lambda, \kappa + \mu, 2\kappa\}.
\end{align}
The assumption (3.8) implies
\begin{align}
(3.10) & \quad m = 1 + \epsilon_0, \quad \epsilon_0 > 0.
\end{align}
Letting \(t' = \frac{d}{dr}\), we solve the Hamilton equation
\begin{align}
(3.11) & \quad \begin{cases}
\dot{t} = \frac{\partial H}{\partial \tau}, & \dot{\tau} = \frac{\partial H}{\partial \zeta}, \\
\dot{z} = -\frac{\partial H}{\partial t'} & \dot{z} = -\frac{\partial H}{\partial t}
\end{cases},
\end{align}
with the condition at infinity
\begin{align}
(3.12) & \quad t(r) - r \to 0, \quad z(r) \to x, \quad \tau(r) \to 1/2, \quad \zeta(r) \to 0,
\end{align}
where $x \in V$. Note that, since $\frac{\partial H}{\partial \tau} \to 2\tau$ as $t \to \infty$, the condition $\tau(r) \to 1/2$ is compatible with $t(r) - r \to 0$. Then, $\tau$, $z$ and $\zeta$ should satisfy the integral equations

$$2\tau - 1 = 2 \int_{r}^{\infty} \frac{\partial H}{\partial t} dr',$$

$$z - x = - \int_{r}^{\infty} \frac{\partial H}{\partial \zeta} dr',$$

$$\zeta = \int_{r}^{\infty} \frac{\partial H}{\partial z} dr',$$

and $t$ should satisfy the integro-differential equation

$$\frac{dt}{dr} = 1 + 2 \int_{r}^{\infty} \frac{\partial H}{\partial t} dr' + \frac{\partial K}{\partial \tau}.$$ 

Letting

$$X(r) = X(r, x) = (t(r) - r, z(r) - x, \tau(r) - 1/2, \zeta(r)),$$

$$\tilde{X}(r) = \tilde{X}(r, x) = (t(r), z(r), \tau(r), \zeta(r)),$$

$$X_\infty(r) = X_\infty(r, x) = (r, x, 1/2, 0),$$

(hence $\tilde{X}(r) = X(r) + X_\infty$), we put

$$U_0(X(r)) = 2 \int_{r}^{\infty} \frac{\partial H}{\partial t} (\tilde{X}(r')) dr' + \frac{\partial K}{\partial \tau} (\tilde{X}(r)),$$

$$U(X(r)) = \int_{r}^{\infty} \left( -U_0(\tilde{X}(r')) - \frac{\partial H}{\partial \zeta} (\tilde{X}(r')) - \frac{\partial H}{\partial t} (\tilde{X}(r')) - \frac{\partial H}{\partial z} (\tilde{X}(r')) \right) dr'.$$

Then, $t$ should satisfy

$$t - r = - \int_{r}^{\infty} U_0(X(r')) dr'.$$

Note that

$$\int_{r}^{\infty} U_0(X(r')) dr' = \int_{r}^{\infty} \left( 2(r' - r) \frac{\partial K}{\partial t} (\tilde{X}(r')) + \frac{\partial K}{\partial \tau} (\tilde{X}(r')) \right) dr'.$$

By (3.14) and (3.15), the differential equation (3.12) with the condition at infinity (3.13) is converted to the integral equation

$$X(r, x) = U(X_\infty(r, x) + X(r, x)).$$

Define the norm $\| \cdot \|$ by

$$\|X\| = \sup_{(r, x) \in (r_0, \infty) \times V} |X(r, x)|.$$

The conditions (3.10) and (3.11) imply that for a sufficiently small $\epsilon, r_0^{-1} > 0$, $U$ is a contraction mapping in the ball

$$B_{\epsilon, r_0} = \{ X \in C((r_0, \infty) \times V); \|X\| \leq \epsilon \}.$$ 

We then have:
Lemma 3.1. There exists a unique solution $X(r, x)$ of the equation (3.16). It satisfies

$$\left| \partial_{\ell} ^{2} \partial_\alpha ^{2} X(r, x) \right| \leq C_\ell r^{-\ell - \alpha}, \quad \forall \ell, \alpha.$$  

Moreover, the differential of the map $(t, z) \rightarrow (r, x)$ is $1 + O(r^{-\alpha}).$

Lemma 3.2. As a 2-form with respect to $r, x$, we have

$$d\tau \wedge dt + \sum_{i=1}^{n-1} d\zeta_i \wedge dz^i = 0.$$  

Proof. Let $y = (t, z), \eta = (\tau, \zeta)$ and $\theta = (r, x)$. Then

$$d\tau \wedge dt + \sum_{i=1}^{n-1} d\zeta_i \wedge dz^i = \sum_{j<k} [\eta, y]_{jk} d\theta^j \wedge d\theta^k,$$

where

$$[\eta, y]_{jk} = \frac{\partial \eta}{\partial \theta^j} \cdot \frac{\partial y}{\partial \theta^k} - \frac{\partial \eta}{\partial \theta^k} \cdot \frac{\partial y}{\partial \theta^j}.$$  

Noting that

$$\frac{\partial}{\partial r} \left( \frac{\partial \eta}{\partial \theta^j} \cdot \frac{\partial y}{\partial \theta^k} \right) = -\frac{\partial^2 H}{\partial y^m \partial \theta^j} \frac{\partial y^m}{\partial \theta^k} + \frac{\partial^2 H}{\partial \eta^m \partial \theta^j} \frac{\partial \eta^m}{\partial \theta^k}$$

is symmetric with respect to $j$ and $k$, we have $\frac{\partial}{\partial r} [\eta, y]_{jk} = 0$. Lemma 3.1 implies $[\eta, y]_{jk} \rightarrow 0$ as $r \rightarrow \infty$. Hence $[\eta, y]_{jk} = 0$, which proves Lemma 3.2. □

By Lemma 3.1, the map $(r, x) \rightarrow (t, z)$ is a global diffeomorphism on $(r_0, \infty) \times M$. We invert it to get $r = r(t, z), \quad x = x(t, z), \quad \tau = \tau(t, z), \quad \zeta = \zeta(t, z)$. Lemma 3.2 implies

$$\frac{\partial \zeta_j}{\partial z^k} = \frac{\partial \zeta_k}{\partial z^j}, \quad \frac{\partial \zeta_j}{\partial z^j} = \frac{\partial \tau}{\partial z^j}, \quad 1 \leq j, k \leq n - 1.$$  

Recalling that

$$\tau(r) - \frac{1}{2} = \int_r^\infty \frac{\partial H}{\partial t}(\tilde{X}(r')) dr' = O(r^{-1-\alpha}),$$

we define

$$\varphi(t, z) = \frac{t}{2} - \int_t^\infty \left( \tau(t', z) - \frac{1}{2} \right) dt'.$$

Lemma 3.3. There exists $t_0 > 0$ such that for $t > t_0$,

(1) $\partial_t \varphi(t, z) = \tau(t, z),$
(2) $\partial_z \varphi(t, z) = \zeta(t, z),$
(3) $H(t, z, \partial_t \varphi(t, z), \partial_z \varphi(t, z)) = 1/4,$
(4) $|\partial_{\ell} \partial_\alpha (\varphi(t, z) - t/2)| \leq C_\ell t^{-\ell - \alpha}, \quad \forall \ell, \alpha,$
(5) $\varphi(t, z) = r(t, z)/2.$

Proof. The assertion (1) is obvious, and (2) follows from

$$\frac{\partial \varphi}{\partial z^j} = -\int_t^\infty \frac{\partial \tau}{\partial z^j} dt' = -\int_t^\infty \frac{\partial \zeta_j}{\partial t'} dt' = \zeta_j(t, z).$$
Since the energy is conserved, \( H(t, z, \tau, \zeta) \) is constant along the orbit, which turns out to be \( 1/4 \) by letting \( r \to \infty \). The assertion (4) follows from (3.18) and (3.10).

Using (1), (2) and (3), we have

\[
\frac{\partial \varphi}{\partial r} = \frac{\partial \varphi}{\partial t} \frac{\partial t}{\partial r} + \frac{\partial \varphi}{\partial z} \frac{\partial z}{\partial r} = \tau \frac{\partial t}{\partial r} + \zeta \frac{\partial z}{\partial r} = \tau \frac{\partial H}{\partial \tau} + \zeta \frac{\partial H}{\partial \zeta} = 2H = 1/2.
\]

Therefore, \( \varphi(t, z) - r/2 \) is independent of \( r \). On the other hand \( \varphi(t, z) - r/2 \to 0 \) as \( r \to \infty \). This proves \( \varphi = r/2 \). \( \square \)

The diffeomorphism \((r, x) \to (t, z)\) induces \( r \)-dependent local coordinates \( z = z(r, x) \) on \( M \). As \( r \to \infty \), they converge to local coordinates \( z(x) \) on \( M \).

**Theorem 3.4.** Assume \((3.4) \sim (3.5)\), and \( h_{ij} \) be as in (3.7). Then, in the coordinate system \((r, x)\), the Riemannian metric (3.3) is written as

\[
ds^2 = (dr)^2 + w(r)h(r, x, dx)
\]

where \( h(r, x, dx) \) is a Riemannian metric on \( M \) and satisfies

\[
(3.20) \quad h_{ij}(r, x) - h_{ij}(z(x)) \in S^{-\min(\nu, \epsilon_0)}.
\]

**Proof.** We put \( y = (t, z) \), \( \overline{y} = (r, x) \). Then, the Hamiltonian is written as

\[
H = g^{ij}(y)\eta_i \eta_j = \overline{g}^{ij}(\overline{y})\overline{\eta}_i \overline{\eta}_j,
\]

where \( \eta = (\tau, \zeta) \). Using Lemma 3.3, we have

\[
\overline{g}^{00} = g^{ij} \frac{\partial \overline{y}^j}{\partial y^i} \frac{\partial \overline{y}^0}{\partial y^j} = g^{ij} \frac{\partial \overline{y}^j}{\partial y^i} \frac{\partial r}{\partial \overline{y}^j} = 4g^{ij} \frac{\partial \varphi}{\partial y^i} \frac{\partial \varphi}{\partial y^j} = 4H = 1,
\]

\[
\overline{g}^{0k} = g^{ij} \frac{\partial \overline{y}^j}{\partial y^i} \frac{\partial \overline{y}^k}{\partial y^j} = g^{ij} \frac{\partial \overline{y}^j}{\partial y^i} \frac{\partial x^k}{\partial \overline{y}^j} = 0, \quad 1 \leq k \leq n - 1.
\]

Here, in the 2nd line, we have used

\[
0 = \frac{\partial x^k}{\partial r} = \frac{\partial x^k}{\partial y^i} \frac{\partial y^i}{\partial r} = \frac{\partial x^k}{\partial y^j} g^{ij} \eta_i = \frac{1}{2} \frac{\partial x^k}{\partial y^j} g^{ij} \frac{\partial \varphi}{\partial y^i}.
\]

Therefore, the Riemannian metric has the form

\[
ds^2 = (dr)^2 + \sum_{1 \leq i, j \leq n - 1} \overline{g}_{ij} dx^i dx^j.
\]

We observe the matrix \((\overline{g}_{ij})_{1 \leq i, j \leq n - 1}\) with

\[
\overline{g}_{ij} = g_{00} \frac{\partial y^i}{\partial \overline{y}^j} \frac{\partial \overline{y}^0}{\partial \overline{y}^j} + 2g_{0k} \frac{\partial y^i}{\partial \overline{y}^j} \frac{\partial \overline{y}^k}{\partial \overline{y}^j} + g_{k\ell} \frac{\partial x^i}{\partial \overline{y}^j} \frac{\partial x^\ell}{\partial \overline{y}^j}.
\]

Since \( g_{00} = O(1) \), \( g_{0k} = O(w) \) and \( g_{k\ell} = w^2 c_{k\ell} \), we have

\[
\overline{g}_{ij} = w^2 (c_{ij} + O(r^{-\epsilon_0})),
\]

from which we can derive the lemma. \( \square \)
The above theorem says that the metric of the form (3.3) can be transformed to (3.19) if
\[
W(t) \sim \exp \left( c_0 t + \frac{\beta}{1 - \alpha} t^{1 - \alpha} \right), \quad \text{or} \quad t^\beta \quad \text{with} \quad \beta > 1/2.
\]
Let us consider the case \( W(t) = t^\beta \) for large \( t \). Then, (3.19) is a short-range perturbation of \((dr)^2 + W(r)^2 h_M\) when \( \min\{\nu, \epsilon_0\} > 1 \), and long-range perturbation when \( \min\{\nu, \epsilon_0\} \leq 1 \). Since \( \kappa = \beta \), (3.19) is a short-range perturbation of \((dr)^2 + W(r)^2 h_M\) only when \( \beta > 1 \). For \( 1/2 < \beta \leq 1 \), it is a long-range perturbation. The border-line case appears when the metric is Euclidean. This fact is pointed out by Bouclet \[14\]. He also mentions that there appears a conformal factor in front of \( h_{ij}(z(x)) \) in Theorem 3.4. This is because he solves the Hamilton equation as an initial value problem with data on a surface \( \{t = t_0\} \). If, as has been done above, we solve it as the Cauchy problem from infinity, the conformal factor does not appear.

4. Rellich-Vekua type theorem

4.1. Volume growth condition. The aim of this section is to derive a decay rate of solutions to the Helmholtz equation near infinity of an end \((0, \infty) \times M\), which is crucial to study the spectral theory for the Laplacian, in particular for the discreteness or the non-existence of eigenvalues embedded in the continuous spectrum. This property is closely related to the volume growth of the manifold at infinity. We consider only the growing metric satisfying the following assumption:

\((VG)\) There exist a non-negative constant \( c_0 \) and positive constants \( \alpha_0, \beta_0, \gamma_0, r_0 \) such that
\[
\begin{align*}
\rho' - \rho &= c_0 \in S^{-\alpha_0}, \\
\frac{\rho'}{\rho} &\geq \beta_0, \quad r > r_0, \\
h^{ij}(r, x) - h_{ij}^M(x) &\in S^{-\gamma_0}.
\end{align*}
\]
Note that (A-2) and (A-3) imply (VG), and that (VG) yields
\[
\rho(r) \geq \rho(r_0) \left( \frac{r}{r_0} \right)^{\beta_0}, \quad r > r_0.
\]
Letting \( h = \det h(r, x, dx) \), we have \( g = \rho^{2(n-1)} h \), and the assumption on \( h^{ij} \) implies \( h' = O(r^{-1-\nu}) \). Therefore, we can reformulate (4.1) in terms of \( g \):

\((VG)'\) There exist a non-negative constant \( c_0 \) and positive constants \( \alpha_0, \beta_0, r_0 \) such that
\[
\begin{align*}
\frac{g'}{4g} - \frac{(n-1)c_0}{2} &\in S^{-\alpha_0}, \\
\frac{g'}{4g} &\geq \frac{(n-1)\beta_0}{2r}, \quad r > r_0, \\
h^{ij}(r, x) - h_{ij}^M(x) &\in S^{-\gamma_0}.
\end{align*}
\]
Let $S(r) = \{r\} \times M$ and $dS(r)$ be the surface element of $S(r)$ induced from the metric $ds^2$. The volume element $dV$ of the end $(0, \infty) \times M$ is then written as

$$dV = drdS(r).$$

Let $\| \cdot \|$ and $(\cdot, \cdot)$ be the norm and the inner product of $L^2(M)$. We say that $u$ is locally in $D(-\Delta_M)$ if $\chi(r)u \in D(-\Delta_M)$ for any $\chi(r) \in C_0^\infty((0, \infty))$. Then the following theorem holds.

**Theorem 4.1.** Assume $(VG)$ with \(\alpha_0 > 0, \quad \beta_0 > 0, \quad \gamma_0 > 0.\) Let $E_0 = ((n-1)c_0/2)^2$. Suppose $u$ is locally in $D(-\Delta_M)$ and satisfies

$$(-\Delta_M - E)u = 0, \quad \text{for} \quad r > R$$

for some constants $E > E_0$ and $R > 0$. If $u$ satisfies

$$\liminf_{r \to \infty} r^\gamma \int_{S(r)} \left( |u'|^2 + |u|^2 \right) dS(r) = 0$$

for a constant $\gamma > 0$, then

$$\int_{r>R} r^m \left( \|u\|^2 + (Bu, u) + \|u\|^2 \right) dV < \infty$$

for any $m > 0$, where the operator $B = B(r)$ is defined in (4.11).

As will be shown later, this theorem yields the discreteness of embedded eigenvalues in the continuous spectrum for metrics having growth order $\beta > 0$. Much more significant is the following Rellich-Vekua type theorem which proves the non-existence of embedded eigenvalues and also plays an important role in the inverse problem, however with the trade-off of losing slowly growing metrics of order $0 < \beta \leq 1/3$.

**Theorem 4.2.** Assume $(VG)$ with \(\alpha_0 > 0, \quad \beta_0 > 1/3, \quad \gamma_0 > 0.\) Let $E_0 = ((n-1)c_0/2)^2$. Suppose $u$ is locally in $D(-\Delta_M)$ and satisfies

$$(-\Delta_M - E)u = 0, \quad \text{for} \quad r > R$$

for some constants $E > E_0$ and $R > 0$. If $u$ satisfies

$$\liminf_{r \to \infty} \int_{S(r)} \left( |u'|^2 + |u|^2 \right) dS(r) = 0,$$

there exists a constant $R_1 > 0$ such that $u = 0$ for $r > R_1$.

For the other results on this type of theorem on non-compact manifolds, see Kumura [70, 71], Ito-Skibsted [58] and the references therein. Note that these works do not deal with the manifold with conic singularities. Below, we derive this theorem from growth properties of solutions to an abstract ordinary differential equation with operator-valued coefficients.
4.2. Abstract differential equations. Let $X$ be a Hilbert space and consider the following differential equation for an $X$-valued function $u(t)$:

$$
- u''(t) + B(t)u(t) + V(t)u(t) - \lambda u(t) = 0, \quad t > 0,
$$

where $\lambda > 0$ is a constant. We assume the following conditions (B) and (V):

(B) For each $t > 0$, $B(t)$ is a non-negative self-adjoint operator on $X$ having the following properties:

- the domain $D = D(B(t))$ of $B(t)$ is independent of $t > 0$,
- the domain $D_1 = D(\sqrt{B(t)})$ of $\sqrt{B(t)}$ is independent of $t > 0$,
- for any $x \in D$, $(B(t)x, x)$ is differentiable with respect to $t > 0$,
- there exists a closed form $b_t(\cdot, \cdot)$ on $D_1$ such that

$$
\frac{d}{dt}(B(t)x, x) = b_t(x, x), \quad \forall x \in D, \quad \forall t > 0,
$$

- there exist positive constants $\delta, C, \epsilon_0$ such that

$$
\frac{d}{dt}(B(t)x, x) \leq -\delta (B(t)x, x) + Ct^{-\epsilon_0}(x, x), \quad \forall x \in D, \quad \forall t > 0.
$$

We denote this quadratic form $b_t(x, y)$ by $(\frac{d}{dt}B(t)x, y)$ or $(B'(t)x, y)$ and write (4.7) as

$$
t\frac{d}{dt}B(t) \leq -\delta B(t) + Ct^{-\epsilon_0}, \quad \forall t > 0.
$$

(V) For each $t > 0$, $V(t)$ is bounded self-adjoint on $X$, $V(t) \in C^1(0, \infty); B(X))$, and satisfies

$$
\|V(t)\| + t\left\|\frac{dV(t)}{dt}\right\| \leq Ct^{-\epsilon_0}, \quad \forall t \geq 1,
$$

for some constants $C > 0$, where $\| \cdot \|$ is the norm in $B(X)$.

By the solution $u(t)$ of (1.10), we mean that $u(t) \in C^1((0, \infty); X)$, $u''(t) \in X$ exists almost everywhere on $(0, \infty)$, $u''(t) \in L^2_{loc}(0, \infty); X)$ and

$$
u'(t) = u'(t_0) + \int_{t_0}^{t} u''(s)ds
$$

holds for all $t > t_0 > 0$. Moreover, $u(t) \in D$ for any $t > 0$, and the equation (1.10) is satisfied on $(0, \infty)$.

**Theorem 4.3.** Assume (B), (V) and for some $\gamma > 0$

$$
\liminf_{t \to \infty} t^\gamma (\|u'(t)\| + \|u(t)\|) = 0.
$$

Then

$$
\int_{0}^{\infty} t^m \left(\|u''\|^2 + (Bu, u) + \|u\|^2\right) dt < \infty
$$

for all $m > 0$, where $B = B(t)$ is defined in (4.7).

The Rellich type theorem is proven by restricting the range of $\delta$.

**Theorem 4.4.** Assume (B) with $\delta > 2/3$, (V) and

$$
\liminf_{t \to \infty}(\|u'(t)\| + \|u(t)\|) = 0.
$$

Then there exists $t_2 > 0$ such that $u(t) = 0$, for $t > t_2$. 

Let us derive Theorems 4.1 and 4.2 from Theorems 4.3 and 4.4. Let $g = \rho^{2(n-1)/2}$, where

$$h = h(r, x) = \det (h_{ij}(r, x)), \quad h_M = h_M(x) = \det (h_{ij,M}(x)).$$

Then, the Laplacian is rewritten as

$$-\Delta_M = -\partial_r^2 - \frac{\partial_r g}{2g} \partial_r + B(r), \quad B(r) = -\frac{1}{\sqrt{g}} \partial_x i (\rho(r)^{-2} \sqrt{g} h^{ij}(r, x) \partial_x j).$$

Letting $' = \partial_r$ and noting that

$$\left( g \right)^{1/4} \left( \partial_r^2 + \frac{g'}{2g} \partial_r \right) g^{-1/4} = \partial_r^2 - \left( \frac{g'}{4g} \right)^2 - \left( \frac{g'}{4g} \right)'$$

we transform $\Delta_M$ as

$$\left( g \right)^{1/4} \Delta_M \left( \frac{g}{h_M} \right)^{-1/4} = -\partial_r^2 + B_0(r) + C(r),$$

$$B_0(r) = \left( \frac{g}{h_M} \right)^{1/4} B(r) \left( \frac{g}{h_M} \right)^{-1/4},$$

$$C(r) = \left( \frac{g'}{4g} \right)^2 + \left( \frac{g'}{4g} \right)'$$

Let $X = L^2(M)$ equipped with the inner product

$$(u, v)_X = \int_M u(x) v(x) \sqrt{h_M(x)} dx.$$

Then, the transformation

$$L^2((0, \infty) \times M) \ni u \to v = \left( \frac{g}{h_M} \right)^{1/4} u \in L^2((0, \infty); X; dr)$$

is unitary, $B_0(r)$ is self-adjoint on $X$ with domain characterized in Theorem 2.2.

We put

$$V(t) = C(t) - E_0, \quad \lambda = E - E_0.$$

Then, by virtue of Theorem 2.2 the solution of the equation $-\Delta_M u = Eu$ is transformed to an $X$-valued solution of the ordinary differential equation

$$(-\partial_t^2 + B_0(t) + V(t)) v = \lambda v, \quad v = \left( g / h_M \right)^{1/4} u.$$

We show that this differential equation satisfies the assumptions (B), (V). Note that by (4.2)

$$\frac{g'}{4g} - \sqrt{E_0} \in S^{-\alpha_0},$$

which implies

$$\|V(t)\| \leq Ct^{-\alpha_0}, \quad \|\partial_t V(t)\| \leq Ct^{-1-\alpha_0}.$$

By Theorem 2.2 $D(B_0(t))$ is independent of $t > 0$, and by (1.17), $D(\sqrt{B_0(t)}) = H^1(M)$. Passing to the quadratic form, we see that $(B_0(t)) x, x$ is differentiable with respect to $t > 0$. 


Lemma 4.5. Let $0 < \delta < 2\beta_0$. Then, there exist positive constants $C, t_0$ such that 

$$\frac{t dB_0(t)}{dt} + \delta B_0(t) \leq Ct^{-\gamma_0}, \quad t > t_0.$$ 

Proof. Rewrite the left-hand side as 

$$\frac{t dB_0(t)}{dt} + \delta B_0(t) = \frac{t dB_0(t)}{dt} + 2\beta_0 B_0(t) - (2\beta_0 - \delta) B_0(t).$$

Since $B_0(r) = (h/h_M)^{1/4} B(r)(h/h_M)^{-1/4}$, we have for $u \in D(B_0(t))$

$$(B_0(t)u, u) = \sum_{i,j} \left( \frac{h^{ij}}{\rho(t)^2} \partial_x^i u, \partial_x^j u \right) + \sum_{|\alpha| \leq 1} \left( b_{1,\alpha} \partial_x^\alpha u, \partial_x^\alpha u \right),$$

where $b_{1,\alpha}$ behaves like 

$$b_{1,\alpha} = O(t^{-1-\gamma_0} \rho(t)^{-2}).$$

Therefore,

$$\left( \frac{dB_0(t)}{dt} u, u \right) = - \sum_{i,j} \left( c^{ij}(t, x) \partial_x^i u, \partial_x^j u \right) + \cdots, \quad c^{ij}(t, x) = \partial_t \left( \frac{h^{ij}}{\rho(t)^2} \right).$$

By (4.1),

$$c^{ij}(t, x) = -2 \left( \frac{h^{ij}}{\rho(t)^2} \right) \rho'(t) + O(t^{-1-\gamma_0} \rho(t)^{-2}).$$

We then have, again using (4.1),

$$c^{ij} \leq - \frac{2\beta_0}{t} \left( \frac{h^{ij}}{\rho(t)^2} \right) + O(t^{-1-\gamma_0} \rho(t)^{-2}).$$

Therefore, we have in the sense of quadratic form

$$\frac{t dB_0(t)}{dt} + 2\beta_0 B_0(t) \leq B_1(t),$$

where $B_1(t)$ is a 2nd order differential operator on $X$ whose coefficients decay like $t^{-\gamma_0} \rho(t)^{-2}$. By the positive definiteness of the matrix $(h^{ij})$, we have

$$B_1(t) \leq (2\beta_0 - \delta) B_0(t) + Ct^{-\gamma_0}$$

for a constant $C > 0$, which proves the lemma. \qed

Therefore, (B) and (V) are justified.

4.3. Proof of Theorem 4.3. The following method based on integration by parts is essentially due to Eidus [32]. Let $(\cdot, \cdot), \parallel \cdot \parallel$ be the inner product and the norm of $X$, respectively. For $0 < a < b < \infty$, we put

$$J_{(a,b)}(u, v) = \int_a^b (u(t), v(t)) \, dt,$$

$$S_t(u, v) = (u(t), v(t)),$$

$$S(u, v) \bigg|_a^b = S_b(u, v) - S_a(u, v).$$

Note the formula

$$S(u, v) \bigg|_a^b = J_{(a,b)}(u', v) + J_{(a,b)}(u, v').$$
Take a real-valued $C^\infty$-function $d(t)$ on $(0, \infty)$ and put $v(t) = e^{d(t)}u(t)$, where $u$ is a solution to (4.6). Then, it satisfies
\begin{equation}
\tag{4.16}
-v''(t) + 2d'(t)v'(t) + (B(t) + q(t))v(t) = 0,
q(t) = V(t) + d''(t) - d'(t)^2 - \lambda.
\end{equation}

In the following arguments, we compute under the additional assumption that $(v'(t), v(t))$, etc. are real-valued, for the sake of simplicity. This is not essential at all. In fact, we have only to take the real part in the following formulas. Or, in the practical applications, say $X = L^2(\mathbf{u})$, we have only to take the real part of $v$.

We start from the following identities.

**Lemma 4.6.** Let $v(t) = e^{d(t)}u(t)$, and $\psi = \psi(t)$ a positive $C^\infty$-function on $(0, \infty)$. Then we have for $0 < a < b$
\begin{equation}
\tag{4.17}
S(\psi'v', v')\bigg|_a^b - S(\psi(B + q)v, v)\bigg|_a^b
= J_{(a, b)}\left((\psi' + 4d'\psi)v', v'\right) - J_{(a, b)}\left((\psi(B + q))'v, v\right),
\end{equation}
\begin{equation}
\tag{4.18}
S(\psi'v, v)\bigg|_a^b - S(\psi d'v, v)\bigg|_a^b - \frac{1}{2}S(\psi' v, v)\bigg|_a^b
= J_{(a, b)}\left((\psi' + 1)v, (d''\psi + d'\psi')v + \frac{1}{2}\psi''\right).
\end{equation}

**Proof.** Take the inner product of (4.16) and $\psi v'$. By integration by parts, we have
\begin{equation*}
S(v', \psi v')\bigg|_a^b = 2J_{(a, b)}(v', \psi v') + J_{(a, b)}(v', \psi v'),
S((B + q)v, \psi v)\bigg|_a^b = 2J_{(a, b)}((B + q)v, \psi v') + J_{(a, b)}((\psi(B + q))'v, v),
\end{equation*}
which yield (4.17). Next note that
\begin{equation*}
J_{(a, b)}(v'', \psi v) = S(v', \psi v)\bigg|_a^b - \frac{1}{2}S(v, \psi' v)\bigg|_a^b + \frac{1}{2}J_{(a, b)}(v, \psi'' v) - J_{(a, b)}(v', \psi v').
\end{equation*}
Using this and taking the inner product of (4.16) with $\psi v$, we obtain
\begin{equation*}
S(v', \psi v)\bigg|_a^b - \frac{1}{2}S(v, \psi' v)\bigg|_a^b + \frac{1}{2}J_{(a, b)}(v, \psi'' v) - J_{(a, b)}(v', \psi v')
= 2J_{(a, b)}(d'v', \psi v) + J_{(a, b)}(\psi(B + q)v, v).
\end{equation*}
By integration by parts,
\begin{equation*}
2J_{(a, b)}(d'v', \psi v) = S(v, d'\psi v)\bigg|_a^b - J_{(a, b)}(v, (d''\psi + d'\psi')v).
\end{equation*}
These three formulas imply (4.18) \qed.

Our main task is to increase the decay order of $\|u(t)\|$ step by step.
Lemma 4.7. Assume that for some $\alpha < \delta$

\begin{equation}
\liminf_{t \to \infty} t^\alpha \left( \|u'(t)\|^2 + \|u(t)\|^2 \right) = 0.
\end{equation}

Then we have

\begin{equation}
\liminf_{t \to \infty} t^\alpha (B(t)u(t), u(t)) = 0.
\end{equation}

Proof. Take $\psi = t^\alpha$, $d = 0$ in (4.17). Noting that $q = V - \lambda$, we have

\begin{equation}
S(t^\alpha u', u')_a - S(t^\alpha (B + V)u, u)_a + \lambda S(t^\alpha u, u)_a
\end{equation}

\begin{equation}
= \alpha J_{(a,b)}(t^\alpha u', u') - J_{(a,b)}((t^\alpha (B + V))'u, u) + \lambda \alpha J_{(a,b)}(t^\alpha u, u).
\end{equation}

Note that by (4.18),

\begin{equation}
-(t^\alpha B)' = -t^{\alpha - 1}(tB' + \delta B) - (\alpha - \delta)t^\alpha B \geq -Ct^{\alpha - \epsilon_0} + (\delta - \alpha)t^{\alpha - 1}B.
\end{equation}

By the assumption (4.8), the right-hand side of (4.20) is estimated from below by

\begin{equation}
\alpha J_{(a,b)}(t^\alpha u', u') - \lambda \alpha J_{(a,b)}(t^\alpha u, u)
\end{equation}

\begin{equation}
+ (\delta - \alpha) J_{(a,b)}(t^{\alpha - 1}Bu, u) + \lambda \alpha J_{(a,b)}(t^{\alpha - 1}u, u).
\end{equation}

Taking $\alpha$ large enough (independently of $u$), this is estimated from below by

\begin{equation}
\kappa_0 \left( J_{(a,b)}(t^{\alpha - 1}u', u') + J_{(a,b)}(t^{\alpha - 1}Bu, u) + J_{(a,b)}(t^{\alpha - 1}u, u) \right),
\end{equation}

where

\begin{equation}
\kappa_0 = \min \{\alpha, \delta - \alpha, \lambda \alpha / 2\}.
\end{equation}

We have by (4.21)

\begin{equation}
S_b(t^\alpha u', u') - S_b(t^\alpha V u, u) + S_a(t^\alpha (B + V)u, u) + \lambda S_b(t^\alpha u, u)
\end{equation}

\begin{equation}
\geq S_b(t^\alpha Bu, u) + S_a(t^\alpha u', u') + \lambda S_a(t^\alpha u, u)
\end{equation}

\begin{equation}
+ \kappa_0 \left( J_{(a,b)}(t^{\alpha - 1}u', u') + J_{(a,b)}(t^{\alpha - 1}Bu, u) + J_{(a,b)}(t^{\alpha - 1}u, u) \right).
\end{equation}

Noting that

\begin{equation}
CS_b(t^{\alpha - \epsilon_0}u, u) \geq -S_b(t^\alpha V u, u), \quad S_b(t^\alpha Bu, u) \geq 0,
\end{equation}

we have by using the assumption of the lemma and letting $b \to \infty$ in (4.21) along a suitable sequence

\begin{equation}
S_a(t^\alpha (B + V)u, u) \geq S_a(t^\alpha u', u') + \lambda S_a(t^\alpha u, u)
\end{equation}

\begin{equation}
+ \kappa_0 \left( J_{(a,\infty)}(t^{\alpha - 1}u', u') + J_{(a,\infty)}(t^{\alpha - 1}Bu, u) + J_{(a,\infty)}(t^{\alpha - 1}u, u) \right).
\end{equation}

This implies $J_{(a,\infty)}(t^{\alpha - 1}Bu, u) < \infty$, hence $\liminf_{t \to \infty} S_t(t^\alpha Bu, u) = 0$. In fact, if $g(t) \geq 0$, $\int_a^\infty g(t)dt < \infty$ implies that $\liminf_{t \to \infty} tg(t) = 0$.

The following lemma follows easily from (4.22).

Lemma 4.8. Assume $\alpha < \delta$ and (4.19). Then, there exist constants $C, a_0 > 0$ such that

\begin{equation}
t^\alpha ((B + V)u, u) \bigg|_{t = a} \geq t^\alpha \left( \|u'\|^2 + \lambda \|u\|^2 \right) \bigg|_{t = a}
\end{equation}

\begin{equation}
+ C \int_a^\infty t^{\alpha - 1} \left( \|u'\|^2 + (Bu, u) + \|u\|^2 \right)dt
\end{equation}
for any \( a > a_0 \).

**Lemma 4.9.** Assume \( \alpha < \delta \) and (4.19). Then, there exist constants \( C, r_0 > 0 \) such that for \( r > r_0 \)

\[
C \int_r^\infty (t - r)t^{\alpha - 1} \left( \|u'\|^2 + (Bu, u) + \|u\|^2 \right) dt
\]

(4.24)

\[
\leq t^{\alpha} \left( \|u'\|^2 + \|u\|^2 \right) \bigg|_{t=r} + \int_r^\infty t^{\alpha - 2}\|u\|^2 dt.
\]

**Proof.** We put

\[
f(t) = t^{\alpha - 1} \left( \|u'\|^2 + (Bu, u) + \|u\|^2 \right).
\]

Integrating (4.23) with respect to \( a \) over \( (r, b) \), we have

\[
C \int_r^b (t - r)f(t) dt + C \int_b^\infty (b - r)f(t) dt
\]

(4.25)

\[
\leq \int_r^b \left( -(t^\alpha u', u') + (t^{\alpha}(B + V)u, u) - \lambda(t^\alpha u, u) \right) dt.
\]

Taking \( \psi = t^\alpha \) and \( d = 0 \) in (4.18), we see that the right-hand side is equal to

\[
S(t^\alpha u', u) \bigg|_r^b - \frac{\alpha}{2} S(t^{\alpha - 1}u, u) \bigg|_r^b
\]

\[
- 2J_{(r, b)}(t^\alpha u', u') + \frac{\alpha(\alpha - 1)}{2} J_{(r, b)}(t^\alpha - 2u, u).
\]

We then have

\[
C \int_r^b (t - r)f(t) dt + 2J_{(r, b)}(t^\alpha u', u')
\]

\[
\leq (t^\alpha u', u) \bigg|_{t=r} - (t^\alpha u', u) \bigg|_{t=b} + \frac{\alpha}{2} (t^{\alpha - 1}u, u) \bigg|_{t=r} + \frac{\alpha(\alpha - 1)}{2} J_{(r, b)}(t^\alpha - 2u, u).
\]

Therefore, letting \( b \to \infty \) along a suitable sequence, we obtain

\[
C \left( \int_r^\infty (t - r)f(t) dt + \int_r^\infty t^\alpha \|u'\|^2 dt \right)
\]

\[
\leq t^{\alpha} \left( \|u'\|^2 + \|u\|^2 \right) \bigg|_{t=r} + \int_r^\infty t^{\alpha - 2}\|u\|^2 dt.
\]

This proves the lemma. \( \square \)

**Lemma 4.10.** Let \( u \) be as in Theorem (4.5). Then

\[
\int_1^\infty t^m \left( \|u'\|^2 + (Bu, u) + \|u\|^2 \right) dt < \infty, \quad \forall m > 0.
\]

**Proof.** We take \( 0 < \alpha < \min\{2\gamma, \delta\} \), and prove this lemma in the form

\[
\int_1^\infty t^{\alpha - 1+m} \left( \|u'\|^2 + (Bu, u) + \|u\|^2 \right) dt < \infty, \quad \forall m \geq 0.
\]

By the assumption of the theorem, (4.19) holds for \( 0 < \alpha < 2\gamma \). In the proof of Lemma (4.7), i.e. in (4.22), we have already proven (4.27) for \( m = 0 \). Let \( f(t) \) be as in (4.25) and put

\[
g(t) = t^{\alpha - 1} \left( \|u'\|^2 + \|u\|^2 \right).
\]
We show that for $m \geq 1$

$$(4.28) \quad C \int_r^\infty \frac{(t-r)^{m+1}}{(m+1)!} f(t)dt \leq \int_r^\infty \frac{(t-r)^{m-1}}{(m-1)!} g(t)dt + \int_r^\infty \frac{(t-r)^m}{m!} h(t)dt$$

and the right-hand side is finite. By (4.24), we have

$$C \int_r^\infty (t-r)f(t)dt \leq rg(r) + \int_r^\infty g(t)dt,$$

where the right-hand side is finite. Integrating this inequality, we obtain (4.28) for $m = 1$, where the right-hand side is finite. Repeating this procedure, we prove (4.28), hence (4.27) for all $m \geq 1$. \hfill \Box

The proof of Theorem 1.13 is now completed.

If we assume $\delta > 2$, which holds when $\rho(r)$ grows like $O(r^\beta)$ with $\beta > 1$, i.e. faster than the Euclidean metric, one can show that $u = 0$ near infinity. In fact, let $u$ be as in Theorem 1.13 and put $d = m log t$, $v = e^d u = t^m u$. Take $0 < \alpha < \min \{2\gamma, \delta\}$. Since $u$ decays rapidly at infinity, one can take $\gamma > 2$. Then, (4.17) is satisfied. In (4.17), we take $\psi = t^\alpha$, $d = m log t$, $a = r$ and let $b \to \infty$. Since $\delta > 2$, one can take $\alpha > 2$. Then, we have

$$-(t^\alpha B)_{t'} \geq -(t^\alpha B - q)^{t'} - (\delta - \alpha)t^{\alpha-1}B + (\alpha - 2)(m^2 + m)t^{-3+\alpha} + \lambda t^\alpha$$

for a constant $\kappa_0 > 0$. Then, we have

$$-S_t(t^\alpha v', v') + S_t(t^\alpha (B+q)v, v)) \geq \kappa_0(J_{(r, \infty)}(t^{\alpha-1}v', v') + J_{(r, \infty)}(t^{\alpha-1}Bv, v) + J_{(r, \infty)}(t^{\alpha-1}v, v)).$$

In particular, there exists $t_0 > 0$ such that

$$-S_t(v', v') + S_t((B+q)v, v) \geq 0, \quad \forall t \geq t_0.$$ 

Using $v = t^m u$, $q = V - \lambda - (m + m^2)/t^2$, we then have

$$-(2m^2 + m)S_t(u, \frac{u}{t}) - 2mS_t(u', \frac{u'}{t}) - S_t(u', u') + S_t(Bu, u) + S_t((V - E)u, u) \geq 0, \quad \forall t \geq t_0.$$ 

Since this holds for any $m > 0$, we see that $u(t) = 0$ for $t \geq t_0$.

This observation shows that the rapid growth of the volume facilitates the spectral analysis. To deal with slowly growing metrics, we need more elaborate consideration.
4.4. Proof of Theorem 4.4. We use the method in Saito [98], which originates from the work of Kato [66]. Although the proof here is apparently different from the one in the previous section, they are actually closely related. In the following, \( \| \cdot \|_X \) is simply written as \( \| \cdot \| \). We show that if \( \text{supp} \ u(t) \) is unbounded,

\[
\liminf_{t \to \infty} (\|u'(t)\|^2 + \|u(t)\|^2) > 0.
\]

To prove this, we consider the following two cases. We put

\[
(Ku)(t) = \|u'(t)\|^2 + \lambda \|u(t)\|^2 - (B(t)u(t), u(t)) - (V(t)u(t), u(t)).
\]

**Case 1.** There exists a sequence \( t_1 < t_2 < \cdots \to \infty \) such that

\[
(Ku)(t_n) > 0, \ n = 1, 2, \cdots.
\]

**Lemma 4.11.** There exist constants \( C_1, T_1 > 0 \) such that

\[
\frac{d}{dt} (Ku)(t) \geq -C_1 (1 + t)^{−1−\epsilon} (Ku)(t), \ \forall t > T_1.
\]

**Proof.** By choosing \( \epsilon \) small enough, we can assume that

\[
\|V'(t)\| \leq C (1 + t)^{−1−2\epsilon}.
\]

By the equation (4.10)

\[
\frac{d}{dt} (Ku)(t) = 2 \Re \left[ (u'', u') + \lambda (u, u') - (Bu, u') - (Vu, u') \right] - ((B' + V')u, u)
\]

\[
= -((B' + V')u, u).
\]

By (4.32), there exists \( t_0 = t_0(\epsilon) > 0 \) such that for \( t > t_0 \)

\[
|(V'(t)u, u)| \leq \frac{\epsilon}{2} (1 + t)^{−1−\epsilon} \|u\|^2.
\]

By Lemma 4.35,

\[
-(B'u, u) \geq \frac{\delta}{t} (Bu, u) - \frac{C}{t^{1+\epsilon}} \|u\|^2.
\]

By virtue of the above estimates, there is \( C_\epsilon > 0 \) such that for \( t > t_0 \)

\[
\frac{d}{dt} (Ku)(t) \geq -Ct^{−1−\epsilon} (\|u''\|^2 + \|u\| \|u'\| + \frac{\epsilon}{2} \|u\|^2) + \frac{\delta}{t} (Bu, u)
\]

\[
\geq -Ct^{−1−\epsilon} \|u''\|^2 - C\epsilon t^{−1−\epsilon} \|u\|^2 + \frac{\delta}{t} (Bu, u).
\]

We rewrite the right-hand side as

\[
-Ct^{−1−\epsilon} (\|u''\|^2 + \lambda \|u\|^2) + (C_\epsilon \lambda - C\epsilon) t^{−1−\epsilon} \|u\|^2 + \frac{\delta}{t} (Bu, u)
\]

\[
= -Ct^{−1−\epsilon} (Ku)(t)
\]

\[
+(C_\epsilon \lambda - C\epsilon) t^{−1−\epsilon} \|u\|^2 - C\epsilon t^{−1−\epsilon} (Vu, u) + \frac{\delta}{t} (Bu, u).
\]

Choose \( C_\epsilon \) large enough so that \( C_\epsilon \lambda - C\epsilon \geq \frac{1}{2} C_\epsilon \lambda \). Using (4.13), choose \( t_0 = t_0(\epsilon, C_\epsilon) \) such that, for \( t > t_0 \),

\[
\frac{\lambda}{2} \|u\|^2 - (Vu, u) \geq 0.
\]

Thus, the 3rd line is non-negative for \( t > t_0 \). Hence the lemma is proved. \( \square \)
Let us prove (4.30) for the Case 1. Let $T_1$ be as in Lemma 4.11. Then for some $T > T_1$, $(Ku)(T) > 0$. We show that $(Ku)(t) \geq 0, \forall t > T$. In fact Lemma 4.11 implies
\[
\frac{d}{dt} \left\{ \exp \left( C_1 \int_T^t (1 + s)^{-1-\epsilon} ds \right) (Ku)(t) \right\} \geq 0, \forall t > T.
\]
Hence,
\[
(Ku)(t) \geq \exp \left( -C_1 \int_T^t (1 + s)^{-1-\epsilon} ds \right) (Ku)(T), \forall t > T.
\]
This then implies that, for $t > t(\lambda)$,
\[
\|u'(t)\|^2 + \lambda \|u(t)\|^2 = Ku(t) + (B(t)u(t), u(t)) + (V(t)u(t), u(t)) \geq \exp \left( -C_1 \int_T^t (1 + s)^{-1-\epsilon} ds \right) (Ku)(T) - Ct^{-\epsilon} \|u(t)\|^2.
\]
Therefore, we arrive at
\[
\liminf_{t \to \infty} (\|u'(t)\|^2 + \|u(t)\|^2) \geq \frac{1}{2} \exp \left( -C_1 \int_T^\infty (1 + s)^{-1-\epsilon} ds \right) (Ku)(T) > 0.
\]
Note that in this case, we do not use the assumption $\delta > 2/3$.

Next let us consider Case 2:

Case 2. There exists $T_1 > 0$ such that $(Ku)(t) \leq 0$ for all $t > T_1$.

To deal with this case, take $\beta, \gamma, m, d(t)$ such that
\[
(4.33) \quad m > 0, \quad \frac{1}{3} < \gamma < 1, \quad 2\gamma < \beta < \delta, \quad d(t) = \frac{m}{1 - \gamma} t^{1-\gamma},
\]
and put
\[
(Nu)(t) = t^\beta \left[ K(e^{d(t)}u) + \frac{m^2 - \log t}{t^{2\gamma}} \|e^{d(t)}u\|^2 \right].
\]

**Lemma 4.12.** If supp $u(t)$ is unbounded, there exist constants $m_1 \geq 1, T_2 \geq T_1$ such that
\[
(Nu)(t) \geq 0, \quad \forall t \geq T_2, \quad \forall m \geq m_1.
\]

**Proof.** Letting $w(t) = e^{d(t)}u(t)$, we have by a direct computation,
\[
w' = e^d w' + mt^{-\gamma} w,
\]
\[
w'' = e^d w'' + mt^{-\gamma} e^d w' + mt^{-\gamma} w' - \gamma mt^{-\gamma-1} w = Bw + Vw - \lambda w + 2mt^{-\gamma} w' - (\gamma mt^{-\gamma-1} + m^2 t^{-2\gamma}) w.
\]
Hence,
\[
\frac{d}{dt} (Kw) = 2 \text{Re} (w'' + \lambda w - Vw - Bw, w') + ((B' + V')w, w) = 4mt^{-\gamma} \|w'\|^2 - 2(\gamma mt^{-\gamma-1} + m^2 t^{-2\gamma}) \text{Re} (w, w') - ((B' + V')w, w).
\]
We then have
\[
\frac{d}{dt}(Nu) = \beta t^{\beta-1} Kw + t^\beta \frac{d}{dt}(Kw) + \frac{(\beta - 2\gamma)(m^2 - \log t) - 1}{t^{2\gamma - \beta + 1}} \|w\|^2
\]
\[
+ 2\left(\frac{m^2 - \log t}{t^{2\gamma - \beta}}\right) \text{Re}(w', w)
\]
Using (4.33), we have
\[
t^{1-\beta} \frac{d}{dt}(Nu)
\]
\[
= (4mt^{1-\gamma} + \beta)\|w'\|^2 + (2\beta\lambda + (\beta - 2\gamma)(m^2 - \log t) - 1)\|w\|^2
\]
\[
- 2(\gamma mt^{-\gamma} + t^{1-2\gamma} \log t) \text{Re}(w', w) - ((\beta V + tV')w, w)
\]
\[
= I_1 + I_2 + I_3.
\]
For large \(t > 0\), \(I_1\) is estimated from below as
\[
I_1 \geq (4mt^{1-\gamma} + \beta)\|w'\|^2 + \left(\frac{\beta\lambda}{2} + (\beta - 2\gamma)t^{-2\gamma}m^2\right)\|w\|^2.
\]
By (4.39), \(I_2\) is estimated from below as
\[
I_2 \geq -2(\gamma mt^{-\gamma} + t^{1-2\gamma} \log t)\|w\|\|w'\| - Ct^{-\epsilon}\|w\|^2
\]
\[
\geq -2m^2t^{-2\gamma}\|w\|^2 - \frac{\gamma^2}{\epsilon}\|w'\|^2
\]
\[
-2t^{1-2\gamma} \log t\|w\|\|w'\| - Ct^{-\epsilon}\|w\|^2,
\]
where \(\epsilon > 0\) is chosen arbitrarily small. Note that the constant \(C\) is independent of \(m\).
Since \(\beta < \delta\), \(I_3\) is estimated from below as
\[
I_3 \geq -Ct^{-\epsilon}\|w\|^2.
\]
Choosing \(\beta - 2\gamma \geq \epsilon\) and putting the above estimates together, we have
\[
t^{1-\beta} \frac{d}{dt}(Nu) \geq 3mt^{1-\gamma}\|w'\|^2 + \frac{\beta\lambda}{3}\|w\|^2 - 2t^{1-2\gamma} \log t\|w\|\|w'\|.
\]
Finally, we use the inequality
\[
t^{1-2\gamma} \log t\|w\|\|w'\| \leq \epsilon t^{1-\gamma}\|w'\|^2 + C\epsilon t^{1-3\gamma}(\log t)^2\|w\|^2
\]
and \(1 - 3\gamma < 0\). Then there is \(t_0 > 0\) independent of \(m\) such that
\[
(4.36) \quad \frac{d}{dt}(Nu)(t) \geq t^{\beta-1}\left(2mt^{1-\gamma}\|w'\|^2 + \frac{\beta\lambda}{4}\|w\|^2\right) \geq 0
\]
for \(t > t_0\).
On the other hand, \(Nu(t)\) can be rewritten as
\[
(Nu)(t) = t^\beta e^{2d}\left[\|mt^{-\gamma}u + u'\|^2 + \lambda \|u\|^2
\]
\[
- (Bu, u) - (Vu, u) + t^{-2\gamma}(m^2 - \log t)\|u\|^2
\]
\[
= t^\beta e^{2d} \left[2t^{-2\gamma}\|u\|^2 + 2t^{-\gamma}\text{Re}(u, u) + (Ku - t^{-2\gamma}\|u\|^2 \log t)\right].
\]
By the assumption of the lemma, supp $u(t)$ is unbounded. Therefore, there is $T_2 > t_0$ such that $\|u(T_2)\| > 0$. By choosing $m_1$ large enough, we then have

$$\langle Nu(T_2) \rangle > 0, \quad \forall m > m_1.$$  \hfill (4.38)

The inequalities (4.36) and (4.38) prove the lemma. \hfill \Box

Since $Ku \leq 0$, Lemma 4.12 and (4.37) show that, for large $t$,

$$2t^{-2\gamma} \|u(t)\|^2 m^2 + 2t^{-\gamma} \text{Re} (u(t), u'(t)) m - t^{-2\gamma} \|u(t)\|^2 \log t \geq 0,$$

which together with

$$\frac{d}{dt} \|u(t)\|^2 = 2 \text{Re} (u(t), u'(t)),$$

yields, for large $t > 0$,

$$\frac{d}{dt} \|u(t)\|^2 \geq t^{-\gamma} \left( \frac{1}{m} \log t - 2m \right) \|u(t)\|^2 \geq 0.$$  \hfill (4.39)

Since the support of $u(t)$ is unbounded, we can choose $T$ large enough so that $\|u(T)\| > 0$. In view of (4.39), we then have

$$\|u(t)\| \geq \|u(T)\| > 0, \quad \forall t > T,$$

which proves (4.30). \hfill \Box

5. Integral identities

The next aim is to derive resolvent estimates for $-\Delta_{\mathcal{M}}$ on one end $(0, \infty) \times M$, which we denoted as $\mathcal{M} = (0, \infty) \times M$ for the sake of simplicity. Our method is based on the two integral identities to be proved in Lemmas 5.2 and 5.3 below. The basic assumption on the metric $ds^2 = (dr)^2 + \rho(r)^2 h(r, x, dx)$ in this section is the existence of the constants $\alpha_0, \gamma_0 > 0$ such that

$$\frac{\rho'(r)}{\rho(r)} - c_0 \in S^{-\alpha_0},$$  \hfill (5.1)

$$h_{ij}^{\alpha}(r, x) - h^{ij}_{M}(x) \in S^{-\gamma_0},$$  \hfill (5.2)

where $S^\kappa$ is defined in Definition 2.9. Let $g = \rho^{2(n-1)} h, h = h(r, x) = \det(h_{ij}(r, x))$.

5.1. Preliminaries. We regard $L^2(\mathcal{M})$ as the $L^2$-space of $L^2(M)$-valued functions over $(0, \infty)$. We did it already in Subsection 4.2 however, we employ here a slightly different formulation in order to take account of the growth order of $\rho(r)$ more explicitly. We define the $L^2$-space over $M$ by

$$\mathbf{h}(r) = L^2(M; \sqrt{h(r, x)dx})$$

with the $r$-dependent inner product and norm

$$(\varphi, \psi)_{\mathbf{h}(r)} = \int_M \varphi(x) \overline{\psi(x)} \sqrt{h(r, x)} dx, \quad \|\varphi\|_{\mathbf{h}(r)} = \sqrt{(\varphi, \varphi)_{\mathbf{h}(r)}}.$$
By integration by parts, we have
\[
I_m = \int_0^\infty (u(r), v(r))_h(r) \rho^{n-1}(r) dr.
\]
Here and in the sequel, we take the branch of \(\sqrt{r}\) which is independent of \(r\) as a set. The inner product of the Hilbert space \(L^2((0, \infty) \times M; \sqrt{g} drdx)\) is rewritten as
\[
(u, v) = \int_0^\infty (u(r), v(r))_h(r) \rho^{n-1}(r) dr.
\]
For \(0 < a < b < \infty\), we put
\[
M_b(u, v) - M_a(u, v),
\]
\[
I(a, b)(u, v) = \int_a^b (u(r), v(r))_h(r) \rho^{n-1}(r) dr = \int_a^b M_r(u, v) dr,
\]
\[
I(u, v) = I_{(0, \infty)}(u, v).
\]
By integration by parts, we have
\[
I(a, b)(u',, v) = M(u, v) - I(a, b)(u, v') - I(a, b)(\frac{g'}{2g} u, v),
\]
which implies that the formal adjoint of \(\partial_r\) is
\[
\partial_r^* = -\partial_r - \frac{g'}{2g},
\]
and, for a real-valued \(C^1\)-function \(\varphi\),
\[
\text{Re} I(a, b)(v', \varphi v) = M\left(\frac{g'}{2} v, v\right) - I(a, b)\left(\frac{g'}{2} + \frac{g'\varphi}{4g} v, v\right).
\]
Recall that the Laplacian on \(M\) is written as
\[
- \Delta_M = -\partial_r^2 - \frac{g'}{2g} \partial_r + B(r), \quad ' = \partial_r,
\]
\[
B(r) = -\rho(r)^{-2} \Lambda(r), \quad \Lambda(r) = \frac{1}{\sqrt{h}} \partial_{x_i} \left( \sqrt{h} h^{ij} \partial_{x_j} \right).
\]
For any \(r > 0\), \(\Lambda(r)\) is self-adjoint on \(h(r)\) with domain described in Theorem 2.2 which is independent of \(r\). We rewrite \(\Delta_M\) into a form which is more convenient for our computation. Put for \(z \in \mathbb{C}\)
\[
E_0 = \left((n-1)c_0/2\right)^2,
\]
\[
k = \sqrt{z - E_0},
\]
\[
Q = \left(\frac{g'}{4g}\right)^2 + \left(\frac{g'}{4g}\right)' - E_0.
\]
Here and in the sequel, we take the branch of \(\sqrt{z}\) on \(\mathbb{C} \setminus [0, \infty)\) in such a way that
\(\text{Im } \sqrt{z} \geq 0\), i.e. \(\sqrt{z} = \sqrt{Te^{i\theta}/2}\) if \(z = re^{i\theta}\), \(0 \leq \theta < 2\pi\). Then, \(-\Delta_M - z\) is rewritten as
\[
-\Delta_M - z = -\left(\partial_r + \frac{g'}{4g}\right)^2 + B(r) + Q - k^2.
\]
The assumption (5.2) implies,
\[
Q \in S^{-\alpha_0}.
\]
Take a complex-valued function $\psi$ and introduce a differential operator
\[
D_r = \partial_r + \frac{g'}{4g} - i\psi.
\]
Then
\[
(5.11) \quad -\Delta_M - z = -D_r^2 - 2i\psi D_r + B(r) - i\psi' + \psi^2 + Q - k^2.
\]

We construct an approximate solution of
\[
-i\psi' + \psi^2 + Q - k^2 = 0
\]
by putting $\psi_{-1} = 0$ and
\[
(5.12) \quad \psi_m(r, x, k) = \chi(r/R_m)\sqrt{k^2 - Q + i\partial_r \psi_{m-1}}, \quad m \geq 0,
\]
where $\chi \in C^\infty(\mathbb{R})$, $\chi(t) = 1 \ (t > 2)$, $\chi(t) = 0 \ (t < 1)$.

**Lemma 5.1.** For $k \neq 0$ and $m \geq 0$, by choosing $R_m$ large enough, $\psi_m$ is $C^\infty$ with respect to $r$ and has the following properties.

1. $\psi_m \in S^0$, $\partial_r \psi_m \in S^{-1-\alpha_0}$.
2. $-i\partial_r \psi_m + (\psi_m)^2 + Q - k^2 \in S^{-m-1-\alpha_0}$.
3. $\psi_m - \sqrt{k^2 - Q} \in S^{-1-\alpha_0}$.

**Proof.** The assertion (1) is proven by induction on $m$. For large $r$ we have
\[
-i\partial_r \psi_m + (\psi_m)^2 + Q - k^2 = -i(\partial_r \psi_m - \partial_r \psi_{m-1}), \quad m \geq 0,
\]
\[
i(\psi_m - \psi_{m-1}) = -\frac{\partial_r \psi_{m-1} - \partial_r \psi_{m-2}}{\psi_m + \psi_{m-1}}, \quad m \geq 1.
\]
One can then derive (2) from (1). The assertion (3) follows from (5.12) and (1). □

We take $m$ large enough, and put
\[
(5.13) \quad \psi(r, x, k) = \psi_m(r, x, k),
\]
\[
(5.14) \quad V = -i\psi' + \psi^2 + Q - k^2,
\]
\[
(5.15) \quad D(k) = \partial_r + \frac{g'}{4g} - i\psi(r, x, k).
\]
For a solution $u$ of the equation
\[
(5.16) \quad (-\Delta_M - z)u = (-\partial_r^2 - \frac{g'}{2g}\partial_r + B(r) - z)u = f,
\]
we put
\[
(5.17) \quad v = D(k)u, \quad w = \sqrt{B(r)}u = \rho(r)^{-1}\sqrt{-\Lambda(r)}u,
\]
where $B(r)$ and $\Lambda(r)$ are defined by (5.8).
5.2. The 1st identity.

**Lemma 5.2.** Let \( u \) be a solution to \((5.16)\), and \( v, w \) as in \((5.17)\). Then, for a non-negative \( C^1 \)-function \( \varphi(v) \), we have

\[
I_{(a,b)}(\varphi', v) - I_{(a,b)}(\varphi'w, w) + M(\varphi w, w)\bigg|_a - M(\varphi v, v)\bigg|_a
\]

\[= -2I_{(a,b)}((\Im \psi)\varphi v, v) - 2I_{(a,b)}((\Im \psi)\varphi w, w) + 2\Re I_{(a,b)}(\varphi f, v) - 2\Re I_{(a,b)}(\varphi Vu, v) + I_{(a,b)}(\varphi B'u, u).
\]

**Proof.** By \((5.11)\), we have

\[-D(k)^2 u - 2i\psi D(k) u + B(r) u = f - Vu,\]

which implies

\[-(\partial_{\tau} + \frac{g'}{4g})v + Bu = f + i\psi v - Vu.\]

Taking the inner product with \( \varphi v \),

\[
I_{(a,b)}(- (\partial_{\tau} + \frac{g'}{4g})v + Bu, \varphi v)
\]

\[= I_{(a,b)}(\varphi f, v) + iI_{(a,b)}(\psi \varphi v, v) - I_{(a,b)}(\varphi Vu, v).
\]

On the other hand, by \((5.7)\),

\[2\Re I_{(a,b)}(- (\partial_{\tau} + \frac{g'}{4g})v, \varphi v) = I_{(a,b)}(\varphi', v) - M(\varphi, v)\bigg|_a.
\]

Using

\[
\Re I_{(a,b)}(Bu, \varphi u') = M(\frac{\varphi}{2} w, w)\bigg|_a - I_{(a,b)}(\frac{\varphi'}{2} w, w)
\]

\[- \Re I_{(a,b)}(\frac{g'}{4g} Bu, u) - I_{(a,b)}(\varphi B'u, u),
\]

we have

\[
\Re I_{(a,b)}(Bu, \varphi v) = M(\frac{\varphi}{2} w, w)\bigg|_a - I_{(a,b)}(\frac{\varphi'}{2} w, w)
\]

\[- I_{(a,b)}(\frac{\varphi}{2} B'u, u) + I_{(a,b)}((\Im \psi) \varphi w, w).
\]

By \((5.11)\) and \((5.23)\), the real part of the left-hand side of \((5.20)\) is equal to

\[
M(\frac{\varphi}{2} w, w)\bigg|_a - M(\frac{\varphi}{2} v, v)\bigg|_a + I_{(a,b)}(\frac{\varphi'}{2} v, v)
\]

\[- I_{(a,b)}(\frac{\varphi'}{2} w, w) - I_{(a,b)}(\frac{\varphi}{2} B'u, u) + I_{(a,b)}((\Im \psi) \varphi w, w).
\]

In view of \((5.20)\) and \((5.24)\), we have

\[
M(\frac{\varphi}{2} w, w)\bigg|_a - M(\frac{\varphi}{2} v, v)\bigg|_a + I_{(a,b)}(\frac{\varphi'}{2} v, v)
\]

\[- I_{(a,b)}(\frac{\varphi'}{2} w, w) - I_{(a,b)}(\frac{\varphi}{2} B'u, u) + I_{(a,b)}((\Im \psi) \varphi w, w)
\]

\[= \Re I_{(a,b)}(\varphi f, v) - I_{(a,b)}((\Im \psi) \varphi v, v) - \Re I_{(a,b)}(\varphi Vu, v).
\]

This proves the lemma.

\[\square\]
5.3. The 2nd identity.

**Lemma 5.3.** The solution \( u \) of (5.16) satisfies

\[
\text{Im} \ M(v, u) \bigg|_a^b = -2 I_{(a,b)}((\text{Re} \psi \text{Im} \psi)u, u) - M((\text{Re} \psi)u, u) \bigg|_a^b + I_{(a,b)}(u, (\text{Re} \psi')u) \\
- \text{Im} I_{(a,b)}(f, u) + \text{Im} I_{(a,b)}(Vu, u).
\]

**Proof.** Note that \( \text{Im} \ M_r(v, u) = \text{Im} \ M_r(u', u) - M_r((\text{Re} \psi)u, u) \), and

\[
\partial_r M_r(v, u) = M_r(v', u) + M_r(v, u') + M_r(v, \frac{g''}{2g} u).
\]

Integrating over \((a, b)\), we have

\[
(5.25) \quad M(v, u) \bigg|_a^b = I_{(a,b)}(v', u) + I_{(a,b)}(v, u') + I_{(a,b)}(\frac{g'}{2g} v, u).
\]

Using the equation (5.19), we have

\[
\text{Im} I_{(a,b)}(v', u) + \text{Im} I_{(a,b)}(\frac{g'}{2g} v, u) = \text{Im} I_{(a,b)}(\frac{g'}{4g} v, u) - \text{Im} I_{(a,b)}(f, u) \\
- \text{Im} I_{(a,b)}(i\psi v, u) + \text{Im} I_{(a,b)}(Vu, u).
\]

Hence, by (5.25),

\[
\text{Im} M(v, u) \bigg|_a^b = \text{Im} I_{(a,b)}(v, u' + \frac{g'}{4g} v + i\psi u) \\
- \text{Im} I_{(a,b)}(f, u) + \text{Im} I_{(a,b)}(Vu, u) \\
= \text{Im} I_{(a,b)}(v, 2i(\text{Re} \psi)u) - \text{Im} I_{(a,b)}(f, u) + \text{Im} I_{(a,b)}(Vu, u).
\]

We have, by integration by parts,

\[
\text{Im} I_{(a,b)}(v, 2i(\text{Re} \psi)u) = -2\text{Re} I_{(a,b)}(u', (\text{Re} \psi)u) - 2I_{(a,b)}(\frac{g'}{4g} u, (\text{Re} \psi)u) \\
- 2 I_{(a,b)}((\text{Re} \psi)\text{Im} \psi)u, u) \\
= -M(u, (\text{Re} \psi)u) \bigg|_a^b + I_{(a,b)}(u, (\text{Re} \psi')u) \\
- 2 I_{(a,b)}((\text{Re} \psi)\text{Im} \psi)u, u).
\]

The lemma then follows from (5.26) and (5.27). \(\square\)

6. A priori-estimates on each end

We begin to estimate the resolvent of the Laplacian on each end. We assume (6.14) and (6.15) for

\[
\alpha_0 > 0, \quad \gamma_0 > 0,
\]

omitting the subscript \( j \). We pick up one end \((0, \infty) \times M, \) and consider the solution \( u \) of \((-\Delta_M - z)u = f\) with support in \((0, \infty) \times M, \) For the sake of simplicity, we denote \((0, \infty) \times M\) as \( M \). We use the same notations as in §5, and define the following Besov type function spaces introduced by Agmon-Hörmander \(\Pi\) in the case of Euclidean space.
6.1. Function spaces. For an interval $I \subset (0, \infty)$, we put

$$L^2(I) = L^2(I; h(r); \rho^{n-1}(r)dr).$$

Let $\mathcal{B}$ be the set of functions $f$ satisfying

$$\|f\|_\mathcal{B} = \sum_{j=0}^{\infty} 2^{j/2} \|f\|_{L^2(I_j)} < \infty,$$

where $I_0 = (0, 1]$, $I_j = (2^j, 2^{j+1})$, $j \geq 1$. The dual space of $\mathcal{B}$ is identified with the set of functions $v(r)$ satisfying

$$\|v\|_* = \sup_{j \geq 0} 2^{-j/2} \|v\|_{L^2(I_j)} < \infty.$$

Letting

$$\|v\|_{\mathcal{B}^*} = \left( \sup_{R > 1} \frac{1}{R} \int_0^R \|v(r)\|^2_{\mathcal{B}(r)} \rho^{n-1}(r)dr \right)^{1/2},$$

one can show the existence of a constant $C > 0$ such that

$$C^{-1} \|v\|_* \leq \|v\|_{\mathcal{B}^*} \leq C \|v\|_*.$$

Therefore, we employ $\| \cdot \|_{\mathcal{B}^*}$ as the norm of the dual space of $\mathcal{B}$. Note that the coupling of $f \in \mathcal{B}$ and $v \in \mathcal{B}^*$ is given by

$$(f, v) = \int_0^\infty (f(r), v(r)) h(r) \rho^{n-1}(r)dr.$$

We introduce a closed subspace $\mathcal{B}_0^*$ of $\mathcal{B}^*$ as follows:

$$\mathcal{B}_0^* \ni v \iff \lim_{R \to \infty} \frac{1}{R} \int_0^R \|v(r)\|^2_{\mathcal{B}(r)} \rho^{n-1}(r)dr = 0.$$

For $s \in \mathbb{R}$, let $L^{2,s}$ be the set of functions $v(r)$ satisfying

$$\|v\|_s = \left( \int_0^\infty \|v(r)\|^2_{\mathcal{B}(r)} (1 + r)^{2s} \rho^{n-1}(r)dr \right)^{1/2} < \infty.$$

For $s = 0$, this norm is denoted by $\| \cdot \|$. For $s > 1/2$, the following inclusion relations hold:

$$L^{2,s} \subset \mathcal{B} \subset L^{2,1/2} \subset L^2 \subset L^{2,-1/2} \subset \mathcal{B}^* \subset L^{2,-s}.$$

For example, that $L^{2,-1/2} \subset \mathcal{B}^* \subset L^{2,-s}$ is proven as follows:

$$\|f\|_{-s}^2 \leq C \sum_{j=0}^{\infty} 2^{-2sj} \|f\|_{L^2(I_j)}^2 \leq C \left( \sum_{j=0}^{\infty} 2^{-j(2s-1)} \right) \sup_{j \geq 0} 2^{-j} \|f\|_{L^2(I_j)}^2 \leq C \sum_{j=0}^{\infty} 2^{-j} \|f\|_{L^2(I_j)}^2 \leq \|f\|_{-1/2}^2.$$

**Lemma 6.1.** (1) $L^{2,-1/2} \subset \mathcal{B}_0^*$.

(2) If $v \in \mathcal{B}_0^*$, we have $\lim_{R \to \infty} M_\tau(v, v) = 0$.

(3) If $u \in \mathcal{B}^*$ and $v \in \mathcal{B}_0^*$, we have $\lim_{R \to \infty} |M_\tau(u, v)| = 0$. 
is defined similarly. For \( r > r_1 \) that \( R > R \),

\[
\frac{1}{R} \int_{R_0}^R \| v(r) \|_{\mathcal{H}}^2 \rho^{n-1}(r) dr \leq \frac{1}{R} \int_{R_0}^R \| v(r) \|_{\mathcal{H}}^2 \rho^{n-1}(r) dr < \epsilon.
\]

Therefore, for \( R > R \),

\[
\frac{1}{R} \int_0^R \| v(r) \|_{\mathcal{H}}^2 \rho^{n-1}(r) dr \leq \frac{1}{R} \int_0^R \| v(r) \|_{\mathcal{H}}^2 \rho^{n-1}(r) dr + \epsilon,
\]

which implies \( \lim_{R \to \infty} \frac{1}{R} \int_0^R \| v(r) \|_{\mathcal{H}}^2 \rho^{n-1}(r) dr \leq \epsilon. \)

If \( \lim_{R \to \infty} M_{\rho}(v, v) > 0 \), there exist constants \( C, r_0 > 0 \) such that \( M_{\rho}(v, v) > C \) for \( r > r_0 \), hence \( v \notin B_0 \). This proves (2).

It follows from the inequality

\[
\frac{1}{R} \int_0^R (u(r), v(r))_{\mathcal{H}} \rho^{n-1}(r) dr 
< \left( \frac{1}{R} \int_0^R \| u(r) \|_{\mathcal{H}}^2 \rho^{n-1}(r) dr \right)^{1/2} \left( \frac{1}{R} \int_0^R \| v(r) \|_{\mathcal{H}}^2 \rho^{n-1}(r) dr \right)^{1/2}.
\]

that \( \frac{1}{R} \int_0^R (u(r), v(r))_{\mathcal{H}} \rho^{n-1}(r) dr \to 0 \), which implies (3). \( \square \)

Finally, for a non-negative integer \( m \), the Sobolev space of order \( m \) on \( \mathcal{M} = (0, \infty) \times M \) is denoted by \( H^m(\mathcal{M}) \). For an interval \( I \subset (0, \infty) \), \( H^m(I \times M) \) is defined similarly. For \( s \in \mathbb{R} \), \( H^{m,s}(\mathcal{M}) \) is the set of functions \( v \) such that \( (1 + r^2)^{s/2} v \in H^m(\mathcal{M}) \) equipped with norm

\[
\| v \|_{H^{m,s}} = \| (1 + r^2)^{s/2} v \|_{H^m(\mathcal{M})}.
\]

Similarly, for \( s \in \mathbb{R} \), we introduce \( \tilde{H}^{2,s}(\mathcal{M}) \)-norm by

\[
\| u \|^2_{\tilde{H}^{2,s}(\mathcal{M})} = \sum_{m \leq 2} \| (1 + r)^s \partial^m_u \|^2_{L^2(\mathcal{M})} + \int_0^\infty (1 + r)^{2s} \| u \|^2_{\tilde{H}^{2,s}(\mathcal{M})} \rho^{n-1}(r) dr,
\]

where \( \| \cdot \|_{\tilde{H}^{2}} \)-norm is defined by (2.44). For an interval \( I \subset (0, \infty) \), \( \tilde{H}^{2,s}(I \times M) \) is defined similarly.

6.2. A-priori estimates. In this subsection, we derive some a-priori estimates for the solution \( u \) to

\[
(-\Delta_{\mathcal{M}} - z) u = f, \quad \text{on} \quad (0, \infty) \times M
\]

satisfying \( u = 0 \) for \( r < 1 \).

Lemma 6.2. (1) For any \( a, \epsilon > 0 \), there exists a constant \( C_{a,\epsilon} > 0 \) such that

\[
\| u \|^2_{\tilde{H}^2((0,a) \times M)} \leq C_{a,\epsilon} \left( \| u \|^2_{L^2((0,a+\epsilon) \times M)} + \| f \|^2_{L^2((0,a+\epsilon) \times M)} \right).
\]
Here, we note that if \( u, f \in L^{2,s} \) and \( u' \in L^{2,s'} \),
\[
\|u\|_{\dot{H}^{2,s}} + \|B(r)u\|_s \leq C_{s,s'} (\|u\|_s + \|f\|_s).
\]
In the above inequalities, the constants \( C_{a,e} \) and \( C_{s,s'} \) are independent of \( z \) if \( z \) varies over a compact set in \( C \).

**Proof.** Take \( \chi \in C^{\infty}(\mathbb{R}) \) such that \( \chi(r) = 1 \) for \( r < a \) and \( \chi(r) = 0 \) for \( r > a + \epsilon \). Applying Theorem 2.11 to \( \chi u \), we obtain (1). To prove (2), we have only to consider the case in which \( s' = s - 1 \). Let \( I_R = (0, Ra) \), \( J_R = (0, R(a + \epsilon)) \) and \( K_R = (Ra, R(a + \epsilon)) \). Applying Theorem 2.11 to \((1+r)\chi(r/R)u\), we have
\[
\|u\|_{\dot{H}^{2,s}(I_R \times M)} \leq C\|Hu\|_{L^{2,s}(J_R \times M)} + \|u\|_{L^{2,s}(J_R \times M)} + \|u'\|_{L^{2,s-1}(J_R \times M)}.
\]
Here, we note \( \|u'\|_{L^{2,s-1}(J_R \times M)} = \|u'\|_{L^{2,s-1}(I_R \times M)} + \|u'\|_{L^{2,s-1}(K_R \times M)} \) and \( 2 \leq (1 + R) \leq \|u\|_{L^{2,s-1}(I_R \times M)} + \|u'\|_{L^{2,s-1}(K_R \times M)} \).

\[
\begin{align*}
\int_0^R \int_M (1+r)^{2(s-1)} u(r)u'(r)\sqrt{g} dx dr &= \int_0^{1+R} (1+r)^{2(s-1)} u(R)r u(R)\sqrt{g} dx + \epsilon \int_0^R \int_M \left((1+r)^{2(s-1)} u(r)\sqrt{g} u'(r)\right) dx dr.
\end{align*}
\]

This implies
\[
\|u'\|_{L^{2,s-1}(I_R \times M)} \leq 2(1+R)^{2(s-1)}|MR(u, u')| + \epsilon \|u''\|_{L^{2,s-1}(I_R \times M)} + C_\epsilon \|u\|_{L^{2,s-1}(J_R \times M)}.
\]
Letting \( R \to \infty \) in (5.8) along a suitable sequence, we then obtain
\[
\|u\|_{\dot{H}^{2,s}(M)} \leq C\|Hu\|_{L^{2,s}(M)} + \|u\|_{L^{2,s}(M)}.
\]
Using the equation (5.10), we also have
\[
\|B(r)u\|_{L^{2,s}(M)} \leq C\|Hu\|_{L^{2,s}(M)} + \|u\|_{L^{2,s}(M)}.
\]
These two inequalities imply (2).

\( \square \)

We define the \( \| \cdot \|_{G^*_m} \) norm by
\[
\|u\|_{G^*_m} = \sum_{|\alpha| \leq m} \|\partial^\alpha u\|_{B^*}.
\]

**Lemma 6.3.** If \( u, u', f \in B^* \), we have
\[
\|u\|_{G^*_1} \leq C\left(\|f\|_{B^*} + \|u\|_{B^*}\right),
\]
where the constant \( C \) is independent of \( z \) when \( z \) varies over a compact set in \( C \).

**Proof.** Take \( \chi \in C^{\infty}(\mathbb{R}) \) such that \( \chi(t) = 1 \) for \( t < 1 \), \( \chi(t) = 0 \) for \( t > 2 \), and put \( u_R = \frac{1}{\sqrt{R}} \chi_R(r/u) \), where \( \chi_R(r) = \chi(r/R) \). Then, we have
\[
(-\Delta_M - z) u_R = \frac{1}{\sqrt{R}} \chi_R f - \frac{1}{\sqrt{R}} \left(2 \chi' R u' + \frac{g'}{2g} \chi' R u + \chi'' R u\right).
\]
Taking the inner product with \( u_R \) and integrating by parts, we have
\[
\frac{1}{R} \|(\chi R u')\|^2 + \frac{1}{R} \| \sqrt{B} \chi R u \|^2 \leq C\left(\|f\|_{B^*}^2 + \|u\|_{B^*}^2 + \frac{1}{R^2} |(c_R(r)u', u)|\right).
\]
For \( R > R_e \) such that
\[
\frac{1}{R^2} |(c_R(r)u', u)| \leq \epsilon (\|u'\|_B^2 + \|u\|_B^2), \quad \text{for} \quad R > R_e.
\]
For \( 1 < R < R_e \), applying Lemma 6.2 (1), we have
\[
\frac{1}{R^2} |(c_R(r)u', u)| \leq C_\epsilon (\|f\|_B^2 + \|u\|_B^2).
\]
We then have
\[
\frac{1}{R} \|(\chi_R u')'' + 2R^2 \chi_R u'' \| \leq \epsilon \|u'\|_B^2 + C_\epsilon (\|f\|_B^2 + \|u\|_B^2),
\]
which yields the lemma.

For energies below \( E_0 = ((n - 1)c_0/2)^2 \), the following inequality holds.

**Lemma 6.4.** Take any compact interval \([c_1, c_2] \subset (-\infty, E_0)\). Let \( s \geq 0, s' = s - \min\{\alpha_0, 1\} \). Then, there exists a constant \( C_s > 0 \) such that for a solution \( u \) to \( (6.7) \), if \( \text{Re} z \in [c_1, c_2] \), and \( u, u' \in L^{2,s'}, \ f \in L^{2,s}, \)
\[
\|u\|_{H^{s'}} \leq C_s (\|f\|_s + \|u\|_{s'}).
\]

**Proof.** By (5.10) and (5.11), we have
\[
-\Delta_M - z = (\partial_t + \frac{q'}{4g})^s (\partial_t + \frac{q'}{4g}) + B(r) + Q + E_0 - z.
\]
Hence for any \( h \in D(H), \)
\[
\text{Re} \, ((-\Delta_M - z)h, h) \geq (E_0 - c_2)\|h\|^2 + (Qh, h).
\]
Take \( \chi \in C_\infty(\mathbb{R}) \) such that \( \chi(t) = 1 \) for \( t < 1 \) and \( \chi(t) = 0 \) for \( t > 2 \). Then, letting \( u_R = \chi(r/R)u \), and \( h = (1 + r)^s u_R \), we have by (5.10)
\[
\|u_R\|_s^2 \leq \epsilon \|u_R\|_{s'}^2 + C_s (\|f\|_s^2 + \|u\|_{s'}^2).
\]
Letting \( R \to \infty \), and using Lemma 6.2 we obtain the lemma.

**6.3. Regular ends.** We begin with the case of regular ends. In addition to (6.1), we assume (0.16), which implies
\[
(6.9) \quad \rho(r)^{-1} \leq C(1 + r)^{-\beta_0}, \quad \beta_0 > 0.
\]
Moreover, there exists a constant \( r_0 > 0 \) such that
\[
(6.10) \quad \rho'(r) > 0 \quad \text{for} \quad r > r_0.
\]
We fix an interval \([c_1, c_2] \subset (E_0, \infty)\), and put
\[
(6.11) \quad J_\pm = \{ z \in \mathbb{C} : c_1 \leq \text{Re} \, z \leq c_2, \ 0 < \pm \text{Im} \, z < 1 \}.
\]
We derive a-priori estimates for a solution \( u \) to (5.10) satisfying
\[
f \in \mathcal{B}, \quad u, u' \in \mathcal{B}^*, \quad z \in J_\pm.
\]
Using (5.13), we put as in the previous section
\[
v = D(k)u, \quad w = \sqrt{B(r)}u,
\]
and assume that
\[(6.12) \quad u = 0 \quad \text{for} \quad r < 1,\]
\[(6.13) \quad v \in B_0^\ast.\]

In the following, \(C\)'s denote constants independent of \(z \in J_\pm\).

**Lemma 6.5.** Let \(0 < \delta < 2\beta_0\). Then, for any non-negative \(C^1\)-function \(\varphi(r)\), we have
\[
I_{(a,b)}(\varphi^'v,v) + I_{(a,b)}((\delta \varphi r^{-1} - \varphi^')w,w) + M(\varphi w,w) \leq M(\varphi v,v) + |I_{(a,b)}(\varphi f,v)| + |I_{(a,b)}(\varphi V u,v)| + I_{(a,b)}(\varphi r^{-1-\gamma_0}u,u) \bigg)\]
\[
\leq C \left( I_{(a,b)}(\varphi r^{-1-\alpha_0}v,v) + I_{(a,b)}(\varphi r^{-1-\alpha_0}w,w) + |I_{(a,b)}(\varphi f,v)| + |I_{(a,b)}(\varphi V u,v)| + I_{(a,b)}(\varphi r^{-1-\gamma_0}u,u) \right). \tag{6.14}
\]

**Proof.** Adding \(I_{(a,b)}(\frac{\delta}{r}Bu,u)\) to the both sides of (5.18)
\[
I_{(a,b)}(\varphi^'v,v) + I_{(a,b)}((\delta \varphi r^{-1} - \varphi^')w,w) + M(\varphi w,w) \leq M(\varphi v,v) + |2I_{(a,b)}((\Im \psi)\varphi v,v)| + |2I_{(a,b)}((\Im \psi)\varphi w,w)|
\]
\[
+ 2\Re I_{(a,b)}(\varphi f,v) - 2\Re I_{(a,b)}(\varphi V u,v) + I_{(a,b)}((\varphi(B^' + \frac{\delta}{r}B) u,u).
\]

The proof of Lemma 4.3 works also for \(B(r)\). This and Lemma 5.1 (3) imply
\[-\Im \psi \leq C r^{-1-\alpha_0}, \quad B' + \frac{\delta}{r}B \leq C r^{-1-\gamma_0}\]
for \(0 < \delta < 2\beta_0\). The lemma then follows from (6.14). \(\square\)

**Lemma 6.6.** Let \(\epsilon_0 = \min(\alpha_0, \gamma_0)\) and \(\frac{1}{2} < s \leq \frac{1}{r}(1 + \epsilon_0)\). Then, we have
\[
\|v\|_{B^\ast} + \|w\|_{-1/2} \leq C(\|f\|_B + \|u\|_{-s}).
\]

**Proof.** Letting \(\varphi = 1\) and \(a = 0\) in Lemma 6.5 we have
\[
\delta I_{(0,b)}(\frac{1}{r}w, w) - M_b(v,v) \leq C(\|v\|_{-s}^2 + \|w\|_{-s}^2 + (f,v) + \|u\|_{-s}^2).
\]

Note that by Lemma 5.1, \(V\) decays sufficiently rapidly. We let \(b \to \infty\) along a suitable sequence in the above inequality. By (6.13) and Lemma 6.1 \(M_b(v,v) \to 0\).

Using Lemma 6.2 we obtain
\[
\|w\|_{-1/2} \leq C(\|f\| + \|u\|_{-s}). \tag{6.15}
\]

Letting \(\varphi = 1, a = r\) and \(b \to \infty\) along a suitable sequence in Lemma 6.5 we have
\[
M_r(v,v) \leq M_r(w,w) + C(\|v\|_{-s}^2 + \|w\|_{-s}^2 + (f,v) + \|u\|_{-s}^2).
\]

By taking the integral mean in \(r\),
\[
\|v\|_{B_r}^2 \leq \|w\|_{B_r}^2 + C(\|v\|_{-s}^2 + \|w\|_{-s}^2 + (f,v) + \|u\|_{-s}^2).
\]

Using Lemma 6.2 (2), we then have
\[
\|v\|_{B_r}^2 \leq C\left(\|w\|_{B_r}^2 + (f,v) + \|u\|_{-s}^2\right).
\]
Finally, using \( \square \) we obtain the lemma.

Since \( \|w\|_{s} \leq C\|w\|_{s-1/2} \), using (6.15), we have
\[
\|v\|_{B^s}^2 \leq C(\|f\| + \|u\|_{s}^2).
\]
Finally, using
\[
|\langle f, v \rangle| \leq \epsilon \|v\|_{B^s} + C \epsilon \|f\|_{B^s}^2,
\]
we obtain the lemma. \( \square \)

Lemma 6.6 can be improved as follows.

**Lemma 6.7.** For any \( 0 < t < \min(2\beta_0, \gamma_0) \), there exists a constant \( C > 0 \) such that if \( \liminf_{r \to \infty} M_r(r^t v, v) = 0 \),
\[
\|v\|_{(t-1)/2} + \|w\|_{(t-1)/2} \leq C(\|f\|_{(t+1)/2} + \|u\|_{(t-1)-(\gamma_0)/2}^{s}).
\]

**Proof.** Take \( t < \delta < 2\beta_0 \) and put \( \varphi = r^t, a = 0 \) in the inequality in Lemma 6.6. We drop the term \( M_b(\varphi v, w) \) from the left-hand side. For large \( b > 0 \), the first two terms of the right-hand side are absorbed into the left-hand side. For the 3rd and 4th terms, we apply the inequalities
\[
|I_{(0,b)}(\varphi f, v)| \leq \epsilon I_{(0,b)}(r^{t-1}v, v) + C \epsilon \|f\|_{s}, \quad s = (t-1-(\gamma_0)/2).
\]
Here, we have used Lemma 5.1 (2). Then, the term \( \epsilon I_{(0,b)}(r^{t-1}v, v) \) is absorbed into the left-hand side. Letting \( b \to \infty \), we obtain the lemma. \( \square \)

**Remark 6.8.** If \( 2\beta_0 \geq \gamma_0 > 1 \), one can take \( 1 < t < \gamma_0 \) in Lemma 6.7. This is important to study the behavior of the resolvent as \( r \to \infty \) (Lemma 6.2), and explains the appearance of border-lines \( \beta_0 = 1/2 \) and \( \gamma_0 = 1 \).

**Lemma 6.9.** For any \( s > 1/2 \), there exists a constant \( C > 0 \) such that
\[
\|u\|_{B^s} \leq C(\|f\|_{B} + \|u\|_{s}).
\]

**Proof.** By Lemma 5.3
\[
- \text{Im} M(v, u)^b_a = M((\text{Re} \psi)u, u)^b_a + 2I_{(a,b)}((\text{Re} \psi)u, u) + I_{(a,b)}(f, u) - \text{Im} I_{(a,b)}(V u, u).
\]
For \( z \in J_+ \), \( \text{Re} \psi \) \text{Im} \( \psi \geq 0 \) by (5.12). Hence
\[
(6.16) \quad M((\text{Re} \psi)u, u)^b_a \leq - \text{Im} M(v, u)^b_a - I_{(a,b)}(u, (\text{Re} \psi)u) + \text{Im} I_{(a,b)}(f, u) + \text{Im} I_{(a,b)}(V u, u).
\]
We let \( 0 < a < 2 < b \) \( r \). By Lemma 5.1, there exist constants \( C, r_0 > 0 \) such that
\[
C \leq \text{Re} \psi, \quad \text{for} \quad r > r_0.
\]
Since \( \psi \in S^{-1-\alpha_0} \) by Lemma 6.1 (1), we have by (6.16). \[ M_r(u, u) \leq C(M_r(v, u) + \|f\|_{B} + \|u\|_{s}^2), \]
which, together with \( |M_r(v, u)| \leq \epsilon M_r(u, u) + C \epsilon M_r(v, v) \), yields
\[
M_r(u, u) \leq C(M_r(v, v) + \|f\|_{B} + \|u\|_{s}^2).
\]

Integrating with respect to \( r \) over \((0, R)\) and dividing by \( R \), we have
\[
\| u \|_{B^s} \leq C(\| v \|_{B^s} + \| f \|_{B^s} + \| u \|_{-s}).
\]
The lemma then follows from Lemma 6.6. \( \square \)

6.4. Cusp ends. We consider the case in which \( \mathcal{M} = (0, \infty) \times M \) is a cusp. We assume as in the beginning of this section. By (6.15), there exists a constant \( C_0 > 0 \) such that
\[
(6.17) \quad \rho(r)^{-1} \geq C_0(1 + r)^{|\beta|}.
\]
Let \( \Lambda(r) \) and \( \Lambda_0 \) be the Laplace-Beltrami operators on \( M \) associated with the metric \( h(r, x, dx) \) and \( h_M(x, dx) \), respectively. We put \( B(r) = -\rho(r)^{-2}\Lambda(r) \). Let \( P_0(r) \) be the projection associated with the 0 eigenvalue of \( \Lambda(r) \), and put \( P_1(r) = 1 - P_0(r) \).

**Lemma 6.10.** There exists \( r_0 > 0 \) such that for \( r > r_0 \), \( P_0(r) \) satisfies
\[
(6.18) \quad \left\| \left( \frac{d}{dr} \right)^m P_0(r) \right\| \leq C_m(1 + r)^{-m-\gamma_0}, \quad \forall m \geq 1,
\]
\[
(6.19) \quad \left\| P_0(r), \frac{g^\gamma}{g} \right\| \leq C_m(1 + r)^{-1-\gamma_0}.
\]
The same inequalities hold for \( P_1(r) \).

**Proof.** Since \( -\Lambda(r) \) and \( -\Lambda_0 \) have compact resolvents, and the 2nd eigenvalue of \( -\Lambda_0 \) is positive, there exists a constant \( \delta_0 > 0 \) and \( r_0 > 0 \) such that the second eigenvalue of \( -\Lambda(r) \) is greater than \( 2\delta_0 \) for all \( r > r_0 \). We take \( \varphi \in C_0^\infty(\mathbb{R}) \) such that \( \varphi(r) = 1 \) for \( |r| < \delta_0/2 \), \( \varphi(r) = 0 \) for \( |r| > \delta_0 \). Then, we have \( P_0(r) = \varphi(-\Lambda(r)) \) for \( r > r_0 \). The assertions for \( P_0(r) \) then follows from Lemma 2.14. Since \( P_1(r) = 1 - P_0(r) \), the lemma also holds for \( P_1(r) \). \( \square \)

**Lemma 6.11.** Let \( u \) be a solution to the equation
\[
\left( -\partial_r^2 - \frac{g'}{2g} \partial_r + B(r) - z \right) u = f, \quad z \in J_\pm
\]
satisfying \( u = 0 \) for \( r < 2r_0 \), and put \( u_0 = P_0(r)u, u_1 = P_1(r)u \).

1. For any \( s' > 1/2 \), there exists a constant \( C > 0 \) such that for \( s = |\beta| + 1 + \gamma_0 - s' \)
\[
\| u_1' \|_s + \| \sqrt{B(r)} u_1 \|_s + \| u_1 \|_s \leq C(\| P_1(r) f \|_s + \| u \|_{-s'}),
\]
\[
\| u_1'' \|_s + \| B(r) u_1 \|_s \leq C(\| P_1(r) f \|_s + \| u \|_{-s'}).
\]

2. For \( s' > 1/2 \)
\[
\| u_0 \|_{B^s} + \| D(k) u_0 \|_{B^s} \leq C(\| P_0(r) f \|_s + \| u \|_{-s'}),
\]

3. For \( 0 < t < \gamma_0 \), there exists a constant \( C > 0 \) such that
\[
\| D(k) u_0 \|_{(t-1)/2} \leq C(\| P_0(r) f \|_{(t+1)/2} + \| u \|_{(t-1-\gamma_0)/2}).
\]

**Proof.** Letting \( B = B(r), P_0 = P_0(r), P_1 = P_1(r), \) we have
\[
- u'' - \frac{g'}{2g} u' - zu = f_0,
\]
\[
f_0 = P_0 f - P_0' u - 2P_0 u' - \frac{g'}{2g} P_0 u + \left[ P_0, \frac{g'}{2g} \right] u',
\]

(6.20)
We have thus proven the assertion (1). In view of Lemma 6.2 (2), we then have as long as
\[ s \]
\[ \text{Noting that } u \]
\[ \text{regular end applies to assertions (2) and (3).} \]
\[ \square \]
Assume that \( u \in L^{2,s} \) for some \( s \). Integration by parts in the equation (6.21) gives
\[ \int (1 + r)^{2s} u_1' \ dx, u_1' \] + \Re (2s(1 + r)^{2s-1} u_1', u_1) + \((1 + r)^{2s} Bu_1, u_1) = \Re z \|u_1\|_s^2 + \Re ((1 + r)^{2s} f_1, u_1), \]
which yields
\[ \|u_1\|_s^2 - 2s\|u_1\|_s\|u_1\|_{s-1} + \|\sqrt{B} u_1\|_s^2 \leq C (\|f_1\|_s^2 + \|u_1\|_s^2). \]
Noting that \( P_1(-\Lambda) \geq \delta P_1 \), by (6.17) we have
\[ (Bu_1, u_1)_{h(r)} \geq C (1 + r)^{2|\beta|} \|u_1\|_{h(r)}^2, \quad C > 0. \]
This and (6.22) imply
\[ \|u_1'\|_s + \|u_1\|_{|\beta|+s} \leq C (\|f_1\|_s + \|u_1\|_s). \]
By Lemma 6.10
\[ \|f_1\|_s \leq C (\|f\|_s + \|u\|_{s-1-\gamma_0} + \|u'\|_{s-1-\gamma_0}). \]
In view of Lemma 6.2 (2), we then have as long as \( s - 1 - \gamma_0 \leq -s' \),
\[ \|u_1'\|_{|\beta|+s} + \|u_1\|_{|\beta|+s} \leq C (\|f\|_{|\beta|+s} + \|u\|_{-s'}). \]
Starting from \( s = -s' \), we obtain for \( s = |\beta| + 1 + \gamma_0 - s' \),
\[ \|u_1'\|_s + \|u_1\|_s \leq C (\|f\|_s + \|u\|_{-s'}). \]
We have also obtained
\[ \|u_1'\|_s + \|\sqrt{B(r)} u_1\|_s \leq C_s (\|P_1 f\|_s + \|u_1\|_{-s'}). \]
Arguing in the same way as in the proof of Lemma 2.5, we can show
\[ \|u'\|_s + \|B(r) u_1\|_s \leq C (\|f_1\|_s + \|u_1\|_s) \leq C (\|f_1\|_s + \|u_1\|_{-s'}). \]
We have thus proven the assertion (1).

As for \( u_0 \), since \( Bu_0 = 0 \), the problem is reduced to the one-dimensional case, and we can argue in the same way as in the previous subsection. This proves the assertions (2) and (3). \( \square \)

Let us note that Lemma 6.4 also holds for the cusp end, since the proof for the regular end applies to \( u_0 \) as well and \( u_1 \) belongs to \( L^{2,-1/2+\epsilon} \) for some \( \epsilon > 0 \) by virtue of Lemma 6.11 (1).

**Lemma 6.12.** For any \( s > 1/2 \), there exists a constant \( C > 0 \) such that
\[ \|u\|_{B^s} \leq C (\|f\|_s + \|u\|_{-s}). \]
7. Spectrum of the Laplacian on $\mathcal{M}$

We are now ready to study the spectral theory of $-\Delta_{\mathcal{M}}$. Let $\mathcal{M}$ be a connected $n$-dimensional CMAG of the form

$$
\mathcal{M} = \mathcal{K} \cup \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_{N+N'},
$$

(7.1)

satisfying (0.14), (0.15), (0.16) and (0.18). In this section, we assume

$$
\alpha_{0,i} > 0, \quad \beta_{0,i} > 0, \quad \gamma_{0,i} > 0
$$

(7.2)

for all regular ends and (A-3) for cusp ends.

7.1. Essential spectrum. Assuming that

$$
\mathcal{K} \cap ((1/2, \infty) \times M_i) = \emptyset, \quad i \neq 0,
$$

we take a partition of unity $\{\chi_i\}_{i=0}^{N+N'}$ on $\mathcal{M}$ such that

$$
\sum_{i=0}^{N+N'} \chi_i = 1, \quad \text{on } \mathcal{M},
\chi_0 \in C_0^\infty(\mathcal{M}), \quad \chi_0 = 1 \quad \text{on } \mathcal{K},
\chi_i = 0 \quad \text{on } (0,2) \times M_i, \quad i \neq 0,
\chi_i = 1 \quad \text{on } (3, \infty) \times M_i, \quad i \neq 0.
$$

(7.3)

We also take $\tilde{\chi}_i \in C^\infty((0, \infty) \times M_i)$ such that

$$
\tilde{\chi}_i = 1, \quad \text{on } (1, \infty) \times M_i,
\tilde{\chi}_i = 0, \quad \text{on } (0,1/2) \times M_i.
$$

For $i = 1, \cdots, N$ and $s \in \mathbb{R}$, the Banach spaces $L^{2,s}(\mathcal{M}_i)$, $B(\mathcal{M}_i)$, $B^*(\mathcal{M}_i)$, $B^0(\mathcal{M}_i)$ are defined in the same way as in Subsection 6.1. For $i = N+1, \cdots, N+N'$, these space are defined similarly with $h(r)$ replaced by $\mathcal{C}$. We put for $f \in L^{2}_{loc}(\mathcal{M})$ and $s \in \mathbb{R}$,

$$
\|f\|_{L^{2,s}(\mathcal{M})} = \|\chi_0 f\|_{L^2(\mathcal{M})} + \sum_{i=1}^{N+N'} \|\chi_i f\|_{L^{2,s}(\mathcal{M}_i)},
$$

$$
\|f\|_{B(\mathcal{M})} = \|\chi_0 f\|_{L^2(\mathcal{M})} + \sum_{i=1}^{N+N'} \|\chi_i f\|_{B(\mathcal{M}_i)},
$$

$$
\|f\|_{B^*(\mathcal{M})} = \|\chi_0 f\|_{L^2(\mathcal{M})} + \sum_{i=1}^{N+N'} \|\chi_i f\|_{B^*(\mathcal{M}_i)},
$$

and also define

$$
f \in B^0(\mathcal{M}) \iff \chi_i f \in B^0(\mathcal{M}_i), \quad 1 \leq i \leq N+N'.
$$

In the following, we often denote these norms by $\|f\|_s$, $\|f\|_B$, $\|f\|_{B^*}$ omitting the end $\mathcal{M}_i$ or $\mathcal{M}$, which will not confuse our arguments. Finally, we define $L^{2}_{comp}(\mathcal{M})$ to be the set of all compactly supported $L^2$-functions on $\mathcal{M}$.

We define

$$
H = -\Delta_{\mathcal{M}}
$$
to be the Friedrichs extension of $-\Delta_M$. Therefore, $D(\sqrt{H}) = H^1(M)$, the Sobolev space of order 1 on $M$. We introduce the following two operators defined on $\tilde{M}_i = (2, \infty) \times M_i$ with Dirichlet boundary condition at $r = 2$:

\[(7.4) \quad H_i = -\Delta_M, \quad H_{\text{free}}(i) = -\partial^2_r - \frac{(n-1)\rho_i'(r)}{\rho_i(r)} \partial_r - \rho_i(r)^2 \Lambda_i,\]

where $\Lambda_i$ is the Laplace-Beltrami operator on $M_i$ equipped with the metric $h_{M_i}(x, dx)$. Let for $z \notin \mathbb{R}$

\[R(z) = (H - z)^{-1}, \quad R_i(z) = (H_i - z)^{-1}, \quad R_{\text{free}}(i)(z) = (H_{\text{free}}(i) - z)^{-1}.\]

Note that

\[R(z) \in \mathcal{B}(L^2(M); H^1(M)), \quad R_i(z), R_{\text{free}}(i)(z) \in \mathcal{B}(L^2(\tilde{M}_i); H^1(\tilde{M}_i)).\]

The following formula holds:

\[(7.5) \quad R(z) = \sum_{i=1}^{N+N'} \chi_i R_i(z) \tilde{\chi}_i + R(z)(\chi_0 - S(z)), \]

\[S(z) = \sum_{i=1}^{N+N'} S_i(z) \tilde{\chi}_i, \quad S_i(z) = [H, \chi_i] R_i(z).\]

In fact, we have

\[\sum_{i=1}^{N+N'} \chi_i R_i(z) \tilde{\chi}_i = \sum_{i=1}^{N+N'} S_i(z) \tilde{\chi}_i + \sum_{i=1}^{N+N'} \chi_i \tilde{\chi}_i,\]

from which (7.5) follows. Similarly, one can show the following formula.

\[(7.6) \quad R(z) = \sum_{i=1}^{N+N'} \chi_i R_{\text{free}}(i)(z) \tilde{\chi}_i + R(z)(\chi_0 - T(z)), \]

\[T(z) = \sum_{i=1}^{N+N'} T_i(z) \tilde{\chi}_i, \]

\[T_i(z) = [H, \chi_i] R_{\text{free}}(i)(z) + \chi_i (H - H_{\text{free}}(i)) \tilde{\chi}_i R_{\text{free}}(i)(z).\]

We put

\[(7.7) \quad E_{0,i} = \left( \frac{(n-1)c_{0,i}}{2} \right)^2, \quad E_{0,tot} = \min_{1 \leq i \leq N+N'} E_{0,i}.\]

**Lemma 7.1.**

1. $\sigma_{ess}(H_i) = [E_{0,i}, \infty)$, $i = 1, \ldots, N + N'$.
2. $\sigma_{ess}(H) = [E_{0,tot}, \infty)$.

**Proof.** Let $0 = \lambda_0^{(i)} < \lambda_1^{(i)} \leq \cdots$ be the eigenvalues of $-\Lambda_i$. Then, $H_{\text{free}}(i)$ is unitarily equivalent to the direct sum $\bigoplus_{k=0}^{\infty} L_{\epsilon}^{(i)}$, where

\[L_{\epsilon}^{(i)} = -\partial^2_r - \frac{(n-1)\rho_i'(r)}{\rho_i(r)} \partial_r + \frac{\lambda_k^{(i)}}{\rho_i(r)^2}.\]
By the transformation $u \rightarrow \rho_i(r)^{(n-1)/2}u$, $\tilde{L}^{(i)}_\ell$ is unitarily equivalent to $\tilde{L}^{(i)}_\ell$, where

$$\tilde{L}^{(i)}_\ell = -\partial_r^2 + \frac{\lambda^{(i)}_\ell}{\rho_i(r)^2} + Q_\ell, \quad Q_\ell = \left(\frac{n-1}{2} \frac{\rho_i'}{\rho_i}\right)^2 + \left(\frac{n-1}{2} \frac{\rho_i'}{\rho_i}\right)^2.$$ 

By (0.14), we have $Q_\ell = E_{0,i} + O(r^{-\alpha_\ell})$ as $r \to \infty$. Since $\rho_i(r) \to \infty$ for the case of regular end, we have $\sigma_e(\tilde{L}^{(i)}_\ell) = [E_{0,i}, \infty)$. For the case of cusp end, we have $\rho_i^{-1}(r) \to \infty$, hence $\sigma_e(\tilde{L}^{(i)}_\ell) = \emptyset$ if $\lambda^{(i)}_\ell > 0$, and $\sigma_e(\tilde{L}^{(i)}_\ell) = [E_{0,i}, \infty)$ if $\lambda^{(i)}_\ell = 0$. This proves that $\sigma_e(H_{\text{free}(i)}) = [E_{0,i}, \infty)$. By Weyl’s theorem for the perturbation of the essential spectrum, we have $\sigma_e(H_i) = \sigma_e(H_{\text{free}(i)})$.

Applying well-known Weyl’s method of singular sequence, we can show $\sigma_e(H_i) \subset \sigma_e(H)$. Therefore $[E_{0,\text{tot}}, \infty) \subset \sigma_e(H)$. To prove the converse inclusion relation, take any compact interval $I \subset (-\infty, E_{0,\text{tot}})$, and $f(\lambda) \in C_0^\infty(\mathbb{R})$ such that $f(\lambda) = 1$ on $I$ and $f(\lambda) = 0$ on $[E_{0,\text{tot}}, \infty)$. Using (7.5), one can show

$$f(H) = \sum_{i=1}^{N+N'} \chi_i f(H_i) \overline{\chi}_i + K,$$

where $K$ is a compact operator (see e.g. the proof of Chap. 3, Theorem 3.2 of [51]). Since $I \cap \sigma_e(H_i) = \emptyset$, $f(H_i)$ is compact. Therefore, $f(H)$ is also compact. This proves that $I \cap \sigma_e(H) = \emptyset$, which implies $\sigma_e(H) \subset [E_{0,\text{tot}}, \infty)$. \hfill $\square$

### 7.2. Embedded eigenvalues

We put

$$\mathcal{T} = \{E_{0,1}, \ldots, E_{0,N+N'}\}.$$

**Theorem 7.2.** If there exists a regular end $\mathcal{M}_i$ with $\beta_i > 1/3$, we have

$$\sigma_p(H) \cap (E_{0,i}, \infty) = \emptyset.$$

**Proof.** By (1.1) and (1.5), we can apply Theorem 1.3. Therefore, if $u$ is an eigenfunction of $-\Delta_{\mathcal{M}}$ with eigenvalue in $(E_{0,i}, \infty)$, $u$ vanishes near infinity of $\mathcal{M}_i$. By the unique continuation theorem, $u$ vanishes identically on $\mathcal{M}$. \hfill $\square$

Recall that $\frac{\rho_i'}{\rho_i} \to c_{0,i}$ on each end. Therefore, if $c_{0,i} > 0$, $\rho_i(r)$ is exponentially growing and $\beta_i > 1/3$ holds, hence there is no embedded eigenvalue in $(E_{0,i}, \infty)$. If $c_{0,i} = 0$ for some regular end and $\beta_i > 1/3$, again there is no embedded eigenvalue in $(0, \infty)$. The remaining cases are the ones in which either $\beta_i \leq 1/3$ for all regular ends or all the ends are cusp. In these cases the essential spectrum is $[0, \infty)$, and the embedded eigenvalues are discrete in the following sense.

**Theorem 7.3.** Assume that $\beta_i \leq 1/3$ for all regular ends. Then, the eigenvalues in $(0, \infty) \setminus \mathcal{T}$ are of finite multiplicities with possible accumulation points at $\mathcal{T}$ and $\infty$.

In particular, this theorem holds when all ends are cusp.

To prove this theorem, we first show that the eigenfunctions associated with embedded eigenvalues decay faster than $L^2$. 


By Rellich’s selection theorem,

\[ \|u\|_{H^{1,s}(\mathcal{M})} \leq C\|u\|_{L^2(\mathcal{M})} \]

holds for any eigenfunction \( u \) of \(-\Delta_M\) with eigenvalue in \( I \).

**Proof.** We have proven that \( u \in L^{2,s} \) for any \( s > 0 \) on the regular end \( \mathcal{M}_i \), moreover, by Lemma 7.4

\[ \|\chi u\|_{H^{1,s}(\mathcal{M}_i)} \leq C_\chi \|u\|_{L^2} \]

where, \( \chi = 1 \) for \( r > 2 \), \( \chi = 0 \) for \( r < 1 \). On the cusp end, multiplying \( \chi \) to \( u \), which we denote by \( u \) and use Lemma 6.11. Then, \( P_i(r)u \) obeys the desired estimate. Let \( \bar{u}_0 = g^{1/4}P_0(r)u \). Then, it satisfies

\[ -\bar{u}_0'' - k^2\bar{u}_0 = P_0(r)f - \left( \frac{g'_i}{4g} + \frac{g''_i}{4g^2} \right)\bar{u}_0, \]

where \( f \) is compactly supported. By virtue of (2.57), \( P_0(r)u \) satisfies the desired inequality. \( \Box \)

Let us now prove Theorem 7.3.

**Proof.** Take a compact interval \( I \subset (0, \infty) \setminus \mathcal{T} \) and suppose that there exists an infinite number of eigenvalues \( \{\lambda_i\}_{i=1}^\infty \subset I \). Let \( \{\varphi_i\}_{i=1}^\infty \) be the orthonormal system of eigenvectors associated with \( \{\lambda_i\}_{i=1}^\infty \). By Lemma 7.4 \( \{\varphi_i\}_{i=1}^\infty \) is bounded in \( H^{1,s}(\mathcal{M}) \) for some \( s > 0 \). This implies that for any \( \epsilon > 0 \) there is a constant \( r_0 > 0 \) such that on each end

\[ \|\varphi_i\|_{L^2(r_0, \infty) \times \mathcal{M}_i} < \epsilon, \quad \forall i \geq 1. \]

By Rellich’s selection theorem, \( \{\varphi_i\}_{i=1}^\infty \) contains a subsequence which is convergent in \( L^2(\mathcal{M}) \), which is a contradiction. \( \Box \)

### 7.3. Radiation condition and uniqueness.

The radiation condition is the boundary condition at infinity to guarantee the uniqueness of solutions to the reduced wave equation. It is closely related to the Rellich-Vekua theorems.

We define \( D_j(k) \) in the same way as in (6.10) on each end \( \mathcal{M}_j \) with \( E_0 \) replaced by \( E_{0,j} \). Let \( \psi_j(k) = \psi_j(r, x, k) \) be an approximate solution of the equation

\[ -i\psi' + \psi^2 + Q_j - k^2 = 0, \quad Q_j = \left( \frac{g_j'}{4g_j} \right)' + \left( \frac{g_j''}{4g_j} \right), \quad -E_{0,j}. \]

By Lemma 5.1 it behaves like

\[ \psi_j(k) = \sqrt{z - E_{0,j}} + O(r^{-\alpha_{0,j}}), \quad \text{as} \quad r \to \infty. \]

For \( z = re^{i\theta}, 0 < \theta < 2\pi \), we defined \( \sqrt{z} = \sqrt{r}e^{i\theta/2} \). Therefore, for \( \lambda > 0 \),

\[ \sqrt{\lambda \pm i\epsilon} \to \pm \sqrt{\lambda}, \quad \text{as} \quad \epsilon \to 0. \]

Hence, for \( \lambda > E_{0,j} \), we have two \( \psi_j \)'s, denoted by \( \psi_j^{(\pm)} \), where

\[ \psi_j^{(\pm)}(\sqrt{\lambda - E_{0,j}}) = \pm \sqrt{\lambda - E_{0,j}} + O(r^{-\alpha_{0,j}}), \quad \text{as} \quad r \to \infty. \]
We put

\begin{equation}
D_j^{(\pm)}(k) = \partial_k + \frac{\gamma'_j}{4g_j} - iv_j^{(\pm)}(k),
\end{equation}

\begin{equation}
\mathcal{E} = \mathcal{T} \cup \sigma_\nu(H).
\end{equation}

**Definition 7.5.** Let \( \lambda \in \sigma_e(H) \setminus \mathcal{E} \). A solution \( u \in \mathcal{B}^*(\mathcal{M}) \) of the equation

\begin{equation}
( - \Delta_{\mathcal{M}} - \lambda ) u = f \quad \text{on} \quad \mathcal{M}
\end{equation}

is said to satisfy the outgoing radiation condition on the end \( \mathcal{M}_j \), if it has the following properties.

1. For \( \lambda < E_0, j \) and \( 1 \leq j \leq N + N' \),
   \( u, u' \in \mathcal{B}_0^*(\mathcal{M}_j) \).

2. For \( \lambda > E_0, j \) and \( 1 \leq j \leq N \),
   \( D_j^{(+)}(k)u \in \mathcal{B}_0^*(\mathcal{M}_j) \), if \( 1/3 < \beta_{0,j} < 0 \),
   \( D_j^{(+)}(k)u \in L^2(1-\epsilon)/2(\mathcal{M}_j) \), if \( 0 < \beta_{0,j} \leq 1/3 \),
   for some \( \epsilon > 0 \).

3. For \( \lambda > E_0, j \) and \( N + 1 \leq j \leq N + N' \),
   \( D_j^{(+)}(k)u \in \mathcal{B}_0^*(\mathcal{M}_j) \).

When \( D_j^{(+)}(k) \) is replaced by \( D_j^{(-)}(k) \), \( u \) is said to satisfy the incoming radiation condition. We say that \( u \) satisfies the outgoing (incoming) radiation condition on \( \mathcal{M} \), if it satisfies the outgoing (incoming) radiation condition on all \( \mathcal{M}_j \).

The purpose of this subsection is to prove the following theorem.

**Theorem 7.6.** Assume that \( \lambda \in \sigma_e(H) \setminus \mathcal{E} \), and let \( u \in \mathcal{B}^*(\mathcal{M}) \) be a solution to the equation \( ( - \Delta_{\mathcal{M}} - \lambda ) u = 0 \) satisfying the radiation condition. Then, \( u = 0 \).

The starting point of the proof of this theorem is the following formula \((7.13)\) in Lemma 7.7. Let \( \{ \chi_j \}_{j=0}^{N+N'} \) be the partition of unity on \( \mathcal{M} \) satisfying \((7.14)\). Take \( \varphi(r) \in C^\infty(\mathbb{R}) \) such that \( \varphi(r) = 1 \) for \( r < 1 \), \( \varphi(r) = 0 \) for \( r > 2 \), and put

\[ \phi(r) = \int_r^\infty \varphi(t)dt, \quad \phi_R(r) = \phi\left(\frac{r}{R}\right). \]

Then, \( \phi_R(r) = 0 \) for \( r > 2R \) and \( \phi'_R(r) = -\frac{1}{\pi} \varphi\left(\frac{r}{R}\right) \).

**Lemma 7.7.** Let \( u \) be a solution to the equation \( ( - \Delta_{\mathcal{M}} - \lambda ) u = 0 \). Then, for any constant \( \sigma \in \mathbb{R} \),

\begin{equation}
\sum_{j=1}^{N+N'} \frac{1}{R} \left( \varphi\left(\frac{r}{R}\right) \right)^{r^\sigma} \chi_j u, u' \right) = \frac{1}{2} \text{Im} \left( \left( [H, r^\sigma (1 - \chi_0)] \phi_R u, u \right) \right).
\end{equation}

**Proof.** Consider the equation

\[ \text{Im} \left( \phi_R r^\sigma \chi_j u, (\Delta_{\mathcal{M}} + \lambda) u \right) = 0. \]

Then by integration by parts,

\[ \text{Im} \left( (\phi_R r^\sigma \chi_j)' u, u' \right) = 0, \]
which implies
\[ \text{Im} \left( \frac{1}{R} \varphi(R^{-1}) r^\sigma \chi_j u, u' \right) = \text{Im} \left( \phi_R(r^\sigma \chi_j)' u, u' \right), \quad \forall j \geq 1. \]

Summing up with respect to \( j \), we have
\[ (7.14) \sum_{j=1}^{N+N'} \frac{1}{R} \left( \varphi(R^{-1}) r^\sigma \chi_j u, u' \right) = \text{Im}(\phi_R(r^\sigma (1 - \chi_0))' u, \partial_r u). \]

Noting that on each end for any smooth function \( w(r) \)
\[ 2w' \partial_r = [w, H] - w'' - g'(w'), \]
we have
\[ \text{Im}(\phi_R w' u, \partial_r u) = \frac{1}{2} \text{Im}([H, w] \phi_R u, u). \]

Letting \( w = r^\sigma (1 - \chi_0) \), and taking notice of \( (7.14) \), we obtain the lemma. \( \square \)

We consider the case of outgoing radiation condition, and divide the proof of Theorem 7.6 into three cases.

**Lemma 7.8.** Let \( u \in B^*(M) \) be a solution to \((-\Delta_M - \lambda)u = 0\) satisfying the radiation condition. Suppose there exists a regular end \( M_j \) such that \( \lambda > E_{0,j} \) and \( \beta_{0,j} > 1/3 \). Then, \( u = 0 \) on \( M \).

**Proof.** Take \( \sigma = 0 \) in \( (7.13) \). Using the equation \((H - \lambda)u = 0\), we have
\[ ([H, -\chi^0_0] \phi_R u, u) = (\chi^0_0[H, \phi_R] u, u). \]

The right-hand side tends to 0 as \( R \to \infty \). Using the radiation condition for the left-hand side of \( (7.13) \) and noting that \( u, u' \in B^0_\sigma(M_j) \) if \( \lambda < E_{0,j} \), we then have
\[ \lim_{R \to \infty} \sum_{j=1}^{N'} \frac{1}{R} \left( \varphi(R^{-1}) \chi_j u, \sqrt{\lambda - E_{0,j}} \chi_j u \right) = 0, \]
where the sum \( \sum_{j} \) ranges over all \( j \) such that \( \lambda > E_{0,j} \). This yields
\[ (7.15) \lim_{R \to \infty} \frac{1}{R} \left( \varphi(R^{-1}) \chi_j u, \chi_j u \right) = 0, \quad \forall j \geq 1. \]

Hence \( u \in B^0_\sigma(M_j), \forall j \geq 1 \). Then, letting \( S(r) = \{r\} \times M_j \), we have
\[ \liminf_{r \to \infty} \int_{S(r)} (|u'|^2 + |u|^2) dS(r) = 0. \]

By Theorem 122, \( u = 0 \) near infinity of \( M_j \) with \( \beta_{0,j} > 1/3 \). Therefore, \( u = 0 \) on \( M \) by the unique continuation theorem. \( \square \)

We next consider the case in which all ends are cusp.

**Lemma 7.9.** Suppose all ends are cusp, and \( \lambda \in \sigma_\sigma(H) \setminus \mathcal{E} \). If \( u \in B^* \) satisfies \((-\Delta_M - \lambda)u = 0\) and the radiation condition, then \( u = 0 \).
PROOF. We pick up one end $M_i$ and drop the subscript $i$. Take $\chi \in C^\infty((0, \infty))$ such that $\chi = 1$ for $r > r_0 + 1$ and $\chi = 0$ for $r < r_0$, where $r_0$ is chosen large enough. Then, $U = \chi u$ satisfies

$$(-\partial_r^2 - \frac{g'}{2g} \partial_r + B(r) - \lambda)U = f,$$

where $f = -\chi'' u - 2\chi'u' - \frac{g'}{2g} \chi'u$. Recalling that now $M = M_i$, we show that $U \in L^2(M_i)$. If $\lambda < E_{0,i}$, this is true. Below, we consider the case in which $\lambda > E_{0,i}$. Again dropping the subscript $i$, let $P_0(r)$ be the projection to the 0-eigenvalue of $\Lambda(r)$ and $P_1(r) = 1 - P_0(r)$. Then, $u_0 = P_0(r)U$ and $u_1 = P_1(r)U$ satisfy

$$(-\partial_r^2 - \frac{g'}{2g} \partial_r + B(r) - \lambda)u_0 = f_0,$$

$$(-\partial_r^2 - \frac{g'}{2g} \partial_r + B(r) - \lambda)u_1 = f_1,$$

$$f_i = P_i(r) f - P_i''(r)U - 2P_i'(r)U' - \frac{g'}{2g} P_i'(r)u + [P_i(r), \frac{g'}{2g}]U', \quad i = 0, 1.$$ 

Letting $v_0 = g^{1/4} u_0$ and $k^2 = \lambda - E_0$, we have

$$-v_0'' - k^2 v_0 = f_2,$$

$$f_2 = g^{1/4} f_0 - \left(\left(\frac{g'}{4g}\right)' + \left(\frac{g'}{4g}\right)^2 - E_0\right)v_0.$$ 

Note that $f$ is compactly supported. Lemma 6.10 implies

$$\|P_i'(r)\| + \|P_i''(r)\| + \|[P_i(r), \frac{g'}{2g}]\| \leq C(1 + r)^{-1-2\epsilon},$$

and the assumption $b.19$ yields

$$\left|\left(\frac{g'}{4g}\right)' + \left(\frac{g'}{4g}\right)^2 - E_0\right| \leq C(1 + r)^{-1-2\epsilon}$$

for a small $\epsilon > 0$. Note also that

$$U \in B^* \subset L^2, (1+\epsilon)/2.$$ 

Therefore, $f_1 \in L^2, (1+\epsilon)/2$. Lemma 6.11 (1) then implies that $u_1 \in L^2, (1+\epsilon)/2$.

Similarly, $f_2 \in L^2, (1+\epsilon)/2$. In the proof of Lemma 7.8, we have already proven that $u_0 \in B^*_0(M)$, which implies $\lim_{r \to \infty} |v_0(r)| = 0$. Then, by Lemma 2.15, $v_0 \in L^2, (\epsilon-1)/2$. This fact yields $f_0 \in L^2, (1+2\epsilon)/2$. Then, by the same argument, we have $u_0 \in L^2(\epsilon-1)/2$. Repeating this procedure, we obtain $u_0 \in L^2(1+\epsilon)/2$. We have thus proven that $U \in L^2$. Therefore, $u \in L^2(M_i)$ on all ends $M_i$. Since $\lambda$ is not an eigenvalue of $-\Delta_M$, we have $u = 0$. \hfill \Box

We consider the remaining case.

**Lemma 7.10.** Assume that $0 < \beta_i \leq 1/3$ on all regular ends. Let $u \in B^*(M)$ be a solution to $(-\Delta_M - \lambda)u = 0$ on $M$ satisfying the radiation condition and $\lambda \in \sigma_e(H) \backslash \mathcal{E}$. Then, $u = 0$. 

Proof. By the arguments in the proof of Lemma 7.9 on all cusp ends $M_i$, $u \in L^2(1+\epsilon)/2(M_i)$, hence $D_i^+(k)u \in L^2(1+\epsilon)/2(M_i)$.

Take $\sigma = \epsilon$ in (7.13), and let $w = r^\sigma(1 - \chi_0)$. By the equation $-\Delta_M u = \lambda u$, we have

\[ ([H, w] \phi_R u, u) = -(w[H, \phi_R] u, u) \]

which tends to 0 as $R \to \infty$. In fact, since $\phi'_R(r) = -\frac{1}{R} \phi(Rr)$,

\[ |(w[H, \phi_R] u, u)| \leq \frac{1}{R} \int_{r<2R} (1 + r)^{n-1} |f(r, x)|^2 \rho^{n-1}(r) dr dx \]

for some $f \in B^*$. Then, by (7.13), we have

\[ \sum_{j=1}^{N+N'} \text{Im} \frac{1}{R} \phi(Rr) r^\sigma \chi_j u, u' \]

By the radiation condition, we then have

\[ \sum_{j=1}^{N+N'} \frac{1}{R} \phi(Rr) r^\sigma \chi_j u, u' = 0, \]

which implies that

\[ \liminf_{r \to \infty} \int_{S(r)} r^\sigma (|u'|^2 + |u|^2) dS(r) = 0. \]

Theorem 4.1 then yields $u \in L^2$, hence $u = 0$. \qed

8. Limiting absorption principle

For a self-adjoint operator $A$ on a Hilbert space $H$, $(A - \lambda)^{-1}$ does not exist for $\lambda \in \sigma(A)$. However, when $\lambda \in \sigma_c(A)$, it often happens that $\lim_{\epsilon \to 0}(A - \lambda \mp i\epsilon)^{-1}$ exists as an operator from $X$ to $Y$, where $X$ and $\mathcal{Y}$ are Banach spaces rigging $H$, i.e. $X \subset H \subset Y$, with dense and continuous imbedding. This is called the limiting absorption principle and used as a fundamental tool in the study of continuous spectrum. The purpose of this section is to prove this limiting absorption principle for the Laplacian $H = -\Delta_M$ on $M$. In this section, we assume (7.2). We put $R(z) = (H - z)^{-1}$. Take a compact interval $I \subset \sigma_c(H) \setminus \mathcal{E}$, and define $J_\pm = \{ z \in \mathbb{C} ; \Re z \in I, 0 < \Im z < 1 \}$.

LEMMA 8.1. Let $s > 1/2$.

1) There exists a constant $C_s > 0$ such that

\[ \| R(z) f \|_{-s} \leq C_s \| f \|_s, \quad z \in J_\pm. \]

2) For any $\lambda \in I$ and $f \in L^2$, there exists a strong limit $s - \lim_{\epsilon \to 0} R(\lambda \pm i\epsilon) f$ in $L^{2-s}$. Moreover, $R(\lambda \pm i\epsilon) f$ is an $L^{2-s}$-valued strongly continuous function of $\lambda \in I$.

3) For $\lambda \in I$, $R(\lambda + i\epsilon) f$ satisfies the outgoing radiation condition, and $R(\lambda - i\epsilon) f$ satisfies the incoming radiation condition.
PROOF. Let us prove (1) for the case $z \in J_+$. If the assertion (1) does not hold, there exist sequences $z_j \in J_+$, $f_j \in L^{2,\infty}$ such that $\|f_j\|_\infty \to 0$, and $u_j = R(z_j)f_j$ satisfies $\|u_j\|_{-\infty} = 1$. Then, there exists a subsequence, which is denoted by $\{z_j\}$ again, such that $z_j \to \lambda$. If $\lambda \notin \mathbb{R}$, we easily arrive at a contradiction. Assume that $\lambda \in I$. Fix any $i = 1, \ldots, N + N'$, and take $s > s' > 1/2$. Note the inequality
\[
\int_R (1 + r)^{-2s} \|\chi_i u_j\|_{h_i(r)}^2 \rho_i(r)^{n-1} dr 
\leq (1 + R)^{-2(s-s')} \int_R (1 + r)^{-2s'} \|\chi_i u_j\|_{h_i(r)}^2 \rho_i(r)^{n-1} dr.
\]
Taking account of Lemmas 6.6, 6.7 and 6.11 we have
(8.1) \[
\int_R (1 + r)^{-2s} \|\chi_i u_j\|_{h_i(r)}^2 \rho_i(r)^{n-1} dr \leq C(1 + R)^{-2(s-s')}, \quad 1/2 < s' < s,
\]
where the constant $C$ does not depend on $j$. By the a-priori estimate (Lemma 6.2), \{u_j\} is bounded in $H^{1-s}(M)$. By Rellich’s selection theorem and (8.1), we can choose a subsequence, which is denoted by $\{u_j\}$ again, and $u \in L^{2,\infty}$ such that $u_j \to u$ in $L^{2,\infty}$. Then $\|u\|_{-\infty} = 1$, and $u$ satisfies
\[
(- \Delta_M - \lambda)u = 0.
\]
By Lemmas 6.6, 6.7 and 6.11 $u$ satisfies the outgoing radiation condition. Theorem 7.6 then implies that $u = 0$, which is a contradiction.

To prove (2), take a sequence $z_j \in J_+$ such that $z_j \to \lambda \in J_+$, and put $u_j = R(z_j)f$. Arguing as above, we see that $\{u_j\}$ contains a subsequence $\{u_j'\}$, which is convergent in $L^{2,\infty}$, and the limit $u$ satisfies the outgoing radiation condition as well as the equation
\[
(- \Delta_M - \lambda)u = f.
\]
Such a solution $u$ is unique. We thus see that any subsequence of $\{u_j\}$ contains a sub-sub sequence which converges to one and the same limit. This shows that $\{u_j\}$ itself converges in $L^{2,\infty}$. The strong continuity is proven similarly. The assertion (3) is already proven. □

We extend Lemma 8.1 to $\mathcal{B}, \mathcal{B}^*$ spaces.

**THEOREM 8.2.** Let $I \subset \sigma_e(H) \setminus \mathcal{E}$ be a compact interval and $J_\pm = \{z \in \mathbb{C}; \text{Re} \ z \in I, \ 0 < \text{Im} \ z < 1\}$.
(1) There exists a constant $C > 0$ such that
\[
\|R(z)f\|_{\mathcal{B}^*} \leq C\|f\|_{\mathcal{B}}, \quad z \in J_\pm.
\]
(2) For any $f \in \mathcal{B}$, and $\lambda \in I$, there exists a $s$-weak limit $\lim_{\varepsilon \to 0} R(\lambda \pm i\varepsilon)f$, i.e. for any $f, g \in \mathcal{B},$
\[
\lim_{\varepsilon \to 0} (R(\lambda \pm i\varepsilon)f, g) = (R(\lambda \pm i0)f, g).
\]
Moreover
\[
\|R(\lambda \pm i0)f\|_{\mathcal{B}^*} \leq C\|f\|_{\mathcal{B}}, \quad \lambda \in I,
\]
and $(R(\lambda \pm i0)f, g)$ is a continuous function of $\lambda \in I$ for any $f, g \in \mathcal{B}$.
(3) For any $f \in \mathcal{B}$, and $\lambda \in I$, $R(\lambda \pm i0)f$ satisfies $D^{(\pm)}(k)R(\lambda \pm i0)f \in \mathcal{B}_0(M_i)$ on each end $M_i$, $i = 1, \ldots, N + N'$.
The aim of this section is to prove the following theorem.

9.1. Asymptotic expansion at infinity. We study the asymptotic behavior of the resolvent \( R(\lambda \pm i0)f \) as \( r \to \infty \) on each end, where \( \lambda \in (E_0, \infty) \setminus \mathcal{E} \). In this section, we consider the metric of long-range behavior, using the method developed for the Schrödinger operator \(-\Delta + V(x)\) in \( \mathbb{R}^n \) (see e.g. \([97], [98], [50]\)). We pick up a regular end \( \mathcal{M}_j \) satisfying

\[
\alpha_{0,j} > 0, \quad \beta_{0,j} > 1/2, \quad \gamma_{0,j} > 1.
\]

The aim of this section is to prove the following theorem.

**Theorem 9.1.** Let \( \mathcal{M}_j \) be a regular end satisfying \([9.1]\), and put

\[
\Phi_j(r, \lambda) = \int_0^r \phi_j(t, \lambda)dt,
\]
Then, for $\lambda > E_{0,j}$, there exists a strong limit

$$
\lim_{r \to \infty} e^{\mp \Phi_j(r, \lambda)} \rho_j(r)^{(n-1)/2} \left( R(\lambda \pm i0) f \right)(r) \quad \text{in} \quad L^2(M_j),
$$

for $f \in L^2_{\text{comp}}(M)$. 

Note that

$$
\frac{(n-1)^2}{4} \left( \frac{\rho_j'}{\rho_j} \right)^2 - E_{0,j} \in \begin{cases} S^{-\alpha_j}, & \text{if } c_{0,j} \neq 0, \\ S^{-2\alpha_j}, & \text{if } c_{0,j} = 0, 
\end{cases}
$$
hence $\Phi_j(r, \lambda) \sim \sqrt{\lambda - E_{0,j}} r = kr$ as $r \to \infty$.

### 9.2. Reduction to each end

Let $H_j$ be the Laplacian on $\tilde{M}_j = (2, \infty) \times M_j$ associated with the metric $(dr)^2 + \rho_j(r)^2 h_j(r, x, dx)$ with Dirichlet boundary condition at $r = 2$. By the well-known perturbation theory, one can show that $\sigma_e(H_j) = [E_{0,j}, \infty)$, and by Theorem 9.2 $H_j$ has no eigenvalues in $(E_{0,j}, \infty)$. The limiting absorption principle in §8 can also be proved for $H_j$, since our concern is only the estimates of the resolvent at infinity. Hence, Theorem 9.2 holds for $H_j$ on $\tilde{M}_j$ as well. We put $R_j(z) = (H_j - z)^{-1}$.

By using the partition of unity $\{\chi_j\}_{j=1}^{N+N'}$ in (7.8), we have for $\lambda \in \sigma_e(H) \setminus \mathcal{E}$

$$
\chi_j R(\lambda \pm i0) = R_j(\lambda \pm i0) \chi_j + R_j(\lambda \pm i0)[H_j, \chi_j] R(\lambda \pm i0).
$$

Hence, the behavior of $R(\lambda \pm i0) f$ on the end $M_j$ is reduced to that of $R_j(\lambda \pm i0) f$. Let $u_j^{(\pm)} = R_j(\lambda \pm i0) f$, where $f \in C_0^\infty(\tilde{M}_j)$. Let $g_j = \det \left( \rho_j(r)^2 h_j(r, x, dx) \right)$ and

$$
Q_j = \left( \frac{g_j'}{4g_j} \right)^2 + \left( \frac{g_j'}{4g_j} \right)' - E_{0,j}.
$$

Note that

$$
D_j^{(\pm)}(k) = \partial_r + \frac{g_j'}{4g_j} - i\psi_j^{(\pm)},
$$

where $\psi_j^{(\pm)}$ is defined by $\psi_m$ in Lemma 5.1. Since $Q_j = \frac{(n-1)^2}{4} \left( \frac{\rho_j'}{\rho_j} \right)^2 - E_{0,j} + O(r^{-1-\epsilon})$ for some $\epsilon > 0$, we have by (9.3)

$$
\psi_j^{(\pm)}(r, x, \lambda) = \pm \phi_j(r, \lambda) + O(r^{-1-\epsilon}).
$$

Letting

$$
\Psi_j^{(\pm)}(r, x, \lambda) = \int_0^r \psi_j^{(\pm)}(t, x, \lambda) \, dt,
$$

we have only to show Theorem 9.1 with $\pm \Phi_j$ replaced by $\Psi_j^{(\pm)}$. 

9.3. Utility of radiation condition. Let us note that if \( \beta_{0,j} > 1, \gamma_{0,j} > 2, \)
the proof of Theorem 9.1 is easy. In fact, in this case, in Lemma 6.7 one can choose
\( 2 < t < \gamma_{0,j} \) so that \((t-1)/2 > 1/2 \) and \((t-1 - \gamma_{0,j})/2 < -1/2. \)
Note that
\[
\partial_r \left( \frac{g_j}{g_j} e^{-i\psi_j^{(\pm)}} u_j^{(\pm)} \right) = g_j^{1/4} e^{-i\psi_j^{(\pm)}} D_j^{(\pm)}(k)u_j^{(\pm)},
\]
and that the right-hand side belongs to \( L^{2,s} (\tilde{M}_j) \) for some \( s > 1/2. \) This yields
\[
\int_2^\infty ||\partial_r \left( \frac{g_j}{g_j} e^{-i\psi_j^{(\pm)}} u_j^{(\pm)} \right)||_{L^2(M_j)} dr \leq C \left( \int_2^\infty r^{2s} ||\partial_r \left( \frac{g_j}{g_j} e^{-i\psi_j^{(\pm)}} u_j^{(\pm)} \right)||_{L^2(M_j)}^2 dr \right)^{1/2},
\]
which proves Theorem 9.1.

We turn to the general case.

Lemma 9.2. Assume (9.1), and for \( f \in L^2_{\text{comp}}(\mathcal{M}), \) let \( u_j^{(\pm)} = R_j(\lambda \pm 0)f. \)
Let \( v_j^{(\pm)} \in H^1_{\text{loc}}(\tilde{M}_j) \cap B^s(\tilde{M}_j) \) be such that
\[
(9.6) \quad D_j^{(\pm)}(k)u_j^{(\pm)}, \sqrt{g_j}v_j^{(\pm)} \in L^{2,-s}(\tilde{M}_j)
\]
for some \( 0 < s < \min \frac{1}{2}(2\beta_j - 1, \gamma_j - 1). \) We put
\[
a_j^{(\pm)}(r) = \int_{M_j} \left( D_j^{(\pm)}(k)u_j^{(\pm)}(r,x) \right) v_j^{(\pm)}(r,x) g_j(r,x) dx.
\]
Then, we have
\[
(9.7) \quad \frac{d}{dr} a_j^{(\pm)}(r) = \mp 2i\phi_j(r,\lambda,a_j^{(\pm)}(r) + F_j^{(\pm)}(r), \quad \int_1^\infty |F_j^{(\pm)}(r)| dr < \infty,
\]
and also
\[
(9.8) \quad \lim_{r \to \infty} a_j^{(\pm)}(r) = 0.
\]

Proof. We prove the + case. Using (5.14) and (5.19), we have
\[
\partial_r \left( D_j^{(+)}(k)u_j^{(+)} \right) = - \frac{g_j'}{4g_j} D_j^{(+)}(k)u_j^{(+)} + B_j u_j^{(+)}
\]
\[
- i\psi_j D_j^{(+)}(k)u_j^{(+)} - f + V_j v_j^{(+)}.
\]
\[
\partial_r \left( v_j^{(+)} \sqrt{g_j} \right) = \left( D_j^{(+)}(k)u_j^{(+)} \right) \sqrt{g_j} + \left( \frac{g_j'}{4g_j} + i\psi_j^{(+)} v_j^{(+)} \right) \sqrt{g_j},
\]
where we put
\[
V_j^{(+)} = -i(\psi_j^{(+)})' + (\psi_j^{(+)})^2 + Q_j - k^2,
\]
\[
Q_j = \left( \frac{g_j'}{4g_j} \right)^2 + \left( \frac{g_j'}{4g_j} \right)' - E_{0,j}.
\]
We then have, by a straightforward computation,
\[
\partial_r \left( \left( D_j^{(+)}(k)u_j^{(+)} \right) \sqrt{g_j} v_j^{(+)} \sqrt{g_j} \right) = - i(\psi_j^{(+)} + \overline{\psi}_j^{(+)})(D_j^{(+)}(k)u_j^{(+)} v_j^{(+)} \sqrt{g_j} + (D_j^{(+)}(k)u_j^{(+)})(D_j^{(+)}(k)u_j^{(+)} \sqrt{g_j} + (B_j u_j^{(+)} v_j^{(+)} f_j^{(+)} v_j^{(+)} \sqrt{g_j} - f_j^{(+)} v_j^{(+)} \sqrt{g_j}.
\]
This implies
\[ F_j^{(+)}(r) = \int_{M_j} G_j(r, x) \sqrt{g_j} dx, \]
\[ G_j(r, x) = -i(\psi_j^{(+)} + \overline{\psi_j^{(+)}} - 2\phi_j(r, \lambda))(D_j^{(+)}(k)u_j^{(+)}v_j^{(+)} + (D_j^{(+)')(k)u_j^{(+)}v_j^{(+)}) + (B_ju_j^{(+)})v_j^{(+)} + V_j^{(+)}u_j^{(+)}v_j^{(+)}) - f u_j^{(+)}. \]
(9.9)

Since \( \psi_j^{(+)} = \phi_j(r, \lambda) + O(r^{-1-\epsilon}) \), we have
\[ \psi_j^{(+)} + \overline{\psi_j^{(+)}} - 2\phi_j(r, \lambda) = O(r^{-1-\epsilon}), \quad V_j^{(+)} = O(r^{-1-\epsilon}). \]

By virtue of the assumption \( \beta_j > 1/2 \) and Lemmas 9.1 and 9.11, we have
\[ D_j^{(+)}(k)u_j^{(+)} + \sqrt{B_j}u_j^{(+)} \in L^{2,s}(\overline{M_j}), \quad \text{for} \quad 0 < s < \min \left\{ \frac{1}{2}(2\beta_j - 1, \gamma_j - 1) \right\}. \]

This and the assumption \( 9.10 \) imply that \( F_j^{(+)}(r) \in L^1((2, \infty)) \), which proves (9.7).

Letting
\[ b_j(r, \lambda) = 2 \int_0^r \phi_j(t, \lambda) dt, \]
we then have \( \frac{d}{dt}(e^{ib_j} a_j^{(+)}(r)) = e^{ib_j} F_j^{(+)}(r) \), which shows the existence of the limit \( \lim_{r \to \infty} e^{ib_j} a_j^{(+)}(r) \). However, this limit is equal to 0, since \( r^{-\alpha} a_j^{(+)}(r) \in L^1((2, \infty)) \) for some \( 0 < \alpha < 1 \). This proves (9.8).

In the course of the proof, we have obtained
\[ a_j^{(+)}(r) = -\int_r^\infty e^{i(b_j(t, \lambda) - b_j(r, \lambda))} F_j^{(+)}(t) dt, \]

hence
\[ |a_j^{(+)}(r)| \leq \int_r^\infty |F_j^{(+)}(t)| dt. \]
(9.10)

**Lemma 9.3.** Let \( u_j^{(\pm)} \) be as in Lemma 9.2. Then, we have
\[ (R_j(\lambda + i0)f - R_j(\lambda - i0)f, f) = \lim_{r \to \infty} 2ik \int_{M_j} |u_j^{(\pm)}(r, x)|^2 \sqrt{g_j(r, x)} dx \]

**Proof.** We prove the case for \( u_j^{(+)} \). Integrating \( u_j^{(+)} \) on the region
\[ \overline{M_j, t} := (2, t) \times M_j, \]
we obtain
\[ (u_j^{(+), f}, f)_{L^2(\overline{M_j}, \leq t)} - (f, u_j^{(+)})_{L^2(\overline{M_j}, \leq t)} \]
\[ = \int_{M_j} \left( (\partial_x u_j^{(+)}) u_j^{(+)} - u_j^{(+)} \partial_x u_j^{(+)} \right) \sqrt{g_j} dx \]
\[ = \int_{M_j} \left( (D_j^{(+)')(k)u_j^{(+)}) u_j^{(+)} - u_j^{(+)} D_j^{(+)}(k)u_j^{(+)} + 2i \Re \overline{\psi_j^{(+)}} |u_j^{(+)}|^2 \right) \sqrt{g_j} dx. \]
(9.11)

We use Lemma 9.2 with \( v_j^{(\pm)} = u_j^{(\pm)} \). Letting \( t \to \infty \), we get the present lemma. □
Let us note that (9.10) and (9.11) yield
\[
\|u_j^{(+)}(r, \cdot)\|_{L^2(M_j)} \leq C \rho_j(r)^{-(n-1)/2} \|f\|_{L^2, r}, \quad \text{if } s = 2\beta_j.
\]

We show the existence of the limit of
\[
\mathcal{F}_j^{(\pm)}(\lambda, r)f = C_j(\lambda)^{-1} \rho_j(r)^{-(n-1)/2} e^{\mp i\Phi_j(r, \lambda)} (R_j(\lambda \pm i0)f)(r, x),
\]
\[
C_j(\lambda) = \left( \frac{\pi}{\sqrt{\lambda - E_{0,j}}} \right)^{1/2}, \quad E_{0,j} = \left( \frac{(n-1)c_{0,j}}{2} \right)^2
\]
as \(r \to \infty\), where \(\Phi_j(r, \lambda)\) is defined by (9.2).

**Lemma 9.4.** For \(f \in L^2_{comp}(\tilde{\mathcal{M}}_j)\), there exists a weak limit
\[
w - \lim_{r \to \infty} \mathcal{F}_j^{(\pm)}(\lambda, r)f = : \mathcal{F}_j^{(\pm)}(\lambda)f, \quad \text{in } L^2(M_j).
\]

**Proof.** We prove the + case. Since \(\sup_{r > 2} \|\mathcal{F}_j^{(\pm)}(\lambda, r)f\|_{L^2(M_j)}\) is finite by (9.12), we only have to show the existence of
\[
\lim_{r \to \infty} (\mathcal{F}_j^{(\pm)}(\lambda, r)f, \varphi)_{L^2(M_j)}
\]
for \(\varphi \in D(\Lambda_j)\), where \(\Lambda_j\) is the Laplace-Beltrami operator on \(M_j\). We put
\[
v = \rho_j(r)^{-(n-1)/2} e^{i\Phi_j} \varphi, \quad \varphi \in D(\Lambda_j).
\]

By a direct calculation, we have,
\[
\begin{align*}
\rho_j^{(n-1)/2} e^{-i\Phi_j} v' &= \left( -\frac{n-1}{2} \frac{\rho_j'}{\rho_j} + i \Phi_j' \right) \varphi, \\
\rho_j^{(n-1)/2} e^{-i\Phi_j} v'' &= \left( \frac{n-1}{4} \left( \frac{\rho_j'}{\rho_j} \right)^2 - \frac{n-1}{2} \left( \frac{\rho_j'}{\rho_j} \right)' - i(n-1) \frac{\rho_j'}{\rho_j} \Phi_j' \\
&\quad + i \Phi_j'' - (\Phi_j')^2 \right) \varphi.
\end{align*}
\]

By our assumption,
\[
\frac{\rho_j'}{\rho_j} = c_{0,j} + O(r^{-\alpha_{0,j}}), \quad \frac{g_j'}{\rho_j} = (n-1)c_{0,j} + O(r^{-\alpha_{0,j}}).
\]

Therefore, modulo a term of \(O(r^{-1-\epsilon})\)
\[
\rho_j^{(n-1)/2} e^{-i\Phi_j} \left( v'' + \frac{g_j'}{\rho_j} v' \right) \equiv -\lambda \varphi
\]
in the sense of \(L^2(M_j)\)-norm. Since \(2\beta_j > 1\), we have
\[
B_j \varphi = O(\rho_j(r)^{-2} \rho_j(r)^{-(n-1)/2}) = O(r^{-2\beta_j} \rho_j(r)^{-(n-1)/2})
\]
\[
= O(r^{-1-\epsilon} \rho_j(r)^{-(n-1)/2}),
\]

hence
\[
(-\Delta_{\mathcal{M}_j} - \lambda)v = O(r^{-1-\epsilon} \rho_j(r)^{-(n-1)/2}).
\]

We let
\[
g = (-\Delta_{\mathcal{M}_j} - \lambda)v,
\]
and integrate by parts as in (9.11) to obtain
\[
(u_j^{(+)}, g)_{ L^2(\mathcal{M}_{t,\leq t})} - (f, v)_{ L^2(\mathcal{M}_{t,\leq t})}
\]
(9.16)
\[
= \int_{M_j} \left( (D_j^{(+)(k)}u_j^{(+)p}) - u_j^{(+)}D_j^{(+)}(k)v + 2i \left( \text{Re} \, \psi_j^{(+)p} \right) u_j^{(+)p} \right) \sqrt{g_j} \, dx.
\]

Letting \( t \to \infty \) and using Lemma 9.2, we have proven the existence of the limit (9.14). \( \square \)

**Lemma 9.5.** Let \( f \in L^2_{\text{comp}}(\mathcal{M}_j) \), and \( u_j^{(+) = R_j(\lambda + i0)f} \).

1. There exist \( \epsilon > 0 \) and a sequence \( \{r_p\}_{p=1}^{\infty} \) tending to infinity such that
\[
r_p^{1+\epsilon} \int_{M_j} \left| (D_j^{(+)(k)}u_j^{(+)p})(r_p, x) \right|^2 \sqrt{g_j(r_p, x)} \, dx < 0.
\]

2. Let \( \{r_p\}_{p=1}^{\infty} \) be as above. Then, for any \( r_q > r_p \) and \( \varphi \in H^1(M_j) \)
\[
\left| (F_j^{(+)}(\lambda, r_q)f - F_j^{(+)}(\lambda, r_q)f, \varphi)_{ L^2(\mathcal{M}_j)} \right|
\leq C(r_p) \left( \|\varphi\|_{ L^2(\mathcal{M}_j)} + r_p^{-\epsilon} \sqrt{ \Lambda_j(r_p) \varphi}_{ L^2(\mathcal{M}_j)} \right),
\]
where the constant \( C(r_p) \) is independent of \( \varphi \), and \( C(r_p) \to 0 \) as \( r_p \to \infty \).

**Proof.** By Lemmas 6.7 and 6.11 there exists an \( \epsilon > 0 \) such that
\[
\int_2^{\infty} \int_{M_j} r^\epsilon |D_j^{(+)(k)}u_j^{(+)p}|^2 \sqrt{g_j} \, dr \, dx < \infty,
\]
which implies (1) immediately.

Let \( v \) and \( g \) be as in the proof of Lemma 9.4. Letting
\[
\mathcal{M}_{s,t} = (s, t) \times M, \quad F|_{s}^{t} = F(t) - F(s),
\]
we have, by integration by parts,
\[
(v, f)_{ L^2(\mathcal{M}_{r,\leq r})} - (g, u_j^{(+)})_{ L^2(\mathcal{M}_{r,\leq r})}
\]
(9.17)
\[
+ \int_{M_j} (D_j^{(+)(k)}v) u_j^{(+)p} \sqrt{g_j} \, dx - \int_{M_j} v D_j^{(+)(k)} u_j^{(+)p} \sqrt{g_j} \, dx
\]
\[
= 2i \int_{M_j} \left( \text{Re} \, \psi_j^{(+)p} \right) u_j^{(+)p} \sqrt{g_j} \, dx.
\]

By Lemma 9.12, \( \|F_j^{(+)}(\lambda, r)f\|_{ L^2(\mathcal{M}_j)} \) is uniformly bounded for \( r > 2 \). Since \( \psi = k + O(r^{-\epsilon}) \), the right-hand side of (9.17) is estimated from below as follows:
\[
2i \int_{M_j} \left( \text{Re} \, \psi_j^{(+)p} \right) u_j^{(+)p} \sqrt{g_j} \, dx
\]
(9.18)
\[
\geq C \left| (F_j^{(+)}(\lambda, r_q)f - F_j^{(+)}(\lambda, r_q)f, \varphi)_{ L^2(\mathcal{M}_j)} \right|
\]
\[
- C' r_p^{-\epsilon} \left( \sup_{q \geq p} \|F_j^{(+)}(\lambda, r_q)f\|_{ L^2(\mathcal{M}_j)} \right) \|\varphi\|_{ L^2(\mathcal{M}_j)}.
\]
We estimate the left-hand side of (9.17) from above. The 4th term is estimated as follows:

\[
| \int_{M_j} vD_j^{(+)\varepsilon}(k)u_j^{(+)\varepsilon} \sqrt{g_j} |_{r_p}^r \, dx | \leq C \sup_{q \geq p} \left( \int_{M_j} |D_j^{(+)\varepsilon}(k)u_j^{(+)\varepsilon}(r_q, x)|^2 \sqrt{g_j(r_q, x)} \, dx \right)^{1/2} \| \varphi \|_{L^2(M_j)}.
\]

(9.19)

Since

\[
D_j^{(+)\varepsilon}(k)v = \left( \frac{\varrho_j'}{\varrho_j} - \frac{n - 1}{2} \rho_j' + i(\Phi_j' - \psi_j) \right) v = O(r^{-1-\varepsilon})v,
\]

the 3rd term is dominated by \(Cr_p^{-1-\varepsilon}\|\varphi\|_{L^2(M_j)}\). The 1st term is estimated as follows

\[
\left| (v, f)_{L^2(M_{r_p', r_q})} \right| \leq Cr_p^{-\varepsilon/2}\|\varphi\|_{L^2(M_j)}\|f\|_{L^2(M_{r_p', r_q})}.
\]

(9.20)

We estimate the 2nd term. Recalling (9.19), we have

\[
g = (-\Delta_{M_j} - \lambda)v = \alpha(r)\varphi + \beta(r)B_j(r)\varphi,
\]

\[
\alpha(r) = O(r^{-1-\varepsilon}\rho_j(r)^{-(n-1)/2}), \quad \beta(r) = O(\rho_j(r)^{-(n-1)/2}).
\]

The term \((\alpha(r)\varphi, u_j^{(+)\varepsilon})_{L^2(M_{r_p', r_q})}\) is estimated as

\[
\left| (\alpha(r)\varphi, u_j^{(+)\varepsilon})_{L^2(M_{r_p', r_q})} \right| \leq Cr_p^{-\varepsilon/2}\|\varphi\|_{L^2(M_j)}\|u_j^{(+)\varepsilon}\|_{-(1+\varepsilon)/2}.
\]

(9.21)

The term \((\beta(r)B_j(r)\varphi, u_j^{(+)\varepsilon})_{L^2(M_{r_p', r_q})}\) is rewritten as

\[
(\beta(r)B_j(r)\varphi, u_j^{(+)\varepsilon})_{L^2(M_{r_p', r_q})} = (\beta(r)\sqrt{B_j(r)\varphi}, \sqrt{B_j(r)u_j^{(+)\varepsilon}})_{L^2(M_{r_p', r_q})}.
\]

By virtue of Lemma 6.7, letting \(t = (s - 1)/2\) where \(0 < s < \min\{2\beta_{0,j}, \gamma_{0,j}\}\), this is estimated from above as

\[
\left| (\beta(r)\sqrt{B_j(r)\varphi}, \sqrt{B_j(r)u_j^{(+)\varepsilon}})_{L^2(M_{r_p', r_q})} \right| \leq C \|\sqrt{A_j(r_p)\varphi}\|_{L^2(M_j)} \|r^{-t}\beta(r)\rho_j(r)^{-1}\|_{L^2((r_p, r_q); \rho_j^{-1})} \|f\|_{L^2(M_j)}
\]

\[
\times \|\sqrt{B_j(r_p)u_j^{(+)\varepsilon}}\|_{L^2(M_{r_p', r_q})} \leq C \|\sqrt{A_j(r_p)\varphi}\|_{L^2(M_j)} \|r^{-t}\beta(r)\rho_j(r)^{-(n+1)/2}\|_{L^2((r_p, r_q); \rho_j^{-1})} \|f\|_{L^2(M_j)}
\]

\[
\leq C \|A_j(r_p)\varphi\|_{L^2(M_j)} \|f\|_{L^2(M_j)} \left( \int_{r_p}^{r_q} r^{-2t-2\beta_{0,j}} \, dr \right)^{1/2}
\]

\[
\leq C r_p^{-\varepsilon} \|A_j(r_p)\varphi\|_{L^2(M_j)} \|f\|_{L^2(M_j)}^{(s+1)/2}.
\]

This and the inequalities (9.18), (9.19), (9.20), (9.21), together with (1), prove the lemma.

**Lemma 9.6.** Let \(f \in L^2_{comp}(\widetilde{M}_j)\). Then, there exists a sequence \(\{r_p\}_{p=1}^\infty\) such that \(\mathcal{F}_j^{(\pm)}(\lambda, r_p)f\) converges to \(\mathcal{F}_j^{(\pm)}(\lambda)f\) strongly on \(L^2(M_j)\).
9. RESOLVENT ASYMPTOTICS - NON PERTURBATIVE APPROACH

Proof. Let \( \{r_p\}_{p=1}^\infty \) be the sequence in Lemma 9.3 (1). We can also assume that
\[
r_p^{1+} \| \sqrt{B_j(r_p)} u_j^{(+)} \|_{L^2(M_j)}^2 \to 0, \quad u_j^{(+)} = R_j(\lambda + i0)f.
\]
Using Lemma 9.5 with \( \varphi \) replaced by \( F_j^{(+)}(\lambda, r_p)f \), we have
\[
\left| \| F_j^{(+)}(\lambda, r_p)f \|_{L^2(M_j)}^2 - (F_j^{(+)}(\lambda, r_p)f, F_j^{(+)}(\lambda, r_p)f)_{L^2(M_j)} \right| \leq C(r_p) \left( \| F_j^{(+)}(\lambda, r_p)f \|_{L^2(M_j)} + r_p^{-2} \| \sqrt{B_j(r_p)} F_j^{(+)}(\lambda, r_p)f \|_{L^2(M_j)} \right).
\]

We first let \( q \) tend to \( \infty \), and then \( p \) to \( \infty \). Then, we see that the norm of \( F_j^{(+)}(\lambda, r_p)f \) converges to that of \( F_j^{(+)}(\lambda)f \). This proves the strong convergence, since we already know its weak convergence in Lemma 9.4.

Lemma 9.7. Let \( f \in L^2_{\text{comp}}(\mathcal{M}_j) \). Then, \( F_j^{(\pm)}(\lambda, r)f \) converges to \( F_j^{(\pm)}(\lambda)f \) strongly on \( L^2(M_j) \), and we have
\[
\frac{1}{2\pi i} (R_j(\lambda + i0)f - R_j(\lambda - i0)f, f) = \| F_j^{(\pm)}(\lambda)f \|_{L^2(M_j)}^2.
\]

Proof. Since \( F_j^{(\pm)}(\lambda)f \) is the weak limit of \( F_j^{(+)}(\lambda, r)f \), we have
\[
\| F_j^{(\pm)}(\lambda)f \|_{L^2(M_j)} \leq \liminf_{r \to \infty} \| F_j^{(+)}(\lambda, r)f \|_{L^2(M_j)}.
\]
Using the sequence in Lemma 9.6 we have
\[
\liminf_{r \to \infty} \| F_j^{(\pm)}(\lambda, r)f \|_{L^2(M_j)} \leq \lim_{r_p \to \infty} \| F_j^{(+)}(\lambda, r_p)f \|_{L^2(M_j)} = \| F_j^{(+)}(\lambda)f \|_{L^2(M_j)}.
\]
By Lemma 9.3 we already know the existence of the limit of \( \| F_j^{(+)}(\lambda, r)f \|_{L^2(M_j)} \). Therefore, we have \( \| F_j^{(\pm)}(\lambda)f \|_{L^2(M_j)} = \lim_{r \to \infty} \| F_j^{(+)}(\lambda, r)f \|_{L^2(M_j)} \). This proves the existence of the strong limit \( \lim_{r \to \infty} F_j^{(\pm)}(\lambda, r)f \) in \( L^2(M_j) \). In view of Lemma 9.4 we obtain the lemma.

9.4. Asymptotic expansion of the resolvent. Lemma 9.7 implies that
\[
\| F_j^{(\pm)}(\lambda)f \|_{L^2(M_j)} \leq C\| f \|_{\mathcal{B}}, \quad f \in L^2_{\text{comp}}(\mathcal{M}),
\]
where the constant \( C \) is independent of \( \lambda \) varying over a compact set in \( \sigma_0(H) \setminus \mathcal{E} \). Therefore, it is uniquely extended on \( \mathcal{B} \) and (9.22) holds also for \( f \in \mathcal{B} \). Making use of (9.5), we compute the asymptotics of \( R(\lambda \pm i0) \) on each end. For \( f, g \in \mathcal{B}^* \), we use the following notation:
\[
f \simeq g \iff f - g \in \mathcal{E}^*_0.
\]

Theorem 9.8. For any \( f \in \mathcal{B} \) and \( \lambda \in (E_{0,j}, \infty) \setminus \mathcal{E} \), we have on the regular ends \( M_j \) satisfying (9.1)
\[
R(\lambda \pm i0)f \simeq C_j(\lambda) \rho_j(r)^{-n/2} e^{\pm i\Phi_j(r, \lambda)} F_j^{(\pm)}(\lambda)f,
\]
where \( C_j(\lambda) \) is defined by (9.13).

Proof. For \( f \in L^2_{\text{comp}}(\mathcal{M}), \) (9.24) is already proven. Since \( R(\lambda \pm i0) \in \mathcal{B}(\mathcal{B}, \mathcal{B}^*) \), in view of (9.22), we have only to approximate \( f \in \mathcal{B} \) by compactly supported functions to prove (9.24).
9.5. Cusp ends. For the cusp ends, one can argue as above without any change. Thus, we have also proven the following theorem.

**Theorem 9.9.** For any \( f \in \mathcal{B} \) and \( \lambda \in (E_{0,j}, \infty) \setminus \mathcal{E} \), we have on the cusp ends \( M_j \),

\[
R(\lambda \pm i0)f \approx C_j(\lambda)P_{0,j} \otimes \rho_j(r)^{-(n-1)/2}e^{\pm i\Phi_j(r,\lambda)} f^{(\pm)}(\lambda)f,
\]

\( P_{0,j} \) being the projection associated with the 0-eigenvalue for \(-\Delta_{M_j}\). Here, \( C_j(\lambda) \) is defined by (9.13).

**Proof.** By Lemma 6.11, \((1 - P_{0,j} \otimes 1)R(\lambda \pm i0)f \in L^2(M_j)\) for the cusp end, from which (9.25) follows. \(\square\)

10. Resolvent asymptotics - Perturbative approach

We study the remaining case in this section. We fix one regular end \( M_j = (0, \infty) \times M_j \), (we shifted the interval \((2, \infty)\) to \((0, \infty)\), which does not matter at all), and observe the asymptotic behavior of the resolvent on \( M_j \) under the assumption

\[
(10.1) \quad c_{0,j} = 0, \quad \alpha_{0,j} > 0, \quad 1/2 \geq \beta_{0,j} > 0, \quad \gamma_{0,j} > 1.
\]

In this section, we drop the subscript \( j \) in \( \rho_j(r), c_{0,j}, \alpha_{0,j}, \beta_{0,j}, \gamma_{0,j} \).

We reduce the problem to the one-dimensional case, and apply the perturbation technique. Hence, letting \( E \geq 0 \) be an arbitrary constant, we start with the equation

\[
(10.2) \quad \left( -\partial_r^2 - \frac{(n-1)\rho'(r)}{\rho(r)} \partial_r + \frac{E}{\rho(r)^2} - z \right) u = f, \quad \text{on} \quad (0, \infty).
\]

10.1. WKB method.

10.1.1. Asymptotic solutions. We seek an asymptotic solution of the equation

\[
(10.3) \quad \left( -\partial_r^2 - \frac{(n-1)\rho'(r)}{\rho(r)} \partial_r + \frac{E}{\rho(r)^2} \right) u = \lambda u, \quad \lambda > 0
\]

in the form \( u = e^{\varphi}a \). A direct computation yields

\[
(10.4) \quad e^{-\varphi} \left( -\partial_r^2 - \frac{(n-1)\rho'}{\rho} \partial_r + \frac{E}{\rho^2} - \lambda \right) e^{\varphi}a
\]

\[
= - \left( (\varphi')^2 + \frac{(n-1)\rho'}{\rho} \varphi' + \lambda - \frac{E}{\rho^2} \right) a - \left\{ 2\varphi' + \frac{(n-1)\rho'}{\rho} \right\} a' + \varphi''a
\]

By the assumptions (A-1) and (A-2), we have

\[
\frac{\rho'}{\rho} \in S^{-\alpha_0}, \quad \partial_r^m \left( \frac{1}{\rho} \right) \in S^{-\beta - m}, \quad \forall m \geq 0.
\]

Letting \( \epsilon = \min(2\alpha_0, 2\beta_0) \), we then have

\[
\left( \frac{(n-1)\rho'}{2\rho} \right)^2 + \frac{E}{\rho^2} \leq C r^{-\epsilon}(1 + E).
\]

We define \( r_0(\lambda, E) \) by

\[
(10.5) \quad r_0(\lambda, E) = \left( \frac{2C(1 + E)}{\lambda} \right)^{1/\epsilon}
\]
10. RESOLVENT ASYMPTOTICS - PERTURBATIVE APPROACH

so that \( Cr^{-\epsilon}(1 + E) < \lambda/2 \) for \( r > r_0(\lambda, E) \). We put

\[
\alpha(\lambda, E, r) = \sqrt{\lambda - \left(\frac{(n-1)\rho'}{2\rho}\right)^2 - \frac{E}{\rho^2}},
\]

(10.6)

\[
\varphi_{\pm} = -\frac{n-1}{2} \log \rho(r) \pm i \int_{r_0(E, \lambda)}^{r} a(\lambda, E, s)ds.
\]

(10.7)

\[
a_{\pm} = \exp \left( \int_{r}^{\infty} \frac{\varphi''_{\pm}}{2\varphi'_{\pm} + (n-1)\rho'/\rho} ds \right).
\]

(10.8)

They satisfy the eikonal equation

\[
(\varphi')^2 + \frac{(n-1)\rho'}{\rho} \varphi' + \lambda - \frac{E}{\rho^2} = 0,
\]

and the transport equation

\[
\left( 2\varphi' + \frac{(n-1)\rho'}{\rho} \right) a' + \varphi'' = 0.
\]

Hence the equation (10.4) becomes

\[
e^{-\varphi} \left( -\partial_r^2 - \frac{(n-1)\rho'}{\rho} \partial_r + \frac{E}{\rho^2} - \lambda \right) e^{\varphi} a = -a''.
\]

Fix a compact interval \( I \subset (0, \infty) \) arbitrarily. In the following, \( C \)'s denote constants independent of \( \lambda \in I, E \geq 0 \) and \( r > r_0(\lambda, E) \). Similarly, the various estimates are uniform with respect to \( \lambda, E \) and \( r \) satisfying these conditions. The following lemma is proven by a direct computation.

**Lemma 10.1.** For \( \alpha, \varphi_{\pm}, a_{\pm} \) defined by (10.6), (10.7), (10.8), we have

\[
\alpha \in S^0, \quad \alpha' \in S^{-1-\epsilon}, \quad \varphi'_{\pm} \in S^0, \quad \varphi''_{\pm} \in S^{-1-\epsilon}, \quad a_{\pm} - 1 \in S^{-\epsilon}.
\]

Noting that

\[
e^{\varphi_{\pm}} = \rho(r)^{-(n-1)/2} \exp \left( \pm i \int_{r_0(E, \lambda)}^{r} \alpha(\lambda, E, s)ds \right),
\]

and summarizing the above computation, we have proven the following lemma.

**Lemma 10.2.** There exist asymptotic solutions \( a_{\pm} e^{\varphi_{\pm}} \) of (10.3) satisfying

\[
|a_{\pm} e^{\varphi_{\pm}} - \rho(r)^{-(n-1)/2} \exp \left( \pm i \int_{r_0(E)}^{r} \alpha(\lambda, E, s)ds \right)| \leq C\rho(r)^{-(n-1)/2} r^{-\epsilon},
\]

(10.9)

\[
\left( -\partial_r^2 - \frac{(n-1)\rho'}{\rho} \partial_r + \frac{E}{\rho^2} - \lambda \right) a_{\pm} e^{\varphi_{\pm}} = -a''_{\pm},
\]

(10.10)

\[
\partial_r^m(a_{\pm} - 1) = O(r^{-m-\epsilon}), \quad m \geq 0,
\]

uniformly for \( r > r_0(\lambda, E) \).
10.1.2. **Exact solutions.** Next let us construct the exact solutions to (10.3) which behave like $a_\pm e^{\varphi_\pm}$ as $r \to \infty$. Putting $a = a_\pm, \varphi = \varphi_\pm, u = ae^{\varphi}(1 + v)$ and using (10.10), we have for $r > r_0(E, \lambda)$

$$v'' + \left(2\frac{a'}{a} + \varphi' + \frac{(n-1)\rho'}{\rho}\right)v' + \frac{a''}{a}v = -\frac{a''}{a}.$$ 

Putting $v = \begin{pmatrix} v \\ v' \end{pmatrix}, f = \begin{pmatrix} 0 \\ -a''/a \end{pmatrix}$, we transform it into the 1st order system:

$$\frac{d}{dr} v = K(r)v + L(r)v + f(r),$$

$$K(r) = \begin{pmatrix} 0 & 1 \\ 0 & -\left(2\frac{a'}{a} + \varphi' + \frac{(n-1)\rho'}{\rho}\right) \end{pmatrix}, \quad L(r) = \begin{pmatrix} 0 & 0 \\ -a''/a & 0 \end{pmatrix}.$$ 

A fundamental matrix for the equation $\frac{d}{dr} v = K(r)v$ is

$$W(r,s) = V(r)V(s)^{-1}, \quad V(r) = \begin{pmatrix} 1 & 0 \\ a^{-2}e^{2\varphi} \rho^{-(n-1)} & 0 \end{pmatrix},$$

where

$$F = \int_{r_0(E,\lambda)}^r a^{-2}e^{2\varphi} \rho^{-(n-1)}ds.$$ 

Then by (10.9),

$$F' = a^{-2}e^{-2\varphi} \rho^{-(n-1)} = \exp \left( \mp 2i \int_{r_0}^r \alpha ds \right) \left( 1 + O(r^{-\epsilon}) \right).$$

Using

$$\frac{1}{\mp 2i\alpha} \frac{d}{dr} e^{\mp 2i \int_{r_0}^r \alpha ds} = e^{\mp 2i \int_{r_0}^r \alpha ds}$$

and integrating by parts, we have $F = O(1)$. Then $W(r,s)$ is uniformly bounded for $r_0(\lambda, E) \leq r \leq s$. The equation (10.11) is rewritten as the integral equation

$$v(r) = -\int_r^\infty W(r,s)L(s)v(s)ds - \int_r^\infty W(r,s)f(s)ds,$$

which is solved by iteration, since $L(r) = O(r^{-1-\epsilon}), f(r) = O(r^{-1-\epsilon})$. We have thus proved the following lemma.

**Lemma 10.3.** There exist exact solutions $\Psi^{(\pm)}(\lambda, r, E)$ to (10.3) on $[r_0(\lambda, E), \infty)$ such that $\Psi^{(\pm)} = a_\pm e^{\varphi_\pm} \left( 1 + O(r^{-\epsilon}) \right)$ as $r \to \infty$.

We extend the solutions $\Psi^{(\pm)}(\lambda, r; E)$ to the whole interval $[0, \infty)$. The following lemma is an immediate consequence.

**Lemma 10.4.** If $u$ satisfies (10.3) and

$$\frac{1}{R} \int_0^R |u(r)|^2 \rho(r)^{n-1} dr \to 0, \quad R \to \infty,$$

then $u$ is identically equal to 0.
Proof. Since \( \Psi^{(\pm)} \) in Lemma [10.3] are linearly independent, \( u \) is written as \( u = c_+ \Psi^{(+)} + c_- \Psi^{(-)} \) for some constants \( c_\pm \). The assumption of the lemma then implies

\[
\frac{1}{R} \int_0^R |c_+ e^{i \int_0^t \alpha ds} + c_- e^{-i \int_0^t \alpha ds}|^2 dt \to 0.
\]

For large \( r \), we can make the change of variable \( r \to t \) by \( \int_0^r \alpha ds = t \). We then have

\[
\frac{1}{R} \int_0^R |c_+ e^{it} + c_- e^{-it}|^2 dt \to 0,
\]

which implies \( c_+ = c_- = 0 \). \( \square \)

10.1.3. Green function. The Green operator \( G^{(\pm)}(\lambda, E) \) for \( u = 0 \) with Dirichlet condition at \( r = 0 \) is defined by

\[
(G^{(\pm)}(\lambda, E)f)(r) = \int_0^\infty G^{(\pm)}(r, s, \lambda, E)f(s)ds,
\]

\[
G^{(\pm)}(r, s, \lambda, E) = \frac{1}{\mp \lambda W^{(\pm)}(\lambda, s, E)} \Psi_0(\lambda, r, E)\Psi^{(\pm)}(\lambda, s, E), \quad 0 < r < s,
\]

\[
W^{(\pm)}(\lambda, r, E) = \Psi_0(\lambda, r, E)\Psi^{(\pm)}(\lambda, r, E)' - \Psi_0(\lambda, r, E)'\Psi^{(\pm)}(\lambda, r, E),
\]

where \( \Psi_0(r) = \Psi_0(\lambda, r, E) \) is the solution of [10.3] satisfying \( \Psi_0(0) = 0, \Psi_0'(0) = 1 \). Since \( W^{(\pm)}' = \frac{-(\mp 1)\rho'}{\rho} W^{(\pm)} \), we have

\[
W^{(\pm)}(\lambda, r, E) = -\Psi^{(\pm)}(\lambda, 0, E)\left(\frac{\rho(0)}{\rho(r)}\right)^{n-1}.
\]

Note that \( \Psi^{(\pm)}(\lambda, 0, E) \neq 0 \). In fact, if it vanishes, \( \Psi^{(\pm)}(\lambda, r, E) \) is a solution to [10.3] satisfying the boundary condition and the radiation condition. Arguing in the same way as in the proof of Lemma [7.8] using Lemma [10.3] (actually much simpler), we see that \( \Psi^{(\pm)}(\lambda, r, E) = 0 \), which is a contradiction.

The following Lemma can be proven easily by using the Green function.

Lemma 10.5. For \( f \in L^2_{comp}((0, \infty)) \), we put

\[
\tilde{f}^{(\pm)}(\lambda, E) = -\int_0^\infty \frac{\Psi_0(\lambda, s, E)}{W^{(\pm)}(\lambda, s, E)} f(s)ds \tag{10.12}
\]

\[
= \frac{\rho(0)^{1-n}}{\Psi^{(\pm)}(\lambda, 0, E)} \int_0^\infty \Psi_0(\lambda, s, E)f(s)s^{n-1}ds.
\]

Then if \( f(r) = 0 \) for \( r > r' \),

\[
(G^{(\pm)}(\lambda, E)f)(r) = \Psi^{(\pm)}(\lambda, r, E)\tilde{f}^{(\pm)}(\lambda, E), \quad r > r'
\]

holds, and the following limit exists

\[
\lim_{r \to \infty} \rho(r)^{(n-1)/2} e^{\mp i \int_0^r \alpha(\lambda, E,t)dt} (G^{(\pm)}(\lambda, E)f)(r) = \tilde{f}^{(\pm)}(\lambda, E).
\]
10.2. Fourier transform. Let \( E \geq 0 \) be a constant, \( L(E) \) the differential operator

\[
L(E) = -\partial_r^2 - \frac{(n-1)\rho'}{\rho} \partial_r + \frac{E^2}{\rho^2}
\]

with Dirichlet boundary condition at \( r = 0 \), and

\[
R_{\text{free}}(z,E) = (L(E) - z)^{-1}.
\]

All results in the previous sections hold for \( L(E) \). In particular, Theorem 8.2 holds for \( L(E) \) with \( E \geq 0 \). Letting

\[
\phi(\alpha, E, r) = \int_{r_0(\alpha, E)}^r \alpha(\lambda, E, s) \, ds,
\]

we define for \( f \in L^2_{\text{comp}}((0, \infty)) \)

\[
F_{\text{free}, E}^{(\pm)}(\lambda) f = \left( \frac{\sqrt{\lambda}}{\pi} \right)^{1/2} \lim_{r \to \infty} \rho(r)^{(n-1)/2} e^{\mp i\phi(\lambda, E, r)} (R_{\text{free}}(\lambda \pm i0, E) f)(r).
\]

The existence of the limit (10.16) is guaranteed by Theorem 10.5.

**Lemma 10.6.** For \( f \in L^2_{\text{comp}}((0, \infty)) \), we have

\[
|F_{\text{free}, E}^{(\pm)}(\lambda) f|^2 = \frac{1}{2\pi i} \left( [R_{\text{free}}(\lambda + i0, E) - R_{\text{free}}(\lambda - i0, E)] f, f \right).
\]

**Proof.** Let \( u_\pm = R_{\text{free}}(\lambda \pm i0, E) f \). Multiply the equation

\[-u''_\pm - \frac{(n-1)\rho'}{\rho} u'_\pm + \left( \frac{E}{\rho^2} - \lambda \right) u_\pm = f\]

by \( \overline{u_\pm} \rho^{n-1} \), integrate by parts over \( (0, r) \), and take the imaginary part. Then we have

\[
\text{Im} u'_\pm(r) \overline{u_\pm}(r) \rho^{n-1} = -\text{Im} \int_0^r \overline{u_\pm} \rho^{n-1} dt.
\]

The left-hand side is equal to

\[
\text{Im} (D_\pm(k) u_\pm) \overline{u_\pm} \rho^{n-1} + \text{Re} \psi_j^{(\pm)} |u_\pm|^2 \rho^{n-1}.
\]

By Lemma 6.11 \( D_\pm(k) u \in L^2 \). Therefore, the 1st term tends to 0 along a suitable sequence \( r_1 < r_2 < \cdots \to \infty \). Taking the limit in (10.17) along \( \{r_n\} \), we then have

\[
\lim_{r \to \infty} |u_\pm|^2 \rho^{n-1} = \mp \text{Im} (f, u_\pm) = \frac{1}{2\pi i} \left( [R_{\text{free}}(\lambda + i0, E) - R_{\text{free}}(\lambda - i0, E)] f, f \right).
\]

Noting \( \lambda = \sqrt{\lambda} \) and (10.10), we prove the lemma.\( \square \)

Lemma 10.6 and Theorem 8.2 imply

\[
|F_{\text{free}, E}^{(\pm)}(\lambda) f| \leq C \|f\|_B,
\]

where the constant \( C \) is independent of \( E \) and \( \lambda \) when they vary over a compact set in \( (0, \infty) \). In view of (10.16), (10.12) and (10.13), we have

\[
F_{\text{free}, E}^{(\pm)}(\lambda) f = \left( \frac{\sqrt{\lambda}}{\pi} \right)^{1/2} \overline{f}^{(\pm)}(\lambda, E) = \left( \frac{\sqrt{\lambda}}{\pi} \right)^{1/2} \frac{\rho(0)^{1-n}}{\Psi^{(\pm)}(\lambda,0,E)} \int_0^\infty \Psi_0(\lambda,s,E)f(s)\rho(s)^{n-1} ds.
\]
This implies that $F_{\text{free}, E}^{(\pm)}(\lambda)^* \in B(\mathbb{C}; \mathcal{B})$ is the operator of multiplication by the function

$$
\left(\frac{\sqrt{\lambda}}{\pi}\right)^{1/2} \left(\frac{\rho(0)^{1-n}}{\Psi^{(\pm)}(\lambda, 0; E)}\right) \Psi_0(\lambda, r, E).
$$

Since $\Psi^{(+)}(\lambda, r, E) = \Psi^{(-)}(\lambda, r, E)$, there exist constants $c_{\pm}(\lambda, E)$ such that

$$
\Psi_0(\lambda, r, E) = c_+(\lambda, E)\Psi^{(+)}(\lambda, r, E) + c_-(\lambda, E)\Psi^{(-)}(\lambda, r, E),
$$

$$
c_+(\lambda, E) = c_-(\lambda, E).
$$

Therefore, letting

$$
\begin{align*}
\omega_+(\lambda, E) &= \left(\frac{\sqrt{\lambda}}{\pi}\right)^{1/2} \left(\frac{\rho(0)^{1-n}}{\Psi^{(-)}(\lambda, 0; E)}\right) c_+(\lambda, E), \\
\omega_-(\lambda, E) &= \left(\frac{\sqrt{\lambda}}{\pi}\right)^{1/2} \left(\frac{\rho(0)^{1-n}}{\Psi^{(-)}(\lambda, 0; E)}\right) c_-(\lambda, E),
\end{align*}
$$

we have the following lemma.

**Lemma 10.7.** $F_{\text{free}, E}^{(-)}(\lambda)^*$ is the operator of multiplication by the function

$$
\omega_+(\lambda, E)\Psi^{(+)}(\lambda, r, E) + \omega_-(\lambda, E)\Psi^{(-)}(\lambda, r, E).
$$

**10.3. Warped product metric.** We equip $(0, \infty) \times M$ with the metric

$$
ds^2 = (dr)^2 + \rho(r)^2 h_M(x, dx),
$$

and let

$$
H_{\text{free}} = -\partial_r^2 - \frac{(n-1)\rho'}{\rho} \partial_r - \frac{\Lambda}{\rho^2},
$$

assuming the Dirichlet boundary condition at $r = 0$, where $\Lambda = \Delta_M$ is the Laplace-Beltrami operator on $M$. We also let

$$
R_{\text{free}}(z) = (H_{\text{free}} - z)^{-1}.
$$

Let $0 = \lambda_0 \leq \lambda_1 \leq \cdots \to \infty$ be the eigenvalues of $-\Lambda$, $P_\ell$ the eigenprojection associated with $\lambda_\ell$, and $\varphi_\ell(x)$ the associated normalized eigenvector. We then have

$$
R_{\text{free}}(\lambda \pm i0) = \sum_{\ell=0}^{\infty} P_\ell \otimes R_{\text{free}}(\lambda \pm i0, \lambda_\ell).
$$

Let $h_\infty = L^2(M)$ and put

$$
F_{\text{free}}^{(\pm)}(\lambda) = \sum_{\ell=0}^{\infty} P_\ell \otimes F_{\text{free}, \lambda_\ell}^{(\pm)}(\lambda),
$$

Letting $E = \lambda_\ell$ in Lemma 10.6 and summing up with respect to $\ell$, we obtain the following lemma.

**Lemma 10.8.** For $f \in \mathcal{B}$ and $\lambda > 0$, we have

$$
\|F_{\text{free}}^{(\pm)}(\lambda)f\|_{h_\infty}^2 = \frac{1}{2\pi i} \left(\|R_{\text{free}}(\lambda \pm i0) - R_{\text{free}}(\lambda - i0)\| f, f \right).
$$
It then follows that
\[ \|F_{\text{free}}^{(\pm)}(\lambda)f\|_{L^2(M)}^2 = \sum_{\ell=0}^{\infty} \|F_{\text{free},\lambda_\ell}(\lambda)f\|_{B}^2 \leq C \|f\|_{B}^2, \]
where the constant $C$ is independent of $\lambda$ when $\lambda$ varies over a compact set in $(0, \infty)$. Take $\chi \in C^\infty(\mathbb{R})$ such that $\chi(r) = 0$ for $r < 1$, and $\chi(r) = 1$ for $r > 2$, and put
\[ c_\ell(\lambda, r) = \left( \frac{\pi}{\lambda} \right)^{1/2} \chi \left( \frac{r}{r_0(\lambda, \lambda_\ell)} \right), \]
where $r_0(\lambda, E)$ is given in (10.5). Recall that $\varphi(\lambda, E, r)$ is defined by (10.15).

**Theorem 10.9.** For $f \in B$, we have
\[ R_{\text{free}}(\lambda \pm i 0) f \simeq \sum_{\ell=0}^{\infty} c_\ell(\lambda, r) \rho(r)^{-(n-1)/2} e^{\pm i\varphi(\lambda, \lambda_\ell, r)} P_\ell \otimes F_{\text{free},\lambda_\ell}(\lambda)f. \]

**Proof.** Since both sides in (10.18) are bounded operators from $B$ to $B^\ast$, we have only to prove it for $f$ of the form $f = \sum_{\ell=0}^{m} \varphi_{\ell}(x)f_{\ell}(r)$, where $f_{\ell}(r) \in L^2_{\text{comp}}((0, \infty))$. Assume that $f_{\ell}(r) = 0$ for $r > a > 0$, $0 \leq \ell \leq m$. Then by (10.15), we have
\[ R_{\text{free}}(\lambda \pm i 0) f = \sum_{\ell=0}^{m} \Psi(\lambda, r; \lambda_\ell)f_{\ell}(\lambda, \lambda_\ell), \quad r > a, \]
from which (10.18) follows immediately.

Note that if $\rho(r) = O(r^3)$, $r_0(\lambda, \lambda_\ell) = O((\lambda_\ell)^{1/2})$. This shows the subtlety of the expansion (10.18).

**10.4. Perturbed metric.** We return to the perturbed metric
\[ ds^2 = (dr)^2 + \rho(r)^2 h(r, x, dx), \]
\[ h(r, x, dx) - h_M(x, dx) \in S^{-\gamma}, \quad \gamma > 1. \]
Let $R_{\text{free}}(z) = (H_{\text{free}} - z)^{-1}$ and $R(z) = (H - z)^{-1}$. Take $\chi \in C^\infty(0, \infty)$ such that $\chi(r) = 0$ for $r < 1$, $\chi(r) = 1$ for $r > 2$. Then, we have
\[ \chi R(z) = R_{\text{free}}(z)V(z), \]
\[ V(z) = \chi + \left( [H_{\text{free}}, \chi] - \chi \bar{V} \right) R(z), \quad \bar{V} = H - H_{\text{free}}. \]
Here we use the assumption $\gamma > 1$ to have
\[ V(\lambda \pm i 0) \in B(B^\ast; B), \]
which implies that, by virtue of (10.19), $R(\lambda \pm i 0)$ has the same asymptotic expansion as in Theorem 10.9 where $f$ of the right-hand sides of (10.18) is replaced by $V(\lambda \pm i 0)f$. Therefore, the following theorem is proved.

**Theorem 10.10.** Let $f \in B$. Then, on the regular end satisfying (10.4)
\[ R(\lambda \pm i 0) f \simeq \sum_{\ell=0}^{\infty} c_\ell(\lambda, r) \rho(r)^{-(n-1)/2} e^{\pm i\varphi(\lambda, \lambda_\ell, r)} P_\ell \otimes F_{\text{free},\lambda_\ell}(\lambda)V(\lambda \pm i 0)f. \]
11. Spectral representation

We return to our original manifold $M = \mathcal{K} \cup M_1 \cup \cdots \cup M_{N+N'}$. From here until the end of §12, we assume
\[
\begin{cases}
\alpha_{0,j} > 0, & \beta_{0,j} > 0, & \gamma_{0,j} > 1 \\
(A - 3) & \text{on cusp ends.}
\end{cases}
\]

In this section, we construct a spectral representation for $H$ by observing the asymptotic expansion of the resolvent at infinity.

11.1. Definition of $\mathcal{F}_j^{(\pm)}(\lambda)$. Let $h_j = L^2(M_j)$ and
\[
h_{\infty,j} = \begin{cases}
h_j & \text{for } 1 \leq j \leq N, \\
\mathbb{C} & \text{for } N + 1 \leq j \leq N + N',
\end{cases}
\]
\[
h_{\infty} = \bigoplus_{j=1}^{N+N'} h_{\infty,j},
\]
(11.1)
\[
\hat{H} = \bigoplus_{j=1}^{N+N'} L^2((E_{0,j}, \infty); h_{\infty,j}; d\lambda).
\]

Let $\lambda_{\ell,j}$ and $P_{\ell,j}$ ($\ell = 0, 1, 2, \ldots$) be the eigenvalues and the associated eigenprojections for $-\Delta_{M_j}$ with respect to the metric $h_{M_j}(x, dx)$.

For $f \in B(M)$ and $\lambda \in \sigma_e(H) \setminus \mathcal{E}$, we put
\[
\mathcal{F}_j^{(\pm)}(\lambda)f = (\mathcal{F}_1^{(\pm)}(\lambda)f, \ldots, F_{N+N'}^{(\pm)}(\lambda)f) \in h_{\infty},
\]
where $\mathcal{F}_j^{(\pm)}(\lambda)$ is defined as follows.

(I) Regular ends with $\beta_{0,j} > 1/2$ : For $1 \leq j \leq N$ and $\lambda > E_{0,j}$, we define it making use of the asymptotic expansion (9.24) in Theorem 9.8:
\[
R(\lambda \pm i0)f \simeq C_j(\lambda)\rho_j(\lambda)^{-(n-1)/2}e^{\pm i\Phi_j(\lambda)} F_j^{(\pm)}(\lambda)f \quad \text{on } M_j,
\]
(11.2)
where
\[
C_j(\lambda) = \left(\frac{\pi}{\sqrt{\lambda - E_{0,j}}}\right)^{1/2},
\]
\[
\Phi_j(\lambda) = \int_0^\rho \phi_j(t, \lambda) dt,
\]
\[
\phi_j(\lambda) = \sqrt{\lambda - \left(\frac{n-1}{4}\frac{\rho_j'(\lambda)}{\rho_j(\lambda)}\right)^2}.
\]

(II) Regular ends with $0 < \beta_{0,j} \leq 1/2$ : For $1 \leq j \leq N$ and $\lambda > E_{0,j}$, we define
\[
\mathcal{F}_j^{(\pm)}(\lambda) = \sum_{\ell=0}^{\infty} \mathcal{F}_{\ell,j}^{(\pm)}(\lambda),
\]
where
\[
\mathcal{F}_{\ell,j}^{(\pm)}(\lambda) = P_{\ell,j} \otimes F_{\ell,j}^{(\pm)}(\lambda)V_j(\lambda \pm i0)
\]
appearing in Theorem 11.10
\[
(11.3) \quad R(\lambda \pm i0)f \simeq \sum_{\ell=0}^{\infty} c_{\ell,j}(\lambda, \rho_j(\lambda)^{-(n-1)/2}e^{\pm i\phi_j(\lambda, \lambda_{\ell,j}, r)} F_{\ell,j}^{(\pm)}(\lambda)f, \quad \text{on } M_j,
\]
\[
(11.4) \quad R(\lambda \pm i0)f \simeq \sum_{\ell=0}^{\infty} c_{\ell,j}(\lambda, \rho_j(\lambda)^{-(n-1)/2}e^{\pm i\phi_j(\lambda, \lambda_{\ell,j}, r)} F_{\ell,j}^{(\pm)}(\lambda)f, \quad \text{on } M_j,
\]
\[
(11.5) \quad R(\lambda \pm i0)f \simeq \sum_{\ell=0}^{\infty} c_{\ell,j}(\lambda, \rho_j(\lambda)^{-(n-1)/2}e^{\pm i\phi_j(\lambda, \lambda_{\ell,j}, r)} F_{\ell,j}^{(\pm)}(\lambda)f, \quad \text{on } M_j,
\]
\[
(11.6) \quad R(\lambda \pm i0)f \simeq \sum_{\ell=0}^{\infty} c_{\ell,j}(\lambda, \rho_j(\lambda)^{-(n-1)/2}e^{\pm i\phi_j(\lambda, \lambda_{\ell,j}, r)} F_{\ell,j}^{(\pm)}(\lambda)f, \quad \text{on } M_j,
\]
Then, we have
\[ c_{\ell,j}(\lambda, r) = \left( \frac{\pi}{\sqrt{\lambda - E_{0,j}}} \right)^{1/2} \lambda \left( \frac{r}{r_0(\lambda, \lambda_{\ell,j})} \right), \]
\[ \varphi_j(\lambda, E, r) = \int_{r_0(\lambda, E)}^r \alpha_j(\lambda, E, s) ds, \]
\[ \alpha_j(\lambda, E, r) = \sqrt{\lambda - \left( \frac{(n-1)\rho_j'}{2\rho_j} \right)^2 - \frac{E}{\rho_j^2}}. \]

(III) Cusp ends: For \( N + 1 \leq j \leq N + N' \) and \( \lambda > E_{0,j} \), we use \((9.23)\) in Theorem \(9.9\)
(11.4) \( R(\lambda \pm i0) f \simeq C_j(\lambda) P_{\lambda,j} \otimes \rho_j(r)^{- \frac{(n-1)}{2} e^{\pm i\phi_j(r, \lambda)} \mathcal{F}_j^{(\pm)}(\lambda) f} \) on \( \mathcal{M}_j \).

Finally, we define
\[ \mathcal{F}_j^{(\pm)}(\lambda) = 0, \quad \text{if} \quad \lambda < E_{0,j}, \quad 1 \leq j \leq N + N'. \]

**Lemma 11.1.** For \( f, g \in B \) and \( \lambda \in \sigma_c(H) \setminus \mathcal{E} \)
\[ \frac{1}{2\pi i} \left( \left[ R(\lambda + i0) - R(\lambda - i0) \right] f, g \right) = \left( \mathcal{F}^{(\pm)}(\lambda)f, \mathcal{F}^{(\pm)}(\lambda)g \right)_{\mathcal{H}_\infty} \]

**Proof.** Take \( \varphi \in C_0^\infty(\mathbb{R}) \) such that \( \varphi(r) = 0 \) for \( r < 1/2 \) and \( r > 3 \), and \( \varphi(r) = 1 \) for \( 1 < r < 2 \). Put
\[ \overline{\varphi}(r) = \int_r^\infty \varphi(t) dt. \]

Take \( \varphi_R \in C_0^\infty(\mathcal{M}) \) such that \( \varphi_R = \overline{\varphi}(0) \) on \( \mathcal{K} \), and on each end \( \mathcal{M}_j \)
\[ \varphi_R(r) = \overline{\varphi}(r/R). \]

Then, we have \( \varphi'_R = -\varphi(r/R)/R \) on each end, hence
(11.5) \[ \left[ -\frac{\partial^2}{\partial r^2} - \frac{(n-1)\rho_j'}{2\rho_j} \frac{\partial}{\partial r}, \varphi_R \right] = \frac{2}{R} \varphi(R) \left( \frac{\partial}{\partial r} + \frac{(n-1)\rho_j'}{2\rho_j} \right) + \frac{1}{R^2} \varphi''(R). \]

Let \( u = R(\lambda + i0) f, v = R(\lambda + i0) g \). Then, we have
(11.6) \[ \left( \varphi_R u, g \right) - \left( f, \varphi_R v \right) = \left( [H, \varphi_R] u, v \right). \]

As \( R \to \infty \), the left-hand side tends to \( \overline{\varphi}(0) \left( [R(\lambda + i0) - R(\lambda - i0)] f, g \right) \). By \((7.10)\), we have on each end,
\[ \frac{\partial}{\partial r} + \frac{(n-1)\rho_j'}{2\rho_j} = D_j^{(+)}(k) + ik + O(r^{-\epsilon}), \quad \epsilon > 0. \]

This, together with \((11.5)\), implies that the right-hand side of \((11.6)\) is asymptotically equal to
\[ \frac{2ik}{R} \left( \varphi(R) u, v \right). \]

By \((11.2), (11.3)\) and \((11.4)\), this is asymptotically equal to
(11.7) \[ 2ik \overline{\varphi}(0) \sqrt{\frac{\pi}{\lambda - E_{0,j}}} \left( \mathcal{F}_j^{(\pm)}(\lambda)f, \mathcal{F}_j^{(\pm)}(\lambda)g \right)_{\mathcal{H}_j}. \]
for the case of regular end with $\beta_j > 1/2$,
\begin{equation}
(11.8) \quad 2ik \varphi(0) \sum_{\ell=0}^{\infty} \frac{\pi}{\sqrt{\lambda - E_{0,j}}} \langle F_{\ell,j}^{(\pm)}(\lambda)f, F_{\ell,j}^{(\pm)}(\lambda)g \rangle_h,
\end{equation}
for the case of regular end with $0 < \beta_j \leq 1/2$, and
\begin{equation}
(11.9) \quad 2ik \varphi(0) \frac{\pi}{\sqrt{\lambda}} F_{0,j}^{(\pm)}(\lambda)f F_{0,j}^{(\pm)}(\lambda)g,
\end{equation}
for the case of cusp end.

Let us prove (11.8). It is easy to show that if $w_1 \in \mathcal{B}_0^*$ and $w_2 \in \mathcal{B}$, then
\[
\frac{1}{R}(\varphi(\frac{r}{R})w_1, w_2) \to 0. \quad \text{By (11.3), } \frac{1}{R}(\varphi(\frac{r}{R})u, v) \text{ is asymptotically equal to}
\]
\[
\frac{1}{R} \sum_\ell \left( \chi(\frac{r}{R}) \sum_{c_{\ell,j}} a_{\ell,j} \right) dt = \sum_\ell \int_0^\infty \varphi(\frac{r}{R}) \chi(\frac{r}{r_0(\lambda, \lambda_{\ell,j})})^2 dr a_{\ell,j},
\]
where $a_{\ell,j} = \langle F_{\ell,j}^{(\pm)}(\lambda)f, F_{\ell,j}^{(\pm)}(\lambda)g \rangle_h$. Let $r = tR$. Since $\sum_\ell |a_{\ell,j}| < \infty$ and $\chi(\frac{tR}{r_0(\lambda, \lambda_{\ell,j})}) \to 1$, the right-hand side converges to
\[
\frac{\pi}{\sqrt{\lambda - E_{0,j}}} \sum_\ell \int_0^\infty \varphi(t dt) a_{\ell,j} = \frac{\pi}{\sqrt{\lambda - E_{0,j}}} \varphi'(0) \sum_\ell a_{\ell,j}.
\]
This proves (11.8). The proof of (11.7) and (11.9) is similar and simpler.

Summing up (11.7), (11.8), (11.9) with respect to $j$ and dividing by $\varphi(0)$, we obtain the lemma for the case of $F^{(\tau)}(\lambda)$.

As a corollary, we have
\begin{equation}
(11.10) \quad \|F^{(\pm)}(\lambda)f\|_{h_\infty} \leq C\|f\|_B,
\end{equation}
where the constant $C$ does not depend on $\lambda$ when $\lambda$ varies over a compact set in $\sigma_v(H) \setminus \mathcal{E}$.

11.2. Generalized Fourier transform. We put
\[
(F^{(\pm)} f)(\lambda) = F^{(\pm)}(\lambda)f
\]
for $f \in \mathcal{B}$. In view of (8.2) and Lemma 11.1 we have, for $f, g \in \mathcal{B}$,
\[
(P_{ac}(H)f, g) = \int_{E_{0,tot}}^{\infty} \langle F^{(\pm)}(\lambda)f, F^{(\pm)}(\lambda)g \rangle_{h_\infty} d\lambda = \langle (F^{(\pm)} f), (F^{(\pm)} g) \rangle_{\hat{H}_t}.
\]
Therefore, $F^{(\pm)}$ is uniquely extended to a partial isometry on $L^2(\mathcal{M})$ with initial set $\mathcal{H}_{ac}(H)$ and final set in $\hat{H}_t$, defined in (11.1), which is denoted by $F^{(\pm)}$ again. We show the following lemma.

**Lemma 11.2.** (1) For any $f \in D(H)$ and a.e. $\lambda \in (E_{0,tot}, \infty)$, we have
\[
(F^{(\pm)} H f)(\lambda) = \lambda (F^{(\pm)} f)(\lambda).
\]
(2) For any bounded Borel function $\alpha(\lambda)$ on $\mathbb{R}$, any $f \in L^2(\mathcal{M})$ and a.e. $\lambda \in (E_{0,tot}, \infty)$, we have
\begin{equation}
(11.11) \quad (F^{(\pm)} \alpha(H)f)(\lambda) = \alpha(\lambda)(F^{(\pm)} f)(\lambda).
\end{equation}
Proof. For $f \in L^2_{\text{comp}}(\mathcal{M}) \cap D(H)$, let $u = R(\lambda + i0)f$ and $v = Hu$. Then, we have $(H - \lambda)u = f$, $v = R(\lambda + i0)Hf$. Observing the spatial asymptotics of $v = Hu = \lambda u + f$, we have $\mathcal{F}(\lambda)f = \lambda \mathcal{F}(\lambda)f$, which proves (1). It then follows that

$$(\lambda - z)(\mathcal{F}(\pm)(H - z)^{-1}f)(\lambda) = (\mathcal{F}(\pm)f)(\lambda)$$

for $z \notin \mathbb{R}$, which shows the assertion (2) for $\alpha(\lambda) = (\lambda - z)^{-1}$. Then, by Stone’s formula, (2) holds for any step function, hence for any bounded Borel function. □

Lemma 11.3. Ran $\mathcal{F}(\pm) = \hat{\mathcal{H}}$.

Proof. We have only to show that the range of $\mathcal{F}(\pm)$ is dense in $\hat{\mathcal{H}}$. For the sake of notational simplicity, we consider the case that $N = N' = 1$, and assume that the volume of $M_2$ is equal to 1. Suppose

$$(\phi_1(\lambda), \phi_2(\lambda)) \in L^2((E_{0,1}, \infty); L^2(M_1); d\lambda) \times L^2((E_{0,2}, \infty); C; d\lambda)$$

is orthogonal to Ran $\mathcal{F}(\pm)$. Let $\lambda_{\ell,1}$, $\ell = 0, 1, 2, \ldots$, be the eigenvalues of $-\Delta_{M_1}$, and $\epsilon_{\ell,1}$ the associated complete orthonormal system of eigenvectors in $L^2(M_1)$. We put

$$(11.12) \quad \phi_{\ell,1}(\lambda) = (\phi_1(\lambda), \epsilon_{\ell,1})_{L^2(M_1)}.$$  

For $\psi \in L^1_{\text{loc}}((E_{0,\text{tot}}, \infty))$, let $\mathcal{L}(\psi)$ be the set of Lebesgue points of $\psi$, i.e.

$$\mathcal{L}(\psi) \ni \lambda \iff \psi(\lambda) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{\lambda - \epsilon}^{\lambda + \epsilon} \psi(t)dt.$$  

It is well-known that $(E_{0,\text{tot}}, \infty) \setminus \mathcal{L}(\psi)$ is a null set. Take an arbitrary point $\mu \in (E_{0,\text{tot}}, \infty)$ satisfying

$$\mu \in \left(\bigcap_{\ell = 0}^{\infty} \mathcal{L}(\phi_{\ell,1})\right) \cap \left(\mathcal{L}(\|\phi_1\|^2_{L^2(M_1)})\right) \cap \left(\mathcal{L}(\phi_2)\right) \cap \left(\mathcal{L}(\|\phi_2\|^2)\right) \cap ((E_{0,\text{tot}}, \infty) \setminus \mathcal{E}).$$

Let $\{\chi_j\}_{j=0}^{\infty}$ be the partition of unity in (7.3). We fix $m$ arbitrarily, and put

$$u_\mu(r) = \chi_1(r)\Psi^{(+)}_1(\mu, r, \lambda_{m,1})\alpha e_{m,1}(x) + \chi_2(r)\Psi^{(+)}_2(\mu, r, 0)\beta,$$

where $\Psi^{(+)}_j(\lambda, r, E)$ is the solution constructed in Lemma 10.3 for the end $\mathcal{M}_j$, and $\alpha, \beta$ are arbitrary constants. We put

$$(H - \mu) u_\mu = g_\mu.$$  

Then, by virtue of Lemma 10.2, $g_\mu \in L^{2,1+\epsilon}(\mathcal{M})$. Since $u_\mu$ is outgoing, we have $u_\mu = R(\mu + i0)g_\mu$. Moreover, letting $\mathcal{F}(\lambda)g_\mu = (C_1(\lambda), C_2(\lambda))$ and observing the behavior of $u_\mu$ at infinity, we see that $(C_1(\lambda), C_2(\lambda))$ is an $L^2(M_1) \times \mathbb{C}$-valued continuous function of $\lambda > 0$ satisfying

$$(11.13) \quad (C_1(\mu), \epsilon_{\ell,1}) = \left(\frac{\pi}{\sqrt{\mu - E_{0,1}}}\right)^{-1/2} \delta_{\ell,m}\alpha, \quad C_2(\mu) = \left(\frac{\pi}{\sqrt{\mu - E_{0,2}}}\right)^{-1/2} \beta.$$  

By the assumption, $(\phi_1(\lambda), \phi_2(\lambda))$ is orthogonal to $\mathcal{F}(\pm)E_H(I)g_\mu$, $I$ being any interval in $(E_{0,\text{tot}}, \infty) \setminus \mathcal{E}$. Hence by Lemma 11.2 (2)

$$\int_I \left((\phi_1(\lambda), C_1(\lambda))_{L^2(M_1)} + \phi_2(\lambda)C_2(\lambda)\right) d\lambda = 0$$

for $I \in (E_{0,\text{tot}}, \infty) \setminus \mathcal{E}$. Hence by Lemma 11.2 (2)

$$\int_I \left((\phi_1(\lambda), C_1(\lambda))_{L^2(M_1)} + \phi_2(\lambda)C_2(\lambda)\right) d\lambda = 0$$

for $I \in (E_{0,\text{tot}}, \infty) \setminus \mathcal{E}$. Hence by Lemma 11.2 (2)
for any interval $I \subset (E_{0, \text{tot}}, \infty) \setminus \mathcal{E}$. Since $C_2(\lambda)$ is continuous, and $\mu$ is a Lebesgue point of $\phi_2(\lambda)$, we have

$$
\frac{1}{2\epsilon} \int_{\mu-\epsilon}^{\mu+\epsilon} \phi_2(\lambda) C_2(\lambda) d\lambda \to \phi_2(\mu) \left( \frac{\pi}{\sqrt{\mu - E_{0,2}}} \right)^{-1/2} \beta.
$$

The 1st term is computed as

$$
\frac{1}{2\epsilon} \int_{\mu-\epsilon}^{\mu+\epsilon} (\phi_1(\lambda), C_1(\lambda) \rangle_{L^2(M_1)} d\lambda = \frac{1}{2\epsilon} \int_{\mu-\epsilon}^{\mu+\epsilon} (\phi_1(\lambda), C_1(\lambda) - C_1(\mu)) \rangle_{L^2(M_1)} d\lambda
$$

$$
+ \frac{1}{2\epsilon} \int_{\mu-\epsilon}^{\mu+\epsilon} (\phi_1(\lambda), C_1(\mu)) \rangle_{L^2(M_1)} d\lambda.
$$

By (11.12) and (11.13), $$(\phi_1(\lambda), C_1(\mu)) \rangle_{L^2(M_1)} = \left( \frac{\pi}{\sqrt{\mu - E_{0,1}}} \right)^{-1/2} \phi_{m,1}(\lambda) \overline{\pi},$$ hence

$$
\frac{1}{2\epsilon} \int_{\mu-\epsilon}^{\mu+\epsilon} (\phi_1(\lambda), C_1(\mu)) \rangle_{L^2(M_1)} d\lambda \to \left( \frac{\pi}{\sqrt{\mu - E_{0,1}}} \right)^{-1/2} \phi_{m,1}(\mu) \overline{\pi}.
$$

We also have

$$
\left| \frac{1}{2\epsilon} \int_{\mu-\epsilon}^{\mu+\epsilon} (\phi_1(\lambda), C_1(\lambda) - C_1(\mu)) \rangle_{L^2(M_1)} d\lambda \right|
$$

$$
\leq \left( \frac{1}{2\epsilon} \int_{\mu-\epsilon}^{\mu+\epsilon} \|\phi_1(\lambda)\|^2_{L^2(M_1)} d\lambda \right)^{1/2} \times \left( \frac{1}{2\epsilon} \int_{\mu-\epsilon}^{\mu+\epsilon} \|C_1(\lambda) - C_1(\mu)\|^2_{L^2(M_1)} d\lambda \right)^{1/2}.
$$

The right-hand side tends to 0, since $\mu$ is a Lebesgue point of $\|\phi_1(\lambda)\|^2_{L^2(M_1)}$, and $C_1(\lambda)$ is an $L^2(M_1)$-valued continuous function of $\lambda > 0$. Therefore, we have

$$
\phi_{m,1}(\mu) \left( \frac{\pi}{\sqrt{\mu - E_{0,1}}} \right)^{-1/2} \overline{\pi} + \phi_2(\mu) \left( \frac{\pi}{\sqrt{\mu - E_{0,2}}} \right)^{-1/2} \beta = 0.
$$

Since $\alpha, \beta$ can be chosen arbitrarily, we have proven that $\phi_1(\mu) = 0, \phi_2(\mu) = 0$. \hfill \Box

Now, we have arrived at the main theorem.

**Theorem 11.4.** (1) The operator $\mathcal{F}^{(\pm)}$ is uniquely extended to a partial isometry with initial set $\mathcal{H}_{ac}(H)$ and final set $\mathcal{H}^*$. (2) For any $\lambda \in (E_{0,\text{tot}}, \infty) \setminus \mathcal{E}$, $\mathcal{F}^{(\pm)}(\lambda)^* \in \mathcal{B}(h_\infty; B^*)$. Moreover, for any $a \in h_\infty$,

$$
(-\Delta_M - \lambda) \mathcal{F}^{(\pm)}(\lambda)^* a = 0.
$$

(3) For any $f \in L^2(M)$, and a compact interval $I \subset (E_{0,j}, \infty) \setminus \mathcal{E}$, the $B^*$-valued integral

$$
\mathcal{F}_j^{(\pm)}(\lambda)^* (\mathcal{F}_j^{(\pm)} f)(\lambda) d\lambda
$$

belongs to $L^2(M)$. Letting $I_k \to (E_{0,j}, \infty) \setminus \mathcal{E}$, the strong limit

$$
\lim_{k \to \infty} \int_{I_k} \mathcal{F}_j^{(\pm)}(\lambda)^* (\mathcal{F}_j^{(\pm)} f)(\lambda) d\lambda = \int_{E_{0,j}} \mathcal{F}_j^{(\pm)}(\lambda)^* (\mathcal{F}_j^{(\pm)} f)(\lambda) d\lambda
$$
For any \( f \in \mathcal{H}_{ac}(H) \), the inversion formula holds:

\[
f = \left( \mathcal{F}(\pm) \right)^* \mathcal{F}(\pm) f = \sum_{j=1}^{\infty} \int_{E_{0,j}} \mathcal{F}_j(\pm)^*(\mathcal{F}_j(\pm)) (\lambda) d\lambda.
\]

**Proof.** The assertion (1) is already proven in Lemma 11.3. By (11.10), \( \mathcal{F}(\pm)(\lambda)^* \in B(\mathcal{H}_{\infty}; B_0^*) \). Taking the adjoint in Lemma 11.2 (1), we obtain (2). We prove (3) and (4) at the same time. Take any compact interval \( I \subset (E_0, \infty) \setminus \mathcal{E} \), and put

\[
u_I = \int_I \mathcal{F}(\pm)(\lambda)^*(\mathcal{F}(\pm) f)(\lambda) f d\lambda,
\]

which belongs to \( B^* \) by (2). Letting \( c_I(\lambda) \) be the characteristic function of \( I \), we have for any \( g \in B \),

\[
(u_I, g) = \int c_I(\lambda)(\mathcal{F}(\pm) f)(\lambda), (\mathcal{F}(\pm) g) d\lambda = (E_H(I) f, g).
\]

Therefore \( u_I = E_H(I) f \in L^2(\mathcal{M}) \). This implies that for any simple function \( \alpha(\lambda) \)

\[
\int \alpha(\lambda) \mathcal{F}(\pm)(\lambda)^*(\mathcal{F}(\pm) f)(\lambda) f d\lambda = \alpha(H) f.
\]

Finally, we approximate \((E_{0,tot}, \infty)\) by a union of compact intervals in \((E_{0,tot}, \infty) \setminus \mathcal{E}\) to complete the proof.

### 12. Helmholtz equation and S-matrix

**12.1. Eigenoperator.** The adjoint of \( \mathcal{F}(\pm)(\lambda) \) is an eigenoperator of \( H \) in the sense that \((-\Delta_M - \lambda) \mathcal{F}(\pm)(\lambda)^* a = 0\) holds for \( a \in \mathcal{h}_{\infty} \). We will derive the S-matrix by observing its asymptotic behavior at infinity.

Letting \( c_j(\lambda) \) be the characteristic function of the interval \((E_{0,j}, \infty)\), we put

\[
\mathcal{h}_{\infty}(\lambda) = \sum_{j=1}^{N+N'} c_j(\lambda) \mathcal{h}_{\infty,j}.
\]

For \( a = (a_1, \cdots, a_{N+N'}) \in \mathcal{h}_{\infty}(\lambda) \), we have

\[
\mathcal{F}(\pm)(\lambda)^* a = \sum_{j=1}^{N+N'} \mathcal{F}_j(\pm)(\lambda)^* a_j.
\]

Let \( \{e_{\ell,j}(x)\}_{\ell=0}^{\infty} \) be a complete orthonormal system of eigenvectors of \(-\Delta_{M_j}\) associated with eigenvalues \( \{\lambda_{\ell,j}\}_{\ell=0}^{\infty} \). In particular, \( e_{0,j}(x) = (\text{vol}(M_j))^{-1/2} \). Let \( D_{finite}(A_j) \) be the set of \( c \in L^2(M_j) \) such that \( (c, e_{\ell,j}) = 0 \) except for a finite number of \( \ell \). We define a subset \( \mathcal{h}_{\infty,j}^{comp} \subset \mathcal{h}_{\infty,j} \) by

\[
\mathcal{h}_{\infty,j}^{comp} = \left\{ D_{finite}(A_j) \text{ for } 1 \leq j \leq N, \mathbb{C} \text{ for } N+1 \leq j \leq N+N' \right\}.
\]

and \( \mathcal{h}_{\infty}^{comp}(\lambda) \) by

\[
\mathcal{h}_{\infty}^{comp}(\lambda) = \sum_{j=1}^{N+N'} c_j(\lambda) \mathcal{h}_{\infty,j}^{comp}.
\]

\(^1\)One needs to be careful about the definition of \( \mathcal{F}(\pm)(\lambda)^* \). We discuss it in the next section.
Taking \( a = (a_1, \ldots, a_{N+N'}) \in h^{\text{comp}}(\lambda) \), and using the partition of unity \( \{\chi_j\}_{j=0}^{N+N'} \), we put \( u^{(\pm)}_j \) as follows.

(I) For regular ends with \( \beta_{0,j} > 1/2 \):
\[
 u^{(\pm)}_j = C_j(\lambda)\chi_j\rho_j(r)^{-(n-1)/2}e^{\pm i\phi_j(r,\lambda)}a_j, \quad 1 \leq j \leq N,
\]

(II) For regular ends with \( 0 < \beta_{0,j} \leq 1/2 \):
\[
 u^{(\pm)}_j = \sum_{\ell=0}^{\infty} e_{\ell,j}(\lambda, r)\chi_j\rho_j(r)^{-(n-1)/2}e^{\pm i\phi_j(\lambda,\ell,\lambda, r)}a_{\ell,j}e_{\ell,j}(x), \quad 1 \leq j \leq N,
\]
where \( a_j = \sum_{\ell=0}^{\infty} a_{\ell,j}e_{\ell,j}(x) \).

(III) For cusp ends:
\[
 u^{(\pm)}_j = C_j(\lambda)\chi_j\rho_j(r)^{-(n-1)/2}e^{\pm i\phi_j(\lambda,\lambda, r)}a_{0,j}e_{0,j}(x), \quad N + 1 \leq j \leq N + N'.
\]

We put
\[
 (12.1) \quad f^{(\pm)}_j = (-\Delta_M - \lambda)u^{(\pm)}_j, \quad f^{(\pm)} = \sum_{j=1}^{N+N'} f^{(\pm)}_j, \quad u^{(\pm)} = \sum_{j=1}^{N+N'} u^{(\pm)}_j.
\]

**Lemma 12.1.** Let \( a, u^{(\pm)}, f^{(\pm)} \) be as above. Then we have
\[
\mp 2\pi i(\mathcal{F}^{(\pm)}(\lambda))^*a = u^{(\pm)} - R(\lambda \mp i0)f^{(\pm)}.
\]

**Proof.** We prove the \((+)\) case. Let \( v = R(\lambda + i0)h \) for \( h \in \mathcal{B} \). Take \( \chi(r) \in C^\infty((0, \infty)) \) such that \( \chi(r) = 1 \) for \( r < 1 \), \( \chi(r) = 0 \) for \( r > 2 \), and put \( \chi(t) = \chi(r/t) \). Then by integration by parts, we have
\[
 (\chi_t u^{(\pm)}_j, h) - (\chi f^{(\pm)}_j, v)
\]
\[
 = \frac{1}{t} \int \chi'(\frac{r}{t}) \left( u^{(\pm)}_j D^{(+)}_j(k)v - (D^{(+)}_j(k)u^{(\pm)}_j)v \right) \sqrt{g_j} dr dx
\]
\[
 + 2i \frac{1}{t} \int \chi'(\frac{r}{t}) (\text{Re} \psi^{(\pm)}_j) u^{(\pm)}_j \sqrt{g_j} dr dx,
\]
where \( \psi^{(\pm)}_j = \sqrt{\lambda - E_{0,j}} + O(r^{-\epsilon}) \), \( \epsilon > 0 \). The first term of the right-hand side of (12.2) vanishes as \( t \to \infty \). To compute the 2nd term, we use the asymptotic expansions (11.2), (11.3), (11.4). Since \( \frac{1}{\tau} \int \chi'(r/\tau) dr = -1 \), it tends to
\[
-2\pi i(a_j, \mathcal{F}^{(\pm)}(\lambda)h)\mathbf{h}_{\infty,j}.
\]

We thus obtain
\[
 (u^{(\pm)}_j, h) - (R(\lambda - i0)f^{(\pm)}_j, h) = -2\pi i(\mathcal{F}^{(\pm)}(\lambda)^*a_j, h).
\]

Summing up with respect to \( j \), we obtain the lemma. \( \square \)

In view of Lemma 12.1 and the asymptotic expansion of the resolvent (Theorems 0.8, 0.9, 10.10), we have the following lemma.
LEMMA 12.2. For any $a \in h_\infty^{omp}(\lambda)$, there exists $b^{(\pm)} \in h_\infty(\lambda)$ such that, letting $U^{(\pm)} \in B^*$ be defined by (1), (2), (3) below, $2\pi i\mathcal{F}^{(\pm)}(\lambda)^*a$ satisfies

$$
(12.3) \quad \mp 2\pi i\mathcal{F}^{(\pm)}(\lambda)^*a - U^{(\pm)} \in B^*_0,
$$

where

(1) for regular ends with $\beta_{0,j} > 1/2$,

$$
U^{(\pm)} = \left(\frac{\pi}{\sqrt{\lambda - E_{0,j}}}\right)^{1/2} \rho_j(r)^{-(n-1)/2} \left(e^{\pm i\Phi_j(r,\lambda)}a_j(x) - e^{\mp i\Phi_j(r,\lambda)}b_j^{(\pm)}(x)\right),
$$

(2) for regular ends with $0 < \beta_{0,j} \leq 1/2$,

$$
U^{(\pm)} = \left(\frac{\pi}{\sqrt{\lambda - E_{0,j}}}\right)^{1/2} \rho_j(r)^{-(n-1)/2} \times \sum_{\ell=0}^{\infty} \chi_{\ell,j}(r) \left(e^{\pm i\varphi_j(\lambda,\lambda_{\ell,j},r)}a_{j,\ell} - e^{\mp i\varphi_j(\lambda,\lambda_{\ell,j},r)}b_j^{(\pm)}\right)e_{j,\ell}(x),
$$

where $\chi_{\ell,j}(r) = \chi\left(\frac{r}{r_0(\lambda,\lambda_{\ell,j})}\right)$ with $\chi(r)$ satisfying $\chi \in C^\infty(\mathbb{R})$, $\chi(r) = 0$ for $r < 1$, $\chi(r) = 1$ for $r > 2$,

(3) for cusp ends

$$
U^{(\pm)} = \left(\frac{\pi}{\sqrt{\lambda - E_{0,j}}}\right)^{1/2} \rho_j(r)^{-(n-1)/2} \left(e^{\pm i\varphi_j(r,\lambda)}a_{j,0} - e^{\mp i\varphi_j(r,\lambda)}b_j^{(\pm)}\right)e_{j,0}(x).
$$

We equip $h_\infty(\lambda)$ with the inner product

$$
(a, b)_{h_\infty(\lambda)} = \sum_{j=1}^{N} (a_j, b_j)_{L^2(M_j)} + \sum_{j=N+1}^{N+N'} a_jb_j \mathrm{vol}(M_j).
$$

LEMMA 12.3. The operator

$$
h_\infty(\lambda) \ni h_\infty^{omp}(\lambda) \ni a \to b^{(\pm)} \in h_\infty(\lambda)
$$

is isometric.

PROOF. We prove the lemma for $\mathcal{F}^{(+)}(\lambda)$. Let $v = -2\pi i\mathcal{F}^{(+)}(\lambda)^*a$. Take $\chi(r) \in C^\infty(\mathbb{R})$ such that $\chi(r) = 1$ for $r < 1$, and $\chi(r) = 0$ for $r > 2$. Let

$$
(12.5) \quad \chi_t = \chi_0 + \sum_{j=1}^{N+N'} \chi_j(r/t),
$$

where $\{\chi_j\}$ is the partition of unity on $M$ as in (73). Since $(-\Delta_M - \lambda)v = 0$, we have by integration by parts

$$
0 = -\Im \langle \chi_t \Delta_M v, v \rangle
$$

$$
= \sum_{j=1}^{N+N'} \frac{1}{t} \int_{M_j} \chi_j'(r/t) \Im \left(\frac{\partial v}{\partial r} \overline{\frac{\partial v}{\partial r}} - i \frac{\partial v}{\partial r} \overline{\frac{\partial v}{\partial r}}\right) \sqrt{g_j} \, dr.\, dx.
$$

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Replacing $v$ by the asymptotic expansion in Lemma 12.2, we compute the resulting integral. Note that

$$
\frac{1}{t} \int_0^\infty \chi'(\frac{r}{t}) e^{i\lambda r} dr \to 0, \quad t \to \infty.
$$

Therefore, for regular ends with $\beta_{0,j} > 1/2$, we have

$$
\frac{1}{t} \int_{M_j} \chi'(\frac{r}{t}) \Im \frac{L}{\pi} \frac{\partial v}{\partial r} \left( \frac{\partial v}{\partial r} - v \right) \sqrt{g_j} dr dx
= \frac{2\pi}{t} \int_0^\infty \chi'(\frac{r}{t}) \left( \|a_j\|^2_{L^2(M_j)} - \|b_j\|^2_{L^2(M_j)} \right) dr + o(1).
$$

For regular ends with $0 < \beta_{0,j} \leq 1/2$, we have by a similar computation

$$
\frac{1}{t} \int_{M_j} \chi'(\frac{r}{t}) \Im \frac{L}{\pi} \frac{\partial v}{\partial r} \left( \frac{\partial v}{\partial r} - v \right) \sqrt{g_j} dr dx
= \frac{2\pi}{t} \int_0^\infty \chi'(\frac{r}{t}) \left( \|a_j\|^2 - \|b_j\|^2 \right) dr + o(1).
$$

For cusp ends,

$$
\frac{1}{t} \int_{M_j} \chi'(\frac{r}{t}) \Im \frac{L}{\pi} \frac{\partial v}{\partial r} \left( \frac{\partial v}{\partial r} - v \right) \sqrt{g_j} dr dx
= \frac{2\pi}{t} \int_0^\infty \chi'(\frac{r}{t}) \left( |a_j|^2 - |b_j|^2 \right) dr \text{Vol}(M_j) + o(1).
$$

Adding these equalities and using $\frac{1}{t} \int_0^\infty \chi'(r/t) dt = -1$, we obtain the lemma. \[\square\]

By Lemma 12.3 the expansion (12.3) in Lemma 12.2 is extended to all $a \in h_\infty(\lambda)$.

**Lemma 12.4.** For any $a \in h_\infty(\lambda)$, we have

$$
\lim_{R \to \infty} \sum_{j=1}^{N+N'} \frac{1}{R} \int_0^R \|F_j^{(\pm)}(\lambda)^* a\|_{h_\infty,\lambda}^2 dr = \sum_{j=1}^{N+N'} \frac{1}{2\pi} \sqrt{\lambda - E_{0,j}} \|a_j\|_{h_\infty,\lambda}^2.
$$

**Proof.** Since $F^{(\pm)}(\lambda)^* \in B(h_\infty; B^*)$, we have only to prove this lemma for $a \in h_\infty^{comp}(\lambda)$. Letting $v = F^{(\pm)}(\lambda)^* a$, we have only to compute

$$
\sum_{j=1}^{N+N'} \frac{1}{R} \int_{(0,R) \times M_j} |x_j b_j|^2 \sqrt{g_j} dr dx.
$$

We can then replace $v$ by the terms in the asymptotic expansion (12.3) in Lemma 12.2. Arguing in the same way as above, we obtain the lemma. \[\square\]

**Lemma 12.5.** There exists a constant $C = C(\lambda) > 0$ such that

$$
C^{-1} \|a\|_{h_\infty(\lambda)} \leq \|F^{(\pm)}(\lambda)^* a\|_{B^*} \leq C \|a\|_{h_\infty(\lambda)}, \quad \forall a \in h_\infty(\lambda).
$$
proof. This lemma follows from the definition of $B^*$ and Lemma 12.2.1.

Lemma 12.6. Let $u \in B^*$ satisfy the equation $(-\Delta \mathcal{M} - \lambda)u = 0$. If $f \in B$ satisfies $\mathcal{F}^{(+)}(\lambda)f = 0$ or $\mathcal{F}^{(-)}(\lambda)f = 0$, then $(u, f) = 0$.

proof. Suppose $\mathcal{F}^{(+)}(\lambda)f = 0$, and let $v = R(\lambda + i0)f$. Let $\chi_i$ be as in (12.5). Then, by integration by parts

$$(\chi_i u, (-\Delta \mathcal{M} - \lambda)v) = \sum_{j=1}^{N+N'} \frac{1}{R} \int_{M_j} \chi_j' \left( \frac{1}{R} \right) \left( u \partial_r v - (\partial_r u) \overline{v} \right) \sqrt{g_j} \, dr \, dx + o(1).$$

Since $\mathcal{F}^{(+)}(\lambda)f = 0$, we have $v \in B_0^*$ and also $\partial_r v \in B_0^*$. Therefore, letting $t \to \infty$, the above integral vanishes, which proves the lemma.

Recall Banach’s closed range theorem (see e.g. 107, p. 205).

Theorem 12.7. Let $X, Y$ be Banach spaces, and $T$ a densely defined closed operator from $X$ to $Y$. Then, the following 4 assertions are equivalent.

1. $R(T)$ is closed.
2. $R(T')$ is closed.
3. $R(T) = N(T')^\perp = \{ y \in Y; \langle y, y' \rangle = 0, \forall y' \in N(T') \}.$
4. $R(T') = N(T)^\perp = \{ x' \in X'; \langle x', x' \rangle = 0, \forall x \in N(T) \}.$

Here, for an operator $T$ on a Banach space $X$, $R(T)$ and $N(T)$ are the range and nullspace for $T$, $T'$ is the dual operator, and $X'$ is the dual space of $X$.

Theorem 12.8. For $\lambda \in (E_{0, \text{tot}}, \infty) \setminus \mathcal{E}$, let

$$\mathcal{N}(\lambda) = \{ u \in B^* : (-\Delta \mathcal{M} - \lambda)u = 0 \}.$$ 

Then, we have

$$\mathcal{F}^{(\pm)}(\lambda)_B = h_\infty(\lambda),$$ 

$$\mathcal{N}(\lambda) = \mathcal{F}^{(\pm)}(\lambda)^* h_\infty(\lambda).$$

Proof. Take $X = B, Y = h_\infty(\lambda)$ and $T = \mathcal{F}^{(\pm)}(\lambda)$ in Theorem 12.7. Lemma 12.5 shows that $R(T')$ is closed. For $a \in h_{\text{comp}}^\text{out}(\lambda)$, define $f$ by (12.1). Then $\mathcal{F}^{(\pm)}(\lambda)f = a$, hence the range of $\mathcal{F}^{(\pm)}(\lambda)$ is dense in $h_\infty(\lambda)$. However, Theorem 12.7.1 shows that it is closed, whence (12.6) follows. Since $\mathcal{F}^{(\pm)}(\lambda)^*$ is an eigenoperator, $\mathcal{N}(\lambda) \supset \mathcal{F}^{(\pm)}(\lambda)^* h_\infty(\lambda)$. If $u \in \mathcal{N}(\lambda)$, Lemma 12.6 implies that $u \perp N(T')$. By Theorem 12.7.4, $u \in R(T')$, which proves (12.7).

12.2. S-matrix. By (12.7), any $u \in \mathcal{N}(\lambda)$ is written as $u = 2\pi i \mathcal{F}^{(-)}(\lambda)^* a$ for some $a = a^{(\text{in})} \in h_\infty(\lambda)$. Lemma 12.2 and the remark after Lemma 12.3 imply that there exists $a^{(\text{out})} \in h_\infty(\lambda)$ such that $u$ has the asymptotic expansion in Lemma 12.2 with $a = a^{(\text{in})}$ and $b = a^{(\text{out})}$. The S-matrix is the mapping between these asymptotic profiles $a^{(\text{in})}$ and $a^{(\text{out})}$. We make this fact more precise in the following theorem.

Theorem 12.9. Let $\lambda \in (E_{0, \text{tot}}, \infty) \setminus \mathcal{E}$. Then, for any $a^{(\text{in})} \in h_\infty(\lambda)$, there exist unique $u \in \mathcal{N}(\lambda)$ and $a^{(\text{out})} \in h_\infty(\lambda)$ such that, letting $a = a^{(\text{in})}$ and $b = a^{(\text{out})}$, $u$ behaves as follows on each end $M_j$. 

13. RADIAL SOLUTIONS ON CUSP END

(1) For regular ends with $\beta_{0,j} > 1/2$,

$$u \simeq \left( \frac{\pi}{\sqrt{\lambda - E_{0,j}}} \right)^{1/2} \rho_j(r)^{-(n-1)/2} \left( e^{-i\phi_j(r,\lambda)} a_j(x) - e^{i\phi_j(r,\lambda)} b_j(x) \right).$$

(2) For regular ends with $0 < \beta_{0,j} \leq 1/2$,

$$u \simeq \left( \frac{\pi}{\sqrt{\lambda - E_{0,j}}} \right)^{1/2} \rho_j(r)^{-(n-1)/2} \times \sum_{\ell=0}^{\infty} \chi_{\ell,j}(r) \left( e^{-i\varphi_j(\lambda,\lambda_{\ell,j},r)} a_{j,\ell} - e^{i\varphi_j(\lambda,\lambda_{\ell,j},r)} b_{j,\ell} \right) e_{\ell,j}(x),$$

where $\chi_{\ell,j}(r) = \chi(r/r_0(\lambda,\lambda_{\ell,j}))$ with $\chi \in C^\infty(\mathbb{R})$, $\chi(r) = 0$ for $r < 1$, $\chi(r) = 1$ for $r > 2$.

(3) For cusp ends

$$u \simeq \left( \frac{\pi}{\sqrt{\lambda - E_{0,j}}} \right)^{1/2} \rho_j(r)^{-(n-1)/2} \left( e^{-i\phi_j(r,\lambda)} a_{j,0} - e^{i\phi_j(r,\lambda)} b_{j,0} \right) e_{j,0}(x).$$

The operator $S(\lambda) : a^{(in)} \to a^{(out)}$ is unitary on $h_\infty(\lambda)$.

**Proof.** First we prove the existence of $u$ and $a^{(out)}$. Take $u = 2\pi i F^{(-)}(\lambda)^* a^{(in)}$. Then by Lemma 12.3 $\|u\|_{B^s} \leq C\|a^{(in)}\|_{h_\infty(\lambda)}$. For any $\epsilon > 0$, take $a_\epsilon \in h_\infty^{comp}(\lambda)$ such that $\|a^{(in)} - a_\epsilon\|_{h_\infty(\lambda)} < \epsilon$. Put $u_\epsilon = 2\pi i F^{(-)}(\lambda)^* a_\epsilon$. Then, by Lemma 12.2 there exists $b_\epsilon \in h_\infty(\lambda)$ such that $u_\epsilon, a_\epsilon, b_\epsilon$ have the asymptotic expansion in the present lemma. By the isometric property in Lemma 12.3, $b_\epsilon$ tends to some $a^{(out)} \in h_\infty(\lambda)$ as $\epsilon \to 0$. Thus, $a^{(in)}, u, a^{(out)}$ have the desired properties.

To show the uniqueness, suppose for a given $a^{(in)}$, there exist two such $u', u''$. Then, $w = u' - u''$ is a solution to the equation $(-\Delta_M - \lambda)w = 0$ satisfying the outgoing radiation condition. Therefore, $w = 0$.

Lemma 12.3 shows that $S(\lambda)$ is isometric. Arguing as above, changing the roles of $a^{(in)}$ and $a^{(out)}$, one can prove that the range of $S(\lambda)$ is dense. This implies the unitarity. \qed

13. Radial solutions on cusp end

In this section, we equip a cusp end with a warped product metric $ds^2 = (dr)^2 + \rho(r)^2 h_M(x, dx)$, and construct (super) exponentially growing or decaying solutions to the equation $(-\Delta_M - \lambda)u = 0$. A typical example of $\rho(r)$ is

$$(13.1) \quad \rho(r) = \begin{cases} 
\rho_{\alpha}, & c_0 < 0, \\
\rho_{\alpha}^{-1} e^{c_1 r^{1-\alpha}}, & c_1 < 0, \quad 0 < \alpha < 1, \\
\rho_{\beta}, & \beta < 0.
\end{cases}$$

We formulate its perturbation in the form of asymptotic series.

For a real constant $\kappa$, let $A_{<\kappa}$ be the set of finite linear combinations of the following terms

$$(\log r)^i r^\alpha, \quad i = 0, 1, 2, \ldots, \quad \alpha < \kappa.$$
Let $\tilde{A}^\kappa$ be the set of real functions $\tilde{f}(r)$ satisfying

$$\tilde{f}(r) - a_0 r^\kappa \in \tilde{A}_{<\kappa},$$

where $a_0$ is a constant. We define $A^\kappa$ to be the set of real functions $f \in C^\infty((0, \infty))$ such that for any $N > 0$, there exists $\tilde{f} \in \tilde{A}^\kappa$ satisfying

$$\partial_r^m (f(r) - \tilde{f}(r)) = O(r^{-N-m}), \quad r \to \infty, \quad \forall m \geq 0.$$

We put for $f \in A^\kappa$

$$a_0(f) = a_0,$$

where $a_0$ is from (13.2). Finally we define $A^\kappa_+$ by

$$A^\kappa_+ = \{ f \in A^\kappa : f(r) > 0 \}.$$

The following lemma is obvious.

**Lemma 13.1.** $\cup_{\kappa} A^\kappa$ is an algebra in the following sense.

$$A^\kappa A^{\kappa'} \subset A^{\kappa+\kappa'},$$

$$\partial_r^m A^\kappa \subset A^{\kappa-m}, \quad \forall m \geq 0,$$

$$\frac{1}{f} \in A_{+}^{-\kappa} \quad \text{if} \quad f \in A^\kappa_+.$$

We solve the equation

$$-\partial_r^2 - \left( \frac{n-1}{\rho(r)} \rho'(r) \right) \partial_r + \frac{B}{\rho(r)^2} - \lambda \right) u = 0,$$

where $B \in [B_0, \infty)$ is a parameter, $B_0$ being a fixed positive constant. In our applications, $B_0$ is the least non-zero eigenvalue of $-\Delta_M$.

**Lemma 13.2.** Let $u = \rho^{-(n-2)/2} w$ in (13.3) and introduce a new variable

$$t = \int_0^r \sqrt{\frac{B}{\rho(\tau)}} d\tau.$$

Then, we have

$$\begin{align*}
- \frac{d^2 w}{dt^2} + (1 + V(t)) w &= 0, \\
V(t) &= \frac{\rho^2}{B} \left( -\lambda + \frac{(n-1)^2}{4} \left( \frac{\rho'}{\rho} \right)^2 + \frac{n-2}{2} \left( \frac{\rho'}{\rho} \right)' \right), \\
\rho' &= d\rho/dr, \quad \rho'' = d^2\rho/dr^2.
\end{align*}$$

**Proof.** We put $u = \rho^{-(n-1)/2} v$ so that

$$-v'' + \left( \frac{B}{\rho^2} - \lambda + \frac{(n-1)^2}{2\rho} \right)^2 + \left( \frac{(n-1)^2}{2\rho} \right)' v = 0.$$

Passing to the variable $t$, and letting

$$q(t) = 1 + \frac{\rho^2}{B} \left( -\lambda + \frac{(n-1)^2}{2\rho} \right)^2 + \left( \frac{(n-1)^2}{2\rho} \right)' ,$$

where $' = d/dr$, we have

$$- \frac{d^2 v}{dt^2} + \frac{\rho'}{\sqrt{B}} \frac{dv}{dt} + q(t) v = 0.$$
We put \( v = e^\varphi w \), and obtain
\[
- \frac{d^2w}{dt^2} + \left( -2 \frac{d\varphi}{dt} + \frac{1}{\sqrt{B}} \frac{d\rho}{dr} \right) \frac{dw}{dt} + \left( q - \frac{d^2\varphi}{dt^2} - \left( \frac{d\varphi}{dt} \right)^2 + \frac{1}{\sqrt{B}} \frac{d\rho}{dr} \frac{d\varphi}{dt} \right) w = 0.
\]
We take \( \varphi = \log \sqrt{\rho} \) so that the 2nd term vanishes. We thus arrive at
\[
\begin{cases}
- \frac{d^2w}{dt^2} + \left( q + \rho'' - 2\rho'' \right) w = 0, \\
u = \rho - \frac{n-1}{2} w, \quad \rho' = \frac{d\rho}{dr}, \quad \rho'' = \frac{d^2\rho}{dr^2}.
\end{cases}
\]
The term in the parenthesis is rewritten as in (13.4).

Let us check the properties of the change of variable \([0, \infty) \ni r \to t \in [0, \infty)\). We put
\[
s = \frac{t}{\sqrt{B}} = \int_0^r \rho(\tau)^{-1} d\tau,
\]
and study the following three cases corresponding to (13.1):
\[
\rho = \begin{cases}
e^{-\phi_1} f_1, & \phi_1 \in A^1, \quad a_0(\phi_1) > 0, \quad f_1 \in A^0_+,
e^{-\phi_2} f_2, & \phi_2 \in A^{1-\alpha}, \quad 0 < \alpha < 1, \quad a_0(\phi_2) > 0, \quad f_2 \in A^\alpha_+,
g_3, & g_3 \in A^\beta_+, \quad \beta < 0.
\end{cases}
\]
We use the following notation. By \( f(r) \sim g(r) \), we mean that
\[
f(r) \to g(r) \text{ as } r \to \infty,
\]
and \( f(r) \asymp g(r) \) means that the following inequalities hold
\[
C^{-1} g(r) \leq f(r) \leq C g(r), \quad r > r_0,
\]
for some constants \( C, r_0 > 0 \).

**Lemma 13.3.** We have as a function of \( r \)
\[
s = s(r) = \begin{cases}e^{\phi_1} g_1, & g_1 \in A^0_+,
e^{\phi_2} g_2, & g_2 \in A^\alpha_+,
g_3, & g_3 \in A^{1-\beta}_+,
\end{cases}
\]
where \( \phi_1, \phi_2 \) are the same functions as in (13.4). In particular, we have
\[
\log s \asymp r, \quad \log s \asymp r^{1-\alpha}, \quad s \asymp r^{1-\beta},
\]
respectively.

**Proof.** Noting that \( 1/\rho \) has the form \( \frac{1}{\rho} = \frac{e^\phi}{f} = \frac{1}{f} \frac{d}{d\tau} e^\phi \), we integrate by parts in (13.5). We then have
\[
s \sim \frac{e^\phi}{f} \phi',
\]
which proves the 1st two cases of (13.9). The 3rd case is easy. For \( 0 < \kappa \leq 1 \) and \( \beta < 0 \), as \( r \to \infty \),
\[
\int_1^r e^{ct} dt \asymp r^{1-\kappa} e^{cr} \quad (c > 0), \quad \int_1^r t^{-\beta} dt \asymp r^{1-\beta}.
\]
Therefore, we have for the 1st two cases
\[
\log s \asymp (1 - \kappa) \log r + r^{\kappa} \asymp r^{\kappa},
\]
where \( \kappa = 1, 1 - \alpha \) and for the 3rd case
\[
s \asymp r^{1 - \beta},
\]
which proves (13.10).

\[\square\]

**Lemma 13.4.** If \( h(r) \in A^{\kappa} \) and \( m \geq 1 \), we have, as a function of \( r > 0 \),
\[
\left( \frac{d}{ds} \right)^m h(r) \in \begin{cases} 
A^{\kappa - 1}, \\
A^{\kappa - 1 - (m-1)\alpha}, \\
A^{\kappa + m(\beta - 1)}. 
\end{cases}
\]

**Proof.** Using \( \frac{d}{ds} = \rho \frac{d}{dr} \), we have
\[
\frac{d}{ds} h(r) = \begin{cases} 
egphantom{\left( \frac{d}{ds} \right)^m \phi_1 h', \\
\phi_2 h', \\
\phi_3 h'. 
\end{cases}
\]

Differentiating this equality, we get the lemma.

\[\square\]

We return to the differential equation (13.4).

**Lemma 13.5.** Fix \( B_0 > 0, C_0 > 1 \) arbitrarily. Then, for \( B \geq B_0, \frac{t}{\sqrt{B}} \geq C_0 \) and \( m \geq 1 \),
\[
\left( \frac{d}{dt} \right)^m V(t) = \begin{cases} 
O\left( \left( \frac{\sqrt{B}}{t} \right)^m t^{-2} \right), \\
O\left( \left( \frac{\sqrt{B}}{t} \right)^m \left( \log \frac{t}{\sqrt{B}} \right)^{2\alpha} t^{-2} \right), \\
O\left( \left( \frac{\sqrt{B}}{t} \right)^m \frac{2\alpha}{1 - 2\alpha} t^{-2} \right). 
\end{cases}
\]

**Proof.** Recalling \( t = \sqrt{B}s \), we put
\[
V(t) = W(s) = \frac{\rho^2}{B} \left( -\lambda + \frac{(n^2 - 2n)}{4} \left( \frac{\rho'}{\rho} \right)^2 + \frac{(n - 2)}{2} \left( \frac{\rho'}{\rho} \right)^4 \right).
\]

Then, by Lemma 13.3 we have
\[
\left( \frac{d}{ds} \right)^m W(s) = \frac{1}{B} \begin{cases} 
egphantom{\left( \frac{d}{ds} \right)^m \phi_1 k_1(r), \\
\phi_2 k_2(r), \\
\phi_3 k_3(r) \in A^{2\beta + m(\beta - 1)}. 
\end{cases}
\]

In view of 13.11, we have
\[
e^{-\phi_1} \sim \frac{1}{s f_1 \phi_1} \asymp \frac{1}{s}, \quad e^{-\phi_2} \sim \frac{1}{s f_2 \phi_2} \asymp \frac{r^\alpha}{s},
\]
By Lemma 13.4 we then have for $m \geq 1$
\[
\left( \frac{d}{ds} \right)^m W(s) = \frac{1}{B} \begin{cases}
O(s^{-(m+2)}), \\
O(s^{-m-2} \log s)^{\frac{2\alpha}{1-\alpha}}, \\
O(s^{-m+\frac{2}\alpha}).
\end{cases}
\]
Using $t = \sqrt{B}s$, we obtain the lemma. $\square$

In the equation (13.4), we put $w = ae^\varphi$. Then
\[
-w'' + (1 + V)w = e^\varphi \left( a \left( 1 + V - (\varphi')^2 \right) - (2a'\varphi' + a\varphi'') - a'' \right).
\]
We take
\[
\varphi(t) = \int_{t_0}^t \sqrt{1 + V(\tau)} \, d\tau,
\]
\[
a(t) = \varphi^{-1/2} = (1 + V(t))^{-1/4},
\]
where $t_0 = t_0(B)$ is chosen so that $|V(t)| \leq 1/2$ for $t \geq t_0$. Then, we have
\[
1 + V - (\varphi')^2 = 0, \quad 2a'\varphi' + a\varphi'' = 0,
\]
and by virtue of Lemma 13.5 there exists $p > 2$ such that
\[
a''(t) = O(t^{-p}).
\]
if $t \geq t_0(B)$ is chosen so that $V(t) = (t^\epsilon(1 + v)$. Then $v$ satisfies
\[
v'' + 2\left( \frac{a'}{a} + \varphi' \right)v' + \frac{a''}{a}v = -\frac{a''}{a},
\]
where we have used (13.12). Putting $f = \begin{pmatrix} v \\ v' \end{pmatrix}$, $g = \begin{pmatrix} 0 \\ -a''/a \end{pmatrix}$, we get the equation
\[
\frac{df}{dt} = A(t)f + B(t)f + g(t),
\]
where
\[
A(t) = \begin{pmatrix} 0 & 1 \\ 0 & -2a'/a - 2\varphi' \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0 & 0 \\ -a''/a & 0 \end{pmatrix}.
\]
The fundamental matrix for the equation $dh/dt = A(t)h$ is
\[
F(t, s) = C(t)C(s)^{-1},
\]
where
\[
C(t) = \begin{pmatrix} 1 & v_0(t) \\ 0 & v_0'(t) \end{pmatrix}, \quad v_0(t) = -\int_t^\infty e^{-2\varphi/a^2} \, d\tau.
\]
Then
\[
F(t, s) = \frac{1}{v_0'(s)} \begin{pmatrix} v_0'(s) & -v_0(s) + v_0(t) \\ 0 & v_0(t) \end{pmatrix}.
\]
Noting that
\[
v_0(t) = e^{-2\varphi}c_0(t), \quad c_0(t) \in A_+^0, \quad \varphi(t) = t + O(t^{1-\epsilon}),
\]
we see that \(F(t, s)\) is bounded for \(t \geq s \geq t_0\). The differential equation \([13.13]\) is now transformed into the integral equation

\[
f(t) = -\int_t^\infty F(t, s)B(s)\varphi(s)ds - \int_t^\infty F(t, s)g(s)ds.
\]

By virtue of \([13.13]\), one can solve it uniquely by iteration, and obtain \(f(t) = O(t^{-1-\epsilon})\). Therefore \([13.4]\) has a solution \(w_+\) which behaves like \(w_+ \sim ae^{\varphi}\). Another solution \(w_-(t)\) is obtained by

\[
w_-(t) = w_+(t) \int_t^\infty w_+(\tau)^{-2}d\tau.
\]

We have thus constructed two solutions \(w_{\pm}\) of \([13.4]\) such that

\[
w_{\pm}(t) \sim a_{\pm}e^{\pm\varphi}, \quad a_+ = a, \quad a_- \sim \frac{1}{a\varphi'}.
\]

We pass to the variable \(r\), and put \(\psi(r) = \varphi(t)\). Then, we have

\[
\frac{d\psi}{dr} = \sqrt{\frac{B}{\rho^2} - \lambda + \frac{(n^2 - 2n)}{4} \left(\frac{\rho'}{\rho}\right)^2 + \frac{(n - 2)}{2} \left(\frac{\rho'}{\rho}\right)'}dr,
\]

and proved the following theorem.

**Theorem 13.6.** Assume \([13.10]\). Then, for any \(B > 0\), there exist solutions \(u_0^{(\pm)}\) of the equation

\[
-u'' - \frac{(n - 1)}{\rho}u' + \left(\frac{B}{\rho^2} - \lambda\right)u = 0
\]

satisfying

\[
u_0^{(\pm)}(r) \sim \rho(r)^{-(n-2)/2}e^{\pm\psi(r)},
\]

\[
\psi(r) = \int_{r_0}^r \sqrt{\frac{B}{\rho^2} - \lambda + \frac{(n^2 - 2n)}{4} \left(\frac{\rho'}{\rho}\right)^2 + \frac{(n - 2)}{2} \left(\frac{\rho'}{\rho}\right)'}dr,
\]

\(r_0 = r_0(B)\) being a sufficiently large constant.

Note that by \([13.11]\), we have the following asymptotics of \(\psi\) as \(r \to \infty\):

\[
\psi(r) \sim \begin{cases}
\frac{e^{\phi_1(r)}}{f_1(r)\phi_1'(r)} & \text{if } e^{\phi_2(r)/\sqrt{B}} \\
\frac{e^{\phi_2(r)}}{f_2(r)\phi_2'(r)} & \text{if } e^{\phi_1(r)/\sqrt{B}} \int f_3(r)^{-1}dr.
\end{cases}
\]

**14. Generalized S-matrix**

We generalize the notion of S-matrix by enlarging the solution space of the equation \((-\Delta_M - \lambda)u = 0\) on the cusp end. To make the distinction clear, we call the scattering data and the S-matrix constructed in §12 *physical*, and call the ones to be introduced here *non-physical*. To construct these non-physical scattering data and S-matrix by the separation of variables, we assume that our end is a pure cusp. Namely, we impose the assumption (A-4-2).
Let \( N + k \leq j \leq N + N' \), and \( 0 = \lambda_{0,j} \leq \lambda_{1,j} \leq \lambda_{2,j} \leq \cdots \) be the eigenvalues of \(-\Delta_{M_j}\) with complete orthonormal system of eigenvectors \( e_{\ell,j}(x) \), \( \ell = 0, 1, 2, \cdots \). We put

\[
\Phi_j(r, B) = \int_{r_0}^r \sqrt{\frac{B}{\rho_j^2} - \lambda + \frac{(n - 2n)}{4} \left( \frac{\rho_j'}{\rho_j} \right)^2 + \frac{(n - 2)}{2} \left( \frac{\rho_j'}{\rho_j} \right)' \} \, dr.
\]

By Theorem 13.6, there exist solutions \( u_{\ell,j,\pm} \) to the equation

\[
(14.1)\quad -u'' - \frac{(n - 1)}{r} \rho_j \left( \rho_j \right)' u' + \left( \frac{\lambda_{\ell,j}}{\rho_j^2} - \lambda \right) u = 0,
\]

which behave like

\[
u_{\ell,j,\pm} \sim \rho_j(r)^{-(n-2)/2} e^{\pm \Phi_j(r, \lambda_{\ell,j})}, \quad r \to \infty.
\]

Take any solution \( u \) of the equation

\[
(-\Delta_{M_j} - \lambda) u = 0, \quad \text{on} \quad M_j, \quad j = N + 1, \cdots, N + N'.
\]

Expanding it by \( e_{\ell,j} \), we have

\[
(14.3)\quad (u(r, \cdot), e_{\ell,j})_{L^2(M_j)} = a_{\ell,j} u_{\ell,j,+}(r) + b_{\ell,j} u_{\ell,j,-}(r).
\]

Here, we introduce two spaces of sequences \( A_{j,\pm} \):

\[
A_{j,\pm} \ni \{c_{\ell,\pm}\}_{\ell=0}^{\infty} \iff \sum_{\ell=0}^{\infty} |c_{\ell,\pm}|^2 |u_{\ell,j,\pm}(r)|^2 < \infty, \quad \forall r > 0.
\]

**Lemma 14.1.** For a solution \( u \) of (14.3), let \( a_{\ell,j}, b_{\ell,j} \) be defined by (14.3). If \( \{a_{\ell,j}\}_{\ell=0}^{\infty} \in A_{j,+} \), then \( \{b_{\ell,j}\}_{\ell=0}^{\infty} \in A_{j,-} \).

**Proof.** Since \( \sum_{\ell \geq 0} (|u(r, \cdot), e_{\ell,j})_{L^2(M_j)}|^2 < \infty \), the lemma follows from (14.3). \( \square \)

Any finite sequence belongs to \( A_{j,\pm} \). For the hyperbolic metric, one can find a more explicit subspace of \( A_{j,\pm} \) by using the asymptotic expansion of modified Bessel functions (see [56]).

Using the partition of unity \( \{\chi_j\} \) in (7.3), we define the generalized incoming solution on the cusp end \( M_j \) by

\[
\Psi_{j}^{(in)} = \chi_j \sum_{\ell=0}^{\infty} a_{\ell,j} u_{\ell,j,+}(r) e_{\ell,j}(x), \quad \{a_{\ell,j}\}_{\ell=0}^{\infty} \in A_{j,+},
\]

which is (super)-exponentially growing as \( r \to \infty \), and the generalized outgoing solution by

\[
\Psi_{j}^{(out)} = \chi_j \sum_{\ell=0}^{\infty} b_{\ell,j} u_{\ell,j,-}(r) e_{\ell,j}(x), \quad \{b_{\ell,j}\}_{\ell=0}^{\infty} \in A_{j,-},
\]

which is (super)-exponentially decaying as \( r \to \infty \). We also define the spaces of generalized scattering data by

\[
h_{\infty}^{(in)}(\lambda) = \left( \bigoplus_{j=1}^{N} c_j(\lambda) L^2(M_j) \right) \oplus \left( \bigoplus_{j=N+1}^{N+k-1} c_j(\lambda) C_j \right) \oplus \left( \bigoplus_{j=N+k}^{N+N'} c_j(\lambda) A_{j,+} \right),
\]

\[
h_{\infty}^{(out)}(\lambda) = \left( \bigoplus_{j=1}^{N} c_j(\lambda) L^2(M_j) \right) \oplus \left( \bigoplus_{j=N+1}^{N+k-1} c_j(\lambda) C_j \right) \oplus \left( \bigoplus_{j=N+k}^{N+N'} c_j(\lambda) A_{j,-} \right).
\]
\[ h^{(\text{out})}_\infty(\lambda) = \left( \bigoplus_{j=1}^{N+k-1} c_j(\lambda)L^2(M_j) \right) \oplus \left( \bigoplus_{j=N+1}^{N+k} c_j(\lambda)C_j \right) \oplus \left( \bigoplus_{j=N+k}^{N+N'} c_j(\lambda)A_j \right), \]

where \( c_j(\lambda) \) is the characteristic function of the interval \((E_0, j, \infty)\).

**Theorem 14.2.** For any generalized incoming data \( a^{(\text{in})}(\lambda) \in h^{(\text{in})}_\infty(\lambda) \), there exist a unique solution \( u \) of the equation \((-\Delta_M - \lambda)u = 0\), and the outgoing data \( a^{(\text{out})}(\lambda) \in h^{(\text{out})}_\infty(\lambda) \) such that

\[ u - \sum_{j=N+k}^{N+N'} \Psi_j^{(\text{in})} \in \mathcal{B}^*, \]

and on the ends \( M_j \), \( 1 \leq j \leq N + k - 1 \), \( u \) has the asymptotic form in Theorem 12.9. Here, \( \Psi_j^{(\text{in})} \) and \( \Psi_j^{(\text{out})} \) are written by (14.4), (14.5) with \( a_{\ell,j} \) replaced by the associated components of \( a_j^{(\text{in})} \) and \( a_j^{(\text{out})} \).

**Proof.** Put \( u^{(\text{in})} = \sum_{j=N+k}^{N+N'} \Psi_j^{(\text{in})} \) and \( u = u^{(\text{in})} - R(\lambda + i0)(-\Delta_M - \lambda)u^{(\text{in})} \). Then, \( u \) has the desired properties. If \( u_1 \) and \( u_2 \) are two such solutions, \( u_1 - u_2 \) is an outgoing solution of the equation \((-\Delta_M - \lambda)u = 0\), hence vanishes identically. □

We call the mapping

\[ S(\lambda) : h^{(\text{in})}_\infty(\lambda) \ni a^{(\text{in})} \rightarrow a^{(\text{out})} \in h^{(\text{out})}_\infty(\lambda) \]

the generalized scattering matrix.

Let us remark here that in Theorem 13.6 the decaying solution \( u_0^{(-)} \) is determined uniquely from its asymptotic behavior near infinity, while the growing solution \( u_0^{(+)} \) is not unique, since \( u_0^{(+)} + cu_0^{(-)} \), \( c \) being any constant, is again an increasing solution. However, it gives no harm to the definition of the generalized S-matrix. In fact, given two incoming data \( \Psi_j^{(\text{in})}, \Psi_j^{(\text{in})}' \), let \( u, u' \) be the associated solutions to the Helmholtz equation. If \( \Psi_j^{(\text{in})} - \Psi_j^{(\text{in})}' \in \mathcal{B}_0^* \), \( u - u' \) is outgoing, hence vanishes identically. Therefore, \( u \) and \( u' \) give the same decaying solution in the end \( M_j \).
CHAPTER 2

Inverse scattering

1. From S-matrix to source-to-solution map

Let us start the reconstruction of the manifold from the (generalized) scattering matrix. We follow the arguments in \[56\] and \[54\] with some modifications. We reduce the problem to the source-to-solution map in the interior domain, see \[13, 48\].

Let \( O \subset M_{\text{reg}} \) be a relatively compact open set. We consider the problem with source \( F \) supported in \( O \) and the radiation condition (see Definition 7.5):

\[
\begin{aligned}
\begin{cases}
(−Δ_M − λ)u = F, & \text{in } M, \\
u \text{ satisfies the radiation condition.}
\end{cases}
\end{aligned}
\]

We extend \( F \) to be 0 outside \( O \). By Theorem 8.2, for \( λ \in \sigma_e(H) \setminus \mathcal{E} \), there exists a unique solution to this equation, denoted by \( u_{\pm}(\lambda) = (−Δ_M − λ − i0)^{-1}F \).

We define the stationary source-to-solution operator by

\[
U_{\mathcal{O}, \pm}(\lambda) : L^2(O) \ni F \rightarrow u_{\pm}(\lambda) ∈ L^2(O).
\]

Now, we enter into the first step of the inverse problem. Suppose we are given two manifolds \( M^{(1)} \) and \( M^{(2)} \) satisfying the assumptions (A-1), (A-2), (A-3) and (A-4-1), (A-4-2). Then, \( M^{(i)} \) is written as

\[
M^{(i)} = \mathcal{K}^{(i)} \cup M_{1,i}^{(i)} \cup \cdots \cup M_{N_i+N_i',i}^{(i)},
\]

where \( \mathcal{K}^{(i)} \) is bounded, and \( M_{j,i}^{(i)} \)'s are non-compact. Note that the number of ends of \( M^{(i)} \) is not assumed to be the same for \( i = 1, 2 \) a-priori. Let \( H_i \) be the Laplacian on \( M^{(i)} \), and \( S^{(i)}(λ) \) the associated S-matrix, which is an \((N_i + N_i') \times (N_i + N_i')\) operator-valued matrix. Let \( S_{pq}^{(i)}(λ) \) be its \((p,q)\)-entry. Let \( \mathcal{E}_i \) be the set of exceptional points \( \mathcal{E} \) for \( H_i \). First we consider the case for regular ends. Let \( M_1^{(i)} \) be the 1st regular end of \( M^{(i)} \).

**Theorem 1.1.** Assume that \( M_1^{(1)} \) and \( M_1^{(2)} \) are isometric. If \( S_{11}^{(1)}(λ) = S_{11}^{(2)}(λ) \) for some \( λ \in (\sigma_e(H_1) \setminus \mathcal{E}_1) \cap (\sigma_e(H_2) \setminus \mathcal{E}_2) \), then \( U_{\mathcal{O}, \pm}^{(1)}(λ) = U_{\mathcal{O}, \pm}^{(2)}(λ) \).

**Proof.** We consider the outgoing case, and omit the subscript +. First we assume that \( \beta_1 > 1/3 \) on \( M_1^{(1)} = M_1^{(2)} \). Take \( a_1 ∈ L^2(M_1^{(1)}) = L^2(M_1^{(2)}) \) and put \( u^{(i)} = \mathcal{F}^{(i,+)}(λ)^*a^{(i)} \), where \( a^{(i)} = (a_1, 0, \cdots, 0) \), and \( \mathcal{F}^{(i,+)}(λ) \) is \( \mathcal{F}^{(i)}(λ) \) for \( M^{(i)} \). Then

\[
(−Δ_M − λ)(u^{(1)} − u^{(2)}) = 0
\]
on $\mathcal{M}_1^{(1)} = \mathcal{M}_1^{(2)}$. In view of the asymptotic expansion in Lemma 12.4 and the assumption $S_1^{(1)}(\lambda) = S_1^{(2)}(\lambda)$, we have $u^{(1)} - u^{(2)} \in B_0$ on $\mathcal{M}_1^{(1)}$. Then $u^{(1)} - u^{(2)} = 0$ by Rellich-Vekua’s theorem (Theorem 4.2). Take $F \in L^2(\mathcal{M}_1^{(1)}) = L^2(\mathcal{M}_1^{(2)})$ with support in $\mathcal{O} \subset \mathcal{M}_{1,\text{reg}}^{(1)} = \mathcal{M}_{1,\text{reg}}^{(2)}$. Let

$$w^{(i)} = R^{(i)}(\lambda + i0)F.$$ 

Then for any $a_1 \in L^2(\mathcal{M}_1^{(1)})$ and $a^{(i)} = (a_1, 0, \ldots, 0)$,

$$\langle \mathcal{F}^{(1,+)}(\lambda)F, a_1 \rangle_{L^2(\mathcal{M}_1)} = \langle F, \mathcal{F}^{(1,+)}(\lambda)^* a_1 \rangle_{L^2(\mathcal{M}_1^{(1)})} = \langle F, u^{(1)} \rangle_{L^2(\mathcal{O})} = \langle F, u^{(2)} \rangle_{L^2(\mathcal{O})} = \langle F, \mathcal{F}^{(2,+)}(\lambda)^* a_1 \rangle_{L^2(\mathcal{M}_2^{(2)})} = \langle \mathcal{F}^{(2,+)}(\lambda)F, a_1 \rangle_{L^2(\mathcal{M}_1^{(1)})}.$$ 

As this holds for all $a_1 \in L^2(\mathcal{M}_1^{(1)})$, this implies

$$\mathcal{F}^{(1,+)}(\lambda)F = \mathcal{F}^{(2,+)}(\lambda)F.$$ 

This implies that the far fields of $w^{(1)}$ and $w^{(2)}$ coincide. Let $v = w^{(1)} - w^{(2)}$. Then

$$(1.5) \quad (-\Delta_M - \lambda)v = 0, \quad \text{in} \quad \mathcal{M}_1^{(1)} = \mathcal{M}_1^{(2)},$$

and the far field of $v$ in the end $\mathcal{M}_1^{(1)} = \mathcal{M}_1^{(2)}$ is zero. Then $v = 0$ by Theorem 4.2.

This implies that $w^{(1)} = w^{(2)}$ in $\mathcal{O}$, and hence $U^{(1)}_{\mathcal{O},\pm}(\lambda)F = U^{(2)}_{\mathcal{O},\pm}(\lambda)F$.

Next assume that $0 < \beta_1 \leq 1/3$. Then, by the assumption (A-4-1), on $\mathcal{M}_1^{(1)} = \mathcal{M}_1^{(2)}$, the metric has the warped product form (1.20). Then, the equation (1.4) can be solved by the separation of the variable: $u^{(1)} - u^{(2)} = \sum_j v_j(r)\varphi_j(x)$, where $\varphi_j(x)$ is the normalized eigenvector of the Laplace-Beltrami operator on $\mathcal{M}_1^{(1)} = \mathcal{M}_1^{(2)}$. Each $v_j(r)$ is a $B_0$-solution of the radial equation. Hence $v_j(r) = 0$, and $u^{(1)} - u^{(2)} = 0$. By the same argument, $v = 0$ for (1.30). This completes the proof of the theorem.

We next consider cusp ends. Assume that (A-4-2) is satisfied for cusp ends $\mathcal{M}_j^{(1)}, \mathcal{M}_j^{(2)}$. Assume further that $\mathcal{M}_j^{(1)}$ and $\mathcal{M}_j^{(2)}$ are isometric. Take a bounded open set $\mathcal{O} \subset \mathcal{M}_j^{(1)} = \mathcal{M}_j^{(2)}$ and define $U^{(i)}_{\mathcal{O},\pm}(\lambda), i = 1, 2$, as in (1.2). Let $\tilde{S}^{(i)}(\lambda) = (\tilde{S}^{(i)}_{pq}(\lambda))$ be the associated generalized S-matrix.

**Theorem 1.2.** Assume that cusp ends $\mathcal{M}_j^{(1)}$ and $\mathcal{M}_j^{(2)}$ are isometric, and that $\tilde{S}^{(1)}_{jj}(\lambda) = \tilde{S}^{(2)}_{jj}(\lambda)$ for some $\lambda \in (\sigma_c(H_1) \setminus \mathcal{E}_1) \cap (\sigma_c(H_2) \setminus \mathcal{E}_2)$. Then, $U^{(1)}_{\mathcal{O},\pm}(\lambda) = U^{(2)}_{\mathcal{O},\pm}(\lambda)$.

**Proof.** Take $F \in L^2(\mathcal{O})$ and extend it to be 0 outside $\mathcal{O}$. Let $u^{(i)} = (H_i - \lambda - i0)^{-1}F$. By using the notation in §14, one can expand it as

$$u^{(i)} = \sum_{\ell \geq 0} \left( a_{\ell,j}^{(i)} u_{\ell,j,+}(r) + b_{\ell,j,-}^{(i)}(r) \right) e_{\ell,j},$$

where
Since \((-\Delta g - \lambda)(u^{(1)} - u^{(2)}) = 0\) on \(\mathcal{M}_j^{(1)} = \mathcal{M}_j^{(2)}\), and \(u^{(1)} - u^{(2)} \in B^*\), we then have
\[
\alpha^{(1)}_{\ell,j} = \alpha^{(2)}_{\ell,j}, \quad \forall \ell.
\]
Since the incoming data and the generalized S-matrices coincide, by virtue of Theorem 14.2 we have \(u^{(1)} = u^{(2)}\) on \(\mathcal{M}_j^{(1)} = \mathcal{M}_j^{(2)}\) and \(b^{(1)}_{\ell,j} = b^{(2)}_{\ell,j}\). This completes the proof. \(\square\)

The physical as well as mathematical legitimacy of the source-to-solution operator is easily seen in the following observation. Let us consider the wave equation in \(\mathbb{R}^3\):
\[
(1.6) \quad v_{tt} - \Delta v = F(t, x),
\]
with the initial condition
\[
v(0) = v_t(0) = 0.
\]
The Duhamel principle gives the following solution
\[
(1.7) \quad v(t, x) = \frac{1}{4\pi} \int_{|y-x|<t} \frac{F(t-|y-x|, y)}{|y-x|} dy.
\]
If \(F(t, x) = f(x)e^{-i\sqrt{\lambda}t}\) \((\lambda > 0)\), it is rewritten as
\[
u(t, x) = e^{-i\sqrt{\lambda}t} \frac{1}{4\pi} \int_{|y-x|<t} \frac{e^{i\sqrt{\lambda}|y-x|}f(y)}{|y-x|} dy.
\]
Therefore, as \(t \to \infty\),
\[
(1.8) \quad v(t, x) \sim e^{-i\sqrt{\lambda}t} \left(-\Delta - \lambda - i0\right)^{-1} f.
\]
This means that if we apply the time-harmonic oscillation in a medium, the wave motion is asymptotically equal to the solution of the Helmholtz equation with time-periodic factor. This is a well-known physical phenomenon called the limiting amplitude principle.

This is also a general fact for self-adjoint operators with absolutely continuous spectrum. Namely, the solution of the abstract wave equation
\[
v_{tt} + Hv = e^{-i\sqrt{\lambda}t}f, \quad u(0) = u_t(0) = 0
\]
behaves like
\[
v(t) \sim e^{-i\sqrt{\lambda}t} \left(H - \lambda - i0\right)^{-1} f, \quad t \to \infty,
\]
if the limiting absorption principle, i.e. the existence of the limit \((H - \lambda + i0)^{-1}\) and its Hölder continuity with respect to \(\lambda\) are guaranteed. Thus, the source-to-solution operator is the observation of the stationary wave by the time-periodic input. See e.g. \cite{32} and the references therein.

One can also consider the time-dependent source-to-solution operator. For a bounded domain \(O \subset \mathcal{M}\), consider the wave equation
\[
(1.9) \quad v_{tt} + Hv = F(t), \quad (t, x) \in \mathbb{R} \times \mathcal{M},
\]
where $F(t) = F(t, x)$ is assumed to be compactly supported in $(0, \infty) \times \mathcal{M}$. Then, the solution of this equation satisfying $v(t) = 0$ for $t < 0$ exists uniquely, which is denoted by $v_{\mathcal{O},+}^{F}(t)$. We fix $I = (0, T)$ arbitrarily, and consider the operator

$$V_{\mathcal{O},+}(T) : L^2(I \times \mathcal{O}) \ni F(t, x) \to v_{\mathcal{O},+}^{F}(t, x) \in L^2(I \times \mathcal{O}).$$

We call it an outgoing time-dependent source-to-solution operator. By the time reversal, one can also define an incoming time-dependent source-to-solution operator $V_{\mathcal{O},-}(T)$. Let $R(z) = (-\Delta_{\mathcal{M}} - z)^{-1}$ and $r_\mathcal{O}$ the operator of restriction to $\mathcal{O}$.

**Lemma 1.3.** Let $S$ be any set of positive measure in $\sigma_e(H) \setminus \mathcal{E}$. Then, for any $f(\lambda) \in C(\mathbb{R})$ such that $f(\lambda) \to 0$ as $\lambda \to \infty$, the knowledge of $r_\mathcal{O}R(\lambda + i0)r_\mathcal{O}$ for all $\lambda \in S$ determines $r_\mathcal{O}f(H)r_\mathcal{O}$.

**Proof.** Since $r_\mathcal{O}R(\lambda + i0)r_\mathcal{O}$ is the boundary value of an analytic function $r_\mathcal{O}R(z)r_\mathcal{O}$, by Fatou’s theorem, the knowledge of $r_\mathcal{O}R(\lambda + i0)r_\mathcal{O}$ for all $\lambda \in S$ determines $r_\mathcal{O}R(z)r_\mathcal{O}$ for all $z \in \mathbb{C} \setminus \mathbb{R}$. To prove the lemma, we have only to consider the case in which $f(\lambda) \in C_0^\infty(\mathbb{R})$. Then, letting $F(\lambda)$ be an almost analytic extension of $f(\lambda)$, we have the representation formula \[ L_{\lambda} \]. The lemma then readily follows.

**Lemma 1.4.** Take any subset $S$ of positive measure in $\sigma_e(H)$. Given a relatively compact open set $\mathcal{O}$ and a constant $T > 0$ arbitrarily, the knowledge of $U_{\mathcal{O},+}(\lambda)$ for all $\lambda \in S$ determines $V_{\mathcal{O},+}(T)$ uniquely.

**Proof.** Let $U(t) = H^{-1/2} \sin(tH^{1/2})$. Then, by Duhamel’s principle, the solution of the equation \[ \frac{\partial v}{\partial t} = \Delta v - v \] satisfying $v(0) = v_t(0) = 0$ is written as

$$v(t) = \int_0^t U(t - s)F(s)ds.$$ \[ (1.10) \]

The Lemma then follows from Lemma 1.3.

As is seen from the proof, Lemma 1.4 holds for 4 choices of the mapping

$$U_{\mathcal{O},\pm}(\lambda) \to V_{\mathcal{O},\pm}(T), \quad U_{\mathcal{O},\pm}(\lambda) \to V_{\mathcal{O},\mp}(T).$$

The converse of Lemma 1.4 is also true.

**Lemma 1.5.** Given a relatively compact open set $\mathcal{O}$, the knowledge of $V_{\mathcal{O},+}(T)$ for all $T > 0$ determines $U_{\mathcal{O},+}(\lambda)$ for all $\lambda \in \sigma_e(H) \setminus \mathcal{E}$ uniquely.

**Proof.** In \[ (1.10) \], take $F(t) = \varphi(t)f(x)$, where $\varphi(t) \in C_0^\infty((0, \infty))$ and $f \in L^2(\mathcal{O})$ extended to be 0 outside $\mathcal{O}$. Let $v(t)$ be given in \[ (1.10) \]. We put for $\epsilon > 0$

$$\tilde{v}_{\epsilon}(k) = \int_{-\infty}^{\infty} e^{i(k+i\epsilon)t}v(t)dt. \quad \text{(1.11)}$$

Since $v(t) = 0$ for $t < 0$, the integral is convergent. Then, $\tilde{v}_{\epsilon}(k)$ satisfies

$$(H - (k + i\epsilon)^2)\tilde{v}_{\epsilon}(k) = -\varphi(k)f.$$ \[ \text{One can show} \]

$$\|\tilde{v}_{\epsilon}(k)\|_{L^2(\mathcal{M})} \leq \frac{C}{\epsilon^2} \|f\|. \quad \text{(1.12)}$$
2. Definitions

2.1. Metric tensor. Let $\mathcal{M}$ be a CMGA, that can be considered as an orbifold with $C^\infty$-smooth coordinates and a non-smooth metric on it. Before we define the distance functions and consider the finite speed of wave propagation, we recall some properties of local coordinates on CMGA.

Let us consider conic coordinates $(\hat{U}, \Gamma, \pi)$ and the set $U = \pi(\hat{U}) \subset \mathcal{M}$. Here $U \subset \mathcal{M}$ and $\hat{U} \subset \mathbb{R}^n$. Recall that $\Phi : \Gamma \setminus \hat{U} \to U$ is a homeomorphism. In $\hat{U}$ we consider coordinates $(y, z)$, where $y = Y(x)$, $Y : \hat{U} \to \hat{W}$ and $z = Z(x)$, $Z = \hat{U} \to \hat{V}$, that is, for $x \in U$

$$x = (Y(x), Z(x)) \in \hat{W} \times \hat{V} \subset \mathbb{R}^k \times \mathbb{R}^{n-k}. $$

The action of elements $\gamma \in \Gamma$ is such that $Y(\gamma x) = Y(x)$, $x \in \hat{U}$, that is, $\gamma$ keeps the $Y$-coordinate coordinate invariant. In $B^{n-k}(0, R_0) \setminus \{0\}$ we also use spherical coordinates $Z(x) = r\omega$, $r = r(x)$, $\omega = \omega(x)$ such that $r \in (0, R_0)$ and $\omega \in S^{n-k-1}$ is a unit vector. We assume that $r(\gamma x) = r(x)$ for all $\gamma \in \Gamma$ and moreover, the operators $\omega_* \gamma : S^{n-k-1} \to S^{n-k-1}$, where $(\omega_* \gamma)(\omega(x)) = \omega(\gamma(x))$, satisfy $\omega_* \gamma \in SO(n-k-1)$.

We assume that on

$$\hat{U}^{reg} = \hat{W} \times (B^{n-k}(0, R_0) \setminus \{0\}), \quad \hat{W} \subset \mathbb{R}^k, \quad B^{n-k}(0, R_0) \subset \mathbb{R}^{n-k},$$

we have a $C^\infty$-smooth metric tensor $\hat{g}$, that in the coordinates $(y, r, \omega)$ has the form

\begin{equation}
\hat{g} = dv^2 + \sum_{j,k=1}^k a_{jk}(y, r, \omega)dy^jdy^k + \sum_{\alpha,\beta=1}^{n-k-1} r^2 b_{\alpha\beta}(y, r, \omega)d\omega^\alpha d\omega^\beta + \\
+ \sum_{j=1}^k \sum_{\alpha=1}^{n-k-1} r h_{j\alpha}(y, r, \omega)dy^j d\omega^\alpha.
\end{equation}
We denote by $g_{S^{n-k-1}}$ the standard metric of $S^{n-k-1}$. We assume that
\begin{equation}
(2.2)
\begin{align*}
& a_{jk}(y, r, \omega) \to \hat{a}_{jk}(y), \\
& b_{\alpha\beta}(y, r, \omega)d\omega^\alpha d\omega^\beta \to \hat{b}_{\alpha\beta}(y, \omega)d\omega^\alpha d\omega^\beta, \\
& h_{j\alpha}(y, r, \omega)dy^j d\omega^\alpha \to 0,
\end{align*}
\end{equation}
uniformly in $(y, \omega)$, in compact subsets of $\tilde{W} \times S^{n-k-1}$, as $r \to 0$. Moreover, we assume that there are $c_0, c_1 > 0$ such that
\begin{equation}
(2.3)
c_0 g_{S^{n-k-1}} \leq \hat{b}_{\alpha\beta}(y, \omega)d\omega^\alpha d\omega^\beta \leq T(y)^2 g_{S^{n-k-1}},
\end{equation}
and $0 < T(y) \leq c_1$. Also, we assume that the metric $\tilde{g}$ is $\Gamma$-invariant, that is, $\gamma_*\tilde{g} = \tilde{g}$ on $\tilde{U}^{reg}$ for all $\gamma \in \Gamma$.

### 2.2. Distance function.

We have that for any $R \in (0, R_0)$ there are $c_0(R), c_1(R) > 0$ such that for all $x \in \tilde{W} \times (B^{n-k}(0, R) \setminus \{0\}) \subset \mathbb{R}^n$ the metric tensor $\tilde{g} = \tilde{g}_{jk}(x)dx^j dx^k$ satisfies
\begin{equation}
(2.4)
c_0(R) g_E \leq \tilde{g} \leq c_1(R) g_E,
\end{equation}
where $g_E$ is the Euclidean metric on $\mathbb{R}^k \times \mathbb{R}^{n-k}$. However, in the Euclidean coordinates $x = (y, z)$ the map $x \mapsto \tilde{g}(x)$ is not Lipschitz due to its behaviour near $z = 0$. This is due to the fact that the radial projection map $z \mapsto P(z)$, that is the matrix valued map that has at $z$ the value
\begin{equation}
P(z) = \left[ \frac{z^j z^k}{|z|^2} \right]_{j,k=1}^{n-k}, \quad P : \mathbb{R}^{n-k} \setminus \{0\} \to \mathbb{R}^{(n-k) \times (n-k)},
\end{equation}
is not Lipschitz on the set $\mathbb{R}^{n-k} \setminus \{0\}$ having the distance function induced from $\mathbb{R}^{n-k}$.

We assume that there are conic coordinates $(\tilde{U}_\ell, \Gamma^{(\ell)}, \pi^{(\ell)})$ such that the sets $U_\ell = \pi^{(\ell)}(\tilde{U}_\ell) \subset \mathcal{M}$, $\ell = 1, 2, \ldots$ are an open covering of $\mathcal{M}$. We recall that $\pi^{(\ell)} = \tilde{\pi}^{(\ell)} \circ \pi^{(\ell)}$ where $\tilde{\pi}^{(\ell)} : \Gamma^{(\ell)} \setminus \tilde{U}_\ell \to \tilde{U}_\ell$ are homeomorphisms and $\pi^{(\ell)} : \Gamma^{(\ell)} \setminus \tilde{U}_\ell \to U_\ell \cap \mathcal{M}^{reg}$ are $C^\infty$-smooth. The sets $\pi^{(\ell)}(\tilde{V})$, where $\tilde{V} \subset \tilde{U}_\ell$, is open, form a basis of the topology of $\mathcal{M}$.

We define metric tensors $\tilde{g}^{(\ell)}$ on $\tilde{U}_\ell$ and
\begin{equation}
g^{(\ell)} = (\pi^{(\ell)})_*\tilde{g}^{(\ell)} \text{ on } U_\ell^{reg}.
\end{equation}
We assume that for indexes $\ell$ and $\ell'$, we have that on the sets
\begin{equation}
U_\ell^{reg} \cap U_{\ell'}^{reg} = (\mu^{(\ell)}(\tilde{U}_{\ell}^{reg})) \cap (\pi^{(\ell')} (\tilde{U}_{\ell'}^{reg}))
\end{equation}
the metric tensors $g^{(\ell)} = (\pi^{(\ell)})_*\tilde{g}^{(\ell)}$ and $g^{(\ell')} = (\pi^{(\ell')})_*\tilde{g}^{(\ell')}$ coincide.

These metric tensors $g^{(\ell)}$ define a smooth metric on regular part $\mathcal{M}^{reg}$ of $\mathcal{M}$ and we denote this metric by $\mu$. We say that $\mu : [0, 1] \to \mathcal{M}$ is piecwise $C^1$-smooth on the lifted local coordinates, if for all $[s_1, s_2] \subset [0, 1]$ such that $\mu([s_1, s_2]) \subset \tilde{U}_\ell$ there is a piececwise $C^1$-smooth path $\tilde{\mu} : [s_1, s_2] \to \tilde{U}_\ell$ such that
\begin{equation}
\pi^{(\ell)}(\tilde{\mu}(s)) = \mu(s), \quad \text{for } s \in [s_1, s_2].
\end{equation}

As $\mathcal{M}$ is the topological closure of $\mathcal{M}^{reg}$, we define the distance on $\mathcal{M}$ by
\begin{equation}
d_\mathcal{M}(x, y) = \inf_{\mu} \text{Length}_\mu(\mu([0, 1]) \cap \mathcal{M}^{reg}),
\end{equation}
where infimum is taken over paths \( \mu : [0, 1] \to \mathcal{M} \) that are piecewise \( C^1 \)-smooth on the lifted local coordinates, and satisfy \( \mu([0, 1]) \cap \mathcal{M}^{\text{sing}} \) is a finite set, and \( \mu(0) = x, \mu(1) = y \). Note that \( \mu([0, 1]) \cap \mathcal{M}^{\text{reg},(\ell)} \) is rectifiable in all local coordinate charts \( U^{\text{reg},(\ell)} \). Also, when \( \mu([s_1, s_2]) \subset U^{\text{reg},(\ell)} \subset \mathcal{M}^{\text{reg}} \), the length is defined using local coordinates as

\[
(2.7) \quad \text{Length}_{g}(\mu([s_1, s_2])) = \int_{s_1}^{s_2} (g_{jk}(\mu(s))\dot{\mu}^j(s)\dot{\mu}^k(s))^{1/2} ds.
\]

Let \( m_0 \in \mathbb{Z}_+ \) be such that \( \frac{1}{m_0} < R_0 \), and let below \( m \geq m_0 \).

### 2.3. Approximation of \( g \) by smooth metric tensors

For constructions below, let us define an auxiliary metric that is smooth on lifted coordinates. Let \( \tilde{g}^{(\ell),\text{smooth}} \) be a \( C^\infty \)-smooth metric tensor defined on \( \tilde{U}_\ell \) such that \( \tilde{g}^{\text{smooth},(\ell)} \geq \tilde{g}^{(\ell)} \) on \( \tilde{U}_\ell^{\text{reg}} \), that means that the positive definite matrices satisfy

\[
(\tilde{g}^{\text{smooth},(\ell)})_{mn} \geq (\tilde{g}^{(\ell)})_{mn} \text{ on } \tilde{U}_\ell^{\text{reg}},
\]

and \( \gamma_c \tilde{g}^{\text{smooth},(\ell)} = \tilde{g}^{\text{smooth},(\ell)} \) for all \( \gamma \in \Gamma^{(\ell)} \). Moreover,

\[
(2.8) \quad c_0(R_\ell) g_E \leq \tilde{g}^{\text{smooth},(\ell)} \leq 2c_1(R_\ell) g_E,
\]

where \( g_E \) is the Euclidean metric on \( \mathbb{R}^k \times \mathbb{R}^{n-k} \). The metric tensors \( \tilde{g}^{\text{smooth},(\ell)} \) define an “orbifold metric” on neighborhoods \( \tilde{U}_\ell \) that we denote by \( g^{\text{smooth},(\ell)} = (\tilde{\phi}^{(\ell)}_*)\tilde{g}^{\text{smooth},(\ell)} \).

We define a metric \( g^{\text{smooth}} \) by summing metric tensors \( g^{\text{smooth},(\ell)} \) together using a locally finite partition of unity \( \phi_\ell : \mathcal{M} \to \mathbb{R} \), such that \( \text{supp}(\phi_\ell) \subset U_\ell \), \( \sum \phi_\ell(x) = 1 \), \( \phi_\ell \geq 0 \), and that there are functions \( \tilde{\phi}_\ell \in C^\infty_0(\tilde{U}_\ell) \) such that \( \phi_\ell(\pi^{(\ell)}(x)) = \tilde{\phi}_\ell(x) \).

Using such partition of unity we define

\[
g^{\text{smooth}}(x) = \sum_{\ell=1}^L \phi_\ell(x)(\pi^{(\ell)}_*)g^{\text{smooth},(\ell)}(x), \quad x \in \mathcal{M}^{\text{reg}}.
\]

Strictly speaking, this metric tensor is defined only on \( \mathcal{M}^{\text{reg}} \). When \( \mu([s_1, s_2]) \subset U_\ell \subset \mathcal{M} \) and there is a piecewise \( C^1 \)-smooth (on the lifted local coordinates) path \( \tilde{\mu} : [s_1, s_2] \to \tilde{U}_\ell \) such that \( \pi^{(\ell)}(\tilde{\mu}(s)) = \mu(s) \) for \( s \in [s_1, s_2] \), we define the length of the path \( \mu([s_1, s_2]) \) with respect to the metric \( g^{\text{smooth}} \) using local coordinates as

\[
(2.9) \quad \text{Length}_{g^{\text{smooth}}}(\mu([s_1, s_2])) = \int_{s_1}^{s_2} (\tilde{g}^{\text{smooth},(\ell)}(\mu(s))\partial_s\tilde{\mu}^j(s)\partial_s\tilde{\mu}^k(s))^{1/2} ds.
\]

Decomposing a path \( \mu : [0, 1] \to \mathcal{M} \), that is piecewise \( C^1 \)-smooth on the lifted local coordinates, to a union of paths which are all supported on some set \( U_\ell \), we can define the length of arbitrary path \( \mu : [0, 1] \to \mathcal{M} \) that is piecewise \( C^1 \)-smooth on the lifted local coordinates. Moreover, we define

\[
(2.10) \quad d_{g^{\text{smooth}}}(x, y) = \inf_{\mu} \text{Length}_{g^{\text{smooth}}}(\mu([0, 1])),
\]

where infimum is taken over paths \( \mu : [0, 1] \to \mathcal{M} \) that are piecewise \( C^1 \)-smooth on the lifted local coordinates, and satisfy \( \mu(0) = x, \mu(1) = y \). We say that a curve \( \mu \) is a geodesic of the metric \( g^{\text{smooth}} \) if it is locally distance minimising.

Let

\[
K_m = \{ x \in \mathcal{M}; \quad d_{\mathcal{M}}(x, \mathcal{M}^{\text{sing}}) < \frac{1}{m} \}.
\]
Next we modify the non-Lipschitz metric $g$ by defining a metric $g^{(m)}$ that is smooth on the lifted coordinates. We define that $g^{(m)}$ is in the set $\mathcal{M} \setminus K_m$ equal to $g$ and in the set $K_m$,

$$g^{(m)}_{jk}(x) = \psi_m(x)\hat{g}^{\text{smooth}}_{jk}(x) + (1 - \psi_m(x))g_{jk}(x),$$

where $\psi_m \in C^\infty(K_m)$ is equal to 1 in an open neighbourhood of $\mathcal{M}^{sing}$ and $0 \leq \psi_m(x) \leq 1$. Then $g^{(m)}$ is $C^\infty$-smooth orbifold metric in $\mathcal{M}$. We assume that $\psi_{m+1}(x) \leq \psi_m(x)$ so that $g^{(m+1)}_{jk}(x) \leq g^{(m)}_{jk}(x)$.

Note that then $g^{(m)} \geq g$ on $\mathcal{M}^{reg}$ and using this we will see that the waves are slower when they propagate following the metric $g^{(m)}$ than $g$. Also, $g^{(m)} \geq g^{(m+1)}$. In particular, the travel time of waves between points $x$ and $y$ with respect to $g^{(m+1)}$, i.e., $d_{g^{(m)}}(x, y) \geq d_{g^{(m+1)}}(x, y)$.

Below, we denote $V_m = \mathcal{M} \setminus K_m$.

**Lemma 2.1.** Let $x, y \in \mathcal{M}$, $x \neq y$, $\varepsilon > 0$ and let $\gamma : [0, 1] \to \mathcal{M}$ be a piecewise $C^1$-smooth path on the lifted local coordinates that connects $x = \gamma(0)$ and $y = \gamma(1)$. Then there is a path $\mu([0, 1])$ that connects $x$ and $y$ so that $\mu \cap \mathcal{M}^{sing}$ is a finite set, and

$$|\text{Length}_{g^{(m)}}(\mu([0, 1])) - \text{Length}_{g^{(m)}}(\gamma([0, 1]))| \leq \varepsilon.$$

**Proof.** We assume that $\gamma$ is parametrised so that it has constant speed with respect to $g^{(m)}$.

For let $s_j \in I$, $j = 1, 2, \ldots, J$ be such that $s_1 = 0 < s_2 < s_3 < \cdots < s_J = 1$ and $[s_j, s_{j+1}] \subset I$ are intervals such that $\gamma(s_i) \in \mathcal{M}^{reg}$ for $i = 2, 3, \ldots, J - 1$, and there are projections from covering neighbourhoods $\pi_\ell : \tilde{U}_\ell \to U_\ell$, $\ell = \ell(j)$ such that $\gamma([s_j, s_{j+1}]) \subset U_{\ell(j)}$ for $j = 1, 2, \ldots, J - 1$. Let $\tilde{\gamma}_j : [s_j, s_{j+1}] \to \tilde{U}_{\ell(j)}$, $j = 1, 2, \ldots, J - 1$ be paths such that $\pi_{\ell(j)}(\tilde{\gamma}_j(s)) = \gamma(s)$, that is, $\tilde{\gamma}_j : [s_j, s_{j+1}] \to \tilde{U}_{\ell(j)}$ is the lift of the path $\gamma : [s_j, s_{j+1}] \to U_{\ell(j)}$.

Let

$$\rho_2 : \tilde{U}_{\ell(j)} = \tilde{W}_{\ell(j)} \times \tilde{V}_{\ell(j)} \to \tilde{V}_{\ell(j)} \subset \mathbb{R}^{n-k},$$

$$\rho_2(y, z) = z$$

be the projection of $x$ to the $z$-coordinate. Note that $n - k \geq 2$. Since $\tilde{\gamma}_j$ is a rectifiable path, we see that $\rho_2(\tilde{\gamma}_j([s_j, s_{j+1}]))$ has the Hausdorff dimension 1. Hence, we see that for any $i \in \mathbb{Z}_+$ there are vectors $v_{j,i} \in \mathbb{R}^{n-k}$, such that $|v_{j,i}| < 1/i$ and that $-v_{j,i} \not\in \rho_2(\tilde{\gamma}_j([s_j, s_{j+1}]))$. This implies that

$$0 \not\in v_{j,i} + \rho_2(\tilde{\gamma}_j([s_j, s_{j+1}]))$$

and for $\bar{v}_{j,i} = (0, v_{j,i}) \in \mathbb{R}^n$

$$(\bar{v}_{j,i} + \tilde{\gamma}_j([s_j, s_{j+1}])) \cap (\tilde{W} \times \{0\}) = \emptyset.$$

Denote

$$\eta_{j,i} : [s_j, s_{j+1}] \to \mathcal{M}, \quad \eta_{j,i}(s) = \pi_{\ell(j)}(\tilde{\eta}_{j,i}(s)), \quad \tilde{\eta}_{j,i}(s) = \bar{v}_{j,i} + \tilde{\gamma}_j(s).$$
For \( j = 1, 2, \ldots, J - 1 \), let \( \tilde{\alpha}_{j,i} : [0, 1] \to U_{\ell(j)} \) be the Euclidean line segment  
\[ \tilde{\alpha}_{j,i}(t) = \tilde{\gamma}_{j,i} + \tilde{\gamma}_{j}(s_j) \]  
that connects \( \tilde{\gamma}_{j}(s_j) \) to \( \tilde{\eta}_{j}(s_j) \). Similarly, let \( \tilde{\beta}_{j,i} : [0, 1] \to U_{\ell(j)} \) be the Euclidean line segment that connects \( \tilde{\eta}_{j}(s_{j+1}) \) to \( \tilde{\gamma}_{j}(s_{j+1}) \). Observe that when \( i \) is large enough, the line segments \( \tilde{\alpha}_{j,i}([0, 1]) \) and \( \tilde{\beta}_{j,i}([0, 1]) \) are subsets of \( U_{\ell(j)} \) and intersect \( \tilde{W} \times \{ 0 \} \) at most at the points \( \tilde{\alpha}_{1,i}(0) \) or \( \tilde{\beta}_{J-1,i}(1) \).

Let \( \alpha_{j,i}(t) = \pi_{\ell(j)}(\tilde{\alpha}_{j,i}(t)) \) and \( \beta_{j,i}(t) = \pi_{\ell(j)}(\tilde{\beta}_{j,i}(t)) \). Then \( \alpha_{j,i}([0, 1]) \) and \( \beta_{j,i}([0, 1]) \) intersect \( M^{\text{sing}} \) at most at the points \( \alpha_{1,i}(0) = \gamma(0) \) or \( \beta_{J-1,i}(1) = \gamma(1) \).

Let now \( \mu_i : [0, 1] \to M \) be a path that is obtained by concatenating the paths \( \alpha_{1,i}, \eta_{1,i}, \beta_{1,i}, \alpha_{2,i}, \eta_{2,i}, \beta_{2,i}, \ldots, \alpha_{J-1,i}, \eta_{J-1,i}, \beta_{J-1,i} \) that connects \( \gamma(0) \) to \( \gamma(1) \), that is,

\[ \mu_i = \bigcup_{j=1}^{J-1} (\alpha_{j,i} \cup \eta_{j,i} \cup \beta_{j,i}) \]

As the metric tensor \( g^{(m)} \) is a smooth orbifold metric in \( M \) we see that  
\[ \lim_{i \to \infty} \text{Length}_{g^{(m)}}(\mu_i([0, 1])) = \text{Length}_{g^{(m)}}(\gamma([0, 1])) \].  
By choosing \( \mu = \mu_i \) with sufficiently large \( i \), we prove the claim.

Let us consider the set \( \mathcal{M} = M^{\text{reg}} \cup M^{\text{sing}} \) as an orbifold with a smooth metric \( g^{(m)} \) defined on the lifted coordinate neighbourhoods.

**Lemma 2.2.** For \( x, y \in M \) we have  
\[ d_{\mathcal{M}}(x, y) \leq d_{g^{(m)}}(x, y). \]

**Proof.** Let \( \gamma \) be a \( g^{(m)} \)-length minimizing curve on \( M \) that connects \( x \) to \( y \). By Lemma 2.1 there is a path \( \mu([0, 1]) \) that connects \( x \) and \( y \), has the property that \( \mu \cap M^{\text{sing}} \) is a finite set, and  
\[ |\text{Length}_{g^{(m)}}(\mu([0, 1])) - d_{g^{(m)}}(x, y)| \leq \varepsilon. \]

Then  
\[ d_{g^{(m)}}(x, y) + \varepsilon \geq \text{Length}_{g^{(m)}}(\mu) \geq \text{Length}_{g^{(m)}}(\mu \cap M^{\text{reg}}) \geq \text{Length}_{g}(\mu \cap M^{\text{reg}}) \geq d_{\mathcal{M}}(x, y). \]

As \( \varepsilon > 0 \) is here arbitrary, the claim follows.

**Lemma 2.3.** For \( x, y \in M \) we have  
\[ d_{\mathcal{M}}(x, y) \geq \lim_{m \to \infty} d_{g^{(m)}}(x, y). \]

**Proof.** By above definitions,

\[ d_{\mathcal{M}}(x, y) = \inf_{\gamma} \text{Length}_{g}(\gamma \cap M^{\text{reg}}) \]

where infimum is taken over piecewise \( C^1 \)-smooth paths \( \gamma : [0, 1] \to \mathcal{M} \) on the lifted coordinates such that \( \gamma(0) = x, \gamma(1) = y, \) and \( \gamma([0, 1]) \cap M^{\text{sing}} \) is a finite set.
Let $\varepsilon > 0$ and choose $\gamma : [0, 1] \to \mathcal{M}$ such that $\gamma(0) = x$, $\gamma(1) = y$, and $\gamma([0, 1]) \cap \mathcal{M}^{\text{sing}}$ is a finite set and

$$\text{Length}_y(\gamma \cap \mathcal{M}^{\text{reg}}) \leq d_{\mathcal{M}}(x, y) + \varepsilon.$$ 

As the path $\gamma$ intersects $\mathcal{M}^{\text{sing}}$ finitely many times, by definition there are $t_1, t_2, \ldots, t_J$, $J \in \mathbb{Z}_+$, such that for $t \in [0, 1]$ we have

$$\gamma(t) \in \mathcal{M}^{\text{sing}}$$

if and only if $t \in \{t_1, t_2, \ldots, t_J\}$.

Then for all $h > 0$ there is $m_0(h, \gamma)$ such that for $m > m_0(h, \gamma)$ we have

$$\text{dist}_y(t, \{t_1, t_2, \ldots, t_J\}) \geq h \implies \gamma(t) \in \mathcal{M} \setminus K_m.$$ 

We see that there is $C_2 = C_2(\gamma)$ such that

$$\sum_{j=1}^{J} \text{Length}_y(\gamma((t_j - h, t_j + h) \cap [0, 1])) \leq C_2Jh$$

Hence, for $m > m_0(h, \gamma)$ we have

$$d_{\mathcal{M}}(x, y) + \varepsilon \geq \text{Length}_y(\gamma \cap \mathcal{M}^{\text{reg}})$$

$$\geq \text{Length}_y(\gamma([0, 1] \setminus \bigcup_{j=1}^{J} (t_j - h, t_j + h)))$$

$$\geq \text{Length}_y(\gamma([0, 1] \setminus \bigcup_{j=1}^{J} (t_j - h, t_j + h))) + \left(\text{Length}_y(\gamma((t_j - h, t_j + h) \cap [0, 1])) - C_2Jh\right)$$

$$\geq \text{Length}_y(\gamma([0, 1])) - C_2Jh.$$ 

Thus,

$$d_{\mathcal{M}}(x, y) + \varepsilon \geq \lim_{m \to \infty} \text{Length}_y(\gamma([0, 1])) - C_2Jh.$$ 

As $h > 0$ is here arbitrary, we have

$$d_{\mathcal{M}}(x, y) + \varepsilon \geq \lim_{m \to \infty} \text{Length}_y(\gamma([0, 1])) \geq \lim_{m \to \infty} d_{\gamma}(x, y)$$

As $\varepsilon > 0$ is arbitrary, we obtain the claim. \hfill \square

For $x, y \in \mathcal{M}^{\text{reg}}$, define

$$d_{\mathcal{M}^{\text{reg}}}(x, y) = \inf_{\gamma} \text{Length}_y(\gamma)$$

where the infimum is taken over rectifiable curves $\gamma : [0, 1] \to \mathcal{M}^{\text{reg}}$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

**Lemma 2.4.** For $x, y \in \mathcal{M}^{\text{reg}}$ we have

$$d_{\mathcal{M}^{\text{reg}}}(x, y) = d_{\mathcal{M}}(x, y).$$
Proof. Assume that \( x \neq y, \varepsilon > 0 \) and let \( \gamma : [0, 1] \to \mathcal{M} \) be a piecewise \( C^1 \)-smooth path that connects \( x = \gamma(0) \) and \( y = \gamma(1) \), has the property that \( \gamma \cap \mathcal{M}^{\text{reg}} \) is a finite set, and

\[
\text{Length}_\gamma(\gamma([0, 1]) \cap \mathcal{M}^{\text{reg}}) \leq d_M(x, y) + \varepsilon.
\]

We assume that \( \gamma \) is parametrised so that it has constant speed \( 1/\text{Length}_\gamma(\gamma([0, 1]) \cap \mathcal{M}^{\text{reg}}) \).

We start the proof by a modification of the proof of Lemma 2.1. As the path \( \gamma \) intersects \( \mathcal{M}^{\text{reg}} \) finitely many times, by definition there are \( t_1, t_2, \ldots, t_J, J \in \mathbb{Z}_+ \), such that for \( t \in [0, 1] \) we have

\[
\gamma(t) \in \mathcal{M}^{\text{reg}} \text{ if and only if } t \in \{t_1, t_2, \ldots, t_J\}.
\]

For each \( j = 1, 2, \ldots, J \), let \([s_j, r_j] \subset I\) be intervals such that \( s_j < t_j < r_j \) and \( \pi_\ell : \tilde{U}_\ell \to U_\ell, \ell = \ell(j) \) be projections from covering neighbourhoods such that \( \gamma([s_j, r_j]) \subset \tilde{U}_\ell \). Let \( \tilde{\gamma}_j : [s_j, r_j] \to \tilde{U}_\ell(j) \) paths such that \( \pi_\ell(j)(\tilde{\gamma}_j(s)) = \gamma(s) \), that is, \( \tilde{\gamma}_j : [s_j, r_j] \to \tilde{U}_\ell(j) \) is the lift of the path \( \gamma : [s_j, r_j] \to U_\ell(j) \).

Let

\[
\rho_2 : \tilde{U}_\ell(j) = \tilde{W}_\ell(j) \times \tilde{V}_\ell(j) \to \tilde{V}_\ell(j) \subset \mathbb{R}^{n-k},
\]

be the projection of \( x \) to the \( z \)-coordinate. Note that \( n - k \geq 2 \). Since \( \tilde{\gamma}_j \) is a rectifiable path, we see that \( \rho_2(\tilde{\gamma}_j([s_j, r_j])) \) has the Hausdorff dimension 1. Hence, for all \( i \in \mathbb{Z}_+ \) that there are vectors \( v_{j,i} \in \mathbb{R}^{n-k} \), such that \( |v_{j,i}| < 1/i \) and that \( -v_{j,i} \not\in \rho_2(\tilde{\gamma}_j([s_j, r_j])) \). This implies that

\[
0 \not\in v_{j,i} + \rho_2(\tilde{\gamma}_j([s_j, r_j]))
\]

and for \( \tilde{v}_{j,i} = (0, v_{j,i}) \in \mathbb{R}^n \)

\[
(\tilde{v}_{j,i} + \tilde{\gamma}_j([s_j, r_j])) \cap (\tilde{W} \times \{0\}) = \emptyset.
\]

Denote

\[
\eta_{j,i} : [s_j, r_j] \to \mathcal{M}, \quad \eta_{j,i}(s) = \pi_\ell(j)(\tilde{\eta}_{j,i}(s)), \quad \tilde{\eta}_{j,i}(s) = \tilde{v}_{j,i} + \tilde{\gamma}_j(s).
\]

For \( j = 1, 2, \ldots, J \), let \( \tilde{\alpha}_{j,i} : [0, 1] \to U_\ell(j) \) be the Euclidean line segment \( \tilde{\alpha}_{j,i}(t) = \tilde{v}_{j,i} + \tilde{\gamma}_j(s_j) \) that connects \( \tilde{\gamma}_j(s_j) \) to \( \tilde{\eta}_{j,i}(s_j) \). Similarly, let \( \tilde{\beta}_{j,i} : [0, 1] \to U_\ell(j) \) be the Euclidean line segment that connects \( \tilde{\eta}_{j,i}(r_j) \) to \( \tilde{\gamma}_j(r_j) \). When \( i \) is large enough, the line segments \( \tilde{\alpha}_{j,i}([0, 1]) \) and \( \tilde{\beta}_{j,i}([0, 1]) \) do not intersect \( \mathbb{R}^k \times \{0\} \).

Let \( \alpha_{j,i}(t) = \pi_\ell(j)(\tilde{\alpha}_{j,i}(t)) \) and \( \beta_{j,i}(t) = \pi_\ell(j)(\tilde{\beta}_{j,i}(t)) \). Then \( \alpha_{j,i}([0, 1]) \) and \( \beta_{j,i}([0, 1]) \) do not intersect \( \mathcal{M}^{\text{reg}} \).

Let now \( \mu_i : [0, 1] \to \mathcal{M} \) be a path that is obtained by concatenating the paths \( \gamma([0, s_1]), \alpha_{1,i}, \eta_{1,i}, \beta_{1,i}, \gamma([r_1, s_2]), \alpha_{2,i}, \ldots, \beta_{J,i} \) and \( \gamma([r_J, 1]) \) that connects \( \gamma(0) \) to \( \gamma(1) \), that is,

\[
\mu_i = \gamma([0, s_1]) \cup \left( \bigcup_{j=1}^{J-1} \alpha_{j,i} \cup \eta_{j,i} \cup \beta_{j,i} \cup \gamma([r_j, s_{j+1}]) \right) \cup \\
\cup \alpha_{J,i} \cup \eta_{J,i} \cup \beta_{J,i} \cup \gamma([r_J, 1]) \subset \mathcal{M}^{\text{reg}}.
\]
The metric tensor $g$ satisfies (2.4) and $g$ is smooth in $\mathcal{M}^{reg}$. Thus when we compute the lengths of curves using (2.7), we see using Lebesgue dominated convergence theorem there are numbers $h_i > 0, \lim_{i \rightarrow \infty} h_i = 0$ such that
\[ \sum_{j=1}^{J-1} |\text{Length}_g(\eta_{j,i}) - \text{Length}_g(\gamma([s_j, r_j]) \cap \mathcal{M}^{reg})| < h_i \]
and there is $C_1$ such that
\[ \sum_{j=1}^{J} (\text{Length}_g(\alpha_{j,i}) + \text{Length}_g(\beta_{j,i})) \leq C_1 \frac{J}{i}. \]
Hence,
\[ |\text{Length}_g(\mu_i) - \text{Length}_g(\gamma([0, 1]) \cap \mathcal{M}^{reg})| < h_i + C_1 \frac{J}{i} \]
Denote $I = [0, 1]$. Note that $\mu_i(I) \subset \mathcal{M}^{reg}$, $\mu_i(0) = \gamma(0) = x$ and $\mu_i(1) = \gamma(1) = y$. These imply
\[ d_{\mathcal{M}}(x, y) + \varepsilon \geq \text{Length}_g(\gamma([0, 1]) \cap \mathcal{M}^{reg}) \]
\[ \geq \text{Length}_g(\mu_i([0, 1])) + h_i + C_1 \frac{J}{i} \]
\[ \geq d_{\mathcal{M}^{reg}}(x, y) + h_i + C_1 \frac{J}{i}. \]
By taking the limit $i \rightarrow \infty$, we obtain
\[ d_{\mathcal{M}}(x, y) + \varepsilon \geq d_{\mathcal{M}^{reg}}(x, y). \]
As $\varepsilon > 0$ is arbitrary, we obtain
\[ d_{\mathcal{M}}(x, y) \geq d_{\mathcal{M}^{reg}}(x, y). \]
The opposite inequality follows from the definitions.

\[ \square \]

3. Finite speed of wave propagation

In this section we consider estimates on how the support of the solutions of the wave equation propagates in time. The propagation of singularities has been analyzed extensively using microlocal analysis, see e.g. [31, 30, 36, 37, 79, 85, 86, 106], and these techniques describe the propagation of the singular support. However, to consider the support of the wave we use more conventional methods with $\Gamma$-convergence.

3.1. Auxiliary results based on $\Gamma$-convergence. Let us now introduce quadratic forms
\[ Q^{(m)}(u, v) = Q^{(m)}_1(u, v) + Q^{(m)}_2(u, v) \]
where
\[ Q^{(m)}_1(u, v) = \int_{\mathcal{M}^{reg}} g^{(m)}_{ij} \partial_i u \partial_j v \det(g^{(m)})^{1/2} dx, \]
\[ Q^{(m)}_2(u, v) = \int_{\mathcal{M}^{reg}} \lambda u \overline{v} \det(g^{(m)})^{1/2} dx, \]
and
\[ Q(u, v) = Q_1(u, v) + Q_2(u, v) \]
where
\[
Q_1(u, v) = \int_{M^{reg}} g^{ij} \partial_i u \partial_j v \det(g)^{1/2} \, dx,
\]
\[
Q_2(u, v) = \int_{M^{reg}} \lambda u v \det(g)^{1/2} \, dx.
\]

All these quadratic forms are defined as (unbounded) closed quadratic forms in \( L^2(M) \) with the domain \( \mathcal{D}(Q^{(m)}) = \mathcal{D}(Q) = H^1(M) \). Note that we next consider \( L^2(M) \) with the inner product given by metric \( g \), that is,
\[
(u, v)_{L^2} = \int_{M^{reg}} u v \det(g)^{1/2} \, dx.
\]

Next we consider quadratic forms as \( \mathbb{R} \cup \{ \infty \} \) valued functions \( u \mapsto Q_1^{(m)}(u, u) \) in \( L^2(M) \) that have value \( \infty \) in \( L^2(M) \setminus H^1(M) \).

Let \( C_0, C_1 > 0 \) be such that for all \( m \geq 1 \),
\[
C_0^{-1} |g^{(m)}|^{-1/2} |g|^{1/2} \leq C_0,
\]
\[
C_1^{-1} I \leq g^{(m)} \leq C_1 I,
\]
\[
C_1^{-1} I \leq g \leq C_1 I.
\]

As for almost all \( x \in M^{reg} \) we have the pointwise limit
\[
\lim_{m \to \infty} g^{(m),ij}(x) \det(g^{(m)}(x))^{1/2} = g^{ij}(x) \det(g(x))^{1/2},
\]
we see that the proof of [22] Prop. 5.14, applied in local coordinates, implies that, \( Q_1^{(m)} \) \( \Gamma \)-converges to \( Q_1 \) in the weak topology of \( H^1(M, |g|^{1/2} \, dx) \) as \( m \to \infty \). As \( Q_1^{(m)} \) are quadratic forms, by [22] Prop. 13.12 this implies that \( Q_1^{(m)} \) \( \Gamma \)-converges to \( Q_1 \) in the strong topology of \( L^2(M, |g|^{1/2} \, dx) \) as \( m \to \infty \). In addition, by Lebesque dominated convergence theorem, \( Q_1^{(m)} \) converges also pointwise to \( Q_1 \).

As
\[
(g^{(m)}_{ij}) \geq (g^{(m+1)}_{ij}) \geq (g_{ij}),
\]
we see that
\[
det(g^{(m)}) \geq det(g^{(m+1)}) \geq det(g),
\]
and hence the sequence \( Q_2^{(m)}(u, u) \) is decreasing. By Lebesque dominated convergence theorem, \( Q_2^{(m)}(u, u) \) converge pointwise to \( Q_2(u, u) \) as \( m \to \infty \), and \( Q_2(u, u) \) is lower semi-continuous in the strong topology of \( L^2(M, |g|^{1/2} \, dx) \), and hence [22] Prop. 5.7] implies that \( Q_2^{(m)} \) \( \Gamma \)-converges to \( Q_2 \) in the strong topology of \( L^2(M, |g|^{1/2} \, dx) \) as \( m \to \infty \).

As the sequences \( Q_i^{(m)}, i = 1, 2 \) both \( \Gamma \)-converges to \( Q_i \) in the strong topology of \( L^2(M, |g|^{1/2} \, dx) \) and pointwise converge as \( m \to \infty \), [22] Prop. 6.25], implies that the sum of the sequences, \( Q^{(m)} \), \( \Gamma \)-converges to the sum \( Q = Q_1 + Q_2 \) in the strong topology of \( L^2(M, |g|^{1/2} \, dx) \) as \( m \to \infty \).
In the Hilbert space \( L^2(\mathcal{M}, |g|^{1/2}dx) \), the symmetric quadratic form \( Q \) is associated, in the sense of [67] Sec. VI, Thm 2.6, to the selfadjoint operator

\[
A^{(m)} = A_1^{(m)} + A_2^{(m)}
\]

in the Hilbert space \( L^2(\mathcal{M}, |g|^{1/2}dx) \) endowed with the volume form of the metric \( g \). We denote the domain of this selfadjoint operator \( A^{(m)} \) by \( \mathcal{D}(A^{(m)}) \). Then, for \( u \in \mathcal{D}(A^{(m)}) \) we have

\[
A_1^{(m)} u(x) = -|g(x)|^{-1/2} \partial_j (|g^{(m)}(x)|^{1/2} g^{(m)}(x) \partial_j u(x)), \quad \text{for } x \in \mathcal{M}^{reg},
\]

\[
A_2^{(m)}(x) = \lambda |g^{(m)}(x)|^{1/2} |g(x)|^{-1/2}, \quad \text{for } x \in \mathcal{M}^{reg}.
\]

In the Hilbert space \( L^2(\mathcal{M}, |g|^{1/2}dx) \), the symmetric quadratic form \( Q \) is associated to the selfadjoint operator

\[
A = A_1 + A_2,
\]

having the domain \( \mathcal{D}(A) \), and for \( u \in \mathcal{D}(A) \) we have

\[
A_1 u(x) = -|g(x)|^{-1/2} \partial_j (|g(x)|^{1/2} g^j(x) \partial_k u(x)) = -\Delta_g u(x), \quad \text{for } x \in \mathcal{M}^{reg},
\]

\[
A_2(u) = \lambda u(x).
\]

By [22] Prop. 13.12 the operators \( A^{(m)} \) \( G \)-converges in the Hilbert space \( L^2(\mathcal{M}, |g|^{1/2}dx) \) to the operator \( A \) in the strong topology of \( L^2(\mathcal{M}, |g|^{1/2}dx) \). Moreover, by [22] Prop. 13.12 then \( A^{(m)} \) converge to \( A \) in the strong resolvent sense in the strong topology of \( L^2(\mathcal{M}, |g|^{1/2}dx) \).

Note that the equation

\[
(-\Delta g^{(m)} + \lambda) u^{(m)} = f
\]

is equivalent to

\[
A^{(m)} u^{(m)} = |g^{(m)}|^{1/2} |g|^{-1/2} f
\]

and if \( \text{supp} \, f \) does not intersect \( \text{supp} \, (g^{(m)}-g) \subset K_m \), we have that \( |g^{(m)}|^{1/2} |g|^{-1/2} f = f \).

Let \( f \in L^2(\mathcal{M}, |g|^{1/2}dx) \) and \( \lambda \in \mathbb{C} \setminus (-\infty, 0] \).

Let \( \varepsilon > 0 \) be arbitrary and choose \( m_0 \) such that

\[
(\int_{K_{m_0}} |f(x)|^2 |g|^{1/2}dx)^{1/2} < \frac{\varepsilon}{4 C_0 \text{dist}(\lambda, \mathbb{R}_-)}.
\]

Let

\[
f_1(x) = \chi_{K_{m_0}}(x) f(x), \quad f_2(x) = f(x) - f_1(x).
\]

Then for all \( m \geq m_0 \)

\[
\|g^{(m)}|^{1/2} |g|^{-1/2} f_1 \|_{L^2(\mathcal{M}, |g|^{1/2}dx)} \leq \frac{\varepsilon}{4 \text{dist}(\lambda, \mathbb{R}_-)},
\]

so that

\[
u_1^{(m)} = (-\Delta g^{(m)} + \lambda)^{-1} f_1, \quad u_1 = (-\Delta_g + \lambda)^{-1} f_1
\]
satisfy
\[ \|u_1^{(m)}\|_{L^2(M, |g|^{1/2}dx)} = \|(-\Delta_{g^{(m)}} + \lambda)^{-1}f_1\|_{L^2(M, |g|^{1/2}dx)} = \|(A^{(m)})^{-1}(|g^{(m)}|^{1/2}|g|^{-1/2}f_1)\|_{L^2(M, |g|^{1/2}dx)} \leq \frac{\varepsilon}{4} \]
and
\[ \|u_1\|_{L^2(M, |g|^{1/2}dx)} < \frac{\varepsilon}{4}. \]
Moreover, as \(K_{n+1} \subset K_n\), we see that for \(m \geq m_0\)
\[ \text{supp}(f_2) \cap K_m = \emptyset, \]
so that
\[ |g^{(m)}|^{1/2}|g|^{-1/2}f_2 = f_2. \]
Next we consider \(m \geq m_0\). As \(A^{(m)}\) converge to \(A\) in the strong resolvent sense in the topology of \(L^2(M, |g|^{1/2}dx)\), by [22, Def. 13.3] we have that
\[ u_2^{(m)} = (-\Delta_{g^{(m)}} + \lambda)^{-1}f_2, \quad u_2 = (-\Delta_g + \lambda)^{-1}f_2 \]
satisfies
\[ u_2^{(m)} = (-\Delta_{g^{(m)}} + \lambda)^{-1}f_2 = (A^{(m)})^{-1}f_2 \to A^{-1}f_2 = (-\Delta_g + \lambda)^{-1}f_2 = u_2 \]
in the strong topology of \(L^2(M, |g|^{1/2}dx)\) as \(m \to \infty\). Thus there is \(m_1 > m_0\) such that for all \(m > m_1\)
\[ \|u_2^{(m)} - u_2\|_{L^2(M, |g|^{1/2}dx)} < \frac{\varepsilon}{2}. \]
As above \(\varepsilon > 0\) is arbitrary and \(u^{(m)} = u_1^{(m)} + u_2^{(m)}\) and \(u = u_1 + u_2\) satisfy
\[ \|u^{(m)} - u\|_{L^2(M, |g|^{1/2}dx)} < \varepsilon \]
for \(m > m_1\), we see that for all \(f \in L^2(M, |g|^{1/2}dx)\) and \(\lambda \in \mathbb{C} \setminus (-\infty, 0]\)
\[ \lim_{m \to \infty} (-\Delta_{g^{(m)}} + \lambda)^{-1}f = (-\Delta_g + \lambda)^{-1}f, \]
strong topology of \(L^2(M, |g|^{1/2}dx)\).
In particular, when \(\lambda = 1\), we have that \((-\Delta_{g^{(m)}} + 1)^{-1}\) converges to \((-\Delta_g + 1)^{-1}\)
strongly in \(L^2(M, |g|^{1/2}dx)\).

Remark. In the \(n\)-dimensional case with \(n \geq 3\) the above considerations can be simplified as follows. Let \(\omega_1, \omega_2, \ldots, \omega_n\) be the eigenvalues of \(g\). Then the eigenvalues of \(\text{det}(g)^{1/2}g^{-1}\) are
\[ s_1 = \omega_1^{-1/2}\omega_2^{1/2}\omega_3^{1/2}\ldots\omega_n^{1/2}, \]
\[ s_2 = \omega_1^{1/2}\omega_2^{-1/2}\omega_3^{1/2}\ldots\omega_n^{1/2}, \]
\[ \ldots \]
\[ s_n = \omega_1^{1/2}\omega_2^{1/2}\omega_3^{-1/2}\ldots\omega_n^{-1/2}, \]
and we see that \(|s_j| > C_0^{-n/2}\). Also, assume that \(g^{\text{smooth}} = \hat{g}\) is constant metric \(\hat{g} = \hat{\omega}I\) in the domain \(K_1\) (Note that gluing different local coordinate charts has to
be added, e.g. by using a partition of unity and summing up to local construction). Then the eigenvalues of $\det(\hat{g})^{1/2}\hat{g}^{-1}$ are constants
\[
\hat{s} = \hat{\omega}^{(n-2)/2}.
\]
If we change the definition of $g^{(m)}$ in local coordinates to be defined using powers of symmetric matrixes, that is,
\[
g_{jk}^{(m)}(x) = g_{jk}^{\text{smooth}}(x)^{\psi_m(x)} g_{jk}(x)^{1-\psi_m(x)},
\]
we see that then $\det(g^{(m)})^{1/2}(g^{(m)})^{-1}$ has the eigenvalues
\[
s_j^{(m)} = \hat{\omega}^{\psi_m(x)/2}\omega_j(x)^{(1-\psi_m(x))/2}.
\]
If $\hat{\omega} < C_0^{-n/(n-2)}$ so that
\[
\hat{s} = \hat{\omega}^{(n-2)/2} < C_0^{-n/2} \leq s_j,
\]
we see that the sequences $s_j^{(m)}$ are decreasing, and thus the the sequence of the positive definite matrixes $\det(g^{(m)})^{1/2}(g^{(m)})^{-1}$ is decreasing as $m \to \infty$. Then the quadratic forms $Q^{(m)}(u, v)$ and the corresponding self-adjoint operators are also decreasing, and we can use the following monotone theorem on the quadratic forms.

Recall that for for two quadratic forms $q_1$ and $q_2$ on a Hilbert space $\mathcal{H}$ the inequality $q_1 \leq q_2$ means that their form domains satisfy $D(q_1) \supset D(q_2)$ and $q_1[u] \leq q_2[u]$.

**Theorem 3.1.** Let $\{q_n\}_{n=1}^{\infty}$ be a sequence of closed, positive definite quadratic forms satisfying $0 \leq q_1 \leq q_2 \leq \cdots$. Suppose that
\[
(3.9) \quad D(q_\infty) = \{u \in \mathcal{H} : \sup_n q_n[u] < \infty\}
\]
is dense in $\mathcal{H}$. Then, the quadratic form $q_\infty$ defined by
\[
(3.10) \quad q_\infty[u] = \lim_{n \to \infty} q_n[u] = \sup_{n \in \mathbb{Z}_+} q_n[u]
\]
with domain $D(q_\infty)$ is closed. Moreover, if $A_n$ and $A_\infty$ are the self-adjoint operators associated with $q_n$ and $q_\infty$, then $A_n \to A_\infty$ in the strong resolvent sense.

For this result, see [94], Theorem S.14, p.373.

In our case, $Q^{(m)}$ and $Q$ have the common domain $H^1(\mathcal{M})$. Since $g_{ij}^{(m)} \to g_{ij}$, we have for $x \in \mathcal{M}_{\text{reg}}$
\[
(3.11) \quad (g^{(m),ij}(x)) \leq (g^{(m+1),ij}(x)) \leq (g^{ij}(x)) \quad \text{and} \quad (g^{(m),ij}(x)) \to (g^{ij}(x)) \text{ as } m \to \infty.
\]
Therefore, we have that $A^{(m)}_{g^{(m)}}$ converges to $A^{-1}$ strongly in $L^2(\mathcal{M}, |g|^{1/2}dx)$.

### 3.2. Strong resolvent convergence

In this section, we study some results related to the strong resolvent convergence in a Hilbert space $\mathcal{H}$.

**Lemma 3.2.** Suppose $A_n, A \geq 1$ are self-adjoint in a Hilbert space $\mathcal{H}$, and $A_n^{-1} \to A^{-1}$ strongly. Then
\[
(A_n - z)^{-1} \to (A - z)^{-1}, \quad n \to \infty
\]
strongly for $z \in \mathbb{C} \setminus [1, \infty)$.
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Proof. Take \( z_0 \not\in [1, \infty) \) and assume that \( (A_n - z_0)^{-1} g \to (A - z_0)^{-1} g \) strongly for any \( g \in \mathcal{H} \). For an arbitrary \( f \in \mathcal{H} \), put

\[
u_n = (A_n - z_0)^{-1} f, \quad u = (A - z_0)^{-1} f.
\]

Then,

\[
(A_n - z_0)^{-2} f = (A_n - z_0)^{-1} u_n = (A_n - z_0)^{-1} (u_n - u) + (A_n - z_0)^{-1} u.
\]

Since \( (A_n - z_0)^{-1} \) is norm convergent, this implies that \( (A_n - z_0)^{-1} (u_n - u) \to 0 \).

Therefore, by the above assumption,

\[
(A_n - z_0)^{-2} f \to (A - z_0)^{-2} f \quad \text{strongly}.
\]

By induction

\[
(A_n - z_0)^{-k} f \to (A - z_0)^{-k} f \quad \forall k \geq 1.
\]

If \( |z - z_0| < \text{dist}(z_0, [1, \infty)) \), we have \( |z - z_0| < \|(A_n - z_0)^{-1}\|^{-1} \) and

\[
(A_n - z)^{-1} = (1 - (z - z_0)(A_n - z_0)^{-1})^{-1} (A_n - z_0)^{-1},
\]

and the Neumann series

\[
(1 - (z - z_0)(A_n - z_0)^{-1})^{-1} = \sum_{k=0}^{\infty} (z - z_0)^k (A_n - z_0)^{-k}
\]

is norm convergent. This implies that \( (A_n - z)^{-1} f \) converges strongly to \( (A - z_0)^{-1} f \) when \( |z - z_0| < \text{dist}(z_0, [1, \infty)) \). Starting from \( z_0 = 0 \), we obtain the desired strong convergence for any \( z \not\in [1, \infty) \) by iterating the above analysis finitely many times with \( z_0 \) that are on a path the connects 0 to \( z \) in \( \mathbb{C} \setminus [1, \infty) \).

**Lemma 3.3.** Let \( A_n, A \geq 1 \) be self-adjoint and \( (A_n - z)^{-1} \to (A - z)^{-1} \) strongly for any \( z \not\in [1, \infty) \). Let \( f(\lambda) \in C(\mathbb{R}) \) satisfy \( f(\lambda) \to 0 \) as \( \lambda \to \infty \). Then, we have

\[
f(A_n) \to f(A) \quad \text{strongly as} \quad n \to \infty.
\]

**Proof.** We first prove this lemma for \( f \in C_c^\infty(\mathbb{R}) \). Let \( F(z) \in C_c^\infty(\mathbb{C}) \) be an almost analytic extension of \( f \) constructed by Lemma 2.12. Using Lemma 2.13, we then have for any \( u \in \mathcal{H} \)

\[
(f(A_n) - f(A))u = \frac{1}{2 \pi i} \int_{\mathcal{C}} \overline{\partial_z F(z)} ((z - A_n)^{-1} - (z - A)^{-1}) u dz d\bar{z}.
\]

Taking \( N = 1 \) and \( s = 3 \), we have

\[
\| \overline{\partial_z F(z)} ((z - A_n)^{-1} - (z - A)^{-1}) u \| \leq C(1 + |z|)^{-3} \| u \|.
\]

Moreover, by the assumption, for any \( z \not\in [1, \infty) \),

\[
\| ((z - A_n)^{-1} - (z - A)^{-1}) u \| \to 0.
\]

By Lebesgue’s dominated convergence theorem, we obtain

\[
\| (f(A_n) - f(A))u \| \to 0.
\]

To prove the general case, let \( \epsilon > 0 \) and put \( f_R(\lambda) = f(\lambda) \) for \( \lambda < R \) and \( f_R(\lambda) = 0 \) for \( \lambda > R \). For a sufficiently large \( R > 0 \), \( |f(\lambda) - g_R(\lambda)| \leq \epsilon \) for all \( \lambda \in \mathbb{R} \). Moreover, we can approximate \( f_R(\lambda) \) by a \( C_c^\infty(\mathbb{R}) \)-function \( g_R(\lambda) \) so that \( \| f_R - g_R \|_{L^\infty(\mathbb{R})} \leq \epsilon. \)
Then the above shows that \( g_R(A_n) - g_R(A) \to 0 \) strongly. As \( \epsilon > 0 \) is arbitrary. This proves the lemma. □

**Lemma 3.4.** Let \( A_n, A \geq 1 \) be self-adjoint and \( (A_n - z)^{-1} \to (A - z)^{-1} \) strongly for any \( z \notin [1, \infty) \). Suppose there exists \( \delta > 0 \) such that for the function \( F(\lambda) = \lambda^\delta \) the self-adjoint operators \( F(A_n) = A_n^\delta \) satisfy \( D(A_n^\delta) = D(A^\delta) \) for all \( n \) and there exists \( u \in D(A^\delta) \) such that

\[
\sup_n \|A_n^\delta u\| < \infty, \quad \forall n.
\]

Then, we have

\[
\|f(A_n)u - f(A)u\| \to 0, \quad \text{as } n \to \infty
\]

for any bounded continuous function \( f \) on \([1, \infty)\).

**Proof.** Take \( \chi_{0,R}(\lambda) \in C(\mathbb{R}) \) such that \( \chi_{0,R}(\lambda) = 1 \) for \( \lambda < R \) and \( \chi_{0,R}(\lambda) = 0 \) for \( \lambda > R + 1 \). Put \( \chi_{\infty,R}(\lambda) = 1 - \chi_{0,R}(\lambda) \). Then, we have

\[
\|\chi_{\infty,R}(A_n)u\| = \|\chi_{\infty,R}(A_n)A_n^\delta A_n^\delta u\| \leq C \sup_\lambda |\chi_{R,\infty}(\lambda)\lambda^{-\delta}| \leq CR^{-\delta}.
\]

The same inequality holds for \( A \). We fix large \( R > 0 \) and put \( g(\lambda) = f(\lambda)\chi_{0,R}(\lambda) \). Then, by Lemma 3.6, \( g(A_n)u \to g(A)u \) strongly. □

We consider an abstract wave equation.

**Definition 3.5.** Let \( A \) be the unbounded self-adjoint operator associated to the quadratic from \( Q \) in \( L^2(\mathcal{M}, g) \) having the domain \( H^1(\mathcal{M}) = D(Q) \). We say that \( u \in C^1([0,T];L^2(\mathcal{M})) \cap C^0([0,T];H^1(\mathcal{M})) \) is a (finite energy) solution of the wave equation

\[
\partial_t^2 u + Au = 0
\]

if there are \( u_0 \in H^1(\mathcal{M}) \) and \( u_1 \in L^2(\mathcal{M}) \) and

\[
u(t) = \cos(tA^{1/2})u_0 + \sin(tA^{1/2})A^{-1/2}u_1.
\]

Here we define \( \sin(tA^{1/2})A^{-1/2} = t \) for \( \lambda = 0 \).

Next we consider corresponding abstract system.

**Lemma 3.6.** Let \( A_n, A \geq 0 \) be self-adjoint in a Hilbert space \( \mathcal{H} \) such that \( (A_n - z)^{-1} \to (A - z)^{-1} \) strongly for any \( z \in \mathbb{C} \setminus [0, \infty) \). Let \( u(t) \) be the solution of the equation

\[
\begin{cases}
\partial_t^2 u + Au = 0, \\
u_{|t=0} = w_0 \in \mathcal{H}, \quad \partial_t u_{|t=0} = w_1 \in H^1(\mathcal{M}) = D(A^{1/2}),
\end{cases}
\]

and \( u_n(t) \) the solution of the same equation with \( A \) replaced by \( A_n \). Assume that \( D(\sqrt{A_n}) = D(\sqrt{A}) \) for all \( n \) and \( w_0 \in D(\sqrt{A}) \) with \( \sup_n \|\sqrt{A_n}w_0\| < \infty \). Then, we have

\[
\|u_n(t) - u(t)\| \to 0 \quad \text{as } n \to \infty
\]

for any \( t \).
Proof. We have

\[
  u(t) = \cos(t\sqrt{A})w_0 + \sin(t\sqrt{A})\sqrt{A}^{-1}w_1,
\]

and the similar formula for \( u_n(t) \). Then, the lemma follows from Lemma 3.4.

Lemma 3.7. Let \( A_n \) and \( A \) be as in Lemma 3.6. Let \( u(t) \) and \( u_n(t) \) be the solutions of the equations

\[
\begin{align*}
  &\partial_t^2 u + Au = f(t), \\
  &u\big|_{t=0} = \partial_t u\big|_{t=0} = 0,
\end{align*}
\]

(3.14)

\[
\begin{align*}
  &\partial_t^2 u_n + A_n u = f_n(t), \\
  &u_n\big|_{t=0} = \partial_t u_n\big|_{t=0} = 0.
\end{align*}
\]

(3.15)

Assume that for any \( T > 0 \) there exists a constant \( C > 0 \) such that \( \|f_n(t)\| \leq C \) for all \( n \) and \( 0 \leq t \leq T \), and \( \|f_n(t) - f(t)\| \to 0 \) as \( n \to \infty \) for any \( t \geq 0 \). Then \( \|u_n(t) - u(t)\| \to 0 \) as \( n \to \infty \) for any \( t \geq 0 \).

Proof. We put

\[
F(t, \lambda) = (\sin(t\sqrt{\lambda}))\sqrt{\lambda}^{-1}.
\]

Then,

\[
u(t) = \int_0^t F(t-s, A)f(s)ds,
\]

and the similar formula holds with \( A \) and \( f(s) \) replaced by \( A_n \) and \( f_n(s) \). The lemma then follows from Lemma 3.6.

3.3. Domains of influence. Let \( u \) satisfy

\[
\begin{align*}
  &\partial_t^2 - \Delta_g u(x, t) = 0, \quad \text{on } \mathcal{M} \times \mathbb{R}_+ \\
  &u(x, 0) = v_0(x) \in L^2(\mathcal{M}), \quad \partial_t u(x, 0) = v_1(x) \in H^1(\mathcal{M}).
\end{align*}
\]

(3.16)

(3.17)

Next we prove the finite propagation of waves. Below, for \( W \subset \mathcal{M} \), let

\[
\mathcal{M}(W, T) = \{ x \in \mathcal{M}; \ d_\mathcal{M}(x, W) < T \}
\]

denote the (open) domain of influence.

Proposition 3.8. Let \( W \subset \mathcal{M} \) be an open, relatively compact set and \( u(t) \) be the solution of initial boundary value problem (3.16)–(3.17) with \( \text{supp}(v_0) \cup \text{supp}(v_1) \subset W \). Then

\[
\text{supp}(u(\cdot, T)) \subset \mathcal{M}(W, T).
\]

Proof. Let \( W \subset \mathcal{M} \) be an open, relatively compact set and

\[
\text{supp}(v_0) \cup \text{supp}(v_1) \subset W.
\]

Let \( u^{(m)} \) satisfy

\[
\begin{align*}
  &\partial_t^2 - \Delta_{g^{(m)}} u^{(m)}(x, t) = 0, \quad \text{on } \mathcal{M} \times \mathbb{R}_+ \\
  &u^{(m)}(x, 0) = v_0(x), \quad \partial_t u^{(m)}(x, 0) = v_1(x),
\end{align*}
\]

(3.19)

(3.20)

where the wave equation is defined in weak sense on all coordinate neighbourhoods.
As we can represent \( v_0 \) and \( v_1 \) as a sum of functions supported in single coordinate neighborhood \( U_\ell \), without loss of generality we can below assume that
\[
W \subset U_\ell.
\]
Also, let
\[
0 < T_0 < \text{dist}_{g^{\text{smooth},(\ell)}}(W,U_\ell).
\]

The standard results on the finite speed of wave propagation is valid on the smooth lifted coordinate neighbourhood \( (\tilde{U}_\ell, g^{(m)}) \), see [49] (Here the metric \( g^{(m)} \) is smooth. We note that the considerations can may be simplified using Lipschitz-smooth metric as the generalized results for the finite speed of wave propagation seem to be valid on Lipschitz-smooth manifold, or with divergence form equations with log-Lipschitz coefficients see e.g. [20], Thm 1.6 and Remark 1.8). Then there is an open set \( W_1 \subset W \) such that \( \text{supp} (v_0) \cup \text{supp} (v_1) \subset W_1 \). Also, there is an open set \( W_2 \subset W_1 \) such that \( \text{supp} (v_0) \cup \text{supp} (v_1) \subset W_2 \). Then
\[
\{ (x, t) \in \mathcal{M} \times [0, T_0); \ d^{(m)}(x, W_2) > t \} 
\subset \{ (x, t) \in \mathcal{M} \times [0, T_0); \ u^{(m)}(x, t) = 0 \},
\]
so that
\[
\text{supp} (u^{(m)}(\cdot, \cdot)) \cap (\mathcal{M} \times [0, T_0)) 
\subset \{ (x, t) \in \mathcal{M} \times [0, T_0); \ d^{(m)}(x, W_2) \leq t \} 
\subset \{ (x, t) \in \mathcal{M} \times [0, T_0); \ d^{(m)}(x, W_1) < t \}.
\]

Let \( u \) satisfy
\[
(\partial_t^2 - \Delta_g)u(x, t) = 0, \quad \text{on } \mathcal{M} \times \mathbb{R}_+ \\
u(x, 0) = v_0(x), \quad \partial_t u(x, 0) = v_1(x).
\]
As propagation of waves can be studied in separately on small time intervals \([jT_0, (j+1)T_0]\), that cover a longer interval \([0, T_1]\), without loss of generality we can consider the case when \( 0 < T \leq T_0 \) and initial data \((v_0, v_1)\) supported in \( W \).

We see using Lemma 3.6 that
\[
\lim_{m \to \infty} u^{(m)} = u \quad \text{in } L^\infty([0, T]; L^2(\mathcal{M})).
\]
This implies
\[
\lim_{m \to \infty} u^{(m)} = u \quad \text{in } L^2([0, T] \times \mathcal{M}).
\]

As above \( g^{(m)} \geq g^{(m+1)} \), we have
\[
\{ (x, t) \in \mathcal{M} \times [0, T]; \ d^{(m)}(x, W_1) < t \} 
\subset \{ (x, t) \in \mathcal{M} \times [0, T]; \ d^{(m+1)}(x, W_1) < t \},
\]
and thus equation (3.21) implies that
\[
\text{supp} (u(\cdot, \cdot)) \cap ([0, T] \times \mathcal{M}) 
\subset \text{cl}\left( \bigcap_{m' > m_0} \bigcup_{m > m'} \text{supp} (u_{m'}(\cdot, \cdot)) \cap ([0, T] \times \mathcal{M}) \right) 
\subset \text{cl}\left( \bigcup_{m \in \mathbb{Z}_+} \{ (x, t) \in \mathcal{M} \times [0, T]; \ d^{(m)}(x, W_1) < t \} \right).
\]

(3.22)
4. Unique continuation

Next we will show that Tataru’s approximate controllability result, see e.g. [102], is valid for CMGAs. As usual, we start with the observability result.

**Theorem 4.1 (Tataru’s Unique Continuation Principle).** Let \( u \in C^0([0, 2T]; H^1(M)) \cap C^1([0, 2T]; L^2(M)) \) satisfy

\[
(\partial_t^2 - \Delta_g) u(x, t) = 0, \quad (x, t) \in \mathcal{M} \times (0, 2T).
\]

Assume, in addition, that \( u(x, t) = 0 \) in \( W \times (0, 2T) \), where \( W \) is an open subset of \( \mathcal{M}^{reg} \). Then,

\[
u(x, t) = 0 \quad \text{in } K(W, T),
\]

where \( K(W, T) \) is the double cone of influence,

\[
(4.1) \quad K(W, T) = \{(x, t) \in \mathcal{M} \times (0, 2T) : d(x, W) < T - |t - T|\}.
\]

**Proof.** Let \( V \subset \mathcal{M}^{reg} \) be open. Assume that \( u \) satisfies on \( \mathcal{M} \times \mathbb{R}_+ \) the wave equation

\[
(\partial_t^2 - \Delta_g)u(x, t) = 0, \quad \text{on } \mathcal{M} \times \mathbb{R}_+;
\]

\[
u(x, 0) = v_0(x), \quad \partial_t u(x, 0) = v_1(x).
\]

Also, assume that \( u(x, t) \) vanishes in the set \( V \times (0, 2T) \).

The restriction of \( u(x, t) \) on \( \mathcal{M}^{reg} \times \mathbb{R}_+ \) satisfies

\[
(\partial_t^2 - \Delta_g)u(x, t) = 0, \quad \text{on } \mathcal{M}^{reg} \times \mathbb{R}_+,
\]

and by applying Tataru’s theorem [102] on the smooth manifold \( \mathcal{M}^{reg} \times \mathbb{R}_+ \) (see [11, 12] for the corresponding stability results), we see that \( u \) vanishes in

\[
\Sigma_{V,T} = \{(x, t) \in \mathcal{M}^{reg} \times \mathbb{R}_+ : d_{\mathcal{M}^{reg},g}(x, V \cap \mathcal{M}^{reg}) - T| < T\}.
\]

As \( V \) is open, we see that \( V \cap \mathcal{M}^{reg} \neq \emptyset \). Next, let \( y \in V \cap \mathcal{M}^{reg} \) and let \( x \in \mathcal{M}^{reg} \) be such that \( d_{\mathcal{M}}(x, y) < T \). Above in Lemma 2.3 we have shown that there is \( m \) such that \( d_{g(m)}(x, y) < T \). Let \( \mu \) be a \( g(m) \)-geodesic that connects \( x \) and \( y \) and has length \( L = d_{g(m)}(x, y) < T \). Let \( \varepsilon = (T - L)/2 \). In the proof of Lemma
we showed that there is a path \( \eta : I = [0, 1] \to \mathcal{M} \) such that \( \eta(0) = x, \eta(1) = y \),
and \( \eta(\text{int}) \subset \mathcal{M}^{\text{reg}} \), and finally, \( g^{(m)} \)-length of \( \eta \) is at most \( L + \varepsilon \). As \( g^{(m)} \geq g \) in \( \mathcal{M}^{\text{reg}} \), this shows that \( g \)-length of \( \eta \) is at most \( L + \varepsilon \). Thus we see that
\[
x \in \{ x' \in \mathcal{M} ; \ d_{\mathcal{M}^{\text{reg}},g}(x',V) < T \}
\]
and we have by Lemma 2.4 that
\[
\{ x' \in \mathcal{M}^{\text{reg}} ; \ d_{\mathcal{M}^{\text{reg}},g}(x',V) < T \} = \{ x' \in \mathcal{M}^{\text{reg}} ; \ d_{\mathcal{M},g}(x,V) < T \}.
\]
This also shows that the solution of the wave equation,
\[
u(x,t),
\]
vanishes in the set
\[
(\mathcal{M}^{\text{reg}} \times \mathbb{R}) \cap \Sigma^{\text{reg}}_{V,T} = \{ (x,t) \in \mathcal{M}^{\text{reg}} \times \mathbb{R} ; \ |d_{\mathcal{M},g}(x,V) - T| < T \}.
\]
As \( \mathcal{M}^{\text{reg}} \) is dense in \( \mathcal{M} \) with respect to the topology defined with metric \( d_{\mathcal{M}} \), we have
\[
\Sigma^{\text{reg}}_{V,T} = \Sigma_{V,T} \cap (\mathcal{M}^{\text{reg}} \times \mathbb{R}_+),
\]
where
\[
\Sigma_{V,T} = \{ (x,t) \in \mathcal{M} \times \mathbb{R}_+ ; \ |d_{\mathcal{M}}(x,V) - T| < T \}.
\]
Thus we see that \( u \) vanishes a.e. in \( \Sigma_{V,T} \), as \( u \) is in \( L^2_{\text{loc}}(\mathcal{M} \times \mathbb{R}) \), this means that \( u \), considered as a distribution, vanishes in the set \( \Sigma_{V,T} \). This means that Tataru’s theorem is valid on \( (\mathcal{M},g) \).

5. Controllability results

Consider the initial value problem
\[
(\partial_t^2 - \Delta_g) u(x,t) = H(x,t), \quad \text{in} \ \mathcal{M} \times \mathbb{R}_+,
\]
\[
u|_{t=0} = 0, \quad \partial_t \nu|_{t=0} = 0,
\]
and denote its solution by \( u^H(x,t) = u(x,t) \).

Next we prove Tataru’s controllability theorem on CMGA.

**Theorem 5.1.** Let \( W \subset \mathcal{M}^{\text{reg}} \) be an open set. Then the set \( \{ u^H(\cdot, T) : H \in C_0^\infty(W \times (0,T)) \} \) is dense in \( L^2(\mathcal{M}(W,T)) \).

**Proof.** Assume that \( \eta \in L^2(\mathcal{M}(W,T)) \) satisfies
\[
(\eta, u^H(T))_{L^2(\mathcal{M},dV_g)} = 0
\]
for all \( H \in C_0^\infty(W \times (0,T)) \).

We consider the approximate initial value problem
\[
(\partial_t^2 - \Delta_g)^{(m)} u^{(m)}(x,t) = H(x,t), \quad \text{in} \ \mathcal{M} \times \mathbb{R}_+,
\]
\[
u^{(m)}|_{t=0} = 0, \quad \partial_t \nu^{(m)}|_{t=0} = 0,
\]
and denote its solution also by \( u^{H,(m)}(x,t) = u^{(m)}(x,t) \).

We consider also the dual problem
\[
(\partial_t^2 - \Delta_g) a(x,t) = 0, \quad \text{in} \ \mathcal{M} \times \mathbb{R},
\]
\[
a(x,T) = 0, \quad \partial_t a(x,T) = \eta(x),
\]
and the approximate dual problems

\begin{equation}
(\partial_t^2 - \Delta g) a^{(m)}(x, t) = 0, \quad \text{in } \mathcal{M} \times \mathbb{R},
\end{equation}

\begin{equation*}
a^{(m)}(x, T) = 0, \quad \partial_t a^{(m)}(x, T) = \eta(x).
\end{equation*}

Then,

\begin{equation*}
a^{(m)}(x, t) \in C(\mathbb{R}, H^1(\mathcal{M})) \cap C^1(\mathbb{R}, L^2(\mathcal{M})) \subset H^1_{\text{loc}}(\mathcal{M} \times \mathbb{R}).
\end{equation*}

By energy conservation for the wave equation, we have

\begin{equation*}
\|\partial_t a^{(m)}(\cdot, t)\|_{L^2(\mathcal{M}, |g|^{1/2}dx)}^2 \leq \|\partial_t a^{(m)}(\cdot, 0)\|_{L^2(\mathcal{M}, |g|^{1/2}dx)}^2 + 0
\end{equation*}

and hence for all \( t \in [0, T] \) we obtain by integrating in the time variable

\begin{equation*}
\|a^{(m)}(\cdot, t)\|_{L^2(\mathcal{M}, |g|^{1/2}dx)} \leq \|\eta\|_{L^2(\mathcal{M}, |g|^{1/2}dx)} + \int_0^t \|a^{(m)}(\cdot, t')\|_{L^2(\mathcal{M}, |g|^{1/2}dx)} dt' \leq (1 + T)\|\eta\|_{L^2(\mathcal{M}, |g|^{1/2}dx)}.
\end{equation*}

Letting \( m \to \infty \), we obtain

\begin{equation}
\|a^{(m)}(\cdot, t)\|_{L^2(\mathcal{M}, |g|^{1/2}dx)} \leq C_0 (1 + T)\|\eta\|_{L^2(\mathcal{M}, |g|^{1/2}dx)}.
\end{equation}

Then, by integrating by parts,

\begin{equation}
\int_{\mathcal{M} \times (0, T)} H(x, t) a^{(m)}(x, t) \, dV_g(x) \, dt
= \int_{W \times (0, T)} H(x, t) a^{(m)}(x, t) \, dV_g(x) \, dt
= \int_{\mathcal{M} \times (0, T)} \left[ \left( \partial_t^2 - \Delta g \right) u^H, (m) (x, t) a^{(m)}(x, t) \right. \\
\left. - u^H, (m) (x, t) \partial_t a^{(m)}(x, t) \right] \, dV_g(x) \, dx
= -\int_{\mathcal{M} \times (0, T)} \left[ \partial_t u^H, (m) (x, 0) a^{(m)}(x, 0) - u^H, (m) (x, 0) \partial_t a^{(m)}(x, 0) \right] \, dV_g(x) \, dx
\end{equation}

where we use equations \((5.3)\) and \((5.5)\). Thus,

\begin{equation}
\int_{\mathcal{M} \times (0, T)} H(x, t) a^{(m)}(x, t) |g^{(m)}(x)|^{1/2} dx \, dt = -\int_{\mathcal{M}} u^H, (m) (x, T) \eta(x) |g^{(m)}(x)|^{1/2} dx.
\end{equation}

By the strong resolvent convergence that for all \( t \in [0, T] \) we have \( a^{(m)}(\cdot, t) \to a(\cdot, t) \) in \( L^2(\mathcal{M}) \); \( dV_g \) and \( u^H, (m)(\cdot, T) \to u^H(\cdot, T) \) in \( L^2(\mathcal{M}) \); \( dV_g \) as \( m \to \infty \).
By applying estimate (5.6) and Lebesque dominated convergence theorem we can take limit of both sides of (5.7) as \( m \to \infty \), and obtain

\[
\int_{\mathcal{M} \times (0,T)} H(x,t)a(x,t) \, dV_g(x) \, dt = -\langle u^H(\cdot,T), \eta(\cdot) \rangle_{L^2(\mathcal{M})}, \quad \text{where we use (5.2) in the last identity.}
\]

Therefore,

\[
\int_{W \times (0,T)} H(x,t)a(x,t) \, dV_g(x) \, dt = 0,
\]

for all \( H \in C_0^\infty(W \times (0,T)) \), and hence \( a(x,t) = 0 \) in \( W \times (0,T) \). Since \( a(x,t) \) satisfies \( a(x,t) = a(x,2T - t) \) we have

\[
a(x,t) = 0 \quad \text{for } (x,t) \in W \times (0,2T).
\]

By Theorem 4.1

\[
a(x,t) = 0 \quad \text{for } (x,t) \in K(W,T),
\]

in particular,

\[
\eta(x) = \partial_t a(x,T) = 0.
\]

This yields that the set \( \{ u^H(\cdot,T) : H \in C_0^\infty(W \times (0,T)) \} \) is dense in \( L^2(\mathcal{M}(W,T)) \).

\( \Box \)

6. Uniqueness of inverse scattering

Next we introduce an additional assumption for the inverse problem we use to show its unique solvability. To this end, we consider the volume factor

\[
\Lambda(x) = \lim_{r \to 0^+} \frac{\text{vol}_M(B_M(x,r) \cap \mathcal{M}^{\text{reg}})}{\text{vol}_{\mathbb{R}^n}(B(0,r))}.
\]

Note that \( \Lambda(x) = 1 \) for all \( x \in \mathcal{M}^{\text{reg}} \).

**Assumption (L).** We assume below that

\[
\Lambda(x) \neq 1, \quad \text{for all } x \in \mathcal{M}^{\text{sing}}.
\]

Note that

\[
\Lambda(x) \leq \frac{T(x)}{\# \Gamma_x},
\]

see (2.3).

We aim to prove the following

**Theorem 6.1.** Suppose we are given two conic manifolds with group action \( \mathcal{M}^{(1)} \) and \( \mathcal{M}^{(2)} \) satisfying the assumptions (A-1) \( \sim \) (A-4), (C-1) \( \sim \) (C-4), (D), (VG), and (L). Let the (1,1) component of the (generalized) scattering matrix coincide:

\[
S^{(1)}_{11}(k) = S^{(2)}_{11}(k), \quad \forall k > 0, \quad k^2 \notin \sigma_p(-\Delta^{(1)}) \cup \sigma_p(-\Delta^{(2)}),
\]

and \( r^{(1)}_1 = r^{(2)}_1 \). Then there is an isometry between \( \mathcal{M}^{(1)} \) and \( \mathcal{M}^{(2)} \) in the following sense.
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(1) There is a homeomorphism \( \Phi : M^{(1)} \rightarrow M^{(2)} \).
(2) \( \Phi(M^{(1)}_{\text{sing}}) = M^{(2)}_{\text{sing}} \).
(3) \( \Phi : M^{(1)} \setminus M^{(1)}_{\text{sing}} \rightarrow M^{(2)} \setminus M^{(2)}_{\text{sing}} \) is a Riemannian isometry.

6.1. Blagovestchenskii’s identity. To prove the uniqueness of the inverse scattering problem we start with some auxiliary results. Let \( M \) be a (compact or complete) conical Riemannian manifold with group action. Let \( O \subset M \) be open. Consider the solution \( u^f(x,t) = u \) of the initial boundary value problem

\[
\begin{align*}
\partial_t^2 u - \Delta_g u &= f, \quad \text{in } M \times \mathbb{R}_+, \\
|u|_{t=0} = \partial_t u |_{t=0} &= 0, \quad \text{in } M.
\end{align*}
\]

Also, we define the source-to-solution map \( V_{O,+} : C^\infty_0(O \times \mathbb{R}_+) \rightarrow C^\infty(O \times (0,T)) \), given by

\[
V_{O,+}(f) = u^f|_{O \times \mathbb{R}_+}
\]

where we denote \( V_{O,\pm}(T) = V_{O,\pm}^T \) and \( V_{O,\pm}^\infty = V_{O,\pm}^\infty \). We denote below also \( U_{O,+}(\lambda) = U_{O,+}^\lambda \).

For considerations below, we observe that when a subset \( O \subset M^{reg} \) and the metric \( g|_O \) on it are given, the hyperbolic source-to-solution operators \( V_{O,+} \) and \( U_{O,+}^\lambda \) determine their Schwartz kernels \( G(x,t; x_0, t_0) = G(x,x_0, t-t_0) \) and \( G(\lambda, x, x_0) \) that satisfy

\[
\begin{align*}
V_{O,+}f(x,t) &= \int_{O \times \mathbb{R}_+} G(x,t; x_0, t_0)f(x_0, t_0)dv_g(x_0)dt_0, \\
U_{O,+}^\lambda F(x) &= \int_O G(\lambda, x, x_0)F(x_0)dv_g(x_0)dt.
\end{align*}
\]

To construct the manifold \( (M, g) \) from local measurements, we use a version of the boundary control method. The method originates from results of Belishev and Kurylev [4, 6] and it has been further developed for different linear equations in [7, 48, 61, 74, 77, 89, 90], see also [16] on the related scattering control method. The numerical implementation of the method has been recently developed in [26, 27, 28]. The present version of the method is based on focusing of the waves so that at a given time moment \( t \) the value \( u(\cdot, t) \) of the wave is concentrated in a neighborhood of a point, see also [75] for related techniques. This focusing technique has recently been used to study non-linear wave equation [73, 78] and the relation of reconstruction methods for the linear and the non-linear equations have been discussed in [74].

**Theorem 6.2.** Let \( (M, g) \) be a compact or complete Riemannian manifold. Let \( T > 0, O \subset M \) be open and bounded.

(i) Let \( f, h \in C^\infty_0(O \times \mathbb{R}_+) \), then

\[
\langle u^f(\cdot, T), u^h(\cdot, T) \rangle_{L^2(O)} = \langle f, JV_{O,+,+}^2h \rangle_{L^2(O \times (0,T))} - \langle V_{O,+,+}^2f, Jh \rangle_{L^2(O \times (0,T))}
\]
where the operator \( J : L^2(O \times (0, 2T)) \to L^2(O \times (0, T)) \) is defined as
\[
J \phi(x, t) = \frac{1}{2} \int_0^{2T-t} \phi(x, s) \, ds.
\]

(ii) Let \( f \in C_0^\infty(O \times \mathbb{R}_+) \), then
\[
\langle u^f(t), 1 \rangle_{L^2(M)} = \int_M u^f(x, t) \, dV(x)
\]
is given by
\[
\langle u^f(t), 1 \rangle_{L^2(M)} = \int_0^t \int_{t'}^t \langle f(t''), 1 \rangle_{L^2(O)} dt'' \, dt'.
\]

**Proof.**

(i) Let \( f, h \in C_0^\infty(O \times \mathbb{R}_+) \) and consider the mapping \( V : [0, 2T] \times [0, 2T] \to \mathbb{R} \),
\[
V(t, s) = \langle u^f(t), u^h(s) \rangle_{L^2(M)}.
\]
Then using Green’s formula we obtain
\[
(\partial_t^2 - \partial_s^2)V(t, s) = (\partial_t^2 - \partial_s^2)\langle u^f(t), u^h(s) \rangle_{L^2(M)}
\]
\[
= \langle f(t), V_\mathcal{O} + h(s) \rangle_{L^2(M)} - \langle V_\mathcal{O} + f(t), h(s) \rangle_{L^2(M)} := F(t, s).
\]
The function \( (t, s) \mapsto F(t, s) \) can be computed, if the the source-to-solution map \( V_\mathcal{O} + \) is given. Note that
\[
V(0, s) = 0, \quad \partial_t V(t, s)|_{t=0} = 0.
\]
Thus \( V \) is the solution of the following \((1+1)\)-dimensional initial value problem:
\[
\begin{aligned}
(\partial_t^2 - \partial_s^2)V(t, s) &= F(t, s), \quad \text{in } (0, 2T) \times \mathbb{R} \\
V|_{t=0} &= \partial_t V|_{t=0} = 0.
\end{aligned}
\]
Recall that the following formula
\[
V(t, s) = \frac{1}{2} \int_0^t \int_{s-\tau}^{s+\tau} F(t - \tau, y) \, dy \, d\tau, \quad s \in \mathbb{R}, \ t \in [0, 2T],
\]
solves (6.8). By the change of variables \( T - s = \tau \), we conclude
\[
V(T, T) = \frac{1}{2} \int_0^T \int_{-\tau}^{2T-\tau} F(\tau, y) \, dy \, d\tau.
\]
\[
= \langle f, JV_{\mathcal{O},+}^2 h \rangle_{L^2(\mathcal{O} \times (0,T))} - \langle V_{\mathcal{O},+}^2 f \bigg|_{\mathcal{O} \times (0,T)}, Jh \rangle_{L^2(\mathcal{O} \times (0,T))}.
\]
This proves (i).

(ii) Let \( f \in C_0^\infty(O \times \mathbb{R}_+) \) and consider the mapping \( I : [0, T] \to \mathbb{R} \),
\[
I(t) = \langle u^f(t), 1 \rangle_{L^2(M)}.
\]
Then using Green’s formula
\[
\partial_t^2 I(t) = (\partial_t^2 u^f(t), 1)_{L^2(M)}
\]
\[
= \langle \Delta u^f(t), 1 \rangle_{L^2(M)} + \langle f(t), 1 \rangle_{L^2(O)} = \langle f(t), 1 \rangle_{L^2(O)}.
\]
Also, we have \( I(0) = \partial_t I(t)|_{t=0} = 0 \). By solving the ordinary differential equation for \( I(t) \) with initial conditions, we obtain the claim. \( \square \)
Thus, using any sequence \((f_j)\) for Riemannian manifolds with conic singularities.

Next we will apply these formulas to compute the volume of the (open) domain of influence

\[(6.10) \quad M(\tilde{O}, T) = \{x \in M : d_M(x, \tilde{O}) < T\}, \quad \tilde{O} \subset O,\]

where \(d_M\) denotes the distance in \(M\) with respect to \(g\). We denote the volume of \(M(\tilde{O}, T) \cap M^{reg}\) by \(\text{Vol}_g(M(\tilde{O}, T))\) and define that \(\text{Vol}_g(M^{sing}) = 0\).

**Lemma 6.3.** Assume that we are given \(O\), the metric \(g|_O\) and the map \(V_{O,+}^{2T}\). Then, for any given open set \(\tilde{O} \subset O\) and \(T > 0\), these data uniquely determine the volume \(\text{Vol}_g(M(\tilde{O}, T))\) of \(M(\tilde{O}, T)\).

**Proof.** Let \(w \in L^2(M)\) be a function such that \(w = 1\) in \(M(\tilde{O}, T)\). For \(f \in C^\infty_0(\tilde{O} \times (0, T))\), real-valued, we define the quadratic functional

\[I_T(f) = \|u^f(\cdot, T) - w\|_{L^2(M)}^2 - \|w\|_{L^2(M)}^2,\]

Since \(\text{supp}(u^f(\cdot, T)) \subset M(\tilde{O}, T)\), we have

\[(6.11) \quad I_T(f) = \|u^f(\cdot, T)\|_{L^2(M)}^2 - 2\langle u^f(\cdot, T), 1 \rangle_{L^2(M)} - \|w\|_{L^2(M)}^2.\]

Hence, by Theorem [3.2] we can compute \(I_T(f)\) for any \(f \in C^\infty_0(\tilde{O} \times (0, T))\) uniquely by using \((O, g|_O)\) and \(V_{O,+}^{2T}\). In the sequel, this is phrased as we can compute \(I_T(f)\).

Now we use again the fact that, for \(f \in C^\infty_0(\tilde{O} \times (0, T))\), \(\text{supp}(u^f(\cdot, T)) \subset M(\tilde{O}, T)\) so that (6.11) yields that

\[(6.12) \quad I_T(f) = \|w\|_{L^2(M)}^2 - \|\chi_M(\tilde{O}, T)\|_{L^2(M)}^2 - \|\chi_M(\tilde{O}, T)\|_{L^2(M)},\]

where \(\chi_M(\tilde{O}, T)\) is the characteristic function of \(M(\tilde{O}, T)\). Thus,

\[(6.13) \quad I_T(f) \geq -\text{Vol}_g(M(\tilde{O}, T)), \quad \text{for all } f \in C^\infty_0(\tilde{O} \times (0, T)).\]

By Tataru’s controllability theorem, Theorem ReTa1, there is a sequence \(h_j \in C^\infty_0(\tilde{O} \times (0, T))\), such that

\[\lim_{j \to \infty} u^{h_j}(\cdot, T) = \chi_M(\tilde{O}, T) \quad \text{in } L^2(M).\]

For this sequence,

\[(6.14) \quad \lim_{j \to \infty} I_T(h_j) = -\text{Vol}_g(M(\tilde{O}, T)).\]

On the other hand, if \(f_j \in C^\infty_0(\tilde{O} \times (0, T))\) is a minimizing sequence for \(I_T\), i.e.,

\[(6.15) \quad \lim_{j \to \infty} I_T(f_j) = m_0 := \inf\{I_T(f) : f \in C^\infty_0(\tilde{O} \times (0, T))\},\]

then, by using (6.13) and (6.14) and the definition (6.12) of \(I_T(f)\) and the property that \(\text{supp}(u^f(\cdot, T)) \subset M(\tilde{O}, T)\) for all \(f \in C^\infty_0(\tilde{O} \times (0, T))\) we see that

\[\lim_{j \to \infty} u^{f_j}(\cdot, T) = \chi_M(\tilde{O}, T) \quad \text{in } L^2(M).\]

Thus, using any sequence \((f_j)\) satisfying (6.15), we can compute

\[\text{Vol}_g(M(\tilde{O}, T)) = \lim_{j \to \infty} \langle u^{f_j}(\cdot, T), u^{f_j}(\cdot, T) \rangle_{L^2(M)}.\]

\[\square\]
6.2. Reconstruction near $\mathcal{M}$. To prove Theorem 6.1 our first aim is to show that $\mathcal{M}^{(1)}_{\text{reg}}$ and $\mathcal{M}^{(2)}_{\text{reg}}$ are isometric. The proof is based on the procedure of the continuation of Green’s functions.

By the above considerations, the scattering operator $S_{11}$ determines in an open set $W \subset \mathcal{M}_{\text{reg}} \cap M_1$, the source-to-solution operator $V_{W,+}$, and thus we next we assume that we are given $(W, g|_W)$ and the operator $V_{W,+}$.

We are going to prove the uniqueness for the inverse problem step by step by constructing relatively open subsets $\mathcal{M}^{(1)}_{\text{rec}} \subset \mathcal{M}^{(1)}_{\text{reg}}$ and $\mathcal{M}^{(2)}_{\text{rec}} \subset \mathcal{M}^{(2)}_{\text{reg}}$, which are isometric and enlarge these sets at each step. In the following, when $\mathcal{M}^{(1)}_{\text{rec}} \subset \mathcal{M}^{(1)}_{\text{reg}} \cap \mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}_{\text{rec}} \subset \mathcal{M}^{(2)}_{\text{reg}} \cap \mathcal{M}^{(2)}$ are relatively open connected sets and

$$\Phi_{\text{rec}} : \mathcal{M}^{(1)}_{\text{rec}} \to \mathcal{M}^{(2)}_{\text{rec}},$$

is a diffeomorphism, we say that the triple $(\mathcal{M}^{(1)}_{\text{rec}}, \mathcal{M}^{(2)}_{\text{rec}}, \Phi_{\text{rec}})$ is admissible if

(i) $\Phi_{\text{rec}} : \mathcal{M}^{(1)}_{\text{rec}} \to \mathcal{M}^{(2)}_{\text{rec}}$ is a diffeomorphism and an isometry, that is, $\Phi_{\text{rec}}^* g^{(1)} = g^{(2)}$,

(ii) the source-to-solution maps $V_{\mathcal{O}^{(1)},+}$ and $V_{\mathcal{O}^{(2)},+}$ are $\Phi_{\text{rec}}$-related on $\mathcal{O}^{(1)}$ and $\mathcal{O}^{(2)}$, that is,

$$V_{\mathcal{O}^{(i)},+}(f \circ \Phi_{\text{rec}}) = (V_{\mathcal{O}^{(i)},+}(f)) \circ \Phi_{\text{rec}}$$

for all $f \in C_0^\infty(\mathcal{O}^{(2)} \times \mathbb{R}_+)$. 

By the above, (6.16) is equivalent to that the source-to-solution operators for the spectral problems $U_{\mathcal{O}^{(1)},+}^{(1)}(\lambda)$ and $U_{\mathcal{O}^{(2)},+}^{(2)}(\lambda)$ are $\Phi_{\text{rec}}$-related for all $\lambda \in \sigma_e(H^{(i)}) \setminus \mathcal{L}^{(i)}$, $i = 1, 2$, that is,

$$U_{\mathcal{O}^{(i)},+}^{(i)}(\lambda) \circ (\Phi_{\text{rec}})^* = (\Phi_{\text{rec}})^* \circ U_{\mathcal{O}^{(2)},+}^{(2)}(\lambda).$$

Furthermore, (6.16) implies that Schwartz kernels of the source-to-solution operators, that is, the time-domain Green’s functions $G^{(i)}(z, x, t)$ on $\mathcal{M}^{(i)}_{\text{rec}}$ satisfy the relation

$$G^{(2)}(\Phi_{\text{rec}}(x), \Phi_{\text{rec}}(y), t) = G^{(1)}(x, y, t)$$

for $x, y \in \mathcal{M}^{(1)}_{\text{rec}}$ and $t \in \mathbb{R}$.

Note that then the values of Green’s functions $G^{(i)}(z, x, y)$ on $\mathcal{M}^{(i)}_{\text{rec}}$ satisfy the relation

$$G^{(2)}(z; \Phi_{\text{rec}}(x), \Phi_{\text{rec}}(y)) = G^{(1)}(z; x, y), \text{ for } x, y \in \mathcal{M}^{(1)}_{\text{rec}}, z \in \mathbb{C} \setminus \mathbb{R}.$$

First we consider Green’s functions in the set $W$.

Our earlier considerations, Theorem 1.4 and Lemma 1.3 show the following lemma:

**Lemma 6.4.** When $W$ is considered both as a subset $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$ and $I : W \to W$ is the identity map, then the triple $(W, W, I)$ is admissible.
6.3. Continuation by Green’s functions. To reconstruct subsets of manifolds $\mathcal{M}^{(i)}$, $i = 1, 2$, by continuing Green’s function, we need the following result telling that the values of Green’s functions identify the points of the manifold.

**Lemma 6.5.** Let $x_1, x_2 \in \mathcal{M}^{(i)}$ be such that

\[(6.20) \quad G^{(i)}(z, x_1, y) = G^{(i)}(z, x_2, y)\]

for all $y \in W$ and some $z \in \mathbb{C} \setminus \mathbb{R}$. Then $x_1 = x_2$.

**Proof.** Using the unique continuation principle for the solutions of elliptic equations, we see that (6.20) implies that $G^{(i)}(z, x_1, y) = G^{(i)}(z, x_2, y)$, for all $y \in \mathcal{M}^{(i)} \setminus \{x_1, x_2\}$. As the map $y \mapsto G^{(i)}(z, x, y)$ is bounded in the compact subsets of $\mathcal{M}^{(i)} \setminus \{x\}$ and tends to infinity as $y$ approaches $x$, this proves that $x_1 = x_2$. □

**Remark 6.6.** Lemma 6.5 has the following important consequence: If the triples $(N_1^{(1)}, N_1^{(2)}, \Phi_1)$ and $(N_2^{(1)}, N_2^{(2)}, \Phi_2)$ are admissible and $N_1^{(1)} \cap N_2^{(1)} \neq \emptyset$, then, by Lemma 6.5, the maps $\Phi_1(x)$ and $\Phi_2(x)$ have to coincide in $N_1^{(1)} \cap N_2^{(1)}$. Moreover, if $N_3^{(i)} = N_1^{(i)} \cup N_2^{(i)}$, $i = 1, 2$, and

\[(6.21) \quad \Phi_3(x) = \begin{cases} \Phi_1(x), & \text{for } x \in N_1^{(i)}, \\ \Phi_2(x), & \text{for } x \in N_2^{(i)}, \end{cases}\]

then, by Lemma 6.5, the map $\Phi_3 : N_3^{(1)} \to N_3^{(2)}$ is bijective and hence a diffeomorphism. This implies that the triple $(N_3^{(1)}, N_3^{(2)}, \Phi_3)$ is admissible.

The procedure of constructing the isometry between $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$ consists of extending the admissible triple $(\mathcal{M}^{(1), \text{rec}}, \mathcal{M}^{(2), \text{rec}}, \Phi^{\text{rec}})$. In the first step, we apply Lemma 6.3 to the triple $(W, W, I)$. In the subsequent steps we extend the sets $\mathcal{M}^{(1), \text{rec}}$ and $\mathcal{M}^{(2), \text{rec}}$ and use sets $\mathcal{O}^{(i)} \subset \mathcal{M}^{(i), \text{rec}}$, defined below, to have the role of the set $W$ above.

Let $q_i \in \mathcal{M}^{(i), \text{rec}}, i = 1, 2$, be such that

\[(6.22) \quad \Phi^{\text{rec}}(q_1) = q_2,\]

and let $d^{(i)} = d_{\mathcal{M}^{(i)}}$ denote the distance on $\mathcal{M}^{(i)}$. Let $R > 0$ be sufficiently small so that

\[(6.23) \quad B^{(i)}(q_i, 4R) \subset \mathcal{M}^{(i), \text{rec}}, \quad i = 1, 2\]

and the set

$$\mathcal{O}^{(i)} = B^{(i)}(q_i, R)$$

is geodesically convex set, has smooth boundary and that all points in the closure of $B^{(i)}(q_i, 2R)$ can be joined by a unique length minimizing curve. Note that when the set $\mathcal{M}^{(i), \text{rec}}$ and the metric on it are known, one can verify for a given $q_i \in \mathcal{M}^{(i), \text{rec}}$ if a given value $R$ satisfies these assumptions. Note that then normal coordinates, centered at $q_i$, are well defined in $B^{(i)}(q_i, 2R)$. Then $\Phi^{\text{rec}}(\mathcal{O}^{(1)}) = \mathcal{O}^{(2)}$ and $\mathcal{M}^{(i), \text{rec}} \setminus \overline{\mathcal{O}^{(i)}}$ are connected.

Below, we say that the source-to-solution maps $V_{\mathcal{O}^{(1)}, +}^{\mathcal{O}^{(2)}, +}$ and $V_{\mathcal{O}^{(2)}, +}^{\mathcal{O}^{(1)}, +}$ are $\Phi^{\text{rec}}$-related on $\mathcal{O}^{(1)}$ and $\mathcal{O}^{(2)}$ if

\[(6.24) \quad V_{\mathcal{O}^{(2)}, +}^{\mathcal{O}^{(1)}, +}(H \circ \Phi^{\text{rec}}) = (V_{\mathcal{O}^{(1)}, +}^{\mathcal{O}^{(2)}, +}(H)) \circ \Phi^{\text{rec}}\]
for all \( H \in C^\infty_0(\mathcal{O}^{(2)} \times \mathbb{R}_+) \).

**Lemma 6.7.** Let \((\mathcal{M}^{(1),rec}, \mathcal{M}^{(2),rec}, \Phi^{rec})\) be an admissible triple and \(\mathcal{O}^{(i)}\), \(i = 1, 2\) be relatively compact subsets of \(\mathcal{M}^{(i),rec}\) such that \(\mathcal{O}^{(2)} = \Phi^{rec}(\mathcal{O}^{(1)})\). Then for all \( T > 0 \) the source-to-solution maps \(V_{\mathcal{O}^{(1)}}^{2T}, +\) and \(V_{\mathcal{O}^{(2)}}^{2T}, +\) are \(\Phi^{rec}\)-related on \(\mathcal{O}^{(1)}\) and \(\mathcal{O}^{(2)}\).

**Proof.** Our earlier considerations, Theorem 1.1 and Lemma 1.4, yield the claim. □

### 6.4. Source-to-solution maps for subdomains of \(\mathcal{M}\) and recognition of singular points.

For \( y \in \partial \mathcal{M}^{(i),reg} \) we define the singular set and cut locus distances

\[
\tau^{(i)}(y) = \min(\tau_{cut}^{(i)}(y), \tau_{sing}^{(i)}(y))
\]

where

\[
\tau_{sing}^{(i)}(y) = \inf_{\xi \in S_y \mathcal{M}^{(i)}} \tau_{sing}^{(i)}(y, \xi), \quad \tau_{sing}^{(i)}(y, \xi) = \inf \{ t > 0; \quad \gamma_{y, \xi}^{(i)}(t) \in \mathcal{M}^{(i)}_{reg} \},
\]

where \(\gamma_{y, \xi}^{(i)}(t)\) is the geodesic on \(\mathcal{M}^{(i)}\), and

\[
\tau_{cut}^{(i)}(y) = \inf_{\xi \in S_y \mathcal{M}^{(i)}} \tau_{cut}^{(i)}(y, \xi), \quad \tau_{cut}^{(i)}(y, \xi) = \inf \{ t \in (0, \tau_{sing}^{(i)}(y, \xi)); \quad d^{(i)}(\gamma_{y, \xi}^{(i)}(t), y) < t \} \cup \{ \tau_{sing}^{(i)}(y, \xi) \}\]

When \( W \subset \mathcal{O}^{(i)} \) are open and \( s > 0 \), we denote the (open) domain of influence by

\[
\mathcal{M}^{(i)}(W, s) = \{ x \in \mathcal{M}^{(i)}; \quad d^{(i)}(x, W) < s \}.
\]

We also denote

\[
\mathcal{M}^{(i)}(W, s) = \{ x \in \mathcal{M}^{(i)} \setminus \mathcal{O}^{(i)}; \quad d^{(i)}(x, W) < s \}.
\]

Next we consider the points \( q_i \in \mathcal{M}^{(i),reg} \) satisfying \( (6.22) \) and \( R \) satisfying assumption in the formula \( (6.23) \) and below it. Moreover, assume that

\[
T < \min_{i=1,2} \tau^{(i)}(q_i) - R.
\]

Below, we consider the sets

\[
\mathcal{N}^{(i)} = \{ p \in \mathcal{M}^{(i)} \setminus \mathcal{O}^{(i)} : d^{(i)}(p, \partial \mathcal{O}^{(i)}) < T \}
\]

and the families of the interior distance functions corresponding to interior points in \(\mathcal{N}^{(i)}\),

\[
R_{\mathcal{O}^{(i)}}(\mathcal{N}^{(i)}):= \{ d^{(i)}(x, \cdot)|_{\mathcal{O}^{(i)}} : \quad x \in \mathcal{N}^{(i)} \} \subset C(\mathcal{O}^{(i)}).
\]

**Theorem 6.8.** Let \((\mathcal{M}^{(1),rec}, \mathcal{M}^{(2),rec}, \Phi^{rec})\) be an admissible triple and \(\mathcal{O}^{(i)}\), \(i = 1, 2\) be relatively compact subsets of \(\mathcal{M}^{(i),rec}\) such that \(\mathcal{O}^{(2)} = \Phi^{rec}(\mathcal{O}^{(1)})\). Then

\[
\{(d^{(2)}(x, \cdot)|_{\mathcal{O}^{(2)}}) \circ \Phi^{rec} : \quad x \in \mathcal{N}^{(2)} \} = \{ d^{(2)}(x', \cdot)|_{\mathcal{O}^{(1)}} : \quad x' \in \mathcal{N}^{(1)} \}.
\]
This is to be proved in several steps.
Next we drop the superindex (i) for a while and consider manifold \((\mathcal{M}, g)\).
Let \(T, \epsilon > 0\). For each \(r > \epsilon\) and \(x \in \mathcal{N}\) we define a set
\[
S_r(x, r) := (T - (r - \epsilon), T) \times B(x, \epsilon), \quad B(x, \epsilon) \subset \mathcal{M}.
\]
(6.29)

We denote for any measurable \(A \subset \mathcal{N}\) the function space
\[
L^2(A) := \{u \in L^2(\mathcal{M}) : \text{supp}(u) \subset \overline{A}\}.
\]
(6.30)

We use the following lemma, developed in [48], to study interior distance functions on smooth manifolds, and give its proof for convenience of the reader.

**Lemma 6.9.** Let \(p, y, z \in \mathcal{O}, \epsilon > 0\) and \(\ell_p, \ell_y, \ell_z \in (\epsilon, T)\). Then the following are equivalent:

(i) We have
\[
B(p, \ell_p) \subset \overline{B(y, \ell_y) \cup B(z, \ell_z)}.
\]
(6.30)

(ii) Suppose that
\[
\forall f \in C_0^\infty(S_r(p, \ell_p)) \exists (f_j^i)_{j=1}^{\infty} \subset C_0^\infty(S_r(y, \ell_y) \cup S_r(z, \ell_z))
\]
such that \(\lim_{j \to \infty} \|u^j(\cdot, T) - u^{j_i}(\cdot, T)\|_{L^2(\mathcal{N})} = 0\).
(6.31)

Here \(u^j, u^{j_i}\) are the solutions of \((6.4)\) with metric \(g\) and with sources \(f\) and \(f_j,\) respectively.

**Proof.** Suppose that \((6.30)\) is valid. Let \(f \in C_0^\infty(S_r(p, \ell_p))\). Then by the finite speed of wave propagation it holds that
\[
\text{supp}(u^j(\cdot, T)) \subset B(p, \ell_p) \subset \overline{B(y, \ell_y) \cup B(z, \ell_z)}
\]
\[
\text{supp}(u^{j_i}(\cdot, T)) \subset B(p, \ell_p) \subset \overline{B(y, \ell_y) \cup B(z, \ell_z)}
\]
\[
\text{supp}(u^j(\cdot, T)) \subset B(p, \ell_p) \subset \overline{B(y, \ell_y) \cup B(z, \ell_z)}
\]
\[
\text{supp}(u^{j_i}(\cdot, T)) \subset B(p, \ell_p) \subset \overline{B(y, \ell_y) \cup B(z, \ell_z)}
\]

Let \(\chi(x)\) be the characteristic function of the ball \(B(y, \ell_y)\) and set
\[
u_{(in)}(x, T) := \chi(x)u^j(x, T), \quad u_{(ext)}^j(x, T) := u^j(x, T) - u_{(in)}^j(x, T).
\]
Then \(u_{(in)}^j(\cdot, T) \in L^2(B(y, \epsilon))\) and \(u_{(ext)}^j(\cdot, T) \in L^2(B(z, \epsilon))\). By approximate controllability there exist sequences \((f_j^i)_{j=1}^{\infty} \subset C_0^\infty(S_r(y, \ell_y))\) and \((f_j^i)_{j=1}^{\infty} \subset C_0^\infty(S_r(z, \ell_z))\) such that sequences \((u_{(in)}^{j_i}(\cdot, T))_{j=1}^{\infty}\) and \((u_{(ext)}^{j_i}(\cdot, T))_{j=1}^{\infty}\) converge to \(u_{(in)}^j(\cdot, T)\) and \(u_{(ext)}^j(\cdot, T)\), respectively, in \(L^2(\mathcal{M})\). Therefore the sequence
\[
f_j = f_{y_j}^j + f_{z_j}^j \in C_0^\infty(S_r(y, \ell_y) \cup S_r(z, \ell_z)), \quad j = 1, 2, \ldots
\]
satisfies \((6.31)\).

Suppose that \((6.30)\) is not valid. Then the open set
\[
U := B(p, \ell_p) \setminus (B(y, \ell_y) \cup B(z, \ell_z))
\]
is not empty. By approximate controllability, we can choose \(f \in C_0^\infty(S_r(p, \ell_p))\) such that \(\|u^j(\cdot, T)\|_{L^2(U)} > 0\). By finite speed of wave propagation it holds that
\[
\inf\{\|u^j(\cdot, T) - u^h(\cdot, T)\|_{L^2(\mathcal{N})} : h \in C_0^\infty(S_r(y, \ell_y) \cup S_r(z, \ell_z))\}
\]
\[
\geq \|u^j(\cdot, T)\|_{U} \|_{L^2(U)} > 0.
\]
(6.31)

Therefore \((6.31)\) is not true. This proves the claim. \(\square\)
Next we consider the interior distance functions related to a point \( p \) (in the unknown part of the manifold) that gives distances to points \( z \) in the set \( \mathcal{O}^{(i)} \).

**Proposition 6.10.** Let \( (\mathcal{M}^{(1), rec}, \mathcal{M}^{(2), rec}, \Phi^{rec}) \) be an admissible triple, and \( \mathcal{O}^{(i)} = B^{(i)}(q_i, R), i = 1, 2 \) be the balls of radius \( R \), centered at \( q_i \) in \( \mathcal{M}^{(i), rec} \), satisfying \( \Phi^{rec}(q_1) = q_2 \) and \( T \) satisfy (6.25). Assume that the source-to-solution maps \( V_{\mathcal{O}^{(1)}, +}^{2T} \) and \( V_{\mathcal{O}^{(2)}, +}^{2T} \) are \( \Phi^{rec} \)-related. Let \( z_i \in \mathcal{O}^{(i)}, i = 1, 2 \), be such that \( z_2 = \Phi^{rec}(z_1) \). Let \( \xi_i \in T_{q_i} \mathcal{M}^{(i)} \) be the unit vector, \( R \leq \bar{r} < R + T \) and \( p_i = \gamma(q_i, \xi_i(\bar{r})) \). Then

\[
d_{g^{(i)}}(p_1, z_1) = d_{g^{(2)}}(p_2, z_2).
\]

**Proof.** Note that we have \( R + T < \tau^{(i)}_{cut}(q_i, \xi_i) \) for \( i = 1, 2 \) and hence \( \bar{r} < \tau^{(i)}_{cut}(q_i, \xi_i) \).

Let \( s \in (0, R) \) be such that \( \gamma_{q_i, \xi_i}([0, s]) \subset \mathcal{O}^{(i)} \). We denote \( x_i = \gamma_{q_i, \xi_i}(s) \).

Let \( r = \bar{r} - s > 0 \), and \( 0 < \epsilon < \epsilon_0 = \min(R + T - \bar{r}, s) \). Below we consider the balls \( B(x_i, r + \epsilon) \subset \mathcal{M}^{(i)} \) and the sets \( S_{\epsilon}(x, \ell) \subset \mathcal{M}^{(i)} \) defined in (6.29).

By Lemma 6.10 the balls \( B(x_i, r + \epsilon) \subset \mathcal{M}^{(i)} \) satisfy the inclusion

\[
B(x_i, r + \epsilon) \subset B(q_i, r + s) \cup B(z_i, t)
\]

if and only if the equation (6.31) is valid with \( p = x_i, y = q_i, z = z_i \in \mathcal{M}^{(i)} \), \( \ell_p = r + \epsilon, \ell_y = r + s \) and \( \ell_z = t \).

Let us consider functions \( f^i \in C^\infty_0(S_{\epsilon}(x_i, \ell_p)) \), \( i = 1, 2 \) and the sequences \( (f_j^i)_{j=1}^\infty \subset C^\infty_0(S_{\epsilon}(q_i, \ell_y) \cup S_{\epsilon}(z_i, \ell_z)) \), \( i = 1, 2 \), that satisfy

\[
f^1 = (\Phi^{rec})^* f^2, \quad f_j^1 = (\Phi^{rec})^* f_j^2.
\]

Using the Blagovestchenskii identity, Theorem 6.2, we can compute for functions \( f^i \) and \( f_j^i \) the norm \( \| u^{f^i}(\cdot, T) - u^{f_j^i}(\cdot, T) \|_{L^2(\mathcal{M}^{(i)})} \) using the local source-to-solution data \( (\mathcal{O}^{(i)}, g|_{\mathcal{O}^{(i)}}, V_{\mathcal{O}^{(i)}, +}^{2T}) \). Hence, as the source-to-solution maps \( V_{\mathcal{O}^{(1)}, +}^{2T} \) and \( V_{\mathcal{O}^{(2)}, +}^{2T} \) are \( \Phi^{rec} \)-related, (6.34) implies

\[
\| u^{f^i}(\cdot, T) - u^{f_j^i}(\cdot, T) \|_{L^2(\mathcal{M}^{(i)})} = \| u^{f^2}(\cdot, T) - u^{f_j^2}(\cdot, T) \|_{L^2(\mathcal{M}^{(2)})}.
\]

Hence for functions \( f^i \in C^\infty_0(S_{\epsilon}(x_i, \ell_p)) \), \( i = 1, 2 \) and the sequences \( (f_j^i)_{j=1}^\infty \subset C^\infty_0(S_{\epsilon}(q_i, \ell_y) \cup S_{\epsilon}(z_i, \ell_z)) \), \( i = 1, 2 \), satisfying (6.34) we have

\[
\lim_{j \to \infty} \| u^{f^i}(\cdot, T) - u^{f_j^i}(\cdot, T) \|_{L^2(\mathcal{M}^{(i)})} = 0 \quad \text{if and only if}
\]

\[
\lim_{j \to \infty} \| u^{f^2}(\cdot, T) - u^{f_j^2}(\cdot, T) \|_{L^2(\mathcal{M}^{(2)})} = 0.
\]

Thus by Lemma 6.9 we have

\[
B(x_1, r + \epsilon) \subset B(q_1, r + s) \cup B(z_1, t) \quad \text{if and only if}
\]

\[
B(x_2, r + \epsilon) \subset B(q_2, r + s) \cup B(z_2, t),
\]

see (6.37).

Let us define, for \( i = 1, 2 \),

\[
t^*_i := \inf \{ t > 0 : \text{Formula (6.38) is valid with index } i \text{ for some } \epsilon \in (0, \epsilon_0) \}.
\]

By (6.37), we have \( t^*_1 = t^*_2 \). Due to this, we denote below \( t^*_i = t^*_2 = t^* \).
Next we will show that
\[ d_{g^{(i)}}(p_i, z_i) = t^*. \]

Suppose \( t \) is such that (6.33) is valid for some \( \varepsilon > 0 \), that we next denote by \( \varepsilon_0 \). Then, we have that (6.33) is valid for all \( \varepsilon \in (0, \varepsilon_0) \). For any \( \varepsilon \in (0, \varepsilon_0) \), let
\[ y_i^\varepsilon = \gamma_{x_i, q_i}(r + \varepsilon) = \gamma_{q_i, \xi_i}(s + r + \varepsilon), \quad \eta_i = \gamma_{q_i, \xi_i}(s). \]
Since the right hand side of (6.33) is a closed set, we have that \( y_i^\varepsilon \in B(q_i, r + s) \cup B(z_i, t) \). As \( x_i = \gamma_{q_i, \xi_i}(s) \) and we assumed that
\[ s + r + \varepsilon = \tilde{r} + \varepsilon < R + T \leq \tilde{r}^* \]
for \( i = 1, 2 \), it holds that
\[ d_{g^{(i)}}(y_i^\varepsilon, q_i) = r + s + \varepsilon > r + s, \]
so that \( y_i^\varepsilon \notin B(q_i, r + s) \). As (6.33) holds, we need to have \( y_i^\varepsilon \in B(z_i, t) \). Thus \( t \geq d_{g^{(i)}}(y_i^\varepsilon, z_i) \). As this holds for all sufficiently small \( \varepsilon > 0 \) and \( y_i^\varepsilon \to p_i \) as \( \varepsilon \to 0 \), we obtain \( t \geq d_{g^{(i)}}(p_i, z_i) \). This yields that \( t^* \geq d_{g^{(i)}}(p_i, z_i) \).

Next, suppose that there exists \( t \in (d_{g^{(i)}}(p_i, z_i), t^*) \). Then for any sufficiently small \( \varepsilon > 0 \) the formula (6.33) is not valid. Choose for every \( k \in \mathbb{N} \) a point
\[ p_i^k \in B(x_i, r + 1/k) \setminus B(q_i, r + s) \cup B(z_i, t). \]
By compactness of \( B(x_i, r + 1) \) we may assume that \( p_i^k \to \tilde{p}_i \in \partial B(x_i, r) \) as \( k \to \infty \). We will show that \( \tilde{p}_i = p_i \). As \( \tilde{p}_i \in \partial B(x_i, r) \) and \( d_{g^{(i)}}(x_i, q_i) \leq s \), we have by triangle inequality \( d_{g^{(i)}}(\tilde{p}_i, q_i) \leq s + r \). Let \( \alpha_i \) be a minimizing geodesic from \( x_i \) to \( \tilde{p}_i \). Suppose first that \( \alpha_i \) is not the geodesic continuation of the geodesic segment \( \gamma_{q_i, \xi_i}([0, s]) \). Since a curve \( \gamma_{q_i, \xi_i}([0, s]) \cup \alpha_i \) has a length \( s + r \) and it is not smooth at \( x_i \), it must hold that \( d_{g^{(i)}}(\tilde{p}_i, q_i) < s + r \). Then for sufficiently large \( k \), we have \( d_{g^{(i)}}(p_i^k, q_i) < s + r \) and it is not possible because of (6.33). This show that \( \alpha_i \) has the geodesic continuation of segment \( \gamma_{q_i, \xi_i}((0, s)) \). This yields that \( \tilde{p}_i = \gamma_{q_i, \xi_i}(s + r) = p_i \).

Since \( p_i \in B(z_i, t) \) we get a contradiction with the assumptions that \( p_i^k \to \tilde{p}_i = p_i \) as \( k \to \infty \) and \( p_i^k \notin B(z_i, t) \). Therefore interval \( (d_{g^{(i)}}(p_i, z_i), t^*) \) is empty. This shows that \( t^* = d_{g^{(i)}}(p_i, z_i) \) for both \( i = 1, 2 \). Hence, \( d_{g^{(i)}}(p_1, z_1) = d_{g^{(i)}}(p_2, z_2) \). \( \square \)

For \( i = 1, 2 \), let \( p_i \in \mathcal{N}^{(i)} \) and \( z_i \in \mathcal{O}^{(i)} \) be such that \( p_2 = \Phi^{rec}(p_1) \) and \( z_2 = \Phi^{rec}(z_1) \).

By definition of the set \( \mathcal{N}^{(i)} \), we see that there exist vectors \( \xi_i \in S_{q_i} \mathcal{M}^{(i)} \) such that \( p_i = \gamma_{q_i, \xi_i}(\tilde{r}) \), for some \( \tilde{r} \in [R, R + T] \). By Proposition 6.10, we see that 6.32 is valid. Therefore Theorem 6.8 is proved. \( \square \)

Consider points \( p_1 \in \mathcal{N}^{(1)} \) and \( p_2 = \Phi^{rec}(p_1) \). Let \( A(p_i) \) be the set of the points \( y_i \in \partial \mathcal{O}^{(i)} \) such that there are no points \( z_i \in \mathcal{O}^{(i)} \) such that \( d^{(i)}(p_i, z_i) = d^{(i)}(p_i, y_i) + d^{(i)}(y_i, z_i) \). Note that if \( y_i \in A(p_i) \), then any shortest geodesics in \( \mathcal{M}^{(i)} \) from \( p_i \) to \( y_i \) do not intersect \( \mathcal{O}^{(i)} \). Let us define the distance function \( d_{\mathcal{N}^{(i)}}(x_i, p_i) \) analogously to (2.10), that is, as the infimum of length of paths connecting \( x_i \) to
\[
d_{\mathcal{N}(i)}(x_i, p_i) = \inf_{y_i \in A(p_i)} d_{\partial \mathcal{O}(i)}(x_i, y_i) + d_{\mathcal{M}(i)}(y_i, p_i).
\]

Here, \(d_{\partial \mathcal{O}(i)}(x_i, y_i)\) is the intrinsic distance of points \(x_i\) and \(y_i\) along the boundary \(\partial \mathcal{O}(i)\). By Theorem 6.8, this and \(d_{\partial \mathcal{O}(i)}(x_1, y_1) = d_{\partial \mathcal{O}(i)}(\Phi^{rec}(x_1), \Phi^{rec}(y_1))\) for \(x_1, y_1 \in \partial \mathcal{O}(i)\) yield that the following:

**Corollary 6.11.** Under assumptions of Theorem 6.8, we have for all \(p_i \in \mathcal{N}(i)\) and \(x_1 \in \partial \mathcal{O}(i)\) that

\[
d_{\mathcal{N}(2)}(\Phi^{rec}(x_1), \Phi^{rec}(p_1)) = d_{\mathcal{N}(1)}(x_1, p_1).
\]

Next we consider how the source-to-solution operator causes bounds for the cut locus functions.

**Lemma 6.12.** Let \(0 < s < R\). Assume that \(r > 0\) is such that

\[
0 < r + s < \tau_{\text{sing}}^{(i)}(q_i).
\]

Let \(\xi_i \in S_{q_i} \mathcal{M}^{(i)}\), and \(x_i = \gamma_{q_i, \xi_i}(s)\).

(i) If \(\tau_{\text{cut}}^{(i)}(q_i, \xi_i) < s + r\), then

\[
\text{there exists } \epsilon > 0 \text{ such that } B(x_i, r + \epsilon) \subset B(q_i, s + r).
\]

(ii) If \(\text{(6.42)}\) is valid then \(\tau_{\text{cut}}^{(i)}(q_i, \xi_i) \leq s + r\).

**Proof.** Denote \(p_i = \gamma_{q_i, \xi_i}(s + r)\).

Suppose that \(\text{(6.42)}\) is valid. Let \(\delta \in (0, \epsilon)\) and consider a point

\[
z_i = \gamma_{q_i, \xi_i}(s + r + \delta) \in B(x_i, r + \epsilon).
\]

By \(\text{(6.42)}\), \(d^{(i)}(z_i, q_i) \leq s + r\). Thus \(\tau_{\text{cut}}^{(i)}(q_i, \xi_i) < s + r + \delta\). Since \(\delta\) was arbitrary we have \(\tau_{\text{cut}}^{(i)}(q_i, \xi_i) \leq s + r\).

Suppose that \(\tau_{\text{cut}}^{(i)}(q_i, \xi_i) < s + r\). We show first that

\[
B(x_i, r) \subset B(q_i, s + r).
\]

By triangle inequality it suffices to show that \(\partial B(x_i, r) \subset B(y_i, s + r)\). Let \(z_i \in \partial B(x_i, r)\). By triangle inequality \(d^{(i)}(z_i, q_i) \leq s + r\). Let \(\alpha\) be a minimizing geodesic from \(x_i\) to \(z_i\). Suppose first that \(\alpha\) is not the geodesic continuation of the geodesic segment \(\gamma_{q_i, \xi_i}([0, s])\). Since a curve \(\gamma_{q_i, \xi_i}([0, s]) \cup \alpha\) has a length \(s + r\) and it is not smooth at \(x_i\), it must hold that \(d^{(i)}(z_i, q_i) < s + r\). Second, suppose \(\alpha\) is the geodesic continuation of segment \(\gamma_{q_i, \xi_i}(0, s)\), then \(z_i = \gamma_{q_i, \xi}(s + r) = p_i\). Since \(\tau_{\text{cut}}^{(i)}(q_i, \xi_i) < s + r\), it holds that \(d^{(i)}(q_i, p_i) < s + r\). As \(z_i \in \partial B(x_i, r)\) above is arbitrary, the formula \(\text{(6.43)}\) follows. Therefore \(\text{dist}(\partial B(x_i, r), \partial B(q_i, s + r)) > 0\) and \(\text{(6.42)}\) is valid.

**Proposition 6.13.** Let \((\mathcal{M}^{(1), rec}, \mathcal{M}^{(2), rec}, \Phi^{rec})\) be an admissible triple, and \(\mathcal{O}^{(i)} = B^{(i)}(q_i, R) \subset \mathcal{M}^{(i), rec}, i = 1, 2\) be the balls of radius \(R\), centered at \(q_i\) in
\[
\mathcal{M}^{(i), rec}, \text{ satisfying } \Phi^{rec}(q_1) = q_2. \text{ Assume that the source-to-solution maps } V_{\mathcal{O}(1),+} \text{ and } V_{\mathcal{O}(2),+} \text{ are } \Phi^{rec}\text{-related. Assume that}
\]

\[
\tau_{cut}^{(1)}(q_1) < \min(\tau_{sing}^{(1)}(q_1), \tau_{sing}^{(2)}(q_2)).
\]

Then

\[
\tau_{cut}^{(2)}(q_2) \geq \tau_{cut}^{(1)}(q_1).
\]

**Proof.** Let \( \xi_1 \in S_{q_1}\mathcal{M}^{(1), rec} \) and \( \xi_2 = (\Phi^{rec})_* \xi_1 \) and let \( r < \tau_{cut}^{(1)}(q_1) \). Let \( 0 < \kappa < \rho \) and denote \( \gamma = \gamma_{q, \xi}(s) \). Also, let

\[
a_i(\xi_1) = \inf \{ s + r > 0 : \text{ Formula (6.42) with } i = 1 \text{ and } s \in (0, \rho) \}.
\]

Then by Lemma 6.12, \( a_1(\xi_1) \geq \tau_{cut}^{(1)}(q_1) \). Note that here \( \xi_1 \) is an arbitrary unit vector and thus \( \inf_{\xi_1} a_1(\xi_1) \geq \tau_{cut}^{(1)}(q_1) \).

Choose \( \epsilon \in (0, \rho - s) \). Then,

\[
B(q_1, \epsilon) \cup B(x_1, \epsilon) \subset \mathcal{O}(i), \quad i = 1, 2.
\]

Now we can proceed as in the proof of Proposition 6.10.

By applying the Blagovestchenskii identity \( 6.4 \) and the fact that the source-to-solution maps \( V_{\mathcal{O}(1),+} \) and \( V_{\mathcal{O}(2),+} \) are \( \Phi^{rec}\text{-related} \) with any \( T > 0 \), we see that formula \( 6.42 \) holds for \( r > 0 \) and \( s \in (0, \rho) \) with the index \( i = 1 \) and only it holds with the index \( i = 2 \). Thus by taking \( z = y = q_1, x = x_i, \ell_y = r + s = \ell_z, \ell_x = r + \epsilon \) and applying Lemma \( 6.13 \) with \( i = 1 \) and \( i = 2 \) we see that \( a_1(\xi_1) = a_2(\xi_2) \).

By Lemma 6.12, \( \tau_{cut}^{(2)}(q_2) \geq \inf_{\xi_2} a_2(\xi_2) = \inf_{\xi_1} a_1(\xi_1) \). This yields the claim.

\[
\quad
\]

**Theorem 6.14.** Let \( (\mathcal{M}^{(1), rec}, \mathcal{M}^{(2), rec}, \Phi^{rec}) \) be an admissible triple and \( \mathcal{O}(i) = B^{(i)}(q_i, R) \subset \mathcal{M}^{(i), rec}, i = 1, 2, \) be a ball centered at \( q_i \) and radius \( R \) satisfying \( 6.22 \) and \( 6.23 \). Then \( \tau^{(1)}(q_1) = \tau^{(2)}(q_2) \). Using the notation \( \tau = \tau^{(1)}(q_1) \), then, for

\[
\tilde{\mathcal{M}}^{(i), rec} = \mathcal{M}^{(i), rec} \cup \mathcal{M}^{(i)}(\partial \mathcal{O}(i), \tau - R), \quad i = 1, 2,
\]

there is a map \( \tilde{\Phi}^{rec} : \tilde{\mathcal{M}}^{(1), rec} \rightarrow \tilde{\mathcal{M}}^{(2), rec} \) which is an extension of \( \Phi^{rec} \). Moreover, the triple \( (\tilde{\mathcal{M}}^{(1), rec}, \mathcal{M}^{(2), rec}, \tilde{\Phi}^{rec}) \) is admissible.

Note that \( B^{(i)}(q_i, \tau) = \mathcal{M}^{(i)}(\partial \mathcal{O}(i), \tau - R) \cup B^{(i)}(q_i, R) \).

**Proof.** Assume opposite to the claim that we would have \( \tau^{(1)}(q_1) > \tau^{(2)}(q_2) \). Let

\[
a = \tau^{(1)}(q_1) - R, \quad b = \tau^{(2)}(q_2) - R, \quad 0 < T < b < a.
\]

Recall that \( \mathcal{N}^{(i)} = B^{(i)}(q^{(i)}, T + R) \setminus O^{(i)} \).

Let us extend the manifolds \( \mathcal{M}^{(i)}(\partial \mathcal{O}(i), T) \subset B^{(i)}(q^{(i)}, T + R) \). First, observe that as \( T + R < \tau^{(1)}(q_1) \), \( B^{(i)}(q^{(i)}, T + R) \) is a diffeomorphic to a ball of an Euclidean space.

On the surfaces \( \Gamma^{(i)} = \partial B^{(i)}(q^{(i)}, R) \) the boundary distance function \( \tau_{\Gamma^{(i)}}^{(i)} \in C(\Gamma^{(i)}) \) corresponding to the point \( p_i \in \mathcal{N}^{(i)} \) is

\[
\tau_{\Gamma^{(i)}}^{(i)}(x) = d_{\mathcal{N}^{(i)}}(x, p_i), \quad \text{for } x \in \Gamma^{(i)}.
\]
By (6.40), we have for all \( p_1 \in \mathcal{N}^{(1)} \) and \( p_2 = \Phi^{rec}(p_1) \)
\[
(6.47) \quad r^{(2)}_{p_2}(\Phi^{rec}(x)) = r^{(1)}_{p_1}(x), \quad \text{for } x \in \Gamma^{(1)}.
\]
For \( x \in \Gamma^{(i)} \) and \( p \in \mathcal{N}^{(i)} \), denote
\[
r^{(i),T}_{p}(x) = \max(r^{(i)}_{p}(x), T)
\]
and let \( R^{T}_{\Gamma^{(i)}} : \mathcal{N}^{(i)} \to C(\Gamma^{(i)}) \) be the map
\[
R^{T}_{\Gamma^{(i)}}(p) = r^{(i),T}_{p}(\cdot).
\]
These and related boundary distance functions have been considered in [24, 25, 29, 60, 68]. Naturally, the functions \( r^{(i)}_{p} \in C(\Gamma^{(i)}) \), \( p \in \mathcal{N}^{(i)} \) determine the range of \( R^{T}_{\Gamma^{(i)}} \), that is, the family functions
\[
R^{T}_{\Gamma^{(i)}}(\mathcal{N}^{(i)}) = \{ r^{(i),T}_{p} \in C(\Gamma^{(i)}) : p \in \mathcal{N}^{(i)} \}.
\]
By (6.41), Subsection 4.2.9, the family \( R^{T}_{\Gamma^{(i)}}(\mathcal{N}^{(i)}) \) of functions determine the topological and differentiable type of the manifold \( \mathcal{N}^{(i)} \) and the isometry type of the Riemannian manifold \( (\mathcal{N}^{(i)}, g^{(i)}|_{\mathcal{N}^{(i)}}) \). Moreover, when we identify the sets \( \Gamma^{(1)} \) and \( \Gamma^{(2)} \) using the map \( \Phi^{rec} \) and denote by \( \nu \), the unit exterior normal vector of \( \Gamma^{(i)} \) and by \( E_{i} : \Gamma^{(i)} \times [0, T) \to \mathcal{N}^{(i)} \) the normal exponential map,
\[
E_{i}(x,t) = \gamma_{x,\nu(x)}(s),
\]
we see that when \( J(x,s) = (\Phi^{rec}(x), s) \) maps \( J : \Gamma^{(1)} \times [0, T) \to \Gamma^{(2)} \times [0, T) \), then the \textit{“collar”} map
\[
(6.48) \quad \tilde{\Phi}_{c} = E_{2} \circ J \circ E_{1}^{-1} : \mathcal{N}^{(1)} \to \mathcal{N}^{(2)}
\]
is a diffeomorphism and an isometry.

For the convenience of the reader, let us sketch the idea of the above cited construction in [64], Subsection 4.2.9. There, the map \( \tilde{\Phi}_{c} : \mathcal{N}^{(1)} \to \mathcal{N}^{(2)} \) gives a diffeomorphism that can be used to identify the sets \( \mathcal{N}^{(1)} \) and \( \mathcal{N}^{(2)} \) and their differentiable structures. Moreover, the gradients of the distance functions,
\[
\nabla_{p}(r^{(i),T}_{p}(x_{i})) = \nabla_{p}(g^{(i)}(p, x_{i})) \in T_{p}\mathcal{M}^{(i)}
\]
are unit length vectors. When \( x_{i} \) moves on the boundary near the closest point to \( p \), these unit vectors run over an open subset of the unit sphere in \( T_{p}\mathcal{M}^{(i)} \). As the differentials of the distance functions satisfy (6.40), we see that
\[
(6.49) \quad (\Phi^{rec})_{*}(\nabla_{p}r^{(1),T}_{p}(x_{i})) = \nabla_{q}r^{(2),T}_{q}(\Phi^{rec}(x_{i})) \bigg|_{q=\Phi^{rec}(p)}.
\]
This implies that the linear map \( (\Phi^{rec})_{*} \), in \( T_{p}\mathcal{M}^{(1)} \) maps an open subset of unit vectors in \( T_{p}\mathcal{M}^{(1)} \) to unit vectors in \( T_{p}\mathcal{M}^{(2)} \), implying that \((\tilde{\Phi}_{c})^{*} g^{(2)} = g^{(1)} \). This implies that \( \tilde{\Phi}_{c} : \mathcal{N}^{(1)} \to \mathcal{N}^{(2)} \) is an isometry.

Now we return to the proof of the claim. As above \( T < b \) is arbitrary, the fact that the map (6.48) is an isometry implies that \( \mathcal{M}^{(1)}(\partial\mathcal{O}^{(1)}, b) \) and \( \mathcal{M}^{(2)}(\partial\mathcal{O}^{(2)}, b) \) are isometric. We can extend this isometry to the sets \( \mathcal{O}^{(1)} \) and \( \mathcal{O}^{(2)} \), and see there is an isometry
\[
(6.50) \quad \tilde{\Phi} : B^{(1)}(q^{(1)}, b + R) \to B(q^{(2)}, b + R),
\]
where we recall that \( B^{(1)}(q, r) \subset \mathcal{M}^{(i)} \) denote the balls. Let \( b' < b \) be so small that \( B^{(1)}(q^{(i)}, b' + R) \subset \mathcal{M}^{(i), rec} \). By the assumptions we made in the claim,

\[
\tilde{\Phi}(x) = \Phi^{rec}(x), \quad x \in B^{(1)}(q^{(1)}, b' + R),
\]

and using Proposition 6.13 with roles of \( i \) and \( \tau \) above, yield the claim in the case (6.54).

By Proposition 6.13, the time domain Green’s functions \( G^{(i)}(x, y, t), i = 1, 2, \) satisfy the relation

\[
G^{(2)}(\tilde{\Phi}(x), \tilde{\Phi}(y), t) = G^{(1)}(x, y, t),
\]

in \( \{(x, y, t) \in B^{(1)}(q^{(1)}, b' + R)^2 \times \mathbb{R}; \ x \neq y\} \). Similarly, to the proof of Lemma 6.18, we use Tataru’s unique continuation first in the \( x \) and \( t \) variables to see that (6.52) is valid in \( \{(x, y, t) \in B^{(1)}(q^{(1)}, b + R) \times B^{(1)}(q^{(1)}, b' + R) \times \mathbb{R}; \ x \neq y\} \). As \( G^{(1)}(x, y, t) = G^{(1)}(y, x, t) \), we can then use Tataru’s unique continuation in the \( y \) and \( t \) variables to see that (6.52) is valid in \( \{(x, y, t) \in B^{(1)}(q^{(1)}, b + R) \times \mathbb{R}; \ x \neq y\} \). Thus the triple \( (B^{(1)}(q^{(1)}, b + R), B^{(2)}(q^{(2)}, b + R), \tilde{\Phi}) \) is admissible.

Using (6.30) and (6.52), it follows from Remark 6.10 that \( \Phi^{rec} \) can be extended by \( \tilde{\Phi} \) as \( \Phi^{rec} \),

\[
\tilde{\Phi}^{rec} : \bar{\mathcal{M}}^{(1), rec} \rightarrow \bar{\mathcal{M}}^{(2), rec};
\]

\[
\bar{\mathcal{M}}^{(i), rec} = \mathcal{M}^{(i), rec} \cup B^{(i)}(q^{(i)}, b + R)
\]

Now, consider the case when

\[
\tau^{(i)}_{cut}(q_1) \leq \tau^{(i)}_{sing}(q_1)
\]

for both \( i = 1, 2 \). Then we see using Proposition 6.13 that \( \tau^{(1)}_{cut}(q_1) \leq \tau^{(2)}_{cut}(q_2) \), and using Proposition 6.13 with roles of \( i = 1 \) and \( i = 2 \) exchanged that \( \tau^{(1)}_{cut}(q_1) \geq \tau^{(2)}_{cut}(q_2) \). Hence, we have \( \tau^{(1)}(q_1) = \tau^{(2)}(q_2) \). Then, applying Corollary 6.11 with all \( T < \tau^{(1)}(q_1) = \tau^{(2)}(q_2) \) and applying results of (6.4), Subsection 4.2.9 as described above, yield the claim in the case (6.51).

Recall that, by our assumption, \( \tau^{(1)}(q_1) > \tau^{(2)}(q_2) \), that is, \( a > b \). Due to (6.23), this implies that \( B^{(1)}(q^{(1)}, b + R) \cap \partial B^{(1)}(q^{(1)}, b + R) \neq \emptyset \). Thus it remains to consider the case when

\[
\tau^{(2)}(q_2) = d^{(2)}(q_2, \mathcal{M}^{(i), sing}) < \tau^{(1)}(q_1) \leq d^{(1)}(q_1, \mathcal{M}^{(1), sing})
\]

Next we show that this is not possible.

As the mapping (6.30) is an isometry between \( B^{(1)}(q^{(1)}, b + R) \) and \( B^{(2)}(q^{(2)}, b + R) \), we see that,

\[
\min(\tau^{(1)}(q^{(1)}), b + R) = \min(\tau^{(2)}(q^{(2)}), b + R).
\]

Next, any point \( p^{(i)} \) in \( \bar{B}^{(i)}(q^{(i)}, b + R) \) can be written in the form \( p^{(i)} = \gamma_{q^{(i)}, \xi^{(i)}}(t) \) where \( \xi \) is a unit vector and \( t \leq \min(\tau^{(1)}(q^{(1)}), b + R) \).

Let

\[
p^{(2)} \in \mathcal{M}^{(2), sing} \cap \partial B^{(2)}(q^{(2)}, b + R).
\]

By the above, there is a point \( \xi^{(2)} \in S_{q^{(2)}}, \mathcal{M}^{(2)} \) such that \( p^{(2)} = \gamma_{q^{(2)}, \xi^{(2)}}(b + R) \). Let \( \xi^{(1)} = d(\Phi^{rec})^{-1}(\xi^{(2)}) \) and consider \( p^{(1)} = \gamma_{q^{(1)}, \xi^{(1)}}(b + R) \). Since \( \tau^{(1)}(q_1) > \tau^{(2)}(q_2) \), that is, \( a > b \), we have \( p^{(1)} \notin \mathcal{M}^{(1), sing} \).
Let 
\[ p_{i}^{(i)} := \gamma_{\xi^{(i)}} (b + R - 2\varepsilon), \quad \varepsilon > 0, \ i = 1, 2. \]

We denote by \( \tilde{\Omega}_{\varepsilon}^{(i)} = B^{(i)}(p_{\varepsilon}^{(i)}, \varepsilon) \) the metric ball in \( M^{(i)} \) of radius \( \varepsilon \). By using (6.58) and choosing \( \varepsilon > 0 \) to be small, \( \tilde{\Omega}_{\varepsilon}^{(i)} \) satisfy the conditions of Lemma 6.7 with \( \tilde{\Omega}_{\varepsilon}^{(i)} \) instead of \( \tilde{O}^{(i)} \).

Then, Lemma 6.7 implies that 
\[(6.57) \quad \text{Source-to-solution operators } V_{\tilde{\Omega}^{(i)},+}^{2T} \text{ for } -\Delta^{(i)} \text{ on } \tilde{\Omega}_{\varepsilon}^{(i)}, \ i = 1, 2, \text{ are } \Phi^{reg}-related. \]

Here \(-\Delta^{(i)}\) is the Laplace operator associated with \( M^{(i)} \). Equation (6.57) together with Lemma 6.3 imply that 
\[
\text{Vol}^{(1)}(\mathcal{M}^{(1)}(\tilde{\Omega}_{\varepsilon}^{(1)}, r - \varepsilon)) = \text{Vol}^{(2)}(\mathcal{M}^{(2)}(\tilde{\Omega}_{\varepsilon}^{(2)}, r - \varepsilon))
\]
when \( r > 0 \). Therefore, 
\[(6.58) \quad \text{Vol}^{(1)}(B^{(1)}(p_{\varepsilon}^{(1)}, r)) = \text{Vol}^{(2)}(B^{(2)}(p_{\varepsilon}^{(2)}, r)). \]

Next, we observe that as \( d^{(i)}(p_{\varepsilon}^{(i)}, p^{(i)}) \leq 2\varepsilon \), we have 
\[
B^{(i)}(p^{(i)}, r - 2\varepsilon) \subset B^{(i)}(p^{(i)}, r) \subset B^{(i)}(p^{(i)}, r + 2\varepsilon), \quad \text{for } r > 2\varepsilon.
\]

Thus, by the continuity of the volume, 
\[
\text{Vol}^{(i)}(B^{(i)}(p^{(i)}, r)) = \lim_{\varepsilon \to 0} \text{Vol}^{(i)}(B^{(i)}(p_{\varepsilon}^{(i)}, r)).
\]

Together with (6.58), this implies that, for \( r > 0 \), 
\[(6.59) \quad \text{Vol}^{(1)}(B^{(1)}(p^{(1)}, r)) = \text{Vol}^{(2)}(B^{(2)}(p^{(2)}, r)). \]

Let us now consider the conic coordinates of \( M^{(i)} \) and the volume factor, see (6.11) and (6.2). It then follows from (6.59), that 
\[
\Lambda^{(i)} = \lim_{r \to 0} \frac{1}{\text{Vol}_{\mathbb{R}^{n}}(B_{\mathbb{R}^{n}}(0, r))} \left[ \text{Vol}^{(i)}(B^{(i)}(p^{(i)}, r)) \right]
\]
and thus we have 
\[(6.60) \quad \Lambda^{(1)} = \Lambda^{(2)}. \]

Denote \( \Lambda^{(1)} = \Lambda(p^{(i)}) \).

Note that, if \( p^{(i)} \in \mathcal{M}^{(i)}_{\text{sing}} \) we have \( \Lambda^{(i)} \neq 1 \) and if \( p^{(i)} \in \mathcal{M}^{(i)}_{\text{reg}} \) then \( \Lambda^{(i)} = 1 \).

As we assumed that \( a = \tau^{(1)}(q_{1}) - R > b = \tau^{(2)}(q_{2}) - R \), we have \( p^{(1)} \in \mathcal{M}^{(1)}_{\text{reg}} \) and thus \( \Lambda^{(1)} = 1 \). Hence, we also have \( \Lambda^{(2)} = 1 \), and thus \( p^{(2)} \in \mathcal{M}^{(2)}_{\text{reg}} \), contradicting (6.60). Thus the above assumption \( \tau^{(1)}(q_{1}) > \tau^{(2)}(q_{2}) \) led to a contradiction. By changing roles of indexes 1 and 2, the above considerations show that we must have \( \tau^{(1)}(q_{1}) = \tau^{(2)}(q_{2}) \).

Thus using Corollary 6.11 with all \( T < \tau^{(1)}(q_{1}) = \tau^{(2)}(q_{2}) \) and applying results of [64], Subsection 4.2.9 we prove the claim of the theorem. □

Let \( \mathcal{A} \) be the collection of admissible triples \((\mathcal{W}^{(1)}, \mathcal{W}^{(2)}, \Phi)\) such that \( W \subset \mathcal{W}^{(i)} \) and \( \mathcal{W}^{(i)} \subset \mathcal{M}^{(i)}_{\text{reg}} \) are connected open sets for both indexes \( i = 1, 2 \). We define
a partial order on \( \mathcal{A} \) by setting \((W_1, W_2, \Phi) \leq (\tilde{W}_1, \tilde{W}_2, \tilde{\Phi})\) if \(W_1 \subseteq \tilde{W}_1\) and \(\Phi = \tilde{\Phi}|_{\tilde{W}_1}\).

Note that, by Remark 6.6 if \((W_1, W_2, \Phi)\) and \((\tilde{W}_1, \tilde{W}_2, \tilde{\Phi})\) are admissible triples, then the extended triple, \((W_1, ex, W_2, ex, \Phi, ex)\), where

\[
W_1, ex = W_1 \cup \tilde{W}_1,
\]

\[
\Phi, ex|_{W_1} = \Phi,
\]

is also an admissible triple. Therefore, by Zorn’s lemma, there exists a maximal element \((W_m^{(1)}, W_m^{(2)}, \Phi_m) \in \mathcal{A}\).

**Lemma 6.15.** The maximal element \((W_m^{(1)}, W_m^{(2)}, \Phi_m)\) of \(\mathcal{A}\) satisfies

\[
W_m^{(1)} = \mathcal{M}^{(1), reg}.
\]

**Proof.** If the claim is not true, there exists \(x_0^{(1)} \in \mathcal{M}^{(1), reg} \cap \partial W_m^{(1)}\). Let \(\mu([0, 1])\) be a smooth path from \(\mu(0) = z \in W\) to \(\mu(1) = x_0^{(1)}\), such that

\[
\mu([0, 1]) \subset \mathcal{M}^{(1), reg}.
\]

Then \(d_0 = d^{(1)}(\mu, \mathcal{M}^{(1), sing}) > 0\). Let \(c = \frac{d_0}{2}\). We can cover \(\mu([0, 1])\) by a finite number of balls \(B_j^{(1)} = B^{(1)}(x_j^{(1)}, c/2) \subset \mathcal{M}^{(1), reg}\) so that

\[
B_j^{(1)} \subset W_m^{(1)}, \ x_j^{(1)} \in B_j^{(1)},
\]

where we order them so that \(x_0^{(1)} \in B_1^{(1)}\). Let \(O_1^{(1)} = B^{(1)}(x_1^{(1)}, R)\) be a small ball such that \(0 < R < c/2\) satisfies (6.22, 6.23), and \(O_1^{(1)} \subset W_m^{(1)}\). As \(d^{(1)}(x_1^{(1)}, \mathcal{M}^{(1), sing}) > \frac{d_0}{2}\), Theorem 6.13 yields that we can extend the admissible triple \((W_m^{(1)}, W_m^{(2)}, \Phi_m)\) onto

\[
\tilde{W}_m^{(i)} = W_m^{(i)} \cup B^{(i)}(x_i^{(1)}, c), \ x_i^{(2)} = \Phi_m(x_i^{(1)}).
\]

As \(x_0^{(1)} \in B(x_1^{(1)}, c)\) and \(x_0^{(1)} \in \partial W_m^{(1)}\), this contradicts the fact that \((W_m^{(1)}, W_m^{(2)}, \Phi_m)\) is a maximal element of \(\mathcal{A}\), which completes the proof of (6.61). □

Lemma 6.13 proves that there is a diffeomorphism

\[
\Phi_m : \mathcal{M}^{(1), reg} \to W_m^{(2)}, \ W_m^{(2)} = \Phi_m(\mathcal{M}^{(1), reg}) \subset \mathcal{M}^{(2), reg},
\]

which is a Riemannian isometry. Changing the role of indexes 1 and 2, we see that there is also a diffeomorphism

\[
\tilde{\Phi}_m : \mathcal{M}^{(2), reg} \to \tilde{W}_m^{(1)}, \ \tilde{W}_m^{(1)} \subset \mathcal{M}^{(1), reg}
\]

which is a Riemannian isometry. Moreover, using Lemma 6.4 we see that \(\tilde{\Phi}_m\) and \(\Phi_m\) coincide with the identity map on \(W\).

Using (6.19) we see that for all \(z \in \mathbb{C} \setminus \mathbb{R}, x \in \mathcal{M}^{(2), reg}\) and \(y \in W\).

\[
G^{(1)}(z; \Phi_m(\tilde{\Phi}_m(x)), y) = G^{(2)}(z; x, y).\]

By Lemma 6.5, this implies that \(\Phi_m(\tilde{\Phi}_m(x)) = x\), that is, \(\Phi_m \circ \tilde{\Phi}_m = I\) on \(\mathcal{M}^{(1), reg}\). Similarly, we see that \(\tilde{\Phi}_m \circ \Phi_m = I\) on \(\mathcal{M}^{(2), reg}\) and hence

\[
W_m^{(2)} = \mathcal{M}^{(2), reg}, \ W_m^{(1)} = \mathcal{M}^{(1), reg}, \ \text{and } \tilde{\Phi}_m = \Phi_m^{-1}.
\]
Summarizing, we have shown that
\[ \Phi_m : (\mathcal{M}^{(1)}_{reg}, g^{(1)}) \rightarrow (\mathcal{M}^{(2)}_{reg}, g^{(2)}) , \]
is a diffeomorphism and an isometry.

Skipping again the superscript \(^{(i)}\), recall that by Lemma 2.3
\[ (6.63) \quad d_{\mathcal{M}}(x, y) = d_{\mathcal{M}_{reg}}(x, y), \quad \text{for any } x, y \in \mathcal{M}_{reg}, \]
where \( d_{\mathcal{M}_{reg}} \) is the distance on \((\mathcal{M}_{reg}, g)\), defined as the infimum of the length of rectifiable paths connecting \( x \) to \( y \).

The identity (6.63) implies that \((\mathcal{M}, d_{\mathcal{M}_{reg}})\), considered as a metric space, is isometric to the completion of the metric space \((\mathcal{M}_{reg}, d_{reg})\). Thus, we can uniquely extend \( \Phi_m \) to a metric isometry
\[ (6.64) \quad \Phi : (\mathcal{M}^{(1)}, d^{(1)}) \rightarrow (\mathcal{M}^{(2)}, d^{(2)}). \]
Again, taking into account that \( \mathcal{M}^{(1)}_{reg} \) is mapped to \( \mathcal{M}^{(2)}_{reg} \) we see that \( \Phi \) maps also singular points to singular points.

These prove conditions (1)–(3) of Theorem 6.1 \( \square \)
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