LAGRANGIAN SUBVARIETIES IN THE CHOW RING OF SOME HYPERKÄHLER VARIETIES

ROBERT LATERVEER

ABSTRACT. Let $X$ be a hyperkähler variety, and let $Z \subset X$ be a Lagrangian subvariety. Conjecturally, $Z$ should have trivial intersection with certain parts of the Chow ring of $X$. We prove this conjecture for certain Hilbert schemes $X$ having a Lagrangian fibration, and $Z \subset X$ a general fibre of the Lagrangian fibration.

1. INTRODUCTION

For a smooth projective variety $X$ over $\mathbb{C}$, let $A^i(X) := CH^i(X)_{\mathbb{Q}}$ denote the Chow groups (i.e. the groups of codimension $i$ algebraic cycles on $X$ with $\mathbb{Q}$-coefficients, modulo rational equivalence).

The world of Chow groups is a huge building site that is still under construction, with many unfinished parts that only exist “in pencil”, i.e. dependent on conjectures [7], [23], [24], [25], [36], [53], [37]. In this building site, one place of particular interest is occupied by hyperkähler varieties (i.e. projective irreducible holomorphic symplectic manifolds [2], [1]). Here, recent years have seen an intense activity of new constructions and significant progress in the understanding of Chow groups [5], [50], [56], [48], [44], [47], [40], [41], [13], [14], [31], [32], [16]. Much of this progress has centered around the following conjecture:

**Conjecture 1.1** (Beauville, Voisin [5], [48]). Let $X$ be a hyperkähler variety. Let $D^*(X) \subset A^*(X)$ denote the $\mathbb{Q}$–subalgebra generated by divisors and Chern classes of $X$. Then the cycle class maps induce injections

$$D^i(X) \hookrightarrow H^{2i}(X, \mathbb{Q}) \quad \forall i.$$  

(cf. [5], [48], [3], [10], [41], [13], [57] for cases where conjecture 1.1 is satisfied.)

The “motivation” underlying conjecture 1.1 is that for a hyperkähler variety $X$, the Chow ring $A^*(X)$ is expected to have a bigrading $A^*_i(X)$, where the piece $A^*_{i,j}(X)$ corresponds to the graded $Gr^F_{i-j}A^*(X)$ for the conjectural Bloch–Beilinson filtration. In particular, it is expected that the subring $A^*_{i,0}(X)$ injects into cohomology, and that $D^*(X) \subset A^*_{i,0}(X)$.

In addition to divisors and Chern classes, what other cycles should be in the subring $A^*_{i,0}(X)$ (assuming this subring exists)? A conjecture of Voisin provides more candidate members:
\section*{Conjecture 1.2 (Voisin \cite{56}).} Let $X$ be a hyperkähler variety of dimension $n = 2m$. Let $Z \subset X$ be a codimension $i$ subvariety swept out by $i$–dimensional constant cycle subvarieties. There exists a subring $A^i_0(X) \subset A^i(X)$ injecting into cohomology, and

$$Z \in A^i_0(X).$$

A constant cycle subvariety is by definition a closed subvariety $T \subset X$ such that the image of the natural map $A_0(T) \to A^i(X)$ has dimension 1. In particular, conjecture \cite{12} stipulates that Lagrangian constant cycle subvarieties (i.e., constant cycle subvarieties of dimension $m$) should lie in $A^m_0(X)$.

Another conjecture concerns the behaviour of Lagrangian subvarieties (i.e. $m$–dimensional subvarieties $Z \subset X$ such that the symplectic form of $X$ restricts to 0 on the regular part of $Z$) with respect to the intersection product. The Lagrangian condition implies that

$$\cup Z : H^{2,0}(X) \to H^{m+2,m}(X)$$

is the zero map. Since $H^{*,0}(X)$ is generated by $H^{2,0}(X)$, we have that

$$\cup Z : H^{j,0}(X) \to H^{m+j,m}(X)$$

is the zero map for all $j > 0$. Since conjecturally, the piece $A^j_0(X)$ is determined by $H^{j,0}(X)$, and the piece $A^m_0(X)$ is determined by $H^{2m+j}(X)$, we arrive at the following conjecture:

\section*{Conjecture 1.3.} Let $X$ be a hyperkähler variety of dimension $2m$. Let $Z \subset X$ be a Lagrangian subvariety. Then the maps

$$A^j_0(X) \xrightarrow{\cup Z} A^{m+j}(X) \to A^{m+j}_0(X),$$

$$A^m_0(X) \xrightarrow{\cup Z} A^{2m}(X) \to A^{2m}_0(X),$$

are zero for all $j > 0$. (Here, the right arrows are projection to the piece $A^i_0(X)$.)

The goal of this note is to provide some examples where conjectures \cite{12} and \cite{13} are satisfied, by looking at Hilbert schemes of $K$3 surfaces. Here, thanks to work of Vial \cite{47} and of Shen–Vial \cite{44}, the Chow groups of $X$ split in a finite number of pieces $A^i_0(X)$.

The first series of examples consists of Hilbert squares $X = S^{[2]}$, where $S$ is a general $K$3 surface of genus $g$. If the integer $g$ satisfies $2g - 2 = 2m^2$ for some integer $m \geq 2$, the Hilbert square $X$ admits a Lagrangian fibration $\phi : X \to \mathbb{P}^2$ \cite{19} (cf. subsection \cite{27}). The general fibre $A$ of $\phi$ is Lagrangian; it thus makes sense to ask whether conjecture \cite{13} is true for $A \subset X$. We give an answer for the first two values of $g$:

\footnote{NB: we will reserve the notation $A^i_0()$ for the bigrading that is constructed unconditionally in \cite{44}, \cite{47} for certain hyperkähler varieties. The notation $A^i_0()$, that occurs only in this introduction, refers to a \textit{conjectural} bigrading with the property that $A^i_0(X)$ is related to the graded $\text{Gr}_F A^i(X)$ for the conjectural Bloch–Beilinson filtration. In short, the (unconditionally existing) bigrading $A^i_0()$ is a candidate for the (only ideally existing) bigrading $A^i_0()$.}
Theorem (=theorem 4.2 and corollary 4.3). Let $X = S^{[2]}$, where $S$ is a general $K3$ surface of genus $g = 5$ or $g = 10$. Let $A \subset X$ be a general fibre of the Lagrangian fibration $\phi$. Then $A \in A^{2}_{hom}(X)$ and

$A \cdot \cdot : A^{2}_{hom}(X) \to A^{4}(X)$

is the zero map.

(In particular, let $b \in A^{4}(X)$ be a 0–cycle of the form $b = A \cdot c$ with $c \in A^{2}(X)$. Then $b$ is rationally trivial if and only if $b$ has degree 0.)

(The $g = 5$ case of theorem 4.2 was already done in [29].)

The second series of examples consists of Hilbert cubes $X = S^{[3]}$, where $S$ is a general $K3$ surface of genus $g$. For $g = 9$, the Hilbert cube $X$ admits a Lagrangian fibration $\phi : X \to P^{3}$ [21] (cf. subsection 2.7). We establish a weak version of conjecture 1.3 for this case:

Theorem (=theorem 4.4 and corollary 4.5). Let $X = S^{[3]}$, where $S$ is a general $K3$ surface of genus 9. Let $A \subset X$ be a general fibre of the Lagrangian fibration $\phi$. Then $A \in A^{3}_{(0)}(X)$ and

$A^{2}_{hom}(X) \to A^{5}(X) \to A^{6}(X)$

is the zero map, for any divisor $D \in A^{1}(X)$.

(In particular, let $b \in A^{6}(X)$ be a 0–cycle of the form $b = A \cdot D \cdot c$, where $D \in A^{1}(X)$ and $c \in A^{2}(X)$. Then $b$ is rationally trivial if and only if $b$ has degree 0.)

Theorems 4.2 and 4.4 are deduced from a more general statement (theorem 4.1). Roughly speaking, this general statement says that if a subvariety $Z$ of $X$ exists relatively (i.e. there exists a subvariety $Z$ in the family $X \to B$ of all Hilbert schemes of $K3$ surfaces of fixed genus $g \leq 10$, such that $Z$ is the restriction of $Z$ to the fibre $X$), then the behaviour of $Z$ in the cohomology ring of $X$ can be translated into consequences about the behaviour of $Z$ in the Chow ring of $X$. This type of statement, highlighting the distinguished behaviour of cycles that exist relatively, is a typical feature of the technique of “spread” of algebraic cycles as developed by Voisin [51], [54], [52], [53], [55], which we employ to prove theorem 4.1.

One ingredient in the proof that may be of independent interest is a “hard Lefschetz” type of statement for certain pieces of the Chow groups of Hilbert schemes:

Theorem (=corollary 3.4). Let $S$ be a $K3$ surface of genus $g \leq 10$, and let $X = S^{[m]}$ be the Hilbert scheme of length $m$ subschemes of $S$. There exists an ample line bundle $L$ on $X$ such that

$\cdot L^{m-1} : A^{2}_{(2)}(X) \to A^{2m}_{(2)}(X)$

is an isomorphism.

This is also proven using the method of “spread”. It would be interesting to prove the results of this note for other hyperkähler varieties. Unfortunately, our method runs into problems for Hilbert schemes of high genus $K3$ surfaces (this is due to the lack of Mukai models for high genus $K3$ surfaces).

\(^2\text{NB: after the present paper was written, the paper [15] appeared, which explores closely related questions. Both the present paper and [15] are inspired by [39].}\)
Conventions. In this article, the word variety will refer to a reduced irreducible scheme of finite type over \(\mathbb{C}\). A subvariety is a (possibly reducible) reduced subscheme which is equidimensional.

All Chow groups will be with rational coefficients: we will always write \(A_j(X)\) for the Chow group of \(j\)-dimensional cycles on \(X\) with \(\mathbb{Q}\)-coefficients; for \(X\) smooth of dimension \(n\) the notations \(A_j(X)\) and \(A_{n-j}(X)\) are used interchangeably.

The notations \(A_j^\text{hom}(X)\), \(A_j^\text{AJ}(X)\) will be used to indicate the subgroups of homologically trivial, resp. Abel–Jacobi trivial cycles. For a morphism \(f : X \to Y\), we will write \(\Gamma_f \in A_*(X \times Y)\) for the graph of \(f\). The contravariant category of Chow motives (i.e., pure motives with respect to rational equivalence as in [43], [37]) will be denoted \(\mathcal{M}_{\text{rat}}\).

We will use \(H^j(X)\) to indicate singular cohomology \(H^j(X, \mathbb{Q})\).

2. Preliminaries

2.1. Quotient varieties.

**Definition 2.1.** A projective quotient variety is a variety
\[
X = Y/G,
\]
where \(Y\) is a smooth projective variety and \(G \subset \text{Aut}(Y)\) is a finite group.

**Proposition 2.2** (Fulton [17]). Let \(X\) be a projective quotient variety of dimension \(n\). Let \(A^*(X)\) denote the operational Chow cohomology ring. The natural map
\[
A^i(X) \to A_{n-i}(X)
\]
is an isomorphism for all \(i\).

**Proof.** This is [17, Example 17.4.10].

**Remark 2.3.** It follows from proposition 2.2 that the formalism of correspondences goes through unchanged for projective quotient varieties (this is also noted in [17, Example 16.1.13]). We can thus consider motives \((X, p, 0) \in \mathcal{M}_{\text{rat}}\), where \(X\) is a projective quotient variety and \(p \in A^n(X \times X)\) is a projector. For a projective quotient variety \(X = Y/G\), one readily proves (using Manin’s identity principle) that there is an isomorphism
\[
h(X) \cong h((Y)^G) := (Y, \Delta_Y^G, 0) \quad \text{in } \mathcal{M}_{\text{rat}},
\]
where \(\Delta_Y^G\) denotes the idempotent \(\frac{1}{|G|} \sum_{g \in G} \Gamma_g\).

2.2. The Fourier decomposition.

**Theorem 2.4** (Shen–Vial [44]). Let \(S\) be a \(K3\) surface, and let \(X = S^{[2]}\) be the Hilbert scheme of length 2 subschemes of \(S\). There is a decomposition
\[
A^i(X) = \bigoplus_{0 \leq j \leq \ell} A_{(j)}^i(X),
\]
with the following properties:
(i) \(A^*_*(X)\) is a bigraded ring;
(ii) \(A_{(j)}^i(X) \subset A^i_{\text{hom}}(X)\) for \(j > 0\).
Proof. This is essentially [44, Theorem 2], combined with the fact that there is a class \( L \in A^2(X \times X) \) lifting the Beauville–Bogomolov class and satisfying certain equalities, which is [44, Part 2]. □

2.3. MCK decomposition.

Definition 2.5 (Murre [36]). Let \( X \) be a smooth projective variety of dimension \( n \). We say that \( X \) has a CK decomposition if there exists a decomposition of the diagonal

\[
\Delta_X = \pi_0^X + \pi_1^X + \cdots + \pi_{2n}^X \quad \text{in} \quad A^n(X \times X),
\]

such that the \( \pi_i^X \) are mutually orthogonal idempotents in \( A^n(X \times X) \) and \( (\pi_i^X)_* H^i(X) = H^i(X) \).

(NB: “CK decomposition” is shorthand for “Chow–Künneth decomposition”.

Remark 2.6. The existence of a CK decomposition for any smooth projective variety is part of Murre’s conjectures [36, 23, 24].

Definition 2.7 (Shen–Vial [44]). Let \( X \) be a smooth projective variety of dimension \( n \). Let \( \Delta^e_{X} \in A^{2n}(X \times X \times X) \) be the class of the small diagonal

\[
\Delta^e_{X} = \{(x, x, x) \mid x \in X \} \subset X \times X \times X.
\]

An MCK decomposition is a CK decomposition \( \{\pi_i^X\} \) of \( X \) that is multiplicative, i.e. it satisfies

\[
\pi_k^X \circ \Delta^e_{X} \circ (\pi_i^X \times \pi_j^X) = 0 \quad \text{in} \quad A^{2n}(X \times X \times X) \quad \text{for all} \quad i + j \neq k.
\]

(NB: “MCK decomposition” is shorthand for “multiplicative Chow–Künneth decomposition”.

A weak MCK decomposition is a CK decomposition \( \{\pi_i^X\} \) of \( X \) that satisfies

\[
(\pi_k^X \circ \Delta^e_{X} \circ (\pi_i^X \times \pi_j^X))_*(a \times b) = 0 \quad \text{for all} \quad a, b \in A^*(X).
\]

Remark 2.8. The small diagonal (seen as a correspondence from \( X \times X \) to \( X \)) induces the multiplication morphism

\[
\Delta^e_{X} : \ h(X) \otimes h(X) \to h(X) \quad \text{in} \quad \mathcal{M}_{\text{rat}}.
\]

Suppose \( X \) has a CK decomposition

\[
h(X) = \bigoplus_{i=0}^{2n} h^i(X) \quad \text{in} \quad \mathcal{M}_{\text{rat}}.
\]

By definition, this decomposition is multiplicative if for any \( i, j \) the composition

\[
h^i(X) \otimes h^j(X) \to h(X) \otimes h(X) \xrightarrow{\Delta^e_{X}} h(X) \quad \text{in} \quad \mathcal{M}_{\text{rat}}
\]

factors through \( h^{i+j}(X) \).

If \( X \) has a weak MCK decomposition, then setting

\[
A_{(i)}^i(X) := (\pi_{2i-j}^X)_* A^i(X),
\]

one obtains a bigraded ring structure on the Chow ring: that is, the intersection product sends \( A_{(i)}^i(X) \otimes A'_{(j')}^{i'}(X) \) to \( A_{(i+j')}(X) \).
It is expected (but not proven!) that for any \( X \) with a weak MCK decomposition, one has

\[
A^j(X) \cong 0 \quad \text{for} \ j < 0 , \quad A^0(X) \cap A^i_{\text{hom}}(X) \cong 0 ;
\]

this is related to Murre’s conjectures B and D, which have been formulated for any CK decomposition [36].

The property of having an MCK decomposition is severely restrictive, and is closely related to Beauville’s “(weak) splitting property” [5]. For more ample discussion, and examples of varieties with an MCK decomposition, we refer to [44, Section 8], as well as [47, 45, 16, 30].

### 2.4. Relative MCK for \( S^m \) and for \( S^{(m)} \).

**Theorem 2.9** (Vial [47]). Let \( S \) be a projective K3 surface, and let \( X = S^m \) be the Hilbert scheme of length \( m \) subschemes of \( S \). Then \( X \) has a self–dual MCK decomposition \( \{ \Pi^X_i \} \).

In particular, \( A^*(X) = A^*(X) \) is a bigraded ring, where

\[
A^i(X) = \bigoplus_{j=2i-2n} A^j(X) ,
\]

and \( A^i_j(X) = 0 \) for \( j \) odd. In case \( m = 2 \), the bigrading \( A^*(X) \) coincides with the one given by the Fourier decomposition of theorem 2.4.

**Proof.** This is [47, Theorems 1 and 2]. The last statement is [44, Theorem 15.8], plus the fact that for \( m = 2 \) the MCK decomposition of [47] coincides with the one of [44]. \( \square \)

**Remark 2.10.** Let \( X \) be as in theorem 2.9 and suppose \( m = 2 \) (i.e. \( X = S^{[2]} \) is a hyperkähler fourfold). Then the bigrading \( A^*(X) \) of theorem 2.9 has an interesting alternative description in terms of a Fourier operator on Chow groups (theorem 2.4). For \( m > 2 \), there is no such “Fourier operator” description of the bigrading \( A^*(S^{[m]}) \); the bigrading is defined exclusively by an MCK decomposition.

Another point particular to \( m = 2 \) is that (thanks to [44]) we know that

\[
A^i_j(S^{[2]}) = 0 \quad \forall \ j < 0 .
\]

This vanishing statement is (conjecturally true but) open for \( S^{[m]} \) with \( m > 2 \).

**Notation 2.11.** Let \( S \to B \) be a family (i.e., a smooth projective morphism). For \( r \in \mathbb{N} \), we write \( S^{r/B} \) for the relative \( r \)-fold fibre product

\[
S^{r/B} := S \times_B S \times_B \cdots \times_B S
\]

(\( r \) copies of \( S \)).

**Proposition 2.12.** Let \( S \to B \) be a family of K3 surfaces. There exist relative correspondences

\[
\Pi_j^{S^m/B} \in A^{2m}(S^m/B \times S^m/B) \quad (j = 0, 2, 4, \ldots, 4m) ,
\]

such that for each \( b \in B \), the restriction

\[
\Pi_j^{S^m/B} \big|_{(S_b)^m} \in A^{2m}((S_b)^m \times (S_b)^m)
\]
defines a self–dual MCK decomposition for \((S_b)^m\).

**Proof.** On any \(K3\) surface \(S_b\), there is the distinguished 0–cycle \(o_{S_b}\) such that \(c_2(S_b) = 24o_{S_b}\) [3]. Let \(p_i : S^{m/B} \to S\), \(i = 1, \ldots, m\), denote the projections to the \(i\)th factor. Let \(T_{S/B}\) denote the relative tangent bundle. The assignment

\[
\Pi^S_0 := (p_1)^* \left( \frac{1}{24} c_2(T_{S/B}) \right) A^2(S \times_B S), \\
\Pi^S_1 := (p_2)^* \left( \frac{1}{24} c_2(T_{S/B}) \right) A^2(S \times_B S), \\
\Pi^S_2 := \Delta_S - \Pi^S_0 - \Pi^S_1
\]

defines (by restriction) an MCK decomposition for each fibre, i.e.

\[
\Pi^{S_b}_j := \Pi^S_j \mid_{S_b \times S_b} \in A^2(S_b \times S_b) \quad (j = 0, 2, 4)
\]
is an MCK decomposition for any \(b \in B\) [44, Example 8.17].

Next, we consider the \(m\)–fold relative fibre product \(S^{m/B}\). Let \(p_{i,j} : S^{2m/B} \to S^{2/B}\) denote projection to the \(i\)-th and \(j\)-th factor. We define

\[
\Pi^{S^{m/B}}_j := \sum_{k_1 + k_2 + \cdots + k_m = j} (p_{1, m+1})^* (\Pi^S_{k_1}) \cdot (p_{2, m+2})^* (\Pi^S_{k_2}) \cdots \cdot (p_{m, 2m})^* (\Pi^S_{k_m})
\]
\[
\in A^{2m}(S^{4m/B}), \quad (j = 0, 2, 4, \ldots, 4m).
\]

By construction, the restriction to each fibre induces an MCK decomposition (the “product MCK decomposition”)

\[
\Pi^{(S_b)^m}_j := \Pi^{S^{m/B}}_j \mid_{(S_b)^{2m}} = \sum_{k_1 + k_2 + \cdots + k_m = j} \Pi^{S_b}_{k_1} \times \Pi^{S_b}_{k_2} \times \cdots \times \Pi^{S_b}_{k_m} \in A^{2m}((S_b)^{4m}),
\]
\[
(j = 0, 2, 4, \ldots, 4m).
\]

\[\square\]

**Remark 2.13.** Let \(S \to B\) be a family of \(K3\) surfaces. Let

\[
S^{(m)} := S^{m/B} / \mathfrak{S}_m
\]
denote the associated family of \(m\)–fold symmetric products (here \(\mathfrak{S}_m\) denotes the symmetric group on \(m\) factors). The construction of the \(\Pi^{S^{m/B}}_j\) is \(\mathfrak{S}_m\)–invariant, and so it induces relative projectors

\[
\Pi^{S^{(m)}}_j \in A^{2m}(S^{(m)} \times_B S^{(m)}).
\]

**Proposition 2.14.** Let \(S \to B\) be a family of \(K3\) surfaces. There exist relative correspondences

\[
\Theta'_1, \ldots, \Theta'_m \in A^{2m}(S^{m/B} \times_B S), \quad \Xi'_1, \ldots, \Xi'_m \in A^2(S \times_B S^{m/B})
\]
such that for each $b \in B$, the composition

$$A^{2m}_{(2)}((S_b)^m) \xrightarrow{\langle (\Theta'_{1}|_{(S_b)^{m+1}}) \ast \ldots (\Theta'_{m}|_{(S_b)^{m+1}}) \rangle} A^2(S_b) \oplus \cdots \oplus A^2(S_b) \xrightarrow{\langle (\Xi'_{1} + \ldots + \Xi'_{m})|_{(S_b)^{m+1}} \rangle} A^{2m}((S_b)^m)$$

is the identity.

**Proof.** As before, let

$$p_{i,j} : S^{2m/B} \rightarrow S^{2/B} \quad (1 \leq i < j \leq 2m)$$

denote projection to the $i$-th and $j$-th factor, and let

$$p_i : S^{m/B} \rightarrow S \quad (1 \leq i \leq m)$$

denote projection to the $i$-th factor.

We now claim that for each $b \in B$, there is equality

$$\left(\prod_{1 \leq j \leq 2m-2} \left(\Gamma_{p_1} \circ \Pi_2^S \circ \Gamma_{p_1} \circ \left((p_{1,m+1})^* (\Delta_S) \cdot \prod_{2 \leq j \leq 2m \atop j \neq m+1} (p_j)^* c_2(T_{S/B})\right) \right)\right)_{(S_b)^{2m}}$$

$$+ \ldots + \left(\Gamma_{p_m} \circ \Pi_2^S \circ \Gamma_{p_m} \circ \left((p_{m,2m})^* (\Delta_S) \cdot \prod_{1 \leq j \leq 2m-1 \atop j \neq m} (p_j)^* c_2(T_{S/B})\right) \right)_{(S_b)^{2m}}$$

in $A^{2m}((S_b)^m \times (S_b)^m)$.

Indeed, using Lieberman’s lemma [17.16.1.1], we find that

$$\left(\Gamma_{p_1} \circ \Pi_2^S \circ \Gamma_{p_1}\right)_{(S_b)^{2m}} = \left(\Gamma_{p_{1,m+1}}(\Pi_2^S)\right)_{(S_b)^{2m}} = \left((p_{1,m+1})^* (\Pi_2^S)\right)_{(S_b)^{2m}},$$

$$\vdots$$

$$\left(\Gamma_{p_m} \circ \Pi_2^S \circ \Gamma_{p_m}\right)_{(S_b)^{2m}} = \left(\Gamma_{p_{m,2m}}(\Pi_2^S)\right)_{(S_b)^{2m}} = \left((p_{m,2m})^* (\Pi_2^S)\right)_{(S_b)^{2m}}.$$
denote the projection on the first $m$ and last $m$ factors. Writing out the definition of composition of correspondences, we find that

\[
\frac{1}{2^{4m-2}} \left( \Gamma_{p_1} \circ \Pi_2^S \circ \Gamma_{p_1} \circ \left( (p_{1,m+1})^* (\Delta S) \cdot \prod_{2 \leq j \leq 2m \atop j \neq m+1} (p_j)^* c_2(T_{S/B}) \right) \right)_{(S_b)^{2m}} =
\]

\[
\frac{1}{2^{4m-2}} \left( (p_{1,m+1})^* (\Pi_2^S) \right) \circ \left( (p_{1,m+1})^* (\Delta S) \cdot \prod_{2 \leq j \leq 2m \atop j \neq m+1} (p_j)^* c_2(T_{S_b}) \right) =
\]

\[
P \left( \left( \Delta S_b \right)_{(1,m+1)} \times S_b \times \cdots \times S_b \right) =
\]

\[
\left( S_b \times \cdots \times S_b \times (\Pi_2^S)_{(m+1,2m+1)} \times S_b \times \cdots \times S_b \right) =
\]

\[
P \left( \left( \Delta S_b \times S_b \right) \cdot S_b \times \cdots \times S_b \right) \times S_b \times \cdots \times S_b \times S_b \times \cdots \times S_b =
\]

\[
\Pi_2^S \times \Pi_4^S \times \cdots \times \Pi_4^S \in A^{2m}((S_b)^{m} \times (S_b)^{m}).
\]

(Here, we use the notation $(C)_{(i,j)}$ to indicate that the cycle $C$ lies in the $i$th and $j$th factor, and likewise for $(D)_{(i,j,k)}$.)

Doing the same for the other summands in $(\Pi)$, one convinces oneself that both sides of $(\Pi)$ are equal to the fibrewise product Chow–Künneth component

\[
\Pi_{4m-2}^{(S_b)^m} = \Pi_2^S \times \Pi_4^S \times \cdots \times \Pi_4^S + \cdots + \Pi_4^S \times \cdots \times \Pi_4^S \times \Pi_2^S \in A^{2m}((S_b)^{m} \times (S_b)^{m}),
\]

thus proving the claim.

Let us now define

\[
\Theta'_i := \frac{1}{2^{4m-2}} \Gamma_{p_i} \circ \left( (p_{i,m+1})^* (\Delta S) \cdot \prod_{j \in [1,2m] \atop j \neq i,m+1} (p_j)^* c_2(T_{S/B}) \right) \in A^{2m}((S_b^m) \times B),
\]

\[
\Xi'_i := \Gamma_{p_i} \circ \Pi_2^S \in A^2(S \times B (S^m/B)),
\]

where $1 \leq i \leq m$. It follows from equation $(\Pi)$ that there is equality

\[
(\Xi'_i \circ \Theta'_1 + \cdots + \Xi'_m \circ \Theta'_m)_{(S_b)^{2m}} = \left( \Pi_{4m-2}^{(S_b)^m} \right)_* \colon
\]

\[
A^i_j((S_b)^m) \to A^i_j((S_b)^m) \quad \forall b \in B \quad \forall (i,j).
\]

Taking $(i,j) = (2m, 2)$, this proves the proposition. $\square$

The following is a version of proposition $2.14$ for the group $A^2_{(2)}((S_b)^m)$:

**Proposition 2.15.** Let $S \to B$ be a family of $K3$ surfaces. There exist relative correspondences

\[
\Theta_1, \ldots, \Theta_m \in A^{2m}(S \times_B (S^m/B)), \quad \Xi_1, \ldots, \Xi_m \in A^2((S^m/B) \times_B S)
\]
such that for each $b \in B$, the composition
\[
A^2((S_b)^m) \xrightarrow{(\Xi_1|_{(S_b)^{m+1}} \times \cdots \times \Xi_m|_{(S_b)^{m+1}})} A^2(S_b) \oplus \cdots \oplus A^2(S_b) \xrightarrow{((\Theta_1 + \cdots + \Theta_m)|_{(S_b)^{m+1}})} A^2((S_b)^m)
\]
is the identity.

**Proof.** One may take
\[
\Theta_i := \iota \Theta_i' \in A^{2m}(S \times_B (S^{m/B})),
\]
\[
\Xi_i := \iota \Xi_i' \in A^2((S^{m/B}) \times_B S) \quad (i = 1, \ldots, m).
\]
By construction, the product MCK decomposition $\{\Pi_i((S_b)^m)\}$ satisfies
\[
\Pi_2^{(S_b)^m} = (\Pi_4^{(S_b)^m})^t \in A^{2m}((S_b)^m \times (S_b)^m).
\]
Hence, the transpose of equation (2) gives the equality
\[
((\Pi_2^{(S_b)^m}))^* = (t(\Pi_4^{(S_b)^m}))^* \in A^m((S_b)^m) \forall b \in B \quad (i, j).
\]
Taking $(i, j) = (2, 2)$, this proves the proposition. 

2.5. **Spread.** The following result, taken from Voisin’s method of “spread” [51], [54], [53], [55], will be an essential ingredient in this note. This result acts as a magic wand, taking a homological equivalence and transmuting it into a rational equivalence.

**Proposition 2.16 (Voisin [51]).** Let $M$ be a smooth projective variety of dimension $r + 2$, and assume $M$ has trivial Chow groups (i.e. $A^*_{\text{hom}}(M) = 0$). Let $L_1, \ldots, L_r$ be very ample line bundles on $M$, and let
\[
\mathcal{V} \to B
\]
be the universal family of smooth complete intersections
\[
Y_b = M \cap D_1 \cap \cdots \cap D_r, \quad D_j \in |L_j|.
\]
Let $R \in A^2(\mathcal{V} \times_B \mathcal{V})$ be a relative correspondence such that
\[
R|_{Y_b \times Y_b} = 0 \in H^4(Y_b \times Y_b) \quad \text{for very general } b \in B.
\]
Then there exists $\delta \in A^2(M \times M)$ such that
\[
R|_{Y_b \times Y_b} = \delta|_{Y_b \times Y_b} \in A^2(Y_b \times Y_b) \quad \forall b \in B.
\]

**Proof.** This follows from the argument of [51]. More in detail: a Leray spectral sequence argument [51] Lemmas 3.11 and 3.12] shows that (after shrinking $B$) one can find $\delta \in A^2(M \times M)$ such that
\[
R - (\delta \times B)|_{\mathcal{V} \times_B \mathcal{V}} \in A^2_{\text{hom}}(\mathcal{V} \times_B \mathcal{V}).
\]
But $\widetilde{\mathcal{Y}} \times_B \mathcal{Y}$ (the blow-up along the relative diagonal) is a Zariski open in a smooth projective variety with trivial Chow groups [51, Proof of Proposition 3.13], and so

$$A^2_{hom}(\mathcal{Y} \times_B \mathcal{Y}) = 0.$$ 

In particular, this forces

$$R - (\delta \times B)|_{\mathcal{Y} \times_B \mathcal{Y}} = 0 \text{ in } A^2(\mathcal{Y} \times_B \mathcal{Y}).$$

Restricting to a fibre, this gives

$$R|_{Y_b \times Y_b} = \delta|_{Y_b \times Y_b} \in A^2(Y_b \times Y_b) \text{ for general } b \in B.$$ 

Finally, a Hilbert schemes argument shows that the same is actually true for all $b \in B$.

Note that in arbitrary dimension $n$, the argument of [51] is dependent on the “Voisin standard conjecture” [51, Conjecture 1.6]. However (as also noted in [51, Theorem 3.14]), the Voisin standard conjecture is satisfied for $n = 2$ and so is not needed as extra assumption.

(Alternatively, one could give a quick proof of proposition 2.16 along the lines of [54, Proposition 1.6], at least under the extra assumption that the surfaces $Y_b$ have non-zero primitive cohomology, which is OK in all cases where we apply proposition 2.16 since we only consider $K3$ surfaces $Y_b$.)

2.6. Families of $K3$ surfaces.

**Notation 2.17.** Let $g \in [2, 10]$. Let

$$\mathbb{P}_g := \begin{cases}
\mathbb{P}(1^3, 3) & \text{if } g = 2 , \\
\mathbb{P}^g(\mathbb{C}) & \text{if } g = 3, 4, 5 , \\
G(2, 5) & \text{if } g = 6 , \\
OG(5, 10) & \text{if } g = 7 , \\
G(2, 6) & \text{if } g = 8 , \\
LG(3, 6) & \text{if } g = 9 , \\
G_{ad}^{2d} & \text{if } g = 10 .
\end{cases}$$

(Here $\mathbb{P}(1^3, 3)$ denotes a weighted projective space, and $G(r, m)$ is the Grassmannian of $r$-dimensional subspaces in an $m$-dimensional vector space. The spaces $OG(r, d)$ and $LG(r, d)$ are the orthogonal, resp. lagrangian Grassmannian. The space $G_{ad}^{2d}$ is the adjoint variety of the
exceptional group $G_2$.) Consider the vector bundle $U_g$ on $\mathbb{P}_g$ defined as

$$U_g := \begin{cases} 
\mathcal{O}(6) & \text{if } g = 2, \\
\mathcal{O}(4) & \text{if } g = 3, \\
\mathcal{O}(3) \oplus \mathcal{O}(2) & \text{if } g = 4, \\
\mathcal{O}(2)^{\oplus 3} & \text{if } g = 5, \\
\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus 3} & \text{if } g = 6, \\
\mathcal{O}(1)^{\oplus 8} & \text{if } g = 7, \\
\mathcal{O}(1)^{\oplus 6} & \text{if } g = 8, \\
\mathcal{O}(1)^{\oplus 4} & \text{if } g = 9, \\
\mathcal{O}(1)^{\oplus 3} & \text{if } g = 10. 
\end{cases}$$

(here $\mathcal{O}(i)$ on a Grassmannian refers to the Plücker embedding).

Let $B_g \subset \mathbb{P} H^0(\mathbb{P}_g, U_g)$ denote the Zariski open parametrizing smooth sections, and let

$$S_g := \{ (x, s) \mid s(x) = 0 \} \subset \mathbb{P}_g \times B_g$$

denote the universal family.

As shown by Mukai [35], a general $K3$ surface of genus $g \in [2, 10]$ is isomorphic to a fibre $S_b$ of the family $S_g \to B_g$ (cf. also [4] and [20, Section 3.1]).

**Notation 2.18.** Let $S \to B$ be one of the families $S_g \to B_g$ of notation 2.17. The family of $m$-fold symmetric products is defined as

$$S^{(m)} := S^{m/B}/\mathcal{S}_m$$

(where $\mathcal{S}_m$ is the symmetric group on $m$ factors).

The family

$$S^{[2]} \to B$$

is defined as follows: take $S \times_B S$ and blow–up the relative diagonal, then take the quotient for the action of $\mathcal{S}_2$ exchanging the two factors. The fibre of $S^{[2]} \to B$ is the Hilbert square $(S_b)^{[2]}$ of the $K3$ surface $S_b$.

Likewise, the family $S^{[3]} \to B$ of Hilbert cubes can be constructed from $S^{3/B}$ by blowing up various partial diagonals, and quotienting for the action of $\mathcal{S}_3$.

2.7. Lagrangian fibrations.

**Proposition 2.19 (Mukai [34]).** Let $S$ be a general $K3$ surface of genus 5, and let $X = S^{[2]}$ be the Hilbert scheme. There exists a Lagrangian fibration

$$\phi : X \to \mathbb{P}^2.$$

**Proof.** The surface $S$ can be defined as the intersection of three quadrics $Q_1, Q_2, Q_3$ in $\mathbb{P}^5(\mathbb{C})$. Let $N \cong (\mathbb{P}^2)^\vee$ be the net of quadrics spanned by $Q_1, Q_2, Q_3$. Any length 2 subscheme $\xi$ in $S$
determines a line $\ell_\xi$ in $\mathbb{P}^5$. Quadrics in $N$ containing the line $\ell_\xi$ form a pencil $P_\xi \cong \mathbb{P}^1$ inside $N$. Dually, this determines a point in $\mathbb{P}^2$, and so we obtain a morphism
\[
\phi: \ X \to \mathbb{P}^2, \quad \xi \mapsto (P_\xi)^{\vee}.
\]
(This fibration $\phi$ is also described in [42, Section 2.1] and [11].) □

The following result generalizes proposition 2.19 to Hilbert squares of other $K3$ surfaces:

**Proposition 2.20** (Hassett–Tschinkel [19]). Let $S$ be a general $K3$ surface of genus $g$, and let $X = S^{[2]}$ be the Hilbert scheme. Assume that $2g - 2 = 2m^2$ for some integer $m > 1$. Then $X$ admits a Lagrangian fibration
\[
\phi: \ X \to \mathbb{P}^2.
\]
This fibration exists relatively, i.e. let $X \to B$ be the universal family of Hilbert squares of $K3$ surfaces of genus $g$ (notation 2.18), and let $X^0 \to B^0$ denote the restriction to $K3$ surfaces of Picard number 1. Then there exists a morphism
\[
\phi_X: \ X^0 \to \mathbb{P}^2 \times B^0
\]
such that the restriction of $\phi_X$ to a fibre $X = X_b$ is the Lagrangian fibration $\phi$.

**Proof.** The construction of $\phi$ is [19] Proposition 7.1.

It remains to see that the fibration exists relatively. This is clear for $m = 2$ from the explicit description of $\phi$ given by proposition 2.19. For $m > 2$, it follows from the deformation theoretic argument proving [19] Proposition 7.1. □

**Proposition 2.21** (Iliev–Ranestad [21]). Let $S$ be a general $K3$ surface of genus 9, and let $X = S^{[3]}$ be the Hilbert cube. Then $X$ admits a Lagrangian fibration
\[
\phi: \ X \to \mathbb{P}^3.
\]
This fibration exists relatively, i.e. let $X \to B$ be the universal family of Hilbert cubes of $K3$ surfaces of genus 9. Then there exists an almost holomorphic fibration
\[
\phi_X: \ X^0 \to \mathbb{P}^3 \times B
\]
such that the restriction of $\phi_X$ to a general fibre $X = X_b$ is the Lagrangian fibration $\phi$.

**Proof.** The construction is inspired by Mukai’s construction (proposition 2.19): the ambient space $\mathbb{P}^5$ in Mukai’s construction is replaced by the lagrangian Grassmannian $LG(3, 6)$, and lines in Mukai’s construction are replaced by twisted cubic curves. Let $N \cong \mathbb{P}^3$ be the space of genus 9 prime Fano threefolds $Y_\xi$ in $LG(3, 6)$ containing $S$. As shown in [21], a general length 3 subscheme $\xi$ in $S$ determines a unique twisted cubic curve $C_\xi$ in $LG(3, 6)$. There is a unique element $Y_\xi$ in $N$ containing $C_\xi$; this determines the morphism $\phi: X \to \mathbb{P}^3$.

As for the second assertion, this follows from the fact that this construction can be done relatively over $B$. The upshot is a rational map
\[
\phi_X: \ X \to \mathbb{P}^3 \times B.
\]
Since it is shown in loc. cit. that the restriction of \( \phi^X \) to a general fibre is a morphism, the map \( \phi^X \) is almost holomorphic.

\[ \square \]

**Remark 2.22.** Generalizations of proposition 2.21 to Hilbert schemes \( S^{[r]} \), for certain other values of the genus of \( S \) and of \( r \), are given in [42] and [33].

### 3. AN INTERMEDIATE RESULT

In this section, we prove a hard Lefschetz result for the Chow groups of \( S^m \) (theorem 3.1) and \( S^{[m]} \) (corollary 3.4), where \( S \) is a low genus \( K3 \) surface. This will be an ingredient in the proof of the main result (theorem 4.1) in the next section.

**Theorem 3.1.** Let \( S \to B \) be the universal family of \( K3 \) surfaces of genus \( g \), where \( 2 \leq g \leq 10 \) (cf. subsection 2.6). Let \( L \in A^1(S^m/B) \) be a line bundle such that the restriction \( L_b \) (to the fibre over \( b \in B \)) is big for very general \( b \in B \). Then

\[ (L_b)^{2m-2} : A^2_\Theta((S_b)^m) \to A^2_\Theta((S_b)^m) \]

is an isomorphism for all \( b \in B \).

Moreover, there exists \( C_b \in A^2((S_b)^m \times (S_b)^m) \) inducing the inverse isomorphism.

**Proof.** This is proven using the technique of spread as developed by Voisin [51], [54]. Let us write

\[ \Gamma_{L^{2m-2}} := (p_1)^*(L^{2m-2}) \cdot \Delta_{S^m/B} \in A^{4m-2}((S^m/B) \times_B (S^m/B)) , \]

where

\[ \Delta_{S^m/B} \subset (S^m/B) \times_B (S^m/B) \]

is the relative diagonal, and

\[ p_1 : (S^m/B) \times_B (S^m/B) \to S^m/B \]

is projection on the first factor. The relative correspondence \( \Gamma_{L^{2m-2}} \) acts on Chow groups as multiplication by \( L^{2m-2} \).

As “input”, we will make use of the following result:

**Proposition 3.2** (L. Fu [12]). Let \( X \) be a smooth projective variety of dimension \( n \) verifying the Lefschetz standard conjecture \( B(X) \). Let \( L \in A^1(X) \) be a big line bundle. Then

\[ \cap L^{n-2} : H^2(X)/N^1H^2(X) \to H^{2n-2}(X)/N^{n-1}H^{2n-2}(X) \]

is an isomorphism. (Here \( N^* \) denotes the coniveau filtration [8], so \( N^iH^2(X) \) is the image of the cycle class map.) Moreover, there is a correspondence \( C \in A^2(X \times X) \) inducing the inverse isomorphism.

**Proof.** The first statement is (a special case of) [12, Theorem 4.11], and the second statement follows from the proof of [12, Theorem 4.11]. Alternatively, for the second statement one could reason as follows: it follows from [12, Lemma 3.3] that

\[ \cap L^{n-2} : H^2(X)/N^1H^2(X) \to H^{2n-2}(X)/N^{n-1}H^{2n-2}(X) \]
is an isomorphism. Since the category of motives for numerical equivalence $\mathcal{M}_{\text{num}}$ is semisimple \cite{22}, it follows that there is an isomorphism of motives

$$h^2(X) \oplus \bigoplus_i \mathbb{L}(m_i) \cong h^{2n-2}(X)(n-2) \oplus \bigoplus_j \mathbb{L}(m_j) \quad \text{in} \quad \mathcal{M}_{\text{num}},$$

where the arrow from $h^2(X)$ to $h^{2n-2}(X)(n-2)$ is given by $\Gamma_{L^{n-2}} \in A^{2n-2}(X \times X)$, and $\mathbb{L}$ denotes the Lefschetz motive. Since homological and numerical equivalence coincide for $X$ and for $\mathbb{L}$, this implies there is also an isomorphism

$$h^2(X) \oplus \bigoplus_i \mathbb{L}(m_i) \cong h^{2n-2}(X)(n-2) \oplus \bigoplus_j \mathbb{L}(m_j) \quad \text{in} \quad \mathcal{M}_{\text{hom}},$$

with the arrow from $h^2(X)$ to $h^{2n-2}(X)(n-2)$ being given by $\Gamma_{L^{n-2}}$. It follows that there exists a correspondence $C$ as required. \hfill \Box

Any fibre $(S_b)^m$ of the family $S^{m/B} \to B$ verifies the Lefschetz standard conjecture (the Lefschetz standard conjecture is known for products of surfaces \cite{26}). Applying proposition 3.2, this means that for all $b \in B$ there exists a correspondence

$$C_b \in A^2((S_b)^m \times (S_b)^m)$$

with the property that the compositions

$$H^2((S_b)^m)/N^1 \xrightarrow{(L_b)^{2m-2}} H^{4m-2}((S_b)^m)/N^{2m-1} \xrightarrow{(C_b)} H^2((S_b)^2)/N^1$$

and

$$H^{4m-2}((S_b)^m)/N^{2m-1} \xrightarrow{(C_b)} H^2((S_b)^m)/N^1 \xrightarrow{(L_b)^{2m-2}} H^4((S_b)^m)/N^{2m-1}$$

are the identity. In other words, for all $b \in B$ there exist

$$\gamma_b, \quad \gamma'_b \in A^{2m}((S_b)^m \times (S_b)^m)$$

supported on $D_b \times D_b \subset (S_b)^m \times (S_b)^m$ for some divisor $D_b \subset (S_b)^m$ and such that

$$\Pi_2 S^{m/B}|(S_b)^m \circ C_b \circ \left( \Pi_2 S^{m/B} \circ \Gamma_{L^{2m-2}} \circ \Pi_2 S^{m/B} \right)|((S_b)^m) = \Pi_2 S^{m/B}|(S_b)^m + \gamma_b,$$

$$\Pi_2 S^{m/B}|(S_b)^m \circ \left( \Gamma_{L^{2m-2}} \circ \Pi_2 S^{m/B} \right)|((S_b)^m) \circ C_b \circ \Pi_2 S^{m/B}|((S_b)^m) = \Pi_2 S^{m/B}|((S_b)^m) + \gamma'_b$$

in $H^{4m}((S_b)^m \times (S_b)^m)$.

Applying a Hilbert schemes argument as in \cite{51} Proposition 3.7 (cf. also \cite{28} Proposition 2.10), we can find a relative correspondence

$$C \in A^2(S^{m/B} \times_B (S^{m/B}))$$

doing the same job as the various $C_b$, i.e. such that for all $b \in B$ one has

$$(\Pi_2 S^{m/B} \circ C \circ \Pi_2 S^{m/B} \circ \Gamma_{L^{2m-2}} \circ \Pi_2 S^{m/B})|(S_b)^m = \Pi_2 S^{m/B}|(S_b)^m + \gamma_b,$$

$$(\Pi_2 S^{m/B} \circ \Gamma_{L^{2m-2}} \circ \Pi_2 S^{m/B} \circ C \circ \Pi_2 S^{m/B})|(S_b)^m = \Pi_2 S^{m/B}|((S_b)^m) + \gamma'_b$$

in $H^{4m}((S_b)^m \times (S_b)^m)$.\hfill \Box
Applying once more the same Hilbert schemes argument \[51, \text{Proposition 3.7}\], we can also find a divisor \( D \subset S^{m/B} \) and relative correspondences

\[ \gamma, \gamma' \in A^{2m} \left( S^{m/B} \times_B S^{m/B} \right) \]

supported on \( D \times_B D \) and doing the same job as the various \( \gamma_b, \text{resp.} \gamma'_b \). That is, \( \gamma \) and \( \gamma' \) are such that for all \( b \in B \) one has

\[
\begin{align*}
(\Pi_2^{S^{m/B}} \circ C \circ \Pi_2^{S^{m/B}} \circ \Gamma_{L2m-2} \circ \Pi_2^{S^{m/B}})|_{(S_b)^m} &= (\Pi_2^{S^{m/B}} + \gamma)|_{(S_b)^m}, \\
(\Pi_4^{S^{m/B}} \circ \Gamma_{L2m-2} \circ \Pi_2^{S^{m/B}} \circ C \circ \Pi_4^{S^{m/B}})|_{(S_b)^m} &= (\Pi_4^{S^{m/B}} + \gamma')|_{(S_b)^m} \\
&\quad \in H^{4m}\left( (S_b)^m \times (S_b)^m \right).
\end{align*}
\]

We now make an effort to rewrite this more compactly: the relative correspondences defined as

\[ \Gamma := \Pi_2^{S^{m/B}} \circ C \circ \Pi_4^{S^{m/B}} \circ \Gamma_{L2m-2} \circ \Pi_2^{S^{m/B}} - \Pi_2^{S^{m/B}} - \gamma, \]

\[ \Gamma' := \Pi_4^{S^{m/B}} \circ \Gamma_{L2m-2} \circ \Pi_2^{S^{m/B}} \circ C \circ \Pi_4^{S^{m/B}} - \Pi_4^{S^{m/B}} - \gamma' \in A^{2m}\left( (S^{m/B}) \times_B (S^{m/B}) \right) \]

have the property that their restriction to any fibre is homologically trivial. That is, writing

\[
\begin{align*}
\Gamma_b := \Gamma|_{(S_b)^m \times (S_b)^m} \\
\Gamma'_b := (\Gamma')|_{(S_b)^m \times (S_b)^m} \in A^{2m}\left( (S_b)^m \times (S_b)^m \right)
\end{align*}
\]

for the restriction to a fibre, we have that

\[ (\Gamma_b, \Gamma'_b) \in A_{hom}^{2m}( (S_b)^m \times (S_b)^m) \quad \forall b \in B. \]

Let us now define the modified relative correspondences

\[
\begin{align*}
\Gamma_1 := \Pi_2^{S^{m/B}} \circ \Gamma \circ \Pi_2^{S^{m/B}}, \\
\Gamma'_1 := \Pi_4^{S^{m/B}} \circ \Gamma \circ \Pi_4^{S^{m/B}} \in A^{2m}( (S^{m/B}) \times_B S^{m/B})
\end{align*}
\]

This modification does not essentially modify the fibrewise rational equivalence class: we have

\[
\begin{align*}
(\Gamma_1)_b &= \Gamma_b + (\gamma_1)_b, \\
(\Gamma'_1)_b &= (\Gamma')_b + (\gamma'_1)_b \quad \text{in} \ A^{2m}( (S_b)^m \times (S_b)^m),
\end{align*}
\]

where \( \gamma_1, \gamma'_1 \in A^{2m}( (S^{m/B}) \times_B S^{m/B}) \) are relative correspondences supported on \( D \times_B D \). (Indeed, this is true because \( (\Pi_i(S_b)^m)^0 = \Pi_i(S_b)^m \) for all \( i \), and the relative correspondences

\[
\begin{align*}
\Pi_2^{S^{m/B}} \circ \gamma \circ \Pi_2^{S^{m/B}}, \quad \Pi_4^{S^{m/B}} \circ \gamma' \circ \Pi_4^{S^{m/B}}
\end{align*}
\]

are still supported on \( D \times_B D \).)

As \( \Gamma \) and \( \Gamma' \) were fibrewise homologically trivial (equation (4)), the same is true for \( \Gamma_1 \) and \( \Gamma'_1 \):

\[ (\Gamma_1)_b, (\Gamma'_1)_b \in A_{hom}^{2m}( (S_b)^m \times (S_b)^m) \quad \forall b \in B, \]
Claim 3.3. We have

\[(\Gamma_1)_b) = 0: A^i_{\text{hom}}((S_b)^m) \to A^i_{\text{hom}}((S_b)^m) \quad \forall b \in B,\]
\[(\Gamma'_1)_b) = 0: A^i_{\text{hom}}((S_b)^m) \to A^i_{\text{hom}}((S_b)^m) \quad \forall b \in B.\]

Let us prove claim 3.3 for \(\Gamma_1\) (the argument for \(\Gamma'_1\) is only notationally different). Using proposition 2.15 one finds there is a fibrewise equality modulo rational equivalence

\[(\Gamma_1)_b) = \left(\left(\sum_{i=1}^m \Xi_i \circ \Theta_i\right) \circ \Gamma \circ \left(\sum_{i=1}^m \Xi_i \circ \Theta_i\right)\right)_b \text{ in } A^{2m}(S_b) \times (S_b)^m) \quad \forall b \in B.\]

To rewrite this, let us define relative correspondences

\[\Gamma_{k,\ell} := \Theta_k \circ \Gamma \circ \Xi_\ell \in A^2(S \times_B S) \quad (1 \leq k, \ell \leq m).\]

With this notation, equality (7) becomes the equality

\[(\Gamma_1)_b) = \left(\sum_{k=1}^m \sum_{\ell=1}^m \Xi_k \circ \Gamma_{k,\ell} \circ \Theta_\ell\right)_b \text{ in } A^{2m}(S_b) \times (S_b)^m) \quad \forall b \in B.\]

As \(\Gamma\) is fibrewise homologically trivial (equation (4)), the same is true for the various \(\Gamma_{k,\ell}\):

\[(\Gamma_{k,\ell})_b) \in A^2_{\text{hom}}(S_b \times S_b) \quad \forall b \in B \quad (1 \leq k, \ell \leq m).\]

This means that we can apply Voisin’s key result, proposition 2.16 to the relative correspondence \(\Gamma_{k,\ell}\). The conclusion is that for each \(1 \leq k, \ell \leq m\), there exists a cycle \(\delta_{k,\ell} \in A^2(\mathbb{P} \times \mathbb{P})\) (where \(\mathbb{P} = \mathbb{P}_g\) is the homogeneous variety as in subsection 2.6) such that

\[(\Gamma_{k,\ell})_b) + (\delta_{k,\ell})_b = 0 \text{ in } A^2(S_b) \times (S_b)^m) \quad \forall b \in B.\]

Since \(\mathbb{P}\) has trivial Chow groups, this implies in particular that

\[(\Gamma_{k,\ell})_b) = 0: A^i_{\text{hom}}(S_b) \to A^i_{\text{hom}}(S_b) \quad \forall b \in B.\]

In view of equality (8), this implies

\[(\Gamma_1)_b) = 0: A^i_{\text{hom}}((S_b)^m) \to A^i_{\text{hom}}((S_b)^m) \quad \forall b \in B,\]

as claimed.

(The argument for \(\Gamma'_1\) is the same; it suffices to replace the use of proposition 2.15 by proposition 2.14.) Claim 3.3 is now proven.

It is now time to wrap up the proof of theorem 3.1. For \(b \in B\) general, the restrictions \((\gamma_1)_b)\), \((\gamma'_1)_b)\) of equation (5) will be supported on \(D_b \times D_b \subset (S_b)^m \times (S_b)^m\), where \(D_b \subset (S_b)^m\) is a divisor. As such, the action

\[\((\gamma_1)_b) : R((S_b)^m) \to R((S_b)^m),\]
\[\((\gamma'_1)_b) : R((S_b)^m) \to R((S_b)^m),\]
is 0 for general $b \in B$, where $R$ is either $A_{homo}^2$ or $A^{2m}$. Combining this observation with equation (5) and claim (3,3), we find that

\[(\Gamma_b)_* = 0: \quad R((S_b)^m) \to R((S_b)^m),\]
\[(\Gamma'_b)_* = 0: \quad R((S_b)^m) \to R((S_b)^m),\]

(where, once more, $R$ is either $A_{homo}^2$ or $A^{2m}$).

In view of the definition (3) of $\Gamma, \Gamma'$ (and using that the cycles $\gamma_b, \gamma'_b$ occurring in (3) are supported in codimension 1 for $b \in B$ general, and so act trivially on $A_{homo}^2$ and on $A^{2m}$), it follows that

\[(\Pi_2^{(S_b)^m} \circ \mathcal{C}_b \circ \Pi_4^{(S_b)^m} \circ (\Gamma_{L^{2m-2}})_b \circ \Pi_2^{(S_b)^m} - \Pi_2^{(S_b)^m})_* = 0: \quad A_{homo}^2((S_b)^m) \to A_{homo}^2((S_b)^m),\]
\[(\Pi_4^{(S_b)^m} \circ (\Gamma_{L^{2m-2}})_b \circ \Pi_2^{(S_b)^m} \circ \mathcal{C}_b \circ \Pi_4^{(S_b)^m} - \Pi_4^{(S_b)^m})_* = 0: \quad A^{2m}((S_b)^m) \to A^{2m}((S_b)^m),\]

for general $b \in B$. Since $\Pi_2^{(S_b)^m}$ acts as the identity on $A_{(2)}((S_b)^m)$, it follows from the first line of (9) that

\[(\Pi_2^{(S_b)^m} \circ \mathcal{C}_b \circ \Pi_4^{(S_b)^m} \circ (\Gamma_{L^{2m-2}})_b)_* = \text{id}: \quad A_{(2)}^2((S_b)^m) \to A_{(2)}^2((S_b)^m);\]

in particular

\[L^{2m-2} \cdot A_{(2)}^2((S_b)^m) \to A_{(2)}^{2m}((S_b)^m)\]

is injective for general $b \in B$. Likewise, it follows from the second line of (9) that

\[(\Pi_4^{(S_b)^m} \circ (\Gamma_{L^{2m-2}})_b \circ \Pi_2^{(S_b)^m} \circ \mathcal{C}_b)_* = \text{id}: \quad A_{(2)}^{2m}((S_b)^m) \to A_{(2)}^{2m}((S_b)^m);\]

for general $b \in B$. However, the image of

\[A_{(2)}^2((S_b)^m) \xrightarrow{L^{2m-2}} A_{(2)}^{2m}((S_b)^m)\]

is contained in $A_{(2)}^{2m}((S_b)^m)$, since $L \in A^1((S_b)^m) = A_{(0)}^1((S_b)^m)$, and so this further simplifies to

\[(\Gamma_{L^{2m-2}})_b \circ \Pi_2^{(S_b)^m} \circ \mathcal{C}_b)_* = \text{id}: \quad A_{(2)}^{2m}((S_b)^m) \to A_{(2)}^{2m}((S_b)^m);\]

for general $b \in B$. In particular,

\[L^{2m-2} \cdot A_{(2)}^2((S_b)^m) \to A_{(2)}^{2m}((S_b)^m)\]

is surjective for general $b \in B$.

Theorem 3.1 is now proven for general $b \in B$. To prove the theorem for all $b \in B$, one observes that the above argument can be made to work “locally around a given $b_0 \in B$”, i.e. given $b_0 \in B$ one can find relative correspondences $\gamma, \gamma', \ldots$ supported in codimension 1 and in general position with respect to the fibre over $b_0$.

Theorem 3.1 can be reformulated in terms of Hilbert schemes:
Corollary 3.4. Let \( S_b \) be a K3 surface of genus \( g \leq 10 \), and let \( X = (S_b)^{[m]} \) be the Hilbert scheme of length \( m \) subschemes of \( S \). Let \( L \in A^1((S_b)^{[m]}/B) \) be a relatively big line bundle, and set

\[
L_X := (f_b)^*(p_b)_*(L_b) \in A^1(X),
\]

where \( p_b: (S_b)^m \to (S_b)^{(m)} \) denotes the projection, and \( f_b: (S_b)^{[m]} \to (S_b)^{(m)} \) denotes the Hilbert–Chow morphism. Then

\[
(L_X)^{m-1} : A^2_{(2)}(X) \to A^2_{(2)}((S_b)^{(m)})
\]

is an isomorphism.

Moreover, there exists a correspondence \( C \in A^2(X \times X) \) inducing the inverse isomorphism.

Proof. Let the symmetric group \( \mathfrak{S}_m \) act on \( S^{m/B} \) by permuting the factors, and let

\[
p: S^{m/B} \to S^{(m)} := S^{m/B}/\mathfrak{S}_m
\]

denote the quotient morphism. Theorem 3.1 applies to the line bundle

\[
L' := p^* p_*(L) = \sum_{\sigma \in \mathfrak{S}} \sigma^*(L) \in A^1(S^{m/B}).
\]

There is a commutative diagram

\[
\begin{align*}
A^2_{(2)}((S_b)^{(m)}) & \xrightarrow{(p_b)^* \uparrow \cong} A^2_{(2)}((S_b)^{[m]}) \\
A^2_{(2)}((S_b)^{(m)}) & \xrightarrow{-(p_b)_*(L_b)^{m-1}} A^2_{(2)}((S_b)^{(m)})
\end{align*}
\]

In view of theorem [3.1](applied to \( L' \)), the lower horizontal arrow is an isomorphism.

It follows from the de Cataldo–Migliorini isomorphism of motives [9] that there is an isomorphism (induced by a correspondence)

\[
A^2(X) \cong A^2((S_b)^{(m)}) \oplus A^1() \oplus A^0(),
\]

and so in particular an isomorphism

\[
A^2_{A_J}(X) \cong A^2_{A_J}((S_b)^{(m)}).
\]

Since \( A^2_{A_J}() \subset A^2_{A_J}((S_b)^{(m)}) \), and the de Cataldo–Migliorini isomorphism respects the bigrading (by construction, the MCK decomposition for \( X \) is induced by one for \( (S_b)^{(m)} \)), this implies that

\[
f^*: A^2_{(2)}((S_b)^{(m)}) \to A^2_{(2)}(X)
\]

is an isomorphism.

Similarly, there is an isomorphism

\[
f^*: A^{2m}((S_b)^{(m)}) \cong A^{2m}(X)
\]

which respects the bigrading.
Corollary 3.4 now follows from what we have said above, in view of the commutative diagram

\[ \begin{array}{ccc}
A^2_{(2)}(X) & \xrightarrow{(LX)^{m-1}} & A^2_{(2)}(X) \\
(f_b)^* \uparrow \cong & & (f_b)^* \uparrow \cong \\
A^2_{(2)}((S_b)^{(m)}) & \xrightarrow{(p_b(L_b))^{m-1}} & A^2_{(2)}((S_b)^{(m)})
\end{array} \]

\[ \square \]

Remark 3.5. Looking at corollary 3.4, one might hope that a similar result is true more generally. Let \( X \) be any hyperkähler variety of dimension \( 2m \), and suppose the Chow ring of \( X \) has a bigraded ring structure \( A^*(X) \). One can ask the following questions:

(i) Let \( L \in A^1(X) \) be an ample line bundle. Is it true that there are isomorphisms

\[ L^{2m-2i+j} : A^i_{(j)}(X) \cong A^{2m-i+j}_{(j)}(X) \text{ for all } 0 \leq 2i - j \leq 2m \]?

(ii) Let \( L \in A^1(X) \) be a big line bundle. Is it true that there are isomorphisms

\[ L^{2m-i} : A^i_{(i)}(X) \cong A^{2m}_{(i)}(X) \text{ for all } 0 \leq i \leq 2m \]?

The answer to the first question is “yes” for generalized Kummer varieties [27]. The answer to both questions is “I don’t know, except for \( i = 2 \) and \( g \) low” for Hilbert schemes of genus \( g \) K3 surfaces.

(The question for \( A^i_{(j)}(S^{[m]}) \) with \( i > 2 \) and \( g \) low becomes more complicated, as one would need an analogon of proposition 2.16 for higher fibre products \( S^{m/B} \) with \( m > 2 \).)

Remark 3.6. Let \( X \) be either \( S^m \) or \( S^{[m]} \) where \( S \) is a K3 surface of genus \( g \leq 10 \). Let \( L \in A^1(X) \) be a line bundle as in theorem 3.1 (resp. as in corollary 3.4). Provided \( L \) is sufficiently ample, there exists a smooth complete intersection surface \( Y \subset X \) defined by the linear system \( |L| \). Theorem 3.7 (resp. corollary 3.4) then implies that \( A^2_{(2)}(X) \) is supported on \( Y \), and that

\[ A^2_{(2)}(X) \to A^2(Y) \]

is injective. This injectivity statement is in agreement with Hartshorne’s “weak Lefschetz” conjecture for Chow groups [18] (we recall that it is expected that \( A^2_{(2)}(X) = A^2_{hom}(X) \) for these \( X \)).

4. Main result

This section proves the main result of this note, theorem 4.1. The proof is based on the method of “spread” of cycles in nice families, as developed by Voisin [51], [54], [52], [53], [55]. The results announced in the introduction (theorems 4.2 and 4.4) are immediate corollaries of theorem 4.1.

Theorem 4.1. Let \( g \in [2, 10] \). Let \( X = S^{(m)} \to B \) denote the universal family of \( m \)-fold symmetric products of genus \( g \) K3 surfaces (notation 2.78). Let \( \Gamma \subset X \) be a codimension \( 2m - 2 \) subvariety, and let \( \Gamma_b \) denote the restriction

\[ \Gamma_b := \Gamma|_{X_b} \in A^m(X_b) \]
Assume that
\[ \cup \Gamma_b = 0 : \ H^{2,0}(X_b) \rightarrow H^{2m,2m-2}(X_b) \]
for very general \( b \in B \). Then
\[ A^2_{(2)}(X_b) \xrightarrow{\Gamma_b} A^{2m}(X_b) \rightarrow A^{2m}_{(2)}(X_b) \]
is the zero map, for all \( b \in B \). (Here the last arrow is projection to the summand \( A^{2m}_{(2)}(X_b) \).)

**Proof.** Let \( f : \tilde{\Gamma} \rightarrow \Gamma \) be a resolution of singularities, and let \( \tau : \tilde{\Gamma} \hookrightarrow X \) denote the composition of \( f \) with the inclusion morphism \( \Gamma \hookrightarrow X \). Let \( p : S^{m/B} \rightarrow X \) denote the quotient morphism. Let us now consider the relative correspondence
\[ \Gamma_0 := C \circ \mathcal{T}_p \circ \Pi^{X}_{2m-2} \circ \mathcal{T}_\tau \circ \Pi^{X}_{2} \circ \Gamma_p \in A^{2m}(S^{m/B} \times_{S} S^{m/B}) , \]
where \( \Pi^{X}_{2} \) is as in remark 2.13 and \( C \in A^{2}(S^{m/B} \times_{S} S^{m/B}) \) is as in the proof of theorem 3.1.

By construction, for any \( b \in B \), the restriction
\[ \Gamma_0|_{(S_b)^m \times (S_b)^m} : A^2((S_b)^m) \xrightarrow{\mathcal{T}_b} A^2(X_b) \xrightarrow{(\Pi^{X}_{2})_*} A^2_{(2)}(X_b) \xrightarrow{\Gamma_b} A^{2m}(X_b) \xrightarrow{(\Pi^{X}_{2m-2})_*} A^2_{(2)}(S_b) \xrightarrow{p_*} A^2_{(2)}((S_b)^m) . \]

We now make the following claim: to prove theorem 4.1 it suffices to prove that
\[ (\Pi^{S^{m/B}}_{2} \circ \Gamma_0)|_{(S_b)^m \times (S_b)^m} =: \Gamma_b : A^2((S_b)^m) \rightarrow A^2_{(2)}((S_b)^m) \quad \forall b \in B . \]

To prove the claim, we first remark that (as noted above)
\[ (\Gamma_0|_{(S_b)^m \times (S_b)^m})_* : A^2((S_b)^m) \subset A^2_{(2)}((S_b)^m) , \]
and so adding \((\Pi^{(S_b)^m}_{2})_*\), doesn’t change anything, i.e. the truth of statement (10) implies that
\[ (\Gamma_0|_{(S_b)^m \times (S_b)^m})_* =: \Gamma_b : A^2_{(2)}((S_b)^m) \rightarrow A^2_{(2)}((S_b)^m) \quad \forall b \in B . \]

Next, we know from theorem 3.1 that \( A^2_{(2)}((S_b)^m) \xrightarrow{(\mathcal{T}_b)_*} A^2_{(2)}((S_b)^m) \) is an isomorphism, and so this implies that also
\[ A^2((S_b)^m) \xrightarrow{p_*} A^2(X_b) \xrightarrow{(\Pi^{X}_{2})_*} A^2_{(2)}(X_b) \xrightarrow{\Gamma_b} A^{2m}(X_b) \xrightarrow{(\Pi^{X}_{2m-2})_*} A^2_{(2)}(S_b) \xrightarrow{p_*} A^2_{(2)}((S_b)^m) \]
is the zero map, for all \( b \in B \). Composing some more on both sides, this implies that also
\[ A^2(X_b) \xrightarrow{p_*} A^2((S_b)^m) \xrightarrow{p_*} A^2(X_b) \xrightarrow{(\Pi^{X}_{2})_*} A^2_{(2)}(X_b) \xrightarrow{\Gamma_b} A^{2m}(X_b) \xrightarrow{(\Pi^{X}_{2m-2})_*} A^2_{(2)}(S_b) \xrightarrow{p_*} A^2_{(2)}((S_b)^m) \xrightarrow{p_*} A^2_{(2)}(X_b) \]
is the zero map, for all $b \in B$. But $p_*p^*$ is a multiple of the identity, and so this implies that actually
\[
A^2(X_b) \xrightarrow{\Pi^{X_b}_2} A^2(\mathbb{P}^1) \xrightarrow{\Gamma_b} A^2_m(X_b) \xrightarrow{\Pi^{X_b}_{2m-2}} A^2_m(\mathbb{P}^1)
\]
is already the zero map, for all $b \in B$. This proves the claim, i.e. we are now reduced to proving statement (10).

The input we have at our disposition is that we know (from the coisotropic assumption) that
\[
(11) \quad (\Gamma_0|_{(S_b)^m \times (S_b)^m})_* = 0: \quad H^{2,0}((S_b)^m) \to H^{2,0}((S_b)^m) \quad \text{for very general } b \in B.
\]

We observe that (11), combined with the Lefschetz (1,1) theorem, implies the following: for very general $b \in B$, there exist a curve $Y_b \subset (S_b)^m$, a divisor $D_b \subset (S_b)^m$ and a cycle $\gamma_b$ supported on $Y_b \times D_b \subset (S_b)^m \times (S_b)^m$, such that
\[
\Gamma_0|_{(S_b)^m \times (S_b)^m} - \gamma_b = 0 \quad \text{in } H^{4m}((S_b)^m \times (S_b)^m).
\]

Thanks to Voisin's key result [51, Proposition 3.7] (cf. also [54, Proposition 4.25]), it is possible to spread out these data. That is, there exist subvarieties $\mathcal{Y} \subset S^{m/B}, \mathcal{D} \subset S^m/B$ of codimension $2m - 1$ resp. 1, and a cycle $\gamma \in A^{2m}(S^{m/B} \times_B S^m/B)$ supported on $\mathcal{Y} \times_B \mathcal{D}$ that does the job of the various $\gamma_b$, i.e. such that
\[
(12) \quad (\Gamma_0 - \gamma)|_{(S_b)^m \times (S_b)^m} = 0 \quad \text{in } H^{4m}((S_b)^m \times (S_b)^m) \quad \text{for very general } b \in B.
\]

In other words, the relative correspondence defined as
\[
\Gamma_1 := \Gamma_0 - \gamma \in A^{2m}(S^{m/B} \times_B S^m/B)
\]
has the property that
\[
(13) \quad \Gamma_1|_{(S_b)^m \times (S_b)^m} = 0 \quad \text{in } H^{4m}((S_b)^m \times (S_b)^m) \quad \text{for very general } b \in B.
\]

It is more convenient to switch to correspondences in $A^2(S \times_B \mathcal{S})$. To this end, we now define relative correspondences
\[
\Gamma^{i,j}_2 := \Xi_i \circ \Gamma_1 \circ \Theta_j \in A^2(S \times_B \mathcal{S}) \quad (1 \leq i, j \leq 2),
\]
where $\Xi_i, \Theta_j$ are as in proposition 2.15. The relative correspondence $\Gamma_1$ being fibrewise homologically trivial (equation (12)), the same holds for the $\Gamma^{i,j}_2$:
\[
\Gamma^{i,j}_2|_{S_b \times S_b} = 0 \quad \text{in } H^4(S_b \times S_b) \quad \text{for very general } b \in B \quad (1 \leq i, j \leq 2).
\]

We can now apply proposition 2.16 to the $\Gamma^{i,j}_2$ (with $M = \mathbb{P}^g$ and $L_r$ as given in subsection 2.6). The conclusion is that there exist cycles $\delta^{i,j} \in A^2(\mathbb{P}^g \times \mathbb{P}^g)$ such that there is a fibrewise rational equivalence
\[
\Gamma^{i,j}_2|_{S_b \times S_b} + \delta^{i,j}|_{S_b \times S_b} = 0 \quad \text{in } A^2(S_b \times S_b) \quad \forall b \in B \quad (1 \leq i, j \leq 2).
\]

In particular, since ($\mathbb{P}^g$ has trivial Chow groups and hence) the restriction $\delta^{i,j}|_{S_b \times S_b}$ acts trivially on $A^*_{\hom}(S_b)$, this implies that
\[
(\Gamma^{i,j}_2|_{S_b \times S_b})_* = 0: \quad A^*_{\hom}(S_b) \to A^*(S_b) \quad \forall b \in B \quad (1 \leq i, j \leq 2).
\]
This implies that also
\[
\left( \Pi_2^{S_m/B} \circ \Gamma_1 \circ \Pi_2^{S_m/B} \right)_{|((S_b)^m \times (S_b)^m)} = \left( \sum_{i,j \in \{1,2\}} \Theta_{ij} \circ \Pi_2^{i,j} \circ \Xi_j \right)_{|S_b \times S_b},
\]
\[
= 0: \ A^*_{\text{hom}} \left( (S_b)^m \right) \rightarrow A^* \left( (S_b)^m \right) \quad \forall b \in B
\]
(here the first equality follows from proposition 2.15). Since \( A^2(2)_b \subset A^2_{\text{hom}}() \), this implies in particular that
\[
A^2(2)_b \left( (S_b)^m \right) \quad \text{is the zero map, for all } b \in B.
\]
For general \( b \in B \), the restriction of the cycle \( \delta \) to the fibre \( (S_b)^m \times (S_b)^m \) will be supported on (curve) \( \times \) (divisor), and so will act trivially on \( A^2((S_b)^m) \) for dimension reasons. That is, for general \( b \in B \) we have equality
\[
(\Gamma_1|_{(S_b)^m \times (S_b)^m})_* = (\Gamma_0|_{(S_b)^m \times (S_b)^m})_*: \ A^2((S_b)^m) \rightarrow A^2((S_b)^m).
\]
The above thus implies that
\[
A^2(2)_b \left( (S_b)^m \right) \quad \text{is the zero map, for general } b \in B.
\]
That is, we have now proven the desired statement (10), and hence theorem 4.1 for general \( b \in B \) (this already suffices to prove theorems 4.2 and 4.4 below).

To extend the statement to all \( b \in B \), one notes that the construction of [51, Proposition 3.7] (which was used above to globalize the various \( \gamma_b \)) can be done locally around a given \( b_0 \in B \).

As special cases of theorem 4.1, we can now prove the results announced in the introduction:

**Theorem 4.2.** Let \( X = S[2] \), where \( S \) is a general K3 surface of genus \( g = 5 \) or \( g = 10 \). Let \( A \subset X \) be a general fibre of the Lagrangian fibration \( \phi: X \rightarrow \mathbb{P}^2 \) (subsection 2.7). Then \( A \in A^2_{(1)}(X) \) and
\[
\cdot A: \ A^2_{\text{hom}}(X) \rightarrow A^4(X)
\]
is the zero map.

**Proof.** The first statement is easy: any point \( p \in \mathbb{P}^2 \) is an intersection of two divisors, and so \( A = \phi^*(p) \in A^2(X) \) is also an intersection of two divisors.

As for the second statement, we have a decomposition
\[
A^2_{\text{hom}}(X) = A^2(2)_b(X) \oplus (A^2(1)_b(X) \cap A^2_{\text{hom}}(X))
\]
(where the second summand is conjecturally zero). We know that
\[
\cdot A: \ A^2(1)_b(X) \cap A^2_{\text{hom}}(X) \rightarrow A^4(X)
\]
is zero (the image lands in \( A^4(X) \cap A^4_{\text{hom}}(X) = 0 \)). It is thus sufficient to prove that
\[
\cdot A: \ A^2(2)_b(X) \rightarrow A^4(X)
\]
is the zero map.
Let
\[ h : X := S^{[2]} \to S^{(2)} \]
denote the Hilbert–Chow morphism, and let \( A' := h(A) \subset S^{(2)} \). Since
\[ h_*(h^*(a) \cdot A) = a \cdot h_*(A) = a \cdot A' \quad \text{in} \quad A^4(S^{(2)}) \quad \forall a \in A^2(S^{(2)}) , \]
and
\[
\begin{align*}
  &h^* : A^2_{(2)}(S^{(2)}) \to A^2_{(2)}(X), \\
  &h_* : A^4(X) \to A^4(S^{(2)})
\end{align*}
\]
are isomorphisms, it suffices to prove that
\[
\cdot A' : A^2_{(2)}(S^{(2)}) \to A^4(S^{(2)})
\]
is the zero map.

We have seen (subsection 2.7) that the fibration \( \phi \) exists relatively, and so \( A \subset X \) exists relatively (i.e. there exists \( A \subset X \) such that \( A \) is the restriction of \( A \) to the fibre \( X \)). It follows that \( A' \) also exists relatively (i.e. there exists \( A' \subset S^{(2)} \) such that \( A' \) is the restriction of \( A' \) to a fibre). The result now follows from theorem 4.1 applied to \( A' \).

**Corollary 4.3.** Let \( X \) and \( A \) be as in theorem 4.2. Let \( b \in A^4(X) \) be a 0–cycle of the form \( b = A \cdot c \), where \( c \in A^2_2(X) \). Then \( b \) is rationally trivial if and only if \( b \) is of degree 0.

**Proof.** One can decompose \( c = c_0 + c_2 \), where \( c_j \in A^2_{(j)}(X) \). Since \( A \cdot c_2 = 0 \) (theorem 4.2), there is equality
\[ b = A \cdot c_0 \quad \in A^4_{(0)}(X) . \]
But \( A^4_{(0)}(X) \) is isomorphic to \( H^4(X) \cong \mathbb{Q} \).

**Theorem 4.4.** Let \( X = S^{[3]} \), where \( S \) is a general K3 surface of genus 9. Let \( A \subset X \) be a general fibre of the Lagrangian fibration \( \phi \) (subsection 2.7). Then \( A \in A^3_{(0)}(X) \) and
\[
\cdot A \cdot D : A^2_{\text{hor}}(X) \to A^6(X)
\]
is the zero map, for any divisor \( D \in A^1(X) \).

**Proof.** This is similar to the proof of theorem 4.2. Again, the fact that \( A \in A^3_{(0)}(X) \) is clear from the fact that a point in \( \mathbb{P}^3 \) can be written as intersection of divisors.

Since the fibration exists relatively the fibre \( A \subset X \) exists relatively (i.e. there exists \( A \subset X \) such that \( A \) is the restriction of \( A \) to the fibre \( X \)). The assumption implies that \( X \) has Picard number 2 and so the divisor \( D \) also exists relatively, i.e. there exists \( D \in A^1(X) \) such that \( D \) is the restriction of \( D \) to the fibre \( X \). We may write \( D \) as a sum
\[ D = \sum_j \lambda_j D_j \quad \text{in} \quad A^1(X) , \]
where \( \lambda_j \in \mathbb{Q} \) and \( D_j \) is effective and in general position with respect to \( A \).
Let \( h: X = \mathcal{S}^{[3]} \to \mathcal{S}^{(3)} \) denote the “relative Hilbert–Chow morphism”. The result now follows upon applying theorem 4.1 to the subvarieties
\[
h(A \cdot D_j) \subset \mathcal{S}^{(3)}.
\]

**Corollary 4.5.** Let \( X \) and \( A \) be as in theorem 4.4. Let \( b \in A^6(X) \) be a 0–cycle of the form \( b = A \cdot D \cdot c \), where \( c \in A^2(X) \). Then \( b \) is rationally trivial if and only if \( b \) is of degree 0.

**Proof.** One can decompose \( c = c_0 + c_2 \), where \( c_j \in A^2_{(j)}(X) \). Since \( A \cdot D \cdot c_2 = 0 \) (theorem 4.4), there is equality
\[
b = A \cdot D \cdot c_0 \in A^6_{(0)}(X).
\]
But \( A^6_{(0)}(X) \) is isomorphic to \( H^6(X) \cong \mathbb{Q} \).

**Remark 4.6.** Let \( X_b \) be the Hilbert square of a very general \( K3 \) surface \( S_b \). Let \( \Gamma_b \subset X_b \) be a coisotropic subvariety of codimension 2. It follows from a result of Voisin [56, Proposition 4.2] that there is a homological equivalence
\[
\Gamma_b = \sum_i \lambda_i C_i \text{ in } H^4(X_b),
\]
where \( \lambda_i \in \mathbb{Q} \) and \( C_i \subset X_b \) is a constant cycle surface.

Suppose we know in addition that \( \Gamma_b \in A^2_{(0)}(X_b) \) (for instance because \( \Gamma_b \) is the fibre of a Lagrangian fibration). Then conjecturally, equality (13) implies there is a rational equivalence
\[
\Gamma_b \cong \sum_i \lambda_i C_i \text{ in } A^2(X_b),
\]
which clearly would imply theorem 4.2.

Unfortunately, I have not been able to prove equality (14). The approach taken here (using the Fourier decomposition of [44]) only yields the weaker statement that
\[
\Gamma_b = \sum_i \lambda_i C_i + R \text{ in } A^2(X_b),
\]
where \( R \) is in the “troublesome part” \( A^2_{(0)}(X_b) \cap A^2_{hom}(X_b) \) (which is conjecturally zero). This is not sufficient to settle theorem 4.2 which is why we needed to work harder to prove theorem 4.2.

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Institut de Recherche Mathématique Avancée, CNRS – Université de Strasbourg, 7 Rue René Descartes, 67084 Strasbourg CEDEX, FRANCE.

E-mail address: robert.laterveer@math.unistra.fr