Abstract

We prove that spacetimes satisfying the vacuum Einstein equations on a manifold of the form $\Sigma \times U(1) \times R$ where $\Sigma$ is a compact surface of genus $G > 1$ and where the Cauchy data is invariant with respect to $U(1)$ and sufficiently small exist for an infinite proper time in the expanding direction.

1 Introduction

In this paper we prove a global in time existence theorem, in the expanding direction, for a family of spatially compact vacuum spacetimes having spacelike $U(1)$ isometry groups. The 4-manifolds we consider have the form $V = M \times R$ where $M$ is an (orientable) circle bundle over a compact higher genus surface $\Sigma$ and where the spacetime metric is assumed to be invariant with respect to the natural action of $U(1)$ along the bundle’s circle fibers. We reduce Einstein’s equations, à la Kaluza-Klein, to a system on the base $\Sigma \times R$ where it takes the form of the 2+1 dimensional Einstein equations coupled to a wave map matter source whose target space is the hyperbolic plane. This wave map represents the true gravitational wave degrees of freedom that have descended from 3+1 dimensions to appear as ”matter” degrees of freedom in 2+1 dimensions. The 2+1 metric itself contributes only a finite number of additional, Teichmüller parameter, degrees of freedom which couple to the wave map and control the conformal geometry of $\Sigma$.

After the constraints have been solved and coordinate conditions imposed, through a well defined elliptic system, nothing remains but the evolution
problem for the wave map / Teichmuller parameter system though the latter has now become non local in the sense that the "background" metric in which the wave map is propagating is now a non local functional of the wave map itself given by the solution of the elliptic system mentioned above. Thus even in the special "polarized" case which we concentrate on here, in which the wave map reduces to a pure wave equation, this wave equation is now both non linear and non local.

In addition to the simplifying assumption of polarization (which obliges us here to treat only trivial bundles, $M = \Sigma \times S^1$) we shall need a smallness condition on the initial data, an assumption that the genus of $\Sigma$ is greater than 1 and a restriction on the initial values allowed for the Teichmuller parameters. It seems straightforward to remove each of these restrictions except for the smallness condition on the initial data. In particular we believe that the methods developed herein can be extended to the treatment of non polarized solutions on non trivial bundles over surfaces including the torus (but not $S^2$) with no restriction on the initial values of the Teichmuller parameters. Some preliminary work in this direction has already been carried out.

We do not know how to remove the small data restriction even in the polarized case but conjecture that long time existence should hold for arbitrary large data since the U(1) isometry assumption seems to suppress the formation of black holes (note that U(1) is here essentially a "translational" and not a "rotational" symmetry since the existence of an axis of rotation would destroy the bundle structure). Of course there is as yet no large data global existence result for smooth wave maps in 2+1 dimensions even on a given background so there is no immediate hope for such a result in our still more non linear (and non local) problem but the polarized case, though non linear and non local as well, seems more promising. One knows how to control the Teichmuller parameters in pure 2+1 gravity and a wave equation on a given curved background offers no special difficulty. But now the "background metric is instead a functional of the evolving scalar field and one needs to control this along with the Teichmuller parameters.

Serious progress on this problem would represent a "quantum jump" forward in one's understanding of long time existence problems for Einstein's equations since, up to now, the only large data global results require simplifying assumptions that effectively reduce the number of spatial dimensions to one (e.g., Gowdy models and their generalizations, plane symmetric gravitational waves, spherically symmetric matter coupled to gravity) or zero (e.g.,
Bianchi models, 2+1 gravity). We hope that this work on small data global existence will lay the groundwork for such an eventual quantum jump.

But why assume a Killing field if only small data results are aimed for in the current project? A small data global existence result already exists (Andersson-Moncrief, in preparation) for Einstein equations on different 3-manifolds of negative Yamabe class which makes no symmetry assumption whatsoever. Shouldn’t those methods be applicable to our problem in which case the U(1) symmetry assumption could be removed. The answer to this question is far from obvious for a somewhat subtle reason. In those cases where small data global existence can be established the conformal geometry of the spatial slices (which represents the propagating gravitational wave degrees of freedom) is tending to a well behaved limit. Therefore the various Sobolev ”constants” (which are in fact functionals of the geometry) which are needed in the associated energy estimates are tending to well behaved limits as well. This simplifying feature is however missing in the current problem since, during the course of our evolution, the conformal geometry of the circle bundles under study is undergoing a kind of Cheeger-Gromov collapse in which the circular fibers shrink to zero length and the various related Sobolev ”constants” may careen out of control making even small data energy estimates much more difficult.

Of the various Thurston types of 3-geometries which compactify to negative or zero Yamabe class manifolds \( \{H^3, H^2 \times R, SL(2, R), Sol, Nil, R^3\} \) only the hyperbolics are immune from such degenerations and the remaining (positive Yamabe class) Thurston types \( \{S^3, S^2 \times R\} \) not only are subject to Cheeger-Gromov type collapse but also to recollapse of the actual physical geometry to ”big crunch” singularities in the future direction. By focusing on negative (or zero) Yamabe class manifolds which exclude (due to Einstein’s equations) the occurrence of maximal hypersurfaces, that would signal the onset of recollapse to a big crunch, we thereby concentrate on spacetimes that can expand indefinitely.

That such Cheeger-Gromov collapse can be expected in solutions of Einstein’s equations can be seen already in the basic compactified Bianchi models wherein all the known solutions of negative Yamabe type except \( H^3 \) exhibit conformal collapse either along circular fibers \( \{H^2 \times R, SL(2, R)\} \), or collapse along \( T^2 \) fibers \( \{Sol\} \), or even total collapse with non zero but bounded curvature of the Gromov ”almost flat” variety \( \{Nil\} \). The solutions we are considering here (of Thurston type \( H^2 \times R \) or eventually \( SL(2, R) \) in non polarized generalizations) extend results exhibiting such behavior to a large
family of spatially inhomogeneous spacetimes. We sidestep the extra complication of degenerating Sobolev constants by imposing $U(1)$ symmetry and carrying out Kaluza Klein reduction to work on a spatial manifold of hyperbolic type (though now a 2-dimensional one) for which, as we shall show, collapse and the corresponding degeneracy of the needed Sobolev constants is suppressed.

The reason why we avoid the base $\Sigma = T^2$ is that the 2-tori themselves tend to collapse under the Einstein flow whereas the higher genus surfaces do not. On the other hand one can probably compute the explicit dependence of the needed Sobolev constants on the Teichmuller parameters for the torus and eventually exploit this to treat the Thurston cases $\{R^3, Nil\}$ which compactify typically to trivial and non trivial $S^1-$bundles over $T^2$. The $Sol$ case (which compactifies to $T^2$ bundles over $S^1$) tends to collapse (as seen from the Bianchi models) the entire $T^2$ fibers. Thus to avoid degenerating Sobolev constants” in this case it seems necessary to impose a full $T^2 = U(1) \times U(1)$ isometry group and Kaluza Klein reduce to an $S^1$ spatial base manifold. This leads to a certain nice generalization of the Gowdy models defined on the ”Sol - twisted torus” but has effectively only one space dimension remaining. We exclude the Thurston types $\{S^2 \times R, S^3\}$ which correspond to trivial and non trivial $S^1$ bundles over $S^2$ respectively since they belong to the positive Yamabe class as we have mentioned and should not exhibit infinite expansion but rather recollapse to big crunch singularities.

The eight Thurston types are the basic building blocks from which other (and conjucturally all) compact 3-manifolds can be built by glueing together along so called incompressible 2-tori or (to obtain non prime manifolds) along essential 2-spheres. Very little is known about the Einstein ”flow” on such more general manifolds but it seems that a natural first step in this direction may be made by studying the Einstein flow on the basic building block manifolds themselves. This program seems tractable provided that a $U(1)$ symmetry is imposed in the $H^2 \times R, Sl(2, R)$ and perhaps $Nil$ and $R^3$ cases, and provided that a $U(1) \times U(1)$ symmetry is imposed in the $Sol$ case. No symmetries are needed in the $H^3$ case due to the absence of Cheeger - Gromov collapse but one can hope to remove the symmetry hypothesis in the other cases by learning how to handle degenerating Sobolev ”constants”. In this respect the $Nil$ and $R^3$ cases may provide some guidance since they seem to require a treatment of degenerating Sobolev constants but only in the setting of 2-dimensions (when $U(1)$ symmetry is imposed).
The basic methods we use involve the construction of higher order energies to control the Sobolev norms of the scalar wave degrees of freedom combined with an application of the "Dirichlet energy" function in Teichmuller space to control the Teichmuller parameters degrees of freedom. A subtlety is that the most obvious definition of wave equation (or, more generally, wave map) energies does not lead to a well defined rate of decay so that corrected energies must be introduced which exploit "information" about the lowest eigenvalue of the spatial laplacian which enters into the wave equation. Since the lowest eigenvalues vary with position in Teichmuller space we find convenient to choose initial data such that, during the course of the evolution, the lowest eigenvalue avoids a well known gap in the spectrum for an arbitrary higher genus surface. If no eigenvalue drifts into this gap (which we enforce by suitable restriction on the initial data) then one can establish a universal rate of decay for the energies. If the lowest eigenvalue drifts into this gap and remains there asymptotically then the rate of decay of these energies will depend upon the asymptotic value of the lowest eigenvalue and will no longer be universal. While it is straightforward to modify the definitions of the corrected energies to take this refinement into account we shall not do so here to avoid further complication of an already involved analysis. An extension of the definition of our corrected energies to the non polarized case and to the treatment of non trivial $S^1$ bundles is also relatively straightforward but for simplicity we shall not pursue that here either.

The sense in which our solutions are global in the expanding direction is that they exhaust the maximal range allowed for the mean curvature function on a manifold of negative Yamabe type, for which a zero mean curvature can only be asymptotically approached. The normal trajectories to our space slices all have an infinite proper time length. We do not attempt to prove causal geodesic completeness but that would be straightforward to do given the estimates we obtain.

Another question concerns the behavior of our solutions in the collapsing direction. Since our energies are decaying in the expanding direction they are growing in the collapsing direction and will eventually escape the region in which we can control their behavior. In particular we cannot use these arguments to show that our solutions extend to their conjectured natural limit as the mean curvature function tends to $-\infty$. There is another approach to the $U(1)$ problem however which, although local in nature, can describe a large family of $U(1)$-symmetric spacetimes by convergent expansions about the big-bang singularities themselves. This method, which is based on work
by S. Kichenassamy and its extensions by A. Rendall and J. Isenberg, can handle vacuum spacetimes that are "velocity dominated" at their big-bang singularities. Work by J. Isenberg and one of us (V.M) shows that the polarized vacuum solutions on $T^3 \times R$ are amenable to this analysis. In fact there are two larger families of "half-polarized" solutions that can also be rigorously treated and shown to have velocity dominated singularities. By contrast the general (non polarized) solution does not seem to be amenable to this kind of analysis and indeed numerical work by B. Berger shows that such solutions should have generically "oscillatory" rather than velocity dominated singularities. The expansion methods which produce these solutions near their velocity dominated singularities are essentially local and should be readily adaptable to other manifolds such as circle bundles over higher genus surfaces. Thus one should be able to generate a large collection of initial data sets for the problem dealt with in this paper which treats the further evolution globally in the expanding direction. Thus the machinery seems to be at hand for treating a large family of $U(1)$ symmetric solutions from their big-bang initial singularities to the limit of infinite expansion.

2 Equations.

The spacetime manifold $V$ is a principal fiber bundle with one dimensional Lie group $G$ and base $\Sigma \times R$, with $\Sigma$ a smooth 2 dimensional manifold which we suppose here to be compact.

The spacetime metric is invariant under the action of $G$, the orbits are the fibers of $V$ and are supposed to be space like. We write it in the form

$$(4) g = e^{-2\gamma} (3) g + e^{2\gamma} (\theta)^2,$$

where $\gamma$ is a scalar function and $(3) g$ a lorentzian metric on $\Sigma \times R$ which reads:

$$(3) g = -N^2 dt^2 + g_{ab}(dx^a + \nu^a dt)(dx^b + \nu^b dt)$$

$N$ and $\nu$ are respectively the lapse and shift of $(3) g$, while

$$g = g_{ab} dx^a dx^b$$

is a riemannian metric on $\Sigma$, depending on $t$.  


The 1-form $\theta$ is a connection on the fiber bundle $V$, represented in coordinates $(x^3, x^\alpha)$ adapted to the bundle structure by

$$\theta = dx^3 + A_\alpha dx^\alpha$$

Note that $A$ is a locally defined 1-form on $\Sigma \times R$.

2.1 Twist potential.

The curvature of the connection locally represented by $A$ is a 2-form $A$ on $\Sigma \times R$, given by

$$F_{\alpha\beta} = (1/2)e^{-3\gamma}\eta_{\alpha\beta\lambda}E^\lambda$$

where $E$ is an arbitrary closed 1-form if the equations $^{(4)}R_{\alpha\beta}=0$ are satisfied. Hence if $\Sigma$ is compact

$$E = d\omega + H$$

where $\omega$ is a scalar function on $V$, called the twist potential, and $H$ a representative of the 1-cohomology class of $\Sigma \times R$, for instance defined by a 1-form on $\Sigma$, harmonic for some given riemannian metric $m$.

2.2 Wave map equation.

The fact that $F$ is a closed form together with the equation $^{(4)}R_{33}=0$ imply (with the choice $H=0$) that the pair $u \equiv (\gamma, \omega)$ satisfies a wave map equation from $(\Sigma \times R, ^{(3)}g)$ into the hyperbolic 2-space, i.e. $R^2$ endowed with the riemannian metric $2(d\gamma)^2 + (1/2)e^{4\gamma}(d\omega)^2$. This wave map equation is a system of hyperbolic type when $^{(3)}g$ is a known lorentzian metric.

In this article we will consider only the polarized case that is we take $\omega$ and $H$ to be zero. Some of the computations and partial results hold however in the general case. It is why we keep the wave map notation wherever possible, since we intend to extend our final result to the general case in later work.

In the polarized case the wave map equation reduces to the wave equation for $\gamma$ in the metric $^{(3)}g$. 

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2.3 3-dimensional Einstein equations

When \((4) R_{3\alpha} = 0\) and \((4) R_{33} = 0\) the Einstein equations \((4) R_{\alpha\beta} = 0\) are equivalent to Einstein equations on the 3-manifold \(\Sigma \times R\) for the metric \((3) g\) with source the stress energy tensor of the wave map:

\[
(3) R_{\alpha\beta} = \partial_\alpha u.\partial_\beta u
\]

where a dot denotes a scalar product in the metric of the hyperbolic 2-space. We continue to use the same notation in the polarized case, that is we set \(\gamma = u\) and

\[
\partial_\alpha u.\partial_\beta u \equiv 2\partial_\alpha \gamma \partial_\beta \gamma.
\]

These Einstein equations decompose into

a. Constraints.

b. Equations for lapse and shift to be satisfied on each \(\Sigma_t\). These equations, as well as the constraints, are of elliptic type.

c. Evolution equations for the Teichmüller parameters, which are ordinary differential equations.

2.3.1 Constraints on \(\Sigma_t\).

One denotes by \(k\) the extrinsic curvature of \(\Sigma_t\) as submanifold of \((\Sigma \times R, (3) g)\); then, with \(\nabla\) the covariant derivative in the metric \(g\),

\[
k_{ab} \equiv (2N)^{-1}(-\partial_t g_{ab} + \nabla_a \nu_b + \nabla_b \nu_a)
\]

The equations (momentum constraint)

\[
(3) R_{0a} \equiv N(-\nabla_b k_a^b + \partial_a \tau) = \partial_b u.\partial_a u
\]

and (hamiltonian constraint, \((3) S_{00} \equiv (3) R_{00} + \frac{1}{2} N^2 (3) R)\)

\[
2N^{-2(3)} S_{00} \equiv R(g) - k_a^b k_a^b + \tau^2 = N^{-2}\partial_b u.\partial_b u + g^{ab} \partial_a u.\partial_b u
\]

do not contain second derivatives transversal to \(\Sigma_t\) of \(g\) or \(u\), they are the constraints. To transform the constraints into an elliptic system one uses the conformal method. We set

\[
g_{ab} = e^{2\lambda} \sigma_{ab},
\]
where \( \sigma \) is a riemannian metric on \( \Sigma \), depending on \( t \), on which we will comment later, and

\[
k_{ab} = h_{ab} + \frac{1}{2} g_{ab} \tau
\]

where \( \tau \) is the \( g \)-trace of \( k \), hence \( h \) is traceless.

We denote by \( D \) a covariant derivation in the metric \( \sigma \). From now on, unless otherwise specified, all operators are in the metric \( \sigma \), and indices are raised or lowered in this metric. We set

\[
u' = N^{-1} \partial_0 u
\]

with \( \partial_0 \) the Pfaff derivative of \( u \), namely

\[
\partial_0 = \partial_t - \nu^a \partial_a \text{ with } \partial_a = \frac{\partial}{\partial x^a}
\]

and

\[
u = e^{2\lambda} \nu'
\]

The momentum constraint reads if \( \tau \) is constant in space, a choice which we will make,

\[
D_b h^b_a = L_a, \quad L_a \equiv -D_a \dot{u}
\]

This is a linear equation for \( h \), independent of \( \lambda \). The general solution is the sum of a transverse traceless tensor \( h_{TT} \equiv q \) and a conformal Lie derivative \( r \). Such tensors are \( L^2 \)-orthogonal on \((\Sigma, \sigma)\).

The hamiltonian constraint reads as the semilinear elliptic equation in \( \lambda \):

\[
\Delta \lambda = f(x, \lambda) \equiv p_1 e^{2\lambda} - p_2 e^{-2\lambda} + p_3,
\]

with

\[
p_1 \equiv \frac{1}{4} \tau^2, \quad p_2 \equiv \frac{1}{2} (| \dot{u} |^2 + | h |^2), \quad p_3 \equiv \frac{1}{2} (R(\sigma) - |Du|^2)
\]
2.3.2 Equations for lapse and shift.

The lapse and shift are gauge parameters for which we obtain elliptic equations on each \( \Sigma_t \) as follows.

We impose that the \( \Sigma_t \)'s have constant (in space) mean curvature, namely that \( \tau \) is a given negative increasing function of \( t \). The lapse \( N \) satisfies then the linear elliptic equation

\[
\Delta N - \alpha N = -e^{2\lambda} \partial_t \tau
\]

with

\[
\alpha \equiv e^{-2\lambda}(|h|^2 + |\dot{u}|^2) + \frac{1}{2} e^{2\lambda} \tau^2
\]

The equation to be satisfied by the shift \( \nu \) results from the knowledge of \( \sigma_t \).

Indeed the definition of \( k \) implies that \( \nu \) satisfies a linear differential equation with an operator \( L \), the conformal Lie derivative, which we first write in the metric \( g \):

\[
(L_g \nu)_{ab} \equiv \nabla_a \nu_b + \nabla_b \nu_a - g_{ab} \nabla_c \nu^c = \phi_{ab}
\]

with

\[
\phi_{ab} \equiv 2Nh_{ab} + \partial_t g_{ab} - \frac{1}{2} g_{ab} g^{cd} \partial_t g_{cd}
\]

then in the metric \( \sigma \)

\[
(L_\sigma n)_{ab} \equiv D_a n_b + D_b n_a - \sigma_{ab} D_c n^c = f_{ab} \text{ with } n_a \equiv \nu_a e^{-2\lambda}
\]

where

\[
f_{ab} \equiv 2Ne^{-2\lambda} h_{ab} + \partial_t \sigma_{ab} - \frac{1}{2} \sigma_{ab} \sigma^{cd} \partial_t \sigma_{cd}
\]

The kernel of the dual of \( L \) is the space of transverse traceless symmetric 2-tensors, i.e. symmetric 2-tensors \( T \) such that

\[
g^{ab} T_{ab} = 0, \quad \nabla^a T_{ab} = 0. \tag{6}
\]

These tensors are usually called TT tensors. The spaces of TT tensors are the same for two conformal metrics.
2.3.3 Teichmuller parameters.

On a compact 2-dimensional manifold of genus $G \geq 2$ the space $T_{\text{eich}}$ of conformally inequivalent riemannian metrics, called Teichmuller space, can be identified (cf. Fisher and Tromba) with $M_{-1}/D_0$, the quotient of the space of metrics with scalar curvature $-1$ by the group of diffeomorphisms homotopic to the identity. $M_{-1} \rightarrow T_{\text{eich}}$ is a trivial fiber bundle whose base can be endowed with the structure of the manifold $R^n$, with $n = 6G - 6$.

We impose to the metric $\sigma_t$ to be in some chosen cross section $Q \rightarrow \psi(Q)$ of the above fiber bundle. Let $Q^I, I = 1,...,n$ be coordinates in $T_{\text{eich}}$, then $\partial \psi / \partial Q^I$ is a known tangent vector to $M_{-1}$ at $\psi(Q)$, that is a traceless symmetric 2-tensor field on $\Sigma$, sum of a transverse traceless tensor field $X_I(Q)$ and of the Lie derivative of a vector field on the manifold $(\Sigma, \psi(Q))$. The tensor fields $X_I(Q), I = 1,...n$ span the space of transverse traceless tensor fields on $(\Sigma, \psi(Q))$. The matrix with elements

$$\int_\Sigma X^a_I X_Jab \mu_{\psi(Q)}$$

is invertible.

**Lemma 1** If we impose to the metric $\sigma_t$ to lie in the chosen cross section, i.e. $\sigma_t \equiv \psi(Q(t))$, the solvability condition for the shift equation determines $dQ^I/dt$ in terms of $h_t$.

Proof.

The time derivative of $\sigma$ is given by

$$\partial_t \sigma = (dQ^I/dt) \partial \psi / \partial Q^I$$

hence it is of the form

$$\partial_t \sigma_{ab} = \frac{dQ^I}{dt} X_{Iab} + C_{ab}$$

where $C$ is a Lie derivative, $L^2$ orthogonal to TT tensors.

The shift equation on $\Sigma_t$ is solvable if and only if its right hand side $f$ is $L^2$ orthogonal to TT tensors, i.e. to each tensor field $X_I$. Theses conditions read

$$\int_{\Sigma_t} f_{ab} X_{Jab} \mu_{\sigma_t} = 0$$
We have seen that \( h \) is the sum of a tensor \( r \) which is in the range of the conformal Killing operator, hence \( L^2 \) orthogonal to TT tensors, and a TT tensor. This last tensor can be written with the use of the basis \( X_I \) of such tensors, the coefficients \( P^I \) depending only on \( t \):

\[
h_{ab}^{TT} = P^I(t) X_{I,ab}
\]

The orthogonality conditions read, using the fact that the transverse tensors \( X_I \) are orthogonal to Lie derivatives and are traceless:

\[
\int_{\Sigma_t} [2 Ne^{-2\lambda} (r_{ab} + P^I X_{I,ab}) + (dQ^I/dt) X_{I,ab}] X_j^{ab} \mu_{\sigma} = 0
\]

The tangent vector \( dQ^I/dt \) to the curve \( t \to Q(t) \) and the tangent vector \( P^I(t) \) to \( T_{eich} \) are therefore linked by the linear system

\[
X_{IJ} \frac{dQ^I}{dt} + Y_{IJ} P^I + Z_J = 0
\]

with

\[
X_{IJ} \equiv \int_{\Sigma_t} X_I^{ab} X_{J,ab} \mu_{\sigma},
\]

\[
Y_{IJ} \equiv \int_{\Sigma_t} 2 Ne^{-2\lambda} X_I^{ab} X_{J,ab} \mu_{\sigma},
\]

\[
Z_J \equiv \int_{\Sigma_t} 2 Ne^{-2\lambda} r_{ab} X_J^{ab} \mu_{\sigma}.
\]

We will now construct an ordinary differential system for the evolution of the \( Q^I \) and \( P^I \) by considering the as yet non solved 3-dimensional Einstein equations

\[(3) R_{ab} = \rho_{ab} \equiv \partial_a u \partial_b u\]

**Lemma 2** The constraint equations together with the lapse and the wave map equations imply that \( N (3) R_{ab} - \rho_{ab} \) with \( \rho_{ab} \equiv \partial_a u \partial_b u \) is a transverse traceless tensor on each \( \Sigma_t \).
Proof. 1. The equations

\[(3)S_{00} = T_{00}\]  \hspace{1cm} (7)

and

\[(3)R_{00} = \rho_{00}\]  \hspace{1cm} (8)

imply

\[(3)R = \rho\]  \hspace{1cm} (9)

since

\[(3)S_{00} - T_{00} \equiv (3)R_{00} - \frac{1}{2}g_{00}^{(3)} R - (\rho_{00} - \frac{1}{2}g_{00} \rho)\]  \hspace{1cm} (10)

hence

\[(3)R^{ab} - \rho^{ab} = (3)S^{ab} - T^{ab}\]

The equations 2 and 3 imply

\[g^{ab}(3)R_{ab} - \rho_{ab}) = 0\]

On the other hand the Bianchi identity in the 3-metric \(g\) gives

\[\nabla^{a}(3)S^{ab} - T^{ab} = 0.\]

An elementary calculus using the connexion coefficients of \((3)g\) and \(g\) shows that, due to equations previously satisfied, this equation reduces to the following divergence in the metric \(g\):

\[\nabla_{a}[N((3)R^{ab} - \rho^{ab})] = 0\]

The tensor \(N((3)R_{ab} - \rho_{ab})\) is therefore a traceless and transverse tensor on \((\Sigma, g)\), and hence also on \((\Sigma, \sigma)\), by conformal invariance of this property for symmetric 2-covariant tensors.

We deduce from this lemma that a necessary and sufficient condition for the previous equations to imply \((3)R_{ab} - \rho_{ab} = 0\) is that the tensor \(N((3)R_{ab} - \rho_{ab})\)
\( \rho_{ab} \) be \( L^2 \) orthogonal to transverse traceless tensors on \((\Sigma_t, \sigma_t)\), i.e. to each of the TT tensors \( X_I \) defined above through the cross section \( \psi \) where we choose \( \sigma_t \), that is

\[
\int_{\Sigma_t} N^{(3)} R_{ab} - \rho_{ab} X^a_b \mu_{\sigma_t} = 0, \quad \text{for } J = 1, 2, \ldots, 6G - 6
\]

We recall that

\[
^{(3)} R_{ab} \equiv R_{ab} - N^{-1} \tilde{\partial}_0 k_{ab} - 2k_{ac} k^c_b + \tau k_{ab} - N^{-1} \nabla_a \partial_b N
\]

with

\[
k_{ab} \equiv P^I X_{I,ab} + r_{ab} + \frac{1}{2} g_{ab} \tau
\]

and \( \tilde{\partial}_0 \) is an operator on time dependent space tensors (cf. C-B and York) defined by, with \( \mathcal{L}_\nu \) the Lie derivative in the direction of \( \nu \),

\[
\tilde{\partial}_0 \equiv \partial_t - \mathcal{L}_\nu
\]

We thus obtain an ordinary differential system of the form

\[
X^I_{JJ} \frac{dP^I}{dt} + \Phi_J(P, \frac{dQ}{dt}) = 0
\]

where \( \Phi \) is a polynomial of degree 2 in \( P \) and \( dQ/dt \) with coefficients depending smoothly on \( Q \) and directly but continuously on \( t \) through the other unknown, namely:

\[
\Phi_J \equiv A_{JKL} P^I P^K + B_{JKL} P^I \frac{dQ^K}{dt} + C_{JJ} P^I + D_J
\]

with

\[
A_{JKL} \equiv \int_{\Sigma_t} 2N e^{2\lambda} X^c_{I,ab} X_{K,dc} X^{ab}_J \mu_{\sigma_t}
\]

\[
B_{JKL} \equiv \int_{\Sigma_t} \frac{\partial X_{Lab}}{\partial Q^K} X^{ab}_J \mu_{\sigma_t}
\]
\[ C_J J \equiv \int_{\Sigma_t} \left[ (-L_{\nu} X_I)_{ab} + 4Ne^{-2\lambda} r_c^c X_{I,ac} - \tau N X_{I,ab} \right] X^{ab}_{J \mu \sigma} \]

and, using integration by parts and the transverse property of the \( X_I \) to eliminate second derivatives of \( \Xi \) (recall that \( \nabla_a \partial_b \Xi \equiv D_a \partial_b \Xi - 2 \partial_a \lambda \partial_b \Xi \))

\[ D_J \equiv \int_{\Sigma_t} \left( -\partial_0 r_{ab} - 2Ne^{-2\lambda} r_c^c + \tau N r_{ab} + 2\partial_b \lambda \partial_b \Xi - \partial_a u. \partial_b u \right) X^{ab}_{J \mu \sigma}. \]

### 3 Homogeneous solution.

**Theorem 3** A particular solution, obtained by taking for \( u \) a constant wave map and for \( h \) the zero tensor, is given by:

\[ (4) g = -4dt^2 + 2t^2 \sigma_{ab} dx^a dx^b + \theta^2 \]

with \( \sigma \) a metric on \( \Sigma \) independent of \( t \) and of scalar curvature \(-1\), and \( \theta \) a flat connexion 1-form on the bundle.

Proof. The wave map equation is satisfied by any constant map. Such a map has zero stress energy tensor. The momentum constraint is then satisfied by \( h = 0 \), hence

\[ k_{ab} = \frac{1}{2} \tau g_{ab} \]

The hamiltonian constraint is satisfied by a constant in space \( \lambda \) given if \( R(\sigma) = -1 \) by

\[ e^{2\lambda} = \frac{2}{\tau^2} \]

the shift equation is then satisfied by \( \nu = 0 \) and the lapse equation by

\[ N = 2 \]

A straightforward computation shows that \( (3) R_{ab} = 0 \). All the equations \( Ricci((4) g) = 0 \) are satisfied.

**Remark 4** The hypothesis imply that the bundle \( M \to \Sigma \) is a trivial bundle.
4 Local existence theorem.

4.1 Cauchy problem.

The unknowns which permit the reconstruction of the spacetime metric in the
gauge $\tau \equiv \tau(t)$, given some smooth cross section $Q \rightarrow \psi(Q)$ of Teichmuller
space $T_{eich}$, are on the one hand $u = \gamma$ satisfying the wave equation in the
metric $(3)g$, on the other hand $\lambda, N$ and $\nu$, which satisfy elliptic equations
on each $\Sigma_t$, and also a curve $Q(t)$ in $T_{eich}$ which determines the metric $\sigma_t \equiv 
\psi(Q(t))$ on $\Sigma_t$. An intermediate unknown is the traceless tensor $h$ which splits
into a transverse part and a conformal Lie derivative of $\sigma_t$ in the direction of
a vector $Y$ which satisfies also an elliptic system on $\Sigma_t$. The transverse part is
determined through a field of tangent vectors to $T_{eich}$ at the points of $Q(t)$.

Definition 5 The Cauchy data on $\Sigma_{t_0}$ denoted $\Sigma_0$ are:

1. A $C^\infty$ riemannian metric $\sigma_0$ which projects onto a point $Q(t_0)$ of $T_{eich}$
and a $C^\infty$ tensor $q_0$ transverse and traceless in the metric $\sigma_0$.

2. Cauchy data for $u$ and $\dot{u}$ on $\Sigma_0$, i.e.

$$u(t_0,.) = u_0 \in H_2, \dot{u}(t_0,.) = \dot{u}_0 \in H_1$$

where $H_s$ is the usual Sobolev space on $(\Sigma, \sigma_0)$.

4.2 Functional spaces.

Definition 6 Let $\sigma_t$ be a curve of $C^\infty$ riemannian metrics on $\Sigma$, uniformly
equivalent to the metric $\sigma_0$ for $t \in [t_0, T]$ and $C^1$ in $t$. Such metrics are called
regular for $t \in [t_0, T]$

1. The spaces $W^p_s(t)$ are the usual Sobolev spaces of tensor fields on the
riemannian manifold $(\Sigma, \sigma_t)$.

By the hypothesis on $\sigma_t$ the norms in $W^p_s(t)$ are uniformly equivalent for
$t \in [t_0, T]$ to the norm in $W^p_s(t_0)$. We set $W^2_s(t) = H_s(t)$. When working on
one slice $\Sigma_t$ we will often omit reference to the $t$ dependence of the norm.

2. The spaces $E^p_s(T)$ are the Banach spaces of $t$ dependent tensor fields $f$ on $\Sigma$

$$E^p_s(T) \equiv C^0([t_0, T], W^p_s) \cap C^1([t_0, T], W^p_{s-1}).$$
with norm

$$\|f\|_{E_s^p(T)} = \sup_{0 \leq t \leq T} (\|f\|_{W_s^p(t)} + \|\partial_t f\|_{W_s^{p-1}(t)}).$$

We set $E_s^2(T) = E_s(T)$.

We will proceed in two steps:

Case a. $Du_0, u_0 \in H_2$
Case b. $Du_0, u_0 \in H_1$

We will need the following lemma (we set $E_s = E_s^2$)

**Lemma 7** Let $\sigma_t$ be a regular metric on $\Sigma \times [t_0, T]$ then

a. If $Du, u \in E_2(T)$ then $Du.Du, u \in E_2(T)$,
b. If $Du, u \in E_1(T)$ then $Du.Du, u \in E_1^p(T) \cap E_0^q(T)$, $1 \leq p < 2$, $1 \leq q < \infty$.

Proof. a. Since in dimension 2 the space $H_2$ is an algebra one has

$$Du.Du, u \in C^0([t_0, T], H_2)$$

On the other hand we have

$$|\partial_t (Du.Du)| = 2|\partial_t Du.Du| \leq 2|\partial_t Du||Du|$$

hence by multiplication properties of Sobolev spaces

$$\partial_t (Du.Du) \in C^0([t_0, T], H_1)$$

b. If $Du \in E_1$ then $Du \in E_0^q \equiv C^0([t_0, T], L^q)$, for all $q < \infty$ by the standard Sobolev embedding theorem, and so does $Du.Du$.

We have

$$|D(Du.Du)| = 2|D^2 u.Du| \leq 2|D^2 u||Du|$$

hence $D(Du.Du) \in E_0^p$ for all $1 \leq p < 2$ since $D^2 u \in E_0$ and $Du \in E_0^q, 1 \leq q < \infty$.

An analogous proof gives the result for the other products.

Using again $|\partial_t (Du.Du)|| \leq 2|\partial_t Du||Du|$ we obtain $\partial_t (Du.Du) \in E_0^p$ for $1 \leq p < 2$ since we have by definition $\partial_t Du \in E_0^2$. Analogous reasoning with $u$ completes the proof.
4.3 Resolution of the elliptic equations for given $Q(t)$, $P(t)$ and $u$

We have supposed chosen a smooth cross section $Q \to \psi(Q)$ of $M_{-1}$ over the Teichmuller space $T_{eich}$. We suppose given a $C^1$ curve $t \to Q(t)$ contained when $t \in [t_0, T]$ in a compact subset of $T_{eich}$, and a continuous set of tangent vectors $P$ to $T_{eich}$ at points of this curve. We are then given by lift to $M_{-1}$ a regular metric $\sigma_t$ for $t \in [t_0, T]$, with scalar curvature -1, together with a smooth symmetric 2-tensor $h_{TT}^t \equiv q_t$ transverse and traceless in the metric $\sigma_t$ and depending continuously on $t$.

4.3.1 Determination of $h$.

We have set

$$h_{ab} = q_{ab} + r_{ab}$$

where $q$ and $r$ are traceless $q$ is transverse and $r$ is a conformal Lie derivative, i.e.

$$D_a q^a_b = 0 \text{ and } q^a_a = 0$$

$$r_{ab} = D_a Y_b + D_b Y_a - \sigma_{ab} D_c Y^c$$

**Determinant of $q$.** The traceless transverse tensor $q$ on $(\Sigma_t, \sigma_t)$ is deduced by lifting its given projection onto the tangent space to Teichmuller space at the point $Q(t)$. It is smooth and depends continuously on $t \in [t_0, T]$. Let us denote by $X_I(Q), I = 1, ..., 6G - 6$, a basis of traceless transverse tensor fields on $(\Sigma, \psi(Q))$ then

$$q_t = X_I(Q(t)) P^I(t)$$

**Determinant of $r$.** The vector $Y$ satisfies on each $\Sigma_t$ the elliptic system with zero kernel (in accordance with the fact that $(\Sigma, \sigma)$ does not admit conformal Killing fields when $R(\sigma) < 0$),

$$D^a r_{ab} \equiv D^a D_a Y_b + \frac{1}{2} R(\sigma) Y_b = L_b \equiv -D_b u. u$$
Case a. \( L \in E_2(T) \). It results from elliptic theory that the system satisfied by \( Y \) has for each \( t \in [0, T] \) one and only one solution in \( H_4(t) \) and there exists a constant depending only on \( \sigma_t \) such that

\[
||r_t||_{H_3(t)} \leq C_{\sigma_t} ||Du.\dot{u}||_{H_2(t)}
\]

The constant \( C_{\sigma_t} \) is invariant under diffeomorphism acting on \( \sigma_t \), that is it depends only on its projection on the Teichmuller space of \( \Sigma \), hence is uniformly bounded under the hypothesis made on \( \sigma_t \). We denote by \( M_{\sigma,T} \) such a constant.

We have since the norms \( W^p_s(t) \) and \( W^p_s \) are uniformly equivalent

\[
||r_t||_{H_3} \leq M_{\sigma,T}||(Du.\dot{u})_t||_{H_2}
\]

Derivations with respect to \( t \) of the equation for \( Y \) show that for a regular \( \sigma_t \) we have

\[
r \in E_3(T), ||r||_{E_3(T)} \leq M_{\sigma,T} ||Du||_{E_2(T)} \times ||\dot{u}||_{E_2(T)}
\]

Case b. \( L \in E^p_1(T) \). The system for \( Y \) has one and only one solution in \( W^p_s(t) \) for each \( t \in [t_0, T] \), then \( r_t \) is in \( W^p_2(t) \) and there exists a constant \( C_{\sigma_t} \) such that

\[
||r_t||_{W^p_2(t)} \leq C_{\sigma_t} ||(Du.\dot{u})_t||_{W^p_1(t)}
\]

One proves also that \( \partial_t r \in W^p_1(t) \) hence \( r \in E^p_2(T) \) and there exists a constant \( M_{\sigma,T} \) such that

\[
||r||_{E^p_2(T)} \leq M_{\sigma,T} ||Du||_{E_1(T)} \times ||\dot{u}||_{E_1(T)}
\]

4.3.2 Case of initial values.

On the initial manifold \( \Sigma_{t_0} \) we have given \( q_0 \in C^\infty \), and \( r_0 \) satisfies the inequality (we abbreviate to \( ||.|| \) the \( L^2 \) norm on \( (\Sigma, \sigma_0) \))

\[
||r_0|| \leq C_{\sigma_0} ||Du_0.\dot{u}_0||
\]

hence \( h_0 \) is small in \( L^2 \) norm if it is so of \( q_0 \) while \( Du_0 \) and \( \dot{u}_0 \) are small in \( H_1 \) norm.
4.3.3 Determination of the conformal factor $\lambda$

On each $\Sigma_t$ the conformal factor $\lambda_t$ satisfies the equation, with $\Delta \equiv \Delta_{\sigma_t}$ the laplacian in the metric $\sigma_t$ (we omit the writing of $t$ to simplify the notation)

$$\Delta \lambda = f(\lambda) \equiv p_1 e^{2\lambda} - p_2 e^{-2\lambda} + p_3$$

where the coefficients $p$ are given by, with $R(\sigma) = -1$,

$$p_1 = \frac{1}{4} \tau^2, p_2 = \frac{1}{2} (|h|^2 + |\dot{u}|^2), p_3 = \frac{1}{2} (R(\sigma) - |Du|^2)$$

**Case a.** We suppose that the coefficients $p$ are given functions in $E_2(T)$. This hypothesis is consistent with $Du, \dot{u} \in E_2(T)$ and $h \in E_2(T)$.

We know from elliptic theory that the semi linear elliptic equation for $\lambda$ on $(\Sigma_t, \sigma_t)$ admits a solution in $H^4(t)$, which is included in $C^2$, if it admits a subsolution $\lambda_-$ and a supersolution $\lambda_+$, i.e. $C^2$ functions such that

$$\Delta \lambda_+ \leq f(\lambda_+), \quad \Delta \lambda_- \geq f(\lambda_-), \quad \lambda_- \leq \lambda_+$$

We construct sub and super solutions as follows.

We define the number $\omega$ to be the real root of the equation

$$P_1 e^{2\omega} - P_2 e^{-2\omega} + P_3 = 0$$

where the $P$’s are the integrals of the $p$’s on $\Sigma_t$.

By the Gauss Bonnet theorem the volume of $(\Sigma_t, \sigma_t)$ is a constant if $R(\sigma_t)$, is constant. We have here $R(\sigma_t) = -1$, hence:

$$V_\sigma \equiv \int_{\Sigma_t} \mu_\sigma = - \int_{\Sigma_t} R(\sigma) \mu_\sigma = -4\pi \chi$$

We find:

$$P_2 = \frac{1}{2} (|h|^2 + |\dot{u}|^2) \geq 0$$

$$P_1 = \frac{1}{4} V_\sigma \tau^2 > 0, P_3 = -\frac{1}{2} (V_\sigma + |Du|^2) < 0$$
hence $e^{2\omega}$ exists, is unique and satisfies
\[ e^{2\omega} \geq 2\tau^{-2}. \]

We define $v \in H_4$ as the solution with mean value zero on $\Sigma_t$ of the linear equation
\[ \Delta v = f(\omega) \equiv p_1 e^{2\omega} - p_2 e^{-2\omega} + p_3 \]
Such a solution exists and is unique, because $f(\omega)$ has mean value zero on $\Sigma_t$

Lemma 8. The functions $\lambda_+ = \omega + v - \min_{\Sigma} v$ and $\lambda_- = \omega + v - \max_{\Sigma} v$ are respectively a super and sub solution of the equation for $\lambda$.

Proof. We have $\lambda_+ \geq \omega$ and $\lambda_- \leq \omega$ hence $f(\lambda_+) \geq f(\omega) \geq f(\lambda_-)$, since $f$ is an increasing function of $\lambda$, while $\Delta \lambda_+ = \Delta \lambda_- = \Delta v = f(\omega)$.

The solution $\lambda \in H_4$ thus obtained for each $t \in [t_0, T]$ is unique, due to the monotony of the function $f$. Its $H_4$ norm depends continuously on $t$. Derivation with respect to $t$ of the equation satisfied by $\lambda$ shows that $\partial_t \lambda \in C^0([t_0, T], H_3)$.

We have proved:

Theorem 9 The equation for $\lambda$ has one and only one solution $\lambda \in E_4(T)$ under the hypothesis a (where $p_i \in E_2(T)$).

Case b.

Theorem 10 The equation for $\lambda$ has one and only one solution $\lambda \in E_3^p(T)$ under the hypothesis b (where $p_i \in E_3^p(T)$).

Proof. Consider a Cauchy sequence of functions $p_2^{(n)} \geq 0, p_3^{(n)} + \frac{1}{2} \leq 0$, both in $E_3(T)$, converging in $E_3^p(T)$ to functions $p_2, p_3 + \frac{1}{2}$. For each $n$ there is a solution $\lambda_{(n)} \in E_4(T)$ of the conformal factor equation. The difference $\lambda_{(n)} - \lambda_{(m)}$ satisfies the equation
\[ \Delta (\lambda_{(n)} - \lambda_{(m)}) = p_1(e^{2\lambda_{(n)}} - e^{2\lambda_{(m)}}) - p_2^{(n)}(e^{-2\lambda_{(n)}} - e^{-2\lambda_{(m)}}) \]
\[ + (p_2^{(m)} - p_2^{(n)})e^{-2\lambda_{(m)}} + p_3^{(n)} - p_3^{(m)} \]

Applying elementary calculus inequalities to the estimate of $(a-b)^{-1}(e^a - e^b)$ and $(a-b)^{-1}(e^{-a} - e^{-b})$ one obtains a well posed linear elliptic equation for $\lambda_{(n)} - \lambda_{(m)}$ and an inequality for its norm in $W_3^p$ for each $t \in [t_0, T]$. We thus have shown the convergence of the sequence to a limit $\lambda$ which satisfies the required equation. One can prove similarly that $\lambda \in E_3^p(T)$. The uniqueness of the solution results from the monotony of $f(\lambda)$.
Bounds for $\lambda$ When $\lambda \in C^2$ one obtains a lower bound by using the maximum principle: at a minimum of $\lambda$ we have $\Delta \lambda \geq 0$. Hence a minimum $\lambda_m$ of $\lambda$ satisfies the inequality

$$e^{2\lambda_m} \geq \frac{2}{\tau^2}, \quad \text{i.e.} \quad e^{-2\lambda_m} \leq \frac{1}{2}\tau^2$$

when $\lambda \in E^p_3$ is solution of the equation it satisfies the same inequality since $W^p_3 \subset C^0$ and $\lambda$ can be obtained as a limit in $W^p_3$ of functions satisfying this inequality.

An analogous argument shows that

$$\lambda_- \leq \lambda \leq \lambda_+$$

with

$$\lambda_- = \omega - \max v + v, \quad \text{and} \quad \lambda_+ = \omega + v - \min v$$

where $v \in E_2 \cap E_3^p$ is the solution with mean value zero on $\Sigma_t$ of the linear equation

$$\Delta v = f(\omega) \equiv p_1 e^{2\omega} - p_2 e^{-2\omega} + p_3$$

with $e^{2\omega}$ the positive solution of the equation

$$P_1 e^{4\omega} + P_3 e^{2\omega} - P_2 = 0$$

Case of initial values. The above construction applies in particular on the initial surface $\Sigma_0$. In this case the functions $u_0$ and $\dot{u}_0$ are considered as given. We have

$$\Delta u_0 = f(\omega_0) \equiv p_{1,0} e^{2\omega_0} - p_{2,0} e^{-2\omega_0} + p_{3,0}$$

with

$$p_{1,0} = \frac{1}{4} \tau^2_0, \quad p_{2,0} = \frac{1}{2} (\| h_0 \|^2 + || \dot{u}_0 ||^2), \quad p_{3,0} = -\frac{1}{2} (1 + \| Du_0 \|^2)$$

We have

$$e^{2\omega_0} = \frac{(V_\sigma + \| Du_0 \|^2)}{V_\sigma \tau^2_0} + \sqrt{(V_\sigma + \| Du_0 \|^2)^2 + 2\tau^2_0(\| h_0 \|^2 + \| \dot{u}_0 \|^2)}$$

we see that $e^{2\omega_0}$ tends to $\frac{2}{\tau^2_0}$ and $\| f(\omega_0) \|$ tends to zero when $q_0$ tends to zero as well as the $H_1$ norms of $Du_0$ and $\dot{u}_0$ (then the $L^2$ norm of $h_0$ tends also to zero).
4.3.4 Determination of the lapse $N$.

The lapse $N$ satisfies the equation

$$\Delta N - \alpha N = -e^{2\lambda} \partial_t \tau$$  \hspace{1cm} (11)$$

with

$$\alpha e^{-2\lambda} = \frac{1}{2} \tau^2 + e^{-4\lambda}(|\dot{u}|^2 + |h|^2) > 0$$

It is a well posed elliptic equation on $(\Sigma_t, \sigma_t)$, when $u, h$ and $\lambda$ are known, which has one and only one solution, always positive, in $E_4(T)$ in case a, in $E_3^p(T)$ in case b. Indeed:

**Case a.** We have $\dot{u} \in E_2(T), h \in E_3(T), \lambda \in E_4(T)$ hence also $e^{2\lambda} \in E_4(T)$ and $\alpha \in E_2(T)$. The equation has then a solution $N \in E_4(T)$.

**Case b.** We have $|\dot{u}|^2 + |h|^2 \in E_1^p(T), e^{2\lambda}, e^{-2\lambda} \in E_3^p(T)$. The equation has a solution $N \in E_3^p(T)$.

**Upper bound of $N$.** At a maximum $x_M$ of $N \in C^2$ we have $(\Delta N)(x_M) \leq 0$ hence this maximum $N_M$ is such that

$$N_M \leq (\alpha^{-1}e^{2\lambda} \partial_t \tau)(x_M)$$

a fortiori

$$N_M \leq \frac{2 \partial_t \tau}{\tau^2}$$

A reasoning analogous to that given for $\lambda$ shows that this upper bound also holds in case b.

4.3.5 Determination of the shift $\nu$.

The definition of $k$ implies that $n \equiv e^{-2\lambda} \nu$ satisfies a linear differential equation involving an operator L, the conformal Lie derivative, with injective symbol:

$$(L_{\sigma_t}n)_{ab} \equiv D_a n_b + D_b n_a - \sigma_{ab} D_c n^c = f_{ab}$$
with
\[ f_{ab} \equiv 2Ne^{-2\lambda}h_{ab} + \partial_i \sigma_{ab} - \frac{1}{2} \sigma_{ab} \sigma^{cd} \partial_t \sigma_{cd} \]

The kernel of the dual of \( L \) is the space of transverse traceless symmetric 2-tensors in the metric \( \sigma_t \), the equation for \( \nu \) admits a solution if and only if \( f \) is \( L^2 \)-orthogonal to all such tensors, i.e.
\[ \int_{\Sigma_t} f_{ab} X^a_I \mu_{\sigma_t} = 0, \text{ for } I = 1, \ldots, 6G-6 \]

This integrability condition will not in general be satisfied with the arbitrary choice of \( P(t) \), that is of \( q_t \equiv h^{TT}_t \). In this subsection \( P(t) \) is not considered as given. We set in the expression of \( f_{ab} \)
\[ h_t = P^I(t) X_I(t) + r_t \]

When \( \sigma_t \) is a known \( C^1 \) function of \( t \) the integrability condition determines \( P(t) \) as a continuous field of tangent vectors to \( \mathcal{T}_{\text{eich}} \) by an invertible system of ordinary linear equations.

When \( h \) is so chosen the equation for \( n \) has a solution, unique since \( L_\sigma \) has a trivial kernel on manifolds with \( R(\sigma) = -1 \). It results from elliptic theory that \( n \in E_4(T) \) in case a, and \( n \in E_3^p(T) \) in case b. The same properties hold for \( \nu \).

### 4.4 Wave equation, local solution.

The wave equation on \((\Sigma \times R)\) in the metric \( (3)^g \) reads
\[ -N^{-1} \partial_0 (N^{-1} \partial_0 u) + Ng^{ab} \nabla_a (N \partial_b u) + N^{-1} \tau \partial_0 u = 0 \]

We suppose that \( \sigma_t \) is a given regular riemannian metric for \( t \in [t_0, T] \) and that \( \lambda, N, \nu \) are given in \( E^p_3(T) \) with \( p > 1 \) and \( N > 0 \). Then we have \( (3)^g \in E^p_3(T) \subset C^1(\Sigma \times [t_0, T]) \) and \( (3)^g \) has hyperbolic signature. It is easy to prove along standard lines that the Cauchy problem with data \( u_0, (\partial_t u)_0 \in H_2 \times H_1 \) has a solution such that \((u, \partial_t u) \in E_2(T) \times E_1(T) \) on \( \Sigma \times [t_0, T] \). The initial value \((\partial_t u)_0 \) is the product of the datum \( \dot{u}_0 \) by \( e^{-2\lambda_0} \), it belongs to \( H_1 \) under the hypothesis made in section 1.1 on the Cauchy data.
4.5 Teichmüller parameters.

We suppose known $h \in E_2^2(T), \lambda, N, \nu \in E_3^3(T)$, $u \in E_2(T)$, and we suppose given $Q \rightarrow \psi(Q)$ a smooth cross section of $M_{-1}$ over $T_{eich}$. The unknown is the curve $t \rightarrow Q(t)$. We have $\sigma_t \equiv \psi(Q(t))$ and

$$\partial_t \sigma_{ab} = \frac{dQ^I}{dt} X_{I,ab} + C_{ab}$$

with $X_I(Q)$ a basis of the space of TT tensors on $(\Sigma, \psi(Q))$ and $C$ a conformal Lie derivative, $L^2$ orthogonal to TT tensors. The curve $t \rightarrow Q(t)$ and the tangent vector $P^I(t)$ to $T_{eich}$ satisfy the ordinary differential system (cf. section 2.3.3)

$$X_{IJ} \frac{dQ^I}{dt} + Y_{IJ} P^I + Z_J = 0$$

and

$$X_{IJ} \frac{dP^I}{dt} + \Phi_J(P, \frac{dQ}{dt}) = 0$$

This quasi linear first order system for $P$ and $Q$ has coefficients continuous in $t$ and smooth in $Q$ and $P$. The matrix of the principal terms, $X_{IJ}$, is invertible. There exists therefore a number $T > 0$ such that the system has one and only one solution in $C^1([t_0,T])$ with given initial data $P_0, Q_0$.

4.6 Local existence theorem.

We can now prove the following theorem

**Theorem 11** The Cauchy problem with data $(u_0, \dot{u}_0) \in H_2 \times H_1$, on $\Sigma_{t_0}$ (denoted $\Sigma_0$) and $Q_0$, a point in $T_{eich}$, $P_0$ a tangent vector to $T_{eich}$, for the Einstein equations with $U(1)$ isometry group (polarized case) has a solution with $\sigma_t$ a regular metric on $\Sigma_t$ for $t \in [t_0,T]$ and $u \in E_2(T), T > t_0$, if $T - t_0$ is small enough. This solution is unique when $\tau$, depending only on $t$, is chosen together with a cross section of $M_{-1}$ over $T_{eich}$.

**Remark 12** One has, for this solution, $\lambda, N, \nu \in E_3^p(T), 1 < p < 2$ and $N > 0$. 

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Proof. The proof is straightforward, using iteration to solve alternatively
the elliptic systems, the wave equation and the differential system satisfied
by Teichmüller parameters, with \( \tau \) a given function of \( t \) and \( \sigma_t \) required to
remain in a chosen cross section of \( M_{-1} \) over \( T_{eich} \). The iteration converges if
\( T - t_0 \) is small enough. The limit can be shown to be a solution of Einstein
equations with \((^{(3)}g)\) in constant mean curvature gauge by standard arguments,
the 2-metric \( g \) is conformal with the factor \( e^{2A} \) to a metric in the chosen cross
section by construction.

This local existence theorem can be extended to the non polarized case.

5 Scheme for a global existence theorem.

As it is well known we will deduce from our local existence theorem a global
one, i.e. on \( \Sigma \times [t_0, \infty) \), if we can prove that the curve \( Q(t) \) remains in a
compact subset of \( T_{eich} \) and that neither the \( H_2 \times H_1 \) norm of \((u(., t), u(., t))\)
nor the \( E^2_3 \) norms of \( \lambda(., t), N(., t), \nu(., t) \) blow up when \( t \in [t_0, \infty) \) while \( N \)
remains strictly positive.

If the spacetime we construct is supported by the manifold \( M \times [t_0, \infty) \) it
will reach a moment of maximum expansion. It will be after an infinite proper
time for observers moving along orthogonal trajectories of the hypersurfaces
\( M_t \equiv M \times \{t\} \) if the lapse function is uniformly bounded below by a strictly
positive number.

Our proof of this fact will rely on various refined estimates, using in
particular corrected energies. The correction of the energies poses special
problems in the non polarized case, which we will treat in another paper.

5.1 Notations.

\(|.|\) and \( |.|_g\) : pointwise norms of scalars or tensors on \( \Sigma \), in the \( \sigma \) or \( g \) metric
\( \| \cdot \| \) and \( \| \cdot \|_p\): \( L^2 \) and \( L^p \) norms in the \( \sigma \) metric
\( \| \cdot \|_g\): \( L^2 \) norm in the \( g \) metric.

A lower case index \( m \) or \( M \) denote respectively the lower or upper bound
of a scalar function on \( \Sigma_t \). It may depends on \( t \).

When we have to make a choice of the time parameter \( t \) we will set

\[
t = -\tau^{-1}
\] (12)
then \( t \) will increase from \( t_0 > 0 \) to infinity when, \( \Sigma_t \) expanding, \( \tau(t) \) increases from \( \tau_0 < 0 \) to zero. With this choice the upper bound on \( N \) of subsection 4.3.4 reads

\[ N \leq 2. \quad (13) \]

**Remark 13** Other admissible choices of \( t \), for instance \( \tau = t, t \in [t_0, 0) \), \( t_0 = \tau_0 < 0 \), would lead to the same geometrical conclusions.

### 5.2 Fundamental inequalities.

**Lemma 1.** Let \( f \) be a scalar function on \( \Sigma \). the following inequalities hold

1.

\[
\| f \|_q \leq e^{-2\lambda_m/q} \| f \|_{L^q(g)}
\]

\[
\| f \|_{L^q(g)} \leq e^{2\lambda_M/q} \| f \|_q
\]

2.

\[
|Df|_g = e^{-\lambda} |Df|, \quad \| Df \|_{L^\infty(g)} \leq e^{-\lambda_m} \| Df \|_{\infty}
\]

and if \( q \geq 2 \)

\[
\| Df \|_{L^q(g)} \leq e^{-\lambda_m(q-2)/q} \| Df \|_q
\]

in particular

\[
\| Df \| = \| Df \|_g
\]

3a.

\[
|D^2f| = e^{2\lambda} |D^2f|_g
\]

\[
\| D^2f \|_{L^q(g)} \leq e^{-2\lambda_m(q-1)/q} \| D^2f \|_q
\]
3b.

\[ \|D^2 f\| \leq e^{\lambda M} \|\Delta_g f\|_g + \frac{1}{\sqrt{2}} \|Df\|_g \]

Proof: The inequalities 1, 2, 3.a are trivial consequences of the identities:

\[ \int_{\Sigma} f^q \mu_\sigma = \int_{\Sigma} f^q e^{-2\lambda} \mu_g \quad \text{since} \quad \mu_g = e^{2\lambda} \mu_\sigma \]

and

\[ g^{ab} D_a f D_b f = e^{-2\lambda} g^{ab} D_a f D_b f \]

and a corresponding equality for \( D^2 f \) or, more generally, for covariant 2-tensors.

To prove 3b we use the identity obtained by two successive partial integrations and the Ricci formula with \( R(\sigma) = -1 \)

\[ \|D^2 u\|^2 = \|\Delta u\|^2 + \frac{1}{2} \|Du\|^2 \]

We have

\[ \Delta u = e^{2\lambda} \Delta_g u, \quad \|e^{2\lambda} \Delta_g u\|_g = \|e^{\lambda} \Delta_g u\|_g \]

The given result follows.

**Lemma 2.**

We denote by \( C_\sigma \) any positive number depending only on \((\Sigma, \sigma)\).

1. Let \( f \) be a scalar function on \( \Sigma \). There exists \( C_\sigma \) such that the \( L^4 \)

norms of \( f \) and \( Df \) are estimated by:

\[ \|f\|_4 \leq C_\sigma \{ e^{-\lambda_m} \|f\|_g + e^{-\frac{1}{2}\lambda_m} \|f\|_g^{\frac{1}{2}} \|Df\|_g^{\frac{1}{2}} \} \]

and

\[ \|Df\|_4 \leq C_\sigma \{ \|Df\|_g + \|Df\|_g^{\frac{1}{2}} e^{\frac{1}{2}\lambda M} \|\Delta_g f\|_g^{\frac{1}{2}} \} \]

2. For any \( q \) such that \( 1 \leq q < \infty \) there exists \( C_\sigma \) such that

\[ \|f\|_q \leq C_\sigma \|f\|_{H^1} \]

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Proof. 1. By the Sobolev inequalities there exists $C_\sigma$ such that
\[ \| f \|_4^2 = \| | f |^2 \| \leq C_\sigma (\| | f |^2 \|_1 + \| D | f |^2 \|_1) \]
Using
\[ D | f |^2 = 2f.Df \]
we obtain
\[ \| | f |^2 \| \leq C_\sigma \| f \| (\| f \| + 2\| Df \|) \]
which gives the first result using the lemma 1.

Analogously
\[ \| Df \|_4^2 = \| | Df |^2 \| \leq C_\sigma \{\| Df \|_2^2 + \| D | Df |^2 \|_1\} \]
leads to the second inequality.

2. The Sobolev embedding theorem and the compactness of $\Sigma$.

6 Energy estimates.

6.1 Bound of the first energy.

The 2+1 dimensional Einstein equations with source the stress energy tensor of the wave map $u$ contain the following equation (hamiltonian constraint)
\[ 2N^{-2}(T_{00} - (3) S_{00}) = N^{-2}\partial_b u.\partial_b u + g^{ab}\partial_a u.\partial_b u + g^{ab}g^{cd}k_{ab}k_{cd} - R - \tau^2 = 0 \]
(14)

Recall the splitting of the covariant 2-tensor $k$ into a trace and a traceless part:
\[ k_{ab} = h_{ab} + \frac{1}{2}g_{ab}\tau \]
(15)

hence
\[ |k|^2 = g^{ac}g^{bd}k_{ab}k_{cd} = |h|^2 + \frac{1}{2}\tau^2 \]
(16)
and the Hamiltonian constraint equation reads

\[ |u'|^2 + |Du|_g^2 + |h|_g^2 = R(g) + \frac{1}{2} \tau^2 \]  

(17)

with

\[ u' \equiv N^{-1} \partial_0 u \]

We define the first energy by the following formula (recall that $|.|_g$ and $\| . \|_g$ denote respectively the pointwise norm and the $L^2$ norm in the metric $g$)

\[ E(t) = \frac{1}{2} \int_{\Sigma_t} (|u'|^2 + |Du|_g^2 + |h|_g^2) \mu_g = \frac{1}{2} (\| u' \|_g^2 + \| Du \|_g^2 + \| h \|_g^2) \]  

(18)

This energy is the first energy of the wave map $u$ completed by the $L^2(g)$ norm of $h$.

We integrate the Hamiltonian constraint on $(\Sigma_t, g)$ using the constancy of $\tau$ and the Gauss Bonnet theorem which reads, with $\chi$ the Euler characteristic of $\Sigma$.

\[ \int_{\Sigma_t} R(g) \mu_g = 4\pi \chi \]

We have then

\[ E(t) = \frac{\tau^2}{4} Vol_g(\Sigma_t) + 2\pi \chi \]  

(19)

with

\[ Vol_g(\Sigma_t) = \int_{\Sigma_t} \mu_g \]

We know from elementary calculus that on a compact manifold

\[ \frac{d Vol_g(\Sigma_t)}{dt} = \frac{1}{2} \int_{\Sigma_t} g^{ab} \frac{\partial g_{ab}}{\partial t} \mu_g = -\tau \int_{\Sigma_t} N \mu_g \]
since
\[ g^{ab} \partial_t g_{ab} = -2N\tau + 2\nabla^a \nu_a \]
We use the equation
\[ N^{-1(3)} R_{00} = \Delta_g N - N|k|_g^2 + \partial_t \tau = |u'|^2 \]
together with the splitting of \( k \) to write after integration, since \( \tau \) is constant in space,
\[ \frac{1}{2} \tau^2 \int_{\Sigma_t} N \mu_g = \frac{d\tau}{dt} \Vol_g(\Sigma_t) - \int_{\Sigma_t} N(|h|_g^2 + |u'|^2)\mu_g \]
We use these results to compute the derivative of \( E(t) \) and we find that it simplifies to:
\[ \frac{dE(t)}{dt} = \frac{1}{2} \tau \int_t (|h|_g^2 + |u'|^2)N\mu_g. \]
We see that \( E(t) \) is a non increasing function of \( t \) if \( \tau \) is negative. The absence of the term \( |Du|_g^2 \) on the right hand side does not permit an estimate of the rate of decay of \( E(t) \).
We will estimate this decay in a forthcoming section.
Note in addition the appearance of \( N \) in the right hand side.

### 6.2 Second energy estimates
In this paragraph indices are raised with \( g \). We denote by \( h_g^{ab} \) the contravariant components of \( h_{ab} \) computed with the metric \( g \).

We define the energy of gradient \( u \) by the formula
\[ E^{(1)}(t) \equiv \int_{\Sigma_t} (J_0 + J_1)\mu_g \]
with
\[ J_1 = \frac{1}{2} | D_u u |^2, \quad J_0 = \frac{1}{2} | Du' |^2 \]
We have for an arbitrary function \( f \):
\[ \frac{d}{dt} \int_{\Sigma_t} f \mu_g = \int_{\Sigma_t} \{ \partial_t f + \frac{1}{2} g^{ab} \partial_t g_{ab} \} \mu_g \]
that is, due to the definition of $k_{ab}$,

$$\frac{d}{dt} \int_{\Sigma_t} f \mu_g = \int_{\Sigma_t} \left\{ \partial_t f - (N \tau - \nabla_a \nu^a) f \right\} \mu_g$$

hence after integration by parts on the compact manifold $\Sigma$, using the expression of $\partial_0$ and replacing $f$ by $J_0 + J_1$ the following formula where the shift does not appear explicitly:

$$\frac{d}{dt} \int_{\Sigma_t} (J_1 + J_0) \mu_g = \int_{\Sigma_t} \left\{ \partial_0 (J_1 + J_0) - N \tau (J_1 + J_0) \right\} \mu_g$$

We first compute

$$\int_{\Sigma_t} \partial_0 J_1 \mu_g = \int_{\Sigma_t} \partial_0 \Delta_g u \Delta_g u \mu_g$$

We define the operator $\tilde{\partial}_0$ on time dependent space tensors by

$$\tilde{\partial}_0 = \partial_0 - L_\nu$$

where $L_\nu$ denotes the Lie derivative in the direction of the shift $\nu$. We have

$$\tilde{\partial}_0 \Delta_g u = g^{ab} \partial_0 \nabla_a \partial_b u + \partial_0 g^{ab} \nabla_a \partial_b u$$

Therefore using

$$\tilde{\partial}_0 g^{ab} = 2N k^{ab} \equiv 2Nh^{ab} + N g^{ab} \tau$$

$$\int_{\Sigma_t} \partial_0 J_1 \mu_g = \int_{\Sigma_t} g^{ab} \tilde{\partial}_0 \nabla_a \partial_b u \Delta_g u \mu_g + X_1$$

with

$$X_1 = \int_{\Sigma_t} \left\{ 2Nh^{ab} \nabla_a \partial_b u, \Delta_g u + 2N \tau J_1 \right\} \mu_g$$

Analogously

$$\int_{\Sigma_t} \partial_0 J_0 \mu_g = \int_{\Sigma_t} g^{ab} \tilde{\partial}_0 \partial_a u' \partial_b u' \mu_g + X_0$$

with

$$X_0 = \int_{\Sigma_t} \left\{ Nh^{ab} \partial_a u' \partial_b u' + N \tau J_0 \right\} \mu_g$$

We use the commutation of the operator $\tilde{\partial}_0$ with the partial derivative $\partial_a$ (cf. C.B-York 1995) together with partial integration to obtain
\[ \int_{\Sigma_t} g^{ab} \tilde{\partial}_0 \partial_a u' \partial_b u' \mu_g = - \int_{\Sigma_t} \partial_0 u' \Delta_g u' \mu_g \]

The function \( u \) satisfies the wave equation on \((\Sigma \times R,(3)\,g)\), namely:

\[ \partial_0 u' = N \Delta_g u + \partial^a N \partial_a u + \tau N u' \]

which gives

\[ \int_{\Sigma_t} g^{ab} \tilde{\partial}_0 \partial_a u' \partial_b u' \mu_g = - \int_{\Sigma_t} N \Delta_g u \Delta_g u' \mu_g + Y_0 \]

with, after another integration by parts

\[ Y_0 \equiv \int_{\Sigma_t} \{ (\nabla_b (\partial^a N \partial_a u) + \tau \partial_b N u') \cdot (\partial^b u') + 2 \tau N J_0 \} \mu_g \]

On the other hand:

\[ g^{ab} \tilde{\partial}_0 \nabla_a \partial_b u \equiv \Delta_g \partial_0 u - g^{ab} \tilde{\partial}_0 \Gamma^c_{ab} \partial_c u \]

with

\[ \Delta_g \partial_0 u \equiv \Delta_g (Nu') \equiv N \Delta u' + 2 \partial^a \partial_a u' + u' \Delta_g N \]

therefore

\[ \int_{\Sigma_t} g^{ab} \tilde{\partial}_0 \nabla_a \partial_b u . \Delta_g u \mu_g = \int_{\Sigma_t} N \Delta_g u \Delta_g u' \mu_g + Y_1 \]

with

\[ Y_1 = \int_{\Sigma_t} \{ -g^{ab} \tilde{\partial}_0 \Gamma^c_{ab} \partial_c u + 2 \partial^a N \partial_a u' + u' \Delta_g N \} \Delta_g u \mu_g \]

which can be written, using the identity

\[ \tilde{\partial}_0 \Gamma^c_{ab} = \nabla^c (Nk_{ab}) - \nabla_a (Nk^c_b) - \nabla_b (Nk^c_a) \]

together with the equation

\[ \nabla_a k^a_b = - \partial_b u.u' \]
\[
Y_1 = \int_{\Sigma_t} \{(2\partial_a Nh^a_g - 2N\partial^c u.u')\partial_c u + 2\partial^a N\partial_a u' + u'\Delta_g N\}.\Delta_g u \mu_g \quad (20)
\]

We see that the terms in third derivatives of \(u\) disappear in the derivative of \(E^{(1)}(t)\). We have obtained

\[
\int_{\Sigma_t} \partial_0 (J_0 + J_1) \mu_g = X_0 + X_1 + Y_0 + Y_1
\]

where the X's and Y' are given by the above formulas. We read from these formulas the following theorem

**Theorem 14** The time derivative of the second energy \(E^{(1)}\) satisfies the equality

\[
\frac{dE^{(1)}}{dt} - 2\tau E^{(1)} = \tau \int_{\Sigma_t} N J_0 + (N - 2)(J_0 + J_1) \mu_g + + Z \quad (21)
\]

The quantity \(Z\) is given by:

\[
Z \equiv \int_{\Sigma_t} \{Nh^{ab}_g \partial_a u'.\partial_b u' + 2Nh^{ab}_g \nabla_a \partial_b u.\Delta_g u + \nabla_b (\partial^a N \partial_a u) + \tau \partial_b Nu'.(\partial^b u')\} \mu_g + Y_1 \quad (22)
\]

For \(\tau \leq 0\), and \(0 < N \leq 2\), the right hand side of (19) is less than \(Z\), which can be estimated with non linear terms in the energies: all the terms which are only quadratic in the derivatives of \(u\), i.e. linear in energy densities, have coefficients which contain \(N - 2, \partial_a N\) or \(h^{ab}_g\), or their derivatives. To estimate these terms we need bounds which will be deduced from estimates on the conformal factor and the lapse \(N\).

In the following paragraphs we will set

\[
E(t) \equiv \varepsilon^2, \quad \text{and} \quad \tau^{-2} E^{(1)}(t) \equiv \varepsilon_1^2
\]

### 7 Estimates for \(h\) in \(H_1\).

#### 7.1 Estimate of \(\| h \|\).
We have defined the auxiliary unknown $h$ by

$$h_{ab} \equiv k_{ab} - \frac{1}{2} g_{ab} \tau$$

Its $L^2$ norm on $(\Sigma, \sigma)$ is bounded in terms of the first energy and an upper bound $\lambda_M$ of the conformal factor since we have

$$\| h \|^2 = \int_{\Sigma_t} \sigma^{ac} \sigma^{bd} h_{ab} h_{cd} \mu_g = \int_{\Sigma_t} e^{2\lambda} g^{ac} g^{bd} h_{ab} h_{cd} \mu_g \leq e^{2\lambda M} \| h \|^2_{L^2(g)}$$

which implies on $\Sigma_t$, by the definition of $E(t)$,

$$\| h \| \leq e^{\lambda M} \epsilon$$

with

$$\epsilon \equiv E^{\frac{1}{2}}(t)$$

### 7.2 Estimate of $\| Dh \|$.

The tensor $h$ satisfies the equations

$$D_a h^a_b = L_b \equiv -\partial_a u. \dot{u}$$

It is the sum of a TT tensor $h_{TT} \equiv q$ and a conformal Lie derivative $r$:

$$r \equiv q + r$$

It results from elliptic theory that on each $\Sigma_t$ the tensor $r$ satisfies the estimate

$$\| r \|_{H^1} \leq C_\sigma \| D u. \dot{u} \| \leq C_\sigma \| Du \|^2 \| \dot{u} \|^2 \| u \|^2$$

We will bound the right hand side of this inequality in terms of the first and second energies of $u$. We have:

$$\| \dot{u} \|^2 \leq e^{4\lambda M} \| u \|^2$$

we have proven in section 4 that

$$\| u' \|^2 \leq C_\sigma e^{-\lambda u} \| u' \|_{L^2(g)} (e^{-\lambda u} \| u' \|_{L^2(g)} + \| D u' \|_{L^2(g)})$$

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We have set
\[ \varepsilon_1 \equiv |\tau|^{-1}\{E^{(1)}(t)\}^{1\over 2} \]

hence, using the lower bound on \( \lambda \) and the definitions of \( \varepsilon \) and \( \varepsilon_1 \) we obtain
\[ \| | u' |^2 \| \leq C_{\sigma}\tau^2(\varepsilon^2 + \varepsilon\varepsilon_1) \]

On the other hand
\[ \| | Du |^2 \| \leq C_{\sigma}\| Du\|_{L^2(g)}(\| Du\|_{L^2(g)} + e^{\lambda M}\| \Delta_g u\|_{L^2(g)}) \]

hence
\[ \| | Du |^2 \| \leq C_{\sigma}\{\varepsilon^2 + \varepsilon\varepsilon_1e^{\lambda M}|\tau|\} \]

It results from these inequalities that
\[ \| r \|^2_{H_1} \leq C_{\sigma}e^{4\lambda M}\tau^2\varepsilon^2(\varepsilon + \varepsilon_1)\{\varepsilon + \varepsilon_1e^{\lambda M}|\tau|\} \]

We now estimate the transverse part \( h_{TT} = q \).

It is known (cf. Andersson and Moncrief) that in dimension 2 the equation
\[ D_a q^a_b = 0, \text{ with } q^a_b = 0 \]

implies
\[ D^c D_c q_{ab} = R(\sigma)q_{ab}. \]

When \( R(\sigma) = -1 \) this equation gives by integration on \( \Sigma_t \) of its contracted product with \( q^{ab} \) the following relation
\[ \| Dq \| = \| q \| \]

more generally any \( H_s \) norm of \( q \) is a multiple of its \( L^2 \) norm.

We have
\[ \| q \| \leq \| h \| + \| r \| \]

therefore
\[ \| Dh \| \leq \| Dq \| + \| Dr \| \leq \| h \| + \| r \|_{H_1} \]

In other words
\[ \| Dh \| \leq e^{\lambda M}\varepsilon\{1 + C_{\sigma}e^{\lambda M}|\tau|(|\varepsilon + \varepsilon_1e^{\lambda M}|\tau|)^{1\over 2}(\varepsilon + \varepsilon_1)^{1\over 2}\} \]

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8 Estimates for the conformal factor.

8.1 First estimates.

Recall that we denote respectively by $\| \cdot \|$ and $\| \cdot \|_p$ the $L^2(\sigma)$ and $L^p(\sigma)$ norms on $\Sigma$ and by $\| \cdot \|_g$ an $L^2(\sigma)$ norm on $\Sigma$.

The conformal factor $\lambda$ satisfies the equation

$$\Delta \lambda = f(\lambda) \equiv p_1 e^{2\lambda} - p_2 e^{-2\lambda} + p_3$$

where the coefficients $p_i$ are functions in $E_0 \cap E_1$, $1 < p < 2$, hypothesis consistent with $Du, \dot{u}, h \in E_1$, given by

$$p_1 = \frac{1}{4} \tau^2, p_2 = \frac{1}{2} \left( \| h \|^2 + \| \dot{u} \|^2 \right), p_3 = \frac{1}{2} (R(\sigma) - \| Du \|^2)$$

Having chosen $R(\sigma) = -1$ we have seen that a lower bound $\lambda_m$ for $\lambda$ is such that

$$e^{-2\lambda_m} \leq \frac{1}{2} \tau^2$$

Also

$$\lambda_- \leq \lambda \leq \lambda_+$$

$$\lambda_- = \omega - maxv + v, \text{ and } \lambda_+ = \omega + v - minv$$

where $v \in E_2 \cap E_3$ is the solution with mean value zero on $\Sigma_t$ of the linear equation

$$\Delta v = f(\omega) \equiv p_1 e^{2\omega} - p_2 e^{-2\omega} + p_3$$

where $e^{2\omega}$, positive solution of the equation

$$P_1 e^{4\omega} + P_3 e^{2\omega} - P_2 = 0$$

is given by, since $P_3 < 0, P_2 \geq 0, \ P_1 = \frac{1}{4} \tau^2 V_\sigma$,

$$e^{2\omega} = \frac{-P_3 + \sqrt{P_3^2 + 4P_1P_2}}{2P_1} = \frac{-P_3(1 + \sqrt{1 + 4P_3^{-2}P_1P_2})}{2P_1}$$
This formula will permit an estimate of $e^{2\omega - \frac{2}{\tau^2}}$, a positive quantity, in terms of the energies. Indeed using the elementary algebra inequality

$$\sqrt{1 + a} \leq 1 + \frac{1}{2}a,$$

when $a \geq 0$ we obtain

$$e^{2\omega} \leq \frac{P_3}{P_1} - \frac{P_2}{P_3}$$

and, using the expressions of $P_2, P_3$ and $P_1 = \frac{1}{4}\tau^2V_\sigma$, together with

$$\| \dot{u} \|^2 \leq e^{2\lambda_M} \| u' \|_g^2,$$

and

$$\| h \|^2 \leq e^{2\lambda_M} \| h \|_g^2$$

we find

$$0 \leq \frac{1}{2}\tau^2 e^{2\omega} - 1 \leq V_\sigma^{-1}\{\| Du \|^2 + \frac{\tau^2}{2}e^{2\lambda_M}(\| u' \|_g^2 + \| h \|_g^2)\}$$

We have set $\varepsilon^2 \equiv E(t)$ and therefore we have

$$0 \leq \frac{1}{2}\tau^2 e^{2\omega} - 1 \equiv \varepsilon_\omega \leq V_\sigma^{-1}\{1 + \frac{\tau^2}{2}e^{2\lambda_M}\varepsilon^2\}$$

We will now give estimates for $\lambda$.

**Lemma 15** Denote by $\lambda_M$ the maximum of $\lambda$, one has

$$0 \leq \lambda_M - \omega \leq 2 \| v \|_{L^\infty}$$

$$0 \leq \omega - \lambda_m \leq 2 \| v \|_{L^\infty}$$

Proof. The result follows from the expressions of $\lambda_-$ and $\lambda_+$:

$$\lambda_M \leq \sup \lambda_+ = \omega + \text{max}v - \text{min}v,$$

and

$$\lambda_m \geq \inf \lambda_- = \omega + \text{min}v - \text{max}v$$

Also

$$\lambda_M - \lambda_m \leq 2\text{max}v - 2\text{min}v \leq 4\text{max}v \leq 4 \| v \|_{L^\infty}$$
Corollary 16 The following inequality holds

\[ 1 \leq e^{\lambda M - \omega} \leq 1 + 2 \| v \|_{L^\infty} e^{2\|v\|_{L^\infty}} \]  \hspace{1cm} (24)

\[ 1 \leq e^{\lambda M - \omega_m} \leq 1 + 4 \| v \|_{L^\infty} e^{4\|v\|_{L^\infty}} \]  \hspace{1cm} (25)

Proof. Elementary calculus

We set

\[ \varepsilon_v \equiv \| v \|_{L^\infty} \]

Denote by \( \varepsilon_{v_0} \) the \( L^\infty \) norm of the function \( v \) computed with initial data. We have shown in the section on local existence that \( \varepsilon_{v_0} \) tends to zero with the initial data \( q_0, Du_0 \) and \( u_0 \).

**Hypothesis H\( c \).** We say that \( v \) satisfies the hypothesis \( H_c \) if there exists a number \( c > \varepsilon_{v_0} \), independent of \( t \), such that \( \varepsilon_v \leq c \).

We suppose also that the initial data are such that \( E(t_0) \equiv \varepsilon_{v_0}^2 \) verifies the inequality (we chose \( \frac{1}{2} \) for simplicity of notations)

\[ V^{-1}_{v_0} \varepsilon_{v_0}^2 (1 + 2ce^{2c})^2 < \frac{1}{2}. \]

Then, since \( E(t) \) is non increasing and the volume \( V_\sigma \) of \( (\Sigma, \sigma) \) is constant by the Gauss Bonnet theorem, it holds for all \( t \) that

\[ V^{-1}_\sigma \varepsilon^2 (1 + 2ce^{2c})^2 < \frac{1}{2}. \]

Recall that we have denoted by \( C_\sigma \) any positive number depending only on \( (\Sigma, \sigma) \).

We denote by \( C \) any positive number depending only on \( c \).

**Theorem 17** When \( \varepsilon_v \leq c \) there exist numbers \( C \) such that the conformal factor \( \lambda \) satisfies the estimates:
1. \[
\frac{1}{2} \tau^2 e^{2\lambda M} \leq 1 + C(\varepsilon^2 + \varepsilon_v)
\]

2. \[
e^{\lambda M - \lambda m} \leq 1 + C\varepsilon_v.
\]

Proof.
1. We find, using the estimate of \(\omega\):
\[
1 \leq \frac{1}{2} \tau^2 e^{2\lambda M} \equiv \frac{1}{2} e^{2(\lambda M - \omega)} \tau^2 e^{2\omega} \leq (1 + 2\varepsilon_v e^{2\varepsilon_v})^2 [1 + V_\sigma^{-1} (1 + \frac{\tau^2}{2} e^{2\lambda M}) \varepsilon^2]
\]
therefore
\[
1 \leq \frac{1}{2} \tau^2 e^{2\lambda M} \leq \frac{(1 + 2\varepsilon_v e^{2\varepsilon_v})^2 (1 + V_\sigma^{-1} \varepsilon^2)}{1 - V_\sigma^{-1} \varepsilon^2 (1 + 2\varepsilon_v e^{2\varepsilon_v})^2}
\]
that is
\[
0 \leq \frac{1}{2} \tau^2 e^{2\lambda M} - 1 \leq \frac{(1 + 2\varepsilon_v e^{2\varepsilon_v})^2 2V_\sigma^{-1} \varepsilon^2 + 4\varepsilon_v e^{2\varepsilon_v} + 4\varepsilon_v^2 e^{4\varepsilon_v}}{1 - V_\sigma^{-1} \varepsilon^2 (1 + 2\varepsilon_v e^{2\varepsilon_v})^2}
\]
The result 1 of the lemma follows then from the hypothesis \(H_c\) and \(H_0\).

2. Is immediate.

8.2 Estimate of \(v\).

The equation satisfied by \(v\) implies
\[
\int_\Sigma |Dv|^2 \mu_\sigma = - \int_\Sigma f(\omega) v \mu_\sigma
\]
hence
\[
\| Dv \|^2 \leq || f(\omega) || || v ||
\]
but the Poincare inequality applied to the function \(v\) which has mean value 0 on \(\Sigma\) gives
\[
\| v \|^2 \leq [\Lambda]^{-1} \| Dv \|^2
\]
where $\Lambda$ is the first (positive) eigenvalue of $-\Delta$ for functions on $\Sigma_t$ with mean
value zero. Therefore on each $\Sigma_t$

$$\| Dv \| \leq [\Lambda]^{-1/2} \| f_0 \|$$

We use Ricci identity and $R(\sigma) = -1$ to obtain

$$\| \Delta v \|^2 = \| D^2 v \|^2 - \frac{1}{2} \| Dv \|^2$$

The equation satisfied by $v$ implies then

$$\| D^2 v \|^2 = \| f(\omega) \|^2 + \frac{1}{2} \| Dv \|^2$$

Assembling these various inequalities implies

$$\| v \|_{H^2} \leq \left[ 1 + \frac{3}{2\Lambda} + \frac{1}{\Lambda^2} \right] \| f(\omega) \|$$

The Sobolev inequality

$$\| v \|_{L^\infty} \leq C_\sigma \| v \|_{H^2}$$

gives then a bound on the $L^\infty$ norm of $v$ on $\Sigma_t$ in terms of the $L^2$ norm of
$f(\omega)$, a Sobolev constant $C_\sigma$ and the lowest eigenvalue $\Lambda$ of $-\Delta$

We now estimate the $L^2$ norm of $f(\omega)$.

$$f(\omega) \equiv f_\omega \equiv p_1 e^{2\omega} - p_2 e^{-2\omega} + p_3$$

We split $f_\omega$ into a constant part and a non constant part $h_\omega$ by setting

$$h_\omega \equiv p_2 e^{-2\omega} + \frac{1}{2} |Du|^2.$$

Since the mean value $\bar{f}_\omega$ of $f(\omega)$ is zero and the mean value of a constant is
equal to itself we have

$$f_\omega \equiv \bar{h}_\omega - h_\omega.$$

By the isoperimetric inequality there exists a constant $I_\sigma$ such that

$$\| f_\omega \| \leq I_\sigma \| Dh_\omega \|_1$$

We want to bound the right hand side in terms of the first and second energies
of the wave map. We have by the definition of $h_\omega$:

$$\| Dh_\omega \|_1 \leq \frac{1}{2} \left( \| D|Du|^2 \|_1 + e^{-2\omega_0}(\| D|h|^2 \|_1 + \| D|\bar{u}|^2 \|_1) \right)$$

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Lemma 18

1. The following estimate holds

\[ \frac{1}{2} ||D|Du|^2||_1 \leq ||Du||_g(e^{\lambda_M}||\Delta_g u||_g + (1/\sqrt{2})||Du||_g) \]

2. It implies under the hypothesis \( H_c \) that

\[ \frac{1}{2} ||D|Du|^2||_1 \leq C(\varepsilon^2 + \varepsilon \varepsilon_1) \]

Proof. 1. We have:

\[ D|Du|^2 = 2Du.D^2u \]

hence

\[ ||D|Du|^2||_1 \leq 2||Du||||D^2u|| \]

Previous elementary calculus gave

\[ ||Du|| = ||Du||_g \equiv ||Du||_{L^2(g)} \]

and

\[ ||D^2u||^2 = ||\Delta u||^2 + \frac{1}{2}||Du||^2 \]

with

\[ \Delta u = e^{2\lambda}\Delta_g u, \text{ and } ||e^{2\lambda}\Delta_g u|| = ||e^{\lambda}\Delta_g u||_g \]

hence we have the inequality

\[ ||D^2u|| \leq e^{\lambda_M}||\Delta_g u||_g + (1/\sqrt{2})||Du||_g \]

which implies the given result 1.

2. Under the hypothesis \( H_c \) we have

\[ e^{\lambda_M}||\tau|| \leq C. \]

the result 2 follows from the definitions of \( \varepsilon \) and \( \varepsilon_1 \).

Lemma 19

The following estimate holds if \( \varepsilon_v \leq c \) (hypothesis \( H_c \))

\[ \frac{1}{2} e^{-2\omega} ||D|h|^2||_1 \leq C'_\sigma(\varepsilon^2 + \varepsilon \varepsilon_1) \]

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Proof. We have:

\[ \| D|h^2|_1 \leq 2 \| h \|_1 \| Dh \| \]

We have shown in a previous section that the \( L^2 \) norm of \( h \) and \( Dh \) can be estimated through the first and second energies. We have found

\[ \| h \| \leq e^{\lambda M} \varepsilon \]

and under the hypothesis \( H_c \)

\[ \| Dh \| \leq e^{\lambda M} \{ \varepsilon + CC \sigma \varepsilon (\varepsilon + \varepsilon_1) \} \]

The given result follows from the bound of \( e^{2(\lambda M - \omega)} \).

We now estimate the last term in \( Dh_\omega \), i.e. \( \frac{1}{2} e^{-2\omega} \| D|u| \|_1 \). We will use the following estimates of \( L^4 \) norms of \( Du, u' \) and \( h \):

**Lemma 20**

1. Under the hypothesis \( H_c \) the \( L^4 \) norms of \( Du, u' \) and \( h \) are estimated by:

\[ \| u' \|_4^2 \equiv \| |u'|^2 \| \leq CC \sigma \tau^2 \{ \varepsilon^2 + \varepsilon \varepsilon_1 \} \]

and

\[ \| Du \|_4^2 \equiv \| |Du|^2 \| \leq CC \sigma \{ \varepsilon^2 + \varepsilon \varepsilon_1 \} \]

2. The \( L^4 \) norm of \( h \) is estimated by

\[ \| h \|_4^2 \equiv \| |h|^2 \| \leq CC \sigma \{ e^{2\lambda M} \varepsilon^2 + C \sigma e^{\lambda M} \varepsilon^2 (\varepsilon + \varepsilon_1) \} \]

Proof.

1. Immediate consequence of the inequalities proved in the final section on local existence, and the definitions.

2. The inequality for the \( L^2 \) norm of \( |h|^2 \) is also proved through the Sobolev inequality

\[ \| |h|^2 \| \leq C \sigma \{ \| h \|^2 + \| Dh|^2 \|_1 \} \]

which gives, using previous results

\[ \| |h|^2 \| \leq C \sigma e^{2\lambda M} \{ \varepsilon^2 + C \varepsilon^2 (\varepsilon + \varepsilon_1) \} \].

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Lemma 21 We have

\[ \frac{1}{2} e^{-2\omega} \| D|\partial_t u|\|_1 \leq CC_\sigma (\varepsilon^2 + \varepsilon \varepsilon_1) \]

Proof. We have

\[ \| D|\partial_t u|\|_1 \leq 2 \| \partial_t \| \| D\partial_t u \| \]

Recall that

\[ \partial_t = e^{2\lambda} u', u' = N^{-1} \partial_0 u, \text{ and } \mu_\sigma = e^{-2\lambda} \mu_g \]

hence

\[ \| \partial_t \| \leq e^{\lambda M} \| u' \|_g \leq e^{\lambda M} \varepsilon \]

we have

\[ D_a \partial_t = e^{2\lambda} [D_a u' + 2u' D_a \lambda] \]

\[ \| D\partial_t \| = \int_{\Sigma} e^{4\lambda} \{ |Du'|^2 + 4|u'|^2 |D\lambda|^2 + 2D^n |u'|^2 |D\partial_n \lambda| \} \mu_\sigma \]

after integration by parts

\[ \| D\partial_t \| = \int_{\Sigma} e^{4\lambda} [ |Du'|^2 - |u'|^2 (4|D\lambda|^2 + 2\Delta \lambda)] \mu_\sigma \]

When \( R(\sigma) = -1 \) we have

\[ 2\Delta \lambda = (\frac{1}{2} e^{2\lambda} \tau^2 - 1) - (e^{-2\lambda} |h|^2 + |Du|^2 + e^{2\lambda} |u'|^2) \]

It results from the estimate of \( \lambda_m \) that on \( \Sigma_t \)

\[ \frac{1}{2} e^{2\lambda} \tau^2 - 1 \geq 0 \]

hence

\[ \| D\partial_t \| \leq \int_{\Sigma} e^{4\lambda} \{ |Du'|^2 + |u'|^2 (|Du|^2 + e^{-2\lambda} |h|^2 + e^{2\lambda} |u'|^2) \} \mu_\sigma \]

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From which we deduce, using the Cauchy-Schwarz inequality
\[ \| D\dot{u} \|^2 \leq e^{4\lambda_M} \{ \| Du' \|^2 + \| u' \|^2 (\| D\dot{u} \|^2 + e^{-2\lambda_m} \| |h| \|^2 + e^{2\lambda_M} \| |u'| \|^2 ) \} \]

which we write, using previous results
\[
\| D\dot{u} \|^2 \leq e^{4\lambda_M} \tau^2 \{ \varepsilon_1^2 + CC_\sigma \varepsilon_2 (\varepsilon + \varepsilon_1) \} [ (e^{2\lambda_M} \tau^2 + 1)(\varepsilon^2 + \varepsilon \varepsilon_1) + e^{-2\lambda_m} e^{2\lambda_M} (\varepsilon^2 + CC_\sigma \varepsilon^2 (\varepsilon + \varepsilon_1)]
\]

Using once more the estimates resulting from the H hypothesis we obtain
\[
\| D\dot{u} \| \leq Ce^{\lambda_M} \{ \varepsilon_1 + C_\sigma \varepsilon_2 + \varepsilon \varepsilon_1 \} \leq CC_\sigma e^{\lambda_M} (\varepsilon + \varepsilon_1)
\]

Assembling inequalities and the bound of \( e^{2(\lambda_M - \omega)} \) leads to the given result.

The following theorem is a straightforward consequence of our lemmas.

**Theorem 22** There exists numbers \( C \) and \( C_\sigma \) such that the \( L^\infty \) norm of \( v \) is bounded by the following inequality
\[
\| v \|_\infty \equiv \varepsilon_v \leq CC_\sigma (\varepsilon^2 + \varepsilon \varepsilon_1)
\]

Proof. Recall that there exists a Sobolev constant \( C_\sigma \) such that
\[
\| v \|_\infty \leq C_\sigma \{ \| D|Du|\|_1 + e^{-2\omega}(\| D|h|\|_1 + \| D|\dot{u}|\|_1) \}
\]

The three terms in the sum have been evaluated in previous lemmas.

### 8.3 Bound on derivatives.

The equation satisfied by \( \lambda \)
\[
\Delta \lambda = f(\lambda) \equiv \frac{1}{4} \tau^2 e^{2\lambda} - \frac{1}{2}(\| h \|^2 + \| \dot{u} \|^2) e^{-2\lambda} - \frac{1}{2} (1 + |Du|^2)
\]
implies after multiplication by \( \lambda - \tilde{\lambda} \) and integration on \( \Sigma \)
\[
\| D\lambda \|^2 \leq \| \lambda - \tilde{\lambda} \| \| f(\lambda) \|
\]

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The Poincare inequality gives
\[ \| \lambda - \bar{\lambda} \| \leq I_\sigma \| D\lambda \| \]
therefore
\[ \| D\lambda \| \leq I_\sigma \| f(\lambda) \| \leq \frac{1}{4}(\tau^2 e^{2\lambda M} - 1)V_{\sigma}^{\frac{1}{2}} + \frac{1}{2}\left\{ \| Du \|^2 + e^{-2\lambda M} \| |h| \|^2 + e^{2\lambda M} \| |u'| \|^2 \right\} \]
while
\[ \| D^2\lambda \|^2 \equiv \| \Delta\lambda \|^2 + \frac{1}{2}\| D\lambda \|^2 \leq (1 + \frac{I_\sigma^2}{2}) \| f(\lambda) \|^2 \]

The L$^2(\sigma)$ norm of $f(\lambda)$ is bounded by the following quantity:
\[ \| f(\lambda) \| \leq \frac{1}{4}(\tau^2 e^{2\lambda M} - 2)V_{\sigma}^{\frac{1}{2}} + \frac{1}{2}\left\{ \| Du \|^2 + e^{-2\lambda M} \| |h| \|^2 + e^{2\lambda M} \| |u'| \|^2 \right\} \]

Previous estimations show that
\[ \| f(\lambda) \| \leq CC_\sigma(\varepsilon^2 + \varepsilon_1) \tag{27} \]
which gives the following theorem.

**Theorem 23** Under the hypothesis $H_c$ the $H_1$ norm of $D\lambda$ satisfies the inequality
\[ \| D\lambda \|_{H_1} \leq CC_\sigma(\varepsilon^2 + \varepsilon_1) \tag{28} \]

9 Estimates in $W^p_s$.

9.1 Estimates for h in $W^p_2$.

The estimates of $h$ in $W^p_2$, with $1 < p < 2$ (for definiteness we will choose $p = \frac{4}{3}$) will be obtained using estimates for the conformal factor $\lambda$ which have been obtained by using the $H_1$ norm of $h$. 

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Theorem 24 Under the $H$ hypothesis there exist positive numbers $C(c)$ and $C_\sigma$ such that the $W^p_2$ norm of $h$, choosing to be specific $p = \frac{4}{3}$, is bounded by

$$\| h \|_{W^p_2} \leq CC_\sigma e^{\lambda M} \{ \varepsilon + (\varepsilon + \varepsilon_1)^2 \}$$

Corollary 25 It holds that

$$|\tau| \| h \|_{\infty} \leq CC_\sigma \{ \varepsilon + (\varepsilon + \varepsilon_1)^2 \}$$

and that

$$\| h \|_{L^\infty(g)} \leq CC_\sigma |\tau| \{ \varepsilon + (\varepsilon + \varepsilon_1)^2 \}$$

Proof. We recall that for any function $f$ on a compact manifold one has, if $p \leq 2$,

$$\| f \|_p \leq V^\frac{1}{p} \| f \|$$

We deduce therefore from the $H_s$ estimate of section 6.2 that ($C_0$ is a given number, $V_\sigma = |4\pi \chi|$ is a constant)

$$\| q \|_{W^p_2} \leq C_0 \| q \| \leq C_0 \| h \| \leq \sqrt{2} C_0 e^{\lambda M} \varepsilon$$

To estimate $h$ in $W^p_2$ it remains to estimate $r$ in $W^p_2$.

It results from elliptic theory that on each $\Sigma_t$ the tensor $r$ satisfies for each $1 < p < \infty$ the following estimate

$$\| r \|_{W^p_2} \leq C_\sigma \| Du.\dot{u} \|_{W^p_1}$$

We choose

$$p = \frac{4}{3}$$

We have

$$\| Du.\dot{u} \|_{\frac{4}{3}} \leq \| Du \| \| \dot{u} \|_4 \leq e^{\lambda M} \varepsilon (\varepsilon^2 + \varepsilon \varepsilon_1)^{\frac{1}{2}}$$

because

$$\| Du \| = \| Du \|_g \leq \varepsilon$$
and, under the $H_c$ hypothesis
\[
\| \dot{u} \|_4 \equiv \| \dot{u} \|^2 \leq e^{2\lambda_M} \| u' \|^2 \leq CC_\sigma e^{\lambda_M} (\varepsilon^2 + \varepsilon \varepsilon_1)^{\frac{1}{2}}
\]

We now estimate
\[
\| D(Du.\dot{u}) \|_4 \leq \| D^2 u \| \| \dot{u} \|_4 + \| Du \|_4 \| D\dot{u} \|
\]

Using previous estimates we obtain by a straightforward calculation
\[
\| D(Du.\dot{u}) \|_4 \leq CC_\sigma e^{\lambda_M} \left\{ (\varepsilon^\frac{1}{2} + \varepsilon_1)^\frac{3}{2} + (\varepsilon^\frac{3}{2} + \varepsilon_1)^\frac{1}{2} \right\}
\]

The result of the theorem follows from the bound of $\varepsilon$ by $\varepsilon + \varepsilon_1$.

Proof of corollary.
1. The Sobolev embedding theorem,
\[
\| h \|_\infty \leq C_\sigma \| h \|_{W^p_2} \quad \text{if} \quad p > 1,
\]
and the estimate of $e^{\lambda_M} |\tau|$.
2.
\[
\| h \|_{L^\infty(g)} = \sup_{\Sigma} \{ g^{ac} g^{bd} h_{ab} h_{cd} \}^{\frac{1}{2}} \leq e^{-2\lambda_m} \| h \|_\infty \leq \frac{1}{2} \tau^2 \| h \|_\infty
\]

9.2 $W^p_3$ estimates for $N$.

9.2.1 $H^2$ estimates of $N$.

**Theorem 26** There exist numbers $C = C(c)$ and $C_\sigma$ such that the $H^2$ norm of $N$ satisfies the inequality
\[
\| 2 - N \|_{H^2} \leq CC_\sigma (\varepsilon^2 + \varepsilon \varepsilon_1)
\]

**Corollary 27** The minimum $N_m$ of $N$ is such that
\[
0 \leq 2 - N_m \leq CC_\sigma (\varepsilon^2 + \varepsilon \varepsilon_1)
\]
Proof. We write the equation satisfied by $N$ in the form

$$\Delta(2 - N) - (2 - N) = \beta$$

with, having chosen the parameter $t$ such that $\partial_t \tau = \tau^2$,

$$\beta \equiv (2 - N)(e^{2\lambda} \frac{1}{2} \tau^2 - 1) - N(e^{2\lambda} |u'|^2 + e^{-2\lambda} |h|^2)$$

The standard elliptic estimate applied to the form given to the lapse equation gives

$$\|2 - N\|_{H^2} \leq C_\sigma \|\beta\|$$

(30)

Since $0 < N \leq 2$ and $e^{-2\lambda} \leq \frac{1}{2} \tau^2$ it holds that

$$\|\beta\| \leq 2(\frac{1}{2} e^{2\lambda} \tau^2 - 1)V_\sigma^{1/2} + 2(e^{2\lambda} M \|u'\|^2 + \frac{1}{2} \tau^2 \|h\|^2)$$

(31)

The $L^4$ norms of $h$ and $u'$ as well as $\frac{1}{2} e^{2\lambda} \tau^2 - 1$ have been estimated in the section conformal factor estimate. We deduce from these estimates the bound

$$\|\beta\| \leq CC_\sigma (\varepsilon^2 + \varepsilon_1).$$

which gives the result of the theorem.

The corollary is a consequence of the Sobolev embedding theorem.

**Theorem 28** Under the hypothesis $H_c$ there exist numbers $C$ depending only on $c$ and $C_\sigma$ such that if $1 < p < 2$, for instance $p = \frac{4}{3}$

$$\varepsilon_{DN} \equiv \|2 - N\|_{W^{1,2}} \leq CC_\sigma (\varepsilon^2 + \varepsilon_1).$$

**Corollary 29** The gradient of $N$ satisfies the inequality:

$$\|DN\|_{L^\infty(g)} \leq CC_\sigma |\tau|(\varepsilon^2 + \varepsilon_1)$$
Proof. We have

\[ |\beta| \leq (2 - N_m)\left(\frac{1}{2}e^{2\lambda m} \tau^2 - 1\right) + 2(e^{2\lambda m}|u'|^2 + \frac{1}{2}\tau^2|h|^2) \]  

(32)

We apply the standard elliptic estimate

\[ \|2 - N\|_{W^{s+2}_{p+2}} \leq C_\sigma \|\beta\|_{W^p_1} \]  

(33)

with now 1 < p < 2, s = 1.

We have for any p ≤ 2,

\[ \|\beta\|_p \leq V^{\frac{1}{p} - \frac{1}{2}} \|\beta\| \]

We have already estimated \( \|\beta\| \).

To estimate \( \|\beta\|_{W^1_p} \) we compute

\[
D\beta \equiv [\left(2 - N\right)e^{2\lambda} \tau^2 - 2N(e^{2\lambda} |u'|^2 - e^{-2\lambda} |h|^2)]D\lambda
- DN\left[\frac{1}{2}e^{2\lambda} \tau^2 - 1 - e^{2\lambda} |u'|^2 - e^{-2\lambda} |h|^2\right] - N[e^{2\lambda} D |u'|^2 + e^{-2\lambda} D |h|^2]
\]

We have therefore, with \( \frac{1}{q} + \frac{1}{q'} = \frac{1}{p} \), and using estimates obtained for \( \lambda \) under the H hypothesis

\[
\|D\beta\|_p \leq CC_\sigma \{(2 - N_m) \|D\lambda\|_p + (\varepsilon^2 + \varepsilon\varepsilon_1) \|DN\|_p +
\left[e^{2\lambda_m} \|u'\|^2 \|q'\| + \frac{1}{2}\tau^2 \|h\|^2 \|q\|\right]4 \|D\lambda\|_q + \|DN\|_q + A\}
\]

with

\[ A \equiv 2[e^{2\lambda_m} \|D |u'|^2\|_p + \frac{1}{2}\tau^2 \|D |h|^2\|_p] \]

To bound the first line we recall that the \( L^p \) norms of \( D\lambda \) and \( DN \) are bounded by their \( L^2 \) norms estimated before. To estimate the second line (except for \( A \)) we choose \( p = \frac{4}{3}, q = 4, q' = 2 \). We find quantities bounded before and the \( L^4 \) norm of \( D\lambda \) and \( DN \) which can be estimated in terms of their \( H_1 \) norms bounded before.

To bound \( A \) we write again, with \( p = \frac{4}{3} \):

\[ \|D|u'|^2\|_p \leq 2 \|u'\|_4 \|Du'\|, \text{ since } \frac{1}{4} + \frac{1}{2} = \frac{1}{p} \]
This inequality and corresponding estimates for \( h \) give:

\[
A \leq CC_\sigma (\varepsilon^2 + \varepsilon_1)
\]

The \( H_1 \) bound found above for \( DN \) and \( D\lambda \) permits the obtention of the given result.

The corollary is a consequence of the Sobolev embedding theorem and the relation between \( \sigma \) and \( g \) norms:

\[
\| DN \|_{L^\infty (g)} \leq e^{-\lambda m} \| DN \|_{\infty} \leq e^{-\lambda m} C_\sigma \| DN \|_{W^2} \leq CC_\sigma |\tau| (\varepsilon^2 + \varepsilon_1)
\]

10 Corrected energy estimates.

We have obtained in section 6 a bound for the first energy and a decay for the second energy. These bounds prove unsufficient to control the behaviour in time of the Teichmüller parameters. The right hand side of the first energy inequality is non positive, as well as the quadratic term of the right hand side of the second energy inequality, but the space derivatives are lacking in those right hand sides which would make them negative definite. The introduction of corrected energies enables one to obtain such a definiteness, compensating some terms by others, and leading to better decay estimates.

10.1 Corrected first energy.

10.1.1 Definition and lower bound.

One defines as follows a corrected first energy where \( \alpha \) is a constant, which we will choose positive:

\[
E_{\alpha}(t) = E(t) - \alpha \tau \int_{\Sigma_t} (u - \bar{u}).u' \mu_g 
\]

(34)

where we have denoted by \( \bar{u} \) the mean value of \( u \), a scalar function, on \( \Sigma_t \):

\[
\bar{u} = \frac{1}{Vol_\sigma \Sigma_t} \int_{\Sigma_t} u \mu_\sigma
\]
An estimate of $E_\alpha$ will give estimates of the $L^2$ norms of the derivatives of $u$ and of $h$ if there exists a $K > 0$, independent of $t$, such that

$$E(t) \leq KE_\alpha(t) \equiv K[E(t) - \alpha \tau \int_{\Sigma_t} (u - \bar{u}).u' \mu_g]$$

(35)

We set

$$I_0 \equiv \frac{1}{2} |u'|^2, \text{ and } I_1 \equiv \frac{1}{2} |Du|^2$$

(36)

and

$$x_0 = \int_{\Sigma_t} I_0 \mu_g \equiv \frac{1}{2} \| u' \|^2_g, \text{ and } x_1 = \frac{1}{2} \| Du \|^2_g$$

We estimate the complementary term through the Cauchy-Schwarz inequality

$$| \int_{\Sigma_t} (u - \bar{u}).u' \mu_g | \leq \| u - \bar{u} \| \| u' \|_g.$$

We will use the Poincaré inequality on the compact manifold $(\Sigma, \sigma)$ to estimate the $L^2(\sigma)$ norm of $u - \bar{u}$:

$$\| u - \bar{u} \|_g \leq e^{\lambda M} \| u - \bar{u} \| \leq e^{\lambda M} \Lambda^{-1/2}_{\sigma} \| Du \|$$

(37)

where $\| . \|_g$ denotes the $L^2$ norm on $\Sigma$ in the metric $\sigma$, $\lambda_M$ is an upper bound of the conformal factor $\lambda$ and $\Lambda_{\sigma}$ is the first positive eigenvalue of the operator $-\Delta \equiv -\Delta_{\sigma}$ acting on functions with mean value zero. Note that $\| Du \| = \| Du \|_g$.

The inequality (27) to satisfy is implied by the two following ones:

$$K \geq 1$$

(38)

and (to be satisfied by all $x_0, x_1 \geq 0$)

$$(K - 1)(x_0 + x_1) - 2 |\alpha \tau| KE^{\lambda M} \Lambda^{-\frac{1}{2}}_{\sigma} x_0^\frac{1}{2} x_1^\frac{1}{2} \geq 0$$

(39)
this quadratic form in the $x$’s will be always non negative if $K \geq 1$ and its discriminant is non positive. This last condition reads

$$aK \leq K - 1$$

with

$$a \equiv \frac{\alpha|\tau|e^{\lambda M}}{\Lambda_{\sigma}^{\frac{1}{2}}}$$

(40)

A necessary and sufficient condition for the existence of $K \geq 1$ and $K$ finite is therefore

$$a^2 \equiv \alpha^2 \tau^2 e^{2\lambda M} \Lambda_{\sigma}^{-1} < 1$$

(41)

Any $K$ such that

$$K \geq \frac{1}{1 - a}$$

(42)

satisfies then the required conditions.

It is known that given a 2-manifold $\Sigma$ of genus $G > 1$ there is an open subset of Teichmüller space such that for metrics $\sigma \in M_{-1}$ projecting on this open set it holds

$$8\Lambda_{\sigma} = 1 + \delta_{\sigma}, \text{ with } \delta_{\sigma} > 0$$

(43)

We now choose

$$\alpha = \frac{1}{4}$$

The condition $a < 1$ then reads

$$\left(\frac{\tau^2 e^{2\lambda M}}{2}\right)\left(\frac{1}{1 + \delta_{\sigma}}\right) < 1,$$

(44)

that is using estimates on the conformal factor

$$C(\varepsilon^2 + C_{\sigma}\varepsilon_1) < \delta_{\sigma}$$
10.1.2 Time derivative of the corrected energy.

We set:

\[
\frac{dE_\alpha}{dt} = \frac{dE}{dt} - R_\alpha
\]

with (the terms explicitly containing the shift \( \nu \) give an exact divergence which integrates to zero)

\[
R_\alpha = \alpha \tau \int_{\Sigma_t} \{ \partial_\nu u' (u - \bar{u}) + u'.\partial_\nu(u - \bar{u}) - N \tau u'.(u - \bar{u}) \} \mu_g
\]

\[
+ \alpha \frac{d\tau}{dt} \int_{\Sigma_t} u'.(u - \bar{u}) \mu_g
\]

(45)

The function \( \gamma \equiv u \) satisfies the wave equation

\[
-N^{-1} \partial_\nu(N^{-1} \partial_\nu u) + N^{-1} \nabla^a (N \partial_a u) + N^{-1} \tau \partial_\nu u = 0
\]

(46)

Some elementary computations and integration by parts show that

\[
R_\alpha = \alpha \tau \int_{\Sigma_t} \{ |u'|^2 - |Du|^2 \} N \mu_g
\]

\[
- \alpha \tau \int_{\Sigma_t} u'. \partial_\nu \bar{u} \mu_g + \alpha \frac{d\tau}{dt} \int_{\Sigma_t} (u - \bar{u}) u' \mu_g
\]

**Lemma.** If \( u \) satisfies the wave equation the quantity

\[
\int_{\Sigma_t} u' \mu_g
\]

is conserved in time

Proof. Integration on \((\Sigma_t, g)\) of the wave equation (multiplied by \( N \)) shows that on a compact manifold, where exact divergences integrate to zero, one has

\[
\frac{d}{dt} \int_{\Sigma_t} u' \mu_g = \int_{\Sigma_t} (\partial_\nu u' - N \tau u') \mu_g = 0
\]
To simplify the proofs we will suppose in all that follows that
\[ \int_{\Sigma_t} u' \mu_g = 0 \tag{47} \]

Then \( R_{\alpha} \) reduces to, since \( \partial_t \tilde{u} \) is constant on \( \Sigma_t \),
\[ R_{\alpha} = \alpha \tau \int_{\Sigma_t} \{ |u'|^2 - |Du|_{g}^2 \} N \mu_g + \alpha \frac{d\tau}{dt} \int_{\Sigma_t} (u - \tilde{u}) u' \mu_g \tag{48} \]

### 10.1.3 Decay of the corrected first energy.

In the corrected energy inequality we have seen appear the quantity \( d\tau/dt \). To obtain a differential inequality we have to make a choice of \( \tau \) as a function of \( t \). We wish to work in the expanding direction of our spacetime, where \( \tau \), with our sign convention for the extrinsic curvature, starts from a negative value \( \tau_0 \) and increases, eventually up to the moment of maximum expansion where \( \tau = 0 \). We have made (section 5, notations) the choice
\[ \tau = -t^{-1}, \quad t \in [t_0, \infty), \quad t_0 > 0, \quad \frac{d\tau}{dt} = \frac{1}{t^2} = \tau^2. \tag{49} \]

We obtain, using the value of \( dE/dt \) and \( R_{\alpha} \), that
\[ \frac{dE_{\alpha}}{dt} = \tau \int_{\Sigma_t} \left\{ \frac{1}{2} |h|^2 + \left( \frac{1}{2} - \alpha \right) |u'|^2 + \alpha |Du|_{g}^2 \right\} N - \alpha \tau u' \mu_g \tag{50} \]
we look for a positive number \( k \) such that the difference
\[ \frac{dE_{\alpha}}{dt} - k\tau E_{\alpha} \]
can be estimated with higher order terms. We choose
\[ \alpha = \frac{1}{4}, k = 1 \]

We have then
\[ \frac{dE_{1/4}}{dt} - \tau E_{1/4} = \tau \int_{\Sigma_t} \left\{ \frac{1}{2} |h|_{g}^2 (N - 1) + \left( \frac{1}{2} N - 1 \right) (I_0 + I_1) \right\} \mu_g \]
Which we write

\[
\frac{dE_{1/4}}{dt} - \tau E_{1/4} = \tau \int_{\Sigma_t} \left\{ \frac{1}{2} h^2(1 + N - 2) + \frac{1}{2} (N - 2)(I_0 + I_1) \right\} \mu_g
\]

The right hand side is the sum of a negative term and a term which can be considered as a non linear term in the energies because we have proved that (cf section 9.2 on \(N\) estimates):

\[
0 \leq 2 - N \leq 2 - N_m \leq CC_\sigma (\varepsilon^2 + \varepsilon \varepsilon_1)
\]

Therefore we obtain the following theorem (remember that \(\tau < 0\)):

**Theorem 30** The corrected first energy with \(\alpha = \frac{1}{4}\) satisfies the differential equation

\[
\frac{dE_{1/4}}{dt} = \tau E_{1/4} + |\tau| A \quad \text{with} \quad A \leq CC_\sigma \varepsilon^2 (\varepsilon^2 + \varepsilon \varepsilon_1)
\] (51)

10.2 Corrected second energy.

10.2.1 Definition and lowerbound.

We define a corrected second energy \(E_\alpha\) by the formula, with \(\alpha\) some constant

\[
E^{(1)}_\alpha(t) = E^{(1)}(t) + C_\alpha
\]

with

\[
C_\alpha = \alpha \tau \int_{\Sigma_t} \Delta_g u.u' \mu_g
\]

This corrected second energy will give bounds on the derivatives of \(Du\) and \(u'\) if there exists a number \(K > 0\) such that:

\[
E^{(1)} \leq KE^{(1)}_\alpha
\] (52)
The hypothesis \( \bar{u}' = 0 \) is not necessary here because on a compact manifold \( \int_{\Sigma_t} \Delta_g u \cdot \bar{u}' \mu_g = 0 \). We obtain the estimate, analogous to one obtained in the previous section,

\[
\int_{\Sigma_t} \Delta_g u \cdot u' \mu_g \leq \| \Delta_g u \|_g \| u' \|_g \leq \| \Delta_g u \|_g e^\lambda \mu \Lambda_\sigma^{-\frac{1}{2}} \| Du' \|_g
\]

The same \( K \) as in the previous section satisfies the required inequality when we choose \( \alpha = \frac{1}{4} \).

**10.2.2 Time derivative of the corrected second energy.**

We have

\[
dC_\alpha / dt = \alpha \tau \int_{\Sigma_t} [\partial_0 \Delta_g u \cdot u' + \Delta_g u \cdot \bar{u}' - N \tau \Delta_g u \cdot u'] \mu_g + \alpha \frac{d\tau}{dt} \int_{\Sigma_t} \Delta_g u \cdot u' \mu_g
\]

We recall that (indices are raised with \( g \) in the next few lines)

\[
\partial_0 \Delta_g u = \Delta_g (Nu') + N \tau \Delta_g u + 2Nh_{g}^{ab} \nabla_a \partial_b u + \partial_c u [2\nabla_a (Nk^{ac}) - \tau \partial^c N]
\]

Partial integration together with the splitting \( k_{ab} = h_{ab} + \frac{1}{2} g_{ab} \tau \), and the equation

\[
(3) R^c_0 \equiv -N \nabla_a k^{ac} = \partial_0 u \cdot \partial^c u
\]

gives:

\[
\int_{\Sigma_t} \partial_0 \Delta_g u \cdot u' \mu_g = \int_{\Sigma_t} \{ -N|Du'|^2_g - \partial^a N \partial_a u' \cdot u' + N \tau \Delta_g u \cdot u' + 2Nh_{g}^{ab} \nabla_a \partial_b u \cdot u' + 2u' \partial_c u (\partial_a Nh_{g}^{ac} - \partial^c u \cdot u') \} \mu_g
\]

On the other hand and if \( u \) satisfies the wave equation we find

\[
\int_{\Sigma_t} \Delta_g u \cdot \bar{u}' \mu_g = \int_{\Sigma_t} \{ N|\Delta_g u|^2 + \partial^a N \partial_a u \cdot \Delta_g u + N \tau u' \cdot \Delta_g u \} \mu_g
\]
These equalities give, if we make the choice \( \tau = -\frac{1}{t} \), hence \( \frac{d\tau}{dt} = \tau^2 \):

\[
\frac{dC_\alpha}{dt} = \alpha \tau \int_{\Sigma_t} \left\{ -N|Du|^2 + N|\Delta_g u|^2 + \partial^a N(\partial_a u. \Delta_g u + u'. \partial_a u') \\
+ 2Nh_g^{ab} \nabla_a \partial_b u. u' + 2u'. \partial_a u(\partial_a Nh_g^{ac} - u'. \partial^c u) + (N+1)\tau \Delta_g u. u' \right\} \mu_g
\]

We have found an equality of the form

\[
\frac{dE_{\alpha}^{(1)}}{dt} = \frac{dE^{(1)}}{dt} + \frac{dC_\alpha}{dt} = \int_{\Sigma_t} \{ \tau P_a + \alpha \tau Q \} \mu_g + Z
\]

with

\[
P_a = N[2(1 - \alpha)J_0 + (1 + 2\alpha)J_1] + (N+1)\alpha \tau \Delta_g u. u'
\]

and

\[
Q = \partial^a N(\partial_a u. \Delta_g u + u'. \partial_a u') \\
+ 2Nh_g^{ab} \nabla_a \partial_b u. u' + 2u'. \partial_a u(\partial_a Nh_g^{ac} - u'. \partial^c u)
\]

We see that \( Q \) contains also terms only quadratic in the first and second derivatives of \( u \), but its integral will be bounded by non linear terms in the energies through previous estimates on \( DN \) and \( h \).

We choose \( \alpha = \frac{1}{4} \). We split the integral of \( P_{1/4} \) into linear and non linear terms in the energies by writing

\[
\int_{\Sigma_t} P_{1/4} \mu_g = 3 \int_{\Sigma_t} (J_0 + J_1 + \frac{1}{4} \tau \Delta_g u. u') \mu_g + U \equiv 3E_{1/4}^{(1)} + U
\]

with non linear terms \( U \) given by

\[
U = \int_{\Sigma_t} (N-2)(\frac{3}{2}(J_0 + J_1) + \frac{1}{4} \tau \Delta_g u. u') \mu_g
\]

We are ready to prove the following theorem

**Theorem 31**  With the choice \( \alpha = \frac{1}{4} \) and \( \tau = -\frac{1}{t} \), \( t > 0 \), the corrected second energy satisfies the inequality

\[
\frac{dE_{1/4}^{(1)}}{dt} = 3\tau E_{1/4}^{(1)} + |\tau|^3 B
\]

where \( B \) a polynomial in \( \varepsilon \) and \( \varepsilon_1 \) with all terms of order at least 3 and coefficients of the form \( CC_\sigma \).
Proof. We have shown that
\[
\frac{dE_{1/4}^{(1)}}{dt} = 3\tau E_{1/4}^{(1)} + Z + \frac{1}{4}\tau \int Q\mu_g + \tau U
\]
We will estimate the various terms in the right hand side.
We obtain, using the bound of $2 - N$ and the definition of $\varepsilon_1$
\[
|\tau U| \leq CC_0|\tau|^3(\varepsilon^2 + \varepsilon\varepsilon_1)(\varepsilon_1^2 + \varepsilon\varepsilon_1)
\]
We now estimate $\int \tau Q\mu_g$, using its expression and the estimates (cf. section 9)
\[
\| h \|_{L^\infty(g)} \leq |\tau|\varepsilon_h, \text{ with } \varepsilon_h = CC_0\{\varepsilon + \varepsilon^{1/2}(\varepsilon + \varepsilon_1)^{3/2}\},
\]
\[
\| DN \|_{L^\infty(g)} \leq |\tau|C_0\varepsilon_{DN}, \text{ with } \varepsilon_{DN} = CC_0(\varepsilon^2 + \varepsilon\varepsilon_1)
\]
we have, with $C_0$ a fixed number
\[
|\tau \int \left\{ \partial^a N(\partial_a u, \Delta_g u + u'\partial_a u') + 2u'\partial_c u(\partial_a N_{pc}) \right\} \mu_g | \\
\leq C_0|\tau|^3(\varepsilon_{DN} + \varepsilon_h)\varepsilon_1 + \varepsilon_{DN}\varepsilon_h\varepsilon^2
\]
while
\[
|\tau \int \Sigma_t 2N p^{ab} \nabla_a \partial_b u.\mu_g | \leq 4\tau^2\varepsilon_h\varepsilon \| \nabla^2 u \|_g
\]
It holds on a 2 dimensional compact manifold
\[
\| \nabla^2 u \|_g^2 = \| \Delta_g u \|_g^2 - \frac{1}{2} \int \Sigma_t R(g)|Du|_g^2 \mu_g
\]
Recall that
\[
R(g) = -\frac{1}{2}\tau^2 + |p|_g^2 + |u'|_g^2 + |Du|_g^2, \text{ with } |Du|_g^2 \leq \frac{\tau^2}{2}|Du|^2
\]
and
\[
\int \Sigma_t R(g)|Du|_g^2 \mu_g = \int \Sigma_t R(g)|Du|_g^2 \mu_g
\]
therefore
\[ \| \nabla^2 u \|_g^2 \leq C_0 \tau^2 [\varepsilon_1^2 + (1 + \varepsilon_h)\varepsilon^2] + [\| u' \|_2^2 + \tau^2 \| Du \|_2^2] \| Du \|_2^2 \]
The bounds on \( L^4 \) norms of \( u' \) and \( Du \) give
\[ \| \nabla^2 u \|_g \leq \tau |\varepsilon| \nabla^2 u, \quad \varepsilon \nabla^2 u = C_0 \{ \varepsilon_1^2 + (1 + \varepsilon_h)\varepsilon^2 + CC_\sigma \varepsilon^2 \} + \frac{1}{2} \]
Finally
\[ |\tau \int_{\Sigma_t} \{ 2(u'.\partial_c u)(u'.\partial_c u) \} \mu_g | \leq 2 |\tau| \| u' \|^2_{L^4(g)} \| Du \|^2_{L^4(g)} \]
We have, using previous estimates,
\[ \| u' \|^2_{L^4(g)} \leq e^{\lambda M} \| u' \|_4 \leq CC_\sigma e^{\lambda M} \tau^2 (\varepsilon^2 + \varepsilon \varepsilon_1) \]
hence
\[ \| u' \|^2_{L^4(g)} \leq CC_\sigma |\tau| (\varepsilon^2 + \varepsilon \varepsilon_1) \]
An inequality of the same type holds for \( \| Du \|_{L^4(g)} \).

The estimate of \( | \int \tau Q \mu_g | \) by the product of \( |\tau|^3 \) with higher than 2 powers of the \( \varepsilon' \)'s follows.

We now estimate \( Z \). We recall that
\[
Z \equiv \int_{\Sigma_t} \{ Np^{ab} \partial_a u' \partial_b u' + 2N^{ab} \nabla_a \partial_b u \Delta_g u + (\nabla_b \partial^a N \partial_a u + \tau \partial_b N u')(\partial^b u') \} \mu_g \\
+ Y_1 \tag{53}
\]

Previous estimates give
\[
|Z| \leq |\tau|^3 \{ C_0 \varepsilon_h \varepsilon_1^2 + \varepsilon_D N (\varepsilon_1^2 + \varepsilon \varepsilon_1) + 4 \varepsilon_1 \varepsilon \nabla^2 u \} + Y_2 + |Y_1| \tag{54}
\]
with
\[
Y_2 \equiv | \int_{\Sigma_t} \{ (\nabla_b \partial^a N) \partial_a u \partial_b u' \} \mu_g | \]

To bound \( Y_2 \) we use the \( L^4 \) norm of \( \nabla^2 N \) estimated in terms of its \( W^3_3 \) norm in the section on lapse estimates. Indeed
\[
Y_2 \leq |\tau| \varepsilon_1 \| \nabla^2 N \|_{L^4(g)} \| Du \|_{L^4(g)}
\]
We have

$$|\nabla^2 N|_g = e^{-2\lambda} |\nabla^2 N|$$

hence

$$\| \nabla^2 N \|_{L^4(g)} \leq e^{-2\frac{3}{2}\lambda_n} \| \nabla^2 N \|_4 \leq C_0 |\tau|^{\frac{3}{2}} \| \nabla^2 N \|_4$$

On the other hand we recall the identity

$$\nabla_a \partial_b N \equiv D_a \partial_b N + \sigma^{cd} \partial_c N \partial_d \lambda - \delta^c_a \partial_b \lambda \partial_c N - \delta^c_b \partial_a \lambda \partial_c N$$

By the Sobolev embedding theorem, with $p = \frac{4}{3}$

$$\| D^2 N \|_4 \leq C_\sigma \| D^2 N \|_{W^{p}_2} \leq C_\sigma \varepsilon_{DN}$$

We also bound

$$\| D\lambda \|_4 \leq C_\sigma \| D\lambda \|_{H^1}$$

with

$$\| D\lambda \|_{H^1} \leq C C_\sigma (\varepsilon^2 + \varepsilon \varepsilon_1)$$

and we obtain

$$\| \nabla^2 N \|_4 \leq C_\sigma \varepsilon_{DN} (1 + C(\varepsilon^2 + \varepsilon \varepsilon_1))$$

Recall that

$$\| Du \|_{L^4(g)} \leq C_0 |\tau|^{\frac{1}{2}} \| Du \|_4 \leq C C_\sigma |\tau|^{\frac{1}{2}} (\varepsilon + \varepsilon^2 \varepsilon_1)$$

Finally

$$Y_2 \leq C C_\sigma |\tau|^{\frac{3}{2}} \varepsilon_{DN}[1 + C(\varepsilon^2 + \varepsilon \varepsilon_1)] \varepsilon_1 (\varepsilon + \varepsilon^2 \varepsilon_1^{\frac{1}{2}})$$

Recall that

$$Y_1 = \int_{\Sigma_t} \{(2\partial_a N p^{ac} - 2N \partial^e u.u')\partial_c u + 2\partial^a N \partial_a u' + u' \Delta_g N \}.\Delta_g u \mu_g \tag{55}$$
hence

\[ |Y_1| \leq |\tau|^3 C \sigma \{ |\varepsilon_{DN} \varepsilon_h \varepsilon_1 + \varepsilon_{DN} \varepsilon_1^2| \} + Y_3 + Y_4 \quad (56) \]

with

\[ Y_3 = \left| \int_{\Sigma_t} \left\{ \left( -2N \partial^c u. u' \right) \partial_c u \right\} \Delta g u \mu_g \right| \]

The term \( Y_3 \) can be estimated using the Holder inequality,

\[ Y_3 \leq 4 |\tau| \varepsilon_1 \| Du \|_{L^6}^2 \| u' \|_{L^6} \]

Elementary calculus gives

\[ \| Du \|_{L^6} \leq e^{-\frac{2}{3} \lambda_m} \| Du \|_6 \leq C_0 |\tau|^\frac{2}{3} \| Du \|_{L^6} \]

and

\[ \| u' \|_{L^6} \leq e^{\frac{1}{3} \lambda_M} \| u' \|_6 \]

The \( L^6 \) norms can be estimated with \( H_1 \) norms using the Sobolev inequality

\[ \| f \|_6 \leq C_\sigma (\| f \| + \| D f \|) \]

applied to \( f = u' \) and \( f = |Du| \) together with the inequality

\[ D |f| \leq |D f| \]

we obtain

\[ Y_3 \leq 4 |\tau| \varepsilon_1 C_\sigma e^{\frac{1}{3} (\lambda_M - \lambda_m)} \| u' \| + \| Du' \| \| Du \|^2 + \| D^2 u \|^2 \]

hence, going back to the energies

\[ Y_3 \leq C C_\sigma |\tau|^3 \varepsilon_1 [\varepsilon + \varepsilon_1] [\varepsilon^2 + \varepsilon_1^2] \]

Finally

\[ Y_4 \equiv \left| \int_{\Sigma_t} u' \Delta g N. \Delta_g u \mu_g \right| \leq |\tau| \varepsilon_1 \| u' \|_{L^4} \| \Delta_g N \|_{L^4} \]
therefore, using laplacian and norms in conformal metrics and the previous estimate of $\| u' \|_{L^4(g)}$

$$Y_4 \leq C_\sigma |\tau|^2 e^{-2\lambda_m} e^{2 \lambda_M} \varepsilon_1 (\varepsilon^2 + \varepsilon \varepsilon_1) \| \Delta N \|_4$$

The bound we have just computed of $\| D^2 N \|_4$ gives also a bound of $\| \Delta N \|_4$, hence

$$Y_4 \leq |\tau|^3 C \varepsilon_1 \varepsilon (\varepsilon^2 + \varepsilon \varepsilon_1) \varepsilon_D N_1 (1 + C (\varepsilon^2 + \varepsilon \varepsilon_1))$$

Gathering the results gives the theorem.

11 Decay of the total energy.

We call total energy the quantity

$$E_{\text{tot}}(t) \equiv E(t) + \tau^{-2} E^{(1)}(t) \equiv \varepsilon^2 + \varepsilon_1^2$$

We define $y(t)$ to be the total corrected energy namely:

$$y(t) \equiv E_{1/4}(t) + \tau^{-2} E_{1/4}^{(1)}$$

We have

$$E_{\text{tot}}(t) \leq \frac{1}{1 - a_t} y(t)$$

with on each $\Sigma_t$

$$a_t \equiv \frac{|\tau| e^{\Lambda_M}}{4 \Lambda_t^4}, \quad \Lambda_t \equiv \Lambda_\sigma$$

(57)

The inequalities obtained for the corrected energies imply, with $\tau = -t^{-1}$

$$\frac{dy}{dt} = \frac{1}{t} [-y + A + B]$$

(58)

where $A$ and $B$ are bounded by polynomials in $\varepsilon$ and $\varepsilon_1$ with terms of degree at least 3.
Lemma 32 Suppose that on \((\Sigma, \sigma)\) there is \(\delta_\sigma > 0\) such that the first positive eigenvalue \(\Lambda_\sigma\) is
\[
(4\Lambda_\sigma)^{-\frac{1}{2}} = \frac{1 - \delta_\sigma}{\sqrt{2}}.
\]
then if the energies are such that
\[
CC_\sigma(\varepsilon^2 + \varepsilon\varepsilon_1) \leq \frac{\delta_\sigma}{2}
\]
then
\[
1 - a_\sigma \geq \frac{\delta_\sigma}{2}
\]
The numbers \(C\) and \(C_\sigma\) are known numbers depending respectively on the number \(c\) of the hypothesis \(H_c\) and on the metric \(\sigma\).

Proof. By the definition of \(a \equiv a_\sigma\) it holds that
\[
1 - a_\sigma = 1 - \frac{|\tau|e^{\lambda M}}{\sqrt{2}} + \frac{\delta_\sigma|\tau|e^{\lambda m}}{\sqrt{2}}
\]
which gives using the lower bound of \(\lambda\) and the lemma 3 of the section 8 ”conformal factor estimates”
\[
1 - a_\sigma \geq \delta_\sigma - CC_\sigma(\varepsilon^2 + \varepsilon\varepsilon_1)
\]
from which the result follows.

Hypothesis \(H_\sigma\): 1. The numbers \(C_\sigma\) are uniformly bounded by a constant \(M\) for all \(t \geq t_0\) for which they exist.

2. There exists a constant \(\delta > 0\) such that the numbers \(\Lambda_\sigma\), the first positive eigenvalues of \(-\Delta_\sigma\), for functions with mean value zero, are such that
\[
(4\Lambda_\sigma)^{-\frac{1}{2}} = \frac{1 - \delta_\sigma}{\sqrt{2}}, \quad \text{with} \quad \delta_\sigma \geq \delta.
\]

Hypothesis \(H_E\). The energies \(\varepsilon_1^2\) and \(\varepsilon_{1,t}^2\) satisfy as long as they exist an inequality of the form
\[
C(c, M)(\varepsilon^2 + \varepsilon\varepsilon_1) \leq \frac{\delta}{2}
\]
where \(C\) is a number depending only on the numbers \(c\) and \(M\).

We will prove the following theorem.
**Theorem 33** Under the hypothesis $H_c, H_E$ and $H_\sigma$ there exists a number $\eta$ such that if the total energy is bounded at time $t_0$ by $\eta$ then it satisfies at time $t = -\tau^{-1} \geq t_0 > 0$ an inequality of the form

$$tE_{\text{tot}}(t) \equiv t(\varepsilon^2 + \varepsilon_1^2) \leq M_{\text{tot}}E_{\text{tot}}(t_0)$$

where $M_{\text{tot}}$ depends only on $\delta$.

**Proof.** Under the hypothesis we have made the polynomials $A$ and $B$ are bounded by polynomials in $y^{3/2}$ with terms of degree at least 3 and bounded coefficients depending only on $c, M, \delta$.

Take $\eta$ such that $y_0 \equiv y(t_0) < 1$. Then all powers of $y_0$ greater than $3/2$ are less than $y_0^{3/2}$ and there exists a constant $M_1$, depending only on $c, \delta$ and $M$ such that

$$(A + B)_{t=t_0} \leq M_1 y_0^{3/2}$$

Take $\eta$ such that moreover

$$y_0^{1/2} < \frac{1}{M_1}$$

hence $\frac{dy}{dt}(t_0) < 0$ and $y$ starts decreasing, therefore continues to satisfy $y < 1$. Therefore

$$A + B \leq M_1 y^{3/2}$$

and $y$ satisfies the differential inequality

$$\frac{dy}{dt} \leq -\frac{1}{t}(y - M_1 y^{3/2}) \quad (59)$$

with always

$$y - M_1 y^{3/2} > 0$$

and, consequently, the differential inequality

$$\frac{dy}{y(1 - M_1 y^{1/2})} + \frac{dt}{t} \leq 0$$

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equivalently
\[ \frac{dz}{z(1 - M_1 z)} + \frac{dt}{2t} \leq 0, \quad \text{with} \quad y = z^2 \]
which gives by integration
\[ \log\{ \frac{z(1 - M_1 z_0)}{(1 - M_1 z)z_0} \} + \log\left( \frac{t}{t_0} \right) \geq 0 \]
that is
\[ \frac{t^{\frac{1}{2}}(1 - M_1 z)}{(1 - M_1 z)t_0^{\frac{1}{2}}z_0} \leq 1 \]
in other words
\[ t^{1/2}z + M_1 z - \frac{t_0^{1/2}z_0}{1 - M_1 z_0} \leq \frac{t_0^{1/2}z_0}{1 - M_1 z_0} \]
a fortiori
\[ ty \leq \frac{t_0 y_0}{(1 - M_1 z_0)^2} \]
We suppose for instance
\[ z_0 \leq \frac{1}{2M_1} \]
then
\[ ty \leq 4t_0 y_0 \]
Recall that under the $H_c$, $H_E$ and $H_\sigma$ hypotheses
\[ E_{tot}(t) \leq \frac{1}{1 - a_t} y(t) \leq \frac{2}{\delta_t} y(t), \]
also
\[ y_0 \leq \frac{1}{1 - a_0} y_0 \leq \frac{2}{\delta_0} E_{tot}(t_0), \]
The inequality for $y$ implies therefore
\[ tE_{tot}(t) \leq M_t E_{tot}(t_0) \]
with, as announced, $M_t$ uniformly bounded:
\[ M_t = \frac{4t_0}{(1 - a_t)(1 - a_0)} \leq \frac{16t_0}{\delta t \delta t_0} \leq \frac{16t_0}{\delta^2} \]
12 Teichmuller parameters.

12.1 Dirichlet energy.

Let $s$ and $\sigma$ be two given metrics on $\Sigma$ and $\Phi$ be a mapping from $\Sigma$ into $\Sigma$. The energy of the mapping $\Phi : (\Sigma, \sigma) \rightarrow (\Sigma, s)$ is by definition the positive quantity:

$$E(\sigma, \Phi) \equiv \int_{\Sigma} \sigma^{ab} \frac{\partial \Phi^A}{\partial x^a} \frac{\partial \Phi^B}{\partial x^b} s_{AB}(\Phi) \mu_\sigma$$

Consider the metric $s$ as fixed. Elementary calculus shows that the energy $E(\sigma, \Phi)$ is invariant under a diffeomorphism $f$ of $\Sigma$ in the following sense

$$E(\sigma, \Phi) = E(f_*\sigma, \Phi \circ f)$$

In the case where $s$ and $\sigma$ both have negative curvature it has been proved by Eells and Sampson that there exists one and only one harmonic map $\Phi_\sigma : (\Sigma, \sigma) \rightarrow (\Sigma, s)$ which is a diffeomorphism homotopic to the identity, i.e. $\Phi_\sigma \in D_0$. Such a harmonic map is equivariant under diffeomorphisms homotopic to the identity, i.e.

$$\Phi_{f_*\sigma} = \Phi_\sigma \circ f, \text{ with } f \in D_0$$

One is then led to the definition:

**Definition 34** Given a metric $s \in M_{-1}$ the Dirichlet energy $D(\sigma)$ of the metric $\sigma \in M_{-1}$ is the energy of the harmonic map $\Phi_\sigma \in D_0$:

$$D(\sigma) \equiv E(\sigma, \Phi_\sigma)$$

It depends on the choice of the fixed metric $s$, but is invariant under the action of diffeomorphisms included in $D_0$ hence defines a positive functional on the Teichmuller space $T_{\text{teich}} \equiv M_{-1}/D_0$.

**Remark 35** The energy of the mapping $\Phi : (\Sigma, \sigma) \rightarrow (\Sigma, s)$ as well as the harmonic map $\Phi_\sigma$ are also invariant under conformal rescalings of $\sigma$. They can be used on the space of riemannian metrics of negative curvature before the rescaling which restricts them to metrics of curvature $-1$. 

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The importance of the Dirichlet energy rests on the following theorem which says that if $D(\sigma)$ remains in a bounded set of $R$ then the equivalence class of $\sigma$ remains in a bounded set of $Teich$.

**Theorem 36** (Eells and Sampson) *The Dirichlet energy is a proper function on Teichmuller space.*

### 12.2 Estimate of the Dirichlet energy.

We will require of the metric $\sigma_t$ that it remains, when $t$ varies, in some cross section of $M_{-1}$ (space of $C^\infty$ metrics with scalar curvature -1) over the Teichmuller space, diffeomorphic to $R^{6G-6}$, $G$ the genus of $\Sigma$.

**Remark.** Following Andersson-Moncrief one can choose the cross section as follows, having given some metric $s \in M_{-1}$. To an arbitrary metric $\zeta \in M_{-1}$ we associate another such metric by its pull back through $\Phi^{-1}_\zeta$

$$\psi(\zeta) = (\Phi^{-1}_\zeta)_* \zeta$$

For any $f \in D_0$ we have

$$\psi(f_* \zeta) = (\Phi^{-1}_f)_* f_* \zeta = \psi(\zeta)$$

hence the metric $\psi$ depends only on the equivalence class $Q$ of $\zeta$ through $D_0$. Thus one gets a cross section of $M_{-1}$ over Teichmuller space, $Q \in Teich \mapsto \psi(Q) \in M_{-1}$. If $Q$ remains in a bounded set of $Teich$ then $\psi(Q)$ remains in a bounded set of $M_{-1}$ i.e. all these metrics are uniformly equivalent.

We will estimate the Dirichlet energy $D(\sigma) \equiv E(\sigma, \Phi_\sigma)$. We have, with $g_{ab} = e^{2\lambda} \sigma_{ab}$

$$E(\sigma, \Phi_\sigma) \equiv \int_\Sigma \sigma^{ab} \partial_a \Phi_\sigma^A \partial_b \Phi_\sigma^B s_{AB}(\Phi_\sigma) \mu_\sigma = \int_\Sigma g^{ab} \partial_a \Phi_\sigma^A \partial_b \Phi_\sigma^B s_{AB}(\Phi_\sigma) \mu_g \equiv E(g, \Phi_\sigma)$$

If $\Phi_\sigma$ is a harmonic map from $(\Sigma, \sigma)$ into $(\Sigma, s)$ it is an extremal of the mapping

$$\Phi \mapsto E(\sigma, \Phi)$$

and also an extremal of the mapping

$$\Phi \mapsto E(g, \Phi)$$
We have, with \( \frac{\partial E}{\partial g} \) and \( \frac{\partial E}{\partial \Phi} \) denoting functional derivatives (linear maps acting respectively on \( \frac{dg}{dt} \) and \( \frac{d\Phi}{dt} \))

\[
\frac{d}{dt} E(g, \Phi) = \frac{\partial E}{\partial g} \frac{dg}{dt} + \frac{\partial E}{\partial \Phi} \frac{d\Phi}{dt}
\]

We compute this derivative at a point \((\sigma, \Phi_{\sigma})\); we have, by the extremality of \( \Phi_{\sigma} \), \( \left( \frac{\partial E}{\partial \Phi} \right)(\sigma, \Phi_{\sigma}) = 0 \). Therefore

\[
\frac{d}{dt} D(\sigma) \equiv \{ \frac{d}{dt} E(g, \Phi) \}_{(g, \Phi_{\sigma})} = \{ \frac{\partial E}{\partial g} \frac{dg}{dt} \}_{(g, \Phi_{\sigma})}
\]

which gives using previous notations and the vanishing of the integral of a divergence on a compact manifold

\[
\frac{d}{dt} D(\sigma) = \int_{\Sigma_t} \{ \tilde{\partial}_0 g^{ab} \partial_a \Phi^A_{\sigma} \partial_b \Phi^B_{\sigma} - N \tau g^{ab} \partial_a \Phi^A_{\sigma} \partial_b \Phi^B_{\sigma} \} s_{AB}(\Phi_{\sigma}) \mu_g
\]

Recall that

\[
\tilde{\partial}_0 g^{ab} = 2N g^{ac} g^{bd} k_{cd} = 2Ne^{-4\lambda} h^{ab} + Ne^{-2\lambda} h^{ab, \tau}
\]

hence

\[
\frac{d}{dt} D(\sigma) = \int_{\Sigma_t} 2Ne^{-2\lambda} h^{ab} \partial_a \Phi^A_{\sigma} \partial_b \Phi^B_{\sigma} s_{AB}(\Phi_{\sigma}) \mu_{\sigma}
\]

Using \( 0 < N \leq 2 \) and \( e^{-2\lambda} \leq \frac{\tau^2}{2} \) we find

\[
\left| \frac{d}{dt} D(\sigma) \right| \leq 2\tau^2 \| h \|_{\infty} D(\sigma)
\]

The bound of \( \| h \|_{\infty} \) found in the section on \( h \) estimates gives:

\[
\left| \frac{d}{dt} D(\sigma) \right| \leq |\tau| C C_{\sigma} [\varepsilon + (\varepsilon + \varepsilon_1)^2] D(\sigma)
\]

We recall the following lemmas.

**Lemma 37** There exists an open subset \( \Omega \) of \( T_{eich} \) such that if the equivalence class of \( \sigma \) is in \( \Omega \) and \( \sigma \) is in a smooth cross section of \( T_{eich} \), then there exists a number \( \delta > 0 \) such that \( \Lambda(\sigma) \geq \frac{1}{8} + \delta \) and all constants \( C_{\sigma} \) are bounded by a fixed number \( M \).
Lemma 38 There exists an interval $I=\langle a,b \rangle$ of $R$ such that if the Dirichlet energy (taken with some metric $s$) $D(\sigma) \in I$ then $\sigma$ projects into $\Omega$. More precisely, there exists $\sigma_0$ projecting in $\Omega$ and given $\sigma_0$ there exists a number $D$ such that if $|D(\sigma) - D(\sigma_0)| \leq D$ then the hypothesis $H_\sigma$ is satisfied.

We will prove the following theorem

Theorem 39 Under the hypothesis $H_c$, $H_E$ and $H_\sigma$ there exists a number $M_D$ depending only on the bounds in these hypothesis such that the Dirichlet energy satisfies the inequality

$$|D(\sigma_t) - D(\sigma_0)| \leq M_D x_0^{\frac{1}{2}} \quad \text{with} \quad x_0 \equiv E_{tot}(t_0)$$

Proof. Under the hypothesis that we have made the Dirichlet energy satisfies the differential inequality (we have set $\tau = -t^{-1}$)

$$\left|\frac{d}{dt} D(\sigma)\right| \leq D(\sigma) CM \left\{ \frac{t^{\frac{1}{2}} [\varepsilon + (\varepsilon + \varepsilon_1)^2]}{t^{\frac{1}{2}}} \right\}$$

We recall the decay found for the total energy

$$t E_{tot}(t) \equiv t (\varepsilon^2 + \varepsilon_1^2) \leq M_{tot} E_{tot}(t_0)$$

with

$$M_{tot} \leq \frac{16 t_0}{\delta^2}$$

We have, using $t \geq t_0$ and $(\varepsilon + \varepsilon_1)^2 \leq 2 E_{tot}$

$$t^{\frac{1}{2}} (\varepsilon + (\varepsilon + \varepsilon_1)^2) \leq t^{\frac{1}{2}} E_{tot}(t) + 2 t^{\frac{1}{2}} E_{tot}(t)$$

Using the decay of the total energy (section 11) and the assumption $x_0 \equiv E_{tot}(t_0) < 1$ we find that there exists a number $M_2$ depending only on $c, M$ and $\delta$ such that

$$\left|\frac{d}{dt} D(\sigma)\right| \leq D(\sigma) \frac{M_2 x_0^{\frac{1}{2}}}{t^{\frac{1}{2}}}$$

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We deduce from this inequality, by elementary calculus, abbreviating $D(\sigma)$ to $D$ and $D(\sigma_0)$ to $D_0$,

$$\frac{d}{dt}|D - D_0| \leq \|\frac{d}{dt}(D - D_0)\| \leq \|D - D_0\| + D_0\frac{M_2 x_0^{\frac{1}{2}}}{t^\frac{3}{2}}$$

By the Gromwall lemma $|D - D_0|$ is for $t \geq t_0$ bounded by the solution of the associated differential equality with initial value zero, which gives

$$|D - D_0| \leq [D_0 M_2 x_0^{\frac{1}{2}} \int_{t_0}^{t} t^{-\frac{3}{2}} dt] \exp(M_2 x_0^{\frac{1}{2}} \int_{t_0}^{t} t^{-\frac{3}{2}} dt)$$

hence, as announced

$$|D_{\sigma_t} - D_{\sigma_0}| \leq M_D x_0^{\frac{1}{2}}$$

with (recall that $x_0 \leq 1$)

$$M_D = D_{\sigma_0} 2M_2 t_0^{-\frac{1}{2}} \exp(2M_2 t_0^{-\frac{1}{2}})$$

### 13 Global existence.

**Theorem 40** Let $(\sigma_0, q_0) \in C^\infty(\Sigma_0)$ and $(u_0, \dot{u}_0) \in H_2(\Sigma_0, \sigma_0) \times H_1(\Sigma_0, \sigma_0)$ be initial data for the polarized Einstein equations with $U(1)$ isometry group on the initial manifold $M_0 \equiv \Sigma_0 \times U(1)$; suppose that $\sigma_0$ is such that $R(\sigma_0) = -1$ and the first positive eigenvalue $\Lambda_0$ of $-\Delta_{\sigma_0}$ (for functions with mean value zero) is such that

$$\Lambda_0 > \frac{1}{8}.$$ 

Then there exists a number $\eta > 0$ such that if

$$E_{tot}(t_0) < \eta$$

these Einstein equations have a solution on $M \times [t_0, \infty)$, with initial values determined by $\sigma_0, q_0, u_0, \dot{u}_0$. The orthogonal trajectories to the space sections $M \times \{t\}$ have an infinite proper length.
Proof. It results from the local existence theorem that we only have to prove that $E_{tot}(t)$ does not blow up. We have in the previous sections made the following hypothesis, to hold for all $t \geq t_0$ for which the involved quantities exist

Hypothesis $H_c$. There exists a number $c > c_0 = \varepsilon v_0 > 0$ such that

1. $\varepsilon v \leq c$

2. $\varepsilon_0 \leq \frac{1}{2(1 + 2ce^2c)}$

Hypothesis $H_D$. The Dirichlet energy is such that

$|D(\sigma) - D(\sigma_0)| \leq d$

where $d > 0$ is a given number such that the above inequality implies the hypothesis $H_\sigma$.

Hypothesis $H_E$. The total energy is such that

$E_{tot}(t) \leq c_E$

where $c_E$ is a number depending only on $c$ and $d$.

Under these hypothesis we have obtained the following result: there are numbers $A_i$ depending only on $c$ and $d$ such that

$\varepsilon v \leq A_1 E_{tot}(t_0)$

and

$tE_{tot}(t) \leq A_2 E_{tot}(t_0)$

and

$|D(\sigma) - D(\sigma_0)| \leq A_3 E_{tot}^{\frac{1}{2}}(t_0)$

Now consider the triple of numbers

$\{X_t \equiv \varepsilon v_t, x_t \equiv E_{tot}(t), Z_t \equiv |D(\sigma_t) - D(\sigma_0)|\}$
We have shown that the hypothesis

\[ X_t \leq c, \ x_t \leq c_E, \ Z_t \leq d \]

and smallness conditions on \( x_0 \), imply the existence of numbers \( A_i \) depending only on \( c, c_E \) and \( d \) such that

\[ X_t \leq A_1 x_0, \ t x_t \leq A_2 x_0, \ Z_t \leq A_3 x_0^\frac{1}{2} \]

Therefore there exists \( \eta > 0 \) such that \( x_0 \leq \eta \) implies that the triple belongs to the subset \( U_1 \subset \mathbb{R}^3 \) defined by the inequalities:

\[ U_1 \equiv \{ X_t < c, \ x_t < c_E, \ Z_t < d \} \]

For such an \( \eta \) the triple either belongs to \( U_1 \) or to the subset \( U_2 \) defined by

\[ U_2 \equiv \{ X_t > c \ or \ x_t > c_E \ or \ Z_t > d \} \]

These subsets are disjoint. We have supposed that for \( t = t_0 \) it holds that \((X_0, x_0, Z_0) \in U_1 \) hence, by continuity in \( t \), \((X_t, x_t, Z_t) \in U_1 \) for all \( t \). We have proved the required a priori bounds.

The orthogonal trajectories to the space sections \( M \times \{ t \} \) have an infinite proper length since the lapse \( \mathcal{N} \) is bounded below by a strictly positive number.

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