Strong laws of large numbers for capacities *

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In this paper, with the notion of independent identically distributed random variables under sub-linear expectations initiated by Peng, we derive three kinds of strong laws of large numbers for capacities. Moreover, these theorems are natural and fairly neat extensions of the classical Kolmogorov’s strong law of large numbers to the case where probability measures are no longer additive. Finally, an important feature of these strong laws of large numbers is to provide a frequentist perspective on capacities.

Keywords: capacity, strong law of large number, independently and identically distributed(IID), sub-linear expectation.

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1 Introduction

The classical strong laws of large numbers (strong LLN) as fundamental limit theorems in probability theory play an important role in the development of probability theory and its applications. The key in the proofs of these limit theorems is the additivity of the probabilities and the expectations. However, such additivity assumption is not reasonable in many areas of applications because many uncertain phenomena can not be well modeled using additive probabilities or additive expectations. More specifically, motivated by some problems in mathematical economics, statistics, quantum mechanics and finance, a number of papers have used non-additive probabilities (called capacities) and nonlinear expectations (for example Choquet integral/expectation, $g$-expectation) to describe and interpret the phenomena (see for example, Chen and Epstein [1], Feynman [4], Gilboa [5], Huber [16], Peng [8,9], Schmeidler [18], Wakker [19], Walley and Fine [20], Wasserman and Kadane [21]). Recently, motivated by the risk measures, super-hedge pricing and modelling uncertainty in finance, Peng [11,12,13,14,15] initiated the

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notion of independent and identically distributed (IID) random variables under sub-linear expectations. Under this framework, he proved the weak law of large numbers and the central limit theorems (CLT). However, Peng’s techniques can not be extended to prove the strong laws of large numbers. In this paper, we develop new approaches to solving this problem. We obtain three strong laws of large numbers for capacities in this framework. All of them are natural and fairly neat extensions of the classical Kolmogorov’s strong law of large numbers, but the proofs here are different from the original proofs of the classical strong law of large numbers.

Now we describe the problem in more detail. For a given set $P$ of multiple prior probability measures on $(\Omega, \mathcal{F})$, we define a pair $(V, v)$ of capacities by

\[ V(A) := \sup_{P \in P} P(A), \quad v(A) := \inf_{P \in P} P(A), \quad \forall A \in \mathcal{F}. \]

The corresponding Choquet integrals/expecations $(C_V, C_v)$ are defined by

\[ C_V[X] := \int_{-\infty}^{\infty} V(X \geq t)dt + \int_{-\infty}^{0} [V(X \geq t) - 1]dt, \]

where $V$ is replaced by $v$ and $V$ respectively.

The pair of so-called maximum-minimum expectations $(E, E)$ are defined by

\[ E[\xi] := \sup_{P \in P} E_P[\xi], \quad E[\xi] := \inf_{P \in P} E_P[\xi]. \]

Here and in the sequel, $E_P$ denotes the classical expectation under probability $P$.

In general, the relation between Choquet integral and maximum -minimum expectations is as follows: For any random variable $X$,

\[ E[X] \leq C_V[X], \quad C_v[X] \leq E[X]. \]

Note that under some very special assumptions on $P$ and $V$, both inequalities could become equalities (see for example Gilboa [5], Huber [16], Schmeidler [18]).

Given a sequence $\{X_i\}_{i=1}^{\infty}$ of IID random variables for capacities, the earlier papers related to strong laws of large numbers are Dow and Werlang [2] and Walley and Fine [20]. However, the more general results for strong laws of large numbers for capacities were given by Marinacci [6,7] and Epstein and Schneider [3]. They show that, on full set, any cluster point of empirical averages lies between the lower Choquet integral $C_v[X_1]$ and the upper Choquet integral $C_V[X_1]$ with probability one under capacity $v$. That is

\[ v\left( \omega \in \Omega : C_v[X_1] \leq \lim \inf_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} X_i(\omega) \leq \lim \sup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} X_i(\omega) \leq C_V[X_1] \right) = 1. \]

Marinacci [6,7] obtains his result under the assumptions that $V$ is a totally monotone capacity on a Polish space $\Omega$, random variables $\{X_n\}_{n \geq 1}$ are bounded or continuous. Epstein and Schneider [3] also show the same result under the assumptions that $V$ is rectangular and the set $P$ is finite.
Since the gap between the Choquet integrals $C_v[X]$ and $C_V[X]$ is bigger than that of the maximum-minimum expectations $\mathcal{E}[X]$ and $\mathbb{E}[X]$ for all $X$, it is of interest to ask whether we can obtain a more precise result if the Choquet integrals/expectations in the above equality are replaced by maximum-minimum expectations. That is

$$v \left( \omega \in \Omega : \mathcal{E}[X_1] \leq \lim\inf_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} X_i(\omega) \leq \lim\sup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} X_i(\omega) \leq \mathbb{E}[X_1] \right) = 1.$$ 

The first result in this paper is to show that the above equality is still true in Peng’s framework even under some weaker assumptions. Furthermore, motivated by this result, we establish two new laws of large numbers. The first is to show that there exist two cluster points of empirical averages which reach the minimum expectation $\mathbb{E}[X_1]$ and the maximum expectation $\mathbb{E}[X_1]$ respectively under capacity $V$. That is

$$V \left( \omega \in \Omega : \lim\inf_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} X_i(\omega) = \mathbb{E}[X_1] \right) = 1;$$

$$V \left( \omega \in \Omega : \lim\sup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} X_i(\omega) = \mathcal{E}[X_1] \right) = 1.$$ 

The second is to prove that the cluster set of empirical averages coincides with the interval between minimum expectation $\mathcal{E}[X_1]$ and maximum expectation $\mathbb{E}[X_1]$. That is, if $C(\{x_n\})$ is the cluster set of $\{x_n\}$, then

$$V \left( \omega \in \Omega : C \left( \left\{ \frac{1}{n} \sum_{i=1}^{n} X_i(\omega) \right\} \right) = [\mathcal{E}[X_1], \mathbb{E}[X_1]] \right) = 1.$$ 

Obviously, if capacity $V$ or $v$ in the above results is a probability measure, all of our main results are natural and fairly neat extension of the classical Kolmogorov’s strong law of large numbers. Moreover, an important feature of our strong laws of large numbers is to provide a frequentist perspective on capacities.

Finally, our results also imply that $[\mathcal{E}[X_1], \mathbb{E}[X_1]]$ is the smallest interval of all intervals in which the limit point of empirical averages lies with probability (capacity $v$) one.

## 2 Notation and Lemmas

In this section, we will recall briefly the notion of IID random variables and sub-linear expectations initiated by Peng.

Let $(\Omega, \mathcal{F})$ be a measurable space, and $\mathcal{H}$ be a set of random variables on $(\Omega, \mathcal{F})$.

**Definition 1** A functional $\mathbb{E}$ on $\mathcal{H} \mapsto (-\infty, +\infty)$ is called a sub-linear expectation, if it satisfies the following properties: for all $X, Y \in \mathcal{H}$,
(a) Monotonicity: $X \geq Y$ implies $\mathbb{E}[X] \geq \mathbb{E}[Y]$.

(b) Constant preserving: $\mathbb{E}[c] = c, \forall c \in \mathbb{R}$.

(c) Sub-additivity: $\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$.

(d) Positive homogeneity: $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X], \forall \lambda \geq 0$.

Given a sub-linear expectation $\mathbb{E}$, let us denote the conjugate expectation $\mathcal{E}$ of sub-linear $\mathbb{E}$ by

$$\mathcal{E}[X] := -\mathbb{E}[-X], \quad \forall X \in \mathcal{H}.$$  

Obviously, for all $X \in \mathcal{H}$, $\mathcal{E}[X] \leq \mathbb{E}[X]$.

Furthermore, let us denote a pair $(\mathcal{V}, v)$ of capacities by

$$\mathcal{V}(A) := \mathbb{E}[I_A], \quad v(A) := \mathcal{E}[I_A], \quad \forall A \in \mathcal{F}.$$  

It is easy to check that

$$\mathcal{V}(A) + v(A^c) = 1, \quad \forall A \in \mathcal{F},$$

where $A^c$ is the complement set of $A$.

**Definition 2** A set function $V: \mathcal{F} \to [0,1]$ is called a continuous capacity if it satisfies

1. $V(\emptyset) = 0, V(\Omega) = 1$.
2. $V(A) \leq V(B)$, whenever $A \subset B$ and $A, B \in \mathcal{F}$.
3. $V(A_n) \uparrow V(A)$, if $A_n \uparrow A$.
4. $V(A_n) \downarrow V(A)$, if $A_n \downarrow A$, where $A_n, A \in \mathcal{F}$.

**Assumption 1** Throughout this paper, we assume that $(\mathcal{V}, v)$ is a pair of continuous capacities generated by sub-linear expectation $\mathbb{E}$ and its conjugate expectation $\mathcal{E}$.

The following example shows that the maximum expectation generated by multiple prior probability measures is a sub-linear expectation and the associated capacities are continuous (see for example, Huber and Strassen [17]).

**Example 1** Let $\mathcal{P}$ be a weakly compact set of probability measures defined on $(\Omega, \mathcal{F})$. For any random variable $\xi$, we denote upper expectation by

$$\mathbb{E}[\xi] = \sup_{Q \in \mathcal{P}} E_Q[\xi].$$

Then $\mathbb{E}[\cdot]$ is a sub-linear expectation. Moreover, the capacities $V$ and $v$ given by

$$\mathcal{V}(A) = \sup_{Q \in \mathcal{P}} Q(A), \quad v(A) = \inf_{Q \in \mathcal{P}} Q(A), \quad \forall A \in \mathcal{F}$$

are continuous.
The following is the notion of IID random variables under sub-linear expectations introduced by Peng.

**Definition 3 (Peng [11])**

**Independence:** Suppose that \( Y_1, Y_2, \ldots, Y_n \) is a sequence of random variables such that \( Y_i \in \mathcal{H} \). Random variable \( Y_n \) is said to be independent of \( X := (Y_1, \ldots, Y_{n-1}) \) under \( \mathbb{E} \), if for each measurable function \( \varphi \) on \( \mathbb{R}^n \) with \( \varphi(X, Y_n) \in \mathcal{H} \) and \( \varphi(x, Y_n) \in \mathcal{H} \) for each \( x \in \mathbb{R}^{n-1} \), we have

\[
\mathbb{E}[\varphi(X, Y_n)] = \mathbb{E}[\varphi(X)],
\]

where \( \varphi(x) := \mathbb{E}[\varphi(x, Y_n)] \) and \( \varphi(X) \in \mathcal{H} \).

**Identical distribution:** Random variables \( X \) and \( Y \) are said to be identically distributed, denoted by \( X \overset{d}{=} Y \), if for each \( \varphi \) such that \( \varphi(X), \varphi(Y) \in \mathcal{H} \),

\[
\mathbb{E}[\varphi(X)] = \mathbb{E}[\varphi(Y)].
\]

**IID random variables:** A sequence of random variables \( \{X_i\}_{i=1}^\infty \) is said to be IID, if \( X_i \overset{d}{=} X_1 \) and \( X_{i+1} \) is independent of \( Y := (X_1, \ldots, X_i) \) for each \( i \geq 1 \).

The following lemma shows the relation between Peng’s independence and pairwise independence in Marinacci [6,7].

**Lemma 1** Suppose that \( X, Y \in \mathcal{H} \) are two random variables. \( \mathbb{E} \) is a sub-linear expectation and \( (\mathbb{V}, v) \) is the pair of capacities generated by \( \mathbb{E} \). If random variable \( X \) is independent of \( Y \) under \( \mathbb{E} \), then \( X \) is also pairwise independent of \( Y \) under capacities \( \mathbb{V} \) and \( v \), i.e. for all subsets \( D \) and \( G \subset \mathbb{R} \),

\[
\mathbb{V}(X \in D, Y \in G) = \mathbb{V}(X \in D)\mathbb{V}(Y \in G)
\]

holds for both capacity \( \mathbb{V} \) and \( v \).

**Proof.** If we choose \( \varphi(x, y) = I_D(x)I_G(y) \) for \( \mathbb{E} \), by the definition of Peng’s independence, it is easy to obtain

\[
\mathbb{V}(X \in D, Y \in G) = \mathbb{V}(X \in D)\mathbb{V}(Y \in G).
\]

Similarly, if we choose \( \varphi(x, y) = -I_D(x)I_G(y) \) for \( \mathbb{E} \), it is easy to obtain

\[
v(X \in D, Y \in G) = v(X \in D)v(Y \in G).
\]

The proof is complete.

Borel-Cantelli Lemma is still true for capacity under some assumptions.
**Lemma 2** Let \( \{A_n, n \geq 1\} \) be a sequence of events in \( \mathcal{F} \) and \((V, v)\) be a pair of capacities generated by sub-linear expectation \( \mathbb{E} \).

1. If \( \sum_{n=1}^{\infty} V(A_n) < \infty \), then \( V\left( \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i \right) = 0 \).

2. Suppose that \( \{A_n, n \geq 1\} \) are pairwise independent with respect to \( v \), i.e.
   \[
   v\left( \bigcap_{i=1}^{\infty} A_{i}^c \right) = \prod_{i=1}^{\infty} v(A_{i}^c).
   \]
   If \( \sum_{n=1}^{\infty} V(A_n) = \infty \), then
   \[
   V\left( \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i \right) = 1.
   \]

**Proof.**

\[
0 \leq V\left( \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i \right) \\
\leq V\left( \bigcup_{i=n}^{\infty} A_i \right) \\
\leq \sum_{i=n}^{\infty} V(A_i) \to 0, \text{ as } n \to \infty.
\]

The proof of (1) is complete.

\[
0 \leq 1 - V\left( \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i \right) \\
= 1 - \lim_{n \to \infty} V\left( \bigcup_{i=n}^{\infty} A_i \right) \\
= \lim_{n \to \infty} [1 - V\left( \bigcup_{i=n}^{\infty} A_i \right)] \\
= \lim_{n \to \infty} v\left( \bigcap_{i=n}^{\infty} A_{i}^c \right) \\
= \lim_{n \to \infty} \prod_{i=n}^{\infty} v\left( A_{i}^c \right) \\
= \lim_{n \to \infty} \prod_{i=n}^{\infty} (1 - V(A_i)) \\
\leq \lim_{n \to \infty} \prod_{i=n}^{\infty} \exp(-V(A_i)) \\
= \lim_{n \to \infty} \exp\left( - \sum_{i=n}^{\infty} V(A_i) \right) = 0.
\]

We complete the proof of (2).

With the notion of IID under sub-linear expectation, Peng shows a weak law of large numbers for sub-linear expectation (see Theorem 5.1 in Peng [14]).
**Lemma 3** (Peng [14]) Let \( \{X_i\}_{i=1}^\infty \) be a sequence of IID random variables with finite means \( \overline{\mu} := \mathbb{E}[X_1], \mu := \mathbb{E}[X_1]. \) Suppose \( \mathbb{E}[|X_1|^{1+\alpha}] < \infty \) for some \( \alpha > 0. \) Then for any bounded, continuous and linear growth function \( \phi, \)

\[
\mathbb{E} \left[ \phi \left( \frac{1}{n} \sum_{i=1}^n X_i \right) \right] \to \sup_{\mu \leq x \leq \overline{\mu}} \phi(x), \quad \text{as } n \to \infty.
\]

**3 The Main Results**

We now prove our main results, let us first prove the following Lemma.

**Lemma 4** Given sub-linear expectation \( \mathbb{E}, \) let \( \{X_i\}_{i=1}^\infty \) be a sequence of independent random variables such that \( \sup_{i \geq 1} \mathbb{E}[|X_i|^{1+\alpha}] < \infty \) for some constant \( \alpha > 0. \) Suppose that there exists a constant \( c > 0 \) such that

\[
|X_n - \mathbb{E}[X_n]| \leq c \frac{n}{\log(1 + n)}, \quad n = 1, 2, \ldots.
\]

Then there exists a sufficiently large number \( m > 1 \) such that

\[
\sup_{n \geq 1} \mathbb{E} \left[ \exp \left( \frac{m \log(1 + n) S_n}{n} \right) \right] < \infty,
\]

where \( S_n := \sum_{i=1}^n [X_i - \mathbb{E}[X_i]]. \)

**Proof.** It is easy to prove that \( \sup_{i \geq 1} \mathbb{E}[|X_i|^{1+\alpha}] < \infty \) implies \( \sup_{i \geq 1} \mathbb{E}[|X_i|] < \infty \) and \( \sup_{i \geq 1} \mathbb{E}[|X_i - \mathbb{E}[X_i]|^{1+\alpha}] < \infty. \)

For given \( \alpha > 0, \) noting that \( \frac{n^\alpha}{\log(1 + n)^{1+\alpha}} \to \infty \) as \( n \to \infty, \) thus, there exist a constant \( n_0 > 0 \) and a sufficiently large number \( m > 1 \) such that

\[
\frac{n^\alpha}{\log(1 + n)^{1+\alpha}} \geq m^{1+\alpha}, \quad n \geq n_0.
\]

So

\[
\left( \frac{m \log(1 + n)}{n} \right)^{1+\alpha} \leq \frac{1}{n}, \quad n \geq n_0. \tag{1}
\]

On the other hand, it can be directly verified that the following inequality holds:

\[
e^x \leq 1 + x + |x|^{1+\alpha} e^{2|x|}, \quad x \in \mathbb{R}, \ 1 \geq \alpha > 0. \tag{2}
\]

Set \( x = \frac{m \log(1 + n)}{n} [X_k - \mathbb{E}[X_k]] \) and put it to (2).

It suffices to obtain that, for each \( n \geq k, \) \( k = 1, 2, \ldots, \)
\[
\exp \left( \frac{m \log(1+n)}{n} [X_k - \mathbb{E}[X_k]] \right) \leq 1 + \frac{m \log(1+n)}{n} [X_k - \mathbb{E}[X_k]] + \left( \frac{m \log(1+n)}{n} \right)^{1+\alpha} |X_k - \mathbb{E}[X_k]|^{1+\alpha} e^{\left( \frac{2m \log(1+n)}{n} |X_k - \mathbb{E}[X_k]| \right)}.
\]

By assumption, it is easy to obtain
\[
\frac{m \log(1+n)}{n} |X_k - \mathbb{E}[X_k]| \leq cm.
\]

This with (1), we have
\[
\exp \left( \frac{m \log(1+n)}{n} [X_k - \mathbb{E}[X_k]] \right) \leq 1 + \frac{m \log(1+n)}{n} [X_k - \mathbb{E}[X_k]] + \frac{|X_k - \mathbb{E}[X_k]|^{1+\alpha}}{n} e^{2cm}.
\]

Taking expectation on both sides of the above equality and setting
\[
L := \sup_{k \geq 1} \mathbb{E}[|X_k - \mathbb{E}[X_k]|^{1+\alpha}],
\]
we have
\[
\mathbb{E} \left[ \exp \left( \frac{m \log(1+n)}{n} [X_k - \mathbb{E}[X_k]] \right) \right] \leq 1 + \frac{L}{n} e^{2cm}.
\]

By the independence of \( \{X_i\}_{i=1}^n \),
\[
\mathbb{E} \left[ \exp \left( \frac{m \log(1+n)}{n} S_n \right) \right] = \prod_{k=1}^n \mathbb{E} \left[ \exp \left( \frac{m \log(1+n)}{n} [X_k - \mathbb{E}[X_k]] \right) \right] \leq (1 + \frac{L}{n} e^{2cm})^n \to e^{Le^{2cm}} < \infty, \text{ as } n \to \infty.
\]

The proof is complete.

**Theorem 1** Let \( \{X_i\}_{i=1}^\infty \) be a sequence of IID random variables for sublinear expectation \( \mathbb{E} \). Suppose \( \mathbb{E}[|X_1|^{1+\alpha}] < \infty \) for some \( \alpha \in (0, 1] \). Set \( \overline{\mu} := \mathbb{E}[X_1], \mu = \mathbb{E}[X_1] \) and \( S_n := \sum_{i=1}^n X_i \). Then

(1)
\[
\mathbb{V} \left( \left\{ \liminf_{n \to \infty} S_n/n < \mu \right\} \cup \left\{ \limsup_{n \to \infty} S_n/n > \overline{\mu} \right\} \right) = 0. \tag{3}
\]

Also
\[
v \left( \mu \leq \liminf_{n \to \infty} S_n/n \leq \limsup_{n \to \infty} S_n/n \leq \overline{\mu} \right) = 1. \tag{4}
\]
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(II) \[ \mathbb{V} \left( \limsup_{n \to \infty} S_n/n = \mu \right) = 1, \quad \mathbb{V} \left( \liminf_{n \to \infty} S_n/n = \mu \right) = 1. \]

(III) Suppose that \( C(\{x_n\}) \) is the cluster set of a sequence of \( \{x_n\} \) in \( \mathbb{R} \), then \[ \mathbb{V} \left( C(\{S_n/n\}) = [\mu, \mu] \right) = 1. \]

**Proof.** First, it is easy to show that argument (3) is equivalent to the conjunction of

\[ \mathbb{V} \left( \limsup_{n \to \infty} S_n/n > \mu \right) = 0, \quad (5) \]

\[ \mathbb{V} \left( \liminf_{n \to \infty} S_n/n < \mu \right) = 0. \quad (6) \]

In fact, writing \( A := \{ \limsup_{n \to \infty} S_n/n > \mu \} \), \( B := \{ \liminf_{n \to \infty} S_n/n < \mu \} \), the equivalence can be proved from the inequality

\[ \max \{ \mathbb{V}(A), \mathbb{V}(B) \} \leq \mathbb{V}(A \cup B) \leq \mathbb{V}(A) + \mathbb{V}(B). \]

We only need to prove (5), since the proof of (6) is similar for the reason of symmetry equation.

We shall finish the proof of (5) by two steps.

**Step 1** Assume that there exists a constant \( c > 0 \) such that \( |X_n - \mu| \leq \frac{cn}{\log(1+n)} \) for \( n \geq 1 \). Thus, \( \{X_i\}_{i=1}^{\infty} \) satisfies the assumptions of Lemma 4.

To prove (5), we shall show that for any \( \epsilon > 0 \),

\[ \mathbb{V} \left( \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \{ S_k/k \geq \mu + \epsilon \} \right) = 0. \quad (7) \]

In fact, by Lemma 4, for \( \epsilon > 0 \), let us choose \( m > 1/\epsilon \) such that

\[ \sup_{n \geq 1} \mathbb{E} \left[ \exp \left( \frac{m \log(1+n)}{n} \sum_{k=1}^{n} (X_k - \mathbb{E}[X_k]) \right) \right] < \infty. \]

By Chebyshev’s inequality,

\[ \mathbb{V}(S_n/n \geq \mu + \epsilon) = \mathbb{V}(\frac{S_n - \mu n}{n} \geq \epsilon) \]

\[ = \mathbb{V} \left( \sum_{k=1}^{n} (X_k - \mu) \geq \epsilon m \log(1+n) \right) \]

\[ \leq e^{-em \log(1+n)} \mathbb{E} \left[ \exp \left( \frac{m \log(1+n)}{n} \sum_{k=1}^{n} (X_k - \mu) \right) \right] \]

\[ \leq \frac{1}{(1+n)^m} \sup_{n \geq 1} \mathbb{E} \left[ \exp \left( \frac{m \log(1+n)}{n} \sum_{k=1}^{n} (X_k - \mu) \right) \right]. \]
Since $\epsilon m > 1$, $\sup_{n \geq 1} \mathbb{E} \left[ \exp \left( \frac{\epsilon \log(1+n)}{n} \sum_{k=1}^{n} (X_k - \mu) \right) \right] < \infty$. Following from the convergence of $\sum_{n=1}^{\infty} \frac{1}{(1+n)^m}$, we have

$$\sum_{n=1}^{\infty} \mathbb{P}(S_n/n \geq \mu + \epsilon) < \infty.$$  

Using the first Borel-Cantelli Lemma, we have

$$\mathbb{P} \left( \limsup_{n \to \infty} S_n/n \geq \mu + \epsilon \right) = 0 \quad \forall \epsilon > 0,$$

which implies

$$\mathbb{P} \left( \limsup_{n \to \infty} S_n/n > \mu \right) = 0,$$

and

$$\mathbb{P} \left( \limsup_{n \to \infty} S_n/n \leq \mu \right) = 1.$$  

Similarly, considering the sequence $\{-X_i\}_{i=1}^{\infty}$, by step 1, it suffices to obtain

$$\mathbb{P} \left( \limsup_{n \to \infty} (-S_n)/n \geq \mathbb{E}[-X_1] \right) = 0.$$

Hence,

$$\mathbb{P} \left( \liminf_{n \to \infty} S_n/n < -\mathbb{E}[-X_1] \right) = 0.$$

But $\mu = -\mathbb{E}[-X_1]$, hence

$$\mathbb{P} \left( \liminf_{n \to \infty} S_n/n < \mu \right) = 0,$$

and

$$\mathbb{P} \left( \liminf_{n \to \infty} S_n/n \geq \mu \right) = 1.$$  

**Step 2.** Write

$$X_n := (X_n - \mu) I_{\{|X_n - \mu| \leq \frac{c n}{\log(1+n)}\}} - \mathbb{E} \left[ (X_n - \mu) I_{\{|X_n - \mu| \leq \frac{c n}{\log(1+n)}\}} \right] + \mu.$$

Immediately, $\mathbb{E}[X_n] = \mu$. Moreover, it is easy to check that $\{X_i\}_{i=1}^{\infty}$ satisfies the assumptions in Lemma 4.

Indeed, obviously for each $n \geq 1$,

$$|X_n - \mu| \leq \frac{2c n}{\log(1+n)}.$$

On the other hand, for each $n \geq 1$, it easy to check that

$$|X_n - \mu| \leq |X_n - \mu| + \mathbb{E}[|X_n - \mu|].$$
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Then
\[ \mathbb{E}[|X_n - \mu|^{1+\alpha}] \leq 2^{1+\alpha} \left( \mathbb{E}[|X_n - \mu|^{1+\alpha}] + \left( \mathbb{E}[|X_n - \mu|]\right)^{1+\alpha} \right) < \infty. \]

Setting \( S_n := \sum_{i=1}^{n} X_i \), immediately,
\[
\frac{1}{n} S_n \leq \frac{1}{n} S_n + \frac{1}{n} \sum_{i=1}^{n} |X_i - \mu| I_{\{|X_i - \mu| > \frac{c_i}{\log(1+i)}\}} + \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[|X_i - \mu| I_{\{|X_i - \mu| > \frac{c_i}{\log(1+i)}\}}]. \tag{8}
\]

Since \( \{X_i - \mu\}_{i=1}^{\infty} \) have common distributions, we have
\[
\sum_{i=1}^{\infty} \mathbb{E}[|X_i - \mu| I_{\{|X_i - \mu| > \frac{c_i}{\log(1+i)}\}}] \leq \sum_{i=1}^{\infty} \frac{\log(1+i)^{\alpha}}{c^{\alpha} i^{1+\alpha}} \mathbb{E}[|X_i - \mu|^{1+\alpha}] = \mathbb{E}[|X_1 - \mu|^{1+\alpha}] \left( \frac{i}{c} \right)^{\alpha} \sum_{i=1}^{\infty} \frac{\log(1+i)^{\alpha}}{i^{2+\alpha}} < \infty.
\]

By Kronecker Lemma,
\[
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[|X_i - \mu| I_{\{|X_i - \mu| > \frac{c_i}{\log(1+i)}\}}] \to 0. \tag{9}
\]

Furthermore, write \( A_i := \{|X_i - \mu| > \frac{c_i}{\log(1+i)}\} \) for \( i \geq 1 \). It suffices now to prove that
\[
\mathbb{V} \left( \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i \right) = 0.
\]

In fact, by Chebyshev’s inequality,
\[
\mathbb{V} \left( |X_i - \mu| > \frac{c_i}{\log(1+i)} \right) \leq \left( \frac{\log(1+i)}{c_i} \right)^{1+\alpha} \mathbb{E}[|X_i - \mu|^{1+\alpha}]
\]

Hence,
\[
\sum_{i=1}^{\infty} \mathbb{V} \left( |X_i - \mu| > \frac{c_i}{\log(1+i)} \right) < \infty
\]

and by the first Borel-Cantelli Lemma, we have \( \mathbb{V} \left( \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i \right) = 0. \)

This implies that \( \omega \notin \bigcup_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i \), the sequence
\[
\sum_{i=1}^{n} \frac{|X_i - \mu| I_{\{|X_i - \mu| > \frac{c_i}{\log(1+i)}\}}}{i}
\]
converges almost surely with respect to \( \mathbb{V} \) as \( n \to \infty \).
Applying Kronecker Lemma again,
\[ \frac{1}{n} \sum_{i=1}^{n} \left( |X_i - \mu| I_{\{|X_i - \mu| > c_i \log(1+i)\}} \right) \to 0, \text{ a.s.} \quad \forall. \]  
(10)

Setting \( \limsup \) on both side of (8), then by (9) and (10), we have
\[ \limsup_{n \to \infty} S_n/n \leq \limsup_{n \to \infty} S_{n}/n, \text{ a.s.} \quad \forall. \]

Since \( \{X_n\}_{n=1}^{\infty} \) satisfies the assumption of Step 1, by Step 1,
\[ V(\limsup_{n \to \infty} S_{n}/n > \mu) = 0, \]
and also
\[ v(\limsup_{n \to \infty} S_{n}/n \leq \mu) = 1. \]

Therefore, the proof of (I) is complete.

To prove (II), if \( \mu = \bar{\mu}, \) it is trivial. Suppose \( \mu > \bar{\mu}, \) we only need to prove that there exists an increasing subsequence \( \{n_k\} \) of \( \{n\} \) such that for any \( 0 < \epsilon < \mu - \bar{\mu}, \)
\[ V\left( \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} \{S_{n_k}/n_k \geq \mu - \epsilon\} \right) = 1. \]  
(11)

Then we have
\[ V\left( \limsup_{k \to \infty} S_{n_k}/n_k \geq \mu \right) = 1. \]

This with (I) suffices to yield the desired result (II).

Indeed, choosing \( n_k = k^k \) for \( k \geq 1 \) and setting \( S_n := \sum_{i=1}^{n} (X_i - \bar{\mu}), \) we have
\[ V\left( \frac{S_{n_k} - S_{n_k-1}}{n_k - n_k-1} \geq \mu - \epsilon \right) = V\left( \frac{S_{n_k} - S_{n_k-1}}{n_k - n_k-1} \geq \mu - \epsilon \right) \]
\[ = V\left( \frac{S_{n_k} - (n_k - n_k-1) \bar{\mu}}{n_k - n_k-1} \geq -\epsilon \right) \]
\[ = V\left( \frac{\bar{\mu} - (n_k - n_k-1) \bar{\mu}}{n_k - n_k-1} \geq -\epsilon \right) \]
\[ \geq \mathbb{E}[\phi(\frac{\bar{\mu} - (n_k - n_k-1) \bar{\mu}}{n_k - n_k-1})] \]

where \( \phi(x) \) is defined by
\[ \phi(x) = \begin{cases} 
1 - e^{-(x+\epsilon)}, & x \geq -\epsilon; \\
0, & x < -\epsilon.
\end{cases} \]

Considering the sequence of IID random variables \( \{X_i - \mu\}_{i=1}^{\infty} \). Obviously
\[ \mathbb{E}[X_i - \mu] = 0, \quad \mathbb{E}[X_i - \mu] = -(\mu - \bar{\mu}). \]
Applying Lemma 3, we have, \( n_k - n_{k-1} \rightarrow \infty \) as \( k \rightarrow \infty \) and

\[
\mathbb{E}[\phi\left(\frac{S_{n_k} - n_{k-1}}{n_k - n_{k-1}}\right)] \rightarrow \sup_{-(\mu - \bar{\mu}) \leq y \leq 0} \phi(y) = \phi(0) = 1 - e^{-\epsilon} > 0.
\]

Thus

\[
\sum_{k=1}^{\infty} \mathbb{V}\left(\frac{S_{n_k} - S_{n_{k-1}}}{n_k - n_{k-1}} \geq \bar{\mu} - \epsilon\right) \geq \sum_{k=1}^{\infty} \mathbb{E}\left[\phi\left(\frac{S_{n_k} - n_{k-1}}{n_k - n_{k-1}}\right)\right] = \infty.
\]

Note the fact that \( \{S_{n_k} - S_{n_{k-1}}\}_{k \geq 1} \) is a sequence of independent random variables for \( k \geq 1 \).

Using the second Borel-Cantelli Lemma, we have

\[
\lim \sup_{k \rightarrow \infty} \frac{S_{n_k} - S_{n_{k-1}}}{n_k - n_{k-1}} \geq \bar{\mu} - \epsilon, \quad \text{a.s.} \ \mathbb{V}.
\]

But

\[
\frac{S_{n_k}}{n_k} \geq \frac{S_{n_k} - S_{n_{k-1}}}{n_k - n_{k-1}} \cdot \frac{n_k - n_{k-1}}{n_k} - \frac{|S_{n_{k-1}}|}{n_k} \cdot \frac{n_k}{n_{k-1}}.
\]

Note the fact that

\[
\frac{n_k - n_{k-1}}{n_k} \rightarrow 1, \quad \frac{n_k}{n_{k-1}} \rightarrow 0, \quad \text{as} \quad k \rightarrow \infty
\]

and from

\[
\lim_{n \rightarrow \infty} S_n / n \leq \bar{\mu}, \quad \lim_{n \rightarrow \infty} (-S_n) / n \leq -\mu,
\]

we have

\[
\lim_{n \rightarrow \infty} |S_n| / n \leq \max\{|\bar{\mu}|, |\mu|\}, \quad \text{a.s.} \ \mathbb{V}.
\]

Hence,

\[
\lim \sup_{k \rightarrow \infty} \frac{S_{n_k}}{n_k} \geq \lim \sup_{k \rightarrow \infty} \frac{S_{n_k} - S_{n_{k-1}}}{n_k - n_{k-1}} \lim_{k \rightarrow \infty} \frac{n_k - n_{k-1}}{n_k} - \lim \sup_{k \rightarrow \infty} \frac{|S_{n_{k-1}}|}{n_k} \lim_{k \rightarrow \infty} \frac{n_k}{n_{k-1}}.
\]

We conclude that

\[
\lim \sup_{k \rightarrow \infty} \frac{S_{n_k}}{n_k} \geq \bar{\mu} - \epsilon, \quad \text{a.s.} \ \mathbb{V}.
\]

Since \( \epsilon \) is arbitrary, we have

\[
\mathbb{V}\left(\lim \sup_{k \rightarrow \infty} S_{n_k} / n_k \geq \bar{\mu}\right) = 1.
\]
By (I), we know \[ V \left( \limsup_{n \to \infty} \frac{S_n}{n} > \mu \right) = 0, \] thus
\[ V \left( \limsup_{n \to \infty} \frac{S_n}{n} = \mu \right) = V \left( \limsup_{n \to \infty} \frac{S_n}{n} \geq \mu \right) \geq V \left( \limsup_{n \to \infty} \frac{S_n}{n} = \mu \right) = 1. \]

Considering the sequence of \( \{-X_n\}_{n=1}^\infty \), we have
\[ V \left( \limsup_{n \to \infty} \frac{-S_n}{n} = \mathbb{E}[-X_1] \right) = 1. \]
Therefore,
\[ V \left( \liminf_{n \to \infty} \frac{S_n}{n} = -\mathbb{E}[-X_1] \right) = 1. \]
But \( \mu = -\mathbb{E}[-X_1] \), thus
\[ V \left( \liminf_{n \to \infty} \frac{S_n}{n} = \mu \right) = 1. \]

The proof of (II) is complete.

To prove (III), obviously by (I), we have
\[ V \left( C(\{S_n/n\}) \subset [\mu, \mu] \right) = 1. \]

We still need to prove
\[ V \left( C(\{S_n/n\}) \supset [\mu, \mu] \right) = 1. \]
If \( \mu = \mu \), it is trivial. Suppose \( \mu > \mu \), we only need to prove that, for any rational number \( b \in (\mu, \mu) \)
\[ V \left( \liminf_{n \to \infty} |S_n/n - b| = 0 \right) = 1. \]
To do so, we only need to prove that there exists an increasing subsequence \( \{n_k\} \) of \( \{n\} \) such that for any rational number \( b \in (\mu, \mu) \)
\[ V \left( \bigcap_{m=1}^\infty \bigcup_{k=m}^\infty \{|S_{n_k}/n_k - b| \leq \epsilon \} \right) = 1. \]

Indeed, for any \( 0 < \epsilon \leq \min\{\mu - b, b - \mu\} \), let us choose \( n_k = k^k \) for \( k \geq 1 \).

Setting \( \overline{S}_n := \sum_{i=1}^n (X_i - b) \), then
\[
V \left( \left| \frac{S_{n_k} - S_{n_k - 1}}{n_k - n_{k-1}} - b \right| \leq \epsilon \right) = V \left( \left| \frac{S_{n_k - n_{k-1}}}{n_k - n_{k-1}} - b \right| \leq \epsilon \right)
= V \left( \left| \frac{S_{n_k - n_{k-1}} - (n_k - n_{k-1})b}{n_k - n_{k-1}} \right| \leq \epsilon \right)
= V \left( \left| \frac{\overline{S}_{n_k - n_{k-1}}}{n_k - n_{k-1}} \right| \leq \epsilon \right)
\geq \mathbb{E}[\phi(\overline{S}_{n_k - n_{k-1}})].
\]
where $\phi(x)$ is defined by

$$
\phi(x) = \begin{cases} 
1 - e^{|x| - \epsilon}, & |x| \leq \epsilon; \\
0, & |x| > \epsilon.
\end{cases}
$$

Considering the sequence of IID random variables $\{X_i - b\}_{i=1}^{\infty}$, obviously

$$
E[X_i - b] = \mu - b > 0, \quad E[X_i - b] = \mu - b < 0.
$$

Applying Lemma 3, we have, as $k \to \infty$,

$$
E[\phi\left(\frac{S_{n_k} - S_{n_k-1}}{n_k - n_{k-1}} - b\right)] \to \sup_{\mu - b \leq y \leq \mu - b} \phi(y) = \phi(0) = 1 - e^{-\epsilon} > 0.
$$

Thus

$$
\sum_{k=1}^{\infty} \mathbb{P}\left(|\frac{S_{n_k} - S_{n_k-1}}{n_k - n_{k-1}} - b| \leq \epsilon\right) \geq \sum_{k=1}^{\infty} E[\phi\left(\frac{S_{n_k} - S_{n_k-1}}{n_k - n_{k-1}}\right)] = \infty.
$$

Note the fact that the sequence of $\{S_{n_k} - S_{n_k-1}\}_{k \geq 1}$ is independent for all $k \geq 1$. Using the second Borel-Cantelli Lemma, we have

$$
\liminf_{k \to \infty} \left|\frac{S_{n_k} - S_{n_k-1}}{n_k - n_{k-1}} - b\right| \leq \epsilon, \quad \text{a.s. } \mathbb{P}.
$$

But

$$
\left|\frac{S_{n_k} - b}{n_k}\right| \leq \left|\frac{S_{n_k} - S_{n_k-1}}{n_k - n_{k-1}} - b\right| \cdot \frac{n_k - n_{k-1}}{n_k} + \left[\frac{|S_{n_k-1}|}{n_{k-1}} + |b|\right] \frac{n_{k-1}}{n_k}, \quad (13)
$$

Noting the following fact,

$$
\frac{n_k - n_{k-1}}{n_k} \to 1, \quad \frac{n_{k-1}}{n_k} \to 0, \text{ as } k \to \infty
$$

and

$$
\limsup_{n \to \infty} \frac{S_n}{n} \leq \overline{\mu}, \quad \limsup_{n \to \infty} \frac{(-S_n)}{n} \leq -\mu,
$$

which imply

$$
\limsup_{n \to \infty} \frac{|S_n|}{n} \leq \max\{|\overline{\mu}|, |\mu|\} < \infty.
$$

Hence, from inequality (13), for any $\epsilon > 0$,

$$
\liminf_{k \to \infty} \left|\frac{S_{n_k} - b}{n_k}\right| \leq \epsilon, \quad \text{a.s. } \mathbb{P}.
$$

That is

$$
\mathbb{P}\left(\limsup_{n \to \infty} \left|\frac{S_n}{n} - b\right| \leq \epsilon\right) = 1.
$$

Since $\epsilon$ is arbitrary, we have

$$
\mathbb{P}\left(\liminf_{n \to \infty} \left|\frac{S_n}{n} - b\right| = 0\right) = 1.
$$

The proof of (III) is complete.
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