Criterion of superfluidity, elementary excitations, and heat capacity singularity in superfluid helium

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We proceed from the premise that the spectrum of elementary excitations in the normal component in the Landau theory of superfluidity should depend on the superfluid helium temperature. This leads to generalization of the Landau superfluidity criterion. On this basis, taking into account available experimental data on inelastic neutron scattering, it is shown that, in addition to phonon–roton excitations, there is another type of elementary excitation in superfluid helium, which we called helons. The energy spectrum with such momentum dependence was first proposed by Landau. The helon energy spectrum shape and its temperature dependence make it possible to explain the singular behavior of the heat capacity of superfluid helium near its phase transition to the normal state.

Subject Index A62, A63, I20

1. Introduction

According to the seminal phenomenological Landau theory [1,2], superfluid helium is a liquid consisting of superfluid and normal components (see also Ref. [3]). The superfluid component moves without friction and is not involved in energy transport in the form of heat. The normal component moves with friction and is involved in heat transport. In this case, according to the Landau theory [1], the normal component is a gas of elementary excitations, which are characterized by the dependence of the energy spectrum \( \epsilon(p) \) on the momentum \( p \). If the flow velocity \( V \) of the superfluid component reaches the critical velocity \( V_{cr} \), determined from the condition

\[
V_{cr} = \min \left( \frac{\epsilon(p)}{p} \right).
\]

superfluidity breakdown occurs. Thus, superfluidity cannot be observed in homogeneous helium at velocities \( V > V_{cr} \). This statement [1], known as the Landau superfluidity criterion, is in qualitative agreement with experimental data on superfluid helium motion in capillaries, although the criterion is obtained for homogeneous systems (see Ref. [4] for more details). To describe quantitatively the motion of superfluid helium in capillaries, the influence of inhomogeneity caused by boundary effects (see, e.g., Refs. [5,6]) should be taken into account. In this paper, we consider the infinite medium model corresponding to the thermodynamic limit for homogeneous helium.

Thus, the Landau superfluidity criterion is in fact the superfluidity breakdown criterion, since it initially assumes the existence of superfluidity. Otherwise, considering that there are well defined
acoustic elementary excitations (phonons) in any liquid, we would obtain a nonzero critical velocity equal to the speed of sound in a corresponding liquid. In the Landau theory, the superfluidity of helium at temperatures $T < T_\lambda$ is conditioned by the existence of the superfluid component with the density $n_s$ that, as is known, essentially depends on temperature (see, e.g., Ref. [4]). On the other hand, the temperature dependence of the critical velocity in the Landau theory is not formulated (L. P. Pitaevski, private communication).

Phonon–roton excitations, predicted by Landau and experimentally measured by neutron scattering, result in the critical velocity exceeding the observed values of superfluid flow breakdown by an order of magnitude. Moreover, we demonstrate that phonon–roton excitations are also characteristic of normal liquids, hence, cannot be directly related to the superfluidity phenomenon. The vortex excitations are connected to the existence of boundaries and cannot play a role in the infinite homogeneous medium. Therefore, the problem of determining the excitations responsible for the superfluidity breakdown arises. We show the necessity of the existence of new elementary excitations in the normal component of superfluid helium, with the spectrum essentially dependent on temperature. These excitations, which we call “helons”, disappear at the transition temperature $T_\lambda$ to the normal state and require generalization of the Landau superfluidity condition. The helon spectrum is similar to the spectrum proposed by Landau in Ref. [1] and, due to the temperature dependence of excitations, gives an explanation for the singular behavior of the heat capacity at $T_\lambda$.

2. Temperature dependence of excitations and generalization of the Landau superfluidity criterion

To clarify the problem, it should be taken into account that the elementary excitation spectrum $\epsilon$ in the normal component is a function of not only the momentum $p$, but also thermodynamic parameters of the system under consideration, e.g., temperature $T$, i.e.,

$$\epsilon = \epsilon(p; T).$$

We note that the temperature dependence of the elementary excitation spectrum was probably first considered in Ref. [7] in calculating the thermodynamic functions of superfluid helium. It is necessary to stress that we suppose the applicability of the quasiparticle concept to the whole temperature range below the critical one. There is also a serious argument for the applicability of this concept to normal liquid below the Debye temperature as well [8].

Hence, the critical velocity $V_{cr}$ determined from relation (1) is also a function of the thermodynamic parameters, $V_{cr} = V_{cr}(T)$. In this connection, questions arise on the temperature behavior of the critical velocity below the critical temperature ($T \to (T_\lambda - 0)$). There are two possibilities: the critical velocity remains finite at $T \to (T_\lambda - 0)$ and superfluidity disappears since $n_s[T \to (T_\lambda - 0)] \to 0$, or the critical velocity also tends to zero at $T \to (T_\lambda - 0)$ simultaneously with $n_s$. The known experimental data [9–13] count in favor of the second opportunity. This hypothesis is supported by the recent theoretical models (see Ref. [14] and references therein). Let us further take into account that the superfluidity phenomenon is absent at the temperature $T > T_\lambda$, where $T_\lambda$ is the superfluid transition temperature, i.e., the liquid is normal. Therefore, it should be accepted that

$$V_{cr}(T > T_\lambda) = 0.$$
Thus, we can formulate the generalized Landau superfluidity criterion precisely as the superfluidity
criterion, rather than the superfluidity breakdown criterion, in the following form: if the spectrum of
elementary excitations in the liquid satisfies the conditions

\[ V_{cr}(T) > 0 \quad \text{for} \quad T < T_\lambda; \quad V_{cr}(T) = 0 \quad \text{for} \quad T > T_\lambda, \tag{4} \]

then the corresponding liquid at temperatures \( T < T_\lambda \) is superfluid; superfluidity breakdown occurs
at velocities \( V > V_{cr} \).

As noted above, there are well defined acoustic elementary excitations in any liquid, both normal
and superfluid; therefore, the phonon spectrum of elementary excitations \( \varepsilon(p) = cp \), where \( c \)
is the speed of sound, does not satisfy the generalized Landau superfluidity criterion (4). This means
that another branch of elementary excitations should exist in the homogeneous case in addition to
phonons, which differs essentially from the phonon spectrum. Thus, we should introduce one more
correction to the formulation of the generalized Landau superfluidity criterion, associated with the
fact that several “branches” of elementary excitations can exist in a liquid. Therefore, among all
possible values of the critical velocity \( V_{cr}^\alpha(T) \) determined by each spectrum (spectrum index \( \alpha \)), our
interest is only in providing a minimum value among \( V_{cr}^\alpha(T) \). Hence, the quantity \( V_{cr}(T) \), appearing
in relation (4), is determined from the condition

\[ V_{cr}(T) = \min_\alpha V_{cr}^\alpha(T); \quad V_{cr}^\alpha(T) = \min \left( \varepsilon^\alpha(p, T)/p \right). \tag{5} \]

We note that, according to (4) and (5), two cases are possible:

- either there is an excitation branch with \( V_{cr}^\alpha = 0 \) in normal liquid, which differs essentially
  from the phonon spectrum and, during the transition to the superfluid state, yields \( V_{cr}^\alpha > 0 \),
- or there is another branch of elementary excitations in the normal component of superfluid
  liquid, which differs essentially from the phonon spectrum and disappears at temperatures
  \( T > T_\lambda \).

Before turning to a discussion of the possible shape of the energy spectrum of the additional
branch of elementary excitations differing essentially from phonons, let us consider the situation
with the phonon–roton spectrum of elementary excitations \( \varepsilon^{\text{ph–rot}}(p) \) in superfluid helium, which
was proposed by Landau in Ref. [2]. The shape of the phonon–roton spectrum of elementary exci-
tations was confirmed in experiments on inelastic neutron scattering in superfluid helium (see, e.g.,
Refs. [15,16]).

Furthermore, numerous experiments on inelastic neutron scattering (see, e.g., Refs. [17–19]) show
that the phonon–roton spectrum of elementary excitations very weakly depends on temperature to
\( T_\lambda = 2.17 \text{K} \) for all values of momenta, including phonon and roton spectral regions. Moreover,
phonon–roton excitations also exist at temperatures \( T > T_\lambda \), where liquid helium is in the normal
state [20]. Thereby, according to the above discussion, there is reason to believe that the phonon–
roton spectrum of elementary excitations is not nearly related to the explanation of the superfluidity
phenomenon in liquid helium. This point of view is confirmed by the experimental results on inelastic
neutron scattering in liquid metals (Fig. 1), where the phonon–roton spectrum of elementary exci-
tations was detected (see, e.g., Refs. [21–24]), which was noticed in Ref. [25]. Similar excitations
were also experimentally detected in the two-dimensional Fermi liquid [26].
Fig. 1. The position of the maximum in the dynamic ion structure factor in liquid rubidium at 320 K: full circles are the experimental data of Copley and Rowe [21], vertical bars are the experimental data of Glaser et al. [23], and the solid curve is the numerical calculation of the dispersion equation [25].

3. “Landau rotons” and helons

Let us now note that Landau, in his first paper [1], proposed to consider, in addition to phonons, elementary excitations, which he initially called “rotons” with the spectrum

$$\varepsilon^r = \Delta^r + \frac{p^2}{2\mu^r}. \quad (6)$$

Let us refer to this type of excitation as “Landau rotons”, in contrast to later introduced rotons [2] being a fraction of the single phonon–roton branch of excitations. In (6), $\Delta^r$ is the energy of the Landau roton with the effective mass $\mu^r$ at the zero momentum $p = 0$. It is clear that the critical velocity $V_{cr}^r$ of Landau rotons, according to (1), (5), and (6) is given by

$$V_{cr}^r = \sqrt{\frac{2\Delta^r}{\mu^r}}. \quad (7)$$

Assuming that $\Delta^r$ in (6) depends on temperature and satisfies the condition

$$\Delta^r(T) = 0 \quad \text{for } T > T_\lambda, \quad (8)$$

the spectrum of such excitations satisfies the generalized Landau superfluidity criterion (4).

We now take into account that, in his second paper on superfluidity [2], Landau proposed a different definition of rotons from that introduced in Ref. [1]. The rotons in Ref. [2] are described by the characteristic momentum $p_0$ and defined by the spectrum $\varepsilon^r = \Delta^r + \frac{(p-p_0)^2}{2\mu^r}$. Precisely this spectrum of excitations is now called rotons, in contrast to the spectrum (6), initially proposed by Landau in Ref. [1].

Let us note that, while satisfying condition (8), elementary excitations with energy spectrum (6) exist only in superfluid helium, in contrast to the phonon–roton spectrum characteristic of liquid. To distinguish elementary excitations with the spectrum (6)–(8) from “rotons” (used in the literature)
in the phonon–roton spectrum and also from vortical “Landau rotons” $\Delta'$ is currently dropped from consideration, and, taking into account that the vortical nature of this new type of excitation in the infinite medium is not obvious, in what follows, we refer to these elementary excitations as “helons” (index $h$).

Therefore, we believe that the spectrum of excitations for helons is defined by relations (6)–(8); instead of superscript $r$ we write ($h$) in the quantities $\Delta'$ and $\mu'$. This is not a formal replacement, since the helon’s spectrum is defined, according to (6), by the roton’s mass $\mu^{(h)}$ and by the gap $\Delta^{(h)}$ related to the critical velocity $V_{\text{cr}}$ (7), since the latter tends to zero at $T = T_{\lambda}$, according to the generalized temperature-dependent Landau criterion. The gap $\Delta^{(h)}$, in contrast to the Landau vortical gap $\Delta'$ in Ref. [1], tends to zero when $T \rightarrow T_{\lambda}$. This means that we suppose that, in parallel with the well known quasiparticles, which are characterized by the phonon–roton excitation spectrum, there are also quasiparticles—helons—with the spectrum (6)–(8).

The existence of helons with the spectrum (6)–(8) is in fact confirmed by experiments on inelastic neutron scattering [27,28] (see Fig. 2), in which, in addition to the maxima in the dynamic structure factor of superfluid helium, corresponding to the “single-atom scattering” spectrum, maxima were detected, whose positions appeared close to the spectrum of the free helium atom $\varepsilon^{(a)}(p) = p^2/2m$ (here $m$ is the helium atom mass) for the region of the momentum-transferred values $q > 0.5$ Å$^{-1}$. The corresponding experimental data were called the spectrum of “single-atom scattering”. It is clear that the spectrum of the free helium atom $\varepsilon^{(a)}(p)$ (as well as other spectra with the same $p$-dependence at small $p$) does not satisfy the generalized Landau superfluidity criterion (4), (5). Therefore, new experiments are required for the smaller values of $q = p/\hbar$ and for different temperatures, which can prove the existence of helons and give an estimate $\Delta^{(h)}(T)$.

In attempts to theoretically explain the experimentally observed maxima in the dynamic structure factor of superfluid helium, corresponding to the “single-atom scattering” spectrum, the possible existence of helons was not taken into consideration (see, e.g., Refs. [29–32] and references therein).
We also note that theoretical models were repeatedly proposed in microscopic descriptions of superfluid helium, in which the spectrum of elementary excitations, similar to the spectrum of helons, arises (see, e.g., Refs. [33–36] and references therein); however, the existence of the corresponding maximum in the dynamic structure factor has not yet been confirmed in these models.

To provide condition (8), there is an appropriate quantity in the Landau theory, i.e., the superfluid component density $n_s$ for which the condition

$$n_s(T > T_\lambda) = 0 \quad \text{at } T > T_\lambda$$

is satisfied. Then we can assume that $\Delta^{(h)} \simeq [n_s]^\gamma, \gamma > 0$ at $T < T_\lambda$.

In this case, for dimensionality reasons, to determine the quantity $\Delta^{(h)}(T)$, several quantities with an energy dimension can be constructed, based on the superfluid component density $n_s$, in particular, $\hbar^2 n_s^{2/3} (T)/m$ and $\hbar^2 L n_s (T)/m$, where $L$ is the so-called scattering length, which is completely defined by the interparticle interaction potential of helium atoms.

4. Thermodynamics and the heat capacity divergence

Thus, there is good reason to believe that, in addition to phonon–roton elementary excitations, there are helons with spectrum (6)–(8) in the normal component of superfluid helium in the absence of boundary effects. Let us consider the consequences of this statement. According to the Landau superfluidity theory [1,2], the free energy per unit volume of superfluid helium at temperature $T$ can be written as

$$F = E_0 + \sum_a F^{(a)}, \quad F^{(a)} = T \int \frac{d^3 p}{(2\pi \hbar)^3} \ln \left\{ 1 - \exp \left[ -\frac{\epsilon^{(a)}(p; T)}{T} \right] \right\}.$$ (10)

Here $E_0$ is the ground state energy per unit volume of superfluid helium, which depends only on its density $n$ equal to the sum of the densities of the superfluid and normal components, $n = n_N + n_s$. The quantity $F^{(a)}$ is the free energy per unit volume of superfluid helium, corresponding to elementary excitations of type $a$ with energy spectra $\epsilon^{(a)}(p; T)$ and corresponding to helons (6)–(8) and phonon–roton excitations.

It immediately follows from (10) that the average (internal) energy per unit volume of superfluid helium is given by

$$E = E_0 + \sum_a E^{(a)}, \quad E^{(a)} = T \int \frac{d^3 p}{(2\pi \hbar)^3} \frac{\epsilon^{(a)}(p; T) - T[\partial \epsilon^{(a)}(p; T)/\partial T]}{\exp \left[ \epsilon^{(a)}(p; T)/T \right] - 1}. $$ (11)

In turn, from (11), it is easy to verify that the heat capacity $c_V = (\partial E / \partial T)_V$ of superfluid helium, by virtue of condition (8), has a peculiarity at the temperature $T = T_\lambda$ for the helon energy spectrum, caused by the temperature dependence of $\Delta^{(h)}(T)$:

$$\lim_{T \to (T_\lambda - 0)} c_V = \infty, \quad \text{for } \lim_{T \to (T_\lambda - 0)} \Delta^{(h)}(T) = 0,$$ (12)

which is widely known in the literature as the $\lambda$-curve of the heat capacity.

The simple calculation for the singular part of the heat capacity for $T < T_\lambda$, which takes into account that the value of the respective integral is determined mainly by small values of the momenta...
leads to the following temperature dependence of $c_V$:

$$\lim_{T \to (T_\lambda - 0)} c_V \to \left\{ \frac{(2\mu^{(h)} T)^{3/2}}{8\pi \hbar^3} \left( \frac{d \Delta^{(h)}(T)}{dT} \right)^2 \left( \frac{T}{\Delta^{(h)}(T)} \right)^{1/2} \right\} \right\}_{T \to (T_\lambda - 0)}. \quad (13)$$

To determine the behavior of the temperature dependence of the singularity of the heat capacity $c_V$, taking into account that $\Delta^{(h)}(T) \sim [n_s(T)]^{\gamma}$, $\gamma > 0$ (see above), it is necessary to determine the temperature dependence of the superfluid component density $n_s(T)$.

In the vicinity of the phase transition from the superfluid to the normal state, according to the experimental data (see Ref. [38] and references therein),

$$n_s \sim |\theta|^{\alpha}, \quad \alpha \simeq 0.671, \quad \theta = \frac{(T - T_\lambda)}{T_\lambda}. \quad (14)$$

Therefore, for $T \to (T_\lambda - 0)$ the following relation for $\Delta^{(h)}(T)$ is fulfilled:

$$\Delta^{(h)}(T) \sim |\theta|^{\alpha \gamma}. \quad (15)$$

Substituting (14) and (15) into (13), one finds the behavior of the temperature dependence of the heat capacity $c_V$ at $T \to (T_\lambda - 0)$ in the vicinity of the phase transition:

$$\ln c_V \sim \left( \frac{3\alpha\gamma}{2} - 2 \right) \ln |\theta| \approx (\gamma - 2) \ln |\theta|. \quad (16)$$

Therefore, according to this consideration, the heat capacity temperature dependence $c_V(T)$ has a singularity for the values

$$\gamma < 2. \quad (17)$$

In particular, for $\gamma = 3/2$, the singularity of the heat capacity $c_V$ has the form

$$\lim_{T \to (T_\lambda - 0)} c_V \sim |\theta|^{-1/2}. \quad (18)$$

Such a result was obtained, e.g., in Ref. [39] (see also Ref. [40]), where, to describe superfluid helium, the roton liquid model was proposed (from the phonon–roton spectrum of elementary excitations), which is similar to the Fermi-liquid theory by Landau. Within this model, the singularity of the heat capacity $c_V$ (18) was caused by roton interaction effects. As noted above, the existence of rotons was also experimentally confirmed in the normal state. The temperature dependence of the heat capacity, obtained in Ref. [39], would be symmetric in the vicinity of the phase transition. At the same time, the experimental data [41–43] (see Fig. 3) show the essential asymmetry of the heat capacity in the vicinity (below and above) of the transition temperature. Furthermore, it should also be mentioned that, to describe the thermodynamical properties of helium, it is usually sufficient to take into account the ideal gas of quasiparticles [1,2,32]. Within the proposed consideration, this asymmetry can be explained by the idea that the gap is probably completely absent above the transition temperature (this situation is similar to the transition from the superconductive to the normal state for superconductors). For the case of weak interaction between the helium atoms this statement can be easily proved, e.g., in the Hartree–Fock approximation.

We note that the results obtained above characterize the anomalous heat capacity behavior in the vicinity of the phase transition from superfluid to normal state, rather than on the line of this phase transition.

The above consideration of the thermodynamic functions is based on the ideal gas model of quasiparticles that consist of two types of elementary excitations, i.e., helons and phonon–roton ones.
A natural question arises about the limitations of this model’s applicability. In particular, it is usually assumed that this kind of model is adequate only for low temperatures $T \ll T_\lambda$. However, the model of the ideal phonon gas is successfully applied to the description not only of solids, but also liquids (see, e.g., Ref. [8] and references therein). This applicability, in fact, is connected with the characteristic Debye temperature for phonons. In our opinion, the use of the quasiparticle concept for describing condensed matter in which the interparticle interaction is strong is based not on the consideration of a low temperature, but mainly on a weak damping of quasiparticles (see, e.g., Refs. [25,31,32] and references therein), which takes place at least below the Debye temperature. Precisely this assumption is the cornerstone for interpreting the inelastic scattering experiments in condensed matter. The positions of maxima for the experimentally measured dynamical structure factor $S(p, \varepsilon)$, as it is accepted [21–28], correspond to the quasiparticle energy $\varepsilon$ with the momentum $p$. When $p$ increases, the damping becomes strong and the maxima disappear [25]. This is the reason why phonons and helons, as quasiparticles defined for small values of $p$, exhibit weak damping and are well defined over a wide range of thermodynamic parameters. At the same time, in contrast to phonons, helons exist only in the superfluid state. Confirmation of this hypothesis demands further development of the microscopic theory of superfluidity, as well as new experiments on low-angle neutron scattering.

5. Conclusions

Thus, according to the above consideration, in addition to elementary excitations with the phonon–roton energy spectrum in superfluid helium, there are helons (6)–(8) that satisfy the generalized Landau superfluidity criterion. The consideration of the temperature dependence of the helon energy spectrum allows explanation of the anomalous behavior of the specific heat of superfluid helium in the vicinity of the phase transition to the normal state. Various theoretical models without use of anomalous averages have been considered in [33–36,44–46] to justify the spectrum of elementary excitations with a gap.

Fig. 3. Heat capacity of liquid $^4$He. Circles are the experimental data of [41–43].
In this connection, it is necessary to mention that the usual way of describing the critical behavior is based on renormalization group theory (RGT), which is a brilliant hypothesis, permitted to predict the critical indices (see Ref. [38] and references therein). However, the development of physical models makes it possible to describe the properties of He II not only in the vicinity of the phase transition (where RGT is applicable) but, on the basis of Eqs. (10)–(11), in a whole region of superfluid-state parameters.

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