GEOMETRIC RIGIDITY ESTIMATES FOR INCOMPATIBLE FIELDS IN DIMENSION \( \geq 3 \)

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Abstract. We prove geometric rigidity inequalities for incompatible fields in dimension higher than 2. We are able to obtain strong scaling-invariant \( L^p \) estimates in the supercritical regime \( p > 1^* = \frac{n}{n-1} \), while for critical exponent \( 1^* \) we have a scaling invariant inequality only for the weak \( L^1 \) norm. Although not optimal, such an estimate in \( L^1 \) is enough in order to infer a useful lemma which gives BV bounds for \( SO(n) \)-valued fields with bounded Curl.

1. Introduction

The geometric rigidity estimate for gradient fields proved in [1] plays a crucial role in nonlinear elasticity theory. However, in the study of lattices with dislocations, a geometric rigidity estimate for incompatible fields (i.e., fields not arising from gradients) becomes necessary (cf. e.g. [3] and [4]). In [3], the authors proved a (scaling invariant) version of the geometric rigidity theorem in [1] for incompatible fields in dimension 2 for the critical exponent.

In this work we give a proof of the analogous result in dimension \( \geq 3 \), for the supercritical regime \( p > 1^* = \frac{n}{n-1} \) (Theorem 4). The approach is to write down an incompatible field as the sum of a compatible term, for which we can use the classical geometric rigidity from [1] and a remainder, which is the \( L^p \) norm of a weakly singular operator (the averaged linear homotopy operator), whose derivative is a Calderón-Zygmund operator. This allows to give the bounds in the supercritical case. On the other hand, for the critical exponent we can still use the weak geometric rigidity estimate proved in [2] in order to find a scaling invariant estimate for the weak-\( L^1 \) norm (Theorem 3). From Theorem 3, we deduce directly in Proposition 1 that the Curl of a matrix field \( A \in L^{1,\infty}(\Omega)^{n \times n} \) (where \( \Omega \) is an open bounded set in \( \mathbb{R}^n \)) taking values in \( SO(n) \) bounds its gradient.

2. Notations and Preliminaries

In what follows, \( C \) will denote a (universal) constant whose value is allowed to change from line to line. We put \( \|x\| = \frac{|x|}{|x|} \), while \( L^p(U, \Lambda^r) \) (\( W^{m,p}(U, \Lambda^r) \)) denotes the space of \( r \)-forms on \( U \) whose coefficients are \( L^p \) (\( W^{m,p} \)) functions. Moreover, recall that we can identify a tensor field \( A \in L^1(\Omega)^{n \times n} \) with a vector of 1-forms of length \( n \), that is with \( \omega = (\omega^i)_{i=1}^n \), \( \omega^i = A^i_j \, dx^j \), and its Curl with \( d\omega \) (or, more precisely, with \( (\star d\omega)^b \)), given by

\[
d\omega^i = \sum_{j<k} \left( \frac{\partial A^i_j}{\partial x^k} - \frac{\partial A^i_k}{\partial x^j} \right) dx^j \wedge dx^k.
\]

We recall that a real-valued function \( f \) from a measure space \((X, \mu)\) is in \( L^{p,\infty}(X, \mu) \) or \( (L^p_{\text{loc}}(X, \mu)) \) if

\[
\|f\|_{L^{p,\infty}(X, \mu)} := \sup_{t>0} t \mu \left( \left\{ x \in X \mid |f(x)| > t \right\} \right)^{\frac{1}{p}} < \infty.
\]

Is easy to check that \( \|\cdot\|_{L^{p,\infty}(X, \mu)} \) is only a quasi-norm, that is the triangle inequality holds just in the weak form

\[
\|f + g\|_{L^{p,\infty}(X, \mu)} \leq C_p \left( \|f\|_{L^{p,\infty}(X, \mu)} + \|g\|_{L^{p,\infty}(X, \mu)} \right).
\]
We write $L^{p, \infty}(\Omega)$ for $L^{p, \infty}(\Omega, |\cdot|)$, when $\Omega \subset \mathbb{R}^n$ and $|\cdot|$ is the Lebesgue measure. We recall the

**Definition 1.** Let $U \subset \mathbb{R}^n$ be a star-shaped domain with respect to the point $y \in U$. The linear homotopy operator at the point $y$ is the operator

$$k_y = k_{y, r} : \Omega^r(U) \to \Omega^{r-1}(U),$$

defined as

$$(k_y \omega)(x) := \int_0^1 s^{r-1} \omega(s x + (1-s) y) \mathcal{L}(x-y) ds,$$

where $(\omega(x) \mathcal{L} v)[v_1, \ldots, v_{n-1}] := \omega(x)[v, v_1, \ldots, v_{n-1}]$. It is well known that the linear homotopy operator satisfies

$$\omega = k_{y, r+1} d\omega + dk_{y, r} \omega \quad \forall \omega \in \Omega^r(U).$$

In order to get more regularity, we consider the following averaged linear homotopy operator on $B := B(0,1)$, which coincides with the one introduced by Iwaniec and Lutoborski in [5], except for the choice of the weight function:

$$T = T_r : \Omega^r(B) \to \Omega^{r-1}(B),$$

$$T \omega(x) := \int_B \varphi(y) (k_y \omega)(x) dy,$$

where $\varphi \in C_c^\infty(B(0,2))$ is a positive cut-off function, with $\varphi \equiv 1$ in $B$ and

$$\max \left\{ ||\varphi||_{L^\infty(\mathbb{R}^n)}, ||\nabla \varphi||_{L^\infty(\mathbb{R}^n)} \right\} \leq 3.$$

Clearly, (1) holds for $T$ as well:

$$\omega = T d\omega + dT \omega.$$

An essential result is the rigidity estimate due to Friesecke, James and Müller:

**Theorem 1 ( [1]).** Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $n \geq 2$, and let $1 < p < \infty$. There exists a constant $C = C(p, \Omega)$ such that for every $u \in W^{1,2}(\Omega)$ there exists a rotation $R \in SO(n)$ such that

$$||\nabla u - R||_{L^p(\Omega)^{n \times n}} \leq C ||\text{dist}(\nabla u, SO(n))||_{L^p(\Omega)^{n \times n}}.$$

For weak-$L^p$ estimate, we shall need the following theorem proved by Conti, Dolzmann and Müller:

**Theorem 2 ( [2]).** Let $p \in (1, \infty)$ and $\Omega \subset \mathbb{R}^n$ be a bounded connected domain. There exists a constant $C > 0$ depending only on $p, n$ and $\Omega$ such that for every $u \in W^{1,1}(\Omega)^n$ such that $\text{dist}(\nabla u, SO(n)) \in L^{p, \infty}(\Omega)^{n \times n}$ there exists a rotation $R \in SO(n)$ such that

$$||\nabla u - R||_{L^{p, \infty}(\Omega)^{n \times n}} \leq C ||\text{dist}(\nabla u, SO(n))||_{L^{p, \infty}(\Omega)^{n \times n}}.$$

We also recall that, as proved in [5], $T$ satisfies (for smooth forms $\omega$) the pointwise bound

$$|T \omega(x)| \leq C_{n, r} \int_B \frac{|\omega(y)|}{|x-y|^{n-1}} dy.$$

Indeed, for $\omega = \omega_\alpha dx^\alpha \in \Omega^r(B)$ we have

$$T \omega(x) = \left( \int_B dy \varphi(y) \int_0^1 t^{r-1} \langle x-y, e_i \rangle \omega_\alpha(tx + (1-t)y) \right) dx^\alpha e_i.$$
We then make the substitution $\Phi(y,t) := (tx + (1-t)y, \frac{t}{1-t}) \equiv (z(t,y), s(t))$, $\Phi : B(0,1) \times (0,1) \rightarrow B(0,1) \times (0, \infty)$, which gives

$$T_\omega(x) = \left( \int_B \omega_n(z) \frac{\langle x - z, e_i \rangle}{|x - z|^n} \int_0^2 s^{-1}(1 + s)^{n-r}\varphi(z + sz - x) \, ds \right) \, dx \, e_i = \left( \int_B K_i^t(z, x - z) \omega_n(z) \, dz \right) \, dx \, e_i,$$

where

$$K_i^t(x, h) := \frac{\langle h, e_i \rangle}{|h|^n} \int_0^2 s^{-1}(1 + s)^{n-r}\varphi(x - sh) \, ds,$$

and we noticed that, since $\varphi$ has compact support, the integral from 0 to $\infty$ actually reduces to an integral over a finite interval. That is, we get (4). It also follows easily from (4) that this gives immediately

$$||T\omega||_{L^1(B)} \leq C_n \frac{||\omega||_{L^1(B)}}{|E|},$$

and thus, using (2), $||A - TdA||_{L^1(B)} \leq C ||dA||_{L^1(B)}$, which extends immediately by density in the case when $dA$ is a vector measure with bounded total variation. Choosing $E = \left\{ x \in B \mid |T\omega(x)| > t \right\}$, for $t > 0$

$$t |E| \leq \int_E |T\omega(x)| \, dx \leq C |E| \frac{1}{t} ||dA||_{L^1(B)}.
Passing to the supremum over \( t > 0 \), we find
\[
\| TdA \|_{L^1, \infty(B)} \leq C_n \| dA \| (B).
\]
Since \( B \) is convex and \( d(A - TdA) = d^2 T A = 0 \), we can find a function \( g \) such that \( dg = A - TdA \). From the estimates proven, it is possible to apply Theorem 2 to \( g \) and find
\[
\| dg - R \|_{L^1, \infty(B)} \leq C \| \text{dist}(dg, SO(n)) \|_{L^1, \infty(B)}.
\]

But
\[
\| dg - R \|_{L^1, \infty(B)} \geq C \| A - R \|_{L^1, \infty(B)} - \| TdA \|_{L^1, \infty(B)}
\]
and
\[
\| \text{dist}(dg, SO(n)) \|_{L^1, \infty(B)} \leq \| \text{dist}(A, SO(n)) \|_{L^1, \infty(B)} + \| TdA \|_{L^1, \infty(B)}.
\]
In particular,
\[
\| A - R \|_{L^1, \infty(B)} \leq C \left( \| \text{dist}(A, SO(n)) \|_{L^1, \infty(B)} + |\text{Curl}(A)| (B) \right).
\]

We now give another estimate for \( L^p \) norms. It requires an \( L^\infty \)-bound on the matrix field \( A \), which is natural in the context of the theory of elasticity.

**Theorem 4.** Let \( n \geq 3 \), \( 1^* := 1^*(n) := \frac{n}{n-1} \), \( p \in [1^*, 2] \) and fix \( M > 0 \). There exists a constant \( C = C(n, M, p) > 0 \), depending only on the dimension \( n \), the exponent \( p \) and the constant \( M \), such that for every \( A \in L^\infty(B) \), with \( \| A \|_\infty \leq M \) and \( \text{Curl}(A) \in M(B, A^2) \), \( B := B(0, 1) \), there exists a corresponding rotation \( R \in SO(n) \) for which, if \( p > 1^* \)
\[
\int_B |A - R|^p \, dx \leq C \left( \int_B \text{dist}^p(A, SO(n)) \, dx + |\text{Curl}(A)|^{1^*} (B) \right),
\]
while, if \( p = 1^* \),
\[
\int_B |A - R|^{1^*} \, dx \leq C \int_B \text{dist}^{1^*}(A, SO(n)) \, dx + C |\text{Curl}(A)|^{1^*} (B) \{ \log (|\text{Curl}(A)| (B)) + 1 \}.
\]

**Remark 1.** The constant \( C \) in (7) is not scaling invariant in the critical regime \( p = 1^* \).

**Proof of Theorem 4.** Without loss of generality, we can assume \( TdA \) not identically constant. Indeed, if \( TdA \) is identically constant, from the identity \( TdA = A - dTA \), we see that \( dA = 0 \), hence the result follows applying Theorem 1 as in the proof of Theorem 2. As in the proof of Theorem 1 and using \( |a - b|^p \geq 2^{1-p} |a|^p - |b|^p \) we find a rotation \( R \in SO(n) \) for which the inequality
\[
\int_B |A - R|^p \, dx \leq C_n \left( \int_B |\text{dist}(A, SO(n))|^p \, dx + \int_B |TdA(x)|^p \, dx \right)
\]
holds. We then just need to estimate the last term in the right hand side of (8). For, fix a \( \Lambda > 1 \) (to be chosen later), and define the integrals
\[
I := \int_{|TdA| > \Lambda} |TdA|^p \, dx, \quad II := \int_{|TdA| \leq \Lambda} |TdA|^p \, dx.
\]
We now give an estimate for \( I \). Firstly, we recall that \( T \) is a bounded operator from \( L^p(B, A^r) \) into \( W^{1,p}(B, A^{r+1}) \), whenever \( p \in (1, \infty) \) (cf. [5] Proposition 4.1]). Moreover, \( TdA = A - dTA \), and \( \nabla T = S_1 + S_2 \), where \( S_1 \) is a “weakly” singular operator which maps continuously \( L^\infty \) into itself, while \( S_2 \) is a Calderón-Zygmund operator (cf. [5] Proposition 4.1]). In particular,
\[
|TdA|_{\text{BMO}} \leq C_n \| A \|_{\infty} \leq C_n M,
\]
where \( C_n > 0 \) is a constant depending only on the dimension. Now, we can write
\[
I = \Lambda^{p-1} \Lambda^r \left( \int_{(TdA) > \Lambda} |(TdA) > \Lambda| \right) + I', \quad I' := \int_\Lambda ^\infty \lambda ^{p-1} \left( |(TdA) > \lambda| \right) \, d\lambda.
\]
Clearly,

\[ \Lambda^* \{ |[TdA] > \Lambda| \} \leq |[TdA]|^*_{L^1,\infty} \leq C |dA| (B)^* . \]

We now take a Calderón-Zygmund decomposition of \( F(x) := |TdA(x)|^p \): namely, we find a function \( g \in L^\infty \), with \( |g|_\infty \leq 2^{-n} \Lambda^p \) and disjoint cubes \( \{ Q_j \}_{j \geq 1} \) such that, if \( b := \sum_{j \geq 1} \chi_{Q_j} F \),

\[
\begin{align*}
F &= g + b, \\
2^{-n} \Lambda^p &< \int_{Q_j} F \, dx \leq \Lambda^p \quad \text{(Jensen \Rightarrow \text{\int_{Q_j} TdA(x)dx} \leq \Lambda)}, \\
\left| \bigcup_{j \geq 1} Q_j \right| &\leq \frac{2^n}{\Lambda^p} \int |TdA|^p \, dx.
\end{align*}
\]

With such a decomposition, outside the cubes \( Q_j \), \( |TdA|^p = |g(x)| \leq 2^{-n} \Lambda^p \leq \Lambda^p \). Hence, using the John-Nirenberg inequality and the elementary estimate

\[
\int_x^\infty \lambda^q e^{-\lambda} \, d\lambda \leq e^{-x}(1 + x), \quad \forall q \leq 1 \text{ and } x \geq 1,
\]

we find that (provided \( p \leq 2 \))

\[
I' = \int_\Lambda^\infty \lambda^{p-1} \sum_{j \geq 1} \left\{ x \in Q_j \left| TdA > \lambda \right| \right\} \, d\lambda \leq \\
\leq \int_\Lambda^\infty \lambda^{p-1} \sum_{j \geq 1} \left\{ x \in Q_j \left| TdA(x) - \int_{Q_j} TdA \, dx > \lambda - \Lambda \right| \right\} \, d\lambda \leq \\
\leq C_1 \int_\Lambda^\infty \lambda^{p-1} \left( \sum_{j \geq 1} |Q_j| \right) \exp \left( -C_2 \frac{\lambda - \Lambda}{||TdA||_{BMO}} \right) \, d\lambda < \\
< C_1 \frac{2^n}{\Lambda^p} \left( \int |TdA|^p \right) e^{C_2 \frac{\lambda - \Lambda}{||TdA||_{BMO}}} \left( \frac{||TdA||_{BMO}}{C_2} \right)^p \int_\Lambda^\infty \lambda^{p-1} e^{-\lambda} \, d\lambda \leq \\
\leq C_1 \frac{2^n}{\Lambda^p} \left( \int |TdA|^p \right) \left( \frac{||TdA||_{BMO}}{C_2} \right)^p \left( 1 + \frac{C_2}{||TdA||_{BMO}} \Lambda \right) \leq \\
\leq C_{n,M} \left( \frac{1 + \Lambda}{\Lambda^p} \right).
\]

Hence, if we choose \( \Lambda \) big enough (depending only on \( n \) and \( M \)) in (10),

\[
I' \leq \frac{1}{2} \int |TdA|^p.
\]

Let us now estimate \( II \). If \( p > 1^* \), we can write

\[
\int_{|TdA| \leq \Lambda} |TdA|^p \, dx = \int_{1<|TdA|\leq \Lambda} |TdA|^p \, dx + \sum_{j \geq 0} \int_{2^{-j-1} < |TdA| \leq 2^{-j}} \\
\leq C \left\{ \Lambda^p |dA|^{1^*} (B) + \sum_{j \geq 0} 2^{-(j+1)p} \left( |TdA| > 2^{-(j+1)} \right) \right\} \leq \\
\leq C \left| dA \right|^{1^*} (B) \left( \Lambda^p + \sum_{j \geq 0} 2^{-j(1^*-p)} \right) \leq \\
\leq C(n,p,M) \left| dA \right|^{1^*} (B),
\]

which gives (13). In the case \( p = 1^* \), we are going to make use of the increasing convex function \( \Psi \), defined as the linear (convex) continuation of \( t \to t^p \) for \( t \geq \Lambda \):

\[
\Psi(t) := \begin{cases} 
1^* \Lambda^{1^*} - t & \text{if } t \leq \Lambda, \\
1^* \Lambda^{1^*} - t + (1 - 1^*) \Lambda^{1^*} & \text{if } t \geq \Lambda.
\end{cases}
\]

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piecewise constant function \( R \) where the rotations \( R \) Consider a tessellation of \( \Phi \) That is, \( \phi \)

Proposition 1. Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set, and suppose \( A \in L^2(\Omega) \) and \( \text{spt}(A) \subseteq \Omega \). Consider a tessellation of \( \mathbb{R}^n \) with cubes \( \{ Q_{i}^{(\ell)} \}_{i \geq 1} = \{ Q(x_i, \ell) \} \) of side \( \ell \), and define \( A_{\ell} \) as the piecewise constant function

\[
A_{\ell} := \sum_{i \geq 1} R_{i}^{(\ell)} X_{Q_i^{(\ell)}},
\]

where the rotations \( R_{i}^{(\ell)} \) are the ones given by Theorem \( 1 \) applied to \( A \) on the balls \( B(x_i, \frac{3}{2} \ell) \). There exists a constant \( C = C(n) > 0 \), depending only on the dimension \( n \), such that

\[
\frac{1}{\ell} ||A - A_{\ell}||_{L^1(\Omega)} + \|DA_{\ell}\|_{L^2(\Omega)} \leq C \left( \frac{n^2}{\ell^2} ||\text{dist}(A, SO(n))||_{L^2(\Omega)} + ||\text{Curl}(A)||_{L^2(\Omega)} \right).
\]

In particular, if \( A \in SO(n) \) almost everywhere,

\[
|DA|_{(\Omega)} \leq C |\text{Curl}(A)|_{(\Omega)}.
\]

That is, \( A \in BV(\Omega, SO(n)) \) provided \( |\text{Curl}(A)|_{(\Omega)} \) is finite.

Proof. By definition, the rotations \( R_{i}^{(\ell)} \) in (13) satisfy

\[
\left\| A - R_{i}^{(\ell)} \right\|_{L^1(\Omega)} \leq C_n \left( \|\text{dist}(A, SO(n))\|_{L^1(\Omega)} + \|\text{Curl}(A)\|_{(\Omega)} \right).
\]

Let \( \phi \in C^1_{c}(\Omega) \). Then

\[
\int A_{\ell} \text{div}(\phi) \, dx \leq \sum_{i,j \text{ s.t. } \partial Q_i^{(\ell)} \cap \partial Q_j^{(\ell)} \neq \emptyset} \ell^{n-1} \left| R_{i}^{(\ell)} - R_{j}^{(\ell)} \right|.
\]
Now, for any two adjacent cubes $Q_i^{(e)}$ and $Q_j^{(e)}$, take the rotation $R_{e,i}^j$ given applying Theorem 5 to the cube $2Q_i^{(e)}$. Then
\[
\left| R_{e,i}^{(e)} - R_{e,j}^{(e)} \right| \leq C_n \left( \left| R_{e,i}^{(e)} - R_{e,j}^j \right| \right) \leq C_n \left( \left| A - R_{e,i}^{(e)} \right| \right) \leq C_n \left( \left| A - R_{e,j}^{(e)} \right| \right) \leq C_n \left( \left| \text{dist}(A, SO(n)) \right| \right) + |\text{Curl}(A)| \left( 4Q_i^{(e)} \right)
\]

Taking the supremum over $\varphi$, since the cubes $4Q_i^{(e)}$ overlap only finitely many times, we obtain
\[
|DA_{\varphi}|(\Omega) \leq C_n \left( \left| \text{dist}(A, SO(n)) \right| \right) + |\text{Curl}(A)| \left( \Omega \right) .
\]

Moreover, from the definition of weak-$L^1$:
\[
\frac{1}{\theta} \int_{Q_i^{(e)}} |\theta - A_{\varphi}| dx \leq C_n \left( \left| A - A_{\varphi} \right| \right) \leq C_n \left( \left| \text{dist}(A, SO(n)) \right| \right) + |\text{Curl}(A)| \left( 4Q_i^{(e)} \right)
\]

This gives in particular (14). Moreover
\[
\left| A - A_{\varphi} \right| \leq C_n \theta \sum_{i \geq 1} \left( \left| A - A_{\varphi} \right| \right) \leq C_n \theta \left( \left| \text{dist}(A, SO(n)) \right| \right) + |\text{Curl}(A)| \left( 2Q_i^{(e)} \right)
\]

That is, $A_{\varphi} \to A$ strongly in $L^1$. Thus, if we let $\theta \to 0$, we obtain provided $A \in SO(n)$ almost everywhere.

\[\square\]

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