A LOSS-AVERSE TWO-PRODUCT ORDERING MODEL WITH INFORMATION UPDATING IN TWO-ECHelon INVENTORY SYSTEM

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ABSTRACT. This paper integrates the prospect theory with two-product ordering problem and adopts Bayesian forecasting model under Brownian motion to propose a loss-averse two-product ordering model with demand information updating in a two-echelon inventory system. We also derive all psychological perceived revenue functions for sixteen supply-demand cases as well as the expected value functions and prospect value function for the loss-averse retailer. To solve this model, a Monte Carlo algorithm is presented to estimate the high dimensional integrals with curved polyhedral integral region of unknown volume. Numerical results show that the optimal order quantities of both high-risk product and low-risk product vary across different psychological reference points, which are also affected by information updating, and the loss-averse retailer benefits considerably from information updating. All results suggest that our model provides a better description of the retailer’s actual ordering behavior than existing models.

1. Introduction. Making optimal ordering decisions for products with probabilistic demand and fixed selling stages is a practically significant challenge frequently faced by sellers and retailers [9, 17]. A considerable research has been devoted to address ordering problems and their various extensions. One important extension is to analyze multi-product newsvendor problem (MPNP) [1, 6, 26]. Another significant extension is a two-echelon ordering model with information updating, where the retailer collects demand information in the first stage to update demand distribution for the second stage using the Bayesian analysis [4]. This is a popular model used by decision makers to effectively mitigate the market risk [10, 14, 15, 25].

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Most of these works are based on expected utility theory [16], where the expected utility (minimizing expected cost or maximizing expected income) is regarded as an optimization objective. However, the research outcomes obtained by experimental economists have shown [2, 5, 19, 20, 22] that decision makers, who have preferences, are rarely risk neutral. For loss-aversion, meaning that a decision maker, who strongly prefers avoiding losses to acquiring same-sized gains, is risk-averse for gains and risk-seeking for losses, it is modelled as one of the key features of the prospect theory in [8]. It provides key insight into the decision maker’s risk preference behavior and has been adopted widely in inventory management [11, 12, 23, 24].

Although the importance of multi-product newsvendor problem, two-echelon order policy with information updating, and the decision maker’s risk behavior in inventory management have been widespread recognized, all the three have not been considered simultaneously in an ordering model. In this paper, we extend the ordering model to the case of a loss-averse retailer, where the loss-averse retailer has two order opportunities with information updating, and she can purchase different products from her supplier. This new model has important implications, both theoretically and practically.

Several papers in the literature have considered ordering models involving two of the three characteristics mentioned above. CVaR (Conditional Value at Risk) is incorporated into a multi-product two-echelon ordering model in [27], which is then formulated as a linear programming problem, and Bayesian forecasting under Brownian motion is adopted to update demand information. In [3], it is assumed that the products have independent demands. The emphasis is on the analysis of the closed-form approximations of the optimal order quantities for a multi-product risk-averse retailer using exponential utility function. A newsvendor game is considered in [24], where two different retailers sell two substitutable products with loss-averse preference. In [28], a loss-averse two-product order policy is modelled based on the prospect theory, where the subjective weight functions and the expected value functions are derived. The influences of the psychological reference point and the loss-averse coefficient on the loss-averse retailer’s optimal two-product ordering decision are examined.

The models mentioned above could describe the retailer’s practical order policies. However, information updating, which is of practical importance, has been ignored in these models. Assuming that the retailer is prudent with zero profit target, a single-product loss-averse ordering model with two order opportunities and market information updating is considered in [13], where the unreachable profit is taken as a penalty to describe the loss aversion of the retailer. The optimal order quantities in each stage by forecast revision are derived and analyzed.

Based on the papers mentioned above, especially [27], [28] and [13], our model considers the optimal order policy for a loss-averse retailer with two order opportunities and information updating. We introduce loss aversion to model the retailer’s risk attitude rather than the common risk aversion as considered in the existing literature, such as [27]. Our model takes into account the influence of information updating, which was ignored in the existing literature, such as [28]. The model considers the case of two products, where the maximum profit target is also taken into account. It differs significantly from the case of single product with the zero profit target considered in the existing literature, such as [13]. Furthermore, taking two-product as an example, besides using Bayesian forecasting model under Brownian motion to update demand information, we also develop a Monte Carlo algorithm
to estimate the high dimensional integrals with curved polyhedral integral region of unknown volume to maximize the prospect value for the loss-averse retailer.

The main purpose of this study is to provide decision references, obtained based on rigorous theoretical analysis, for the decision maker in the real world. In the loss-averse two-product ordering problem, the effect of both psychological reference point and information updating on the optimal order quantities are taken into consideration. In Section 2, we describe the problem followed by definitions and main assumptions. The Bayesian forecasting model under Brownian motion is presented in Section 3, and a loss-averse two-product ordering model based on the prospect theory is proposed in Section 4. Section 5 is devoted to the development of a Monte Carlo algorithm to estimate and solve the high dimensional integral with a curved polyhedral integral region of unknown volume. This algorithm is then used to solve the model we have proposed. Numerical results are reported to examine the optimal order policies in Section 6, and conclusions are drawn in the last section.

2. Problem description. The following notations and assumptions will be used throughout.

Indices:
\( i = 1, 2 \) index of products
\( j = 1, 2 \) index of order stages
\( k \) index of the supply-demand cases. \( k = d, e; d, e = 1, 2, 3, 4 \).

Definitions:
\( x_{ij} \) order quantity of product \( i \) in stage \( j \), decision variable
\( y_{ij} \) demand for product \( i \) in stage \( j \), random variable
\( p_{ij} \) retail price of product \( i \) in stage \( j \)
\( c_{ij} \) wholesale price of product \( i \) in stage \( j \)
\( r_i \) unit salvage value of product \( i \) in stage 2, \( r_i < c_{ij} < p_{ij} \)
\( s_{ij} \) unit shortage cost of product \( i \) in stage \( j \)
\( L \) total length of the sales season
\( \tau \) length of stage 1
\( u_{ij}^U \) mean of the updated demand distribution of product \( i \) in stage \( j \)
\( \sigma_{ij}^{U^2} \) variance of the updated demand distribution of product \( i \) in stage \( j \)
\( f_{Y_{i1}}(\cdot) \) demand probability density function of product \( i \) in stage 1
\( f_{Y_{i2}|Y_{i1}}(\cdot) \) demand conditional probability density function of product \( i \) in stage 2
\( \pi_0 \) psychological reference point of the retailer
\( \lambda \) loss aversion coefficient to measure the retailer’s sensitivity to losses
\( \alpha, \beta \) risk attitude coefficients of the retailer
\( \Delta \pi_k \) psychological perceived revenue of the retailer
\( V^+(\Delta \pi_k) \) value function in case \( k \) where the retailer perceives gains
\( V^-(\Delta \pi_k) \) value function in case \( k \) where the retailer perceives losses
\( U(x) \) prospect value function of the retailer, \( x = (x_{11}, x_{21}, x_{12}, x_{22}) \)
\( f_n \xrightarrow{p} f \) \( f_n \) converges in probability to \( f \)
\( f(x_{11}, y_{21}, y_{12}, y_{22}) \) updated probability density function
Consider a situation, where a retailer purchases two products from her supplier. As shown in FIGURE 1, the retailer, who has two order opportunities, first obtains a forecast on the demand over stage 1 and orders at $t_0$, then updates the demand distribution in stage 2 according to the observed demand information over stage 1 and orders at $t_2$.

The main assumptions are as follows:

i. The retailer is loss-averse.

ii. The order time is always pre-specified.

iii. The retail price for each product is exogenous.

iv. The demand for each product is independent.

v. The lead time in stage 1 is $t_1 - t_0$. The lead time in stage 2 is ignored, that is, the delivery happens instantaneously after a product is ordered in stage 2.

3. Bayesian forecasting model under Brownian motion. As the demand for product $i$ can never be less than zero, we assume that the demand for product $i$ follows Geometric Brownian motion (The validity of this assumption has been established in [18] and [21]), that is, the demand process of product $i$ is

$$dy_i = \gamma_i y_idt + \sigma_i y_idZ_i$$

where $\gamma_i$ and $\sigma_i$ are the expected demand growth rate and the standard deviation of demand for product $i$, respectively, and $Z_i$ is a standard Wiener Process. Then, $\theta_i = \gamma_i y_i$ is the drift parameter. From [7], the total demand for product $i$ in stage 1 follows a normal distribution with mean $\tau \theta_i$ and variance $\tau \sigma_i^2$, i.e.,

$$f_{Y_{i1}}(y_{i1} | \theta_i) \sim N(\tau \theta_i, \tau \sigma_i^2)$$

Suppose that the prior distribution of product $i$, $h_i(\theta_i)$, is a mixture of the prior distributions $h_{ij}(\theta_i)$, $j = 1, 2$, i.e.,

$$h_i(\theta_i) = \sum_{j=1}^{2} t_{ij} h_{ij}(\theta_i)$$

where $h_{ij}(\theta_i)$ follows a normal distribution, $h_{ij}(\theta_i) \sim N(u_{ij}, \nu_{ij}^2)$; $t_{ij}$ is the weighted coefficient with $t_{i1} + t_{i2} = 1$ and $t_{i1}, t_{i2} \geq 0$. Therefore, the probability density function for the demand for product $i$ in stage 1 is

$$f_{Y_{i1}}(y_{i1}) = \sum_{j=1}^{2} \frac{t_{ij}}{\sigma'_{ij}} \phi \left( \frac{y_{i1} - \tau u_{ij}}{\sigma'_{ij}} \right)$$

(1)
where $\sigma_{ij}^2 = \tau\sigma_i^2 + \tau^2\nu_{ij}^2$, and $\phi(\cdot)$ is the probability density function of a standard normal distribution. So in stage 1, the mean demand is

$$u_{i1}^{IU} = \tau \sum_{j=1}^{2} t_{ij} u_{ij} \quad (2)$$

and the variance is

$$\sigma_{i1}^{IU2} = \tau\sigma_i^2 + \tau^2 \left[ \sum_{j=1}^{2} t_{ij}(u_{ij}^2 + \nu_{ij}^2) - \left( \sum_{j=1}^{2} t_{ij} u_{ij} \right)^2 \right] \quad (3)$$

With the observed demand information in stage 1, we can obtain a posterior distribution for the demand rate for product $i$ in stage 2. To do so, define $m(\rho_i)$ as the marginal distribution of the demand rate in stage 1 and let $\bar{\sigma}_{i}^2 = \sigma_i^2/\tau + \nu_{ij}^2$. From (1), we obtain

$$m(\rho_i) = \sum_{j=1}^{2} \frac{t_{ij}}{\bar{\sigma}_{ij}} \phi \left( \frac{\rho_i - u_{ij}}{\bar{\sigma}_{ij}} \right) \quad (4)$$

Given that the total demand for product $i$ in stage 1 is $y_{i1}$, the average demand rate is $\bar{\rho}_i = y_{i1}/\tau$, which is a sufficient statistic for estimating $\theta_i$.

Let $m_i(\bar{\rho}_i) = \phi((\bar{\rho}_i - u_{ij})/\bar{\sigma}_{ij}) / \bar{\sigma}_{ij}$. With the observed demand rate $\bar{\rho}_i$, the posterior distribution of $\theta_i$ is

$$P_i(\theta_i/\bar{\rho}_i) = \sum_{j=1}^{2} w_{ij} P_i(\theta_i/\bar{\rho}_i)$$

where $P_i(\theta_i/\bar{\rho}_i) \sim N(\tilde{u}_{ij}, \tilde{\nu}_{ij}^2)$, while $\tilde{u}_{ij} = (\sigma_i^2 u_{ij} + \tau \nu_{ij}^2 \bar{\rho}_i) / (\sigma_i^2 + \tau \nu_{ij}^2)$ and $\tilde{\nu}_{ij}^2 = \sigma_i^2 \nu_{ij}^2 / (\sigma_i^2 + \tau \nu_{ij}^2)$ are, respectively, the mean and variance of the posterior distributions for each stage in the mixture, and $w_{ij}$ is the updated weight given to each distribution in the mixture.

$$w_{ij} = \frac{t_{ij} m_{ij}(\bar{\rho}_i)}{\sum_{j=1}^{2} t_{ij} m_{ij}(\bar{\rho}_i)}$$

Let $\tilde{u}_{ij} = (L - \tau) \bar{\rho}_i$, $\tilde{\sigma}_{ij}((L - \tau)) = (L - \tau) \sigma_i^2$, and $\tilde{\nu}_{ij}^2 = (L - \tau)\nu_{ij}^2$. Then, similar to (1), the conditional demand of product $i$ in stage 2 has a probability density function

$$f_{Y_{i2}|Y_{i1}}(y_{i2}|y_{i1}) = \sum_{j=1}^{2} \frac{w_{ij}}{\tilde{\sigma}_{ij}} \phi \left( \frac{y_{i2} - \tilde{u}_{ij}}{\tilde{\sigma}_{ij}} \right) \quad (5)$$

So in stage 2, the updated mean demand of product $i$ is

$$u_{i2}^{IU} = (L - \tau) \sum_{j=1}^{2} w_{ij} \tilde{u}_{ij} \quad (6)$$

and the variance is

$$\sigma_{i2}^{IU2} = (L - \tau)\sigma_i^2 + (L - \tau)^2 \left[ \sum_{j=1}^{2} w_{ij}(\tilde{u}_{ij}^2 + \tilde{\nu}_{ij}^2) - \left( \sum_{j=1}^{2} w_{ij} \tilde{u}_{ij} \right)^2 \right] \quad (7)$$
4. Loss-averse two-product ordering model.

4.1. Psychological perceived revenue function. In stage 1, the profit of the loss-averse retailer is

\[
\pi_1 = p_{11} \min(x_{11}, y_{11}) + p_{21} \min(x_{21}, y_{21}) - c_{11} x_{11} - c_{21} x_{21} - s_{11} (y_{11} - x_{11})^+ - s_{21} (y_{21} - x_{21})^+
\]

where \((\cdot)^+ = \max\{\cdot, 0\}\).

The unsold product in stage 1 will be sold at full price rather than discounted price in stage 2. So the unit salvage value of unsold product in stage 1 is equal to the retail price in stage 2.

In stage 2, the loss-averse retailer can observe the order quantity \(x_{12}\) and the demand \(y_{12}\) in stage 1 to derive a new demand distribution function. Therefore, her profit in stage 2 is

\[
\pi_2 = p_{12} \min\left(x_{12} + (x_{11} - y_{11})^+, y_{12}\right) + p_{22} \min\left(x_{22} + (x_{21} - y_{21})^+, y_{22}\right) - c_{12} x_{12} - c_{22} x_{22} - s_{12} (y_{12} - x_{12} - (x_{11} - y_{11}))^+ - s_{22} (y_{22} - x_{22} - (x_{21} - y_{21}))^+
\]

\[
+ r_1 (x_{12} + (x_{11} - y_{11})^+ - y_{12})^+ + r_2 (x_{22} + (x_{21} - y_{21})^+ - y_{22})^+
\]

Then the psychological perceived revenue of the loss-averse retailer is

\[
\Delta \pi = \pi_1 + \pi_2 - \pi_0
\]

\[
= \sum_{i=1}^{2} \left[p_{1i} \min(x_{i1}, y_{i1}) - c_{1i} x_{i1} - s_{1i} (y_{i1} - x_{i1})^+\right] + \sum_{i=1}^{2} \left[p_{2i} \min\left(x_{i2} + (x_{i1} - y_{i1})^+, y_{i2}\right) - c_{2i} x_{i2}\right]
\]

\[
- \sum_{i=1}^{2} \left[s_{1i} (y_{i2} - x_{i2} - (x_{i1} - y_{i1}))^+ - r_i (x_{i2} + (x_{i1} - y_{i1})^+ - y_{i2})^+\right] - \pi_0
\]

(8)

There are four supply-demand relationships when the loss-averse retailer purchases and sells two products in each stage. So the psychological perceived revenue function in (8) varies across the following sixteen supply-demand cases:

1. Both products are undersupplied in stage 1
   Case 1.1 both products are undersupplied in stage 2
   Case 1.2 product 1 is undersupplied and product 2 is oversupplied in stage 2
   Case 1.3 product 1 is oversupplied and product 2 is undersupplied in stage 2
   Case 1.4 both products are oversupplied in stage 2

2. Product 1 is undersupplied and product 2 is oversupplied in stage 1
   Case 2.1 both products are undersupplied in stage 2
   Case 2.2 product 1 is undersupplied and product 2 is oversupplied in stage 2
   Case 2.3 product 1 is oversupplied and product 2 is undersupplied in stage 2
   Case 2.4 both products are oversupplied in stage 2

3. Product 1 is oversupplied and product 2 is undersupplied in stage 1
   Case 3.1 both products are undersupplied in stage 2
   Case 3.2 product 1 is undersupplied and product 2 is oversupplied in stage 2
   Case 3.3 product 1 is oversupplied and product 2 is undersupplied in stage 2
   Case 3.4 both products are oversupplied in stage 2
4. Both products are oversupplied in stage 1
   Case 4.1 both products are undersupplied in stage 2
   Case 4.2 product 1 is undersupplied and product 2 is oversupplied in stage 2
   Case 4.3 product 1 is oversupplied and product 2 is undersupplied in stage 2
   Case 4.4 both products are oversupplied in stage 2

The psychological perceived revenue functions of the loss-averse retailer for all the cases are as follows:

\[ \Delta \pi_{1.1} = \sum_{i=1}^{2} \sum_{j=1}^{2} [(p_{ij} - c_{ij} + s_{ij})x_{ij} - s_{ij}y_{ij}] - \pi_0 \]

\[ \Delta \pi_{1.2} = \sum_{i=1}^{2} [(p_{i1} - c_{i1} + s_{i1})x_{i1} - s_{i1}y_{i1}] + (p_{12} - c_{12} + s_{12})x_{12} - s_{12}y_{12} \]
\[ + (p_{22} - r_2)y_{22} - (c_{22} - r_2)x_{22} - \pi_0 \]

\[ \Delta \pi_{1.3} = \sum_{i=1}^{2} [(p_{i1} - c_{i1} + s_{i1})x_{i1} - s_{i1}y_{i1}] + (p_{12} - r_1)y_{12} - (c_{12} - r_1)x_{12} \]
\[ + (p_{22} - c_{22} + s_{22})x_{22} - s_{22}y_{22} - \pi_0 \]

\[ \Delta \pi_{1.4} = \sum_{i=1}^{2} [(p_{i1} - c_{i1} + s_{i1})x_{i1} - s_{i1}y_{i1}] + (p_{12} - r_1)y_{12} - (c_{12} - r_1)x_{12} \]
\[ + (p_{22} - c_{22} + s_{22})x_{22} - s_{22}y_{22} - \pi_0 \]

\[ \Delta \pi_{2.1} = (p_{11} - c_{11} + s_{11})x_{11} - s_{11}y_{11} + (p_{22} - c_{21} + s_{22})x_{21} + \]
\[ \sum_{i=1}^{2} [(p_{i2} - c_{i2} + s_{i2})x_{i2} - s_{i2}y_{i2}] + (p_{21} - p_{22} - s_{22})y_{21} - \pi_0 \]

\[ \Delta \pi_{2.2} = \sum_{j=1}^{2} [(p_{1j} - c_{1j} + s_{1j})x_{1j} - s_{1j}y_{1j} + p_{2j}y_{2j} - c_{2j}x_{2j}] \]
\[ + r_2(x_{21} + x_{22} - y_{21} - y_{22}) - \pi_0 \]

\[ \Delta \pi_{2.3} = (p_{11} - c_{11} + s_{11})x_{11} - s_{11}y_{11} + (p_{21} - p_{22} - s_{22})y_{21} + (p_{12} - r_1)y_{12} - \]
\[ (c_{12} - r_1)x_{12} + (p_{22} + s_{22} - c_{21})x_{21} + (p_{22} - c_{22} + s_{22})x_{22} - s_{22}y_{22} - \pi_0 \]

\[ \Delta \pi_{2.4} = (p_{11} - c_{11} + s_{11})x_{11} - s_{11}y_{11} + (p_{21} - r_1)y_{21} - (c_{21} - r_2)x_{21} + \]
\[ \sum_{i=1}^{2} [(p_{i2} - r_i)y_{i2} - (c_{i2} - r_i)x_{i2}] - \pi_0 \]

\[ \Delta \pi_{3.1} = (p_{11} - p_{12} - s_{12})y_{11} + (p_{12} - c_{11} + s_{12})x_{11} + (p_{21} - c_{21} + s_{21})x_{21} + \]
\[ \sum_{i=1}^{2} [(p_{i2} - c_{i2} + s_{i2})x_{i2} - s_{i2}y_{i2}] - s_{21}y_{21} - \pi_0 \]

\[ \Delta \pi_{3.2} = (p_{11} - p_{12} - s_{12})y_{11} + (p_{12} - c_{11} + s_{12})x_{11} + (p_{21} - c_{21} + s_{21})x_{21} - s_{21}y_{21} \]
\[ + (p_{12} - c_{12} + s_{12})x_{12} - s_{12}y_{12} - (c_{22} - r_2)x_{22} + (p_{22} - r_2)y_{22} - \pi_0 \]

\[ \Delta \pi_{3.3} = \sum_{j=1}^{2} [(p_{1j} - r_1)y_{1j} - (c_{1j} - r_1)x_{1j} + (p_{2j} - c_{2j} + s_{2j})x_{2j} - s_{2j}y_{2j}] - \pi_0 \]

\[ \Delta \pi_{3.4} = (p_{11} - r_1)y_{11} - (c_{11} - r_1)x_{11} + (p_{21} - c_{21} + s_{21})x_{21} - s_{21}y_{21} + \]
\[
\Delta \pi_{4.1} = \sum_{i=1}^{2} [(p_{i1} - r_i)y_{i1} - (c_{i1} - r_i)x_{i1}] - \pi_0
\]
\[
\Delta \pi_{4.2} = \sum_{i=1}^{2} [(p_{i1}y_{i1} - c_{i1}x_{i1}) + (p_{i2} - c_{i2} + s_{i2})x_{i2} + (p_{i2} + s_{i2})(x_{i1} - y_{i1}) - s_{i2}y_{i2}] - \pi_0
\]
\[
\Delta \pi_{4.3} = \sum_{i=1}^{2} [(p_{i1}y_{i1} - c_{i1}x_{i1}) + r_1(x_{i1} - y_{i1}) + (p_{i2} + s_{i2})x_{i2} - (p_{i2} + s_{i2})y_{i2}]
\]
\[
\Delta \pi_{4.4} = \sum_{i=1}^{2} \sum_{j=1}^{2} [(p_{ij} - r_i)y_{ij} - (c_{ij} - r_i)x_{ij}] - \pi_0
\]

4.2. **Expected value function and prospect value function.** According to the loss-averse behavior model in the prospect theory proposed in [8], the expected value functions for this loss-averse retailer can be written as follows:

\[
\begin{cases}
E(V^+_k(\Delta \pi_k)) = \int_{G^+_k} \cdots \int (\Delta \pi_k)^\alpha f(y_{11}, y_{21}, y_{12}, y_{22}), & \text{if } \Delta \pi_k \geq 0 \\
E(V^-_k(\Delta \pi_k)) = \int_{G^-_k} \cdots \int (-\Delta \pi_k)^\beta f(y_{11}, y_{21}, y_{12}, y_{22}), & \text{if } \Delta \pi_k < 0
\end{cases}
\]

(9)

where \(E(\cdot)\) is the expected value function, \(V^+_k(\Delta \pi_k)\) and \(V^-_k(\Delta \pi_k)\) are the value function in Case \(k\) where the loss-averse retailer perceives gains and losses, respectively; \(G^+_k\) and \(G^-_k\), which will be derived in Appendix, are the feasible regions of demand which are the integral regions of the expected value functions \(E(V^+_k(\Delta \pi_k))\) and \(E(V^-_k(\Delta \pi_k))\), in Case \(k\), where the loss-averse retailer perceives gains and losses, respectively.

During the analysis mentioned above, we obtain the prospect value function, the objective function of the loss-averse retailer with two order opportunities by substituting the updated demand distribution functions derived in Section 3, the expected value functions \(E(V^+_k(\Delta \pi_k))\) and \(E(V^-_k(\Delta \pi_k))\) expressed by (9) into (10).

Therefore, the optimization model of this loss-averse two-product ordering problem with information updating in two-echelon inventory system can be written as:

\[
\begin{align*}
\max & \quad U(x) = \sum [E(V^+_k(\Delta \pi_k)) + E(V^-_k(\Delta \pi_k))] \\
\text{s.t.} & \quad x_{11}, x_{21}, x_{12}, x_{22} \geq 0
\end{align*}
\]

(10)

Note that we do not have information on the range of integration of each function. However, it is known that the domain of each integration must be a polyhedron specified by a certain formula. Clearly, it is difficult to solve the optimization problem (10) using general purpose algorithms. Thus, we shall develop a Monte Carlo algorithm to estimate the high dimensional integral involving polyhedral integral region.
5. Solution approach.

5.1. Monte Carlo algorithm for high dimensional integral with polyhedral integral region. As mentioned above, the optimization problem (10) involves some four dimensional integrals without information on the upper and lower bounds of each integral. Therefore, we develop a Monte Carlo algorithm to estimate the values of these integrals.

Here, we perform a detailed analysis of \( E \left( V_1^1 (\Delta \pi_{1,1}) \right) \) for Case 1.1. The solutions for other cases can be estimated similarly.

The integral region \( G_{1,1}^+ \) of \( E \left( V_1^1 (\Delta \pi_{1,1}) \right) \) is unknown. However, the constraint condition can be deduced from formula (11) as given below.

\[
\Delta \pi_{1,1} = \sum_{i=1}^{2} \sum_{j=1}^{2} \left[ (p_{ij} - c_{ij} + s_{ij}) x_{ij} - s_{ij} y_{ij} \right] - \pi_0 \geq 0
\]

\[
\Rightarrow \begin{cases}
  x_{11} \leq y_{11}, & x_{21} \leq y_{21} \\
  x_{12} \leq y_{12}, & x_{22} \leq y_{22} \\
  \sum_{i=1}^{2} \sum_{j=1}^{2} s_{ij} y_{ij} \leq \sum_{i=1}^{2} \sum_{j=1}^{2} (p_{ij} - c_{ij} + s_{ij}) x_{ij} - \pi_0 
\end{cases}
\]

Let

\[
a = \max \{ u_{11}^U + 3 \sigma_{11}^U, u_{21}^U + 3 \sigma_{21}^U, u_{12}^U + 3 \sigma_{12}^U, u_{22}^U + 3 \sigma_{22}^U \}
\]

and

\[
b = \min \{ x_{11}, x_{21}, x_{12}, x_{22} \}
\]

So, there is a hypercube

\[ T = [a, b]^4 \]

such that \( G_{1,1}^+ \subset T \). This implies that \( G_{1,1}^+ \), whose volume \( VL \) is unknown, is within this hypercube \( T \).

The process for solving this integral using the Monte Carlo algorithm is detailed as follows:

Assume that

\[
y^{(1)} = \left( y_{11}^{(1)}, y_{21}^{(1)}, y_{12}^{(1)}, y_{22}^{(1)} \right),
\]

\[
y^{(2)} = \left( y_{11}^{(2)}, y_{21}^{(2)}, y_{12}^{(2)}, y_{22}^{(2)} \right),
\]

\[
\vdots
\]

\[
y^{(l)} = \left( y_{11}^{(l)}, y_{21}^{(l)}, y_{12}^{(l)}, y_{22}^{(l)} \right).
\]

are independent random vectors within \( T = [a, b]^4 \).

Every new incoming random vector \( y^{(l)} \) will be checked to see whether or not \( y^{(l)} \in G_{1,1}^+ \): if \( y^{(l)} \in G_{1,1}^+ \), then \( y^{(f)} = y^{(l)} \); otherwise, \( y^{(l)} \) is abandoned. This check will be repeated for the next random vector \( y^{(l+1)} \): if \( y^{(l+1)} \in G_{1,1}^+ \), then \( y^{(f+1)} = y^{(l+1)} \); otherwise, \( y^{(l+1)} \) is also abandoned. All subsequent vectors will be checked in this way.
Next, assume that there are \( f_b \) number of random vectors \( y^{(1)}, y^{(2)}, \ldots, y^{(f_b)} \) such that they all belong to \( G_{1,1} \). Then, the estimator \( VG \) of \( VL \) and the estimator \( \theta E \) of \( E(V_{1,1}^+(\Delta \pi_{1,1})) \) are

\[
\theta E = \frac{VG}{f_b} \sum_{g=1}^{f_b} (\Delta \pi_{1,1})^\alpha f(y_{11}, y_{21}, y_{12}, y_{22})
\]

\[
= \frac{(b-a)^4}{l} \sum_{g=1}^{f_b} (\Delta \pi_{1,1})^\alpha f(y_{11}, y_{21}, y_{12}, y_{22}), \quad g = 1, 2, \ldots, f_b
\]

Obviously, \( VG \) is an unbiased estimate of \( VL \) with \( E(VG) = VL \).

**Proposition 1.** \( \theta E \), which converges to 1 in probability, is an asymptotically unbiased estimator of \( E(V_{1,1}^+(\Delta \pi_{1,1})) \).

**Proof of Proposition 1.**

\[
E(\theta E) = \frac{f_b}{l} (b-a)^4 E \left[ (\Delta \pi_{1,1})^\alpha f(y_{11}, y_{21}, y_{12}, y_{22}) \right]
\]

\[
= \frac{f_b}{l} (b-a)^4 \int_{G_{1,1}} \cdots \int (\Delta \pi_{1,1})^\alpha f(y_{11}, y_{21}, y_{12}, y_{22}) \cdot \frac{1}{VL} dy_{11} dy_{21} dy_{12} dy_{22}
\]

\[
= \frac{f_b}{l} (b-a)^4 \frac{1}{VL} E \left( V_{1,1}^+(\Delta \pi_{1,1}) \right)
\]

According to the Wiener-khinchin law of large numbers,

\[
\lim_{l \to \infty} P \left( \left| \frac{f_b}{l} (b-a)^4 \frac{1}{VL} E \left( V_{1,1}^+(\Delta \pi_{1,1}) \right) - E \left( V_{1,1}^+(\Delta \pi_{1,1}) \right) \right| < \varepsilon \right) = 1
\]

Therefore, \( \theta E \) is an asymptotically unbiased estimator of \( E(V_{1,1}^+(\Delta \pi_{1,1})) \).

Let

\[
\tilde{f} \left( y^{(g)} \right) = \begin{cases} \sum_{i=1}^{2} \sum_{j=1}^{2} [(p_{ij} - c_{ij} + s_{ij})x_{ij} - s_{ij}y_{ij}^{(g)}] - \pi_0 \end{cases}^\alpha f(y_{11}^{(g)}, y_{21}^{(g)}, y_{12}^{(g)}, y_{22}^{(g)})
\]

Given that \( \frac{f_b}{l} (b-a)^4 \) converges in probability to \( VL \), it follows that

\[
\frac{f_b}{l} (b-a)^4 \overset{p}{\to} VL
\]

\[
\frac{1}{f_b} \sum_{g=1}^{f_b} \tilde{f} \left( y^{(g)} \right) \overset{p}{\to} \frac{1}{VL}, \quad g = 1, 2, \ldots, f_b
\]

Therefore,

\[
\theta E = \frac{(b-a)^4}{l} \sum_{g=1}^{f_b} \tilde{f} \left( y^{(g)} \right) = \frac{f_b}{l} (b-a)^4 \frac{1}{f_b} \sum_{g=1}^{f_b} \tilde{f} \left( y^{(g)} \right) \overset{p}{\to} VL \cdot \frac{1}{VL} = 1
\]

\[
g = 1, 2, \ldots, f_b
\]

\[\square\]
5.2. Algorithm for estimating high dimensional integral. The specific algorithm for estimating $E \{V_{11}^{+}(\Delta \pi_{1})\}$ is detailed as follows:

- **Step 1.** Input $l$ number of required random vectors.
- **Step 2.** Initialize variables: $ss = 0$, $f_{b} = 0$, $l_{0} = 0$.
- **Step 3.** Set $l_{1} = l_{0} + 1$. If $l_{1} \leq l$, then go to step 6; otherwise return to step 3.
- **Step 4.** Invoke random function subroutine to generate pseudo random vector $y^{(l)}$.
- **Step 5.** If $y^{(l)} \in G_{11}^{+}$, then go to step 6; otherwise return to step 3.
- **Step 6.** Calculate $f^{*}(y^{(l)})$.
- **Step 7.** Set $ss = ss + f^{*}(y^{(l)})$, $f_{b} = f_{b} + 1$, and then turn to step 3.
- **Step 8.** Calculate $\theta E = \frac{(b-a)^{4}}{l} \cdot ss$.
- **Step 9.** Output $l$, $f_{b}$, $\theta E$.

The solution for each expected value function for other cases can be estimated similarly.

In practice, by entering all data, including parameters $\alpha$, $\beta$, $\lambda$ into corresponding formulas, the decision maker can obtain the optimal order quantity for each product in each stage, $x^{*} = (x_{11}^{*}, x_{21}^{*}, x_{12}^{*}, x_{22}^{*})$, and the maximum prospect value $U^{*}(x^{*})$ using some software such as Matlab R2014a.

6. Numerical results. A company (retailer) purchases two products, high-risk product $A$ and low-risk product $B$, whose demands are independent from each other, and sells them in two stages. The total length of sale season $L = 20$, the length of the first stage $\tau = 10$, the retail price $p_{Aj} = p_{Bj} = 10$, the wholesale price $c_{Aj} = c_{Bj} = 7$, the unit shortage cost $s_{Aj} = s_{Bj} = 10$, and the unit salvage value $r_{A} = r_{B} = 3$.

6.1. The updated demand information values of two products. The variances of the expected demands for both products are $\sigma_{A}^{2} = 100, \sigma_{B}^{2} = 30$, the weight coefficients $t_{1} = 0.7, t_{2} = 0.3$, and $u_{Aj} = u_{Bj} = 20, \nu_{A}^{2} = 150, \nu_{B}^{2} = 30$ for the prior distribution of each product in two stages. Then we obtain the updated demand information values of each product (see TABLE 1), according to formulas (2), (3), (6) and (7).

| $u_{A1}$ | $\sigma_{A1}^{2}$ | $u_{A2}$ | $\sigma_{A2}^{2}$ | $u_{B1}$ | $\sigma_{B1}^{2}$ | $u_{B2}$ | $\sigma_{B2}^{2}$ |
|----------|------------------|----------|------------------|----------|------------------|----------|------------------|
| 200      | 123.69           | 400      | 63.72            | 200      | 57.44            | 400      | 59.79            |

Next, we examine the impacts of psychological reference point and information updating on the optimal order quantity for each product in both stages. All results are found by the proposed Monte Carlo algorithm in Section 5 using Matlab R2014a.

6.2. The impact of psychological reference point on optimal order quantity. With $\alpha = 0.88, \beta = 2.25$, and $\pi_{0} = \{0, 1000, 2000, 3000, 4000, 5000, 8000, 10000, 30000, 50000\}$, the optimal order quantity for each product is shown in TABLE 2 and 3.

The following results come from TABLE 2 and TABLE 3.

(1) Both the optimal order quantities of high-risk product $A$ and low-risk product $B$ vary across different psychological reference points: In stage 1, the optimal order quantities for the two products are similar, and the difference between them is quite
small. Compared with this result, the optimal order quantities for the two products differ substantially in stage 2. Furthermore, the updated demand information improves significantly the optimal order quantities for both products and the profit of the retailer. In other words, the retailer has benefited significantly from demand information updating.

TABLE 2. Optimal Order Quantity with Different Psychological Reference Points and Information Updating

| $\pi_0$ | $x^*_A$ | $x^*_B$ | $x^*_A$ | $x^*_B$ | $U^*(x^*)$ |
|---------|---------|---------|---------|---------|------------|
| 0       | 271     | 281     | 427     | 429     | 3283.9     |
| 1000    | 270     | 278     | 426     | 428     | 2475.8     |
| 2000    | 265     | 273     | 421     | 427     | 1347.2     |
| 3000    | 270     | 268     | 413     | 427     | 643.9      |
| 4000    | 298     | 280     | 410     | 403     | -235.7     |
| 5000    | 315     | 285     | 460     | 305     | -785.4     |
| 8000    | 335     | 290     | 459     | 303     | -1436.7    |
| 10000   | 333     | 285     | 457     | 302     | -2578.8    |
| 30000   | 331     | 283     | 455     | 301     | -3521.6    |
| 50000   | 333     | 288     | 454     | 300     | -4076.4    |

TABLE 3. Optimal Order Quantity with Different Psychological Reference Points and No Information Updating

| $\pi_0$ | $x^*_A$ | $x^*_B$ | $x^*_A$ | $x^*_B$ | $U^*(x^*)$ |
|---------|---------|---------|---------|---------|------------|
| 0       | 40      | 23      | 15      | 25      | 8.5853     |
| 1000    | 36      | 22      | 14      | 25      | 0.1422     |
| 2000    | 37      | 23      | 16      | 23      | 0.1422     |
| 3000    | 39      | 24      | 18      | 21      | 0.1422     |
| 4000    | 41      | 25      | 20      | 20      | 0.1422     |
| 5000    | 43      | 25      | 22      | 18      | 0.1422     |
| 8000    | 45      | 28      | 23      | 15      | 0.1422     |
| 10000   | 43      | 27      | 25      | 13      | 0.1422     |
| 30000   | 41      | 25      | 26      | 12      | 0.1422     |
| 50000   | 43      | 27      | 27      | 11      | 0.1422     |

(2) In stage 1, the mean optimal order quantity for product A is 302.1, varying within the range (-12.28%~10.89%). The mean optimal order quantity for product B is 281.1, varying within the range (-4.66%~3.17%). In stage 2, the mean optimal order quantity for product A is 483.2, varying within the range (-6.43%~4.97%), and the mean optimal order quantity for product B is 281.1, varying within the range (-17.24%~18.34%). These results show that when the risk levels of the two products are different, the impact due to the change of psychological reference points on the optimal order quantity is more obvious (relatively) for high-risk product A than on the optimal order quantity for low-risk product B in stage 1. That is, a high-risk product is more sensitive to the change of the psychological reference points than a low-risk product. However, in stage 2, the change of the optimal
order quantity for low-risk product B is more obvious. Furthermore, the volatility of the optimal order quantity for high-risk product A decreases markedly while the optimal order quantity for low-risk product B increases.

(3) TABLE 2 also shows the impact of different psychological reference points on the optimal order quantities for both products with information updating. When the retailer's prospect value is positive, she is risk-averse, and the optimal order quantities for both products decrease as the psychological reference point increases in each stage. When the psychological reference point is increased, the range of the optimal order quantities for high-risk product A is decreased more than that for low-risk product B. Furthermore, the optimal order quantity for low-risk product B is always greater than that for high-risk product A in each stage. This shows that the retailer is risk-averse in regards of gains, which is consistent with the decision maker's risk-averse behavior about gains according to the prospect theory.

When the retailer’s prospect value is negative, it is risk-seeking. In stage 1, when the psychological reference point is small, the optimal order quantities for both products increase as the psychological reference point increases. This result reflects the risk-seeking behavior of the retailer. Increasing the psychological reference point leads to much larger increases in the range of the optimal order quantity for high-risk product A than that for the low-risk product B. However, when the psychological reference point reaches a certain level, the retailer will generally decrease the order quantities for both products, even though it is risk-seeking. The optimal order quantity for high-risk product A is always more than that for low-risk product B. In stage 2, the optimal order quantity for high-risk product A increases initially and subsequently decreases, while the optimal order quantity for low-risk product B decreases continuously. The optimal order quantity for high-risk product A is still greater than the optimal order quantity for low-risk product B. These results show that the retailer is risk-seeking in regards of losses, which is also in line with the result obtained in the literature based on the prospect theory.

7. Conclusion. In this paper, the prospect theory was introduced to the study of a two-echelon two-product ordering model for a loss-averse retailer with information updating. A Bayesian forecasting model under Brownian motion was used to update demand distribution for the second stage through analyzing the demand information collected in the first stage: It is assumed that the demand for each product follows a Geometric Brownian motion. The mean and variance of the demand in the first stage are obtained, the updated mean and variance of the demand are then obtained in the second stage through Bayesian analysis.

Based on the prospect theory, all the psychological perceived revenue functions and the expected value functions, which can lead to the prospect value function, are derived for the loss-averse two-product retailer for sixteen different supply-demand cases. Taking one case as an example, both products are undersupplied in each stage. A Monte Carlo algorithm is developed to estimate the high dimensional integrals with curved polyhedral integral region of unknown volume. On this basis, the optimal solution that maximizes the prospect value for this retailer is obtained.

Numerical results obtained are used to examine the influences of both psychological reference point and information updating on the optimal order quantities and the maximized prospect value. The optimal order policies with and without information updating are compared and analyzed. Some interesting results are obtained and they are shown in TABLE 2 and TABLE 3.
First, with increasing psychological reference points, the optimal order quantities of high-risk product and low-risk product are quite similar to each other in the first stage, but show an obvious difference from each other in the second stage.

Second, with increasing psychological reference points and updated information, the loss-averse retailer, whose risk-seeking behavior about losses becomes more and more obvious, will gradually order more high-risk product and less low-risk product.

Third, the optimal order quantities for both products and the loss-averse retailer’s prospect value increase significantly due to the availability of the updated demand information. That is, demand information updating is helpful for improving the loss-averse retailer’s profit, which is consistent with reality.

All the results show that our model provides a better description of the retailer’s actual ordering behavior than the existing models in the literature.

Our model can be extended in many directions. Consider the case of two products and two-echelon order as an example and assume that the demands of both products are independent. For this case, multi-product order policies where products are substitutable or complementary in multi-echelon inventory system with information updating are interesting research topics. In reality, many conditions and decisions always change over time such as costs. Thus, the dynamic order decision making should also be investigated in future research.

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Appendix A. The integral region $G^+_k$ of the expected value function $E\left( V^+_k(\Delta \pi_k) \right)$ in Case $k(k = d.e and d, e = 1, 2, 3, 4)$, where the loss-averse retailer perceives gains.

$$G^+_{1,1} : \begin{cases} y_{11} \geq x_{11}, & y_{21} \geq x_{21} \\ y_{12} \geq x_{12}, & y_{22} \geq x_{22} \\ \sum_{i=1}^{2} \sum_{j=1}^{2} s_{ij} y_{ij} \leq \sum_{i=1}^{2} \sum_{j=1}^{2} (p_{ij} - c_{ij} + s_{ij}) x_{ij} - \pi_0 \end{cases}$$

$$G^+_{1,2} : \begin{cases} y_{11} \geq x_{11}, & y_{21} \geq x_{21} \\ y_{12} \geq x_{12}, & y_{22} < x_{22} \\ \sum_{i=1}^{2} s_{1i} y_{i1} + s_{12} y_{12} - (p_{22} - r_2) y_{22} \leq \sum_{i=1}^{2} (p_{i1} - c_{i1} + s_{i1}) x_{i1} + (p_{12} - c_{12} + s_{12}) x_{12} - (c_{22} - r_2) x_{22} - \pi_0 \end{cases}$$

$$G^+_{1,3} : \begin{cases} y_{11} \geq x_{11}, & y_{21} \geq x_{21} \\ y_{12} < x_{12}, & y_{22} \geq x_{22} \\ \sum_{i=1}^{2} s_{1i} y_{i1} - (p_{12} - r_1) y_{12} + s_{22} y_{22} \leq \sum_{i=1}^{2} (p_{i1} - c_{i1} + s_{i1}) x_{i1} + (p_{22} - c_{22} + s_{22}) x_{22} + (r_1 - c_{12}) x_{12} - \pi_0 \end{cases}$$
A LOSS-AVERSE TWO-PRODUCT ORDERING MODEL

\[
G_{1,4}^+ : \begin{cases} 
  y_{11} \geq x_{11}, \quad y_{21} \geq x_{21} \\
  y_{12} < x_{12}, \quad y_{22} < x_{22} \\
  \sum_{i=1}^2 (s_i y_{1i} - (p_{i2} - r_i) y_{i2}) \leq \sum_{i=1}^2 ((p_{i1} - c_{i1} + s_{i1}) x_{i1} - (c_{i2} - r_i) x_{i2}) - \pi_0 \\
  y_{11} \geq x_{11}, \quad y_{21} < x_{21} \\
  y_{12} \geq x_{12}, \quad y_{22} + y_{21} \geq x_{22} + x_{21} 
\end{cases}
\]

\[
G_{2,1}^+ : \begin{cases} 
  \sum_{i=1}^2 s_{i2} y_{i2} + s_{11} y_{11} - (p_{21} - p_{22} - s_{22}) y_{21} \leq (p_{11} - c_{11} + s_{11}) x_{11} + \sum_{i=1}^2 (p_{i2} - c_{i2} + s_{i2}) x_{i2} + (p_{22} - c_{21} + s_{22}) x_{21} - \pi_0 \\
  y_{11} \geq x_{11}, \quad y_{21} \geq x_{21} \\
  y_{12} \geq x_{12}, \quad y_{22} + y_{21} \geq x_{22} + x_{21} 
\end{cases}
\]

\[
G_{2,2}^+ : \begin{cases} 
  \sum_{j=1}^2 (s_{1j} y_{1j} - p_{2j} y_{2j}) + r_2 (y_{21} + y_{22}) \leq r_2 (x_{21} + x_{22}) + \sum_{j=1}^2 [(p_{1j} - c_{1j} + s_{1j}) x_{1j} - c_{2j} x_{2j}] - \pi_0 \\
  y_{11} \geq x_{11}, \quad y_{21} \geq x_{21} \\
  y_{12} < x_{12}, \quad y_{22} + y_{21} < x_{22} + x_{21} 
\end{cases}
\]

\[
G_{2,3}^+ : \begin{cases} 
  s_{11} y_{11} - (p_{21} - r_2) y_{21} - \sum_{i=1}^2 (p_{i2} - r_i) y_{i2} \leq (r_2 - c_{21}) x_{21} + \sum_{i=1}^2 (r_i - c_{i2}) x_{i2} + (p_{11} - c_{11} + s_{11}) x_{11} - \pi_0 \\
  y_{11} \geq x_{11}, \quad y_{21} \geq x_{21} \\
  y_{12} < x_{12}, \quad y_{22} + y_{21} < x_{22} + x_{21} 
\end{cases}
\]

\[
G_{2,4}^+ : \begin{cases} 
  \sum_{i=1}^2 s_{i2} y_{i2} - (p_{11} - p_{12} - s_{12}) y_{11} + s_{21} y_{21} \leq (p_{12} - c_{11} + s_{12}) x_{11} + \sum_{i=1}^2 (p_{i2} - c_{i2} + s_{i2}) x_{i2} + (p_{21} - c_{21} + s_{21}) x_{21} - \pi_0 \\
  y_{11} \geq x_{11}, \quad y_{21} \geq x_{21} \\
  y_{12} + y_{11} \geq x_{12} + x_{11}, \quad y_{22} \geq x_{22} 
\end{cases}
\]

\[
G_{3,1}^+ : \begin{cases} 
  \sum_{i=1}^2 s_{i2} y_{i2} - (p_{11} - p_{12} - s_{12}) y_{11} + s_{21} y_{21} \leq (p_{12} - c_{11} + s_{12}) x_{11} + \sum_{i=1}^2 (p_{i2} - c_{i2} + s_{i2}) x_{i2} + (p_{21} - c_{21} + s_{21}) x_{21} - \pi_0 \\
  y_{11} \geq x_{11}, \quad y_{21} \geq x_{21} \\
  y_{12} + y_{11} \geq x_{12} + x_{11}, \quad y_{22} < x_{22} 
\end{cases}
\]

\[
G_{3,2}^+ : \begin{cases} 
  \sum_{j=1}^2 [s_{2j} y_{2j} - (p_{1j} - r_1) y_{1j}] \leq \sum_{j=1}^2 [(r_1 - c_{1j}) x_{1j} + (p_{2j} - c_{2j} + s_{2j}) x_{2j}] - \pi_0 \\
  y_{11} < x_{11}, \quad y_{21} \geq x_{21} \\
  y_{12} + y_{11} < x_{12} + x_{11}, \quad y_{22} \geq x_{22} 
\end{cases}
\]

\[
G_{3,3}^+ : \begin{cases} 
  \sum_{j=1}^2 [s_{2j} y_{2j} - (p_{1j} - r_1) y_{1j}] \leq \sum_{j=1}^2 [(r_1 - c_{1j}) x_{1j} + (p_{2j} - c_{2j} + s_{2j}) x_{2j}] - \pi_0 \\
  y_{11} < x_{11}, \quad y_{21} \geq x_{21} \\
  y_{12} + y_{11} < x_{12} + x_{11}, \quad y_{22} < x_{22} 
\end{cases}
\]

\[
G_{3,4}^+ : \begin{cases} 
  s_{21} y_{21} - (p_{11} - r_1) y_{11} - \sum_{i=1}^2 (p_{i2} - r_i) y_{i2} \leq (p_{21} - c_{21} + s_{21}) x_{21} + \sum_{i=1}^2 (r_i - c_{i2}) x_{i2} + (r_1 - c_{11}) x_{11} - \pi_0 \\
  y_{11} < x_{11}, \quad y_{21} \geq x_{21} \\
  y_{12} + y_{11} < x_{12} + x_{11}, \quad y_{22} < x_{22} 
\end{cases}
\]
\[ \begin{align*}
G^+_1 : & \quad \begin{cases}
y_{11} < x_{11}, \ y_{21} < x_{21} \\
y_{12} + y_{11} \geq x_{12} + x_{11}, \ y_{22} + y_{21} \geq x_{22} + x_{21}
\end{cases} \\
\sum_{i=1}^{2} \sum_{j=1}^{2} [p_{ij} - c_{ij} + s_{ij}]x_{ij} - \pi_0
\end{align*} \]

\[ \begin{align*}
G^+_2 : & \quad \begin{cases}
y_{11} < x_{11}, \ y_{21} \geq x_{21} \\
y_{12} + y_{11} \geq x_{12} + x_{11}, \ y_{22} + y_{21} < x_{22} + x_{21}
\end{cases} \\
(p_{12} + s_{12} - p_{11})y_{11} - (p_{21} - r_2)y_{21} + s_{12}y_{12} - (p_{22} - r_2)y_{22} \leq (p_{12} + s_{12} - c_{11})x_{11} + (r_2 - c_{22})x_{22} - \pi_0
\end{align*} \]

\[ \begin{align*}
G^+_3 : & \quad \begin{cases}
y_{11} < x_{11}, \ y_{21} < x_{21} \\
y_{12} + y_{11} < x_{12} + x_{11}, \ y_{22} + y_{21} \geq x_{22} + x_{21}
\end{cases} \\
(p_{22} + s_{22} - p_{21})y_{21} - \sum_{j=2}^{2} (p_{1j} - r_1)y_{ij} + s_{22}y_{22} \leq (r_1 - c_{11})x_{11} + (p_{22} - c_{22} + s_{22})x_{22} - \pi_0
\end{align*} \]

\[ \begin{align*}
G^+_4 : & \quad \begin{cases}
y_{11} < x_{11}, \ y_{21} < x_{21} \\
y_{12} + y_{11} < x_{12} + x_{11}, \ y_{22} + y_{21} < x_{22} + x_{21}
\end{cases} \\
\sum_{i=1}^{2} \sum_{j=1}^{2} (c_{ij} - r_i)x_{ij} \leq \sum_{i=1}^{2} \sum_{j=1}^{2} (p_{ij} - r_i)y_{ij} - \pi_0
\end{align*} \]

**Appendix B.** The integral region \( G_k^- \) of the expected value function \( E(V_k^-(\Delta \pi_k)) \) in Case \( k = d.e \ and \ d, \ e = 1, 2, 3, 4 \), where the loss-averse retailer perceives losses.
\[ G_{2,1} : \begin{cases} y_{11} \geq x_{11}, \ y_{21} < x_{21} \\ y_{12} \geq x_{12}, \ y_{22} + y_{21} \geq x_{22} + x_{21} \\ 2 \sum_{i=1}^{2} s_{i2}y_{i2} + s_{11}y_{11} - \left(p_{21} - p_{22} + s_{22}\right)y_{21} > 2 \sum_{i=1}^{2} \left(p_{i2} - c_{i2} + s_{i2}\right)x_{i2} + \\
\left(p_{11} - c_{11} + s_{11}\right)x_{11} + \left(p_{22} + c_{21} - s_{22}\right)x_{22} - \pi_0 \\
y_{11} \geq x_{11}, \ y_{21} \geq x_{21} \\ y_{12} \geq x_{12}, \ y_{22} + y_{21} < x_{22} + x_{21} \\
y_{11} \geq x_{11}, \ y_{21} \geq x_{21} \\ y_{12} < x_{12}, \ y_{22} + y_{21} \geq x_{22} + x_{21} \\
s_{11}y_{11} - \left(p_{21} - p_{22} + s_{22}\right)y_{21} - \left(p_{12} - r_{1}\right)y_{12} + s_{22}y_{22} \leq \left(r_{1} - c_{12}\right)x_{12} + \\
\left(p_{11} - c_{11} + s_{11}\right)x_{11} + \left(p_{22} + s_{22} - c_{21}\right)x_{21} + \left(p_{22} - c_{22} + s_{22}\right)x_{22} - \pi_0 \\
y_{11} \geq x_{11}, \ y_{21} \geq x_{21} \\ y_{12} < x_{12}, \ y_{22} + y_{21} < x_{22} + x_{21} \\
y_{11} \geq x_{11}, \ y_{21} \geq x_{21} \\ y_{12} < x_{12}, \ y_{22} + y_{21} > x_{21} + x_{22} \\
s_{11}y_{11} - \left(p_{21} - r_{2}\right)y_{21} - \sum_{i=1}^{2} \left(p_{i2} - r_{i}\right)y_{2i} > \left(r_{2} - c_{21}\right)x_{21} + \\
2 \sum_{i=1}^{2} \left(r_{i} - c_{i2}\right)x_{i2} + \left(p_{11} - c_{11} + s_{11}\right)x_{11} - \pi_0 \\
y_{11} < x_{11}, \ y_{21} \geq x_{21} \\ y_{12} + y_{11} \geq x_{12} + x_{11}, \ y_{22} \geq x_{22} \\
2 \sum_{i=1}^{2} s_{i2}y_{i2} - \left(p_{11} - p_{12} - s_{12}\right)y_{11} + s_{21}y_{21} > 2 \sum_{i=1}^{2} \left(p_{i2} - c_{i2} + s_{i2}\right)x_{i2} + \\
\left(p_{12} - c_{11} + s_{12}\right)x_{11} + \left(p_{21} - c_{21} + s_{21}\right)x_{21} - \pi_0 \\
y_{11} < x_{11}, \ y_{21} \geq x_{21} \\ y_{12} + y_{11} \geq x_{12} + x_{11}, \ y_{22} < x_{22} \\
s_{21}y_{21} - \left(p_{11} - p_{12} - s_{12}\right)y_{11} + s_{21}y_{21} \geq \left(p_{22} - c_{22}\right)x_{22} + \\
\left(p_{12} - c_{11} + s_{12}\right)x_{11} + \left(p_{21} - c_{21} + s_{21}\right)x_{21} + \left(p_{12} - c_{12} + s_{12}\right)x_{12} - \pi_0 \\
y_{11} < x_{11}, \ y_{21} \geq x_{21} \\ y_{12} + y_{11} < x_{12} + x_{11}, \ y_{22} \geq x_{22} \\
2 \sum_{j=1}^{2} \left[s_{2j}y_{2j} - \left(p_{1j} - r_{1}\right)y_{1j}\right] > 2 \sum_{j=1}^{2} \left[r_{1} - c_{1j}\right]x_{1j} + \left(p_{2j} - c_{2j} + s_{2j}\right)x_{2j} - \pi_0 \\
y_{11} < x_{11}, \ y_{21} \geq x_{21} \\ y_{12} + y_{11} < x_{12} + x_{11}, \ y_{22} < x_{22} \\
s_{21}y_{21} - \left(p_{11} - r_{1}\right)y_{11} - \sum_{i=1}^{2} \left(p_{i2} - r_{i}\right)y_{i2} > \left(p_{21} - c_{21} + s_{21}\right)x_{21} + \\
2 \sum_{i=1}^{2} \left(r_{i} - c_{i2}\right)x_{i2} + \left(r_{1} - c_{11}\right)x_{11} - \pi_0 \\
y_{11} < x_{11}, \ y_{21} < x_{21} \\ y_{12} + y_{11} < x_{12} + x_{11}, \ y_{22} < x_{22} \\
s_{21}y_{21} - \left(p_{11} - r_{1}\right)y_{11} - \sum_{i=1}^{2} \left(p_{i2} - r_{i}\right)y_{i2} > \left(p_{21} - c_{21} + s_{21}\right)x_{21} + \\
2 \sum_{i=1}^{2} \left(r_{i} - c_{i2}\right)x_{i2} + \left(r_{1} - c_{11}\right)x_{11} - \pi_0 \\
y_{11} < x_{11}, \ y_{21} < x_{21} \\ y_{12} + y_{11} \geq x_{12} + x_{11}, \ y_{22} + y_{21} \geq x_{22} + x_{21} \\
2 \sum_{i=1}^{2} \left[p_{i2} - p_{i1} + s_{i2}\right]y_{i1} + s_{i2}y_{i2} > \\
2 \sum_{i=1}^{2} \left[p_{i2} - c_{i2} + s_{i2}\right]x_{i2} + \left(p_{i2} + s_{i2} - c_{i1}\right)x_{i1} - \pi_0 \end{cases} \]
\[ G_{4,2} : \begin{cases} y_{11} < x_{11}, \ y_{21} \geq x_{21} \\ y_{12} + y_{11} \geq x_{12} + x_{11}, \ y_{22} + y_{21} < x_{22} + x_{21} \\ (p_{11} + s_{12} - p_{11})y_{11} - (p_{21} - r_{1})y_{21} + s_{12}y_{12} - (p_{22} - r_{1})y_{22} > \\ (p_{12} + s_{12} - c_{11})x_{11} + (r_{2} - c_{21})x_{21} + (p_{12} - c_{12} + s_{12})x_{12} - \\ (c_{22} - r_{2})x_{22} - \pi_{0} \end{cases} \]

\[ G_{4,3} : \begin{cases} y_{11} < x_{11}, \ y_{21} < x_{21} \\ y_{12} + y_{11} < x_{12} + x_{11}, \ y_{22} + y_{21} \geq x_{22} + x_{21} \\ (p_{22} + s_{22} - p_{21})y_{21} - \frac{2}{2} \sum_{j=2} (p_{1j} - r_{1})y_{1j} + s_{22}y_{22} > (r_{1} - c_{11})x_{11} + \\ (p_{22} - c_{21} + s_{22})x_{21} - (c_{12} - r_{1})x_{12} + (p_{22} - c_{22} + s_{22})x_{22} - \pi_{0} \end{cases} \]

\[ G_{4,4} : \begin{cases} y_{11} < x_{11}, \ y_{21} < x_{21} \\ y_{12} + y_{11} < x_{12} + x_{11}, \ y_{22} + y_{21} < x_{22} + x_{21} \\ \frac{2}{2} \sum_{i=1} \frac{2}{j=1} (c_{ij} - r_{i})x_{ij} > \frac{2}{2} \sum_{i=1} \frac{2}{j=1} (p_{ij} - r_{i})y_{ij} - \pi_{0} \end{cases} \]

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