Universal models for effective constitutive relations of laminated composites with finite strains

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Abstract. The results of the development of the theory of constructing effective constitutive relations for laminated composites with finite deformations are presented. The theory is based on the application of the method of asymptotic averaging for periodic structures and the use of universal models of constitutive relations for composite layers, proposed by Yu.I. Dimitrienko. Universal models allow us to formulate the constitutive relations simultaneously for all the main conjugated stress – strain tensors pairs. The method of asymptotic averaging makes it possible to obtain a relationship between the effective constitutive relations of the composite and its constituent layers. Cases of incompressible layers and layers with compressibility are considered. A numerical algorithm for constructing effective deformation diagrams of layered composites and calculating the material constants in the effective constitutive relations of an anisotropic composite is proposed. Numerical examples of method implementation and numerical algorithms are given.

1. Introduction
There are a limited quantity of the methods for calculating the deformation diagrams of composites with nonlinearly elastic characteristics of individual phases [1-5], much smaller than for linearly elastic composites [6-10]. The most promising method for calculating effective deformation diagrams is the method of asymptotic averaging [11,12]. In the works of Yu.I. Dimitrienko, this method was used to calculate the nonlinear-elastic properties of layered composites with large deformations [13,14]. In this paper, we propose the further development of this method for calculating the deformation diagrams of incompressible layered composites with finite deformations. For the defining relations of the phases of the composite, the so-called universal representations of models of incompressible nonlinear-elastic media with finite deformations, proposed by Yu. Dimitrienko [13,14], are used. These models make it possible to solve the problems of the non-linear elasticity theory at once for several classes of media models with finite deformations within a single methodological approach.

2. Initial non-linear elasticity problem for incompressible composites with periodic structure and finite deformations. Consider a layered composite, which in the initial configuration \( K \) is a region - a system of parallel periodically repeating \( N \) layers orthogonal to the direction \( 3 \overrightarrow{O} \), where \( X^i \) are Lagrangian coordinates. We introduce \( \kappa = l/L << 1 \) - a small parameter, as the ratio \( l \) of the thickness of the periodicity cell (PC) to the total thickness of the composite \( L \), and also introduce \( \zeta \) - the local Lagrangian coordinate in: \( \zeta = \overline{X}^3/\kappa, \quad \overline{X}^i = X^i/L \). We formulate for the layered composite the problem of the nonlinear elasticity theory in the Lagrangian description in the general formulation [14] using universal models of media with finite deformations for incompressible elastic media.
\[ \nabla_0 P^{ij}_0 = 0, \quad X^0_i \in V^0, \]
\[ P^{ij}_0 = -pF^{-1}_{ij} + \mathcal{F}^{(n)}_{ij}(F_{k}^{i}, \xi), \quad X^0_i \in V \cup \Sigma, \]
\[ F_{i}^{k} = \delta_{i}^{k} + \nabla_0 u_{k}^{i}, \quad X^0_i \in V \cup \Sigma, \]
\[ \det(F_{i}^{k}) = 1 \]
\[ \n_i[P^{ij}_0] = 0, \quad [u^{i}] = 0, \quad X^0_i \in \Sigma_0, \]
\[ \n_i P^{ij}_0 = t^{i}_j, \quad X^0_i \in \Sigma_1, \quad u^{i}_j = u^{i}_j, \quad X^0_i \in \Sigma_2, \]

here \( \Sigma_0 \) - the interfaces of \( \alpha \) and \( \beta \) components of the composite, \( \Sigma_1 \) and \( \Sigma_2 \) - the outer surfaces of the composite, \( [P^{ij}_0] \) - the discontinuity of the functions, \( p \) - the hydrostatic pressure, \( P^{ij}_0 \) - the components of the Piola-Kirchhoff stress tensor, \( F_{i}^{k} \) - the strain gradient, \( u_{k}^{i} \) - the displacement vector, \( n_i \) - the normal vector, \( t^{i}_j \) - vector of specified displacements, \( \nabla_0 = \frac{\partial}{\partial X^i} \) - nabla-operator. In relation (1), we denote the tensor \( \mathcal{F}^{(n)}_{ij} \) describing the nonlinearly elastic properties of the phases of the composite according to the models Bn of elastic incompressible media with finite deformations [13,14].

3. Asymptotic expansions.

We seek a solution of the problem (1) for a composite of a periodic structure in the form of asymptotic expansions in the parameter.

\[ u_{k}^{i}(X^i, \xi) = u^{k(i)}(X^i) + \kappa u^{k(1)}(X^i, \xi) + \kappa^2 \ldots \]
\[ p(X^i, \xi) = p^{0(0)}(X^i, \xi) + \kappa p^{0(1)}(X^i, \xi) + \kappa^2 \ldots \]

Substituting (2) into (1), we find the asymptotic expansion for the deformation gradient defining the relations and the Piola-Kirchhoff tensor

\[ F_{i}^{k} = F_{i}^{k(0)}(X^i, \xi) + \kappa F_{i}^{k(1)}(X^i, \xi) + \kappa^2 \ldots \]
\[ F_{i}^{k(0)}(X^i, \xi) = \delta_{i}^{k} + u^{k(0)}_j + u^{k(1)}_j \delta_{j3}, \quad F_{i}^{k(1)} = u^{k(1)}_j + u^{k(2)}_j \delta_{j3}, \]
\[ P^{i} = P^{i(0)}(X^k, \xi) + \kappa P^{i(1)}(X^k, \xi) + \kappa^2 \ldots, \]
\[ P^{0(0)} = -p^{0(0)}(F^{-1})^{0(0)} + \mathcal{F}_{ij}^{(n)}(F_{i}^{k(0)}, \xi). \]

4. Local problems of nonlinear elasticity theory for incompressible media.

Substituting expansions (3) into system (1) after equating the terms at identical powers to zero, we obtain a recurrent sequence of local problems of non-linear elasticity theory for incompressible media. The problem has the following form

\[ P^{0(0)}_{i3} = 0, \]
\[ P^{0(0)} = -p^{0(0)}(F^{-1})^{0(0)} + \mathcal{F}_{ij}^{(n)}(F_{i}^{k(0)}, \xi), \]
\[ F_{i}^{k(0)}(X^i, \xi) = F_{i}^{k} + u^{k(1)}_j \delta_{j3}, \]
\[ \det(F^{(0(k)}) = 1 \]
\[ [P^{0(0)}] = 0, \quad [u^{k(1)}] = 0, \quad \xi = \xi^0, \quad \alpha = 1, \ldots, n-1 \]
\[ \left\{ u^{k(1)} \right\} = 0, \quad \left\{ [u^{k(1)}]_j \right\} = 0. \]
Here we have introduced the averaging operation over the PC \( V_\xi \quad \langle u^{(1)} \rangle = \int_{-0.5}^{0.5} u^{(1)}(1,1,1) d_\xi \), and denotes the \( F_{ij}^k = \delta_{ij} + u^{(1)}_{ij} \) - average gradient of the deformation of the composite, and also \( \overline{p} \) - the mean hydrostatic pressure.

5. The averaged problem of nonlinear elasticity theory for a composite.

Averaging the system of equations (1), we obtain the averaged nonlinear elasticity problem for a composite

\[
\langle P^{(0)} \rangle = 0, \\
\det(F_{ij}^k) = 1, \\
\langle P^{(0)} \rangle = -\overline{p}(F_{ij}^{-1}) + \langle \mathcal{A}^{(0)}(F_{ij}^{k(0)}, \xi) \rangle, \\
F_{ij}^k = \delta_{ij} + u^{(1)}_{ij}, \\
0 < P^{(0)} > = < t^I >, \quad X^I \in \Sigma_1, \quad u^{(0)} = < u^I >, \quad X^I \in \Sigma_2.
\]

6. Results of numerical simulation.

As specific models of nonlinearly elastic media, a model of incompressible media of class Bv [13] (\( n = 5 \)) (the Mooney model) was considered. The cell of the periodicity of the composite consisted of 3 layers. Figure 1, as an example, shows graphs of composite functions \( P_{ij}^{(0)} = \bar{P}_{ij}^0(F_{ij}^k) \) under uniaxial tension, constructed using the algorithm described above for individual composite layers and for the composite itself.

![Figure 1](image)

Figure 1 - Diagrams of deformation for a layered composite and individual constituents of its layers, constructed using the developed method

7. Conclusions

A variant of the method of asymptotic averaging is proposed for layered elastic incompressible composites with finite deformations and a periodic structure. A universal representation of the defining relations for a complex of different models of media with finite deformations is used.

The algorithm of the problem on the periodicity cell for layered incompressible composite materials with finite deformations is developed. This algorithm allows one to calculate effective deformation diagrams of layered composites with finite deformations connecting the components of the averaged Piola-Kirchhoff stress tensors and the strain gradient as well as the stresses in the composite layers.
The presented example of the construction of the deformation diagram of a composite with finite deformations under uniaxial tension shows the feasibility of the proposed algorithm, and its possibilities for modeling the nonlinear elastic properties of composites based on the properties of its constituent layers.

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