Linear and angular momentum of electromagnetic fields generated by an arbitrary distribution of charge and current densities at rest

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Starting from Stratton-Panofsky-Phillips-Jefimenko equations for the electric and magnetic fields generated by completely arbitrary charge and current density distributions at rest, we derive far-zone approximations for the fields, containing all components, dominant as well as sub-dominant. Using these approximate formulas, we derive general formulas for the total electromagnetic linear momentum and angular momentum, valid at large distances from arbitrary, non-moving charge and current sources.

I. INTRODUCTION

Standard classical electrodynamics offers different methods for calculating electromagnetic fields and physical observables derived from them. Usually the methods amount to solving the Maxwell equations in one form or another, or to find the potentials, choose a gauge and then perform the calculations.

In recent years, the use of alternative methods, based on integral equations that relate the fields directly to their charge and current sources, have gained attraction; see Ref. 1 and references cited therein. Here we show how these methods can be used to directly derive general expressions for the linear and angular momentum of the electromagnetic field that approximate these quantities at large distances from the sources. Among other things, these expressions show that both linear momentum and angular momentum can be used to transfer information wirelessly in free space.

In Chapter VIII of his 1941 textbook, Stratton² calculates, without the intervention of potentials, temporal Fourier transform expressions for the retarded electric and magnetic fields, \( \mathbf{E}(t, \mathbf{x}) \) and \( \mathbf{B}(t, \mathbf{x}) \), respectively, generated by arbitrary distributions of charge and current densities at rest relative to the observer. In Chapter 14 of the second edition of their textbook on electrodynamics, published in 1962, Panofsky and Phillips³ present a variant form of these expressions, and give them also in ordinary space-time coordinates. Four years later, Jefimenko published his electrodynamics textbook⁴ where the Panofsky and Phillips expressions for the retarded \( \mathbf{E} \) and \( \mathbf{B} \) fields were given in Chapter 15. These expressions are sometimes referred to as the Jefimenko equations,⁵,⁶ but should, perhaps, rather be called the Stratton-Panofsky-Phillips-Jefimenko (SPPJ) equations, a name we will use throughout.

Introducing the notation \( \mathbf{x} \) for the observer’s coordinate, \( \mathbf{x}' \) for the source coordinate, and

\[
\mathbf{t} = t - \frac{|\mathbf{x} - \mathbf{x}'|}{c} \tag{1}
\]

for the retarded time⁷ relative to the source point where \( t \) is the observer’s time and \( c \) is the speed of light, the SPPJ equations for the retarded electric and magnetic fields can, in obvious notation, be written

\[
\mathbf{E}(t, \mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int_{V'} d^3\mathbf{x}' \frac{\rho(t_{\text{ret}}, \mathbf{x}') \mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^2} d^3\mathbf{x}' + \frac{1}{4\pi\varepsilon_0 c} \int_{V'} d^3\mathbf{x}' \frac{\rho(t_{\text{ret}}, \mathbf{x}') \mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}' \tag{2a}
\]

and

\[
\mathbf{B}(t, \mathbf{x}) = \frac{1}{4\pi\varepsilon_0 c^2} \int_{V'} d^3\mathbf{x}' \frac{\mathbf{j}(t_{\text{ret}}, \mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^2} d^3\mathbf{x}' + \frac{1}{4\pi\varepsilon_0 c} \int_{V'} d^3\mathbf{x}' \frac{\mathbf{j}(t_{\text{ret}}, \mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}' \tag{2b}
\]

respectively. By introducing the vector

\[
\mathbf{T}(t_{\text{ret}}, \mathbf{x}', \mathbf{x}) = \frac{\rho(t_{\text{ret}}, \mathbf{x}') |\mathbf{x} - \mathbf{x}'|}{|\mathbf{x} - \mathbf{x}'|^2} \mathbf{x} - \mathbf{x}' \tag{3}
\]

\[
+ \frac{1}{c |\mathbf{x} - \mathbf{x}'|^2} \left( \mathbf{j}(t_{\text{ret}}, \mathbf{x}') \times \mathbf{x} - \mathbf{x}' \right) \times \mathbf{x} - \mathbf{x}' \tag{3a}
\]

\[
+ \frac{1}{c |\mathbf{x} - \mathbf{x}'|^2} \left( \mathbf{j}(t_{\text{ret}}, \mathbf{x}) \times \mathbf{x} - \mathbf{x}' \right) \times \mathbf{x} - \mathbf{x}' \tag{3b}
\]

\[
+ \frac{1}{c^2 |\mathbf{x} - \mathbf{x}'|^2} \left( \mathbf{j}(t_{\text{ret}}, \mathbf{x}) \times \mathbf{x} - \mathbf{x}' \right) \times \mathbf{x} - \mathbf{x}' \tag{3c}
\]
the SPPJ equations (2) can be cast into the more compact and symmetric form

\[ E(t, x) = \frac{1}{4\pi\varepsilon_0} \int_{V'} d^3x' T(t_{ret}, x', x) \]  
\[ B(t, x) = \frac{1}{4\pi\varepsilon_0} \int_{V'} d^3x' \frac{x - x'}{|x - x'|^3} \times T(t_{ret}, x', x) \]

so that Eq. (2a) can be written in the alternative form

\[ E(t, x) = \frac{1}{4\pi\varepsilon_0} \int_{V'} d^3x' \frac{\rho(t_{ret}, x') (x - x')}{|x - x'|^3} \]
\[ + \frac{1}{4\pi\varepsilon_0} \int_{V'} d^3x' \frac{[j(t_{ret}, x')] \cdot (x - x')}{|x - x'|^4} \]
\[ + \frac{1}{4\pi\varepsilon_0 c} \int_{V'} d^3x' \frac{[j(t_{ret}, x')] \times (x - x')}{|x - x'|^4} \]
\[ + \frac{1}{4\pi\varepsilon_0 c^2} \int_{V'} d^3x' \frac{[j(t_{ret}, x')] \times (x - x')}{|x - x'|^3} \times (x - x') \]

which more clearly exhibits the relation between the various components of the retarded \( E \) field.

In Section II, far-zone formulas for the \( E \) and \( B \) fields, based on the SPPJ equations, are introduced. These equations are used in Section III and IV to derive far-zone expressions for the electromagnetic linear momentum and angular momentum, respectively. In Section V, a summary and conclusions are made.

II. THE FIELDS AT LARGE DISTANCES FROM THE SOURCE

Following Panofsky and Phillips, we introduce the temporal Fourier component representations of Eq. (5) for \( E(t, x) \) and of Eq. (2b) for \( B(t, x) \),

\[ E_{\omega}(x) = \frac{1}{4\pi\varepsilon_0} \left( \int_{V'} d^3x' \rho_{\omega}(x') e^{ik|x-x'|} (x - x') \right) \]
\[ + \frac{1}{c} \int_{V'} d^3x' \left[ j_{\omega}(x') e^{ik|x-x'|} \cdot (x - x') \right] (x - x') \]
\[ + \frac{1}{c} \int_{V'} d^3x' \left[ j_{\omega}(x') e^{ik|x-x'|} \times (x - x') \right] \]
\[ - \frac{ik}{c} \int_{V'} d^3x' \left[ j_{\omega}(x') e^{ik|x-x'|} \times (x - x') \right] \times (x - x') \]

and

\[ B_{\omega}(x) = \frac{1}{4\pi\varepsilon_0 c^2} \left( \int_{V'} d^3x' \left[ j_{\omega}(x') e^{ik|x-x'|} \times (x - x') \right] \right) \]
\[ + \int_{V'} d^3x' \frac{(-ik) j_{\omega}(x') e^{ik|x-x'|} \times (x - x')}{|x - x'|^3} \]

respectively.

As illustrated in Fig. 1, the observation point \( x \) is assumed to be located far away from the sources, which in turn are assumed to be localized near a point \( x_0 \) inside a volume \( V' \) that has such a limited spatial extent that \( \sup |x' - x_0| \ll \inf |x - x'| \), and the integration surface \( S \), centered on \( x_0 \), has a large enough radius \( |x - x_0| \gg \sup |x' - x_0| \). Then one can make the usual far-zone approximation

\[ e^{ik|x-x'|} \approx \frac{e^{ik|x-x_0| - ik(x' - x_0)}}{|x - x_0|} \]

The corresponding approximate retarded time is

\[ t_{ret}' = t' - \frac{|x - x_0|}{c} \]

where

\[ t' \approx t + \frac{\hat{k} \cdot (x' - x_0)}{c} \]

so that Eq. (5) for the electric field and Eq. (2b) for the magnetic field can, in complex notation, be approximated by

\[ E(t, x) \approx \frac{1}{4\pi\varepsilon_0} \left( \int_{V'} d^3x' \rho(t, x') \hat{k} \right) \]
\[ + \frac{1}{4\pi\varepsilon_0 c} \left( \int_{V'} d^3x' [j(t, x') \cdot \hat{k}] \hat{k} \right) \]
\[ + \frac{1}{4\pi\varepsilon_0 c} \left( \int_{V'} d^3x' [j(t, x') \times \hat{k}] \times \hat{k} \right) \]
\[ + \frac{1}{4\pi\varepsilon_0 c^2} \left( \int_{V'} d^3x' [j(t, x') \times \hat{k}] \times \hat{k} \right) \]
and

\[ \mathbf{B}(t, x) \approx \frac{1}{4\pi\varepsilon_0 c^2} \frac{e^{ik|x-x_0|}}{|x-x_0|} \int_{V'} d^3x' \mathbf{j}(t', x') \times \hat{k} + \frac{1}{4\pi\varepsilon_0 c^3} \frac{e^{ik|x-x_0|}}{|x-x_0|} \int_{V'} d^3x' \mathbf{j}(t', x') \times \hat{k} \]  

(10b)

as obtained from inverse Fourier transforming of Eqs. (6) with Eq. (7) inserted.

In many cases, when one wants to calculate electromagnetic observables in the far zone, the approximate Eqs. (10) are accurate enough. At the same time they are easier to evaluate, and give results which are physically more lucid than the SPPJ equations (2). Below, we follow this scheme to derive general, untruncated expressions for the linear and angular momentum, valid far away from the source volume \( V' \).

### III. LINEAR MOMENTUM

The linear momentum conservation law \( \frac{dp_{\text{mech}}}{dt} + \frac{dp_{\text{field}}}{dt} + \oint_S d^2x \hat{n} \cdot \mathbf{T} = 0 \) (11)
describes the balance between the time rate of change of the mechanical linear momentum \( p_{\text{mech}} \), i.e., the mechanical force, the time rate of change of the electromagnetic field linear momentum \( p_{\text{field}} \), and the flow of linear momentum across the surface \( S \) described by the linear momentum flux tensor density \( \mathbf{T} \) (the negative of Maxwell’s stress tensor) of the electromagnetic field. For example, this law describes how the translational motion of charges (e.g., a transmitting antenna current) at one point in space induces a translational motion, via force action, of remote charges (e.g., a receiving antenna current). This is one of several physical mechanisms by which information can be transferred electromagnetically through free space and the primary physical basis for current radio astronomy and other radio-based science, as well as today’s wireless communications technology.

The linear momentum of the electromagnetic field is defined as \( \hat{\mathbf{z}} \)

\[ p_{\text{field}} = \int_{V'} d^3x' \hat{\mathbf{g}}_{\text{field}} \]  

(12)

where

\[ \hat{\mathbf{g}}_{\text{field}} = \varepsilon_0 (\mathbf{E} \times \mathbf{B}) = \frac{S}{c^2} \]  

(13)
is the electromagnetic field linear momentum density and \( S \) the Poynting vector.

Applying the paraxial approximation\(^5\)

\[ \hat{k} = \frac{x-x'}{|x-x'|} \approx \frac{x-x_0}{|x-x_0|} = \hat{n} \]  

(14)
of the wave vector direction, and moving the constant unit vector \( \hat{n} \), from \( x_0 \) to the observation point \( x \), outside each of the six constituent integrals in the following way

\[ \int_{V'} d^3x' \rho(t', x') \hat{n} \approx \left( \int_{V'} d^3x' \rho(t', x') \right) \hat{n} = q(t') \hat{n} \]  

(15a)

\[ \int_{V'} d^3x' \hat{j}(t', x') \times \hat{n} \approx \left( \int_{V'} d^3x' \hat{j}(t', x') \right) \times \hat{n} = I(t') \times \hat{n} \]  

(15b)

\[ \int_{V'} d^3x' \hat{j}(t', x') \cdot \hat{k} \approx \left[ \left( \int_{V'} d^3x' \hat{j}(t', x') \right) \cdot \hat{n} \right] \hat{n} = (\mathbf{I} \cdot \hat{n}) \hat{n} = \hat{I}(t') \hat{n} \]  

(15d)

\[ \int_{V'} d^3x' \hat{j}(t', x') \times \hat{k} \approx \left[ \left( \int_{V'} d^3x' \hat{j}(t', x') \right) \times \hat{n} \right] \hat{n} = (\mathbf{I} \times \hat{n}) \hat{n} = \hat{I}_n(t') \hat{n} \]  

(15e)

one obtains, after some vector algebraic simplifications, the following general, untruncated expression for the cycle averaged linear momentum density

\[ \langle g_{\text{field}} \rangle = \frac{1}{32\pi^2\varepsilon_0 c^3} \left( \frac{|I|^2 - |I_n|^2}{c^2 |x-x_0|^2} \hat{n} \right. \]

\[ + \frac{2\text{Re}\{\mathbf{I} \cdot \mathbf{I}^* - \mathbf{I}_n \mathbf{I}^*_n\} - \text{Re}\{\langle (cq+I_n)\mathbf{I}^*_n\rangle\hat{n}}}{|x-x_0|^3} \]

\[ + \frac{\text{Re}\{\langle cq+I_n\rangle \mathbf{I}^*_n\} + \text{Re}\{\langle cq+I_n\rangle \mathbf{I}^*_n\}}{|x-x_0|^4} \]  

(16)

This approximate expression is valid far away from the source volume \( V' \).

At very large distances \( r \equiv |x-x_0| \) from the source volume \( V' \), we see that the linear momentum density, and consequently also the Poynting vector, is accurately represented by the first term, which is radial (along \( \hat{n} \)) and falls off as \( 1/r^3 \), while the successive non-radial terms fall off as \( 1/r^5 \) or as \( 1/r^4 \), respectively. However, at finite distances the Poynting vector is not radial but has transverse components. Of course, these transverse components, which make the Poynting vector spiral around \( \hat{n} \), are difficult to observe because of their smallness. Furthermore, since the non-radial terms fall off faster than \( 1/r^2 \), the spiraling of the Poynting vector will diminish with distance from the source.


IV. ANGULAR MOMENTUM

The angular momentum conservation law\(^8\)

\[
\frac{dJ_{\text{mech}}(x_0)}{dt} + \frac{dJ_{\text{field}}(x_0)}{dt} + \oint_S d^3\gamma \vec{n}' \cdot \mathbf{K}(x_0) = 0 \tag{17}
\]
describes the balance between the time rate of change of the mechanical angular momentum \(J_{\text{mech}}\), i.e., mechanical torque, the time rate of change of the electromagnetic field angular momentum \(J_{\text{field}}\), and the flow of angular momentum across the surface \(S\) described by the angular momentum flux tensor \(\mathbf{K} = (\mathbf{x} - x_0) \times \mathbf{T}\), all about the point \(x_0\). This law describes how the rotational (spin, orbital) motion of charges at one point in space induces a rotational motion, via torque action, of charges at other points in space. This is an additional physical mechanism by which information can be transferred electromagnetically through free space but one that is used only partially and sparingly in today’s radio-based research and wireless communication technology.

The field angular momentum about a point \(x_0\) is defined as\(^3,8,9\)

\[
J_{\text{field}}(x_0) = \int_{\Omega'} d^3\gamma \, \mathbf{h}_{\text{field}}(x_0) \tag{18}
\]

where

\[
\mathbf{h}_{\text{field}}(x_0) = (\mathbf{x} - x_0) \times \mathbf{g}_{\text{field}} \tag{19}
\]
is the electromagnetic angular momentum density. For a single temporal Fourier component in complex notation and a beam geometry, Eq. (18) can be written\(^{10,11}\)

\[
J_{\text{field}} = \mathbf{L}_{\text{field}} + \mathbf{\Sigma}_{\text{field}} \tag{20}
\]

where

\[
\mathbf{L}_{\text{field}} = -\frac{\varepsilon_0}{2\omega} \int_{\Omega'} d^3\gamma \, E_{\gamma}^* \left[(\mathbf{x}' - x_0) \times \nabla\right] E_i \tag{21a}
\]

\[
\mathbf{\Sigma}_{\text{field}} = -\frac{\mu_0}{2\omega} \int_{\Omega'} d^3\gamma \, (E^* \times \mathbf{E}) \tag{21b}
\]
The first term is the electromagnetic orbital angular momentum (OAM), which describes the vorticity of the EM field, and the second term is the electromagnetic spin angular momentum (SAM), which describes wave polarization (left or right circular).

Since \(\mathbf{x} - x_0 = |\mathbf{x} - x_0| \hat{\mathbf{n}}\), the vector product in Eq. (19) gives vanishing contributions when operating on the radial terms (terms parallel to \(\hat{\mathbf{n}}\)) in Eq. (16). As a result, the complete cycle averaged far-zone expression for a frequency component \(\omega\) of the electromagnetic angular momentum density generated by arbitrary charge and current sources is simply

\[
\left\langle \mathbf{h}_{\text{field}}(x_0) \right\rangle = \frac{1}{32\pi^2\varepsilon_0 c^3} \left( \frac{\hat{\mathbf{n}} \times \text{Re} \left\{ (c q + I_n) \hat{\mathbf{i}} \right\}}{c |\mathbf{x} - x_0|^2} + \frac{\hat{\mathbf{n}} \times \text{Re} \left\{ (c q + I_n) \hat{\mathbf{i}} \right\}}{|\mathbf{x} - x_0|^3} \right) \tag{22}
\]

We see that at very large distances \(r\), the angular momentum density falls off as \(1/r^2\), i.e., it has precisely the same behavior in the far zone as the linear momentum density and can therefore also transfer information wirelessly over large distances. The only difference is that while the direction of the linear momentum (Poynting vector) becomes purely radial at infinity, the angular momentum becomes perpendicular to the linear momentum, i.e. purely transverse, there.

V. SUMMARY AND CONCLUSIONS

We have shown how the SPPJ equations for the retarded electric and magnetic fields, generated by arbitrary charge and current distributions, can be used to derive general far-zone formulas for the concomitant electromagnetic linear and angular momentum.

From the far-zone approximations obtained for the linear momentum and angular momentum densities, Eq. (16) and Eq. (22), respectively, it is easy to see that both these physical quantities, to leading order, fall off as \(1/r^2\) with distance \(r\) from the source. Consequently, when integrated over a surface element \(r^2 \sin(\theta) d\theta d\phi\) of a large spherical shell, centered on the source, they behave as constants. Physically, this means that both linear and angular momentum can be carried all the way to infinity. While the time rate of change of the linear momentum provides the force that causes the charges in the EM sensor (e.g., a receiving antenna) perform translational (oscillating) motions, the angular momentum gives rise to a rotational (spinning and/or orbiting) motion.

However, while the linear momentum is in the far zone determined essentially by the dominant \(\mathbf{E}\) and \(\mathbf{B}\) far-zone fields that fall off as \(r^{-1}\), the far-zone angular momentum is not determined by far-zone fields, but rather by near-zone components of the \(\mathbf{E}\) field, that fall off as \(r^{-2}\), and the far-zone \(\mathbf{B}\) field. In fact, the far-zone \(\mathbf{E}\) field does not contribute to the far-zone angular momentum at all! This somewhat surprising result is a generalization to arbitrary EM fields of a result, originally derived by Abraham\(^{12}\) already in 1914, for the special case of pure dipole fields. To quote from page 916 of Abraham\(^{12}\) (using modern notation),\(^{13}\)

“Der elektrische Vektor \(\mathbf{E}\) hat (s. Gl. 6c unten) eine radiale Komponente, die freilich mit wachsendem \(r\) wie \(r^{-2}\) abnimmt, während die zum Fahrstrahl senkrechten Komponenten von \(\mathbf{E}\) wie \(r^{-1}\) abnehmen. Mit wachsender Entfernung von der Lichtquelle werden somit die Lichtwellen transversal, der Impuls wird parallel dem Fahrstrahl; es könnte darum nach (4) scheinen, als ob der Drehimpuls der Wellenzonen gleich null sei. Doch sieht man, daß dem nicht so ist, wenn man die Ordnung der fraglichen Größen bestimmt; wenn auch die longitudinalen Komponente von \(\mathbf{E}\) von der Ordnung \(r^{-2}\) ist, so ist doch das skalare Produkt von \(\mathbf{r}\) und \(\mathbf{E}\) von der Ordnung \(r^{-1}\), ebenso wie \(\mathbf{H}\). Aus (4c) folgt demnach, daß die Dichte des Impulsmomentes, ebenso wie die
The fact that electromagnetic linear and angular momentum can be transferred wirelessly over long distances (they are both irreversibly lost at infinity), means, e.g., that both of them can be used in wireless communications. Today, only the linear momentum and, in some cases the two orthogonal states of the spin part of the angular momentum (SAM, wave polarization), are used in wireless communication protocols, including such modern concepts as MIMO. However, wave polarization techniques do not always work reliably in real-world communication settings. A common problem is the depolarization of the radio beam, due to reflections and other interactions with the surroundings that can cause a conversion of polarization (SAM) into orbital angular momentum (OAM). The SAM is thereby diminished or lost, while the total angular momentum (SAM plus OAM) is still conserved. The use of OAM, in addition to SAM (polarization), could therefore be a remedy for this deficiency. Besides, and perhaps more importantly, OAM spans a denumerably infinite Hilbert space, providing, in principle, an infinite number of orthogonal basis states that can be used as the “letters” of a higher dimensional information alphabet. By adding OAM encoding to an EM beam, this beam can therefore transfer more wireless information per unit time and unit frequency than a radio beam carrying only linear momentum (power), or linear momentum plus SAM (polarization) as is the case for current communication techniques.

The possibility to use OAM encoding of EM beams for efficient free-space information transfer has been successfully demonstrated in several experiments, even at the single-photon level. A scheme for the use of OAM at radio frequencies in the gigahertz range and below, where digital radio methods are readily applicable to make the method feasible for practical technology applications, has been developed.

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