INTEGRAL DELIGNE COHOMOLOGY FOR REAL VARIETIES

PEDRO F. DOS SANTOS AND PAULO LIMA-FILHO

Abstract. We develop an integral version of Deligne cohomology for smooth proper real varieties $Y$. For this purpose the role played by singular cohomology in the complex case has to be replaced by ordinary bigraded $\mathcal{G}$-equivariant cohomology, where $\mathcal{G} := \text{Gal}(\mathbb{C}/\mathbb{R})$. This is the $\mathcal{G}$-equivariant counterpart of singular cohomology; cf. [LMMS]. We establish the basic properties of the theory and give a geometric interpretation for the groups in dimension 2 in weights 1 and 2.

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Date: May 2008.
The first author was supported in part by FCT (Portugal) through program POCTI.
1. Introduction

Integral Deligne cohomology \( H^n_{D/\mathbb{C}}(Y; \mathbb{Z}(p)) \) for a complex variety \( Y \) has been widely studied in the literature. When \( Y \) is smooth and proper, it is defined as the hypercohomology group \( H^n(Y, \mathbb{Z}(p)_{D/\mathbb{C}}) \) of the complex

\[
Z(p)_{D/\mathbb{C}} : 0 \to \mathbb{Z}(p) \to 0 \to \Omega^1 \to \cdots \to \Omega^{p-1},
\]

where \( \mathbb{Z}(p) \) is the constant sheaf \( (2\pi i)^p \mathbb{Z} \) and \( \Omega^j \) is the sheaf of holomorphic \( j \)-forms. More generally, one can extend this setting to arbitrary complex varieties using suitable compactifications and simplicial resolutions, along with forms with logarithmic poles, and the resulting theory is called Deligne-Beălinson cohomology; see [Beă84]. An excellent account can be found in [EV88]. The theory is complemented by a homological counterpart and together they are shown to satisfy Bloch-Ogus’ formalism [BO74]; cf. [Gil84] and [Jan88].

In this paper we develop an integral version of Deligne cohomology for smooth proper real varieties. For this purpose the role played by singular cohomology \( H^n_{\text{sing}}(Y; \mathbb{Z}(p)) \) in the complex case has to be replaced by ordinary bigraded \( \mathcal{G} \)-equivariant cohomology \( H^{n,p}_{\text{Br}}(Y(\mathbb{C}), \mathbb{Z}) \), where \( \mathcal{G} := \text{Gal}(\mathbb{C}/\mathbb{R}) \). This is the \( \mathcal{G} \)-equivariant counterpart of singular cohomology; cf. [LMM81].

We must emphasize that the ordinary equivariant cohomology of a \( \mathcal{G} \)-space \( X \) is not obtained as the singular cohomology of the Borel construction \( E\mathcal{G} \times_{\mathcal{G}} X \). In fact, the “Borel version” of equivariant cohomology is just \( H^{n,0}_{\text{Br}}(E\mathcal{G} \times_{\mathcal{G}} X, \mathbb{Z}) \) and, more generally, \( H^{n,p}_{\text{Br}}(E\mathcal{G} \times_{\mathcal{G}} X, \mathbb{Z}) \cong H^n(E\mathcal{G} \times_{\mathcal{G}} X; \mathbb{Z}(p)) \), where the latter denotes cohomology with twisted coefficients \( \mathbb{Z}(p) \).

The bigraded version stems from the \( RO(\mathcal{G}) \)-graded equivariant cohomology theories developed by J. Peter May et al. in [LMM81], [LMSM86] and [May96]. With weight \( p = 0 \) ordinary equivariant cohomology was developed in [Bre67] and hence, for simplicity, we call it Bredon cohomology even with non-zero weights, thus explaining the notation \( H^{n,p}_{\text{Br}}(X, \mathbb{Z}) \).

From a motivic standpoint, this difference can be expressed by saying that ordinary equivariant cohomology is to its Borel version as motivic cohomology is to its étale counterpart (étale-motivic cohomology; cf. [MVW06]). In fact, this setting is much more that a mere analogy, once one observes that the topological realization of the \( \mathbb{A}^1 \)-homotopy category of schemes over \( \mathbb{R} \) lands in the \( \mathcal{G} \)-equivariant homotopy category. This realization carries motivic to ordinary bigraded equivariant cohomology, and carries étale-motivic cohomology to Borel equivariant cohomology (with twisted coefficients). See [MV99] and [DI04] for details.

Let \( An_{/\mathbb{R}} \) denote the category of real holomorphic manifolds, whose objects are pairs \( (M, \sigma) \) consisting of a holomorphic manifold \( M \) together with an anti-holomorphic involution \( \sigma \), and whose morphisms are holomorphic maps commuting with the involutions. We consider those objects as having an action of \( \mathcal{G} \) and give \( An_{/\mathbb{R}} \) the structure of a site using equivariant open covers. In order to introduce Deligne cohomology for proper real
holomorphic manifolds, we construct a real version $\mathbb{Z}(p)_{\mathbb{D}/\mathbb{R}}$ of the Deligne complex $\mathbb{D}$ on the site $A_{\mathbb{N}/\mathbb{R}}$ and define

$$H^n_{\mathbb{D}/\mathbb{R}}(M; \mathbb{Z}(p)) := \mathbb{H}^n(M_{eq}; \mathbb{Z}(p)_{\mathbb{D}/\mathbb{R}}).$$

The construction of $\mathbb{Z}(p)_{\mathbb{D}/\mathbb{R}}$ involves a replacement of the constant sheaf $\mathbb{Z}(p)$ by a complex $\mathbb{Z}(p)_{\mathbb{B}r}$ which computes ordinary equivariant cohomology; cf. [Yan08]. The key technical construction is a morphism of complexes

$$\tau_p : \mathbb{Z}(p)_{\mathbb{B}r} \to \mathbb{E}^*,$$

where $\mathbb{E}^*$ is the complex of sheaves on $A_{\mathbb{N}/\mathbb{R}}$ such that $\mathbb{E}^j(X)$ consists of those smooth complex-valued differential $j$-forms invariant under the simultaneous action of $\mathbb{S}$ on $X \in A_{\mathbb{N}/\mathbb{R}}$ and $\mathbb{C}$. Using this complex we define

$$\mathbb{Z}(p)_{\mathbb{D}/\mathbb{R}} := \text{Cone}\left(\mathbb{Z}(p)_{\mathbb{B}r} \oplus F^p\mathbb{E}^* \xrightarrow{\tau_p - 1} \mathbb{E}^*\right)[-1]$$

where $F^p\mathbb{E}^*$ is the $p$-th piece of the Hodge filtration.

Amongst the basic properties of the theory are the evident long exact sequences and the fact that it recovers the usual theory for complex varieties. More precisely, given a complex projective variety $Y_{\mathbb{C}}$, let and $Y_{\mathbb{C}/\mathbb{R}}$ denote the associated real variety obtained by restriction of scalars.

**Proposition 3.2.** Let $Y_{\mathbb{C}}$ be a proper complex holomorphic manifold and let $X$ be a proper smooth real algebraic variety.

i. One has natural isomorphisms

$$H^i_{\mathbb{D}/\mathbb{R}}(Y_{\mathbb{C}/\mathbb{R}}; A(p)) \cong H^i_{\mathbb{D}/\mathbb{C}}(Y_{\mathbb{C}}; A(p)),$$

where the latter denotes the usual Deligne cohomology of the complex manifold $Y_{\mathbb{C}}$. (See Remark 2.1.)

ii. If $X_{\mathbb{C}}$ is the complex variety obtained from $X$ by base extension, the corresponding map of real varieties $X_{\mathbb{C}/\mathbb{R}} \to X$ induces natural homomorphisms

$$H^i_{\mathbb{D}/\mathbb{R}}(X; A(p)) \to H^i_{\mathbb{D}/\mathbb{C}}(X_{\mathbb{C}}; A(p))^\mathbb{S},$$

where the latter denotes the invariants of the Deligne cohomology of the complex variety $X_{\mathbb{C}}$.

iii. One has a long exact sequence

$$\cdots \to H^{j-1}_{\text{sing}}(X(\mathbb{C}); \mathbb{C})^\mathbb{S} \to H^j_{\mathbb{D}/\mathbb{R}}(X; A(p)) \xrightarrow{\nu}$$

$$\to H^j_{\mathbb{B}r}(X(\mathbb{C}); A) \oplus \left(F^p H^j_{\text{sing}}(X(\mathbb{C}); \mathbb{C})\right)^\mathbb{S} \xrightarrow{p^p - 1} H^j_{\text{sing}}(X(\mathbb{C}); \mathbb{C})^\mathbb{S} \to \cdots$$

Here $X(\mathbb{C})$ denotes the set of complex points of $X$ with the analytic topology and $\left\{F^p H^j(X(\mathbb{C}); \mathbb{C})\right\}^\mathbb{S}$ denotes the invariants of the $p$-th level of the Hodge filtration on singular cohomology, under the simultaneous action of $\mathbb{S}$ on $X(\mathbb{C})$ and on the coefficients $\mathbb{C}$ of the cohomology.

We define Deligne cohomology with negative weights $p < 0$ to coincide with ordinary equivariant cohomology. Using this convention, we give
\( \oplus_{n,p} H^p_{D/R}(X; \mathbb{Z}(p)) \) a bigraded ring structure compatible with various functors in the theory and having many computational properties. For example, the bigraded group structure of the cohomology of a point \( X = \text{Spec}(\mathbb{R}) \) is displayed in Table 1 above, and its ring structure is displayed in Table 2, subsection 3.2.

Using this product structure we obtain formulae for the Deligne cohomology ring of some relevant examples, such as:

**Corollary 3.9.** Under the hypothesis of Proposition 3.8, one has an isomorphism of bigraded rings

\[
H^*_{D/R}(X; A(\ast))[T]/\langle T^{p+1} \rangle \cong H^*_{D/R}(X \times \mathbb{P}^p; A(\ast))
\]

when \( T \) is given the bigrading \((2, 1)\).

A more general projective bundle formula together with a theory of characteristic classes appear in a forthcoming paper.

The cases of weights \( p = 1 \) and \( p = 2 \) have interesting geometric interpretations. We first show that \( \mathbb{Z}(1)_{D/R} \) is quasi-isomorphic to \( \mathcal{O}^\times[-1] \), cf. Corollary 4.6 and derive as a consequence an exponential sequence relating the cohomology of \( \mathbb{Z}(1)_{Br} \), \( \mathcal{O} \) and \( \mathcal{O}^\times \):

**Corollary 4.7.** Let \( X \) be a smooth proper real algebraic variety. Then there is a long exact sequence

\[
\rightarrow H^*_\mathcal{Br}(X(\mathbb{C}); \mathbb{Z}) \rightarrow H^*(X; \mathcal{O}_X) \rightarrow H^*(X; \mathcal{O}_X^\times) \rightarrow H^*_{\mathcal{Br}+1}(X(\mathbb{C}); \mathbb{Z}) \rightarrow
\]
where \( \vartheta \) denotes the composite
\[
H^*_{Br}(X(\mathbb{C}); \mathbb{Z}) \xrightarrow{\vartheta} H^*_{\mathbb{C}}(X; \mathbb{C})^\mathfrak{E} \rightarrow H^*_{\mathbb{C}}(X; \mathfrak{C}_X) = H^*_{Br}(X(\mathbb{C})),
\]
and the latter denotes the invariants of the Dolbeault cohomology of the complex manifold \( X(\mathbb{C}) \).

When \( X \) is a curve, the exponential sequence then gives:

**Proposition 4.10.** Let \( X \) be an irreducible, smooth, projective curve over \( \mathbb{R} \), of genus \( g \). Then \( \text{Pic}(X) \cong \text{Pic}_0(X\mathbb{C})^\mathfrak{E} \times H^2_{Br}(X(\mathbb{C}), \mathbb{Z}) \).

As a corollary to this proposition we provide a new proof of Weichold’s Theorem — a classical result in real algebraic geometry which determines the Picard group of a real algebraic curve.

In order to give a geometric interpretation of \( H^2_{\mathcal{D}/\mathbb{R}}(X; \mathbb{Z}(2)) \) for a real projective variety \( X \) we first provide in Proposition 4.3 alternative interpretations of the Bredon cohomology group \( H^2_{Br}(Y, \mathbb{Z}) \) for any \( \mathfrak{E} \)-manifold \( Y \). Using Atiyah’s terminology in [Ati66], a **Real vector bundle** \( (E, \tau) \) on a \( \mathfrak{E} \)-manifold \( (Y, \sigma) \) consists of a complex vector bundle \( E \) on \( Y \) together with an isomorphism \( \tau : \sigma^*E \rightarrow E \) satisfying \( \tau \circ \sigma^*\tau = \text{Id} \). Now, consider the set \( \mathcal{L}_2(Y) \) of equivalence classes of pairs \( (L, \varrho) \) satisfying:

1. \( L \) is a (smooth) complex line bundle on \( Y \);
2. \( \varrho : L \otimes \sigma^*L \rightarrow 1_Y \) is an isomorphism of Real line bundles, where \( L \otimes \sigma^*L \) carries the tautological Real line bundle structure.

It follows that \( \mathcal{L}_2(Y) \) becomes a group under the tensor product of line bundles and we show that this group is naturally isomorphic to \( H^2_{Br}(Y, \mathbb{Z}) \).

Now, let \( S = \pi_0(Y^\mathfrak{E}) \) denote the set of connected components of the fixed point set \( Y^\mathfrak{E} \), and identify \( H^0(Y^\mathfrak{E}, \mathbb{Z}) \equiv (\mathbb{Z}^\times)^S \). Given \( (L, \varrho) \in H^2_{Br}(Y, \mathbb{Z}) \), the restriction of \( \varrho \) to \( L|_{Y^\mathfrak{E}} \) becomes a non-degenerate hermitian pairing, and hence it has a well-defined signature \( \mathfrak{N}_{(L, \varrho)} \in (\mathbb{Z}^\times)^S \). We call
\[
\mathfrak{N} : H^2_{Br}(Y, \mathbb{Z}) \rightarrow (\mathbb{Z}^\times)^S
\]
the **equivariant signature map** of \( Y \) and the image \( \mathfrak{N}_{\text{tor}}(Y) \subseteq (\mathbb{Z}^\times)^S \) of the torsion subgroup \( H^2_{Br}(Y, \mathbb{Z})_{\text{tor}} \) under \( \mathfrak{N} \) is called the **equivariant signature group** of \( Y \). In the case where \( Y = X(\mathbb{C}) \) for a real algebraic variety \( X \) with \( S = \pi_0(X(\mathbb{R})) \), we denote the equivariant signature group of \( X(\mathbb{C}) \) simply by \( \mathfrak{N}_{\text{tor}}(X) \). For example, if \( X \) is a real algebraic curve, then \( \mathfrak{N}_{\text{tor}}(X) \) is the Brauer group of \( X \); see Section 3.

Given a proper real variety \( X \), let \( PW^\nabla(X) \) denote the set of isomorphism classes \( (L, \nabla, \varrho) \) of triples where \( L \) is a holomorphic line bundle over \( X(\mathbb{C}) \), \( \nabla \) is a holomorphic connection on \( L \) and \( \varrho : L \otimes \sigma^*L \rightarrow 1 \) is a holomorphic isomorphism of Real line bundles satisfying the following properties:

1. The restriction of \( \varrho \) to \( X(\mathbb{R}) \) is a positive-definite hermitian metric.
2. As a section of \( (L \otimes \sigma^*L)^\nabla \), \( \varrho \) is parallel with respect to the connection induced by \( \nabla \).
One sees that the tensor product endows $P^\nabla(W)(X)$ with a group structure which makes $P^\nabla(W)(X)$ the kernel of $\Psi$:

**Theorem 5.10.** If $X$ is a smooth real projective variety then one has a natural short exact sequence

$$0 \to P^\nabla(W)(X) \to H^2_{D/R}(X; \mathbb{Z}(2)) \xrightarrow{\Psi} \mathcal{N}_{tor}(X) \to 0.$$ 

This paper is organized as follows. Section 2 contains background information and the key technical ingredients for the paper, including Proposition 2.8 which constructs the map of complexes $\tau_p: A(p)_{Br} \to \mathcal{E}^*$. In Section 3 we construct Deligne complexes $A(p)_{D/R}$ for any subring $A \subset \mathbb{R}$ and define the corresponding Deligne cohomology for proper smooth real varieties. In this section we prove basic properties, introduce the product structure and provide basic examples. In Section 4 we study the weight $p = 1$ case, proving the quasi-isomorphism $\mathbb{Z}(1)_{D/R} \simeq \mathbb{O}^*[-1]$ together with some applications. In Section 5 we study the group $H^2_{D/R}(X; \mathbb{Z}(2))$ and associated interpretations of $H^2_{Br}(X(\mathbb{C}), \mathbb{Z})$. The proof of the main result, Theorem 5.10, is delegated to Appendix B. Section 6 contains a remark about number fields, where we give a ring homomorphism from the Milnor $K$-theory of a number field to the “diagonal” subring of integral Deligne cohomology, and we observe that the classical regulator of a number field can be described in terms of the image of the change-of-coefficients homomorphism between Deligne cohomology with integral and real coefficients, respectively; similar computations can be made for arbitrary Artin motives. In Appendix A we describe the relationship between Esnault-Viehweg’s “Borel version” of Deligne cohomology for real varieties and the theory discussed in this paper.

In a forthcoming series of papers we first extend this theory to an integral Deligne-Beilinson cohomology theory for arbitrary real varieties, and study its relation to a corresponding notion of Mixed Hodge Structures. Then we provide a natural and explicit cycle map from motivic cohomology to Deligne cohomology, directly using the approach in [MVW06]. This is an alternative description, even in the complex case, of the maps discussed in [Blo86, KLMS06, KL07]. The corresponding real intermediate Jacobians and their relations to real algebraic cycles are also under study.

**Acknowledgements.** The first author would like to thank Texas A&M University and the second author would like to thank the IST (Instituto Superior Técnico, Lisbon) for their respective warm hospitality during the elaboration of parts of this work; and the second author wants to thank Spencer Bloch for inspiring conversation and pointed questions during a visit to the University of Chicago.
2. Ordinary equivariant cohomology and sheaves

We first introduce the various categories used throughout this article. Let \( \mathcal{G} := \text{Gal}(\mathbb{C}/\mathbb{R}) \) denote the Galois group of \( \mathbb{C} \) over \( \mathbb{R} \).

\( \text{a) The category of smooth manifolds with smooth } \mathcal{G} \text{-action and equivariant smooth morphisms is denoted } \mathcal{G} \text{-Man. Let } Cov(X) \text{ be the set of coverings of } X \in \mathcal{G} \text{-Man by open } \mathcal{G} \text{-invariant subsets. These coverings give } \mathcal{G} \text{-Man a site structure whose restriction to } X \text{ is denoted } X_{eq}. \)

\( \text{b) A real holomorphic manifold } (M, \sigma) \text{ is a smooth complex holomorphic manifold } M \text{ endowed with an anti-holomorphic involution } \sigma: M \to M, \text{ and morphisms between two such objects are holomorphic equivariant maps. These comprise the category } A_n/\mathbb{R} \text{ of real holomorphic manifolds, with its evident site structure.} \)

\( \text{c) We denote by } Sm/\mathbb{R} \text{ the category of smooth real algebraic varieties.} \)

\( \text{d) The categories of smooth holomorphic manifolds and complex algebraic varieties are denoted } A_n/\mathbb{C} \text{ and } Sm/\mathbb{C}, \text{ respectively.} \)

Remark 2.1. \( \text{(1) If } M \text{ is a complex manifold, denote } M/\mathbb{R} := M \sqcup \overline{M}, \text{ where } \overline{M} \text{ denotes } M \text{ with the opposite complex structure. The map } \sigma: M/\mathbb{R} \to M/\mathbb{R} \text{ sending a point in one copy of } M \text{ to the same point in the other copy of } M \text{ is an anti-holomorphic involution. The assignment } M \mapsto M/\mathbb{R} \text{ is a functor } A_n/\mathbb{C} \to A_n/\mathbb{R}. \)

\( \text{(2) Given a real variety } X \in Sm/\mathbb{R}, \text{ let } X(\mathbb{C}) \text{ denote its set of complex-valued points with the analytic topology. The natural action of } \mathcal{G} \text{ on } X(\mathbb{C}) \text{ which a morphism of sites from } Sm/\mathbb{R} \text{ into } A_n/\mathbb{R}. \)

\( \text{(3) We will denote a complex algebraic variety always as } X_{\mathbb{C}}, \text{ and we use } X_{\mathbb{C}}/\mathbb{R} \text{ to denote } X_{\mathbb{C}} \text{ seen as a real variety. It follows that the set of complex valued points } X_{\mathbb{C}}/\mathbb{R}(\mathbb{C}) \text{ (over } \mathbb{R} \text{) coincides with } X_{\mathbb{C}}(\mathbb{C}) \sqcup X_{\mathbb{C}}(\overline{\mathbb{C}}). \text{ The constructions above give a commuting diagram of functors:} \)

\[
\begin{array}{ccc}
Sm/\mathbb{C} & \longrightarrow & Sm/\mathbb{R} \\
\downarrow & & \downarrow \\
An/\mathbb{C} & \longrightarrow & An/\mathbb{R}.
\end{array}
\]

2.1. Ordinary equivariant cohomology. In [Bre67] Bredon defines an equivariant cohomology theory \( H^n_G(X; M) \) for \( G \)-spaces, where \( G \) is a finite group and \( M \) is a contravariant coefficient system. When \( M \) is a Mackey functor, P. May et al. [LMM81] showed that this theory can be uniquely extended to an \( RO(G) \)-graded theory \( \{ H^n_G(X; M), \alpha \in RO(G) \} \), called \( RO(G) \)-graded ordinary equivariant cohomology theory, where \( RO(G) \) denotes the orthogonal representation ring of \( G \). When \( G = \mathcal{G} \), one has \( RO(\mathcal{G}) = \mathbb{Z} \cdot 1 \oplus \mathbb{Z} \cdot \xi \), where \( 1 \) is the trivial representation and \( \xi \) is the sign representation. In this paper we use the motivic notation:

\[
H^n_{Br}(X, M) := H^n_{\mathcal{G}}(X, M) = H^{n-p+1+p}((X; M), \mathcal{G}),
\]
and call $H_{Br}^{n,p}(X, M)$ bigraded Bredon cohomology.

In the homotopy theoretic approach, one proves the existence of equivariant Eilenberg-MacLane spaces $K(M_+,(n,p))$ that classify Bredon cohomology (for $n \geq p \geq 0$). In other words, $H_{Br}^{n,p}(X, M) = [X_+, K(M_+,(n,p))]_G$, where the latter denotes the set of based equivariant homotopy classes of maps. When $n < p$ one uses the suspension axiom to define the corresponding cohomology groups; see [LMM81].

A quick way to construct $K(Z,(n,p))$ is the following. Let $S_{n,p}$ denote the one-point compactification $\{((n-p) \cdot 1 \oplus p \cdot \xi) \cup \{\infty}\}$ of the indicated representation, and define $Z_0(S_{n,p}) := Z(S_{n,p})/Z(\{\infty\})$, where $Z(S_{n,p})$ denotes the free abelian group on $S_{n,p}$, suitably topologized. Then $Z_0(S_{n,p})$ is an Eilenberg-MacLane space $K(Z,(n,p));$ cf. [dS03b].

Example 2.2. In order to describe the bigraded cohomology ring of a point $B := \bigoplus_{p,n} H_{Br}^{n,p}(pt, Z)$, first consider indeterminates $\varepsilon, \varepsilon^{-1}, \tau, \tau^{-1}$ satisfying $\deg \varepsilon = (1,1), \deg \varepsilon^{-1} = (-1,-1), \deg \tau = (0,2)$ and $\deg \tau^{-1} = (0,-2)$. Henceforth, $\varepsilon$ and $\varepsilon^{-1}$ will always satisfy $2\varepsilon = 0 = 2\varepsilon^{-1}$.

As an abelian group, $B$ can be written as a direct sum

$$B := \mathbb{Z}[\varepsilon, \tau] \cdot 1 \oplus \mathbb{Z}[\tau^{-1}] \cdot \alpha \oplus F_2[\varepsilon^{-1}, \tau^{-1}] \cdot \theta$$

where each summand is a free bigraded module over the corresponding ring and $F_2$ is the field with two elements (hence $2\theta = 0$). The respective bidegrees of the generators $1, \alpha$ and $\theta$ are $(0,0), (0,-2)$ and $(0,-3)$.

The product structure on $B$ is completely determined by the following relations

$$\alpha \cdot \tau = 2, \quad \alpha \cdot \theta = \alpha \cdot \varepsilon = \theta \cdot \tau = \theta \cdot \varepsilon = 0.$$ 

Note that $B$ is not finitely generated as a ring, and that $B$ has no homogeneous elements in degrees $(p,q)$ when $p \cdot q < 0$.

We now present an alternative sheaf-theoretic construction of Bredon cohomology which is more suitable for our purposes. The details of such construction will appear in [Yan08].

Given $U \in \mathcal{S}$-$\text{Man}$, let $\hat{U}$ denote the full subcategory of $\mathcal{S}$-$\text{Man} \downarrow U$ consisting of equivariant finite covering maps $\pi_S: \mathcal{S} \to U$. In particular, $\{pt\}$ is the category $\mathcal{S}$-$\text{Fin} \subseteq \mathcal{S}$-$\text{Man}$ of finite $\mathcal{S}$-sets.

A topological $G$-Module $M$ represents an abelian Mackey presheaf on $\mathcal{S}$-$\text{Man}$, in other words, the contravariant functor $M: \mathcal{S}$-$\text{Man}^{op} \to Ab$ sending $U \mapsto M(U) := \text{Hom}_{\mathcal{S}, X_{op}}(U, M)$ is also covariant for maps in $\hat{U}$, for all $U \in \mathcal{S}$-$\text{Man}$, and satisfies the following property. Given a pull-back square

$$
\begin{array}{ccc}
Z & \xrightarrow{\gamma} & X \\
\varphi \downarrow & & \downarrow f \\
Y & \xrightarrow{g} & U
\end{array}
$$

We have $\varphi \circ \gamma = f \circ g$. The category $\mathcal{S}$-$\text{Man}$ is a Grothendieck topos, and the above construction shows that the Bredon cohomology ring of a point is a sheaf on $\mathcal{S}$-$\text{Man}$.
with \( f \in \hat{U} \), the diagram

\[
\begin{array}{cc}
M(Z) \xleftarrow{\gamma^*} M(X) \\
\varphi_\ast \downarrow & \downarrow f_\ast \\
M(Y) \xleftarrow{g^*} M(U)
\end{array}
\]

commutes. The case where \( M \) is a subring a \( \mathbb{R} \) with trivial \( \mathcal{G} \)-module structure will play a special role in this work.

**Definition 2.3.** Let \( \mathcal{F} \) be an abelian presheaf on \( \mathcal{G}\text{-Man} \), and let \( M \) be a topological \( \mathcal{G} \)-module. Given \( U \in \mathcal{G}\text{-Man} \), let \( \mathcal{F} \otimes \hat{U} M \) denote the coend:

\[
\{ \bigoplus_{\pi_S : S \to U} \mathcal{F}(S) \otimes M(S) \} / K_{\mathcal{F}, M}(U) \text{ in } Ab,
\]

where \( \pi_S \in \hat{U} \) and \( K_{\mathcal{F}, M}(U) \) is the subgroup generated by elements of the form:

\[
(\phi^\ast \alpha') \otimes m - \alpha' \otimes \phi_\ast M^\ast m,
\]

when \( \pi_S \xleftarrow{\phi} S' \xrightarrow{\pi_S'} U \) is a morphism in \( \hat{U} \), \( \alpha' \in \mathcal{F}(S') \) and \( m \in M(S) \). It is easy to see that the assignment \( U \mapsto \mathcal{F} \otimes \hat{U} M \) is a contravariant functor from \( \mathcal{G}\text{-Man} \) to \( Ab \). Denote by \( \mathcal{F} \int M \) the resulting abelian presheaf on \( \mathcal{G}\text{-Man} \), i.e. \( \mathcal{F} \int M(U) := \mathcal{F} \otimes \hat{U} M \).

Let \( \Delta^n \) denote the standard topological \( n \)-simplex with the trivial \( \mathcal{G} \)-action. Using the co-simplicial structure on \( \{ \Delta^\ast \mid n \geq 0 \} \) one creates a simplicial abelian presheaf \( \mathcal{C}^\ast(\mathcal{F}) \) associated to any presheaf \( \mathcal{F} \), whose \( n \)-th term is

\[
\mathcal{C}^n(\mathcal{F}) : U \mapsto \mathcal{F}(\Delta^n \times U).
\]

Denote the associated complex of sheaves by \( (\mathcal{C}^\ast(\mathcal{F}), d_n) \) and use the convention in [SGA72, XVII 1.1.5] to define a cochain complex \( (\mathcal{C}^\ast(\mathcal{F}), d^\ast) \) where \( \mathcal{C}^n(\mathcal{F}) := \mathcal{C}^{-n}(\mathcal{F}) \) and \( d^n : \mathcal{C}^n(\mathcal{F}) \to \mathcal{C}^{n+1}(\mathcal{F}) \) is defined by \( d^n = (-1)^n d_{-n} \).

A \( \mathcal{G} \)-manifold \( X \) defines an abelian presheaf on \( \mathcal{G}\text{-Man} \)

\[
\mathbb{Z}X : U \mapsto \mathbb{Z}\text{Hom}_{\mathcal{G}\text{-Man}}(U, X),
\]

sending \( U \) to the free abelian group on the set of smooth equivariant maps from \( U \) to \( X \). Yoneda Lemma identifies \( \text{Hom}_{\text{AbPreSh}}(\mathbb{Z}X, \mathcal{F}) = \mathcal{F}(X) \) for any \( \mathcal{F} \). In particular, if \( \mathcal{F} \) is any abelian presheaf and \( X \in \mathcal{G}\text{-Man} \), the presheaf \( \text{Hom}(\mathbb{Z}X, \mathcal{F}) \) sends \( U \) to \( \mathcal{F}(X \times U) \).

**Proposition 2.4.**

i. The assignment \( \mathcal{F} \mapsto \mathcal{C}^\ast(\mathcal{F}; M) \) is covariant on \( \mathcal{F} \).

ii. Let \( I = [0, 1] \) denote the unit interval with the trivial \( \mathcal{G} \)-action. For any abelian presheaf \( \mathcal{F} \) let \( i_0^\ast, i_1^\ast : \text{Hom}(\mathbb{Z}I, \mathcal{C}^\ast(\mathcal{F})) \to \mathcal{C}^\ast(\mathcal{F}) \), be the maps of complexes induced by evaluation at the end-points. Then there is a
homotopy $h_F$ between $i_0^*$ and $i_1^*$ which is natural on $F$. In particular, the complexes $C^*(F)$ have homotopy-invariant cohomology presheaves.

In what follows denote

$$
(C^*)^{p-1}_i := C^x \times \cdots \times 1 \times \cdots \times C^x \subset C^{\times p},
$$

where 1 appears in the $i$-th coordinate.

**Definition 2.5.** Given a $\mathcal{G}$-manifold $X$, let

$$
J_{X,p} : \bigoplus_{i=1}^p C^*(Z((C^*)^{p-1}_i \times X)) \to C^*(Z(C^{\times p} \times X))
$$

be the map induced by the inclusions and denote

$$
(7) \quad C^*(Z_0(S^{p-p} \wedge X_+)) := \text{Cone}(J_{X,p}).
$$

Write $C^*(Z_0(S^{p-p}))$ when $X = \emptyset$ is the empty manifold. Let $A \subset \mathbb{R}$ be a subring endowed with the discrete topology. The $p$-th Bredon complex with coefficients in $A$ is the complex of presheaves

$$
(8) \quad A(p)_{\mathcal{B}r} := C^*(Z_0(S^{p-p})) \int A \ [-p],
$$

where the coend is taken levelwise.

The following result is proven in [Yan08]

**Theorem 2.6.** Let $X$ be a $\mathcal{G}$-manifold and let $A \subset \mathbb{R}$ be a subring, endowed with the discrete topology. Then for all $p \geq 0$ and $n \in \mathbb{Z}$ there is a natural isomorphism $\bar{H}^n(X_{eq}; A(p)_{\mathcal{B}r}) \cong H^p_{Br}(X, A)$ between the Čech hypercohomology of $X_{eq}$ with values in $A(p)_{\mathcal{B}r}$ and the equivariant cohomology group $H^p_{Br}(X, A)$.

For all $p, n \in \mathbb{Z}$ there is a forgetful functor

$$
(9) \quad \varphi : H^{n,p}_{Br}(X, A) \to H^n_{sing}(X; A(p))^{\mathcal{G}},
$$

where $A(p)$ is the $\mathcal{G}$-submodule $A(p) := (2\pi i)^p A \subset \mathbb{C}$ and the invariants $H^n_{sing}(X; A(p))^{\mathcal{G}}$ of the singular cohomology of $X$ with coefficients in $A(p)$ are taken under the simultaneous action of $\mathcal{G}$ on $X$ and $A(p)$.

In the following section we present a realization of the composition

$$
(10) \quad H^{n,p}_{Br}(X, A) \to H^n_{sing}(X; A(p))^{\mathcal{G}} \to H^n_{sing}(X; \mathbb{C})^{\mathcal{G}},
$$

where the latter is the change of coefficients map.

**2.2. Integration and change of coefficient functors.** Let $\mathcal{A}^p$ denote the sheaf of smooth, complex valued differential $p$-forms on $\mathcal{G}$-manifolds. Given $X \in \mathcal{G}$-$\text{Man}$, denote

$$
\mathcal{E}^p(X) = \{ \theta \in \mathcal{A}^p(X) \mid \sigma^*(\theta) = \theta \}.
$$

In other words, $\mathcal{E}^p$ is the subsheaf of $\mathcal{A}^p$ consisting of those $p$-forms invariant under the simultaneous action of $\mathcal{G}$ on $X$ and $\mathbb{C}$. 
Let \( \pi: E \to B \) be a locally trivial bundle, where \( B \) be a smooth manifold, \( E \) is an oriented manifold-with-corners and the fiber is an orientable \( n \)-dimensional manifold-with-boundary \( F \) (and with corners). Let

\[
\pi_1: \mathcal{A}^{p+n}(E) \to \mathcal{A}^p(B).
\]

denote the integration along the fiber homomorphism. If \( \pi \) is a map in \( \mathfrak{S}\text{-Man} \) then \( \pi_1 \) preserves invariants, thus sending \( \mathcal{E}^{p+n}(E) \) to \( \mathcal{E}^p(B) \). The following properties are well-known.

**Properties 2.7.** Let \( \omega \) be a \((n+p)\)-form and let \( \pi': \partial E \to X \) denote the restriction of \( \pi \) to the boundary of \( E \).

i. (Projection formula) \( \pi_1(\pi^*\theta \wedge \omega) = \theta \wedge \pi_1^!(\omega) \)

ii. (Boundary formula) \( d\pi_1^!(\omega) = \pi_1^!(d\omega) + (-1)^p \pi_1^!(\omega|_{\partial E}) \)

iii. (Pull-back formula) Given a pull-back square

\[
\begin{array}{ccc}
E' & \xrightarrow{f} & E \\
\downarrow{\pi'} & & \downarrow{\pi} \\
X' & \xrightarrow{f} & X
\end{array}
\]

then \( f^* \circ \pi_1 = \pi_1' \circ f'^* \).

iv. (Functoriality) If \( X \xrightarrow{f} Y \xrightarrow{g} Z \) are smooth fibrations with compact fibers, then \( (g \circ f)^! = g^! \circ f^! \).

v. (Product formula) Let

\[
\begin{array}{ccc}
E'' & \xrightarrow{\rho} & E \\
\downarrow{\pi''} & & \downarrow{\pi} \\
E' & \xrightarrow{\pi'} & X
\end{array}
\]

be a pull-back square where both \( \pi \) and \( \pi' \) are fibrations with fiber dimensions \( n \) and \( n' \), respectively. Given \( \omega \in \mathcal{A}^{p+n}(E) \) and \( \omega' \in \mathcal{A}^{q+n'}(E') \) one has \( \pi_1''(\rho_1^*\omega \wedge \rho_2^*\omega') = (-1)^{pq} \pi_1(\omega) \wedge \pi_1'(\omega') \).

Integration along the fiber can be used to construct maps of complexes

\[
\tau_p: A(p)_{br} \to \mathcal{E}^*,
\]

as follows. First consider \( U \in \mathfrak{S}\text{-Man} \) and \( 0 \leq j \leq p \). An element in \( A(p)_{br}^j(U) \) is represented by sums of pairs of the form \( \alpha \otimes m \), where \( \alpha = (a, f) \) with \( a, f \) and \( m \) equivariant maps satisfying

1. \( a: \Delta^{p-j-1} \times S \to (\mathbb{C}^\times)^{p-1} \subset \mathbb{C}^\times^p \) is smooth and \( \pi: S \to U \) is a map in \( \hat{U} \);

2. \( f: \Delta^{p-j} \times S \to (\mathbb{C}^\times)^p \) is a smooth map;

3. \( m: S \to A \in A(S) \) is a locally constant.
In the diagram

\[ U \xrightarrow{p_1} \Delta^{p-j} \times U \xrightarrow{1 \times \pi} \Delta^{p-j} \times S \xrightarrow{p_2} S \longrightarrow A \]

\( \hat{\pi} \) is a locally trivial fibration with fiber dimension \( p-j \), and we consider the locally constant map \( m \circ p_2 = p_2^* m \) as an element in \( E^0(\Delta^{p-j} \times S) \). Denote

\[ \omega_p := \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_p}{t_p} \in E^p(\{\mathbb{C}^\times\}^p) \]

and define

\[ \tau^j(\alpha \otimes m) = \hat{\pi}_1(\{p_2^* m \cdot f^* \omega_p \}) \in E^j(U) \]

where \( \hat{\pi}_1 : E^p(\Delta^{p-j} \times S) \longrightarrow E^j(U) \) is the integration along the fiber homomorphism. This can be extended to a homomorphism

\[ \tau^j : \bigoplus_{S \in \hat{U}} \{E^p(\Delta^{p-j} \times S), E^j(U) \} \longrightarrow \mathcal{E}^j(U). \]

**Proposition 2.8.** For each \( 0 \leq j \leq p \), the map \( \tau^j \) above factors through \( E^p(\Delta^{p-j} \times S), E^j(U) \). Furthermore, these maps induce a morphism of complexes of presheaves

\[ \tau_p : A(p)^{\mathbb{B}_r} \longrightarrow \mathcal{E}^*. \]

**Proof.** Given a morphism

\[ S \xrightarrow{\phi} S' \]

in \( \hat{U} \), consider the associated diagram

\[ \Delta^{p-j} \times S \xrightarrow{1 \times \phi} \Delta^{p-j} \times S' \]

Pick \( \alpha = (a, f) \in \text{Cone}(J_p)^{p-j}(S') \) and \( m \in A(S) \); cf. Definition 2.5. By definition,

\[ \tau^j(f^* \alpha \otimes m) = \hat{\pi}_1(p^* m \cdot (1 \times \phi)^* f^* \omega_p). \]
It follows from Properties i)–iv) of integration along the fibers that for all \( \theta \in \mathcal{E}^*(\Delta^{p-j} \times S) \) one has \( \pi_1\{p^*\theta \} = \pi_1\{(p^*(\phi_s m) \cdot \theta) \} \), since the top square in (16) is a pull-back diagram. In particular, for \( \theta = f^*\omega_p \) one obtains
\[
\tau^j(\alpha \otimes \phi_s m) = \pi_1(p^*(\phi_s m) \cdot f^*\omega_p) = \tau^j(\phi^*\alpha \otimes m).
\]
This proves the first assertion in the proposition.

To prove the second assertion, let \( \partial_i : \Delta^{p-j-1} \hookrightarrow \Delta^{p-j} \) be the inclusion of the \( i \)-th face, and pick \( \alpha \otimes m \in \text{Cone}(J_{p-j}(S) \otimes \Delta(S)) \), as before, representing an element in \( A(p)^*_j \mathbb{B} \). Then, using the sign convention relating the differential \( D \) of \( A(p)^*_j \mathbb{B} \) with the differential \( d \) of \( \mathcal{C}_*(\mathbb{C}^{\times p}) \) one gets
\[
(17) \quad \tau^{j+1}(D(\alpha \otimes m)) = \tau^{j+1}((-d_p^*a, a + d_p^*f) \otimes m)
\]
\[
= \pi_1(p^* m \cdot a^*\omega_p) + (-1)^j \sum_{i=1}^n (-1)^i \pi_1(\tilde{p}^* m \cdot \partial_i^* f^*\omega_p),
\]
where \( \tilde{p} : \Delta^{p-j-1} \times S \to S \) is the projection. Since \( a : \Delta^{p-j-1} \times S \to \mathbb{C}^{\times p} \) factors through some \( (\mathbb{C}^{\times p})^{p-1}_i \subset \mathbb{C}^{\times p} \), one has \( a^*\omega_p = 0 \). On the other hand,
\[
(18) \quad \sum_{i=1}^n (-1)^i \pi_1(\tilde{p}^* m \cdot \partial_i^* f^*\omega_p) = \sum_{i=1}^n (-1)^i \pi_1((\partial_i \times 1)^* p^* m \cdot f^*\omega_p)
\]
\[
= \pi_1(\{p^* m \cdot f^*\omega_p\}_{|\partial(\Delta^{p-j} \times S)}) = (-1)^j d \pi_1(p^* m \cdot f^*\omega_p);
\]
cf. the boundary formula (2.7). It follows from (17) and (18) that \( \tau^{j+1} \circ D = d \circ \tau^j \), for \( 1 \leq j \leq p \).

The two remaining cases are:
\[
(19) \quad \begin{array}{ccc}
0 & \longrightarrow & \mathcal{E}^{p+1}(U) \\
\downarrow & & \downarrow d \\
A(p)^*_j \mathbb{B}(U) & \longrightarrow & \mathcal{E}^p(U)
\end{array}
\]
and
\[
(20) \quad \begin{array}{ccc}
A(p)^*_j \mathbb{B}(U) & \longrightarrow & \mathcal{E}^0(U) \\
\downarrow d^{-1} & & \\
A(p)^{-1} \mathbb{B}(U) & \longrightarrow & 0
\end{array}
\]

Given \( f : S \to \mathbb{C}^{\times p} \), \( m : S \to A \) with \( S \in \widehat{U} \), then the boundary formula for integration along the fibers gives \( d \pi_1(m \cdot f^*\omega_p) = 0 \), thus showing that (19) commutes.
To show that \( [20] \) commutes, pick \( f: S \times \Delta^{p+1} \to \mathbb{C} \times P \), \( m: S \to A \). Then:

\[
\tau_0(D[f \otimes m]) = \tau_0 \left\{ \sum_{k=0}^{p+1} (-1)^k f \circ (1 \times \partial_k) \otimes m \right\} \\
= \sum_{k=0}^{p+1} (-1)^k \hat{\pi}_1 (m \cdot (1 \times \partial_k)^* f^* \omega_p) \\
= \sum_{k=0}^{p+1} (-1)^k \hat{\pi}_1 ((1 \times \partial_k)^* \{m \cdot f^* \omega_p\}) \\
= d\hat{\pi}_1 (m \cdot f^* \omega_p) \pm \hat{\pi}_1 (d\{m \cdot f^* \omega_p\}) = 0.
\]

The functoriality of the maps \( \tau^j \) with respect to \( U \) should be evident. \( \square \)

**Remark 2.9.** Recall that if \( U \) is a \( \mathcal{G} \)-manifold, one obtains a natural \( \mathcal{G} \)-isomorphism \( U^{\text{triv}} \times \mathcal{G} \cong X \), \( U \times \mathcal{G} \) by sending \( (x, g) \) to \((gx, g)\). On the other hand, \( \mathcal{E}^p(U^{\text{triv}} \times \mathcal{G}) \) is the subgroup of \( \mathcal{A}^p(U^{\text{triv}} \times \mathcal{G}) \cong \mathcal{A}^p(U^{\text{triv}}) \times \mathcal{A}^p(U^{\text{triv}}) \) invariant under the involution that sends \( (\omega_1, \omega_\sigma) \) to \((\omega_\sigma, \omega_1)\).

This observation shows that we have a functor

\[
F: \mathcal{E}^p(U \times \mathcal{G}) \longrightarrow \mathcal{A}^p(X)
\]

satisfying the following properties:

1. \( F \) commutes with differentials;
2. If \( \iota: U \times \mathcal{G} \to U \times \mathcal{G} \) is the \( \mathcal{G} \)-homeomorphism sending \( (x, \alpha) \) to \((x, \sigma \alpha)\), then \( F(\iota^* \omega) = \sigma^* F(\omega) \);
3. \( F \circ p^* = Id \), where \( p: U \times \mathcal{G} \to U \) is the projection.

To describe the product structure on \( A(\ast)_{\mathbb{B}^r} \), let \( \Gamma_{n,m} := \{ \sigma: \Delta^{n+m} \to \Delta^n \times \Delta^m \mid n \geq 0, m \geq 0 \} \) be the triangulation of \( \Delta^n \times \Delta^m \) inducing the Alexander-Whitney diagonal approximation; cf. \( [ES52, \text{p. 68}] \). Given \( p, q \geq 0 \), the (external) pairing of presheaves \( \mathcal{Z}(\mathbb{C} \times P) \otimes \mathcal{Z}(\mathbb{C} \times Q) \to \mathcal{Z}(\mathbb{C} \times P+Q) \) yields a pairing of complexes

\[
\mathcal{E}^*(\mathcal{Z}(\mathbb{C} \times P)) \otimes \mathcal{E}^*(\mathcal{Z}(\mathbb{C} \times Q)) \to \mathcal{E}^*(\mathcal{Z}(\mathbb{C} \times P+Q)) \tag{21}
\]

in the usual manner. Denoting \( \mathcal{E}^*(\mathcal{Z}(\mathbb{C}^{r,x})) := \bigoplus_{i=1}^r \mathcal{E}^*(\mathcal{Z}(\mathbb{C}^{r,x}_i)) \), one sees that this pairing sends both \( \mathcal{E}^*(\mathcal{Z}(\mathbb{C}^{r,x})) \otimes \mathcal{E}^*(\mathcal{Z}(\mathbb{C}^{r,y})) \) and \( \mathcal{E}^*(\mathcal{Z}(\mathbb{C}^{r,x})) \otimes \mathcal{E}^*(\mathcal{Z}(\mathbb{C}^{r,y})) \) to \( \mathcal{E}^*(\mathcal{Z}(\mathbb{C}^{r,x+y})) \), and hence it induces a pairing of complexes

\[
\mathcal{E}^*(\mathcal{Z}_0(S^{p,p})) \otimes \mathcal{E}^*(\mathcal{Z}_0(S^{q,q})) \to \mathcal{E}^*(\mathcal{Z}_0(S^{p+q,p+q})) \tag{22}
\]

Finally, the multiplication \( \bigotimes \mathcal{A} \to \mathcal{A} \) together with the appropriate sign conventions yields a pairing of complexes \( \mu: A(p)_{\mathbb{B}^r} \otimes A(q)_{\mathbb{B}^r} \to A(p+q)_{\mathbb{B}^r} \).

See the proof of Theorem 2.10 below.
Theorem 2.10. The maps of complexes \( \tau \) are compatible with multiplication. In other words, for every \( p, q \geq 0 \) and \( i, j \in \mathbb{Z} \) one has a commutative diagram of presheaves:

\[
\begin{array}{ccc}
A(p)_B \otimes A(q)_B & \xrightarrow{\mu} & A(p+q)_B \\
\tau_p \otimes \tau_q & & \tau_{p+q} \\
\mathcal{E}^i \otimes \mathcal{E}^j & \wedge & \mathcal{E}^{i+j}
\end{array}
\]

Proof. Given \( U \in \mathcal{G}\text{-Man} \), elements in \( A(p)_B(U) \) and \( A(q)_B(U) \) are represented respectively by \( \alpha \otimes m = (a, f) \otimes m \) and \( \beta \otimes n = (b, g) \otimes n \), where

1. \( \pi_S: S \to U \) and \( \pi_T: T \to U \) are in \( \widehat{U} \) and \( \pi: S \times_U T \to U \) denotes their fibered product;
2. \( a: \Delta^{p-i-1} \times S \to \mathbb{C}_{x_{p-1}} \), \( f: \Delta^p \times S \to \mathbb{C}^{x_{p}} \) smooth equivariant, and \( m: S \to A \) locally constant;
3. \( b: \Delta^{q-j-1} \times T \to \mathbb{C}_{x_{q-1}} \), \( g: \Delta^q \times T \to \mathbb{C}^{x_{q}} \) smooth and equivariant, and \( n: T \to A \) locally constant.

The relevant maps in the constructions that follow are summarized in the diagram:

\[
\begin{array}{ccc}
\Delta^{p+q-i-j} \times S \times_U T & \xrightarrow{s} & S \times_U T \\
\sigma \times 1 & & \rho_S \\
\Delta^p \times \Delta^q \times S \times_U T & \xrightarrow{\rho_T} & S \times_U T \\
& \Delta^q \times T & \xrightarrow{n} A \\
(\Delta^p \times S) \times (\Delta^q \times T) & \xrightarrow{f \times g} & \mathbb{C}^{x_p} \times \mathbb{C}^{x_q} \equiv \mathbb{C}^{x_{p+q}}
\end{array}
\]

where \( \sigma: \Delta^{p+q-i-j} \to \Delta^p \times \Delta^q \) is in \( \Gamma_{p-i,q-j} \).

By definition, \( \mu (\alpha \otimes m, \beta \otimes n) = (a \cup_U g + f \cup_U b, f \cup_U g) \otimes m \star n \), where \( f \cup_U g := (-1)^{i(p+i)} \sum_{\sigma \in \Gamma_{p-i,q-j}} (-1)^{\sigma} (f \times g) \circ (\sigma \times 1) \), and \( m \star n: S \times_U T \to A \) is defined as \( m \star n(s, t) = m(s)n(t) \in A \). The elements \( a \cup_U g \) and \( f \cup_U b \) are defined similarly.
Let \( \Delta \) be a subring endowed with the discrete topology.

The theorem now follows from (23) and (24).

Remark 2.11. The forgetful functor \( \phi: H^*_\text{Br}(X; A) \to H^*_\text{sing}(X; \mathbb{C}) \) described in [10] is the ring homomorphism induced by the morphisms of complexes \( \tau_*: A(*)\text{Br} \to \mathcal{E}^* \).

3. Deligne cohomology for real varieties

In this section we restrict our attention to \( \mathbb{A}^n/\mathbb{R} \), the category of real holomorphic manifolds, seen as a site with the topology induced by \( \mathcal{S}\text{-Man} \). In particular, heretofore the complexes \( A(p)\text{Br}, \mathcal{E}^* \) and \( A^* \) will be restricted to \( \mathbb{A}^n/\mathbb{R} \).

The Hodge decomposition \( \mathbb{A}^n = \bigoplus_{i+j=n} A^{i,j} \) is invariant with respect to the \( \mathcal{S} \)-action on \( \mathbb{A}^n \) and this gives a Hodge filtration \( \{F^p\mathcal{E}^*\} \) on the complex \( \mathcal{E}^* \) of invariant smooth forms.

Definition 3.1. Let \( A \subset \mathbb{R} \) be a subring endowed with the discrete topology.

ii. Given \( p \geq 0 \), define the \( p \)-th equivariant Deligne complex \( A(p)\text{D/}\mathbb{R} \) on \( \mathbb{A}^n/\mathbb{R} \) as:

\[
A(p)\text{D/}\mathbb{R} := \text{Cone} \left( A(p)\text{Br} \oplus F^p\mathcal{E}^* \xrightarrow{\iota_p} \mathcal{E}^* \right)[-1],
\]

where \( \iota_p(\alpha, \omega) = \tau_p(\alpha) - \omega \); cf. (15).

\[
H^i_{\text{D/}\mathbb{R}}(X; A(p)) := \tilde{H}^i \left( X_{\text{eq}}; A(p)\text{D/}\mathbb{R} \right),
\]

where \( X_{\text{eq}} \) denotes the equivariant site of \( X \).
iii. If \( p < 0 \), define 
\[
H^i_{D/R}(X; A(p)) := H^i_{Br}(X(C), A).
\]
In other words, Deligne cohomology with negative weights is defined so as to coincide with Bredon cohomology.

iv. The following diagram introduces notation for the various natural maps arising from the cone 
\[
\begin{array}{ccc}
H^n_{Br}(X(C), A) & \xrightarrow{\nu} & H^n_{sing}(X(C); A(p)) \\
\downarrow & & \downarrow \\
F^p H^n_{Br}(X(C), A) & \implies & H^n_{sing}(X(C); Z(p))
\end{array}
\]

Proposition 3.2. Let \( Y_C \) be a proper complex holomorphic manifold and let \( X \) be a proper smooth real algebraic variety.

i. One has natural isomorphisms
\[
H^i_{D/R}(Y_C/R; A(p)) \cong H^i_{D/C}(Y_C; A(p)),
\]
where the latter denotes the usual Deligne cohomology of the complex manifold \( Y_C \). (See Remark 2.1.)

ii. If \( X_C \) is the complex variety obtained from \( X \) by base extension, the corresponding map of real varieties \( X_C/R \to X \) induces natural homomorphisms
\[
H^i_{D/R}(X; A(p)) \to H^i_{D/C}(X_C; A(p))^{\mathcal{E}},
\]
where the latter denotes the invariants of the Deligne cohomology of the complex variety \( X_C \).

iii. One has a long exact sequence
\[
\cdots \to H^{i-1}_{sing}(X(C); C)^{\mathcal{E}} \to H^i_{D/R}(X; A(p)) \xrightarrow{\nu} H^i_{Br}(X(C), A) \oplus \{ F^p H^i_{sing}(X(C); C) \}^{\mathcal{E}} \xrightarrow{\nu_{p+1}} H^i_{sing}(X(C); Z(p))^{\mathcal{E}} \to \cdots
\]

Remark 3.3. The map \( H^i_{D/R}(X; A(p)) \to H^i_{D/C}(X_C; A(p))^{\mathcal{E}} \), is not an isomorphism in general. However, it is always an isomorphism when \( 1/2 \in A \).

Example 3.4. Denote
\[
\mathcal{T}^{i,p} := H^i_{D/R}(\text{Spec } R; Z(p)) \quad \mathcal{T}^{i,p} := H^i_{D/R}(\text{Spec } R; R(p)) \quad \text{and} \quad \mathcal{T}^{i,p} := H^i_{D/C}(\text{Spec } R; Z(p)),
\]
and let $D_{i,p} \rightarrow B_{i,p} := H_{R}^{i,p}(pt, \mathbb{Z})$ denote the natural map between the Deligne and Bredon cohomology groups of a point, respectively. See Example 2.2. The following statements follow from the exact sequence in the Proposition above and from the definitions. Fix $p \geq 0$.

i. The vanishing of singular cohomology in negative degrees gives isomorphisms $D_{-i,p} \cong B_{-i,p} = 0$, for $i > 0$.

ii. $D_{i,0} \cong B_{i,0} \cong \begin{cases} \mathbb{Z}(0), & \text{if } i = 0 \\ 0, & \text{otherwise} \end{cases}$.

iii. For $p > 0$ one has an exact sequence

$$0 \rightarrow D_{0,p} \rightarrow B_{0,p} \rightarrow H_{0}(pt; \mathbb{C}) = \mathbb{R} \rightarrow D_{1,p} \rightarrow B_{1,p} \rightarrow 0.$$ 

iv. For $i \neq 0, 1$ one has $D_{i,p} \cong B_{i,p}$.

Now,

$$B_{0,p} \cong \begin{cases} \mathbb{Z}(2k), & \text{if } p = 2k \\ 0, & \text{if } p \text{ is odd} \end{cases}$$

and

$$B_{1,p} \cong \begin{cases} \mathbb{Z}/2\mathbb{Z}, & \text{if } p = 2k + 1 \\ 0, & \text{if } p \text{ is even} \end{cases}.$$ 

Hence, $D_{0,p} \cong 0$, for all $p > 0$, and for $k \geq 1$ one has short exact sequences:

$$(e_k) \quad 0 \to \mathbb{Z}(2k) \to \mathbb{R} \to D_{1,2k} \to 0$$

and

$$(o_k) \quad 0 \to \mathbb{R} \to D_{1,2k-1} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$ 

Using similar exact sequences, one concludes that

$$(29) \quad D_{R}^{1,2k-1} = \mathbb{R} \hookrightarrow D_{C}^{1,2k-1} = \mathbb{C}$$

is the inclusion as fixed point set, and the change of coefficients homomorphism is given by

$$(30) \quad D_{R}^{1,2k-1} = \mathbb{R}^{\times} \longrightarrow D_{R}^{1,2k-1} \quad x \mapsto \log |x|.$$ 

It follows that $D_{1,2k} = \mathbb{R}/\mathbb{Z}(2k)$ and that $D_{1,2k-1} \cong \mathbb{R}^{\times}$. See Table 1 for a display of the bigraded group structure of $D_{*,*}$.
3.1. **Product structure I: positive weights.** The constructions in this section are modelled after [EVSS].

**Definition 3.5.** Given $0 \leq \alpha \leq 1$ and $p, q \geq 0$, define a pairing of complexes of presheaves

$$\cup: A(p)^i_{D/R} \otimes A(q)^j_{D/R} \to A(p + q)^{i+j}_{D/R}$$

by

$$(a, \theta, \omega) \cup (a', \theta', \omega') := (a \cdot a', \theta \wedge \theta', \omega \wedge \Xi_\alpha [a', \theta'] + (-1)^i \Xi_{1-\alpha} [a, \theta]),$$

where $(a, \theta, \omega) \in A(p)^i_{Br} \oplus F_p \mathcal{E}^i \oplus \mathcal{E}^{i-1}$, $(a', \theta', \omega') \in A(q)^j_{Br} \oplus F_q \mathcal{E}^j \oplus \mathcal{E}^{j-1}$ and for $(u, \eta) \in A(p)^i_{Br} \oplus F_p \mathcal{E}^i$ one defines

$$\Xi_t(u, \eta) := (1 - t) \tau_p(u) + t \eta.$$

The following results are straightforward and their proofs are left to the reader.

**Proposition 3.6.** Fix $p, q \geq 0$ and let $X$ be a proper real holomorphic manifold.

1. The pairings $\cup$ are homotopic to each other, for all $0 \leq \alpha \leq 1$, and they induce pairings

$$\cup: H^i_{D/R}(X; A(p)) \otimes H^j_{D/R}(X; A(q)) \to H^{i+j}_{D/R}(X; A(p + q)),$$

satisfying $a \cup b = (-1)^{ij} b \cup a$.

2. The natural maps $H^*_D(X; A(p)) \to H^*_{Br}(X; A)$, for $p \geq 0$, are compatible with the respective multiplicative structures of Deligne and Bredon cohomology.

3. The natural map $\nu: H^*_D(X; A(0)) \to H^*_{Br}(X(\mathbb{C}); A)$ is a ring isomorphism.

Well-known facts from Bredon cohomology allow the computation of a few more examples.

**Example 3.7.** Let $X \in \mathcal{M}_n/R$ be a real cellular proper algebraic variety, and let $CH^*(X)$ denote its Chow ring, seen as a bigraded ring where $CH^p(X)$ has degree $(2p, p)$. As a bigraded ring, the Bredon cohomology of $X$ with $\mathbb{Z}$-coefficients is given by $H^*(X; \mathbb{Z}) \cong CH^*(X) \otimes \mathcal{B}^{*,*}$, cf. [dSLP07]. Since $CH^*(X)$ is free and Bredon cohomology is a geometric cohomology theory in the sense of [Kar00], it follows that $H^*_{Br}(X; A) \cong CH^*(X) \otimes \mathcal{B}^{*,*}$, where $\mathcal{B}^*_A := H^*_{Br}(pt, A)$ and $A \subset R$ is a subring of $\mathbb{R}$. Furthermore, the singular cohomology of $X(\mathbb{C})$ with $\mathbb{C}$-coefficients is invariant and of Hodge type $(p, p)$. Hence, the long exact sequence in the Proposition 3.2 and the above observations give natural isomorphisms:

$$H^{2p}_{D/R}(X; A(p)) \cong H^{2p}_{Br}(X(\mathbb{C}); A) \cong H^{2p}(X(\mathbb{C}); A(p))^S \cong CH^p(X) \otimes A.$$
In particular, when $X = \mathbb{P}^p$, one has $H_B^{i\ast}(\mathbb{P}^p, A) \cong B^\ast_A[h]/\langle h^{p+1} \rangle$, where $h \in H_B^{2,1}(\mathbb{P}^p, \mathbb{Z})$ is the first Chern class of the hyperplane bundle in Bredon cohomology. Define $\xi \in H_{D/R}^2(\mathbb{P}^p; A(1))$ as the element corresponding to $h$ via the isomorphisms \ref{eq:iso}. It follows that $H_{D/R}^{2j}(\mathbb{P}^p; A(j)) \cong \mathbb{Z}$ is generated by $\xi^j$, for all $0 \leq j \leq p$ and is 0 for $j > p$.

Let $\xi \in H_{D/R}^2(X \times \mathbb{P}^p; A(1))$ denote the pull-back under the projection $X \times \mathbb{P}^p \to \mathbb{P}^p$ of the class $\xi \in H_{D/R}^2(\mathbb{P}^p; A(1))$ defined above, and let $\pi : X \times \mathbb{P}^p \to X$ denote the projection onto $X$.

Given $\alpha \in H_{D/R}^i(X \times \mathbb{P}^p; A(-r))$, with $r, k \geq 0$, define $\alpha \cup \xi^k \in H_{D/R}^{i+2k}(X \times \mathbb{P}^p; A(k - r))$ as follows:

\begin{equation}
\alpha \cup \xi^k := \begin{cases} 
\alpha \cdot h^k, & \text{if } k \leq r \\
(\alpha \cdot h^r) \cup \xi^{k-r}, & \text{if } k \geq r.
\end{cases}
\end{equation}

Here we are identifying Deligne and Bredon cohomology with weights $\leq 0$, and being careful to differentiate between $h$ and $\xi$ and between the product $\alpha \cdot h^k$ in Bredon cohomology and the $\cup$ product for Deligne cohomology with positive weights.

The following result is a preliminary version of a more general projective bundle formula.

**Proposition 3.8.** Let $X$ be a proper real holomorphic manifold. Given integers $p \geq 0$ and $i, q \in \mathbb{Z}$, the map

$$
\psi : \bigoplus_{j=0}^p H_{D/R}^{i-2j}(X; A(q-j)) \longrightarrow H_{D/R}^i(X \times \mathbb{P}^p; A(q))
$$

$$(a_0, \ldots, a_p) \longmapsto \pi^*a_0 + \pi^*a_1 \cup \xi + \cdots + \pi^*a_p \cup \xi^p$$

is an isomorphism.

**Proof.** Denote by $\xi_C$ the image of $\xi$ under the cycle map into singular cohomology with complex coefficients. Since $\xi_C$ is of Hodge type $(p, p)$ and invariant under $S$, the map $a \mapsto \pi^*a \cup \xi_C^i$ from $H^{i-2j}(X(\mathbb{C}); \mathbb{C})$ to $H^i(X(\mathbb{C}) \times \mathbb{P}^p(\mathbb{C}); \mathbb{C})$ sends $F^{2-j}H^{i-2j}(X(\mathbb{C}); \mathbb{C})^S$ to $F^qH^i(X(\mathbb{C}) \times \mathbb{P}^p(\mathbb{C}); \mathbb{C})^S$.

Recall that Chern classes are compatible under the forgetful functor from Bredon cohomology to singular cohomology and that the projective bundle formula holds in both theories; cf. [IS03]. The result now follows from the definition of $\cup \xi$ in Deligne cohomology and the five-lemma suitably applied to the long exact sequences in Proposition 3.2. \hfill $\Box$

### 3.2. Product structure II: arbitrary weights

Given $p, q \geq 0$ define the product

\begin{equation}
\cup : H_{D/R}^i(X; A(-p)) \otimes H_{D/R}^j(X; A(-q)) \to H_{D/R}^{i+j}(X; A(-p-q))
\end{equation}

simply as the product in Bredon cohomology. Given $0 \leq |r| \leq p$, one uses \ref{eq:cup} to define

$$
\ast : H_{D/R}^i(X; A(-p)) \otimes H_{D/R}^j(X; A(p-r)) \to H_{D/R}^{i+j+2p}(X \times \mathbb{P}^p; A(p-r))
$$
by sending \( a \otimes b \mapsto a \star b := (a \cup \xi^p) \cup \pi^*b \). Now, let \( \pi_\uparrow: H_{D/R}^{i+j+2p}(X \times \mathbb{P}^p; A(p-r)) \rightarrow H_{D/R}^{i+j}(X; A(-r)) \) denote the composition of \( \psi^{-1} \) (see Proposition 3.8) with the projection onto the last factor. Sending \( a \otimes b \mapsto \pi_\uparrow(a \star b) \) defines the product

\[
H_{D/R}^i(X; A(-p)) \otimes H_{D/R}^j(X; A(p-r)) \rightarrow H_{D/R}^{i+j}(X; A(-r)),
\]

Combining Proposition 3.6, (34) and (35) one defines a product on Deligne cohomology which makes \( H_{D/R}^*(X; A(*)) \rightarrow H_{Br}^*(X(\mathbb{C}); A) \) a natural map of bigraded rings.

**Corollary 3.9.** Under the hypothesis of Proposition 3.8 one has an isomorphism of bigraded rings

\[
H_{D/R}^*(X; A(*))[T]/\langle T^{p+1} \rangle \cong H_{D/R}^*(X \times \mathbb{P}^p; A(*))
\]

when \( T \) is given the bigrading \((2,1)\).

The ring structure of \( D_{\star}^* \) can be easily read from Table 2. See Example 2.2 for notation.

## 4. The exponential sequence

In this section we show that \( \mathbb{Z}(1)_{D/R} \) is quasi-isomorphic to \( \mathcal{O}^\times [-1] \). As an immediate corollary, we obtain an exponential sequence relating the cohomology of \( \mathbb{Z}(1)_{Br}, \mathcal{O} \) and \( \mathcal{O}^\times \). Using this sequence we obtain a new proof of Weichold’s Theorem – a classical result in real algebraic geometry.
Proposition 4.5. There is a quasi-isomorphism \( \mathcal{O}_X \rightarrow \mathcal{O}_X^{0,1} \rightarrow \mathcal{O}_X^{0,2} \rightarrow \ldots \)

where \( \mathcal{O}_X \subset \mathcal{O}_X^{0,0} \) denotes the subsheaf of nowhere zero functions and \( \mathcal{O}_X^{p,q} \subset \mathcal{O}_X^{p+q} \) denotes the subsheaf on invariant \( (p+q) \)-forms of Hodge type \( (p,q) \).

Remark 4.2. (1) It is easy to check that \( \mathcal{O}^\times \) is a resolution of \( \mathcal{O}^\times \).

Similarly, \( \mathcal{O} \rightarrow \mathcal{O}^{0,0} \rightarrow \mathcal{O}^{0,1} \rightarrow \ldots \) is a soft resolution of \( \mathcal{O} \).

(2) The exponential map \( \mathcal{O} \rightarrow \mathcal{O}^\times \) extends to a map \( \exp : \mathcal{E}^{0,*} \rightarrow \mathcal{R}^* \) between the respective resolutions.

Definition 4.3. Recall that an element in \( \mathbb{Z}(1)_{2r}^1(U) \) is represented by sums of pairs of the form \( f \otimes m \), with \( f \) and \( m \) equivariant maps such that

1. \( f : S \rightarrow \mathbb{C}^\times \) is smooth and \( \pi : S \rightarrow U \) is in \( \tilde{U} \);
2. \( m : S \rightarrow \mathbb{Z} \in \mathbb{Z}(S) \) is locally constant.

Let \( \eta : \mathbb{Z}(1)_{2r}^1 \rightarrow \mathcal{E}^{0,0} \) be the map sending \( f \otimes m \in \mathbb{Z}(1)_{2r}^1(U) \) to

\[
\eta(f \otimes m) = \tilde{\pi}(f^m).
\]

Alternatively, \( \eta(f \otimes m)(u) = \prod_{s \in \pi^{-1}(u)} f(s)^{m(s)} \).

Remark 4.4. It follows from this definition that, given \( \sigma \in \mathbb{Z}(1)_{2r}^0 \) and \( \alpha \in \mathbb{Z}(1)_{2r}^1 \), we have \( \tau_0^0 \sigma = \log \eta(\sigma(1)) - \log \eta(\sigma(0)) \) and \( \tau_1^1 \alpha = d \log \eta \alpha \).

Proposition 4.5. There is a quasi-isomorphism \( \zeta : \mathbb{Z}(1)_{2r} \rightarrow \mathcal{R}^*[1] \) such that the composite \( \mathcal{E}^*[-1] \rightarrow \mathbb{Z}(1)_{2r} \rightarrow \mathcal{R}^*[1] \) induces the map \( \exp : \mathcal{O}_X[-1] \rightarrow \mathcal{O}_X^\times[-1] \), and the composition \( \mathcal{E}_X^{0,*}[-1] \rightarrow \mathbb{Z}(1)_{2r} \rightarrow \mathcal{R}^*[1] \) induces the map \( \exp : \mathcal{O}_X[-1] \rightarrow \mathcal{O}_X^\times[-1] \).

Proof. Let \( X \) be a projective real variety and let \( \zeta : \mathbb{Z}(1)_{2r} \rightarrow \mathcal{R}^*[1] \) denote the map of complexes displayed in the following diagram:

\[
\begin{array}{cccccccc}
\ldots & \rightarrow & \mathbb{Z}(1)_{2r}^0 & \rightarrow & \mathbb{Z}(1)_{2r}^1 & \rightarrow & \mathcal{E}^{1,0} & \rightarrow & \mathcal{E}^{0,1} & \rightarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\ldots & \rightarrow & \mathbb{R}^0 & \rightarrow & \mathcal{E}^{1,0} & \rightarrow & \mathcal{E}^{0,1} & \rightarrow & \ldots \\
\end{array}
\]

where \( \eta \cdot \exp \) denotes the map \( (f, \omega, h) \mapsto \eta(f) \exp h \), and \( p_{0,1}^0 \) denotes the projection from \( \mathcal{E}^1 \) to \( \mathcal{E}^{0,1} \). Similarly, for \( i > 1 \) we set \( \zeta^i = p_{0,i}^i \), where \( p_{0,i}^i : \mathcal{E}^i \rightarrow \mathcal{E}^{0,i} \) is the projection.

It is easy to check \( \zeta \) is a map of cochain complexes and that the cohomology presheaves of both complexes are concentrated in dimension 1. Hence it suffices to check that \( H^1(\zeta) \) is an isomorphism on the stalks.

Surjectivity: Note that \( H^1(\mathcal{R}^*[1]) = H^0(\mathcal{R}^*) \cong \mathcal{O}^\times \) and let \( g \in \mathcal{O}^\times_u \), \( u \in X \). Let \( h \in \mathcal{O}_u \) be such that \( \exp h = g \). Set \( \alpha = 0 \) in \( \mathbb{Z}(1)_{2r,u} \) (so that \( \alpha \) can be
represented, for example, by 1 ⊗ 0 and set \( w = \frac{\partial w}{g} \in E_u^{1,0} \). Then we have, 
\[
d^0_{\mathbb{Z}(1)_{\mathbb{D}/\mathbb{R}}} (\alpha, \omega, h) = 0 \quad \text{and} \quad \zeta u (\alpha, \omega, h) = g.
\]

**Injectivity**: Let \((\alpha, \omega, h) \in \mathbb{Z}(1)_{\mathbb{D}/\mathbb{R}}, u \in X\), be such that \(\zeta u (\alpha, \omega, h) = 0 \) and \(d^0_{\mathbb{Z}(1)_{\mathbb{D}/\mathbb{R}}} (\alpha, \omega, h) = 0\). The first equality just means that \(-h\) is a logarithm for \(\eta(\alpha)\). From the second equality we get
\[
\omega = -dh - \tau^1_1 \alpha = -(dh + d \log \eta(\alpha)) = 0.
\]

Let \(\sum_i f_i \otimes m_i\) be a representative for \(\alpha\) (cf. Definition 4.5). Choose \(\log f_i\) so that \(\sum_i m_i \log f_i = -h\), shrinking neighborhoods if necessary, and define
\[
\sigma := \left( \Delta^1 \ni t \mapsto \exp \left[ -t \sum_i m_i \log f_i \right] \right) \in \mathbb{Z}(1)^0_{\mathbb{D}/\mathbb{R}, u}.
\]

Then, by our choice of \(\log f_i\), we have \(d^1_{\mathbb{Z}(1)_{\mathbb{D}/\mathbb{R}}} (\sigma) = (\alpha, \omega, h)\).

The remaining assertions are evident. \(\square\)

**Corollary 4.6.** The complexes \(\mathbb{Z}(1)_{\mathbb{D}/\mathbb{R}}\) and \(\mathfrak{O}_X^\times[-1]\) are quasi-isomorphic.

**Corollary 4.7 (Exponential Sequence).** Let \(X\) be a smooth proper real algebraic variety. Then there is a long exact sequence
\[
\to H^*_{\text{Br}}(X(\mathbb{C}); \mathbb{Z}) \overset{\vartheta}{\to} H^*(X; \mathfrak{O}_X) \overset{\exp}{\to} H^*(X; \mathfrak{O}_X) \to H^*_{\text{Br}}(X(\mathbb{C}); \mathbb{Z}) \to
\]
where \(\vartheta\) denotes the composite
\[
H^*_{\text{Br}}(X(\mathbb{C}); \mathbb{Z}) \overset{\tau}{\to} H^*(X(\mathbb{C}); \mathfrak{C})^{\mathfrak{O}} \to H^*(X; \mathfrak{O}_X) = H^*_{\text{Br}}(X(\mathbb{C}))^{\mathfrak{O}},
\]
and the latter denotes the invariants of the Dolbeault cohomology of the complex manifold \(X(\mathbb{C})\).

**Proof.** From the hypercohomology long exact sequence of the cone and Lemma 4.5 we get the following exact sequence
\[
\to H^*_{\text{Br}}(X(\mathbb{C}); \mathbb{Z}) \oplus \tilde{H}^*(X; F^1 E_X^*) \overset{\tau}{\to} \tilde{H}^*(X; E_X^*) \to H^*(X; \mathfrak{O}_X^\times) \to
\]
which, in turn, gives
\[
H^*_{\text{Br}}(X(\mathbb{C}); \mathbb{Z}) \overset{\tau}{\to} \text{coker} \left( \tilde{H}^*(X; F^1 E_X^*) \to \tilde{H}^*(X; E_X^*) \right) \to H^*(X; \mathfrak{O}_X^\times).
\]

Now, since \(\text{Cone}(F^1 E^* \to E^*) \simeq E^{0,*}\) and since \(\tilde{H}^*(X; F^1 E_X^*) \to \tilde{H}^*(X; E_X^*)\) is injective, we have
\[
\text{coker} \left( \tilde{H}^*(X; F^1 E_X^*) \to \tilde{H}^*(X; E_X^*) \right) \cong \tilde{H}^*(X; E_X^{0,*}) \cong H^*(X; \mathfrak{O}_X).
\]

The assertion about the map \(\vartheta\) follows immediately from the construction of this sequence. \(\square\)

**Remark 4.8.** A similar exact sequence appears in \([\text{Kra91}]\).
4.1. **An application.** Given a real variety $X$, denote $S = \pi_0(X(\mathbb{R}))$. Hence $H^0(X(\mathbb{R}); \mathbb{Z}^\times) \cong (\mathbb{Z}^\times)^S$ and $\tilde{H}^0(X(\mathbb{R}); \mathbb{Z}^\times) \cong (\mathbb{Z}^\times)^S/\mathbb{Z}^\times$, where $\mathbb{Z}^\times \subset (\mathbb{Z}^\times)^S$ is the subgroup of constant functions.

**Lemma 4.9.** Let $X$ be an irreducible, smooth, projective curve over $\mathbb{R}$, of genus $g$. Let $c$ denote the number of connected components of $X(\mathbb{R})$. Then

$$H^2_{Br}(X(\mathbb{C}); \mathbb{Z}) \cong \mathbb{Z} \times (\mathbb{Z}^\times)^S/\mathbb{Z}^\times \cong \mathbb{Z} \times \left\{ \begin{array}{ll} (\mathbb{Z}/2)^{c-1} & \text{if } c \neq 0 \\ 0 & \text{if } c = 0 \end{array} \right\}$$

**Proof.** By Poincaré duality $H^2_{Br}(X(\mathbb{Z}); \mathbb{Z}) \cong H_0^S(X(\mathbb{C}); \mathbb{Z})$, where the last group denotes $\mathbb{Z}$-graded Bredon homology. The following exact sequence is well known (see [LLFM03])

$$0 \to H^0_{\text{sing}}(X(\mathbb{C})/\mathcal{G}; \mathbb{Z}) \to H^0_{Br}(X(\mathbb{C}); \mathbb{Z}) \to H_0(X(\mathbb{R}); \mathbb{Z}^\times) \to 0.$$  

If $X(\mathbb{R}) = \emptyset$ then the sequence gives $H^0_{\text{sing}}(X(\mathbb{C})/\mathcal{G}) \cong \mathbb{Z}$ since under the above assumptions $X(\mathbb{C})/\mathcal{G}$ is connected.

If $X(\mathbb{R}) \neq \emptyset$ then $H^0_{Br}(X(\mathbb{C}); \mathbb{Z}) \cong \mathbb{Z} \times \tilde{H}^0_{Br}(X(\mathbb{C}); \mathbb{Z})$, where the $\tilde{H}^0_{Br}(\cdot; \mathbb{Z})$ denotes reduced Bredon homology. There is a reduced version of the sequence above which gives $H^0_{Br}(X(\mathbb{C}); \mathbb{Z}) \cong \tilde{H}_0(X(\mathbb{R}); \mathbb{Z}^\times) \cong (\mathbb{Z}^\times)^{c-1}$, because $\tilde{H}^0_{\text{sing}}(X(\mathbb{C})/\mathcal{G}; \mathbb{Z}) = 0$.  

Denote $V := H^1(X_C, \emptyset)$ and let $\Lambda \subset V$ be the lattice

$$\Lambda := \text{Im}(j_C: H^1_{\text{sing}}(X_C, \mathbb{Z}(1)) \to H^1(X_C, \emptyset)),$$

so that $\text{Pic}(X_C) \cong V/\Lambda$. Taking fixed points gives a short exact sequence

$$(36) \quad 0 \to \Lambda^\mathcal{G} \to V^\mathcal{G} \to \text{Pic}_0(X_C)^\mathcal{G} \to H^1(\mathcal{G}, \Lambda).$$

This shows that $V^\mathcal{G}/\Lambda^\mathcal{G}$ is the connected component $\text{Pic}_0(X_C)^\mathcal{G}$. Now, Proposition 4.1 shows that $H^1_{Br}(X(\mathbb{C}), \mathcal{A}) \cong H^1_{\text{bos}}(X(\mathbb{C}), \mathcal{A})$, hence one can use the Leray-Serre spectral sequence to conclude that the image of the natural map $j: H^1_{Br}(X(\mathbb{C}), \mathcal{Z}) \to H^1(X_C, \emptyset)$ is precisely $\Lambda^\mathcal{G}$.

**Proposition 4.10.** Let $X$ be an irreducible, smooth, projective curve over $\mathbb{R}$, of genus $g$. Then

$$\text{Pic}(X) \cong \text{Pic}_0(X_C)^\mathcal{G} \times H^2_{Br}(X(\mathbb{C}), \mathbb{Z}).$$

**Proof.** By the previous results, $\text{Pic}(X)$ fits in the following exact sequence

$$H^1_{Br}(X(\mathbb{C}), \mathcal{Z}) \xrightarrow{j} H^1(X_C, \emptyset)^\mathcal{G} \to \text{Pic}(X) \xrightarrow{\text{c_1}} H^2_{Br}(X(\mathbb{C}), \mathbb{Z}) \to 0.$$

The result follows.

**Corollary 4.11** (Weichold’s Theorem [PW91, Proposition 1.1]). With $X$ as above, let $c$ denote the number of connected components of $X(\mathbb{R})$. Then

$$\text{Pic}(X) \cong \mathbb{Z} \times (\mathbb{R}/\mathbb{Z})^g \times \left\{ \begin{array}{ll} (\mathbb{Z}/2)^{c-1} & \text{if } c \neq 0 \\ 0 & \text{if } c = 0 \end{array} \right\}$$

**Proof.** This follows directly from the proposition together with Lemma 4.9.
5. The group $H^2_{D/\mathbb{R}}(X; \mathbb{Z}(2))$

In this section we derive a geometric interpretation of the integral cohomology group $H^2_{D/\mathbb{R}}(X; \mathbb{Z}(2))$ for a real projective variety $X$. As a motivation we start with a geometric interpretation of the Bredon cohomology group $H^2_{Br}(Y, \mathbb{Z})$ of an arbitrary $\mathcal{G}$-manifold $Y$.

5.1. Variations on the theme of $H^2_{Br}(Y, \mathbb{Z})$. As a $\mathcal{G}$-manifold, the sphere $S^{2,2}$ is isomorphic to $\mathbb{P}^1(\mathbb{C})$ with the action induced by a linear involution $\sigma$. Describing the action in homogeneous coordinates by $\sigma: [x_0 : x_1] \mapsto [x_0 : -x_1]$, one sees that $SP_{\infty}(\mathbb{P}^1(\mathbb{C})) \equiv \mathbb{P}^\infty(\mathbb{C})$ inherits the linear involution $\sigma([x_0 : x_1 : x_2 : \cdots]) = [x_0 : -x_1 : x_2 : -x_3 : \cdots]$ and that this is an equivariant $K(\mathbb{Z}, (2,2))$; cf. [dS03b]. In particular, one can define a linear isomorphism of tautological bundles $\tau: \sigma^*\mathcal{O}(-1) \to \mathcal{O}(-1)$ satisfying $\tau \circ (\sigma^*\tau) = 1$.

Given a $\mathcal{G}$-space $Y$, with involution $\sigma$, let $P_1(Y)$ denote the set of pairs $(L, \tau)$ where:

- $t1)$ $L$ is a (smooth) complex line bundle on $Y$
- $t2)$ $\tau: \sigma^*L \to L$ is a bundle isomorphism;
- $t3)$ $\tau \circ (\sigma^*\tau) = 1$.

Define an equivalence relation on $P_1(L)$ by $(L, \tau) \sim_1 (L', \tau')$ if there exists an isomorphism $\phi: L \to L'$ such that $\phi \circ \tau = \tau' \circ \sigma^*\phi$.

**Lemma 5.1.** The tensor product of line bundles induces a group structure on the set $L_1(Y) := P_1(Y)/\sim_1$ of equivalence classes of pairs satisfying $t1)$–$t3)$. Furthermore, this group is naturally isomorphic to $H^2_{Br}(Y, \mathbb{Z})$.

**Proof.** The first assertion is clear and the last one follows from the fact that $\mathbb{CP}^{\infty}$ with the linear involution $\sigma$ described above is an equivariant $K(\mathbb{Z}, (2,2))$. $\square$

Recall that a Real vector bundle $(E, \tau)$ on a $\mathcal{G}$-manifold $(Y, \sigma)$ consists of a complex vector bundle $E$ on $Y$ together with an isomorphism $\tau: \sigma^*E \to E$ satisfying $\tau \circ (\sigma^*\tau) = 1$. Now, consider the set $P_2(Y)$ consisting of pairs $(L, q)$ satisfying:

- $p1)$ $L$ is a (smooth) complex line bundle on $Y$;
- $p2)$ $q: L \otimes \sigma^*L \to 1_Y$ is an isomorphism of Real line bundles, where $L \otimes \sigma^*L$ carries the tautological Real line bundle structure;

Denote $(L, q) \sim_2 (L', q')$ iff there is an isomorphism $\phi: L \to L'$ satisfying $q' \circ (\phi \otimes \sigma^*\phi) = q$ and observe that this is an equivalence relation on $P_2(Y)$.

**Lemma 5.2.** The tensor product also induces a group structure on the set $L_2(Y) := P_2(Y)/\sim_2$ of isomorphism classes of pairs $(L, q)$ satisfying $p1)$–$p2$.

Finally, consider the complex of sheaves $G^0 \xrightarrow{a} G^1$ on $\mathcal{G}$-$\text{Man}$ where

\[(37) \quad G^0(U) = \{ f: U \to \mathbb{C}^\times \mid f \text{ is smooth} \} \]
where $a: G^0 \to G^1$ is the “transfer map” $a(f) = f \cdot \overline{\sigma f}$.

Proposition 5.3. There are natural isomorphisms

$$H^{2,2}_{Br}(Y, \mathbb{Z}) \cong H^{2,2}_{tor}(Y, \mathbb{Z}) \cong L_1(Y) \cong L_2(Y) \cong H^2(Y_{eq}; G^0) \to G^1).$$

Proof. The first two isomorphisms follow from Lemma 5.1 and Proposition A.1, respectively, and the last isomorphism is a tautology.

Given $(L, \tau) \in P_1(Y)$, pick a hermitian metric $h: L \otimes L^* \to 1_Y$ on $L$ and define $q^h_y: L \otimes \sigma^*L \to 1_Y$ as the composition $L \otimes \sigma^* L \xrightarrow{1\otimes \tau} L \otimes L \to 1_Y$, where $\tau: \sigma^*L \to L$ is the map induced by $\tau$. It is easy to see that $q^h_y$ is an isomorphism of Real line bundles, and hence we obtain an element $(L, q^h_y) \in P_2(Y)$.

Suppose that $\psi: L' \to L$ induces an equivalence $(L', \tau') \sim_1 (L, \tau)$ and pick a hermitian metric $h$ on $L$. One sees that $(L', q^{\psi^*h}) \sim_2 (L, q^h_l)$. It follows that one has a well-defined homomorphism $L_1(Y) \to L_2(Y)$ sending $[L, \tau]$ to $\langle L, q^h_l \rangle$, where $h$ is any choice of metric on $L$. The construction of the inverse homomorphism is evident. \hfill $\square$

Let $S = \pi_0(Y^\Sigma)$ denote the set of connected components of the fixed point set $Y^\Sigma$, and identify $H^0(Y^\Sigma; \mathbb{Z}^\times) \cong (\mathbb{Z}^\times)^S$. Observe that the Künneth formula yields a natural isomorphism $H^{2,2}_{tor}(Y^\Sigma, \mathbb{Z}) \cong H^2(Y^\Sigma \times B\Sigma; \mathbb{Z}(2)) \cong H^0(Y^\Sigma, \mathbb{Z}^\times) \oplus H^2_{sing}(Y^\Sigma; \mathbb{Z}(2))$. Using the first identification in Proposition 5.3 one considers the composition

$$H^{2,2}_{Br}(Y, \mathbb{Z}) \cong H^{2,2}_{tor}(Y; \mathbb{Z}) \to H^{2,2}_{tor}(Y^\Sigma; \mathbb{Z}) \to H^0(Y^\Sigma; \mathbb{Z}^\times)$$

of the restriction map followed by the evident projection to obtain a natural homomorphism

$$\mathcal{N}: H^{2,2}_{Br}(Y, \mathbb{Z}) \to (\mathbb{Z}^\times)^S.$$

This map has a natural geometric interpretation when one uses the identification $H^{2,2}_{Br}(Y, \mathbb{Z}) \cong L_2(Y)$. Given $\langle L, q \rangle \in L_2(Y)$, the restriction of $q$ to $L_{Y^\Sigma}$ becomes a non-degenerate hermitian pairing, and hence it has a well-defined signature $\mathcal{N}(L, q) \in (\mathbb{Z}^\times)^S$. It is easy to see that this is another description of (39).

Definition 5.4. We call $\mathcal{N}: H^{2,2}_{Br}(Y, \mathbb{Z}) \to (\mathbb{Z}^\times)^S$ the equivariant signature map of $Y$. The image $\mathcal{N}_{tor}(Y) \subseteq (\mathbb{Z}^\times)^S$ of the torsion subgroup $H^{2,2}_{Br}(Y, \mathbb{Z})_{tor}$ under $\mathcal{N}$ is called the equivariant signature group of $Y$. In the case where $Y = X(\mathbb{C})$ for a real algebraic variety $X$ with $S = \pi_0(X(\mathbb{R}))$, we denote the equivariant signature group of $X(\mathbb{C})$ simply by $\mathcal{N}_{tor}(X)$.

Example 5.5. When $X$ is a projective algebraic curve, it follows from the cohomology sequence of the pair $(E\Sigma \times_\Sigma X(\mathbb{C}), B\Sigma \times X(\mathbb{R}))$ that $\mathcal{N}$ is an isomorphism. As a consequence, one obtains isomorphisms $H^{2,2}_{Br}(X(\mathbb{C}), \mathbb{Z}) \cong$
Br(X) \cong \mathcal{N}_{\text{tor}}(X), \text{ where } Br(X) \text{ is the Brauer group of } X, \text{ since } Br(X) \cong (\mathbb{Z}^\times)^2 \text{ when } X \text{ is an algebraic curve; cf. [Wit33].}

5.2. The Deligne group $H^2_{DR}(X; \mathbb{Z}(2))$. The Hodge filtration on singular cohomology induces a filtration on Bredon cohomology, with

$$F^jH^0_{\text{Br}}(X(\mathbb{C}), \underline{\mathbb{Z}}) := \varphi^{-1}j^{-1}F^jH^0_{\text{sing}}(X(\mathbb{C}), \mathbb{C})$$

see diagram (27) for notation.

**Proposition 5.6.** For any smooth real projective variety $X$, one has

$$F^2H^2_{Br}(X(\mathbb{C}), \underline{\mathbb{Z}}) = H^2_{Br}(X(\mathbb{C}), \underline{\mathbb{Z}})_{\text{tor}} = \text{im}(\varphi),$$

where $\varphi$ is the cycle map from Deligne to Bredon cohomology (27). In particular the image of the composition

$$\Psi : H^2_{DR}(X; \mathbb{Z}(2)) \xrightarrow{\varphi} H^2_{Br}(X(\mathbb{C}), \underline{\mathbb{Z}}) \xrightarrow{\cong} H^0(X(\mathbb{R}), \mathbb{Z}^\times)$$

is the equivariant signature group $\mathcal{N}_{\text{tor}}(X)$.

**Proof.** Consider diagram (27) with $p = n = 2$. Since the middle row is exact, one concludes that $\text{im}(\varphi) = F^2H^2_{Br}(X, \underline{\mathbb{Z}})$, by definition. On the other hand, $j \circ \varphi : H^2_{Br}(X(\mathbb{C}), \underline{\mathbb{Z}}) \rightarrow H^2(X(\mathbb{C}); \mathbb{C})$ factors as $H^2_{Br}(X(\mathbb{C}), \underline{\mathbb{Z}}) \rightarrow H^2_{Br}(X(\mathbb{C}), \underline{\mathbb{Z}})_{\text{tor}} \hookrightarrow H^2_{Br}(X(\mathbb{C}), \underline{\mathbb{Z}}) \otimes \mathbb{Q} \hookrightarrow H^2_{\text{sing}}(X(\mathbb{C}); \mathbb{Z}(2)) \otimes \mathbb{Q} \hookrightarrow H^2(X(\mathbb{C}); \mathbb{C})$. The injectivity of $H^2_{Br}(X(\mathbb{C}), \underline{\mathbb{Z}}) \otimes \mathbb{Q} \hookrightarrow H^2_{\text{sing}}(X(\mathbb{C}); \mathbb{Z}(2)) \otimes \mathbb{Q}$ follows from the isomorphism $H^2_{Br}(X(\mathbb{C}), \underline{\mathbb{Z}}) \cong H^2_{\text{bor}}(X(\mathbb{C}); \underline{\mathbb{Z}})$ and well-known facts in equivariant cohomology. Since

$$H^2_{\text{sing}}(X(\mathbb{C}); \mathbb{Z}(2)) \otimes \mathbb{Q} \cap F^2H^2(X(\mathbb{C}); \mathbb{C}) = 0,$$

one concludes that

$$F^2H^2_{Br}(X(\mathbb{C}), \underline{\mathbb{Z}}) = H^2_{Br}(X(\mathbb{C}), \underline{\mathbb{Z}})_{\text{tor}}.$$

The next goal is to provide a geometric interpretation to the kernel of the surjection $\Psi : H^2_{DR}(X; \mathbb{Z}(2)) \rightarrow \mathcal{N}_{\text{tor}}(X)$. To this purpose, consider triples $(L, \nabla, \mathbf{q})$ where $L$ is a holomorphic line bundle over $X(\mathbb{C})$, $\nabla$ is a holomorphic connection on $L$ and $\mathbf{q} : L \otimes \sigma^*L \rightarrow 1$ is a holomorphic isomorphism of Real line bundles satisfying the following properties:

1. The restriction of $\mathbf{q}$ to $X(\mathbb{R})$ is a positive-definite hermitian metric on $L|_{X(\mathbb{R})}$.

2. As a section of $(L \otimes \sigma^*L)^\vee$, $\mathbf{q}$ is parallel with respect to the connection induced by $\nabla$.

A morphism between two such triples $f : (L, \nabla, \mathbf{q}) \rightarrow (L', \nabla', \mathbf{q}')$ consists of a line bundle map $f : L \rightarrow L'$ such that $\mathbf{q}' \circ (f \otimes \sigma'f) = \mathbf{q}$ and $\nabla' \circ f = (1 \otimes f) \circ \nabla$. 


Definition 5.7. Given a real variety \( X \), let \( \text{PW}^\nabla(X) \) denote the set of isomorphism classes \( \langle L, \nabla, \mathfrak{q} \rangle \) of triples as above. This is a group under the operation
\[
\langle L, \nabla, \mathfrak{q} \rangle \circ \langle L', \nabla', \mathfrak{q}' \rangle := \langle L \otimes L', \nabla \otimes 1 + 1 \otimes \nabla', \mathfrak{q} \cdot \mathfrak{q}' \rangle,
\]
which we call the differential Picard-Witt group of \( X \).

The group \( \text{PW}^\nabla(X) \) has an alternative definition in terms of a complex \( \mathfrak{p}^* : \mathfrak{p}^0 \xrightarrow{D} \mathfrak{p}^1 \xrightarrow{D} \mathfrak{p}^2 \) of presheaves on \( \mathcal{A}n_{/\mathbb{R}} \). Given a real analytic variety \( U \), define
\[
\mathfrak{p}^0(U) := \mathcal{O}^\times_C(U) = \{ f : U \to \mathbb{C}^\times \mid f \text{ is holomorphic} \},
\]
and
\[
\mathfrak{p}^1(U) := \Omega^1_C(U) \oplus \mathcal{O}^\times_{\mathbb{R}^+}(U),
\]
where \( \Omega^1_C(U) \) denotes the holomorphic 1-forms on \( U \) and \( \mathcal{O}^\times_{\mathbb{R}^+}(U) \) denotes the subgroup of \( \mathcal{O}^\times_C(U) \) consisting of those holomorphic Real functions \( f \) (i.e. \( \sigma^* f = f \)) which are positive on the real locus \( U(\mathbb{R}) := \{ u \in U \mid u \text{ is real} \} \). Finally, define \( \mathfrak{p}^2(U) \) as the group of Real holomorphic 1-forms on \( U \), in other words
\[
\mathfrak{p}^2(U) := \Omega^1_R(U) := \{ \psi \in \Omega^1_C(U) \mid \sigma^* \psi = \psi \}.
\]
Define \( D : \mathfrak{p}^0(U) \to \mathfrak{p}^1(U) \) as \( \mathfrak{p}^1(U) := (dg/g, g \cdot \sigma^* g) \) and \( D : \mathfrak{p}^1(U) \to \mathfrak{p}^2(U) \) as \( D(\psi, f) =: \psi + \sigma^* \psi - df/f \).

Proposition 5.8. \( \text{PW}^\nabla(X) \) is naturally isomorphic to \( \mathbb{H}^1(X_{\text{eq}}; \mathfrak{p}^*) \).

Proof. This is straightforward. \( \square \)

Remark 5.9. Given \( \langle L, \nabla, \mathfrak{q} \rangle \) in \( \text{PW}^\nabla(X) \) one can always find an equivariant cover \( U \) and a cocycle \( \mathfrak{c} \) in \( \check{\mathcal{C}}^1(U; \mathfrak{p}^*) \) representing \( \langle L, \nabla, \mathfrak{q} \rangle \) that has the form \( \mathfrak{c} = \left( (g_{a_0 a_1}) ; (\psi_{a_0}), (1) \right) \). In other words, \([\mathfrak{c}]\) is determined by \((g_{a_0 a_1}), (\psi_{a_0})\) satisfying:

i: \( g_{a_0 a_1} \in \Omega^0_C(U_{a_0 a_1}) \) and \( \delta(g_{a_0 a_1}) = 1 \) (gives the cocycle condition for a holomorphic line bundle \( L \));

ii: \( \frac{dg_{a_0 a_1}}{g_{a_0 a_1}} = \delta(\psi_{a_0}) \) (gives the holomorphic connection on \( L \));

iii: \( g_{a_0 a_1} \cdot \sigma^* g_{a_0 a_1} = 1 \) (gives the hermitian form \( \mathfrak{q} \));

iv: \( \psi_{a_0} \in \Omega^1_C(U) \) and \( \psi_{a_0} + \sigma^* \psi_{a_0} = 0 \), i.e. \( \psi_{a_0} \) is a holomorphic anti-invariant 1-form (\( \mathfrak{q} \) is parallel).

The main result of this section is the following.

Theorem 5.10. If \( X \) is a smooth real projective variety then one has a natural short exact sequence
\[
0 \to \text{PW}^\nabla(X) \to H^2_{D/\mathbb{R}}(X; \mathbb{Z}(2)) \xrightarrow{\varphi} \mathcal{K}_{\text{tor}}(X) \to 0.
\]

Proof. See Appendix \( \square \)
6. A REMARK ON NUMBER FIELDS

Let $F$ be a number field and let $\Gamma_R$ and $\Gamma_C$ denote the sets of real and complex embeddings of $F$, respectively. One can write $\Gamma_C = \Gamma^+_C \times \mathcal{S}$, where $\Gamma^+_C$ contains one chosen element in each $\mathcal{S}$-orbit of $\Gamma_C$.

Abusing language, write $H^r_{D/R}(F; A(p))$ instead of $H^r_{D/R}(X; A(p))$, where $X := Spec(F \otimes_\mathbb{Q} \mathbb{R})$. Observe that $X(\mathbb{C}) = \Gamma_R \coprod \Gamma_C$, and hence

$$H^r_{D/R}(F; A(p)) = H^r_{D/R}(\Gamma_R; A(p)) \times H^r_{D/R}(\Gamma_C; A(p))$$

$$(40) \quad \equiv H^r_{D/R}(\ast; A(p))^{\Gamma_R} \times H^r_{D/C}(\ast; A(p))^{\Gamma_C^+}$$

where $A$ is a subring of $\mathbb{R}$ and $\Gamma_R = \{ \varphi_1, \ldots, \varphi_s \}$ and $\Gamma_C^+ = \{ \eta_1, \ldots, \eta_t \}$. In particular,

$$H^1_{D/R}(F; \mathbb{Z}(1)) \equiv (\mathbb{R}^x)^{\Gamma_R} \times (\mathbb{C}^x)^{\Gamma_C^+} \cong (\mathbb{R}^x)^s \times (\mathbb{C}^x)^t$$

and

$$H^1_{D/R}(F; \mathbb{R}(1)) \equiv \mathbb{R}^{\Gamma_R} \times \mathbb{R}^{\Gamma_C^+} \cong \mathbb{R}^s \times \mathbb{R}^t;$$

cf. Example 3.4. Taking adjoints to the evaluation maps $F^x \times \Gamma_R \to \mathbb{R}^x$ and $F^x \times \Gamma_C^+ \to \mathbb{C}^x$ gives a monomorphism

$$(42) \quad F^x \to (\mathbb{R}^x)^{\Gamma_R} \times (\mathbb{C}^x)^{\Gamma_C^+} \equiv H^1_{D/R}(F; \mathbb{Z}(1)).$$

Since $\bigoplus_{p \geq 0} H^p_{D/R}(F; \mathbb{Z}(p))$ is a graded commutative ring this map induces a homomorphism

$$(43) \quad \rho: T(F^x) \to \bigoplus_{p \geq 0} H^p_{D/R}(F; \mathbb{Z}(p)),$$

where $T(F^x)$ is the tensor algebra of $F^x$. Using the commutativity of the diagram

$$\begin{array}{ccc}
F^x \otimes F^x & \overset{\rho}{\longrightarrow} & H^1_{D/R}(F; \mathbb{Z}(1)) \otimes H^1_{D/C}(F; \mathbb{Z}(1)) \longrightarrow H^2_{D/R}(F; \mathbb{Z}(2)) \\
\downarrow_{e \otimes \phi} & & \downarrow_{\phi} \\
H^{1,1}_{Br}(F, \mathbb{Z}) \otimes H^{1,1}_{Br}(F, \mathbb{Z}) & \cong & H^{2,2}_{Br}(F, \mathbb{Z}),
\end{array}$$

together with the description of the ring structure of the Bredon cohomology of a point [22] one concludes that if $a \neq 0, 1$, then $\phi(a \otimes (1 - a)) = 0$. It follows that $\phi$ descends to a homomorphism

$$(44) \quad \tilde{\phi}: K^M_*(F) \to \bigoplus_{p \geq 0} H^p_{D/R}(F; \mathbb{Z}(p)),$$

from the Milnor $K$-theory ring of $F$ to the “diagonal” subring of the integral Deligne cohomology of $F$.

**Remark 6.1.** i. If follows from the work of Bass and Tate that

$$K^M_*(\mathbb{R})/2K^M_2(\mathbb{R}) \cong \oplus_i D^{i,i}$$

and

$$K^M_*(\mathbb{R})/2K^M_1(\mathbb{R}) \cong \mathbb{Z}/2[\varepsilon] \cong \oplus_i B^{i,i}.$$
ii. Since \( \pi: \text{Spec}(F \otimes \mathbb{Q}) \to \text{Spec}(\mathbb{R}) \) is a finite cover, one has an additive transfer homomorphism
\[
\pi_1: H^1_{\mathbb{D}/\mathbb{R}}(F; \mathbb{R}(1)) \to H^1_{\mathbb{D}/\mathbb{R}}(\mathbb{R}; \mathbb{R}(1))
\]
\[
(x_1, \ldots, x_s, y_1, \ldots, y_t) \mapsto x_1 + \cdots + x_s + 2y_1 + \cdots + 2y_t;
\]
see (41).

iii. In subsequent work we will show that the homomorphism (44) is a particular case of a natural transformation between the motivic cohomology of a real variety and its integral Deligne cohomology.

It follows from (41) and (30) that the composition
\[
F^\times \to H^1_{\mathbb{D}/\mathbb{R}}(F; \mathbb{Z}(1)) \to H^1_{\mathbb{D}/\mathbb{R}}(F; \mathbb{R}(1))
\]
is given by
\[
(46) \quad F^\times \to \mathbb{R}^s \times \mathbb{R}^t
\]
\[
x \mapsto (\log |\varphi_1(x)|, \ldots, \log |\varphi_s(x)|; \log |\eta_1(x)|, \ldots, \log |\eta_t(x)|).
\]
Basic class field theory shows that the image of the units \( \alpha_F^\circ \) of the rings of integers of \( F \) under this map is a lattice \( L \) in the hyperplane \( H : x_1 + \cdots + x_s + 2y_1 + \cdots + 2y_t = 0 \), i.e., the kernel of the transfer homomorphism (45).

Therefore, the Euclidean volume of this lattice in \( H \) is given by \( \text{Vol}(L) = \sqrt{\frac{2\pi}{2}} R \), where \( R \) is the classical regulator of \( F \).

**Appendix A. The Borel/Esnault-Viehweg version**

Given any equivariant cohomology theory \( h^* \) on \( \mathcal{S} \)-spaces, one can define its corresponding **Borel version** \( h^*_{\text{bor}} \) as \( h^*_{\text{bor}}(U) := h^*(U \times E\mathcal{S}) \), where \( E\mathcal{S} \) is a contractible \( \mathcal{S} - \text{CW-complex} \) on which \( \mathcal{S} \) acts freely. In particular, one has an associated Borel cohomology theory \( H^p_{\text{bor}}(X, A) \). This theory is (0,2) periodic and can be more easily calculated than Bredon cohomology, via Leray-Serre spectral sequences
\[
E^{r,s}(p) := H^r(\mathcal{S}, K^s(X; A(p))) \Rightarrow H^{r+s,p}_{\text{bor}}(X(\mathbb{C}), A).
\]

It is easy to see that the natural map \( \mathbb{Z}_0(S^p) \to F(E\mathcal{S}, \mathbb{Z}_0(S^p)) \) is an equivariant homotopy equivalence, thus giving the following result.

**Proposition A.1.** Let \( Y \) be a \( \mathcal{S} \)-space. For all \( p \geq 0 \) and \( n \leq p \) one has a natural isomorphism \( H^{n,p}_{\text{bor}}(Y, A) \cong H^{n,p}_{\text{bor}}(Y, A) \).

In order to translate this construction into our context, denote \( X := (\text{Spec} \mathbb{C})_{/\mathbb{R}} \) and let \( E_\bullet \mathcal{S} := N(X \to \text{Spec} \mathbb{R}) \) be the nerve of the cover \( X \to \text{Spec} \mathbb{R} \). This is a smooth simplicial real projective variety with the property that \( E_\bullet \mathcal{S}(\mathbb{C}) \) is a simplicial object in \( A\mathcal{N}_{/\mathbb{R}} \) whose geometric realization \( |E_\bullet \mathcal{S}(\mathbb{C})| \) is a model for \( E\mathcal{S} \). In particular, \( \mathcal{Z}(E_\bullet \mathcal{S}(\mathbb{C})) \) defines a simplicial abelian presheaf on \( \mathcal{S} \text{-Man} \) whose associated complex (graded in negative degrees) is denoted \( \mathcal{Z}(E\mathcal{S})^* \).
Definition A.2. Given a complex of presheaves \( \mathcal{F}^* \) on \( \mathcal{S} \text{-Man} \), define its associated Borel complex as

\[
\mathcal{F}^*_{\text{bor}} := \text{Hom}(\mathcal{Z}(E\mathcal{S})^*, \mathcal{F}^*),
\]

and let \( \iota_{\mathcal{F}^*} : \mathcal{F}^* \to \mathcal{F}^*_{\text{bor}} \) denote the natural map induced by the projection \( E_\bullet \mathcal{S} \to \text{Spec} \mathbb{R} \). In particular, given \( p \in \mathbb{Z} \) one can define the Borel version of the Deligne complex as \( A(p)_{D/\mathbb{R}}^{\text{bor}} := (A(p)_{D/\mathbb{R}})_{\text{bor}} \) and the Borel version of the Bredon complex as \( A(p)_{Br}^{\text{bor}} \). Correspondingly, given \( X \in \mathcal{A}_{n/\mathbb{R}} \) define its Borel version of Deligne cohomology as

\[
H^i_{D/\mathbb{R}, \text{bor}}(X; A(p)) := \mathbb{H}^i(X_{\text{eq}}; A(p)_{D/\mathbb{R}}^{\text{bor}}),
\]

and the Borel version of Bredon cohomology as

\[
H^i_{\text{bor}}(X; A) := \mathbb{H}^i(X_{\text{eq}}; A(p)_{Br}^{\text{bor}}).
\]

It follows from the definitions that

\[
A(p)_{D/\mathbb{R}}^{\text{bor}} = \text{Cone} \left( A(p)_{Br}^{\text{bor}} \oplus F^p \mathcal{E}^*_{\text{bor}} \to \mathcal{E}^*_{\text{bor}} \right) [-1].
\]

Proposition A.3. Given \( p \in \mathbb{Z} \) one has map of exact triangles on \( \mathcal{A}_{n/\mathbb{R}} \):

\[
\begin{array}{cccc}
A(p)_{D/\mathbb{R}} & \rightarrow & A(p)_{Br} \oplus F^p \mathcal{E}^* & \rightarrow & \mathcal{E}^* & \rightarrow & A(p)_{D/\mathbb{R}}[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A(p)_{D/\mathbb{R}}^{\text{bor}} & \rightarrow & A(p)_{Br}^{\text{bor}} \oplus F^p \mathcal{E}^*_{\text{bor}} & \rightarrow & \mathcal{E}^*_{\text{bor}} & \rightarrow & A(p)_{D/\mathbb{R}}^{\text{bor}}[1].
\end{array}
\]

Corollary A.4. Let \( X \) be a real analytic manifold such that \( X_{\mathcal{S}} = \emptyset \). Then \( \iota : H^i_{D/\mathbb{R}}(X; A(p)) \to H^i_{D/\mathbb{R}, \text{bor}}(X; A(p)) \) is an isomorphism for all \( i \) and \( p \).

Proof. Since multiplication by 2 is invertible in \( \mathcal{E}^* \) and preserves the filtration \( \{ F^p \mathcal{E}^* \} \) one concludes that \( \iota : F^p \mathcal{E}^* \to F^p \mathcal{E}^*_{\text{bor}} \) is a quasi-isomorphism for all \( p \in \mathbb{Z} \). The result now follows from the five-lemma and the fact the same result holds for Bredon cohomology.

Remark A.5. When \( X \) is a projective smooth real variety, the associated Borel version of the Deligne cohomology groups \( H^i_{D/\mathbb{R}, \text{bor}}(X; A(p)) \) defined above coincide with the Deligne cohomology for real varieties introduced by Esnault and Viehweg in [E-V88]. However, in general, the groups \( H^i_{D/\mathbb{R}}(X; A(p)) \) and \( H^i_{D/\mathbb{R}, \text{bor}}(X; A(p)) \) are rather distinct.

Appendix B. Proof of Theorem 5.10

We will need the following two technical lemmas.

Lemma B.1. Let \( \iota : \mathbb{Z} \to \mathcal{E}^0 \) denote the natural inclusion of sheaves on \( \mathcal{S} \text{-Man} \). Given any real number \( \lambda \) one can find a map of presheaves \( \xi_\lambda : \mathbb{Z} \to \mathbb{Z}(2)_{Br}^0 \) such that for each \( U \in \mathcal{S} \text{-Man} \), the composition

\[
\mathbb{Z}(U) \xrightarrow{\xi_\lambda} \mathbb{Z}(2)_{Br}^0(U) \xrightarrow{\iota} \mathcal{E}^0(U)
\]

coincides with \( \lambda \cdot \iota \).
Proof. Let \( a > 0 \) be a positive real number and let \( I_a \) denote the interval \([a, 1]\) if \( a < 1 \) and \([1, a]\) if \( a \geq 1 \). Let \( \phi_{a,i} : \Delta^2 \to \mathbb{R}^x \times \mathbb{R}^x \subset \mathbb{C}^x \times \mathbb{C}^x \), \( i = 1, 2 \) be smooth maps that give an oriented triangulation of the rectangle \( I_a \times I_a \subset \mathbb{R}^x \times \mathbb{R}^x \). It follows that
\[
\int_{\Delta^2} (\phi_{a,1}^* \omega_2 + \phi_{a,2}^* \omega_2) = (\log a)^2.
\]

Let \( p_U : U \to \ast \) denote the projection to the point, where \( U \) is a \( \mathcal{G} \)-manifold. Given \( \nu \in \mathbb{Z}(U) \), define \( \xi_\lambda(\nu) \in \mathbb{Z}(2)_{\mathcal{B}_r}(U) \) by
\[
\xi_\lambda(\nu) := \begin{cases} 
(p_U^* \phi_{a,1} + p_U^* \phi_{a,2}) \otimes \nu & \text{if } \lambda > 0 \text{ and } a = \exp \sqrt{\lambda} \\
-(p_U^* \phi_{a,1} + p_U^* \phi_{a,2}) \otimes \nu & \text{if } \lambda < 0 \text{ and } a = \exp \sqrt{|\lambda|}.
\end{cases}
\]

It is clear that \( \xi_\lambda \) is a homomorphism satisfying the desired conditions. \( \square \)

Lemma B.2 (The period argument). Given \( \alpha \in \mathbb{Z}(p)_{\mathcal{B}_r}(U)^0 \) satisfying \( d^B \alpha = 0 \), then \( \tau(\alpha) \in \mathbb{Z}(p)(U) \). In other words, \( \tau(\alpha) \in E^0(U) \) is a locally constant equivariant function with values in \( \mathbb{Z}(p) \).

Proof. The cocycle \( \alpha \) is represented by an element of the form \((a, \sum_i f_i \otimes \nu_i)\), where \( f_i : S_i \times \Delta^p \to (\mathbb{C}^x)^p \) is smooth and equivariant, \( p_i : S_i \to U \) is an equivariant covering map, and \( \nu_i : S_i \to \mathbb{Z} \) is equivariant and locally constant. Since \( d\tau(\alpha) = \tau(d^B \alpha) = 0 \), we know that \( \tau(\alpha) \) is an equivariant locally constant function.

Given \( x_0 \in U \) and \( y \in p_i^{-1}(x_0) \subset U_i \) the restriction of \( f_i \) to \( y \times \Delta^p \) is a smooth proper map, and hence one obtains a smooth integral \( p \)-simplex \( f_i#[y \times \Delta^p] \) in \((\mathbb{C}^x)^p\) with boundary \( \partial f_i#[y \times \Delta^p] = f_i#[y \times \partial \Delta^p] \). It follows that
\[
T_{a,x_0} := \sum_i \sum_{y \in \pi_i^{-1}(x_0)} \sum \nu_i(y) f_i#[y \times \Delta^p]
\]
is a smooth integral \( p \)-cycle on \((\mathbb{C}^x)^p\) and a simple inspection shows that \( \int_{T_{a,x_0}} \omega_p = \tau(\alpha)(x_0) \). On the other hand, since \( T_{a,x_0} \) represents an integral homology class in \((\mathbb{C}^x)^p\), then \( \int_{T_{a,x_0}} \omega_p \) is a period of \( \omega_p \) over an integral homology class and hence it lies in \( \mathbb{Z}(p) \). \( \square \)

B.1. Cocycles for Bredon cohomology. As a preparation to the main arguments of next section, we describe the isomorphism
\[
\Phi : \check{H}^2(X(\mathcal{C})_{eq}; \mathbb{Z}(2)_{\mathcal{B}_r}) \to \check{H}^1(X(\mathcal{C})_{eq}; G^0 \to G^1)
\]
in terms of Čech cocycles; cf. Proposition 5.3

If \( U \) is a \( \mathcal{G} \)-manifold, we denote by \( U_{\text{triv}} \) the same space with the trivial \( \mathcal{G} \)-action. Given integers \( n \geq j \geq 0 \) let \( D^{n,j} \subset (n-j) \cdot 1 \oplus j \cdot \xi \) denote the unit ball in \( \mathbb{R}^n \) with the action induced by the representation. We say that a \( \mathcal{G} \)-manifold \( U \) has Type \( \mathcal{G} \) if it is equivariantly isomorphic to \( (D^{n,0})_{\text{triv}} \times \mathcal{G} \). We say that \( U \) has Type \( P \) if \( U \cong D^{n,j} \), for some \( j \geq 0 \).

Let \( Y \) be a \( \mathcal{G} \)-manifold of dimension \( n \). A good cover for \( Y \) is an open cover \( \mathcal{V} = \{ V_\alpha \mid \alpha \in \Lambda \} \) such that all non-empty intersections are contractible. We
may even assume that these intersections are homeomorphic to disks. We say that \( Y \) is equivariantly good if the group permutes the open sets in the cover. A cover with these properties always yields an equivariant good cover, i.e., a cover by \( \mathcal{G} \)-invariant open sets having the property that all elements \( U_{\alpha_0 \cdots \alpha_k} := U_{\alpha_0} \cap \cdots \cap U_{\alpha_k} \) in the nerve of \( \mathcal{U} \) have either Type \( \mathcal{G} \) or Type \( P \). Furthermore, if one of the elements \( \alpha_0, \ldots, \alpha_k \) has Type \( \mathcal{G} \), then the intersection \( \alpha_0 \cdots \alpha_k \) is required to have Type \( \mathcal{G} \). Also, if all \( \alpha_0, \ldots, \alpha_k \) have Type \( P \), then intersection \( \alpha_0 \cdots \alpha_k \) must also have Type \( P \). Abusing language, we say that the index \( \alpha_0 \cdots \alpha_k \) has Type \( \mathcal{G} \) or Type \( P \), accordingly.

**Remark B.3.** Using totally convex balls for a Riemannian metric so that \( \sigma \) acts via isometries, one sees that any \( \mathcal{G} \)-manifold \( Y \) has an equivariant good cover. Also, these covers form a cofinal family amongst the family of all equivariant covers of \( Y \).

**Lemma B.4** (The local obstruction argument). Let \( Y \) be a \( \mathcal{G} \)-manifold whose path-components are all contractible. Then, for all \( p \geq 0 \), the complex \( \mathbb{Z}(p)_B(Y \times \mathcal{G}) \) is acyclic, i.e., \( H^j(\mathbb{Z}(p)_B(Y \times \mathcal{G})) = 0 \) for all \( j \neq 0 \). In particular, \( \mathbb{Z}(p)_B(U) \) is acyclic if \( U \) is a \( \mathcal{G} \)-manifold of Type \( \mathcal{G} \) and \( \mathbb{Z}(p)_B(U \times \mathcal{G}) \) is acyclic if \( U \) has Type \( P \).

**Proof.** Using the isomorphism \( U \times \mathcal{G} \cong U^{triv} \times \mathcal{G} \) in Remark [B.3](ii), pick one point in each path component of \( Y \) and obtain an equivariant strong deformation retraction \( Y \times \mathcal{G} \simeq \pi_0(Y) \times \mathcal{G} \), where \( \pi_0(Y) \) is given the discrete topology. It follows from Proposition [2.4](ii) that one has a quasi-isomorphism \( \mathbb{Z}(p)_B(Y \times \mathcal{G}) \cong \mathbb{Z}(p)_B(\pi_0(Y) \times \mathcal{G}) \). Now, the cohomology of the latter complex gives the bigraded Bredon cohomology groups \( H^*_{Br}(\pi_0(Y) \times \mathcal{G}, \mathbb{Z}) \cong H^*_{sing}(\pi_0(Y); \mathbb{Z}(p)) \). The result follows. \( \square \)

Let \( \mathcal{U} := \{ U_\lambda \mid \lambda \in \Lambda \} \) be an “equivariant good cover” of \( Y \) and let \( h = (h_{\alpha_0 \cdots \alpha_j}) \mid i + j = 2, j \geq 0 \) be a Čech cocycle representing an element \( [h] \in \mathbb{H}^2(Y_{eq}; \mathbb{Z}(2)_B) \). The cocycle condition gives:

\[
(48) \quad d^B h_{\alpha_0}^2 = 0 \quad \text{and} \quad \delta h_{\alpha_0 \cdots \alpha_j}^i = (-1)^i d^B h_{\alpha_0 \cdots \alpha_{j+1}}^{i-1} \quad \text{for all} \quad i \leq 1,
\]

where \( d^B \) and \( \delta \) are the differentials in the Bredon and Čech complexes, respectively.

**MAIN GOAL:** We will find a representative \( (g_{\alpha_0 \alpha_1}, (\rho_{\alpha_0})) \in \mathcal{C}(\mathcal{U}, G^0 \to G^1) \) for \( \Phi([\mathcal{U}]) \), satisfying:

\[
\delta(g_{\alpha_0 \alpha_1}) = 1; \quad (g_{\alpha_0 \alpha_1} \cdot \sigma^* g_{\alpha_1 \alpha_2}) = \delta(\rho_{\alpha_0}); \quad (\sigma^* \rho_{\alpha_0}) = (\rho_{\alpha_0}).
\]
Remark B.5. Let \([h]\) be as above. Given a fixed point \(x_0 \in Y^\mathcal{S}\), let \(U_{a_0}\) be an element of the cover (necessarily of Type \(P\)) containing \(x_0\). Since \(Z(p)^{2r}\) has homotopy invariant cohomology presheaves, one has natural isomorphisms \(H^j(Z(2)^{2r}(U_{a_0})) \cong \mathcal{B}^j\). It follows from \(\text{(48)}\) that \(h^2_{a_0}\) represents a class in \(H^2(Z(2)^{2r}(U_{a_0})) \cong \mathcal{B}^{2,2} \cong \mathbb{Z}^x\), thus giving an element in \(\mathbb{Z}^x\). Notice that this element is the same for any point \(x \in U_{a_0} \cap Y^\mathcal{S}\) and it is easy to see that it depends only on the class \([h]\) \(\in H^{2,2}_{Br}(Y^\mathcal{S};\mathbb{Z})\). Hence, the resulting map \(S \to \mathbb{Z}^x\) depends only on \([h]\), and this is an additional description of the signature map in terms of Čech-Bredon cocycles.

The identity \(\delta(h^0_{\alpha_0\alpha_1\alpha_2}) = (d^B h_{\alpha_0\alpha_1\alpha_2}^{-1} - \tau h^0_{\alpha_0\alpha_1\alpha_2}) = 0\) and hence, since \(\mathcal{E}^0\) is a soft sheaf, one can find \(f^0_{\alpha_0\alpha_1} \in \mathcal{E}^0(U_{a_0\alpha_1})\) such that

\[
\tau h^0_{\alpha_0\alpha_1\alpha_2} = f^0_{\alpha_0\alpha_1}.
\]

Let \(p: Y \times \mathcal{S} \to Y\) denote the projection. Given any \(\alpha_0 \cdots \alpha_j\) in the nerve of the covering \(\mathcal{U}\), we use the same notation \(p: U_{\alpha_0 \cdots \alpha_j} \times \mathcal{S} \to U_{\alpha_0 \cdots \alpha_j}\) to denote the corresponding projection. For any presheaf \(\mathcal{F}\) on \(\mathcal{S} \cdot \text{Man}\) and \(h \in \mathcal{F}(U_{\alpha_0 \cdots \alpha_j})\) let \(p^*h \in \mathcal{F}(U_{\alpha_0 \cdots \alpha_j} \times \mathcal{S})\) denote the pull-back of \(h\) under \(p\).

**Type \(\mathcal{S}\) case:**

Recall that if \(\alpha_k\) is of Type \(\mathcal{S}\), for some \(k = 1, \ldots, j\), then so is \(U_{\alpha_0 \cdots \alpha_j}\).

**Step 1:** If \(a_0\) is of Type \(\mathcal{S}\), then the local obstruction argument (Lemma B.4) implies that one can find \(t^1_{a_0} \in Z(2)^1_{Br}(U_{a_0})\) such that

\[
d^B t^1_{a_0} = h^2_{a_0}.
\]

**Step 2:** Assume that both \(a_0\) and \(\alpha_1\) are of Type \(\mathcal{S}\). Since

\[
d^B (t^1_{a_0} - \delta(t^1_{a_0})) = \delta(h^2_{a_0} - \delta(h^2_{a_0})) = 0,
\]

by the cocycle condition, the local obstruction argument guarantees the existence of \(t^0_{\alpha_0\alpha_1} \in Z(2)^1_{Br}(U_{\alpha_0\alpha_1})\) such that

\[
d^B t^0_{\alpha_0\alpha_1} = (h^1_{\alpha_0\alpha_1}) - \delta(t^1_{a_0}).
\]

**Step 3:** Here we only consider \(\alpha_0 \cdots \alpha_j\) in the nerve of the cover \(\mathcal{U}\) for which all \(\alpha_k\)'s are of Type \(\mathcal{S}\). Observe that

\[
d^B ((h^0_{\alpha_0\alpha_1\alpha_2}) + \delta(t^0_{\alpha_0\alpha_1})) = -\delta(h^1_{\alpha_0\alpha_1\alpha_2}) + \delta(d^B t^0_{\alpha_0\alpha_1})
\]

\[
= -\delta(h^1_{\alpha_0\alpha_1\alpha_2}) + \delta((h^1_{\alpha_0\alpha_1\alpha_2} - \delta(t^1_{a_0})) = 0.
\]

Applying \(\tau\) one obtains \(d((\tau h^0_{\alpha_0\alpha_1\alpha_2} + \delta(\tau t^0_{\alpha_0\alpha_1})) = 0\), and hence Lemma B.2 (the periods argument) shows that

\[
(\nu_{\alpha_0\alpha_1\alpha_2}) := (\tau h^0_{\alpha_0\alpha_1\alpha_2}) + \delta(\tau t^0_{\alpha_0\alpha_1})
\]

consists of locally constant equivariant functions \(\nu_{\alpha_0\alpha_1\alpha_2}: U_{\alpha_0\alpha_1\alpha_2} \to \mathbb{Z}(2)\). Using \(\text{(48)}\) one writes

\[
(\nu_{\alpha_0\alpha_1\alpha_2}) = \delta(f^0_{\alpha_0\alpha_1} + \tau t^0_{\alpha_0\alpha_1}).
\]
Step 4: Denote $\tilde{g}_{\alpha_0\alpha_1} := f^0_{\alpha_0\alpha_1} + \tau f^0_{\alpha_0\alpha_1}$ and choose a square root $i$ of $-1$. Define

$$g_{\alpha_0\alpha_1} = \exp \left( \frac{1}{2\pi i} \tilde{g}_{\alpha_0\alpha_1} \right) \quad \text{and} \quad \rho_{\alpha_0} = 1.$$  

The cocycle condition

$$g_{\alpha_0\alpha_1} g_{\alpha_1\alpha_2} g_{\alpha_2\alpha_0} = 1$$

when all $\alpha$'s are of Type $\mathcal{S}$ follows from (53). Since $\tilde{g}_{\alpha_0\alpha_1} - \sigma^* \tilde{g}_{\alpha_0\alpha_1} = 0$ one concludes that

$$g_{\alpha_0\alpha_1} \cdot \sigma^* g_{\alpha_0\alpha_1} = 1 = \rho_\beta / \rho_\alpha$$

on $U_{\alpha_0\alpha_1}$ and by definition

$$\sigma^* \rho_\alpha = \rho_\alpha.$$

**Type $P$ case:**

Recall that if $\alpha_k$ is of Type $P$, for all $k = 1, \ldots, j$, then so is $U_{\alpha_0 \cdots \alpha_j}$.

**Step 1:** If $\alpha_0$ is of Type $P$, then the local obstruction argument (Lemma B.4) gives some $\tilde{l}_{\alpha_0} \in \mathbb{Z}(2)_{\mathfrak{g}}(U_{\alpha_0} \times \mathcal{S})$ such that $d^B \tilde{l}_{\alpha_0} = p^* h^2_{\alpha_0}$.

Given any $\alpha_0$ define

$$T^1_{\alpha_0} = \begin{cases} \tilde{l}_{\alpha_0}^1, & \text{if } \alpha_0 \text{ is of Type } \mathcal{S} \\ \tilde{l}_{\alpha_0}, & \text{if } \alpha_0 \text{ is of Type } P, \end{cases}$$

and observe that $d^B T^1_{\alpha_0} = p^* h^2_{\alpha_0}$.

**Step 2:** If either $\alpha_0$ or $\alpha_1$ is of Type $P$ then

$$d^B \left( (p^* h^0_{\alpha_0 \alpha_1}) - \delta(T^1_{\alpha_0}) \right) = 0.$$  

Since $B^{1,2} = 0$, it follows from the local obstruction argument (see Remark B.5) that one obtains $\tilde{l}^0_{\alpha_0 \alpha_1} \in \mathbb{Z}(2)_{\mathfrak{g}}(U_{\alpha_0 \alpha_1} \times \mathcal{S})$ such that

$$d^B \tilde{l}^0_{\alpha_0 \alpha_1} = (p^* h^1_{\alpha_0 \alpha_1}) - \delta(T^1_{\alpha_0}).$$

**Step 3:** Here we consider those $\alpha_0 \cdots \alpha_j$ in the nerve of the cover $\mathcal{U}$ for which one of the $\alpha_k$'s is of Type $P$. In this case one has

$$d^B \left( (p^* h^0_{\alpha_0 \alpha_1 \alpha_2}) + \delta(\tilde{l}^0_{\alpha_0 \alpha_1 \alpha_2}) \right) = 0.$$  

Applying $\tau$ one obtains $d \left( (\tau p^* h^0_{\alpha_0 \alpha_1 \alpha_2}) + \delta(\tau \tilde{l}^0_{\alpha_0 \alpha_1 \alpha_2}) \right) = 0$. It follows from Lemma B.2 that $(\nu_{\alpha_0 \alpha_1 \alpha_2}) := (\tau p^* h^0_{\alpha_0 \alpha_1 \alpha_2}) + \delta(\tau \tilde{l}^0_{\alpha_0 \alpha_1 \alpha_2})$ consists of equivariant locally constant functions $U_{\alpha_0 \alpha_1 \alpha_2} \times \mathcal{S} \to \mathbb{Z}(2)$. In view of Remark 2.9 we can also consider $\nu_{\alpha_0 \alpha_1 \alpha_2}$ as a non-equivariant locally constant function from $U_{\alpha_0 \alpha_1 \alpha_2}$ to $\mathbb{Z}(2)$.

**Step 4:** Let $\iota: U \times \mathcal{S} \to U \times \mathcal{S}$ be as in Remark 2.9. Observe that

$$d^B \left( *_{\alpha_0} T^1_{\alpha_0} - T^1_{\alpha_0} \right) = *_{\alpha_0} d^B T^1_{\alpha_0} - d^B T^1_{\alpha_0} = h^2_{\alpha_0} - h^2_{\alpha_0} = 0.$$  

It follows from the local obstruction argument that

$$*_{\alpha_0} T^1_{\alpha_0} - T^1_{\alpha_0} = d^B \tilde{l}_{\alpha_0}.$$
for some $\tilde{\gamma}_{a_0}^0 \in \mathbb{Z}(2,2)_{br}(U_{a_0} \times \mathcal{G})$. In particular, one has $d^B (\tilde{\gamma}_{a_0}^0 + \iota^*\tilde{\gamma}_{a_0}^0) = 0$, and the period argument shows that

\begin{equation}
\tau (\tilde{\gamma}_{a_0}^0) + \iota^* \tau (\tilde{\gamma}_{a_0}^0) = \tilde{C}_{a_0},
\end{equation}

for some equivariant locally constant function $\tilde{C}_{a_0}$ on $U_{a_0} \times \mathcal{G}$ with values in $\mathbb{Z}(2)$.

Define $\tilde{g}_{a_0a_1} := p^* f_{a_0a_1} + \tau t_{a_0a_1}$. Then

\begin{equation}
(\tilde{g}_{a_0a_1}) - \iota^* (\tilde{g}_{a_0a_1}) - \delta (\tau (\tilde{\gamma}_{a_0}^0)) = \tau t_{a_0a_1}^0 - \iota^* \tau t_{a_0a_1}^0 - \delta (\tau (\tilde{\gamma}_{a_0}^0)) = \tau (t_{a_0a_1}^0 - \iota^* t_{a_0a_1}^0 - \delta (\tilde{\gamma}_{a_0}^0)).
\end{equation}

On the other hand, equations \((60)\) and \((62)\) give

\begin{equation}
d^B (t_{a_0a_1}^0 - \iota^* t_{a_0a_1}^0 - \delta (\tilde{\gamma}_{a_0}^0)) = \delta (-T_{a_0}^1 + \iota^* T_{a_0}^1 - d^B (\tilde{\gamma}_{a_0}^0)) = 0,
\end{equation}

and hence Lemma \[\ref{lem:1}\] shows that $(\tilde{g}_{a_0a_1}) - \iota^* (\tilde{g}_{a_0a_1}) - \delta (\tau (\tilde{\gamma}_{a_0}^0))$ consists of equivariant locally constant functions from $U_{a_0a_1} \times \mathcal{G}$ to $\mathbb{Z}(2)$.

Using the notation in Remark \[\ref{rem:2}\] define $\tilde{g}_{a_0a_1} = F(\tilde{g}_{a_0a_1}) = f_{a_0a_1} + F(\tau (t_{a_0a_1}^0))$, and $\tilde{\gamma}_{a_0}^0 := F(\tau (\tilde{\gamma}_{a_0}^0))$. With the same choice of square root $i$ of $-1$ as in the Type $S$ case, define

\begin{equation}
g_{a_0a_1} = \exp \left( \frac{1}{2\pi i} \tilde{g}_{a_0a_1} \right) \quad \text{and} \quad \rho_{a_0} = \exp \left( \frac{1}{2\pi i} \tilde{\gamma}_{a_0}^0 \right).
\end{equation}

Step 3 shows that

\begin{equation}
g_{a_0a_1} g_{a_1a_2} g_{a_2a_0} = 1,
\end{equation}

and \((64)\) together with subsequent remarks and properties of the functor $F$ show that

\begin{equation}
g_{a_0a_1} - \sigma^* g_{a_0a_1} = \rho_{a_1} / \rho_{a_0}
\end{equation}
on $U_{a_0a_1}$. Finally, \((63)\) and Remark \[\ref{rem:2}\] give $F(\tau (\tilde{\gamma}_{a_0}^0)) + \sigma^* F(\tau (\tilde{\gamma}_{a_0}^0)) = F(\tilde{C}_{a_0}) := C_{a_0}$. In particular, $C_{a_0}$ is an equivariant locally constant function on $U_{a_0}$ with values in $\mathbb{Z}(2)$, and hence

\begin{equation}
\left( \frac{1}{2\pi i} \tilde{\gamma}_{a_0}^0 \right) - \sigma^* \left( \frac{1}{2\pi i} \tilde{\gamma}_{a_0}^0 \right) = \frac{1}{2\pi i} \left( F(\tau (\tilde{\gamma}_{a_0}^0)) + \sigma^* F(\tau (\tilde{\gamma}_{a_0}^0)) \right) \in \mathbb{Z}(1)(U_{a_0}).
\end{equation}

This gives

\begin{equation}
\overline{\sigma^* \rho_{a_0}} = \rho_{a_0}.
\end{equation}

In other words, $(g_{a_0a_1}, \rho_{a_0})$ defines a 1-cocycle in $\tilde{C}^*(U, G^0 \to G^1)$. It is easy to check that this process would send a Čech coboundary to a coboundary in $\tilde{C}^*(U, G^0 \to G^1)$, realizing the isomorphism \[\ref{iso:1}\].
B.2. The proof. We now prove Theorem 5.10

Proof. Let \( \mathcal{U} \) be an equivariant good cover of \( X(\mathbb{C}) \) and fix a cocycle \( c = (e^{r,s})_{r+s=2} \in C^2(\mathcal{U}; \mathbb{Z}(2)_{\mathbb{D}/\mathbb{R}}) \) representing a class in \( H^2_{\mathbb{D}/\mathbb{R}}(X; \mathbb{Z}(2)) \). Since \( e^{r,s} = 0 \) for \( r > 2 \), the cocycle condition is given by

\[
Dc^{2,0} = 0 \quad \text{and} \quad Dc^{r-1,s+1} = (-1)^s \delta e^{r,s}, \ r \leq 1.
\]

Recall that \( \mathbb{Z}(2)_{\mathbb{D}/\mathbb{R}} = \text{Cone} \left( \mathbb{Z}(2)_{\mathbb{R}} \oplus F^2 \mathcal{E}^* \rightarrow \mathcal{E}^* \right) [-1] \).

Write

\[
\begin{align*}
\epsilon^{2,0} &= \left( (h_{a_0}^2), (\omega_{a_0}^2), (\theta^1_{a_0}) \right) \\
\epsilon^{1,1} &= \left( (h_{a_0a_1}^1), 0, (\theta^0_{a_0a_1}) \right) \\
\epsilon^{0,2} &= \left( (h_{a_0a_1a_2}^0), 0, 0 \right) \\
\epsilon^{-i,2+i} &= \left( (h_{a_0...a_2+i}^{-i}), 0, 0 \right), \ i \geq 1.
\end{align*}
\]

Since \( D\epsilon^{2,0} = 0 \), one has for all \( a_0 \in \Lambda \):

\[
\begin{align*}
d^B h_{a_0}^2 &= 0, \ \text{(by definition)}; \\
d^0 \omega_{a_0}^2 &= 0; \\
\tau(h_{a_0}^2) - \omega_{a_0}^2 + d\theta_{a_0}^1 &= 0.
\end{align*}
\]

Similarly, \( D\epsilon^{1,1} = \delta \epsilon^{2,0} \) gives the identities:

\[
\begin{align*}
(d^B h_{a_0a_1}^1) &= \delta(h_{a_0}^2) \\
0 &= \delta(\omega_{a_0}) \\
-(\tau h_{a_0a_1}^1) - (d\theta_{a_0a_1}^0) &= \delta(\theta^1_{a_0})\ 
\]

\( D\epsilon^{0,2} = -\delta \epsilon^{1,1} \) gives:

\[
\begin{align*}
(d^B h_{a_0a_1a_2}^0) &= -\delta(h_{a_0a_1}^1) \\
(\tau h_{a_0a_1a_2}^0) &= \delta(\theta_{a_0a_1}^0)
\end{align*}
\]

and for all \( r \leq 0 \) one has:

\[
(d^B h_{a_0...a_{3+r}}^{-r-1}) = (-1)^r \delta(h_{a_0...a_{3+r}}^{-r}).
\]

The assignment \( c := (e^{r,s}) \mapsto h := (h_{a_0...a_{-i}}) \) gives the cycle map from Deligne to Bredon cohomology. Assume that \( c \in \ker \Psi \), hence \( \Psi([c]) = \mathbb{N}([h]) = 0 \). Therefore the Čech-Bredon cocycle \( h \) is “unobstructed” and we can apply the arguments in TYPE & CASE above; see [54]. Furthermore, the data in the Čech-Deligne complex gives a natural choice for the \( f^{0}_{a_0a_1} \) introduced in [49]. More precisely, one can choose \( f^{0}_{a_0a_1} := \theta_{a_0a_1}^0 \); cf. [82].

It follows that one can take \( g_{a_0a_1} := \exp \left( \frac{1}{2\pi} \tau \hat{g}_{a_0a_1} \right) \), with

\[
\hat{g}_{a_0a_1} = \theta_{a_0a_1}^0 + \tau(t_{a_0a_1}^0),
\]

to obtain a cocycle for the line bundle associated to \( [h] \); cf. [54]. However, one can find an equivalent holomorphic cocycle as follows.
Write the 1-form \( a_{\alpha} = \theta_{\alpha} + \tau t_{\alpha} = a_{\alpha}^{1,0} + a_{\alpha}^{0,1} \) as a sum of their (1, 0) and (0, 1) parts, respectively, where \( t_{\alpha}^{1,0} \) is introduced in (50). Since \( da_{\alpha} = d(\theta_{\alpha} + \tau t_{\alpha}) = d\theta_{\alpha} + d\tau(h_{\alpha}^2) = \omega_{\alpha}^2 \), cf. (50) and (77), and \( \omega_{\alpha}^2 \) is a form of type (2, 0) one concludes that

\[
\begin{align*}
\bar{\partial}a_{\alpha}^{0,1} &= 0 \\
\partial a_{\alpha}^{0,1} &= -\bar{\partial}a_{\alpha}^{1,0} \\
\partial a_{\alpha}^{1,0} &= \omega_{\alpha}^2
\end{align*}
\]

It follows from the \( \bar{\partial} \)-Poincaré lemma that one can find \( f_{\alpha}^{0} \in E^0(U_{\alpha}) \) (equivariant) such that

\[
\bar{\partial}f_{\alpha}^{0} = a_{\alpha}^{0,1}.
\]

Now define

\[
(\tilde{g}_{\alpha_{0}}) := (\hat{g}_{\alpha_{0}}) + \delta(f_{\alpha}^{0}).
\]

Hence,

\[
\begin{align*}
\bar{\partial}\tilde{g}_{\alpha_{0}} &= \{d\hat{g}_{\alpha_{0}}\}^{0,1} + \delta(\bar{\partial}f_{\alpha}^{0}) = \{d\theta_{\alpha_{0}} + \tau(t_{\alpha_{0}})\}^{0,1} + \bar{\partial}f_{\alpha}^{0} \\
&= \{d\theta_{\alpha_{0}} + \tau(h_{\alpha_{0}}^1) - \delta(t_{\alpha_{0}})\}^{0,1} + \bar{\partial}f_{\alpha}^{0} \\
&= \{-\delta(\theta_{\alpha_{0}}^{1}) - (\tau h_{\alpha_{0}}^1) + (\tau h_{\alpha_{0}}^1) - \delta(t_{\alpha_{0}})\}^{0,1} + \bar{\partial}f_{\alpha}^{0} \\
&= -\delta(a_{\alpha_{0}}^{0,1}) + \delta(\bar{\partial}f_{\alpha}^{0}) = 0,
\end{align*}
\]

cf. \( \text{[SS]} \).

Defining

\[
\tilde{g}_{\alpha_{0}} := \exp\left(\frac{1}{2\pi i} \hat{g}_{\alpha_{0}}\right)
\]

one obtains a holomorphic structure \((\tilde{g}_{\alpha_{0}})\) for \( L \). See Remark \( \text{[5.9]}(i) \).

Now, define

\[
\psi_{\alpha_{0}} := \frac{1}{2\pi i} (\bar{\partial}f_{\alpha_{0}} - a_{\alpha_{0}}^{1,0}) = \frac{1}{2\pi i} (df_{\alpha_{0}} - \theta_{\alpha_{0}}^{1} - \tau t_{\alpha_{0}}^{1}).
\]
It follows from [88] and [89] that $\psi_{\alpha_0}$ is a holomorphic 1-form. Furthermore,

$$\left( \frac{dg_{\alpha_0 \alpha_1}}{g_{\alpha_0 \alpha_1}} \right) = \frac{1}{2\pi i} \{(d\hat{g}_{\alpha_0 \alpha_1}) + \delta(df_{\alpha_0})\}$$

$$= \frac{1}{2\pi i} \{(d\theta_{\alpha_0 \alpha_1}^0 + (\tau d^B t_{\alpha_0 \alpha_1}^0) + \delta(\partial f_{\alpha_0}) + \delta(\bar{\partial} f_{\alpha_0})\}$$

$$= \frac{1}{2\pi i} \{-\delta(\theta_{\alpha_0}^1) - (\tau h_{\alpha_0 \alpha_1}^1) + \tau((h_{\alpha_0 \alpha_1}^1) - (\delta(t_{\alpha_0}^1))$$

$$+ \delta(\partial f_{\alpha_0}) + \delta(\bar{\partial} f_{\alpha_0})\}$$

$$= \frac{1}{2\pi i} \left\{-\delta((\theta_{\alpha_0}^1 + (\tau t_{\alpha_0}^1)) + \delta(\partial f_{\alpha_0}) + \delta(a_{\alpha_0}^{0,1})\right\}$$

$$= \frac{1}{2\pi i} \left\{\delta(\partial f_{\alpha_0} - a_{\alpha_0}^{1,0})\right\}$$

$$= \delta(\psi_{\alpha_0}).$$

See Remark 5.9(ii).

**Remark B.6.** It follows from [91], [50] and [77] that

$$d\psi_{\alpha_0} = -\frac{1}{2\pi i} (d\theta_{\alpha_0}^1 + d\tau(t_{\alpha_0}^1)) = -\frac{1}{2\pi i} (d\theta_{\alpha_0}^1 + \tau(d^B t_{\alpha_0}^1))$$

$$= -\frac{1}{2\pi i} (d\theta_{\alpha_0}^1 + \tau(h_{\alpha_0}^2)) = -\frac{1}{2\pi i} \omega_{\alpha_0}^2.$$

Therefore, $(\psi_{\alpha_0})$ defines a holomorphic connection $\nabla$ on the holomorphic line bundle $L$ associated to $(g_{\alpha_0 \alpha_1})$. Since both $\hat{g}_{\alpha_0 \alpha_1}$ and $f_{\alpha_0}$ are equivariant functions, it follows that

$$\overline{\sigma^* g_{\alpha_0 \alpha_1}} \cdot g_{\alpha_0 \alpha_1} = 1,$$

and this defines a holomorphic isomorphism $q: L \otimes \overline{\sigma^* L} \to 1$ which becomes a positive definite hermitian form on $X(\mathbb{R})$. (See Remark 5.9(iii).) Finally, the identity $\overline{\sigma^* \psi_{\alpha_0}} + \psi_{\alpha_0} = 0$ shows that $q$ is parallel with respect to the connection on $L \otimes \overline{\sigma^* L}$ induced by $\nabla$. See Remark 5.9(iv).

We have thus associated to $c$, with $[c] \in \ker \Psi$, a triple $(L, q, \nabla)$ of elements satisfying the conditions in Definition 5.7. This gives a well-defined homomorphism

$$\Phi: \ker \Psi \longrightarrow PW^\nabla(X).$$

We now proceed to show that this is in fact an isomorphism.
Lemma B.7. There is a natural transformation Θ: \(PW\nabla(\cdot) \rightarrow H^{2,2}_{Br}(\cdot, \mathbb{Z})\) such that for all \(U \in \mathcal{S}\) Man the following diagram commutes

\[
\begin{array}{ccc}
\ker \Psi_U & \xrightarrow{i} & H^2_{D/\mathbb{R}}(U; \mathbb{Z}(2)) \\
\Phi & \downarrow & \Phi \\
PW\nabla(U) & \xrightarrow{\Theta_U} & H^{2,2}_{Br}(U, \mathbb{Z}).
\end{array}
\]

Proof. Using the interpretation of \(H^{2,2}_{Br}(U, \mathbb{Z})\) as equivalence classes \([L, q]\) of pairs as in Proposition 6.3 then \(\Theta_U\) simply sends \(\langle L, \nabla, q \rangle\) to \([L, q]\). \(\square\)

Injectivity of \(\Phi\): Let \(c' = (h', \omega', \theta')\) be a Čech cocycle representing a class \([c'] \in \ker \Psi_U\), and suppose that \(\Phi_U([c']) = 0\). Since \(0 = \Theta_U \circ \Phi_U([c']) = \varphi \circ i([c])\), one concludes that \([h'] = 0 \in H^{2,2}_{Br}(U, \mathbb{Z})\) and hence, one can find \(t \in \text{Tot}(\hat{C}^*(\mathcal{L}, \mathbb{Z}(2)_{Br}))\) such that \(d_B t = h\). It follows that \(c := c' - D(t, 0, 0) = (0, \omega, \theta)\) represents \([c']\) and is simply given by \((\omega^2_{a_0}), (\theta^0_{a_0a_1})\) and \((\theta^1_{a_0})\), where \(\omega^2_{a_0} \in F^2E^2(U)\), \(\theta^0_{a_0a_1} \in \mathcal{E}^0(U_{a_0a_1})\), and \(\theta^1_{a_0} \in \mathcal{E}^1(U_{a_0})\).

Let \((g_{a_0a_1}), (\psi_{a_0})\) be constructed as in (90) and (91), representing \(\Phi([c])\), and observe that \(h = 0\) allows one to take \(t^1_{a_0} = 0\), \(t^0_{a_0a_1} = 0\) and \(f^0_{a_0a_1} = 0\) (cf. STEPS 1 and 2, and (92)) in their definition. Assuming that \(\Phi([c]) = 0\), one can find \((\rho_{a_0})\) satisfying:

\[
\begin{align}
\rho_{a_0} \cdot \sigma^* \rho_{a_0} &= 1, \quad \text{and} \quad \rho_{a_0} \text{ is holomorphic} \\
(g_{a_0a_1}) &= \delta(\rho_{a_0}) \\
\psi_{a_0} &= \frac{d\rho_{a_0}}{\rho_{a_0}}
\end{align}
\]

The latter equation gives \(0 = d\psi_{a_0} = -\frac{1}{2\pi i} \omega^2_{a_0}\), cf. Remark B.6.

Now, the cocycle condition on \(c = (0, 0, \theta)\) boils down to

\[
\begin{align}
(d\theta^1_{a_0}) = 0 \\
(d\theta^0_{a_0a_1}) + \delta(\theta^1_{a_0}) = 0 \\
\delta(\theta^0_{a_0a_1}) = 0.
\end{align}
\]

Choose \(\hat{\rho}_{a_0}\) such that \(\exp \hat{\rho}_{a_0} = \rho_{a_0}\) and \(\hat{\rho}_{a_0} + \sigma^* \hat{\rho}_{a_0} = 0\), and let \(f^0_{a_0}\) be as in (88). Hence, by definition one has

\[
(g_{a_0a_1}) = \left(\exp \frac{1}{2\pi i} \left\{ (\theta^0_{a_0a_1}) + \delta(f^0_{a_0}) \right\} \right) = \delta(\rho_{a_0})
\]

and

\[
\psi_{a_0} = \frac{1}{2\pi i} \left( df^0_{a_0} - \theta^1_{a_0} \right) = \frac{d\rho_{a_0}}{\rho_{a_0}} = d\hat{\rho}_{a_0}.
\]

It follows that one can find \(n_{a_0a_1} \in \mathbb{Z}(U_{a_0a_1})\) such that

\[
\frac{1}{2\pi i} \left\{ (\theta^0_{a_0a_1}) + \delta(f^0_{a_0}) \right\} = \delta(\hat{\rho}_{a_0}) + (2\pi i \ n_{a_0a_1}).
\]
Using Lemma B.1 define
\[ \xi_{\alpha_{01}} := \xi_{(2\pi i)^2}(n_{\alpha_{01}}) \in \mathbb{Z}(2)_{2r}(U_{\alpha_{01}}) \]
and define
\[ b^0_{\alpha} = 2\pi i \rho_{\alpha} - f^0_{\alpha} \in \mathfrak{C}^0(U_{\alpha_{01}}). \]
It is easy to see that \( \delta(\xi_{\alpha_{01}}) = 0 \) and that (101) and (102) give
\[ \tau(\xi_{\alpha_{01}}) + \delta(b^0_{\alpha}) = (\theta^0_{\alpha_{01}}). \]
One obtains a 1-cochain \( \tau \) := \( (\xi_{\alpha_{01}}, 0, (b^0_{\alpha_{01}})) \) \( \in \) \( \text{Tot} \left( \check{\mathcal{C}}^*(\mathbb{L}, \mathbb{Z}(2)_{2r}) \right) \)
that satisfies \( D(t) = c \); cf. (99), (102), (103) and (91). Therefore \( [c] = 0 \),
thus showing the injectivity of \( \Phi \).

**Surjectivity of \( \Phi \):**

Represent \( \langle L, \nabla, q \rangle \in \mathbb{P}W^\vee(X) \) by a cocycle \((G_{\alpha_{01}}), (\psi_{\alpha_{0}})\) satisfying
the conditions of Remark 5.9 and let \( h \in \text{Tot} \left( \check{\mathcal{C}}^*(\mathbb{L}, \mathbb{Z}(2)_{2r}) \right) \)
representing the element \( \Theta(\langle L, \nabla, q \rangle) = [L, q] \in H^2_{\text{BPL}}(X, \mathbb{Z}) \). Note that
this cocycle is “unobstructed”, in the sense of Type \( \mathcal{S} \) case. Hence one can find \( t^0_{\alpha_{0}}, t^0_{\alpha_{01}} \) and \( f^0_{\alpha_{01}} \) as in (50), (51) and (49), respectively. By
definition, \( g_{\alpha_{01}} := f^0_{\alpha_{01}} + \tau(t^0_{\alpha_{01}}) \) so that \( g_{\alpha_{01}} := \exp \left( \frac{1}{2\pi} \hat{g}_{\alpha_{01}} \right) \)
is a cocycle representing the isomorphism class of \( L \) as a smooth line bundle
and satisfying the condition \( g_{\alpha_{01}} \cdot \sigma^* g_{\alpha_{01}} = 1 \), which gives the desired \( q^* \),
positive definite over \( X(\mathbb{R}) \).

Therefore, one can find smooth functions \( \rho_{\alpha_{0}} \) such that
\[ \left( \begin{array}{c} G_{\alpha_{01}} \\ g_{\alpha_{01}} \end{array} \right) = \delta(\rho_{\alpha_{0}}) \quad \text{and} \quad \rho_{\alpha_{0}} \cdot \sigma^* \rho_{\alpha_{0}} = 1. \]

Now, find \( \hat{G}_{\alpha_{01}} \) and \( \hat{\rho}_{\alpha_{0}} \) such that \( \exp \hat{G}_{\alpha_{01}} = G_{\alpha_{01}}, \exp \hat{\rho}_{\alpha_{0}} = \rho_{\alpha_{0}} \)
and satisfying \( \hat{G}_{\alpha_{01}} + \sigma^* \hat{G}_{\alpha_{01}} = 0, \hat{\rho}_{\alpha_{0}} + \sigma^* \hat{\rho}_{\alpha_{0}} = 0. \)
It follows from (104) that one can find \( n_{\alpha_{01}} \in \mathbb{Z}(U_{\alpha_{01}}) \) such that
\( (\hat{G}_{\alpha_{01}}) = \frac{1}{2\pi i} \{ \hat{g}_{\alpha_{01}} + \delta(\rho_{\alpha_{0}}) \} + (2\pi i)n_{\alpha_{01}}. \) Hence:
\[ 2\pi i \hat{G}_{\alpha_{01}} = f^0_{\alpha_{01}} + \tau f^0_{\alpha_{01}} + \delta(\hat{\rho}_{\alpha_{0}}) + (2\pi i)^2 n_{\alpha_{01}}. \]

Want to find \( \omega_{\alpha_{0}}, \theta^1_{\alpha_{0}}, \theta^0_{\alpha_{01}} \) satisfying:

\[ \begin{align*}
(C.1) & \quad d\omega_{\alpha_{0}} = 0 \\
(C.2) & \quad \delta(\omega_{\alpha_{0}}) = 0 \\
(C.3) & \quad \delta(\theta^0_{\alpha_{01}}) = (\tau h^0_{\alpha_{01}}) \\
(C.4) & \quad (d\theta^0_{\alpha_{01}}) + \delta(\theta^1_{\alpha_{0}}) + \tau(h^1_{\alpha_{01}}) = 0 \\
(C.5) & \quad \omega_{\alpha_{0}} = d\theta^1_{\alpha_{0}} + \tau h^2_{\alpha_{0}}. 
\end{align*} \]

Note that (C.5) implies (C.1).
It follows from (105) that
\[
2\pi i \delta(\psi_{\alpha_0}) = 2\pi id\hat{G}_{\alpha_0\alpha_1} = df^{0}_{\alpha_0\alpha_1} + \tau (dR^{1}_{\alpha_0\alpha_1}) + \delta(d\hat{\rho}_{\alpha_0}) \\
= df^{0}_{\alpha_0\alpha_1} + \tau (h^{1}_{\alpha_0\alpha_1} - \delta(t^{1}_{\alpha_0})) + \delta(d\hat{\rho}_{\alpha_0}) \\
= df^{0}_{\alpha_0\alpha_1} + \tau h^{1}_{\alpha_0\alpha_1} + \delta (-\tau t^{1}_{\alpha_0} + d\hat{\rho}_{\alpha_0}).
\]
Therefore,
\[
0 = (df^{0}_{\alpha_0\alpha_1}) + (\tau h^{1}_{\alpha_0\alpha_1}) + \delta (-2\pi i \psi_{\alpha_0} - \tau t^{1}_{\alpha_0} + d\hat{\rho}_{\alpha_0})
\]
Define
\[
\theta^{1}_{\alpha_0} := -2\pi i \psi_{\alpha_0} - \tau t^{1}_{\alpha_0} + d\hat{\rho}_{\alpha_0}
\]
and observe that the latter is an invariant closed form of Hodge type (2, 0), since \(\psi_{\alpha_0}\) is holomorphic. We now proceed to show that these forms satisfy (C.1) - (C.5).

- \(d\theta^{1}_{\alpha_0} = -2\pi id\psi_{\alpha_0} - \tau (dR^{1}_{\alpha_0}) = \omega_{\alpha_0} - \tau (h^{2}_{\alpha_0})\). This gives (C.5) and (C.1), as well.

- \(\delta(\omega_{\alpha_0}) = -2\pi i \delta(d\psi_{\alpha_0}) = -2\pi i \delta(\psi_{\alpha_0}) = -2\pi i d\left(\frac{d\hat{G}_{\alpha_0\alpha_1}}{G_{\alpha_0\alpha_1}}\right) = -2\pi i d\hat{G}_{\alpha_0\alpha_1} = 0\). This gives (C.2).

- \(\delta(\theta^{0}_{\alpha_0\alpha_1}) = \delta(f^{0}_{\alpha_0\alpha_1}) = (\tau h^{0}_{\alpha_0\alpha_1\alpha_2})\). This gives (C.3).

- It follows from (106) and (107) that \(0 = (df^{0}_{\alpha_0\alpha_1}) + (\tau h^{1}_{\alpha_0\alpha_1}) + \delta(\theta^{1}_{\alpha_0\alpha_1})\) which, together with (108), implies (C.4).

It follows that \(c := (h, \omega, \theta)\) gives a cocycle in \(\text{Tot} \left(\check{C}^{*}(U_{\alpha_0}; \mathbb{Z}(2)_{D/R})\right)\) such that \(|c| \in \ker \Psi\).

Finally, one needs to verify that \(\Phi([c]) = \langle L, \nabla, q \rangle\). At this point, this is a mere tautology. Following the steps in the definition of \(\Phi\), one constructs \((\gamma_{\alpha_0\alpha_1}), (\xi_{\alpha_0})\) representing \(\Phi([c])\).

We first find \(f^{0}_{\alpha_0} \in \mathcal{E}^{0}(U_{\alpha_0})\) such that \(\delta f^{0}_{\alpha_0} = \left\{ \theta^{1}_{\alpha_0} + \tau (t^{1}_{\alpha_0}) \right\}^{0,1};\) cf. (88).

Note that, by definition (107), one has \(\theta^{1}_{\alpha_0} + \tau (t^{1}_{\alpha_0}) = -2\pi i \psi_{\alpha_0} + d\hat{\rho}_{\alpha_0}\), and since \(\Psi_{\alpha_0}\) has Hodge type (1, 0), one concludes that \(\left\{ \theta^{1}_{\alpha_0} + \tau (t^{1}_{\alpha_0}) \right\}^{0,1} = d\hat{\rho}_{\alpha_0}\).

Hence, we can choose \(f^{0}_{\alpha_0} = \hat{\rho}_{\alpha_0}\).

By definition,
\[
\gamma_{\alpha_0\alpha_1} = \exp \left( \frac{1}{2\pi i} \left\{ \hat{g}_{\alpha_0\alpha_1} + \delta(\hat{\rho}_{\alpha_0}) \right\} \right) \\
= \exp \left( \frac{1}{2\pi i} \left\{ \hat{g}_{\alpha_0\alpha_1} + \delta(\hat{\rho}_{\alpha_0}) \right\} + (2\pi i n_{\alpha_0\alpha_1}) \right) \\
= \exp(\hat{G}_{\alpha_0\alpha_1}) = G_{\alpha_0\alpha_1}.
\]
Also,
\[
\xi_{\alpha_0} = \frac{1}{2\pi i} (\partial \hat{\rho}_{\alpha_0} - \{\theta^1_{\alpha_0} + \tau t^1_{\alpha_0}\})^{1,0} = \frac{1}{2\pi i} \left( d\hat{\rho}_{\alpha_0} - \{\theta^1_{\alpha_0} + \tau t^1_{\alpha_0}\} \right)
\]
\[
= \frac{1}{2\pi i} (2\pi i \psi_{\alpha_0}) = \psi_{\alpha_0}.
\]

□

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Departamento de Matemática, Instituto Superior Técnico, Portugal
E-mail address: pedro.f.santos@math.ist.utl.pt

Department of Mathematics, Texas A&M University, USA
E-mail address: plfilho@math.tamu.edu