DOUBLE AFFINE HECKE ALGEBRAS AND CALOGERO-MOSER SPACES

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Abstract. In this paper we prove that the spherical subalgebra $eH_{1,\tau}e$ of the double affine Hecke algebra $H_{1,\tau}$ is an integral Cohen-Macaulay algebra isomorphic to the center $Z$ of $H_{1,\tau}$, and $H_{1,\tau}$ is a Cohen-Macaulay $eH_{1,\tau}e$-module with the property $H_{1,\tau} = \text{End}_{eH_{1,\tau}e}(H_{1,\tau}e)$. In the case of the root system $A_{n-1}$ the variety $\text{Spec}(Z)$ is smooth and coincides with the completion of the configuration space of the relativistic analog of the trigonometric Calogero-Moser system. This implies the result of Cherednik that the module $eH_{1,\tau}$ is projective and all irreducible finite dimensional representations of $H_{1,\tau}$ are regular representation of the finite Hecke algebra.

Introduction

Ivan Cherednik in his pioneering paper [1] introduced the double affine Hecke algebras. These algebras play a crucial role in the proof of Macdonald Conjectures [2] and are a natural generalization of affine Hecke algebras, which are an object of great importance in representation theory.

In the paper [3] Pavel Etingof and Victor Ginzburg studied the rational degeneration of a double affine Hecke algebra. They discovered that in the case when this algebra has a nontrivial center, the spectrum of the center is isomorphic to the so called Calogero-Moser space, and this isomorphism respects the Poisson structure. The Calogero-Moser space first appeared in [4] as a completed configuration space for the Calogero-Moser integrable system. Recently attention to this object was aroused by the paper [5].

The isomorphism between the spectrum of the center of the degenerate double affine Hecke algebra and the Calogero-Moser space gives an interpretation of the degenerate double affine Hecke algebra as an Azumaya algebra in the case when the Calogero-Moser space is smooth.

In the present paper we study the double affine Hecke algebra $H$ with $q = 1$. In this case the algebra has a nontrivial center. We establish a Poisson isomorphism between the spectrum of the center $Z(H)$ of $H$ and a relativistic analog of the Calogero-Moser space in the case of the root system $A_{n-1}$. The relativistic analog of the Calogero-Moser space is a completed configuration space for the so called Ruijsenaars-Shneider (or briefly RS) integrable system [6].

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For the general algebra \( H \) (with \( q = 1 \)) we prove that the ring \( \mathbb{Z}(H) \) has no zero divisors, and that it is a normal, Cohen-Macaulay ring isomorphic to the spherical subalgebra \( eHe \) (where \( e \) is the symmetrizer in the finite Hecke algebra). We also prove the equality \( \mathbb{Z}(H) = \text{End}_{eHe}(He) \), which allows us, in the case of the root system \( A_{n-1} \), to interpret \( H \) as an Azumaya algebra.

The techniques of the paper work also in the degenerate case and furnish a simpler proof of the results of [3]. Furthermore, there exists an intermediate degeneration of the double affine Hecke algebra which lies between the double affine Hecke algebra and the rational degeneration of this algebra. We call this algebra the trigonometric degeneration of the double affine Hecke algebra. The corresponding degeneration of the Calogero-Moser space yields the configuration space for the trigonometric Calogero-Moser system (sometimes this space is called the trigonometric Calogero-Moser space). The results of the paper hold for this intermediate degeneration and are given in the last section.

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1. Definitions

1.1. Definition of the double affine Hecke algebra corresponding to \( GL(n, \mathbb{C}) \). We denote this algebra by the symbol \( H_{q,t} \). It is generated by the elements \( T_i, 1 \leq i \leq n-1, \pi, X_i^{\pm 1}, 1 \leq i \leq n \) with relations

\[
\begin{align*}
(1) & \quad X_iX_j = X_jX_i, \quad (1 \leq i, j \leq n), \\
(2) & \quad T_iX_iT_i = X_{i+1}, \quad (1 \leq i < n), \\
(3) & \quad T_iX_j = X_jT_i, \quad \text{if } j - i \neq 0, 1 \\
(4) & \quad [T_i, T_j] = 0, \quad \text{if } |i - j| > 1 \\
(5) & \quad T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \quad (1 \leq i < n), \\
(6) & \quad \pi X_i = X_{i+1}\pi, \quad (1 \leq i \leq n - 1), \quad \pi X_n = q^{-1}X_1\pi, \\
(7) & \quad \pi T_i = T_{i+1}\pi, \quad \pi^n T_j = T_j\pi^n, \quad (1 \leq i < n - 1, 1 \leq j < n), \\
(8) & \quad (T_i - \tau)(T_i + \tau^{-1}) = 0, \quad (1 \leq i \leq n).
\end{align*}
\]

Remark 1.1. To identify this definition with the standard definition from the papers of Cherednik one should replace \( \tau \) by \( t^{\frac{1}{2}} \) and \( q \) by \( q^{\frac{1}{2}} \). Also, some definitions use the element \( T_0 = \pi T_{n-1}\pi^{-1} \).

Remark 1.2. The double affine Hecke algebra corresponding to \( SL(n, \mathbb{C}) \) is a quotient of the subalgebra of \( H_{q,\tau} \) generated by \( X_i/X_{i+1}, T_i, \pi, 1 \leq i \leq n - 1, \)
by one extra relation:
\[ \pi^n = 1. \]

1.2. **Definition of the Calogero-Moser space.** Let \( E \) be an \( n \)-dimensional vector space (over \( \mathbb{C} \)). We denote by the symbol \( CM'_\tau \) the subset of \( GL(E) \times GL(E) \times E \times E^* \) consisting of the elements \((X,Y,U,V)\) satisfying the equation
\[
X^{-1}Y^{-1}XY\tau - \tau^{-1} = U \otimes V.
\]
Obviously it is an affine variety.

The group \( GL(n, \mathbb{C}) = GL(E) \) acts on it by conjugation:
\[
(X,Y,U,V) \rightarrow (gXg^{-1}, gYg^{-1}, gU, gV), \quad g \in GL(E).
\]
Later we will show that this action is free if \( \tau \) is not a root of unity. So the naive quotient by the action (i.e. the spectrum of the ring of \( GL(E) \) invariant functions) yields an affine variety, and the quotient is nonsingular if \( CM'_\tau \) is.

**Definition.** The quotient of \( CM'_\tau \) by the action \( GL(E) \) is called the Calogero-Moser space. We use the notation \( CM_\tau \) for this space.

Below we always suppose that \( \tau \) is not a root of unity.

2. **Properties of the Calogero-Moser space**

The goal of this section is to prove that \( CM_\tau \) is a smooth irreducible algebraic variety of dimension \( 2n \). We also introduce coordinates on its dense subset. The methods of this section are analogous to the ones from the paper [5]. In principal smoothness of \( CM_\tau \) follows from the results of the paper [7], the authors of [7] use the moduli space of the vector bundles on the punctured torus. For convenience of reader we give a direct elementary proof.

2.1. **Smoothness of the Calogero-Moser space.** First we prove a simple lemma on which all the following statements are based.

**Lemma 2.1.** If \((X,Y,U,V) \in CM'_\tau \) and \([R, X] = [R, Y] = 0, R \in \mathfrak{gl}(E)\) then \( R = \lambda Id \) for some \( \lambda \in \mathbb{C} \).

**Proof.** Let \( W \subset E \) be a nonzero subspace which is invariant under the action of \( X, Y \) and \( R \). We denote by \( \bar{X} \) and \( \bar{Y} \) the restriction of the operators \( X, Y \) to this subspace. It follows from equation (9) that there are two possibilities.

In the first case \( W \subset V^\perp \), where \( V^\perp \) is the notation for the annihilator. In this case (9) implies
\[ \bar{X}^{-1}\bar{Y}^{-1}\bar{X}\bar{Y} = \tau^{-2}Id. \]

But the determinant of LHS is equal to 1, hence we get a contradiction.

In the second case \( W \nsubseteq V^\perp, U \in W \). In this case (9) implies
\[ \bar{X}^{-1}\bar{Y}^{-1}\bar{X}\bar{Y} - U\bar{V} = \tau^{-2}Id, \]
Hence $X$ as the bilinear form $\text{tr}_{X,Y}$ for all $W$, we can rewrite the last condition in the form:

$$
\tau
$$

We know from equation (9) that the spectrum of $X$ is diagonal with the spectrum $\tau^{-2}, \tau^{-2}, \ldots, \tau^{-2}, \tau^{2k}$ where $k = \dim W$. But we know from equation [9] that the spectrum of $X^{-1}Y^{-1}XY$ is equal to $\tau^{-2}, \tau^{-2}, \ldots, \tau^{2n}$. Thus we get $W = E$.

The fact that the only common nonzero invariant subspace of $X, Y$ and $R$ is the whole $E$ immediately implies the statement of the lemma. Indeed, let $\lambda$ be an eigenvalue of $R$, then the corresponding eigenspace $W_\lambda$ is invariant under the action of $X$ and $Y$, hence it coincides with $E$.

**Corollary 2.1.** The action of $GL(E)$ on $CM'_\tau$ is free.

**Lemma 2.2.** $CM'_\tau$ is smooth.

**Proof.** Let us introduce the map $\Psi: GL(E) \times GL(E) \times E \times E^* \to gl(E)$:

$$
\Psi(X, Y, U, V) = X^{-1}Y^{-1}XY - U \otimes V.
$$

It is enough to show that $d\Psi$ is epimorphic at a point $(X, Y, U, V) \in CM'_\tau$. Let $x, y \in gl(E)$, $u \in E, v \in E^*$ and $X(t) = Xe^{xt}$, $Y(t) = Ye^{yt}$, $U(t) = U + tu$, $V(t) = V + tv$. Then

$$
d\Psi(x, y, u, v) = d\Psi(X(t), Y(t), U(t), V(t))|_{t=0} =
$$

$$
-x^{-1}X^{-1}XY + X^{-1}Y^{-1}XY - x^{-1}y^{-1}XY + X^{-1}Y^{-1}XY - U \otimes v - u \otimes V.
$$

If $d\Psi$ is not an epimorphism, then there exists $0 \neq R \in gl(E)$ such that

$$
\text{tr}(d\Psi(x, y, u, v)R) = 0
$$

for all $x, y \in gl(E)$, $u \in E, v \in E^*$. Using the cyclic invariance of the trace, we can rewrite the last condition in the form:

$$
\text{tr}(x(\text{YRX}^{-1}X - X^{-1}X^{-1}XYR)) + \text{tr}(y(\text{YRX}^{-1}XY - Y^{-1}XYRX^{-1})) - v(RU) - VR(u) = 0.
$$

As the bilinear form $\text{tr}(xy)$ is nondegenerate, the last equation implies

(10) $YRX^{-1}X - X^{-1}X^{-1}XYR = 0,$

(11) $RX^{-1}Y^{-1}XY - Y^{-1}XYRX^{-1} = 0,$

(12) $RU = 0, \quad VR = 0.$

These equations together with equation [9] imply $[R, X] = [R, Y] = 0$. Indeed, let us derive the first equation.

Multiplying on the right formula [10] by $R$ we get

(13) $X^{-1}Y^{-1}XYR = \tau^{-2}R.$

Hence

$$
\tau^{-2}XRX^{-1} = Y^{-1}XYRX^{-1} = RX^{-1}XY = R(\tau^{-1}U \otimes V + \tau^{-2}Id) = \tau^{-2}Id,
$$

where $0 \neq \bar{V}$ is the restriction of $V$ to the subspace $W$. Since $\det(\bar{X}Y, \bar{X}^{-1}Y^{-1}) = 1$, the last equation implies that there is a basis in $W$ in which $\bar{X}^{-1}Y^{-1}XY$ is diagonal with the spectrum $\tau^{-2}, \tau^{-2}, \ldots, \tau^{-2}, \tau^{2k}$ where $k = \dim W$. But we know from equation [9] that the spectrum of $X^{-1}Y^{-1}XY$ is equal to $\tau^{-2}, \tau^{-2}, \ldots, \tau^{2n}$. Thus we get $W = E$. 

□
Corollary 2.2. \( CM_\tau \) is smooth algebraic variety, and all its irreducible components have dimension \( 2n \).

2.2. Local coordinates on \( CM_\tau \). It is easy to see that matrices \( X,Y \in \text{gl}(n,\mathbb{C}) \),

\[
X = \text{diag}(\lambda_1, \ldots, \lambda_n), \quad Y_{ii} = q_i, \quad i = 1, \ldots, n, \quad Y_{ij} = \frac{(\tau - \tau^{-1})q_i \lambda_j}{(\tau \lambda_i - \tau^{-1} \lambda_j)}, \quad 1 \leq i \neq j \leq n, \quad (16)
\]

satisfy the equation

\[
\text{rk}(\tau XY - \tau^{-1} YX) = 1, \quad (17)
\]

for all \( \lambda \in (\mathbb{C}^*)^n \setminus D_\tau, \quad q \in (\mathbb{C}^*)^n \) where

\[
D_\tau = \{ \lambda | \delta_\tau(\lambda) = \prod_{i \neq j} (\tau \lambda_i - \tau^{-1} \lambda_j) = 0 \}. \quad (18)
\]

There is a well known formula: if \( M = (M_{ij}) \), where \( M_{ij} = (\lambda_i - \mu_j)^{-1}, \quad 1 \leq i,j \leq n \), then

\[
\det(M) = \prod_{i<j}(\lambda_i - \lambda_j)(\mu_j - \mu_i) / \prod_{i,j}(\lambda_i - \mu_j). \quad (19)
\]

To prove this formula one can proceed by the induction on \( n \) using the Gaussian method of calculation of the determinant for the step of the induction.

Applying the last formula to the matrix \( Y \) we see that \( \det(Y) \) is nonzero if and only if \( \lambda_i \neq \lambda_j, \quad i \neq j \).

Let us denote by \( \pi_{12}^\prime: \text{CM}^\prime_\tau \rightarrow \text{GL}(E) \times \text{GL}(E) \) the projection on the first two coordinates. The previous reasoning shows that \( (X,Y) \in \pi_{12}(\text{CM}^\prime_\tau), \quad \lambda \in (\mathbb{C}^*)^n \setminus (D_\tau \cup D), \quad q \in (\mathbb{C}^*)^n \) where

\[
D = \{ \lambda | \delta(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j) = 0 \}. \quad (20)
\]

Now we can state

**Proposition 2.1.** Let \( (X,Y,U,V) \in \text{CM}^\prime_\tau \) and \( X \) be diagonalizable with the different eigenvalues \( \lambda_i, \quad i = 1, \ldots, n \) such that \( \tau \lambda_i \neq \tau^{-1} \lambda_j \). Then the \( \text{GL}(n,\mathbb{C}) \) orbit of \( (X,Y,U,V) \) contains a representative satisfying equations \( V = \lambda^t \) and \( (14)-(16) \) for some \( q \in (\mathbb{C}^*)^n \). Such a representative is unique up to (simultaneous) permutation of the parameters \( (\lambda_i,q_i) \).

**Proof.** Equation \( (17) \) is equivalent to the system

\[
\frac{(\tau \lambda_i - \tau^{-1} \lambda_j)Y_{ij}}{\tau - \tau^{-1}} = p_is_j, \quad 1 \leq i,j \leq n, \quad (17)
\]
if \( X = \text{diag}(\lambda_1, \ldots, \lambda_n) \). If there exists \( i \) such that \( s_i = 0 \) then \( Y_{ij} = 0 \), \( j = 1, \ldots, n \) and \( \det(Y) = 0 \). Thus we have \( s_i \neq 0 \). Analogously we get \( p_i \neq 0 \).

Let us fix a solution of \( (13) \) lying in the \( GL(n, E) \) orbit of \((X, Y, U, V)\). Putting \( q_i = p_i(s_i)/\lambda_i \) we get the desired representative with \( X \) given by formula \( (14) \), \( Y \) by formulas \( (15), (16) \) and \( U = (\tau - \tau^{-1})X^{-1}Y^{-1}q \). □

This proposition together with Corollary \( 2.2 \) implies that \((\lambda, q)\) are local coordinates on the open subset \( U \subset CM_{\tau} \). In the next section we show that this subset is dense.

2.3. Irreducibility of \( CM_{\tau} \). In this subsection we prove

**Proposition 2.2.** The variety \( CM_{\tau} \) is irreducible.

Let us consider the projection on the first component \( \pi_1^t : CM_{\tau}^t \rightarrow GL(E) \). After the taking the quotient by the action of \( GL(E) \) this map becomes a map \( \pi_1^t : CM_{\tau} \rightarrow JNF \), where \( JNF \) is a stack but we can think about it as the set of Jordan normal forms of matrices (we do not need the stack structure).

Inside \( JNF \) there is an open part \( \tilde{U} \) corresponding to diagonal matrices with eigenvalues \( \{\lambda_1, \ldots, \lambda_t\} \) such that \( \lambda_i \neq \lambda_j, \tau \lambda_i \neq \tau^{-1} \lambda_j \) for \( i \neq j \). The subset \( \pi_1^{-1}(\tilde{U}) \) was described in the previous section. It is obviously connected. If we show that \( \dim \pi^{-1}(JNF \setminus \tilde{U}) < 2n \) then Corollary \( 2.2 \) implies the irreducibility.

Let us denote by \( J_k(\lambda) \) the Jordan block of size \( k \) with the eigenvalue \( \lambda \) and by the symbol \( J_k(\lambda) \) the matrix \( \text{diag}(J_k^1(\lambda), \ldots, J_k^t(\lambda)) \), \( \vec{k} \in \mathbb{N}^t \) and \( k^i \geq k^{i+1}, i = 1, \ldots, t - 1 \). Let us formulate without a proof an elementary statement from linear algebra.

**Lemma 2.3.** The dimension of \( \text{Stab}(J_k(\lambda)) = \{X \in GL(n, \mathbb{C}) | [X, J_k(\lambda)] = 0\} \)
is equal to \( \sum_{1 \leq i, j \leq 1} \min\{k^i, k^j\} \).

Let us denote by \( J_k(\lambda) \) the matrix \( \text{diag}(J_k^1(\lambda), J_k^2(\lambda \tau^2), \ldots, J_k^r(\lambda \tau^{2r})) \), \( \vec{k} \in \mathbb{N}^t \). We use notations \( |\vec{k}| = \sum_{j=1}^{t_j} k^j, |k| = \sum_{j=1}^r |\vec{k}| \).

Let \( \lambda_1, \ldots, \lambda_s \in \mathbb{C} \) be such that \( \lambda_i/\lambda_j \neq \tau^{2c}, c \in \mathbb{Z}, |c| \leq n \) and

\[
J = \text{diag}(J_{k_1}(\lambda_1), \ldots, J_{k_s}(\lambda_s)).
\]

We denote by \( \pi_3^4 : CM_{\tau}^t \rightarrow E \times E^* \) the slightly modified projection on the last two components: \( \pi_3^4(X, Y, U, V) = (YXU, V) \). The fiber of the map \( \pi_3^4 \) over the point \((U, V)\) of the subset \( \tilde{J} = \pi_3^4((\pi_1^t)^{-1}(J)) \) consists of the points \((J, Y + F, J^{-1}(Y + F)^{-1}U, V)\) where \( F \) is an element of the kernel of the linear map:

\[
S_J(F) = \tau JF - \tau^{-1}FJ, \quad F \in \mathfrak{gl}(E),
\]
$Y + F$ is invertible, and $(J, Y, J^{-1}Y^{-1}U, V) \in CM'_t$. Obviously $(\pi_{31})^{-1}(U, V)$ is a Zariski open nonempty subset inside $ker(S_J)$ hence they have the same dimension.

First let us study the map $S_J$ in the simple case when in the equation (19) we have $s = 1$ and $k_1 = k = (\tilde{k}_1, \ldots, \tilde{k}_r)$, $\tilde{k}_i \in \mathbb{N}^d$, $1 \leq i \leq r$. In this situation we denote by $F_{ij}^{st} \in Mat(k_s^i, k^j_{l})$, $1 \leq s, t \leq r$ the matrix with the entries $F_{ij}^{st} = F_{ij}^r$, $p' = \sum_{l=1}^{s-1} |\tilde{k}_l| + \sum_{l=1}^{t-1} k^l_s + p$, $q' = \sum_{l=1}^{t-1} |\tilde{k}_l| + \sum_{l=1}^{s-1} k^l_s + q$. In these notations the following lemma holds

**Lemma 2.4.** Let $J$ be the matrix given by (12) with $s = 1$ and $k_1 = k = (\tilde{k}_1, \ldots, \tilde{k}_r)$. Then $F \in ker S_J$ if and only if

(20) \hspace{1cm} F_{ij}^{st} = 0, \hspace{1cm} \text{if } t - s \neq 1,

(21) \hspace{1cm} F_{ij}^{s,s+1} = (\sum_{l=0}^{k^j_s-1} c_{ij,l}^s J_{k^j_s}^l(0)) D_{r}^{k^l_s k^j_s} \hspace{1cm} \text{if } k^i_s \leq k^j_{s+1},

(22) \hspace{1cm} F_{ij}^{s,s+1} = D_{r}^{k^l_s k^j_s} (\sum_{l=0}^{k^j_s-1} c_{ij,l}^s J_{k^j_s}^l(0)) \hspace{1cm} \text{if } k^i_s > k^j_{s+1},

where $c_{ij,l}^s \in \mathbb{C}$, $J_{k_s}^l(0)$ (and $J_{k_s}^l(0)$) is the $l$-th power of the Jordan block matrix, and $D_{r}^{k^l_s k^j_s} \in Mat(k^l_s, k^j_{s+1})$ is given by formula

$$D_{r}^{k^l_s k^j_s} = \delta_{p+k^l_s+q+k^j_s, r} 2^{-2p} \hspace{1cm} \text{if } k^i_s \leq k^j_{s+1}$$

$$D_{r}^{k^l_s k^j_s} = \delta_{p,q, r} 2^{-2p} \hspace{1cm} \text{if } k^i_s > k^j_{s+1}.$$  

**Proof.** The system of linear equations $S_J(F) = 0$ is equivalent to the collection of linear systems:

$$\tau J_{k^j_s} (\lambda^{2s-2}) F_{ij}^{st} - \tau^{-1} F_{ij}^{st} \lambda_{k^j_s} (\lambda^{2t-2}) = 0, \hspace{1cm} 1 \leq s, t \leq r,$$

because $J$ has a block structure. The equations for the entries of $F_{ij}^{st}$ are of the simple form:

(23) \hspace{1cm} F_{ij}^{st} (\lambda^{2s-1} - \tau^{2t-3}) = \tau (\delta_{p,k^i_s, q-1}) F_{ij}^{st} \lambda_{k^j_s+1,q} - \tau^{-1} (\delta_{p,1, q-1}) F_{ij}^{st} \lambda_{p+1,q-1} - 1.$

First consider the case $t - s \neq 1$. Then $\tau^{2s-1} - \tau^{2t-3} \neq 0$ and equations (23) express the entries of the $i$-th diagonal through the entries of the $(i-1)$-th diagonal. It easy to see that in this case (23) implies $F_{ij;1,1}^{st} = 0$, that is, the first diagonal is zero. Moving from the left to the right we get that all the diagonals of $F_{ij}^{st}$ are zero.

If $s+1 = t$ then equation (23) is a linear relation between the neighboring entries on the diagonal. It is easy to derive equations (21), (22) from this fact.

Indeed, let us consider the case $k^j_s \leq k^j_{s+1}$. Then equation (23) for $p = k^i_s$, $1 < q \leq k^i_s$ says $F_{ij;1,q-1}^{st} = 0$. Moving along the diagonal from the bottom
to the top and using equation (23) we get that the first \( k^s_i - 1 \) diagonals of the matrix \( F^{s,s+1}_{ij} \) are zero. For the rest of the diagonals equation (23) implies \( F^{s,s+1}_{ij,p+1,q+p} = F^{s,s+1}_{ij,1,q} \tau^{-2p} \). Putting \( c^{s,i}_{i,j} = F^{s,s+1}_{ij,1,l+k^i_j-k^i_{s+1}} \) we get equation (21).

□

Obviously \( Z \in \text{Im} S_J \) if and only if \( \text{tr}(ZF) = 0 \) for all \( F \in \text{ker} \bar{S}_J \), \( \bar{S}_J(F) = \tau^{-1}JF - \tau FJ \). The space \( \text{ker} \bar{S}_J \) has a description similar to the one of \( \text{ker} S_J \) (to get \( \text{ker} \bar{S}_J \) from \( \text{ker} S_J \) it is enough to change the order of the Jordan blocks in \( J \)) and one can easily derive

**Corollary 2.3.** \( Z \in \text{Im} S_J \) if and only if following equations hold

\[
\sum_{l=0}^{u-1} Z^{s,s+1}_{ij;\nu_{i-l}u-l} = 0, \quad u = 1, \ldots, \min\{k^i_s, k^j_{s+1}\},
\]

where \( s = 1, \ldots, r - 1 \).

The lowest nonzero diagonal of a rank one matrix contains only one nonzero entry. As \( \hat{J} \subset \text{Im} S_J \cap \{ \text{matrices of rank 1} \} \) the following statement holds

**Corollary 2.4.** \((U,V) \in \hat{J} = \pi_{34}^{-1}((\pi_1')^{-1}(J)) \) if and only if \( Z = U \otimes V \) satisfies the equation

\[
Z^{s,s+1}_{ij,pq} = 0 \text{ if } p - q \geq \min\{0, k^i_s - k^j_{s+1}\}, \quad s = 1, \ldots, r - 1.
\]

Lemma 2.4 gives us the formula for the dimension of the kernel

\[
\dim \text{ker} S_J = \sum_{s=1}^{r-1} \sum_{i,j} \min\{k^i_s, k^j_{s+1}\}.
\]

We know that \( GL(n, E) \) acts on \( CM'_\tau \) freely. Hence if we want to estimate the dimension of the fiber of \( \pi_{34} \) over \( \hat{J} \) we should estimate \( \dim \text{Stab}(J) - \dim \text{ker} S_J \). This difference is positive:

**Lemma 2.5.** Let \( k^s_s \in \mathbb{N}^d_s \), \( s = 1, \ldots, r \), \( k^i_s \geq k^i_{s+1} \) then the following inequality holds

\[
\sum_{s=1}^{r} \sum_{i,j} \min\{k^i_s, k^j_{s+1}\} - \sum_{s=1}^{r-1} \sum_{i,j} \min\{k^i_s, k^j_{s+1}\} > 0,
\]

if there exists \( s \) such that \( k^s_s \neq 0 \).

**Proof.** Because of the inequality \( k^i_s \geq k^i_{s+1} \) we can rewrite LHS of the inequality in the form

\[
\sum_{\nu=1}^{r} \left( \sum_{s=1}^{r} (x^\nu_s)^2 - \sum_{s=1}^{r-1} x^\nu_s x^\nu_{s+1} \right),
\]

\( x^\nu_s = \# \{ i \in \mathbb{N} | k^i_s \geq \nu \}. \)
But the first expression is a sum of positive definite quadratic forms. Thus we get the lemma.

The following statement is crucial for estimating of \( \dim(\pi_1^{-1}(JNF \setminus \tilde{U})) \):

**Proposition 2.3.** If \( J \) is given by (12) with \( s = 1 \) and \( k_1 = k = (\tilde{k}_1, \ldots, \tilde{k}_r) \), then \( \dim \pi_1^{-1}(J) < 2n - 1 \) when either \( r > 1 \) or \( k_1 > 1 \).

**Proof.** In the case \( r > 1 \) Corollary 2.4 implies that \( \dim \pi_1^{-1}(JNF \setminus \tilde{U}) \leq 2n - 1 \). The theorem on the dimension of the fibers and previous reasoning imply:

\[
\dim \pi_1^{-1}(J) \leq \dim \pi_1'(\pi_1^{-1}(J)) + \dim \ker S_J - \dim \text{Stab}(J).
\]

Together with the inequality from Lemma 2.5 it proves the statement.

Another case (i.e. \( k = \tilde{k}_1 \)) is even easier because in this case we have

\[
\dim \pi_1^{-1}(J) \leq 2n - \dim \text{Stab}(J) < 2n - 1.
\]

The case when in formula (19) \( s > 1 \) can be easily reduced to the previous case. For that let us introduce the embedding \( i_l : gl(|k_l|, C) \to gl(n, C) \) and the projection \( pr_l : gl(n, C) \to gl(|k_l|, C) : i_l(Y)_{p', q'} = Y_{pq}, pr_l(Y)_{pq} = Y_{p', q'}, \)

\[
p' = p + \sum_{m=1}^{l-1} |k_m|, q' = q + \sum_{m=1}^{l-1} |k_m|, 0 \leq p, q \leq |k_l|, \text{ and } i_l(Y)_{ij} = 0 \text{ for the rest of the entries of } i_l(Y).
\]

Using arguments analogous to the ones from Lemma 2.4 one gets

**Lemma 2.6.** Let \( J \) be given by formula (12). Then

1. \( \ker S_J = \oplus_{i=1}^s i_l(\ker S_{k_i}(\lambda_i)) \)
2. for \( l = 1, \ldots, s \), \( pr_l(\text{Im} S_J) \subset \text{Im} S_{k_l} \).

This lemma immediately implies

**Proposition 2.4.** Let \( J \) be given by formula (12) and exists \( l, 1 \leq l \leq s \) such that \( |k_l| > 1 \) then \( \dim \pi_1^{-1}(J) < 2n - s \).

And we eventually achieved the goal of the subsection:

**Proof of Proposition 2.3.** Indeed Proposition 2.4 implies \( \dim \pi_1^{-1}(JNF \setminus \tilde{U}) < 2n \). Hence by Corollary 2.2 \( \pi_1^{-1}(JNF \setminus \tilde{U}) \) lies inside the Zariski closure of \( \pi_1^{-1}(\tilde{U}) \). But \( \pi_1^{-1}(\tilde{U}) \) is irreducible.

2.4. The Poisson structure on the CM space. In the paper [7] the Poisson structure on the space \( CM_r \) was constructed. This Poisson structure on \( CM_r \) yields the RS integrable system which is the relativistic analog of the trigonometric Calogero-Moser system.
On the open part \( U \) of \( CM_{\tau} \) described in the subsection 2.2 the Poisson bracket \( \{ \cdot, \cdot \}_{FR} \) takes the form (see Appendix of [7] for the proof):

\[
\{ \lambda_i, \lambda_j \}_{FR} = 0, \quad \{ \lambda_i, q_i \}_{FR} = \lambda_i q_i \delta_{ij}, \\
\{ q_i, q_j \}_{FR} = \frac{(\tau - \tau^{-1})^2 q_i q_j (\lambda_i + \lambda_j) \lambda_i \lambda_j}{(\tau \lambda_i - \tau^{-1} \lambda_j) (\tau \lambda_j - \tau^{-1} \lambda_i) (\lambda_i - \lambda_j)}.
\]

**Remark 2.1.** The formulas in [7] contain the misprint, the authors lost the factor \((\tau^2 - 1)^2\) in the expression for \( \{ q_i, q_j \}_{FR} \).

Using the Hamiltonian reduction on the combinatorial model of the space of flat connections on the torus without a point the authors of [7] prove that the Poisson structure \( \{ \cdot, \cdot \}_{FR} \) has a holomorphic extension from \( U \) to the whole \( CM_{\tau} \), and this Poisson structure is nondegenerate (i.e. \( CM_{\tau} \) is a symplectic variety). Another way to see this Poisson structure is to use Quasi-Poisson reduction [8]. In this picture the Poisson structure is the result of the reduction of the natural Quasi-Poisson structure on the product \( GL(n, \mathbb{C}) \times GL(n, \mathbb{C}) \) and it is immediate that this Poisson structure is symplectic.

### 3. Finite dimensional representation of \( H_{1,\tau} \)

In this subsection we construct a family of finite dimensional representations of \( H_{1,\tau} \). Later we will show that this family forms an open dense set inside the space of all finite dimensional representations. The main tool of this section is the faithful representation of \( H_{1,\tau} \) which is the quasiclassical limit of the standard realization of \( H_{q,\tau} \) as a subring of the ring of reflection difference operators [2].

#### 3.1. Limit of the Lusztig-Demazure operators

Let us introduce the ring \( \tilde{R} = \mathbb{C}[P_{1}^{\pm 1}, \ldots, P_{n}^{\pm 1}, X_{1}^{\pm 1}, \ldots, X_{n}^{\pm 1}]_{\delta(X)} \# S_{n} \), where the subscript \( \delta(X) \) means localization by the ideal generated by \( \delta(X) = \prod_{1 \leq i < j \leq n} (X_i - X_j) \) and \# is a notation for the smash product. Let us explain what the smash product is. For brevity we will use notation \( \mathbb{C}[P^{\pm 1}, X^{\pm 1}] \) for the ring \( \mathbb{C}[P_{1}^{\pm 1}, \ldots, P_{n}^{\pm 1}, X_{1}^{\pm 1}, \ldots, X_{n}^{\pm 1}] \).

An element of the ring \( \tilde{R} \) has the form \( \sum_{w \in S_{n}} F_{w}(P, X) w \). The group \( S_{n} \) acts on the ring \( R = \mathbb{C}[P^{\pm 1}, X^{\pm 1}]_{\delta(X)} \) by the formulas

\[
P_{i}^{w} = P_{w^{-1}(i)}, \quad X_{i}^{w} = X_{w^{-1}(i)},
\]

and

\[
F(P, X) w F'(P, X) w' = F(P, X) (F')^{w}(P, X) w w'.
\]
Proposition 3.1. The following formulas give an injective homomorphism of $H_{1,\tau} \to \tilde{R}$:

$$X^\mu \mapsto X^\mu,$$

$$T_i \mapsto \tau s_i + \frac{\tau - \tau^{-1}}{X_i/X_{i+1} - 1}(s_i - 1), \quad i = 1, \ldots, n - 1,$$

$$\pi \mapsto P_1^{-1}c,$$

where $s_i = (i, i + 1) \in S_n$ is a transposition and $c \in S_n$ is a cyclic transformation: $c(i) = i + 1, i = 1, \ldots, n - 1, c(n) = 1$.

The homomorphism from the proposition is a quasiclassical limit of the Lusztig-Demazure representation [2]. For brevity we call this homomorphism the Lusztig-Demazure representation.

Remark 3.1. Actually the paper [2] contains the proof for the case $q \neq 1$. The proof in the case $q = 1$ can be obtained from this proof by mechanical replacement of shifts operators $\tau(\lambda), \lambda \in \mathbb{Z}^n$ by their quasiclassical limits $P^\lambda$. The reader may do this operation with Lecture 5 from the exposition [9].

3.2. The representation $V_{\mu,\nu}$. Let $(\mu, \nu) \in (\mathbb{C}^*)^{2n}$ and $\chi_{\mu,\nu} \simeq \mathbb{C}$ be a one dimensional $R$-module (character): $\chi_{\mu,\nu}(R(P,X)) = R(\mu, \nu)$. We can induce a finite dimensional module $V_{\mu,\nu}$ from this module:

$$V_{\mu,\nu} = \tilde{R} \otimes_R \chi_{\mu,\nu}.$$

This module has a $\mathbb{C}$ basis $w \otimes 1, w \in S_n$, hence dim$V_{\mu,\nu} = n!$.

Proposition 3.2. If $\nu_i \neq \nu_j, i \neq j$ then the $H_{1,\tau}$-module $V_{\mu,\nu}$ is irreducible.

Proof. The module $V_{\mu,\nu}$ has a natural $H_{\delta(X)}$ module structure. From the Lusztig-Demazure representation we see that $H_{\delta(X)} \simeq \mathbb{C}[P^{\pm 1}, X^{\pm 1}]_{\delta(X)}\#S_n$. The group $S_n$ acts freely on the variety Spec$\mathbb{C}[P^{\pm 1}, X^{\pm 1}]_{\delta(X)}$ hence the algebra $H_{\delta(X)}$ is Morita equivalent to the algebra $\mathbb{C}[P^{\pm 1}, X^{\pm 1}]_{\delta(X)}S_n$. In particular, the module $V_{\mu,\nu}$ corresponds to the one-dimensional representation: $P \mapsto P(\mu, \nu)$. Thus $V_{\mu,\nu}$ is an irreducible $H_{\delta(X)}$-module and hence an irreducible $H$-module.

3.3. The action of the finite Hecke algebra. The elements $T_i, i = 1, \ldots, n - 1$ generate an algebra of dimension $n!$ which is called the finite Hecke algebra. We will denote it by the symbol $A^n_\tau$.

If $e$ is the unit in $S_n$ then by the action of elements $T_i$ we can get from the vector $e \otimes 1$ the whole space $V_{\mu,\nu}$. Hence the map $j: A^n_\tau \to V_{\mu,\nu}, j(T_{i_1} \cdots T_{i_k}) = T_{i_1} \cdots T_{i_k} e \otimes 1$ is an isomorphism of (left) $A^n_\tau$ modules.

Definition. We denote the subset of all finite dimensional irreducible $H_{1,\tau}$-modules which are regular $A^n_\tau$-modules by the symbol $\operatorname{Irrep}^n_\tau$. 

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Let us denote the subset of \( \text{Irrep}^{n!} \) consisting of \( V_{\mu, \nu} \) \( \mu, \nu \in (\mathbb{C}^*)^n, \delta(\nu) \neq 0 \) by \( \mathcal{U} \). Later we will show that all finite dimensional irreducible modules are from \( \text{Irrep}^{n!} \).

3.4. \textbf{The GL}(2, \mathbb{Z}) action on double affine Hecke algebras.} One of the most important properties of the double affine Hecke algebra \( H_{q, \tau} \) is the existence of the action of \( \text{GL}(2, \mathbb{Z}) \) \([3]\). To explain how this group acts on the double affine Hecke algebra we need to introduce pairwise commutative elements \( Y_i \in H_{q, \tau} \):

\[
Y_i = T_1 \ldots T_{n-i} \pi^{-1} T_{n-i+1} \ldots T_{n-1} \quad i = 1, \ldots, n - 1.
\]

These elements satisfy the relations

\[
T_i Y_{i+1} T_i = Y_i, \quad (1 \leq i < n)
\]

\[
T_i Y_j = Y_j T_i, \quad \text{if } j - i \neq 0, 1.
\]

The group \( \text{GL}(2, \mathbb{Z}) \) is generated by the elements:

\[
\varepsilon = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
\]

These generators act by the following formulas:

\[
\varepsilon : X_i \mapsto Y_i, Y_i \mapsto X_i, T_i \mapsto T_i^{-1},
\]

\[
\sigma : X_i \mapsto X_i, Y_i \mapsto X_i Y_i \pi^{-1}, T_i \mapsto T_i,
\]

where \( \varepsilon : H_{q, \tau} \to H_{q^{-1}, \tau^{-1}}, \sigma : H_{q, \tau} \to H_{q, \tau} \). The transformation \( \varepsilon \) is called the Fourier-Cherednik transform.

Using these transformations we can construct new finite dimensional representations. Indeed if \( \gamma \in \text{GL}(2, \mathbb{Z}) \) is such that \( \gamma(H_{1, \tau}) = H_{1, \tau'} \) and \( \phi' : H_{1, \tau'} \to \text{GL}(V'_{\mu, \nu}) \) is the corresponding representation of \( H_{1, \tau'} \) (here \( \tau' \) is either \( \tau \) or \( \tau^{-1} \)) then the map \( \phi' \circ \gamma \) is a representation of \( H_{1, \tau} \). We denote the set of such representations by \( \gamma(\mathcal{U}) \).

4. \textbf{The map from Irrep}^{n!} \textbf{to CM}_r

In this section we construct a map \( \Phi : \text{Irrep}^{n!} \to \text{CM}_r \). Later we will show that it is an isomorphism. Constructions of this section generalize constructions of section 11 of \([3]\).

4.1. \textbf{Construction of the map.} Let us denote by \( A_{\tau}^{n-1} \subset A_{\tau}^{n} \) the subalgebra generated by the elements \( T_2, \ldots, T_{n-1} \). It is the finite Hecke algebra of rank \( n - 2 \). The element \( v \) of an \( A_{\tau}^{n} \)-module is said to be \( A_{\tau}^{n-1} \)-invariant if \( xv = \tau v \) for all \( x \in A_{\tau}^{n-1} \).

The \( H_{1, \tau} \)-module \( V \in \text{Irrep}^{n!} \) by definition is a regular \( A_{\tau}^{n} \)-module. Hence the space \( V.A_{\tau}^{n-1} \) of \( A_{\tau}^{n-1} \)-invariants has dimension \( n \). The relations inside \( H_{1, \tau} \) and \([26]\) imply that \( X_1 \) and \( Y_1 \) commute with the action of \( A_{\tau}^{n-1} \). Thus if we fix a basis in \( V \) we get \( X_1|_V.A_{\tau}^{n-1}, Y_1|_V.A_{\tau}^{n-1} \in \text{GL}(n, \mathbb{C}) \). The following statement is a key statement of the section.
**Proposition 4.1.** Let $V \in \text{Irrep}_n^{\text{ad}}$ then the operators $\bar{X}_1 = X_1|_{V_{\text{ad}}^{n-1}}$, $\bar{Y}_1 = Y_1|_{V_{\text{ad}}^{n-1}}$ satisfy the equation:

$$rk(\bar{X}_1 \bar{Y}_1 X_1^{-1} Y_1^{-1} - \tau^{-2}Id) = 1.$$ 

Obviously the space $CM_\tau$ is isomorphic to the quotient of the space of solutions of (17) by the action of $GL(n, \mathbb{C})$. Thus the last proposition proves that the map $\Phi: \text{Irrep}_n^{\text{ad}} \to CM_\tau$, $\Phi(V) = (\bar{X}_1, \bar{Y}_1)$ is well defined.

In the rest of the section we prove Proposition 4.1. It is done in two steps. First we prove

**Lemma 4.1.** The elements $X_1, Y_1 \in H_{1,\tau}$ satisfy the relation

$$X_1 Y_1 X_1^{-1} Y_1^{-1} = T_1 T_2 \ldots T_{n-2} T_{n-1}^2 T_{n-2} \ldots T_1.$$ 

This is done in the next subsection using the geometric interpretation of the double affine Hecke algebra. The proof of this lemma was communicated to the author by Ivan Cherednik. The last step is the analysis of the LHS of (27) using the quasiclassical limit $\tau \to 1$. It is done in the last subsection.

**4.2. The double affine braid group.** The double affine Hecke algebra admits a simple topological interpretation [1]. This construction is especially simple in the case $q = 1$. In this case the algebra $H_{1,\tau}$ is a quotient of the so called double affine braid group $B_n$.

For better understanding of this group the reader may have in mind the picture analogous to the geometric interpretation of the usual braid group but in the case when the points live on the two dimensional torus.

In this picture the elements $T_i$, $i = 1, \ldots, n-1$ correspond to the paths which permute of the $i$-th and the $i+1$-th nearby points (just like in the case of the usual braid group) and $X_i, Y_i$ correspond to the paths in which the $i$-th point goes along the parallel, respectively the meridian of the torus. In this geometric setting it is obvious that formula (27) holds in $B_n$. The double affine Hecke algebra is a quotient of $B_n$ by the relations (8). Hence (27) holds in $H_{1,\tau}$. Below we give formal definitions to justify this reasoning.

Let

$$U = \{ z \in \mathbb{C}^n | z_k - z_l \notin \mathbb{Z} + i\mathbb{Z}, k \neq l \},$$

and $\bar{W} = S_n \ltimes (\mathbb{Z} \oplus \mathbb{Z} i)$ acts on $z \in U$ by the formula:

$$\bar{w}(z) = w(z + a + ib), \quad \bar{w} = w(a + ib), \quad a, b \in \mathbb{Z}^n, \quad w \in S_n.$$ 

We fix a point $z^0$ such that its real and imaginary part is sufficiently small.

**Definition.** Paths $\gamma \subset U$ joining $z^0$ with points from $\{ \bar{w}(z^0), \bar{w} \in \bar{W} \}$ modulo homotopy and the action of $\bar{W}$ form the double affine braid group $B_n$ with the multiplication induced by the usual composing operation for the paths.
This group is generated by the elements:
\[ T_j = t_j(\psi) = z^0 + (\exp(\pi i \psi) - 1)(z_j - z_{j+1})(e_j - e_{j+1}) \]
\[ X_j = x_j(\psi) = z^0 + \psi e_j, \quad Y_j = y_j(\psi) = z^0 + \psi e_{j+1}, \]
\[ \tau = T_1 \ldots T_{n-1} Y_1, \]
where \( 0 \leq \psi \leq 1 \).

**Proposition 4.2.** The group \( \mathfrak{B}_n \) is generated by elements \( X_i, Y_i, i = 1, \ldots, n, T_j, j = 1, \ldots, n - 1 \) (and \( \pi \)) with defining relations \( \{1, \ldots, n\} \) with \( q = 1 \).

Thus \( H_{1, \tau} \) is a quotient of \( \mathfrak{B}_n \) by relations \( \{5\} \) and Lemma \( \{4\} \) follows.

### 4.3. The spectrum of \( Z = T_1 \ldots T_{n-2} T_{n-1}^2 T_{n-2} \ldots T_1 \).

For a representation \( V \) is from \( Irrep^n \) there is an isomorphism \( V \simeq A^n_\tau \) of left \( A^n_\tau \)-modules. Hence the right multiplication on \( A^n_\tau \) induces a structure of a right \( A^n_\tau \)-module on \( V \) and as a consequence on \( V^{A^n_{\tau^{-1}}} \).

The right \( A^n_\tau \)-module \( V^{A^n_{\tau^{-1}}} \) is a sum of the \( n - 1 \) dimensional vector representation and one-dimensional representation because it is true for \( \tau = 1 \). Obviously, the operator \( Z \) (acting by the left multiplication) commutes with the right action of \( A^n_\tau \). Hence by the Schur lemma \( Z \) acts by a constant on \( A^n_\tau \)-irreducible components of the right \( A^n_\tau \)-module \( V^{A^n_{\tau^{-1}}} \). That is, there exists a basis in the module in which \( Z \) is diagonal and of the form \( \text{diag}(\lambda_1(\tau), \lambda_2(\tau), \ldots, \lambda_n(\tau)) \). Thus we only need to calculate \( \lambda_1(\tau), \lambda_2(\tau) \).

The module \( V^{A^n_{\tau^{-1}}} \) exists for all \( \tau \neq 0 \). As the operator \( Z \) is invertible for all nonzero values of \( \tau \), we have \( \lambda_1(\tau) \neq 0, \lambda_2(\tau) \neq 0 \).

The functions \( \lambda_1(\tau) \) are single valued. Indeed for \( n = 2 \) it is obvious. So let us suppose \( n > 2 \), then the eigenvalues \( \lambda_i(\tau) \) have the different multiplicities. Hence the Galois group of the extension of the field of rational functions by \( \lambda_1(\tau), \lambda_2(\tau) \) is trivial because it cannot exchange \( \lambda_1 \) and \( \lambda_2 \). Thus the functions \( \lambda_i(\tau) \) are rational and we have \( \lambda_i(\tau) = C_i \tau^{k_i}, i = 1, 2 \).

When \( \tau = 1 \), the algebra \( A^n_\tau \) becomes the group algebra of \( S_n \), and \( Z = 1 \). Thus we have \( C_1 = C_2 = 1 \). The calculation of \( k_1, k_2 \) uses the quasiclassical limit reasoning.

If \( \tau = e^h \) then we can write the expansion of \( T_i \) in terms of \( h \)
\[ T_i = s_i + h \tilde{s}_i + O(h^2), \quad i = 1, \ldots, n - 1, \]
where \( s_i = (i, i+1) \) is a usual transposition. Relation \( \{5\} \) inside \( H_{1, \tau} \) implies
\[ s_i \tilde{s}_i + \tilde{s}_i s_i = 2s_i, \quad i = 1, \ldots, n - 1. \]

Let us calculate the first nontrivial term \( \tilde{Z} \) of the expansion of \( Z = 1 + h \tilde{Z} + O(h^2) \):
\[ \tilde{Z} = \sum_{i=1}^{n-1} s_1 \ldots s_{i-1} (\tilde{s}_i s_i + \tilde{s}_i s_i) s_{i-1} \ldots s_1 = 2 \sum_{i=1}^{n-1} s_{1i}. \]
where \(s_{1i} = s_1 \ldots s_{i-1} \cdot s_i \ldots s_l\) is a permutation of 1 and \(i\).

The operator \(Z/2\) acts on \(\mathbb{C}[S_n]^{S_n}\) (by the left multiplication) and in the basis \(e_i = (\sum_{w' \in S_{n-1}} w')s_{1i}\) it has the matrix \(J - \text{Id}, J_{ij} = 1, 1 \leq i, j \leq n\). Hence \(\text{Spec}(Z/2) = (n-1, -1, \ldots, -1)\). On the other hand \(\text{Spec}(Z) = (k_1, k_2, \ldots, k_2)\). Thus \(k_1 = 2(n-1), k_2 = -2\) and we proved Proposition 4.1.

### 4.4. The map \(\Phi\) on the subset \(\mathcal{U} \subset \text{Irrep}^{\text{aff}}\).

It is possible to calculate \(\Phi(V_{\mu, \nu})\) explicitly. Indeed let us fix a basis in \(V_{\mu, \nu}^{A_1} : e_i = (\sum_{w' \in S_{n-1}} w')s_{1i}, i = 1, \ldots, n\).

**Proposition 4.3.** For the matrices of the operators \(\tilde{X}_1\) and \(\tilde{Y}_1\) written in the basis \(e_i\) the following equations hold

\[
\tilde{X}_1 = \text{diag}(\nu_1, \ldots, \nu_n)
\]

\[
\tilde{Y}_{ii} = \mu_i \prod_{j \neq i} \frac{(\tau^{-1} \nu_j - \tau \nu_i)}{(\nu_j - \nu_i)}, \quad i = 1, \ldots, n.
\]

**Proof.** The first equation is obvious. The second formula is a result of direct calculation using formulas (24) for \(Y_1\) and explicit formulas for \(T_1\).

Indeed let make this calculation for \(i = 1\). The expansion of the product expression for \(Y_1\) consists of the terms of the form \(s_{i_1,j_1} \ldots s_{i_r,j_r} e^{-1} F(X) P_1\), where \(i_1 < j_1, j_m < i_{m+1}\), \(l = 1, \ldots, r, m = 1, \ldots, r-1\) and \(F \in \mathbb{C}[X^{\pm 1}]_{\delta(X)}\). We know that \(Y_1 e_1\) is a linear combination of \(e_i, i = 1, \ldots, n\). The terms of the expansion of \(Y_1 e_1\) which contribute to the coefficient before \(e_1\) satisfy the equation \(s_{i_1,j_1} \ldots s_{i_r,j_r} e^{-1}(1) = 1\). This is possible only in the case \(r = 1\), \(i_1 = 1, j_1 = n\). Thus rewriting \(T_1\) in the form:

\[
T_i = \frac{(\tau X_i - \tau^{-1} X_{i+1})}{X_i - X_{i+1}} s_i + \frac{X_{i+1}(\tau^{-1} - \tau)}{X_i - X_{i+1}},
\]

we see that

\[
Y_1 e_1 = \left( \prod_{i=1}^{n-1} \frac{(\tau X_i - \tau^{-1} X_{i+1})}{X_i - X_{i+1}} s_i \right) e^{-1} e_1 + R,
\]

where \(R\) is a linear combination of \(e_j\) with \(j > 1\). This formula immediately implies the last formula from the proposition for \(i = 1\).

Let \(D_\tau\) be a subset of \(\mathcal{U}\) consisting of the representations of the form \(V_{\mu, \nu}\) such that \(\delta_\tau(\nu) = \prod_{i,j} (\tau \nu_i - \tau^{-1} \nu_j) = 0\).

It is actually not easy to compute all coefficients \(\tilde{Y}_1\) using explicit formulas for \(Y_1\) and \(T_1\) but we do not need them. Because by proposition 2.1 if the pair \((X, Y)\) satisfies equation (11) and \(X\) is diagonal with eigenvalues satisfying the conditions of Proposition 2.1 then the corresponding \(GL(E)\)-orbit is uniquely determined by the diagonal elements of \(X\) and \(Y\) (because the stabilizer of \(X\) consists of diagonal matrices which do not change diagonal elements of \(Y\) and we can extract \(q\) from these elements). This reasoning implies
Corollary 4.1. The map $\Phi$ is an isomorphism on the subset $U_0 = U \setminus D_\tau$, and local coordinates $\lambda, q$ on $CM_\tau$ are expressed through coordinates $\mu, \nu$ on $U_0 \subset \text{Irrep}_n$ by the formulas

$$\lambda_i = \nu_i, \quad q_i = \mu_i \prod_{j \neq i} \frac{(\tau^{-1} \nu_j - \tau \nu_i)}{(\nu_j - \nu_i)}.$$

5. Results on the general double affine Hecke algebra

Let $R = \{\alpha\}$ be a root system (possibly nonreduced) of type $A, B, BC, \ldots, F, G, W$ the Weyl group generated by the reflections $s_\alpha, \alpha \in R$. The extended affine Weyl group $\tilde{W}$ is a semidirect product $W \rtimes P$, where $P$ is a weight lattice (i.e. $b \in P$ if $2(b, \alpha)/(\alpha, \alpha) \in \mathbb{Z}$ for all $\alpha \in R$).

The affine Hecke algebra $\hat{H}_\tau$ is a deformation of the group algebra $\mathbb{C}[\tilde{W}]$ with deformation parameters $\tau_\alpha, \tau_{w(\alpha)} = \tau_\alpha, \alpha \in R, w \in W$ (for the exact definition of the affine Hecke algebra see [3]). The double affine Hecke algebra $H_{q,\tau}$ is a nontrivial extension of the affine Hecke algebra $\hat{H}_\tau$ by the group algebra $\mathbb{C}[P^\vee]$ of the coweight lattice $P^\vee$ ($b \in P^\vee$ if $(b, \alpha) \in \mathbb{Z}$ for all $\alpha \in R$). This extension has one parameter $q$ which is the shift parameter in the Lusztig-Demazure representation of this algebra. We consider algebras with $q = 1$ and we denote them by $H$. For the exact definition of the double affine Hecke algebra and formulas for the Lusztig-Demazure representation see the original paper [11] or survey [9].

We use the notation $\delta(X)$ for the Weyl denominator for the root system $R$. By symbol $\mathbb{C}[X^{\pm 1}]$ we denote the group algebra of the weight lattice $P$ lying inside the affine Hecke algebra $\hat{H}_\tau$ and by symbol $\mathbb{C}[Y^{\pm 1}]$ we denote group algebra $\mathbb{C}[P^\vee] \subset H$ which extends $\hat{H}_\tau$.

There is an injective homomorphism $g: \hat{H}_\tau \to \mathbb{C}[P^{\pm 1}, X^{\pm 1}]_{\delta(X)} \# W$ via the quasiclassical Lusztig-Demazure operators $w \mapsto w, X_b \mapsto X^b, T_{s_\alpha} = T_\alpha \mapsto D_\alpha$, where $\alpha \in R, b \in P$. The formulas for the embedding are very similar to the formulas from the previous section. Let $A$ be the corresponding finite Hecke algebra, and $e$ the symmetrizer in $A$:

$$e = \sum_{w \in W} \tau^{l(w)} T_w / \left( \sum_{w \in W} \tau^{2l(w)} \right),$$

where $T_w = T_{i_1} \ldots T_{i_{l(w)}}$ if $w = s_{i_1} \ldots s_{i_{l(w)}}$ is a reduced expression for $w$.

In this section we will need the following PBW type result

Proposition 5.1. Each element $h \in H$ can be uniquely presented in the forms:

$$h = \sum_{w \in W} f_w(X) T_w g_w(Y),$$

$$h = \sum_{w \in W} g'_w(Y) T_w f'_w(X).$$
5.1. **Formulation of the theorem.** The goal of this section is to study the center \( Z \) of \( H \) and corresponding scheme \( \text{Spec}(Z) \). It turns out that \( Z \) is isomorphic to the subalgebra \( eHe \) and we can reduce the study of \( Z \) to the study of \( eHe \).

We remind the definition of a Cohen-Macaulay algebra.

**Definition.** A finitely generated commutative \( \mathbb{C} \)-algebra \( A \) is called Cohen-Macaulay if it contains a subalgebra of the form \( \mathcal{O}(V) \) such that \( A \) is a free \( \mathcal{O}(V) \)-module of finite rank, and \( V \) is a smooth affine algebraic variety.

For the definition of a Cohen-Macaulay module see [10] (Chapter 4 p. 18). In this section we prove the following

**Theorem 5.1.** For any double affine Hecke algebra \( H \) the following is true:

1. \( eHe \) is commutative.
2. \( M = \text{Spec}(eHe) \) is an irreducible Cohen-Macaulay and normal variety.
3. The right \( eHe \) module \( He \) is Cohen-Macaulay.
4. The left action of \( H \) on \( He \) induces an isomorphism of algebras \( H \cong \text{End}_{eHe}(He) \).
5. The map \( \eta : z \rightarrow ze \) is an isomorphism \( Z \rightarrow eHe \). Thus, \( M = \text{Spec}(Z) \).

We call the isomorphism \( \eta \) the Satake isomorphism (by analogy with [3]).

5.2. **Proofs of theorem 5.1.**

**Lemma 5.1.** \( Z \) contains \( \mathbb{C}[X^{\pm 1}]^W \otimes \mathbb{C}[Y^{\pm 1}]^W \).

**Proof.** \( \mathbb{C}[X^{\pm 1}]^W \) clearly lies in the center of \( \mathbb{C}[P^{\pm 1}, X^{\pm 1}]_{\delta^{-1}(X)}#W \), and therefore in the center of \( H \). The fact that \( \mathbb{C}[Y^{\pm 1}]^W \) is contained in \( Z \) follows from the existence of the Fourier-Cherednik morphism [2] (i.e. the action of the element \( \varepsilon \) of \( GL(2, \mathbb{Z}) \) which is described in the previous section in the case of the root system \( A_{n-1} \)).

Indeed, the morphism \( \varepsilon \) is an isomorphism between the double affine Hecke algebra \( H' \) with parameter \( \tau^{-1} \) and the double affine Hecke algebra \( H \). This morphism maps the subring \( \mathbb{C}[X^{\pm 1}]^W \subset H' \) onto the subring \( \mathbb{C}[Y^{\pm 1}]^W \) of \( H \).

Now the statement follows from the PBW theorem for \( H \). \( \square \)

**Lemma 5.2.** \( eHe \) is commutative, without zero divisors.

**Proof.** Let us prove that the subalgebra \( eH_{\delta(X)}e \) of \( H_{\delta(X)} \cong \mathbb{C}[P^{\pm 1}, X^{\pm 1}]_{\delta(X)}#W \) is commutative and without zero divisors. Obviously it implies the statement.

An element \( z \in H_{\delta(X)} \) has a unique representation in the form \( z = \sum_{w \in W} Q_u T_w \); that is, \( H_{\delta(X)} \) is isomorphic to \( \mathbb{C}[P^{\pm 1}, X^{\pm 1}]_{\delta(X)} \otimes A \) as a right \( A \)-module. If \( z \in eH_{\delta(X)}e \) then \( zT_\alpha = \tau_\alpha z \) for all \( \alpha \in R \) because \( eT_\alpha = \tau_\alpha e \). Hence \( z \) is an \( A \)-invariant element of the right \( A \)-module
eH_{δ(X)}e \subset \mathbb{C}[P^{±1}, X^{±1}]_{δ(X)} \otimes A. As \mathbb{C}(P, X) \otimes A is a regular A-module (over the field \mathbb{C}(P, X)) \mathbb{C}(P, X) \otimes e is a unique copy of the trivial representation. It implies that \( z = Qe, Q \in \mathbb{C}[P^{±1}, X^{±1}]_{δ(X)}. \)

Finally for \( z = Qe \in eH_{δ(X)}e \) we have \((T_α - τ_α)Qe = 0\). The simple calculation using the explicit expression for \( T_α \) yields:
\[
(T_α - τ_α)Qe = P_α(s_α - 1)Qe = P_α(s_α(Q) - Q)e,
\]
where \( P_α \in \mathbb{C}[X^{±1}]_{δ(X)} \) and \( α \) is a simple root. This implies \( Q \in \mathbb{C}[P^{±1}, X^{±1}]_{δ(X)}e \) and \( eH_{δ(X)}e \simeq \mathbb{C}[P^{±1}, X^{±1}]_{δ(X)}. \)

The algebra \( H \) has a natural \( \mathbb{C}[X^{±1}]W \otimes \mathbb{C}[Y^{±1}]W \)-module structure: the element \( p \otimes q \) acts on \( x \in H \) by the formula \( (p \otimes q)x = pxq \).

**Lemma 5.3.** \( H \) is a projective finitely generated \( \mathbb{C}[X^{±1}]W \otimes \mathbb{C}[Y^{±1}]W \)-module.

**Proof.** Let us first show that \( \mathbb{C}[X^{±1}] \) is a projective finitely generated \( \mathbb{C}[X^{±1}]W \) module. Finite generation is clear, since \( W \) is a finite group. Also, it is well known that \( \mathbb{C}[X]W \) is a polynomial ring (it is generated by the characters of the fundamental representations of the corresponding simply connected group). Since \( \mathbb{C}[X^{±1}] \) is a regular ring, by Serre’s theorem ([10], chapter 4, p. 37, proposition 22) \( \mathbb{C}[X^{±1}] \) must be locally free over \( \mathbb{C}[Y^{±1}]W \) (in fact, by Steinberg-Pittie [11] theorem it is free, but we will not use it). For the same reasons \( \mathbb{C}[Y^{±1}] \) is locally free over \( \mathbb{C}[Y^{±1}]W \).

Now the claim follows from the PBW factorization from Proposition 5.1 \( H = \mathbb{C}[X^{±1}] \otimes A \otimes \mathbb{C}[Y^{±1}] \). \( \square \)

**Lemma 5.4.** \( He \) and \( eHe \) are projective finitely generated modules over \( \mathbb{C}[X^{±1}]W \otimes \mathbb{C}[Y^{±1}]W \).

**Proof.** The finite generation follows from the Hilbert-Noether lemma and Lemma 5.3. The projectivity is true because \( He \) and \( eHe \) are direct summands in \( H \). \( \square \)

**Lemma 5.5.**
1. \( H_{δ(X)} \simeq \mathbb{C}[X^{±1}, P^{±1}]_{δ(X)}W \)
2. The map \( η: Z(H_{δ(X)}) \to \mathbb{C}[P^{±1}, X^{±1}]W_{δ(X)}e \), induced by multiplication by \( e \) is an isomorphism.
3. The left \( H_{δ(X)} \)-action on \( H_{δ(X)} \) induces the isomorphism \( H_{δ(X)} \simeq \text{End}_{eH_{δ(X)}}(H_{δ(X)}). \)

**Proof.** The first and second items of the lemma follow from the representation of \( H \) by the quasiclassical Lusztig-Demazure operators. The third item is equivalent to the isomorphism
\[
\mathbb{C}[P^{±1}, X^{±1}]_{δ(X)}W \simeq \text{End}_{\mathbb{C}[P^{±1}, X^{±1}]_{δ(X)}}(\mathbb{C}[P^{±1}, X^{±1}]_{δ(X)}).
\]
We will proceed analogously to the proof of theorem 1.5 from [3].

If \( a: \mathbb{C}[P^{±1}, X^{±1}]_{δ(X)} \to \mathbb{C}[P^{±1}, X^{±1}]_{δ(X)} \) is \( \mathbb{C}[P^{±1}, X^{±1}]_{δ(X)} \)-linear then it defines a \( \mathbb{C}(P, X)W \)-linear map \( \mathbb{C}(P, X) \to \mathbb{C}(P, X). \) The isomorphism
\[ C(P, X) \# W \simeq \text{End}_{\mathcal{C}(P, X)}(\mathcal{C}(P, X)) \] implies \[ a = \sum_{w \in W} a_w w, \quad a_w \in C(P, X). \]

It is clear that the functions \( a_w \) are regular on \((\mathbb{C}^*)^n \times (\mathbb{C}^*)^n \setminus \Delta\) where \( \Delta \) is the subset of the points of \((\mathbb{C}^*)^n \times (\mathbb{C}^*)^n\) with a nontrivial stabilizer in \( W \) and \( D = \{ X \in \mathbb{C}^* | \delta(X) = 0 \} \). But \( \Delta \subset D \), hence \( a_w \in \mathcal{C}[P^\pm, X^\pm]_{\delta(X)} \).

**Proof of Theorem 5.1**: The first item follows from Lemma 5.2.

**Proof of (2)**: \( M = \text{Spec}(eHe) \) is an irreducible affine variety by Lemma 5.2. The subalgebra \((\mathbb{C}[X^\pm]^W \otimes \mathbb{C}[Y^\pm]^W)e\) is polynomial. Hence to prove that \( M \) is Cohen-Macaulay it is sufficient to show that \( eHe \) is a locally free module of finite rank over its subalgebra \((\mathbb{C}[X^\pm]^W \otimes \mathbb{C}[Y^\pm]^W)e\). But the module is projective and finitely generated by Lemma 5.4.

It is easy to see by localizing with respect to \( e\delta(X) \) or \( e\delta(Y) \) that \( M \) is smooth away from a codimension 2 subset. Indeed, by the first item of Lemma 5.3 after localizing with respect to \( e\delta(X) \) the image of \( eHe \) under the injection \( g \) becomes \( e\mathbb{C}[X, Y]_{\delta(X)}e \simeq \mathbb{C}[X, Y]_{\delta(X)}^W \), which is the ring of regular functions on a smooth affine variety. The statement for the localization with respect to \( e\delta(Y) \) follows from the existence of the Fourier-Cherednik transform. But an irreducible Cohen-Macaulay variety that is smooth outside of a codimension 2 subset is normal (12.2.2).

**Proof of (3)**: \( eHe \) is finitely generated over \( \mathbb{C}[X^\pm]^W \otimes \mathbb{C}[Y^\pm]^W \). Hence by Theorem 2.1 of [13] \( eH \) is Cohen-Macaulay over \( eHe \) if and only if it is Cohen-Macaulay over \( \mathbb{C}[X^\pm]^W \otimes \mathbb{C}[Y^\pm]^W \).

We know that \( He \simeq \mathbb{C}[X^\pm, Y^\pm] \) as a \( \mathbb{C}[X^\pm]^W \otimes \mathbb{C}[Y^\pm]^W \)-module and \( He \) is projective over \( \mathbb{C}[X^\pm]^W \otimes \mathbb{C}[Y^\pm]^W \). As \( \mathbb{C}[X^\pm]^W \otimes \mathbb{C}[Y^\pm]^W \) is a polynomial ring, the module \( \mathbb{C}[X^\pm, Y^\pm] \) is Cohen-Macaulay if and only if it is projective. So Lemma 5.4 implies the statement.

**Proof of (4)**: We have an obvious homomorphism \( f : H \to \text{End}_{eHe}(eHe) \). It is clearly injective because it is injective after localization by the ideal \((\delta(X))\).

Let us denote \( \text{End}_{eHe}(eHe) \) by \( H' \). Regard \( H' \supset H \) as \( \mathbb{C}[X^\pm]^W \otimes \mathbb{C}[Y^\pm]^W \)-modules. \( H' \) is torsion free because \( He \) is a torsion free \( \mathbb{C}[X^\pm]^W \otimes \mathbb{C}[Y^\pm]^W \)-module (by the PBW theorem). As \( He \) is finitely generated over \( eHe \), \( H' \) is finitely generated \( \mathbb{C}[X^\pm]^W \otimes \mathbb{C}[Y^\pm]^W \)-module. Also, \( H \) is finitely generated projective, and \( H'/H \) is supported in codimension 2. Indeed, the last part of Lemma 5.5 implies that \( \text{Ext}^1_{\mathbb{C}[X^\pm]^W \otimes \mathbb{C}[Y^\pm]^W}(H_{\delta(X)}) \) is isomorphic to \( H_{\delta(X)}^1 \) as a \( eHe_{\delta(X)} \) module. Similarly, the module \( H_{\delta(Y)} \) is isomorphic to \( H'_{\delta(Y)} \) as a \( eHe_{\delta(Y)} \) module because we can use (the same way as in the proof of Lemma 5.2) the Fourier-Cherednik transform \( \varepsilon \) from subsection 5.4.

The module \( H' \) represents some class in \( \text{Ext}^1(H'/H, H) \), which must be zero since \( H'/H \) is finitely generated and lives in codimension 2 and \( H \) is projective. Thus, \( H' = H \oplus H'/H \) and the summand \( H'/H \) is torsion. But \( H' \) is a torsion free \( eHe \) module, hence \( H'/H = 0 \) and \( H' = H \).

**Proof of (5)**: It is clear that \( \eta \) is injective, by looking at the Lusztig-Demazure representation. Indeed the equation \( ze = 0 \) implies \( zp = 0 \) for any \( p \in \mathbb{C}[X]^W \), hence by the PBW theorem \( z = 0 \).
It remains to show that \( \eta \) is surjective. Since \( eHe \) is commutative, every element \( a \in eHe \) defines an endomorphism of \( He \) over \( eHe \) (by right multiplication). So by statement \( 4 \) a defines an element \( z_a \in H \). This element commutes with \( H \). Indeed, the right multiplication by \( a \) is an endomorphism of the right \( eHe \)-module which commutes with left multiplication by elements of \( H \) hence by the fourth part of the theorem \([z_a, h] = 0\) for all \( h \in H \). For any \( x \in H \), \( z_a x = xa \), so \( x z_a = xa \), i.e. \( x(z_a e - a) = 0 \). Since \( eHe \) has no zero divisors, we find \( \eta(z_a) = a \), as desired. \( \square \)

6. The Results in the Case of the Root System \( A_{n-1} \)

In this section \( H = H_{1, \tau} \) is the double Hecke algebra corresponding to \( GL(n, \mathbb{C}) \).

A point \((\mu, \nu) \in (\mathbb{C}^*)^n \times ((\mathbb{C}^*)^n \setminus D)\) defines a \( \mathbb{C}[P^{\pm 1}, X^{\pm 1}]_{\delta(X)} \)-character \( \chi(\mu, \nu) : \chi(\mu, \nu)(Q(P, X)) = Q(\mu, \nu) \). The embedding \( Z \hookrightarrow Z_{\delta(X)} \cong \mathbb{C}[P^{\pm 1}, X^{\pm 1}]_{\delta(X)} \) allows us to restrict this character to \( Z \). We use the same notation for this character.

**Lemma 6.1.** For any point \((\mu, \nu) \in (\mathbb{C}^*)^n \times ((\mathbb{C}^*)^n \setminus D)\) we have
\[
He \otimes_{eHe} \chi(\mu, \nu) \cong V(\mu, \nu).
\]

**Proof.** The \( H \)-module \( V_{\mu, \nu} \) has a natural structure of an \( H_{\delta(X)} \)-module. Let us study finite dimensional irreducible \( H_{\delta(X)} \)-modules.

By Lemma 5.5, the ring \( eH_{\delta(X)} \) is a regular ring. As the action of \( S_n \) on \((\mathbb{C}^*)^n \times ((\mathbb{C}^*)^n \setminus D)\) is free, the ring \( \mathbb{C}[P^{\pm 1}, X^{\pm 1}]_{\delta(X)} \) is a projective \( eH_{\delta(X)} \)-module and defines the vector bundle \( F \) over \( \mathbb{C}^n \times ((\mathbb{C}^*)^n \setminus D) = Spec(eH_{\delta(X)}) \). Hence by the last item of Lemma 5.5 \( H_{\delta(X)} = End(F) \) is an Azumaya algebra and by the basic property of Azumaya algebras any irreducible \( H_{\delta(X)} \)-module is of the form \( H_{\delta(X)} \otimes_{eH_{\delta(X)}} \chi(\mu', \nu') \) for some point \((\mu', \nu') \in (\mathbb{C}^*)^n \times ((\mathbb{C}^*)^n \setminus D)\).

Obviously any irreducible \( H_{\delta(X)} \)-module is irreducible as a \( H \)-module. Also we have an obvious isomorphism of \( H \) modules \( H_{\delta(X)} \otimes_{eH_{\delta(X)}} \chi(\mu', \nu') \cong He \otimes_{eHe} \chi(\mu', \nu') \). Thus the previous paragraph implies \( V_{\mu, \nu} \cong He \otimes_{eHe} \chi(\mu', \nu') \). Comparing the action of the center on the both sides yields the statement. \( \square \)

The previous lemma implies that there is a map \( \Upsilon \) from the open part \( Spec(Z_{\delta(X)}) \) of \( Spec(Z) \) to the space \( CM_r \): \( \Upsilon(\mu, \nu) = \Phi(V_{\mu, \nu}) \), where \( \Phi \) is the map constructed at the section 4. As \( Spec(Z_{\delta(X)}) \) is an open dense subset in \( Spec(Z) \), we can define a rational map \( \Upsilon: Spec(Z) \rightarrow CM_r \).

**Theorem 6.1.** The map \( \Upsilon: Spec(Z) \rightarrow CM_r \) is a regular isomorphism of the algebraic varieties. In particular \( Spec(Z) \) is smooth.

**Proof.** The previous lemma and Corollary 4 imply that \( \Upsilon \) is a regular isomorphism on \( Spec(Z_{\delta(X)}) \). The Fourier-Cherednik transform from the section 5 allows us to state the same for the open subset \( Spec(Z_{\Upsilon(\delta(X))}) \).
Indeed, the Fourier-Cherednik transform $\varepsilon$ maps the double affine Hecke algebra $H_{1,\tau}$ to $H_{1,\tau^{-1}}$ and it induces the map $\varepsilon_{CM}: CM_{\tau} \to CM_{\tau^{-1}}$, $\varepsilon_{CM}(X,Y,U,V) = (Y,X, -Y^{-1}X^{-1}YXU,V)$. By the construction we have $\varepsilon_{CM} \circ \Upsilon = \Upsilon \circ \varepsilon$. Thus the restriction of the morphism $\varepsilon_{CM}^{-1} \circ \Upsilon \circ \varepsilon$ to $\text{Spec}(Z_{\delta(Y)\delta_{\tau}(Y)})$ is a regular isomorphism.

Now, we know from the Theorem 5.1 that $\text{Spec}(Z)$ is normal. As the complement of $\text{Spec}(Z_{\delta(X)\delta_{\tau}(X)}) \cup \text{Spec}(Z_{\delta(Y)\delta_{\tau}(Y)})$ has codimension 2 (because $\text{Spec}(Z)$ is irreducible by Theorem 5.1), we can extend $\Upsilon$ to a regular map on the whole $\text{Spec}(Z)$. The extended map is dominant because by Proposition 2.2 the variety $CM_{\tau}$ is irreducible.

Thus $\Upsilon$ is a regular birational map which is an isomorphism outside of the subset of codimension 2. But we know that $CM_{\tau}$ is smooth and $\text{Spec}(Z)$ is normal, hence (by theorem 5 section 5 of chapter 2 of [15]) the map $\Upsilon^{-1}$ is regular and as a consequence is an isomorphism.

Corollary 6.1. $He$ is a projective eHe-module.

Proof. We proved for any $R$ that $He$ is a Cohen-Macaulay module over $eHe$. Since $M = \text{Spec}(eHe)$ is smooth, the result follows from corollary 2 from chapter 4 of [10].

Thus $He$ defines the vector bundle $E$ on $\text{Spec}(eHe)$, with fibers of dimension $n!$.

Corollary 6.2. For the double affine Hecke algebra $H = H_{1,\tau}$ the following is true:

1. $H = \text{End}E$ where $E$ is a vector bundle over $\text{Spec}(Z)$ i.e. $H$ is an Azumaya algebra.
2. Every irreducible representation of $H$ is of the form $V_z = He \otimes_{eHe} \chi_z$, $z \in M = \text{Spec}(Z)$.
3. $V_z$ has dimension $n!$ and is a regular representation of $A^n_{\tau}$.

Proof. The first item follows from Theorem 5.1. The second item is a general property of Azumaya algebras. The third item follows from the fact that it is true for the generic point $z \in \text{Spec}(Z)$.

Remark 6.1. This corollary was proved in 2000 by Cherednik using different methods [16].

The ring $Z \simeq eH_{1,\tau}e$ has a natural noncommutative deformation $eH_{q,\tau}e$. Hence this ring has a natural Poisson structure $\{\cdot, \cdot\}$. The variety $CM_{\tau}$ also has a Poisson structure described in subsection 2.3. It turns out that the isomorphism $\Phi$ respects these Poisson structures.

Theorem 6.2. The isomorphism $\Phi$ is an isomorphism of Poisson varieties, that is the following formula holds

$\{\cdot, \cdot\}_{FR} = \{\cdot, \cdot\}$. 

**Proof.** It is enough to prove that it is an isomorphism of Poisson varieties on the open set $\mathcal{U}$. For $q = e^h \neq 1$ we have an embedding $g_q; H_{q,τ} \to \mathbb{D}_q \# S_n$ via Lusztig-Demazure reflection difference operators. Here $\mathbb{D}_q$ is a localization of the Weyl algebra with generators $X^\pm_i$, $\hat{P}^\pm_i$, $i = 1, \ldots, n$ and relations:

\[
[X_i, X_j] = 0, \quad [\hat{P}_i, \hat{P}_j] = 0, \quad X_j \hat{P}_i - q^{δ_{ij}} \hat{P}_i X_j = 0,
\]

by the ideal $(δ(X))$. When $q = 1$, the noncommutative ring $\mathbb{D}_q$ becomes the commutative ring $\mathbb{C}[P^\pm, X^\pm]_{δ(X)}$ and the corresponding Poisson structure on this ring is given by the formulas:

\[
\{X_i, X_j\} = 0, \quad \{P_i, P_j\} = 0, \quad \{X_i, P_j\} = δ_{ij}X_iP_j.
\]

The $H_{1,τ}$-module $V_{μ,ν}$ has a natural $\mathbb{C}[P^\pm, X^\pm]_{δ(X)} \# S_n$ structure. It is easy to see that in the basis $1 \otimes w$, $w ∈ W$ operators $P_i$, $X_j$ are diagonal. In particular $P_i(1 \otimes e) = μ_i(1 \otimes e)$ and $X_i(1 \otimes e) = ν_i(1 \otimes e)$, hence we have the following Poisson bracket on $\mathcal{U}$:

\[
(28) \quad \{ν_i, ν_j\} = 0, \quad \{μ_i, μ_j\} = 0 \quad \{ν_i, μ_j\} = δ_{ij}ν_iμ_j.
\]

The comparison of the formulas for the Poisson bracket on $\mathcal{U} \subset CM_τ$ from subsection 4.4 and explicit formulas for the map $Φ|\mathcal{U}$ from subsection 4.4 give the formula. Indeed, we can express the functions $λ_i$, $q_k$ through the functions $μ_s$, $ν_t$ and using (28) calculate the Poisson brackets $\{λ_i, λ_k\}$, $\{λ_i, q_k\}$, $\{q_i, q_k\}$. We prove a formula for the last bracket:

\[
\{q_i, q_k\} = q_iq_k\left(ν_k \frac{∂ ln(q_i)}{∂ν_k} - ν_i \frac{∂ ln(q_k)}{∂ν_i}\right) =
\]

\[
q_iq_k\left(ν_k \left(\frac{τ}{τ^{-1}ν_i - τν_k} + \frac{1}{ν_i - ν_k}\right) - ν_i \left(-\frac{τ}{τ^{-1}ν_k - τν_i} + \frac{1}{ν_k - ν_i}\right)\right) =
\]

\[
\frac{(τ^{-1} - τ)^2 q_iq_k(ν_k + ν_i)ν_iν_k}{(ν_i - ν_k)(τ^{-1}ν_k - τν_i)(τ^{-1}ν_i - τν_k)}.
\]

\[\Box\]

7. The rational and trigonometric cases

In this section we explain how to degenerate results from the main body of the paper to obtain an easier proof of the results of [2] on the rational double affine Hecke algebra. We also give the version of the results of the paper for the trigonometric Hecke algebra and explain how to modify the proof from the paper for this case.

We give the modifications of the results from the main body of the text only for the root system $A_{n-1}$ but similar things can be done for any root system $R$. Moreover, in the rational case we can replace the Weyl group $W$ by a finite Coxeter group (see [3]). Proofs of these results almost identically repeat proofs for (nondegenerate) double affine Hecke algebras.
7.1. Definition of the rational and trigonometric double affine Hecke algebras. Below we give a definition of the rational and trigonometric double affine Hecke algebra.

Definition. [3][7] The rational double affine Hecke algebra $H_{t,c}^{\text{rat}}$ is generated by elements $s_{ij}$, $1 \leq i \neq j \leq n$, $x_i, y_j$, $1 \leq i, j \leq n$. The elements $s_{ij}$, $1 \leq i, j \leq n$ generate the subalgebra inside $H_{t,c}^{\text{rat}}$ isomorphic to the group algebra of the symmetric group $S_n$, and $s_{ij}$ corresponds to the transposition $(ij)$. In addition generators of $H_{t,c}^{\text{rat}}$ satisfy the relations

$$x_i s_{ij} = s_{ij} x_j, \quad y_i s_{ij} = s_{ij} y_j, \quad 1 \leq i, j \leq n,$$

$$[x_k, s_{ij}] = 0, \quad [y_k, s_{ij}] = 0, \quad k \notin \{i, j\}, \quad 1 \leq i, j, k \leq n,$$

$$[y_i, x_j] = c s_{ij}, \quad 1 \leq i \neq j \leq n,$$

$$[x_i, x_j] = 0 = [y_i, y_j], \quad 1 \leq i, j \leq n,$$

$$[y_k, x_k] = t - c \sum_{i \neq k} s_{ik}, \quad 1 \leq k \leq n.$$

Definition. The trigonometric double affine Hecke algebra $H_{t,c}^{\text{trig}}$ is generated by elements $s_{ij}$, $1 \leq i \neq j \leq n$, $X_i^{\pm 1}$, $y_j$, $1 \leq i, j \leq n$. The elements $s_{ij}$, $1 \leq i, j \leq n$ generate the subalgebra inside $H_{t,c}^{\text{rat}}$ isomorphic to the group algebra of the symmetric group $S_n$, and $s_{ij}$ corresponds to the transposition $(ij)$. In addition the generators of $H_{t,c}^{\text{trig}}$ satisfy the relations

$$X_i s_{ij} = s_{ij} X_j, \quad 1 \leq i, j \leq n,$$

$$s_{ij} y_i - y_j s_{ij} = c \text{ if } j > i, \quad s_{ij} y_i - y_j s_{ij} = -c \text{ if } j < i,$$

$$[X_k, s_{ij}] = 0, \quad [y_k, s_{ij}] = 0 \text{ if } k \notin \{i, j\}, \quad 1 \leq i, j, k \leq n,$$

$$[X_i, X_j] = 0 = [y_i, y_j], \quad 1 \leq i, j \leq n,$$

$$X_j^{-1} y_i X_j - y_i = c s_{ij} \text{ if } j > i, \quad X_j^{-1} y_i X_j - y_i = X_j X_j^{-1} c s_{ij} \text{ if } j < i,$$

$$X_j^{-1} y_k X_k - y_k = t - c (\sum_{i < k} s_{ik} + \sum_{i > k} X_i X_k^{-1} s_{ik}), \quad 1 \leq k \leq n.$$

Remark 7.1. Let $\hat{H}$ be the $\mathbb{C}[c, t][[h]]$-algebra topologically generated (in the $h$-adic topology) by $X_i$, $y_i$, $s_{i,i+1}$ with $T_i = s_{i,i+1} e^{h s_{i,i+1}}$, $i = 1, \ldots, n - 1$, $Y_i = e^{h y_i}$, $X_i$, $i = 1, \ldots, n$ satisfying the relations for the double affine Hecke algebra $H_{q,\tau}$, $q = e^{th}$, $\tau = e^{th}$. It coincides with an appropriate completion of the double affine Hecke algebra $H_{q,\tau}$ in the $h$-adic topology. Moreover, one can show that $\hat{H}$ is flat over $\mathbb{C}[[h]]$ and $\hat{H}/h\hat{H} = H_{t,c}^{\text{trig}}$. Analogously, if $\hat{H}^{\text{trig}}$ is the $\mathbb{C}[c, t][[h]]$-algebra topologically generated by $s_{ij}$, $y_i$, $x_j$, $1 \leq i \leq n$ with $s_{ij}, y_i, X_j = e^{h x_j}$, $i, j = 1, \ldots, n$, satisfying the relations for the trigonometric double affine Hecke algebra $H_{t,c}^{\text{trig}}$ then the algebra $\hat{H}^{\text{trig}}$ is flat over $\mathbb{C}[[h]]$ and $H_{t,c}^{\text{rat}} = \hat{H}^{\text{trig}}/h\hat{H}^{\text{trig}}$.

7.2. Representation by Dunkl operators. Let $\mathcal{D}_{t,c}^{\text{rat}}$ be the localization of the $n$-dimensional Weyl algebra $A_{t,c}^{\text{rat}}$ by the ideal generated by $\delta(x)$. 


The Weyl algebra $\mathcal{A}_i^{rat}$ is generated by elements $x_i, p_i$, $1 \leq i \leq n$ modulo relations:

$$[x_i, x_j] = 0 = [p_i, p_j], \quad [x_i, p_j] = t\delta_{ij}, \quad 1 \leq i, j \leq n.$$ 

Let us denote by $\mathcal{D}_i^{trig}$ the trigonometric version of the algebra $\mathcal{A}_i^{rat}$. This algebra is localization by $(\delta(X))$ of the algebra $\mathcal{A}_i^{trig}$ with generators $p_i, X_i^{\pm 1}$, $i = 1, \ldots, n$ modulo relations:

$$(29) \quad [X_i, X_j] = 0 = [p_i, p_j], \quad [X_i, p_j] = t\delta_{ij}X_i, \quad 1 \leq i, j \leq n.$$ 

It is easy to see that the ring $\mathcal{A}_i^{trig}$ is isomorphic to the ring of differential operators on the torus $(\mathbb{C}^*)^n$.

**Proposition 7.1.** The homomorphisms $g^{rat}: H_{t,c}^{rat} \to D_i^{rat} # S_n$, $g^{trig}: H^{trig} \to D_i^{trig} # S_n$ defined by the formulas

$$g^{rat}(y_i) = p_i + c\sum_{j \neq i} \frac{1}{x_i - x_j}(s_{ij} - 1),$$

$$g^{rat}(x_i) = x_i, \quad g^{rat}(w) = w,$$

$$g^{trig}(y_i) = p_i + c\sum_{j<i} \frac{X_i}{X_i - X_j}(s_{ij} - 1) + c\sum_{j>i} \frac{X_j}{X_i - X_j}(s_{ij} - 1),$$

$$g^{trig}(X_i) = X_i, \quad g^{trig}(w) = w,$$

$(i = 1, \ldots, n)$ is injective.

This proposition allows to prove the PBW type result for these algebras.

7.3. **Calogero-Moser spaces.** In this subsection we give a definition of the Calogero-Moser space in the rational and trigonometric cases.

Let $CM_i^{rat}$ be the subset of $\mathfrak{gl}(n, \mathbb{C}) \times \mathfrak{gl}(n, \mathbb{C})$ consisting of the elements $(x, y)$ satisfying the equation

$$rk([x, y] + Id) = 1.$$ 

By $CM_i^{trig} \subset GL(n, \mathbb{C}) \times \mathfrak{gl}(n, \mathbb{C})$ we denote the subset of pairs $(X, y)$ satisfying:

$$rk(X^{-1}yX - y + Id) = 1.$$ 

The group $GL(n, \mathbb{C})$ acts on the spaces $CM_i^{rat}$ and $CM_i^{trig}$ by conjugation. This action is free.

**Definition.** The quotient of $CM_i^{rat}$ ($CM_i^{trig}$) by the action of $GL(n, \mathbb{C})$ is called the rational (trigonometric) Calogero-Moser space. We use the notation $CM_i^{rat}$ (respectively $CM_i^{trig}$) for this space.

**Proposition 7.2.** The rational (trigonometric) Calogero-Moser space $CM_i^{rat}$ ($CM_i^{trig}$) is an irreducible smooth variety of dimension $2n$. 

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For the rational Calogero-Moser space this statement is proved in section 1 of [5]. The proof in the trigonometric case almost identically repeats the proof in the rational case.

The Calogero-Moser spaces $CM_{rat}$ and $CM_{trig}$ are the configuration spaces for the rational and trigonometric integrable Calogero-Moser systems. The Poisson structures corresponding to these systems are the results of the Hamiltonian reduction of the natural Poisson structures on the spaces $\mathfrak{gl}(n, \mathbb{C}) \oplus \mathfrak{gl}^*(n, \mathbb{C})$ and $T^*GL(n, \mathbb{C})$ (see [13]).

7.4. The main result for the rational and trigonometric double-affine Hecke algebras. As we mentioned in the first subsection, the algebras $H_{0,c}^{rat}$, $H_{0,c}^{trig}$ are in some sense quasiclassical limits of the double affine Hecke algebra $H_{1,\tau}$. Naturally, the theorems from the previous section have their rational and trigonometric analogs:

**Theorem 7.1.** Let $H$ be one of three described algebras: $H_{1,\tau}$, $H_{0,c}^{trig}$, $H_{0,c}^{rat}$, $CM$ is the corresponding Calogero-Moser space, and $e$ is the symmetrizer (in the finite Hecke algebra if $H = H_{1,q}$ and in the symmetric group otherwise). Then the following is true:

1. The map $h: z \mapsto ze$ is an isomorphism between $Z(H)$ and $eHe$.
2. Spec($Z(H)$) is an irreducible smooth variety naturally isomorphic to $CM$.
3. The Poisson structure on $CM$ which comes from the noncommutative deformation $eH_{q,\tau}c$ ($eH_{t,c}^{trig}$, $eH_{t,c}^{rat}$ respectively) of $eHe$ coincides (up to a constant) with the (Quasi) Poisson structure on $CM$ coming from the (Quasi) Hamiltonian reduction.
4. The left $eHe$-module $He$ is projective and $H = \text{End}_{eHe}(He)$.

In particular the algebras $H_{0,c}^{rat}$ and $H_{0,c}^{trig}$ are Azumaya algebras and for these algebras the statement of Corollary [13,2] holds with $A_n^\tau$ replaced by $S_n$.

The proof of the theorem in the case $H = H_{0,c}^{rat}$ is completely parallel to the case $H = H_{1,\tau}$.

In the trigonometric case the only difficulty is that the group $GL(2, \mathbb{Z})$ does not act on $H_{0,c}^{trig}$ and we do not have any analog of the Fourier-Cherednik transform. But instead of the Fourier-Cherednik transform one can use the faithful representation $\bar{g}$ of $H_{t,c}^{trig}$. The representation $\bar{g}$ is the "bispectral dual" to $g^{trig}$; that is, the role of $X_i$, $1 \leq i \leq n$ is played by $y_i$, $1 \leq i \leq n$.

Let us describe the representation $\bar{g}$. The homomorphism $\bar{g}: H_{t,c}^{trig} \to \mathbb{C}[P^{\pm 1}, y]^\delta(y)\delta^\# S_n$ is defined by the formulas

$$s_{i,i+1} \mapsto \bar{T}_i = s_{i,i+1} + \frac{c}{y_i - y_{i+1}}(s_{i,i+1} - 1), \quad 1 \leq i \leq n - 1,$$

$$y_i \mapsto y_i, \quad 1 \leq i \leq n,$$

$$X_i \mapsto \bar{T}_1 \ldots \bar{T}_{n-i}P_i\bar{T}_{n-i+1} \ldots \bar{T}_{n-1}, \quad 1 \leq i \leq n,$$
where \( w \in S_n, w(1) = n, w(i) = i - 1, i = 2, \ldots, n. \)

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