A SHORT NOTE ON SIGN CHANGES OF FOURIER COEFFICIENTS OF CUSP FORM

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ABSTRACT. In this note we investigate the sign changes for the sequence \( \{a(n^j)\} \) for any \( j \in \mathbb{N} \) of the Fourier coefficients of cusp form for the full modular group \( SL_2(\mathbb{Z}) \).

1. INTRODUCTION

Sign changes of Fourier coefficients of cusp forms in one or in several variables have been studied in various aspects by many authors. It is known that, if the Fourier coefficients of a cusp form are real then they change signs infinitely often \([2]\). Further, many quantitative results for the number of sign changes for the sequence of the Fourier coefficients have been established. The sign changes of the subsequence of the Fourier coefficients at prime numbers was first studied by M. Ram Murty \([11]\). Later, Meher et. al. in \([9]\) studied the problem for the subsequence \( \{a(n^j)\}_{n \geq 1} (j = 2, 3, 4) \). W.Kohnen and Y.Martin in 2014 \([4]\) proved that the subsequence \( \{a(p^n)\}_{n \geq 0} \) has infinitely many sign changes for almost all primes \( p \) and \( j \in \mathbb{N} \).

Here we investigate the sign changes of the subsequence \( \{a(n^j)\}_{n \geq 1} \) for \( j = 5, 6, 7, 8 \). We also genaralize the result of \([9]\) by showing that the subsequence \( \{a(n^j)\}_{n \geq 1} \) has infinitely many sign changes for all \( j \) under certain conditions.

1.1. Notations. Let \( \Gamma = SL_2(\mathbb{Z}) \) and \( S_k(\Gamma) \) denotes the space of cusp forms of weight \( k \) on \( \Gamma \) with \( k \geq 4 \) an even integer. Let \( f \in S_k(\Gamma) \) be a Hecke eigenform with standard Fourier expansion \( f(z) = \sum_{n \geq 1} a(n)q^n \).

We denote the normalized Fourier coefficients by \( \lambda(n) \) and

\[
\lambda(n) = \frac{a(n)}{n^{(k-1)/2}}.
\]

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The \( j \) th symmetric power L-function attached to \( f \in S_k(\Gamma) \) is,

\[
L(\text{sym}^j f, s) := \prod_p \prod_{m=0}^j (1 - \alpha(p)^j - \beta(p)^m p^{-s})^{-1} \quad (1.1)
\]

where \( \alpha(p) + \beta(p) = \lambda(p) \) and \( \alpha(p)\beta(p) = 1 \). We assume \( L(\text{sym}^j f, s) \) is automorphic cuspidal in the hypothesis of our second result. We refer section 2 of [6] for detailed discussion on automorphic cuspidality of such \( l \) functions.

2. Results and the proofs

We prove the following two results.

**Theorem 2.1.** Let \( f \in S_k(\Gamma) \) be a nonzero Hecke eigenform with normalized Fourier coefficients \( \lambda(n) \in \mathbb{R} \). Then \( \{\lambda(n^j)\}_{n \geq 1} \) has infinitely many sign changes for \( j \in \{5, 6, 7, 8\} \).

**Theorem 2.2.** Let \( f \in S_k(\Gamma) \) be a nonzero Hecke eigenform with normalized Fourier coefficients \( \lambda(n) \in \mathbb{R} \). If \( L(\text{sym}^j f, s) \) is automorphic cuspidal then for any \( \{\lambda(n^j)\}_{n \geq 1} \) has infinitely many sign changes for all \( j \in \mathbb{N} \).

2.1. Proof of Theorem 2.1

The following two lemmas will be required to prove the results.

**Lemma 2.1** (G.Lu ([7], Theorem 1.2)). Let \( f \in S_k(\Gamma) \) be nonzero Hecke eigenform with normalized Fourier coefficients \( \lambda(n) \in \mathbb{R} \). Then for any \( j \in \mathbb{N} \) there exist a suitable constant \( c_1 \) depending on \( f \) and \( j \) such that,

\[
\sum_{n \leq x} \lambda^2(n^j)d(n - 1) = c_1 x \log x (1 + o(1)) \quad (2.1)
\]

where \( d(n) \) is the divisor function.

**Lemma 2.2** (G.Lu and H.Tang ([8], Theorem 1.1)). Let \( f \in S_k(\Gamma) \) be nonzero Hecke eigenform with normalized Fourier coefficients \( \lambda(n) \in \mathbb{R} \). Then for \( j \in \{5, 6, 7, 8\} \) there exist a suitable constant \( c_2 > 0 \) such that,

\[
\sum_{n \leq x} \lambda(n^j) \ll x \exp(-c_2 \sqrt{\log x}). \quad (2.2)
\]

**Proof.** (Proof of Theorem 2.1): If possible, let us assume that the sequence \( \{\lambda(n^j)\}_{n \geq 1} \) for \( j \in \{5, 6, 7, 8\} \) are of constant sign say positive for all \( n \in (x, 2x] \) in some interval.
Now from lemma 2.1, we get,
\[
\sum_{x<n \leq 2x} \lambda^2(n^j)d(n-1) = c_1 2x \log 2x - c_1 x \log x + o(x \log x)
\]
\[
= c_1 x \log x \left(\frac{2 \log 2x}{\log x} - 1\right) + o(x \log x)
\]
\[
\gg x \log x. \quad (2.3)
\]
On the other hand, by using lemma 2.2 and Delign’s bound (cf. [1]) on \( \lambda(n) \) with sufficiently \( \epsilon > 0 \), we get,
\[
\sum_{x<n \leq 2x} \lambda^2(n^j)d(n-1) \ll \sum_{x<n \leq 2x} \lambda(n^j)d(n-1)
\]
\[
\ll x^{2\epsilon} \sum_{x<n \leq 2x} \lambda(n^j)
\]
\[
\ll x^{2\epsilon} \left\{ 2xe^{-c_2 \sqrt{\log 2x}} + xe^{-c_2 \sqrt{\log x}} \right\}
\]
\[
\ll x^{2\epsilon} x e^{-c_2 \sqrt{\log x}}. \quad (2.4)
\]
Now, by comparing the bounds of \( \sum_{x<n \leq 2x} \lambda^2(n^j)d(n-1) \) in (2.3) and (2.4), we arrive at a contradiction. Therefore, at least one \( \lambda(n^j) \) for \( n \in (x, 2x] \) is negative. Hence the sequence \( \{\lambda(n^j)\}_{n \geq 1} \) has infinitely many sign changes for \( j \in \{5, 6, 7, 8\} \).

\[\square\]

2.2. Proof of Theorem 2.2
Now we complete the proof of the second theorem. We need to use lemma 2.1 and the following lemma to prove this result.

Lemma 2.3 (Lau and Lu([6], Theorem 1)). Let \( f \in S_k(\Gamma) \) be nonzero Hecke eigenform with normalized Fourier coefficients \( \lambda(n) \in \mathbb{R} \). Suppose \( L(\text{sym}^j f, s) \) is automorphic cuspidal for \( j \in \mathbb{N} \). Then for any \( j \geq 3 \) and \( j \in \mathbb{N} \),
\[
\sum_{n \leq x} \lambda(n^j) \ll x^{\frac{1}{2} + \epsilon}. \quad (2.5)
\]

Proof. (Proof of Theorem 2.2): If possible, let us assume that the sequence \( \{\lambda(n^j)\}_{n \geq 1} \) for \( j \geq 3 \) and \( j \in \mathbb{N} \) are of constant sign say positive for all \( n \in (x, 2x] \).
Again as in the previous case by using lemma 2.3 we get,
\[ \sum_{x<n\leq 2x} \lambda^2(n^j)d(n-1) = \sum_{x<n\leq 2x} \lambda(n^j)\lambda(n^j)d(n-1) \]
\[ \ll x^{2e} \sum_{x<n\leq 2x} \lambda(n^j) \]
\[ \ll x^{2e} \left( (2x)^{\frac{j}{2}} + x^{\frac{j}{4}} \right) \]
\[ \ll x^{2e} x^{\frac{j}{2}}. \] (2.6)
Equation (2.3) gives the lower bound of \( \sum_{x<n\leq 2x} \lambda^2(n^j)d(n-1) \) for any \( j \in \mathbb{N} \). Now, by comparing the bounds of \( \sum_{x<n\leq 2x} \lambda^2(n^j)d(n-1) \) in equation (2.3) and (2.6), we arrive at a contradiction. Hence, we can conclude that our theorem is true for \( j \geq 3 \). The theorem holds true in the cases \( j = 1, 2 \) due to the works in [4] and [9]. □

Remark 1. The condition of automorphic cuspidality of \( L(\text{sym}^j f, s) \) is necessary to conclude the result of infinitely many sign changes of the subsequence \( \{\lambda(n^j)\}_{n \geq 1} \) in theorem 2.2. The result holds unconditionally for \( j \in \{1, 2, \cdots, 8\} \) by theorem 2.1 and due to the works in [4] and [9]. Hence, it is still open to prove the result without assuming automorphic cuspidality of \( L(\text{sym}^j f, s) \) for \( j > 8 \).

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References

[1] P. Deligne, *La Conjecture de Weil, I*, Inst. Hautes Études Sci. Publ. Math., 43 (1974), 273-307. MR0340258(49:5013)
[2] M. Knopp, W. Kohnen and W. Pribikin, *On the Signs Of Fourier coefficients Of cusp forms*, The Ramanujan Journal, 7 (2003), 269–277.
[3] N. Koblitz, *Introduction to elliptic curves and modular forms*
[4] W. Kohnen and Y. Martin, *Sign Changes of Fourier Coefficients of Cusp forms supported on prime power indices*, International Journal Of Number Theory, 10(8) (2014), 1921–1927.
[5] H. Lao and A. Sankaranarayanan, *The Distribution of Fourier Coefficients of Cusp Forms Over Sparse Sequences*, Acta Arithmetica, 163.2 (2014), 101-110
[6] Y. Lau and G. Lu, *Sums of Fourier Coefficients of Cusp Forms*, Q J Math, 62(3) (2011), 687-716. MR2825478
[7] G. Lu, *Shifted Convolution Sums of Fourier Coefficients With Divisor Functions*, Acta Math. Hungarica, 146(1) (2015), 86-97.
[8] G. Lu and H. Tang, *Sums of Fourier Coefficients Related to Hecke Eigencuspforms*, Ramanujan J, 37: 309 (2015). doi:10.1007/s11139-014-9581-8

[9] J. Meher, K. Shankhadhar and G. K. Viswanadham, *A Short Note On Sign Changes*, Proc. Indian Acad. Sci. Math. Sci., 123 (2013), 315–320.

[10] T. Miyake, *Modular Forms*, Springer, Berlin-Heidelberg-New York, 1989. MR1021004(90M:11062)

[11] M. Ram Murty, *Oscillations of the Fourier coefficients of Modular forms*, Math. Ann., 262 (1983), 431–446.

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