Self-Consistent Approach to Quenched Impurity Effects on Quantum Phase Transitions

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Abstract. Double expansion renormalization group treatments are still employed in literature to describe the effects of symmetry-conserving quenched disorder on quantum phase transitions as an alternative to more or less phenomenological models. If one assumes $\varepsilon = 4 - d$ and the dimensionality $\varepsilon_\tau$ of the imaginary-time space involved in a generalized quantum action as simultaneously small parameters, a stable random fixed point is found which governs a new disorder-induced quantum criticality usually considered meaningful also after extrapolation to the physical time-like dimension $\varepsilon_\tau = 1$. In contrast, by considering $\varepsilon_\tau = 1$ at the beginning, a runaway takes place in the parameter space for dimensionalities $d < 4$. To give some insight into these contrasting results, we employ here a simple self-consistent approach for case of short-range correlated quenched disorder which allows us to analyze analytically what happens by continuous variation of the fictitious time-like dimension $\varepsilon_\tau$. We find that a $\varepsilon_\tau^*(n) < 1$ exists, depending on the symmetry index $n$, above which the disorder inhibits the occurrence of a conventional quantum phase transition. This suggests that the usual procedure to extrapolate the small $\varepsilon_\tau$-predictions to the value of interest $\varepsilon_\tau = 1$ may have no real physical meaning.

The physical properties of many quantum systems of current interest, as for instance anomalous non-Fermi liquid behaviors and unconventional forms of superconductivity and magnetism, seem to be controlled, over a wide region of the phase diagram, by the presence of a quantum critical point (QCP) [1, 2]. Quantum phase transitions (QPT) can be strongly affected by the presence of disorder [3-7], leading to new behaviors, as for example quantum Griffiths phenomena [3-6]. Even though a lot of effort has been devoted to understand the effects of quenched disorder on quantum criticality, some crucial aspects still remain to be clarified. Conventional renormalization group (RG) treatments are unable to give reliable information about the proper behavior at temperature $T = 0$ of quantum models for $d < 4$ and the effective physical meaning of the related fixed point (FP) instability remains an open question [1, 7]. In order to give insight into this puzzling problem, perturbative “double expansion” RG approaches [8, 9] are generally employed where the dimensionality $\varepsilon_\tau$ of a fictitious imaginary-time space involved in functional representations is assumed as an additional expansion parameter. In this picture, the ($T = 0$)-physical predictions should be obtained by extrapolation of the small-$\varepsilon_\tau$ data to the case of interest $\varepsilon_\tau = 1$, implying a sharp second order QPT with new critical exponents and oscillatory corrections to the power law scaling [8, 9]. Although the “double expansion” approach is used in several situations, there are indications that the correctness of the extrapolation philosophy to the physical value $\varepsilon_\tau = 1$ is at least questionable. Indeed, some studies showed that the $\varepsilon_\tau$-power series does not converge [10, 11] and a value of $\varepsilon_\tau$ exists...
where the RG theory becomes singular and the random FP unphysical [12]. Besides, there are alternative approaches [13-15] which yield different findings assuming \( \varepsilon_r = 1 \) at the beginning.

In this paper we apply a self-consistent approach to a quantum action assuming \( 0 \leq \varepsilon_r \leq 1 \) with the aim to give some insight into the origin of the FP instability found for \( \varepsilon_r = 1 \) [7] and into the physical meaning of the extrapolation predictions from the double expansion approaches.

We consider a quantum model in presence of quenched disorder described by the action

\[
H\{\psi, \varphi\} = \int_0^{1/T} d^3 r \int d^d x \left\{ (\nabla_\chi \psi(\mathbf{x}, \tau))^2 + (\nabla_\tau \psi(\mathbf{x}, \tau))^2 + \right. \\
\left. [r_0 + \varphi(\mathbf{x})] \psi^2(\mathbf{x}, \tau) + \frac{u_0}{8} \psi^4(\mathbf{x}, \tau) \right\},
\]

(1)

where \( \psi \equiv \{ \psi^j; j = 1, ..., n \} \) is the local \( n \)-vector order parameter field, \( \varphi(\mathbf{x}) \) is the random potential in the \( d \)-dimensional space with Gaussian quenched averages \( \varphi(\mathbf{x}) = 0 \) and \( \langle \varphi(\mathbf{x}) \varphi(\mathbf{x}') \rangle = \Delta_0 \delta(\mathbf{x} - \mathbf{x}') \), \( \tau \) is an \( \varepsilon_r \)-dimensional imaginary-time vector and the parameters \( r_0 \) and \( u_0 \) are related to the microscopic system under study. Here we consider systems with transverse-Ising-like intrinsic dynamics \( \langle (\nabla_\tau \psi)^2 \rangle \rightarrow |\partial_\tau \psi|^2 \) for \( \varepsilon_r = 1 \). Other quantum systems, reviewed in Ref. [7], can be discussed similarly.

Our aim is to calculate the physical susceptibility \( \chi \) of the random model (1) whose divergence constitutes a reliable signal of criticality. We work in the Fourier \((\mathbf{k}, \omega_n)\)-space where \( \mathbf{k} \) denotes the usual \( d \)-dimensional wave vector and \( \omega_n = 2\pi nT \), with \( n \equiv \{ n_i = 0, \pm 1, \pm 2, ..., i = 1, ..., \varepsilon_r \} \), are the \( \varepsilon_r \)-dimensional Matsubara bosonic frequencies. Thus the susceptibility is defined as

\[
\chi = \lim_{\mathbf{k}, \omega_n \rightarrow 0} G_R(\mathbf{k}, \omega_n) ,
\]

(2)

where the retarded response function \( G_R(\mathbf{k}, \omega_n) \) is obtained as the analytical continuation with \( i\omega_n \rightarrow \omega \) to the real frequency axis of the quenched averaged temperature propagator \( G(q) = \langle |\psi(q)|^2 \rangle_{av} \), with \( q \equiv \mathbf{k}, \omega_n \). Here \( \langle \cdots \rangle \) denotes the ensemble average weighted by the action (1) for a fixed configuration of disorder.

We determine the propagator \( G(q) \) using the Dyson equation

\[
G^{-1}(q) = r + q^2 + \Sigma(q, r) ,
\]

(3)

where \( r = \chi^{-1} \) and \( \Sigma(q, r) \) is the self-energy part [16]. This is achieved using the Hartree-Fock approximation (which is assumed to become exact in the large \( n \) limit) with [17] or without [16] the replica trick [18]. Then, the limit process (2) yields the self-consistent equation

\[
r = r_0 + \frac{n + 2}{4} u_0 \int \frac{d^dk}{(2\pi)^d} \left[ T^{\varepsilon_r} \sum_{\omega_n} \frac{1}{r + k^2 + \omega_n^2} \right] + \
\left. - \Delta_0 \int \frac{d^dk}{(2\pi)^d} \frac{1}{r + k^2} \right. .
\]

(4)

where a multidimensional frequency sum is involved.

Here we consider the \((T = 0)\)-case which is of main interest for us. Then taking into account the transformation \( T^{\varepsilon_r} \sum_{\omega_n} \rightarrow \sum_{T = 0} \int \frac{d^d \omega}{(2\pi)^d} \), eq. (4) becomes:

\[
r = r_0 + \frac{n + 2}{4} u_0 F_{d,\varepsilon_r}(r) - \Delta_0 G_{d}(r) ,
\]

(5)

where

\[
F_{d,\varepsilon_r}(r) = \frac{\Gamma(1 - \frac{\varepsilon_r}{2})}{(4\pi)^{\frac{d}{2}}} \int \frac{d^dk}{(2\pi)^d} \frac{1}{(r + k^2)^{1 - \frac{\varepsilon_r}{2}}} , \quad \text{and} \quad G_{d}(r) = \int \frac{d^dk}{(2\pi)^d} \frac{1}{r + k^2} .
\]

(6)
In eqs. (6) a cutoff $\Lambda = 1$ over wave vectors is assumed.

The divergence of susceptibility ($r = 0$) select in eq. (5) the QCP value

$$ r_{0c} = - \frac{n + 2}{4} u_0 F_{d,\varepsilon,0}(0) + \Delta_0 G_d(0) $$

of the parameter $r_0$ which defines, for $d > 2$, a line in the ($r_0, u_0, \Delta_0$)-space along which $\chi = \infty$.

Since we want to look for solutions of eq. (5) as $\chi \to +\infty$ for $r_0 \to r_{0c}^+$, it is convenient to rewrite eq. (5) as

$$ r = g + \frac{n + 2}{4} u_0 F_{d,\varepsilon,0}(r) - \Delta_0 G_d(r), $$

where $g = r_0 - r_{0c}$, $F_{d,\varepsilon}(r) = F_{d,\varepsilon,0}(r) - F_{d,\varepsilon,0}(0)$ and $G_d(r) = G_d(r) - G_d(0)$.

We now look for physical solutions of diverging susceptibility $r \to 0^+$ as $g \to 0^+$, studying the self-consistent eq. (8) for different ranges of dimensionality.

For $\varepsilon_r = 0$, we recover the classical weak disorder result $\chi \propto g^{-\gamma}$ with $\gamma = 2/(d - 2)$ for $2 < d < 4$ and $\gamma = 1$ for $d \geq 4$ with logarithmic corrections at $d = 4$.

For finite $\varepsilon_r$ (including $\varepsilon_r = 1$), we have that for $2 < d < 4$ the $\Delta_0$-term in eq. (5) dominates as $r \to 0^+$ and we get the solution $r \propto (-g)^{2/(d-2)}$, i.e. a divergent complex susceptibility as $r_0 \to r_{0c}^+$, which signals a thermodynamic instability.

Only for $d \geq 4$ a real susceptibility is found to occur with $r \propto g$ for $d > 4$ (and $r \propto g(\ln g^{-1})$ for $d = 4$.)

Notice that, if we assume $g \to 0^-$, a divergent real susceptibility appears also for $2 < d < 4$ but its physical meaning is unclear since it happens in a region of the parameter space where the action (1) is not adequate to describe a possible ordered phase for $r_0 < r_{0c}$.

The previous results allow us to argue the existence of a borderline imaginary-time dimensionality $\varepsilon_r^*$ such that for $0 \leq \varepsilon_r < \varepsilon_r^*$ a second order QPT occurs, with a divergent real susceptibility as $r_0 \to r_{0c}^+$, and for $\varepsilon_r^* < \varepsilon_r \leq 1$ a thermodynamic instability takes place along the line (7) (in this case eq. (7) defines a line of thermodynamic instability).

Indeed eq. (8) shows that a crossover between the two previous regimes may occur when the $u_0$ and $\Delta_0$ terms are comparable and one has the mean field solution $r \simeq g > 0$. Then, the borderline value $\varepsilon_r^*$ of $\varepsilon_r$ should be determined by the equation:

$$\frac{n + 2}{4} u_0 F_{d,\varepsilon,0}(g) = \Delta_0 G_d(g).$$

The existence of $\varepsilon_r^*$ for $2 < d < 4$ appears evident from Figs. 1. In this figures, the two regions of thermodynamic stability and instability of a second order QPT are plotted in the $(n, \varepsilon_r)$-plane for some values of the parameters involved in eq. (9). Here, the separation line determines $\varepsilon_r^*$ as a function of $n$ for selected values of $r = g$, $\Delta_0/u_0$ and $d$. It is worth noting that the qualitative features in these figures do not change for different values of $2 < d < 4$ and small $r$ varying the ratio $\Delta_0/u_0$. In particular, the numerical results do not change significantly varying $d$ and one has $\varepsilon_r^* \ll 1$ when $r = g \ll 1$, i.e. very close to the critical line (7).

An analytical solution of eq. (9) for $\varepsilon_r^* \ll 1$ (when $0 < g \ll 1$) can be easily obtained. We get, for $2 < d < 4$ (essentially independent of $d$):

$$\varepsilon_r^* \approx 2 \frac{\ln \left( \frac{n + 2}{4} \frac{u_0}{\Delta_0} \right)}{\ln(g^{-1})},$$

which in the large-$n$ limit with $u_0 = \pi_0/n$, reduces to

$$\varepsilon_r^* = 2 \frac{\ln \left( \frac{\pi_0}{\Delta_0} \right)}{\ln(g^{-1})}. $$
Figure 1. Diagram in the space $n - \varepsilon_{\tau}$ where are shown regions of stability and instability of a second order QPT for $d = 3$, $r = 10^{-4}$ and two different values of the ratio $\Delta_0/u_0$.

Of course, a positive value of $\varepsilon_{\tau}^*$ requires that $\Delta_0/u_0 < (n+2)/4$ (or $\Delta_0/\mu_0 < 1/4$ as $n \to \infty$). Besides, when $\Delta_0/u_0 = (n+2)/4$ (or $\Delta_0/\mu_0 = 1/4$ as $n \to \infty$), we have $\varepsilon_{\tau}^* = 0$, consistently with the numerical results.

In summary, our simple self-consistent approach seems to suggest the existence of a borderline value $\varepsilon_{\tau}^*$ of the imaginary-time dimensionality $\varepsilon_{\tau}$ and hence of a region in the parameter space where, for small $\varepsilon_{\tau} < \varepsilon_{\tau}^*$, the random model (1) exhibits a second order QPT. Besides, for $\varepsilon_{\tau}^* < \varepsilon_{\tau} \leq 1$ a thermodynamic instability occurs which could be a signal of a smeared transition, a disorder-induced first order QPT, a glassy-like transition, or absence of transition at all. These results appear to confirm the idea that, although the double expansion RG predictions for $\varepsilon_{\tau} \ll 1$ may have a mathematical sense, the extrapolation of data to $\varepsilon_{\tau} = 1$ may be misleading or without physical meaning.

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