SIMPLICITY OF TANGENT BUNDLES ON THE MODULI SPACES OF SYMPLECTIC AND ORTHOGONAL BUNDLES OVER A CURVE

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ABSTRACT. The variety of minimal rational tangents associated to Hecke curves was used by J.-M. Hwang [8] to prove the simplicity of the tangent bundle on the moduli of vector bundles over a curve. In this paper, we use the tangent maps of the symplectic and orthogonal Hecke curves to prove an analogous result for symplectic and orthogonal bundles. In particular, we show the nondegeneracy of the associated variety of minimal rational tangents, which implies the simplicity of the tangent bundle on the moduli spaces of symplectic and orthogonal bundles over a curve. We also show that for large enough genus, the tangent map is an embedding for a general symplectic or orthogonal bundle.

RÉSUMÉ. La variétés des tangentes des courbes minimales rationnelles associées aux courbes de Hecke, a été utilisée par J.-M. Hwang [8] pour prouver la simplicité du fibré tangent à l’espace de modules des fibrés vectoriels sur une courbe. Nous utilisons les applications tangentes des courbes de Hecke symplectiques et orthogonales pour démontrer un résultat analogue pour les fibrés symplectiques et orthogonaux. En particulier, nous prouvons que la variété des tangentes des courbes rationnelles minimaux associée est nondégénérée ; ce qui implique la simplicité des fibrés tangents des espaces de modules des fibrés symplectiques et orthogonaux sur une courbe. Nous montrons d’ailleurs, pour genre suffisamment grand, que l’application tangente est un plongement pour un fibré symplectique ou orthogonal générique.

1. Introduction

Let $C$ be a smooth projective curve of genus $g \geq 2$ over the complex numbers. Let $\mathcal{M} := SU_C(n, d)$ be the moduli space of semistable vector bundles over $C$ of rank $n$ with fixed determinant of degree $d$. Note that $\mathcal{M}$ is a Fano variety of Picard number 1 and moreover smooth if $n$ and $d$ are coprime.

It was shown in [8, Corollary 1] that for $g \geq 4$, the tangent bundle of the smooth part $\mathcal{M}^0 \subset \mathcal{M}$ is simple. The strategy was to exploit certain minimal rational curves called Hecke curves and the associated variety of minimal rational tangents. More precisely, it is shown that the variety of minimal rational tangents $C_W$ at a generic point $W \in \mathcal{M}$ is non-degenerate in $\mathbb{P}(T_W \mathcal{M})$ and this implies the simplicity of the tangent bundle (cf. Proposition 2.6).

The goal of this paper is to prove the analogous result for the moduli spaces $\mathcal{M}SU_C(n, L)$ of symplectic bundles and $\mathcal{M}O_C(n, L)$ of orthogonal bundles. The symplectic and orthogonal versions of Hecke curves were constructed in [3]. In the same paper, these curves were shown to be the minimal rational curves in the ambient varieties. Based on this, we establish the following results in this paper:

- The smoothness of the symplectic and orthogonal Hecke curves (§3)
- The nondegeneracy of the tangent map on the variety of minimal rational tangents of these Hecke curves (§4)

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• The very-ampleness of the associated complete linear system (§5)

In particular, as a corollary of the nondegeneracy in §4, we show that the tangent bundles of \( \mathcal{M}_{SC}(n, L) \) and \( \mathcal{MO}_C(n, L) \) are simple, under a certain genus bound.

We would like to add a word of warning for the arguments that will follow. Inside the moduli space \( SU_C(n, d) \) of vector bundles, the locus of symplectic/orthogonal bundles form a closed subvariety, and a symplectic/orthogonal Hecke curve can be thought of as either a special kind of Hecke curve on \( SU_C(n, d) \) or its variation. So one might expect that the results for Hecke curves in [8], [9], or [13] directly imply the same results for the symplectic or orthogonal setting.

But in most discussions of a Hecke curve on \( SU_C(n, d) \), one assumes that it passes through a generic point, such as a \((1, 1)\)-stable bundle [12, Definition 5.1]. And it is unclear if a generic point of \( \mathcal{M}_{SC}(n, L) \) and/or \( \mathcal{MO}_C(n, L) \) is \((1, 1)\)-stable as a vector bundle. By simple dimension comparison, it is still possible that the subvarieties \( \mathcal{M}_{SC}(n, L) \) and/or \( \mathcal{MO}_C(n, L) \) are entirely contained in the non-\((1, 1)\)-stable locus. For this reason, we cannot tell from the outset if the symplectic and orthogonal Hecke curves share the same properties as the Hecke curves passing through \((1, 1)\)-stable locus. This is why we later devise arguments based on \( \delta \)-stability (cf. [3, §4.1]) on \( \mathcal{M}_{SC}(n, L) \) and \( \mathcal{MO}_C(n, L) \).

2. Preliminary results

In this section, we gather the notations and preliminary results relevant to our discussion. Let \( C \) be a smooth projective curve of genus \( g \geq 2 \).

Notation 2.1. Given a subspace \( \Lambda \subset V^* \) of a dual vector space, \( \Lambda^\perp \subset V \) denotes the kernel:

\[
\Lambda^\perp = \{ v \in V : \lambda(v) = 0 \text{ for all } \lambda \in \Lambda \}.
\]

Also for a subspace \( U \subset V \), \( U^\perp \subset V^* \) denotes the annihilator:

\[
U^\perp = \{ \phi \in V^* : \phi(u) = 0 \text{ for all } u \in U \}.
\]

When \( V \) is equipped with a bilinear form \( \omega : V \otimes V \rightarrow \mathbb{C} \), we define

\[
\ker(\omega) := \{ v_0 \in V : \omega(v_0, v) = 0 \text{ for all } v \in V \}
\]

2.1. Hecke modification. Let \( W \) be a vector bundle over \( C \). Choose a subspace \( \Lambda \subset W^*_x \) for some \( x \in C \). The Hecke modification \( W^\Lambda \) of \( W \) along \( \Lambda \) is given by the kernel of the composition map \( W \rightarrow W_x \rightarrow W_x/\Lambda^\perp \). There is an exact sequence of sheaves:

\[
0 \rightarrow W^\Lambda \xrightarrow{\phi} W \rightarrow (W_x/\Lambda^\perp) \otimes \mathcal{O}_x \rightarrow 0
\]

whose restriction to the fiber at \( x \) is given by

\[
0 \rightarrow \operatorname{Ker}(\phi_x) \rightarrow W^\Lambda \rightarrow (W_x/\Lambda^\perp) \otimes \mathcal{O}_x \rightarrow 0.
\]

Then the locally free sheaf \( W^\Lambda \) corresponds to a vector bundle with \( \det(W^\Lambda) = \det(W) \otimes \mathcal{O}_C(-kx) \), where \( k \) is the dimension of \( \Lambda \) in \( W^*_x \).

2.2. Hecke curves on \( SU_{C}(n, d) \). The main reference for this subsection is [8].

Let \( W \) be a vector bundle over \( C \). For a subspace \( \theta \subset W^*_x \) of dimension one, by abuse of notation, let \( W^\theta \) denote the Hecke modification of \( W \) along \( \theta \). Then we have

\[
0 \rightarrow W^\theta \rightarrow W \rightarrow (W_x/\theta^\perp) \otimes \mathcal{O}_x \rightarrow 0.
\]
For a subspace $\ell \subset W_x^\Theta$ of dimension one, the Hecke modification $V^\ell$ of $V := (W^\theta)^*$ along $\ell$ fits into the exact sequence
\begin{equation}
0 \to V^\ell \xrightarrow{\beta} V \to (V_x/\ell^\perp) \otimes \mathcal{O}_x \to 0.
\end{equation}
In particular for $\ell_0 := \text{Ker}(W_x^\Theta \to W_x)$, we have $(V^{\ell_0})^* \simeq W$.

For any two-dimensional subspace $U$ with $\ell_0 \subset U \subset W_x^\Theta$, the subspace $U^\perp$ has codimension two in $V_x$, and is contained in $\ell_0^\perp$. Hence the family
\[\{(V^{\ell})^* : [\ell] \in \mathbb{P}(U)\}\]
parameterized by $\mathbb{P}(U) = \mathbb{P}(V_x/\ell_0^\perp) \cong \mathbb{P}^1$, is a deformation of $W$. If we choose a generic $W \in SU_C(n, d)$, then this family gives a smooth rational curve through $W$ called a Hecke curve. It was shown in [13] that the Hecke curves have minimal degree among the rational curves passing through a generic $W \in SU_C(n, d)$.

Note that the parameter space of Hecke curves passing through $W \in SU_C(n, d)$ is given by a double fibration
\[\mathbb{P}(T^*_x) \to \mathbb{P}(W^*) \xrightarrow{\pi} C,
\]
where $T_x$ is the vertical tangent bundle of $\pi : \mathbb{P}(W^*) \to C$. In the previous notation, this corresponds to the composition map
\[(U/\ell_0 \subset W_x^\Theta/\ell_0) \mapsto (\theta \subset W_x^\perp) \mapsto x.
\]
In particular for $n = 2$, this double fibration boils down to the ruled surface $\mathbb{P}(W^*)$.

2.3. Kodaira–Spencer map. The main references of this subsection are [11] and [12]. Consider the above family $\{V^{\ell} : [\ell] \in \mathbb{P}(U)\}$ as a deformation of $V^{\ell_0} \cong W^\star$. Since the map $\beta_x : V_x^{\ell_0} \to V_x$ surjects onto $\ell_0^\perp$, we have the induced pull-back map
\[\hat{\beta}_x : \text{Hom}(\ell_0^\perp/U^\perp, V') \to \text{Hom}(V_x^{\ell_0}, V')
\]
for any vector space $V'$.

**Proposition 2.2.** The Kodaira–Spencer map of the family $\{V^{\ell} : [\ell] \in \mathbb{P}(U)\}$ is given by
\[\text{Hom}(\ell_0^\perp/U^\perp, V_x/\ell_0^\perp) \xrightarrow{\hat{\beta}_x} \text{Hom}(V_x^{\ell_0}, V_x/\ell_0^\perp) \xrightarrow{\delta} H^1(C, \text{End}(V^{\ell_0})),
\]
where $\delta$ is induced from the sequence (7).

Later we will also need to consider a slight generalization of the above family, where the Hecke modification is taken for subspaces of dimension two. For a subspace $\Theta \subset W_x^\star$ of dimension 2, the Hecke modification $W_\Theta$ of $W$ along $\Theta$ can be put into the following exact sequence:
\[0 \to W^\Theta \to W \to (W_x/\Theta^\perp) \otimes \mathcal{O}_x \to 0.
\]
Let $\varphi_0$ denote the kernel of $W^\Theta \to W_x$. The following is a straightforward generalization of Proposition 2.2.

**Proposition 2.3.** Let $U$ be a subspace of dimension 4 with $\varphi_0 \subset U \subset W_x^\Theta$; that is, $U^\perp \subset \varphi_0^\perp \subset (W_x^\Theta)^* := V_x$. Then the family $\{V^\varphi : [\varphi] \in Gr(2, V_x/\ell_0^\perp)\}$ is a deformation of $W^\star = V^{\varphi_0}$, and its Kodaira–Spencer map
\[T_{\varphi_0}(Gr(2, V_x/\ell_0^\perp)) = \text{Hom}(\varphi_0^\perp/U^\perp, V_x/\varphi_0^\perp) \to H^1(C, \text{End}(V^{\varphi_0}))
\]
is given by the composition
\[\text{Hom}(\varphi_0^\perp/U^\perp, V_x/\varphi_0^\perp) \xrightarrow{\hat{\beta}_x} \text{Hom}(V_x^{\varphi_0}, V_x/\varphi_0^\perp) \xrightarrow{\delta} H^1(C, \text{End}(V^{\varphi_0})),
\]
where $\hat{\beta}_x$ and $\delta$ are induced from
\[0 \to V^{\varphi_0} \xrightarrow{\beta} V \to (V_x/\varphi_0^\perp) \otimes \mathcal{O}_x \to 0.
\]
2.4. Symplectic Hecke curves on $\mathcal{M}_C(n, L)$. In this subsection, we recall the construction of symplectic Hecke curves, following [3], to which we refer the reader for the details.

For a line bundle $L$ on $C$, an $L$-valued symplectic bundle of rank $n$ is a vector bundle $W$ of (even) rank $n$ equipped with an $L$-valued symplectic form $\omega: W \otimes W \to L$. Let $\mathcal{M}_C(n, L)$ be the moduli space of $L$-valued symplectic bundles of rank $n$. By the morphism forgetting the symplectic forms, the moduli space $\mathcal{M}_C(n, L)$ can be thought of as a subvariety of $SU_C(n, L)$, where $\ell = \deg(L)$. To avoid the coincidence $\mathcal{M}_C(2, L) = SU_C(2, \ell)$, we assume $n \geq 4$ throughout this paper.

The construction of symplectic Hecke curves on $\mathcal{M}_C(n, L)$ closely follows the previous construction of Hecke curves on $SU_C(n, d)$, keeping track of the deformation of symplectic forms. For a subspace $\theta \subset W_x^*$ of dimension one, let $W^\theta$ be the Hecke modification of $W$ along $\theta$, fitting into the sequence

$$0 \to W^\theta \to W \to (W_x/\theta^\perp) \otimes \mathcal{O}_x \to 0.$$ 

Noting that every 1-dimensional subspace $\theta$ is isotropic, we get an induced $L^*(x)$-valued skew-symmetric form on $V := (W^\theta)^*$:

$$\omega^\theta: (W^\theta)^* \to W^\theta \otimes L^*(x).$$

Then $\ker \omega^\theta_x$ has codimension two in $V_x = (W^\theta)_x^*$. For a subspace $\ell \subset W^\theta_x$ of dimension one, we have the sequence

$$0 \to V^\ell \to V \to (V_x/\ell^\perp) \otimes \mathcal{O}_x \to 0.$$

Then the bundle $V^\ell$ has a skew-symmetric form induced from $\omega^\theta$, and it is an $L^*$-valued nondegenerate (symplectic) form if and only if $\ker \omega^\theta_x \subset \ell^\perp$. Now the family

$$\{V^\ell : \ker \omega^\theta_x \subset \ell^\perp \subset V_x\}$$

of $L$-valued symplectic bundles are parameterized by $\mathbb{P}(V_x/\ker \omega^\theta_x) \simeq \mathbb{P}^1$. In particular, if $\ell_0$ is in the kernel of $W^\theta_x \to W_x$, then $V^{\ell_0} \cong W^*$. Under the assumption that $g \geq 3$ and $W \in \mathcal{M}_C(n, L)$ is a generic point, by [3] Lemma 4.5 the dual family

$$\{((V^\ell)^*)^\perp : \ell \in \mathbb{P}(V_x/\ker \omega^\theta_x)\}$$

gives a rational curve on $\mathcal{M}_C(n, L)$ passing through $W$, called a symplectic Hecke curve. Also it was shown in [3] Theorem 5.2 that these curves have minimal degree among the rational curves passing through a generic point $W \in \mathcal{M}_C(n, L)$.

Later we need the following fact.

**Proposition 2.4.** Assume $g \geq 3$. For a generic point $W \in \mathcal{M}_C(n, L)$, every symplectic Hecke curve passing through $W$ is contained in the smooth locus of $\mathcal{M}_C(n, L)$. 

**Proof.** By [3] Lemma 4.5, any symplectic Hecke curve passing through a generic point $W$ stays inside the locus of stable symplectic bundles. To see that it is contained in the smooth locus, it suffices to show that it does not touch the locus of non-regularly stable symplectic bundles which are of the form $W_1 \perp W_2$ for some stable symplectic subbundles $W_1$ and $W_2$.

This can be checked by dimension count: The locus of non-regularly stable bundles is contained in a finite union of the images of $\mathcal{M}_C(n_1, L) \times \mathcal{M}_C(n_2, L)$, where $n_1 + n_2 = n$. Since symplectic Hecke curves passing through a point $W_0$ in this subvariety are parameterized by $\mathbb{P}(W^*_0)$, it suffices to check the inequality:

$$\dim \mathbb{P}(W^*_0) + \dim \mathcal{M}_C(n_1, L) + \dim \mathcal{M}_C(n_2, L) < \dim \mathcal{M}_C(n, L).$$
for any even integers \( n_1, n_2 \) with \( n_1 + n_2 = n \). This boils down to \( n < n_1 n_2 (g - 1) \), which holds for \( g \geq 3 \).

\[ \square \]

2.5. Orthogonal Hecke curves on \( \mathcal{MO}_C(n, L) \). Again, the main reference in this subsection for a construction of orthogonal Hecke curves will be [3].

For a line bundle \( L \) on \( C \), an \( L \)-valued orthogonal bundle of rank \( n \) is a vector bundle \( W \) of rank \( n \) equipped with an \( L \)-valued orthogonal form \( b: W \otimes W \rightarrow L \). Let \( \mathcal{MO}_C(n, L) \) be the moduli space of \( L \)-valued orthogonal bundles of rank \( n \). The moduli space \( \mathcal{MO}_C(2r, L) \) has several irreducible components, due to the invariants \( \det(W) \) and the 2nd Stiefel–Whitney class \( w_2(W) \) (see [2] §2). By the morphism forgetting the orthogonal forms, each irreducible component of the moduli space \( \mathcal{MO}_C(n, L) \) is sent to a subvariety of \( \text{SU}_C(n, \frac{1}{2} n \ell) \), where \( \ell = \deg(L) \).

As in [3], we assume \( n \geq 5 \) throughout this paper. The reason behind this convention is that the moduli space \( \mathcal{MO}_C(n, L) \) has Picard number one for \( n \geq 5 \), while \( \mathcal{MO}_C(4, L) \) has Picard number two. Accordingly, the minimality of the orthogonal Hecke curves was discussed in [3] for \( n \geq 5 \). We remark that there is a standard construction of orthogonal bundles of low rank from vector bundles, described in [5].

The construction of orthogonal Hecke curves on \( \mathcal{MS}_C(n, L) \) is a little bit different from that of Hecke curves on \( \text{SU}_C(n, d) \): the dimension of the involved subspaces are doubled.

For an isotropic subspace \( \Theta \subset W_x^* \) of dimension two, let \( W_\Theta \) be the Hecke modification:

\[
0 \rightarrow W_\Theta \rightarrow W \rightarrow (W_x/\Theta^\perp) \otimes \mathcal{O}_x \rightarrow 0.
\]

Then there is an \( L^*(x) \)-valued symmetric form

\[
b^\Theta: (W_\Theta)^* \rightarrow W_\Theta \otimes L^*(x)
\]

on \( V := (W_\Theta)^* \) such that \( \ker b^\Theta \) has codimension four in \( V_x = (W_\Theta)^*_x \).

For an isotropic subspace \( \varphi \subset W_\Theta \) of dimension two with \( \varphi^\perp \subset V_x \), we have the Hecke modification

\[
0 \rightarrow V^\varphi \rightarrow V \rightarrow (V_x/\varphi^\perp) \otimes \mathcal{O}_x \rightarrow 0.
\]

Then \( V^\varphi \) is equipped with a symmetric form induced from \( b^\Theta \), and it is an \( L^* \)-valued nondegenerate (orthogonal) form if and only if \( \ker b^\Theta_x \subset \varphi^\perp \). In particular when \( \varphi_0 \) is in the kernel of \( W_x \rightarrow W_\Theta \), we have \( V^\varphi_0 \cong W^* \).

Note that the space of two-dimensional isotropic subspaces \( \varphi \subset W^\Theta_x \) such that

\[
\ker b^\Theta_x \subset \varphi^\perp \subset V_x = (W^\Theta)_x^*
\]

is the isotropic Grassmannian of 2-dimensional subspaces of \( V_x/\ker b^\Theta_x \cong \mathbb{C}^4 \), which is a disjoint union of two projective lines. Let \( \text{IG}(2, V_x/\ker b^\Theta_x) \) be the line containing the point \( \varphi_0^\perp / \ker b^\Theta_x \). Then the family

\[
\{V^\varphi : \varphi \in \text{IG}(2, V_x/\ker b^\Theta_x)\}
\]

of \( L^* \)-valued orthogonal bundles gives a deformation of \( W^* \).

Under the assumption that \( g \geq 5, n \geq 5 \) and that \( W \in \mathcal{MO}_C(n, L) \) is a generic point, by [3] Lemma 4.7 the dual family \( \{(V^\varphi)^*\} \) gives a rational curve on \( \mathcal{MO}_C(n, L) \) passing through \( W \), called an orthogonal Hecke curve. Also it was shown in [3] theorem 5.3] that these curves have minimal degree among the rational curves passing through a generic point \( W \in \mathcal{MO}_C(n, L) \).

Again, we show the following.

**Proposition 2.5.** Assume \( g \geq 5 \) and \( n \geq 5 \). For a generic point \( W \in \mathcal{MO}_C(n, L) \), every orthogonal Hecke curve passing through \( W \) is contained in the smooth locus of \( \mathcal{MO}_C(n, L) \).
Proof. By [3, Lemma 4.7], any orthogonal Hecke curve passing through a generic point \( W \) stays inside the locus of stable orthogonal bundles. To see that it is contained in the smooth locus, it suffices to show that it does not touch the locus of non-regularly stable orthogonal bundles which are of the form \( W_1 \perp W_2 \) for some stable orthogonal subbundles \( W_1 \) and \( W_2 \).

This can be checked by dimension count: The locus of non-regularly stable bundles is contained in a finite union of the images of \( MO_C(n_1, L) \times MO_C(n_2, L) \), where \( n_1 + n_2 = n \). Since orthogonal Hecke curves passing through a point \([W_0]\) in this subvariety are parameterized by \( IG(2, W_0^*) \), it suffices to check the inequality:

\[
\dim IG(2, W_0^*) + \dim MO_C(n_1, L) + \dim MO_C(n_2, L) < \dim MO_C(n, L)
\]

for any integers \( n_1, n_2 \) with \( n_1 + n_2 = n \). Since \( \dim IG(2, W_0^*) = 2n - 6 \), this boils down to \( 2n - 6 < n_1n_2(g - 1) \), which holds for \( g, n \geq 5 \).

2.6. Minimal rational curves. Let \( M \) be a projective variety. Let \( K \) be an irreducible component of the Hilbert scheme of complete curves on \( M \) such that generic members of \( K \) cover an open subset of the smooth locus \( M^\circ \) of \( M \). For a generic point \( x \in M \), denote \( K_x \) the subscheme of \( K \) consisting of members of \( K \) passing through \( x \). Assume that for a generic point \( x \in M \), every member of \( K_x \) is an irreducible smooth rational curve contained in \( M^\circ \) and \( K_x \) is an irreducible complete variety. In this case, we call \( K \) a minimal rational component of \( M \).

A covering family of rational curves having minimal degree gives a minimal rational component. More precisely, an irreducible component \( K \) is a minimal rational component of \( M \) if it satisfies the following conditions:

(i) For a generic \( x \in M \), every member of \( K_x \) is irreducible smooth rational curve contained in \( M^\circ \).
(ii) The locus swept out by the curves in \( K \) is dense in \( M \).
(iii) For a fixed ample line bundle \( \xi \) on \( M \), the degree of members of \( K \) with respect to \( \xi \) is minimal among the curves in an irreducible family satisfying (1) and (2).

Let \( x \in M \) be a generic point. Define the tangent map \( \tau_x : K_x \longrightarrow \mathbb{P}(T_x(M)) \) by

\[
\tau_x([R]) = [T_x R] \in \mathbb{P}(T_x(M)),
\]

where \( R \) is a smooth rational curve in \( M^\circ \) passing through \( x \). The closure \( C_x \) of the image \( \tau_x(K_x) \subset \mathbb{P}(T_x(M)) \) is called the variety of minimal rational tangents (VMRT for short) at \( x \) associated with \( K \).

The following is [3, Theorem 2], which connects the theory of VMRT and the simplicity of the tangent bundle.

**Proposition 2.6.** Let \( M \) be a Fano variety which has a minimal rational component \( K \). If the VMRT \( C_x \) at a generic point \( x \in M \) is non-degenerate in \( \mathbb{P}T_x M \), then the tangent bundle \( T^\circ M \) is simple.

For the moduli space of vector bundles \( M = SU_C(n, d) \), it is proven in [3] that for \( g \geq 4 \), the irreducible component \( K \) of the Hilbert scheme of \( M \) containing Hecke curves is a minimal rational component of \( M \). In this case, given a generic point \( W \in M \), \( K_W \) is given by

\[
K_W = \bigcup_{[\theta] \in \mathbb{P}(W^*)} \mathbb{P}(W^\theta/\ell_0) \cong \mathbb{P}(T^*_\pi)
\]

where \( T_x \) is the vertical tangent bundle of \( \pi : \mathbb{P}(W^*) \rightarrow C \). In particular for \( n = 2 \), we have \( K_W = \mathbb{P}(W^*) \).
Moreover it is shown in [3, Theorem 3.1, Theorem 3.7] that the tangent map at a
generic point \( W \in \mathcal{M} \) is biregular to the image for \( g \geq 5 \) and birational for \( g = 4 \).

Finally we discuss the case of \( \mathcal{M}_C(n, L) \) and \( \mathcal{M}_O(n, L) \). By the result [3, Theorem 5.2] on the minimality of degree, we can see that there is a minimal rational component \( K \) of \( \mathcal{M}_C(n, L) \) containing symplectic Hecke curves such that \( K_W \) for a generic element \( W \) is given by

\[
K_W = \mathbb{P}(W^*).
\]

Similarly by [3, Theorem 5.3], there is a minimal rational component \( K \) of \( \mathcal{M}_O(n, L) \) containing orthogonal Hecke curves such that \( K_W \) for a generic element \([W]\) is given by

\[
K_W = IG(2, W^*).
\]

3. Smoothness of Hecke curves

In this section, we show the smoothness of the symplectic and orthogonal Hecke curves.

**Proposition 3.1.** Assume \( g \geq 4 \) and \( n \geq 4 \). Then any symplectic Hecke curve passing through a generic point \( W \in \mathcal{M}_C(n, L) \) is smooth.

**Proof.** From the construction, the family

\[
\{ V^\ell : \ker \omega_x^\theta \subset \ell \subset V_x \}
\]
gives a deformation of \( V^\ell \) along a subspace \( U := (\ker \omega_x^y)^\perp \cong \mathbb{C}^2 \). By Proposition 2.3 the Kodaira–Spencer map

\[
T_\ell(\mathbb{P}(V_x/U^\perp)) = \text{Hom}(\ell^\perp/U^\perp, V_x/\ell^\perp) \to H^1(C, \text{End}(V^\ell))
\]

associated to this family is given by the composition

\[
\text{Hom}(\ell^\perp/U^\perp, V_x/\ell^\perp) \xrightarrow{\beta_x} \text{Hom}(V^\ell_x, V_x/\ell^\perp) \xrightarrow{\delta} H^1(C, \text{End}(V^\ell)),
\]

where \( \beta_x \) and \( \delta \) are induced from

\[
0 \to V^\ell \xrightarrow{\beta_x} V \to (V_x/\ell^\perp) \otimes \mathcal{O}_x \to 0.
\]

Note that this can be geometrically understood as the composition map

\[
T_\ell(\mathbb{P}(V_x/U^\perp)) \xrightarrow{\beta_x} T_{[\beta]} \text{Quot}(V) \xrightarrow{\delta} H^1(C, \text{ad}(V^\ell)) = T_{[V^\ell]} \mathcal{M}_C(n, L^*),
\]

where the tangent space \( T_{[\beta]} \text{Quot}(V) \) of the Quot scheme of \( V \) is given by

\[
T_{[\beta]} \text{Quot}(V) = H^0(C, \text{Hom}(V^\ell, (V_x/\ell^\perp) \otimes \mathcal{O}_x)) \cong \text{Hom}(V^\ell_x, V_x/\ell^\perp).
\]

Hence to show that the map

\[
\phi_U : \mathbb{P}(U) \cong \mathbb{P}^1 \to \mathcal{M}_C(n, L)
\]

which gives the symplectic Hecke curve \([2] \) is an immersion, we need to show that the map \([3] \) is injective.

Since \( \beta_x \) is injective by definition of Quot schemes, we need to check that \( \delta \) is injective. The map \( \delta \) fits into the long exact sequence associated to \([5] \) tensored by \((V^\ell)^*\):

\[
0 \to H^0(C, \text{End}(V^\ell)) \to H^0(C, (V^\ell)^* \otimes V) \to \text{Hom}(V^\ell, (V_x/\ell^\perp)) \xrightarrow{\delta} H^1(C, \text{End}(V^\ell))
\]

Hence \( \delta \) is injective everywhere if we know:

- \( \dim H^0(C, \text{End}(V^\ell)) = 1 \) for all \( \ell \) and
- \( \dim H^0(C, (V^\ell)^* \otimes V) = 1 \) for all \( \ell \).
The first condition holds if $V^{\ell}$ is regularly stable. By [3, Lemma 4.2], the second condition holds if every point $V$ is a generic point and $g \geq 3$.

Now it remains to show that the map $\phi_U$ is injective. It was shown in [3, Lemma 4.5] that $\phi_U$ is generically injective if $g \geq 3$ and $n \geq 4$. Its proof can be slightly modified to show the injectiveness (under a stronger bound on $g$ and $n$). The point of the proof was to choose $W = (V^{t_0})^*$ as a “1-stable” symplectic bundle (see [3, §4.1]). By the same argument, if we choose $W$ to be 2-stable, then every bundle $V^{\ell}$ can be shown to be 1-stable, and hence $V^{\ell_1} \cong V^{\ell_2}$ implies $\ell_1 = \ell_2$. By [3, Lemma 4.1] a generic point of $M_3(n, L)$ is 2-stable for $g \geq 4, n \geq 4$ and we are done. □

**Proposition 3.2.** Assume $g \geq 5$ and $n \geq 5$. Then any orthogonal Hecke curve passing through a generic point $W \in MO_C(n, L)$ is smooth.

**Proof.** From the construction of orthogonal Hecke curves, the family

$$\{V^p : \text{ Ker } \omega^p_x \subset \varphi^{-1} \subset V_x\}$$

(6) gives a deformation of $V^p$ along a subspace $U := (\text{ Ker } \omega^p_x)^\perp \cong C^l$. By Proposition 4.3 the Kodaira–Spencer map

$$T_x Gr(2, V_x/U^\perp) = \text{ Hom}(\varphi^{-1}/U^\perp, V_x/\varphi^{-1}) \rightarrow H^1(C, End(V^p))$$

(7) associated to this family is given by the composition

$$\text{ Hom}(\varphi^{-1}/U^\perp, V_x/\varphi^{-1}) \overset{\beta_x}{\rightarrow} \text{ Hom}(V_x^p, V_x/\varphi^{-1}) \overset{\delta}{\rightarrow} H^1(C, End(V^p)),$$

where $\beta_x$ and $\delta$ are induced from

$$0 \rightarrow V^p \overset{\beta}{\rightarrow} V \rightarrow (V_x/\varphi^{-1}) \otimes \mathcal{O}_x \rightarrow 0.$$

(9) Note that this can be geometrically understood as the composition map

$$T_\ell Gr(2, V_x/U^\perp) \overset{\beta_x}{\rightarrow} T_\ell[\beta] \text{ Quot}(V) \overset{\delta}{\rightarrow} H^1(C, \text{ ad}(V^p)) = T_{[V^p]} MO_C(n, L^*),$$

where the tangent space $T_{[\beta]} \text{ Quot}(V)$ of the Quot scheme of $V$ is given by

$$T_{[\beta]} \text{ Quot}(V) = H^0(C, \text{ Hom}(V^p, (V_x/\varphi^{-1}) \otimes \mathcal{O}_x)) \cong \text{ Hom}(V_x^p, V_x/\varphi^{-1}).$$

Hence to show the immersedness of the orthogonal Hecke curve (6), it suffices to show that the map (7) is injective.

Since $\beta_x$ is injective by definition of Quot schemes, we need to check that $\delta$ is injective. The map $\delta$ fits into the long exact sequence associated to (9) tensored by $(V^p)^*$:

$$0 \rightarrow H^0(C, \text{ End}(V^p)) \rightarrow H^0(C, (V^p)^* \otimes V) \rightarrow \text{ Hom}(V^p, (V_x/\varphi^{-1})) \overset{\delta}{\rightarrow} H^1(C, \text{ End}(V^p))$$

Hence $\delta$ is injective everywhere if we know:

- $\dim H^0(C, \text{ End}(V^p)) = 1$ for all $\varphi$ and
- $\dim H^0(C, (V^p)^* \otimes V) = 1$ for all $\varphi$.

The first condition holds if $V^p$ is regularly stable. The second condition holds if $V$ is general and $g \geq 3$ by [3, Lemma 4.2].

Now to show that the injectiveness, as in the symplectic case, it suffices to choose $W$ to be 3-stable in order that every bundle $V^{\ell}$ is 2-stable. By [3, Lemma 4.1] a generic point of $MO_C(n, L)$ is 3-stable for $g \geq 5, n \geq 5$ and we are done. □
4. Nondegeneracy of the tangent map

In this section, we discuss the tangent map of the variety of minimal rational tangents for the moduli spaces \( \mathcal{M}S_C(2r, L) \) and \( \mathcal{M}O_C(n, L) \). In particular, we study the complete linear system which defines the tangent map. This confirms that the image of the tangent map is nondegenerate, and as a consequence we get the simplicity of the tangent bundle of the moduli space. Basically we follow the computations in [7] of the Kodaira–Spencer map of the Hecke curves on the moduli space \( SU_C(2, d) \).

4.1. Symplectic bundles. The tangent space of \( SU_C(n, d) \) at a stable bundle \( W \) is given by \( H^1(C, \text{End}_0(W)) \), where \( \text{End}_0(W) \) is the vector bundle of traceless endomorphisms of \( W \). By a similar argument as [11] Lemma 2.2, the tangent space of \( \mathcal{M}S_C(n, L) \) at a regularly stable symplectic bundle \( W \) is given by \( H^1(C, L \otimes \text{Sym}^2 W^*) \). In this context, we put \( ad(W) = L \otimes \text{Sym}^2 W^* \).

**Proposition 4.1.** Assume \( g \geq 4 \) and \( n \geq 4 \) as in Proposition [7.1]. Let \( \mathcal{K} \) be the minimal rational component consisting of symplectic Hecke curves on \( \mathcal{M}S_C(2r, L) \). Then for a generic \( W \in \mathcal{M}S_C(2r, L) \), the tangent map

\[
\tau_W : \mathcal{K}_W = \mathbb{P}W^* \rightarrow \mathbb{P}T_W \mathcal{M}S_C(n, L) = \mathbb{P}H^1(C, \text{Sym}^2 W^* \otimes L)
\]

is the composition \( \Phi_W \circ \iota \), where \( \Phi_W \) is given by the complete linear system \( |O(1) \otimes \pi^* K_C| \). Also, \( \iota(\theta) = \theta \otimes (\omega(\theta, \cdot)) \) for \( \theta \in W^*_x \), and the image of \( \iota(\theta) \) in \( \mathbb{P}(H^1(C, \text{Sym}^2 W^* \otimes L)) \) is given by the linear functional \( H^0(C, \text{Sym}^2 W \otimes L^* \otimes K_C) \rightarrow K_{C,x} \) taking the trace of endomorphisms of \( W_x \).

\[
\Psi_W : \mathbb{P}(\text{Sym}^2 W^* \otimes L) \rightarrow \mathbb{P}H^1(C, \text{Sym}^2 W^* \otimes L)
\]

**Remark 4.2.** The upper arrow \( \Psi_W \) is the natural mapping of the ruled variety \( \mathbb{P}(\text{End}_0(W)) \), which is not necessarily defined everywhere at this stage, but it will turn out in \( \S 5 \) to be a morphism and furthermore an embedding under certain genus assumption. On the other hand, the lower arrow \( \Phi_W \) is a morphism by Proposition 3.1.

**Proof.** Recall that \( \ker \omega^\theta \) is a subspace of \( V_x \) of codimension two, and after we put \( U^\perp = \ker \omega^\theta \subset V_x \), the family \( \{ V^\ell : [\ell] \in \mathbb{P}(V_x/U^\perp) \} \) is a deformation of \( W^* = V^{\ell_0} \).

Applying Proposition 2.2 to the kernel of \( \omega^\theta \), we get that the Kodaira-Spencer map

\[
T_{\ell_0}(\mathbb{P}(V_x/U^\perp)) = \text{Hom}(\ell_0^\perp/U^\perp, V_x/\ell_0^\perp) \rightarrow H^1(C, \text{End}(V^{\ell_0}))
\]

for the family \( \{ V^\ell : [\ell] \in \mathbb{P}(V_x/U^\perp) \} \), is given by the composition

\[
\text{Hom}(\ell_0^\perp/U^\perp, V_x/\ell_0^\perp) \xrightarrow{\delta} \text{Hom}(V^{\ell_0}, V_x/\ell_0^\perp) \xrightarrow{\delta} H^1(C, \text{End}(V^{\ell_0}))
\]

where \( \delta_x \) and \( \delta \) are induced from

\[
0 \rightarrow V^{\ell_0} \xrightarrow{\delta} V \rightarrow (V_x/\ell_0^\perp) \otimes \mathcal{O}_x \rightarrow 0.
\]

Furthermore, as in the case of the moduli space of vector bundles discussed in [7], the element \( \delta(\delta_x(v)) \) in \( H^1(C, \text{End}(V^{\ell_0})) \) for \( v \in \text{Hom}(\ell_0^\perp/U^\perp, V_x/\ell_0^\perp) \), is represented by the cocycle \( \left\{ \frac{e^5 \otimes e_1}{z} \right\} \) on \( U_0 \cap U_j \). Here, \( \{ U_0, U_1, \ldots, U_N \} \) is a coordinate covering of \( C \) such that
all the involved vector bundles are trivial on $\mathcal{U}_0$ and $\mathcal{U}_i$ for $1 \leq j \leq N$,
- $x \in \mathcal{U}_0$ and $x \not\in \mathcal{U}_j$ for $1 \leq j \leq N$ so that on each $\mathcal{U}_j$, we identify $V^{t_0}$ and $V$ via $V^{t_0} \overset{\beta_j}{\to} V$,
- $z$ is a coordinate on $\mathcal{U}_0$ centered at $x$,
- $\{e_1, e_2, \ldots, e_n\}$ and $\{f_1, f_2, \ldots, f_n\}$ are frames of $V^{t_0}|_{\mathcal{U}_0}$ and $V|_{\mathcal{U}_0}$, respectively such that $e_{1,x} \in \theta \subset V^{t_0}_x = W^*_x$, $f_{2,x} \in \ell_0^1/U^\perp \subset V_x/U^\perp = (W^\theta)_x/U^\perp$ and
- $\beta$ sends $e_1, e_2, \ldots, e_n$ to $f_1, f_2, \ldots, f_n$.

Thus $\delta(\hat{\beta}_x(v))$ corresponds to the image of $\iota(\theta) = \theta \otimes \omega(\theta, \cdot)$ via the duality $H^1(C, \text{End}_0(W)) \simeq H^0(C, K_C \otimes \text{End}_0(W))^*$ induced by the residue pairing. \hfill $\square$

### 4.2. Orthogonal bundles.
By a similar argument as [1, Lemma 2.2], the tangent space of $\mathcal{M}_C(n, L)$ at a regularly stable orthogonal bundle $W$ is given by $H^1(C, L \otimes \wedge^2 W^*)$.

In this context, we put $\text{ad}(W) = L \otimes \wedge^2 W^*$.

**Proposition 4.3.** Assume $g \geq 5$ and $n \geq 5$ as in Proposition 2.2. Let $\mathcal{K}$ be the minimal rational component consisting of orthogonal Hecke curves on $\mathcal{M}_C(n, L)$. Then for a generic $W \in \mathcal{M}_C(n, L)$, the tangent map $\tau_W$

\[
\tau_W : \mathcal{K}_W = IG(2, W^*) \to \mathbb{P}H^1(C, \wedge^2 W^* \otimes L)
\]

is the composition $\Phi_W \circ \iota$, where $\Phi_W$ is given by the complete linear system $|\mathcal{O}(1) \otimes \pi^* K_C|$. Also, $\iota([v_1 \wedge v_2] = b(v_1, \cdot) \otimes v_2 - b(v_2, \cdot) \otimes v_1$ for $[v_1 \wedge v_2] \in IG(2, W^*)$ and the image of $\iota([v_1 \wedge v_2]$ in $\mathbb{P}(H^1(C, \wedge^2 W^*) \otimes L))$ is given by the linear functional $H^0(C, \wedge^2 W^* \otimes L^* \otimes K_C) \to K_{C,x}$ taking the trace of endomorphisms of $W_x$.

\[
\begin{array}{ccc}
\mathcal{K}_W = IG(2, W^*) & \xrightarrow{\iota} & \mathbb{P}(\text{End}_0(W)) \\
\downarrow \Phi_W & & \downarrow \Psi_W \\
\mathbb{P}(\wedge^2 W^* \otimes L) & \xrightarrow{\Phi_W} & \mathbb{P}H^1(C, \wedge^2 W^* \otimes L)
\end{array}
\]

**Remark 4.4.** As before, the upper arrow $\Psi_W$ is not necessarily defined everywhere at this stage, but it will turn out in §5 to be a morphism and furthermore an embedding under certain genus assumption. On the other hand, the lower arrow $\Phi_W$ is a morphism by Proposition 3.2.

**Proof.** Recall that the kernel of $b^\Theta$ is a subspace of codimension 4, and if we put $U^\perp = \text{Ker}(b^\Theta) \subset V_x = (W^\Theta)^*$, then the family $\{V^p : [\varphi] \in IG(2, V_x/U^\perp)\}$ is a deformation of $W^* = V^{\varphi_0}$. As in the proof of Proposition 3.2, we apply Proposition 2.3 to the kernel of $b^\Theta$ and we get that the Kodaira-Spencer map

\[
T^{\varphi_0}(Gr(2, V_x/U^\perp)) = \text{Hom}(\varphi^1_0/U^\perp, V_x/\varphi^1_0) \to H^1(C, \text{End}(V^{\varphi_0}))
\]

for the family $\{V^p : [\varphi] \in Gr(2, V_x/U^\perp)\}$ is given by the composition

\[
\text{Hom}(\varphi^1_0/U^\perp, V_x/\varphi^1_0) \overset{\hat{\beta}_x}{\to} \text{Hom}(V^{\varphi_0}, V_x/\varphi^1_0) \overset{\delta}{\to} H^1(C, \text{End}(V^{\varphi_0})),
\]

where $\hat{\beta}_x$ and $\delta$ are induced from

\[
0 \to V^{\varphi_0} \xrightarrow{\hat{\beta}_x} V \to (V_x/\varphi^1_0) \otimes O_x \to 0.
\]

Choose a coordinate covering $\{\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_N\}$ of $C$ such that

- all the involved vector bundles are trivial on $\mathcal{U}_0$ and $\mathcal{U}_j$ for $1 \leq j \leq N$,
- $x \in \mathcal{U}_0$ and $x \not\in \mathcal{U}_j$ for $1 \leq j \leq N$ so that on each $\mathcal{U}_j$, we identify $V^{\varphi_0} = W^*$ and $V = (W^\Theta)^*$ by $V^{\varphi_0} \overset{\hat{\beta}_x}{\to} V$. 

10
\begin{itemize}
  \item $z$ is a coordinate on $\mathcal{U}_0$ centered at $x$,
  \item $\{e_1, e_2, \ldots, e_n\}$ and $\{f_1, f_2, \ldots, f_n\}$ are frames of $V^{\perp_0}_{|\mathcal{U}_0}$ and $V_{|\mathcal{U}_0}$ respectively such that $e_{1,x}, e_{2,x} \in \Theta \subset V^{\perp_0}_x = W^*_x$, $f_{3,x}, f_{4,x} \in \mathfrak{g}_0^/U^\perp \subset V_x/U^\perp = (W^*_x)^*/U^\perp$ and
  \item $\beta$ sends $e_1, e_2, e_3, e_4$ to $z f_1, z f_2, f_3, f_4$.
\end{itemize}

Then the tangent space $T_{\mathfrak{g}_0}(IG(2, V_x/U^\perp))$ is generated by

$$v = f^*_3, x \otimes f_{1,x} - f^*_4, x \otimes f_{2,x} \in \text{Hom}(\mathfrak{g}_0^/U^\perp, V_x/\mathfrak{g}_0^/).$$

Furthermore, $v \circ \beta$ maps $e_{1,x}, e_{2,x}, e_{3,x}, e_{4,x}$ to $0, 0, f_{1,x}, f_{2,x}$ up to a constant multiple. Thus $\widehat{\beta}_x(v)$ can be extended to $\widehat{v}$ with $\widehat{v}(e_1) = 0$ and $\widehat{v}(e_2) = 0$ and $\widehat{v}(e_3) = f_1$ and $\widehat{v}(e_4) = f_2$. Then $\delta(\widehat{\beta}_x(v))$ is defined by the cocycle

$$\widehat{v}_j = (\beta|_{\mathcal{U}_0 \cap \mathcal{U}_j})^{-1} \circ \widehat{v} \in H^0(\mathcal{U}_0 \cap \mathcal{U}_j, \text{End} V^{\mathcal{U}_0}),$$

where $\beta|_{\mathcal{U}_0 \cap \mathcal{U}_j} : V^{\mathcal{U}_0}_{|\mathcal{U}_0 \cap \mathcal{U}_j} \to V^{\mathcal{U}_0}_{|\mathcal{U}_0 \cap \mathcal{U}_j}$ is the isomorphism. Therefore, $\delta(\widehat{\beta}_x(v))$ is represented by the cocycle \(\left\{ \frac{e^*_3 \otimes e_1 - e^*_4 \otimes e_2}{z} \right\} \) on $\mathcal{U}_0 \cap \mathcal{U}_j$, where $\{e^*_1, e^*_2, \ldots, e^*_n\}$ is the dual frame.

Thus $\delta(\widehat{\beta}_x(v))$ corresponds to the image of $i(v_1 \wedge v_2) = b(v_1, \cdot) \otimes v_2 - b(v_2, \cdot) \otimes v_1$ via the duality $H^1(C, \text{End}_0(W)) \simeq H^0(C, K_C \otimes \text{End}_0(W))^*$ induced by the residue pairing. 

\[\square\]

4.3. Simplicity of the tangent bundles. Now we can prove the simplicity of the tangent bundles. Let us denote by $\mathcal{M}^g$ the smooth locus of the moduli space $\mathcal{M}SC(2r, L)$ $(\mathcal{M}OC(n, L)$, respectively) of $L$-valued symplectic (orthogonal, respectively) bundles on $C$ of genus $g$.

We exclude the case of symplectic bundles of rank two, which are nothing but vector bundles of rank two. In this case, the simplicity of the tangent bundle of $SU_C(2, d)$ has been proven in [8]. Moreover when $d$ is odd, the stability of the tangent bundle has been proven in [7]. Also recall that the rank of orthogonal bundles are assumed to be at least 5 to guarantee the minimality of the orthogonal Hecke curves (see §2.5).

**Theorem 4.5.** Assume that $g \geq 4, n \geq 4$ for symplectic case, and $g \geq 5, n \geq 5$ for orthogonal case. Then the tangent bundle $T(\mathcal{M}^g)$ is simple.

**Proof.** By Proposition 4.1 and Proposition 4.3, the tangent maps of symplectic and orthogonal Hecke curves are given by the corresponding complete linear systems. Hence for each case, the VMRT at a generic point is non-degenerate, and the wanted result follows from Proposition 2.6. \[\square\]

5. Biregularity of the tangent map

Let $W$ be an $L$-valued symplectic or orthogonal bundle over $C$. Throughout this section, we will assume that an orthogonal bundle of even rank, say $2r$, admits an isotropic subbundle of rank $r$. As shown in [2, Lemma 2.5], this is equivalent to that $\det(W) = L^r$. Accordingly, $\mathcal{M}OC(2r, L)$ denotes one of the moduli components which parameterizes $L$-valued orthogonal bundles of rank $2r$ with determinant $L^r$. On the other hand, every orthogonal bundle of odd rank, say $2r + 1$, admits an isotropic subbundle of rank $r$ by [2, Lemma 2.7]. Hence $\mathcal{M}OC(2r + 1, L)$ denotes any moduli component which parameterizes $L$-valued orthogonal bundles of rank $2r + 1$. 


As discussed in the previous section, we write
\[
\text{ad} W = \begin{cases} 
\text{Sym}^2 W \otimes L^* & \text{if } W \text{ is symplectic;} \\
\wedge^2 W \otimes L^* & \text{if } W \text{ is orthogonal.}
\end{cases}
\]
In either case, \( \text{ad} W \) is a self-dual subbundle of \( \text{End}_0 W \).

Let \( \pi: \mathbb{P}(\text{ad} W) \to C \) be the associated projective bundle of \( \text{ad} W \). We consider the natural map
\[
\Phi_W: \mathbb{P}(\text{ad} W) \to \mathbb{P}H^0(C, K_C \otimes \text{ad} W)^*,
\]
given by the complete linear system \( |O_{\mathbb{P}(\text{ad} W)}(1) \otimes \pi^* K_C| \). Our goal will be to prove that \( \Phi_W \) is an embedding. In fact, we show a stronger statement: the map
\[
\Psi_W: \mathbb{P}(\text{End}_0 W) \to \mathbb{P}H^0(C, K_C \otimes \text{End}_0 W)^*
\]
is an embedding. More precisely:

**Theorem 5.1.** Let \( \ell := \deg(L) \in \{0, 1\} \). The map \( \Psi_W \) is an embedding for:

1. a generic \( W \in \mathcal{M}_{SC}(2r, L) \) if \( r \geq 2 \) and \( g \geq 5 + 2\ell \),
2. a generic \( W \in \mathcal{M}_{OC}(2r, L) \) if either \( (r = 3, g \geq 8) \) or \( (r \geq 4, g \geq 7) \),
3. a generic \( W \in \mathcal{M}_{OC}(2r + 1, L) \) if either \( (r = 2, g \geq 14) \) or \( (r \geq 3, g \geq 9) \).

**Corollary 5.2.** Let \( M \) be one of the moduli spaces in the above theorem. Let
\[
\tau_W: K_x \to \mathbb{P}(T_W M)
\]
be the tangent map of the minimal rational component \( K \) associated to the symplectic or orthogonal Hecke curves. Then \( \tau_W \) is an embedding of the corresponding VMRT under the same assumption as in Theorem 5.1.

**Proof.** This follows from the fact that \( \Psi_W \), hence \( \Phi_W \), is an embedding, together with the pictures of Proposition 4.1 and Proposition 4.3. \( \square \)

The remaining parts of this section are devoted to proving Theorem 5.1. Following [9, Proof of Theorem 3.1], we shall use the fact that \( \Psi_W \) is an embedding if and only if
\[
h^0(O_C(D) \otimes \text{End}_0 W) = 0 \text{ for all } D \in C(2),
\]
where \( C(2) \) parameterizes the effective divisors of degree two.

We first show the following:

**Lemma 5.3.** Let \( E \to B \times C \) be a family of vector bundles over \( C \). Then the subset
\[
\left\{ b \in B : h^0(O_C(D) \otimes E_b) = 0 \text{ for all } D \in C(2) \right\}
\]
is open in \( B \), where \( E_b = E|_{\{b\} \times C} \).

**Proof.** The complement of the locus in question is
\[
\left\{ b \in B : h^0(O_C(D) \otimes E_b) \geq 1 \text{ for some } D \in C(2) \right\}.
\]
This is the image in \( B \) of the closed set
\[
\left\{ (b, D) \in B \times C(2) : h^0(O_C(D) \otimes E_b) \geq 1 \right\}
\]
by the projection from \( B \times C(2) \). As \( C(2) \) is projective and in particular complete, this projection is closed. The statement follows. \( \square \)

Applying Lemma 5.3 to a suitable étale cover of each of the moduli spaces in question, we see that it suffices to exhibit a single bundle \( V \) with property (10). We now assemble some vanishing results which we shall use to this end.
Lemma 5.4. Let $E_1$ and $E_2$ be vector bundles such that for all $D \in C(2)$ we have

(i) $h^0(\mathcal{O}_C(D) \otimes \text{End}_0 E_1) = 0$
(ii) $h^0(\mathcal{O}_C(D) \otimes \text{End}_0 E_2) = 0$
(iii) $h^0(\text{Hom}(E_1, E_2(D))) = 0$
(iv) $h^0(\text{Hom}(E_2, E_1(D))) = 0$

Let $\Pi \subseteq H^1(\text{Hom}(E_2, E_1))$ be a subspace of dimension at least $2 \cdot \text{rk}(E_1) \cdot \text{rk}(E_2) + 3$. Then if $0 \to E_1 \to W \to E_2 \to 0$ is an extension whose class $\delta$ is a general element of $\Pi$, we have $h^0(\mathcal{O}_C(D) \otimes \text{End}_0 W) = 0$ for all $D \in C(2)$.

Proof. Let $W$ be an extension whose class is a general element of $\Pi$. Let $D$ be an element of $C(2)$, and suppose that $\alpha : W \to W(D)$ is a nonzero map. We shall show that under the hypotheses above, $\alpha = \text{Id}_W \otimes s$ for some $s \in H^0(\mathcal{O}_C(D))$. This will suffice to prove the statement.

By (iii), the restriction $\alpha|_{E_1}$ factorizes via $E_1(D)$. Therefore, we have a diagram

$$
\begin{array}{cccccc}
0 & \to & E_1 & \xrightarrow{q} & E_2 & \to 0 \\
\downarrow & & \downarrow & & \downarrow & \\
0 & \to & E_1(D) & \xrightarrow{\alpha} & E_2(D) & \to 0.
\end{array}
$$

By (i), then, $\alpha|_{E_1} = \text{Id}_{E_1} \otimes s$ for some $s \in H^0(\mathcal{O}_C(D))$. (If $C$ is nonhyperelliptic then $(s) = D).$ Therefore, $\alpha - \text{Id}_W \otimes s$ is a map $W \to W(D)$ vanishing on $E_1$. Hence

$$(11) \quad \alpha - \text{Id}_W \otimes s = \beta \circ q$$

for some $\beta \in H^0(\text{Hom}(E_2, W(D))).$

If $\beta \neq 0$ then by (iv) we see that $q' \circ \beta$ is a nonzero map $E_2 \to E_2(D)$. By (ii), we have

$$q' \circ \beta = \text{Id}_{E_2} \otimes s_{D'}$$

for some $D' \in |D|$ and $s_{D'} \in H^0(\mathcal{O}_C(D))$ satisfying $(s_{D'}) = D'$. (If $C$ is nonhyperelliptic then $D' = D)$. In particular, the map $\text{Id}_{E_2} \otimes s_{D'} : E_2 \to E_2(D')$ lifts to $W$. This means that

$$(\delta(W) \in \ker (\text{Id}_{E_2} \otimes s_{D'} : H^1(\text{Hom}(E_2(D), E_1(D))) \to H^1(\text{Hom}(E_2(D - D'), E_1(D))))).$$

In view of the exact sequence

$$0 \to E_2 \otimes E_1 \to E_1 \oplus E_1(D') \to E_2 \otimes E_1(D')|_{D'} \to 0,$$

this kernel has dimension at most $2 \cdot \text{rk}(E_2) \cdot \text{rk}(E_1)$. The union

$$\bigcup_{D' \in C(2)} \ker ((\text{Id}_{E_2} \otimes s_{D'})^*)$$

is therefore of dimension at most $2 \cdot \text{rk}(E_2) \cdot \text{rk}(E_1) + 2 < \dim \Pi$. But since $\delta(W)$ was assumed to be general in $\Pi$, we may assume that the map $\text{Id}_{E_2} \otimes s_{D'}$ does not lift to $W$. Therefore, we must have $\beta = 0$.

By (11), we obtain $\alpha = \text{Id}_W \otimes s$, as desired.

Lemma 5.5. Suppose $t$ and $s$ are integers with $s \geq 0$ and $t + 2s < g$. Let $N$ be a generic line bundle in $\text{Pic}^t(C)$. Then $h^0(N(D)) = 0$ for all $D \in C(s)$, where $C(s)$ parameterizes the effective divisors of degree $s$.

Proof. If $t + s < 0$ then this is clear. Otherwise: If $h^0(C, N(D)) \neq 0$ for some $D \in C(s)$, then $N$ is of the form $\mathcal{O}_C(D_1 - D)$ for some $D_1 \in C(t+s)$ and $D \in C(s)$. Hence the locus of such $N$ is of dimension at most $t + 2s$ in $\text{Pic}^t(C)$. (We take $C(0) = \{\mathcal{O}_C\}$.) By hypothesis, this locus is not dense in $\text{Pic}^t(C)$. The statement follows. \hfill \Box
Lemma 5.6. Let \( r, s, t \) be integers with \( r \geq 2 \), \( s \geq 0 \) and \( t + rs < (r - 1)(g - 1) \). Let \( G \) be a generic stable bundle of rank \( r \) and degree \( t \). Then \( h^0(G(D)) = 0 \) for all \( D \in C^{(s)} \).

Proof. A nonzero section of \( G(D) \) gives a sheaf injection \( O_C(-D) \to G \). This implies that the first Segre invariant of \( G \) is bounded by

\[
 s_1(F) \leq \deg(G) - \text{rk}(G) \cdot \deg(O_C(-D)) = t + rs.
\]

But as \( G \) is generic, by [10, Satz 2.2] we have

\[
 s_1(G) \geq (r - 1)(g - 1).
\]

Hence we have an inequality \((r - 1)(g - 1) \leq t + rs\), which contradicts the assumption. Thus \( h^0(C, G(D)) = 0 \) for all \( D \). \( \square \)

Lemma 5.7. Let \( E \) be a generic stable bundle over \( C \) of rank \( r \geq 1 \) and degree \( e \).

(1) If \( e \geq 0, s \geq 1 \) and \( 2e + 2s < g \), then \( h^0(C, E \otimes E(D)) = 0 \) for all \( D \in C^{(s)} \).

(2) If \( e < 0, s \geq 1 \) and \( 2s < g \), then \( h^0(C, E \otimes E(D)) = 0 \) for all \( D \in C^{(s)} \).

Proof. By Lemma 5.3 the locus of points \( E \) which satisfy the desired vanishing property is open. Thus it suffices to exhibit a single \( E \) with the desired property. (Note that even if such an \( E \) is unstable, we can continuously deform it to get a generic stable one with the same vanishing property.)

We first assume \( e \geq 0 \) and proceed by induction on \( r \). For \( r = 1 \), this holds by Lemma 5.5. Suppose now that \( r \geq 2 \). By induction hypothesis, there is a stable vector bundle \( F \) of rank \( r - 1 \) and degree \( e \) with the desired vanishing property, which can be assumed to be generic in the moduli. Fix a generic line bundle \( L_0 \in \text{Pic}^0(C) \) and put \( E = L_0 \oplus F \). Then \( E \otimes E(D) \) has four direct summands, and it suffices to show all of them have no sections.

By induction we may assume that \( h^0(C, F \otimes F(D)) = 0 \) for all \( D \in C^{(2)} \). Also we note that \( h^0(C, (D_0)^2(D)) = 0 \) for all \( D \in C^{(2)} \) by Lemma 5.5. Finally we show that \( h^0(C, F \otimes L_0(D)) = 0 \) for all \( D \). If \( F \) is a line bundle, the condition for the vanishing is \( e + 2s < g \) by Lemma 5.5 and this holds by assumption. If \( \text{rk}(F) = r - 1 \geq 2 \), then the condition for the vanishing is \( e + (r - 1)s < (r - 2)(g - 1) \) by Lemma 5.6. Also this holds by assumption, noting that \( s \geq 1 \) and \( r = \text{rk}(F) + 1 \geq 3 \).

It follows that for a generic extension \( E \) given above, we have \( h^0(C, E \otimes E(D)) = 0 \) for all \( D \in C^{(s)} \).

To finish, we need to find a bundle of degree \( e < 0 \) with the vanishing property. By the part (1), we may choose a stable bundle \( E \) of rank \( r \) and degree 0 satisfying \( h^0(E \otimes E(D)) = 0 \) for all \( D \in C^{(s)} \). Let \( \tilde{E} \) be obtained by an elementary transformation \( 0 \to \tilde{E} \to E \to \tau \to 0 \), where \( \deg(\tau) = e \). Then \( \tilde{E} \otimes \tilde{E}(D) \) is a subsheaf of \( E \otimes E(D) \), and by the vanishing result for \( E \), we obtain \( h^0(C, \tilde{E} \otimes \tilde{E}(D)) = 0 \). This completes the proof. \( \square \)

Also we record the following cohomology vanishing result [9, Proposition 3.2], which will be used later:

Proposition 5.8. For a general stable vector bundle \( F \) of arbitrary rank and degree, \( H^0(C, (\text{End}_0 F)(D)) = 0 \) for any effective divisor \( D \) of degree \( d \) whenever \( g \geq \frac{2d}{3} \).

Now we apply these results to show the desired vanishing property for a generic symplectic and orthogonal bundle of even rank. Consider an extension

\[
(*) \quad 0 \to E \to W \to E^* \otimes L \to 0.
\]

for \( E \in SU(r, e) \) which is a generic stable bundle. Recall that, by [6, Criterion 2.1], we get a symplectic bundle \( W \) if we choose \((*)\) in \( H^1(\text{Sym}^2 E \otimes L^*) \), and an orthogonal
bundle if we choose (\ast) in $H^1(\wedge^2 E \otimes L^*)$. Also in both cases, $E \to W$ is an isotropic subbundle. (From now on whenever we discuss orthogonal bundles of even rank, we consider those bundles with an isotropic subbundle of the half rank only.)

Once we find a symplectic/orthogonal bundle $W$ in this extension which has the desired vanishing property (10), by deformation this will show that the vanishing property holds for a general stable symplectic/orthogonal bundle with the same topological invariants. The only topological invariants of a symplectic bundle are rank and degree, while we need to additionally consider the 2nd Stiefel–Whitney class for an orthogonal bundles. Note that we may assume $L = \mathcal{O}_C$ when $\deg(L)$ is even and $L = \mathcal{O}_C(x)$ for some $x \in C$ when $\deg(L)$ is odd.

If $L = \mathcal{O}_C$ so that $\deg(W) = 0$, the moduli space $\mathcal{M}_C(2r, \mathcal{O}_C)$ has two components classified by the 2nd Stiefel–Whitney class $w_2(W)$ such that the degree of any rank $r$ isotropic subbundle of $W$ has the same parity as $w_2(W)$. On the other hand, if $L = \mathcal{O}_C(x)$ so that $\deg(W) = r$, then by [2] §2 every orthogonal bundle $W \in \mathcal{M}_C(2r, \mathcal{O}_C(x))$ has rank $r$ isotropic subbundles both of even degree and of odd degree.

Hence it suffices to show that a bundle $W$ obtained by a generic extension (\ast) has the vanishing property (10) in each of the following cases:

- For symplectic bundles: $e = \deg E = 0$ and $L = \mathcal{O}_C$ or $\mathcal{O}_C(x)$.
- For orthogonal bundles: $(L = \mathcal{O}_C$ and $e = -1, 0)$ or $(L = \mathcal{O}_C(x)$ and $e = 0)$.

Proposition 5.9. Let $\ell = \deg(L) \in \{0, 1\}$.

1. For $r \geq 2$, suppose $g \geq 5+2\ell$. If (\ast) is a symplectic extension defined by a generic class in $H^1(\text{Sym}^2 E \otimes L^*)$ where $e := \deg E = 0$, then $h^0(\mathcal{O}_C(D) \otimes \text{End}_0 W) = 0$ for all $D \in C^{(2)}$.

2. Suppose either $(r = 3, g \geq 8)$ or $(r \geq 4, g \geq 7)$. If (\ast) is an orthogonal extension defined by a generic class in $H^1(\wedge^2 E \otimes L^*)$ where $e := \deg E \in \{-1, 0\}$, then $h^0(\mathcal{O}_C(D) \otimes \text{End}_0 W) = 0$ for all $D \in C^{(2)}$.

Proof. For the vanishing, let us apply Lemma 5.7 with $E_1 = E^*$ and $E_2 = E \otimes L$. As $E$ is general in moduli and $g \geq 5$, conditions (i) and (ii) follow from Proposition 5.8.

Conditions (iii) and (iv) read $h^0(E \otimes E \otimes L(D)) = 0$ and $h^0(E^* \otimes E^* \otimes L^*(D)) = 0$ for any $D \in C^{(2)}$, respectively. These vanishing conditions are checked by Lemma 5.7 under the assumption $g \geq 5 + 2\ell$ in each case for $L = \mathcal{O}_C$ or $L = \mathcal{O}_C(x)$. To get a conclusion from Lemma 5.7, it will suffice to check that $h^1(\text{Sym}^2 E \otimes L^*)$ and $h^1(\wedge^2 E \otimes L^*)$ are bigger than $2r^2 + 2$, respectively.

In the symplectic case, by Riemann–Roch the desired inequality would follow from

$$2r^2 + 3 \leq \frac{1}{2} r(r + 1)(g - 1 + \deg(L)).$$

This reads

$$\frac{g - 1 + \ell}{2} \geq \frac{2r^2 + 3}{r(r + 1)},$$

which holds for $r \geq 2$ and $g \geq 5 - \ell$.

In the orthogonal case, the desired inequality would follow from

$$2r^2 + 3 \leq -(r - 1)e + \frac{1}{2} r(r - 1)(g - 1 + \deg(L)).$$

This reads

$$\frac{g - 1 + \ell}{2} \geq \frac{2r^2 + 3 + (r - 1)e}{r(r - 1)},$$

which holds if either $(r = 3, g \geq 8)$ or $(r \geq 4, g \geq 7)$. \qed
Now we discuss the case of orthogonal bundles of odd rank. In this case, we require some more vanishing results. To get a better genus bound, we assume here $e = \deg E \in \{-2, -1\}$ instead of $\{-1, 0\}$.

**Lemma 5.10.** Suppose $g \geq 8$. Suppose $r \geq 2$, and let $E$ be a generic stable bundle of rank $r$ and degree $e$, where $e \in \{-2, -1\}$. Let $0 \to E \to F \to O_C \to 0$ be a generic extension. Then $h^0(O_C(D) \otimes \text{End}_0 F) = 0$ for all $D \in C^{(2)}$.

**Proof.** Let us apply Lemma 5.4 with $E_1 = E$ and $E_2 = O_C$. Condition (i) follows as above from [9, Proposition 3.2] since $E$ is general and $g \geq 5$. Condition (ii) is trivial. Conditions (iii) and (iv) follow from Lemma 5.11 under the assumption $g \geq 8$.

It remains to check that $\dim H^1(\text{Hom}(O_C, E)) > 2r + 3$. As $E$ is stable of negative degree, $h^0(E) = 0$. By Riemann–Roch, $h^1(E) = -e + r(g - 1) \geq r(g - 1) + 1$. Using the inequalities $r \geq 2$ and $g \geq 5$, one checks that this exceeds $2r + 3$. The statement now follows from Lemma 5.4. \qed

**Lemma 5.11.** Let $E$ and $F$ be as in Lemma 5.10 and suppose $g \geq 8$. Then for all $D \in C^{(2)}$ we have $h^0(E^* \otimes F^*(D)) = 0$ and $h^0(F \otimes E(D)) = 0$.

**Proof.** By construction of $F$, for each $D \in C^{(2)}$ we have exact sequences
\[
H^0(E \otimes E(D)) \to H^0(F \otimes E(D)) \to H^0(E(D)) \to \cdots
\]
and
\[
H^0(E^*(D)) \to H^0(E^* \otimes F^*(D)) \to H^0(E^* \otimes E^*(D)) \to \cdots
\]
By Lemma 5.7, we have $h^0(E \otimes E(D)) = 0 = h^0(E^* \otimes E^*(D))$ for all $D \in C^{(2)}$. The vanishing $h^0(E^*(D)) = 0 = h^0(E(D))$ is a consequence of Lemma 5.6. The statement follows. \qed

By [2, Lemma 2.4], for any $L$-valued orthogonal bundle $W$ of odd rank, there is a line bundle $N$ such that $W \otimes N$ is an $O_C$-valued orthogonal bundle of trivial determinant. So we may work only for the moduli component of $O_C$-valued orthogonal bundles of trivial determinant. As in the even rank case, the moduli space $M_{O_C}(2r + 1, O_C)$ has two components classified by the 2nd Stiefel–Whitney class $w_2(W)$. By [1, Theorem 3.1], the degree of any rank $n$ isotropic subbundle of $W$ has the same parity as $w_2(W)$. To construct such orthogonal bundles of rank $2r + 1$ as extensions, we use some results from [1, §3]. Let $0 \to E \to F \to O_C \to 0$ be an extension as above, and let $\Pi_j$ be the subspace of $H^1(F \otimes E)$ as defined in [1, §3], which contains $H^1(C, \wedge^2 E)$ as a subspace of codimension 1. By [1, Lemma 3.2], an extension $0 \to E \to W \to F^* \to 0$ defined by a class contained in $\Pi_j \setminus H^1(C, \wedge^2 E)$ is an orthogonal bundle.

**Proposition 5.12.** Suppose either $(r = 2, g \geq 14)$ or $(r \geq 3, g \geq 9)$. Let $0 \to E \to W \to F^* \to 0$ be a stable orthogonal bundle of rank $2r + 1$ as above, whose extension whose class is general in $\Pi_j$. Then $h^0(O_C(D) \otimes \text{End}_0 W) = 0$ for all $D \in C^{(2)}$.

**Proof.** Again, we use Lemma 5.4 this time with $E_1 = E$ of rank $r$, degree $e$ and $E_2 = F^*$ of rank $r + 1$, degree $-e$ where $e \in \{-2, -1\}$. Condition (i) follows from Proposition 5.8 as before. Condition (ii) follows from Lemma 5.10. Conditions (iii) and (iv) follow from Lemma 5.11.

Lastly, we must show that $\dim(\Pi_j) \geq 2 \cdot \text{rk}(E) \cdot \text{rk}(F^*) + 3 = 2r(r + 1) + 3$. By Riemann–Roch,
\[
\dim(\Pi_j) = h^1(\wedge^2 E) + 1 \geq -(r - 1)e + \frac{r(r - 1)}{2}(g - 1) + 1.
\]
Hence it suffices to have
\[-(r-1)e + \frac{r(r-1)}{2}(g-1) + 1 \geq 2r(r+1) + 3.\]
Simplifying this by using \(e \leq -1\), we get the wanted inequality if
\[
\frac{r(r-1)}{2}(g-1) \geq 2r^2 + r + 3.
\]
This holds for \((r = 2, g \geq 14)\) or \((r \geq 3, g \geq 9)\). \(\square\)

**Proof of Theorem 5.1.** By Lemma 5.3 it suffices to exhibit a single element \(W_0\) of each of the moduli spaces satisfying the vanishing property (10). This follows from Proposition 5.9 and 5.12. \(\square\)

**Remark 5.13.** (1) The genus bound in Theorem 5.1 is not optimal, and can be improved simply by computing inequalities above more accurately. For instance, the above inequality (12) reads \(g \geq \frac{2(2r^2 + r + 3)}{r(r-1)} + 1\), which becomes \(g \geq 6\) in the limit \(r \to \infty\).
(2) We did not consider orthogonal bundles of rank \(\leq 4\) according to the convention in §2.5. But all the arguments in §5 are valid for arbitrary rank, hence we can apply the same argument to get a very amenples result for orthogonal bundles of rank \(\leq 4\). For example, we could state Proposition 5.9 (2) for \(r = 2\), in which case the genus assumption would be \(g \geq 12\). Hence the map \(\Psi_W\) is an embedding for a generic \(W \in MO_C(4, L)\) if \(g \geq 12\).
(3) Theorem 5.1 shows that the VMRT of \(MS_C(n, L)\) and \(MO_C(n, L)\) at a generic point \(W\) is biregular to \(P(W^*)\) and \(IG(2, W)\) respectively, under the assumption on the genus. As remarked in [3, Remark 6.2], this improves the involved genus bound in [3, §6].

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