Spacetime Duality and Superduality

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Abstract

We introduce a new class of duality symmetries amongst quantum field theories. The new class is based upon global spacetime symmetries, such as Poincaré invariance and supersymmetry, in the same way as the existing duality transformations are based on global internal symmetries. We illustrate these new duality transformations by dualizing several scalar and spin-half field theories in 1+1 spacetime dimensions, involving nonsupersymmetric as well as (1, 1) and (2, 2) supersymmetric models. For (2, 2) models the new duality transformations can interchange chiral and twisted chiral multiplets.
1. Introduction

Duality symmetries are dramatically changing our understanding of quantum field theories by demonstrating the equivalence of many different models which previously had been thought to be entirely distinct. This is having enormous implications for our understanding of string theory, with more and more string models being discovered to be duals of one another [1]. At this writing evidence is accumulating in support of a duality between some types of string theories and four-dimensional gauge theories [2]. Duality also has more prosaic implications, implying in two dimensions the equivalence between fermions and bosons [3], and in $D$ dimensions the equivalence between massless antisymmetric tensor fields of rank $r$ and rank $D - r - 2$ [4], and so on.

In its most concrete form, duality may be considered to be the following algorithm for constructing an alternate description of a given field theory, $\mathcal{F}$, provided that $\mathcal{F}$ has a global internal symmetry group, $G$ [5], [6], [7]. The duality algorithm instructs us to gauge the symmetry $G$, but also to impose a constraint that eliminates the corresponding gauge field strength. This constraint is to be implemented by introducing a Lagrange multiplier field $\Lambda$. Thus, by construction, integrating over $\Lambda$ and fixing a gauge for $G$ reproduces the original model, $\mathcal{F}$. The dual, $\tilde{\mathcal{F}}$, is found by changing the order of functional integration, leaving for last the field $\Lambda$, which is the fundamental field of the dual theory.

As our understanding improves, many of the equivalences among field theories prove to be special cases of this duality algorithm. Some equivalences — such as fermionization (as opposed to bosonization), or the string-theoretic equivalence known as mirror symmetry — have resisted such an understanding however, and at present do not appear to be based on any pre-existing internal global symmetries. For mirror symmetry the situation is even worse. Although there is strong evidence in its favour, an explicit proof of the validity of mirror symmetry still eludes us, interesting insights [8] notwithstanding. Despite the failure of recent valiant efforts [9], one suspects that a better understanding of such symmetries in terms of the duality algorithm must shed light on their origins and domains of validity.

Motivated by these considerations, it is interesting to try to extend the duality for-
malism beyond its present limits and to explore its broadest consequences. In this article we make a first step towards a generalization of the duality formalism. Up until now only global internal symmetries have been used to dualize a theory, even though the quantum field theories of interest also have spacetime symmetries, such as Poincaré invariance or supersymmetry. Our purpose here is to propose a new class of duality transformations which are based on these spacetime symmetries. Besides its intrinsic formal interest as a more general way to dualize arbitrary field theories, we expect that our extension may be of use in better understanding some of the open questions mentioned above.

When duality is based on a spacetime symmetry, the gauging means coupling the original model to gravity (or supergravity). The constraint which removes the gauge degrees of freedom now imposes flat spacetime (or a vanishing gravitino, or both). As in the usual construction, it is the interchange of the order of functional integrations which gives rise to the dual theory.

In principle this very general prescription provides a concrete construction of a dual theory for any relativistic field theory, with potentially far-reaching consequences. In particular, global internal symmetries are not required; our procedure should be applicable, for example, to string backgrounds without isometries, such as Calabi-Yau compactifications and conformal field theories based on cosets $G/H$, having a non-Abelian group $H$.

We remark in passing that our new duality transformation should not be confused with the well known $T$-duality of strings propagating in curved spacetimes. For $T$-duality the original theory is the 2D worldsheet sigma model and the metric of the curved ‘target’ space appears as a coupling of the worldsheet fields. Even though $T$-duality is based on the existence of isometries of the target metric, within the 2D worldsheet field-theory context this isometry is an ordinary internal symmetry. The duality we propose, on the other hand, is based on 2D Lorentz invariance or supersymmetry on the worldsheet itself.

We present our ideas as follows: In the next section, §2, we review one of the simplest duality transformations, based on the symmetry $X \rightarrow X + \text{constant}$ for a scalar field $X$, as a paradigm of the steps we follow for the spacetime symmetries. §3 through §6 then present spacetime duality through a series of simple examples, all involving massless fields.
in 1+1 dimensions. §3 starts with the simplest possible case, the dualization of a scalar field coupled to a background gravitational field in 1+1 dimensions. In §4 we generalize the result of §3 to find the dual of a Dirac fermion in 1+1 dimensions. This turns out to provide an alternative bosonization method that differs in detail from the standard one. In §5 we discuss superduality, which is the extension of our formalism to dualization using supersymmetry itself as the initial global symmetry. Again we illustrate the method using the simplest case of a (1, 1) supersymmetric Wess-Zumino model, although the dual theory is in this case equivalent to the original one. §6 examines the same construction for a (2, 2) supersymmetric Wess-Zumino model, where we find that duality maps chiral and twisted chiral multiplets into one another. In §7 we summarize our results. In the appendix we discuss some ambiguities in the expression for the (2,2) component superconformal anomaly.

2. The Duality Algorithm

In order to better describe the technical details of spacetime duality, we pause here to first discuss ordinary duality in its simplest setting. Consider therefore a free massless real scalar field in 1+1 dimensions, with action \( S = -\frac{1}{2} \int \partial_\mu X \partial^\mu X \). We compute the generating functional for this model in the presence of an external background gauge field, \( a_\mu \):

\[
Z[a_\mu] = \int [\mathcal{D}X] \exp \left\{ -\frac{i}{2} \int d^2 x \left( \partial_\mu X - a_\mu \right) \left( \partial^\mu X - a^\mu \right) \right\}. \tag{1}
\]

The generating functional defined by (1) is invariant under the background gauge symmetry: \( a_\mu \rightarrow a_\mu + \partial_\mu \omega \), for arbitrary \( \omega \), as may be seen by performing the change of integration variable: \( X \rightarrow X + \omega \).

We include the coupling to \( a_\mu \) in order to have an argument on which the result depends after performing the path integral. This is important since in what follows we ignore overall constants throughout when performing functional integrals. Physically, differentiation of \( Z \) with respect to \( a^\mu \) gives the correlation functions for the operator \( \partial_\mu X \), which is also the Noether current of the symmetry for which \( a_\mu \) is the gauge potential.
When considering the dependence on background fields, such as \( a_\mu \), it is important to keep in mind that some dependence may appear implicitly in the definition of the functional integral measure, particularly for the background gravitational fields we encounter in subsequent sections. This complication does not arise for the simple system considered here.

In order to dualize this system we follow the following steps:

1. Gauge the background symmetry \( X \to X + \omega \) by introducing a dynamical gauge field \( A_\mu \) (i.e. one over which a functional integral is to be performed).

2. Choose a gauge-fixing condition, \( f = 0 \), (together with the corresponding Fadeev-Popov-DeWitt determinant, \( J_{FP} \)), as is required to evaluate the path integral over \( A_\mu \).

3. Impose a gauge-covariant constraint which implies \( A_\mu \) is pure gauge. This constraint, together with the gauge condition just described, is designed to ensure that the path integral over \( A_\mu \) is equivalent to evaluating the integrand at the configuration \( A_\mu = 0 \).

4. Rewrite the constraint of item 3 by introducing a Lagrange multiplier (\( \Lambda \)) whose path integration imposes this constraint. Integrating over \( \Lambda \), and then over \( A_\mu \), therefore reproduces the original theory, eq. (1).

5. Finally, perform the path integral in a different order: integrate first over \( X \) and \( A_\mu \), leaving the integral over \( \Lambda \) unperformed. The result is the ‘dual’ theory, with \( \Lambda \) playing the role of the dual field variable.

For the present example, steps 1 through 4 amount to rewriting eq. (1) in the following way:

\[
Z[a_\mu] = \int [\mathcal{D}X] [\mathcal{D}A_\mu] [\mathcal{D}\Lambda] \Delta[f] J_{FP} \times \\
\exp \left\{ \frac{-i}{2} \int d^2x \left[ (\partial_\mu X - a_\mu - A_\mu) (\partial^\mu X - a^\mu - A^\mu) + 2\Lambda \varepsilon^{\mu\nu} \partial_\mu A_\nu \right] \right\}. \tag{2}
\]

Here \( \Delta[f] \) is the functional delta function which imposes the gauge condition \( f = 0 \), and \( J_{FP} \) is the associated Fadeev-Popov-DeWitt determinant.
It is clear that integrating over $\Lambda$ gives a functional delta function which imposes the constraint $\varepsilon^{\mu\nu} \partial_\mu A_\nu = 0$, i.e. $A_\mu$ has vanishing field strength. Using the gauge fixing condition $f \equiv \partial^\mu A_\mu = 0$ and ignoring overall $a_\mu$-independent constants, the integration over $A_\mu$, barring topological complications,\(^1\) is accomplished by simply setting $A_\mu = 0$ everywhere, thus recovering the original generating functional (1).

On the other hand, first integrating $X$ and $A_\mu$, it is more convenient to use the gauge $f \equiv X = 0$ to perform the $X$ integration. Then the remaining integral over $A_\mu$ is Gaussian, giving the dual result:

$$Z[a_\mu] = \int [D\Lambda] \exp \left\{ -\frac{i}{2} \int d^2 x \left[ \partial_\mu \Lambda \partial^\mu \Lambda - 2\varepsilon^{\mu\nu} a_\mu \partial_\nu \Lambda \right] \right\}. \quad (3)$$

The significance of the dual formulation lies in the observation that the coupling to $a_\mu$ differs in eq. (3) from that of eq. (1). In particular, the difference in the term linear in $a_\mu$ in the respective actions indicates that the field operators dualize according to the standard relation:

$$\partial_\mu X \leftrightarrow \varepsilon_{\mu\nu} \partial^\nu \Lambda. \quad (4)$$

In addition, notice that the action for $X$ contains the quadratic term, $a_\mu a^\mu$, but no such term appears in the dual action for $\Lambda$. This also has physical implications, since twice differentiating eqs. (1) and (3) implies:

$$(-i)^2 \left. \frac{\delta^2 Z}{\delta a^\mu(x)a^\nu(y)} \right|_{a_\mu=0} = \langle \varepsilon_{\mu\alpha} \partial^\alpha \Lambda(x) \varepsilon_{\nu\beta} \partial^\beta \Lambda(y) \rangle = \langle \partial_\mu X(x) \partial_\nu X(y) \rangle + i\eta_{\mu\nu} \delta^2(x - y),$$

where $\eta_{\mu\nu}$ is the usual Minkowski-space metric. $\langle \cdots \rangle$ here indicates the covariant $T^*$ product, which is related to the garden-variety time-ordered ($T$) product by, for example, \(\langle 0|T^*[\partial_\mu X(x)\partial_\nu X(y)]|0 \rangle \equiv \partial_\mu \partial_\nu \langle 0|T[X(x)X(y)]|0 \rangle\).

The $\delta$-function contact term in the last of the equalities in eq. (5) is just what is required for this equation to make sense. After all, the correspondence, eq. (4), implies\(^1\) See, however, refs. [10] for a discussion of duality on spaces with nontrivial topology.
that time derivatives, $\partial_t X$, dualize to space derivatives, $\partial_x \Lambda$, and while time derivatives get $\delta$-function contributions when the derivatives hit the time ordering, space derivatives do not. The contact term of eq. (5) is just what is required to make both sides of eq. (5) agree.

3. Spacetime Duality I: The Scalar Field

We now extend the duality algorithm to spacetime symmetries. To describe the procedure we focus first on the simplest case, that of a massless real scalar field in 1+1 dimensions. Although this model proves to be self-dual under the construction we outline, the same does not prove to be true for some of the models considered in subsequent sections.

As in the previous example, we must choose a background field as the argument of the generating functional. In this and later examples we choose it to be a background gravitational field $h_{\mu\nu}$. With this choice the action is

$$S[h, X] = -\frac{1}{2} \int d^2 x \sqrt{-h} h^{\mu\nu} \partial_\mu X \partial_\nu X = \frac{1}{2} \int d^2 x \sqrt{-h} \Box h X,$$

where $\Box h = h^{\mu\nu} \nabla_\mu \nabla_\nu = \frac{1}{\sqrt{-h}} \partial_\mu (\sqrt{-h} h^{\mu\nu} \partial_\nu)$, and the suffix $h$ emphasizes its dependence on the metric $h_{\mu\nu}$. We denote the same quantity without subscripts built using the Minkowski-space metric $\eta_{\mu\nu}$ by $\Box = \eta^{\mu\nu} \partial_\mu \partial_\nu$.

The functional integral on which we perform the duality algorithm is

$$Z[h_{\mu\nu}] = \int [DX]_h e^{iS[h, X]}.$$

The subscript ‘$h$’ here is a reminder that the measure, $[DX]_h$, depends implicitly on the field $h_{\mu\nu}$. This dependence can be found explicitly using any of a number of methods for defining a measure [11], [12], [13], [14] invariant under general coordinate transformations (GCTs). Any such measure is subject to a conformal anomaly, and since any metric in 1+1 dimensions is conformally flat, the conformal anomaly may be used to explicitly display
the metric dependence of $[\mathcal{D}X]_h$. Thus, adopting coordinates for which $h_{\mu\nu} = e^\varphi \eta_{\mu\nu}$, we have $[\mathcal{D}X]_h = [\mathcal{D}X]_\eta \exp(iS_L[\eta, \varphi])$, where generally $S_L$ denotes the Liouville action

$$S_L[h, \phi] \equiv -\frac{1}{48\pi} \int d^2x \sqrt{-h} \left( -\frac{1}{2} h^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \mathcal{R}_h \phi + \mu^2 e^\phi \right)$$

$$= -\frac{1}{96\pi} \left\{ \int d^2x \sqrt{-g} \left( \mathcal{R}_g \frac{1}{\Box_h} \mathcal{R}_g + \mu^2 \right) - \int d^2x \sqrt{-h} \left( \mathcal{R}_h \frac{1}{\Box_h} \mathcal{R}_h \right) \right\}. \quad (8)$$

Here $\mathcal{R}_h$ is the curvature scalar defined from the metric $h_{\mu\nu}$, and $\mathcal{R}_g$ is the same quantity constructed from the conformally-related metric $g_{\mu\nu} = e^\varphi h_{\mu\nu}$. We adjust the regularization-dependent scale $\mu$ such as to cancel any such term appearing elsewhere in the path integral, and so we omit it in what follows. As usual, $\Box_h^{-1} \mathcal{R}_h$ denotes the convolution $\int d^2y \sqrt{-h} \ G_h(x, y) \mathcal{R}_h(y)$, of $\mathcal{R}_h$ with the Feynman Green’s function for $\Box_h$: $\Box_h G_h(x, y) = \delta^2(x - y)/\sqrt{-h}$.

We note that the conformal invariance of the action, eq. (6), permits the functional integral over $X$ in eq. (7) to be explicitly evaluated, yielding

$$[\det(-\Box_h)]^{-1/2} = Z[h_{\mu\nu}] = Z[e^\varphi \eta_{\mu\nu}] = Z[\eta_{\mu\nu}] e^{iS_L[\eta, \varphi]} = [\det(-\Box)]^{-1/2} e^{iS_L[\eta, \varphi]}$$

$$= [\det(-\Box)]^{-1/2} \exp \left\{ -\frac{i}{96\pi} \int d^2x \sqrt{-h} \left( \mathcal{R}_h \frac{1}{\Box_h} \mathcal{R}_h \right) \right\}. \quad (9)$$

With these preliminaries in hand, we now turn to the dualization of this model, following the steps outlined in §2. We base the duality on the spacetime symmetries of the model, which we gauge by coupling the system to a dynamical gravitational field, $g_{\mu\nu}$, over which we must functionally integrate. To this we also add a generally-covariant constraint which forces the dynamical field $g_{\mu\nu}$ to be gauge-equivalent to the background metric $h_{\mu\nu}$, thereby making the gauged system identical to the original, eq. (7). Duality is then achieved by interchanging the order of functional integrations.

Therefore we start with the following gauged functional integral

$$Z[h_{\mu\nu}] = \int [\mathcal{D}X]_g [\mathcal{D}g_{\mu\nu}]_g \Delta_{LM}(g_{\mu\nu}, h_{\mu\nu}) e^{iS[g, X]}, \quad (10)$$
where $\Delta_{LM}(g_{\mu\nu}, h_{\mu\nu})$ implements the constraint which forces $g_{\mu\nu}$ to agree with $h_{\mu\nu}$. (We will have more to say about this constraint shortly.) To define the measure $[\mathcal{D}g_{\mu\nu}]$, we write the dynamical metric as a combined coordinate and conformal transformation of the background, $g_{\mu\nu} = (e^\phi h_{\mu\nu})^\xi$, and so write $[\mathcal{D}g_{\mu\nu}]$ as $[\mathcal{D}\phi][\mathcal{D}\xi^\mu]_g[\mathcal{D}f][J_{FP}]_g$. Here $f^\mu = 0$ denotes the coordinate condition which fixes $\xi^\mu$, and for which we choose conformal gauge ($i.e.$ we choose coordinates so that $g_{\mu\nu} = e^\phi h_{\mu\nu}$). With this choice the Fadeev-Popov-DeWitt determinant becomes $[J_{FP}]_g = \det[-\Box_g + \frac{i}{2} \mathcal{R}_g]^{1/2}$, where $\Box_g$ is the Laplacian operating on vector fields. (For later reference we record the contribution of this determinant to the conformal anomaly, $[J_{FP}]_h e^{-2\delta S_L[h_{\mu\nu}, \phi]}$.)

Combining the above definitions permits eq. (10) to be written in the following way

$$Z[h_{\mu\nu}] = \int [\mathcal{D}X]_{e^\phi h} [\mathcal{D}\phi]_{e^\phi h} [J_{FP}]_{e^\phi h} \Delta_{LM}(e^\phi h_{\mu\nu}, h_{\mu\nu}) e^{iS[e^\phi h_{\mu\nu}, X]}.$$

(11)

At this point we turn to the construction of a suitable constraint term, $\Delta_{LM}(g_{\mu\nu}, h_{\mu\nu})$. We are guided in this construction by two requirements. First, it must be proportional to $\Delta[\phi]$, in order to remove the integration over $\phi$ in eq. (11) by setting $\phi = 0$. Second, it must remove the $\phi$-independent factor $[J_{FP}]_h$ in this equation, since this does not appear in the original expression eq. (7) for $Z[h_{\mu\nu}]$. These two conditions do not suffice to fix $\Delta_{LM}$ completely, since they leave the freedom to multiply by an arbitrary function which approaches unity as $\phi \to 0$. We use this freedom to combine as many factors of $h_{\mu\nu}$ and $\phi$ together into $g_{\mu\nu}$’s as possible, leading to the following choice

$$\Delta_{LM}[g_{\mu\nu}, h_{\mu\nu}] = \int [\mathcal{D}\Lambda]_g \exp \left\{ -i \int d^2x \left( \sqrt{-g} \mathcal{R}_g - \sqrt{-h} \mathcal{R}_h \right) \Lambda \right\} \frac{\det(-\Box_g)}{[J_{FP}]_g},$$

(12)

so that

$$[J_{FP}]_{e^\phi h} \Delta_{LM}(e^\phi h_{\mu\nu}, h_{\mu\nu}) = \det(-\Box_{e^\phi h}) \int [\mathcal{D}\Lambda]_{e^\phi h} \exp \left\{ -i \int d^2x \sqrt{-h} \left( \Lambda \Box_h \phi \right) \right\}$$

$$= \det(-\Box_h) \int [\mathcal{D}\Lambda]_h \exp \left\{ -i \int d^2x \sqrt{-h} \left( \Lambda \Box_h \phi \right) \right\} e^{-iS_L[h, \phi]}$$

$$= \Delta[\phi] e^{-iS_L[h, \phi]}.$$

(13)
These manipulations use the conformal anomaly, \([\mathcal{D}\Lambda]_{e^{\phi}h} = [\mathcal{D}\Lambda]_h e^{iS_L[h_{\mu\nu},\phi]}\), the relation between curvatures for conformally related metrics in two dimensions \(\sqrt{-g} R_g - \sqrt{-h} R_h = \sqrt{-h} \Box_h \phi\), as well as eq. (9). In the third equality, the quantity \(\Delta[\phi]\) is the functional delta function, and this equation follows from the previous lines after performing the change of variables \(\Lambda \to -\Box_h \Lambda\).

To verify that eq. (11) is equivalent to our starting point eq. (7), we insert the last of eqs. (13) into eq. (11) and use the functional delta function to perform the \(\phi\) path integral, thus setting \(\phi = 0\) everywhere in the integrand. What remains is precisely eq. (7).

In order to obtain the dual we again use eqs. (13) for \(\Delta_{LM}\) in eq. (11), but this time perform the integrals over \(X\) and \(\phi\). We have

\[
Z[h_{\mu\nu}] = \det(-\Box_h) \int [\mathcal{D}X]_h [\mathcal{D}\phi]_h [\mathcal{D}\Lambda]_h \exp \left\{ i \int d^2x \sqrt{-h} \left[ \frac{1}{2} X \Box_h X - \frac{1}{48\pi} \left( \phi \Box_h \phi + R_h \phi \right) - \Lambda \Box_h \phi \right] \right\},
\]

\[
= \det(-\Box_h) \int [\mathcal{D}X]_h [\mathcal{D}\phi]_h [\mathcal{D}\Lambda]_h \exp \left\{ i \int d^2x \sqrt{-h} \left( X \Box_h X - \frac{1}{48\pi} \left( \phi \Box_h \phi + R_h + 48\pi \Lambda \right) \right) \right\},
\]

(14)

where the last equality is obtained from the first by completing the square. Finally, we perform the remaining two Gaussian integrations over \(X\) and over \(\phi\), producing thereby two factors of \(\det(-\Box_h)^{-1/2}\) which precisely cancel the determinant which appears on the right-hand-side of eq. (14). After rescaling \(\Lambda \to \Lambda / \sqrt{48\pi}\), we are left with the dual expression for \(Z[h_{\mu\nu}]\)

\[
Z[h_{\mu\nu}] = \int [\mathcal{D}\Lambda]_h \exp \left\{ \frac{i}{2} \int d^2x \sqrt{-h} \left( \Lambda + \frac{1}{\sqrt{48\pi}} \frac{1}{\Box_h} R_h \right) \Box_h \left( \Lambda + \frac{1}{\sqrt{48\pi}} \frac{1}{\Box_h} R_h \right) \right\}
\]

\[
\rightarrow e^{\phi_\eta} \int [\mathcal{D}\Lambda]_\eta \exp \left\{ \frac{i}{2} \int d^2x \left( \Lambda + \frac{\varphi}{\sqrt{48\pi}} \right) \Box_h \left( \Lambda + \frac{\varphi}{\sqrt{48\pi}} \right) + iS_L[\eta_{\mu\nu}, \varphi] \right\},
\]

(15)

where the last equality applies in a conformally-flat background, \(h_{\mu\nu} = e^{\phi_\eta} \eta_{\mu\nu}\). After
shifting $\Lambda$ to absorb $\frac{1}{\sqrt{48\pi}} \Box^1_{\mu} R_{\mu}$, or equivalently $\frac{1}{\sqrt{48\pi}} \varphi$, we recover in eq. (15) the original massless scalar theory.

We see that although this example has the advantage of extreme simplicity, it has the drawback that spacetime duality does not do anything particularly interesting, simply mapping the massless scalar field theory back onto itself. This drawback is not shared by the next example, to which we now turn.

4. Spacetime Duality II: Bosonization of Fermions

As our next example we apply spacetime duality to a free massless fermion in 1+1 dimensions. Since the dual variable is bosonic, this transformation cannot lead to the same theory as the one with which we start.

We begin with a massless Dirac spinor $\chi$ in the presence of a curved background zweibein $e^a_\mu$ which is related to the background metric in the usual way, $h_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$.

$$Z[e^a_\mu] = \int [D\chi]_e [D\chi]_e \exp \left\{ iS[e^a_\mu, \chi, \chi] \right\},$$
where
$$S[e^a_\mu, \chi, \chi] = -\int d^2 x e^{i\gamma^\mu D_\mu \chi},$$

and $e = \text{det}[e^a_\mu] = \sqrt{-h}$, $\gamma^\mu = \gamma^a e^a_\mu$, and $D_\mu$ denotes the covariant derivative acting on spinors.

As for the scalar case, we start the duality program by introducing a dynamical gravitational field represented by the zweibein $f^a_\mu = (e^a_\mu/2 e^b_\mu \theta^a_b)^{\xi}$, where $\theta^a_b$ is a local Lorentz transformation (LLT) and (as before) $\xi_\mu$ parametrizes diffeomorphisms. The dynamic metric is $g_{\mu\nu} = f^a_\mu f^b_\nu \eta_{ab}$. We are led to the following extended, gauge-invariant, version of eq. (16)

$$Z[e^a_\mu] = \int [D\chi]_f [D\chi]_f [Df^a_\mu]_f \Delta_{LM}[f^a_\mu, e^a_\mu] \exp \left\{ iS[f^a_\mu, \chi, \chi] \right\},$$
where
$$\Delta_{gauge \ fix \ GCT} \Delta_{gauge \ fix \ LLT} [J_{FP}],$$

and the covariant measure for the zweibein integration is given by

$$[Df^a_\mu]_f = [D\phi]_f [D\xi]_f [D\theta^a_b]_f \Delta[\text{gauge fix GCT}] \Delta[\text{gauge fix LLT}] [J_{FP}]_f$$
Choosing conformal gauge, i.e. $\xi^\mu = x^\mu$ and $\theta_b^a = \delta_b^a$, ensures that $f^a_\mu = e^{\phi/2} e^a_\mu$. The path integral eq. (17) becomes

$$Z[e^a_\mu] = \int [D\varphi] e^{\phi/2} e [D\chi] e^{\phi/2} e [D\phi] e^{\phi/2} e [J_{FP}] e^{\phi/2} e \times$$

$$\Delta_{LM}[e^{\phi/2} e^a_\mu, e^a_\mu] \exp \left\{ iS[e^{\phi/2} e^a_\mu, \varphi, \chi, \chi] \right\}. \quad (19)$$

The Fadeev-Popov-DeWitt determinant, $J_{FP}$, now ensures gauge invariance with respect to both GCTs and LLTs, but since LLT gauge fixing is algebraic and the corresponding Jacobian a determinant of an orthogonal matrix, $J_{FP}$ is the same as for the scalar theory of §3.

As usual, the $\phi$ dependence of all measures may be obtained purely by keeping track of the conformal anomaly, which for a Dirac fermion is given by

$$[D\varphi] e^{\phi/2} e [D\chi] e^{\phi/2} e = [D\varphi] e^{\frac{1}{2}S_L[h_{\mu\nu}, \phi]} \quad [D\chi] e^{\phi/2} e = [D\chi] e^{\frac{1}{2}S_L[h_{\mu\nu}, \phi]} \quad (20)$$

Using this directly in eq. (16) and specializing to a conformally-flat metric $e^a_\mu = e^{\varphi/2} \delta^a_\mu$ gives the standard relation between the determinant of the Dirac operator and the determinant of the scalar Laplacian

$$\frac{\det(i\nabla)}{\det(\nabla)} = \frac{Z[e^a_\mu]}{Z[\delta^a_\mu]} = \frac{Z[e^{\varphi/2} \delta^a_\mu]}{Z[\delta^a_\mu]} = e^{iS_L[h_{\mu\nu}, \phi]} = \frac{[\det(-\Box_h)]^{-1/2}}{[\det(-\Box)]^{-1/2}}. \quad (21)$$

We once again choose either eq. (12) or (13) to define $\Delta_{LM}[e^{\phi/2} e^a_\mu, e^a_\mu] = \Delta_{LM}[e^{\phi} h_{\mu\nu}, h_{\mu\nu}]$. After substitution of eq. (13) into eq. (19), the generating functional becomes

$$Z[e^a_\mu] = [\det(-\bigtriangledown_h)] \int [D\varphi] e^{\phi/2} e [D\chi] e^{\phi/2} e [D\phi] e^{\phi/2} e [D\Lambda]_h \times$$

$$\exp \left\{ iS[e^a_\mu, \varphi, \chi, \chi] + iS_L[h_{\mu\nu}, \phi] - i \int d^2 x \sqrt{-h} \phi \Box_h \Lambda \right\}. \quad (22)$$

Performing the $\Lambda$ and $\phi$ integrations, we recover the starting point, eq. (16). Instead, evaluating the path integrals over $\varphi, \chi, \chi$, and $\phi$, the dual theory becomes

$$Z[e^a_\mu] = \det(i\nabla) \det(-\Box_h)^{1/2} \int [D\Lambda]_h \times$$

$$\exp \left\{ \frac{i}{2} \int d^2 x \sqrt{-h} \left( \Lambda + \frac{1}{48\pi} \frac{1}{\Box_h} \mathcal{R}_h \right) \Box_h \left( \Lambda + \frac{1}{48\pi} \frac{1}{\Box_h} \mathcal{R}_h \right) \right\}. \quad (23)$$
Inspection of eq. (21) shows that the $e^a_\mu$ dependence of the functional determinants in front of this expression cancels.

Eq. (23) is the image of the free Dirac fermion under spacetime duality. After a shift, we obtain the path integral for a massless scalar field. This may be regarded as a way of expressing bosonization as a duality transform, which is an alternative to that of ref. [3].

5. Superduality I: The (1,1) Supersymmetric Wess-Zumino Model

We now turn to examples where the spacetime symmetry on which the duality is based is supersymmetry. This introduces the novel feature that some of the dual variables may now be fermions. We present two examples of supersymmetry-based duality, or superduality. In this section we construct the superdual of the (1, 1) supersymmetric Wess-Zumino (WZ) model with the goal of exhibiting the method in the simplest possible setting. We find a result similar to what was found in §3 above: the dual model is the same as the starting theory. For this reason we present a more interesting second example, the (2, 2) supersymmetric WZ model, in the next section.

The general procedure for superduality once more follows the paradigm of §2. Starting with a globally-supersymmetric model, we gauge the global supersymmetry by coupling the model to supergravity, and then enforce a gauge-fixing condition which eliminates the resulting dynamical supergravity (SUGRA) degrees of freedom. We implement this constraint using an entire supermultiplet of Lagrange-multiplier fields. To reach the dual formulation, we integrate the original matter fields and the SUGRA multiplet to leave the path integral in terms of the Lagrange multipliers.

We start with the action of the (1,1)-supersymmetric WZ model in flat Minkowski space, described by a scalar multiplet $(A, \chi, N)$ with action

$$S_0 = \frac{1}{2} \int d^2 x \left( A \Box A - i \overline{\chi} \gamma^\rho \partial_\rho \chi + N^2 \right)$$

(24)

where $A$ and $N$ are real scalar fields and $\chi$ is a Majorana fermion.
In what follows it will be convenient to introduce a compact matrix notation, as in ref. [16], which is inspired by the superspace formulation of supersymmetry. We therefore define

\[ \rho := \begin{pmatrix} A \\ N \\ \chi \end{pmatrix}, \quad \mathbf{T} := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ C & 0 & 0 \end{pmatrix}, \quad \Theta_0 := \begin{pmatrix} 0 & 0 & -i\gamma^\rho \partial_\rho \\ 0 & 1 & 0 \\ \Box & 0 & 0 \end{pmatrix} \] (25)

where \( C \) is the charge conjugation matrix \(-\gamma^0\). With these definitions the action (24) can be rewritten in the compact form

\[ S_0 = \frac{1}{2} \int d^2 x \, \bar{\rho} \Theta_0 \rho, \] (26)

where the conjugate \( \bar{\rho} \) is defined as \( \rho^t T \), the superscript ‘\( t \)’ denoting transposition.

We next couple to a background supergravity multiplet \( B = \{ e_a^\mu, \psi_B^\mu, S^B \} \), in order to have arguments to follow after performing the path integral over the WZ multiplet. The resulting action takes the form

\[ S(\rho, B) = \frac{1}{2} \int d^2 x \, e \left\{ A \Box A - i\chi D_\rho \gamma^\rho \chi + N^2 + i\kappa\chi \gamma^\mu \gamma^\nu \psi_B^\mu \partial_\nu A - \frac{\kappa^2}{8} \chi \psi_B^\nu \gamma^\mu \gamma^\nu \psi_B^\mu \right\}, \] (27)

where \( \kappa \) is the supergravity coupling constant. This last expression can also be rewritten, using matrix notation,

\[ S = \frac{1}{2} \int d^2 x \, \bar{\rho} \Theta \rho, \] (28)

where now

\[ \Theta := |e| \begin{pmatrix} 2i\kappa \gamma^\mu \gamma^\nu \psi_B^\mu \partial_\nu & 0 & -i\gamma^\rho \partial_\rho - \frac{\kappa^2}{8} \psi_B^\mu \gamma^\mu \gamma^\nu \psi_B^\nu \\ 0 & 1 & 0 \\ \Box & 0 & -\frac{i\kappa}{2|e|} \partial_\mu |e| \psi_B^\mu \gamma^\mu \gamma^\nu \end{pmatrix}. \] (29)

(In our notation, derivatives in \( \Theta \) act on everything which stands to their right.) The quantum system of interest is given by the following path integral:

\[ Z(B) \equiv Z[e_a^\mu, \psi_B^\mu, S^B] = \int [\mathcal{D}\rho]_B \exp \{ iS(\rho, B) \} \] (30)
where the measure is required to be locally supersymmetric (see [16] for one possible construction).

In superconformal gauge the supergravity multiplet has the form
\begin{align*}
e^a_\mu &\rightarrow e^a_\phi \delta^a_\mu \\
\psi^B_\mu &\rightarrow \frac{1}{2} \gamma_\mu \psi^B \\
S^B &\rightarrow S^B, \tag{31}
\end{align*}

Because the superconformal gauge supergravity multiplet has the same field content as does a scalar matter multiplet, it is convenient to group these fields into a background-field multiplet $B = (\phi^B, \psi^B, S^B)$. For later use we record the superconformal gauge limit of the operator of eq. (29):

\begin{align*}
\Theta^B_{SC} &= \begin{pmatrix}
0 & 0 & -ie^{\phi^B} \gamma^\rho \partial_\rho \\
0 & e^{2\phi^B} & 0 \\
\Box & 0 & 0
\end{pmatrix} \tag{32}
\end{align*}

Although equation (32) seems to imply that the path integral has lost its dependence on $\psi^B$ and $S^B$, this is not true once the SUGRA-dependence of the path-integral measures is taken into account.

To proceed with the duality construction, we start now with the scalar multiplet $\rho$ coupled to a dynamical supergravity multiplet $T = (f^a_\mu, \psi^T_\mu, S^\nu)$, and a generating functional defined by the path integral

\begin{align*}
Z[B] &= \int [D\rho][D\mu][DT]\Delta_{LM}[T,B] \exp \{iS[\rho,T]\} \tag{33}
\end{align*}

where $\Delta_{LM}$ is the constraint which the duality procedure introduces to trivialize the integral over the dynamical SUGRA multiplet (i.e. by setting $T = B$). As before we use the latitude in choosing this constraint to ensure that all of the SUGRA dependence of the path-integral measures is precisely compensated, ensuring that the original theory is recovered. To describe the $T$ integration in detail, we parametrize the dynamical supergravity fields in terms of diffeomorphisms, Lorentz transformations, and local supersymmetry.

\footnote{Notice we deliberately choose here, for later convenience, a conformal factor for the zweibein which is twice what we used in previous sections.}
transformations acting on the background fields $B$. The integration symbol $[\mathcal{D}T]_T$ stands for integration over the corresponding transformation parameters, as well as a gauge-fixing condition and the Fadeev-Popov determinant $J_{FP}$. This is done most easily, as in the ordinary gravitational case, by choosing superconformal gauge for the background fields and imposing superconformal gauge for the dynamical fields, in which case one has essentially a linear splitting $f^a = e^{\phi^Q} e^a, \psi^T = \delta^a_{\mu} (\psi^B + \psi^Q), S^T = S^B + S^Q$. We group the superconformal gauge dynamical SUGRA fields into another scalar multiplet $Q = (\phi^Q, \psi^Q, S^Q)$. With this choice the total SUGRA multiplet enters into the WZ action only as the sum of a background and a dynamical contribution $B + Q$.

We choose the Lagrange multiplier fields as members of another scalar multiplet $\Lambda = (L, \eta, F)$. We are led to the following representation of eq. (30):

$$Z[B] = \int [\mathcal{D}Q[B + Q][\mathcal{D}Q][\mathcal{D}T[B + Q][\mathcal{D}T] \text{sdet}[\Theta^{B + Q}_{SC}] \exp \left\{ i S[\rho; B + Q] - i \int d^2 x \bar{Q} \Theta^{B + Q}_{SC} \Lambda \right\}$$

(34)

where the superdeterminant is the remnant of $J_{FP} \Delta_{LM}$. Using superconformal invariance one can replace $\Theta^{B + Q}_{SC}$ with $\Theta^{B + Q}_{SC}$ in the Lagrange multiplier term for ease of performing the duality algorithm.

To verify that eqs. (34) and (30) are equivalent, we perform explicitly the $[\mathcal{D}T]$ integral, obtaining the functional delta function $\Delta[\Theta^{B + Q}_{SC}] = \text{sdet}[\Theta^{B + Q}_{SC}]^{-1} \Delta[Q]$. Eq. (30) is then obtained by using this delta function to integrate out $Q$.

To reach the dual formulation, we integrate all fields except the Lagrange multiplier multiplet. The matter multiplet integrates to give $\text{sdet}[\Theta^{B + Q}_{SC}]^{-1/2}$ which changes the power of the superdeterminant term in the path integral. In order to perform the $Q$ integration, we need to make explicit the dependence of the measures on this variable. As in the scalar case, we do so by performing a super-Weyl transformation and rewriting the

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3 This point requires some elaboration: in a general gauge, starting with the dynamical supergravity multiplet one would introduce the background supergravity by a standard, but nonlinear, background-quantum splitting. In superconformal gauge however, where the superspace description is by means of a scalar compensator superfield, the splitting is linear.

4 In superspace we have the relation for the spinor vielbein $E_\alpha (\text{total}) = e^Q E_\alpha (\text{background})$ and $E_T^{-1} R_T - E_B^{-1} R_B = -4 E_B^{-1} \nabla^2 B Q$. The Lagrange multiplier action $\int d^2 x d^2 \theta E_B^{-1} A \nabla B Q$ reduces at the component level to the term in eq. (34).
superdeterminant as in eq. (9)

\[
[DQ]_{B+Q} [D\Lambda]_{B+Q} \text{sdet} \left[ \Theta_{\text{sc}}^{B+Q} \right]^{1/2} = [DQ]_0 [D\Lambda]_0 \text{sdet} \left[ \Theta_0 \right]^{1/2} \exp \left[ iS_{SL}(B + Q) \right],
\]

where the subscript ‘0’ on the measures indicates that all of the dependence on the background SUGRA fields has been made explicit, leaving measures which depend only on the flat metric. Here \( S_{SL}(B + Q) \) denotes the super-Liouville action. For the superconformally-flat SUGRA configurations we are considering (keeping in mind our unconventionally normalized conformal factor, \( g_{\mu\nu} = e^{2\phi} \eta_{\mu\nu} \)) the super-Liouville action is given by [18]:

\[
S_{SL}(B) = -\frac{1}{8\pi} \int d^2 x \left( \frac{1}{2} \phi \Box \phi - i \bar{\psi} \gamma^\mu \partial_\mu \psi + \frac{1}{2} S^2 \right)
= -\frac{1}{16\pi} \int d^2 x B \Theta_0 B.
\]

These steps lead to the following expression

\[
Z[B] = \int [DQ]_0 [D\Lambda]_0 \text{sdet} \left[ \Theta_0 \right]^{1/2} \exp \left\{ -i \int d^2 x \bar{Q} \Theta_0 \Lambda - \frac{i}{16\pi} \int d^2 x (\bar{B} + \bar{Q}) \Theta_0 (B + Q) \right\}
= \int [DQ]_0 [D\Lambda]_0 \text{sdet} \left[ \Theta_0 \right]^{1/2} \exp \left\{ -\frac{i}{16\pi} \int d^2 x \left( \bar{Q} + \bar{B} + 8\pi \bar{\Lambda} \right) \Theta_0 \left( Q + B + 8\pi \Lambda \right) + \frac{i}{16\pi} \int d^2 x \left( \bar{B} + 8\pi \bar{\Lambda} \right) \Theta_0 \left( B + 8\pi \Lambda \right) - \frac{i}{16\pi} \int d^2 x \bar{B} \Theta_0 B \right\},
\]

where the last line is obtained by completing the squares. Performing the \( Q \) path integral removes the \( Q \)-dependent first term from the action, and produces a superdeterminant which cancels the superdeterminant which is already present. Recognizing the last term in the action as the super-Liouville action, we may absorb it to rewrite the \( \Lambda \) measure in terms of the SUGRA background, \( B \). The remaining term is the dual action, which takes a canonical form after appropriately shifting and rescaling the multiplet \( \Lambda \). We are left with the final dual form

\[
Z[B] = \int [D\Lambda]_B \exp \left\{ \frac{i}{2} \int d^2 x \bar{\Lambda} \Theta_{\text{sc}}^B \Lambda \right\}.
\]
Our result is similar to the scalar example of §3; the dual action is identical to the original theory with which we started.

6. Superduality II: The (2, 2) Supersymmetric Wess-Zumino Model

For our final example we choose the simplest model for which the superduality transformation may be explicitly performed, and yet for which the result differs nontrivially from the original model. We consider the (2, 2) generalization of the previous example – a single massless WZ multiplet coupled only to background SUGRA fields. The feature which makes the (2,2) example more interesting is the fact that (2,2) scalar multiplets come in more than one type [19], [20], [22]. We shall find that superduality can map one type of multiplet into another.

The basic matter multiplet for (2,2) supersymmetry has field content \((\phi, \psi, F)\), where \(\phi\) and \(F\) are complex scalars and \(\psi\) is a Dirac spinor. In (2,2) superspace these fields may be grouped into a scalar superfield \(\Phi\) satisfying a supersymmetric constraint. Two types of constraints are relevant for us. For global (2,2) supersymmetry a chiral scalar superfield is one which satisfies \(D_+ \Phi = D_- \Phi = 0\). Here \(D_\pm\) are the supercovariant spinor derivatives, whose complex conjugates are denoted \(D_{\dot{\pm}}\). A twisted-chiral scalar superfield is defined by \(D_+ \Phi = D_- \Phi = 0\). Similar definitions apply for local (2,2) supersymmetry, with the derivatives \(D_\alpha\) and \(D_{\dot{\alpha}}\) replaced by their locally supercovariant counterparts, \(\nabla_\alpha\) and \(\nabla_{\dot{\alpha}}\).

To apply the duality algorithm we require the coupling of these multiplets to (2,2) supergravity. The irreducible (2,2) supergravity multiplet has the field content \((e_\mu^a, \xi_\mu, A_\mu, G)\), where \(e_\mu^a\) is the zweibein, \(\xi_\mu\) is a Dirac gravitino, \(G\) is a complex scalar auxiliary field and \(A_\mu\) is a gauge potential. \(A_\mu\) gauges one of the two \(U(1)\) internal symmetries of the (2,2) supersymmetry algebra. These symmetries act as vector and axial symmetries on the various fermions within (2,2) supermultiplets. There are two distinct (2,2) supergravity multiplets, denoted by \(U_V(1)\) or \(U_A(1)\) supergravity depending on which one of these symmetries is gauged by the field \(A_\mu\) [23], [24]. For both of these supergravities the Ricci scalar \(R(e_\mu^a)\) lies within a scalar superfield \(R\). For \(U_A(1)\) supergravity this superfield is chiral, while for \(U_V(1)\) supergravity it is twisted-chiral.
In principle there are four \((2,2)\) WZ models to consider, depending on whether the matter multiplet is chiral \((C)\) or twisted-chiral \((T)\), and whether we use \(U_A(1)\) or \(U_V(1)\) supergravity. We denote these four possibilities by \(CA\), \(CV\), \(TA\) and \(TV\). The lagrangian for each of these four possibilities is known, both in superspace \([21], [22]\) and in components \([17], [25]\). In superspace the action distinguishes between the four cases, whereas the component action (in the absence of masses and self-interactions) does not. It is given by

\[
S = \frac{1}{2} \int d^2x|e| \left\{-g^{\mu\nu}\partial_\mu \phi \partial_\nu \phi^* - i\bar{\psi}\gamma^\mu D_\mu \psi - 2A_\mu \bar{\psi}\gamma^\mu \psi - 2 \left(\partial_\mu \phi^* + \bar{\psi}\xi_\mu\right) \bar{\xi}_\rho \gamma^\mu \gamma^\rho \psi + \\
-2 \left(\partial_\mu \phi + \bar{\xi}_\mu \psi\right) \bar{\psi}\gamma^\rho \gamma^\mu \xi_\rho + F^* F\right\}.
\]

(39)

The rest of this section applies superduality to the four versions of the massless model with no self-couplings. We show that superduality interchanges these multiplets in the following way

\[
\begin{align*}
CA & \rightarrow CA & TA & \rightarrow CA \\
CV & \rightarrow TV & TV & \rightarrow TV.
\end{align*}
\]

(40)

Because the component actions do not distinguish between the various kinds of multiplets, we examine the consequences of superduality in a superspace formulation. There are, however, some subtleties concerning the path-integral measures for \((2,2)\) superspace \([26]\), and consequently we shall proceed in two steps. In order to show that the duality procedure produces a well-defined and local result — which need not be generally true — we first perform the duality algorithm on the component action, keeping explicit track of the cancellation of all nonlocal functional determinants. We follow this component calculation by presenting the duality argument directly in superspace yielding the results as indicated in eq. (40).

6.1) \((2,2)\) Superduality in Components

As in previous sections, we start with matter coupled to background SUGRA

\[
Z(B) = \int [\mathcal{D}(M, \bar{M})]_B \exp \left\{+i \int d^2x \bar{M} \Theta^B M\right\},
\]

(41)
where the action is taken from eq. (39), but is written in a compact notation which is similar to that used in the (1,1) case. $M$ collectively denotes the matter multiplet, $(\phi, \psi, F)$, and a bar on $M$ indicates hermitian conjugation and multiplication by a generalization of the charge conjugation matrix $T$. The operator $\Theta^B$ depends on the background supergravity multiplet $(e^a_\mu, \xi_\mu, A_\mu, G)$. In superconformal gauge $^5$ this multiplet reduces to the conformal factor $\sigma$ of the metric, a Dirac spinor $\lambda$ from the gravitino, $\xi_\mu = \frac{1}{2} \gamma_\mu \lambda$, the transverse component of the gauge field, $A_\mu = \frac{1}{2} \epsilon_{\mu\nu} \partial^\nu \rho$, and the auxiliary field $G$, which we collectively denote by $B = (\sigma, \lambda, \rho, G)$. The fields $\sigma, \rho$ can be combined into a single complex scalar field. Notice that the SUGRA field content in this gauge is the same as for the matter multiplet.

We imagine the $B$-dependence of the measure in eq. (41) to be defined to ensure invariance with respect to local (2,2) supersymmetry. We can infer this dependence from the superconformal anomaly of this model [17]: \[ [\mathcal{D}(M, \bar{M})]_B = [\mathcal{D}(M, \bar{M})]_0 \exp\left\{ iS_{SL}(B) \right\}, \]
where $^6$

\[
S_{SL}(B, \bar{B}) = -\frac{1}{4\pi} \int d^2x \left( -\frac{1}{2} \partial_a \sigma \partial^a \sigma - \frac{1}{2} \partial_a \rho \partial^a \rho - \frac{i}{2} \bar{\lambda} \gamma^a \partial_a \lambda + \frac{1}{2} G^a G \right) = -\frac{1}{4\pi} \int d^2x \, \bar{B} \Theta_0 B,
\]
which is once again simply proportional to the kinetic action for a matter multiplet.

Superduality proceeds by introducing dynamical supergravity fields. After imposing superconformal gauge we are left with the matter multiplet coupled to a total SUGRA multiplet $B + Q$. We must also constrain away the $Q$ degrees of freedom, to ensure consistency with the original form, eq. (41). We do so using a multiplet of Lagrange multipliers, denoted $\Lambda = (L_1 + iL_2, \eta, \mathcal{G})$. Note that $\Lambda$ has the same field content as have the superconformal supergravity multiplets, $B, Q$, and the matter multiplet, $M$. We are

---

$^5$ The terminology is not strictly correct. In superspace, superconformal gauge is defined by setting the prepotential $H^a$ to zero and keeping the conformal compensator superfield. Here one first goes to WZ gauge by setting the conformal compensator to 1 and gauging away some components of $H^a$, and then going to component conformal gauge. See also the discussion in appendix B of [27] .

$^6$ See appendix for a discussion of the sign of the second term.
led in this way to the expression:

\[
Z(B, \bar{B}) = \int \left[ \mathcal{D}(M, \bar{M}) \right]_{B+Q} \left[ \mathcal{D}(Q, \bar{Q}) \right]_{B+Q} \left[ \mathcal{D}(\Lambda, \bar{\Lambda}) \right]_{B+Q} (\text{sdet} [\Theta^{B+Q}_{SC}])^2 \times \exp \left\{ + i \int d^2x \bar{M} \Theta^{B+Q}_{SC} M + i \int d^2x (\bar{Q} \Theta^{B}_{SC} \Lambda + \bar{\Lambda} \Theta^{B}_{SC} Q) \right\}. \tag{43}
\]

Here the subscript on the field operator \( \Theta_{SC} \) is a reminder that it is obtained from eq. (39) in superconformal gauge. Again, because of superconformal invariance of the action, one can replace \( \Theta^{B}_{SC} \) by \( \Theta^{B+Q}_{SC} \) in the Lagrange multiplier term.

To perform the relevant path integrals, one can make all SUGRA dependence explicit by performing a superconformal transformation. When this is done for eq. (43), the transformation of the measures for \( Q \) and \( \Lambda \) cancels a factor from the superdeterminant, leaving only a single factor of the anomaly action coming from \( \left[ \mathcal{D}(M, \bar{M}) \right] \). We find

\[
Z(B, \bar{B}) = \int \left[ \mathcal{D}(M, \bar{M}) \right]_0 \left[ \mathcal{D}(Q, \bar{Q}) \right]_0 \left[ \mathcal{D}(\Lambda, \bar{\Lambda}) \right]_0 (\text{sdet} [\Theta_0])^2 \times \exp \left\{ + i \int d^2x \bar{M} \Theta_0 M + i \int d^2x (\bar{Q} \Theta_0 \Lambda + \bar{\Lambda} \Theta_0 Q) + i S_{SL}(B + Q) \right\}. \tag{44}
\]

We first verify the equivalence of eq. (44) with our starting expression, eq. (41). To do so, we change variables \( \Lambda \rightarrow \Theta_0 \Lambda \), and integrate to obtain the functional delta function \( \Delta[Q] \). Using this to perform the \( Q \) integration leaves

\[
Z(B, \bar{B}) = \int \left[ \mathcal{D}(M, \bar{M}) \right]_0 \exp \left\{ + i \int d^2x \bar{M} \Theta_0 M + i S_{SL}(B + Q) \right\}. \tag{45}
\]

This reproduces eq. (41) once \( S_{SL}(B) \) is rescaled into the measure using a superconformal transformation.

To obtain the dual, we evaluate the path integral over \( (M, \bar{M}) \) to obtain \( \text{sdet} [\Theta_0]^{-1} \) which partially cancels the explicit superdeterminant already present in eq. (44). This leaves

\[
Z(B, \bar{B}) = \int \left[ \mathcal{D}(Q, \bar{Q}) \right]_0 \left[ \mathcal{D}(\Lambda, \bar{\Lambda}) \right]_0 \text{sdet} [\Theta_0] \times \exp \left\{ i \int d^2x (\bar{Q} \Theta_0 \Lambda + \bar{\Lambda} \Theta_0 Q) - \frac{i}{4\pi} \int d^2x (\bar{B} + \bar{Q}) \Theta_0 (B + Q) \right\}. \tag{46}
\]
Completing the squares in the exponent allows it to be rewritten as

\[
iS = -\frac{i}{4\pi} \int d^2x \left( Q + \bar{B} - 4\pi \Lambda \right) \Theta_0 \left( Q + B - 4\pi \Lambda \right) + \frac{i}{4\pi} \int d^2x \left( B - 4\pi \Lambda \right) \Theta_0 \left( B - 4\pi \Lambda \right) - \frac{i}{4\pi} \int d^2x \bar{B} \Theta_0 B
\]

(47)

Rescaling \( \Lambda \to \Lambda/\sqrt{4\pi} \), and performing the \( Q \) path integration, we obtain

\[
Z(B, \bar{B}) = \int [\mathcal{D}(\Lambda, \bar{\Lambda})]_0 \exp \left\{ -\frac{i}{4\pi} \int d^2x \bar{B} \Theta_0 B \right\} \times \exp \left\{ i \int d^2x \left( \bar{\Lambda} - \frac{\bar{B}}{\sqrt{4\pi}} \right) \Theta_0 \left( \Lambda - \frac{B}{\sqrt{4\pi}} \right) \right\},
\]

(48)

which, after shifting the dual multiplet \( \Lambda \) to absorb \( B/\sqrt{4\pi} \) and rescaling the first exponential back into the measure \( [\mathcal{D}(\Lambda, \bar{\Lambda})]_0 \), we recognize as the generating functional for a massless \((2,2)\) matter multiplet:

\[
Z(B, \bar{B}) = \int [\mathcal{D}(\Lambda, \bar{\Lambda})]_B \exp \left\{ i \int d^2x \bar{\Lambda} \Theta_{SC}^\Lambda \right\}
\]

(49)

Superficially, it appears that we have obtained our starting action, eq. (41), although this need not be true because the component action does not distinguish between chiral and twisted-chiral multiplets. The distinction might become visible if one were to use more complicated actions, such as having more than one matter multiplet, or including self-interactions. Alternatively, one can examine the situation in superspace, as we do next.

6.2) Superduality in Superspace

To see how the mappings in eq. (40) arise we examine the \((2,2)\)-invariant form of the Lagrange multiplier action used in §3, which involved terms of the form \( \sqrt{-g} \Lambda \mathcal{R}_g \). Recall that in superspace \( \mathcal{R}_g \) lives in a scalar superfield \( R \), which is chiral in \( U_A(1) \) supergravity and twisted-chiral in \( U_V(1) \) supergravity. The \((2,2)\)-supersymmetric generalization of the condition that imposes \( \sqrt{-g} \mathcal{R}_g - \sqrt{-h} \mathcal{R}_h = 0 \) involves a Lagrange-multiplier
superfield with the same chirality properties as $R$ itself, as follows: In $U_A(1)$ supergravity and superconformal gauge we write for the full superspace vielbein $E^T_\alpha = e^Q E_B$, where the chiral superfield $Q$ plays the analogous role to $\phi$ in §3. Thus, the constraint $E^{-1}_T R_T - E^{-1}_B R_B = E^{-1}_B \bar{\nabla}\bar{Q} = 0$ ($E$ is the density superfield which supercovariantizes the chiral integration) is enforced by a chiral integral

$$\int d^2x d^2\theta E^{-1}_B \Lambda \nabla^2 B \bar{Q} = \int d^2x d^4\theta E^{-1} \Lambda \bar{Q}$$

where $\Lambda$ is a chiral superfield and $E^{-1}$ is the (nonchiral) superdeterminant of the vielbein for the full superspace integral.\(^7\) Correspondingly in $U_V(1)$ theory, $R$, the Lagrange multiplier $\Lambda$, and the density are all twisted chiral. Because the Lagrange multipliers end up being the dual fields, superduality takes both $C$ and $T$ multiplets into $C$ multiplets in $U_A(1)$ supergravity. Conversely in $U_V(1)$ supergravity, superduality always produces a $T$ multiplet as in eq. (40). We now demonstrate this explicitly.

Consider, for example, a twisted-chiral superfield $\mathcal{X}$ in $U_A(1)$ supergravity. The invariant generating functional is defined by

$$Z = \int \left[D(\mathcal{X}, \bar{\mathcal{X}})\right]_{E_A} \exp \left\{ -i \int d^2x d^4\theta E^{-1} \bar{\mathcal{X}} \mathcal{X} \right\}$$

Performing the functional integral over $\mathcal{X}$ involves some subtleties that we will not address here (see however [26]). In $U_A(1)$ supergravity one finds [22]:

$$Z = [\det \Box_+]^{-\frac{1}{2}} = \exp \left[ \frac{i}{8\pi} \int d^2x d^4\theta E^{-1} \left( \bar{R} \frac{1}{\Box_+} R \right) \right] \to \exp \left[ -\frac{2i}{\pi} \int d^2x d^4\theta \bar{\Sigma} \Sigma \right], \quad \text{(in superconformal gauge),}$$

Here $\Box_+ = \nabla^2 \nabla^2$ is the superspace d’Alembertian acting on a chiral superfield and, generically in superconformal gauge, $R = -4 \nabla^2 \Sigma$, $\bar{R} = 4 \nabla^2 \Sigma$, where $\Sigma$ is a chiral (compensator) superfield (denoted $\sigma$ in refs. [21], [22]).

\(^7\) Our conventions in this section follow those of refs. [21] and [22].
We now start dualizing. We choose the gauge for dynamical supergravity such that
$E_\alpha (\text{dynamical}) = e^Q E_\alpha (\text{background})$ and in addition we put the background in conformal
gauge $E_\alpha (\text{background}) = e^B D_\alpha$ so that again we are dealing with a linear splitting $B + Q$
in terms of chiral superfields. As discussed above, we also introduce a Lagrange-multiplier
chiral superfield $\Lambda$ as in eq. (50), We choose the following gauged version of eq. (51):

$$Z(B, \bar{B}) = \int \left[ \mathcal{D}(X, \bar{X}) \right]_{B+Q} \left[ \mathcal{D}(Q, \bar{Q}) \right]_{B+Q} \left[ \mathcal{D}(\Lambda, \bar{\Lambda}) \right]_{B+Q} \det \left[ \square^+ + Q \right]$$

$$\times \exp \left\{ -i \int \! d^2 x \, d^4 \theta \, \bar{X} \dot{X} + i \int \! d^2 x \, d^4 \theta \, (\bar{Q} \Lambda + \Lambda \bar{Q}) \right\}$$

(53)

where the determinant was chosen by the requirement that eq. (53) reduce to eq. (51), once
the $\Lambda$ and $Q$ are functionally integrated. To see this, one rewrites the Lagrange multiplier
term as $i \int d^2 x d^2 \theta \, \mathcal{E}^{-1} \bar{Q} \bar{\nabla}^2 \Lambda + i \int d^2 x d^2 \theta \, \bar{\mathcal{E}}^{-1} \bar{Q} \nabla^2 \Lambda$ and makes the change of variables
$\Lambda \to \bar{\nabla}^2 \Lambda', \bar{\Lambda} \to \nabla^2 \Lambda'$ with Jacobian $(\det \nabla^2 \times \det \nabla^2)^{-1} = (\det \square^+)^{-1}$. This cancels the
determinant factor already present and produces delta-functions that set $Q = 0$, etc. This
guarantees the required equivalence of eqs. (53) and (51).

To reach the dual formulation we instead integrate $X$ and $Q$, leaving only $\Lambda$ as the dual
field. As in eq. (52), the $X$ integral gives a factor of $(\det \left[ \square^+ + Q \right])^{-\frac{1}{2}}$ and Weyl rescaling the $Q$
and $\Lambda$ measures gives $[\mathcal{D}(Q, \bar{Q})]_{B+Q} [\mathcal{D}(\Lambda, \bar{\Lambda})]_{B+Q} = [\mathcal{D}(Q, \bar{Q})]_0 [\mathcal{D}(\Lambda, \bar{\Lambda})]_0 (\det \left[ \square^+ + Q \right])^{-1}$. We obtain

$$Z(B, \bar{B}) = \int \left[ \mathcal{D}(Q, \bar{Q}) \right]_0 \left[ \mathcal{D}(\Lambda, \bar{\Lambda}) \right]_0 \exp \left\{ i \int \! d^2 x \, d^4 \theta \, (\bar{Q} \Lambda + \Lambda \bar{Q}) \right\} \times \exp \left\{ -\frac{2i}{\pi} \int \! d^2 x \, d^4 \theta \, (\bar{Q} + \bar{B})(Q + B) \right\}$$

(54)

where the last exponential is a representation of $(\det \left[ \square^+ + Q \right])^{-\frac{1}{2}}$, c.f. eq. (52).

Suitably completing squares, performing the Gaussian $Q$ integral and rescaling $\Lambda$
finally gives:

$$Z(B, \bar{B}) = \int \left[ \mathcal{D}(\Lambda, \bar{\Lambda}) \right]_{(B)} \exp \left\{ i \int \! d^2 x d^4 \theta \left( \bar{\Lambda} - \frac{2B}{\pi} \right) \left( \Lambda - \frac{2B}{\pi} \right) \right\}$$

(55)
which we recognize, after a shift, as the superspace generating functional for the massless chiral multiplet $\Lambda$.

We have transformed the twisted-chiral starting multiplet, $\mathcal{X}$, into a chiral one. As is clear from the derivation, so long as we stick to $U_A(1)$ supergravity, a chiral dual variable $\Lambda$ would also have followed if we had started with chiral $\Phi$. Similar manipulations for $U_V(1)$ supergravity fill out the rest of the relationships of eq. (40).

7. Conclusions

With this last calculation we have completed what we set out to accomplish. Four simple (1+1)-dimensional examples have been the vehicles for showing how duality transformations can be performed based on spacetime symmetries, rather than on internal symmetries.

For the examples of §3 and §4, the duality transformation was based on the general covariance of the starting model. In §3, application to a free scalar field gave a trivial result: the dual and starting models are identical. A nontrivial result is obtained in §4 by starting with a free fermion. Here the dual is a free boson, giving bosonization in yet another guise.

The examples in §5 and §6 focus on supersymmetric models, where supersymmetry is used as the symmetry on which duality is based. When applied to the massless (1,1)-supersymmetric WZ model in §5, we find the dual is equivalent to the starting theory. More interesting consequences arise when superduality is applied to (2,2)-invariant models in §6. In this case we find that both chiral and twisted-chiral multiplets dualize to chiral multiplets if the $U_A(1)$ formulation of supergravity is used. Using instead $U_V(1)$ supergravity one finds both chiral and twisted-chiral multiplets go to a twisted-chiral dual. Because mirror symmetry for Calabi-Yau manifolds involves the interchange of chiral and twisted-chiral multiplets, it would be interesting to find a closer connection of our results with this case, by investigating superduality for self-interacting (2,2) scalar multiplets.
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Appendix A. Signs in the super-Liouville Action

We discuss here a sign mismatch between the component (2, 2) Liouville action that we have used in this work, and the one in refs. [29]. The difference lies in the relative sign between, for example, the $\partial_\mu \rho \partial^\mu \rho$ and $\partial_\mu \sigma \partial^\mu \sigma$ term in eq. (42). In our action all signs are the same, as also follows from superspace considerations, giving an anomaly multiplet which is proportional to (but opposite in overall sign from) the kinetic action of a matter multiplet. By contrast, the results in refs. [29] assign the opposite sign to the $\partial_\mu \rho \partial^\mu \rho$ term of the anomaly action.

We believe this discrepancy has to do with the interpretation of the anomaly action, which was used in the euclidean-signature calculations of refs. [29]. To see what is going on we note that the $\rho$-dependent term really starts its life as the two-dimensional anomaly action for a gauge potential, $A_\mu$ [30]:

$$L_{\text{anom}} = \frac{1}{4\pi} F^{\mu\nu} \left( \frac{1}{\Box} \right) F_{\mu\nu}. \quad (56)$$

Eq. (42) is obtained from this when $A_\mu$ is restricted (in Minkowski signature) to be transverse:

$$A_\mu = \frac{1}{2} \epsilon_{\mu\nu} \partial^\nu \rho. \quad (57)$$

Now comes the key point. If the same replacement, eq. (57), were made in Euclidean signature, as is done in refs. [29], then one instead obtains a $\rho$ kinetic term having the
opposite sign. One obtains opposite-sign actions depending on whether or not eq. (57) is applied in Minkowski or Euclidean signature.

Which sign is correct? In a string-theory context, where the fundamental path-integral formulation involves the Euclidean action, one is led to the assignments of refs. [29]. On the other hand, if the fundamental theory is defined as a Minkowski space field theory, it is the Minkowski sign which is right. Euclidean conventions are generally defined to reproduce Minkowski-space results. The correct substitution which restricts $A_\mu$ to be transverse in Euclidean signature is

$$A_m = \frac{i}{2} \epsilon_{mn} \partial^n \rho,$$

(58)

where the key difference from eq. (57) is the factor of ‘$i$’. Besides ensuring the equivalence of the Minkowski- and Euclidean-signature anomaly actions, this factor of ‘$i$’ is required for the unitarity of the Euclidean action. That is, terms linear in $A_m$, such as $i \bar{\psi} \gamma^m \psi A_m$, do not satisfy the Osterwalder-Schraeder (OS) positivity condition [31] unless eq. (58) is used instead of eq. (57). (The OS condition is the Euclidean equivalent of the Minkowski-signature condition of the reality of the action, as required by unitarity. A similar argument in four dimensions implies the standard result that the $C P$-violating $\theta$-term of QCD has an imaginary coefficient in Euclidean signature.) We conclude that our eq. (42) is the correct expression for the super-Liouville action.
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