Distributional Metrics and the Action Principle of Einstein-Hilbert Gravity

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Abstract

In this work, a subclass of the generalized Kerr-Schild class of space-times is specified, with respect to which the Ricci tensor (regardless of the position of indices) proves to be linear in the so-called profile function of the geometry. Considering Colombeau’s nonlinear theory of generalized functions, this result is extended to apply to an associated class of distributional Kerr-Schild geometries, and then used to formulate a variational principle for these singular spacetimes. More specifically, it is shown in this regard that a variation of a suitably regularized Einstein-Hilbert action can be performed even if the metric of one of the corresponding generalized Kerr-Schild representatives contains a generalized delta function that converges in a suitable limit to a delta distribution.

Key words: generalized Kerr-Schild class, Colombeau algebra, action principle

Introduction

The general theory of relativity is a nonlinear theory of gravity. The mathematical theory of distributions, on the other hand, is a linear theory that uses a variety of techniques which cannot be implemented in the nonlinear framework of Lorentzian geometry. This is also true with regard to Colombeau’s nonlinear theory of generalized functions [8, 9, 14, 17, 25], which, though capable of solving an impressive spectrum of problems associated with the treatment of distributions in gravitational physics does not always allow a rigorous treatment of the simultaneously singular and nonlinear field equations of theory.

In view of this fact, it is surprising that in Einstein’s theory well-defined distributional metrics and curvature expressions have been found in the past. These offer the possibility of characterizing singular energy-momentum distributions and thus can be used to solve the highly non-trivial problem of how to describe gravitational fields of point-like particles in general relativity.

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Perhaps the most prominent classes of solutions of Einstein’s equations characterizing the gravitational fields of said point-like sources are, on the one hand, the class of impulsive pp-wave spacetimes and, on the other hand, the class of gravitational shock wave spacetimes in black hole and cosmological backgrounds. The most famous representatives of the first class are the Aichelburg-Sexl geometry [1] and the models of Lousto and Sanchez [28], which arose as ultrarelativistic limits of boosted black hole spacetimes of Schwarzschild and Reissner-Nordström geometries. The most famous representatives of the second class, on the other hand, are the shock wave geometries found by Dray and ’t Hooft [10] and Sfetsos [44], which, as recently discovered, can be obtained as a special case of a more general class of gravitational shock wave geometries in stationary black hole backgrounds [20].

The representatives of both of these classes have in common that they have a delta-like profile, which implies that their geometries are regular everywhere, with the exception of a special null hypersurface on which the confined field of the ultrarelativistic particle is concentrated. Unfortunately, due to the nonlinear structure of the field equations of the theory, said representatives also have in common that they lead to very problematic curvature expressions, which are pathological insofar as they contain ill-defined products of delta distributions.

More specifically, ill-defined ‘squares of the delta distribution’ have occurred in the past in the course of the calculation of gravitational shock wave spacetimes in black hole backgrounds as a result of the (rather careless) application of Penrose’s ‘scissors-and-paste’ method [38]. This left the authors of the cited works with no choice but to ‘blithely ignore’ all problematic terms and solve only the remaining meaningful part of Einstein’s field equations.

This, of course, did not solve the problem. Fortunately, however, a few years later it became apparent that the generalized Kerr-Schild framework can be used to show that (in a specific sense) the results obtained are meaningful not only from a physical, but also from a mathematical point of view [2, 4]. The main reason for this is that the said geometric framework has the special property that the mixed deformed Einstein tensor is linear in the so-called profile function of geometry; a circumstance that can be used to both determine the geometric structure of the metric of spacetime and to avoid the dangerous ill-defined terms arising in the course of the calculation. This trick was then also used to determine the exact structure of the shock wave geometries calculated in [20].

However, the problem with this idea of treating the subject is that it still proves to be impossible to deduce the corresponding distributional geometries from Einstein’s equations with lowered and raised indices. In addition, it turns out to be impossible to obtain the field equations by varying the Einstein-Hilbert action, so that one could get the impression that there are specific models in general relativity that cannot be deduced from the principle of stationary action.

In response to this particular shortcoming of the theory, it shall be proven in the following that both the class of impulsive pp-wave spacetimes as well as the class of gravitational shock wave spacetimes are contained in the superordinate class of spacetimes, which represents a specific subclass of the generalized Kerr-
Schild class, to be discussed in section one of this work. The representatives of this class share the main geometric properties that their associated Kerr-Schild null vector fields are the generators of a foliation of null hypersurfaces and that the corresponding profile functions remain constant under the flow of these Kerr-Schild generators. This has the effect that not only the Ricci tensor with mixed indices, but also that with lowered and raised indices are linear in the profile function of the geometry. Based on these findings, it is concluded that not only can corresponding linear field equations be formulated, but also that it should be possible to perform a variation of the Einstein-Hilbert action with a degree of mathematical rigor sufficiently high to obtain a meaningful result in the end.

The strategy to perform said variation, which is explored in section two of this work, is to use methods from the theory of Colombeau algebras of manifold-valued generalized functions; a mathematical theory tailor-made for dealing with singular problems in various geometrical settings used in gravitational physics. More precisely, the main idea in this context is not to consider a delta distribution directly, but to consider instead a so-called strict delta net, which converges to said distribution in a suitable limit. The consideration of such a delta net proves both beneficial and necessary in this regard owing to the fact that it allows one to derive mathematically well-defined distributional field equations whose distributional character emerges (in the limit previously mentioned) only after the variation of the regularized Einstein-Hilbert action has been completed. The avoidance of encountering any ill-defined 'delta-square' terms is thereby achieved via focussing exclusively on generalized Kerr-Schild spacetimes with linear deformed Ricci tensor.

The physical significance of the approach is illustrated using special, selected examples of distributional spacetimes in the third and final section of this work. The focus is placed, however, exclusively on impulsive pp-wave spacetimes and gravitational shock wave spacetimes in black hole and cosmological backgrounds.

1 Generalized Kerr-Schild Framework

All spacetimes to be considered in this work are generalized Kerr-Schild spacetimes belonging to the so-called generalized Kerr-Schild class. Spacetimes lying in this class have the property that their metrics can be decomposed in the form

\[ \tilde{g}_{ab} = g_{ab} + f l_al_b, \]

where \( g_{ab} \) represents the so-called background or seed metric, \( f \) represents the so-called profile function and \( l_a = g_{ab}l^b \) the so-called Kerr-Schild null co-vector field. An important factor in this decomposition is that the associated Kerr-Schild null vector field \( l^a \) is given in such a way that the null geometric constraints \( \tilde{g}_{ab}l^a_l^b = g_{ab}l^a_l^b = 0 \) and \( (l\nabla)l^a = (l\nabla)l^b = 0 \) are met, so that a congruence of null geodesics is formed whose associated vector field \( l^a \) is to be assumed as affinely parametrized for simplicity’s sake.
The main property of any generalized Kerr-Schild class spacetime is that its mixed Ricci tensor

\[ \tilde{R}_{ab} = R_{ab} - \frac{1}{2} f R_c{}^c l^a l^b - \frac{1}{2} f R^c{}_{b} l^c l^a + \frac{1}{2} \nabla_c \nabla^a (f l^c l^b) + \nabla^c \nabla_b (f l_c l^a) - \nabla_c \nabla^c (f l^a l_b), \]  

its Ricci scalar

\[ \tilde{R} = R - f R^d{}_c l_d l^c + \nabla_d \nabla^d (f l^c l^a), \]  

and therefore its mixed Einstein tensor

\[ \tilde{G}^a{}_{b} = \tilde{R}^a{}_{b} - \frac{1}{2} \tilde{R} \tilde{g}^a{}_{b} = G^a{}_{b} + \rho^a{}_{b}, \]  

with

\[ \rho^a{}_{b} = - \frac{1}{2} f R_c{}^c l^a l_b - \frac{1}{2} f R^c{}_{b} l_c l^a + \frac{1}{2} \delta^a{}_{b} (f R^d{}_{c} l_d l^c - \nabla_d \nabla^c (f l^d l_c)) + \frac{1}{2} (\nabla^a (f l^c l_b) + \nabla^c \nabla_b (f l_c l^a) - \nabla_c \nabla^c (f l^a l_b)) \]  

are all linear in the profile function \( f \). Obviously, this particular property of the generalized Kerr-Schild framework proves to be extremely useful in practice for finding exact solutions to Einstein’s equations.

Obviously, decomposition relation (1) allows one to associate two different metrical structures \( \tilde{g}_{ab} \) and \( g_{ab} \) with a given manifold \( \tilde{M} \). The situation is similar with the inverse metrics, which are related with each other by means of the decomposition relation

\[ \tilde{g}^{ab} = g^{ab} - f l^a l^b. \]  

However, the Ricci and Einstein tensors with raised and lowered indices unfortunately do not have the same remarkable properties. Instead, they usually turn out to be nonlinear in the profile function \( f \). Of course, the same holds true for the Riemann curvature tensor of the geometry.

Nevertheless, it would be desirable to know under which exact circumstances at least the Ricci with lowered and raised indices is linear in \( f \). To investigate this, one may use relations (1) and (6) to set up the affine connection

\[ C^a{}_{bc} = \frac{1}{2} \nabla_b (f l^a l_c) + \frac{1}{2} \nabla_c (f l^a l_b) - \frac{1}{2} \nabla^a (f l_b l_c) + \frac{1}{2} f D f l^a l_b l_c, \]  

which relates the pair of covariant derivative operators \( \tilde{\nabla}_a \) associated with \( \tilde{g}_{ab} \) and \( \nabla_a \) and associated with \( g_{ab} \). Given this definition, the deformed Riemann curvature tensor of the geometry can be set up in the next step, which has the form

\[ \tilde{R}^a{}_{bcd} = R^a{}_{bcd} + E^a{}_{bcd}, \]
provided that the abbreviation $E_{bcd} = 2 \nabla_c C_{ab} + 2 C_{ad} C_{cb}$ is used in the present context. Using the fact that $C_{ab} = 0$ holds true with respect to any generalized Kerr-Schild class metric, the deformed Ricci tensor with lowered indices reads

$$\tilde{R}_{ab} = R_{ab} + E_{ab},$$

(9)

where $E_{ab} = \nabla_c C_{ac} + C_{ad} C_{cb}$ applies.

To ensure that this object as well as the Einstein tensor of the geometry are linear in the profile function $f$, it must be ensured that the conditions

$$C_{ad} C_{cb} \equiv 0$$

(10)

and

$$\nabla_c C_{ab} l^a \equiv \nabla_c C_{ab} l^b \equiv 0$$

(11)

as well as

$$\tilde{R}_{ab} l^a l^b \equiv 0$$

(12)

are met, where the conditions (11) and (12) result from the consistency condition

$$\tilde{R}_{ab} = \tilde{g}_{ac} \tilde{R}_{cb}.$$ 

However, using the result

$$C_{ad} C_{cb} = \frac{1}{2} \left\{ (l \nabla) f \right\}^2 + f^2 \nabla^c l^d \nabla_d l^c + f^2 \nabla_{[a} l_{b]} \nabla^{a} l^{d} \right\} l_{a} l_{b},$$

(13)

it immediately becomes clear that both the Ricci and the Einstein tensors with lowered indices are linear in the profile function if the conditions

$$\tilde{\nabla}_{[alb]} = \nabla_{[alb]} \equiv 0$$

(14)

and

$$L_l f = (l \nabla) f \equiv 0$$

(15)

are met, where $L_l$ is the Lie derivative with respect to $l^a$. It must therefore be assumed that the profile function of the geometry can be selected exactly in such a way that it vanishes along the flow of the vector field $l^a$. Furthermore, said vector field must be chosen exactly in such a way that it represents the generator of a foliation of spacetime in lightlike hypersurfaces, generally referred to as null foliation of spacetime\footnote{Note that a large number of different constructions of said null foliations has been given in the literature over the years \cite{13, 15, 35}, some of which are based on very different mathematical and physical assumptions. A quite recent construction, which is strongly based on previous results on so-called double null foliations of spacetime, is discussed in \cite{21}.}. It is therefore clear that the class of spacetimes to be considered is a specific subclass of the Robinson-Trautmann class of spacetimes, which is a class of spacetimes, admitting a geodesic, shearfree, twistfree and diverging congruence of null curves.

Assuming now that this is the case, one finds that the nonlinear part of the deformation of the Riemann tensor of the geometry is given by the expression

$$C_{e[c} C_{d]b} = 2 f \nabla^c f l_{[c} \nabla_d l_{e]} l^a l_b, \quad (16)$$
whereas the deformed Ricci takes the form

\[ \tilde{R}_{ab} = R_{ab} + \frac{1}{2} \nabla_c \nabla_a (f^c l_b) + \frac{1}{2} \nabla_c \nabla_b (f^c l_a) - \frac{1}{2} \nabla^2 (l_a l_b). \] (17)

Hence, looking at relation (16), it can be concluded that \( C_{[c} C_{d]}^{a} = 0 \) is met if and only if there exists a null frame \((l^a, k^a, m^a, \bar{m}^a)\) such that \((m \nabla) l^a \propto l^a\) applies and there is a profile function \( f \) in relation to which not only \((l \nabla) f = 0\), but also \((l \nabla) (m \nabla) f = (m \nabla) (l \nabla) f = 0\) is fulfilled. This is because in such a case

\[ \nabla^e f l_e \nabla_d l_c = l_d (l \nabla) \nabla_c f = 0 \] (18)

applies, which, however, implies that the deformed Riemann tensor must be of the form

\[ \tilde{R}^{a}_{bcd} = R^{a}_{bcd} + 2 \nabla_{[c} C^{a}_{d]} b. \] (19)

This result represents the very last needed for further investigations.

As shall be shown in the following, the results obtained are of great importance for the derivability of special classes of distributional solutions of Einstein’s field equations from a suitably regularized Einstein-Hilbert action, since they allow one to bypass the problem of performing nonlinear operations on distributional objects. The definition of said regularized action functional is thereby achieved in this context on the basis of well-established methods of Colombeau’s theory of generalized functions, which allow one to avoid ill-defined products of distributions and to derive well-defined Euler-Lagrange equations from a variation of the regularized Lagrangian of the theory. This shall be explained in more detail below.

## 2 Colombeau Algebras and the Einstein-Hilbert Action

Based on the results obtained in the previous section, one may now proceed by considering the special case of a generalized Kerr-Schild spacetime with smooth background metric and a profile distribution containing a delta-like singularity being concentrated on a single null hypersurface of spacetime. More precisely, given some double null coordinate system \((u, v, \theta, \phi)\), one may consider the case of a Kerr-Schild metric (1) with profile distribution \( f = f(u, v, \theta, \phi) \) of the form

\[ f = f_{0\delta}, \] (20)

where \( \delta \equiv \delta(u) \) is Dirac’s delta distribution and \( f_{0} = f_{0}(v, \theta, \phi) \) is a function of the remaining coordinates, from now on to be referred to as reduced profile function. It then turns out that \((k \nabla) f = f_{0}(k \nabla) \delta \) must be valid if \((k \nabla) u = 1\), where \( k^a \) is the null vector field non-tangential to the \( u = 0 \)-null hypersurface \( \mathcal{X} \).
Without referring to a concrete geometric model at this point, it shall however be assumed for the sake of simplicity that there is a normalized null frame \((l^a, k^a, m^a, \bar{m}^a)\), which can be chosen in such a way that not only conditions (12), (14) and (15), but also the conditions \((m\nabla)l^a \propto l^a\) and \((l\nabla)(m\nabla)f = (m\nabla)(l\nabla)f = 0\) are met; although it is completely sufficient if this applies only locally at \(X\).

Keeping in mind that the calculation of the curvature tensor and its invariants requires to perform nonlinear operations involving both the metric and its inverse, both of which contain a Kerr-Schild deformation proportional to a delta distribution, great care must be taken at this point; especially in view of the fact that the definition of the Einstein-Hilbert action for a generalized Kerr-Schild spacetime with profile function of the form (20) requires the calculation of the square root of the determinant of the Kerr-Schild metric.

Consequently, to avoid serious mathematical deficiencies related to the consideration of a delta distribution in the Kerr-Schild metric (1), the strategy of approaching the problem of how to derive distributional field equations from the Einstein-Hilbert action in the face of the low regularity of the profile function will be to resort to Colombeau’s theory of generalized functions \[8, 9, 16, 17, 25, 26, 28, 27\]. The basic idea behind this strategy is that Colombeau’s theory provides a suitable framework for a mathematically rigorous treatment of problems associated with the differentiation and execution of nonlinear operations on singular quantities that actually arise as products of distributions on either Riemannian or Lorentzian manifolds.

For the reader not so familiar with said theory, a brief introduction will now be given; although only the most important facts and pieces of non-redundant information will be covered in the following. To get a better overview of the theory, of course, it is advisable to consult more detailed and mathematically precise treatments of the subject, such as, for example, given in \[14\].

Since it provides a flexible and efficient way of modelling singularities in general relativity, the focus of the introduction shall be placed right away on the so-called special (or simplified) Colombeau framework, dealing with so-called special Colombeau algebras of manifold-valued generalized functions. Given a paracompact \(C^\infty\)-manifold \(X\), the center of attention of this special (or simplified) Colombeau framework is the so-called special Colombeau algebra \(\mathcal{G}(X)\), which is a commutative, associative and unital differential algebra that contains the vector space of Schwartz distributions as a linear subspace, and the space of smooth functions as a faithful subalgebra. As such, it is an algebra consisting of one-parameter families of \(C^\infty\)-functions \((f_\varepsilon(x))_{\varepsilon \in (0, 1]}\), which are subject to certain growth conditions in \(\varepsilon\). To be more precise, \(\mathcal{G}(X)\) results from forming the quotient algebra \(E_m(X)/N(X)\) of the algebra of nets of moderate functions \(E_m(X) = \{(f_\varepsilon)_\varepsilon \in C^\infty(X)^{(0, 1]} : \forall K \subset\subset X \forall P \in P(M) \exists l \sup_{x \in K} |P f_\varepsilon(x)| = O(\varepsilon^{-l})\}\) by the ideal of nets of so-called negligible functions \(N(X) = \{(f_\varepsilon)_\varepsilon \in \) \[2\] As shall be clarified below, special Colombeau algebras are to be distinguished from so-called full Colombeau algebras.
$C^\infty(X)^{[0,1]} : \forall K \subseteq X \forall m \forall P \in \mathcal{P}(M) \sup_{x \in K} |P f_\varepsilon(x)| = O(\varepsilon^m)$, where, in this context, $\mathcal{P}(X)$ denotes the space of all linear differential operators on the manifold $X$.

Probably the most significant advantage of working with the Colombeau framework is that it extends the standard repertoire of operations available for theories of distributions and smooth functions, respectively. In particular, said mathematical framework can be used not only in a strictly linear, but also in a nonlinear context, where conventional linear distribution theory has its natural limitations [42]. The main reason for this is that the corresponding algebras of generalized functions yield expressions that are singular in a fixed, but in principle arbitrary real (regularization) parameter $\varepsilon$, which coincide with Schwartz distributions in the limit $\varepsilon \to 0$ (if such a limit exists). In this sense, said elements of $G(X)$ can be identified as regularizations of distributions, which, as has long been known in theoretical physics, is very satisfying in the sense that regularizations of distributions are much easier to handle in practice than the distributions with which they are associated.

Unfortunately, however, the special algebra $G(X)$ usually suffers from the disadvantage that its elements do not allow one to make a single, unique choice for the parameter $\varepsilon$. This general absence of a preferred regularization parameter is accompanied by the lack of a preferred regularization method (in the sense that there is no canonical embedding of distributions into $G(X)$), which in turn is the reason why there cannot be such a thing as a single canonical Colombeau algebra. Rather, the situation is such that there are many different types of Colombeau algebras, whose constructions revolve around the same principles, but often are only loosely related to each other, which has the disadvantage that the results obtained for one Colombeau algebra often cannot be formally transferred to another. Nevertheless, there are repeated situations in which certain mathematical or physical assumptions can be made on the basis of which a preferred construction can be selected.

A well-known situation, in which such a preferred choice can actually be made, is given, for example, if $X \subseteq \mathbb{R}^n$ open. The standard procedure to embed both continuous functions and compactly supported distributions into $G(X)$ in such a case is to consider an appropriate mollifier $\rho$ satisfying

\begin{enumerate}
  \item $\int \rho(x) d^n x = 1$, and
  \item $\int \rho(x) x^\alpha d^n x = 0 \forall |\alpha| \geq 0$,
\end{enumerate}

which can be used to set $\rho_\varepsilon(x) \equiv \varepsilon^{-n} \rho \left( \frac{x}{\varepsilon} \right)$.

Given this choice, the embedding of elements of $\mathcal{D}'(X)$ - the space of Schwartz distributions on $X$ - into $G(X)$ is then accomplished by considering generalized functions $f_\varepsilon(x) = \int \rho_\varepsilon(x - y) f(y) d^n y$ that converge to distributions in the limit $\varepsilon \to 0$. In case that $\text{supp}(\rho)$ is not compact, a sheaf-theoretic construction (which is based on the observation that the functor $X \to G(X)$ defines a fine
sheaf of differential algebras (in the category of complex vector spaces)) must be used to fix the regularization and to choose a specific preferred mollifier \( \rho(x) \) for the convolution, which is based on the consideration of a partition of unity subordinate to the charts of some atlas. The embedding \( \mathcal{D}'(X) \hookrightarrow \mathcal{G}(X) \) is then called canonical and \( \mathcal{G}(X) \) is no longer referred to as special, but as full Colombeau algebra.

In more general cases, however, in which \( M \) is not strictly presupposed to be an open subset of \( \mathbb{R}^n \), the situation is more involved. This is not least because in these more general cases the procedure of embedding \( \mathcal{D}'(X) \) into \( \mathcal{G}(X) \) via convolution with a preferred mollifier (an idea that represents one of the main building blocks of the theory in \( \mathbb{R}^n \)) is usually prevented by diffeomorphism invariance. Respectively, to put it more accurately, the problem arises that Colombeau algebras which allow for a canonical embedding of distributions generally lack the feature of diffeomorphism invariance.

A way out of this dilemma is to exploit the fact that de Rham regularizations are available through convolution with a mollifier in charts. However, the use of this finding entails the disadvantage that in order to obtain a covariant result it is necessary to explicitly check the coordinate invariance of the results obtained on a case-by-case basis. For this reason, it seems more natural and elegant to consider full diffeomorphism invariant Colombeau algebras instead, which entail a canonical embedding of distributions and yet allow for covariant regularization procedures in the modeling of singularities. Such algebras have indeed been discovered a while ago and studied intensively in the literature ever since [17, 30, 47, 45]. An introduction to the technical machinery associated with the existence of such algebras shall however be avoided at this stage in favour of the consideration of special Colombeau algebras of manifold-valued generalized functions.

The main reason for this is that special Colombeau algebras, although they lack a canonical embedding of the space of distributions, not only allow one to model singularities in a nonlinear context broadly and efficiently, but also lend themselves in a very natural way to geometric applications. As a result, they offer a suitable framework in any situation in which one is prepared to refrain from such a canonical embedding, that is, in particular, when considering models that are given in relation to a fixed coordinate system. Such models will be the subject of this work in due course.

The interplay between generalized functions and distributions is most conveniently formalized in terms of the notion of weak equality or association. A generalized function \( (f_\varepsilon(x))_\varepsilon \) and a distribution \( T \) are called locally associated if

\[
\lim_{\varepsilon \to 0} \int f_\varepsilon \nu \equiv \lim_{\varepsilon \to 0} \langle f_\varepsilon, \nu \rangle \equiv \langle T, \nu \rangle
\]

for all compactly supported one-densities \( \nu \) on \( X \). In such a case, one writes \( f \approx T \). On the other hand, two generalized functions \( (f_\varepsilon(x))_\varepsilon \) and \( (g_\varepsilon(x))_\varepsilon \) are associated if \( f - g \approx 0 \). Hence, as can readily be seen, association behaves like equality on the level of distributions. It is an equivalence relation compatible with addition and differentiation and it allows multiplication with \( C^\infty \) functions.
However, as is well known, it does not respect multiplication of Colombeau objects.

The simplest way to illustrate this is to consider classic examples in \( \mathbb{R}^n \); a case, in which the association relation \( f \approx T \) for a generalized function \( (f_\varepsilon(x))_\varepsilon \) and a distribution \( T \) formulated in (21) gives the expression

\[
\lim_{\varepsilon \to 0} \langle f_\varepsilon, \varphi \rangle = \langle T, \varphi \rangle,
\]

which can be written down somewhat less compactly in the form

\[
\lim_{\varepsilon \to 0} \int f_\varepsilon(x)\varphi(x)d^n x = \int T(x)\varphi(x)d^n x
\]

for all \( \varphi(x) \in C_0^\infty \). Perhaps the most well-known example occurs if one tries to calculate the powers of the \( \theta(x) \) function, which upon naive multiplication would lead to serious contradictions. Specifically, if one tries to conclude

\[
\theta^n(x) = \theta(x) \Rightarrow n\theta(x)^n-1\theta'(x) = \theta'(x),
\]

one immediately finds that this cannot hold for arbitrary \( n \), since it would imply the validity of

\[
(\theta(x)^n)' = n\theta(x)\theta'(x) = \theta'(x) = (\theta(x)^n+1)' = (n+1)\theta(x)\theta'(x)
\]

and thus would force one to erroneously conclude that \( \theta'(x) = 0 \). However, since one would also expect that \( \theta'(x) = \delta(x) \), this would also imply that \( \delta(x) = 0 \), which is obviously nonsense.

Of course, from the point of view of Colombeau theory, the situation is different. There, one rather has

\[
\theta^n(x) \approx \theta(x) \Rightarrow n\theta(x)^n-1\theta'(x) \approx \theta'(x),
\]

which, after using the fact that \( \theta'(x) \approx \delta(x) \), where \( \delta(x) \) is Dirac’s delta distribution, leads to the results

\[
\theta(x) \cdot \theta'(x) \approx \frac{1}{2} \delta(x)
\]

and

\[
\theta^n(x) \cdot \theta'(x) \approx \theta(x) \cdot \theta'(x) \approx \frac{1}{n+1} \delta(x).
\]

Hence, one finds that

\[
\theta(x) \cdot \theta'(x) \approx \theta(x) \cdot \delta(x) \approx A\delta(x)
\]

for some constant \( A \), which makes it perfectly clear that it would be both wrong and misleading to naively conclude that in the Colombeau algebra \( \theta \) times \( \delta \) is just \( \frac{1}{2}\delta \). Instead, as it turns out, association enables one to model \( \theta \) times \( \delta \) in a large number of ways, which shows that the situation is much more nuanced and the problem is much more diverse than one would expect at first glance.
Anyway, after this brief introduction to Colombeau theory, it may be the time to return to the main subject of this work, which is the problem of formulating a well-defined variational principle for distributional Kerr-Schild metrics.

For the purpose of dealing with this subject, the first step will be to consider a singular generalized Kerr-Schild spacetime \((\tilde{M}, \tilde{g})\) with properties very similar to those of the class of generalized Kerr-Schild spacetimes presented in section one, whereas the main difference shall be that the profile function of the geometry is assumed to be of the form \((20)\). The next step is to regularize the delta distribution appearing in \((20)\) by a so-called strict delta net \((\delta_\varepsilon)_{\varepsilon \in (0,1]} \in C^\infty(\tilde{M})^{(0,1]}\), sometimes called a model delta net, i.e. a net that has to meet the following conditions:

\[
a) \text{supp}(\delta_\varepsilon) \to \{0\} \quad (\varepsilon \to 0) \\
b) \int \delta_\varepsilon(x)dx \to 1 \quad (\varepsilon \to 0) \text{ and} \\
c) \exists \eta > 0 \exists C \geq 0 : \int |\delta_\varepsilon(x)|dx \leq C \forall \varepsilon \in (0,\eta).
\]

More specifically, based on the observation that any such net converges to a delta distribution as \(\varepsilon \to 0\), the regularized profile function \(f_\varepsilon = f_\varepsilon(u,v,\theta,\phi)\) of the geometry shall be chosen in such a way that

\[f_\varepsilon \equiv f_0 \delta_\varepsilon, \quad (30)\]

where \(f_0 = f_0(v,\theta,\phi)\) and it shall be assumed that \((\delta_\varepsilon)_{\varepsilon}\) is a function of \(u\) only.

In somewhat sloppy notation, one can then write

\[
\tilde{g}^\varepsilon_{ab} \equiv g_{ab} + f_\varepsilon l^a l^b, \quad (31)
\]

and

\[
\tilde{g}^{\varepsilon}_{a b} \equiv g^{ab} - f_\varepsilon l^a l^b. \quad (32)
\]

Note that the same idea was used in \([3, 24, 47]\) to solve the geodesic and the geodesic deviation equations for impulsive pp-wave spacetimes.

Next, in order to be able to perform a variation of the Einstein-Hilbert action for distributional metrics later on, the validity of \((14)\) shall be required. In addition, it shall be assumed that there holds

\[ (l \nabla) f_\varepsilon = 0 \quad (33) \]

and

\[ \tilde{R}^\varepsilon_{ab} l^a l^b = 0, \quad (34) \]

which can hold if and only if \((l \nabla) f_0 = 0\) and \(R^a_{b a} l^b = 0\) holds locally on \(X\) as well.

\[\text{Note that, to simplify notations, condition } a) \text{ is often replaced by an alternative condition } a') \text{ which requires that } \text{supp}(\delta_\varepsilon) \subseteq [-\varepsilon,\varepsilon] \forall \varepsilon \in (0,1).\]
Based on these assumptions, one comes to the conclusion that conditions (10 - 12) are met and thus relation (17) is valid, which implies that the deformed distributional Ricci and Einstein tensors are linear in the generalized profile function \((f_\varepsilon)\). In addition, based on the validity of \((m\nabla)^a \propto l^a\) and \((l\nabla)(m\nabla)f_0 = (m\nabla)(l\nabla)f_0 = 0\), it is found that relation (18) and thus relation (19) applies in this context as well, so that it can be concluded that also the deformed distributional Riemann tensor is linear in the generalized profile function \((f_\varepsilon)\). However, this ensures that by construction the limit \(\varepsilon \to 0\) yields reasonable singular expressions and that, in principle, 'standard' linear distribution theory could be used in order to solve the field equations of the theory.

This shall turn out to be of great importance for the variation of the Einstein-Hilbert action in the following.

As is well known, in the standard smooth case, one uses the fact that the total action of the system consists of two parts; a pure geometric and – in the case that the gravitational field is coupled to a material source - an additional matter part, so that there holds
\[
S = S_G + S_M. \tag{35}
\]
In the given generalized Kerr-Schild context, one has \(S = S[\tilde{g}], S_G = S_G[\tilde{g}] \equiv \frac{1}{16\pi} \int_M \tilde{R} \omega \) and \(S_M = S_M[\tilde{g}] \equiv \int_M \tilde{T} \omega\), where \(\tilde{R} \equiv R_\varepsilon \equiv \tilde{R}_{ab} \tilde{g}^{ab}\) is the scalar curvature, \(\tilde{T} \equiv \tilde{T}_\varepsilon \equiv \tilde{T}_{ab} \tilde{g}^{ab}\) is the scalar scalar energy-momentum density and \(\omega \equiv \omega_0 = \tilde{\omega} \equiv \tilde{\omega}\) is the scalar volume form being defined with respect to the determinant \(\tilde{g} = g\) of the Kerr-Schild metric \(\tilde{g}_{ab}\) of spacetime \(\tilde{M}\). Here, as usual, a variation of both parts of the action yields (up to an irrelevant total divergence term)
\[
\delta S = \frac{1}{8\pi} \int_M \tilde{G}_{ab} \delta \tilde{g}^{ab} \omega + \int_M \tilde{T}_{ab} \delta \tilde{g}^{ab} \omega. \tag{36}
\]
By requiring then that \(\delta S = 0\), the field equations of the theory result.

In the singular case, on the other hand, the situation is similar, but not identical. Here again the total action consists of a purely geometric and a matter part, but in the sense of distributions, which means that the total action is associated with these parts in the following sense
\[
S \approx S_G + S_M, \tag{37}
\]
where \(S_G \equiv \frac{1}{8\pi} \lim_{\varepsilon \to 0} \int_M \tilde{R} \omega, \nu\) and \(S_M \equiv \lim_{\varepsilon \to 0} \int_M \tilde{T} \omega, \nu\) are defined with respect to the Kerr-Schild deformed generalized Ricci and Laue scalars \((\tilde{R}_\varepsilon)_c\) and \((\tilde{T}_\varepsilon)_c\) (which can be calculated from metric (31)) and compactly supported one-density \(\nu\) on \(M\). Using these definitions, a combined variation of both the geometric and the stress-energy part yields the result
\[ \delta S \approx \delta S_G + \delta S_M, \]  
which may be re-written in the form

\[
\delta S \equiv \lim_{\varepsilon \to 0} \left\{ \frac{1}{8\pi} \left( \int_M \left( \tilde{G}^\varepsilon_{ab} \delta \tilde{g}^\varepsilon_{ab} + \delta \tilde{R}^\varepsilon_{ab} \tilde{g}^\varepsilon_{ab} \right) \omega, \nu \right) + \left( \int_M \tilde{T}^\varepsilon_{ab} \delta \tilde{g}^\varepsilon_{ab} \omega, \nu \right) \right\},
\]

where the second second part of the first integral \( \left( \int_M \delta \tilde{R}^\varepsilon_{ab} \tilde{g}^\varepsilon_{ab} \omega, \nu \right) \) leads to a total divergence and thus to a boundary term, which, however, shall be assumed to be zero for the sake of simplicity.

Due to the fact that nets of generalized functions rather than distributions are considered in this context, the given variation of the action may turn out to be reasonable in the sense that the objects considered are mathematically well-defined. This is not least because \( \omega^\varepsilon \equiv \omega_{\tilde{g}^\varepsilon} = \omega_{\tilde{g}} \equiv \omega \) applies for the volume form of a generalized Kerr-Schild spacetime, so that there is no dependence on the regularization in this case. And also the very dangerous looking terms \( \tilde{G}^\varepsilon_{ab} \delta \tilde{g}^\varepsilon_{ab} \) and \( \tilde{T}^\varepsilon_{ab} \delta \tilde{g}^\varepsilon_{ab} \) turn out to be harmless in the final analysis, as \( \tilde{G}^\varepsilon_{ab} \delta \tilde{g}^\varepsilon_{ab} \equiv \tilde{G}^\varepsilon_{ab} \delta \tilde{g}^\varepsilon_{ab} \) and \( \tilde{T}^\varepsilon_{ab} \delta \tilde{g}^\varepsilon_{ab} \equiv \tilde{T}^\varepsilon_{ab} \delta \tilde{g}^\varepsilon_{ab} \) applies to them if the validity of \( \tilde{G}^\varepsilon_{ab} \delta l^a l^b = \tilde{T}^\varepsilon_{ab} \delta l^a l^b = 0 \) is required in the present context, which is what is to be done in the following. However, this implies that all dangerous, ill-defined terms in (39) turn out to be zero under the given circumstances.

Hence, given this setting, it can now finally be required - in accordance with the principle of stationary action - that

\[ \delta S \approx 0. \]  

For Kerr-Schild spacetimes with geometric properties discussed above, this variational principle then implies the validity of the distributional field equations

\[ \tilde{G}_{ab} \approx 8\pi \tilde{T}_{ab}, \]  

where \( \tilde{G}_{ab} \) and \( \tilde{T}_{ab} \) are the embeddings of the Einstein and energy-momentum tensors into the Colombeau algebra. In this context, it may be noted that one is in principle free to choose any (suitable) regularization of the action. However, to fix a specific regularization and therefore to guarantee that the regularized Einstein-Hilbert action is diffeomorphism invariant in the same way as in the smooth case, one may decide to work not with the special, but with the full diffeomorphism invariant Colombeau algebra treated in [15, 17].

Examples for distributional spacetimes, in relation to which a variational principle of this kind can actually be formulated, shall be discussed in the next and final section of this work.
3 Distributional Field Equations and Curvature

The variational principle discussed in the previous section does not apply to distributional Kerr-Schild spacetimes, but only to selected ones, whose properties are very close to those discussed in section one. Therefore, an overview of specific models shall now be given, in relation to which said variation can be performed, whereupon the main focus will be placed on setting up the field equations 'the right way' via using Colombeau methods discussed in the previous section. On the basis of the results to be determined, the consistency of the method developed in the previous section for varying the regularized total action of general relativity (gravity plus matter) is demonstrated.

A class of spacetimes to which these methods can be applied is the class of impulsive plane fronted gravitational waves with parallel rays or, for short, impulsive pp-waves, which is a class of singular spacetimes where the curvature is concentrated on a null hypersurface. As first discovered by Penrose [38], these spacetimes arise in the so-called impulsive limit of so-called sandwich pp-waves for infinitesimal time intervals, or, as he further noted, as a byproduct of the so-called 'scissors-and-paste' procedure, often alternatively referred to as 'cut-and-paste' procedure in the literature.

The line element of spacetimes belonging to this class can be written in the form

$$ds^2 = fdu^2 - 2dvdu + dy^2 + dz^2,$$

(42)

where

$$f(v, u, y, z)$$

is the so-called profile function of the geometry. More precisely, the main characteristic of representatives of this class is that their profile function can be written in the form

$$f(u, y, z) = \delta(u)f_0(x, y),$$

(43)

where

$$f_0(x, y)$$

is the reduced profile function and

$$\delta(u)$$

is Dirac's delta distribution. Consequently, as can be concluded from the observation that the metric related to line element (45) can be written in the form

$$g_{ab} = \eta_{ab} + f_0 \delta_{a\mathcal{U}} b,$$

(44)

the given spacetime is flat everywhere except for the null hyperplane \(u = 0\), where the delta-like impulse is located. Representatives of the given class of geometries therefore belong to the superordinate Kerr-Schild class of spacetimes.

As first discovered by Ehlers and Kundt [11, 22, 23], the family of pp-wave spacetimes, however, also belongs to another superordinate family of spacetimes at the same time, the so-called Kundt family; a family of non-twisting, shear-free and non-expanding geometries, whose line element is of the form

$$ds^2 = Hdu^2 - 2dvdu + 2W_b dx^b dv + q_{bc} dx^b dx^c,$$

(45)

where

$$H = H(v, u, x^2, x^3),$$

$$W_b = W_b(v, u, x^2, x^3)$$

and

$$q_{bc} = q_{bc}(v, x^2, x^3)$$

applies by definition.
The main characteristic of this class is that there always exists a null vector field \( l^a = \partial v^a \), which is the generator of a null foliation of spacetime in non-expanding null hypersurfaces, so that it can be concluded that condition (14) is always globally met. In addition, \( f \) can always be chosen in such a way for pp-wave geometries that condition (15) is met as well. Besides that, it can be arranged that \((m \nabla)l^a \propto l^a \) and \((l \nabla) [(m \nabla)f] = (m \nabla)(l \nabla)f = 0 \) also holds for these spacetimes and the Ricci tensor \( R_{ab} \) of the geometry is always given in such a way that \( R_{ab} \propto l^a l^b \), so that it can be concluded that not only conditions (10 – 12) are met, but also relation (18) turns out to be valid, which, however, implies that the validity of (17) and (19).

The exact same applies in the case of impulsive pp-waves; however, only in a distributional sense. This implies that not only the deformed Ricci tensor with lowered or raised indices is linear in the profile function, but also the deformed part of the Riemann curvature tensor. In addition, using that there holds \( \omega_{x} \equiv \omega_{y} \equiv \omega \) for the volume form of a generalized Kerr-Schild spacetime, it becomes possible to set up a regularized gravitational action (matter plus gravity) of the form (37) and to repeat the steps discussed in the previous section, which lead to distributional field equations of the form (41).

Specific models in general relativity to which this approach can be applied include the arguably most well-known model for an impulsive pp-wave spacetime, the geometric model of Aichelburg and Sexl, whose reduced profile function is given by the expression

\[
f_0(y, z) = 8p \ln \sqrt{y^2 + z^2}.
\]

(46)

This spacetime, which was originally found by the authors by calculating the ultrarelativistic limit of a Lorentz boosted Schwarzschild geometry in isotropic coordinates, characterizes the field of a point-like particle moving close to the speed of light. It represents an exact solution to Einstein’s equations, which in the given case reduce to a single differential equation for the reduced profile function of the form

\[
\Delta_{\eta} f_0 = -16\pi p \delta^{(2)},
\]

(47)

where \( \delta^{(2)} \equiv \delta^{(2)}(y, z) \).

A similar model, to which the ideas formulated in section two also apply, was found by Lousto and Sanchez, who calculated the ultrarelativistic limit of the boosted Reissner-Nordström geometry. Assuming that \( e = \gamma \frac{p}{c} \), they had to require for this purpose the charge \( e \) to vanish in the said limit. Based on this constraint, they found a solution of Einstein’s equations of the form (47) with reduced profile function

\[
f_0(y, z) = 8p \ln \sqrt{y^2 + z^2} + \frac{3\pi p^2}{2\sqrt{y^2 + z^2}}.
\]

(48)

The solution obtained again represents a pp-wave and is flat everywhere except on the null plane \( u = 0 \). More precisely, as was confirmed by Steinbauer in using a calculation in \( \mathcal{G} \), it was found that all the components of the
electromagnetic field and all but one of the components of the energy-momentum tensor are associated to zero.

The ultrarelativistic limit of the Kerr metric has been calculated by several authors [5, 6, 7, 12, 29, 30]. The same limit has also been applied to other non-flat backgrounds and used for a number of different boosted sources, such as cosmological constant sources, cosmic strings, domain walls and monopoles to obtain ultrarelativistic impulsive pp-wave spacetimes [19, 31, 32, 39, 40, 41], which have been used to describe (quantum) scattering processes of highly energetic particles [30, 33]. Of course, the methods developed in section two of this paper work for all these approaches in a similar way.

In any case, taking into account the results obtained in section one of this work, it becomes clear that there is another class of geometries to which these methods can be applied, namely the class of gravitational shock wave geometries in black hole and cosmological backgrounds.

The most famous representatives of this class were all found on the basis of Penrose’s cut-and-paste procedure, one of the forerunners of today’s thin shell approaches [34, 43, 44]. In particular, the field of a spherical shock wave caused by a massless particle moving at the speed of light along the horizon of a Schwarzschild black hole was derived by Dray and ’t Hooft [10] on the basis of Penrose’s methods. Using exactly the same method, Sfetsos calculated a similar geometry for the Reissner-Nordström case [44] and Lousto and Sanchez specified a spherical shock wave for the Kottler alias Schwarzschild-de Sitter case.

Due to their similarity, all these approaches shall be discussed in a single effort in the following. The reason why this is possible is the following: Using the fact that the line element of any static spherically symmetric spacetime can be brought into the form

\[-2A^2 dU dV + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),\]  

where \( r = r(UV) \) and \( A = A(r(UV)) \) are implicit functions of \( U \) and \( V \), the line elements of the Dray-’t Hooft, Sfetsos and Lousto-Sanchez shock wave geometries written down as follows

\[ds^2 = 2A^2 f_0 \delta dU^2 - 2A^2 dU dV + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)\]  

where \( \delta = \delta(U) \) is Dirac’s delta distribution. Hence, it can be concluded that the metrics corresponding to these line elements belong individually to the generalized Kerr-Schild classes

\[\tilde{g}_{ab} = g_{ab} + 2A^2 f_0 \delta l_a l_b,\]  

of the Schwarzschild, Reissner-Nordström and Kottler backgrounds, where in each case one has \( l_a = g_{ab} l^b = -dU_a \).

What all three cases additionally have in common is that the profile functions of the respective shock wave geometries can be obtained via solving Einstein’s equations, whereupon in all three cases ill-defined ‘delta-square’ terms occurred in the course of the calculation, which, from a mathematical point of view,
made no sense at all and therefore had to be ignored by the authors in each and every case. For this reason, in particular, it was shown several years later by Balasin on the basis of the generalized Kerr-Schild framework that the problematic terms do not occur in Einstein’s equations with mixed indices [4], which allowed him to rigorously deduce said shock wave spacetimes from their associated backgrounds. Thus, as can straightforwardly be deduced from Balasin’s results, the mixed field equations of the generalized Kerr-Schild class lead to a single differential equation for the reduced profile function of the form

\[(\Delta_{s_2} - c)f_0 = 2\pi b\delta,\]  

where \(\delta \equiv \delta(\cos\theta - 1)\) is Dirac’s delta distribution and \(b\) and \(c\) are constants, whereas \(c\) is given by \(c = 2r_+(\kappa - A\eta r_+)\) in the Schwarzschild-de Sitter case, \(c = 2r_+\kappa\) in the Reissner-Nordström case and by \(c = 1\) in the Schwarzschild case.

The resulting equation can be solved by expanding the reduced profile function on the left hand side and the delta function on the right hand side simultaneously in Legendre polynomials. Using here the fact that \(\delta(x) = \sum_{l=0}^{\infty} (l + \frac{1}{2})P_l(x),\) one obtains the solution

\[f_0(\theta) = -b\sum_{l=0}^{\infty} \frac{l + \frac{1}{2}}{l(l + 1) + c} P_l(\cos\theta)\]  

by solving the corresponding eigenvalue problem. An integral expression for this solution can then be found by considering the generating function of the Legendre polynomials

\[\sum_{l=0}^{\infty} \frac{l + \frac{1}{2}}{l(l + 1) + c} P_l(\cos\theta)e^{-sl} = \frac{1}{\sqrt{1 - 2\cos\theta e^{-s} + e^{-2s}}}\]  

in addition to the fact that

\[\frac{l + \frac{1}{2}}{l(l + 1) + \alpha^2 + \frac{1}{4}} = \int_0^{\infty} e^{-s(l + \frac{1}{2})} \cos(\alpha s)ds.\]  

This yields a result of the form

\[f_0(\theta) = -b\frac{a_0}{\sqrt{2}} \int_0^{\infty} \frac{\cos(\sqrt{c - \frac{1}{2}s})}{\sqrt{\cosh s - \cos\theta}}ds,\]  

which for each individual value of \(r = r(UV)\) and \(A = A(r(UV))\) and \(c\) gives the precise form of the shock wave geometries of Dray and ‘t Hooft, Sfetsos and Lousto and Sanchez.\(^4\)

\(^4\)To be exact, Balasin did not demonstrate explicitly the validity of his method for all these cases, but only for the geometry of Dray and ‘t Hooft. However, his results are completely general and therefore, of course, turn out to be valid for the cases mentioned above as well.
The only problem with this idea of treating the subject is that it still proves to be impossible to deduce the corresponding distributional geometries from Einstein’s equations with lowered and raised indices. Besides that, one could come to the conclusion that it is still impossible to set up a regularized action of the form (37) for such spacetimes and to obtain distributional field equations of the form (41) from its variation.

However, based on the methods developed in section two, it becomes clear that this is in fact not true. Using once more the fact that $\omega \equiv \omega_{\tilde{g}} = \omega_{g} \equiv \omega$ for the volume form of generalized Kerr-Schild spacetimes and that condition (14) is met and that also (33) turns out to be valid if it is required that $\partial_V A|_{U=0} = \partial_V r|_{U=0} = 0$, it can be checked that conditions (12), (14), (15) and (18) are met and therefore relations (17) and (19) turn out to be valid in the given setting, which is not least due to the fact that $R_{ab}^{\alpha \beta} = 0$ holds true for the three different background fields.

Thus, it can be concluded that Balasin’s ideas can be extended by a more specific choice of the Kerr-Schild approach to the effect that said problematic ‘delta-square’ terms never occur in the field equations of the theory and even the deformed Riemannian curvature tensor (if calculated appropriately) does not contain any ill-defined non-linear delta-terms. As a result, it becomes possible to set up a regularized gravitational action (matter plus gravity) of the form (37) and to repeat the steps discussed in the previous section, which lead to distributional field equations of the form (41).

Finally, note that quite recently Balasin’s results were used as a starting point for calculating the field of a gravitational shock wave caused by a massless particle moving at the speed of light along the exterior event horizon of a Kerr-Newman black hole [20]. Using the same geometrical assumptions as in the present work, a much more general class of spacetimes was deduced in this context, to which the variation principle for distributional Kerr-Schild metrics developed in section two can also be applied without further ado.

**Discussion**

In the present work, a subclass of the generalized Kerr-Schild spacetime class was specified with respect to which the variational principle of general relativity can be generalized - on the basis of Colombeau’s theory of generalized functions - to singular situations, in which one would usually not expect the principle of stationary action to be valid. Considering specific metrics in this context, which contain a generalized delta (profile) function converging to a delta distribution in an appropriate limit, it was shown how said variational principle leads to mathematically exact distributional field equations. As further shown for the cases of impulsive pp-wave spacetimes and various gravitational shock wave geometries in black holes and cosmological backgrounds, the solutions of these distributional field equations fit perfectly into the physically expected picture, as they contain no undefined 'delta-square' terms or the like, proving yet again why Colombeau’s theory of generalized functions offers not only a solid
mathematical framework, but an indispensable machinery for accommodating calculations involving distributional metrics in general relativity as well.

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