Have your cake and eat it too: increasing returns while lowering large risks!

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Abstract

Based on a faithful representation of the heavy tail multivariate distribution of asset returns introduced previously (Sornette et al., 1998, 1999) that we extend to the case of asymmetric return distributions, we generalize the return-risk efficient frontier concept to incorporate the dimensions of large risks embedded in the tail of the asset distributions. We demonstrate that it is often possible to *increase* the portfolio return while *decreasing* the large risks as quantified by the fourth and higher order cumulants. Exact theoretical formulas are validated by empirical tests.

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1 Introduction

One of the most fundamental tenets of economic theory and practice is that returns above the so-called riskless rate come with increased risks. This is the basis of Markovitz's portfolio theory (e.g. Markovitz, 1959) and of the CAPM (e.g. Merton, 1990). Reciprocally, investors want to be compensated for taking risk, that is, they want to earn a return high enough to make them comfortable with the level of risk they are assuming. It is thus a fundamental premise of efficient markets that “one cannot have both the cake and eat it too”, i.e. one cannot increase the return and lower the risk at the same time. This result stems simply from the linear (resp. quadratic) dependence of the average return (respectively variance) of a portfolio return on the weights of its constituent assets leading to a parabolic efficient frontier in the return-risk diagram.

In the real world, the variance of portfolio returns provide only a limited quantification of incurred risks, as the distributions of returns have “fat tails” (e.g. Lux, 1996, Gopikrishnan et al., 1998, Lux and Marchesi, 1999) and the dependences between assets are only imperfectly accounted for by the correlation matrix (e.g. Litterman and Winkelmann, 1998). Value-at-Risk (e.g. Jorion, 1997) and other measures of risks (e.g. Artzner et al., 1996, Sornette, 1998, Bouchaud et al., 1998, Sornette et al., 1998, 1999) have been developed to account for the larger moves allowed by non-Gaussian distributions.

Here, we generalize our previously introduced representation of the heavy tail multivariate distribution of asset returns (Sornette et al., 1998, 1999) to the case of asymmetric return distributions. We calculate theoretically and test empirically the cumulants of a portfolio and generalize the return-risk efficient frontier concept to incorporate the dimensions of large risks embedded in the tail of the asset distributions. We demonstrate the novel remarkable result that it is often possible to improve on the optimal mean-variance portfolio by increasing the return while decreasing the large risks quantified by the fourth and higher order cumulants. This is related to and generalizes our previous rigorous result (Sornette et al., 1998, 1999) that minimizing the variance, i.e. the relatively “small” risks, often increases larger risks as measured by higher normalized cumulants and the Value-at-risk. Thus, putting the emphasis on the risk quantified by the volatility can be both misleading because large risks are still looming and in addition damage profitability.

2 The asymmetric modified Weibull distribution

In order to make our approach concrete, we assume that price returns $\delta x$ are distributed according to the following probability distribution function (pdf)

$$P(\delta x) = \begin{cases} Q \frac{\gamma_+}{\sqrt{\pi}} \frac{1}{\chi_+^{\gamma_+/2}} |\delta x|^{\gamma_+ - 1} \exp \left( - \left( \frac{\delta x}{\chi_+} \right)^{\gamma_+} \right) & \text{for } 0 < \delta x , \\
\frac{1-Q}{\sqrt{\pi}} \frac{\gamma_-}{\chi_-^{\gamma_-/2}} |\delta x|^{\gamma_- - 1} \exp \left( - \left( \frac{\delta x}{\chi_-} \right)^{\gamma_-} \right) & \text{for } \delta x < 0 \end{cases} \quad (1)$$

Q is the probability for observing a positive return, the $\chi$’s are the characteristic returns and the exponent $\gamma$’s control the fatness of the pdf tails, which can be different for positive and negative returns.
For \( Q = 1/2 \), \( \chi_+ = \chi_- \) and \( \gamma_+ = \gamma_- \), we recover the symmetric modified Weibull pdf studied by Sornette et al. (1998, 1999) and the special case \( \gamma_+ = \gamma_- = 2 \) recovers the standard normal law. The case when the exponents \( \gamma \) are smaller than one corresponds to a “stretched” exponential with a tail fatter than an exponential and thus much fatter than a Gaussian, but still thinner than a power law. Stretched exponential pdf’s have been found to provide a parsimonious and accurate fit to the full range of currency price variations at daily intermediate time scales (Laherrère and Sornette, 1998). This stretched exponential model is also validated theoretically by the recent demonstration that the tail of pdf’s of products of a finite number of random variables is generically a stretched exponential (Frisch and Sornette, 1997), in which the exponent \( \gamma \) is proportional to the inverse of the number of generations (or products) in a multiplicative process.

3 Nonlinear change of variable

Let us pose

\[
\begin{align*}
y_+ & = (\delta x)^{\gamma_+/2} \quad \text{for } \delta x > 0 , \\
y_- & = -|\delta x|^{\gamma_-/2} \quad \text{for } \delta x < 0 .
\end{align*}
\]

Inversely, we have

\[
\begin{align*}
\delta x & = y_+^{q_+} \quad \text{for } \delta x > 0 , \quad \text{with } q_+ \equiv \frac{2}{\gamma_+} , \\
\delta x & = -|y_-|^{q_-} \quad \text{for } \delta x < 0 , \quad \text{with } q_- \equiv \frac{2}{\gamma_-} .
\end{align*}
\]

The change of variable (3,4) from \( \delta x \) to \( y \) leads to a Gaussian pdf for the \( y \)-variable defined in each semi-infinite domain:

\[
\begin{align*}
P(y+) & = \frac{2Q}{\sqrt{2\pi } \sigma_+} \exp \left( -\frac{y_+^2}{2\sigma_+^2} \right) , \quad \text{where } \sigma_+^2 = \frac{1}{2} \chi_+^{\gamma_+} , \\
P(y-) & = \frac{2(1-Q)}{\sqrt{2\pi } \sigma_-} \exp \left( -\frac{y_-^2}{2\sigma_-^2} \right) , \quad \text{where } \sigma_-^2 = \frac{1}{2} \chi_-^{\gamma_-} .
\end{align*}
\]

Using a maximization entropy principle, one can then show (Sornette et al., 1998, 1999) that the correlations between the \( y \) variables of different assets provide the most efficient and parsimonious multivariable representation. This transformation has also been used for the analysis of particle physics experiments (Karlen, 1998) and much earlier for the treatment of bivariate gamma distributions (Moran, 1969). It can also be viewed as a concrete implementation of the copula representation of dependence between assets (e.g. Embrechts et al., 1998, 1999). Generalizations to other non-Gaussian pdf’s are discussed in Sornette et al. (1999).

We have made empirical tests on three assets, using annualized daily returns of stock prices of Chevron (CHV) and Exxon (XON) in the period Jan. 1970 - Mar. 1999, and of the Malaysian Ringit (MYR) against the US dollar in the period Jan. 1971 - Oct. 1998. The CHV-XON pair
is among the most strongly connected group of stocks in the S&P 500 index while the Malaysian
Ringgit is essentially uncorrelated to the Chevron and Exxon stocks. These extreme cases allow
us to test the influence of correlations. Especially for strongly correlated stocks, we have shown
(Sornette et al., 1998) that a change of variable like Eq. (3, 4) leads to a covariance matrix which is
much more stable compared to the usual covariance matrix.

Fig. 1 shows in a log-log plot the \( y(r) \) transformation (3,4) calculated from the empirical positive
and negative returns of the Chevron and Exxon stocks and for the Malaysian Ringgit against the
US dollar (MYR). Assuming that price returns are distributed according to an asymmetric modified
Weibull (1,2), the slope of the \( y(r) \)-plot gives for large \( |r| \)-values the exponents \( \gamma_+/2 \) and \( \gamma_-/2 \). The
positive and negative returns of each asset are seen to have almost the same slope for large \( r \) values,
and consequently we will assume for each asset that \( \gamma_+ = \gamma_- \equiv \gamma \) in the sequel. The linearity of
the \( y(r) \) plots for large \( r \) values show that the large tails of the pdf’s are indeed to a very good
approximation distributed according to a modified Weibull distribution Eq. (1,2), with \( \gamma \approx 1.4 \)
(CHV), \( \gamma \approx 1.2 \) (XON) and \( \gamma \approx 0.62 \) (MYR). For small and intermediate \( r \) values, the \( y(r) \) curves
have a slope close to 1 (indicated by the \( y = r \) line), which means that small and intermediate
returns are distributed according to a Gaussian distribution. Because of the finite resolution of the
data (the data has a lower bound for the return), \( y(r) \) approaches a constant value for the smallest
values of \( r \).

4 Portfolio theory for the diagonal case

In this short letter, we present the theory for the diagonal case where assets are uncorrelated.
This is already sufficient to illustrate the most important results. Especially in the case of fat tails
(exponents \( c < 1 \)), correlations are less important than a precise determination of the tails (Sornette
et al., 1998). We will however present some empirical tests with uncorrelated and with correlated
assets, in order to illustrate the importance of correlations. Sornette et al. (1999) treat the case
of correlated assets with symmetric distributions with the same exponent \( \gamma \). Generalization to the
asymmetric case and with different exponents \( \gamma \) will be reported elsewhere.

The discrete time estimation of the returns \( \delta x_i(t) \) are \( \delta x_i(t) \equiv \delta p_i(t)/p_i(t) = (p_i(t+1)−p_i(t))/p_i(t) \),
where \( p_i(t) \) is the price of asset \( i \) at time \( t \). The total variation of the value of the portfolio made
of \( N \) assets between time \( t − 1 \) and \( t \) reads

\[
\delta S(t) = \sum_{i=1}^{N} W_i \delta p_i(t) = \sum_{i=1}^{N} w_i \delta x_i(t) ,
\]

(9)

where \( W_i \) is the number of shares invested in asset \( i \) and \( w_i = W_i p_i \) is the weight in capital invested
in the \( i \)th asset at time \( t \) in the portfolio. We will assume normalization, i.e. \( \sum_{i=1}^{N} w_i = 1 \), thus
leading to a dynamical reallocation of the assets in the portfolio.

The expression (9) can be expressed in terms of the variables \( y_i \)’s defined by (3,4) as follows

\[
\delta S(t) = \sum_{i=1}^{N} w_i \epsilon_i |y_{\epsilon_i}|^{\gamma_{\epsilon_i}} ,
\]

(10)
where $\epsilon_i$ is the sign of $\delta x_i$. All the properties of the portfolio are contained in the probability distribution $P_S(\delta S(t))$ of $\delta S(t)$. We would thus like to characterize it, knowing the multivariate distribution of the $\delta x_i$’s (or equivalently the multivariate Gaussian distribution of the $y_i$’s) for the different assets. The general formal solution reads 
\[
P_S(\delta S) = C \prod_{i=1}^{N} \left( \int d\gamma_i \right) e^{-\frac{1}{2} \gamma^T \gamma} \delta \left( \delta S(t) - \sum_{i=1}^{N} w_i \epsilon_i |y_i|^q_{+} \right).
\] (11)

Taking the Fourier transform $\hat{P}_S(k) \equiv \int_{-\infty}^{+\infty} d\delta S \; P_S(\delta S) \; e^{-ik\delta S}$ of (11) gives 
\[
\hat{P}_S(k) = \prod_{i=1}^{N} \left( \int d\gamma_i \right) e^{-\frac{1}{2} \gamma^T \gamma} \sum_{i=1}^{N} w_i \epsilon_i |y_i|^q_{+} \cdot \]
\[
= \prod_{i=1}^{N} \left[ 2(1 - Q_i) \int_{-\infty}^{0} \frac{d\gamma_i}{\sqrt{2\pi} \sigma_i} \exp \left( -\frac{\gamma_i^2}{2\sigma_i^2} - ikw_i |y_i|^q_{+} \right) + 2Q_i \int_{0}^{+\infty} \frac{d\gamma_i}{\sqrt{2\pi} \sigma_i} \exp \left( -\frac{\gamma_i^2}{2\sigma_i^2} + ikw_i |y_i|^q_{+} \right) \right].
\] (13)

Using the explicit expression of the form of the distributions (15), we get 
\[
\hat{P}_S(k) = 2 \prod_{i=1}^{N} \left[ \sum_{m=0}^{+\infty} \frac{(ikw_i)^m}{m!} \left( (-1)^m (1 - Q_i) \sigma_i^{mq_{+}} \langle y^{mq_{+}} \rangle_{+} + Q_i \sigma_i^{mq_{-}} \langle y^{mq_{-}} \rangle_{-} \right) \right],
\] (14)

where 
\[
\langle y^{mq} \rangle_{\pm} = \int_{0}^{+\infty} \frac{dy}{\sqrt{2\pi}} y^\alpha e^{-\frac{y^2}{2}} = \frac{2^{\alpha-1}}{\sqrt{\pi}} \Gamma \left( \frac{\alpha}{2} + \frac{1}{2} \right),
\] (15)

and $\Gamma$ is the Gamma function. Replacing in (14), we obtain 
\[
\hat{P}_S(k) = \prod_{i=1}^{N} \left[ \sum_{m=0}^{+\infty} \frac{(ikw_i)^m}{m!} M_i(m) \right],
\] (16)

where 
\[
M_i(m) = \frac{1}{\sqrt{\pi}} \left( (-1)^m (1 - Q_i)^2 \sigma_i^{mq_{-}} \Gamma \left( \frac{mq_{-}}{2} + \frac{1}{2} \right) + Q_i \sigma_i^{mq_{+}} \Gamma \left( \frac{mq_{+}}{2} + \frac{1}{2} \right) \right).
\] (17)

For symmetric distributions with $q_{i+} = q_{i-}$, i.e. $\gamma_{i+} = \gamma_{i-}$, $\sigma_{i+} = \sigma_{i-}$ and $Q_i = 1/2$, we retrieve our previous result (Sornette et al., 1999) that all the odd order terms in the sum over $m$ cancel out: 
\[
\hat{P}_S(k) = \prod_{i=1}^{N} \left[ \sum_{m=0}^{+\infty} \frac{(ikw_i)^{2n}}{(2n)! \sqrt{\pi}} \sigma_i^{2mq_i} \Gamma \left( nq_i + \frac{1}{2} \right) \right].
\] (18)
The expression $\sum_{m=0}^{+\infty} \frac{(ikw_i)^m}{m!} M_i(m)$ in (16) is similar to the expansion of a characteristic function in terms of moments. We need to get the corresponding expansion in terms of cumulants, i.e. find the coefficients $c_n$ such that

$$\sum_{m=0}^{+\infty} \frac{(ikw_i)^m}{m!} M_i(m) = \exp \left( \sum_{n=1}^{+\infty} \frac{(ik)^n}{n!} c_1(n) \right).$$  \hspace{1cm} (19)

By identifying the same powers of $k$ term by term, we get the cumulants. Then, using the product in (16) of the exponentials from $i = 1$ to $N$, we obtain the cumulants of the portfolio distribution as

$$c_1 = \sum_{i=1}^{N} w_i M_i(1),$$  \hspace{1cm} (20)

$$c_2 = \sum_{i=1}^{N} w_i^2 \left( M_i(2) - M_i(1)^2 \right),$$  \hspace{1cm} (21)

$$c_3 = \sum_{i=1}^{N} w_i^3 \left( M_i(3) - 3M_i(1)M_i(2) + 2M_i(1)^3 \right),$$  \hspace{1cm} (22)

$$c_4 = \sum_{i=1}^{N} w_i^4 \left( M_i(4) - 3M_i(2)^2 - 4M_i(1)M_i(3) + 12M_i(1)^2M_i(2) - 6M_i(1)^4 \right),$$  \hspace{1cm} (23)

$$c_5 = \sum_{i=1}^{N} w_i^5 \left( M_i(5) - 5M_i(4)M_i(1) - 10M_i(3)M_i(2) + 20M_i(3)M_i(1)^2 + 30M_i(2)^2M_i(1) - 60M_i(2)M_i(1)^3 + 24M_i(1)^5 \right),$$  \hspace{1cm} (24)

$$c_6 = \sum_{i=1}^{N} w_i^6 \left( M_i(6) - 6M_i(5)M_i(1) - 15M_i(4)M_i(2) + 30M_i(4)M_i(1)^2 - 10M_i(3)^2 + 120M_i(3)M_i(1)M_i(1)^2 - 120M_i(3)M_i(1)^3 + 30M_i(2)^3 - 270M_i(2)^2M_i(1)^2 + 360M_i(2)M_i(1)^4 - 120M_i(1)^6 \right).$$  \hspace{1cm} (25)

Higher order cumulants are obtained by using the formulas given for instance by Stuart and Ord (1994). The first cumulant $c_1$ provides the average gain $\langle \delta S \rangle$ and the second cumulant $c_2$ is the variance of the portfolio gain. The higher order cumulants as well as the excess kurtosis $\kappa = c_4/c_2^2$ quantify larger risks occurring with smaller probabilities but larger impact.

Fig. 3 presents a comparison of the empirical determined $c_n$’s and those determined from the equations (20-25), for a portfolio constituted of the Malaysian Ringgit (MYR) and the Chevron stock (CHV). This choice is made because MYR is essential uncorrelated to CHV and the above calculation should thus apply directly. For an extension of the theory to correlated assets, see Sornette et al. (1999). To perform the empirical test shown in figure 3, we first determined the exponents $\gamma_+ = \gamma_- \equiv \gamma$ from a regression of the linear parts of the $y(r)$ functions for large values of $|r|$ shown in figure 1. We then use these $\gamma$’s to estimate the coefficients $\chi_{i+}, \chi_{i-}$ from the empirical averages $\chi_{i\pm} = \langle (\delta x_{\pm}^i)^\gamma \rangle_{\pm}$. The notation $\langle \quad \rangle_{\pm}$ represents an average taken with respect to positive/negative returns of the data. The asymmetric weight parameter $Q_i$ is determined from
the asset \(i\) as the ratio of the number of positive returns over the total number of returns. The error bars shown in the figure are determined from the observation that the main source of error comes from a mis specification of the tail exponent \(\gamma\)'s and we assume conservatively an error of \(\pm 0.05\) on the \(\gamma\) values. Fig. 2 shows a very good agreement between theory and the direct empirical determination of the cumulants. There is some discrepancy for the third order cumulant \(c_3\), which reflects our simplification to use symmetric tails with \(\gamma_+ = \gamma_- \equiv \gamma\) in our calculations Eq. (24-23). As a consequence, the sole contribution to the odd-order cumulants stems from the difference between \(\chi_{i+}\) and \(\chi_{i-}\) and between \(Q_i\) and 1/2. An additional asymmetry in the shape of the tail captured by \(\gamma_+ \neq \gamma_-\), however small, can easily make the agreement adequate between the theoretical and empirical \(c_3\). We have chosen not to incorporate this additional complexity in order to keep the number of degrees of freedom as small as possible. The even-order cumulants and the excess kurtosis \(\kappa\) are much less sensitive to the asymmetry in the exponents \(\gamma_+, \gamma_-\).

The portfolio with minimum variance \(c_2\) has the optimal weight \(w_1 = 9.5\%\), where the index 1 stands for the Chevron stock, i.e. the weight \(w_2 = 1 - w_1\) of the Malaysian Ringgit is 90.5\%. In comparison, the portfolio with minimum fourth cumulant has an investment ratio of \(w_1 = 38\%\) in Chevron and \(w_2 = 62\%\) in the Malaysian Ringgit. It is clear that the minimum variance portfolio has a rather large fourth cumulant, i.e. minimizing the small risks quantified by the second order cumulant comes at the cost of increasing the largest risks quantified by the fourth order cumulant (Sornette et al., 1998, 1999).

Fig. 3 illustrates another even more interesting phenomenon. We compare the daily returns and the cumulative wealth of two portfolios. The first \(c_1 - c_2\) portfolio has a minimum variance \(c_2\) (Chevron weight \(w_1 = 0.095\) and Malaysian Ringgit weight \(w_2 = 0.905\)). The second \(c_1 - c_4\) portfolio has a minimum fourth-order cumulant (Chevron weight \(w_1 = 0.38\) and Malaysian Ringgit weight \(w_2 = 0.62\)). The horizontal dotted lines in the daily return plots are the maximum values sampled for the returns of the \(c_1 - c_4\) portfolio. Notice that the daily returns of the minimum variance portfolio exceeds these bounds. This illustrates vividly that, while most of the time the fluctuation of the returns are smaller for the \(c_1 - c_2\) portfolio, fluctuations with larger amplitudes and thus larger risks are observed in this minimum variance portfolio: again, minimizing small risks can lead to a dangerous increase of large risks (Sornette et al., 1998, 1999). Furthermore, the cumulative wealth of the \(c_1 - c_2\) portfolio with \(w_1 = 0.095\) is drastically inferior to that accrued in the \(c_1 - c_4\) portfolio with \(w_1 = 0.38\). In other words, you can have your cake and eat it too: decrease the large risk (those that count for the safety of investment houses and for regulatory agencies) and increase the profit! This example illustrates how misleading can be the focus on the variance as a suitable measure of risks and how limited is the use of standard portfolio optimization techniques. Not only they do not provide a suitable quantification of the really dangerous market moves, in addition they miss important profit opportunities.

Fig. 4 is the same as Fig. 2 for a portfolio constituted of the Exxon and the Chevron stocks. Due to the very large correlation between the two assets, the departure between theory and experiments is a measure of the importance of correlations that have been neglected in the above formulas, especially in this case where the exponents \(\gamma\) for the pdfs of the two stocks are relatively large around 1.4 and 1.2 respectively, i.e. the pdf tails are relatively “thin”. This constitutes a worst-case scenerio for the application of the above theory that is best justified for exponents \(\gamma < 1\) (recall that the standard Gaussian regime corresponds to \(\gamma = 2\)). Notwithstanding this limitation, the
results conform qualitatively to our previous discussion: the best variance gives a substantially larger risk for large moves and the return is sub-optimal.

5 Efficient Portfolio Frontiers

Based on our previous calculation, it is straightforward to construct the optimal mean-variance portfolios from the knowledge of the cumulants $c_1$ and $c_2$ as a function of the asset weights $w_i$. Similarly, we introduce the optimal $c_1 - c_4$ portfolios.

For a given mean return $c_1$, the portfolios that minimize the risks expressed through $c_2$ given by Eq. (21) or by $c_4$ given by Eq. (23)) are determined from the conditions

$$\frac{\partial}{\partial \omega_j} \left[ c_2 - \lambda_1 c_1 - \lambda_2 \sum_i \omega_i \right] \bigg|_{\omega_j = \omega_j^*} = 0 ,$$

$$\frac{\partial}{\partial \omega_j} \left[ c_4 - \lambda_1 c_1 - \lambda_2 \sum_i \omega_i \right] \bigg|_{\omega_j = \omega_j^*} = 0 ,$$

where the $\omega_j^*$ denote the weights for an optimal portfolio. From the normalization condition

$$\sum_i \omega_i = 1 ,$$

one of the Lagrange multipliers among $\lambda_1, \lambda_2$ can be eliminated. Let us define $cn$ such that the expressions (20,21,23) read

$$c_1 \equiv \sum_i \omega_i c_{1i} ,$$

$$c_2 \equiv \sum_i \omega_i^2 c_{2i} ,$$

$$c_4 \equiv \sum_i \omega_i^4 c_{4i} .$$

The efficient frontier for the mean-variance $c_1 - c_2$ portfolios is given by:

$$c_1 = \frac{1}{2\lambda_1} (A - B^2/D) + B/D ,$$

$$c_2 = \frac{1}{4\lambda_1^2} (A - B^2/D) + 1/D ,$$

$$A \equiv \sum_i \frac{c_{1i}^2}{c_{2i}} ,$$

$$B \equiv \sum_i \frac{c_{1i}}{c_{2i}} ,$$

$$D \equiv \sum_i \frac{1}{c_{2i}} .$$
Varying $\lambda_1$ then traces out the efficient frontier. Likewise the efficient frontier for the $c_1 - c_4$ portfolios is given by:

\[ c_1 \equiv \sum_i \omega_i^* c_{1i}, \]  
\[ c_4 \equiv \sum_i (\omega_i^*)^4 c_{4i}, \quad \text{with} \]  
\[ \omega_i^* = \frac{1}{\sum_i \pm |(c_{1i} - \lambda_2)/(4c_{4i})|^{1/3}} \pm \frac{(c_{1i} - \lambda_2)}{4c_{4j}} |^{4/3}, \]  

with $+$ if $c_{1j} > \lambda_2$ and $-$ otherwise.

Fig. 5 shows the efficient frontiers for portfolios constituted of the three assets CHV-XON-MYR. The lines are derived from the theoretical prediction given by Eq. (27) using the exponents determined from Fig. 1. The solid line shows the mean-variance efficient frontier normalized to the minimum variance and the dotted line shows the $c_1 - c_4$ efficient frontier normalized to the minimum fourth-order cumulant determined from the theory assuming no correlations between the assets. The $+$ (resp. $-$) are the empirical mean-variance (resp. $c_1 - c_4$) portfolios constructed by scanning the weights $w_1$ (Chevron), $w_2$ (Exxon) and $w_3$ (Malaysian Ringgit) in the interval $[0, 1]$ by steps of 0.02 with the condition of normalization (28). Both family define a set of accessible portfolios and the frontier of each domain define the corresponding empirical efficient frontiers. Note that by allowing negative weights (short position), the domains within the parabola are progressively filled up, corresponding to accessible portfolios with “short” positions.

The agreement is not good quantitatively between theory and empirical tests due to the strong correlations between Chevron and Exxon which is neglected in the theory (see figure 4). However, there is good qualitative agreement: the theory and empirical tests are essentially translated vertically, with the same characteristics. The most important feature is that the $c_1 - c_4$ portofolio with minimum fourth-order cumulant (small “large risks”) has a significantly larger return $c_1$ than the portfolio with the minimum variable. For instance in the historical data, the return for the minimum variance occurs for $w_1 = 0.032, w_2 = 0.084, w_3 = 0.884$ for which the mean annualized return is $c_1 = 3.1\%$ and the fourth-order cumulant is $c_4/c_{4\text{min}} = 2.22$, i.e. more than twice the minimum possible value. The minimum of $c_4$ is reached for $w_1 = 0.292, w_2 = 0.084, w_3 = 0.624$ for which the mean annualized return is $c_1 = 7.2\%$, i.e. more than double the return for optimal the mean-variance portfolio. Its variance is $c_2/c_{2\text{min}} = 1.73$ which is a relatively moderate increase of “small risks”. The results presented here can be easily generalized to higher cumulants with similar conclusions.

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Figure 1: $y(r)$-transformation defined by equations (3,4) for the period from January 1971 to October 1998. + corresponds to positive returns and o to negative returns. The daily returns $r$ are expressed in annualized percentage. a) Chevron stock (CHV), b) Exxon stock (XON), c) Malaysian Ringgit against US dollar (MYR).
Figure 2: Comparison of the empirically determined cumulants $c_n$ and excess kurtosis $\kappa$ (fat solid line) to the theory Eq. (20-23) (thin solid line) using the exponents $\gamma_i$ determine from Fig. 1 for a portfolio constituted of the Malaysian Ringgit and the Chevron stock. The cumulants are plotted as a function of the asset weight $w_1$, where the index 1 corresponds to CHV, with the normalization $w_1 + w_2 = 1$. Thus, the weight of the Malaysian Ringgit is $w_2 = 1 - w_1$. The error bars shown are obtained assuming an uncertainty in the determination of the exponents $\gamma_i = \gamma_i \pm 0.05$. 
Figure 3: Annualized daily returns (in percent) and cumulative wealth (starting with a unit wealth at time zero) for the two portfolios corresponding to the minimum variance with Chevron weight $w_1 = 0.095$ and minimum fourth-order cumulant $c_4$ with Chevron weight $w_1 = 0.38$, determined from figure 2.
Figure 4: Same as figure 2 for a portfolio constituted of the Exxon and the Chevron stocks. The cumulants are plotted as a function of the Chevron weight $w_1$ and the weight of the Exxon stock is $w_2 = 1 - w_1$. 
Figure 5: Efficient frontiers for the three-asset portfolio CHV-XON-MYR derived from theory Eq. (27) using the exponents $\gamma_i$’s determined from Fig. (1). The solid line shows the mean-variance efficient frontier normalized to the minimum variance and the dotted line shows the $c_1 - c_4$ efficient frontier normalized to the minimum fourth-order cumulant determined from the theory assuming no correlations between the assets. The + (resp. o) are the empirical mean-variance ($c_1 - c_4$) (resp. $c_1 - c_4$) portfolios constructed by scanning the weights $w_1$ (Chevron), $w_2$ (Exxon) and $w_3$ (Malaysian Ringgit) in the interval $[0, 1]$ by steps of 0.02 while still implementing the condition of normalization (28). Both family define a set of accessible portfolios excluding any “short” positions and the frontier of each domain define the corresponding empirical frontiers.