Hamiltonicity of bi-power of bipartite graphs, for finite and infinite cases *

Binlong Li
Department of Applied Mathematics, Northwestern Polytechnical University, Xi’an, Shaanxi 710072, P.R. China

Abstract: For a graph $G$, the $t$-th power $G^t$ is the graph on $V(G)$ such that two vertices are adjacent if and only if they have distance at most $t$ in $G$; and the $t$-th bi-power $G^t_B$ is the graph on $V(G)$ such that two vertices are adjacent if and only if their distance in $G$ is odd at most $t$. Fleischner’s theorem states that the square of every 2-connected finite graph has a Hamiltonian cycle. Georgakopoulos prove that the square of every 2-connected infinite locally finite graph has a Hamiltonian circle. In this paper, we consider the Hamiltonicity of the bi-power of bipartite graphs. We show that for every connected finite bipartite graph $G$ with a perfect matching, $G^3_B$ has a Hamiltonian cycle. We also show that if $G$ is a connected infinite locally finite bipartite graph with a perfect matching, then $G^3_B$ has a Hamiltonian circle.

Keywords: Bipartite graph; infinite graph; Hamiltonian cycle; Hamiltonian circle

1 Introduction

A graph $G$ is Hamiltonian if it has a Hamiltonian cycle, i.e., a cycle containing all vertices of $G$. The $t$-th power $G^t$ of $G$ is the graph on $V(G)$ such that two vertices are adjacent in $G^t$ if and only if they have distance at most $t$ in $G$. The following classical theorems concern the Hamiltonicity of the power of graphs:

Theorem 1 (Fleischner [9]). If $G$ is a 2-connected finite graph, then $G^2$ is Hamiltonian.

Theorem 2 (Sekanina [16]). If $G$ is a connected finite graph of order at least 3, then $G^3$ is Hamiltonian.

We consider the analogues of the above theorems on bipartite graphs. We first notice that a bipartite graph is Hamiltonian only if it is balanced, i.e., its two partition sets have the same size. Generally, the power of a bipartite graph may not be bipartite. In order to find a graph operation for bipartite graphs, we pose the bi-power (or bi-power for short) of graphs.

For a (bipartite or non-bipartite) graph $G$, we define the $t$-th bi-power $G^t_B$ as the graph on $V(G)$ with edge set

$$E(G^t_B) = \{xy : d_G(x, y) \text{ is odd at most } k\},$$

where $d_G(x, y)$ is the distance between $x$ and $y$ in $G$. Note that $G^2_B = G$, $G^4_B = G^3_B$, etc. It is nature to ask that is there $k, t$ such that the $t$-th bi-power of every $k$-connected balanced bipartite graph is Hamiltonian. The answer is negative by the following construction.

*Supported by NSFC (11601429, 11671320). E-mail: libinlong@mail.nwpu.edu.cn.
Let $s \geq t$ be even, and let $V_0, V_1, \ldots, V_{s+1}$ be disjoint sets of vertices such that $|V_0| = |V_{s+1}| > \frac{sk}{2}$ and $|V_i| = k$ for $1 \leq k \leq s$. Let $G$ be the graph on $\bigcup_{i=0}^{s+1} V_i$ by adding all possible edges between $V_i$ and $V_{i+1}$, $0 \leq i \leq s$. Thus $G$ is $k$-connected. Since the distance between the vertices in $V_0$ and in $V_{s+1}$ in $G$ is more than $t$, $V_0 \cup V_{s+1}$ is an independence set of $G^B$. It follows that $G^B$ is not Hamiltonian.

So we need some additional conditions to get Hamiltonian graphs.

**Theorem 3.** Let $G$ be a connected finite bipartite graph of order at least 4. If $G$ has a perfect matching, then $G^3_B$ is Hamiltonian.

We remark that the condition ‘$G$ has a perfect matching’ in Theorem 3 cannot be replaced by ‘$G^3_B$ has a perfect matching’. Let the bi-star $S_{k,k}$, $k \geq 3$, be the tree with two vertices of degree $k + 1$ and all other vertices of degree 1; and let $G$ be the graph obtained by subdividing each pendant edge of $S_{k,k}$ twice. One can check that $G^3_B$ has a perfect matching but is not Hamiltonian.

Now we turn to the infinite graphs. Thomassen [19] generalized Theorem 1 to locally finite graphs with one end. Diestel [4] launched the ambitious project of extending results on finite Hamiltonian cycles to Hamiltonian circles in infinite graphs. Diestel [4] then conjectured that the square of any 2-connected locally finite graph has a Hamiltonian circle. Georgakopoulos [10] confirmed Diestel’s conjecture. Since it is necessary to introduce a lot of terminology and notations in order to state the definition of Hamiltonian circles of infinite graphs, we refrain from stating the concept explicitly in this introductory section. Here we present Georgakopoulos’s theorem on infinite graphs concerning our topic. We will explain the concepts in Section 3 (see also [5], Chapter 8). We apologize for the inconvenience this may cause.

**Theorem 4** (Georgakopoulos [10]). Suppose that $G$ is an infinite locally finite graph.

1. If $G$ is 2-connected, then $G^2$ has a Hamiltonian circle.
2. If $G$ is connected, then $G^3$ has a Hamiltonian circle.

Several other results in the area of Hamiltonian circles of infinite graphs can be found in [2, 3, 11, 12, 15]. Our main result of the paper is the infinite extension of Theorem 3.

**Theorem 5.** Let $G$ be a connected infinite locally finite bipartite graph. If $G$ has a perfect matching, then $G^3_B$ has a Hamiltonian circle.

The rest of the paper is organized as follows. In Section 2, we exhibit the proof of Theorem 5 with a lemma that will be also used for our infinite proof. In Section 3, after introducing the basic terminology and notations, we give some lemmas and techniques for dealing with the infinite Hamiltonian problems, following which we complete the proof of Theorem 5.

## 2 Finite graphs

For the purpose of the case of infinite graphs, we first give some new definitions and a lemma.

Let $G$ be a graph, $A, B \subseteq V(G)$ be disjoint, and $P = v_0v_1 \ldots v_p$ be a nontrivial path of $G$. We say that $P$ is an $(A, B)$-path if $V(P) \cap A = \{v_0\}$ and $V(P) \cap B = \{v_p\}$; and $P$ is an $A$-path if $V(P) \cap A = \{v_0, v_p\}$. For two disjoint subgraph $H, K$ of $G$, a $(V(H), V(K))$-path $(V(H)$-path) is also called an $(H, K)$-path ($H$-path). Let $F$ be a path or a cycle, $T$ be a tree of $G$, $e \in E(T)$ and $T_1, T_2$ be
the two components of \( T - e \). We say that \( F \) crosses the edge \( e \) \( k \) times with respect to \( T \) if \( F \) contains \( k \) \((T_1, T_2)\)-paths.

Now we prove the following lemma.

Lemma 1. If \( T \) is a finite tree and \( M \) is a perfect matching of \( T \), then for every edge \( xy \in M \), \( T_B^3 \) has a Hamiltonian \((x, y)\)-path crossing every edge \( e \in E(T) \setminus M \) exactly twice with respect to \( T \).

Proof. We use induction on the order of \( T \). The assertion is trivially true if \( T \) has only two vertices. So we assume that \(|V(T)| \geq 4\). Since \( T \) is a tree and \( xy \in E(T) \), every component of \( T - \{x, y\} \) has a neighbor of either \( x \) or \( y \), but not both. Let \( \mathcal{H}^1 = \{H_1^1, \ldots, H_l^1\} \) be the set of components of \( G - \{x, y\} \) that have a neighbor of \( x \) and \( \mathcal{H}^2 = \{H_1^2, \ldots, H_l^2\} \) be the set of components of \( G - \{x, y\} \) that have a neighbor of \( y \). For each \( H_i^1 \in \mathcal{H}^1 \), let \( x_i^1 \) be such that \( x \) neighbored \( y_i^1 \); for each \( H_i^2 \in \mathcal{H}^2 \), let \( x_i^2 \) be such that \( y \) neighbored \( x_i^2 \). By induction hypothesis, \( (H_i^1)^3_B \) has a Hamiltonian \((x_i^1, y_i^1)\)-path \( P_i^1 \) that crosses every edge \( e \in E(H_i^1) \setminus M \) exactly twice with respect to \( H_i^1 \); and \( (H_i^2)^3_B \) has a Hamiltonian \((x_i^2, y_i^2)\)-path \( P_i^2 \) that crosses every edge \( e \in E(H_i^2) \setminus M \) exactly twice with respect to \( H_i^2 \). If \( \mathcal{H}^1 = \emptyset \), then \( P = x_1^1 y_1^1 P_1^1 x_2^1 y_2^1 P_2^1 \cdots y_l^1 P_l^1 x_1^1 y_1^1 P_1^1 x_2^1 \cdots y_l^1 P_l^1 x_1^1 y_1^1 P_1^1 x_2^1 \cdots y_l^1 P_l^1 \) is a Hamiltonian \((x, y)\)-path of \( G_B^3 \) that crosses every edge \( e \in E(T) \setminus M \) exactly twice with respect to \( T \). The case of \( \mathcal{H}^2 = \emptyset \) is similar. If neither \( \mathcal{H}^1 \) nor \( \mathcal{H}^2 \) is empty, then \( P = x_1^1 y_1^1 P_1^1 x_2^1 \cdots y_l^1 P_l^1 x_1^1 y_1^1 P_1^1 x_2^1 \cdots y_l^1 P_l^1 x_1^1 y_1^1 P_1^1 x_2^1 \cdots y_l^1 P_l^1 \) is a Hamiltonian \((x, y)\)-path of \( G_B^3 \) that crosses every edge \( e \in E(T) \setminus M \) exactly twice with respect to \( T \).

We say that a balanced bipartite graph \( G \) is Hamilton-laceable if for any two vertices \( x, y \) in distinct bipartition sets, \( G \) has a Hamiltonian \((x, y)\)-path (i.e., an \((x, y)\)-path containing all vertices of \( G \)). The concept Hamilton-laceability was introduced by Simmons [17, 18], and sometimes it is called Hamilton-biconnectedness (see [7, 8] for examples). Note that every Hamilton-laceable balanced bipartite graph (apart from \( K_2 \)) is Hamiltonian. Now we prove the following result, which is stronger than Theorem 3.

Theorem 6. If \( G \) is a connected finite bipartite graph that has a perfect matching, then \( G_B^3 \) is Hamilton-laceable.

Proof. We use induction on the order of \( G \). The assertion is trivial if \( G \) has only two vertices. So we assume that \(|V(G)| \geq 4\). If \( G \) is not a tree, then it has a spanning tree with a perfect matching, which can be obtained by taking a perfect matching of \( G \) and adding edges one by one avoiding creating cycles, until no edges can be added. So we need only consider the case that \( G \) is a tree. Let \( M \) be a perfect matching of \( G \).

Let \( x \in X \), \( y \in Y \) be any two vertices, where \( X, Y \) are the two bipartition sets of \( G \). We will find a Hamiltonian \((x, y)\)-path in \( G_B^3 \). Recall that we assume that \( G \) is a tree. If \( xy \in M \), then we are done by Lemma 1. So we assume that \( xy \notin M \). It follows that the unique \((x, y)\)-path of \( G \) contains some edges \( e = x' y' \in E(G) \setminus M \), where \( x' \in X \) and \( y' \in Y \). Thus \( G - x' y' \) has exactly two components one of which contains \( x \) and the other contains \( y \). Let \( H_1, H_2 \) be the two components of \( G - x' y' \) containing \( x \) and \( y \), respectively.

If \( y' \in V(H_1) \) and \( x' \in V(H_2) \), then by induction hypothesis, \( (H_1)^3_B \) has a Hamiltonian \((x', y')\)-path \( P_1 \), and \( (H_2)^3_B \) has a Hamiltonian \((x', y')\)-path \( P_2 \). Thus \( P = P_1 y' x' P_2 \) is an Hamiltonian \((x, y)\)-path of \( G_B^3 \). If \( x' \in V(H_1) \) and \( y' \in V(H_2) \), then let \( y'', x'' \) be the neighbors of \( x', y' \) in \( M \), respectively. By induction hypothesis, \( (H_1)^3_B \) has a Hamiltonian \((x', y'')\)-path \( P_1 \), and \( (H_2)^3_B \) has a Hamiltonian \((x'', y')\)-path \( P_2 \). Note that \( x'' y'' \in E(G_B^3) \), implying that \( P = P_1 y'' x'' P_2 \) is a Hamiltonian \((x, y)\)-path of \( G_B^3 \).
3 Infinite graphs

3.1 Basic terminology and notations

Now we consider the infinite graphs. We first give the terminology concerning circles of infinite graphs.

An (infinite) graph $G$ is locally finite if every vertex of $G$ has finite degree. In this section, we always assume that $G$ is a locally finite graph. A 1-way infinite path is called a ray, and the subrays of a ray are its tails. Two rays of $G$ are equivalent if for every finite set $S \subseteq V(G)$, there is a component of $G - S$ containing tails of both rays. We write $R_1 \approx_G R_2$ if $R_1$ and $R_2$ are equivalent in $G$. The corresponding equivalence classes of rays are the ends of $G$. We denote by $\Omega(G)$ the set of ends of $G$.

Let $\alpha \in \Omega(G)$ and $S \subseteq V(G)$ be a finite set. We denote by $C_G(S, \alpha)$ the unique component of $G - S$ that containing a ray (and a tail of every ray) in $\alpha$. We let $\Omega_G(S, \alpha)$ be the set of all ends $\beta$ with $C_G(S, \beta) = C_G(S, \alpha)$. When no confusion occurs, we will denote $C_G(S, \alpha)$ and $\Omega_G(S, \alpha)$ by $C(S, \alpha)$ and $\Omega(S, \alpha)$, respectively.

To built a topological space $|G|$ we associate each edge $uv \in E(G)$ with a homeomorphic image of the unit real interval $[0, 1]$, where 0,1 map to $u,v$ and different edges may only intersect at common endpoints. Basic open neighborhoods of points that are vertices or inner points of edges are defined in the usual way, that is, in the topology of the 1-complex. For an end $\alpha$ we let the basic neighborhood $\hat{C}(S, \alpha) = C(S, \alpha) \cup \Omega(S, \alpha) \cup E(S, \alpha)$, where $E(S, \alpha)$ is the set of all inner points of the edges between $C(S, \alpha)$ and $S$. This completes the definition of $|G|$, called the Freudenthal compactification of $G$. In [5] it is shown that if $G$ is connected and locally finite, then $|G|$ is a compact Hausdorff space.

An arc of $G$ a homeomorphic map of the unit interval $[0, 1]$ in $|G|$; and a circle is a homeomorphic map of the unit circle $S^1$ in $|G|$. A circle of $G$ is Hamiltonian if it meets every vertex (and then every end) of $G$.

We define a curve of $G$ as a continuous map of the unit interval $[0, 1]$ in $|G|$. A curve is closed if 0,1 map to the same point; and is Hamiltonian if it is closed and meets every vertex of $G$ exactly once. In other words, a Hamiltonian curve is a continuous map of the unit circle $S^1$ in $|G|$ that meets every vertex of $G$ exactly once. Note that a Hamiltonian circle is a Hamiltonian curve but not vice versa.

3.2 Faithful subgraphs

For a finite graphs $G$, if $G$ has a spanning subgraph $H$ that is Hamiltonian, then $G$ itself is Hamiltonian. But this is not true for infinite graphs, in the meaning that $H$ having a Hamiltonian circle does not imply $G$ has one. The main reason is that we have to guarantee injectivity at the ends in Hamiltonian circles. Now we define a type of subgraphs that are stable on Hamiltonian circles. We say a subgraph $H$ of $G$ is faithful if

1. every end of $G$ contains a ray of $H$; and
2. for any two rays $R_1, R_2$ of $H$, $R_1 \approx_H R_2$ if and only if $R_1 \approx_G R_2$.

If $H \subseteq G$, then for every finite set $S \subseteq V(H)$, each component of $H - S$ is contained in a component of $G - S$. Thus the condition (2) can be replaced by ‘for any two rays $R_1, R_2$ of $H$, $R_1 \approx_G R_2$ implies $R_1 \approx_H R_2$’.

Lemma 2. Let $H$ be a faithful spanning subgraph of $G$. If $H$ has a Hamiltonian circle, then $G$ has a Hamiltonian circle.
Proof. We define a map \( \pi : \Omega(H) \to \Omega(G) \) such that for the end \( \alpha \) of \( H \), \( \pi(\alpha) \) is the end of \( G \) containing all the rays in \( \alpha \). By the definition of the faithful subgraphs, \( \pi \) is a bijection between \( \Omega(H) \) and \( \Omega(G) \) (see also [10]). Let \( \alpha \in \Omega(H) \) and \( S \subseteq V(G) \) be finite. Since \( H \leq G \), the component \( C_H(S,\alpha) \) is contained in \( C_G(S,\pi(\alpha)) \). If there is an end \( \beta \in \Omega_H(S,\alpha) \), then every ray in \( \beta \) has a tail contained in \( C_H(S,\alpha) \), which is contained in \( C_G(S,\pi(\alpha)) \). This implies that \( \pi(\beta) \in \Omega_G(S,\pi(\alpha)) \). It follows that \( \pi(\Omega_H(S,\alpha)) \subseteq \Omega_G(S,\pi(\alpha)) \).

Now let \( \sigma_H : S^1 \to |H| \) be a Hamiltonian circle of \( H \). We define \( \sigma_G : S^1 \to |G| \) such that

\[
\sigma_G(p) = \begin{cases} 
\pi(\sigma_H(p)), & \text{if } \sigma_H(p) \in \Omega(H); \\
\sigma_H(p), & \text{otherwise}.
\end{cases}
\]

Clearly the map \( \sigma_G \) is injective and meets all vertices of \( V(G) \). Now we prove that it is continuous.

Since \( \sigma_H \) is homeomorphic, \( \sigma_G \) is continuous at point \( p \) if \( \sigma_H(p) \) is a vertex or is an inner point of an edge. Now we assume that \( \sigma_H(p) = \alpha \in \Omega(H) \). Let \( p = (p_i)_{i=0}^\infty \) be a sequence of points in \( S^1 \) converging to \( p \) and let \( S \subseteq V(G) \) be a finite set. Since \( \sigma_H \) is continuous, the neighborhood \( C_H(S,\alpha) \) of \( \alpha \) contains almost all terms of \( (\sigma(p_i))_{i=0}^\infty \) (that is, there exists \( j \) such that \( \sigma(p_i) \in C_H(S,\alpha) \) for all \( i \geq j \)). Recall that \( C_H(S,\alpha) \subseteq C_G(S,\pi(\alpha)) \), \( E_H(S,\alpha) \subseteq E_G(S,\pi(\alpha)) \) and \( \pi(\Omega_H(S,\alpha)) \subseteq \Omega_G(S,\pi(\alpha)) \). It follows that \( \hat{C}_G(S,\pi(\alpha)) \) contains all terms of \( (\sigma(p_i))_{i=0}^\infty \). Thus \( \sigma_G \) is homeomorphic and then is a Hamiltonian circle of \( G \). \( \square \)

Lemma 3. Suppose that \( K \leq H \leq G \). If \( H \) is faithful to \( G \) and \( K \) is faithful to \( H \), then \( K \) is faithful to \( G \).

Proof. Let \( \alpha_G \) be an arbitrary end of \( G \). Since \( H \) is faithful to \( G \), \( H \) has a ray \( R_H \in \alpha_G \). Let \( \alpha_H \) be the end of \( H \) with \( R_H \in \alpha_H \). Since \( K \) is faithful to \( H \), \( K \) has a ray \( R_K \in \alpha_H \). Thus \( R_K \approx_H R_H \), implying that \( R_K \approx_G R_H \), that is, \( R_K \in \alpha_G \).

Now let \( R_1, R_2 \) be two rays of \( K \). Since \( K \) is faithful to \( H \), \( R_1 \approx_K R_2 \) if and only if \( R_1 \approx_H R_2 \). Since \( H \) is faithful to \( G \), \( R_1 \approx_H R_2 \) if and only if \( R_1 \approx_G R_2 \). This implies that \( R_1 \approx_K R_2 \) if and only if \( R_1 \approx_K R_2 \). It follows that \( K \) is faithful to \( G \). \( \square \)

A comb of \( G \) is the union of a ray \( R \) with infinitely many disjoint finite paths having precisely their first vertex on \( R \); the last vertices of the paths are the teeth of the comb; and the ray \( R \) is the spine of the comb. We will use the following Star-Comb Lemma in our proof.

Lemma 4 (Diestel, see [3]). If \( U \) is an infinite set of vertices in a connected graph, then the graph contains either a comb with all teeth in \( U \) or a subdivision of an infinite star with all leaves in \( U \).

Since a locally finite graph \( G \) contains no infinite stars, Lemma 4 always yields a comb of \( G \). If \( H \) is a connected spanning subgraph of \( G \), then for every ray \( R \) of \( G \), the spine \( R' \) of a comb of \( H \) with all teeth in \( V(R) \) is a ray in \( H \) with \( R \approx_G R' \). Therefore the connected spanning subgraph \( H \) is faithful to \( G \) if and only if for any two rays \( R_1, R_2 \) of \( H \), \( R_1 \approx_G R_2 \) implies \( R_1 \approx_H R_2 \).

Lemma 5. Let \( H \) be a spanning subgraph of \( G \) and \( K \) be a spanning subgraph of \( H \). If \( K \) is connected and faithful to \( G \), then \( H \) is faithful to \( G \) and \( K \) is faithful to \( H \).

Proof. Let \( R_1, R_2 \) be two rays of \( H \) with \( R_1 \approx_G R_2 \). Let \( \alpha_G \) be the end of \( G \) containing \( R_1, R_2 \), let \( R \in \alpha_G \) be a ray of \( K \). By Lemma 4 \( K \) has a comb with all teeth in \( V(R_1) \). Let \( R'_1 \) be the spine of
the comb. Thus $R'_i$ is a ray of $K$ and $R_1 \approx_H R'_1$. Since $H \leq G$, $R_1 \approx_G R'_1$ and then $R \approx_G R'_1$. Since $K$ is faithful to $G$, $R \approx_K R'_1$. Since $K \leq H$, $R \approx_H R'_1$, and then $R \approx_H R_1$. By a similar analysis, we have $R \approx_H R_2$, and thus $R_1 \approx_H R_2$. This implies that $H$ is faithful to $G$.

Now let $R_1, R_2$ be two rays of $K$ with $R_1 \approx_H R_2$. Since $H$ is faithful to $G$, $R_1 \approx_G R_2$. Since $K$ is faithful to $G$, $R_1 \approx_K R_2$. It follows that $K$ is faithful to $H$.

Lemma 6. Let $\mathcal{P}$ be a partition of $V(G)$ such that $G[P]$ is connected and finite for every $P \in \mathcal{P}$, and $\mathcal{G}$ be the graph on $\mathcal{P}$ such that for $P_1, P_2 \in \mathcal{P}$, $P_1P_2 \in E(\mathcal{G})$ if and only if $E(G(P_1, P_2)) \neq \emptyset$, and let $T$ be a spanning tree of $\mathcal{G}$. For every partition set $P \in \mathcal{P}$, let $T_P$ be a spanning tree of $G[P]$; and for every edge $f = P_1P_2 \in E(\mathcal{G})$, let $e_f$ be an edge in $E(G(P_1, P_2))$. Let $T$ be the spanning tree of $\mathcal{G}$ with edge set

$$\{e_f : f \in E(T)\} \cup \bigcup_{P \in \mathcal{P}} E(T_P).$$

If $T$ is faithful to $\mathcal{G}$, then $T$ is faithful to $G$.

Proof. For every ray $R = v_0v_1v_2 \ldots$ of $G$, we define a ray $\rho(R)$ of $\mathcal{G}$ as follows: Let $P_0 \in \mathcal{P}$ with $v_0 \in P_0$ and $\phi(0)$ be the maximum integer with $v_{\phi(0)} \in R$ ($\phi(0)$ exists since $P_0$ is finite). Suppose we have already defined $P_{i-1}$ and $\phi(i-1)$, $i \geq 1$. Let $P_i \in \mathcal{P}$ such that $v_{\phi(i-1)+1} \in P_i$ and $\phi(i)$ be the maximum integer with $v_{\phi(i)} \in P_i$. Clearly $v_{\phi(i-1)} \in P_{i-1}$, $v_{\phi(i)+1} \in P_i$, implying that $E_G(P_{i-1}, P_i) \neq \emptyset$, and $P_{i-1}P_i \in E(\mathcal{G})$, $i \geq 1$. Now it follows that $\rho(R) = P_0P_1P_2 \ldots$ is a ray of $\mathcal{G}$. Note that if $R$ is a ray of $T$, then $\rho(R)$ is a ray of $T$.

We claim that if $R_1 \approx_G R_2$, then $\rho(R_1) \approx_{\mathcal{G}} \rho(R_2)$. Let $S \subseteq \mathcal{P} = V(\mathcal{G})$ be an arbitrary finite set. Set $S = \bigcup_{P \in S} P$. Since each $P \in \mathcal{P}$ is finite, $S$ is finite. If $R_1 \approx_G R_2$, then there is a component $C$ of $G - S$ that contains a tail $R'_i$ of $R_i$, $i = 1, 2$. Recall that each $P \in \mathcal{P}$ induces a connected finite subgraph of $G$. Thus the subgraph $C$ of $\mathcal{G}$ induced by $\{P \in \mathcal{P} : P \subseteq V(C)\}$ is a component of $\mathcal{G} - S$. Since $C$ contains all the vertices $P$ with $P \cap V(R'_i) \neq \emptyset$, $C$ contains a tail of $\rho(R_i)$ for $i = 1, 2$. It follows that $\rho(R_1) \approx_{\mathcal{G}} \rho(R_2)$.

For every ray $R = P_0P_1P_2 \ldots$ of $T$, we define a ray $\phi(R)$ of $T$ as follows: Let $u_0 \in P_0$ be a fixed vertex. For $i \geq 0$, let $e_{P_iP_{i+1}} = v_{i}u_{i+1}$ be the unique edge in $E_G(P_i, P_{i+1})$, let $R_i$ be the unique $(u_i, v_i)$-path of $T_i$. It follows that $\phi(R) = u_0R_0u_1R_1u_2R_2u_3 \ldots$ is a ray of $T$. Note that for every ray $R$ of $T$, $\rho(\phi(R)) = R$; and for every ray $R$ of $T$, $R$ and $\phi(\rho(R))$ differ only by a finite initial segments (and so $R \approx_T \phi(\rho(R))$).

We claim that if $R_1 \approx_T R_2$, then $\rho(R_1) \approx_T \rho(R_2)$. Let $S \subseteq V(G)$ be an arbitrary finite set. Set $S = \{P \in \mathcal{P} : P \cap S \neq \emptyset\}$. So $S$ is a finite subset of $V(\mathcal{G})$. If $R_1 \approx_T R_2$, then there is a component $C$ of $T - S$ that contains a tail $R'_i$ of $R_i$, $i = 1, 2$. Let $C$ be the subgraph of $T$ induced by $\bigcup_{P \in V(C)} P$. It follows that $C$ is contained in a component of $T - S$. Since $C$ contains all vertices in $\bigcup_{P \in V(C)} P$, $C$ contains a tail of $\phi(R_i)$ for $i = 1, 2$. It follows that $\rho(R_1) \approx_T \rho(R_2)$.

Now we prove the lemma. Suppose that $T$ is faithful to $\mathcal{G}$, and $R_1, R_2$ are two rays of $T$ such that $R_1 \approx_G R_2$. It follows that $\rho(R_1), \rho(R_2)$ are two rays of $T$ with $\rho(R_1) \approx_{\mathcal{G}} \rho(R_2)$. Since $T$ is faithful to $\mathcal{G}$, $\rho(R_1) \approx_T \rho(R_2)$, and thus $\phi(\rho(R_1)) \approx_T \phi(\rho(R_2))$. Recall that $R_1 \approx_T \phi(\rho(R_1))$ and $R_2 \approx_T \phi(\rho(R_2))$. We have $R_1 \approx_T R_2$, implying that $T$ is faithful to $G$.

Lemma 7. For any connected graph $G$ and integer $t \geq 1$, $G$ is faithful to $G^t$ and $G^t_B$.  

6
Proof. Suppose that $R_1, R_2$ are two rays of $G$ with $R_1 \approx_{G^t} R_2$. Let $S \subseteq V(G)$ be an arbitrary finite set, and set $S' = S \cup N_G(S)$. Clearly $S'$ is finite, and thus there is a component $C'$ of $G^t - S'$ that contains tails of both $R_1$ and $R_2$. For any two adjacent vertices $u, v \in V(G) \setminus S'$, $G$ has a $(u, v)$-path $P$ of length at most $t$. Since both $u, v$ have distance more than $t$ from $S$, $V(P) \cap S = \emptyset$. It follows that $u, v$ are contained in a common component of $G - S$. This implies that all vertices in $V(C')$ are contained in a common component $C$ of $G - S$. Thus $C$ contains tails of both $R_1$ and $R_2$, implying that $G$ is faithful to $G'$.

Recall that $G'_{B}$ is a spanning subgraph of $G^t$. By Lemma 5, $G$ is faithful to $G'_{B}$ as well. □

A rooted tree $T$ of $G$ is normal if the end-vertices of every $T$-path in $G$ are comparable in the tree-order of $T$. Note that if $T$ is spanning, then every $T$-path is an edge of $G$. The normal rays of $T$ are those starting at the root of $T$. From the following lemma, one can see that a normal spanning tree of $G$ is faithful to $G$.

Lemma 8 (Diestel, see [5]). If $T$ is a normal spanning tree of $G$, then every end of $G$ contains exactly one normal ray of $T$.

One can see that the normal spanning tree has a nice property for the infinite graphs. From the following theorem, we can always find a normal spanning tree in connected locally finite graphs.

Theorem 7 (Jung [13]). Every countable connected graph has a normal spanning tree.

3.3 Degree of ends

The (vertex-)degree of an end $\alpha \in \Omega(G)$ is the maximum number of vertex-disjoint rays in $\alpha$; and the edge-degree of $\alpha$ is the maximum number of edge-disjoint rays in $\alpha$. We refer the reader to [1] for some properties on the end degrees of graphs. Before giving our lemma concerning the degree of ends, we first list the following König’s Infinite Lemma.

Lemma 9 (König, see [5]). Let $V_0, V_1, V_2, \ldots$ be an infinite sequence of disjoint non-empty finite sets, and let $G$ be a graph on $\bigcup_{i=0}^{\infty} V_i$. Assume that every vertex in $V_i$ has a neighbor in $V_{i-1}$, $i \geq 1$. Then $G$ has a ray $R = v_0v_1v_2\ldots$ with $v_i \in V_i$ for all $i \geq 0$.

Lemma 10. Let $A, B \subseteq V(G)$ be disjoint, and $\alpha \in \Omega(G)$.

1. $G$ has $k$ vertex-disjoint $(A, B)$-paths if and only if $|G|$ has $k$ vertex-disjoint $(A, B)$-curves.

2. $\alpha$ has degree at least $k$ if and only if $|G|$ has $k$ vertex-disjoint nontrivial curves ending in $\alpha$.

Proof. (1) The necessity of the assertion is trivial since a (topological) path of $G$ is also a curve of $|G|$. Now we prove the sufficiency of the assertion. Suppose that $G$ has no $k$ vertex-disjoint $(A, B)$-paths. By Menger’s Theorem, there is a set $S \subseteq V(G)$ with $|S| < k$ such that $G - S$ has no $(A, B)$-path. If $|G|$ has $k$ vertex-disjoint $(A, B)$-curves, then one of them is contained in $|G - S|$. It follows that some component of $G - S$ contains some vertices of both $A$ and $B$, and thus $G - S$ has an $(A, B)$-path (see also [3]), a contradiction.

(2) The necessity of the assertion is trivial since a ray in $\alpha$ is a curve of $|G|$ ending in $\alpha$. Now we prove the sufficiency of the assertion. Clearly any nontrivial curve ending in $\alpha$ contains some vertices. For convenience we assume that $|G|$ has $k$ vertex-disjoint curves between some vertices and $\alpha$. Let $S_0$ be the set of the starting vertices of the $k$ curves. For $i \geq 1$, set $S_i = S_{i-1} \cup N(S_{i-1})$. Thus $S_i$
is finite and \(|G|\) has \(k\) vertex-disjoint curves between \(S_0\) and \(C(S_i, \alpha)\), for all \(i \geq 0\). By \((1)\), \(G\) has \(k\) vertex-disjoint paths between \(S_0\) and \(C(S_i, \alpha)\). Let \(V_i\) be the set of the unions of \(k\) vertex-disjoint paths between \(S_0\) and \(C(S_i, \alpha)\). Since every path between \(S_0\) and \(C(S_i, \alpha)\) is contained in \(S_{i+1}\), which is finite, we can see that \(V_i\) is finite for every \(i \geq 0\).

We define a graph \(G\) on \(\bigcup_{i=0}^{\infty} V_i\) such that \(U_{i-1} \in V_{i-1}\) is adjacent to \(U_i \in V_i\) if and only if the \(k\) paths of \(U_{i-1}\) are the subpaths of the \(k\) paths of \(U_i\). Clearly every vertex in \(V_i\) has a neighbor in \(V_{i-1}\).

By Lemma 3, \(G\) has a ray \(R = U_0 U_1 U_2 \ldots\) with \(U_i \in V_i\), \(i \geq 0\). It follows that \(\bigcup_{i=0}^{\infty} U_i\) is the union of \(k\) vertex-disjoint rays in \(\alpha\), implying that the degree of \(\alpha\) is at least \(k\).

**Lemma 11.** Let \(T\) be a faithful spanning tree of \(G\) and \(F \subseteq E(T)\) such that every component of \(T - F\) is finite. If for every edge \(e \in F\), \(G\) has at most \(k\) edges between the two components \(T_1, T_2\) of \(T - e\), then every end of \(G\) has degree at most \(k\).

**Proof.** Let \(\alpha_G\) be an arbitrary end of \(G\), \(R\) be a ray of \(T\) contained in \(\alpha_G\), and \(\alpha_T\) be the end of \(T\) containing \(R\). We first claim that for every ray \(R' \in \alpha_G\) and every finite subtree \(T_0\) of \(T\), the component \(C = C_T(V(T_0), \alpha_T)\) contains almost all vertices of \(R'\). Suppose otherwise that \(R'\) has infinitely many vertices contained in \(T - C\). Note that \(T - C\) is connected. By Lemma 4, \(T - C\) has a comb with all teeth in \(V(R')\). Let \(R''\) be the spine of the comb. It follows that \(R''\) is a ray of \(T\) and \(R' \approx G R''\). Since \(R \approx G R', R \approx G R''\). Since \(T\) is faithful to \(G\), \(R \approx_G R''\), contradicting the fact that \(R''\) has no tail in \(C\).

Now we prove the lemma. Let \(\alpha_G, \alpha_T\) be defined as above. Suppose that \(\alpha_G\) has degree at least \(k + 1\). Let \(S_0\) be the starting vertices of \(k + 1\) vertex-disjoint rays in \(\alpha\), \(T_0\) be a subtree of \(T\) containing \(S_0\) and \(H\) be the set of the components \(H\) of \(T - F\) with \(V(H) \cap V(T_0) \neq \emptyset\). Set \(S_1 = \bigcup_{H \in H} V(H)\), and \(T_1 = T[S_1]\). Clearly \(S_1 \supseteq S_0\) is finite and \(T_1\) is a finite subtree of \(T\). Recall that every ray in \(\alpha_G\) contains some vertices of \(C_T(S_1, \alpha_T)\). It follows that \(G\) has \(k + 1\) vertex-disjoint paths between \(S_1\) and \(C_T(S_1, \alpha_T)\). Let \(e\) be the unique edge of \(T\) between \(S_1\) and \(C_T(S_1, \alpha_T)\). Clearly \(e \in F\) and thus \(E_G(S_1, C_T(S_1, \alpha_T)) \leq k\), a contradiction.

### 3.4 Hamiltonian curves and Hamiltonian circles

In [14], the authors obtained some necessary and sufficient conditions for a graph \(G\) to have a Hamiltonian curve. We list one of the conditions which we will use in our paper.

**Theorem 8** (Kündgen et al. [14]). *The graph \(G\) has a Hamiltonian curve if and only if every finite set \(S \subseteq V(G)\) is contained in a cycle of \(G\).*

Clearly if a Hamiltonian curve meets every end exactly once, then it is also a Hamiltonian circle.

**Lemma 12.** If every end of \(G\) has degree at most 3, then every Hamiltonian curve of \(G\) is also a Hamiltonian circle.

**Proof.** It sufficient to show that the Hamiltonian curve passes through each end exactly once. Suppose that it passes through an end \(\alpha\) at least twice. It is clearly that \(G\) has four vertex-disjoint curves ending in \(\alpha\). By Lemma 10, \(\alpha\) has degree at least 4, a contradiction.

**Theorem 9.** Let \(T\) be a faithful spanning tree of \(G\), and \(F \subseteq E(T)\) such that every component of \(T - F\) is finite. Suppose that for every subtree \(T'\) of \(T\), \(G\) has a cycle \(C'\) such that

1. \(V(T') \subseteq V(C')\), and
(2) \( C' \) crosses each edges in \( F \cap E(T') \) exactly twice respect to \( T \).

Then \( G \) has a Hamiltonian circle.

**Proof.** Let \( V(G) = \{ v_i : i \geq 0 \} \). For \( i \geq 0 \), let \( T_i \) be a subtree of \( T \) containing all vertices of \( \{ v_0, \ldots, v_i \} \), and \( C_i \) be a cycle of \( G \) with \( V(T_i) \subseteq V(C_i) \) and \( C_i \) crosses each edges in \( F \cap E(T_i) \) exactly twice respect to \( T \). Set \( C = (C_i)_{i=0}^\infty \). In the following, we will define a sequence of infinite subsequences of \( C \), a sequence of finite subsets of \( E(G) \), and a sequence of finite subsets of \( F \).

First let \( C^0 = (C^0_i)_{i=0}^\infty = C \) and \( E^0 = F^0 = \emptyset \). Suppose now we have already defined \( C^{i-1} \), \( E^{i-1} \) and \( F^{i-1} \).

Consider the first cycle \( C^{i-1}_0 \) in \( C^{i-1} \). Let \( \mathcal{H}_i \) be the set of components \( H \) of \( T - F \) such that \( V(H) \cap V(C^{i-1}_0) \neq \emptyset \). Set \( S_i = \bigcup_{H \in \mathcal{H}_i} V(H) \), \( S'_i = S_i \cup N(S_i) \), \( E_i = E_G(S_i, G - S_i) \), and \( F^i = E_T(S_i, G - S_i) \). Clearly \( F^i \subseteq F \). Since \( C^{i-1}_0 \) is a cycle and each component of \( T - F \) is finite, we see that \( S_i \) is finite. Since \( G \) is locally finite, we have that \( S'_i \), \( E_i \) and \( F^i \) are finite.

Note that there are only finitely many cycles in \( C \) (and then, in \( C^{i-1} \)) that does not contain all vertices in \( S'_i \). It follows that there are infinitely many cycles in \( C^{i-1} \) that contains all vertices of \( S'_i \). Recall that \( E_i \) is finite, and has only finitely many of subsets. So there is a set \( E^i \subseteq E_i \) such that for infinitely many cycles \( C \) in \( C^{i-1} \), \( E(C) \cap E_i = E^i \). Let \( C^i \) be the subsequence of \( C^{i-1} \) consists of all the cycles \( C \) with \( S'_i \subseteq V(C) \) and \( E(C) \cap E_i = E^i \).

From the above construction, we can see that every cycle in \( C^i \) containing all vertices of \( C^{i-1}_0 \), and for \( i, j \geq 0 \),

\[
E_i \cap E(C^i_0) = \begin{cases} 
\emptyset, & \text{if } j < i; \\
E^i, & \text{if } j \geq i.
\end{cases}
\]

Recall that every cycle in \( C^i \) (and then \( C^0_i \)) crosses every edge in \( F^i \) exactly twice respect to \( T \). It follows that for any edge \( e \) of \( F^i \), \( E^i \) contains exactly two edges in \( E_G(T_1, T_2) \), where \( T_1, T_2 \) are the two components of \( T - e \).

Now let \( F' = \bigcup_{i=0}^\infty F^i \) and \( G' \) be the spanning subgraph of \( G \) with edge set

\[
E(G') = E(T) \cup \bigcup_{i=0}^\infty E(C^i_0).
\]

It follows that for every edge \( e \in F' \), \( G' \) has at most three edges between the two components of \( T - e \).

We claim that every component of \( T - F' \) is finite. Suppose otherwise that there is an infinite component \( H' \) of \( T - F' \). Let \( v \in V(H') \). Note that there are only finitely many of cycles in \( C \) not containing \( v \). It follows that there exists \( i \) with \( v \in V(C^{i-1}_0) \). Let \( S_i \) be defined as above. Since \( S_i \) is finite, there is some edge in \( E_T(S_i, G - S_i) \cap E(H') \), which is contained in \( F^i \), a contradiction. Thus we conclude that every component of \( T - F' \) is finite.

By Lemma 11 every end of \( G' \) has degree at most 3. Clearly every finite subset of \( V(G') \) is contained in a cycle of \( G' \). By Theorem 3 \( G' \) has a Hamiltonian curve. By Lemma 12 \( G' \) has a Hamiltonian circle. By Lemma 13 \( G' \) is faithful to \( G \). By Lemma 2 \( G \) has a Hamiltonian circle.

\[ \square \]

### 3.5 Proof of Theorem 5

Let \( M \) be a perfect matching of \( G \). We define a graph \( G \) on \( M \) such that for any two edges \( e_1, e_2 \in M \), \( e_1 e_2 \in E(G) \) if and only if \( G \) has an edge between \( e_1 \) and \( e_2 \). Clearly \( G \) is connected and locally finite.
By Theorem 7, $G$ has a normal tree $T$, which is faithful to $G$ by Lemma 8. By Lemma 6, $G$ has a faithful spanning tree $T$ containing all edges in $M$. By Lemma 7, $T$ is faithful to $T_B^3$.

Let $F = E(T) \setminus M$. So every component of $T - F$ consists an edge in $M$. Let $T'$ be an arbitrary subtree of $T$. By Lemma 1 ($(T')_B^3$) has a Hamiltonian cycle $C'$ that crosses every edge in $F \cap E(T')$ exactly twice respect to $T'$ (and then respect to $T$ since $C'$ contains no vertices outside $T'$). By Theorem 9, $T_B^3$ has a Hamiltonian circle.

By Lemmas 3 and 7, $T$ is faithful to $G_B^3$. By Lemma 5, $T_B^3$ is faithful to $G_B^3$. By Lemma 2, $G_B^3$ has a Hamiltonian circle.

The proof is complete.

References

[1] H. Bruhn, M. Stein, On end degrees and infinite circuits in locally finite graphs, Combinatorica 27 (2007) 269-291.

[2] H. Bruhn, X. Yu, Hamilton cycles in planar locally finite graphs, SIAM J. Discrete Math. 22 (2008) 1381-1392.

[3] Q. Cui, J. Wang, X. Yu, Hamilton circles in infinite planar graphs, J. Combin. Theory Ser. B 99 (2009) 110-138.

[4] R. Diestel, The cycle space of an infinite graph, Combin. Probab. Comput. 14 (2005) 59-79.

[5] R. Diestel, Graph Theory, 5th Edition (Springer, New York, 2016).

[6] R. Diestel, D. Kühn, Topological paths, cycles and spanning trees in infinite graphs, European J. Combin. 25 (2004) 835-862.

[7] O. Favaron, P. Mago, C. Maulino, O. Ordaz, Hamiltonian properties of bipartite graphs and digraphs with bipertite independence 2, SIAM J. Discrete Math. 6 (2) (1993) 189-196.

[8] O. Favaron, P. Mago, O. Ordaz, On the bipartite independence number of a balanced bipartite graph, Discrete Math. 121 (1993) 55-63.

[9] H. Fleischner, The square of every two-connected graph is hamiltonian, J. Combin. Theory Ser. B 16 (1974) 29-34.

[10] A. Georgakopoulos, Infinite Hamilton cycles in squares of locally finite graphs, Advances Math. 220 (2009) 670-705.

[11] K. Heuer, A sufficient condition for hamiltonicity in locally finite graphs, European J. Combin. 45 (2015) 97-114.

[12] K. Heuer, A sufficient local degree condition for Hamiltonicity in locally finite claw-free graphs, European J. Combin. 55 (2016) 82-99.

[13] H.A. Jung, Wurzelbäume und unendliche wege in graphen, Math. Nachr. 41 (1969) 1-22.
[14] A. Kündgen, B. Li, C. Thomassen, Cycles through all finite vertex sets in infinite graphs, European J. Combin. 65 (2017) 259-275.

[15] F. Lehner, On spanning tree packings of highly edge connected graphs, J. Combin. Theory Ser. B 105 (2014) 93-126.

[16] M. Sekanina, On an ordering of the set of vertices of a connected graph, Publ. Fac. Sc. Univ. Brno 412 (1960) 137-142.

[17] G.J. Simmons, Almost all $n$-dimensional rectangular lattices are hamilton-laceable, in: Proceedings of the Ninth Southeastern Conference on Combinatorics, Graph Theory, and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1978) pp. 649-661.

[18] G.J. Simmons, Maximal non-hamilton-laceable graphs, J. Graph Theory 5 (4) (1981) 407-415.

[19] C. Thomassen, Hamiltonian paths in squares of infinite locally finite blocks, Ann. Discrete Math. 3 (1978) 269-277.