Quantum Dark Soliton: 
Non-Perturbative Diffusion of Phase and Position

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I. INTRODUCTION

A dark soliton in a quasi one dimensional (1D) atomic Bose Einstein condensate (BEC) is a particle-like solution of the Gross-Pitaevskii equation [1]. Dark solitons were observed in two experiments [2, 3]. A classical soliton in a quasi-1D condensate in a harmonic trap behaves like a (negative mass) particle in a harmonic potential with a frequency equal to trap frequency divided by \(\sqrt{2} \) [1]. This results in a (negative frequency) anomalous mode in the spectrum of the Bogoliubov theory. This mode describes small fluctuations of the soliton around the center of the trap. A soliton wave packet which is initially localized in the center of the trap is going to disperse [7]. The width of the wave packet grows until it becomes comparable to the width of the soliton - the healing length. In Refs. [7] this dispersion was estimated to happen, for reasonable experimental parameters, on a time scale of 10ms. This quantum “instability”, present even at zero temperature, adds to the list of more classical decay channels [8, 9].

The calculations in Ref. [7] were done within Bogoliubov theory which is a linearized theory of small quantum fluctuations around a classical soliton localized in the center of the trap. This theory breaks down when fluctuations grow large because for large fluctuations one cannot neglect interactions between Bogoliubov modes. The perturbative theory breaks down when the width of the soliton wave packet becomes comparable with the healing length i.e. after around 10ms from soliton creation. To extend the soliton diffusion beyond this point one has to treat the diffusion in a non-perturbative way.

In this paper I develop non-perturbative theory of soliton diffusion that includes both position and phase fluctuations. To avoid some technical problems, and also to afford more pedagogical presentation, I consider a soliton in uniform 1D condensate confined to a box of finite length. The finite length of the box makes quantum and thermal depletion from the 1D condensate finite. In fact finite quasi-1D condensates were observed in experiments [2, 4] in harmonic traps. A quasi-1D condensate in the Thomas-Fermi regime can be considered locally uniform on length scales comparable to the soliton width. What is more atom chips [5] make possible confinement of cold atoms in a quasi-1D box.

II. DARK SOLITON IN A BOX

The dark soliton [1] is a stationary solution of the Gross-Pitaevskii equation in 1D

\[
ih\partial_t \phi = -\frac{\hbar^2}{2m}\partial_x^2 \phi + g|\phi|^2 \phi - \mu \phi. \tag{1}
\]

Here \( m \) is atomic mass and \( g \) is effective 1D interaction strength. The system is placed in a box by imposing the boundary conditions \( \phi(x = 0) = 0 \) and \( \phi(x = L) \) at the walls of the box. The stationary dark soliton is

\[
\phi_0(x) = \begin{cases} 
-e^{-i\theta}\sqrt{\rho}\tanh\frac{x}{\xi}, & x_L < x \\
e^{-i\theta}\sqrt{\rho}\tanh\frac{x_L - x}{\xi}, & x_L < x < x_R \\
e^{-i\theta}\sqrt{\rho}\tanh\frac{x_R - x}{\xi}, & x_R < x
\end{cases}
\]

see Figure 1. Here \( \rho \) is (linear) density of the condensate, \( \xi = \hbar/\sqrt{\rho g m} \) is the healing length, \( c = \sqrt{\rho g m} \) is the speed of light, \( x_L \) and \( x_R \) are placed six healing lengths from the left and right walls of the box.

Figure 1. The function \( f_0(x) = e^{i\theta}\sqrt{\rho}\tanh\frac{x}{\xi} \) and \( x_L \) and \( x_R \) are placed six healing lengths from the left and right walls of the box.
of sound, and θ is arbitrary global phase. I made a convenient choice of μ = gρ. I assume that the width of the soliton is much less than the size of the box, ξ ≪ l, and that the position of the soliton q is at safe distance of a few healing lengths from the walls.

This stationary solution is degenerate with respect to the soliton position q and to the global phase θ of the condensate wave function.

III. SMALL FLUCTUATIONS AROUND DARK SOLITON

The Gross-Pitaevskii equation (1) can be linearized in small fluctuations δφ(t, x) around the classical stationary solution (2):

\[ i\hbar \partial_t \left( \frac{\delta \phi}{\delta \phi^*} \right) = \mathcal{L} \left( \frac{\delta \phi}{\delta \phi^*} \right), \]

Here the linear differential operator is

\[ \mathcal{L} = \left( +\mathcal{H}_{GP} + g|\phi_0(x)|^2 \right) \frac{\partial^2}{\partial x^2} - \frac{g\phi_0^2(x)}{2} - \mathcal{H}_{GP} - g|\phi_0(x)|^2. \]

\[ \mathcal{H}_{GP} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + g|\phi_0(x)|^2 - \mu. \]

The right eigenmodes of the non-hermitian \( \mathcal{L} \) are solutions of the Bogoliubov-de Gennes equations

\[ \mathcal{L} \left( \begin{array}{c} u \\ v \end{array} \right) = \epsilon \left( \begin{array}{c} u \\ v \end{array} \right). \]

Every right eigenmode has a corresponding left eigenmode \( (u^*, -v^*) \). The right eigenmodes can be classified as phonon modes \( (\epsilon > 0) \) and zero modes \( (\epsilon = 0) \).

A. Phonons

For later convenience I write first continuous spectrum \( \epsilon_k \) and eigenfunctions for phonons on the background \( \phi_0(x) = e^{-i\theta} \tanh \frac{x-q}{\xi} \) extending to \( x = \pm \infty \),

\[ \epsilon_k = \sqrt{\frac{\hbar^2 e^2 k^2}{2m} + \left( \frac{\hbar^2 k^2}{2m} \right)^2}, \]

\[ u_k(x,q) = \frac{g\rho}{\sqrt{4\pi \xi}} e^{ikx} e^{-i\theta} \times \left[ \left( \frac{\hbar^2 e^2}{2m} \right) \frac{k \xi}{2} + i \tanh \frac{x-q}{\xi} \right], \]

\[ v_k(x,q) = \frac{g\rho}{\sqrt{4\pi \xi}} e^{ikx} e^{i\theta} \times \left[ \left( \frac{\hbar^2 e^2}{2m} \right) \frac{k \xi}{2} + i \tanh \frac{x-q}{\xi} \right]. \]

The soliton does not scatter phonons but shifts their phase: as \( u_k(x) \) or \( v_k(x) \) is passing from left to right the function \( \frac{k \xi}{2} + i \tanh \frac{x-q}{\xi} \) is changing phase by \( \Delta \phi_k = 2 \arctan \left( \frac{2 \frac{k \xi}{2}}{\xi} \right) \).

B. Phonons in a box

In a box the wavevector \( k \) is quantized. In figure 1 we can see a half-antikink at \( x = 0 \), a kink at \( x = q \), and another half-antikink at \( x = l \). In this background total phaseshift of a phonon passing from the left wall to the right wall of the box is \( \frac{1}{2} \Delta \phi_k + 2 \Delta \phi_k = \frac{1}{2} \Delta \phi_k = 0 \) and the quantization condition is \( k_n l + 2 \Delta \phi_k = n \pi \). Discrete Bogoliubov modes are

\[ \epsilon_n = \epsilon_{kn}, \]

\[ u_n(x) = \begin{cases} u_{-k_n}(x,0) + u_{k_n}(x,0) & \text{if } x < x_L, \\ \alpha_n u_{k_n}(x,q) + \alpha_n^* u_{-k_n}(x,q) & \text{if } x_L < x < x_R, \\ \alpha_n^2 u_{-k_n}(x,-l) + \alpha_n^2 u_{k_n}(x,-l) & \text{if } x_R < x, \end{cases} \]

where \( \alpha_n = -e^{i \Delta \phi_{kn}} \), plus a formula for \( v_n(x) \) obtained by replacing \( u \) with \( v \). The modes are \( 0 < x < l \). With proper normalization phonon modes satisfy the orthogonality relation:

\[ \langle u_m | u_n \rangle - \langle v_m | v_n \rangle = \delta_{mn}. \]

C. Zero modes

In addition to phonons there are two zero modes with \( \epsilon = 0 \). One originates [6] from the global \( U(1) \) gauge invariance \( \phi e^{-i\theta} \rightarrow \phi e^{-(\theta + \epsilon)} \) broken by the classical solution (2):

\[ \left( \begin{array}{c} u \phi \\ v \phi \end{array} \right) = i \hbar \frac{\partial}{\partial \theta} \left( \begin{array}{c} \phi_0 \\ \phi_0^* \end{array} \right), \]

and the other from the translational invariance \( q \rightarrow q + \epsilon \) broken by the solution (2):

\[ \left( \begin{array}{c} u \phi \\ v \phi \end{array} \right) = i \hbar \frac{\partial}{\partial q} \left( \begin{array}{c} \phi_0 \\ \phi_0^* \end{array} \right). \]

The zero modes are orthogonal: \( \langle u\phi | u\phi \rangle - \langle v\phi | v\phi \rangle = 0 \).

Unlike phonons both zero frequency modes also have zero norms: \( \langle u | u \rangle - \langle v | v \rangle = 0 \). As a result, phonon modes together with zero modes do not span the whole Hilbert space in the functional space \( (\delta \phi, \delta \phi^*) \). For example, to find a coordinate of \( (\delta \phi, \delta \phi^*) \) in the direction of the zero mode \( (u\phi, v\phi) \) one has to project \( (\delta \phi, \delta \phi^*) \) on an adjoint vector \( \langle u\phi | v\phi \rangle \)

\[ \left( \begin{array}{c} \langle u\phi | u\phi \rangle - \langle v\phi | v\phi \rangle \end{array} \right) \left( \begin{array}{c} \delta \phi \\ \delta \phi^* \end{array} \right), \]

which has unit overlap with the zero mode, \( \langle u\phi | u\phi \rangle - \langle v\phi | v\phi \rangle = 1 \), but is orthogonal to all other modes.

In a similar way, the translational mode \( (u\phi, v\phi) \) requires an adjoint mode \( (u\phi', v\phi') \) normalized so that \( \langle u\phi | u\phi' \rangle - \langle v\phi | v\phi' \rangle = 1 \), but orthogonal to all other modes including \( (u\phi', v\phi') \). In summary, two adjoint modes are missing in order to span the whole Hilbert space of \( (\delta \phi, \delta \phi^*) \).

D. Adjoint modes

In the case of the dark soliton (2) a vector \( (u\phi, v\phi) \) adjoint to a zero mode \( (u, v) \) turns out to be a solution of the
inhomogeneous equation [6]

\[ \mathcal{L} \left( \frac{\partial}{\partial t} \right) = \frac{1}{M} \left( u \right. \left. \frac{\partial}{\partial x} v \right) \]

with a constant \( M \) chosen so that the overlap \( \langle \psi_{q}^{ad} | u \rangle - \langle \psi_{q}^{ad} | v \rangle = 1 \). Eq. (6) warrants that the adjoint vector \( (u^{ad}, v^{ad}) \) is an eigenstate of \( \mathcal{L}^2 \) with eigenvalue 0. As such it is orthogonal to all phonon modes because phonon modes are eigenstates of \( \mathcal{L}^2 \) with non-zero eigenvalues \( \epsilon^{\pm} \).

To find the adjoint vector to the gauge mode (4) it is good to start form a stationary Gross-Pitaevskii equation solved by \( \phi_0 \) in Eq. (2)

\[ 0 = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \phi_0 + g|\phi_0^2| \phi_0 - \mu \phi_0 . \]

Taking derivative of this equation and its complex conjugate with respect to \( \rho \) gives

\[ \mathcal{L} \left( \frac{\partial}{\partial \rho} \phi_0 \right) = \frac{\partial}{\partial \rho} \mu \left( \phi_0 - \phi_0^* \right) . \]

Comparing this equation with (6) and (4) and using \( \partial \rho \mu = g \) gives

\[ \left( \frac{u^{ad}}{v^{ad}_q} \right) = \frac{1}{\hbar g M_0} \left( \partial_{\phi_0} \phi_0 \right) \left( \phi_0^* \phi_0 \right) \quad \text{(8)} \]

The normalization condition \( \langle \psi_{q}^{ad} | u_{q} \rangle - \langle \psi_{q}^{ad} | v_{q} \rangle = 1 \) requires that \( M_0 = \frac{1}{g \hbar} \frac{\partial}{\partial \phi_0} \). Here \( N_0 = \langle \phi_0 | \phi_0 \rangle \) is average number of atoms in the condensate mode. With this \( M_0 \) one recovers the general formula [6]

\[ \left( \frac{u^{ad}}{v^{ad}_q} \right) = \frac{\partial}{\partial N_0} \left( \phi_0 \phi_0^* \right) . \]

To get the adjoint vector to the translational mode (5) we verify first that

\[ \mathcal{L} \left( \frac{e^{-i\theta}}{e^{i\theta}} \right) \frac{i\sqrt{\rho}}{c} \frac{I(x)}{I(x)} = i\hbar \frac{\partial}{\partial \phi_0} \left( \phi_0 \right) \quad \text{(9)} \]

Here the envelope function is

\[ I(x) = \begin{cases} \tanh \frac{x}{\xi}, & x < x_L \\ 1, & x_L < x < x_R \\ \tanh \left( \frac{y-x}{\xi} \right), & x_R < x \end{cases} \quad \text{(10)} \]

Comparing this equation with Eqs. (6,5) gives

\[ \left( \frac{u^{ad}}{v^{ad}_q} \right) = \frac{1}{M_q} \left( \frac{e^{-i\theta}}{e^{i\theta}} \right) i\sqrt{\rho} \frac{I(x)}{I(x)} \quad \text{(11)} \]

The normalization condition \( \langle \psi_{q}^{ad} | u_{q} \rangle - \langle \psi_{q}^{ad} | v_{q} \rangle = 1 \) requires \( M_q = -i\hbar \rho / c \). However, the overlap between two adjoint modes should vanish but it does not: \( \langle \psi_{q}^{ad} | u_{q}^{ad} \rangle - \langle \psi_{q}^{ad} | v_{q}^{ad} \rangle = iR \) with real \( R = \frac{2g - 1}{\hbar g M_0 M_q} \). This problem can be easily fixed because the solution of the inhomogeneous equation (6) is not unique - we can always add a zero mode of \( \mathcal{L} \) to the solution. Using this freedom I replace the adjoint gauge mode as

\[ \left( \frac{u^{ad}}{v^{ad}_q} \right) \rightarrow \left( \frac{u^{ad}}{v^{ad}_q} \right) - iR \left( \frac{u_q}{v_q} \right) . \quad \text{(12)} \]

The new adjoint gauge mode has no overlap with the adjoint translational mode.

### E. Decomposition of unity

Putting together phonons and zero modes with their adjoint partners results in a decomposition of the unit operator as

\[ \hat{1} = \frac{|u_{q}^a \rangle \langle v_{q}^a|}{|v_{q}^a \rangle \langle v_{q}^a|} + \frac{|u_{q}^a \rangle \langle v_{q}^a|}{|v_{q}^a \rangle \langle v_{q}^a|} + \cdots \sum_{n=1}^{\infty} \frac{|u_{n}^a \rangle \langle v_{n}^a|}{|v_{n}^a \rangle \langle v_{n}^a|} + \frac{|v_{n}^a \rangle \langle v_{n}^a|}{|v_{n}^a \rangle \langle v_{n}^a|} + \frac{|v_{n}^a \rangle \langle v_{n}^a|}{|v_{n}^a \rangle \langle v_{n}^a|} . \]

This decomposition is a foundation of the Bogoliubov theory.

### IV. Perturbative Theory

The Gross-Pitaevskii equation (1) is a classical version of a second quantized theory with a Hamiltonian

\[ \hat{H} = \int dx_0 \left( \frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V_0 \psi \psi \right) . \quad \text{(13)} \]

Here \( \hat{\psi}(x) \) is a bosonic field operator. In standard perturbative treatment [6] of zero modes the field operator is expanded in small quantum fluctuations \( \delta \psi \) around the classical solution \( \phi_0 \) with fixed \( q = q_0 \) and \( \theta = 0 \): \( \psi = \phi_0 + \delta \psi \). The fluctuation operator is expanded as

\[ \left( \frac{\delta \psi}{\delta \psi^*} \right) = \sum_n \hat{b}_n \left( \frac{u_n}{v_n} \right) + \hat{b}_n^* \left( \frac{v_n}{u_n} \right) + \hat{P}_n \left( \frac{u_n^a}{v_n^a} \right) + \hat{P}_n^* \left( \frac{v_n^a}{u_n^a} \right) \]

\[ \frac{\delta}{\delta \psi^*} \left( \frac{u_{q}^a}{v_{q}^a} \right) \left( \frac{v_{q}^a}{u_{q}^a} \right) + \frac{\delta}{\delta \psi^*} \left( \frac{u_{q}^a}{v_{q}^a} \right) \left( \frac{v_{q}^a}{u_{q}^a} \right) . \quad \text{(15)} \]

Here the subscript 0 means a mode with \( q = q_0 \) and \( \theta = 0 \).

Fluctuating position \( \delta q - q_0 \) and phase \( \delta \theta \) are assumed small. \( \hat{P}_q \) and \( \hat{P}_q^* \) are momenta conjugate to \( q \) and \( \theta \): \( \{ q, \hat{P}_q \} = i\hbar \) and \( \{ \theta, \hat{P}_q \} = i\hbar \). \( \hat{b}_n \) is a phonon annihilation operator: \[ \left[ \hat{b}_m, \hat{b}_n^* \right] = \delta_{mn} \]. These commutation relations plus the decomposition of unity (13) give the desired \( \delta \psi(x), \delta \psi^*(y) = \delta(x-y) \).

In the Bogoliubov theory the Hamiltonian (14) is expanded to second order in \( \delta \psi \). The linear term vanishes because the
classical $\phi_0$ is a solution of the Gross-Pitaevskii equation. The leading second order term is a perturbative Bogoliubov Hamiltonian

$$H_{\text{pert.}} = \int dx \left( \delta \dot{\psi}^\dagger, -\delta \dot{\psi} \right) \mathcal{L} \left( \delta \dot{\psi} \right) = \sum_n \epsilon_n \hat{b}_n^\dagger \hat{b}_n + \frac{\hat{P}_q^2}{2M_q} + \frac{\hat{P}_\theta^2}{2M_\theta}$$

(16)

As is well known [6], this theory is predicting its own demise. The perturbative Hamiltonian (16) predicts indefinite spreading of phase and position with time,

$$\langle \dot{\theta}^2 \rangle \sim t \, , \langle (\dot{q} - q_0)^2 \rangle \sim t \, ,$$

while at the same time the derivation of (16) requires them to remain small. In the next Section I introduce non-perturbative treatment of the zero modes that does not suffer from this inconsistency.

V. NON-PERTURBATIVE THEORY

The key point is observation that any classical field can be expanded as

$$\begin{pmatrix} \phi \\ \phi^* \end{pmatrix} = \begin{pmatrix} \phi_0 \\ \phi_0^* \end{pmatrix} + \gamma_\theta \begin{pmatrix} u_{\theta \theta}^d \\ v_{\theta \theta}^d \end{pmatrix} + \gamma_q \begin{pmatrix} u_{q q}^d \\ v_{q q}^d \end{pmatrix} + \sum_n b_n \begin{pmatrix} u_n \\ v_n \end{pmatrix} + b_n^* \begin{pmatrix} u_n^* \\ v_n^* \end{pmatrix}$$

(17)

Here $b_n$’s are complex Bogoliubov amplitudes. For the lower component to be a complex conjugate of the upper component the coordinates $\gamma_\theta$ and $\gamma_q$ must be real, compare Eqs.(8,11,12). The real collective coordinates $\theta$ and $q$ are implicit in definitions of $\phi_0(x)$, adjoint modes and Bogoliubov modes. Unlike in the perturbative treatment (15), here $\theta$ and $q$ are not assumed to be small, but they can take non-perturbatively large values.

A. Effective Hamiltonian

The Gross-Pitaevskii equation (1) follows from a Lagrangian

$$L = \int_0^t \! dx \left( i \hbar \phi^* \partial_t \phi - \frac{\hbar^2}{2m} |\partial_x \phi|^2 - g |\phi|^4 + \mu |\phi|^2 \right)$$

Substitution of Eq.(17) to $L$, expansion to second order in $\gamma$ and $b$, and subsequent integration over $x$ give an effective Lagrangian for the collective coordinates:

$$L_{\text{eff}} = \sum_n \left( i \hbar b_n^* \dot{b}_n - \epsilon_n b_n^* \dot{b}_n \right) + \hbar N \dot{\theta} + P \dot{q} - \frac{\gamma_\theta^2}{2M_\theta} - \frac{\gamma_q^2}{2M_q}$$

(18)

Here $N$ is a total number of atoms (conjugate to $\theta$)

$$N = \frac{1}{2} i \langle [\phi | \partial_\theta \phi] - \text{c.c.} \rangle = N_0 + \frac{\gamma_\theta}{\hbar} + \mathcal{O}(\gamma_\theta, b_\gamma, b_b) \, ,$$

and $P$ is center of mass momentum (conjugate to $q$)

$$P = \frac{1}{2} i \hbar \langle [\phi | \partial_q \phi] - \text{c.c.} \rangle = \gamma_q + \mathcal{O}(\gamma_\gamma, b_\gamma, b_b) \, .$$

Legendre transformation leads to an effective Hamiltonian

$$H_{\text{eff}} = \sum_n \epsilon_n b_n^* \dot{b}_n + \frac{\gamma_\theta^2}{2M_\theta} + \frac{\gamma_q^2}{2M_q} \, .$$

(20)

To complete the transformation $\gamma$’s must be expressed as functions of the canonical momenta $N$ and $P$: $\gamma_\theta^2 \approx \hbar^2 (N - N_0)^2$ and $\gamma_q^2 \approx P^2$. These approximate $\gamma_\theta$ and $\gamma_q$ result in a non-perturbative Bogoliubov Hamiltonian

$$H_{\text{eff}} = \sum_n \epsilon_n b_n^* \dot{b}_n + \frac{\hbar^2 (N - N_0)^2}{2M_\theta} + \frac{P^2}{2M_q} \, .$$

(21)

B. Quantum dark soliton

$H_{\text{eff}}$ can be quantized by replacing $c$-numbers with operators. Non-zero commutators are: $[\hat{b}_k, \hat{b}_p^\dagger] = \delta(k - p), [\hat{\theta}, \hat{N}] = i$, and $[\hat{q}, \hat{P}] = i \hbar$. With constant $N_0$ we also have $[\hat{\theta}, \hat{N} - \hat{N}_0] = i$. A quantum Hamiltonian in coordinate representation where $\hat{N} - \hat{N}_0 = -i \hbar \partial_\theta$ and $\hat{P} = -i \hbar \partial_q$ is

$$\hat{H}_{\text{eff}} = \sum_n \epsilon_n \hat{b}_n^\dagger \hat{b}_n - \frac{\hbar^2}{2M_\theta} \partial_\theta^2 - \frac{\hbar^2}{2M_q} \partial_q^2 \, .$$

(22)

This Hamiltonian is a sum of phonon terms, a phase diffusion term, and a soliton diffusion term with negative mass $M_q < 0$.

In the framework of the non-perturbative theory of zero modes one is free to work in a subspace of Hilbert space with definite total number of atoms $\hat{N}$. A state with definite $\hat{N} = \hat{N}_0$ has a wave function $\propto \exp(i \hat{N} \theta)$ which covers the whole range of $\theta \in [-\pi, +\pi]$ including non-perturbatively large values of $\theta$. Such a wave function, and consequently also a state with definite $\hat{N}$, is beyond reach for the perturbative theory. In the subspace with $\hat{N} = \hat{N}_0$ the Hamiltonian becomes

$$\hat{H}^N = \sum_n \epsilon_n \hat{b}_n^\dagger \hat{b}_n - \frac{\hbar^2}{2M_\theta} \partial_\theta^2 \, .$$

(23)

In the same way, a wave packet for soliton position $q$ can disperse covering most of the box without contradicting assumptions of the theory. For example, initial width $\Delta q_0$ of a gaussian wave packet grows like

$$\Delta q(t) = \sqrt{\Delta q_0^2 + \frac{\hbar^2 t^2}{4 M_q^2 \Delta q_0^2}} \approx \frac{ct}{8 \rho \Delta q_0} \, .$$

(24)
Two condensates of solitons. For this question we must calculate overlap between these solitons. This dispersion of soliton position becomes comparable to the soliton width, \( \Delta q = \xi \), at the time \( t = \tau_{\text{diffusion}} = 8\hbar\Delta q_0/g \) which depends on the initial dispersion \( \Delta q_0 \). In between \( t = 0 \) and \( t = \tau_{\text{diffusion}} \), a hole in single particle density distribution fills up with atoms, compare Fig. 2.

It turns out that there is non-zero minimal uncertainty of soliton position \( \Delta q_{\text{min}} > 0 \). Suppose that we have two condensates with a dark soliton, but with different soliton positions \( q_1 \) and \( q_2 \). What is the minimal distance between the solitons \( |q_1 - q_2| \) when it becomes possible to distinguish these two condensates by a suitable quantum measurement? To answer this question we must calculate overlap between these two condensates of \( N \) atoms and see how it decays with the intersoliton distance,

\[
\left( \frac{\langle \phi_0^{q=q_1} | \phi_0^{q=q_2} \rangle}{\langle \phi_0 | \phi_0 \rangle} \right)^N \approx \left( 1 - \frac{2(q_1 - q_2)^2}{3\xi} \right)^N \approx \left( 1 - \frac{2\rho(q_1 - q_2)^2}{3N\xi} \right)^N \approx 1 \quad N \gg 1
\]

Two condensates become orthogonal and in principle distinguishable when the intersoliton distance becomes greater than \( \Delta q_{\text{min}} = \sqrt{\frac{2\rho}{3}} \). This is a fundamental limitation derived only from properties of the quantum states and not of any particular measurement technique. This fundamental limitation means that the initial gaussian wave packet cannot be localized better than \( \Delta q_0 = \Delta q_{\text{min}} \). This minimal dispersion leads to the minimal soliton diffusion time

\[
\tau_{\text{diffusion}} = \frac{8\xi}{c} \sqrt{\frac{3\rho\xi}{2}}.
\]

It is interesting to evaluate \( \tau_{\text{min}} \) for the parameters of the Hanover experiment [2]. This condensate can be approximated by a quasi-1D harmonic trap like in Ref.[7]. In the present paper we consider uniform condensate in a box with linear density \( \rho \). Using the linear density in the center of the effective quasi-1D trap as \( \rho \) we obtain \( \tau_{\text{min}} = 8.0\text{ms} \). This is close to the time \( O(10\text{ms}) \) when solitons are observed to lose contrast in that experiment.

Finally, I make a brief comment on solitons moving with finite velocity \( v \) with respect to condensate. Condensate wave function for \( x \in \{x_L, x_R\} \) is \( \psi_0 = e^{-i\theta} \sqrt{\pi} (\beta + \alpha \tanh \alpha \sqrt{1 - \beta^2}) \) with \( \beta = v/c \) and \( \alpha = \sqrt{1 - \beta^2} \). Calculation of zero modes and adjoint modes follows the same lines as for \( v = 0 \). Mass of a moving soliton is \( M^v_q = \alpha M_q \) and the overlap between condensates decays when the intersoliton distance becomes comparable to \( \Delta q_{\text{min}}^v = \Delta q_{\text{min}}/\alpha^{3/2} \). The minimal soliton diffusion time is

\[
\tau_{\text{diffusion}}^v = \frac{\tau_{\text{diffusion}}}{\sqrt{1 - \frac{v^2}{c^2}}}.
\]

For Hannover solitons [2] moving with velocities \( v/c = 0.4, \ldots, 0.8 \) the minimal time is \( \tau_{\text{min}}^v = 8.7 \ldots 13.3\text{ms} \).

VI. CONCLUSION

This paper develops non-perturbative theory of zero modes in the Bogoliubov theory of atomic BEC. In the non-perturbative approach phase fluctuations and fluctuations of soliton position are not restricted to be small. This theory predicts that an initially well localized soliton wave packet is going to disperse beyond the soliton width.

When parameters of the present model are fit to the parameters of Hannover experiment [2], then dispersion of soliton position becomes comparable to the soliton width after 10ms from soliton creation. This number is consistent with earlier perturbative studies of soliton diffusion in a harmonic trap [7]. This diffusion time is also consistent with the time when the dark soliton appears to loose contrast or gray in Hannover experiment [2].

Interaction of the soliton with a thermal cloud was offered as an explanation [8] of the observed graying. The quantum diffusion described here and in Refs.[7] is a mechanism that operates even in the \( T = 0 \) quantum limit when the thermal cloud is turned off. The quantum diffusion mechanism could be tested in an experiment with variable temperature. Another dissipation mechanism due to non-uniform condensate density was suggested in Ref.[9]. The influence of the inhomogeneity could be eliminated in an experiment with a static soliton \( (v = 0) \) in the center of the trap. In principle it is possible to make an experiment where the other mechanisms [8, 9] are turned off leaving the quantum diffusion as the only known instability of the dark soliton.

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