$p$-ELEMENTARY SUBGROUPS OF THE CREMONA
GROUP OF RANK 3

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Abstract. For the subgroups of the Cremona group $\text{Cr}_3(\mathbb{C})$ having the form $(\mu_p)^s$, where $p$ is prime, we obtain an upper bound for $s$. Our bound is sharp if $p \geq 17$.

1. Introduction

Let $k$ be an algebraically closed field. The Cremona group $\text{Cr}_n(k)$ is the group of birational transformations of $\mathbb{P}^n_k$, or equivalently the group of $k$-automorphisms of the field $k(x_1, \ldots, x_n)$. Finite subgroups of $\text{Cr}_2(\mathbb{C})$ are completely classified (see [DI09] and references therein). In contrast, subgroups of $\text{Cr}_n(k)$ for $n \geq 3$ are not studied well (cf. [Pro09]).

In the present paper we study some kind of abelian subgroups of $\text{Cr}_3(\mathbb{C})$. Let $p$ be a prime number. We say that a group $G$ is $p$-elementary if $G \cong (\mu_p)^s$ for some positive integer $s$. In this case $s$ is called the rank of $G$ and denoted by $\text{rk} G$.

Theorem 1.1 ([Bea07]). Let $p$ be a prime $\neq \text{char}(k)$ and let $G \subset \text{Cr}_2(k)$ be a $p$-elementary subgroup. Then:

$$\text{rk} G \leq 2 + \delta_{p,3} + 2\delta_{p,2}$$

where $\delta_{i,j}$ is Kronecker’s delta. Moreover, for any such $p$ this bound is attained for some $G$. These “maximal” groups $G$ are classified up to conjugacy in $\text{Cr}_2(k)$.

More generally, instead of $\text{Cr}_n(k)$ we also can consider the group $\text{Bir}(X)$ of birational automorphisms of an arbitrary rationally connected variety $X$. Our main result is the following

Theorem 1.2. Let $X$ be a rationally connected threefold defined over a field of characteristic 0, let $p$ be a prime, and let $G \subset \text{Bir}(X)$ be a
$p$-elementary subgroup. Then

$$\text{rk } G \leq \begin{cases} 
7 & \text{if } p = 2, \\
5 & \text{if } p = 3, \\
4 & \text{if } p = 5, 7, 11, \text{ or } 13, \\
3 & \text{if } p \geq 17.
\end{cases}$$

For any prime $p \geq 17$ this bound is attained for some subgroup $G \subset \text{Cr}_3(\mathbb{C})$. (However we do not assert that the bound (1.3) is sharp for $p \leq 13$).

**Remark 1.4.** (i) Note that $\text{Cr}_1(\mathbb{k}) \simeq \text{PGL}_2(\mathbb{k})$. Hence for any prime $p \neq \text{char}(\mathbb{k})$ and any $p$-elementary subgroup $G \subset \text{Cr}_1(\mathbb{k})$, we have $\text{rk } G \leq 1 + \delta_{p,2}$ (see, e.g., [Bea07, Lemma 2.1]).

(ii) Since $\text{Cr}_1(\mathbb{k}) \times \text{Cr}_2(\mathbb{k})$ admits (a lot of) embeddings to $\text{Cr}_3(\mathbb{k})$, the group $\text{Cr}_3(\mathbb{k})$ contains a $p$-elementary subgroup $G$ of rank $3 + \delta_{p,3} + 3\delta_{p,2}$. This shows the last assertion of our theorem.

The following consequence of Theorem 1.2 was proposed by A. Beauville.

**Corollary 1.5.** The group $\text{Cr}_3(\mathbb{C})$ is not isomorphic to $\text{Cr}_n(\mathbb{C})$ for $n \neq 3$ as an abstract group.

**Proof.** Denote by $\xi(n,p)$ the maximal rank of a $p$-elementary group contained in $\text{Cr}_n(\mathbb{C})$. Then $\xi(2,17) = 2 < \xi(3,17) = 3$ and $\xi(n,17) \geq n$ by Theorems 1.1 and 1.2. 

Our method is a generalization of the method used for study of finite subgroups of $\text{Cr}_2(\mathbb{k})$ [Bea07], [DI09]. Similarly to [Pro09] we use the equivariant three-dimensional minimal model program. This way easily allows us to reduce the problem to the study of automorphism groups of some (not necessarily smooth) Fano threefolds.

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## 2. Preliminaries

Clearly, we may assume that $\mathbb{k} = \mathbb{C}$. All the groups in this paper are multiplicative. In particular, we denote a cyclic group of order $n$ by $\mu_n$. 


2.1. Terminal singularities. We need a few facts on the classification of three-dimensional terminal singularities (see [Mor85, Rei87]). Let $(X \ni P)$ be a germ of a three-dimensional terminal singularity. Then $(X \ni P)$ is isolated, i.e., $\text{Sing}(X) = \{ P \}$. The index of $(X \ni P)$ is the minimal positive integer $r$ such that $rK_X$ is Cartier. If $r = 1$, then $(X \ni P)$ is Gorenstein. In this case $(X \ni P)$ is analytically isomorphic to a hypersurface singularity in $\mathbb{C}^4$ of multiplicity 2. Moreover, any Weil $\mathbb{Q}$-Cartier divisor $D$ on $(X \ni P)$ is Cartier. If $r > 1$, then there is a cyclic, étale outside of $P$ cover $\pi : (X^2 \ni P^2) \to (X \ni P)$ of degree $r$ such that $(X^2 \ni P^2)$ is a Gorenstein terminal singularity (or a smooth point). This $\pi$ is called the index-one cover of $(X \ni P)$.

**Theorem 2.2** ([Mor85, Rei87]). In the above notation $(X^2 \ni P^2)$ is analytically $\mu_r$-isomorphic to a hypersurface in $\mathbb{C}^4$ with $\mu_r$-semi-invariant\footnote{In invariant theory people often say “relative invariant” rather than “semi-invariant”. We prefer to use the terminology of [Mor85].} coordinates $x_1, \ldots, x_4$, and the action is given by 

$$(x_1, \ldots, x_4) \mapsto (\varepsilon^{a_1}x_1, \ldots, \varepsilon^{a_4}x_4)$$

for some primitive $r$-th root of unity $\varepsilon$, where one of the following holds:

(i) $(a_1, \ldots, a_4) \equiv (1, -1, a_2, 0) \mod r$, $\gcd(a_2, r) = 1$,

(ii) $r = 4$ and $(a_1, \ldots, a_4) \equiv (1, -1, 1, 2) \mod 4$.

**Definition 2.3.** A $G$-variety is a variety $X$ provided with a biregular faithful action of a finite group $G$. We say that a normal $G$-variety $X$ is $G\mathbb{Q}$-factorial if any $G$-invariant Weil divisor on $X$ is $\mathbb{Q}$-Cartier. A projective normal $G$-variety $X$ is called $G\mathbb{Q}$-Fano if it is $G\mathbb{Q}$-factorial, has at worst terminal singularities, $-K_X$ is ample, and $\text{rk Pic}(X)^G = 1$.

**Lemma 2.4.** Let $(X \ni P)$ be a germ of a threefold terminal singularity and let $G \subset \text{Aut}(X \ni P)$ be a $p$-elementary subgroup. Then $\text{rk} G \leq 3 + \delta_{2,p}$.

**Proof.** Assume that $\text{rk} G \geq 4 + \delta_{2,p}$. First we consider the case where $(X \ni P)$ is Gorenstein. The group $G$ acts faithfully on the Zariski tangent space $T_{P,X}$, so $G \subset \text{GL}(T_{P,X})$, where $\dim T_{P,X} = 3$ or 4. If $\dim T_{P,X} = 3$, then $G$ is contained in a maximal torus of $\text{GL}_3(\mathbb{C})$, so $\text{rk} G \leq 3$ and we are done. Thus we may assume that $\dim T_{P,X} = 4$. Take semi-invariant coordinates $x_1, \ldots, x_4$ in $T_{P,X}$. There is a $G$-equivariant analytic embedding $(X \ni P) \subset \mathbb{C}^4_{x_1, \ldots, x_4}$. As above, $\text{rk} G \leq 4$. Thus we may assume that $\text{rk} G \leq 4$ and $p > 2$. Let $\phi(x_1, \ldots, x_4) = 0$ be an equation of $X$, where $\phi$ is a $G$-semi-invariant function. Regard $\phi$ as a power series and write $\phi = \sum_d \phi_d$, where $\phi_d$ is the sum of all monomials of degree $d$. Since the action of $G$ on $x_1, \ldots, x_4$ is linear, all
the \( \phi_d \)'s are semi-invariants of the same \( G \)-weight \( w = wt \phi_d \). Hence, for any \( \phi_d, \phi_d' \neq 0 \) we have \( d - d' \equiv 0 \mod p \). Since \( (X \ni P) \) is a terminal singularity, \( \phi_2 \neq 0 \) and so \( \phi_3 = 0 \). Recall that \( G \simeq (\mu_p)^4 \), \( p \geq 3 \). In this case, \( \phi_2 \) must be a monomial. Thus up to permutations of coordinates and scalar multiplication we get either \( \phi_2 = x_1^2 \) or \( \phi_2 = x_1x_2 \). In particular, we have rk \( \phi_2 \leq 2 \) and \( \phi_3 = 0 \). This contradicts the classification of terminal singularities [Mor85], [Rei87].

Now assume that \( (X \ni P) \) is non-Gorenstein of index \( r > 1 \). Consider the index-one cover \( \pi: (X^2 \ni P^2) \to (X \ni P) \) (see [2.1]). Here \( (X^2 \ni P^2) \) is a Gorenstein terminal point and the map \( X^2 \setminus \{P^2\} \to X \setminus \{P\} \) can be regarded as the topological universal cover. Hence there exists a natural lifting \( G^2 \subset Aut(X^2 \ni P^2) \) fitting to the following exact sequence

\begin{equation}
1 \longrightarrow \mu_r \longrightarrow G^2 \longrightarrow G \longrightarrow 1.
\end{equation}

It is sufficient to show that there exists a subgroup \( G^* \subset G^2 \) isomorphic to \( G \) (but we do not assert that the sequence splits). Indeed, in this case \( G^* \simeq G \) acts faithfully on the terminal Gorenstein singularity \( (X^2 \ni P^2) \) and we can apply the above considered case. We may assume that \( G^2 \) is not abelian (otherwise a subgroup \( G^* \simeq G \) obviously exists). The group \( G^2 \) permutes eigenspaces of \( \mu_r \). By Theorem [2.2] the subspace \( T := \{x_4 = 0\} \subset \mathbb{C}^4_{x_1, \ldots, x_4} \) is \( G^2 \)-invariant and \( \mu_r \) acts on any eigenspace \( T_1 \subset T \) faithfully. On the other hand, by (2.5) we see that the derived subgroup \( [G^2, G^2] \) is contained in \( \mu_r \). In particular, \( [G^2, G^2] \) is abelian and also acts on any eigenspace \( T_1 \subset T \) faithfully.

Since \( \dim T = 3 \), this implies that the representation of \( G^2 \) on \( T \) is irreducible (otherwise \( T \) has a one-dimensional subrepresentation, say \( T_1 \), and the kernel of the map \( G \to GL(T_1) \simeq \mathbb{C}^* \) must contain \( [G^2, G^2] \)). Hence eigenspaces of \( \mu_r \) have the same dimension and so \( \mu_r \) acts on \( T \) by scalar multiplication. By Theorem [2.2] this is possible only if \( r = 2 \).

Let \( G^2_p \subset G^2 \) be a Sylow \( p \)-subgroup. If \( \mu_r \cap G^2_p = \{1\} \), then \( G^2_p \simeq G \) and we are done. Thus we assume that \( \mu_r \subset G^2_p \), so \( p = r = 2 \) and \( G^2_p = G^2 \). But then \( G^2 \) is a 2-group, so the dimension of its irreducible representation must be a power of 2. Hence \( T \) is reducible, a contradiction. \( \square \)

**Lemma 2.6.** Let \( X \) be a \( G \)-threefold with isolated singularities.

(i) If there is a curve \( C \subset X \) of \( G \)-fixed points, then \( \text{rk} \ G \leq 2 \).

(ii) If there is a surface \( S \subset X \) of \( G \)-fixed points, then \( \text{rk} \ G \leq 1 \).

If moreover \( S \) is singular along a curve, then \( G = \{1\} \).

**Sketch of the Proof.** Consider the action of \( G \) on the tangent space to \( X \) at a general point of \( C \) (resp. \( S \)). \( \square \)
**G-equivariant minimal model program.** Let $X$ be a rationally connected three-dimensional algebraic variety and let $G \subset \text{Bir}(X)$ be a finite subgroup. By shrinking $X$ we may assume that $G$ acts on $X$ biregularly. The quotient $Y = X/G$ is quasiprojective, so there exists a projective completion $\hat{Y} \supset Y$. Let $X$ be the normalization of $\hat{Y}$ in the function field $\mathbb{C}(X)$. Then $\hat{X}$ is a projective variety birational to $X$ admitting a biregular action of $G$. There is an equivariant resolution of singularities $\tilde{X} \to \hat{X}$, see [AW97]. Run the $G$-equivariant minimal model program: $\tilde{X} \to \bar{X}$, see [Mor88, 0.3.14]. Running this program we stay in the category of projective normal varieties with at worst terminal $GQ$-factorial singularities. Since $X$ is rationally connected, on the final step we get a Fano-Mori fibration $f: \bar{X} \to Z$. Here $\dim Z < \dim X$, $Z$ is normal, $f$ has connected fibers, the anticanonical Weil divisor $-K_{\bar{X}}$ is ample over $Z$, and the relative $G$-invariant Picard number $\rho(\bar{X})^G$ is one. Obviously, we have the following possibilities:

(i) $Z$ is a rational surface and a general fiber $F = f^{-1}(y)$ is a conic;
(ii) $Z \simeq \mathbb{P}^1$ and a general fiber $F = f^{-1}(y)$ is a smooth del Pezzo surface;
(iii) $Z$ is a point and $\bar{X}$ is a $GQ$-Fano threefold.

**Proposition 2.7.** In the above notation assume that $Z$ is not a point. Then $\text{rk} G \leq 3 + \delta_{p,3} + 3\delta_{p,2}$. In particular, $\text{(1.3)}$ holds.

**Proof.** Let $G_0 \subset G$ be the kernel of the homomorphism $G \to \text{Aut}(Z)$. The group $G_1 := G/G_0$ acts effectively on $Z$ and $G_0$ acts effectively on a general fiber $F \subset X$ of $f$. Hence, $G_1 \subset \text{Aut}(Z)$ and $G_0 \subset \text{Aut}(F)$. Clearly, $G_0$ and $G_1$ are $p$-elementary groups with $\text{rk} G_0 + \text{rk} G_1 = \text{rk} G$. Assume that $Z \simeq \mathbb{P}^1$. Then $\text{rk} G_1 \leq 1 + \delta_{p,2}$. By Theorem $\text{(1.1)}$ we obtain $\text{rk} G_0 \leq 2 + \delta_{p,3} + 2\delta_{p,2}$. This proves our assertion in the case $Z \simeq \mathbb{P}^1$. The case $\dim Z = 2$ is treated similarly. $\square$

**2.8. Main assumption.** Thus from now on we assume that we are in the case (iii). Replacing $X$ with $\hat{X}$ we may assume that our original $X$ is a $GQ$-Fano threefold.

The group $G$ acts naturally on $H^0(X, -K_X)$. If $H^0(X, -K_X) \neq 0$, then there exists a $G$-semi-invariant section $s \in H^0(X, -K_X)$ (because $G$ is an abelian group). This section gives us an invariant member $S \in |-K_X|$. Let $S$ be an invariant Weil divisor such that $-(K_X + S)$ is nef. Then the pair $(X, S)$ is log canonical (LC).
Proof. Assume that the pair \((X, S)\) is not LC. Since \(S\) is \(G\)-invariant and \(\rho(X)^G = 1\), we see that \(S\) is numerically proportional to \(K_X\). Since \(-(K_X + S)\) is nef, \(S\) is ample. We apply quite standard connectedness arguments of Shokurov [Sho93] (see, e.g., [MP09, Prop. 2.6]): for a suitable \(G\)-invariant boundary \(D\), the pair \((X, D)\) is LC, the divisor \(- (K_X + D)\) is ample, and the minimal locus \(V\) of log canonical singularities is also \(G\)-invariant. Moreover, \(V\) is either a point or a smooth rational curve. By Lemma 2.1 we may assume that \(G\) has no fixed points. Hence, \(V \cong \mathbb{P}^1\) and we have a map \(\varsigma : G \to \text{Aut}(\mathbb{P}^1)\). If \(p > 2\), then \(\varsigma(G)\) is a cyclic group, so \(G\) has a fixed point, a contradiction.

Let \(p = 2\) and let \(G_0 = \ker \varsigma\). By Lemma 2.6 \(\text{rk} G_0 \leq 2\). Therefore \(\text{rk} \varsigma(G_0) \geq 3\). Again we get a contradiction. \(\square\)

**Lemma 2.10.** Let \(X\) be a \(G_{\mathbb{Q}}\)-Fano threefold, where \(G\) is a \(p\)-elementary group with

\[
(2.11) \quad \text{rk} G \geq \begin{cases} 
7 & \text{if } p = 2, \\
5 & \text{if } p = 3, \\
4 & \text{if } p \geq 5.
\end{cases}
\]

Let \(S \in |-K_X|\) be a \(G\)-invariant member. Then we have

(i) Any component \(S_i \subset S\) is either rational or birationally ruled over an elliptic curve.

(ii) The group \(G\) acts transitively on the components of \(S\).

(iii) For the stabilizer \(G_{S_i}\) we have \(\text{rk} G_{S_i} \leq \delta_{p,2} + 4\).

(iv) The surface \(S\) is reducible (and reduced).

Proof. By Lemma 2.9 the pair \((X, S)\) is LC. Assume that \(S\) is normal (and irreducible). By the adjunction formula \(K_S \sim 0\). We claim that \(S\) has at worst Du Val singularities. Indeed, otherwise by the Connectedness Principle [Sho93, Th. 6.9] \(S\) has at most two non-Du Val points. If \(p > 2\), these points must be \(G\)-fixed. This contradicts Lemma 2.4. Otherwise \(p = 2\) and these points are fixed for an index two subgroup \(G^* \subset G\). Again we get a contradiction by Lemma 2.4. Thus we may assume that \(S\) has at worst Du Val singularities. Let \(\Gamma\) be the image of \(G\) in \(\text{Aut}(S)\). By Lemma 2.6 \(\text{rk} G \leq \text{rk} \Gamma + 1\). Let \(\tilde{S} \to S\) be the minimal resolution. Here \(\tilde{S}\) is a smooth K3 surface. The natural representation of \(\Gamma\) on \(H^{2,0}(\tilde{S})\) induces the exact sequence (see [Nik80])

\[1 \to \Gamma_0 \to \Gamma \to \Gamma_1 \to 1,\]

where \(\Gamma_0\) (resp. \(\Gamma_1\)) is the kernel (resp. image) of the representation of \(\Gamma\) on \(H^{2,0}(\tilde{S})\). The group \(\Gamma_1\) is cyclic. Hence either \(\Gamma_1 = \{1\}\) or...
Γ₁ \simeq \mu_p$. In the second case by [Nik80, Cor. 3.2] \( p \leq 19 \). Further, according to [Nik80, Th. 4.5] we have

\[
\text{rk} \Gamma_0 \leq \begin{cases} 
4 & \text{if } p = 2 \\
2 & \text{if } p = 3 \\
1 & \text{if } p = 5 \text{ or } 7 \\
0 & \text{if } p > 7.
\end{cases}
\]

Combining this we obtain a contradiction with (2.11).

Now assume that \( S \) is not normal. Let \( S_i \subset S \) be an irreducible component (the case \( S_i = S \) is not excluded). Let \( \nu : S' \to S_i \) be the normalization. Write \( 0 \sim \nu^*(K_X + S_i) = K_{S'} + D' \), where \( D' \) is the different, see [Sho93, §3]. Here \( D' \) is an effective reduced divisor and the pair is LC [Sho93, 3.2]. Since \( S \) is not normal, \( D' \neq 0 \). Consider the minimal resolution \( \mu : \tilde{S} \to S' \) and let \( \tilde{D} \) be the crepant pull-back of \( D' \), that is, \( \mu_* \tilde{D} = D' \) and

\[
K_{\tilde{S}} + \tilde{D} = \mu^*(K_{S'} + D') \sim 0.
\]

Here \( \tilde{D} \) is again an effective reduced divisor. Hence \( \tilde{S} \) is a ruled surface. If it is not rational, consider the Albanese map \( \alpha : \tilde{S} \to C \). Clearly \( \alpha \) is \( \Gamma \)-equivariant and the action of \( \Gamma \) on \( C \) is not trivial. Let \( \tilde{D}_1 \subset \tilde{D} \) be an \( \alpha \)-horizontal component. By Adjunction \( \tilde{D}_1 \) is an elliptic curve. So is \( C \). This proves (i).

If the action on components \( S_i \subset S \) is not transitive, we have an invariant divisor \( S' < S \). Since \( X \) is \( G\mathbb{Q} \)-factorial and \( \rho(X)^G = 1 \), we can take \( S' \) so that \( -(K_X + 2S') \) is nef. This contradicts Lemma 2.9. So, (ii) is proved.

Now we prove (iii). Let \( \Gamma \) be the image of \( G_{S_i} \) in \( \text{Aut}(S_i) \). By Lemma 2.6 \( \text{rk} \, G_{S_i} \leq \text{rk} \, \Gamma + 1 \). If \( S_i \) is rational, then we get the assertion by Theorem 1.1. Assume that \( S_i \) is birationally ruled surface over an elliptic curve. As above, let \( \tilde{S}_i \to S_i \) be the composition of the normalization and the minimal resolution, and let \( \alpha : \tilde{S}_i \to C \) be the Albanese map. Then \( \Gamma \) acts faithfully on \( \tilde{S}_i \) and \( \alpha \) is \( \Gamma \)-equivariant. Thus we have a homomorphism \( \alpha_* : \Gamma \to \text{Aut}(C) \). Here \( \text{rk} \, \Gamma \leq \text{rk} \, \alpha_*(\Gamma) + 1 + \delta_{p,2} \). Note that \( \alpha_*(\Gamma) \) is a \( p \)-elementary subgroup of the automorphism group of an elliptic curve. Hence, \( \text{rk} \, \alpha_*(\Gamma) \leq 2 \). This implies (iii).

It remains to prove (iv). Assume that \( S \) is irreducible. By (iii) the surface \( S \) is not rational. So, \( S \) is birational to a ruled surface over an elliptic curve. By Lemma 2.6 the group \( G \) acts on \( S \) faithfully. Hence, in the above notation, \( \text{rk} \, G = \text{rk} \, \Gamma \leq \text{rk} \, \alpha_*(\Gamma) + 1 + \delta_{p,2} \leq 3 + \delta_{p,2} \), a contradiction. \( \Box \)
3. Proof of Theorem 1.2

3.1. In this section we prove Theorem 1.2. As in 2.8 we assume that $X$ is a $G\mathbb{Q}$-Fano threefold, where $G$ be a $p$-elementary subgroup of $\text{Aut}(X)$.

First we consider the case where $X$ non-Gorenstein, i.e., it has at least one point of index $>1$.

**Proposition 3.2.** Let $G$ be a $p$-elementary group and let $X$ be a non-Gorenstein $G\mathbb{Q}$-Fano threefold. Then $\text{rk } G \leq \begin{cases} 7 & \text{if } p = 2, \\ 5 & \text{if } p = 3, \\ 4 & \text{if } p = 5, 7, 11, 13, \\ 3 & \text{if } p \geq 17. \end{cases}$

**Proof.** Let $P_i$ be a point of index $r > 1$ and let $P_1, \ldots, P_l$ be its $G$-orbit. Here $l = p^t$ for some $t$ with $t \geq s - \delta_{2p} - 3$, where $s = \text{rk } G$ (see Lemma 2.4). By the orbifold Riemann-Roch formula [Rei87] and a form of Bogomolov-Miyaoka inequality [Kaw92, KMMT00] we have

$$\sum \left( r_{P_i} - \frac{1}{r_{P_i}} \right) < 24.$$  

Since $r_i - 1/r_i \geq 3/2$, we have $3l/2 < 24$ and so

$$p^{s-\delta_{2p}-3} \leq l < 16.$$  

This gives us the desired inequality. $\square$

From now on we assume that our $G\mathbb{Q}$-Fano threefold $X$ is Gorenstein, i.e., $K_X$ is a Cartier divisor. Recall (see, e.g., [IP99]) that the Picard group of a Fano variety $X$ with at worst (log) terminal singularities is a torsion free finitely generated abelian group ($\simeq H^2(X, \mathbb{Z})$). Then we can define the Fano index of $X$ as the maximal positive integer that divides $-K_X$ in $\text{Pic}(X)$.

**Proposition-Definition 3.3** (see, e.g., [IP99]). Let $X$ be a Fano threefold with at worst terminal Gorenstein singularities. The positive integer $-K_X^3$ is called the degree of $X$. We can write $-K_X^3 = 2g-2$, where $g$ is an integer $\geq 2$ called the genus of $X$. Then $\dim |-K_X| = g+1 \geq 3$.

**Corollary-Notation 3.4.** In notation 3.1 the linear system $|-K_X|$ is not empty, so there exists a $G$-invariant member $S \in |-K_X|$. Write $S = \sum_{i=1}^N S_i$, where $S_i$ are irreducible components.
Theorem 3.5 ([Nam97]). Let $X$ be a Fano threefold with terminal Gorenstein singularities. Then $X$ is smoothable, that is, there is a flat family $X_t$ such that $X_0 \cong X$ and a general member $X_t$ is a smooth Fano threefold of the same degree, Fano index and Picard number. Furthermore, the number of singular points is bounded as follows:

$$|\text{Sing}(X)| \leq 20 - \rho(X_t) + h^{1,2}(X_t).$$

where $h^{1,2}(X_t)$ is the Hodge number.

Combining the above theorem with the classification of smooth Fano threefolds [Isk80], [MM82] (see also [IP99]) we get the following

Theorem 3.7. Let $X$ be a Fano threefold with at worst terminal Gorenstein singularities and let $X_t$ be its smoothing. Let $g$ and $q$ be the genus and Fano index of $X$, respectively.

(i) $q \leq 4$.

(ii) If $q = 4$, then $X \cong \mathbb{P}^3$.

(iii) If $q = 3$, then $X$ is a quadric in $\mathbb{P}^4$ (with $\dim \text{Sing}(X) \leq 0$).

(iv) If $q = 2$, then $\rho(X) \leq 3$ and $-K_X^3 = 8d$, where $1 \leq d \leq 7$. Moreover $\rho(X) = 1$ if and only if $d \leq 5$.

(v) If $q = 1$ and $\rho(X) = 1$, then there are the following possibilities:

| $g$  | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 12  |
|------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $h^{1,2}(X_t)$ | 52  | 30  | 20  | 14  | 10  | 7   | 5   | 3   | 2   | 0   |

Lemma 3.8. Let $G$ be a $p$-elementary group and let $X$ be a Gorenstein $G$-$\mathbb{Q}$-Fano threefold. If the linear system $|-K_X|$ is not base point free, then $\text{rk}G \leq 3 + \delta_{p,2}$.

\textit{Proof.} Assume that $\text{Bs}|-K_X| \neq \emptyset$. Clearly, $\text{Bs}|-K_X|$ is $G$-invariant. By [Isk80], [Shi89] $\text{Bs}|-K_X|$ is either a single point or a smooth rational curve. In the first case the assertion immediately follows by Lemma 2.1. In the second case $G$ acts on the curve $C = \text{Bs}|-K_X|$. Since $C \cong \mathbb{P}^1$, the assertion follows by Lemma 2.6. \qed

Proposition 3.9. Let $G$ be a $p$-elementary group, where $p \geq 5$, and let $X$ be a Gorenstein $G$-$\mathbb{Q}$-Fano threefold. Then

$$\text{rk}G \leq \begin{cases} 4 & \text{if } p = 5, 7, 11, 13, \\ 3 & \text{if } p \geq 17. \end{cases}$$

\textit{Proof.} Assume that the above inequality does not hold. We use the notation of 3.4. In particular, $N$ denotes the number of components of $S = \sum S_i \in |-K_X|$. By Lemma 2.10 $N = p^l$, where $l \geq 1$. Hence
$p$ divides $-K_X^3 = 2g - 2 = ((-K_X)^2 \cdot S_i)N$. First we claim that $\rho(X) = 1$. Indeed, if $\rho(X) > 1$, then the natural representation of $G$ on $\text{Pic}_Q(X) := \text{Pic}(X) \otimes \mathbb{Q}$ is decomposed as $\text{Pic}_Q(X) = V_1 \oplus V$, where $V_1$ is a trivial subrepresentation generated by the class of $-K_X$ and $V$ is a subrepresentation such that $V^G = 0$. Since $G$ is a $p$-elementary group, $\dim V \geq p - 1$. Hence, $\rho(X) \geq p \geq 5$ and by the classification [MM82] we have two possibilities:

- $-K_X^3 = 6(11 - \rho(X))$, $5 \leq \rho(X) \leq 10$, or
- $-K_X^3 = 28$, $\rho(X) = 5$.

In the last case $p = 5$, so $-K_X^3 \equiv 0 \mod p$, a contradiction. In the first case $p$ divides $-K_X^3$ only if $p = 5$. Then $\rho(X) = 6$. So, $\dim V = 5$ and $V^G \neq 0$. Again we get a contradiction.

Therefore, $\rho(X) = 1$. Let $q$ be the Fano index of $X$. We claim that $X$ is singular. Indeed, otherwise all the $S_i$ are Cartier divisors. Then $-K_X = NS_1$, where $N \geq p$, and so $q \geq 5$. This contradicts (i) of Theorem 3.7. Hence $X$ is singular. By Lemma 2.4 and our assumption we have $|\text{Sing}(X)| \geq p$. In particular, $q \leq 2$ (see Theorem 3.7). If $q = 1$, then by Theorem 3.7 either $2 \leq g \leq 10$ or $g = 12$. Thus $N = p$ and we get the following possibilities: $(p,g) = (5,6), (7,8)$, or $(11,12)$. Moreover, $(-K_X)^2 \cdot S_i = (2g - 2)/N = 2$. Therefore, the restriction $|-K_X|_{S_i}$ of the (base point free) anticanonical linear system defines either an isomorphism to a quadric $S_i \to Q \subset \mathbb{P}^3$ or a double cover $S_i \to \mathbb{P}^2$. In both cases the image is rational, so we get a map $G_i \to \text{Cr}_2(\mathbb{C})$ whose kernel is of rank $\leq 1$ by Lemma 2.6 and because $p > 2$. Then by Theorem 1.1 $\text{rk} G_{S_i} \leq 3$. Hence, $\text{rk} G \leq 4$ which contradicts our assumption.

Finally, consider the case $q = 2$. Then $-K_X = 2H$ for some ample Cartier divisor $H$ and $d := H^3 \leq 7$. Therefore, $NS_1 \cdot H^2 = S \cdot H^2 = 2d$. Since $\rho(X) = 1$, by Theorem 3.7 we get $p = d = 5$. Then we apply (3.6). In this case, $h^{1,2}(X) = 0$ (see [IP99]). So, $|\text{Sing}(X)| \leq 19$. On the other hand, $|\text{Sing}(X)| \geq 25$ by Lemma 2.4 and our assumption. The contradiction proves the proposition.

We need the following result which is a very weak form of much more general Shokurov’s toric conjecture [McK01, Pro03].

**Lemma 3.10.** Let $V$ be a smooth Fano threefold and let $D \in |-K_V|$ be a divisor such that the pair $(V, D)$ is LC. Then $D$ has at most $3 + \rho(V)$ irreducible components.

**Proof.** Write $D = \sum_{i=1}^n D_i$. If $\rho(V) = 1$, then all the $D_i$ are linearly proportional: $D_i \sim n_i H$, where $H$ is an ample generator of $\text{Pic}(V)$. Then $-K_V \sim \sum n_i H$ and by Theorem 3.7 we have $\sum n_i = q \leq 4$. 

If $V$ is a blowup of a curve on another smooth Fano threefold $W$, then we can proceed by induction replacing $V$ with $W$. Thus we assume that $V$ cannot be obtained by blowing up of a curve on another smooth Fano threefold. In this situation $V$ is called primitive ([MM83]). According to [MM83, Th. 1.6] we have $\rho(V) \leq 3$ and $V$ has a conic bundle structure $f : V \to Z$, where $Z \cong \mathbb{P}^2$ (resp. $Z \cong \mathbb{P}^1 \times \mathbb{P}^1$) if $\rho(V) = 2$ (resp. $\rho(V) = 3$). Let $\ell$ be a general fiber. Then $2 = -K_V \cdot \ell = \sum D_i \cdot \ell$. Hence $D$ has at most two $f$-horizontal components and at least $n - 2$ vertical ones. Now let $h : V \to W$ be an extremal contraction other than $f$ and let $\ell'$ be any curve in a non-trivial fiber of $h$. For any $f$-vertical component $D_i \subset D$ we have $D_i = f^{-1}(\Gamma_i)$, where $\Gamma_i \subset Z$ is a curve, so $D_i \cdot \ell' = \Gamma_i \cdot f_* \ell' \geq 0$. If $\rho(V) = 2$, then $D_i \cdot \ell' \geq 1$. Hence, $-K_V \cdot \ell' \geq n - 2$. On the other hand, $-K_V \cdot \ell' \leq 3$ (see [MM83 §3]). This immediately gives us $n \leq 5$ as claimed. Finally consider the case $\rho(V) = 3$. Assume that $n \geq 7$. Then we can take $h$ so that $\ell'$ meets at least three $f$-vertical components, say $D_1, D_2, D_3$. As above, $-K_V \cdot \ell' \geq 3$ and by the classification of extremal rays (see [MM83 §3]) $h$ is a del Pezzo fibration. This contradicts our assumption $\rho(V) = 3$.

\[\square\]

**Proposition 3.11.** Let $G$ be a 2-elementary group and let $X$ be a Gorenstein $G\mathbb{Q}$-Fano threefold. Then $\text{rk} G \leq 7$.

**Proof.** Assume that $\text{rk} G \geq 8$. By Lemma 2.10 we have $\text{rk} G_{S_i} \leq 5$. Hence, $N \geq 8$. If $X$ is smooth, then by Lemma 3.10 we have $\rho(X) \geq 5$. If furthermore $X \cong Y \times \mathbb{P}^1$, where $Y$ is a del Pezzo surface, then the projection $X \to Y$ must be $G$-equivariant. This contradicts $\rho(X)^G = 1$. Therefore, $\rho(X) = 5$ and $-K_X^3 = 28$ or 36 (see [MM82]). On the other hand, $-K_X^3$ is divisible by $N$, a contradiction.

Thus $X$ is singular. Assume that $|\text{Sing}(X)| \geq 32$. Then for a smoothing $X_t$ of $X$ by (3.6) we have $h^{1,2}(X_t) \geq 13$. Since $N$ divides $-K_X^3 = -K_{X_t}^3$, using the classification of Fano threefolds [Isk80, MM82] (see also [IP99]) we get:

$$\rho(X) = 1, \quad -K_X^3 = 8, \quad N = 8, \quad |\text{Sing}(X)| = 32.$$ 

Consider the representation of $G$ on $H^0(X, -K_X)$. Since

$$7 = \dim H^0(X, -K_X) < \text{rk} G,$$

this representation is not faithful (otherwise $G$ is contained in a maximal torus of $\text{GL}(H^0(X, -K_X)) = \text{GL}_7(\mathbb{C})$). Therefore, the linear system $|-K_X|$ is not very ample. On the other hand, $|-K_X|$ is base point free (see Lemma 3.8). Hence $|-K_X|$ defines a double cover $X \to Y \subset \mathbb{P}^6$ [Isk80]. Here $Y$ is a variety of degree 4 in $\mathbb{P}^6$, a variety of
minimal degree. If \( Y \) is smooth, then according to the Enriques theorem (see, e.g., [Isk80, Th. 3.11]) \( Y \) is a rational scroll \( \mathbb{P}_2(\mathcal{E}) \), where \( \mathcal{E} \) is a rank 3 vector bundle on \( \mathbb{P}^1 \). Then \( X \) has a \( G \)-equivariant projection to a curve. This contradicts \( \rho(X)^G = 1 \). Hence \( Y \) is singular. In this case, \( Y \) is a cone (again by the the Enriques theorem [Isk80, Th. 3.11]). If its vertex \( O \in Y \) is zero-dimensional, then \( \dim T_{O,Y} = 6 \). On the other hand, \( X \) has only hypersurface singularities (see 2.1). Therefore the double cover \( X \to Y \) is not étale over \( O \) and so \( G \) has a fixed point on \( X \). This contradicts Lemma 2.4. Thus \( Y \) is a cone over a rational normal curve of degree 4 with vertex along a line. Then \( X \) cannot have isolated singularities, a contradiction.

Therefore, \( |\text{Sing}(X)| < 32 \). Then for any point \( P \in \text{Sing}(X) \) by Lemma 2.4 we have \( \text{rk} G_P \geq 4 \). Hence the orbit of \( P \) contains 16 elements and coincides with \( \text{Sing}(X) \), i.e. the action of \( G \) on \( \text{Sing}(X) \) is transitive. Since \( S \cap \text{Sing}(X) \neq \emptyset \), we have \( \text{Sing}(X) \subset S \). On the other hand, our choice of \( S \) in 2.8 is not unique: there is a basis \( s^{(1)}, \ldots, s^{(g+2)} \in H^0(X, -K_X) \) consisting of eigensections. This basis gives us \( G \)-invariant divisors \( S^{(1)}, \ldots, S^{(g+2)} \) generating \( |-K_X| \). By the above \( \text{Sing}(X) \subset S^{(i)} \) for all \( i \). Thus \( \text{Sing}(X) \subset \cap S^{(i)} = Bs |-K_X| \). This contradicts Lemma 3.8. Proposition 3.11 is proved. □

**Proposition 3.12.** Let \( G \) be an 3-elementary group and let \( X \) be a Gorenstein \( GQ \)-Fano threefold. Then \( \text{rk} G \leq 3 \).

**Proof.** Assume that \( \text{rk} G \geq 6 \). By Lemma 2.10 we have \( \text{rk} G_P \leq 5 \). Hence, \( N \geq 9 \). If \( X \) is smooth, then by Lemma 3.10 we have \( \rho(X) \geq 6 \) and so \( X \simeq Y \times \mathbb{P}^1 \), where \( Y \) is a del Pezzo surface [MM82]. Then the projection \( X \to Y \) must be \( G \)-equivariant. This contradicts \( \rho(X)^G = 1 \). Therefore, \( X \) is singular. By Lemma 2.4 \( |\text{Sing}(X)| \geq 3^{6-3} = 27 \). Hence, for a smoothing \( X_t \) of \( X \) by (3.6) we have \( h^{1,2}(X_t) \geq 7 + \rho(X) \). Recall that \( N \) divides \(-K_X^3 = -K_{X_t}^3 \). Then we use the classification of smooth Fano threefolds [Isk80], [MM82] and get a contradiction. □

Now Theorem 1.2 is a consequence of Propositions 3.2, 3.9, 3.11 and 3.12.

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