Multiloop Integrand Reduction for Dimensionally Regulated Amplitudes

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We present the integrand reduction via multivariate polynomial division as a natural technique to encode the unitarity conditions of Feynman amplitudes. We derive a recursive formula for the integrand reduction, valid for arbitrary dimensionally regulated loop integrals with any number of loops and external legs, which can be used to obtain the decomposition of any integrand analytically with a finite number of algebraic operations. The general results are illustrated by applications to two-loop Feynman diagrams in QED and QCD, showing that the proposed reduction algorithm can also be seamlessly applied to integrands with denominators appearing with arbitrary powers.

Introduction – In the perturbative approach to quantum field theories, the elements of the scattering matrix, which are the scattering amplitudes, can be expressed in terms of Feynman diagrams. The latter generally represent multiple integrals whose integrand is a rational function of the integration variables. Scattering amplitudes are analytic functions of the kinematic variables of the interacting particles, hence they are determined by their singularities, whose location in the complex plane is specified by a set of algebraic equations. The analysis of the singularity structure can be used to define the discontinuities of a Feynman integral across the branch cuts attached to the Landau singularities. They are encoded in the Cutkosky formula and correspond to the unitarity conditions of the scattering amplitude. In the canonical formalism, the unitarity cut conditions have been used for the evaluation of the scattering amplitudes through dispersive Cauchy’s integral representations. However, the dispersive approach is well-known to suffer from ambiguities which limit its applicability for the quantitative evaluation of generic Feynman integrals in gauge theories.

In the more modern interpretation of unitarity, cut conditions and analyticity are successfully exploited for decomposing scattering amplitudes in terms of independent functions – rather than for their direct evaluation. The basic functions entering the amplitudes decomposition are univocally characterized by their singularities. The singularity structure can be accessed before integration, at the integrand level [1,2]. Therefore, the decomposition of the integrated amplitudes can be deduced from the the decomposition of the corresponding integrands. The integrand-reduction methods [1,7] rely on the existence of a relation between the numerator and the denominators of each Feynman integral. A generic numerator can be expressed as a combination of products of denominators, multiplied by polynomial coefficients, which correspond to the residues at the multiple cuts of the diagrams. The multiple-cut conditions, generally fulfilled for complex values of the integration variables, can be viewed as projectors isolating each residue. The latter, depicted as an on-shell cut diagram, represents the amplitude factorized into a product of simpler amplitudes, either with fewer loops or a lower number of legs.

The residues are multivariate polynomials in those components of the propagating momenta which correspond to irreducible scalar products (ISPs), that cannot be decomposed in terms of denominators. The ISPs either yield spurious contributions, which vanish upon integration, or generate the basic integrals entering the amplitude decomposition [2,4].

Within the integrand reduction methods, the problem of decomposing any scattering amplitude in terms of independent integrals is therefore reduced to the algebraic problem of reconstructing the residues at its multiple cuts.

In Refs. [6,7] the determination of the residues at the multiple cuts has been formulated as a problem of multivariate polynomial division, and solved using algebraic geometry techniques. These techniques allowed one to prove that the integrand decomposition, originally formulated for one-loop amplitudes [1], is valid and applicable at any order in perturbation theory, irrespective of the complexity of the topology of the involved diagrams, being them massless or massive, planar or non-planar. This novel reduction algorithm has been applied to the decomposition of supersymmetric amplitudes at two and three loops [8,9]. Also, it has been used for the identification of the two-loop integrand basis in four dimen-
varieties \[11, 12\].

In Ref. \[7\], we found an *integrand-recursion formula* for the iterative decomposition of scattering amplitudes, based on successive divisions of the numerators modulo the Gröbner basis of the ideals generated by the cut denominators. The integrand recurrence relation may be applied in two ways.

The first approach, that we define *fit-on-the-cuts*, requires the knowledge of the parametric residues and of the parametric (families of) solutions of all possible multiple cuts. The parameters of the residues are determined by evaluating the numerator at the solutions of the multiple cuts, as many times as the number of the unknown coefficients. This approach is the canonical way to achieve the integrand decomposition of scattering amplitudes at one loop \[1\], and it has been implemented in public codes like CUTTOOLS \[13\], and SamuRai \[14\]. In this approach the parametrization of the residues can be found by applying the integrand-recursion formula to the most generic numerator function, with parametric coefficients.

Alternatively, as we show in this letter, the reduction formula can be applied directly to the numerator, within what we define as the *divide-and-conquer* approach. In this case, the decomposition of the amplitude is obtained by successive polynomial divisions, which at each step generate the actual residues. In this way, the decomposition of any integrand is obtained analytically, with a finite number of algebraic operations, without requiring the knowledge of the varieties of solutions of the multiple cuts, nor the one of the parametric form of the residues.

In the following, we describe the coherent mathematical framework underlying the integrand decomposition, interpreting the unitarity-cut conditions as *equivalence classes* of polynomials. We present the *divide-and-conquer* approach through its systematic application to the decomposition of some classes of two-loop diagrams. The examples show the main features of the proposed reduction algorithm, which can be applied to generic dimensionally regulated Feynman integrals with multiple denominators, namely denominators appearing with arbitrary powers. To the best of our knowledge, this is the first application of integrand-reduction algorithms directly to diagrams with multiple propagators.

With this communication we finally aim at presenting the integrand reduction via multivariate polynomial division as a natural technique to encode the unitarity conditions of Feynman amplitudes. Indeed Cauchy’s integration, which is the underlying concept of unitarity integrals and, more generally, of discontinuities formulas, when applied to rational integrands corresponds to partial fraction, which is the objective of the polynomial division.

**Integrand reduction formula** – The extension of the integrand recurrence relation required to accommodate multiple propagators is straightforward. An arbitrary graph with \(\ell\) loops represents a \(d\)-dimensional integral of the type

\[
\int d^d q_1 \cdots d^d q_\ell I_{i_1 \cdots i_n}^{a_1 \cdots a_n},
\]

\[
I_{i_1 \cdots i_n}^{a_1 \cdots a_n} \equiv \frac{N_{i_1 \cdots i_n}^{a_1 \cdots a_n}}{D_{i_1}^{a_1} \cdots D_{i_n}^{a_n}},
\]

(1)

with \(i_1, \ldots, i_n\) distinct indices. The numerator and the denominators are polynomials in a set of coordinates \(z\), i.e., they are in the polynomial ring \(P[z]\). We define the ideal

\[
J_{i_1 \cdots i_n} = \left\{ \sum_{k=1}^{n} h_k(z) D_{i_k}(z) : h_k(z) \in P[z] \right\}.
\]

(2)

Given a monomial ordering, we define the *normal form* of a polynomial \(p(z)\) with respect to the ideal \(J\) as

\[
[p(z)]_J = p(z) \mod \mathcal{G}_{i_1 \cdots i_n},
\]

(3)

i.e., the normal form of \(p\) is the remainder of its division modulo a Gröbner basis \(\mathcal{G}\) of \(J\). Two polynomials \(p(z), q(z) \in P[z]\) are *congruent modulo \(J\) iff their difference can be written in terms of the denominators, i.e.

\[
p(z) \sim q(z) \quad \text{iff} \quad p(z) - q(z) \in J_{i_1 \cdots i_n}.
\]

The congruence modulo \(J\) is an equivalence relation and the set of all its equivalence classes is the *quotient ring* \(P[z]/J\). The properties of the Gröbner basis ensure that

\[
p(z) \sim q(z) \quad \text{iff} \quad [p(z)]_J = [q(z)]_J.
\]

Therefore, the normal form of the elements of the equivalence classes establish a natural correspondence between \(P[z]/J\) and \(P[z]\).

The numerator \(N\) of Eq. (1) is a polynomial in \(z\) and can be decomposed by performing the division

\[
N_{i_1 \cdots i_n}^{a_1 \cdots a_n} / \mathcal{G}_{i_1 \cdots i_n}^{a_1 \cdots a_n},
\]

(4)

Eq. (2) allows one to write its decomposition as

\[
N_{i_1 \cdots i_n}^{a_1 \cdots a_n} = \Gamma_{i_1 \cdots i_n} +
\]
The normal form of the numerator is not in the ideal $\mathcal{J}$, thus it cannot be expressed in terms of the denominators and it is identified with the residue of the multiple cut $D_{i_1}^{a_1} \cdots = D_{i_n}^{a_n} = 0$, 

$$|N_{i_1} \cdots i_n \cdots i_n J_{j_1 i_2} \cdots i_n | = \Delta_{i_1} \cdots i_n \cdots i_n,$$  

(6)

belonging to the quotient ring $P[z]/\mathcal{J}$. The term $\Gamma_i$, instead, belongs to the ideal $J$, thus it can be written as

$$\Gamma_{i_1} \cdots i_1 \cdots i_n = \sum_{k=1}^{n} N_{i_1} \cdots i_k \cdots i_n \cdots i_n = \Delta_{i_1} \cdots i_n \cdots i_n,$$  

(7)

Substituting Eqs. (5), (6), and (7) in Eq. (1), we obtain

$$I_{i_1} \cdots i_n = \sum_{k=1}^{n} I_{i_1} \cdots i_k \cdots i_n \cdots i_n + \frac{\Delta_{i_1} \cdots i_n \cdots i_n}{D_{i_1}^{a_1} \cdots D_{i_n}^{a_n}},$$  

(8)

which is a non-homogeneous recurrence relation expressing a given integrand in terms of integrands with fewer denominators. It is the generalization of the recurrence relation of Ref. [7], valid for arbitrary powers of the denominators.

The integrand of the diagram (a) is

$$I_{12345}^{(a)} = \frac{1}{3 - 2\epsilon} \frac{N_{12345}^{(a)}}{D_1 D_2 D_3 D_4 D_5},$$  

(9)

while its denominators are

$$D_1 = q_1^2 - m^2, \quad D_2 = (q_1 + k)^2 - m^2, \quad D_3 = q_2^2 - m^2, \quad D_4 = (q_2 + k)^2 - m^2, \quad D_5 = (q_1 - q_2)^2.$$

According to our algorithm, the first step of the reduction requires the division $N_{12354}^{(a)}/G_{12345}$, whose result reads as

$$\frac{N_{12354}^{(a)}}{G_{12345}} = \Delta_{12354} + \Delta_{12345} D_4 + \Delta_{2345} D_1 + \Delta_{1345} D_2 + \Delta_{1245} D_3 + \Delta_{1234} D_5 + \Delta_{1235} D_4 + \Delta_{1234} D_5 + \Delta_{1235} D_4 + \Delta_{1234} D_5.$$  

(10)

In the second step, the numerators $N_{i_1 i_2 i_3 i_4}$ are reduced performing the division $N_{i_1 i_2 i_3 i_4}/G_{i_1 i_2 i_3 i_4}$.

$$\frac{N_{i_1 i_2 i_3 i_4}^{(a)}}{G_{i_1 i_2 i_3 i_4}} = \Delta_{i_1 i_2 i_3 i_4} + \Delta_{i_1 i_2 i_3} D_4 + \Delta_{i_1 i_2 i_3} D_1 + \Delta_{i_1 i_2 i_3} D_2 + \Delta_{i_1 i_2 i_3} D_3 + \Delta_{i_1 i_2 i_3} D_5 + \Delta_{i_1 i_2 i_3} D_4 + \Delta_{i_1 i_2 i_3} D_5 + \Delta_{i_1 i_2 i_3} D_4 + \Delta_{i_1 i_2 i_3} D_5.$$  

(11)

The complete decomposition of $N_{12345}^{(a)}$ is obtained by iterating the procedure twice,

$$\frac{N_{12345}^{(a)}}{G_{12345}} = \Delta_{12345} + \Delta_{1235} D_4 + \Delta_{1234} D_1 + \Delta_{1234} D_2 + \Delta_{1235} D_3 + \Delta_{1234} D_5 + \Delta_{1235} D_4 + \Delta_{1234} D_5 + \Delta_{1235} D_4 + \Delta_{1234} D_5 + \Delta_{1235} D_4 + \Delta_{1234} D_5 + \Delta_{1235} D_4 + \Delta_{1234} D_5 + \Delta_{1235} D_4 + \Delta_{1234} D_5 + \Delta_{1235} D_4 + \Delta_{1234} D_5 + \Delta_{1235} D_4 + \Delta_{1234} D_5.$$  

(12)

The residues in Eq. (12) read as follows:

$$\Delta_{12345} = 8 \left(4 m^4 - k^4 + k^2 (k^2 - 2 m^2) \epsilon \right).$$
\[ \Delta_{1234} = -4 \left[ (4m^2 + k^2(3 - \epsilon - 2\epsilon^2)) + 4(1 - \epsilon) \left( \mu_{12}^2 - \frac{(q_1 \cdot k)(q_2 \cdot k)}{k^2} \right) \right. \\
- \left. \left( (q_1 \cdot e_3)(q_2 \cdot e_4) - (q_1 \cdot e_4)(q_2 \cdot e_3) \right) \right]. \]

\[ \Delta_{1235} = \Delta_{2345} = \Delta_{145} = 8 \left( m^2 + k^2(1 - \epsilon) \right), \]
\[ \Delta_{123} = \Delta_{124} = \Delta_{134} = \Delta_{234} = 4 \left( 1 - \epsilon \right), \]
\[ \Delta_{125} = \Delta_{135} = \Delta_{245} = \Delta_{345} = -8 \left( 1 - \epsilon \right), \]
\[ \Delta_{145} = \Delta_{235} = 8 \epsilon \left( 1 - \epsilon \right), \]
\[ \Delta_{13} = \Delta_{24} = -\Delta_{14} = -\Delta_{23} = \frac{4(1 - \epsilon)}{k^2}. \] (13)

The diagram (b) contains a double propagator,
\[ T^{(b)}_{11234} = \frac{1}{3 - 2\epsilon} \frac{N^{(b)}_{11234}}{D_1^2 D_2 D_3 D_4}, \] (14)
where the denominators are
\[ D_1 = q_1^2 - m^2, \quad D_2 = (q_1 - k)^2 - m^2, \]
\[ D_3 = q_2^2, \quad D_4 = (q_1 + q_2)^2 - m^2. \]

The first step of the reduction requires the division \( N^{(b)}_{11234}/G_{11234} \) which, because of Eq. (2), is equivalent to the division \( N^{(b)}_{11234}/G_{1234} \).
\[ N^{(b)}_{11234} = \Delta_{1234} + N_{1234} D_1 + N_{1123} D_4 \]
\[ + N_{1134} D_2 + N_{1124} D_3. \] (15)

In the second step we perform the divisions \( N^{(b)}_{111234}/G_{111234} \), obtaining
\[ N^{(b)}_{11234} = \Delta_{1234} + \Delta_{1234} D_1 + \Delta_{1123} D_4 + \Delta_{1134} D_2 \]
\[ + \Delta_{1243} + N_{1134} D_2 D_4 + N_{1142} D_3 + N_{234} D_1^2. \] (16)

Due to Eq. (2), the division \( N^{(b)}_{111234}/G_{111234} \) is equivalent to \( N^{(b)}_{111234}/G_{111234} \). The reduction is completed by performing the divisions \( N^{(b)}_{111234}/G_{111234} \), along the lines of the previous steps, obtaining
\[ N^{(b)}_{11234} = \Delta_{1234} + \Delta_{1234} D_1 + \Delta_{1123} D_4 + \Delta_{1134} D_2 \]
\[ + \Delta_{1243} + N_{1134} D_2 D_4 + N_{1142} D_3 + N_{234} D_1^2, \] (17)

in terms of the residues
\[ \Delta_{11234} = 16m^2 \left( k^2 + 2m^2 - k^2\epsilon \right), \]
\[ \Delta_{1234} = 16 \left[ (q_2 \cdot k)(1 - \epsilon^2) + m^2 \right], \]
\[ \Delta_{124} = -\Delta_{123} = 8 \left( 1 - \epsilon \right) \left[ k^2(1 - \epsilon) + 2m^2 \right], \]
\[ \Delta_{113} = -16m^2 \left( 1 - \epsilon \right), \]
\[ \Delta_{114} = -\Delta_{234} = 8 \left( 1 - \epsilon \right)^2. \] (18)

The integrand of the diagram (c) is obtained by performing the replacement \( T^{(c)}_{11234} = T^{(b)}_{11234} \mid k \to -k \).

We remark that the residues can also be expressed in terms of normal forms. For instance, in the case of \( T^{(b)} \), \( \Delta_{1123} \) and \( \Delta_{113} \) can be written as
\[ \Delta_{1123} = \frac{\left[ N^{(b)}_{11234} - \Delta_{11234} \right]}{[D_1]_{J_{123}}}, \]
\[ \Delta_{113} = \frac{\left[ N^{(b)}_{11234} - \Delta_{11234} - \Delta_{1123} D_4 - \Delta_{1134} D_2 \right]}{[D_2 D_4]_{J_{13}}} J_{13}. \]

Since \( \Delta_{1234} D_1 \in J_{1234}, \) the residue \( \Delta_{1234} \) can be obtained using
\[ \Delta_{1234} = \left[ \frac{N^{(b)}_{11234} - \Delta_{11234}}{[D_1] J_{1234}} \right]_{J_{1234}} \],
where \( J_{1234} \equiv \langle D_1^2, D_2, D_3, D_4 \rangle \subset J_{1234} \).

Diagrams for Higgs production via gluon fusion. – We also consider the three-point diagrams in the second row of Figure 2 which enter the two-loop QCD corrections to the Higgs production via gluon fusion in the heavy top limit 16. In this case, the variables \( z \) are \( \mu_1^2, \mu_2^2, \mu_3^2 \) and the components of the four-vectors \( q_i \) in the basis of massless vectors \( \{ k_1, k_2, e_3, e_4 \} \), such that \( k_i \cdot e_j = 0 \) and \( e_3 \cdot e_4 \neq 0 \). Within the divide-and-conquer approach, the integrand of the generic diagram is decomposed as
\[ T^{(e)} = \sum_{k=2}^6 \sum_{i_1 \ldots i_k} \frac{\Delta_{i_1 \ldots i_k}}{D_{i_1} \ldots D_{i_k}}, \quad x = d, e, f. \] (19)

For the diagram (d) the second sum runs over the unordered selections without repetition of \( \{ 1, 1, 2, 3, 4, 5 \} \), while for the diagram (e) and (f) it runs over the unordered selections without repetition of \( \{ 1, \ldots, 6 \} \). The expression of the residues are lengthy and are omitted, however they are available upon request.

The reduction algorithm described in this letter has been automated in a python package which uses MACAULAY2 17 and FORM 18. The numerators of the presented examples have been generated with QGRAF 19 and FORM and independently with FEYNARTS 20, FEYNCALC 21, and FORMCALC 22.

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