CONFINED ELASTICAE AND THE BUCKLING OF CYLINDRICAL SHELLS

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Abstract. For curves of prescribed length embedded into the unit disc in two dimensions, we obtain scaling results for the minimal elastic energy as the length just exceeds $2\pi$ and in the large length limit. In the small excess length case, we prove convergence to a fourth order obstacle type problem with integral constraint on the real line which we then solve. From the solution, we obtain the first order coefficient $\Theta \approx 37$ in the energy expansion $2\pi + \Theta \delta^{1/3} + o(\delta^{1/3})$ when a curve has length $2\pi + \delta$. We present an application of the scaling result to buckling in two-layer cylindrical shells where we can determine an explicit bifurcation point between compression and buckling in terms of universal constants and material parameters scaling with the thickness of the inner shell.

Contents

1. Introduction 1
2. Review of Elastic Curves 4
3. Minimisation Problem at $2\pi$ 7
4. Minimisation Problem on the Line 14
4.1. Existence of Minimisers 15
4.2. Functions of Low Energy 24
4.3. A Problem with Delamination 30
4.4. Application to the Buckling of Cylindrical Shells 32
5. The Large Length Limit 35
6. Curves in Three Dimensions 37
7. Conclusion 38
Acknowledgements 40
Appendix A. Proofs of Basic Properties 40
References 42

1. Introduction

For a curve $\gamma$ in two or three dimensions, we define the elastic energy

$$W(\gamma) = \int_{\gamma} \kappa^2 \, d\mathcal{H}^1$$

where $\kappa$ is the curvature of $\gamma$ and $\mathcal{H}^1$ is the one-dimensional Hausdorff measure. This energy is a simple model for elastic beams, the leading order elastic energy of (almost) straight thin sheets, and is proposed in image segmentation to reconstruct objects partially occluded from the viewer which a perimeter-based functional may not capture. Below, we will consider a specific application in two-layer cylindrical shells (such as tubes composed of two different materials).
Our notation is derived from the corresponding energy on surfaces, which is usually referred to as Willmore's energy. For the Willmore functional, Müller and Röger have considered the following problem: \textit{Minimise} \( W \) \textit{among all surfaces} \( \Sigma \) \textit{embedded in the unit ball} \( B_1(0) \) \textit{in three dimensions which have prescribed area} \( S > 0 \). In [MR14], they prove that

\[
\limsup_{S \to \infty} \inf_{|\Sigma| = S} [W(\Sigma) - S] < \infty, \quad \inf_{|\Sigma| = S} W(\Sigma) = S \iff S \in 4\pi \mathbb{Z}
\]

and, perhaps most interestingly, that there exist constants \( c, C > 0 \) and \( \delta_0 > 0 \) such that

\[
4\pi + c\delta^{1/2} \leq \inf_{|\Sigma| = 4\pi + \delta} W(\Sigma) \leq 4\pi + C\delta^{1/2} \quad \forall 0 < \delta < \delta_0.
\]

In this article, we obtain the analogous results for curves in the plane. While the proof in [MR14] invokes rigidity estimates for nearly umbilical surfaces due to De Lellis and Müller, our arguments are elementary by comparison and we characterize the leading order term in the energy explicitly instead of just giving scaling bounds.

In analogy to [MR14], we introduce the space of admissible curves

\[
M_L = \left\{ \gamma \in C^\infty(S^1; B_1(0)) \mid |\gamma'| = \frac{L}{2\pi}, \gamma \text{ is embedded} \right\}
\]

for \( L > 0 \) and consider the problem of minimizing \( W \) in \( M_L \). This can be thought of as a geometric higher order obstacle problem where the obstacle is given by the domain boundary \( \partial B_1(0) \) and the curve itself due to the non-self intersection constraint. We have a technical advantage over the setting considered in [MR14] since curves unlike surfaces admit arc-length parametrisations and we can avoid the language of geometric measure theory entirely. In the short area regime, we show the following.

**Theorem 1.1.** There exists \( 0 < \delta < \delta_0 \) and a constant \( \Theta > 0 \) such that

\[
\inf_{\gamma \in M_{2\pi + \delta}} W(\gamma) = 2\pi + \Theta\delta^{1/3} + o(\delta^{1/3}).
\]

So in the regime where the length of the curve just exceeds the parameter where it can comfortably fit into the domain \( B_1(0) \) as a circle, \( W \) shows a steep increase in terms of the excess length. The qualitative behaviour is therefore comparable to that in the two-dimensional case while the order of the rapid growth is different.

The key argument in proving the theorem is showing that it is asymptotically equivalent to a higher order obstacle type problem with an integral constraint on the real line which we obtain by a careful expansion procedure. This problem also characterises the constant \( \Theta \).

Denote the linearised length functional energy functionals by

\[
L, \mathcal{E} : C^\infty_c(\mathbb{R}) \to \mathbb{R}, \quad L(\phi) = \int_{\mathbb{R}} \frac{(\phi')^2}{2} - \phi \, ds, \quad \mathcal{E}(\phi) = \int_{\mathbb{R}} |\phi''|^2 \, ds.
\]

The non-linear space \( M_L \) is replaced by the manifold

\[
M = \left\{ \phi \in C^\infty_c(\mathbb{R}) \mid \phi \geq 0, \int_{\mathbb{R}} \frac{(\phi')^2}{2} - \phi \, ds = 1 \right\}.
\]

**Theorem 1.2.** The energy \( \mathcal{E} \) has a minimiser \( \pi \) in the larger class

\[
\overline{M} = \left\{ \phi \in W^{2,2}(\mathbb{R}) \cap L^1(\mathbb{R}) \mid \phi \geq 0, \int_{\mathbb{R}} \frac{(\phi')^2}{2} - \phi \, ds = 1 \right\}
\]

that satisfies the following properties:

- \( 1 < \mathcal{E}(\pi) < \infty \).
- \( \pi \in C^2_c(\mathbb{R}) \setminus C^3(\mathbb{R}) \).
- \( \pi \) is compactly supported and the set \( \{ \pi > 0 \} \) is connected.
- \( \pi \) is even.
- \( \pi \) increases from 0 to its maximum in a monotone fashion.
There is an interesting scaling relation in the problem. Namely, for given $\phi$ the function $\phi_\rho(x) = \rho^{2/3} \phi(\rho^{-1/3} x)$ satisfies

$$L(\phi_\rho) = \rho \cdot L(\phi), \quad \mathcal{E}(\phi_\rho) = \rho^{1/3} \cdot \mathcal{E}(\phi)$$

which explains the emergence of the problem on the line from the geometric problem: For small $\delta$, $L(\phi_\delta)$ approximates the excess length of a curve which is given by a slight perturbation of a circle, described by a $\phi_\delta$-shaped bump in radial direction. The condition $\phi \geq 0$ is required to ensure that the corresponding curve lies inside the circle. The parameter $\Theta := \inf_{\phi \in \mathcal{M}} \mathcal{E}(\phi)$ connects Theorems 1.1 and 1.2 and the same connection together with the scaling property explains the emergence of the optimal order $1/3$.

Note that the regularity $W^{3, \infty} \setminus C^3$ is strictly higher than the optimal regularity in more classical obstacle problems. The minimiser is obtained as the limit of minimisers of similar problems over compact intervals and can be described fairly explicitly as the solution of an Euler-Lagrange equation, which in one dimension is just an ODE. This non-homogeneous linear fourth order ODE can be solved explicitly, and we can analyse it further to explicitly obtain the minimiser

$$\nu(x) = \begin{cases} a - \frac{x^2}{2} + \alpha \cos(\mu x) & |x| < r \\ 0 & \text{else} \end{cases}$$

with parameters $r = \sqrt{6} \approx 1.82$ and $\mu \approx 2.47$ such that $\mu r$ is the first positive solution of the equation $\tan(\mu) = \rho$. The remaining parameters $a \approx 1.81$, $\alpha \approx 0.75$ and an energy $\Theta \approx 36.69$ can be computed analytically from $\mu$ and $r$. We give a graphical representation below in Figure 1.

Besides energy minimisation, we describe regularity properties of functions satisfying energy bounds and develop a general variational machinery for the problem. We also obtain analogous results for the minimisation problems of the energies

$$\mathcal{W}_\alpha(\gamma) = \mathcal{W}(\gamma) + \alpha \mathcal{H}^1\left(\{\gamma \notin \partial B_1(0)\}\right), \quad \mathcal{E}_\alpha(\phi) = \mathcal{E}(\phi) + \alpha \mathcal{H}^1\left(\{\phi \neq 0\}\right)$$

in the respective classes $\mathcal{M}_{2\pi + \delta}$ and $\mathcal{M}$. Many results carry over, except that minimisers of $\mathcal{E}_\alpha$ are only $C^{1,1}$-smooth and not $C^2$-smooth if $\alpha > 0$. We apply the results of Theorem 1.1 to a model of the following problem: Imagine two cylindrical shells, one contained in the other, of which the outer one has the same height but smaller area (say, a pipe made of two layers where the outer one contracts more at low temperatures). The inner layer has two options to comply with the constraint forced by the outer layer: compression or buckling. Assuming that all shells remain cylindrical and that the outer layer is a lot more rigid than the inner one, we show that bifurcation to buckling would be expected at

$$\delta = \lambda_0^{-3/5} \left( \frac{\Theta c_{\text{mat}}}{\frac{4\pi}{3} r_o^3} \right)^{3/4} h^{6/5}$$

where $\delta$ is the excess preferred length of the planar profile of the inner shell, $h$ is the thickness of the inner shell, $r_o$ is the radius of the (circular) outer shell, $c_{\text{mat}}$ is a material constant, the universal constant $\Theta$ is as above and $\lambda_0 \approx 1.034$ is the parameter such that the function

$$f(s) = (s - 1)^2 + \lambda s^{1/3}$$

has its minimum at 0 if $\lambda > \lambda_0$ and at a positive point if $\lambda < \lambda_0$. If we include an adhesive between the shells in the model by considering a bending energy $\mathcal{W}_\alpha$ with $\alpha > 0$ which penalises delamination, the buckling regime changes from $h^{6/5}$ to $(ah)^{2/5}$.

We also characterise the large length limit. Since in general $\mathcal{W}(\gamma) \geq \mathcal{H}^1(\gamma)$ in analogy to the two-dimensional case [MR13], we only need to show an estimate of the form $\mathcal{W}(\gamma) \leq L + e_L$ for curves of length $L \geq 1$.

**Theorem 1.3.** In the large length limit, we observe that

$$\limsup_{L \to \infty} \inf_{\gamma \in \mathcal{M}_L} \frac{\mathcal{W}(\gamma) - L}{\sqrt{L}} < \infty.$$
Finally, we demonstrate that the sharp energy increase is a two-dimensional phenomenon which can be avoided in three (or more) dimensions by out-of-plane buckling.

**Theorem 1.4.** Using the same notation as above, but assuming that $B_1(0)$ is the unit ball in three (or more) dimensions, there exists a constant $1 < C < \frac{2}{\pi}$ and $\delta_0 > 0$ such that

$$2\pi + \delta \leq \inf_{\gamma \in \mathcal{M}_{L_2+\delta}} W(\gamma) \leq 2\pi + C\delta \quad \forall \delta > \delta_0.$$  

Furthermore,

$$\limsup_{L \to \infty} \left( \inf_{\gamma \in \mathcal{M}_L} W(\gamma) - L \right) < \infty.$$  

Theorems 1.3 and 1.4 are proved by explicit constructions of energy competitors. The article is structured as follows. In Section 2, we review some classical results on elastic curves and collect a few results on their variational structure. Section 3 is devoted to the study of the minimisation problem for small excess length and the proof of Theorem 1.1 where we derive the linearised obstacle problem on the line, which is then treated in Section 4. In this section, we also consider the problem with a delamination penalty and applications to the buckling of elastic shells. Finally, in Sections 5 and 6 we discuss the large length limit and the situation for space curves. The sections are essentially independent and can be read separately, with the exception of Section 2, which is needed for all but Section 4. All results are discussed and put into context in Section 7. In the appendix, we give quick proofs of some of the results from Section 2.

2. Review of Elastic Curves

The energy $W$ is geometric in nature, i.e. independent of the parametrisation of a curve. This allows an *extrinsic* approach through generalised objects in geometric measure theory known as varifolds. While this technicality is mostly unavoidable for higher dimensional versions of this problem, in one dimension, we have viable alternatives. In particular, if a curve is parametrised by unit speed, $|\gamma'| = 1$, then its curvature vector is given by

$$\vec{H}_{\gamma}(s) = \gamma''(s)$$

and its curvature is $\kappa_{\gamma}(s) = \pm |\vec{H}_{\gamma}(s)| = \pm |\gamma''(s)|$. In particular, the elastic energy of a curve can be written as

$$W(\gamma) = \int_{S^1_1} |\gamma''|^2 ds$$

if $\gamma$ is parametrised by arc-length on a circle of length $L$. This is a valuable tool in the calculus of variations, allowing us to relate the problem to the Sobolev space $W^{2,2}(S^1_1; \mathbb{R}^n)$. In a later chapter, we will use a radial parametrisation of curves, so we note that the curvature of a curve in general parametrisation is given by

$$\vec{H}_{\gamma} = \frac{\gamma'' - \langle \gamma'', \frac{\gamma'}{|\gamma'|} \rangle \frac{\gamma'}{|\gamma'|}}{|\gamma'|^2}$$

as easily confirmed by the chain rule. This makes the elastic energy

$$W(\gamma) = \int_{S^1_1} \frac{|\gamma'' - \langle \gamma'', \frac{\gamma'}{|\gamma'|} \rangle \frac{\gamma'}{|\gamma'|}|^2}{|\gamma'|^2} |\gamma'| ds$$

$$= \int_{S^1_1} \frac{|\gamma''|^2 - 2 \langle \gamma'', \frac{\gamma'}{|\gamma'|} \rangle \frac{\gamma'}{|\gamma'|} \langle \frac{\gamma'}{|\gamma'|}, \gamma'' \rangle + \langle \gamma'', \frac{\gamma'}{|\gamma'|} \rangle^2}{|\gamma'|^3} ds$$

$$= \int_{S^1_1} \frac{|\gamma''|^2 - \langle \gamma'', \frac{\gamma'}{|\gamma'|} \rangle^2}{|\gamma'|^3} ds.$$
Due to the geometric nature of the energy, we will not distinguish between the trace of a curve in $\mathbb{R}^n$ and its parametrisations and reparametrisations. Similarly, we identify the circle $S^1_L$ of length $L$ with the periodic interval $[0, L]$ or $\mathbb{R}/L\mathbb{Z}$.

Let us review classical results for elastic curves in any dimension $d \geq 2$ which we prove for the reader’s convenience in the appendix.

**Lemma 2.1.**

1. Let $\gamma$ be any curve and $\alpha \neq 0$. Then $W(\alpha \gamma) = \frac{W(\gamma)}{|\alpha|}$.

2. Let $L > 0$ and $\gamma$ be a $W^{2,2}$-curve of length $L$. Then $W(\gamma) \geq \frac{4\pi^2}{L}$ and equality holds if and only if $\gamma$ is a circle.

3. For any $W^{2,2}$-curve $\gamma$, energy and length are related by $W(\gamma) \geq \frac{4\pi^2}{\text{length}}$.

4. If there exists a point $x \in \gamma$ of multiplicity $k$, i.e., there exists $x \in \mathbb{R}^n$ such that

$$x = \gamma(t_1) = \cdots = \gamma(t_k)$$

for $k$ distinct parameters $t_1, \ldots, t_k \in S^1$, then

$$W(\gamma) \geq \frac{C k^2}{L}$$

for a constant $C > \pi^2$.

This means that the unique minimiser (up to Euclidean motion) of $W$ among $W^{2,2}$-curves with given length is the once covered circle. Given the rescaling property $W(\alpha \gamma) = \frac{W(\gamma)}{|\alpha|}$, the functional $W$ has no critical points since we can always reduce the energy of a curve by making it larger, corresponding to the variation in radial direction. However, there are critical points under a length constraint, or equivalently critical points of the scale-invariant functional

$$\tilde{W}(\gamma) = H^1(\gamma) \cdot W(\gamma).$$

The following result is deeper and characterises the critical points of $\tilde{W}$. These curves are often referred to as (Euler) elasticae.

**Theorem 2.2.** [AKS13 Theorem 1.3] Let $\gamma$ be a critical point of $\tilde{W}$. Then one of the three following holds.

1. $\gamma$ is a once or multiply covered circle.

2. $\gamma$ is a particular once covered figure eight curve.

3. $\gamma$ is a multiple cover of the same figure eight curve.

In the first two cases, $\gamma$ is a stable critical point, in the third one, it is unstable.

There are other elasticae (critical points of $\tilde{W}$ under compact perturbations) which are not periodic; in fact, planar elasticae were classified into 9 different families already by Euler in 1744 [Lev08]. Only one of the closed elasticae, the once covered circle, is approximable by embedded curves.

**Lemma 2.3.** Let $\gamma \in C^\infty(S^1; \mathbb{R}^2)$ be a closed elastica and $\gamma_n \in C^1(S^1; \mathbb{R}^2)$ a sequence of embedded curves such that $\gamma_n \rightarrow \gamma$ in $C^1$. Then $\gamma$ is the once covered circle.

A proof can be found in the appendix. The next result has been proved for (varifold-)surfaces in the three-dimensional unit ball in [MR14 Theorem 1] and essentially expresses a curve’s elastic energy in terms of how much it deviates from being the unit circle. We repeat the proof for the convenience of the reader in this simpler setting.

**Lemma 2.4.** Let $\gamma \in C^2(S^1_L; \mathbb{R}^n)$ be a curve in $n$ dimensions such that $|\gamma| \leq 1$, $|\gamma'| \equiv 1$ where $S^1_L$ denotes the circle of length $L$. Then

$$W(\gamma) = 2\mathcal{H}^1(\gamma) - \int_{S^1_L} |\gamma'|^2 \, ds + \int_{S^1_L} |\gamma'' + \gamma|^2 \, ds \geq \mathcal{H}^1(\gamma) + \int_{S^1_L} |\gamma'' + \gamma|^2 \, ds$$
Proof. We compute
\[ W(\gamma) = \int_{S^1} |\gamma''|^2 \, ds \]
\[ = \int_{S^1} |\gamma'' + |\gamma||^2 - 2\langle \gamma'', \gamma \rangle - |\gamma|^2 \, ds \]
\[ = \int_{S^1} |\gamma'' + |\gamma||^2 + 2|\gamma'|^2 - |\gamma|^2 \, ds \]
\[ \geq \int_{S^1} |\gamma'' + |\gamma||^2 + 2 - 1 \, ds \]
\[ = \int_{S^1} |\gamma'' + |\gamma||^2 \, ds + L. \]
\[ \square \]

Note that together with Lemma 2.1, this implies that
\[ W(\gamma) \geq \max \left\{ H^1(\gamma), \frac{4\pi^2}{H^1(\gamma)} \right\} \]
for elastic curves confined to the unit ball in any dimension.

Remark 2.5. In parameter-invariant formulation the estimate appears as
\[ W(\gamma) = \int \left( |\gamma'' - \left\langle \frac{\gamma'}{|\gamma'|} \right\rangle \frac{\gamma'}{|\gamma'|} |^2 + \gamma' \right)^2 \, ds. \]

Let us consider the problem of minimising the elastic energy \( W \) in the class of curves which are embedded into a domain \( \Omega \subset \mathbb{R}^2 \) with prescribed length. Some properties of the problem which were originally proved in [BM04, DMR11] are described in the following Lemma.

Lemma 2.6. Let \( \Omega \subset \mathbb{R}^2 \) be an open set with Lipschitz boundary and \( L > 0 \). We set
\[ M_L = \left\{ \gamma \in C^\infty(S^1; \Omega) \mid |\gamma'| = \frac{L}{2\pi}, \gamma \text{ is embedded} \right\}. \]

Denote by \( \overline{M}_L \) the closure of \( M_L \) in the \( W^{2,2} \)-weak topology. The following are true.

1. \( \overline{M}_L \) coincides with the closure of \( M_L \) in the \( W^{2,2} \)-strong topology.
2. There exists a minimiser \( \overline{\gamma} \) of \( W \) in \( \overline{M}_L \).
3. \( \inf_{\gamma \in M_L} W(\gamma) = \min_{\gamma \in \overline{M}_L} W(\gamma) \).
4. The minimiser is either a circle, touches the boundary, or has at least one multiple point.

Essentially, the theorem shows that a curve which arises as the weak limit of smooth embedded curves with prescribed length and bounded energy can also be approximated strongly in the \( W^{2,2} \)-topology. This property is well-known for convex sets in Banach-spaces, but the embeddedness constraint is highly non-convex. At minimisers, it shows that the problem does not exhibit the Levrentiev-gap phenomenon where smoothness is incompatible with low energy.
3. Minimisation Problem at $2\pi$

Lemma 3.1. Let $\delta_n \to 0$ and $\gamma_n \in \mathcal{M}_{2\pi + \delta_n}$ be a sequence such that $W(\gamma_n) = \inf_{\gamma \in \mathcal{M}_{2\pi + \delta_n}} W(\gamma)$. Then

1. there exists $C > 0$ such that $W(\gamma_n) \leq 2\pi + C\delta_n^{1/3}$,
2. $\gamma_n$ converges to the unit circle strongly in $W^{2,2}(S^1; \mathbb{R}^2)$ (up to reparametrisation) and
3. for every $n$ there exists a parameter $s \in S^1$ such that $|\gamma_n(s)| = 1$, i.e. $\gamma_n \not\subset B_1(0)$.

Proof. **Energy bound.** Clearly, it suffices to construct curve $\tilde{\gamma}_n \in \mathcal{M}_{2\pi + \delta_n}$ such that $W(\tilde{\gamma}_n) \leq 2\pi + C\delta_n^{1/3}$. This is by far the longest part of the proof and concluded in Lemmas 3.2 and 3.3.

**Convergence to the unit circle.** By compactness, up to a subsequence, we see that there exists a curve $\gamma \in \mathcal{M}_{2\pi}$ such that $\gamma_n \rightharpoonup \gamma$ weakly in $W^{2,2}$ and thus strongly in $C^1$ – in particular, $\gamma$ is parametred by arc-length and has length $2\pi$. Furthermore,

$$W(\gamma) \leq \liminf_{n \to \infty} W(\gamma_n) = 2\pi.$$ 

As a consequence, $\gamma$ is a curve of length $2\pi$ and energy $2\pi$, which can only be realised by a circle.

Since $W(\gamma_n) \to W(\gamma)$, by a common Hilbert space argument we find that $\gamma_n \to \gamma$ strongly in $W^{2,2}$. Furthermore, since the limiting object is the unique circle $\partial B_1(0)$ of length $2\pi$ in $B_1(0)$, we can (after reparametrising the curves if necessary) show that the whole sequence converges.

**Touching the circle.** Assume that $\gamma_n \subset B_1(0)$. Since we assumed $\gamma_n$ to be a minimiser, $\gamma_n$ must be a critical point of $W$ under the length constraint without the confinement constraint. Since $W(\gamma_n) \leq 2\pi + C\delta_n^{1/3}$, we can see from Statement 4 in Lemma 2.4 that $\gamma_n$ has no double points. Together this means that $\gamma_n$ is embedded in $B_1(0)$ and we can take variations of $\gamma_n$ in all directions. But then, due to Theorem 2.2 $\gamma_n$ must be a once covered circle or a figure eight. Both the multiply covered circle and any cover of the figure eight are ruled out in Lemma 2.3 (or by the fact that they have double points) so $\gamma_n$ has to be a once covered circle. However, $\gamma_n \subset B_1(0)$ and $\mathcal{H}^1(\gamma_n) > 2\pi$ which means that $\gamma_n$ cannot be a circle. We have reached a contradiction.

Since $\gamma_n$ is $C^1$-close to a circle for $n$ large enough, we can write it as a normal graph over the unit circle, i.e. there exists a function $\phi_n \in W^{2,2}(S^1; [0, 1/2])$ such that $\gamma_n(s) = (1 - \phi_n(s)) \begin{pmatrix} \cos s \\ \sin s \end{pmatrix}$ up to reparametrisation. The energy can now be re-written in terms of $\phi_n$. In the following, we will drop the subscript $n$ and simply write $\gamma, \phi, \delta$ instead of $\gamma_n, \phi_n, \delta_n$. When varying $\delta$, we may make the dependence explicit by writing $\phi_\delta, \gamma_\delta$.

We now compute arc-length element, curvature, length and energy of $\gamma$ in terms of $\phi$ for general curves presented in radial form.

$$\gamma' = (1 - \phi) \begin{pmatrix} -\sin s \\ \cos s \end{pmatrix} - \phi' \begin{pmatrix} \cos s \\ \sin s \end{pmatrix}$$

$$|\gamma'| = \sqrt{(1 - \phi)^2 + (\phi')^2}$$

$$\gamma'' = (1 - \phi) \begin{pmatrix} -\cos s \\ -\sin s \end{pmatrix} - 2\phi' \begin{pmatrix} -\sin s \\ \cos s \end{pmatrix} - \phi'' \begin{pmatrix} \cos s \\ \sin s \end{pmatrix}$$

$$= (\phi - 1 - \phi'') \begin{pmatrix} \cos s \\ \sin s \end{pmatrix} - 2\phi' \begin{pmatrix} -\sin s \\ \cos s \end{pmatrix}$$

$$\langle \gamma'', \gamma' \rangle = (\phi - 1 - \phi'')(\phi') + (-2\phi')(1 - \phi)$$

$$= (\phi - 1 + \phi'')\phi'$$
\[ H = \gamma'' - \langle \gamma'', \frac{\gamma'}{|\gamma'|} \rangle \frac{\gamma'}{|\gamma'|} \]

\[
= (\phi - 1 - \phi''') \left( \frac{\cos s}{\sin s} \right) - 2\phi' \left( -\sin s \right) \cos s \\
- \frac{(\phi - 1 + \phi''')\phi'}{(1 - \phi)^2 + (\phi')^2} \left( (1 - \phi) \left( -\sin s \right) \cos s \right) - \phi' \left( \cos s \right) \sin s \\
= \left[ \phi - 1 - \phi'' - \frac{(1 - \phi)(\phi - 1 + \phi'')(\phi')}{(1 - \phi)^2 + (\phi')^2} \left( \cos s \right) \sin s \right] \\
- 2 + \frac{(\phi - 1 + \phi'')(\phi')}{(1 - \phi)^2 + (\phi')^2} \phi' \left( -\sin s \right) \cos s \\
\]

\[ \mathcal{H}^1(\gamma) = \int_{S^1} |\gamma'| \, ds \]

\[ = \int_{S^1} \sqrt{(1 - \phi)^2 + (\phi')^2} \, ds \]

\[ \mathcal{W}(\gamma) = \int_{S^1} \left| \frac{\gamma'' - \langle \gamma'', \frac{\gamma'}{|\gamma'|} \rangle}{|\gamma'|^2} \right| |\gamma'| \, ds + \int_{S^1} (2 - |\gamma|^2) \, |\gamma'| \, ds \]

\[ = \int_{S^1} \left[ \phi - 1 - \phi'' - \frac{(1 - \phi)(\phi - 1 + \phi'')(\phi')}{(1 - \phi)^2 + (\phi')^2} + 1 - \phi \right] \left( \cos s \right) \sin s \left[ \frac{2 + \frac{(\phi - 1 + \phi'')(\phi')}{(1 - \phi)^2 + (\phi')^2}}{(1 - \phi)^2 + (\phi')^2} \right] \phi' \left( -\sin s \right) \cos s \]

Before arguing for a general function \( \phi \) which is \( C^1 \)-close to 0, let us make a specific ansatz \( \psi_\delta(s) = \delta^\alpha \psi(\delta^2 s) \) for some non-negative function \( \psi \in C_c^\infty(\mathbb{R}) \) which, for small enough \( \delta \), induces a function \( \psi_\delta \in W^{2,2}(S^1) \), once rescaled sufficiently to force the support into an interval of length 2\( \pi \). We calculate

\[ 2\pi + \delta \equiv \mathcal{H}^1(\gamma \psi_\delta) \]

\[ = \int_{S^1} \sqrt{(1 - \delta^\alpha \psi)^2 + (\delta^{2\alpha - \beta} \psi')^2} (\delta^{-\beta}) \, ds \]

\[ = \int_{S^1} 1 + \frac{1}{2} (-2\delta^\alpha \psi + (\delta^\alpha \psi)^2 + (\delta^{2\alpha - \beta} \psi')^2) (\delta^{-\beta}) \, ds \]

\[ + O \left( \left( -2\delta^\alpha \psi + (\delta^\alpha \psi)^2 + (\delta^{2\alpha - \beta} \psi')^2 \right)^2 (\delta^{-\beta}) \right) \, ds \]

\[ = 2\pi + \frac{1}{2} \int_{S^1} \delta^{2\alpha - \beta} \left( \frac{\psi'(x)^2}{2} - \delta^{\alpha + \beta} \psi(x) + \delta^{2\alpha + \beta} \psi^2(x) \right) \, dx + O \left( \delta^{2\alpha} + \delta^{2(2\alpha - \beta)} \right) \cdot \delta^\beta \]

where the error term is multiplied by the length of the support of the functions. We see that the dominant term must be \( O(\delta) \), and since the second term is always negative as \( \psi \geq 0 \) and always dominates the third term, we see that

\[ \alpha + \beta \geq 2\alpha - \beta = 1 \quad \Rightarrow \quad \beta = 2\alpha - 1, \quad \beta \geq \frac{\alpha}{2}. \]
Lemma 3.2. Let \( \rho > 0 \), a function with \( \int_R \frac{(\psi')^2}{2} \, dx = 1 \) will match up to leading order and if \( \beta = 2\alpha \), a function satisfying \( \int_R \frac{(\psi')^2}{2} - \psi \, dx = 1 \) matches the right length up to leading order. In order to obtain the optimal order, we want to maximise \( \alpha \) which means

\[
\frac{\alpha}{2} = \beta = 2\alpha - 1 \quad \Rightarrow \quad \alpha = \frac{2}{3}, \quad \beta = \frac{1}{3}.
\]

For any \( \rho > 0 \), we therefore set \( \psi_\rho(x) = \rho^{2/3}\psi(\rho^{-1/3}x) \). We now see that the error term is

\[
O \left( \delta^{2\alpha+\beta} + \delta^2 + \delta^{2(2\alpha-\beta)} \right) \cdot \delta^\beta = O \left( \delta^{4/3} \right) \cdot \delta^{1/3} = O(\delta^{5/3}).
\]

Since the length of \( \gamma_{\psi_\rho}(s) := (1 - \psi_\rho(s))(\cos s, \sin s) \) depends continuously on the scaling parameter \( \rho \), we find the following.

**Lemma 3.2.** Let \( \psi \in C_\infty^\infty(R) \) be a function such that \( \int_R \frac{(\psi')^2}{2} - \psi \, dx = 1 \). Then there exists a \( \delta_0 > 0 \) such that for all \( \delta < \delta_0 \) there exists \( \rho(\delta) > 0 \) such that

\[
H^1(\gamma_{\psi_\rho(s)}) = 2\pi + \delta, \quad \limsup_{\delta \to 0} \frac{|\rho(\delta) - \delta|}{\delta^{5/3}} < \infty.
\]

Using this construction we bound the infimum energy from above. This also concludes the proof of Lemma 3.1.

**Lemma 3.3.** Under the same assumptions as in the previous Lemma, we have

\[
W(\gamma_{\psi_\rho(s)}) = 2\pi + \delta^{1/3} \int_R |\psi''|^2(s) \, ds + O(\delta^{5/3}).
\]

**Proof.**

\[
W(\gamma_{\psi_\rho(s)}) = \int_{S^1} \left[ \left( \rho^\alpha \psi' - 1 - \rho^{2\alpha-3\beta} \psi'' - \frac{2(\rho^\alpha \psi' - 1 + \rho^{2\alpha-3\beta} \psi'' + \rho^{\alpha-\beta} \psi')}{(1 - \rho^\alpha \psi')^2 + (\rho^{\alpha-\beta} \psi')^2} \right) \left( 1 + \rho^\alpha \psi' \right) \right]^2 \, ds
\]

\[
= \int_{S^1} \left[ \left( -1 + \psi'' + \rho^{1/3} \psi' - \rho^{1/3} \psi'' + O(\rho^{2/3}) \right) + \frac{2\rho^{1/3} \psi' + O(\rho^{2/3})}{1 + O(\rho^{2/3})} \right] \left( 1 + O(\rho^{2/3}) \right) \, ds + H^1(\gamma)
\]

\[
= \int_{S^1} \left[ \left( (\psi'')^2 + \rho^{1/3} (\psi' - \psi'')' + O(\rho^{2/3}) \right)^2 + 4 \rho^{2/3} (\rho')^2 + O(\rho) \right] \left( 1 + O(\rho^{2/3}) \right) \, ds + H^1(\gamma)
\]

\[
= \int_{S^1} \left( 1 - \rho^{1/3} \psi' \right) |\psi''|^2 + O(\rho^{2/3}) \, ds + H^1(\gamma)
\]

since \( \int_R \psi' \psi'' \, ds = \int_R \frac{d}{dx} (\psi')^2 \, ds = 0 \). We now note that all terms (including the term labeled as \( O(\rho^{2/3}) \)) are functions of \( \rho^{-1/3}s \), so after a change of variables, we find that

\[
W(\gamma_{\psi_\rho(s)}) = 2\pi + \left( \delta + O(\delta^{5/3}) \right)^{1/3} \int_R |\psi''|^2(s) \, ds + \left( \delta + O(\delta^{5/3}) \right)^{2/3} \int_R |\psi''|^2 \psi' \, ds + O(\delta).
\]
Since $5/9 < 2/3$, we are unable to refine the estimate by considering the second order term.

Note that, if $\psi$ is an even function, then $\psi''$ is also even and $\psi'$ is odd, so $\int_{\mathbb{R}} \psi'^2 \, ds = 0$ which eliminates the next order part of the error estimate in $\rho$, but not in $\delta$. In the following, we will show that the order $\delta^{1/3}$ is in fact optimal.

**Lemma 3.4.** There exists a $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$ the following holds: Let $\gamma$ be the minimiser of $W$ in $\mathcal{M}_{2\pi + \delta}$. Then $W(\gamma) = 2\pi + \Theta \delta^{1/3} + o(\delta^{1/3})$ where

$$\Theta := \inf \left\{ \int_{\mathbb{R}} |\phi''|^2 \, dx \mid \phi \in C^\infty_c(\mathbb{R}), \phi \geq 0, \int_{\mathbb{R}} \frac{(\phi')^2}{2} - \phi \, dx = 1 \right\}.$$ 

In the following section we will prove that $\Theta > 0$ and further properties of the minimisation problem for $\phi$ on the line.

**Proof. Upper bound.** We have already seen that curves of the form $\gamma^{{\psi_\epsilon}(s)} \in \mathcal{M}_{2\pi + \delta}$ satisfy

$$\lim_{\delta \to 0} \frac{W(\gamma^{{\psi_\epsilon}(s)}) - 2\pi}{\delta^{1/3}} = \int_{\mathbb{R}} |\phi''|^2 \, ds.$$ 

For any $\epsilon > 0$ we can choose the function $\phi$ in the admissible class such that $\int_{\mathbb{R}} |\phi''|^2 \, ds \leq \Theta + \epsilon$, so we can in particular construct a family of curves satisfying

$$\limsup_{\delta \to 0} \frac{W(\gamma^{{\psi_\epsilon}(s)}) - 2\pi}{\delta^{1/3}} \leq \Theta + \epsilon.$$ 

A diagonal sequence argument shows that

$$\limsup_{\delta \to 0} \frac{\inf_{\gamma \in \mathcal{M}_{2\pi + \delta}} W(\gamma) - 2\pi}{\delta^{1/3}} \leq \Theta.$$ 

**Lower bound.** Observe that, since $\gamma$ is $W^{2,2}$-close to the circle, it is also $C^1$-close to the circle, which means that $\phi \in W^{2,2}(S^1)$ is $C^1$-close to 0 as can be verified by examining the explicit expressions for $\gamma, \gamma'$ in terms of $\phi$. Once we analyse the complicated expression for $W(\gamma)$, we will see that $\phi$ is also $W^{2,2}$-close to zero.

**Preliminary estimates.** Observe that

$$2\pi + \delta = H^1(\gamma_s) = \int_{S^1} \sqrt{1 - 2\phi + \phi^2 + (\phi')^2} \, ds \leq \int_{S^1} 1 + \frac{-2\phi + \phi^2 + (\phi')^2}{2} \, ds$$

$$= 2\pi + \int_{S^1} \frac{(\phi')^2}{2} - \phi + \frac{\phi^2}{2} \, ds$$

since the square root function is concave. We deduce that

$$\|\phi\|_{L^1} = \int_{S^1} \phi \, ds \leq \frac{1}{2} \int (\phi')^2 + \phi^2 \, ds \leq C \int_{S^1} (\phi')^2 \, ds = C \|\phi''\|_{L^2}$$

where we used that $\phi \geq 0$ and that there exists a point $s_0 \in S^1$ such that $\phi(s_0) = 0$ since $\gamma \not\subset B_1(0)$ by Lemma 3.1 to apply Poincaré’s inequality. This relationship is non-homogeneous in $\phi$ and will provide the estimates we need, together with more classical estimates below. The constant $C$ – as all constants in the following – depends only on the circle $S^1$ and can be made explicit. Furthermore

$$\int_{S^1} (\phi')^2 \, ds = -\int_{S^1} \phi \phi'' \, ds \leq \|\phi\|_{L^2} \|\phi''\|_{L^2},$$

so

$$\int_{S^1} \phi^2 \, ds \leq \|\phi\|_{L^\infty} \|\phi\|_{L^1} \leq C \|\phi\|_{L^\infty} \|\phi''\|_{L^2} \leq C \|\phi\|_{L^\infty} \|\phi''\|_{L^2} \|\phi''\|_{L^2}$$

which then implies that

$$\int_{S^1} \phi^2 \, ds \leq C \|\phi\|_{L^\infty} \|\phi''\|_{L^2} \leq C \|\phi''\|_{L^2} \|\phi''\|_{L^2} \leq C \|\phi\|_{L^2} \|\phi''\|_{L^2} \|\phi''\|_{L^2}$$
so
\[ \|\phi\|_{L^2} \leq C \|\phi''\|_{L^2}^\frac{3}{2} \quad \Rightarrow \quad \int_{S^1} \phi^2 \, ds \leq \left( \int_{S^1} (\phi'')^2 \, ds \right)^3 \]
and finally
\[ \|\phi'\|_{L^2}^2 \leq \|\phi\|_{L^2} \|\phi''\|_{L^2} \leq C \|\phi''\|_{L^2}^\frac{3}{4}, \quad \|\phi\|_{L^2} \leq C \|\phi'\|_{L^2} \leq C \|\phi''\|_{L^2}^\frac{3}{4}. \]

**Excess energy.** We calculate the excess energy \( \mathcal{W}_{\delta_s}^{(4)} = \mathcal{W}(\gamma) - \mathcal{H}^1(\gamma). \)

\[
\mathcal{W}_{\delta_s}^{(4)} = \int_{S^1} \left[ \left( \frac{\phi - 1 - \phi'' - \frac{(1 - \phi)\phi' + \phi''}{(1 - \phi)^2 + (\phi')^2}}{1 - \phi} \right)^2 + \frac{2 + \frac{(\phi' - 1 + \phi'')\phi'}{(1 - \phi)^2 + (\phi')^2}}{2\phi - \phi^2} \right] \sqrt{(1 - \phi)^2 + (\phi')^2} \, ds
\]

\[
\geq \int_{S^1} \left( \frac{\phi - 1 - \phi'' - \frac{(1 - \phi)\phi' + \phi''}{(1 - \phi)^2 + (\phi')^2}}{1 - \phi} \right)^2 \sqrt{(1 - \phi)^2 + (\phi')^2} \, ds
\]

\[
= \int_{S^1} \left( \frac{(1 - \phi) [(1 - \phi)^2 + (\phi')^2] - \frac{(1 - \phi)^2\phi'}{(1 - \phi)^2 + (\phi')^2}}{1 + \frac{(1 - \phi)^2\phi'}{(1 - \phi)^2 + (\phi')^2}} \right)^2 \sqrt{(1 - \phi)^2 + (\phi')^2} \, ds
\]

\[
=: \int_{S^1} (f_\phi + (1 + g_\phi)\phi''^2) (1 + h_\phi) \, ds
\]

where \( f_\phi, g_\phi, h_\phi \) all go to zero in \( L^\infty \) as \( \delta \to 0 \) and specifically
\[
f_\phi = (1 - \phi) \left( \phi^2 - 2\phi + (\phi')^2 \right) + \frac{(1 - \phi)^2\phi'}{(1 - \phi)^2 + (\phi')^2}.
\]

We thus compute
\[
\mathcal{W}_{\delta_s}^{(4)} \geq \int_{S^1} ((1 + g_\phi)^2(\phi'')^2 + 2f_\phi(1 + g_\phi)\phi'' + f_\phi^2)(1 + h_\phi) \, ds
\]

\[
\geq \int_{S^1} \left( (1 + g_\phi)^2|\phi''|^2 - \frac{1}{\varepsilon(1 + g_\phi)^2(\phi'')^2 - \frac{1}{\varepsilon}f_\phi^2} \right)(1 + h_\phi) \, ds
\]

\[
\geq \int_{S^1} (1 + g_\phi)^2(1 + h_\phi)(1 - \varepsilon)|\phi''|^2 - \frac{f_\phi}{\varepsilon} (1 + h_\phi) \, ds.
\]

In particular, we find that
\[
(\Theta + 1)^{1/3} \geq \mathcal{W}_{\delta_s}^{(4)} \geq \frac{1}{16} \int_{S^1} |\phi''|^2 \, ds \geq \frac{1}{16} \int_{S^1} |\phi''|^2 \, ds
\]

for \( \phi = \phi_\delta \) and small enough \( \delta \) since \( f_\phi \to 0 \) in \( L^\infty(S^1) \) as \( \delta \to 0 \) so we find that \( \|\phi''\|_{L^2} \to 0 \) as \( \delta \to 0. \)

We observe that
\[
\int_{S^1} f_\phi^2 \, ds \leq C \int_{S^1} \phi^4 + \phi^2 + (\phi')^4 + (\phi')^2 \, ds
\]

\[
\leq C \int_{S^1} \phi^2 + (\phi')^2 \, ds
\]

\[
\leq C \left( \|\phi''\|_{L^2}^2 + \|\phi''\|_{L^2}^4 \right),
\]
and thus – for \( \delta \) so small that \(|h_\delta| \leq \varepsilon \) – we have

\[
\Theta \delta^{1/3} + o(\delta^{1/3}) \geq W_{2s}^{(\delta)}(\phi) \geq (1 - 2\varepsilon) \int_{S^1} |\phi''|^2 \, ds - \frac{2}{\varepsilon} \int_{S^1} |f_\delta|^2 \, ds \\
\geq (1 - 2\varepsilon) \int_{S^1} |\phi''|^2 \, ds - \frac{C}{\varepsilon} \left( \int_{S^1} |\phi''|^2 \, ds \right)^2.
\]

Note that the domain

\[
\left\{ X \in \mathbb{R} \mid (1 - 2\varepsilon)X - \frac{C}{\varepsilon} X^2 < \delta^{1/3} \right\} = (-\infty, a_\delta) \cup (b_\delta, \infty)
\]

has two connected components but that \( b_\delta > c_\varepsilon \) independently of \( \delta \), and since \( \int_{S^1} |\phi''|^2 \, ds \to 0 \), we find that \( \int_{S^1} |\phi''|^2 \, ds \leq a_\delta \leq C \delta^{1/3} \) for all small enough \( \delta \) for a constant \( C \) which depends on \( \Theta \) and \( \varepsilon \) but not \( \delta \).

Using this estimate, we obtain

\[
\liminf_{\delta \to 0} \frac{W_{2s}^{(\delta)}}{\delta^{1/3}} \geq (1 - 2\varepsilon) \liminf_{\delta \to 0} \frac{\int_{S^1} |\phi''|^2 \, ds}{\delta^{1/3}}
\]

for all \( \varepsilon > 0 \) and thus

\[
\liminf_{\delta \to 0} \frac{W_{2s}^{(\delta)}}{\delta^{1/3}} \geq \liminf_{\delta \to 0} \frac{\int_{S^1} |\phi''|^2 \, ds}{\delta^{1/3}}.
\]

**Length functional.** We have seen before that

\[
2\pi + \delta = \int_{S^1} \sqrt{(1 - \phi_\delta)^2 + (\phi_\delta')^2} \, ds \leq 2\pi + \int_{S^1} \frac{(\phi_\delta')^2}{2} - \phi_\delta + \frac{\phi_\delta^2}{2} \, ds
\]

which means that for all \( \varepsilon > 0 \) and all sufficiently small \( \delta \) we have

\[
\delta \leq \int_{S^1} \frac{(\phi_\delta')^2}{2} - \left[ 1 - \frac{\|\phi_\delta\|_{L^\infty}}{2} \right] \phi_\delta \, ds \leq \int_{S^1} \frac{(\phi_\delta')^2}{2} - [1 - \varepsilon]\phi_\delta \, ds
\]

since \( \phi_\delta \to 0 \) in \( L^\infty(S^1) \) and \( \phi \geq 0 \). We observe that, after taking \( \delta \to 0 \), this estimate must hold for all \( \varepsilon > 0 \), in particular in the limit \( \varepsilon \to 0 \).

**Conclusion.** Recall that there exists \( s_0 \in S^1 \) such that \( \phi(s_0) = 0 \) since \( \gamma \not\subset B_1(0) \). In principle, \( s_0 \) may depend on \( \phi \), but due to the rotational invariance of \( \phi \) we may fix a single point \( s_0 \). Put together, our argument shows that

\[
\liminf_{\delta \to 0} \frac{W_{2s}^{(\delta)}}{\delta^{1/3}} \geq \liminf_{\varepsilon \to 0} \liminf_{\delta \to 0} \left\{ \int_{S^1} |\phi''|^2 \, ds \mid \phi \geq 0, \int_{S^1} \frac{(\phi')^2}{2} - (1 - \varepsilon)\phi \, ds \geq \delta, \phi(s_0) = 0 \right\}
\]

for all \( \varepsilon > 0 \). Since \( \phi(s_0) = 0 \), we can extend \( \phi \) to a function defined on \( \mathbb{R} \) by

\[
\phi(x) = \begin{cases} 
\phi(s_0 + x) & x \in (0, 2\pi) \\
0 & \text{else}
\end{cases}
\]

which we also denote by \( \phi \). By construction, \( \phi \) is \( W^{2,2} \)-smooth and compactly supported, so

\[
\liminf_{\delta \to 0} \frac{W_{2s}^{(\delta)}}{\delta^{1/3}} \geq \liminf_{\varepsilon \to 0} \liminf_{\delta \to 0} \left\{ \int_{\mathbb{R}} |\phi''|^2 \, ds \mid \phi \in W^{2,2}(\mathbb{R}) \cap C_\varepsilon(\mathbb{R}), \phi \geq 0, \int_{\mathbb{R}} \frac{(\phi')^2}{2} - (1 - \varepsilon)\phi \, ds \geq \delta \right\}.
\]

On the whole real line, we can use scaling to normalise the minimisation problem. Since this is of more fundamental importance, we will treat this in separate statements and conclude the proof in Lemmas 3.5 and Corollary 3.6.

Let us expose a simple scaling relationship in the minimisation problem.

**Lemma 3.5.** Let \( \phi \in W^{2,2}(\mathbb{R}) \cap L^1(\mathbb{R}) \) and define \( \phi_\rho(x) = \rho^{2/3} \phi(\rho^{-1/3}x) \) for \( \rho \neq 0 \). Then

\[
\int_{\mathbb{R}} \frac{(\phi_\rho')^2}{2} - \phi_\rho \, dx = \rho \int_{\mathbb{R}} \frac{(\phi')^2}{2} - \phi \, dx, \quad \int_{\mathbb{R}} |\phi_\rho''|^2 \, dx = \rho^{1/3} \int_{\mathbb{R}} |\phi''|^2 \, dx
\]
Proof. A simple change of variables shows that
\[
\int_{\mathbb{R}} \frac{(\phi')^2}{2} - \phi \, dx = \int_{\mathbb{R}} \left( \frac{(\rho^{2/3} / 3 - 1/3) \phi'}{2} - \rho^{2/3} \phi \right) (\rho^{-1/3} x) \, dx
\]
\[
= \int_{\mathbb{R}} \rho^{2/3} \left( \frac{(\phi')^2}{2} - \phi \right) \rho^{-1/3} x \, dx
\]
\[
= \rho \int_{\mathbb{R}} \left( \frac{(\phi')^2}{2} - \phi \right) \rho^{-1/3} \, dx
\]
\[
= \rho \int_{\mathbb{R}} \left( \frac{(\phi')^2}{2} - \phi \right) \, dz
\]
\[
\int_{\mathbb{R}} (\phi''')^2 \, dz = \int_{\mathbb{R}} \left( \rho^{2/3} \rho^{-2/3} \phi'' \right)^2 \rho^{-1/3} \, dx
\]
\[
= \rho^{1/3} \int_{\mathbb{R}} (\phi'')^2 \rho^{-1/3} \, dx
\]
\[
= \rho^{1/3} \int_{\mathbb{R}} (\phi'')^2 \, dz.
\]
\]
\]
\]
\]

\]
\]
\]

Corollary 3.6. Let \(0 < \varepsilon < 1\). Then we have
\[
\inf \left\{ \frac{\int_{S^1} |\phi|^2 \, ds}{\delta^{1/3}} \mid \phi \in W^{2,2}(\mathbb{R}) \cap C^1_c(\mathbb{R}), \ \phi \geq 0, \ \int_{S^1} \frac{(\phi')^2}{2} - (1 - \varepsilon) \phi \, ds \geq \delta \right\}
\]
\[
= (1 - \varepsilon) \frac{\delta}{\delta} \inf \left\{ \int_{S^1} |\phi'|^2 \, ds \mid \phi \in W^{2,2}(\mathbb{R}) \cap C^1_c(\mathbb{R}), \ \phi \geq 0, \ \int_{S^1} \frac{(\phi')^2}{2} - \phi \, ds = 1 \right\}.
\]
The same identity holds if we consider \(L^1\) instead of \(C^1_c\).
Proof. Take any \(\phi \in W^{2,2}(\mathbb{R}) \cap L^1(\mathbb{R})\) such that \(\phi \geq 0\) and \(\int_{\mathbb{R}} \frac{(\phi')^2}{2} - (1 - \varepsilon) \phi \, ds \geq \delta\). Introduce \(\psi = \frac{\phi}{1 - \varepsilon}\) and observe that
\[
\int_{\mathbb{R}} \left( \frac{\psi'}{2} \right)^2 - \psi \, ds = \int_{\mathbb{R}} \left( \frac{1}{(1 - \varepsilon)^2} \frac{(\phi')^2}{2} - \frac{\phi}{1 - \varepsilon} \right) \, ds
\]
\[
= \frac{1}{(1 - \varepsilon)^2} \int_{\mathbb{R}} \frac{(\phi')^2}{2} - (1 - \varepsilon) \phi \, ds \geq \frac{\delta}{1 - \varepsilon}
\]
\[
\int_{\mathbb{R}} \left( \frac{\psi''}{2} \right)^2 = \frac{1}{(1 - \varepsilon)^2} \int_{\mathbb{R}} (\phi'')^2 \, ds.
\]
Denote
\[
\rho := \left( \frac{1}{(1 - \varepsilon)^2} \int_{\mathbb{R}} \frac{(\phi')^2}{2} - (1 - \varepsilon) \phi \, ds \right)^{-1} \leq \frac{(1 - \varepsilon)^2}{\delta}.
\]
Then the rescaled function \(\psi_\rho\) satisfies
\[
\int_{\mathbb{R}} \left( \frac{\psi_\rho'}{2} \right)^2 - \psi_\rho \, ds = \rho \int_{\mathbb{R}} \left( \frac{\psi'}{2} \right)^2 - \psi \, ds = 1
\]
and
\[
\int_{\mathbb{R}} (\psi_\rho'')^2 \, ds = \rho^{1/3} \int_{\mathbb{R}} (\phi'')^2 \, ds
\]
with equality if \(\int_{\mathbb{R}} (\phi')^2 - (1 - \varepsilon) \phi \, ds = \delta\). Since the process is entirely reversible (assuming we rescaled to length \(\geq 1\) instead of \(= 1\)), we have proved equality. □
We will show in the next section that the infimum is in fact positive, which shows that we have successfully identified the first order expansion of the minimal elastic energy with small excess length. Combining the results of this section, we have

\[
\lim_{\delta \to 0} \inf_{\gamma} \frac{W^{(\delta)}_{\gamma}}{\delta^{1/3}} \geq \sup_{\varepsilon > 0} \lim_{\delta \to 0} \inf_{\varepsilon} \left\{ \int_{\mathbb{R}} |\phi''|^2 \, ds \left| \phi \in W^{2,2}(\mathbb{R}) \cap C^1_\varepsilon(\mathbb{R}), \phi \geq 0, \int_{\mathbb{R}} (\phi')^2 - (1 - \varepsilon) \phi \, ds \geq \delta \right\}
\]

\[
\geq \lim_{\varepsilon \to 0} (1 - \varepsilon) \frac{\delta}{\pi} \inf \left\{ \int_{S^1} |\phi''|^2 \, ds \left| \phi \in W^{2,2}(\mathbb{R}) \cap C^1_\varepsilon(\mathbb{R}), \phi \geq 0, \int_{S^1} (\phi')^2 - \phi \, ds = 1 \right\}
\]

\[
= \inf \left\{ \int_{S^1} |\phi''|^2 \, ds \left| \phi \in W^{2,2}(\mathbb{R}) \cap C^1_\varepsilon(\mathbb{R}), \phi \geq 0, \int_{S^1} (\phi')^2 - \phi \, ds = 1 \right\}.
\]

We have thus proved the first of our main results, Theorem 1.1:

\[
\lim_{\delta \to 0} \inf_{\gamma \in M_{2R + \delta}} W(\gamma) - 2\pi \frac{\delta^{1/3}}{R} = \inf \left\{ \phi \in W^{2,2}(\mathbb{R}) \cap C^1_\varepsilon(\mathbb{R}) \mid \phi \geq 0, \int_{\mathbb{R}} (\phi')^2 - \phi \, dx = 1 \right\}.
\]

**Remark 3.7.** Note that the sequence in the proof of the upper bound satisfies

\[
\|\phi_\delta\|_{L^1} = O(\delta), \quad \|\phi_\delta\|_{L^2}^2 = O(\delta^{4/3}), \quad \|\phi_\delta''\|_{L^2}^2 = O(\delta), \quad \|\phi_\delta''\|_{L^2} = O(\delta^{1/3})
\]

while we only establish the sub-optimal orders

\[
\|\phi_\delta\|_{L^1} = O(\delta^{2/3}), \quad \|\phi_\delta\|_{L^2} = O(\delta), \quad \|\phi_\delta''\|_{L^2} = O(\delta^{2/3}), \quad \|\phi_\delta''\|_{L^2} = O(\delta^{1/3})
\]

in the proof of the lower bound.

**Remark 3.8.** We have seen that the energy minimiser \(\gamma\) of length \(2\pi + \delta\) is \(W^{2,2}\)-close to a circle (when parametrised in radial fashion). However, note that the set bounded by \(\gamma\) is contained in the unit disk and has a boundary of length \(\mathcal{H}^1(\gamma) > 2\pi\), which means that the set cannot be convex. In particular, it has negative curvature at a point, and \(\gamma\) is not \(C^2\)-close to the unit circle.

**Remark 3.9.** Assume that \(\gamma \subset B_R(0)\) and \(\mathcal{H}^1(\gamma) = 2\pi R + \delta\) for some small \(\delta > 0\). Then we consider the curve \(\gamma_R = \frac{\delta}{R} \subset B_1(0)\) of length \(\mathcal{H}^1(\gamma_R) = 2\pi + \frac{\delta}{R}\). The result above shows that \(W(\gamma) \geq 2\pi + \Theta \left(\frac{\delta}{R}\right)^{1/3}\) (to leading order) and thus

\[
W(\gamma) = \frac{1}{R} W(\gamma_R) \geq \frac{2\pi}{R} + \frac{\Theta}{R^{4/3}} \frac{\delta^{1/3}}{R^{4/3}}
\]

(again to leading order) which means that the qualitative behaviour remains unchanged, but the prefactor \(\Theta R^{-4/3}\) decreases quickly with increasing radius/decreasing boundary curvature. In domains with non-constant boundary curvature, we would therefore expect buckling of \(\gamma\) at the points of lowest boundary curvature, at least if the domain is similar enough to a circle (e.g. an ellipse with very similar major and minor axes).

### 4. Minimisation Problem on the Line

From the geometric problem of the previous section, in the asymptotic regime we inherit the minimisation problem of a boring energy over an interesting domain. We show that the competition between the terms \((\phi')^2\) – which needs to be large in an integrated sense to compete with \(\phi\), but cannot have large localized energy since \((\phi')^2\) needs to be small – leads to non-trivial behaviour.

Define the non-linear domain

\[
M = \left\{ \phi \in C^\infty(\mathbb{R}) \mid \phi \geq 0, \int_{\mathbb{R}} (\phi')^2 - \phi \, ds = 1 \right\},
\]

its closure

\[
\overline{M} = \left\{ \phi \in W^{2,2}(\mathbb{R}) \cap L^1(\mathbb{R}) \mid \phi \geq 0, \int_{\mathbb{R}} (\phi')^2 - \phi \, ds = 1 \right\},
\]
and the energy function
\[ E(\phi) = \int_{\mathbb{R}} |\phi''|^2 \, ds. \]
Recall that we denote
\[ \Theta = \inf_{\phi \in M} E(\phi) = \inf_{\phi \in M} \mathcal{E}(\phi). \]

4.1. Existence of Minimisers. In this section, we establish that energy minimisers exist and find them explicitly, together with the constant \( \Theta \) from the previous section. Since it suffices show that \( \Theta > 0 \) to obtain the energy scaling result from the previous section, we note that this section can be skipped by a reader only interested in the correct order of scaling. A much shorter proof that \( \Theta > 0 \) is given below in Lemma 4.4 and Corollary 4.5. We begin by showing that it is energetically favourable to create a single bump rather than many small bumps, assuming that \( \Theta > 0 \).

**Lemma 4.1.** Assume that \( \phi \in C^\infty_c(\mathbb{R}) \) is a function such that \( \phi \geq 0 \) and \( \phi = \phi_1 + \phi_2 \) where \( \phi_1 \phi_2 \equiv 0 \) and
\[ \int_{\mathbb{R}} \frac{(\phi_1')^2}{2} - \phi_1 \, dx = t, \quad \int_{\mathbb{R}} \frac{(\phi_2')^2}{2} - \phi_2 \, dx. \]
Then
\[ E(\phi) \geq \Theta \left[ t^{1/3} + (1 - t)^{1/3} \right] > \Theta \]
if \( t \in (0, 1) \) and
\[ E(\phi) \geq \Theta \max\{t^{1/3}, (1 - t)^{1/3}\} \]
else. \( E(\phi) > \Theta \) unless \( \phi_1 \equiv 0 \) or \( \phi_2 \equiv 0 \).

**Proof.** If \( t \in (0, 1) \), then by the rescaling property of Lemma 3.5, we have
\[ E(\phi) = E(\phi_1) + E(\phi_2) \geq t^{1/3} \Theta + (1 - t)^{1/3} \Theta \]
since \( \phi_1 \) and \( \phi_2 \) cannot be simultaneously non-zero and thus do not interact. If \( t < 0 \), then \( 1 - t > 0 \) and thus
\[ E(\phi) \geq E(\phi_2) \geq (1 - t)^{1/3} \Theta \]
and similarly if \( t > 1 \). \( \Box \)

We deduce a few further a priori properties.

**Lemma 4.2.** Assume that there exists a function \( \pi \in W^{2,2}(\mathbb{R}) \cap L^1(\mathbb{R}) \) such that
\[ \int_{\mathbb{R}} \frac{(\pi')^2}{2} - \pi \, dx = 1, \quad E(\pi) = \Theta. \]
Then
1. The set \( \{\pi > 0\} \) is connected and
2. \( \pi \in C^{\infty}_c(\{\pi > 0\}) \) solves the Euler-Lagrange equation
\[ \phi^{(4)} + \frac{\Theta}{6} (\phi'' + 1) = 0 \]
on the set \( \{\pi > 0\} \).

**Proof.** The first statement follows directly from Lemma 4.1. For the second statement, we observe that \( \pi \) is a critical point of the scale-invariant functional \( \int_{\mathbb{R}} \frac{\mathcal{E}(\phi)}{\mathcal{L}(\phi)^{1/3}} \) under the constraint \( \pi \geq 0 \). On the
set \{\pi > 0\} – which is open due to regularity – we can take a variation in any direction \(\psi \in C^\infty_c(\{\pi\})\), leading to

\[
0 = \delta \left( \frac{\mathcal{E}}{L^{1/3}} \right)(\pi, \psi) = \frac{\delta \mathcal{E}(\pi, \psi)}{L(\phi)^{1/3}} + \mathcal{E}(\bar{\pi}) \cdot \frac{-1}{3} L(\bar{\pi})^{-4/3} \delta L(\bar{\pi}; \psi)
\]

\[
= \frac{1}{L(\phi)^{1/3}} \left( \delta \mathcal{E}(\pi, \psi) - \frac{\mathcal{E}(\pi, \psi)}{3 L(\pi)} \delta L(\pi, \psi) \right)
\]

\[
= \delta \mathcal{E}(\bar{\pi}, \psi) - \frac{\Theta}{3} \delta L(\bar{\pi}, \psi)
\]

\[
= \int_R 2\phi'' \psi'' - \frac{\Theta}{3} (\phi' \psi' - \psi) \, ds
\]

\[
= 2 \int_R \left( \phi^{(4)} + \frac{\Theta}{6} (\phi'' + 1) \right) \psi \, ds
\]

in the weak sense. Classical regularity theory implies that \(\phi\) is infinitely smooth on \(\{\phi > 0\}\). \(\square\)

Now, we are ready to establish the existence of a minimiser and give an explicit characterisation. The following arguments are fairly direct and classical.

**Lemma 4.3.** There exists a function \(\overline{\pi} \in C^{2,1}(\mathbb{R})\) such that

\[
\int_R \frac{(|\overline{\pi}|^2)}{2} - \overline{\pi} \, dx = 1, \quad \mathcal{E}(\overline{\pi}) = \inf_{\phi \in M} \mathcal{E}(\phi).
\]

The function is \(\overline{\pi}\) is even, compactly supported, not \(C^3\)-smooth on \(\mathbb{R}\) and given by

\[
\overline{\pi}(x) = \begin{cases} a - \frac{x^2}{2} + \alpha \cos(\mu x) & x \in (-r, r) \\ 0 & \text{else} \end{cases}
\]

for parameters \(a \approx 1.81, \alpha \approx 0.75, \mu \approx 2.47, r \approx 1.82\). Its energy is \(\Theta \approx 36.69\).

**Proof.** **Set-up.** For \(R > 0\), consider

\[
M_R = \left\{ \phi \in W^{2,2}_0(-R, R) \mid \phi \geq 0, \int_{-R}^R \frac{(\phi')^2}{2} - \phi \, ds = 1 \right\}.
\]

Then, by the direct method of the calculus of variations, there exists a minimiser \(\phi_R \in M_R\) of \(\mathcal{E}\) where by an abuse of notation we denote

\[
\mathcal{E} : M_R \to \mathbb{R}, \quad \mathcal{E}(\phi) = \frac{1}{2} \int_{-R}^R |\phi''|^2 \, ds
\]

as the same functional on the new domain (with a normalising factor of \(1/2\) included here for convenience). On the set \(\{\phi_R > 0\}\), we may take variations of \(\phi\) in any direction and see that the Euler-Lagrange equation

\[
\int_{\{\phi_R > 0\}} \phi'' \psi'' + \lambda (\phi' \psi' - \psi) \, ds = 0
\]

holds where \(\lambda\) is a Lagrange multiplier stemming from the constraint and \(\psi \in W^{2,2}_0(-R, R)\). Since \(r\) may coincide with \(R\), we lose the scaling argument of the previous lemma to identify the Lagrange multiplier or its sign. After integration by parts, this is equivalent to the ODE

\[
\phi^{(4)} - \lambda (\phi'' + 1) = 0 \quad \text{on } \{\phi_R > 0\}.
\]

Since \(\phi_R\) is continuous, the set \(\{\phi_R > 0\}\) is open, and we can focus on an individual connected component \((a, b)\) or, after a translation, \((-r, r)\). Clearly, the Lagrange multiplier can vary from connected component to connected component.
Solving the Euler-Lagrange equation. We can distinguish two cases: \( \lambda = 0 \) and \( \lambda \neq 0 \). If \( \lambda = 0 \), \( \phi^{(4)} = 0 \) and thus any solution \( \phi \) is a third degree polynomial. Knowing that \( \phi(r) = \phi(-r) = \phi'(r) = \phi'(-r) = 0 \) since \( \phi \in W_0^{2,2}(-r,r) \), we find that \( \phi \) can only be the constant zero function. Hence \( \lambda \neq 0 \).

If \( \lambda \neq 0 \), the ODE seizes to be homogeneous. A particular solution of the equation is given by

\[
\phi_R(x) = -\frac{x^2}{2}
\]

and the homogeneous ODE

\[
(\phi'' - \lambda \phi'') = \phi^{(4)} - \lambda \phi'' = 0
\]

has the general solution

\[
\phi(x) = \begin{cases} 
  a + bx + \alpha \cos(\mu x) + \beta \sin(\mu x) & \lambda < 0 \\
  a + bx + \alpha \cosh(\mu x) + \beta \sinh(\mu x) & \lambda > 0 
\end{cases}
\]

where \( \mu = \sqrt{|\lambda|} \). Adding the particular and the general solution, we have a natural decomposition of \( \phi \) into an odd and an even part

\[
\phi_{\text{even}}(x) = \begin{cases} 
  a + \alpha \cos(\mu x) - \frac{x^2}{2} & \lambda < 0 \\
  a + \alpha \cosh(\mu x) - \frac{x^2}{2} & \lambda > 0 
\end{cases}, \quad \phi_{\text{odd}}(x) = \begin{cases} 
  bx + \beta \sin(\mu x) & \lambda < 0 \\
  bx + \beta \sinh(\mu x) & \lambda > 0 
\end{cases}
\]

We denote \( \phi_{\text{even}}^{(k)} \) as the \( k \)-th derivative of the even part (not the even part of the \( k \)-th derivative). Due to symmetry, both the even and the odd part of \( \phi \) need to satisfy the matching conditions

\[
\phi_{\text{even}}(r) = \phi'_{\text{even}}(r) = \phi_{\text{odd}}(r) = \phi'_{\text{odd}}(r)
\]

separately, i.e.

\[
\begin{align*}
0 &= \phi_{\text{even}}(r) = \begin{cases} 
  a + \alpha \cos(\mu r) - \frac{r^2}{2} & \lambda < 0 \\
  a + \alpha \cosh(\mu r) - \frac{r^2}{2} & \lambda > 0 
\end{cases} \\
0 &= \phi'_{\text{even}}(r) = \begin{cases} 
  -\alpha \mu \sin(\mu r) - r & \lambda < 0 \\
  \alpha \mu \sinh(\mu r) - r & \lambda > 0 
\end{cases} \\
0 &= \phi_{\text{odd}}(r) = \begin{cases} 
  br + \beta \sin(\mu r) & \lambda < 0 \\
  br + \beta \sinh(\mu r) & \lambda > 0 
\end{cases} \\
0 &= \phi'_{\text{odd}}(r) = \begin{cases} 
  b + \beta \mu \cos(\mu r) & \lambda < 0 \\
  b + \beta \mu \cosh(\mu r) & \lambda > 0 
\end{cases}
\]

For the even part, we note that

\[
\alpha = \begin{cases} 
  \frac{\mu \sin(\mu r)}{\mu \sinh(\mu r)} & \lambda < 0 \\
  \frac{\mu \sinh(\mu r)}{\mu \sin(\mu r)} & \lambda > 0 
\end{cases}, \quad a = \begin{cases} 
  \frac{\mu^2}{2} - \alpha \cos(\mu r) & \lambda < 0 \\
  \frac{\mu^2}{2} - \alpha \cosh(\mu r) & \lambda > 0 
\end{cases}
\]

Thus for the even part, the matching conditions can be satisfied whenever \( \mu r \notin \pi \mathbb{Z} \) if \( \lambda < 0 \) and always if \( \lambda > 0 \).

Symmetry. For the odd part, the matching conditions imply that

\[
\frac{b}{\beta} = -\frac{\sin(\mu r)}{r} = -\mu \cos(\mu r) \quad \Rightarrow \quad \frac{\sin(\mu r)}{\mu r} = \cos(\mu r)
\]

if \( \beta \neq 0 \), \( \lambda < 0 \) and

\[
\frac{b}{\beta} = -\frac{\sinh(\mu r)}{r} = -\mu \cosh(\mu r) \quad \Rightarrow \quad \frac{\sinh(\mu r)}{\mu r} = \cosh(\mu r)
\]

if \( \beta \neq 0 \) and \( \lambda > 0 \). If \( \lambda > 0 \), the condition can never be satisfied because

\[
\frac{\sinh(\rho)}{\rho} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)!} < \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = \cosh(\rho)
\]
for all \( \rho > 0 \) while for \( \lambda < 0 \), there are countably many solutions of the equation \( \frac{\sin \rho}{\rho} - \cos \rho = 0 \).

In the following, we will generally use the parameter \( \rho = \mu r \) instead of \( \mu = \frac{r}{\rho} \) as it simplifies many expressions. Note that, if \( \sin \rho = \cos \rho \), then

\[
\phi''(r) = -1 \alpha \mu^2 \cos(\mu r)
\]

unless \( \beta = 0 \). Thus, if \( \phi''(r) = \phi'''(r) > 0 \), then \( \phi''(-r) < 0 \) and vice versa. Since \( \phi(\pm r) = \phi'(-r) = 0 \), this would imply that \( \phi \) cannot be non-negative. Thus, whether \( \lambda > 0 \) or \( \lambda < 0 \), we have shown that \( \phi \) has to be an even function, \( \phi = \phi_{\text{even}} \).

Note that the odd part is excluded for two different reasons, depending on whether \( \lambda > 0 \) or \( \lambda < 0 \). If \( \lambda > 0 \), the operator \( \phi^{(4)} - \lambda \phi'' \) is positive definite on \( W^{2,2} \) (since we integrate by parts twice in the first and once in the second term), so that it cannot have a non-trivial kernel and the purely even solution of the boundary value problem that we constructed is unique. If \( \lambda < 0 \), it depends on the relationship between \( \lambda \) and the length of the interval whether we can find a non-trivial solution to the homogeneous problem. It is only the sign constraint on \( \phi \) that excludes them.

**Calculating energy and length.** We calculate energy and length of \( \phi \), first in the case \( \lambda < 0 \):

\[
\mathcal{E}(\phi) = \int_{-r}^{r} |\phi''|^2 \, dx
\]

\[
= \int_{-r}^{r} | \alpha \mu^2 \cos(\mu x) + 1 |^2 \, dx
\]

\[
= \int_{-r}^{r} 1 + 2 \alpha \mu^2 \cos(\mu x) + \alpha^2 \mu^4 \cos^2 \mu x \, dx
\]

\[
= 2 \left[ r + 2 \alpha \mu \sin(\mu r) + \alpha^2 \mu^4 \left( \frac{r}{2} + \frac{\sin(\mu r) \cos(\mu r)}{2 \mu} \right) \right]
\]

\[
= \left[ 2 + \alpha^2 \mu^4 \right] r + 4 \alpha \mu \sin(\mu r) + \alpha^2 \mu^3 \sin(\mu r) \cos(\mu r)
\]

\[
= \left[ 2 + \frac{\mu^2 \sin^2(\mu r)}{\mu^2 \sin^2(\mu r)} \right] r + 4(-r) + \left( \frac{-r}{\mu \sin(\mu r)} \right)^2 \mu^3 \sin(\mu r) \cos(\mu r)
\]

\[
= 2 r + \frac{(\mu r)^2}{\sin^2(\mu r)} r - 4 r + \frac{\mu r}{\sin(\mu r)} \cos(\mu r) r
\]

\[
= \left( \frac{(\mu r)^2}{\sin^2(\mu r)} + \frac{\mu r}{\sin(\mu r)} \cos(\mu r) - 2 \right) r
\]

\[
= \left( \frac{\rho^2}{\sin^2(\rho)} + \frac{\rho}{\sin(\rho)} \cos(\rho) - 2 \right) r
\]
where we substituted the variable \( \rho \) for \( \mu r \). Similarly, we compute

\[
L(\phi) = \int_{-r}^{r} \frac{(\phi')^2}{2} - \phi \, dx
\]

\[
= \int_{-r}^{r} \frac{1}{2} \left[ -x - \alpha \mu \sin(\mu x) \right]^2 - \left[ a + \alpha \cos \mu x - \frac{x^2}{2} \right] \, dx
\]

\[
= \int_{-r}^{r} \frac{\mu^2}{2} + \alpha \mu x \sin(\mu x) + \frac{1}{2} \alpha^2 \mu^2 \sin^2(\mu x) - a - \alpha \cos \mu x + \frac{x^2}{2} \, dx
\]

\[
= 2 \left[ \frac{r^3}{3} + \frac{\sin(\mu r) - \mu r \cos(\mu r)}{\mu} + \frac{\alpha^2 \mu^2}{2} \left( \frac{r}{2} - \frac{\sin(\mu r) \cos(\mu r)}{2\mu} \right) - ar - \frac{\alpha}{\mu} \sin(\mu r) \right]
\]

\[
= 2 \left[ \frac{r^3}{3} + \alpha \frac{\sin(\mu r) - \mu r \cos(\mu r)}{\mu} + \alpha^2 \frac{\mu^2}{2} \left( \frac{r}{2} - \frac{\sin(\mu r) \cos(\mu r)}{2\mu} \right) - ar - \frac{\alpha}{\mu} \sin(\mu r) \right]
\]

\[
= 2 \left[ \frac{r^3}{3} - \alpha r \cos(\mu r) + \frac{\alpha^2 \mu^2}{4} r - \frac{\alpha^2 \mu^2}{4} \sin(\mu r) \cos(\mu r) - ar \right]
\]

\[
= 2 \left[ \frac{r^3}{3} + \frac{r^2 \cos(\mu r)}{\mu \sin(\mu r)} + \left( \frac{r}{\mu \sin(\mu r)} \right)^2 \frac{\mu^2}{4} r - \left( \frac{r}{\mu \sin(\mu r)} \right)^2 \frac{\mu}{4} \sin(\mu r) \cos(\mu r)
\]

\[
- \frac{r^2}{2} + \frac{r}{\mu \cot(\mu r)} \right] r
\]

\[
= 2 \left[ \frac{r^3}{3} + \frac{r^2 \cot(\mu r)}{\mu} + \frac{r^3}{4} \cot(\mu r) - \frac{r^3}{4} \cot(\mu r) - \frac{r^3}{2} + \frac{r^2 \cot(\mu r)}{\mu} \right]
\]

\[
= \left[ -\frac{r}{3} + \frac{r}{2 \sin^2(\mu r)} - \frac{\cot(\mu r)}{2\mu} \right] r^2
\]

\[
= \left[ -\frac{r}{3} + \frac{r}{2 \sin^2(\rho)} - \frac{\cot(\rho)}{2\frac{\rho}{2}} \right] r^2
\]

\[
= \left[ -\frac{1}{3} + \frac{1}{2 \sin(\rho)} \left( \frac{1}{\sin(\rho)} - \frac{\cos(\rho)}{\rho} \right) \right] r^3
\]

Finally, we need to repeat the calculations for the case of a positive Lagrange multiplier. A direct calculation yields

\[
E(\phi) = \left[ \frac{\rho^2}{\sinh^2(\rho)} + \frac{\rho}{\sinh(\rho)} \cosh(\rho) - 2 \right] r
\]

\[
L(\phi) = \left[ -\frac{1}{3} + \frac{1}{2 \sinh(\rho)} \left( \frac{\cosh(\rho)}{\rho} - \frac{1}{\sinh(\rho)} \right) \right] r^3
\]

with calculations very similar to the case \( \lambda < 0 \) – the only differences are signs that need to be carefully taken into account.

**Estimating** \( r \). Assume for a contradiction that over intervals \([-R_k, R_k] \), we have a sequence of minimisers with a component such that \( r_k \to \infty \). Then necessarily

\[
\lim_{k \to \infty} \left( \frac{\rho_k^2}{\sinh^2(\rho_k)} + \frac{\rho_k}{\sinh(\rho_k)} \cosh(\rho_k) - 2 \right) = 0
\]

or – if \( \lambda > 0 \) for infinitely many minimisers –

\[
\lim_{k \to \infty} \left( \frac{\rho_k^2}{\sinh^2(\rho_k)} + \frac{\rho_k}{\sinh(\rho_k)} \cosh(\rho_k) - 2 \right) = 0.
\]

Consider the second case first. Then we know that \( \cosh(\rho) > \sinh(\rho) \) for all \( \rho > 0 \)

\[
\frac{\rho_k^2}{\sinh^2(\rho_k)} + \frac{\rho_k}{\sinh(\rho_k)} \cosh(\rho_k) - 2 \geq \rho_k - 2
\]
which means that $0 < \rho_k < 3$ for almost all $k \in \mathbb{N}$. By compactness, there exists $\overline{\rho} \in [0, 3]$ such that $\rho_k \to \overline{\rho}$ (up to a subsequence) and

$$\frac{\overline{\rho}}{\sinh^2 \overline{\rho}} + \frac{\overline{\rho}}{\sinh \overline{\rho}} \cosh \overline{\rho} - 2 = 0.$$  

If $\overline{\rho} > 0$, then the energy of a function $\phi$ associated to $\overline{\rho}$ and any $r > 0$ would be zero, but as this is not the case, $\overline{\rho}$ must be 0.

In the first case, on the other hand, we know that $|\rho| \sin \rho \geq \rho$ for all $\rho > 0$ and since $X^2 - X - 2 \geq 1$ for all $|X| \geq 2$, we find that $\rho \leq \left|\frac{\rho \sin \rho}{\rho \sin \rho}\right| \leq 2$. We reach the same conclusion as before. Now observe that if $\lambda < 0$ we have

$$\phi(0) = a + \alpha = \frac{r^2}{2} + \frac{r}{\mu} \cot(\mu r) - \frac{r}{\mu \sin(\mu r)}$$

$$= \left(\frac{1}{2} + \frac{\cos \rho - 1}{\rho \sin \rho}\right) r^2$$

$$= \left(\frac{1}{2} + \frac{-\rho^2 + \frac{1}{6} \rho^4 + O(\rho^6)}{\rho(\rho - \frac{3}{6} + O(\rho^5))}\right) r^2$$

$$= \left(\frac{\rho^2}{24} + O(\rho^4)\right) r^2,$$

so $\phi(0) < 0$ if $\rho$ is too small which poses a contradiction. Similarly, we can compute that

$$L(\phi) = \left(\frac{2}{45} \rho^2 + O(\rho^4)\right) r^3$$

for small $\rho$ if $\lambda > 0$, so if $\lambda > 0$ and $\rho$ is very small, we find that $L(\phi) < 0$, leading to a contradiction again. We conclude that $r$ is uniformly bounded (and $\rho$ is uniformly bounded away from 0).

**Minimisers on the real line.** Let $\varepsilon > 0$ and $\psi \in M$ be a function such that $\mathcal{E}(\psi) < \Theta + \varepsilon$. Since $\psi \in M$ is compactly supported, we see that there exists $R > 0$ such that $\text{supp}(\psi) \subset (-R, R)$ and thus in particular

$$\inf_{\phi \in M_R} \mathcal{E}(\phi) \leq \mathcal{E}(\psi) < \Theta + \varepsilon.$$

We conclude that letting $R \to \infty$, we recover the original energy infimum:

$$\lim_{R \to \infty} \inf_{\phi \in M_R} \mathcal{E}(\phi) = \Theta$$

where the limit exists since the quantity is monotone decreasing in $R$. Now let $\phi_k$ be the minimiser of $\mathcal{E}$ in $W^{2,2}_0(-k, k)$. We know that $\lim_{k \to \infty} \mathcal{E}(\phi_k) = \Theta$, so by Lemma 4.1 there exists a connected component $I_k$ of $\{\phi_k > 0\}$ such that

$$\int_{I_k} \frac{(\phi_k')^2}{2} - \phi_k \, dx \to 1.$$

After a translation, we have $I_k = (-r_k, r_k)$ and we introduce the restriction $\psi_k = \phi_k|_{I_k}$. We observe that $\psi_k$ is a function of the type $\psi_k = \phi_k^\pm$, for suitable parameters $\rho_k, r_k$ and a choice of either $\lambda > 0$ or $\lambda < 0$. Since $r_k \not\to +\infty$ by the previous step in the proof, we find that $\psi_k \in W^{2,2}_0(-R, R)$ for some suitably large $R$ and

$$\int_{-R}^R |\psi_k''|^2 \, dx \leq \int_{-R}^R |\phi_k''|^2 \, dx \leq \Theta + \varepsilon_k$$
where \( \varepsilon_k \to 0 \). On the bounded set \((-R, R)\) and with zero boundary values, this controls the entire \( W^{2,2} \)-norm. In particular, there exists a function \( \pi \in W^{2,2}_0(-R, R) \) such that \( \psi_k \to \pi \) in \( W^{2,2}(-R, R) \) and thus

\[
E(\pi) = \int_{-R}^{R} |\pi'|^2 \, dx \leq \liminf_{k \to \infty} \int_{-R}^{R} |\psi_k'|^2 \, dx \leq \Theta.
\]

Furthermore, since \( \psi_k \to \pi \) strongly in \( W^{1,2}(-R, R) \), we find that

\[
L(\pi) = \lim_{k \to \infty} L(\psi_k) = \lim_{k \to \infty} \int_{I_k} \frac{\phi_k'^2}{2} - \phi_k \, dx = 1
\]

by our choice of \( I_k \). In total, \( \pi \) satisfies

\[
\pi \in W^{2,2}_0(-R, R) \subset W^{2,2}(\mathbb{R}) \cap L^1(\mathbb{R}), \quad L(\pi) = 1 \quad \Rightarrow \quad \pi \in \overline{M}
\]

and \( E(\pi) = \Theta \) which means that \( \pi \) is a minimiser of \( E \) in \( \overline{M} \).

**Direct consequences.** Since the energy \( E \) admits a minimiser \( \pi \in \overline{M} \), we conclude from the arguments above that \( \pi \) is compactly supported and thus can be found as the minimiser of \( E \) over \( \overline{M} \), that the Lagrange multiplier is \( \lambda = -\frac{\pi}{\pi} < 0 \) and that the set \( \{ \pi > 0 \} \) is connected.

**Smoothness.** We quickly observe that the Lagrange multiplier \( \lambda \neq 0 \) satisfies

\[
0 \neq \lambda = \frac{1}{2r} \int_{-r}^{r} \lambda \, dx = \frac{1}{2r} \int_{-r}^{r} \phi(4) - \lambda \phi'' \, dx = \frac{\phi(3)(r) - \phi(3)(-r)}{2r}
\]

since \( \phi'(\pm r) = 0 \) as \( \phi \) is globally \( C^1 \)-smooth. This means that either \( \lim_{x \to -r} \phi(3)(x) \neq 0 \) or \( \lim_{x \to -r} \phi(3)(x) \neq 0 \) which implies immediately that \( \phi \) cannot be globally \( C^3 \)-smooth.

Note that \( \phi(x) = 0 \) for all \( x > r \), so if \( \phi \) is \( C^2 \)-smooth, then necessarily \( \lim_{x \to -r} \phi''(x) = 0 \). We calculate

\[
\lim_{x \to -r} \phi''(x) = -1 - \alpha \mu^2 \cos(\mu r) = -1 + \mu^2 \frac{r}{\mu} \frac{\mu}{\cot(\mu r)} = 0 \quad \Leftrightarrow \quad \frac{\sin(\mu r)}{\mu r} = \cos(\mu r)
\]

as we already encountered when considering the odd part of \( \phi \). Let us now use the characterisation of the Lagrange multiplier from Lemma 4.2 to compute that

\[
\mu^2 = \left( \frac{\rho}{r} \right)^2 = \frac{E(\phi)}{6L(\phi)} = \frac{\left( \frac{\rho^2}{\sin^2(\rho)} + \frac{\rho}{\sin(\rho)} \cos(\rho) - 2 \right) r}{6 \left[ \frac{1}{3} + \frac{1}{2 \sin(\rho)} \left( \frac{1}{\sin(\rho)} - \frac{\cos(\rho)}{\rho} \right) \right] r^3}
\]

which means that

\[
\rho^2 = \frac{\frac{\rho^2}{\sin^2(\rho)} + \frac{\rho}{\sin(\rho)} \cos(\rho) - 2}{6 \left[ \frac{1}{3} + \frac{1}{2 \sin(\rho)} \left( \frac{1}{\sin(\rho)} - \frac{\cos(\rho)}{\rho} \right) \right]}
\]

or

\[
6 \left[ \frac{-\rho^2}{3} + \frac{\rho}{2 \sin(\rho)} \left( \frac{-\rho}{\sin(\rho)} - \cos(\rho) \right) \right] = \frac{\rho^2}{\sin^2(\rho)} + \frac{\rho}{\sin(\rho)} \cos(\rho) - 2.
\]

Further algebra shows that this is equivalent to

\[
-2 \rho^2 + 3 \left( \frac{\rho}{\sin(\rho)} \right)^2 - 3 \cos(\rho) \left( \frac{\rho}{\sin(\rho)} \right) = \left( \frac{\rho}{\sin(\rho)} \right)^2 + \cos(\rho) \left( \frac{\rho}{\sin(\rho)} \right) - 2
\]
and finally

\[ 2 \left( \frac{\rho}{\sin \rho} \right)^2 - 2 \cos \rho \left( \frac{\rho}{\sin \rho} \right) + 1 - \rho^2 \right] = 0. \]

We compute that

\[ \rho^2 + (1 - \rho^2) \sin^2 \rho = 2 \rho \cos \rho \sin \rho \]
\[ \Rightarrow \rho^4 + 2 \rho^2 (1 - \rho^2) \sin^2 \rho + (1 - \rho^2)^2 \sin^4 \rho = 4 \rho^2 \cos^2 \rho \sin^2 \rho \]
\[ \Rightarrow \rho^4 + 2 \rho^2 (1 - \rho^2) \sin^2 \rho + (1 - \rho^2)^2 \sin^4 \rho = 4 \rho^2 (1 - \sin^2 \rho) \sin^2 \rho \]
\[ \Rightarrow \rho^4 + 2 \rho^2 (1 - \rho^2) \sin^2 \rho - 4 \rho^2 \sin^2 \rho + (1 - 2 \rho^2 + \rho^4) \sin^4 \rho + 4 \rho^2 \sin^2 \rho = 0 \]
\[ \Rightarrow \rho^4 + 2 (-1 - \rho^2) \sin^2 \rho + (1 + \rho^2)^2 \sin^4 \rho = 0 \]
\[ \Rightarrow \rho^4 - 2 (1 + \rho^2) \sin^2 \rho + (1 + \rho^2)^2 \sin^4 \rho = 0 \]
\[ \Rightarrow \left[ \rho^2 - (1 + \rho^2) \sin^2 \rho \right]^2 = 0 \]

so necessarily

\[ \sin^2 \rho = \frac{\rho^2}{1 + \rho^2} \quad \Rightarrow \quad \cos^2 \rho = 1 - \sin^2 \rho = \frac{1}{1 + \rho^2} = \frac{\sin^2 \rho}{\rho^2} \]

such that \( \cos \rho = \frac{\sin \rho}{\rho} \) is satisfied at least up to a sign. If \( \cos \rho = \frac{\sin \rho}{\rho} \), we get

\[ \left( \frac{\rho}{\sin \rho} \right)^2 - 2 \cos \rho \left( \frac{\rho}{\sin \rho} \right) + 1 - \rho^2 = \left( \frac{\rho}{\sin \rho} \right)^2 - 1 - \rho^2 = \frac{\rho^2}{1 + \rho^2} - (1 + \rho^2) = 0, \]

so the original equation is satisfied. If, on the other hand, \( \cos \rho = -\frac{\sin \rho}{\rho} \), we find a contradiction assuming that

\[ 0 = \left( \frac{\rho}{\sin \rho} \right)^2 - 2 \cos \rho \left( \frac{\rho}{\sin \rho} \right) + 1 - \rho^2 = \left( \frac{\rho}{\sin \rho} \right)^2 + 3 - \rho^2 \quad \Rightarrow \quad \sin^2 \rho = \frac{\rho^2}{\rho^2 - 3} \neq \frac{\rho^2}{\rho^2 + 1}. \]

So the minimiser must satisfy \( \tan \rho = \rho \) which by our previous computations implies that \( \overline{u} \) is \( C^2 \)-smooth. Since \( \overline{u} \) is \( C^\infty \)-smooth on \( \{ \overline{u} > 0 \} \) and \( \{ \overline{u} = 0 \}^o \), we find that \( \overline{u} \) has a bounded weak third derivative, i.e. \( \overline{u} \in W^{3, \infty}(\mathbb{R}) = C^{2,1}(\mathbb{R}) \).
Finding the minimiser. The minimiser $\phi$ of $E$ in $\mathcal{M}$ is given by $\phi = \phi_{\rho,r}$ for parameters $\rho, r$ which satisfy $\tan \rho = \rho$ and a fortiori $\sin^2 \rho = \frac{\rho^2}{1+\rho^2}$. We can therefore re-write length and energy as

$$L(\phi_{\rho,r}) = \left[ -\frac{1}{3} + \frac{1}{2} \left( \frac{1}{\sin^2 \rho} - \frac{\cos \rho}{\rho \sin \rho} \right) \right] r^3$$
$$= \left[ -\frac{1}{3} + \frac{1}{2} \left( \frac{1}{\rho^2} - \frac{1}{\rho^2} \right) \right] r^3$$
$$= \left[ -\frac{1}{3} + \frac{1}{2} \left( \frac{1}{2\rho^2} - \frac{1}{2\rho^2} \right) \right] r^3$$
$$= \frac{r^3}{6}$$

$$E(\phi_{\rho,r}) = \left[ \frac{\rho^2}{\sin^2(\rho)} + \frac{\rho}{\sin \rho} \cos(\rho) - 2 \right] r$$
$$= \left[ \frac{\rho^2}{1+\rho^2} + 1 - 2 \right] r$$
$$= [1 + \rho^2 - 1] r$$
$$= \rho^2 r.$$

Given $\rho$, we need to find $r$ such that

$$1 = L(\phi_{\rho,r}) = \frac{r^3}{6} = 1 \quad \Rightarrow \quad r = 6^{1/3}$$

and calculate the energy

$$E(\phi_{\rho,r}) = \rho^2 r = 6^{1/3} \rho^2$$

As this function is increasing in $\rho$, we need to find the first positive solution $\rho$ of $\tan(\rho) = \rho$ for the global minimiser. Since $\tan(\rho) > \rho$ for $\rho \in (0, \pi/2)$ and $\tan(\rho) < 0$ for $\rho \in (\pi/2, \pi)$, we find that $\pi < \rho < \frac{3\pi}{2}$. Numerically, we find

$$\rho \approx 4.4934.$$

We then calculate

$$\Theta = 6^{1/3} \rho^2 \approx 36.6890$$
$$\overline{\rho} = 6^{1/3} \approx 1.8171$$
$$\overline{\rho} = \frac{\pi}{2} \approx 2.4728$$
$$\overline{\rho} = -\frac{\rho}{\tan(\rho)} \approx 0.7528$$
$$\overline{\sigma} = \left( \frac{1}{\rho} + \frac{\cos(\rho)}{\rho} \right) \approx 1.8145.$$

It is easy to see that $\overline{\rho}'(r) = \overline{\rho}'(0)$ and if $\overline{\rho}'$ has a local extremum in $(-r, 0)$, then

$$\overline{\rho}''(x) = 0 \quad \Rightarrow \quad -\alpha \mu^2 \cos(\mu x) - 1 = 0 \quad \Rightarrow \quad \cos(\mu x) = -\frac{1}{\alpha \mu^2} < 0.$$

Since $0 < |\mu x| < \mu r < \frac{3\pi}{2}$, this equation has at most two solutions in $[-\mu r, 0]$. As we already know, one of the solutions is $-\mu r$ itself since $\overline{\rho}'(-r) = 0$ and observing that $\overline{\rho}'''(-r) > 0$, we get that $\overline{\rho}$ is increasing at $-r$. This means that $\overline{\rho}$ has a local maximum and no local minima in $(-r, 0)$ and thus that $\overline{\rho}$ is increasing on $(-r, 0)$. The same argument shows that $\overline{\rho}$ is decreasing on $(0, r)$. In particular, we deduce that $\overline{\rho} > 0$ on $(-r, r)$.

□
4.2. Functions of Low Energy. While we have established the existence of minimisers, we continue to study the variational structure of the obstacle problem in this section. Our arguments were based entirely on the Euler-Lagrange equation being a linear ODE with constant coefficients. The arguments presented below appear more stable and applicable in more general situations.

Many common embeddings are not obvious since our domain is the whole real line. In this section, we use the interaction between the non-linear domain \( M \) and the energy functional to understand the structure of admissible functions. Lemmas 4.4, 4.6, 4.7 and 4.10 and Corollary 4.5 concern the properties of functions in \( M \) satisfying an energy bound without an assumption of minimality. Among others, the analysis yields a simple proof that \( \Theta > 0 \) which is sufficient to establish the order of energy scaling in Theorem 1.1, but not to find the leading order coefficient explicitly.

This section can be skipped by a reader only interested in energy minimisers.

**Lemma 4.4.** Let \( 0 \leq \phi \in W^{2,2}(\mathbb{R}) \cap L^1(\mathbb{R}) \) and define

\[
I_+ = \left\{ \frac{(\phi')^2}{2} - \phi > 0 \right\} = \left\{ x \in \mathbb{R} \mid \frac{(\phi')^2(x)}{2} - \phi(x) > 0 \right\}.
\]

Then the following hold.

(1) \( \mathcal{H}^1(I^+) \leq \int_\mathbb{R} |\phi''|^2 \, ds \).

(2) The height and slope of \( \phi \) are related by

\[
|\phi'(x)|^3 \leq \left( 3 \int_\mathbb{R} |\phi''|^2 \, dy \right) \phi(x) \quad \forall \ x \in \mathbb{R}.
\]

(3) The length integrand is bounded by

\[
\frac{(\phi')^2}{2} - \phi \leq \left( \int_\mathbb{R} |\phi''|^2 \, ds \right)^2.
\]

**Proof. First Property.** Let \( \phi \in \overline{M} \). Since \( \phi, \phi' \) are continuous, the set

\[
I_+ = \left\{ (\phi')^2 - 2\phi > 0 \right\} = \left\{ x \in \mathbb{R} \mid \frac{(\phi')^2(x)}{2} - \phi(x) \geq 0 \right\} = \bigcup_{n \in \mathbb{Z}} (a_n, b_n)
\]
is a union of disjoint open intervals with endpoints \(a_n, b_n\) satisfying
\[
\frac{(\phi')^2}{2}(a_n) - \phi(a_n) = \frac{(\phi')^2}{2}(b_n) - \phi(b_n) = 0.
\]
Since \(\mathbb{R}\) is second countable, the union is at most countable. Clearly, inside an interval there cannot be any point where \(\phi'\) vanishes, so we find that on any interval \((a_n, b_n)\) the function \(\phi\) is either increasing or decreasing. In particular, \(\phi(a) > 0\) or \(\phi(b) > 0\). Let us fix \(n\) and consider one such interval of length \(\ell = b - a\) where we assume without loss of generality that \(\phi\) is increasing and that \(\phi(a) > 0\). We denote \(h := \phi(a)\) and \(\rho := \sqrt{2h}\) and use a translation to normalize \(a = \rho, b = \rho + \ell\). Since the ODE
\[
\begin{cases}
  f' = \sqrt{2f} & x > \rho \\
  f = h \left(\frac{\rho^2}{2}\right) & x = \rho
\end{cases}
\]
is solved by \(f(x) = \frac{x^2}{2}\) and \(\phi' \geq \sqrt{2\phi}\) on the interval \((a, b)\), we find that
\[
\phi(x) \geq \frac{x^2}{2} \quad \forall \ x \in [a, b]
\]
by the comparison principle for ODEs. It follows that
\[
\phi'(a) = \rho, \quad \phi'(b) = \sqrt{2\phi(b)} \geq \sqrt{2 \frac{(\rho + \ell)^2}{2}} = \rho + \ell
\]
whence
\[
\ell = \phi'(b) - \phi'(a) = \int_a^b \phi'' \, ds \leq |b - a|^{1/2} \left( \int_a^b |\phi''|^2 \, ds \right)^{1/2} \Rightarrow \ell \leq \int_a^b |\phi''|^2 \, ds.
\]
Adding up the terms over the intervals \((a_n, b_n)\), we find that
\[
\mathcal{H}^1(I_+) \leq \int_{I^+} |\phi''|^2 \, ds \leq \int_{\mathbb{R}} |\phi''|^2 \, ds.
\]

**Second property.** Denote \(\Xi := \mathcal{E}(\phi)^{1/2}\). Without loss of generality, we may assume that \(\phi(0) > 0\). We denote \(\phi(0) = h\) and \(\phi'(0) = a\) Denoting \(l_{h,a}(x) = h + as\) we observe that

\[
(4.1) \quad \left|\phi'(x) - a\right| = \left|\int_0^x \phi''(s) \, ds\right| \leq \left( \int_0^x |\phi''(x)|^2 \, dx \right)^{1/2} |x|^{1/2} \leq \Xi |x|^{1/2}
\]
\[
\left|\phi(x) - l_{h,a}(x)\right| \leq \int_0^x |\phi'(s) - a| \, ds \leq \Xi \int_0^x |s|^{1/2} \, ds = \frac{2\Xi}{3} |x|^{3/2}.
\]
Since \(\phi \geq 0\), this gives us a compatibility condition on the height \(h\) and the slope \(a\):

\[
0 \leq \phi(x) \leq l_{h,a}(x) + |\phi(x) - l_{h,a}(x)| \leq h + ax + \frac{2\Xi}{3} |x|^{3/2}.
\]

To find the minimum of the expression on the right, we may assume that \(a < 0\) so that the minimum is positive. Taking derivatives, we find that the minimum is assumed when
\[
a + \Xi |x|^{1/2} = 0 \quad \Rightarrow \quad |x| = \frac{a^2}{2\Xi} \quad \Rightarrow \quad h + ax + \frac{2\Xi}{3} |x|^{3/2} = h + \frac{a^3}{2\Xi} - \frac{2\Xi}{3} \frac{a^3}{2\Xi} = h + \frac{a^3}{3\Xi^2}.
\]
It follows that we have to require \(h \geq \frac{|a|^3}{3\Xi}\) for \(\phi\) to be positive, or in other words
\[
\frac{|\phi'(0)|^3}{3\Xi^2} \leq \phi(0).
\]
Since we can repeat the argument at any point \(x\), we have established the first property.
Step 3. If \( x \in I_+ \) we have
\[
0 \leq \frac{(\phi')^2(x)}{2} - \phi(x) \leq \left( \frac{1}{2} - \frac{|\phi'(x)|}{3\Xi^2} \right) (\phi')^2(x) \quad \Rightarrow \quad |\phi'(x)| \leq \frac{3\Xi^2}{2}.
\]
If we want to maximise the length integrand, we may consider
\[
g(a) := \left( \frac{1}{2} - \frac{|a|}{3\Xi^2} \right) a^2
\]
and observe that \( g \) has to have a global maximum at a point \( a \neq 0 \) and without loss of generality we may assume that \( a > 0 \). Thus
\[
g'(a) = -\frac{1}{3\Xi^2} a^2 + \left( \frac{1}{2} - \frac{a}{3\Xi^2} \right) 2a = -\frac{a^2}{\Xi^2} + a = \left( 1 - \frac{a}{\Xi^2} \right) a
\]
which means that \( a = \Xi^2 \) and
\[
g(a) = \left( \frac{1}{2} - \frac{\Xi^2}{3\Xi^2} \right) \Xi^4 = \frac{\Xi^4}{6}.
\]

Corollary 4.5. From Lemma 4.4 it follows that \( 1 < \frac{6}{3} \leq \inf_{\phi \in M} E(\phi) \).

Proof. Observe that
\[
1 = \int_{\mathbb{R}} \frac{(\phi')^2}{2} - \phi \, dx
\leq \int_{I_+} \frac{(\phi')^2}{2} - \phi \, dx
\leq H^1(I_+) \cdot \sup_{x \in I_+} \left( \frac{(\phi')^2}{2} - \phi \right) (x)
\leq \Xi^2 \cdot \frac{\Xi^4}{6}
\]
such that \( \Xi^6 \geq 6 \). \(\square\)

While this result is not new and the estimate is not particularly accurate, it is obtained in a much faster and simpler way than the lengthy characterisation of a minimiser above, and the argument only makes use of embedding theorems and a comparison principle for ODEs.

We now establish that for functions in \( \overline{M} \), a bound on \( E \) induces Lipschitz-bounds similar to those which are known on finite domains, but with a non-linear dependence on the energy corresponding to the non-linear domain.

Lemma 4.6. Let \( \Xi > 0 \) and \( \phi \in \overline{M} \) such that \( E(\phi) \leq \Xi^2 \). Then
\[
\sup_{x \in \mathbb{R}} \phi(x) \leq \sqrt{\frac{3}{8}} \Xi^4 \quad \text{and} \quad \sup_{x \in \mathbb{R}} |\phi|(x) \leq \frac{3\Xi^2}{2}.
\]

In particular, \( \phi \) is a bounded Lipschitz function whose constants can be estimated explicitly in terms of the energy.

Proof. Idea. At a maximum, the derivative vanishes so that \( \frac{(\phi')^2}{2} - \phi < 0 \). If the maximum is very high, the domain where the length integrand is negative is very large and we generate a large amount of negative length. Since the total length is positive, this forces high energy by the same argument as above. Once we have an \( L^\infty \)-bound, it is easy to deduce the \( W^{1,\infty} \)-bound by considering the sets \( I_+, I_- \) separately.
Step 1. Since $\phi \in L^1(\mathbb{R}) \cap C^0(\mathbb{R})$, we find that $\phi$ does not diverge at $\pm \infty$, hence it achieves a local maximum somewhere in $\mathbb{R}$ which we denote by $H$. Without loss of generality, we may assume that

$$\phi(0) = H, \quad \phi'(0) = 0.$$ 

Using the Hölder bounds from (45), we find that

$$\frac{(\phi')^2(x)^2}{2} - \phi(x) \leq \frac{\Xi^2|x|^2}{2} - H + \frac{3\Xi}{3} |x|^{3/2} = \frac{2\Xi}{3} |x|^{3/2} + \frac{\Xi^2}{2} |x| - H < -\frac{H}{3}$$

if

$$|x| < R := \min \left\{ \left( \frac{H}{2\Xi} \right)^{\frac{2}{3}}, \frac{2H}{3\Xi^2} \right\}$$

so

$$\int_{-R}^R \frac{(\phi')^2(x)^2}{2} - \phi(x) \, dx \leq 2R \cdot \frac{-H}{3} = -\frac{2H}{3} \min \left\{ \left( \frac{H}{2\Xi} \right)^{\frac{2}{3}}, \frac{2H}{3\Xi^2} \right\}.$$ 

It follows that

$$1 = \int_R \frac{(\phi')^2}{2} - \phi \, dx$$

$$= \int_{-R}^R \frac{(\phi')^2}{2} - \phi \, dx + \int_{I_+} \frac{(\phi')^2}{2} - \phi \, dx$$

$$\leq \int_{-R}^R \frac{(\phi')^2}{2} - \phi \, dx + |I_+| \sup_{I_+} \left( \frac{(\phi')^2}{2} - \phi \right)$$

$$\leq -\frac{2H}{3} \min \left\{ \left( \frac{H}{2\Xi} \right)^{\frac{2}{3}}, \frac{2H}{3\Xi^2} \right\} + \Xi^2 \cdot \frac{\Xi^1}{6}$$

so either

$$1 + \frac{2H}{3} \left( \frac{H}{2\Xi} \right)^{2/3} \leq \frac{\Xi^6}{6} \quad \Rightarrow \quad H^2 \leq \frac{3}{2} 2^{2/3} \frac{\Xi^6+2/3}{6} = 2^{-4/3} \Xi^6+2/3 \quad \Rightarrow \quad H \leq 2^{-4/5} \Xi^{(18+2)/5}$$

or

$$1 + \frac{2H}{3} \frac{2H}{3\Xi^2} \leq \frac{\Xi^6}{6} \quad \Rightarrow \quad H^2 \leq \frac{9}{4} \frac{\Xi^8}{6} = \frac{3\Xi^8}{8} \quad \Rightarrow \quad H \leq \sqrt{\frac{3}{8}} \Xi^4$$

In either case, we have derived an $L^\infty$-bound on $\phi$ in terms of the energy. Since both bounds scale the same way in $\Xi$, we only need to choose the larger constant $\sqrt{3}/8$. Since this holds for all local extrema of $\phi$, we have found an $L^\infty$-bound.

Step 3. Let us now show that the $L^\infty$-bound on $\phi$ implies an $L^\infty$-bound on $\phi'$. If $x \in I_+$, we have already shown that

$$0 \leq \frac{(\phi')^2(x)^2}{2} - \phi(x) \leq \left( \frac{1}{2} - \frac{|\phi'(x)|}{3\Xi^2} \right) (\phi')^2(x) \quad \Rightarrow \quad |\phi'(x)| \leq \frac{3\Xi^2}{2}$$

by the minimal height estimate. Otherwise, by definition

$$\left( \frac{(\phi')^2}{2} - \phi \right)(x) \leq 0 \quad \Rightarrow \quad |\phi'(x)| \leq \sqrt{2 \phi(x)}.$$ 

Thus

$$\sup_{x \in \mathbb{R}} |\phi'(x)| \leq \max \left\{ \frac{3\Xi^2}{2}, \sqrt{2 \sup \phi(x)} \right\} \leq \max \left\{ \frac{3}{2}, (3/8)^{1/4} \right\} \Xi^2 = \frac{3\Xi^2}{2}.$$ 

We can also obtain an $L^1$-bound from the same arguments.
Lemma 4.7. Under the same conditions as Lemma 4.6, we know that

$$\|\phi\|_{L^1} \leq \left(\frac{1}{3} + 4 \sqrt{\frac{3}{8}}\right) \Xi^6$$

and the $L^2$-bounds

$$\|\phi\|_{L^2} \leq \sqrt{\frac{3}{8}} \left(\frac{1}{3} + 4 \sqrt{\frac{3}{8}}\right)^{\frac{1}{2}} \Xi^5$$
$$\|\phi'\|_{L^2} \leq \frac{3}{8} \left(\frac{1}{3} + 4 \sqrt{\frac{3}{8}}\right)^{\frac{1}{4}} \Xi^3$$
$$\|\phi''\|_{L^2} \leq \Xi$$

hold.

Proof. $L^1$-bound. Note that by the same proof as before, we find that the set

$$\tilde{I}^+ = \{ x \in \mathbb{R} \ | \ (\phi')^2 (x) > \phi(x) \}$$

(missing a factor of 2 compared to $I^+$) is bounded by

$$\frac{\mathcal{H}^1(\tilde{I}^+)}{4} \leq \int_{\tilde{I}^+} |\phi''|^2 \, ds$$

so

$$1 = \int_{\tilde{I}^+} \frac{(\phi')^2}{2} - \phi \, dx + \int_{\tilde{I}^-} \frac{(\phi')^2}{2} - \phi \, dx$$
$$\leq \mathcal{H}^1(\tilde{I}^+) \cdot \max_{x \in \mathbb{R}} \left(\frac{(\phi')^2}{2} - \phi\right) - \int_{I^-} \phi \, dx$$
$$\leq \Xi^6 - \int_{I^-} \frac{\phi}{2} \, dx.$$
**H²-bound.** The bound \( \|\phi''\|_{L^2} \leq \Xi \) is immediate from the choice of the constant \( \Xi \) in Lemma 4.6.

The \( L^2 \)-bound on \( \phi \) is obtained by interpolation:

\[
\|\phi\|_{L^2} \leq \|\phi\|_{L^1}^{1/2} \|\phi\|_{L^\infty}^{1/2} \leq \left( \frac{3}{2} \Xi^4 \right)^{1/2} \left( \left( \frac{1}{3} + 4 \sqrt{\frac{3}{8}} \right)^6 \right)^{1/2} = \sqrt{\frac{3}{8}} \left( \frac{1}{3} + 4 \sqrt{\frac{3}{8}} \right)^{3/2} \Xi^3.
\]

The estimate on \( \phi' \) follows by integration by parts:

\[
\int_R (\phi')^2 \, dx = -\int_R \phi \phi'' \, dx \leq \|\phi\|_{L^2} \|\phi''\|_{L^2} \leq \sqrt{\frac{3}{8}} \left( \frac{1}{3} + 4 \sqrt{\frac{3}{8}} \right)^{3/2} \Xi^6.
\]

\[\square\]

**Remark 4.8.** Note that the functional \( L \) drives the fact that the \( \|\phi\|_{L^1} \) and \( \|\phi''\|_{L^2} \) are comparable, which explains the non-linear dependence of certain norms of \( \phi \) on the energy encoded in the non-linear structure of \( M \). In standard notation, the estimates read as follows: Let \( \phi \in M \), then

\[
\|\phi\|_{L^1} \leq C \|\phi''\|_{L^2}^6, \quad \|\phi\|_{L^2} \leq C \|\phi''\|_{L^2}^5, \quad \|\phi\|_{L^\infty} \leq C \|\phi''\|_{L^2}^2
\]

and

\[
\|\phi'\|_{L^2} \leq C \|\phi''\|_{L^2}^3, \quad \|\phi\|_{L^\infty} \leq C \|\phi''\|_{L^2}^2.
\]

**Remark 4.9.** We have seen that sets of the form \( \overline{M} = \{ \phi \in M \mid E(\phi) \leq \Xi^2 \} \) are uniformly bounded in \( W^{2,2} \) and \( L^1 \). Note that this allows the extraction of weakly convergent subsequences – at least in \( W^{2,2} \) – but that \( \overline{M} \) is not weakly closed since for any function \( \phi \in \overline{M} \) the translations \( \phi_n(x) = \phi(x - n) \) lie in \( \overline{M} \) converge to 0 weakly.

If we invest slightly more work, we can create a function with three bumps – one high one and two lower ones. If we keep the high bump fixed and send the smaller bumps of to \( \pm \infty \), we can create a sequence in \( \overline{M} \) whose weak limit is given by only the larger bump and may not lie in \( \overline{M} \). Clearly, this can be done while keeping the maximum and centre of mass of \( \phi \) fixed at 0.

Finally, we show that functions of low energy cannot be confined to a small band close to zero. The idea is that if the function \( \phi \) is bounded by a small constant \( h \), it must oscillate quickly between 0 and \( h \), creating many sharp local extrema and a high energy.

**Lemma 4.10.** Assume that \( \phi \in \overline{M} \) satisfies \( \phi \leq h \) for some \( h > 0 \). Then

\[
E(\phi) \geq \left( \frac{3^{2/3}}{2} \right)^{-\frac{3}{2}} h^{-2/5}.
\]

**Proof.** Again, we denote \( \Xi^2 = E(\phi) \). Recall that \( \frac{\phi''(x)^2}{2} \leq \phi(x) \) due to (2) in Lemma 4.4 so

\[
\frac{(\phi')^2}{2} - \phi \leq \left( \frac{3\Xi^2 \phi}{2} \right)^{\frac{3}{2}} - \phi \leq \left( \frac{3\Xi^2 h}{2} \right)^{\frac{3}{2}} - \phi
\]

whence

\[
1 \leq \int_R \frac{(\phi')^2}{2} - \phi \, dx \leq \left( \frac{3\Xi^2 h}{2} \right)^{\frac{3}{2}} - \phi \leq \left( \frac{3\Xi^2 h}{2} \right)^{\frac{3}{2}} - \phi
\]

whence

\[
1 \leq \int_R \frac{(\phi')^2}{2} - \phi \, dx \leq \left( \frac{3\Xi^2 h}{2} \right)^{\frac{3}{2}} - \phi \leq \left( \frac{3\Xi^2 h}{2} \right)^{\frac{3}{2}} - \phi
\]

such that

\[
h^{-2/3} \leq \frac{1}{2} \Xi^{10/3} \Rightarrow h^{-2/5} \leq \left( \frac{3^{2/3}}{2} \right)^{\frac{3}{2}} \Xi^2
\]
This simple argument shows the qualitatively right behaviour (energy blowup) but misses the order by several orders of magnitude due to its very reduced geometric information.

**Example 4.11.** Take any function \( \phi \in \overline{\mathcal{M}} \) and \( \delta > 0 \) where for simplicity we assume that \( h = \delta^{2/3} \) and \( \delta = \frac{1}{N} \) for some large integer \( N \). The rescaled function \( \phi = \phi_\delta \) satisfies \( L(\phi_\delta) = \delta = \frac{1}{N} \) and has compact support, so we can place \( N \) copies of \( \phi_\delta \) next to each other on the line. Their combined length adds up to 1 and their combined energy to

\[
\frac{1}{\delta} \cdot \delta^{1/3} \mathcal{E}(\phi) = \delta^{-2/3} \mathcal{E}(\phi) = \frac{\mathcal{E}(\phi)}{h}.
\]

Using this construction for the minimiser \( \phi \) of \( \mathcal{E} \) and using slightly too large \( N \) for general \( h \), one can determine this to be the exact order of blow-up.

### 4.3. A Problem with Delamination.

The fact that a minimiser \( \phi \) of \( \mathcal{E} \) is compactly supported suggests that \( \gamma \) would attach to the unit circle \( \partial B_r(0) \) except on a segment of length proportional to \( \delta^{1/3} \). For a future application, we consider a related problem in which the delamination from the unit circle is penalised: **minimise the functional**

\[
\mathcal{W}_\alpha(\gamma) = \mathcal{W}(\gamma) + \alpha \mathcal{H}^1(S^1 \setminus \gamma)
\]

in the class \( \mathcal{M}_L \). By the exact same analysis as above, we find the following.

**Theorem 4.12.** There exists a constant \( \Theta_\alpha > 0 \) such that

\[
\inf_{\gamma \in \mathcal{M}_L} \mathcal{W}(\gamma) = 2\pi + \Theta_\alpha \delta^{1/3} + o(\delta^{1/3})
\]

which is given by

\[
\Theta_\alpha = \inf \left\{ \int_{S^1} |\phi''|^2 \, ds + \alpha \mathcal{H}^1(\{ \phi > 0 \}) \, \bigg| \phi \in M \right\}.
\]

Note that the functional \( \mathcal{E}_\alpha(\phi) = \mathcal{E}(\phi) + \alpha \mathcal{H}^1(\{ \phi > 0 \}) \) has the same scaling property \( \mathcal{E}_\alpha(\phi_\rho) = \rho^{1/3} \mathcal{E}_\alpha(\phi) \) as the original functional \( \mathcal{E} \). The variational analysis for this problem is actually easier since there is an a priori bound on the size of the support of \( \phi \) in terms of the energy \( \mathcal{E}_\alpha \) while no such bound was available purely in terms of the energy \( \mathcal{E} = \mathcal{E}_0 \). Much of the previous analysis remains in tact and \( \mathcal{E}_\alpha \) has a minimiser in \( \overline{\mathcal{M}} \), but since the Lagrange multiplier now satisfies

\[
\left( \frac{\rho}{r} \right)^2 = \mu^2 = \frac{\mathcal{E}_\alpha(\phi)}{6 \mathcal{L}(\phi)} = \frac{\left( \frac{\rho^2}{2 \sin^2 \rho} + \frac{\rho}{\sin \rho} - 2 + \alpha \right) \rho}{6 \left[ -1 + \frac{1}{2 \sin^3 \rho} - \frac{\cos \rho}{2p \sin \rho} \right] r^3},
\]

the optimal value for \( \rho \) changes. As a corollary, the minimiser \( \phi_\alpha \in \overline{\mathcal{M}} \) of \( \mathcal{E}_\alpha \) is only \( C^{1,1} = W^{2,\infty} \) and not \( C^2 \)-smooth. Since the analysis does not simplify as nicely as before, we do not compute minimisers explicitly in this setting, but we obtain scaling results for \( \Theta_\alpha \) with \( \alpha \). Note that always \( \Theta \leq \Theta_\alpha \leq \Theta + 4\alpha \) by using the minimiser \( \overline{\phi} \) of \( \mathcal{E} \) as an energy competitor for \( \mathcal{E}_\alpha \).

**Theorem 4.13.** We have

\[
\lim_{\alpha \to \infty} \frac{\Theta_\alpha}{\alpha^{2/3}} = \frac{3\pi^{2/3}}{2}.
\]

**Proof.** Assume that \( \phi \in \overline{\mathcal{M}} \) is supported on an interval \([-r, r] \). Then

\[
1 = \int_{-r}^r \frac{(\phi')^2}{2} - \phi \, dx \leq \frac{1}{2} \int_{-r}^r (\phi')^2 \, dx \leq \frac{1}{2} \left( \frac{2r}{\pi} \right)^2 \int_{-r}^r |\phi''|^2 \, dx
\]

with equality if and only if \( \phi \) is the cos-shaped transition

\[
\phi(x) = \begin{cases} 
\left( \frac{8\pi}{\pi} \right)^{1/2} \left( 1 + \cos \left( \frac{\pi}{2\pi} x \right) \right) & |x| < r \\
0 & |x| > r
\end{cases}
\]
by very similar arguments as above, noting that $\phi$ has to satisfy the following properties:

1. $\phi \geq 0$,
2. $\phi(r) = \phi(-r) = 0$ and
3. $\phi'(r) = \phi'(-r) = 0$

The normalising factor occurs since

$$\frac{1}{2} \int_{-r}^{r} \frac{\pi^2}{(2r)^2} \sin^2 \left( \frac{\pi}{2r} x \right) \, dx = \frac{\pi}{4r} \int_{-\pi/2}^{\pi/2} \sin^2(y) \, dy = \frac{\pi}{4r} \frac{\pi}{2} = \frac{\pi^2}{8r}.$$ 

whence

$$\int_{-r}^{r} (\phi'')^2 \, dx = \frac{8r}{\pi^2} \int_{-r}^{r} \left( \frac{\pi}{2r} \right)^2 \cos \left( \frac{\pi}{2r} x \right) \, dx = \frac{\pi}{r^2} \int_{-\pi/2}^{\pi/2} \cos^2(y) \, dy = \frac{\pi}{\pi^2} \frac{\pi}{2} = \frac{\pi^2}{2r^2}$$

meaning that for any admissible function we have

$$\int_{-r}^{r} (\phi')^2 \, dx \leq \left( \frac{2r}{\pi} \right)^2 \int_{-r}^{r} (\phi'')^2 \, dx$$

This implies that

$$W_\alpha(\phi) \geq \frac{\pi^2}{2r^2} + 2\pi r.$$ 

If we optimise this over $r$, we find that

$$-\frac{\pi^2}{r^3} + \frac{\pi}{\alpha} = 0 \quad \Rightarrow \quad r = \left( \frac{\pi}{\alpha} \right)^{\frac{1}{3}}$$

whence

$$W_\alpha(\phi) \geq \frac{\pi^2}{2} \left( \frac{\pi^2}{\alpha} \right)^{-\frac{2}{3}} + \alpha \left( \frac{\pi^2}{\alpha} \right)^{-\frac{4}{3}} = \left( \frac{\pi^2 \pi^{-4/3}}{2} + \pi^{2/3} \right) \alpha^{2/3} = \frac{3\pi^{2/3}}{2} \alpha^{2/3}$$

For the opposite inequality, choose any positive bump function $\eta \in C_c^\infty(-1, 1)$ such that $\int_{-1}^{1} (\eta')^2 \, dx > 2$ and set $\eta_\alpha = \alpha^{-1/6} \eta \left( \alpha^{1/3} x \right)$. Then $\eta_\alpha \in C_c^\infty(-\alpha^{-1/3}, \alpha^{-1/3})$, 

$$L(\eta_\alpha) = \int_{\mathbb{R}} \left( \frac{\alpha^{-\frac{1}{3}} + \frac{1}{3} \eta'}{2} - \alpha^{-\frac{1}{3}} \eta \right) \left( \alpha^{1/3} x \right) \, dx = \alpha^2 \left( \frac{1}{3} + \frac{1}{3} \right) \int_{\mathbb{R}} \frac{(\eta')^2}{2} \, dx - \alpha^{-\frac{1}{3}} \int_{\mathbb{R}} \eta \, dx$$

$$= \int_{\mathbb{R}} \frac{(\eta')^2}{2} \, dx - \alpha^{-1/2} \int_{\mathbb{R}} \eta \, dx$$
such that $L(\eta_\alpha) \geq 1$ for large enough $\alpha > 0$. While $\eta_\alpha \not\in M$ for large $\alpha$, we find that $\mathcal{E}_\alpha(\eta_\alpha) \geq \Theta_\alpha$ since $L(\eta_\alpha) \geq 1$ by the same argument as in Corollary 3.6 so

$$\Theta_\alpha \leq \mathcal{E}_\alpha(\eta_\alpha)$$

$$= \int_{\mathbb{R}} |\eta_\alpha''|^2 \, dx + \alpha \cdot 2\alpha^{-1/3}$$

$$= \alpha^{2(\frac{1}{6} + \frac{1}{2} + \frac{1}{6}) - \frac{1}{6}} \int_{\mathbb{R}} |\eta''|^2 \, dx + \alpha^{2/3}$$

$$= \left( \int_{\mathbb{R}} |\eta''|^2 \, dx + 1 \right) \alpha^{2/3}.$$ 

If we more specifically choose functions $\eta$ which approximate the cos-profile on an interval $[-\pi^{2/3}, \pi^{2/3}]$, we obtain asymptotically equality of the upper and lower bound in the limit. □

4.4. Application to the Buckling of Cylindrical Shells. In this section, we derive a simple one-dimensional model for two-layer cylindrical shells and apply Theorem 44 to get conditions for when the inner layer will buckle away from the outer one in certain scaling regimes. The setting we have in mind is a pipe or tube with an outer layer which contracts more at low temperatures than the inner layer. While the model is simplistic, its benefit is that it provides explicit parameters in terms of universal constants and material properties. During this section, we remain entirely on a formal level.

The elastic energy of a thin shell can be decomposed into an energy contribution due to stretching which scales asymptotically with the thickness $h$ of the plate and an energy contribution due to bending which scales with $h^3$. For a cylindrical shell, i.e. a shell of the form

$$\Sigma = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) = \gamma(s) \text{ for } s \in S^1, -H < z < H \}$$

the energy can be determined in terms of the planar profile $\gamma$:

$$\mathcal{E}_{\text{stretch}} = c_{\text{stretch}} (2H) h \int_{S^1} \left| \gamma' \right| \frac{L}{2\pi} \, d\mathcal{H}^1$$

$$\mathcal{E}_{\text{bend}} = h^3 \int_\Sigma \chi_H \left| \vec{H} \right|^2 + \chi_K \kappa \, d\mathcal{H}^2$$

$$= \frac{\chi_H h^3}{4} (2H) \int_{S^1} \kappa \, d\mathcal{H}^1$$

where $\chi_H, \chi_K$ are material parameters, $\vec{H}$ and $K$ denote the mean curvature vector and Gauss curvature of $\Sigma$ respectively. This bending energy is commonly known as the Helfrich functional and has been derived rigorously as a $\Gamma$-limit of three-dimensional elasticity in [FJM02]. Since $\Sigma$ has a straight direction in $z$, the Gauss curvature of the cylindrical shell vanishes identically, and the mean curvature of $\Sigma$ is the average of the curvature vector $\vec{N}$ of $\gamma$ and 0. The stretching energy is minimised for an arc-length parametrised curve, so we consider the normalised elastic energy

$$\mathcal{E}_{\text{el}}(\Sigma) = c_{\text{stretch}} \int_{S^1} \left| \gamma' \right| \frac{L}{2\pi} \, d\mathcal{H}^1$$

$$= c_{\text{stretch}} \left[ \frac{\mathcal{H}^1(\gamma)}{2\pi} - \frac{L}{2\pi} \right] ^2 \mathcal{H}^1(\gamma) + \frac{\chi_H h^2}{4} \int_{S^1} \kappa^2 \, d\mathcal{H}^1$$

$$= c_{\text{stretch}} \frac{(2\pi)^2}{2} \left[ \frac{\mathcal{H}^1(\gamma)}{2\pi} - \frac{L}{2\pi} \right] ^2 \mathcal{H}^1(\gamma) + \frac{\chi_H h^2}{4} \mathcal{W}(\gamma)$$

$$\approx c_{\text{stretch}} \frac{L}{(2\pi)^2} \left( \frac{\mathcal{H}^1(\gamma)}{2\pi} - \frac{L}{2\pi} \right) ^2 \mathcal{H}^1(\gamma)$$

$$= \frac{\chi_H h^2}{4} \mathcal{W}(\gamma)$$

which is a purely geometric functional of $\gamma$. We now consider the physical situation of a two-layer cylinder whose layers are composed of materials with different physical properties. We model the
layers separately by shells $\Sigma_i, \Sigma_o$ (the inner and the outer layer) given by two planar profiles $\gamma_i, \gamma_o$. If $H^1(\gamma_i), H^1(\gamma_o)$ and $W(\gamma_i), W(\gamma_o)$ are all approximately $2\pi$, we know that $\gamma_i, \gamma_o$ are $W^{2,2}$-close to the unit circle (up to translation), and in particular that $\gamma_o$ is a Jordan curve that bounds an open set $E_o$.

The concepts of inner layer and outer layer now translate to $\gamma_i \subset \overline{E_o}$.

We can model the elastic energy of our two-layer cylinder by

$$E_{el} = E_{el, \gamma_o} + E_{el, \gamma_i} + E_{interaction, \gamma_i, \gamma_o}$$

$$= C_o \left( |H^1(\gamma_o) - L_o|^2 + \varepsilon_o W(\gamma_o) \right)$$

$$+ C_i \left( |H^1(\gamma_i) - L_i|^2 + \varepsilon_i W(\gamma_i) \right)$$

$$+ \alpha H^1(\gamma_o \setminus \gamma_i)$$

where $C_i, C_o$ are material parameters of the inner and outer shell which model resistance to stretching and $\varepsilon_i, \varepsilon_o$ are parameters which encode resistance to bending and thickness of the shell. The coupling parameter $\alpha \in [0, \infty)$ models that $\Sigma_i, \Sigma_o$ are connected by an adhesive and the contribution to the elastic energy is proportional to the area of delaminating.

In this article, let us consider the asymptotic case $C_o \to \infty$ in which the outer shell is a lot more rigid than the inner one. Then $\gamma_o$ must minimise the energy

$$C_o \left( |H^1(\gamma_o) - L_o|^2 + \varepsilon_o W(\gamma_o) \right).$$

Since the first part of the functional only depends on the length of $\gamma_o$ and the second part is minimised for a circle, we see that $\gamma_o$ is a circle of radius

$$r_o = \arg\min_{r>0} \left( (r - L_o)^2 + \frac{4\pi^2 \varepsilon_o}{r} \right) \in \left( L_o, L_o + \frac{2\pi^2 \varepsilon_o}{L_o^2} \right)$$

since the first term prefers $r$ to be close to $L_o$ and the second term prefers $r$ to be large. The precise constant is determined by verifying that the derivative of the function to be minimised is negative at $L_o$ and positive at $\frac{2\pi^2 \varepsilon_o}{L_o^2}$. We are then left to find $\gamma_i \subset B_{r_o}(0)$ such that $\gamma_i$ minimises

$$C_i \left[ |H^1(\gamma_i) - L_i|^2 + \varepsilon_i W(\gamma_i) \right] + \alpha H^1(\gamma_o \setminus \gamma_i).$$

The interesting case for us is when $L_i > r_o$, i.e. when the outer shell contracts more for low temperature than the inner shell. The inner shell has three options:

1. compression,
2. buckling,
3. fracture.

Assuming that fracture does not occur, we try to distinguish whether buckling or compression is energetically favourable, always assuming that all shells remain cylindrical. This corresponds to buckling by ridge formation, while blistering is excluded from this analysis. This assumption is reasonable since an initially Gauss-flat cylindrical shell wants to remain Gauss-flat due to the non-stretching (isometry) constraint, which suggests a cylindrical shape (assuming high enough regularity for curvature arguments).
Denote $L_i = r_o + \delta$, $H^1(\gamma_i) = r_o + t$ for some $t \geq 0$. Then, from the previous analysis, we see that to leading order we have

\[
C_i \left[ |H^1(\gamma_i) - L_i|^2 + \varepsilon_i W(\gamma_i) \right] + \alpha H^1(\gamma) = C_i \left[ |H^1(\gamma_i) - L_i|^2 + \varepsilon_i \left( W(\gamma_i) + \frac{\alpha}{C_i \varepsilon_i} H^1(S^1 \setminus \gamma) \right) \right]
\]

\[
\approx C_i \left[ (t - \delta)^2 + \varepsilon_i \left( \frac{4\pi^2}{r_o} + \frac{\Theta \tilde{\alpha} \varepsilon_i}{r_o^{4/3}} t^{1/3} \right) \right]
\]

\[
= C_i \left[ \frac{4\pi^2 \varepsilon_i}{r_o} + (t - \delta)^2 + \frac{\Theta \tilde{\alpha} \varepsilon_i}{r_o^{4/3}} t^{1/3} \right]
\]

\[
= C_i \left[ \frac{4\pi^2 \varepsilon_i}{r_o} + \delta^2 \left( \frac{t}{\delta} - 1 \right)^2 + \delta^2 \frac{\Theta \tilde{\alpha} \varepsilon_i}{r_o^{4/3}} \left( \frac{t}{\delta} \right)^{1/3} \right]
\]

\[
= C_i \left[ \frac{4\pi^2 \varepsilon_i}{r_o} + \delta^2 \left[ \left( \frac{t}{\delta} - 1 \right)^2 + \frac{\Theta \tilde{\alpha} \varepsilon_i}{r_o^{4/3}} \delta^{5/3} \left( \frac{t}{\delta} \right)^{1/3} \right] \right]
\]

as we introduce the scaled parameter $\tilde{\alpha} = \frac{\alpha}{C_i \varepsilon_i}$. The question whether buckling or compression is energetically favourable thus reduces to the question whether the function

\[
e_\lambda(s) = (s - 1)^2 + \lambda s^{1/3}
\]

has its minimum at 0 or a positive number $s = \frac{t}{\delta}$ given the parameter $\lambda = \frac{\Theta \tilde{\alpha} \varepsilon_i}{r_o^{4/3} \delta^{5/3}}$. We observe the following:

1. $e_\lambda(0) = 1$ for all $\lambda > 0$,
2. $\min_{s \in \mathbb{R}} e_\lambda(s)$ is increasing in $\lambda$ and
3. $e_\lambda(0) = e_\lambda(1) = 1$ if $\lambda = 1$, so buckling is favourable for $\lambda \leq 1$ (because $e'_\lambda(1) = 2 \cdot (1-1) + \frac{1}{3} > 0$, $e_\lambda$ does not assume its minimum at 1, which makes the minimum lower).

However, we see that

\[
e'_\lambda(s) = 2(s - 1) + \frac{\lambda}{3} s^{-2/3}
\]

satisfies $\lim_{s \to 0} e'_\lambda(s) = \lim_{s \to \infty} e'_\lambda(s) = \infty$, so $e'_\lambda$ assumes a minimum at a point $s \in (0, \infty)$ where

\[
0 = e''_\lambda(s) = 2 - \frac{2\lambda}{9} s^{-5/3} \quad \Rightarrow \quad s = \left( \frac{\lambda}{9} \right)^{2/5}
\]
which means that
\[ e'_\lambda(s) = 2s + \lambda s^{-2/3} - 2 = 2\left(\frac{\lambda}{9}\right)^\frac{2}{3} + 3\lambda \left(\frac{\lambda}{9}\right)^{-\frac{2}{3}} - 2 = 5\left(\frac{\lambda}{9}\right)^{\frac{2}{3}} - 2, \]
so if \( \lambda > 9\left(\frac{2}{5}\right)^{3/5} \approx 5.2 \), we have \( e'_\lambda > 0 \) on \((0, \infty)\) and the minimum is zero. We therefore find that there must exist a critical threshold \( \lambda_0 \in (1, 5.2) \) such that the global minimum of \( e_\lambda \) is assumed at a positive values \( s \) for \( \lambda < \lambda_0 \) and at 0 for \( \lambda > \lambda_0 \). Numerically, we find that \( \lambda_0 \in (1.0341, 1.0342) \) (see also Figure [2]. We thus expect bifurcation to buckling at \( \lambda_0 \), compression if
\[ \lambda = \frac{\Theta_\delta \varepsilon_i}{r_0^{\frac{1}{3}} \delta^{\frac{5}{3}}} < \lambda_0 \quad \Leftrightarrow \quad \delta^{5/3} < \frac{1}{\lambda_0} \frac{\Theta_\delta \varepsilon_i}{r_0^{\frac{1}{3}}} \quad \Leftrightarrow \quad \delta < \left(\frac{1}{\lambda_0} \frac{\Theta_\delta \varepsilon_i}{r_0^{\frac{1}{3}}}\right)^{\frac{3}{5}} \]
and buckling if the strict opposite inequality holds. Recall that, if \( \alpha = 0 \) we have \( \Theta_\delta = \Theta_0, \varepsilon_i = \frac{\lambda H \pi^2}{c_{\text{stretch}} L} h^2 \) and thus we expect to see bifurcation to buckling if the preferred excess length \( \delta \) satisfies
\[ \delta = \left(\frac{1}{\lambda_0} \frac{\Theta_\delta \pi^2 \chi H}{c_{\text{stretch}} (r_0 + \delta) r_0^{\frac{4}{3}}}\right)^{\frac{3}{5}} h^{\frac{3}{5}} \approx \left(\frac{\Theta_\delta \pi^2 \chi H}{\lambda_0 c_{\text{stretch}} r_0^{\frac{7}{3}}}\right)^{\frac{3}{5}} h^{\frac{3}{5}}. \]
In the presence of an adhesive, we see that \( \tilde{\alpha} = \frac{\alpha}{c_{\text{stretch}} L} = \frac{4\pi \alpha}{\chi H} \) which is large for small \( h \), so
\[ \Theta_\delta \approx \frac{4\pi^2/3}{2} \tilde{\alpha}^{3/3} = 3 \left(\frac{4\pi \alpha}{\chi H}\right)^{2/3} h^{-4/3} \]
such that we expect bifurcation to buckling at a preferred excess length
\[ \delta = \left(\frac{1}{\lambda_0} \frac{\Theta_\delta \pi^2 \chi H}{c_{\text{stretch}} (r_0 + \delta) r_0^{\frac{4}{3}}}\right)^{\frac{3}{5}} \approx \left(\frac{4\pi \alpha}{\chi H}\right)^{2/5} \left(\frac{3}{2\lambda_0 c_{\text{stretch}} r_0^{\frac{7}{3}}}\right)^{3/5} \]
with compression below this threshold and possible buckling above the threshold. In the setting of strong adhesion \( \alpha \sim h^{-1} \), we leave the regime of small \( \delta \) and the asymptotic analysis becomes invalid.

Note that we assumed \( \gamma \) to be arc-length parametrised after buckling. This is a sensible assumption in the case \( \alpha = 0 \) where tangential slip along the exterior shell is possible, but an over-simplification in the presence of an adhesive, meaning that the buckling cost would be higher than assumed here. We recall, however, that the first derivatives of the buckling profile decay as \( \delta^{1/3} \) in \( L^\infty \) and as \( \delta^{1/2} \) in \( L^2 \), which suggests that the stretching effect along the buckling profile should not influence the total energy to leading order.

In the other asymptotic regime \( C_i \to \infty \), the inner shell is given by a circular profile and the outer shell attaches to the inner shell everywhere, with or without adhesive. Whether the outer profile remains a circle in the true competitive regime \( 1 < C_i, C_o \ll \infty \) remains open.

5. The Large Length Limit

Lemma 2.4 implies that \( W(\gamma) \geq H^1(\gamma) \) for all curves \( \gamma \subset B_1(0) \) with equality if and only if \( \gamma \) is a (multiply covered) circle of radius 1. In particular, if \( H^1(\gamma) \neq 2\pi \), we have \( W(\gamma) > H^1(\gamma) \) since a multiply covered circle cannot be approximated by embedded curves. In this section, we construct a family of curves \( \gamma_L \) of length \( L \) for sufficiently large \( L \) such that
\[ \lim_{L \to \infty} \frac{W(\gamma_L)}{L} = 1, \]
recovering the optimal scaling to leading order. We show slightly more, namely that
\[ \limsup_{L \to \infty} \frac{W(\gamma_L) - L}{L^{1/2}} < \infty, \]
but do not characterise the first order term more precisely. The idea of constructing $\gamma_L$ is as follows:

1. The elastic energy of a multiply covered circle of radius $\rho_L$ (not necessarily a closed curve) is $L \rho_L^{-2}$.

2. While a multiply covered circle cannot be approximated by embedded curves with bounded energy, we can approximate a multiply covered circle with end tied of in two loops by spiralling curves with two loops – see Figure 3.

3. The energy of the inner loop-like appendage is just roughly constant in $L$, whereas the energy of the outer appendage is inversely proportional to the space between the spiral and the domain boundary $1 - \rho_L$. We approximate the energy of the spiral by

$$L \cdot \frac{1}{\rho_L^2} \approx L \cdot (1 + 3 (1 - \rho_L)) = L + 3L (1 - \rho_L).$$

Trying to match the orders of the leading excess energy terms, we have to satisfy

$$1 - \rho_L \sim (1 - \rho_L)^2 \Rightarrow (1 - \rho_L)^2 \sim \frac{1}{L} \Rightarrow \rho_L \approx 1 - cL^{-1/2}.$$ 

for a suitable constant $c$.

The optimal $c$ would have to be found by optimising over the shape of loops and balancing the terms. We do not see applications for this fine structure and do not execute this step.

Proof of Theorem 1.3. Of course, it suffices to construct curves $\gamma_L \in \mathcal{M}_L$ such that $\mathcal{W}(\gamma_L) \leq L + C \left( \sqrt{L} + 1 \right)$. Make the following ansatz: $\gamma = \gamma_1 \oplus \gamma_2 \oplus \gamma_3 \oplus \gamma_4$ where

$$\begin{aligned}
\gamma_1 : [0, \ell] &\to \mathbb{R}^2, \quad \gamma_1(s) = \left( \rho_L + \sigma_L f \left( \frac{s}{\ell} \right) \right) \left( \cos \left( \frac{s}{\ell} \right) \sin \left( \frac{s}{\ell} \right) \right), \\
\gamma_3 : [0, \ell] &\to \mathbb{R}^2, \quad \gamma_3(s) = \left( \rho_L + \frac{\sigma_L}{2} + \sigma_L f \left( \frac{\ell - s}{\ell} \right) \right) \left( \cos \left( \frac{\ell - s}{\ell} \right) \sin \left( \frac{\ell - s}{\ell} \right) \right),
\end{aligned}$$

$0 < \rho_L < 1$, $0 < \sigma_L < \frac{1 - \rho_L}{2}$ and $f \in C^\infty([0,1],[0,1])$ is a strictly monotone increasing function satisfying

$$f(0) = 0, \quad f(1) = 1, \quad f^{(k)}(0) = f^{(k)}(1) = 0 \quad \forall \ k \geq 1.$$

By construction, the curves $\gamma_1, \gamma_3$ are embedded and do not touch. Finally, we take two curves $\tilde{\gamma}_2, \tilde{\gamma}_4 : [0,1] \to \mathbb{R}^2$ for the loops. We choose

$$\begin{aligned}
\tilde{\gamma}_2 : [-1,1] &\to \mathbb{R}^2, \quad \tilde{\gamma}_2(s) = g(s) = \left( \cos \left( f(|s|) - 1 \right) \sin \left( f(|s|) - 1 \right) \right), \\
\tilde{\gamma}_4(s) &\to \mathbb{R}^2, \quad \gamma_4(s) = \left( \cos \left( f(|s|) - 1 \right) \sin \left( f(|s|) - 1 \right) \right).
\end{aligned}$$
where \( g : [-1,1] \rightarrow [0,1] \) satisfies

\[
g(-1) = g(1) = 1, \quad g^{(k)}(-1) = g^{(k)}(1) = 0 \quad \forall k \geq 1, \quad g(s) < g(1-s) \quad \forall s \in (0,1)
\]

for the inner loop. We then set

\[
\gamma_2(s) = \left( \rho + \frac{\sigma_L}{2} f \left( \frac{s + 1}{2} \right) \right) \tilde{\gamma}_2(s).
\]

The outer loop can be constructed similarly. We take a bump of the form

\[
\tilde{\gamma}_4(s) = \left( 1 - f(|s|) \right)
\]

and connect it to the curve segments \( \gamma_1, \gamma_2 \) by a translation and a rotation

\[
\gamma_4(s) = v_0 + O \left( \left( 1 - \frac{\sigma_L}{2} f \left( \frac{s + 1}{2} \right) \right) \left( 1 - \rho_L + \frac{\sigma_L}{2} \right) \tilde{\gamma}_4(s) \right).
\]

Finally, we can take the limit \( \sigma_L \rightarrow 0 \) under which \( \gamma_1, \gamma_3 \) approach a multiple cover of a circle of radius \( 1 - \rho_L \), \( \gamma_2 \) approaches \( (1 - \rho_L) \tilde{\gamma}_2 \) and \( \gamma_4 \) approaches \( v_0 + \rho_L \tilde{\gamma}_4 \) such that

\[
\lim_{\sigma_L \rightarrow 0} \mathcal{H}^1(\gamma) = 2\ell \rho_L + \rho_L \mathcal{H}^1(\tilde{\gamma}_2) + (1 - \rho_L) \mathcal{H}^1(\tilde{\gamma}_4)
\]

\[
\lim_{\sigma_L \rightarrow 0} \mathcal{W}(\gamma) = \frac{2\ell}{\rho_L^2} + \frac{\mathcal{W}(\tilde{\gamma}_2)}{\rho_L} + \frac{\mathcal{W}(\tilde{\gamma}_4)}{1 - \rho_L}
\]

\[
\approx 2\ell + 3\ell (1 - \rho_L) + \frac{\mathcal{W}(\tilde{\gamma}_4)}{\rho_L} + \mathcal{W}(\tilde{\gamma}_2) + \frac{\mathcal{W}(\tilde{\gamma}_4)}{1 - \rho_L}
\]

which requires us to choose \( \ell = \frac{L}{2} + O(\rho_L^{-1}) \) to match the length constraint. To balance the orders \( (1 - \rho_L)^{-1}, \ell (1 - \rho_L) \) in the error term, we need to choose \( 1 - \rho_L = O(L^{-1/2}) \).

\[\square\]

### 6. Curves in Three Dimensions

Finally we show that, to an extent, the phenomena described above were two-dimensional and can be avoided if curves are permitted to buckle out of plane.

**Proof of Theorem** [4]. In the case of small excess length \( L = 2\pi + \delta \) for small \( \delta \), it suffices to show that there exists a smooth curve \( \gamma_\delta \) embedded into \( B_1(0) \) such that \( \mathcal{H}^1(\gamma_\delta) = 2\pi + \delta \), \( \mathcal{W}(\gamma_\delta) \leq 2\pi + C\delta \). As a competitor, consider the curve

\[
\gamma_\eta(s) = \sqrt{1-\eta^2} \begin{pmatrix} \cos s \\ \sin s \\ 0 \end{pmatrix} + \eta \sqrt{2} \begin{pmatrix} 0 \\ 0 \\ \cos(m\eta) \end{pmatrix}
\]
which satisfies $|\gamma|^2 = 1 - \eta^2 + \frac{m^2}{2} \cos^2(ms) < 1$ and

$$
\mathcal{H}^1(\gamma) = \int_0^{2\pi} 1 - \eta^2 + \frac{m^2}{2} \cos^2(ms) \, ds
$$

$$
= \int_0^{2\pi} 1 + \left( \frac{m^2}{2} \cos^2(ms) - 1 \right) \eta^2 + O(\eta^4) \, ds
$$

$$
= 2\pi + \left( \frac{m}{2} \int_0^{2\pi} \cos^2(ms) m \, ds - 2\pi \right) \eta^2 + O(\eta^4)
$$

$$
= 2\pi + \left( \frac{m}{2} \int_0^{2m\pi} \cos^2(s') \, ds' - 2\pi \right) \eta^2 + O(\eta^4)
$$

$$
= 2\pi + \left( \frac{m}{2} m \pi - 2\pi \right) \eta^2 + O(\eta^4)
$$

$$
= 2\pi + \left( \frac{m^2}{2} - 2 \right) \pi \eta^2 + O(\eta^4),
$$

so if we choose $m = 2$, we find that $\mathcal{H}^1(\gamma) = 2\pi + 2\pi \eta^2 + O(\eta^4)$ so we can choose $\eta = \sqrt{\frac{\delta}{2\pi}} + O(\delta)$ such that $\mathcal{H}^1(\gamma) = 2\pi + \delta$. We further find by the same calculation that

$$
\int_0^{2\pi} |\gamma'|^2 \, ds = 2\pi + \left( \frac{m^2}{2} + 1 \right) \pi \eta^2 + O(\eta^4) = 2\pi + 9\pi \eta^2 + O(\eta^4)
$$

which coincides with the elastic energy up to leading order, by much the same calculation as before, so

$$
\mathcal{W}(\gamma) = 2\pi + \frac{9\pi}{2\pi} \delta + O(\delta^2).
$$

Finally, for the long length limit, note that we can use the energy competitor $\gamma = \gamma_1 \oplus \gamma_2 \oplus \gamma_3 \oplus \gamma_4$ where $\gamma_1, \gamma_2, \gamma_3$ are as before in the $\{x_3 = 0\}$ coordinate plane and we set

$$
\gamma_4(s) = \left( \rho_L + \frac{\sigma_L}{2} (1 + f(s)) \right) \gamma_2(s) + \varepsilon_L h(s) e_3
$$

where $g \in C^\infty(\mathbb{R})$ satisfies $h(s) = 0$ for $s \leq 0$ and $s \geq 1$, but $h(s) > 0$ for $s \in (0,1)$. The curve is embedded into the unit ball if we choose $\varepsilon_L$ small enough. This time, we can also take $\rho_L \to 1$ since there is no longer an energy contribution proportional to $(1 - \rho_L)^{-1}$. The limiting length is

$$
\lim_{\rho_L, \varepsilon_L \to 0} \mathcal{H}^1(\gamma) = 2 (\ell + \mathcal{H}^1(\gamma_2)), \quad \lim_{\rho_L, \varepsilon_L \to 0} \mathcal{H}^1(\gamma) = 2 (\ell + \mathcal{W}(\gamma_2))
$$

and we just have to choose $\ell = L - \mathcal{H}^1(\gamma_2)$ to match the constraints. \qed

7. Conclusion

We can identify four parameter regimes for the following problem: Minimise $\mathcal{W}(\gamma) = \int_0^L \kappa^2 \, d\mathcal{H}^1$ among all curves of length $L$ which are embedded into the two-dimensional unit disc. In all regimes, minimisers exist and are non-unique, with the sole exception of $L = 2\pi$ where the unit circle is the unique minimiser.

(1) $L \leq 2\pi$. In this regime, minimisers are given by circles of length $L$ and have energy $\mathcal{W} = \frac{4\pi}{L}$. In particular, the energy is decreasing with increasing length. minimisers for $L < 2\pi$ have a translational degree of non-uniqueness. The same is true for curves in higher dimensions, where another rotational degree of non-uniqueness is introduced.

(2) $L < 2\pi < 2\pi + \delta_0$ for some sufficiently small $\delta_0 > 0$. In this regime, the minimum energy scales like

$$
\min_{|\gamma|=L} \mathcal{W}(\gamma) = 2\pi + \Theta(L - 2\pi)^{1/3} + o(L - 2\pi)^{1/3}
$$
Our approximation of a minimiser of excess length $\delta = 2^{-9}$ (left), $\delta = 2^{-7}$ (middle) and $\delta = 2^{-5}$ (right).

where

$$\Theta := \inf \left\{ \int |\phi''|^2 \, ds \mid \int |(\phi')^2| - \phi \, ds = 1 \right\} \approx 37.$$

We expect minimisers to be shaped like minimisers of the associated problem on the real line (see Figure 1) in radial direction, suitably rescaled. In particular, we expect them to attach to the circle except on a segment of length $\sim (L - 2\pi)^{1/3}$ where they form a single bump of height $\sim (L - 2\pi)^{2/3}$. While we have not proved that the energy increases with increasing length, the highest order term does and we expect the statement to be true, qualitatively differing from the previous regime.

Minimisers cannot be circles (or unions of circles) in this regime and thus do not possess radial symmetry. However, the set of minimisers is rotationally symmetric, so minimisers cannot be unique. We have thus entered a truly non-linear regime. Since the solution of the associated minimisation problem on the real line is symmetric, we expect the rotational symmetry to be replaced by at least a reflectional symmetry for the individual minimisers.

For curves in three dimensions, the infimum energy scales as $\inf_{|\gamma| = 2\pi + \delta} W(\gamma) - 2\pi \sim \delta$ instead and we observe out-of-plane buckling. Again, up to highest order the energy is increasing with increasing length. Minimisers are almost planar (since they are $C^1$-close to a circle).

(3) $2\pi + \delta_0 < L \ll \infty$. In this regime, we have no results. We expect a higher degree of stability here under small changes of length. If $L$ is small enough, a minimiser must touch the boundary of the unit disk but cannot have points of higher multiplicity. As $L$ increases beyond a second threshold, a minimiser must touch the boundary or have points of self-contact (possibly both). We conjecture that any minimiser also in this regime touches the boundary and that the energy is increasing with increasing length.

This regime seems more amenable to numerical computations, utilising for example phase-field methods developed in [DMR11] and improved in [DLWT17]. This is the only regime in which truly three-dimensional curves may appear as energy minimisers.

(4) $L \to \infty$. In this regime, the energy minimum scales linearly with $L$ and the remainder term is bounded by $O(\sqrt{L})$ in two dimensions, $O(1)$ in higher dimensions. Minimising curves in two dimensions are expected to be spiralling approximations of a multiply covered circle of radius $1 - CL^{-1/2}$ with two loops for closedness, one large and inside the interior circle, the other small and between the approximated circle and the domain boundary. In higher dimensions, we expect the same planar spiralling profile in the limit, except that the exterior loop can be brought into the circle by out-of-plane buckling.

Again, since minimisers cannot be rotationally symmetric, they cannot be unique.
While our analysis was for curves in the unit disk, a simple scaling argument extends our results to disks of any radius. It stands to conjecture that attaching to the boundary would remain optimal in convex domains $C^2$-close to a disk and that in domains with non-constant boundary curvature buckling should happen at the point of lowest boundary curvature since the constant $\Theta_R$ of energy increase for small excess length in disks $B_R(0)$ is given by $\frac{\Theta}{R^{7/3}} = \Theta \kappa^{4/3}$ so that the prefactor decreases rapidly with decreasing curvature.

If curves are allowed to be slightly compressible, we expect to see either buckling away from the boundary or compression along the boundary, depending on the competition between the stretching and bending energy contributions and the amount of excess length. Without adhesion between the boundary or compression along the boundary, depending on the competition between the stretching and bending energy contributions and the amount of excess length. Without adhesion between the boundary and the curve, we expect to see bifurcation to buckling as a curve’s preferred excess length exceeds

$$\delta_{\text{crit}} = \left(\frac{\Theta \pi^2 \chi_H}{\lambda_0 \chi_{\text{stretch}} r_o^7 \delta^{-7/5}}\right)^{1/7} h^{\delta^{-7/5}} \approx 33.62 \frac{\lambda_0^{3/5}}{\chi_{\text{stretch}} r_o^{7/5}} h^{8/5}$$

where $\chi_H, c_{\text{stretch}}$ are material parameters, $r_o$ is the radius of the disk that the curve is confined to, $\Theta$ is as above and $\lambda_0$ is an explicit parameter, and $h$ is the thickness of a membrane modelled by the curve (or the diameter of the cross-section of a rod modelled by $\gamma$). If an adhesion $\alpha > 0$ is included and the functional $\alpha$ is considered instead, we expect bifurcation to buckling as $\delta$ exceeds

$$\delta_{\text{crit}} = \frac{4^{2/5} 2^{2/5} \pi^{8/5}}{(2\lambda_0)^{2/5}} \frac{\lambda_0^{1/5}}{\chi_{\text{stretch}} r_o^{7/5}} (\alpha h)^{2/5} \approx 12.61 \frac{\lambda_0^{1/5}}{\chi_{\text{stretch}} r_o^{7/5}} (\alpha h)^{2/5}$$

where $\alpha$ models the strength of the adhesion.

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**Appendix A. Proofs of Basic Properties**

Let us begin by proving the basic scaling and energy estimates.

**Proof of Lemma 2.1. First claim.** This follows easily from the observation that $\kappa_{\alpha\gamma} = \frac{1}{\alpha} \kappa_{\gamma}$ at corresponding points $\alpha x \in \alpha \gamma, x \in \gamma$ and that the length measure scales linearly with $\alpha$. Alternatively, if $\gamma$ is parametrized by arclength on the interval $[0, L]$, we can parametrize $\alpha \gamma$ by arc-length on the interval $[0, \alpha L]$ by $(\alpha \gamma)(s) = \alpha \cdot \gamma(s/\alpha)$ and calculate

$$W(\alpha \gamma) = \int_0^{\alpha L} \left(\frac{1}{\alpha} \alpha \frac{1}{\alpha^2} \kappa'' \right)^2 (s/\alpha) \, ds = \frac{1}{\alpha} \int_0^{L} \left| \kappa'' \right|^2 \, ds.$$  

**Second claim.** Without loss of generality, we may assume that $\gamma$ is parametrised by arc-length, $\gamma : [0, L] \to \mathbb{R}^d$ and $\gamma(0) = \gamma(L)$. Then

$$W(\gamma) = \int_\gamma \kappa^2 \, d\mathcal{H}^1 = \int_0^L \left| \kappa'' \right|^2 \, ds$$

and $\int_0^L \gamma_i' \, ds = \gamma_i(L) - \gamma_i(0) = 0$ for all $i = 1, \ldots, d$. In particular, there exists a point $s_i \in [0, L]$ such that $\gamma_i'(s_i) = 0$. Since $\gamma_i'$ is $L$-periodic, we can without loss of generality assume that $s_i = 0 = L$ modulo $L$. By the Poincaré inequality in one dimension, we find that

$$\int_0^L \left| \gamma_i'' \right|^2 \, ds \leq \left( \frac{L}{2\pi} \right)^2 \int_0^L \left| \gamma_i' \right|^2 \, ds \quad \forall \ i = 1, \ldots, d$$
and equality holds precisely if \( \gamma_i \) is a multiple of the first \( L \)-periodic eigenfunction of the Dirichlet-Laplacian \( \gamma_i(s) = \alpha_i \sin \left( \frac{2\pi s}{L} \right) \). Adding these individual estimates up, we find that

\[
L = \int_0^L |\gamma'|^2 \, ds = \sum_{i=1}^d \int_0^L (\gamma_i')^2 \, ds \leq \sum_{i=1}^d \frac{L^2}{4\pi^2} \int_0^L (\gamma_i'')^2 \, ds = \frac{L^2}{4\pi^2} \mathcal{W}(\gamma).
\]

For equality to hold, we require equality to hold in Poincaré’s inequality which is achieved in the original parametrisation if and only if

\[
\gamma_i(s) = \alpha_i \cos \left( \frac{2\pi s}{L} \right) + \beta_i \sin \left( \frac{2\pi s}{L} \right)
\]
is a linear combination appropriately scaled of \( \sin, \cos \) functions for all \( i \) (possibly up to translation by a constant vector). Thus automatically, \( \gamma \in C^\infty \). After a rotation, we may assume that

\[
\gamma'(0) = e_1, \quad \gamma''(0) = \lambda e_2
\]
for some \( \lambda \in \mathbb{R} \). It follows that

\[
\gamma_1(s) = \frac{L}{2\pi} \sin \left( \frac{2\pi s}{L} \right), \quad \gamma_2(s) = -\frac{\lambda L}{2\pi} \cos \left( \frac{2\pi s}{L} \right)
\]
and \( \gamma_3 = \cdots = \gamma_d \equiv 0 \) since otherwise either the first or second derivative would be visible. Since we assumed \( \gamma \) to be parametrised by arc-length, we find that \( \lambda = \pm 1 \). It follows that \( \gamma \) is a planar circle in \( d \)-dimensional space.

**Third claim.** This is essentially a convenient way to phrase the second claim.

**Fourth claim.** Without loss of generality, we assume that \( t_1 = 0 \). We can now decompose the curve \( \gamma \) into segments \( \gamma^j = \gamma|_{[t_j, t_{j+1}]} \) where we identify \( L = t_{k+1} - t_0 \) modulo \( L \) − note that \( \gamma^j \) is a curve segment, not a coordinate function. Then, applying Poincaré’s inequality to the coordinate functions \( \gamma^j_i \) as before, we find that

\[
\int_{t_j}^{t_{j+1}} |(\gamma^j_i)'|^2 \, ds \leq \frac{|t_{j+1} - t_j|^2}{\pi^2} \int_{t_j}^{t_{j+1}} |(\gamma^j_i)'|^2 \, ds
\]

with equality if and only if \( \gamma^j_i \) is a multiple of the first eigenfunction of the Neumann Laplacian with zero integral \( \gamma^j_i(s) = \alpha_{ij} \cos \left( \frac{2\pi s}{L} + t_j \right) \) for all \( j = 1, \ldots, k \) and \( i = 1, \ldots, n \). However, since \( (\gamma^j_i)' \) is no longer periodic, we lose the translational degree of freedom since \( \int_0^\pi \cos(x) \, dx = 0 \), but \( \int_0^\pi \sin(x) \, dx = 1 \neq 0 \). In particular, if \( n = 2 \) and \( (\gamma^j_1)' = \cos \), to preserve the unit length constraint we would need \( (\gamma^j_2)' = \pm \sin \) pointwise. As there is no continuous selection of \( \sin \) on \([0, \pi]\) which integrates to 0, we find that the infimum in Poincaré’s inequality is not attainable in the two-dimensional geometric problem. We conclude that

\[
C := \inf_{\gamma(0)=\gamma(1)=0} \mathcal{W}(\gamma) > \frac{\pi^2}{\mathcal{H}^1(\gamma)}
\]
and obtain

\[
\mathcal{W}(\gamma) = \sum_{j=1}^k \int_{t_j}^{t_{j+1}} |(\gamma')''|^2 \, ds
\geq \sum_{j=1}^k \frac{C}{|t_{j+1} - t_j|^2} \int_{t_j}^{t_{j+1}} |(\gamma')'|^2 \, ds
= C \sum_{j=1}^k \frac{1}{|t_{j+1} - t_j|^2}.
\]
The sum on the right becomes minimal for equi-distant points $t_j = \frac{j-1}{k} L$ giving rise to the estimate

$$W(\gamma) \geq C \sum_{j=1}^{n} \frac{1}{L/k} = \frac{C k^2}{L}.$$ 

Referring the reader to a more classical treatment of the fact that the only closed elasticae are the circle, a figure eight curve and their periodic covers, we prove that only the once covered circle can be approximated by embedded curves.

**Proof of Lemma 2.3.** The transversal self-crossing of the figure eight is easily excluded when writing the curves locally as graphs and using the intermediate value theorem. The multiply covered circle, which only has tangential self-contact, is slightly harder to exclude.

Any curve $\gamma$ which is $W^{2,2}$ or more generally $C^1$-close to an $m$-fold covered circle can be written as a radial graph

$$\gamma(s) = r(s) \begin{pmatrix} \cos s \\ \sin s \end{pmatrix}$$

for a $2\pi m$-periodic function $r$ which is $C^1$-close to the constant 1-function, applying a general statement about writing a surface as a normal graph over a $C^1$-close surface. If $m > 1$, we consider the shifted function $\tilde{r}(s) = r(s + 2\pi)$ and pick an interval $[a, b] \subset [0, 2\pi m]$ such that $r(a) = \max r$, $r(b) = \min r$. By the intermediate value theorem

$$\tilde{r}(a) - r(a) = r(a + 2\pi) - \max r \leq 0, \quad \tilde{r}(b) - r(b) = \tilde{r}(b) - \min r \geq 0$$

imply that there exists $s \in [a, b]$ such that $\tilde{r}(s) - r(s) = 0$ since $r, \tilde{r}$ are continuous. Then

$$\gamma(s + 2\pi) = r(s + 2\pi) \begin{pmatrix} \cos(s + 2\pi) \\ \sin(s + 2\pi) \end{pmatrix} = \tilde{r}(s) \begin{pmatrix} \cos s \\ \sin s \end{pmatrix} = r(s) \begin{pmatrix} \cos s \\ \sin s \end{pmatrix} = \gamma(s),$$

which means that $\gamma$ is not embedded. 

**Remark A.1.** While we chose an elementary argument, there are more powerful tools that would cover larger classes of curves. Assuming that a curve $\gamma$ of length $L$ is parametrised by unit speed, we can write $\gamma'(s) = \begin{pmatrix} \cos \omega(s), \sin \omega(s) \end{pmatrix}$ and compute that the curvature of $\gamma$ is $\kappa = \omega'(s)$. It follows that

$$\int_{\gamma} \kappa \, d\mathcal{H}^1 = \int_{0}^{L} \omega'(s) \, ds = \omega(L) - \omega(0) \in 2\pi \mathbb{Z}$$

since $\gamma'(0) = \gamma'(L)$. The function $\omega$ is called a Gauss representation of $\gamma$. The quantity $\omega(L) - \omega(0) \in 2\pi \mathbb{Z}$ which measures how often the tangent turns is the Whitney index of the curve.

It is clear that for the circle and all curves $C^1$-close to a circle, we have $\int_{\gamma} \kappa \, d\mathcal{H}^1 = 2\pi$. Since the space of embedded curves is connected (every embedded curve becomes a round circle under curve shortening flow), all embedded curves must have Whitney index $2\pi$, while an $m$-fold covered circle has Whitney index $2\pi m$ and any figure eight curve has Whitney index 0.

The same result on the Whitney index of an embedded curve can be obtained by using the Gauss-Bonnet theorem on the disk bounded by the curve $\gamma$ due to Jordan’s curve theorem instead of the connectedness of the space of embedded curves.

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