We have studied here the geometrodynamics of relativistic electron vortex beams from the perspective of the geometric phase associated with the scalar electron encircling the vortex line. It is pointed out that the electron vortex beam carrying orbital angular momentum is a natural consequence of the skyrmion model of a fermion. This follows from the quantization procedure of a fermion in the framework of Nelson’s stochastic mechanics when a direction vector (vortex line) is introduced to depict the spin degrees of freedom. In this formalism, a fermion is depicted as a scalar particle encircling a vortex line. It is here shown that when the Berry phase acquired by the scalar electron encircling the vortex line involves quantized Dirac monopole, we have paraxial (non-paraxial) beam when the vortex line is parallel (orthogonal) to the wavefront propagation direction. Non-paraxial beams incorporate spin–orbit interaction. When the vortex line is tilted with respect to the propagation direction, the Berry phase involves non-quantized monopole. The temporal variation of the direction of the tilted vortices is studied here taking into account the renormalization group flow of the monopole charge and it is predicted that this gives rise to the spin Hall effect.

1. Introduction

In a seminal paper, Nye & Berry [1] pointed out that in three-dimensional space the wavefronts in general contain dislocation lines when the phase becomes singular and currents coil around the vortex line. These are known as optical vortices. The quantized
vortex strength corresponds to a topological charge. When the vortex line is parallel (orthogonal) to the wave propagation direction the corresponding situation is characterized as screw (edge) dislocations. It is also possible to have mixed screw–edge dislocations characterized by the vortex lines which are tilted with respect to the propagation direction. Screw dislocations arise for monochromatic waves, whereas mixed screw–edge dislocations require temporal variations. Allen et al. [2] demonstrated that optical beams in free space bearing screw dislocation possess quantized orbital angular momentum (OAM) directed along the beam axis. This suggests that for monochromatic vortex beams, the intrinsic OAM is collinear to the momentum and its projection on the beam axis takes quantized values. Bliokh & Nori [3] have shown that it is also possible for a wavefront with mixed screw–edge dislocations having tilted vortex lines to carry well-defined OAM in an arbitrary direction. These correspond to time-diffracting or non-diffracting spatio-temporal vortex beams which are polychromatic in nature.

Recently, electron vortex beams with OAM have been produced experimentally [4–6]. This bears a close analogy between optical and matter waves. Electron vortex beams are generally visualized as scalar electrons orbiting around vortex lines. Indeed, Bliokh et al. [7] earlier predicted the existence of free space vortex beams for non-relativistic scalar electrons. Bliokh et al. [8] studied the relativistic electron vortex beams representing the angular momentum eigenstates of a free Dirac electron and constructed exact Bessel beam solutions. These authors considering the spin–orbit interaction (SOI) gave a self-consistent description of the OAM and spin angular momentum (SAM) properties of the Dirac electron. Bessel beams in general represent monoenergetic plane waves having constant momentum generating a fixed polar angle with the z-axis. In the limit of vanishing SOI, the solutions are eigenstates of both OAM and SAM. This occurs for the polar angle \( \theta = 0 \) when we have paraxial beams and also in the non-relativistic case implying momentum \( p \to 0 \). However, when we switch on SOI, in general, we have non-paraxial beams which are eigenvalues of the total angular momentum but not of the OAM and SAM separately.

In this note, we study the geometrodynamics of relativistic electron vortex beams from the perspective of the geometric phase [9] acquired by the scalar electron encircling the vortex line. It has been pointed out earlier [10,11] that the quantization of a fermion in the framework of Nelson’s stochastic quantization procedure [12,13] can be achieved when we introduce a direction vector at the space–time point which appears as a vortex line depicting the spin degrees of freedom. In this scenario, a fermion is depicted as a scalar particle encircling a vortex line which is topologically equivalent to a magnetic flux line. This effectively gives rise to a gauge theoretic extension of the space–time coordinate as well as momentum and the spin appears as an SU(2) gauge bundle. In this framework, a massive fermion appears as a skyrmion [14,15]. The specific fermionic properties, such as the sign change of the wave function after a \( 2\pi \) rotation and the spin–statistics relation, are manifestations of the Berry phase acquired by the scalar electron orbiting around the vortex line [16]. When the scalar particle rotating around the vortex line acquires certain quantized OAM \( l \), the total phase acquired by it is \( 2\pi l + \phi_B \), \( \phi_B \) being the Berry phase. For a quantized OAM, with \( l \in \mathbb{Z} \), the effective phase for such a system is \( \phi_B \) as the factor \( 2\pi l \) leads to a trivial phase. So when the system bears certain quantized OAM, the specific fermionic properties generated through the Berry phase are not disturbed. In this framework, a relativistic electron vortex beam carrying quantized OAM appears to be a natural consequence. The geometrodynamics of such beams correspond to situations when vortex lines are oriented along different directions. In fact, the Bessel beam spectrum forms a cone of plane waves with a fixed polar angle \( \theta \) with the quantization axis (z-axis). When the vortex line is along the z-axis (\( \theta = 0 \)), which corresponds to the wavefront propagation direction, we have paraxial beams. Non-paraxial beams are generated when we switch on SOI. The Berry phase acquired by the scalar electron orbiting around the vortex line involves quantized Dirac monopole when the polar angle \( \theta \) with the z-axis is zero and \( \pi/2 \). But for other angular orientations of the vortex line, the concerned monopole is non-quantized. It has been pointed out that the monopole charge in \( 3 + 1 \) dimensions is equivalent to the central charge of the conformal field theory in \( 1 + 1 \) dimensions [17,18]. As the central charge undergoes renormalization group (RG) flow as envisaged by Zamolodchikov [19], the monopole charge also undergoes RG flow. This induces a
time variation of the non-quantized monopole charge which eventually leads to the anomalous velocity giving rise to the spin Hall effect. This is the situation which occurs when the vortex line is tilted with respect to the wavefront propagation direction.

In §2, we recapitulate certain features of the skyrmion model of a fermion and discuss its implications in the generation of relativistic electron vortex beams. In §3, we consider the spin–orbit coupling, and in §4 we study the situation for tilted vortex lines with respect to the wavefront propagation direction when the beam carries OAM in an arbitrary direction.

2. Skyrmion model of a fermion, the geometric phase and relativistic electron vortex beam

It has been shown in some earlier papers [10,11] that in Nelson’s stochastic quantization procedure [12,13], the quantization of a fermion can be achieved when we introduce an internal variable that appears as a direction vector. This direction vector essentially gives rise to the spin degrees of freedom. In fact, this gives rise to the SU(2) gauge theory, and demanding Hermiticity we may take the gauge field belonging to the unitary group SU(2). In this scenario, spin degrees of freedom are represented as SU(2) gauge bundle. This effectively represents a gauge theoretic extension of the space–time coordinate as well as momentum which can be written as the gauge covariant operator acting on functions in phase space

\[ Q_\mu = -i \left( \frac{\partial}{\partial p_\mu} + A_\mu(p) \right) \quad \text{and} \quad P_\mu = i \left( \frac{\partial}{\partial q_\mu} + B_\mu(q) \right), \]

where \( A_\mu(B_\mu) \) are the momentum (spatial coordinate)-dependent SU(2) gauge field. Here, \( q_\mu(p_\mu) \) denotes the mean position (momentum) of the external observable space. In this formalism, a massive fermion appears as a skyrmion [14,15]. It is noted that the configuration variables as well as the momentum variables given by equation (2.1) represent non-commutative geometry and the non-commutativity parameter is given by

\[ [Q_\mu, Q_\nu] = \mathcal{F}_{\mu\nu}(p) \quad \text{and} \quad [P_\mu, P_\nu] = \mathcal{F}_{\mu\nu}(q), \]

where \( \mathcal{F}_{\mu\nu} \) is the corresponding field strength given by \( \mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \). The functional dependence of this non-commutativity parameter corresponds to the existence of monopoles [20,21]. In particular, the spatial components of the momentum variable can be taken to satisfy

\[ [p_i, p_j] = i\mu \epsilon_{ijk} \frac{x_k}{r^3}, \]

where \( \mu \) represents the monopole strength.

The general property of a non-Abelian gauge theory is that when the topological \( \theta \)-term is incorporated in the theory a vortex line effectively represents a loop space, when a loop can be visualized as an orbit [24]. This suggests that when the topological \( \theta \)-term is incorporated in the theory a vortex line corresponding to a magnetic flux line is enclosed by the loop. When a scalar particle encircles the loop enclosing the magnetic flux line, it acquires a geometric phase (Berry phase) apart from the usual

\[ L = \theta^* \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}, \]
dynamical phase. This geometric phase is given by \( \phi_B = 2\pi \mu \), where \( \mu \) is the monopole charge associated with the magnetic flux line and \( \mu = 1/2 \) corresponds to one magnetic flux line [25]. So for \( \mu = 1/2 \), the particle acquires the phase \( e^{i\pi} \), which is the phase associated with a fermion when it undergoes a \( 2\pi \) rotation. Thus, a fermion may be visualized as a scalar particle encircling a magnetic flux line. In this formalism, a fermion represents a skyrmion.

In this skyrmionic picture of an electron, the electron vortex beam is a natural consequence. Indeed, it is possible to have a cone of plane waves with a vortex line having a fixed polar angle \( \theta \) with the \( z \)-axis. The scalar electron encircling the vortex line can carry quantized OAM. In fact, the experimental observation of the electron vortex beam carrying OAM [4–6] substantiates the skyrmion model of an electron. As the vortex line gives rise to the spin degrees of freedom, this essentially represents a spin vortex. As the characteristic feature of a fermion, for example the sign change of the wave function after a \( 2\pi \) rotation, is a manifestation of the Berry phase acquired by the scalar electron orbiting around the vortex line, the fermionic properties do not change when the scalar electron carries quantized OAM. This follows the fact that total phase in this case is \( 2\pi l + \phi_B \), where \( l \) is the OAM and \( \phi_B \) is the Berry phase acquired by the scalar particle when it traverses the loop around the vortex line. For quantized \( l \) as \( l \in \mathbb{Z} \), the factor \( 2\pi l \) gives rise to a trivial contribution, and so the effective phase is essentially \( \phi_B \). So the electron vortex beam can carry quantized OAM without disturbing the fermionic feature. In this context, we may add that the vortex beams carrying large values of OAM have a very large region around the \( z \)-axis, where the wave function is effectively zero [6]. This gives rise to the situation when OAM carrying the scalar electron appears to be detached from the spin degrees of freedom.

In our formulation, the Berry phase acquired by the scalar electron encircling the vortex line is \( 2\pi \mu \), \( \mu \) being the monopole charge. When the monopole is located at the origin of a unit sphere, the Berry phase is given by \( \phi_B = \mu \Omega(C) \), where \( \Omega(C) \) is the solid angle subtended by the closed contour at the origin which is given by

\[
\Omega(C) = \int_C (1 - \cos \theta) \, d\phi = 2\pi (1 - \cos \theta),
\]

where \( \theta \) is the polar angle of the vortex line with the quantization axis (\( z \)-axis). So for \( \mu = 1/2 \), we have the phase

\[
\phi_B = \pi (1 - \cos \theta).
\]

This corresponds to the flux associated with the monopole passing through the surface spanning the closed contour. In fact, we have

\[
\mu \Omega(C) = e\phi|\Sigma,
\]

where \( \phi|\Sigma \) is the flux through the surface \( \Sigma \) and can be written as

\[
\phi|\Sigma = \int_\Sigma B \cdot d\Sigma.
\]

Transforming to a reference frame where the scalar electron is considered to be fixed and the vortex state (spin state) moves in the field of the magnetic monopole around a closed path, \( \phi_B \) in equation (2.6) corresponds to the geometric phase acquired by the vortex state. The angle \( \theta \) represents the deviation of the vortex line from the \( z \)-axis. Equating this phase \( \phi_B \) in equation (2.6) with \( 2\pi \mu \), which is the geometric phase acquired by the scalar electron moving around the vortex line in a closed path, we find that the effective monopole charge associated with a vortex line having polar angle \( \theta \) with the \( z \)-axis is given by

\[
\mu = \frac{1}{2} (1 - \cos \theta).
\]

In this connection, one may note that for a particle moving around a closed path, adiabaticity is not an essential criterion for acquiring the Berry phase. Indeed, Aharonov & Anandan [26] have pointed out that the Berry phase is acquired even when the system is non-adiabatic in nature. Relation (2.9) shows that for \( \theta = 0 \) and \( \pi/2 \), it takes quantized values but for other angles \( 0 < \theta < \pi/2 \), \( \mu \) is non-quantized. We note that when the vortex line representing the spin axis
is along the \( z \)-direction, i.e. when the vortex line is parallel to the wave propagation direction (implying \( \theta = 0 \)), we have the paraxial vortex beam. For \( \theta = \pi/2 \), the vortex line is orthogonal to the wave propagation direction. For other values of \( \theta \) corresponding to non-quantized monopole charge, the vortex line is tilted in an arbitrary direction. This implies the deviation of the spin axis from the \( z \)-axis and represents the anisotropic feature associated with the system. These three states correspond to the screw, edge and mixed edge–screw dislocations in optical beams.

### 3. Relativistic electron vortex beam and spin–orbit interaction

A paraxial relativistic electron vortex beam carrying quantized OAM involves a scalar electron and the vortex line along the axis of propagation implying that the polar angle \( \theta = 0 \). The corresponding beams essentially represent a superposition of monoenergetic plane waves. In this case, OAM and SAM are separately conserved. However, we can introduce SOI which will lead to non-paraxial vortex beams. Indeed, Bliokh \textit{et al.} \cite{Bliokh2013} studied the situation by solving the Dirac equation using cylindrical coordinates. For convenience, we here recapitulate some of the results derived by these authors. We consider the Dirac equation \((c = \hbar = 1)\)

\[
i \partial_t \psi = (\alpha \cdot p + \beta m)\psi,
\]

where \( \alpha \) and \( \beta \) are the \( 4 \times 4 \) Dirac matrices, \( p = -i \partial_r \) is the momentum operator and \( m \) is the electron mass. The positive energy eigenstates are

\[
\psi_p(r, t) = W(p) \exp[i(p \cdot r - Et)],
\]

with energy

\[
E = \sqrt{p^2 + m^2} \quad \text{and} \quad W = \left( \frac{1}{\sqrt{2}} \right) \left( \frac{\sqrt{1 + (m/E)w}}{\sqrt{1 + (m/E)w}} \right).
\]

Here, \( \sigma \) are the Pauli matrices, \( \kappa = p/p \) is the momentum direction vector and \( w \) is the two-component spinor characterizing the electron polarization in the rest frame with \( E = m \). Using cylindrical coordinates \((r, \varphi, z)\) in real space and \((p_{\perp}, \varphi, p_{\parallel}) = (p \sin \theta, \phi, p \cos \theta)\) in momentum space, we write the Fourier spectrum

\[
\tilde{\psi}_l(p) = \frac{1}{i |p_{\perp}|} W(p) \delta(p_{\perp} - p_{\perp 0}) \exp(il\phi),
\]

where \( p_{\perp 0} = p \sin \theta_0, \ p_{\parallel 0} = p \cos \theta_0, \ \theta_0 \) being the fixed polar angle with the \( z \)-axis. Beams represent superposition of plane waves which are uniformly distributed over the azimuthal angle \( \phi \in (0, 2\pi) \) with a vortex phase dependence \( \exp(il\phi), \ l = 0, \pm 1, \pm 2 \ldots \). The beam field is given by the Fourier integral of \( \tilde{\psi}_l(p)\)

\[
\psi_l(r, t) = \frac{\exp(i\Phi)}{2\pi i l} \int_0^{2\pi} W(p) \exp[i\xi \cos(\phi - \varphi) + il\phi] \, d\phi,
\]

where \( \Phi = (p_{\parallel 0} - Et) \) and \( \xi = p_{\perp 0}r \). Assuming that the polarization amplitudes are the same for all the plane waves we have

\[
\psi_l = \exp \frac{i\phi}{\sqrt{2}} \left[ \begin{pmatrix} \sqrt{1 + \frac{m}{E}w} \\
\sqrt{1 - \frac{m}{E}\sigma_z \cos \theta_0 w} \end{pmatrix} e^{il\psi_l}(\xi) + i \begin{pmatrix} 0 \\
-\beta \sqrt{\Delta} \end{pmatrix} e^{i(l-1)\psi_l}j_{l-1}(\xi) \right] \\
+ i \begin{pmatrix} 0 \\
0 \\
\alpha \sqrt{\Delta} \end{pmatrix} e^{i(l+1)\psi_l}j_{l+1}(\xi),
\]

(3.5)
where $\Delta = (1 - m/E) \sin^2 \theta_0$. The first term in the square bracket represents a scalar-like Bessel beam of the order of $l$: $\psi_l \propto J_l(\xi) e^{i(w \cdot \mathbf{z})}$. The terms proportional to $\sqrt{\Delta}$ describe the polarization-dependent coupling implying SOI.

In the present framework, we note that when the configuration of an electron is viewed as a scalar particle encircling a vortex line in a specific direction, the solution of the Dirac equation in cylindrical coordinates exhibits the vortex-dependent properties explicitly. It is noted that when OAM $l$ is taken to be zero so that the SOI is switched off, solution (3.5) reduces to the term in the relativistic limit $m/E \rightarrow 0$

$$\psi = \frac{e^{i(pz-\mathbf{E}t)}}{\sqrt{\Delta}} \left( \begin{array}{c} w \\ \sigma_z w \end{array} \right),$$

(3.6)

where $w$ represents the electron polarization. As the vortex line gives rise to the spin degrees of freedom, we note that the spinorial term involving $w$ corresponds to the contribution of the vortex line where the plane wave $e^{i(pz-\mathbf{E}t)}$ represents the contribution of the scalar electron orbiting around the vortex line which is taken to be along the $z$-axis. When the polar angle of the vortex line with the $z$-axis $\theta \neq 0$ and the scalar electron orbiting around it carries any arbitrary OAM with $l \in \mathbb{Z}$, the first term in the square bracket in (3.5) in the relativistic limit $m/E \rightarrow 0$ reduces to

$$\psi_l = \frac{\exp i\phi}{\sqrt{2}} e^{il\theta} \left( \begin{array}{c} w \\ \sigma_z \cos \theta w \end{array} \right) J_l(\xi).$$

(3.7)

Here, the term involving $w$ corresponds to the contribution of the vortex line and the factor $(\exp i(\phi + l\phi)/\sqrt{2})J_l(\xi)$ corresponds to the contribution of the scalar electron. The second and third term in the square bracket in (3.5) corresponds to the contribution of the SOI.

It is noted that the SOI factor $\Delta$ is determined by the Berry phase. In fact in the relativistic limit $m/E \rightarrow 0$, we find $\Delta = \sin^2 \theta = 4\mu(\mu - 1)$ which follows from equation (2.9). Here, $\mu$ is the monopole charge associated with the Berry phase. For $\mu = 0(1)$, corresponding to $\theta = 0(\pi)$, we have $\Delta = 0$, which indicates that the vortex line is parallel (anti-parallel) to the $z$-axis and SOI vanishes. For $\mu = \frac{1}{2}$, corresponding to $\theta = \pi/2$, the vortex line is orthogonal to the $z$-axis. However, $\Delta$ involves non-quantized monopole charge for other values of $\theta$. It may be recalled here that the association of SOI with the Berry phase has been studied by other authors in some earlier works. It has been pointed out that when we treat the orbital degrees of freedom and the spin degrees of freedom as slowly and fast-varying variables, respectively, the effective Hamiltonian for the slow degrees of freedom may contain a non-trivial gauge potential which represents the Berry connection. The SOI is derived by making an adiabatic approximation to the Dirac equation for an electron moving in a smooth external potential in which the orbital degrees of freedom are treated as slowly varying with respect to the spin degrees of freedom [27]. Bliokh et al. [8] have described the OAM and SAM with Berry phase corrections and predicted SOI in relativistic electron vortex beams. The SOI term has been incorporated through the gauge theoretic methods by several authors [28–30]. This effectively leads to the interaction of the spin with a magnetic field in momentum space. In the present formalism, the SOI is obtained through the Berry phase, which essentially corresponds to a monopole and thus manifests the presence of a magnetic field. In view of this fact, we note that this formalism effectively has the same underlying physics as in the gauge theoretic formalism adopted by other authors.

In the present formalism, we study the SOI from the gauge theoretic extension of the space coordinates as discussed in the previous section. In three-dimensional space, we can write from (2.1)

$$Q = q + A(p), \quad A(p) \in SU(2).$$

(3.8)

The generalized OAM can now be written as

$$\mathbf{L} = Q \times p = q \times p + A(p) \times p = \mathbf{L} + \mathbf{L'},$$

(3.9)

with

$$\mathbf{L}' = A(p) \times p.$$

(3.10)
The gauge field \( \mathcal{A}(p) \) can be written as

\[
\mathcal{A}(p) = A(p) \times \sigma, \tag{3.11}
\]

with \( A(p) = \mu(p/p^2) \), \( \mu \) being the monopole charge in the momentum space. From this, we find

\[
L' = \mathcal{A}(p) \times p = A(p) \times \sigma \times p = \frac{\mu}{p^2} \times \sigma \times p. \tag{3.12}
\]

Now substituting \( p/p = \kappa \), \( \kappa \) being the unit vector, we have

\[
L' = -\mu \kappa \times (\kappa \times \sigma). \tag{3.13}
\]

The expectation value of \( \sigma \) i.e \( \langle \sigma \rangle \) is given by \( \langle \sigma \rangle = \langle \psi | \sigma | \psi \rangle / \langle \psi | \psi \rangle \), where \( \psi \) is a two-component spinor \( \psi = (\psi_1, \psi_2) \) with \( \langle \psi | \psi \rangle = 1 \). This gives

\[
\langle \sigma \rangle = \langle \psi | \sigma | \psi \rangle = n, \tag{3.14}
\]

with \( n^2 = 1 \). We can write from (3.13)

\[
\langle L' \rangle = -\mu n. \tag{3.15}
\]

Now taking into account the mapping \( (\hbar = 1) \)

\[
L \rightarrow l \hat{z} \quad \text{and} \quad S \rightarrow s \hat{z} \tag{3.16}
\]

and using the relation \( \tilde{L} + \tilde{S} = L + S \) we find from equation (3.15) and (3.16)

\[
\langle \tilde{L} \rangle = (l - \mu) \hat{z} \quad \text{and} \quad \langle \tilde{S} \rangle = (s + \mu) \hat{z}. \tag{3.17}
\]

This suggests that a part of the angular momentum is transformed from the SAM to OAM implying SOI. It may be mentioned that the quantized value of \( \mu = 1/2 \) corresponds to the relation \( |\mu| = s \). It can be noted that for non-quantized values of \( \mu \), the expectation value \( \langle \tilde{L} \rangle \) and \( \langle \tilde{S} \rangle \) can take arbitrary values.

For the quantized values of \( \mu \), the expression for \( \langle \tilde{L} \rangle \) expresses the relation for the angular momentum in the presence of a magnetic monopole. Indeed when the OAM \( l = 0 \), the quantized value for \( \mu = 1/2 \) suggests the total angular momentum 1/2, indicating that the intrinsic angular momentum of the system is 1/2, which is the SAM of an electron.

Essentially, when the scalar electron traverses a closed loop around the vortex line it acquires the Berry phase as

\[
\phi_B = \oint A(p) \, dp = 2\pi \mu. \tag{3.18}
\]

Expressions (3.17) and (3.18) explicitly exhibit that the Berry phase plays a significant role in the SOI.

From the Dirac equation, the general features of SOI can be studied from the Foldy–Wouthuysen momentum representation \([31,32]\) separating the positive and negative energy components. This effectively leads to the non-commutativity of space when the spatial coordinates are given by \( R = r + A \), with \( A = ((p \times \sigma)/2p^2)(1 - m/E) \), where \( A \) is the non-Abelian Berry connection but not a monopole. However, \( A \) is here a momentum-dependent gauge field which incorporates spin degrees of freedom through the relation \( S = 1/2\sigma \), \( S \) being the spin operator and \( \sigma \) is the vector of Pauli matrices. The Berry phase is obtained by integrating \( A \) along the contour of the Bessel beam spectrum in momentum space and is found to be \( \phi_B = \oint A \, dp = 2\pi \Delta s \) \([8]\). Here, \( \Delta s \) is the spin variable which modifies the OAM, so that the expectation \( \langle L \rangle \) is now given by \( \langle L \rangle = (l + \Delta s) \hat{z} \), with \( \langle L_z \rangle = (l + \Delta s) \). In our formulation, the Berry connection corresponds to a monopole and the monopole charge is linked with the helicity. In fact, the angular momentum of the charged particle in the field of a magnetic monopole is given by \( J = L - \mu \hat{r} \), where \( L \) is the OAM and \( \mu \) is the monopole charge. When OAM is zero, the total angular momentum of the system which is effectively the spin degrees of freedom of the system is given by \( |\mu| \) with \( S_z = \pm \mu \). In view of this, the Berry phase obtained in terms of the spin variable \( \Delta s \) in Dirac equation formalism can be taken to correspond to the phase obtained in terms of
the monopole charge $\mu$. Thus, we observe that the Berry connection derived in Dirac equation formalism essentially leads to the same physics as envisaged in the present formalism.

4. Tilted vortex and spin Hall effect

We have elucidated in §2 that when the polar angle $\theta$ of the vortex line with the $z$-axis is not 0 and $\pi/2$, the Berry phase involves non-quantized value of the monopole charge, which we denote as $\tilde{\mu}$. In earlier papers [17,18], it has been pointed out that the monopole charge in $3+1$ dimensions is equivalent to the central charge $c$ of the conformal field theory in $1+1$ dimensions. Zamolodchikov [19] has shown that the central charge $c$ undergoes the RG flow. In analogy to this, it has been pointed out that the monopole charge $\mu$ undergoes an RG flow. This suggests that when $\mu$ depends on a certain parameter $\lambda$, we have

- $\mu$ is stationary at fixed points $\lambda^*$ of the RG flow.
- At the fixed points, $\mu(\lambda^*)$ is equal to the monopole charge $\mu$ given by the quantized values $0, \pm 1/2, \pm 1, \ldots$.
- $\mu$ decreases along the RG flow i.e $L(\partial \mu/\partial L) \leq 0$, where $L$ is a length scale.

Now transforming the length scale to the time scale ($L = ct$), we can write

$$\frac{\partial \mu}{\partial t} \leq 0,$$

which implies that we can consider $\mu$ as the time-dependent parameter. In fact when $\mu$ takes a value on the RG flow and is non-quantized (denoted by $\tilde{\mu}$), we can take it as a function of time $\tilde{\mu}(t)$ and at certain fixed value of time it takes the quantized value $\mu$.

It is now clear that the explicit time dependence of the monopole charge, $\tilde{\mu}(t)$ effectively makes the corresponding gauge field explicitly time dependent. An electric field $E$ is now generated through its derivative $\partial A/\partial t$. An electric field accelerates electrons so that the momentum carries explicit time dependence. We denote the time-dependent momentum as $k$. In this case, we can introduce a non-inertial coordinate frame with basis vectors $(v, w, t)$ attached to the local direction of momentum $t = k/k$. This coordinate frame rotates as $k$ varies with time. Such rotation with respect to a motionless (laboratory) coordinate frame describes a precession of the triad $(v, w, t)$, with some angular velocity. Now taking the direction of the vortex line at an instant of time as the local z-axis, which represents the direction of propagation of the wavefront, we note that this corresponds to the paraxial beam in the local frame. In this local non-inertial frame, the local monopole charge will correspond to a pseudo-spin. Indeed, the expectation value of the spin operator

$$\langle S \rangle = \frac{1}{2} \frac{\langle \psi | \sigma | \psi \rangle}{\langle \psi | \psi \rangle}$$

undergoes precession with the precession of the coordinate frame. This suggests that the polarization state depends on the choice of the coordinate frame [33]. When the direction of the vortex line is taken to be the local z-axis in the non-inertial frame, the local value of $\tilde{\mu}$ is changed to the quantized value $|\mu| = 1/2$ owing to the precession of the spin vector and thus corresponds to the pseudo-spin in this frame. The pseudo-spin vector $S$ is parallel to the momentum vector $k$.

Now transforming the momentum $p$ by the time-dependent momentum $k$ in the gauge potential, we can write the Berry curvature as

$$\Omega(k) = \mu \frac{k}{k^3}.$$  

(4.3)

In fact, the time dependence of $\tilde{\mu}$ is here incorporated through the time dependence of $k$. This curvature will give rise to an anomalous velocity given by

$$v \parallel = k \times \Omega(k).$$

(4.4)
This anomalous velocity gives rise to the spin Hall effect. Indeed, substituting the expression of \( \Omega(k) \) given by equation (4.3) in (4.4), we can write the anomalous velocity as

\[
v_a = \mu \dot{k} \times k^3. \tag{4.5}
\]

Thus, the anomalous velocity is perpendicular to the pseudo-spin vector and points along opposite directions depending on the chirality \( s_z = \pm \frac{1}{2} \) corresponding to \( \mu > 0 (< 0) \). This separation of the spins gives rise to the spin Hall effect [34]. Thus, a tilted vortex line with respect to the propagation direction in the inertial frame carrying OAM will give rise to relativistic spin Hall effect. This essentially describes the polarization-dependent shift of the wave trajectory.

The SOI in the non-relativistic limit can be derived from the Dirac equation by introducing the FWT [35] separating the positive and negative energy components in the Dirac equation. The transformation \( \psi' = U_{\text{FW}}(p)\psi \) with \( U_{\text{FW}} = (1/\sqrt{2})(\sqrt{1 + m/E - \beta \alpha \cdot \kappa \sqrt{1 - m/E}}) \) diagonalizes the Dirac Hamiltonian \( U_{\text{FW}}^\dagger(\alpha, p + \beta m)U_{\text{FW}} = \beta E \) and also yields \( W = U_{\text{FW}}^\dagger W = (w, 0)^T \) for plane wave (3.2). Using the projection on the positive energy subspace which excludes the negative energy levels, the electron position operator is given by Bliokh [31] and Berard & Mohrbach [32]

\[
R = r + A(p) \tag{4.6}
\]

with

\[
A(p) = \frac{p \times \sigma}{2 p^2} \left( 1 - \frac{m}{E} \right). \tag{4.7}
\]

The non-relativistic spin operator is given by \( S = \frac{1}{2} \sigma \) and the SOI is introduced through the operator

\[
\Delta = A(p) \times p = - \left( 1 - \frac{m}{E} \right) \kappa \times (\kappa \times S). \tag{4.8}
\]

The spin-dependent connection \( A(p) \) gives rise to the Berry phase when the integration is performed along the contour of the Bessel beam spectrum in momentum space [8]

\[
\phi_B = \oint A(p) dp = 2\pi \Delta s. \tag{4.9}
\]

We can now have simple mapping

\[
L \to l\hat{z}, \quad S \to s\hat{z} \quad \text{and} \quad \Delta \to \Delta s. \tag{4.10}
\]

The SOI is now determined by the spin to OAM conversion analogous to that depicted in equation (3.17) where \( \mu \) is replaced by \( -\Delta s \).

In our framework, when the relativistic electron is depicted as a scalar particle moving around a vortex line and the space–time coordinate is given by a gauge theoretic extension, where the corresponding gauge field \( A(p) \in SU(2) \), the non-relativistic effect is obtained in the sharp point limit. In the sharp point limit, the residual effect of the non-Abelian gauge field is manifested through an Abelian gauge field [36]. The Abelian gauge field in the momentum space gives rise to a momentum-dependent magnetic field. The SOI, which is derived in the non-relativistic limit through the momentum-dependent gauge field obtained through the FWT can now be incorporated through the momentum-dependent magnetic field. The non-relativistic Hamiltonian (for detailed derivation see the appendix) in the presence of the SOI term can now be written as [37–39]

\[
H = \frac{p^2}{2m} + V - g\sigma \cdot B(p), \tag{4.11}
\]

where \( g \) is the coupling constant and \( V \) is the electrostatic potential. Time dependence of the momentum will make the magnetic field \( B(p) \) time dependent. We can have the explicit time dependence of the momentum when we switch over to the interaction picture. We can split the
Hamiltonian in $H$ into two parts $H_0 + H_1$, so that for $H_0$ we take

$$H_0 = \varepsilon E \cdot r$$

(4.12)

and

$$H_1 = \frac{p^2}{2m} - g \sigma B(p).$$

(4.13)

Here, $\varepsilon$ is the electric field. In the interaction picture, an operator $O$ in the Schrodinger picture is transformed as

$$\tilde{O}(t) = e^{iH_0 t} O e^{-iH_0 t}.$$  

(4.14)

It is noted that the state vector $|\psi(t)\rangle$ in the Schrodinger picture is now transformed as

$$|\tilde{\psi}(t)\rangle = e^{iH_0 t} |\psi(t)\rangle.$$  

(4.15)

For the momentum operator, it gives

$$k(t) = p - \varepsilon E t.$$  

(4.16)

We can now rewrite the expression for the anomalous velocity generated by the Berry curvature $\Omega(k)$ given by equation (4.5) involving explicitly the non-relativistic spin vector $S$ and the electric field $E$. In fact from equation (4.16) we have,

$$\dot{k} = -\varepsilon E.$$  

(4.17)

The monopole charge $\mu$ effectively represents the electron helicity as this corresponds to $s_z = \pm \frac{1}{2}$ in the local frame for $\mu > 0$ ($\mu < 0$) we can substitute $\mu$ in equation (4.5) by the expression of the helicity

$$\mu = \frac{S \cdot k}{2|k|},$$  

(4.18)

Inserting this in equation (4.5), the expression of the anomalous velocity yields

$$v_a = \frac{S \times \varepsilon E}{2k^2}.$$  

(4.19)

This gives rise to the spin current along the direction of $v_a$ when two opposite orientations of the spin move in opposite directions.

The expression of $v_a$ in (4.5) can be rewritten in terms of the unit vector $t = k/|k|$ and its time derivative $\dot{t}$ as [33]

$$v_a = \mu \frac{\dot{k} \times k}{k^3} = \mu \dot{t} \times t.$$  

(4.20)

Denoting $\dot{t}/|\dot{t}| = n$, we note that the spin current is orthogonal to the local plane $(t, n)$. Thus, we observe that for a tilted vortex with respect to the wave propagation direction though the Berry phase acquired by the orbiting scalar electron may be viewed as an artefact of a rotating coordinate frame, the spin Hall effect is a Coriolis-type transverse deflection as the spin current is orthogonal to the local plane $(t, n)$. Furthermore, the Berry phase involves non-quantized monopole charge when the vortex line remains non-rotating in an inertial frame. But in a rotating frame, one can take the quantized value $|\mu| = 1/2$ by choosing the proper coordinate frame when the vortex line is taken to be in the propagation direction. However, the spin Hall effect represents a real deflection of the wave trajectory as the spin current moves out of plane in the orthogonal direction of the local $(t, n)$ plane and thus becomes independent of the local coordinate frame.

In addition, one may indicate here that a well-defined gauge field in time space which has the physical significance of an effective magnetic field accounts for the spin Hall effect in the Rashba system in the adiabatic limit [40]. This magnetic field is also found to be the underlying origin of the anomalous velocity owing to the curvature in momentum space. In fact the anomalous velocity owing to the Berry curvature in momentum space is a direct manifestation of the effective magnetic field from the gauge field in the time space. Thus in our analysis, the time dependence of the monopole charge, which gives rise to the spin Hall effect through the generation of anomalous
velocity from the Berry curvature, effectively corresponds to an analogous situation as developed through the introduction of a gauge field in time space [40].

5. Discussion

We have studied here the geometrodynamics of the relativistic electron vortex beam from the perspective of the geometric phase associated with the scalar electron encircling the vortex line. It is observed that the vortex beam is a natural consequence of the skyrmion model of an electron as investigated in the quantization scheme of a fermion. In this scenario, a fermion is visualized as a scalar particle encircling a vortex line which is topologically equivalent to a magnetic flux line. The geometric phase acquired by the scalar particle moving around the vortex line in a closed loop essentially gives rise to the specific properties of a fermion, such as the sign change of the wave function after a 2\pi rotation and the spin–statistics relation. The electron vortex beam can carry the OAM as the introduction of the quantized OAM given by \( l \in \mathbb{Z} \) contributes only trivially to the total phase so that the effective phase is the concerned geometric phase. This ensures the fermionic properties of such a system. It is pointed out that we have paraxial beam when the vortex line is parallel to the wavefront propagation direction. In this case both the OAM and the SAM are separately conserved. However, when the spin–orbit coupling is switched on, we have non-paraxial beam. When the vortex line is orthogonal to the propagation direction the monopole related to the Berry phase is quantized. Interestingly, when the vortex line is tilted with respect to the propagation direction of the wavefront, the related Berry phase involves non-quantized monopole. From an analysis of the RG flow of the monopole charge, it is argued that in this case we have temporal variation of the direction of the vortex line giving rise to the spin Hall effect which can be observed experimentally.

However, these features associated with electron vortex beams are analogous to optical vortex beams. Indeed, the paraxial beam corresponds to the screw wavefront dislocation and the non-paraxial beam associated with the quantized monopole charge represents edge dislocation when the vortex line is orthogonal to the wave propagation direction. The tilted vortex lines with respect to the wave propagation direction which involve non-quantized monopole charge correspond to the mixed edge–screw dislocations in optical beams. These incorporate temporal variations. It has been pointed out by Bliokh & Nori [3] that one can avoid temporal diffraction, when these dislocations represent spatio-temporal vortex beams which can be achieved through the Lorentz transformation of the corresponding spatial beams. However for mathematical consistency, one has to take into account the ‘relativistic Hall effect’ caused by a shift in the transverse direction of the OAM carrying object observed in a moving frame. We have argued above that in the case of electron vortex beams with tilted vortices, the temporal variation leads to the spin Hall effect.

Finally, one can conclude that the dynamics of relativistic electron vortex beam is determined by the Berry phase acquired by the scalar electron orbiting around the vortex line. When this phase involves quantized monopole, we have paraxial and non-paraxial beams such that in the former (latter) case the vortex line is parallel (orthogonal) to the propagation direction. On the other hand, when this phase involves non-quantized monopole we have tilted vortex lines which have temporal variation and cause the spin Hall effect. This non-quantized monopole is realized through the RG flow of the monopole charge and hence does not envisage the contribution of the Dirac string.

Appendix A. Foldy–Wouthuysen transformation

The Dirac Hamiltonian for a particle of charge e is given by

\[
H = \beta mc^2 + \gamma (\alpha \cdot p) + eV(r),
\]

where \( \beta, \alpha \) are usual Dirac matrices and \( \Sigma \) is the spin operator for 4-spinor. To obtain a Hamiltonian in the low energy limit, one has to apply a series of FWT on the Hamiltonian (A 1).
The Dirac wave function is a four-component spinor with the up and down spin electron and hole components. Generically, the energy gap between the electron and hole is much larger than the energy scales associated with condensed matter systems. One can achieve this by block diagonalization method of the Dirac Hamiltonian exploiting FWT. Hamiltonian (A 1) can be divided into block diagonal and off diagonal parts denoted by $\epsilon$ and $O$, respectively,

$$H = \beta mc^2 + O + \epsilon \quad \text{and} \quad O = \alpha p \cdot \epsilon = eV(r),$$

where $\beta = \gamma_0$ and $\alpha_i = \gamma_0 \gamma_i$. Applying FWT on $H$ yields,

$$H_{FW} = \beta \left( mc^2 + \frac{p^2}{2m} \right) + \epsilon - \frac{1}{8m^2c^4} [O, [O, \epsilon]].$$

(A 3)

Calculating various terms of the Hamiltonian ($O^2$, $[O, \epsilon]$ and $[O, [O, \epsilon]]$), following relation $(\alpha A)(\alpha B) = AB + i \Sigma (A \times B)$ we can write the FW-transformed Hamiltonian upto $1/c^2$ terms as

$$H_{FW} = \beta \left( mc^2 + \frac{p^2}{2m} \right) + V(r) - \frac{\hbar^2}{8m^2c^2} (\nabla \cdot E) - \frac{i\hbar^2}{8m^2c^2} \nabla \times E - \frac{\hbar}{4m^2c^2} \sigma \cdot (E \times p).$$

(A 4)

Consideration of constant electric field can help us leaving the terms with $(\nabla \times E)$ and $(\nabla E)$. Finally, we land up with the Hamiltonian for the upper component of Dirac spinor as

$$H_{FW} = \left( mc^2 + \frac{p^2}{2m} \right) + V(r) - \frac{\hbar}{4m^2c^2} \sigma \cdot (E \times p).$$

(A 5)

This FW-transformed Hamiltonian gives the dynamics of an electron (or hole with proper sign of $\epsilon$) in the positive energy part of the full energy spectrum. Here, $\sigma$ is the Pauli spin matrix.

For electrons moving through the lattice, the electric field $E$ is Lorentz transformed to an effective magnetic field $(p \times E) \approx B(p)$ in the rest frame of electron. Thus from (A 5), we can write [37–39] the Hamiltonian as

$$H_{FW} = \frac{p^2}{2m} + V(r) - g\sigma \cdot B(p),$$

where the rest energy term is neglected and $g$ is the coupling strength.

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