AN ELEMENTARY PROOF OF THE EULER’S FORMULA USING THE CAUCHY’S METHOD

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Abstract. The use of Cauchy’s method in proving the well-known Euler formula is an object of many controversies. The purpose of this paper is to prove that the Cauchy’s method applies for convex polyhedra and not only for them, but also for surfaces such as the torus, the projective plane, the Klein bottle and the pinched torus.

Introduction

The Euler’s formula says that for any convex polyhedron the alternating sum

\[ n_0 - n_1 + n_2, \]

is equal to 2, where the numbers \( n_i \) are respectively the number of vertices \( n_0 \), the number of edges \( n_1 \) and the number of triangles \( n_2 \) of the polyhedron. There are many controversies about the paternity of the formula, also about who gave the first correct proof.

In the section 1, we provide precisely some elements about the history of the formula as well as about the first topological proof given by Cauchy. Some authors criticize the Cauchy’s proof, saying that the proof needs deep topological results that were proved after Cauchy’s epoch: “Não se pode, portanto, esperar obter uma demonstração elementar do Teorema de Euler, com a hipótese de que o poliedro é homeomorfo a uma esfera, como fazem Hilbert-Cohn Vossen e Courant-Robbins” (Lima, [Li2]). Notice that the proof provided by Hilbert-Cohn Vossen [HC] and Courant-Robbins [CR] is the one of Cauchy.

In the section 2, we provide an elementary proof which shows that only with a lifting technique and the use of sub-triangulations, the Cauchy’s proof works without using any other result. More precisely, consider a triangulated polygon in the plane, with possible identifications of the simplices on its boundary, we prove that the alternating sum \( [1] \) of the polygon is equal to the one of its boundary plus 1 (Theorem 2.1). The idea of our proof is, starting from the hole formed by the removal of a simplex, to extend the hole by successive puddles. The process is illustrated by the construction of a suitable pyramid. A direct consequence of the theorem is an elementary proof of the Euler’s formula using only the Cauchy’s method.

As applications of our theorem 2.1 in the section 3, we use also these tools for proving that for a triangulable surface \( S \) like the torus, the projective plane, the Klein bottle and even for the pinched torus, the alternating sum \( [1] \) does not depend on the triangulation of the surface. To be completely honest, for applications (other than the sphere) in the section 3 we use also the idea of “cutting” surfaces that, in general, has been introduced by Alexander Veblen in a seminar in 1915 (see [Bra]). Of course, one can ask why we do not apply the theorem 2.1 and the reasoning to come to all (smooth) orientable and non-orientable surfaces. The reason is very simple: we want to provide proofs as it was possible to do at the time of Cauchy. It is only in 1925 that T. Radó [Rad] proved the triangulation theorem for surfaces, that was more or less admitted at the Cauchy’s epoch. The classification Theorem for compact surfaces and the representation under the “normal” form has been proved for the first time in a rigorous way.
by H.R. Brahana [Bra] (1921). It is clear that using our theorem 2.1 and the representation of surfaces under the normal form, we obtain immediately the Euler-Poincaré characteristic of any compact surface. However, that is like a dog biting his own tail. That is the reason why we do not present the result for surfaces in general but only what is possible to do with the Cauchy’s method on some elementary surfaces.

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1. History

1.1. Before Cauchy. The name “Euler’s formula” comes from an announcement of Leonhard Euler in November 14th of 1750 in a letter to a friend, Goldbach of the following result:

**Theorem 1.1.** Let $K$ be a convex polyhedron, with the numbers $n_0$ of vertices, $n_1$ of edges and $n_2$ of 2-dimensional polygons, then

$$n_0 - n_1 + n_2 = 2. \tag{2}$$

There are many different possible definitions of polyhedra depending on authors. The discussion concerns to the dimension of a polyhedron: Is a polyhedron a solid object of dimension three or only the superficial part? In this paper, we call “polyhedron” the three dimensional solid figure. A polyhedron is a figure constructed by polygons in such a way that each segment is the common face of exactly two 2-dimensional polygons and each vertex is the common face of at least three segments (see [Ri], Chapter 2 for the discussion).

There are many questions and controversies about Euler’s formula. Here, let us discuss the two following questions: Was Euler the first mathematician stating the formula? Who provided the first proof of the formula?

Let us discuss the first question: *Who was the first to state the formula?*

Some authors (see [Eve], §3.12; [Lie], p. 90) write that it is possible that Archimedes (≈ 287 AC, ≈ 212 AC) already knew the formula. Some authors say that the formula was known by Descartes (1596 – 1650). In fact, Descartes, in a manuscript [De], “De solidorum elementis”, proved the following result:

**Theorem 1.2.** If we take the right angle as the unit, then the sum of the angles of all the 2-dimensional polygons of a convex polyhedron will be equal to $4(n_0 - 2)$.

**Proof of the fact “the formula (2) is equivalent to theorem 1.2.”** On the one hand, the theorem of Descartes says that the sum of the angles of all the 2-dimensional faces is $2(n_0 - 2)\pi$. On the other hand, let us denote by $i = 1, \ldots, n_2$ the 2-dimensional faces of the convex polyhedron. For each face $i$, let us denote by $k_i$ the number of vertices, which is also the number of edges of the face. We use, for each face, the following property: In a convex polygon with $k_i$ edges, the sum of all the angles equals $(k_i - 2)\pi$. Since each edge of the polyhedron appears in two faces of the polyhedron, then $\sum_{i=1}^{n_2} k_i = 2n_1$. Hence the sum of the angles of all the faces of the polyhedron equals $\sum_{i=1}^{n_2} (k_i - 2)\pi$, that is $(2n_1 - 2n_2)\pi$. We obtain the formula $$2(n_0 - 2)\pi. \tag{2}$$

Descartes did not publish his manuscript in his epoch. The original version of the manuscript disappeared, but a copy was rediscovered in 1860 among papers left by Leibnitz (1646-1716). This copy suffered some accidents in particular an immersion in the Seine river in Paris (see [Fa], [dJ1]).
Some authors say that Descartes discovered the formula (2) as an application of his theorem 1.2. This is a reason why sometimes the formula (2) is called “Descartes-Euler formula”. Other authors, for example, Malkevich [Mal1], affirmed that “Though Descartes did discover facts about 3-dimensional polyhedra that would have enabled him to deduce Euler’s formula, he did not take this extra step. With hindsight it is often difficult to see how a talented mathematician of an earlier era did not make a step forward that with today’s insights seems natural, however, it often happens.” However, we know that some Descartes’ papers disappeared, so nobody can decide if Descartes was knowing the formula or not and the response to the first question has not been known until nowadays.

Let us discuss now the second question: Who was the first to provide a correct proof to the formula (2)?

Two years after writing the formula, Euler provided a proof [Eu2, Eu3] consisting in removing step by step each vertex of the polyhedron together with the pyramid of which it is the vertex.

Here we provide the example of the cube, copied from [Ri]: In figure 1 (b), one eliminates the vertex A as well as the (white) pyramid of which A is a vertex. This operation does not change the alternating sum \(n_0 - n_1 + n_2\). In figure 1 (c) we perform the same process in order to eliminate the vertex B, and so on, till we obtain a tetrahedron. For the tetrahedron, one has \(n_0 - n_1 + n_2 = 2\), so we obtain the formula.

![Figure 1. The Euler’s proof: Successive elimination of a vertex as well as the pyramid in which it is the vertex.](image)

But this proof is not correct. In the book [Ri], Richeson provides a clear description of the Euler’s proof, as well as problems in the proof. According to Richeson, these problems were solved by Samelson and by Francese and Richardson (Sam, FR).

The first correct proof was provided by Legendre [Leg] in the first edition of his book Éléments de Géométrie (1794) (see [Ri], Chapter 10 for a presentation of the Legendre’s proof). The Legendre’s argument was geometric in the same way than the proof of Descartes for the theorem 1.2. The only difference between Descartes’ argument and Legendre’s argument is that Descartes argued on the sphere presentation of the polyhedron (polar polyhedron), while Legendre argued on the polyhedron itself. The passage of the polyhedron \(K\) (Legendre) to the polar polyhedron of \(K\) (Descartes) makes a permutation of \(n_0\) and \(n_2\), without modifying \(n_1\). This is the reason why some authors say that the proof of the Euler’s formula (2) should be called “Descartes-Legendre’s proof”.

1.2. Cauchy’s epoch: Cauchy’s method - The first combinatorial and topological proof. In February of 1811, being 22 years old, Cauchy, who was already an engineer of Ponts et Chaussées, gave a talk entitled Recherches sur les polyèdres in École Polytechnique, in Paris. This talk was published in 1813 in Journal de l’École Polytechnique [Ca1], as the first combinatorial and topological proof of the Euler’s
The first step of the Cauchy’s proof is to construct a planar representation.

**Definition 1.3.** A planar representation of a compact and without boundary surface $S$ is a triple $(K, K_0, \varphi)$ where:

1. $K$ is a 2-dimensional polygon in $\mathbb{R}^2$,
2. the segments and vertices of the boundary of $K$ are named and oriented with possible identifications. We denote by $K_0$ the boundary of $K$ with the given identifications,
3. $\varphi : |K| \to S$ is a homeomorphism of the geometrical realisation of $K$ (taking care of identifications of the segments and vertices on the boundary $K_0$ in $S$).

It seems that Cauchy was the first person using the idea of planar representation. Let us provide now the Cauchy’s proof.

**Proof.** Given a convex polyhedron $\hat{K}$. We choose a 2-dimensional face $P$ of the polyhedron. We remove this face. The first idea of Cauchy is to construct the associated planar representation $K$ of the polyhedron with respect to the choice of the removed face. Lakatos [Lak] explains the Cauchy’s construction in the following way: Put a camera above the removed face of the polyhedron, the planar representation will appear on the photography. Notice that this idea of planar representation is similar to the stereographic projection.

![Figure 2. Planar representation according to Cauchy. The pyramid with vertex $O$ is supposed translucent.](image)

The hole formed by the removed face appears around of the planar representation (see figure 3).

The next step of the Cauchy’s proof is to define a triangulation of $K$ by a subdivision of all the polygons. Notice that, by the triangulating process, the alternating sum $n_0 - n_1 + n_2$ does not change. The boundary of the hole is constituted of edges with the following property: Each edge is a face of a triangle that has only this edge as the common edge with the hole. For example, in the figure 4 (a), the triangle $\sigma$ has the edge $\tau$ as the common edge with the hole. The extension of the hole consists of two operations that we describe in the following.
The “operation I” consists of removing from the polyhedron $K$ such triangle $\sigma$ together with the corresponding edge $\tau$ and then the hole is extended. The operation I does not change the sum $n_0 - n_1 + n_2$, because $n_0$ does not change while $n_1$ and $n_2$ decrease by 1.

When a situation like the one of the figure 4 (b) appears, where one triangle (here $\sigma$) has two common edges $\tau_1$ and $\tau_2$ with the hole, we use the “operation II” consisting of removing from the polyhedron $K$ such triangle $\sigma$ together with the two edges $\tau_1$ and $\tau_2$ and with the vertex $a$ that is the common vertex of $\tau_1$ and $\tau_2$. Then, the hole is extended.

The operation II also does not change the sum $n_0 - n_1 + n_2$, since $n_0$ and $n_2$ decrease by 1 and $n_1$ decreases by 2.

If we take care of keeping the boundary such that it is homemorphic to a circle, then the hole is extended, using the two above operations, until we have only one triangle. In this triangle, we have $n_0 - n_1 + n_2 = 3 - 3 + 1 = 1$. Since we removed an (open) polygon at the beginning, we have already +1 in the account of the sum $n_0 - n_1 + n_2$. Hence for any convex polyhedron, we have $n_0 - n_1 + n_2 = +2$. □

1.3. After Cauchy. Some authors, in particular Lakatos [Lak], criticize the Cauchy’s proof. In his book (see [Lak], pages 11 and 12), Lakatos provided a counter-example to the Cauchy’s process. Here, we adapt the Lakatos counter-example to our example of figure 2. Extending the hole by removing triangles with the indicated order in the figure 5 (a), we use the operations I and II of Cauchy until the ninth
triangle. If we remove the tenth triangle, the hole disconnects the rest of the figure (see figure 5 (b)): The
eleventh and twelfth triangles are no longer connected. Moreover, we observe that if we remove the tenth triangle, we do not remove any
vertex, but we remove two edges and one triangle, then the sum \( n_0 - n_1 + n_2 \) is not preserved.

Hence, we need to be very careful about the removal’s order of the triangles since the situations that
the hole disconnects the polyhedron \( K \) can happen. Moreover, the boundary of the hole is no longer a
curve homeomorphic to a circle, it has multiple points.

![Figure 5. Removal order according to Lakatos.](image)

In the paper \([L2]\), Lima formalized the arguments of Lakatos and explicited situations of the figures
In the figures (a), (b) and (c), the extension of the hole, obtained by the removal of the triangle \( \sigma \) from
the polyhedron \( K \), on the one hand, changes the sum \( n_0 - n_1 + n_2 \) and, on the other hand, disconnects
the polyhedron \( K \). The Lakatos’ example corresponds to the situation (a) in the figure of Lima. In the
examples of Lima, the boundary of the hole is a curve with multiple points.

We observe that the figure (d) is also a case where the boundary is a curve having multiple points.
However the sum \( n_0 - n_1 + n_2 \) is preserved when we remove from \( K \) the triangle \( \sigma \), since we remove
two vertices \( x_2 \) and \( x_3 \), three edges \((x_1,x_2), (x_2,x_3), (x_1,x_3)\) and the triangle \( \sigma \). It seems that Lima did
not realize that this case is admissible. The situation of the figure (d) can be also used in the process of
Cauchy. In this paper, we call this operation as ‘operation III’. This operation appeared also in \([CG]\).

![Figure 6. The counter-examples of Lima: The hole \( B \) is in grey and the simplices to be removed are in dashed and dotted gray.](image)
Note that with operation III, the Lakatos’ example is no longer a counter-example: come back to figure 5(a), after removing the ninth triangle, one can remove the twelfth triangle by operation II, then remove the tenth triangle by operation III. We have only the eleventh triangle for which \( n_0 - n_1 + n_2 = +1 \) so we are done!

Some authors suggest a strategy to define an order of removal of the triangles that allows to use the Cauchy’s method to obtain the result. For example, Kirk [Kirk] suggests the following strategy: “There are two important rules to follow when doing this. Firstly, we must always perform [Operation II] when it is possible to do so; if there is a choice between [Operation I] and [Operation II] we must always choose [Operation II]. If we do not, the network may break up into separate pieces. Secondly, we must only remove faces one at a time.” We provide, in figure 7(b), an example whose process follows the rules of the strategy defined by Kirk but the process disconnects the polyhedron. So these rules are not sufficient. It is easy to build an example showing that they are not necessary.

Figure 7. A counter-example to the “strategy” suggested by Kirk. Triangles are removed in the order of numbers. The index I or II below each number means that the triangle is removed using the corresponding operation I or II, respectively. After removing the seventh triangle, the boundary of the hole is no longer homeomorphic to a circle.

The example provided in figure 7 is also a counter-example showing that even with the operation III, the Cauchy’s process does not work with the chosen removal’s order. In fact, after removing the seventh triangle, we can continue till the tenth triangle, but we cannot remove it, because in that case we remove one vertex, three segments and one triangle. Notice that the vertices A and B do not belong to the hole.

After Cauchy, many authors proposed alternating proofs of the Euler’s formula (2), using original arguments. See the site of Eppstein [Epp] containing 20 different proofs, using tools that appear only after the Cauchy’s epoch. In particular, some proofs use the Jordan Lemma (Jordan [Jo], 1866). In fact, we will see that Jordan curves will appear in the proof of our theorem 2.1 as an artifact.

However, to our knowledge, no one has given an adapted strategy for removing the triangles allowing to use only the Cauchy’s method and tools known in Cauchy’s time. That is the goal of our paper.
1.4. **Generalization of the formula** (2). The formula (2) was extended by many authors, in particular by Lhuilier, firstly for orientable surfaces of genus $g$, as follows:

(1.4) \[ n_0 - n_1 + n_2 = 2 - 2g. \]

In the non-orientable case, the formula is given by (see [Mas]):

\[ n_0 - n_1 + n_2 = 2 - g. \]

The general result was provided by Poincaré [Po2, Po3] who proved that, for any triangulation of a polyhedron $X$ of dimension $k$, calling by $n_i$ the number of simplices of dimension $i$, the sum

(3) \[ \chi(X) = \sum_{i=0}^{k} (-1)^i n_i \]

does not depend on the triangulation of $X$. This sum is called the Euler-Poincaré characteristic of $X$.

Due to the dimension convention for polyhedra in the section 1.1, one remarks that the “Euler’s formula” should be better written as Euler-Poincaré characteristic of the convex 3-dimensional polyhedra, as

\[ n_0 - n_1 + n_2 - n_3 = +1. \]

Of course, $n_3$ is +1 anyway, but this writing of Euler’s formula seems more suitable.

2. **The Cauchy’s method in the proof of Euler’s formula**

The classical planar representations of surfaces such as the sphere, the torus, the projective plane and the Klein bottle are examples of the following situation: The surface is homeomorphic to the geometric representation of a polygon $K$, itself homeomorphic to a disc $D$, such that there are possible identifications of simplices on the boundary $K_0$. 

![Figure 8. Planar representations of the sphere.](image-url)
Theorem 2.1. Let $K$ be a triangulated polygon in $\mathbb{R}^2$, homeomorphic to a disc $D$, with possible identifications of simplices on the boundary $K_0$ of $K$. We have

$$\chi(K) = \chi(K_0) + 1.$$ 

We emphasize that the important point of our proof is that, from the given triangulation, we provide a sub-triangulation such that we can prove the theorem using only the Cauchy’s method (section 1.2), without using other tools.

Proof of the theorem 2.1 using only the Cauchy’s method. Given in the plane $\mathbb{R}^2$ a polygon $K$ triangulated and homeomorphic to a disc $D$, with possible identifications of the simplices on the boundary $K_0$ of $K$, the proof consists of six following steps.

1) Step 1: The first step is to construct a “lifting” of $K$ into a pyramidal shape. Here, we call the “pyramid” only the superficial part (dimension 2) of the pyramid, i.e. the union of the faces of the pyramid without the base.

We can assume that the origin 0 of $\mathbb{R}^2$ lies inside a 2-dimensional simplex $\sigma_0$ being in the interior of the polygon $K$. 

![Figure 10. The triangulation of the polygon $K$.](image)
Let us consider the Euclidean metric in $\mathbb{R}^2$. We can assume that the distances from the origin to the vertices of the triangulation are different, otherwise a small perturbation will not change the structure of the simplicial complex and the proof of the theorem can be processed by the same way.

Let us denote by $a_1, a_2, a_3$ the vertices of $\sigma_0$ and by $b_1, \ldots, b_k$ the vertices of the triangulation of $K_0$. The other vertices are denoted by the following way: We call $y_1$ the vertex nearest to the origin 0 and $y_2, \ldots, y_n$ the vertices in the increasing order of distances from 0.

![Figure 11. The lifting on the polygon.](image)

We construct a pyramid $\Pi$ in $\mathbb{R}^3$ lying above $K$ by fixing the boundary $K_0$ as the base of the pyramid in the horizontal plane $Q = \mathbb{R}^2$ in $\mathbb{R}^3$. For $i = 0, \ldots, n + 1$, we consider the planes $P_i$ parallel to $Q$ having distance $n - i + 1$ relatively to the base plane $Q$ in $\mathbb{R}^3$. Now, for $i = 1, \ldots, n$, we call $x_i$ the orthogonal projection of the point $y_i$ to the plane $P_i$. In the plane $P_0$, we denote by $u_1, u_2, u_3$ the orthogonal projections of the points $a_1, a_2, a_3$ (see figure 12 (a)).

![Figure 12. The pyramid $\Pi$ and the decomposition $L'$ of the pyramid $\Pi$.](image)

The triangulation of the polygon induces a triangulation $L$ on the pyramid $\Pi$ lifting each simplex $[b_i, y_j]$ to $[b_i, x_j]$, each $[y_i, y_j]$ to $[x_i, x_j]$ and each $[y_i, a_j]$ to $[x_i, u_j]$. In the same way, we lift also the 2-dimensional simplices.
2) Step 2: Let us construct a sub-decomposition $L'$ of the triangulation $L$, such that the intersections of the planes with the pyramid are triangulated in the following way: Let us define new vertices of $L'$ as the intersection of 1-dimensional simplices of $L$ with the planes $P_i$. In the same way, we define also new 1-dimensional simplices of $L'$ as the intersections of 2-dimensional simplices of $L$ with the planes $P_i$. The decomposition $L'$ of the pyramid contains vertices, edges (1-dimensional simplices), and faces which can be triangles or quadrilaterals. The sum $n_0 - n_1 + n_2$ is the same for the triangulation $L$ and the decomposition $L'$ (see figure 12 (b)).

3) Step 3: Let us define a sub-triangulation $L''$ of $L$ in the following way: each quadrilateral is divided into two triangles. The sum $n_0 - n_1 + n_2$ is the same for the triangulations $L$ and $L''$ (see figure 13).

![Figure 13. The sub-triangulation $L''$ of the pyramid $\Pi$.](image13)

![Figure 14. The sub-triangulation $K''$ of the polygon.](image14)
4) Step 4: We will prove that the intersection of $L''$ with each plane $P_i$ is a curve homeomorphic to a circle. Firstly, we show that the projection of $L''$ to the plane $Q$ provides a sub-triangulation $K''$ of the polygon $K$ (figure 14). In fact, by construction, since there is no vertical edges in the pyramid, then the orthogonal projection $\pi$ of the pyramid to $Q$ is a bijection between the triangulations $L''$ and $K''$. Notice that each vertex of $K''$ corresponds to a vertex of $L''$, and in the same way for the edges and the triangles of $K''$ and $L''$, respectively. Moreover, in subdivision $L''$, each edge is the common edge of exactly two triangles, and so in $K''$. That implies that $K''$ is a triangulation.

Now, we prove, by induction, that the intersection of $L''$ with each plane $P_i$ is a curve homeomorphic to a circle. We see that $L \cap P_0$, which is the boundary of the triangle $\sigma_0$ denoted by $B_0$, is homeomorphic to a circle. Assume that $B_i$ is homeomorphic to a circle but $B_{i+1}$ is not homeomorphic to a circle. Then $L''$ has multiple points in the plane $P_{i+1}$ (see figure 15). By projection, $K''$ is no longer a triangulation. That provides a contradiction.

![Figure 15. Not admisible picture: The intersection $B_i$ of $L$ with each plane $P_i$ can not have multiple points.](image)

5) Step 5: Apply the Cauchy’s method (section 1.2): Now, let us apply the Cauchy’s method on the pyramid by starting to remove the triangle $\sigma_0$. Assume that we already removed all triangles above the plane $P_i$. We will prove that if we remove all the triangles in the band situated between $P_i$ and $P_{i+1}$, the sum $n_0 - n_1 + n_2$ does not change. This fact can be processed since the open band between $P_i$ and $P_{i+1}$ does not possess vertices, like the following: Let us fix a triangle $(\alpha_0, \alpha_1, \beta_0)$ of the band between $B_i$ and $B_{i+1}$, where the vertices $\alpha_0$ and $\alpha_1$ belong to $B_i$ and $\beta_0$ belongs to $B_{i+1}$. Firstly, we remove the triangle $(\alpha_0, \alpha_1, \beta_0)$ by the operation I, without changing the sum $n_0 - n_1 + n_2$. Now, the edge $(\alpha_1, \beta_0)$ is an edge of either the triangle $(\alpha_1, \beta_0, \beta_1)$, where $\beta_1 \in B_{i+1}$ (see figure 16(a)), or of the triangle $(\alpha_1, \alpha_2, \beta_0)$, where $\alpha_2 \in B_i$ (see figure 16(b)). In the first case, the triangle $(\alpha_1, \beta_0, \beta_1)$ can be removed by the operation I of the Cauchy’s process, and in the second case the triangle $(\alpha_1, \alpha_2, \beta_0)$ can be removed by the operation II. In both of these two cases, the sum $n_0 - n_1 + n_2$ is not changed.

We continue the process for all the triangles of the band, which are all of the two above cases, until we reach the last vertices of $B_i$ and $B_{i+1}$ situated before going back to $\alpha_0$ and $\beta_0$, respectively. We call these vertices $\alpha_k$ and $\beta_j$. We have two possible situations (a) and (b) (see figure 16). In the situation (a), the last remaining triangles are $(\alpha_k, \alpha_0, \beta_j)$ and $(\alpha_0, \beta_j, \beta_0)$. In this case, these triangles can be removed in this order using the operation II. In the situation (b), the remaining triangles $(\alpha_k, \beta_j, \beta_0)$ and $(\alpha_k, \alpha_0, \beta_0)$ can be removed in this order using also the operation II. In the two cases, the sum $n_0 - n_1 + n_2$ is not changed.
6) Step 6: The conclusion.

The process holds until the boundary $B_{n+1}$ of the hole, which is the boundary $K_0$ of $K$, and is also the intersection of $L''$ with the plane $P_{n+1}$.

Let us denote by $n^K_0$, $n^K_1$ and $n^K_2$ respectively the numbers of vertices, edges and triangles of the triangulation $K$ and let us use the same notations for the sub-triangulation $K''$ and the triangulation $K_0$. We have

$$n^K_0 - n^K_1 + n^K_2 = n''_0 - n''_1 + n''_2 = n^K_0 - n^K_0 + 1.$$ 

The first equality follows from the fact that the sub-triangulation process does not change the alternating sum and the second equality comes from the fact that we removed the triangle $\sigma_0$ at the beginning and that $n^K_2 = 0$. Here we take the identifications of simplices of $K_0$ into account.

Given a triangulation $K$ of the polygon, the result does not depend on the choices made. □

Proof of Theorem 1.1 using the Cauchy’s method. Let $\hat{K}$ be a convex polyhedron. We proceed the planar representation $K$ of $\hat{K}$, according to the first step of the Cauchy’s proof (see figure 2). Notice that $K$ is a polygon without any identification of simplices on its boundary $K_0$. Then $n^K_0 - n^K_1 + n^K_2 = 0$. The theorem 2.1 implies that $n^K_0 - n^K_1 + n^K_2 = +2$, taking into account the removed polygon $P$ in the first step of the Cauchy’s proof. Since the theorem 2.1 is proved using only the Cauchy’s method, then the Euler’s formula is also proved by using only the Cauchy’s method. □

Before going further with applications, let us provide some remarks on the proof of Theorem 2.1.

Remark 2.2. The stereographic projection proof of the Euler’s formula is a particular case of the proof.

Remark 2.3. There are other ways to define an order of vertices of the polygon to be able to draw the pyramid, without using the Euclidean distance in $\mathbb{R}^2$.

One possible way is to use the notion of distance between two vertices as the least number of edges in an edge path joining them. As in the proof of Theorem 2.1 choose a 2-dimensional simplex $\sigma_0$ in the interior of the polygon $K$ and define distance 0 for the three vertices of $\sigma_0$. Deciding any order between vertices whose distance to vertices of $\sigma_0$ is 1, one continues the ordering deciding any order between vertices whose distance to vertices of $\sigma_0$ is 2, etc. Then one proceeds to the construction of the pyramid.

Another way would be to start the proof of Theorem 1.1 with the convex polyhedron and to order the 2-dimensional faces according to the shelling process (see [Zie] and [BM]). The 2-dimensional faces of the

Figure 16. Going from $B_i$ to $B_{i+1}$. 

(a) $B_i$ 

(b) $B_{i+1}$
polyhedron are dual of the vertices of the polar polyhedron. One obtains an order on the vertices of the polar polyhedron. One continues the proof using the polar polyhedron instead of the original polyhedron, knowing that the sum \( n_0 - n_1 + n_2 \) is the same for the polyhedron and its polar. Notice that the shelling is a tool which was defined well after Cauchy, so it is not acceptable in our context. We mention it just for the sake of completeness.

**Remark 2.4.** In the step 4 of the proof, the projection on the plane \( Q \) of the intersection of each plane \( P_i \) with the pyramid is a Jordan curve passing through \( y_i \). Moreover, in step 5 of our proof, we make clear that if the boundary of the extended hole is homeomorphic to a circle, then the Cauchy’s process works. It is then possible to proceed in the step 5 either with the sub-triangulation \( L'' \) of the pyramid, or with the sub-triangulation \( K'' \) of the polygon \( K \).

**Remark 2.5.** In the step 5 of the proof we use only the operations I and II of Cauchy. Observing that if we change the removal’s order of the last remaining triangles, for example, in the situation (a), if we remove the triangle \( (\alpha_0, \beta_j, \beta_0) \) and thereafter the triangle \( (\alpha_k, \alpha_0, \beta_j) \), we will use firstly the operation I of Cauchy and then the operation that we called operation III in the section 1.3 (see figure 6 (d)). Here also, we do not change the sum \( n_0 - n_1 + n_2 \).

**Remark 2.6.** Like we emphasize at the beginning of this section, we use in the proof only the Cauchy’s method (section 1.2) without other tools. We know very well that there exist “modern and faster” ways to prove the theorem 2.1. However, these proofs use tools appearing after the Cauchy’s epoch, in particular some proofs use the Jordan Lemma, which, as we have seen, appears as an artifact in our proof.

### 3. Applications

In the following, using the theorem 2.1 as the main tool, we prove that the sum \( n_0 - n_1 + n_2 \) does not depend on the triangulation in the case of the sphere, the torus, the projective plane, the Klein bottle and even for a singular surface: the pinched torus. In each case, we use a planar representation of the surface homeomorphic to a disc with possible identifications on the boundary, and the following lemma of “cutting surfaces” idea that, in general, has been introduced by Alexander Veblen in a seminar in 1915 (see [Bra]). This idea is well developed in the book by Hilbert and Cohn-Vossen [HC], in particular for the surfaces that we give as examples.

The following “cutting” lemma will be used in the forthcoming proofs.

**Lemma 3.1.** Let \( T \) be a triangulation of a compact surface. Let \( \Gamma \) be a continuous simple curve in \( S \). There exists a sub-triangulation \( T' \) of \( T \), with curvilinear simplices, compatible with the curve (that means \( \Gamma \) is an union of segments of \( T' \)) in such the way that the number \( n_0 - n_1 + n_2 \) is the same for \( T \) and \( T' \).

**Proof.** First of all, we can assume that the curve \( \Gamma \) is transversal to all the edges of \( T \), i.e the intersection of \( \Gamma \) with each edge is a finite number of points. Otherwise, a small perturbation of \( \Gamma \) allows us to obtain the transversality.

We choose a base point (the starting point) \( x_0 \) on the curve, as well as an orientation of the curve. If the curve is not closed, we define the base point as one of two extremities. The following process does not depend on neither the starting point, nor the orientation of the curve.

A sub-triangulation \( T' \) is built simplex by simplex following the orientation of the curve \( \Gamma \). The first subdivided simplex is the one \( \sigma_0 \) containing the base point. Let \( y \) be the first point where the curve leaves
$\sigma_0$. The (curvilinear) segment $(x_0, y)$ will be an edge of $T'$ as well as segments connecting $x_0$ to vertices of $\sigma_0$, one of these can be $(x_0, y)$ if the point $y$ is a vertex of $\sigma_0$ (figure 17).

Now, it is enough to perform the construction for a simplex $\sigma = (a, b, c)$ in which the curve $\Gamma$ enters. In the following construction, we assume that all the simplices that the curve meets between the base point and the simplex $\sigma$ (with the given orientation) are already subdivided. The “entry” point of $\Gamma$ in the simplex $\sigma$ can be either a vertex, or a point $d$ located on an edge of $\sigma$.

If the entry point of the curve $\Gamma$ is a vertex $a$, the curve can exit at one point $d$ located either in the opposite edge, or in an edge containing the vertex $a$, or at another vertex. In the first case (figure 18 (i)), we divide the triangle $(a, b, c)$ into two (curvilinear) triangles $(a, b, d)$ and $(a, d, c)$. In the second case (figure 18 (ii)), let $e \in (a, c)$ be the point in which the curve $\Gamma$ exits from the triangle, we choose one point $f$ on the curve, located between $a$ and $e$. We divide the triangle $(a, b, c)$ into four (curvilinear) triangles $(a, b, f)$, $(b, f, c)$, $(b, e, c)$ and $(a, f, e)$. Finally, in the last case (figure 18 (iii)), assume that the exit point is the vertex $c$, we choose one point $f$ on the curve, located between $a$ and $c$. We divide the triangle $(a, b, c)$ into three (curvilinear) triangles $(a, b, f)$, $(b, f, c)$ and $(a, f, c)$.

Notice that, in all the three cases, the choice of sub-triangulation is not unique, but the sum $n_0 - n_1 + n_2$ remains unchanged independently of the choice.

If the entry point of the curve $\Gamma$ is located in one edge, we denote by $d$ the entry point and by $(a, c)$ the edge of $\sigma$ containing $d$. The next exit point of $\Gamma$ can be either in an edge different from $(a, c)$ (for example $(a, b)$), or in the same edge $(a, c)$, or a vertex (see figure 19).
In the first situation (figure 19 (i)), we define, for example, a sub-triangulation of the triangle \((a, b, c)\) formed by the (curvilinear) triangles \((a, d, e)\), \((c, e, d)\) and \((c, e, b)\).

In the second situation, for example if the curve \(\Gamma\) enters and exits by two points \(d\) and \(e\) situated on the same segment \((a, c)\), we choose a point \(f\) on the curve located between \(d\) and \(e\) (figure 19 (ii)). We define a sub-triangulation of the triangle \((a, b, c)\) as being formed by five (curvilinear) triangles \((a, b, d)\), \((b, d, f)\), \((b, f, e)\), \((b, e, c)\) and \((d, f, e)\).

The two last situations of the figure 19 (iii) and (iv)) are similar to the cases of the figure 18 (ii) and (i) respectively.

In the four situations, the choice of sub-triangulations is not unique, but the sum \(n_0 - n_1 + n_2\) remains unchanged independently of the choice.

The process continues for all the 2-dimensional simplices crossed by the curve \(\Gamma\), because the number of simplices is finite, even after the subdivision. □

3.1. The sphere case. In the following, we provide an alternating proof of the Euler’s formula for the sphere using the theorem 2.1 as the main tool, with the idea of “cutting surfaces” as an additional tool.

Let \(T\) be a triangulation of the sphere \(S^2\). We consider four curves on the sphere: The equator \(E\) (or any parallel) and three curves \(\gamma_1, \gamma_2\) and \(\gamma_3\), going from the North pole \(N\) to the curve \(E\) along to meridians. Let us denote by \(a_i\) the intersection points \(\gamma_i \cap E\), where \(i = 1, 2, 3\). Using the lemma 3.1 we can construct a subdivision \(T'\) of the triangulation \(T\) compatible with the four curves, i.e. such that the union of the four curves is a subcomplex of \(T'\). The lemma 3.1 shows that the sum \(n_0 - n_1 + n_2\) remains the same.
AN ELEMENTARY PROOF OF THE EULER’S FORMULA USING THE CAUCHY’S METHOD

Now, we cut the sphere along the curves \( \gamma_1, \gamma_2 \) and \( \gamma_3 \), in such a way that the projection provides a polygon \( K \) in the plane containing the equator (see figure 20). Notice that the projection of \( T' \) gives us a sub-triangulation \( K' \) of \( K \). We obtain a planar representation of the sphere, homeomorphic to a disc with identifications of the simplices on the boundary \( K_0 \) of \( K \) corresponding to the cuts.

The theorem 2.1 says that the sum \( n_0^T - n_1^T + n_2^T \) of the triangulation \( T \) is equal to the sum \( n_0^{K_0} - n_1^{K_0} + 1 \), where the sum \( n_0^{K_0} - n_1^{K_0} \) is calculated by the boundary of the figure. Notice that, using the same notations on the sphere and on the planar representation, the vertex \( N \) is common to all the curves \( \gamma_i \) and must be identified. Beside this vertex \( N \), the number of vertices in each curve \( \gamma_i \) is equal to the number of the edges (see figure 20). Then for the boundary of the planar representation, we have \( n_0^{K_0} - n_1^{K_0} = +1 \) and for the triangulation, we have \( n_0^T - n_1^T + n_2^T = +2 \).

3.2. The torus case. Let \( T \) be a triangulation of the torus \( T = S^1 \times S^1 \). We choose a meridian \( M = S^1 \times \{0\} \) and a parallel \( P = \{0\} \times S^1 \). They cross at one point \( A = \{0\} \times \{0\} \). Observing that, without loss of generality, we can choose them transversally to all the edges (1-dimensional simplices of \( T \)). We define a sub-triangulation \( T' \) of \( T \), in the following way (see figure 21): Each triangle \( \sigma \) (2-dimensional simplex) of \( T \) meeting \( M \) or \( P \) is divided in such the way that \( \sigma \cap M \) (or \( \sigma \cap P \)) is an edge of \( T' \). The lemma 3.1 shows that the sum \( n_0 - n_1 + n_2 \) remains the same for \( T \) and \( T' \).
Now, cutting the torus along \( M \) and \( P \), we obtain a planar representation \( K \) of the torus which is homeomorphic to a disc, with identifications on the boundary \( K_0 \) corresponding to the cuts. Hence, by the lemma \( \ref{lem3.1} \) the number \( n_0 - n_1 + n_2 \) remains unchanged. Using the theorem \( \ref{thm2.1} \) we have:

\[
 n_0^T - n_1^T + n_2^T = n_0^{T'} - n_1^{T'} + n_2^{T'} = n_0^K - n_1^K + n_2^K = n_0^{K_0} - n_1^{K_0} + 1.
\]

Now, with the identifications in the boundary \( K_0 \), we have \( n_1^{K_0} = n_0^{K_0} + 1 \). Finally

\[
 n_0^T - n_1^T + n_2^T = 0
\]

for any triangulation of the torus.

The same proof holds for the torus of genus \( g \). For example, let us take a torus of genus 3. Given a triangulation \( T \) of the torus, we fix a point \( A \) and around each "hole" of the torus, we fix a "meridian" (\( a_1, a_2, a_3 \) in the figure \( \ref{fig23} \)) and a "parallel" (\( b_1, b_2, b_3 \) in the figure \( \ref{fig23} \)). We construct a sub-triangulation \( T' \) of \( T \) by the same method than the case of the torus. Cutting the torus of genus 3 along the meridians and the parallels, we obtain a planar representation of the torus of genus 3, triangulated by the triangulation.
$T'$. By the same calculus than the torus case, one obtains $n_0 - n_1 + n_2 = -4$. That is an example of the Lhuilier’s formula \[1.4\]

![Figure 23. Planar representation of the torus of genus 3. In order to have a light figure, we do not draw here a triangulation $T'$ in such a way that each edge $a_i$ and $b_i$ are subdivided in at least three segments.](image)

3.3. **The projective plane case.** The projective plane is represented by a sphere whose diametrically opposite points are identified. A triangulation of the projective plane is given by a triangulation of the sphere which is symmetric with respect to the center of the sphere. Let us consider the sphere in $\mathbb{R}^3$ (see figure \[24\](a)) and let $T$ be such triangulation of the projective plane.

The intersection of $T$ with the equator defines a triangulation $J$ of the equator, that is symmetric with respect to the center of the sphere. Let us define a sub-triangulation $T'$ of $T$ such that simplices of $J$ are simplices of $T'$ and such that $T'$ is symmetric with respect to the center of the sphere (see figure \[24\](b)). By the lemma \[3.1\] the sum $n_0 - n_1 + n_2$ is the same for $T$ and $T'$.

![Figure 24. Triangulation $T$ of the projective plane Sub-triangulation $T'$ of $T$.](image)
Now, the orthogonal projection of the northern semi-sphere to the plane $0xy$ provides a triangulation of the disc $D$ of radius 1, centered at the origin, whose triangulation of the boundary is symmetric with respect to the center of the disc. With the identification of simplices, we have $n_0 - n_1 = 0$ on the boundary. Then, by the theorem 2.1 we have

$$n_0^T - n_1^T + n_2^T = +1$$

for any triangulation of the projective plane.

3.4. The Klein bottle case. The case of Klein bottle is similar to the case of torus. Given a triangulation $T$ of the Klein bottle, we choose a meridian $M$ a parallel $P$. Let us define a sub-triangulation $T'$ of $T$ compatible with $M$ and $P$. The cut along $M$ and $P$ provides a planar representation of the Klein bottle as a rectangle triangulated with identifications on the boundary. On the boundary, we have $n_1 = n_0 + 1$. Then, by the theorem 2.1 we have

$$n_0^T - n_1^T + n_2^T = 0$$

for any triangulation of the Klein bottle.
3.5. **The pinched torus case.** Not every surface with singularities admits a planar representation. The pinched torus is an example of a singular surface admitting such planar representation.

Let us recall that the pinched torus is a surface in \( \mathbb{R}^3 \) defined by the following cartesian parameterization:

\[
\begin{align*}
  x &= (r_1 + r_2 \cos(v) \cos\left(\frac{1}{2}u\right)) \cos(u) \\
  y &= (r_1 + r_2 \cos(v) \cos\left(\frac{1}{2}u\right)) \sin(u) \\
  z &= r_2 \sin(v) \cos\left(\frac{1}{2}u\right)
\end{align*}
\]

where \( r_1 \) and \( r_2 \) are respectively the big and small radii.

Let \( T \) be a triangulation of the pinched torus. We choose a “parallel” \( P \) passing by the singular point \( A \) of the pinched torus and we define a sub-triangulation \( T' \) of \( T \) compatible with \( P \) using the lemma 3.1. By cutting along \( P \), one obtains a planar representation of the pinched torus (figure 29) with identifications on the boundary. One observes that the point \( A \) is duplicated. On the boundary, we have \( n_0 - n_1 = 0 \). Then, by the theorem 2.1, we have

\[
n_0^T - n_1^T + n_2^T = +1
\]

for any triangulation of the pinched torus.
Figure 29. A planar representation of the pinched torus.

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