Abstract:
Compactifications on tori may seem to have revealed their beauty long ago but the mystery of 11d Supergravity remains and fresh attempts at a conceptual breakthrough are worth the effort and quite timely. We shall concentrate here on the analogy with Instanton Mathematics and discuss some open questions and work in progress.
1 Introduction

This short presentation covers only one section of the talk and has been expanded accordingly. We refer to other conference talks by the author for other aspects. It has been a fascinating task to rewrite the equations of toroidally compactified maximal supergravity as self-duality equations similar to the celebrated Instanton equations of four dimensional Yang-Mills theory. Let us recall that the latter are first order equations obeyed by a subset of the solutions to the full system of classical equations which are usually written as second order equations in the gauge potentials. These equations require euclidean or (2,2) signature in four dimensions.

In two dimensions analogous self-duality equations can also be defined for sigma models with target space a K"ahler manifold (N=2), and if one considers again euclidean signature one needs a “twisted” self-duality equation to compensate for the minus sign in the square of the Hodge dualisation. Depending on the signature of spacetime and in particular on the dimension there is sometimes the need and always the possibility of a twist of the self-duality equations by an invariant operator of the symmetry group of square $\pm 1$ and it will appear below.

The typical equations of relativistic physics are second order but they can always be rewritten as first order systems. In fact the main result here will be to rewrite the bosonic matter equations of all massless forms of toroidally compactified 11d Supergravity in first order form so as to become self-duality equations.

This was successively realised for the vectors in 4d N=8 SUGRA, for the symmetric space sigma models obtained after reduction to two dimensions of all SUGRA theories and it generalises independently of supersymmetry there, then for the dilaton field of curved space sigma models reduced from three to two dimensions (this allows to enlarge the symmetry from the affine algebra to its extension by the diffeomorphisms of the circle) and finally to all forms and all dimensions of M-gravity including odd dimensions of spacetime.

2 Supersymmetry and Instantons

There is a deep connection between self-duality equations and existence of unbroken supersymmetry. Let us recall that the so-called BPS condition started life as a solvable limit of electromagnetic dyon solutions where the similarity between the adjoint Higgs field and a spacelike extra component of the Yang-Mill potential becomes exact, there was no explicit fermion in this picture. It is the stationary version of the famous instantonic self-duality equation of pure Euclidean Yang-Mills theory.

Subsequently and case by case a suitable supersymmetric extension of each theory admitting “self-dual” solutions was constructed in which a Killing spinor is a covariantly constant spinor can be interpreted as an unbroken supersymmetry of the bosonic background which implies the saturation of the Bogomol’ny bound. The first analysis of this phenomenon was given in [1] in the case of rigid supersymmetry. So a bosonic self-duality equation becomes the condition of preservation of some supersymmetry and stability can be reinterpreted as the property of the supersymmetry algebra that some bosonic generators are squares of fermionic ones. This is not a totally well defined algorithm as the Killing spinor equation may involve covariantisation terms beyond the Lorentz connection. One should now attack the problem head on and analyze the possibility to embed any bosonic “self-duality” problem in a larger theory, possibly with more bosonic fields as
well, and to decide a priori the maximal possible number of supersymmetries that can be realised once the bosonic field content has been decided. In the bosonic case the converse of dimensional reduction has been first coined group disintegration and then oxidation, we are advocating now the study of natural supersymmetrisation ie a superoxidation mechanism in other words the maximal addition to a bosonic theory of fermionic dimensions whenever possible. One can ask the same question in the study of calibrations and make some progress there.

In a way the next sections address the opposite problem: we are going to show that all the bosonic matter equations of toroidally compactified 11d SUGRA can be rewritten as self-duality equations of a generalised but universal type once one doubles the field content. It is a standard procedure in the analysis of differential systems to introduce auxiliary variables to render the system first order. The nontrivial observation maybe is now that our rather intricate systems are always defined by a finite dimensional superalgebra (ie $\mathbb{Z}_2$-graded Lie algebra) and have a universal form. The occurrence of fermionic symmetries is amusing for bosonic equations but can be understood from the odd character of odd degree gauge potentials like the three form of 11d SUGRA \cite{4}.

We shall call self-duality equation any equation relating some curvatures $F$ and of the form

$$F = \ast S F,$$

where $S$ is an operator of square plus or minus one that compensates for the same property of the Hodge duality, more precisely $S$ exchanges the generators of the superalgebra associated to gauge potentials and those associated to their magnetic duals.

### 3 Middle dimension

The prototype example is of course the 4d Maxwell equations written in terms of electric and magnetic potentials with dual field strengths. Similarly in 2d the principal sigma model or more generally the symmetric space sigma models can be rewritten in the above form, at least for the propagating degrees of freedom. We recall that the typical structure is that of a coset space $KG \backslash G$ where $KG$ is the maximal compact subgroup of $G$. There are two descriptions, first the gauge fixed one where one chooses a representative of each coset but the better one restores the $KG$ gauge invariance and allows the symmetry under $G$ to become manifest. In the latter case however the self-duality (in 2d at this stage) involves the components of the field strength orthogonal to $KG$ only

$$F = (dg.g^{-1})^\perp.$$

In fact a harmonic scalar function and its conjugate form a first order self dual pair and one can render the $SL(2,R)$ invariance subgroup of the 2d conformal group manifest by the same trick.

The main example that led to my conviction that self-duality was a general feature is the case of the 28 vector potentials of 4d $N = 8$ SUGRA that cannot form a representation of the duality symmetry group $G = E_7(7)$ unless one combines them with the 28 (Hodge) duals. The scalar fields in that theory obey the equations of the sigma model $KG \backslash G$ again and the self-duality equation for the vectors reads in that case

$$g.F = \ast Sg.F,$$
where \( g \) stands for the 56 dimensional matrix representation of \( G \) and \( S \) has to be an invariant operator for \( KG = SU(8) \) \([3]\). This structure has been extended to the compactifications of 11d SUGRA on a 3-torus and on a 5-torus in \([4]\ and references therein for the field strengths of degree half that of the spacetime volume form.

4 All forms

From there it was natural to try an extension to all fields, and we succeeded for all bosonic forms leaving aside for the time being the graviton and the fermions. We expect the fermions to transform only under the compact subgroup \( KG \) and under the Lorentz (spin) group. We are now going to exhibit a vast generalisation of \( G \) or at least of its Borel subgroup. This is quite typical of broken symmetries in polynomial situations in which the components of some group element \( g \) appear also polynomially in its inverse \( g^{-1} \) which occurs also as we have seen in the equations. The way to permit this is of course nilpotence and this is why the coset spaces appear usually in their Iwasawa parametrisation. One must restore the local \( KG \) invariance to have simple formulas for the fermionic couplings and for the full action of \( G \). We refer to \([3]\) for the compactified cases but we shall illustrate our general structure with the 11 dimensional example; the 4-form field strength has a dual that has a non abelian piece. A compact way to encode the equation of motion and the Bianchi identity is to define a supergroup element and its field strength or curvature by

\[
E = \exp(A_3 T) \exp(A_6' T')
\]

\[
F = dEE^{-1}.
\]

This is a generalised sigma model structure, one pair of generators for each form and its dual, a theory is then specified by the choice of a super Lie algebra law. The action of the involution \( S \) is simply the exchange of \( T \) and \( T' \). 11d SUGRA is defined by the superalgebra

\[
\{ T, T \}_+ = T'
\]

and its equations of motion are simply (1).

5 Partial integrability of gravitational theories

In two dimensions the so-called totally integrable Hamiltonian systems are the compatibility conditions for linear (Lax) systems that generate infinite systems of commuting conservation laws and allow a non-local transformation to action-angle variables. They usually involve a scattering parameter. The four dimensional self-dual Yang-Mills equations do admit such a Lax pair where in effect the scattering parameter breaks the 4d covariance by selecting the corresponding anti-self-dual null plane with flat connection. Various truncations relate 4d self-dual Yang-Mills equations to most 2d Lorentz invariant integrable systems. One such system is the reduction of pure Einstein gravity from 4 to 2 dimensions on a compact torus by ignoring the torus coordinates, the so-called Ge- roch group acts again nonlocally on solutions or more precisely on a space of potentials that covers the space of solutions. On the other hand there is chaos in some sectors of
Einstein theory and the coexistence of integrability and chaos is presumably due to the noncompactness of the symmetry groups involved.\footnote{Indeed negative curvature spaces witness chaotic geodesic flows. In this eprint version we comment that the Coxeter group of the overextension \footnote{The overextension of a Dynkin diagram is its affinisation followed by the addition of yet one more vertex with a single bond to the affine one.} of the 3-dimensional duality group controls the BKL chaotic cosmologies \footnote{The overextension of a Dynkin diagram is its affinisation followed by the addition of yet one more vertex with a single bond to the affine one.}.}

An important question is to examine the “integrability” of our twisted self-dual systems, one problem of course the lack of covariance of the known regular Hamiltonian formulations. It should be distinguished from the same question for 4d gravitational instantons which has been answered in the affirmative by R. Ward. We may also recall that the linearised gravitino equations of N=1 SUGRA have Einstein’s equations as compatibility condition.

6 Magic triangle

We would very much like to understand better the origin of the duality symmetries. At the classical level and in the low energy approximation to M-theory one understands some aspects of these exceptional groups for instance the regularity in the rank \( r \) of the group for instance \( r \) is \((11-d)\) in the maximal SUGRA case (listed as \( N=7 \) below despite the usual denomination \( N=8 \) for a number \( 4N \) of supersymmetry charges). It was noticed in 1980 that there is a similar regularity at fixed spacetime dimension and varying number \( N \) of quartet of supersymmetry generators. There is an approximate symmetry of the duality groups under the interchange of \( 8-N \) and \( d-2 \), for instance the \( N=7 \) series of \( E_r \) groups which are the split real forms are replaced in dimension 3 by other real forms of the same complex Lie groups namely \( E_{N+1}(C) \).

It turns out \footnote{Indeed negative curvature spaces witness chaotic geodesic flows. In this eprint version we comment that the Coxeter group of the overextension \footnote{The overextension of a Dynkin diagram is its affinisation followed by the addition of yet one more vertex with a single bond to the affine one.} of the 3-dimensional duality group controls the BKL chaotic cosmologies \footnote{The overextension of a Dynkin diagram is its affinisation followed by the addition of yet one more vertex with a single bond to the affine one.}.} that the Magic is maximal if one gets the inspiration to try to oxidise curved space three dimensional sigma models for split real form noncompact symmetric spaces, in other words for the scalar manifolds of toroidally compactified 11d SUGRA. Doing this one abandons in general supersymmetry that was related to the other real forms mentioned above of \( E_{N+1}(C) \). However the oxidation triangle one obtains by lifting these three dimensional sigma models when possible to higher dimensions is completely symmetric under the above symmetry exchanging dimension and SUSY number. The reason is not clear yet but is being actively searched for.

In fact these symmetries are low energy symmetries. It is believed that at the quantum level the discreteness of the charges in 4 dimensions for instance should break the symmetry down to an arithmetic subgroup. This is a generalisation of the breaking of \( SL(2,R) \) to \( SL(2,Z) \) that occurs for instance in type IIB SUGRA in 10 dimensions.

7 del Pezzo surfaces

Finally let us point out that the series of groups \( E_k \) for \( k = 3, \ldots, 8 \), namely \( A_1 \times A_2, A_4, D_5, E_6, E_7, E_8 \) is well known in Mathematics, their Cartan matrices arise as the intersection matrices of the exceptional divisors on some projective smooth complex surface obtained from the complex projective plane by blowing up a finite number of suitably placed points. Technically one usually requires the singular points to be in general position and the “anticanonical sheaf to be ample”.

The cases \( k = 0, 1, 2 \) are less obvious and the factor \( R \) of scaling symmetries disappears both in the discrete quantum theory and probably in the algebraic geometry, or at least it has a more subtle interpretation in both cases. Nevertheless the IIA and IIB U-duality
groups seem to correspond also to two del Pezzo surfaces of degree 8 (private communication of C. Vafa) and the projective plane $CP^2$ has degree 9. A detailed discussion can be found for instance in [8]. The degree $D$ is in fact the degree of the projective surface for instance the smooth cubic surface of degree 3 has 27 lines and is associated to the Cartan matrix of $E_6$, more generally the rank of the group is $r = 9 - D = 11 - d$ ie $d - 2 = D$ (one could add by magic symmetry $D = 8 - N!$).

Work is in progress to understand also the other columns of the triangle $(N = 1, ..., 6)$ in this way. They follow very similar patterns and do seem to come together. One would also like to relax the hypotheses to allow for the affine groups corresponding to two dimensions of spacetime and maybe more...

As usual with ADE problems the Cartan matrix is a rather universal object but it is sometimes difficult to relate two instances of the occurrence of simply laced groups. One example is the Grothendieck-Brieskorn realisation of ADE singularities so to speak on the groups themselves, Arnold has been the champion of a systematic study of other cases. Here we have been enlarging the Borel subgroup of $E_r$ to a rather big Lie superalgebra, its connection with algebraic geometry remains to be discovered.

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