The Cost of Two-dimensional Rearrangement

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INTRODUCTION Consider a two dimensional torus $T^2 = \mathbb{R}^2 / \mathbb{Z}^2$, with coordinates $x = (x_1, x_2) \in [0, 1) \times [0, 1)$. Let $A = \{(x_1, x_2) | 0 \leq x_2 < 1/2\} \subset T^2$ be a subset, an diffeomorphism $\Phi : T^2 \to T^2$ is called a rearrangement of $A$.

We say that $\Phi$ mixes the set $A$ up to scale $\varepsilon$ if the following holds: there is a fixed real number $\kappa \in (0, 1/2)$, for any ball $B_\varepsilon(x)$ centered at a point $x \in T^2$ with radius $\varepsilon$, we have

$$\kappa \text{Area}(B_\varepsilon(x)) \leq \text{Area}(B_\varepsilon(x) \cap \Phi(A)) \leq (1 - \kappa)\text{Area}(B_\varepsilon(x)) \quad (1)$$

For the rearrangement $\Phi$, let

$$e(\Phi) = \frac{1}{2}[(\Phi_{x_1})^2 + (\Phi_{x_2})^2 + (\Phi_{x_1})^2 + (\Phi_{x_2})^2]$$
be the energy density, and we define the cost of the rearrangement as the energy of $\Phi$:

$$E(\Phi) = \int_{\mathbb{T}^2} e(\Phi) d\sigma.$$ 

The main result in this paper is the following theorem.

**Theorem.** Let $\Phi(x) : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is a diffeomorphism and it mixes the set $A$ up to scale $\varepsilon$. If $\Phi$ satisfies

$$0 < \kappa' \leq |\text{det}[\nabla_x \Phi]|$$

Then there exists a constant $C$ which depends on $\kappa, \kappa'$ only, such that

$$E(\Phi) \geq \frac{C}{\varepsilon^2}.$$ 

If $F : [0, 1] \times \mathbb{T}^2 \rightarrow \mathbb{R}^2$ is a time dependent smooth vector field on $\mathbb{T}^2$, and $\Phi_t : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ the flow associated with the vector field $F$, i.e., $\Phi_t$ is the solution of the following initial value problem

$$\begin{cases}
\dot{\Phi}_t(x) = F(t, \Phi_t(x)) \\
\Phi_0(x) = x
\end{cases}$$

Let $\Phi_1(\cdot) = \Phi(\cdot)$ be the value of the flow at time $t = 1$.

In [Br], Bressan made the following conjecture:

**Conjecture.** If the flow $\Phi_t$ generated by smooth vector field $F$ is nearly incompressible, i.e., for some constant $\kappa' > 0$, we have

$$\kappa' \text{Area}(\Omega) \leq \text{Area}(\Phi_t(\Omega)) \leq \frac{1}{\kappa'} \text{Area}(\Omega),$$

for any measurable set $\Omega \subset \mathbb{T}^2$ and $t \in [0, 1]$, and $\Phi$ mixes the set $A$ up to scale $\varepsilon$, then there is a constant $C$ depends on $\kappa$ and $\kappa'$ only, such that

$$\int_0^1 \int_{\mathbb{T}^2} |\nabla_x F| d\sigma dt \geq C |\log \varepsilon|.$$ 

As a corollary, we will prove the following corollary which is in the same manner as Bressan’s conjecture.

**Corollary.** Let $F = F(t, x)$ be a smooth vector field on $\mathbb{T}^2$, and assume that the associated flow $\Phi_t(x)$ satisfies

$$0 < \kappa' \leq |\text{det}[\nabla_x \Phi_t]|,$$

and $\Phi = \Phi_1$ mixes the set $A$ up to scale $\varepsilon$. Then there exists constant $C$ depends on $\kappa, \kappa'$ only, such that

$$\frac{C}{\varepsilon^2} \leq \int_0^1 \int_{\mathbb{T}^2} e^{\sqrt{\text{Area}(\Omega)}} |\nabla_x F| d\sigma dt$$
Proof of the Main Theorem and Corollary

We first prove the theorem.

**Theorem.** Let \( \Phi(x) : T^2 \to T^2 \) is a diffeomorphism and it mixes the set \( A \) up to scale \( \varepsilon \). If \( \Phi \) satisfies

\[
0 < \kappa' \leq |\det[\nabla_x \Phi]|
\]

Then there exists a constant \( C \) which depends on \( \kappa, \kappa' \) only, such that

\[
E(\Phi) \geq \frac{C}{\varepsilon^2}.
\]

**Proof.** On the set \( A \subset T^2 \), we have

\[
\int_A e(\Phi)d\sigma \geq \frac{1}{2} \int_0^{1/2} \left( \int_0^1 [(\Phi^1_{x_1})^2 + (\Phi^2_{x_1})^2]dx_1 \right) dx_2,
\]

and by Hölder inequality we have

\[
\int_0^1 [(\Phi^1_{x_1})^2 + (\Phi^2_{x_1})^2]dx_1 \geq \left( \int_0^1 [(\Phi^1_{x_1})^2 + (\Phi^2_{x_1})^2]^{1/2} dx_1 \right)^2
\]

If we fix \( x_2 = s \), then we can view \( \Phi(x_1, s) \) as a curve \( C_s \subset T^2 \). Thus,

\[
l(s) = \int_0^1 [(\Phi^1_{x_1}(x_1, s))^2 + (\Phi^2_{x_1}(x_1, s))^2]^{1/2} dx_1
\]

is the length of the curve \( C_s \). Then

\[
\int_A e(\Phi) \geq \frac{1}{2} \int_0^{1/2} l^2(s)ds = \frac{1}{2} \int_0^{1/4} \left[ l^2 + l^2 \left( \frac{1}{2} - s \right) \right] ds
\]

\[
\geq \frac{1}{4} \int_0^{1/4} \left[ l(s) + l \left( \frac{1}{2} - s \right) \right]^2 ds.
\]

Let \( A_s = \{(x_1, x_2) | s < x_2 < 1/2 - s \} \subset T^2 \), and \( l(s) + l(1/2 - s) \) is just the length of boundary \( \partial(\Phi(A_s)) \). Now we will estimate the length of \( \partial(\Phi(A_s)) \) in terms of \( \varepsilon \).

For a given point \( y \in \Phi(A_s) \), let \( B_\varepsilon(y) \) be the ball centered at \( y \) with radius \( \varepsilon \), if \( \kappa \pi \varepsilon^2 > \text{Aera}(\Phi(A_s) \cap B_\varepsilon(y)) \), then for any \( r > \varepsilon/\sqrt{2} \), \( \partial B_r(y) \) has at least two points intersects with \( \partial(\Phi(A_s)) \), and this will imply that

\[
\text{length}(\partial(\Phi(A_s)) \cap B_\varepsilon(y)) \geq 2(\varepsilon - \varepsilon/\sqrt{2}) = (2 - \sqrt{2})\varepsilon.
\]

Otherwise, \( \partial(B_r(y)) \cap \partial(\Phi(A_s)) = \emptyset \). Two curves in \( \partial(\Phi(A_s)) \) are both homologically non-trivial, and \( y \in B_r(y) \cap \Phi(A_s) \), so we must have \( B_r(y) \subset \Phi(A_s) \). This contradicts to the assumption \( \kappa \pi \varepsilon^2 > \text{Aera}(\Phi(A_s) \cap B_\varepsilon(y)) \).
If $\kappa \pi \varepsilon^2 \leq \text{Area}(\Phi(A_s) \cap B_\varepsilon(y))$, by highly mixing condition on $\Phi$, we have

$$\kappa \pi \varepsilon^2 \leq \text{Area}(\Phi(A_s) \cap B_\varepsilon(y)) \leq \text{Area}(\Phi(A) \cap B_\varepsilon(y)) \leq (1 - \kappa)\pi \varepsilon^2.$$ 

The minimal curve which separates two regions of areas $\text{Area}(\Phi(A_s) \cap B_\varepsilon(y))$ and $\pi \varepsilon^2 - \text{Area}(\Phi(A_s) \cap B_\varepsilon(y))$ is a circular arc perpendicular to $\partial B_\varepsilon(y)$, so there is a constant $m'_\kappa$ depends on $\kappa$ only, such that

$$\text{length}(\partial(\Phi(A_s)) \cap B_\varepsilon(y)) \geq m'_\kappa \varepsilon.$$ 

Combine these two cases together, let $m_\kappa = \min\{m'_\kappa, (2 - \sqrt{2})\}$, we get a low bound estimation for the length of $\partial(\Phi(A_s)) \cap B_\varepsilon(y)$:

$$\text{length}(\partial(\Phi(A_s)) \cap B_\varepsilon(y)) \geq m_\kappa \varepsilon.$$ 

Now we pack the set $\Phi(A_s)$ by a maximal set of balls $\{B_\varepsilon(y_i) | y_i \in \Phi(A_s)\}$ and any two balls in the set are disjoint. Let $n$ be the number of balls in this maximal set. We have

$$l(s) + l\left(\frac{1}{2} - s\right) \geq m_\kappa n \varepsilon.$$ 

On the other hand, we notice that balls $\{B_{2\varepsilon}(y_i)\}$ will cover $\Phi(A_s)$. If not, suppose $y_0 \in \Phi(A_s)$ cannot be covered by $\{B_{2\varepsilon}(y_i)\}$, that means the distance between $y_0$ and all the $B_\varepsilon(y_i)$ is larger than $\varepsilon$. It contradict the maximality of $\{B_\varepsilon(y_i) | y_i \in \Phi(A_s)\}$. Thus we have

$$4\pi \varepsilon^2 n \geq \text{Area}(\Phi(A_s)).$$ 

By condition that $\kappa' < |\det(\nabla_x \Phi)|$, we have

$$\text{Area}(\Phi(A_s)) \geq \kappa' \text{Area}(A_s) = \kappa'(1/2 - 2s).$$
Then

$$4\pi \varepsilon^2 n \geq \kappa'(1/2 - 2s).$$

Hence

$$l(s) + l\left(\frac{1}{2} - s\right) \geq \frac{\kappa'm_\kappa}{4\pi \varepsilon} \left(\frac{1}{2} - 2s\right),$$

and then

$$\int_A e(\Phi)d\sigma \geq \frac{1}{4} \int_0^\frac{1}{4} \left[\frac{\kappa'm_\kappa}{4\pi \varepsilon} \left(\frac{1}{2} - 2s\right)\right]^2 ds = \frac{1}{48} \left(\frac{\kappa'm_\kappa}{8\pi \varepsilon}\right)^2.$$ 

When \(\Phi\) mixes set \(A\) up to scale \(\varepsilon\), it mixes \(\mathbb{T}^2 - A\) as well. Similarly, we have

$$\int_{\mathbb{T}^2 - A} e(\Phi) \geq \frac{1}{48} \left(\frac{M_\kappa \kappa'}{8\pi \varepsilon}\right)^2.$$ 

Thus

$$\int_{\mathbb{T}^2} e(\Phi) \geq \frac{1}{24} \left(\frac{M_\kappa \kappa'}{8\pi \varepsilon}\right)^2 = \frac{C}{\varepsilon^2}$$

(11)

where \(C\) depends on \(\kappa, \kappa'\) only, and we complete the proof of the theorem.

Now we turn to the proof of the corollary.

**Proof of the Corollary.** We take differentiation with respect to \(x\) on both sides of the ordinary differential equation \(\dot{\Phi}_t(x) = F(t, \Phi_t(x))\). According to chain rule, we get

$$\partial_t(\nabla_x \Phi_t(x)) = \nabla_x F(t, \Phi_t(x)) \nabla_x \Phi_t(x).$$

Let

$$|\nabla_x F| = \left(\sum_{i,j=1}^2 (\partial_{x_i} F^j)^2\right)^{1/2}$$
be the variation of the vector field \(F\). Then we have

$$\partial_t e(\Phi_t) = \sum_{i,j=1}^2 (\partial_{x_i} \Phi^j_t)(\partial_t (\partial_{x_i} \Phi^j_t)) \leq \sqrt{6} |\nabla_x F(t, \Phi_t(x))| e(\Phi_t).$$

Then

$$\partial_t \log e(\Phi_t) = \frac{\partial_t e(\Phi_t)}{e(\Phi_t)} \leq \sqrt{6} |\nabla_x F(t, \Phi_t(x))|.$$

Integrating over 0 to 1 from both side we obtain

$$\log e(\Phi) \leq \sqrt{6} \int_0^1 |\nabla_x F(t, \Phi_t(x))| dt.$$

By Jensen’s inequality

$$e(\Phi) \leq \exp \left[\sqrt{6} \int_0^1 |\nabla_x F(t, \Phi_t(x))| dt \right] \leq \int_0^1 e^{\sqrt{6} |\nabla_x F(t, \Phi_t(x))|} dt.$$
Integrating over $\mathbb{T}^2$ we have

\[
\int_{\mathbb{T}^2} e(\Phi) d\sigma \leq \int_0^1 \int_{\mathbb{T}^2} e^{\sqrt{6}\|\nabla_x F(t, \Phi_t(x))\|} d\sigma dt \\
\leq \int_0^1 \int_{\mathbb{T}^2} e^{\sqrt{6}\|\nabla_x F(t, \Phi_t(x))\|} |\det(\nabla_x \Phi_t)|^{-1} d\sigma dt \\
\leq \frac{1}{\kappa'} \int_0^1 \int_{\mathbb{T}^2} e^{\sqrt{6}\|\nabla_x F(t, x)\|} d\sigma dt. \tag{12}
\]

This completes the proof of the corollary. \qed

**Reference**

[Br] A. Bressan, A lemma and a conjecture on the cost of rearrangements, *Rend. Sem. Mat. Univ. Padova*, **110**(2003), 97-102.