Bubble Dynamics Equations in Newton Fluid

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Abstract. For the high-speed flow of Newton fluid, bubble is produced and expanded when it moves toward the surface of fluid. Bubble dynamics is a very important research field to understand the intrinsic feature of bubble production and motion. This research formulates the bubble expansion by expansion-local rotation transformation, which can be calculated by the measured velocity field. Then, the related dynamic equations are established to describe the interaction between the fluid and the bubble. The research shows that the bubble production condition can be expressed by critical vortex value and fluid pressure; and the bubble expansion rate can be obtained by solving the non-linear dynamic equation of bubble motion. The results may help the related research as it shows a special kind of fluid motion in theoretic sense. As an application example, the nanofiber radium-voltage relation and threshold voltage-surface tension relation in electrospinning process are discussed.

1. Introduction

The bubble in incompressible flow is common when the flow has rotational component. Usually, the small bubbles will be combined into big bubble. The bubble usually moves to the boundary of fluid and even may stay at there. For a pipe transportation system, this will greatly reduce the efficiency of the system.

Although the bubble motion is common in daily life also, its motion equation had not been established well. This problem is caused by two problems. One problem is the exact mathematical representation of rotation flow; another problem is the condition of bubble production. Only after these two problems are solved, the motion equation of bubble can be rationally established.

Bubble production in compressible fluid is strikingly related with the local rotation flow. The produced bubble will move toward the boundary of fluid or free surface. Its moving speed and direction is determined by the fluid velocity field, bubble size, and the intrinsic parameter of fluid feature. As there are many available experimental and theoretic researches on the motion of bubble, a concise theoretic formulation should be constructed with sufficient confidence. Based on the Navier-Stokes equation, for Newton fluid, using Chen’s S+R decomposition [1], the condition of bubble production is firstly established. Then, the motion equation of bubble is derived from the Navier-Stokes equation. As an example, some typical phenomena of bubble motion in compressible flow for Newton fluid are discussed.

2. Bubble Velocity Field Description

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For a bubble, if one establishes Lagrangian coordinators for the bubble boundary, then the motion of bubble can be expressed by the instant gauge field variation. Taking the coordinators on the bubble rotational plane as \((x^1, x^2)\) and the normal of rotation plane as \((x^3)\), then a bubble Lagrangian coordinators system can be defined with instant basic gauge vectors \(\hat{g}_i(x,t)\). Referring to the Newton fluid configuration at a special time \(t = 0\), the current basic gauge vectors can be expressed by the initial basic gauge vectors through introducing points-set transformation [1] tensor \(F^j_i(x,t)\) defined by:

\[
\hat{g}_i(x,t) = F^j_i(x,t)\hat{g}_j(x,0)
\]

(1)

From geometrical field point to see, the bubble expansion means that the bubble instant motion can be expressed by the rotational motion tensor [2] \(R^j_i(x,t)\) defined as:

\[
F^j_i(x,t) = \frac{1}{\cos\theta(x,t)}R^j_i(x,t)
\]

(2)

Where, \(\theta\) is the instant local rotational angular, \(R^j_i\) is an orthogonal rotation tensor. For a bubble, on average sense, the bubble velocity field can be approximated as:

\[
\frac{\partial \hat{u}^1}{\partial x^1} = \frac{\partial \hat{u}^1}{\partial x^3} = 0, \quad \frac{\partial \hat{u}^2}{\partial x^2} = 0, \quad \frac{\partial \hat{u}^1}{\partial x^3} + \frac{\partial \hat{u}^2}{\partial x^1} = 0
\]

(3-1)

\[
\frac{\partial \hat{u}^2}{\partial x^2} = \frac{\partial \hat{u}^2}{\partial x^3} = 0, \quad \frac{\partial \hat{u}^3}{\partial x^1} = \frac{\partial \hat{u}^3}{\partial x^2} = 0
\]

(3-2)

Where, \(u^i\) is the velocity field of bubble defined by a suitable time scale \(T\). In this geometrical description, the bubble is approximated as a continuum and the initial gauge field is selected as standard rectangular coordinator system. This is supported by actual observation. Hence, it is viewed as a fact. Defining the local rotation angular [2] \(\theta(x,t)\) \((\theta < \frac{\pi}{2})\) by equation:

\[
\left(\frac{1}{\cos\theta}\right)^2 = 1 + \frac{1}{4}\left(\frac{\partial \hat{u}^1}{\partial x^3} - \frac{\partial \hat{u}^2}{\partial x^2}\right)^2
\]

(4)

As the local rotation on bubble rotation plane can be expressed as:

\[
R^j_i = \begin{bmatrix}
\cos\theta & \sin\theta & 0 \\
-\sin\theta & \cos\theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

(5)

Through mathematic treatment, it is easy to find out that the Equation (2) will require that:

\[
\frac{\partial \hat{u}^3}{\partial x^1} = \frac{1}{\cos\theta} - 1
\]

(6)

It means that, referring to the Newton fluid, the bubble has a velocity gradient on its normal direction of local rotational plane. For a bubble, on average sense, its gauge field is determined by the local rotation angular as:

\[
g_{ij} = \left(\frac{1}{\cos\theta}\right)^2 g_{ij}^0
\]

(7)

**3. Stress on Bubble Surface and Its Balance Equation**

On the other hand, as the Euler rotation angular \(\alpha\) of bubble surface referring to fluid element is determined by the equation:

\[
\sin\alpha = \frac{1}{2}\left(\frac{\partial V^1}{\partial x^2} - \frac{\partial V^2}{\partial x^1}\right)
\]

(8)

Where, \(V^i\) is the velocity of fluid element on bubble surface. Roughly, it is referred as local fluid vortex. On this bubble-fluid surface, the tangent velocity field should be continuous. Hence, one has geometrical continuity equation:
\[
\left(\frac{1}{\cos \theta}\right)^2 = 1 + (\sin \alpha)^2
\]  
(9)

The \( \alpha \) can be calculated by Newton fluid velocity field at the bubble boundary. On this sense, the observational bubble expansion data supplies the value for the calculation of vortex of Newton fluid. For Newton fluid, introducing the physical constant \( \lambda, \mu \) and static pressure \( p \), the stress field on bubble surface can be defined as:

\[
\sigma_{ij} = [(\lambda + 2\mu) \cdot \left(\frac{1}{\cos \theta} - 1\right) - p] \delta_{ij}
\]  
(10)

Inserting the Equation (9) into it, one has the bubble-fluid stress equation for bubble surface:

\[
\sigma_{ij} = [(\lambda + 2\mu) \cdot \left(\sqrt{1 + (\sin \alpha)^2} - 1\right) - p] \delta_{ij}
\]  
(11)

In mechanics, the positive stress means expansion. So, based on this equation, the bubble expansion condition is expressed by the fluid vortex:

\[
(\lambda + 2\mu) \cdot \left(\sqrt{1 + (\sin \alpha)^2} - 1\right) - p > 0
\]  
(12)

Therefore, the critical condition for bubble production is expressed by the rotational components of fluid velocity field as:

\[
(\lambda + 2\mu) \cdot \sqrt{1 + \frac{1}{4} \left(\frac{\partial V^1}{\partial x^1} - \frac{\partial V^2}{\partial x^2}\right)^2 - 1} = p
\]  
(13)

It shows that only when the vortex field of Newton fluid is big enough, the bubble can be produced. This is well observed in fluid flow phenomenon.

4. Bubble Dynamics Equation

On average sense, the Newton fluid motion is described by the Navier-Stokes equation in Euler spatial coordinator system \((X^1, X^2, X^3)\) as:

\[
\frac{\partial \sigma_{ij}}{\partial X^j} = \rho \frac{\partial V^i}{\partial t} + \rho V^j \frac{\partial V^i}{\partial X^j}
\]  
(14)

For the spatial position taken by bubbles, the stress can be expressed by the local bubble rotational angular \( \alpha(X, t) \) parameter and the local static pressure \( p(X) \). Hence, taking the Newton fluid instant configuration as the instant Lagranian coordinator system, the bubble motion equation is:

\[
(\lambda + 2\mu) \sqrt{1 + (\sin \alpha)^2} - \rho \frac{\partial V^1}{\partial t} = \rho \frac{\partial V^1}{\partial t} + \rho V^2 \frac{\partial V^1}{\partial X^2}
\]  
(15)

Using Equations (3-1), (3-2), and (6), one has approximation:

\[
(\lambda + 2\mu) \sqrt{1 + (\sin \alpha)^2} = \rho \frac{\partial V^1}{\partial t} + \rho V^2 \frac{\partial V^1}{\partial X^2}
\]  
(16-1)

\[
(\lambda + 2\mu) \sqrt{1 + (\sin \alpha)^2} = \rho \frac{\partial V^2}{\partial t} + \rho V^1 \frac{\partial V^2}{\partial X^1}
\]  
(16-2)

\[
(\lambda + 2\mu) \sqrt{1 + (\sin \alpha)^2} = \rho \frac{\partial V^3}{\partial t} + \rho V^1 \frac{\partial V^3}{\partial X^3}
\]  
(16-3)

Where, the bubble parameter is defined by equation \( \sin \alpha = \frac{1}{2} \left(\frac{\partial V^1}{\partial x^1} - \frac{\partial V^2}{\partial x^2}\right) \). This is the bubble dynamic equation for bubble moving toward \( X^3 \) direction. With suitable initial and boundary conditions, the equations can be closed. Hence, it forms a complete description about bubble motion. By bubble production condition Equation (13), the critical bubble production equations are:

\[
\frac{\partial V^1}{\partial t} + V^2 \frac{\partial V^1}{\partial X^2} = 0
\]  
(17-1)
\[
\frac{\partial V^2}{\partial t} + V^1 \frac{\partial V^2}{\partial X^1} = 0 \quad (17-2)
\]
\[
\frac{\partial V^3}{\partial t} + V^1 \frac{\partial V^3}{\partial X^1} = 0 \quad (17-3)
\]

The third Equation (17-3) has a form solution, which gives the bubble front surface motion rule as:
\[
V^3 = (B + C V^3) \cdot e^{-Ct} \quad (18)
\]

By Equation (6), the bubble size variation is:
\[
R = \frac{1}{\cos \theta} - 1 = BC \cdot e^{-Ct} \quad (19)
\]

Where, B and C are constants waiting to be determined by initial and boundary condition. For positive C, the bubble size is exponential decaying with time. For negative C, the bubble size is exponential expanding with time. This bubble size exponential decaying/expanding phenomenon is well observed in Newton fluid [3-4]. By Equations (6) and this equation, the bubble trace is on the rotational axe direction and it is straight line. This is well observed in experiments [4].

5. Comparing with Available result

For the small bubble chains, on average sense, a bubble can be viewed as a small ball with radium \( R \).

The average total stress of a ball surface contribution can be calculated by the following equation:
\[
\sigma = (\lambda + 2\mu) \cdot \left( \frac{1}{\cos \theta} - 1 \right) \cdot 4\pi R^2 \quad (20)
\]

If this stress is completely balanced with the fluid pressure and the bubble stops its expansion, one has the balance equation:
\[
(\lambda + 2\mu) \cdot \left( \frac{1}{\cos \theta} - 1 \right) \cdot 4\pi R^2 = K \cdot p \quad (21)
\]

Where, \( K \approx 1 \) is a suitable parameter related with fluid physical feature. For very small \( \theta \) variation (acoustic bubble [3]), one has approximation:
\[
\theta^2 \approx \frac{K \cdot p}{2(\lambda + 2\mu)\pi R^2} \quad (22)
\]

As the local rotational angular \( \theta \) is, physically, equivalent with the frequency multiplied by time scale \( T \), this equation is similar with the Reyleigh-Plesset equation [5] in the sense that it gives out the correct frequency relation for pressure and bubble size.

Sure, the above discussion is pure theoretically. What benefit will be available is waiting further research.

6. Application in Electrospinning of Nanofibers

In electrospinning process, initially, the fluid forms a drop at the exit of the capillary and its size is determined by surface tension \( \gamma \). When an external electrical field is applied, the coupling of surface charge and the external electrical field creates a tangential stress, resulting in the deformation of the droplet into a conical shape. Once the electrical exceeds the critical value needed to overcome the surface tension a fluid jet ejects from the apex of the cone. Taking the jet ejection as the \( x^3 \) coordinator, this process can be described by the local expansion-rotation deformation Equation (2)-(6) in this research. The local rotation is the spin defined in related researches.

Based on electrical field theory, the droplet surface charge density \( q \), the surface velocity field \( \vec{V} \), and the external electrical field \( \vec{E} \) have the Ohm law relation:
\[
\vec{E} = \frac{q}{\nu} \vec{V} \quad (23)
\]

Where, \( \nu \) is the conductivity of the bubble surface. For fixed geometrical setting in experiments, it is convenient to introduce the unit-electrical field geometrical parameter \( \omega_{x^2} \) (It is a fixed parameter...
for fixed geometrical setting in an experiment. Theoretically, it can be obtained by solving the
electrical field equation for the geometrical setting.). Then, for a dropt with radium \( R \), the electrical
field strength produced by applied voltage \( V_C \), one has:

\[
\frac{1}{2} \left( \frac{\partial E^1}{\partial x^1} - \frac{\partial E^2}{\partial x^2} \right) = \frac{V_C}{R} \cdot \partial \theta_2
\]  

(24)

For Newton fluid, introducing the physical constant \( \lambda, \mu \) and letting the inward surface pressure
difference can be approximated by the surface tension \( \gamma \) and the droplet radium \( R \) as \( \Delta p = \gamma / R \), the
critical condition for bubble production Equations (12) and (13) can be modified as the critical
condition for jet ejection, as the bubble produced in the internal fluid of droplet will produce an
expansion force. Based on the Equation (23) and Equation (12), the critical condition for jet ejection is:

\[
(\lambda + 2\mu) \left( 1 + \frac{1}{4} \frac{q}{\nu\omega} \left( \frac{\partial E^1}{\partial x^2} - \frac{\partial E^2}{\partial x^1} \right)^2 - 1 \right) > \frac{\gamma}{R}
\]  

(25)

It shows that only when the stress produce by the external electrical field is big enough to
overcome the surface tension, the jet ejection can happen.

So, the Equation (25) can be rewritten as the threshold voltage-surface tension relation in
electrospinning of nanofibers as:

\[
(V_{\text{threshold}})^2 = \left( \frac{q \cdot R}{\nu\omega^2} \right)^2 \left[ (\frac{\gamma}{\lambda + 2\mu} + 1)^2 - 1 \right]
\]  

(26)

For the real polymer fluid, \( \lambda + 2\mu \) is big enough and surface pressure difference is very small. One
has condition: \( \frac{\gamma}{(\lambda + 2\mu)R} \ll 1 \). So, the threshold voltage-surface tension relation is approximated as:

\[
(V_{\text{threshold}})^2 \approx \left( \frac{q}{\nu\omega^2} \right)^2 \frac{2R \cdot \gamma}{\lambda + 2\mu}
\]  

(27)

This equation is similar with the well-known Taylor threshold voltage-surface tension relation.

Once the jet ejection is produced, the macro deformation along the jet direction \( x^1 \) is described by
Equation (6). It means that the jet ejection has a macro velocity gradient on its jet direction. By the
geometrical meaning of fluid deformation described by the related Equations (3-1) and (3-2), the
macro deformation of jet fluid is the fiber elongation along the eject direction.

Physically, the bubble is initiated around polymer fibers. When the bubble is expanded, the vacuum
space will be taken by the polymer fiber. As the jet ejection will roughly have the bubble size as its
section size, the final fiber size is the bubble size. For a nanofiber with initial radium \( d_0 \), on average
sense, the bubble gauge field is determined by the local intrinsic rotation angular \( \theta \) as Equation (7).
So, combining with Equations (4) and (25), the nanofiber radium after jet ejection is:

\[
d = \left( \frac{1}{\cos \theta} \right) \cdot d_0 = d_0 \cdot \left( 1 + \frac{\nu\omega^2}{q} \cdot \frac{V_C}{R} \right)^{1/2}
\]  

(28)

This equation is the nanofiber radium-voltage relation for electrospinning process. It shows that
the nanofiber radium is increased with the applied voltage. This phenomenon is observed in
experiments [6]. Therefore, to reduce the fiber radium, the applied voltage should be as small as
possible. Once the required fiber radium parameter is fixed for an industrial purpose, to reduce the
threshold voltage, one must find a way to reduce the surface tension and increase the polymer fluid
viscosity. As the parameter \( \lambda \) is mainly related with the nanofiber strength and the viscosity parameter
is mainly with the solvent, hence there is no intrinsic contradicts between reducing the applied voltage
and increasing the mechanical strength of nanofibers. The related topics have been well treated by
professor He Ji-Huan [7-8] with numerical simulation and other theoretic treatment.

Combining Equations (27) and (28), one can get the nanofiber radium-voltage relation expressed
by the applied voltage and the threshold voltage as:
It shows that the radius of the nanofibers can be adjusted by the applied voltage. Hence, theoretically, electrospinning can produce nanofibers by selecting a proper solvent for a proper geometrical setting as well as controlling the intrinsic conductivity, the viscosity and the surface tension of the solution [9]. Theoretically, the minimum radius of nanofiber is determined by the fluid feature and initial droplet radius as:

$$d = d_0 \cdot \sqrt{1 + \frac{2 \cdot \gamma}{(\lambda + 2\mu)R} \cdot \left(\frac{V_c}{V_{\text{threshold}}}\right)^2}$$ \quad (29)

The above results show that the electro-spinning process can be well described by the theoretic formulation proposed in this research.

However, as the nonlinearity of fluid conductivity with the electrical field may exist and the surface charge density lightly depends on the electrical field, some correction should be introduced to verify the accuracy of Equations (26)- (29).

References
[1] Chen Zhida 2000 Rational Mechanics (Chongqing: Chongqing Publication) (In Chinese) p352
[2] Xiao Jianhua 2005 E-print (arXiv: physics/0511170)
[3] Manasseh R, et al. 2004 J. Sound & Vibration 278 807
[4] Yamamoto Y and F. Murai 2004 Pro. the Fourth International Conference on Fluid Mechanics (Dalian, China) (Beijing: Tsinghua University Press) p45
[5] Leighton T G 1994 The Acoustic Bubble (London: Academic Press) p613
[6] Demir M M, et al. 2002 Polymer 43 3303
[7] He JH, Wan YQ and Yu JY 2004 Int. J. Nonlinear Sci. 5 253
[8] Liu Y and He JH. 2007 Int. J. Nonlinear Sci. 8 393
[9] Jamila Shawon and Changmo Sung 2004 J. Mater. Sci. 39 4605