Sidney A. Morris, & David T. Yost. (2020). Observations on the Separable Quotient Problem for Banach Spaces. Axioms, 9(1).

Available online at https://doi.org/10.3390/axioms9010007.
Observations on the Separable Quotient Problem for Banach Spaces

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Abstract: The longstanding Banach–Mazur separable quotient problem asks whether every infinite-dimensional Banach space has a quotient (Banach) space that is both infinite-dimensional and separable. Although it remains open in general, an affirmative answer is known in many special cases, including (1) reflexive Banach spaces, (2) weakly compactly generated (WCG) spaces, and (3) Banach spaces which are dual spaces. Obviously (1) is a special case of both (2) and (3), but neither (2) nor (3) is a special case of the other. A more general result proved here includes all three of these cases. More precisely, we call an infinite-dimensional Banach space dual-like, if there is another Banach space $E$, a continuous linear operator $T$ from the dual space $E^*$ onto a dense subspace of $X$, such that the closure of the kernel of $T$ (in the relative weak* topology) has infinite codimension in $E^*$. It is shown that every dual-like Banach space has an infinite-dimensional separable quotient.

Keywords: Banach space; separable space; quotient space; weakly compactly generated; dual space; separable quotient problem; Markushevich base; biorthogonal system

We work in the category of Banach spaces, where the quotient by a closed (i.e., complete) subspace is always another Banach space. The Banach–Mazur separable quotient problem, which asks whether every infinite-dimensional Banach space has a quotient space which is both separable and infinite-dimensional, has remained unsolved for 85 years (the dual problem, finding a separable infinite-dimensional subspace in a given Banach space, is almost trivial). Reflexive Banach spaces constitute one case which is easily resolved. If $R$ is reflexive and infinite-dimensional, then so is its dual $R^*$. Choose any infinite-dimensional separable subspace $S \subset R^*$. Then $S$ is the annihilator $M^0$ of some subspace $M$ of $R$, and $(R/M)^* \cong M^0 = S$ is separable, whence $R/M$ is also separable.

For a comprehensive account of known results, we refer to [1–3]. These give an affirmative answer in a large number of special cases, of which we just mention one omnibus result now (Corollary 17, [3]): If a Banach space $X$ or its dual $X^*$ contains a subspace isomorphic to either $c_0$ or $\ell_1$, then $X$ has an infinite-dimensional separable quotient. This covers most known concrete examples of Banach spaces, in particular the classical function spaces, as each is reflexive or has a subspace isomorphic to either $c_0$ or $\ell_1$. We will focus on two natural generalisations of reflexive spaces, namely weakly compactly generated (WCG) spaces and dual spaces, and examine what they have in common.

Weakly compactly generated (WCG) spaces were introduced to the world by Amir and Lindenstrauss [4]: a Banach space is WCG if it is generated by (i.e., is the closed linear span of) a weakly compact subset. This includes all reflexive Banach spaces, because the unit ball of a reflexive space is weakly compact. Amir and Lindenstrauss showed that WCG spaces admit many projections, in particular, every separable subspace of a WCG space is contained in a complemented separable subspace. (For a survey of this topic see (Sections 3 and 4, [5]); and for a modern
Proposition 1. For a Banach space $X$, the following are equivalent:

(i) $X$ is weakly compactly generated.

(ii) There is a Banach space $Y$, and a weak* to weak continuous linear injection $T : Y^* \to X$, with dense range.

(iii) There is a Banach space $Y$, and a weak* to weak continuous linear injection $T : X^* \to Y$.

(iv) There is a Banach space $Y$, and a weak* to weak continuous linear injection $T : X^* \to Y$, with dense range.

(v) There is a Banach space $Y$, and a weak* to weak continuous linear operator $T : Y^* \to X$, with dense range.

Proof. (sketch)

(i) $\Rightarrow$ (iii): There are several possible choices for $Y$ and $T$. The first historically, albeit with the most difficult proof, is that $Y$ can be $c_0(\Gamma)$ for a suitably large set $\Gamma$ (Proposition 2, [4]).

The simplest argument is perhaps the following, which appears in the proof of (Theorem 2.3, [10]). If $K$ is a weakly compact generating subset of the Banach space $X$, consider the restriction operator $T : X^* \to C(K)$. This is clearly continuous from the topology of uniform convergence on weakly compact subsets of $X$ (i.e., the Mackey* topology $\tau(X^*, X)$) to the norm topology on $C(K)$. It must therefore be continuous in the corresponding weak topologies. But the dual of $X^*$ under $\tau(X^*, X)$ is just $X$ (p. 62, Theorem 7, [11]), so $T$ is weak* to weak continuous. Since $K$ generates $X$, $T$ is also injective.

Another particularly interesting possibility [12] is that $Y$ can be a reflexive Banach space.

(iii) $\Rightarrow$ (iv): Simply replace $Y$ by the closure of the range of $T$.

(iv) $\Rightarrow$ (ii): Note that the adjoint $T^* : Y^* \to X$ will be weak* to weak continuous, injective, and have dense range.

(ii) $\Rightarrow$ (v): This is obvious.

(v) $\Rightarrow$ (i): The unit ball of $Y$ is weak* compact, so its image under $T$ will be a weakly compact generating set. □
Note that if we replace dense range by surjective in condition (ii) above, it becomes a characterisation of reflexivity.

We now introduce a class of Banach spaces which, by virtue of the preceding result, includes all WCG spaces and all dual spaces, and show that all of its members have separable quotients.

**Definition 1.** A Banach space $X$ is said to be dual-like if there is another Banach space $E$ and a continuous linear operator $T$ from the dual space $E^*$ onto a dense subspace of $X$, such that the kernel $W$ of $T$ is not too large, in the sense that its closure in the weak*-topology on $E^*$ has infinite codimension in $E^*$.

**Remark 1.** Clearly every dual Banach space is dual-like, as is every WCG space.

**Remark 2.** If $X$ and $E$ are Banach spaces and there exists a one-to-one continuous linear operator from $E^*$ onto a dense subspace of $X$, then $X$ is dual-like.

Before presenting our main result, we highlight the following beautiful result of Saxon and Wilansky [1]. Recall that a (closed linear) subspace $A$ of a Banach space $X$ is said to be quasicomplemented if there is another subspace $B$ with $A \cap B = \{0\}$ and $A + B$ dense in $X$. A complemented subspace is clearly quasicomplemented; a proper quasicomplemented subspace is one which is not complemented.

**Proposition 2.** For a Banach space $X$, the following are equivalent:

(i) $X$ has an infinite-dimensional separable quotient Banach space.

(ii) $X$ has a dense nonbarrelled subspace.

(iii) $X$ has a separable infinite-dimensional quasicomplemented subspace.

(iv) $X$ has a proper quasicomplemented subspace.

**Theorem 1.** Any infinite-dimensional dual-like Banach space has a quotient Banach space which is infinite-dimensional and separable.

**Proof.** Let $X$ be dual-like, then there exist a Banach space $E$ and a continuous linear operator $T : E^* \to X$ such that $T(E^*)$ is dense in $X$ and the weak*-closure of the kernel $W$ of $T$ has infinite codimension in $E^*$.

Firstly consider the case that $T$ is surjective. Let $F = \{ f \in E : w(f) = 0, \text{ for all } w \in W \}$ be the annihilator of $W$ in $E$. Then let $V = \{ v \in E^* : v(f) = 0 \text{ for all } f \in F \}$ be the annihilator in $E^*$ of $F$. By the Bipolar Theorem (p. 35, Theorem 4 [11]), $V$ is the weak*-closure of $W$, and by our assumption we have that $V$ has infinite codimension in $E^*$. By the open mapping theorem $X \cong E^*/W$. Now $E^*/W$ has $E^*/V$ as a quotient space, and $E^*/V$ is isomorphic to $F^*$. As an infinite-dimensional dual Banach space, by [9], $F^*$ has an infinite-dimensional separable quotient Banach space, and therefore $X$ does too.

Now we consider the case that $T$ is not surjective. The conclusion follows immediately from (Corollary 3.4 [2]); let us repeat the short argument. Since the image $T(E^*)$ is a dense proper subspace, it must be an incomplete normed space. The open mapping theorem for continuous operators mapping a Banach space onto a barrelled locally convex space (p. 116, Theorem 7 and Corollary 1, [11]) then ensures that $T(E^*)$ is not barrelled. Proposition 2 now completes the proof. $\square$

**Corollary 1.** Let $X$ be an infinite-dimensional Banach space which is either reflexive, or weakly compactly generated (WCG), or a dual space. Then $X$ has a quotient Banach space which is infinite-dimensional and separable.

It is well known that Banach spaces with suitable biorthogonal systems, in particular Markushevich bases, admit separable quotients. We show that the idea of dual-like leads to this
A fundamental and total biorthogonal system

**Author Contributions:** The authors contributed equally to the research. All authors have read and agreed to the published version of the manuscript.

**Acknowledgments:** The authors thank the referees for their thoroughness and suggestions which improved the presentation of this paper.

**Conflicts of Interest:** The authors declare no conflict of interest.
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