Unusual scalar products in Hilbert space of Quantum Mechanics: non-Hermitian square-well model with two coupled channels

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Abstract

A pseudo-Hermitian square-well model “with spin” is proposed, solved and discussed. The domain of parameters is determined where all the bound-state energies remain real and where the necessary transition from the original elementary non-physical indefinite pseudo-metric to another, more involved but correct positive-definite physical metric is possible.

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1 Introduction

Qualitative phenomenological description of quantum phenomena often relies on a drastically simplified model. Most typically, one derives an “effective” reduced Hamiltonian $H_{\text{eff}}$ from a given microscopic $H_{\text{orig}}$ by some approximation procedure. In one of applications of such an approach in nuclear physics, people map the fermions governed by a complicated Hamiltonian $H_{\text{orig}}$ on a bona fide equivalent bosonic system controlled by $H_{\text{eff}}$ [1].

Even though the latter recipe may lead to non-Hermitian $H_{\text{eff}}$ in general, the authors of the review [1] emphasized that the work with $H_{\text{eff}} \neq H_{\text{eff}}^\dagger$ need not contradict the principles of Quantum Mechanics. A core of the message lies in the observation that one is free to introduce an alternative metric $\Theta \neq I$ in Hilbert space. This operator defines the new scalar product and the new norm in our Hilbert space in such a way that

$$H_{\text{eff}}^\dagger = \Theta H_{\text{eff}} \Theta^{-1}, \quad \Theta = \Theta^\dagger > 0. \quad (1)$$

In this language, our Hamiltonian may be called “quasi-Hermitian” (the word meaning just Hermitian and, hence, physical with respect to the new metric). It becomes allowed to represent an observable (energy). Let us only note that it is in fact just conventional to call $H$ “the operator of energy” since it may also be interpreted as “an occupation number”, etc. After all, the latter re-classification is not too unusual, say, within the so called supersymmetric quantum mechanics [2].

In the late nineties, a renewed interest in the quasi-Hermitian non-Hermitian models has been inspired by Bessis [3] and by Bender and Boettcher [4] who revealed that one of important non-Hermitian models in field theory seemed to possess the discrete and real “bound-state-like” spectrum. This hypothesis (which has rigorously been confirmed a few years later [5]) opened the question of the possible physical interpretation of the model because its $H_{\text{eff}}$ only happened to satisfy a weaker form of eq. (1),

$$H_{\text{eff}}^\dagger = \mathcal{P} H_{\text{eff}} \mathcal{P}^{-1}, \quad \mathcal{P}^{-1} = \mathcal{P} = \mathcal{P}^\dagger, \quad (2)$$

with the role of an indefinite “pseudo-metric” played by the operator of parity [6].

In the latter new family of models (conventionally called, for some historical reasons [7], $\mathcal{PT}$-symmetric) people only succeeded in constructing physical metric $\Theta$ very recently [8]. One must remember that $H_{\text{eff}} \neq H_{\text{eff}}^\dagger$ so that the standard Schrödinger equation must be considered together with the parallel Hermitian-
conjugate problem,

\[ H_{\text{eff}} |\psi\rangle = E |\psi\rangle, \quad H_{\text{eff}}^\dagger |\psi\rangle\rangle = E |\psi\rangle\rangle. \quad (3) \]

Moreover, many \(\mathcal{PT}\)–symmetric models proved to behave, in many a respect, against our current intuition [9]. Hence, up to a few partial differential exceptions [10], people prefer working with the one-dimensional effective Hamiltonians. Hence, it is not too surprising that the constructions of the correct positive definite metric \(\Theta\) found their most transparent presentations in exactly solvable models. For example, one may recollect ref. [11] where an entirely elementary “schematic” square-well model of refs. [12] has been used and studied and where its measurable and physical aspects have been described in detail.

Our present letter was immediately motivated by some specific features of the transition \(\mathcal{P} \rightarrow \Theta\) in the context of coupled-channel Schrödinger equations, perceived here as residing somewhere in between the unsolvable (= “realistic”) and solvable extremes. Their introduction may be based on various physical assumptions as well as on some formal considerations in representation theory [13] and/or on constructions belonging to relativistic quantum mechanics [14] etc.

2 Coupled-channel problems

In some of the contemporary papers devoted to \(\mathcal{PT}\)–symmetric Quantum Mechanics [15] the operator \(\mathcal{P}\) of eq. (2) does not coincide with parity and it need not even be chosen as involutive. For this reason, let us now change its symbol from \(\mathcal{P}\), say, to \(\theta\). Although the rigorous mathematical specification of this \(\theta \neq \theta^{-1}\) need not be easy in general [16], a generic requirement is that this auxiliary indefinite pseudo-metric operator as well as the related Hamiltonian \(H_{\text{eff}}\) remain sufficiently elementary [17].

2.1 Two particles coupled in a one-dimensional deep box

The most elementary coupled-channel model are, undoubtedly, the two-channel models and, in particular, their one-dimensional example

\[ H_{\text{eff}} = \begin{pmatrix} -\frac{d^2}{dx^2} & 0 \\ 0 & -\frac{d^2}{dx^2} \end{pmatrix} + V_{\text{eff}}, \quad V_{\text{eff}} = \begin{pmatrix} V_a(x) & W_b(x) \\ W_a(x) & V_b(x) \end{pmatrix} \quad (4) \]

written in units \(\hbar = 2m = 1\). These operators lie somewhere in between the ordinary and partial differential ones and offer a certain combination of merits of the solvability
(so characteristic for ordinary differential equations) with a richer structure of wave functions.

In what follows we shall only search for such two-channel bound states $|\psi\rangle$ which have the standard asymptotic Dirichlet boundary conditions mimicked by their suitable large-$L$ approximation

$$
\langle \pm L | \psi \rangle = \begin{pmatrix}
\langle \pm L | \psi_a \rangle \\
\langle \pm L | \psi_b \rangle
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}.
$$

In this setting, a non-trivial core of our message will lie in the assumption that while the character of the spectrum $\{E_n\}$ will be assumed “entirely standard” (i.e., real, discrete and bounded below), the potential term itself will possess an unusual, asymmetric and manifestly non-Hermitian form with $V_a \neq V_a^\dagger$, etc.

### 2.2 Models with $\theta$–pseudo-Hermiticity and $\mathcal{PT}$–symmetry

Our key motivation stems form the success of a number of the ordinary differential non-Hermitian Hamiltonians which were offered within the framework of the so called quasi-Hermitian [1], $\mathcal{PT}$–symmetric [4] or $\mathcal{P}$–pseudo-Hermitian [18] Quantum Mechanics. In this context, rather surprisingly, coupled-channel models were not yet studied in sufficient detail. Our present letter has been written just to fill this gap.

For the sake of a maximal transparency of our forthcoming arguments we shall violate the current Hermiticity as drastically as possible and postulate the $\theta$–pseudo-Hermiticity property

$$
H_{\text{eff}}^\dagger = \theta H_{\text{eff}} \theta^{-1}, \quad \theta = \theta^\dagger
$$

using the following parity-dependent (or, if you wish, generalized-parity-dependent) pseudo-metric

$$
\theta = \theta^\dagger = \begin{pmatrix}
0 & \mathcal{P} \\
\mathcal{P} & 0
\end{pmatrix}, \quad \theta^{-1} = \begin{pmatrix}
0 & \mathcal{P}^{-1} \\
\mathcal{P}^{-1} & 0
\end{pmatrix}.
$$

As long as these operators commute with the kinetic (i.e., differential) part of our $H_{\text{eff}}$ (4), the related $\theta$–pseudo-Hermiticity condition (6) degenerates to the following two $\mathcal{P}$–pseudo-Hermiticity relations and one definition,

$$
W_a^\dagger = \mathcal{P}W_a\mathcal{P}^{-1}, \quad W_b^\dagger = \mathcal{P}W_b\mathcal{P}^{-1}, \quad V_b = \mathcal{P}^{-1}V_a^\dagger\mathcal{P}.
$$

In our present note just a “minimal” model will be considered, with the re-scaled coordinate $x$ [such that $L = 1$ in eq. (5)] and with the current parity operator such that $\mathcal{P}\varphi(x) = \varphi(-x)$. 

3
3 Exactly solvable square-well example

In one of the simplest versions of the above Hamiltonian (4) let us consider the following purely imaginary square-well realization of the off-diagonal $\mathcal{P}$–pseudo-Hermitian potentials,

\[ \begin{align*}
\text{Re} W_{a,b}(x) &= 0, \quad x \in (-1, 1), \\
\text{Im} W_a(x) &= Z, \quad \text{Im} W_b(x) = Y, \quad x \in (-1, 0), \\
\text{Im} W_a(x) &= -Z, \quad \text{Im} W_b(x) = -Y, \quad x \in (0, 1),
\end{align*} \]

accompanied by the trivial intra-channel interactions, $V_a = V_b = 0$.

3.1 Trigonometric wave functions

The obvious ansatz

\[ \begin{align*}
\varphi(x) &= \langle x|\psi_a \rangle = \begin{cases} 
A \sin \kappa_L(x + 1), & x \in (-1, 0), \\
C \sin \kappa_R(1 - x), & x \in (0, 1),
\end{cases} \\
\chi(x) &= \langle x|\psi_b \rangle = \begin{cases} 
B \sin \kappa_L(x + 1), & x \in (-1, 0), \\
D \sin \kappa_R(1 - x), & x \in (0, 1),
\end{cases}
\end{align*} \]

may be normalized as usual, with $\varphi(x) = \varphi^*(-x)$ and $\chi(x) = \chi^*(-x)$ giving $C = A^*$, $D = B^*$ and $\kappa_L = \kappa_R^* = \kappa = s - it$ where, say, $s > 0$. Its insertion in the differential Schrödinger eqs. (3) leads to the complex solvability condition

\[ \begin{pmatrix} \kappa^2 - E & iZ \\
iY & \kappa^2 - E \end{pmatrix} \begin{pmatrix} A \\
B \end{pmatrix} = 0. \]

The related complex secular equation may be re-read as two real conditions,

\[ \begin{align*}
2st &= \pm \sqrt{YZ}, \quad E = s^2 - t^2 & \text{for } YZ > 0, \\
t &= 0, \quad E = s^2 \pm \sqrt{-YZ} & \text{for } YZ < 0.
\end{align*} \]

We note that our matrix problem (11) is in fact Hermitian at $Y = -Z$. Hence, let us only study the more challenging former option with $Y > 0, Z > 0$ in what follows.

3.2 Matching conditions at $x = 0$

What we have to postulate is the continuity of both the wave functions $\varphi(x)$ and $\chi(x)$ and of their first derivatives at $x = 0$. Two of the resulting four complex equations

\[ \begin{align*}
A \sin \kappa &= A^* \sin \kappa^*, \\
B \sin \kappa &= B^* \sin \kappa^*, \\
A \kappa \cos \kappa &= -A^* \kappa^* \cos \kappa^*, \\
B \kappa \cos \kappa &= -B^* \kappa^* \cos \kappa^*.
\end{align*} \]
specify the ratio of coefficients \( A/B = 2st/Y \) as real. The remaining two equations

\[
\begin{pmatrix}
\sin \kappa & -\sin \kappa^* \\
\kappa \cos \kappa & \kappa^* \cos \kappa^*
\end{pmatrix}
\begin{pmatrix}
A \\
A^*
\end{pmatrix} = 0
\]

(13)
define one of these complex coefficients. The non triviality of this solution is guaranteed by the elementary secular equation \( \text{Re} (\kappa^{-1} \tan \kappa) = 0 \). The later condition has the simplified equivalent form

\[
s \sin 2s + t \sinh 2t = 0
\]

(14)
with the structure of solutions known from the single-well constructions [19],

\[
s = s_n = \frac{(n + 1)\pi}{2} + (-1)^n \varepsilon_n, \quad n = 0, 1, \ldots
\]

(15)
where quantities \( \varepsilon_n \) remain small and positive at large \( n \) or small \( \sqrt{YZ} \). In contrast to the single-well case, the present real energy levels \( E_n = s_n^2 - YZ/(4s_n^2) \) are doubly degenerate since \( t \) in the above-mentioned ratio \( A/B = 2st/Y = \pm \sqrt{Z/Y} \) may acquire both signs,

\[
|\psi_n^{(\sigma)}\rangle = \begin{pmatrix}
|\varphi_n\rangle \cdot \sqrt{Z} \\
|\varphi_n\rangle \cdot \sigma \sqrt{Y}
\end{pmatrix}, \quad \sigma = \pm 1, \quad n = 0, 1, \ldots
\]

(16)
The construction is completed.

4 Discussion

4.1 The existence of the set of two commuting observables

The key merit of our choice of the example (9) is methodical since its Hamiltonian \( H_{\text{eff}} \) commutes with the operator which might play the role of an independent spin-like observable in our system,

\[
\Omega = \begin{pmatrix}
0 & \sqrt{Z/Y} \\
\sqrt{Y/Z} & 0
\end{pmatrix}.
\]

(17)
Indeed, the \( \theta-\text{pseudo-Hermiticity} \) \( \Omega^\dagger = \theta \Omega \theta^{-1} \) is readily verified as one of the welcome intuitive arguments supporting the possible consistency of such an interpretation.
We saw that both our pseudo-Hermitian candidates $H = H_{\text{eff}}$ and $\Omega$ for observables possess the real spectra (remember: just two points $\sigma = \pm 1$ in the latter case), at not too large couplings $Y \geq 0$ and $Z \geq 0$ at least (more precisely, at all of them such that $\sqrt{YZ} < Z_{\text{crit}} \approx 4.48$ [12]). In this regime, our wave functions may be perceived as functions of the (real) variables $E$ (or, equivalently, $n$) and $\sigma$. From such a point of view their set becomes complete once the operators in question (i.e., $H$ and $\Omega$ in our model) form a complete set of commuting operators in a given indefinite metric.

### 4.2 The basis in Hilbert space for the degenerate spectrum

In the majority of studies concerned with $\mathcal{PT}$–symmetric quantum mechanics the energy spectra happen to be non-degenerate. In contrast, bound states are usually classified by more quantum numbers in practice [1, 20]. In this sense, the degeneracy of energies and the emergence of the second quantum number $\sigma$ might further enhance the pragmatic as well as theoretical appeal of our present example.

In the present unusual non-Hermitian setting, the eigenvectors of $H^\dag$ and $\Omega^\dag$ (or, equivalently, the left eigenvectors of $H$ and $\Omega$) will be also needed. We must solve the following extended set of Schrödinger equations,

\[
H |E,\sigma\rangle = E |E,\sigma\rangle, \quad \Omega |E,\sigma\rangle = \sigma |E,\sigma\rangle, \\
\langle\langle E,\sigma|H = E \langle\langle E,\sigma|, \quad \langle\langle E,\sigma|\Omega = \sigma \langle\langle E,\sigma|. 
\]

Fortunately, the latter pair only means that

\[
H^\dag |E,\sigma\rangle = E^* |E,\sigma\rangle, \quad \Omega^\dag |E,\sigma\rangle = \sigma^* |E,\sigma\rangle
\]

so that, due to the pseudo-Hermiticity (6) and due to the independence and completeness of our set of wave functions (be it proved or assumed) we have

\[
|E,\sigma\rangle = \theta |E^*,\sigma^*\rangle q_{E\sigma}, \quad E = E_1, E_2, \ldots, \quad \sigma = \pm 1.
\]  

(18)

We are just left with a freedom in a complex normalization constant in the explicit definition of all the missing solutions.

Now, it is easy to derive the biorthogonality relations among our wave functions,

\[
\langle\langle E',\sigma'|E,\sigma|E' - E = 0, \quad \langle\langle E',\sigma'|E,\sigma|\sigma' - \sigma = 0.
\]
We see that only the diagonal overlaps may remain non-vanishing and enter the completeness relations

\[ I = \sum_{E, \sigma} |E, \sigma \rangle \frac{1}{\langle \langle E, \sigma | E, \sigma \rangle} \langle \langle E, \sigma | \langle \langle E, \sigma | \rangle \langle \langle E, \sigma |\rangle } \]

as well as the following two spectral representation formulae,

\[ H = \sum_{E, \sigma} |E, \sigma \rangle \frac{E}{\langle \langle E, \sigma | E, \sigma \rangle} \langle \langle E, \sigma |, \quad \Omega = \sum_{E, \sigma} |E, \sigma \rangle \frac{\sigma}{\langle \langle E, \sigma | E, \sigma \rangle} \langle \langle E, \sigma | \]  \hspace{1cm} (19)

where the later one is rather formal of course.

5 The transition from indefinite \( \theta \) to physical \( \Theta \)

In terms of the metric \( \Theta \), the bound-state coupled-channel wave functions of our model acquire the standard probabilistic interpretation. Indeed, the scalar product

\[ (|\psi_1\rangle \odot |\psi_2\rangle) = \langle \psi_1 | \Theta | \psi_2 \rangle = \langle \psi_1 | \psi_2 \rangle_{(physical)} \]  \hspace{1cm} (20)

generates the norm, \( ||\psi|| = \sqrt{\langle \psi | \psi \rangle_{(physical)}} \), and enables us to treat all the quasi-Hermitian operators \( A \) with the property \( A^\dagger = \Theta A \Theta^{-1} \) as observables. Such a usage of this word makes good sense because the expectation values \( \langle \psi | A | \psi \rangle_{(physical)} \) are mathematically unambiguously defined,

\[ (|\psi_1\rangle \odot |A \psi_2\rangle) \equiv (|A \psi_1\rangle \odot |\psi_2\rangle) . \]  \hspace{1cm} (21)

In our particular square-well model just a re-interpretation of our \( \theta \)–pseudo-Hermitian Hamiltonian \( H_{\text{eff}} \) and spin \( \Omega \) as quasi-Hermitian operators with respect to \( \Theta \) is needed.

5.1 A formula for the metric

Let us recollect that we started our considerations from a given (i.e., with a strong preference, very simple) indefinite metric (i.e., pseudo-metric) operator \( \theta \) and from a \( \theta \)–pseudo-Hermitian Hamiltonian \( H_{\text{eff}} \) [cf. eq. (6)]. Now, having performed all the constructions of the bound states we are left with the ultimate task of finding the physical metric, i.e., a Hermitian and positive definite solution \( \Theta = \Theta^\dagger > 0 \) of eq. (1). In terms of the above formulae (19) it is easy to see, immediately, that we must have

\[ \Theta = \sum_{E, \sigma, F, \tau} |F, \tau \rangle \langle \langle E, \sigma | \]

where the later one is rather formal of course.
where the (in general, fairly ambiguous [1]) choice of the matrix $R$ must remain compatible with eq. (1) and with its analogue for $\Omega$. This remains true if and only if

$$R_{E,\sigma, F, \tau} (E^* - F) = 0, \quad R_{E,\sigma, F, \tau} (\sigma - \tau) = 0.$$  

Once the spectrum of energies is assumed real we arrive at the compact formula

$$\Theta = \sum_{E, \sigma} |E, \sigma\rangle S_{E, \sigma} \langle E, \sigma|$$  

(22)

which represents the menu of all the eligible pseudo- and metrics parametrized by the infinite sequence of the non-vanishing parameters $S_{E, \sigma}$ where $\sigma = \pm 1$ and, by assumption, $E = E_0, E_1, \ldots$ are all real. Easily we also deduce that

$$\Theta^{-1} = \sum_{E, \sigma} |E, \sigma\rangle \frac{1/S_{E, \sigma}}{\langle E, \sigma | E, \sigma \rangle} \cdot \langle E, \sigma | E, \sigma \rangle \cdot \langle E, \sigma|.$$  

The obligatory invertibility and Hermiticity of $\Theta$ is guaranteed when all the parameters $S_{E, \sigma}$ remain real and non-vanishing. Finally, its positivity (i.e., tractability as a physical metric) will be achieved whenever all $S_{E, \sigma}$ remain positive.

### 5.2 Quasi-parity

In our particular model of section 3, the reality of the energies was comparatively easy to prove. In such a situation, people usually work with eq. (18) and employ very particular $q = q_{E,\sigma} = \pm 1$, calling such a “new quantum number” quasi-parity [6, 21] or charge [22]. It enters the formula

$$\langle E, \sigma | E, \sigma \rangle = q_{E,\sigma} \langle E, \sigma | \theta | E, \sigma \rangle, \quad E = E_1, E_2, \ldots, \quad \sigma = \pm 1, \quad q_{E,\sigma} = \pm 1.$$  

In our model where the proportionality of both the components of our wave functions $|\psi_n^{(\sigma)}\rangle$ to the same single-channel ket $|\varphi_n\rangle$ is a useful artifact, we may insert eqs. (7) and (16) and arrive at an even more compact relation

$$\langle \langle \psi_n^{(\sigma)} | \psi_n^{(\sigma)} \rangle = q_{E,\sigma} \cdot \langle \psi_n^{(\sigma)} | \theta | \psi_n^{(\sigma)} \rangle = \sigma q_{E,\sigma} \cdot \langle \varphi_n | P | \varphi_n \rangle \cdot \sqrt{4YZ}.$$  

Obviously, we may prescribe the overall sign of this overlap since it is controlled

- by $\sigma = \pm 1$, i.e., by the optional sign-convention accepted in eq. (16),
- by the overlap $\langle \varphi_n | P | \varphi_n \rangle$ which “measures” the parity of the upper-channel wave function in eq. (10) and varies with $n = 0, 1, \ldots$.  


and by the quasi-parity \( q_{E_n} = \pm 1 \) which is our free choice.

Due to the mere two-by-two matrix character of the spin \( \Omega \), our key definition (22) of the metric may be now further reduced to the sum

\[
\Theta = \sum_{n=0}^{\infty} \mathcal{P} | \varphi_n \rangle \left( \frac{Y (S_{E_n} + S_{E_m}) \sqrt{YZ} (S_{E_n} - S_{E_m})}{\sqrt{YZ} (S_{E_n} + S_{E_m})} \right) \langle \varphi_n | \mathcal{P} \right)
\]

which is a two-by-two matrix with respect to the spin (17) and depends on the pairs of the positive free parameters \( S_{E_n} > 0 \).

6 Summary and outlook

6.1 An efficiency of perturbation expansions

One of the key advantages of our present model is that the coordinate representation (10) of its wave functions is piecewise trigonometric. In addition, perturbation ansatz of the form

\[
\varepsilon_n = \sum_{k=1}^{K} \left[ \frac{YZ}{(n+1)^2 \pi^2} \right]^k \cdot \sum_{t=1}^{T(k)} \frac{c_{k,t}}{(n+1)^t \pi^t}
\]

may be used to solve eq. (14) by iterations. Then, for the sufficiently high excitations \( n \geq n_0 \gg 1 \) and/or for the sufficiently small geometric-mean measure \( \sqrt{YZ} \) of the non-Hermiticity of our \( H_{\text{eff}} \) we may derive and work, say, with the formula

\[
\varepsilon_n = \left[ \frac{2YZ}{(n+1)^3 \pi^3} + \frac{4Y^2Z^2}{3(n+1)^5 \pi^5} \right] \left[ 1 + \mathcal{O} \left( \frac{1}{(n+1)^4} \right) \right] + \mathcal{O} \left( \frac{Y^3Z^3}{(n+1)^7} \right)
\]

showing that the convergence in \( 1/(n+1) \) proves extremely rapid.

It would be easy to demonstrate that one of the important consequences of the steady growth of the latter quantities with growing \( YZ \) would be a merger and the subsequent complexification of \( s_0 \) and \( s_1 \) (or of \( s_2 \) and \( s_3 \) etc) at a sufficiently large \( YZ \). At this critical point (or rather critical curve \( Y = \text{const}/Z \)), the reality of the spectrum of our parity-pseudohermitian \( H_{\text{eff}} \) gets spontaneously broken. This possibility opens a number of questions which were not discussed here at all.

Another immediate consequence of eq. (24) is that the role of the non-Hermiticity decreases very quickly at the higher excitations. This means that in our present model a fairly reliable approximation of the metric \( \Theta_{\text{approx}} \) will sufficiently significantly differ from the unit operator \( \Theta_{\text{trivial}} = I \) just in a finite-dimensional subspace spanned, say, by the \( N \) lowest excitations of the \( Y = Z = 0 \) system of the two completely decoupled deep (and, of course, Hermitian) square wells.
6.2 Point-interaction models

In the future, our present choice and study of our example could prove insufficient. Then, it need not necessarily be followed just by its various generalizations with a suitable piece-wise form of the forces. Indeed, with the number of the admissible discontinuities, one may expect an increase of difficulties of a purely technical nature. We believe that at least a partial reduction of such a obstacle could be achieved when one switches to the class of the point interactions, say, of the $\mathcal{PT}$-symmetric form

$$W_a(x) = \sum_{\ell=1}^{M_a} [i \alpha_\ell \delta (x - a_\ell) - i \alpha_\ell \delta (x + a_\ell)] ,$$  \hspace{1cm} (25)

$$W_b(x) = \sum_{j=1}^{M_b} [i \beta_j \delta (x - b_j) - i \beta_j \delta (x + b_j)] ,$$  \hspace{1cm} (26)

$$V_a(x) = V_b^*(-x) = \sum_{n=1}^{N} i \gamma_n \delta (x - g_n)$$ \hspace{1cm} (27)

with the purely imaginary delta functions located at certain ordered sets of the points

$$0 < a_1 < \ldots < a_{M_a} < 1, \quad 0 < b_1 < \ldots < b_{M_b} < 1, \quad -1 < g_1 < \ldots < g_{N} < 1$$

and proportional to some real constants $\alpha_\ell$, $\beta_j$ and $\gamma_n$.

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