The $S^1_t \times S^1_s$-valued lightcone Gauss map of a Lorentzian surface in semi-Euclidean 4-space

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Abstract

We define the notions of $S^1_t \times S^1_s$-valued lightcone Gauss maps, lightcone pedal surface and Lorentzian lightcone height function of Lorentzian surface in semi-Euclidean 4-space and established the relationships between singularities of these objects and geometric invariants of the surface as applications of standard techniques of singularity theory for the Lorentzian lightcone height function.

1 Introduction

In [8, 9], S.Izumiya et al studied singularities of lightcone Gauss maps and lightlike hypersurfaces of spacelike surface in Minkowski 4-space, and established the relationships between such singularities and geometric invariants of these surfaces under the action of Lorentz group. Our aim in this paper is to develop the analogous study for Lorentzian surface in semi-Euclidean 4-space $\mathbb{R}^4_2$. To do this we need to develop first the local differential geometry of Lorentzian surface in semi-Euclidean 4-space $\mathbb{R}^4_2$ in a similar way than the classically done surfaces in Euclidean 4-space [15]. As it was to be expected, the situation presents certain peculiarities when compared with the Euclidean case. For instance, in our case it is always possible to choose two lightlike normal directions along the Lorentzian surface a frame of its normal bundle. By using this, we define a Lorentzian invariant $K_4(1, \pm 1)$ and call it the lightlike Gauss-Kronecker curvature of the Lorentzian surface. We introduce the notion of lightcone height function and use it to show that the $S^1_t \times S^1_s$-valued lightcone Gauss map has a singular point if and only if the lightlike Gauss-Kronecker curvature vanishes at such point. Moreover, we show that the $S^1_t \times S^1_s$-valued lightcone Gauss map is a constant map if and only if the Lorentzian surface is contained in a lightlike hyperplane, so we can view the singularities of the $S^1_t \times S^1_s$-valued lightcone Gauss map as an estimate of the contacts of the surface with lightlike hyperplanes.

We shall assume throughout the whole paper that all the maps and manifolds are $C^\infty$ unless the contrary is explicitly stated.

Let $\mathbb{R}^4 = \{(x_1, x_2, x_3, x_4)|x_i \in \mathbb{R} \ (i = 1, 2, 3, 4) \}$ be a 4-dimensional vector space. For any vectors $\mathbf{x} = (x_1, x_2, x_3, x_4)$ and $\mathbf{y} = (y_1, y_2, y_3, y_4)$ in $\mathbb{R}^4$, the pseudo scalar product of $\mathbf{x}$ and $\mathbf{y}$ is

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defined to be $\langle x, y \rangle = -x_1y_1 - x_2y_2 + x_3y_3 + x_4y_4$. We call $(\mathbb{R}^4, \langle \cdot, \cdot \rangle)$ a semi-Euclidean 4-space and write $\mathbb{R}_s^4$ instead of $(\mathbb{R}^4, \langle \cdot, \cdot \rangle)$.

We say that a vector $x$ in $\mathbb{R}_s^4 \setminus \{0\}$ is spacelike, lightlike or timelike if $\langle x, x \rangle > 0$, $0$ or $< 0$ respectively. The norm of the vector $x \in \mathbb{R}_s^4$ is defined by $\|x\| = \sqrt{\langle x, x \rangle}$. For a lightlike vector $n \in \mathbb{R}_s^4$ and a real number $c$, we define the lightlike hyperplane with pseudo normal $n$ by

$$LHP(n, c) = \{ x \in \mathbb{R}_s^4 | \langle x, n \rangle = c \}.$$  

Let $X : U \to \mathbb{R}_s^4$ an immersion, where $U \subset \mathbb{R}^2$ is an open subset. We denote that $M = X(U)$ and identify $M$ and $U$ by the immersion $X$.

We say that $M$ is a Lorentzian surface if the tangent space $T_pM$ of $M$ is a Lorentzian surface for any point $p \in M$. In this case, the normal space $N_pM$ is a Lorentzian plane. Let $\{e_3(x, y), e_4(x, y); p = (x, y)\}$ be an pseudo-orthonormal frame of the tangent space $T_pM$ and $\{e_1(x, y), e_2(x, y); p = (x, y)\}$ a pseudo-orthonormal frame of $N_pM$, where, $e_1(p), e_3(p)$ are unit timelike vectors and $e_2, e_4$ are unit spacelike vectors.

We shall now establish the fundamental formula for a Lorentzian 2-space in $\mathbb{R}_s^4$ by means of similar notions to those of [9].

We can write $dX = \sum_{i=1}^{4} \omega_i e_i$ and $de_i = \sum_{j=1}^{4} \omega_{ij} e_j$; $i = 1, 2, 3, 4$, where $\omega_i$ and $\omega_{ij}$ are 1-forms given by $\omega_i = \delta(e_i)\langle dX, e_i \rangle$ and $\omega_{ij} = \delta(e_j)\langle de_i, e_j \rangle$,

with $\delta(e_i) = \langle e_i, e_i \rangle = \begin{cases} 1, & i = 1, 2, 4; \\ -1, & i = 3. \end{cases}$

We have the Codazzi type equations:

$$\begin{cases} d\omega_i = \sum_{j=1}^{4} \delta(e_i)\delta(e_j)\omega_{ij} \wedge \omega_j; \\ d\omega_{ij} = \sum_{k=1}^{4} \omega_{ik} \wedge \omega_{kj}, \end{cases}$$

where $d$ is exterior derivative.

Since $\langle e_i, e_j \rangle = \delta_{ij}\delta(e_j)$ (where $\delta_{ij}$ is Kronecker’s delta), we get

$$\omega_{ij} = -\delta(e_i)\delta(e_j)\omega_{ji}. \tag{2}$$

In particular, $\omega_{ii} = 0$; $i = 1, 2, 3, 4$. It follows from the fact $\langle dX, e_1 \rangle = \langle dX, e_2 \rangle = 0$ that

$$\omega_1 = \omega_2 = 0. \tag{3}$$

Therefore we have

$$\begin{cases} 0 = d\omega_1 = \delta(e_j)\sum_{j=1}^{4} \omega_{1j} \wedge \omega_j = \delta(e_j)\sum_{j=3}^{4} \omega_{1j} \wedge \omega_j = -\omega_{13} \wedge \omega_3 + \omega_{14} \wedge \omega_4; \\ 0 = d\omega_2 = \delta(e_j)\sum_{j=1}^{4} \omega_{2j} \wedge \omega_j = \delta(e_j)\sum_{j=3}^{4} \omega_{2j} \wedge \omega_j = -\omega_{23} \wedge \omega_3 + \omega_{24} \wedge \omega_4. \tag{4} \end{cases}$$

By Cartan’s lemma, we can write

$$\begin{cases} \omega_{13} = a\omega_3 + b\omega_4; \\ \omega_{23} = \bar{a}\omega_3 + \bar{b}\omega_4; \\ \omega_{14} = -b\omega_3 - a\omega_4; \\ \omega_{24} = -\bar{b}\omega_3 - \bar{a}\omega_4, \end{cases} \tag{5}$$

for appropriate functions $a, b, c, \bar{a}, \bar{b}, \bar{c}$. 

2
Since $\langle d\mathbf{X}, e_1 \rangle = \langle d\mathbf{X}, e_2 \rangle = 0$,

$$
\langle d^2 \mathbf{X}, e_1 \rangle = -\langle d\mathbf{X}, de_1 \rangle = \langle d\mathbf{X}, de_2 \rangle = -\left( \sum_{i=1}^{4} \omega_i e_i, \sum_{j=1}^{4} \omega_j e_j \right) = -\left( \sum_{i=3}^{4} \omega_i e_i, \sum_{j=2}^{4} \omega_j e_j \right)
$$

$$
= -(\omega_3 \omega_{13} + \omega_4 \omega_{14})
= \omega_3 (a \omega_3 + b \omega_4) + \omega_4 (b \omega_1 + c \omega_4)
= a \omega_3^2 + 2b \omega_3 \omega_4 + c \omega_4^2.
$$

As the same $\langle d^2 \mathbf{X}, e_2 \rangle = \pi \omega_3^2 + 2\pi \omega_3 \omega_4 + \pi \omega_4^2$.

Then we have a vector-valued quadratic form

$$
-\langle d^2 \mathbf{X}, e_1 \rangle e_1 + \langle d^2 \mathbf{X}, e_2 \rangle e_2 = -(a \omega_3^2 + c \omega_4^2 + 2b \omega_3 \omega_4) e_1 + (a \omega_3^2 + c \omega_4^2 + 2b \omega_3 \omega_4) e_2,
$$

which is called the second fundamental form of the Lorentz surface.

By using equations (2) and a straightforward calculation leads us to the following equations:

$$
d \begin{pmatrix}
e_1 + e_2 \\
e_1 - e_2 \\
e_3 \\
e_4
\end{pmatrix} =
\begin{pmatrix}
0 & \omega_{12} & \omega_{13} + \omega_{23} & \omega_{14} + \omega_{24} \\
-\omega_{12} & 0 & \omega_{13} - \omega_{23} & \omega_{14} - \omega_{24} \\
-(\omega_{13} + \omega_{23})/2 & (\omega_{23} - \omega_{13})/2 & 0 & \omega_{34} \\
(\omega_{14} + \omega_{24})/2 & (\omega_{14} - \omega_{24})/2 & \omega_{34} & 0
\end{pmatrix}
\begin{pmatrix}
e_1 - e_2 \\
e_1 + e_2 \\
e_3 \\
e_4
\end{pmatrix}
$$

On the other hand, we define

$$
\mathcal{LC}_p = \{ \mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid (x_1 - p_1)^2 - (x_2 - p_2)^2 + (x_3 - p_3)^2 + (x_4 - p_4)^2 = 0 \}
$$

and

$$
S_1^1 \times S_1^1 = \{ \mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathcal{LC}_0 \mid x_1^2 + x_2^2 = 1, x_1 \geq 0, x_2 \geq 0 \},
$$

where $p = (p_1, p_2, p_3, p_4) \in \mathbb{R}^4$, $S_1^1$ denotes the timelike circle and $S_1^1$ denotes the spacelike circle.

We call $\mathcal{LC}_p^\ast = \mathcal{LC}_p \setminus \{p\}$ a lightcone at the vertex $p$. Given any lightlike vector $\mathbf{x} = (x_1, x_2, x_3, x_4)$, we have $\overline{\mathbf{x}} = \left( \frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \frac{x_2}{\sqrt{x_1^2 + x_2^2}}, \frac{x_3}{\sqrt{x_1^2 + x_2^2}}, \frac{x_4}{\sqrt{x_1^2 + x_2^2}} \right) \in S_1^1 \times S_1^1$.

Let $e_1 = (a_1, a_2, a_3, a_4)$, $e_2 = (b_1, b_2, b_3, b_4)$, and $\xi^\pm = \sqrt{(a_1 - b_1)^2 \pm (a_2 - b_2)^2}$, then we have the following fundamental formula:

$$
d \begin{pmatrix}
e_1 + e_2 \\
e_1 - e_2 \\
e_3 \\
e_4
\end{pmatrix} =
\begin{pmatrix}
0 & -\omega_{12} - \frac{d\xi^+}{\xi^+} & \omega_{13} + \omega_{23} - \frac{d\xi^+}{\xi^+} & \omega_{14} + \omega_{24} - \frac{d\xi^+}{\xi^+} \\
-\omega_{12} - \frac{d\xi^-}{\xi^-} & 0 & \omega_{13} - \omega_{23} - \frac{d\xi^-}{\xi^-} & \omega_{14} - \omega_{24} - \frac{d\xi^-}{\xi^-} \\
-(\omega_{13} + \omega_{23})/2 & (\omega_{23} - \omega_{13})/2 & 0 & \omega_{34} \\
(\omega_{14} + \omega_{24})/2 & (\omega_{14} - \omega_{24})/2 & \omega_{34} & 0
\end{pmatrix}
\begin{pmatrix}
e_1 - e_2 \\
e_1 + e_2 \\
e_3 \\
e_4
\end{pmatrix}
$$

Given $v = xe_1 + ye_2 \in N_p M$, we have $dv = dxe_1 + xde_1 + dye_2 + yde_2$, and then

$$
\langle dv, e_3 \rangle \wedge \langle dv, e_4 \rangle = K_i(x, y) \omega_3 \wedge \omega_4,
$$

where the function $K_i$ as follows:

$$
K_i(x, y) = (ax + \tilde{a}y)(cx + \tilde{c}y) - (bx + \tilde{b}y)^2.
$$
On the other hand, we define two maps

\[ LG^2_M : U \rightarrow S^1_T \times S^1_s \]

by \( LG^2_M(x, y) = e_1 \pm e_2(x, y) \). Each one of these maps shall be called \( S^1_T \times S^1_s \)-valued lightcone Gauss map of \( X(U) = M \).

Now we introduce the notion of Lorentzian lightcone height functions on the Lorentzian surface in \( \mathbb{R}^4_2 \) which is useful for the study of singularities of \( S^1_T \times S^1_s \)-valued lightcone Gauss maps.

For a Lorentzian surface \( M = X(U) \in \mathbb{R}^4_2 \), we now define a function

\[ H : U \times S^1_T \times S^1_s \rightarrow \mathbb{R} \]

by \( H((x, y), \lambda) = (X(x, y), \lambda) \), where \( \lambda = (\cos \theta, \sin \theta, \lambda_3, \lambda_4) \in S^1_T \times S^1_s \). We call \( H \) the Lorentzian lightcone height function on the surface \( M \). We denote that \( h_{\lambda_0}(x, y) = H(x, y, \lambda_0) \), for any fixed \( \lambda_0 \in S^1_T \times S^1_s \). Then we have the following proposition.

**Proposition 1.1** Let \( M \) be a Lorentzian surface in \( \mathbb{R}^4_2 \) and \( H : U \times S^1_T \times S^1_s \rightarrow \mathbb{R} \) a Lorentzian lightcone height function. Then we have the following assertions:

1. \( (\partial h_{\lambda}/\partial x)(p_0) = (\partial h_{\lambda}/\partial y)(p_0) = 0 \) if and only if \( \lambda = \mu(e_1 \pm e_2)(p_0) = e_1 \pm e_2(p_0) \),

   where \( e_1(p_0) = (a_1, a_2, a_3, a_4) \), \( e_2(p_0) = (b_1, b_2, b_3, b_4) \) and \( \mu = \frac{1}{\sqrt{(a_1 \pm b_1)^2 + (a_2 \pm b_2)^2}} \); for any point \( p_0 \) in \( M \),

2. \( (\partial h_{\lambda}/\partial x)(p_0) = (\partial h_{\lambda}/\partial y)(p_0) = \det H(h_{\lambda})(p_0) = 0 \) if and only if \( \lambda = e_1 \pm e_2(p_0) \) and \( K_i(1, \pm 1)(p_0) = 0 \).

Here, \( \det H(h_{\lambda})(x, y) \) is the determinant of the Hessian matrix of \( h_{\lambda} \) at \((x, y)\).

**Proof.** By a straight forward calculation, \( (\partial h_{\lambda}/\partial x)(p_0) = (\partial h_{\lambda}/\partial y)(p_0) = 0 \) if and only if

\[ \langle X_x, \lambda \rangle(p_0) = \langle X_y, \lambda \rangle(p_0) = 0. \]

It is equivalent to the condition that \( \lambda \in N_{p_0}M \) and \( \lambda \in S^1_T \times S^1_s \). This means that \( \lambda = \mu(e_1 \pm e_2) = e_1 \pm e_2 \).

On the other hand, we now choose local coordinates such that \( X \) is given by the Monge form \( X(x, y) = (f_1(x, y), x, f_2(x, y), y) \) and \( e_1(p_0) = (1, 0, 0, 0) \) and \( e_2(p_0) = (0, 0, 1, 0) \). Since

\[ \det H(h_{\lambda})(x, y) = \begin{vmatrix} \langle X_{xx}, \lambda \rangle & \langle X_{xy}, \lambda \rangle \\ \langle X_{xy}, \lambda \rangle & \langle X_{yy}, \lambda \rangle \end{vmatrix} = 0 \]

and \( \lambda(p_0) = (1, 0, \pm 1, 0) \), we have

\[ \begin{vmatrix} \langle (f_{1xx}, 0, f_{2xx}, 0), \lambda(p_0) \rangle & \langle (f_{1xy}, 0, f_{2xy}, 0), \lambda(p_0) \rangle \\ \langle (f_{1yx}, 0, f_{2yx}, 0), \lambda(p_0) \rangle & \langle (f_{1yy}, 0, f_{2yy}, 0), \lambda(p_0) \rangle \end{vmatrix} = \begin{vmatrix} a \pm \bar{a} & b \pm \bar{b} \\ b \pm \bar{b} & c \pm \bar{c} \end{vmatrix} = 0. \]

This is equivalent to the condition that \( K_i(1, \pm 1)(x, y) = 0 \) and \( \lambda(p_0) = e_1 \pm e_2 \). \( \square \)

As a corollary of the above proposition, we have the following theorem.
Theorem 1.2 Under the same assumption as the assumption of the above proposition, the following conditions are equivalent:

1. \( p \in M \) is a degenerate singular point of Lorentzian lightcone height function \( h_\lambda \).
2. There is \( \lambda \in S^1_l \times S^1_s \) such that \( (p, \lambda) \) is a singular point of the \( S^1_l \times S^1_s \)-valued lightcone Gauss map \( LG^+_M \).
3. \( K_i(1, \pm 1)(p) = 0 \).

Proof. We denote that

\[
\Sigma(H) = \left\{ (p, \lambda) \in U \times S^1_l \times S^1_s \mid \frac{\partial h_\lambda}{\partial x}(p) = \frac{\partial h_\lambda}{\partial y}(p) = 0 \right\}.
\]

By above proposition, (1), we have

\[
\Sigma(H) = \left\{ (p, \lambda) \in U \times S^1_l \times S^1_s \mid \lambda = e_1 \pm e_2(p) \right\}.
\]

We now consider the canonical projection \( \pi : U \times S^1_l \times S^1_s \longrightarrow S^1_l \times S^1_s \), then \( \pi|\Sigma(H) \) can be identified to the \( S^1_l \times S^1_s \)-valued lightcone Gauss map \( LG^+_M \). Under this identification, we can show that the condition (1) is equivalent to the condition (2).

Above proposition, (2) means that the condition (2) is equivalent to the condition (3).

Theorem 1.3 Let \( M \) be a Lorentzian surface in \( \mathbb{R}^5 \).

1. The \( S^1_l \times S^1_s \)-valued lightcone Gauss maps \( LG^+_M \) (respectively, \( LG^-_M \)) is constant if and only if there exists a unique lightlike hyperplane \( LHP(v^+, c^+) \) (respectively, \( LHP(v^-, c^-) \)) such that \( M \subset LHP(v^+, c^+) \) (respectively, \( M \subset LHP(v^-, c^-) \)), where \( v^\pm = e_1 \pm e_2(x, y) \) and \( (X(x, y), v^\pm) = e^\pm \) for any \( (x, y) \in M \).
2. Both of the \( S^1_l \times S^1_s \)-valued lightcone Gauss maps \( LG^+_M \) and \( LG^-_M \) are constant if and only if \( M \) is a Lorentzian 2-plane. In this case, the intersection of lightlike hyperplanes

\[
LHP(e_1 + e_2, c^+) \cap LHP(e_1 - e_2, c^-)
\]

is the Lorentzian 2-plane \( M \).

Proof. (1) For convenience, we consider the case when \( LG^+_M(x, y) = e_1 + e_2(x, y) \) is constant, so that we have

\[
d(X, e_1 + e_2) = d(X, e_1) + d(X, e_2) = 0.
\]

Therefore, \( \langle X, e_1 + e_2 \rangle \equiv c^+ \). This means that \( M = X(U) \subset LHP(v^+, c^+) \), where \( v^+ = e_1 + e_2(x, y) \). For the converse assertion, suppose that there exists a lightlike vector \( v \) and a real number \( c \) such that \( X(U) = M \subset LHP(v, c) \). Since \( \langle X(x, y), v \rangle = c \), we have \( d(X(x, y), v) = 0 \). This means that \( v \) is a lightlike normal vector of \( M \). Thus we have \( \bar{v} = e_1 + e_2(x, y) \). This completes the proof of the assertion (1).

Since \( v^+ \notin LHP(v^-, c^-) \) and \( v^- \notin LHP(v^+, c^+) \), \( LHP(v^+, c^+) \) and \( LHP(v^-, c^-) \) intersect transversally. By the assertion (1), both of the \( S^1_l \times S^1_s \)-valued lightcone Gauss maps \( LG^+_M \) and \( LG^-_M \) are constant if and only if \( M \subset LHP(v^+, c^+) \cap LHP(v^-, c^-) \). Here, the intersection is a Lorentzian 2-plane. Thus we have the assertion (2).

We say that a point \( p_0 = (x_0, y_0) \) is a Lorentzian lightlike parabolic point of \( M \) if \( K_i(1, 1)(p_0) = 0 \) or \( K_i(1, -1)(p_0) = 0 \).
2 The lightcone pedal surface

In this section we consider a singular hyperplane in the lightcone $LC_0$ associated to $M$ whose singularities correspond to singularities of the $S_1^1 \times S_2^2$-valued lightcone Gauss map of $M$. We now define a family of functions

$$\tilde{H}: U \times LC_0 \rightarrow \mathbb{R}$$

by

$$\tilde{H}(x, y, v) = \langle X(x, y), \tilde{v} \rangle - \sqrt{v_1^2 + v_2^2},$$

where $v = (v_1, v_2, v_3, v_4)$. We call $\tilde{H}$ the extended Lorentzian lightcone height function of $M = X(U)$. As a corollary of above proposition, we have the following proposition.

**Proposition 2.1** Let $M$ be a Lorentzian surface in $\mathbb{R}_2^4$ and $\tilde{H}: M \times LC_0 \rightarrow \mathbb{R}$ the extended Lorentzian lightcone height function of $M$. For $p_0 = (x_0, y_0)$ and $v_0 \in LC_0$, we have the following:

1. $\tilde{H}(p_0, v_0) = (\partial \tilde{H}/\partial x)(p_0, v_0) = (\partial \tilde{H}/\partial y)(p_0, v_0) = 0$ if and only if

   $$\tilde{v}_0 = e_1 \pm e_2(p_0) \text{ and } \sqrt{v_1^2 + v_2^2} = \langle X(p_0), e_1 \pm e_2(p_0) \rangle.$$

2. $\tilde{H}(p_0, v_0) = \frac{\partial \tilde{H}}{\partial x}(p_0, v_0) = \frac{\partial \tilde{H}}{\partial y}(p_0, v_0) = \det \tilde{H}(\tilde{v}_0)(p_0) = 0$

if and only if

   $$\tilde{v}_0 = e_1 \pm e_2(p_0), \sqrt{v_1^2 + v_2^2} = \langle X(p_0), e_1 \pm e_2(p_0) \rangle \text{ and } K_i(1, \pm 1)(p_0) = 0.$$

Here, for a fixed $v \in LC_0$, $\tilde{H}((x, y), v) = \tilde{h}_v(x, y)$.

The assertion of proposition 2.1 means that the discriminant set of the extended Lorentzian lightcone height function $\tilde{H}$ is given by

$$D_{\tilde{H}} = \left\{ v \mid v = (X(x, y), e_1 \pm e_2(x, y))(e_1 \pm e_2)(x, y) \text{ for some } (x, y) \in U \right\}.$$

Therefore we now define a pair of singular surface in $LC_0$ by

$$LP^\pm_M(p) = LP^\pm_M(x, y) = \langle X(x, y), e_1 \pm e_2(x, y) \rangle(e_1 \pm e_2)(x, y).$$

We call each $LP^\pm$ the lightcone pedal surface of $X(U) = M$. A singularity of the lightcone pedal surface exactly corresponds to a singularity of the $S_1^1 \times S_2^2$-valued lightcone Gauss map.

We define a pair of hyperplane $LH^\pm_M: M \times \mathbb{R} \rightarrow \mathbb{R}_2^4$ by

$$LH^\pm_M(p, u) = LH^\pm_M(x, y, u) = X(x, y) + u(e_1 \pm e_2)(x, y),$$

where $p = X(x, y)$ we call $LH^\pm_M$ the lightlike hyperplane along $M$.

We now explain the reason why such a correspondence exists from the view point of Symplectic and Contact geometry. We consider a point $v = (v_1, v_2, v_3, v_4) \in LC_0$, then we have a relation $v_1 = \sqrt{-v_2^2 + v_3^2 + v_4^2}$. We adopt the coordinate $(v_2, v_3, v_4)$ of the manifold $LC_0$. We now consider
the projective cotangent bundle $\pi : PT^*(LC_0) \to LC_0$ with the canonical contact structure. We review geometric properties of this space. Consider the tangent bundle $\tau : TPT^*(LC_0) \to PT^*(LC_0)$ and the differential map $d\pi : TPT^*(LC_0) \to TLC_0$ of $\pi$. For any $X \in TPT^*(LC_0)$, there exists an element $\alpha \in T^*(LC_0)$ such that $\tau(X) = [\alpha]$. For an element $V \in T_2(LC_0)$, the property $\alpha(V) = 0$ does not depend on the choice of representative of the class $[\alpha]$. Thus we can define the canonical contact structure on $PT^*(LC_0)$ by

$$K = \{ X \in TPT^*(LC_0) | \tau(X)(d\pi(X)) = 0 \}.$$  

Since we consider the coordinate $(v_2, v_3, v_4)$, we have the trivialization $PT^*(LC_0) \cong LC_0 \times P(\mathbb{R}^2)^*$, we call

$$((v_2, v_3, v_4), [\xi_2 : \xi_3 : \xi_4])$$

a homogeneous coordinate, where $[\xi_2 : \xi_3 : \xi_4]$ is the homogeneous coordinate of the dual projective space $P(\mathbb{R}^2)^*$.

It is easy to show that $X \in K(x, [\xi])$ if and only if $\sum_{i=2}^4 \mu_i \xi_i = 0$, where $d\pi(X) = \sum_{i=2}^4 \mu_i \frac{\partial}{\partial \xi_i}$. An immersion $i : L \to PT^*(LC_0)$ is said to be a Legendrian immersion if $\dim L = 2$ and $d\pi(T_qL) \subset K_{i(q)}$ for any $q \in L$. We also call the map $\pi \circ i$ the Legendrian map and the set $W(i) = \text{image } \pi \circ i$ the wave front of $i$. Moreover, $i$ (or, the image of $i$) is called the Legendrian lift of $W(i)$.

In order to study the lightcone pedal surface, we give a quick survey on the Legendrian singularity theory mainly due to Arnol’d-Zakalyukin [1] [19]. Although the general theory has been described for general dimension, we only consider the 3-dimensional case for the purpose. Let $F : (\mathbb{R}^k \times \mathbb{R}^3, 0) \to (\mathbb{R}, 0)$ be a function germ. We say that $F$ is a Morse family if the mapping

$$\Delta^*F = \left(F, \frac{\partial F}{\partial q_1}, \ldots, \frac{\partial F}{\partial q_k}\right) : (\mathbb{R}^k \times \mathbb{R}^3, 0) \to (\mathbb{R} \times \mathbb{R}^k, 0)$$

is non-singular, where $(q, x) = (q_1, \ldots, q_k, x_1, x_3) \in (\mathbb{R}^k \times \mathbb{R}^3, 0)$. In this case we have a smooth 2-dimensional submanifold

$$\Sigma_*(F) = \left\{ (q, x) \in (\mathbb{R}^k \times \mathbb{R}^3, 0) | F(q, x) = \frac{\partial F}{\partial q_1}(q, x) = \cdots = \frac{\partial F}{\partial q_k}(q, x) = 0 \right\}$$

and the map germ $\Phi_F : (\Sigma_*(F), 0) \to PT^*\mathbb{R}^3$ defined by

$$\Phi_F(q, x) = \left(x, \frac{\partial F}{\partial x_1}(q, x) : \frac{\partial F}{\partial x_2}(q, x) : \frac{\partial F}{\partial x_3}(q, x)\right)$$

is a Legendrian immersion. Then we have the following fundamental theorem of Arnol’d-Zakalyukin [1] [19].

**Proposition 2.2** All Legendrian submanifold germs in $PT^*\mathbb{R}^3$ are constructed by the above method.

We call $F$ a generating family of $\Phi_F$. Therefore the corresponding wave front is

$$W(\Phi_F) = \left\{ x \in \mathbb{R}^3 | \text{there exists } q \in \mathbb{R}^k \text{ such that } F(q, x) = \frac{\partial F}{\partial q_1}(q, x) = \cdots = \frac{\partial F}{\partial q_k}(q, x) = 0 \right\}.$$  

By definition, we have $\mathcal{D}_F = W(\Phi_F)$. By the previous arguments, the lightcone pedal surface $LP_{H_{19}^N}$ is the discriminant set of the extended Lorentzian lightcone height function $H$. We have the following proposition.

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Proposition 2.3 The extended Lorentzian lightcone height function $\tilde{H}$ is a Morse family.

Proof. We define another family of function

$$\tilde{H} : U \times S^1_t \times S^2_s \times \mathbb{R} \rightarrow \mathbb{R}$$

by $\tilde{H}((x, y), w, r) = \langle X(x, y), w \rangle - r$. We consider a $C^\infty$-diffeomorphism

$$\Phi : U \times S^1_t \times S^2_s \times \mathbb{R} \rightarrow LC_0$$

defined by $\Phi((x, y), w, r) = ((x, y), rw)$. Then we have $\tilde{H} = \tilde{H} \circ \Phi$. It is enough to show that $\tilde{H}$ is a Morse family. For any $w = (\cos \theta, \sin \theta, w_3, w_4) \in S^1_t \times S^2_s$, we have $w_3 = \sqrt{1 - w_4^2}$, so that

$$\tilde{H}((x, y), w, r) = -x_1(p) \cos \theta - x_2(p) \sin \theta + x_3(p) \sqrt{1 - w_4^2} + x_4(p)w_4 - r,$$

where $X(x, y) = X(p) = (x_1(p), x_2(p), x_3(p), x_4(p))$. We now prove that the mapping

$$\Delta^* \tilde{H} = \left( \frac{\partial \tilde{H}}{\partial x}, \frac{\partial \tilde{H}}{\partial y} \right)$$

is non-singular at $w \in \mathcal{D}_H$. The Jacobian matrix of $\Delta^* \tilde{H}$ is given as follows:

$$\begin{pmatrix}
\langle X_x, w \rangle & \langle X_y, w \rangle \\
\langle X_{xx}, w \rangle & \langle X_{xy}, w \rangle \\
\langle X_{xy}, w \rangle & \langle X_{yy}, w \rangle
\end{pmatrix}
= \begin{pmatrix}
x_1 \sin \theta - x_2 \cos \theta & -x_3 \frac{w_4}{w_4^3} + x_4 & -1 \\
x_{1,x} \sin \theta - x_{2,x} \cos \theta & -x_{3,x} \frac{w_4}{w_4^3} + x_{4,x} & 0 \\
x_{1,y} \sin \theta - x_{2,y} \cos \theta & -x_{3,y} \frac{w_4}{w_4^3} + x_{4,y} & 0
\end{pmatrix}.$$

By a straightforward calculation, the determinant of the matrix

$$A = \begin{pmatrix}
x_{1,x} \sin \theta - x_{2,x} \cos \theta & -x_{3,x} \frac{w_4}{w_4^3} + x_{4,x} \\
x_{1,y} \sin \theta - x_{2,y} \cos \theta & -x_{3,y} \frac{w_4}{w_4^3} + x_{4,y}
\end{pmatrix}$$

is equal to

$$\frac{1}{2w_3} \begin{vmatrix}
\sin \theta & \cos \theta & -w_3 & w_4 \\
-x_{1,x} & x_{2,x} & x_{3,x} & x_{4,x} \\
x_{1,y} & x_{2,y} & x_{3,y} & x_{4,y}
\end{vmatrix}.$$

If $\det A = 0$ then $(\cos \theta, \sin \theta, -w_3, w_4) \in T_pM$. So that $(\cos \theta, \sin \theta, -w_1, w_2) \in N_pM \cap S^1_t \times S^2_s$. It is impossible because $w = (\cos \theta, \sin \theta, w_1, w_2) \in N_pM \cap S^1_t \times S^2_s$ and $N_pM$ is a Lorentzian 2-plane. Hence $\det A \neq 0$.

By proposition 2.3, we remark that the lightcone pedal surface $LP^L_{M^r}$ are wave fronts and the extended Lorentzian lightcone height function $\tilde{H}$ gives generating families of the Legendrian lifts of $LP^L_{M^r}$. 

8
3 Contact with lightlike hyperplanes

In this section, we consider the geometric meanings of the singularities of the $S^1 \times S^2$-valued lightcone Gauss map (respectively, the lightcone pedal surface) of $X(U) = M$. We consider the contact between Lorentzian surface and lightlike hyperplane like as the classical differential geometry. In the first place, we briefly review the theory of contact due to Montaldi [20]. Let $X_i, Y_i (i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. We say that the contact of $X_1$ and $Y_1$ at $y_1$ is same type as the contact of $X_2$ and $Y_2$ at $y_2$ if there is a diffeomorphism germ $\Phi : (\mathbb{R}^n, y_1) \rightarrow (\mathbb{R}^n, y_2)$ such that $\Phi(X_1) = X_2$ and $\Phi(Y_1) = Y_2$. In this case, we write $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$. It is clear that in the definition $\mathbb{R}^n$ could be replaced by any manifold. In his paper [20], Montaldi gives a characterization of the notion of contact by using the terminology of singularity theory.

Theorem 3.1 Let $X_i, Y_i (i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. Let $g_i : (X_i, x_i) \rightarrow (\mathbb{R}^n, y_i)$ be immersion germs and $f_i : (\mathbb{R}^n, y_i) \rightarrow (\mathbb{R}^p, 0)$ be submersion germs with $(Y_i, y_i) = (f_i^{-1}(0), y_i)$. Then

$$K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$$

if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $K$-equivalent.

We now consider a function $H : \mathbb{R}^4_2 \times LC_0 \rightarrow \mathbb{R}$ defined by $H(x, v) = \langle x, \bar{v} \rangle - \sqrt{v_1^2 + v_2^2}$. For any $v_0 \in LC_0$, we denote $\bar{h}_{v_0}(x) = H(x, v_0)$ and we have a lightlike hyperplane $\bar{h}_{v_0}^{-1}(0) = LHP(\bar{v}_0, \sqrt{v_0^2 + v_0^2})$. For any $p_0 = (x_0, y_0) \in U$, we consider the lightlike vector $v^\pm_0 = e_1 \pm e_2(p_0)$ and $c^\pm = \langle X(p_0), v^\pm_0 \rangle$, then we have

$$H_{v^\pm_0} \circ X(p_0) = H(X \times id_{LC_0})(p_0, v^\pm_0) = H((x_0, y_0), \bar{v}_0^\pm) - c^\pm = 0.$$

We also have relations that

$$\frac{\partial H_{v^\pm_0} \circ X}{\partial x}(p_0) = \frac{\partial H}{\partial x}(p_0, v^\pm_0) = 0,$$

and

$$\frac{\partial H_{v^\pm_0} \circ X}{\partial y}(p_0) = \frac{\partial H}{\partial y}(p_0, v^\pm_0) = 0.$$

This means that the lightlike hyperplane $\bar{h}_{v_0}^{-1}(0) = LHP(\bar{v}_0^\pm, c^\pm)$ is tangent to $M = X(U)$ at $p_0$. In this case, we call each $LHP(\bar{v}_0^\pm, c^\pm)$ the tangent lightlike hyperplane of $M = X(U)$ at $p_0 = X(x_0, y_0)$. Moreover, the intersection

$$LHP(\bar{v}_0^+, c^+) \cap LHP(\bar{v}_0^-, c^-)$$

is the tangent plane of $M$ at $p_0$. Let $v_1, v_2$ be lightlike vectors. If $v_1, v_2$ are linearly dependent, then corresponding lightlike hyperplanes $LHP(v_1, c_1)$ and $LHP(v_2, c_2)$ are parallel. Then we have the following simple lemma.

Lemma 3.2 Let $X : U \rightarrow \mathbb{R}^4_2$ be an immersion with $X(U)$ is a Lorentzian surface and $\sigma = \pm$. Consider two points $p_1 = X(x_1, y_1), p_2 = X(x_2, y_2)$. Then we have the following assertions:

1. $LG_M^\sigma(p_1) = LG_M^\sigma(p_2)$ if and only if $LHP(v_1^\pm, c_1^\pm)$ and $LHP(v_2^\pm, c_2^\pm)$ are parallel.
2. $LP_M^\sigma(p_1) = LP_M^\sigma(p_2)$ if and only if $LHP(v_1^\pm, c_1^\pm) = LHP(v_2^\pm, c_2^\pm)$.

Here, $v_1^\pm = e_1 \pm e_2(p_i)$ and $c_i^\pm = \langle X(x_i, y_i), v_i^\pm \rangle$ for $i = 1, 2$.
On the other hand, for any map \( f : N \rightarrow P \), we denote \( \Sigma(f) \) the set of singular points of \( f \) and \( D(f) = f(\Sigma(f)) \). In this case we call \( f|_{\Sigma(f)} : \Sigma(f) \rightarrow D(f) \) the critical part of the mapping \( f \). For any Morse family \( F : (\mathbb{R}^k \times \mathbb{R}^3, 0) \rightarrow (\mathbb{R}, 0) \), \( (F^{-1}(0), 0) \) is a smooth hypersurface, so that we define a smooth map germ \( \pi_F : (F^{-1}(0), 0) \rightarrow (\mathbb{R}^3, 0) \) by \( \pi_F(q, x) = x \). We can easily show that \( \Sigma_k(F) = \Sigma(\pi_F) \). Therefore, the corresponding Legendrian map \( \pi \circ \Phi_F \) is the critical part of \( \pi_F \).

We now introduce an equivalence relation among Legendrian immersion germs. Let \( i : (L, p) \subset (PT^* \mathbb{R}^3, p) \) and \( i' : (L', p') \subset (PT^* \mathbb{R}^3, p') \) be Legendrian immersion germs. Then we say that \( i \) and \( i' \) are Legendrian equivalent if there exists a contact diffeomorphism germ \( H : (PT^* \mathbb{R}^3, p) \rightarrow (PT^* \mathbb{R}^3, p') \) such that \( H \) preserves fibers of \( \pi \) and that \( H(L) = L' \). A Legendrian immersion germ into \( PT^* \mathbb{R}^3 \) at a point is said to be Legendrian stable if for every map with the given germ there is a neighbourhood in the space of Legendrian immersions (in the Whitney \( C^\infty \) topology) and a neighbourhood of the original point such that each Legendrian immersion belonging to the first neighbourhood has in the second neighbourhood a point at which its germ is Legendrian equivalent to the original germ.

Since the Legendrian lift \( i : (L, p) \subset (PT^* \mathbb{R}^3, p) \) is uniquely determined on the regular part of the wave front \( W(i) \), we have the following simple but significant property of Legendrian immersion germs:

**Proposition 3.3** Let \( i : (L, p) \subset (PT^* \mathbb{R}^3, p) \) and \( i' : (L', p') \subset (PT^* \mathbb{R}^3, p') \) be Legendrian immersion germs such that regular sets of \( \pi \circ i, \pi \circ i' \) are dense respectively. Then \( i, i' \) are Legendrian equivalent if and only if wave front sets \( W(i), W(i') \) are diffeomorphic as set germs.

This result has been firstly pointed out by Zakalyukin [27]. The assumption in the above proposition is a generic condition for \( i, i' \). Especially, if \( i, i' \) are Legendrian stable, then these satisfy the assumption.

We can interpret the Legendrian equivalence by using the notion of generating families. We denote \( \mathcal{E}_n \), the local ring of function germs \( (\mathbb{R}^n, 0) \rightarrow \mathbb{R} \) with the unique maximal ideal \( \mathfrak{m}_n = \{ h \in \mathcal{E}_n \mid h(0) = 0 \} \).

Let \( F, G : (\mathbb{R}^k \times \mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0) \) be function germs. We say that \( F \) and \( G \) are \( P-K \)-equivalent if there exists a diffeomorphism germ \( \Psi : (\mathbb{R}^k \times \mathbb{R}^n, 0) \rightarrow (\mathbb{R}^k \times \mathbb{R}^n, 0) \) of the form \( \Psi(x, u) = (\psi_1(q, x), \psi_2(x)) \) for \( (q, x) \in (\mathbb{R}^k \times \mathbb{R}^n, 0) \) such that \( \Psi^*((F)_{\mathcal{E}_{k+n}}) = (G)_{\mathcal{E}_{k+n}} \). Here \( \Psi^*: \mathcal{E}_{k+n} \rightarrow \mathcal{E}_{k+n} \) is the pull back \( \mathbb{R} \)-algebra isomorphism defined by \( \Psi^*(h) = h \circ \Psi \).

Let \( F : (\mathbb{R}^k \times \mathbb{R}^3, 0) \rightarrow (\mathbb{R}, 0) \) a function germ. We say that \( F \) is a \( K \)-versal deformation of \( f = F|_{\mathbb{R}^k \times \{0\}} \) if

\[
\mathcal{E}_k = T_e(\mathcal{K})(f) + \left\langle \frac{\partial F}{\partial x_1} \bigg|_{\mathbb{R}^k \times \{0\}}, \frac{\partial F}{\partial x_2} \bigg|_{\mathbb{R}^k \times \{0\}}, \frac{\partial F}{\partial x_3} \bigg|_{\mathbb{R}^k \times \{0\}} \right\rangle_{\mathbb{R}}
\]

where

\[
T_e(\mathcal{K})(f) = \left\langle \frac{\partial f}{\partial q_1}, \ldots, \frac{\partial f}{\partial q_k}, f \right\rangle_{\mathcal{E}_k}.
\]

(See [9].)

The main result in Arnol’d-Zakalyukin’s theory [14] is as follows:

**Theorem 3.4** Let \( F, G : (\mathbb{R}^k \times \mathbb{R}^3, 0) \rightarrow (\mathbb{R}, 0) \) be Morse families. Then

1. \( \Phi_F \) and \( \Phi_G \) are Legendrian equivalent if and only if \( F, G \) are \( P-K \)-equivalent,
2. \( \Phi_F \) is Legendrian stable if and only if \( F \) is a \( K \)-versal deformation of \( F|_{\mathbb{R}^k \times \{0\}} \).
Since $F,G$ are function germs on the common space germ $(\mathbb{R}^{k} \times \mathbb{R}^{3}, 0)$, we do no need the notion of stably P-K-equivalences under this situation. By the uniqueness result of the K-versal deformation of a function germ, Lemma 3.2 and Proposition 3.3, we have the following classification result of Legendrian stable germs. For any map germ $f : (\mathbb{R}^{n}, 0) \to (\mathbb{R}^{p}, 0)$, we define the local ring of $f$ by $Q(f) = \mathcal{E}_n / f^* (\mathcal{M}_p) \mathcal{E}_n$.

**Proposition 3.5** Let $F,G : (\mathbb{R}^{k} \times \mathbb{R}^{3}, 0) \to (\mathbb{R}, 0)$ be Morse families. Suppose that $\Phi_F, \Phi_G$ are Legendrian stable. Then the following conditions are equivalent.

1. $(W(\Phi_F), 0)$ and $(W(\Phi_G), 0)$ are diffeomorphic as germs.
2. $\Phi_F$ and $\Phi_G$ are Legendrian equivalent.
3. $Q(f)$ and $Q(g)$ are isomorphic as $\mathbb{R}$-algebras, where $f = F|_{\mathbb{R}^{k} \times \mathbb{R}^{3}(0)}$, $g = G|_{\mathbb{R}^{k} \times \mathbb{R}^{3}(0)}$.

We now have tools for the study of the contact between Lorentzian surface and lightlike hyperplanes. Let $LP^\sigma_{M,1} : (U, (x_i, y_i)) \to (LC_0, v_i^\sigma)$ ($i = 1, 2$) be two lightcone pedal surface germs of Lorentzian surface germs $X_i : (U, (x_i, y_i)) \to (\mathbb{R}_2^4, p_i)$, where $\sigma = \pm$. We say that $LP^\sigma_{M,1}$ and $LP^\sigma_{M,2}$ are $A$-equivalent if there exist diffeomorphism germs $\phi : (U, (x_1, y_1)) \to (U, (x_2, y_2))$ and $\Phi : (LC_0, v_1^\sigma) \to (LC_0, v_2^\sigma)$ such that $\Phi \circ LP^\sigma_{M,1} = LP^\sigma_{M,2} \circ \phi$. If the both of the regular sets of $LP^\sigma_{M,1}$ and $LP^\sigma_{M,2}$ are dense in $(U, (x_i, y_i))$, it follows from proposition 3.5 that $LP^\sigma_{M,1}$ and $LP^\sigma_{M,2}$ are $A$-equivalent if and only if the corresponding Legendrian lift germs are Legendrian equivalent. This condition is also equivalent to the condition that two generating families $\tilde{H}_1$ and $\tilde{H}_2$ are $P$-$K$-equivalent by Theorem 3.4. Here, $\tilde{H}_i : (U \times LC_0, ((x_i, y_i), v_i^\sigma(U))) \to \mathbb{R}$ is the extended Lorentzian lightcone height function germ of $X_i$.

On the other hand, we denote that $\tilde{h}_{i,v_i^\sigma}(u) = \tilde{H}_i(u, v_i^\sigma)$, then we have $\tilde{h}^{-1}_{i,v_i^\sigma}(u) = h_{i,v_i^\sigma} \circ X_i(u)$. By Theorem 3.1, $K(X_1(U), LHP(v_1^\sigma, -1), v_1^\sigma) = K(X_2(U), LHP(v_2^\sigma, -1), v_2^\sigma)$ if and only if $\tilde{h}_{1,v_1}$ and $\tilde{h}_{1,v_2}$ are K-equivalent. Therefore, we can apply the previous arguments to our situation. We denote $Q^\sigma(X_i, (x_0, y_0))$ the local ring of the function germ $\tilde{h}_{i,v_i} : (U, (x_0, y_0)) \to \mathbb{R}$, where $v_i^\sigma = LP^\sigma_{M}(x_0, y_0)$. We remark that we can explicitly write the local ring as follows:

$$Q^\pm(X_i, (x_0, y_0)) = C^\infty_{(x_0, y_0)}(U) \langle \langle X(x, y), e_1 \pm e_2(x_0, y_0) \rangle \rangle_{C^\infty_{(x_0, y_0)}(U)},$$

where $C^\infty_{(x_0, y_0)}(U)$ is the local ring of function germs at $(x_0, y_0)$ with the unique maximal ideal $\mathfrak{m}_{(x_0, y_0)}(U)$.

**Theorem 3.6** Let $X_i : (U, (x_i, y_i)) \to (\mathbb{R}_2^4, X_i(x_i, y_i))$ ($i = 1, 2$) be an immersion germs with $X(U) = M$ is a Lorentzian surface such that the corresponding Legendrian lift germs are Legendrian stable and $\sigma = \pm$. Then the following conditions are equivalent:

1. lightcone pedal surface germs $LP^\sigma_{M,1}$ and $LP^\sigma_{M,2}$ are $A$-equivalent.
2. $\tilde{H}_1$ and $\tilde{H}_2$ are $P$-$K$-equivalent.
3. $\tilde{h}_{1,v_1}$ and $\tilde{h}_{1,v_2}$ are K-equivalent.
4. $K(X_1(U), LHP(v_1^\sigma, c_1^\sigma), v_1^\sigma) = K(x_2(U), LHP(v_2^\sigma, c_2^\sigma), v_2^\sigma)$
5. $Q^\sigma(X_1, (x_1, y_1))$ and $Q^\sigma(X_2, (x_2, y_2))$ are isomorphic as $\mathbb{R}$-algebras.

**Proof.** By the previous arguments (mainly by Theorem 3.1), it has been already shown that conditions (3) and (4) are equivalent. Other assertions follow from proposition 3.5. \qed
Given an immersion germ $X : (U, (x_0, y_0)) \rightarrow (\mathbb{R}^4, X(x_0, y_0))$ with $X(U) = M$ is a Lorentzian surface, we call each set
\[ (X^{-1}(LHP(v^+, c^+)), (x_0, y_0)) \]
a tangent lightlike hyperplane indicatrix germ of $X$, where $v^+ = e_1 \pm e_2(x_0, y_0)$ and $c^+ = \langle X(x_0, y_0), v^+ \rangle$. Moreover, by the above results, we can borrow some basic invariants from the singularity theory on function germs. We need $K$-invariants for function germ. The local ring of a function germ is a complete $K$-invariant for generic function germs. It is, however, not a numerical invariant. The $K$-codimension (or, Tyurina number) of a function germ is a numerical $K$-invariant of function germs [12]. We denote that
\[ \text{L-ord}^\sigma (X, (x_0, y_0)) = \dim_{\mathbb{R}} \frac{C^\infty_{(x_0, y_0)}(U)}{\langle h_{v_0^+}(x, y), h_{v_0^+}(x, y), h_{v_0^+}(x, y) \rangle}. \]

Usually $\text{L-ord}^\sigma (x, u_0)$ is called the $K$-codimension of $h_{v_0^+}$, where $\sigma = \pm$. However, we call it the order of contact with the tangent lightlike hyperplane at $X(x_0, y_0)$. We also have the notion of corank of function germs.
\[ \text{L-corank}^\sigma (X, (x_0, y_0)) = 2 - \text{rank Hess}(h_{v_0^+}(x_0, y_0)), \]
where $v_0^+ = e_1 \pm e_2(x_0, y_0)$.

By proposition 2.1, $X(x_0, y_0)$ is a $L^\sigma$-parabolic point if and only if $\text{L-corank}^\sigma (X, (x_0, y_0)) \geq 1$. Moreover $X(x_0, y_0)$ is a lightlike umbilic point if and only if $\text{L-corank}^\sigma (X, (x_0, y_0)) = 2$.

On the other hand, a function germ $f : (\mathbb{R}^{n-1}, a) \rightarrow \mathbb{R}$ has the $A_k$-type singularity if $f$ is $K$-equivalent to the germ $\pm u_1^2 \pm \cdots \pm u_{n-2}^2 + u_{n-1}^{k+1}$. If $\text{L-corank}^\sigma (X, (x_0, y_0)) = 1$, the extended Lorentzian lightcone height function $\tilde{h}_{v_0}$ has the $A_k$-type singularity at $(x_0, y_0)$ in generic. In this case we have $\text{L-ord}^\sigma (x, u_0) = k$. This number $k$ is equal to the order of contact in the classical sense (cf., [11]). This is the reason why we call $\text{L-ord}^\sigma (X, (x_0, y_0))$ the order of contact with the tangent lightlike hyperplane at $X(x_0, y_0)$.

As a corollary of the theorem 3.6, we have the following result.

**Corollary 3.7** Under the assumptions of Corollary 3.6, if the tangent lightlike hyperplane indicatrix germ $LHP_M^\sigma$ and $LHP_M^\tau$ are $A$-equivalent, then tangent lightcone indicatrix germs $X^{-1}(LHP(v^+_1, c_1^+))$ and $X^{-1}(LHP(v^+_2, c_2^+))$ are diffeomorphic as set germs.

**Proof.** Notice that the tangent lightlike hyperplane indicatrix germ of $X$, is the zero level set of $h_{i, \lambda}$. Since $K$-equivalence among function germs preserves the zero-level sets of function germs, the assertion follows from theorem 3.6. \qed

### 4 Classification of singularities of $S^1_t \times S^2_s$-valued lightcone Gauss maps and lightcone pedalsurfaces

In this section we consider generic singularities of $S^1_t \times S^2_s$-valued lightcone Gauss maps and lightcone pedal surfaces. We consider the space of Lorentzian embeddings $\text{Emb}_L(U, \mathbb{R}^4)$ with Whitney $C^\infty$-topology, where $U \subset \mathbb{R}^2$ is an open subset. We have the following theorem.
Theorem 4.1 There exists an open dense subset \( \mathcal{O} \subset \text{Emb}_L(U, \mathbb{R}^4_+) \) such that for any \( X \in \mathcal{O} \), the following conditions hold:

1. Each lightlike parabolic set \( \mathcal{K}(1, \sigma)^{-1}(0) \) is a regular curve. We call such a curve the lightlike parabolic curve.

2. The lightcone pedal surface \( LP^*_M \) along the lightlike parabolic curve is the cuspidal edge except isolated points. At these points \( LP^*_M \) is the swallowtail.

Here, \( \sigma = \pm \) and a map germ \( f : (\mathbb{R}^2, a) \longrightarrow (\mathbb{R}^3, b) \) is called a cuspidal edge if it is \( \mathcal{A} \)-equivalent to the germ \( (u_1, u_2, u^2_3) \) (cf., Fig. 1) and a swallowtail if it is \( \mathcal{A} \)-equivalent to the germ \((3u_1^4 + u_2^2 u_2, 4u_1^3 + 2u_1 u_2, u_2)\)

For the proof of Theorem 4.1, we consider the function \( \mathcal{H} : \mathbb{R}^4_+ \times \text{LC}_0 \longrightarrow \mathbb{R} \) which is given in §3. We claim that \( \mathcal{H}_v \) is a submersion for any \( v \in \text{LC}_0 \), where \( \mathcal{H}_v(x) = \mathcal{H}(x, v) \). For any \( X \in \text{Emb}_L(U, \mathbb{R}^4_+) \), we have \( \tilde{\mathcal{H}} = \mathcal{H} \circ (X \times \text{id}_{\text{LC}_0}) \). We also have the \( \ell \)-jet extension

\[
j^\ell_1 \tilde{\mathcal{H}} : U \times \text{LC}_0 \longrightarrow J^\ell(U, \mathbb{R})
\]

defined by \( j^\ell_1 \tilde{\mathcal{H}}((x, y), v) = j^\ell_1 \mathcal{H}_v(x, y) \). We consider the trivialization \( J^\ell(U, \mathbb{R}) \equiv U \times \mathbb{R} \times J^\ell(2, 1) \). For any submanifold \( Q \subset J^\ell(2, 1) \), we denote that \( \tilde{Q} = U \times \{0\} \times Q \). Then we have the following proposition as a corollary of Lemma 6 in Wassermann [17]. (See also Montaldi [14] and Looijenga [11]).

Proposition 4.2 Let \( Q \) be a submanifold of \( J^\ell(2, 1) \). Then the set

\[
T_Q = \{ x \in \text{Emb}_L(U, \mathbb{R}^4_+) \mid j^\ell_1 H \text{ is transversal to } \tilde{Q} \}
\]

is a residual subset of \( \text{Emb}_s(U, \mathbb{R}^4_+) \). If \( Q \) is a closed subset, then \( T_Q \) is open.

If we consider \( \mathcal{K} \)-orbits in \( J^\ell(2, 1) \), we obtain the proof of Theorem 4.1, so that we omit the detailed discussion. The assertion of Theorem 4.1 can be interpreted that the Legendrian lift of the lightcone pedal surface \( LP^*_M \) of \( X \in \mathcal{O} \) is Legendrian stable at each point. Since the Legendrian lift is the Legendrian covering of the Lagrangian lift of \( LG^*_M \), it has been known that the corresponding singularities of \( LG^*_M \) are folds or cusps [1]. Hence, we have the following corollary.

Corollary 4.3 Let \( \mathcal{O} \subset \text{Emb}_L(U, \mathbb{R}^4_+) \) be the same open dense subset as in Theorem 4.1. For any \( X \in \mathcal{O} \), the followings hold:

1. A lightlike parabolic point \((x_0, y_0) \in U\) is a fold of the \( S^1_1 \times S^2_2 \)-valued lightcone Gauss map \( LG^*_M \) if and only if it is a cuspidal edge of the lightcone pedal surface \( LP^*_M \).

2. A lightlike parabolic point \((x_0, y_0) \in U\) is a cusp of the \( S^1_1 \times S^2_2 \)-valued lightcone Gauss map \( LG^*_M \) if and only if it is a swallowtail of the lightcone pedal surface \( LP^*_M \).

Here, a map germ \( f : (\mathbb{R}^2, a) \longrightarrow (\mathbb{R}^3, b) \) is called a fold if it is \( \mathcal{A} \)-equivalent to the germ \((u_1, u_2, u_3)\) and a cusp if it is \( \mathcal{A} \)-equivalent to the germ \((u_1, u_2^2 + u_1 u_2)\).

Following the terminology of Whitney [18], we say that a surface \( X : U \longrightarrow \mathbb{R}^4_+ \) has the excellent lightcone pedal surface \( LP^*_M \) if the Legendrian lift of \( LP^*_M \) is a stable Legendrian immersion at each point. In this case, the lightcone pedal surface \( LP^*_M \) has only cuspidal edges and swallowtails as singularities. Proposition 4.1 asserts that a Lorentzian surface with the excellent lightcone pedal surface is generic in the space of all Lorentzian surface in \( \mathbb{R}^4_+ \). We now consider the geometric meanings of cuspidal edges and swallowtails of the lightcone pedal surface. We have the following results analogous to the results of Banchoff et al [2].
Theorem 4.4 Let \( LP^\sigma_M : (U, (x_0, y_0)) \rightarrow (\mathbb{R}^2_+, \nu_0) \) be the excellent lightcone pedal surface of a Lorentzian surface \( X \) and \( \tilde{h}^\sigma_{v_0} : (U, (x_0, y_0)) \rightarrow \mathbb{R} \) be the extended lightcone height function germ at \( v_0^\sigma = e_1 \pm e_2(\nu_0) \), where \( \sigma = \pm \). Then we have the following:

1. \((x_0, y_0)\) is a lightlike parabolic point of \( X \) if and only if \( L\text{-corank}^\sigma(X, (x_0, y_0)) = 1 \).

2. If \((x_0, y_0)\) is a lightlike parabolic point of \( X \), then \( \tilde{h}^\sigma_{v_0} \) has the \( A_k \)-type singularity for \( k = 2, 3 \).

3. Suppose that \((x_0, y_0)\) is a lightlike parabolic point of \( X \). Then the following conditions are equivalent:

   a) \( LP^\sigma_M \) has a cuspidal edge at \((x_0, y_0)\)
   b) \( \tilde{h}^\sigma_{v_0} \) has the \( A_2 \)-type singularity.
   c) \( L\text{-ord}^\sigma(X, (x_0, y_0)) = 2 \).

   d) The tangent lightlike hyperplane indicatrix is a ordinary cusp, where a curve \( C \subset \mathbb{R}^2 \) is called a ordinary cusp if it is diffeomorphic to the curve given by \( \{(u_1, u_2) \mid u_1^2 - u_2^2 = 0 \} \).

   e) For each \( \varepsilon > 0 \), there exist two distinct points \((x_i, y_i) \in U \) \((i = 1, 2)\) such that

   \[ \|(x_0, y_0) - (x_i, y_i)\| < \varepsilon \]

   for \( i = 1, 2 \), both of \((x_i, y_i)\) are not lightlike parabolic points and the tangent lightlike hyperplanes to \( M = x(U) \) at \((x_i, y_i)\) are parallel.

4. Suppose that \((x_0, y_0)\) is a lightlike parabolic point of \( X \). Then the following conditions are equivalent:

   a) \( LP^\sigma_M \) has a swallowtail at \((x_0, y_0)\)
   b) \( \tilde{h}^\sigma_{v_0} \) has the \( A_3 \)-type singularity.
   c) \( L\text{-ord}^\sigma(X, (x_0, y_0)) = 3 \).

   d) The tangent lightlike hyperplane indicatrix is a point or a tachnodal, where a curve \( C \subset \mathbb{R}^2 \) is called a tachnodal if it is diffeomorphic to the curve given by \( \{(u_1, u_2) \mid u_1^2 - u_2^2 = 0 \} \).

   e) For each \( \varepsilon > 0 \), there exist three distinct points \((x_i, y_i) \in U \) \((i = 1, 2, 3)\) such that

   \[ \|(x_0, y_0) - (x_i, y_i)\| < \varepsilon \]

   for \( i = 1, 2, 3 \) and the tangent lightlike hyperplanes to \( M = x(U) \) at \((x_i, y_i)\) are parallel.

   f) For each \( \varepsilon > 0 \), there exist two distinct points \((x_i, y_i) \in U \) \((i = 1, 2)\) such that

   \[ \|(x_0, y_0) - (x_i, y_i)\| < \varepsilon \]

   for \( i = 1, 2 \) and the tangent lightlike hyperplanes to \( M = x(U) \) at \((x_i, y_i)\) are equal.

Proof. We have shown that \((x_0, y_0)\) is a lightlike parabolic point if and only if

\[ L\text{-corank}^\sigma(X, (x_0, y_0)) \geq 1. \]

We have \( L\text{-corank}^\sigma(X, (x_0, y_0)) \leq 2 \). Since the extended lightcone height function germ \( \tilde{H} : (U \times L\text{C}_0, ((x_0, y_0), \nu_0)) \rightarrow \mathbb{R} \) can be considered as a generating family of the Legendrian lift of \( LP^\sigma_M \), \( \tilde{h}^\sigma_{v_0} \) has only the \( A_k \)-type singularities \((k = 1, 2, 3)\). This means that the corank of the Hessian matrix of \( \tilde{h}^\sigma_{v_0} \) at a lightlike parabolic point is 1. The assertion (2) also follows. By the same reason, the conditions (3);(a),(b),(c) (respectively, (4); (a),(b),(c)) are equivalent. If the height function germ \( \tilde{h}^\sigma_{v_0} \) has the \( A_2 \)-type singularity, then it is \( \mathcal{K} \)-equivalent to the germ \( \pm u_1^2 + u_2^2 \).
Since the $K$-equivalence preserve the diffeomorphism type of zero level sets, the tangent lightlike hyperplane indicatrix is diffeomorphic to the curve given by $\pm u_1^2 + u_2^3 = 0$. This is the ordinary cusp. The normal form for the $A_3$-type singularity is given by $\pm u_1^2 + u_2^4$, so that the tangent lightlike hyperplane indicatrix is diffeomorphic to the curve $\pm u_1^2 + u_2^3 = 0$. This means that the condition (3),(d) (respectively, (4),(d)) is also equivalent to the other conditions.

Suppose that $(x_0, y_0)$ is a lightlike parabolic point, then the $S_1^1 \times S_2^2$-valued lightcone Gauss map has only folds or cusps. If the point $(x_0, y_0)$ is a fold point, there is a neighbourhood of $(x_0, y_0)$ on which the $S_1^1 \times S_2^2$-valued lightcone Gauss map is 2 to 1 except at the lightlike parabolic curve (i.e, fold curve). By Lemma 3.2, the condition (3), (e) is satisfied. If the point $(x_0, y_0)$ is a cusp, the critical value set is an ordinary cusp. By the normal form, we can understand that the $S_1^1 \times S_2^2$-valued lightcone Gauss map is 3 to 1 inside region of the critical vale. Moreover, the point $(x_0, y_0)$ is in the closure of the region. This means that the condition (4),e holds. We can also observe that near by a cusp point, there are 2 to 1 points which approach to $(x_0, y_0)$. However, one of those points are always lightlike parabolic points. Since other singularities do not appear for in this case, so that the condition (3),(e) (respectively, (4),(e)) characterizes a fold (respectively, a cusp).

If we consider the lightcone pedal surface in stead of the lightcone Gauss map, the only singularities are cuspidaledges or swallowtails. For the swallowtail point $(x_0, y_0)$, there is a self intersection curve approaching to $(x_0, y_0)$. On this curve, there are two distinct point $(x_i, y_i) (i = 1, 2)$ such that $LP_M(x_1, y_1) = LP_M(x_2, y_2)$. By Lemma 3.2, this means that tangent lightlike hyperplane to $M = x(U)$ at $(x_1, y_1)$ are equal. Since there are no other singularities in this case, the condition (4),(f) characterize a swallowtail point of $LP_M$. This completes the proof. \hfill $\Box$

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