REALISATION AND SPLITTING FOR POINCARÉ DUALITY PAIRS IN DIMENSION THREE

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Dedicated to the memory of Andrew Ranicki, a wonderful colleague and a real mensch.

Abstract. In earlier work we presented necessary conditions for a fundamental triple to be that of a 3-dimensional Poincaré duality pair with aspherical boundary components. We provide a construction which shows that the necessary conditions are sufficient.

1. Introduction

Poincaré duality pairs of dimension $n$, or $PD^n$–pairs, are homotopy generalisation of $n$–dimensional manifolds with boundary. A $PD^n$–pair consists of a pair of CW–complexes, $(X, \partial X)$, a homomorphism $\omega : \pi_1(X) \to \mathbb{Z}/2\mathbb{Z}$ and a homology class $[X, \partial X] \in H_n(X, \partial X; \mathbb{Z}^\omega)$ satisfying the relative version of Poincaré duality exhibited by $n$–dimensional manifolds with boundary. We call $\partial X$ the boundary, $\omega$ the orientation character and $[X, \partial X]$ the fundamental class of the $PD^n$–pair. A $PD^n$–pair with empty boundary yields a $PD^n$–complex.

To describe fundamental triples of $PD^3$–pairs take a $PD^3$–pair $((X, \partial X), \omega, [X, \partial X])$ with aspherical boundary components. The set of homomorphisms $\{\kappa_i : \pi(\partial X_i) \to \pi_1(X)\}_{i \in J}$ induced by the inclusions of the boundary components is called the peripheral or $\pi_1$-system of $(X, \partial X)$. Attaching cells of dimension $\geq 3$ to $X$ we obtain an Eilenberg-Mac Lane pair $(K, \partial X)$ of type $K(\{\kappa_i\}_{i \in J}; 1)$ and an inclusion of pairs $c : (X, \partial X) \to (K, \partial X)$. Then $\{\kappa_i\}_{i \in J}, \omega, c_*([X, \partial X])$ is the fundamental triple of the $PD^3$–pair $((X, \partial X), \omega, [X, \partial X])$.

The generalisation of Hendriks’ classification result for $PD^3$–complexes [6] states that the oriented homotopy type of a $PD^3$–pair with aspherical boundary components is uniquely determined by the isomorphism class of its fundamental triple [3]. A fundamental triple $\{\kappa_i\}_{i \in J}, \omega, \mu$ is $\pi_1$–injective if $\kappa_i$ is injective for all $i \in J$ and it is realisable if it is the fundamental triple of a $PD^3$–pair.

In [3] we generalised Turaev’s realisation condition for $PD^3$–complexes. Central to the statement of necessary conditions for a fundamental triple to be realisable was that the image of $\mu$ under a specific homomorphism, $\nu$, which we call the Turaev map, be an isomorphism in the stable module category of $\mathbb{Z}[G]$, that is to say, a homotopy equivalence of $\mathbb{Z}[G]$-modules.

To prove that the necessary conditions are sufficient we had to restrict attention to $\pi_1$–injective fundamental triples and wrote “As the assumption of $\pi_1$–injectivity is indispensable for the method used, the question remains whether the realisation theorem holds without it.”

Hillman provided alternative necessary conditions via Crisp’s Algebraic Loop Theorem which are also sufficient under additional conditions [7].

For the construction of a $PD^3$–complex realising a given fundamental triple Turaev used a representative of $\nu(\mu)$ to construct a realisation at the level of chain complexes and an attaching map of 3-cells. In [3] we followed Turaev’s approach to construct the relative chain complex of the realisation and depended on $\pi_1$–injectivity to obtain an attaching map. Here we incorporate boundary components to construct the chain complex of the realisation and an attaching map for 3-cells without requiring $\pi_1$–injectivity.
Our main result, Theorem 1 in Section 3, finally settles the question raised in [3] by showing that the necessary conditions in [3] are sufficient without restriction.

The results on decomposition and splitting for PD³-pairs [3] rely on the realisation result and were thus restricted to π₁-injective PD³-pairs. As π₁-injectivity is not required beyond the application of the realisation theorem, Theorem 1 removes this restriction. We discuss the resulting splitting theorems in Section 5.

The proof of Theorem 1 follows in two steps. First we construct a candidate for the realisation of a given triple in Section 3, then we verify that this candidate is indeed a PD³-pair realising the given triple in Section 4.

Section 2 provides the necessary background and notation, including the definitions of PDⁿ-complexes, PDⁿ-pairs and fundamental triples.

2. Background and Notation

2.1. Algebraic Preliminaries. Let Λ be the integral group ring, \( \mathbb{Z}[G] \), of the group \( G \). We write \( I \) for the augmentation ideal, the kernel of the augmentation map

\[
\text{aug}: \Lambda \to \mathbb{Z}, \quad \sum_{g \in G} n_g g \mapsto \sum_{g \in G} n_g
\]

where \( \mathbb{Z} \) is a \( \Lambda \)-bi-module with trivial \( \Lambda \) action. Each cohomology class \( \omega \in H^1(G; \mathbb{Z}/2\mathbb{Z}) \) may be viewed as a group homomorphism \( \omega: G \to \mathbb{Z}/2\mathbb{Z} = \{0, 1\} \) and yields an anti-isomorphism

\[
\omega: \Lambda \to \Lambda, \quad \lambda \mapsto \sum_{g \in G} (-1)^{\omega(g)} n_g g^{-1}
\]

Consequently, a right \( \Lambda \)-module, \( A \), yields the conjugate left \( \Lambda \)-module, \( \omega A \), with action given by

\[
\lambda \cdot a := \lambda \cdot a
\]

for \( \lambda \in \Lambda, a \in A \). Plainly, the conjugate defines a functor from the category of left \( \Lambda \)-modules to the category of right \( \Lambda \)-modules. Similarly, a left \( \Lambda \)-module \( B \) yields the conjugate right \( \Lambda \)-module, \( B^\omega \).

If \( B \) is a left \( \Lambda \)-module and \( M \) a \( \Lambda \)-bi-module, then \( \text{Hom}_\Lambda(B, M) \) is a right \( \Lambda \)-module with action given by

\[
\varphi.\lambda: B \to M, \quad b \mapsto \varphi(b).\lambda
\]

The dual of the left \( \Lambda \)-module \( B \) is the left \( \Lambda \)-module \( B^* = \omega \text{Hom}_\Lambda(B, \Lambda) \). The construction of the dual defines an endofunctor on the category of left \( \Lambda \)-modules.

Evaluation defines a natural transformation, \( \varepsilon \), from the identity functor to the double dual functor, where for the left \( \Lambda \)-module \( B \),

\[
\varepsilon_B: B \to B^{**} = \omega \text{Hom}_\Lambda(\omega \text{Hom}_\Lambda(B, \Lambda), \Lambda), \quad b \mapsto \overline{\overline{b}}
\]

with \( \overline{\overline{b}} \) defined by

\[
\overline{\overline{b}}: \omega \text{Hom}(B, \Lambda) \to \Lambda, \quad \psi \mapsto \overline{\overline{\psi(b)}}
\]

The left \( \Lambda \)-module, \( A \), defines the natural transformation, \( \eta \), from the functor \( A^\omega \otimes_\Lambda \_ \) to the functor \( \text{Hom}_\Lambda(\omega \text{Hom}_\Lambda(\_, \Lambda), \Lambda) \) where, for the left \( \Lambda \)-module \( B \),

\[
\eta_B: A^\omega \otimes_\Lambda B \to \text{Hom}_\Lambda(B^*, A) = \text{Hom}_\Lambda(\omega \text{Hom}_\Lambda(B, \Lambda), A)
\]

is given by

\[
\eta_B(a \otimes b): \psi \mapsto \overline{\overline{\psi(b)}.a}
\]

for \( a \otimes b \in A^\omega \otimes_\Lambda B \). When restricted to the category of finitely generated free \( \Lambda \)-modules both natural transformations yield natural equivalences which we also denote by \( \varepsilon \) and \( \eta \).

The \( \Lambda \)-morphisms \( f, g: A_1 \to A_2 \) are homotopic if and only if the \( \Lambda \)-morphism \( f - g: A_1 \to A_2 \) factors through a projective \( \Lambda \)-module \( P \). Associated with \( \Lambda \) is its stable module category, whose
objects are all $\Lambda$-modules and whose morphisms are all homotopy classes of $\Lambda$-morphisms. Thus, an isomorphism in the stable module category of $\Lambda$ is a homotopy equivalence of $\Lambda$-modules.

2.2. Poincaré Duality Complexes. Here we work in the category of connected, well pointed $CW$–complexes and pointed maps. We write $X^{[k]}$ for the $k$-skeleton of $X$, suppressing the base point from our notation.

From now, we write $G = \pi_1(X)$ for the fundamental group of $X$, and $\Lambda = \mathbb{Z}[G]$ for its integral group ring. We write $u: \tilde{X} \to X$ for the universal cover of $X$, fixing a base point for $\tilde{X}$ in $u^{-1}(\ast)$, and $\tilde{C}(X)$ for the cellular chain complex of $\tilde{X}$ viewed as a complex of left $\Lambda$-modules. Note that $\tilde{C}_n(X) \cong H_n(\tilde{X}, \tilde{X}[n-1])$.

The homology and cohomology of $X$ we work with are the abelian groups

$$H_q(X; A) := H_q(A \otimes_\Lambda \tilde{C}(X))$$

$$H^q(X; B) := H^q(\text{Hom}_\Lambda(\tilde{C}(X), B))$$

where $A$ is a right $\Lambda$-module and $B$ is a left $\Lambda$-module.

An $n$-dimensional Poincaré duality complex (PD$^n$–complex) comprises a connected $CW$–complex, $X$, whose fundamental group, $G$, is finitely presentable, together with an orientation character, $\omega = \omega_X \in H^1(G; \mathbb{Z}/2\mathbb{Z})$, viewed as a group homomorphism $G \to \mathbb{Z}/2\mathbb{Z}$, and a fundamental class, $[X] \in H_n(X; \mathbb{Z}^\omega)$, such that for every $r \in \mathbb{Z}$ and left $\Lambda$-module $M$, the cap product with $[X]$,

$$\alpha \leadsto [X]: H^r(X; M) \to H_{n-r}(X; M^\omega), \quad \alpha \mapsto \alpha \cap [X]$$

is an isomorphism of abelian groups. We denote this by $(X, \omega, [X])$ or simply by $X$.

An oriented map of PD$^n$–complexes $X_1$ and $X_2$ is a map $f: X_1 \to X_2$ with $f_*([X_1]) = [X_2]$.

2.3. Poincaré Duality Pairs. Next we turn to the category of pairs $(X, Y)$ of $CW$–complexes, where $X$ is connected and well pointed. With the notation as above, we write $\tilde{C}(Y)$ for the subcomplex of the chain complex $\tilde{C}(X)$ of left $\Lambda$-modules generated by cells not contained in $u^{-1}(Y)$ and $\tilde{C}(X, Y)$ for the relative cellular complex generated by the cells not contained in $u^{-1}(Y)$. We obtain the short exact sequence of free chain complexes of left $\Lambda$-modules

$$\tilde{C}(Y) \xrightarrow{\alpha} \tilde{C}(X) \xrightarrow{\beta} \tilde{C}(X, Y)$$

The relative homology and cohomology of the pair $(X, Y)$ we work with are the abelian groups

$$H_q(X, Y; A) := H_q(A \otimes_\Lambda \tilde{C}(X, Y))$$

$$H^q(X, Y; B) := H^q(\text{Hom}_\Lambda(\tilde{C}(X, Y), B))$$

where $A$ is a right $\Lambda$-module and $B$ is a left $\Lambda$-module. Note that $H_q(X; A) = H_q(X, \emptyset; A)$ and $H^q(X; B) = H^q(X, \emptyset; B)$.

An $n$-dimensional Poincaré duality pair (PD$^n$-pair) comprises a pair $(X, \partial X)$ of $CW$–complexes, an orientation character, $\omega : G \to \mathbb{Z}/2\mathbb{Z}$, and a fundamental class, $[X, \partial X] \in H_n(X, \partial X; \mathbb{Z}^\omega)$, such that the fundamental group, $G$, is finitely presentable, each connected component $Y_i$ of $\partial X$ is a PD$^{n-1}$–complex $(Y_i, \omega_i, [Y_i])$ with $\omega_i = \omega|_{Y_i}$ and $[Y_i]$ equal to the image of $[X, \partial X]$ in $H_2(Y_i, \mathbb{Z}^\omega)$ under the connecting homomorphism and, for every $r \in \mathbb{Z}$ and left $\Lambda$-module $M$, the cap product with $[X, \partial X]$,

$$\alpha \leadsto [X]: H^r(X, \partial X; M) \to H_{n-r}(X; M^\omega), \quad \alpha \mapsto \alpha \cap [X]$$

is an isomorphism of abelian groups. We denote this by $((X, \partial X), \omega, [X, \partial X])$ or simply by $(X, \partial X)$.

An oriented map of PD$^n$–pairs $(X_1, \partial X_1)$ and $(X_2, \partial X_2)$ is a map of pairs, $f : (X_1, \partial X_1) \to (X_2, \partial X_2)$, with $f_*([X_1, \partial X_1]) = [X_2, \partial X_2]$. 
2.4. **Fundamental Triples of PD³–Pairs.** We take a pair of CW–complexes, \((X, \partial X)\), with \(X\) connected. We fix base points in \(X\) and in each of the connected components \(\partial X_i, i \in J,\) of \(\partial X\). We write \(G\) and \(G_i\) for the fundamental groups with respect to the chosen base points in \(X\) and \(\partial X_i\), respectively. Further, we fix paths \(\gamma_i\) from the base point of \(X\) to the base points of \(\partial X_i\), for every \(i \in J\). Then the paths \(\gamma_i\) together with the inclusions \(\partial X_i \hookrightarrow X, i \in J\), yield a family of group homomorphisms \(\{\kappa_i : G_i \to G\}_{i \in J}\), also called the \(\pi_1\)-system of \((X, \partial X)\).

Given a family of group homomorphisms, \(\{\kappa_i : G_i \to G\}_{i \in J}\), an **Eilenberg–Mac Lane pair** of type \(K(\{\kappa_i : G_i \to G\}_{i \in J}; 1)\) is a pair \((K, \partial K)\) such that \(X\) is an Eilenberg–Mac Lane complex of type \(K(G; 1)\), the connected components \((\partial K_i)_{i \in J}\) of \(\partial K\) are Eilenberg–Mac Lane complexes of type \(K(G_i; 1)\) and the \(\pi_1\)-system of \((X, \partial X)\) is isomorphic to \(\{\kappa_i : G_i \to G\}_{i \in J}\). Note that we do not require the homomorphisms \(\kappa_i\) to be injective as in the standard definition given by Bieri–Eckmann [1]. For any \(\pi_1\)-system, \(\{\kappa_i : G_i \to G\}_{i \in J}\), there is an Eilenberg–Mac Lane pair of type \(K(\{\kappa_i : G_i \to G\}_{i \in J}; 1)\), which is uniquely determined up to homotopy equivalence of pairs.

A **fundamental triple of PD³–pairs** is a family of group homomorphisms, \(\{\kappa_i : G_i \to G\}_{i \in J}\), where \(G_i\) is the fundamental group of an aspherical \(PD^2\)-complex for all \(i \in J\), together with an Eilenberg–Mac Lane pair, \((K, \partial K)\), of type \(K(\{\kappa_i : G_i \to G\}_{i \in J}; 1)\), a homomorphism \(\omega : G \to \mathbb{Z}/2\mathbb{Z}\) and a homology class \(\mu \in H_3(K, \partial K; \mathbb{Z})\). A **homomorphism from the fundamental triple** \(\{\kappa_i : G_i \to G\}_{i \in J}, \omega, \mu\) to the fundamental triple \(\{\kappa'_i : G'_i \to G', \omega', \mu'\}\) is a homomorphism \(\varphi : G \to G'\) together with a family of homomorphisms \(\{\varphi_i : G_i \to G'_i\}_{i \in J}\) such that the diagram

\[
\begin{array}{ccc}
G & \xrightarrow{\varphi} & G' \\
\kappa_i & \downarrow & \kappa'_i \\
G_i & \xrightarrow{\varphi_i} & G'_i
\end{array}
\]

commutes, \(\omega' = \omega \circ \varphi\) and the homology homomorphism of the respective Eilenberg-Mac Lane pairs maps \(\mu\) to \(\mu'\). The fundamental triple is called **\(\pi_1\)-injective** if \(\kappa_i\) is injective for every \(i \in J\).

For any pair of CW–complexes, \((X, \partial X)\), there is a classifying map of pairs

\[c_X : (X, \partial X) \to K(\{\kappa_i : G_i \to G\}_{i \in J}; 1),\]

which is uniquely determined up to homotopy of pairs and induces an isomorphism of \(\pi_1\)-systems. The fundamental triple \(\{\kappa_i : G_i \to G\}_{i \in J}, \omega, c_X([X, \partial X])\) of the PD³–pair \((X, \partial X)\) is unique up to isomorphism of fundamental triples.

An oriented homotopy equivalence of PD³–pairs, \(f : (X, \partial X) \to (Y, \partial Y)\), induces an isomorphism of their fundamental triples. The Classification Theorem [3] states that the converse holds for PD³–pairs with aspherical boundaries.

3. **Constructing a Candidate for Realisation**

First we describe the conditions for realisation. Then we state the main result and construct a candidate for realisation.

Assume that the family \(\{\kappa_i : G_i \to G\}_{i \in J}\) of group homomorphisms is the \(\pi_1\)-system of a PD³–pair with aspherical boundary components. Then, by definition,

1. \(G\) is finitely presented and there are finitely many boundary components, that is, the set \(J\) is finite;
2. the connected components of \(\partial X\) are aspherical PD²–complexes and hence homotopy equivalent to closed aspherical surfaces \(S_i\) for all \(i \in J\), by results of Eckmann, Müller and Linnell ([5] and [4]).

Taking an Eilenberg-Mac Lane space \(K'\) of type \(K(G; 1)\), there is a map \(\partial X \to K'\) inducing \(\{\kappa_i : G_i \to G\}_{i \in J}\) and the mapping cylinder construction yields a CW–complex \(K\) such that \((K, \partial X)\) is an an Eilenberg-Mac Lane pair of type \(K(\{G_i \to G\}_{i \in J}; 1)\).
Further assume that \( \omega : G \to \mathbb{Z}/2\mathbb{Z} \) is a homomorphism and \( \mu \in H_3(K, \partial X; \mathbb{Z}^\omega) \), such that \( \{ \kappa_i : G_i \to G_i \}_{i \in J, \omega, \mu} \) is the fundamental triple of a PD\(^3\)-pair. Then, again by definition,

1. the homomorphism \( \omega \) restricts to the orientation characters \( \omega_i \) of the surface \( S_i \) for all \( i \in J \);
2. \( \delta_\ast \mu = [\partial X] = \sum_{i \in J} [\partial X_i] \), where \([\partial X_i]\) is the fundamental class of the PD\(^2\)-complex \( \partial X_i \) for all \( i \in J \), and \( \delta_\ast \) is the connecting homomorphism of \( \mathbb{Z}^\omega \otimes_A \tilde{C}(\partial X) \to \mathbb{Z}^\omega \otimes_A \tilde{C}(K) \to \mathbb{Z}^\omega \otimes_A \tilde{C}(K, \partial X) \).

Poincaré duality is manifest on the level of chains, namely, for a PD\(^3\)-complex of free left \( \Lambda \)-modules called the PD\(^3\)-pair, \((X, \partial X)\), product with a representative of the fundamental class, \([T]\), and \( \omega \hat{\cdot} \). Writing Theorem 4.7 in [3] is a generalisation of Turaev’s realisation condition to IC\(^3\)-pairs. It uses the Turaev map \( \nu \) and states that

\[ \nu(\mu) \text{ is an isomorphism in the stable module category of } \Lambda = \mathbb{Z}[G]. \]

Our main result states that conditions (1) to (5) are sufficient.
Theorem 1. Let \( \{ \kappa_i : G_i \to G \}_{i \in J} \) be a finite family of group homomorphisms, where \( G \) is finitely presentable and \( G_i \) is the fundamental group of an aspherical closed surface for every \( i \in J \). Let \((K, \partial X)\) be an Eilenberg–Mac Lane pair of type \( K(\{\kappa_i\}_{i \in J}; 1) \) such that the \( \Lambda \)-modules \( F_{\mu}^h K \) are surfaces for all \( i \in J \). Take a homology class \( \mu \in H_3(K, \partial X; \mathbb{Z}^w) \) such that \( \delta_\mu = [\partial X] = \sum_{i \in J} [\partial X_i] \), where \([\partial X_i]\) is the fundamental class of the PD\(2\)-complex \( \partial X_i, i \in J \), and \( \delta_\mu \) is the connecting homomorphism of \( \mathbb{Z}^w \otimes_\Lambda \hat{C}(\partial X) \to \mathbb{Z}^w \otimes_\Lambda \hat{C}(K, \partial X) \).

Then \( \{ \{\kappa_i\}_{i \in J}, \omega, \mu \} \) is realised by a PD\(3\)-pair \((X, \partial X)\) if and only if \( \nu(\mu) \) is an isomorphism in the stable module category of \( \Lambda = \mathbb{Z}[G] \).

Restricting to pairs with empty boundaries recovers Turaev’s Realisation Theorem for PD\(3\)-complexes [8].

Now take a family of group homomorphisms \( \{ \kappa_i : G_i \to G \}_{i \in J} \), where \( G \) is finitely presented and \( G_i \) is a surface group for all \( i \in J \). Take aspherical closed surfaces \( S_i \) with \( \pi_1(S_i) = G_i \) and a homomorphism \( \omega : G \to \mathbb{Z}/2\mathbb{Z} \), such that \( \omega \circ \kappa_i = \omega_i \), where \( \omega_i \) is the orientation character of \( S_i \) for all \( i \in J \), and an Eilenberg–Mac Lane space \( K^2 \) of type \( K(G, 1) \). The family of homomorphisms \( \{ \kappa_i \}_{i \in J} \) induces a map \( f : \prod_{i \in J} S_i \to K^2 \).

The family of homomorphisms \( \{ \kappa_i \}_{i \in J} \) induces a map \( f : \prod_{i \in J} S_i \to K^2 \). Let \( K' \) be the mapping cylinder of \( f \) and identify \( \partial X := \prod_{i \in J} S_i \) with its image under the inclusion in \( K' \) to obtain the Eilenberg-Mac Lane pair \((K', \partial X)\) of type \( K(\{\kappa_i : G_i \to G\}_{i \in J}; 1) \) and the short exact sequence

\[
\begin{align*}
\hat{C}(\partial X) &\longrightarrow \hat{C}(K') \longrightarrow \hat{C}(K', \partial X)
\end{align*}
\]

of chain complexes of modules over \( \Lambda = \mathbb{Z}[G] \). As the connected components \( S_i \) of \( \partial X \) are closed surfaces, we may assume that \( \hat{C}_2(\partial X) = \bigoplus_{i \in J} \mathbb{Z}[x_i] \), where \( 1 \otimes x_i \) represents the fundamental class of \( S_i \) for all \( i \in J \).

Take a homology class \( \mu \in H_3(K', \partial X; \mathbb{Z}^w) \), such that \( \nu(\mu) \) is a class of homotopy equivalences and \( \delta_\mu = [\partial X] = \sum_{i \in J} [1 \otimes x_i] \), where \( \delta_\mu \) is the connecting homomorphism of

\[
\begin{align*}
\mathbb{Z}^w \otimes_\Lambda \hat{C}(\partial X) &\longrightarrow \mathbb{Z}^w \otimes_\Lambda \hat{C}(K') \longrightarrow \mathbb{Z}^w \otimes_\Lambda \hat{C}(K', \partial X)
\end{align*}
\]

Since \( G \) is assumed finitely presentable, we may also assume that \( K' \) has finite 2–skeleton \( (K')[2] \), so that the \( \Lambda \)–modules \( \hat{C}_2(K', \partial X) \) and \( F^2(\hat{C}(K', \partial X)) \) are finitely generated.

Let \( h : F^2(\hat{C}(K', \partial X)) \longrightarrow I \) be a \( \Lambda \)–morphism representing \( \nu(\mu) \). By Theorem 4.1 and Observation 1 in [3], \( h \) factors as

\[
F^2(\hat{C}(K', \partial X)) \longrightarrow F^2(\hat{C}(K', \partial X)) \oplus \Lambda^m \longrightarrow I \oplus P \longrightarrow I
\]

for some projective \( \Lambda \)–module, \( P \), and \( m \in \mathbb{N} \). Let \( B = (e^0 \cup e^2) \cup e^3 \) be the 3-dimensional ball and replace \( K' \) by the Eilenberg–Mac Lane space \( K = K' \cup \left( \bigvee_{i=1}^m B \right) \). Then \( F^2(\hat{C}(K, \partial X)) = F^2(\hat{C}(K', \partial X)) \oplus \Lambda^m \) and the factorisation of \( h \) becomes

\[
\begin{align*}
h : F^2(\hat{C}(K, \partial X)) &\longrightarrow I \oplus P \stackrel{pr_1}{\longrightarrow} I
\end{align*}
\]

with \( j \) surjective. We consider the \( \Lambda \)–morphism

\[
\varphi : \hat{C}^2(K, \partial X) \longrightarrow F^2(\hat{C}(K, \partial X)) \longrightarrow I \oplus P \stackrel{pr_1}{\longrightarrow} I \stackrel{i}{\longrightarrow} \Lambda
\]

where \( p \) is the projection onto the co-kernel, and \( i : I \hookrightarrow \Lambda \) is the inclusion.

Let \( i_\partial \) and \( i_{rel} \) be the natural inclusions of \( \hat{C}_2(\partial X) \) and \( \hat{C}_2(K, \partial X) \) in the direct sum \( \hat{C}_2(K) = \hat{C}_2(\partial X) \oplus \hat{C}_2(K, \partial X) \), respectively, and let \( p_\partial \) and \( p_{rel} \) be the corresponding natural projections. Note that \( \hat{C}_3(K, \partial X) \cong \hat{C}_3(K) \) and take \( y \in \hat{C}_3(K) \) with \( \mu = [1 \otimes p_{rel}(y)] \). Then \( (\hat{C}^2(K) ; p_\partial^*, p_{rel}^*) \)
is the direct sum of $\widehat{C}^2(\partial X)$ and $\widehat{C}^2(K, \partial X)$ and the maps $\varphi_\partial := \omega \varphi_{\partial}(d)\overline{\varphi}$ and $\varphi$ determine a unique map $\psi : \widehat{C}^2(K) \to \Lambda$ such that $\varphi = \psi \circ p^*_r$ and $\varphi_\partial = \psi \circ p^*_\partial$.

As $\widehat{C}^2(K)$ is a finitely generated free $\Lambda$-module, the natural map $\varepsilon : \widehat{C}^2(K) \to \widehat{C}^2(K)**$ is an isomorphism and we define

$$d := \varepsilon^{-1} \circ \psi^* : \Lambda^* \to \widehat{C}^2(K)$$

**Proposition 2.** $d_2 \circ d = 0$.

**Proof.** Note that it is sufficient to show $\psi \circ d_2^* = 0$, as this implies

$$0 = (\psi \circ d_2^*)^* = d_2^* \circ \psi^* = \varepsilon \circ d_2 \circ \varepsilon^{-1} \circ \psi^* = \varepsilon \circ d_2 \circ d$$

where the third equality uses that $\varepsilon$ is a natural equivalence. As $\varepsilon$ is an isomorphism, we conclude $d_2 \circ d = 0$.

To show that $\psi \circ d_2 = 0$, take $\alpha \in \widehat{C}^1(K)$. Note that $i_\partial \circ p_\partial + i_{rel} \circ p_{rel} = \text{id}_{\widehat{C}^2(K)}$ and put $x := p_{rel}y$. Then, for $\beta = \alpha \circ i_{rel} \in \widehat{C}^2(K, \partial X)$,

$$\varphi(\beta) = \hat{\nu} \circ \delta_3([1 \otimes x])([\beta]) = \hat{\beta}(p_{rel}(d_3(y)))$$

Thus

$$(\psi \circ d_2^*)(\alpha) = \varphi_\partial(\alpha \circ d_2 \circ i_\partial) + \varphi(\alpha \circ d_2 \circ i_{rel}) = \alpha \circ d_2 \circ d_3(y) = 0$$

Since $\hat{K}[2]$ is 1-connected, the Hurewicz homomorphism is an isomorphism so that

$$\text{im} \ d \subseteq \ker \ d_2 = H_2(\hat{K}[2]) \cong \pi_2(\hat{K}[2])$$

Thus we may attach a 3-cell to $K[2]$ to obtain a CW-complex $X$ whose universal cover has cellular chain complex

$$\Lambda^* \xrightarrow{d} \widehat{C}^2(K) \xrightarrow{d_2} \widehat{C}^1(K) \xrightarrow{} \widehat{C}_0(K)$$

As $\pi_2(K) = 0$, the inclusion of $(K[2], \partial X)$ in $(K, \partial X)$ extends to a map $f : (X, \partial X) \to (K, \partial X)$.

The next section is devoted to showing that $(X, \partial X)$ realises the fundamental triple $\{(\kappa_i)_{i \in J}, \omega, \mu\}$.

4. Verifying Realisation

To show that the pair $(X, \partial X)$ constructed in Section 3 is a $PD^3$-pair with fundamental triple $\{(\kappa_i)_{i \in J}, \omega, \mu\}$ it is enough to prove

**Proposition 3.**

(i) $H_3(X, \partial X; \mathbb{Z}^\omega) \cong \mathbb{Z}$ generated by $[X, \partial X] = [1 \otimes \text{id}]$

(ii) $\delta_4[X, \partial X] = [\partial X]$

(iii) $f_*([X, \partial X]) = \mu$

(iv) $\cap[X, \partial X] : H^r(X; \mathbb{Z}^\omega) \to H_{r-3}(X, \partial X; \Lambda)$ is an isomorphism for every $r \in \mathbb{Z}$. 

Proof. (i) The relative cellular chain complex $\tilde{C}(X, \partial X)$ is given by

$$\Lambda^* \xrightarrow{d_{rel}} \tilde{C}_2(K, \partial X) \xrightarrow{d_{2rel}} \tilde{C}_1(K, \partial X) \rightarrow \tilde{C}_0(K, \partial X)$$

where

$$d_{rel} = p_{rel} \circ d = p_{rel} \circ \varepsilon^{-1} \circ \psi^* = \varepsilon^{-1} \circ p_{rel}^* \circ \psi^* = \varepsilon^{-1} \circ (\psi \circ p_{rel}^*) = \varepsilon^{-1} \circ \varphi^*$$

As $\tilde{C}(X, \partial X)$ a chain complex of finitely generated free modules, the natural transformation $\eta$ is a natural equivalence. We write $\zeta^+$ for the morphism $\text{Hom}_A(B, \mathbb{Z}) \rightarrow \text{Hom}_A(A, \mathbb{Z})$ induced by $\zeta: A \rightarrow B$ and note that $\varepsilon^+ \circ \varepsilon = \text{id}^+$ to obtain

$$1 \otimes d_{rel} = \eta^{-1} \circ (\varphi^+ \circ (\varepsilon^{-1})^+ \circ \eta) = \eta^{-1} \circ \varphi^+ \circ \varepsilon^+ \circ \eta$$

Thus $H_3(X, \partial X; \mathbb{Z}^\omega) \cong \ker(\varphi^+)$. By $\zeta^+ \varphi = i \circ \text{pr}_1 \circ j \circ p$. Since $p, j$ and $\text{pr}_1$ are surjective, $p^+, j^+$ and $\text{pr}_1^+$ are injective, so that

$$\ker(\varphi^+) = \ker(i \circ \text{pr}_1 \circ j \circ p)^+ \cong \ker(i^+)$$

But $I$ is generated by elements $1 - g$ ($g \in G$) and, for all $g \in G, \alpha \in \text{Hom}_A(\Lambda, \mathbb{Z})$, we obtain $i^+(\alpha)(1 - g) = (\alpha \circ i)(1 - g) = 0$. Hence, $\ker(i^+) = \text{Hom}_A(\Lambda, \mathbb{Z}) \cong \mathbb{Z}$ is generated by $\text{aug}$ and

$$H_3(X, \partial X; \mathbb{Z}^\omega) \cong \ker \varphi^+ = \ker i^+ \cong \mathbb{Z}$$

For $\lambda \in \Lambda$ we compute $(\varepsilon^+ \circ \eta \circ (1 \otimes \text{id}))(\lambda) = \text{aug}(\lambda)$, where $\text{id} = \text{id}_\Lambda$, and conclude that $H_3(X, \partial X; \mathbb{Z}^\omega)$ is generated by $[X, \partial X] = [1 \otimes \text{id}]$.

(ii) First compute

$$p_\partial(d\text{id}) = (\varepsilon^{-1} \circ (\text{ev}_{p_\partial(d3y)})^*)(\text{id}) = p_\partial(d3y)$$

Hence

$$\delta_*([X, \partial X]) = [1 \otimes p_\partial(d3y)] = \delta_*(\mu) = [\partial X]$$

(iii) The restriction of the map $f : (X, \partial X) \rightarrow (K, \partial X)$ to the 2-skeleta is the identity, so that (ii) and naturality of $\delta_*$ yield

$$\delta_*(f_*(1 \otimes \text{id}) - \mu) = f_*\delta_*(1 \otimes \text{id}) - \delta_*(\mu) = 0$$

Now put $z = f_*(\text{id}) - y \in \tilde{C}_3(K) = \tilde{C}_3(K\partial, X)$, so that $1 \otimes z$ is a cycle in $\mathbb{Z}^\omega \otimes \tilde{C}_3(K)$. The map $\nu([1 \otimes z])$ is represented by

$$F^2(\tilde{C}(K)) \rightarrow I, \quad \alpha \mapsto \overline{\alpha(d3z)}$$

Recall that $\nu(\mu) = \nu([1 \otimes p_{rel}(y)])$ is represented by $h$ and, by (ii), $p_\partial(d\text{id}) = p_\partial(d3y)$ so that, for $[\alpha] \in F^2(\tilde{C}(K))$

$$\overline{\alpha(d3z)} = \overline{(\alpha \circ i_{rel} \circ p_{rel}(d3z))} = \overline{(\alpha \circ i_{rel})(\varepsilon^{-1} \circ (\psi \circ p_{rel}^*)^*(\text{id})) - h((\alpha \circ i_{rel}))} = \overline{(\alpha \circ i_{rel})(\varepsilon^{-1}(\varphi)) - h((\alpha \circ i_{rel}))} = \overline{(i \circ h \circ p)(\alpha \circ i_{rel}) - h((\alpha \circ i_{rel}))} = 0$$

Hence $\nu([1 \otimes z]) = 0$. As $\nu$ is injective by Lemma 2.5 in [8], we conclude $[1 \otimes z] = 0$ and thus

$$f_*([X, \partial X]) = f_*([1 \otimes \text{id}]) = \mu$$

(iv) The definition of $\varphi$ in [3] implies $H^2(X, \partial X; \Lambda^\omega) = 0$. Since $H_1(X; \Lambda) = H_1(\tilde{C}(X)) = 0$, the homomorphism

$$\cap[X, \partial X] : H^2(X, \partial X; \Lambda^\omega) \rightarrow H(X; \Lambda)$$

is an isomorphism.
As $\hat{C}(X, \partial X)$ a chain complex of finitely generated free modules, the natural transformation $\varepsilon$ is a natural equivalence and we obtain
\[ H^3(X, \partial X; \omega \Lambda) \cong \omega \operatorname{Hom}_\Lambda(\Lambda^*, \omega \Lambda)/\operatorname{im} \varphi^{**} \cong \Lambda/I \cong \mathbb{Z} \]
The class $[\gamma]$ of the image of $1 \in \Lambda$ under the (twisted) evaluation isomorphism generates $H^3(X, \partial X; \omega \Lambda)$ and so, by Lemma 4.4 of [3],
\[ [\gamma] \cap [X, \partial X] = [\gamma] \cap [1 \otimes \text{id}] = [\gamma(\text{id})].e_0 = [\text{id}(1).e_0] = [e_0] \]
where $e_0$ is a chain representing the basepoint. Hence $\cap [X, \partial X] : H^3(X, \partial X; \omega \Lambda) \to H_0(X; \Lambda)$ is an isomorphism.

Since $\partial X$ is a PD$^2$-complex, $\cap [\partial X] : H^r(\partial X; \omega \Lambda^e) \to H_{2-r}(\partial X; \Lambda)$ is an isomorphism for every $r \in \mathbb{Z}$. Thus the Cap Product Ladder of $(X, \partial X)$ (Theorem 2.2 in [3]) with $y = [X, \partial X]$, and the Five Lemma imply that $\cap [X, \partial X] : H^r(X; \omega \Lambda^e) \to H_{r-3}(X, \partial X; \Lambda)$ is an isomorphism for $r = 2$ and $r = 3$. Identifying $\omega \Lambda$ with $\Lambda$ and $\Lambda \otimes_A A$ with $A$, we obtain the chain homotopy equivalence

\[
\begin{array}{ccccccccc}
0 & \overset{\text{im} \partial^*_1}{\longrightarrow} & \hat{C}^2(X) & \overset{\cap [1 \otimes x]}{\longrightarrow} & \hat{C}^3(X) & \overset{\partial^*_1}{\longrightarrow} & 0 \\
0 & \overset{\text{im} \partial^*_1}{\longrightarrow} & \hat{C}^1(X, \partial X) & \overset{\cap [1 \otimes x]}{\longrightarrow} & \hat{C}^2(X, \partial X) & \overset{\partial^*_1}{\longrightarrow} & 0
\end{array}
\]

Applying $\omega \operatorname{Hom}_\Lambda(-, \Lambda)$ yields the chain homotopy equivalence

\[
\begin{array}{ccccccccc}
0 & \overset{\text{im} \partial^*_1}{\longrightarrow} & \hat{C}^0(X, \partial X) & \overset{\cap [1 \otimes x]}{\longrightarrow} & \hat{C}^1(X, \partial X) & \overset{\partial^*_1}{\longrightarrow} & (\text{im} \partial^*_1)^* & \overset{0}{\longrightarrow} \\
0 & \overset{\text{im} \partial^*_1}{\longrightarrow} & \hat{C}^3(X)^* & \overset{\cap [1 \otimes x]^*}{\longrightarrow} & \hat{C}^2(X)^* & \overset{\partial^*_1}{\longrightarrow} & (\text{im} \partial^*_1)^* & \overset{0}{\longrightarrow}
\end{array}
\]

which shows that $(\cap [1 \otimes x])^*$ induces homology isomorphisms. By Lemma 2.1 in [3], $\cap (1 \otimes x)$ induces isomorphisms in homology if and only if $(\cap [1 \otimes x])^*$ does. Thus the homomorphism
\[ \cap [X, \partial X] : H^k(X, \partial X; \omega \Lambda) \to H_{3-k}(X; \Lambda) \]
is an isomorphism for $k = 0$ and $k = 1$. Again using the Cap Product Ladder of $(X, \partial X)$ with $y = [X, \partial X]$ and the Five Lemma, we conclude that
\[ \cap [X, \partial X] : H^r(X; \omega \Lambda) \to H_{r-3}(X, \partial X; \Lambda) \]
is an isomorphism for $r = 0$ and $r = 1$ and hence for every $r \in \mathbb{Z}$. $\square$

This concludes the proof of Theorem $\Box$

5. Splitting

There are two distinct notions of connected sum for $PD^3$–pairs reflecting the situation of manifolds with boundary, the interior connected sum and the connected sum along boundary components. The partial splitting results for $PD^3$–pairs with injective $\pi_1$–system and aspherical boundary components obtained in [3] only use $\pi_1$–injectivity to allow for the application of the realisation theorem. Thus Theorem $\Box$ immediately yields splitting results for $PD^3$–pairs with aspherical boundary components.

5.1. Interior Connected Sum. For the definition of the interior connected sum we use that, for every $PD^3$–pair, $(X, \partial X)$, there is a homologically 2–dimensional $CW$–complex $K$ and a map $f : S^2 \to K$, such that
\[ X \sim K \cup_f e^3 \]
where the pair $(K, f)$ is unique up to homotopy by Proposition 5.1 in [2]. Given two $PD^3$–pairs $(X_1, \partial X_1)$ and $(X_2, \partial X_2)$, write $X_\ell = K_\ell \cup_{f_\ell} e_3^\ell$ for $\ell = 1, 2$, and let $\iota_\ell : K_\ell \to K_1 \vee K_2$ be the inclusion of the first and second factor respectively. Then $f_\ell := \iota_\ell \circ f_\ell : S^2 \to K_1 \vee K_2$ determines
an element of $\pi_2(\mathcal{K}_1 \lor \mathcal{K}_2)$, and we put $f_1 + f_2 := \hat{f}_1 + \hat{f}_2$. Up to oriented homotopy equivalence, the pair

$$(X, \partial X) := (X_1, \partial X_1) \lor (X_2, \partial X_2) := ((K_1 \lor K_2) \cup f_1 + f_2, \partial X_1 \cup \partial X_2)$$

is a $PD^3$–pair uniquely determined by $(X_i, \partial X_i)$, $i = 1, 2$. It is called the interior connected sum of $(X_1, \partial X_1)$ and $(X_2, \partial X_2)$ [3].

If the $PD^3$–pair $(X, \partial X)$ is oriented homotopy equivalent to the interior connected sum of two $PD^3$–pairs $(X_k, \partial X_k)$ with $\pi_1$–systems $\{\kappa_{kt} : G_{kt} \to G_k\}_{t \in J_k}$ for $k = 1, 2$, then the $\pi_1$–system of $(X, \partial X)$ is isomorphic to $\{\kappa_t \circ \kappa_{kt} : G_{kt} \to G_1 \ast G_2\}_{t \in J_1 \times J_2, k=1,2}$, where $\kappa_t : G_k \to G_1 \ast G_2$ denotes the inclusion of the factor in the free product of groups for $k = 1, 2$. We then say that the $\pi_1$–system of $(X, \partial X)$ decomposes as free product. The converse holds for $PD^3$–pairs with aspherical boundaries.

**Theorem 4.** Let $(X, \partial X)$ be a $PD^3$–pair with aspherical boundary components. Then $(X, \partial X)$ is a non-trivial interior connected sum of two $PD^3$–pairs if and only if its $\pi_1$–system is a non-trivial free product of two $\pi_1$–systems.

**Proof.** Use Theorem [1] and the proof of Decomposition I in [3].

5.2. **Boundary Connected Sum.** Let $(X_1, \partial X_1)$ and $(X_2, \partial X_2)$ be two $PD^3$–pairs with connected boundary components $\{\partial X_1i\}_{i \in J_1}$ and $\{\partial X_2j\}_{j \in J_2}$ respectively. Choosing $\ell_k \in J_k$, $k = 1, 2$, we may assume that $\partial X_{1\ell_1}$ and $\partial X_{2\ell_2}$ are collaried in $X_1$ and $X_2$ respectively, and that there are discs $\partial X_{1\ell_1} \subseteq \partial X_{1\ell_1}$ and $\partial X_{2\ell_2} \subseteq \partial X_{2\ell_2}$. We denote the chains corresponding to $e_1^f$ and $e_2^f$ by $e_1^f$ and $e_2^f$ respectively, and the quotient of $X_1\coprod X_2$ obtained by identifying $e_1^f$ and $e_2^f$ via an orientation reversing map by $X_1\coprod X_2/\sim$. For subsets $A_i \subseteq X_i$, $i = 1, 2$, we denote the image of $A_i\coprod A_2$ under the canonical projection $\pi : X_1\coprod X_2 \to X_1\coprod X_2/\sim$ by $A_1\coprod A_2/\sim$. If we assume that $\partial X_{1\ell_1}$ and $\partial X_{2\ell_2}$ are closed surfaces, then $\partial X_{1\ell_1}\cup\partial X_{2\ell_2} := (\partial X_{1\ell_1} \setminus e_1^f) \coprod (\partial X_{2\ell_2} \setminus e_2^f)/\sim$ is homotopy equivalent to the connected sum of $\partial X_{1\ell_1}$ and $\partial X_{2\ell_2}$ as 2–manifolds.

Up to oriented homotopy equivalence, the pair

$$(X_1, \partial X_1)\lor_{\ell_1, \ell_2} (X_2, \partial X_2) := (X_1\coprod X_2/\sim, (\partial X_1 \setminus e_1^f) \coprod (\partial X_2 \setminus e_2^f)/\sim)$$

is a $PD^3$–pair determined by $(X_i, \partial X_i)$, $i = 1, 2$, and $(\ell_1, \ell_2) \in J_1 \times J_2$. It is called the boundary connected sum of $(X_1, \partial X_1)$ and $(X_2, \partial X_2)$ along the boundary components $\partial X_{1\ell_1}$ and $\partial X_{2\ell_2}$.

If the $PD^3$–pair $(X, \partial X)$ is oriented homotopy equivalent to the boundary connected sum of two $PD^3$–pairs $(X_k, \partial X_k)$ with $\pi_1$–systems $\{\kappa_{kt} : G_{kt} \to G_k\}_{t \in J_k}$, $k = 1, 2$, along the boundary components $\partial X_{1\ell_1}$ and $\partial X_{2\ell_2}$, the $\pi_1$–system of $(X, \partial X)$ is isomorphic to $\{\kappa_t : H \to G_1 \ast G_2, \kappa_t \circ \kappa_{kt} : G_{kt} \to G_1 \ast G_2\}_{t \in J_1 \times J_2, k=1,2}$, where $H := \pi_1(\partial X_{1\ell_1}\cup\partial X_{2\ell_2})$ and $\kappa : H \to G_1 \ast G_2$ is induced by the inclusion of the connected sum of the boundary components $\partial X_{1\ell_1}$ and $\partial X_{2\ell_2}$ in the connected sum of the pair. We then say that the $\pi_1$–system of $(X, \partial X)$ decomposes as free product along $G_{1\ell_1}$ and $G_{2\ell_2}$. The converse holds for $PD^3$–pairs with non-empty aspherical boundaries.

**Theorem 5.** Let $(X, \partial X)$ be a $PD^3$–pair with non–empty aspherical boundary components. Then $(X, \partial X)$ is a non-trivial boundary connected sum of two $PD^3$–pairs $(X_k, \partial X_k)$, $k = 1, 2$, along $\partial X_{1\ell_1}$ and $\partial X_{2\ell_2}$, if and only if its $\pi_1$–system is a non-trivial free product of two $\pi_1$–systems along $\pi_1(\partial X_{1\ell_1})$ and $\pi_1(\partial X_{2\ell_2})$.

**Proof.** Use Theorem [1] and the proof of Decomposition II in [3].

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