Malliavin differentiability and regularity of densities in semi-linear stochastic delay equations driven by weighted fractional Brownian motion

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Abstract

In this work, we will show the existence and uniqueness of the solution to semi-linear stochastic differential equations driven by weighted fractional Brownian motion with delay. We also prove smoothness of the density of the solution with respect to Lebesgue’s measure on $\mathbb{R}^d$ for $d \geq 1$.

Keywords: stochastic delay differential equations, Malliavin calculus, weighted fractional Brownian motion, regular density.

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1. Introduction

Theory of stochastic differential equations (SDEs) driven by fractional Brownian motion (fBm) is widely studied using different approach (see for instance [7, 13, 23, 28–30]). These models have many applications in finance, telecommunications, image processing and turbulence; [4, 24, 33, 36].

One of great interest area in study of fractional SDEs is on investigating the existence, uniqueness and regularity of the density of solutions to SDEs. When $H > \frac{1}{4}$, the existence and uniqueness of the solution are obtained by Lyons in [21], Zahle in [40], Nualart-Rascanu in [30] and by using Young’s integration theory in [5]. The problem of existence and uniqueness of solution is considered in [22] for a Hurst parameter $H > 1/3$, and extended to $H > 1/4$ in [37]. Recently, when the drift is locally Lipschitz and unbounded in the neighborhood of the origin, particularly to the mean-reverting stochastic volatility models in finance, the existence and positivity of a unique solution have studied in [41].

In the presence of delay in SDEs, Wei and Wang [38] considered the problem of the existence
and uniqueness of the solution to stochastic functional differential equations with infinite delay. Caraballo et al. [13] have studied the existence, uniqueness and exponential asymptotic behavior of mild solutions to stochastic delay evolution equations driven by a fractional Brownian motion. In 2015, Boudaoui et al. [11] showed the existence of mild solutions to stochastic impulsive evolution equations with time delays via a new fixed point analysis approach. It is worth mentioning that in these two last articles, the diffusion coefficient functions are dependent only on the parameter $t$ and are not dependent on the state process. More studies have discussed in [15] and then in [12] on stochastic delayed differential equations (SDDEs) with fractional Brownian motion when the diffusion functions depend upon the past value of the state process with time delay $\tau$ (actually, as $B = 0$ in SDDE (1)).

The Malliavin calculus is a useful tool to study the regularity of the densities of the solutions to SDEs. We refer the reader to [26] and [27] for more details of this theory. The problem of smoothness of density of the solutions to stochastic differential equations driven by a fractional Brownian motion with Hurst parameter greater than $1/2$ in the one-dimensional case is solved in [25] by using Doss-Süssman methods and in [31] and [19] under ellipticity assumptions. Rovira and Ferrante [15] established the existence and regularity of density of SDEs with delay via Young’s integration. The authors considered a SDE with the diffusion coefficient function depends upon only the past value of solution with time delay; i.e., it depends on $X(t)$ and time delay $\tau$. In 2012, the authors [20] have shown the existence of a $C^\infty$-density to a general class of Young delay equations driven by fBm with Hurst parameter $H > 1/2$.

The weighted fractional Brownian motion (wfBm) is a general expression for the fractional Brownian motion introduced in Bojdecki et al. [8]. Due to the memory effects of various phenomena in the real world (such as a wide variety of natural and financial markets), it would be reasonable to replace fBm by weighted fBm. In this area, a few studies can be found in some especial subjects; see for instance Bojdecki et al. [8, 10], Garzón [16], Shen et al. [34, 35], Yan et al. [39], and the references therein.

In current paper, we consider the following semi-linear SDDE of the form

\[
\begin{align*}
\frac{dx(t)}{dt} &= \left( Ax(t) + f(x(t-\tau)) \right) dt \quad + \quad \left( Bx(t) + \sigma(x(t-\tau)) \right) \delta^{a,b} B^{a,b}(t), \quad t \in [0, T], \\
x(t) &= \xi(t), \quad t \in [-\tau, 0],
\end{align*}
\]  

(1)
where $A$ is a $d \times d$-matrix and the stochastic integral is a type of Skorohod integral with respect to weighted fBm. We recall some basic elements of this stochastic calculus in section 3.

In the case of weighted fractional Brownian motion, to the best of our knowledge, there is no paper which study the existence and smoothness of the density of the solution to linear SDDE (1). We are interested to consider Malliavin differentiability and smoothness of the density of the solution to SDDE (1) under the usual globally Lipschitz conditions on coefficient functions.

**Assumption 1.1.** Let $K_0$ is the largest integer number that $K_0 \tau < T$ and $M$ is an integer number such that $K_0 + 1 < 2M$.

- Functions $f$ and $\sigma$ are continuous and $2M$-times differentiable, whose their derivatives are bounded with some constant $K_2$.

- There exist some positive constants $L$ and $K_1$ such that for every $x, y \in \mathbb{R}^d$ and $t, s \in [0, T]$
  
  $|f(y_1) - f(y_2)|^2 + |\sigma(y) - \sigma(x)|^2 \leq K_1|y_1 - y_2|^2, \quad |f(y)|^2 + |\sigma(x)|^2 \leq L \left(1 + |y|^2\right).$

In addition, to obtain smoothness of the density, we also assume that the function $\sigma$ has a lower bound greater than zero, confirming a Hörmander’s type condition.

Our main approach to prove these problems is inspired from [15]. More precisely, we shall construct the solution of SDDE (1) step by step within the intervals $[i\tau, (i + 1)\tau]$, for any integer $i = 1, \cdots, K_0$, and then in the interval $[K_0 \tau, T]$. We successively determine a comprehensive exposition for its Malliavin derivatives throughout the steps. This exposition will allow us to achieve some upper bounds for Malliavin derivatives of the solution.

In order to obtain these results, the need for suitable upper bound inequality on supremum of stochastic Skorokhod integral generated by wfBm is essential. This inequality is a general type of $L^p$-maximal inequality in [3] on stochastic integral with respect to wfBm (Theorem 4.1).

The organization of the article is as follows: In sections 2 and 3, we recall some basic elements of weighted fractional Brownian motion and Malliavin calculus based on this type of stochastic integral. In section 4, we will prove a version of maximal inequality to achieve main results. The existence and uniqueness and Malliavin differentiability of the solution will be established in section 5. Uniformly boundedness of the moments of the solution and its Malliavin derivatives are investigated in section 6. Finally, we discuss the problem of regularity of the density in section 7.
2. Weighted fractional Brownian motion

For all parameters $a, b$ with $a > -1$, $|b| < 1$ and $|b| < a + 1$, a weighted fractional Brownian motion, denoted by $B^{a,b}$, on the complete probability space $(\Omega, F, P)$ is a mean zero Gaussian process with simple covariance function\:[35]:

$$R^{a,b}(t, s) = E[B^{a,b}(t)B^{a,b}(s)] = \int_0^{\alpha \wedge t} u^a[(t - u)^b + (s - u)^b]du, \quad s, t \geq 0.$$  

This process is $\frac{a+b+1}{2}$-self-similar, long-range dependence with Hölder paths. It is neither a semimartingale nor a Markov process if $b \neq 0$, and for some constant $c_{a,b}$ and for any $s, t \geq 0$

$$c_{a,b}(t \wedge s)^{a+b} \leq E\left[\left(B^{a,b}(t) - B^{a,b}(s)\right)^2\right] \leq K_{a,b}(t \wedge s)^{a+b}, \quad (2)$$

where $K_{a,b} = \frac{b}{2a + 1 + b}$. Some surveys and references could be found in [2, 26, 35].

When $-1 < b < 1$ and $a = 0$, covariance of the process coincides particularly with the covariance of the fractional Brownian motion with Hurst index $\frac{b+1}{2}$;\[6\]:

$$E[B^{a,b}(t)B^{a,b}(s)] = \frac{1}{b + 1}[t^{b+1} + s^{b+1} - |s - t|^{b+1}].$$

Let $H$ be the Hilbert space defined as the closure of the linear space $E$ of indicator functions \{1_{(0,t]}, t \in [0, T]\} with respect to the inner product

$$\langle 1_{(0,s)}1_{(0,t]} \rangle_H = R^{a,b}(t, s).$$

Consider the Gaussian processes $R^{a,b}(u)$ on $H$ such that for every $u_1, u_2 \in H$, $\mathbb{E}(R^{a,b}(u_1)R^{a,b}(u_2)) = \langle u_1, u_2 \rangle_H$. In \[35\] have been shown that $H = \{u : [0, T] \to \mathbb{R}; \|u\|_H < \infty\}$, where

$$\|u\|_H^2 = \int_0^T \int_0^T u(s)u(t)\phi(t, s)dtds := \int_0^T \int_0^T u(s)u(t)b(t \wedge s)^{a}(t \wedge s - t \wedge s)^{b-1}dtds, \quad (3)$$

The map $u \in E \to R^{a,b}(u)$ is an isometry from $E$ to the Gaussian space generated by $B^{a,b}$ and it can be extended to $H$. We should note that The subspace $[H]$ of measurable functions $u$ equipped by the norm

$$\|u\|_{[H]}^2 = \int_0^T \int_0^T |u(s)||u(t)||\phi(t, s)dtds < \infty, \quad (4)$$

is a Banach space and $E$ is dense in $[H]$. Moreover, Pipiras and Taqqu\[32\] have shown that as $a + b < 1$

$$L^2([0, T]) \subset L^{2/(a+b+1)}(\mathbb{R}) \subset [H] \subset H. \quad (5)$$
3. Preliminaries on Malliavin calculus

We briefly recall some Malliavin criteria on fractional Brownian motion and weighted fractional Brownian motion in \([14, 26, 35]\). When \(b > 0\), we denote by \(\mathcal{S}\) the set of smooth functionals of the form

\[
F = f \left( B^{a,b}(u_1), B^{a,b}(u_2), \cdots B^{a,b}(u_n) \right),
\]

where \(f \in C_0^\infty(\mathbb{R}^n)\) (\(f\) and all its derivatives are bounded) and \(u_i \in H\), \(i = 1, 2, \cdots, n\). For every \(F \in \mathcal{S}\), define

\[
D^{a,b} F = \sum_{i=0}^n \frac{\partial f}{\partial x_i} \left( B^{a,b}(u_1), B^{a,b}(u_2), \cdots B^{a,b}(u_n) \right) u_i.
\]

The derivative operator \(D^{a,b}\) is a closable operator from \(L^p(\Omega)\) into \(L^p(\Omega, H)\) for every \(p \geq 1\). We denote by \(\mathbb{D}^{1,p}\) the closure of \(\mathcal{S}\) respect to the norm

\[
\|F\|_{1,p}^p = E|F|^p + E\|D^{a,b} F\|_H^p.
\]

For any \(k \geq 1\), \(k\)-times iteration of the derivative operator is expressed by \(D^{a,b} \cdots D^{a,b} F\) \((k\)-times). For given Hilbert space \(V\), the corresponding sobolev space \(\mathbb{D}^{1,p}(V)\) is the domain of the derivative operator of \(V\)-valued random variable.

The adjoint of the derivative operator, called \(\delta^{a,b}(\varphi)\), is characterized by the duality relationship

\[
E \left[ F \delta^{a,b}(\varphi) \right] = E \left[ D^{a,b} F, \varphi \right]_H,
\]

for any \(F \in \mathcal{S}\). For every stochastic process \(\varphi \in \mathbb{D}^{1,2}(H)\) indefinite Skorohod integral is expressed as

\[
\delta^{a,b}(\varphi) = \int_0^T \varphi(s) \delta^{a,b} B^{a,b}(s).
\]

In a similar argument of \([26, 35]\)

\[
\int_0^T \varphi(s) dB^{a,b}(s) = \int_0^T \varphi(s) \delta^{a,b} B^{a,b}(s) + b \int_0^T \int_0^T D^{a,b}_\xi \varphi(s)(t \wedge s)^a(t \vee s - t \wedge s)^{b-1} dtds, \tag{6}
\]

providing the second summand is finite. A classical relation is that,

\[
\mathbb{E} \left[ |D^{a,b}(\varphi)|^2 \right] = \mathbb{E}\|\varphi\|_{H_n}^2 + \mathbb{E} \int_{[0,T]^4} D^{a,b}_\xi \varphi(r) D^{a,b}_\eta \varphi(\eta) \varphi(s) \phi(\eta, r) \phi(\xi, s) ds dr d\xi d\eta
\]

\[
\leq \mathbb{E}\|\varphi\|_{H_n}^2 + \mathbb{E} \int_{[0,T]^4} |D^{a,b}_\xi \varphi(r)||D^{a,b}_\eta \varphi(s)||\varphi(\eta, r)||\phi(\xi, s)| ds dr d\xi d\eta. \tag{7}
\]
For any $p > 1$ we denote by $L^1_{(a+b+1)/2}$ the set of processes $\varphi$ in $\mathbb{D}^{1,2}([H])$ such that
\[
\|\varphi\|_{L^1_{(a+b+1)/2}}^p = \mathbb{E}\|\varphi\|_{L^2(\mathbb{R}^d)}^p + \mathbb{E}\|D^{a,b}\varphi\|_{L^2(\mathbb{R}^d)}^p < \infty.
\]
In view of (5) and (7), one can easily show that for every $\varphi \in L^1_{(a+b+1)/2}$
\[
\mathbb{E}\left(\|D^{a,b}\varphi\|^p\right) \leq C_{(a+b+1)/2,p} \left(\mathbb{E}\|\varphi\|_{L^2(\mathbb{R}^d)}^p + \mathbb{E}\|D^{a,b}\varphi\|_{L^2(\mathbb{R}^d)}^p\right)
\]
(8)

4. $L^p$-maximum estimation of Skorokhod integral driven by weighted fractional integrals

In this section, we shall establish $L^p$-maximal estimation of the divergence process driven by weighted fractional Brownian motion for every $1 < p_0 < \infty$. To do this, we will apply some upper bound inequality in (8) to Hardy type operator $T_{\alpha,\beta}$ of any function $f \in L^p((a_1,b_1); v)$ defined by
\[
T_{\alpha,\beta}f(x) = \int_{a_1}^{x} \frac{u(s)W^\alpha(s)f(s)w(s)ds}{(W(x) - W(s))^{\alpha-\beta}}, \quad x \in [0, T],
\]
where $W$ is a non-negative, strictly increasing and locally absolutely continuous function on interval $I = (a_1, b_1)$ and $\frac{dW(s)}{ds} = w(s)$, and also $u$ is almost everywhere positive locally integrable function as well as $v$ is positive kernel function.

Let $\frac{1}{p_0} + \frac{1}{p} = 1$ and for every $p_0 \leq q < \infty$ denote
\[
A_{\alpha,\beta} = \sup_{s \in J} \left(\int_{a_1}^{s} u^q(s)W^\alpha(s)w(s)ds\right)^{\frac{1}{q}} \left(\int_{s}^{b_1} W^{q(\alpha-1)}(s)ds\right)^{\frac{1}{q}}.
\]
In Theorem 3.3 of (11) have been shown that for $0 < \alpha < 1$ and $\beta \geq 0$ there exists some constant $C$ such that
\[
\left(\int_{a_1}^{b_1} \left(T_{\alpha,\beta}f(x)\right)^q v(x)dx\right)^{\frac{1}{q}} \leq C \left(\int_{a_1}^{b_1} (f(x))^{p_0} w(x)dx\right)^{\frac{1}{q}}.
\]
(9)
if and only if $A_{\alpha,\beta} < \infty$. Now, we establish a $L^p$-maximal estimation for the indefinite integral
\[
\int_{0}^{t} \varphi(s)\partial^{a,b}B^{a,b}(s).
\]

**Theorem 4.1.** For $a > -1$, $|b| < 1$ such that $a + b < 1$, let $u = \{u(t), t \in [0, T]\}$ be a stochastic
process in $L^p\left((a+b+1)/2\right)$. Then for every $p > \frac{4}{a+b+1}$, there exists some constant $C_1$ such that

$$
\mathbb{E}\left(\sup_{t \in [0,T]} \left| \int_0^t u(s)\delta^{u,h} B^{u,h}_s \right|^p \right) \leq C_1 T \left\{ T \mathbb{E} \left( \frac{2^p}{a+b+1} \right) \right\}^{p(a+b+1)\lambda} 
\left( \int_0^T \mathbb{E} \left( \frac{2^p}{a+b+1} \right) \right)^{p(a+b+1)}.
$$

where the constant $C_1$ depends on $(a + b + 1)/2, p$ and $T$.

Proof. The proof is motivated by the proof of Theorem 4 in [3]. Set a positive constant $\lambda$ such that $\frac{a+b+1}{2} < \lambda < 1$ and put $a_0 = a + b + 1$ which is a positive number. Now, using the equality

$$
c_{a,b,\lambda} = \int_0^\infty \tau^4(t - \theta)^{-a_0} \lambda^a \theta^{-1} d\theta < \infty,
$$

we know that

$$
\int_0^\infty u(s)\delta^{u,h} B^{u,h}(s) = c_{a,b,\lambda}^{-1} \int_0^\infty u(s) \int_0^\infty \tau^4 (t - \tau)^{-a_0} \lambda^a (\tau - s)^{-1} d\tau \theta^{-1} B^{u,h}(s).
$$

One can apply Fubini’s stochastic theorem to result

$$
\int_0^\infty u(s)\delta^{u,h} B^{u,h}(s) = c_{a,b,\lambda}^{-1} \int_0^\infty \tau^4 (t - \tau)^{-a_0} \mathbb{E} \left( \int_0^\infty u(s) (\tau - s)^{-1} \delta^{u,h} B^{u,h}(s) \right) d\tau.
$$

From Hölder’s inequality for some other constant $c$

$$
\left| \int_0^\infty u(s)\delta^{u,h} B^{u,h}(s) \right|^p \leq c^{-1} \int_0^\infty \left| \int_0^\infty u(s)\delta^{u,h} B^{u,h}(s) \right|^p d\tau.
$$

Therefore,

$$
\mathbb{E}\left( \sup_{t \in [0,T]} \left| \int_0^\infty u(s)\delta^{u,h} B^{u,h}(s) \right|^p \right) \leq c^{-1} \mathbb{E} \left( \int_0^\infty \left| \int_0^\infty u(s)\delta^{u,h} B^{u,h}(s) \right|^p d\tau \right).
$$

Using now inequality (9) we obtain

$$
\mathbb{E}\left( \sup_{t \in [0,T]} \left| \int_0^\infty u(s)\delta^{u,h} B^{u,h}(s) \right|^p \right) \leq c^{-1} \int_0^T \left( \int_0^T \mathbb{E} \left( \frac{2^p}{a+b+1} \right) \right)^{p(a+b+1)/2} d\tau.
$$

$$
= c^{-1} C_{a,b,\lambda} (I_1 + I_2).
$$
To bound the terms \( I_1 \) and \( I_2 \), we apply the Hardy type operator inequality (9) for \( q = \frac{p(a+b+1)}{2} \), \( p_0 = \frac{\ln \ln \alpha}{\ln \alpha} \), \( \alpha = 1 + 2 \frac{(k-1)}{\ln \alpha} \), \( \beta = 2 \), \( u(.) = 1 \), \( \psi(.) = \frac{1}{t} \) and \( W(s) = s \). Under our assumptions and the way of choosing \( \lambda \), can has \( 1 < p_0 < q \) and \( 0 < \alpha < 1 \) as well as \( A_{\alpha, \beta} < \infty \). Consequently,

\[
I_1 \leq C T \left( \int_0^T \left| \mathbb{E} u(r) \right|^{\frac{2q}{p_0}} \, dr \right)^{\frac{p_0}{2q}},
\]

\[
I_2 \leq C \mathbb{E} \left( \int_0^T \left( \int_0^T \left| D_t^{1/b} u(r) \right|^{2/\alpha} \, ds \right) \, dr \right)^{\frac{p_0}{2}},
\]

which completes the proof.

According to the proof of this theorem, for any partition \( 0 \leq t_1 \cdots \leq t_N = T \) and every \( 0 \leq k \leq N \), the following corollary can be covered, if the weighted function \( \psi \) in the proof of Theorem 3.4 replaced by \( \psi = \frac{1}{t_{k+1} - t_k} \).

**Corollary 4.2.** Under conditions of Theorem 3.4 for every \( 0 \leq k \leq N \) and any partition \( 0 \leq t_1 \cdots \leq t_N = T \), there exists some constant which we denote again \( C_1 \) such that

\[
\mathbb{E} \left( \sup_{\tau \in [t_k, t_{k+1}]} \left| \int_0^t u(s) \delta^{a,b} \mathcal{B}_{t_k}^{a,b} \, ds \right|^p \right) \leq C_1 (t_{k+1} - t_k) \left( \int_{t_k}^{t_{k+1}} \left| \mathbb{E} u(s) \right|^{\frac{2q}{p_0}} \, ds \right)^{\frac{p_0}{2q}} \cdot \frac{\mu(a+b+1)}{2},
\]

\[
+ \mathbb{E} \left( \int_{t_k}^{t_{k+1}} \left( \int_{t_k}^{t_{k+1}} \left| D_t^{1/b} u(r) \right|^{2/\alpha} \, dr \right) \, ds \right)^{\frac{p_0}{2}} \cdot \frac{\mu(a+b+1)}{2}.
\]

**Corollary 4.3.** When the function \( u(.) \) is a deterministic function, it is obviously that \( D_t u(s) = 0 \) and therefore for every \( p > 4/(a+b+1) \)

\[
\mathbb{E} \left( \sup_{\tau \in [0, T]} \left| \int_0^t u(s) \delta^{a,b} \mathcal{B}^{a,b}(s) \, ds \right|^p \right) \leq C_1 T \left( \int_0^T \left| u(s) \right|^{\frac{2q}{p_0}} \, ds \right)^{\frac{p_0}{2q}} \cdot \frac{\mu(a+b+1)}{2},
\]

and using Young’s inequality implies that

\[
\mathbb{E} \left( \sup_{\tau \in [0, T]} \left| \int_0^t u(s) \delta^{a,b} \mathcal{B}^{a,b}(s) \, ds \right|^p \right) \leq C_1 T^2 \left( \int_0^T \left| u(s) \right|^p \, ds \right).
\]

5. Existence and Malliavin differentiability of the solution

As an analytic level, firsst we show the existence of the solution of Equation (1) step by step in the intervals \([i\tau, (i+1)\tau]\) for any integer \( i = 1, \cdots, K_0 \), and then in the interval \([K_0\tau, T]\). Malliavin
differentiability of the solution in the sense of stochastic Skorokhod integral will be concluded, recursively. The uniqueness of the solution can be proved throughout any step.

We assume the following conditions on the function \(\xi_0(.)\) and also recall two proposition from [14] for future use.

**Assumption 5.1.** The measurable function \(\xi_0(.)\) is Malliavin Differentiable up to the order \(2M+1\) and for every \(p \geq 1\), there exists some constant \(C_{\xi,p}\) such that

\[
\mathbb{E}\left( \sup_{0 \leq r \leq t} \xi_0(r - \tau)^p \right) \leq C_{\xi,p} < \infty,
\]

(11)

Let us recall the Ito’s formula has been introduced in [14] which is essential in our modification. Proceeding the proof of Ito’s formula therein show that it would be also hold in weighted stochastic integral with inner product \(\langle \ldots \rangle\) defined in [3] with the kernel function \(\delta(\ldots)\), as the authors have mentioned in their paper. So, we shall rewrite Ito’s formula for \(wFbm\) instead of fBm. Recall \(\mathcal{L}(0, T)\) as the set of Malliavin differentiable stochastic processes \(G\) such that \(\mathbb{E}[G]_H + \mathbb{E}[D^{\mu,b}G]_H < \infty\), and for any sequence of partition \(\pi: 0 = \tau_0 \leq \cdots \leq \tau_n = T\) of \([0, T]\) such that \(|\pi| \rightarrow 0\) as \(n \rightarrow \infty\)

\[
\sum_{i=0}^{n} \mathbb{E}\left( \int_{\tau_i}^{\tau_{i+1}} (D^{\mu,b}_r G - D^{\mu,b}_{\tau_i} G) \phi(r, s) dr ds \right)^2 + \mathbb{E}[\|G\|^2]_H < \infty
\]

**Proposition 5.2.** (Duncan [14], Theorem 4.3) Let \([F_s, s \in [0, T]] \in \mathcal{L}(0, T)\) be a stochastic process such that for some \(\alpha_3 > 1 - \frac{\mu+b}{2}\),

\[
\mathbb{E}\left( |F_{s_1} - F_{s_2}|^2 \right) \leq c_F|s_1 - s_2|^{2\alpha_3}
\]

where \(|s_1 - s_2| \leq \delta\) for some \(\delta > 0\) and

\[
\lim_{|n-s_i| \rightarrow 0} \mathbb{E}\left( |D^{\mu,b}_i F (F_{s_1} - F_{s_2})|^2 \right) = 0.
\]

in which \((D^{\mu,b}_i)^\phi F = \int_0^T \int_0^T D^{\mu,b}_r F \phi(r, s) dr\). Assume that \(\mathbb{E}(\sup_{0 \leq s \leq T} |G_s|) < \infty\) and \(f : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}\) is a function in \(C^{1,2}_b(\mathbb{R}_+ \times \mathbb{R})\). If \(\eta_t = \xi + \int_0^t G_s ds + \int_0^t F_\delta\eta_s B^\delta_s, \ \xi \in \mathbb{R}\) for \(t \in [0, T]\) and \(\frac{\partial f}{\partial x}(s, \eta_t) F_\delta, \eta_t \in \mathcal{L}(0, T)\). Then for \(t \in [0, T]\)

\[
f(t, \eta_t) = f(0, \xi) + \int_0^t \frac{\partial f}{\partial x}(s, \eta_t) G_s ds + \sum_{k=1}^n \int_0^t \frac{\partial f}{\partial x}(s, \eta_t) G_s ds
\]

\[
+ \int_0^t \frac{\partial f}{\partial x}(s, \eta_t) F_\delta \eta_s B^\delta_s + \int_0^t \frac{\partial^2 f}{\partial x^2}(s, \eta_t) F_\delta(D^{\mu,b}_s)^\delta \eta_s ds.
\]
To show main results, we start with showing that, the SDE (12) has a unique solution. Thanks to Proposition 5.2, following as in the proof of Theorem 2.5 and Lemma 2.2 in [18], one can conclude that the solution of (12) is \( t \in [0, T] \) and \( \frac{\partial f}{\partial x}(s, \eta, \xi)F^s_{t} \in \mathcal{L}(0, T) \). Then for \( t \in [0, T] \)

\[
\begin{align*}
f(t, \eta_1, \cdots, \eta_n) &= f(0, \xi_1, \cdots, \xi_n) + \int_0^t \frac{\partial f}{\partial t}(s, \eta, \xi)ds + \sum_{k=1}^n \int_0^t \frac{\partial f}{\partial \eta_k}(s, \eta, \xi)G^k_{t}ds \\
&\quad + \sum_{k=1}^n \int_0^t \frac{\partial f}{\partial \xi_k}(s, \eta, \xi)F^k_{t}B_{\eta, \xi} + \sum_{k,l=1}^n \int_0^t \frac{\partial^2 f}{\partial \eta_k \partial \xi_l}(s, \eta, \xi)F^k_{t}B_{\eta, \xi}^l ds,
\end{align*}
\]

It is worth mentioning that, Theorem 4.2. in [14] can be also rewritten in the sense of weighted fractional Brownian motion, clearly.

**Proposition 5.4.** If \( \{F, s \in [0, T]\} \in \mathcal{L}(0, T) \) and \( \sup_{0 \leq s \leq T} \mathbb{E}(\|D^{\alpha, \beta}_{s}F\|_{2}^2) < \infty \). Then for \( s, t \in [0, T] \)

\[
\left(D^{\alpha, \beta}_{s}F\right) \left(\int_0^s F_{u}^{\alpha, \beta}B_{\alpha, \beta}^u \right) = \left(\int_0^s D^{\alpha, \beta}_{u}F_{s}^{\alpha, \beta}B_{\alpha, \beta}^u \right) + \int_0^s F_{u}^\alpha d\phi(u)du, \quad a.s.
\]

Now, consider the following linear SDE driven by weighted fractional Brownian motion

\[
d\psi(t) = A\psi(t)dt + B\psi(t)\sigma_{\alpha, \beta}B^{\alpha, \beta}(t), \quad \text{as} \quad t \in [0, T], \quad \text{and} \quad \psi(0) = 1. \quad (12)
\]

To show main results, we start with showing that, the SDE (12) has a unique solution \( \psi(.) \) with an exponential exposition and it has Malliavin derivatives which can be presented as a function of the solution \( \psi(.) \).

Thanks to Proposition 5.2, following as in the proof of Theorem 2.5 and Lemma 2.2 in [18], one can conclude that the solution of (12) is

\[
\psi(t) = \exp\left\{At + B\psi(t)\sigma_{\alpha, \beta}B^{\alpha, \beta}(t)\right\} - \frac{1}{2}B^2 \int_0^t \int_0^{s'} \phi(s, s')dsds' \}
\]

and \( \psi'(t) = \psi(t)B_{1_{0 \leq s \leq T}} \) and equivalently \( \left(D^{\alpha, \beta}_{s}\psi(t) = \psi(t)B_{s}\int_0^s \phi(r, s)ds \right) \). This fact leads to the conclusion that, the process \( \psi^{-1} \) is the solution to the SDE

\[
d\psi^{-1}(t) = \left(-A + \psi^{-1}(t)\psi^{-1}(t)\right)dt - B\psi^{-1}(t)\psi^{-1}(t) \sigma_{\alpha, \beta}B^{\alpha, \beta}(t), \quad t \in [0, T],
\]

\[
\psi^{-1}(0) = 1,
\]

and \( \psi^{-1}(t) = -\psi^{-1}(t)B_{1_{0 \leq s \leq T}} \), which is equivalent to \( \left(D^{\alpha, \beta}_{s}\psi^{-1}(t) = -\psi^{-1}(t)B_{s}\int_0^s \phi(r, s)ds \right) \).
Theorem 5.5. The unique solution $\psi(.)$ has uniformly bounded moments; i.e., there exists some positive constant $C_p$ such that

$$\mathbb{E}\left( \sup_{0 \leq t \leq T} |\psi(t)|^p \right) \leq C_p$$

Proof. According to the relation $D_t^{r,b} \psi(t) = \psi(t) B_{1_{[r,b]}}$ and inequality (7) in connection with inclusion (5) and then Gronwall’s inequality, the uniqueness of the solution $\psi$ is resulted. The boundedness of the $p$-moments of $\psi(.)$ follows from Theorem 3.3 in [14] and Proposition 5.2, and deduces that there exists some constant $C_p$ such that

$$\mathbb{E}\left( \sup_{0 \leq t \leq T} |\psi(t)|^p \right) \leq \exp\left\{ pAT + \frac{1}{2} \left( p^2 - p \right) \int_0^T \int \int \phi(s,s')dldl \right\} \leq C_p.$$

Proposition 5.6. For every $p \geq 1$, there exists some constant $C_{\psi,p}$ such that

$$\mathbb{E}\left( \sup_{0 \leq t \leq T} |\psi(r)^{-1}|^p \right) + \mathbb{E}\left( \int_0^T \int_0^T |D_t^{r,b} \psi^{-1}(r)|^p dr ds \right) \leq C_{\psi,p} < \infty,$$

As a consequence, the stochastic processes $\psi(.)$ and $\psi^{-1}(.)$ are in $L(0,T)$ and satisfy the condition of Theorem 5.2 with $\alpha_3 = (1 + b)/2$, applying Equations (7) and (2).

Now, we are ready to construct the solution of Equation (1) in the following theorem and obtain its Malliavin derivative.

Theorem 5.7. Under Assumption [11] and the first part of Assumption [5] SDE (11) admits a unique solution on $[-\tau,T]$ which is also Malliavin differentiable up to the order $2M$.

Proof. We will prove the assertion in four steps for the convenience of readers. The uniqueness of the solution throughout any step stands on a similar proof of Theorem 5.5 to show the uniqueness of $\psi(.)$ to SDE (12).

step 1. For $t \in [0, \tau]$, define the Gaussian process

$$M_0(t) = \int_0^t f(\xi_0(s-\tau))ds - B \int_0^t \int_0^\infty \sigma(\xi_0(s-\tau))d\phi(u,s)du + \int_0^t \sigma(\xi_0(s-\tau))d^{a,b}B^{a,b}(s)$$

and consider the following stochastic differential equation

$$X_0(t) = \xi_0(0) + \int_0^t AX_0(s)ds + \int_0^t BX_0(s)d^{a,b}B^{a,b}(t) + M_0(t) \quad t \in [0, \tau]$$
To introduce the solution $X_0$, we first denote $Z_0(t) := \xi_0(0) + \int_0^t \psi^{-1}(s)dM_0(s)$. In view of Proposition 5.4, since $\psi^{-1} \in L[0, T]$ and for every $-\tau \leq s < 0$ and $r \geq 0$, we know $D_t^{r,b}\xi_0(s) = 0$, therefore for any $0 \leq r \leq t$

$$D_t^{r,b}Z_0(t) = \psi^{-1}(r)\sigma(\xi_0(r - \tau)) - \int_r^t B\psi^{-1}(s)\left(f(\xi_0(s - \tau)) - \int_0^s \sigma(\xi_0(s - \tau))\phi(u, s)du\right)ds$$

$$= \psi^{-1}(r)\sigma(\xi_0(r - \tau)) - BZ_0(t) + BZ_0(t).$$

It deduce that $D_t^{r,b}Z_0(t) = \psi^{-1}(r)\sigma(\xi_0(t - \tau))$, or equivalently

$$(D_t^{r,b})^bZ_0(t) = \psi^{-1}(t)\int_0^t \sigma(\xi_0(u - \tau))\phi(u, t)du.$$

Applying Ito’s formula for $U_0(t) := e^{-B\int_0^t \psi(s)ds}\psi(t)Z_0(t)$ and then substituting $(D_t^{r,b})^b\psi(t)$ and $(D_t^{r,b})^bZ_0(t)$ result

$$dU_0(t) = e^{-B\int_0^t \psi(s)ds}\left(B\int_0^t \phi(t, u)du\psi(t)Z_0(t)dt + A\psi(t)Z_0(t)dt + B\psi(t)\sigma(\xi_0(t - \tau))dt + B\psi(t)(D_t^{r,b})^bZ_0(t)dt\right)$$

$$= \left[AU_0(t) + f(\xi_0(s - \tau))\right]dt + \left[BU_0(t) + \sigma(\xi_0(s - \tau))\right]\delta^{a,b}B^{a,b}(t).$$

It show that for every $0 \leq t \leq \tau$, the process $e^{-B\int_0^t \psi(s)ds}\psi(t)Z_0(t)$ is a solution to SDE (1).

Next, due to Proposition 5.4 and the relationship of $D_t^{r,b}$ and $(D_t^{r,b})^b$ in Proposition 5.2 one deduce that $x(t)$ in the time interval $[0, \tau]$ has a weak derivative satisfying

$$D_t^{r,b}X_0(t) = BX_0(r) + \int_r^t AD_t^{r,b}X_0(s)ds + \int_r^t BD_t^{r,b}X_0(s)\delta^{a,b}B^{a,b}(s), \quad t \in [0, \tau],$$

for every $0 < r \leq t$.

**step 2.** For every $k = 1, \cdots, K_0 - 1$ and for all $k\tau \leq t \leq (k + 1)\tau$, define again the Gaussian processes

$$M(t) = \int_{k\tau}^t f(X_k(s - \tau)ds - B\int_{k\tau}^t \sigma(X_k(s - \tau))\phi(u, s)du + \int_{k\tau}^t \sigma(X_k(s - \tau))\delta^{a,b}B^{a,b}(t),$$

and consider the following stochastic differential equations

$$X_k(t) = X_{k-1}(k\tau) + \int_{k\tau}^t AX_k(s)ds + \int_{k\tau}^t BX_k(s)\delta^{a,b}B^{a,b}(t) + M_{k-1}(t), \quad t \in [k\tau, (k + 1)\tau].$$
similarly, if we define $Z_k(t) := X_{k-1}(k\tau) + \int_0^t \psi^{-1}(s)dM_{k-1}(s)$, as $\psi^{-1} \in \mathcal{L}[0, T]$ and for every $-k\tau \leq s < (k+1)\tau$ and $r \geq k\tau$ we know $D_r^{a,b}X_{k-1}(s) = 0$, then for any $k\tau \leq r \leq t$

$$D_r^{a,b}Z_k(t) = 0 - \int_r^t B\psi^{-1}(s)\left(f(X_{k-1}(s - \tau)) - \int_0^s \sigma(X_{k-1}(s - \tau))\phi(u, s)du\right)ds$$

$$- \int_r^t B\psi^{-1}(s)\sigma(X_{k-1}(s - \tau))\delta^{a,b}B_r^{a}(t) + \psi^{-1}(r)\sigma(X_{k-1}(r - \tau)) = \psi^{-1}(r)\sigma(X_{k-1}(r - \tau)) - BZ_k(t) + BZ_k(r).$$

Clearly, it deduces that $D_r^{a,b}Z_k(t) = \psi^{-1}(t)\sigma(X_{k-1}(t - \tau))$, or equivalently

$$(D_r^{a,b})^\phi Z_k(t) = \psi^{-1}(t)\int_0^t \sigma(X_{k-1}(u - \tau))\phi(u, t)du. \quad (13)$$

We employ Ito’s formula \textbf{[5.4]} for $U_k := e^{-B\int_0^t \phi(s, a)du}B_t^{a} \psi(t)Z_k(t)$ and substitute \textbf{[13]} and get that

$$dU_k(t) = e^{-B\int_0^t \phi(s, a)du}B_t^{a}\left[f(B_t^{a}, t)\psi(t)Z_k(t)dt + A(\psi(t)Z_k(t)dt + \Phi(\psi(t)Z_k(t)\phi^{a,b}B_r^{a}(t)\right)$$

$$+ \psi(t)\psi^{-1}(t)dM_{k-1}(t) + (D_r^{a,b})^\phi \psi(t)\psi^{-1}(t)\sigma(X_{k-1}(t - \tau))dt + B\psi(t)(D_r^{a,b})^\phi Z_k(t)dt\right]$$

$$= \left[U_k(t) + f((X_{k-1}(t - \tau))dt + \left(BU_k(t) + \sigma(X_{k-1}(t - \tau))\phi^{a,b}B_r^{a}(t)\right)\right].$$

Thus SDE \textbf{[1]} has the solution $e^{-B\int_0^t \phi(s, a)du}B_t^{a} \psi(t)Z_k(t)$ for every $k\tau \leq t \leq (k+1)\tau$. Next, Proposition\textbf{[5.4]} results this solution has also a Malliavin derivative satisfying

$$D_r^{a,b}X_k(t) = D_r^{a,b}X_{k-1}(k\tau) + BX_k(t) + \sigma(X_{k-1}(r - \tau))1_{k\tau \leq t < r}$$

$$+ \int_{k\tau < t} (AD_r^{a,b}X_k(s) + f'(X_{k-1}(s - \tau))D_r^{a,b}X_{k-1}(s - \tau)1_{k\tau < t})ds$$

$$+ \int_{k\tau < t} \left(BD_r^{a,b}X_k(s) + \sigma'(X_{k-1}(s - \tau))D_r^{a,b}X_{k-1}(s - \tau)1_{k\tau < t}\right)\phi^{a,b}B_r^{a}(s), \quad t \in [k\tau, (k+1)\tau], \quad (14)$$

for every $0 \leq r \leq t - \tau$ and also

$$D_r^{a,b}X_k(t) = BX_k(t) + \int_{k\tau < t} AD_r^{a,b}X_k(s)ds + \int_{k\tau < t} BD_r^{a,b}X_k(s)\phi^{a,b}B_r^{a}(s), \quad t \in [k\tau, (k+1)\tau], \quad (15)$$

for every $t - \tau < r \leq t$.

**Step 3.** Perform step 2 for $k = K_0$ and result that for every $t \in [K_0\tau, T]$, SDE \textbf{[1]} has a unique solution $e^{-B\int_0^t \phi(s, a)du}B_t^{a} \psi(t)Z_{k_0}(t)$ for every $K_0\tau \leq t \leq T$ with Malliavin derivative satisfying \textbf{[14]} and \textbf{[15]}. In this sense, it is sufficient to define

$$x(t) = \sum_{k=0}^{K_0-1} X_k(t)1_{k\tau \leq t \leq (k+1)\tau} + X_{k_0}(t)1_{k_0\tau \leq t < T}. \quad (16)$$
Therefore, for every $0 \leq r \leq t - \tau$, $D_r^{\mu, b} x(t)$ should satisfy

$$D_r^{\mu, b} x(t) = D_r^{\mu, b} x(0) + B x(r) + \sigma(x(r - \tau)) 1_{[r, r + \tau - \tau]}$$

$$+ \int_r^t \{ A D_r^{\mu, b} x(s) + f'(x(s - \tau)) D_r^{\mu, b} x(s - \tau) 1_{[s, s + \tau - t]} \} \, ds$$

$$+ \int_r^t \{ B D_r^{\mu, b} x(s) + \sigma'(x(s - \tau)) D_r^{\mu, b} x(s - \tau) 1_{[s, s + \tau - t]} \} \sigma^{(b)} B^b(s), \quad t \in [0, T], \quad (17)$$

and for every $t - \tau < r \leq t$,

$$D_r^{\mu, b} x(t) = B x(r) + \int_r^t A D_r^{\mu, b} x(s) \, ds + \int_r^t B D_r^{\mu, b} x(s) \sigma^{(b)} B^b(s), \quad t \in [0, T]. \quad (18)$$

**Step 4.** We continue the steps 1, 2 and 3 for Equation (17) instead of Equation (1) to derive Malliavin differentiability of $D_r^{\mu, b} x(t)$. Finally we repeat this procedure up to the order $2M$, the order of differentiability of the functions $f$ and $g$, to deduce the assertion.

We end this section by giving a recursively expression for higher order Malliavin derivatives of the solution $x(t)$, provided we continue differentiating of Equations (14) or (17). Our expression deal with the case $0 \leq r_1, \ldots, r_l \leq t - \tau$, the other cases have the same computation. We note that our modification will be useful to obtain some bounds for their moments in the next section.

First, let us define the processes $H_{r_1, \ldots, r_l}(s), F_{r_1, \ldots, r_l}(s)$ for every $2 \leq l \leq 2M$ as follows for simplicity.

$$1_{(1, -j)} := 1_{r_j > (k-l)\tau} \prod_{i=1}^{l-1} 1_{r_i + \tau < r_j}, \quad 1_{(2, -j)} := 1_{r_j > (k-l)\tau} \prod_{i=2}^{l} 1_{r_i + \tau < r_j}$$

$$1_{(1, -j)} := 1_{r_j > (k-l)\tau} \prod_{i=1}^{l-1} 1_{r_i + \tau < r_j}, \quad 2 \leq j \leq l - 1$$

$$H_{r_1, \ldots, r_l}(k) := BD_{r_1}^{\mu, b} \cdots D_{r_k}^{\mu, b} X_k(r_1) + D_{r_1}^{\mu, b} \cdots D_{r_l}^{\mu, b}(x(X_{k-1}(r_1 - \tau))) 1_{(2, -j)} + BD_{r_2}^{\mu, b} \cdots D_{r_l}^{\mu, b} X_k(r_1)$$

$$+ D_{r_1}^{\mu, b} \cdots D_{r_l}^{\mu, b}(x(X_{k-1}(r_1 - \tau))) 1_{(1, -j)} + B \sum_{j=2}^{l-1} D_{r_2}^{\mu, b} \cdots D_{r_l}^{\mu, b} D_{r_{l-j}}^{\mu, b} \cdots D_{r_1}^{\mu, b} X_k(r_1)$$

$$+ \sum_{j=2}^{l-1} D_{r_1}^{\mu, b} \cdots D_{r_{l-j+1}}^{\mu, b} D_{r_{l-j+1}}^{\mu, b} \cdots D_{r_1}^{\mu, b}(x(X_{k-1}(r_j - \tau))) 1_{(1, -j)},$$

$$F_{r_1, \ldots, r_l}(X_{k-1}, s) := D_{r_1}^{\mu, b} \cdots D_{r_l}^{\mu, b}(f(X_{k-1}(s - \tau))), \quad G_{r_1, \ldots, r_l}(X_{k-1}, s) := D_{r_1}^{\mu, b} \cdots D_{r_l}^{\mu, b}(\sigma(X_{k-1}(s - \tau))).$$
Then $l$-th derivative for every $k\tau \leq t \leq (k+1)\tau$ as $k = 1, \cdots, K_0$ and also every $0 \leq r_1, \cdots, r_l \leq t - \tau$ satisfies

$$D^{r_1}_t \cdots D^{r_l}_t X_k(t) = D^{r_1}_t \cdots D^{r_l}_t X_{k-1}(k\tau) + H_{r_1r_1}(k)$$

$$+ \int_{[\tau \leq s \leq T]} \left\{ \sum_{i=1}^l D^{r_i}_t \cdots D^{r_l}_t X_k(s) + F_{r_1r_1}(X_{k-1}, s) \prod_{i=1}^l 1_{t+s\leq \tau} \right\} ds$$

$$+ \int_{[\tau \leq s \leq T]} \left\{ \sum_{i=1}^l D^{r_i}_t \cdots D^{r_l}_t X_k(s) + G_{r_1r_1}(X_{k-1}, s) \prod_{i=1}^l 1_{t+s\leq \tau} \right\} \delta^{a,b} B^{a,b}(s),$$

(19)

and for every $K_0 t \leq t \leq T$

$$D^{r_1}_t \cdots D^{r_l}_t X_k(t) = D^{r_1}_t \cdots D^{r_l}_t X_{k-1}(k\tau) + H_{r_1r_1}(k)$$

$$+ \int_{[\tau \leq s \leq T]} \left\{ \sum_{i=1}^l D^{r_i}_t \cdots D^{r_l}_t X_k(s) + F_{r_1r_1}(X_{k-1}, s) \prod_{i=1}^l 1_{t+s\leq \tau} \right\} ds$$

$$+ \int_{[\tau \leq s \leq T]} \left\{ \sum_{i=1}^l D^{r_i}_t \cdots D^{r_l}_t X_k(s) + G_{r_1r_1}(X_{k-1}, s) \prod_{i=1}^l 1_{t+s\leq \tau} \right\} \delta^{a,b} B^{a,b}(s).$$

According to Step 3 in the proof of Theorem 5.7 for every $0 \leq r_1, \cdots, r_l \leq t - \tau$ we have

$$D^{r_1}_t \cdots D^{r_l}_t X(t) = \sum_{k=0}^{K_0-1} D^{r_1}_t \cdots D^{r_l}_t X_k(1) 1_{kT \leq (k+1)\tau} + D^{r_1}_t \cdots D^{r_l}_t X_k(t) 1_{K_0T \leq T},$$

(20)

where $X_k(t) = e^{-B \int_0^t \phi(s,a)da} \phi(t) Z_{k,r_1\cdots r_l}(t)$ in which

$$Z_{k,r_1\cdots r_l}(t) := D^{r_1}_t \cdots D^{r_l}_t X_{k-1}(k\tau) + H_{r_1r_1}(k) + \int_{[\tau \leq s \leq T]} \psi^{-1}(s) dM_{k-1,r_1\cdots r_l}(s)$$

and

$$M_{k,r_1\cdots r_l}(t) = \int_{[\tau \leq s \leq T]} F_{r_1r_1}(X_k, s) ds - B \int_{[\tau \leq s \leq T]} \int_{[\tau \leq r \leq s]} G_{r_1r_1}(X_k, r) \phi(u, s) duds$$

$$+ \int_{[\tau \leq s \leq T]} G_{r_1r_1}(X_k, s) \delta^{a,b} B^{a,b}(t).$$

6. Bounds of moments to solution and its derivatives

Construction of the solution of Equation (1) in the proof of Theorem 5.7 allow us to show that this solution and its Malliavin derivatives have uniformly bounded $p$-moments for every $p \geq 1$. 

15
Theorem 6.1. For every $p \geq 2$, under Assumptions [14] and [5.7] there exists some positive constant $C_{0,p}$ such that

$$\mathbb{E}\left( \sup_{0 \leq t \leq T} |x(t)|^p \right) \leq C_{0,p} \quad (21)$$

Proof. Proceeding the steps in the proof of Theorem 5.7 induction on $k = 0, \ldots, K_0$ and also the definition of $x(t)$ in Equation (16). To do this, for every $p \geq 2$, from Assumption [1.1] and the fact $D_x \xi_0(s) = 0$ for all $r \geq 0$ we obtain

$$N_0(r) := \mathbb{E} \left( \int_0^r \left( \int_0^r \left| D_x \psi^{-1}(r) \sigma(\xi_0(r - \tau)) \right|^p \right)^{\frac{p}{2}} dr \right)^{\frac{2}{p+2}} ds \leq \tau^2 \mathbb{E} \left( \int_0^r \int_0^r |D_x \psi^{-1}(s) \sigma(\xi_0(s - \tau))|^p ds dr \right) \leq L \tau^2 \mathbb{E} \left( \sup_{0 \leq s \leq r} (1 + \xi_0(r - \tau))^2 \int_0^r \int_0^r |D_x \psi^{-1}(r)|^p ds dr \right) \leq 2^{p-2} L \tau^2 \left( 1 + \mathbb{E} \left( \sup_{0 \leq s \leq r} |\xi_0(r - \tau)|^p \right) \right) + \frac{1}{2} L \tau^4 \mathbb{E} \left( \int_0^r \int_0^r |D_x \psi^{-1}(r)|^p ds dr \right).$$

Substituting $dM_0$ into the definition of $Z_0$ and then applying Corollary 4.2 and Jensen’s inequality result

$$\mathbb{E}\left( \sup_{0 \leq s \leq r} |Z_0(t)|^p \right) \leq 2^p \mathbb{E}(|\xi_0(0)|^p) + \mathbb{E}\left( \sup_{0 \leq s \leq r} \int_0^r |\psi^{-1}(s) f(\xi_0(s - \tau)) ds|^p \right) + \mathbb{E}\left( \sup_{0 \leq s \leq r} \int_0^r |\psi^{-1}(s) \sigma(\xi_0(s - \tau)) | |D_x \psi^{-1}(s)|^p ds \right)$$

$$\leq 2^p \left( \mathbb{E}(|\xi_0(0)|^p) + 2^{p-1} L \tau^{2p} \int_0^r \left( 1 + \mathbb{E}(|\xi_0(s - \tau)|^2) \right) ds + \frac{1}{2} \mathbb{E} \left( \sup_{0 \leq s \leq r} |\psi^{-1}(s)|^2 \right) \right) + C_1 \tau \left( \int_0^r \mathbb{E}(|\psi^{-1}(s) \sigma(\xi_0(s - \tau))|^{\frac{2}{\alpha}} \right) ds + N_0(r) \right)$$

$$\leq 2^p \left( \mathbb{E}(|\xi_0(0)|^p) + 2^{p-1} L \tau^{2p} \int_0^r \left( 1 + \mathbb{E}(|\xi_0(s - \tau)|^2) \right) ds + \frac{1}{2} \mathbb{E} \left( \sup_{0 \leq s \leq r} |\psi^{-1}(s)|^2 \right) \right) + C_1 \tau^2 \left( \int_0^r \mathbb{E}(|\psi^{-1}(s) \sigma(\xi_0(s - \tau))|^{\frac{2}{\alpha}} \right) ds + C_1 \tau N_0(r) \right)$$

$$\leq 2^p \left( \mathbb{E}(|\xi_0(0)|^p) + 2^{p-1} L \tau^{2p} \int_0^r \left( 1 + \mathbb{E}(|\xi_0(s - \tau)|^2) \right) ds + \frac{1}{2} \mathbb{E} \left( \sup_{0 \leq s \leq r} |\psi^{-1}(s)|^2 \right) \right) + C_1 \tau^2 \left( \int_0^r \mathbb{E}(|\psi^{-1}(s)|^2) ds + \frac{1}{2} C_1 \tau^2 L \tau^2 \int_0^r \mathbb{E}(1 + |\xi_0(s - \tau)|^2) ds + C_1 \tau N_0(r) \right).$$

(23)
where we used the fact \((z + y)^p \leq 2^p(z^p + y^p)\) several times in the above inequalities. Finally, substitute (22) in (23) and then use Assumption 5.1 and Proposition 5.6 to deduce that for every 
On replacing \(\xi_0\) by \(X_k\) and repeating a similar computation recursively on \(k\), one derive that for every \(k = 1, \ldots, K_0 - 1\)

\[
N_k(kt) := \mathbb{E}\left( \int_{k\tau}^{(k+1)\tau} \left( \int_{k\tau}^{(k+1)\tau} D_s^{\theta} \left( \psi^{-1}(r)\sigma(X_{k-1}(r - \tau)) \right)^{2(a+b+1)} dr \right) \frac{dr}{\mathbb{E}^{(a+b+1)}} \right) \]  

\[
\leq \tau^2 \mathbb{E}\left( \int_{k\tau}^{(k+1)\tau} \left( \int_{k\tau}^{(k+1)\tau} |\psi^{-1}(r)|^2 \right) + 2 \mathbb{E}\left( \int_{k\tau}^{(k+1)\tau} |D^{\theta} X_{k-1}(r - \tau)|^2 dr \right) \right)^p ds 
\]

\[
+ 2^{p-1} \mathbb{E}\left( \sup_{k\tau \leq s \leq (k+1)\tau} |\psi^{-1}(s)|^2 \right) + 2^{p-1} \mathbb{E}\left( \int_{k\tau}^{(k+1)\tau} |D^{\theta} \psi^{-1}(r)|^2 ds \right) \]  

and also

\[
\mathbb{E}\left( \sup_{k\tau \leq s \leq (k+1)\tau} |X_k(s)|^p \right) \leq 2^p \mathbb{E}\left( \int_{k\tau}^{(k+1)\tau} \left( 1 + \mathbb{E}(|X_{k-1}(s - \tau)|^p) \right) ds \right) 
\]

\[
+ \frac{1}{2} \mathbb{E}\left( \sup_{k\tau \leq s \leq (k+1)\tau} |\psi^{-1}(s)|^2 \right) + \frac{1}{2} C_1 \tau^2 \mathbb{E}(\psi^{-1}(s)^2) ds 
\]

\[
+ \frac{1}{2} C_1 \tau^2 2^p \int_{k\tau}^{(k+1)\tau} \mathbb{E}(1 + |X_{k-1}(s - \tau)|^p) ds + C_1 \tau N_k(kt). \]  

Hence, Proposition 5.6 and induction on \(k\) deduce the boundedness of moments of \(x(\cdot)\) in \([0, K_0\tau]\) and finally by repeating this computation for every \(K_0\tau \leq t \leq T\), the claim can be obtained.

In sequence, since the following computations and results can be exactly repeat for every \(K_0\tau \leq t \leq T\) and \(t - \tau \leq r \leq t\), we just demonstrate the results on \(0 \leq t \leq K_0\tau\) and \(0 \leq r \leq t - \tau\) as follows.

From the definition of the functions \(F_{\tau_{r-\tau}}(X_k, t)\) and \(G_{\tau_{r-\tau}}(X_k, t)\), we understand that these processes depend on higher derivatives of the functions \(f\) and \(\sigma\), respectively, and Malliavin derivatives of \(X_{k-1}\) up to the order \(l\). Then

\[
|F_{\tau_{r-\tau}}(X_{k-1}, t)|^p = \mathcal{P}_1 \left( f'(X_{k-1}), \ldots, f^{(l)}(X_{k-1}), D^{\theta} X_{k-1}, \ldots, D^{\theta} X_{k-1} \right)(t), \]  

\[
|G_{\tau_{r-\tau}}(X_{k-1}, t)|^p = \mathcal{P}_2 \left( \sigma'(X_{k-1}), \ldots, \sigma^{(l)}(X_{k-1}), D^{\theta} X_{k-1}, \ldots, D^{\theta} X_{k-1} \right)(t), \]
where \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) are polynomial functions. From Assumption \( \square.1 \) we derive that for every \( p \geq 2 \), 

\[
0 \leq r_1, \ldots, r_T \leq T \quad \text{and} \quad 0 \leq t_1 \leq t_2 \leq T 
\]

\[
\mathbb{E}\left( \sup_{t_1 \leq t_2} |F_{r_1}^{(X_k(t))}|^p \right) + \mathbb{E}\left( \sup_{t_1 \leq t_2} |G_{r_1}^{(X_k(t))}|^p \right) \leq 2^p K_2 \sum_{j=2}^i \mathbb{E}\left( \sup_{t_1 \leq t_2} |D_{r_1}^{q_j} \cdots D_{r_1}^{q_j} X_{k-1}(t-\tau)|^p \right) 
\]

(24)

where integer numbers \( q_j \) satisfy \( 1 \leq q_j \leq p \). Furthermore, when \( l < 2M \), for every \( 0 \leq r \leq t-\tau \)

\[
\mathbb{E}\left( \sup_{t_1 \leq t_2} |D_{r_1}^{q_j} G_{r_1}^{(X_k(t))}|^p \right) \leq 2^p K_2 \sum_{j=1}^{l-1} \mathbb{E}\left( \sup_{t_1 \leq t_2} |D_{r_1}^{q_j} \cdots D_{r_1}^{q_j} X_{k-1}(t-\tau)|^p \right) 
\]

(25)

in which integer numbers \( q_{j,G} \) satisfy \( 1 \leq q_{j,G} \leq p \). Define the vectors

\[
V_i := (\sigma^{(X_k-1)(r-1)}(1), \ldots, \sigma^{(X_k-1)(r-\tau)}),
\]

\[
W_{ij} := (D_{r_1}^{q_j} \cdots D_{r_1}^{q_j} X_k(r_1), \ldots, D_{r_1}^{q_j} \cdots D_{r_1}^{q_j} X_k(r_1)),
\]

\[
W_{ij} := (D_{r_1}^{q_j} \cdots D_{r_1}^{q_j} X_k-1(r_1), \cdots, D_{r_1}^{q_j} \cdots D_{r_1}^{q_j} X_k-1(r_1) - \tau),
\]

\[
w_{ij} := (D_{r_1}^{q_j} \cdots D_{r_1}^{q_j} X_k-1(r_1), \cdots, D_{r_1}^{q_j} \cdots D_{r_1}^{q_j} X_k-1(r_1) - \tau),
\]

\[
v_{ij} := (D_{r_1}^{q_j} \cdots D_{r_1}^{q_j} X_k-1(r_1) - \tau), \cdots, D_{r_1}^{q_j} \cdots D_{r_1}^{q_j} X_k-1(r_1) - \tau),
\]

\[
v_{ij} := (D_{r_1}^{q_j} \cdots D_{r_1}^{q_j} X_k-1(r_1) - \tau), \cdots, D_{r_1}^{q_j} \cdots D_{r_1}^{q_j} X_k-1(r_1) - \tau),
\]

for every \( 1 \leq i \leq l \), \( 2 \leq j \leq l \), \( 3 \leq j' \leq l \) and \( 1 \leq j'' \leq l-1 \). Therefore \( |H_{r_1}^{(r_2)}| \) would be a polynomial composition of the elements of these vectors; i.e., there exists some polynomial \( \mathcal{P}_3 \) such that

\[
|H_{r_1}^{(r_2)}| = \mathcal{P}_3(V_1, \ldots, V_l, W_{2,1}, W_{2,2}, \ldots, W_{l,1}, W_{l,2}, w_{2,3}, \ldots, w_{l,3}, w_{1,1}, v_{1,1}, v_{1,1}, \ldots, v_{l-1}, v_{l-1}, v_{l-1}).
\]

We again conclude from Assumption \( \square.1 \) that for every \( p \geq 2 \) there exists some constant \( c_{p,H} \) such that

\[
\mathbb{E}(|H_{r_1}^{(r_2)}|) \leq c_{p,H} \left( \sum_{i=1}^{l} \mathbb{E}(|V_i|^p) + \sum_{j=1}^{l} \mathbb{E}(|V_j|^p) + \sum_{j=2}^{l} \mathbb{E}(\mathbb{E}(W_{j,j}^{(p)q_{j,j}}) + \mathbb{E}(W_{j,j}^{(p)q_{j,j}})) \right)
\]

\[
+ \sum_{j=3}^{l} \sum_{j=2}^{l} \mathbb{E}(\mathbb{E}(W_{j,j}^{(p)q_{j,j}}))
\]

(26)

where all integer numbers \( q_{j,j}, q_{j,j}, q_{j,j}, q_{j,j}, q_{j,j}, q_{j,j}, q_{j,j}, q_{j,j} \) are between 1 and \( p \).

It is worth mentioning that the order of differentiability in components of vectors
V_t, W_{j_1}, W_{j_2}, w_{j'}, w_{j''}, u^{(1)}_t, v^{(1)}_t, v^{(1)}_j, v^{(1)}_{j'}$ are utmost of the order $l - 1$. This fact helps us to find some upper bounds for the moments of $D'^{b}_{1} \ldots D'^{b}_{l} x(t)$ that will be demonstrated in Theorem 6.2.

Now, we are ready to prove some upper bounds for the moments of higher derivatives of the solution to SDDE (1).

**Theorem 6.2.** Under Assumptions [17] and [57] for every $p \geq 2$ and $1 \leq l \leq M$ there exists some positive constant $C'_{l,p}$ such that $l$-th Malliavin derivative of the solution to SDE (1) satisfies the following inequality.

$$
\mathbb{E} \left( \sup_{0 \leq t \leq T} \sup_{0 \leq u_1 \leq \ldots \leq u_l \leq t} |D'^{b}_{u_1} \ldots D'^{b}_{u_l} x(t)|^p \right) \leq C'_{l,p} \quad (27)
$$

**Proof.** We show the assertion by induction on $k$. We first take the following computation for every $k = 0, \ldots, K_0$ and $1 \leq l < 2M$. Assume $\sup_{0 \leq u_1 \leq \ldots \leq u_l \leq t} |D'^{b}_{u_1} \ldots D'^{b}_{u_l} x(t)|^p$ is achieved in $u_1 = r_1, \ldots, u_l = r_l$, which we omitted dependency of $r_i$ to $t$ for simplicity without loss of generality. From Equations (25) and (24)

$$
2^{-p+1} \mathcal{N}_{h_1, \ldots, h_2} (k \tau) := 2^{-p+1} \mathbb{E} \left( \int_{\tau^{(k+1)\tau}}^{\tau^{(k+1)\tau}} \left( \int_{\tau^{(k+1)\tau}}^{\tau^{(k+1)\tau}} |D'^{b}_{h_1} \psi^{-1}(s) G_{r_{1-1}} (X_{k-1}, s) |^{2(p+1)} \frac{\nu^{(k+1)\tau} \nu^{(k+1)\tau}}{\nu^{(k+1)\tau} \nu^{(k+1)\tau}} \frac{d \nu^{(k+1)\tau}}{d \nu^{(k+1)\tau}} \right) dr \right)
$$

$$
\leq \tau^2 \mathbb{E} \left( \sup_{0 \leq s \leq T} |\psi^{-1}(s)|^2 \right) + \tau^2 \mathbb{E} \left( \int_{\tau^{(k+1)\tau}}^{\tau^{(k+1)\tau}} \int_{\tau^{(k+1)\tau}}^{\tau^{(k+1)\tau}} |D'^{b}_{h_1} G_{r_{1-1}} (X_{k-1}, s) |^{2p} dr ds \right)
$$

$$
+ \tau^2 \mathbb{E} \left( \int_{\tau^{(k+1)\tau}}^{\tau^{(k+1)\tau}} \int_{\tau^{(k+1)\tau}}^{\tau^{(k+1)\tau}} |D'^{b}_{h_1} \psi^{-1}(s) |^{2p} dr ds \right) + \tau^2 \mathbb{E} \left( \int_{\tau^{(k+1)\tau}}^{\tau^{(k+1)\tau}} \int_{\tau^{(k+1)\tau}}^{\tau^{(k+1)\tau}} |G_{r_{1-1}} (X_{k-1}, s) |^{2p} dr ds \right)
$$

$$
\leq \tau^2 \mathbb{E} \left( \sup_{0 \leq s \leq T} |\psi^{-1}(s)|^2 \right) + \tau^2 \mathbb{E} \left( \int_{\tau^{(k+1)\tau}}^{\tau^{(k+1)\tau}} \int_{\tau^{(k+1)\tau}}^{\tau^{(k+1)\tau}} |D'^{b}_{h_1} \psi^{-1}(s) |^{2p} dr ds \right)
$$

$$
+ 2^{2p} \mathcal{K}_2^2 \sum_{j=1}^{l+1} \mathbb{E} \left( \sup_{k \tau \leq s \leq (k+1)\tau} |D'^{b}_{h_1} \ldots D'^{b}_{h_{l+1}} x_{k-1} (s - \tau) |^{2p} \right), \quad (28)
$$

19
On the other hand, from Theorem 4.1,

\[
\mathbb{E}\left( \sup_{k \leq (k+1)\tau} |Z_{k,r_1,\ldots,r_l}(t)|^p \right) \leq 3^p \left\{ 2^p \mathbb{E}(|D_{r_1}^{\ast} \cdots D_{r_l}^{\ast} X_{k-1}(kt)|^p) + 2^p \mathbb{E}(|H_{r_1-r_1}(k)|^p) \right. \\
+ \frac{1}{2} \mathbb{E} \left( \int_{k\tau}^{(k+1)\tau} |\psi^{-1}(s)|^{2p} \, ds \right) \\
+ \left. \mathbb{E} \left( \sup_{k \leq (k+1)\tau} \int_{k\tau}^{(k+1)\tau} \psi^{-1}(s) G_{r_1-r_1}(X_{k-1}, s) \delta^{ab} \mathbb{E}(s)^p \right) \right\} \\
\leq 3^p \left\{ 2^p \mathbb{E}(|D_{r_1}^{\ast} \cdots D_{r_l}^{\ast} X_{k-1}(kt)|^p) + 2^p \mathbb{E}(|H_{r_1-r_1}(k)|^p) \right. \\
+ \frac{1}{2} \tau \mathbb{E} \left( \sup_{k \leq (k+1)\tau} |F_{r_1-r_1}(X_{k-1}, s)|^{2p} \right) + \frac{1}{2} (1 + C_1 \tau) \mathbb{E} \left( \int_{k\tau}^{(k+1)\tau} |\psi^{-1}(s)|^{2p} \, ds \right) \\
+ \frac{1}{2} C_1 \tau \left( \mathbb{E}(|G_{r_1-r_1}(X_{k-1}, s)|^{2p}) + C_1 \tau D N_{k,r_1,\ldots,r_l}(kt) \right). \tag{29} \]

Now, substitute Equations (26), (24) and (28) in (29) and result

\[
\mathbb{E}\left( \sup_{k \leq (k+1)\tau} |Z_{k,r_1,\ldots,r_l}(t)|^p \right) \]

\[
\leq 3^p \left\{ 2^p \mathbb{E}(|D_{r_1}^{\ast} \cdots D_{r_l}^{\ast} X_{k-1}(kt)|^p) + 2^p c_p H_i \sum_{j=1}^{l} \sum_{j=1}^{l-1} \sum_{j=1}^{l} \mathbb{E}(|W_{j,r_1}^{(\cdot)}|^{p \cdot} + |W_{j,r_1}^{(\cdot)}|^{p \cdot}) \right. \\
+ \frac{1}{2} \tau (1 + C_1 \tau) \mathbb{E} \left( \sup_{s \leq T} |\psi^{-1}(s)|^{2p} \right) \\
+ \frac{2^{p+1}}{2} (1 + C_1 \tau) \tau K_2 \left. \sum_{j=2}^{l} \mathbb{E} \left( \sup_{k \leq (k+1)\tau} |D_{r_1}^{\ast} \cdots D_{r_2}^{\ast} X_{k-1}(t)|^{(p+1)} \right) \right\} \\
+ \frac{1}{2} C_1 \tau \left. \sum_{p=1}^{2} \left( \mathbb{E}\left( \sup_{k \leq (k+1)\tau} |\psi^{-1}(s)|^{2p} \right) + 2^p C_1 \tau \mathbb{E} \left( \int_{k\tau}^{(k+1)\tau} |\psi^{-1}(s)|^{2p} \, ds \right) \right) \right\} \\
+ \frac{2^{p+1}}{2} C_1 \tau \left. (2^p K_2)^{p+1} \sum_{j=2}^{l} \mathbb{E} \left( \sup_{k \leq (k+1)\tau} |D_{r_1}^{\ast} \cdots D_{r_2}^{\ast} X_{k-1}(s-\tau)|^{2p \cdot} \right) \right\}. \tag{30} \]

Here it is crucial to mention that in this computation the order of derivatives for all terms in the right hand side of Equation (30) are utmost of the order \(l\), except of the last term which involves \(D_{r_1}^{\ast} \cdots D_{r_l}^{\ast} X_{k-1}(s-\tau)\). So, we first assume that \(k = 0\). Applying Assumption 5.1 Proposition 5.5 in Equation (30) and induction on \(l\) deduce that the process \(X_l\) is \(2M\)-times differentiable and their derivatives have bounded \(p\)-moments. Following by induction on \(k\) to result that the process \(X_{k+1}\) has Malliavin derivatives of one order less than those of \(X_k\) and their derivatives have bounded \(p\)-moments.
have uniformly bounded moments. Since $K_0 + 1 < 2M$, it is sufficient to proceed previous stage $M$ times to obtain the assertion.

7. Regularity of the density

To achieve the regularity of the density of solution to SDDE (1), we assume that the functions $f$ and $\sigma$ are infinitely differentiable with bounded derivatives; in fact $M = \infty$ in Assumption 1.1. Also we consider the following hypothesis.

**Hypothesis H:** There exists some constant $M_0$ such that $|\sigma(x)| > M_0$ for all $x$.

**Theorem 7.1.** Under Hypothesis H and assumption $M = \infty$, for every $t \in [0, T]$ the solution of the SDDE (1) has an infinitely differentiable density with respect to Lebesgue’s measure on $\mathbb{R}$.

**Proof.** Fixed $t \in [0, T]$. Theorem 5.7 guarantees the infinitely Malliavin differentiability of the solution $x(t)$ as $M = \infty$. To prove the second part, we have to show that $\mathbb{E}\left(\int_0^T |D_u X(t)|^2 du\right) < \infty$.

By using Malliavin’s criterion it is sufficient to check that for every $p \geq 1$ there exists some $\epsilon_0 > 0$ such that for all $\epsilon_1 \leq \epsilon_0$;

$$P\left(\int_0^T |D_u X(t)|^2 du < \epsilon_1\right) < \epsilon_1^p$$

To this end, we know

$$P\left(\int_0^T |D_u X(t)|^2 du < \epsilon_1\right) \leq P\left(\int_{\tau_1}^u |\phi(t)|^2 |\phi(u)|^{-2} |Bx(u) + \sigma(x(u - \tau)1_{\tau < 1}|^2 du < \epsilon_1\right)$$

$$\leq P(\mathcal{A}_0, \inf_{0 \leq u \leq T} \psi(r) > \delta_0, \sup_{0 \leq u \leq T} \psi(r) < \delta^{-1}) + P(\sup_{0 \leq u \leq T} \psi(r) \leq \delta) + P(\sup_{0 \leq u \leq T} \psi(r) \geq \delta^{-1})$$

$$:= P_{1, \epsilon_1} + P_{2, \epsilon_1} + P_{3, \epsilon_1},$$

with $\mathcal{A}_0 = \left\{\int_{\tau_1}^u |\phi(t)|^2 |\phi(u)|^{-2} |Bx(u) + \sigma(x(u - \tau)1_{\tau < 1}|^2 du < \epsilon_1\right\}$. Let $\delta = \epsilon_1^{\frac{1}{p}}$ and apply Chebyshev’s inequality to show that $P_{3, \epsilon_1} \leq \epsilon_1^p$. Indeed,

$$P_{3, \epsilon_1} \leq \delta^{\frac{1}{p}} \mathbb{E}\left(\sup_{0 \leq u \leq T} |\psi(t)|^{2p}\right) \leq \delta^{\frac{1}{p}} C_{\delta p}.$$  

From Proposition 5.6, the infimum part being a simple modification of this argument for $\psi^{-1}(\cdot)$.
and therefore $P_{2,\epsilon_1} \leq \epsilon_1^p$. On the other hand,

$$P_{1,\epsilon_1} \leq P\left( \int_{t-\epsilon_1^p}^{t} |Bx(u) + \sigma(x(u - \tau)1_{r>\tau})|^2 du < \epsilon_1^q \delta^{-4} \right)$$

$$\leq P\left( \int_{t-\epsilon_1^p}^{t} |Bx(u) + \sigma(x(u - \tau)1_{r>\tau})|^2 du < \epsilon_1^q \delta^{-4}, \int_{t-\epsilon_1^p}^{t} |Bx(u)| \leq \epsilon_1^q \right)$$

$$+ P\left( \int_{t-\epsilon_1^p}^{t} |Bx(u)| > \epsilon_1^q \right).$$

Since $|\sigma(y)| > 0$ for all $y$, when $\theta < \frac{1}{2}$ one can easily see that the first summand in above inequality is vanished. By using Chebyshev’s inequality and applying Equation (21), for every $q > 1$

$$P\left( \int_{t-\epsilon_1^p}^{t} |Bx(u)| > \epsilon_1^q \right) \leq \frac{1}{\epsilon_1^q} E\left( \sup_{t-\epsilon_1^p \leq u \leq t} |Bx(u)|^q \right) \leq B^q C_{1,q} \epsilon_1^{q(\theta - \gamma)}.$$

So, choosing $0 < \gamma < \theta < \frac{1}{2}$ result the assertion.

8. Conclusion

In this article, we demonstrate the problem of existence and uniqueness of solution to a stochastic differential equations with delay driven by weighted fractional Brownian motion. We introduce the solution step by step when proceeding with delay in time. This solution is Malliavin differentiable of higher order and its Malliavin derivatives have uniformly bounded moments. The solution has an infinitely differentiable density with respect to Lebesgue’s measure on $\mathbb{R}^d$ for $d \geq 1$. Our result allow one apply this result in many applications such as numerical methods, for instance one can easily check that Euler approximation process of the SDDE (1), in continuous version, has Malliavin derivatives with uniformly bounded moments. Also, one can result that the law of Euler approximation process is smooth, which is essential to show weak convergence of approximated process to the true solution.

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