EXAMPLES OF FINITELY DETERMINED MAP-GERMS OF CORANK 2 FROM $n$-SPACE TO $(n+1)$-SPACE

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Abstract. We produce new examples supporting the Mond conjecture which can be stated as follows. The number of parameters needed for a miniversal unfolding of a finitely determined map-germ from $n$-space to $(n+1)$-space is less than (or equal to if the map-germ is weighted homogeneous) the rank of the $n$th homology group of the image of a stable perturbation of the map-germ. In this paper, we give the first examples of finitely determined map-germs of corank 2 from 3-space to 4-space satisfying the conjecture. We introduce a method for generating series of finitely determined map-germs in dimensions $(n, n + 1)$ from a given finitely determined map-germ in dimensions $(n - 1, n)$. We present more examples in the dimensions $(4, 5)$ and $(5, 6)$, and verify the conjecture for them.

1. Introduction

One of the intriguing problems in Singularity Theory is to relate algebraic properties of holomorphic map-germs with topological properties of the images of their stable perturbations. The image of a stabilisation of a finitely $A$-determined map-germ from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}^{n+1}, 0)$ has the homotopy type of a wedge of $n$-spheres if $(n, n + 1)$ is in the range of Mather’s nice dimensions, i.e. $n < 15$, ([24]) – the existence of stabilisations is not guaranteed outside the nice dimensions; however, a similar statement can be proved for topological stabilisations which exist even for $n \geq 15$ ([5, Section 4]). The number of spheres in the wedge is an $A$-invariant of the map-germ which is called the image Milnor number and denoted by $\mu_I$. Pellikaan and de Jong (unpublished) then de Jong and van Straten ([7]) and later Mond ([24]) proved the following. For any finitely $A$-determined map-germ $f$ from a surface to 3-space,

$$A_c\text{-codim}(f) \leq \mu_I(f)$$

and with equality if $f$ is weighted homogeneous where $A_c\text{-codim}(f)$ is the dimension of the base of a mini-$A_c$-versal unfolding of $f$. A similar result for map-germs from $(\mathbb{C}, 0)$ to $(\mathbb{C}^2, 0)$ was also proved by Mond ([25]). Motivated by these results, Mond suggested the following generalisation.

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Conjecture 1.1 ([24], Mond Conjecture). Let $f: (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ be a finitely $\mathcal{A}$-determined map-germ. Then

\begin{equation}
\mathcal{A}_e{-}\text{codim}(f) \leq \mu_I(f)
\end{equation}

and with equality if $f$ is weighted homogeneous and $n < 15$.

We claim that Conjecture 1.1 would follow from the following statement.

Conjecture 1.2. Let $F$ be a stable $d$-parameter unfolding of a finitely $\mathcal{A}$-determined map-germ $f: (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$, and $H$ be the defining equation of the image of $F$. Assume that $G: (\mathbb{C}^{n+1} \times \mathbb{C}, 0) \to (\mathbb{C}^{n+1} \times \mathbb{C}^d, 0)$ is a germ transverse to $F$. Then $NK_{H,e/c}G$ is a Cohen-Macaulay module of dimension 1.

Our claim is motivated by Damon and Mond’s work [5] where they proved a phenomenon similar to (1) for holomorphic map-germs $(\mathbb{C}^n, 0)$ and $(\mathbb{C}^p, 0)$ where $(n, p)$ are nice dimensions and $n \geq p$. For a finitely $\mathcal{A}$-determined map-germ in these dimensions, the discriminant of a stabilisation intersected with a Milnor ball about the origin has the homotopy type of a wedge of $(p - 1)$-dimensional spheres ([24, Theorem 4.6]). The number $\mu_\Delta$ of spheres in the wedge is called the discriminant Milnor number. The main result is that

\begin{equation}
\mathcal{A}_e{-}\text{codim}(f) \leq \mu_\Delta(f)
\end{equation}

and with equality if $f$ is weighted homogeneous.

Here we run through the main steps of the proof of (2) to clarify Conjecture 1.2. Consider $f: (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ as a pullback of a stable $d$-parameter unfolding $F$ by an immersion $g: (\mathbb{C}^p, 0) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$ transverse to $F$. Assume that the discriminant $V$ of $F$ is defined by some $H \in \mathcal{O}_{\mathbb{C}^{p+d}, 0}$. For any 1-parameter deformation $G(y, t) = (g_t(y), t)$ of $g$, $NK_{H,e/c}G$ is a Cohen-Macaulay module of dimension 1. This is deduced from the fact that $V$ is a free divisor ([5, Proposition 5.2]). Consequently, $\mathcal{K}_{H}$-equivalence has a free deformation theory; that is,

\[ NK_{H,e,g} = \sum_{(y,t) \in \text{Supp}(NK_{H,e,c}G)} (NK_{H,e,g_t})_y \]

([5, Corollary 5.7]). Now, if $y$ is such that $g_t(y) \in V$, then

\[ (NK_{H,e,g_t})_y = (N\mathcal{A}_e f_t)_y = 0 \]

since $f_t$ is stable. Let $h_t = H \circ g_t$ and $J_{h_t}$ be the Jacobian ideal of $h_t$. If $g_t(y) \notin V$

\[ ev_H: (NK_{H,e,g_t})_y \to \mathcal{O}_{\mathbb{C}^p, y}/J_{h_t} \]

\[ \sum_{i=1}^{p+d} \alpha_i \frac{\partial}{\partial Y_i} \mapsto \sum_{i=1}^{p+d} \alpha_i \frac{\partial H}{\partial Y_i} \circ g_t \mod J_{h_t} \]

is an isomorphism ([5, Lemma 5.6]). So the only contribution to the $\mathcal{K}_{H,e}$-dimension of $g$ comes from the points $y \in \mathbb{C}^p$ with $g_t(y) \notin V$. The sum of the Milnor numbers $\dim_C\mathcal{O}_{\mathbb{C}^p, y}/J_{h_t}$ over all such $y$ is equal to $\mu_\Delta(f)$. Hence one gets (2).
When \( p = n + 1 \), the discriminant coincides with the image of the map-germ. Furthermore, Damon and Mond’s argument would prove Conjecture 1.1 if Conjecture 1.2 held.

A classification of finitely \( \mathcal{A} \)-determined map-germs can be pursued as an alternative way to attack Conjecture 1.1. However, as the dimension and the corank gets higher, classifying such map-germs or even finding examples becomes a difficult task.

In this paper we focus on finding new examples of finitely \( \mathcal{A} \)-determined corank \( \geq 2 \) map-germs in order to test Conjecture 1.1. It is not our aim to obtain a classification. In [15], Houston and Kirk gave a classification of map-germs of corank 1 from \((\mathbb{C}^3,0)\) to \((\mathbb{C}^4,0)\) with the strata of \( \mathcal{A} \)-codimension \( \leq 4 \), and showed that all the examples in their list satisfy the conjecture. We start with investigating the next interesting case: finitely \( \mathcal{A} \)-determined map-germs of corank 2 in the dimensions \((3,4)\). In Section 3, we prove some equivalent conditions to finite \( \mathcal{A} \)-determinacy for map-germs of any corank in the dimensions \((3,4)\). These conditions can easily be checked by a computer algebra program such as SINGULAR ([8]) or Macaulay2 ([12]). We finish the section with two sets of examples.

In Section 4, we study finite \( \mathcal{A} \)-determinacy of 1-parameter unfoldings defined by a base change operation on stable unfoldings. We refer to them as reductions. A reduction of a \( d \)-parameter unfolding \( F: (\mathbb{C}^n \times \mathbb{C}^d,0) \to (\mathbb{C}^p \times \mathbb{C}^d,0) \) by a germ \( \gamma: (\mathbb{C},0) \ni u \mapsto \gamma(u) \in (\mathbb{C}^d,0) \) is the pullback of \( F \) by \( \hat{g}: (Y,u) \mapsto (Y,\gamma(u)) \), \( Y \in (\mathbb{C}^p,0) \). This definition coincides with Houston’s definition of augmentations for \( d = 1 \) ([14]). We prove that a reduction is finitely \( \mathcal{A} \)-determined if \( F \) is a stable \( d \)-parameter unfolding of a finitely \( \mathcal{A} \)-determined map-germ \( f \) and \( \gamma \) intersects the \( KV \)-discriminant of the identity map on \((\mathbb{C}^p \times \mathbb{C}^d,0)\) only at the origin. In this setup, if \( f \) is defined from \((\mathbb{C}^n,0)\) to \((\mathbb{C}^p,0)\) then the reduction is a map from \((\mathbb{C}^{n+1},0)\) to \((\mathbb{C}^{p+1},0)\). We prove that a reduction satisfies Conjecture 1.2 if, additionally, \( f \) does and \( \gamma \) has multiplicity \( \geq 2 \). We also present series of finitely \( \mathcal{A} \)-determined map-germs of corank 2 supporting the conjecture in dimensions \((4,5)\) and \((5,6)\). Finally, we discuss the lack of a correspondence between finitely \( \mathcal{A} \) determined map-germs in the dimensions \((n-1,n)\) and the ones in \((n,n+1)\) (see Remarks 4.11 and 4.12).

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2. Terminology and Notations

Our terminology is standard, but the details can be found in [28] or [18]. We denote the space of holomorphic map-germs \( f: (\mathbb{C}^n,0) \to (\mathbb{C}^p,0) \) by \( \mathcal{E}_{n,p}^0 \). The group \( \mathcal{A} := \text{Diff}(\mathbb{C}^n,0) \times \text{Diff}(\mathbb{C}^p,0) \) of local diffeomorphisms acts on \( \mathcal{E}_{n,p}^0 \) by \((\phi,\psi) \cdot \mathcal{E}_{n,p}^0 \).
$f \mapsto \psi \circ f \circ \phi^{-1}$ for all $(\phi, \psi) \in \mathcal{A}$. We say that $f, g \in \mathcal{E}^0_{n,p}$ are $\mathcal{A}$-equivalent if $g \in \mathcal{A} \cdot f$. A map-germ $f \in \mathcal{E}_{n,p}^0$ is $\ell$-determined if every map-germ $g \in \mathcal{E}_{n,p}^0$ with the same $\ell$-jet (at $0$) as $f$ is $\mathcal{A}$-equivalent to $f$. Furthermore, $f$ is finitely $\mathcal{A}$-determined (or $\mathcal{A}$-finite) if it is $\mathcal{A}$-determined for some $\ell < \infty$. A map-germ is $\mathcal{A}$-stable if any of its unfoldings is $\mathcal{A}$-equivalent to the trivial unfolding $f \times 1$. By fundamental results of Mather, finite determinacy is equivalent to the finite dimensionality of $N_{\mathcal{A}} f := f^*(\Theta_{\mathcal{C}^p,0})/tf(\Theta_{\mathcal{C}^n,0}) + f^{-1}(\Theta_{\mathcal{C}^p,0})$, and thus (if $f$ is not stable) to $0 \in \mathbb{C}^p$ being an isolated point of instability of $f$. We set $\mathcal{A}_c$-codim$(f) := \dim_{\mathbb{C}} N_{\mathcal{A}_c} f$.

Following [3], for $g \in \mathcal{E}^0_{s,p}$, $(V,0) \subseteq (\mathbb{C}^p,0)$ and $H \in \mathcal{O}_{\mathbb{C}^p,0}$, the normal spaces with respect to $K_V$ and $K_{H}$-equivalences are given respectively by

$$NK_{V,e}g := g^*(\Theta_{\mathcal{C}^p,0})/tg(\Theta_{\mathcal{C}^n,0}) + g^*\text{Der}(-\log V)$$

and

$$NK_{H,e}g := g^*(\Theta_{\mathcal{C}^p,0})/tg(\Theta_{\mathcal{C}^n,0}) + g^*\text{Der}(-\log H)$$

where Der(-log V) (resp. Der(-log H)) is the module of logarithmic vector fields tangent to V (resp. to the level sets of $H$). To be more precise,

$$\text{Der}(-\log V) := \{\xi \in \Theta_{\mathcal{C}^p,0} \mid \xi(I(V)) \subseteq I(V)\}$$

and

$$\text{Der}(-\log H) := \{\xi \in \Theta_{\mathcal{C}^p,0} \mid \xi(H) = 0\}$$

in which $I(V)$ is the ideal of germs vanishing on $V$.

For a $d$-parameter deformation $G$ of $g$, the relative normal space is

$$NK_{H,e/Cd}G := G^*(\Theta_{\mathcal{C}^p,d,0})/tG(\theta_{\mathcal{C}^p,\mathbb{C}^d/Cd}) + G^*\text{Der}(-\log H).$$

Let $\rho$: $(\mathbb{C}^p \times \mathbb{C}^d,0) \rightarrow (\mathbb{C}^d,0)$ be the standard projection. The $K_V$-discriminant of $G$ is defined to be $D_V(G) := \rho(\text{Supp}(NK_{V,e/Cd}G))$.

Assume that $f \in \mathcal{E}_{n,p}^0$ is finite and equals the pullback of a stable unfolding $F \in \mathcal{E}_{n+d,p+d}^0$ by an immersion $g \in \mathcal{E}_{p,p+d}^0$ transverse to $F$. Let $V$ be the image of $F$ and $H$ its defining equation. Then

$$N_{\mathcal{A}_c} f \cong NK_{V,e}g$$

([26] Theorem 8.1]). When $f$ is weighted homogeneous, $NK_{V,e}g = NK_{H,e}g$ ([5] Corollary 3.18]).

For a $f \in \mathcal{E}_{n,p}^0$ with $n < p$, the ramification ideal is $R_f := \bigwedge^n df$, the ideal of $n \times n$-minors of the differential matrix of $f$. The corank is the $\mathbb{C}$-vector space dimension of the kernel of $df(0)$. Let $Q(f) := \mathcal{O}_{\mathbb{C}^p,0}/f^*m_{\mathbb{C}^p,0}$ be the local algebra and $q(f) := \dim_{\mathbb{C}} Q(f)$ the multiplicity of $f$. 

An important tool for studying the geometry of maps is the notion of multiple point spaces. Given a map \( f: X \to Y \), we set
\[
\mathcal{D}^k(f) = \{(x_1, \ldots, x_k) \in X^k | f(x_1) = \cdots = f(x_k), x_i \neq x_j \text{ if } i \neq j\}
\]
and define the \( k \)'th source multiple point space of \( f \), \( D^k(f) \), by
\[
D^k(f) = \text{closure } \mathcal{D}^k(f)
\]
(where the closure in taken in \( X^k \)) provided \( \mathcal{D}^k(f) \) is not empty. We extend this definition to germs of maps by taking the limit over representatives; if \( f \in \mathcal{E}_{n,p}^0 \) is finite, the local conical structure guarantees that we obtain in this way a well defined germ at \( 0 \in (\mathbb{C}^n)^k \).

In the corank 1 case, \( D^k(f) \) is given an analytic structure which is compatible with unfolding, and this leads to a definition of \( D^k(f) \) even in cases where \( \mathcal{D}^k(f) \) is empty. However, if corank \( \geq 2 \), we only have an explicit description for this analytic structure for \( k = 2 \) which we will recall shortly. We refer the reader to [23] for a description of the ideals defining \( D^k(f) \) in corank 1 case.

For the moment we want to distinguish the set-theoretic description from the analytic one. So we will denote by \( \hat{D}^2(f) \) the variety given by the ideal sheaf
\[
\mathcal{I}_2(f) := (f \times f)^* I_{\Delta_p} + \bigwedge^n \alpha
\]
where \( I_{\Delta_p} \) is the ideal of the diagonal of \( \mathbb{C}^p \times \mathbb{C}^p \) and \( \alpha := (\alpha_{ij}) \) is the matrix with entries coming from the equations
\[
f_i(x_1) - f_i(x_2) = \sum_{i=1}^p \alpha_{ij}(x_1, x_2) \cdot (x_{1j} - x_{2j})
\]
for \( i = 1, \ldots, p \) ([23]).

For a finite \( f \), \( D^k(f) \) is the fibre over \( 0 \in \mathbb{C}^d \) of \( D^k(F) \) where \( F \in \mathcal{E}_{n+d,p+d}^0 \) is a parametrised stable unfolding of \( f \) ([10]). If \( f \) is stable \( \hat{D}^2(f) \) is reduced. Hence, we can adopt the following definition for the double point space.

**Definition 2.1.** Let \( f \in \mathcal{E}_{n,n+1}^0 \) be finite and generically one-to-one. Let \( F: (\mathbb{C}^n \times \mathbb{C}^d, 0) \to (\mathbb{C}^{n+1} \times \mathbb{C}^d, 0) \), \( F(x, u) = (\tilde{F}_u(x), u) \), be a stable unfolding of \( f \). Then we set \( D^2(f) = \hat{D}^2(F) \cap \{u_1 = u_2 = 0\} \) and \( \pi_1^2(f) = \pi_2^1(f) \).

It is straightforward to check that the definition is independent of the choice of the stable unfolding.

For a finite \( f \in \mathcal{E}_{n,p}^0 \), the \( k \)'th target multiple point space is the set
\[
M_k(f) = \text{closure}\{y \in (\mathbb{C}^p, 0) | \|f^{-1}(y)\| \geq k\}
\]
(where preimages are counted with multiplicity) with analytic structure defined by the \((k-1)\)'st Fitting ideal \( \text{Fitt}_{k-1}(f, \mathcal{O}_{\mathbb{C}^n, 0}) \).

There is a natural projection \( \pi_{k-1}^k(f): D^k(f) \to D^{k-1}(f) \) which forgets \( k \)'th component, for all \( k \geq 2 \). Let \( \epsilon^k = f \circ \pi_{k-1}^2 \circ \pi_{k-2}^3 \circ \cdots \circ \pi_{k-1}^k \). For \( k > \ell \) we define
\( D^k(f) \) to be the image in \( D^k(f) \) of \( D^k(f) \) under the composite \( \pi^{k+1}_1 \circ \cdots \circ \pi^k_{k-1} \). Then we have set-theoretic equalities \( e^k(D^k(f)) = M_k(f) \) and \( f^{-1}M_k(f) = D^k(f) \) for all \( k \geq 1 \). Moreover, \( D^k(f) = M_{k-1}(\pi^2_k(f)) \) by the principal of iteration (see [**11**], Remark 2.7 (iii)). If \( f \in \mathcal{E}^0_{n,p} \) has corank 1 then \( q(\pi^k_{k-1}) = q(f) - k + 1 \) for \( k = 1, \ldots, q(f) \); however, if the corank is not 1 then \( q(\pi^2_k) \leq q(f) \) (**2**, Section 2.2).

**Proposition 2.2.** Let \( f \in \mathcal{E}^0_{n,n+1} \) be a finite and generically one-to-one map-germ. Then \( \bar{D}^2(f) \) is a Cohen-Macaulay space of codimension \( n + 1 \).

**Proof.** Let \( R_0 \) be any ring with identity and \( A = (a_{ij}) \) a generic \((n+1) \times n\)-matrix over the ring \( R = R_0[a_{ij} \mid 1 \leq i \leq n + 1, 1 \leq j \leq n] \). Let \( t_i = \sum_{j=1}^n a_{ij}s_j \) for \( s_j \in R_0 \) and \( i = 1, \ldots, n + 1 \). The quotient \( R[s_1, \ldots, s_n]/\bigwedge^n A + (t_1, \ldots, t_{n+1}) \) is Cohen-Macaulay of codimension \( n + 1 \) by [**6**], Theorem 2.7 – there one needs to make the following substitutions \( n_0 = n + 1 \), \( n_1 = n \), \( n_2 = 1 \) for \( j \geq 2 \), \( X^{(1)} = A \), \( X^{(2)} = [s_1 \ldots s_n]^t \), and \( k_1 = n - 1 \) so that their \( \mathcal{E}(k_1) \) equals to \( \bigwedge^n A + (t_1, \ldots, t_{n+1}) \).

Let \( X = \mathbb{C}^n \times \mathbb{C}^n \), \( Y = \text{Spec}(\mathbb{C}[s_1, \ldots, s_n, \alpha_{ij} \mid 1 \leq i \leq n + 1, 1 \leq j \leq n]) \) and \( Z := V(\bigwedge^n A + (t_1, \ldots, t_{n+1})) \). Define \( \Phi: X \to Y \) by

\[
\Phi: (x_1, x_2) \mapsto (\alpha_{11}, \ldots, \alpha_{n+1,n}, x_{11} - x_{21}, \ldots, x_{1n} - x_{2n})
\]

where \( \alpha_{ij} \) are given by (**5**). Then \( \Phi^{-1}(Z) = D^2(f) \).

As \( f \) is finite and generically one-to-one, the image of \( \pi^2_k \) has codimension 1 in \((\mathbb{C}^n, 0)\) (**27**, Proposition 3.5). So, \( D^2(f) \) has codimension \( n + 1 \). It follows from \( \Phi^{-1}(Z) \cong (X \times Z) \cap \text{graph}(\Phi) \) that \( \text{codim}_X Z \geq \text{codim}_X \Phi^{-1}(Z) \). Since we have the equality, \( \text{graph}(\Phi) \) is defined by a regular sequence (**21**, Theorem 17.4) whence the result.

In the case where \( p = n + 1 \) and \( n < 6 \), stable map-germs have corank 1.

**Proposition 2.3.** If \( f \in \mathcal{E}^0_{n,n+1} \) is a stable map-germ of corank \( \geq 2 \) then \( n \geq 6 \).

**Proof.** If \( f \) is stable and of corank 2 then \( j^1f: (\mathbb{C}^n, 0) \to J^1(n, n + 1) \) is transverse to \( \Sigma^2 \), the space of matrices of corank 2, at 0 by [**13**], §1.2, Chapter XV. By definition,

\[
\Sigma^2 + df(T_0\mathbb{C}^n) = J^1(n, n + 1).
\]

So \( \text{codim} \Sigma^2 \leq n \). However, \( \Sigma^2 \) is a vector space of codimension 6 by [**13**], §5.1, Chapter VII. Therefore, we must have \( n \geq 6 \).

**Proposition 2.4.** If \( f \in \mathcal{E}^0_{n,n+1} \) be an \( \mathcal{A} \)-finite map-germ of corank \( \geq 2 \) and \( n < 6 \), then \( D^2(f) \) has codimension \( n + 1 \) and at most an isolated singularity at 0.

**Proof.** Let \( (x_1, x_2) \in D^2(f) \setminus \{0\} \) and \( \tilde{f} \) be a representative of the multi-germ of \( f \) at \( \{x_1, x_2\} \). Since \( f \) is \( \mathcal{A} \)-finite, \( \tilde{f} \) is stable by the Mather-Gaffney criterion (see [**20**], p. 241), [**9**] or [**23**], Theorem 1.4). Moreover, each branch is stable by [**19**], Proposition 1.6. So, the statement follows from Proposition 2.3 and [**17**]...
Proposition 2.13] which states that the \( k \)'th multiple point space of a multi-germ of corank 1 is smooth.

\[ \square \]

3. MAP-GERMS FROM 3-SPACE TO 4-SPACE

In these dimensions, all stable mono-germs have corank 1 (Proposition [2.3]). According to Mather’s classification in [19], a stable corank 1 map-germ \( f \) is a stable corank 1 map-germ having a local algebra isomorphic to \( \mathbb{C}(z) / (z^{\ell+1}) \) for \( \ell \leq n/(p-n+1) \). For \( (n, p) = (3, 4) \), we get \( \ell \leq 1 \). In fact, if \( \ell = 0 \), \( f \) is an immersion; if \( \ell = 1 \), \( f \) is \( \mathcal{A} \)-equivalent to a constant 1-parameter unfolding of a cross-cap (or shortly a cross-cap), e.g. to \((x, y, z) \mapsto (x, y, z^2, yz)\). This classification enables us to characterise the geometry of finitely \( \mathcal{A} \)-determined map-germs away from the origin.

**Theorem 3.1.** A finite \( f \) is \( \mathcal{E}_{3,4}^0 \) is finitely \( \mathcal{A} \)-determined if and only if

1. \( D^2(f) \) has at most an isolated singularity at the origin,
2. \( D^1_k(f) \) is of dimension \( 4-k \) or empty for \( k = 1, \ldots, 4 \),
3. \( \text{Sing}(D^1_k(f)) \) and \( D^1_k(f) \) reduced and agree outside the origin for \( k = 2, 3 \),
4. \( V(R_f) \) has codimension \( \geq 2 \).
5. \( V(R_f) \cap \text{Sing}(D^2(f)) \) is \( \{0\} \) or empty.

Here, \( D^1_k(f) \) is defined by \( \text{Fitt}_0((\pi^1_k)_* \mathcal{O}_{D^2(f), 0}) \) and \( D^2_k(f) \) by \( f^* \text{Fitt}_{k-1}(f_* \mathcal{O}_{\mathbb{C}^3, 0}) \) for \( k = 3, 4 \). Moreover, \( \text{Sing}(-) \) is the singular locus of the variety defined by \((3-r) \times (3-r)\)-minors of the Jacobian matrix of its ideal where \( r \) is the codimension (cf. [13] Chapter I).

**Proof.** Assume that \( f \) is \( \mathcal{A} \)-finite. Then it is stable away from the origin; that is, the multi-germ \( f : (\mathbb{C}^3, S) \to (\mathbb{C}^4, y) \) where \( 0 \neq y \in \mathbb{C}^4 \) and \( S := f^{-1}(y) \) is stable ([23] Theorem 1.4]). By Mather’s results, each branch \( f^{(i)} : (\mathbb{C}^3, x_i) \to (\mathbb{C}^4, y) \) is an immersion or a cross-cap for all \( x_i \in S \). Now,

1. \( (i) \) clearly follows from Proposition [2.4]
2. \( (ii), (iii) \). By [27] Proposition 3.5, \( f^* \text{Fitt}_1(f_* \mathcal{O}_{\mathbb{C}^3, 0}) \) is a principal ideal. Therefore, \( D^2_1(f) = \mathbb{C}^{10} \) only when \( f \) fails to be degree 1 onto its image. In that case \( f \) is nowhere stable so it cannot be finitely \( \mathcal{A} \)-determined. Hence we must have \( \text{dim} D^2_1(f) = 2 \).

We have \( \text{Sing}(\text{im}(f)) = f(\text{Sing}(\text{dom}(f))) \cup M_2(f) \). A similar equality applies with \( \pi^2_1(f) \) in place of \( f \):

\[
\text{Sing}(D^1_2(f)) = \text{Sing}(\pi^2_1(f)) = \pi^2_1(\text{Sing}(D^2(f))) \cup \pi^2_1(D^2(\pi^2_1)).
\]

By the Principal of iteration ([11] Remark 2.7 (iii)), \( D^2(\pi^2_1) \cong D^3(\pi^2_1) \). Therefore \( \pi^2_1(D^2(\pi^2_1)) \cong D^3_1(f) \). So, set theoretically

\[
\text{Sing}(D^2_1(f)) = \pi^2_1(\text{Sing}(D^2(f))) \cup D^3_1(f) \subseteq \{0\} \cup D^3_1(f).
\]

By [17] Proposition 2.13, \( D^3(f) \) is 1-dimensional and smooth. It follows from [10] that \( D^2_1(f) \) is a reduced hypersurface and that \( \text{Sing}(D^2_1(f)) \) and \( D^3_1(f) \) agree.
outside the origin. Similarly,

\[
(7) \quad \text{Sing}(D^3_1(f)) = \pi_1^3(\text{Sing}(D^3(f))) \cup D^4_1(f) \subseteq \{0\} \cup D^4_1(f).
\]

A map-germ forms a quadruple point away from the origin only if there is a line of quadruple points in the image. But a line of quadruple points is not stable. Hence, \(D^4_1(f) = \{0\}\) or empty. From (7), we get \(\text{Sing}(D^3_1(f)) = \{0\}\) or empty.

(iv) follows from [3, Remark 2.2] which states that map-germs in \(\mathcal{E}^0_{n,n+1}\) are generically immersive, i.e. almost all have a critical set of codimension \(\geq 2\).

(v). Consider [4]. We have \(\alpha(x, x) = d_xf\). If \(x \in V(R_f), \ (x, x) \in (\pi^3_1)^{-1}(x)\). If \(0 \neq x \in \text{Sing}(D^3_1(f)) \cap V(R_f), x\) has multiplicity \(\geq 2\) in \(\text{Sing}(D^3_1(f)) = D^4_1(f)\). In this case \(f\) defines an unstable triple point at \(f(x)\). Hence no such \(x\) exists.

This concludes the first part of the proof. For the second part, we will show that (i)-(v) imply isolated instability at 0.

Let \(0 \neq y \in \mathbb{C}^4\) and \(S := f^{-1}(y)\). Consider the multi-germ \(\tilde{f} : (\mathbb{C}^3, S) \to (\mathbb{C}^4, y)\).

The geometric multiplicity of any \(x \in S\) is less than 3. This can be seen as follows. If the multiplicity \(\text{mult}(x)\) of \(x\) is greater than or equal to 3 then \(x \in D^3_1(f)\). By (iii), \(x \in \text{Sing}(D^2_1(f))\). On the other hand, \((x, x) \in D^2(f)\) whence \(x \in V(R_f)\). But this contradicts (v). So, \(\text{mult}(x) \leq 2\) for any \(x \in S\).

Notice that if \(\text{mult}(x) = 1\) then the branch at \(x\) is an immersion. If \(\text{mult}(x) = 2\) then \(x\) is in \(V(R_f)\) which has dimension \(\leq 1\). Since \(x \neq 0\), \(x \notin \text{Sing}(D^2_1(f))\) by (v). So, the branch at \(x\) is a cross-cap.

Now, let us study the cases based on \(S\). First of all, the number of points in \(S\) is less than 5 as \(D^5 = \emptyset\) in these dimensions. In fact, it is less than equal to 3 because the only quadruple point is the origin.

Another case that cannot occur is that \(S = \{x_1, x_2, x_3\}\) where at least one of \(x_1, x_2, x_3\) has multiplicity 2: if one of \(x_i\)’s, say \(x_1\), has multiplicity 2 then \(x_1 \in V(R_f)\) as well as \(x_1 \in D^2_1(f)\). But this a contradiction by (v). So the only possible cases (up to a permutation) are the following.

Case 1. \(S = \{x\}\) and \(\text{mult}(x) = 1\); i.e. \(\tilde{f}\) is a mono-germ and an immersion.

Case 2. \(S = \{x\}\) and \(\text{mult}(x) = 2\); i.e. \(\tilde{f}\) is a mono-germ and a cross-cap.

Case 3. \(S = \{x_1, x_2\}\) and \(\text{mult}(x_i) = 1\) for all \(i = 1, 2\); i.e. \(\tilde{f}\) is a bi-germ, each branch is an immersion.

Case 4. \(S = \{x_1, x_2\}\) and \(\text{mult}(x_i) = 2\) for all \(i = 1, 2\); i.e. \(\tilde{f}\) is a bi-germ, each branch is a cross-cap.

Case 5. \(S = \{x_1, x_2\}\) and \(\text{mult}(x_1) = 2\) and \(\text{mult}(x_1) = 1\); i.e. \(\tilde{f}\) is a bi-germ, the first branch is a cross-cap and the other is an immersion.

Case 6. \(S = \{x_1, x_2, x_3\}\) and \(\text{mult}(x_i) = 1\) for all \(i = 1, 2, 3\); i.e. \(\tilde{f}\) is a 3-germ, each branch is an immersion.

It remains to show the stability of \(\tilde{f}\) in Cases 3-6.

If both \(f^{(1)}\) and \(f^{(2)}\) are immersions but not transversal then their images coincide and they form a 3-dimensional double locus which contradicts (ii). So, they
must be transversal whence $\bar{f}$ in Case 3 is a normal crossing. By a similar consideration, we can conclude that the multi-germ in Case 6 forms a line of ordinary triple points which is also stable.

For the other two cases, we will again use [19, Proposition 1.6]. To be more precise, we will show that (1) each $f^{(i)}$ is stable, and (2) $f^{(1)} \times \cdots \times f^{(m)}: (A_1 \times \cdots \times A_m, S) \to (\mathbb{C}^4)^m$ is transverse to the diagonal $\{(y, \ldots, y) \in (\mathbb{C}^4)^m\}$ where $A_i$ is the isosingular locus of $f^{(i)}$ and $m$ is the number of points in $S$. Recall that the isosingular locus of a map is the set of points in the domain at which the germ of the map is equivalent to the map in question. Clearly, the isosingular locus equals the domain for stable map-germs. Also for $m = 2$, (2) is equivalent to $f^{(1)}|_{A_1}$ being transverse to $f^{(2)}|_{A_2}$.

Let $\bar{f}$ be a bi-germ with $f^{(1)}$ a cross-cap and $f^{(2)}$ an immersion (or again a cross-cap). If $f^{(1)}$ and $f^{(2)}$ are not transversal then $D^2(\bar{f})$ has a singularity which contradicts (i). To see this, put $f^{(1)}$ into a standard form; e.g.

$$f^{(1)}: (x, y, z) \mapsto (x, xy, y^2, z).$$

By standard classification methods, one can show that $f^{(2)}$ has the form

$$f^{(2)}: (t, u, v) \mapsto (t, u, h(t, u, v), v)$$

for some $h \in m^2_{(C^3, x_2)}$. Then

$$D^2(\bar{f}) = D^2(f^{(1)}) \cup V(x - t, xy - u, y^2 - h(t, u, v), z - v),$$

of which the second component is singular. This concludes the proof of the theorem. \hfill \Box

**Corollary 3.2.** Assume that $\bar{f}: (\mathbb{C}^3, S) \to (\mathbb{C}^4, y)$ is a representative of the multi-germ of a finitely determined $f \in E_{3,4}^0$ at $S := f^{-1}(y)$ where $y \neq 0$. There are four standard forms that $\bar{f}$ can take -- a normal crossing (8), an immersion and a cross-cap meeting transversely (9), two cross-caps meeting transversely (10) and a line of ordinary triple points (11).

\begin{equation}
\bar{f} \sim_\mathcal{A} \begin{cases}
(x, y, z) \mapsto (x, y, z, 0) \\
(t, u, v) \mapsto (t, u, 0, v)
\end{cases},
\end{equation}

\begin{equation}
\bar{f} \sim_\mathcal{A} \begin{cases}
(x, y, z) \mapsto (x, xy, y^2, z) \\
(t, u, v) \mapsto (t, u, 0)
\end{cases},
\end{equation}

\begin{equation}
\bar{f} \sim_\mathcal{A} \begin{cases}
(x, y, z) \mapsto (x, xy, y^2, z) \\
(t, u, v) \mapsto (uv, t, u, v^2)
\end{cases},
\end{equation}

\begin{equation}
\bar{f} \sim_\mathcal{A} \begin{cases}
(x, y, z) \mapsto (x, y, z, 0) \\
(t, u, v) \mapsto (t, u, 0, v)
\end{cases},
\end{equation}

\begin{equation}
\bar{f} \sim_\mathcal{A} \begin{cases}
(t, u, v) \mapsto (t, u, 0, v) \\
(a, b, c) \mapsto (a, 0, b, c)
\end{cases}.
\end{equation}
Remark 3.3. In [16], Kleiman, Lipman and Ulrich proved that if $F: X \to Y$ is a finite map of corank 1 between locally Noetherian schemes of dimensions $n$ and $n+1$,

\[ \text{Fitt}_i(F^*F_*\mathcal{O}_X) = \text{Fitt}_{i-1}((\pi_1^2),\mathcal{O}_{D^2(F)}) \]

for all $i \geq 1$ (under some additional hypotheses: $F$ is of flat dimension 1, and $Y$ satisfies Serre’s condition $(S_2)$). The equality, for a finite and generically one-to-one map-germ of corank 1 in $\mathcal{E}^0_{n,n+1}$, is proved in [2] with a different approach.

We are able to prove the following equalities similar to (12) for map-germs of corank 2.

**Proposition 3.4.** Let $f \in \mathcal{E}^0_{3,4}$ be $A$-finite and of corank $\geq 2$. Then

\[ f^*\text{Fitt}_1(f_*\mathcal{O}_{C^3,0}) = \text{Fitt}_0((\pi_1^2),\mathcal{O}_{D^2(f)}) \]
\[ (f^*\text{Fitt}_k(f_*\mathcal{O}_{C^3,0}))_x = \text{Fitt}_{k-1}((\pi_1^2),\mathcal{O}_{D^2(f)})_x \]

for all $0 \neq x \in (\mathbb{C}^3,0)$ and $k \geq 2$. Moreover, all ideals in (13) and (14) are reduced.

**Proof.** The equalities and reducedness at the stalks away from 0 can easily be seen by calculating the presentations of pushforwards ([27, §2]) for (8)-(11). We can extend the equality

\[ (f^*\text{Fitt}_1(f_*\mathcal{O}_{C^3,0}))_x = \text{Fitt}_0((\pi_1^2),\mathcal{O}_{D^2(f)})_x \]

to the origin as follows. By [27, Proposition 3.5], $f^*\text{Fitt}_1(f_*\mathcal{O}_{C^3,0})$ is a principal ideal. Also, $\text{Fitt}_0((\pi_1^2),\mathcal{O}_{D^2(f)})$ is principal as it is the determinant of a certain square matrix ([27, Lemma 2.1]). As both are reduced away from the origin, they are reduced everywhere because they are Cohen-Macaulay. Therefore $f^*\text{Fitt}_1(f_*\mathcal{O}_{C^3,0}) = \text{Fitt}_0((\pi_1^2),\mathcal{O}_{D^2(f)})$. \qed

The following examples suggests that for map-germs of corank $\geq 2$, in general (12) should hold only for $i = 1, 2$.

**Example 3.5.** For the stable map-germ

\[ h: (y, z, u) \mapsto (y^2 + u_1z, z^2 + u_2y, u_3yz + u_4y + u_5z, u), \]

one calculates that

\[ \text{Fitt}_i(h^*h_*\mathcal{O}_{C^7,0}) = \text{Fitt}_{i-1}((\pi_1^2(h)),\mathcal{O}_{D^2(h)}) \]

for $i = 1, 2$, but

\[ \text{Fitt}_j(h^*h_*\mathcal{O}_{C^7,0}) \neq \text{Fitt}_{j-1}((\pi_1^2(h)),\mathcal{O}_{D^2(h)}) \]

for $j = 3, 4$. The same equalities, and inequalities, hold for the stable map-germ

\[ \hat{h}: (y, z, v) \mapsto (y^3 + v_1z + v_2y, yz + v_3z, z^2 + v_4y + v_5y^2 + v_6z, v). \]

Notice that $Q(h) = \mathbb{C} \cdot \{1, y, z, yz\}$ and $Q(\pi_1^2(h)) = \mathbb{C} \cdot \{1, y_2, z_2\}$; $Q(\hat{h}) = \mathbb{C} \cdot \{1, y, y^2, z\}$ and $Q(\pi_1^2(\hat{h})) = \mathbb{C} \cdot \{1, y_2, y^2_2, z_2\}$.
3.1. Examples and non-examples.

**Proposition 3.6.** The map-germ

\[ \hat{A}_\ell : (x, y, z) \mapsto (x, y^\ell + xz + x^{2\ell - 2}y, yz, z^2 + y^{2\ell - 1}) \]

is finitely A-determined for \( \ell = 2, \ldots, 6 \) and has the data in Table 1.

**Table 1.** The first set of examples

| Label | \( A_e \)-codimension | Weights   | Conjecture 1.1 |
|-------|------------------------|-----------|----------------|
| \( A_2 \) | 18 | (1, 2, 3) | True |
| \( A_3 \) | 186 | (1, 2, 5) | True |
| \( A_4 \) | 844 | (1, 2, 7) | True |
| \( A_5 \) | \(< \infty \) | (1, 2, 9) | ? |
| \( A_6 \) | \(< \infty \) | (1, 2, 11) | ? |

**Proof.** Finite \( A \)-determinacy of each map-germ is checked by Theorem 3.1, and \( A_e \)-codimensions are calculated on SINGULAR using the identity (3). By another calculation we see that \( \hat{A}_\ell \) satisfies Conjecture 1.2 for \( \ell = 2, 3, 4 \). Due to computer memory restrictions, we are yet to calculate \( A_e \)-codimensions and verify the conjecture for \( k > 4 \) and the finite determinacy for \( \ell > 6 \).

**Proposition 3.7.** Let

\[ \hat{B}_{2\ell+1}^\pm: (x, y, z) \mapsto (x, y^2 + xz, z^2 + xy, y^{2\ell+1} \pm y^{2\ell-1}z^2 + z^{2\ell+1}) \]

for \( \ell \geq 1 \). Then, \( \hat{B}_{3}^\pm, \hat{B}_{5}^\pm, \hat{B}_{7}^\pm, \hat{B}_{9}^\pm, \hat{B}_{11}^\pm, \hat{B}_{13}^\pm \) are \( A \)-finite with the data in Table 2.

**Table 2.** The second set of examples

| Label | \( A_e \)-codimension | Weights   | Conjecture 1.1 |
|-------|------------------------|-----------|----------------|
| \( B_{3}^\pm \) | 33 | (1, 1, 1) | True |
| \( B_{5}^\pm \) | 252 | (1, 1, 1) | True |
| \( B_{7}^\pm \) | 837 | (1, 1, 1) | True |
| \( B_{9}^\pm \) | 1968 | (1, 1, 1) | True |
| \( B_{11}^\pm \) | 3825 | (1, 1, 1) | True |
| \( B_{13}^\pm \) | 6588 | (1, 1, 1) | ? |

**Proof.** Finite \( A \)-determinacy of each map-germ is checked by Theorem 3.1. The double point spaces for \( \hat{B}_{3}^\pm \) and \( \hat{B}_{13}^\pm \) are not isolated singularities, hence they are not \( A \)-finite. We calculate \( A_e \)-codimensions on SINGULAR using the identity (3), and also see that they satisfy Conjecture 1.2. Due to computer memory restrictions, we are yet to verify the conjecture for \( \ell > 5 \) and the finite determinacy for \( \ell > 6 \).
Proposition 3.8. There are no finitely $\mathcal{A}$-determined homogeneous map-germs of corank 2 in $E_{d,4}^0$ with degrees $(1,2,2,2d)$, $d \geq 1$.

Proof. A finite homogeneous map-germ of corank 2 with degrees $(1,2,2,2d)$ is $\mathcal{A}$-equivalent to

$$h: (x,y,z) \mapsto (x,y^2 + axz, z^2 + bxy, P_{2d}(x,y,z))$$

where $a, b \in \mathbb{C}$ and $P_{2d}(x,y,z)$ is a homogeneous polynomial of degree $2d$. Assume that $d = 1$. Then,

$$h \sim_{\mathcal{A}} h': (x,y,z) \mapsto (x,y^2 + axz, z^2 + bxy, cxy + dxz + eyz)$$

with $c, d, e \in \mathbb{C}$. A calculation on SINGULAR shows that $D^2(h')$ is not an isolated singularity whence $h'$ is not finitely $\mathcal{A}$-determined. Now, assume that $d \geq 2$. We will show that $h$ forms a line of quadruple points in the image whence it is not $\mathcal{A}$-finite. Let us consider two lines $L_+ : t \mapsto (0,t,\alpha t)$ and $L_- : t \mapsto (0,t,-\alpha t)$ where $\alpha \in \mathbb{C} \setminus \{0\}$. We have

$$h(0,t,\alpha t) = (0,t^2, \alpha^2 t^2, P_{2d}(0,t,\alpha t))$$

$$h(0,t,-\alpha t) = (0,t^2, \alpha^2 t^2, P_{2d}(0,t,-\alpha t)).$$

Write $P_{2d}(x,y,z) = \sum_{l+m+n=2d} a_{lmn} x^l y^m z^n$ where $a_{lmn} \in \mathbb{C}$, for all $l, m, n$. So,

$$P_{2d}(0,t,\alpha t) = P_{2d}(0,t,-\alpha t) \iff \sum_{m+n=2d} a_{0mn} \alpha^n = \sum_{m+n=2d} a_{0mn} (-\alpha)^n$$

$$\iff \sum_{m+n=2d} a_{0mn} \alpha^{2n-1} = 0. \quad (15)$$

Let $\alpha_1$ be a solution of (15). Then

$$h(0,t,\alpha_1 t) = h(0,t,-\alpha_1 t) = h(0,-t,-\alpha_1 t) = h(0,-t,\alpha_1 t)$$

whence the result. \qed

Remark 3.9. Notice that $\mathcal{A}_e$-codimension of our examples are quite high. The smallest codimension we have encountered so far is $\mathcal{A}_e$-codim($\hat{A}_2$) = 18 among the map-germs of corank 2 in these dimensions (see also Proposition 4.4 – the first map-germ in the series there, $f_1$, has codimension 33.). It would be very interesting to see if there exist weighted homogeneous map-germs of lower $\mathcal{A}_e$-codimension in the corank 2 case; even more, if there exist any finitely $\mathcal{A}$-determined map-germs of corank 3 in any dimensions.

4. Generating new examples from old

In this section, we introduce a method for generating new examples by a special base change operation on stable unfoldings of finitely determined map-germs. First, we fix some notation. The chosen coordinate systems on $(\mathbb{C}^p,0)$ and $(\mathbb{C}^d,0)$ are denoted by $\mathbf{Y} = (Y_1, \ldots, Y_p)$ and $\mathbf{U} = (U_1, \ldots, U_d)$, respectively. Let $\rho: (\mathbb{C}^p \times \mathbb{C}^d)$.
\( \mathbb{C}^d, 0 \) \rightarrow (\mathbb{C}^d, 0) be the standard projection and let \( F \in \mathcal{E}_{n+d,p+d}^0 \). We may represent any \( \xi \in \Theta(F) \) as \( \xi = (\xi_1, \xi_2) \) where \( \xi_1 \) is the \( Y \)-component and \( \xi_2 \) the \( U \)-component. We set \( \Theta(F)/\rho := \{ \xi_2 \mid \xi = (\xi_1, \xi_2) \in \Theta(F) \} \) which is isomorphic to \( (\mathcal{O}_{\mathbb{C}^p,0})^d \).

**Definition 4.1** (cf. XIII, 1.4, [18]). Let \( F \in \mathcal{E}_{n+d,p+d}^0 \) be a map-germ given by \( F(x, u) = (F_u(x), u) \) for \( u \in \mathbb{C}^d \), and \( \gamma : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^d, 0) \) be a holomorphic map-germ. We define the reduction \( R_\gamma(F) \) of \( F \) by \( \gamma \) to be the map-germ

\[
R_\gamma(F) : (\mathbb{C}^n \times \mathbb{C}, 0) \rightarrow (\mathbb{C}^p \times \mathbb{C}, 0) \\
(x, w) \mapsto (F_\gamma(w)(x), w).
\]

Notice that the corank of \( R_\gamma(F) \) equals the corank of \( F \). Moreover, if \( F \) is a \( d \)-parameter unfolding of an \( f \in \mathcal{E}_{n,p}^0 \) then a reduction is a 1-parameter unfolding of \( f \). The following theorem states the criteria for finite determinacy of reductions.

**Theorem 4.2.** Let \( F \in \mathcal{E}_{n+d,p+d}^0 \) be a parametrised stable unfolding of an \( \mathcal{A} \)-finite \( f \in \mathcal{E}_{n,p}^0 \) and \( V \) the image of \( F \). Let \( G \) be the identity on \( (\mathbb{C}^p \times \mathbb{C}^d, 0) \). Assume that \( \gamma : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^d, 0) \) is a non-constant map-germ parametrising a curve which intersects \( D_V(G) \) only at the origin. Then, \( R_\gamma(F) \) is also \( \mathcal{A} \)-finite.

In this setting, we will refer to \( f \) as the initial map-germ of \( R_\gamma(F) \).

**Proof.** Consider \( R_\gamma(F) \) as a pull-back of \( F \) by

\[
\hat{g} : (\mathbb{C}^p \times \mathbb{C}, 0) \rightarrow (\mathbb{C}^p \times \mathbb{C}^d, 0) \\
(Y, w) \mapsto (Y, \gamma_1(w), \ldots, \gamma_d(w)).
\]

Notice that \( \hat{g} \) is transverse to \( F \) since \( g : Y \rightarrow (Y, 0) \) is. So, \( \mathcal{N}A_{e,R_\gamma(F)} \simeq \mathcal{N}K_{V,e,\hat{g}} \). We will show that the support of \( \mathcal{N}K_{V,e,\hat{g}} \) consists of the origin at most.

We have

\[
\mathcal{N}K_{V,e,\hat{g}} \simeq \frac{(\mathcal{O}_{\mathbb{C}^p \times \mathbb{C},0})^{p+d}}{(\frac{\partial \hat{g}}{\partial Y_1}, \ldots, \frac{\partial \hat{g}}{\partial Y_p}, \frac{\partial \hat{g}}{\partial w}) \mathcal{O}_{\mathbb{C}^p \times \mathbb{C},0} + \hat{g}^* \text{Der}(-\text{log } V)}.
\]

As \( \frac{\partial \hat{g}}{\partial Y_i} = \frac{\partial}{\partial Y_i} \) for all \( i = 1, \ldots, p \),

\[
\mathcal{N}K_{V,e,\hat{g}} \simeq \frac{\Theta(\hat{g})/\rho}{(\frac{\partial \hat{g}}{\partial w}) \mathcal{O}_{\mathbb{C}^p \times \mathbb{C},0} + \hat{g}^* t \rho (\text{Der}(-\text{log } V))}.\]

On the other hand,

\[
\mathcal{N}K_{V,e/\mathbb{C}^d G} \simeq \frac{\Theta(G)/\rho}{t \rho (\text{Der}(-\text{log } V))}.
\]

We have

\[
\mathcal{N}K_{V,e/\mathbb{C}^d G} \otimes_{\mathcal{O}_{\mathbb{C}^p \times \mathbb{C}^d,0}} \mathcal{O}_{\mathbb{C}^p \times \mathbb{C}^d,0} (U_1 - \gamma_1(w), \ldots, U_d - \gamma_d(w)) \simeq \frac{\Theta(\hat{g})/\rho}{\hat{g}^* t \rho (\text{Der}(-\text{log } V))}.
\]
So
\[ \supp \left( \frac{\Theta(\hat{\gamma})/\rho}{\hat{\gamma}^{*} t_{\rho} (\der (-\log V))} \right) \cong \rho \times 1 \left( (\supp (NK_{V,e}/\mathbb{C}dG) \times \mathbb{C}) \cap (\mathbb{C}^p \times \graph(\gamma)) \right). \]

We have $NK_{V,e}/\mathbb{C}dG/(U) \cong NK_{V,e}g$. Thus,
\[ \supp (NK_{V,e}/\mathbb{C}dG) \cap (\mathbb{C}^p \times \{0\}) = \{0\}. \]
As $\im(\gamma) \cap D_{V}(G) = \{0\}$, (18) is also $\{0\}$. By (17), $\supp (NK_{V,e}\hat{\gamma})$ is $\{0\}$ or empty; equivalently, $R_{\gamma}(F)$ is $\mathcal{A}$-finite.

We expect that the converse of Theorem 4.2 also holds. We leave the proof for further study. We will use the following proposition to check Conjecture 1.2 for reductions.

**Proposition 4.3.** Let $H \in \mathcal{O}_{p,p+d}$, $g \in \mathcal{E}_{p,p+d}$ be the inclusion and $\hat{\gamma}$ given by (10). Assume that $q(\gamma) \geq 2$. If $NK_{H,e}/\mathbb{C}G$ is a free $\mathcal{O}_{C,0}$-module for a 1-parameter deformation $G$ of $g$ then there exists a 1-parameter deformation $\hat{G}$ of $\hat{\gamma}$ for which $NK_{H,e}/\mathbb{C}G$ is also a free $\mathcal{O}_{C,0}$-module.

**Proof.** Without loss of generality we may assume that $G$ is given by
\[ G: (Y, v) \mapsto (Y_{1}, \ldots, Y_{p}, \alpha_{1}(Y, v), \ldots, \alpha_{d}(Y, v)) \]
for $\alpha_{i} \in \mathcal{O}_{p,p+d}$, $i = 1, \ldots, d$. Define
\[ \hat{G}: (Y, w, v) \mapsto (Y_{1}, \ldots, Y_{p}, \gamma_{1}(w) + \alpha_{1}(Y, v), \ldots, \gamma_{d}(w) + \alpha_{d}(Y, v)) \]
so that $\hat{G}$ is also a deformation of $G$. Since
\[ NK_{H,e}/\mathbb{C}G/(v) NK_{H,e}/\mathbb{C}\hat{G} \cong NK_{H,e}\hat{\gamma}, \]
$NK_{H,e}/\mathbb{C}G$ has dimension less than or equal to 1. Now, $\frac{\partial G}{\partial Y_{i}} = \frac{\partial \hat{G}}{\partial Y_{i}}$ for all $i = 1, \ldots, p$ and $\frac{\partial G}{\partial \nu} \in \mathfrak{m}_{C,0} \hat{G}^{*}(\Theta_{p,p+d,0})$. So,
\[ NK_{H,e}/\mathbb{C}\hat{G}/(w) NK_{H,e}/\mathbb{C}\hat{G} \cong NK_{H,e}/\mathbb{C}G \]
whence $\dim NK_{H,e}/\mathbb{C}\hat{G} \geq 1$. Therefore, $\dim NK_{H,e}/\mathbb{C}\hat{G} = 1$ and $v$ is $NK_{H,e}/\mathbb{C}\hat{G}$-regular by [21, Theorem 17.4]. Hence, $NK_{H,e}/\mathbb{C}\hat{G}$ is a Cohen-Macaulay module of dimension 1. Since
\[ \rho^{-1}(0) \cap \supp \left( NK_{H,e}/\mathbb{C}\hat{G} \right) = \supp (NK_{H,e}\hat{\gamma}) = \{0\} \]
$\rho$ is a finite map on $\supp \left( NK_{H,e}/\mathbb{C}\hat{G} \right)$. Thus, $\rho_{*}(NK_{H,e}/\mathbb{C}\hat{G})$ is a finite $\mathcal{O}_{C,0}$-module; moreover it is Cohen-Macaulay since the Cohen-Macaulay property is conserved under finite pushforwards. Now the result easily follows from the Auslander-Buchsbaum formula
\[ \projdim_{\mathcal{O}_{C,0}} \left( \rho_{*}(NK_{H,e}/\mathbb{C}\hat{G}) \right) + \depth_{\mathcal{O}_{C,0}} \left( \rho_{*}(NK_{H,e}/\mathbb{C}\hat{G}) \right) = \depth_{\mathcal{O}_{C,0}} (\mathcal{O}_{C,0}) \]
\[\text{Proposition 4.3 for } h \text{ Proposition 4.5.}\]

The series \(M\) and others.
\[\Box\]

\[\partial\]

Let have \(A\) has \(\gamma\) by \(21\), Theorem 19.1). \[\Box\]

One can check that \(f\) is \(A\)-determined \(A\)-finite for \(\ell \geq 1\) and satisfies Conjecture 1.1. \[\Box\]

Proof. Notice that \(f_\ell\) is a reduction of
\[
F(x, y, u) = (x^2 + u_1y, y^2 + u_2x + u_3y, x^3 + x^2y + xy^2 - y^3 + u_4x + u_5y, u)
\]
by \(\gamma: z \mapsto (z, z, 0, 0, 0)\); \(F\) is a stable unfolding of
\[
h: (x, y) \mapsto (x^2, y^2, x^3 + x^2y + xy^2 - y^3)
\]
which is \(A\)-finite \([3]\). Let \(G\) be the identity on \((C^2 \times C^4, 0)\). The image of \(\gamma\) cuts \(D_V(G)\) only at \(\{0\}\). Therefore, finite \(A\)-determinacy follows from Theorem 4.2

One can check that \(f_1\) satisfies Conjecture 1.2. The statement for \(\ell \geq 2\) follows from Proposition 4.3 for \(h\) satisfies the conjecture for \(n = 2\) by the theorem of Mond and others. \[\Box\]

We give examples in higher dimensions in the rest of this section.

\[\text{Proposition 4.4. The map-germ}\]

\[\text{(21) } f_\ell: (x, y, z) \mapsto (x^2 + z^\ell y, y^2 - z^\ell x, x^3 + x^2y + xy^2 - y^3, z)\]
is \(A\)-determined for \(\ell \geq 1\) and satisfies Conjecture 1.1. \[\Box\]

Proof. Notice that \(\hat{C}_\ell\) is a reduction of
\[
\hat{F}_{\hat{A}_2}(x, y, z, u) = (x, y^2 + xz + x^2y + u_1y, yz + u_2y, z^2 + y^3 + u_3y, u)
\]
by \(\gamma_{\hat{C}}: w \mapsto (0, w^\ell, 0)\). And \(\hat{A}_2\) is a stable unfolding of \(\hat{A}_2\) from Proposition 3.6.

Let \(G\) be the identity on \((C^3 \times C^3, 0)\). Any curve \(\gamma: w \mapsto (0, \gamma_2(w), 0)\), where \(\gamma_2(w)\) is not constant, intersects \(D_V(G)\) only at the origin. Hence, \(\hat{C}_\ell\) is also \(A\)-determined by Theorem 4.2. Now we will show that \(NK_{V,e}\hat{g}_\ell\) has dimension \(30\ell - 18\) over \(C\) where \(\hat{g}_\ell: (X, Y, Z, W) \mapsto (X, Y, Z, W, 0, w^\ell, 0)\) for all \(\ell \geq 1\). We have
\[
NK_{V,e}\hat{g}_\ell \approx \text{dim}(\Theta(F)^3)\frac{(O_{C^5,0})^3}{(\partial^2)\partial_1\partial_3^2 + \hat{g}_\ell^*t\rho(\text{Der}(-\log V))}
\]

Let \(\frac{\partial}{\partial_1}, \frac{\partial}{\partial_2}, \frac{\partial}{\partial_3}\) denote the standard basis for \(\Theta(F)^3\). A Groebner basis for \(M_0 := \hat{g}_\ell^*t\rho(\text{Der}(-\log V))\) with respect to the reverse lexicographic order with priority

\[\text{[21] Theorem 19.1}].\]
given to the coefficients is given by the following vector fields.

\[ m_1 := X^8 \frac{\partial}{\partial U_1}, \]

\[ m_2 := X^9 \frac{\partial}{\partial U_2}, \]

\[ m_3 := -\frac{277}{229} X^7 \frac{\partial}{\partial U_1} - \frac{25}{458} X^8 \frac{\partial}{\partial U_2} + X^9 \frac{\partial}{\partial U_3}, \]

\[ m_4 := \frac{5}{2} X^2 \frac{\partial}{\partial U_1} - \frac{1}{4} X^3 \frac{\partial}{\partial U_2} + \left( Y - \frac{1}{2} X^4 \right) \frac{\partial}{\partial U_3}, \]

\[ m_5 := \left( X^2 Y + \frac{74}{39} X^6 \right) \frac{\partial}{\partial U_1} - \frac{61}{312} X^7 \frac{\partial}{\partial U_2} - \frac{59}{156} X^8 \frac{\partial}{\partial U_3}, \]

\[ m_6 := \left( -\frac{8}{7} X Y - \frac{36}{7} X^5 \right) \frac{\partial}{\partial U_1} + \left( X^2 Y - \frac{4}{7} X^6 \right) \frac{\partial}{\partial U_2} + \frac{40}{7} X^7 \frac{\partial}{\partial U_3}, \]

\[ m_7 := \left( Y^2 + \frac{3439}{32} X^4 Y \right) \frac{\partial}{\partial U_1} - \frac{19473}{256} X^5 Y \frac{\partial}{\partial U_2} - \frac{82731}{128} X^6 Y \frac{\partial}{\partial U_3}, \]

\[ m_8 := \left( \frac{147}{2} X^3 Y + \frac{12939}{16} X^7 \right) \frac{\partial}{\partial U_1} + \left( Y^2 - \frac{817}{16} X^4 Y + \frac{4331}{32} X^8 \right) \frac{\partial}{\partial U_2} + \left( -\frac{3483}{8} X^5 Y + \frac{1173}{16} X^9 \right) \frac{\partial}{\partial U_3}, \]

\[ m_9 := \left( Z + \frac{29}{2} X Y - \frac{2549}{16} X^5 \right) \frac{\partial}{\partial U_1} + \left( -\frac{159}{16} X^2 Y + \frac{861}{32} X^6 \right) \frac{\partial}{\partial U_2} + \left( -\frac{693}{8} X^3 Y - \frac{243}{16} X^7 \right) \frac{\partial}{\partial U_3}, \]

\[ m_{10} := \left( 2 Y + \frac{29}{4} X^4 \right) \frac{\partial}{\partial U_1} + \left( Z + \frac{7}{4} X Y - \frac{1}{8} X^5 \right) \frac{\partial}{\partial U_2} - \left( \frac{3}{2} X^2 Y + \frac{9}{4} X^6 \right) \frac{\partial}{\partial U_3}, \]

\[ m_{11} := \frac{17}{2} X^3 \frac{\partial}{\partial U_1} + \left( \frac{1}{2} Y - \frac{3}{4} X^4 \right) \frac{\partial}{\partial U_2} + \left( Z + 3 X Y - \frac{3}{2} X^5 \right) \frac{\partial}{\partial U_3}, \]

\[ m_{12} := \left( W + \frac{189}{40} X Z + \frac{201}{55} X^2 Y - \frac{4853}{80} X^6 \right) \frac{\partial}{\partial U_1} + \left( \frac{927}{110} X^7 - \frac{15}{176} X^2 Z - \frac{177}{44} X^3 Y \right) \frac{\partial}{\partial U_2} - \left( \frac{4491}{880} X^3 Z + \frac{3087}{110} X^4 Y \right) \frac{\partial}{\partial U_3}, \]

\[ m_{13} := \left( - Z + \frac{1677}{88} X Y - \frac{18877}{32} X^3 \right) \frac{\partial}{\partial U_1} - \left( \frac{11609}{352} X^2 Z + \frac{5919}{22} X^3 Y \right) \frac{\partial}{\partial U_3} + \left( W + \frac{305}{88} X Z - \frac{3215}{88} X^2 Y + \frac{3623}{44} X^6 \right) \frac{\partial}{\partial U_2}, \]

\[ m_{14} := \left( \frac{7}{33} Y + \frac{55}{24} X^4 \right) \frac{\partial}{\partial U_1} + \left( \frac{20}{33} Z - \frac{19}{132} X Y - \frac{13}{33} X^5 \right) \frac{\partial}{\partial U_2} + \left( W - \frac{47}{88} X Z + \frac{41}{22} X^2 Y \right) \frac{\partial}{\partial U_3}. \]
\[ m_{15} := (w + \frac{57}{4}X^3) \frac{\partial}{\partial U_1} + (Y - 2X^4) \frac{\partial}{\partial U_2} + \left( \frac{3}{4}Z + 6XY \right) \frac{\partial}{\partial U_3}, \]
\[ m_{16} := w \frac{\partial}{\partial U_2}, \]
\[ m_{17} := -\frac{3}{2}X \frac{\partial}{\partial U_1} - \frac{1}{4}X^2 \frac{\partial}{\partial U_2} + \left( w + \frac{3}{2}X^3 \right) \frac{\partial}{\partial U_3}. \]

Let \( \iota: (X,Y,Z,W,w) \mapsto (X,Y,Z,W,w) \) so that \( \hat{g}_{\ell} = \iota^* \hat{g}_1 \). By an application of Buchberger’s Algorithm, we find that \( m_{1}, \ldots, m_{14}, \iota^* m_{15}, \iota^* m_{17}, m_{18} \) together with the following elements form a Groebner basis for \( M := \left( \frac{\partial \hat{g}_1}{\partial w} \right) \mathcal{O}_{\mathbb{C}^5,0} + \hat{g}_{\ell} t \rho (\text{Der}(-\log V)) \) (see [11, Appendix D.1] for detailed calculations).

\[ m_{19} := (Yw^\ell - 1 + \frac{11}{2}X^4w^\ell - 1) \frac{\partial}{\partial U_1} - \frac{3}{2}X^6w^\ell \frac{\partial}{\partial U_3}, \]
\[ m_{20} := \frac{2}{7}X^5w^\ell - 1 \frac{\partial}{\partial U_1} + X^7w^\ell - 1 \frac{\partial}{\partial U_3}, \]
\[ m_{21} := X^5w^\ell - 1 \frac{\partial}{\partial U_1}. \]

By a standard argument from commutative algebra,

\[ \dim_{\mathbb{C}} NK_{V,e} \hat{g}_{\ell} = \dim_{\mathbb{C}} \frac{\mathcal{O}^3_{\mathbb{C}^5,0}}{(\text{LT}(M)) \mathcal{O}_{\mathbb{C}^5,0}}. \]

So,

\[ \dim_{\mathbb{C}} NK_{V,e} \hat{g}_{\ell} = \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^5,0}}{(X^8, X^2Y, Y^2, Z, W, X^5w^\ell - 1, Yw^\ell - 1, w^\ell)} \mathcal{O}_{\mathbb{C}^5,0} + \]
\[ + \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^5,0}}{(X^9, X^2Y, Y^2, Z, W, w^\ell - 1)} \mathcal{O}_{\mathbb{C}^5,0} + \]
\[ + \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^5,0}}{(X^9, Y, Z, W, X^7w^\ell - 1, w^\ell)} \mathcal{O}_{\mathbb{C}^5,0} = \]
\[ (10\ell - 5) + (11\ell - 11) + (9\ell - 2) = 30\ell - 18. \]

We observe that \( NK_{H,e/\mathbb{C}G} \) is a 1-dimensional Cohen-Macaulay module for \( \ell = 1 \) (here \( G \) is a 1-parameter unfolding of \( \hat{g}_1 \) and \( H \) is the defining equation of \( V \), as usual). Since \( \hat{A}_2 \) satisfies the conjecture; so does \( \hat{C}_\ell \) for \( \ell \geq 2 \) (Proposition 4.3).

By similar calculations, we can prove the following two propositions.

**Proposition 4.6.** The map-germ \( f_{\ell} \) defined by (21) has \( A_e \)-codimension \( 45\ell - 12 \).

**Proposition 4.7.** The map-germs shown in Tables 3 are finitely \( A \)-determined and satisfy the conjecture for all \( \ell > 0 \).
See Table 4 for the list of unfoldings from which the series in Table 3 are deduced.

Remark 4.8. The smallest codimension we have got so far is $\mathcal{A}_e$-codim$(\hat{C}_1) = 12$ in the dimensions $(4, 5)$ and $\mathcal{A}_e$-codim$(\hat{M}_{1,1}) = 13$ in $(5, 6)$.

Table 3. $\mathcal{A}$-finite map-germs in $\mathcal{E}^0_{4,5}$ and $\mathcal{E}^0_{5,6}$.

| Label | Reduction | $\mathcal{A}_e$-codim | Initial map |
|-------|-----------|------------------------|-------------|
| $\hat{D}_\ell$ | $(x, y^2 + xz + x^2 y + w^\ell y, yz, z^2 + y^3 + w^2 y, w)$ | $45\ell - 18$ | $A_2$ |
| $\hat{E}_\ell$ | $(x, y^3 + xz + x^4 y + w^\ell y, yz, z^2 + y^5 + w^2 y, w)$ | $536 - 186$ | $A_3$ |
| $\hat{K}_\ell$ | $(x, y^2 + xz, z^2 + xy, y^3 + yz^2 + z^3 + w^\ell z, w)$ | $51\ell - 33$ | $\hat{B}_3^\ell$ |
| $\hat{L}_\ell$ | $(x, y^2 + xz, z^2 + xy, y^5 + y^3 z^2 + z^5 + w^\ell y - w^\ell z, w)$ | $372 - 252$ | $\hat{B}_5^\ell$ |
| $\hat{M}_{1,\ell}$ | $(x, y^2 + xz + x^2 y + v^\ell y, yz + wy, z^2 + y^3 + v^2 y, w, v)$ | $25\ell - 12$ | $C_1$ |
| $\hat{M}_{2,\ell}$ | $(x, y^2 + xz + x^2 y + v^\ell y, yz + w^\ell y, z^2 + y^3 + v^2 y, w, v)$ | $95\ell - 42$ | $C_2$ |
| $\hat{M}_{3,\ell}$ | $(x, y^2 + xz + x^2 y + v^\ell y, yz + w^\ell y, z^2 + y^3 + v^2 y, w, v)$ | $165\ell - 72$ | $C_3$ |
| $\hat{N}_\ell$ | $(x, y^2 + xz + x^2 y + w^\ell y, yz + v^\ell z, z^2 + y^5 + w^2 y + v^4 y + v^3 y^2, w, v)$ | $1750\ell - 350$ | $\hat{E}_1$ |
| $\hat{P}_\ell$ | $(x, y^2 + xz + v^\ell z, z^2 + xy, y^3 + yz^2 + z^3 + wz + v^2 y, w, v)$ | $42\ell - 18$ | $\hat{K}_1$ |

Remark 4.9. The map-germs $\hat{M}_{1,\ell}$ and $\hat{M}_{2,\ell}$ are actually parts of the series

$$(23) \quad \hat{M}_{k,\ell}: (x, y, z, w, v) \mapsto (x, y^2 + xz + x^2 y + v^k y, yz + w^\ell y, z^2 + y^3 + v^2 y, w, v)$$

which is finitely $\mathcal{A}$-determined with $\mathcal{A}_e$-codim = $(70k - 30)\ell - 45k + 18$.

Table 4. Stable unfoldings.

| Label | Unfolding |
|-------|-----------|
| $F_{\hat{A}_2}$ | $(x, y^2 + xz + x^2 y + u_{11} y, yz + u_{12} y, z^2 + y^3 + u_{13} y, u_1)$ |
| $F_{\hat{A}_3}$ | $(x, y^3 + xz + x^2 y + u_{21} y + u_{22} y^2, yz + u_{23} y, z^2 + y^3 + u_{24} y + u_{25} y^2, u_2)$ |
| $F_{\hat{B}_3}$ | $(x, y^2 + xz + u_{31} z, z^2 + xy + u_{32} z, y^3 + yz^2 + z^3 + u_{33} y + u_{34} z, u_3)$ |
| $F_{\hat{B}_6}$ | $(x, y^2 + xz + u_{41} z, z^2 + xy, y^3 + yz^2 + z^3 + u_{42} y + u_{43} z + u_{44} y, u_4)$ |
| $F_{\hat{C}_\ell}$ | $(x, y^2 + xz + x^2 y + u_{51} y, yz + w^\ell y + u_{52} y, z^2 + y^3 + u_{53} y, u_5)$ |
| $F_{\hat{E}_1}$ | $(x, y^3 + xz + x^2 y + wy, yz + u_{61} y + u_{62} z, z^2 + y^5 + u_{63} y + u_{64} y^2, u_6)$ |
| $F_{\hat{E}_7}$ | $(x, y^2 + xz + w_{71} z, z^2 + xy + u_{72} z, y^3 + yz^2 + z^3 + wz + w_{73} y, w, u_7)$ |

Remark 4.10. The formula for $\mathcal{A}_e$-codimension for each germ in Table 3 is a linear form in one variable and the constant term is equal to the codimension of the initial map-germ. It would be interesting to see if this holds in general.
**Remark 4.11.** There exist finitely $\mathcal{A}$-determined map-germs which cannot be obtained by the method of Theorem 3.11 e.g. $\tilde{A}_\ell$ and $\tilde{B}_{2\ell+1}$ of Propositions 3.6 and 3.7. If $\tilde{A}_\ell$ were a reduction it would come from a stable unfolding of the map-germ $(y, z) \mapsto (y^\ell, y^2z^2, z^3)$, $\tilde{B}_{2\ell+1}$ from a stable unfolding of the map-germ $(y, z) \mapsto (y^2, y^2z^2 + y^2z^2z^2 + z^2z^2)$. But none of the germs is $\mathcal{A}$-finite since their $D_{11}$ is not an isolated singularity.

**Remark 4.12.** All map-germs from Houston and Kirk’s list of simple singularities of corank 1 ([15, Table 1]) but one $(Q_k)$ can be constructed from finitely $\mathcal{A}$-determined map-germs in $\mathcal{E}^0_{2,3}$ by our method. We list them in Table 5 together with their initial map-germs. The labels for the initial map-germs are from Mond’s list [22].

| Label | Map-germ | Initial map | Mond’s reference |
|-------|----------|------------|------------------|
| $A_k$ | $(x, y, z^2, z^2 + x^2z ± x^2z)$ | $(x, z^2, z^2 + x^2z)$ | $S_1$ |
| $B_k$ | $(x, y, z^2, z^2 + x^2z ± y^2z)$ | $(x, z^2, z^2 + x^2z)$ | $B_2$ |
| $C_k$ | $(x, y, z^2, x^2z + y^2z ± y^2z)$ | $(x, z^2, x^2z ± y^2z)$ | $C_2$ |
| $D_k$ | $(x, y, z^2, x^2z + y^2z ± z^2)$ | $(x, z^2, x^2z ± y^2z)$ | $D_2$ |
| $E_k$ | $(x, y, z^2, x^2z + y^2z ± z^2)$ | $(x, z^2, x^2z ± y^2z)$ | $E_1$ |
| $F_k$ | $(x, y, z^2, x^2z + y^2z ± z^2)$ | $(x, z^2, x^2z ± y^2z)$ | $F_1$ |
| $G_k$ | $(x, y, z^2, x^2z + y^2z ± z^2)$ | $(x, z^2, x^2z ± y^2z)$ | $G_1$ |
| $H_k$ | $(x, y, z^2, x^2z + y^2z ± z^2)$ | $(x, z^2, x^2z ± y^2z)$ | $H_1$ |
| $I_k$ | $(x, y, z^2, x^2z + y^2z ± z^2)$ | $(x, z^2, x^2z ± y^2z)$ | $I_1$ |
| $J_k$ | $(x, y, z^2, x^2z + y^2z ± z^2)$ | $(x, z^2, x^2z ± y^2z)$ | $J_1$ |
| $K_k$ | $(x, y, z^2, x^2z + y^2z ± z^2)$ | $(x, z^2, x^2z ± y^2z)$ | $K_1$ |
| $L_k$ | $(x, y, z^2, x^2z + y^2z ± z^2)$ | $(x, z^2, x^2z ± y^2z)$ | $L_1$ |
| $M_k$ | $(x, y, z^2, x^2z + y^2z ± z^2)$ | $(x, z^2, x^2z ± y^2z)$ | $M_1$ |
| $N_k$ | $(x, y, z^2, x^2z + y^2z ± z^2)$ | $(x, z^2, x^2z ± y^2z)$ | $N_1$ |
| $O_k$ | $(x, y, z^2, x^2z + y^2z ± z^2)$ | $(x, z^2, x^2z ± y^2z)$ | $O_1$ |
| $P_k$ | $(x, y, z^2, x^2z + y^2z ± z^2)$ | $(x, z^2, x^2z ± y^2z)$ | $P_1$ |
| $Q_k$ | $(x, y, z^2, x^2z + y^2z ± z^2)$ | $(x, z^2, x^2z ± y^2z)$ | $Q_1$ |
| $R_k$ | $(x, y, z^2, x^2z + y^2z ± z^2)$ | $(x, z^2, x^2z ± y^2z)$ | $R_1$ |
| $S_k$ | $(x, y, z^2, x^2z + y^2z ± z^2)$ | $(x, z^2, x^2z ± y^2z)$ | $S_1$ |

It is natural to ask if there is an operation (other than augmentations, concatenations and reductions) that would allow us to map equivalence classes in $\mathcal{E}^0_{n-1,n}$ to the ones in $\mathcal{E}^0_{n,n+1}$. Knowing all possible constructions might help us prove Conjecture 1.1 by an induction on dimension for the classes of finitely $\mathcal{A}$-determined...
map-germs coming from lower dimensions since it was already proven for \( n = 2 \). We leave this problem for further study.

References

1. A. Altintas, *Multiple point spaces and finitely determined map-germs*, Ph.D. thesis, The University of Warwick, 2011, http://wrap.warwick.ac.uk/38086/.
2. A. Altintas and D. Mond, *Free resolutions for multiple point spaces*, arXiv:1111.2909v1, 2011.
3. J. W. Bruce and W. L. Marar, *Images and varieties*, J. Math. Sci. 82 (1996), no. 5, 3633–3641.
4. J. Damon, *Deformations of sections of singularities and Gorenstein surface singularities*, American Journal of Mathematics 109 (1987), 695–721.
5. J. Damon and D. Mond, *A-codimension and the vanishing topology of discriminants*, Invent. Math. 106 (1991), 217–242.
6. C. De Concini and E. Strickland, *On the variety of complexes*, Adv. Math 41 (1981), 57–77.
7. T. de Jong and D. van Straten, *Disentanglements*, Singularity Theory and Applications (Warwick 1989) (D. Mond and J. Montaldi, eds.), Lecture Notes in Mathematics, vol. 1462, Springer, 1991.
8. W. Decker, G.-M. Greuel, G. Pfister, and H. Schönemann, SINGULAR 3-1-2 — A computer algebra system for polynomial computations, (2010), http://www.singular.uni-kl.de.
9. T. Gaffney, *Properties of finitely determined germs*, Ph.D. thesis, Bandeis University, 1975.
10. *Multiple points and associated loci*, Singularities, Part 1, Proc. Sympos. Pure Math., no. 40, American Mathematical Society, RI, 1983, pp. 429–437.
11. V. Goryunov and D. Mond, *Vanishing cohomology of singularities of mappings*, Compositio Mathematica 89 (1993), 45–80.
12. D. R. Grayson and M. E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, http://www.math.uiuc.edu/Macaulay2/.
13. R. Hartshorne, *Algebraic geometry*, 8th ed., Graduate Text in Mathematics, no. 52, Springer-Verlag, New York, 1997.
14. K. Houston, *On singularities of foldings maps and augmentations*, Math. Scand. 82 (1998), 191–206.
15. K. Houston and N. Kirk, *On the classification and geometry of corank 1 map-germs from three-space to four-space*, Singularity Theory (Liverpool 1996), London Maths. Soc. Lecture Notes Series, vol. 263, Cambridge University Press, 1999.
16. S. Kleiman, J. Lipman, and B. Ulrich, *The multiple-point schemes of a finite curvilinear map of codimension one*, Ark. Mat. 34 (1996), 285–326.
17. W. L. Marar and D. Mond, *Multiple point schemes for corank 1 maps*, J. London Math. Soc. 39 (1989), no. 2, 553–567.
18. J. Martinet, *Singularities of smooth functions and maps*, Cambridge University Press, 1982.
19. J. Mather, *Stability of \( C^\infty \)-mappings IV: Classification of stable germs by r-algebras*, Inst. Hautes Etudes Sci. Publ. Math. 37 (1969), 223–248.
20. *Generic projections*, Ann. of Math. 98 (1973), 226–245.
21. H. Matsumura, *Commutative ring theory*, Cambridge University Press, 1989.
22. D. Mond, *On the classification of germs of maps from \( \mathbb{R}^2 \) to \( \mathbb{R}^3 \)*, Proc. London Math. Soc. 50 (1985), 333–369.
23. *Some remarks on the geometry and classification of germs of maps from surfaces to 3-spaces*, Topology 26 (1987), 361–383.
24. , Vanishing cycles for analytic maps, Singularity Theory and Applications (Warwick 1989) (D. Mond and J. Montaldi, eds.), Lecture Notes in Mathematics, vol. 1462, Springer, New York, 1991.

25. , Looking at bent wires: $A_r$-codimension and the vanishing topology of parametrized curve singularities, Math. Proc. Cambridge Philos. Soc. 117 (1995), 213–222.

26. , Differential forms on free and almost free divisors, Proc. London Math. Soc. 81 (2000), no. 3, 587–617.

27. D. Mond and R. Pellikaan, Fitting ideals and multiple points of analytic mappings, Algebraic Geometry and Complex Analysis (Pátzcuaro, 1987), Lecture Notes in Mathematics, vol. 1414, Springer, Berlin, 1989.

28. C. T. C. Wall, Finite determinacy of smooth map-germs, Bull. Lond. Math. Soc. 13 (1981), 481–539.

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