Abstract. We propose some new mixed finite element methods for the time dependent stochastic Stokes equations with multiplicative noise, which use the Helmholtz decomposition of the driving multiplicative noise. It is known [16] that the pressure solution has a low regularity, which manifests in sub-optimal convergence rates for well-known \textit{inf-sup} stable mixed finite element methods in numerical simulations, see [11]. We show that eliminating this gradient part from the noise in the numerical scheme leads to optimally convergent mixed finite element methods, and that this conceptual idea may be used to retool numerical methods that are well-known in the deterministic setting, including pressure stabilization methods, so that their optimal convergence properties can still be maintained in the stochastic setting. Computational experiments are also provided to validate the theoretical results and to illustrate the conceptual usefulness of the proposed numerical approach.

Key words. Stochastic Stokes equations, multiplicative noise, Wiener process, Itô stochastic integral, mixed finite element methods, \textit{inf-sup} condition, error estimates, Helmholtz decomposition, pressure stabilization

AMS subject classifications. 65N12, 65N15, 65N30,

1. Introduction. This paper is concerned with fully discrete mixed finite element approximations of the following time-dependent stochastic Stokes equations with multiplicative noise for viscous incompressible fluids covering the domain $D = (0, L)^d$ for $d = 2, 3$:

\begin{align}
(1.1a) & \quad \, du = [\Delta u - \nabla p]dt + B(u)dW(t) \quad \text{in } DT := (0, T) \times D, \\
(1.1b) & \quad \text{div } u = 0 \quad \text{in } DT, \\
(1.1c) & \quad u(0) = u_0 \quad \text{in } D,
\end{align}

where $u$ and $p$, respectively, denote the velocity field and the pressure of the fluid which are spatially periodic with period $L > 0$ in each coordinate direction. For the sake of simplicity and ease of presentation, we assume $\{W(t); t \geq 0\}$ to be an $\mathbb{R}$-valued Wiener process; see section 2 for further details.

When $B \equiv 0$, (1.1) is the well-known (deterministic) Stokes system; one motivation for studying (1.1a)–(1.1b) with “random force” $B(u)\frac{dW}{dt}$ is to develop mathematical models of this type for turbulent fluids [2, 13]. In addition to their importance in applied sciences and engineering, the Stokes equations are a well-known PDE model with saddle point structure, which requires special numerical discretizations to construct optimally convergent methods; it should be noted that although the involved deterministic Stokes operator is linear, system (1.1a)–(1.1b) is nonlinear due to the nonlinear function $B$.

The numerical analysis of the deterministic Stokes problem is well-established in the literature, see [2, 12, 17]. Well-known numerical methods include \textit{exactly}
divergence-free methods, which approximate the velocity in exactly divergence-free finite element spaces; mixed finite element methods, where the (discrete) inf-sup condition is the key criterion that distinguishes stable pairings of finite element ansatz spaces for the velocity (with more degrees of freedom) and the pressure (with less degrees of freedom); mixed methods allow a more flexible, broader application if compared to exactly divergence-free methods, thus putting them in the center of research on numerical methods for saddle point problems in the last decades. Another class of related numerical methods are stabilization methods which were initiated in \cite{15}, where the incompressibility constraint \((1.1b)\) is relaxed into

\begin{equation}
\label{1.2}
\text{div } \varepsilon u - \varepsilon \Delta p = 0 \quad \text{in } D_T,
\end{equation}

This relaxation allows for stable pairings of equal order (nodal-based) finite element ansatz spaces for both, velocity and pressure (putting \(\varepsilon = O(h^2)\), where \(h > 0\) is the spatial mesh size). We remark that optimal order error estimates had been obtained for all three classes of finite element methods in the deterministic setting (cf. \cite{12} \cite{3}), where

- \text{inf-sup} stable mixed finite element methods require the \(H^1\)-regularity of the pressure in order to optimally bound the best-approximation error for the pressure, which leads to optimal order convergence; cf. \cite{3} \cite{12} \cite{17},
- stabilization methods require the \(H^1\)-regularity of the pressure for convergence; cf. \cite{15} \cite{18}.

This work contributes to the numerical analysis of the stochastic Stokes problem \((1.1)\) (i.e., \(B \neq 0\)). By \cite{16}, the (temporal) regularity of the pressure \(p \in L^1(\Omega; W^{-1,\infty}(0, T; H^1(D)/\mathbb{R}))\) is limited due to the driving noise. In order to motivate its impact onto the pressure, we here discuss the related question regarding \(k\)-independent stability estimates for the pair of random variables \((u^{n+1}, p^{n+1})\) of the following time-implicit discretization of \((1.1)\) on a uniform mesh of \([0, T]\) with the mesh size \(k > 0\):

\begin{align}
(1.3a) \quad u^{n+1} - k\Delta u^{n+1} + k\nabla p^{n+1} &= u^n + B(u^n)\Delta_{n+1}W \quad \text{in } D, \\
(1.3b) \quad \text{div } u^{n+1} &= 0 \quad \text{in } D, \\
\end{align}

where \(\Delta_{n+1}W := W(t_{n+1}) - W(t_n) \sim \mathcal{N}(0, k) = \sqrt{k}\mathcal{N}(0, 1)\). A crucial observation for the motivation of this paper is that the pressure gradient on the left-hand side is scaled by \(k\), while the noise term is of order \(O(\sqrt{k})\). Let us assume that the estimate \([3.5]\) in Lemma \([3.1]\) for \(\{\Delta u^n\}_n\) taking values in \(L^2(D; \mathbb{R}^d)\) is already shown, and we now look for a uniform bound for \(\{\nabla p^n\}_n\) taking values in \(L^2(D; \mathbb{R}^d)\). The strategy for deriving such a stability estimate is to fix one \(\omega \in \Omega\), and to multiply \((1.3)\) with \(\nabla p^{n+1}(\omega)\): the leading terms on both sides vanish due to the incompressibility constraint, and we end up with

\begin{equation}
\label{1.4}
k^2/2\|\nabla p^{n+1}(\omega)\|^2 \leq k^2/2\|\Delta u^{n+1}(\omega)\|^2 + \left(\langle B(u^n(\omega))\rangle \Delta_{n+1}W(\omega) \cdot \nabla p^{n+1}(\omega)\right).
\end{equation}

Note that the term on the right-hand side does not vanish since \(\text{div } B(u^n) \neq 0\) for a general (Lipschitz) nonlinear mapping \(B\). We now take expectations \(E[\cdot]\) on both sides, sum over all time steps, and use \([3.5]\), the facts that \(B(u^n)\) and \(\Delta_{n+1}W\) are independent and \(E[\|\Delta_{n+1}W\|^2] \leq Ck\), and Young’s inequality (with \(\alpha > 0\)) to obtain
the estimate
\[
\frac{k}{2} \sum_{n=0}^{N} \mathbb{E}[\|\nabla p^{n+1}\|^2] \leq C + \frac{1}{4\alpha} \sum_{n=0}^{N} \mathbb{E}[\|\nabla p^{n+1}\|^2].
\]

Taking \(\alpha = \frac{1}{k}\) allows to absorb the last term on the right-hand side to the one on the left, but the remaining term is \(\sum_{n=0}^{N} \mathbb{E}[\|\mathbf{B}(\mathbf{u}^n)\|^2] \propto \mathcal{O}(k^{-1})\), therefore, we end up with the following \(k\)-dependent estimate:

\[
(1.5) \quad \frac{k}{2} \sum_{n=0}^{N} \mathbb{E}[\|\nabla p^{n+1}\|^2] \leq \frac{C}{k}.
\]

The above consideration crucially affects the error analysis of a space-time discretization of (1.1a)–(1.1b):

- **Exactly divergence-free methods** require restricted settings of data, including the dimension, topology, and regularity of the spatial domain \(D\). However, an optimal order error estimate can be proved for the velocity approximation, see [6], which uses the fact that no pressure is involved in the analysis.

- The error estimate for the velocity approximation of *inf-sup* stable mixed finite element methods in [11] was obtained based on a stability bound of type (1.5) to bound the related best-approximation error for the pressure that appears in (an auxiliary temporal discretization of) (1.1), thus leading to a sub-optimal error estimate for the velocity of order \(O(k^{1/2} + h^{1/4})\). The computational studies in [11] suggest that this error bound is sharp.

The first goal of the paper is to construct optimally convergent *inf-sup* stable mixed finite element methods, with “minimum” extra effort. Our main idea, which is partly borrowed from [5], is to perform the Helmholtz decomposition for the noise term at each time step first, and then to determine the new velocity and pressure iterates simultaneously via the mixed finite element method. Below we shall use the semi-discrete time-stepping scheme (1.3) to motivate our strategy. Introducing the Helmholtz decomposition of \(\mathbf{B}\) as follows

\[
(1.6) \quad \mathbf{B}(\mathbf{u}^n) = \nabla \xi^n + \eta^n, \quad \text{where} \quad \text{div} \, \eta^n = 0,
\]

and setting \(r^{n+1} := p^{n+1} - k^{-1}\Delta_{n+1} W \xi^n\), then (1.3) can be rewritten as

\[
(1.7a) \quad \mathbf{u}^{n+1} - k\Delta \mathbf{u}^{n+1} + k\nabla r^{n+1} = \mathbf{u}^n + \eta^n \Delta_{n+1} W \quad \text{in} \ D,
\]

\[
(1.7b) \quad \text{div} \, \mathbf{u}^{n+1} = 0 \quad \text{in} \ D.
\]

In contrast to estimate (1.5) for \(p^{n+1}\), it can be shown that the new pressure \(r^{n+1}\) satisfies the following improved stability estimate (see Lemma 3.1):

\[
(1.8) \quad \frac{k}{2} \sum_{n=0}^{N} \mathbb{E}[\|\nabla r^{n+1}\|^2] \leq C,
\]

which is a consequence of the divergence-free property of the modified noise term (i.e., the last term on the right-hand side of (1.7a)). Conceptually, this improved stability
for the new pressure $r^{n+1}$ is obtained by removing the stochastic pressure $\xi^n$ from the driving noise in (1.3a). As it will be detailed in Section 4 any inf-sup stable mixed finite element discretization of (1.7) then gives optimally convergent velocity approximations (see Theorem 4.4), whose proof essentially relies on (1.8). We also present optimal error estimates for (temporal averages of) the pressure approximations in $L^2$, which improve corresponding suboptimal estimates in [11].

We therefore conclude by saying that it is essential to identify the proper role of the semi-discrete pressures, namely, $\{p^n\}_n$ in (1.3) vs. $\{r^n\}$ in (1.7), for inf-sup stable mixed finite element methods for (1.1) in order to construct optimally convergent mixed methods. Moreover, this insight also suggests how to construct optimally convergent stabilization methods for (1.1) which circumvent the inf-sup stability criterion for mixed element methods, and hence allow a more efficient discretization such as

\begin{align}
(1.9a) \quad & u_{n+1}^\varepsilon - k\Delta u_{n+1}^\varepsilon + k\nabla r_{\varepsilon}^{n+1} = u_0 + \eta^\varepsilon \Delta n_1 W & \text{in } D, \\
(1.9b) \quad & \text{div } u_{n+1}^\varepsilon - \varepsilon \Delta n_1 = 0 & \text{in } D,
\end{align}

for which $\varepsilon = O(h^2)$ will be shown to be the optimal choice in section 5. The error analysis in section 5 verifies optimal order convergence for a standard finite element discretization of (1.9) which employs the same finite element space for approximating both, $u_{n+1}^\varepsilon$ and $r_{\varepsilon}^{n+1}$; see Theorem 5.2. Corresponding computational studies in section 6 support the conclusion that the choice of pressure in the stabilization is crucial for achieving an optimally convergent stabilization method for (1.1).

The remainder of this paper is organized as follows. In Section 2 we give exact assumptions on the data in (1.1), and recall the definition and known properties of the (strong) variational solution for problem (1.1). In sections 3 and 4 we analyze the Helmholtz decomposition enhanced Euler-Maruyama time-stepping scheme (1.6)–(1.7) and its mixed finite element approximations, and establish the optimal convergence for both. Section 5 establishes optimal convergence for the stabilized scheme (1.9) and its equal-order finite element approximations. Two-dimensional numerical experiments and computational studies are given in section 6 to validate the theoretical error bounds, and to computationally evidence that a proper selection of the pressure for the construction of optimally convergent mixed methods is indeed necessary.

2. Preliminaries.

2.1. Notations. Standard function and space notation will be adopted in this paper. For example, $H_{\text{per}}^\ell(D, \mathbb{R}^d)$ ($\ell \geq 0$) denotes the subspace of the Sobolev space $H^\ell(D, \mathbb{R}^d)$ consisting of $\mathbb{R}^d$-valued periodic functions with period $L$ in each spatial coordinate direction, and $\langle \cdot , \cdot \rangle_D$ denote the standard $L^2$-inner product, with induced norm $\| \cdot \|$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ be a filtered probability space with the probability measure $\mathbb{P}$, the $\sigma$-algebra $\mathcal{F}$ and the continuous filtration $\{\mathcal{F}_t\} \subset \mathcal{F}$. For a random variable $v$ defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, let $\mathbb{E}[v]$ denote the expected value of $v$. For a vector space $X$ with norm $\| \cdot \|_X$, and $1 \leq p < \infty$, we define the Bochner space $L^p(\Omega, X); \|v\|_{L^p(\Omega, X)} := (\mathbb{E}[\|v\|_X^p])^{\frac{1}{p}}$.

We also define

$$
\mathbb{H} := \{ v \in L_{\text{per}}^2(D; \mathbb{R}^d); \text{div } v = 0 \text{ in } D \},
$$

$$
\mathcal{V} := \{ v \in H_{\text{per}}^1(D; \mathbb{R}^d); \text{div } v = 0 \text{ in } D \}.
$$

We recall from [12] that the (orthogonal) Helmholtz projection $\mathbf{P}_{\mathbb{H}} : L_{\text{per}}^2(D; \mathbb{R}^d) \to \mathbb{H}$ is defined by $\mathbf{P}_{\mathbb{H}} v = \eta$ for every $v \in L_{\text{per}}^2(D; \mathbb{R}^d)$, where $(\eta, \xi) \in \mathbb{H} \times H_{\text{per}}^1(D)/\mathbb{R}$.
is a unique tuple such that

\[ \mathbf{v} = \mathbf{\eta} + \nabla \xi, \]

and \( \xi \in H^1_{\text{per}}(D)/\mathbb{R} \) solves the following Poisson problem (cf. [1]):

\[ (\nabla \xi, \nabla q) = (\mathbf{v}, \nabla q) \quad \forall q \in H^1_{\text{per}}(D). \tag{2.1} \]

In this paper we denote by \( \mathbf{A} := \mathbf{P}_H \Delta : H^2(D; \mathbb{R}^d) \to \mathbb{H} \) the Stokes operator.

We assume that \( \mathbf{B} : L^2(\Omega; H^1_{\text{per}}(D; \mathbb{R}^d)) \to L^2(\Omega; H^1_{\text{per}}(D; \mathbb{R}^d)) \) is Lipschitz continuous and has linear growth, \( i.e. \), there exists a constant \( C > 0 \) such that for all \( \mathbf{v}, \mathbf{w} \in L^2_{\text{per}}(D; \mathbb{R}^d) \),

\[ \|\mathbf{B}(\mathbf{v}) - \mathbf{B}(\mathbf{w})\| \leq C\|\mathbf{v} - \mathbf{w}\|, \tag{2.2a} \]
\[ \|\mathbf{B}(\mathbf{v})\| \leq C(1 + \|\mathbf{v}\|), \tag{2.2b} \]
\[ \|\partial_\mathbf{B}\|_* \leq C, \tag{2.2c} \]

where \( \partial_\mathbf{B} \) denotes the Gateaux derivative of \( \mathbf{B} \), and \( \| \cdot \| \) is its operator norm.

### 2.2. Variational formulation of the stochastic Stokes equations.

We first recall the solution concept for (1.1), and refer to [7, 8] for its existence and uniqueness.

**Definition 2.1.** Given \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P}) \), let \( W \) be an \( \mathbb{R} \)-valued Wiener process on it. Suppose \( \mathbf{u}_0 \in L^2(\Omega, \mathcal{V}) \). An \( \{\mathcal{F}_t\} \)-adapted stochastic process \( \{\mathbf{u}(t); 0 \leq t \leq T\} \) is called a variational solution of (1.1) if \( \mathbf{u} \in L^2(\Omega; C([0, T]; \mathcal{V})) \cap L^2(0, T; H^2_{\text{per}}(D; \mathbb{R}^d)) \), and satisfies \( \mathbb{P} \)-a.s. for all \( t \in (0, T] \)

\[ \mathbf{u}(t) = \mathbf{u}_0 + \int_0^t \nabla \mathbf{u}(s) \, ds + \int_0^t \mathbf{B}(\mathbf{u}(s)) \, dW(s) \quad \forall \mathbf{v} \in \mathcal{V}. \tag{2.3} \]

The following estimates from [5, 11] establish the H"older continuity in time of the variational solution in various spatial norms.

**Theorem 2.2.** Additionally suppose \( \mathbf{u}_0 \in L^2(\Omega; \mathcal{V} \cap H^2_{\text{per}}(D; \mathbb{R}^d)) \). There exist constants \( C_i \equiv C_i(T) > 0, i = 1, 2 \), such that the variational solution to problem (1.1) satisfies for \( s, t \in [0, T] \)

\[ \mathbb{E}[\|\mathbf{u}(t) - \mathbf{u}(s)\|^2] + \mathbb{E}\left[ \int_s^t \|\nabla (\mathbf{u}(\tau) - \mathbf{u}(s))\|^2 \, d\tau \right] \leq C_1|t - s|, \tag{2.4a} \]
\[ \mathbb{E}[\|\nabla (\mathbf{u}(t) - \mathbf{u}(s))\|^2] + \mathbb{E}\left[ \int_s^t \|\mathbf{A}(\mathbf{u}(\tau) - \mathbf{u}(s))\|^2 \, d\tau \right] \leq C_2|t - s|. \tag{2.4b} \]

### 2.3. Definition and role of the pressure.

The Definition [2.1] only addresses the velocity \( \mathbf{u} \) in the stochastic PDE (1.1), a corresponding pressure which satisfies a proper formulation (see Theorem 2.3 below) may be constructed after the existence of a velocity field \( \mathbf{u} \) has been established. We therefore consider processes

\[ \mathbf{U}(t) := \int_0^t \mathbf{u}(s) \, ds, \quad \text{and} \quad \mathbf{F}(t) := \int_0^t \mathbf{B}(\mathbf{u}(s)) \, dW(s). \]

Evidently, \( \mathbf{U} \in L^2(\Omega, L^2(0, T; H^2_{\text{per}}(D; \mathbb{R}^d))) \) and \( \mathbf{F} \in L^2(\Omega, L^2(0, T; L^2_{\text{per}}(D; \mathbb{R}^d))) \), and 2.3 therefore implies

\[ (\mathbf{u}(t) - \Delta \mathbf{U}(t) - \mathbf{u}_0 - \mathbf{F}(t), \mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathcal{V}, \ t \in (0, T), \mathbb{P}\text{-a.s.} \tag{2.5} \]
By the Helmholtz decomposition [16] Theorem 4.1 and Remark 4.3, there exists a unique $P \in L^2(\Omega, L^2(0, T; H^1_{\text{per}}(D))/\mathbb{R})$ such that

$$\nabla P(t) = -\left[ u(t) - \Delta U(t) - u_0 - F(t) \right] \quad \forall t \in (0, T), \mathbb{P}\text{-a.s.}$$

in distributional sense. It is shown in [16] Section 5, that its distributional time derivative $p := \partial_t P \in L^1(\Omega; W^{-1,\infty}(0, T; H^1_{\text{per}}(D)/\mathbb{R}))$. As a consequence, we have the following result.

**Theorem 2.3.** Let $\{u(t); 0 \leq t \leq T\}$ be the variational solution of (1.1). There exists a unique adapted process $P \in L^2(\Omega, L^2(0, T; H^1_{\text{per}}(D)/\mathbb{R}))$ such that $(u, P)$ satisfies $\mathbb{P}$-a.s. for all $t \in (0, T]$.

$$\begin{align*}
(2.7a) & \quad (u(t), v) + \int_0^t (\nabla u(s), \nabla v) \, ds - (\text{div} \, v, P(t)) = (u_0, v) + \int_0^t (B(u(s)), v) \, dW(s) \quad \forall v \in H^1_{\text{per}}(D; \mathbb{R}^d), \\
(2.7b) & \quad (\text{div} \, u, q) = 0 \quad \forall q \in L^2_{\text{per}}(D).
\end{align*}$$

System (2.7) can be regarded as a mixed formulation for the stochastic Stokes system (1.1), where the (time-averaged) pressure $P$ is defined. Below, we also define another time-averaged “pressure”

$$R(t) := P(t) - \int_0^t \xi(s) \, dW(s),$$

where we use the Helmholtz decomposition $B(u(t)) = \eta(t) + \nabla \xi(t)$, where $\xi \in H^1_{\text{per}}(D)/\mathbb{R}$ $\mathbb{P}$-a.s. such that

$$\nabla \xi(t), \nabla \phi = (B(u(t)), \nabla \phi) \quad \forall \phi \in H^1_{\text{per}}(D).$$

Then, (2.6) can be rewritten as

$$\nabla R(t) = -\left[ u(t) - \Delta U(t) - u_0 - \int_0^t \eta(s) \, dW(s) \right] \quad \forall t \in (0, T), \mathbb{P}\text{-a.s.}$$

The time averaged “pressure” $\{R(t); 0 \leq t \leq T\}$ will also be a target process to be approximated in our numerical methods.

### 3. Semi-discretization in time.

In this section we study the stability and convergence properties of a Helmholtz decomposition enhanced Euler-Maruyama time discretization scheme that is based on (1.7), where the stochastic pressure is removed from the noise term via the Helmholtz decomposition; but its $\nabla$-valued velocity approximation $\{u^{n+1}\}_n$ still solves the original Euler-Maruyama scheme (1.3).

**3.1. Formulation of the time-stepping scheme.** In the following, let $N$ be a positive integer, $k = \frac{T}{N}$, and $t_n = nk$ for $n = 0, 1, \ldots, N$ be a uniform mesh that covers $[0, T]$.

**Algorithm 1**

Let $u^0 = u_0$. For $n = 0, 1, \ldots, N - 1$ do the following steps:
Step 1: Find \( \xi^n \in L^2(\Omega, H^1_{\text{per}}(D)/\mathbb{R}) \) by solving
\[
(\nabla \xi^n, \nabla \phi) = (B(u^n), \nabla \phi) \quad \forall \phi \in H^1_{\text{per}}(D).
\]

Step 2: Set \( \eta^n := B(u^n) - \nabla \xi^n \), and find \((u^{n+1}, r^{n+1}) \in L^2(\Omega, \mathbb{V} \times L^2_{\text{per}}(D)/\mathbb{R})\) by solving
\[
\begin{align*}
(u^{n+1}, v) + k(\nabla u^{n+1}, \nabla v) - k(\text{div}\, v, r^{n+1}) &= (u^n, v) + (\eta^n \Delta_{n+1} W, v) \quad \forall v \in H^1_{\text{per}}(D, \mathbb{R}^d), \\
(\text{div}\, u^{n+1}, q) &= 0 \quad \forall q \in L^2_{\text{per}}(D).
\end{align*}
\]

Step 3: Define \( p^{n+1} := r^{n+1} + k^{-1} \xi^n \Delta_{n+1} W \).

Remark 1. By the elliptic regularity theory, see [12, p. 13], the solution of (3.1) is in \( \xi^n \in L^2(\Omega, H^2_{\text{per}}(D)/\mathbb{R}) \), and satisfies Lebesgue-a.e.
\[
-\Delta \xi^n = -\text{div}\, B(u^n) \quad \text{in } D.
\]

Moreover there exists a constant \( C \equiv C(D) > 0 \) such that
\[
\|\xi^n\|_{H^2/\mathbb{R}} \leq C \|\text{div}\, B(u^n)\|.
\]

The solvability of Algorithm 1 is clear because a linear coercive elliptic PDE problem is solved at each step. Step 1 in Algorithm 1 requires to solve a Poisson problem \([3.1]\), which only slightly increases the computational cost if a fast solver is used to solve them. The iterates \( \{(u_n, r_n)\}_n \) and \( \{p_n\}_n \) defined in Step 2 and 3 aim to approximate \( \{(u(t), r(t)) : 0 \leq t \leq T\} \) and \( \{p(t) : 0 \leq t \leq T\} \), respectively. See subsection 3.4 for details.

3.2. Stability estimates. In this subsection we present some stability estimates for the time-stepping scheme given in Algorithm 1. All these estimates, in particular the estimate for \( \{r^{n+1}\}_n \), will play an important role in establishing optimal order error estimates for the fully mixed finite element discretization to be given in the next section.

Lemma 3.1. Let \( \{(u^{n+1}, r^{n+1})\}_n \) be generated by Algorithm 1. There exists a constant \( C_3 \equiv C(D_T, \|u_0\|_{L^2(\Omega; \mathbb{V})}) > 0 \), such that
\[
\begin{align*}
\max_{1 \leq n \leq N} & \mathbb{E}[\|\nabla u^n\|^2] + \mathbb{E}\left[\sum_{n=1}^N \|\nabla(u^n - u^{n-1})\|^2\right] + \mathbb{E}\left[\sum_{n=1}^N \|Au^n\|^2\right] \leq C_3, \\
\mathbb{E}\left[\sum_{n=1}^N \|r^n\|^2\right] & \leq C_3.
\end{align*}
\]

Proof. See [6, 11] for a proof of estimate (3.5); estimate (3.6) was already proved in section 1 after (1.7) was introduced. We note that the periodicity of \( B(u^n) \) was crucially used in the proof of (3.5) to avoid the boundary integral terms arising from integration by parts in the noise term.
3.3. Error estimate for the velocity approximation. Since the velocity approximation \( \{u^{n+1}\}_n \) generated by Algorithm 1 also solves the original Euler-Maruyama time-stepping scheme (1.3), the following optimal order error estimate for \( \{u^n\}_n \) was established in [6, 11].

Theorem 3.2. Let \( \{u^{n+1}, P^{n+1}\}_n \) be generated by Algorithm 1. There exists a constant \( C_4 \equiv C(D_T, C_1, C_2) > 0 \), such that

\[
(3.7) \quad \max_{1 \leq n \leq N} \left( E\left[ \left\| u(t_n) - u^n \right\|^2 \right] \right)^{1/2} + \left( E\left[ k \sum_{n=1}^{N} \| \nabla (u(t_n) - u^n) \|^2 \right] \right)^{1/2} \leq C_4 k^{2/3}.
\]

We note that the proof of the above error estimate crucially uses the fact that \( u^n \) is exactly divergence-free for each \( 0 \leq n \leq N \).

3.4. Error estimates for the pressure approximations. An optimal order error estimate was obtained in [11] for \( \{P(t_n)\}_n \) via the Euler-Maruyama time-stepping scheme (1.3). For the reader’s convenience, we here give its proof.

Theorem 3.3. Let \( \{p^n, 1 \leq n \leq N \} \) be the pressure in (1.3), and \( \{P(t); 0 \leq t \leq T\} \) be defined in Theorem 3.2. There exists \( C_5 \equiv C(D_T, C_1, C_2, \beta) > 0 \), such that

\[
(3.8) \quad \left( E\left[ \left\| P(t_m) - k \sum_{n=1}^{m} p^n \right\|^2 \right] \right)^{1/2} \leq C_5 k^{2/3}, \quad m = 1, 2, \ldots, N.
\]

Proof. Consider (1.3), and take the sum over steps \( 0 \leq n \leq m - 1 \). We denote \( U^m := k \sum_{n=0}^{m-1} u^{n+1} \) and \( P^m := k \sum_{n=0}^{m-1} p^{n+1} \), and therefore obtain

\[
(3.9) \quad u^m - u^0 - \Delta U^m + \nabla P^m = \sum_{n=0}^{m-1} B(u^n) \Delta n+1 W.
\]

We subtract this equation from (2.6) at time \( t = t_m \), and denote \( E^m_U := U(t_m) - U^m \in L^2(\Omega; H^1_{\text{per}}(D; \mathbb{R}^d)) \), and \( E^m_P := P(t_m) - P^m \in L^2(\Omega; L^2_{\text{per}}(D)) \). By the stability of the divergence operator, there exists \( \beta > 0 \), such that

\[
(3.10) \quad \frac{1}{\beta} \| E^m_P \| \leq \sup_{v \in H^1_{\text{per}}(D; \mathbb{R}^d)} \frac{\langle E^m_P, \text{div } v \rangle}{\| \nabla v \|}
\]

\[
\leq \| u(t_m) - u^m \| + \| \nabla E^m_U \| + \left\| \sum_{n=0}^{m-1} (B(u(t_n)) - B(u^n)) \Delta n+1 W \right\|
\]

\[
\quad + \left\| \sum_{n=0}^{m-1} \int_{t_n}^{t_{n+1}} B(u(s)) - B(u(t_n)) \, dW(s) \right\| =: I + \ldots + IV.
\]

Taking squares on both sides, and then applying expectations, Theorem 3.2 in combination with H"older’s inequality leads to

\[
\frac{1}{\beta^2} E[\| E^m_P \|^2] \leq C_4 k + E[\| II \|^2] + E[\| IV \|^2].
\]

By Itô’s isometry, and (2.2a), as well as (2.4a), and Theorem 3.2 we find the bounds

\[
E[\| II \|^2] + E[\| IV \|^2] \leq CE \left[ k \sum_{n=0}^{m-1} \| u(t_n) - u^n \|^2 \right] + C_1 k \leq C k,
\]
which leads to the desired estimate (3.8). □

We now consider the pressure $R^m := k \sum_{n=0}^{m-1} r^{n+1}$, where $\{r^n\}_n$ is defined by Algorithm 1. Using the new notation (3.9), we can write

$$u^m - u^0 - \Delta U^m + \nabla R^m = \sum_{n=0}^{m-1} (B(u^n) - \nabla \xi^n) \Delta n+1 W.$$  \hfill (3.11)

We again subtract this equation from (2.9) at time $t = t_m$, and adapt the error notation in (3.10),

$$\frac{1}{\beta} ||E^m|| \leq \sup_{\mathbf{v} \in H^1_{\text{per}}(D; \mathbb{R}^d)} \frac{\langle E^m, \text{div} \mathbf{v} \rangle}{||\nabla \mathbf{v}||} \leq I + \ldots + IV + V,$$

where

$$V := \left| \sum_{n=0}^{m-1} \nabla (\xi(t_n) - \xi^n) \Delta n+1 W \right| + \left| \sum_{n=0}^{m-1} \int_{t_n}^{t_{n+1}} \nabla (\xi(s) - \nabla \xi(t_n)) \, dW(s) \right| := V_1 + V_2.$$

By a stability result for the Poisson problems (2.8), (3.1), and property (2.2a), we easily obtain, thanks to Theorem 3.2,

$$E[||V_1||^2] \leq C E \left[ k \sum_{n=0}^{m-1} ||u(t_n) - u^n||^2 \right] \leq C k.$$  \hfill (3.12)

Similarly, we get $E[||V_2||^2] \leq C k$. We collect this result below.

**Corollary 3.4.** Let $\{r^n; 1 \leq n \leq N\}$ be the discrete process from Algorithm 1. There exists $C_0 \equiv C(\beta, C_1, C_2) > 0$, such that

$$E[\|R(t_m) - k \sum_{n=1}^{m} r^n\|^2]^{\frac{1}{2}} \leq C_0 k^{\frac{1}{2}}, \quad m = 1, 2, \ldots, N.$$  \hfill (4.12)

### 4. Fully discrete, $inf$-$sup$ stable mixed finite element method.

In this section, we discretize Algorithm 1 in space via an $inf$-$sup$ stable mixed finite element method. We choose the prototypical Taylor-Hood mixed finite element (see, e.g., [12, 3]) as an example and give a detailed error analysis for the resulted fully discrete method, but we remark that the convergence analysis below also applies to general $inf$-$sup$ stable mixed finite elements.

**4.1. Preliminaries.** Let $\mathcal{T}_h$ be a quasi-uniform triangular or rectangular mesh of $D \subset \mathbb{R}^d$ with mesh size $0 < h \ll 1$. We define the following finite element spaces:

$$X_h = \{ v_h \in H^1_{\text{per}}(D; \mathbb{R}^d); v_h|_K \in P_2(K, \mathbb{R}^d) \quad \forall \ K \in \mathcal{T}_h \},$$

$$W_h = \{ q_h \in H^1_{\text{per}}(D)/\mathbb{R}; q_h|_K \in P_1(K) \quad \forall \ K \in \mathcal{T}_h \},$$

$$S_h = \{ \phi_h \in H^1_{\text{per}}(D)/\mathbb{R}; \phi_h|_K \in P_\ell(K) \quad \forall \ K \in \mathcal{T}_h \},$$

where $P_\ell(K; \mathbb{R}^d) (\ell \geq 1)$ denotes the set of $\mathbb{R}^d$-valued polynomials of degree less than or equal to $\ell$ over the element $K \in \mathcal{T}_h$. In general, we require that $S_h \subseteq W_h$, in particular, we choose $\ell = 1$ so that $S_h = W_h$ in this section.
We recall that the pair \((X_h, W_h)\) satisfies the (discrete) inf-sup condition: there exists an \(h\)-independent constant \(\gamma > 0\) such that
\[
\sup_{v_h \in X_h} \frac{(\text{div } v_h, q_h)}{\|\nabla v_h\|} \geq \gamma \|q_h\| \quad \forall q_h \in W_h.
\]

Next, let \(\rho_h : L^2_{\text{per}}(D) \to W_h\) resp. \(R_h : H^1_{\text{per}}(D)/\mathbb{R} \to S_h\) denote the \(L^2\)-resp. the Ritz-projection operators which are defined by
\[
\begin{align*}
(\phi - \rho_h \phi, \chi_h) &= 0 \quad \forall \phi \in L^2_{\text{per}}(D), \, \chi_h \in W_h, \\
(\nabla \psi - R_h \psi, \nabla \zeta_h) &= 0 \quad \forall \psi \in H^1_{\text{per}}(D)/\mathbb{R}, \, \zeta_h \in S_h.
\end{align*}
\]

Then, the following approximation properties are well known (cf. \([17, 19, 24]\)):
\[
\begin{align*}
\|\phi - \rho_h \phi\| + h\|\nabla (\phi - \rho_h \phi)\| &\leq C_7 h^s \|\phi\|_{H^s} \quad \forall \phi \in H^s_{\text{per}}(D), \\
\|\psi - R_h \psi\| + h\|\nabla (\psi - R_h \psi)\| &\leq C_7 h^s \|\psi\|_{H^s} \quad \forall \psi \in H^s_{\text{per}}(D)/\mathbb{R},
\end{align*}
\]
for \(s = 1, 2\). Here, \(C_7\) is a positive constant independent of \(h\).

We also consider the space \(V_h \subset X_h\) of discretely divergence-free functions,
\[
V_h := \{ v_h \in X_h; \, (\text{div } v_h, q_h) = 0 \quad \forall q_h \in W_h \},
\]
and define the \(L^2_{\text{per}}(D; \mathbb{R}^d)\)-projection operator \(P_h : L^2_{\text{per}}(D; \mathbb{R}^d) \to V_h\) by
\[
(v - P_h v, w_h) = 0 \quad \forall v \in L^2_{\text{per}}(D; \mathbb{R}^d), \, w_h \in V_h.
\]

The following approximation properties are well-known (cf. \([17]\)):
\[
\begin{align*}
\|v - P_h v\| + h\|\nabla (v - P_h v)\| &\leq C_7 h^s \|v\|_{H^s} \quad \forall v \in V \cap H^s_{\text{per}}(D; \mathbb{R}^d)
\end{align*}
\]
for \(s = 1, 2\). Here, \(C_7\) is again a positive constant independent of \(h\).

### 4.2. Formulation of the fully discrete mixed finite element method

The fully discrete, inf-sup stable finite element below is a spatial discretization of Algorithm 1. We note that since \(V_h \not\subset V\), in general, the mixed finite element discretization requires improved stability estimates for the semi-discrete pressure \(\{r^{n+1}\}_n\) as given in Lemma 3.1 in order to ensure optimal convergence properties.

**Algorithm 2**

Let \(u^n_0 \in L^2(\Omega; X_h)\). For \(n = 0, 1, \ldots, N - 1\), we do the following steps:

**Step 1:** Determine \(\xi^n_h \in L^2(\Omega; S_h)\) by solving
\[
(\nabla \xi^n_h, \nabla \phi_h) = (B(u^n_h), \nabla \phi_h) \quad \forall \phi_h \in S_h.
\]

**Step 2:** Set \(\eta^n_h := B(u^n_h) - \nabla \xi^n_h\). Find \((u^{n+1}_h, r^{n+1}_h) \in L^2(\Omega, \nabla \times W_h)\) by solving
\[
\begin{align*}
(\eta^{n+1}_h, v_h) &+ k(\nabla u^{n+1}_h, \nabla v_h) - k(\text{div } v_h, r^{n+1}_h) \\
&= (u^n_h, v_h) + (\eta^n_h, \nabla \Delta \xi_{n+1} W, v_h) \quad \forall v_h \in X_h,
\end{align*}
\]
\[
(\text{div } u^{n+1}_h, q_h) = 0 \quad \forall q_h \in W_h.
\]

**Step 3:** Define the \(W_h\)-valued random variable \(p^{n+1}_h = r^{n+1}_h + k^{-1} \xi^n_h \Delta \xi_{n+1} W\).

**Remark 2.** Because of (4.7), we have \((\eta^n_h, \nabla \phi_h) = 0\) for all \(\phi_h \in S_h\), \(\mathbb{P}\)-a.s. We also note that each of Step 1 and Step 2 solves a linear problem which is clearly well-posed; in particular, the well-posedness of (4.8) is ensured by the inf-sup property (4.1) of the mixed finite element spaces \(X_h\) and \(W_h\).
4.3. Error estimate for the velocity approximation. The main result of this section is to prove the following optimal estimate for the velocity error $u^n - u^n_h$.

**Theorem 4.1.** Suppose that

$$\mathbb{E}[\|u^0 - u^n_h\|^2] \leq Ch^2.$$  

Let $\{(u^n, r^n); 1 \leq n \leq N\}$ and $\{(u^n_h, r^n_h); 1 \leq n \leq N\}$ be respectively the solutions of Algorithm 1 and 2. Then there holds

$$\max_{1 \leq n \leq N} \left( \mathbb{E}[\|u^n - u^n_h\|^2] \right)^{\frac{1}{2}} + \left( \mathbb{E} \left[ k \sum_{n=1}^{N} \|\nabla(u^n - u^n_h)\|^2 \right] \right)^{\frac{1}{2}} \leq C_8 h,$$

where $C_8 \equiv C(D_T, C_3)$ is a positive constant independent of $h$.

**Proof.** Define $e^n_u = u^n - u^n_h$ and $e^n_r = r^n - r^n_h$. It is easy to check that $\{(e^n_u, e^n_r)\}_n$ satisfies the following error equations $\mathbb{P}$-a.s. for all tuple $(v_h, q_h) \in X_h \times W_h$,

$$e^{n+1}_u - e^n_u, v_h) + k(\nabla e^{n+1}_u, \nabla v_h) - k(e^{n+1}_r, \text{div } v_h) = \left( [\eta^n - \eta^n_h] \Delta n+1 W; v_h \right),$$

$$\text{div } e^{n+1}_u, q_h) = 0.$$

Now for any fixed $\omega \in \Omega$, setting $v_h = P_h e^{n+1}_u(\omega) \in \mathbb{V}_h$ in (4.10) yields

$$e^{n+1}_u - e^n_u, P_h e^{n+1}_u) + k(\nabla e^{n+1}_u, \nabla P_h e^{n+1}_u) - k(e^{n+1}_r, \text{div } P_h e^{n+1}_u) = \left( [\eta^n - \eta^n_h] \Delta n+1 W; P_h e^{n+1}_u \right).$$

We now estimate each term on the left-hand side of (4.12) from below. First, by the definition of $P_h$ we get

$$e^{n+1}_u - e^n_u, P_h e^{n+1}_u) = (P_h e^{n+1}_u - e^n_u, P_h e^{n+1}_u)$$

$$= \frac{1}{2} \left( \|P_h e^{n+1}_u\|^2 - \|P_h e^n_u\|^2 \right) + \frac{1}{2} \|P_h e^{n+1}_u - e^n_u\|^2.$$

Next, using the fact that $P_h u^{n+1}_u = u^{n+1}_h$ and Schwarz inequality, we obtain

$$k(\nabla e^{n+1}_u, \nabla P_h e^{n+1}_u) = k\|\nabla e^{n+1}_u\|^2 - k(\nabla e^{n+1}_u, \nabla [u^{n+1}_u - P_h u^{n+1}_u])$$

$$\geq \frac{k}{2}\|\nabla e^{n+1}_u\|^2 - \frac{k}{2}\|\nabla [u^{n+1}_u - P_h u^{n+1}_u]\|^2$$

$$\geq \frac{k}{2}\|\nabla e^{n+1}_u\|^2 - C C_7^2 k^2 h^2\|u^{n+1}_u\|^2_{H^2},$$

where we have used (4.6) to get the last inequality.

For the next term in (4.12), using the fact that $P_h e^{n+1}_u$ takes values in $\mathbb{V}_h$, and estimates (4.4), (4.5), and (4.6), we get

$$-k(e^{n+1}_r, \text{div } P_h e^{n+1}_u) = -k(r^{n+1}_r - \rho_h r^{n+1}_r, \text{div } P_h e^{n+1}_u)$$

$$\geq -k(r^{n+1}_r - \rho_h r^{n+1}_r, \|\text{div } P_h e^{n+1}_u\|)$$

$$\geq -C(1 + C_7) C_7 h k \|\nabla r^{n+1}_r\| \|\nabla e^{n+1}_u\|$$

$$\geq -\frac{k}{4}\|\nabla e^{n+1}_u\|^2 - C(1 + C_7)^2 C_7^2 h^2 k \|\nabla r^{n+1}_r\|^2. $$
Finally, we bound the only term on the right-hand side of (4.12) from above. By
the independence of the increments \{\Delta_{n+1}W\}_n, and its distribution, we get

\[(4.16) \quad \mathbb{E}\left[\left(\left|\eta^n - \eta^m_h\right| \Delta\nabla \mathcal{W}, P_h e^{n+1}_u \right)\right] = \mathbb{E} \left[\left|\eta^n - \eta^m_h\right| \Delta\nabla \mathcal{W}, P_h (e^{n+1}_u - e^m_u)\right] \leq k\mathbb{E}\left[\left|\eta^n - \eta^m_h\right|^2\right] + \frac{1}{4} \mathbb{E}\left[\left|P_h (e^{n+1}_u - e^m_u)\right|^2\right],\]

and because of (2.2) and (4.6), there holds

\[(4.17) \quad \left|\eta^n - \eta^m_h\right|^2 \leq 2\|B(u^n) - B(u^m)\|^2 + 2\|\nabla(\xi^n - \xi^m_h)\|^2 \leq 2C_T^2\|B(u^n)\|^2 + 2\|\nabla(\xi^n - \xi^m_h)\|^2.
\]

To control \(\|\nabla(\xi^n - \xi^m_h)\|^2\), we recall the definitions of \(\xi^n\) and \(\xi^m_h\) to get

\[(\nabla(\xi^n - \xi^m_h), \nabla \phi_h) = (B(u^n) - B(u^m), \nabla \phi_h) \quad \forall \phi_h \in S_h.\]

Setting \(\phi_h = R_h[\xi^n - \xi^m_h] = (\xi^n - \xi^m_h) - (\xi^n - R_h[\xi^n])\), properties (2.2a) and (4.5) yield

\[\|\nabla(\xi^n - \xi^m_h)\|^2 \leq \|\nabla(\xi^n - \xi^m_h), \nabla[\xi^n - R_h[\xi^n]]\| + C_C T \|\varepsilon^n\| \|\nabla(\xi^n - \xi^m_h)\| \leq \frac{1}{2}\|\nabla(\xi^n - \xi^m_h)\|^2 + C_T^2 h^2 \|\varepsilon^n\|^2 + C_C T \|\varepsilon^n\|^2.
\]

Hence, by (3.4) in Remark 1 (4.6), and (2.2a), we get

\[(4.18) \quad \|\nabla(\xi^n - \xi^m_h)\|^2 \leq C_T^2 h^2 \|\xi^n\|_{H^2/R} + C_T^2 \|\varepsilon^n\| \leq C_T^2 (\|\nabla B(u^n)\|^2 + C_T^2 \|\varepsilon^n\|^2) \leq C_T^2 \left(\|\nabla B(u^n)\|^2 + C_T^2 h^2 \|\varepsilon^n\|^2 + C_T^2 h^2 \|\nabla u^n\|^2 + \|P_h e^m_u\|^2\right).
\]

We insert estimates (4.13)–(4.18) into (4.12), take the expectation, and apply the
summation operator \(\sum_{n=0}^{m} \) for any \(0 \leq m \leq N - 1\) to conclude

\[(4.19) \quad \frac{1}{2} \mathbb{E}\left[\|P_h e^{m+1}_u\|^2\right] + \frac{1}{4} \sum_{n=0}^{m} \mathbb{E}\left[\|P_h (e^{n+1}_u - e^n_u)\|^2\right] + \frac{1}{4} \mathbb{E}\left[\| \nabla e^{m+1}_u\|^2\right] \leq C_T^2 k \sum_{n=0}^{m} \mathbb{E}\left[\|P_h e^n_u\|^2\right] + C_T^2 h^2,
\]

where \(C_T^2 \equiv C(D_T)\) is a positive constant independent of \(h\).

Applying the discrete Gronwall inequality to (4.19) then leads to

\[(4.20) \quad \frac{1}{2} \mathbb{E}\left[\|P_h e^{m+1}_u\|^2\right] + \frac{1}{4} \sum_{n=0}^{m} \mathbb{E}\left[\|P_h (e^{n+1}_u - e^n_u)\|^2\right] + \frac{1}{4} \mathbb{E}\left[\| \nabla e^{m+1}_u\|^2\right] \leq C_T^2 \exp(C_T^2 T) h^2 \quad (1 \leq m \leq N).\]

Finally, the desired estimate (4.9) follows from an application of the triangle inequality on \(e^{m+1}_u = (u^{m+1} - P_h u^{m+1}) + P_h e^{m+1}_u\) and using (4.20) and (4.6). The
proof is complete. \( \Box \)
4.4. Error estimates for the pressure approximations. In this subsection, we derive some error estimates for both, \( r^n - r_h^n \) and \( p^n - p_h^n \). The argumentation parallels the one in section 3.4, and uses the inf-sup condition \( (1.1) \), in particular.

**Theorem 4.2.** Suppose that

\[
\mathbb{E} \left[ \| u^0 - u_h^0 \|^2 \right] \leq C h^2.
\]

Let \( \{ (u^n, r^n); 1 \leq n \leq N \} \) and \( \{ (u_h^n, r_h^n); 1 \leq n \leq N \} \) be respectively the solutions of Algorithm 1 and 2. There exists \( C_9 = C(D_T, C_3, \gamma) > 0 \) independent of \( h \), such that

\[
\left( \mathbb{E} \left[ \| k \sum_{n=1}^{N} (r^n - r_h^n) \|^2 \right] \right)^{\frac{1}{2}} \leq C_9 h.
\]

**Proof.** Summing (4.10) (after lowering the index by one) over \( 1 \leq n \leq m \leq N \) leads to

\[
(e^m_u, v_h) + k \sum_{n=1}^{m} (\nabla e^n_u, \nabla v_h) - \sum_{n=1}^{m} (\text{div } v_h, e^n_r)
= (e^0_u, v_h) + \sum_{n=1}^{m} (\eta^{n-1} - \eta^{n-1}_h) W, v_h) \quad \forall v_h \in \mathbb{X}_h.
\]

By (4.1), we conclude (compare with (3.10))

\[
\frac{1}{2\gamma} \left\| k \sum_{n=1}^{m} e^n_r \right\|^2 \leq \left\| e^0_u \right\|^2 + \left\| e^0_u \right\|^2 + \left\| \sum_{n=1}^{m} \nabla e^n_u \right\|^2 + \left\| \sum_{n=1}^{m} (\eta^{n-1} - \eta^{n-1}_h) W \right\|.
\]

Taking squares, and expectations, estimate (4.17) and Theorem 4.1 settles the result. \( \square \)

The following result now is a simply corollary of Theorem 4.2.

**Corollary 4.3.** Let \( \{ p_h^n \} \) be the solution in Algorithm 2. Then there exists \( C_{10} = C(D_T, C_3, \gamma) > 0 \) independent of \( h \), such that

\[
\left( \mathbb{E} \left[ \| k \sum_{n=1}^{N} (p^n - p_h^n) \|^2 \right] \right)^{\frac{1}{2}} \leq C_{10} h.
\]

4.5. Space-time error estimates for Algorithm 2. Theorems 3.2, 3.3, 4.1, 4.2, and Corollaries 3.4, 4.3 now provide the following global error estimates.

**Theorem 4.4.** Let \( (u, P) \) solve (1.1), and \( \{ (u^n_h, r^n_h, p^n_h); 1 \leq n \leq N \} \) solves Algorithm 2. There exists a constant \( C = C(D_T, C_1, C_2, C_3, \beta, \gamma) > 0 \), such that

\[
\begin{align*}
(i) \quad & \max_{1 \leq n \leq N} \left( \mathbb{E} \left[ \| u(t_n) - u^n_h \|^2 \right] \right)^{\frac{1}{2}} + \left( \mathbb{E} \left[ k \sum_{n=1}^{N} \| \nabla (u(t_n) - u^n_h) \|^2 \right] \right)^{\frac{1}{2}} \leq C (k^{\frac{3}{2}} + h), \\
(ii) \quad & \left( \mathbb{E} \left[ \| R(t_m) - k \sum_{n=1}^{m} r^n_h \|^2 \right] \right)^{\frac{1}{2}} + \left( \mathbb{E} \left[ \| P(t_m) - k \sum_{n=1}^{m} p^n_h \|^2 \right] \right)^{\frac{1}{2}} \leq C (k^{\frac{3}{2}} + h),
\end{align*}
\]

for all \( 1 \leq m \leq N \).
5. Stabilization methods for \((1.1)\). The scheme in section 4 requires \textit{inf-sup} stable pairings \((X_h, W_h)\), for which the Taylor-Hood mixed finite element is one example. By recalling its definition in subsection 4.1, we observe that the dimension of \(X_h\) exceeds that of \(W_h\). The motivation for the stabilization methods in [13] is to relax the \textit{inf-sup} stability criterion for pairings of ansatz spaces in order to allow for equal-order ansatz spaces for both, velocity and pressure approximates; see [13, 3, 12, 9] for further details.

Below we replace \(X_h\) defined in subsection 4.1 by
\[
Y_h = \{ v_h \in H^1_{\text{per}}(D; \mathbb{R}^d) ; v_h|_K \in P_1(K, \mathbb{R}^d) \, \forall \, K \in T_h \},
\]
which satisfies the following approximation property (cf. [9]):
\[
\| v - Q_h v \|_H + h \| \nabla (v - Q_h v) \| \leq C_{11} h^s \| v \|_{H^s} \quad \forall \, v \in H^s_{\text{per}}(D; \mathbb{R}^d)
\]
for \(s = 1, 2\). Here, \(C_{11} > 0\) is a constant independent of \(h\). Moreover, let \(W_h\) be the same as in section 4, and \(R_h\) denote the Ritz projection from \(H^1_{\text{per}}(D)/\mathbb{R}\) to \(W_h\). Again, we take \(S_h = W_h\) in this section.

In this section, we consider the equal-order pairing \((Y_h, W_h)\) to discretize \((1.1)\) based on \((1.9a)-(1.9b)\), which violates the \textit{inf-sup} condition; in fact, the following estimate is known to hold (cf. [13]): there exists \(\delta > 0\) independent of \(h > 0\), such that
\[
\| v - Q_h v \|_H + h \| \nabla (v - Q_h v) \| \leq C_{11} h^s \| v \|_{H^s} \quad \forall \, v \in H^s_{\text{per}}(D; \mathbb{R}^d)
\]
for \(s = 1, 2\). Here, \(C_{11} > 0\) is a constant independent of \(h > 0\), such that
\[
\frac{1}{\delta^2} \| Q_h \|_H^2 \leq \sup_{v_h \in Y_h} \frac{|(q_h, \nabla v_h)|^2}{\| \nabla v_h \|^2} + h^2 \| \nabla q_h \|_H^2 \quad \forall \, q_h \in W_h.
\]

5.2 can be regarded as the reason why this pairing still performs optimally when applied to the Stokes problem, where \(\varepsilon = O(h^2)\). Below we show that such a strategy can be again successful for the stochastic Stokes problem \((1.1)\), if the proper pressure is chosen for the perturbation, and that using the Helmholtz projection of the noise provides such an approach.

To prepare for the analysis, we start with a modification of Algorithm 1 that perturbs the incompressibility constraint.

\underline{Algorithm 3}

Let \(0 < \varepsilon \ll 1\) and \(u_\varepsilon^0 = u_0\). For \(n = 0, 1, \ldots, N - 1\), do the following steps:

\underline{Step 1:} Find \(\xi_\varepsilon^n \in L^2(\Omega, H^1_{\text{per}}(D)/\mathbb{R})\) by solving
\[
(\nabla \xi_\varepsilon^n, \nabla \phi) = (B(u_\varepsilon^n), \nabla \phi) \quad \forall \phi \in H^1_{\text{per}}(D).
\]

\underline{Step 2:} Set \(\eta_\varepsilon^n := B(u_\varepsilon^n) - \nabla \xi_\varepsilon^n\), and find \((u_\varepsilon^{n+1}, p_\varepsilon^{n+1}) \in L^2(\Omega, H^1_{\text{per}}(D; \mathbb{R}^d) \times H^1_{\text{per}}(D)/\mathbb{R})\) by solving
\[
\begin{align}
(\nabla u_\varepsilon^{n+1}, v) + k(\nabla u_\varepsilon^{n+1}, \nabla v) - k(\nabla v,p_\varepsilon^{n+1}) &= (u_\varepsilon^n, v) + (\eta_\varepsilon^n \Delta_{n+1} W, v) \quad \forall \, v \in H^1_{\text{per}}(D; \mathbb{R}^d), \\
(\nabla u_\varepsilon^{n+1}, q) + \varepsilon(\nabla r_\varepsilon^{n+1}, \nabla q) &= 0 \quad \forall \, q \in H^1_{\text{per}}(D).
\end{align}
\]
Step 3: Define $p^{n+1}_\varepsilon := r^{n+1}_\varepsilon + k^{-1} \xi_n \Delta_{n+1} W$.

Because each step involves a coercive linear problem, Algorithm 3 has a unique solution. The first energy estimate can be obtained from (5.4) by fixing one $\omega \in \Omega$ and choosing $(v, q) = (u^{n+1}_\varepsilon, r^{n+1}_\varepsilon(\omega))$, we then obtain the identity

\[ \frac{1}{2} \left( \|u^{n+1}_\varepsilon\|^2 - \|u^n_\varepsilon\|^2 + \|u^{n+1}_\varepsilon - u^n_\varepsilon\|^2 \right) + k \|\nabla u^{n+1}_\varepsilon\|^2 + k \varepsilon \|\nabla r^{n+1}_\varepsilon\|^2 = \langle \eta^n_{\varepsilon} \Delta_{n+1} W, u^{n+1}_\varepsilon \rangle. \]

(5.5)

Taking expectations, applying the summation operator $\sum_{n=0}^N$ for any $0 \leq m \leq N$, and using the independence of the increments $\{\Delta_{n+1} W\}_n$ yield

\[ \mathbb{E}[\langle \eta^n_{\varepsilon} \Delta_{n+1} W, u^{n+1}_\varepsilon \rangle] = \mathbb{E}[\langle \eta^n_{\varepsilon} \Delta_{n+1} W, u^{n+1}_\varepsilon - u^n_\varepsilon \rangle] \]

\[ \leq k \mathbb{E}[\|\eta^n_\varepsilon\|^2] + \frac{1}{4} \mathbb{E}[\|u^{n+1}_\varepsilon - u^n_\varepsilon\|^2]. \]

(5.6)

Because of (5.3) and (2.2b), we have

\[ \mathbb{E}[\|\nabla_\varepsilon u^n_\varepsilon\|^2] \leq \mathbb{E}[\|B(u^n_\varepsilon)\|^2] \leq C(1 + \mathbb{E}[\|u^n_\varepsilon\|^2]), \]

and therefore $\mathbb{E}[\|\eta^n_\varepsilon\|^2] \leq C(1 + \mathbb{E}[\|u^n_\varepsilon\|^2])$ in (5.6). We insert these auxiliary estimates into (5.5), take expectations, sum over all iteration steps, and use the discrete Gronwall inequality to get

\[ \max_{0 \leq n \leq N} \frac{1}{2} \mathbb{E}[\|u^{n+1}_\varepsilon\|^2] + k \sum_{n=0}^N \mathbb{E}[\|\nabla u^{n+1}_\varepsilon\|^2 + \|\nabla r^{n+1}_\varepsilon\|^2] \leq C_F \mathbb{E}[\|u_0\|^2]. \]

(5.7)

Note that the estimate for $\{\nabla r^{n+1}_\varepsilon\}$ is scaled by $\varepsilon > 0$, which is too weak in the following to verify optimal error estimates for a spatial discretization of Algorithm 3. The following stability result therefore sharpens the estimate (5.7): its proof crucially exploits again the fact that each $\eta^n_\varepsilon$ is a $\mathbb{H}$-valued random variable.

**Lemma 5.1.** Let $\{u^n_\varepsilon, r^n_\varepsilon\}_n$ be the solution of Algorithm 3. Then there exists a positive constant $C_{11} \equiv C(D_F, C_3)$ independent of $\varepsilon$ and $k$, such that

\[ \max_{1 \leq n \leq N} \frac{1}{2} \mathbb{E}[\|\nabla^2 u^n_\varepsilon\|^2] + k \sum_{n=1}^N \mathbb{E}[\|\Delta u^n_\varepsilon\|^2] + k \sum_{n=1}^N \mathbb{E}[\|\nabla r^n_\varepsilon\|^2] \leq C_{11}. \]

(5.8)

**Proof. Step 1:** We adapt the argumentation from [5, Thm. 3.1], and interpret problem (1.3) — with $\eta^n$ being replaced by $\eta^n_\varepsilon$ — as perturbation of problem (1.7). Subtracting the corresponding equations of both systems and denoting $e^{n+1}_u := u^{n+1}_\varepsilon - u^{n+1}_r$ resp. $e^{n+1}_r := r^{n+1}_\varepsilon - r^{n+1}_r$ yield

\[ e^{n+1}_u - k \Delta e^{n+1}_u + k \nabla e^{n+1}_r = e^{n}_u + [\eta^n - \eta^n_\varepsilon] \Delta_{n+1} W \quad \text{in } D, \]

\[ \text{div } e^{n+1}_u - \varepsilon \Delta e^{n+1}_r = -\varepsilon \Delta r^{n+1}_r \quad \text{in } D. \]

(5.9a)

(5.9b)

Now fix one $\omega \in \Omega$, test (5.9a) with $v = e^{n+1}_u(\omega)$, and (5.9b) with $q = e^{n+1}_r(\omega)$, and afterwards sum both equations, we then conclude

\[ \frac{1}{2} \left( \|e^{n+1}_\varepsilon\|^2 - \|e^n_\varepsilon\|^2 + \|e^{n+1}_u - e^n_u\|^2 \right) + k \|\nabla e^{n+1}_\varepsilon\|^2 + k \varepsilon \|\nabla r^{n+1}_\varepsilon\|^2 \]

\[ = \langle [\eta^n - \eta^n_\varepsilon] \Delta_{n+1} W, e^{n+1}_\varepsilon \rangle + k \varepsilon (\nabla r^{n+1}_\varepsilon, \nabla e^{n+1}_\varepsilon). \]

(5.10)
Using Young’s inequality, hiding one part of the last term into the corresponding term on the left-hand side, using the independence of $\Delta_{n+1} W$ and $e^u_n$, $\Delta_{n+1} W$ as well as of $[\eta^n - \eta^u_n]$, and utilizing (2.2), we obtain

\[
E \left[ (\eta^n - \eta^u_n) [\Delta_{n+1} W, e^u_n + 1 - e^u_n] \right] = E \left[ (\eta^n - \eta^u_n) [\Delta_{n+1} W, e^u_n + 1 - e^u_n] \right] \\
\leq k E \left[ ||\eta^n - \eta^u_n||^2 \right] + \frac{1}{2} E \left[ ||e^u_n + 1 - e^u_n||^2 \right] \\
\leq C_T k \left( E \left[ ||e^u_n||^2 \right] + E \left[ ||(\xi^n - \xi^u_n)||^2 \right] \right) + \frac{1}{2} E \left[ ||e^u_n + 1 - e^u_n||^2 \right].
\]

Subtracting (5.3) from (3.1) and using (2.2), we get

\[
\|\nabla (\xi^n - \xi^u_n)\| \leq \|B(u^n) - B(u^u)\| \leq C_T \|e^u_n\|.
\]

Hence, we then conclude from (5.10) with the help of the discrete Gronwall inequality, and (3.6) that

\[
\frac{1}{2} E \left[ ||e^u_n + 1||^2 \right] + \sum_{m=0}^m E \left[ ||e^u_n + 1 - e^u_n||^2 \right] + k \sum_{m=0}^m E \left[ ||\nabla e^u_n + 1 + \varepsilon ||\nabla e^u_n + 1||^2 \right] \\
\leq C_T \varepsilon k \sum_{m=0}^m E \left[ ||\nabla r^u_n + 1||^2 \right] \leq C_T \varepsilon \quad (0 \leq m \leq N).
\]

Consequently, by (3.6) we conclude that

\[
k \sum_{n=0}^N E \left[ ||\nabla e^u_n + 1||^2 \right] \leq C_T \quad \text{implies} \quad k \sum_{n=0}^N E \left[ ||\nabla r^u_n + 1||^2 \right] \leq C_T.
\]

**Step 2:** Fix one $\omega \in \Omega$ in (1.9a), multiply the equation with $-\Delta u^{n+1}(\omega)$, integrate, perform an integration by parts on the last term, and use the periodicity of $\eta^u_n$ and $u^{n+1}$, we get

\[
\frac{1}{2} E \left[ \|\nabla u^{n+1}_\omega\|^2 - \|\nabla u^n_\omega\|^2 + \|\nabla (u^{n+1}_\omega - u^n_\omega)\|^2 + k\|\Delta u^{n+1}_\omega\|^2 \right] \\
\leq k E \left[ \|\nabla r^{n+1}_\omega\|^2 \right] + E \left[ \int_D \Delta_{n+1} W \nabla \eta^u_n \cdot \nabla (u^{n+1}_\omega - u^n_\omega) \, dx \right].
\]

To bound the last term above, we use Schwarz inequality, the fact that $\eta^u_n = B(u^n) - \nabla \xi^u_n$, (2.2), and (3.4) to get

\[
E \left[ \int_D \Delta_{n+1} W \nabla \eta^u_n \cdot \nabla (u^{n+1}_\omega - u^n_\omega) \, dx \right] \leq \frac{1}{4} E \left[ \|\nabla (u^{n+1}_\omega - u^n_\omega)\|^2 \right] \\
+ C_T k E \left[ \|\nabla u^n_\omega\|^2 \right].
\]

Substituting (5.12) into (5.11), summing over all time steps, and using (5.7) and the result of **Step 1**, we get the desired estimate (5.8). The proof is complete.

From **Step 1** of the above proof, we also obtain the following result.

**Theorem 5.2.** Let $\{u^{n+1}, r^{n+1}\}_n$ and $\{u^n, r^n\}_n$ be the solutions of Algorithm 1 and 3, respectively. Then there exists a constant $C_{12} \equiv C(D_T, C_3) > 0$ independent of $\varepsilon$ and $k$, such that

\[
\max_{1 \leq n \leq N} E \left[ \|u^n - u^n_\omega\|^2 \right] + k \sum_{n=1}^N E \left[ \|\nabla (u^n - u^n_\omega)\|^2 \right] \\
+ \varepsilon \|\nabla (r^n - r^n_\omega)\|^2 \leq C_{12} \varepsilon.
\]
We are ready to bound the error between the pressures \(\{r^n\}_n\) and \(\{r^n_\varepsilon\}_n\); the proof of it uses (5.9a) after summation in time, and follows along the lines of the proof of Theorem 3.3, using the stability of the divergence operator (cf. estimate (3.10)), and Theorem 5.2.

**Theorem 5.3.** Let \(\{r^n; 1 \leq n \leq N\}\) be generated by Algorithm 1 and \(\{r^n_\varepsilon; 1 \leq n \leq N\}\) by Algorithm 3. There exists \(C_{12} \equiv C(D_T, C_3, \beta) > 0\), such that for \(m = 1, 2, \ldots, N\),

\[
\left(\mathbb{E}\left[\left|\sum_{n=1}^{m} r^n - k \sum_{n=1}^{m} r^n_\varepsilon\right|^2 + \left|k \sum_{n=1}^{m} p^n - k \sum_{n=1}^{m} p^n_\varepsilon\right|^2\right]\right)^{\frac{1}{2}} \leq C_{12}\varepsilon.
\]

Next, we present the following modification of Algorithm 2.

**Algorithm 4**

Let \(0 < \varepsilon \ll 1\) and \(u^0_{\varepsilon, h} \in L^2(\Omega; \mathbb{Y}_h)\). For \(n = 0, 1, \ldots, N - 1\), do the following steps:

**Step 1:** Determine \(\xi^n_{\varepsilon, h} \in L^2(\Omega; S_h)\) from

\[
(\nabla u^n_{\varepsilon, h}, \nabla \phi_h) = (B(u^n_{\varepsilon, h}), \nabla \phi_h) \quad \forall \phi_h \in S_h.
\]  

**Step 2:** Set \(\eta^n_{\varepsilon, h} = B(u^n_{\varepsilon, h}) - \nabla u^n_{\varepsilon, h}\). Find \((u^{n+1} \in L^2(\Omega, \mathbb{Y}_h \times W_h)\) by solving

\[
\left(\begin{array}{c}
u^{n+1}, v_h + k(\nabla u^{n+1}, \nabla v_h) - k(\text{div} v_h, r^{n+1})_n \in \mathbb{Y}_h \times W_h, \\
(\text{div} u^{n+1}), q_h + \varepsilon(\nabla u^{n+1}, \nabla q_h) = 0 \quad \forall q_h \in W_h.
\end{array}\right)
\]

**Step 3:** Define the \(W_h\)-valued random variable \(p^{n+1}_h = r^{n+1} + k^{-1}\xi^n_{\varepsilon,h} \Delta_{n+1} W\).

The main result of this section is the following estimate for the velocity error \(u^0 - u^0_{\varepsilon, h}\).

**Theorem 5.4.** Suppose

\[
\mathbb{E}\left[\|u^0 - u^0_{\varepsilon, h}\|^2\right] \leq Ch^2.
\]

Let \(\{(u^n, r^n) ; 1 \leq n \leq N\}\) and \(\{(u^n_{\varepsilon, h}, r^n_{\varepsilon, h}) ; 1 \leq n \leq N\}\) be the solutions of Algorithm 3 and 4, respectively. Then

\[
\max_{1 \leq n \leq N} \left(\mathbb{E}\left[\|u^n - u^n_{\varepsilon, h}\|^2\right]\right)^{\frac{1}{2}} + \left(\mathbb{E}\left[k \sum_{n=1}^{N} \|\nabla(u^n - u^n_{\varepsilon, h})\|^2\right]\right)^{\frac{1}{2}} \leq C_{13}\left(h + \frac{h^2}{\sqrt{\varepsilon}}\right),
\]

where \(C_{13} \equiv C(D_T, C_3)\) is a positive constant independent of \(h \) and \(\varepsilon\). This estimate suggests that \(\varepsilon = \mathcal{O}(h^2)\) is the optimal choice of \(\varepsilon\).

**Proof.** Let \(e^{n+1}_u := u^{n+1} - u^{n+1}_{\varepsilon, h}\) and \(e^{n+1}_r := r^{n+1} - r^{n+1}_{\varepsilon, h}\). Then \(\{(e^n_u, e^n_r)\}_n\) satisfies the following error equations \(\mathbb{P}\)-a.s. for all tuple \((v_h, q_h) \in \mathbb{Y}_h \times W_h,\)

\[
(e^{n+1}_u - e^{n}_u, v_h) + k(\nabla e^{n+1}_u, \nabla v_h) + k(\text{div} e^{n+1}_u, v_h) = ((\eta^n - \eta^n_{\varepsilon, h})_h \Delta_{n+1} W, v_h),
\]

\[
(\text{div} e^{n+1}_r, q_h) + \varepsilon(\nabla e^{n+1}_r, \nabla q_h) = 0.
\]
Now consider (5.17)–(5.18) for \( \omega \in \Omega \) fixed, and choose

\[
(v_h, q_h) = (Q_h e_u^{n+1}(\omega), \tilde{R}_h e_r^{n+1}(\omega)) \in \mathbb{V}_h \times W_h;
\]
we then deduce

\[
(5.19) \quad (e_u^{n+1} - e_u^n, Q_h e_u^{n+1}) + k(\nabla e_u^{n+1}, \nabla Q_h e_u^{n+1}) - k(e_r^{n+1}, \text{div } Q_h e_u^{n+1}) = (\eta_e^n - \eta_e^{n,h})|_{\Delta_{n+1} W, Q_h e_u^{n+1}}.
\]

We can adopt the corresponding arguments in (4.13) and (4.14), and use Lemma 5.1 to treat the first two terms in (5.19), and also the argument around (4.16) may easily be adopted to the present setting. But a different treatment is required to deal with the last term on the left-hand side of (5.19) because it involves the error in the pressure. We rewrite this term as follows:

\[
\Pi := (\tilde{R}_h e_r^{n+1}, \text{div } e_u^{n+1} + (Q_h u_r^{n+1} - u_r^{n+1}))
\]

\[
= (\tilde{R}_h e_r^{n+1}, e_u^{n+1}) + (\varepsilon e_r^{n+1} - \tilde{R}_h e_r^{n+1}, \text{div } e_u^{n+1}) + (e_r^{n+1}, \text{div } [Q_h u_r^{n+1} - u_r^{n+1}])
\]

\[
=: \Pi_1 + \Pi_2 + \Pi_3.
\]

We estimate \( \Pi_2 \) with the help of (4.5) and using Lemma 5.1:

\[
\mathbb{E}[\Pi_2] \leq C h^2 \mathbb{E}[\|\nabla e_r^{n+1}\|^2] + \frac{1}{4} \mathbb{E}[\|\nabla h e_u^{n+1}\|^2].
\]

Integrating by parts in \( \Pi_3 \), using (5.1) and again Lemma 5.1 yield

\[
|\Pi_3| = |(\tilde{R}_h e_r^{n+1}, \text{div } [Q_h u_r^{n+1} - u_r^{n+1}])| + |(\varepsilon e_r^{n+1} - \tilde{R}_h e_r^{n+1}, \text{div } [Q_h u_r^{n+1} - u_r^{n+1}])|
\]

\[
\leq \frac{\varepsilon}{4} \|\text{div } e_r^{n+1}\|^2 + \frac{C h^4}{\varepsilon} \|\Delta u_r^{n+1}\|^2 + C h^2 \|\nabla e_r^{n+1}\| \|\Delta u_r^{n+1}\|.
\]

Because of (5.18), we have

\[
\Pi_1 = -\varepsilon (\nabla e_r^{n+1}, \text{div } e_r^{n+1}) = -\varepsilon \|\nabla \tilde{R}_h e_r^{n+1}\|^2.
\]

Putting the above auxiliary estimates together, we obtain that there exists some \( h \)- and \( \varepsilon \)-independent constant \( C_{11} > 0 \) such that

\[
(5.20) \quad \frac{1}{2} \mathbb{E}[\|Q_h e_u^{n+1}\|^2] + \frac{1}{4} \sum_{n=0}^m \mathbb{E}[\|Q_h (e_u^{n+1} - e_u^n)\|^2]
\]

\[
+ \frac{1}{4} k \sum_{n=0}^m \mathbb{E}[\|\nabla e_u^{n+1}\|^2 + \varepsilon \|\nabla \tilde{R}_h e_r^{n+1}\|^2] \leq C_{11} \left( h^2 + \frac{h^4}{\varepsilon} \right)
\]

for every \( 0 \leq m \leq N \). The desired estimates (5.16) then follows from an application of the discrete Gronwall inequality. The proof is complete.

The last result gives an estimate for the pressure approximation error. Using (5.2), equation (5.17) after taking a summation in \( n \), and Lemma 5.1, we obtain

**Theorem 5.5.** Let \( \{r_r^n; 1 \leq n \leq N\} \) be the pressure in Algorithm 3, and \( \{r_r^{n,h}; 1 \leq n \leq N\} \) be the pressure in Algorithm 4. There exists \( C_{14} \equiv C(D_T, C_3, \delta) > 0 \), such that for all \( 1 \leq m \leq N \),

\[
\left( \mathbb{E} \left[ k \sum_{n=1}^m r_r^n - k \sum_{n=1}^m r_r^{n,h} \right]^2 + \mathbb{E} \left[ k \sum_{n=1}^m p_r^n - k \sum_{n=1}^m p_r^{n,h} \right]^2 \right)^{1/2} \leq C_{14} \left( h + \frac{h^2}{\sqrt{\varepsilon}} \right).
\]
To sum up the results in this section, we have shown the following error estimates for Algorithm 4.

**Theorem 5.6.** Let \((u, P)\) be the solution of (1.1) and \(\{(u^n_{\varepsilon,h}, r^n_{\varepsilon,h}, p^n_{\varepsilon,h}); 1 \leq n \leq N\}\) be the solution of Algorithm 4. There exists a constant \(C \equiv C(D_T, C_3, \beta, \delta) > 0\), such that

\[
(i) \quad \max_{1 \leq n \leq N} \left( E \left[ \|u(t_n) - u^n_{\varepsilon,h}\|^2 \right] \right)^{\frac{1}{2}} + \left( E \left[ k \sum_{n=1}^{N} \|\nabla(u(t_n) - u^n_{\varepsilon,h})\|^2 \right] \right)^{\frac{1}{2}} \leq C \left( k^{\frac{3}{2}} + h + \frac{h^2}{\sqrt{\varepsilon}} \right),
\]

\[
(ii) \quad \left( E \left[ \|R(t_m) - k \sum_{n=1}^{m} r^n_{\varepsilon,h}\|^2 \right] \right)^{\frac{1}{2}} + \left( E \left[ \|P(t_m) - k \sum_{n=1}^{m} p^n_{\varepsilon,h}\|^2 \right] \right)^{\frac{1}{2}} \leq C \left( k^{\frac{1}{2}} + h + \frac{h^2}{\sqrt{\varepsilon}} \right).
\]

### 6. Computational experiments.

We present computational results to validate the theoretical error estimates in Theorems 4.4 and 5.6, and evidence how crucial the numerical treatment of the pressure part in the noise is to obtain an optimally convergent mixed method for (1.1). Our computations are done using the software packages FreeFem++ [14] and Matlab, and the physical domain of all experiments is taken to be \(D = (0, 1)^2\), i.e., \(L = 1\).

Specifically, we use Algorithm 2 to compute the solution of the following initial-(Dirichlet) boundary value problem:

\[
\begin{align*}
(6.1a) \quad d\mathbf{u} &= [\Delta \mathbf{u} - \nabla p] dt + \mathbf{B}(\mathbf{u}) dW(t) \quad \text{in } D_T := (0, T) \times D, \\
(6.1b) \quad \text{div } \mathbf{u} &= 0 \quad \text{in } D_T, \\
(6.1c) \quad \mathbf{u} &= 0 \quad \text{on } \partial D_T := (0, T) \times \partial D, \\
(6.1d) \quad \mathbf{u}(0) &= \mathbf{u}_0 \quad \text{in } D,
\end{align*}
\]

and use Algorithm 4 to compute the solution of the pressure stabilization of the above system which is obtained by replacing (6.1b) by (6.2a)–(6.2b) below.

\[
\begin{align*}
(6.2a) \quad \text{div } \mathbf{u} - \varepsilon \Delta p &= 0 \quad \text{in } D_T, \\
(6.2b) \quad \partial_n p &= 0 \quad \text{on } \partial D_T,
\end{align*}
\]

where \(\partial_n p\) stands for the normal derivative of \(p\).

**Test 1.** Let \(\mathbf{u}_0 = (0, 0)\) and \(\mathbf{B}(u_1, u_2) = \left( (u_1^2 + 1)^{\frac{1}{3}}, (u_2^2 + 1)^{\frac{1}{3}} \right)\), which is non-solenoidal. We choose \(W\) in (1.1) to be a \(\mathbb{R}^d\)-valued Wiener process, with increment

\[
W^J(t_{n+1}, x) - W^J(t_n, x) = k \sum_{j=1}^{J} \sum_{j_1,j_2=1}^{J} \sqrt{\lambda_{j_1,j_2}} g_{j_1,j_2}(x) \xi^m_{j_1,j_2},
\]

where \(x = (x_1, x_2) \in D, \quad \xi^m_{j_1,j_2} \sim N(0, 1), \quad \lambda_{j_1,j_2} = \frac{1}{j_1 + j_2}, \quad \text{and}
\]

\[
g_{j_1,j_2}(x) = 2 \sin(j_1 \pi x_1) \sin(j_2 \pi x_2).
\]
We use the following parameters: \( J = 4 \) and \( T = 1 \), and take \( N_p = 501 \) to be the number of realizations in this test.

Let \( k_0 \) and \( k \) denote the fine and regular time step sizes which are used to generate the numerical true solution and a computed solution, clearly \( k_0 \ll k \). Moreover, \((u_h^n(\tau), p_h^n(\tau))\) denote the numerical solution at the time step \( t_n \) using the time step size \( \tau \); below, \( \tau = k_0 \) or \( k \). For any \( 1 \leq n \leq N \), we use the following numerical integration formulas:

\[
\mathcal{E}_{u,0}^n := \left( \mathbb{E} \left[ \| u(t_n) - u_h^n(k) \|^2 \right] \right)^{\frac{1}{2}} \approx \left( \frac{1}{N_p} \sum_{\ell=1}^{N_p} \| u_h^n(k_0, \omega_\ell) - u_h^n(k, \omega_\ell) \|^2 \right)^{\frac{1}{2}},
\]

\[
\mathcal{E}_{u,1}^n := \left( \mathbb{E} \left[ \| \nabla(u(t_n) - u_h^n(k)) \|^2 \right] \right)^{\frac{1}{2}} \approx \left( \frac{1}{N_p} \sum_{\ell=1}^{N_p} \| \nabla(u_h^n(k_0, \omega_\ell) - u_h^n(k, \omega_\ell)) \|^2 \right)^{\frac{1}{2}},
\]

\[
\mathcal{E}_{p,av}^{N} := \left( \mathbb{E} \left[ \left\| \int_0^T p(s) \, ds - k \sum_{\ell=1}^{N} p_h^n(k) \right\|^2 \right] \right)^{\frac{1}{2}}
\]

\[
\approx \left( \frac{1}{N_p} \sum_{\ell=1}^{N_p} \left\| k_0 \sum_{n=1}^{N} p_h^n(k_0, \omega_\ell) - k \sum_{n=1}^{N} p_h^n(k, \omega_\ell) \right\|^2 \right)^{\frac{1}{2}},
\]

and

\[
\mathcal{E}_{p,0}^n := \left( \mathbb{E} \left[ \| p(t_n) - p_h^n(k) \|^2 \right] \right)^{\frac{1}{2}} \approx \left( \frac{1}{N_p} \sum_{\ell=1}^{N_p} \| p_h^n(k_0, \omega_\ell) - p_h^n(k, \omega_\ell) \|^2 \right)^{\frac{1}{2}}.
\]

The definitions of \( \mathcal{E}_{r,av}^{N} \) and \( \mathcal{E}_{r,0}^{N} \) are similar.

We then implement Algorithm 2 and verify the convergence orders of the time and spatial discretizations proved in Theorem 4.4.

To generate a numerical exact solution for computing the orders of convergence, we use \( k_0 = \frac{1}{500} \) and \( h_0 = \frac{1}{100} \) as fine mesh sizes to compute such a solution. Then, to compute the convergence order of the time discretization for the velocity, we fix \( h = \frac{1}{400} \) and then compute the numerical solution with following time mesh sizes: \( k = \frac{1}{5}, \frac{1}{10}, \frac{1}{20}, \frac{1}{40} \). The errors in the \( L^2 \)-norm (\( \mathcal{E}_{u,0}^n \)) and \( H^1 \)-norm (\( \mathcal{E}_{u,1}^n \)) are shown in Table 6.1. The numerical results verify the convergence order \( O(k^2) \) which is stated in Theorem 4.4.

| \( k \)     | \( \mathcal{E}_{u,0}^n \) | order | \( \mathcal{E}_{u,1}^n \) | order |
|------------|-------------------|------|-------------------|------|
| 1/5        | 0.16253           |      | 0.25558           |      |
| 1/10       | 0.11521           | 0.496| 0.18050           | 0.5018|
| 1/20       | 0.08145           | 0.5002| 0.12580           | 0.5209|
| 1/40       | 0.05730           | 0.5073| 0.08758           | 0.5225|

Table 6.1: Time discretization errors for the velocity \( \{ u_h^n \}_n \).

Tables 6.2 and 6.3 display respectively the \( L^2 \)-norm errors \( \mathcal{E}_{r,av}^{N} \) and \( \mathcal{E}_{r,0}^{N} \) \((\alpha = r \text{ and } p)\) of the time-averaged pressure approximations using time mesh sizes: \( k = \frac{1}{5}, \frac{1}{10}, \frac{1}{20}, \frac{1}{40} \). The numerical results indicate the convergence rate \( O(k^2) \) which
was predicted in Theorem 4.4. We also present the standard $L^2$-norm errors $E_r^N$ and $E_p^{N,0}$ in Table 6.2 and 6.3 respectively for comparison purposes, for which we observe a significantly slower rate. It should be noted that our convergence theory does not conclude such a convergence behavior.

| $k$  | $E_{r,av}^N$ | order | $E_{r,0}^N$ | order |
|------|--------------|-------|-------------|-------|
| 1/5  | 0.06352      |       | 0.08013     |       |
| 1/10 | 0.04486      | 0.519 | 0.06231     | 0.3629|
| 1/20 | 0.03161      | 0.504 | 0.04842     | 0.3639|
| 1/40 | 0.02219      | 0.510 | 0.03734     | 0.3745|

Table 6.2
Algorithm 2: Time discretization errors for the pressure $\{r^h\}_n$.

| $k$  | $E_{p,av}^N$ | order | $E_{p,0}^N$ | order |
|------|--------------|-------|-------------|-------|
| 1/5  | 0.00217      |       | 0.0967      |       |
| 1/10 | 0.00154      | 0.494 | 0.0722      | 0.3211|
| 1/20 | 0.00109      | 0.498 | 0.0579      | 0.3184|
| 1/40 | 0.00077      | 0.501 | 0.0461      | 0.3288|

Table 6.3
Algorithm 2: Time discretization errors for the pressure approximation $\{p^h\}_n$.

To verify the convergence rate $O(h)$ for the velocity approximation, we fix $k = \frac{1}{200}$ and use different spatial mesh sizes $h = \frac{1}{5}, \frac{1}{10}, \frac{1}{20}, \frac{1}{40}$ to compute the errors $E_{u,0}^N$ and $E_{u,1}^N$. Table 6.4 contains the computational results which verify first order convergence rate for both as stated in Theorem 4.4.

| $h$  | $E_{u,0}^N$ | order | $E_{u,1}^N$ | order |
|------|-------------|-------|-------------|-------|
| 1/5  | 0.07981     |       | 0.50832     |       |
| 1/10 | 0.04034     | 0.984 | 0.25315     | 1.0057|
| 1/20 | 0.02016     | 1.000 | 0.12662     | 0.9995|
| 1/40 | 0.01007     | 1.001 | 0.06322     | 1.0021|

Table 6.4
Algorithm 2: Spatial discretization errors for the velocity approximation $\{u^h\}_n$.

To verify the convergence rate for the pressure approximation, we fix $k = \frac{1}{200}$ and use different spatial mesh sizes: $h = \frac{1}{5}, \frac{1}{10}, \frac{1}{20}, \frac{1}{40}$. Tables 6.5 and 6.6 display the error $E_{p,av}^N$ of the pressure approximation. It is evident that $E_{p,av}^N$ converges linearly in $h$ as stated in Theorem 4.4. For comparison purposes, we also compute the error $E_{p,0}^N$ and include it in Table 6.5 and 6.6. The numerical results suggest that the error $E_{p,0}^N$ converges with a slower rate.

Test 2. In this test, we use Algorithm 2 to solve the driven cavity problem with stochastic forcing, which is described by system (1.1a)–(1.1b) with the following non-homogeneous boundary condition:

$$u(x_1, x_2) = \begin{cases} 
  (1, 0), & x_2 = 1, 0 < x_1 < 1, \\
  0, & \text{otherwise}.
\end{cases}$$
Let $W$ be the Wiener process as in (6.3), and $\mathbf{B} \equiv (1, 1)^T$, i.e., the noise is additive. We use the following parameters in the test: $T = 1$, $h = \frac{1}{20}$, $k = 0.01$, and the number of the realizations is $N_p = 1001$.

Figure 6.1 displays the expected values of the computed stochastic velocity $\mathbf{u}_h^N$ and pressure $p_h^N$; expectedly, they behave similarly as their deterministic counterparts do. On the other hand, individual realizations of the computed stochastic velocity $\mathbf{u}_h^N$ and pressure $p_h^N$ given in Figures 6.2–6.4 show quite different behaviors from their deterministic counterparts.

![Fig. 6.1. Test 2. (a) The expected value of $\{\mathbf{u}_h^N\}_n$. (b) Level-lines of the expected value of $\{p_h^N\}_n$. (c) The streamlines of the expected value of $\{\mathbf{u}_h^N\}_n$.](image)

**Test 3.** In this test, we study the stabilization method in section 5. Specifically, we implement Algorithm 4 with the same function $\mathbf{B}$ as in Test 1, and $\{W(t); 0 \leq t \leq T\}$ is chosen as an $\mathbb{R}$-valued Wiener process. We also add a constant forcing term $\mathbf{f} \equiv (1, 1)^T$ to $\mathbf{(1.1)}$ in order to construct an exact solution to system $\mathbf{(1.1)}$. We also take $\mathbf{u}_0 = (0, 0)$, $T = 1$, the number of realizations $N_p = 800$, and the minimum time step $k_0 = \frac{1}{4096}$. The computations are done on a uniform mesh of $D$ with the mesh size $h = \frac{1}{100}$.

In order to verify the optimal convergence rate $O(h)$ of Theorem 5.4, we fix $k = h = \frac{1}{256}$ and $\varepsilon = h^2$, and then compute the numerical solutions for different values of $h$. The standard $L^2$-errors $\mathcal{E}_{\mathbf{u}_h^0}$ and $\mathcal{E}_{p_h^0}$ for the velocity and pressure approximations are presented in Table 6.7. The numerical results verify the first order convergence rate for the spatial approximation of the velocity as stated in Theorem 5.4.
For comparison purposes, we also implement the ‘standard’ stabilization method, which is based on (1.2) instead of (1.9a)–(1.9b), with the same noise and parameters as above. Table 6.8 displays the $L^2$-errors $\mathcal{E}^{N}_{u,0}$ and $\mathcal{E}^{N}_{p,0}$ of the velocity and pressure approximations. The numerical results indicate that the velocity approximation is also convergent but at a slower rate. This confirms the advantages of the proposed Helmholtz decomposition enhanced stabilization method (Algorithm 4) over the ‘standard’ stabilization method.

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Fig. 6.4. Third realization of (a) the velocity \( \{u^N_n\} \); (b) the pressure \( \{p^N_n\} \); (c) the streamline of \( \{u^N_n\} \).

| \( h \) | \( \mathcal{E}^N_{u,0} \) | order | \( \mathcal{E}^N_{p,0} \) | order |
|-------|----------------|-------|----------------|-------|
| 1/5   | 0.018392       |       | 0.147406       |       |
| 1/10  | 0.009083       | 1.0178| 0.092913       | 0.6658|
| 1/20  | 0.004095       | 1.1493| 0.052611       | 0.8205|
| 1/40  | 0.002279       | 0.8454| 0.044723       | 0.2344|

Table 6.7

Algorithm 4: Spatial discretization errors for the velocity \( \{u^0_{\varepsilon,h}\} \) and pressure \( \{p^0_{\varepsilon,h}\} \).

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Table 6.8

| $h$     | $\mathcal{E}_{u,0}^n$ | order | $\mathcal{E}_{p,0}^n$ | order |
|---------|-----------------------|-------|-----------------------|-------|
| 1/5     | 0.037658              |       | 0.735843              |       |
| 1/10    | 0.025586              | 0.5576| 0.888352              | -0.2717|
| 1/20    | 0.019342              | 0.4036| 0.579818              | 0.6155 |
| 1/40    | 0.011412              | 0.7611| 0.442691              | 0.3893 |

*Standard stabilization method: Spatial discretization errors for the velocity $\{u_{n,h}^n\}$ and pressure $\{p_{n,h}^n\}$.***