THE RUNNING COUPLING METHOD WITH NEXT-TO-LEADING ORDER ACCURACY AND PION, KAON ELM FORM FACTORS

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Abstract

The pion and kaon electromagnetic form factors $F_M(Q^2)$ are calculated at the leading order of pQCD using the running coupling constant method. In calculations the leading and next-to-leading order terms in $\alpha_S((1-x)(1-y)Q^2)$ expansion in terms of $\alpha_S(Q^2)$ are taken into account. The resummed expression for $F_M(Q^2)$ is found. Results of numerical calculations for the pion (asymptotic distribution amplitude) are presented.

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During last years a considerable progress were made in understanding of infrared (ir) renormalon effects in various inclusive and exclusive processes [1-4]. It is well known that all-order resummation of ir renormalons corresponds to the calculation of the one-loop Feynman diagrams with the running coupling constant $\alpha_S(-k^2)$ at the vertices, or to calculation of the same diagrams with non-zero gluon mass. Both these approaches are generalization of the Brodsky, Lepage and Mackenzie (BLM) scale-setting method [5]. Studies of ir renormalon effects have also opened interesting prospects for evaluation of higher twist corrections to processes' different characteristics [6].

As was proved in our works [2-4], exclusive processes have additional source of ir renormalon corrections. Indeed, integration over longitudinal fraction of hadron constituents in the expression, for example, of the electromagnetic (elm) form factor generates ir renormalon effects.

In this letter we extend our previous consideration of the pion and kaon elm form factors in the context of the running coupling constant method.

A meson $M$ form factor in the framework of pQCD has the following form [7]

$$F_M(Q^2) = \int_0^1 \int_0^1 dx dy \phi_M^* (y, Q^2) T_H \left( x, y; Q^2, \alpha_S \left( \hat{Q}^2 \right) \right) \phi_M (x, Q^2).$$  \hspace{1cm} (1)

Here $Q^2 = -q^2$ is the square of the virtual photon’s four-momentum, $\phi_M (x, Q^2)$ is the meson distribution amplitude. In Eq.(1) $T_H \left( x, y; Q^2, \alpha_S \left( \hat{Q}^2 \right) \right)$ is the hard-scattering amplitude of the subprocess $q\bar{q}^* + \gamma^* \rightarrow q\bar{q}$, which at the leading order of pQCD is [7]

$$T_H \left( x, y; Q^2, \alpha_S \left( \hat{Q}^2 \right) \right) = \frac{16\pi C_F}{Q^2} \left[ \frac{2}{3} \frac{\alpha_S ((1-x)(1-y)Q^2)}{(1-x)(1-y)} + \frac{1}{3} \frac{\alpha_S(xyQ^2)}{xy} \right].$$  \hspace{1cm} (2)

In Eq.(2) $C_F = \frac{4}{3}$ is the color factor, $\hat{Q}^2$ is taken as the square of the momentum transfer of the exchanged hard gluon in corresponding Feynman diagrams for $F_M(Q^2)$. Unlike the frozen coupling approximation, where for calculation of $F_M(Q^2)$ one fixes the argument of $\alpha_S ((1-x)(1-y)Q^2) \rightarrow \alpha_S (Q^2/4)$, in the running coupling method we use Eq.(2) in Eq.(1) and, as a result, encounter with new troubles. Indeed, it is evident that $\alpha_S \left( \hat{Q}^2 \right)$ in Eq.(2) suffers from ir singularities associated with the behaviour of the $\alpha_S \left( \hat{Q}^2 \right)$ in the regions $x \rightarrow 0, y \rightarrow 0; x \rightarrow 1, y \rightarrow 1$. Therefore, $F_M(Q^2)$ can be found only after proper regularization of $\alpha_S \left( \hat{Q}^2 \right)$ in these end-point regions. For solving of this problem let us relate the running coupling $\alpha_S (\lambda Q^2)$ in terms of $\alpha_S (Q^2)$ by means of the renormalization group equation. The renormalization group equation for the running coupling $\alpha \equiv \alpha_S / \pi$ has the form

$$\frac{\partial \alpha(\lambda Q^2)}{\partial \ln \lambda} = -b_2 \left[ \alpha(\lambda Q^2) \right]^2 - b_3 \left[ \alpha(\lambda Q^2) \right]^3,$$  \hspace{1cm} (3)
where
\[
b_2 = \frac{1}{12} (33 - 2n_f), \quad b_3 = \frac{1}{48} (306 - 38n_f).
\]
The solution of Eq.(3) with initial condition \(\alpha(\lambda) |_{\lambda=1} = \alpha \equiv \alpha_S(Q^2)/\pi\) is [8]
\[
\frac{\alpha(\lambda)}{\alpha} = \left[ 1 + \alpha b_2 \ln \lambda - \frac{\alpha b_3}{b_2} \left( \ln \alpha(\lambda) - \ln \frac{b_2/b_3 + \alpha(\lambda)}{b_2/b_3 + \alpha} \right) \right]^{-1}. \tag{4}
\]
This transcendental equation can be solved iteratively by keeping the leading \(\alpha^k \ln^k \lambda\) and next-to-leading \(\alpha^k \ln^{k-1} \lambda\) powers. For \(\lambda = (1-x)(1-y)\) these terms are given by
\[
\alpha \left( (1-x)(1-y)Q^2 \right) \simeq \frac{\alpha(Q^2)}{1 + \alpha(Q^2)b_2 \ln(1-x)(1-y)}
- \frac{\alpha^2(Q^2)b_3 \ln [1 + \alpha(Q^2)b_2 \ln(1-x)(1-y)]}{b_2 \left[ 1 + \alpha(Q^2)b_2 \ln(1-x)(1-y) \right]^2}. \tag{5}
\]
The first term in Eq.(5) is the solution of the renormalization group equation (3) with leading power accuracy, whereas the whole expression (5) is the solution of Eq.(3) with next-to-leading power accuracy.

In our previous papers [3,4] for calculation of the pion and kaon form factors we used only the first term from Eq.(5). In this work for evaluation of \(F_M(Q^2)\) we use both of them. Let us clarify our approach by considering the pion form factor and pion’s simplest distribution amplitude \(\phi_{asy}(x)\)
\[
\phi_{asy}(x) = \sqrt{3} f_{\pi} x(1-x). \tag{6}
\]
Generalization of obtained results for the pion’s other distribution amplitudes as well as for the kaon is straightforward.

After substitution of Eqs.(5),(6) into Eq.(1), the pion form factor takes the form
\[
Q^2 F_{\pi}(Q^2) \simeq \frac{(8\pi f_{\pi})^2}{b_2} \int_0^1 \int_0^1 \frac{xydxdy}{t + \ln(1-x) + \ln(1-y)}
- \frac{(8\pi f_{\pi})^2}{b_2} \int_0^1 \int_0^1 \frac{xy \{ \ln [t + \ln(1-x) + \ln(1-y)] - \ln t \} dxdy}{[t + \ln(1-x) + \ln(1-y)]^2}. \tag{7}
\]
where \(t = 1/\alpha b_2\).

The resummed expression for \(F_{\pi}(Q^2)\) can be found using an approach advocated in Ref.[3]. For these purposes it is instructive to change variables from \(x, y\) in Eq.(7) to \(z = \ln(1-x), w = \ln(1-y)\) and apply the inverse Laplace transforms [9]
\[
\frac{1}{(t + z + w)^\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty \exp \{-u (t + z + w)\} u^{\nu-1}du, \quad Re \nu > 0, \tag{8}
\]
and
\[ \ln \left( \frac{t + z + w}{(t + z + w)^2} \right) = \int_0^\infty \exp \left[ -u (t + z + w) \right] (1 - C - \ln u) u \, du. \] (9)

In Eq.(9) \( C \approx 0.577216 \) is the Euler-Mascheroni constant. After some manipulations one gets

\[ Q^2 F_\pi(Q^2) \approx \frac{(8\pi f_\pi)^2}{b_2^2} \int_0^\infty \exp(-tu) B \left[ Q^2 F_\pi \right] (u) du \]
\[ - \left( \frac{8\pi f_\pi}{b_2} \right)^2 \frac{b_3}{b_2^2} \int_0^\infty \exp(-tu) u \left[ 1 - C - \ln t - \ln u \right] B \left[ Q^2 F_\pi \right] (u) du, \] (10)

where \( B \left[ Q^2 F_\pi \right] (u) \) is given by expression
\[ B \left[ Q^2 F_\pi \right] (u) = \frac{1}{(1 - u)^2} + \frac{1}{(2 - u)^2} - \frac{2}{1 - u} + \frac{2}{2 - u}, \] (11)

and is the Borel transform of the pion elm form factor obtained in our work [3]. It is worth noting that Eq.(10) is the general expression valid for both the pion and kaon; one needs only to replace in Eq.(10) the Borel transform \( B \left[ Q^2 F_\pi \right] (u) \) and \( f_\pi \) with corresponding ones \( B \left[ Q^2 F_K \right] (u) \) and \( f_K \) from Ref.[4]. The first term in Eq.(10) is the pion form factor found in Ref.[3] at the leading order of pQCD in the framework of the running coupling method using only the first term from Eq.(5). Of course, the second term in Eq.(10) is the contribution to the form factor coming from the second term of Eq.(5). But interpretation of the integrand in this term (without \( \exp(-tu) \)) as a "traditional" Borel transform of corresponding perturbative series is problematical, because some regularization prescription at \( u = 0 \) for recovering of perturbative series is needed. Here we bypass this problem and concentrate on calculation of the resummed expression for \( F_M(Q^2) \). As we shall see later, integration in Eq.(10) using the principal value prescription [8,10] removes ir renormalon poles at \( u = 1, 2... \) and gives correct results for \( [Q^2 F_M(Q^2)]^{res} \).

For the pion we get
\[ [Q^2 F_\pi(Q^2)]^{res} = [Q^2 F_\pi(Q^2)]^{res}_1 - \left( \frac{8\pi f_\pi}{b_2} \right)^2 \frac{b_3}{b_2^2} \{ (C + \ln t) [f_1(0) + f_2(0)] \]
\[ + (1 - C - \ln t) [(t - 2)f_1(1) + (t + 2)f_2(1)] + g_1(0) + g_2(0) \]
\[ - (t - 2)g_1(1) - (t + 2)g_2(1) \}. \] (12)
Here $[Q^2 F_\pi(Q^2)]_1^{\text{res}}$ is the resummed elm form factor of the pion obtained in Ref.[3] by means of the leading term of Eq.(5),

$$[Q^2 F_\pi(Q^2)]_1^{\text{res}} = \left(\frac{8\pi f_\pi}{b_2}\right)^2 \left[ -\frac{3}{2} + (t-2)f_1(0) + (t+2)f_2(0) \right]. \tag{13}$$

In Eqs.(12,13) we introduce following notations,

$$f_k(n) = \text{P.V.} \int_0^\infty \frac{\exp(-tu)u^n du}{k-u},$$

$$g_k(n) = \text{P.V.} \int_0^\infty \frac{\exp(-tu)\ln(u)u^n du}{k-u}. \tag{14}$$

For the pion’s Chernyak-Zhitnitsky (CZ) distribution amplitude [11] in the form

$$\phi_{\text{CZ}}(x) = 5\phi_{\text{asy}}(x) (2x-1)^2,$$

we get

$$[Q^2 F_\pi(Q^2)]_1^{\text{res}} = [Q^2 F_\pi(Q^2)]_1^{\text{res}} - \left(\frac{40\pi f_\pi}{b_2}\right)^2 \frac{b_2}{b_2} \left\{ (1-C-\ln t) \left[ (t-\frac{14}{3})f_1(1) 
+ 25(t-2)f_2(1) + 8(8t+1)f_3(1) + 4(4t+\frac{35}{3})f_4(1) \right] 
+ (C+\ln t)[f_1(0) + 25f_2(0) 
+ 64f_3(0) + 16f_4(0)] + g_1(0) + 25g_2(0) + 64g_3(0) + 16g_4(0) - (t-\frac{14}{3})g_1(1) 
- 25(t-2)g_2(1) - 8(8t+1)g_3(1) - 4(4t+\frac{35}{3})g_4(1) \right\},$$

with

$$[Q^2 F_\pi(Q^2)]_1^{\text{res}} = \left(\frac{40\pi f_\pi}{b_2}\right)^2 \left[ -\frac{233}{6} + (t-\frac{14}{3})f_1(0) + 25(t-2)f_2(0) 
+ 8(8t+1)f_3(0) + 4(4t+\frac{35}{3})f_4(0) \right].$$

Expressions for the resummed pion form factor obtained using more general than the asymptotic and CZ distribution amplitudes [12] are rather lengthy and will be published elsewhere.
For the kaon we find

\[
\left[ Q^2 F_K(Q^2) \right]_{\text{res}} = \left[ Q^2 F_K(Q^2) \right]_1
\]

\[
- \left( \frac{40\pi f_\pi}{b_2} \right)^2 \frac{b_3}{b_2} \sum_{k=1}^{5} \left\{ (1 - C - \ln t) \left( t m_k + n_k \right) f_k(1)
\right. \\
+ \left. (C + \ln t) m_k f_k(0) - (t m_k + n_k) g_k(1) + m_k g_k(0) \right\}.
\]

The values of \((m_k, n_k)\) as well as \([Q^2 F_K(Q^2)]_{\text{res}}\) can be found in Ref.[4].

As an example, results of numerical calculations for \([Q^2 F_\pi(Q^2)]_{\text{res}}\) carried out using the pion’s asymptotic distribution amplitude are shown in Fig.1. Here the ratio \(R = [Q^2 F_\pi(Q^2)]_{\text{res}}/[Q^2 F_\pi(Q^2)]^0\), where \([Q^2 F_\pi(Q^2)]^0\) is the pion form factor calculated in the frozen coupling approximation using the same distribution amplitude, is depicted. As is seen, the next-to-leading order correction changes the shape of the curve in the region of small \(Q^2\) (2 GeV\(^2\) \(\leq Q^2 < 6\) GeV\(^2\)), where the whole result is smaller than the leading order one. At \(Q^2 = 2\) GeV\(^2\) the correction amounts to \(\sim 15\%\) of the leading order result decreasing toward \(Q^2 = 6\) GeV\(^2\). At the end of the considered values of \(Q^2\) the next-to-leading order correction is positive and equals to 3 – 6\% of the leading order result. The similar calculations can be also fulfilled for the pion using its alternative distribution amplitudes [12] and for the kaon.

As was pointed out in Refs.[3-5], the ir renormalon corrections can be hidden into the scale of \(\alpha_S(Q^2)\) at the leading order form factor \([Q^2 F_M(Q^2)]^0\). In the studied case of the pion (asymptotic distribution amplitude) we find

\[
\alpha_S(Q^2) \rightarrow \alpha_S(e^{f(Q^2)} Q^2),
\]

\[
f(Q^2) \simeq -6.71 + 16.67\alpha_S(Q^2), \tag{15}
\]

where in numerical fitting we have used Eq.(12).

It is known [3,4] that ir renormalon effects enhance the perturbative predictions for the pion, kaon form factors approximately two times. Our recent studies confirm that next-to-leading order term in Eq.(5) does not change the picture considerably. This feature of ir renormalons may help one in solution of a contradiction between theoretical interpretations of experimental data for the photon-to-pion transition form factor \(F_{\gamma\pi}(Q^2)\) [13] from one side and for the pion form factor \(F_\pi(Q^2)\) [14] from another side. Thus, in works [15,16] the authors noted that the scaling and normalization of the photon-to-pion transition form factor tends to favor of the pion asymptotic-like distribution amplitude. But then prediction for \(F_\pi(Q^2)\) obtained using the same distribution amplitude is lower than the data by approximately a factor of 2. We think that in this discussion a crucial point is a chosen method of integration in Eq.(1). Indeed, unlike \(F_\pi(Q^2)\) the expression...
for $F_{\gamma\pi}(Q^2)$ at the leading order of pQCD does not contain an integration over $\alpha_S(xQ^2)$. In other words, the running coupling method being applied to Eq.(1) and to the expression for $F_{\gamma\pi}$ (see, Refs.[15,16]) enhances the perturbative result for the pion elm form factor and, at the same time, does not change $F_{\gamma\pi}$. This allows us to suppose that in the context of pQCD the same pion distribution amplitude may explain experimental data for both $F_{\gamma\pi}(Q^2)$ and $F_{\pi}(Q^2)$. This problem is a subject of separate publication.

\footnote{Our statements concerning $F_{\gamma\pi}(Q^2)$ remain true also in the light of ir renormalon corrections computed for $F_{\gamma\pi}$ in Ref.[17], because they have another source than ones considered in our work, namely, Feynman diagrams with bubble chains. At the leading order of pQCD the only source of ir renormalons is the running coupling constant and integration in Eq.(1) over longitudinal momenta of a meson $M$.}
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FIGURE CAPTION

Fig. 1 The ratio \( R = \left[ Q^2 F_\pi(Q^2) \right]^{\text{res}} / \left[ Q^2 F_\pi(Q^2) \right]^{0} \) as a function of \( Q^2 \). In calculation the pion asymptotic distribution amplitude \( \phi_{\alpha \gamma}(x) \) is used. The curve 1 is \( R \) found using the whole result for the resummed form factor Eq.(12), whereas the dashed curve is the ratio \( R = \left[ Q^2 F_\pi(Q^2) \right]^{\text{res}} / \left[ Q^2 F_\pi(Q^2) \right]^{0} \). The curve 2 corresponds to \( R \) with \( [Q^2 F_\pi(Q^2)]^{0} \) in the frozen coupling approximation and after scale-setting procedure (15). In calculations the QCD scale parameter \( \Lambda \) has been taken equal to 0.1 GeV.
