The condition number of a randomly perturbed matrix

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ABSTRACT
Let $M$ be an arbitrary $n \times n$ matrix. We study the condition number a random perturbation $M + N_n$ of $M$, where $N_n$ is a random matrix. It is shown that, under very general conditions on $M$ and $N_n$, the condition number of $M + N_n$ is polynomial in $n$ with very high probability. The main novelty here is that we allow $N_n$ to have discrete distribution.

1. INTRODUCTION
1.1 The condition number
Let $M$ be an $n \times n$ matrix,
$$\sigma_1(M) := \sup_{x \in \mathbb{R}^n, \|x\| = 1} \|Mx\|$$
is the largest singular value of $M$ (this parameter is also often called the operator norm of $M$).
If $M$ is invertible, the condition number $\kappa(M)$ is defined as
$$\kappa(M) := \sigma_1(M)/\sigma_1(M^{-1}).$$

The condition number plays a crucial role in numerical linear algebra. The accuracy and stability of most algorithms used to solve the equation $Mx = b$ depend on $\kappa(M)$. The exact solution $x = M^{-1}b$ in theory, can be computed quickly (by Gaussian elimination, say). However, in practice computers can only present a finite subset of real numbers and this leads to two difficulties. The represented numbers cannot be arbitrary large of small, and there are gaps between them. A quantity which is frequently used in numerical analysis is $\epsilon_{\text{machine}}$ which is half of the distance from 1 to the nearest represented number. A fundamental result in numerical analysis [1] asserts that if one denotes by $\tilde{x}$ the result computed by computers, then the relative error $\|\tilde{x} - x\|/\|x\|$ satisfies
$$\frac{\|\tilde{x} - x\|}{\|x\|} = O(\epsilon_{\text{machine}} \kappa(M)).$$

We call $M$ well conditioned if $\kappa(M)$ is small. For quantitative purposes, we say that an $n \times n$ matrix $M$ is well conditioned if its condition number is polynomially bounded in $n$ ($\kappa(M) \leq n^C$ for some constant $C$ independent of $n$).

1.2 Effect of noise
An important issue in the theory of computing is noise, as almost all computational processes are effected by it. By the word noise, we would like to represent all kinds of errors occurring in a process, due to both humans and machines, including errors in measuring, errors caused by truncations, errors committed in transmitting and inputting the data, etc.

It happens frequently that while we are interested in a solving a certain equation, because of the noise the computer actually ends up with solving a slightly perturbed version of it. Our work is motivated by the following phenomenon, proposed by Spielman and Teng [9]

P1: For every input instance it is unlikely that a slight random perturbation of that instance has large condition number.

If the input is a matrix, we can reformulate this in a more quantitative way as follows

P2: Let $M$ be an arbitrary $n \times n$ matrix and $N_n$ a random $n \times n$ matrix. Then with high probability $M + N_n$ is well conditioned.

The crucial point here is that $M$ itself may have large condition number. The above phenomenon gives an explanation to the fact (which has been observed numerically for some time—see [5]) that one rarely encounters ill-conditioned matrices in practice. This is also the core of Spielman-Teng smooth analysis which we will discuss in more details in Section 4.

The goal of this paper is to show that under very general assumptions on $M$ and $N_n$, $M + N_n$ indeed has small condition number with overwhelming probability. The main novelty here is that we allow the random matrix $N_n$ to have discrete distribution. This is a natural assumption for random variables involved in digital processes. On the other

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hand, very little has been known, prior to this paper, about this case. Random discrete matrices are indeed much more
difficult to analyze than their continuous counterparts and
our analysis is significantly different from those used earlier
for the continuous models. In particular, it relies heavily on
a new development in additive combinatorics, the so-called
Inverse Littlewood-Offord theory (see Section 3).

1.3 A necessary assumption
Suppose that we would like to show that \( M + N_n \) is well
conditioned. This requires to bound both \( \| M + N_n \| \) and
\( \| (M + N_n)^{-1} \| \) by a polynomial in \( n \). Let us look at the first
norm. By the triangle inequality

\[
\| M \| - \| N_n \| \leq \| M + N_n \| \leq \| M \| + \| N_n \|.
\]

In most models for random matrices, \( \| N_n \| \) is \( O(\sqrt{n}) \) with
very high probability. Thus \( \| M + N_n \| \) is often dominated by \( \| M \| \). So in order to make \( \kappa(M + N_n) = n^{O(1)} \), it is
natural to assume that \( \| M \| = n^{O(1)} \). In fact, as

\[
\| M \|^2 = \sigma_1^2 \leq \sum_{ij} m_{ij}^2 = \sum_{i=1}^n \sigma_i^2 \leq n \sigma_1^2 = n \| M \|^2,
\]

where \( m_{ij} \) are the entries of \( M \), this assumption is equivalent
to saying that all entries of \( M \) are polynomially bounded.
We will make this assumption about \( M \) in the rest of the
paper. The main task now is to bound the second norm, \( \| (M + N_n)^{-1} \| \), from above.

2. THE RESULTS

2.1 Continuous noise
The case when entries of \( N_n \) are i.i.d Gaussian random vari-
able (with mean zero and variance one) has been studied
by various authors \[3,8\]. In particular, Sankar, Spielman
and Teng \[8\] proved

**Theorem 2.2.** Let \( M \) be an arbitrary \( n \times n \) matrix.
Then for any \( \epsilon > 0 \),

\[
P(\| (M + N_n)^{-1} \| \geq x) = O(\sqrt{n} x).
\]

It is well known that there are positive constants \( c_1 \) and \( c_2 \)
such that \( P(\| N_n \| \geq c_1 \sqrt{n}) \leq e^{-c_2 n} \).

**Corollary 2.3.** Let \( B > C + 3/2 \) be positive constants.
Let \( M \) be an arbitrary \( n \times n \) matrix whose entries have
absolute value at most \( n^C \). Then

\[
P(\kappa(M + N_n) \geq n^B) = O(n^{-B+C+3/2}).
\]

**Proof.** By the assumption on \( M \) and the fact about
\( \| N_n \|, \| M + N_n \| = O(n^{C+1}) \) with probability \( 1 - \exp(-\Omega(n)) \).
By Theorem 2.2, \( \| (M + N_n)^{-1} \| \leq n^{-B-C-1} \) with probability
\( O(n^{-B+3+C+1/2}) \). Thus the claim follows by the union
bound. \( \square \)

2.4 Discrete noise: Bernoulli case
Let us now consider random variables with discrete sup-
ports. By rescaling, we can assume that their supports lie
on \( \mathbb{Z} \) or \( \mathbb{Z}^d \) for some \( d \). The most basic model among ran-
dom discrete matrices is the Bernoulli matrix, whose entries
are i.i.d Bernoulli random variables (taking values \(-1\) and
\( 1 \) with probability \( 1/2 \)).

Bounding the norm of the inverse of a random discrete ma-
trix is a difficult task, and the techniques used for the con-
tinuous case are no longer applicable. In fact, it is already
not trivial to prove that a random Bernoulli matrix is al-
most surely invertible. Efficient bounds on the norm of the
inverse of a Bernoulli random matrix were obtained only
very recently \[7,12\].

Our first result here is the discrete analogue of Theorem
2.5, where the Gaussian noise is replaced by the Bernoulli
noise.

**Theorem 2.5.** For any constants \( A \) and \( C \) there is a con-
stant \( B \) such that the following holds. Let \( M \) be an integer \( n \times n \) matrix whose entries (in absolute values) are bounded from above by \( n^C \) and \( N_n \) be the \( n \times n \) random Bernoulli matrix. Then

\[
P(\| (M + N_n)^{-1} \| \geq n^B) \leq n^{-A}.
\]

**Corollary 2.6.** For any constants \( A \) and \( C \) there is a constant \( B \) such that the following holds. Let \( M \) be an
arbitrary \( n \times n \) matrix whose entries (in absolute values) are bounded from above by \( n^C \) and \( N_n \) be the \( n \times n \) random
Bernoulli matrix. Then

\[
P(\kappa(M + N_n) \geq n^B) \leq n^{-A}.
\]

**Remark 2.7.** It is useful to have the right hand side be
\( n^{-A} \) rather than just \( o(1) \). The reason is that in certain
applications (see for instance Section 3), we need to show
that polynomially many matrices have, simultaneously, small
condition numbers. The bound \( n^{-A} \) guarantees that we can
achieve this by a straightforward union-bound argument.

Theorem 2.5 is a special case of a general theorem, which,
among others, asserts that the same conclusion still holds when
we replace the Bernoulli random variable by arbitrary
symmetric random discrete variables. We present this the-
orem in the next subsection.

2.8 Arbitrary discrete noise
**Notation.** For a real number \( x \), we use \( e(x) \) to denote

\[
\exp(2\pi i x) = \cos 2\pi x + i \sin 2\pi x.
\]

**Definition 2.9.** Let \( \mu \leq \frac{1}{2} \) and \( D \) be positive con-
stants. A random variable \( \xi \) is \((\mu, D)\)-bounded if there is an
integer \( 1 \leq k \leq D \) such that for any \( t \)

\[
|E(e(\xi t))| \leq (1 - \mu) + \mu \cos 2\pi t.
\]

A random vector (matrix) is \((\mu, D)\)-bounded if its coordi-
nates (entries) are independent \((\mu, D)\)-bounded random vari-
able.

**Remark 2.10.** We need to assume \( \mu \leq \frac{1}{2} \) to guarantee
that \( 1 - \mu + \mu \cos 2\pi t \) is non-negative for all \( t \).

**Theorem 2.11.** For any positive constants \( \mu \leq \frac{1}{2}, D \) and there is a constant \( B \) such that the following holds.
Let \( M \) be a fixed integer \( n \times n \) matrix whose entries have
absolute values at most \( n^C \). Let \( N_n \) be an \( n \times n \) \((\mu, D)\)
bounded random matrix whose entries have absolute values
at most \( n^C \) (with probability one). Then

\[
P(\|M + N_n\| \leq n^{-B}) \leq n^{-A}.
\]
Remark 2.12. It is useful to note that the entries of $N_n$ are not required to have the same distribution. This allows the possibility that the noise at a certain location has a correlation with the corresponding entry of the original matrix $M$. For instance, it might be natural to expect that the noise occurring to a larger entry of $M$ have larger variance.

The following lemma provides a sufficient condition for $(\mu, D)$-boundedness.

**Lemma 2.13.** Let $\xi$ be a symmetric discrete random variable and assume that there is a positive integer $s$ such that $P(\xi = s) \geq \epsilon$. Then $\xi$ is $(\epsilon/2, 2s)$-bounded.

**Proof.** (Proof of Lemma 2.13) By the symmetry of $\xi$ and the triangle inequality

$$|E(e(\xi t))| = |\sum_{m=1}^{\infty} P(\xi = m) \cos 2\pi mt| \leq (1-2\epsilon) + 2\epsilon \cos 2\pi st.$$

Using the elementary inequality $|\cos x| \leq \frac{1}{2} + \frac{1}{2} \cos 2x$ with $x = 2\pi st$, we have

$$(1 - 2\epsilon) + 2\epsilon \cos 2\pi st \leq (1 - \frac{\epsilon}{2}) + \frac{\epsilon}{2} \cos 4\pi st,$$

concluding the proof. \qed

With this lemma, one can easily check that most basic variables are $(\mu, D)$-bounded for some constants $\mu$ and $D$. Let us list a few examples:

- (Bernoulli) $\xi$ is 1 or $-1$ with probability 1/2. We can take $\epsilon = 1/2$ and $s = 1$.
- (Lazy coin flip) $\xi = 0$ with probability $1 - \alpha$ and 1 or $-1$ with probability $\alpha/2$. We can take $\epsilon = \alpha/2$ and $s = 1$.
- (Discretized Gaussian) Define $\xi$ as follows: $P(\xi = m) = P(m - 1/2 \leq \Xi \leq m + 1/2)$, where $\Xi$ is standard Gaussian. We can take $\epsilon = P(1/2 \leq \Xi \leq 3/2)$ and $s = 1$.
- As a generalization of the previous example, one can consider the discretization of any symmetric random variable.

**2.14 The general result**

Now we are going to present an even more general result, which implies Theorem 2.11. In this result, we do not require that the entries of the random matrix be independent.

**Definition 2.15.** Let $\mu \leq 1/2$ and $C, K$ be positive constants. A random vector $X$ of length $n$ is said to be of type $(\mu, C, K)$ if

- (boundedness) With probability one, all coordinates of $X$ are integer with absolute value at most $n^C$.
- (non-degeneracy) For any unit vector $y$, $P(|X \cdot y| \leq n^{-1/2}) \leq 1 - \mu/2$. (This means that $X$ is not concentrated near a hyperplane.)
- (concentration) There are positive integers $a_1, \ldots, a_m$ with $\text{lcm}(a_1, \ldots, a_m) \leq n^K$ such that for any vector $v \in \mathbb{Z}^n$,

$$\sup_{a \in \mathbb{Z}} P(X \cdot v = a) \leq \int_0^1 \prod_{i \in \{1, \ldots, n\} \setminus E} ((1-\mu) + \mu \cos 2\pi a_i v_i t) \, dt,$$

where $\text{lcm}(a_1, \ldots, a_m)$ (least common multiple) is the smallest positive integer divisible by all $a_i$.

**Remark 2.16.** Here and later, one should not take the absolute constants such as $-2$ and 2 too seriously. We make no attempt to optimize these constants. The first two conditions in the definition are quite intuitive. The third and critical condition comes from Fourier analysis and the reader will have a better understanding of it after reading the next section.

**Definition 2.17.** A collection of $n$ random vectors $Y_1, \ldots, Y_n$ in $\mathbb{R}^n$ is strongly linearly independent if for any non-zero vector $y \in \mathbb{R}^n$ and any $1 \leq i \leq n$,

$$P(Y_1, \ldots, Y_n \text{ independent} | Y_i = y) \leq \exp(-\Omega(n)).$$

**Theorem 2.18.** (Main Theorem) For every positive constants $\mu \leq 1/2, A, C, K$ there is a positive constant $B$ such that the following holds. Let $M_n$ be a random matrix with the following two properties

- The row vectors of $M_n$ are independent random vectors of type $(\mu, C, K)$.
- The column vectors of $M_n$ are strongly linearly independent.

Then

$$P(\sigma_n(M_n) \leq n^{-B}) \leq n^{-A}.$$ **Remark 2.19.** Actually in the concentration property, one can omit a few coordinates in the product. To be more precise, we can make the following weaker assumption:

- There is a subset $E \subseteq \{1, \ldots, n\}$ of at most $n^{9/2}$ elements and positive integers $a_i, i \in \{1, \ldots, n\} \setminus E$ with $\text{lcm}$ at most $n^K$ such that for any vector $v \in \mathbb{Z}^n$,

$$\sup_{a \in \mathbb{Z}} P(X \cdot v = a) \leq \int_0^1 \prod_{i \in \{1, \ldots, n\} \setminus E} ((1-\mu) + \mu \cos 2\pi a_i v_i t) \, dt,$$

where $\text{lcm}(a_1, \ldots, a_m)$ (least common multiple) is the smallest positive integer divisible by all $a_i$.

**3. PROOF OF THEOREM 2.11**

In order to derive Theorem 2.11 from Theorem 2.18, we first need to verify that the matrix in Theorem 2.11 is of type $(\mu, C, K)$ for some constants $\mu$, $C$ and $K$. This will be done in the first two subsections. Next, we need to verify the strong linear independence. This will be done in the last subsection.
3.1 Checking the concentration property

In this subsection, we verify the concentration property in the definition of $(\mu, C, K)$-type. This is based on the following lemma.

**Lemma 3.2.** Let $Z$ be an arbitrary integer vector and $X$ be a random $(\mu, D)$-bounded vector, both of length $n$. Then there exist positive integers $a_1, \ldots, a_n$ at most $D$ such that for any vector $v \in \mathbb{Z}^n$

$$\sup_{a \in \mathbb{Z}} P((Z + X) \cdot v = a) \leq \int_0^1 \prod_{i=1}^n \left((1 - \mu) + \mu \cos 2\pi a_i v_i \right) dt.$$

Proof. As $a$ can take any value, it suffices to prove the statement for $Z = 0$. For an integer $x$, the indicator $I_{x=0}$ of the event $x = 0$ can be expressed, using Fourier analysis, as

$$I_{x=0} = \int_0^1 e(xt) dt.$$

Let $\xi_i, 1 \leq i \leq n$ be the coordinates of $X$. The event $X \cdot v = a$ can be rewritten as $\sum_{i=1}^n \xi_i v_i - a = 0$. Thus

$$P(X \cdot v = a) = E(I_{\sum_{i=1}^n \xi_i v_i - a = 0}) = E\left( \int_0^1 e\left(\sum_{i=1}^n \xi_i v_i (1-a)\right) dt \right).$$

As the $\xi_i$ are independent, the last expectation is equal to

$$\int_0^1 \exp(-2\pi a t) \prod_{i=1}^n |E(e(\xi_i v_i t))| dt \leq \int_0^1 \prod_{i=1}^n |E(e(\xi_i v_i t))| dt.$$

As $\xi_i$ is $(\mu, D)$-bounded, there is a positive integer $a_i \leq D$ such that

$$|E(e(2\pi a_i v_i t))| \leq (1 - \mu) + \mu \cos 2\pi a_i v_i t,$$

completing the proof. ☐

3.3 Checking the non-degeneracy property

Let $y$ be a unit vector in $\mathbb{R}^n$ and $X$ be a random $(\mu, D)$-bounded vector of length $n$ and $Z$ be an arbitrary integer vector of length $n$. We want to show that

$$P(|(Z + X) \cdot y| \leq n^{-2}) \leq 1 - \mu/2.$$

If $(Z + X) \cdot y$ has absolute value at most $n^{-2}$, then $X \cdot ny$ has absolute value at most $n^{-1}$. As $y$ is a unit vector, one of the coordinate of $ny$ has absolute value larger than 1. Assume, without loss of generality, that the first coordinate $y_1$ of $ny$ is such large. Recall that $X = (\xi_1, \ldots, \xi_n)$ where the $\xi_i$ are independent $(\mu, D)$-bounded random variables. Condition on $\xi_2, \ldots, \xi_n$, it suffices to show that for any interval $I$ of length $2n^{-1}$

$$P(\xi_1 v_1 \in I) \leq 1 - \mu/2.$$

But since $\xi$ take only integer values and $|y_1| \geq 1$, the values of $\xi_1 v_1$ would be at least one apart. Assume, for a contradiction, that $P(\xi_1 v_1 \in I) > 1 - \mu/2$. This would imply that there is a number $s$ such that $P(\xi_1 = s) > 1 - \mu/2$. Then by the triangle inequality

$$|E(e(\xi_1 t))| \geq |e(\xi_1 t)(1 - \mu/2) - \mu/2| \geq 1 - \mu,$$

for any $t$. On the other hand, as $\xi_1$ is $(\mu, D)$-bounded

$$|E(e(\xi_1 t))| \leq (1 - \mu) + \mu \cos 2\pi a_1 t$$

for some $a_1 \leq D$. Taking $t$ such that $\cos 2\pi a_1 t = -1$, we obtain a contradiction and conclude the proof.

3.4 Checking the strong linear independence

The strong linear independence of the column vectors of a random $(\mu, D)$-bounded matrix is a consequence of the following theorem, which can be proved by refining the proof of [11] Theorem 1.6.

**Theorem 3.5.** Let $\mu \leq 1/2$ and $D, l$ be positive constants. Then there is a positive constant $\varepsilon = \varepsilon(\mu, D, l)$ such that the following holds. For any set $Y$ of $l$ independent vectors from $\mathbb{R}^n$ and $n - l$ independent random $(\mu, D)$-bounded vectors of length $n$, the probability that they are linearly dependent is at most $(1 - \varepsilon)^n$.

**Remark 3.6.** This theorem is a generalization of a well known theorem of Kahn, Komlós and Szemerédi [7] which asserts that the probability that a random Bernoulli matrix is singular is exponentially small. To see this, recall that a random Bernoulli vector is $(1/4, 2)$-bounded and in Theorem 5.3 take $l = 1$ and fix $y$ be the all one vector.

4. SMOOTH COMPLEXITY WITH DISCRETE NOISE

Running times of algorithms are frequently estimated by worst-case analysis. But in practice, it has been observed that many algorithms perform significantly better than the estimates obtained from the worst-case analysis. Few years ago, Spielman and Teng [9, 10] came up with an ingenious explanation for this fact. The rough idea behind their argument is as follows. Even if the input $I$ is the worst-case one (which, in theory, would require a long running time), because of the noise, the computer actually works on some slightly randomly perturbed version of $I$. Next, one would show that the running time on a slightly randomly perturbed input, with high probability, is much smaller than the worst-case one. The smooth complexity of an algorithm is the maximum over its input of the expected running time of the algorithm under slight perturbations of that input. The puzzling question here is, of course: why the perturbed input is typically better than the original (worst-case) one? In some sense, the "magic" lies in the Phenomenon

It's the worst-case noise. This random noise guarantees that the condition number of the perturbed input is small (so the perturbed input is likely to be well conditioned), no matter how ill conditioned the original input may be. The bound on the condition number then can be used to derive a bound on the running time of the algorithm.

In their work [9, 10, 8], Spielman and Teng (and coauthors) assumed Gaussian noise (or more generally continuous noise). Theorem 2.2 played a significant role in their proofs.

An important (and largely open) problem is to obtain smooth complexity bounds when the noise is discrete. (We would like to thank Spielman for communicating this problem.) In fact, it is not clear how computers would compute with Gaussian (and other continuous) distributions without discretizing them. This problem seems to pose a considerable mathematical challenge. Naturally, the first step would be to obtain estimates for the condition number with discrete noise. This step has now been accomplished in this paper. However, these estimates themselves are not always sufficient. To be more specific, the situation looks as follows:

- There are problems where an efficient bound on the condition number leads directly to an efficient com-
plexity bound. In such a situation, we obtain a smooth complexity bound with discrete noise in the obvious manner. This seems to be the case, e.g., with the problems involving the Gaussian Elimination in \[3\]. In the proofs in \[3\], the critical fact was that all \(n - 1\) minors of a random perturbed matrix are all well conditioned, with high probability. This can be obtained using our results combined with the union bound (see the remark after Theorem 2.10).

- There are situations where beside the estimate on the condition number, further properties of the noise is used. An important example is the simplex method in linear programming. In the smooth analysis of this algorithm with Gaussian noise \[10\], the fact that the distribution is continuous was exploited at several places. Thus, even with the discrete version of the condition number estimates in hand, it is still not clear to us how to obtain a smooth complexity bound with discrete noise in this problem.

5. KEY INGREDIENTS

In this section, we present our key ingredients in the proof of Theorem 2.18

5.1 Generalized arithmetic progressions and their discretization

One should take care to distinguish the subset \(kA\) from the dilate \(k \cdot A\), defined for any real \(k\) as

\[ k \cdot A := \{ka | a \in A\}. \]

Let \(P\) be a GAP of integers of rank \(d\) and volume \(V\). Our first key ingredient is a theorem that shows that given any specified scale parameter \(R_0\), one can “discretize” \(P\) near the scale \(R_0\). More precisely, one can cover \(P\) by the sum of a coarse progression and a small progression, where the diameter of the small progression is much smaller (by an arbitrarily specified factor of \(S\)) than the spacing of the coarse progression, and that both of these quantities are close to \(R_0\) (up to a bounded power of \(SV\)).

Theorem 5.2 (Discretization). \[12\] For every constant \(d\) there is a constant \(d'\) such that the following hold. Let \(P \subset \mathbb{Z}\) be a symmetric generalized arithmetic progression of rank \(d\) and volume \(V\). Let \(R_0, S\) be positive integers. Then there exists a number \(R \geq 1\) and two generalized progressions \(P_{\text{small}}, P_{\text{sparse}}\) of rational numbers with the following properties:

- (Scale) We have \(R \leq (SV)^{d'}R_0\).
- (Smallness) \(P_{\text{small}}\) has rank at most \(d\), volume at most \(V\), and takes values in \([-R/S, R/S]\).
- (Sparseness) \(P_{\text{sparse}}\) has rank at most \(d\), volume at most \(V\), and any two distinct elements of \(SP_{\text{sparse}}\) are separated by at least \(RS\).
- (Covering) We have \(P \subset P_{\text{small}} + P_{\text{sparse}}\).

5.3 Inverse Littlewood-Offord theorem

Our second key ingredient is a theorem which characterizes all sets \(v = \{v_1, \ldots, v_n\}\) such that \(\int_0^1 \prod_{i=1}^n (1 - \mu + \mu \cos 2\pi v_i t) d\mu\) is large. This is a refinement of \[12\] Theorem 2.5] (see Remark 2.8 from this paper) and will enable us to exploit the non-concentration property from Definition 2.15 in a critical way.

Theorem 5.4. Let \(0 < \mu \leq 1\) and \(A, \alpha > 0\) be arbitrary. Then there is a positive constant \(A'\) such that the following holds. Assume that \(v = \{v_1, \ldots, v_n\}\) is a multiset of integers satisfying

\[ \int_0^1 \prod_{i=1}^n (1 - \mu + \mu \cos 2\pi v_i t) d\mu \geq n^{-A}. \]

Then there is a GAP \(Q\) of rank at most \(A'\) and volume at most \(n^\alpha\) which contains all but at most \(n^\alpha\) elements of \(v\) (counting multiplicity). Furthermore, there is a integer \(1 \leq s \leq n^\alpha\) such that \(su \in v\) for each generator \(u\) of \(Q\).

With the two key tools in hand, we are now ready to prove Theorem 2.18

6. PROOF OF THEOREM 2.18

Let \(B > 10\) be a large number (depending on the type of \(M_\alpha\)) to be chosen later. If \(\sigma_{\alpha}M_\alpha < n^{-\beta}\) then there exists a unit vector \(v\) such that

\[ \|M_\alpha v\| < n^{-\beta}. \]

By rounding each coordinate \(v_i\) to the nearest multiple of \(n^{-2}\), we can find a vector \(\tilde{v} \in n^{-2} \cdot \mathbb{Z}^n\) of magnitude \(0.9 \leq \|	ilde{v}\| \leq 1.1\) such that

\[ \|M_\alpha \tilde{v}\| \leq 2n^{-\beta}. \]

Writing \(w := n^{B+2} \tilde{v}\), we thus can find an integer vector \(\tilde{w} \in \mathbb{Z}^n\) of magnitude \(9n^{B+2} \leq \|\tilde{w}\| \leq 1.1n^{B+2}\) such that

\[ \|M_{\alpha} \tilde{w}\| \leq 2n^{2}. \]

Let \(\Omega\) be the set of integer vectors \(w \in \mathbb{Z}^n\) of magnitude \(9n^{B+2} \leq \|w\| \leq 1.1n^{B+2}\). It suffices to show the probability bound

\[ P(\text{there is some } w \in \Omega \text{ such that } \|M_{\alpha} w\| \leq 2n^{2}) \leq n^{-A}. \]

We now partition the elements \(w = (w_1, \ldots, w_n)\) of \(\Omega\) into three sets:

- We say that \(w\) is rich if

\[ \sup_{a \in \mathbb{Z}^n, \|a\| \leq n} P(X_i \cdot w = a) \geq n^{-A-4}, \]

where \(X_i\) are the row vectors of \(M_{\alpha}\). Otherwise we say that \(w\) is poor. Let \(\Omega_1\) be the set of poor \(w\)’s.

- A rich \(w\) is singular \(w\) if fewer than \(n^{0.2}\) of its coordinates have absolute value \(n^{B/2}\) or greater. Let \(\Omega_2\) be the set of rich and singular \(w\)’s.

- A rich \(w\) is non-singular \(w\), if at least \(n^{0.2}\) of its coordinates have absolute value \(n^{B/2}\) or greater. Let \(\Omega_3\) be the set of rich and non-singular \(w\)’s.

Remark 6.1. Again one should not take the absolute constants \(-4, 1/2, 0.2\) too seriously.

The desired estimate follows directly from the following lemma and the union bound.
Lemma 6.2 (Estimate for poor w).

\( P(\text{there is some } w \in \Omega_1 \text{ such that } \| M_n w \| \leq 2n^2) = o(n^{-A}) \).

Lemma 6.3 (Estimate for rich singular w).

\( P(\text{there is some } w \in \Omega_2 \text{ such that } \| M_n w \| \leq 2n^2) = o(n^{-A}) \).

Lemma 6.4 (Estimate for rich non-singular w).

\( P(\text{there is some } w \in \Omega_3 \text{ such that } \| M_n^2 w \| \leq 2n^2) = o(n^{-A}) \).

The proofs of Lemmas 6.2 and 6.3 are relatively simple and rely on well-known methods. The proof of Lemma 6.4 which is essentially the heart of the matter, is more difficult and requires the tools provided in Section 6.

7. PROOF OF LEMMA 7.2

We use a conditioning argument, following [7]. (An argument of the same spirit was used by Komlós to prove the bound \( O(n^{-1/2}) \) for the singularity problem [2].) Let \( M \) be a matrix such that there is \( w \in \Omega_1 \) satisfying \( \| M w \| \leq 2n^2 \). Since \( M^{-1} \) and its transpose have the same spectral norm, there is a vector \( w' \) which has the same norm as \( w \) such that \( \| w'M \| \leq 2n^2 \). Let \( u = w'M \) and \( X_i \) be the row vectors of \( M \). Then

\[
    u = \sum_{i=1}^{n} w'_i X_i
\]

where \( w'_i \) are the coordinates of \( w' \). Now consider \( M = M_n \). By paying a factor of \( n \) in the probability (whenever this phrase is used, keep in mind that we will use the union bound to conclude the proof), we can assume that \( w'_i \) has the largest absolute value among the \( w'_i \). We expose the first \( n-1 \) rows \( X_1, \ldots, X_{n-1} \) of \( M_n \). If there is \( w \in \Omega_1 \) satisfying \( \| M w \| \leq 2n^2 \), then there is a vector \( y \in \Omega_1 \), depending only on the first \( n-1 \) rows such that

\[
    \left( \sum_{i=1}^{n-1} (X_i \cdot y)^2 \right)^{1/2} \leq 2n^2.
\]

We can write \( X_n = \frac{1}{w'_n} (u - \sum_{i=1}^{n-1} w'_i X_i) \).

Thus,

\[
    |X_n \cdot y| = \left| \frac{1}{w'_n} |u \cdot y - \sum_{i=1}^{n-1} w'_i X_i \cdot y| \right|.
\]

The right hand side, by the triangle inequality, is at most

\[
    \frac{1}{|w'_n|} (|u| |y| + \| w' \| \left( \sum_{i=1}^{n-1} (X_i \cdot y)^2 \right)^{1/2}).
\]

By assumption \( |w'_n| \geq n^{-1/2} |w'| \). Furthermore, as \( |u| \leq 2n^2 \), \( |u||y| \leq 2n^2 |y| \leq 3n^2 |w'| \) as \( |w'| \leq |w| \) and both \( y \) and \( w \) belong to \( \Omega_1 \). (Any two vectors in \( \Omega_1 \) have roughly the same length.) Finally \( \left( \sum_{i=1}^{n-1} (X_i \cdot y)^2 \right)^{1/2} \leq 2n^2 \). Putting all these together, we have

\[
    |X_n \cdot y| \leq 5n^{5/2}.
\]

Recall that both \( X_n \) and \( y \) are integer vectors, so \( X_n \cdot y \) is an integer. The probability that \( |X_n \cdot y| \leq 5n^{5/2} \) is at most

\[
    (10n^{5/2} + 1) \sup_{a \in \mathbb{Z}} P(X_n \cdot y = a) \leq n^{-A-4}. \]

On the other hand, \( y \) is poor, so by definition \( \sup_{a \in \mathbb{Z}} P(X_n \cdot y = a) \leq n^{-A-4} \). Thus, it follows that

\[
    P(\text{there is some } w \in \Omega_1 \text{ such that } \| M_n w \| \leq 2n^2) \leq n^{-A-4}(10n^{5/2} + 1) = o(n^{-A}),
\]

where the extra factor \( n \) comes from the assumption that \( w_n \) has the largest absolute value. This completes the proof.

8. PROOF OF LEMMA 7.3

We use an argument from [6]. The key point will be that the set \( \Omega_2 \) of rich non-singular vectors has sufficiently low entropy that one can proceed using the union bound. A set \( N \) of vectors on the \( n \)-dimensional unit sphere \( S_{n-1} \) is said to be an \( \epsilon \)-net if for any \( x \in S_{n-1} \), there is \( y \in N \) such that \( |x - y| \leq \epsilon \). A standard greedy argument shows

Lemma 8.1. For any \( n \) and \( \epsilon \leq 1 \), there exists an \( \epsilon \)-net of cardinality at most \( O(1/\epsilon)^n \).

We need another lemma, showing that for any unit vector \( y \), very likely \( \| M_n y \| \) is polynomially large.

Lemma 8.2. For any unit vector \( y \)

\[
    P(\| M_n y \| \leq n^{-2}) = \exp(-\Omega(n)).
\]

Proof. If \( \| M_n y \| \leq n^{-2} \), then \( |X_i \cdot y| \leq n^{-2} \) for all index \( 1 \leq i \leq n \). However, by the assumption of the theorem, for any fixed \( i \), the probability that \( |X_i \cdot y| \leq n^{-2} \) is at most \( 1 - \mu/2 \). Thus,

\[
    P(\| M_n y \| \leq n^{-2}) \leq (1 - \mu/2)^n = \exp(-\Omega(n))
\]

concluding the proof. \( \square \)

For a vector \( w \in \Omega_2 \), let \( w' \) be its normalization \( w'/\|w\| \). Thus, \( w' \) is an unit vector with at most \( n^{-3/2} \) coordinates with absolute values larger or equal \( n^{-5/2} \). By choosing \( B \geq 2C + 20 \), we can assume that \( w' \) belong to \( \Omega_2 \). The collection of vectors at most \( n^{-3/2} \) coordinates with absolute values larger or equal \( n^{-5/2} \). If \( \| M w' \| \geq 2n^2 \), then \( \| M w' \| \leq 3n^{-B} \) as \( \| w' \| \geq n^{-3/2} \). Thus, it suffices to give an exponential bound on the event that there is \( w' \in \Omega_2 \) such that \( \| M_n w' \| \leq 3n^{-B} \). By paying a factor of \( \exp(\alpha(n)) \) in probability, we can assume that the large coordinates (with absolute value at least \( n^{-C-10} \)) are among the first \( l := n^{0.2} \) coordinates. Consider an \( n^{-C-5} \)-net \( N \) in \( S_{l-1} \). For each vector \( y \in N \), let \( y' \) be the \( n \)-dimensional vector obtained from \( y \) by letting the last \( n-l \) coordinates be zeros, and let \( N' \) be the set of all such vectors obtained. These vectors have magnitude between 0.9 and 1.1, and from Lemma 6.3 we have \( |N'| \leq O(n^{0.1}) \). Now consider a rich singular vector \( w' \in \Omega_2 \) and let \( w' \) be the \( l \)-dimensional vector formed by the first \( l \) coordinates of this vector. As the remaining coordinates are small \( \| w'' \| = 1 + O(n^{-C-9}) \). There is a vector \( y \in N \) such that

\[
    \| y - w'' \| \leq n^{-C-5} + O(n^{-C-9}).
\]

It follows that there is a vector \( y' \in N' \) such that

\[
    \| y' - w' \| \leq n^{-C-5} + O(n^{-C-9}) \leq 2n^{-C-5}.
\]
If $M$ has norm at most $n^{C+1}$, then
\[ \|Mw\| \geq \|My\| - 2n^{-C-5}n^{C+1} = \|My\| - 2n^{-4}. \]
It follows that if $\|Mw\| \leq 3n^{-B}$ for some $B \geq 2$, then $\|My\| \leq 5n^{-4}$. Now take $M = M_{a_i}$. For each fixed $y$’s, the probability that $\|M_{a_i}y\| \leq 5n^{-4} \leq n^{-2}$ is at most $\exp(-\Omega(n))$, by Lemma 5.2. Furthermore, the number of $y$’s is subexponential (at most $O(n^{C+3})O(n)^{3n^{-2} = \exp(o(n))}$). The claim follows by the union bound.

9. PROOF OF LEMMA 7.4

This is the most difficult part of the proof, where we will need all the tools provided in Section 5. Informally, the strategy is to use the inverse Littlewood-Offord theorem to place the integers $w_1, \ldots, w_m$ in a progression, which we then discretize using Theorem 5.2. This allows us to replace the event $\|M_{a_i}w\| \leq 2n^2$ by some dependence event involving the columns of $M_{a_i}$, whose probability is very small by the strong linear independence assumption of the theorem.

We now turn to the details. By the inverse theorems and the non-concentration property from Definition 2.19, there is a constant $A$ such that for each $w \in \Omega$, there exists a symmetric GAP $Q$ of integers of rank at most $d$ and volume at most $n^{A}$ and non-zero integers $a_1, \ldots, a_n$ with least common multiple at most $n^{B}$ such that $Q$ contains all but $\lfloor n^{0.1} \rfloor$ of the integers $a_1w_1, \ldots, a_nw_n$. Furthermore, the generators of $Q$ are of the form $a_jw_j/s$ for some $1 \leq s \leq n^{A}$. Notice that if $a_jw_j \in Q$ then $w_j \in Q^+ := \{x/a | x \in Q, a \in \mathbb{Z}, a \neq 0, |a| \leq n^{A}\}$. Using the description of $Q$ and the fact that $w_1, \ldots, w_n$ and $a_1, \ldots, a_n$ are polynomially bounded in $n$, one can see that the total number of possible $Q$ is $n^{O(1)} = \exp(o(n))$. Next, by a factor of $\binom{n}{\lfloor n^{0.1} \rfloor} \leq n^{n^{0.1}} = \exp(o(n))$

we may assume that it is the last $\lfloor n^{0.1} \rfloor$ integers $a_{m+1}w_{m+1}, \ldots, a_nw_n$ which possibly lie outside $Q$, where we set $m := n - \lfloor n^{0.1} \rfloor$. As each of the $w_j$ has absolute value at most $1.1n^{B+2}$, the number of ways to fix these exceptional elements is at most $(2.1n^{B+2})^{\lfloor n^{0.1} \rfloor} = \exp(o(n))$. Overall, it costs a factor only $\exp(o(n))$ (keep in mind that we intend to use the union bound) to fix $Q$, the positions and the values of the exceptional elements of $w_j$. Notice that $M_{a_i}w = w_1Y_1 + \ldots + w_mY_m$, where $Y_i$ is the $i$th column of $M_{a_i}$. Fixing $w_{m+1}, \ldots, w_n$ and set $Y := \sum_{i=m+1}^n w_iY_i$. This way we can rewrite $M_{a_i}w$ as

$M_{a_i}w = w_1Y_1 + \ldots + w_mY_m + Y.$

For any number $y$, define $F_y$ be the event that there exists $w_1, \ldots, w_m$ in the set $Q^+$, where at least one of the $w_j$ has absolute value larger or equal $n^{B-10}$, such that

$|w_1Y_1 + \ldots + w_mY_m + y| \leq 2n^2.$

It suffices to prove that for any $y$

$\mathcal{P}(F_y) \leq \exp(-\Omega(n)).$

We now apply Theorem 5.2 to the GAP $Q$ with $R_0 := n^{B/3}$ and $S := n^A$, where $L = C + K + 2$ ($C$ and $K$ are the constants in Definition 2.14). By choosing $B$ sufficiently large, we can guarantee that $B/3$ is considerably larger than $L$. Recall that the volume of $Q$ is at most $n^{A'}$, where $A'$ is a constant depending on $A$ and $\mu$. We can find a number $R = n^{B/3 + O(A')/L(1)}$ and symmetric GAPs $Q_{\text{small}}$, $Q_{\text{small}}$ of rank at most $d' = d'/(d', A')$ and volume at most $n^{A'}$ such that

- $Q \subseteq Q_{\text{small}} + Q_{\text{small}}$
- $Q_{\text{small}} \subseteq [-n^{-L}R, n^{-L}R]$
- The elements of $n^LQ_{\text{small}}$ are $n^L$-separated.

Since $Q$ (and hence $n^LQ$) contains $a_1w_1, \ldots, a_mw_m$ (for some set $\{a_1, \ldots, a_m\}$ we can therefore write

$w_j = a_j^{-1}(w_j^{\text{small}} + w_j^{\text{small}})$

for all $1 \leq j \leq m$, where $w_j^{\text{small}} \in Q_{\text{small}}$ and $w_j^{\text{small}} \in Q_{\text{small}}$. In fact, this decomposition is unique. Suppose that the event $F_y$ holds. Let $y = (y_1, \ldots, y_n)$ and $\eta_{i,j}$ denote the entry of $M_{a_i}$ at row $i$ and column $j$. We have

$w_1\eta_{1,1} + \ldots + w_m\eta_{m,m} = y_1 + O(n^\eta).$

for all $1 \leq i \leq n$. Split the $w_j$ into sparse and small components and estimating the small components. The contribution coming from the small components is

$\sum_{j=1}^m a_j^{-1} w_j^{\text{small}} \eta_{i,j} = O(n^{-L+C+1}R)$

since $\eta_{i,j}$ are bounded from above by $n^{C}$, $w_j^{\text{small}}$ is bounded from above by $n^{-LR}$ and $a_j$ are positive integers. By the triangle inequality, it follows that

$a_1^{-1} w_1^{\text{small}} \eta_{i,1} + \ldots + a_m^{-1} w_m^{\text{small}} \eta_{i,m} = y_i + O(n^{-L+C+1}R)$

for all $1 \leq i \leq n$.

Set $T := \text{lcm}(a_1, \ldots, a_m)$. The previous estimate implies

$b_1 w_1^{\text{small}} \eta_{i,1} + \ldots + b_m w_m^{\text{small}} \eta_{i,m} = T y_i + O(T n^{-L+C+1}R)$

where $b_i = T/a_i$. Now we use the assumption that $T \leq n^K$ from Definition 2.13. This assumption yields that $b_i \leq n^K$ and the left-hand side lies in

$n^{K+1}Q_{\text{small}} \subset n^{K+1}Q_{\text{small}} \subset n^LQ$, which is known to be $n^L$-separated. Furthermore,

$O(T^n^{-L+C+1}R) = O(n^{K-L+C+1}R) = O(n^{R-1})$

by the definition of $L$. Thus there is a unique value for the right-hand side, call it $y'_i$, which depends only on $y$ and $Q$ such that

$b_1 w_1^{\text{small}} \eta_{i,1} + \ldots + b_m w_m^{\text{small}} \eta_{i,m} = y'_i$.

The point is that we have now eliminated the $O()$ errors, and have thus essentially converted the singular value problem to a problem about dependence. Note also that since one of the $w_1, \ldots, w_m$ is known to have magnitude at least $n^{B/2}$ (which will be much larger than $n^{B/3}$ given that we set $B > 6L = 6(C + K + 2)$), we see that at least one of the $w_j^{\text{small}}, \ldots, w_m^{\text{small}}$ is non-zero.

Let $y = (y_1, \ldots, y_n)$. The equation

$b_1 w_1^{\text{small}} \eta_{i,1} + \ldots + b_m w_m^{\text{small}} \eta_{i,m} = y'_i$.
implies that the first $m$ columns of $M_n$ span $y'$. For any fixed non-zero $y'$, the probability that this happens is exponentially small by the strong linear independence assumption. This completes the proof.

10. FROZEN ENTRIES

We now give an explanation to Remark 2.19. This remark is based on the fact that in the previous proof one is allowed to have as many as $n^{1-\epsilon}$ coordinates outside the set $Q'$, for any positive constant $\epsilon < 1$. Indeed, these extra coordinates contribute a factor of $\left(\frac{n}{n^{1-\epsilon}}\right)$ which is $\exp(o(n))$. This factor will be swallowed by the exponential bound we have at the end of the proof. (In the proof we, for convenience, set $\epsilon = 0.9$ and have $n^{1}$ exceptional coordinates, but the actual value of $\epsilon$ plays no role.) The main point here is that we can set aside the "frozen" coordinates even before applying the Inverse Littlewood-Offord theorem.

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