Averaging Inhomogeneous Universes: Volume, Angle, Line of Sight

Eric V. Linder
University of Massachusetts, Department of Physics and Astronomy, Amherst, MA 01003

ABSTRACT

Cosmologies that match in a volume averaged sense need not generally have the same light propagation behaviors. In particular a universe with inhomogeneity may not demonstrate the Friedmann-Robertson-Walker distance-redshift relation even after volume averaging to FRW spacetime. Even the Dyer-Roeder prescription for incorporating inhomogeneity within a universe equivalent to FRW in an angle averaged sense does not guarantee FRW behavior in general. To legitimately use the FRW distance-redshift relation to interpret observations, the physical conditions must match in a line of sight sense (defined herein: most observations do), since light probes the mass distribution or geometry on all length scales.

1. Introduction

Since many cosmological quantities depend on the distance-redshift relation it is of great observational relevance to ensure that the Friedmann-Robertson-Walker (FRW) expression is the correct one to use. For our universe, containing local inhomogeneities, it is generally assumed that implementing some sort of large scale averaging procedure causes the behavior of light propagation to revert to FRW. In the absence of a unique and rigorous mathematical prescription for averaging (Buchert 1997; Buchert & Ehlers 1997), due to the nonlinear complexities of general relativity, one tends to rely on simplified models and reasonability arguments.

One commonly adopted model is that of Dyer and Roeder (1972, 1973: DR), which restricts the inhomogeneities to randomly distributed compact clumps and compensates for the higher density within the clumps with a lower density of smoothly distributed matter – so the overall density in a volume containing on average many clumps is on average the same as the equivalent smooth FRW model. Nevertheless, the angular diameter distance-redshift relations differ in the two cases and can cause, for example, “shrinking” of angular scales (Linder 1988b) in an inhomogeneous universe. A typical line of sight will avoid the clumps and hence the obvious gravitational lensing effects, making it difficult for the observer to discern whether the universe is truly smooth FRW or actually clumpy DR, and hence which distance relation is appropriate.

While simplistic, this model allows us to calculate easily the distance-redshift relation for such an inhomogeneous universe, and evaluate a criterion for what type of averaging can restore the FRW result (Linder 1998); increasing the angular size of the observation beam encompasses more and more clumps and causes a transition toward the FRW behavior. (For numerical investigation of distances in clumpy universes see Fukushige & Makino 1994; Watanabe & Tomita 1990).

One can ask whether general inhomogeneous models that in some sense are “close” to FRW can be realistically averaged to transform their inhomogeneous distance-redshift
relations into the FRW one in some limit. The interesting results of Mustapha, Bassett, Hellaby, & Ellis (1998: MBHE) show that even a full sky average, i.e. isotropization of an inhomogeneous universe, may not give the FRW relation generally.

MBHE work within a spherically symmetric, Tolman-Bondi model and find that the FRW results are also not recovered even at source depths much greater than the inhomogeneity depth. Because the calculations are already in an isotropic framework one cannot appeal to angular averaging to remove any discrepancies from FRW. And volume averaging, over a large region of space that has overall properties equal to its FRW analog, is found insufficient to yield equal light propagation behaviors.

These problems with averaging arise because the light is affected by the spacetime, the gravitational field of the matter along its path, on all scales and not just by some coarse grained quantity. [Dynamics, on the other hand, such as the expansion of a region or growth of density perturbations, could be expected to be reasonably handled by a properly volume smoothed description (see Bildhauer & Futamase 1991; Buchert & Ehlers 1997).]

This paper examines what averaging procedures can be applied to various inhomogeneous universes in order to attain the FRW distance-redshift behavior. Section 2 first investigates some formal concerns regarding large scale dynamics and flux conservation conditions, then section 3 considers volume (§3.1) and angle (§3.2) averaging, and introduces line of sight averaging (§3.3), and section 4 summarizes the divergence/transition conditions to FRW.

2. Dynamical Deviation and Flux Conservation

If we want the deviation of the distance-redshift behavior of the inhomogeneous model from FRW to be large enough to be observationally significant, we must check whether such inhomogeneity will affect the dynamics of the volume we are concerned with. The DR model has an equation for the light beam area, or equivalently angular diameter distance \( r \), of

\[
\frac{r''}{r} = -(1/2)R_{ab}k^ak^b = -(3/2)(1+z)^2H^2(z)\alpha \Omega(z),
\]

with respect to affine parameter \( \lambda \), or

\[
\ddot{r} + [3 + q(z)](1+z)^{-1}\dot{r} + (3/2)(1+z)^{-2}\alpha \Omega(z) r = 0,
\]

with respect to redshift \( z \). Here \( R_{ab} \) is the Ricci tensor, \( k^a \) the photon four momentum, \( H(z) \) the Hubble parameter, \( q(z) \) the deceleration parameter, and \( \Omega(z) \) the density parameter. The smoothness parameter \( \alpha \) is the ratio of smoothly distributed mass density to the total mass density. (The forms of eqs. 1-2 do not change for a universe with components other than dust – see Linder 1988a.)

However, if the background FRW dynamics is affected, we must abandon the DR use of the global values for the Hubble, deceleration, and large scale density parameters and use instead the local values that the light experiences along its path. That is, while in the DR model the difference of the local density from the FRW background was sufficiently represented in the effective density \( \alpha \Omega \), now the density inhomogeneity also enters significantly into (the local) \( H \) and \( q \).
At first glance, it is not as bad as one might think. The beam equation in terms of the Ricci tensor (first part of equation 1) still holds (we are treating the case of the inhomogeneous universe being sufficiently close to FRW that shear can still be neglected—a circular beam remains circular—thus allowing a single quantity \( r \) to describe the angular diameter distance). Likewise, its conversion to being written in terms of the energy-momentum tensor:

\[
\frac{r''}{r} = -4\pi T_{ab} k^a k^b,
\]

still holds since the Einstein equations do and \( k^a \) is still null. One might even be able to get away with taking the form of \( T_{ab} \) to be that of an isotropic, homogeneous perfect fluid, putting in the inhomogeneity spatial dependence by hand. That is, write the RHS as \(-4\pi (1 + z)^2 \rho(z)\), equivalent to the second part of equation (1).

However, we cannot simply apply this sort of patch onto the FRW model. Now the relation between affine parameter and redshift, \( d\lambda/dz = (1 + z)^{-2} H(z) \), is not the global FRW one (nor is it isotropic if the inhomogeneity field is not). Thus we cannot directly compare to the FRW situation in terms of the observable, the redshift, because we have different dynamics; the redshift to the source will differ if the observer is in a (dynamically significant) inhomogeneity.

Furthermore, if we try matching boundaries between an inhomogeneous and homogeneous region, we see the whole model breaks down: from the Friedmann equations a jump in density implies a discontinuity in the expansion rate \( \dot{a} \) at the boundary and the two regions will (unphysically) lose contact. The Robertson-Walker metric has broken down and we must instead use another that takes into account the impact of the inhomogeneities, e.g. the Tolman-Bondi case. This is what MBHE did and they found, not surprisingly, that no reduction to FRW behavior occurred. Different dynamics implies different light behavior regardless of similarities in the large scale volume averages.

Similarly, note that questions of flux conservation become more complicated when the dynamics of the models differ; one cannot simply compare areas at some fixed redshift \( z \). So if we hope to obtain an asymptotic agreement with FRW behavior then we must keep our inhomogeneities small enough that the dynamics is undisturbed. In this case the DR model may be a reasonable approximation. Let us examine further the role of different types of averaging.

### 3. Averaging

#### 3.1 Volume Averaging

As we have just seen, in comparing two dynamically different universes there should be no requirement or expectation that they produce the same large scale light propagation behavior. One example of this is the MBHE result. But another significant point of that result is the breakdown of volume averaging. Smoothing over the inhomogeneity scale reduces the gross physical conditions and dynamics to FRW, but a difference in distance behavior remains. Thus volume averaging is not sufficient to enforce equal distance-redshift relations, because the light also probes the small scale structure of the spacetime.

However, MBHE take a highly nongeneric situation of spherical symmetry of the inhomogeneity around the observer. But it is precisely the observer who is comparing
distances and light propagation behaviors. The observer is indeed in a privileged position and one cannot carry out such Copernican averaging. While volume averaging may be correct within an ergodic framework, this is fundamentally unobservable—we observe at a single location.

So such explicit volume averaging seems overly restrictive. To avoid this let us examine the distance-redshift relation in a more generic model. Consider the DR model, which possesses nonexplicit volume averaging, i.e., every substantial volume is taken to look like FRW under coarse graining. Imagine one case where the observer lives in a finite inhomogeneous region and another case where the observer lives in a FRW region but surrounded by an inhomogeneous shell at some distance.

Within the DR model the beam equation for the light propagation (see eq. 2) depends only on the global expansion rate $H(z)$, deceleration parameter $q(z)$, density parameter $\Omega(z)$ (all of which are related), and the local inhomogeneity pattern through the density clumping parameter along the line of sight, $\alpha$. We will sometimes find it useful to employ the generalization of DR to arbitrarily varying clumpiness $\alpha(z)$ (Linder 1988a). [Except under isotropy or angle averaging (see the next section), $\alpha(z)$ will really be a function of position on the sky as well: $\alpha(z; \vec{\theta}, \vec{\phi})$. However, the DR model encompasses an implicit isotropy through its implicit volume averaging. In any case, we do not need that: it suffices for us to consider a given line of sight with some $\alpha(z)$; another line of sight could have $\alpha'(z)$. We do not assume the observer lives in any special position.]

Linder (1988a) gives a closed form integral solution for the angular distance $r(z)$ in terms of $\alpha(z)$, the global parameters, and the FRW relative distance $r_{\text{FRW}}(z', z)$. Alternatively one can simply solve the beam equation (2), or even obtain an analytic solution if one considers discrete regions with different, constant $\alpha$ ($\alpha = 1$ for FRW) and matches them at the boundaries.

If one uses discrete regions one needs to match the distances and their first derivatives (equal to the local Hubble flow) at the boundary redshift (there is no ambiguity in the redshift since the global dynamics of the regions are the same). While this is fine for the analytic solution, it presents problems for the closed form solution. The beam equation is a second order equation in the distance. Despite matching the zeroth and first derivatives, the discontinuity in the density ($\alpha$) at the boundary requires a discontinuity in the second derivative of the distance. This invalidates the integral solution since then $\ddot{r}(z)$ or $\alpha(z)$ is undefined at the boundary. Of course it is trivial to fix this by taking a rapid but smooth change in $\alpha$ between the desired regions.

We take a toy model of a transition in $\alpha$ from $\alpha_1$ in the vicinity of the observer to $\alpha_2$ far from the observer along the line of sight. We calculate the distance-redshift relation using both a step in $\alpha$ at $z_*$, evaluating the distance by the analytic matching procedure, and a smoother transition in $\alpha$,

$$\alpha(z) = \alpha_1 \left[1 + e^{(z-z_*)/\Delta}\right]^{-1} + \alpha_2 \left[1 + e^{-(z-z_*)/\Delta}\right]^{-1},$$

numerically solving the beam equation. Here $\Delta$ is the transition width. As $\Delta \to 0$ the two approaches agree.

As a first case consider a sphere of inhomogeneity $\alpha = 0$ surrounding the observer out to redshift $z_*$, with beyond $z_*$ the universe perfectly FRW ($\alpha = 1$). (Again, the sphericity
is not important, as we could just consider $\alpha(z)$ along some particular line of sight.) Even for $z_*$ small the distance to a source at $z > z_*$ is not found to approach the FRW distance even asymptotically as $z \gg z_*$, but maintains a constant fractional offset of order $z_2^2$. While general inhomogeneous universes would require $z_*$ to be small in order to recover FRW dynamics (if those were used in the calculation, as here), this is not required in the DR model since its implicit volume averaging, i.e. density compensation, automatically gives FRW dynamics. In any case, this has the important implication that even for a universe that agrees with FRW in both dynamics and as a volume average there can be a situation where no line of sight obeys the FRW relation.

As the second case suppose that the observer lives in a FRW region but a small locality along a line of sight has inhomogeneity given by the clumpiness $\alpha$ of the DR model. Here the observer is certainly not in a special position. If the clumpiness, say $\alpha = 0$, extends from $z_1$ to $z_1 + \Delta z$, then the solution for the angular distance to sources beyond the small patch of inhomogeneity again does not converge to the FRW result. Instead it asymptotically has a constant fractional offset of order $\Delta z/(1 + z_1)$. That it does not agree with FRW does not depend on each line of sight having the same inhomogeneity location $z_1$, i.e. the observer being at the center of a spherical shell of inhomogeneity, so the observer truly is not singled out in any Copernican sense. So long as every line of sight had a clumpy region somewhere along it, the (anisotropic) distance-redshift relation will nowhere be FRW.

These cases show that despite universes looking like FRW under volume averaging, their distance-redshift relations may not be FRW.

### 3.2 Angular Averaging

Taken at face value, the DR model seems to imply that since each line of sight has a distance greater than FRW at the same redshift, even an angle averaging procedure cannot restore the FRW behavior. However, as mentioned in Section 1, the DR approach ignores those lines of sight passing near the inhomogeneity clumps. Once those are included, then the full sky average distance relation does agree with FRW, and flux conservation holds since the universes are dynamically equivalent. [Note that flux conservation requires $< r_a^{-2} >= r_{frw}^{-2}$; there is no requirement that $< r_a >= r_{frw}$, though of course if $r_a = r_{frw}$ for each line of sight then flux conservation is automatic.]

Even without a full sky average the DR distance relation approaches FRW upon angle averaging over a sufficiently large angular scale, i.e. a wide enough observation beam. One criterion for this scale is given by Linder (1998). Here we briefly mention two further points regarding the transition of the DR relation to FRW.

In that paper, the angular averaging scale was equated to an angular scale where the beam had a certain probability to include a clump and hence feel the full density, smooth plus clumpy. This was done under Poisson statistics. Here we note that if the clumps cluster, as galaxies for example, then the excess mass density in an angular region is proportional to the angular two point mass correlation function $w(\theta)$, evaluated at the source depth. This helps determine the angular transition scale to FRW behavior. Secondly, note that the shear contribution to the effective smoothness $\alpha$, equation (12) in that paper, dies off as $(1 + z)^{-5}$ for constant shear and as $(1 + z)^{-7}$ for shear in the linear density perturbation regime.
3.3 Line of Sight Averaging

As discussed in the previous section, while the DR model does implicitly volume average to a FRW universe, and does approach the FRW distance relation for sufficiently large angular averages, it does not along a given line of sight. Let us ask if there is any way to restore FRW behavior for every line of sight (short of taking a purely FRW universe).

From the closed form expression of Linder (1988a) for the deviation of the angular distance from the FRW behavior,

\[ r(z) - r_{frw}(z) = \frac{3}{2} \Omega \int_0^z ds \frac{1}{(1 + s)(1 + \Omega s)^{1/2}} [1 - \alpha(s)] r(s) r_{frw}(s, z), \]

in a dust universe, one sees that the integrand must oscillate in such a way as to cause the RHS to vanish. For a localized inhomogeneity with density smoothness parameter \( \alpha \), this can be done by requiring a neighboring inhomogeneity to compensate for the density deviation from FRW suffered by the beam. This condition on the two regions becomes, for small inhomogeneity extents \( \Delta z \),

\[ (1 - \alpha_1) \Delta z_1 = -(1 - \alpha_2) \Delta z_2. \]

We will call this density compensation or line of sight averaging.

This is not a particularly elegant or transparent condition; it does not correspond to a simple physical situation like requiring the same projected mass along the line of sight as the FRW model. In fact, that property of having the same projected mass in no way guarantees that the light propagation behavior will be similar – the run of density is what is important. Furthermore, there is no particular reason to believe that a realistic inhomogeneous universe would naturally include density compensation.

The DR model does not generically satisfy this condition for the density along the line of sight, but we can evaluate a model with compensated DR inhomogeneities, i.e. ones arranged to obey equation (6). Along every line of sight we allow there to be a DR inhomogeneity region where \( \alpha \) is less than one. We then compensate for this “lost mass” by taking an adjacent region to have higher than global density, arranging the size (extent in redshift) of this region according to equation (6) to make up exactly for the mass deficit from FRW experienced by the light ray in the first region.

There are a couple of technical points to clarify before proceeding to the results. The first region, with \( \alpha < 1 \), does not actually lack any mass; the DR model says the inhomogeneous region has the same mass and density in the volume averaged sense as the equivalent FRW volume. So there seems no a priori reason why such a compensation would be required for the light ray – within the framework of volume averaging being a legitimate treatment for evaluating light propagation behavior. Nevertheless, we will see that compensation is key, so the volume averaging procedure is lacking.

Secondly, the compensating region will have \( \alpha > 1 \), that is the light ray experiences a density above the FRW average. Mathematically, it is fine to treat this by means of the usual DR clump interpretation but with clumps of negative mass. This is physically a bit unsettling but there are two ways around it. One, one can make the clumps small but not pointlike, as the usual DR model uses, and then they can simply have density sufficiently
below the FRW value that this mass deficit offsets the mass excess from the smoothly distributed above average density matter. Second, one can ignore the interpretation of the density deviations as being in clumps plus a smooth component and simply treat $\alpha \Omega$ in the beam equation as giving the actual density along the light path in this inhomogeneity region. Then there is no problem having $\alpha$ less than or greater than one.

[Note that in the usual solution to the distance-redshift relation the expression breaks down for $\alpha > 25/24$. We are free to take our compensating region to have $1 < \alpha < 25/24$ with a correspondingly greater redshift extent, but we will find that our final solution is independent of the $\alpha$ adopted. So we need not specify it or worry about the upper limit, in line with the second interpretation above.]

We calculate two situations: 1) the observer is in a region with $\alpha = 0$, extending out to $\Delta z$, which is then compensated by an adjoining region according to the above prescription; 2) the observer is in a FRW region but at some $z_1$ the light ray passes through an inhomogeneity with $\alpha = 0$ and extent $\Delta z$, which is compensated by an adjoining region according to the above prescription. In both cases the universe is taken to be FRW outside of the localized inhomogeneities. These cases are the analogs of the uncompensated ones calculated above in section 3.1.

With the compensations, however, we find that the distance-redshift behavior beyond the inhomogeneity agrees exactly with the FRW result. Note that no assumption was made of spherical symmetry or isotropy; the compensation prescription works independently for each line of sight, as designed. As mentioned in section 3.2, flux conservation is then automatic. In this light, one can consider the large angle averaging approach to FRW of Linder (1998) as another method of ensuring compensation: widening the line of sight, i.e. the beam, allows the effective density smoothness to approach the FRW value naturally within the DR model, without the need for a secondary inhomogeneity region. In some sense they are complementary; averaging along the line of sight through compensation, or averaging transverse to the line of sight by beam widening.

For an alternate view of line of sight averaging, we can use the postNewtonian perturbed approach of Jacobs, Linder, & Wagoner (1993). Here, light propagation can be calculated from a metric realistically describing an inhomogeneous universe so long as the gravitational potential $\phi$ of the inhomogeneities is small compared to unity and its first derivatives small compared to the Hubble scale $H_0^{-1}$ (this condition also ensures that the dynamics are driven by the background, FRW model). Given the metric, one can derive the angular distance deviation to be

$$r(z) - r_{frw}(z) = -H_0^{-2} \int_1^{1+z} du \, u^{-1/2} (\nabla^2 \phi) r(u) r_{frw}(u, 1 + z),$$

in a flat, dust background. The right hand side, giving the deviation from FRW distance behavior, arises from the perturbed Ricci tensor.

The condition for approaching FRW is that the integration over the path length must reduce the possibly large (compared to background) but fluctuating effective density deviation, $\nabla^2 \phi$. This would occur naturally unless the density fluctuations were arranged coherently over the path length. Otherwise, the RHS would be expected to be of order $\nabla \phi$, which is small by original assumption. That is, one has sufficient effective mass
compensation (oscillation), as in the above line of sight averaging procedure, that the distance deviation becomes small. So in a realistically inhomogeneous universe one expects the distance-redshift relation to approach FRW for path lengths much larger than the inhomogeneity coherence scale.

4. Conclusion

The results of the three categories of models for an inhomogeneous universe can be summed up concisely:

- A model that is FRW in a volume averaged sense but with different dynamics (e.g. Tolman-Bondi) does not yield FRW results for light propagation.
- A model that is FRW in a volume averaged sense and also agrees dynamically (e.g. Dyer-Roeder) but does not match in a line of sight averaged sense does not generally yield FRW results for light propagation. However, it can if angle averaging provides effective line of sight density compensation.
- A model that is FRW in a line of sight averaged sense (same local mass) and agrees dynamically (compensated Dyer-Roeder) will yield FRW results for light propagation. This seems to be a rather ad hoc model though. Depth averaging, however, where the path length exceeds the inhomogeneity coherence length, can create effective compensations naturally.

Because light propagation probes the spacetime along the path on all scales, a coarse graining by volume averaging over a locally inhomogeneous universe will not necessarily yield the light propagation of that average spacetime. That is, a large scale volume averaging that smooths the universe to FRW, while maybe sufficient for dynamical (e.g. expansion and growth of perturbation) calculations, is not appropriate for recovering FRW light propagation behavior. One must adopt a line of sight averaging that brings the path conditions to FRW in order to match the FRW result.

This can be achieved through 1) ensuring that density perturbations along the line of sight are compensated, 2) smoothing the perturbations by using beams sufficiently wide to sample beyond the extent of the inhomogeneity (essentially the wide beam just redefines what is meant by line of sight), or 3) local volume averaging where the inhomogeneities can be smoothed on scales small enough to retain the background dynamics and provide effective mass compensation within the beam. Again, the local mass agreement with the background is required because light probes the local, not just average, gravitational field.

As one example, for the Tolman-Bondi model of MBHE none of these hold: the dynamics is not FRW and the effective coherence scale is infinite due to the special position of the observer. Generally, if the physical conditions of an observation, in angular width and path characteristics, do not match one of the above situations then one should not expect to be able to legitimately use a FRW distance-redshift relation to interpret that observation.

This work was supported by NASA grants NAG5-3525, NAG5-3922, and NAG5-4064.
REFERENCES

Bildhauer, S. & Futamase, T. 1991, MNRAS, 249, 126
Buchert, T. 1997, in Proc. 2nd SFB Workshop on Astro-particle Physics, ed. R. Bender et al., p. 71 (astro-ph/9706214)
Buchert, T. & Ehlers, J. 1997, A&A, 320, 1
Dyer, C.C. & Roeder, R.C. 1972, ApJ, 174, L115
Dyer, C.C. & Roeder, R.C. 1972, ApJ, 180, L31
Fukushige, T. & Makino, J. 1994, ApJ, 436, L111
Jacobs, M.W., Linder, E.V., & Wagoner, R.V. 1993, PRD, 48, 4623
Linder, E.V. 1988a, A&A, 206, 190
Linder, E.V. 1988b, A&A, 206, 199
Linder, E.V. 1998, ApJ, 497, in press (astro-ph/9707343)
Mustapha, N., Bassett, B.A., Hellaby, C., & Ellis, G.F.R. 1997, submitted to Class. Q. Grav., gr-qc/9708043
Watanabe, K. & Tomita, K. 1990, ApJ, 355, 1