Remarks on Bounds of Normalized Laplacian Eigenvalues of Graphs

Emina I. Milovanović, Igor Ž. Milovanović
Faculty of Electronic Engineering, A. Medvedeva 14, P.O.Box 73, 18000 Niš, Serbia

Abstract
Let $G$ be a connected undirected graph with $n$, $n \geq 3$, vertices and $m$ edges. Denote by $\rho_1 \geq \rho_2 \geq \cdots > \rho_n = 0$ the normalized Laplacian eigenvalues of $G$. Upper and lower bounds of $\rho_i$, $i = 1, 2, \ldots, n - 1$, are determined in terms of $n$ and general Randić index, $R_{-1}$.

Keywords: Normalized Laplacian eigenvalues, general Randić index, inequalities.
AMS subject classifications 05C50

1. Introduction and preliminaries

Let $G = (V, E)$, $V = \{1, 2, \ldots, n\}$, be $n$-vertex undirected connected graph, with $m$ edges and vertex degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n > 0$, $d_i = d(i)$. Denote by $A$ adjacency matrix of $G$ and by $D = \text{diag}\{d_1, d_2, \ldots, d_m\}$ a diagonal matrix of its vertex degrees. Then, $L = D - A$ is Laplacian matrix of $G$. Since it is assumed that $G$ is connected, matrix $D$ is nonsingular, and thus, $D^{-1}$ always exists. Matrix $L^* = \sqrt{D}^{-1}LD^{-1/2} = I - D^{-1/2}AD^{-1/2}$ is called normalized Laplacian matrix of $G$. Eigenvalues of $L^*$, $\rho_1 \geq \rho_2 \geq \cdots > \rho_n = 0$, are normalized Laplacian eigenvalues of graph $G$ (see [3]).

Well known properties of these eigenvalues are [11]

$$
\sum_{i=1}^{n-1} \rho_i = n \quad \text{and} \quad \sum_{i=1}^{n-1} \rho_i^2 = n + 2R_{-1},
$$

where $R_{-1}$ is general Randić index [1, 2, 8, 11]. Well known inequalities valid for $\rho_1$ and $\rho_{n-1}$ are [3]

$$
\rho_1 \geq \frac{n}{n - 1} \quad \text{and} \quad \rho_{n-1} \leq \frac{n}{n - 1},
$$
with equality holding if and only if $G \cong K_n$. In [1–6, 9, 10] several upper/lower bounds for $\rho_1$ and $\rho_{n-1}$ were reported. In the present paper we consider upper and lower bounds for $\rho_i$, $i = 1, 2, \ldots, n-1$.

In the text that follows we first recall some results from spectral graph theory and polynomials needed for our work.

Shi [9] (see also [4]) proved the following result for general Randić index.

**Lemma 1.1.** [9] Let $G$ be a connected undirected graph with $n$ vertices and $m$ edges. Then

$$\frac{n}{2d_1} \leq R_{-1} \leq \frac{n}{2d_n}. \tag{3}$$

Equality holds if and only if $G$ is a regular graph.

Lupas [7] considered a class of real polynomials $P_n(a_1, a_2)$, i.e. the polynomials of type $P_n(x) = x_n + a_1x^{n-1} + a_2x^{n-2} + b_3x^{n-3} + \cdots + b_n$, where $a_1$ and $a_2$ are fixed real numbers. Denote by $x_1 \geq x_2 \geq \cdots \geq x_n$ zeros of polynomial $P_n(x)$, whereby

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \quad \text{and} \quad \Delta = n \sum_{i=1}^{n} x_i^2 - \left( \sum_{i=1}^{n} x_i \right)^2. \tag{4}$$

The following bounds for zeros of polynomial $P_n(x)$ were proved in [7].

**Lemma 1.2.** [7] If $P_n(x) \in \mathcal{P}_n(a_1, a_2)$ then

$$\bar{x} + \frac{1}{n} \sqrt{\frac{\Delta}{n-1}} \leq x_1 \leq \bar{x} + \frac{1}{n} \sqrt{(n-1)\Delta},$$

$$\bar{x} - \frac{1}{n} \sqrt{\frac{i-1}{n-i+1} \Delta} \leq x_i \leq \bar{x} + \frac{1}{n} \sqrt{\frac{n-i}{i} \Delta}, \quad 2 \leq i \leq n-1 \tag{5}$$

$$\bar{x} - \frac{1}{n} \sqrt{(n-1)\Delta} \leq x_n \leq \bar{x} - \frac{1}{n} \sqrt{\frac{\Delta}{n-1}}.$$

2. **Main results**

In the following theorem we determine upper and lower bounds for normalized Laplacian eigenvalues, $\rho_i$, $i = 1, 2, \ldots, n-1$ in terms of $n$ and $R_{-1}$.

**Theorem 2.1.** Let $G$ be a connected undirected graph with $n$, $n \geq 3$, vertices and $m$ edges. Then
\[
\frac{n}{n-1} + \frac{1}{n-1} \sqrt{\frac{2(n-1)R_{-1} - n}{n-2}} \leq \rho_1 \leq \frac{n}{n-1} + \frac{1}{n-1} \sqrt{(n-2)(2(n-1)R_{-1} - n)},
\]

\[
\leq \frac{n}{n-1} + \frac{1}{n-1} \sqrt{(n-2)(2(n-1)R_{-1} - n)}, \quad 2 \leq i \leq n - 2
\]

\[
\frac{n}{n-1} - \frac{1}{n-1} \sqrt{n - i + 1} \frac{2(n-1)R_{-1} - n}{2(n-1)R_{-1} - n} \leq \rho_i \leq \frac{n}{n-1} - \frac{1}{n-1} \sqrt{2(n-1)R_{-1} - n - 2}.
\]

\[
\leq \frac{n}{n-1} - \frac{1}{n-1} \sqrt{2(n-1)R_{-1} - n - 2}.
\]

In all three cases equalities hold if and only if \( G \cong K_n \).

**Proof.** The characteristic polynomial of the normalized Laplacian matrix \( L^* \) of graph \( G \) is

\[
\varphi_n(x) = x\varphi_{n-1}(x) = x \left( x^{n-1} + a_1 x^{n-2} + a_2 x^{n-3} + b_3 x^{n-4} + \cdots + b_{n-1} \right),
\]

whereby

\[
a_1 = -\sum_{i=1}^{n-1} \rho_i = -n \quad \text{and} \quad a_2 = \frac{1}{2} \left( \left( \sum_{i=1}^{n-1} \rho_i \right)^2 - \sum_{i=1}^{n-1} \rho_i^2 \right) = \frac{n(n-1)}{2} - R_{-1}.
\]

This means that polynomial \( \varphi_{n-1}(x) \) belongs to a class of polynomial \( \mathcal{P}_{n-1}(-n, \frac{n(n-1)}{2} - R_{-1}) \). According to (4) for the zeros, \( \rho_1 \geq \rho_2 \geq \cdots \geq \rho_{n-1} > 0 \), of polynomial \( \varphi_{n-1}(x) \) the following equalities are valid

\[
\bar{x} = \frac{1}{n-1} \sum_{i=1}^{n-1} \rho_i = \frac{n}{n-1},
\]

\[
\Delta = (n - 1) \sum_{i=1}^{n-1} \rho_i^2 - \left( \sum_{i=1}^{n-1} \rho_i \right)^2 = 2(n - 1)R_{-1} - n.
\]

Now according to (5) and from (5) for \( n := n - 1, x_i := \rho_i, i = 1, 2, \ldots, n - 1 \) we obtain the required result.
Remark 2.1. Left-side inequality in (6) and right-side inequality in (7) were proved in [10].

Corollary 1. Let $G$ be a connected undirected graph with $n$, $n \geq 3$, vertices and $m$ edges. Then

$$\frac{n}{n-1} + \frac{1}{n-1} \sqrt{\frac{n(n-1-d_1)}{(n-2)d_1}} \leq \rho_1 \leq \frac{n}{n-1} + \frac{1}{n-1} \sqrt{\frac{n(n-2)(n-1-d_n)}{d_n}}$$

(9)

$$\frac{n}{n-1} - \frac{1}{n-1} \sqrt{\frac{n(i-1)(n-1-d_n)}{(n-i)d_n}} \leq \rho_i \leq \frac{n}{n-1} - \frac{1}{n-1} \sqrt{\frac{n(n-i-1)(n-1-d_n)}{id_n}}, 2 \leq i \leq n-2$$

$$\frac{n}{n-1} - \frac{1}{n-1} \sqrt{\frac{n(n-2)(n-1-d_n)}{d_n}} \leq \rho_{n-1} \leq \frac{n}{n-1} - \frac{1}{n-1} \sqrt{\frac{n(n-1-d_1)}{(n-2)d_1}}.$$  

(10)

In all three cases equalities hold if and only if $G \cong K_n$.

Remark 2.2. Let us note that left-side inequalities in (6) and (9) as well as right-side inequalities in (7) and (10) are stronger than those in (2).

Remark 2.3. Based on the left inequality in (6) and the right inequality in (7) the following inequalities are obtained, respectively

$$R_{-1} \leq \frac{1}{2} (n-1)(n-2) \left( \rho_1 - \frac{n}{n-1} \right)^2 + \frac{n}{2(n-1)}$$

and

$$R_{-1} \leq \frac{1}{2} (n-1)(n-2) \left( \frac{n}{n-1} - \rho_{n-1} \right)^2 + \frac{n}{2(n-1)},$$

which were proved in [10] [Theorem 1.1].
Corollary 2. Let $G$ be a connected undirected graph with $n$, $n \geq 3$, vertices and $m$ edges. Then
\begin{equation}
R_{-1} \geq \frac{n-1}{2(n-2)} \left( \rho_1 - \frac{n}{n-1} \right)^2 + \frac{n}{2(n-1)} \tag{11}
\end{equation}
and
\begin{equation}
R_{-1} \geq \frac{n-1}{2(n-2)} \left( \frac{n}{n-1} - \rho_{n-1} \right)^2 + \frac{n}{2(n-1)} \tag{12}
\end{equation}
Equalities hold if and only if $G \cong K_n$.

Remark 2.4. The inequality (11) was proved in [10] [Theorem 1.2], under slightly stronger condition. Also, it is not difficult to see that inequalities (11) and (12) are stronger than inequality $R_{-1} \geq \frac{n}{2(n-1)}$ proved in [6].

References

[1] M. Cavers, The normalized Laplacian matrix and general Randić index of graphs, A. Thesis, University of Regina, Saskatchewan, 2010.

[2] M. Cavers, S. Fallat, S. Kirkland, On the normalized Laplacian energy and general Randić index $R_{-1}$ of graphs, Linear Algebra Appl., 433 (2010), 172–190.

[3] F.R. K. Chung, Spectral graph theory, Amer. Math. Soc., Providence, 1997.

[4] R. Gu, F. Huang, X. Li, General Randić matrix and general Randić energy, Trans. Combin., 3 (3) (2014), 21–33.

[5] J. Li, J-M. Guo, W. C. Shiu, Bounds on normalized Laplacian eigenvalues of graphs, J. Ineq. Appl., (2014), 2014:316.

[6] X. Li, Y. Yang, Sharp bounds for the general Randić index, MATCH Commun. Math. Comput. Chem., 52 (2004), 147–156.

[7] A. Lupas, Inequalities for the roots of a class of polynomials, Univ. Beograd Publ. Elektroteh. Fak. Ser. Mat. Fiz. 577-598 (1977), 79–85.

[8] M. Randić, On characterization of molecular branching, J. Am. Chem. Soc., 97 (1975), 6609–6615.

[9] L. Shi, Bounds on Randić indices, Discr. Appl. Math., 309 (2009), 5238–5241.
[10] G. Yu, L. Peng, Randić index and eigenvalues of graphs, Rocky Mountain J. Math., 40 (2) (2010), 713–721.

[11] P. Zumstein, Comparison of spectral methods through the adjacency matrix and the Laplacian of a graph, Th. Diploma, ETH Zürich, 2005.