Scattering of Charged Tensor Bosons
in
Gauge and Superstring Theories

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Abstract

We calculate the leading-order scattering amplitude of one vector and two tensor gauge bosons in a recently proposed non-Abelian tensor gauge field theory and open superstring theory. The linear in momenta part of the superstring amplitude has identical Lorentz structure with the gauge theory, while its cubic in momenta part can be identified with an effective Lagrangian which is constructed using generalized non-Abelian field strength tensors.

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1 Introduction

An infinite tower of particles of high spin naturally appears in the spectrum of different string theories. In the zero slope limit massless states of open and closed strings can be identified with vector - Yang-Mills and tensor - graviton gauge quanta \[1, 2, 3, 4, 6, 7\]. Massive string states can be described by string field theory developed in \[8, 9, 10, 12, 11, 13, 14\]. Nevertheless to represent the Lagrangian and equations in terms of components of tensor fields still remains a challenge \[10\]. In this respect higher spin field theories have received large attention \[15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28\] together with the recent development of interacting field theories of high spins \[29, 30, 31, 32, 33, 34, 35, 36\].

As we mentioned the massless states of open superstring theory with Chan-Paton charges \[37\] have been identified with the Yang-Mills gauge quanta \[1, 6, 7, 38, 39, 40, 41\] and it is of great importance to identify also massive higher spin string states with states of some Lagrangian quantum field theory.

One can imagine that massive states of open string may be described by some extension of Yang-Mills theory to non-Abelian tensor gauge field theory. Such extension of Yang-Mills theory which includes charged tensor gauge fields was suggested recently in \[42, 43, 44, 45\]. Not much is known about physical properties of this gauge field theory and our intension is to compare tree-level scattering amplitudes of tensor gauge bosons in non-Abelian tensor gauge field theory and open superstring theory.

Non-Abelian tensor gauge fields are defined as rank-(s+1) tensor potentials \( A^{a}_{\mu_{1}...\lambda_{s}} \). The gauge invariant Lagrangian describing dynamical tensor gauge bosons of all ranks has the form \[42, 43, 44, 45\]

\[
\mathcal{L} = \mathcal{L}_{YM} + \mathcal{L}_{2} + \mathcal{L}_{3} + ..., \quad (1.1)
\]

where \( \mathcal{L}_{YM} \) is the Yang-Mills Lagrangian and defines cubic and quartic interactions with dimensionless coupling constant. For the lower-rank tensors, the Lagrangian has the following form \[42, 43, 44\]:

\[
\mathcal{L}_{2} = -\frac{1}{4} G^{a}_{\mu_{1}\lambda_{1}} G^{a}_{\mu_{2}\lambda_{2}} - \frac{1}{4} G^{a}_{\mu_{2}} G^{a}_{\mu_{1}\lambda_{1}\lambda_{2}} + \frac{1}{4} G^{a}_{\mu_{1}\lambda_{1}} G^{a}_{\mu_{2}\lambda_{2}} + \frac{1}{4} G^{a}_{\mu_{2}} G^{a}_{\mu_{1}\lambda_{1}\lambda_{2}}, \quad (1.2)
\]

\(^{1}\)Tensor gauge fields \( A^{a}_{\mu_{1}...\lambda_{s}}(x) \), \( s = 0, 1, 2, ... \) are totally symmetric with respect to the indices \( \lambda_{1}...\lambda_{s} \), but with no a priori symmetry with respect to the first index \( \mu \). In particular, we have \( A^{a}_{\mu\lambda} \neq A^{a}_{\lambda\mu} \) and \( A^{a}_{\mu\rho} = A^{a}_{\mu\rho} \neq A^{a}_{\lambda\rho} \). The adjoint group index \( a = 1, ..., N^{2} - 1 \) in the case of \( SU(N) \) gauge group.

\(^{2}\)In D-dimensions the coupling constant has dimension \((4 - D)/2\).
where the generalized gauge field strengths are:

\[ G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc} A^b_\mu A^c_\nu, \]

\[ G^a_{\mu\nu,\lambda} = \partial_\mu A^a_{\nu\lambda} - \partial_\nu A^a_{\mu\lambda} + gf^{abc} (A^b_\mu A^c_\nu + A^b_\nu A^c_\mu), \]

\[ G^a_{\mu\nu,\lambda\rho} = \partial_\mu A^a_{\nu\lambda\rho} - \partial_\nu A^a_{\mu\lambda\rho} + gf^{abc} (A^b_\mu A^c_{\nu\lambda\rho} + A^b_{\nu\lambda\rho} A^c_\mu + A^b_\nu A^c_{\mu\lambda\rho}). \]

The Lagrangian forms \( \mathcal{L}_s \) for higher-rank fields can be found in Refs.\[42, 43, 44\].

Here we shall focus our attention on the lower-rank tensor gauge field \( A^a_{\mu\lambda} \), which decomposes into a symmetric tensor \( T^S \) of spin two and an antisymmetric tensor \( T^A \), Poincaré dual of spin zero, charged gauge bosons \[44\]. The Feynman rules for these propagating modes and their interaction vertices can be extracted from the Lagrangian (1.2) and allow to calculate tree-level scattering amplitudes for processes involving vector and tensor gauge bosons \[44\].

In the spectrum of open superstring theory with Chan-Paton charges there is also a massless vector gauge boson \( V \) on the first excited level and a rank-two massive tensor boson \( T_S \) at the second level. The emission vertices for these states are defined as follows (in the zero and -1 ghost picture for \( V \) and \( T_S \), respectively) \[6, 7, 38\]:

\[
V : \quad \varepsilon^\mu(k)(\hat{X}^\mu - 2i\alpha' k \cdot \psi\psi^\mu)e^{ikX} \quad \alpha' k^2 = 0
\]

\[
T_S : \quad \varepsilon_{\mu\nu}(k)\psi^\mu(\hat{X}^\nu - 2i\alpha' k \cdot \psi\psi^\nu)e^{ikX} \quad \alpha' k^2 = -1. \quad (1.4)
\]

and allow to calculate different tree level scattering amplitudes involving vector and tensor bosons \( V \) and \( T_S \). Our intension is to compare tree-level scattering amplitudes of tensor gauge bosons in the above non-Abelian tensor gauge field theory and in open superstring theory.

We have found that the linear in momenta part of the 3-point superstring amplitude has similar Lorentz structure with the one in gauge theory defined by the Lagrangian \( \mathcal{L}_2 \) (1.2) and that the cubic in momenta part of the superstring amplitude can be identified with an effective Lagrangian \( \mathcal{L}_\theta \) (6.23) and \( \mathcal{L}'_\theta \) (6.24) constructed using non-Abelian field strength tensors (1.3). This result suggests that most probably non-Abelian tensor gauge field theory describes a sub-sector of excited states of open superstring theory with higher helicities, similar to a Yang-Mills theory describing the first excited state. More complicated amplitudes should be analyzed in order to solidify this proposal.
2 VTT Amplitude in Tensor Gauge Theory

In massless tensor gauge field theory, the on-shell tensor-vector-tensor amplitude VTT is

\[ M_{\text{gauge theory}} = \varepsilon_{\alpha\dot{\alpha}}(k_1)e_\beta(k_3)\varepsilon_{\gamma\gamma'}(k_2) \mathcal{F}^{\alpha\dot{\alpha}\beta\gamma\gamma'}(k_1, k_3, k_2) \delta(k_1 + k_2 + k_3) \] (2.5)

where

\[ \mathcal{F}^{\alpha\dot{\alpha}\beta\gamma\gamma'}(k_1, k_3, k_2) = k_1^\beta \left( \eta^{\alpha\gamma}\eta^{\dot{\alpha}\gamma'} + \eta^{\alpha\gamma'}\eta^{\dot{\alpha}\gamma} \right) \]

\[ + \frac{1}{4} k_1^\alpha \left( \eta^{\beta\gamma}\eta^{\alpha\gamma'} + \eta^{\beta\gamma'}\eta^{\alpha\gamma} \right) \]

\[ + \frac{1}{4} k_1^\alpha \left( \eta^{\beta\dot{\alpha}\gamma'} + \eta^{\beta\dot{\alpha}\gamma} \right) \]

\[ + \frac{1}{4} k_1^{\dot{\alpha}} \left( \eta^{\beta\gamma}\eta^{\alpha\gamma'} + \eta^{\beta\gamma'}\eta^{\alpha\gamma} \right) \]

\[ + \frac{1}{4} k_1^{\dot{\alpha}} \left( \eta^{\beta\dot{\alpha}\gamma'} + \eta^{\beta\dot{\alpha}\gamma} \right), \] (2.6)

where \( k_i^2 = 0 \) \( i = 1, 2, 3 \) and \( k_{ij} = k_i - k_j \). It is important that in this massless theory the momentum conservation \( \delta(k_1 + k_2 + k_3) \) has a nontrivial solution

\[ k_1 = (\omega, 0, 0, r), \quad k_3 = (\omega, 0, 0, r), \quad k_2 = (-2\omega, 0, 0, -2r) \]

\((\omega^2 = r^2)\) that can be deformed by a complex parameter \( z \) [47, 48, 49, 50, 51, 52, 53],

\[ k_1 = (\omega, z, iz, r), \quad k_3 = (\omega, -z, -iz, r), \quad k_2 = (-2\omega, 0, 0, -2r). \] (2.7)

Thus, the above expression for VTT \( \mathcal{F}^{\alpha\dot{\alpha}\beta\gamma\gamma'}(k_1, k_3, k_2) \) has a nonzero phase space of validity which is parameterized by the complex parameter \( z \) (2.7). Depending on polarizations of scattered particles one can see that there are only four non-zero helicity amplitudes \( M(+2, +1, -2), \ M(-2, +1, +2), \ M(+2, -1, -2) \) and \( M(-2, -1, +2) \).

Contracting Lorentz indices one can see that in tensor gauge theory the amplitude (2.5), (2.6) can explicitly be written in the form

\[ M_{\text{gauge theory}} = \varepsilon_{\alpha\dot{\alpha}}(k)e_\beta(p)e_{\gamma\gamma'}(q)\mathcal{F}^{\alpha\dot{\alpha}\beta\gamma\gamma'}(k, p, q)\delta(k + p + q) = \]

\[ = 4(k \cdot e_p) \cdot (\varepsilon_k \cdot e_q) - 2(p \cdot \varepsilon_k \cdot e_p \cdot e_q) + 2(e_p \cdot \varepsilon_k \cdot e_q \cdot p). \] (2.8)

Furthermore, one should take into account that all particles are massless \( k^2 = p^2 = q^2 = 0 \) and that the momentum conservation \( k + p + q = 0 \) gives \( k \cdot p = p \cdot q = q \cdot k = 0 \). Thus the
VTT amplitude is nontrivial only if one considers complex momenta (2.7) or the space-time signature $\eta^{\mu\nu} = (- + - +)$. The polarization vectors and tensors we shall take are then in the form

$$e^+_k = \frac{1}{\sqrt{2}} (\frac{z}{\omega}, 1, -i, -\frac{z}{r}), \quad e^+_p = \frac{1}{\sqrt{2}} (-\frac{z}{\omega}, 1, -i, \frac{z}{r}), \quad e^-_q = \frac{1}{\sqrt{2}} (0, 1, i, 0)$$

where $\varepsilon_{\alpha\alpha'}(k) = e_\alpha(k)e_{\alpha'}(k)$ and $\varepsilon_{\gamma\gamma'}(q) = e_{\gamma}(q)e_{\gamma'}(q)$. We can calculate now the amplitude (2.8) using the relations $e_k \cdot e_q = e_p \cdot e_q = 1, \quad k \cdot e_p = -p \cdot e_k = 2\sqrt{2}z$ and $e_k \cdot e_p = 0$

$$M(+2, +1, -2)_{\text{gauge theory}} = 4(k \cdot e_p) (e_k \cdot e_q) - 2(p \cdot e_k \cdot e_q \cdot e_p) + 2(e_p \cdot e_k \cdot e_q \cdot p) = 12\sqrt{2}z,$$

so that the VTT amplitude has non-trivial analytical continuation and is proportional to the deformation parameter $z$. In the same way one can compute other polarization amplitudes.

### 3 Mass-shell gauge invariance of VTT in tensor gauge theory

The expression for the VTT amplitude (2.8) is on mass-shell gauge invariant

$$e_\beta(p) \rightarrow e_\beta(p) + \xi p_\beta \quad \varepsilon_{\alpha\alpha'}(k) \rightarrow \varepsilon_{\alpha\alpha'}(k) + k_\alpha \xi_{\alpha'}, \quad k^2 = 0, \quad k \cdot \xi = 0$$

where $\xi$ and $\xi_{\alpha}$ are gauge parameters. Indeed the gauge variation of (2.8) under (3.10) $\delta e_p \sim p$ is

$$\delta M_{\text{gauge theory}} = 4(k \cdot p) (e_k \cdot e_q) - 2(p \cdot e_k \cdot e_q \cdot e_p) + 2(e_p \cdot e_k \cdot e_q \cdot p) = 0$$

and under (3.10) $\delta e_k \sim k \otimes \xi + \xi \otimes k$ is

$$\delta M_{\text{gauge theory}} = 8(k \cdot e_p) (k \cdot e_q \cdot \xi) - 2(p \cdot k) (\xi \cdot e_q \cdot e_p) - 2(p \cdot \xi) (k \cdot e_q \cdot e_p) + 2(e_p \cdot k) (\xi \cdot e_q \cdot p) + 2(e_p \cdot \xi) (k \cdot e_q \cdot p) = 0,$$

because $p \cdot k = 0, \quad k \cdot e_q = p \cdot e_q = 0$. Thus, in tensor gauge field theory, the VTT amplitude gets non-trivial values in four cases $M(+2, +1, -2), \quad M(-2, +1, +2), \quad M(+2, -1, -2), \quad M(-2, -1, +2)$ and is explicitly gauge invariant quantity. This completes the analysis of VTT scattering amplitude in tensor gauge field theory.
4 VTT Amplitude in Superstring Theory

In open superstring theory, the linear in momenta part (7.37), (7.40) of the full VTT amplitude (7.35), (7.36) for massless vector and massive tensors in ten dimensions is given by the expression

\[ M_{\text{string theory}} = \varepsilon_{a\dot{a}}(k_1)e_{\beta}(k_3)\varepsilon_{\gamma\gamma'}(k_2) F^{a\alpha\beta\gamma\gamma'}(k_1, k_3, k_2) \delta(k_1 + k_2 + k_3), \quad (4.13) \]

where

\[
F^{a\alpha\beta\gamma\gamma'}(k_1, k_3, k_2) = + k_{12}^\beta (\eta^{\alpha\gamma}\eta^{\alpha'\gamma'} + \eta^{\alpha\gamma'} \eta^{\alpha'\gamma}) \\
+ k_{23}^{\alpha} (\eta^{\beta\gamma}\eta^{\alpha\gamma'} + \eta^{\beta\gamma'} \eta^{\alpha\gamma}) \\
+ k_{23}^{\alpha'} (\eta^{\beta\gamma}\eta^{\alpha\gamma'} + \eta^{\beta\gamma'} \eta^{\alpha\gamma}) \\
+ k_{31}^\gamma (\eta^{\alpha\beta}\eta^{\alpha'\gamma'} + \eta^{\alpha'\beta} \eta^{\alpha\gamma'}) \\
+ k_{31}^\gamma' (\eta^{\alpha\beta}\eta^{\alpha'\gamma} + \eta^{\alpha'\beta} \eta^{\alpha\gamma}). \quad (4.14)
\]

Formally comparing these amplitudes in tensor gauge theory (2.6) and in string theory (4.14) one can see that they have identical Lorentz structure and are linear in momenta. But there is a difference in coefficients between these two expressions in last four terms: 1/4 in gauge theory and 1 in string theory. It seems that this may contradict to the gauge invariance of both scattering amplitudes. But one should keep in mind that in string theory tensor particles are massive and momentum conservation equation has no solutions unlike the massless case. Nevertheless we shall try to define the amplitude in string theory by taking a special limit and then to demonstrate that it is also on mass-shell gauge invariant.

Indeed in string theory tensor states are massive \( m_T^2 = 1/\alpha' \) and the vector boson is massless \( m_V^2 = 0 \), therefore the momentum conservation \( \delta(k_1 + k_2 + k_3) \) has no solutions at all. The expression for the amplitude (4.13) has therefore a formal character because it is multiplied by a delta function which vanishes identically. The idea is to find a reasonable extension of the string scattering amplitude considering some non-trivial limit that will allow to define it away from the zeros of the delta function. Let us first consider the wave

\[\text{The details of the calculation are given in section 7 and in Appendix.}\]
function of a massive tensor

\[
e_{\alpha \alpha'} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\epsilon^2 = r^2 + m_T^2
\]  \hspace{1cm} (4.15)

where the first two wave functions correspond to helicities ±2, the next two correspond to helicities ±1 and the last one to 0. In the limit \( m_T \to \infty \) and \( r \to 0 \) wave functions have a well defined limit and can be used to select only helicity ±2 states as in the tensor gauge field theory. Thus we are selecting only a subclass of superstring amplitudes with higher helicities.

Now considering the momentum conservation equation for the choice

\[
k_1 = (m_T, 0, 0, 0, \ldots), \quad k_3 = (\omega, 0, 0, r, \ldots), \quad k_2 = (-\sqrt{r^2 + m_T^2}, 0, 0, -r, \ldots),
\]

and its complex deformation

\[
k_1 = (m_T, z, iz, 0, \ldots), \quad k_3 = (\omega, -z, -iz, r, \ldots), \quad k_2 = (-\sqrt{r^2 + m_T^2}, 0, 0, -r, \ldots)
\]

one can see that the equation fulfills if \( 2m_T r = 0 \). Therefore, if we take the limit \( m_T \to \infty \), \( r \to 0 \), so that \( m_T r \to 0 \), then the momentum conservation indeed can be fulfilled. Thus it seems possible to define this amplitude in superstring model. Physically, this limit corresponds to the interaction between infinitely heavy tensor bosons \( (m_T \to \infty) \) and massless vector bosons which are in the deep infrared region of the spectrum \( (r \to 0) \). It is like an exchange interaction between very heavy ions at rest and photons of tiny energy.

Thus in superstring theory the amplitude (4.13), (4.14) can be written in the form

\[
M_{\text{string theory}} = \varepsilon_{\alpha \bar{\alpha}}(k)\varepsilon_{\beta \bar{\beta}}(p)\varepsilon_{\gamma \bar{\gamma}}(q)F^{\alpha \bar{\alpha} \beta \bar{\beta} \gamma \bar{\gamma}}(k; p, q) =
4(k \cdot e_p) \cdot (\varepsilon_k \cdot \varepsilon_q) - 8(p \cdot \varepsilon_k \cdot \varepsilon_q \cdot e_p) + 8(e_p \cdot \varepsilon_k \cdot \varepsilon_q \cdot p).
\]  \hspace{1cm} (4.16)
The momenta we shall take as above:

\[ k = (m_T, z, iz, 0, ...), \quad p = (\omega, -z, -iz, r, ...), \quad q = (-\sqrt{r^2 + m_T^2}, 0, 0, -r, ...), \quad (4.17) \]

(\(\omega^2 = r^2\)) and the polarization tensors can be chosen as follows

\[
e_k = \frac{1}{\sqrt{2}}(\frac{2z}{m_T}, 1, -i, -\frac{2z}{m_T}, ...), \quad e_p = \frac{1}{\sqrt{2}}(-\frac{z}{\omega}, 1, -i, \frac{z}{r}, ...), \quad e_q = \frac{1}{\sqrt{2}}(0, 1, i, 0, ...)
\]

where \(\varepsilon_{a\alpha}(k) = e_{a}(k)\varepsilon_{\alpha}(k), \quad \varepsilon_{\gamma\zeta}(q) = e_{\gamma}(q)\varepsilon_{\zeta}(q)\). We can calculate now the amplitude \((4.16)\) using the relations \(e_k \cdot e_p = 0, \quad e_k \cdot e_q = 1, \quad e_p \cdot e_q = 1, \quad k \cdot e_p = z(2 + m_T/r)/\sqrt{2}, \quad p \cdot e_k = -z(2 + 4r/m_T)/\sqrt{2}\). Thus we shall get nontrivial analytical continuation of the VTT amplitude in string theory

\[
M(+2, +1, -2)_{\text{string theory}} = 4(k \cdot e_p) (\varepsilon_k \cdot \varepsilon_q) - 8(p \cdot \varepsilon_k \cdot \varepsilon_q \cdot e_p) + 8(e_p \cdot \varepsilon_k \cdot \varepsilon_q \cdot p) = 12\sqrt{2}z + 2\sqrt{2}z\left(\frac{m_T}{r} + 8\frac{r}{m_T}\right), \quad (4.18)
\]

which has a part which is identical with the massless tensor gauge theory \(12\sqrt{2}z\) in \((2.9)\) and an additional part which depends on the mass of the tensor particle \(m_T\). One should take now the limit \(m_T \to \infty, \quad r \to 0, \quad m_T r \to 0, \quad \text{keeping} \quad z\frac{m_T}{r} = Z \text{ fixed, so that} \quad M(+2, +1, -2)_{\text{string}} = 2\sqrt{2}Z\).

5 \hspace{1cm} \textbf{Mass-shell gauge invariance in superstring theory}

This expression is also on mass-shell gauge invariant

\[
e_{\beta}(p) \to e_{\beta}(p) + \xi p_{\beta}
\]

\[
\varepsilon_{a\alpha'}(k) \to \varepsilon_{a\alpha'}(k) + k_{\alpha'}\xi_{\alpha}, \quad k^2 = -m_T^2, \quad k \cdot \xi = 0 \quad (5.19)
\]

where \(\xi\) and \(\xi_{\alpha}\) are gauge parameters. The gauge variation of \((4.16)\) under \((5.19)\) \(\delta e_p \sim p\) is

\[
\delta M_{\text{string theory}} = 4(k \cdot p) (\varepsilon_k \cdot \varepsilon_q) - 8(p \cdot \varepsilon_k \cdot \varepsilon_q \cdot p) + 8(p \cdot \varepsilon_k \cdot \varepsilon_q \cdot e_p) = 4rm_T \to 0 \quad (5.20)
\]

and under \((5.19)\) \(\delta \varepsilon_k \sim k \otimes \xi + \xi \otimes k\)

\[
\delta M_{\text{string theory}} = 8(k \cdot e_p) (k \cdot \varepsilon_q \cdot \xi) - 8(p \cdot k) (\xi \cdot \varepsilon_q \cdot e_p) - 8(p \cdot \xi) (k \cdot \varepsilon_q \cdot p) + 8(e_p \cdot k) (\xi \cdot \varepsilon_q \cdot p) + 8(e_p \cdot \xi) (k \cdot \varepsilon_q \cdot p) = -8rm_T (\xi \cdot \varepsilon_q \cdot e_p) \to 0, \quad (5.21)
\]

because \(p \cdot k = rm_T \to 0, \quad k \cdot e_q = 0, \quad p \cdot e_q = 0\). Thus, this expression is also gauge invariant on mass-shell for gauge variations \(\xi_{\alpha}\) in any direction of the ten dimensional space-time.
6 Effective Action in Terms of Tensor Gauge Fields

In the full superstring amplitude (7.35), (7.36) together with the linear part (7.37) we have also an additional term which is cubic $k^3$ in momenta

\[
\alpha' \{ -\eta^{'\alpha'\gamma}k_3^\alpha k_1^\beta k_2^\gamma + \eta^{\beta'\gamma}k_3^\beta k_2^\alpha k_1^\gamma - \eta^{\beta'\alpha}k_3^\beta k_2^\gamma k_1^\alpha + \\
+ (k_2 \cdot k_3) k_2^\alpha (\eta^{\alpha'\gamma} - \eta^{\alpha'\beta}) + (k_1 \cdot k_3) k_3^\gamma (\eta^{\alpha'\beta} - \eta^{\alpha'\gamma}) + \\
+ (k_1 \cdot k_2) (\eta^{\beta'\gamma} - \eta^{\beta'\alpha}) k_3^\alpha + (\eta^{\alpha'\beta} - \eta^{\alpha'\gamma}) k_3^\alpha \\
+ (\eta^{\alpha'\gamma} - \eta^{\alpha'\beta}) k_3^\gamma + (\eta^{\alpha'\beta} - \eta^{\alpha'\gamma}) k_3^\gamma \\
+ (\eta^{\alpha'\gamma} - \eta^{\alpha'\beta}) k_3^\beta \}.
\]

(6.22)

In particular it contains scalar products $k_1 \cdot k_2$, $k_1 \cdot k_3$ and $k_2 \cdot k_3$. In the three particle scattering amplitude, which we are considering here, they can take only fixed values $k_1 \cdot k_3 = k_2 \cdot k_3 = 0$, $2\alpha' k_1 \cdot k_2 = -\alpha' (k_1^2 + k_2^2) = 2$, therefore they do not appear in the final expression (7.36). Nevertheless let us keep them all, as they are, in order to examine if they can be reproduced by an effective Lagrangian which is constructed using generalized field strength tensors (1.3).

Naturally we should try to associate these cubic terms with a gauge invariant effective Lagrangian which has higher derivatives. Indeed, there are two independent gauge invariant forms which can be constructed in tensor gauge field theory using the field strengths (1.3) and are cubic in derivatives

\[
\mathcal{L}_\partial = \alpha' [ Tr(G_{\mu\nu,\lambda}G_{\nu\rho,\lambda}) + \frac{1}{2} Tr(G_{\mu\nu}G_{\nu\rho,\lambda}\lambda G_{\rho\mu}) ]
\]

(6.23)

and

\[
\mathcal{L}_\partial' = \alpha' [ -Tr(G_{\mu\nu,\lambda}G_{\nu\rho,\lambda}) + Tr(G_{\mu\lambda,\rho}G_{\mu\nu}) + Tr(G_{\nu,\lambda}G_{\nu,\rho}) + \\
+ Tr(G_{\mu,\lambda}G_{\mu,\rho}) + Tr(G_{\nu,\lambda}G_{\nu,\rho}) + Tr(G_{\mu,\lambda}G_{\mu,\rho}) + \\
+ Tr(G_{\mu,\lambda}G_{\mu,\rho}) + Tr(G_{\nu,\lambda}G_{\nu,\rho}) + Tr(G_{\mu,\lambda}G_{\mu,\rho}) + \\
+ 2Tr(G_{\mu,\lambda}G_{\mu,\rho}) - Tr(G_{\mu,\lambda}G_{\mu,\rho}) ]
\]

(6.24)

It is interesting that reproducing the higher derivative part (6.22) of the VTT vertex, there are no “traces” of any higher derivative string (gravity) vertex VVT between two vectors and a tensor (two photons and a graviton). What is also striking is that one reproduces
all terms with scalar products of momenta $k_1 \cdot k_2$, $k_1 \cdot k_3$ and $k_2 \cdot k_3$ in (6.22). In our on-mass-shell scattering amplitude they have fixed values and did not “show up”, but they will certainly contribute to other more complicated amplitudes. Therefore it is important that they are present in the effective Lagrangian.

In the next section we shall present the actual calculation of the superstring scattering amplitude (4.13), (7.35). As our calculation shows, the $(\alpha')^4 k^5$ terms are absent in superstring theory.

7 Open Type I Superstring Tree-Level Amplitudes

To set up notation let us begin with the simplest example of the tree-level scattering amplitude for three on-shell massless vector bosons. The vertex operator has the following form [6, 7]:

$$\mathcal{V}^0 = e_\alpha(k)(X^\alpha - 2i\alpha' k \cdot \psi\psi) e^{ikX}(y)$$

$$\mathcal{V}^{-1} = e_\alpha(k) e^{-\phi} \psi^\alpha e^{ikX}(y)$$

and we shall represent the disk as the upper half-plane so that the boundary coordinate $y$ is real $y \in [-\infty, +\infty]$. The tree amplitude can take the form

$$\mathcal{V}_{\mu_1\mu_2\mu_3}^{\nu_1}(k_1, k_2, k_3) = F^{\mu_1\mu_2\mu_3}(k_1, k_2, k_3) tr(\lambda^\alpha \lambda^\beta \lambda^\gamma) + F^{\mu_2\mu_3\mu_1}(k_2, k_1, k_3) tr(\lambda^\beta \lambda^\alpha \lambda^\gamma),$$

where $\lambda^\alpha$ are isotopic matrices and the matrix element $F$ is given below

$$F^{\mu_1\mu_2\mu_3}(k_1, k_2, k_3) = \langle c \mathcal{V}^{-1}(y_1) c \mathcal{V}^{-1}(y_2) c \mathcal{V}^0(y_3) =$$

$$= \langle ce^{-\phi} \psi^\mu e^{ik_1X}(y_1) ce^{-\phi} \psi^\mu e^{ik_2X}(y_2) c(X^\mu - 2i\alpha' k_3 \cdot \psi\psi) e^{ik_3X}(y_3) >$$

$$= y_{12} y_{23} y_{13}^{-1}$$

$$\{ F_{y_3}^{\mu_3} \eta^{\mu_1\mu_2} y_{12}^{-1} + 2i\alpha' k_{\mu_3}^{1} y_{13}^{-1} y^{\mu_2\mu_3} y_{23}^{-1} - 2i\alpha' k_{\mu_2}^{1} y_{23}^{-1} y^{\mu_1\mu_3} y_{13}^{-1} \}. $$

Here $y_{ij} = y_i - y_j$, $y_3 < y_2 < y_1$ and we have to sum over two orderings of the vertex operators on the disk. The vector function $F_{y_3}^{\mu_3}$ is given by the expression

$$F_{y_3}^{\mu_3} = -2i\alpha' \left( \frac{k_{\mu_3}^{1}}{y_3 - y_1} + \frac{k_{\mu_3}^{2}}{y_3 - y_2} \right) = -2i\alpha' \frac{k_{\mu_3}^{1} y_{12}}{y_{13} y_{23}}. $$

\[1\] In eq. (7.25) and below, the superscript -1 stands for the $(−1)$-ghost picture.
The relevant correlation functions are
\[ < c(y_1)c(y_2)c(y_3) >= y_{12} y_{23} y_{13}, \quad < e^{-\phi}(y_1)e^{-\phi}(y_2) >= y_{12}^{-1}, \]
while the contraction of world-sheet fermions is
\[ < \psi^\mu(y_1)\psi^\nu(y_2) >= \eta^{\mu\nu} y_{12}^{-1}. \]

All bosons are on mass-shell \( \alpha' k_1^2 = \alpha' k_2^2 = \alpha' k_3^2 = 0 \) and \( k_1 + k_2 + k_3 = 0 \). Their wave functions are \( e_{\mu_1}(k_1), \ e_{\mu_2}(k_2), \ e_{\mu_3}(k_3) \) and are transversal to the corresponding momenta \( k_i \cdot e(k_i) = 0, \ i = 1, 2, 3 \). One sees that the matrix element \( F^{\mu_1\mu_2\mu_3} \) is linear in momentum
\[
[F^{\mu_1\mu_2\mu_3}] \sim \alpha' k. \quad (7.26)
\]

Unlike the bosonic open string amplitude, there is no \( k^3 \) term and so no \( G^3 \) term in the low energy effective action. Thus for the \( F^{\mu_1\mu_2\mu_3}(k_1, k_2, k_3)tr(\lambda^{a_1}\lambda^{a_2}\lambda^{a_3}) \) we have
\[
2i\alpha' [k_3^{\mu_1}\eta^{\mu_2\mu_3} - k_3^{\mu_2}\eta^{\mu_1\mu_3} - k_3^{\mu_3}\eta^{\mu_1\mu_2}] \ tr(\lambda^{a_1}\lambda^{a_2}\lambda^{a_3}). \quad (7.27)
\]

Adding the equal term
\[
2i\alpha' [-k_1^{\mu_1}\eta^{\mu_2\mu_3} + k_1^{\mu_2}\eta^{\mu_1\mu_3} + k_1^{\mu_3}\eta^{\mu_1\mu_2}] \ tr(\lambda^{a_1}\lambda^{a_2}\lambda^{a_3})
\]
and the reversed cyclic orientation amplitude \( a_1, \mu_1, k_1 \leftrightarrow a_2, \mu_2, k_2 \), we can get the total matrix element:
\[
i\alpha' [(k_3 - k_2)^{\mu_1}\eta^{\mu_2\mu_3} + (k_1 - k_3)^{\mu_2}\eta^{\mu_1\mu_3} + (k_2 - k_1)^{\mu_3}\eta^{\mu_1\mu_2}] \ tr([\lambda^{a_1}, \lambda^{a_2}]\lambda^{a_3}). \quad (7.28)
\]

This expression coincides with the Yang-Mills vertex projected to the mass-shell.

7.1 Tree-Level Amplitude for Two Symmetric Tensors and a Vector

The vertex operator for the symmetric rank-2 tensor boson \( T_S \) on the second level is
\[
\mathcal{V}^{-1} = \varepsilon_{\alpha\alpha'}(k)e^{-\phi}\psi^\alpha(\dot{X}^\alpha - 2i\alpha' k \cdot \psi^\alpha')e^{ikX}(y)
\]
and together with the vertex (7.25) can be used to calculate now the scattering amplitude between a vector and two tensor bosons:
\[
\mathcal{V}^{\alpha\beta\gamma}(k, p, q) = F^{\alpha\beta\gamma}(k, p, q) \ tr(\lambda^a\lambda^b\lambda^c) + F^{\gamma\alpha\beta}(q, p, k) \ tr(\lambda^c\lambda^b\lambda^a) \quad (7.30)
\]

**One should use momentum conservation and transversality of the wave functions.**
where their wave functions are:

\[ \varepsilon_{\alpha'}(k), \quad \varepsilon_{\beta}(p), \quad \varepsilon_{\gamma'}(q). \]  

(7.31)

We shall define for convenience \( k_1 \equiv k, k_2 \equiv q, k_3 \equiv p, \) and \( k_1 + k_2 + k_3 = 0. \) The mass-shell conditions are

\[ \alpha' k_1^2 = \alpha' k_2^2 = -1, \quad \alpha' k_3^2 = 0 \]  

(7.32)

and therefore it follows that

\[ k_1 \cdot k_3 = k_2 \cdot k_3 = 0, \quad 2\alpha' k_1 \cdot k_2 = -\alpha'(k_1^2 + k_2^2) = 2. \]  

(7.33)

We have to calculate the correlation function:

\[ < ce^{-\phi} \psi^\alpha(\hat{X}^\alpha' - 2i\alpha' k \cdot \psi^{\alpha'})e^{ikX}(y_1) : ce^{-\phi} \psi^\gamma(\hat{X}^\gamma' - 2i\alpha' q \cdot \psi^{\gamma'})e^{iqX}(y_2) : \]

\[ : c(\hat{X}^\beta - 2i\alpha' p \cdot \psi^{\beta})e^{ipX}(y_3) : > \]  

(7.34)

We shall split it into four terms. The first one gives (the details of the calculation are given in the Appendix)

\[ < ce^{-\phi} \psi^\alpha \hat{X}^\alpha e^{ik_1X}(y_1) ce^{-\phi} \psi^\gamma \hat{X}^\gamma e^{ik_2X}(y_2) c(\hat{X}^\beta - 2i\alpha' k_3 \cdot \psi^{\beta})e^{ik_3X}(y_3) >= \]

\[ = i(2\alpha')^2[\eta^{\alpha\gamma}(\eta^{\beta\gamma} k_2^\alpha + \eta^{\alpha'\beta} k_3^\gamma + \eta^{\alpha'\gamma} k_1^\beta) + \eta^{\alpha'\gamma'}(\eta^{\alpha\beta} k_2^3 + \eta^{\beta\gamma} k_2^\beta)] \]

\[ (-2i\alpha')^3[\eta^{\alpha\gamma} k_1^\beta + \eta^{\alpha\beta} k_3^\gamma + \eta^{\beta\gamma} k_2^\alpha]k_1^\alpha k_3^\gamma \]

and contains linear as well as cubic in momentum expressions. The other three remaining terms have only cubic in momentum expressions. Indeed the second one gives

\[ < ce^{-\phi} \psi^\alpha \hat{X}^\alpha e^{ik_1X}(y_1) ce^{-\phi} \psi^\gamma(-2i\alpha')k_2 \cdot \psi^{\gamma'} e^{ik_2X}(y_2) c(\hat{X}^\beta - 2i\alpha' k_3 \cdot \psi^{\beta})e^{ik_3X}(y_3) >= \]

\[ = (-2i\alpha')^3k_2^\alpha[\eta^{\alpha\gamma} k_2^3 k_3^\gamma + \eta^{\alpha'\gamma} k_2^\alpha k_3^\gamma - \eta^{\alpha'\gamma} k_2^\beta k_3^\gamma + k_2 \cdot k_3(\eta^{\alpha'\gamma} \eta^{\beta\gamma} - \eta^{\alpha'\gamma} \eta^{\beta\gamma'})] \]

and contains only cubic in momentum terms. A new feature of this expression is that it contains a scalar product \((k_2 \cdot k_3). \) The third one gives

\[ < ce^{-\phi} \psi^\alpha(-2i\alpha')k_1 \cdot \psi^{\alpha'} e^{ik_1X}(y_1) ce^{-\phi} \psi^\gamma \hat{X}^\gamma' e^{ik_2X}(y_2) c(\hat{X}^\beta - 2i\alpha' k_3 \cdot \psi^{\beta})e^{ik_3X}(y_3) >= \]

\[ = (-2i\alpha')^3k_3^\gamma[\eta^{\alpha\gamma} k_1^\beta k_3^\gamma - \eta^{\alpha'\gamma} k_3^\beta k_1^\gamma + \eta^{\beta\gamma} k_3^\gamma k_1^\alpha - \eta^{\gamma\gamma'} k_1^\beta k_3^\gamma + k_1 \cdot k_3(\eta^{\alpha\beta} \eta^{\alpha'\gamma} - \eta^{\alpha\gamma} \eta^{\alpha'\beta})] \]
and also contains only cubic in momentum expressions as well as a scalar product \((k_1 \cdot k_3)\). Finally, the last term gives

\[
\begin{align*}
< c e^{-\phi \psi^\alpha (-2i\alpha^\prime) k_1 \cdot \psi \psi^\alpha e^{ik_1 X}(y_1) c e^{-\phi \psi^\gamma (-2i\alpha^\prime) k_2 \cdot \psi \psi^\gamma} c(\dot{X}^\beta - 2i\alpha^\prime k_3 \cdot \psi \psi^\beta)e^{ik_3 X}(y_3) > & = (-2i\alpha^\prime)^3 \\
& \{ + k_1^3_1 [\eta^{\alpha\gamma} (-k_1 \cdot k_2 \eta^{\alpha'\gamma} + k_1^\prime k_2^\prime) - k_2^2 (-k_2^\gamma \eta^{\alpha'\gamma} + k_1^\prime \eta^{\alpha'\gamma}) + \eta^{\alpha'\gamma} (-k_1^\prime k_2^\prime + k_1 \cdot k_2 \eta^{\alpha'\gamma})] \\
& - k_3^3_2 [\eta^{\beta\gamma} (-k_1 \cdot k_2 \eta^{\alpha'\gamma} + k_1^\prime k_2^\prime) - k_2^2 (-k_2^\gamma \eta^{\alpha'\gamma} + k_1^\prime \eta^{\alpha'\gamma}) + \eta^{\alpha'\gamma} (-k_1^\prime k_2^\prime + k_1 \cdot k_2 \eta^{\alpha'\gamma})] \\
& - k_3^3_2 [\eta^{\alpha\beta} (k_1 \cdot k_2 \eta^{\alpha'\gamma} - k_1^\prime k_2^\prime) - k_2^2 (-k_2^\gamma \eta^{\alpha'\gamma} + k_1^\prime \eta^{\alpha'\gamma}) + \eta^{\alpha'\gamma} (-k_1^\prime k_2^\prime + k_1 \cdot k_2 \eta^{\alpha'\gamma})] \\
& + k_3^3_2 [\eta^{\alpha\beta} (k_1 \cdot k_2 \eta^{\alpha'\gamma} - k_1^\prime k_2^\prime) - k_2^2 (-k_2^\gamma \eta^{\alpha'\gamma} + k_1^\prime \eta^{\alpha'\gamma}) + \eta^{\alpha'\gamma} (-k_1^\prime k_2^\prime + k_1 \cdot k_2 \eta^{\alpha'\gamma})] \\
& + k_3^3_2 [\eta^{\alpha\beta} (k_1 \cdot k_2 \eta^{\alpha'\gamma} - k_1^\prime k_2^\prime) - k_2^2 (-k_2^\gamma \eta^{\alpha'\gamma} + k_1^\prime \eta^{\alpha'\gamma}) + \eta^{\alpha'\gamma} (-k_1^\prime k_2^\prime + k_1 \cdot k_2 \eta^{\alpha'\gamma})]] \\
& + k_3^3_2 [\eta^{\alpha\beta} (k_1 \cdot k_2 \eta^{\alpha'\gamma} - k_1^\prime k_2^\prime) - k_2^2 (-k_2^\gamma \eta^{\alpha'\gamma} + k_1^\prime \eta^{\alpha'\gamma}) + \eta^{\alpha'\gamma} (-k_1^\prime k_2^\prime + k_1 \cdot k_2 \eta^{\alpha'\gamma})]
\end{align*}
\]

and again contains only cubic in momentum expressions as well as a scalar product \((k_1 \cdot k_2)\). The scalar products \(k_1 \cdot k_3\) and \(k_2 \cdot k_3\) can be dropped because of \((7.33)\). Summing all remaining terms together one can see that many terms which are cubic in momentum cancel each other so that we are left with the expression

\[
\begin{align*}
& i(2\alpha^\prime)^2 [\eta^{\alpha'\gamma} (\eta^{\beta\gamma} k_2^\prime + \eta^{\alpha'\beta} k_3^\prime + \eta^{\alpha'\gamma} k_1^\prime) + \eta^{\alpha'\gamma} (\eta^{\alpha\beta} k_3^\gamma + \eta^{\beta\gamma} k_2^\prime)] + \\
& (-2i\alpha^\prime)^3 \{ - \eta^{\alpha'\gamma} k_2^\prime k_3^\prime k_1^\prime + \eta^{\alpha'\gamma} k_2^\prime k_3^\prime k_1^\prime - \eta^{\beta\gamma} k_2^\prime k_3^\prime k_1^\prime \} + \\
& (k_1 \cdot k_2)[ + (\eta^{\beta\gamma} \eta^{\alpha'\gamma} - \eta^{\alpha'\gamma} \eta^{\beta\gamma}) k_3^\prime + (\eta^{\alpha\gamma} \eta^{\beta\gamma} - \eta^{\beta\gamma} \eta^{\alpha\gamma}) k_3^\prime \\
& + (\eta^{\alpha\gamma} \eta^{\beta\gamma} - \eta^{\beta\gamma} \eta^{\alpha\gamma}) k_3^\prime + (\eta^{\alpha\gamma} \eta^{\beta\gamma} - \eta^{\beta\gamma} \eta^{\alpha\gamma}) k_3^\prime \\
& + (\eta^{\alpha'\gamma} \eta^{\alpha'\gamma} - \eta^{\alpha'\gamma} \eta^{\alpha'\gamma}) k_3^\prime ] \}.
\end{align*}
\]

Note that the terms proportional to the nonzero product \(2\alpha^\prime k_1 \cdot k_2 = 2\) \((7.33)\) can also be dropped because they are antisymmetric with respect to the indices \(\alpha\alpha^\prime\) and \(\gamma\gamma^\prime\) while the wave functions \(\varepsilon_{\alpha\alpha^\prime}\) and \(\varepsilon_{\gamma\gamma^\prime}\) are symmetric.

Thus, we arrive to the following expression

\[
\begin{align*}
& i(2\alpha^\prime)^2 [\eta^{\alpha'\gamma} (\eta^{\beta\gamma} k_2^\prime + \eta^{\alpha'\beta} k_3^\prime + \eta^{\alpha'\gamma} k_1^\prime) + \eta^{\alpha'\gamma} (\eta^{\alpha\beta} k_3^\gamma + \eta^{\beta\gamma} k_2^\prime)] + \\
& (-2i\alpha^\prime)^3 [ - \eta^{\alpha'\gamma} k_2^\prime k_3^\prime k_1^\prime + \eta^{\alpha'\gamma} k_2^\prime k_3^\prime k_1^\prime - \eta^{\beta\gamma} k_2^\prime k_3^\prime k_1^\prime],
\end{align*}
\]

which is linear and cubic in momenta. Its linear in momenta part is

\[
4[\eta^{\alpha\gamma} (\eta^{\beta\gamma} q^\alpha + \eta^{\alpha'\beta} p^\gamma + \eta^{\alpha'\gamma} k^3) + \eta^{\alpha'\gamma} (\eta^{\alpha\beta} p^\gamma + \eta^{\beta\gamma} q^\alpha)].
\]

(7.37)
It should be symmetrized over $\alpha\alpha'$ and $\gamma\gamma'$
\[
+ (\eta^\gamma_\alpha \eta^{\alpha'}_\gamma + \eta^{\alpha'}_\gamma \eta^\gamma_\alpha)k^\beta \\
+ (\eta^{\beta\gamma}_\alpha \eta^{\alpha'}_\gamma + \eta^{\beta\gamma}_\gamma \eta^\alpha_\alpha')q^\alpha \\
+ (\eta^{\beta\gamma}_\alpha \eta^{\alpha'}_\gamma + \eta^{\beta\gamma}_\gamma \eta^\alpha_\alpha')q^\alpha' \\
+ (\eta^{\beta\gamma}_\alpha \eta^{\alpha'}_\gamma + \eta^{\beta\gamma}_\gamma \eta^\alpha_\alpha')p^\gamma \\
+ (\eta^{\beta\gamma}_\alpha \eta^{\alpha'}_\gamma + \eta^{\beta\gamma}_\gamma \eta^\alpha_\alpha')p^\gamma'
\]
and we can add to it an equal term
\[
- (\eta^\gamma_\alpha \eta^{\alpha'}_\gamma + \eta^{\alpha'}_\gamma \eta^\gamma_\alpha)q^\beta \\
- (\eta^{\beta\gamma}_\alpha \eta^{\alpha'}_\gamma + \eta^{\beta\gamma}_\gamma \eta^\alpha_\alpha')p^\alpha \\
- (\eta^{\beta\gamma}_\alpha \eta^{\alpha'}_\gamma + \eta^{\beta\gamma}_\gamma \eta^\alpha_\alpha')p^\alpha' \\
- (\eta^{\beta\gamma}_\alpha \eta^{\alpha'}_\gamma + \eta^{\beta\gamma}_\gamma \eta^\alpha_\alpha')k^\gamma \\
- (\eta^{\beta\gamma}_\alpha \eta^{\alpha'}_\gamma + \eta^{\beta\gamma}_\gamma \eta^\alpha_\alpha')k^\gamma'
\]
in order to get a symmetric expression:
\[
(1) F^{\alpha\alpha'}\beta\gamma' (k, p, q) = + (k - q)^\beta (\eta^\gamma_\alpha \eta^{\alpha'}_\gamma + \eta^{\alpha'}_\gamma \eta^\gamma_\alpha) \\
+ (q - p)^\alpha (\eta^{\beta\gamma}_\alpha \eta^{\alpha'}_\gamma + \eta^{\beta\gamma}_\gamma \eta^\alpha_\alpha') \\
+ (q - p)^\alpha' (\eta^{\beta\gamma}_\alpha \eta^{\alpha'}_\gamma + \eta^{\beta\gamma}_\gamma \eta^\alpha_\alpha') \\
+ (p - k)^\gamma (\eta^{\beta\gamma}_\alpha \eta^{\alpha'}_\gamma + \eta^{\beta\gamma}_\gamma \eta^\alpha_\alpha') \\
+ (p - k)^\gamma' (\eta^{\beta\gamma}_\alpha \eta^{\alpha'}_\gamma + \eta^{\beta\gamma}_\gamma \eta^\alpha_\alpha').
\]
Substituting this result into the expression (7.30) with the terms in the reversed cyclic orientation $a, (\alpha, \alpha')$, $k \leftrightarrow c, (\gamma, \gamma')$, $q$, we get:
\[
\nu^{\alpha\alpha'}_{abc} \beta\gamma' (k, p, q) = tr(\{\lambda^a, \lambda^b\} \lambda^c) F^{\alpha\alpha'}\beta\gamma' (k, p, q).
\]
The remaining part of the vertex VTT (7.36), which is cubic in momenta, (3) $F$ is
\[
(3) F^{\alpha\alpha'}\beta\gamma' (k, p, q) = 8\alpha' \left[ - \eta^\gamma_\alpha k^\beta \eta^{\alpha'}_\gamma k_1^\beta + \eta^{\beta\gamma}_\alpha k_2^\gamma k^\alpha_\alpha' + \eta^{\beta\gamma}_\gamma k_3^\alpha k^\gamma_\alpha' - \eta^{\beta\alpha'}_\gamma k_3^\alpha k_2^\gamma k_1^\beta \right] 
\]
and can be written in the form
\[
(3) M_{\text{string}} = \varepsilon_{\alpha\alpha'}(k_1)\varepsilon_\beta(k_3)\varepsilon_{\gamma\gamma'}(k_2) (3) F^{\alpha\alpha'}\beta\gamma' (k, p, q) = 8\alpha' \left[ \frac{1}{2} (p \cdot \varepsilon_k \cdot \varepsilon_q \cdot p) ((q - k) \cdot e_p) - (p \cdot \varepsilon_k \cdot p)(k \cdot \varepsilon_q \cdot e_p) + (q \cdot \varepsilon_k \cdot e_p)(p \cdot \varepsilon_q \cdot p) \right].
\]
We can calculate its value in the limit considered in section 4, that gives
\[ (3) M(+2, +1, -2)_{\text{string}} = \varepsilon_{\alpha\dot{\alpha}}(k_1)\varepsilon_{\beta\dot{\beta}}(k_3)\varepsilon_{\gamma\dot{\gamma}}(k_2) \ (3) F^{\alpha\beta\gamma\dot{\gamma}}(k, p, q) = 0, \] (7.44)
because \( k \cdot e_q = 0, \ p \cdot e_q = 0 \). Its gauge variation under (3.10) \( \delta e_p \sim p \) vanishes
\[ \delta (3) M_{\text{string}} = 0 \]
and under (5.19) \( \delta \varepsilon \sim k \otimes \xi + \xi \otimes k \) vanishes as well
\[ \delta (3) M_{\text{string}} = 0, \]
because of the same relations \( k \cdot e_q = 0, \ p \cdot e_q = 0 \).

The cubic in momenta part of the VTT vertex can be associated with an effective action which have higher derivative terms constructed by generalized field strength tensors (1.3).

The gauge invariant effective action which is cubic in field strength tensors is
\[ \mathcal{L}_{\partial} = G_{\mu\nu,\lambda}G_{\nu\rho,\lambda} + \frac{1}{2}G_{\mu\nu}G_{\nu\rho,\lambda\lambda}G_{\rho\mu}. \] (7.45)
Indeed, as one can easily check, its gauge variation vanishes
\[ \delta \mathcal{L}_{\partial} = \left( [G_{\mu\nu,\lambda} \xi] + [G_{\mu\nu} \xi_\lambda] \right)G_{\nu\rho,\lambda}G_{\rho\mu,\lambda} + G_{\mu\nu,\lambda}G_{\nu\rho,\lambda}G_{\rho\mu,\lambda} + G_{\mu\nu,\lambda}G_{\nu\rho,\lambda\lambda}G_{\rho\mu} + G_{\mu\nu,\lambda\lambda}G_{\nu\rho,\lambda}\xi + 2G_{\nu\rho,\lambda\lambda}G_{\rho\mu} + G_{\mu\nu}G_{\nu\rho,\lambda\lambda}G_{\rho\mu} = 0 \]
On the other hand, the second invariant we have found has the form
\[ \mathcal{L}'_{\partial} = -Tr(G_{\mu\nu,\lambda}G_{\nu\rho,\lambda}G_{\rho\mu}) + Tr(G_{\mu\nu,\lambda\lambda}G_{\nu\rho,\lambda\lambda}G_{\rho\mu}) + 2Tr(G_{\mu\nu,\lambda\lambda}G_{\nu\rho,\lambda\lambda}G_{\rho\mu}) + 2Tr(G_{\mu\nu,\lambda\lambda}G_{\nu\rho,\lambda\lambda}G_{\rho\mu}) - Tr(G_{\mu\nu}G_{\nu\rho,\lambda\lambda}G_{\rho\mu}) . \] (7.46)

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Appendix

Let us consider four terms of the interaction vertex VTT. We have

$$\prod_{i<j} |y_i - y_j|^{2\alpha_k k_i} = |y_{12}|^2 \tag{7.47}$$

because the mass-shell conditions are $\alpha' k_1^2 = \alpha' k_2^2 = -1$, $\alpha' k_3^2 = 0$, implying that $k_1 \cdot k_3 = k_1 \cdot k_3 = 0$, $2\alpha' k_1 \cdot k_2 = -\alpha'(k_1^2 + k_2^2) = 2$. The first term gives

$$y_{12}^2 <: ce^{-\phi} \psi \dot{X} \psi e^{ik_1 X}(y_1) : \cdot ce^{-\phi} \psi \dot{X} \psi e^{ik_2 X}(y_2) : \cdot c(\dot{X}^3 - 2ia' k_3 \cdot \psi \psi \dot{X} e^{ik_3 X}(y_3) : >=$$

$$= y_{12}^2 y_{12} y_{23} y_{13} y_{12}^{-1} \left\{ \eta^{\alpha \gamma} \frac{F_{y_1} \dot{F}_{y_2} F_{y_3}}{y_{12}^2} - 2\alpha' \left[ F_{y_1} \eta^{\alpha \beta} \frac{\eta^{\beta \gamma} k_3}{y_{13}^2} + F_{y_2} \eta^{\alpha \beta} \frac{\eta^{\beta \gamma} k_3}{y_{13}^2} \right] \right\} -$$

$$- 2i\alpha' \left[ F_{y_1} \dot{F}_{y_2} - 2\alpha' \frac{\eta^{\alpha \gamma} k_3}{y_{12}^2} \right] \left\{ \eta^{\alpha \beta} k_3 - \eta^{\gamma \beta} k_3 \right\} =$$

$$= (-2i\alpha')^3 [\eta^{\alpha \gamma} k_3^2 + \eta^{\alpha \beta} k_3^3 + \eta^{\gamma \beta} k_3^3] +$$

$$+ i(2\alpha')^2 [\eta^{\alpha \gamma} (\eta^{\beta \gamma} k_3^2 + \eta^{\alpha \beta} k_3^3 + \eta^{\gamma \beta} k_3^3)] (7.48)$$

The second term gives

$$y_{12}^2 < ce^{-\phi} \psi \dot{X} \psi e^{ik_1 X}(y_1) ce^{-\phi} \psi \dot{X} \psi e^{ik_2 X}(y_2) c(\dot{X}^3 - 2ia' k_3 \cdot \psi \psi \dot{X} e^{ik_3 X}(y_3) : >=$$

$$= y_{12}^2 y_{12} y_{23} y_{13} y_{12}^{-1} (-2i\alpha')^2 F_{y_1} \left\{ \eta^{\alpha \gamma} \frac{y_{12}^2}{y_{12}^2} \left[ \frac{y_{12}^2}{y_{23}^2} + \frac{k_3^2 k_3'}{y_{23}^2} \right] - \frac{k_2^2 k_2'}{y_{12}^2} \left[ \frac{k_3^2 \eta^{\beta \gamma}}{y_{23}^2} + \frac{k_3^2 \eta^{\gamma \beta}}{y_{23}^2} \right] \right\} +$$

$$+ \eta^{\alpha \gamma} \left\{ \frac{k_3^2 k_3'}{y_{12}^2} + \frac{k_2^2 k_2'}{y_{12}^2} \right\} =$$

$$= (-2i\alpha')^3 k_3^2 [\eta^{\alpha \gamma} k_3^2 k_3' + \eta^{\alpha \beta} k_3^3 k_3' - \eta^{\gamma \beta} k_3^3 k_3' - \eta^{\gamma \beta} k_3^3 k_3'] +$$

$$+ k_2 \cdot k_3 (\eta^{\alpha \gamma} \eta^{\beta \gamma} - \eta^{\gamma \beta} \eta^{\gamma \beta} \eta^{\beta \gamma}) \tag{7.49}$$

The third one gives

$$y_{12}^2 < ce^{-\phi} \psi \alpha (-2i\alpha') k_1 \cdot \psi \psi \dot{X} \psi e^{ik_1 X}(y_1) ce^{-\phi} \psi \dot{X} \psi e^{ik_2 X}(y_2) c(\dot{X}^3 - 2ia' k_3 \cdot \psi \psi \dot{X} e^{ik_3 X}(y_3) : >=$$

$$= y_{12}^2 y_{12} y_{23} y_{13} y_{12}^{-1} (-2i\alpha')^2 F_{y_2} \left\{ \eta^{\alpha \gamma} \frac{y_{12}^2}{y_{12}^2} \left[ \frac{k_1^2 \eta^{\beta \alpha}}{y_{23}^2} + \frac{k_1^2 \eta^{\beta \alpha}}{y_{23}^2} \right] - \frac{k_1^2 \eta^{\beta \alpha}}{y_{12}^2} \left[ \frac{k_3^2 \eta^{\gamma \beta}}{y_{23}^2} + \frac{k_3^2 \eta^{\gamma \beta}}{y_{23}^2} \right] \right\} +$$

$$+ \eta^{\alpha \gamma} \left\{ \frac{k_1^2 \eta^{\beta \alpha}}{y_{12}^2} + \frac{k_1^2 \eta^{\beta \alpha}}{y_{12}^2} \right\} =$$

$$= (-2i\alpha')^3 k_3^2 [\eta^{\alpha \gamma} k_1^2 k_3' + \eta^{\alpha \beta} k_3^3 k_3' - \eta^{\gamma \beta} k_3^3 k_3' - \eta^{\gamma \beta} k_3^3 k_3'] +$$

$$+ k_2 \cdot k_3 (\eta^{\alpha \gamma} \eta^{\beta \gamma} - \eta^{\gamma \beta} \eta^{\gamma \beta} \eta^{\beta \gamma}) \tag{7.50}$$

and the last forth term gives

$$y_{12}^2 < ce^{-\phi} \psi \alpha (2i\alpha') k_2 \cdot \psi \psi \dot{X} \psi e^{ik_1 X}(y_1) ce^{-\phi} \psi \dot{X} \psi e^{ik_2 X}(y_2) c(\dot{X}^3 - 2ia' k_3 \cdot \psi \psi \dot{X} e^{ik_3 X}(y_3) >=$$
After some simple algebra, one can get the expressions given in the main text.

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