A $\mathbb{Z} \times \mathbb{Z}$ topological invariant for triple-point Nexus fermions

Ankur Das,$^{1,∗}$ Eyal Cornfeld,$^1$ and Sumiran Pujar$^2,†$

$^1$Department of Condensed Matter Physics, Weizmann Institute of Science, Rehovot 7610001, Israel
$^2$Department of Physics, Indian Institute of Technology Bombay, Mumbai, MH 400076, India

A three-dimensional Nexus triple-point can not be enclosed by surrounding gapped bands in one lower dimension unlike Weyl points. We write down a $\mathbb{Z} \times \mathbb{Z}$ topological invariant that characterizes such triple-point band topology. This is akin to how the Chern invariant characterizes the topology of Weyl nodes by characterizing the surrounding gapped surface; however, it necessarily goes beyond the technology of classifying gapped bands because of the gapless Nexus structure. Our characterization applies to internal symmetries which map the triple-point to a distinct partner triple-point in the Brillouin zone. The $\mathbb{Z} \times \mathbb{Z}$ nature of the invariant is further confirmed by abstract algebraic techniques. Finally, we write down models where this invariant characterizes the Nexus topology with pairs of half-integers and connect it to observable physical quantities.

Electronic band topology has emerged as an important and exciting area of investigation in the recent decades following the realization of topological insulators as having distinct phenomenology [1–8] than conventional band insulators. Non-trivial topology of band structures arising out of the underlying wavefunction geometry drives this phenomenon. This has made the topological characterization and classification of bands into an active field. This often is done with the help of topological invariants that can distinguish different band topologies in a digital or yes-no way. A very familiar example is the integral Chern invariant [9]. More generally, they are described by different discrete groups [10–16].

Until now, such topological classification has focused on gapped or insulating bands [1, 11–13], e.g. the $\mathbb{Z}_2$ topological insulator [17, 18]. This insulator-based classification is also used to characterize non-trivial band degeneracies in one higher dimension, e.g. Weyl points in 3d using 2d Chern classification [19], 2d Dirac points using 1d chiral insulator classification, etc. Recently, there has emerged a new class of 3d semimetals in potential candidate materials that host three-fold band degeneracies called Nexus triple-points [20–28] where this method can not be applied. This is so since the insulator-based method rests on the ability to enclose or surround the non-trivial degenerate point by a surface (in one lower dimension) where the bands are gapped. This is not true of Nexus points because they generically have doubly-degenerate line degeneracies or nodal-lines emanating from the triply-degenerate point (see Fig. 1) that necessarily intersect with any enclosing surface. This precludes a surrounding gapped surface and thereby the application of known techniques for gapped bands.

Several attempts have been made to understand the topology of such Nexus point nodes in the 3d band structure, however a full classification has not yet been achieved. The original papers (e.g. Refs. 20, 21, and 25) had focused on a $\mathbb{Z}_2$ invariant in a still lower dimension with two doubly-degenerate nodal-lines (dashed line) emanating from the triple-point. This is a chosen 2d projection containing the nodal-lines. Other projections without the nodal-lines look like the right panel (b). Panel (b) shows an example of a (“spin-1”) three-fold degenerate point which lacks the Nexus structure in any projection.

FIG. 1. Panel (a) shows a sketch of a 3d Nexus triple-point with two doubly-degenerate nodal-lines (dashed line) emanating from the triple-point. This is a chosen 2d projection containing the nodal-lines. Other projections without the nodal-lines look like the right panel (b). Panel (b) shows an example of a (“spin-1”) three-fold degenerate point which lacks the Nexus structure in any projection.

In this paper, we will fill this gap for a specific set of symmetries to be specified after we describe the invariant. Let us begin with the central idea of the paper. By the well-known Chern theorem [31], the Chern number of a
closed manifold has to be an integer. One way to evaluate the Chern number, \( \text{Ch}(n) \), of a gapped band, \( n \), is by the Berry connection, \( \mathcal{A} \), and curvature, \( \Omega \),

\[
\text{Ch}(n) = \frac{1}{2\pi} \int_{k \in \mathcal{M}} \Omega_{n} d^{2}k, \quad (1a)
\]
\[
\Omega_{n} = \frac{\partial}{\partial n} A_{2}(n) - \frac{\partial}{\partial n} A_{1}(n), \quad (1b)
\]
\[
A_{i}(n) = i(n | \frac{\partial}{\partial n} | n). \quad (1c)
\]

If the manifold, \( \mathcal{M} \), is not closed, but rather has a boundary, then the Stokes theorem relates the Berry curvature integral to the the Berry connection integral along the boundary of the manifold,

\[
\gamma_{n} = \int_{k \in \mathcal{M}} \Omega_{n} d^{2}k = \int_{k \in \partial \mathcal{M}} \mathcal{A}(n) \cdot dk. \quad (2)
\]

This quantity, in general, does not have to be quantized. In this paper we study the case where our surface of interest is a punctured manifold. In particular, take a surface in the \( \mathcal{BZ} \) which crosses one or more nodal-line band-crossings. The relevant bands cannot be defined on the nodal-line(s). The surface should thus be considered as punctured, with a puncture imagined as an infinitesimal boundary circling the associated nodal-line (see Fig. 2).

Let us consider first the contribution of a single nodal-line between two bands, say \( | n \rangle \) and \( | n + 1 \rangle \), and its associated puncture to Eq. 2. If this nodal-line has a linear band-crossing then the wavefunction \( \Psi_{n}(k(-)) \) is analytically connected with \( \Psi_{n+1}(k(s)) \), where \( \{k(s)\} \subset \mathcal{M} \) is any curve traversing the nodal-line at \( s = 0 \) [30]. Moreover, since \( \Psi_{n}(k(s)) \) is orthogonal to \( \Psi_{n+1}(k(s)) \) it follows that \( \Psi_{n}(k(s = 0^{-})) \) is orthogonal to \( \Psi_{n}(k(s = 0^{+})) \) for all such curves \( k(s) \), i.e., the wavefunction \( \Psi_{n}(k) \) is orthogonal at antipodal points along the infinitesimal puncture circling the nodal-line (Fig. 2).

Consider now the geometry of the two bands at the nodal-line. At any point on the nodal-line, the two bands span a 2d subspace of the Hilbert space. This 2d subspace has the geometry of a Bloch-sphere (also known as the Riemann-sphere, or the complex projective line \( \mathbb{CP}^{1} \)). Perpendicular states are represented as antipodal points on the Bloch-sphere, and in particular, the image on the Bloch-sphere of the puncture discussed above is an antipodal trajectory (Fig. 2). What is the Berry flux associated with this antipodal trajectory? Since the Bloch-sphere includes an integer Berry flux, and any antipodal trajectory necessarily divides a sphere into two equal halves, therefore the antipodal trajectory divides the net Berry flux of the Bloch-sphere into two equal halves. This implies that the Berry flux associated with the antipodal trajectory is half-integer (in units of flux quantum). That is, a nodal-line with a linear band-crossing gives a half-integer contribution to the Chern number on the punctured manifold. In particular, a manifold which is punctured by an odd number of such nodal-lines will have a net half-integer Chern number. A natural case where such structures appear are Nexus triple-points.

The above discussion leads to a \( \mathbb{Z} \times \mathbb{Z} \) classification for the Nexus triple-point as follows: all the three bands involved in the Nexus point must have (some multiple of) half-integer Chern numbers in presence of linear band-crossings [32]. The sum of the “punctured” Chern numbers over all the three bands must be an integer (zero if these are the only three bands in the picture) implying there are only two independent half-integers. As half-integers can be mapped to \( \mathbb{Z} \), therefore the classification of the three bands is \( \mathbb{Z} \times \mathbb{Z} \). This is the central result of the paper: the \( \mathbb{Z} \times \mathbb{Z} \) “punctured Chern set” as a topological invariant for Nexus triple-points.

This invariant is appropriate for a specific choice of internal symmetries. Either time reversal or particle-hole may be present, but not both together. More generally, we restrict ourselves to the case when \( \text{all} \) the internal symmetries map (the vicinity of) the Nexus point to a different Nexus point in the \( \mathcal{BZ} \) implying that there is at least one distinct partner of the Nexus triple-point. Let us call this as the distinct symmetry-related partner existence (DSPE) condition or restriction. This then excludes a Nexus point at the \( \Gamma \) point in the \( \mathcal{BZ} \), but does not restrict to only a single pair of Nexus points in the \( \mathcal{BZ} \). In the rest of the paper, we will first briefly describe an abstract algebraic way of deriving the above result by considering the Nexus wavefunction space as an abstract punctured manifold. We will then discuss model settings where these results are directly applicable, along with connections to physical observables and the rationale for the DSPE restriction.
The general idea is to use exact sequences to calculate the homotopy groups of the mappings \([33, 34]\) that are relevant to the Nexus (single-particle) band topology. Perhaps a more familiar (many-body) example would be the classes of mappings relevant for a filled subset of gapped bands mapped onto a general sphere in the sought-after dimension \([5, 35]\). Its group structure can be arrived at by analyzing the constraints on it due to elements on either side of the corresponding long exact sequence \([34]\). The classification of the topological insulators (namely the periodic table of topological insulators) was similarly achieved using long exact sequences (e.g. Hopf fibration \([35, 36]\)). A single-particle example would be a set of non-degenerate single-particle bands, say three bands, whose manifold structure would be described by the flag manifold \(F\Omega \simeq U(3)/U(1)^3\). For such a manifold, we can conclude \(\pi_2(F\Omega) = \mathbb{Z} \times \mathbb{Z}\) using the corresponding long-exact sequence by exploiting its fibration structure as briefly described in Ref. \(37\).

Here, we have additional structure on the single-particle Nexus bands due to the emanating nodal-lines. The presence of gapless points on the manifold induces the antipodal boundary condition described previously which does not yield a simple fibration structure. This prevents from directly writing down a long-exact sequence. To handle this, we treat the classification problem by means of a homotopy pullback \([38]\), and thereby construct a fibration on the space of mappings from the vicinity of the triple-point to the band manifold and use its long exact sequence \([34, 39, 40]\) to unravel the classification. This leads to the \(\mathbb{Z} \times \mathbb{Z}\) homotopy group for the band manifolds surrounding the Nexus triple-point. The details of this calculation are described in Ref. \(37\).

Next, we discuss possible model realizations of the above ideas. A way to obtain a viable model consistent with the DSPE condition is to take a 2d model of spinless electrons with three bands and time reversal symmetry that has the Nexus structure with fine-tuned parameters \([41]\), and build up into 3d. In order to stabilize the triple-point, we consider a crystalline system with \(C_3\) ditrigonal pyramidal symmetry and a sublattice anti-symmetry \(\Pi\). Let us further assume that the chiral symmetry is broken (in accordance with DSPE condition), and also broken reflection symmetry. However, let the product \(\Pi_\sigma v\) be unbroken. The (anti-)symmetries thus act on the momenta as follows,

\[
O_{C_3} = \begin{pmatrix}
\cos \frac{2\pi}{3} & -\sin \frac{2\pi}{3} & 0 \\
\sin \frac{2\pi}{3} & \cos \frac{2\pi}{3} & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad O_{\Pi\sigma v} = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

such that

\[
gH(k)g^{-1} = H(O_\sigma k),
\]

with \(g\) and \(O_\sigma\) being the representations of the symmetry transformation in the Hilbert space and the momentum space respectively. We pick the following actions of these symmetry operations on the Hilbert space,

\[
C_3 = \begin{pmatrix}
\frac{\sqrt{3}}{2} & 0 & 0 \\
0 & -\frac{\sqrt{3}}{2} & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \Pi_\sigma v = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

At a generic point \(k_0\) on three-fold rotational \((C_3)\) symmetry axis chosen to be say along the \(z\)-direction, i.e. \(k_0 = k_z \hat{z}\), the most generic Hamiltonian can be linearized in its vicinity \((p = k - k_0)\) as follows,

\[
H(p) = \begin{pmatrix}
\alpha p_x - \epsilon & \beta(p_x + ip_y) & \zeta(p_x - ip_y) \\
\beta(p_x - ip_y) & \epsilon - \alpha p_x & \zeta^*(p_x + ip_y) \\
\zeta^*(p_x + ip_y) & \zeta^*(p_x - ip_y) & 0
\end{pmatrix}.
\]

Here, \(\alpha, \beta, \) and \(\epsilon\) are constant real parameters, and \(\zeta\) is a constant complex parameter for a given \(k_0\). Their values naturally depend on the point around which we are linearizing, i.e. \(\alpha \equiv \alpha(k_0), \beta \equiv \beta(k_0), \zeta \equiv \zeta(k_0)\) and \(\epsilon \equiv \epsilon(k_0)\). In presence of a constant electric field parallel to the \(C_3\) symmetry axis given by the vector potential, \(A(t) = Et \hat{z}\), we get the following time-dependent Hamiltonian after minimal substitution,

\[
H(p, t) = \begin{pmatrix}
\alpha p_x + \alpha \epsilon Et - \epsilon & \beta(p_x + ip_y) & \zeta(p_x - ip_y) \\
\beta(p_x - ip_y) & \epsilon - \alpha p_x & \zeta^*(p_x + ip_y) \\
\zeta^*(p_x + ip_y) & \zeta^*(p_x - ip_y) & 0
\end{pmatrix}.
\]

Assuming that the electric field is small, we may apply the adiabatic approximation such that at time \(t = \frac{\epsilon(k_0)}{\alpha(\epsilon/|\zeta|)\beta}\), the corresponding point \(k_0\) on the symmetry axis would become an isolated three-fold degenerate point (not a Nexus triple-point but a “spin-1” chiral fermion as in Fig. 2 (b)). Since this is true for any point on the symmetry axis, some point \(k_0^*\) may generically also have the property that \(|\beta| = |\zeta|\). Thus at \(t^*\) corresponding to \(k_0^*\), a Nexus triple-point is obtained at \(k_0^*\) with three nodal-lines with linear band-crossings emanating from it, and therefore characterized by a half-integer punctured Chern set. This is summarized in Table \(I\) below.

Alternatively, we note that the eigenvalues of rotation symmetry representations in the Hilbert space are complex (Eq. 5 involving \(C_3\)). This implies that the wavefunctions transform as (orbital) angular momentum \(L = \)
1. This can be readily seen by looking at the diagonal entries of Eq. 6. We may thus apply a constant magnetic field \( B \) parallel to the three-fold symmetry axis, which will Zeeman couple as \( B_z L_z \). If one assumes that this term dominates over minimal coupling terms, then the Hamiltonian is now essentially as in Eq. 7 with the following replacement, \( \alpha eE t \to \mu_0 B \). By very similar arguments, we will again get triple-points with half-integer punctured Chern sets when \( B = \epsilon(k_3^a) / \mu_0 \) (Table 1). One can understand the half-integer nature of the punctured Chern set as a topological transition between two non-Nexus triple-points with integer Chern numbers which are an odd integer apart. We can finally write a “tight-binding” model based on the above by the following replacements

\[
(p_x + ip_y) \to \sum_{\ell} e^{\frac{2\pi i}{3}} \sin(R_1 \cdot OC_3^{-1} k) \tag{8a}
\]

\[
\epsilon \to \eta \sum_{\ell} \cos(R_1 \cdot OC_3^{-1} k) \tag{8b}
\]

involving \( C_3 \), and \( \alpha p_x \to \epsilon(k_z), \beta \to \beta(k_z), \zeta \to \zeta(k_z) \). Here \( \ell \in \{1, 2, 3\} \), \( R_1 \) is the primitive lattice vector in the \( x \)-direction, \( \eta \) is a real parameter, \( \epsilon(k_z) \) is a real function of \( k_z \), and \( \beta(k_z) \) and \( \zeta(k_z) \) are respectively real and complex functions of \( k_z \) whose absolute values can intersect at some generic momentum (see Ref. 37 for the explicit formula).

The \( Z \times Z \) topological invariant delineated in this paper may be relevant in other contexts, e.g. optical [42] or phononic [43] band-structures. We should also note that in the Nexus fermion models of Ref. 21 and 25, there are quadratic band-touchings (instead of linear band-crossings) where the nodal-lines intersect the enclosing surface. For a quadratic band-touching, the antipodal condition gets lifted which leads back to a flag manifold of dimension 3. With the DSPE condition, the classification will thus be \( \pi_2(\text{Fl}_3) \) which is again \( Z \times Z \) given by integer Chern sets. For Refs. 21 and 25, there is the additional symmetry (time reversal times a specific reflection) which does not satisfy the DSPE condition. The Berry connection will thus get further “constrained” due to this extra symmetry in the vicinity of the Nexus point. This makes the punctured Chern set defined in this paper inapplicable as the topological invariant [44] for these cases. It is known that such DSPE violating symmetries affect the classification [16] and thereby the associated topological invariants.

We conclude by discussing the relation between the punctured Chern set that has been identified as a topological invariant for Nexus band topology and physical quantities. Because of the DSPE condition, we expect at least another “partner” Nexus point inside the BZ (similar to Weyl points). Thus we expect that there will be Fermi arcs on the surface [45]. Their details depend on the filling details and also the punctured Chern set. Due to its half-integral nature when there are odd number of emanating nodal-lines, appropriate surface cuts will show “half-integral conductance” per cut (in units of \( e^2/\hbar \)). An equivalent consequence is a half-integral anomalous Hall effect contribution when the chemical potential is nearby a Nexus point due to its topological nature [46]. When the chemical potential is slightly below (above) the Nexus point in our model, then the Fermi surface will consist of two hole (electron) Fermi pockets near the triple-point that linearly touch each other on the neighboring nodal-lines. This gives a Hall conductance that depends [47] on the Berry curvature integral over the two pockets and the Berry connection integral on the boundary (the infinitesimal loop at the puncture as sketched in Fig. 2 in our case). The boundary terms over the two pockets cancel out, while the total Berry curvature integral will equal that of the bottom (top) sphere, which is a half-integer as shown in Table 1. There will also similarly be an integral anomalous Hall conductance contribution for the other entries in Table 1.

Finally, a curious consequence of the main idea of defining topological invariants on gapless punctured manifolds may be realized in a 2d situation. Let us imagine a model with linear band-crossings such that their chiralities or windings “add up” over the BZ [48] as schematized in Fig. 3. In such a scenario, we will again have bulk band topology characterized by punctured Chern sets which can give Hall conductance. For example in the sketch of Fig. 3, a chemical potential of quarter-filling gives point-like Fermi surfaces and thus a quantum of Hall conductance at this filling. However, this would not be quantized but rather will continuously change with the chemical potential due to the Fermi surface.

The gapless methods involving punctured manifolds and the punctured Chern set as a topological invariant formulated in this work provide a new perspective on topological semimetals as also new calculational tools. Our methods go through in presence of certain symmetries as described before, which opens up questions as to the nature and formulation of topological invariants to characterize Nexus points with symmetries beyond the
DSPE condition. This will indeed be necessary for several existing triple-point semi-metal candidates \[40\].

The authors acknowledge fruitful discussions with S. Carmeli, T. Holder, E. Berg, A. Stern, J. S. Hofmann. AD was supported by the German-Israeli Foundation Grant No. 1-I505-303.10/2019, CRC 183, and the GIF. AD also thanks Israel planning and budgeting committee (PBC) and Weizmann Institute of Science, Israel Dean of Faculty fellowship, and Koshland Foundation for financial support. This work was supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) project grant 277101999 within the CRC network TRR 183. S.P. acknowledges financial support from SERB-DST, India via grant no. SRG/2019/001419, and in the final stages of writing by grant no. CRG/2021/003024.

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* ankur.das@weizmann.ac.il
† sumiran@phy.iitb.ac.in

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Supplement to $A \times Z$ topological invariant for triple-point Nexus fermions

Ankur Das, 1 Eyal Cornfeld, 1 and Sumiran Pujari 2

1 Department of Condensed Matter Physics, Weizmann Institute of Science, Rehovot 7610001, Israel
2 Department of Physics, Indian Institute of Technology Bombay, Mumbai, MH 400076, India

Some details connected to the main text are presented here. We discuss the homotopy groups of the different manifolds and specifically the case of three connected surrounding surfaces for Nexus triple-points. We also present a lattice model with Nexus points and various energy cuts to show the associated Fermi arcs.

Below, we present the abstract algebraic proof that the classification of Nexus triple-points with linear band-crossings in presence of the DSPE condition (see main text) is given by $Z \times Z$. We start with the simpler case of flag manifolds in Sec. I before going to the case of punctured Nexus manifolds in Sec. II. In Sec. III, we describe a lattice realization and associated Fermi arcs.

I. HOMOTOPIES OF FLAG MANIFOLDS

We start with the complex complete flag manifold, denoted by

\[ Fl_3 \overset{\text{def}}{=} U(3)/U(1)^3 \simeq SU(3)/U(1)^2. \]  

A fibration, $F \to E \to B$, satisfies

\[ \cdots \to \pi_{n+1}(B) \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \cdots \]  

By $U(1)^2 \to SU(3) \to Fl_3$ we have

\[ \begin{array}{cccccc}
\pi_2(SU(3)) & \to & \pi_2(Fl_3) & \to & \pi_1(U(1)^2) & \to & \pi_1(SU(3)) & \to & \pi_1(Fl_3) & \to & \pi_0(U(1)^2) \\
\mathbb{Z}^2 & \to & \mathbb{Z}^2 & \to & \mathbb{Z}^2 & \to & \mathbb{Z}^2 & \to & \mathbb{Z}^2 & \to & \mathbb{Z}^2 \\
0 & \to & \mathbb{Z}^2 & \to & \mathbb{Z}^2 & \to & \mathbb{Z}^2 & \to & \mathbb{Z}^2 & \to & 0 \\
0 & \to & 0 & \to & 0 & \to & 0 & \to & 0 & \to & 0
\end{array} \]  

Here, the dashed arrows are deducted by the structure of the long exact sequence. We conclude that $\pi_1(Fl_3) = 0$ and $\pi_2(Fl_3) = \mathbb{Z}^2$.

II. THE THREE CONNECTED SPHERES

If $Z$ is a connected space then the homotopy pullback

\[ \begin{array}{ccc}
P & \to & X \times Z \\
\downarrow & & \downarrow \\
Y & \to & Z
\end{array} \]  

is a fibration, $\Omega Z \to P \to X \times Y$, and $P = X \times Z Y = \{(x, \gamma(t), y) : \gamma(0) = z(x), \gamma(1) = z(y)\}$.

A. Warmup: two connected spheres

Notice that $Fl_2 \simeq \mathbb{C}P^1 \simeq S^2$. 
The two connected spheres can be viewed as a disk \( \partial(D^2) = S^1 \). There is a \( \mathbb{Z}_2 \)-action on the boundary sphere which takes each point to its antipodal point. Under this action, the two basis vectors in \( \text{Fl}_2 \) change places which is the antipodal action on \( \mathbb{C}P^1 \).

Therefore there is a homotopy pullback

\[
\text{Map}^{\text{Conditions}}(D^2, \mathbb{C}P^1) \xrightarrow{\text{antipodal}} \text{Map}(S^1, \mathbb{C}P^1) \]

such that the number of topologically distinct maps from the disc to \( \mathbb{C}P^1 \) that obey the antipodality conditions on the boundary are given by \( \pi_0(P) \) with \( P = \text{Map}^{\text{Conditions}}(D^2, \mathbb{C}P^1) \). The homotopy pullback must be defined relative to a selection of basepoints. For the spaces of maps from the disc to \( \mathbb{C}P^1 \) we pick as a basepoint the hemisphere map which maps a disc to one of the hemispheres.

The exact sequence reads

\[
\pi_1(X) \times \pi_1(Y) \xrightarrow{\pi_0(\Omega Z)} \pi_0(P) \xrightarrow{\pi_0(\Omega Z)} \pi_0(X) \times \pi_0(Y)
\]

We thus conclude that \( \pi_0(P) = \mathbb{Z} \).

Check: We know that \( \pi_0(P) = \pi_0(\text{Map}^{\text{Conditions}}(D^2, \mathbb{C}P^1)) = \pi_0(\text{Map}^{\text{antipodal}}_{\mathbb{Z}_2}(S^2, \mathbb{C}P^1)) = \{\mathbb{Z}\} - \{2\mathbb{Z}\} \) as labeled by the pushforward \( \pi_2(S^2) \to \pi_2(\mathbb{C}P^1) \).

\[ \text{B. Calculation} \]

The three connected spheres can be viewed as a cylinder \( \partial(S^1 \times I) = S^1_u \cup S^1_d \). On each sphere, one of the vectors \( v_1 \perp v_2 \perp v_3 \in \text{Fl}_3 \) is constant. These vectors are thus points in \( \mathbb{C}P^2 \subset \text{Fl}_3 \) and they restrict the other pairs to lie in the orthogonal \( \text{Fl}_2 \cong \mathbb{C}P^1 \cong S^2 \). There is a \( \mathbb{Z}_2 \)-action on the boundary spheres which takes each point to its antipodal point. Under this action, the two free basis vectors in \( \text{Fl}_3 \) change places. That is, for \( x_{u,d} \in S^1_u \) we have

\[
x_u; (v_1, v_2, v_3) \mapsto -x_u; (v_2, v_1, v_3), \quad x_d; (v_1, v_2, v_3) \mapsto -x_d; (v_1, v_3, v_2).
\]

Therefore there is a homotopy pullback

\[
\text{Map}^{\text{Conditions}}(S^1 \times I, \text{Fl}_3) \xrightarrow{\text{antipodal}} \text{Map}(S^1, \text{Fl}_3) \]

such that the number of topologically distinct maps from the cylinder to \( \mathbb{C}P^2 \) that obey the antipodality conditions on the boundary are given by \( \pi_0(P) \) with \( P = \text{Map}^{\text{Conditions}}(S^1 \times I, \text{Fl}_3) \). Here basepoints are picked akin to the previous section.
The exact sequence reads

\[
\begin{array}{c}
\pi_1(X) \times \pi_1(Y) \xrightarrow{\pi_0(\Omega Z)} \pi_0(P) \xrightarrow{\pi_0(X) \times \pi_0(Y)} \\
\mathbb{Z}^2 \times ((0 \times \mathbb{Z}) \times (0 \times \mathbb{Z})) \xrightarrow{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}} \mathbb{Z}^2 \times \mathbb{Z}^2 \xrightarrow{\{ \mathbb{Z}^2 \}} \{ \mathbb{Z} \} \xrightarrow{0 \times 0} \\
\end{array}
\]  

(10)

Thus, we finally conclude that \( \pi_0(P) = \mathbb{Z}^2 \).

### III. LATTICE DETAILS

Here we write down the explicit formula for the tight-binding Hamiltonian starting from the continuum model that was described in the main text. It is

\[
H(k) = \begin{pmatrix}
\epsilon(k_z) - \eta \sum_\ell \cos(R_1 \cdot O_{C_3}^{-\ell} k) & \beta(k_z) \sum_\ell e^{\frac{2\pi i \ell}{3}} \sin(R_1 \cdot O_{C_3}^{-\ell} k) & \zeta(k_z) \sum_\ell e^{\frac{2\pi i \ell}{3}} \sin(R_1 \cdot O_{C_3}^{-\ell} k) \\
\beta(k_z) \sum_\ell e^{\frac{2\pi i \ell}{3}} \sin(R_1 \cdot O_{C_3}^{-\ell} k) & -\epsilon(k_z) + \eta \sum_\ell \cos(R_1 \cdot O_{C_3}^{-\ell} k) & \zeta(k_z) \sum_\ell e^{\frac{2\pi i \ell}{3}} \sin(R_1 \cdot O_{C_3}^{-\ell} k) \\
\zeta^*(k_z) \sum_\ell e^{\frac{2\pi i \ell}{3}} \sin(R_1 \cdot O_{C_3}^{-\ell} k) & \zeta^*(k_z) \sum_\ell e^{\frac{2\pi i \ell}{3}} \sin(R_1 \cdot O_{C_3}^{-\ell} k) & 0
\end{pmatrix},
\]  

(11)

where \( \ell \in \{1, 2, 3\} \), \( R_1 \) is the primitive lattice vector in the \( r_1 \) direction, \( \eta \) is a real parameter, \( \epsilon(k_z) \) is a real function of \( k_z \), and \( \beta(k_z) \) and \( \zeta(k_z) \) are respectively real and complex functions of \( k_z \) whose absolute values intersect at some generic momentum (one may choose one but not both to be constant functions). If one denotes by \( k_z' \) the momentum which satisfies \(|\beta(k_z')| < |\zeta(k_z')|\) the triple-point transition would occur at magnetic fields \( B = [3\eta - \epsilon(k_z')]/\mu \). This model is inspired by those in Ref. 3, specifically the \( \Lambda_3 \) modification in the \( k_z \) direction, however with the \( C_3 \) symmetry axis lying along \( k_z \) as well.

![FIG. 1. From left to right, we present equal energy surface state density at three different increasing energies which shows the Fermi arc structure connecting two pockets.](image-url)

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