Arov–Krein Entropy Functionals and Indefinite Interpolation Problems

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Abstract. We generalize the notion of the Arov–Krein entropy functional for the case of generalized Nevanlinna functions and obtain a representation of these functionals on solutions of indefinite interpolation problems. The case of indefinite Carathéodory problem and application to Szegö limit formula for this nonclassical case are considered in greater detail.

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1. Introduction

1. In their important work [1], D.Z. Arov and M.G. Krein studied matrix-valued functions \( \omega(\lambda) \) (given as linear fractional transformations), such that \( \omega(\lambda) + \omega(\lambda)^* \geq 0 \) on the unit disk \( \mathbb{D} = \{ \lambda : |\lambda| < 1 \} \), where \( \omega(\lambda)^* \) stands for the complex conjugate transpose of \( \omega(\lambda) \). (See also related results in [2,29].) The class of such matrix-valued functions (matrix functions) is called Carathéodory class \( C \) (or \( C_0 \)) and \( \omega \in C \) admits Herglotz representation

\[
\omega(\lambda) = i\nu + \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + \lambda}{e^{i\theta} - \lambda} d\mu(\theta) \quad (\nu = \nu^*),
\]

where \( i \) above stands for the imaginary unit and \( \mu(\theta) \) is a nondecreasing matrix function. Clearly, the real part

\[
\Re \left( \frac{e^{i\theta} + \lambda}{e^{i\theta} - \lambda} \right) = \frac{1 - |\lambda|^2}{|e^{i\theta} - \lambda|^2} \geq 0,
\]

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and so $\omega(\lambda) + \omega(\lambda)^* \geq 0$ is immediate from (1.1) and (1.2). The Arov–Krein entropy functionals are given by the formula:

$$E(\omega, \tilde{\lambda}) = -\frac{1}{4\pi} \int_0^{2\pi} |e^{i\theta} - \tilde{\lambda}|^{-2}(1 - |\tilde{\lambda}|^2) \ln \det (\mu' (\theta)) d\theta \quad (|\tilde{\lambda}| < 1).$$

(1.3)

Further notions and results on Arov–Krein and related Burg’s entropy and further references are given in [4,5,7,11,19].

Here, we consider a natural generalization of the Arov–Krein entropy functional for the case of the solutions of indefinite interpolation problems belonging to the Krein–Langer (generalized Caratheodory) class $C_\kappa$, where $C_0 = C$. We take also solutions of indefinite interpolation problems belonging to the generalized Nevanlinna class $N_\kappa$, transform them into $C_\kappa$ and consider the entropy (since the entropy formula for the functions in $C_\kappa$ is more convenient for our purposes). See Definition 1.1 of the classes $N_\kappa$ and $C_\kappa$ and a more detailed discussion below. The entropy formula is closely related to the Szegő limit formula in the indefinite case [30,31] (see also interesting recent papers [8,9]). The entropy formula could be considered in the context of generalization of discrete Dirac system theory (see [12,32] and references therein).

2. The solutions of the so called indefinite interpolation problems are often described in terms of linear fractional transformations. In this paper, we deal with the case when the matrix functions, which are obtained using these linear fractional transformations, are defined in the open upper half-plane $C_+$. Moreover, these functions belong to the classes $N_\kappa$, where $\kappa \geq 1$ (see the definition of $N_\kappa$ below).

**Definition 1.1.** The generalized Nevanlinna class $N_\kappa$ is the set of meromorphic $p \times p$ matrix functions $\varphi(z)$ ($z \in C_+$) such that the kernel

$$((\varphi(z) - \varphi(\zeta)^*)/(z - \bar{\zeta})$$

has $\kappa$ negative squares. That is, for any $n \in \mathbb{N}$ and any set $z_1, z_2, \ldots, z_n \in C_+$ the matrix $\{(\varphi(z_i) - \varphi(z_k)^*)/(z_i - \bar{z}_k)\}_{i,k=1}^n$ has at most $\kappa$ negative eigenvalues, and for at least one choice $n, z_1, z_2, \ldots, z_n$ it has exactly $\kappa$ negative eigenvalues.

The generalized Caratheodory class $C_\kappa$ is the set of meromorphic $p \times p$ matrix functions $\omega(\lambda)$ ($|\lambda| < 1$) such that the kernel $(\omega(\lambda) + \omega(\zeta)^*)/(1 - \lambda\bar{\zeta})$ has $\kappa$ negative squares. That is, for any $n \in \mathbb{N}$ and any set $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{D}$ the matrix $\{((\omega(\lambda_i) + \omega(\zeta_k)^*)/(1 - \lambda_i\bar{\zeta}_k)\}_{i,k=1}^n$ has at most $\kappa$ negative eigenvalues, and for at least one choice $n, \lambda_1, \lambda_2, \ldots, \lambda_n$ it has exactly $\kappa$ negative eigenvalues.

Here $\mathbb{N}$ denotes the set of positive integers and $\bar{z}$ is the complex conjugate of $z$. The classes $N_\kappa$ and $C_\kappa$ have been studied in a series of seminal Krein–Langer papers, for instance [17,18] (see also the articles [6,24] and references therein).

**Remark 1.2.** Sometimes, we use the notations $N_p^\kappa$ and $C_p^\kappa$ instead of $N_\kappa$ and $C_\kappa$, respectively, to stress the order $p$ of the matrix functions in the classes.
In Sect. 2, we introduce the entropy functional for the generalized Nevanlinna functions and obtain (in Theorem 2.8) a representation of this functional on the solutions of indefinite interpolation problems. In Sect. 3, we consider in detail the indefinite Carathéodory problem and related block Toeplitz matrices. Section 4 is dedicated to the application of the results from Sect. 3 in the proof of indefinite Szegő limit formula (see formula (4.14)).

The representation of the generalized Nevanlinna functions is important for us, and we present some basic results in Appendix A. The mapping of the functions from $N_{\kappa}$ into $C_{\kappa}$ is shortly discussed in Appendix C. The “operator identities” formalism for solving indefinite interpolation problems from [30] (scalar function case) and [25, 26, 31] (matrix function case), which is based on the approaches to sign-definite interpolation problems described in [23, 37], is given in Appendix B. Although our main entropy results are restricted (for simplicity) to the case of two-sided Nudelman-Takagi interpolation problems, one can see that they admit interesting generalizations.

Some notations have been introduced above. Moreover, $N_{0}$ denotes the set of nonnegative integers, $\mathbb{R}$ denotes the real axis, $T$ stands for the unit circle (i.e., $T = \{ \lambda : |\lambda| = 1 \}$), $\mathbb{C}$ stands for the complex plane, and $\mathbb{C}_{\pm}$ ($\mathbb{C}_{+}$) stands for the open lower (upper) half-plane. For $\Delta \subset \mathbb{R}$, $\chi_{\Delta}$ is the characteristic function of $\Delta$, that is, $\chi_{\Delta}(t) = 1$ for $t \in \Delta$ and $\chi_{\Delta}(t) = 0$ for $t \notin \Delta$. The spectrum of a bounded operator or a square matrix $A$ is denoted by $\sigma(A)$ and $I_p$ stands for the $p \times p$ identity matrix.

The set of linear bounded operators acting from the Hilbert space $\mathcal{G}$ into Hilbert space $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{G}, \mathcal{H})$ and the notation $\mathcal{B}(\mathcal{H}, \mathcal{H})$ we simplify to $\mathcal{B}(\mathcal{H})$. The set of Hermitian operators (or matrices) $S$ from $\mathcal{B}(\mathcal{H})$, such that the spectrum of $S$ contains precisely $\kappa$ (counting multiplicities) negative eigenvalues ($\kappa < \infty$), is denoted by $\mathcal{P}_{\kappa}$. We use also the notation $\kappa_{S}$ for the mentioned above $\kappa$.

2. Entropy

Definition 2.1. The entropy functionals on the matrix function $\varphi \in N_{\kappa}$ are given by the formula

$$E(\varphi, \tilde{\lambda}) = -\frac{1}{4\pi} \int_{0}^{2\pi} |e^{i\theta} - \tilde{\lambda}|^{-2} (1 - |\tilde{\lambda}|^2) \ln \det (\mu'(\theta)) d\theta \quad (|\tilde{\lambda}| < 1),$$

(2.1)

where $\mu$ is the matrix function from the Krein–Langer representation (C.2) of $\omega(\lambda)$, and $\omega$ and $\lambda$ are constructed from $\varphi$ and $z$, respectively, using the mapping of $N_{\kappa}$ into $C_{\kappa}$:

$$\omega(\lambda) = -i\varphi(z), \quad \text{where} \quad \lambda = \frac{z - v}{z - \overline{v}} \quad (v \in \mathbb{C}_{+}).$$

(2.2)

Note that the right-hand sides of (1.3) and (2.1) formally coincide although the requirements on $\mu$ in (1.3) and (2.1) differ. See also Appendix C dedicated to the mapping (2.2). In particular, we have formula (C.13):

$$\Re(\omega(e^{i\theta})) = \mu'(\theta).$$

(2.3)
In view of (2.3), we rewrite (2.1) in the form
\[ E(\varphi, \tilde{\lambda}) = -\frac{1}{4\pi} \int_{0}^{2\pi} |e^{i\theta} - \tilde{\lambda}|^{-2}(1 - |\tilde{\lambda}|^2) \ln \det (\Re(e^{i\theta})) d\theta. \] (2.4)

We will consider functions \( \varphi \) given by the linear fractional transformations (B.17):
\[ \varphi(z) = \varphi(z, P, Q) = i(a(z)P(z) + b(z)Q(z))(c(z)P(z) + d(z)Q(z))^{-1}, \]
(2.5)
where \( a, b, c, d \) and \( P, Q \) are \( p \times p \) matrix functions, which are meromorphic in \( \mathbb{C}_+ \). The coefficients \( a, b, c, d \) of the linear fractional transformation are given by the formula (B.15):
\[ U(z) := w_A(z)^* = I_{2p} - iz\Pi^*(I - zA^*)^{-1}S^{-1}\Pi J = \begin{bmatrix} a(z) & b(z) \\ c(z) & d(z) \end{bmatrix}, \] (2.7)
where \( A \in B(\mathcal{H}), \Pi = \begin{bmatrix} \Phi_1 & \Phi_2 \end{bmatrix}, \Phi_1 \in B(\mathbb{C}^p, \mathcal{H}), \Phi_2 \in B(\mathbb{C}^p, \mathcal{H}) \) and the following relations hold:
\[ AS - SA^* = i(\Phi_1 \Phi_2^* + \Phi_2 \Phi_1^*), \]
(2.8)
\[ S^{-1} \in B(\mathcal{H}). \]
(2.9)
The so called frame \( \mathcal{U} \) has the property (B.16):
\[ \mathcal{U}(\bar{\xi})^* J \mathcal{U}(z) \equiv J. \] (2.10)
The pairs \( \{P, Q\} \) satisfy the inequalities
\[ P(z)^*Q(z) + Q(z)^*P(z) \geq 0, \quad P(z)^*P(z) + Q(z)^*Q(z) > 0, \] (2.11)
extcept for a set of isolated points, and are called nonsingular pairs with the property-J. Solutions of interpolation problems may be presented in the form (2.5)–(2.11) (see Appendix B).

According to (2.5), (2.6), and (2.7) we have
\[ i(\varphi(z)^* - \varphi(z)) = ((c(z)P(z) + d(z)Q(z))^{-1})^* \times \left[ P(z)^* \right] \left[ a(z)^* \right] \left[ b(z) \right] \left[ c(z) \right. \left. d(z) \right] \left[ P(z) \right] \left[ Q(z) \right] \]
\[ + \left[ P(z)^* \right] \left[ c(z)^* \right] \left[ d(z)^* \right] \left[ a(z) \right] \left[ b(z) \right] \left[ P(z) \right] \left[ Q(z) \right] \]
\[ \times (c(z)P(z) + d(z)Q(z))^{-1} \]
\[ = ((c(z)P(z) + d(z)Q(z))^{-1})^* \times \left[ P(z)^* \right] \left[ Q(z)^* \right] \mathcal{U}(z)^* J \mathcal{U}(z) \left[ P(z) \right] \left[ Q(z) \right] \]
\[ \times (c(z)P(z) + d(z)Q(z))^{-1}. \] (2.12)
It follows from (2.10) that
\[ \mathcal{U}(\xi)^* J \mathcal{U}(\xi) \equiv J \quad (\xi \in \mathbb{R}). \] (2.13)
Moreover, (2.2) implies that
\[ z = (\nu \lambda - \nu) / (\lambda - 1), \quad (2.14) \]
and we set, correspondingly,
\[ \xi = \xi(\theta) = (\nu e^{i\theta} - \nu) / (e^{i\theta} - 1). \quad (2.15) \]

Using (2.12), (2.13) and (2.15), we rewrite (2.4) in the form
\[
E(\varphi, \tilde{\lambda}) = -\frac{1}{4\pi} \int_0^{2\pi} |e^{i\theta} - \tilde{\lambda}|^{-2} (1 - |\tilde{\lambda}|^2) \times \left( -2 \ln |\det (c(\xi)P(\xi) + d(\xi)Q(\xi))| \right.
\]
\[ + \ln \det \left( \frac{1}{2} [P(\xi)^* Q(\xi)^* J \left[ P(\xi) \right] Q(\xi)] \right) \bigg) \, d\theta. \quad (2.16) \]

Let us study the case of the following nonsingular pairs with the property-\( J \):
\[ P(z) = \psi(z) \in N_0 = N, \quad Q \equiv iI_\rho. \quad (2.17) \]

Each nonsingular pair with the property-\( J \), such that \( \det Q(z) \neq 0 \), may be substituted by the equivalent pair (generating the same \( \varphi(z) \)) of the form (2.17). Using (2.17), we rewrite (2.16) in the form
\[
E(\varphi, \tilde{\lambda}) = E(\psi, \bar{\lambda})
\]
\[ + \frac{1}{2\pi} \int_0^{2\pi} |e^{i\theta} - \bar{\lambda}|^{-2} (1 - |\bar{\lambda}|^2) \ln |\det (c(\xi)\psi(\xi) + i\xi)| \, d\theta, \quad (2.18) \]
where \( \xi(\theta) \) is defined in (2.15).

**Remark 2.2.** Formula (2.18) is a generalization for the sign-indefinite case of the formula [2, (11)] (see the corresponding Theorems 2 and 3 in [2]).

**Remark 2.3.** If one wants to consider entropy for all the solutions of indefinite interpolation problems (which is equivalent to the general case of the pairs \( \{P, Q\} \)) one may easily switch from the matrix functions \( \psi \in N \) to contractive matrix functions \( \varphi \). See Remark 3.8, Definition 3.9 and formula (3.39) as well as the considerations after (3.39).

**Remark 2.4.** Our next considerations are similar to the corresponding considerations in the proof of the Szegö limit theorem (indefinite case), see [31, pp. 480-482]. In particular, we use the fact that the entries of \( \omega(\lambda) - T(\lambda) \), where \( \omega \in C_\infty \) and \( T \) is taken from the representation (C.2) of \( \omega \), belong to some Hardy class \( H_\delta \) (\( \delta > 0 \)). This fact follows from the expressions [26, (2.8)] for \( \frac{e^{i\theta} + \lambda}{e^{i\theta} - \lambda} + L(a_i, \rho_i, \theta, \lambda) \) and from the convergence inequality (C.3). As a result, for the entries \( \omega_{ij} \) of \( \omega \) we have for some \( \delta > 0 \) the inequalities:
\[
\lim_{r \to 1-0} \int_0^{2\pi} |\omega_{ij}(re^{i\theta})|^\delta \, d\theta < \infty. \quad (2.19) \]

Further we assume that
\[ \mathcal{H} = \mathbb{C}^n, \quad (2.20) \]
and so \( A \) and \( S \) are \( n \times n \) matrices.
Notation 2.5. Let us introduce the notation
\[ q(\lambda) = \left( \det \left( A^* - \frac{1}{z(\lambda)} I_n \right) \right)^p \det \left( c(z(\lambda))\psi(z(\lambda)) + \text{id}(z(\lambda)) \right) \]  
where \( z(\lambda) \) is given in (2.14).

The number of different eigenvalues of \( A \) in \( \mathbb{C}_+ \) we denote by \( \vartheta \).

Proposition 2.6. Let \( \mathcal{H} = \mathbb{C}^n, S \in \mathcal{P}_\kappa, \det S \neq 0 \) and (2.8) hold. Assume that \( A \) satisfies the conditions
\[ \sigma(A) \cap \sigma(A^*) = \emptyset; \quad \sigma(A) \text{ is a finite set,} \]  
and \( \ker \Phi_2 = \{0\} \). Then, the number \( \kappa \) of different zeros of \( q(\lambda) \) in \( \mathbb{D} \) is finite. Moreover, we have
\[ \kappa \leq \kappa + \vartheta. \]  
Proof. Similar to [34] and [31, (1.17)] we introduce the matrix function
\[ A_{\psi} := A - \Phi_2(\Phi_1^* - \psi(1/\overline{z})^* \Phi_2^*)S^{-1} \]  
\( (z \in \mathbb{C}_+ \cup \mathbb{C}_-) \), (2.24)
where
\[ \psi(z) := \psi(\overline{z})^* \text{ for } z \in \mathbb{C}_- \]  
(2.25)
From (2.8) and (2.24) we derive
\[ A_{\psi}S - SA_{\psi}^* = \Phi_2(\psi(1/\overline{z})^* - \psi(1/\overline{z})^*) \Phi_2^*. \]  
(2.26)
In this proof we will consider \( A_{\psi} \) and \( \psi(z) \) for \( z \in \mathbb{C}_+ \). Assume first that
\[ \psi \equiv \text{const}, \quad \text{i}(\psi^* - \psi) > 0. \]  
(2.27)
In this case, \( A_{\psi}^* \) is \( S \)-dissipative and does not have real eigenvalues. The first fact follows from (2.27) and the second fact is proved by contradiction. Indeed, assuming that
\[ A_{\psi}^*f = cf \quad (f \in \mathbb{C}^n, \quad f \neq 0, \quad c = \overline{c}), \]  
(2.28)
and multiplying both parts of (2.26) by \( f^* \) from the left and by \( f \) from the right, we obtain
\[ 0 = f^* \Phi_2(\psi^* - \psi)^* \Phi_2^*f \]  
(2.29)
Recall that \( \text{i}(\psi^* - \psi) \) is sign-definite, and so equality (2.29) yields \( \Phi_2^*f = 0 \). Hence, taking into account (2.24) and (2.28), we derive \( A_{\psi}^*f = A^*f = cf \). Since (in view of (2.22)), \( A^* \) does not have real eigenvalues, we arrive to a contradiction.

We showed that \( A_{\psi}^* \) is \( S \)-dissipative and does not have real eigenvalues. Recall also that \( S \in \mathcal{P}_\kappa \). Thus, \( A_{\psi}^* \) has \( \kappa \) eigenvalues in \( \mathbb{C}_- \) counting multiplicities.

In the same way as in [31, p. 477] it follows that \( \det(A_{\psi} - zI_n) \) has no more than \( \kappa \) zeros (counting multiplicities) in \( \mathbb{C}_+ \) for all \( \psi(z) \in N_0 \). We note that \( \psi \) may now depend on \( z \) and \( \det(A_{\psi}(z) - zI_n) \) means \( \det(A_{\psi}(z) - zI_n) \).

According to (2.24) we have the equality
\[ (A_{\psi} - zI_n)^{-1}\Phi_2 = (A - zI_n)^{-1}\Phi_2 \]
\[ = (A_{\psi} - zI_n)^{-1}\Phi_2(i\Phi_1^* - \psi(1/\overline{z})^* \Phi_2^*)S^{-1}(A - zI_n)^{-1}\Phi_2, \]  
(2.30)
which we rewrite in the form
\[(A_\psi - zI_n)^{-1}\Phi_2 G(z) = (A - zI_n)^{-1}\Phi_2, \quad (2.31)\]
where
\[G(z) = I_p - (i\Phi_1^* - \psi(1/\bar{z})^*\Phi_2^*) S^{-1}(A - zI_n)^{-1}\Phi_2. \quad (2.32)\]

On the other hand formula (2.7) implies that
\[c(z)\psi(z) + id(z) = i(I_p + \Phi_2^*(A^* - (1/z)I_n)^{-1}S^{-1}(i\Phi_1 + \Phi_2\psi(z))) = iG(1/\bar{z})^*. \quad (2.33)\]

Finally, we rewrite (2.31) in the form
\[G(1/\bar{z})^*\Phi_2^*(A_\psi^* - \frac{1}{z}I_n)^{-1} = \Phi_2^*(A^* - \frac{1}{z}I_n)^{-1}. \quad (2.34)\]

Recall that the number of different zeros of \(\det(A - zI_n)\) in \(\mathbb{C}_+\) equals \(\vartheta\) and that \(\det(A_\psi - zI_n)\) has no more than \(\vartheta + \kappa\) zeros (counting multiplicities) in \(\mathbb{C}_+\). Hence, the same holds for \(\det(A^* - \frac{1}{z}I_n)\) and \(\det(A_\psi^* - \frac{1}{z}I_n)\), respectively. Taking into account relations (2.32), (2.34) and \(\ker\Phi_2 = \{0\}\), we see that \(G(1/\bar{z})^*\) is holomorphic and does not have zeros in \(\mathbb{C}_+\) excluding, possibly zeros of \(\det(A^* - \frac{1}{z}I_n)\) and \(\det(A_\psi^* - \frac{1}{z}I_n)\). Hence, (2.33) yields that the number of different zeros of
\[
\left(\det(A^* - \frac{1}{z}I_n)\right)^p \det(c(z)\psi(z) + id(z))
\]
in \(\mathbb{C}_+\) is less or equal \(\vartheta + \kappa\). Clearly, the same is valid, if we switch from \(z\) to \(z(\lambda)\), where \(z(\lambda)\) is given by (2.14), and consider zeros of the function \(q(\lambda)\) (see (2.21)) in \(\mathbb{D}\). That is, (2.23) is proved. \(\Box\)

**Notation 2.7.** Different zeros of \(q(\lambda)\) are denoted by \(\lambda_1, \lambda_2, \ldots, \lambda_\kappa\) and their multiplicities are denoted by \(\eta_1, \eta_2, \ldots, \eta_\kappa\), respectively.

It follows from (2.7) that \(q(\lambda)\) is analytic in \(\mathbb{D}\). Moreover, since the entries of \(\psi(z(\lambda))\) belong to \(H_\delta\), formula (2.7) yields that \(q(\lambda)\) belongs to some Hardy class \(H_\delta\) as well. Hence, using Proposition 2.6 and Notation 2.7 we write down \(q\) in the form
\[q(\lambda) = B(\lambda)D(\lambda), \quad B(\lambda) = \prod_{i=1}^{\kappa} ((\lambda - \lambda_i)/(1 - \bar{\lambda_i}\lambda)), \quad (2.35)\]
where \(B(\lambda)\) is the Blaschke product and \(D(\lambda)\) belongs to \(H_\delta\) for some \(\delta > 0\) and does not have zeros in \(\mathbb{C}_+\).

**Theorem 2.8.** Assume that \(S \in \mathcal{P}_{\vartheta, \kappa}\), that \(\det S \neq 0\), and that (2.20) holds. Let the matrices \(A, S, \Phi_1\) and \(\Phi_2\) satisfy the matrix identity (2.8), let \(A\) satisfy (2.22), and let relations \(\ker\Phi_2 = \{0\}\) and (2.17) be valid.
Then, the entropy functional on \( \varphi \in \mathcal{N}(U) \) (where \( U \) is given by (2.7)) satisfies the equality

\[
E(\varphi, \tilde{\lambda}) = E(\psi, \tilde{\lambda}) + \ln \left| \det \left( c(z(\tilde{\lambda}))\psi(z(\tilde{\lambda})) + id(z(\tilde{\lambda})) \right) \right|
\]

\[+ p \ln \left| \det \left( A^* - \frac{1}{z(\tilde{\lambda})} I_n \right) \right| - \ln |B(\tilde{\lambda})| \]

\[- \frac{p}{2\pi} \int_0^{2\pi} |e^{i\theta} - \tilde{\lambda}|^{-2}(1 - |\tilde{\lambda}|^2) \ln \left| \det \left( A^* - \frac{1}{\xi(\theta)} I_n \right) \right| d\theta \]

(2.36)

(if only the terms on the right-hand side above are finite).

Moreover, if we set \( \psi(z) := \psi(z^*) \) for \( z \in \mathbb{C}_- \) and require additionally that

\[ \det \left( c(z)\psi(z) + id(z) \right) \neq 0 \quad \text{for} \quad z \quad \text{such that} \quad 1/z \in \sigma(A), \quad (2.37) \]

then the matrix functions \( \varphi \) are solutions of indefinite interpolations problems (B.23) and formula (2.36) provides the values of entropy functionals for these solutions.

**Proof.** Together with the matrix function \( c(z)\psi(z) + id(z) \) we consider the matrix function \( (c(z)\psi(z) + id(z))^{-1} \). It is immediate from (2.7) and (2.5) that

\[
\begin{bmatrix} \varphi(z) \\ iI_p \end{bmatrix} = iU(z) \begin{bmatrix} \psi(z) \\ iI_p \end{bmatrix} (c(z)\psi(z) + id(z))^{-1} \quad (2.38)
\]

Relation (2.10) implies that \( U(z)^{-1} = JU(z^*)^*J \), and we rewrite (2.38) as

\[ JU(z^*)^*J \begin{bmatrix} \varphi(z) \\ iI_p \end{bmatrix} = \begin{bmatrix} i\psi(z) \\ -I_p \end{bmatrix} (c(z)\psi(z) + id(z))^{-1}. \quad (2.39) \]

In particular, we have

\[ - \begin{bmatrix} I_p & 0 \end{bmatrix} U(z^*) \begin{bmatrix} iI_p \\ \varphi(z) \end{bmatrix} = (c(z)\psi(z) + id(z))^{-1}. \quad (2.40) \]

According to (2.2), (2.21), (2.35) and (2.40) we have

\[ D(\lambda)^{-1} = \frac{(-i)^p B(\lambda)}{\left( \det \left( A^* - \frac{1}{z(\lambda)} I_n \right) \right)^p} \det \left( \begin{bmatrix} I_p & 0 \end{bmatrix} U(z(\lambda))^* \begin{bmatrix} I_p \\ \omega(\lambda) \end{bmatrix} \right). \quad (2.41) \]

Using (2.7), (2.19) and (2.41) we obtain

\[ \lim_{r \to 1-0} \int_0^{2\pi} \left| 1/D(re^{i\theta}) \right|^{\delta} d\theta < \infty \quad (2.42) \]

for some \( \delta > 0 \). Formula (2.42) means that \( D(\lambda)^{-1} \) belongs some Hardy class, and we recall that \( D(\lambda) \) belongs some Hardy class as well. Thus, \( D(\lambda) \) is an outer function. It is immediate from the parameter representation of the outer function (see, e.g. [16, p. 76]) that

\[ \ln |D(\tilde{\lambda})| = \frac{1}{2\pi} \int_0^{2\pi} |e^{i\theta} - \tilde{\lambda}|^{-2}(1 - |\tilde{\lambda}|^2) \ln |D(e^{i\theta})| d\theta. \quad (2.43) \]
Next, using (2.21) and (2.35) we derive
\[
\det \left( c(\xi(\theta))\psi(\xi(\theta)) + \text{id}(\xi(\theta)) \right) = \frac{B(e^{i\theta})D(e^{i\theta})}{\left( \det \left( A^* - \frac{1}{\xi(\theta)}I_n \right) \right)^p}, \tag{2.44}
\]

where \( \xi(\theta) \) is given in (2.15). We note that according to (2.35) the equality
\[
|B(e^{i\theta})| = 1 \tag{2.45}
\]
is valid. In view of (2.43)–(2.45), we rewrite (2.18) in the form
\[
E(\varphi, \tilde{\lambda}) = E(\psi, \tilde{\lambda}) + \ln |q(\tilde{\lambda})| - \ln |B(\tilde{\lambda})| - \frac{p}{2\pi} \int_0^{2\pi} |e^{i\theta} - \tilde{\lambda}|^{-2}(1 - |\tilde{\lambda}|^2) \ln \left| \det \left( A^* - \frac{1}{\xi(\theta)}I_n \right) \right| d\theta, \tag{2.46}
\]

Now, (2.21) and (2.46) imply (2.36).

Taking into account [25, Theorem 5.2] (for the case \( p > 1 \)), we see that the conditions of Theorems B.6 and B.9 are fulfilled if (2.37) holds. In other words, if (2.37) holds, the matrix functions \( \varphi \) are solutions of indefinite interpolations problems (B.23).

**Remark 2.9.** In the proof of Proposition 2.6, we studied (starting from (2.27)) the case of \( z \in \mathbb{C}_+ \). In order to study the case \( z \in \mathbb{C}_- \), we can substitute the inequality in (2.27) with \( i(\psi - \psi^*) > 0 \). Then, \( A^*_\psi \) will be \(-S\)-dissipative. Quite similar to the considerations after (2.27) one shows that
\[
\det(A_{\psi(z)} - zI_n)
\]
has no more than \( \sim \) zeros (counting multiplicities) in \( \mathbb{C}_- \). Here, \( \psi \in N_0 \) (in \( \mathbb{C}_+ \)) and is introduced via (2.25) in \( \mathbb{C}_- \). Clearly, relations (2.31)–(2.34) hold in \( \mathbb{C}_+ \cup \mathbb{C}_- \).

### 3. Entropy Functionals and Indefinite Caratheodory Problem

Consider the case of block Toeplitz matrices, that is, the case
\[
S = S(n) = \{s_{j-i}\}_{i,j=1}^n \in \mathcal{P}_k, \quad \det S \neq 0, \tag{3.1}
\]
where \( s_k \) are \( p \times p \) blocks. Block Toeplitz matrices \( S \) are unique solutions of the matrix identities (introduced and applied to several problems in \([27, 28, 31]\)):
\[
AS - SA^* = i\Pi\Pi^*; \quad \Pi = [\Phi_1 \Phi_2], \tag{3.2}
\]
where

\[
A = \{a_{j-i}\}_{i,j=1}^n, \quad a_k = \begin{cases} 
0 & \text{for } k > 0 \\
\frac{i}{2} I_p & \text{for } k = 0 \\
\frac{i}{2} I_p & \text{for } k < 0 
\end{cases}, \quad J = \begin{bmatrix} 0 & I_p \\ I_p & 0 \end{bmatrix}; \quad (3.3)
\]

\[
\Phi_2 = \begin{bmatrix} I_p \\ I_p \\ \cdots \\ I_p \end{bmatrix}, \quad \Phi_1 = \begin{bmatrix} s_0/2 \\ s_0/2 + s_{-1} \\ \cdots \\ s_0/2 + s_{-1} + \cdots + s_{1-n} \end{bmatrix} + i\Phi_2\nu, \quad \nu = \nu^*; \quad (3.4)
\]

\[
A = A(n), \quad \Pi = \Pi(n), \quad \Phi_1 = \Phi_1(n), \quad \Phi_2 = \Phi_2(n). \quad (3.5)
\]

Somewhat different identities appeared in the well-known paper [13]. For related results and discussions on the operator identities or operators and matrices with the displacement structure see, for instance, [14,15,33,35,36]. The next proposition is immediate from (3.1)–(3.4).

**Proposition 3.1.** The matrices \(A, S, \Phi_1\) and \(\Phi_2\) given by (3.1), (3.3) and (3.4) satisfy conditions on these matrices from Theorems B.6 and 2.8. The corresponding space \(\mathcal{H}\), where \(A\) and \(S\) act, is finite dimensional. More precisely, \(\mathcal{H} = \mathbb{C}^{pn}\).

Let us add index \(n\) in the notation \(P_\kappa\) and write \(S(n) \in P_\kappa, n\) instead of \(S(n) \in P_\kappa\) when we consider \(pn \times pn\) matrices. We say that \(\{s_{j-i}\}_{i,j=1}^{\infty} \in P_\kappa, n\) if all the reductions \(\{s_{j-i}\}_{i,j=1}^{\hat{n}}\) \((n_0 \leq \hat{n} < \infty)\) starting from some \(n_0\) belong to \(P_\kappa, \hat{n}\).

**Definition 3.2.** Matrices \(S(\hat{n}) = \{s_{j-i}\}_{i,j=1}^{\hat{n}} \,(n < \hat{n} \leq \infty)\) are called extensions of the matrix \(S = S(n) = \{s_{j-i}\}_{i,j=1}^{n}\).

The following result may be derived as a special case of Theorem B.6 and presents a reformulation of [31, Theorem 4.1].

**Theorem 3.3.** Let \(S = \{s_{j-i}\}_{i,j=1}^{n} \in P_\kappa, n\) be an invertible block Toeplitz matrix. Assume that the matrix functions \(\varphi(z, P, Q)\) are given by (2.5) and (2.7), where \(\Pi = [\Phi_1 \Phi_2]\), the matrices \(A, \Phi_1\) and \(\Phi_2\) are given by (3.3) and (3.4), and the pairs \(\{P, Q\}\) are nonsingular pairs with the property-J satisfying inequality

\[
\det \left( \begin{bmatrix} 0 & \cdots & 0 & I_p \\
\cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & I_p \\
\end{bmatrix} \right) S^{-1} \Pi J \begin{bmatrix} P(2i) \\ Q(2i) \end{bmatrix} \neq 0. \quad (3.6)
\]

Then, the functions \(-i\varphi \left(2i1-\lambda \over 1+\lambda\right)\) admit Taylor expansions with the same first \(n\) Taylor matrix coefficients determined by \(S\):

\[
-i\varphi \left(2i1-\lambda \over 1+\lambda\right) = \left(\begin{array}{c} s_{0} \\ s_{1} \\ \cdots \\ s_{n-1} \end{array} \right) + \lambda s_0 + \cdots + s_{n-1} \lambda^{n-1} + \cdots. \quad (3.7)
\]
Moreover, further Taylor coefficients of the functions \( \varphi \) generate extensions \( S(\tilde{n}) = \{s_{j-i}\}_{i,j=1}^{\tilde{n}} \) of \( S \) belonging to the classes \( \mathcal{P}_{\kappa,\tilde{n}} \):

\[
- i\varphi \left( 2i \frac{1 - \lambda}{1 + \lambda} \right) = \left( \frac{s_0}{2} - iv \right) + s_1 \lambda + \cdots + s_{n-1} \lambda^{n-1} + s_n \lambda^n + \cdots + s_{\tilde{n}} \lambda^{\tilde{n}} + \cdots
\]

\[(3.8)\]

\((n < \tilde{n} \leq \tilde{n})\), and all the extensions of \( S \) belonging to the classes \( \mathcal{P}_{\kappa,\tilde{n}} \) are generated in this way.

While reformulating [31, Theorem 4.1] as Theorem 3.3 we took into account that the coefficients \( \{w_{ij}(\lambda)\}_{i,j=1}^{n} \) (see [31, (1.6)]) of the linear fractional transformation \( \varphi_\alpha(\lambda) \) of the form [31, (1.8)] and \( U(z) \) given by (2.7) are connected by the relation:

\[
J \left( \left\{ \frac{1 - (i/2)z}{1 + (i/2)z} \right\}_{i,j=1}^{n} \right)^* J = U(z).
\]

Formula (3.9) implies the equalities \( \varphi_\alpha(\lambda)^* = -i\varphi \left( 2i \frac{1 - \lambda}{1 + \lambda}, P, Q \right) \), where the matrix functions \( \varphi_\alpha(\lambda) \) are given by [31, (1.8)], the matrix functions \( \varphi(z, P, Q) \) are given by (2.5), and there is a simple one to one correspondence between the matrix functions \( \alpha(\lambda) \) and nonsingular pairs \( \{P(z), Q(z)\} \) with the property-

Next, let us consider condition (3.6) in greater detail. We start with a simple auxiliary lemma.

**Lemma 3.4.** Let \( f_1, f_2 \) and \( g_1, g_2 \) be \( p \times 2p \) matrices, let all four matrices have rank \( p \) and assume that the equalities

\[
f_1g_1^* = f_2g_2^* = 0
\]

hold. Then, the inequalities

\[
det(f_1f_2^*) \neq 0 \quad \text{and} \quad det(g_1g_2^*) \neq 0
\]

are equivalent.

**Proof.** Let us show that the first inequality in (3.11) yields the second (clearly, the fact that the second inequality yields the first is proved in the same way). Indeed, if \( det(g_1g_2^*) = 0 \) there is \( h \in \mathbb{C}^p \), \( h \neq 0 \) such that \( h g_1 g_2^* = 0 \). Hence, in view of \( f_2g_2^* = 0 \), there is \( \hat{h} \neq 0 \) such that \( h g_1 = \hat{h} f_2 \), and therefore \( f_1g_1^* = 0 \) implies that \( f_1f_2^* \hat{h}^* = 0 \), which contradicts \( det(f_1f_2^*) \neq 0 \). \qed

**Lemma 3.5.** Let the matrices \( A, \Phi_1 \) and \( \Phi_2 \) by given by (3.3) and (3.4), let \( S \) be given by the equalities in (3.1), and let the relations \( S \in \mathcal{P}_{\kappa,n} \) and \( det S \neq 0 \) hold. Assume that the pair \( \{P, Q\} \) is given by (2.17). Then, condition (3.6) is equivalent to the condition

\[
det \left( c(-2i)\psi(2i)^* + id(-2i) \right) \neq 0.
\]

\[(3.12)\]
Proof. According to [31, p. 452] we have

\[
(A - \frac{1}{z}I_{pn})^{-1} \Phi_2 = -\frac{z}{1 - (i/2)z} \begin{bmatrix} I_p \\
\frac{1+(i/2)z}{1-(i/2)z} I_p \\
\vdots \\
\frac{1+(i/2)z}{1-(i/2)z}^{n-1} I_p \end{bmatrix}.
\]  
(3.13)

Setting

\[
Y = Y(n) := [0 \ldots 0 I_p] S(n)^{-1} \Pi(n)
\]  
(3.14)

and using (2.7) and (3.13), we see that

\[
\lim_{z \to -2i} (1 - (i/2)z)^n [c(z) \ d(z)]^* = 2^n JY^*.
\]  
(3.15)

Moreover, formulas (2.7), (2.10) and (3.15) imply the equality

\[
[c(-2i) \ d(-2i)] S^{-1} = [0 \ I_p] - [I_p \ 0 \ldots 0] S^{-1} \Pi J.
\]  
(3.17)

We will need the equalities

\[
\text{rank} [c(-2i) \ d(-2i)] = \text{rank} Y = p.
\]  
(3.18)

In order to show that rank \( Y = p \), we partition the following matrices into the \( p \times p \) blocks:

\[
[0 \ldots 0 I_p] S^{-1} = [t_{n1} \ t_{n2} \ldots t_{nn}],
\]  
(3.19)

\[
YJ\Pi^*S^{-1} = [q_{n,1} \ q_{n,2} \ldots q_{n,n}].
\]  
(3.20)

It easily follows from (3.2) and is immediate from [28, (12)] that

\[
t_{nk} = q_{n,k} - q_{n,k+1} \quad (q_{n,n+1} := 0).
\]  
(3.21)

Thus, if rank \( Y \neq p \), we have \( hY = 0 \) for some \( h \in \mathbb{C}^p \), \( h \neq 0 \), and formulas (3.19)–(3.21) yield

\[
[0 \ldots 0 h] S^{-1} = 0,
\]  
(3.22)

which contradicts the invertibility of \( S \). Hence, indeed, rank \( Y = p \).

Using the equality rank \( Y = p \), one can show that there is an extension \( S(n+1) \in \mathcal{P}_{\kappa,n+1} \) such that det \( S(n+1) \neq 0 \) (see, e.g., [28, Lemma 8]). Now, relations (3.17) and [31, (2.16)] yield the equality

\[
\text{rank} [c(-2i) \ d(-2i)] = p,
\]  
(3.23)

and (3.18) is proved.

Finally we set

\[
f_1 = Y, \ f_2 = [-iI_p \ \psi(2i)^*], \ g_1 = [c(-2i) \ d(-2i)], \ g_2 = [\psi(2i) - iI_p].
\]

Taking into account (3.16) and (3.18), we see that \( f_1, f_2, g_1 \) and \( g_2 \) satisfy conditions of Lemma 3.4. It follows that the inequalities

\[
\det \left( [0 \ldots 0 I_p] S^{-1} \Pi J \begin{bmatrix} \psi(2i) \\ iI_p \end{bmatrix} \right) \neq 0
\]  
(3.23)

and (3.12) are equivalent. \( \square \)
In order to derive the entropy formula and include the important case \( \tilde{\lambda} = 0 \) one could remove singularities of \( \det \left( c(z(\lambda)) \psi(z(\lambda)) + id(z(\lambda)) \right) \) inside \( \mathbb{D} \) in a simpler way than in (2.21) and introduce \( q(\lambda) \) as the product:
\[
q(\lambda) = \det \left( A^* - \frac{1}{z(\lambda)} I_{pn} \right) \det \left( c(z(\lambda)) \psi(z(\lambda)) + id(z(\lambda)) \right). \tag{3.24}
\]
We note that in the case (3.1)–(3.4) of Toeplitz matrices (and in view of (2.14), where \( \upsilon = 2i \)) the following equality holds:
\[
\det \left( A^* - \frac{1}{z(\lambda)} I_{pn} \right) = \left( \frac{\lambda}{i(\lambda + 1)} \right)^{pn}. \tag{3.25}
\]
Hence, it will be even more convenient to consider \( \tilde{q}(\lambda) \) of the form
\[
\tilde{q}(\lambda) = \det \left( \lambda^n \left( c(z(\lambda)) \psi(z(\lambda)) + id(z(\lambda)) \right) \right) \tag{3.26}
\]
instead of \( q(\lambda) \) given by (3.24). Now, from (2.7), (2.14), (3.13) and (3.26) we derive
\[
\tilde{q}(\lambda) = \det \left( \begin{bmatrix} 0 & \lambda^n I_p \\ \lambda^{n-1} I_p & -\lambda^{n-2} I_p & \ldots & (\lambda - 1)^{n-1} I_p \end{bmatrix} S^{-1} \Pi J \begin{bmatrix} \psi(z(\lambda)) \end{bmatrix} \right). \tag{3.27}
\]
We introduce \( \tilde{B}(\lambda) \) as the Blaschke product in the representation of \( \tilde{q}(\lambda) \), that is, \( \tilde{B}(\lambda) \) in terms of zeros \( \lambda_i \in \mathbb{D} \) of \( \tilde{q}(\lambda) \) (counting multiplicities) is given by the formula
\[
\tilde{B}(\lambda) = \prod_{i=1}^{\kappa} \left( \frac{(\lambda - \lambda_i)/(1 - \lambda_i \lambda)}{1 - \lambda_i \lambda} \right). \tag{3.28}
\]
In a similar to the proof of Theorem 2.8 way one can show that
\[
\tilde{q}(\lambda) = \tilde{B}(\lambda) D(\lambda), \tag{3.29}
\]
where \( D(\lambda) \) is an outer function.

**Theorem 3.6.** Let the matrices \( A, \Phi_1 \) and \( \Phi_2 \) be given by (3.3) and (3.4), let \( S \) be given by the equalities in (3.1), and let the relations \( S \in \mathcal{P}_{\kappa,n} \) and \( \det S \neq 0 \) hold. Assume that the pair \( \{ P, Q \} \) has the form \( \{ \psi, iI_p \} \) (as in (2.17)) and that (3.23) is valid. Set \( \upsilon = 2i \) in (2.2), (2.14) and (2.15). Then, \( \varphi(z, \psi, iI_p) \) defined by (2.5) is a solution of the interpolation problem (B.23) (as well as the indefinite Carathéodory problem described in Theorem 3.3) and its entropy is given by the formula:
\[
E(\varphi, \tilde{\lambda}) = E(\psi, \tilde{\lambda}) + \ln |\tilde{q}(\tilde{\lambda})| - \ln |\tilde{B}(\tilde{\lambda})|, \tag{3.30}
\]
where \( \tilde{q} \) and \( \tilde{B} \) are given by (3.26) and (3.28), respectively.

**Proof.** It is easy to see that the conditions of Theorem 3.3 are fulfilled. According to Lemma 3.5, inequality (3.23) is equivalent to (3.12), which in our case coincides with (2.37). Thus, \( \varphi \) satisfies the conditions of Theorem 2.8.
as well. Therefore, \( \varphi \) is a solution of the interpolation problem (B.23) and of the indefinite Carathéodory problem described in Theorem 3.3.

Taking into account that \( |\tilde{B}(e^{i\theta})| = 1 \), that \( |\lambda| = 1 \) and that equalities (3.26) and (3.29) hold, in the same way as in the proof of (2.36), we rewrite (2.18) in the form (3.30).

Formulas (3.14) and (3.27) yield

\[
\tilde{q}(0) = (-1)^{p+n-1} \det \left( Y \begin{pmatrix} i I_p & \psi(2i) \end{pmatrix} \right).
\]

(3.31)

**Remark 3.7.** Relations (3.23) and (3.31) imply that \( \tilde{q}(\lambda) \) is well-defined at \( \lambda = 0 \) and moreover \( \tilde{q}(0) \neq 0 \). Since \( \tilde{q}(0) \neq 0 \) and \( \tilde{B}(0) \neq 0 \), one can consider entropy \( E(\varphi, \tilde{\lambda}) \) at \( \tilde{\lambda} = 0 \).

**Remark 3.8.** Sometimes, it is more convenient to use contractive \( p \times p \) matrix functions \( \varphi \) instead of the nonsingular pairs \( \{P, Q\} \) with the property-\( J \). For this purpose, one introduces \( 2p \times 2p \) matrix \( \mathcal{W} \) with the following property:

\[
\mathcal{W} := \frac{1}{\sqrt{2}} \begin{bmatrix} I_p & -I_p \\ I_p & I_p \end{bmatrix}, \quad \mathcal{W}^*J\mathcal{W} = j, \quad j := \begin{bmatrix} I_p & 0 \\ 0 & -I_p \end{bmatrix}.
\]

(3.32)

We set:

\[
\begin{bmatrix} \widehat{P}(z) \\ \widehat{Q}(z) \end{bmatrix} := \mathcal{W}^{-1} \begin{bmatrix} P(z) \\ Q(z) \end{bmatrix}, \quad \phi(z) := \widehat{Q}(z)\widehat{P}(z)^{-1}.
\]

(3.33)

From (3.32) and (3.33), it is easy to see that

\[
\text{rank} \begin{bmatrix} \widehat{P}(z) \\ \widehat{Q}(z) \end{bmatrix} = p, \quad \left[ \widehat{P}(z)^* \widehat{Q}(z)^* \right] j \begin{bmatrix} \widehat{P}(z) \\ \widehat{Q}(z) \end{bmatrix} \geq 0,
\]

and so \( \widehat{P}(z) \) is invertible and \( \phi(z) \) is contractive when the pair \( \{P(z), Q(z)\} \) is nonsingular with the property-\( J \). Moreover, relations (2.5), (3.32) and (3.33) imply that

\[
\varphi(z, P, Q) = \tilde{\varphi}(z, \phi),
\]

where

\[
\tilde{\varphi}(z, \phi) = i(\tilde{a}(z) + \tilde{b}(z)\phi(z))(\tilde{c}(z) + \tilde{d}(z)\phi(z))^{-1},
\]

(3.35)

\[
\widehat{U}(z) = \begin{bmatrix} \tilde{a}(z) & \tilde{b}(z) \\ \tilde{c}(z) & \tilde{d}(z) \end{bmatrix} := U(z)\mathcal{W}.
\]

(3.36)

Clearly, inequality (2.6) is equivalent to the inequality

\[
\det(\tilde{c}(z) + \tilde{d}(z)\phi(z)) \neq 0,
\]

(3.37)

and so \( \mathcal{N}(\mathcal{U}) \) (see Notation B.4) coincides with the set of functions \( \tilde{\varphi}(z, \phi) \), where \( \phi \) are contractive in \( \mathbb{C}_+ \) and (3.37) holds.
Definition 3.9. According to [2, (1)], the entropy $\hat{E}$ of the $p \times p$ matrix function $g(\lambda)$ (which is contractive in $\mathbb{D}$) is given by the formula

$$\hat{E}(g, \tilde{\lambda}) = -\frac{1}{4\pi} \int_0^{2\pi} |e^{i\theta} - \tilde{\lambda}|^{-2}(1 - |\tilde{\lambda}|^2) \ln \det (I_p - g(e^{i\theta})^* g(e^{i\theta})) d\theta.$$  

(3.38)

Taking into account (3.32), (3.33), (3.36) and (3.38) we rewrite (2.16) in the form

$$E(\varphi, \tilde{\lambda}) = \hat{E}(\phi(z(\lambda)), \tilde{\lambda}) + \left(p \ln 2 \right)/2 + \frac{1}{2\pi} \int_0^{2\pi} |e^{i\theta} - \tilde{\lambda}|^{-2}(1 - |\tilde{\lambda}|^2) \ln \det (\hat{c}(\xi) + \hat{d}(\xi)\phi(\xi)) |d\theta,$$

(3.39)

where $\xi(\theta)$ is given by (2.15).

In the case of indefinite Caratheodory problem, we (similar to (3.26) and (3.28)) set

$$\hat{q}(\lambda) = \det \left( \lambda^n (\hat{c}(z(\lambda)) + \hat{d}(z(\lambda))\phi(z(\lambda))) \right),$$

(3.40)

$$\hat{B}(\lambda) = \prod_{i=1}^{\tilde{\kappa}} \left( (\lambda - \lambda_i)/(1 - \overline{\lambda_i}\lambda) \right),$$

(3.41)

where $\lambda_i \in \mathbb{D}$ are zeros of $\hat{q}(\lambda)$ (counting multiplicities). Next, we factorise $\hat{q}$: $\hat{q}(\lambda) = \hat{B}(\lambda)D(\lambda)$ and in the same way as (2.41) we obtain the equality

$$D(\lambda)^{-1} = \frac{\hat{B}(\lambda)}{\lambda^n} \det \left( [I_p \ 0] \tilde{U}(z(\lambda))^* \left[ I_p \ \omega(\lambda) \right] \right).$$

(3.42)

Using (2.19) and (3.42), one can see that $D(\lambda)$ is again an outer function. Hence, taking into account (3.39), we have a somewhat stronger version of Theorem 3.6.

Theorem 3.10. Let the matrices $A$, $\Phi_1$ and $\Phi_2$ be given by (3.3) and (3.4), let $S$ be given by the equalities in (3.1), and let the relations $S \in \mathcal{P}_{\kappa,n}$ and $\det S \neq 0$ hold. Set $v = 2i$ in (2.2), (2.14) and (2.15). Assume that the $p \times p$ matrix functions $\phi(z)$ are contractive in $\mathbb{C}_+$ and satisfy the inequality

$$\det \left( [0 \ldots 0 \ I_p] S^{-1} \Pi J \mathcal{W} \left[ I_p \ \phi(2i) \right] \right) \neq 0.$$  

(3.43)

Then, the set of matrix functions $\hat{\varphi}(z, \phi)$ of the form (3.35) describes (via Taylor coefficients of $\hat{\varphi}(z(-\lambda), \phi)$ as in Theorem 3.3) all the solutions of the indefinite Caratheodory problem. The entropy functional on these matrix functions $\hat{\varphi}$ is given by the formula:

$$E(\hat{\varphi}, \tilde{\lambda}) = \hat{E}(\phi(z(\lambda)), \tilde{\lambda}) + (p \ln 2)/2 + \ln |\hat{q}(\tilde{\lambda})| - \ln |\hat{B}(\tilde{\lambda})|.$$  

(3.44)
4. Nonclassical Szegő Limit Formula

In order to introduce the entropy functionals, we transformed functions \( \varphi(z) \in N_{\kappa} \) into the functions \( \omega(\lambda) \) belonging to the class \( C_{\kappa} \). It is easy to see that the relation \( \omega(\lambda) \in C_{\kappa} \) yields \( \omega(-\lambda) \in C_{\kappa} \). In precisely the same way as before, one can deal with the entropy \( E_* \) generated by the transformation of \( \varphi \) into \( \omega_*(\lambda) := \omega(-\lambda) \). In particular, relation (2.4) takes the form

\[
E_*(\varphi, \lambda) = -\frac{1}{4\pi} \int_0^{2\pi} |e^{i\theta} - \lambda|^2 (1 - |\lambda|^2) \ln \det (\Re(\omega_*(e^{i\theta}))) d\theta. \tag{4.1}
\]

It is convenient to consider \( E_* \) instead of \( E \) in the case of Carathéodory problem. Indeed setting \( \psi = 2i \varphi \) (2.14) we obtain

\[
\omega_*(\lambda) := \omega(-\lambda) = -i \varphi \left( \frac{1 - \lambda}{1 + \lambda} \right), \tag{4.2}
\]

that is \( \omega_* \) coincides with the function on the left-hand sides of important formulas (3.7) and (3.8). Using (4.1) and the same arguments as in the proof of Theorem 3.6, one rewrites (3.30) in the form

\[
E_*(\varphi, \lambda) = E_*(\psi, \lambda) + \ln |\tilde{q}(\lambda)| - \ln |\tilde{B}(\lambda)|. \tag{4.3}
\]

Now, let us fix \( n, S(n) \in \mathcal{P}_{\kappa,n} \) and \( \psi(z) \in N_0 \). In the same way as \( \varphi \) generates block Toeplitz matrices \( S(i) \) \( (i > 0) \) using Taylor coefficients from (3.8), the matrix function \( \psi \) generates block Toeplitz matrices \( \tilde{S}(i) = \{ \tilde{s}_{k-j} \}_{j,k=1} \) \( (i > 0) \). Since \( \psi \in N_0 \), we have \( \tilde{S}(i) \geq 0 \) for all \( i > 0 \). The notations, which we introduce using \( S(i) \) (e.g., \( \Pi(i) \) and \( Y(i) \)) will obtain an accent "breve" if we substitute \( S(i) \) with \( \tilde{S}(i) \) (and we will write \( \tilde{\Pi}(i) \) and \( \tilde{Y}(i) \) in that case). Introduce also the notations:

\[
\det S(i) = \Lambda_i, \quad \det \tilde{S}(i) = \tilde{\Lambda}_i. \tag{4.4}
\]

When \( \tilde{S}(i) > 0 \), the famous first Szegő limit formula is valid:

\[
\lim_{i \to \infty} \frac{\tilde{\Lambda}_i}{\Lambda_{i-1}} = \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \ln \left( \det(f(\theta)) \right) d\theta \right\}.
\]

\[
= \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \ln \left( \det(f(2\pi - \theta)) \right) d\theta \right\} \tag{4.5}
\]

\[
f(\theta) := i \left( \psi \left( \frac{2i e^{i\theta} - 1}{e^{i\theta} + 1} \right)^* - \psi \left( \frac{2i e^{i\theta} - 1}{e^{i\theta} + 1} \right) \right). \tag{4.6}
\]

On the other hand, formulas (4.1) and (4.2) imply that

\[
E_*(\psi, 0) = -\frac{1}{4\pi} \int_0^{2\pi} \ln \det \left( f(2\pi - \theta)/2 \right) d\theta. \tag{4.7}
\]

According to (4.5) and (4.7) we have

\[
\lim_{i \to \infty} \frac{\tilde{\Lambda}_i}{\Lambda_{i-1}} = 2^p \exp \{-2E_*(\psi, 0)\}. \tag{4.8}
\]
The following important correspondences exist between the matrices generated by $S(n + i)$ and $\hat{S}(i)$ (see [31, (3.5), (3.6), (3.40)])

$$\hat{Y}(i)^* = Y(n + i)^*\Omega; \quad \hat{t}_{ii} = \Omega^*t_{n+i, n+i}\Omega,$$

(4.9)

where the matrix $\Omega$ does not depend on $i$ and $\hat{t}_{ii}$ ($t_{n+i, n+i}$) is the right lower block of $\hat{S}(i)^{-1}$ (of $S(n + i)^{-1}$):

$$\hat{t}_{ii} := [0 \ldots 0 I_p] \hat{S}(i)^{-1} [0 \ldots 0 I_p]^*.$$

From (3.31) and [31, (4.39)] it follows that

$$|\tilde{q}(0)|^2 = 1/\det(\Omega^*\Omega).$$

(4.10)

Moreover, we have (see, e.g., [31, (4.31)]):

$$\det(\hat{t}_{ii}) = \frac{\hat{\Lambda}_{i-1}}{\hat{\Lambda}_i}, \quad \det(t_{n+i, n+i}) = \frac{\Lambda_{n+i-1}}{\Lambda_{n+i}}.$$  

(4.11)

Relations (4.8)–(4.11) yield

$$\lim_{i \to \infty} \frac{\Lambda_i}{\Lambda_{i-1}} = 2^p \exp \left\{ -2E_*(\psi, 0) \right\} |\tilde{q}(0)|^{-2}. $$

(4.12)

Using the entropy formula (4.3) (at $\tilde{\lambda} = 0$) and taking into account (3.28), we rewrite (4.12) in the form

$$\lim_{i \to \infty} \frac{\Lambda_i}{\Lambda_{i-1}} = 2^p \exp \left\{ -2E_*(\varphi, 0) \right\} |\tilde{B}(0)|^{-2} = 2^p \exp \left\{ -2E_*(\varphi, 0) \right\} \prod_{j=1}^{\tilde{n}} |\lambda_j|^{-2}. $$

(4.13)

From the formula above, in view of (4.1) and (4.2) we derive the nonclassical Szegő limit formula for matrices $S(n) \in \mathcal{P}(n)$ generated by $\varphi\left(2i\frac{1-\lambda}{1+\lambda}\right)$:

$$\lim_{i \to \infty} \frac{\Lambda_i}{\Lambda_{i-1}} = 2^p \prod_{j=1}^{\tilde{n}} |\lambda_j|^{-2} \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \ln \left( \det \Im \left( \varphi \left(2i\frac{1-e^{i\theta}}{1+e^{i\theta}}\right) \right) \right) d\theta \right\}, $$

(4.14)

where $\lambda_j$ are the poles of $\det \varphi\left(2i\frac{1-\lambda}{1+\lambda}\right)$ counting multiplicities. For further details see [31, Section 4]. On the asymptotics of determinants in some other important nonclassical cases see, for instance, [3,10,20].

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Appendix A. Representation of $\varphi(z) \in \mathbb{N}_x$

The matrix function $\varphi(z) \in \mathbb{N}_x^p$ admits representation [6] (see also [24]):

$$\varphi(z) = \int_{-\infty}^{\infty} \left( \frac{1}{t-z} + \sum_{i=0}^{r} K_i(t, z) \right) d\tau(t) + R_0(z)$$

$$- \sum_{i=1}^{r} R_i \left( \frac{1}{z-\alpha_i} \right) \sum_{j=1}^{m} \left( M_j \left( \frac{1}{z-\beta_j} \right) + M_j \left( \frac{1}{z-\beta_j} \right)^* \right),$$

(A.1)

where $\alpha_1 < \alpha_2 < \ldots < \alpha_r$ are real numbers, $\beta_1, \ldots, \beta_m \in \mathbb{C}_+$ are distinct numbers;

(i) the real line $\mathbb{R}$ is a union of the sets $\Delta_0, \Delta_1, \ldots, \Delta_r$, such that $\Delta_1, \ldots, \Delta_r$ are bounded open intervals having disjoint closures, $\Delta_0$ is there complement, and $\alpha_i \in \Delta_i$ $(1 \leq i \leq r)$;

$$K_i(t, z) = \sum_{k=1}^{2\rho_i} \frac{(t-\alpha_i)^{k-1}}{(z-\alpha_i)^k} \chi_{\Delta_i}(t) \quad \text{for} \quad 1 \leq i \leq r$$

(A.2)

(recall the definition of the characteristic function $\chi_{\Delta}$ from the introduction),

$$K_0(t, z) = -\left( t (1+z^2)^{\rho_0} + (t+z) \sum_{k=1}^{\rho_0} \frac{(1+z^2)^{k-1}}{(1+t^2)^k} \right) \chi_{\Delta_0}(t),$$

$$\rho_0 \in \mathbb{N}_0, \quad \rho_1, \ldots, \rho_r \in \mathbb{N};$$

(A.3)

(ii) $\tau(t)$ is a nondecreasing (on each of the intervals $(-\infty, \alpha_0), (\alpha_i, \alpha_{i+1})$, where $0 \leq i < r$, and $(\alpha_r, \infty)$) $p \times p$ matrix function such that the following integral converges:

$$\int_{-\infty}^{\infty} \frac{(t-\alpha_1)^{2\rho_1} \cdots (t-\alpha_r)^{2\rho_r}}{(1+t^2)^{\rho_0+\tau}} \frac{d\tau(t)}{(1+t^2)^{\rho_0+\tau}} < \infty;$$

(A.4)

(iii) for each $0 \leq i \leq r$, the $p \times p$ matrix function $R_i(z)$ is a matrix polynomial of degree at most $2\rho_i + 1$ having self-adjoint matrix coefficients, such that the coefficient $\Omega_i$ in the term $\Omega_i z^{2\rho_i+1}$ of maximal degree in $R_i$ is nonnegative ($\Omega_i \geq 0$), and the equalities $R_1(0) = \cdots = R_r(0) = 0$ hold;

(iv) for each $1 \leq j \leq m$, the $p \times p$ matrix function $M_j(z)$ is a matrix polynomial $\neq 0$ such that $M_j(0) = 0$.

Remark A.1. It is easy to see that

$$\frac{1}{t-z} + K_i(t, z) = \frac{1}{t-z} \left( \frac{t-\alpha_i}{z-\alpha_i} \right)^{2\rho_i} \quad \text{for} \quad t \in \Delta_i \quad (1 \leq i \leq r),$$

(A.5)

$$\frac{1}{t-z} + K_0(t, z) = \frac{1+tz}{t-z} \left( \frac{1+z^2}{1+t^2} \right)^{\rho_0} \quad \text{for} \quad t \in \Delta_0.$$  

(A.6)

Notation A.2. The degree of $M_j$ is denoted by $\zeta_j$.  

Remark A.3. For \( p = 1 \), without loss of generality (see [17, Theorem 3.1]) we require that
\[
\sum_{i=0}^{r} \rho_i + \sum_{j=1}^{m} \zeta_j = \kappa. \tag{A.7}
\]

Appendix B. Indefinite interpolation problem

Let the operator (or matrix) \( S \in \mathcal{P}_\kappa \) be given. In what follows we assume that for some \( A \in \mathcal{B}(\mathcal{H}) \) and \( p \in \mathbb{N} \) the operator identity
\[
AS - SA^* = i(\Phi_1 \Phi_2^* + \Phi_2 \Phi_1^*) \quad (\Phi_1 \in \mathcal{B}(\mathcal{C}^p, \mathcal{H}), \quad \Phi_2 \in \mathcal{B}(\mathcal{C}^p, \mathcal{H}))
\]
holds. Fixed operators \( A \) and \( \Phi_2 \) determine a class of so called structured operators satisfying (B.1) (e.g., Toeplitz or Loewner matrices, operators with difference kernels and so on). We assume also that \( S \) is invertible and
\[
S^{-1} \in \mathcal{B}(\mathcal{H}). \tag{B.2}
\]

1. We introduce the operators \( S_\varphi \) and \( \Phi_\varphi \) in terms of the representation (A.1) of \( \varphi \in N_\kappa \). For this purpose we need some preparations. Let \( A \) and \( \Phi_2 \) be fixed and the matrix function \( F_j \) and \( \gamma_j \in \mathbb{C} \) \((j > 0)\) be given. Then we set
\[
\mathcal{F}_j = \text{Res}_{z=\gamma_j}(I - zA)^{-1} \Phi_2 F_j(z) \Phi_2^*(I - zA^*)^{-1} \quad (j > 0), \tag{B.3}
\]
\[
\hat{\mathcal{F}}_j = \text{Res}_{z=\gamma_j}(I - zA)^{-1} A \Phi_2 F_j(z) \quad (j > 0); \tag{B.4}
\]
\[
\mathcal{F}_0 = \text{Res}_{z=0}(A - zI)^{-1} \Phi_2 F_0(1/z) \Phi_2^*(A^* - zI)^{-1}, \tag{B.5}
\]
\[
\hat{\mathcal{F}}_0 = -\text{Res}_{z=0}((A - zI)^{-1} A \Phi_2 F_0(1/z)/z). \tag{B.6}
\]

Definition B.1. The Krein–Langer data corresponding to the representation (A.1) is the set
\[
\mathcal{D} = \{ \tau(t); \quad \alpha_1, \ldots, \alpha_r; \quad \beta_1, \ldots, \beta_m; \quad \rho_0, \rho_1, \ldots, \rho_r; \quad \Delta_0, \Delta_1, \ldots, \Delta_r; \quad R_0, R_1, \ldots, R_r; \quad M_1, \ldots, M_m \}. \tag{B.7}
\]

Clearly the data \( \mathcal{D}(\varphi) \) corresponding to \( \varphi \in N_\kappa \) is not defined uniquely although the arbitrariness is not so great (see Remark B.3). In particular, the points \( \beta_1, \ldots, \beta_m \) and the functions \( M_1, \ldots, M_m \) are fixed.

The operators and operator functions discussed below are generated by the data \( \mathcal{D}(\varphi) \) using (B.3)–(B.6).

Definition B.2. When \( F_j(z) = R_j(1/(z - \alpha_j)) \) and \( \gamma_j = \alpha_j \), we denote the corresponding \( \mathcal{F}_j \) and \( \hat{\mathcal{F}}_j \) by \( \mathcal{R}_j \) and \( \hat{\mathcal{R}}_j \), respectively; when \( F_j(z) = M_j(1/(z - \beta_j)) \) and \( \gamma_j = \beta_j \), we denote the corresponding \( \mathcal{F}_j \) and \( \hat{\mathcal{F}}_j \) by \( \mathcal{M}_j \) and \( \hat{\mathcal{M}}_1j \), respectively, and when \( F_j(z) = M_j(1/(\overline{z} - \beta_j))^* \) and \( \gamma_j = \overline{\beta}_j \) we denote the corresponding \( \hat{\mathcal{F}}_j \) by \( \hat{\mathcal{M}}_{2j} \). When \( F_j(z) = K_j(t, z) \) and \( \gamma_j = \alpha_j \), we denote the corresponding \( \hat{\mathcal{F}}_j \) by \( \hat{\mathcal{K}}_j(t, z) \). When \( F_0(z) = R_0(z) \), we denote
the corresponding \( F_0 \) and \( \hat{F}_0 \) by \( R_0 \) and \( \hat{R}_0 \), respectively. Finally, setting \( F_0(z) = K_0(t, z) \), where \( t \) is an additional parameter, we put
\[
\hat{R}_0(t) := -F_0(t).
\]
The matrix functions \( \tau_j(t) \) are introduced by the similar to (B.3) formulas:
\[
d\tau_j(t) = \text{Res}_{z = a_j} \left( K_j(t, z)(I - zA)^{-1}\Phi_2(d\tau(t))\Phi_2^*(I - zA^*)^{-1} \right) \quad (B.8)
\]
for \( j > 0 \), and
\[
d\tau_0(t) = -\text{Res}_{z = 0} \left( K_0(t, 1/z)(A - zI)^{-1}\Phi_2(d\tau(t))\Phi_2^*(A^* - zI)^{-1} \right). \quad (B.9)
\]

Let \( A, \Phi_2 \) and the representation (A.1) of \( \varphi(z) \) be given. Then, taking into account Definition B.2, we introduce the operators
\[
S_{\varphi} = \int_{-\infty}^{\infty} \left( (I - tA)^{-1}\Phi_2(d\tau(t))\Phi_2^*(I - tA^*)^{-1} - \sum_{i=0}^{r} d\tau_j(t) \right)
\]
\[
+ \sum_{i=0}^{r} \mathcal{R}_j - \sum_{j=1}^{m} (\mathcal{M}_j + \mathcal{M}_j^*) \quad (B.10)
\]
\[
i\Phi_{\varphi} = \int_{-\infty}^{\infty} \left( (I - tA)^{-1}A\Phi_2 - \sum_{i=0}^{r} \hat{R}_i \right) d\tau(t)
\]
\[
+ \sum_{i=0}^{r} \hat{R}_i - \sum_{j=1}^{m} (\hat{M}_{1j} + \hat{M}_{2j}) \quad (B.11)
\]

Remark B.3. According to [25, Section 3] the operators \( S_{\varphi} \) and \( \Phi_{\varphi} \) are well-defined and the integrals in (B.10) and (B.11) weakly converge under conditions (2.22) and
\[
1/\beta_j \notin \sigma(A), \quad 1/\overline{\beta_j} \notin \sigma(A). \quad (B.12)
\]
Moreover, Theorems 3.4 and 3.5 in [25] state that \( S_{\varphi} \) and \( \Phi_{\varphi} \) do not depend on the choice of the domains \( \Delta_0, \ldots, \Delta_r \), that \( S_{\varphi} = S_{\varphi}^* \in \mathcal{P}_\kappa \), where \( \kappa < \infty \), and the operator identity
\[
AS_{\varphi} - S_{\varphi}A^* = i(\Phi_{\varphi}\Phi_2^* + \Phi_2\Phi_{\varphi}^*) \quad (B.13)
\]
holds.

2. The transfer matrix function in L. Sakhnovich’s form is given here by the equality
\[
w_A(z) = I_{2p} + iz\Pi S^{-1}(I - zA)^{-1}\Pi, \quad \Pi := \begin{bmatrix} \Phi_1 & \Phi_2 \end{bmatrix}, \quad J := \begin{bmatrix} 0 & I_p \\ I_p & 0 \end{bmatrix},
\]
where \( A, S \) and \( \Pi \) satisfy (B.1), \( I \) is the identity operator and \( I_p \) is the \( p \times p \) identity matrix. (Such transfer matrix functions were first introduced and studied in [35], see also [33,37] and the references therein.) We will need the matrix function
\[
U(z) := w_A(z)^* = I_{2p} - iz\Pi^*(I - zA^*)^{-1}S^{-1}\Pi J = \begin{bmatrix} a(z) & b(z) \\ c(z) & d(z) \end{bmatrix}, \quad (B.15)
\]
where \(a, b, c\) and \(d\) are \(p \times p\) blocks of \(U\). By virtue of the properties of the transfer matrix functions (see, e.g. \([33, (1.84)]\)) we have a useful equality

\[
U(\overline{z})^* J U(z) \equiv J. \tag{B.16}
\]

The linear fractional transformations \(\varphi(z) = \varphi(z, P, Q)\), which we are interested in, are given by the formula

\[
\varphi(z) = i(a(z)P(z) + b(z)Q(z))(c(z)P(z) + d(z)Q(z))^{-1}, \tag{B.17}
\]

\[
\det(c(z)P(z) + d(z)Q(z)) \neq 0, \tag{B.18}
\]

where the \(p \times p\) matrix functions \(P\) and \(Q\) are meromorphic in \(\mathbb{C}_+\) and satisfy the inequalities

\[
P(z)^*Q(z) + Q(z)^*P(z) \geq 0, \quad P(z)^*P(z) + Q(z)^*Q(z) > 0, \tag{B.19}
\]

except for a set of isolated points. One says that such pairs \(\{P, Q\}\) are non-singular pairs with the property-\(J\) (they are also called Caratheodory pairs \([23]\)). Given a so called frame \(U\), let us introduce \(N(U)\).

**Notation B.4.** The set of matrix functions \(\varphi(z)\) of the form (B.17), where \(\{P, Q\}\) are nonsingular pairs with the property-\(J\) (which satisfy (B.18)), is denoted by \(N(U)\).

In this paper, we consider the simpler interpolation cases, where the following conditions are valid:

\[
\sigma(A) \cap \sigma(A^*) = \emptyset; \quad \sigma(A) \quad \text{is a finite set.} \tag{B.20}
\]

**3.** The function \(\varphi(z)\) given in \(\mathbb{C}_+\) is determined (further in the text) in \(\mathbb{C}_-\) by the relation

\[
\varphi(z) := \varphi(\overline{z})^* \quad \text{for} \quad z \in \mathbb{C}_-. \tag{B.21}
\]

Now, we introduce \(B_{\varphi}(z)\) in \(\mathbb{C}_+ \cup \mathbb{C}_-\) by the equality

\[
B_{\varphi}(z) := (I - zA)^{-1}(\Phi_1 - i\Phi_2\varphi(z)). \tag{B.22}
\]

**Notation B.5.** We denote by \(\mathcal{E}\) the class of matrix functions \(\varphi(z)\) which are analytic in \(\mathbb{C}_+ \cup \mathbb{C}_-\) (excluding, possibly, isolated points) and may have only removable singularities in the points \(z\) such that \(1/z \in \sigma(A)\).

The next interpolation theorem directly follows from \([25, Theorems 4.4, 5.1]\).

**Theorem B.6.** Let the operators \(A, S, \Phi_1\) and \(\Phi_2\) satisfy the operator identity (B.1). Assume that \(S \in \mathcal{P}_\kappa S\), that conditions (B.2) and (B.20) are fulfilled and that \(U\) is given by (B.15). Then the following two statements are valid.

(i) If \(\varphi \in N(U)\) (in \(\mathbb{C}_+\)), \(\varphi \in \mathcal{E}\) (see Notation B.5) and the function \(B_{\varphi}(z)\) is analytic at every \(z\) such that \(1/z \in \sigma(A)\), then \(\varphi \in N_\kappa (\kappa \leq \kappa S)\), (B.12) is valid and the equalities

\[
S = S_{\varphi}, \quad \Phi_1 = \Phi_{\varphi}, \tag{B.23}
\]

where \(S_{\varphi}\) and \(\Phi_{\varphi}\) are given by (B.10) and (B.11), respectively, are satisfied.
(ii) Conversely, if some \( p \times p \) matrix function \( \varphi(z) \) belongs \( N_{\kappa} \) and relations (B.12) and (B.23) are fulfilled, then \( \varphi(z) \in \mathcal{N}(U) \), \( \varphi(z) \in \mathcal{E} \) and \( \mathbf{B}_{\varphi}(z) \) is analytic at every \( z \) such that \( 1/z \in \sigma(A) \).

**Definition B.7.** Let the conditions of Theorem B.6 hold. Then, the matrix functions \( \varphi(z) \) considered in Theorem B.6 are called the solutions of the interpolation problem (B.23).

Clearly, one can consider the linear fractional transformation (B.17) in both half-planes \( \mathbb{C}_+ \) and \( \mathbb{C}_- \) as is done in [25,30] and is explained for the case \( p = 1 \) below. We note also that in the scalar case \( p = 1 \) the pairs \( \{P, Q\} \) look somewhat simpler (see Remark B.8 and [30]).

**Remark B.8.** Assuming that \( p = 1 \) in this remark, we note that without loss of generality one may consider linear fractional transformations (B.17) generated by the pairs \( P(z) = \psi(z) \in N \) and \( Q(z) \equiv i \) completed with the pair \( P(z) \equiv 1 \) and \( Q(z) \equiv 0 \). In other words, we have

\[
\mathcal{N}(U) = \{ \varphi(z) : \varphi(z) = i(a(z)\psi(z) + ib(z))(c(z)\psi(z) + id(z))^{-1}, \quad \psi(z) \in N, \quad c(z)\psi(z) + id(z) \neq 0 \} \cup \mathcal{N}_\infty, \tag{B.24}
\]

where

\[
\mathcal{N}_\infty = \{ ia(z)/c(z) \} \quad \text{if} \quad c(z) \neq 0; \quad \mathcal{N}_\infty = \emptyset \quad \text{if} \quad c(z) \equiv 0. \tag{B.25}
\]

Next, we set \( \psi(\overline{z}) = \overline{\psi(z)} \). Formula (B.16) may be rewritten in the form

\[
U(\overline{z}) = (J U(z)^* J)^{-1} = J(U(z)^{-1})^* J. \tag{B.26}
\]

When \( p = 1 \), formula (B.26) takes the form

\[
U(\overline{z}) = \left( \frac{1}{\det(U(z))} \right) \begin{bmatrix} a(z) & -b(z) \\ -c(z) & d(z) \end{bmatrix}. \tag{B.27}
\]

Recall that

\[
\varphi(z) = \overline{\varphi(\overline{z})}, \quad \psi(z) = \overline{\psi(\overline{z})} \quad \text{for} \quad z \in \mathbb{C}_-. \tag{B.28}
\]

In view of (B.27) and (B.28), the functions given in (B.24) for \( z \in \mathbb{C}_+ \) have the same form in \( \mathbb{C}_- \), that is

\[
\varphi(z) = i(a(z)\psi(z) + ib(z))(c(z)\psi(z) + id(z))^{-1} \quad \text{or} \quad \varphi(z) = ia(z)/c(z) \quad \text{for} \quad z \in \mathbb{C}_+ \cup \mathbb{C}_-. \tag{B.29}
\]

Instead of the condition \( \varphi(z) \in \mathcal{E} \) in Theorem B.6, we will use the condition \( c(z)\psi(z) + id(z) \neq 0 \) when \( 1/z \in \sigma(A) \) in the next theorem.

**Theorem B.9.** Let \( p = 1 \) and let the operators \( A, S, \Phi_1 \) and \( \Phi_2 \) satisfy the operator identity (B.1). Assume that \( S \in \mathcal{P}_{\infty} \) and that conditions (B.2) and (B.20) are fulfilled. Then the following two statements are valid.

(i) If \( \varphi \in \mathcal{N}(U) \) and \( c(z)\psi(z) + id(z) \neq 0 \) \( (c(z) \neq 0 \) when \( \varphi = ia(z)/c(z) \)) for all \( z \) such that \( 1/z \in \sigma(A) \), then \( \varphi \in N_{\infty} \) and the equalities (B.12) and (B.23) are satisfied.

(ii) Conversely, if some function \( \varphi(z) \) belongs \( N_{\infty} \) and relations (B.12) and (B.23) are fulfilled, then \( \varphi(z) \in \mathcal{N}(U) \).
Appendix C. Mapping of $N^p_{\infty}$ into $C^p_{\infty}$

There is a simple one to one mapping between the matrix functions $\varphi(z) \in N^p_{\infty}$ and matrix functions $\omega(\lambda) \in C^p_{\infty}$ (see, e.g., [17] or [26, p. 344]):

$$\omega(\lambda) = -i \varphi(z), \quad \text{where } \lambda = \frac{z - v}{z - \bar{v}} \quad (v \in \mathbb{C}_+). \quad (C.1)$$

Correspondingly, the representation (A.1) of $\varphi(z)$ is equivalent to the following representation of $\omega(\lambda)$:

$$\omega(\lambda) = \frac{1}{2\pi} \int_{[0,2\pi]\setminus\{a_1,\ldots,a_r\}} \left( \frac{e^{i\theta} + \lambda}{e^{i\theta} - \lambda} + \sum_{i=0}^{r} L(a_i, \rho_i, \theta, \lambda) \chi_{\Delta_i} \right) d\mu(\theta) + T(\lambda), \quad \text{where } \mu(\theta) \text{ is a nondecreasing}$$

intervals $(0, a_1), (a_i, a_{i+1})$ for $0 < i < r$

and $(a_r, 2\pi)$ matrix function, and the following integral converges:

$$\int_{[0,2\pi]\setminus\{a_1,\ldots,a_r\}} \theta^{2\rho_0} (\theta - a_1)^{2\rho_1} \cdots (\theta - a_r)^{2\rho_r} (2\pi - \theta)^{2\rho_0} d\mu(\theta) < \infty; \quad (C.3)$$

e^{i\alpha_0} = 1 \quad (\text{i.e., } a_0 = 0 \text{ or } a_0 = 2\pi); \quad e^{i\alpha_i} = \frac{\alpha_i - \epsilon}{\alpha_i + \epsilon} \quad (0 < \alpha_i < 2\pi) \quad \text{for } i > 0,$

where the set $\alpha_i$ is taken from the representation (A.1) of $\varphi(z)$; $\chi_{\tilde{\Delta}_i}$ is a characteristic function and for $i > 0$ the set $\tilde{\Delta}_i$ is an open interval on $[0, 2\pi]$ containing $\alpha_i$; $\tilde{\Delta}_0 = [0, 2\pi] \setminus \bigcup_{i=1}^{r} \tilde{\Delta}_i$;

$$T(\lambda) = -i R_0(z) + i \sum_{i=1}^{r} R_i \left( \frac{1}{z - \alpha_i} \right)$$

$$-i \sum_{j=1}^{m} \left( M_j \left( \frac{1}{z - \beta_j} \right) + M_j \left( \frac{1}{\bar{z} - \beta_j} \right)^* \right); \quad (C.4)$$

for $e^{i\alpha} \neq 1$ we have

$$L(a, \rho, \theta, \lambda) = \frac{2e^{i\theta}(e^{i\alpha} - 1)}{(e^{i\theta} - 1)^2} \sum_{k=0}^{2\rho_1 - 1} \left( \frac{e^{i\theta} - e^{i\alpha}}{e^{i\theta} - 1} \right)^k \left( \frac{\lambda - 1}{\lambda - e^{i\alpha}} \right)^{k+1} - \frac{e^{i\theta} + 1}{e^{i\theta} - 1}, \quad (C.5)$$

and for $e^{i\alpha} = 1$ we have

$$L(a, \rho, \theta, \lambda) = \frac{(e^{i\theta} + \lambda)(\lambda e^{i\theta} - 1)}{e^{i\theta}(1 - \lambda)^2} \sum_{k=0}^{\rho_1 - 1} \lambda^k (e^{i\theta} - 1)^{2k} \quad (C.6)$$

According to [26, formula (2.10)] we have

$$\frac{e^{i\theta} + \lambda}{e^{i\theta} - \lambda} + L(a, \rho, \theta, \lambda) = O(\theta - a)^{2\rho} \quad \text{for } \theta \to a \quad (\lambda \in \mathbb{D}), \quad (C.7)$$

where $a$ and $\rho$ take the values $a_i$ and $\rho_i$ $(1 \leq i \leq r)$, respectively, as well as $a = 0$, $\rho = \rho_0$ and $a = 2\pi$, $\rho = \rho_0$. We have also (see [26, formula (2.9)]) a useful relation

$$L(a, \rho, \theta, \lambda) = -L(a, \rho, \theta, 1/\lambda). \quad (C.8)$$
Consider representation (C.2) in greater detail. Fix some arbitrary closed interval \([\ell_1, \ell_2]\) belonging to one of the intervals \((0, \alpha_1), (\alpha_i, \alpha_{i+1})\), where \(0 < i < r\), or \((\alpha_r, 2\pi)\), and note that the matrix function \(\mu(\theta)\) is bounded on \([\ell_1, \ell_2]\). Hence, we write down \(\omega(\lambda)\) as a sum of two functions generated by the representation (C.2):

\[
\omega(\lambda) = \omega_0(\lambda) + \tilde{\omega}(\lambda), \quad \tilde{\omega}(\lambda) := \frac{1}{2\pi} \int_{\ell_1}^{\ell_2} e^{i\theta} + \lambda e^{i\theta} - \lambda d\mu(\theta). \tag{C.9}
\]

It is easy to see that \(\tilde{\omega} \in C\). Taking into account Smirnov’s and Nevanlinna’s theorems, we derive that the entries of \(\tilde{\omega}(\lambda)\) belong to the Hardy class \(H_\delta\) for each \(1 > \delta > 0\), and the non-tangential limits \(\lim_{\lambda \to \exp\{it\}} \tilde{\omega}(\lambda)\) \((t = \ell)\) exist and are finite almost everywhere. (One could use also theorem for analytic functions with positive real part from [16, p. 58].) In view of the representation (C.2) of \(\omega\) and equalities (1.2), (C.4) and (C.8), it is easy to see that the function \(\omega_0(\lambda) = \omega(\lambda) - \tilde{\omega}(\lambda)\) has the following property:

\[
\lim_{\lambda \to \exp\{it\}} \Re(\omega_0(\lambda)) = 0 \quad \text{for} \quad t \in (\ell_1, \ell_2). \tag{C.10}
\]

Here we used the relations (C.3) and (C.7) as well. Hence, according to (C.9) the following equality holds for the non-tangential limits:

\[
\lim_{\lambda \to \exp\{it\}} \Re(\omega(\lambda)) = \lim_{\lambda \to \exp\{it\}} \Re(\tilde{\omega}(\lambda)) \quad \text{for} \quad t \in (\ell_1, \ell_2). \tag{C.11}
\]

Therefore, we denote the limits in (C.11) as \(\Re(\omega(e^{it}))\) and \(\Re(\tilde{\omega}(e^{it}))\), and (C.11) takes the form

\[
\Re(\omega(e^{it})) = \Re(\tilde{\omega}(e^{it})) \quad \text{for} \quad t \in (\ell_1, \ell_2). \tag{C.12}
\]

Now, using the definition of \(\tilde{\omega}\) in (C.9) and Fatou’s theorem (see, e.g., [22, p. 39]) we have \(\Re(\tilde{\omega}(e^{it})) = \mu'(t)\) for \(t \in (\ell_1, \ell_2)\). Thus, almost everywhere on \((0, 2\pi)\) equality (C.12) yields

\[
\Re(\omega(e^{i\theta})) = \mu'(\theta). \tag{C.13}
\]

References

[1] Arov, D.Z., Krein, M.G.: The problem of finding the minimum entropy in indeterminate problems of continuation (Russian). Funktsional. Anal. i Prilozhen. 15, 61–64 (1981)

[2] Arov, D.Z., Krein, M.G.: Calculation of entropy functionals and their minima in indeterminate continuation problems (Russian). Acta Sci. Math. 45, 33–50 (1983)

[3] Basor, E.L., Ehrhardt, T.: Some identities for determinants of structured matrices. Special issue on structured and infinite systems of linear equations. Linear Algebra Appl. 343/344, 5–19 (2002)

[4] Bultheel, A., Müller, K.: On several aspects of J-inner functions in Schur analysis. Bull. Belg. Math. Soc. 5, 603–648 (1998)

[5] Burg, J.P.: Maximum Entropy Spectral Analysis. PhD thesis, Stanford University (1975)

[6] Daho, K., Langer, H.: Matrix functions of the class \(N_\kappa\). Math. Nachr. 120, 275–294 (1985)
[7] Dym, H., Gohberg, I.: On an extension problem, generalized Fourier analysis, and an entropy formula. Integr. Equ. Oper. Theory 3, 143–215 (1980)
[8] Derevyagin, M., Simanek, B.: Szegő’s theorem for a nonclassical case. J. Funct. Anal. 272, 2487–2503 (2017)
[9] Derevyagin, M., Simanek, B.: Asymptotics for polynomials orthogonal in an indefinite metric. J. Math. Anal. Appl. 460, 777–793 (2018)
[10] Ehrhardt, T.: The asymptotics of a Bessel-kernel determinant which arises in random matrix theory. Adv. Math. 225, 3088–3133 (2010)
[11] Fritzsche, B., Fuchs, S., Kirstein, B.: An inverse entropy optimization problem for matrix-valued Carathéodory functions. Optimization 29, 1–32 (1994)
[12] Fritzsche, B., Kirstein, B., Roitberg, I.Ya., Sakhnovich, A.L.: Discrete Dirac systems on the semiaxis: rational reflection coefficients and Weyl functions. J. Differ. Equ. Appl. 25, 294–304 (2019)
[13] Kailath, T., Kung, S.Y., Morf, M.: Displacement ranks of a matrix. Bull. Am. Math. Soc. (N.S.) 1, 769–773 (1979)
[14] Kailath, T., Sayed, A.H.: Displacement structure: theory and applications. SIAM Rev. 37, 297–386 (1995)
[15] Kailath, T., Sayed, A.H. (eds.): Fast Reliable Algorithms for Matrices with Structure. SIAM, Philadelphia (1999)
[16] Koosis, P.: Introduction to $H_p$ Spaces, 2nd edn with Two Appendices by V. P. Havin. Cambridge University Press, Cambridge (1998)
[17] Krein, M.G., Langer, H.: Über einige Fortsetzungsprobleme, die eng mit der Theorie hermitescher Operatoren im Raume $Π_κ$ zusammenhängen. I. Einige Funktionenklassen und ihre Darstellungen. Math. Nachr. 77, 187–236 (1977)
[18] Krein, M.G., Langer, H.: On some continuation problems which are closely related to the theory of operators in spaces $Π_κ$. IV. Continuous analogues of orthogonal polynomials on the unit circle with respect to an indefinite weight and related continuation problems for some classes of functions. J. Oper. Theory 13, 299–417 (1985)
[19] Leon, J.R., Marcantognini, S.A.M.: Parameterization of the extrapolations in the Krein–Schwartz theorem and the entropy maximizer: the scalar case. Complex Anal. Oper. Theory 8, 327–348 (2014)
[20] Lyons, R.: Szegő limit theorems. Geom. Funct. Anal. 13, 574–590 (2003)
[21] Makarov, N., Poltoratski, A.: Meromorphic inner functions, Toeplitz kernels and the uncertainty principle. In: Benedicks, M., Jones, P.W., Smirnov, S. (eds.) Perspectives in Analysis. Mathematical Physics Studies, vol. 27, pp. 185–252. Springer, Berlin (2005)
[22] Nikolski, N.K.: Operators, Functions, and Systems: An Easy Reading, Vol. 1. Hardy, Hankel, and Toeplitz. American Mathematical Society, Providence (2002)
[23] Potapov, V.P.: Collected Papers of V. P. Potapov. T. Ando, Hokkaido University, Sapporo (1982)
[24] Rovnyak, J., Sakhnovich, L.A.: On the Krein–Langer integral representation of generalized Nevanlinna functions. Electron. J. Linear Algebra 11, 1–15 (2004)
[25] Rovnyak, J., Sakhnovich, L.A.: On indefinite cases of operator identities which arise in interpolation theory. In: Bakonyi, M., Gheondea, A., Putinar, M.,
Rovnyak, J. (ed.) The Extended Field of Operator Theory. Operator Theory: Advances and Applications, vol. 171, pp. 281–322. Basel, Birkhäuser (2007)

[26] Rovnyak, J., Sakhnovich, L.A.: On indefinite cases of operator identities which arise in interpolation theory. II. In: Alpay, D., Kirstein, B. (eds.) Recent Advances in Inverse Scattering, Schur Analysis and Stochastic Processes. Operator Theory: Advances and Applications, vol. 244, pp. 341–378. Cham, Birkhäuser/Springer (2015)

[27] Sakhnovich, A.L.: A certain method of inverting Toeplitz matrices. Mat. Issled. 8, 180–186 (1973)

[28] Sakhnovich, A.L.: Continuation of Block Toeplitz Matrices. Functional Analysis (Ul’yanovsk) No. 14, pp. 116–127 (1980)

[29] Sakhnovich, A.L.: On a class of extremal problems. Math. USSR-Izv. 30, 411–418 (1988)

[30] Sakhnovich, A.L.: Modification of V. P. Potapov’s scheme in the indefinite case. In: Gohberg, I., Sakhnovich, L.A. (eds.) Matrix and Operator Valued Functions. Operator Theory: Advances and Applications, vol. 72, pp. 185–201. Basel, Birkhäuser (1994)

[31] Sakhnovich, A.L.: Toeplitz matrices with an exponential growth of entries and the first Szegő limit theorem. J. Funct. Anal. 171, 449–482 (2000)

[32] Sakhnovich, A.L.: New ”Verblunsky-type” coefficients of block Toeplitz and Hankel matrices and of corresponding Dirac and canonical systems. J. Approx. Theory 237, 186–209 (2019)

[33] Sakhnovich, A.L., Sakhnovich, L.A., Roitberg, I.Ya.: Inverse Problems and Nonlinear Evolution Equations. Solutions, Darboux Matrices and Weyl–Titchmarsh Functions. De Gruyter, Berlin (2013)

[34] Sakhnovich, L.A.: The operator Bezoutiant in the theory of the separation of roots of entire functions. Funct. Anal. Appl. 10, 45–51 (1976)

[35] Sakhnovich, L.A.: On the factorization of the transfer matrix function. Sov. Math. Dokl. 1(7), 203–207 (1976)

[36] Sakhnovich, L.A.: Equations with a difference kernel on a finite interval. Russ. Math. Surv. 35, 81–152 (1980)

[37] Sakhnovich, L.A.: Interpolation Theory and Its Applications. Kluwer, Dordrecht (1997)

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