Convex regularization of discrete-valued inverse problems

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Motivation: discrete optimization

\[
\min_{u \in U} \mathcal{F}(u) + \frac{\alpha}{2} \|u\|^2
\]

- \(\mathcal{F}\) discrepancy term (involving PDEs)
- \(U\) discrete set,

\[
U = \{ u \in L^p(\Omega) : u(x) \in \{u_1, \ldots, u_d\} \text{ a.e.} \}
\]

- \(u_1, \ldots, u_d\) given voltages, velocities, materials, ...
  (assumed here: ranking by magnitude possible!)

- motivation: topology optimization, medical imaging
**Motivation: penalty**

- **convex relaxation**: replace $U$ by convex hull
- works only for $d = 2$, cf. bang-bang control ($\alpha = 0$)
- $\rightsquigarrow$ promote $u(x) \in \{u_1, \ldots, u_d\}$ by **convex pointwise penalty**
  \[
  \mathcal{G}(u) = \int_{\Omega} g(u(x)) \, dx
  \]
- generalize $L^1$ norm: **polyhedral epigraph** with vertices $u_1, \ldots, u_d$
- **not** exact relaxation/penalization (in general)!
Motivation: penalty

- generalize $L^1$ norm: polyhedral epigraph with vertices $u_1, \ldots, u_d$

- motivation: convex envelope of $\frac{1}{2}||u||^2 + \delta_U$

- multi-bang (generalized bang-bang) control

- $\Rightarrow$ non-smooth optimization in function spaces
1 Overview

2 Approach
   - Convex analysis
   - Moreau–Yosida regularization
   - Semismooth Newton method
   - Multi-bang penalty

3 Multi-bang regularization
   - Regularization properties
   - Structure and numerical solution

4 Nonlinear problems
Fenchel duality

\[ F : V \to \bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\} \text{ convex, } \quad V \text{ Banach space, } V^* \text{ dual space} \]

- **subdifferential**

\[ \partial F(\bar{v}) = \left\{ v^* \in V^* : \langle v^*, v - \bar{v}\rangle_{V^*,V} \leq F(v) - F(\bar{v}) \text{ for all } v \in V \right\} \]

- **Fenchel conjugate** (always convex)

\[ F^* : V^* \to \bar{\mathbb{R}}, \quad F^*(v^*) = \sup_{v \in V} \langle v^*, v \rangle_{V^*,V} - F(v) \]

- "convex inverse function theorem":

\[ v^* \in \partial F(v) \iff v \in \partial F^*(v^*) \]
Fenchel duality: application

\[ F(\tilde{u}) + G(\tilde{u}) = \min_u F(u) + G(u) \]

1. Fermat principle: \( 0 \in \partial (F(\tilde{u}) + G(\tilde{u})) \)

2. Sum rule: \( 0 \in \partial F(\tilde{u}) + \partial G(\tilde{u}) \), i.e., there is \( \bar{p} \in V^* \) with

\[
\begin{aligned}
-\bar{p} & \in \partial F(\tilde{u}) \\
\bar{p} & \in \partial G(\tilde{u})
\end{aligned}
\]

3. Fenchel duality:

\[
\begin{aligned}
-\bar{p} & \in \partial F(\tilde{u}) \\
\tilde{u} & \in \partial G^*(\bar{p})
\end{aligned}
\]
Regularization

\[ \mathcal{G} \text{ non-smooth} \leadsto \text{subdifferential } \partial \mathcal{G}^* \text{ set-valued} \leadsto \text{regularize} \]

\( u, p \in L^2(\Omega) \) Hilbert space \( \leadsto \) consider for \( \gamma > 0 \)

Proximal mapping

\[
\text{prox}_{\gamma \mathcal{G}^*}(p) = \arg \min_w \mathcal{G}^*(w) + \frac{1}{2\gamma} \| w - p \|^2
\]

- single-valued, Lipschitz continuous
- coincides with \text{resolvent} \((\text{Id} + \gamma \partial \mathcal{G}^*)^{-1}(p)\)
- (also required for primal-dual first-order methods)
### Proximal mapping

\[
\text{prox}_{\gamma G^*}(p) = \arg \min_w G^*(w) + \frac{1}{2\gamma} \| w - p \|^2
\]

### Complementarity formulation of \( u \in \partial G^*(p) \)

\[
u = \frac{1}{\gamma} \left( (p + \gamma u) - \text{prox}_{\gamma G^*}(p + \gamma u) \right)
\]

- **equivalent** for every \( \gamma > 0 \)
- **single-valued, Lipschitz continuous, implicit**
Proximal mapping

\[ \text{prox}_{\gamma \mathcal{G}^*}(p) = \arg \min_w \mathcal{G}^*(w) + \frac{1}{2\gamma} \|w - p\|^2 \]

Moreau–Yosida regularization of \( u \in \partial \mathcal{G}^*(p) \)

\[ u = \frac{1}{\gamma} \left( p - \text{prox}_{\gamma \mathcal{G}^*}(p) \right) =: \partial \mathcal{G}^*_\gamma(p) \]

- \( \partial \mathcal{G}^*_\gamma = \partial \left( \mathcal{G} + \frac{\gamma}{2} \| \cdot \|^2 \right)^* \rightarrow \partial \mathcal{G}^* \text{ as } \gamma \rightarrow 0 \)

- single-valued, Lipschitz continuous, explicit
  \( \sim \text{nonsmooth operator equation, Newton method} \)
Semismooth Newton method

\( f \) locally Lipschitz, piecewise \( C^1 \):

\[
f(v) = 0, \quad f : \mathbb{R}^n \rightarrow \mathbb{R}
\]

Newton derivative

\[
D_N f(v) \delta v \in \partial_C f(v) \delta v
\]

Clarke generalized gradient: convex hull of piecewise derivatives

semismooth Newton method

\[
D_N f(v^k) \delta v = -f(v^k), \quad v^{k+1} = v^k + \delta v
\]

converges locally superlinearly
Semismooth Newton method

$f$ locally Lipschitz, piecewise $C^1$:

\[ F(u) = 0, \quad F : L^r(\Omega) \rightarrow L^s(\Omega), \quad [F(u)](x) = f(u(x)) \]

Newton derivative

\[ [D_NF(u)\delta u](x) \in \partial_C f(\delta u(x))\delta u(x) \]

any measurable selection of Clarke generalized gradient

semismooth Newton method

\[ D_NF(u^k)\delta u = -F(u^k), \quad u^{k+1} = u^k + \delta u \]

converges locally superlinearly if $r > s$
Numerical solution: summary

For (non)convex $G : L^2(\Omega) \rightarrow \mathbb{R}$, $G(u) = \int_\Omega g(u(x)) \, dx$,

**Approach:** pointwise

1. compute subdifferential $\partial g$ (or Fenchel conjugate $g^*$)
2. compute subdifferential $\partial g^*$
3. compute proximal mapping $\text{prox}_{\gamma g^*}$
4. compute Moreau–Yosida regularization $\partial g^*_\gamma$
5. compute Newton derivative $D_N \partial g^*_\gamma$

$\Rightarrow$ semismooth Newton method, continuation in $\gamma$ for superposition operator $[\partial G^*_\gamma(p)](x) = \partial g^*_\gamma(p(x))$
Multi-bang penalty

\[ g : \mathbb{R} \to \mathbb{R}, \quad v \mapsto \begin{cases} \frac{1}{2} ((u_i + u_{i+1})v - u_iu_{i+1}) & v \in [u_i, u_{i+1}] \\ \infty & \text{else} \end{cases} \]

piecewise differentiable \( \sim \) subdifferential convex hull of derivatives

\[ \partial g(v) = \begin{cases} \left( -\infty, \frac{1}{2}(u_1 + u_2) \right) & v = u_1 \\ \left\{ \frac{1}{2}(u_i + u_{i+1}) \right\} & v \in (u_i, u_{i+1}) \quad 1 \leq i < d \\ \left[ \frac{1}{2}(u_{i-1} + u_i), \frac{1}{2}(u_i + u_{i+1}) \right] & v = u_i \quad 1 < i < d \\ \left[ \frac{1}{2}(u_{d-1} + u_d), \infty \right) & v = u_d \end{cases} \]
Multi-bang penalty

\[ \partial g(v) = \begin{cases} 
(\infty, \frac{1}{2}(u_1 + u_2)] & v = u_1 \\
\{\frac{1}{2}(u_i + u_{i+1})\} & v \in (u_i, u_{i+1}) \quad 1 \leq i < d \\
\left[\frac{1}{2}(u_{i-1} + u_i), \frac{1}{2}(u_i + u_{i+1})\right] & v = u_i \quad 1 < i < d \\
\left[\frac{1}{2}(u_{d-1} + u_d), \infty\right) & v = u_d 
\end{cases} \]

convex inverse function theorem:

\[ \partial g^*(q) \in \begin{cases} 
\{u_1\} & q \in (\infty, \frac{1}{2}(u_1 + u_2)) \\
[u_i, u_{i+1}] & q = \frac{1}{2}(u_i + u_{i+1}), \quad 1 \leq i < d \\
\{u_i\} & q \in (\frac{1}{2}(u_{i-1} + u_i), \frac{1}{2}(u_i + u_{i+1})) \quad 1 < i < d, \\
\{u_d\} & q \in (\frac{1}{2}(u_{d-1} + u_d), \infty) 
\end{cases} \]
Multi-bang penalty: sketch

(a) $g(u_1 = 0, u_2 = 1, u_3 = 2)$

(b) $\partial g(u_1 = 0, u_2 = 1, u_3 = 2)$
Multi-bang penalty: sketch

\( \frac{\partial g}{\partial u_1} (u_1 = 0, u_2 = 1, u_3 = 2) \)

\( \frac{\partial g^*}{\partial u_1} (u_1 = 0, u_2 = 1, u_3 = 2) \)
Proximal mapping \( \text{prox}_{\gamma g^*}(q) = w \) iff \( q \in \{w\} + \gamma \partial g^*(w) \)

case-wise inspection of subdifferential:

\[
\partial g^*_\gamma(q) = \frac{1}{\gamma} \left( q - \text{prox}_{\gamma g^*}(q) \right) = \begin{cases} u_i & q \in Q^\gamma_i \\ \frac{1}{\gamma} \left( q - \frac{1}{2}(u_i + u_{i+1}) \right) & q \in Q^\gamma_{i,i+1} \end{cases}
\]

\[
Q^\gamma_i = \left( \frac{1}{2}(u_{i-1} + u_i) + \gamma u_i, \frac{1}{2}(u_i + u_{i+1}) + \gamma u_i \right)
\]

\[
Q^\gamma_{i,i+1} = \left[ \frac{1}{2}(u_i + u_{i+1}) + \gamma u_i, \frac{1}{2}(u_i + u_{i+1}) + \gamma u_{i+1} \right]
\]
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Multi-bang regularization

\[
\min_{u \in L^2(\Omega)} \frac{1}{2} \| Ku - y^\delta \|^2_Y + \alpha \mathcal{G}(u)
\]

- \( K : L^2(\Omega) \rightarrow Y \) (linear) forward mapping, weakly closed
- \( y^\delta \in L^2(\Omega) \) noisy data with \( \| y - y^\delta \|^Y \leq \delta \)
- \( u_1 < \cdots < u_d \) given parameter values \( (d > 2) \)
- \( \mathcal{G} \) multi-bang penalty
Multi-bang regularization

\[
\min_{u \in L^2(\Omega)} \frac{1}{2} \| Ku - y^\delta \|_Y^2 + \alpha \mathcal{G}(u)
\]

- **\( \mathcal{G} \) multi-bang penalty** convex:
  1. existence of solution \( u^\delta_\alpha \) for every \( \alpha > 0 \)
  2. \( \delta \to 0 \) implies \( u^\delta_\alpha \rightharpoonup u_\alpha \) for every \( \alpha > 0 \)
  3. \( \delta \to 0, \alpha \to 0, \delta \alpha^{-2} \to 0 \) implies \( u^\delta_\alpha \rightharpoonup u^\dagger \)

(standard arguments, e.g. [Burger/Osher 04, Ito/Jin 14])
Multi-bang regularization

\[
\min_{u \in L^2(\Omega)} \frac{1}{2} \| Ku - y^\delta \|_Y^2 + \alpha G(u)
\]

- **standard source condition:** \( p^\dagger := K^* w \in \partial G(u^\dagger) \) for \( w \in Y \),

  1. **a priori choice** \( \alpha(\delta) = c \delta \)
  2. **a posteriori choice** \( \| Ku_{a(\delta)}^\delta - y^\delta \|_Y \leq \tau \delta \)

\( \leadsto \) **convergence rate**

\[
d_{G}^{p^\dagger}(u_{a}^\delta, u^\dagger) \leq C \delta
\]

in Bregman distance

\[
d_{G}^{p_1}(u_2, u_1) = G(u_2) - G(u_1) - \langle p_1, u_2 - u_1 \rangle_{X}, \quad p_1 \in \partial G(u_1)
\]
**Multi-bang regularization**

**Pointwise** definition of Bregman distance, $\partial g$:

- $u^\dagger(x) = u_i$ and $p^\dagger \not\in \{\frac{1}{2}(u_{i-1} + u_i), \frac{1}{2}(u_i, u_{i+1})\}$ implies
  \[
  d^p_g(x)(u^\delta_{\alpha(\delta)}(x), u^\dagger(x)) \to 0 \quad \text{for } \delta \to 0
  \]

- $u^\dagger(x) \in (u_i, u_{i+1})$ implies
  \[
  d^p_g(x)(u(x), u^\dagger(x)) = 0 \quad \text{for any } u(x) \in [u_i, u_{i+1}]
  \]

- $\sim \ u^\delta_{\alpha(\delta)} \to u^\dagger$ pointwise a.e. iff $u^\dagger(x) \in \{u_1, \ldots, u_d\}$ a.e.

- (convergence not uniform $\sim \to$ no pointwise rates)
Optimality system

\[ \tilde{p} = \frac{1}{\alpha} K^* (y^\delta - K \tilde{u}) \]

\[ \tilde{u} \in \partial G^* (\tilde{p}) = \begin{cases} 
\{u_i\} & \tilde{p}(x) \in Q_i \\
[u_i, u_{i+1}] & \tilde{p}(x) \in \overline{Q_i} \cap \overline{Q_{i+1}}
\end{cases} \]

- \( \leadsto \) unique solution \( (\tilde{u}, \tilde{p}) \in L^2(\Omega) \times L^2(\Omega) \)
- singular arc \( S = \{x : \tilde{u}(x) \notin \{u_i\}\} \subset \{x : \tilde{p}(x) = \frac{1}{2}(u_i + u_{i+1})\} \)
- for suitable \( K \), \( \tilde{p}(x) \) constant implies \( [y^\delta - K \tilde{u}](x) = 0 \)
  (e.g., \( K = A^{-1} \) for \( A \) pure second-order elliptic)

\( \leadsto |\{x : K \tilde{u}(x) = y^\delta(x)\}| = 0 \Rightarrow \tilde{u} \in \{u_1, \ldots u_d\} \) a.e. (true multi-bang)
Regularized optimality system

\[
\begin{cases}
    p_\gamma = \frac{1}{\alpha} K^* (y^\delta - Ku_\gamma) \\
    u_\gamma = \partial g^*_\gamma(p_\gamma)
\end{cases}
\]

- optimality conditions for \( \mathcal{F}(u) + \alpha \mathcal{G}(u) + \frac{\gamma}{2} \|u\|^2 \)
- \( \rightsquigarrow \) unique solution \((u_\gamma, p_\gamma)\)
- \((u_\gamma, p_\gamma) \rightharpoonup (\bar{u}, \bar{p}) \) as \( \gamma \to 0 \)
- \( \partial g^*_\gamma \) Lipschitz continuous, piecewise \( C^1 \), norm gap \( V \hookrightarrow L^2(\Omega) \)
- \( \rightsquigarrow \) semismooth Newton method
Regularized optimality system

\[
\begin{align*}
\begin{cases}
p_y = \frac{1}{\alpha} K^*(y^\delta - Ku_y) \\
u_y = \partial G_y^*(p_y)
\end{cases}
\end{align*}
\]

- \(\sim\) semismooth Newton method
- inverse source problem: \(K = A^{-1}\), \(A\) elliptic differential operator
- introduce \(y_y = Ku_y\), eliminate \(u_y = G_y^*(p_y)\)

\[
\begin{align*}
\begin{cases}
A^*p_y = \frac{1}{\alpha}(y^\delta - y_y) \\
Ay_y = G_y^*(p_y)
\end{cases}
\end{align*}
\]
Semismooth Newton method

\[
\begin{pmatrix}
\frac{1}{\alpha} \text{Id} & A^* \\
A & -D_N \mathcal{G}_Y^*(p)
\end{pmatrix}
\begin{pmatrix}
\delta y \\
\delta p
\end{pmatrix}
= -\begin{pmatrix}
A^* p + \frac{1}{\alpha} (y - y^\gamma) \\
Ay - \mathcal{G}_Y^*(p)
\end{pmatrix}
\]

\[D_N \mathcal{G}_Y^*(p) \delta p](x) = \begin{cases}
\frac{1}{\gamma} \delta p(x) & p(x) \in Q^\gamma_{i,i+1} \\
0 & \text{else}
\end{cases}\]

- symmetric, but: local convergence
- \(\Rightarrow\) continuation in \(\gamma \to 0\)
- \(\Rightarrow\) backtracking line search based on residual norm
- only number of sets \(Q^\gamma_i\) depends on \(d \sim\) linear complexity
Example: linear inverse problem

- $\Omega = [0,1]^2, \quad A = -\Delta$

- $u^\dagger(x) = u_1 + u_2 \chi\{x:(x_1-0.45)^2+(x_2-0.55)^2<0.1\}(x) + (u_3 - u_2) \chi\{x:(x_1-0.4)^2+(x_2-0.6)^2<0.02\}(x)$

- $d = 3, \quad u_1 = 0, \quad u_2 = 0.1, \quad u_3 \in \{0.15, 0.12\}$

- $y^\delta = y^\dagger + \xi, \quad \xi \in \mathcal{N}(y^\dagger, \delta \|y^\dagger\|_\infty)$

- finite element discretization: uniform grid, $256 \times 256$ nodes

- $\alpha = \alpha(\delta)$ by Morozov discrepancy principle

- terminate at $\gamma < 10^{-12}$
Numerical example: \( u_3 = 0.15 \)

(a) \( u^\dagger \)

(b) \( u_\delta^\alpha, \delta \approx 1.89 \cdot 10^{-1} \)
Numerical example: $u_3 = 0.15$

- **(c)** $u^\dagger$
- **(d)** $u_\alpha^\delta$, $\delta \approx 2.37 \cdot 10^{-2}$

Overview  Approach  Multi-bang regularization  Nonlinear problems
Numerical example: $u_3 = 0.15$

(e) $u^\dagger$

(f) $u_\delta^\delta$, $\delta \approx 3.69 \cdot 10^{-4}$
Numerical example: \( u_3 = 0.11 \)

\( (a) \ u^\dagger \)

\( (b) \ u_\alpha^\delta, \delta \approx 1.68 \cdot 10^{-1} \)
Numerical example: $u_3 = 0.11$

(c) $u^\dagger$

(d) $u^\delta$, $\delta \approx 2.17 \cdot 10^{-2}$
Numerical example: \( u_3 = 0.11 \)

(e) \( u^\dagger \)

(f) \( u_\delta^\alpha, \delta \approx 3.29 \cdot 10^{-4} \)
Numerical example: $u_3(x) = 0.12(1 - x_1)$

(a) $u^\dagger$

(b) $u^\delta, \delta \approx 2.11 \cdot 10^{-2}$
Numerical example: $u_3(x) = 0.12(1 - x_1)$

(c) $u^\dagger$

(d) $u^\delta_\alpha, \delta \approx 3.29 \cdot 10^{-4}$
Numerical example: \( u_3(x) = 0.12(1 - x_1) \)

\[(e)\, u^\dagger\]

\[(f)\, u_\alpha^\delta, \delta \approx 1.29 \cdot 10^{-6}\]
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Nonlinear forward mapping

Forward mapping $S : u \mapsto y$ nonlinear:

- approach applicable if $S$
  - weak-to-weak continuous
  - twice Fréchet-differentiable

- example: $u \mapsto y$ solving $-\Delta y + uy = f$

- existence, optimality conditions

\[
\begin{align*}
-\bar{p} &= S'(\bar{u})^*(S(\bar{u}) - y^\delta) \\
\bar{u} &\in \partial G^*(\bar{p})
\end{align*}
\]

- semismooth Newton method (regularity condition technical)
Example: nonlinear inverse problem

- $S : u \mapsto y$ solving
  
  $$-\Delta y + uy = f$$

- approach applicable, but $\mathcal{F}$ nonconvex

- numerical example: $\Omega = [0, 1]^2$, $f \equiv 1$

- $u^\dagger(x) = u_1 + (u_2 - u_1) \chi_{\{x: (x_1 - 0.45)^2 + (x_2 - 0.55)^2 < 0.1\}}(x) + (u_3 - u_2 - u_1) \chi_{\{x: (x_1 - 0.4)^2 + (x_2 - 0.6)^2 < 0.02\}}(x)$

- $y^\delta = S(u^\dagger) + \xi$

- $\alpha = 3 \cdot 10^{-5}$, $\gamma \rightarrow 10^{-12}$
Numerical example: nonlinear problem

(a) noisy data $y^\delta$
Numerical example: nonlinear problem

(b) $u^\dagger$

(c) $u^\delta$
Nonlinear forward mapping

Goal: application to EIT

- $S : u \mapsto y$ solving
  $$-\nabla \cdot (u \nabla y) = f$$

- difficulty: $\bar{u} \in L^\infty(\Omega) \leadsto S$ not weakly-∗ closed
  1. lack of existence of minimizer ($\bar{y} \neq S(\bar{u})$, cf. homogenization)
  2. lack of convergence $\gamma \rightarrow 0$
  3. lack of Newton differentiability of $H_\gamma$ (no norm gap)

- remedies: higher regularity of $y$ or $u$ by
  1. local smoothing: consider $-\nabla \cdot \left( \int_{B_\varepsilon(x)} u(s) \, ds \nabla y \right)$
  2. TV regularization: add $\|D\!u\|_M$ $\leadsto u \in BV(\Omega) \cap L^\infty(\Omega) \hookrightarrow_c L^p(\Omega)$

Overview Approach Multi-bang regularization Nonlinear problems
**TV regularization**

**Difficulty:**

- existence requires box constraints $\rightsquigarrow$ use penalty

\[ G(u) + TV(u) + \delta_{[u_1,u_d]}(u) \]

- but: $TV(u) + \delta_{[u_1,u_d]}(u)$ not continuous on $L^p(\Omega), p < \infty$

- but: multipliers $\xi \in \partial TV(u), q \in \partial G(u)$ not pointwise on $BV, L^\infty$

- $\rightsquigarrow$ replace box constraints by ($C^{1,1}$) projection of $u \in L^1(\Omega)$

\[ [\Phi(u)](x) = \text{proj}_\varepsilon^{[u_1,u_d]}(u(x)) \quad \text{a.e. } x \in \Omega \]
TV regularization: existence

\[
\begin{aligned}
\min_{u \in BV(\Omega)} & \quad \frac{1}{2} \| y - z \|_{L^2(\Omega)}^2 + \alpha G(u) + \beta TV(u) \\
\text{s.t.} & \quad -\nabla \cdot (\Phi(u) \nabla y) = f \text{ in } \Omega \\
& \quad y = 0 \text{ on } \partial \Omega
\end{aligned}
\]

- **existence** of optimal \( \bar{u} \in BV(\Omega) \cap L^\infty(\Omega) \) for \( \varepsilon \geq 0 \)

- tracking term Fréchet differentiable in \( \Phi(u) \in L^\infty \) for \( \varepsilon > 0 \)

- regularity of state, adjoint \( \rightsquigarrow \) derivative in \( L^r(\Omega), r > 1 \) (instead of \( L^\infty(\Omega)^* \))

- \( \rightsquigarrow \) sum rule applicable, subgradients in \( L^r(\Omega), r > 1 \)
TV regularization: optimality conditions

\[
\begin{cases}
0 = F'(\Phi(\bar{u}))\Phi'(\bar{u}) + a\bar{q} + \beta\bar{\xi} \\
\bar{u} \in \partial G^*(\bar{q}) \\
\bar{\xi} \in \partial TV(\bar{u})
\end{cases}
\]

- \(F'(\Phi(\bar{u})) = (\nabla \bar{y} \cdot \nabla \bar{p}) \in L^r(\Omega)\) (optimal state, adjoint)
- \(\bar{q} \in L^r(\Omega), r > 1 \rightsquigarrow \) pointwise multi-bang
- \(\bar{\xi} \in L^r(\Omega), r > 1 \rightsquigarrow \) characterization via full trace [Bredies/Holler '12]
- \(\rightsquigarrow \) pointwise optimality conditions
- semi-smooth Newton (after discretization, regularization)
Numerical example: total variation

(a) $u^\dagger$

(b) $\alpha = 5 \cdot 10^{-4}, \beta = 0$
Numerical example: total variation

(c) $u^\dagger$

(d) $\alpha = 5 \cdot 10^{-4}, \beta = 10^{-5}$
**Conclusion**

**Convex relaxation of discrete regularization:**
- well-posed regularization method
- pointwise convergence under general assumptions
- strong structural regularization
- efficient numerical solution (*superlinear convergence*)

**Outlook:**
- regularization properties, parameter choice
- nonlinear inverse problems: EIT
- combination with TV regularization
- other hybrid discrete–continuous problems

**Preprint, MATLAB/Python codes:**
http://www.uni-due.de/mathematik/agclason/clason_pub.php