The G4i analogue of a G3i calculus

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Abstract

This paper provides a method to obtain terminating analytic calculi for a large class of intuitionistic modal logics. For a given logic $L$ with a cut-free calculus $G$ that is an extension of $G3ip$, the method produces a terminating analytic calculus that is an extension of $G4ip$ and equivalent to $G$. $G4ip$ has been introduced by Dyckhoff in 1992 as a terminating analogue of the calculus $G3ip$ for intuitionistic propositional logic. Thus this paper can be viewed as an extension of Dyckhoff’s work to intuitionistic modal logic.

Keywords: intuitionistic modal logic, intermediate logic, sequent calculus, terminating proof systems

MSC: 03B05, 03B45, 03F03

1 Introduction

One of the standard calculi without structural rules for IPC is $G3ip$ (Figure 2.1), which is the propositional part of the calculus $G3i$ from (Troelstra and Schwichtenberg, 1996). This is an elegant analytic calculus, but it has the unfortunate feature that unrestricted proof search is not terminating in it. The reason for this lies in its left implication rule, in which the principal formula occurs in one of the premises. In (Dyckhoff, 1992) a calculus $G4ip$ (Figure 2.2) was introduced that is the result of replacing the single left implication rule in $G3ip$ by four left implication rules, each corresponding to the outermost logical symbol of the antecedent of the principal implication. This system was shown to be terminating and equivalent to $G3ip$, where a calculus is terminating if there exists an order on sequents under which in all rules the premises come before the conclusion in that order.

The modest aim in this short note is to extend Dyckhoff’s result to intuitionistic modal logics. For any extension $G3iX$ of $G3ip$ by modal rules $R$, a calculus, called $G4iX$, is defined that is an extension of $G4ip$ by $R$ and several additional rules determined by $R$. It is shown that under some mild conditions $G3iX$ and $G4iX$ are equivalent. Moreover, if the rules in $R$ are terminating in an order that is an extension (to modal logic) of Dyckhoff’s original order on sequents, then $G4iX$ terminates in that order. Thus for logics that have such a calculus $G3iX$ that is an extension of $G3ip$ by terminating rules, establishing the equivalence of $G3iX$ and $G4iX$ indeed is a method to obtain a terminating calculus for such logics.

The interest in terminating calculi lies in the fact that they can be a useful in establishing certain properties of a logic, such as decidability, or uniform interpolation, where the syntactic approach developed in (Iemhoff, 2020; Jalali and Tabatabai, 2018) defines interpolants on the basis of the rules of a calculus, for which it is essential that the calculus is terminating. This paper’s method to obtain a terminating calculus for a given logic uses a (usually nonterminating) calculus based on $G3ip$. Since many intuitionistic modal logics have a calculus based on $G3ip$, we therefore hope that this paper will be a convenient tool in settings where terminating calculi are required.

This paper grew out of the research carried out in (van der Giessen and Iemhoff, 2019, 2020, Iemhoff, 2020), in which $G4ip$ based calculi are developed for intuitionistic versions of the logics $K$, $KD$, Gödel-Löb Logic $GL$ and Strong Löb Logic $SL$. The results for the first two logics follow

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1Originally, the calculus was called LJT by Dyckhoff, but in this paper we use the name $G4ip$ from (Troelstra and Schwichtenberg, 1996), which seems to be more common nowadays.
With such a rule we associate the following implication rule
\[ \Gamma, \varphi \Rightarrow \psi \quad R\land \]
\[ \Gamma \Rightarrow \varphi \land \psi \]
\[ \Gamma, \varphi, \psi \Rightarrow \Delta \quad L\land \]
\[ \Gamma, \varphi \Rightarrow \psi \quad L\land \]
\[ \Gamma \Rightarrow \varphi \land \psi \quad R\land \]
\[ \Gamma \Rightarrow \varphi \lor \varphi_1 \quad L\lor \]
\[ \Gamma, \varphi \Rightarrow \psi \quad R\lor \]
\[ \Gamma, \varphi \Rightarrow \Delta \quad L\lor \]
\[ \Gamma \Rightarrow \varphi \lor \psi \quad R\lor \]
\[ \Gamma, \varphi \lor \psi \Rightarrow \Delta \quad L\lor \]
\[ \Gamma, \varphi \Rightarrow \psi \quad R\lor \]
\[ \Gamma, \varphi \lor \psi \Rightarrow \Delta \quad L\lor \]
\[ \Gamma, \varphi \Rightarrow \psi \quad L\lor \]
\[ \Gamma, \varphi \lor \psi \Rightarrow \Delta \quad L\lor \]

Figure 2.1: The Gentzen calculus G3ip

from the results in this note, while the results for the other two logics require a slight adjustment of the method presented in this note, as explained in Section 3.

The paper is built-up as follows. Section 2 introduces G3iX and G4iX, the order ≪ on sequents based on Dyckhoff’s weight function, and the further notions needed in the paper. Section 3 contains the main theorem, the equivalence of G3iX and G4iX. Section 4 summarizes the results and discusses possible extensions.

2 Preliminaries

We consider (modal) propositional logics in a language \( \mathcal{L} \) that contains a constant \( \bot \), propositional variables or atoms \( p, q, r, \ldots \), modal operators \( \Box, \Diamond \), and the connectives \( \neg, \land, \lor, \rightarrow \), where \( \neg \varphi \) is defined as \( \varphi \rightarrow \bot \). \( \bot \) is by definition not an atom. \( \Box \) ranges over the modal operators.

We denote finite multisets of formulas by \( \Gamma, \Pi, \Delta, \Sigma \). We denote by \( \Gamma \cup \Pi \) the multiset that contains only formulas \( \varphi \) that belong to \( \Gamma \) or \( \Pi \) and the number of occurrences of \( \varphi \) in \( \Gamma \cup \Pi \) is the sum of the occurrences of \( \varphi \) in \( \Gamma \) and in \( \Pi \). Furthermore (\( a \) for antecedent, \( s \) for succedent):

\[ (\Gamma \Rightarrow \Delta)^a \equiv_{df} \Gamma \quad (\Gamma \Rightarrow \Delta)^s \equiv_{df} \Delta \quad \Box \Gamma \equiv_{df} \{ \Box \varphi \mid \varphi \in \Gamma \}. \]

We only consider single-conclusion sequents, which are expressions \( (\Gamma \Rightarrow \Delta) \), where \( \Delta \) contains at most one formula, and which are interpreted as \( I(\Gamma \Rightarrow \Delta) = (\forall \Delta \rightarrow \Delta) \). In a sequent, \( \Gamma, \Pi \) is short for \( \Gamma \cup \Pi \). When sequents are used in the setting of formulas, we often write \( S \) for \( I(S) \), such as in \( \vdash S_i \), which thus denotes \( \vdash \bigvee_i S_i \).

The degree of a formula \( \varphi \) is inductively defined by \( d(\bot) = 0 \), \( d(p) = 1 \), \( d(\Box \varphi) = d(\varphi) + 1 \), and \( d(\varphi \lor \psi) = d(\varphi) + d(\psi) + 1 \) for \( o \in \{ \land, \lor, \rightarrow \} \). In the setting of G4iX systems we need an order on sequents based on a weight function, which is a function \( w(\cdot) \) that assigns positive numbers to formulas in such a way that all atoms and \( \bot \) have weight 1 and all other formulas have a weight above 1. With a weight function \( w \) we associate the following order on sequents: \( S_0 \vartriangleleft_w S_1 \) if and only if \( S_0 \subseteq S_1 \), \( S_1 \subseteq S_0 \), where \( \vartriangleleft_w \) is the order on multisets determined by \( w \) as in (Dershowitz and Manna 1979) (where they in fact define \( \gg \)): for multisets \( \Gamma, \Delta \) we have \( \Delta \vartriangleleft_w \Gamma \) if \( \Delta \) is the result of replacing one or more formulas in \( \Gamma \) by zero or more formulas of lower weight.

2.1 Calculi G3iX and G4iX

In this paper, a right modal rule is a rule of the following form, where \( \Box \) is one of the modal operators:

\[ \frac{S_1, \ldots, S_n}{\Gamma \Rightarrow \Box \varphi} \quad \mathcal{R} \]

With such a rule we associate the following implication rule

\[ \Gamma, \Box \varphi \Rightarrow \psi \Rightarrow \Delta \quad \mathcal{R} \rightarrow \]

If G3iX (\( X \) a finite string) is the name of a calculus that consists of G3ip plus a set of modal rules \( \mathcal{R} \), then G4iX is the calculus G4ip extended by the rules in \( \mathcal{R} \) plus the rules \( \mathcal{R}^+ \) for those \( \mathcal{R} \in \mathcal{R} \) that are right modal rules.
\[
\begin{align*}
\Gamma, p \Rightarrow p & \quad Ax \quad (p \text{ an atom}) \\
\Gamma \Rightarrow \varphi & \quad \Gamma \Rightarrow \psi \quad R\land \\
\Gamma & \Rightarrow \varphi \land \psi \quad R\land \\
\Gamma & \Rightarrow \varphi_0 \lor \varphi_1 \quad (i = 0, 1) \\
\Gamma, \varphi, \psi & \Rightarrow \Delta \quad L\lor \\
\Gamma, \varphi_0 \lor \psi & \Rightarrow \Delta \quad L\lor \\
\Gamma, \varphi \Rightarrow \psi & \quad R \rightarrow \\
\Gamma, \varphi \rightarrow (\psi \rightarrow \gamma) & \Rightarrow \Delta \quad L\land \rightarrow \\
\Gamma, \psi \rightarrow \gamma & \Rightarrow \psi, \Gamma \Rightarrow \Delta \quad L\rightarrow \\
\Gamma, \psi \rightarrow \gamma & \Rightarrow \varphi \rightarrow \psi, \Gamma \Rightarrow \Delta \\
\end{align*}
\]

Figure 2.2: The Gentzen calculus G4ip

**Example 1** Let G3iX denote G3ip plus the rule

\[
\Pi, \Pi \Rightarrow \varphi \quad \mathcal{R}_X
\]

Then G4iX consists of G4ip, \(\mathcal{R}_X\), and the rule

\[
\Pi, \Pi, \Pi, \Pi \Rightarrow \varphi \quad \Pi, \Pi, \Pi, \Pi \Rightarrow \psi \quad \Delta \quad \mathcal{R}_X
\]

A rule is *nonflat* if its conclusion contains at least one connective or modal operator, and it is not an axiom, i.e. it has nonempty premises. A set of rules or a calculus is nonflat if all of its rules are.

Note that a calculus G3iX or G4iX is nonflat if all rules in \(\mathcal{R}\) are nonflat.

Given a sequent calculus \(G\) and a sequent \(S, \vdash G\), denotes that \(S\) is derivable in \(G\).

### 2.2 Terminating calculi

Given an order \(\ll\) on sequents, a rule is *terminating in* \(\ll\) if its premises come before the conclusion in order \(\ll\). A calculus is *terminating* if there exists a weight function \(w\) such that all rules of \(G\) terminate in the order \(\ll_w\) (see the beginning of Section 2 for the definition of \(\ll_w\)).

**Example 2** A natural weight function based on the weight function from (Dyckhoff, 1992) is inductively defined as follows: the weight of an atom and the constant \(\bot\) is 1, \(w_D(\varphi \circ \psi) = w_D(\varphi) + w_D(\psi) + i\), where \(i = 1\) in case \(\circ \in \{\lor, \rightarrow\}\) and \(i = 2\) otherwise, and \(w_D(\varphi \circ \varphi) = w_D(\varphi) + 1\).

Let \(\ll_D\) denote \(\ll_w\), and call it the *Dyckhoff order*. It is not hard to see that all rules of G4ip terminate in \(\ll_D\). The following are examples of well-known modal rules that terminate in \(\ll_D\) (writing \(\Box\) for \(\Diamond\)).

\[
\begin{align*}
\Pi, \Pi, \Pi, \Pi \Rightarrow \varphi & \quad \mathcal{R}_K \\
\Pi, \Pi, \Pi, \Pi, \Pi \Rightarrow \varphi & \quad \mathcal{R}_D \\
\Pi, \Pi, \Pi, \Pi, \Pi \Rightarrow \varphi & \quad \mathcal{R}_T
\end{align*}
\]

**Remark 1** If the rules in \(\mathcal{R}\) are terminating in the Dyckhoff order, then G4iX is terminating in the Dyckhoff order.

### 2.3 Structural rules

A calculus \(G\) is *closed under weakening* if the following two rules are admissible:

\[
\begin{align*}
\Gamma & \Rightarrow \Delta \quad \Delta \\
\Gamma, \varphi & \Rightarrow \Delta \quad \Gamma, \varphi
\end{align*}
\]
It is closed under contraction if the following rule is admissible:

\[ \frac{\Gamma, \varphi, \varphi \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta} \]

\( G \) has cut-elimination if the Cut rule is admissible:

\[ \frac{\Gamma_1 \Rightarrow \varphi, \Gamma_2, \varphi \Rightarrow \Delta}{\Gamma_1, \Gamma_2 \Rightarrow \Delta} \text{ Cut} \]

\( G \) is closed under the structural rules if it is closed under weakening and contraction and has cut-elimination. \( G \) is closed under Implication Inversion if the following rule is admissible:

\[ \frac{\Gamma, \varphi \rightarrow \psi \Rightarrow \Delta}{\Gamma, \psi \Rightarrow \Delta} \]

A rule is invertible if the derivability of the conclusion implies the derivability of the premises.

Because of the power of the Cut rule, the following lemma is straightforward.

**Lemma 1** If \( G_{3iX} \) is closed under the structural rules, then the rules \( R \land, L \land, L \lor, R \to, \) and \( Lp \to \) are invertible and \( G_{3iX} \) is closed under Implication Inversion.

### 3 Equivalence of \( G_{3iX} \) and \( G_{4iX} \)

In this section, let \( G_{3iX} \) be a calculus that consists of \( G_{3ip} \) plus a set of modal rules \( R \). Recall that \( G_{4iX} \) is the calculus \( G_{4ip} \) extended by the rules in \( R \) plus the rules \( \mathcal{R}^\circ \) for those \( \mathcal{R} \in R \) that are right modal rules. We show that \( G_{4iX} \) and \( G_{3iX} \) are equivalent, for which we first have to prove a normal form theorem (Lemma 2) for derivations in \( G_{3iX} \).

#### 3.1 Strict proofs in \( G_{3iX} \)

A multiset is irreducible if it has no element that is a disjunction or a conjunction or falsum and for no atom \( p \) does it contain both \( p \to \psi \) and \( p \). A sequent \( S \) is irreducible if \( S^n \) is. A proof is sensible if its last inference does not have a principal formula on the left of the form \( p \to \psi \) for some atom \( p \) and formula \( \psi \). A proof in \( G_{3iX} \) is strict if in the last inference, in case it is an instance of \( L \to \) with principal formula \( \circ \varphi \to \psi \), the left premise is an axiom or the conclusion of an application of a right modal rule.

**Remark 2** If \( R \) is nonflat, then in any strict proof ending with an instance of \( L \to \) with principal formula \( \circ \varphi \to \psi \) and conclusion \( S \), because the formula in the succedent of the left premise is \( \circ \varphi \), in case the left premise is an instance of an axiom it can only be an instance of \( L \perp \). This implies that if \( S \) is irreducible, the left premise cannot be an instance of an axiom and thus is required to be the conclusion of an application of a right modal rule.

**Lemma 2** If \( G_{3iX} \) is closed under Implication Inversion, then every irreducible sequent that is provable in \( G_{3iX} \) has a sensible strict proof in \( G_{3iX} \).

**Proof** This is proved in the same way as the corresponding lemma (Lemma 1) in Dykhoof (1992). Arguing by contradiction, assume that among all provable irreducible sequents that have no sensible strict proofs, \( S \) is such a sequent with the shortest proof, \( \mathcal{D} \), where the length of a proof is the length of its leftmost branch. Thus the last inference in the proof is an application

\[ \frac{\mathcal{D}_1}{\Gamma, \varphi \rightarrow \psi \Rightarrow \varphi} \quad \frac{\mathcal{D}_2}{\Gamma, \psi \Rightarrow \Delta} \]

of \( L \to \), where \( \varphi \) is an atom or a boxed formula. Since \( S^n \) is irreducible, \( \perp \notin S^n \) and if \( \varphi \) is an atom, \( \varphi \notin S^n \). Therefore the left premise cannot be an axiom and hence is the conclusion of a rule, say \( \mathcal{R} \). Since the succedent of the conclusion of \( \mathcal{R} \) consists of an atom or a boxed formula, \( \mathcal{R} \) is a left

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2In Iemhoff (2018) the requirement that the principal formula be on the left was erroneously omitted.
rule or a right modal rule. The latter case cannot occur, since the proof then would be strict and sensible. Thus $\mathcal{R}$ is a left rule.

We proceed as in (Dyckhoff [1992]). Sequent $(\Gamma, \varphi \rightarrow \psi \Rightarrow \varphi)$ is irreducible and has a shorter proof than $S$. Thus its subproof $\mathcal{D}_1$ is strict and sensible. Since the sequent is irreducible and $\varphi$ is an atom or a boxed formula, the last inference of $\mathcal{D}_1$ is $\rightarrow \Delta$ with a principal formula $\varphi' \Rightarrow \psi'$ such that $\varphi'$ is not an atom. Let $\mathcal{D}'$ be the proof of the left premise $(\Gamma, \varphi \rightarrow \psi \Rightarrow \varphi')$. Thus the last part of $\mathcal{D}$ looks as follows, where $\Pi, \varphi' \rightarrow \psi' \Rightarrow \Gamma$.

\[
\frac{\Pi, \varphi \rightarrow \psi, \varphi' \rightarrow \psi' \Rightarrow \varphi'}{\Pi, \varphi \rightarrow \psi, \varphi' \rightarrow \psi' \Rightarrow \varphi} \quad \frac{\Pi, \varphi \rightarrow \psi, \varphi' \rightarrow \psi' \Rightarrow \varphi}{\Pi, \psi, \varphi' \rightarrow \psi' \Rightarrow \Delta}
\]

Consider the following proof of $S$.

\[
\frac{\Pi, \varphi \rightarrow \psi, \varphi' \rightarrow \psi' \Rightarrow \varphi'}{\Pi, \varphi \rightarrow \psi, \varphi' \rightarrow \psi' \Rightarrow \varphi} \quad \frac{\Pi, \varphi \rightarrow \psi, \varphi' \rightarrow \psi' \Rightarrow \varphi}{\Pi, \psi, \varphi' \rightarrow \psi' \Rightarrow \Delta}
\]

The existence of $\mathcal{D}''$ follows from Lemma [1] (Implication Inversion) and the existence of $\mathcal{D}_2$. The obtained proof is strict and sensible: In case $\varphi'$ is not a boxed formula, this is straightforward. In case $\varphi'$ is a boxed formula, it follows from the fact that was observed above, namely that $\mathcal{D}_1$ is strict and sensible.

### 3.2 Equivalence Theorem

**Theorem 1** If $G3iX$ is nonflat, closed under the structural rules, and $G4iX$ is terminating and closed under weakening, then $G3iX$ and $G4iX$ are equivalent (derive exactly the same sequents).

**Proof** The proof is an adaptation of the proof of Theorem 1 in (Dyckhoff 1992). Under the assumptions in the theorem we have to show that for all sequents $S$, $\vdash_{G3iX} S$ if and only if $\vdash_{G4iX} S$. The proof of the direction from right to left is straightforward because $G3iX$ is closed under the structural rules, but let’s fill in some of the details. We use induction to the height of the proof of a sequent in $G4iX$, where the “height” of a derivation is the length of its longest branch, where branches consisting of one node are considered to be of height 1.

Suppose $\vdash_{G4iX} S$. If $S$ is an instance of an axiom, then clearly $\vdash_{G3iX} S$ as well. Suppose $S$ is not an instance of an axiom and consider the last inference of the proof of $S$. We distinguish according to the rule $\mathcal{R}$ of which the last inference is an instance.

If the rule is $Lp \rightarrow$, then $S$ is of the form $\Gamma, p, p \rightarrow \varphi \Rightarrow \Delta$. The premise is $\Gamma, p, \varphi \Rightarrow \Delta$, which, by the induction hypothesis, is derivable in $G3iX$. It is not hard to show that $\Gamma, p, p \rightarrow \varphi \Rightarrow \varphi$ is also derivable in $G3iX$. Closure under Cut and Contraction shows that so is $S$.

If the rule is $L\rightarrow$, then $S$ is of the form $\Gamma, (\varphi \rightarrow \psi) \rightarrow \gamma \Rightarrow \Delta$ and the premises are $\Gamma, \gamma \Rightarrow \Delta$ and $\Gamma, \psi \rightarrow \gamma \Rightarrow \varphi \rightarrow \psi$. The premises are derivable in $G3iX$ by the induction hypothesis. It is not difficult to show that then $\Gamma, (\varphi \rightarrow \psi) \rightarrow \gamma, \varphi \Rightarrow \psi$ is derivable in $G3iX$ as well. Hence so is $\Gamma, (\varphi \rightarrow \psi) \rightarrow \gamma \Rightarrow \varphi \rightarrow \psi$. An application of $L \rightarrow$ proves that $S$ is derivable in $G3iX$.

The remaining cases are left to the reader.

The other direction is proved by induction on the order $\ll$ with respect to which $G4iX$ is terminating. So suppose $G3iX \vdash S$. Sequents lowest in the order do not contain connectives or modal operators by definition of the weight function underlying $\ll$. Since the calculi are nonflat, such sequents have to be instances of axioms, and since $G3iX$ and $G4iX$ have the same axioms, $S$ is provable in $G4iX$.

We turn to the case that $S$ is not the lowest in the order. If $S''$ contains a conjunction, $S = (\Gamma, \varphi_1 \land \varphi_2 \Rightarrow \Delta)$, then $S' = (\Gamma, \varphi_1 \Rightarrow \Delta)$ is provable in $G3iX$ by Lemma [1]. As $G4iX$ contains $L \land$ and $G4iX$ is terminating, $S' \ll S$ follows. Hence $S'$ is provable in $G4iX$ by the induction hypothesis. Thus so is $(\Gamma, \varphi_1 \land \varphi_2 \Rightarrow \Delta)$. A disjunction in $S''$ as well as the case that both $p$ and $p \rightarrow \varphi$ belong to $S''$, can be treated in the same way.

Thus only the case that $S$ is irreducible remains, and by Lemma [2] we may assume its proof in $G3iX$ to be sensible and strict. Thus its last inference is an application of a rule, $\mathcal{R}$, that is either a
nonmodal right rule, a modal rule or \( L \rightarrow \). In the first two cases, \( R \) belongs to both calculi and the fact that \( G4iX \) is terminating implies that the premise(s) of \( R \) is lower in the order \( \ll \) than \( S \). Thus the induction hypothesis applies. We turn to the third case. Suppose that the principal formula of the last inference is \( (\gamma \rightarrow \psi) \) and \( S = (\Gamma, \gamma \rightarrow \psi \Rightarrow \Delta) \). Since the proof is sensible, \( \gamma \) is not atomic. We distinguish according to the main connective of \( \gamma \).

If \( \gamma = \bot \), then \( (\Gamma \Rightarrow \Delta) \) is derivable in \( G3iX \) because of the closure under \( \text{Cut} \). Since \( (\Gamma \Rightarrow \Delta) \ll S \), \( (\Gamma \Rightarrow \Delta) \) is derivable in \( G4iX \) by the induction hypothesis. As \( G4iX \) is closed under weakening, \( S \) is derivable in \( G4iX \) too.

If \( \gamma = \varphi_1 \land \varphi_2 \), then \( S' = (\Gamma, \varphi_1 \rightarrow (\varphi_2 \rightarrow \psi) \Rightarrow \Delta) \) is derivable in \( G3iX \) because of the closure under \( \text{Cut} \). The fact that \( G4iX \) is terminating and contains \( L \land \rightarrow \) implies \( S' \ll S \). Hence \( S' \) is derivable in \( G4iX \) by the induction hypothesis. Thus so is \( (\Gamma, \varphi_1 \land \varphi_2 \rightarrow \psi \Rightarrow \Delta) \). The case that \( \gamma = \varphi_1 \lor \varphi_2 \) is analogous.

If \( \gamma = \varphi_1 \rightarrow \varphi_2 \), then because \( \gamma \rightarrow \psi \) is the principal formula, both premises \( S_1 = (\Gamma, \varphi_1 \rightarrow (\varphi_2 \rightarrow \psi) \Rightarrow \Delta) \) and \( (\Gamma, \gamma \rightarrow \psi \Rightarrow \gamma) \) are derivable in \( G3iX \). Thus so is \( (\Gamma, \varphi_1 \rightarrow \varphi_2 \rightarrow \psi \Rightarrow \Delta) \) by the induction hypothesis. Since \( G4iX \) is terminating and \( S_1 \) and \( S_2 \) are the premises of \( L \rightarrow \rightarrow \), they both are lower in the order \( \ll \) than \( S \). Therefore they are derivable in \( G4iX \) by the induction hypothesis. And thus so is \( S \).

If \( \gamma = \Box \varphi \), then Remark 2 and the fact that the proof is strict and \( S \) is irreducible implies that the left premise is the conclusion of an application of a right modal rule \( R \) with premises \( S_1, \ldots, S_n \).

Thus the derivation looks as follows:

\[
\begin{array}{c}
\frac{D_1 \quad \cdots \quad D_n}{S_1 \quad \cdots \quad S_n} \quad \frac{R \quad D_0}{\Gamma, \varphi \rightarrow \psi \Rightarrow \Delta} \\
\end{array}
\]

Therefore \( G4iX \) contains the rule

\[
\frac{S_1 \quad \cdots \quad S_n \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \rightarrow \psi \Rightarrow \Delta} \quad \rightarrow
\]

Since \( G4iX \) is terminating, \( (\Gamma, \psi \Rightarrow \Delta) \ll S \) and \( S_i \ll S \) follow. By the induction hypothesis, the \( S_i \), as well as \( (\Gamma, \psi \Rightarrow \Delta) \) are derivable in \( G4iX \), say with derivations \( D_i' \) and \( D_0' \), respectively. Since \( \rightarrow \rightarrow \) belongs to the calculus, the following is a proof of \( (\Gamma, \varphi \rightarrow \psi \Rightarrow \Delta) \) in \( G4iX \):

\[
\begin{array}{c}
\frac{D_i' \quad \cdots \quad D_n' \quad D_i'}{S_1 \quad \cdots \quad S_n \quad \Gamma, \psi \Rightarrow \Delta} \quad \rightarrow
\end{array}
\]

**Corollary 1** If \( G3iX \) is nonflat, closed under the structural rules, and \( G4iX \) is terminating and closed under weakening, then the structural rules are admissible in \( G4iX \).

Theorem 1 and Corollary 1 combined with the observation in Remark 1 give the following.

**Corollary 2** If \( G3iX \) is nonflat, closed under the structural rules, terminating in the Dyckhoff order, and \( G4iX \) is closed under weakening, then \( G3iX \) and \( G4iX \) are equivalent, \( G4iX \) is terminating, and the structural rules are admissible in \( G4iX \).

## 4 Conclusion

It has been shown that for any calculus \( G3iX \) that consists of \( G3ip \) plus a set of nonflat modal rules \( R \), there exists a calculus \( G4iX \) that is equivalent to \( G3iX \), provided \( G3iX \) is nonflat, closed under the structural rules and \( G4iX \) is terminating and closed under weakening. In the setting of intuitionistic modal logics, one usually requires one sequent calculi to be closed under the structural rules, so that requirement on \( G3iX \) is relatively innocent. Likewise for the closure under weakening of \( G4iX \). The requirement that \( G4iX \) be terminating is not innocent. Although many common modal rules
are terminating in the Dyckhoff order, some well-known rules are not. Examples are the standard rules for transitivity, Gödel-Löb Logic and Strong Löb Logic (writing $\Box$ for $\Diamond$):

$$
\begin{align*}
\Gamma, \Box \Gamma & \Rightarrow \varphi & R_{K4} \\
\Pi, \Box \Gamma & \Rightarrow \Box \varphi & R_{GL} \\
\Pi, \Box \Gamma, \Box \varphi & \Rightarrow \varphi & R_{SL}
\end{align*}
$$

Therefore, to apply the method in this paper to such logics, an order different from the Dyckhoff order has to be found with respect to which $\mathsf{G4iX}$ is terminating. Gödel-Löb Logic and Strong Löb Logic are examples for which that can be done, as shown in (van der Giessen and Iemhoff, 2019, 2020) using an ingenious order introduced by Bilkova (2006).

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