On a conjecture of Vorst

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Introduction

A ring $R$ is defined to be $K_n$-regular, if the map $K_n(R) \to K_n(R[t_1, \ldots, t_r])$ induced by the canonical inclusion is an isomorphism for all $r \geq 0$ [1, Definition 2.2]. It was proved by Quillen [12, Corollary of Theorem 8] that a (left) regular noetherian ring is $K_n$-regular for all integers $n$. A conjecture of Vorst [13, Conjecture] predicts that, conversely, if $R$ is a commutative ring of dimension $d$ essentially of finite type over a field $k$, then $K_{d+1}$-regularity implies regularity. Recently, Cortiñas, Haesemeyer, and Weibel showed that the conjecture holds, if the field $k$ has characteristic zero [2, Theorem 0.1]. In this paper, we prove the following slightly weaker result, if $k$ is an infinite perfect field of characteristic $p > 0$ and strong resolution of singularities holds over $k$ in the sense of Section 1 below.

**Theorem A.** Let $k$ be an infinite perfect field of characteristic $p > 0$ such that strong resolution of singularities holds over $k$. Let $R$ be a localization of a $d$-dimensional commutative $k$-algebra of finite type and suppose that $R$ is $K_{d+1}$-regular. Then $R$ is a regular ring.

We also prove a number of results for more general fields of characteristic $p > 0$. For instance, we show in Theorem 3.2 that, if strong resolution of singularities holds over all infinite perfect fields of characteristic $p$, then for every field $k$ that contains an infinite perfect subfield of characteristic $p$ and every $k$-algebra $R$ essentially of finite type, $K_q$-regularity for all $q$ implies regularity.

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We give a brief outline of the proof of Theorem A. Let \( m \subset R \) be a maximal ideal, and let \( d_m = \dim(R_m) \) and \( e_m = \dim_{R/m}(m/m^2) \) be the dimension and embedding dimension, respectively. One always has \( d_m \leq e_m \) and the ring \( R \) is said to be regular if \( d_m = e_m \) for every maximal ideal \( m \subset R \). Now, we show in Theorem 1.2 below that if \( R \) is \( K_{d+1} \)-regular, then the group \( K_{d+1}(R_m)/pK_{d+1}(R_m) \) is zero for every maximal ideal \( m \subset R \). We further show in Theorem 2.1 below that for every maximal ideal \( m \subset R \), the group \( K_q(R_m)/pK_q(R_m) \) is non-zero for all \( 0 \leq q \leq e_m \). Together the two theorems show that \( d_m \geq e_m \) as desired. Theorem A follows.

1 \( K \)-theory

In this section, we prove Theorem 1.2 below. We say that strong resolution of singularities holds over the (necessarily perfect) field \( k \) if for every integral scheme \( X \) separated and of finite type over \( k \), there exists a sequence of blow-ups

\[
X_n \rightarrow X_{n-1} \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 = X
\]

such that the reduced scheme \( X_n^{\text{red}} \) is smooth over \( k \); the center \( Y_i \) of of the blow-up \( X_{i+1} \rightarrow X_i \) is connected and smooth over \( k \); the closed embedding of \( Y_i \) in \( X_i \) is normally flat; and \( Y_i \) is nowhere dense in \( X_i \).

**Proposition 1.1.** Let \( k \) be an infinite perfect field of characteristic \( p > 0 \) and assume that strong resolution of singularities holds over \( k \). Let \( X \) be the limit of a cofiltered diagram \( \{X_i\} \) with affine transition maps of \( d \)-dimensional schemes separated and of finite type over \( k \). Then \( KH_q(X,\mathbb{Z}/p\mathbb{Z}) \) vanishes, for \( q > d \).

**Proof.** It follows from [5 Sect. IV.8.5] that for all integers \( q \), the canonical map

\[
\colimit_i K_q(X_i,\mathbb{Z}/p\mathbb{Z}) \rightarrow K_q(X,\mathbb{Z}/p\mathbb{Z})
\]

is an isomorphism. Therefore, using the natural spectral sequence

\[
E_1^{i,j} = N_{i,j}(U,\mathbb{Z}/p\mathbb{Z}) \Rightarrow KH_{i+j}(U,\mathbb{Z}/p\mathbb{Z}),
\]

we conclude that for all integers \( q \), the canonical map

\[
\colimit_i KH_q(X_i,\mathbb{Z}/p\mathbb{Z}) \rightarrow KH_q(X,\mathbb{Z}/p\mathbb{Z})
\]

is an isomorphism. Hence, we may assume that \( X \) itself is a \( d \)-dimensional scheme separated and of finite type over \( k \). In fact, we may even assume that \( X \) is integral. Indeed, it follows from [15 Theorem 2.3] that for all integers \( q \), the canonical map

\[
KH_q(X,\mathbb{Z}/p\mathbb{Z}) \rightarrow KH_q(X^{\text{red}},\mathbb{Z}/p\mathbb{Z})
\]

is an isomorphism, so we may assume that \( X \) is reduced. Moreover, if \( X_1 \subset X \) is an irreducible component and \( X_2 \subset X \) the closure of \( X \setminus X_1 \), then \( X_{12} = X_1 \cap X_2 \) has smaller dimension than \( X \) and by [15 Corollary 4.10] there is a long exact sequence

\[
\cdots \rightarrow KH_q(X,\mathbb{Z}/p\mathbb{Z}) \rightarrow KH_q(X_1,\mathbb{Z}/p\mathbb{Z}) \oplus KH_q(X_2,\mathbb{Z}/p\mathbb{Z}) \rightarrow KH_q(X_{12},\mathbb{Z}/p\mathbb{Z}) \rightarrow \cdots
\]
Therefore, a downward induction on the number of irreducible components shows that we can assume $X$ to be integral. So we let $X$ be integral and proceed by induction on $d \geq 0$. In the case $d = 0$, $X$ is a finite disjoint union of prime spectra of fields $k_\alpha$ with $[k_\alpha:k] < \infty$. It follows that the canonical maps

$$KH_q(X,\mathbb{Z}/p\mathbb{Z}) \leftarrow K_q(X,\mathbb{Z}/p\mathbb{Z}) \to \prod_\alpha K_q(k_\alpha,\mathbb{Z}/p\mathbb{Z})$$

are isomorphisms, and since the fields $k_\alpha$ again are perfect of characteristic $p > 0$, the right-hand group is zero, for $q > 0$ as desired [9]. So we let $d \geq 1$ and assume that the statement has been proved for smaller $d$. By the assumption that resolution of singularities holds over $k$, there exists a proper bi-rational morphism $X' \to X$ from a scheme $X'$ smooth over $k$. We may further assume that $X'$ is of dimension $d$. We choose a closed subscheme $Y$ of $X$ that has dimension at most $d - 1$ and contains the singular set of $X$ and consider the cartesian square

$$
\begin{array}{ccc}
Y' & \rightarrow & X' \\
\downarrow & & \downarrow \\
Y & \rightarrow & X
\end{array}
$$

Since the field $k$ is assumed to be an infinite perfect field such that strong resolution of singularities holds over $k$, the proof of [6] Theorem 3.5 shows that the cartesian square above induces a long exact sequence

$$\cdots \rightarrow KH_q(X,\mathbb{Z}/p\mathbb{Z}) \rightarrow KH_q(X',\mathbb{Z}/p\mathbb{Z}) \oplus KH_q(Y,\mathbb{Z}/p\mathbb{Z}) \rightarrow KH_q(Y',\mathbb{Z}/p\mathbb{Z}) \rightarrow \cdots$$

Now, the schemes $Y$ and $Y'$ are of dimension at most $d - 1$ and are separated and of finite type over $k$. Therefore, the groups $KH_q(Y,\mathbb{Z}/p\mathbb{Z})$ and $KH_q(Y',\mathbb{Z}/p\mathbb{Z})$ vanish, for $q > d - 1$, by the inductive hypothesis. Finally, since the scheme $X'$ is smooth over $k$, the canonical map defines an isomorphism

$$K_q(X',\mathbb{Z}/p\mathbb{Z}) \xrightarrow{\sim} KH_q(X',\mathbb{Z}/p\mathbb{Z}),$$

and by [4] Theorem 8.4] the common group vanishes for $q > d$. We conclude from the long exact sequence that $KH_q(X,\mathbb{Z}/p\mathbb{Z})$ vanishes, for $q > d$, as desired.

\begin{theorem}
Let $k$ be an infinite perfect field of positive characteristic $p$ such that strong resolution of singularities holds over $k$. Let $R$ be a localization of a $d$-dimensional $k$-algebra of finite type and assume that $R$ is $K_{d+1}$-regular. Then the group $K_{d+1}(R)/pK_{d+1}(R)$ is zero.
\end{theorem}

\begin{proof}
Since we assume $R$ is $K_{d+1}$-regular, a theorem of Vorst [13] Corollary 2.1] shows that $R$ is $K_q$-regular for all $q \leq d + 1$, or equivalently, that the groups $N_sK_q(R)$ vanish for all $s > 0$ and $q \leq d + 1$. The coefficient exact sequence

$$0 \rightarrow N_sK_q(R)/pN_sK_q(R) \rightarrow N_sK_q(R,\mathbb{Z}/p\mathbb{Z}) \rightarrow Tor^Z_1(N_{s-1}K_{q-1}(R),\mathbb{Z}/p\mathbb{Z}) \rightarrow 0$$

then shows that the groups $N_sK_q(R,\mathbb{Z}/p\mathbb{Z})$ vanish for $s > 0$ and $q \leq d + 1$. Therefore, we conclude from the spectral sequence

$$E^1_{s,t} = N_sK_t(R,\mathbb{Z}/p\mathbb{Z}) \Rightarrow KH_{s+t}(R,\mathbb{Z}/p\mathbb{Z})$$

and by [4] Corollary 8.4] the common group vanishes for $q > d$. We conclude from the long exact sequence that $KH_q(X,\mathbb{Z}/p\mathbb{Z})$ vanishes, for $q > d$, as desired.
\end{proof}
that the canonical map

\[ K_q(R, \mathbb{Z}/p\mathbb{Z}) \rightarrow KH_q(R, \mathbb{Z}/p\mathbb{Z}) \]

is an isomorphism for \( q \leq d + 1 \). Now, for \( q = d + 1 \), Proposition 1.1 shows that the common group is zero, and hence, the coefficient sequence

\[ 0 \rightarrow K_{d+1}(R)/pK_{d+1}(R) \rightarrow K_{d+1}(R, \mathbb{Z}/p\mathbb{Z}) \rightarrow \text{Tor}(K_d(R), \mathbb{Z}/p\mathbb{Z}) \rightarrow 0 \]

shows that the group \( K_{d+1}(R)/pK_{d+1}(R) \) is zero as stated. \( \Box \)

2 Hochschild homology

In this section, we prove the following general result.

**Theorem 2.1.** Let \( \kappa \) be a commutative ring, let \( r \) be a positive integer, and let \( A \) be the \( \kappa \)-algebra \( A = \kappa[x_1, \ldots, x_r]/(x_i x_j \mid 1 \leq i \leq j \leq r) \). Then, for all \( 1 \leq q \leq r \), the image of the symbol \( \{1 + x_1, \ldots, 1 + x_q\} \) by the composition

\[ K_q(A) \rightarrow \text{HH}_q(A) \rightarrow \text{HH}_q(A/\kappa) \]

of the Dennis trace map and the canonical map from absolute Hochschild homology to Hochschild homology relative to the ground ring \( \kappa \) is non-trivial.

To prove Theorem 2.1, we first evaluate the groups \( \text{HH}_*(A/\kappa) \) that are target of the map of the statement. By definition, these are the homology groups of the chain complex associated with the cyclic \( \kappa \)-module \( \text{HH}(A/\kappa)[\cdot] \) defined by

\[ \text{HH}(A/\kappa)[n] = A \otimes \cdots \otimes \kappa A \quad (n + 1 \text{ factors}) \]

with cyclic structure maps

\[ d_i : \text{HH}(A/\kappa)[n] \rightarrow \text{HH}(A/\kappa)[n-1] \quad (0 \leq i \leq n) \]

\[ s_i : \text{HH}(A/\kappa)[n] \rightarrow \text{HH}(A/\kappa)[n+1] \quad (0 \leq i \leq n) \]

\[ t_n : \text{HH}(A/\kappa)[n] \rightarrow \text{HH}(A/\kappa)[n] \]

defined by

\[ d_i(a_0 \otimes \cdots \otimes a_n) = \begin{cases} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n & (0 \leq i < n) \\ a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} & (i = n) \end{cases} \]

\[ s_i(a_0 \otimes \cdots \otimes a_n) = a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n \]

\[ t_n(a_0 \otimes \cdots \otimes a_n) = a_n \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} \]

The cyclic \( \kappa \)-module \( \text{HH}(A/\kappa)[\cdot] \) admits a direct sum decomposition as follows. Recall that a word of length \( m \) with letters in a set \( S \) is defined to be a function

\[ \omega : \{1, 2, \ldots, m\} \rightarrow S. \]

The cyclic group \( C_m \) of order \( m \) acts on the set \( \{1, 2, \ldots, m\} \) by cyclic permutation of the elements. We define a cyclical word of length \( m \) with letters in \( S \) to be an orbit
for the induced action on the set of words of length \( m \) with letters in \( S \). We write \([\omega]\) for the orbit through \( \omega \) and call the length of the orbit the period of \([\omega]\). In particular, the set that consists of the empty word is a cyclic word \([0]\) of length 0 and period 1.

Then the cyclic \( \kappa \)-module \( HH(A/\kappa)[-] \) decomposes as the direct sum

\[
HH(A/\kappa)[-] = \bigoplus_{[\omega]} HH(A/\kappa; [\omega])[-],
\]

where the direct sum ranges over all cyclical words with letters in \( \{x_1, \ldots, x_r\} \), where the summand \( HH(A/\kappa; [0])[-] \) is the sub-cyclic \( \kappa \)-module generated by the 0-simplex 1, and where the summand \( HH(A/\kappa; [\omega])[-] \) with \( \omega = (x_{i_1}, \ldots, x_{i_m}) \), \( m \geq 1 \), is the sub-cyclic \( \kappa \)-module generated by the \((m - 1)\)-simplex \( x_{i_1} \otimes \cdots \otimes x_{i_m} \).

**Lemma 2.2.** Let \( \kappa \) be a commutative ring, let \( r \) be a positive integer, and let \( A \) be the \( \kappa \)-algebra \( A = \kappa[x_1, \ldots, x_r]/(x_i x_j | 1 \leq i \leq j \leq r) \). Let \( \omega = (x_{i_1}, \ldots, x_{i_m}) \) be a word with letters in \( \{x_{i_1}, \ldots, x_r\} \) of length \( m \geq 0 \) and period \( \ell \geq 1 \).

1. If \( m = 0 \), then \( HH_0(A/\kappa; [\omega]) \) is the free \( \kappa \)-module of rank one generated by the class of the cycle 1 and the remaining homology groups are zero.

2. If \( m \) is odd or \( \ell \) is even, then \( HH_{m-1}(A/\kappa; [\omega]) \) and \( HH_m(A/\kappa; [\omega]) \) are free \( \kappa \)-modules of rank one generated by the classes of the cycles \( x_{i_1} \otimes \cdots \otimes x_{i_m} \) and \( \sum_{0 \leq a < \ell} (-1)^{m-1} u_i t_m s_{m-1} t_{m-1}^a (x_{i_1} \otimes \cdots \otimes x_{i_m}) \), respectively, and the remaining homology groups are zero.

3. If \( m \geq 2 \) is even and \( \ell \) is odd, then \( HH_{m-1}(A/\kappa; [\omega]) \) is isomorphic to \( \kappa/2\kappa \) generated by the class of the cycle \( x_{i_1} \otimes \cdots \otimes x_{i_m} \), there is an isomorphism of the 2-torsion sub-\( \kappa \)-module \( \kappa[2] \subset \kappa \) onto \( HH_m(A/\kappa; [\omega]) \) that takes \( a \in \kappa[2] \) to the class of the cycle \( \sum_{0 \leq a < \ell} (-1)^{m-1} u_i t_m s_{m-1} t_{m-1}^a (x_{i_1} \otimes \cdots \otimes x_{i_m}) \), and the remaining homology groups are zero.

**Proof.** Let \( D_r \) be the chain complex given by the quotient of the chain complex associated with the simplicial \( \kappa \)-module \( HH(A/\kappa; [\omega])[-] \) by the subcomplex of degenerate simplices. We recall that the canonical projection induces an isomorphism of \( HH_0(A/\kappa; [\omega]) \) onto \( H_0(D_r) \); see for example [16] Theorem 8.3.8. We evaluate the chain complex \( D_r \) in the three cases (1)--(3).

First, in the case (1), \( D_0 \) is the free \( \kappa \)-module generated by 1 and \( D_q \) is zero, for \( q > 0 \). This proves statement (1).

Next, in the case (2), let \( C_\ell \) be the cyclic group of order \( \ell \), and let \( \tau \) be a generator. We define \( D'_\ell \) to be the chain complex with \( D'_\ell = \kappa(C_\ell) \), if \( q = m - 1 \) or \( q = m \), and zero, otherwise, and with differential \( d': D'_m \rightarrow D'_{m-1} \) given by multiplication by \( 1 - \tau \). Then the map \( \alpha: D'_\ell \rightarrow D_r \) defined by

\[
\alpha_{m-1}(\tau^a) = (-1)^{m-1} u_i t_{m-1} (x_{i_0} \otimes \cdots \otimes x_{i_m})
\]

\[
\alpha_m(\tau^a) = (-1)^{m-1} u_i t_m s_{m-1} t_{m-1}^a (x_{i_0} \otimes \cdots \otimes x_{i_m})
\]

is an isomorphism of chain complexes, since \( m - 1 \ell \) is even. Now, the homology groups \( H_{m-1}(D'_\ell) \) and \( H_m(D'_\ell) \) are free \( \kappa \)-modules of rank 1 generated by the class of 1 and the norm element \( N = 1 + \tau + \cdots + \tau^{\ell-1} \), respectively. This proves the statement (2).
Finally, in the case (3), let $C_\ell$ be the cyclic group of order $\ell$, and let $\tau$ be a generator. We define $D^\mu_\ell$ to be the chain complex with $D^\mu_q = \kappa[C_\ell]$, if $q = m - 1$ or $q = m$, and zero, otherwise, and with differential $d^\mu_\ell: D^\mu_m \to D^\mu_{m-1}$ given by multiplication by $1 + \tau$. Then the map $\beta: D^\mu_\ell \to D_\ell$, defined by

\[
\beta_{m-1}(x) = (-1)^{\mu x_{m-1}} (x_0 \otimes \cdots \otimes x_m)
\]

\[
\beta_m(x) = (-1)^{\mu x} 1^{m-1} (x_0 \otimes \cdots \otimes x_m)
\]

is an isomorphism of chain complexes, since $m$ is even. Hence, to prove statement (3), it suffices to show that the following sequence of $\kappa$-modules is exact.

\[
0 \to \kappa[2] \xrightarrow{N} \kappa[C_\ell] \xrightarrow{1 + \tau} \kappa[C_\ell] \xrightarrow{\varepsilon} \kappa/2\kappa \to 0.
\]

To this end, we consider the following commutative diagram with exact rows.

\[
\begin{array}{c}
0 \to I[C_\ell] \to \kappa[C_\ell] \xrightarrow{1 + \tau} \kappa[C_\ell] \xrightarrow{\varepsilon} \kappa \to 0 \\
\downarrow \quad \downarrow 1 + \tau \quad \downarrow 1 + \tau \quad \downarrow 2 \\
0 \to I[C_\ell] \to \kappa[C_\ell] \xrightarrow{\varepsilon} \kappa \to 0
\end{array}
\]

The augmentation ideal $I[C_\ell]$ is equal to the sub-$k[C_\ell]$-module generated by $1 - \tau$. Since $\ell$ is odd, $\tau^2$ is a generator of $C_\ell$, and hence, $1 - \tau^2 = (1 + \tau)(1 - \tau)$ is a generator of $I[C_\ell]$. This shows that the left-hand vertical map $1 + \tau$ is an isomorphism. Finally, the following diagram commutes.

\[
\begin{array}{ccc}
\kappa[2] & \xrightarrow{N} & \kappa[2] \\
\downarrow \varepsilon & & \downarrow \varepsilon \\
\kappa[C_\ell] & \xrightarrow{\varepsilon} & \kappa
\end{array}
\]

Indeed, $\varepsilon \circ N$ is equal to multiplication by $\ell$ which is congruent to 1 modulo 2. This shows that the sequence in question is exact. Statement (3) follows.

**Remark 2.3.** For $\kappa$ a field of characteristic zero, the Hochschild homology of the $\kappa$-algebra $A$ in Lemma 2.2 was first evaluated by Lindenstrauss [10, Theorem 3.1] who also determined the product structure of the graded $\kappa$-algebra $\text{HH}_q(A/\kappa)$.

**Proof of Theorem 2.7.** We let $\omega$ be the word $(x_1, \ldots, x_q)$ and consider the following coposition of the map of the statement and the projection onto the summand $[\omega]$.

\[
K_q(A) \to \text{HH}_q(A) \to \text{HH}_q(A/\kappa) \xrightarrow{\text{pr}_{[\omega]}} \text{HH}_q(A/\kappa, [\omega])
\]

The Dennis trace map is a map of graded rings and takes the symbol $\{1 + x_i\}$ to the Hochschild homology class $d \log(1 + x_i)$ represented by the cycle $1 \otimes x_i - x_i \otimes x_i$; see for example [3, Corollary 6.4.1], [8, Proposition 2.3.1], and [7, Proposition 1.4.5]. Hence, $\{1 + x_1, \ldots, 1 + x_q\}$ is mapped to $d \log(1 + x_1) \ldots d \log(1 + x_q)$. The product on Hochschild homology is given by the shuffle product $\ast$, and moreover,

\[
\text{pr}_{[\omega]}(d \log(1 + x_1) \ast \cdots \ast d \log(1 + x_q)) = \text{pr}_{[\omega]}((1 \otimes x_1) \ast \cdots \ast (1 \otimes x_q))
\]
since summands that include a factor \( x_i \otimes x_i \) are annihiliated by \( pr_{(a)} \). Now,
\[
(1 \otimes x_1) \ast \cdots \ast (1 \otimes x_q) = \sum_{\sigma} \text{sgn}(\sigma)1 \otimes x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(q)},
\]
where the sum ranges over all permutations of \( \{1, 2, \ldots, q\} \), and hence,
\[
pr_{(a)}((1 \otimes x_1) \ast \cdots \ast (1 \otimes x_q)) = \sum_{\tau} \text{sgn}(\tau)1 \otimes x_{\tau(1)} \otimes \cdots \otimes x_{\tau(q)},
\]
where the sum range over all cyclic permutations of \( \{1, 2, \ldots, q\} \). By Lemma 2.2 (2), this class is the generator of \( HH_q(A/k; \{a\}) \). The theorem follows.

\[\square\]

3 Proof of Theorem A

In this section, we prove Theorem A of the introduction and a number of generalizations of this result.

Proof of Theorem A It suffices to show that for every maximal ideal \( m \subset R \), the local ring \( R_m \) is regular. The assumption that \( R \) is \( K_{d+1} \)-regular implies by [14] Theorem 2.1 and [13] Corollary 2.1 that the local ring \( R_m \) is \( K_q \)-regular for all \( q \leq d + 1 \). The local ring \( R_m \) has dimension \( d_m \leq d \). We first argue that we may assume that \( d_m = d \). Let \( I \subset R \) be the intersection of the minimal prime ideals \( p_1, \ldots, p_n \subset R \) that are not contained in \( m \). We claim that \( m + I = R \). For if not, the ideal \( m + I \) would be contained in a maximal ideal of \( R \) which necessarily would be \( m \). Now, for each \( 1 \leq i \leq n \), we choose \( y_i \in p_i \) with \( y_i \notin m \). Then \( y = y_1 \ldots y_n \) is in \( I \), but not in \( m \). The claim follows. Now, by the Chinese remainder theorem, there exists \( r \in R \) such that \( r \equiv 1 \mod m \) and \( r \equiv 0 \mod I \). We define \( R' = R[1/r] \) and \( m' = mR' \). Then \( m' \subset R' \) is a maximal ideal, since \( R'/m' = (R/m)[1/r] = R/m, \) and the canonical map \( R_m \to R_m' \) is an isomorphism. Moreover, the \( k \)-algebra \( R' \) is of finite type, and since every minimal prime ideal of \( R' \) is contained in \( m' \), we have \( \dim R' = \dim R_m' = d_m \). Therefore, we may assume that \( d = d_m \). Hence, Theorem 1.2 shows that
\[
K_{d_m + 1}(R_m)/pK_{d_m + 1}(R_m) = 0.
\]

We choose a set of generators \( x_1, \ldots, x_r \) of the maximal ideal of the local ring \( R_m \). Then \( r \geq d_m \) with equality if and only if \( R_m \) is regular. By [11] Theorem 28.3], we may choose a \( k \)-algebra section of the canonical projection \( R_m/m^2R_m \to R/m = k \). These choices give rise to a \( k \)-algebra isomorphism
\[
A = k[x_1, \ldots, x_r]/(x_ix_j \mid 1 \leq i \leq j \leq r) \cong R_m/m^2R_m.
\]
Hence, Theorem 2.1 shows that for all \( 1 \leq q \leq r \), the symbol
\[
\{1 + x_1, \ldots, 1 + x_q\} \in K_q(R_m)/pK_q(R_m)
\]
has non-trivial image in \( K_q(A)/pK_q(A) \), and therefore, is non-zero. Since the group \( K_{d_m + 1}(R_m)/pK_{d_m + 1}(R_m) \) is zero, we conclude that \( r \leq d_m \) which shows that \( R_m \) is a regular local ring. This completes the proof.

\[\square\]
Theorem 3.1. Let $k$ be a field of positive characteristic $p$ that is finitely generated over an infinite perfect subfield $k'$, and assume that strong resolution of singularities holds over $k'$. Let $R$ be a localization of a $d$-dimensional commutative $k$-algebra of finite type and suppose that $R$ is $K_{d+r+1}$-regular where $r$ is the transcendence degree of $k$ over $k'$. Then $R$ is a regular ring.

Proof. We can write $R$ as the localization $f: R' \to R = S^{-1}R'$ of a $(d+r)$-dimensional commutative $k'$-algebra $R'$ of finite type with respect to a multiplicative subset $S \subset R'$. Let $p \subset R$ be a prime ideal. Then, by [14, Theorem 2.1], the local ring $R_p$ again is $K_{d+r+1}$-regular. Now, let $p' = f^{-1}(p) \subset R'$. Then the map $f$ induces an isomorphism of $R'_{p'}$ onto $R_p$. Therefore, we conclude from Theorem A that $R_p$ is a regular ring. This proves that $R$ is a regular ring as stated.

Theorem 3.2. Let $p$ be a prime number and assume that strong resolution of singularities holds over all infinite perfect fields of characteristic $p$. Let $k$ be any field that contains an infinite perfect subfield of characteristic $p$, let $R$ be a commutative $k$-algebra essentially of finite type, and assume that $R$ is $K_q$-regular for all $q$. Then $R$ is a regular ring.

Proof. We can write $R$ as a localization of $R' \otimes_{k'} k$ where $k'$ is a finitely generated field that contains an infinite perfect subfield and where $R'$ is a commutative $k'$-algebra of finite type. Then we can write $R$ as the filtered colimit $R = \lim_{\alpha} R' \otimes_{k'} k_{\alpha}$ where $k_{\alpha}$ runs through the finitely generated extensions of $k'$ contained in $k$. It follows from Theorem [31] that the rings $R' \otimes_{k'} k_{\alpha}$ are all regular. Therefore the ring $R$ is regular by [5, Prop. IV.5.13.7].

References

1. H. Bass, Some problems in "classical" algebraic K-theory, Algebraic K-theory, II: "Classical" algebraic K-theory and connections with arithmetic (Proc. Conf., Battelle Memorial Inst., Seattle Wash., 1972), Lecture Notes in Math., vol. 342, Springer-Verlag, Berlin, 1973, pp. 3–73.
2. G. Cortiñas, C. Haesemeyer, and C. A. Weibel, K-regularity, cdh-fibrant Hochschild homology, and a conjecture of Vorst, J. Amer. Math. Soc. 21 (2008), 547–561.
3. T. Geisser and L. Hesselholt, Topological cyclic homology of schemes, K-theory (Seattle, 1997), Proc. Symp. Pure Math., vol. 67, 1999, pp. 41–87.
4. T. Geisser and M. Levine, The K-theory of fields in characteristic p, Invent. Math. 139 (2000), 459–493.
5. A. Grothendieck and J. Dieudonné, Éléments de géométrie algébrique, Inst. Hautes Études Sci. Publ. Math. 4, 8, 11, 17, 20, 24, 28, 32 (1960–1967).
6. C. Haesemeyer, Descent properties of homotopy K-theory, Duke Math. J. 125 (2004), 589–620.
7. L. Hesselholt, On the $p$-typical curves in Quillen’s $K$-theory, Acta Math. 177 (1997), 1–53.
8. L. Hesselholt and I. Madsen, On the $K$-theory of local fields, Ann. of Math. 158 (2003), 1–113.
9. C. Kratzer, $\lambda$-structure en K-théorie algébrique, Comment. Math. Helv. 55 (1980), 233–254.
10. A. Lindenstrauss, The Hochschild homology of fatpoints, Israel J. Math. 133 (2003), 177–188.
11. H. Matsumura, Commutative ring theory, Cambridge studies in advanced mathematics, vol. 8, Cambridge University Press, 1986.
12. D. Quillen, Higher algebraic K-theory I, Algebraic K-theory I: Higher $K$-theories (Battelle Memorial Inst., Seattle, Washington, 1972), Lecture Notes in Math., vol. 341, Springer-Verlag, New York, 1973.
13. T. Vorst, *Localization of the K-theory of polynomial extensions*, Math. Ann. **244** (1979), 33–53.
14. ———, *A survey on the K-theory of polynomial extensions*, Algebraic K-theory, number theory, Geometry and Analysis (Proceedings, Bielefeld 1982), Lecture Notes in Math., vol. 1046, Springer-Verlag, Berlin, 1984, pp. 422–441.
15. C. A. Weibel, *Homotopy algebraic K-theory*, Algebraic K-theory and number theory (Honolulu, HI, 1987), Contemp. Math., vol. 83, Amer. Math. Soc., Providence, RI, 1989, pp. 461–488.
16. ———, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.