RUELLE RESONANCES FROM COHOMOLOGICAL EQUATIONS

GIOVANNI FORNI

ABSTRACT. These notes are based on lectures given by the author at the Summer School on Teichmüller dynamics, mapping class groups and applications in Grenoble, France, in June 2018 and at the Oberwolfach Seminar on Anisotropic Spaces and their Applications to Hyperbolic and Parabolic Systems in June 2019. We derive results about the so-called Ruelle resonances and the asymptotics of correlations for several classes of systems from known results on cohomological equations and invariant distributions for the respective unstable vector fields. In particular, we consider pseudo-Anosov diffeomorphisms on surfaces of higher genus, for horocycle flows on surfaces of constant negative curvature and for partially hyperbolic automorphisms of Heisenberg 3-dimensional nilmanifolds. Ruelle resonances for pseudo-Anosov maps with applications to the cohomological equation for their unstable translation flows was recently studied in depth by F. Faure, S. Gouëzel and E. Lanneau [FGL] by methods based on the analysis of the transfer operator of the pseudo-Anosov map. Ruelle resonances for geodesic flows on hyperbolic compact manifolds of any dimension and of partially hyperbolic automorphisms of Heisenberg 3-dimensional nilmanifolds are studied by general results of Dyatlov, Faure and Guillarmou [DFG] and Faure and Tsujii [FT15] based on methods of semi-classical analysis. These works do not derive results on cohomological equations for unstable flows or horospherical foliations of these systems.

1. INTRODUCTION

Cohomological equations appear in several questions in dynamical systems. In fact, they arise naturally in questions related to the triviality of time changes and the associated cocycles, and this connection motivates their name. They are related to linearized equations coming from conjugacy problems, by the KAM method or other inverse function theorem. These results are classical for linear flows on tori, and were generalized to translation flows by S. Marmi, P. Moussa and J.-C. Yoccoz [MMY12]. For renormalizable systems, obstructions to existence of solutions of the cohomological equation are related to the phenomenon of deviation of ergodic averages for uniquely ergodic systems discovered by A. Zorich [Zor97] (see also M. Kontsevich and A. Zorich [KZ97]), and later investigated in the work of the author [F02] and A. Bufetov [Bu14]. A similar approach based on the theory of unitary representation extended the description of deviation of ergodic averages to other renormalizable algebraic systems, such as horocycle flows on compact hyperbolic surfaces [FF03] and Heisenberg nilflows [FF06] (see also [BuF14] and the surveys [F14], [F15] by the author).

In all of the above works solutions of the cohomological equation were constructed by methods of harmonic analysis, and thus limited to the case of systems of algebraic nature, and with the exception of translation flows, to homogeneous flows. This is in contrast with the world of hyperbolic and partially hyperbolic accessible systems where a full theory has been developed. Indeed, for the hyperbolic case the theory usually goes under the name of Livsic theory, while in the partially hyperbolic accessible case it was initiated by A. Katok and A. Kononenko [KK96] and later developed by A. Wilkinson [W13]. In the uniquely ergodic, parabolic, case, the first solution of a cohomological equation for a non-homogeneous system was given by the author in [F97] by methods of harmonic analysis. A breakthrough came in 2005 when S. Marmi, P. Moussa and J.-C. Yoccoz [MMY05] were able to prove a related result on cohomological equation of interval exchange transformations by a dynamical approach based on renormalization. In their paper they
were able to deal with the case of invariant directions of pseudo-Anosov maps, which were left as an open problem in [F97]. It was the first solution of a cohomological equation by dynamical methods for a class of parabolic non-homogeneous uniquely ergodic systems.

Recently, the methods based on the analysis of the transfer operator developed by V. Baladi, S. Gouëzel, C. Liverani and others have been refined to the point of being able to deliver dynamical analyses of the cohomological equations of more general systems, following a conceptual scheme similar to that of the work of Marmi, Moussa and Yoccoz [MMY05] quoted above. Roughly speaking the idea is to refine the asymptotics on decay of correlations, or rather the analysis of the associated transfer operator, to the point where it becomes possible to derive refined asymptotic estimates of the ergodic integrals. From these asymptotic “expansions” for ergodic integrals the existence of solutions follows by a version of the Gottschalk-Hedlund theorem, according to which functions are coboundaries if they have “bounded” ergodic integrals (in the $C^0$ or $L^2$ topology).

A “proof of concept” paper by P. Giulietti and C. Liverani [GL] treats the somewhat artificial case of flows along invariant foliations of Anosov maps of the 2-torus. The above-mentioned work of F. Faure, S. Gouëzel and E. Lanneau treats the case of translation flows invariant under a pseudo-Anosov. The results of this paper are not surprising, and, as the author will argue in these lectures, could have been derived without too much effort from known results on the cohomological equation, with the exception of improved loss of regularity and the unified treatment of neutral eigenvectors. However, the methods are extremely significant, since they are general enough to be applied, at least in principle, to non-linear pseudo-Anosov maps and to other general, non algebraic, systems.

In these notes we are going to explain in detail, in section 2, how to derive many informations about the Ruelle resonances of a pseudo-Anosov map from knowledge of the obstructions to the existence of solution to the cohomological equation for its unstable direction.

A similar program, which is just outline in these notes, in section 4 can be carried out for the horocycle flow on hyperbolic (negative constant curvature) surfaces based on the work of L. Flaminio and the author [FP03]. Ruelle resonances and asymptotic for geodesic flows on general hyperbolic manifolds were described in depth by S. Dyatlov, F. Faure and C. Guillarmou [DFG]. For horocycle flows in variable negative curvature, partial results have been proved by A. Adam [Ad] also following the Giulietti-Liverani approach.

Finally, we outline, in section 5 the computation of Ruelle resonances for partially hyperbolic automorphisms of the Heisenberg group based on the solution of cohomological equations for Heisenberg nilflows given in [FF06]. Since this is a partially hyperbolic system, the transfer operator approach presents additional difficulties. However, Ruelle resonances were computed in much greater generality by F. Faure and M. Tsujii [FT15], who developed results of Faure [Fau07], for equivariant (isometric) extensions of symplectic Anosov diffeomorphisms to $U(1)$ principal bundles.

From the point of view of parabolic flows, the translation flows stabilized by a pseudo-Anosov diffeomorphism are exceptional among all translation flows on compact orientable surfaces, and Heisenberg nilflows stabilized by a Heisenberg automorphism are rare among all Heisenberg nilflows. In order to develop a theory for generic (in the sense of measure) translation flows or Heisenberg nilflows, the notion of a transfer cocycle over a renormalization dynamics has to be developed and analyzed. Results in Teichmüller dynamics based the study of the Lyapunov structure of the so-called Kontsevich–Zorich cocycle allow us to derive in section 3 some results about the Lyapunov spectra of certain transfer cocycles related to the deviation of ergodic averages for generic translation flows. A similar analysis is carried out in the Heisenberg case, based on the results of [FF03], as outlined in section 6 of these notes.
2. RUELLE RESONANCES FOR PSEUDO-ANOSOV DIFFEOMORPHISMS

Let \( \Phi : M \to M \) denote a pseudo-Anosov diffeomorphism with orientable stable/unstable foliations and let \( \{ X, Y \} \) denote the generators of translation flows along the unstable/stable foliations. There exists \( \lambda > 1 \) such that

\[
\Phi_\ast(X) = \lambda X \quad \text{and} \quad \Phi_\ast(Y) = \lambda^{-1} Y.
\]

Let \( \omega \) denote the \( \Phi \)-invariant area-form. The form \( \omega \) is also invariant for the translation flows of the translation surface given by the pair \( \{ X, Y \} \). Let \( \Sigma \) denote the set of cone points of the translation surface. The area form \( \omega \) vanishes at \( \Sigma \). Let \( S_{X,Y}(M) \) denote the space of all smooth functions on \( M \), which at a cone \( p \in \Sigma \) of order \( k \geq 1 \), are locally the pull-back of a smooth function under the local branched covering chart \( z \to z^{k+1}/(k+1) \) for the translation structure on a neighborhood of \( p \).

For any pair \( f, g \) of sufficiently smooth complex-valued functions on \( M \), we are interested in the asymptotic for the decay of the correlations

\[
C(f, g, n) = \langle f \circ \Phi^n, g \rangle_{L^2(M, \omega)}.
\]

Let

\[
\mu_1 := \lambda > \mu_2 \geq \cdots \geq |\mu_{2g-1}| > \mu_2 := \lambda^{-1}
\]
denote the spectrum \( \sigma(\Phi) \) of \( \Phi_\ast \) on \( H^1(M, \mathbb{R}) \). Since \( \Phi_\ast \) is a symplectic map on \( H^1(M, \mathbb{R}) \) it follows that

\[
\mu_{2g-i+1} = \mu_i^{-1}, \quad \text{for all } i \in \{2, \ldots, 2g-1\}.
\]

The following theorem was recently proved by F. Faure, S. Gouëzel and E. Lanneau by methods based on the analysis of the transfer operator.

**Theorem 2.1.** The set \( \mathcal{R} \) of Ruelle resonances can be described as follows:

\[ \mathcal{R} = \{ 1 \} \cup \{ \mu_1 \lambda^{-i} \mid i \in \{ 2, \ldots, 2g-1 \} \text{ and } j \in \mathbb{N} \setminus \{ 0 \} \}. \]

All spectral values \( \mu_1 \lambda^{-j}, \ldots, \mu_2 \lambda^{-j} \) have multiplicity \( j \geq 1 \). The following asymptotics holds.

For all functions \( f, g \in C^0(M \setminus \Sigma) \) we have an asymptotic expansion

\[
C(f, g, n) \approx \sum_{\rho \in \mathcal{R}} \sum_{i=1}^{l_\rho} c_{\rho,i}(f, g) n^i \rho^n.
\]

**Definition 2.2.** A function \( f \in L^2(M, \omega) \) is an iterated coboundary of order \( k \geq 1 \) with transfer function \( u \in L^2(M, \omega) \) if \( u \) is a weak solution of the equation

\[
X^k u = f.
\]

**Lemma 2.3.** For any \( k \)-iterated coboundary \( f \in L^2(M, \omega) \) with transfer function \( u \in L^2(M, \omega) \) and for any \( g \in L^2(M, \omega) \) such that \( X^k g \in L^2(M, \omega) \) we have the estimate

\[
|C(f, g, n)| \leq \lambda^{-kn} |u|_0 |X^k g|_0.
\]

**Proof.** The statement follows immediately from the following identities:

\[
\langle (X^k u) \circ \Phi^n, g \rangle_{L^2(M, \omega)} = \lambda^{-kn} \langle \Phi^n(X^k u), g \rangle_{L^2(M, \omega)} = \lambda^{-kn} \langle X^k (u \circ \Phi^n), g \rangle_{L^2(M, \omega)} = (-1)^k \lambda^{-kn} \langle u \circ \Phi^n, X^k g \rangle_{L^2(M, \omega)}.
\]

\[ \square \]

In view of the lemma, a natural question is how to characterize iterated coboundaries for the unstable (or stable) vector field. Coboundaries were characterized in [F97], [F07] for generic translation flows, and in [MMY05] also in case of invariant foliations of pseudo-Anosov diffeomorphisms.
For all $s \in \mathbb{R}$, let $W^{s}_{X,Y}(M)$ denote the $L^2$ Sobolev spaces of the translation surface $\{X,Y\}$ (see \cite{F07} and \cite{F02} for definitions and basic properties of these spaces for integer exponent, and \cite{F07}, for the subtler case of real, non-integer exponent).

**Theorem 2.4.** There exists a finite dimensional space $J^s_X(M) \subset W^{-s}_{X,Y}(M)$ of $X$-invariant distributions such that for any $f \in W^{s}_{X,Y}(M)$ (with $s > s_0$) such that

$$D(f) = 0,$$

for all $D \in J^s_X(M)$,

is an $X$-coboundary with zero-average transfer function $u \in W^{s}_{X,Y}(M)$ for all $t < s - s_0$. In addition there exists a constant $C_{s,t} > 0$ such that for all $t < s - s_0$,

$$|u|_t \leq C_{s,t} |f|_s.$$  

**Remark 2.5.** The loss of derivatives in Sobolev spaces was estimated carefully in \cite{F07} to be

$$3^+$$

for almost all directions on any given translation surface, and $1^+$ under the assumption of hyperbolicity of the Kontsevich–Zorich renormalization cocycle, for almost all translation flows. In the Hölder class a loss of $1 + \delta(s)$ (with $\delta(s) \to 0^+$ for $s \to 1^+$) was proved by Marmi and Yoccoz \cite{MY16} under similar hypotheses. Finally, the work of Faure, Gouëzel and Lanneau should lead to a Hölder loss of $1+$ in the “periodic” (pseudo-Anosov) case. However, their results are explicitly stated only for spaces with integer exponents.

Let us recall that $S^s_{X,Y}(M)$ denotes the space of all smooth functions on $M$, which at a cone $p \in \Sigma$ of order $k \geq 1$, are locally the pull-back of a smooth function under the local branched covering chart $z \to z^{k+1}/(k+1)$ for the translation structure on a neighborhood of $p$. The dual space $S^s_{X,Y}(M)$ is called the space of tempered currents for the translation structure.

Let $J^s_X \subset S^s_{X,Y}(M)$ denote the space of all tempered $X$-invariant distributions. There exists a map $C : J^s_X \to Z(M)$ into the space $Z(M)$ of all closed $1$-currents on $M$, defined as

$$C(D) := D \cdot i_X \omega.$$  

The range of the map $C : J^s_X(M) \to Z(M)$ is the subspace $\mathcal{B}_X(M)$ of tempered basic currents for the unstable foliation of the pseudo-Anosov map $\Phi$, that is, the subspace of currents $C$ such that

$$L_X C = i_X C = 0$$

($L_X$ denotes the operator of Lie derivative and $i_X$ the contraction on currents).

The de Rham cohomology map $\mathcal{R} : Z(M) \to H^1(M, \mathbb{R})$, restricted to the subspace $\mathcal{B}_X(M)$ of basic currents has range

$$H^1_X(M, \mathbb{R}) := \{ c \in H^1(M, \mathbb{R}) | c \wedge i_X \omega = 0 \}.$$  

Let now $J^{-s}_X(M) \subset J^s_X(M)$ denote the subspace of invariant distributions of finite order $s > 0$ and let $\mathcal{B}^{-s}_X(M)$ denote the corresponding space of basic currents.

**Lemma 2.6.** \cite{F02} For any $C \in \mathcal{B}^{-s}_X(M)$ such that $[C] = 0$ in $H^1(M, \mathbb{R})$, there exists $C' \in \mathcal{B}^{-s+1}_X(M)$ such that $C = L_Y C'$.

**Proof.** By the de Rham theorem there exists $U \in W^{-s+1}_X(M)$ such that $dU = C$. We have to prove that $C' = i_X U \in \mathcal{B}_X(M)$ and that $L_Y C' = C$. We have

$$dL_X U = d i_X U = L_X dU = L_X C = 0,$$

which implies $L_X U$ is constant, hence it vanishes. Thus $U \in J^{-s+1}_X(M)$ and we have

$$dC' = d i_X U = L_X U - i_X dU = -i_X C = 0.$$  

Since $C'$ is closed and $i_X C' = 0$, it follows that $C' \in \mathcal{B}_X(M)$. 

\qed
Lemma 2.7. The spectrum of $\Phi_s$ on the space $\mathcal{B}_X(M)$ of basic currents is

$$\sigma_{\mathcal{B}_X(M)}(\Phi_s) := \{ \lambda \} \cup \{ \mu \lambda^{-j} \mid i \in \{ 2, \ldots, 2g-1 \} \text{ and } j \in \mathbb{N} \}.$$ 

Consequently, the spectrum of $\Phi_s$ on the space $\mathcal{J}_X(M)$ of invariant distributions is

$$\sigma_{\mathcal{J}_X(M)}(\Phi_s) := \{ 1 \} \cup \{ \mu \lambda^{-j-1} \mid i \in \{ 2, \ldots, 2g-1 \} \text{ and } j \in \mathbb{N} \}.$$ 

Proof. Let $\mathcal{L}_Y: \mathcal{B}_X(M) \to \mathcal{B}_X(M)$ denote the Lie derivative operator with respect to the vector field $Y$ on $M$. By the previous lemma, the maps $\mathcal{L}_Y: \mathcal{B}_X(M) \to \mathcal{B}_X(M)$ and the de Rham cohomology map $\mathcal{H}: \mathcal{B}_X(M) \to H^1_X(M, \mathbb{R})$ give a $\Phi_s$ equivariant exact sequence

$$0 \to \mathcal{C}_X \omega \to \mathcal{B}_X^{-s+1}(M) \to \mathcal{B}_X^{-s}(M) \to H^1_X(M, \mathbb{C}) \to 0.$$ 

Finally, for all $C \in \mathcal{B}_X(M)$, we have, for all $i, j \in \mathbb{N}$,

$$(\Phi_s - \mu I)^i \mathcal{L}_Y^j(C) = \lambda^{-ij} \mathcal{L}_Y^j(\Phi_s - \lambda^j \mu I)^i(C).$$

Finally, as explained above, the map $C: \mathcal{J}_X(M) \to \mathcal{B}_X(M)$ defined as

$$C(D) = D|_X \omega, \quad \text{for all } D \in \mathcal{J}_X(M),$$

is an isomorphism. We clearly have

$$C(\Phi_s D) = (\Phi_s D)|_X \omega = \lambda^{-1} \Phi_s(D|_X \omega) = \lambda^{-1} \Phi_s(C(D)),$$

hence the proof is completed. \qed

Obstructions on functions to be iterated coboundaries can be constructed as follows. A coboundary $f \in W^s_{K,Y}(M)$ (with $s > 2s_0$) with smooth transfer function $u \in W^s_{K,Y}(M)$ (with $s - s_0 > t > s_0$) is a 2-iterated coboundary with $L^2$ transfer function if $u$ is itself a coboundary with smooth transfer function if and only if

$$D(u) = 0, \quad \text{for all } D \in \mathcal{J}_X^t(M).$$

We can therefore define a linear functional as follows. Let

$$G_X: \text{Ker} \mathcal{J}_X^t(M) \to W^s_{K,Y}(M)$$

denote the Green operator such that $u := G_X(f)$ is the zero-average solution of the cohomological equation $Xu = f$. For every $D \in \mathcal{J}_X^t(M)$, we define

$$D'(f) = D(G_X(f)), \quad \text{for all } f \in \text{Ker} \mathcal{J}_X^t(M) \subset W^s_{K,Y}(M).$$

The above functional can be extended to the whole space $W^s_{K,Y}(M)$ as zero on the orthogonal complement of $\text{Ker} \mathcal{J}_X^t(M)$. In fact, it follows from the above results that if $D \in W^s_{K,Y}(M)$, then $D' \in W^s_{K,Y}(M)$ for any $s - t > s_0$. In fact, for $f \in \text{Ker} \mathcal{J}_X^t(M)$,

$$|D'(f)| = |D(G_X(f))| \leq |D|_{s,t} |G_X(f)|_t \leq C_{s,t} |f|_t.$$ 

This construction can be iterated. Another point of view is based on the remark that

$$X D'(f) = D'(X f) = D(G_X(X f)) = D(f), \quad \text{for all } f \in W^s_{K,Y}(M).$$

It follows that $D'$ can be defined as a distributional solution $D'$ of the equation

$$X D' = D.$$ 

The solution of the above equation is unique up to the addition of $X$-invariant distributions, and we can define $D' \in W^s_{K,Y}(M)$ as the unique solution orthogonal to the subspace $\mathcal{J}_X^t(M)$ of invariant distributions.

For all $k \in \mathbb{N}$, let $\mathcal{J}_X^{s+k}(M) \subset W^s_{K,Y}(M)$ denote the subspace

$$\mathcal{J}_X^{s+k}(M) := \{ D \in W^s_{K,Y}(M) \mid X^k D = 0 \}.$$ 

We have the following results on the iterated cohomological equation
Lemma 2.8. Any function \( f \in W^s_{X,Y}(M) \) of zero average with \( s > ks_0 + t \) is a \( k \)-iterated coboundary with transfer function \( u \in W^r_{X,Y}(M) \) if

\[ D(f) = 0, \quad \text{for all } D \in J^{-s}_{X,Y}(M). \]

In addition, there exists a constant \( C_{s,t}^{(k)} > 0 \) such that

\[ |u| \leq C_{s,t}^{(k)} |f|. \]

Proof. We argue by induction on \( k \in N \setminus \{0\} \). For \( k = 1 \) we have the result on the solutions of the cohomological equation. Let us assume that the result holds for \( k \in N \setminus \{0\} \). Let \( u_k \in W^r_{X,Y}(M) \) (with \( r < s - ks_0 \)) be the zero-average solution of the iterated cohomological equation \( X^k u_k = f \) for a function \( f \in \ker J^{-s}_{X,Y}(M) \).

We claim that for all \( D \in J^{-s}_{X,Y}(M) \) we have \( D(u_k) = 0 \). By definition, the set \( J^{-s}_{X,Y}(M) \subset X^k J^{-s}_{X,Y}(M) \), and for all \( D_{k+1} \in J^{-s}_{X,Y}(M) \), we have

\[ X^k D_{k+1}(u_k) = (-1)^k D_{k+1}(X^k u_k) = (-1)^k D_{k+1}(f) = 0, \]

hence the claim is proved. By the result on the cohomological equation, there exists \( u := u_{k+1} \in W^r_{X,Y}(M) \) (with \( t < r - s_0 \)) such that \( Xu_{k+1} = u_k \). The argument is complete except for the a priori bounds. We have

\[ |u| \leq C_{s,t} |u_k| \leq C_{s,t} C_{s,t}^{(k)} |f|. \]

Let \( J_{X,m}(M) \) denote the following distributional space

\[ J_{X,m}(M) := \bigcup_{k \in N \setminus \{0\}} J_{X,k}(M) = \bigcup_{k \in N \setminus \{0\}} \{ D \in S_{X,Y}(M) | X^k D = 0 \}. \]

By definition we have the following inclusions:

\[ \mathcal{L}_X \mathcal{J}_{X,k}^{-s}(M) = \mathcal{J}_{X,k-1}^{-s}(M) \quad \text{and} \quad \mathcal{L}_Y \mathcal{J}_{X,k}^{-s}(M) \subset \mathcal{J}_{X,k-1}^{-s}(M). \]

The operators Lie derivative operators \( \mathcal{L}_X \) and \( \mathcal{L}_Y \) are a creation and an annihilation operators for the spectrum of \( \Phi_s \) on \( J_{X,m}(M) \), in fact

\[ \Phi_s \circ \mathcal{L}_X = \lambda \mathcal{L}_X \circ \Phi_s \quad \text{and} \quad \Phi_s \circ \mathcal{L}_Y = \lambda^{-1} \mathcal{L}_Y \circ \Phi_s. \]

By the above description of the spectrum of \( \Phi_s \) on the space \( J_{X}(M) \) of invariant distributions, we derive the following

Lemma 2.9. For every \( k \in N \setminus \{0\} \), the spectrum of \( \Phi_s \) on the space \( J_{X,k}(M) \) of generalized invariant distributions is the set

\[ \sigma_{J_{X,k}(M)}(\Phi_s) := \{ 1 \} \cup \{ \mu_j \lambda^{-j} | i \in \{2, \ldots, 2g-1\} \text{ and } j \in \{1, \ldots, k\} \}. \]

Consequently, the spectrum of \( \Phi_s \) on the space \( J_{X,m}(M) \) is the set

\[ \sigma_{J_{X,m}(M)}(\Phi_s) := \{ 1 \} \cup \{ \mu_i \lambda^{-i} | i \in \{2, \ldots, 2g-1\} \text{ and } i \in N \setminus \{0\} \}. \]

All spectral values \( \mu_2 \lambda^{-2}, \ldots, \mu_{2g-1} \lambda^{-1} \) have multiplicity exactly equal to \( 1 \).

Proof. We have proved that the spectrum of \( \Phi_s \) on the space \( J_{X}(M) \) is the above set. Since \( L_X : J_{X,2}(M) \to J_{X,1}(M) = J_{X}(M) \) is surjective with kernel \( C \omega \) and acts as a “creation” operator on the spectrum, it follows that the spectrum of \( \Phi_s \) on \( J_{X,2}(M)/J_{X,1}(M) \) is the set

\[ \{ \mu_i \lambda^{-j-2} | i \in \{2, \ldots, 2g-1\} \text{ and } j \in N \}. \]

By induction we can prove that the spectrum of \( \Phi_s \) on \( J_{X,k+1}(M)/J_{X,k}(M) \) is the set

\[ \{ \mu_i \lambda^{-j-(k+1)} | i \in \{2, \ldots, 2g-1\} \text{ and } js \in N \}. \]
It follows that the multiplicity of a spectral value $\mu \lambda^{-l-1}$ equals the set of integer solutions of the equation $j+k = l+1$ with $j \in \mathbb{N}$ and $k \in \mathbb{N} \setminus \{0\}$, that is, it is equal to $l+1$. In other terms, given an eigenvector $D \in \mathcal{I}_{X \omega} (M)$ with eigenvalue $\mu \lambda^{-l-1}$ a basis of the corresponding eigenspace can be written as $$\{D, \mathcal{L}_X \mathcal{L}_Y D, \ldots, (\mathcal{L}_X \mathcal{L}_Y)^l D\}.$$

In fact, since $\Phi_* (D) = \mu \lambda^{-l-1} D$, the identity $$\Phi_* (\mathcal{L}_X^{l+1} D) = \lambda^{l+1} \mathcal{L}_X^{l+1} \Phi_* (D) = \mu \lambda^{l+1} D$$

implies (since $\mu$ does not belong to the spectrum of $\Phi_*$ on $\mathcal{I}_{X \omega} (M)$), that $\mathcal{L}_X^{l+1} D = 0$, hence $(\mathcal{L}_X \mathcal{L}_Y)^l D = \mathcal{L}_X^{l+1} \mathcal{L}_X^{l+1} D = 0$. It can be verified that the above system is indeed linearly independent, hence it is a basis of the eigenspace. \qed

From the above analysis we can derive an asymptotic for the correlations as follows. For every $g \in W_{X,Y} (M)$ let $\mathcal{C}_g \in W_{X,Y} (M)$ defined as $$\mathcal{C}_g (f) = \langle f, g \rangle_{L^2 (M, \omega)}.$$

By definition it follows that $$\Phi_* (\mathcal{C}_g) = \mathcal{C}_g (f \circ \Phi^b) = \mathcal{C} (f, g, n).$$

We recall that for every $s > k\delta_0$, every function $f \in \text{Ker} \mathcal{I}_{X^s}^\perp (M) \subset W_{X,Y} (M)$ is a k-iterated coboundary with transfer function $u \in W_{X,Y} (M)$ for all $t < s - k\delta_0$.

Let $\mathcal{R}_{X}^{-s} \subset \mathcal{I}_{X^s}^\perp (M)$ denote a set of (generalized) eigenvectors (Ruelle eigenstates) for the linear action of $\Phi_*$ on $\mathcal{I}_{X^s}^\perp (M)$. The distribution $\mathcal{C}_g$ can then be expanded as $$\Phi_* (\mathcal{C}_g) = \sum_{D \in \mathcal{R}_{X}^{-s}} C^{(\perp)}_{n} \cdot D + R^{(n)}_{g},$$

with remainder distributions $R^{(n)}_{g} \in \mathcal{I}_{X}^{-s} (M) \subset W_{X,Y}^{-s} (M)$.

**Lemma 2.10.** For any $s > k\delta_0$, there exists a constant $C_s > 0$ such that, for all $g \in L^2 (M, \omega)$ with $X^k g \in L^2 (M, \omega)$ and for all $n \in \mathbb{N}$, we have $$|R^{(n)}_{g}| \leq C_s |X^k g| \lambda^{-kn}.$$

**Proof.** For any $f \in W_{X,Y} (M)$ we have the orthogonal decomposition $$f = f_0 + f_1 \quad \text{with} \quad f_0 \in [\text{Ker} (\mathcal{R}_k^{-s})] \perp \text{and} \ f_1 \in \text{Ker} (\mathcal{R}_k^{-s}).$$

Since by construction $R_{g} \in (\mathcal{R}_k^{-s}) \perp$ and $f_0 \in \text{Ker} (\mathcal{R}_k^{-s})$, it follows that $$R^{(n)}_{g} (f) = R^{(n)}_{g} (f_0 + f_1) = R^{(n)}_{g} (f_1) = [\Phi_* (\mathcal{C}_g)] (f_1).$$

Since the function $f_1 \in \text{Ker} (\mathcal{R}_k^{-s})$, it is a k-iterated coboundary with transfer function $u_1 \in L^2 (M, \omega)$. It follows that $$|R^{(n)}_{g} (f_1)| \leq \|\mathcal{C} (f_1, g, n)\| \leq \lambda^{-kn} |u_1| |X^k g| \leq C_s \lambda^{-kn} |X^k g| |f_1|.$$ 

Since by orthogonality $|f_1| \leq |f|$ we finally derive the stated bound. \qed

**Lemma 2.11.** For any $D \in \mathcal{R}_k^{-s}$, let $\lambda_D = \mu \lambda^{-h} \in \mathbb{C}$, with $\mu \in \{\mu_2, \ldots, \mu_g\}$, denote the corresponding eigenvalue. If $|\lambda^{k-h} \mu| > 1$, there exist $c_D (g) \in \mathbb{C}$ and $C_D (g) > 0$ such that $$|c^{(n)}_{D} (g) - c_D (g) \lambda^{n}_D| \leq C_D (g) (\lambda^{k-h} \mu)^{-n}.$$
Proof. By definition we have
\[ \Phi_{s}^{n+1}(\xi) = \sum_{D \in \mathcal{X}_{s}^{-1}}^{(n+1)}(g)D + R_{s}^{(n+1)} = \sum_{D \in \mathcal{X}_{s}^{-1}}^{(n)}(g)D + \Phi_{s}(R_{s}^{(n)}) \]
\[ = \sum_{D \in \mathcal{X}_{s}^{-1}}^{(n)}(g)\lambda_{D}D + \Phi_{s}(R_{s}^{(n)}). \]
It follows by the above identities and the previous lemma that, for every \( n \in \mathbb{N} \), there exists \( r_{n} \in \mathbb{C} \) with \( |r_{n}(g)| \leq C_{r}(g)\lambda^{-kn} \) such that
\[ c_{D}^{(n+1)}(g) = \lambda_{D}c_{D}^{(n)}(g) + r_{n}(g). \]
By solving the difference equation we can write
\[ c_{D}^{(n)}(g) = \lambda_{D}^{n} \left( c_{D}^{(0)}(g) + \sum_{l=0}^{n-1} \lambda_{D}^{-l-1}r_{l}(g) \right). \]
By the estimate on the remainder term we have
\[ |\lambda_{D}^{-l}r_{l}(g)| \leq C_{r}(g)|\lambda_{D}\lambda^{l}|^{-l}, \]
and, since \( |\lambda_{D}\lambda^{l}| = |\mu\lambda^{-h}\lambda^{l}| = |\lambda^{-h}\mu| > 1 \), the series in the above formula is a convergent geometric series, hence the statement follows.

\[ \square \]

Remark 2.12. There is a symmetry, for all \( f, g \in L^{2}(M, \omega) \) and all \( n \in \mathbb{N} \),
\[ (f \circ \Phi^{n}, g)_{L^{2}(M, \omega)} = (f, g \circ \Phi^{-n})_{L^{2}(M, \omega)}. \]
It follows that in the above expansion the coefficients \( c_{D} \) are given by generalized invariant distributions for the stable translation flow \( Y \) on \( M \). In other terms, for all coefficients \( c(f, g) \) in the Ruelle-type expansion of correlations there exists \( k \in \mathbb{N} \setminus \{0\} \) such that, for all \( f, g \in \mathcal{S}_{XY}(M) \),
\[ c(Y^{k}f, g) = c(f, X^{k}g) = 0. \]
This duality is related to that discovered by Bufetov [Bu14] in his work on limit distributions of ergodic averages (see also [BuF14] for horocycle flows.)

Problem 2.13. Generalize the results of Faure, Gouëzel and Lenneau [FGL] to the case of non-linear pseudo-Anosov maps of surfaces, in particular to smooth perturbations of pseudo-Anosov maps by sufficiently small perturbations supported on the complement of the singularity set.

The author has proved in [F] that, assuming that the non-linear pseudo-Anosov map has a Margulis measure, then all the Ruelle resonances in the interval \( [e^{h_{\text{hor}}}, 1) \) are determined by the action of the map on the first cohomology (or homology) of the surface.

3. Transfer cocycles and generic translation flows

Let \( \mathcal{H}_{g} \) denote the space of Abelian holomorphic differentials (1-forms) on Riemann surfaces on a topological (smooth) surface \( S \) of genus \( g \geq 2 \). We recall that there is a natural identification between translation structures \( \{X, Y\} \), given by pairs of commuting transverse vector fields with appropriate normal forms at the singularities, and Abelian holomorphic differentials \( h \in \mathcal{H}_{g} \) on Riemann surfaces.

For any matrix \( A \in GL(2, \mathbb{R}) \) and \( h \in \mathcal{H}_{g} \) and let \( Ah \) denote the closed complex-valued 1-form
\[ Ah = [a \Re(h) + b \Im(h)] + i[c \Re(h) + d \Im(h)], \quad \text{for } A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \]
Alternatively, in the language of pairs of (commuting) vector fields \( \{X,Y\} \) we can define the \( SL(2,\mathbb{R}) \) action as follows:

\[
A \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} aX + bY \\ cX + dY \end{pmatrix}, \quad \text{for } A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]

There exists a unique complex structure on \( S \), that is, a unique Riemann surface \( M_{Ah} \), such that \( Ah \) is holomorphic on \( M_{Ah} \). This is the well-known standard definition of the action of the group \( GL(2,\mathbb{R}) \) on \( \mathcal{H}_g \).

This action of \( GL(2,\mathbb{R}) \) commutes with the diagonal action of the group \( \text{Diff}^+(S) \) of orientation preserving diffeomorphisms of the surface \( S \), hence it induces an action on the quotient space \( \mathcal{H}_g := (\mathcal{H}_g \times S)/\text{Diff}^+(S) \):

\[
A[h] = [Ah], \quad \text{for all } (A,[h]) \in GL(2,\mathbb{R}) \times \mathcal{H}_g.
\]

The subaction of the diagonal subgroup \( (g_i) < SL(2,\mathbb{R}) \) on \( \mathcal{H}_g \) is known as the Teichmüller geodesic flow, the actions of the unipotent subgroups of \( SL(2,\mathbb{R}) \) are known as the Teichmüller horocycle flows.

We introduce the notion of a transfer cocycle. For every \( h \in \mathcal{H}_g \), let \( S'_g(M) \) denote a space of tempered distributions (or currents) on \( M \) defined in terms of the translation structure determined by \( h \) on the underlying Riemann surface \( M \). Let \( S'_g(M) \) denote the bundle

\[
S'_g(M) = \{(h,D) | D \in S'_h(M)\}.
\]

It is possible to extend the \( GL(2,\mathbb{R}) \) action on \( \mathcal{H}_g \) to the bundle \( S'_g(M) \) by parallel transport with respect to the trivial connection, that is,

\[
A(h,D) = (Ah,D), \quad \text{for all } (h,D) \in S'_g(M).
\]

Since the above action of \( GL(2,\mathbb{R}) \) commutes with the diagonal action of \( \text{Diff}^+(S) \) on \( S'_g \) defined as

\[
\phi(h,D) = (\phi^*(h),\phi_*(D)), \quad \text{for all } (h,D) \in S'_g(M),
\]

it follows that the action of \( GL(2,\mathbb{R}) \) on \( S'_g \) passes to the quotient to an action on the vector bundle \( S'_g : = S'_g/\text{Diff}^+(S) \), over the action of \( GL(2,\mathbb{R}) \) on \( S'_g \). The subaction of the of the diagonal subgroup \( (g_i) < SL(2,\mathbb{R}) \) on \( \mathcal{H}_g \) is by definition the transfer cocycle \( (\mathcal{L}_i) \) over the Teichmüller flow.

The above construction gives the Kontsevich–Zorich cocycle (a finite dimensional cocycle) when \( S'_g(M) \) is replaced with the cohomology \( H^1(S,\mathbb{R}) \) or \( H^1(S,\Sigma,\mathbb{R}) \) (or their complexifications) for all \( h \in \mathcal{H}_g \). We are interested in the case when \( S'_g(M) \) is a Banach (Hilbert) bundle of distributions or (closed) currents. The general question is whether the cocycle has well-defined Lyapunov exponents and whether it has a spectral gap. In fact, we are interested in the Oseledets decomposition of particular distributions, or currents, such as those given by rectifiable arcs or by "correlations".

For instance, our generalization of the Ruelle resonance problem to this setting is as follows. For any function \( g \in L^2(S,\omega_h) \) on \( S \) we can consider the distribution \( C_h(g) \in W^{-1}_h(S) \) defined as

\[
C_h(g)(f) = \langle f,g \rangle_{L^2(S,\omega_h)}.
\]

The distribution \( C_h(g) \) is in fact an absolutely continuous measure, but we ask what is its asymptotic behavior, for instance under the Teichmüller flow (that is, under the action of the distributional cocycle \( (\mathcal{L}_i) \) defined above).

In [F02] the author considered the above construction with \( S'_g(M) \) equal to the bundle \( \mathcal{Z}_g^{-1}(M) \) with fiber the space of closed 1-currents in the Sobolev space \( W^{-1}_h(M) \). It was proved there that the transfer cocycle has Lyapunov exponents equal to the Kontsevich–Zorich exponents with respect to any KZ-hyperbolic \( SL(2,\mathbb{R}) \)-invariant measure (with almost any fiber generated by basic
currents for the horizontal and vertical vector fields). Results on the deviation of ergodic averages were then derived by a short argument based on the de Rham theorem.

We outline this argument below. Let $\mathcal{Z}_g^{-1}(M)$ and $\mathcal{E}_g^{-1}(M)$ denote respectively the bundles of closed and exact currents with coefficients in the Sobolev bundle $W_g^{-1}(M)$ with fiber at every $h = (X, Y) \in \mathcal{H}_g$ the dual Sobolev space $W_{X,Y}^{-1}(M)$.

Under the hypothesis that the Kontsevich–Zorich cocycle is non-uniformly hyperbolic, by a representation theorem, for almost all $h = (X, Y) \in \mathcal{H}_g$, we have

$$\mathcal{Z}_h^{-1}(M) := \mathcal{B}_X^{-1}(M) \oplus \mathcal{B}_Y^{-1}(M) \oplus \mathcal{E}_h^{-1}(M).$$

Let $\mu$ be a Borel probability measure, invariant under the Teichmüller flow and let

$$1 = \lambda_1^\mu > \lambda_2^\mu \geq \cdots \geq \lambda_g^\mu (\geq 0) \geq -\lambda_g^\mu \geq \cdots \geq -\lambda_1^\mu = -1$$

denote the Kontsevich–Zorich exponents (we recall that $\lambda_2^\mu < 1$ was proved by W. Veech [V86] for some class of measures, and in [F02] in general; that $\lambda_g^\mu > 0$ was proved in [F02], [F11] for the canonical measures and for other $SL(2, \mathbb{R})$-invariant measures; the simplicity of the spectrum was proved by Avila and Viana [AV07] for the canonical measures).

Theorem 3.1. [F02] Let us assume that the Kontsevich–Zorich cocycle is non-uniformly hyperbolic. The transfer cocycle on the bundle $\mathcal{Z}_g^{-1}(M)$ has Lyapunov spectrum

$$1 = \lambda_1^\mu > \lambda_2^\mu \geq \cdots \geq \lambda_g^\mu (\geq 0) \geq -\lambda_g^\mu \geq \cdots \geq -\lambda_1^\mu = -1.$$

The Oseledets sub-bundle of the exponent 0 is the sub-bundle of exact currents (which is infinite dimensional).

Since arc of orbits of the translation flows are, as currents, at a distance from closed currents bounded by the diameter of the translation surface, it follows that their behavior under the transfer cocycle is described by the above Lyapunov spectrum.

Corollary 3.2. For all $i \in \{1, \ldots, g\}$, let $D_{i,1}^h, \ldots, D_{i,m_i}^h$ denote a system of invariant distributions associated to the basic currents in the Oseledets space for the exponents $\lambda_i^\mu > 0$. For all functions $f \in W_{X,Y}^1(M)$ such that $D_{j,1}(f) = \cdots = D_{j,m_j}(f) = 0$, for all $1 \leq j \leq i < g$, we have

$$\limsup_{T \to +\infty} \frac{1}{\log T} \log \left| \int_0^T f \circ \Phi_t^X(x) dt \right| \leq \lambda_{i+1},$$

and equality holds if there exists a distribution $D_{i+1}^h$ with basic currents of Lyapunov exponents $\lambda_{i+1} \geq 0$ such that $D_{i+1}^h(f) \neq 0$. If $D_{j,1}(f) = \cdots = D_{j,m_j}(f) = 0$, for all $1 \leq j \leq i \leq g$, then

$$\limsup_{T \to +\infty} \frac{1}{\log T} \log \left| \int_0^T f \circ \Phi_t^X(x) dt \right| = 0.$$

After recalling the above results on ergodic averages, we return to the generalization of Ruelle eigenstates to the generic translation flows. By the results on solutions of the cohomological equation for almost all translation flows (see [F97], [F07], [MMY05], [MY16]), it is possible to construct for almost all translation surface $(X, Y)$ the space of generalized (iterated) invariant distributions $\mathcal{J}_{X,k}(M)$ (for all $k \in \mathbb{N} \setminus \{0\}$) and $\mathcal{J}_{X,w}(M)$ and the corresponding bundles $\mathcal{J}_k(M)$ and $\mathcal{J}_w(M)$ over the moduli space. We have the following result

Theorem 3.3. Let us assume that the KZ cocycle is non-uniformly hyperbolic. The Oseledets spectrum of the transfer cocycle over the the bundle $\mathcal{J}_w(M)$ has exponents

$$\{1\} \cup \{\pm \lambda_i - j | i = 2, \ldots, g, j \in \mathbb{N} \setminus \{0\}\}.$$

The Lyapunov exponent 1 is simple and corresponds to the subbundle of $\mathcal{J}_w(M)$ given by the invariant area on $M$. In addition, the Lyapunov exponent $\pm \lambda_i - j$ has multiplicity exactly $j$, for all $i \in \{2, \ldots, g\}$ and $j \in \mathbb{N} \setminus \{0\}$. 
Problem 3.4. Generalize the results of Faure, Gouëzel and Lanneau [FGL] on the deviation of ergodic averages and cohomological equations for the unstable foliations of pseudo-Anosov maps to (measure) generic translation flows on higher genus surfaces.

It should be possible to prove a version the above result on the basis of the general analytic techniques pioneered by Giulietti and Liverani [GL] and exploited in the work of Faure, Gouëzel and Lanneau. In particular, for every translation surface \( h = (X,Y) \), let \( Y_t \) denote a distributional space introduced in the work of Faure, Gouëzel and Lanneau. These spaces are a straightforward adaptation of to translation surfaces of the anisotropic Banach spaces introduced by S. Gouëzel and C. Liverani [GouL06]. It should be possible to prove that the corresponding transfer cocycle is a linear cocycle of quasi-compact operators on a Banach bundle. (We recall that a quasi-compact operator is an operator equal to a sum of a compact operator and an operator with “small” spectral radius). For quasi-compact cocycles on Banach bundles, the multiplicative ergodic theorem holds (see for instance [GQ15] or [Bl16] and references therein), hence we would derive that the translation such that for any\( f \) is an 1-coboundary with zero-average transfer function \( u \) and \( v \), commutation relations hold:

\[
[X, U] = U, \quad [X, V] = -V, \quad [U, V] = 2X.
\]

Theorem 4.1. [FF03] There exists a space \( Z_U(M) \) of \( U \)-invariant distributions of countable dimension such that for any \( f \in W^s(M) \) (with \( s > 1 \)) such that

\[
D(f) = 0, \quad \text{for all } D \in Z_U(M),
\]

is an \( U \)-coboundary with zero-average transfer function \( u \in L^2(M, \omega) \). In addition, for all \( t < s - 1 \) there exists a constant \( C_{s,t} > 0 \) such that

\[
|u|^t \leq C_{s,t} |f|^s.
\]

The space \( Z_U(M) \) has a basis of generalized eigenvectors for the linear operator Lie derivative \( \mathcal{L}_X \) along the geodesic flow with spectrum

\[
\sigma_{Z_U(M)} = \left\{ -\frac{1 \pm \sqrt{1 - 4\mu}}{2} \mid \mu \in \sigma(\triangle) \right\} \cup (-N).
\]

The linear operator \( \mathcal{L}_X \) on \( Z_U(M) \) is diagonalizable, with eigenvalues of finite multiplicity, with the possible exception of finitely many \( 2 \times 2 \) Jordan blocks for the eigenvalue \( 1/4 \) whenever \( 1/4 \in \sigma(\triangle) \). The multiplicity of the eigenvalues is determined by the spectral multiplicities of the eigenvalues of the Laplace operator and by the dimensions of the spaces of holomorphic n-differentials on the hyperbolic surface \( S \).
By the above result $k$-iterated coboundaries coincide with the kernel of the space of $k$-invariant distributions

$$\mathcal{J}_{U,k}(M) = \{ D \in \mathcal{E}'(M) | U^kD = 0 \}.$$  

**Lemma 4.2.** The space $\mathcal{J}_{U,k}(M)$ can be described as follows:

$$\mathcal{J}_{U,k}(M) = \bigoplus_{j=0}^{k} \mathcal{L}_V^j \mathcal{J}_U(M).$$

The space $\mathcal{J}_{U,k}(M)$ has a basis of generalized eigenvectors for the linear operator Lie derivative $\mathcal{L}_X$ along the geodesic flow with spectrum

$$\sigma_{\mathcal{J}_U(M)} = \{0\} \cup \bigcup_{j=0}^{k} \{ \lambda - j | \lambda \in \sigma_{\mathcal{J}_U(M)} \}.$$  

For every $\lambda \in \sigma_{\mathcal{J}_U(M)}$ and every $j \in \{0, \ldots, k\}$, let $E_X(\lambda - j)$ denote the generalized eigenspace of the operator $\mathcal{L}_X$ with eigenvalue $\lambda - j \in \mathbb{C}$. The operators $\mathcal{L}_U : E_X(\lambda - j - 1) \to E_X(\lambda - j)$ and $\mathcal{L}_V : E_X(\lambda - j) \to E_X(\lambda - j - 1)$ are isomorphisms of finite dimensional vector spaces. The operator $\mathcal{L}_X$ is diagonalizable with the exception of the eigenvalues $1/4 - j$, whenever $1/4 \in \sigma(\triangle)$, for which it has finitely many $2 \times 2$ Jordan blocks.

**Proof.** By the commutation relations we have

$$[\mathcal{L}_X, \mathcal{L}_V] = -\mathcal{L}_V \quad \text{and} \quad [\mathcal{L}_X, \mathcal{L}_U] = \mathcal{L}_U.$$ 

It follows that

$$(\mathcal{L}_X - \lambda \text{Id}) \mathcal{L}_V = [\mathcal{L}_X - \lambda \text{Id}, \mathcal{L}_V]$$

$$+ \mathcal{L}_V(\mathcal{L}_X - \lambda \text{Id}) = \mathcal{L}_V(\mathcal{L}_X - (\lambda + 1)\text{Id}),$$

and by induction, for all $j \in \mathbb{N}$,

$$(\mathcal{L}_X - \lambda \text{Id})^j \mathcal{L}_V = \mathcal{L}_V((\mathcal{L}_X - (\lambda + 1)\text{Id})^j).$$

In fact, by induction hypothesis we have

$$(\mathcal{L}_X - \lambda \text{Id})^j \mathcal{L}_V = (\mathcal{L}_X - \lambda \text{Id})(\mathcal{L}_X - \lambda \text{Id})^{j-1} \mathcal{L}_V$$

$$= (\mathcal{L}_X - \lambda \text{Id}) \mathcal{L}_V(\mathcal{L}_X - (\lambda + 1)\text{Id})^j$$

$$= \mathcal{L}_V(\mathcal{L}_X - (\lambda + 1)\text{Id})(\mathcal{L}_X - (\lambda + 1)\text{Id})^{j-1}$$

$$= \mathcal{L}_V(\mathcal{L}_X - (\lambda + 1)\text{Id})^j.$$  

Similarly, from the commutation relations we derive that, for all $j \in \mathbb{N}$,

$$(\mathcal{L}_X - \lambda \text{Id})^j \mathcal{L}_U = \mathcal{L}_V(\mathcal{L}_X - (\lambda - 1)\text{Id})^j.$$  

By formulas (1) and (2) it follows that if $D \in \mathcal{E}'(M)$ is any distributional generalized eigenvector for the geodesic flow of eigenvalue $\lambda \in \mathbb{C}$ and algebraic multiplicity $m \geq 1$, then $\mathcal{L}_V D$ and $\mathcal{L}_U D$ are distributional generalized eigenvectors for the geodesic flow of eigenvalues respectively $\lambda - 1 \in \mathbb{C}$ and $\lambda + 1 \in \mathbb{C}$ and algebraic multiplicity $m \geq 1$.

The Lie derivative horocyclic operators $\mathcal{L}_V$ and $\mathcal{L}_U$ act therefore as annihilation and creation operators on the distributional pure point spectrum of the geodesic flow Lie derivative operator $\mathcal{L}_X$, and they preserve the algebraic multiplicity of eigenvalues.

It follows immediately by the definition of the iterated invariant distributions that, for all $k \in \mathbb{N} \setminus \{0\}$, the operator $\mathcal{L}_U : \mathcal{J}_{U,k+1}(M) \to \mathcal{J}_{U,k}(M)$ is well-defined. We prove below that the operator $\mathcal{L}_V : \mathcal{J}_{U,k}(M) \to \mathcal{J}_{U,k+1}(M)$ is also well-defined.

By the commutation relations, it follows by induction that the operator $\mathcal{L}_X : \mathcal{J}_{U,k}(M) \to \mathcal{J}_{U,k}(M)$ is well-defined. In fact, for any $D \in \mathcal{J}_{U,k}(M)$ we have

$$\mathcal{L}_U^k \mathcal{L}_X D = \mathcal{L}_U^{k-1} \mathcal{L}_U \mathcal{L}_X D = \mathcal{L}_U^{k-1} [\mathcal{L}_U, \mathcal{L}_X] D + \mathcal{L}_U^{k-1} \mathcal{L}_X \mathcal{L}_U D = -\mathcal{L}_U^k D + \mathcal{L}_U^{k-1} \mathcal{L}_X \mathcal{L}_U D.$$
Thus for \( k = 1 \) we immediately derive that, if \( D \in \mathcal{J}_{U,1}(M) = \mathcal{J}_U(M) \), then \( \mathcal{L}_U \mathcal{L}_X D = 0 \), hence \( \mathcal{L}_X D \in \mathcal{J}_{U,1}(M) \). In general, we have that, if \( D \in \mathcal{J}_{U,k}(M) \), then \( \mathcal{L}_U D \in \mathcal{J}_{U,k-1}(M) \) and, by the induction hypothesis, also \( \mathcal{L}_X \mathcal{L}_U D \in \mathcal{J}_{U,k-1}(M) \). It then follows by the above identity that \( \mathcal{L}_U^k \mathcal{L}_X D = 0 \), that is, \( \mathcal{L}_X D \in \mathcal{J}_{U,k}(M) \). We have thus proved that \( \mathcal{L}_X : \mathcal{J}_{U,k}(M) \to \mathcal{J}_{U,k+1}(M) \) is well-defined.

Next, we prove by induction that the operator \( \mathcal{L}_V : \mathcal{J}_{U,k}(M) \to \mathcal{J}_{U,k+1}(M) \) is well-defined. By the commutation relations we have

\[
\mathcal{L}_U^{k+1} \mathcal{L}_V D = \mathcal{L}_U^k \mathcal{L}_U \mathcal{L}_V D = \mathcal{L}_U^k [\mathcal{L}_U, \mathcal{L}_V] D + \mathcal{L}_U^k \mathcal{L}_V \mathcal{L}_U D = 2 \mathcal{L}_U^k \mathcal{L}_X D + \mathcal{L}_U^k \mathcal{L}_V \mathcal{L}_U D.
\]

Thus for \( k = 1 \), if \( D \in \mathcal{J}_{U,1}(M) = \mathcal{J}_U(M) \), since \( \mathcal{L}_X D \in \mathcal{J}_{U,1}(M) \) we have \( \mathcal{L}_U^1 \mathcal{L}_X D = 0 \), that is, \( \mathcal{L}_U D \in \mathcal{J}_{U,1}(M) \). In general, if \( D \in \mathcal{J}_{U,k}(M) \), then \( \mathcal{L}_X D \in \mathcal{J}_{U,k}(M) \) and, by induction hypothesis, \( \mathcal{L}_V \mathcal{L}_U D \in \mathcal{J}_{U,k}(M) \). It follows by the above identity that \( \mathcal{L}_U^{k+1} \mathcal{L}_V D = 0 \), that is, \( \mathcal{L}_V D \in \mathcal{J}_{U,k+1}(M) \). We have thus proved that \( \mathcal{L}_V : \mathcal{J}_{U,k}(M) \to \mathcal{J}_{U,k+1}(M) \) is well-defined.

Finally, we prove that for any generalized eigenspace \( E_X(\lambda) \) of the operator \( \mathcal{L}_X \), transverse to the subspace \( \mathcal{J}_U(M) \) of invariant distributions, the operators \( \mathcal{L}_U : E_X(\lambda) \to E_X(\lambda + 1) \) and \( \mathcal{L}_V : E_X(\lambda + 1) \to E_X(\lambda) \) are isomorphisms of finite dimensional vector spaces. By construction the operator \( \mathcal{L}_U : \mathcal{J}_{U,k+1}(M) \to \mathcal{J}_{U,k}(M) \) is surjective and it is creation operator (which adds +1 to the spectrum). It follows that the restriction of \( \mathcal{L}_U \) to every generalized eigenspace \( E_X(\lambda) \) of \( \mathcal{L}_X \) transversal to the subspace \( \mathcal{J}_U(M) \) of invariant distribution is an isomorphism with range equal to the eigenspace \( E(\lambda + 1) \). The operator \( \mathcal{L}_V \) is injective on each generalized eigenspace \( E_X(\lambda + 1) \) of \( \mathcal{L}_X \) orthogonal to constant functions, since the kernel of \( \mathcal{L}_V \) on \( E_X(\lambda + 1) \) would consists of smooth invariant functions, which have to be constant by the ergodicity of horocycle flows. Hence, it is also an isomorphism.

\[ \square \]

From the results on cohomological equations (Theorem 4.1) and from the description of the space of iterated coboundaries and of the spectrum of the geodesic flow on them, we can derive the following statement of the Ruelle resonances and Ruelle asymptotic for geodesic flows of compact hyperbolic surfaces.

**Theorem 4.3.** The set of Ruelle resonances of the geodesic flow \( \phi^X_t \) of a compact hyperbolic surface is

\[ \{1\} \cup \{ \exp \left( -\left( \frac{1 + \sqrt{1 - 4\mu}}{2} j \right) \right) | \mu \in \sigma(\triangle) \text{ and } j \in \mathbb{N} \} \cup \{ e^{-nt} | n \in \mathbb{N} \}. \]

There are no Jordan blocks except for the eigenvalues with \( \mu = 1/4 \), whenever \( 1/4 \in \sigma(\triangle) \). In this case, there are finitely many \( 2 \times 2 \) Jordan blocks with eigenvalue \( e^{-\left(\frac{1}{2} - j\right)t} \), for all \( j \in \mathbb{N} \). The Ruelle asymptotic takes the following form:

\[
\langle f \circ \phi^X_t, g \rangle \approx \left( \int_M f d\text{vol} \right) \left( \int_M g d\text{vol} \right) + \sum_{\mu \in \sigma(\triangle) \setminus \{1/4\}} \sum_{j \in \mathbb{N}} e^+_{\mu,j}(f,g,t) e^{-\left(\frac{1+\sqrt{1-4\mu}}{2}j\right)t} + \sum_{j \in \mathbb{N}} e^+_{\mu,j}(f,g,t) e^{-\left(\frac{1}{2} - j\right)t} + \sum_{n \in \mathbb{N}} e_n(f,g,t) e^{-nt}.
\]

A generalization of the above theorem to the geodesic flows on compact hyperbolic spaces in any dimensions has been carried out by S. Dyatlov, F. Faure and C. Guillarmou in [DFG].

**Problem 4.4.** Extend the above theorem to geodesic flows on surfaces of non-constant negative curvature and to Anosov flows in dimension 3.

Partial or conditional results on this problem have been obtained by A. Adam [Ad] and F. Faure and C. Guillarmou [FG18].
5. Ruelle resonances for (partially hyperbolic) Heisenberg automorphisms

Let $N$ denote the 3-dimensional Heisenberg group and let $M := \Gamma \backslash N$ denote a Heisenberg nilmanifold, that is, the quotient of $N$ over a (necessarily co-compact) lattice $\Gamma < N$.

Let $\Phi : M \to M$ denote a partially hyperbolic automorphism of a Heisenberg nilmanifold $M$ and let $\{X, Y, Z\}$ denote the corresponding Heisenberg frame of vector fields on $M$. The Heisenberg commutation relations hold:

$$[X, Y] = Z, \quad [X, Z] = [Y, Z] = 0.$$  

By the assumption, there exists $\lambda > 1$ such that

$$\Phi_*(X) = \lambda X, \quad \Phi_*(Y) = \lambda^{-1}Y \quad \text{and} \quad \Phi_*(Z) = Z.$$  

Let $\omega$ denote the $\Phi$-invariant volume-form. The form $\omega$ is also invariant for all nilflows on $M$. For any pair $f, g$ of sufficiently smooth complex-valued functions on $M$, we are interested in the asymptotic for the decay of the correlations

$$C(f, g; n) = \langle f \circ \Phi^n, g \rangle_{L^2(M, \omega)}.$$  

As in the case of pseudo-Anosov diffeomorphisms, the key step is to characterize iterated coboundaries. Coboundaries were characterized in [FF06] and [FF07] (see also [F14]).

**Theorem 5.1.** [FF06] There exists a space $\mathcal{I}_X(M)$ of $X$-invariant distributions of countable dimension such that for any $f \in W^s(M)$ (with $s > 1$) such that

$$D(f) = 0, \quad \text{for all } D \in \mathcal{I}_X(M),$$

is an $X$-coboundary with zero-average transfer function $u \in L^2(M, \omega)$. In addition, for all $t < s - 1$ there exists a constant $C_{t, s} > 0$ such that

$$|u|_t \leq C_{t, s} |f|_s.$$  

**Lemma 5.2.** [FF06] There exists a basis $\{D_{z,i}|z \in \mathbb{Z} \setminus \{0\}, i = 1, \ldots, |z|\}$ of $\mathcal{I}_X(M)$ and unit complex numbers $\{u_{z,i}|z \in \mathbb{Z} \setminus \{0\}, i = 1, \ldots, |z|\}$ such that, for all $z \in \mathbb{Z} \setminus \{0\}$ and $i \in \{1, \ldots, |z|\}$, we have $ZD_{z,i} = izD_{z,i}$ and

$$\Phi_*(D_{z,i}) = u_{z,i} \lambda^{-1/2}D_{z,i}.$$  

The following result on the Ruelle resonances of partially hyperbolic Heisenberg automorphisms appears a special case of in the work of F. Faure and M. Tsuji [FT15] (see Remark 1.3.5, (4)). Results on cohomological equation for the unstable flow from the methods of Faure and Tsuji, in the spirit of the Giulietti, Liverani [GL] and Faure, Gouëzel, Lanneau [FGL], were recently derived, in the linear case, by O. Butterley and L. Simonelli [BS]. It is an interesting problem to generalize their work to the case of partially hyperbolic non-linear of maps of Heisenberg nilmanifolds treated in the work of Faure and Tsuji [FT15].

**Theorem 5.3.** The set of Ruelle resonances of any partially hyperbolic Heisenberg automorphism is

$$\{1\} \cup \{u_{z,i} \lambda^{-k-1/2}|z \in \mathbb{Z} \setminus \{0\}, i \in \{1, \ldots, |z|\}, k \in \mathbb{N}\}.$$  

There exists a basis of the space of “Ruelle eigenstates” $\{D^{(k)}_{z,i}\}$ such that

$$\Phi_*(D^{(k)}_{z,i}) = u_{z,i} \lambda^{-k-1/2}D^{(k)}_{z,i} + \sum_{j=k} c_{k,j}(\lambda) u_{z,i} \lambda^{-j-1/2} D^{(j)}_{z,i}.$$  

The coefficients $c_{k,j}(\lambda)$ are not all equal to zero.

**Proof.** We consider solutions $D^{(k)}_{z,i}$ of the iterated coboundary equations

$$X^k D^{(k)}_{z,i} = D_{z,i}.$$  

By an immediate computation we have

$$X^k[\Phi_\xi(D^{(k)}_{z,j}) - \lambda^{-k-1/2}D^{(k)}_{z,j}] = 0.$$  

These solutions and the action of the automorphism $\Phi$ on them can be explicitly computed by representation theory. In fact, every irreducible representation space $H$ is unitarily equivalent to $L^2(\mathbb{R}, dx)$ and the derived representation $\mathcal{D}_\xi$ is given by the following formulas:

$$\mathcal{D}_\xi(X) = \frac{d}{dx}, \quad \mathcal{D}_\xi(Y) = i\xi \quad \text{and} \quad \mathcal{D}_\xi(Z) = izI.$$  

Let $\mathcal{F}_H : H \to L^2(\mathbb{R}, dx)$ denote the unitary equivalence. The space of invariant distributions in each representation is 1-dimensional, and it is generated by the functional

$$D_H(f) = \int_{\mathbb{R}} \mathcal{F}_H(f)(x) dx, \quad \text{for all} \ f \in W^s_{XZ}(H) \subset H, \ \text{with} \ s > 1/2.$$

Since the Green operator $G_X$ of the cohomological equation $Xu = f$ can be written in each irreducible representation as

$$[\mathcal{F}_H G_X(f)](x) = \int_{-\infty}^{x} \mathcal{F}_H(f)(\xi) d\xi = -\int_{\infty}^{x} \mathcal{F}_H(f)(\xi) d\xi,$$

the distributions $D^{(k)}_H$ are given formally by the formulas

$$D^{(k)}_H(f) := \int_{\mathbb{R}} \int_{-\infty}^{\xi_1} \cdots \int_{-\infty}^{\xi_k} \mathcal{F}_H(f)(\xi_0) d\xi_0 d\xi_1 \cdots d\xi_k.$$  

In fact, the integral in the above formula for $D^{(k)}_H$ is convergent (even for infinitely differentiable functions) only on the joint kernel of $D^{(0)}_H, \ldots, D^{(k-1)}_H$.

Let then $\{\chi_a\} \subset C_0^\infty(\mathbb{R})$ be a system of functions such that, for all $a \leq b$,

$$\int_{\mathbb{R}} \int_{-\infty}^{\xi_1} \cdots \int_{-\infty}^{\xi_k} \chi_b(\xi_0) d\xi_0 d\xi_1 \cdots d\xi_a = \delta_{ab}$$

and let $P^{(j)}_H : H \to L^2(\mathbb{R}, dx)$ denote the projectors recursively defined as

$$\begin{cases} P^{(0)}_H(f) = \mathcal{F}_H(f) - D_H(f) \chi_0, \\ P^{(j+1)}_H(f) = P^{(j)}_H(f) - D^{(j+1)}_H(P^{(j)}_H(f)) \chi_{j+1}. \end{cases}$$

By construction we have that, for all $j \in \mathbb{N}$ and for all $f \in W^s(H), s > j + 1/2$,

$$D^{(0)}_H(P^{(j)}_H(f)) = \cdots = D^{(j)}_H(P^{(j)}_H(f)) = 0.$$  

It follows that $D^{(j+1)}_H$ can be defined, for all $j \in \mathbb{N}$, by the formula

$$D^{(j+1)}_H(f) := \int_{\mathbb{R}} \int_{-\infty}^{\xi_{j+1}} \cdots \int_{-\infty}^{\xi_k} P^{(j)}_H(f)(\xi_0) d\xi_0 d\xi_1 \cdots d\xi_{j+1}.$$  

The volume preserving, hence unitary, action of the automorphism $\Phi$ on $L^2(M)$ preserves the isotypical components of the regular representation, that is, the eigenspaces of the central circle action. For every $z \in \mathbb{Z} \setminus \{0\}$, there exists a splitting of the isotypical components $H_z$ into irreducible components

$$H_z := \bigoplus_{i=1}^{\infty} H_{z,i},$$  

such that for each $i \in \{1, \ldots, |z|\}$ the component $H_{z,i}$ is invariant under the action of $\Phi$, hence it exists a unit complex number $u_{z,i} \in U(1)$ such that the operator $\Phi^*$ can be written in representation as follows:

$$\mathcal{F}_{H_{z,i}}(\Phi^*(f))(x) = u_{z,i} \lambda^{1/2} \mathcal{F}_{H_{z,i}}(f)(\lambda x), \quad \text{for all} \ x \in \mathbb{R}.$$
For every \( z \in \mathbb{Z} \setminus \{0\} \), every \( i \in \{1, \ldots, |z|\} \) and every \( j, k \geq 0 \), let us adopt the notation

\[
\mathcal{F}_{z,i} := \mathcal{F}_{H_{z,i}}, \quad p^{(j)}_{H_{z,i}} := p^{(j)}_{z,i} \quad \text{and} \quad D^{(k)}_{z,i} := D^{(k)}_{H_{z,i}}.
\]

A direct calculation then shows that, as claimed,

\[
\Phi_{*}(D^{(k)}_{z,i}) = u_{z,i} \lambda^{-k - 1/2} D^{(k)}_{z,i} + \sum_{j \leq k} c_{k,j}(\lambda) u_{z,i} \lambda^{-j - 1/2} D^{(j)}_{z,i}.
\]

The constants \( c_{k,j}(\lambda) \) can be explicitly computed in representation.

For instance, for all \( f \in W^{\infty}(H_{z,i}) \) we have

\[
D^{(0)}_{z,i}(f \circ \Phi) = \int_{\mathbb{R}} u_{z,i} \lambda^{-1/2} \mathcal{F}_{z,i}(f)(\lambda x) dx = u_{z,i} \lambda^{-1/2} D^{(0)}_{z,i}(f),
\]

hence the invariant distribution \( D^{(0)}_{z,i} \) is indeed an eigendistribution.

For \( j = 1 \) by definition we have

\[
D^{(1)}_{z,i}(f \circ \Phi) = \int_{\mathbb{R}} \int_{\mathbb{R}} [u_{z,i} \lambda^{1/2} \mathcal{F}_{z,i}(f)(\lambda \xi) - u_{z,i} \lambda^{-1/2} D^{(0)}_{z,i}(f) \chi_0(\xi)] d\xi dx
\]

\[
= u_{z,i} \int_{\mathbb{R}} \int_{\mathbb{R}} \lambda^{1/2} [\mathcal{F}_{z,i}(f)(\lambda \xi) - D^{(0)}_{z,i}(f) \chi_0(\lambda \xi)] d\xi dx
\]

\[
+ u_{z,i} \lambda^{-1/2} D^{(0)}_{z,i}(f) \int_{\mathbb{R}} \int_{\mathbb{R}} [\lambda \chi_0(\lambda \xi) - \chi_0(\xi)] d\xi dx
\]

By change of variable we have

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} \lambda^{1/2} [\mathcal{F}_{z,i}(f)(\lambda \xi) - D^{(0)}_{z,i}(f) \chi_0(\lambda \xi)] d\xi dx = \lambda^{-3/2} \int_{\mathbb{R}} \int_{\mathbb{R}} \lambda^{1/2} [\mathcal{F}_{z,i}(f)(\lambda \xi) - D^{(0)}_{z,i}(f) \chi_0(\lambda \xi)] d\xi dx = \lambda^{-3/2} D^{(1)}_{z,i}(f)
\]

and

\[
c_{1,0}(\lambda) := \int_{\mathbb{R}} \int_{\mathbb{R}} [\lambda \chi_0(\lambda \xi) - \chi_0(\lambda \xi)] d\xi dx = \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_0(\lambda \xi) d\xi dx.
\]

We therefore have that, as claimed,

\[
\Phi_{*}(D^{(1)}_{z,i}) = u_{z,i} \lambda^{-3/2} D^{(1)}_{z,i} + c_{1,0}(\lambda) u_{z,i} \lambda^{-1/2} D^{(0)}_{z,i}.
\]

However, we remark that when \( \chi_0 \) is chosen to be an even function, then

\[
c_{1,0}(\lambda) = \int_{\mathbb{R}} \int_{\mathbb{R}} \lambda \chi_0(\lambda \xi) d\xi dx = 0.
\]

By definition we then have \( P^{(1)}_{z,i}(f) = P^{(0)}_{z,i}(f) - D^{(1)}_{z,i}(f) \chi_1 \) with

\[
\int_{\mathbb{R}} \chi_1(\lambda \xi_0) d\xi_0 = 0, \quad \text{and} \quad \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_1(\lambda \xi_0) d\xi_0 d\xi_1 = 1.
\]

It follows that

\[
P^{(1)}_{z,i}(f \circ \Phi)(x) = u_{z,i} \lambda^{1/2} \mathcal{F}_{z,i}(f)(\lambda x) - u_{z,i} \lambda^{-1/2} D^{(0)}_{z,i}(f) \chi_0 - u_{z,i} \lambda^{-3/2} D^{(1)}_{z,i}(f) \chi_1.
\]

Up to the unit complex factor \( u_{z,i} \), we can write \( P^{(1)}_{z,i}(f \circ \Phi)(x) \) as the sum of three terms. The first term is

\[
\lambda^{1/2} [\mathcal{F}_{z,i}(f) - D^{(0)}_{z,i}(f) \chi_0 - D^{(1)}_{z,i}(f) \chi_1](\lambda x),
\]

which after triple integration contributes to \( D^{(2)}_{z,i}(f \circ \Phi) \) a term

\[
\lambda^{1/2 - 3} D^{(2)}_{z,i}(f) = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} [\mathcal{F}_{z,i}(f) - D^{(0)}_{z,i}(f) \chi_0 - D^{(1)}_{z,i}(f) \chi_1] \chi_1(\lambda \xi_0) d\xi_0 d\xi_1 d\xi_2.
\]
The second term is equal to $\lambda^{-1/2}D_{c,i}^{(0)}(f)(\lambda \chi_0(\lambda x) - \chi_0(x))$. The corresponding coefficient
\[
c_{2,0}(\lambda) = \int_{\mathbb{R}} \int_{-\infty}^{\xi_2} \int_{-\infty}^{\xi_1} \lambda \chi_0(\lambda \xi_0) - \chi_0(\xi_0) \pi_0 d\xi_0 d\xi_1 d\xi_2 
\]
The third term is equal to $\lambda^{-3/2}D_{c,i}^{(1)}(f)(\lambda^2 \chi_1(\lambda x) - \chi_1(x))$. The corresponding coefficient
\[
c_{2,1}(\lambda) = \int_{\mathbb{R}} \int_{-\infty}^{\xi_2} \int_{-\infty}^{\xi_1} \lambda^2 \chi_1(\lambda \xi_0) - \chi_1(\xi_0) \pi_0 d\xi_0 d\xi_1 d\xi_2 
\]
It follows that we have
\[
\Phi_*(D_{c,i}^{(2)}) = u_{c,i} \lambda^{-1/2}D_{c,i}^{(2)} + c_{2,0}(\lambda)u_{c,i} \lambda^{-1/2}D_{c,i}^{(0)} 
\]

6. Transfer cocycles and generic nilflows

In this section we describe a transfer cocycle adapted to generic nilflows on Heisenberg nilmanifolds. We refer to the paper [FF06] and to the survey [F14] for additional details.

The deformation of Heisenberg structures on a Heisenberg nilmanifold $M = \Gamma \setminus \mathbb{N}$ is the space $\mathcal{D}_M$ of all Heisenberg frames $\{X, Y, Z\}$, that is, all frames such that
\[
[X, Y] = Z, \quad [X, Z] = [Y, Z] = 0 
\]
It can be proved that there exists an isomorphism between $\mathcal{D}_M$ and the the group $\text{Aut}_Z(\mathbb{N})$ of automorphisms of $\mathbb{N}$ which fix the central vector field $Z$. The automorphism is not canonical, since it depends on the choice of a base point $\{X_0, Y_0, Z\}$. Given any $a \in \text{Aut}_Z(\mathbb{N})$ the frame $\{a_s(x_0), a_s(y_0), a_s(z)\}$ is a Heisenberg frame, and the map
\[
a \to \{a_s(x_0), a_s(y_0), a_s(z)\} = \{a_s(x_0), a_s(y_0), Z\}
\]
defines an isomorphism of $\text{Aut}_Z(\mathbb{N})$ onto $\mathcal{D}_M$. The subgroup of coordinate changes is the subgroup $\text{Aut}_T(\mathbb{N}) < \text{Aut}_Z(\mathbb{N})$ of automorphisms which also fix the lattice $\Gamma$. Each element of $\text{Aut}_T(\mathbb{N})$ induces a diffeomorphism on $M$.

The group $\text{Aut}_Z(\mathbb{N})$ acts on itself by right or left multiplication. The moduli space of Heisenberg frames on $M$ is the space $\mathcal{M} := \text{Aut}_T(\mathbb{N})/\text{Aut}_Z(\mathbb{N})$. We note that $\text{Aut}_Z(\mathbb{N})$ is isomorphic to a $SL(2, \mathbb{R}) \times \mathbb{R}^2$ and $\text{Aut}_T(\mathbb{N})$ to a finite index subgroup of $SL(2, \mathbb{Z}) \times \mathbb{Z}^2$, so that $\mathcal{M}$ is isomorphic to a toral bundle (with fiber $\mathbb{T}^2$) over a finite cover of $SL(2, \mathbb{Z}) \setminus SL(2, \mathbb{R})$, the unit tangent bundle of the modular surface.

The renormalization flow of Heisenberg nilflows is defined as follows: let $a_{\mathbb{R}}$ denote the one-parameter group defined as follows:
\[
a_t(X_0, Y_0, Z) = (e^{tX_0}, e^{-tY_0}, Z), \quad \text{for all } t \in \mathbb{R}. 
\]
The group $a_{\mathbb{R}}$ acts on $\text{Aut}_Z(\mathbb{N})$, hence on the moduli space $\mathcal{M}$, by right multiplication. In terms, of Heisenberg triple the action of $a_{\mathbb{R}}$ can be described as follows:
\[
a_t(X, Y, Z) = (e^{tX}, e^{-tY}, Z), \quad \text{for all } \{X, Y, Z\} \in \mathcal{D} \text{ and } t \in \mathbb{R}. 
\]
In fact, if $\{X, Y, Z\} = a_s(X_0, Y_0, Z)$ then
\[
(aa_t)_s(X_0, Y_0, Z) = a_s(e^{tX_0}, e^{-tY_0}, Z) = (e^{tX}, e^{-tY}, Z). 
\]
We note that by definition the renormalization flow projects onto the hyperbolic geodesic flow (diagonal flow) on a finite cover of $SL(2, \mathbb{Z}) \setminus SL(2, \mathbb{R})$. It can be proved (see [FF06]) that $a_{\mathbb{R}}$ is an Anosov flow of the 5-dimensional moduli space. In addition, there is a one-to-one correspondence between periodic orbits of $a_{\mathbb{R}}$ and partially hyperbolic Heisenberg automorphisms, similar to the one between periodic orbits of the Teichmüller flow. In fact, if $\{X, Y, Z\} = a_s(X_0, Y_0, Z)$ is a periodic point for $a_{\mathbb{R}}$, there exists a $T > 0$ and an element $\Phi \in \text{Aut}_T(\mathbb{N})$ such that $aa_T = \Phi a$, hence
\[
\Phi(X, Y, Z) = (e^T X, e^{-T} Y, Z), 
\]
hence $\Phi$ induces a partially hyperbolic automorphism of $M$.

We define, for every $s \in \mathbb{R}$, a Sobolev bundle $W^s(M)$ over the moduli space $M$. The Sobolev bundle $W^s(M)$ is the projection to $M$ of the bundle over the deformation space $\mathcal{D}_M$ with fiber the Sobolev space $W^s_{X,Y,Z}(M)$ at every point $\{X,Y,Z\} \in \mathcal{D}_M$. The Sobolev space $W^s_{X,Y,Z}(M)$ is defined as the domain of the self-adjoint operator $(I -(X^2 + Y^2 + Z^2))^{s/2}$ endowed with the graph norm. For $s > 0$, we also define the sub-bundle $\mathcal{J}^s(M) \subset W^{-s}(M)$ of $X$-invariant distributions, with fiber

$$
\mathcal{J}^s_{X,Y,Z}(M) := \{ D \in \mathcal{J}^s_{X,Y,Z}(M) | \mathcal{L}_X D = 0 \}.
$$

We define the renormalization cocycle $\rho_{\mathbb{R}}$ on the bundle $W^s(M)$ over $M$ as the projection of the trivial cocycle on $W^s(M)$ over $\mathcal{D}_M$, that is, of the cocycle

$$
\text{Id} : W^s_{X,Y,Z}(M) \to W^s_{a_1(X,Y,Z)}(M),
$$
given by the identification of the vector spaces $W^s_{X,Y,Z}(M)$ and $W^s_{a_1(X,Y,Z)}(M)$.

Our main result in [FF06] is a statement on the Lyapunov spectrum of the restriction of the renormalization cocycle $\rho_{\mathbb{R}}$ to the bundle $\mathcal{J}^s(M)$.

**Theorem 6.1.** [FF06] For any $s > 1/2$, the Lyapunov spectrum of the cocycle $\rho_{\mathbb{R}}|\mathcal{J}^s(M)$ (with respect to any probability $a_\mathbb{R}$-invariant measure on the moduli space $M$) consists, in addition to the simple exponent 1, of the single Lyapunov exponent $1/2$ with infinite multiplicity.

The above theorem generalizes Lemma [5.2] from periodic orbits to all probability invariant measures. As a consequence of Theorem [6.1] we have the following result on the deviation of ergodic averages of nilflows.

**Corollary 6.2.** For almost all $\{X,Y,Z\}$ with respect to any $a_\mathbb{R}$-invariant measure the following holds. For any $s > 5/2$, and for any $\varepsilon > 0$, there exists a constant $C_{X,Y,Z} > 0$ such that, for any $f \in W^s(M)$ of zero average, for any $(x,t) \in M \times \mathbb{R}_+$,

$$
\left| \int_0^T f \circ \phi_t^X(x) \, dt \right| \leq C_{X,Y,Z} \| f \|_s T^{1/2+\varepsilon}.
$$

In the spirit of these notes, we can derive from Theorem [6.1] a result on the Lyapunov spectrum of $\rho_{\mathbb{R}}$ on the bundle $\mathcal{J}^s_{M}$ of iterated invariant distribution, for all $k \in \mathbb{N}$, and on the bundle $\mathcal{J}^\infty_{M}$. For instance, let $\mathcal{J}^\infty_{M}$ denote the subbundle with fiber

$$
\mathcal{J}^\infty_{M} = \bigcup_{k \in \mathbb{N}} \{ D \in \mathcal{J}^\infty(M) | X^k D = 0 \}.
$$

**Theorem 6.3.** The Lyapunov spectrum of the cocycle $\rho_{\mathbb{R}}|\mathcal{J}^\infty_{M}$ (with respect to any probability $a_\mathbb{R}$-invariant measure on the moduli space $M$) consists of the set

$$
\{1\} \cup \{1/2 - k | k \in \mathbb{N}\}.
$$

The Lyapunov exponent 1 is simple and corresponds to the subbundle of $\mathcal{J}^\infty_{M}$ given by the invariant volume on $M$, while all the Lyapunov exponents $1/2 - k$ have infinite multiplicity.

The above theorem generalizes Theorem [5.3] from periodic orbits to all probability invariant measures and can be derived from Theorem [6.1] by a similar argument.

We conclude with the following

**Problem 6.4.** Prove a version of the above theorem for a transfer cocycle over a bundle of anisotropic currents over the renormalization flow on the moduli space of Heisenberg manifolds.
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Department of Mathematics, University of Maryland, College Park, MD USA

E-mail address: gforni@math.umd.edu