Development of a unified high-order nonhydrostatic multi-moment constrained finite volume dynamical core: derivation of flux-form governing equations in the general curvilinear coordinate system

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Abstract

In the manuscript we have derived the flux-form atmospheric governing equations in the general curvilinear coordinate system which is used by a high-order nonhydrostatic multi-moment constrained finite volume (MCV) dynamical core, and given the explicit formulations in the shallow-atmosphere approximation. In general curvilinear coordinate $x(i = 1, 2, 3)$, unlike the Cartesian coordinate, the base vectors are not constants either in magnitude or direction. Following the representations such as base vectors, vector and tensor and so on in general curvilinear coordinate, we can obtain the differential relations of base vectors, the gradient and divergence operator etc. which are the component parts of the atmospheric governing equation. Then we apply them in the two specific curvilinear coordinate system: the spherical polar and cubed-sphere coordinates that are adopted in high-order nonhydrostatic MCV dynamical core. By switching the geometrics such as the metric tensors (covariant and contravariant), Jacobian of the transformation, the Christoffel symbol of the second kind between the spherical polar and cubed-sphere coordinates, the resulting flux-form governing equations in the specific coordinate system can be easily achieved. Of course, the Cartesian coordinate can be recovered. Noted that the projection metric tensors like spherical polar system and Cartesian coordinate become simple due to orthogonal properties of coordinate.

Keywords: flux-form governing equations, general curvilinear coordinate system, atmospheric governing equations

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1. 3D compressible non-hydrostatic Euler equation set

An inviscid, no-heat conducting fluid in 3-dimensional motion is governed by the local conservation of mass density $\rho$, momentum density $\rho\mathbf{u}$ and potential temperature $\theta$. The geometric form of these equation on the rotating Earth (angular velocity $\Omega$), independent of any coordinate basis, can be written as

$$\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ \rho \mathbf{u} \end{pmatrix} + \nabla \cdot \begin{pmatrix} \rho \mathbf{u} \\ \rho \mathbf{T} \end{pmatrix} = \begin{pmatrix} 0 \\ -\rho \nabla \Phi - 2 \Omega \times \mathbf{u} \end{pmatrix}$$

(1)

where the momentum tensor is

$$\mathbf{T} = \rho \mathbf{u} \otimes \mathbf{u} + G \rho \mathbf{p}$$

(2)

$\mathbf{G}$ is metric tensor due to coordinate transformation, and potential $\Phi$ satisfies the Poisson equation $\Delta \Phi = 4\pi g_0 \rho$, and $g_0$ is the universal gravitational constant.

More specifically the momentum tensor could be rewritten as

$$\mathbf{T}^i = T^i_{\alpha \beta} a^\alpha a^\beta$$

(3)

$$T^i_{\alpha \beta} = \rho u^i + G^i_{\alpha \beta} p$$

(4)

where $a_\alpha$ and $a_\beta$ are the covariant base vectors, $T^i_{\alpha \beta}$ is the contravariant components of $\mathbf{T}$, $G^i_{\alpha \beta}$ is the contravariant metric in the curvilinear coordinates and the indices $i$ and $j$ span by (1, 2, 3) or $(\xi, \eta, r)$ of the coordinate line $(x^1, x^2, x^3)$.

2. The nonhydrostatic governing equations in general curvilinear form

Based on the representations in appendix Appendix A, we now can express the governing equations (1) in general curvilinear form as

$$\frac{\partial \rho}{\partial t} + \frac{1}{\sqrt{\mathbf{G}}} \left[ \frac{\partial (\sqrt{\mathbf{G}} \rho u^i)}{\partial x^i} \right] = 0,$$

(5)

$$\frac{\partial \rho u^i}{\partial t} + \frac{1}{\sqrt{\mathbf{G}}} \frac{\partial}{\partial x^j} \left[ \sqrt{\mathbf{G}} (\rho u^i u^j + G^i_{\alpha \beta} p) \right] + \Gamma^i_{jk} (\rho u^j u^k + G^j_{\alpha \beta} p) = F_c - \rho g G^{i3},$$

(6)

$$\frac{\partial \rho \theta}{\partial t} + \frac{1}{\sqrt{\mathbf{G}}} \left[ \frac{\partial (\sqrt{\mathbf{G}} \rho \theta)}{\partial x^i} \right] = 0,$$

(7)

$$\frac{\partial \rho q_k}{\partial t} + \frac{1}{\sqrt{\mathbf{G}}} \left[ \frac{\partial (\sqrt{\mathbf{G}} \rho q_k u^i)}{\partial x^i} \right] = 0,$$

(8)

where $u^i$ is contravariant velocity in curvilinear coordinates, $G_{ij}$ being fundamental metric tensor, $\sqrt{\mathbf{G}} = \det(G_{ij})^{1/2}$ is the Jacobian of the transformation, $G^{ij} = G_{ij}^t$, and $i, j, k \in (1, 2, 3)$ or $(\xi, \eta, r)$, $q_k$ is the moisture species. The Christoffel symbol of the second kind $\Gamma^i_{jk}$, namely contravariant derivative of the covariant basis, is

$$\Gamma^i_{jk} = \frac{1}{2} G^{im} \left[ \frac{\partial G_{km}}{\partial x^j} + \frac{\partial G_{jm}}{\partial x^k} - \frac{\partial G_{jk}}{\partial x^m} \right].$$

(9)

the divergence of a tensor has the form in the curvilinear coordinates

$$\text{div} \mathbf{T} = \left( \frac{1}{\sqrt{\mathbf{G}}} \frac{\partial}{\partial x^j} \left( \sqrt{\mathbf{G}} T^{ij} \right) + \Gamma^i_{mk} T^{jk} \right) a^i.$$

(10)
Noted that

\[ \Gamma_{mk}^{i} T^{mk} = \sum_{m=1}^{3} \sum_{k=1}^{3} \Gamma_{mk}^{i} T^{mk} \]  

(11)

In the appendix Appendix A, we have given the representation in the curvilinear coordinates so that the knowledges of the curvilinear coordinates are defaulted to be known in the following. In the appendix Appendix B, the geometric formulations such as metric tensor, Christoffel symbol etc. in the spherical polar and cubed-sphere coordinates are presented. Here something is to be cleared.

• Coriolis force

The Coriolis force is defined by

\[ F_{C} = -2\Omega \times \rho u \]  

(12)

where \( \Omega \) is the Earth angular velocity pointing from the Earth’s origin to pole. It used to adopt the contravariant component to express the vector, thus

\[ \Omega = \omega^{1}a_{1} + \omega^{2}a_{2} + \omega^{3}a_{3} \]

(13)

\[ = \sum_{i=1}^{3} \omega^{i} a_{i} = \omega^{i} a_{i} \]  

(Summation used)

\[ = \Omega \cos \phi e_{\phi} + \Omega \sin \phi e_{r} \]  

(in the spherical coordinate).

In fact, due to the same covariant base vector in the radical direction of sphere (as in Eq. (B4) and (B26)), it reads

\[ \omega^{3} = \Omega \sin \phi \]  

(14)

The Coriolis force now has the form

\[ -2\Omega \times \rho u = -2\rho \omega^{i} u^{i} (a_{i} \times a_{k}) \]

(15)

\[ = -2\rho \omega^{i} u^{i} e_{ij} a^{j} \]  

using Eq. (A.76)

(16)

\[ = -2\rho \sqrt{G} \begin{vmatrix} a^{1} & a^{2} & a^{3} \\ \omega^{1} & \omega^{2} & \omega^{3} \\ u^{1} & u^{2} & u^{3} \end{vmatrix} \]

(17)

\[ = -2\rho \sqrt{G} \left[ (\omega^{2}u^{3} - \omega^{3}u^{2})a^{1} - (\omega^{3}u^{1} - \omega^{1}u^{3})a^{2} + (\omega^{1}u^{2} - \omega^{2}u^{1})a^{3} \right] \]  

using Eq. (A.24)

(18)

\[ = -2\rho \sqrt{G} \left[(\omega^{2}u^{3} - \omega^{3}u^{2})G^{1i}a_{i} - (\omega^{3}u^{1} - \omega^{1}u^{3})G^{2i}a_{i} + (\omega^{1}u^{2} - \omega^{2}u^{1})G^{3i}a_{i} \right] \]

(19)

\[ = -2\rho \sqrt{G} \left[(\omega^{2}u^{3} - \omega^{3}u^{2})G^{1i} - (\omega^{3}u^{1} - \omega^{1}u^{3})G^{2i} + (\omega^{1}u^{2} - \omega^{2}u^{1})G^{3i} \right] a_{i} \]  

(Summation used)

(20)

\[ = F_{C}^{i} a_{i} \]

(21)

where

\[ F_{C}^{i} = -2\rho \sqrt{G} \left[(\omega^{2}u^{3} - \omega^{3}u^{2})G^{1i} - (\omega^{3}u^{1} - \omega^{1}u^{3})G^{2i} + (\omega^{1}u^{2} - \omega^{2}u^{1})G^{3i} \right] \]

(22)

When the shallow-atmosphere approximations are made, \( \Omega \approx \omega^{3} a_{3} = (f/2)a_{3} \) (where \( f = 2\Omega \sin \phi \) called Coriolis parameter) due to only maintaining the vertical component projection of angular velocity \( \Omega \) where it
is necessary condition to conserve the energy \((f_c = 2\Omega \cos \varphi \text{ is dropped})\) \cite{5}. In this case the Coriolis force becomes

\[
F_C^i = \rho f \sqrt{G} (-u^i G^{2j} + u^2 G^{ij})
\]  
(23)
due to \(\omega^1 = \omega^2 = 0\).

- Gravity force
The gravitational source term of the momentum equation has the generic form

\[
F_G = -\rho g \left( \frac{R^2}{r^3} \right) \mathbf{e}_r
\]  
(24)
with radial base vector \(\mathbf{e}_r\) in the spherical coordinates and \(R\) is the Earth radius. Noted that \(a_3 = \mathbf{e}_r\) for radial base vector in the curvilinear coordinates. In the shallow-atmosphere approximations, the gravitational source term yields

\[
F_G = -\rho g \mathbf{e}_r
\]  
(25)

2.1. Splitting of reference state
As commonly applied in atmospheric models, the thermodynamic variables are split into a reference state and deviations. The reference state satisfies the stratification balance, i.e the hydrostatic relation in the vertical direction \(\text{(z)}\). The thermodynamic variables are then written as

\[
\rho(x, t) = \bar{\rho}(x) + \rho'(x, t)
\]  
(26)
\[
p(x, t) = \bar{p}(x) + p'(x, t)
\]  
(27)
\[
(\rho\theta)(x, t) = (\bar{\rho}\theta)(x) + (\rho\theta)'(x, t)
\]  
(28)
where the reference pressure \(\bar{p}(r)\) and density \(\bar{\rho}(r)\) are in local hydrostatic balance,

\[
\frac{\partial \bar{p}}{\partial r} = -\bar{\rho} g.
\]  
(29)

The nonhydrostatic governing equations (Eq. (5)-(8)) with perturbation variables have the form

\[
\frac{\partial \rho'}{\partial t} + \frac{1}{\sqrt{G}} \left[ \frac{\partial (\sqrt{G} \rho u^i)}{\partial x^i} \right] = 0,
\]  
(30)
\[
\frac{\partial \rho' u^i}{\partial t} + \frac{1}{\sqrt{G}} \left[ \frac{\partial (\sqrt{G} \rho u^i u^j)}{\partial x^j} \right] = F_H^i + F_M^i + F_C^i - \rho' g G^{2i},
\]  
(31)
\[
\frac{\partial (\rho\theta)'}{\partial t} + \frac{1}{\sqrt{G}} \left[ \frac{\partial (\sqrt{G} \rho\theta u^i)}{\partial x^i} \right] = 0,
\]  
(32)
\[
\frac{\partial \rho q_k'}{\partial t} + \frac{1}{\sqrt{G}} \left[ \frac{\partial (\sqrt{G} \rho q_k u^i)}{\partial x^i} \right] = 0,
\]  
(33)
where

\[
F_H^i = -G^{ij} \frac{\partial \bar{p}}{\partial x^j}
\]  
(34)
\[
F_M^i = -\Gamma_{\rho}^{ijk} (\rho u^i u^k + G^{jk} p')
\]  
(35)
is the horizontal variation of the hydrostatic background pressure and the source term due to curvilinear geometry by perturbation pressure, respectively. In Eq. (34), the nonconservative form of gradient of the hydrostatic background pressure.
pressure is used while with the conservative form of gradient used for the perturbation pressure, implying from Eq. (1) that

\[
\text{grad } p = \text{grad } (\mathcal{P} + p') = \text{grad } \mathcal{P} + \text{grad } p' = \left\{ \left[ G^i \frac{\partial \mathcal{P}}{\partial x^i} \right]_{\text{nonconservative}} + \left[ \frac{1}{\sqrt{G}} \partial \left( \sqrt{G} G^i p' \right) \right]_{\text{conservative}} \right\} a_i. \tag{36}
\]

We rewrite the momentum equation in the component form as

\[
\frac{\partial u_i}{\partial t} + \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^j} \left[ \sqrt{G} (\rho u_i u^j + G^{ij} p') \right] = F^i_H + F^i_M + F^i_C \tag{37}
\]

\[
\frac{\partial u_2}{\partial t} + \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^1} \left[ \sqrt{G} (\rho u_2 u^1 + G^{12} p') \right] = F^2_H + F^2_M + F^2_C \tag{38}
\]

\[
\frac{\partial u_3}{\partial t} + \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^1} \left[ \sqrt{G} (\rho u_3 u^1 + G^{13} p') \right] = F^3_H + F^3_M + F^3_C - \rho' g \left( \frac{R^2}{c^2} \right) G^{33} \tag{39}
\]

Note that the linearization of EOS is adopted so that \( p' = \epsilon_0 (\rho \theta)' \) where \( \epsilon_0 = R_d P_0 \). Let \( r = R + z \) (\( R \) is the Earth radius) and \( dx^3 = dr = dz \) where \( z \) is the geometry altitude, so the coordinate axes are \((x^1, x^2, x^3) = (\xi, \eta, z)\). Also all \( r \) in the geometric tensors in the appendix Appendix A and Appendix B are replaced by the constant \( R \). The nonhydrostatic equation of shallow-atmosphere approximation can be recast into

\[
\frac{\partial \theta'}{\partial t} + \frac{1}{\sqrt{G}} \left[ \frac{\partial}{\partial x^j} \left( \sqrt{G} \rho u^j \right) \right] = 0, \tag{40}
\]

\[
\frac{\partial u_1}{\partial t} + \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^j} \left[ \sqrt{G} (\rho u_1 u^j + G^{1j} p') \right] = F^1_H + F^1_M + \rho f \sqrt{G} (-u_1^2 G^{23} + u_2^2 G^{11}) \tag{41}
\]

\[
\frac{\partial u_2}{\partial t} + \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^1} \left[ \sqrt{G} (\rho u_2 u^1 + G^{21} p') \right] = F^2_H + F^2_M + \rho f \sqrt{G} (-u_1^2 G^{22} + u_2^2 G^{12}) \tag{42}
\]

\[
\frac{\partial u_3}{\partial t} + \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^1} \left[ \sqrt{G} (\rho u_3 u^1 + G^{13} p') \right] = -\rho' g \tag{43}
\]

\[
\frac{\partial (\rho \theta')}{\partial t} + \frac{1}{\sqrt{G}} \left[ \frac{\partial}{\partial x^j} \left( \sqrt{G} \rho \theta u^j \right) \right] = 0, \tag{44}
\]

\[
\frac{\partial p_{ij}}{\partial t} + \frac{1}{\sqrt{G}} \left[ \frac{\partial}{\partial x^j} \left( \sqrt{G} p_{ij} u^j \right) \right] = 0, \tag{45}
\]

where

\[
F_M^{(1, 2)} = \begin{pmatrix}
-\Gamma_1^{i} \rho u_i u^k \\
-\Gamma_2^{i} \rho u_i u^k
\end{pmatrix} \quad \text{for the cubed-sphere coordinates} \tag{46}
\]

\[
= \begin{pmatrix}
-\Gamma_1^{i} \rho u_i u^k \\
-\Gamma_2^{i} \rho u_i u^k - p' \tan \varphi / R^2
\end{pmatrix} \quad \text{for the spherical polar coordinates} \tag{47}
\]

and

\[
F_H^{(1, 2)} = \begin{pmatrix}
-G^{11} \frac{\partial \theta}{\partial x^1} + G^{12} \frac{\partial \theta}{\partial x^2} \\
-G^{21} \frac{\partial \theta}{\partial x^1} + G^{22} \frac{\partial \theta}{\partial x^2}
\end{pmatrix} \tag{48}
\]
Now we obtain the specific equations of the shallow-atmosphere approximation in the general curvilinear coordinates for the spherical polar and cubed-sphere systems:

1. **Mass equation**

\[
\frac{\partial \rho'}{\partial t} + \frac{1}{\sqrt{G}} \left[ \frac{\partial (\sqrt{G} \rho' u^\xi)}{\partial \xi} + \frac{\partial (\sqrt{G} \rho' u^\eta)}{\partial \eta} + \frac{\partial (\sqrt{G} \rho' w)}{\partial z} \right] = 0
\]  

(49)

2. **\(u^\xi\)-momentum equation**

\[
\frac{\partial \rho' u^\xi}{\partial t} + \frac{1}{\sqrt{G}} \left\{ \frac{\partial}{\partial \xi} \left[ \sqrt{G} (\rho' u^\xi u^\xi + G^{11} p') \right] + \frac{\partial}{\partial \eta} \left[ \sqrt{G} (\rho' u^\xi u^\eta + G^{12} p') \right] + \frac{\partial}{\partial z} \left[ \sqrt{G} (\rho' w) \right] \right\} = - \left( G^{11} \frac{\partial p}{\partial \xi} + G^{12} \frac{\partial p}{\partial \eta} \right) - \left( \Gamma_{11}^2 \rho' u^\xi + 2 \Gamma_{12}^2 \rho' u^\eta + \Gamma_{22}^2 \rho' u^\eta + \delta_p \right) + \rho f \sqrt{G} (-u^\xi G^{21} + u^\eta G^{11})
\]

(50)

3. **\(u^\eta\)-momentum equation**

\[
\frac{\partial \rho' u^\eta}{\partial t} + \frac{1}{\sqrt{G}} \left\{ \frac{\partial}{\partial \xi} \left[ \sqrt{G} (\rho' u^\xi u^\eta + G^{12} p') \right] + \frac{\partial}{\partial \eta} \left[ \sqrt{G} (\rho' u^\eta u^\eta + G^{22} p') \right] + \frac{\partial}{\partial z} \left[ \sqrt{G} (\rho' w) \right] \right\} = - \left( G^{21} \frac{\partial p}{\partial \xi} + G^{22} \frac{\partial p}{\partial \eta} \right) - \left( \Gamma_{11}^2 \rho' u^\eta + 2 \Gamma_{12}^2 \rho' u^\xi + \Gamma_{22}^2 \rho' u^\xi + \delta_p \right)
\]

(51)

4. **w-momentum equation**

\[
\frac{\partial \rho' w}{\partial t} + \frac{1}{\sqrt{G}} \left\{ \frac{\partial}{\partial \xi} \left[ \sqrt{G} (\rho' u^\xi w) \right] + \frac{\partial}{\partial \eta} \left[ \sqrt{G} (\rho' u^\eta w) \right] + \frac{\partial}{\partial z} \left[ \sqrt{G} (\rho' w^2) \right] \right\} = -\rho' g
\]

(52)

5. **Potential temperature equation**

\[
\frac{\partial (\rho' \theta')}{\partial t} + \frac{1}{\sqrt{G}} \left[ \frac{\partial (\sqrt{G} \rho' \theta' u^\xi)}{\partial \xi} + \frac{\partial (\sqrt{G} \rho' \theta' u^\eta)}{\partial \eta} + \frac{\partial (\sqrt{G} \rho' \theta' w)}{\partial z} \right] = 0
\]

(53)

6. **The kth-moisture/tracer equation**

\[
\frac{\partial (\rho' q_k)}{\partial t} + \frac{1}{\sqrt{G}} \left[ \frac{\partial (\sqrt{G} \rho' q_k u^\xi)}{\partial \xi} + \frac{\partial (\sqrt{G} \rho' q_k u^\eta)}{\partial \eta} + \frac{\partial (\sqrt{G} \rho' q_k w)}{\partial z} \right] = 0
\]

(54)

In the \(u^\theta\)-momentum equation \(\delta_p\) has the value

\[
\delta_p = \begin{cases} 
0, & \text{for the cubed-sphere coordinates} \\
\rho' \tan \varphi/R^2, & \text{for the spherical polar coordinates}
\end{cases}
\]

(55)

Here we have replaced the superscript indices \((1, 2, 3)\) by \((\xi, \eta, z)\) when representing the contravariant velocity and the coordinate axes \((x^1, x^2, x^3)\) by \((\xi, \eta, z)\). Obviously, the horizontal curvilinear coordinates become the spherical polar system when \((\xi, \eta) = (\lambda, \varphi)\) while they are the cubed-sphere coordinates if \((\xi, \eta) = (\alpha, \beta)\).

### 2.3 Governing equations with the effects of topography

In the presence of topography, the height-based terrain-following coordinate introduced by [2] is utilized to map the physical space \((x^1, x^2, z) = (\xi, \eta, z)\) into the computational domain \((x^1, x^2, \zeta) = (\xi, \eta, \zeta)\) via the transformation relationship \(\zeta = \zeta(x^1, x^2, z) = \zeta(\xi, \eta, z)\). Before the transformation of vertical coordinate, we should have the vision of separating the 3D curvilinear coordinates into 2D spherical/cubed-sphere surface coordinate \((\mathbf{a}_1, \mathbf{a}_2))\) and radial coordinate \((\mathbf{a}_3)\). For convenience, we denote the curvilinear geometric quantities as \((\zeta)\), and the geometric quantities of vertical coordinate transformation as \((\zeta)\). The total Jacobian of transformation can be derived in the following
manner: the first metric Jacobian associated with spherical/cubed-sphere transformation \((\xi, \eta, z)\) is \(\sqrt{G_\zeta}\); the second Jacobian with the vertical transformation \(z \rightarrow \zeta\) has \(\sqrt{G_v}\). Therefore, the final composite Jacobian of transformation is \(\sqrt{G} = \sqrt{G_\zeta}\sqrt{G_v}\). The conservation form by the chain rule \([1]\) can be expressed in the vertical coordinate transform as

\[
\sqrt{G_v} \left( \frac{\partial \rho}{\partial \xi} \right)_z = \frac{\partial}{\partial \xi} \left( \sqrt{G_v} \phi \right)_z + \frac{\partial}{\partial \zeta} \left( \sqrt{G_v} G_\xi^{13} \phi \right)_z + \frac{\partial}{\partial \eta} \left( \sqrt{G_v} G_\eta^{13} \phi \right)_z + \frac{\partial}{\partial \zeta} \left( \frac{\partial G_v}{\partial \zeta} \right)_z (56)
\]

\[
\sqrt{G_v} \left( \frac{\partial \rho}{\partial \eta} \right)_z = \frac{\partial}{\partial \eta} \left( \sqrt{G_v} \phi \right)_z + \frac{\partial}{\partial \zeta} \left( \sqrt{G_v} G_\eta^{23} \phi \right)_z + \frac{\partial}{\partial \zeta} \left( \frac{\partial G_v}{\partial \eta} \right)_z (57)
\]

\[
\sqrt{G_v} \left( \frac{\partial \rho}{\partial \zeta} \right)_z = \frac{\partial}{\partial \zeta} \left( \sqrt{G_v} \phi \right)_z + \frac{\partial}{\partial \zeta} \left( \sqrt{G_v} G_\zeta^{23} \phi \right)_z + \frac{\partial}{\partial \zeta} \left( \frac{\partial G_v}{\partial \zeta} \right)_z (58)
\]

where

\[
\sqrt{G_v} = \frac{\partial \zeta}{\partial \xi}, G_\xi^{13} = \frac{\partial \zeta}{\partial \xi}, G_\eta^{23} = \frac{\partial \zeta}{\partial \eta} (59)
\]

From the definition of the transformed vertical velocity, we have

\[
\frac{d\tilde{\zeta}}{dt} = \frac{d\zeta}{dt} + u \frac{d\zeta}{dx^1} + \eta \frac{d\zeta}{dx^2} + v \frac{d\zeta}{dx^3} (60)
\]

\[
= u \frac{d\zeta}{dx^1} + \eta \frac{d\zeta}{dx^2} + v \frac{d\zeta}{dx^3} (61)
\]

\[
= G_\xi^{13} u^\xi + G_\eta^{23} u^\eta + \frac{1}{\sqrt{G_v}} w (62)
\]

\[
= \frac{1}{\sqrt{G_v}} \left( w + \sqrt{G_v} G_\xi^{13} u^\xi + \sqrt{G_v} G_\eta^{23} u^\eta \right) (63)
\]

Now we obtain the transformed governing equation as follows

- **Mass equation**

\[
\frac{\partial \rho'}{\partial t} + \frac{1}{\sqrt{\rho'} \sqrt{G_v}} \left[ \frac{\partial \left( \sqrt{G_v} \rho' u^\xi \right)}{\partial \xi} + \frac{\partial \left( \sqrt{G_v} \rho' u^\eta \right)}{\partial \eta} + \frac{\partial \left( \sqrt{G_v} \rho' u^\zeta \right)}{\partial \zeta} \right] = 0 (64)
\]

- **\(u^\xi\)-momentum equation**

\[
\frac{\partial \rho u^\xi}{\partial t} + \frac{1}{\sqrt{\rho}} \left[ \frac{\partial \left( \sqrt{\rho} u^\xi u^\xi \right)}{\partial \xi} + \frac{\partial \left( \sqrt{\rho} u^\eta u^\xi \right)}{\partial \eta} + \frac{\partial \left( \sqrt{\rho} u^\zeta u^\xi \right)}{\partial \zeta} \right] = F_\xi^1 + F_\xi^2 + \rho f \sqrt{G_v} (-u^\xi G_\xi^{11} + u^\eta G_\xi^{12}) (65)
\]

- **\(u^\eta\)-momentum equation**

\[
\frac{\partial \rho u^\eta}{\partial t} + \frac{1}{\sqrt{\rho}} \left[ \frac{\partial \left( \sqrt{\rho} u^\eta u^\eta \right)}{\partial \xi} + \frac{\partial \left( \sqrt{\rho} u^\eta u^\eta \right)}{\partial \eta} + \frac{\partial \left( \sqrt{\rho} u^\zeta u^\eta \right)}{\partial \zeta} \right] = F_\eta^1 + F_\eta^2 + \rho f \sqrt{G_v} (-u^\zeta G_\xi^{21} + u^\eta G_\xi^{22}) (66)
\]

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• w-momentum equation

\[
\frac{\partial p}{\partial t} + \frac{1}{\sqrt{\gamma}} \left[ \frac{\partial}{\partial \xi} \left( \sqrt{G} \rho w u^\xi \right) + \frac{\partial}{\partial \eta} \left( \sqrt{G} \rho w u^\eta \right) + \frac{\partial}{\partial \zeta} \left( \sqrt{G} \rho w u^\zeta \right) \right] = -p' g \tag{67}
\]

• Potential temperature equation

\[
\frac{\partial (\rho \theta)^v}{\partial t} + \frac{1}{\sqrt{\gamma}} \left[ \frac{\partial}{\partial \xi} \left( \sqrt{G} \rho \theta u^\xi \right) + \frac{\partial}{\partial \eta} \left( \sqrt{G} \rho \theta u^\eta \right) + \frac{\partial}{\partial \zeta} \left( \sqrt{G} \rho \theta u^\zeta \right) \right] = 0 \tag{68}
\]

• The kth-moisture/tracer equation

\[
\frac{\partial (\rho q_k)}{\partial t} + \frac{1}{\sqrt{\gamma}} \left[ \frac{\partial}{\partial \xi} \left( \sqrt{G} \rho q_k u^\xi \right) + \frac{\partial}{\partial \eta} \left( \sqrt{G} \rho q_k u^\eta \right) + \frac{\partial}{\partial \zeta} \left( \sqrt{G} \rho q_k u^\zeta \right) \right] = 0 \tag{69}
\]

where

\[
F^1_M = - \left( \Gamma^1_{11} \rho u^\xi \theta' + 2 \Gamma^1_{12} \rho u^\xi \theta^\eta + \Gamma^1_{13} \rho u^\xi \theta^\zeta \right) \tag{70}
\]

\[
F^2_M = - \left( \Gamma^2_{11} \rho u^\xi \theta' + 2 \Gamma^2_{12} \rho u^\xi \theta^\eta + \Gamma^2_{13} \rho u^\xi \theta^\zeta + \delta_p \right) \tag{71}
\]

The above nonhydrostatic governing equations can be written in the compact flux form

\[
\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{q})}{\partial \xi} + \frac{\partial \mathbf{g}(\mathbf{q})}{\partial \eta} + \frac{\partial \mathbf{h}(\mathbf{q})}{\partial \zeta} = \mathbf{S}(\mathbf{q}) \tag{74}
\]

where

\[
\mathbf{q} = (\sqrt{G} p', \sqrt{G} p\theta, \sqrt{G} p\theta^\eta, \sqrt{G} p\theta^\zeta, \sqrt{G} p\theta^\xi, \sqrt{G} p q_k)^T \tag{75}
\]

\[
\mathbf{f}(\mathbf{q}) = \begin{pmatrix}
\sqrt{G} p u^\xi \\
\sqrt{G} p u^\eta + G^{11} p' \\
\sqrt{G} p u^\zeta + G^{12} p' \\
\sqrt{G} p u^\xi + G^{21} p' \\
\sqrt{G} p u^\eta \\
\sqrt{G} p u^\zeta \\
\end{pmatrix} \tag{76}
\]

\[
\mathbf{g}(\mathbf{q}) = \begin{pmatrix}
\sqrt{G} p u^x \\
\sqrt{G} p u^\eta + G^{12} p' \\
\sqrt{G} p u^\zeta + G^{22} p' \\
\sqrt{G} p u^\xi + G^{11} p' \\
\sqrt{G} p u^\eta \\
\sqrt{G} p u^\zeta \\
\end{pmatrix} \tag{77}
\]
and

\[
\mathbf{h}(q) = \left(\begin{array}{c}
\sqrt{G_p \phi_w} \\
\sqrt{G_p \phi_w} \\
\sqrt{G_p \phi_w} \\
\sqrt{G_p \phi_w} \\
\sqrt{G_p \phi_w}
\end{array}\right)
\]

(78)

\[
\mathbf{S}(q) = \sqrt{G} \left(\begin{array}{c}
F^1_{h} + F^1_{M} + \rho f \sqrt{G}, (-u^2 G^2_c + u^2 G^1_c) \\
F^2_{h} + F^2_{M} + \rho f \sqrt{G}, (-u^2 G^2_c + u^2 G^1_c) \\
-p' g \\
0 \\
0
\end{array}\right)
\]

(79)

2.4. Flux Jacobian

The flux Jacobian in the x-direction is given by

\[
\mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{q}} = \left(\begin{array}{cccccc}
0 & 1 & 0 & 0 & 0 & 0 \\
-u^2 u^c & 2u^c & 0 & 0 & G^1_{c1} \epsilon_0 & 0 \\
-u^2 u^0 & u^0 & u^c & 0 & G^2_{c2} \epsilon_0 & 0 \\
-u^2 w & w & 0 & u^c & 0 & 0 \\
-u^2 \theta & \theta & 0 & 0 & 0 & u^c
\end{array}\right).
\]

(80)

The eigenvalues are \((u^c - \sqrt{G^1_{c1}} a, u^c, u^c, u^c, u^c + \sqrt{G^2_{c2}} a)\) where \(a = \sqrt{\epsilon_0 \mu}\) is sound speed, and the corresponding right eigenvectors \(\mathbf{R}_x\) and left eigenvectors \(\mathbf{L}_x\), respectively, are

\[
\mathbf{R}_x = \left(\begin{array}{cccccc}
1 & 0 & 0 & 0 & 1 & 1 \\
\sqrt{G^2_{c2}} a - G^1_{c1} a & 0 & 1 & 0 & \sqrt{G^2_{c2}} a & 0 \\
w & 0 & 1 & 0 & w & 0 \\
\theta & 0 & 0 & 0 & \theta & 0
\end{array}\right).
\]

(81)

and

\[
\mathbf{L}_x = \mathbf{R}_x^{-1} = \left(\begin{array}{cccccc}
u^c & -1 & 0 & 0 & 1 & -1 \\
2 \sqrt{G^2_{c2}} a & -2 \sqrt{G^2_{c2}} a & 0 & 0 & 1 & -1 \\
1 & 0 & 0 & 0 & -1 & 0 \\
\frac{G^2_{c2} \phi_w}{\epsilon_0} & -\frac{G^2_{c2} \phi_w}{\epsilon_0} & 0 & 0 & 1 & -1 \\
0 & 0 & 1 & 0 & -1 & 0 \\
-\frac{u^c}{2 \sqrt{G^2_{c2}} a} & \frac{1}{2 \sqrt{G^2_{c2}} a} & 0 & 0 & 0 & -1
\end{array}\right).
\]

(82)

The flux Jacobian in the y-direction reads

\[
\mathbf{B} = \frac{\partial \mathbf{g}}{\partial \mathbf{q}} = \left(\begin{array}{cccccc}
0 & 0 & 1 & 0 & 0 & 0 \\
-u^2 u^0 & u^0 & u^c & 0 & G^1_{c2} \epsilon_0 & 0 \\
-u^2 u^0 & 0 & 2u^0 & 0 & G^2_{c2} \epsilon_0 & 0 \\
-u^2 w & w & u^0 & 0 & 0 & 0 \\
-u^2 \theta & \theta & 0 & u^0 & 0 & 0
\end{array}\right)
\]

(83)
The eigenvalues are \((u^0 - \sqrt{G_{\ell}^2}a, u^0, u^0, u^0, u^0 + \sqrt{G_{\ell}^2}a)\) where \(a = \sqrt{\epsilon_0}b\) is sound speed, and the corresponding right eigenvectors \(R_r\) and left eigenvectors \(L_r\), respectively, are

\[
R_r = \begin{pmatrix}
1 & 0 & 1 & 0 & 1 \\
\sqrt{G_{\ell}^2} - G_{\ell}^1 & 0 & 0 & 1 & \sqrt{G_{\ell}^2} + G_{\ell}^1 \\
u^0 - \sqrt{G_{\ell}^2}a & 0 & u^0 & 0 & u^0 + \sqrt{G_{\ell}^2}a \\
\theta & 0 & 0 & 0 & \theta \\
\end{pmatrix}
\] (84)

and

\[
L_r = R_r^{-1} = \begin{pmatrix}
u^0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\
0 & 0 & 0 & 1 & -\frac{1}{2} \\
1 & 0 & 0 & 0 & -\frac{1}{2} \\
-\frac{1}{2} \sqrt{G_{\ell}^2} & 1 & G_{\ell}^{12} & 0 & -\frac{1}{2} \sqrt{G_{\ell}^2} \\
-\frac{1}{2} \sqrt{G_{\ell}^2} & 1 & G_{\ell}^{13} & 0 & -\frac{1}{2} \sqrt{G_{\ell}^2} \\
0 & 0 & 0 & 0 & \frac{1}{2} \\
\end{pmatrix}
\] (85)

The flux Jacobian in the \(\zeta\)-direction reads

\[
C = \frac{\partial h}{\partial q} = \begin{pmatrix} 0 & G_y e_0 & G_x e_0 & 0 & 0 \\
-\frac{1}{2} \sqrt{G_{\ell}^2} & \frac{1}{2} \sqrt{G_{\ell}^2} & \frac{1}{2} \sqrt{G_{\ell}^2} & \frac{1}{2} \sqrt{G_{\ell}^2} & G_y e_0 \\
-\frac{1}{2} \sqrt{G_{\ell}^2} & \frac{1}{2} \sqrt{G_{\ell}^2} & \frac{1}{2} \sqrt{G_{\ell}^2} & \frac{1}{2} \sqrt{G_{\ell}^2} & G_x e_0 \\
-\frac{1}{2} \sqrt{G_{\ell}^2} & \frac{1}{2} \sqrt{G_{\ell}^2} & \frac{1}{2} \sqrt{G_{\ell}^2} & \frac{1}{2} \sqrt{G_{\ell}^2} & 0 \end{pmatrix}
\] (86)

where \(G_{vX} = \sqrt{G_{l}} G_{vX}^{13}, G_{vY} = \sqrt{G_{l}} G_{vX}^{23}, G_{X} = G_{vX}^{13} G_{vY}^{11} + G_{vY}^{23} G_{vX}^{12}\) and \(G_{vY} = G_{vX}^{13} G_{vY}^{21} + G_{vY}^{23} G_{vX}^{22}\). The eigenvalues are \((\tilde{w}, \tilde{v}, \tilde{v}, -\mu \sqrt{\nu_0} a, \tilde{w} + \mu \sqrt{\nu_0} a)\) where \(a = \sqrt{\epsilon_0}b\) is sound speed and \(M = 1 + \sqrt{G_{x}} G_{vX} G_{X} + \sqrt{G_{y}} G_{vY} G_{Y}\). The corresponding right eigenvectors \(R_c\) and left eigenvectors \(L_c\), respectively, are

\[
R_c = \begin{pmatrix}
\frac{1}{\sqrt{G_{x} \tilde{w}}} & G_{y} \tilde{w} & G_{x} \tilde{w} & 1 & 1 \\
0 & 0 & 1 & \frac{1}{\sqrt{G_{x} \tilde{w}}} \\
0 & 1 & 0 & \frac{1}{\sqrt{G_{y} \tilde{v}}} \\
1 & 0 & 0 & \frac{1}{\sqrt{G_{x} \tilde{w}}} \\
0 & 0 & 0 & \frac{1}{\sqrt{G_{y} \tilde{v}}} \\
\end{pmatrix}
\] (87)

and

\[
L_c = R_c^{-1} = \begin{pmatrix}
\frac{\sqrt{G_{x} \tilde{w}}}{\sqrt{G_{x} \tilde{w}}} & -\frac{G_{y} \tilde{w} \sqrt{G_{x} \tilde{w}}}{\sqrt{G_{x} \tilde{w}}} & \frac{G_{x} \tilde{w} \sqrt{G_{x} \tilde{w}}}{\sqrt{G_{x} \tilde{w}}} & 1 & -\frac{\tilde{w}}{\sqrt{G_{x} \tilde{w}}} \\
-\frac{G_{y} \tilde{w} \sqrt{G_{x} \tilde{w}}}{\sqrt{G_{x} \tilde{w}}} & \frac{G_{x} \tilde{w} \sqrt{G_{x} \tilde{w}}}{\sqrt{G_{x} \tilde{w}}} & \frac{G_{y} \tilde{w} \sqrt{G_{x} \tilde{w}}}{\sqrt{G_{x} \tilde{w}}} & 1 & -\frac{\tilde{w}}{\sqrt{G_{x} \tilde{w}}} \\
\frac{G_{y} \tilde{w} \sqrt{G_{x} \tilde{w}}}{\sqrt{G_{x} \tilde{w}}} & \frac{G_{x} \tilde{w} \sqrt{G_{x} \tilde{w}}}{\sqrt{G_{x} \tilde{w}}} & \frac{G_{y} \tilde{w} \sqrt{G_{x} \tilde{w}}}{\sqrt{G_{x} \tilde{w}}} & 1 & -\frac{\tilde{w}}{\sqrt{G_{x} \tilde{w}}} \\
-\frac{G_{y} \tilde{w} \sqrt{G_{x} \tilde{w}}}{\sqrt{G_{x} \tilde{w}}} & -\frac{G_{x} \tilde{w} \sqrt{G_{x} \tilde{w}}}{\sqrt{G_{x} \tilde{w}}} & -\frac{G_{y} \tilde{w} \sqrt{G_{x} \tilde{w}}}{\sqrt{G_{x} \tilde{w}}} & 1 & -\frac{\tilde{w}}{\sqrt{G_{x} \tilde{w}}} \\
\frac{G_{y} \tilde{w} \sqrt{G_{x} \tilde{w}}}{\sqrt{G_{x} \tilde{w}}} & \frac{G_{x} \tilde{w} \sqrt{G_{x} \tilde{w}}}{\sqrt{G_{x} \tilde{w}}} & -\frac{G_{y} \tilde{w} \sqrt{G_{x} \tilde{w}}}{\sqrt{G_{x} \tilde{w}}} & 1 & -\frac{\tilde{w}}{\sqrt{G_{x} \tilde{w}}} \\
\end{pmatrix}
\] (88)

3. summary

Based on differential geometry approach, we have derived the flux-form atmospheric governing equation in the general curvilinear coordinate system which are currently utilized in a high-order nonhydrostatic MCV dynamical
core. The explicit flux-form atmospheric governing equations in the shallow-atmosphere approximation are given. In the unified MCV dynamical core framework, the horizontal curvilinear coordinates become the spherical polar system when \((\xi, \eta) = (\lambda, \varphi)\) where \(\lambda\) and \(\varphi\) represent the longitude and latitude directions, while they are the cubed-sphere coordinates if \((\xi, \eta) = (\alpha, \beta)\) where \(\alpha\) and \(\beta\) denote the cube coordinates. Of course, the coordinate system directly reduces to Cartesian coordinate when \((\xi, \eta) = (x, y)\) where \(x\) and \(y\) represent the natural coordinate. It is noted that the projection metric tensors like spherical polar system and Cartesian coordinate become simple due to orthogonal properties of coordinate. In the current nonhydrostatic MCV framework, it is easy to switch to one of the coordinate systems: spherical polar system, cubed-sphere system and Cartesian system by simply changing the projection relations. In addition, the flux Jacobian of the three coordinate system are given in this manuscript.
Appendix

Appendix A. Representation in curvilinear coordinates

In general curvilinear coordinate \( x^i(i = 1, 2, 3) \), unlike the Cartesian coordinate, the base vectors are not constants either in magnitude or direction. Here we use the standard practice to express the representation in curvilinear coordinates \([3, 4]\).

Appendix A.1. Base Vectors

- **Covariant base vectors** are defined by
  \[
  a_i = \frac{\partial r}{\partial x^i} \tag{A.1}
  \]

- **Contravariant base vectors** are defined by
  \[
  a^i = \nabla x^i \tag{A.2}
  \]

where \( \mathbf{r} \) is the position vector. They have the relation as follows

\[
 a^i \cdot a_j = \delta^i_j, \quad i, j = 1, 2, 3 \tag{A.3}
\]

where \( \delta^i_j \) is the Kronecker symbol

\[
 \delta^i_j = \begin{cases} 
  1 & \text{if} \quad i = j \\
  0 & \text{if} \quad i \neq j 
\end{cases} \tag{A.4}
\]

- **Scale Factors**

The covariant and contravariant base vectors are not unit vectors. In particular, consider the covariant base vectors and introduce the unit triad \( \hat{a}_i \), with \(|a_i| = \sqrt{\hat{a}_i \cdot \hat{a}_i} \)

\[
\hat{a}_i = \frac{a_i}{|a_i|} = \frac{a_i}{\sqrt{G_{ii}}} \quad \text{(no summation)} \tag{A.5}
\]

The lengths of the covariant base vectors are usually denoted by \( h \) and are called the scale factors

\[
h_i = |a_i| = \sqrt{G_{ii}} \quad \text{(no summation)} \tag{A.6}
\]

Appendix A.2. Vector and Tensor

- **In terms of the two base vectors**, a vector \( \mathbf{u} \) can now be expressed by using the summation convention as
  \[
  \mathbf{u} = u^i a_i, \tag{A.7}
  \]

where \( u^i \) and \( u_i \) are the so-called **contravariant** and **covariant** component of a vector.

- **Parallel and orthogonal projections**
  \[
  u_i = \mathbf{u} \cdot a_i, \tag{A.9}
  
  u^i = \mathbf{u} \cdot a^i. \tag{A.10}
  \]

From the viewing of projection, they are the parallel and orthogonal projections of a vector.
Similarly a second-order tensor $\mathbf{T}$ (or $\mathbf{T}^\leftrightarrow$) is now represented as

$$
\mathbf{T} = T_{ij} \mathbf{a}_i \mathbf{a}_j 
$$

(A.11)

$$
= T_{ij} a^i a^j 
$$

(A.12)

$$
= T^j_i a_i a^j 
$$

(A.13)

where $T_{ij}$ and $T^j_i$ are the contravariant and covariant components of $\mathbf{T}$, respectively, and the superscript index $i$ stands for the contravariant and subscript $j$ stands for the covariant nature of $T^j_i$ and $T_{ij}$.

- Physical Components of a Vector

The contravariant and covariant components of a vector do not have the same physical significance in a curvilinear coordinate system as they do in a rectangular Cartesian system; actually they often have different dimensions. For instance, the increment of a position vector $\mathbf{r}$ has the contravariant components $(dr, d\theta, dz)$ in cylindrical coordinates

$$
dr = dra_1 + d\theta a_2 + dz a_3 
$$

(A.14)

where $(x^1, x^2, x^3) = (r, \theta, z)$. Here, $d\theta$ does not have the same dimensions as the others. The physical components in this case are $(dr, r d\theta, dz)$.

The physical components $\tilde{u}^i$ of a vector $\mathbf{u}$ are defined to be the components along the covariant base vectors (and hence are obtained from the contravariant components), referred to unit vectors. Thus,

$$
\mathbf{u} = u^i \mathbf{a}_i 
$$

(A.15)

$$
= \sum_{i=1}^{3} u^i h_i \mathbf{\hat{a}}_i 
$$

(A.16)

$$
\equiv \tilde{u}^i \mathbf{\hat{a}}_i 
$$

(A.17)

and

$$
\tilde{u}^i = u^i h_i = u^i \sqrt{G_{ii}} \quad \text{(no summation)} 
$$

(A.18)

is called physical components of a vector.

Appendix A.3. Fundamental Metric Components

Using the two types of basis vectors, we can form the scalars

$$
G_{ij} = \mathbf{a}_i \cdot \mathbf{a}_j = \mathbf{a}_j \cdot \mathbf{a}_i 
$$

(A.19)

$$
G^{ij} = \mathbf{a}^i \cdot \mathbf{a}^j = \mathbf{a}^j \cdot \mathbf{a}^i 
$$

(A.20)

which are the fundamental metric components of the space in which the curvilinear coordinates have been introduced. The components $G_{ij}$ and $G^{ij}$ are the covariant and contravariant components, respectively, of a tensor, called the metric tensor and both symmetric. They have a important property

$$
G_{ij} G^{jk} = \delta^k_i, 
$$

(A.21)

where $\delta^k_i$ in tensor form is

$$
\delta^k_i = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}. 
$$

(A.22)
The following formulas, which are of great importance, can be established by

\[ a_i = G_{ij} a^j \]  
(A.23)
\[ a^i = G^{ij} a_j \]  
(A.24)
\[ u_i = G_{ik} u^k \]  
(A.25)
\[ u^i = G^{ik} u_k \]  
(A.26)

The determinant of the covariant metric tensor is denoted by

\[ G = \det(G_{ij}) \]  
(A.27)

Generally the Jacobian of transformation in the curvilinear coordinates is defined as \( \sqrt{G} \).

Appendix A.4. Elemental Displacement Vector

In a coordinate system \( x^i \), the position vector \( \mathbf{r} \) and its increment at any point can be written as

\[ d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial x^i} dx^i \]  
(A.28)
\[ = a_i dx^i \]  
(A.29)

The magnitude of \( d\mathbf{r} \), denoted as \( ds \), is defined by

\[ (ds)^2 = d\mathbf{r} \cdot d\mathbf{r} \]  
(A.30)
\[ = (a_i \cdot a_j) dx^i dx^j \]  
(A.31)
\[ = G_{ij} dx^i dx^j \]  
(A.32)

- Arc length element

An increment of arc length on a coordinate line along which \( x^i \) varied is given by

\[ ds^i = |a_i| dx^i \]  
(A.33)

- Surface area element

An increment of area on a coordinate surface of constant \( x^i \) is given by

\[ dS^i = |a_j \times a_k| dx^j dx^k \]  
(A.34)
\[ = \sqrt{G_{ij} G_{kk} - G_{jk}^2} dx^i dx^k \]  
(A.35)

- Volume element An increment of volume is given by

\[ dV = a_1 \cdot (a_2 \times a_3) dx^1 dx^2 dx^3 \]  
(A.36)
\[ = \sqrt{G} dx^1 dx^2 dx^3 \]  
(A.37)

Appendix A.5. Differentiation of Base Vectors

The operations of grad, curl, and div on vectors and tensors require a knowledge of partial derivatives of the base vectors in the next. Here some connections among the base vectors are presented.
The derivative of covariant base vector

The first one is

\[ \frac{\partial \mathbf{a}_i}{\partial x^j} = \frac{\partial}{\partial x^j} \left( \frac{\partial \mathbf{r}}{\partial x^i} \right) = \frac{\partial}{\partial x^i} \left( \frac{\partial \mathbf{r}}{\partial x^j} \right) = \frac{\partial \mathbf{a}_j}{\partial x^i} \]

(A.38)

(A.39)

Based on the definition of metric tensor, i.e., \( G_{ij} = \mathbf{a}_i \cdot \mathbf{a}_j \), we differentiate covariant metric tensor and get by mathematical operations

\[ \frac{\partial \mathbf{a}_i}{\partial x^j} \mathbf{a}_k = [i j, k] \]  

(A.40)

which implies

\[ \frac{\partial \mathbf{a}_i}{\partial x^j} = [i j, k] \mathbf{a}^k \]

(A.41)

where

\[ [i j, k] = \frac{1}{2} \left( \frac{\partial G_{ik}}{\partial x^j} + \frac{\partial G_{jk}}{\partial x^i} - \frac{\partial G_{ij}}{\partial x^k} \right) \]

(A.42)

are called Christoffel symbols of the first kind.

Both sides of Eq. (A.41) are multiplied scalarly by \( \mathbf{a}^m \) to get

\[ \frac{\partial \mathbf{a}_i}{\partial x^j} \mathbf{a}^m = \Gamma^m_{ij} \]

(A.43)

which implies

\[ \frac{\partial \mathbf{a}_i}{\partial x^j} = \Gamma^m_{ij} \mathbf{a}^m \]

(A.44)

where

\[ \Gamma^m_{ij} = G^{mk} [i j, k] \]

(A.45)

are called Christoffel symbols of the second kind. Sometimes it is also denoted by

\[ \Gamma^m_{ij} = \begin{cases} m & i \ j \ \end{cases} \]

(A.46)

The derivative of contravariant base vector

In a similar way, we obtain the derivative of contravariant base vector

\[ \frac{\partial \mathbf{a}^i}{\partial x^j} \mathbf{a}_j = -\Gamma^i_{jk} \]

(A.47)

which implies

\[ \frac{\partial \mathbf{a}^i}{\partial x^j} = -\Gamma^i_{jk} \mathbf{a}^j \]

(A.48)

Noted that the relations hold

\[ [i j, k] = [j i, k] \]

\[ \Gamma^i_{jk} = \Gamma^i_{kj} \]
• Useful expressions

When setting $m = i$ in the Eq. (A.45) and carrying out summation over the repeated indices, it reaches that

$$\Gamma^i_{ij} = \frac{1}{2G} \frac{\partial G}{\partial x^j} = \frac{\partial \ln \sqrt{G}}{\partial x^j}. \quad \text{(A.51)}$$

From Eq. (A.48) and Eq. (A.51), we have the formula

$$\frac{\partial}{\partial x^j} (\sqrt{G} a^i) = 0. \quad \text{(A.52)}$$

This identity is also derived by Thompson et al. (1985), c.f. their Eq. (40).

Appendix A.6. Gradient operator

In curvilinear coordinates, the grad ($\nabla$) operator is

$$\text{grad} = \frac{\partial}{\partial x^k} a^k \quad \text{(A.53)}$$

Note that grad is defined in terms of covariant components and the contravariant basis.

• The gradient of a vector

The gradient of a vector $u$ read via the definition of gradient operator is

$$\text{grad} u = \frac{\partial u}{\partial x^i} a^i \quad \text{(A.54)}$$

– The covariant derivative of contravariant component of a vector

Substituting $u = u^k a_k$ in $\partial u/\partial x^i$ and using Eq. (A.44) (involving the derivative of base vector), we obtain

$$\frac{\partial u}{\partial x^i} = u^i_{,k} a_k \quad \text{(A.55)}$$

where

$$u^i_{,k} = \frac{\partial u^k}{\partial x^i} + u^r \Gamma^k_{ir} \quad \text{(A.56)}$$

are called the covariant derivative of the contravariant components. Thus

$$\text{grads} \, u = u^i_{,k} a_k a^i \quad \text{(A.57)}$$

– The covariant derivative of the covariant components of a vector

Substituting $u = u_k a_k$ in $\partial u/\partial x^i$ and using Eq. (A.48), we get

$$\frac{\partial u}{\partial x^i} = u_k a^k \quad \text{(A.58)}$$

$$\text{grads} \, u = u_k a^k a^i \quad \text{(A.59)}$$

where

$$u_k_{,i} = \frac{\partial u_k}{\partial x^i} - u_r \Gamma^r_{ik} \quad \text{(A.60)}$$

is called the covariant derivative of the covariant components.
• Gradient of a scalar

For a scalar, the gradient is simply

\[ \text{grad } \phi = \frac{\partial \phi}{\partial x^i} a^i = \phi a^i \] (A.61)

by using Eq. (A.24), the nonconservative form of the gradient reaches

\[ \text{grad } \phi = \frac{\partial \phi}{\partial x^i} a^i = G^{ij} \frac{\partial \phi}{\partial x^j} a^i \] (A.62)

Through using Eq. (A.52), the conservative form of the gradient is

\[ \text{grad } \phi = \frac{1}{\sqrt{G}} \frac{\partial (\sqrt{G} a^i \phi)}{\partial x^j} \] (A.63)

using Eq. (A.24), it becomes in the covariant base vector

\[ \text{grad } \phi = \frac{1}{\sqrt{G}} \frac{\partial (\sqrt{G} G^{ij} a^i \phi)}{\partial x^j} \] (A.64)

Furthermore, by partially differentiating the above equation and using Eq. (A.44) it reaches

\[ \text{grad } \phi = \frac{1}{\sqrt{G}} \frac{\partial (\sqrt{G} G^{ij} a^i \phi)}{\partial x^j} a^i + G^{mn} \phi \Gamma^i_{mn} a^i \] (A.65)

Appendix A.7. Divergence and Curl of a Vector

• Divergence of a vector

The divergence of a vector \( \mathbf{u} \) is defined by

\[ \text{div } \mathbf{u} = \nabla \cdot \mathbf{u} = \frac{\partial \mathbf{u}}{\partial x^i} \cdot a^i. \] (A.66)

Based on Eq. (A.3) and (A.55), we can obtain

\[ \text{div } \mathbf{u} = u'_i \] (A.67)

where by using Eq. (A.51) and Eq. (A.56)

\[ u'_i = \frac{\partial u^i}{\partial x^j} + u_j \frac{\partial}{\partial x^j} (\ln \sqrt{G}) \] (A.68)

Thus,

\[ \text{div } \mathbf{u} = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i} (\sqrt{G} u^i) = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i} (\sqrt{G} \mathbf{a}^i) \] (A.69)

If covariant components of \( \mathbf{u} \) are used, then by Eq. (A.58) we get

\[ \text{div } \mathbf{u} = G^{ik} u_{ik} \] (A.70)
• **Curl of a vector**

The definition of Curl of a vector is

\[
\text{Curl } \mathbf{u} = \nabla \times \mathbf{u} = \lim_{\Delta V \to 0} \frac{\int dS \mathbf{n} \times \mathbf{u} \Delta V}{\Delta V} \tag{A.71}
\]

where \( V \) is the volume and \( \mathbf{n} \) is outward normal unit of the surface \( S \). The integrant of curl of a vector over a surface can be defined as

\[
\int dS \mathbf{n} \cdot \text{Curl } \mathbf{u} = \oint_C d\mathbf{r} \cdot \mathbf{u}, \tag{A.72}
\]

where \( C \) is the perimeter of the surface \( S \). It will be probably used later. Another important thing is that the curl of gradient disappear, that is,

\[
\nabla \times \nabla \phi = 0 \tag{A.73}
\]

where \( \phi \) is a scalar. If \( \nabla \times \mathbf{u} = 0 \), then \( \mathbf{u} \) can be written as \( \nabla \phi \).

Introducing permutation symbols \( \epsilon_{ijk} \) and \( e^{ijk} \), one can in general write

\[
\epsilon_{ijk} \equiv \mathbf{a}_i \cdot (\mathbf{a}_j \times \mathbf{a}_k) = \epsilon_{ijk} \sqrt{G} \tag{A.74}
\]
\[
e^{ijk} \equiv \mathbf{a}^i \cdot (\mathbf{a}^j \times \mathbf{a}^k) = \epsilon^{ijk} \frac{1}{\sqrt{G}} \tag{A.75}
\]

then the following identities read

\[
\mathbf{a}_j \times \mathbf{a}_k = \epsilon_{ijk} \mathbf{a}^i = \sqrt{G} \epsilon_{ijk} \mathbf{a}^i \tag{A.76}
\]
\[
\mathbf{a}^i \times \mathbf{a}^k = \epsilon^{ijk} \mathbf{a}_i = \frac{1}{\sqrt{G}} \epsilon^{ijk} \mathbf{a}_i \tag{A.77}
\]

where both \( \epsilon_{ijk} \) and \( e^{ijk} \) are the permutation symbols, and they have the values

\[
\epsilon_{ijk} = \epsilon^{ijk} = \begin{cases} 
1, & \text{if } (i, j, k) \text{ is } (1, 2, 3), (3, 1, 2), \text{ or } (2, 3, 1) \\
-1, & \text{if } (i, j, k) \text{ is } (1, 3, 2), (3, 2, 1), \text{ or } (2, 1, 3) \\
0, & \text{otherwise.} 
\end{cases} \tag{A.78}
\]

Noted that the relations hold

\[
e^{ijk} e_{pqr} = \epsilon^{ijk} e_{pqr}, \quad \epsilon^{ijk} e_{pqr} = \delta^i_p \delta^j_q - \delta^j_p \delta^i_q \tag{A.79}
\]

From Eq. (A.76) and Eq. (A.77) we deduce that

\[
\mathbf{a}_j = \frac{1}{2 \sqrt{G}} \epsilon_{ijk} (\mathbf{a}_i \times \mathbf{a}_k) \tag{A.80}
\]
\[
\mathbf{a}_i = \frac{\sqrt{G}}{2} \epsilon_{ijk} (\mathbf{a}^j \times \mathbf{a}^k) \tag{A.81}
\]

thus,

\[
\text{Curl } \mathbf{u} = \mathbf{a}^j \times \frac{\partial \mathbf{u}}{\partial x^i} \tag{A.82}
\]

Using the above developed formulas, we obtain

\[
\text{Curl } \mathbf{u} = \frac{1}{\sqrt{G}} \epsilon^{ijk} u_k \mathbf{a}_i \tag{A.83}
\]
Thus, the contravariant components of curl $u$ are

$$(\text{Curl } u)^i = \frac{1}{\sqrt{G}} \left( \frac{\partial u_k}{\partial x^j} - \frac{\partial u_j}{\partial x^k} \right)$$  \hspace{1cm} (A.84)$$

where $i, j, k$ are cyclic.

- **The Cross Product of vectors**

  The cross product of vectors can be written as

  $$
  u \times v = e_{ijk} u^i v^j a^k = \frac{1}{\sqrt{G}} \begin{vmatrix} a^1 & a^2 & a^3 \\ u^1 & u^2 & u^3 \\ v^1 & v^2 & v^3 \end{vmatrix}
  $$

  where $i, j, k$ are cyclic.

  Using Eq. (A.76) and Eq. (A.77) are used. In another way, we have

  $$
  u \times v = (u^i a_i) \times (v^j a_j)
  $$

  $$
  = u^i v^j (a_i \times a_j)
  $$

  $$
  = u^i v^j e_{ijk} a^k
  $$

  using (A.76)

  $$
  = u^i v^j e_{ijk} G^{jk} a_k
  $$

  using (A.24)

  $$
  = u^i v^j \sqrt{G} e_{ijk} G^{jk} a_k
  $$

  where $n$ is the indices of covariant base vectors and also represents the order of contravariant components. For convenience, we reformulate it as

  $$
  u \times v = u^i v^j \sqrt{G} e_{ijk} G^{jk} a_k
  $$

  \hspace{1cm} (A.92)

  \section*{Appendix A.8. Divergence of Second-Order Tensors}

  The divergence of a tensor $T$ is defined as

  $$
  \text{div } T = \frac{\partial T}{\partial x^k} a^k
  $$

  Using the previously defined expressions for the derivatives of base vectors, we have the following results:

  - **Contravariant component of a tensor**

    $$
    T = T^{ij} a_i a_j
    $$

    $$
    \text{div } T = T_{,k}^i a_i
    $$

  where the covariant derivative is

  $$
  T_{,k}^i = \frac{\partial T^{ij}}{\partial x^k} + \Gamma^i_{mn} T^{mj} + \Gamma^j_{mk} T^{im}
  $$

  using (A.96)

  Being the contraction of $\Gamma^j_{mk}$ in Eq. (A.96) by setting $j = k$

  $$
  \Gamma^k_{mk} = \frac{\partial \ln \sqrt{G}}{\partial x^m} = \frac{1}{\sqrt{G}} \frac{\partial \sqrt{G}}{\partial x^m}
  $$

  \hspace{1cm} (A.97)

  the same as Eq. (A.51). Thus,

  $$
  \text{div } T = T_{,k}^i a_i = \left( \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^k} \left( \sqrt{G} T^{ik} \right) + \Gamma^i_{mk} T^{mk} \right) a_i
  $$

  \hspace{1cm} (A.98)
Covariant component of a tensor

\[ T = T^i_j a^i a^j \]  
(A.99)

\[ \text{div} \ T = G^{jk} T_{ij,k} \]  
(A.100)

The covariant derivative is

\[ T_{ij,k} = \frac{\partial T_{ij}}{\partial x^k} - \Gamma^m_{ik} T_{mj} - \Gamma^m_{jk} T_{im} \]  
(A.101)

Appendix B. Geometric summary in the spherical polar and cubed-sphere coordinates

Appendix B.1. The metric tensor

Consider that the radial base vector is orthogonal to the surface of constant \( r \) in the spherical and cubed coordinate and has the unit length, the metric tensor such as \( G_{ij} \) or \( G^{ij} \) can be decomposed into a 2D component along with a unit radial component

\[ G_{ij} = \begin{pmatrix} \overline{G}_{ij} & 0 \\ 0 & 1 \end{pmatrix}, \quad G^{ij} = \begin{pmatrix} \overline{G}^{ij} & 0 \\ 0 & 1 \end{pmatrix}, \]  
(B1)

where \( \overline{G}_{ij} \) and \( \overline{G}^{ij} \) are a 2D metric tensor on the constant \( r \) surface.

Appendix B.2. Geometrics in the spherical coordinates

Appendix B.2.1. Base vectors in the spherical polar system

The covariant base vectors in the spherical polar coordinate are

\[ a_1 = r \cos \varphi e_\lambda \]  
(B2)

\[ a_2 = r e_\varphi \]  
(B3)

\[ a_3 = e_r \]  
(B4)

where \( (e_\lambda, e_\varphi, e_r) \) are the local normal unit vectors along the \( \lambda, \varphi \) and \( r \) coordinate direction on sphere. Consider the vector wind \( u = u e_1 + v e_\varphi + w e_r \) on sphere, we have

\[ u = u^1 a_1 + u^2 a_2 + u^3 a_3 \]  
(B5)

\[ u e_\lambda + v e_\varphi + w e_r = u^1 r \cos \varphi e_\lambda + u^2 r e_\varphi + u^3 e_r, \]  
(B6)

In the matrix form, we get

\[ \begin{pmatrix} u \\ v \\ w \end{pmatrix} = M \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix} \]  
(B7)

where

\[ M = \begin{pmatrix} r \cos \varphi & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 1 \end{pmatrix} \]  
(B8)

Inversing the Eq. (B7), the contravariant velocity components are

\[ (u^1, u^2, u^3) = (u^1, u^2, u^3) = \left( \frac{u}{r \cos \varphi}, \frac{v}{r}, w \right) \]  
(B9)

20
Appendix B.2.2. Metrics tensor

Once we have the spherical transformation matrix (B8), the covariant metric tensor is defined by

\[ G_{ij} = M^T M \]

\[ = \begin{pmatrix} r^2 \cos^2 \phi & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

Then the contravariant metric tensor can be attained

\[ G^{ij} = (G_{ij})^{-1} \]

\[ = M^{-1} M^{-T} \]

\[ = \begin{pmatrix} \frac{1}{r^2 \cos^2 \phi} & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

The Jacobian of the transformation in the spherical polar system is

\[ \sqrt{G} = \sqrt{\det|G_{ij}|} = a_1 \cdot (a_2 \times a_3) = r^2 \cos \varphi. \]

Noted that \( r = R \) (\( R \) is the Earth radius) in the shallow-atmosphere approximation.

Appendix B.2.3. The Christoffel symbol of the second kind

From the definition of Eq. (A.45), we obtain the expression of the Christoffel symbols of the second kind in the deep atmosphere

\[ \Gamma^1 = \begin{pmatrix} 0 & -\tan \varphi & \frac{1}{r} \\ -\tan \varphi & 0 & 0 \\ \frac{1}{r} & 0 & 0 \end{pmatrix} \]

\[ \Gamma^2 = \begin{pmatrix} \sin \varphi \cos \varphi & 0 & 0 \\ 0 & 0 & \frac{1}{r} \\ 0 & \frac{1}{r} & 0 \end{pmatrix} \]

\[ \Gamma^3 = \begin{pmatrix} -r \cos^2 \varphi & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

In the shallow-atmosphere approximation, \( r \) is constant \( r = R \) so that the changed Christoffel symbols take the form

\[ \Gamma^1 = \begin{pmatrix} 0 & -\tan \varphi & 0 \\ -\tan \varphi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

\[ \Gamma^2 = \begin{pmatrix} \sin \varphi \cos \varphi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

\[ \Gamma^3 = 0 \]
Noted that the product of Christoffel symbol and contravariant metric of the spherical polar system in the shallow- atmosphere approximation reads

\[
G^{\alpha \beta} T_{\alpha \beta} = \begin{pmatrix}
0 & \frac{\tan \varphi}{R^2} \\
\frac{\tan \varphi}{R^2} & 0
\end{pmatrix},
\]

(B22)

however, they in the deep-atmosphere approximation are

\[
G^{\alpha \beta} T_{\alpha \beta} = \begin{pmatrix}
0 & \frac{\tan \varphi}{r^2} \\
-\frac{2}{r} & 0
\end{pmatrix}.
\]

(B23)

Appendix B.3. Geometrics in the cubed-sphere system

Appendix B.3.1. Base vectors in the cubed-sphere coordinates

The covariant base vectors in the cubed-sphere coordinate by using Eq. (A.28) have the form

\[
a_1 = r_t = e_\lambda r \cos \varphi \frac{dt}{d\xi} + e_\varphi \frac{d\varphi}{d\xi}
\]

(B24)

\[
a_2 = r_\eta = e_\lambda r \cos \varphi \frac{dt}{d\eta} + e_\varphi \frac{d\varphi}{d\eta}
\]

(B25)

\[
a_3 = r_r = e_r
\]

(B26)

where \((\xi, \eta) = (x_1, x_2) = (\alpha, \beta) \in [-\frac{\pi}{4}, \frac{\pi}{4}] \times [-\frac{\pi}{4}, \frac{\pi}{4}]\) hold. Consider the wind vector \(u = u e_\lambda + v e_\varphi + w e_r\) on sphere, the contravariant components of wind vector in the cubed coordinate are related by

\[
u e_\lambda + v e_\varphi + w e_r = u^1 a_1 + u^2 a_2 + u^3 a_3
\]

(B27)

Put in matrix form

\[
\begin{pmatrix}
u \\
v \\
w
\end{pmatrix} = M
\begin{pmatrix}
u^1 \\
v^2 \\
v^3
\end{pmatrix}
\]

(B28)

where

\[
M = \begin{pmatrix}
r \cos \varphi \lambda_t & r \cos \varphi \lambda_\eta & 0 \\
r \varphi_t & r \varphi_\eta & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

(B29)

Appendix B.3.2. Metric tensor

Once we have the spherical transformation matrix (B29), the covariant metric tensor is defined by

\[
G_{ij} = M^T M
\]

(B30)

\[
= \begin{pmatrix}
G_{ij} & 0 \\
0 & 1
\end{pmatrix},
\]

(B31)

where

\[
G_{ij} = \frac{r^2(1 + X^2)(1 + Y^2)}{\delta^2} \begin{pmatrix}
1 + X^2 & -XY \\
-XY & 1 + Y^2
\end{pmatrix}
\]

(B32)

noted that \(X = \tan(x_1), Y = \tan(x_2)\) and \(\delta = \sqrt{1 + X^2 + Y^2}\) are defined.
Then the contravariant metric tensor can be attained
\[
G^{ij} = (G_{ij})^{-1} = M^{-1}M^{-T} = \begin{pmatrix} G_{ij} & 0 \\ 0 & 1 \end{pmatrix},
\]
where
\[
\sqrt{G} = \frac{\delta^2}{r^2(1 + X^2)(1 + Y^2)} \begin{pmatrix} 1 + Y^2 & XY \\ XY & 1 + X^2 \end{pmatrix}.
\]

The Jacobian of the transformation in the cubed-sphere coordinates is
\[
\sqrt{G} = \sqrt{\det |G_{ij}|} = a_1 \cdot (a_2 \times a_3) = \frac{r^2(1 + X^2)(1 + Y^2)}{\delta^3}.
\]

Noted that all \( r = R \) (R is the Earth radius) in the shallow-atmosphere approximation.

Appendix B.3.3. The Christoffel symbol of the second kind
From the definition of Eq. (A.45), we write the Christoffel symbols of the second kind in the deep atmosphere as
\[
\Gamma^1 = \begin{pmatrix} \frac{2XY^2}{\sigma^2} & \frac{-Y(Y + X^2)}{\sigma^2} & \frac{1}{7} \\ \frac{1}{7} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]
\[
\Gamma^2 = \begin{pmatrix} 0 & \frac{-X(X + X^2)}{\sigma^2} & 0 \\ \frac{-X(X + X^2)}{\sigma^2} & \frac{\sigma^2Y}{3} & \frac{1}{7} \\ 0 & \frac{\sigma^2Y}{3} & 0 \end{pmatrix},
\]
\[
\Gamma^3 = \frac{r(1 + X^2)(1 + Y^2)}{\delta^3} \begin{pmatrix} -(1 + X^2) & XY & 0 \\ XY & -(1 + Y^2) & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

In the shallow-atmosphere approximation, \( r \) becomes constant \( R \) and the Christoffel symbol takes the form
\[
\Gamma^1 = \begin{pmatrix} \frac{2XY^2}{\sigma^2} & \frac{-Y(Y + X^2)}{\sigma^2} & 0 \\ \frac{-X(X + X^2)}{\sigma^2} & \frac{\sigma^2Y}{3} & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]
\[
\Gamma^2 = \begin{pmatrix} 0 & \frac{-X(X + X^2)}{\sigma^2} & 0 \\ \frac{-X(X + X^2)}{\sigma^2} & \frac{\sigma^2Y}{3} & 0 \\ 0 & \frac{\sigma^2Y}{3} & 0 \end{pmatrix},
\]
\[
\Gamma^3 = 0.
\]

Noted that the product of Christoffel symbol and contravariant metric under the gnomonic mapping in the shallow-atmosphere approximation reads
\[
G^{ij}\Gamma^k_{ij} = 0,
\]
however, they in the deep-atmosphere approximation are
\[
G^{ij}\Gamma^k_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{2}{\sigma} \end{pmatrix}.
\]
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