Towards an Extremal Network Theory – Robust GDoF Gain of Transmitter Cooperation over TIN

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Abstract—We study the GDoF gain of transmitter cooperation (TC) over power control and treating interference as noise (TIN) for 3 regimes – a TIN regime where TIN is GDoF optimal for the $K$ user IC, a CTIN regime where the GDoF region achieved by TIN is convex without time-sharing, and an SLS regime where the GDoF region achieved for 3 regimes – a TIN regime where the GDoF region achieved by TIN is convex without time-sharing, and a CTIN regime where a simple layered superposition scheme is optimal in the $K$ user MISO BC for $K \leq 3$. Under finite precision CSIT, appealing to extreme network theory we obtain the following results. In the TIN regime as well as the CTIN regime, the extremal GDoF gain from TC over TIN is $\Theta(1)$. In fact, the gain is at most a factor of 2 in the CTIN regime and exactly $3/2$ in the TIN regime for $K > 1$. In the SLS regime, the extremal GDoF gain is $\Theta(\log(K))$.

I. INTRODUCTION

With the emergence of aligned images bounds [2], significant progress has been made in the understanding of robust fundamental limits of wireless networks through Generalized Degrees of Freedom (GDoF) characterizations under the assumption of finite precision channel state information at the transmitters (CSIT), especially for smaller or highly symmetric networks. A critical barrier in extending these insights to larger and asymmetric networks is the inherent combinatorial complexity of such settings. Motivated by other fields such as extremal combinatorics and extremal graph theory, we explore the possibility of an extremal network theory, i.e., a study of extremal networks within particular classes of interest. As our test application, we study the GDoF benefits of transmitter cooperation over the simple scheme of power control and treating interference as Gaussian noise (TIN) for three regimes of interest where the interference is weak. The question is intriguing because while in general transmitter cooperation can be quite powerful, finite precision CSIT and weak interference favor TIN. The three regimes that we explore include a TIN regime previously identified by Geng et al. in [3] where TIN was previously shown to be GDoF optimal for the $K$ user interference channel, a CTIN regime previously identified by Yi and Caire in [4] where the GDoF region achievable by TIN turns out to be convex without the need for time-sharing, and an SLS regime previously identified by Davoodi and Jafar in [5] where a simple layered superposition (SLS) scheme is shown to be optimal in the $K$ user MISO BC, but only for $K \leq 3$. It remains an intriguing possibility that TIN may not be far from optimal in the CTIN regime, and that SLS schemes may be close to optimal even for larger networks in the SLS regime. Under finite precision CSIT, appealing to extreme network theory we obtain the following results. In the TIN regime as well as the CTIN regime, we show that the extremal GDoF gain from transmitter cooperation over TIN is $\Theta(1)$, i.e., it is bounded above by a constant regardless of the number of users $K$. In fact, the extremal GDoF gain is at most a factor of 2 in the CTIN regime, which automatically implies that TIN is GDoF optimal within a factor of 2 in the CTIN regime. In the TIN regime, the extremal GDoF gain of transmitter cooperation over TIN is shown to be exactly a factor of $3/2$, regardless of the number of users $K$, provided $K > 2$. However, in the SLS regime, the extremal GDoF gain of transmitter cooperation over TIN is shown to be exactly a factor of $3/2$, regardless of the number of users $K$, provided $K > 2$. However, in the SLS regime, the extremal GDoF gain of transmitter cooperation over TIN is $\Theta(\log_2(K))$, i.e., it scales logarithmically with the number of users. Remarkably, an SLS scheme suffices to achieve this extremal GDoF gain. Last but not the least, as a byproduct of our analysis we prove a useful cyclic partition property of the sum GDoF achievable by TIN in the SLS regime.

Notations: For positive integers $X$ and $Y$, define $[X : Y] = \{X, X + 1, \cdots, Y\}$, and $[X] = \{1 : X\}$. The notation $[a]_{K \times K}$ represents a $K \times K$ matrix whose $(i,j)$th element is $a_{ij}$. The cardinality of a set $S$ is denoted by $|S|$. For functions $f(K)$ and $g(K)$, denote $f(K) = \Theta(g(K))$ if $\lim_{K \to \infty} f(K)/g(K) = c$ for some finite constant $c > 0$.

II. SYSTEM MODEL

For GDoF studies, the $K$ user interference channel is modeled as

$$Y_k(t) = \sum_{i=1}^{K} \bar{P}^{\alpha_{ki}}G_{ki}(t)X_i(t) + Z_k(t) \quad \forall k \in [K].$$

(1)

During the $t^{th}$ channel use, $X_i(t), Y_k(t), Z_k(t) \in \mathbb{C}$ are the symbol transmitted by User $i$, symbol received by User $k$, and the zero mean unit variance additive white Gaussian noise (AWGN) at User $k$, respectively. $\bar{P} \triangleq \sqrt{P}$, is a nominal parameter that approaches infinity to define the GDoF limit. The exponent $\alpha_{ki} \geq 0$ is the channel strength of the link between Transmitter $i$ and Receiver $k$. The channel coefficients $G_{ki}(t)$ are known perfectly to the receivers but only available to finite precision at the transmitters. The finite precision CSIT assumption implies that from the transmitter’s perspective, the joint and conditional probability density functions of the channel coefficients exist and are bounded (see [2] for details of the bounded density assumption). In the $K$ user IC, there are
$K$ independent messages, one for each user, and each message is independently encoded by its corresponding transmitter. The definitions of achievable rate tuples and capacity region, $C_c(P)$ are standard, see e.g., [2]. The GDoF region of the $K$ user interference channel is defined as

$$D_{sc} = \left\{ (d_k)_{k \in [K]} \left| d_k = \lim_{P \to 0} \frac{R_k(P)}{\log(P)}, \quad (R_k(P))_{k \in [K]} \in C_c(P) \right. \right\}. \quad (2)$$

The maximum sum-GDoF value is denoted $D_{sc,sc}$.

For the $K$ user MISO BC obtained by full cooperation among all $K$ transmitters, the GDoF region is denoted $D_{bc}$ and the maximum sum-GDoF value is denoted $D_{sc,bc}$.

### III. Definitions

**Definition 1 (TIN Regime).** Define

$$A_{\text{TIN}} = \left\{ [\alpha]_{K \times K} \in \mathbb{R}^{K \times K}_+ \left| \alpha_{ii} \geq \alpha_{ij} + \alpha_{ki}, \quad \forall i, j, k \in [K], i \notin \{j, k\} \right. \right\}.$$  

The significance of the TIN regime is that in this regime, it was shown by Geng et al. in [3] that TIN is GDoF-optimal.

**Definition 2 (CTIN Regime).** Define

$$A_{\text{CTIN}} = \left\{ [\alpha]_{K \times K} \in \mathbb{R}^{K \times K}_+ \left| \alpha_{ii} \geq \max(\alpha_{ij} + \alpha_{ji}), \quad \forall i, j, k \in [K], i \notin \{j, k\} \right. \right\}.$$  

For the CTIN regime, it was shown by Yi and Caire in [4] that the GDoF region achievable with TIN (also known as $D_{\text{TIN}}$, see Definition 10), is convex, without the need for time-sharing, and equal to the polyhedral TIN region over the set of all $K$ users (see Definition 8).

**Definition 3 (SLS Regime).** Define

$$A_{\text{SLS}} = \left\{ [\alpha]_{K \times K} \in \mathbb{R}^{K \times K}_+ \left| \alpha_{ii} \geq \max(\alpha_{ij} + \alpha_{ji}), \quad \forall i, j, k \in [K], i \notin \{j, k\} \right. \right\}.$$  

In the SLS regime, it was shown by Davoodi and Jafar in [5] that a simple layered superposition scheme is GDoF-optimal for the MISO BC obtained by allowing transmitter cooperation in a $K$ user interference channel. Note that the result of [5] is limited to $K \leq 3$; however, the regime is defined for all $K$. Also note that the SLS regime includes the CTIN regime, which includes the TIN regime.

**Definition 4 (Cycle $\pi$).** A cycle $\pi$ of length $M > 1$, denoted

$$\pi = i_1 \to i_2 \to \cdots \to i_{M-1} \to i_M \not\in$$  

is an ordered collection of links in the $K \times K$ interference network, that includes the desired link between Transmitter $i_m$ and Receiver $i_m$, and the interfering link between Transmitter $i_m$ and Receiver $i_{m+1}$, for all $m \in \{1 : M\}$, where we set $i_{M+1} = i_1$, and the indices $i_1, i_2, \cdots, i_M \in [K]$ are all distinct. A cycle of length $M = 1$ is called a trivial cycle, represented simply as $\pi = i_1 \not\in$, and it includes only the desired link between Transmitter $i_1$ and Receiver $i_1$. Also define the following terms related to the cycle $\pi$.

1. Define $\pi(m) = i_m$ as the $m$th element of the cycle, e.g., $\pi(1) = i_1, \pi(2) = i_2$. Thus, the cycle may be equivalently represented as $\pi = \pi(1) \to \cdots \to \pi(M) \not\in$.
2. Define $\{\pi\} = \{i_1, i_2, \cdots, i_M\}$, i.e., $\{\pi\}$ represents the set of users involved in the cycle $\pi$.
3. Cycles $\pi_1, \pi_2, \cdots, \pi_n$ are disjoint if the sets $\{\pi_1\}, \{\pi_2\}, \cdots, \{\pi_n\}$ are disjoint.
4. Cycles $\pi_1, \pi_2, \cdots, \pi_n$ comprise a cyclic partition of the set $S \subseteq [K]$, if they are disjoint and $\bigcup_{i=1}^n \{\pi_i\} = S$.

The significance of cycles is that they correspond to bounds on the sum-GDoF of the users involved in the cycle. For the interference channel, each cycle $\pi$ leads to a cycle bound $\sum_{k \in \pi} d_k \leq \Delta_\pi$ (see Definition 7) which is a bound on the GDoF region achievable by a restricted form of TIN, called polyhedral TIN (Definition 8). For the broadcast channel, each cycle $\pi$ leads to a bound

$$\sum_{k \in \pi} d_k \leq \Delta_\pi + \alpha_{\pi(i+1)\pi(i)}$$  

for any $i \leq M$, the length of the cycle [1]. Unlike the interference channel, the bounds for the BC are information theoretic bounds on the optimal GDoF region. These bounds are the key to all the results in this work.

**Definition 5 (Combined Cycles).** For disjoint cycles

$$\pi_1 = i_1 \to \cdots \to i_{M_1} \not\in$$  

$$\pi_2 = j_1 \to \cdots \to j_{M_2} \not\in$$  

the combined cycle, denoted $\pi_{1,2} = \pi_1 \to \pi_2 \not\in$ is defined as

$$\pi_{1,2} = \pi_1 \to \pi_2 \not\in$$  

$$= i_1 \to \cdots \to i_{M_1} \to j_1 \to \cdots \to j_{M_2} \not\in$$  

Note that $\pi_{1,2}$ is in general different from $\pi_{2,1}$. Combinations of more than 2 cycles are similarly defined.

It is shown in the full paper [1] that in the SLS regime, if $\pi_1, \pi_2, \cdots, \pi_n$ are $n > 1$ disjoint cycles, and

$$\pi_{1,2, \cdots, n} = \pi_1 \to \pi_2 \to \cdots \to \pi_n \not\in$$  

is their combination, then

$$\Delta_{\pi_{1,2, \cdots, n}} \leq \Delta_{\pi_1} + \Delta_{\pi_2} + \cdots + \Delta_{\pi_n} + \Delta_\pi$$  

where $\pi = \pi(1) \to \pi(2) \to \cdots \to \pi(n) \not\in$.

**Definition 6 ($\delta_{ij}$).** For $i, j \in [K]$, define

$$\delta_{ij} = \begin{cases} \alpha_{ij} - \alpha_{ji}, & i \neq j, \\ 0, & i = j. \end{cases}$$  

**Definition 7 ($\Delta_{\pi}$).** For any cycle $\pi$ of length $M$, $\pi = i_1 \to i_2 \to \cdots \to i_M \not\in$, define

$$\Delta_\pi = \begin{cases} \sum_{j=1}^M \delta_{i_j,i_{j+1}}, & \text{if } M > 1, \\ \alpha_{i_1i_1}, & \text{if } M = 1. \end{cases}$$  

where $i_{M+1} = i_1$. 
**Definition 8** \((\mathcal{D}_{P,TIN}(S))\). For any subset of users, \(S \subseteq [K]\), the polyhedral-TIN region, \(\mathcal{D}_{P,TIN}(S)\), is defined as a set collecting the tuples \((d_1, d_2, \ldots, d_K)\) satisfying
\[
0 = d_k, \quad \forall k \in [K]\setminus S, \quad (11)
\]
\[
0 \leq d_k, \quad \forall k \in S, \quad (12)
\]
\[
\sum_{k \in [\pi]} d_k \leq \Delta_{\pi}, \quad \{\pi\} \subset S \quad (13)
\]
for all cycles \(\pi\) over subsets of \(S\). The sum-GDoF of polyhedral-TIN over the set \(S\) are defined as \(\mathcal{D}_{\Sigma,P,TIN}(S) = \max_{\mathcal{D}_{P,TIN}(S)} \sum_{k \in [S]} d_k\). If \(S = [K]\), then we will simply write \(\mathcal{D}_{\Sigma,P,TIN}([K]) = \mathcal{D}_{\Sigma,P,TIN}\).

**Definition 9** (P-optimal Cyclic Partition of \(S\)). A cyclic partition of a subset of users \(S \subset [K]\), say into the \(n\) disjoint cycles \(\pi_1, \pi_2, \ldots, \pi_n\), is said to be p-optimal if
\[
\mathcal{D}_{\Sigma,P,TIN}(S) = \Delta_{\pi_1} + \Delta_{\pi_2} + \cdots + \Delta_{\pi_n}. \quad (14)
\]

In general a p-optimal cyclic partition does not exist. However, as shown in the full paper [1], such partitions exist for all networks in the SLS regime.

**Definition 10** \((\mathcal{D}_{TINA})\). The TINA region is defined as
\[
\mathcal{D}_{TINA} = \bigcup_{S \subseteq [K]} \mathcal{D}_{P,TIN}(S). \quad (15)
\]

The sum-GDoF over the TINA region are defined as
\[
\mathcal{D}_{\Sigma,TINA} = \max_{\mathcal{D}_{TINA}} \sum_{k \in [K]} d_k. \quad (16)
\]
Thus the TINA region is a union of polyhedral TIN regions. In general this union does not produce a convex region. For example, consider the 2 user interference channel where all \(\alpha_{ij}\) values are equal to \(1\). Incidentally, this channel is in the SLS regime. For this channel, \(\mathcal{D}_{P,TIN}([\{i\}]) = \{(d_1, d_2) : 0 \leq d_1 \leq 1, d_2 = 0\}\), \(\mathcal{D}_{P,TIN}([\{2\}]) = \{(d_1, d_2) : d_1 = 0, 0 \leq d_2 \leq 1\}\), \(\mathcal{D}_{P,TIN}([\{2\}]) = \{(d_1, d_2) : d_1 = 0, d_2 = 0\}\). The union of these three regions, \(\mathcal{D}_{\Sigma,TINA}(\{1\}) \cup \mathcal{D}_{\Sigma,TINA}(\{2\}) \cup \mathcal{D}_{\Sigma,TINA}(\{1, 2\})\), is not convex. However, remarkably, the region \(\mathcal{D}_{TINA}\) is convex for channels in the TIN regime as shown by Geng et al. in [3], and for channels in the CTIN regime as shown by Yi and Caire in [4].

**IV. Extremal GDoF Gain from Transmitter Cooperation in the TIN and CTIN Regimes**

The main result for these regimes appears in Theorem 1. Remarkably, while the optimality of TIN is not known in the CTIN regime, Theorem 1 implies that TIN is within a factor of 2 from optimality in this regime.

**Theorem 1.** For arbitrary number of users, \(K > 1\),
\[
\max_{[\alpha]_{K \times K} \in \mathcal{A}_{TINA}} \frac{\mathcal{D}_{\Sigma,TINA}}{\mathcal{D}_{\Sigma,TINA}} = 3/2, \quad (17)
\]
and
\[
\max_{[\alpha]_{K \times K} \in \mathcal{A}_{CTIN}} \frac{\mathcal{D}_{\Sigma,BCT}}{\mathcal{D}_{\Sigma,TINA}} \leq 2. \quad (18)
\]

**A. Proof of Theorem 1**

Let us provide here the proof for the TIN regime. The proof for the CTIN regime is a bit more involved and is relegated to the full paper [1]. First, let us prove the upper bound, i.e., in the TIN-regime, \(\mathcal{D}_{\Sigma,BC} \leq 1.5\mathcal{D}_{\Sigma,TIN}\). Let \(\pi_i \rightarrow \pi_{i+1} \rightarrow \cdots \rightarrow \pi_{i+M} \) by any cycle of length \(M > 1\), so that the sum-GDoF value of the IC restricted to just the users in \(\{\pi_i\}\) is bounded above by \(\Delta_\pi\) (because TIN is GDoF optimal in the TIN regime [3]). This implies that \(\Delta_\pi \geq \alpha_{i,i+1}\), because \(\alpha_{i,i+1}\) GDoF are trivially achievable by simply allowing only user \(\pi_i\) to transmit. For the same \(M\) users, the sum-GDoF in the BC are bounded in two ways as [1],
\[
\mathcal{D}_{\Sigma,BC}(\{\pi_i\}) \leq \sum_{j=1}^{M} \alpha_{i,j+1} = \Delta_\pi + \alpha_{i,i+1} \quad (19)
\]
\[
\mathcal{D}_{\Sigma,BC}(\{\pi_i\}) \leq \sum_{j=1}^{M} \alpha_{i,j+1} = \Delta_\pi + \alpha_{i,i+1} \quad (20)
\]
Adding the bounds, we have \(2\mathcal{D}_{\Sigma,BC}(\{\pi_i\}) \leq 2\Delta_\pi + \alpha_{i,i+1} + \alpha_{i,i+1} \leq 2\Delta_\pi + \alpha_{i,i+1} \leq 3\Delta_\pi\), where we made use of the fact that in the TIN-regime, \(\alpha_{i,i+1} \leq \alpha_{i,i+1} \leq \Delta_\pi\). Also for a trivial cycle, \(\pi_i\), of length \(M = 1\), say comprised of only user \(m\), we have \(\mathcal{D}_{\Sigma,BC}(\{\pi_i\}) = \mathcal{D}_{\Sigma,BC}(\{\pi\}) = \alpha_{m,m} = \Delta_\pi\), so here also \(\mathcal{D}_{\Sigma,BC}(\{\pi\}) \leq 1.5\Delta_\pi\). Therefore for every cycle \(\pi\) we have \(\mathcal{D}_{\Sigma,BC}(\{\pi\}) \leq 1.5\Delta_\pi\). Now, let us consider the total GDoF of all \(K\) users. Since \([\alpha]_{K \times K} \in \mathcal{A}_{TINA}\), from [6] we know that \(\mathcal{D}_{\Sigma,TINA}\) is given by a cycle partition, comprised of, say the \(N\) cycles \(\pi_1, \pi_2, \ldots, \pi_N\), i.e., \(\mathcal{D}_{\Sigma,TINA} = \sum_{n=1}^{N} \Delta_\pi\). Therefore,
\[
\mathcal{D}_{\Sigma,BC} \leq \sum_{n=1}^{N} \mathcal{D}_{\Sigma,BC}(\{\pi_n\}) \leq \sum_{n=1}^{N} 1.5\Delta_\pi = 1.5\Delta_{\Sigma,TINA}. \quad (21)
\]
This completes the proof of the upper bound.

The lower bound follows directly from existing results. For \(K = 2\) users consider the channel with \(\alpha_{11} = \alpha_{22} = 1, \alpha_{12} = 0.5\), for which \(\mathcal{D}_{\Sigma,BC} = 1\). However, for \(K \geq 3\) it is trivial to generate such \([\alpha]_{K \times K} \in \mathcal{A}_{TINA}\), e.g., simply by adding users \(K \geq 3\) such that all \(\alpha_{ij}\) associated with these additional users are zero. This completes the proof of Theorem 1.

**V. Extremal GDoF Gain from Transmitter Cooperation in the SLS Regime**

**Theorem 2.**
\[
\max_{[\alpha]_{K \times K} \in \mathcal{A}_{SLS}} \frac{\mathcal{D}_{\Sigma,BC}}{\mathcal{D}_{\Sigma,TINA}} = \Theta(\log(K)). \quad (22)
\]

**A. Proof of Theorem 2: Upper Bound**

We will need the following iterative procedure. Stage \(\lambda\) of the procedure, \(\lambda \in [0 : \Lambda]\), is characterized by a subset of users, \(S_n \subset [K]\), a cyclic partition of \(S_n\) into \(N_\lambda\) disjoint cycles \(\pi_1^\lambda, \pi_2^\lambda, \ldots, \pi_{N_\lambda}^\lambda\), and a cyclic partition of \([K]\) into \(N_\lambda\) disjoint cycles \(\pi_1^\lambda, \pi_2^\lambda, \ldots, \pi_{N_\lambda}^\lambda\). The procedure stops in stage \(\lambda = \Lambda\) as soon as we find \(N_\Lambda = 1\).
Stage 0 is the initialization stage. The procedure is initialized with the set \( S_0 = [K] \), the set of all users. Let \( \pi_1^0, \pi_2^0, \ldots, \pi_N^0 \) be a p-optimal cyclic partition of \( S_0 \) having at most one trivial cycle. Such a cyclic partition produces the tight sum-GDoF bound for polyhedral TIN over \( S_0 \) so that
\[
D_{\Sigma,TIN}(S_0) = \Delta_{\pi^0_1} + \Delta_{\pi^0_2} + \cdots + \Delta_{\pi^0_{N_0}}.
\] (23)
Choose \( (\pi_1^0, \pi_2^0, \ldots, \pi_N^0) = (\pi_1^0, \pi_2^0, \ldots, \pi_N^0) \). This completes the initialization stage. Note that because the p-optimal cyclic partition cannot have more than one trivial cycle, we must have \( N_0 \leq (K + 1)/2 \). If \( N_0 = 1 \), then \( \Lambda = 0 \) and the procedure stops here. If not, then we move to the next stage.

Stage 1 begins by defining the set of users,
\[
S_1 = \{ \pi_1^1(1), \pi_2^1(1), \ldots, \pi_N^1(1) \}.
\] (24)
Let \( \pi_1^1, \pi_2^1, \ldots, \pi_N^1 \) be a p-optimal cyclic partition of \( S_1 \) with at most one trivial cycle so that
\[
D_{\Sigma,P,TIN}(S_1) = \Delta_{\pi_1^1} + \Delta_{\pi_2^1} + \cdots + \Delta_{\pi_{N_1}^1}.
\] (25)
Note that these cycles only span \( S_1 \). For each of these cycles, \( \pi_n^1, n \in [1 : N_1] \), we will create a combined cycle, \( \pi_n^1 \) such that the \( N_1 \) combined cycles will be a cyclic partition of \( [K] \). This is done as follows. Let us write the \( n^{th} \) cycle, \( \pi_n^1 \), explicitly as,
\[
\pi_n^1 = \pi_n^0(1) \rightarrow \pi_{n_2}^0(1) \rightarrow \cdots \rightarrow \pi_{n_{N_0}^0}(1) \tag{26}
\]
Then the corresponding combined cycle is defined as
\[
\pi^1_n = \pi_{n_1}^0 \rightarrow \pi_{n_2}^0 \rightarrow \cdots \rightarrow \pi_{n_{N_0}^0} \tag{27}
\]
Note that \( S_1 \) has \( N_0 \) users, and the p-optimal cyclic partition does not have more than one trivial cycle, so we must have \( N_1 \leq (N_0 + 1)/2 \). Furthemore, as shown in [1] and [2],
\[
\Delta_{\pi^1_n} \leq \Delta_{\pi_{n_1}} + \Delta_{\pi_{n_2}} + \cdots + \Delta_{\pi_{n_{N_0}^0}} \tag{28}
\]
Summing over all \( n \in [1 : N_1] \) we have
\[
\sum_{i=1}^{N_1} \Delta_{\pi^1_i} \leq \sum_{i=1}^{N_0} \Delta_{\pi^0_i} + \sum_{i=1}^{N_1} \Delta_{\pi^0_i} \tag{29}
\]
\[
= \sum_{i=1}^{N_0} \Delta_{\pi^0_i} + D_{\Sigma,P,TIN}(S_1) \tag{30}
\]
\[
\leq \sum_{i=1}^{N_0} \Delta_{\pi^0_i} + D_{\Sigma,TIN} \tag{31}
\]
If \( N_1 = 1 \), then we set \( \Lambda = 1 \) and the procedure stops here. If not, then we proceed to the next stage. The procedure now simply repeats, incrementing \( \lambda \) at each stage until \( N_{\Lambda} = 1 \) where we set \( \lambda = \Lambda \) and the procedure stops. This completes the description of the procedure.

The bound can be bounded by using \( N_{\Lambda+1} \leq (N_{\Lambda} + 1)/2, N_0 \leq (K + 1)/2 \) and \( N_{\Lambda-1} \geq 2 \), as follows. \( N_{\Lambda-1} \geq 2 \) \( \Rightarrow N_{\Lambda-2} \geq 3 \) \( \Rightarrow N_{\Lambda-3} \geq 5 \) \( \Rightarrow \cdots \Rightarrow N_0 \geq 2^{\Lambda-1} + 1 \) \( \Rightarrow K \geq 2^{\Lambda+1} + 1 \) \( \Rightarrow \Lambda \leq \log_2(K-1) \).

Finally, we complete the proof of the upper bound as follows.
\[
D_{\Sigma,TIN} \geq D_{\Sigma,P,TIN}(S_0) \tag{32}
\]
\[
= \Delta_{\pi^0_1} + \Delta_{\pi^0_2} + \cdots + \Delta_{\pi^0_{N_0}} \tag{33}
\]
\[
\geq \Delta_{\pi^1_1} + \Delta_{\pi^1_2} + \cdots + \Delta_{\pi^1_{N_1}} - D_{\Sigma,TIN} \tag{34}
\]
\[
\geq \Delta_{\pi^1_1} + \Delta_{\pi^1_2} + \cdots + \Delta_{\pi^1_{N_1}} - 2D_{\Sigma,TIN} \tag{35}
\]
\[
\cdots \geq \Delta_{\pi^1_1} - \Delta D_{\Sigma,TIN} \tag{36}
\]
\[
\geq D_{\Sigma,NC} - D_{\Sigma,TIN} - D_{\Sigma,TIN} \tag{37}
\]
Additional details are available in [1]. Substituting the bound for \( \Lambda \) we obtain
\[
\frac{D_{\Sigma,NC}}{D_{\Sigma,TIN}} \leq 2 + \log_2(K - 1) = \Theta(\log_2(K)) \tag{38}
\]
and the proof of the upper bound is complete.

\[\square\]

B. Proof of Theorem 2: Lower Bound

For the lower bound, let us define a class of interference networks, \( \mathcal{N}(n, \nu) \), that is parameterized by the two numbers, \( n \in \mathbb{N}, \nu \in \mathbb{R}, 0 \leq \nu \leq 1 \). The number of users \( K(n) = 2^n \), all desired channel strengths \( \alpha_{kk} = 1 \), and cross-channel strengths satisfy \( \alpha_{ij} = \alpha_{ji}^{(n)} = \frac{\nu}{2^n} \) if user \( i \) and user \( j \) are siblings (share a common parent), \( \frac{\nu}{2^n} \) if they share the same grandparent (but not the same parent), and the largest possible value of \( \delta_{ij}^{(n)} \) in \( \mathcal{N}(n, \nu) \) is \( \nu/2 \), between users whose closest common ancestor is the root node. Fig. 1 is an example of an \( \mathcal{N}(3,1) \) network. It is easy to verify that an \( \mathcal{N}(n, \nu) \) network is indeed in the SLS regime [1]. We are primarily interested in the network \( \mathcal{N}(n, \nu) \) for \( \nu = 1 \).

Consider the root node of the binary tree representation of \( \mathcal{N}(n, \nu) \). If the root node is eliminated, then the tree splits into two binary trees. For ease of reference, let us denote these two networks as Left(\( \mathcal{N}(n, \nu) \)) and Right(\( \mathcal{N}(n, \nu) \)). Note that each of the networks Left(\( \mathcal{N}(n, \nu) \)) and Right(\( \mathcal{N}(n, \nu) \)) is an \( \mathcal{N}(n-1, \nu/2) \) network [1]. Let \( D_{\Sigma,TIN} \) represent the optimal sum-GDoF value over the \( D_{\Sigma,TIN}^{[n]} \) region for \( \mathcal{N}(n, \nu) \). Since \( \Delta_{\Sigma} \) scale linearly with \( \nu \) for non-trivial cycles, it can be shown that
\[
D_{\Sigma,TIN}^{[n]} \leq \max\left\{1, \frac{1}{2} D_{\Sigma,TIN}^{[2]} \right\} \tag{39}
\]
Now, since isolating the left and right subnetworks of \( \mathcal{N}(n, \nu) \) from each other’s interference does not hurt either of them,
\[
D_{\Sigma,TIN}^{[n]} \leq D_{\Sigma,TIN}^{[n-1, \nu/2]} + D_{\Sigma,TIN}^{[n-1, \nu/2]} = 2D_{\Sigma,TIN}^{[n-1, \nu/2]} \tag{40}
\]
\[
\leq 2 \max\left\{1, \frac{1}{2} D_{\Sigma,TIN}^{[n-1, \nu/2]} \right\} = \max(2, D_{\Sigma,TIN}^{[n-1, \nu/2]}) \tag{41}
\]
\[
\leq \cdots \leq \max(2, D_{\Sigma,TIN}^{[1, \nu/2]}) = 2. \tag{42}
\]
Thus, TIN cannot achieve more than 2 sum-GDoF for our network.

However, by allowing transmitter cooperation in this network, a sum-GDoF value of $1 + \frac{1}{2} \log_2(K)$ is achievable by a hierarchical SLS scheme (see illustration in Fig. 1 for the $\mathcal{N}^{[3,1]}$ example). Starting with the top level, denoted level $n$, which corresponds to all the users in the original network $\mathcal{N}^{[n,1]}$, a common message carrying $1/2$ GDoF is sent to all $K = 2^n$ users. The message is decoded by each receiver while treating all other messages as noise, and subtracted by each receiver before decoding the next message. At each subsequent level $\ell = n - 1, n - 2, \ldots, 1$ there are $2^{n-\ell}$ subnetworks $\mathcal{N}^{[\ell,1/2^{n-\ell}]}$. A total of $2^{n-\ell}$ independent common messages are sent, one within each subnetwork, each carrying $1/2^{n-\ell+1}$ GDoF, to be decoded by the $2^\ell$ users within that network. The transmit power levels are chosen (see Fig. 1) so that these messages appear below the noise floor in their undesired subnetworks. The total GDoF carried by the level $\ell$ messages is $1/2^{n-\ell+1} \times 2^{n-\ell} = 1/2$. This pattern continues for all $n - 1$ levels. Finally at level $\ell = 0$, there are $2^n$ subnetworks comprised of individual users with no more lower

level messages; so the noise floor is 0, and it is possible to achieve $1/2^n \times 1/2^n = 1/2^n$ GDoF per user for a total of 1 GDoF, thus achieving a total of $1 + \frac{1}{2} \log_2(K)$ GDoF. Incidentally, this is the optimal sum-GDoF value with transmitter cooperation in our network. A detailed explanation of the achievable scheme is available in the full paper [1].

VI. CONCLUSION

The results presented in this work lead to a number of interesting questions where external analysis could be insightful to gain a deeper understanding of the benefits of transmitter cooperation. For example, is it possible to achieve more than logarithmic GDoF gain by transmitter cooperation over TIN in a general weak interference regime where the only constraint is that the direct channels are stronger than cross channels? What is the extremal sum-GDoF gain of a $K$ user MISO BC over the corresponding $K$ user IC in the general weak interference regime? Or, even in the SLS-regime? In general, it seems external analysis is useful to gauge the relative benefits of a myriad of factors such as multiple antennas, power control, rate-splitting and network coherence – all intriguing issues for which the current understanding is extremely limited. Indeed, the main message of this work is to underscore the importance of extremal analysis in order to advance our understanding of fundamental limits of large wireless networks beyond symmetric settings, where the curse of dimensionality stands in the way.

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