RESIDUALLY FINITE ALGORITHMICALLY FINITE GROUPS, THEIR SUBGROUPS AND DIRECT PRODUCTS
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We construct an infinite finitely generated recursively presented residually finite algorithmically finite group $G$ answering thereby a question of Myasnikov and Osin. Moreover, $G$ is “very infinite” and “very algorithmically finite” in the sense that $G$ contains an infinite abelian normal subgroup while all finite Cartesian powers of $G$ are algorithmically finite (i.e., for any positive integer $n$, there is no algorithm which writes out an infinite sequence of pairwise different elements of $G^n$). We also state several related problems.

0. Introduction

In [MO11], it was constructed the first example of finitely generated recursively presented infinite group which is \textit{algorithmically finite}, in the sense that there is no algorithm that writes out an infinite sequence of pairwise different elements of this group. Groups having these properties (i.e. finitely generated recursively presented infinite and algorithmically finite) are called \textit{Dehn monsters} in [MO11].

The Dehn monsters constructed in [MO11] also have an additional finiteness-infiniteness property: they have infinite residually finite homomorphic images. This led the authors of [MO11] to the following question: Do there exist residually finite Dehn monsters? We answer this question positively\(^*)\) but ask another question.

\textbf{Question 1.} Is the direct product of two algorithmically finite groups always algorithmically finite?

We cannot answer this question (and conjecture that the answer is negative) but the Dehn monster constructed in this paper has a nice property: all its finite Cartesian powers are algorithmically finite. Even more intriguing, in our opinion, question sounds as follows.

\textbf{Question 2.} Is the wreath product of two algorithmically finite groups always algorithmically finite? Is it true at least that the wreath product of a finite group (e.g., the two-element groups) and an algorithmically finite group is always algorithmically finite?

We do not know the answer and cannot even answer the “opposite question”.

\textbf{Question 3.} Does there exist a Dehn monster $D$ such that the wreath product $\mathbb{Z}_2 \wr D$ is algorithmically finite?

Note, however, that the monster constructed in this paper is in some sense similar to such a wreath product: it is a semidirect product of an infinite elementary abelian normal subgroup and another monster.

\textbf{Main theorem.} There exists an infinite finitely generated recursively presented residually finite group $G$ such that all its finite Cartesian powers are algorithmically finite, i.e., for any positive integer $n$, there is no algorithm that writes out an infinite sequence of pairwise different elements of the group $G^n$.

Moreover, the group $G$ can be chosen containing an infinite direct power of the cyclic groups $\mathbb{Z}_p$ as a normal subgroup (where $p$ is an arbitrary given prime) and the corresponding extension splits: $G = H \rtimes \left( \bigoplus_{i=1}^{\infty} \langle a \rangle_p \right)$.

The question about possible subgroups of Dehn monsters deserves a special attention. Clearly, all finitely generated subgroups of monsters are algorithmically finite themselves; in particular, all cyclic subgroups are finite and, therefore, e.g., all solvable finitely generated subgroups are also finite.

\textbf{Question 4.} Which groups (or which abelian groups) can be embedded in an algorithmically finite group? Which groups can be embedded as normal subgroups?

The main theorem is a corollary of the following fact about algebras, which is of independent interest.

\textbf{Theorem on strongly algorithmically finite algebras.} Over any finite field, there exists an infinite finitely generated recursively presented residually finite associative algebra $A$ (with unity) whose all finite Cartesian powers are algorithmically finite, i.e., for any positive integer $n$, there is no algorithm which writes out an infinite number of pairwise different elements of the algebra $A^n$.

Moreover, the algebra $A$ is generated by a finite set of nilpotent elements.

Our approach is based on the ideas of [MO11], i.e. on the application of the Golod–Shafarevich theorem. However, our proof of the existence of Dehn monsters is simpler than that in [MO11], in spite of the fact that we should care

\(^*\)\text{When this work have been written, we discovered that an answer to this question is contained also in [KhM14].}
about additional properties of the monsters under construction (though, the residual finiteness is obtained at no cost within our approach).

1. Infinite-dimensionality test

Consider the free associative algebra $F \langle X \rangle$ (with unity) with finite basis $X$ over a field $F$. This algebra consists of polynomials in non-commuting variables with coefficient from $F$. We always understand the degree $\deg u$ of a polynomial $u \in F \langle X \rangle$ as the minimal degree of the monomials of this polynomial. For example, $\deg(xy - yx + xy^{2018}x) = 2$.

The following convenient test for infinite-dimensionality of a graded algebra is a corollary the well-known result of Golod–Shafarevich [GSh64] and belongs, apparently, to M. Ershov, see [Er12], Corollary 2.2.

**Infinite-dimensionality test.** If $r_n$ is the number of elements of degree $n$ in a set $R$ consisting of homogeneous elements of the finitely generated free associative algebra $F \langle X \rangle$, where $r_0 = r_1 = 0$, and the series

$$1 - |X|t + H_R(t) \overset{\text{def}}{=} 1 - |X|t + \sum_{i=2}^{\infty} r_i t^i$$

converges to a negative number for some $t \in (0, 1)$, then the quotient algebra $A = F \langle X \rangle / (R)$ is infinite-dimensional.

We need also the following obvious fact.

**Lemma 1.** If a field $F$ and a set $X$ are finite, then, for any positive integers $n$ and $d$, there are only finitely many $n$-tuples $(u_1, \ldots, u_n)$, $(u_2, \ldots, u_n)$, $\ldots$ of elements of the free associative algebra $F \langle X \rangle$ such that, for any different $i$ and $l$, there is $s$ such that $\deg(u_i - u_l) < d$.

**Proof.** This inequality means that all tuples represent different elements of the algebra $(F \langle X \rangle / (X^d)^n$, which is obviously finite. Here, $(X)$ is the ideal generated by the basis and, hence, $(X)^d$ consists of all polynomials of degree at least $d$.

2. Construction of the algebra $A$

Take a positive integer $\alpha$ and a recursive everywhere defined function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ (see the next section for a particular choice of $\alpha$ and $f$) and consider also a recursive enumeration $P_1, P_2, \ldots$ of all programmes with empty input and output alphabet consisting of a finite field $F$, a finite set $X$ containing at least two elements, and three additional symbols: “+” (plus), “,” (comma) and “;” (semicolon). The output sequence of each such programme is treated as a sequence of tuples of elements of the free associative algebra $F \langle X \rangle$: elements of each tuples are separated by commas and the tuples are separated by semicolons; successively written symbols from $F \sqcup X$ are treated as the product, redundant pluses and commas are ignored. For example, for $F = \mathbb{Z}_3 = \{0, 1, 2\}$ and $X = \{x, y\}$, the sequence

$$++ xy2y+, \ldots, 221 + + + + + + 0xy1yy + + + + + + + + + + + + + xex2112 + + + + + + yxyxy22 + + +$$

is treated as four tuples (two of them are empty, and one is incomplete):

$$(2xy^2, 2); (\); (x^3 + yxyxy + \ldots)$$

We shall construct algebra $A$ in the form $A = F \langle X \rangle / (R)$. The algorithm writing down the set of relations $R$ looks simple.

**Main algorithm.** Initially, the set $R$ consists of monomials $x^n$ for all $x \in X$. Further, on a step $k$, we run the programme $N(k)$ (in parallel with all other running programmes) and go to the step $k + 1$.

The programme $N(k)$ (i.e. the programme $N$ inputting a positive integer $k$) performs the following tasks.

**Programme $N(k)$:**

1. Run the program $P_k$ (in parallel with all other running programmes).
2. Trace the work of $P_k$: when $P_k$ writes a semicolon, $N$ acts as follows:
   a) interrupt (pause) the programme $P_k$;
   b) check that all tuples of elements of the algebra $F \langle X \rangle$ outputted by $P_k$ so far have the same length $n$, i.e. the output of $P_k$ has the form:

$$u_{i1}, \ldots, u_{i1}; u_{i2}, \ldots, u_{i2}; \ldots; u_{in}, \ldots, u_{in};$$

for some $u_{ij} \in F \langle X \rangle$ and some $n \in \mathbb{N}$;

if the output does not have this form, then $N$ kills the programme $P_k$ and terminates;
3. Infinite-dimensionality of the algebra $A$

Note that, at the every moment, there is a finite number of programmes $P_i$ working in parallel (at most $k$ on $k$-th step of the main algorithm) and the same number of copies of the programme $N$ (each copy traces one programme $P_i$). Moreover, each copy of the program $N$ either
- works eternally and adds nothing to the set $R$,
- terminates on the step 2b) and, in this case, also adds nothing to the set of relators $R$,
- or terminates on the step 2d) and, in this case, adds a finite set of homogeneous relators $w_1, w_2, \ldots$ of a large degree: $\deg w_i \geq f(n, k)$ (where $k$ is the number of this copy of $N$); the number of added relators of the each particular degree is at most $n$. More precisely, we have the inequality:

$$r_i(k) \leq \begin{cases} 0, & \text{if } i < f(n(k), k); \\ n(k), & \text{if } i \geq f(n(k), k); \end{cases}$$

where $r_i(k)$ is the number of relators of degree $i$ added by the $k$-th copy of the programme $N$ and $n(k)$ is the a length of tuples outputted by $P_k$ (if $P_k$ writes out tuples of different lengths, or some incorrect output, or does not write anything, then we assume $n(k) = \infty$).

3. Infinite-dimensionality of the algebra $A$

To apply the infinite-dimensionality test, we have to estimate the sum of the Golod–Shafarevich series:

$$H_R(t) = \sum_{i=2}^{\infty} r_it^i = |X|^\alpha + \sum_{k=1}^{\infty} \left( \sum_{i=2}^{\infty} r_i(k)t^i \right) \leq |X|^\alpha + \sum_{k=1}^{\infty} \left( \sum_{i=f(n(k), k)}^{\infty} n(k)t^i \right) = \sum_{k=1}^{\infty} t^{f(n(k), k)} n(k) \frac{1}{1-t}.$$

Setting $t = \frac{1}{2}$ and taking into account that $t < 2^x$ for $x \in \mathbb{N}$, we obtain

$$H_R \left( \frac{1}{2} \right) = \frac{|X|}{2^{\alpha}} + 2 \sum_{k=1}^{\infty} \left( \frac{n(k)}{2^{f(n(k), k)}} \right) < 1 - |X| + 2 \sum_{k=1}^{\infty} \left( \frac{1}{2^{f(n(k), k)}} \right).$$

Now, put $f(n, k) = n + k + 2$ and $\alpha = |X| + 1$ and obtain

$$H_R \left( \frac{1}{2} \right) < 1 + 2 \sum_{k=1}^{\infty} \left( \frac{1}{4 \cdot 2^k} \right) = 1,$$

i.e. $1 - \frac{1}{2}|X| + H_R \left( \frac{1}{2} \right) < 0$ if $|X| > 4$.

According to the test from Section 1, the graded algebra $A = F \langle X \rangle / (R)$ is infinite-dimensional. Certainly, this algebra is residually finite because any finitely generated graded algebra over finite field is approximated by its finite quotient algebras $A/(X)^n$.

4. Algorithmic finiteness of Cartesian powers of $A$

Suppose that there is a programme $P$ writing out an infinite sequence of pairwise different elements of the algebra $A^n$. Certainly, this programme can be assumed to have the output alphabet $X \sqcup F \sqcup \{+, u_1, u_2, \ldots, u_\lambda \}$ and writes out pairwise different elements of the algebra $A^n$ in the prescribed format:

$$(u_{11}, \ldots, u_{n_1}; u_{12}, \ldots, u_{n_2}; \ldots), \text{ where } u_{ij} \in F \langle X \rangle$$

(any programme can be transformed into this form).

The programme $P$ is assigned a number $k$ by our enumeration of programmes, i.e. $P = P_k$. Then the programme $N(n)$ is launched on $k$-th step of our main algorithm (in parallel with other working programmes); $N(k)$, in its turn, launches the programme $P_k = P$ and trace it. Two cases are possible.

Case I: $\deg(u_{si} - u_{sl}) \geq f(n, k)$ for some distinct $i$ and $l$ and all $s$. We assume that $i < l$ and $l$ is the minimal number with these properties. Then, after writing out the $l$-th semicolon, the programme $P = P_k$ is interrupted on step 2a) of the tracer-programme $N(k)$. Further, on steps 2b) and 2c), the verification is completed successfully and, on step 2d), all homogeneous components of the differences $u_{si} - u_{sl}$ are added to the set $R$ (for all $s \in \{1, \ldots, n\}$). This means that the tuples $(u_{i1}, \ldots, u_{i\lambda})$ and $(u_{i1}, \ldots, u_{i\lambda})$ outputted by the programme $P$ represent the same element of the algebra $A^n$ and we come to a contradiction.

Case II: for any distinct $i$ and $l$, there is $s$ such that $\deg(u_{si} - u_{sl}) < f(n, k)$. This contradicts Lemma 1.

The theorem on strongly algorithmically finite algebras is proven.
5. Proof of the main theorem
Take the algebra $A$ over a residue field $\mathbb{Z}_p$ with finite generating set $X$ from the theorem on strongly algorithmically finite algebras and consider the set of matrices

$$G = \begin{pmatrix} H & A \\ 0 & 1 \end{pmatrix},$$

where $H$ is the subgroup of the multiplicative group of $A$ generated by all elements of the form $1 + x$, where $x \in X$ (these elements are invertible, because elements of $X$ are nilpotent). Clearly, $G$ is a group:

$$\begin{pmatrix} h & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h' & a' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} hh' & a + ha' \\ 0 & 1 \end{pmatrix},$$

and $G$ is a semidirect product of

the normal subgroup $\begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \simeq \bigoplus_{i=1}^{\infty} \mathbb{Z}_p$ and the non-normal subgroup $\begin{pmatrix} H & 0 \\ 0 & 1 \end{pmatrix} \simeq H$.

The group $G$ is finitely generated ($(|X| + 1)$-generated) as it is generated by matrices $\begin{pmatrix} 1 + X & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, because, obviously, elements of the set $1 + X$ generate $A$ as a ring. All Cartesian powers $G^n$ are algorithmically finite because all Cartesian powers of the algebra $A$ are algorithmically finite. Clearly, the group $G$ is recursively presented since the algebra $A$ is recursively presented. The residual finiteness of the group $G$ also follows immediately from the residual finiteness of the algebra $A$, because the group of invertible matrices (and the algebra of all matrices) over residually finite algebra is residually finite. This completes the proof of the main theorem.

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