Signal and Noise in Correlation Matrix

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Abstract

Using random matrix technique we determine an exact relation between the eigenvalue spectrum of the covariance matrix and of its estimator. This relation can be used in practice to compute eigenvalue invariants of the covariance (correlation) matrix. Results can be applied in various problems where one experimentally estimates correlations in a system with many degrees of freedom, like for instance those in statistical physics, lattice measurements of field theory, genetics, quantitative finance and other applications of multivariate statistics.

Key words: random matrix theory, correlation matrix, eigenvalue spectrum
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Statistical systems with many degrees of freedom appear in numerous research areas. One of the most fundamental issues in studies of such systems is the determination of correlations. In practice, one encounters frequently the following situation: one samples the system many times by carrying out independent measurements. For each sample one estimates values of the elements of the covariance matrix, and then takes the average over a set of samples. The statistical uncertainty of the average of individual elements of the matrix generically decreases with the number of independent measurements \( T \) as \( \sim 1/\sqrt{T} \). There are \( N(N + 1)/2 \) independent elements of the correlation matrix for a system with \( N \) degrees of freedom. Thus, naively, the total uncertainty encoded in the correlation matrix may be expected to be proportional to \( N(N + 1)/2 \) and \( 1/\sqrt{T} \), and therefore to be large for 'non-local' quantities which depend on many elements of the correlation matrix even if one performs a large number of measurements, of the order of the number of degrees of freedom in the system. It turns out that this naive expectation is far from true. Such 'non-local' quantities occur, in particular, in the eigenvalue analysis of the correlation matrix [1]. A question which we address in this paper is how
the spectrum of the experimentally measured covariance (two-point correlation) matrix is related to the spectrum of the genuine correlation matrix for the system.

To be specific, consider a statistical system consisting of $N$ real degrees of freedom $x_i, i = 1, \ldots, N$ with a stationary probability distribution:

$$p(x_1, \ldots, x_N) \prod_{n=1}^{N} dx_n$$

such that

$$\int x_i p(x_1, \ldots, x_N) \prod_{n=1}^{N} dx_n = 0 \quad \forall i.$$  (2)

The covariance matrix for the system is defined as

$$C_{ij} = \int x_i x_j p(x_1, \ldots, x_N) \prod_{n=1}^{N} dx_n.$$  (3)

Further, assume that the system belongs to the Gaussian universality class. Under this assumption the probability distribution can be approximated by

$$p(x_1, \ldots, x_N) \prod_{n=1}^{N} dx_n = \left[ (2\pi)^N \det C \right]^{-1/2} \exp \left( -\frac{1}{2} \sum_{ij} x_i C_{ij}^{-1} x_j \right) \prod_{n=1}^{N} dx_n.$$  (4)

where $C_{ij}$ is a covariance matrix (3) of the system. By construction it is a symmetric, positive-definite matrix. In fact, for a wide class of models, the Gaussian approximation well describes the large $N$ behavior of the system as a consequence of the central limit theorem. Deviations from the Gaussian behavior can result either from the presence of fat (heavy) tails in the probability distribution or from collective excitations of many degrees of freedom. None of these effects will be discussed here.

Experimentally, the correlation matrix is computed as follows. One performs a series of $T$ independent measurements. Assume $T > N$. The measured values $x_n$ form a rectangular $N \times T$ matrix $X$ with elements $X_{nt}$, where $X_{nt}$ is the measured value of the $n^{th}$ degree of freedom $x_n$ in the $t^{th}$ experiment $t = 1, \ldots, T$. The experimental correlation matrix is computed using the following estimator

$$c_{ij} = \frac{1}{T} \sum_{t=1}^{T} X_{it} X_{jt} = \frac{1}{T} \{XX^\tau\}_{ij}.$$  (5)

where $X^\tau$ is the transpose of $X$. We expect that for $T \to \infty$ the estimated values $c_{ij}$ will approach the elements $C_{ij}$. More precisely, if the measurements are independent, the probability distribution of measuring a matrix $X$ of values
$X_{nt}$ is a product of probabilities for individual measurements

$$P(X)DX = \prod_{t=1}^{T} \left( p(X_t, \ldots X_{Nt}) \prod_{n=1}^{N} dX_{nt} \right)$$  \hspace{1cm} (6)

where

$$DX = \prod_{n,t=1}^{N,T} dX_{nt}.$$  \hspace{1cm} (7)

In particular, for the Gaussian approximation

$$P(X)DX = \mathcal{N} \exp \left( -\frac{1}{2} \sum_{t=1}^{T} X_{it}C_{ij}^{-1}X_{jt} \right) \prod_{n,t=1}^{N,T} dX_{nt}$$

$$= \mathcal{N} \exp \left( -\frac{1}{2} \text{Tr} \ X^\tau C^{-1}X \right) DX$$  \hspace{1cm} (8)

where $\mathcal{N}$ is a normalization factor which ensures that $\int P(X)DX = 1$. In this particular case $\mathcal{N} = [(2\pi)^N \det C]^{-T/2}$. All averages over measured values $X_{nt}$ are calculated with this probability measure. We shall denote these averages by $\langle \ldots \rangle$. In particular we see that

$$\langle X_{it}X_{j't'} \rangle = C_{ij}\delta_{tt'}.$$  \hspace{1cm} (9)

This relation reflects the assumed absence of correlations between measurements. In general, if measurements are correlated, the right-hand side of the last equation can be expressed by a matrix $C_{it,j't'}$ in double indices.

After these introductory remarks, we come back to the problem of relating the spectrum of the covariance matrix $C$ to the spectrum of its estimator $c$. We denote the eigenvalues of the matrix $C$ by $\Lambda_n \; n = 1, \ldots, N$. For a given set of eigenvalues we can calculate matrix invariants, like for example the spectral moments

$$M_k = \frac{1}{N} \text{Tr} \ C^k = \frac{1}{N} \sum_{n=1}^{N} \Lambda_n^k = \int d\Lambda \; \rho_0(\Lambda)\Lambda^k$$ \hspace{1cm} (10)

where the density of eigenvalues $\rho_0(\Lambda)$ is defined as

$$\rho_0(\Lambda) = \frac{1}{N} \sum_{n=1}^{N} \delta(\Lambda - \Lambda_n).$$  \hspace{1cm} (11)

The question is how these quantities are related to the analogous quantities defined for the estimator of the correlation matrix $c$

$$m_k = \frac{1}{N} \langle \text{Tr} \ c^k \rangle = \int d\lambda \rho(\lambda)\lambda^k$$  \hspace{1cm} (12)
where the eigenvalue density of the matrix estimator is

\[ \rho(\lambda) = \frac{1}{N} \left\langle \sum_{n=1}^{N} \delta(\lambda - \lambda_n) \right\rangle. \]  

(13)

We expect that the dependence of the estimated spectrum \( \rho(\lambda) \) and the genuine spectrum \( \rho_0(\Lambda) \) should be controlled by \( T \) and \( N \). Indeed, as we shall see later, it turns out that for \( N \to \infty \) this dependence is governed by the parameter \( r = N/T \), which we assume to be finite.

In order to derive the relation between the spectral properties of the covariance matrix and its estimator it is convenient to define resolvents:

\[ G(Z) = \left( Z I_N - C \right)^{-1} \]  

(14)

and

\[ g(z) = \left\langle \left( z I_N - c \right)^{-1} \right\rangle = \left\langle \left( z I_N - \frac{1}{T} XX^T \right)^{-1} \right\rangle \]  

(15)

where \( Z \) and \( z \) are complex variables. The symbol \( I_N \) stands for the \( N \times N \) unit matrix. Expanding the resolvents in \( 1/Z \) (or \( 1/z \)) one sees that they can be interpreted as generating functions for the moments

\[ M(Z) = \frac{1}{N} \text{Tr}[ZG(Z)] - 1 = \sum_{k=1}^{\infty} \frac{1}{Z^k} M_k \]  

(16)

and

\[ m(z) = \frac{1}{N} \text{Tr}[zg(z)] - 1 = \sum_{k=1}^{\infty} \frac{1}{z^k} m_k . \]  

(17)

From the relation between \( M(Z) \) and \( m(z) \) one can determine the corresponding relation between the eigenvalue spectra \( \rho_0(\Lambda) \) and \( \rho(\lambda) \). Indeed, taking the imaginary part of \( \text{Tr} g(z)/N \) (and \( \text{Tr} G(Z)/N \)) for \( z = \lambda + i0^+ \) (or \( Z = \Lambda + i0^+ \)), where \( \lambda \) is real, we can directly calculate the eigenvalue densities \( \rho(\lambda) \) (and \( \rho_0(\Lambda) \)):

\[ \rho(\lambda) = -\frac{1}{\pi} \text{Im} \left\langle \frac{1}{N} \text{Tr} g(\lambda + i0^+) \right\rangle \]  

(18)

as follows from the standard relation for distributions: \((x + i0^+)^{-1} = \text{PV}x^{-1} - i\pi\delta(x)\), where PV stands for principal value.

The fundamental relation between the generating functions (16) and (17) is derived in the Appendix by means of a diagrammatic technique [3] for calculating integrals (15) with the Gaussian measure (8). The large \( N \) limit corresponds to the planar limit in which only planar diagrams contribute. This significantly simplifies considerations and allows one to write down closed formulae for the resolvents.
This fundamental relation between the generating functions (16) and (17) reads

\[ m(z) = M(Z) \] (19)

where the complex number \( Z \) is related to \( z \) by the conformal map (56):

\[ Z = \frac{z}{1 + rm(z)} \] (20)

or equivalently, if we invert the last relation for \( z = z(Z) \), as:

\[ z = Z \left(1 + rM(Z)\right). \] (21)

The equations (19,20) were already announced in our earlier work [2]. They can for example be used to compute moments of the genuine correlation function \( C \) from the experimentally measured moments of the estimator \( c \). Indeed, combining (19) and (20) we obtain the following equation:

\[ m(z) = M \left(\frac{z}{1 + rm(z)}\right) \] (22)

which gives a compact relation between moments \( m_k \) and \( M_k \):

\[ \sum_{k=1}^{\infty} \frac{m_k}{z^k} = \sum_{k=1}^{\infty} \frac{M_k}{z^k} \left(1 + r \sum_{l=1}^{\infty} \frac{m_l}{z^l}\right)^k \] (23)

from which we can recursively express \( m_k \) by \( M_l \), \( l = 1, \ldots, k \)

\[ m_1 = M_1 \]
\[ m_2 = M_2 + rM_1^2 \]
\[ m_3 = M_3 + 3rM_1M_2 + r^2M_1^3 \] (24)

\[
\ldots
\]

or inversely: \( M_k \) by \( m_l \), \( l = 1, \ldots, k \):

\[ M_1 = m_1 \]
\[ M_2 = m_2 - rm_1^2 \]
\[ M_3 = m_3 - 3rm_1m_2 + 2r^2m_1^3 \] (25)

\[
\ldots
\]

Let us observe that for \( r < 1 \) the functions \( M(Z) \) and \( m(z) \) can also be expanded around \( z = Z = 0 \). In this case

\[ M(Z) = -\sum_{k=0}^{\infty} Z^k M_{-k}, \] (26)
where
\[ M_{-k} = \frac{1}{N} \text{Tr} C^{-k}. \]  
(27)

Similarly
\[ m(z) = -\sum_{k=0}^{\infty} z^k m_{-k}, \]  
(28)

where
\[ m_{-k} = \frac{1}{N} \langle \text{Tr} c^{-k} \rangle. \]  
(29)

Using the same manipulation as before we obtain
\[
\sum_{k=1}^{\infty} M_{-k} Z^k = \sum_{k=1}^{\infty} m_{-k} Z^k (1 - r - r \sum_{l=1}^{\infty} M_{-l} Z^l)^k \]  
(30)

and hence:
\[
M_{-1} = (1 - r)m_{-1} \\
M_{-2} = (1 - r)^2 m_{-2} - r(1 - r)m_{-1}^2 \\
M_{-3} = (1 - r)^3 m_{-3} - r(1 - r)^2 m_{-1} m_{-2} - r^2 (1 - r)m_{-1}^3 \\
\ldots .
\]  
(31)

The relations between moments can be used directly in practical applications to clean the spectrum of the correlation matrix. Before discussing this let us make a comment. The formulae (19) and (20) encode full information about the relation between the eigenvalue spectrum \( \rho_0(\Lambda) \) and \( \rho(\lambda) \) for a given \( r \). In particular, if one knows the spectrum \( \rho_0(\Lambda) \) of the correlation matrix \( C \) one can exactly determine for a given \( r \) the shape of the spectrum \( \rho(\lambda) \) of the estimator dressed by statistical fluctuations. One does it as follows. From the eigenvalue spectrum \( \rho_0(\Lambda) \) one deduces an explicit form of the function \( M(Z) \) and of the right hand side of the equation (21). Inverting the equation (21) for \( Z \) one finds the dependence \( Z = Z(z) \). Inserting it to the equation (19) one determines the function \( m(z) \). Taking the imaginary part along the cuts of the map \( m(z) \) on the real axis (18) one eventually finds \( \rho(\lambda) \). One can easily write a numerical program which realizes this procedure. In few cases the solution is possible analytically. Let us shortly discuss them.

Consider the correlation matrix \( C \) whose spectrum is given by a sequence of degenerate eigenvalues \( \mu_i, \ i = 1, \ldots, K \) with degeneracies \( n_i \). Consequently, defining \( p_i = n_i/N, \sum_i p_i = 1 \), we have
\[
M(Z) = \sum_{i=1}^{K} \frac{p_i \mu_i}{Z - \mu_i}. \]  
(32)

This form is particularly simple to discuss. One should however keep in mind that the relations (19,20) remain valid also in a more general case, for instance,
when in the limit $N \to \infty$ the spectrum of $\rho_0(\Lambda)$ is not a sum of delta functions but approaches some continuous distribution. The map (21) now reads

$$z = Z \left(1 + r \sum_{i=1}^{K} \frac{p_i \mu_i}{Z - \mu_i}\right).$$

(33)

Clearly, if we solve this equation for $Z = Z(z)$ we obtain a multi-valued function, except in the case $r = 0$ for which we have a simple relation $z = Z$. The “physical” Riemann sheet of the map $Z = Z(z)$ is singled out by the condition $Z \to z$ for $z \to \infty$. On this sheet the complex $z$-plane is mapped on a part of the $Z$ plane without a simply or multiply connected region surrounding the poles at $Z = \mu_i$.

As an illustration, let us consider the simplest case, where $K = 1$. In this case we have only one eigenvalue $\mu_1 = \mu$ and $p_1 = 1$ and $M(Z) = \mu/(Z - \mu)$. The map (20) has the following form

$$z = Z + r \frac{Z\mu}{Z - \mu}.$$  

(34)

If one rewrites the right-hand side of this equation using polar coordinates $(R, \phi)$ around the pole: $Z - \mu = Re^{i\phi}$:

$$z = Re^{i\phi} + \frac{r\mu^2}{R} e^{-i\phi} + \mu(1 + r)$$

(35)

one can see that the equation is invariant under the “duality” transformation:

$$R \longleftrightarrow \frac{r\mu^2}{R}, \quad \phi \longleftrightarrow -\phi$$

(36)

which maps the inside of the circle $|Z - \mu| = \mu \sqrt{r}$ onto the outside and vice versa. Obviously, the outside corresponds to the “physical” Riemann sheet of the inverse map $Z = Z(z)$

$$Z = \frac{1}{2} \left((1 - r)\mu + z + \sqrt{(z - \mu_+)(z - \mu_-)}\right),$$

(37)

since in this region $Z \sim z$ for $z \to \infty$. The two constants in the last equation are $\mu_\pm = \mu(1 \pm \sqrt{r})^2$. Along the cut $\mu_- < z < \mu_+$ on the real axis the map $Z = Z(z)$ becomes complex and ambiguous: it has a phase (sign) ambiguity which is related to the fact that the cut is mapped into the limiting circle where the two Riemann sheets meet.

From (19) we can easily find the generating function $m(z)$ and then from (18) the spectral density of the correlation matrix $c$:

$$\rho(\lambda) = \frac{1}{2\pi \mu r} \frac{\sqrt{(\mu_+ - \lambda)(\lambda - \mu_-)}}{\lambda}.$$  

(38)
This is a well known result in random matrix theory [4] for the spectral distribution of the Wishart ensemble. It is interesting to interpret this result as a statistical smearing of the initial spectral density $\rho_0(\Lambda)$ given by the delta function localized at $\mu$ into a wide peak $\rho(\lambda)$ supported by the cut $[\mu-,\mu+]$, due to a finite series of measurements. The larger $r$ the larger is the width of the resulting distribution $\rho(\lambda)$. For $r = 1$ the lower limit of the distribution is at $\mu_- = 0$ signaling the appearance of zero modes in the matrix $c$. One can show by considering anti-Wishart matrices that above the limiting value $r = 1$ the zero mode sector of $c$ grows with $r$ reflecting an increasing indeterminacy of the spectrum of the covariance matrix $C$ when the underlying statistical sample becomes too short.

As a second example we consider the case, where the genuine covariance matrix $C$ has two different eigenvalues $\mu_1, \mu_2$ with relative weights $p_1, p_2, p_1 + p_2 = 1$. In this case we can also find an explicit form of the map $Z(z)$ solving the corresponding cubic (Cardano) equation. Depending on the parameters $\mu_i, p_i$ the map $Z(z)$ has one or two cuts on the real axis on the $z$ plane, which means that the corresponding eigenvalue distribution $\rho(\lambda)$ has a support on one or two intervals (fig.1). It is a simple exercise to find the critical value $r_c$ of $r$ at which a single cut solution splits into a two-cut one:

$$r_c = \frac{(\mu_2 - \mu_1)^2}{((p_1 \mu_1^2)^{1/3} + (p_2 \mu_2^2)^{1/3})^3}.$$  \hfill \text{(39)}$$

For instance, this formula gives $r_c \approx 0.01$ if the correlation matrix $C$ has two eigenvalues $\mu_1 = 1$ and $\mu_2 = 1.1$ and the corresponding weights $p_1 = p_2 = 1/2$. Thus in this case, to observe a bimodal signal in the measured spectrum one has to perform $T$ measurements with $T$ of order $100N$. If the gap $\Delta \mu$ between the eigenvalues $\mu_1$ and $\mu_2$ is larger the binomial structure is visible already for smaller $T$. As an example we show in fig.1 typical shapes of the distribution for $\mu_1 = 1, \mu_2 = 2, p_1 = p_2 = 1/2$ for $r < r_c$, $r = r_c$ and $r > r_c$. The shape of the spectral functions for $r$ in the range from 1 to, say, $0.2 - 0.3$ resembles that for a single eigenvalue. When $r$ approaches $r_c$ it starts to deviate from this shape developing a double peak structure which eventually splits into two separate parts at $r = r_c$. If $r$ is further decreased the two peaks get narrower. They eventually entirely localize at $\mu_1 = 1$ and $\mu_2 = 2$ for $r = 0$, that is for an infinitely long sample for which the spectrum of the underlying correlation matrix is recovered.

The method can be straightforwardly generalized from $K = 1, 2$ to arbitrary $K, \mu_1, \ldots, \mu_K$ with $\sum p_i = 1$, although only the $K = 3$ case is solvable analytically (quartic Ferrari equation). In other cases one can use a numerical implementation of the general procedure, which we described before, to determine the shape of the spectrum of the estimator $\rho(\lambda)$ from any given distribution $\rho_0(\Lambda)$ and for any $r$. 

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Fig. 1. The figures represent spectra of the eigenvalue distributions $\rho(\lambda)$ of the experimental correlation matrix measured in a series of measurements for $r = 0.01, 0.115, 0.3$, respectively. The underlying correlation matrix has two eigenvalues $\mu_1 = 1$ and $\mu_2 = 2$ with the weights $p_1 = p_2 = 0.5$. At the critical value $r = r_c = 0.115$ (eq. 39) the spectrum splits. The spectral densities are calculated analytically.

In practice one is however interested in the opposite problem that is in the determination of the spectrum $\rho_0(\Lambda)$ of the genuine correlation matrix $C$ from the distribution of the measured eigenvalues. This is a difficult problem for the following reason. Having one sample of $T$ measurements of a system with $N$ degrees of freedom one has as a result only $N$ eigenvalues of the correlation matrix. From $N$ random numbers it is impossible to reconstruct accurately the exact form of the underlying distribution $\rho(\lambda)$ according to which they are distributed. Thus the input function $\rho(\lambda)$ for the procedure leading from $\rho(\lambda)$ to $\rho_0(\Lambda)$ has a large statistical uncertainty. In effect the output function
\( \rho_0(\Lambda) \) is not well controlled.

In some applications one is not interested in the exact shape of the spectral function \( \rho_0(\Lambda) \) but in its moments. In such cases one can directly make use of the relations (25) between the measured moments \( m_k \) and the moments of the original distribution \( \rho_0(\Lambda) \) to determine the latter ones.

Moreover, the relations (25) between the moments may in some cases significantly improve the determination of the eigenvalues \( \Lambda \) of the original distribution. Assume that on top of the purely statistical information obtained by independent measurements of a system consisting of \( N \)-degrees of freedom we have at our disposal some additional non-statistical knowledge about the system. For example, assume that we know that degrees of freedom can be grouped into \( K \) sectors of degenerate independent constituents. Each sector is represented by an eigenvalue \( \mu_k \) and a degeneracy \( n_k \), or equivalently by the fraction \( p_k = n_k/N \) of eigenvectors related to this eigenvalue. In other words, we assume that the resolvent \( M(Z) \) (16) of the correlation function \( C \) is given by the formula (32) with a specific \( K \). Thus the problem of determining eigenvalues \( \Lambda \) is reduced to the problem of determining parameters \( \mu_k, p_k \). If \( K \ll N \) the problem has much less unknowns. This non-statistical knowledge of the system can be used as follows. We can explicitly express the moments \( M_k \) by the yet unknown parameters \( p_j, \mu_j \). We denote the corresponding functions by \( M_k^{th}(p_j, \mu_j) \). On the other hand we can measure experimentally the moments \( m_k \) and from the relations (25) we can find the corresponding values which we denote by \( M_k^{exp}(m_j) \). Using the jack-knife procedure we can also estimate the statistical errors \( \Delta_k \) of \( M_k^{exp} \)'s. Minimizing \( \chi^2(p_j, \mu_j) \):

\[
\chi^2 = \sum_{k=1}^{L} \left( \frac{M_k^{th}(p_j, \mu_j) - M_k^{exp}}{\Delta_k} \right)^2
\]

we can eventually find optimal values of the parameters \( p_j, \mu_j \). Let us make a few comments. Obviously, \( L \) must be equal or larger than the number of unknown parameters. If it is equal then the minimization of \( \chi^2 \) amounts to solving equations \( M_k^{th}(p_j, \mu_j) = M_k^{exp} \). In practice, if possible, \( L \) should be taken larger than the number of free parameters. In this case the weights \( 1/\Delta_k^2 \) in \( \chi^2 \) take care of the gradually decreasing importance of the higher moments which are usually estimated with larger errors.

Let us illustrate the method at work using the following exercise as an example. The exercise has two parts. First we generate \( T \times N \) matrix \( X \) from the Gaussian distribution (8) with a given covariance matrix \( C \) which has \( K \) eigenvalues \( \mu_k \) with the weights \( p_k, k = 1, \ldots, K \). Then we treat the parameters \( \mu_k, p_k \) as unknown, and experimentally determine their optimal values using the method of moments. In the end we compare the measured values with those used in the generator. We repeat this procedure many times to estimate the
Fig. 2. The figures represent eigenvalue distributions measured in $n = 10^5$ measurements for $N = 100$, $T = 333$ (upper); the cleaned spectrum using the additional information that there are exactly two eigenvalues (middle); and using the information that there are two equally probable eigenvalues in the correlation matrix (lower). The peaks in the last two figures localize around correct values $\mu_1 = 1$ and $\mu_2 = 2$ despite the number of measurements is relatively small: $N/T = r = 0.3$.

error one makes, when one estimates the spectrum in a single measurement.

As a first example we consider the case $K = 2$, $\mu_1 = 1$, $\mu_2 = 2$ and $p_1 = p_2 = 0.5$ as before and with $N = 100$ and $r = N/T = 0.3$. The theoretical curve of the distribution $\rho(\lambda)$ is shown again in fig. 2 where it is compared with the
Fig. 3. The figure represents eigenvalue distributions measured in $n = 10^5$ measurements for $N = 100$, $T = 333$ (solid line) and the cleaned spectrum using additional information that there are three equally probable eigenvalues (dashed line); The peaks in the right figure localize around correct values $\mu_1 = 1$, $\mu_2 = 2$ and $\mu_3 = 3$ despite the asymmetry parameter is $r = 0.3$.

experimentally determined distribution. The experimental curve is obtained by averaging over $n = 10^5$ independent experiments. In each experiment the matrix $X$ was generated and the eigenvalues of $c$ were calculated. Thus the resulting experimental histogram was constructed out of $n \cdot N = 10^7$ eigenvalues. One sees that the experimental curve fits very well to the theoretically predicted. Actually, one cannot distinguish by bare eye the two curves on the first plot in fig. 2. If we knew only this curve we would not be able to conclude from it that the underlying covariance matrix has only two eigenvalues. Now assume that we know that there are exactly two eigenvalues and use the method of moments to compute them. We have to measure at least $L = 3$ moments $m_l$, $l = 1, 2, 3$ in order to determine by minimizing $\chi^2$ (40) the three unknown parameters $\mu_1$, $\mu_2$, $p_1$. The resulting spectrum $\rho_0(\lambda)$ averaged over $n$ experiments is shown in the next plot in fig. 2. The spectrum clearly shows the double peak structure with the correct localization of peaks around $\mu_1 = 1$ and $\mu_2 = 2$. The areas under the peaks are approximately equal which means that $p_1 \approx p_2 \approx 0.5$, as expected. Assume that on top of the information that there are exactly two eigenvalues we additionally know that both are equally probable $p_1 = p_2 = 0.5$. In effect, we have only two unknown parameters $\mu_1$ and $\mu_2$. As we see in fig. 2 this additional information leads to the further sharpening of the resulting spectrum around the expected values $\mu_1 = 1$ and $\mu_2 = 2$. Another example, for $K = 3$ and $\mu_1 = 1 \mu_2 = 2$ and $\mu_3 = 3$ is shown in fig. 3. The method of moments works very well and allows us to localize the positions of eigenvalues of the underlying covariance matrix $C$. It is clear that additional information considerably reduces the error of the estimate.

To summarize: In the limit $N \to \infty$ we give an exact relation between the spectrum of the true correlation matrix between the degrees of freedom in a statistical system and the averaged spectrum of the natural estimator of this
correlation matrix used in a statistical sampling. The latter depends on the parameter \( r = N/T \). For finite \( r \) the spectrum of the estimator is smeared and hides behind the statistical noise the information about the positions of eigenvalues of the exact correlation matrix. The averaged spectrum is very similar to that of a spectrum obtained from a single matrix \( c_{ij} \) (5) used in practical applications. The exact relation can be formulated as an infinite sequence of linear relations between the moments of the true correlation matrix and of its averaged estimator. In principle both sets are equivalent. In practice, the estimator is known only as a single realization of the measuring process and the estimated moments are known only approximately. The relations between the moments can be nevertheless used to determine the spectrum of the true correlation matrix from a single estimator. We illustrate the accuracy of the method of moments on simple examples. The method of moments can be used in practical applications. To give an example, the spectral analysis of the covariance matrix plays the central role in the portfolio assessment. The knowledge of the covariance matrix is crucial for the optimal asset allocation. Typically one constructs the covariance matrix for \( N \) of order 500 assets’ price changes (e.g. SP500 data) and estimates it using a sample of say four years of a daily data. In this case \( T \) is of order 1000 and hence \( r = 0.5 \). As we have learned, in this case one almost entirely looses the signal coming from the eigenvalues of the genuine covariance matrix. On the other hand, it is known that in practice there are only few collective sectors on the market. These sectors are represented by \( K \ll N \) significant eigenvectors and related eigenvalues. One can use the method of moments to localize them [5].

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Note added

After completing our work we became aware of the following papers where some of the issues raised in our work are also discussed: V.A. Marchenko and L.A. Pastur, Math. USSR-Sb 1 (1967) 457; J.W. Silverstein and Z.D. Bai, J. Multivariate Anal. 54 (1995) 175; F. Lillo and R.N. Mantegna, cond-mat/0305546.

Appendix

We apply a diagrammatic method [3] to calculate the integral (15). Expanding the expression under the average (15) we obtain a series of polynomials in \( X \),
which can be integrated term by term for the Gaussian measure $P(X)DX$ (8). It is convenient to write the matrix $c$ as a product of three matrices 
$c = X \frac{1}{T} X^\tau$, the first of which is an $N \times T$ matrix, the second is $T \times T$, and the third $T \times N$. This reveals a sandwich structure of the terms of the expansion:

$$g(z) = \left\langle \frac{1}{z - c} \right\rangle = \left\langle \sum_{k \geq 0} \frac{c^k}{z^{k+1}} \right\rangle$$

$$= \left\langle \frac{1}{z} \right\rangle + \frac{1}{z} \frac{1}{T} X^\tau \frac{1}{z} + \frac{1}{z} \frac{1}{T} X^\tau \frac{1}{T} \frac{1}{T} X^\tau \frac{1}{z} + \ldots \right\rangle$$

The following graphical representation is used in the last equation. The matrix $X$ has two types of indices $X_{it}$, $i = 1, \ldots, N$ and $t = 1, \ldots, T$. We shall call the first an index of the $N$-type, and the second an index of the $T$-type. Indices of the $N$-type are drawn as filled circles and of the $T$-type as empty circles, a matrix $\frac{1}{z} \frac{T}{z}$ having both indices of the $N$-type is drawn as a solid line joining two filled circles corresponding to these indices, while a matrix $\frac{T}{T}$ as a dashed line which joins two empty circles representing two indices of the $T$-type of the matrix. An insertion of $X$ is drawn as an ordered pair of neighboring circles consisting of a filled circle followed by an open one, while an insertion of $X^\tau$ as an ordered pair of an open and a filled circle.

The $X$ insertions have to be integrated with the Gaussian measure $P(X)DX$. The terms $\frac{1}{z}$ and $\frac{T}{z}$ are constant from the point of view of the integration. The integration of the $X$ insertions amounts to calculating the correlation functions $\langle X_{it} X_{j't'} \rangle$. Using the Wick theorem we consecutively calculate $2k$-point correlation functions: the two-point correlation function

$$\langle X_{it} X_{j't'} \rangle = C_{ij} \delta_{tt'}$$

the four-point

$$\langle X_{iti1} X_{i2t2} X_{i3t3} X_{i4t4} \rangle = \langle X_{iti1} X_{i2t1} \rangle \langle X_{i3t3} X_{i4t4} \rangle$$
$$+ \langle X_{iti1} X_{i3t3} \rangle \langle X_{i2t2} X_{i4t4} \rangle$$
$$+ \langle X_{iti1} X_{i4t4} \rangle \langle X_{i2t2} X_{i3t3} \rangle$$

and higher correlation functions. All odd correlation functions vanish. All even functions can be expressed as sums of all distinct products of two-point functions given by (42). This observation leads to a particularly simple graphical representation: a two-point correlation function is drawn as a double line consisting of a solid line which connects two filled circles, and of a dashed line which connects two empty circles representing indices of $X$ and $X^\tau$. We associate the contribution $C_{ij}$ with the solid line and $\delta_{tt'}$ with the dashed line. We will draw the double lines as semi-circles. Using this convention the four-point
correlation function can be drawn as

\[ \langle X_{i_1 t_1} X_{i_2 t_2} X_{i_3 t_3} X_{i_4 t_4} \rangle = \]

\[ \begin{array}{c}
\includegraphics[width=0.5\textwidth]{diagram1.png}
\end{array} \]  \hspace{1cm} (44)

Putting together equations (41) and (44) we eventually arrive at the diagrammatic representation of \( g(z) \):

\[ \begin{array}{c}
\includegraphics[width=0.5\textwidth]{diagram2.png}
\end{array} = \] \hspace{1cm} (45)

We are interested in the limit \( N \to \infty, \ r = N/T = \text{const.} \) In this limit only planar graphs contribute to \( g(z) \), while non-planar diagrams are suppressed by factors proportional to powers of \( 1/N \to 0 \). This property follows from a simple counting argument. Each closed internal line of a diagram gives a factor \( N \) (or \( T \)) coming from taking a trace. Thus a closed solid line contributes a factor proportional to \( N \), and a closed dashed line to \( T \). Because we are interested in the limit of a constant ratio \( N/T \), \( T \) is proportional to \( N \). Each horizontal dashed line introduces a factor \( 1/T \sim 1/N \). For a planar diagram the number of closed loops is equal to the number of dashed horizontal lines and thus the powers of \( N \) and of \( 1/N \) cancel giving a contribution of order unity, while for a non-planar diagram, like for instance the last one displayed in (45), the power of \( 1/N \) coming from the number of dashed horizontal lines is larger than the power of \( N \) coming from the number of closed loops, and thus effectively the contribution of the diagram has at least one unbalanced factor \( 1/N \) which makes it vanish in the limit \( N \to \infty, \ r = N/T = \text{const.} \)

From here on we neglect the non-planar diagrams. In parallel to the generating function \( g(z) \) one can define a generating function:

\[ g_*(z) = \left\langle \frac{1}{T \mathbb{1} T - \frac{1}{2} \mathbf{X}^\tau \mathbf{X}} \right\rangle \] (46)

which contains all diagrams of the same shape as in (45) but with dashed and solid lines replaced. It is convenient to additionally introduce a class of one-line irreducible diagrams having the property that they cannot be divided into two separate diagrams by cutting a single horizontal line. Denote the sum of one-line irreducible diagrams with two external vertices of the \( N \)-type by \( \Sigma(z) \), and with two external vertices of the \( T \)-type by \( \Sigma_*(z) \). The sum
of diagrams $g(z)$ can be expressed as a geometric series of the $\Sigma(z)$ blocks connected by solid horizontal lines and similarly the sum of diagrams $g_*(z)$ by the $\Sigma_*(z)$ blocks connected by dashed lines:

\[
g(z) = \left\langle \frac{1}{z - \frac{T}{z}XX_T} \right\rangle = \frac{1}{z} + \frac{1}{z} \Sigma(z) \frac{1}{z} + \frac{1}{z} \Sigma(z) \frac{1}{z} \Sigma(z) \frac{1}{z} + \ldots = \frac{1}{z I_N - \Sigma(z)}
\]

\[
g_*(z) = \left\langle \frac{1}{T 1_T - \frac{1}{z}X^XX} \right\rangle = \frac{1}{T 1_T - \Sigma_*(z)}
\]

The last equations can be represented graphically as:

The reason why it is useful to introduce the one-line irreducible sums $\Sigma(z)$ and $\Sigma_*(z)$ is that in the large $N$ limit one can write down two additional, independent equations which relate $\Sigma(z)$ and $\Sigma_*(z)$ to $g(z)$ and $g_*(z)$. These are the following Dyson-Schwinger relations:

\[
\Sigma(z) = C \text{Tr} \left[ g_*(z) \right]
\]

\[
\Sigma_*(z) = 1_T \text{Tr} \left[ g(z) C \right]
\]

which follow from the observation that any planar diagram in the sum $g(z)$ or $g_*(z)$ can be transformed into a one-line irreducible planar diagram by adding an arc joining its external points.

In this way one obtains a closed set of equations: one has four equations for four unknown matrices $g$, $\Sigma$, $g_*$ and $\Sigma_*$:

\[
g(z) = \frac{1}{z - \Sigma(z)}
\]

\[
g_*(z) = \frac{1}{T - \Sigma_*(z)}
\]

\[
\Sigma(z) = C \text{Tr} \left[ g_*(z) \right]
\]

\[
\Sigma_*(z) = 1_T \text{Tr} \left[ g(z) C \right]
\]

One can eliminate $\Sigma$, $g_*$ and $\Sigma_*$ and solve it for $g(z)$ as a function of $C$. 


It is convenient to write the result using a variable $Z$ defined by

$$z g(z) = Z G(Z)$$

(54)

where $G(Z)$ is given by equation (14). This amounts to choosing $Z$ as

$$Z = \frac{z}{\text{Tr} g(z)}.$$ 

(55)

as follows from (50) and (52). The equations (51) and (53) complete the solution with the result

$$Z = \frac{z}{1 + r\left(-1 + \frac{1}{N} \text{Tr}[z g(z)]\right)}.$$ 

(56)

The equations (54) and (56) relate the resolvent $g(z)$ calculated in a series of independent experiments to the genuine resolvent $G(z)$ for the system.

In the limit of infinitely many measurements $T = \infty$, and $N$ fixed, which corresponds to $r = 0$, one obtains $Z = z$ and hence $g(z) = G(z)$. As expected, in this limit, spectral densities $\rho(\lambda)$ and $\rho_0(\Lambda)$ are identical.

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