FINITE ORBITS IN RANDOM SUBSHIFTS OF FINITE TYPE

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ABSTRACT. For each \(n, d \in \mathbb{N}\) and \(0 < \alpha < 1\), we define a random subset of \(A^{(1, 2, \ldots, n)^d}\) by independently including each element with probability \(\alpha\) and excluding it with probability \(1 - \alpha\), and consider the associated random subshift of finite type. Extending results of McGoff \[2\] and of McGoff and Pavlov \[3\], we prove there exists \(\alpha_0 = \alpha(d, |A|) > 0\) such that for \(\alpha < \alpha_0\) and with probability tending to 1 as \(n \to \infty\), this random subshift will contain only finitely many elements. In the case \(d = 1\), we obtain the best possible such \(\alpha_0, 1/|A|\).

1. INTRODUCTION

Fix a finite color set \(A\) with at least two elements. Let \(\Omega_n^d\) be the power set of \(A^{(1, 2, \ldots, n)^d}\). Given \(\omega \in \Omega_n^d\), let \(X_\omega\) be the set of colorings \(\eta \in A^{\mathbb{Z}^d}\) such that for every \(j \in \mathbb{Z}^d\) there exists \(\beta \in \omega\) such that \(\eta(i + j) = \beta(i)\) for all \(i \in \{1, 2, \ldots, n\}^d\). That is, \(X_\omega\) is the \(\mathbb{Z}^d\)-subshift of finite type that has \(\omega\) as its set of allowed patterns. For convenience, we will abuse notation slightly and, when \(\gamma_1 : D \to A\) and \(\gamma_2 : D + j \to A\) satisfy \(\gamma_1(i) = \gamma_2(i + j)\) for all \(i \in D\), we will say \(\gamma_1 = \gamma_2\). Thus, \(\eta \in X_\omega\) if and only if each of its restrictions to a subset of the form \(j + \{1, 2, \ldots, n\}^d\) is equal to an element of \(\omega\). We will also say that a coloring \(\gamma : B \to A\) (where \(B \subset \mathbb{Z}^d\)) is \(\omega\)-legal if for each \(j \in \mathbb{Z}^d\) such that \(\{1, 2, \ldots, n\}^d + j \subset B\), \(\gamma|_{\{1, 2, \ldots, n\}^d + j} \in \omega\). Note that this is weaker than requiring that \(\gamma = \eta|_B\) for some \(\eta \in X_\omega\), as it is possible that \(\gamma\) is \(\omega\)-legal but cannot be extended to an \(\omega\)-legal coloring of \(\mathbb{Z}^d\).

Note that if \(\omega = A^{(1, 2, \ldots, n)^d}\), then \(X_\omega\) is the full shift \(A^{\mathbb{Z}^d}\). If \(\omega = \emptyset\), then \(X_\omega\) is empty, but it also possible for \(X_\omega\) to be empty when \(\omega\) is nonempty. For example, let \(d = 1, n = 2, A = \{0, 1\}\), and \(\omega = \{\beta\}\), where \(\beta(1) = 0\) and \(\beta(2) = 1\). Then if there exists \(\eta \in X_\omega\), \(\eta(1) \neq 1\), since otherwise \(\eta|_{\{1, 2\}} \neq \beta\), and \(\eta(1) \neq 0\), since otherwise \(\eta|_{\{0, 1\}} \neq \beta\). But \(\eta(1) \in A = \{0, 1\}\), so we have a contradiction and \(X_\omega\) is empty.

Put the usual topology on the full \(\mathbb{Z}^d\)-shift via the metric
\[
\rho(\eta_1, \eta_2) = 2^{-\min(|m| : \eta_1(m) \neq \eta_2(m))}.
\]
Then put the subspace topology on each subshift \(X_\omega\). Thus, together with the translations \(T^n : X_\omega \to X_\omega\) given by \(T^n(\eta)(i) = \eta(i + n)\), \(X_\omega\) is a \(\mathbb{Z}^d\)-topological dynamical system and we may study dynamical properties such as periodicity, entropy, and directional entropy.

We will study these properties for random subshifts in the following sense. For each \(d, n \in \mathbb{N}\) and \(0 < \alpha < 1\), define a probability measure \(\mu_{\alpha, n}\) on \(\Omega_n^d\) by independently including each element of \(A^{(1, 2, \ldots, n)^d}\) with probability \(\alpha\), and excluding it with probability \(1 - \alpha\). Thus, for any \(\omega \in \Omega_n^d\), \(\mu_{\alpha, n}(\{\omega\}) = \alpha^{|\omega|}(1 - \alpha)^{|A|^d - |\omega|}\). Here and throughout the paper, \(|\cdot|\) is used to denote the cardinality of a finite set.
Note that $\mu_{\alpha,n}$ also depends on $d$ and $\mathcal{A}$, but we suppress these in order to keep the notation manageable.

Notice that the number of $\omega$-legal colorings of $\{1, 2, \ldots, n\}^d$ has binomial distribution with parameters $|\mathcal{A}|^d$ and $\alpha$. Thus, using Chernoff’s inequality, it is easy to see that for any $\alpha \in (0, 1)$ and any $0 < \beta < \alpha$, there exists $c > 0$ such that the number of legal colorings of $\{1, 2, \ldots, n\}^d$ is at least $\beta|\mathcal{A}|^d$ with probability greater than $1 - e^{-c|\mathcal{A}|^d}$. One might expect that the abundance of legal blocks leads to positive entropy, but it was shown by K. McGoff in [2] that if $d = 1$, then $\frac{1}{|\mathcal{A}|}$ is the critical value for $\alpha$ in the sense that the probability of $X_\omega$ having positive entropy tends to 0 as $n$ tends to $\infty$ if $\alpha < 1/|\mathcal{A}|$ and tends to 1 if $\alpha > 1/|\mathcal{A}|$. K. McGoff and R. Pavlov proved a weaker version of this result for $d > 1$, which implies that if $\alpha \leq 1/|\mathcal{A}|$, then for every $\varepsilon > 0$ the probability that $X_\omega$ has entropy at least $\varepsilon$ tends to 0. So if $\alpha$ is small and $n$ is large, then the entropy of $X_\omega$ is, with very high probability, a small positive number or zero. Their result does not distinguish between these two possibilities though so it is natural to ask about the probability that the entropy is positive in the case $d > 1$. Furthermore, for all $d \in \mathbb{N}$ it is possible for $X_\omega$ to have zero entropy, while still containing aperiodic elements \(^1\).

(Here and throughout, we say a coloring $\eta: \mathbb{Z}^d \to \mathcal{A}$ is aperiodic if it has no period vectors, i.e. there does not exist $p \in \mathbb{Z}^d \setminus \{0\}$ such that $\eta(j + p) = \eta(j)$ for all $j \in \mathbb{Z}^d$.) Thus, it is also natural to investigate the likelihood that $X_\omega$ has at least one period vector, as well as the likelihood of the stronger condition that $|X_\omega|$ is finite. We are able to determine both the entropy and finiteness for small enough $\alpha$ by showing the following.

**Theorem 1.1.** For any $d$, $|\mathcal{A}| \in \mathbb{N}$, there exists $\alpha_0 = \alpha_0(d, |\mathcal{A}|) > 0$ such that for $0 < \alpha < \alpha_0$,

$$
\lim_{n \to \infty} \mu_{\alpha,n}(|X_\omega| < \infty) = 1.
$$

In the case $d = 1$, we can take $\alpha_0(1, |\mathcal{A}|) = \frac{1}{|\mathcal{A}|}$.

**Remark 1.2.** Clearly, $X_\omega$ is finite if and only if there exists $p \in \mathbb{N}$ such that every $\eta \in X_\omega$ is periodic in each cardinal direction with period less than $p$.

We will prove that this latter property holds with probability tending to 1 when $0 < \alpha < \alpha_0$.

Since the entropy of a finite subshift is clearly zero, the aforementioned result of McGoff implies that the $\alpha_0$ in Theorem 1.1 is optimal in the case $d = 1$. For $d > 1$, our $\alpha_0$ will be below the critical value of $1/|\mathcal{A}|$, which leaves a gap in the parameter space. This leads to several open questions, which we discuss in §4.

The paper is organized as follows: In [2] we prove the $d = 1$ case of Theorem 1.1. The argument in this case is different from the one for arbitrary dimension and provides a larger $\alpha_0$ than the general argument allows us to obtain. In [3] we present the proof of Theorem 1.1 for $d \in \mathbb{N}$ arbitrary. Finally, in [4] we discuss further directions and open questions.

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\(^1\)For example, let $\omega = \{\beta_1, \beta_2, \beta_3\} \in \Omega^2_1$ where $\beta_1(1) = 0, \beta_1(2) = 0, \beta_2(1) = 0, \beta_2(2) = 1, \beta_3(1) = 1, \beta_3(2) = 1.$
2. The case \( d = 1 \)

To prove the theorem in the case \( d = 1 \) (and obtain the optimal value of \( \alpha_0 \) in this case), we will use the above-mentioned result that for \( \alpha < |A|^{-1} \), the probability that \( X_\omega \) has positive entropy tends to \( 0 \) as \( n \to \infty \).

In our context, topological entropy can be defined using the notion of complexity. Given a finite set \( K \subset \mathbb{Z}^d \), define the complexity of \( K \) with respect to \( \omega \) to be

\[
P_\omega(K) \overset{\text{def}}{=} |\{ \eta|_K : \eta \in X_\omega \}|. \]

In the case that \( K = \mathbb{Z}^d \cap ([1, k_1] \times [1, k_2] \times \cdots \times [1, k_d]) \), we will write \( P_\omega(k) = P_\omega(k_1, k_2, \ldots, k_d) \) and in the case \( K = \mathbb{Z}^d \cap [1, k]^d \), we will write \( P_\omega(K) = P_\omega(k) \).

**Definition 2.1.** If \( \omega \in \Omega_1^d \), the (topological) entropy of the system \( X_\omega \) is

\[
h(X_\omega) = \lim_{k \to \infty} \frac{\log(P_\omega(k))}{k^d}.
\]

It is straightforward to check that this definition coincides with the general definition of topological entropy when \( X_\omega \) is endowed with a topology as described in the introduction.

**Proof of Theorem 1.1** for \( d = 1 \). Let \( d = 1 \) and \( \alpha < \frac{1}{|A|} \). By remark 1.2, it suffices to show that, with probability tending to 1, every element of \( X_\omega \) is periodic with period at most \( n \). We prove this in two steps: First we show that \( X_\omega \) will, with probability tending to 1, contain no periodic colorings with period greater than \( n-1 \); then we show that the existence of an aperiodic coloring would imply, with probability tending to 1, the existence of a periodic coloring of large period, completing the proof. Let \( \omega \in \Omega_1^d \) and suppose \( \eta \in X_\omega \) is periodic with minimal period \( p \geq n \). We first claim that there exist \( \ell \geq n \) and \( j_0 \in \mathbb{Z} \) such that

(i) \( \eta(j_0 + j + \ell) = \eta(j_0 + j) \) for each \( 1 \leq j \leq n \)

(ii) For any \( 1 \leq \ell' < \ell \) there exists \( 1 \leq j \leq n \) such that \( \eta(j_0 + j + \ell') \neq \eta(j_0 + j) \).

To see this, for each \( j_0 \in \mathbb{Z} \) let \( p(j_0) \leq p \) be the smallest natural number such that

\( \eta(j_0 + j + p(j_0)) = \eta(j_0 + j) \) for each \( 1 \leq j \leq n \).

Choose \( j_0 \in \mathbb{Z} \) that maximizes \( p(j_0) \) and assume for contradiction that \( p(j_0) < n \). If \( \eta \) is periodic with period \( p(j_0) \), then \( p(j_0) = p \geq n \) and we are done. Otherwise, there exists \( j_1 \neq j_0 \) such that \( p(j_1) < p(j_0) \). Without loss of generality we may assume \( j_1 > j_0 \) and, by renaming \( j_0 \) if necessary, that \( j_1 = j_0 + 1 \). Let \( p_0 = p(j_0) \) and \( p_1 = p(j_1) = p(j_0 + 1) \). Then for \( 1 \leq j \leq n \), we have \( \eta(j_0 + j) = \eta(j_0 + p_0 + j) \) and \( \eta(j_0 + 1 + j) = \eta(j_0 + 1 + p_1 + j) \).

Hence, \( \eta(j_0 + j) = \eta(j_0 + p_1 + j) \) for \( 2 \leq j \leq n \). Furthermore, since \( p_1 < p_0 < n \), we have

\[
\eta(j_0 + 1 + p_1) = \eta(j_0 + 1 + p_0 + p_1) = \eta(j_0 + 1 + p_0) = \eta(j_0 + 1),
\]

and hence \( \eta(j_0 + j) = \eta(j_0 + j + p_1) \) for \( 1 \leq j \leq n \), contradicting the minimality of \( p(j_0) \). Thus, there exists \( j_0 \in \mathbb{Z} \) satisfying conditions (i) and (ii) above.

Now we split into two cases, and show that either possibility has probability approaching \( 0 \) as \( n \to \infty \).

**Case 1:** There are no integers \( j_0 \leq j_1 < j_2 \leq j_0 + \ell - 1 \) such that \( \eta(j_1 + j) = \eta(j_2 + j) \) for each \( 1 \leq j \leq n \).

In this case, we have that there exists an \( \omega \)-legal coloring \( \alpha \) of \( \{1, 2, \ldots, \ell + n\} \) such that the colorings \( \alpha|_{\{j+1,\ldots,j+n\}} (j = 0, 1, \ldots, \ell - 1) \) are all distinct and \( \alpha(j) = \alpha(j + \ell) \) for each \( 1 \leq j \leq n \). But for each \( \ell \) there are at most
Case 2: There exist $j_0 \leq j_1 < j_2 \leq j_0 + \ell - 1$ such that $\eta(j_1 + j) = \eta(j_2 + j)$ for each $1 \leq j \leq n$.

In this case, we claim that $X_\omega$ must have positive entropy. To see this, let $\beta = \eta|_{\{j_0 + 1, j_0 + 2, \ldots, j_0 + \ell + n\}}$, and define

$$
\gamma : \{j_0 + 1, j_0 + 2, \ldots, j_0 + \ell + n - (j_2 - j_1)\} \rightarrow A
$$

via $\gamma(j) = \eta(j)$ for $j_0 + 1 \leq j \leq j_1 + n$ and $\gamma(j) = \eta(j + (j_2 - j_1))$ for $j_1 + n < j \leq \ell + n - (j_2 - j_1)$. Then $\gamma$ is also $\omega$-legal, and furthermore any coloring “stitched together” out of translates of $\beta$ and $\gamma$ is also $\omega$-legal. That is, given any sequence $k = (k_i)_{i \in \mathbb{Z}}$ with the property that for all $i \in \mathbb{Z}$, $k_{i+1} - k_i = g_i \in \{\ell, \ell - (j_2 - j_1)\}$, we may define an $\omega$-legal coloring $\eta_k : \mathbb{Z} \rightarrow A$ in the following way: For $k_i < j \leq k_{i+1}$, let $\eta_k(j) = \beta(j - k_i)$ if $g_i = \ell$ and let $\eta_k(j) = \gamma(j - k_i)$ if $g_i = \ell - (j_2 - j_1)$. Note that by our choice of $j_0$ and $\ell$, for each $L > \ell$ the set of $j_3 \in \{k_0 + 1, \ldots, k_0 + L\}$ such that $\eta|_k(j_3 + j) = \eta(j_0 + j)$ for $j \in \{1, 2, \ldots, n\}$ is exactly $I(k, L) = \{k_i : k_0 < k_i \leq k_0 + L\}$. Hence, if $k$ and $k'$ are two sequences as above which also satisfy $k_0 = k'_0 = 0$, then $\eta_k$ and $\eta_{k'}$ have distinct restrictions to $\{1, \ldots, \ell\}$ whenever $I(k, L) \neq I(k', L)$. Since there are at least $2^{|L/\ell|}$ finite sequences $0 = k_0 < k_1 < \cdots < k_m$ which satisfy $k_{i+1} - k_i \in \{\ell, \ell - (j_2 - j_1)\}$ and $\sum_{i=1}^m (k_{i+1} - k_i) \leq L$, it follows that $|\{\eta|_{\{1, 2, \ldots, \ell\}} : \eta \in X_\omega\}| \geq 2^{|L/\ell|}$, so the entropy of $X_\omega$ is positive. By [2], the probability of this event tends to 0.

Hence, with probability tending to 1, $X_\omega$ does not contain any periodic colorings of period at least $n$. But if $X_\omega$ contains an aperiodic coloring $\eta$ then there must exist $j_0 \in \mathbb{Z}$ such that $\inf\{j_i : \eta(j_0 + j_i) = \eta(j_0 + j)\}$ for each $1 \leq j \leq n \geq j_0 + n$. If this infimum is a finite integer $j_1$, then we obtain a periodic coloring in $X_\omega$ with period at least $n$ by setting $\eta'(j_1 - j_0) m + j) = \eta(j_0 + j)$ for each $m \in \mathbb{Z}$ and $1 \leq j \leq (j_1 - j_0)$, so by the above we may assume the infimum is infinite for each such $j_0$. Since there are only finitely many colorings of $\{1, 2, \ldots, n\}$, there is a maximal such $j_0$, which implies that $\eta|_{\{0, 1, \ldots, n\}}$ is periodic. Since this induces a periodic element of $X_\omega$, we may assume the minimal period $p$ is strictly less than $n$, by the above. Thus, there exists an $\omega$-legal coloring $\gamma$ of $\{1, 2, \ldots, 2n\}$ such that

(i) $\gamma(j) = \gamma(j + p)$ for $n - p < j \leq 2n - p$

(ii) $\gamma(n - p) \neq \gamma(n)$

Note that such a coloring is determined by $p$ and by its restriction to $\{1, 2, \ldots, n\}$, so there are at most $n|A|^n$ of them. We claim that the colorings $\beta_j : \{1, 2, \ldots, n\} \rightarrow A$ given by $\beta_j(i) = \gamma(j + i)$ for $0 \leq j \leq n - 1$ are all distinct. To see this, let $m(\beta) = \max\{i : \beta(i) \neq \beta(i + p)\}$. For $0 \leq j \leq n - p - 1$, $m(\beta_j) = n - j - p$, so these are all distinct. For $n - p \leq j \leq n - 1$, $\{i : \beta_j(i) \neq \beta_j(i + p)\}$ is the empty set, so $\beta_j$ is distinct from $\beta_j'$ for any $0 \leq j \leq n - p - 1 < j' \leq n - 1$. We claim that, if $n - p \leq j < j' \leq n - 1$, then $\beta_j$ and $\beta_{j'}$ are distinct as well. Indeed, suppose there exist $n - p \leq j < j' \leq n - 1$ with $\beta_j = \beta_{j'}$ and let $p' = j' - j < p$. Then for
are at least $\omega \in \beta$ of the form $\gamma(i) = \gamma(i + p')$. Now, let $i > n - p$ be arbitrary. Since $p < n$ there exists $k \in \mathbb{Z}$ such that $i + kp \in [j + 1, j + n]$ and so
\[
\gamma(i) = \gamma(i + kp) = \gamma(i + kp + p') = \gamma(i + p').
\]
Since $i > n - p$ was arbitrary, this violates the minimality of the period $p$. Hence, the $\beta_j$ are all distinct for $0 \leq j \leq n - 1$, so the probability of such a coloring of $\{1, 2, \ldots, 2n\}$ existing is at most $n|A|^n\alpha^n$. Since this tends to zero, the probability that $X_\omega$ contains an aperiodic coloring does as well. \hfill \Box

**Remark 2.2.** As in [2], we may analogously define a random subshift of a fixed SFT $X$. If the entropy of $X$ is $\log \lambda$, then for large $\ell$ the number of words of length $\ell$ that appear in $X$ is approximately $\lambda^\ell$. It is proved in [2] that if $\alpha < 1/\lambda$, then the probability that $X_\omega$ has positive entropy tends to 0 as $n \to \infty$. Using these two facts and making the obvious changes to the above proof, we see that for $\alpha < 1/\lambda$, the probability that a random SFT $X_\omega \subset X$ is finite tends to 1 as $n \to \infty$. And again by [2], this value $1/\lambda$ is optimal.

### 3. Higher Dimensions

In higher dimensions, we will prove that for large $n$, the probability that $X_\omega$ contains a coloring which is not periodic in $d$ linearly independent directions tends to zero. Throughout this section, we will keep the notation more manageable by saying that a coloring $\eta: \mathbb{Z}^d \to A$ has a certain property on a set $B \subset \mathbb{R}^d$ if it has that property on $B \cap \mathbb{Z}^d$. We will also refer to colorings of $B$ by which we mean colorings of $B \cap \mathbb{Z}^d$.

We begin by proving that, with high probability, colorings in $X_\omega$ are locally periodic in $d$ linearly independent directions. The following lemma provides periodicity in one direction and illustrates the main idea of the more complicated general argument.

**Lemma 3.1.** Fix $d \in \mathbb{N}$. For $\varepsilon > 0$, let $D_\varepsilon$ be the set of $\omega \in \Omega^d_n$ for which there is an $\omega$-legal coloring $\beta$ of $[1, n + 2\varepsilon n]^d$ such that the restrictions $\beta|_{x+[1+\varepsilon n,n+\varepsilon n]^d}$ are distinct for all integer vectors $x$ with $\|x\| < \frac{\varepsilon n}{2}$. Then for $\alpha < |A|^{-(2d)^d(1+2\varepsilon)^d}$, \[ \lim_{n \to \infty} \mu_{n,\alpha}(D_\varepsilon) = 0. \]

**Proof.** Let $A$ denote the set of $\beta$: $[1, n + 2\varepsilon n]^d \to A$ such that the restrictions $\beta|_{x+[1+\varepsilon n,n+\varepsilon n]^d}$ are distinct for integer vectors $x$ with $\|x\| < \frac{\varepsilon n}{2}$. Note that there are at least $\frac{1}{(2d^d)(\varepsilon n)^d}$ such vectors $x$. Given $\beta \in A$, let $D_\varepsilon(\beta)$ denote the set of $\omega \in \Omega$ such that $\beta$ is $\omega$-legal. Since there are at least $\left(\frac{\varepsilon n}{2}\right)^d$ distinct colorings of the form $\beta|_{x+[1+\varepsilon n,n+\varepsilon n]^d}$, all of which must be contained in $\omega$, we see that $\mu_{n,\alpha}(D_\varepsilon(\beta)) \leq \alpha^{(2d)^d(\varepsilon n)^d}$. Since $D_\varepsilon = \bigcup_{\beta \in A} D_\varepsilon(\beta)$, we have
\[
\mu_{n,\alpha}(D_\varepsilon) = \sum_{\beta \in A} \mu_{n,\alpha}(D_\varepsilon(\beta)) \leq |A|\alpha^{(2d)^d(\varepsilon n)^d} \leq |A|^{1+2\varepsilon n}d|A|^{(1+2\varepsilon)^d}n^d
\]
If $\alpha < |A|^{-(2d)^d(\frac{1+2\varepsilon}{2})^d}$, then the right hand side tends to 0, which completes the proof. \hfill \Box
Note that this implies that, with probability tending to 1, every legal coloring of \([1 + 2en, n]^d\) will have a period vector of length at most \(\varepsilon n\). We want to extend this result and prove that, with probability tending to 1, we can find \(d\) period vectors which are small in magnitude and “almost orthogonal.” Before we do this, we first prove that the latter property implies linear independence.

**Lemma 3.2.** Let \(p_1, \ldots, p_m \in \mathbb{R}^d\) satisfy \(|p_i \cdot p_j| \leq \frac{1}{2m^2}||p_i|| ||p_j||\) for every \(1 \leq i < j \leq m\). Then \(p_1, \ldots, p_m\) are linearly independent.

**Proof.** Let \(p_1, \ldots, p_k \in \mathbb{R}^d\) satisfy \(|p_i \cdot p_j| \leq \frac{\delta}{||p_i|| ||p_j||}\) for every \(1 \leq i < j \leq m\) and suppose they are linearly dependent. We will show that \(\delta > \frac{1}{2m^2}\). Linear dependence implies \(p_k = \sum_{i=1}^{k-1} a_i p_i\) for some \(k \leq m\) and some \(a_i \in \mathbb{R}\). Let \(A = \max_{1 \leq i \leq k} ||a_i p_i||\). Clearly, \(||p_k|| < mA\), so we have

\[
||p_k||^2 = \sum_{i=1}^{k-1} a_i p_i \cdot p_k \leq \sum_{i=1}^{k-1} ||a_i|| ||p_i|| ||p_k|| < \delta m^2 A^2
\]

But we also have

\[
||p_k||^2 = \left(\sum_{i=1}^{k-1} a_i p_i\right) \cdot \left(\sum_{i=1}^{k-1} a_i p_i\right) = \sum_{i=1}^{k-1} a_i^2 ||p_i||^2 + \sum_{i \neq j} a_i a_j p_i \cdot p_j.
\]

Combining these two inequalities we obtain

\[
A^2 - \delta m^2 A^2 \leq \sum_{i=1}^{k-1} a_i^2 ||p_i||^2 + \sum_{i \neq j} a_i a_j p_i \cdot p_j = ||p_k||^2 < \delta m^2 A^2.
\]

It follows that \(1 < 2m^2 \delta\), as desired. \(\square\)

**Lemma 3.3.** Fix \(d \in \mathbb{N}, 0 < \varepsilon < 1/2\), and \(0 < \delta < \frac{1}{2d^2}\). Let \(F_n = F_n(\varepsilon, \delta)\) be the set of \(\omega\) such that for every \(\beta \in X_\omega\) and every \(x \in \mathbb{Z}^d\) there exist nonzero integer vectors \(p_1, \ldots, p_d\) such that

(i) \(||p_i|| \leq \varepsilon n\) for all \(1 \leq i \leq d\)

(ii) \(|p_i \cdot p_j| \leq ||p_i|| ||p_j|| \delta\) for all \(1 \leq i < j \leq d\)

(iii) \(\beta(x + p_i) = \beta(x + j)\) for all \(j \in x + [1 + 2\varepsilon n, n]^d\) and \(1 \leq i \leq d\).

Then, there exists \(\alpha_0 = \alpha(|A|, d, \varepsilon, \delta) > 0\) such that if \(0 < \alpha < \alpha_0\),

\[
\lim_{n \to \infty} \mu_{\alpha,n}(\Omega_n \setminus F_n) = 0.
\]

**Proof.** Let \(\alpha_0 = \min\{||A||^{-\frac{1}{2d+1}}d, ||A||^{-\frac{2d}{d+1}}d\}\). We begin by assigning a quantity \(P(\beta)\) to each \(\beta \in A^{Z^d}\). For each \(x \in \mathbb{Z}^d\), let \(P(x)\) be the minimum value of \(n^{d-k} \prod_{i=1}^k ||p_i||\), taken over all sets of vectors \(\{p_1, \ldots, p_k\}\) which satisfy

(a) \(||p_1|| \leq ||p_2|| \leq \cdots \leq ||p_k|| \leq \varepsilon n\)

(b) \(\beta(x + j) = \beta(x + j)\) for all \(j \in [1 + 2\varepsilon n, n]^d\) and \(1 \leq i \leq k\)

(c) For each \(1 \leq j \leq k\), \(p_j\) is a nonzero integer vector of minimal length among those satisfying the above two properties as well as \(|p_i \cdot p_j| \leq \delta ||p_i||^2 \leq \delta ||p_i|| ||p_j||\) for all \(1 \leq i < j\).
If no such vectors exist, set $P(x) = \infty$. Note that for any $\omega \not\in D_\varepsilon$ and any $\beta \in X_\omega$, $P(x)$ is bounded above by $\varepsilon n^d$ for all $x \in \mathbb{Z}^d$. So by Lemma 3.1 and our choice of $\alpha_0$, we may assume $P(x) \leq \varepsilon n^d$ for each $\beta \in X_\omega$ and $x \in \mathbb{Z}^d$. Fix $x_0 \in \mathbb{Z}^d$ such that $P(\beta) \overset{\text{def}}{=} P(x_0) \geq P(x)$ for all $x \in \mathbb{Z}^d$, and let $p_1, \ldots, p_k$ be vectors satisfying \([a])\overset{(c)}{=} with $P(x_0) = n^{-d} \prod_{i=1}^k \|p_i\|$. By Lemma 3.2, the vectors $p_1, \ldots, p_k$ are linearly independent. We claim that the probability that $P(\beta) \geq \varepsilon n$ for some $\beta \in X_\omega$ tends to zero, from which the lemma follows. To show this, we consider two cases: $k < d$ and $k = d$. We claim that, with probability tending to 1, the first case doesn’t occur, and the second occurs only with $P(x_0) < \varepsilon n$.

**Case 1: $k < d$**

Let $q_1, \ldots, q_{d-k}$ be unit vectors orthogonal to the span of $p_1, \ldots, p_k$ and orthogonal to each other, and let $j_0 = (\varepsilon n, \varepsilon n, \ldots, \varepsilon n)$. We claim that the colors $\beta_{x_0 + [1,n]^d + j_0 + j}$ must be distinct for integer vectors $j = \sum_{i=1}^k a_i p_i + \sum_{i=1}^{d-k} b_i q_i$, where $0 \leq a_i < \delta/d$ and $0 \leq b_i < \frac{\varepsilon n}{d \delta}$ are real numbers. Indeed, if not then there exist vectors $j_1$ and $j_2$ of this form such that

$$
\beta_{x_0 + [1,n]^d + j_0 + j_1} = \beta_{x_0 + [1,n]^d + j_0 + j_2}.
$$

Letting $p_{k+1} = j_2 - j_1$, it follows that

- $\|p_{k+1}\| < d\varepsilon n + d\frac{\varepsilon n}{2d} < \varepsilon n$,
- $\beta_{x_0 + j + p_{k+1}} = \beta_{x_0 + j}$ for all $j \in [1 + 2\varepsilon n, n]^d$,
- $\|p_{k+1} \cdot p_i\| \leq \sum_{i=1}^k a_i |p_j \cdot p_i| \leq \frac{\varepsilon n}{d} \|p_i\|^2 = \delta \|p_i\|^2$ for every $1 \leq i \leq k$.

Now, suppose $\|p_i\| > \|p_{k+1}\|$ for some $1 \leq i \leq k$ and let $i_0$ be the least such index. Then $|p_i \cdot p_{k+1}| \leq \delta \|p_i\|^2 \leq \delta \|p_{k+1}\| \|p_i\|$ for all $1 \leq i < i_0$, which violates the assumption that $p_{i_0}$ is of minimal length among such vectors. Thus, $|p_{k+1} \cdot p_i| \leq \delta \|p_i\|^2 \leq \delta \|p_{k+1}\| \|p_i\|$ for all $1 \leq i \leq k$, so (replacing $p_{k+1}$ with a shorter vector satisfying the above properties if necessary) $\{p_1, \ldots, p_{k+1}\}$ satisfies conditions \([a])\overset{(c)}{=} above, violating the minimality of $n^{-d} \prod_{i=1}^k \|p_i\|$ in the definition of $P(x_0)$.

But notice that, since the vectors $p_1, \ldots, p_k, q_1, \ldots, q_{d-k}$ are linearly independent, the number of integer vectors $j = \sum_{i=1}^k a_i p_i + \sum_{i=1}^{d-k} b_i q_i$, where $0 \leq a_i < \delta/d$ and $0 \leq b_i < \frac{\varepsilon n}{d \delta}$, is at least

$$
\left( \prod_{i=1}^k (\delta/d) \|p_i\| \right) \left( \prod_{i=1}^{d-k} \frac{\varepsilon n}{2d} \right) = \frac{(\delta/d)^k (\varepsilon/2d)^{d-k}}{P(\beta)}
$$

Thus, if the colors $\beta_{x_0 + [1,n]^d + j_0 + j}$ are distinct for $j = \sum_{i=1}^k a_i p_i + \sum_{i=1}^{d-k} b_i q_i, (0 \leq a_i < \frac{\delta}{d}, 0 \leq b_i < \frac{\varepsilon n}{d \delta})$, then by the maximality of $P(x_0)$ there is an $\omega$-legal coloring of $[1, n + 2\varepsilon n]^d$ such that

- $[1, n + 2\varepsilon n]^d$ is the union of $2^d$ translates of $[1 + 2\varepsilon n, n]^d$, each of which having period vectors $p_1, \ldots, p_k$ with $\|p_i\| < \varepsilon n$ satisfying $n^{-d} \prod_{i=1}^k \|p_i\| \leq P(\beta)$,
- $[1, n + 2\varepsilon n]^d$ contains at least $\left( \frac{\delta}{d} \right)^d \left( \frac{\varepsilon n}{d \delta} \right)^d P(\beta)$ translates of $[1, n]^d$ on which the restrictions of this legal coloring are distinct.

Now, there are most $(2\varepsilon n)^d + 1 \leq (3\varepsilon n)^d$ choices of a set of at most $d$ integer vectors with norm at most $\varepsilon n$. And once these period vectors are fixed, there are most $|\mathcal{A}|\left( \sqrt{\varepsilon n} \right)^{-d} \prod_{i=1}^k \|p_i\|$ colorings of $[1, n]^d$ that
are periodic with those period vectors. Hence, the first property bounds the number of such colorings of \([1, n + 2\varepsilon n]^d\) by \((3\varepsilon n)^{d^2} |A| (2\sqrt{7})^d P(\beta)\). The second property implies that the probability of any one of these being \(\omega\)-legal is at most \(\alpha \left( \frac{d^2}{\varepsilon n} \right)^d P(\beta)\). So the probability that such a coloring is \(\omega\)-legal, and hence the probability that Case 1 occurs, is at most \((3\varepsilon n)^{d^2} |A| (2\sqrt{7})^d P(\beta) \alpha \left( \frac{d^2}{\varepsilon n} \right)^d P(\beta)\).

Since every integer vector has length at least 1 and \(k < n\), \(P(\beta) \geq n\). Since \(\alpha < |A|^{-\left( \frac{4d^2/3}{\varepsilon n} \right)^d}\), we have

\[
(3\varepsilon n)^{d^2} |A| (2\sqrt{7})^d P(\beta) \alpha \left( \frac{d^2}{\varepsilon n} \right)^d P(\beta) \leq (3\varepsilon n)^{d^2} \left( \alpha \left( \frac{d^2}{\varepsilon n} \right)^d |A| (2\sqrt{7})^d \right)^n,
\]

which tends to zero.

**Case 2:** \(k = d\)

Again, by the maximality of \(P(x_0)\), there are at most \((3\varepsilon n)^{d^2} |A| (2\sqrt{7})^d P(x_0)\) possible colorings of \(x_0 + [1, n + 2\varepsilon n]^d\). Consider the colorings \(\beta|_{x_0 + [1, n]^d + j_0 + j}\) for \(j = \sum_{i=1}^d a_i P_i\), where \(0 \leq a_i < \frac{1}{3d}\) for \(1 \leq i < d\) and \(0 \leq a_d \leq \frac{1}{3}\). If these colorings are not all distinct, say \(\beta|_{x_0 + [1, n]^d + j_0 + j_1 + j + j'_d}\) for every \(j \in [1, n]^d\), \(p'_d\) satisfies property \([\text{b}]\) above as well. Finally, we have

\[
|p'_d - p_i| \leq \frac{\delta}{3d} |p_i|^2 + \frac{\delta}{3d} |p'_i|^2 + \frac{1}{3} \|p_i\|^2 \leq \delta \|p_i\|^2.
\]

Now, if \(\|p'_d\| < \|p_i\|\) for some \(1 \leq i < d\), then, arguing as in Case 1, the minimality in \([\text{c}]\) would be violated for some \(1 \leq j < d\), so we must have \(\|p'_d - p_i\| \leq \delta \|p_i\|^2 \leq \delta \|p'_i\|\), violating the minimality of \(p_d\).

Hence, the colorings \(\beta|_{x_0 + [1, n]^d + j_0 + j}\) must be distinct for \(j = \sum_{i=1}^d a_i P_i\). But then we have a length coloring of \([1, n + 2\varepsilon n]^d\) such that \(\frac{d}{3d^2} P(x_0)\) different translates of \([1, n]^d\) within \(x_0 + [1, n + 2\varepsilon n]^d\) are colored differently. The probability of this event is at most

\[
(3\varepsilon n)^{d^2} |A| (2\sqrt{7})^d P(x_0) \alpha \left( \frac{d^2}{\varepsilon n^d} \right)^d P(x_0).
\]

If \(\alpha < |A|^{-\left( \frac{4d^2/3}{\varepsilon n} \right)^d}\) and \(P(x_0) \geq \varepsilon n\), then this is at most

\[
(3\varepsilon n)^{d^2} \left( |A| (2\sqrt{7})^d \alpha \left( \frac{d^2}{\varepsilon n^d} \right)^d \right)^n,
\]

which tends to zero. By our choice of \(\alpha\), it follows that with probability tending to 1, \(P(\beta) < \varepsilon n\), and therefore \(P(x) < \varepsilon n\) for all \(x \in \mathbb{Z}^d\). The lemma then follows from the definition of \(P\), since if \(\{p'_1, \ldots, p'_k\}\) are the period vectors in the definition of \(P(x)\), we have \(p'_i \in \mathbb{Z}^d\) which implies \(\varepsilon n > n^{d-k'} \prod_{i=1}^k \|p'_i\| \geq n^{d-k'}\) and hence \(k' = d\). \(\square\)
The proof of the general case of Theorem 1.1 consists of a “local-to-global” argument extending the local periodicity guaranteed in the previous lemma to all of \( \mathbb{Z}^d \).

**Proof of Theorem 1.1.** Fix \( \varepsilon < (2d)^{-5/2} \) and \( \delta = \frac{1}{2d} \). Let \( \alpha_0 = \alpha_0(\|A\|, d, \varepsilon, \delta) \) be as in Lemma 3.3 and let \( 0 < \alpha < \alpha_0 \). (Note that \( \varepsilon \) and \( \delta \) both depend only on \( d \), so \( \alpha_0 \) is a function of \( \|A\| \) and \( d \) as in the statement of the theorem.) By Lemma 3.3 and Remark 1.2 it will suffice to show that whenever \( \omega \in F_n(\varepsilon, \delta) \), every orbit in \( X_\omega \) is periodic in each cardinal direction with period at most \( (2d)^n \). Let \( \omega \in F_n(\varepsilon, \delta) \) and let \( \eta \in X_\omega \). Since \( \omega \in F_n(\varepsilon, \delta) \), by Lemma 3.3 there exist linearly independent integer vectors \( p_1, \ldots, p_d \) with \( \|p_i\| \leq \varepsilon n \) such that for any \( 1 \leq i \leq d \) and any \( j \in B \) we have \( \eta(j + p_i) = \eta(j) \). Suppose for contradiction that \( \eta \) is not periodic with the same period vectors on \( B + \mathbb{R}_{\geq 0} p_1 \). Then there exists a minimal \( t \in [0, \infty) \) such that \( \eta(j + \sum_p p_i) \neq \eta(j) \) for some integer vector \( j \in B + tp_1 \). Let \( j_0 \in (B + tp_1) \setminus \bigcup_{t < t'} (B + t'p_1) \) satisfy \( \eta(j_0 - p_i) = \eta(j_0) \), and let \( B' \) be an integer translate of \( [1 + 3\varepsilon n, n - \varepsilon n]^d \) with center at most distance \( \sqrt{d} \) from \( j_0 \). We may assume \( n \) is large enough that \( \sqrt{d} < \frac{1}{2} n \). Since \( \delta < \frac{1}{2(d-1)} \), there exist linearly independent integer vectors \( p'_1, \ldots, p'_d \) with \( \|p'_i\| \leq \varepsilon n \) such that

\[
\eta(j + p'_i) = \eta(j - p'_i) = \eta(j)
\]

for every \( j \in B' \) and

\[
|p'_i \cdot p'_j| \leq \frac{\|p'_i\| \|p'_j\|}{2(d-1)} \quad \text{for } 1 \leq i < j \leq d.
\]

Indeed, Lemma 3.3 asserts that we may find such vectors which satisfy \( \eta(j) = \eta(j + p'_i) \) on a given translate of \( [1 + 2\varepsilon n, n]^d \), which implies \( \eta(j + p'_i) = \eta(j - p'_i) = \varepsilon n \) on the corresponding translate of \( [1 + 3\varepsilon n, n - \varepsilon n]^d \). We claim there exists a sequence of indices \( 1 \leq i_j \leq d \) and integers \( \sigma_j \in \{-1, 1\} \) for \( 1 \leq j \leq J \) such that

(i) For each \( 1 \leq \ell \leq J \), \( j_0 + \sum_{j=1}^{\ell} \sigma_j p'_i \in B' \) and \( j_0 - p_1 + \sum_{j=1}^{\ell} \sigma_j p'_i \in B' \)

(ii) \( j_0 + \sum_{j=1}^{J} \sigma_j p'_i \in B + tp_1 \) for some \( t' < t \).

If we prove this, then we will have

\[
\eta(j_0) = \eta\left(j_0 + \sum_{j=1}^{J} \sigma_j p'_i\right) = \eta\left(j_0 + \sum_{j=1}^{J} \sigma_j p'_i - p_1\right) = \eta(j_0 - p_1),
\]

which is a contradiction. To prove the claim, let \( y' \) be the center of \( B + tp_1 \) and let \( y = j_0 + 2d^2 \varepsilon n \frac{y' - j_0}{\|y' - j_0\|} \). Note that the distance from \( y \) to the boundary of \( B + tp_1 \) is at most \( \frac{2d^2 \varepsilon n}{\|y' - j_0\|} = 2d^{3/2} \varepsilon n \) and hence

\[
B(y, 2d^{3/2} \varepsilon n) \subset B + tp_1.
\]

There exist \( x_1, \ldots, x_d \in \mathbb{R} \) such that \( j_0 + \sum_{i=1}^{d} x_i p'_i = y \). We will choose our indices so that \( i_j = i \) for approximately \( |x_i| \) values of \( j \), but to ensure that (ii) holds we need
to show that the coefficients $x_i$ are small relative to the distance $\|y - j_0\| = 2d^2\varepsilon n$.

Note that
\[
\left\| \sum_{i=1}^{d} x_i p'_i \right\|^2 = \sum_{i=1}^{d} x_i^2 \|p'_i\|^2 + \sum_{i \neq j} (x_i x_j p'_i \cdot p'_j).
\]

By (1), we have that $\|x_i x_j p'_i \cdot p'_j\| \leq \frac{|x_i| |x_j| \|p'_i\| \|p'_j\|}{2(d-1)}$, so
\[
\|y - j_0\|^2 \geq \sum_{i=1}^{d} x_i^2 \|p'_i\|^2 - \frac{1}{2(d-1)} \sum_{i \neq j} (|x_i| \|p'_i\|)(|x_j| \|p'_j\|).
\]

Using Lagrange multipliers it is easy to see that, subject to the constraint that the right-hand side is at most $2d^2\varepsilon n$, $\sum_{i=1}^{d} |x_i| \|p'_i\|$ is maximized when
\[
|x_1| \|p'_1\| = |x_2| \|p'_2\| = \ldots = |x_d| \|p'_d\| \overset{\text{def}}{=} x,
\]
in which case we have
\[
2d^2\varepsilon n \geq \sqrt{dx^2 - \frac{d}{2} x^2} = \frac{d}{\sqrt{2d}} x = \frac{1}{\sqrt{2d}} \sum_{i=1}^{d} (|x_i| \|p'_i\|).
\]

Now define $z = \sum_{i=1}^{d} \sigma_i p'_i$, where $\sigma_i = |x_i|$ if $x_i \geq 0$ and $\sigma_i = |x_i|$ if $x_i \leq 0$. Let
\[
J = \sum_{i=1}^{d} |x_i| \text{ and for } \sum_{i=1}^{m-1} |x_i| < j \leq \sum_{i=1}^{m} |x_i| \text{ set } i_j = m \text{ and } \sigma_j = \frac{|x_i|}{|x_m|}.
\]

Then $z = \sum_{i=1}^{j} \sigma_j p'_i$ and $\|z - y\| \leq \sum_{i=1}^{d} \|p'_i\| \leq d\varepsilon n$, so by (2) $z$ is contained in the interior of $B + t\xi$, so (iii) holds. Also, for each $1 \leq \ell \leq J$,
\[
\left\| j_0 - \sum_{i=1}^{\ell} \sigma_j p'_i \right\| \leq \sum_{i=1}^{d} \left\| \sigma_i \|p'_i\| \leq \sum_{i=1}^{d} |x_i| \|p'_i\| \leq 2^{3/2} d^{5/2} \varepsilon n.
\]

Since we assume $2^{3/2} d^{5/2} \varepsilon < 1/2$, we have that the distance from $\sum_{i=1}^{\ell} \sigma_j p'_i$ to the center of $B'$ is at most $\frac{\sqrt{n}}{2} + \frac{\sqrt{d}}{2} < \frac{1}{2} n + \frac{1}{2} n = n$, so (ii) follows and the proof of the claim is complete. Hence, $\eta$ is periodic with the same period vectors on $B + \mathbb{R}_{\geq 0}\xi$.

Using the same argument, we see that $\eta$ is periodic with period vectors $p_1, \ldots, p_d$ on $B + \mathbb{R} p_1 + \mathbb{R} p_2 + \ldots + \mathbb{R} p_d$ and hence on all of $\mathbb{Z}^d$.

Now, if $p_i = (p_{i,1}^{(0)}, \ldots, p_{i,d}^{(0)}) \in \mathbb{Z}^d$, then for each $1 \leq i \leq d - 1$, let
\[
p_i^{(1)} = (p_{i,1}^{(1)}, \ldots, p_{i,d}^{(1)}) \overset{\text{def}}{=} p_{i,1} p_d - p_{d,1} p_i.
\]

We obtain $d - 1$ period vectors with $e_1$-component equal to zero and $\|p_i^{(1)}\| \leq 2(\varepsilon n)^2 \leq (2\varepsilon n)^2$. Now suppose for $1 \leq k \leq d - 1$ we have $d - k$ period vectors for $\eta$ satisfying $\|p_i^{(k)}\| \leq (2^k \varepsilon n)^{2^k}$ and $e_i \cdot p_i^{(k)} = 0$ for $1 \leq i \leq k$. For $1 \leq i \leq d - (k + 1)$, define
\[
p_i^{(k+1)} = (p_{i,1}^{(k+1)}, \ldots, p_{i,d}^{(k+1)}) \overset{\text{def}}{=} p_{i,k+1} p_{d-k} - p_{d-k+1} p_i^{(k)}.
\]

These are $d - (k + 1)$ period vectors with $\|p_i^{(k+1)}\| \leq 2(2^k \varepsilon n)^{2^{k+1}} \leq (2^{k+1} \varepsilon n)^{2^{k+1}}$ and $e_i \cdot p_i^{(k+1)} = 0$ for $1 \leq i \leq k + 1$. By finite induction, we obtain a period vector parallel to $e_d$ with period at most $(2d \varepsilon n)^{2d}$. Arguing similarly, we can produce a period vector in each of the cardinal directions with period at most $(2d \varepsilon n)^{2d}$. By Remark 1.2 this completes the proof.
4. Open questions

In [3], it is shown that if $\alpha < 1/|A|$, then for each $\varepsilon > 0$ the probability that the entropy of $X_\omega$ is at least $\varepsilon$ tends to zero as $n$ tends to infinity. However it is still possible that the probability of $X_\omega$ having zero entropy tends to zero as well. Even if the entropy is generically zero, this leaves open questions about directional entropy and periodicity. As with the definition of entropy given above, we may also define directional entropy using complexity.

**Definition 4.1.** If $\omega \in \Omega^d_2$; $m \in \{1, 2, \ldots, d - 1\}; u_1, \ldots, u_m$ are linearly independent unit vectors in $\mathbb{R}^d$; and $V = \text{span}(u_1, \ldots, u_m)$, set

$$R_{V,k,t} = \left\{ \sum_{i=1}^{m} a_i u_i + \sum_{i=m+1}^{d} b_i u_i : a_i \in [0, k], b_i \in [0, t] \right\},$$

where $u_i$ ($m + 1 \leq i \leq d$) are unit vectors orthogonal to $V$ which complete a basis for $\mathbb{R}^d$. The $m$-dimension (topological) directional entropy in direction $V$ is

$$h_V(X_\omega) = \sup_{t>0} \lim_{k \to \infty} \frac{\log(P_\omega(R_{V,k,t} \cap \mathbb{Z}^d))}{k^m}.$$

Again, it is straightforward to show that this definition coincides with the general definition of topological directional entropy given in [4]. If $h(X_\omega) > 0$, then $h_V(X_\omega) = \infty$ for all proper subspaces $V \subset \mathbb{R}^d$. But if $h(X_\omega) = 0$, the directional entropy may be zero, positive, or even infinite in any given direction. Ledrappier’s three-dot system (see [1]) provides an example of a zero-entropy SFT with positive but finite directional entropy in all directions. The system corresponding to $\omega = \{\beta\} \in \Omega^2_2$, where $\beta(i,j) = 1$ for $1 \leq i, j \leq 2$ (i.e. the SFT containing only a single constant coloring of $\mathbb{Z}^2$) is of course an example where the directional entropy is zero in all directions. The following example shows that it is possible to have infinite entropy in all directions as well.

**Example 4.2.** Set $A = \{0, 1, 2, 3\}$. Let

$$\omega_1 = \{\eta \in \{0, 1\}^{[1,2]} : \eta(i,1) = 0 \Rightarrow \eta(i,2) = 0\},$$

$$\omega_2 = \{\eta \in \{2,3\}^{[1,2]} : \eta(1,i) = 2 \Rightarrow \eta(2,i) = 2\},$$

and $\omega = \omega_1 \cup \omega_2$. Then $X_\omega$ has zero entropy but has infinite directional entropy in all directions.

**Proof.** To prove both claims, let us find $P_\omega(k,w)$ for arbitrary $k, w \in \mathbb{N}$. Note that for any $\eta \in X_\omega$, either $\text{im} \eta \subset \{0,1\}$ or $\text{im} \eta \subset \{2,3\}$. We first consider the former case. Then for any $(i, j) \in \mathbb{Z}^2$, if $\eta(i,j) = 0$ then $\eta(i, j') = 0$ for all $j' \geq j$. Of course it follows also that if $\eta(i,j) = 1$ then $\eta(i, j') = 1$ for all $j' \leq j$. Hence $\eta_{|[i] \times [1,w]}$ is completely determined by $\min\{1 \leq j \leq w : u(i,j) = 0\}$, where this min is taken to be 0 if the set is empty. Furthermore, if $\alpha : \{1, \ldots, w\} \to \{0,1\}$ satisfies, for some $j_i \in \{1, \ldots, w + 1\}$, $\alpha_i(j) = 0$ for $j \geq j_i$ and $\alpha_i(j) = 1$ for $j \leq j_i - 1$, then it is clear from the definition of $\omega$ that $\alpha(i,j) = \alpha_i(j)$ is an $\omega$-legal coloring of $[1,k] \times [1,w]$. Similarly, if $\text{im} \eta = \{2,3\}$ then $\eta_{|[i] \times [1,w]}$ is determined by $\min\{1 \leq i \leq k : \eta(i,j) = 2\}$ for $1 \leq j \leq w$. Thus, $P_\omega(k,w) = (w+1)^k + (k+1)^w$. From this it follows that

$$h(X_\omega) = \lim_{k \to \infty} \frac{\log(P_\omega(k,k))}{k^2} = \lim_{k \to \infty} \frac{\log(2(k+1)^k)}{k^2} = \lim_{k \to \infty} \left[ \frac{\log 2}{k^2} + \frac{\log(k+1)}{k} \right] = 0.$$
We also obtain
\[ h_{e_1}(X_\omega) = \sup_{w \in \mathbb{N}} \lim_{k \to \infty} \frac{\log(P_\omega(k, w))}{k} \geq \sup_{w \in \mathbb{N}} \lim_{k \to \infty} \frac{\log(w+1)^k}{k} = \sup_{w \in \mathbb{N}} \log(w+1) = \infty \]
and similarly
\[ h_{e_2}(X_\omega) = \sup_{w \in \mathbb{N}} \lim_{k \to \infty} \frac{\log(P_\omega(w, k))}{k} \geq \sup_{w \in \mathbb{N}} \lim_{k \to \infty} \frac{\log((w+1)^{ck})}{k} = \sup_{w \in \mathbb{N}} c \log(w+1) = \infty. \]

If \( v \) is some unit vector not parallel to \( e_2 \), then there is a constant \( c > 0 \) such that if \( L_k \) is the line segment connecting \(-\frac{k}{2}v\) to \( \frac{k}{2}v \), we have that \( L_k(\omega) \cap \mathbb{Z}^2 \) consists of at least \( ck \) vertical lines of length at least \( w \) and hence
\[ h_v(X_\omega) = \sup_{w \in \mathbb{N}} \lim_{k \to \infty} \frac{\log(P_{\omega}(L_k(\omega)))}{k} \geq \sup_{w \in \mathbb{N}} \lim_{k \to \infty} \frac{\log((w+1)^{ck})}{k} = \sup_{w \in \mathbb{N}} c \log(w+1) = \infty. \]

\[ \square \]

Fix \( A \) and let \( \alpha_d = \alpha(d, |A|) \) be as in Theorem 1.1. We ask the following.

**Question 4.3.** For \( \alpha_d \leq \alpha < \frac{1}{|A|} \), what is

- \( \lim_{n \to \infty} \mu_{\alpha,n}(\omega : h(X_\omega) = 0) \)?
- \( \lim_{n \to \infty} \mu_{\alpha,n}(\omega : h_v(X_\omega) = 0 \text{ for some } V) \)?
- \( \lim_{n \to \infty} \mu_{\alpha,n}(\omega : h_v(X_\omega) = 0 \text{ for all } V) \)?
- \( \lim_{n \to \infty} \mu_{\alpha,n}(\omega : h_v(X_\omega) = \infty \text{ for some } V) \)?
- \( \lim_{n \to \infty} \mu_{\alpha,n}(\omega : h_v(X_\omega) = \infty \text{ for all } V) \)?
- \( \lim_{n \to \infty} \mu_{\alpha,n}(\omega : |X_\omega| < \infty) \)?
- \( \lim_{n \to \infty} \mu_{\alpha,n}(\omega : X_\omega \text{ does not contain any infinite orbits}) \)?
- \( \lim_{n \to \infty} \mu_{\alpha,n}(\omega : \text{every element of } X_\omega \text{ has at least one period vector}) \)?

Another natural question concerns the critical value of the parameter \( \alpha_0 = \frac{1}{|A|} \) with \( d = 1 \). Neither our result nor the results in [2] include this case, so it is natural to ask the following.

**Question 4.4.** For \( d = 1 \) and \( \alpha = \frac{1}{|A|} \), what is

- \( \lim_{n \to \infty} \mu_{\alpha,n}(\omega : h(X_\omega) = 0) \)?
- \( \lim_{n \to \infty} \mu_{\alpha,n}(\omega : |X_\omega| < \infty) \)?

Note that in several places in the proof of the \( d = 1 \) case of Theorem 1.1 we use the fact that \( (\alpha_0 |A|^n \text{ and even } n(\alpha_0 |A|^n) \text{ approaches 0 as } n \to \infty, \text{ so the present methods do not seem to yield any information in the case } \alpha = \frac{1}{|A|}. \)

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