ON THE INADEQUACY OF THE PROJECTIVE STRUCTURE WITH RESPECT TO THE UNIVALENCE AXIOM

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Abstract. In this article the author endows the functor category $[B(Z_2), \text{Gpd}]$ with the structure of a type-theoretic fibration category with a universe using the projective fibrations. It offers a new model of Martin-Löf type theory with dependent sums, dependent products, identity types and a universe. It turns out that this universe, the natural candidate that lifts the univalent universe of small discrete groupoids in the groupoid model of Hofmann and Streicher, is not univalent.

1. Introduction

In the seventies Per Martin-Löf set a framework out, suitable for constructive mathematics, called Martin-Löf Type Theory (MLTT for short). It is well known that MLTT enjoys very nice computational properties that make it suitable for the formalization of mathematics with a proof assistant. Recently Vladimir Voevodsky added an axiom to MLTT, the so-called Univalence Axiom (UA for short). Given a type-theoretic universe, UA roughly asserts an equivalence between the identity type of any two small types (i.e. two elements of the universe) and the type of weak equivalences between them. This brave new world, MLTT together with UA, was coined Univalent Foundations (UF for short).

Voevodsky found an interpretation of UF in the category of simplicial sets using Kan simplicial sets, where the universe is interpreted as the base of a universal Kan fibration (cf. [KL12] for details). Through the notion of a type-theoretic fibration category, models of UF were later pursued by Michael Shulman [Shu15b, Shu15a, Shu17]. The line of research initiated by Michael Shulman consists in the exploration of the stability of UA, in particular in the following sense: given a type-theoretic fibration category $\mathcal{C}$ together with a univalent universe, one wants to lift this type-theoretic fibration category with its univalent universe to the functor category $[\mathcal{D}, \mathcal{C}]$, where the index category $\mathcal{D}$ is a small category. This goal was achieved in some specific cases.

First, in [Shu15b] Shulman succeeded when $\mathcal{D}$ is an inverse category by using the so-called Reedy model structure on the functor category. Second, in [Shu15a] Shulman succeeded with the same model structure when $\mathcal{C}$ is the category $\text{sSet}$ of simplicial sets and $\mathcal{D}$ is any elegant Reedy category. Note that inverse categories are particular cases of elegant Reedy categories that are themselves particular cases of (strict) Reedy categories. Since Reedy categories do not allow non-trivial isomorphism, this kind of index categories has severe constraints. Moreover, it is useful to note that in both cases Shulman used the Reedy model structure. However, the difficulty in handling non-trivial isomorphisms in the index category seems a challenge to

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the usefulness of this model structure with respect to the stability of UA. Around the same time Shulman in [Sin17] and the author in his PhD thesis [Bor15] tried different alternative model structures.

In this article the author reports on some results exposed in the chapter 4 of his PhD thesis. We wanted to explore the possibility of using the so-called *projective model structure* to endow a functor category with the structure of a type-theoretic fibration category with a univalent universe. Attractively, in the projective model structure fibrations, the class of maps in a type-theoretic fibration category that interprets dependent types, are simply defined objectwise. Starting from the groupoid model [HS98] of Hofmann and Streicher with a univalent universe of small discrete groupoids, as a test case we treated the 2-dimensional case where $C$ is the category $\Gpd$ of groupoids and $D$ is $B(\mathbb{Z}_2)$, namely the groupoid associated with the group with two elements that presents in this context the interesting technical challenge of containing a non-trivial automorphism. We discovered that the projective fibrations allow one to endow the functor category $[B(\mathbb{Z}_2), \Gpd]$ with the structure of a type-theoretic fibration category with a universe. But, while this universe is arguably the natural universe that lifts, with respect to the projective setting, the univalent universe of small discrete groupoids in $\Gpd$, it turns out that it is not univalent. Even a weaker form of UA, namely function extensionality, does not hold in this new type-theoretic fibration category.

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2. The projective model structure on $\Gpd^{\mathbb{Z}_2}$ made explicit

We will denote the functor category $[B(\mathbb{Z}_2), \Gpd]$ simply by $\Gpd^{\mathbb{Z}_2}$. The reader should note that an object in $\Gpd^{\mathbb{Z}_2}$ is nothing but a groupoid equipped with an involution, and a morphism in $\Gpd^{\mathbb{Z}_2}$ is nothing but an equivariant functor, namely a functor between groupoids compatible with the involutions on the domain and codomain. Such a groupoid will be denoted by a capital letter, $A$ for instance, and the corresponding Greek letter $\alpha$ will be used to refer to its involution (except when stated otherwise).

Since the natural model structure on $\Gpd$ [Rez, Str00] is cofibrantly generated and $B(\mathbb{Z}_2)$ is a small category, there exists the projective model structure [Lur09, proposition A.2.8.2] on $\Gpd^{\mathbb{Z}_2}$. Hereinafter by an objectwise weak equivalence (resp. an objectwise fibration) one means a map whose underlying map of groupoids is a weak equivalence (resp. a fibration) in $\Gpd$.

Recall that one can describe this projective model structure by :

- Weak equivalences are the objectwise weak equivalences.
- Fibrations are the objectwise fibrations.
- Cofibrations are those maps with the left lifting property with respect to acyclic fibrations (fibrations which are simultaneously weak equivalences).

**Notation 2.1.** As usual in category theory, the initial object and the terminal object of $\Gpd$ will be denoted by $0$ and $1$ respectively. We will use the letter $I$ for the groupoid with two distinct points and one isomorphism $\phi$ between them. We denote by $i$ the obvious inclusion $i : 1 \hookrightarrow I$.

We have an obvious functor from $B(\mathbb{Z}_2)$ to $1$ and an obvious inclusion from $1$ to
These two functors induce by precomposition the two following functors,

\[ \cdot : \mathbf{Gpd}^{\mathbb{Z}_2} \to \mathbf{Gpd} \]
\[ G \mapsto \tilde{G} \]

namely the forgetful functor that maps a groupoid \( G \) equipped with an involution to its underlying groupoid;

\[ ! : \mathbf{Gpd} \to \mathbf{Gpd}^{\mathbb{Z}_2} \]
\[ G \mapsto G! \]

that maps a groupoid to the same groupoid together with the identity involution. The forgetful functor has a left adjoint denoted \( S \) that maps a groupoid \( G \) to its underlying groupoid;

\[ \iota : \mathbf{Gpd} \to \mathbf{Gpd}^{\mathbb{Z}_2} \]
\[ G \mapsto G\iota \]

where \( G^{\mathbb{Z}_2} \) is the subgroupoid of \( G \) of fixed points and fixed morphisms under the \( \mathbb{Z}_2 \)-action. Note that \( G^{\mathbb{Z}_2} \) is \( \lim G \).

Since limits and colimits are pointwise in a presheaf category, \( 0! \) and \( 1! \) (shorten \( 0 \) and \( 1 \) when no confusion is possible) are the corresponding initial and terminal objects in \( \mathbf{Gpd}^{\mathbb{Z}_2} \).

Last, given a groupoid \( G \) together with an involution, we will denote by \( G_f \) the full subgroupoid of \( G \) consisting of its fixed points.

Knowing the generating acyclic cofibrations in \( \mathbf{Gpd} \), by looking at the construction of the projective model structure one finds the generating acyclic cofibrations with respect to the projective model structure on \( \mathbf{Gpd}^{\mathbb{Z}_2} \) [Lur09, proposition A.2.8.2]. Indeed, a set (actually it is a singleton in that case) of generating acyclic cofibrations is given by the following inclusion:

\[ S(i) : S(1) \hookrightarrow S(1) \]

Proposition 2.2. Let \( f : A \to B \) be a morphism in \( \mathbf{Gpd}^{\mathbb{Z}_2} \). The following are equivalent:

(i) \( f \) is an acyclic cofibration.
(ii) \( f \) is an acyclic cofibration and induces a bijection between the set of fixed points of \( A \) and the set of fixed points of \( B \).
(iii) \( f \) is an acyclic cofibration and induces an isomorphism between \( A^{\mathbb{Z}_2} \) and \( B^{\mathbb{Z}_2} \).
(iv) \( f \) is an acyclic cofibration and induces an isomorphism between \( A_f \) and \( B_f \).

Proof. We prove successively (i) \( \Rightarrow \) (ii), (ii) \( \Rightarrow \) (iii), (iii) \( \Rightarrow \) (iv) and (iv) \( \Rightarrow \) (i).

• (i) \( \Rightarrow \) (ii): assume that \( f \) is an acyclic cofibration. It is well-known that \( f \) is an acyclic cofibration [Hir03, proposition 11.6.2]. Moreover, note that if \( x \) is a fixed point of \( A \), then one has \( f(x) = f(\alpha(x)) = \beta(f(x)) \). Hence, \( f(x) \) is a fixed point of \( B \). We also know that \( f \) is injective on objects as a cofibration between groupoids. We need to prove that any fixed point in \( B \) is the image of a fixed point in \( A \). To achieve this, the reader can check this fact for the generating acyclic cofibration \( S(i) \) and the stability of this fact under pushouts, transfinite compositions and retraction. One concludes that \( f \) induces a bijection between the fixed points in \( A \) and the fixed points in \( B \).
(ii) ⇒ (iii): it is a straightforward consequence of \( f \) being fully faithful.

(iii) ⇒ (iv): idem.

(iv) ⇒ (i): Since \( f \) is an acyclic cofibration of groupoids where \( A_f \) and \( B_f \) are isomorphic, \( f \) is isomorphic to the inclusion of a full subgroupoid of \( B \) equivalent to \( B \) where \( A_f \) and \( B_f \) are equal. Let \( ((\text{Ob}B \setminus \text{Ob}A)/\mathbb{Z}_2, \leq) \) be the set of orbits of \((\text{Ob}B \setminus \text{Ob}A)\) under the \( \mathbb{Z}_2 \)-action together with a well-ordering.

Let \( \lambda \) be the order type of this well-ordering and \( g: ((\text{Ob}B \setminus \text{Ob}A)/\mathbb{Z}_2) \to \lambda \) an order-preserving bijection. We will construct a \( \lambda \)-sequence \( X \) of pushouts of \( S(i) \), where we add the elements of \((\text{Ob}B \setminus \text{Ob}A)\) to \( A \) by following our well-ordering. Take \( X_0 := A \). For \( \gamma \) such that \( \gamma + 1 < \lambda \), \( X_{\gamma+1} \) is defined as the following pushout. Let \( s \) be the element of \((\text{Ob}B \setminus \text{Ob}A)/\mathbb{Z}_2\) that corresponds to \( \gamma + 1 \) under \( g \). Actually, \( s \) is a set with two distinct elements \( \{x, \beta(x)\} \).

Since \( f \) is essentially surjective, there exists an isomorphism \( \varphi: y \to x \) with \( y \in A \), and we make the following pushout

\[
\begin{array}{cc}
S(1) & \xrightarrow{l} X_\gamma \\
\downarrow & \downarrow \\
S(0) & X_{\gamma+1}
\end{array}
\]

, where \( l(0) \) is \( y \) and \( l(1) \) is \( \alpha(y) \). Last, if \( \gamma < \lambda \) is a limit ordinal, then \( X_\gamma \) is \( \text{colim} X_\delta \). For every \( \gamma < \lambda \), \( X_\gamma \) is a full subgroupoid of \( B \) stable under the involution \( \beta \), and \( f \) is the transfinite composition of the \( \lambda \)-sequence \( X \). So, \( f \) is an acyclic cofibration.

\[\square\]

**Proposition 2.3.** Let \( G \) be a groupoid equipped with an involution and \( G' \) a subgroupoid of \( G \) stable under that involution such that \( G'_f = G_f \) and the inclusion map from \( G' \) to \( G \) is an equivalence of groupoids. Then the inclusion map is an acyclic cofibration.

**Proof.** Since by assumption the inclusion is an objectwise acyclic cofibration and \( G'_f = G_f \), it is straightforward by 2.2. \( \square \)

### 3. \textbf{Gpd}^{\mathbb{Z}_2} AS A TYPE-THEORETIC FIBRATION CATEGORY

We recall below the definition of a type-theoretic fibration category \cite[Definition 7.1]{Shu17}.

**Definition 3.1.** A **type-theoretic fibration category** is a category \( \mathcal{C} \) with:

1. A terminal object \( 1 \).
2. A subcategory of **fibrations** containing all the isomorphisms and all the morphisms with codomain \( 1 \). A morphism is called an **acyclic cofibration** if it has the left lifting property with respect to all fibrations.
3. All pullbacks of fibrations exist and are fibrations.
4. For every fibration \( g : A \to B \), the pullback functor \( g^* : \mathcal{C}/B \to \mathcal{C}/A \) has a partial right adjoint \( \Pi_g \), defined at all fibrations over \( A \), and whose values are fibrations over \( B \). This implies that acyclic cofibrations are stable under pullback along \( g \).
5. Every morphism factors as an acyclic cofibration followed by a fibration.
Remark 3.2. A type-theoretic fibration category corresponds to the categorical structure necessary for interpreting a type theory with a unit type, dependent sums, dependent products, and intensional identity types.

Notation 3.3. In a type-theoretic fibration category we denote a fibration by a two-headed arrow \( \rightarrow \) and an acyclic cofibration by \( \sim \).

Lemma 3.4. In the natural model structure on \( \text{Gpd} \) acyclic cofibrations are stable by pullback along any fibration.

Proof. Let \( g \) be a fibration from \( A \) to \( B \) and \( f \) an acyclic cofibration from \( C \) to \( B \). Consider the pullback \( g^*f \) of \( f \) along \( g \),

\[
\begin{array}{ccc}
A \times_B C & \longrightarrow & C \\
\downarrow g^*f & & \downarrow f \\
A & \longrightarrow & B
\end{array}
\]

. First, \( g^*f \) is an injective-on-objects functor. Indeed, let \((x, y)\) and \((x', y')\) be two objects of \( A \times_B C \) with \( x = x' \). In that case one has

\[
f(y) = g(x) = g(x') = f(y')
\]

, hence \( f(y) = f(y') \) and one concludes \( y = y' \) because \( f \) is injective on objects. Second, we prove that \( g^*f \) is a weak equivalence, namely an equivalence of groupoids. The functor \( g^*f \) is essentially surjective. Indeed, let \( y \) be any object of \( A \), then \( g(y) \) is an object of \( B \), hence there exist \( x \) in \( C \) and an isomorphism \( \phi \) in \( B \) from \( f(x) \) to \( g(y) \). Since \( g \) is a fibration, there exists \( \phi^{-1} \) a lift in \( A \) of \( \phi^{-1} \) at \( y \). Let us denote by \( z \) the codomain of this lift. One has \( g(z) = f(x) \), hence \((z, x)\) is an element of \( A \times_B C \) and \( \phi^{-1} \) is an isomorphism in \( A \) from \( g^*f(z, x) = z \) to \( y \). Now, we prove that \( g^*f \) is a fully faithful functor. Let \((x, y), (x', y')\) be two elements in \( A \times_B C \). We need to prove that the map induced by \( g^*f \) from the homset \( A \times_B C((x, y), (x', y')) \) to the homset \( A(x, x') \) that maps a morphism \((\phi, \psi)\) to \( \phi \) is injective. Let us prove that the induced map is injective. Let \((\phi, \psi), (\phi', \psi')\) be two elements in the first homset such that \( \phi = \phi' \). Since \( f \) is fully faithful, the map induced by \( f \) from \( C(y, y') \) to \( B(f(y), f(y')) \) is in particular injective. So, from

\[
f(\psi) = g(\phi) = g(\phi') = f(\psi')
\]

, one concludes \( \psi = \psi' \). We now prove the surjectivity of the map under consideration. Let \( \phi \) be an element in \( A(x, x') \) and consider \( g(\phi) \) in \( B(g(x), g(x')) = B(f(y), f(y')) \). By surjectivity of the map induced by \( f \), there exists a (unique) map \( \psi \) in \( C(y, y') \) with \( f(\psi) = g(\phi) \). Hence, \((\phi, \psi)\) is an element of \( A \times_B C((x, y), (x', y')) \) with \((g^*f)(\phi, \psi) = \phi \). So, \( g^*f \) is fully faithful, and being also an injective-on-objects functor it is an acyclic cofibration. \(\square\)

We prove the analogous result for \( \text{Gpd}^{\text{Z2}} \) with respect to the projective model structure.
Lemma 3.5. In the projective model structure on $\text{Gpd}^{Z_2}$ acyclic cofibrations are stable under pullback along any fibration.

Proof. Consider the following pullback,

\[
\begin{array}{ccc}
A \times_B C & \longrightarrow & C \\
\downarrow g^* f & \quad & \downarrow f \\
A & \longrightarrow & B
\end{array}
\]

Since the underlying morphism of $g$ is a fibration of groupoids and the underlying morphism of $f$ is an acyclic cofibration of groupoids, we conclude by 3.4 that the underlying morphism of $g^* f$ is an acyclic cofibration of groupoids. Thanks to 2.2 it suffices to prove the surjectivity of $g^* f$ onto the fixed points. Let $x$ be a fixed point in $A$. Then $g(x)$ is a fixed point in $B$. Since $f$ is an acyclic cofibration, by 2.2 there exists a (unique) fixed point $y$ in $C$ with $f(y) = g(x)$. Hence, $(x, y)$ is a fixed point in $A \times_B C$ whose image by $g^* f$ is $x$. □

Lemma 3.6. The pullback functor along a fibration preserves acyclic cofibrations with respect to the projective model structure on $\text{Gpd}^{Z_2}$.

Proof. Let $g$ be a fibration in $\text{Gpd}^{Z_2}$ from $A$ to $B$ and consider $g^*$ the pullback functor along $g$ between the slice categories $\text{Gpd}^{Z_2}/B$ and $\text{Gpd}^{Z_2}/A$. Let $\phi$ be an acyclic cofibration in $\text{Gpd}^{Z_2}/B$,

\[
\begin{array}{ccc}
C & \xrightarrow{\phi} & D \\
\downarrow f & \quad & \downarrow h \\
B
\end{array}
\]

The pullback functor maps $\phi$ to the following dotted arrow $g^* \phi$ in the slice category $\text{Gpd}^{Z_2}/A$,

\[
\begin{array}{ccc}
A \times_B C & \longrightarrow & C \\
\downarrow g^* f & \quad & \downarrow f \\
A \times_B D & \longrightarrow & D \\
\downarrow \quad g^* h & \quad & \downarrow h \\
A & \longrightarrow & B
\end{array}
\]

Since $g^* \phi$ is the pullback of $\phi$ along the fibration from $A \times_B D$ to $D$ (since this last map is a pullback of the fibration $g$, it is itself a fibration), it follows from 3.5 that $g^* \phi$ is an acyclic cofibration. □

Theorem 3.7. For every fibration $g: A \rightarrow B$ in $\text{Gpd}^{Z_2}$, the pullback functor

\[g^*: \text{Gpd}^{Z_2}/B \rightarrow \text{Gpd}^{Z_2}/A\]

has a right adjoint $\Pi_g$, and $\Pi_g$ maps fibrations over $A$ to fibrations over $B$. 
Proof. We start by introducing the following notation. If \( u \) is any isomorphism in \( B \), then \( g^* u \) denote the following pullback in \( \mathbf{Gpd} \):

\[
\begin{array}{ccc}
A \times_B & \overset{\text{I}}{\longrightarrow} & I \\
\downarrow{g^* u} & & \downarrow{u} \\
A & \longrightarrow & B
\end{array}
\]

, where \( u \) denotes the functor that maps the non-trivial isomorphism \( \phi : 0 \rightarrow 1 \) of \( \text{I} \) to the isomorphism \( u \) in \( B \). Let \( f : C \rightarrow A \) be a morphism in \( \mathbf{Gpd}^{\mathbb{Z}/2} \). Define \( \text{dom}(\Pi_g f) \) as the groupoid whose collection of objects is denoted \( (\text{dom}(\Pi_g f))_0 \) and whose objects are the pairs \((y, s)\) with \( y \in B \) and \( s : g^{-1}\{y\} \rightarrow C \) a partial section of \( f \), where \( g^{-1}\{y\} \) is the subgroupoid of \( A \) whose objects are objects of \( A \) above \( y \) and morphisms are morphisms of \( A \) above the identity \( 1_y \). Define its collection \( (\text{dom}(\Pi_g f))_1 \) of morphisms as the set of pairs \((u, v)\) with \( u \) a morphism in \( B \) and \( v : g^* u \rightarrow f \) a morphism in \( \mathbf{Gpd}/A \). Define two functions \( s, t \) as follows,

\[
s : (\text{dom}(\Pi_g f))_1 \longrightarrow (\text{dom}(\Pi_g f))_0 (u, v) \longmapsto (\text{dom } u, \ v_{|A \times_B \{0\}})
\]

and

\[
t : (\text{dom}(\Pi_g f))_1 \longrightarrow (\text{dom}(\Pi_g f))_0 (u, v) \longmapsto (\text{cod } u, \ v_{|A \times_B \{1\}})
\]

. Define a partial function

\[\circ : (\text{dom}(\Pi_g f))_1 \times (\text{dom}(\Pi_g f))_1 \rightarrow (\text{dom}(\Pi_g f))_1\]

as follows. Given \((u, v), (u', v') \in (\text{dom}(\Pi_g f))_1\) such that \( s(u', v') = t(u, v) \), define \( v'' : g^*(u' \circ u) \rightarrow f \) in \( \mathbf{Gpd}/A \) by

\[
\begin{align*}
v''(x, 0) &= v(x, 0) \text{ for } (x, 0) \in A \times_B \text{I} \\
v''(x, 1) &= v'(x, 1) \text{ for } (x, 1) \in A \times_B \text{I} \\
v''(h, 1_0) &= v(h, 1_0) \text{ for } (h, 1_0) \in A \times_B \text{I} \\
v''(h, 1_1) &= v'(h, 1_1) \text{ for } (h, 1_1) \in A \times_B \text{I}
\end{align*}
\]

. It remains to define \( v''(h : x \rightarrow x', \phi) \) with \( g(h) = u' \circ u \). Let \( \tilde{u} \) be a lift of \( u \) at \( x \) by \( g \) (i.e. \( \text{dom } \tilde{u} = x \) and \( g(\tilde{u}) = u \)). Since \( g \) is a fibration of groupoids, such a lift exists. One takes

\[v''(h, \phi) = v'(h \circ \tilde{u}^{-1}, \phi) \circ v(\tilde{u}, \phi)\]

, which is a well-defined composition in \( C \). For the sake of readability and for the purpose of avoiding lengthy but straightforward computations, we do not give further details and the reader can check that the defined composition in \( \text{dom}(\Pi_g f) \) is associative. At least, note that \( v''(h, \phi) \) does not depend on the choice of the lift \( \tilde{u} \). Indeed, from the assumption \( s(u', v') = t(u, v) \) one concludes first that \( v'_{|A \times_B \{0\}} = v_{|A \times_B \{1\}} \). Let \( \hat{u} \) be an other lift of \( u \) at \( x \). Then, one has the following
sequence of equalities,
\[\begin{align*}
v'(h \circ \hat{u}^{-1}, \phi) &\circ v(\hat{u}, \phi) \circ [v'(h \circ \hat{u}^{-1}, \phi) \circ v(\hat{u}, \phi)]^{-1} \\
&= v'(h \circ \hat{u}^{-1}, \phi) \circ v(\hat{u}, \phi) \circ v(\hat{u}, \phi)^{-1} \circ v'(h \circ \hat{u}^{-1}, \phi)^{-1} \\
&= v'(h \circ \hat{u}^{-1}, \phi) \circ v(\hat{u}, \phi) \circ v(\hat{u}^{-1}, \phi^{-1}) \circ v'(\hat{u} \circ h^{-1}, \phi^{-1}) \\
&= v'(h \circ \hat{u}^{-1}, \phi) \circ v'(\hat{u} \circ h^{-1}, 1_1) \circ v'(\hat{u} \circ h^{-1}, \phi^{-1}) \\
&= v'(h \circ \hat{u}^{-1}, \phi) \circ v'(\hat{u} \circ h^{-1}, \phi^{-1}) \\
&= v'(1_{x'}, 1_1) \\
&= 1_{v'(x', 1)}
\end{align*}\]

. Last, we define a map \(\text{id}: (\text{dom}(\Pi_g f))_0 \rightarrow (\text{dom}(\Pi_g f))_1\). For \((y, s) \in (\text{dom}(\Pi_g f))_0\) take \(\text{id}_{(y, s)} := (1_y, s)\), which is a slight abuse of notation since by the second member in this pair we really mean the functor between \(g^*(1_y)\) and \(f\) in \(\text{Gpd}/A\) that maps \((x, i)\) (with \(i = 0, 1\)) to \(s(x)\) and \((h, -)\) to \(s(h)\). The reader can check that \(\text{dom}(\Pi_g f)\) is a groupoid with \((u, v)^{-1}\) given by:

\[
\begin{align*}
\text{fst}[(u, v)^{-1}] &:= u^{-1} \\
\text{snd}[(u, v)^{-1}](x, i) &:= v(x, 1 - i) \\
\text{snd}[(u, v)^{-1}](h, 1_i) &:= v(h, 1_{1-i}) \\
\text{snd}[(u, v)^{-1}](h, \phi) &:= v(h, \phi^{-1})
\end{align*}
\]

, where \(\text{fst}\) and \(\text{snd}\) denote the first and the second projections respectively. Recall that we use a Greek letter to denote the involution of a groupoid denoted by the corresponding uppercase letter. For instance \(\alpha\) denotes the involution of the groupoid \(A\). Now, one can equip \(\text{dom}(\Pi_g f)\) with an involution denoted \(\pi_g f:\)

\[
\begin{align*}
\pi_g f: \text{dom}(\Pi_g f) &\rightarrow \text{dom}(\Pi_g f) \\
(y, s) &\mapsto (\beta(y), \pi_g f(s)) \\
(u, v) &\mapsto (\beta(u), \pi_g f(v))
\end{align*}
\]

with

\[
\begin{align*}
\pi_g f(s): g^{-1}\{\beta(y)\} &\rightarrow C \\
x &\mapsto \gamma(s(\alpha(x))) \\
h &\mapsto \gamma(s(\alpha(h)))
\end{align*}
\]

and

\[
\begin{align*}
\pi_g f(v): g^*\{\{\beta(u)\}\} &\rightarrow f \\
(x, i) &\mapsto \gamma(v(\alpha(x), i)) \\
(h, \omega) &\mapsto \gamma(v(\alpha(h), \omega))
\end{align*}
\]

. Last, one defines \(\Pi_g f\) as follows,

\[
\begin{align*}
\Pi_g f: \text{dom}(\Pi_g f) &\rightarrow B \\
(y, s) &\mapsto y \\
(u, v) &\mapsto u
\end{align*}
\]
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This is straightforward to check that $\Pi_g f$ is equivariant. Next, we define $\Pi_g$ on morphisms. Let $i$ be a morphism from $f$ to $h$ in the slice category $\text{Gpd}^{Z_2}/A$,

$$
\begin{array}{c}
\begin{array}{c}
C \\
\downarrow f \\
A
\end{array}
\end{array}
\xrightarrow{i}
\begin{array}{c}
\begin{array}{c}
D \\
\downarrow h
\end{array}
\end{array}
$$

We define $\Pi_g i$ from $\Pi_g f$ to $\Pi_g h$ as follows. For any element $(y, s)$ in $\text{dom}(\Pi_g f)$ we take

$$(\Pi_g i)(y, s) = (y, i \circ s)$$

$$(\Pi_g i)(u, v) = (u, i \circ v)$$

It remains to check that $\Pi_g$ is "the" right adjoint to $g^*$. Let $h: D \to B$ be a morphism in $\text{Gpd}^{Z_2}$ and define a natural bijection $\varphi_{g,f,h}$ (shorten by $\varphi$):

$$
\varphi: \text{Hom}_{\text{Gpd}^{Z_2}/B}(g^* h, f) \to \text{Hom}_{\text{Gpd}^{Z_2}/A}(g^* h, f)
$$

where for $x \in D$ one has $\varphi(v)(x) = (h(x), s_x)$ with

$$s_x: g^{-1}\{h(x)\} \to C
\begin{array}{c}
z \\
\mapsto v(z, x)
\end{array}
\begin{array}{c}
t \\
\mapsto v(t, 1_x)
\end{array}$$

and $\varphi(v)(u: x \to x') = (h(u), w_u)$ for $u$ in $D$ with $w_u: g^*(h(u)) \to f$ in $\text{Gpd}/A$ defined by

$$w_u(z, 0) := s_x(z)$$

$$w_u(z, 1) := s_{x'}(z)$$

$$w_u(1, 0) := s_x(1)$$

$$w_u(1, 1) := s_{x'}(1)$$

$$w_u(0, \phi) := v(\phi, u)$$

It is a matter of easy computations to check that $\varphi(v)$ is equivariant. We have to check that $\varphi$ is a bijection. Let define $\varphi^{-1}$ as follows,

$$
\varphi^{-1}: \text{Hom}_{\text{Gpd}^{Z_2}/A}(g^* h, f) \to \text{Hom}_{\text{Gpd}^{Z_2}/B}(h, \Pi_g f)
$$

where $\varphi^{-1}(k)(z, x) := \text{snd}(k(x))(z)$ for every $z \in A$ and $x \in D$ such that $g(z) = h(x)$ and $\varphi^{-1}(k)(t: z \to z', u: x \to x') := \text{snd}(k(u))(t, \phi)$ for every morphisms
$t$ in $A$ and $u$ in $D$ such that $g(t) = h(u)$. One can easily check that $\varphi^{-1}(k)$ is equivariant, and furthermore
\[
\varphi^{-1} \circ \varphi = \text{id} \\
\varphi \circ \varphi^{-1} = \text{id}
\]
, and the bijection $\varphi$ is natural in its arguments. Finally, by 3.6 and by adjunction one concludes that $\Pi_g$ preserves fibrations in the slice category. As a consequence, in $\text{Gpd}^{Z_2}$ $\Pi_g$ maps fibrations over $A$ to fibrations over $B$.

□

Remark 3.8. When the involutions involved in the statement of theorem 3.7 are identities, we recover a theorem [Gir64, lemma 4.3, theorem 4.4] by Jean Giraud.

Corollary 3.9. The category $\text{Gpd}^{Z_2}$ has the structure of a type-theoretic fibration category with respect to projective fibrations.

Proof. The required conditions (1), (2), (3) and (5) are straightforward. The theorem 3.7 allows us to conclude that (4) holds. □

4. A universe in $\text{Gpd}^{Z_2}$

We recall the notion of a universe [Shu15b, Definition 6.12] in a type-theoretic fibration category.

Definition 4.1. A fibration $p : \tilde{U} \to U$ in a type-theoretic fibration category $\mathcal{C}$ is a universe if the following hold.

(i) Pullbacks of $p$ are closed under composition and contain the identities.
(ii) If $f : B \to A$ and $g : A \to C$ are pullbacks of $p$, so is $\Pi_g f \to C$.
(iii) If $A \to C$ and $B \to C$ are pullbacks of $p$, then any morphism $f : A \to B$ over $C$ factors as an acyclic cofibration followed by a pullback of $p$.

Definition 4.2. Given a universe $p : \tilde{U} \to U$ in a type-theoretic fibration category, a small fibration, or a $U$-small fibration, is a pullback of $p$.

Remark 4.3. A universe in a type-theoretic fibration category interprets a universe type in type theory.

We now move on to constructing universes in the type-theoretic fibration category on $\text{Gpd}^{Z_2}$ given in 3.9. Note that the groupoid model [HS98] of type theory can be reformulated [Shu15b, Examples 2.16] in terms of a type-theoretic fibration category using the natural model structure on $\text{Gpd}$. Given any inaccessible cardinal $\kappa$, in this type-theoretic fibration structure on $\text{Gpd}$ there exists a (univalent) universe $p : \widetilde{V}_\kappa \to V_\kappa$, where $V_\kappa$ is the groupoid whose objects are $\kappa$-small discrete groupoids with isomorphisms between them, $\widetilde{V}_\kappa$ the corresponding groupoid of pointed discrete groupoids and $p$ the obvious projection. The $\kappa$-smallness means that the set of objects of a discrete groupoid has cardinality strictly less than $\kappa$. The $\kappa$-small fibrations are precisely the discrete fibrations of groupoids with $\kappa$-small fibers. So, projective fibrations being objectwise fibrations, a natural candidate for a (univalent) universe in $\text{Gpd}^{Z_2}$ would be a universal fibration that classifies projective fibrations that are objectwise discrete fibrations of groupoids with $\kappa$-small fibers.

Below we define $\tilde{U}, U$ and $p : \tilde{U} \to U$ in $\text{Gpd}^{Z_2}$. For the rest of this section $\kappa$ is an inaccessible cardinal.
• The objects of the groupoid $\tilde{U}$ are dependent tuples of the form $(A, B, \varphi, a)$, where $A, B$ are $\kappa$-small discrete groupoids, $\varphi: A \to B$ is an isomorphism in $\mathbf{Gpd}$, and $a$ is an object of $A$.

• The morphisms in $\tilde{U}$ between $(A, B, \varphi, a)$ and $(C, D, \psi, c)$ are pairs of the form $(\rho: A \to C, \tau: B \to D)$ such that $\psi \circ \rho = \tau \circ \varphi$ and $\rho(a) = c$.

The composition in $\tilde{U}$ is given by

$$(\rho', \tau') \circ (\rho, \tau) := (\rho' \circ \rho, \tau' \circ \tau)$$

Note that $\tilde{U}$ is a groupoid. Indeed, the inverse of the morphism $(\rho, \tau)$ is given by

$$(\rho, \tau)^{-1} := (\rho^{-1}, \tau^{-1})$$

We equip $\tilde{U}$ with the involution $\tilde{v}$ as follows,

$$\tilde{v}: \tilde{U} \to \tilde{U}$$

$$(A, B, \varphi, a) \mapsto (B, A, \varphi^{-1}, \varphi(a))$$

$$(\rho, \tau) \mapsto (\tau, \rho)$$

One denotes by $U$ the “unpointed” version of $\tilde{U}$, i.e. objects are of the form $(A, B, \varphi)$ and morphisms of the form $(\rho, \tau)$, with its corresponding involution $v$.

We define the morphism $p$ in $\mathbf{Gpd}^{\mathbb{Z}_2}$ as the projection

$$p: \tilde{U} \to U$$

$$(A, B, \varphi, a) \mapsto (A, B, \varphi)$$

$$(\rho, \tau) \mapsto (\rho, \tau)$$

We want to prove that $p: \tilde{U} \to U$ is a universe in the type-theoretic fibration category 3.9.

**Definition 4.4.** In the natural model structure on $\mathbf{Gpd}$, a discrete fibration of groupoids is a fibration satisfying the property that given any isomorphism $\varphi$ in the target groupoid and any object $x$ in the fiber of $\text{dom}(\varphi)$, there exists a unique lift of $\varphi$ at $x$ in the domain groupoid.

The map that sends any such lifting problem to its unique solution is called a (split) cleavage of $f$.

**Lemma 4.5.** The $U$-small fibrations in $\mathbf{Gpd}^{\mathbb{Z}_2}$ are precisely the fibrations whose underlying morphism of groupoids $p$ is a discrete fibration.

**Proof.** The projective fibrations being objectwise, the terminal object being pointwise and every groupoid being fibrant with respect to the natural model structure on $\mathbf{Gpd}$, every object in $\mathbf{Gpd}^{\mathbb{Z}_2}$ is fibrant with respect to the projective model structure. In particular, the groupoids $\tilde{U}$ and $U$ are fibrant objects. Moreover, $p$ is an objectwise discrete fibration and we define its unique split cleavage $c_p$ as follows. Given $(\rho, \tau)$ an isomorphism in $U$ and $(\text{dom}(\rho, \tau), x)$ an element in the $p$-fiber of $\text{dom}(\rho, \tau)$, we have no choice but to take $c_p((\rho, \tau), (\text{dom}(\rho, \tau), x)) := (\rho, \tau)$ seen as a morphism in $\tilde{U}$ between $(\text{dom}(\rho, \tau), x)$ and $(\text{cod}(\rho, \tau), \rho(x))$.

**Lemma 4.6.** The $U$-small fibrations in $\mathbf{Gpd}^{\mathbb{Z}_2}$ are precisely the fibrations whose underlying morphisms of groupoids are discrete fibrations with $\kappa$-small fibers.
Proof. First, assume that $f: A \to B$ is a pullback of $p$,

$$
\begin{array}{c}
A & \xrightarrow{p^*g} & \bar{U} \\
\downarrow f & \downarrow & \\
B & \xrightarrow{g} & U
\end{array}
$$

Let $\varphi: x \to y$ be an isomorphism in $B$ and $z \in A$ such that $f(z) = x$. One
denotes by $c_f$ the intended unique cleavage of $f$. One takes

$$
c_f(\varphi, z) := (\varphi, c_p(g(\varphi), p^*g(z)))
= (\varphi, g(\varphi))
$$

which is an isomorphism in $A$ above $\varphi$ by $f$ with domain $x$ (note that we identify
$A$ with the isomorphic groupoid $B \times_U \bar{U}$). The uniqueness of $c_f$ is a consequence
of the uniqueness of $c_p$. The reader can check that the fibers of $f$ are $\kappa$-small.

Conversely, assume that $f$ is a discrete fibration of groupoids with $\kappa$-small fibers.
We denote $c_f$ its cleavage. One has to display $f$ as a pullback of $p$ in $\mathbf{Gpd}^{\bar{U}}$
along a morphism $g$. We define $g$ as follows,

$$
g: B \to U \\
x \mapsto (f^{-1}\{x\}, f^{-1}\{\beta(x)\}, \alpha_x) \\
x \xrightarrow{\sigma} y \mapsto (\rho_\sigma, \tau_\sigma)
$$

where by $f^{-1}\{x\}$ (resp. $f^{-1}\{\beta(x)\}$) we denote the subgroupoid of $A$
whose objects are objects of $A$ above $x$ (resp. $\beta(x)$) and morphisms are morphisms in $A$
above $1_x$ (resp. $1_{\beta(x)}$). Since $f$ is a discrete fibration with $\kappa$-small fibers, these groupoids
are discrete and $\kappa$-small. Moreover, for $x \in B$ we define $\alpha_x$ as the isomorphism
obtained from the restriction of $\alpha$ to $f^{-1}\{x\}$. Given $\sigma: x \to y$ in $B$, we define $\rho_\sigma$
as follows

$$
\rho_\sigma: f^{-1}\{x\} \xrightarrow{z} f^{-1}\{y\} \\
z \mapsto \text{cod}(c_f(\sigma, z))
$$

In the same way one has

$$
\tau_\sigma: f^{-1}\{\beta(x)\} \xrightarrow{z} f^{-1}\{\beta(y)\} \\
z \mapsto \text{cod}(c_f(\beta(\sigma), z))
$$

The reader can easily check that $\rho_\sigma$ and $\tau_\sigma$ are isomorphisms, that $\alpha_y \circ \rho_\sigma$
is equal to $\tau_\sigma \circ \alpha_x$, and $g$ is functorial and equivariant. It remains to check that
$A$ is isomorphic to $B \times_U \bar{U}$ above $B$, i.e. we need to provide an isomorphism
$\chi: A \to B \times_U \bar{U}$ such that $pr_1 \circ \chi = f$, where $pr_1: B \times_U \bar{U} \to B$ is the first
projection. Let define $\chi$ as follows,

$$
\begin{array}{c}
\chi: A \to B \times_U \bar{U} \\
x \mapsto (f(x), f^{-1}\{f(x)\}, f^{-1}\{\beta(f(x))\}, \alpha_f(x), x) \\
x \xrightarrow{\sigma} y \mapsto (f(\sigma), \rho_f(\sigma), \tau_f(\sigma))
\end{array}
$$
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The functor $\chi$ is equivariant and it is actually an isomorphism with $\chi^{-1}$ given by

$$
\begin{align*}
\chi^{-1} & : B \times_U \tilde{U} \longrightarrow A \\
(x, f^{-1}\{x\}, f^{-1}\{\beta(x)\}, \alpha_x, z) & \longmapsto z \\
(\sigma, \rho_\sigma, \tau_\sigma) & \longmapsto c_f(\sigma, \omega)
\end{align*}
$$

, where $\omega$ denotes the last element of the tuple $\text{dom}(\rho_\sigma, \tau_\sigma)$. □

Remark 4.7. The previous lemma expresses in which sense our wannabe universe $p : \tilde{U} \to U$ in $\text{Gpd}^{\mathbb{Z}^2}$ is the natural candidate with respect to the projective model structure that lifts the (univalent) universe $p : \tilde{V}_\kappa \to V_\kappa$ in $\text{Gpd}$.

Lemma 4.8. In $\text{Gpd}^{\mathbb{Z}^2}$ small fibrations are closed under composition and contain the identities.

Proof. Since according to 4.6 small fibrations are the objectwise discrete fibrations of groupoids with small fibers, it is straightforward. □

Lemma 4.9. If $f$ and $g$ are small fibrations in $\text{Gpd}^{\mathbb{Z}^2}$, so is $\Pi_g f$.

Proof. It suffices to prove that $\Pi_g f$ is a $V_\kappa$-small fibration of groupoids. But $\Pi_g f$ is $\Pi_g f$. Since $g, f$ are $V_\kappa$-small fibrations by assumption and $V_\kappa$ is a universe in $\text{Gpd}^{\mathbb{Z}^2}$, $\Pi_g f$ is a $V_\kappa$-small fibration between groupoids. □

Lemma 4.10. For any $U$-small fibration $f$, the diagonal map $\Delta_f$ is a $U$-small fibration.

Proof. Since pullbacks are pointwise in $\text{Gpd}^{\mathbb{Z}^2}$, one has $\Delta_f = \Delta_f$. It suffices to prove that $\Delta_f$ is a $V_\kappa$-small fibration of groupoids, namely a discrete fibration with small fibers. A lifting problem for $\Delta_f$ with respect to the generating acyclic cofibration $i$ is nothing but a pair of isomorphisms $(\varphi, \psi)$ in $E^2$ such that $f(\varphi) = f(\psi)$ and $\text{dom}(\varphi) = \text{dom}(\psi)$. Since $f$ is a discrete fibration, one has $\varphi = \psi$. So, $\Delta_f$ is a discrete fibration and its fibers are obviously small because any fiber is either empty or a singleton. □

Theorem 4.11. The morphism $p : \tilde{U} \to U$ is a universe in the type-theoretic fibration category $\text{Gpd}^{\mathbb{Z}^2}$.

Proof. The lemmas 4.8 and 4.9 take care of (i) and (ii) respectively. According to [Shu15b, Remark 6.13], (iii) is equivalent (under (i)) to the fact that any small fibration has a small path fibration. But by 4.10 any small fibration $f$ has indeed a small path fibration. □

Now to go further, we need to recall what it means for a universe in a type-theoretic fibration category to be univalent (see also [Shu15b, section 7]). Let $\text{Type}$ be a universe in the type theory under consideration. Given two small types, i.e. two elements of $\text{Type}$, there is the type of weak equivalences between them. In a type-theoretic fibration category with a universe, this dependent type is represented by a fibration $E \to U \times U$. Moreover, there is a natural map $U \to E$ that sends a type to its identity equivalence. By (5) one can factor the diagonal map $\delta : U \to
$U \times U$ as an acyclic cofibration followed by a fibration in the following commutative diagram,

\[
\begin{array}{ccc}
U & \longrightarrow & E \\
\downarrow & \searrow & \downarrow \\
PU & \longrightarrow & U \times U
\end{array}
\]

The universe $p: \tilde{U} \to U$ is univalent if the map $U \to E$ is a right homotopy equivalence, or equivalently (by the 2-out-of-3 property and the fact that $U$ is fibrant like any object of a type-theoretic fibration category) if the dashed map is a right homotopy equivalence.

5. Right homotopy equivalences in $\mathbf{Gpd}_{2}$

Now, we develop a few basic facts about right homotopy equivalences, then we give an explicit characterization of right homotopy equivalences with respect to the projective structure on $\mathbf{Gpd}_{2}$.

**Proposition 5.1.** If $\mathcal{C}$ is a type-theoretic fibration category and $f: A \Rightarrow B$ is an acyclic cofibration, then $f$ is a right homotopy equivalence.

**Proof.** One has the following lifting problem

\[
\begin{array}{ccc}
A & \longrightarrow & A \\
\downarrow & \searrow & \downarrow \\
B & \longrightarrow & 1
\end{array}
\]

where $A$ is fibrant like any object of a type-theoretic fibration category. Since $f$ is an acyclic cofibration and $A$ is fibrant, there exists a diagonal filler $g$

\[
\begin{array}{ccc}
A & \longrightarrow & A \\
\downarrow & \searrow & \downarrow \\
B & \longrightarrow & 1
\end{array}
\]

So, one immediately concludes that $g \circ f \sim 1_A$. Since one has in particular $g \circ f = 1_A$, one can display the following lifting problem

\[
\begin{array}{ccc}
A & \sim & B & \sim & PB \\
\downarrow & \sim & \downarrow & \sim & \downarrow \\
B & \sim & PB & \sim & B \times B
\end{array}
\]

where $PB$ is any path object for $B$. So, there exists a diagonal filler $h$

\[
\begin{array}{ccc}
A & \sim & B & \sim & PB \\
\downarrow & \sim & \downarrow & \sim & \downarrow \\
B & \sim & PB & \sim & B \times B
\end{array}
\]

Such a diagonal filler $h$ is in particular a right homotopy between $f \circ g$ and $1_B$, hence $f \circ g \sim 1_B$. □
Remark 5.2. In particular, if $f$ is an acyclic cofibration in the type-theoretic fibration structure on $\text{Gpd}^{Z^2}$ given in 3.9, then $f$ is a right homotopy equivalence.

Remark 5.3. Since not all objects are cofibrant in the projective model structure on $\text{Gpd}^{Z^2}$, right homotopy equivalences are not the same as the weak equivalences of the model structure.

Definition 5.4. Let $A$ be a groupoid together with an involution $\alpha$. One says that $A$ is **weakly connected** if and only if for every pair $(x, y)$ in $\text{Ob}(A)^2$ either $x$ and $y$ are in the same connected component of $A$ or $x$ and $\alpha(y)$ are in the same connected component.

Lemma 5.5. Every groupoid together with an involution is (isomorphic to) a coproduct in $\text{Gpd}^{Z^2}$ of weakly connected groupoids with involutions.

Proof. Let $A$ be a groupoid together with an involution $\alpha$. Given $x$ in $\text{Ob}(A)$, we denote by $A_x$ the connected component of $x$ in the groupoid $A$. Now, we denote by $A_x^{Z^2}$ the full subgroupoid of $A$ whose set of objects is $\text{Ob}(A_x) \cup \text{Ob}(A_{\alpha(x)})$. This full subgroupoid, which we call the weak connected component of $x$, has a natural involution induced from the involution $\alpha$. By choosing a representative for each set $\text{Ob}(A_x) \cup \text{Ob}(A_{\alpha(x)})$, one can display $A$ as a coproduct of its weakly connected components. \(\square\)

Notation 5.6. In the rest of this section we will use the following notations:

$$S(1) := 1 \coprod 1 := 0 \quad 0'$$

$$S(I) := 1 \coprod I := 0 \xrightarrow{\phi} 1$$

$$0' \xrightarrow{\phi'} 1'$$

, both equipped with the swap involution.

Lemma 5.7. Let $A$ be a groupoid together with an involution $\alpha$ and

$$\emptyset \subset B \subseteq C$$

two full subgroupoids of $A$ stable under $\alpha$ such that $\text{Ob}(C) = \text{Ob}(B) \cup \{x, \alpha(x)\}$ with $x \in A$ and $\alpha(x) \neq x$. Moreover, assume that $A$ is weakly connected. Let $z$ be an element of $B$. By weak connectedness there exist an element $y$ of $\{z, \alpha(z)\}$ and an isomorphism $\psi$ from $y$ to $x$. Then the following square is a pushout square in $\text{Gpd}^{Z^2}$,

$$\begin{array}{c}
S(1) \xrightarrow{l} B \\
S(i) \downarrow \quad \downarrow \\
S(I) \xrightarrow{k} C
\end{array}$$

, where

$$l(0) := y \\
l(0') := \alpha(y)$$

$$k(\phi) := \psi \\
k(\phi') := \alpha(\psi)$$
Proof. We will check that this square satisfies the universal property of a pushout. So, consider the following commutative square

\[
\begin{array}{ccc}
S(1) & \xrightarrow{l} & B \\
| & m & | \\
S(i) & \xrightarrow{n} & H
\end{array}
\]

. We want to prove that there exists a unique map \( j: C \to H \) such that \( j_B = m \) and \( j \circ k = n \). We define \( j \) as follows. Take \( j_B = m \), \( j(x) = n(1) \), \( j(\alpha(x)) = n(1') \) and \( j(\psi) = n(\phi) \), \( j(\alpha(\psi)) = n(\phi') \). It remains to define \( j \) successively on morphisms from any \( z \in B \) to \( x \), on morphisms from any \( z \in B \) to \( \alpha(x) \), on the automorphisms of \( x \) and \( \alpha(x) \) and on morphisms from \( x \) to \( \alpha(x) \). Let \( f \) be a morphism from \( z \in B \) to \( x \). Note that \( B \) being a full subgroupoid and \( z \) and \( y \) being elements of \( B \), the morphism \( \psi^{-1} \circ f \) belongs to \( B \). Hence, take \( j(f) = j(\psi) \circ j(\psi^{-1} \circ f) \). Now, let \( f \) be a morphism from any \( z \in B \) to \( \alpha(x) \), to make sure that \( j \) is compatible with the involutions take \( j(f) = \eta(j(\alpha(f))) \), where \( \eta \) is the involution on \( H \). Next, let \( f \) be an automorphism of \( x \), take \( j(f) = j(\psi) \circ j(\psi^{-1} \circ f) \). Again, for the sake of the compatibility with the involutions, take \( j(f) = \eta(j(\alpha(f))) \) for any automorphism \( f \) of \( \alpha(x) \). Last, for any morphism \( f \) from \( x \) to \( \alpha(x) \), take \( j(f) = j(f \circ \psi) \circ j(\psi)^{-1} \).

The reader can easily check that \( j \) is unique.

\[ \square \]

Lemma 5.8. Let \( A \) be a groupoid together with an involution such that \( A_{\iota} = A \) and \( w: B \to C \) a projective acyclic cofibration in \( \text{Gpd}^{G^2} \). Then for any morphism \( v: A \to C \) there exists a map \( \hat{v}: A \to B \) such that \( w \circ \hat{v} = v \).

Proof. We define \( \hat{v} \) as follows. Let \( x \) be an element of \( \text{Ob}(A) \). By the characterization of acyclic cofibrations 2.2, there exists a unique \( y \in B_{\iota} \) such that \( v(x) = w(y) \). Take \( \hat{v}(x) = y \). Now, let \( f: x \to x' \) be a morphism in \( A \). Since \( w \) is fully faithful, the induced map from \( B(y,y') \) to \( C(v(x),v(x')) \) is a bijection. Hence, there exists a unique map \( \hat{v}(f) \) such that \( w(\hat{v}(f)) = v(f) \). Note that \( w(\beta(\hat{v}(f))) = v(\alpha(f)) \), so by injectivity one has \( \hat{v}(\alpha(f)) = \beta(\hat{v}(f)) \) as expected.

\[ \square \]

Lemma 5.9. Let \( f, g: A \to B \) be two right homotopic maps in \( \text{Gpd}^{G^2} \) such that \( A_{\iota} = A \). Then one has \( f = g \).

Proof. Indeed, let \( PB \) be a path object for \( B \)

\[
\begin{array}{ccc}
B \xrightarrow{\sim} & \xrightarrow{w} & PB \\
\Delta & \xrightarrow{} & B \times B \\
& \xrightarrow{h} & PB \\
< f, g > & \xrightarrow{} & B \times B
\end{array}
\]

, and \( h \) a right homotopy between \( f \) and \( g \)

\[
\begin{array}{ccc}
& \xrightarrow{w} & PB \\
\Delta & \xrightarrow{} & B \times B \\
& \xrightarrow{h} & PB \\
< f, g > & \xrightarrow{} & B \times B
\end{array}
\]

. By 5.8 applied with \( C = PB \) and \( v = h \), there exists \( \hat{h} \) such that \( w \circ \hat{h} = h \). So, we have \( \Delta \circ \hat{h} = < f, g > \), hence \( \hat{h} = f = g \).

\[ \square \]
We give the following characterization of the right homotopy equivalences in \( \text{Gpd}^{2z} \) with respect to the projective structure.

**Theorem 5.10.** Let \( f: A \to B \) be a morphism in \( \text{Gpd}^{2z} \). The following are equivalent:

(i) \( f \) is a right homotopy equivalence.

(ii) \( f \) is an equivalence of groupoids and induces an isomorphism between the full subgroups of fixed points \( A_f \) and \( B_f \).

(iii) \( f \) is an equivalence of groupoids and induces an isomorphism between the subgroupoids of fixed points and fixed morphisms \( A^{2z}_f \) and \( B^{2z}_f \).

(iv) \( f \) is an equivalence of groupoids and induces a bijection between the set of fixed points in \( A \) and the set of fixed points in \( B \).

**Proof.** We will successively prove (i) \( \Rightarrow \) (iv), (iv) \( \Rightarrow \) (iii), (iii) \( \Rightarrow \) (ii) and (ii) \( \Rightarrow \) (i).

We prove (i) \( \Rightarrow \) (iv). Assume (i), so there exists \( g: B \to A \) in \( \text{Gpd}^{2z} \) such that \( f \circ g \sim 1_B \) and \( g \circ f \sim 1_A \). Hence, we have \( (f \circ g)|_{B^{2z}} \sim 1_{B^{2z}} \) and \( (g \circ f)|_{A^{2z}} \sim 1_{A^{2z}} \).

So, by 5.9 one has \( (f \circ g)|_{B^{2z}} = 1_{B^{2z}} \) and \( (g \circ f)|_{A^{2z}} = 1_{A^{2z}} \). We conclude that \( f^{2z}_B: A^{2z}_f \to B^{2z}_f \) is an isomorphism. Hence, in particular \( f \) induces a bijection between the set of fixed points in \( A \) and the set of fixed points in \( B \).

We prove (iv) \( \Rightarrow \) (iii). It is straightforward using the fact that \( f \) is in particular a fully faithful functor.

Next, we prove (iii) \( \Rightarrow \) (ii). Note that \( f|_{A_f} \) is bijective on objects, since \( f^{2z}_B \) is so by assumption. Moreover, \( f \) is fully faithful, hence \( f|_{A_f} \) is an isomorphism.

Last, we prove (ii) \( \Rightarrow \) (i). Note that by 5.5 we can assume without loss of generality that \( A \) is weakly connected. Also, one can assume that \( f \) is surjective. Indeed, first note that one can factorize \( f \) through its image \( \text{Im}f \), the full subgroupoid of \( B \) whose objects are of the form \( f(x) \) for some \( x \in A \). The groupoid \( \text{Im}f \) can be equipped with an involution thanks to \( \beta \). Indeed, given \( y \in B \) such that there exists \( x \in A \) and \( f(x) = y \), then \( f(\alpha(x)) = \beta(f(x)) = \beta(y) \).

Second, we prove that the inclusion \( \text{Im}f \hookrightarrow B \) is a projective acyclic cofibration. Indeed, since \( (\text{Im}f)|_{B} = B_f \) and \( \text{Im}f \) is equivalent to \( B \), we conclude by 2.2. So, thanks to 5.2 this inclusion is a right homotopy equivalence. One concludes by the 2-out-of-3 property that \( f: A \to B \) is a right homotopy equivalence if and only if \( A \to \text{Im}f \) is so. The morphism \( A \to \text{Im}f \) is still an equivalence of groupoids by the 2-out-of-3 property and this morphism still induces an isomorphism between \( A_f \) and \( (\text{Im}f)_f \), since \( (\text{Im}f)|_{B} = B_f \). So, without loss of generality one can assume that our map \( f \) is also surjective onto the objects (hence onto the morphisms). One wants to prove that \( f \) is a homotopy equivalence. Below we will provide a morphism \( g: B \to A \) in \( \text{Gpd}^{2z} \) such that \( f \circ g = 1_B \) and \( g \circ f \sim 1_A \). Also, note that the factorizations can be chosen functorially (i.e. the factorization of any morphism as an acyclic cofibration followed by a fibration), since the small object argument applies in the projective model category. In the rest of this proof we use the letter \( P \) to refer to a functor for path objects. To achieve our goal we rely on Zorn’s lemma, namely we construct a preordered set of partial right homotopy equivalences, then we apply Zorn’s lemma to get a maximal element and last we prove that this maximal element is the required (total) right homotopy equivalence. One defines a set \( S \) as the set of triples

\[
(A', \leq A, g': f(A') \to A', h': A' \to PA')
\]
such that $A'$ is a full subgroupoid of $A$ with $\alpha' = \alpha|_{A'}$ and $A_f \subseteq A'$ and the following squares commute

\[
\begin{array}{ccc}
  f(A') & \xrightarrow{g'} & A' \\
  B & \xrightarrow{f'} & \bigcup_{x \in A'} A_f
\end{array}
\]

\[
\begin{array}{ccc}
  A' & \xrightarrow{\times} & A' \\
  A' & \xrightarrow{\times} & A' \\
  A' & \xrightarrow{\times} & A'
\end{array}
\]

where by $f(A')$ we denote the full subgroupoid of $B$ whose objects are the $f(x)$'s with $x \in A'$. This last groupoid is equipped with an involution, namely $\beta_{f(A')}$, since $\beta(f(x)) = f(\alpha(x)) = f(\alpha'(x))$ with $\alpha'(x) \in A'$ whenever $x \in A'$. One equips $S$ with the structure of a preordered set as follows

\[(A', g', h') \leq (A'', g'', h'') \iff (A' \subseteq A'', g''|_{f(A')} = g', Pi \circ h' = h'' \circ i)\]

, where $i$ above denotes the inclusion from $A'$ to $A''$. The reader can easily check the reflexivity and transitivity of $\leq$. Let $\mathcal{C} \subseteq S$ be a chain of $S$.

- First case, assume that $\mathcal{C} = \emptyset$. Take $A' = A_f$. In this case $f(A') = f(A_f) = B_f$, since $f$ induces an isomorphism between $A_f$ and $B_f$. Take $f|_{A_f}$ for $g'$, then $g' \circ f' = 1_{A_f}$ and $< g' \circ f', 1_{A'} > = \Delta$. Hence, take for $h'$ the acyclic cofibration that comes with the path object $PA'$.
- Second case, assume $\mathcal{C} \neq \emptyset$. One takes $A'_e := \colim_{(A', g', h') \in \mathcal{C}} A'$. More specifically, $A'_e$ is the full subgroupoid of $A$ whose set of objects is given by

\[\text{Ob}(A'_e) = \bigcup_{(A', g', h') \in \mathcal{C}} \text{Ob}(A')\]

. This is easy to check that $A'_e$ contains $A_f$ (since $\mathcal{C} \neq \emptyset$) and is equipped with the restriction of $\alpha$ as an involution. In this case $f(A'_e)$ is the full subgroupoid of $B$ whose set of objects is

\[\text{Ob}(f(A'_e)) = \bigcup_{(A', g', h') \in \mathcal{C}} f(A')\]

. We are looking for $g'_e : f(A'_e) \to A'_e$. Take $g'_e = \bigcup_{(A', g', h') \in \mathcal{C}} g'$, the functor whose underlying functions are obtained by the union of the graphs of the underlying functions of the $g'$'s, which makes sense since $\mathcal{C}$ is totally ordered and for $(A', g', h') \leq (A'', g'', h'')$ two elements of $\mathcal{C}$, one has $g''|_{f(A')} = g'$. Now, we are looking for a right homotopy $h'_e$ between $g'_e \circ f|_{A'_e}$ and $1_{A'_e}$. For each $(A', g', h') \in \mathcal{C}$ one has an inclusion $A' \to A'_e$. Hence, by functoriality one has a map $PA'_e \to PA'_e$ and by precomposition of this last morphism with $h' : A' \to PA'$ we get a map from $A'$ to $PA'_e$. By taking the colimit over these morphisms one gets a map $h'_e : A'_e \to PA'_e$. The reader can easily check that $h'_e$ has the required property.

By Zorn’s lemma $S$ has a maximal element $(A_{max}, g_{max}, h_{max})$. We want to prove the equality $A_{max} = A$. Assume $A_{max} \neq A$. We can distinguish two cases.

- First case, let assume the equality $A_{max} = \emptyset$. Since $A_{max} \neq A$, there exists $x \in A$ such that $x$ is not a fixed point because $A_f \subseteq A_{max}$. Now, consider $A$ the full subgroupoid of $A$ whose set of objects is $\{x, \alpha(x)\}$. This groupoid has a natural involution, namely the restriction of $\alpha$. Since $A_f \subseteq A_{max}$ and $f|_{A_f}$ is
an isomorphism, we conclude that neither $A$ nor $B$ has a fixed point. We are looking for $\tilde{g}: f(\tilde{A}) \to \tilde{A}$ such that the diagram

\[
\begin{array}{ccc}
\tilde{A} & \xrightarrow{g} & \tilde{A} \\
\downarrow & & \downarrow \\
B & \xrightarrow{f} & f(\tilde{A})
\end{array}
\]

commutes. Note that $f(\tilde{A})$ is the full subgroupoid of $B$ with two distinct objects $f(x)$ and $f(\alpha(x))$. Since $f$ is fully faithful, its restriction $f_{\tilde{A}}: \tilde{A} \to f(\tilde{A})$ is an isomorphism, so take its inverse for $\tilde{g}$. The groupoid $\tilde{A}$ together with $\tilde{g}$ and the obvious right homotopy contradicts the maximality of $(A_{\max}; g_{\max}; h_{\max})$.

• Second case, let us assume $A_{\max} \neq \emptyset$. Under the assumption $A_{\max} \neq A$, there exists $x \in \text{Ob}(A) \setminus \text{Ob}(A_{\max})$ with $x \notin \alpha(x)$ because $A_{\ell} \subseteq A_{\max}$. We denote by $\tilde{A}$ the full subgroupoid of $A$ generated by $\text{Ob}(A_{\max}) \cup \{x, \alpha(x)\}$. We denote by $B_{\max}$ the full subgroupoid of $B$ generated by the set of objects $\{f(z) | z \in \text{Ob}(A_{\max})\}$. We need to distinguish two subcases depending on whether $f(x)$ belongs to $B_{\max}$.

– First subcase, assume that $f(x) \notin B_{\max}$.

One has $f(x) \notin B_{\ell}$, since $B_{\ell} \subseteq B_{\max}$. We denote by $\tilde{B}$ the full subgroupoid of $B$ generated by $\text{Ob}(B_{\max}) \cup \{f(x), \beta(f(x))\}$. Let $z$ be an element of $A_{\max}$. By weak connectedness, there exist an element $y$ in $\{z, \alpha(z)\}$ and an isomorphism $\psi$ in $A$ from $g_{\max}(f(y))$ to $x$. Thanks to lemma 5.7 (applied to $\psi$) the following square is a pushout square in $\text{Gpd}_{Z^2}$

\[
\begin{array}{ccc}
S(1) & \xrightarrow{l} & A_{\max} \\
S(i) \downarrow & & \downarrow \\
S(1) & \xrightarrow{k} & \tilde{A}
\end{array}
\]

. Again, thanks to lemma 5.7 (applied to $f(\psi)$) the following square is a pushout square,

\[
\begin{array}{ccc}
S(1) & \xrightarrow{l} & B_{\max} \\
S(i) \downarrow & & \downarrow \\
S(1) & \xrightarrow{k} & \tilde{B}
\end{array}
\]

. Now, we want to use the universal property of the pushout square above to provide $\tilde{g}$ as required. One has the following commutative square,

\[
\begin{array}{ccc}
S(1) & \xrightarrow{l} & B_{\max} \\
S(i) \downarrow & & \downarrow \circ g_{\max} \\
S(1) & \xrightarrow{j} & \tilde{A}
\end{array}
\]

, where $l$ is the inclusion from $A_{\max}$ to $\tilde{A}$ and $j$ is defined by $j(\phi) = \psi$ and $j(\phi') = \alpha(\psi)$. Thanks to the universal property of the pushout, we
get a map $\tilde{g}$ as follows

\[
\begin{array}{c}
S(1) \xrightarrow{l} B_{\text{max}} \\
S(i) \downarrow \\
S(I) \xrightarrow{k} B
\end{array}
\]

The reader can easily check that $f \circ \tilde{g}$ is the inclusion from $\tilde{B}$ to $B$. Last, we need to provide a right homotopy $\tilde{h}$ between $\tilde{g} \circ f \mid \tilde{A}$ and $1 \mid \tilde{A}$. Consider the following square,

\[
\begin{array}{c}
A_{\text{max}} \xrightarrow{h_{\text{max}}} PA_{\text{max}} \xrightarrow{p_t} PA \\
\downarrow \downarrow \\
A \xrightarrow{<g \circ f \mid \tilde{A}, 1 \mid \tilde{A}>} \tilde{A} \times \tilde{A}
\end{array}
\]

By 2.2 the inclusion $t$ is an acyclic cofibration. The above square can be rewritten as

\[
\begin{array}{c}
A_{\text{max}} \xrightarrow{h_{\text{max}}} PA_{\text{max}} \xrightarrow{p_t} PA \\
\downarrow \downarrow \\
A \xrightarrow{<g \circ f \mid \tilde{A}, 1 \mid \tilde{A}>} \tilde{A} \times \tilde{A}
\end{array}
\]

To prove the commutativity of this square, it suffices to prove that its bottom triangle commutes,

\[
\begin{array}{c}
A_{\text{max}} \xrightarrow{<g_{\text{max}} \circ f_{\text{max}}, 1 \mid A_{\text{max}}>} A_{\text{max}} \times A_{\text{max}} \\
\downarrow \downarrow \\
A \xrightarrow{<g \circ f \mid \tilde{A}, 1 \mid \tilde{A}>} \tilde{A} \times \tilde{A}
\end{array}
\]

Indeed, this diagram commutes because $g_{\mid \mu_{\text{max}}} = g_{\text{max}}$ and $(f_{\mid \tilde{A}})_{\mid A_{\text{max}}} = f_{\mid A_{\text{max}}} := f_{\text{max}}$. So, one has a diagonal filler $\tilde{h}$

\[
\begin{array}{c}
A_{\text{max}} \xrightarrow{h_{\text{max}}} PA_{\text{max}} \xrightarrow{p_t} PA \\
\downarrow \downarrow \\
A \xrightarrow{<g \circ f \mid \tilde{A}, 1 \mid \tilde{A}>} \tilde{A} \times \tilde{A}
\end{array}
\]
ON THE INADEQUACY OF THE PROJECTIVE STRUCTURE WITH RESPECT TO THE UNIVERSE AXIOM

and \( \tilde{h} \) is the right homotopy we are looking for. As a consequence, we have \((A_{\text{max}}, g_{\text{max}}, h_{\text{max}}) < (\tilde{A}, \tilde{g}, \tilde{h})\) in \( S \), which contradicts the maximality of \((A_{\text{max}}, g_{\text{max}}, h_{\text{max}})\).

Second subcase, assume that \( f(x) \) belongs to \( B_{\text{max}} \). In this case the full subgroupoid of \( B \) generated by \( \text{Ob}(B_{\text{max}}) \cup \{ f(x), \beta(f(x)) \} \) is still \( B_{\text{max}} \). We still denote by \( \iota \) the inclusion from \( A_{\text{max}} \) to \( \tilde{A} \). Take \( \tilde{g} = \iota \circ g_{\text{max}} \) that makes the following square commutes

\[
\begin{array}{ccc}
B_{\text{max}} & \xrightarrow{\tilde{g}} & \tilde{A} \\
\downarrow & & \downarrow \\
B & \xrightarrow{f_{\tilde{A}}} & \tilde{A}
\end{array}
\]

By a previous argument the following square commutes,

\[
\begin{array}{ccc}
A_{\text{max}} & \xrightarrow{h_{\text{max}}} & PA_{\text{max}} \\
\iota \downarrow & & \downarrow \\
A & \xrightarrow{<\tilde{g} \circ f_{\tilde{A}}, 1_{\tilde{A}}>} & \tilde{A} \times \tilde{A}
\end{array}
\]

Since the inclusion \( \iota \) is an acyclic cofibration, we get the desired right homotopy \( \tilde{h} \) as a diagonal filler

\[
\begin{array}{ccc}
A_{\text{max}} & \xrightarrow{h_{\text{max}}} & PA_{\text{max}} \\
\iota \downarrow & & \downarrow \\
A & \xrightarrow{<\tilde{g} \circ f_{\tilde{A}}, 1_{\tilde{A}}>} & \tilde{A} \times \tilde{A}
\end{array}
\]

, and we conclude in the same way. Thus, eventually \( A_{\text{max}} = A \) and \((A_{\text{max}}, g_{\text{max}}, h_{\text{max}})\) displays \( f \) as a right homotopy equivalence.

6. The Failure of Univalence

Tracing through the interpretation of type theory, one finds that the fibration \( E \to U \times U \), interpreting the dependent type of weak equivalences, is such that the set of objects of the fiber over a pair \((x, y) \in U \times U\) is the set of isomorphisms in \( U \) between \( x \) and \( y \). Moreover, the involution on \( E \) maps \((x, y, \varphi)\), where \( \varphi \) is an isomorphism from \( x \) to \( y \), to \((v(x), v(y), v(\varphi))\) (with \( v \) the involution on \( U \)).

**Proposition 6.1.** Univalence does not hold for the universe \( p: \tilde{U} \to U \) (cf. 4) in the type-theoretic fibration category given in 3.9.

**Proof.** The morphism \( U \to E \) (cf. the end of section 4) is defined as follows

\[
\begin{align*}
U & \to E \\
x & \mapsto (x, x, 1_x)
\end{align*}
\]

i.e. it maps an object \( x \) to the identity isomorphism of \( x \) in \( U \). Note that this morphism is not surjective onto the fixed points of \( E \). Indeed, it is easy to find a non-trivial fixed point of \( E \). For instance, take the following elements of \( U : (N, N, 1_N) \),
(2N, 2N, 1_{2N}), and the isomorphism (2n, 2n) between them, where 2n denotes the bijection from N to 2N that maps n to 2n. Then this triplet is a fixed point of E, where the third component is not the identity. So, it does not belong to the image of the morphism above. Note that we can even take two identical small groupoids and still find a fixed point of E that does not belong to the image of U → E. Indeed, consider (Z, Z, 1_Z) and the automorphism (−n, −n), where −n denotes the bijection from Z to Z that maps n to −n. So, according to 5.10 the map U → E is not a right homotopy equivalence, hence univalence does not hold in the projective type-theoretic fibration structure on \( \text{Gpd}^{22} \).

Below we investigate whether function extensionality holds. For details about the meaning of function extensionality in a type-theoretic fibration category see [Shu15b, section 5]. In particular, according to [Shu15b, lemma 5.9] function extensionality holds in the internal language of a type-theoretic fibration category if and only if dependent products along fibrations preserve acyclic fibrations (i.e. fibrations that are also right homotopy equivalences).

**Proposition 6.2.** Function extensionality does not hold in the internal type theory of the projective type-theoretic fibration structure on \( \text{Gpd}^{22} \).

**Proof.** It suffices to prove that there exist a fibration \( g \) and a fibration \( f \) such that \( f \) is a right homotopy equivalence and \( \Pi_g f \) is not a right homotopy equivalence. In order to achieve this, consider \( g: S(1) \to 1 \) and \( f: S(I) \to S(1) \) (see 5.6) in \( \text{Gpd}^{22} \). The reader can easily check that \( f \) is fully faithful and surjective (and so it is an acyclic fibration in \( \text{Gpd} \)), and \( f \) restricted to fixed points is the identity (since \( (S(I))_f = (S(1))_f = \emptyset \)). So, according to 5.10 \( f \) is a right homotopy equivalence. Now, since \( \Pi_g f \) goes from \( \text{dom}(\Pi_g f) \) to \( 1 \), it suffices to prove that \( \text{dom}(\Pi_g f) \) has at least two fixed points. A fixed point of \( \text{dom}(\Pi_g f) \) over \( 1 \) is nothing but a section \( s \) of \( f \) such that \( \pi_g f(s) = s \) (where \( \pi_g f \) is the involution on \( \Pi_g f \), see 3). But we have two such sections \( s_1 \) and \( s_2 \). Indeed, take

\[
s_1: S(1) \to S(I) \\
0 \mapsto 0 \\
0' \mapsto 0'
\]

and

\[
s_2: S(1) \to S(I) \\
0 \mapsto 1 \\
0' \mapsto 1'
\]

\[\square\]

**Remark 6.3.** We recall that the univalence axiom implies function extensionality. However, since the above proposition involves non-discrete groupoids, it does not give us an alternative proof that univalence does not hold for the universe \( p: \tilde{U} \to U \). Also, note that according to [Shu15b, Remark 5.10] function extensionality holds in the natural type-theoretic fibration structure on \( \text{Gpd} \). So, with the projective type-theoretic fibration structure on \( \text{Gpd}^{22} \) function extensionality is also broken.
7. Conclusion

Our work suggests that projective fibrations are a good choice in order to lift a bare type-theoretic fibration structure from a category $\mathcal{C}$ to a functor category $[\mathcal{D}, \mathcal{C}]$ even in the presence of non-trivial isomorphisms in $\mathcal{D}$, and eventually to provide the additional structure needed for universes. But projective fibrations don’t seem to be adequate for the stability of the univalence property. The fact that projective fibrations prove to be unsuitable for univalence lies in the strongness of the projective homotopy equivalences.

Nevertheless, the model presented in this article provides a new model of type theory with dependent sums, dependent products, identity types and a universe. Moreover, to the best of our knowledge it was the first model derived from a Quillen model structure where not all objects are cofibrant. If it should happen in a model that not all objects are fibrant-cofibrant, then our method of proof makes it clear that even when a whole model structure is available at hand only the classes of fibrations, acyclic cofibrations and right homotopy equivalences are relevant in the context of type theory.

Last, this model together with the model in [Bor17], using the Quillen equivalent injective model structure on the same bare category, gives a counterexample to a tentative model invariance problem suggested by Michael Shulman.

REFERENCES

[Bor15] Anthony Bordg. On lifting univalence to the equivariant setting. PhD thesis, Université Nice Sophia Antipolis, 2015. arXiv:1512.04083. (On p. 2)
[Bor17] Anthony Bordg. On a model invariance problem in Homotopy Type Theory. https://arxiv.org/abs/1712.03409, 2017. (On p. 23)
[Gir64] J. Giraud. Méthode de la descente. Bull. Math. Soc. Mémorie, 2, 1964. (On p. 10)
[Hir03] Philip S. Hirschhorn. Model Categories and their Localizations, volume 99 of Mathematical Surveys and Monographs. American Mathematical Society, 2003. (On p. 3)
[HS98] Martin Hofmann and Thomas Streicher. The groupoid interpretation of type theory. In Twenty-five years of constructive type theory (Venice, 1995), volume 36 of Oxford Logic Guides, pages 83–111. Oxford Univ. Press, New York, 1998. (On pp. 2 and 10)
[KL12] Chris Kapulkin and Peter LeFanu Lumsdaine. The simplicial model of univalent foundations (after Voevodsky). arXiv:1211.2851, 2012. (On p. 1)
[Lur09] Jacob Lurie. Higher topos theory. Number 170 in Annals of Mathematics Studies. Princeton University Press, 2009. (On pp. 2 and 3)
[Rez] Charles Rezk. A model category for categories. http://www.math.uiuc.edu/~rezk/papers.html. (On p. 2)
[Shu15a] Michael Shulman. The univalence axiom for elegant Reedy presheaves. Homology, Homotopy, and Applications, 17(2):81–106, 2015. arXiv:1307.6248. (On p. 1)
[Shu15b] Michael Shulman. Univalence for inverse diagrams and homotopy canonicity. Mathematical Structures in Computer Science, 25:1203–1277, 6 2015. arXiv:1203.3253. (On pp. 1, 10, 13, and 22)
[Shu17] Michael Shulman. The univalence axiom for EI diagrams. Homology, Homotopy and Applications, 19(2):219–249, 2017. arXiv:1508.02410. (On pp. 1, 2, and 4)
[Str00] Neil Strickland. K(n)-local duality for finite groups and groupoids. Topology, 39(4):733–772, 2000. (On p. 2)

1https://ncatlab.org/homotopytypetheory/show/model-invariance+problem