Fourier expansion along geodesics on Riemann surfaces

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Abstract. For an eigenfunction of the Laplacian on a hyperbolic Riemann surface, the coefficients of the Fourier expansion are described as intertwining functionals. All intertwiners are classified. A refined growth estimate for the coefficients is given and a summation formula is proved.

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Introduction

For an automorphic function, the invariance under parabolic elements is used to give the standard Fourier expansion, the coefficients of which define the $L$-function of the form. In this paper, we instead consider the Fourier-expansion along a hyperbolic element. In other terms, let $Y$ be a hyperbolic Riemann surface and let $c$ be a closed geodesic in $Y$. We are interested in the Fourier coefficients

$$c_k(f) = \int_0^1 f(c(l_c t)]e^{2\pi i k t} dt$$

of a smooth function $f \in C^\infty(Y)$. Here $l_c$ is the length of the geodesic $c$. Under the assumption that $f$ be an eigenfunction of the Laplace operator on $Y$ with eigenvalue $\alpha$, one can relate $c_k$ to an intertwining integral $I^\alpha_k(f)$, which depends on $\alpha$ and $f$, but not on $c$. There is an automorphic coefficient $a_k \in \mathbb{C}$, such that

$$c_k = a_k I^\alpha_k.$$

The present paper contains three main results:

- in Theorem 2.3 one finds a classification of all intertwining functionals on the dual of the group $\text{PGL}_2(\mathbb{R})$.
- In Theorem 2.5 there is given the growth estimate

$$a_k = O(|k|^{\frac{1}{2}})$$

as $|k| \to \infty$. The proof uses the technique of analytic continuation developed by Bernstein and Reznikov in [BR99].
- In Theorem 4.2 finally, a summation formula is proved, which involves the coefficients $a_k$ and the spectral decomposition in the compact case. The proof relies on the uniqueness of invariant trilinear forms as in [BR04]. The sum formula is of the form

$$\sum_{k \in \mathbb{Z}} |a_k|^2 \hat{\alpha}(k) = \sum_j c_j \int_{\mathbb{R}^2} W_j(t, x) \alpha(\hat{t}_x) dt dx,$$

where $\alpha$ is a test function, the decomposition of the $G$-representation on $L^2(\Gamma \backslash G)$ is $\bigoplus_j \pi_j$ and the constants $c_j$ and the explicit functions $W_j$ depend on $\pi_j$. Finally $\hat{t}_x = \frac{1}{2} \log \left| \frac{(e^{ix}+x)(x-1)}{(e^{ix}+x-1)x} \right|$. It is hoped that the choice of specific test functions will lead to more precise growth estimates for the $a_k$. 

We explain the construction of the factors $a_k$ in a bit more detail. Let $X$ be the universal covering of $Y$ and $\Gamma$ its fundamental group. Then $\Gamma$ acts on $X$ by isometries and $Y$ is the quotient $\Gamma \backslash X$. So $\Gamma$ injects into the isometry group $G$ of $X$, which acts transitively on $X$, i.e., $X \cong G/K$ for a maximal compact subgroup $K$. Let $(\pi, V_\pi)$ be an irreducible unitary representation of the group $G$ and let $\eta : V_\pi \to L^2(\Gamma \backslash G)$ be an isometric linear $G$-map. Let $P_K : L^2(\Gamma \backslash G) \to L^2(\Gamma \backslash G)^K = L^2(\Gamma \backslash G/K) = L^2(Y)$ denote the orthogonal projection onto the subspace of $K$-invariants. Demanding that $f \in C^\infty(Y)$ be an eigenfunction of the Laplacian amounts to the same as demanding $f$ to lie in the image of $P_K \circ \eta$ for some $\pi$ and some $\eta$. The functional $I^\gamma_k = c_k \circ P_K \circ \eta$ on $V_\pi$ then has an intertwining property with respect to a split torus $A$ inside $G$. By a uniqueness result, proven in Section 2, this implies that $I^\gamma_k$ is a multiple of a standard intertwiner $I^\alpha_{\pi,k}$ on $V_\pi$, which we named $I^\alpha_k$ above. So we get the existence of a factor $a_k \in \mathbb{C}$ with $I^\gamma_k = a_k I^\alpha_{\pi,k} = a_k I^\alpha_k$ as above.

In Section 1 we describe the setting in greater precision. In Section 2 we classify the intertwining functionals that show up in the context and define the standard intertwiners that give rise to the factors $a_k$ above. We also show the growth estimate of the factors $a_k$. In Section 3 we show how the Fourier expansion along a geodesic expands to an expansion on the whole space and in Section 4 we show how to derive the summation formula from the uniqueness of triple products.

1 Generalized period integrals

In this paper we use the group $GL_2(\mathbb{R})$, the elements of which we write as matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and the group $G = PGL_2(\mathbb{R}) = GL_2(\mathbb{R})/\mathbb{R}^\times$, the elements of which we write in the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Here we will usually arrange the determinant of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to be 1 or $-1$. The connected component of $G$ is $G^0 = PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/(\pm 1)$. The group $G^0$ acts on the upper half plane $\mathbb{H}$ by linear fractionalals and this action extends to an action of $G$ in a way that $G$ is identified with the group of all hyperbolic isometries on $\mathbb{H}$. The stabilizer in $G$ of the point $i \in \mathbb{H}$ is the maximal compact subgroup $K = PO(2) = O(2)/\pm 1$ of $G$. So $\mathbb{H}$ is identified with $G/K$.

Let $\Gamma$ be a discrete subgroup of the group $G$. Later we will assume $\Gamma$ to be of finite covolume. For simplicity, we will assume $\Gamma$ to be torsion-free and that $\Gamma \subset G^0$. This implies that $\Gamma$ is the fundamental group of $Y = \Gamma \backslash \mathbb{H}$ and
the latter is a Riemann surface equipped with the hyperbolic metric.

For a closed geodesic $c$ in $Y$ let $l(c)$ denote its length. The period integral
\[ I_c(f) = \int_0^{l(c)} f(c(t)) \, dt \]
is the zeroth coefficient of the Fourier-expansion of the function $t \mapsto f(c(t))$. Therefore the higher coefficients can be viewed as "generalised period integrals".

The real Lie-algebra of $G$ is $\mathfrak{g}_\mathbb{R} = \mathfrak{sl}_2(\mathbb{R})$, the Lie-algebra of all real $2 \times 2$ matrices of trace zero. For $X, Y \in \mathfrak{g}_\mathbb{R}$ let $b(X, Y) = \frac{1}{2} \text{tr} (XY)$. Then $b$ is an invariant symmetric bilinear form. Let $\mathfrak{e}_\mathbb{R} \subset \mathfrak{g}_\mathbb{R}$ be the Lie algebra of $K$. Then $b$ is negative definite on $\mathfrak{e}_\mathbb{R}$ and positive definite on its orthogonal complement $\mathfrak{p}_\mathbb{R}$. Let $a^0 = \mathbb{R}(1_{-1})$, and let $A = \exp(a^0)$ be the corresponding subgroup of $G$. Let $a^+ = \mathbb{R}_{>0}(1_{-1})$ be the positive cone and let $A^+ = \exp(a^+)$.

A closed geodesic $c$ in $Y$ gives rise to a conjugacy class $[\gamma]$ in $\Gamma$ of elements which "close" $c$. Any such $\gamma \in \Gamma$ is hyperbolic in the sense that it is conjugate in $G$ to an element of the form $a_{t_0} = \text{diag}(e^{t_0}, e^{-t_0}) \in A \setminus \{1\}$. We insist that $a_{t_0} \in A^+$, i.e., $t_0 > 0$ to make it unique. We now assume that $\gamma$ is primitive, i.e., $\gamma$ is no power $\tau^n$ for any $\tau \in \Gamma$ and $n \geq 2$. This is equivalent to the geodesic $c$ being primitive, i.e., $c$ is no power of any shorter geodesic. The characters of the compact abelian group $A/\langle a_\gamma \rangle$ are given by those $\mu \in a^*$ with $\mu(\log a_\gamma) = 2\pi i$. Let $\mu_\gamma$ be the unique element of $a^*$ with $\mu_\gamma(\log a_\gamma) = 2\pi i$. Then $A/\langle a_\gamma \rangle = \mathbb{Z}_{\mu_\gamma}$. Later we will use the notation

$$\hat{\mu} = \frac{1}{2\pi i} \mu.$$
Then the map $t \mapsto f^\sigma(a_t x)$ with $a_t = \text{diag}(e^t, e^{-t})$ is periodic of period $t_0$ and thus has a Fourier-expansion

$$f^\sigma(a_t x) = \sum_{k \in \mathbb{Z}} e^{2\pi i k t/t_0} \frac{1}{t_0} \int_0^{t_0} f^\sigma(a_t x) e^{-2\pi i k t/t_0} dt.$$  

For $k \in \mathbb{Z}$ let

$$I_k^\gamma : C^\infty(\Gamma \backslash G) \to \mathbb{C}$$

$$f \mapsto \frac{1}{t_0} \int_0^{t_0} f^\sigma(a_t x) e^{-2\pi i k t/t_0} dt.$$  

Note that $I_k^\gamma$ depends on the choice of $\sigma$. Geometrically, this corresponds to choosing a base-point on the closed orbit $c$. This dependence is not severe, as $\sigma$ is determined up to multiplication from the right by elements of $A$.

If we replace $\sigma$ by $\sigma a_0$, then $I_k^\gamma$ is replaced by $a_0^{k \mu \gamma} I_k^\gamma$. So in particular, the absolute value $|I_k^\gamma|$ is uniquely determined by $k$ and $\gamma$. Further, if $\gamma$ is replaced by a $\Gamma$-conjugate, say $\gamma' = \tau \tau^{-1}$ then one can choose $\sigma_{\gamma'}$ to be equal to $\tau \sigma_{\gamma}$ and with this choice one gets $I_k^{\gamma'} = I_k^\gamma$.

The form $b$ determines a Haar measure $dg$ on $G$. Let $R$ denote the unitary $G$-representation on $L^2(\Gamma \backslash G)$ given by right translations. We are particularly interested in the subspace $L^2_{\text{cusp}}(\Gamma \backslash G)$ of cusp forms. This representation space decomposes discretely,

$$L^2_{\text{cusp}}(\Gamma \backslash G) \cong \bigoplus_{\pi \in \hat{\Gamma}} N_\Gamma(\pi) \pi,$$

where the sum runs over the unitary dual $\hat{\Gamma}$ of $G$ and the multiplicities $N_\Gamma(\pi)$ are finite. Here and later we understand the direct sum to be a completed direct sum in the appropriate topology. We define $C^\infty_{\text{cusp}}(\Gamma \backslash G)$ to be the intersection of $C^\infty(\Gamma \backslash G)$ with the space of cusp forms. Then it turns out that $C^\infty_{\text{cusp}}(\Gamma \backslash G)$ is the set of smooth vectors in the representation space $L^2_{\text{cusp}}(\Gamma \backslash G)$, i.e.,

$$C^\infty_{\text{cusp}}(\Gamma \backslash G) = L^2_{\text{cusp}}(\Gamma \backslash G)^\infty = \bigoplus_{\pi \in \hat{\Gamma}} N_\Gamma(\pi) \pi^\infty,$$

where $\pi^\infty$ is the representation on the Fréchet space of smooth vectors. The linear functional $I_k^\gamma$ satisfies

$$I_k^\gamma(R(a)\varphi) = a^{k \mu \gamma} I_k^\gamma(\varphi)$$

for every $a \in A$. This means that $I_k^\gamma$ is an intertwining functional.
2 Intertwining functionals

We first shall give a description of the admissible and unitary duals of the group $G$. For any topological group $G$, let $\hat{G}$ denote the unitary dual of $G$, that is, the set of unitary equivalence classes of irreducible unitary representations of $G$.

Next let $G$ denote a semisimple Lie group with finite center and finitely many connected components. Then $G$ has a maximal compact subgroup $K$ which is unique up to conjugation. A representation $(\pi, V_{\pi})$ of $G$ is called admissible, if for every $\tau \in \hat{K}$ the isotype $V_{\pi}(\tau)$ is finite-dimensional. In that case the space $V_{\pi,K}$ of $K$-finite vectors in $V_{\pi}$ forms a $(g, K)$-module, where $g$ is the complexified Lie algebra of $G$. Two admissible representations are called infinitesimally equivalent if their $(g, K)$-modules of $K$-finite vectors are isomorphic. The admissible dual $\hat{G}_{\text{adm}}$ of $G$ is the set of infinitesimal equivalence classes of irreducible admissible representations of $G$.

Due to results of Harish-Chandra, every irreducible unitary representation of $G$ is admissible and two unitary admissible representations are unitarily equivalent if and only if they are infinitesimally equivalent. Thus the unitary dual $\hat{G}$ can be considered a subset of the admissible dual $\hat{G}_{\text{adm}}$.

Now consider $G = \text{PGL}_2(\mathbb{R})$. There is a canonical character

$$\chi : G \to \{\pm 1\}; \quad g \mapsto \text{sign}(\det(g)),$$

taking the values $\pm 1$ and having the connected component $G^0$ for kernel. For a representation $\pi$ of $G$ we define the $\chi$-twist $\chi\pi$ of $\pi$, also written $\chi \otimes \pi$ as the representation with the same space $V_{\pi}$ as $\pi$ but defined as

$$\chi\pi(x) = \chi(x)\pi(x).$$

Let $P$ denote the parabolic subgroup of $G$ consisting of all upper triangular matrices. Then $P = MAN$, where $N$ is the group of all upper triangular matrices with ones on the diagonal. For $\lambda \in \mathbb{C}$ and $a = \begin{bmatrix} e^t & \cdot \\ e^{-t} & 1 \end{bmatrix} \in A$ we write

$$a^\lambda = e^{\lambda t}.$$

Let $\pi_\lambda$ be the corresponding principal series representation, which we normalize to live on the space of functions $\varphi : G \to \mathbb{C}$ satisfying $\varphi(manx) = a^{\lambda+1}\varphi(x)$. Let $V_\lambda$ be the space of $\pi_\lambda$ and let $V_\lambda^\infty$ be the space of smooth
vectors in it. These can be viewed as smooth sections of the line bundle $E_\lambda$ over $P \backslash G$ given by the $P$-representation $(\pi, V_\pi) \mapsto a^{\lambda+1}$. Note that restriction of functions to $K$ identifies $V_\lambda^\infty$ with the space $C^\infty(M \backslash K)$, so in particular, for $\varphi \in V_\lambda^\infty$, the function $\varphi|_K$ is independent of $\lambda$.

**Proposition 2.1.** Let $G = \text{PGL}_2(\mathbb{R})$. The admissible dual $\hat{G}_{\text{adm}}$ of $G$ consists of

(a) $\pi_\lambda$, $\lambda \in \mathbb{C}$, $\lambda \notin 1 + 2\mathbb{Z}$,

(b) $\mathcal{D}_{2n}$ for $n = 1, 2, 3, \ldots$ the standard discrete series representations,

(c) $\delta_m$ for $m = 0, 2, 4, \ldots$ where $\delta_m$ is the $(m + 1)$-dimensional representation on the space of all homogeneous polynomials $p(X, Y)$ of degree $m$,

**Proof.** This can be deduced from the description of the unitary dual of $\text{SL}_2(\mathbb{R})$ given, for example, in [Kna01].

Let $T = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$ be the non-trivial element of $M$. As $T^2 = 1$, for every representation $(\pi, V_\pi)$, the space $V_\pi$ splits as a direct sum $V_\pi = V_\pi^+ \oplus V_\pi^-$.
where $V_{π}^\pm$ is the $\pm 1$-eigenspace of $π(T)$. Twisting by $χ$ interchanges the roles of $V_{π}^+$ and $V_{π}^-$. 

Let $μ ∈ a^*$ and let $(π, V_π)$ be a representation of $G$. A continuous linear functional $l : V_π^∞ → C$ is called a $μ$-intertwiner, if

$$l(π(a)v) = a^μl(v)$$

holds for every $a ∈ A$ and every $v ∈ V_π$. Let $V_π^∞(μ)$ be the space of all $μ$-intertwiners. Note that $V_π^∞(μ) = V_π^∞(χπ(μ))$, where we consider the $χ$-twist $χπ$ as a representation with the same representation space as $π$.

Let $w_0 = [1 \ -1]$ and $n_0 = [1 \ 1]$. Then $w_0$ is a representative of the non-trivial element of the Weyl-group $W(G, A)$.

The base space $P \setminus G$ of the bundle $E_λ$ consists of three orbits under the group $AM$, namely the open orbit $[w_0n_0]$, and the two closed orbits $[1]$, $[w_0]$ which are indeed points.

We now define a standard intertwiner on the representation $π_λ$ for $\text{Re}(λ) > -1$. Let

$$I^\text{st}_{λ,μ}(φ) = I^\text{st}_{π_λ,μ}(φ) = \int_A φ(w_0n_0a)a^{-μ}da, \quad φ ∈ V_λ^∞.$$ 

If $\text{supp} φ ⊂ [w_0n_0]$, then the integral $I^\text{st}_{λ,μ}$ is extended over a compact set, hence it exists.

**Lemma 2.2.** Let $μ ∈ C$. If $\text{Re}(λ) > -1 - \text{Re}(μ)$, then the integral $I^\text{st}_{λ,μ}(φ)$ exists for every $φ ∈ V_λ^∞$ and defines a $μ$-intertwiner. The map $λ ↦ I^\text{st}_{λ,μ}$ extends to a meromorphic operator-valued function with poles exactly at

$$λ = -μ - 1 - 4k, \quad k ∈ N_0$$

and

$$λ = μ - 3 - 4k, \quad k ∈ N_0.$$

Outside the poles, $I^\text{st}_{λ,μ}$ spans the one dimensional space of intertwiners supported on the open orbit. At the poles this space is zero.

**Proof.** Let $a : G → A$, $n : G → N$, $k : G → K$ be the smooth maps defined by the Iwasawa decomposition $g = ank = a(g)n(g)k(g)$ for $g ∈ G$. For
\[ \varphi \in V^\infty_{\lambda} \text{ one has} \]

\[
I_{\lambda,\mu}^{st}(\varphi) = \int_{A} a(w_{0}n_{0}a)\lambda^{+1} a^{-\lambda-1-\mu} \varphi(k(w_{0}n_{0}a)) da \\
= \int_{A} a(w_{0}n_{0}^{a})\lambda^{+1} a^{-\lambda-1-\mu} \varphi(k(w_{0}n_{0}a)) da,
\]

where \( n_{0}^{a} = a^{-1}n_{0}a \). Noting that if \( a = \begin{pmatrix} e^{t} & e^{-t} \\ e^{-t} & e^{t} \end{pmatrix} \), then we have \( n_{0}^{a} = \begin{pmatrix} 1 & e^{-2t} \\ e^{-2t} & 1 \end{pmatrix} \), we get \( w_{0}n_{0}^{a} = \begin{pmatrix} 0 & \sqrt{e^{-4t+1}} \\ \sqrt{e^{-4t+1}} & 1 \end{pmatrix} \) and thus \( k(w_{0}n_{0}^{a}) = \frac{1}{\sqrt{e^{-4t+1}} \begin{pmatrix} e^{-2t} & -1 \\ -1 & e^{-2t} \end{pmatrix}} \). We now define special test functions. For \( k, l \in \mathbb{N}_{0} \) let \( \varphi_{k,l} : \text{SO}(2) \to \mathbb{C} \) be defined by

\[ \varphi_{k,l}(d_{c,d}) = e^{kd}. \]

If \( k + l \) is even, \( \varphi_{k,l} \) defines an element of \( V^\infty_{\lambda} \). The theory of Taylor-series tells us that every \( \varphi \in V^\infty_{\lambda} \) can be written as

\[ \varphi = \sum_{0 \leq k,l \leq N} c_{k,l} \varphi_{k,l} + R_{N}(\varphi), \]

where \( c_{k,l} \in \mathbb{C} \) and the function \( R_{N}(\varphi) \in V^\infty_{\lambda} \cong C^{\infty}(M\backslash K) \) vanishes to order \( N \) at 1 and \( w_{0} \). The integral \( I_{\lambda,\mu}^{st}(\varphi) \), as written above, makes sense for \( \varphi = \varphi_{k,l} \) also in the case when \( k + l \) is not even. We use this fact for convenience. For \( k, l \in \mathbb{N}_{0} \) we compute

\[
I_{\lambda,\mu}^{st}(\varphi_{k,l}) = \int_{A} a(w_{0}n_{0}^{a})\lambda^{+1} a^{-\lambda-1-\mu} \varphi_{k,l}(k(w_{0}n_{0}^{a})) da \\
= \int_{R} a\begin{pmatrix} 0 & -1 \\ 1 & e^{-2t} \end{pmatrix}^{\lambda^{+1}} e^{-(\lambda+1+\mu)t} \varphi_{k,l}\begin{pmatrix} 0 & -1 \\ 1 & e^{-2t} \end{pmatrix} dt \\
= \int_{R} (e^{-4t} + 1)^{-\lambda+1} e^{-(\lambda+1+\mu)t} \varphi_{k,l} \left(\frac{1}{\sqrt{e^{-4t} + 1}} \begin{pmatrix} e^{-2t} & -1 \\ -1 & e^{-2t} \end{pmatrix}\right) dt \\
= \int_{R} (e^{-4t} + 1)^{-\lambda + 2l + 1 + \mu} e^{-(\lambda+2l+1+\mu)t} dt \\
= \frac{1}{4} \int_{R} (e^{-t} + 1)^{-\lambda + 2l + 1 + \mu} e^{-\lambda - 2l + 1 + \mu} dt \\
= \frac{1}{4} B\left(\frac{\lambda + 2l + 1 + \mu}{4}, \frac{\lambda + 2k + 1 - \mu}{4}\right),
\]

where \( B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \) is Euler’s Beta-function. We conclude

\[ I_{\lambda,\mu}^{st}(\varphi_{k,l}) = \frac{\Gamma\left(\frac{\lambda + 2l + 1 + \mu}{4}\right) \Gamma\left(\frac{\lambda + 2k + 1 - \mu}{4}\right)}{\Gamma\left(\frac{\lambda + 2l + 1 + \mu}{4}\right)}. \]
Note that
\[ I_{\lambda,\mu}^{st}(\varphi_{k+1,l+1}) = I_{\lambda+2,\mu}^{st}(\varphi_{k,l}). \]
The space \( \varphi_{N,N}C^\infty(M \setminus K) \) is the space of all \( \varphi \in V^\infty_\lambda \) which vanish to order \( \geq N \) at 1 and \( w_0 \). For any \( \varphi \) in this space, the integral \( I_{\lambda,\mu}^{st}(\varphi) \) converges if \( \text{Re}(\lambda) > -N - 1 - \text{Re}(\mu) \). Therefore we get analytic continuation of the map \( \lambda \mapsto I_{\lambda,\mu}^{st} \) as claimed. The lemma follows.

We next consider intertwiners which are supported on the closed orbits [1] and \([w_0]\). Let \( S_{[1],0} : V^\infty_\lambda \to \mathbb{C} \) denote the distribution
\[
S_{[1],0}(\varphi) = \varphi(1).
\]
Then \( S_{[1],0} \circ R(a) = a^{\lambda+1}S_{[1],0} \), so \( S_{[1],0} \) is an \( \mu \)-intertwiner for \( \mu = \lambda + 1 \).

We next consider higher derivatives of this distribution. For \( X \in \mathfrak{g} \), the Lie algebra of \( G \), and \( f \in C^\infty(G) \), we let
\[
R_X f(y) = \frac{d}{dt} \bigg|_{t=0} f(y \exp(tX)).
\]
Let \( \bar{N} = \theta(N) \) and let \( \bar{n}_\mathbb{R} \) be its Lie algebra. Then the tangent space of \( P \setminus G \) at the unit is isomorphic to \( \bar{n}_\mathbb{R} \). Let \( X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) and \( \bar{X} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \). Then \( n_\mathbb{R} = \mathbb{R}X \) and \( \bar{n}_\mathbb{R} = \mathbb{R}\bar{X} \). For \( k \in \mathbb{N} \) set
\[
S_{[1],k}(\varphi) = R^k_X \varphi(1).
\]
Then \( S_{[1],k} \circ R(a) = a^{\lambda+1+2k}S_{[1],k} \). Since these span the space of all distributions supported at 1 we see that we get a non-zero \( \mu \)-intertwiner supported on 1 if and only if
\[
\mu = \lambda + 1 + 2k
\]
for some \( k \in \mathbb{N}_0 \). If this condition is satisfied, then the space of intertwiners supported on \([1]\) is one dimensional.

We turn to the other closed orbit \([w_0]\). In this case we define
\[
S_{[w_0],k}(\varphi) = R^k_X \varphi(w_0).
\]
Then \( S_{[w_0],k} \circ R(a) = a^{-\lambda-1-2k}S_{[w_0],k} \) and we conclude that there exists a non-zero intertwiner supported on \([w_0]\) if and only if
\[
\mu = -\lambda - 1 - 2k, \quad k \in \mathbb{N}_0,
\]
in which case the space of intertwiners supported on \([w_0]\) is one dimensional.
Theorem 2.3. (a) For \( \lambda, \mu \in \mathbb{C} \) with \( \mu \neq 0 \) we have
\[
\dim V^{\infty}_{\lambda}(\mu) = 1.
\]
In this case, if \( \lambda \notin \pm(\mu - 1) - 2\mathbb{N}_0 \), then \( V^{\infty}_{\lambda}(\mu) \) is spanned by \( I_{\lambda\mu}^{st} \). If \( \lambda = \mu - 1 - 2k \) with \( k \in \mathbb{N}_0 \), then \( V^{\infty}_{\lambda}(\mu) \) is spanned by \( S_{[1],k} \) and if \( \lambda = -\mu - 1 - 2k \) with \( k \in \mathbb{N}_0 \), then \( V^{\infty}_{\lambda}(\mu) \) is spanned by \( S_{[w_0],k} \).

The same holds for the \( \chi \)-twist.

(b) For \( \lambda \in \mathbb{C} \) and \( \mu = 0 \) we have
\[
\dim V^{\infty}_{\lambda}(0) = \begin{cases} 
2 & \lambda \in -1 - 2\mathbb{N}_0, \\
1 & \text{otherwise.}
\end{cases}
\]
If \( \lambda \notin -1 - 2\mathbb{N}_0 \), then \( V^{\infty}_{\lambda}(0) \) is spanned by \( I_{\lambda,0}^{st} \). If \( \lambda = -1 - 2k \) with \( k \in \mathbb{N}_0 \), then \( V^{\infty}_{\lambda}(0) \) is spanned by \( S_{[w_0],k}, S_{[1],k} \).

The same holds for the \( \chi \)-twist.

(c) For the finite-dimensional representations we have \( \dim \delta_{2n}(\mu) = 0 \) if \( \mu \neq -2n, -2n + 2, \ldots, 2n \) and \( \dim \delta_{2n}(\mu) = 1 \) if \( \mu \in \{-2n, -2n + 2, \ldots, 2n\} \).

(d) Let \( \mu \in \mathbb{C}, n \in \mathbb{N} \). Then we have an exact sequence
\[
0 \to \mathcal{D}^{\infty}_{2n}(\mu) \to V^{\infty}_{1-2n}(\mu) \to \delta_{2n-2}(\mu) \to 0.
\]
If \( \mu \notin \{-2n + 2, \ldots, 2n - 2\} \), then \( \delta_{2n-2}(\mu) = 0 \) and therefore \( \mathcal{D}^{\infty}_{2n}(\mu) \cong V^{\infty}_{1-2n}(\mu) \) is one dimensional.

If \( \mu \in \{-2n + 2, \ldots, 2n - 2\} \) but \( \mu \neq 0 \), then \( \mathcal{D}^{\infty}_{2n}(\mu) = 0 \) and finally \( \mathcal{D}^{\infty}_{2n}(0) \) is one-dimensional and is spanned by \( S_{[1],n-1} - S_{[w_0],n-1} \).

Proof. (a) and (b) are clear by the above. For (c) recall that \( \delta_{2n} \) has the basis \( e_{2j-2n} = X^j Y^{2n-j} \) for \( j = 0, \ldots, 2n \) and the group \( A \) acts by \( \delta_{2n}(a)e_{2j-2n} = a^{2j-2n}e_{2j-2n} \). This proves (c).

For (d) we consider the exact sequence
\[
0 \to \delta_{2n-2} \to \pi_{1-2n} \to \mathcal{D}_{2n} \to 0,
\]
which induces the exact sequence of intertwiners
\[
0 \to \mathcal{D}^{\infty}_{2n}(\mu) \to V^{\infty}_{1-2n}(\mu) \to \delta_{2n-2}(\mu).
\]
This proves the first assertion, i.e., the case $\delta_{2n-2}(\mu) = 0$. If $\mu \in \{-2n + 2, \ldots, 2n - 2\}$, which means $\delta_{2n-2}(\mu) \neq 0$, then we have to show that the map $V_{1-2n}^\infty(\mu) \to \delta_{2n-2}(\mu)$ is non-zero, for it is automatically onto then, as the target space is one-dimensional. The above exact sequence dualizes to the exact sequence

$$0 \to D_{2n} \to \pi_{2n-1} \to \delta_{2n-2} \to 0,$$

which yields an exact sequence

$$0 \to \delta_{2n-2}(\mu) \to V_{2n-1}^\infty(\mu) \to D_{2n}^\infty(\mu).$$

So the arrow $\delta_{2n-2}(\mu) \to V_{2n-1}^\infty(\mu)$ is non-zero, hence its dual $V_{1-2n}^\infty(\mu) \to \delta_{2n-2}(\mu)$ likewise.

Finally, for $\mu = 0$ we show that the kernel of the restriction map $V_{1-2n}^\infty(0) \to \delta_{2n-2}(0)$ is spanned by $S_{[w_0],n-1} - S_{[1],n-1}$. Recall that $R_X$ and $R_{\bar{X}}$ are the weight-change operators in highest weight theory, so $R_{X}^{-1}e_0$ is a multiple of $e_{2n-2}$ and $R_{\bar{X}}^{-1}e_0$ is a multiple of $e_{2-2n}$. Let $\theta(x) = x^{-t}$ be the transpose followed by the inversion, i.e., $\theta$ is the Cartan-involution on $G$ with fixed point set $K$. We have $\theta(X) = \bar{X}$. It is readily verified that the map $\Psi$ with $\Psi(f)(x) = f(w_0\theta(x))$ is an isomorphism between the representations $\pi_\lambda$ and $\pi_{\lambda} \circ \theta$. Thus it follows that $S_{[w_0],n-1}(\Psi(\varphi)) = S_{[1],n-1}(\varphi)$. As $\Psi(\pi(a)\varphi) = \pi(a^{-1})\Psi(\varphi)$ and $\Psi^2 = \text{Id}$ it follows that $\Psi(e_0) = \pm e_0$. We want to show $\Psi(e_0) = e_0$. For this we let $(d^c_{-c}) \in \text{SO}(2)$ with $c, d > 0$. The $A$-invariance of $e_0$ shows that for $a > 0$ we have

$$e_0 \left( \frac{1}{\sqrt{a^2c^2 + d^2/a^2}} \begin{pmatrix} ad - c/a \\ ac - d/a \end{pmatrix} \right) = (c^2a^2 + d^2/a^2)^{\frac{n-1}{2}} e_0 \left( \begin{pmatrix} d & -c \\ c & d \end{pmatrix} \right).$$

So the $A$-orbit is mapped to positive multiples of $e_0 \left( \begin{pmatrix} d & -c \\ c & d \end{pmatrix} \right)$. By the $M$-invariance we get on the other hand,

$$e_0 \left( \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} d & -c \\ c & d \end{pmatrix} \right) = e_0 \left( \begin{pmatrix} -c & -d \\ d & c \end{pmatrix} \right) = e_0 \left( \begin{pmatrix} c & -d \\ d & c \end{pmatrix} \right).$$

As $(d^c_{-c})$ lies in the same $A$-orbit as $(d^c_{-c})$, the claim follows. We therefore conclude that $S_{[w_0],n-1}(e_0) = S_{[1],n-1}(e_0)$. 

A sequence $(c_j)_{j \in \mathbb{N}}$ of complex numbers is said to be of moderate growth, if there exist $N \in \mathbb{N}$ such that

$$|c_j| = O(j^N),$$
as \( j \to \infty \). The sequence is called rapidly decreasing, if for every \( N \in \mathbb{N} \) one has
\[
|c_j| = O(j^{-N})
\]
as \( j \to \infty \). The product of two moderately growing sequences is moderately growing and the product of a moderately growing sequence and a rapidly decreasing sequence is rapidly decreasing.

**Proposition 2.4.** Let \( \text{Re}(\lambda) > -1 \) and \( f \in V_\lambda^\infty \) as well as \( \mu \in i\mathbb{R} \setminus \{0\} \) be given. Then the sequence \( (I_{\lambda,k\mu}^\ast(f))_{k \in \mathbb{Z}} \) is rapidly decreasing.

**Proof.** Recall \( H_1 = \begin{pmatrix} 1 & -1 \\ \end{pmatrix} \). Using integration by parts, we compute for \( N \in \mathbb{N} \),
\[
I_{\lambda,k\mu}^\ast(\varphi) = \int_A \varphi(w_0n_0a)a^{-k\mu} \, da \\
= \int_\mathbb{R} \varphi(w_0n_0 \exp(tH_1)e^{-k\mu t}) \, dt \\
= \left( \frac{1}{k\mu} \right)^N \int_\mathbb{R} R_{H_1}^N \varphi(w_0n_0a)a^{-k\mu} \, da.
\]
As \( \text{Re}(\mu) = 0 \) we have \( |a^{-k\mu}| = 1 \) and the claim follows. \( \square \)

By an automorphic representation \( (\pi, V_\pi, \eta) \) we mean an irreducible unitary representation \( (\pi, V_\pi) \) of \( G \) together with an isometric \( G \)-equivariant linear map \( \eta: V_\pi \to L^2(\Gamma \setminus G) \). Then \( \eta \) maps the space \( V_\pi^\infty \) of smooth vectors into \( C^\infty(\Gamma \setminus G) \). Let \( k \in \mathbb{Z} \). The map \( I_k^\gamma \circ \eta \) is an intertwiner for \( k\mu_\gamma \). So, for instance, let \( \pi = \pi_\lambda \in \hat{G} \), then the space of intertwiners on \( V_\lambda \) is spanned by \( I_{\lambda,k\mu_\gamma} \). Therefore there exists \( a_k^{\eta;\gamma} \in \mathbb{C} \) such that
\[
I_k^\gamma \circ \eta = a_k^{\eta;\gamma} I_{\lambda,k\mu_\gamma}^\ast.
\]
However, if \( \pi = D_{2n} \) is a discrete series representation and \( k = 0 \), then there are \( b_0^{\eta;\gamma}, c_0^{\eta;\gamma} \in \mathbb{C} \) such that
\[
I_0^\gamma \circ \eta = b_0^{\eta;\gamma} S_{[1],n-1} + c_0^{\eta;\gamma} S_{[w_0],n-1},
\]
where \( c_0^{\eta;\gamma} = -b_0^{\eta;\gamma} \) if \( n \) is even and analogously for the \( \chi \)-twist. Finally, If \( \pi \) is the trivial representation, we consider \( \pi \) as a subrepresentation of \( \pi_{-1} \), so this case does not need extra treatment.
Theorem 2.5. (a) There exists a constant $C_\eta > 0$, depending on $\gamma$, such that

$$|a^{\eta,\gamma}_k| \leq C_{\gamma} \left( 1 + |\lambda|^{1/4} \right) \left( 1 + |\lambda^2 - k^2\mu^2|^{1/4} \right) e^{\pi|k\mu|}$$

holds for every cuspidal automorphic representation $\eta : V_\lambda \to L^2_{\text{cusp}}(\Gamma \backslash G)$ and every $k \in \mathbb{Z}$.

(b) For a fixed automorphic representation $\eta : V_\lambda \to L^2_{\text{cusp}}(\Gamma \backslash G)$ there exists a constant $C_{\eta,\gamma} > 0$ such that

$$|a^{\eta,\gamma}_k| \leq C_{\eta,\gamma} (1 + |k|^{1/4})$$

holds for every $k \in \mathbb{Z}$.

(c) There exists a constant $D_{\gamma} > 0$, depending on $\gamma$, such that

$$|b^{\eta,\gamma}_0|, |c^{\eta,\gamma}_0| \leq D_{\gamma} n^{5-n}$$

holds for every cuspidal automorphic representation $\eta : D_{2n} \to L^2_{\text{cusp}}(\Gamma \backslash G)$.

Proof. (a) Let $\varphi = \varphi_\lambda \in V_\lambda$ be the unique $K$-invariant function which on $K$ takes the value 1 and let $f = \eta(\varphi)$. Then

$$I^{\gamma}_k(f) = a^{\eta,\gamma}_k I^{k\mu}_{\lambda}(\varphi) = a^{\eta,\gamma}_k I^{k\mu}_{\lambda}(f_{0,0}) = a^{\eta,\gamma}_k \frac{\Gamma \left( \frac{\lambda+1+k\mu}{4} \right) \Gamma \left( \frac{\lambda+1-k\mu}{4} \right)}{\Gamma \left( \frac{\lambda+1}{2} \right)}$$

If $\Gamma$ is cocompact, by [SS89] the sup norm of $f$ satisfies

$$\|f\|_\infty = O \left( |\lambda|^{1/4} \right).$$

If $\Gamma$ is not cocompact, one finds in [Iwa02], Sec. 13.2, that

$$|f(z)| = O_x (|z|^{1/4}),$$

where the implied constant depends continuously on $z$. Therefore, this estimate holds uniformly on the closed geodesic attached to $\gamma$. According to [GR07] 8.328.1, for fixed real $x$ and for $|y| \to \infty$ one has

$$|\Gamma(x + iy)| \sim \sqrt{2\pi} e^{-\frac{x}{2}|y|} |y|^{x-\frac{3}{2}}.$$ 

This implies the claim.
(b) We start out as in the above proof, except that we don’t use the estimate $|f(z)| \ll z^{|\lambda| \frac{3}{4}}$. We thus get

$$|a_k^{\eta,\gamma}| \ll |I_k^{\gamma}(f)|(1 + |\lambda|^2 - k^2 |\mu|^2 |\frac{3}{4})e^{\frac{\pi}{4}|k\mu|}.$$ 

It suffices to show that for fixed $\gamma$ and $\eta$ one has

$$|I_k^{\gamma}(f)| \ll e^{-\frac{\pi}{4}|k\mu|}.$$ 

We will show this using the technique of analytic continuation of representations from [BR99]. Let $X \in \text{sl}_2(\mathbb{R})$ with $\gamma = \exp(X)$. After conjugating $\Gamma$, we may assume $X = \text{diag}(A, -A)$ for some $A > 0$. Then $\gamma = \text{diag}(e^A, e^{-A})$ and $\mu = \frac{2\pi i}{A}$. We have

$$I_k^{\gamma}(f) = \int_0^1 f(\exp(tX))e^{-2\pi i k t} dt.$$ 

It is easy to see that the function $t \mapsto \pi_\lambda(a_t)\phi$ with $a_t = \begin{bmatrix} e^{At} & e^{-At} \end{bmatrix}$ extends to a holomorphic function from $\{ \text{Im} z < \frac{\pi}{4A} \}$ to $V_\lambda$. It follows that the function $f(\exp(tX)) = \eta(\pi_\lambda(a_t)\phi(1))$ extends to a holomorphic function on the set of all $z = x + iy \in \mathbb{C}$ with $|y| < \frac{\pi}{4A}$. We get a continuous extension to $|y| \leq \frac{\pi}{4A}$. For $k \geq 0$ we get by a shift of the contour integral that

$$I_k^{\gamma}(f) = \int_0^1 f(\exp(tX))e^{-2\pi i k t} dt$$

$$= \int_0^1 f(\exp((t - i \frac{\pi}{4A})X))e^{-2\pi i k t} dt e^{-\frac{\pi^2}{4A} k} = \text{const} \cdot e^{-\frac{\pi}{4}|k\mu|}.$$ 

For $k < 0$ we similarly move the contour to $i\frac{\pi}{4A}$.

(c) Let $n \in \mathbb{N}$ and let $\varphi_n : K \to \mathbb{C}$ be given by

$$\varphi_n \left( \begin{bmatrix} d & c \\ e & d \end{bmatrix} \right) = (ci + d)^{2n}.$$ 

Then $\varphi_n$ and its complex conjugate span the lowest $K$-type in $\mathcal{D}_{2n} \cong \pi_{1 - 2n}/\delta_{2n - 2}$. A computation shows

$$R_X^* \varphi_n = i n(\varphi_n + \varphi_{n+1}),$$

$$R_X \varphi_n = -i n(\varphi_n - \varphi_{n+1}).$$ 

We use induction in $k \in \mathbb{N}_0$ to show that

$$S_{[1],k}(\varphi_n) = (-1)^{k+n} S_{[\varphi_0],k}(\varphi_n).$$
For $k = 0$ we have

$$S_{[1],k}(\varphi_n) = \varphi_n(1) = 1 = (-1)^n \varphi(w_0) = (-1)^{k+n} S_{[w_0],k}(\varphi_n).$$

The step $k \mapsto k + 1$ is

$$S_{[1],k+1}(\varphi_n) = R_X^{k+1} \varphi_n(1)$$

$$= 
\int (R_X^k \varphi_n + \varphi_{n+1})(1)

= \int (S_{[1],k}(\varphi_n) + S_{[1],k}(\varphi_{n+1})

= \int ((-1)^{k+n} S_{[w_0],k}(\varphi_n) + (-1)^{k+n+1} S_{[w_0],k}(\varphi_{n+1})

= (-1)^{k+n+1} \left( -\int S_{[w_0],k}(\varphi_n) + S_{[w_0],k}(\varphi_{n+1}) \right)

= (-1)^{k+n+1} \left( -\int R_X^k \varphi_n(w_0) + R_X^{k+1} \varphi_{n+1}(w_0) \right)

= (-1)^{k+n+1} \left( R_X^{k+1} \varphi_n(w_0) \right)

= (-1)^{k+n+1} \left( S_{[w_0],k+1}(\varphi_n) \right).$$

Next we show that

$$|S_{[1],n-1}(\varphi_n)| \geq n^{n-1} \quad \text{and} \quad |S_{[1],n-1}(\varphi_{n+1})| \geq n^{n-1}$$

This follows from $\varphi_m(1) = 1$ and the fact that

$$R_X^k \varphi_n = i^k (n^k \varphi_n + (\ast)),$$

where $(\ast)$ denotes a linear combination of $\varphi_m$, $m \geq n + 1$ with positive coefficients. Now the equality

$$I_0^\gamma(\eta(\varphi_n)) = b_0^{\eta_\gamma} S_{[1],n-1}(\varphi_n) + c_0^{\eta_\gamma} S_{[w_0],n-1}(\varphi_n)$$

implies $I_0^\gamma(\eta(\varphi_n)) = (b_0^{\eta_\gamma} - c_0^{\eta_\gamma}) S_{[1],n-1}(\varphi_n)$ and thus

$$|b_0^{\eta_\gamma} - c_0^{\eta_\gamma}| \ll \frac{|I_0^\gamma(\eta(\varphi_n))|}{n^{n-1}}.$$
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independent of \( \eta \). To see this, let \( \Delta \) denote the group Laplacian, i.e.,
\[
\Delta = -C + 2CK
\]
where \( C \) is the Casimir-operator and \( CK \) is the Casimir operator of the group \( K \). Then for \( \eta : D_{2n} \to L^2_{\text{cusp}}(\Gamma \backslash G) \) a computation shows \( \Delta \eta(\varphi_n) = n(n-1)\eta(\varphi_n) \). Since \( \Delta \) is elliptic, positive definite and of order two on the 3-dimensional manifold \( \Gamma \backslash G \), the operator \((1 + \Delta)^{-1}\) has a continuous kernel \( k_0(x, y) \). It acts on the space of cusp forms through the continuous kernel
\[
k_0(x, y) = \sum_{j=1}^{\infty} (1 + \lambda_j)^{-2} \phi_j(x) \overline{\phi_j(y)},
\]
where \( \phi_j \) is an orthonormal basis of \( L^2_{\text{cusp}}(\Gamma \backslash G) \) consisting of \( \Delta \)-eigenfunctions and \( \lambda_j \) is the eigenvalue of \( \phi_j \). Selberg has shown in the Göttingen lectures, that \( k_0 \) actually is an \( L^2 \)-kernel. We have
\[
|\eta(\varphi_n)(x)| = (1 + n(n - 1))^2 |(1 + \Delta)^{-2} \eta(\varphi_n)(x)|
\]
\[
= (1 + n(n - 1))^2 \left| \int_{\Gamma \backslash G} k_0(x, y) \eta(\varphi_n)(y) \, dy \right|
\]
\[
\leq (1 + n(n - 1))^2 \left( \int_{\Gamma \backslash G} |k_0(x, y)|^2 \, dy \right)^{1/2}.
\]
As \( k_0 \) is an \( L^2 \)-kernel, the latter integral is finite almost everywhere in \( x \) and is locally bounded outside a set of measure zero. Hence the continuous function \( \eta(\varphi_n) \) is locally bounded by a constant times \( (1 + n(n - 1))^2 \), so it is locally \( O(n^4) \) and the same holds for the period integral \( I_0(\eta(\varphi_n)) \).

3 Fourier expansion of Maaß forms

Let \( f : \Gamma \backslash \mathbb{H} \to \mathbb{C} \) be a Maaß form, i.e., \( f \in L^2(\Gamma \backslash \mathbb{H}) \) is an eigenform of the hyperbolic Laplacian, say \( \Delta f = (\frac{1}{4} - \lambda^2) f \) for \( \lambda \in \mathbb{C} \). Then there exists an automorphic representation \( (\pi, V_\pi, \eta) \) with \( \pi = \pi_\lambda \) such that \( f = \eta(\varphi_0) \), where \( \varphi_0 \in V_\pi \) is a \( K \)-invariant function. By scaling, one can achieve \( \varphi_0(K) = \{1\} \). One then gets the Fourier expansion
\[
f(x) = \sum_{k \in \mathbb{Z}} a_k^{\eta, \gamma} I_{\lambda_k, \mu_\gamma}^{\text{st}} (\pi(\sigma^{-1} x) \varphi_0).
\]
Note that the automorphic representation \( \eta \), which determines the form \( f \), enters on the right hand side only through the coefficients \( (a_k^{\eta, \gamma})_{k \in \mathbb{Z}} \). In particular, if \( x = \sigma a_t \) for \( t \in \mathbb{R} \), one has
\[
I_{\lambda_k, \mu_\gamma}^{\text{st}} (\pi(\sigma^{-1} x) \varphi_0) = I_{\lambda_k, \mu_\gamma}^{\text{st}} (\pi(a_t) \varphi_0) =
\]
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\[ a^\mu I^\iota_{\lambda,k\mu_\gamma}(\varphi_0), \text{ and therefore } f \left( \sigma \left( \frac{e^t}{e^{-t}} \right) \right) \text{ equals} \]

\[ \frac{1}{\Gamma \left( \frac{\lambda+1}{2} \right)} \sum_{k \in \mathbb{Z}} a^\eta \gamma e^{2\pi i k \tilde{\mu} t} \Gamma \left( \frac{\lambda + 2\pi ik\tilde{\mu} + 1}{4} \right) \Gamma \left( \frac{\lambda - 2\pi ik\tilde{\mu} + 1}{4} \right). \]

4 Triple products

In this section we assume \( \Gamma \) to be cocompact. Then \( L^2_{\text{cusp}}(\Gamma \backslash G) = L^2(\Gamma \backslash G) \).

Let \((\pi, V_\pi, \eta)\) be an automorphic representation with \( \pi = \pi_\lambda \) for some \( \lambda \in \mathbb{C} \). Since \( \pi \) is unitary, there is an anti-linear isomorphism to the dual \( c : V_\pi \to V_\pi^* \). Let \( \tilde{\pi} \) denote the dual representation on \( V_\pi^* = V_\tilde{\pi} \). Let \( \cdot \) be the complex conjugation on \( L^2(\Gamma \backslash G) \) and let \( \tilde{\eta} \) be the composition of the maps

\[ V_\tilde{\pi} \xrightarrow{c^{-1}} V_\pi \xrightarrow{\eta} L^2(\Gamma \backslash G) \xrightarrow{\cdot} L^2(\Gamma \backslash G) \]

Then \( \tilde{\eta} \) is a \( G \)-equivariant linear isometry of \( V_\tilde{\pi} \) into \( L^2(\Gamma \backslash G) \), so \((\tilde{\pi}, V_\tilde{\pi}, \tilde{\eta})\) is an automorphic representation as well.

Let \( \Delta : \Gamma \backslash G \to \Gamma \backslash G \times \Gamma \backslash G \) be the diagonal map. Let \( \Delta^* : C^\infty(\Gamma \backslash G \times \Gamma \backslash G) \to C^\infty(\Gamma \backslash G) \) be the corresponding pullback map and let \( E = V_\pi^\infty \otimes V_\tilde{\pi}^\infty \), where \( \otimes \) denotes the projective completion of the algebraic tensor product.

Let \( \eta_E : E \to C^\infty(\Gamma \backslash G) \otimes C^\infty(\Gamma \backslash G) \cong C^\infty(\Gamma \backslash G \times \Gamma \backslash G) \)

be given by \( \eta \otimes \tilde{\eta} \). For \( \gamma \) as in the first section and \( k \in \mathbb{Z} \) we get an induced functional on \( E \),

\[ l^k_{\Delta(\gamma)} = \hat{I}^\gamma_\iota \circ \Delta^* \circ \eta_E. \]

In other words, for \( w \in E \) we have

\[ l_{\Delta(\gamma)}(w) = \frac{1}{l(\gamma)} \int_{\Lambda/\langle a_\gamma \rangle} \eta_E(w)(\sigma_\gamma a, \sigma_\gamma a) da. \]

This has the Fourier series expansion,

\[ l_{\Delta(\gamma)}(w) = \sum_{k \in \mathbb{Z}} I^\gamma_\iota \circ \eta \otimes I^*_\iota_{-k} \circ \tilde{\eta}(w) \]

\[ = \sum_{k \in \mathbb{Z}} a^\eta \gamma a^{\eta \gamma}_{-k} I^\iota_{\lambda,k\mu_\gamma} \otimes I^*_{-\lambda k\mu_\gamma}(w) \]

\[ = \sum_{k \in \mathbb{Z}} |a^\eta \gamma_a|^2 \hat{w}(k, -k), \]
where the last line defines $\hat{w}$ and we have used the fact that $\hat{a}_{k}^{\gamma} = \overline{a_{k}^{-\gamma}}$.

Let $(\tau, V_{\tau})$ be another element of $\hat{G}$. According to [Mol79] there is a canonical $G$-invariant continuous functional

$$T_{\tau}^{\text{st}} : E \otimes V_{\tau}^{\infty} \to \mathbb{C},$$

and any other such functional is a scalar multiple of $T_{\tau}^{\text{st}}$. This induces a canonical $G$-equivariant continuous map

$$T_{\tau}^{\text{aut}} : E \to V_{\tau}^{\infty}.$$

On the other hand we have $\Delta^{*} \circ \eta_{E} : E \to L^{2}(\Gamma \backslash G)^{\infty}$. For an automorphic representation $(\tau, V_{\tau}, \eta_{\tau})$ we have an orthogonal projection $\text{Pr}_{\eta_{\tau}} : L^{2}(\Gamma \backslash G) \to V_{\tau}$ and thus we get a map

$$T_{\eta_{\tau}}^{\text{aut}} = \text{Pr}_{\eta_{\tau}} \circ \Delta^{*} \circ \eta_{E}$$

from $E$ to $V_{\tau}^{\infty}$. Hence there is a coefficient $c(\eta, \eta_{\tau}) \in \mathbb{C}$ such that

$$T_{\eta_{\tau}}^{\text{aut}} = c(\eta, \eta_{\tau})T_{\tau}^{\text{st}}.$$

Fix a complete family $(\eta_{j})$ of normalized, pairwise orthogonal automorphic representations $\eta_{j} : \pi_{j} \to L^{2}(\Gamma \backslash G)$. Then the spectral expansion of $\Delta^{*} \circ \eta_{E}$ is

$$\Delta^{*} \circ \eta_{E} = \sum_{j} c(\eta, \eta_{j})T_{\pi_{j}}^{\text{st}}.$$

And hence, for $w \in E$,

$$l_{\Delta(\gamma)}(w) = \sum_{j : \pi_{j} \notin \hat{G}_{\text{ds}}} c(\eta, \eta_{j})a_{0}^{\eta_{j}, \gamma} I_{\pi_{j}}^{\text{st}}(T_{\pi_{j}}^{\text{st}}(w))$$

$$+ \sum_{j : \pi_{j} \in \hat{G}_{\text{ds}} \atop \pi_{j} \cong D_{2n}} c(\eta, \eta_{j})b_{0}^{\eta_{j}, \gamma} \left( S_{[1],n-1}(T_{\pi_{j}}^{\text{st}}(w)) - S_{[w_{0},n-1]}(T_{\pi_{j}}^{\text{st}}(w)) \right),$$

where $\hat{G}_{\text{ds}} \subset \hat{G}$ is the set of all discrete series representations of $G$. So we conclude

**Lemma 4.1.**

$$\sum_{k \in \mathbb{Z}} |a_{k}^{\eta, \gamma}|^{2} \hat{w}(k, -k) = \sum_{j : \pi_{j} \notin \hat{G}_{\text{ds}}} c(\eta, \eta_{j})a_{0}^{\eta_{j}, \gamma} I_{\pi_{j}}^{\text{st}}(T_{\pi_{j}}^{\text{st}}(w))$$

$$+ \sum_{j : \pi_{j} \in \hat{G}_{\text{ds}} \atop \pi_{j} \cong D_{2n}} c(\eta, \eta_{j})b_{0}^{\eta_{j}, \gamma} \left( S_{[1],n-1}(T_{\pi_{j}}^{\text{st}}(w)) - S_{[w_{0},n-1]}(T_{\pi_{j}}^{\text{st}}(w)) \right).$$
Likewise, for $\phi$ set

$$\int_{\mathbb{R}^d} \text{FOURIER EXPANSION}$$

where

$$W_j(t, x) = \left( \frac{(e^{2t} + x)x}{(e^{2t} + x - 1)(x - 1)} \right)^{\frac{1}{2}} + \left( \frac{(e^{2t} + x)x}{(e^{2t} + x - 1)(x - 1)} \right)^{\frac{1}{2}} e^{(\lambda_j + 1)t|x|^\lambda - 1} e^{2t + x - \lambda - 1},$$

$$\hat{t}_x = \frac{1}{2} \log \frac{(e^{2t} + x)(x - 1)}{(e^{2t} + x - 1)x}.$$  

Proof. Let $H_1 = \{1 - 1\}$. For $x \in \mathbb{R}$ write $n(x) = (1 \frac{1}{2})$. For $\varphi \in C_c^\infty(\mathbb{R})$ we set

$$f_\varphi(w_0 n(x)) = \varphi(-\frac{1}{2} \log |x|).$$

For given $\lambda \in \mathfrak{a}^*$ the function $f_\varphi$ extends uniquely to an element of $V_\lambda^\infty$. For $a = \exp(tH_1) \in A$ we have $a^\lambda = e^{\lambda t}$. We define $\tilde{\lambda} = \frac{\lambda}{2\pi i}$, so that $a^\lambda = e^{2\pi i \tilde{\lambda} t}$. We normalize the Haar measure $da$ on $A$ such that $\int_A g(a) da = \int_\mathbb{R} g(\exp(tH_1)) dt$. Assuming that $n_0 = (1 \frac{1}{2})$, we compute for $k \in \mathbb{Z}$,

$$I_{\tilde{\lambda}, k\mu_\gamma}(f_\varphi) = \int_A f_\varphi(w_0 n_0^a a^{-\lambda - 1 - k\mu_\gamma}) da$$

$$= \int_\mathbb{R} \varphi(t) e^{-t(2\pi i \tilde{\lambda} + 2\pi i k\mu_\gamma)} dt.$$

Likewise, for $\phi \in C_c^\infty(\mathbb{R}^2)$ let $w_\phi \in E$ be defined by $w_\phi(w_0 n(x), w_0 n(y)) = \phi(-\frac{1}{2} \log |x|, -\frac{1}{2} \log |y|)$. Here we identify $E$ with $V_\lambda^\infty \otimes V_\lambda^\infty$. We get

$$\hat{w}_\phi(k, -k) = \int_{\mathbb{R}^2} \phi(t, s) e^{i(2\pi i \tilde{\lambda} + 2\pi i k\mu_\gamma) - (s + t)} dt ds.$$
Note that \( w_\phi(w_0n(x), w_0n(y)) \) vanishes in a neighborhood of \( \{xy = 0\} \) as well as in a neighborhood of \( \{x = \infty\} \cup \{y = \infty\} \), which means that \( w_\phi \) indeed lies in \( E \) and that

\[
S_{[1],n-1}(T_{\pi_j}^{st}(w_\phi)) = S_{[w_0],n-1}(T_{\pi_j}^{st}(w_\phi)) = 0
\]

for every \( n \in \mathbb{N} \) and every \( \pi_j \cong D_{2n} \). So for these test functions only the first sum on the right hand side of Lemma 4.1 is present.

**Lemma 4.3.** Let \( g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) and \( g' = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \) be in \( G \). If \( cc'd'd = 0 \), then \( w_\phi(g,g') = 0 \). Otherwise,

\[
w_\phi(g,g') = |c|^{-\lambda-1}|c'|^{\lambda-1} \phi \left( \frac{1}{2} \log \left| \frac{c}{d} \right|, \frac{1}{2} \log \left| \frac{c'}{d'} \right| \right).
\]

**Proof.** Replacing \( g \) by \( \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} g \) or \( g' \) by \( \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} g' \) does not change \( w_\phi(g,g') \).

We therefore can restrict our attention to the case \( g, g' \in G^0 \). The Ansatz \( g = anw_0n(x) \) with \( a = \left( \frac{y}{1/y} \right) \) and \( n = \left( \frac{1}{z} \right) \) leads to \( y = 1/c \) and \( x = d/c \) as well as \( z = ac \). This gives the claim. \( \square \)

With \( w = w_\phi \) as above, we want to compute \( T^{st}_{\pi_j}(\pi_j^\star(w)) \). We first consider the case \( \pi_j = \pi_{\lambda_j} \) for some \( \lambda_j \in \mathbb{C} \). Recall that the functional \( T^{st} : \pi \otimes \hat{\pi} \otimes \hat{\pi}_j \to \mathbb{C} \) maps a given \( \varphi = w \otimes f \) to

\[
T^{st}(\varphi) = \int_G \varphi(w_0n_0y, w_0y, y) dy = \int_{ANK} w(w_0n_0ank, w_0ank)f(ank) da dn dk = \int_K \int_{ANK} a^{-\lambda_j+1}w(w_0n_0ank, w_0ank)da dn f(k) dk.
\]

The induced map \( T^{st}_{\pi_j} : E \to \pi_j \) is defined via the pairing

\[
\hat{\pi}_j \otimes \pi_j = \pi_{-\lambda_j} \otimes \pi_{\lambda_j} \to \mathbb{C}
\]

given by

\[
(f \otimes h) \mapsto \int_K f(k)h(k) dk.
\]

The resulting map \( T^{st}_{\pi_j} : \pi \otimes \hat{\pi} \to \pi_j \) therefore is given by

\[
T^{st}_{\pi_j}(w)(k) = \int_{ANK} a^{-\lambda_j+1}w(w_0n_0ank, w_0ank)da dn.
\]
We have
\[ T_0^\pi (T_\pi ^{st}(w)) = \int_A a(w_0n_0a_1)^{\lambda} w(w_0n_0an \hat{k}(w_0n_0a_1), w_0an \hat{k}(w_0n_0a_1)) \, da \]
which equals
\[ \int_A \int_{AN} a(w_0n_0a_1)^{\lambda} a^{-\lambda} w(w_0n_0an, w_0an, w_0n_0a_1) \, da \, dn \, da_1. \]

We write \( \hat{k}(w_0n_0a_1) = \frac{an}{w_0n_0a_1} \) and use the change of variables \( an \mapsto \frac{an}{w_0n_0a_1} \) in the \( AN \)-integral. For this we use the formula
\[ \int_H f(xy) \, dx = \Delta(y^{-1}) \int_H f(x) \, dx \]
for Haar integration over the group \( H = AN \) and the fact that the modular function \( \Delta \) of \( AN \) equals \( \Delta(a \mu n) = e^{-2t} \).

In this way we see that \( T_0^\pi (T_\pi ^{st}(w)) \) equals
\[ \int_A \int_{AN} a(w_0n_0a_1)^{\lambda} \, da \, dn \, da_1. \]

Writing \( \overline{\pi}_x = \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} = w_0n_xw_0 \) this equals
\[ \int_A \int_{AN} a(w_0n_0a_1)^{\lambda} a^{-\lambda} w(\overline{\pi}_0a^{-1}n_0a_1, \overline{\pi}_0n_0a_1) \, da \, dn \, da_1. \]

Writing \( a = \begin{pmatrix} e^t & -e^{-t} \\ e^{-t} & e^t \end{pmatrix}, \quad n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad a_1 = \begin{pmatrix} e^s & -e^{-s} \\ e^{-s} & e^s \end{pmatrix} \) we get
\[ \overline{\pi}_0^{-1}n_0a_1 = \begin{pmatrix} -e^{-t} & -e^{-t} \\ e^s(e^t x + e^{-t}) & e^{-s}(e^t(x-1) + e^{-t}) \end{pmatrix} \]
and
\[ \overline{\pi}_0n_0a_1 = \begin{pmatrix} -e^s & -e^s \\ e^s x & e^{-s}(x-1) \end{pmatrix}. \]

By Lemma 4.3 we conclude that \( w_\phi(\overline{\pi}_0a^{-1}n_0a_1, \overline{\pi}_0n_0a_1) \) equals
\[ e^{-2s}|e^t x + e^{-t} - \lambda|^{-1}|x|^{\lambda-1} \phi \left( s + \frac{1}{2} \log \left| \frac{e^{-2t} + x}{e^{-2t} + x - 1} \right|, s + \frac{1}{2} \log \left| \frac{x}{x - 1} \right| \right). \]

We now pick \( \alpha, \beta \in C^\infty_c(\mathbb{R}) \) and set
\[ \phi(x, y) = \alpha(x - y) \beta \left( \frac{x + y}{2} \right). \]

With this choice, we get
\[ \hat{w}_\phi(k, -k) = \hat{\alpha}(\lambda + k\mu)\beta \left( \frac{1}{\pi i} \right). \]
and \( w_\phi(p_\varphi a^{-1} a, n_\varphi a_1) \) equals
\[
e^{-2s} e^t x + e^{-t} e^{-\lambda-1} |x|^{\lambda-1} \alpha \left( \frac{1}{2} \log \left| \frac{(e^{-2t} + x)(x - 1)}{(e^{-2t} + x - 1)x} \right| \right)
\times \beta \left( s + \frac{1}{4} \log \left| \frac{(e^{-2t} + x)x}{(e^{-2t} + x - 1)(x - 1)} \right| \right).
\]
We conclude that \( I_0^\alpha (T^\alpha_T(w)) \) equals
\[
\int_{\mathbb{R}^3} (e^{2s} + e^{-2s}) \frac{\lambda - 1}{t} e^{(\lambda - \lambda_1)T} e^{-2s} e^t x + e^{-t} e^{-\lambda-1} |x|^{\lambda-1}
\times \alpha \left( \frac{1}{2} \log \left| \frac{(e^{-2t} + x)(x - 1)}{(e^{-2t} + x - 1)x} \right| \right)
\times \beta \left( s + \frac{1}{4} \log \left| \frac{(e^{-2t} + x)x}{(e^{-2t} + x - 1)(x - 1)} \right| \right) ds \, dx \, dt.
\]
After the change of variables \( s \mapsto s - \frac{1}{4} \log \left| \frac{(e^{-2t} + x)x}{(e^{-2t} + x - 1)(x - 1)} \right| \) we end up with
\[
\int_{\mathbb{R}^3} e^{2s} \left| \frac{(e^{-2t} + x)x}{(e^{-2t} + x - 1)(x - 1)} \right|^{\frac{1}{2}} + e^{-2s} \left| \frac{(e^{-2t} + x)x}{(e^{-2t} + x - 1)(x - 1)} \right|^{-\frac{1}{2}} \lambda \frac{1}{2}
\times e^{(\lambda - \lambda_1)T} |x|^{\lambda-1} e^t x + e^{-t} e^{-\lambda-1}
\times \alpha \left( \frac{1}{2} \log \left| \frac{(e^{-2t} + x)(x - 1)}{(e^{-2t} + x - 1)x} \right| \right) \beta(s) ds \, dt \, dx.
\]
Assuming \( \beta \geq 0 \) and \( \int_{\mathbb{R}} \beta(s) ds = 1 \) we replace \( \beta \) with \( \beta_T(x) = T \beta(Tx) \) and let \( T \to \infty \) to get the claim. The interchange of limit and sum resp. integral is justified by the Theorem of dominated convergence as follows. Using results from [BR04] one deduces that
\[
c(\eta, \eta_j) = O(|\lambda_j|^{2+\varepsilon}).
\]
Weyl’s asymptotic law says that \( |\lambda_j| = O(j) \) and with Theorem 2.5 we get
\[
a_{\eta_j}^{\eta_j} c(\eta, \eta_j) = O|\lambda_j|^3 = O(j^3).
\]
So we need an estimate \( O(|\lambda_j|^{-5}) \) of the above integral, with \( \beta = \beta_T \), which is uniform in \( T \). One achieves this by iterated use of the fact that \( 2\lambda_j e^{2T\lambda_j} = \frac{d}{dx} e^{2T\lambda_j} \) and using integration by parts. This, actually, is the place where it is needed that \( \alpha \) be vanishing in a neighborhood of zero, as otherwise there would appear boundary terms of this integration by parts. The Theorem follows. □
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