Complex supermanifolds of low odd dimension
and the example of the complex projective line

M. Kalus

Abstract

Complex supermanifold structures being deformations of the exterior algebra of a holomorphic vector bundle, have been parametrized by group orbits on non-abelian cohomology (see [Gre82]). For the case of odd dimension 4 and 5 an identification of these cohomologies with a subset of abelian cohomologies being computable with less effort, is provided in this article. Furthermore for a rank \( \leq 3 \) sub vector bundle \( F \rightarrow M \) of a holomorphic vector bundle \( E \rightarrow M \), a reduction of a (possibly non-split) supermanifold structure associated to \( \Lambda E \) to a structure associated to \( \Lambda F \) is defined. In the case \( E = F \oplus F' \) with vector bundle \( F' \) of rank \( \leq 2 \), the complete cohomological information of a supermanifold structure associated to \( E \) is given in terms of cohomologies compatible with the decomposition of \( E \). As an application, parameter spaces for complex supermanifold structures associated to any rank 3 or 4 vector bundle on the complex projective line \( \mathbb{P}^1(\mathbb{C}) \) are determined.

Complex non-split supermanifolds arise as deformations of a split complex supermanifold \((M, \mathcal{O}_{\Lambda E})\) constructed from a complex vector bundle \( E \rightarrow M \). They are parametrized by orbits of the group of bundle automorphism \( H^0(M, Aut(E)) \) on a certain in general non-abelian cohomology \( H^1(M, G_E) \) (see [Gre82]). The cochains of this cohomology can be expressed as the exponential of elements in a certain abelian cochain complex \( C^1(M, Der^{(2)}(\Lambda E)) \) (see [Rot82]). However the length of the involved exponential series is \( k \) for \( E \) of rank \( 2k \) or \( 2k + 1 \), increasing the complexity of computations for every second step in odd dimension. In particular \( H^1(M, G_E) \) is zero up to odd dimension 1, abelian up to odd dimension 3 and in general non-abelian beyond this limit.

A method for relating supermanifolds of higher odd dimension to abelian cohomologies can hence considerably simplify computations. The main object of interest in this article is relating the lowest dimensional non-abelian cases of odd dimension 4 and 5 to the abelian case. For this the non-abelian cohomology classifying supermanifold structures of odd dimension 4 or 5 is embedded as a subset into the abelian cohomology \( H^1(M, Der^{(2)}(\Lambda E)) \) (second section). This inclusion depends on a fixed map \( D \) relating cochains with values in a subsheaf \( Der_2(\Lambda E) \subset Der^{(2)}(\Lambda E) \) that are appropriate for building supermanifolds, to cochains with values in a complement \( Der_4(\Lambda E) \subset Der^{(2)}(\Lambda E) \). However the image of the inclusion and the \( H^0(M, Aut(E)) \)-action on it do not depend on the choice of \( D \).

Furthermore it is proved that a reduction of the odd dimension is in general well defined for any subbundle \( F \subset E \) of rank \( \leq 3 \). For the case \( E = F \oplus F' \) with rank \( \leq 2 \) vector bundle \( F' \) the cohomological data for a classification of supermanifold structures is given in terms of a sum of abelian cohomologies defined compatibly with the decomposition of \( E \) (third section).
This decomposition of cohomologies is of good use for the analysis of the $H^0(M, Aut(E))$-orbit structure on $H^1(M, G_E)$ since it is preserved by $H^0(M, Aut(F) \times Aut(F'))$.

As an application we parametrize the supermanifold structures of odd dimension 4 on the manifold $M = \mathbb{P}^1(\mathbb{C})$ using also the classification of the abelian case of odd dimension 3 (fourth section). The complete classification of supermanifold structures of odd dimension 3 is hence also included in this article. Special cases of odd dimension 3 and 4 were discussed and classified before in [BO96a], [BO96b], [Vis13a] and [Vis13b].

1 Non-Split supermanifold structures

The first section contains an introduction to the topic of complex non-split supermanifolds fixing the notation. Details can be found e.g. in [Gre82] and [Rot82].

Let $\mathcal{M} = (M, \mathcal{O}_M)$ be a complex supermanifold with underlying complex manifold $M$, sheaf of superfunctions $\mathcal{O}_M$ and projection onto numerical holomorphic functions $pr : \mathcal{O}_M \to \mathcal{O}_M$. Setting $\mathcal{O}_M^{nil} := \text{Ker}(pr)$ the sheaf $E = \mathcal{O}_M^{nil}/(\mathcal{O}_M^{nil})^2$ defines a holomorphic vector bundle on $M$. Denote its automorphisms by $Aut(E)$, its sheaf of sections by $\mathcal{O}_E$, its full exterior power by $\Lambda E$ and the sheaf of automorphisms of algebras on $\mathcal{O}_E$ preserving the $\mathbb{Z}/2\mathbb{Z}$-grading (but not necessarily the $\mathcal{O}_M$-module structure) by $Aut(\Lambda E)$. The rank of $E$ is the odd dimension of $M$. Following [Gre82] denote by $G_E \subset Aut(\Lambda E)$ the subsheaf of groups given by elements $\varphi \in Aut(\Lambda E)$ satisfying

$$(\varphi - Id)(\mathcal{O}_{\Lambda j E}) \subset \bigoplus_{k \geq 1} \mathcal{O}_{\Lambda j+2k E} \quad \forall \ j \geq 0 .$$

It is proved in [Gre82] that the isomorphy classes of complex supermanifolds associated to a given vector bundle $E \to M$ are in $1 : 1$ correspondence to the $H^0(M, Aut(E))$-orbits by conjugation on the Čech cohomology $H^1(M, G_E)$. Note that this cohomology is meant with respect to composition of maps with identity as neutral element. So $H^1(M, G_E)$ is nothing more but a pointed set. The orbit of the identity in $H^1(M, G_E)$ corresponds to the unique split supermanifold structure associated to $E \to M$ given by $\mathcal{O}_M = \mathcal{O}_E$.

Following [Rot82] let $\text{Der}^{(2)}(\Lambda E)$ denote the sheaf of even derivations on the sheaf of $\mathbb{Z}/2\mathbb{Z}$-graded algebras $\mathcal{O}_E$ satisfying

$$w(\mathcal{O}_{\Lambda j E}) \subset \bigoplus_{k \geq 0} \mathcal{O}_{\Lambda j+2k E} \quad \forall \ j \geq 0 .$$

It is shown in [Rot82] that the exponential map maps $\text{Der}^{(2)}(\Lambda E)$ isomorphically onto $G_E$. The sheaf $\text{Der}^{(2)}(\Lambda E)$ itself decomposes

$$\text{Der}^{(2)}(\Lambda E) = \bigoplus_{k=1}^{\infty} \text{Der}_{2k}(\Lambda E),$$

where $\text{Der}_{2k}(\Lambda E)$ is the sheaf of even derivations satisfying $w(\mathcal{O}_{\Lambda j E}) \subset \mathcal{O}_{\Lambda j+2k E}$ for all $j \geq 0$. More precisely, for fixed $2k$ a derivation in $\text{Der}_{2k}(\Lambda E)$ is given by its values on the sections in the subbundle $\Lambda^0 E \oplus \Lambda^1 E \subset \Lambda E$. This yields the identification:

$$\text{Der}_{2k}(\Lambda E) \cong \text{Der}(\Lambda^0 E, \Lambda^{2k} E) \oplus \text{Hom}_{\mathcal{O}_M}(\Lambda^1 E, \Lambda^{2k+1} E)$$

with $\text{Der}(\Lambda^0 E, \Lambda^{2k} E) = \text{Der}(\mathcal{O}_M) \otimes \mathcal{O}_{\Lambda^{2k} E}$. (2)

Here $\text{Hom}_{\mathcal{O}_M}(\Lambda^1 E, \Lambda^{2k+1} E)$ denotes the sheaf of homomorphisms of locally free $\mathcal{O}_M$-modules. The appropriate cohomology on $\text{Der}^{(2)}(\Lambda E)$ is the usual abelian Čech cohomology with respect to the $\mathcal{O}_M$-module structure.
Remark 1. (1) In odd dimension 0 and 1 the sheaf $\text{Der}^{(2)}(\Lambda E)$ is trivial and hence there are only split supermanifold structures.
(2) In odd dimension 2 and 3 it is $\text{Der}^{(2)}(\Lambda E) = \text{Der}_2(\Lambda E)$. The exponential mapping is just adding the identity and the composition in $\text{Der}_2(\Lambda E)$ is zero. Hence the supermanifold structures on $M$ associated to $E$ correspond to the orbits of $H^0(M, \text{Aut}(E))$ by conjugation on $H^1(M, \text{Der}_2(\Lambda E))$.
(3) In odd dimension $\geq 4$ the cohomologies $H^1(M, G_E)$ and $H^1(M, \text{Der}^{(2)}(\Lambda E))$ are in general not isomorphic any more.

In the following $d$ denotes the coboundary operator of the non-abelian cochain complex of $G_E$, while $\delta$ denotes the respective operator for the abelian complex of $\text{Der}^{(3)}(\Lambda E)$.

2 Non-abelian cohomology in odd dimension 4 and 5

The non-abelian cohomology $H^1(M, G_E)$ for $E$ of rank 4 or 5, is identified with a subset of the abelian cohomology $H^1(M, \text{Der}^{(2)}(\Lambda E))$.

In this section fix $rk(E) \in \{4, 5\}$. For a cocycle $\exp(u_2 + u_4) \in Z^1(M, G_E)$ where $u_2, u_4 \in C^1(M, \text{Der}_2(\Lambda E))$, it is by direct calculation necessary that $u_2 \in Z^1(M, \text{Der}_2(\Lambda E))$. Furthermore define

$$c_{u_2} = pr_{\text{End}_4(\Lambda E)}(d \exp(u_2)) \in C^2(M, \text{End}_4(\Lambda E)),$$

where the notion of $\text{End}^{(2)}(\Lambda E) = \bigoplus_{k=1}^{\infty} \text{End}_{2k}(\Lambda E)$ in the sheaf of complex linear endomorphisms of $\mathcal{O}_{\Lambda E}$ is defined analogously to $[1]$. From the cocycle condition on $\exp(u_2 + u_4)$ it follows that $c_{u_2} = -\delta u_4$ and hence $c_{u_2}$ is a coboundary of derivations. Denote:

$$\tilde{Z}^1(M, \text{Der}_2(\Lambda E)) := \{u_2 \in Z^1(M, \text{Der}_2(\Lambda E)) \mid c_{u_2} \in B^2(M, \text{Der}_4(\Lambda E))\}.$$

For $u_2 + u_4 \in C^1(M, \text{Der}^{(2)}(\Lambda E))$ and $v_2 + v_4 \in C^0(M, \text{Der}^{(2)}(\Lambda E))$ it follows by direct calculation that $[1]$

$$\exp(v_2 + v_4). \exp(u_2 + u_4) = \exp(u_2 + dv_2 + u_4 + dv_4 + F(u_2, v_2)) \quad (3)$$

$$\text{with } F(v_2, u_2) := \frac{1}{2}([v_{2,i} + v_{2,j}, u_{2,ij}] - [v_{2,i}, v_{2,j}]_{ij}).$$

The cochains $C^0(M, \text{Der}_2(\Lambda E))$ act on $Z^1(M, G_E)$ by $\exp(v_2).a$ described by $[1]$ setting $v_4 = 0$ and $\exp(u_2 + u_4) \in Z^1(M, G_E)$. Hence the $C^0(M, \text{Der}_2(\Lambda E))$-action on $Z^1(M, \text{Der}_2(\Lambda E))$ by adding a coboundary, restricts to an action on $\tilde{Z}^1(M, \text{Der}_2(\Lambda E))$.

A map $D : H^0(M, \text{Aut}(E)) \times C^0(M, \text{Der}_2(\Lambda E)) \times \tilde{Z}^1(M, \text{Der}_2(\Lambda E)) \to C^1(M, \text{Der}_4(\Lambda E))$ such that

$$d(D(\varphi, v_2, u_2)) = c_{u_2} \quad (4)$$

for all $\varphi \in H^0(M, \text{Aut}(E))$, $v_2 \in C^0(M, \text{Der}_2(\Lambda E))$ and $u_2 \in \tilde{Z}^1(M, \text{Der}_2(\Lambda E))$, is called compatible if it satisfies:

$$\varphi.D(\varphi, v_2, u_2) = D(Id, 0, (\varphi, u_2) + d(\varphi, v_2)) + F(\varphi, v_2, \varphi, u_2) \quad (5)$$

for all $\varphi \in H^0(M, \text{Aut}(E))$, $v_2 \in C^0(M, \text{Der}_2(\Lambda E))$ and $u_2 \in \tilde{Z}^1(M, \text{Der}_2(\Lambda E))$.

\textsuperscript{1}Here $\exp(v_2 + v_4). \exp(u_2 + u_4)$ denotes $(\exp(v_{2,i} + v_{4,i}) \exp(u_{2,ij} + u_{4,ij}) \exp(-v_{2,j} - v_{4,j}))_{ij}$. 

3
Lemma 1. A compatible map $D$ always exists.

Proof. Let $D(Id,0,\cdot) : \tilde{Z}_1(M,Der_2(\Lambda E)) \to C^1(M,Der_4(\Lambda E))$ be any map satisfying \([4]\) for the third argument and continue it via \([5]\) to $H^0(M,\text{Aut}(E)) \times C^0(M,Der_2(\Lambda E)) \times \tilde{Z}_1(M,Der_2(\Lambda E))$. From \([3]\) setting $u_4 = v_4 = 0$ it follows by direct calculation that $c_{u_2} = c_{u_2 + d\theta_2} + dF(v_2, u_2)$ for all $v_2 \in C^0(M,Der_2(\Lambda E))$ and $u_2 \in \tilde{Z}_1(M,Der_2(\Lambda E))$. Using this and \([3]\), \([4]\) holds for $\varphi = Id$. Since $u_2 \mapsto c_{u_2}$, $d$ and $F$ are $H^0(M,\text{Aut}(E))$-equivariant, \([4]\) holds for the continued map $D$. \hfill \Box

Note that if $\exp(u_2 + u_4) \in Z^1(M,G_E)$ then $D(\varphi,v_2,u_2) + u_4 \in Z^1(M,Der_4(\Lambda E))$, and set:

$$
\begin{align*}
\mu_D : H^0(M,\text{Aut}(E)) &\times C^0(M,G_E) \times Z^1(M,\Lambda E) \\
&\quad \rightarrow Z^1(M,Der_2(\Lambda E)) \oplus Z^1(M,Der_4(\Lambda E)) \\
(\varphi,\exp(v_2 + v_4),\exp(u_2 + u_4)) &\quad \mapsto \quad (u_2, D(\varphi,v_2,u_2) + u_4)
\end{align*}
$$

With Lemma \([\text{II}]\) it follows:

**Proposition 1.** Assume that $\text{rk}(E) \in \{4,5\}$ and that $D$ is compatible. Then the map $\mu_D$ induces an $H^0(M,\text{Aut}(E))$-equivariant map

$$
\sigma_D : H^1(M,G_E) \rightarrow H^1(M,Der_2(\Lambda E)) \oplus H^1(M,Der_4(\Lambda E))
$$

being a bijection onto \((\tilde{Z}_1(M,Der_2(\Lambda E))/B^1(M,Der_2(\Lambda E))) \oplus H^1(M,Der_4(\Lambda E))\).

Proof. It is with \([3]\) and $H^0(M,\text{Aut}(E))$-equivariance of $\exp$, $d$ and $F$:

$$
\begin{align*}
\mu_D(Id,0,\varphi,\exp(v_2 + v_4),\exp(u_2 + u_4))
&= ((\varphi,u_2) + d(\varphi,v_2), D(Id,0,(\varphi,u_2) + d(\varphi,v_2)) + (\varphi,u_4) + d(\varphi,v_4) + F(\varphi,v_2,\varphi,u_2))
\end{align*}
$$

Differing via equation \([5]\) from $\varphi,\mu_D(\varphi,\exp(v_2 + v_4),\exp(u_2 + u_4))$ only by the coboundary $(d(\varphi,v_2),d(\varphi,v_4))$. The proposition follows. \hfill \Box

3 Cohomology for decomposable vector bundles

Assume in this section that $F \subset E$ is a complex sub vector bundle of rank $\leq 3$. In the first part of this section $E$ may have any rank $\geq \text{rk}(F)$ and a projection morphism $pr_F : E \to F$ is fixed and extended to $\Lambda pr_F : \mathcal{O}_{\Lambda E} \rightarrow \mathcal{O}_{\Lambda F}$. The goal is a restriction of a supermanifold structure on $E$ to a supermanifold structure on $F$, and secondly expressing the cohomological data in the case $E = F \oplus F'$ with $\text{rk}(F') \leq 2$ in terms of abelian cohomologies compatible with the decomposition.

Let $[\alpha] \in H^1(M,G_E)$ be a cohomology class represented by $\alpha \in Z^1(M,G_E)$. Define the cochain

$$
\alpha_F := \Lambda pr_F \circ \alpha|_{\Lambda F} \in C^1(M,\text{End}(\Lambda F))
$$

This cochain induces a supermanifold structure associated to $F$:

**Lemma 2.** The cochain $\alpha_F$ lies in $Z^1(M,G_F)$ and the map

$$
H^1(M,G_E) \rightarrow H^1(M,G_F), \quad [\alpha] \mapsto [\alpha_F]
$$

is well-defined.
Proof. Writing $\alpha_{ij}$ as the exponential of $\sum_{k=1}^{\infty} u_{2k,ij}$ with $u_{2k,ij} \in \text{Der}(M, \text{Der}_{2k}(\Lambda E))$ yields $(\alpha_F)_{ij}$ as the exponential of $\Lambda \rho_{EF} \circ u_{2ij|\Lambda F}$ since the $\text{Der}_4(\Lambda F)$-term vanishes. Hence $d(\alpha_F))_{ijk} = \Lambda \rho_{EF} \circ (Id + u_{2,ij} + u_{2,jk} - u_{2,jk})|\Lambda F = \Lambda \rho_{EF} \circ (d(\alpha))_{ijk}|\Lambda F = 0$. In a similar way it is obtained that the map $\alpha \to \alpha_F$ maps coboundaries to coboundaries.

Remark 2. (1) Note that $\alpha_F$ in general does not define the structure of a subsupermanifold. (2) It was used that the composition of endomorphisms, that increase the degree by 1, is zero up to odd dimension 3. The Lemma does in general not hold for higher rank subbundles. (3) In the special case that the cocycle $\alpha$ can be chosen such that $\log(\alpha) \in \text{Der}(\Lambda E)$, the Lemma follows for subbundles up to rank $4\ell - 1$.

Approaching odd dimension 4 and 5, from now on assume the case $E = F \oplus F'$ with $rk(F) \leq 3$ and $rk(F') \leq 2$. This yields a decomposition

$$\Lambda E = X \oplus Y \oplus Z, \quad \text{with} \quad X = \Lambda F, \quad Y = \Lambda F \otimes F', \quad Z = \Lambda F \otimes \Lambda^2 F'$$

Let $G_T := \exp(\text{Der}_2(T))$ for $T = X, Y, Z$. For $\alpha \in C^1(M, G_E)$ regard the cochains

$$\begin{align*}
\alpha_T &:= (pr_T \circ \alpha|_T) \in C^1(M, G_T) \quad \text{for} \quad T = X, Y, Z, \\
u_{2,XY} + u_{4,XY} &:= (pr_Y \circ \alpha|_X) \in C^1(M, \text{Der}_2(X, Y)) \oplus C^1(M, \text{Der}_4(X, Y)), \\
u_{2,XZ} + u_{4,XZ} &:= (pr_Z \circ \alpha|_X) \in C^1(M, \text{Der}_2(X, Z)) \oplus C^1(M, \text{Der}_4(X, Z)), \\
u_{2,YZ} + u_{4,YZ} &:= (pr_Z \circ \alpha|_Y) \in C^1(M, \text{Der}_2(Y, Z)) \oplus C^1(M, \text{Der}_4(Y, Z)), \\
u_{2,YX} &:= (pr_X \circ \alpha|_Y) \in C^1(M, \text{Der}_2(Y, X)), \\
u_{2,ZY} &:= (pr_Y \circ \alpha|_Z) \in C^1(M, \text{Der}_2(Z, Y)).
\end{align*}$$

(6)

Note that the term $(pr_X \circ \alpha|_Z)$ missing in the list, vanishes for reasons of degree. All eleven mentioned cochain complexes, those of the first line with respect to composition, the remaining with respect to the sum of maps, are abelian. Continuing all eleven cochains by zero on the complement of their domain of definition respectively, their sum equals $\alpha$. It follows from Proposition and arguments similar to those in the proof of Lemma 2.

Proposition 2. For a complex manifold $M$ and the sum of a rank $\leq 3$ vector bundle $F$ and a rank $\leq 2$ vector bundle $F'$ denoted $E = F \oplus F'$, fix a map $D$ as in section 3 and decompose $D = D_{XY} + D_{XZ} + D_{YZ}$ with $DP_Q(\varphi, v_2, u_2) := pr_Q \circ D(\varphi, v_2, u_2)|_P$. The map of cochains

$$\alpha \mapsto \left( \begin{array}{c} \alpha_T, \ u_{2,RS}, \ D_{PQ}(Id, 0, u_2) + u_{4,PQ} \\ RsTe\{XY,Z, R\neq S, (R,S)\neq(Z,X) \} \\
(P,Q)\in\{(X,Y),(X,Z),(Y,Z)\} \end{array} \right)$$

induces a map of cohomologies from $H^1(M, G_E)$ to the direct sum $\oplus H$ of the eleven abelian cohomologies of the cochain complexes in (4). The induced map yields a bijection between $H^1(M, G_E)$ and the subset of elements in $\oplus H$ that can be represented by cocycles of the type $(\hat{\alpha}_T, \ u_{2,RS}, \ u_{4,PQ})$ satisfying $c_u \in B^2(M, \text{Der}_4(\Lambda E))$ with $u = \sum T \log(\hat{\alpha}_T) + \sum_{R,S} \hat{u}_{2,RS}$.

Remark 3. The inclusion $H^1(M, G_E) \hookrightarrow \oplus H$ is by Proposition 1 equivariant under the action of $H^0(M, \text{Aut}(F) \times \text{Aut}(F')) \subset H^0(M, \text{Aut}(E))$ acting diagonally on $\oplus H$. Hence the Proposition is of good use for the classification of supermanifold structures being parametrized by $H^0(M, \text{Aut}(E))$-orbits on $H^1(M, G_E)$.

\footnote{This case was pointed out to be of special interest in Rot92.}
For the case of \( rk(F') = 1 \) denoting the line bundle \( F' \) by \( L \), the result of Proposition 2 can be simplified. Most of the cochains in (6) vanish. The remaining are:

\[
\alpha_F := \alpha_X, \quad \alpha_L := \alpha_Y, \quad u_F := u_{2,YX} \quad \text{and} \quad u_L = u_{2,L} + u_{4,L} := u_{2,XY} + u_{4,XY}
\]

Note that for \( \alpha = \alpha_F + \alpha_L + u_F + u_L \) the cochains \( u_F \) and \( u_{4,L} \) have no influence on \( c_u \).

**Corollary 1.** For a complex manifold \( M \) and the sum of a rank \( \leq 3 \) vector bundle \( F \) and a line bundle \( L \) denoted \( E = F \oplus L \) fix a map \( D \) as in section 3. Then the elements \([\alpha]\) in the cohomology \( H^1(M,G_E) \) correspond bijectively to the well defined classes

\[
\begin{align*}
([\alpha_F], [\alpha_L], [u_F], [u_{2,L}], [D(Id,0,u) + u_{4,L}]) & \in H^1(M,G_F) \oplus H^1(M,G_{\Lambda F \otimes L}) \\
& \quad \oplus H^1(M,\text{Hom}_{\mathcal{O}_M}(L,\Lambda^3 F)) \oplus H^1(M,\text{Der}_2(\Lambda F,\Lambda F \otimes L)) \oplus H^1(M,\text{Der}_4(\Lambda F,\Lambda F \otimes L))
\end{align*}
\]

that satisfy \( c_u \in B^2(M,\text{Der}_4(\Lambda F,\Lambda F \otimes L)) \) with \( u = \alpha_F + \alpha_L + u_{2,L} - \text{Id}_{\Lambda E} \).

The methods developed above are now applied to supermanifold structures of odd dimension 4 on \( \mathbb{P}^1(\mathbb{C}) \).

### 4 Supermanifold structures on \( \mathbb{P}^1(\mathbb{C}) \)

For the complex manifold \( M = \mathbb{P}^1(\mathbb{C}) \) all supermanifold structures of odd dimension 3 are parametrized in a first step. These are described by group orbits on abelian cohomologies. Using the methods of the previous sections, parameter spaces for all supermanifold structures of odd dimension 4 are given. All parameter spaces will naturally appear as group orbits on finite dimensional vector spaces.

Let \( \mathcal{O}(k) \) for \( k \in \mathbb{Z} \) denote the line bundle on \( \mathbb{P}^1(\mathbb{C}) \) with divisor \( k \cdot [0 : 1] \). Then it is:

\[
H^0(M,\mathcal{O}(k)) \cong \mathbb{C}^{\max\{0,k+1\}}, \quad H^1(M,\mathcal{O}(k)) \cong \mathbb{C}^{\max\{0,-k-1\}}
\]

Note that any vector bundle on \( \mathbb{P}^1(\mathbb{C}) \) can be decomposed into a direct sum of line bundles (see [Gro56]) which are each isomorphic to one of the \( \mathcal{O}(k) \). For a given vector bundle \( E \to M \) of finite rank fix such a decomposition \( E = \bigoplus_{i=1}^m \mathcal{O}(l_i) \) with \( l_i \in \mathbb{Z} \). This yields a decomposition of \( \Lambda E \) as a vector bundle

\[
\Lambda E = \mathcal{O}(0) \oplus \bigoplus_{k=1}^m \bigoplus_{1 \leq i_1 \leq \cdots \leq i_k \leq m} \mathcal{O}(l_{i_1}) \otimes \cdots \otimes \mathcal{O}(l_{i_k}) .
\]

Note that as a line bundle \( \mathcal{O}(l_{i_1}) \otimes \cdots \otimes \mathcal{O}(l_{i_k}) \cong \mathcal{O}(l_{i_1} + \cdots + l_{i_k}) \), but for the algebra structure on \( \Lambda E \) it is important to distinguish between isomorphic summands in (7).

Since a global section in \( \text{Aut}(E) \) maps a line bundle in \( E \) to a line bundle of the same isomorphy class, it is setting \( m_i = \#\{l_j \mid l_j = i\} \):

\[
H^0(M,\text{Aut}(E)) \cong \bigoplus_{i=-\infty}^{\infty} H^0(M,GL(m_i,\mathbb{C})) \cong \bigoplus_{i=-\infty}^{\infty} GL(m_i,\mathbb{C})
\]

Denote the standard and dual action of \( GL(n,\mathbb{C}) \) by \( \rho \), resp. \( \rho^* \).
4.1 Odd dimension 3

Assume now that \( m = 3 \). Decompose \( E = \mathcal{O}(l_1) \oplus \mathcal{O}(l_2) \oplus \mathcal{O}(l_3) \) with \( l_1, l_2, l_3 \in \mathbb{Z} \) and \( l_1 \leq l_2 \leq l_3 \). It is \( H^1(M, G_E) \cong H^1(M, \text{Der}_2(\Lambda^E)) \). Using [2] and further that \( \text{Hom}_{\mathcal{O}_M}(\Lambda^1 E, \Lambda^3 E) \) consists of multiplication operators in \( \mathcal{O}_{\Lambda^2 E} \), it is:

\[
H^1(M, \text{Der}_2(\Lambda^E)) \cong \bigoplus_{1 \leq i < j \leq 3} \left( H^1(M, \mathcal{O}(l_i + l_j + 2)) \oplus H^1(M, \mathcal{O}(l_i + l_j)) \right)
\]

Hence it is:

\[
H^1(M, \text{Der}_2(\Lambda^E)) \cong \bigoplus_{1 \leq i < j \leq 3} \left( \mathbb{C}^{c_{ij}} \oplus \mathbb{C}^{d_{ij}} \right) \quad \text{with}
\]

\[
c_{ij} := \dim(H^1(M, \mathcal{O}(l_i + l_j + 2))) = \max\{0, -l_i - l_j - 3\}
\]

\[
d_{ij} := \dim(H^1(M, \mathcal{O}(l_i + l_j))) = \max\{0, -l_i - l_j - 1\}
\]

Furthermore it follows:

**Lemma 3.** The group \( H^0(M, \text{Aut}(E)) \) acts diagonally on \( H^1(M, \text{Der}_2(\Lambda^E)) \) by multiples of invariant subspaces of \( \rho \wedge \rho \).

For the parametrization of supermanifold structures identify the orbits of the diagonal action of \( GL(k, \mathbb{C}) \) on \( (\mathbb{C}^k)^n \) with the elements in

\[
\mathbb{V}_k^n(\mathbb{C}) := \{ V \subset \mathbb{C}^n \text{ complex sub vector space} \mid \dim(V) \leq k \}
\]

for \( 0 \leq k, n \). Set \( \mathbb{V}_k^n = \{ 0 \} \) for \( n < 0 \).

The following theorem is deduced from Lemma 3. The case \( l_1 = l_2 = l_3 \) was done in [Vis13b].

**Theorem 1.** Any non-split supermanifold structure of odd dimension 3 on \( \mathbb{P}^1(\mathbb{C}) \) can be associated to a vector bundle \( E = \mathcal{O}(l_1) \oplus \mathcal{O}(l_2) \oplus \mathcal{O}(l_3) \) with \( (l_1, l_2, l_3) \in \mathbb{Z}^3 \), \( l_1 \leq l_2 \leq l_3 \) and \( l_1 + l_2 \leq -2 \). The supermanifold structures associated to \( E \) are bijectively parametrized by:

\[
\mathbb{V}_3^{c_{12} + d_{12}}(\mathbb{C}) \quad \text{in the case} \quad l_1 = l_2 = l_3,
\]

\[
\mathbb{V}_3^{c_{12} + d_{12}}(\mathbb{C}) \times \mathbb{V}_2^{c_{12} + d_{13}}(\mathbb{C}) \quad \text{in the case} \quad l_1 = l_2 < l_3,
\]

\[
\mathbb{V}_2^{c_{12} + d_{12}}(\mathbb{C}) \times \mathbb{V}_1^{c_{12} + d_{23}}(\mathbb{C}) \quad \text{in the case} \quad l_1 < l_2 = l_3 \quad \text{and}
\]

\[
\mathbb{V}_1^{c_{12} + d_{12}}(\mathbb{C}) \times \mathbb{V}_1^{c_{12} + d_{13}}(\mathbb{C}) \times \mathbb{V}_1^{c_{23} + d_{23}}(\mathbb{C}) \quad \text{in the case} \quad l_1 < l_2 < l_3.
\]

The split supermanifold structure is in any case given by the tuple of trivial subspaces.

**Proof.** Case \( l_1 = l_2 = l_3 \): In this case it is \( H^0(M, \text{Aut}(E)) \cong GL(3, \mathbb{C}) \). Its action on the vector space \( H^1(M, G_E) \cong \mathbb{C}^{3(c_{12}+d_{12})} \) is given by \( A \mapsto M_A \otimes \text{Id}_{c_{12}+d_{12}} \), where \( M_A \) is the matrix of minors of \( A \). Note that \( \{ M_A \mid A \in GL(3, \mathbb{C}) \} = GL(3, \mathbb{C}) \).

Case \( l_1 = l_2 < l_3 \), resp. \( l_1 < l_2 = l_3 \): In this case \( H^0(M, \text{Aut}(E)) \cong GL(2, \mathbb{C}) \times \mathbb{C}^\times \), respectively \( \mathbb{C}^\times \times GL(2, \mathbb{C}) \). For \( l_1 = l_2 < l_3 \) its action on \( H^1(M, G_E) \cong \mathbb{C}^{c_{12}+d_{12}} \oplus \mathbb{C}^{2(c_{13}+d_{13})} \) is \( (A, d) \mapsto (\det(A) \otimes \text{Id}_{c_{12}+d_{12}}) \oplus (d \cdot A \otimes \text{Id}_{c_{13}+d_{13}}) \). The case \( l_1 < l_2 = l_3 \) follows analogously.

Case \( l_1 < l_2 < l_3 \): In this case it is \( H^0(M, \text{Aut}(E)) \cong \text{diag}(3, \mathbb{C}) \subset GL(3, \mathbb{C}) \). Its action on the identified vector space \( H^1(M, G_E) \cong \mathbb{C}^{c_{12}+d_{12}} \oplus \mathbb{C}^{c_{13}+d_{13}} \oplus \mathbb{C}^{c_{23}+d_{23}} \) is given by \( (\lambda_1, \lambda_2, \lambda_3) \mapsto \lambda_1 \cdot \lambda_2 \cdot \text{Id}_{c_{12}+d_{12}} \oplus \lambda_1 \cdot \lambda_3 \cdot \text{Id}_{c_{13}+d_{13}} \oplus \lambda_2 \cdot \lambda_3 \cdot \text{Id}_{c_{23}+d_{23}} \). \( \square \)
4.2 Odd dimension 4

In the case of \( \text{rk}(E) = 4 \), the \( \mathcal{O}_M \)-module \( \text{Hom}_{\mathcal{O}_M}(\Lambda^1 E, \Lambda^3 E) \) in (2) is generated by contractions in \( \mathcal{O}_E^* \) followed by a multiplication operator in \( \mathcal{O}_E^{\Lambda^3 E} \). Furthermore \( H^1(M, \text{Der}_4(\Lambda^E)) \cong H^1(M, \text{Der} (\mathcal{O}_M, \Lambda^4 E)) \). It follows:

**Lemma 4.** The group \( H^0(M, \text{Aut}(E)) \subset GL(4, \mathbb{C}) \) acts diagonally on \( H^1(M, \text{Der}_2(\Lambda^E)) \) by multiples of invariant subspaces of \( \rho \wedge \rho \), resp. \( \rho \wedge \rho \wedge \rho \wedge \rho^* \) according to the decomposition in (2). On \( H^1(M, \text{Der}_4(\Lambda^E)) \) the group acts by the determinant.

The three cases of a “fourfold product of a line bundle“, “two couples of line bundles“ and a “distinct line bundle“ are now analyzed separately. In the first two cases Proposition 1 is applied directly, the generic third case uses Corollary 1 and the cohomologies in the parametrization of structures of odd dimension 3.

Set \( \mathbb{C}^n = \{0\} \) for \( n < 0 \). Note that \( \tilde{Z}^1(M, \text{Der}_2(\Lambda^E)) = Z^1(M, \text{Der}_2(\Lambda^E)) \) on \( \mathbb{P}^1(\mathbb{C}) \) and fix \( D \equiv 0 \) in Proposition 1 and Corollary 1.

**Fourfold sum of a line bundle**

Assuming \( E = 4 \mathcal{O}(l) \) it is with respect to the decomposition in (2):

\[
H^1(M, \text{Der}_2(\Lambda^E)) \cong H^1(M, \mathcal{O}(2l + 2))^6 \oplus H^1(M, \mathcal{O}(2l))^8
\]

\[
H^1(M, \text{Der}_4(\Lambda^E)) = H^1(M, \mathcal{O}(4l + 2))
\]

So it follows from Proposition 1 and Lemma 4.

**Theorem 2.** Non-split supermanifold structures on \( \mathbb{P}^1(\mathbb{C}) \) associated to a vector bundle \( E = 4 \mathcal{O}(l) \) can only appear if \( l \leq -1 \). The supermanifold structures associated to \( E \) are parametrized by the \( GL(4, \mathbb{C}) \)-orbits of the diagonal action given by \( \rho \wedge \rho, \rho \wedge \rho \wedge \rho \wedge \rho^* \) and the determinant on the three summands of the vector space:

\[
\mathbb{C}^6(-2l-3) \oplus \mathbb{C}^{16}(-2l-1) \oplus \mathbb{C}^{-4l-3}
\]

*The split case corresponds to the orbit of zero.*

**Two couples**

Assume now that \( E = 2 \mathcal{O}(l) \oplus 2 \mathcal{O}(l') \) with \( l < l' \). In this case \( H^0(M, \text{Aut}(E)) \cong GL(2, \mathbb{C}) \times GL(2, \mathbb{C}) \). Denote the standard and determinant action of the factors by \( \rho_i, \text{resp. det}_i, i = 1,2 \).

It is with respect to the decomposition in (2) – here line by line by:

\[
H^1(M, \text{Der}_2(\Lambda^E)) \cong H^1(M, \mathcal{O}(2l + 2)) \oplus (H^1(M, \mathcal{O}(l + l' + 2)))^4 \oplus H^1(M, \mathcal{O}(2l' + 2))
\]

\[
\oplus H^1(M, \mathcal{O}(2l))^4 \oplus (H^1(M, \mathcal{O}(l + l')))^8 \oplus H^1(M, \mathcal{O}(2l'')))^4
\]

\[
H^1(M, \text{Der}_4(\Lambda^E)) \cong H^1(M, \mathcal{O}(2(l + l') + 2))
\]

Proposition 1 and Lemma 4 yield:
The supermanifold structures of odd dimension

Theorem 4. Non-split supermanifold structures on $\mathbb{P}^1(\mathbb{C})$ associated to a vector bundle $E = 2\mathcal{O}(l) \oplus 2\mathcal{O}(l')$ with $l < l'$ only appear if $l \leq -1$. The supermanifold structures associated to $E$ are parametrized by the $GL(2, \mathbb{C}) \times GL(2, \mathbb{C})$-orbits of the diagonal action given by $\det_1, \rho_1 \otimes \rho_2, \det_2, \rho_1 \otimes \rho_1^*, (\det_1 \cdot \rho_2 \otimes \rho_1^*) \oplus (\det_2 \cdot \rho_1 \otimes \rho_2^*)$, $\det_2 \cdot \rho_1 \otimes \rho_1^*$ and $\det_1 \cdot \det_2$ on the summands of the vector space:

$$C^{-2l-3} \oplus C^4(-l-l'-3) \oplus C^{-2l'+3} \oplus C^4(-2l-1) \oplus C^8(-l-l'-1) \oplus C^4(-2l'-1) \oplus C^{-2l-2l'}$$

The split case corresponds to the orbit of zero.

A distinct line bundle

Decompose a rank 4 vector bundle $E = F \oplus L$ with $F = \mathcal{O}(l_1) \oplus \mathcal{O}(l_2) \oplus \mathcal{O}(l_3)$ ordered to $l_1 \leq l_2 \leq l_3$ and $L = \mathcal{O}(l)$ with $l \neq l_i$ for all $i = 1, 2, 3$. Following Corollary 1 the relevant cohomologies involved in a classification of supermanifold structures are $H^1(M, G_F) \cong H^1(M, G_{L,F})$ given by (8) and further

$$H^1(M, H_{\text{Hom}}_{\mathcal{O}_M}(L, \Lambda^3 F)) \cong H^1(M, \mathcal{O}(l_1 + l_2 + l_3 - l)) \cong C^d$$

with $c := dim(H^1(M, \mathcal{O}(l_1 + l_2 + l_3 - l))) = \max\{0, -l_1 - l_2 - l_3 + l - 1\}$

and $H^1(M, Der_2(\Lambda F, \Lambda F \otimes L))$ and $H^1(M, Der_4(\Lambda F, \Lambda F \otimes L))$. It is with respect to the decomposition in (2) – here line by line:

$$H^1(M, Der_2(\Lambda F, \Lambda F \otimes L)) \cong \bigoplus_{1 \leq i \leq 3} \left( H^1(M, \mathcal{O}(l_1 + l + 2)) \right)$$

$$\oplus H^1(M, \mathcal{O}(l_1 + l + 2))^2 \oplus H^1(M, \mathcal{O}(l_1 + l + 2 + l_2))$$

So $H^1(M, Der_2(\Lambda F, \Lambda F \otimes L)) \cong \bigoplus_{1 \leq i \leq 3} \left( C^{c_i} \oplus C^{2d_i + d_i'} \right)$ with:

$$c_i := \text{dim}(H^1(M, \mathcal{O}(l_1 + l + 2))) = \max\{0, -l_i - l - 3\}$$

$$d_i := \text{dim}(H^1(M, \mathcal{O}(l_1 + l))) = \max\{0, -l_i - l - 1\}$$

$$d_i' := \text{dim}(H^1(M, \mathcal{O}(l_1 + l + 2 + l - 2l_i))) = \max\{0, -l_1 - l_2 - l_3 - l_2l_i - 1\}$$

Finally

$$H^1(M, Der_4(\Lambda F, \Lambda F \otimes L)) \cong H^1(M, \mathcal{O}(l_1 + l_2 + l_3 + l_2)) \cong C^d$$

with $d := \text{dim}(H^1(M, \mathcal{O}(l_1 + l_2 + l_3 + l_2))) = \max\{0, -l_1 - l_2 - l_3 - l - 3\}$

It is $H^0(M, Aut(E)) = H^0(M, Aut(F) \times Aut(L))$. Corollary 1 Lemma 3 and Lemma 4 yield:

Theorem 4. The supermanifold structures of odd dimension 4 on $\mathbb{P}^1(\mathbb{C})$ associated to a vector bundle $E = \mathcal{O}(l_1) \oplus \mathcal{O}(l_2) \oplus \mathcal{O}(l_3) \oplus \mathcal{O}(l)$ with $l_1 \leq l_2 \leq l_3$ and $l \neq l_i$ for all $i = 1, 2, 3$ are bijectively parametrized by the orbits of $H^0(M, Aut(F)) \times \mathbb{C}^\times \subset GL(3, \mathbb{C}) \times \mathbb{C}^\times$ on

$$\left( \bigoplus_{1 \leq i,j \leq 3} C^{c_{ij} + d_{ij}} \right)^2 \oplus C^c \oplus \bigoplus_{1 \leq i \leq 3} C^{c_i} \oplus \bigoplus_{1 \leq i \leq 3} C^{2d_i + d_i'} \oplus C^d$$

The diagonal action of $H^0(M, Aut(F))$ is by the restrictions of $\rho \wedge \rho$ on the first, of $\rho$ on the third, of $\rho \wedge \rho \otimes \rho^*$ on the fourth, and by the determinant action on the second and fifth summand. The diagonal $\mathbb{C}^\times$-action is trivial on the first, dual on the second and standard on the three remaining summands. The split case corresponds to the orbit of zero.
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Matthias Kalus
Fakultät für Mathematik
Ruhr-Universität Bochum,
Universitätsstraße 150
D-44801 Bochum, Germany
Matthias.Kalus@rub.de