BALANCE OF THE VORTICITY DIRECTION AND THE VORTICITY MAGNITUDE IN 3D FRACTIONAL NAVIER-STOKES EQUATIONS

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ABSTRACT. Fractional Navier-Stokes equations—featuring a fractional Laplacian—provide a 'bridge' between the Euler equations (zero diffusion) and the Navier-Stokes equations (full diffusion). The problem of whether an initially smooth flow can spontaneously develop a singularity is a fundamental problem in mathematical physics, open for the full range of models—from Euler to Navier-Stokes. The purpose of this work is to present a hybrid, geometric-analytic regularity criterion for solutions to the 3D fractional Navier-Stokes equations expressed as a balance—in the average sense—between the vorticity direction and the vorticity magnitude, key geometric and analytic descriptors of the flow, respectively.

1. INTRODUCTION

The fractional Navier-Stokes equations (NSE) describing the motion of a three-dimensional (3D) Newtonian fluid with a reduced diffusion read

\begin{equation}
\partial_t u + (u \cdot \nabla) u = -\nu (-\Delta)^\beta u - \nabla p + f
\end{equation}

supplemented with the incompressibility condition \( \nabla \cdot u = 0 \). The diffusion parameter \( \beta \) is restricted to the interval \((0, 1)\) (the case \( \beta = 1 \) corresponds to the Navier-Stokes equations). Here, the vector field \( u \) is the velocity of the fluid, the scalar field \( p \) is the pressure, a positive constant \( \nu \) is the viscosity, and the vector field \( f \) is the external force.

Henceforth, for simplicity of the exposition, \( \nu \) will be set to 1, \( f \) taken to be a potential force, and the spatial domain will be the whole space. In this case, the fractional Laplacian, \((-\Delta)^\beta\) is simply a Fourier multiplier with the symbol \( |\xi|^{2\beta} \).

In order to better understand geometry of the flow mathematically, it is beneficial to study the vorticity-velocity formulation of the 3D NSE,

\begin{equation}
\partial_t \omega + (u \cdot \nabla) \omega = -(-\Delta)^\beta \omega + (\omega \cdot \nabla) u
\end{equation}

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where the vorticity field $\omega$ is given by $\omega = \text{curl } u$. The incompressibility implies that $u$ can be reconstructed from $\omega$ by solving $\Delta u = -\text{curl } \omega$, leading to the Biot-Savart law, and closing the system. The LHS in (2) is the transport of the vorticity by the velocity, the first term on the RHS is the fractional diffusion, and the second one is the vortex-stretching term; there are the three major physical mechanisms in the system.

The mathematical theory of the fractional NSE is mostly parallel to the mathematical theory of the Navier-Stokes equations. It is known that the fractional NSE allow a construction of the Leray-type weak solutions for any $\beta$ in $(0, 1)$ (see, e.g., [CoDeLDeR] for a sketch of the proof). In order to be able to apply a standard fixed-point argument and generate the mild solutions (at least locally-in-time), one needs the parameter $\beta$ confined to $(\frac{1}{2}, 1)$ (see, e.g., [DL09] where locally-in-time spatially analytic solutions were constructed). Various regularity criteria are also available. For example, it was shown in [Ch07] that if the (distributional) vorticity $\omega$ of a weak solution $u$ satisfies

$$
\omega \in L^q(0, T; L^p) \quad \text{with} \quad \frac{3}{2p} + \frac{\beta}{q} \leq \beta
$$

where $\frac{3}{\beta} < p \leq \infty$, then the solution is regular on $(0, T]$ (in the endpoint case $q = \infty$, in order to avoid smallness in the ‘standard’ argument, the inequality in (3) should be strict; a more elaborate argument—in the spirit of [ESS03]—allows for the inclusive inequality).

In the Navier-Stokes case, a geometric regularity theory—based on coherence of the vorticity field—was pioneered in [Co94] where it was shown that the stretching factor in the evolution of the vorticity magnitude has a singular integral representation featuring a geometric kernel regularized by the local coherence of the vorticity direction field $\xi$. It was then showed in [CoFe93] that the Lipschitz-coherence of $\xi$ suffices to rule out singularity formation, establishing the first geometric regularity criterion in the realm of the ‘geometric depletion of the nonlinearity’. Among the follow-up works, the Lipschitz coherence was replaced by the $\frac{1}{2}$-Hölder coherence in [daVeigaBe02], and a complete spatiotemporal localization was demonstrated in [Gr09].

It is worth noting that the regularity theory based on the local coherence of the vorticity direction has deep roots in the computational simulations of turbulent flows which indicate that a dominant morphological signature of the regions of intense vorticity is the one of the vortex filaments (featuring a high degree of local coherence of the vorticity direction; see, e.g., [JWSR93, S81, SJO91, VM94]). In addition, long before the age of high-resolution computational simulations, G.I. Taylor—chiefly based on the experimental measurements of turbulent flows past a grid—conjectured that stretching of the vortex filaments was the principal physical mechanism behind the phenomenon of turbulent dissipation ([Tay37]).

The article [Ch07] presented a regularity criterion for the fractional NSE in which $\xi$ belongs to a suitable class of mixed Lebesgue-in temporal variable and Triebel-Lizorkin-in spatial variable spaces while $\omega$ belongs to an interrelated class of the spatiotemporal Lebesgue spaces. A related work [Na19] extended the range of some of the functional parameters describing the two classes. The articles were in the spirit akin to the results previously obtained in [GrRu04] in the case of the NSE.
The regularity criteria in the aforementioned (preceding paragraph) articles quantify a balance between the vorticity direction and the vorticity magnitude sufficient to rule out a finite time blow-up in the form of two separate—although interconnected—conditions on $\xi$ and $|\omega|$. A natural question is whether it is possible to formulate a single, hybrid geometric-analytic regularity condition in the form of a spatiotemporal average of the suitably defined degree of coherence of $\xi$ weighted against a power of $|\omega|$ (in the spirit of the NSE article [GrGu10-1]). This would provide a qualitatively rarified measure of the balance between $\xi$ and $|\omega|$ needed to prevent a possible singularity formation.

The goal of this note is to give a positive answer to the above question. More precisely, defining a pointwise measure of the coherence of the vorticity direction by

$$
\rho_{\gamma}(x, t) = \sup_{y \neq x} \frac{|\sin \varphi(\xi(x + y, t), \xi(x, t))|}{|y|^{\gamma}},
$$

our main result is summarized as follows.

**Theorem 1.** Let $\omega \in C([0, T), L^p)$ be a (smooth) solution to the fractional NSE (2) for some $p > \frac{3}{\beta}$. Assume that $\omega$ satisfies

$$
\int_0^T \left\{ \int (\rho_{\gamma}(x, t)|\omega(x, t)|^{p_1} dx \right\}^{\frac{2}{p_1}} dt < \infty
$$

where the parameters $\gamma, p_1$ and $a$ conform to the scaling-invariant condition

$$
p_1(\gamma + 2a) - 3 = \beta p_1
$$

(in addition to several restrictions to naturally transpire in the proof). Then $T$ is not a blow-up time.

The next section will first recall some basic concepts in the theory of the geometric depletion of the nonlinearity and then present the proof of the above theorem.

2. Balance of $\xi$ and $|\omega|

A rigorous study of the geometric depletion of the nonlinearity in solutions to the 3D NSE is based on a singular integral representation for the stretching factor in the evolution of the vorticity magnitude $\alpha$ featuring a geometric kernel $D$ depleted by coherence of the vorticity direction ([Co94]),

$$(\partial_t + u \cdot \nabla - \Delta)|\omega|^2 + |\nabla \omega|^2 = \alpha|\omega|^2$$

where

$$\alpha(x, t) = \frac{3}{4\pi} P.V. \int D(\hat{y}, \xi(x + y, t), \xi(x, t)) |\omega(x + y, t)| \frac{1}{|y|^3} dy;$$

here, $\hat{y}$ is the unit vector in the $y$-direction, $\xi$ is the vorticity direction and the kernel $D$ is defined by

$$D(e_1, e_2, e_3) = (e_1 \cdot e_3) (e_1 \cdot (e_2 \times e_3))$$
for any triple of unit vectors $e_1, e_2$ and $e_3$.

Note that

$$|D (\dot{y}, \xi(x + y, t), \xi(x, t))| \leq |\sin \varphi(\xi(x + y, t), \xi(x, t))|$$

and—consequently—a coherence condition of the form

(4) $$|\sin \varphi(\xi(x + y, t), \xi(x, t))| \leq c|y|^\delta$$

for some $\delta \in (0, 1)$ will regularize the critical singularity of $\frac{1}{|y|^3}$ in the integral. This is the essence of the method.

**Proof of Theorem 1.** Multiplying the equations (2) by $\omega |\omega|^{p-2}$ and integrating over the whole space yields

(5) $$\frac{1}{p} \frac{d}{dt} \|\omega(t)\|_{L^p}^p + \int (\sqrt{-\Delta})^{2\beta} \omega \cdot |\omega|^{p-2} \, dx = \int (\omega \cdot \nabla) u \cdot |\omega|^{p-2} \, dx$$

(the advection term drops as a result of the divergence-free constraint).

Due to the pointwise identity $(\omega \cdot \nabla) u \cdot \omega = \alpha |\omega|^2$, the RHS in (5) is equal to

$$\int \alpha |\omega|^p \, dx.$$

A positivity lemma (Lemma 3.3) in [N05] implies a lower bound on the fractional diffusion term,

$$\int (\sqrt{-\Delta})^{2\beta} \omega \cdot |\omega|^{p-2} \, dx \geq \frac{2}{p} \int \left| (\sqrt{-\Delta})^{\beta} (|\omega|^\frac{2}{3}) \right|^2 \, dx,$$

while the Sobolev Embedding Theorem for the fractional derivatives yields

$$\int \left| (\sqrt{-\Delta})^{\beta} (|\omega|^\frac{2}{3}) \right|^2 \, dx \geq c_\beta \left( \int |\omega|^{\frac{3p}{3-2\beta}} \, dx \right)^{\frac{3-2\beta}{3}} = c_\beta \|\omega(t)\|_{L^{\frac{3p}{3-2\beta}}}^p.$$

Collecting all of the above, we arrive at the following differential inequality on $(0, T)$,

(6) $$\frac{d}{dt} \|\omega(t)\|_{L^p}^p + \|\omega(t)\|_{L^{\frac{3p}{3-2\beta}}}^p \leq c_{p, \beta} \int \alpha |\omega|^p \, dx.$$
It will be convenient to estimate the RHS as follows,

\[
(7) \quad c_{p, \beta} \left| \int \alpha(x, t) |\omega(x, t)|^p \, dx \right| \leq c_{p, \beta} \int (\rho_\gamma(x, t) |\omega(x, t)|^a) \left( \int \frac{1}{|y|^{3-\gamma}} |\omega(x + y, t)| \, dy \right) |\omega(x, t)|^b \, dx
\]

where \( a + b = p \).

Applying the Hölder inequality with the exponents \( \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1 \), the quantity above can be bounded by

\[
(8) \quad c_{p, \beta} \left\| \rho_\gamma |\omega|^a \right\|_{L^{p_1}} \left\| \frac{1}{|y|^{3-\gamma}} * |\omega| \right\|_{L^{p_2}} \left\| |\omega|^b \right\|_{L^{p_3}}.
\]

A strong form of the Young inequality for convolutions (estimating one of the functions in the weak Lebesgue space) yields the following bound on the middle norm,

\[
\left\| \frac{1}{|y|^{3-\gamma}} * |\omega| \right\|_{L^{p_2}} \leq c \|\omega\|_{L^s}
\]

where \( \frac{1}{p_2} + 1 = \frac{3 - \gamma}{3} + \frac{1}{s} \).

In addition, the last norm in (8) can be rewritten as

\[
\left\| |\omega|^b \right\|_{L^{p_3}} = \|\omega\|^b_{L^{b p_3}}.
\]

Summarizing, the above estimates on the RHS of (6) imply

\[
(9) \quad \frac{d}{dt} \|\omega(t)\|^p_{L^p} + \|\omega(t)\|_{L^{3p/2 - 2\beta}}^{3p} \leq c_{p, \beta} \left\| \rho_\gamma |\omega|^a \right\|_{L^{p_1}} \|\omega\|_{L^s} \|\omega\|^b_{L^{b p_3}}.
\]

The next step is to interpolate the last two norms on the RHS between the \( L^p \) and the \( L^{3p/2 - 2\beta} \)-norms of \(\omega\),

\[
\|\omega\|_{L^s} \leq \|\omega\|_{L^p}^{\alpha'} \|\omega\|_{L^{3p/2 - 2\beta}}^{1-\alpha'}
\]

where \( \frac{1}{s} = \alpha \frac{1}{p} + (1 - \alpha') \frac{3 - 2\beta}{3p} \), and

\[
\|\omega\|^b_{L^{b p_3}} \leq \|\omega\|_{L^p}^{b(1-\alpha)} \|\omega\|_{L^{3p/2 - 2\beta}}^{b(1-\alpha)}
\]
where \( \frac{1}{bp_3} = \frac{\alpha}{p} + (1 - \alpha) \frac{3 - 2\beta}{3p} \),

leading to

\[
\frac{d}{dt} \| \omega(t) \|_{L^p}^p + \| \omega(t) \|_{L^{3p}}^{\frac{3p}{3 - 2\beta}} \leq c_{p,\beta} \| \rho_\gamma |\omega| \|_{L^1} \| \omega \|_{L^p}^{\alpha' + b\alpha} \| \omega \|_{L^p}^{(1 - \alpha') + b(1 - \alpha)} \\
\leq \tilde{c}_{p,\beta} \| \rho_\gamma |\omega| \|_{L^1}^2 \| \omega \|_{L^p}^{2(\alpha' + b\alpha)} + \frac{1}{2} \| \omega \|_{L^p}^{2\left((1 - \alpha') + b(1 - \alpha)\right)}.
\]

Setting \( 2(\alpha' + b\alpha) = p = 2((1 - \alpha') + b(1 - \alpha)) \) yields the final form of our differential inequality on \((0, T)\),

\[
\frac{d}{dt} \| \omega(t) \|_{L^p}^p \leq \tilde{c}_{p,\beta} \| \rho_\gamma |\omega| \|_{L^1}^2 \| \omega \|_{L^p}^p.
\]

Consequently,

\[
\| \omega(t) \|_{L^p}^p \leq \| \omega_0 \|_{L^p}^p e \left( \tilde{c}_{p,\beta} \int_0^T \| \rho_\gamma |\omega| \|_{L^1}^2 \, dt \right)
\]

for all \( 0 < t < T \).

Since \( \omega_0 \in L^p \), \( \sup_{t \in (0, T)} \| \omega(t) \|_{L^p} < \infty \) provided

\[
\int_0^T \| \rho_\gamma |\omega| \|_{L^1}^2 \, dt < \infty
\]

which—in turn—implies that \( T \) is not a blow-up time utilizing the regularity criterion (3). The constraint

\[
p_1(\gamma + 2\alpha) - 3 = \beta p_1
\]

is imposed in order to make our hybrid geometric-analytic regularity condition scaling-invariant with respect to the (unique) intrinsic scaling of the fractional NSE; namely, if a pair \( \omega(x, t), u(x, t) \) is a solution to (2), so is the rescaled pair \( \omega_\lambda(x, t) = \lambda^{2\beta} \omega(\lambda x, \lambda^{2\beta} t), u_\lambda(x, t) = \lambda^{2\beta - 1} u(\lambda x, \lambda^{2\beta} t) \), for any \( \lambda > 0 \). This type of scaling invariance is useful as it allows one to study the scaling-invariant quantity in view on a parabolic (corresponding to the fractional diffusion) spatiotemporal cylinder of arbitrary size. This concludes the proof.

**Remark 2.** For a given fractional diffusion parameter \( \beta \), there are ten ‘floating parameters’ in the proof, constrained by eight equations, resulting in a two-parameter family of the hybrid geometric-analytic conditions.
Remark 3. As noted in the introduction, if $\beta > \frac{1}{2}$, the fractional NSE are locally-in-time well posed in $L^p$ for $p$ large enough (cf. [DL09]), and one can replace the assumption $\omega \in C([0, T), L^p)$ in the theorem simply by an assumption that $\omega_0 \in L^p$ and designating $T$ to be the first (possible) blow-up time.

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REFERENCES

[Ch07] D. Chae, On the regularity conditions for the Navier-Stokes and related equations, Rev. Mat. Iberoamericana 23, 371 (2007).
[CoDeLDeR] M. Colombo, C. De Lellis, and L. De Rosa, Ill-posedness of Leray solutions for the ipodissipative Navier-Stokes equations, Comm. Math. Phys. 362, 659 (2018).
[Co94] P. Constantin, Geometric statistics in turbulence, SIAM Rev. 36, 73 (1994).
[CoFe93] P. Constantin and C. Fefferman, Direction of vorticity and the problem of global regularity for the Navier-Stokes equations, Indiana Univ. Math. J. 42, 775 (1993).
[daVeigaBe02] H. Beirao da Veiga and L.C. Berselli, On the regularizing effect of the vorticity direction in incompressible viscous flows, Diff. Int. Eqs. 15, 345 (2002).
[DL09] H. Dong and D. Li, Optimal local smoothing and analyticity rate estimates for the generalized Navier-Stokes equations, Commun. Math. Sci. 7, 67 (2009).
[ESS03] L. Escauriaza, G. Seregin, and V. Shverak, $L_{3,\infty}$-solutions of Navier-Stokes equations and backward uniqueness, Uspekhi Mat. Nauk. 58, 3 (2003).
[GiMi] Y. Giga and T. Miyakawa, Navier-Stokes flow in $\mathbb{R}^3$ with measures as initial vorticity and Morrey spaces, Commun. in Partial Diff. Equations 14, 577 (1989).
[Gr09] Z. Grujić, Localization and geometric depletion of vortex-stretching in the 3D NSE, Comm. Math. Phys. 290, 861 (2009).
[GrGu10-1] Z. Grujić and R. Guberović, Localization of analytic regularity criteria on the vorticity and balance between the vorticity magnitude and coherence of the vorticity direction in the 3D NSE, Comm. Math. Phys. 298, 407 (2010).
[GrRu04] Z. Grujić and A. Ruzmaikina, Interpolation between algebraic and geometric conditions for smoothness of the vorticity in the 3D NSE, Indiana Univ. Math. J. 43, 1073 (2004).
[JW93] J. Jimenez, A.A. Wray, P.G. Saffman and R.S. Rogallo, The structure of intense vorticity in isotropic turbulence, J. Fluid Mech. 255, 65 (1993).
[Ki] S. Kida, Three-dimensional periodic flows with high symmetry, J. Phys. Soc. Jpn. 54, 2132 (1985).
[LR] P. G. Lemarie-Rieusset, Recent developments in the Navier-Stokes problem. Chapman and Hall/CRC, 2002.
[Mey] Y. Meyer, Oscillating Patterns in Some Nonlinear Evolution Equations, Lecture notes in mathematics 1871, C. I. M. E. Foundation Subseries: Mathematical Foundation of Turbulent Viscous Flows, Springer 2003.
[Na19] K. Nakai, Direction of vorticity and a refined regularity criterion for the Navier-Stokes equation with fractional Laplacian, J. Math. Fluid Mech. 21, 21 (2019).
[N05] J. Ning, The maximum principle and the global attractor for the dissipative 2D quasi-geostrophic equations, Comm. Math. Phys. 255, 161 (2005).
[S81] E. Siggia, Numerical Study of Small Scale Intermittency in Three-Dimensional Turbulence, J. Fluid Mech. 107, 375 (1981).
[SJO91] Z.-S. She, E. Jackson and S. Orszag, Structure and dynamics of homogeneous turbulence: models and simulations, Proc. R. Soc. Lond. A 434, 101 (1991).
[St] E.M. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals, Princeton University Press, 1993.

[Tay37] G. I. Taylor, Production and dissipation of vorticity in a turbulent fluid, Proc. Roy. Soc., A164, 15 (1937).

[VM94] A. Vincent and M. Meneguzzi, The dynamics of vorticity tubes in homogeneous turbulence, J. Fluid Mech. 225, 245 (1994).

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