Quantum Error-Control Codes
1.1 Introduction

Information is physical. It is sensible to use quantum mechanics as a basis of computation and information processing [19]. Here at the intersection of information theory, computing, and physics, mathematicians and computer scientists must think in terms of the quantum physical realizations of messages. The often philosophical debates among physicists over the nature and interpretations of quantum mechanics shift to harnessing its power for information processing and testing the theory for completeness.

One cannot directly access information stored and processed in massively entangled quantum systems without destroying the content. Turning large-scale quantum computing into practical reality is massively challenging. To start with, it requires techniques for error control that are much more complex than those implemented effectively in classical systems. As a quantum system grows in size and circuit depth, error control becomes ever more important.

Quantum error-control is a set of methods to protect quantum information from unwanted environmental interactions, known as decoherence. Classically, one encodes information-carrying vectors into a larger space to allow for sufficient redundancy for error detection and correction. In the quantum setup, information is stored in a subspace embedded in a larger Hilbert space, which is a finite dimensional, normed, vector space over the field of complex numbers \( \mathbb{C} \). Codewords are quantum states and errors are operators.

The good news is that noise, if it can be kept below a certain level, is not an obstacle to resilient quantum computation. This crucial insight is arrived at based on seminal results that form the so-called threshold theorems. Theoretical references include the exposition of Knill et al. in [34], the work of Preskill
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in [42], the results shown by Steane in [49], and the paper of Ahoronov and Ben-Or [1]. A comprehensive review on related experiments is available in [9].

The possibility of correcting errors in quantum systems was shown, e.g., in the early works of Shor [46], Steane [47] and Laflamme et al. [35]. While the quantum codes that these pioneers proposed may nowadays seem to be rather trivial in performance, their construction highlighted both the main obstacles and their respective workarounds. Measurement collapses the information contained in the state into something useless. One should measure the error, not the data. Since repetition is ruled out due to the no-cloning theorem [53], we use redundancy from spreading the states to avoid repetition. There are multiple types of errors, as we will soon see. The key is to start by correcting the phase errors and, then, use the Hadamard transform to exchange the bit flips and the phase errors. Quantum errors are continuous. Controlling them seemed to be too daunting a task. It turned out that handling a set of discrete error operators, represented by tensor product of Pauli matrices, allows for the control of every C-linear combination of these operators.

Advances continue to be made as effort intensifies to scale quantum computing up. Research in quantum error-correcting codes (QECs) has attracted the sustained attention of established researchers and students alike. Several excellent online lecture notes, surveys, and books are available. Developments up to 2011 have been well-documented in [36]. It is impossible to describe the technical details of every important research direction in QECs. We focus on quantum stabilizer codes and their variants. The decidedly biased take here is for the audience with more applied mathematics background, including coding theorist, information theorist, researchers in discrete mathematics and finite geometries. No knowledge of quantum mechanics is required beyond the very basic. This chapter is meant to serve as an entry point for those who want to understand and get involved in building upon this foundational aspect of quantum information processing and computation, which have been tipped to be indispensable in future technologies.

A quantum stabilizer code is designed so that errors with high probability of occurring transform information-carrying states to an error space which is orthogonal to the original space. The beauty lies in how natural the determination of the location and type of each error in the system is. Correction becomes a simple application of the type of error at the very location.

1.2 Preliminaries

Consider the field extensions $\mathbb{F}_p$ to $\mathbb{F}_{q^r}$ to $\mathbb{F}_{q^{rm}}$, for positive integers $r$ and $m$. For $\alpha \in \mathbb{F}_{q^m}$, the trace mapping from $\mathbb{F}_{q^m}$ to $\mathbb{F}_q$ is given by

$$\text{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\alpha) = \alpha + \alpha^q + \ldots + \alpha^{q^{m-1}} \in \mathbb{F}_q.$$
The trace of $\alpha$ is the sum of its conjugates. If the extension $\mathbb{F}_q$ of $\mathbb{F}_p$ is contextually clear, the notation $\text{Tr}$ is sufficient. Properties of the trace map can be found in standard textbooks, e.g., [37] Chapter 2. The key idea is that $\text{Tr}_{\mathbb{F}_q^n}/\mathbb{F}_q$ serves as a description for all linear transformations from $\mathbb{F}_q^n$ to $\mathbb{F}_q$.

Let $G$ be a finite abelian group, written multiplicatively, with the identity $1_G$. Let $U$ be the multiplicative group of complex numbers of modulus 1, i.e., the unit circle of radius 1 on the complex plane $\mathbb{C}$. A character $\chi : G \mapsto U$ is a homomorphism. For any $g \in G$, the images of $\chi$ are $|G|$-th roots of unity since $(\chi(g))^{|G|} = \chi(g^{|G|}) = 1$. Let $\overline{c}$ denote the complex conjugate of $c$. Then $\chi(g^{-1}) = (\chi(g))^{-1} = \overline{\chi(g)}$. The only trivial character is $\chi_0 : g \mapsto 1$ for all $g \in G$. One can associate $\chi$ and $\overline{\chi}$ by using $\overline{\chi}(g) = \overline{\chi(g)}$. The set of all characters of $G$ forms, under composition $\circ$, an abelian group $\hat{G}$.

For $g, h \in G$ and $\chi, \Psi \in \hat{G}$ we have two orthogonality relations

$$\sum_{g \in G} \chi(g) \overline{\Psi(g)} = \begin{cases} 0, & \text{if } \chi \neq \Psi \\ |G|, & \text{if } \chi = \Psi \end{cases} \quad \text{and} \quad \sum_{\chi \in \hat{G}} \chi(g) \chi(h^{-1}) = \begin{cases} 0, & \text{if } g \neq h \\ |G|, & \text{if } g = h \end{cases}.$$ (1.1)

The additive character $\chi_1 : c \mapsto e^{\frac{2\pi}{q} \text{Tr}(c)}$, for all $c \in \mathbb{F}_q$, is called the canonical character. For a chosen $b \in \mathbb{F}_q$ and for all $c \in \mathbb{F}_q$,

$$\chi_b := \mathbb{F}_q \mapsto U \text{ sending } c \mapsto \chi_1(b \cdot c) = e^{\frac{2\pi}{q} \text{Tr}(b \cdot c)}$$

is a character of $(\mathbb{F}_q, +)$. Every character of $(\mathbb{F}_q, +)$ can, in fact, be expressed in this manner. The extension to $(\mathbb{F}_q^n, +)$ is straightforward.

**Theorem 1.2.1.** Let $\zeta := e^{\frac{2\pi}{q}}$ and $\text{Tr}$ be the trace map with $q = p^m$. Let $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ be vectors in $\mathbb{F}_q^n$. For each $a$,

$$\lambda_a : \mathbb{F}_q^n \mapsto \{1, \zeta, \zeta^2, \ldots, \zeta^{p-1}\}, \text{ sending } b \mapsto \zeta^{\text{Tr}(a \cdot b)} = \zeta^{\text{Tr}(a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)},$$

for all $b \in \mathbb{F}_q$, is a character of $(\mathbb{F}_q^n, +)$. Hence, $\hat{\mathbb{F}}_q^n = \{\lambda_a : a \in \mathbb{F}_q^n\}$.

A **qubit**, a term coined by Schumacher in [45], is the canonical quantum system consisting of two distinct levels. The states of a qubit live in $\mathbb{C}^2$ and are defined by their continuous amplitudes. A **qudit** refers to a system of $q \geq 3$ distinct levels, with a **qutrit** commonly used when $q = 3$. Physicists prefer the “bra” $\langle \cdot |$ and “ket” $| \cdot \rangle$ notation to describe quantum systems. A $|\varphi\rangle$ is a (column) vector while $\langle \psi |$ is the vector dual of $|\psi\rangle$.

**Definition 1 (Quantum systems).** A qubit is a nonzero vector in $\mathbb{C}^2$, usually with basis $\{|0\rangle, |1\rangle\}$. It is written in vector form as $|\varphi\rangle := \alpha |0\rangle + \beta |1\rangle$, or in matrix form as $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, with $\|\alpha\|^2 + |\beta|^2 = 1$.

An $n$-qubit system or vector is a nonzero element in $(\mathbb{C}^2)^\otimes n \cong \mathbb{C}^{2^n}$. Let $a = (a_1, \ldots, a_n) \in \mathbb{F}_2^n$. The standard $\mathbb{C}$-basis is

$$\{|a_1 a_2 \ldots a_n\rangle := |a_1\rangle \otimes |a_2\rangle \otimes \ldots \otimes |a_n\rangle : a \in \mathbb{F}_2^n\}.$$
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An arbitrary nonzero vector in $\mathbb{C}^{2^n}$ is written 
\[ |\psi\rangle = \sum_{a \in \mathbb{F}_2^n} c_a |a\rangle, \text{ with } c_a \in \mathbb{C} \text{ and } \frac{1}{2^n} \sum_{a \in \mathbb{F}_2^n} |c_a|^2 = 1. \]

The normalization is optional since $|\psi\rangle$ and $\alpha |\psi\rangle$ are considered the same state for nonzero $\alpha \in \mathbb{C}$.

The inner product of $|\psi\rangle := \sum_{a \in \mathbb{F}_2^n} c_a |a\rangle$ and $|\varphi\rangle := \sum_{\mathbb{F}_2^n} b_a |a\rangle$ is
\[ \langle \psi | \varphi \rangle = \sum_{a \in \mathbb{F}_2^n} c_a b_a. \]

Their (Kronecker) tensor product is written as $|\varphi\rangle \otimes |\psi\rangle$ and is often abbreviated to $|\varphi\psi\rangle$. The states $|\psi\rangle$ and $|\varphi\rangle$ are orthogonal or distinguishable if
\[ \langle \psi | \varphi \rangle = 0. \]

Let $A$ be a $2^n \times 2^n$ complex unitary matrix with conjugate transpose $A^\dagger$. The (Hermitian) inner product of $|\varphi\rangle$ and $A |\psi\rangle$ is equal to that of $A^\dagger |\psi\rangle$ and $|\varphi\rangle$. Henceforth, $i := \sqrt{-1}$.

**Definition 2 (Qubit error operators).** A qubit error operator is a unitary $\mathbb{C}$-linear operator acting on $\mathbb{C}^{2^n}$ qubit by qubit. It can be expressed by a unitary matrix with respect to the basis $\{|0\rangle, |1\rangle\}$. The three nontrivial errors acting on a qubit are known as the **Pauli matrices**: 
\[ \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \sigma_y = i \sigma_x \sigma_z = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}. \quad (1.2) \]

The actions of the error operators on a qubit $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle \in \mathbb{C}^2$ can be considered based on their types. The **trivial operator** $I_2$ leaves the qubit unchanged. The **bit-flip error** $\sigma_x$ flips the probabilities
\[ \sigma_x |\varphi\rangle = \beta |0\rangle + \alpha |1\rangle \text{ or } \sigma_x \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}. \]

The **phase-flip error** $\sigma_z$ modifies the angular measures
\[ \sigma_z |\varphi\rangle = \alpha |0\rangle - \beta |1\rangle \text{ or } \sigma_z \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ -\beta \end{bmatrix}. \]

The **combination error** $\sigma_y$ contains both bit-flip and phase-flip, implying
\[ \sigma_y |\varphi\rangle = -i \beta |0\rangle + i \alpha |1\rangle \text{ or } \sigma_y \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -i \beta \\ i \alpha \end{bmatrix}. \]

It is immediate to confirm that $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I_2$ and $\sigma_x \sigma_z = -\sigma_z \sigma_x$.

The Pauli matrices generate a group of order 16. Each of its elements can be uniquely represented as $i^\lambda w$, with $\lambda \in \{0, 1, 2, 3\}$ and $w \in \{I_2, \sigma_x, \sigma_z, \sigma_y\}$.
1.3 The Stabilizer Formalism

The most common route from classical coding theory to QEC is via the stabilizer formalism, from which numerous specific constructions emerge. Classical codes can not be used as quantum codes but can model the error operators in some quantum channels. The capabilities of a QEC can then be inferred from the properties of the corresponding classical codes. The main tools come from character theory and symplectic geometry over finite fields. Our focus is on the qubit setup since it is the most deployment-feasible and because the general qudit case naturally follows from it.

Let \( a = (a_1, a_2, \ldots, a_n) \in \mathbb{F}_2^n \), \( \lambda \in \{0, 1, 2, 3\} \), and \( w_j \in \{I_2, \sigma_x, \sigma_z, \sigma_y\} \). A quantum error operator on \( \mathbb{C}^{2^n} \) is of the form \( E := i^\lambda w_1 \otimes w_2 \otimes \ldots \otimes w_n \). It is a \( \mathbb{C} \)-linear unitary operator acting on a \( \mathbb{C}^{2^n} \)-basis \( \{|a\rangle = |a_1\rangle \otimes |a_2\rangle \otimes \ldots \otimes |a_n\rangle\} \) by \( E|a\rangle := i^\lambda (w_1 |a_1\rangle \otimes w_2 |a_2\rangle \otimes \ldots \otimes w_n |a_n\rangle) \). The set of error operators

\[ \mathcal{E}_n := \{i^\lambda w_1 \otimes w_2 \otimes \ldots \otimes w_n\} \]

is a non-abelian group of cardinality \( 4^{n+1} \). Given \( E := i^\lambda w_1 \otimes w_2 \otimes \ldots \otimes w_n \) and \( E' := i^{\lambda'} w'_1 \otimes w'_2 \otimes \ldots \otimes w'_n \) in \( \mathcal{E}_n \), we have

\[ EE' = i^{\lambda+\lambda'} (w_1 w'_1) \otimes (w_2 w'_2) \otimes \ldots \otimes (w_n w'_n) \]

\[ = i^{\lambda+\lambda'} w''_1 \otimes w''_2 \otimes \ldots \otimes w''_n, \]

where \( w_j w'_j = i^{\lambda''} w''_j \) and \( \lambda'' = \sum_{j=1}^{n} \lambda''_j \).

Expanding \( EE' \) makes it clear that \( EE' = \pm 1 \) \( EE' \).

**Example 1.** Given \( n = 2 \), \( E = I_2 \otimes \sigma_x \) and \( E' = \sigma_z \otimes \sigma_y \), we have \( EE' = \sigma_z \otimes \sigma_x \sigma_y = \sigma_z \otimes i \sigma_z = i \sigma_z \otimes \sigma_z \) and \( E'E = \sigma_z \otimes \sigma_y \sigma_x = \sigma_z \otimes i^3 \sigma_z = i^3 \sigma_z \otimes \sigma_z \).

The center of \( \mathcal{E}_n \) is \( C(\mathcal{E}_n) := \{i^\lambda I_2 \otimes I_2 \otimes \ldots \otimes I_2\} \). Let \( \overline{\mathcal{E}_n} \) denote the quotient group \( \mathcal{E}_n/C(\mathcal{E}_n) \) of cardinality \( |\overline{\mathcal{E}_n}| = 4^n \). This group is an abelian 2-elementary group \( \cong (\mathbb{F}_2^{2^n}, +) \), since \( \overline{E'} = I_2 \otimes \cdots \otimes I_2 = I_{2^n} \) for any \( E \in \overline{\mathcal{E}_n} \).

We switch notation to define the product of error operators in terms of an inner product of their vector representatives. We write \( E = i^\lambda w_1 \otimes w_2 \otimes \ldots \otimes w_n \) as \( E = i^{\lambda+\epsilon} X(a)Z(b) \), where \( a = (a_1, \ldots, a_n) \), \( b = (b_1, \ldots, b_n) \in \mathbb{F}_2^n \) and \( \epsilon := |\{1 \leq i \leq n \mid w_i = \sigma_y\}| \), by replacing \( (a_i, b_i) \) with \((0,0)\) if \( w_i = I_2 \), by \((1,0)\) if \( w_i = \sigma_x \), by \((0,1)\) if \( w_i = \sigma_z \), and by \((1,1)\) if \( w_i = \sigma_y \).

The respective actions of \( X(a) \) and \( Z(b) \) on any vector \( |v\rangle \in \mathbb{C}^{2^n} \), for \( v \in \mathbb{F}_2^n \), are \( X(a) |v\rangle = |a + v\rangle \) and \( Z(b) |v\rangle = (-1)^b v \langle v| \). The matrix for \( X(a) \) is a symmetric \( \{0, 1\} \) matrix. It represents a permutation consisting of \( 2^{n-1} \) transpositions. The matrix for \( Z(b) \) is diagonal with diagonal entries \pm 1. Hence, writing the operators in \( \overline{\mathcal{E}_n} \) as \( E := i^\lambda X(a)Z(b) \) and
Thus, a subgroup $G$ distinguishable, i.e., must remain distinguishable, for all $E$. The symplectic inner product of $(a|b)$ and $(a'|b')$ in $\mathbb{F}_2^n$ is

$$\langle (a|b), (a'|b') \rangle_s = a \cdot b + a' \cdot b$$

(1.3)

or, in matrix form,

$$\langle (a|b), (a'|b') \rangle_s = [a \ b] \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} [a' \ b']^\top.$$ 

The symplectic dual of $C \subseteq \mathbb{F}_2^n$ is $C^\perp = \{ u \in \mathbb{F}_2^n : (u, c) = 0 \forall c \in C \}$. Thus, a subgroup $G$ of $\mathcal{E}_n$ is abelian if and only if $G$ is a symplectic self-orthogonal subspace of $\mathcal{E}_n \equiv \mathbb{F}_2^n$.

**Example 2.** Continuing from Example 4, we write $E = X((0,1))Z((0,0))$ and $E' = iX((0,1))Z((1,1))$. We choose the ordering $(0,0), (0,1), (1,0), (1,1)$ of $\mathbb{F}_2^n$ and the corresponding ordering for the basis of $\mathbb{C}^4$. The matrix for $X((0,1))$ agrees with $I_2 \otimes \sigma_z$, the matrix for $Z((0,0))$ is $I_4$, and the matrix for $Z((1,1))$ is diagonal with diagonal entries $1, -1, -1, 1$. Multiplying matrices confirms that $\sigma_z \otimes \sigma_y$ is indeed $iX((0,1))Z((1,1))$.

The quantum weight of an error operator $E = i^jX(a)Z(b) \in \mathcal{E}_n$ is

$$w_Q(E) := w_Q(\overline{E}) = w_Q(a|b) = |\{ 1 \leq i \leq n : a_i = 1 \text{ or } b_i = 1 \}|$$

$$= |\{ 1 \leq i \leq n : w_i \neq I_2 \}|.$$ 

By definition, $w_Q(EE') \leq w_Q(E) + w_Q(E')$, for any $E, E' \in \mathcal{E}_n$. We can define the set of all error operators of weight at most $\delta$ in $\mathcal{E}_n$ and determine its cardinality. Let

$$\mathcal{E}_n(\delta) := \{ E \in \mathcal{E}_n : w_Q(E) \leq \delta \} \quad \text{and} \quad \overline{\mathcal{E}}_n(\delta) = \{ \overline{E} \in \overline{\mathcal{E}}_n : w_Q(\overline{E}) \leq \delta \}.$$ 

Then $|\mathcal{E}_n(\delta)| = 4 \sum_{j=0}^{\delta} 3^j \binom{n}{j}$ and $|\overline{\mathcal{E}}_n(\delta)| = \sum_{j=0}^{\delta} 3^j \binom{n}{j}$.

In the classical setup, both errors and codewords are vectors over the same field. In the quantum setup, errors are linear combinations of the tensor products of Pauli matrices. A qubit code $Q \subseteq \mathbb{C}^2^n$ has three parameters: its length $n$, dimension $K$ over $\mathbb{C}$, and minimum distance $d = d(Q)$. We use

$$(n, K, d(k)) \text{ or } [n, k, d] \text{ with } k = \log_2 K$$

to signify that $Q$ describes the encoding of $k$ logical qubits as $n$ physical qubits, with $d$ being the smallest number of simultaneous errors that can transform a valid codeword into another.

**Definition 3** (Knill-Laflamme condition). A quantum code $Q$ can correct up to $\ell$ quantum errors if the following hold. If $|\varphi\rangle, |\psi\rangle \in Q$ are distinguishable, i.e., $\langle \varphi | \psi \rangle = 0$, then $\langle \varphi | E_1E_2 | \psi \rangle = 0$, i.e., $E_1 |\varphi\rangle$ and $E_2 |\psi\rangle$ must remain distinguishable, for all $E_1, E_2 \in \mathcal{E}_n(\ell)$. The minimum distance of $Q$ is $d := d(Q)$ if $\langle \varphi | E | \psi \rangle = 0$ for all $E \in \mathcal{E}_n(d-1)$ and for all distinguishable $|\varphi\rangle, |\psi\rangle \in Q$. 


Given an \((n, K, d)\)-qubit code \(Q\) and an \(E \in \mathcal{E}_n\), \(EQ\) is a subspace of \(\mathbb{C}^{2^n}\). The fact that \(Q\) corrects errors of weight up to \(\ell = \lceil \frac{d-1}{2} \rceil\) does not imply that the subspaces \(\{EQ : \overline{E} \in \mathcal{E}_n(\ell)\}\) are orthogonal to each other. It is possible that a codeword \(|\psi\rangle\) is fixed by some \(E \neq I_{2^n}\), say, when \(|\psi\rangle\) is an eigenvector of \(E\) satisfying \(E|\psi\rangle = \alpha|\psi\rangle\) for some nonzero \(\alpha \in \mathbb{C}\). If the subspaces \(\{\overline{E}Q : \overline{E} \in \mathcal{E}_n(\ell)\}\) are orthogonal to each other, then \(Q\) is said to be pure. Otherwise, the code is degenerate or impure.

To formally define a qubit stabilizer code, we choose an abelian group \(G\), which is a subgroup of \(\mathcal{E}_n\), and associate \(G\) with a classical code \(C \subset \mathbb{F}_2^{2^n}\), which is self-orthogonal under the symplectic inner product. The action of \(G\) partitions \(\mathbb{F}_2^{2^n}\) into a direct sum of \(\chi\)-eigenspaces \(Q(\chi)\) with \(\chi \in \hat{G}\). The properties of \(Q := Q(\chi)\) follow from the properties of \(C\) and \(C^\perp\). The stabilizer formalism, first introduced by Gottesman in his thesis \([23]\) and described in the language of group algebra by Calderbank et al. in \([8]\), remains the most widely-studied approach to control quantum errors. Ketkar et al. generalized the formalism to qudit codes derived from classical codes over \(\mathbb{F}_q\) in \([31]\).

Let \(G\) be a finite abelian group acting on a finite dimensional \(\mathbb{C}\)-vector space \(V\). Each \(g \in G\) is a Hermitian operator of \(V\) and, for any \(g, g' \in G\) and for all \(|\psi\rangle \in V\), \((gg')|\psi\rangle = g(g'(|\psi\rangle))\) and \(g^{-1}(|\psi\rangle) = |\psi\rangle\). Let \(G\) be the character group of \(G\). For any \(\chi \in \hat{G}\), the map \(L_{\chi} := \frac{1}{|G|} \sum_{g \in G} \chi(g) g\) is a linear operator over \(V\). The set \(\{L_{\chi} : \chi \in \hat{G}\}\) is the system of orthogonal primitive idempotent operators.

**Proposition 1.3.1.** \(L_{\chi}\) is idempotent, i.e., \(L_{\chi}^2 = L_{\chi}\) and \(L_{\chi} L_{\chi'} = 0\) if \(\chi \neq \chi'\).

The operators in the system sum to the identity \(\sum_{\chi \in \hat{G}} L_{\chi} = \mathbb{1}\). For all \(g \in G\), we have \(gL_{\chi} = \chi(g) L_{\chi}\).

**Proof.** In \(G\), let \(gh = a\), i.e., \(h = ag^{-1}\). Using \(\overline{\chi} = \chi^{-1}\) and the orthogonality of characters, we write

\[
L_{\chi} L_{\chi'} = \frac{1}{|G|^2} \sum_{g \in G} \overline{\chi}(g) g \sum_{h \in G} \overline{\chi}(h) h = \frac{1}{|G|^2} \sum_{a \in G} \overline{\chi}(a) \sum_{g \in G} \overline{\chi}((ag^{-1}) a) = \frac{1}{|G|^2} \sum_{a \in G} \overline{\chi}(a) a \sum_{g \in G} \overline{\chi}((\chi')(g)). \tag{1.4}
\]

The third equality comes from collecting terms that contain only \(a\) and only \(g\). By the first orthogonality relation in Equation \((1.1)\), one arrives at

\[
L_{\chi} L_{\chi'} = \frac{1}{|G|} \sum_{a \in G} \overline{\chi}(a) a = \begin{cases} 0, & \text{if } \chi \neq \chi', \\ L_{\chi}, & \text{by definition.} \end{cases}
\]

We verify the second assertion by using the second orthogonality relation in Equation \((1.1)\). Since \(\overline{\chi}(1) = 1\), we obtain

\[
\sum_{\chi \in \hat{G}} L_{\chi} = \frac{1}{|G|} \sum_{\chi \in \hat{G}} \sum_{g \in G} \overline{\chi}(g) g = \frac{1}{|G|} \sum_{g \in G} g \sum_{\chi \in \hat{G}} \overline{\chi}(g) \overline{\chi}(1) = 1.
\]
Using $gh = a$, the definition of $L_\chi$, and the equality $\chi(g^{-1}) = \chi(g)$, one gets

$$g L_\chi = \frac{1}{|G|} \sum_{h \in G} \chi(h) g h = \frac{1}{|G|} \sum_{a \in G} \chi(a \cdot g^{-1}) a$$

$$= \frac{1}{|G|} \chi(g^{-1}) \sum_{a \in G} \chi(a) = \chi(g) L_\chi.$$

\[ \square \]

**Proposition 1.3.2.** For each $\chi \in \hat{G}$, let $V(\chi) := L_\chi V = \{L_\chi(|\psi\rangle) : |\psi\rangle \in V\}$. For $|\psi\rangle \in V(\chi)$ and $g \in G$, we have $g |\psi\rangle = \chi(g) |\psi\rangle$. Thus, $V(\chi)$ is a common eigenspace of all operators in $G$. A direct decomposition $V = \bigoplus_{\chi \in \hat{G}} V(\chi)$ ensures that each $|\psi\rangle \in V$ has a unique expression

$$|\psi\rangle = \sum_{\chi \in \hat{G}} |\psi\rangle_\chi, \text{ where } |\psi\rangle_\chi \in V(\chi).$$

**Proof.** For $|\psi\rangle \in V(\chi)$ and $g \in G$, there exists $|w\rangle \in V$ such that

$$g |\psi\rangle = g L_\chi(|w\rangle) = \chi(g) L_\chi(|w\rangle) = \chi(g) |\psi\rangle,$$

confirming the first assertion.

For each $|\psi\rangle \in V$, we write $|\psi\rangle = \left( \sum_{\chi \in \hat{G}} L_\chi \right) |\psi\rangle = \sum_{\chi \in \hat{G}} |\psi\rangle_\chi$, where $|\psi\rangle_\chi := L_\chi(|\psi\rangle) \in V(\chi)$. On the other hand, if $|\psi\rangle = \sum_{\chi \in \hat{G}} |u_\chi\rangle$ for $|u_\chi\rangle \in V(\chi)$, then $|u_\chi\rangle = L_\chi |w_\chi\rangle$ for some $|w_\chi\rangle \in V$. Since $\{L_\chi : \chi \in \hat{G}\}$ has the orthogonality property, for every $\chi \in \hat{G}$, we have

$$|\psi\rangle_\chi = L_\chi |\psi\rangle = L_\chi \left( \sum_{\chi \in \hat{G}} |u_\chi\rangle \right) = L_\chi \left( \sum_{\chi' \in \hat{G}} L_{\chi'} |w_{\chi'}\rangle \right)$$

$$= \sum_{\chi' \in \hat{G}} (L_{\chi} L_{\chi'} |w_{\chi'}\rangle) = L_{\chi} |w_\chi\rangle = |u_\chi\rangle.$$

Thus, $V = \bigoplus_{\chi \in \hat{G}} V(\chi)$. \[ \square \]

It is also a well-known fact that $V(\chi)$ and $V(\chi')$ are Hermitian orthogonal for all $\chi \neq \chi' \in \hat{G}$ when all $g \in G$ are unitary linear operators on $V$.

All the tools to connect qubit stabilizer codes to classical codes are now in place. We choose $G := \langle g_1, g_2, \ldots, g_k \rangle$ to be an abelian subgroup of $E_n$ with $g_j := i^{a_j} X(a_j) Z(b_j)$ for $1 \leq j \leq k$, where $a_j, b_j \in \mathbb{F}_2^n$ and $\lambda_j \equiv a_j \cdot b_j \pmod{2}$. Since $\sigma_x$ and $\sigma_z$ are Hermitian unitary matrices, $X(a_j)$ and $Z(b_j)$ are also Hermitian matrices. The basis element $g_j$ is Hermitian since

$$g_j^\dagger := \overline{g_j}^\dagger = (-i)^{a_j} Z(b_j)^\dagger X(a_j)^\dagger = (-i)^{a_j} Z(b_j) X(a_j)$$

$$= i^{a_j} (-1)^{a_j \cdot b_j} (-1)^{a_j \cdot b_j} X(a_j) Z(b_j) = i^{a_j} X(a_j) Z(b_j). \quad (1.5)$$
Theorem 1.3.3. Let \( C \) be an \( n - k \)-dimensional self-orthogonal subspace of \( \mathbb{F}_2^{2n} \) under the symplectic inner product. Let \( d := \min\{w_Q(C^⊥, C) : v ∈ C^⊥, C\} \). Then there is an \([n, k, d]\)-qubit stabilizer code \( Q \).

Proof. We lift \( C := \overline{G} \) to an abelian subgroup \( G \) of \( \mathcal{E}_n \), with \( G ⊆ \mathbb{F}_2^{n-k} \). Then \( \mathbb{C}^{2n} = \bigoplus_{g ∈ \overline{G}} Q(\chi) \), where \( Q(\chi) = L_{\chi}\mathbb{C}^{2n} \), is the subspace

\[
\{|v⟩ ∈ \mathbb{C}^{2n} : g|v⟩ = χ(g)|v⟩ \text{ for all } g ∈ G\}.
\]

Showing that each \( Q(\chi) \) is an \([n, k, d]\)-qubit code means proving

\[
\text{dim}_C Q(\chi) = 2^k \text{ and } d(Q(\chi)) ≥ w_Q(C^⊥, C).
\]

Consider the action of \( \mathcal{E}_n \) on \( \{Q(\chi) : χ ∈ \overline{G}\} \). For any \(|v⟩ ∈ Q(\chi)\) and \( g ∈ G \), we have \( g|v⟩ = χ(g)|v⟩ \). Thus, for any \( E ∈ \mathcal{E}_n \) and any \( g ∈ G \),

\[
g(E|v⟩) = (-1)^{ϕ(E)}, E(g|v⟩) = (-1)^{ϕ(E)}χ(g)E|v⟩.
\]

Since \( χ_E : G → \{±1\} \) and \( χ_E(g) = (-1)^{ϕ(E)} \) is a character of \( G \), we have

\[
g(E|v⟩) = χ_E(g)E|v⟩ = χ'(g)E|v⟩ , \text{ for all } g ∈ G.
\]

This implies \( E|v⟩ ∈ Q(χ') \) and \( E : Q(χ) → Q(χ') \), where \( χ' := χ_Eχ \). Since \( \mathcal{E}_n \) is a group, \( E \) is a bijection, making \( \text{dim}_C Q(χ) = \text{dim}_C Q(χ') \). As \( E \) runs through \( \mathcal{E}_n \), \( χ_E \) takes all characters of \( G \), ensuring that \( \text{dim}_C Q(χ) \) is the same for all \( χ ∈ \overline{G} \). Thus, \( \text{dim}_C Q(χ) = 2^{n-(n-k)} = 2^k \) for any \( χ ∈ \overline{G} \).

We now show that, if \( E ∈ \mathcal{E}_{d-1} \) and \( |v_1⟩, |v_2⟩ ∈ Q(χ) \) with \( ⟨v_1|v_2⟩ = 0 \), then \( ⟨v_1|E_1E_2|v_2⟩ = 0 \), where \( E := E_1E_2 \). If \( E ∈ C \), then \( ⟨v_1|E|v_2⟩ = χ_E(E)|v_1⟩|v_2⟩ = 0 \). Otherwise, \( E ∉ C \). From \( w_Q(E) = w_Q(E) ≤ d - 1 \) and the assumption \( (C^⊥, C) ∩ \mathcal{E}_{d-1} = ∅ \), we know \( E ∉ C^⊥ \). Hence, there exists \( E' ∈ C \) such that \( E E' = -E' E \). Then, for \( |v_2⟩ ∈ Q(χ) \), we have

\[
E'E|v_2⟩ = -E E'|v_2⟩ = -χ(E')E|v_2⟩, \text{ with } -χ(E') ≠ χ(E').
\]

Therefore, \( E|v_2⟩ ∈ Q(χ') \), with \( χ' ≠ χ \). Since \( |v_1⟩ ∈ Q(χ) \) and \( Q(χ) \) is orthogonal to \( Q(χ') \), we confirm \( ⟨v_1|E|v_2⟩ = 0 \).

Example 3. We exhibit a \([5, 1, 3]\)-qubit stabilizer code \( Q \). Consider a subspace \( C ⊆ \mathbb{F}_2^9 \) with generator matrix

\[
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4
\end{bmatrix}.
\]

One reads \( v_1 = (a|b) \) as having \( a = (1, 1, 0, 0, 0) \) and \( b = (0, 0, 1, 0, 1) \). The
code \( C \) is symplectic self-orthogonal, with \( \dim_{\mathbb{F}_q} C = 4 \) and \( \dim_{\mathbb{F}_q} C^\perp = 6 \), i.e., the codimension is 2. To extend the basis for \( C \) to a basis for \( C^\perp \), we use \((0,0,0,0,0,1,1,1,1,1)\) and \((1,1,1,1,0,0,0,0,0,0)\). Since \( w_Q(C) = 4 \) and \( w_Q(C^\perp) = 3 \), one obtains \( w_Q(C^\perp \setminus C) = 3 \). We can write the \([5,1,3]-\)code \( Q = Q(\chi_0) \) explicitly by using \( G = \langle g_1, g_2, g_3, g_4 \rangle \), with \( g_1 = \sigma_x \otimes \sigma_x \otimes \sigma_z \otimes I_2 \otimes \sigma_z, g_2 = \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes I_2 \otimes \sigma_z, g_3 = I_2 \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z \), and \( g_4 = \sigma_z \otimes I_2 \otimes \sigma_z \otimes \sigma_x \otimes \sigma_x \).

Since \( k = 1 \) and \( \dim_{\mathbb{F}_q} Q = 2^k = 2 \), two independent vectors in \( \mathbb{C}^{32} \) form a basis of \( Q = \{ |v⟩ ∈ \mathbb{C}^{32} : g|v⟩ = \chi_0(g)|v⟩ \; \forall g ∈ G \} \). \( Q \) consists of vectors which are fixed by all \( g \in G \). After some computation, we conclude that \( Q \) can be generated by \( |v_0⟩ = \sum_{g ∈ G} |00000⟩ \) and \( |v_1⟩ = \sum_{g ∈ G} |11111⟩ \).

With minor modifications, the qubit stabilizer formalism extends to the general qudit case. A complete treatment is available in [31]. We outline the main steps here. An \( n \)-qudit system is a nonzero element in \( (\mathbb{C}^q)^{\otimes n} \cong \mathbb{C}^n \).

Let \( a = (a_1, \ldots, a_n) ∈ \mathbb{F}^n_q \). The standard \( \mathbb{C} \)-basis is

\[
\{ |a_1a_2\ldots a_n⟩ := |a_1⟩ \otimes |a_2⟩ \otimes \ldots \otimes |a_n⟩ : a ∈ \mathbb{F}^n_q \}\]

and an arbitrary vector in \( \mathbb{C}^q \) is written \( |ψ⟩ = \sum_{a ∈ \mathbb{F}_q^n} c_a |a⟩ \), with \( c_a ∈ \mathbb{C} \) and

\[ q^{-n} \sum_{a ∈ \mathbb{F}_q^n} |c_a|^{2} = 1. \]

Let \( a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) ∈ \mathbb{F}^n_q \), and \( ω := e^{2πi/q^n} \), where \( q = p^m \). The error operators form \( \mathcal{E}_n := \{ ω^β X(a)Z(b) : a, b ∈ \mathbb{F}^n_q, β ∈ \mathbb{F}_p \} \) of cardinality \( p^{2n} \). The respective actions of \( X(a) \) and \( Z(b) \) on \( |v⟩ ∈ \mathbb{C}^n \) are \( X(a)|v⟩ = |a + v⟩ \) and \( Z(b)|v⟩ = (ω)\text{Tr}(b\cdot v)|v⟩ \). Hence, for \( E := ω^β X(a)Z(b) \) and \( E′ := ω^β X(a′)Z(b′) \) in \( \mathcal{E}_n \), one gets \( EE′ = ω^{β(a−a′)} E′ E \). The symplectic weight of \( (a|b) \) is the quantum weight of \( E \).

The (trace) symplectic inner product of \( (a|b) \) and \( (a′|b′) \) in \( \mathbb{F}_q^{2n} \) is

\[ ⟨(a|b), (a′|b′)⟩_s = \text{Tr}(b \cdot a′ − b′ \cdot a). \] (1.6)

The symplectic dual of \( C ⊆ \mathbb{F}_q^{2n} \) is \( C^⊥ = \{ u ∈ \mathbb{F}_q^{2n} : ⟨u, c⟩_s = 0 \; \forall c ∈ C \} \).

As in the qubit case, in the general qudit setup, a subgroup \( G \) of \( \mathcal{E}_n \) is abelian if and only if \( \overline{G} \) is a symplectic self-orthogonal subspace of \( \mathcal{E}_n \cong \mathbb{F}_q^{2n} \). The analogue of Theorem [1.3.3] follows.

**Theorem 1.3.4.** Let \( C \) be an \( n − k \)-dimensional self-orthogonal subspace of \( \mathbb{F}_q^{2n} \) under the (trace) symplectic inner product. Let \( a := w_Q(C^⊥ \setminus C) = \min\{w_Q(v) : v ∈ C^⊥ \setminus C \} \). Then there is an \([n,k,d]\)-qudit stabilizer code \( Q \).

"With group and eigenstate, we’ve learned to fix
Your quantum errors with our quantum tricks."

Daniel Gottesman
1.4 Constructions via Classical Codes

Any stabilizer code $Q$ is fully characterized by its stabilizer group, that specifies the set of errors that $Q$ can correct. Any linear combination of the operators in the error set is correctable, allowing $Q$ to correct a continuous set of operators. For this reason, the best-known qubit codes in the online table [24] maintained by M. Grassl are given in terms of their stabilizer generators. The stabilizer approach has massive advantages over other frameworks, some of which will be mentioned below. It describes a large set of QECs, complete with their encoding and decoding mechanism, in a very compact form.

A valid codeword of $Q$ is a $+1$ eigenvector of all the stabilizer generators. An error $E$, expressed as a tensor product of Pauli operators, anticommutes with some of the stabilizer generators and commutes with others. It sends a codeword to an eigenstate of the stabilizer generators. The eigenvalue remains $+1$ for all operators that commute with $E$ but becomes $-1$ for those generators that anticommute with $E$. From the resulting error syndrome, one knows which Pauli operators acts on which qubits. Applying the respective Pauli operators on the corresponding locations corrects the error. Suppose that the location of error is known, but the type is not, then this is a quantum erasure. By the Knill-Laflamme condition, correcting $\ell$ general errors means correcting $2\ell$ erasures.

The encoding and syndrome reading circuits can be written using only three quantum gates, namely the Hadamard gate, the phase $S$ gate, and the CNOT gate, whose respective matrices are

\[
H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad \text{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.
\]

A treatment on the circuit implementations is available, e.g., in [25].

We now look into suitable classical codes that fully describe the set of correctable errors. All constructions are applications of the stabilizer formalism. Since all of the inner products used are nondegenerate, i.e., $(C^\perp)^\perp = C$, one can interchange self-orthogonality and dual-containment, provided that the derived parameters are adjusted accordingly.

First, we consider a generic construction of $q$-ary quantum codes via additive (i.e., $\mathbb{F}_q$-linear) codes over $\mathbb{F}_{q^2}$. Let $\{1, \gamma\}$ be a basis of $\mathbb{F}_{q^2}$ over
Theorem 1.4.1. Let \( C \subseteq \mathbb{F}_q^2 \) be an \( \mathbb{F}_q \)-additive code such that \( C \subseteq C_{alt}^+ \), with \( |C| = q^{n-k} \). Then there exists an \( [n,k,d]_q \) quantum code \( Q \) with

\[
d(Q) = w_H(C_{alt}^+ \setminus C) = \min\{w_H(v) : v \in C_{alt}^+ \setminus C\}.
\]

If \( C \) is \( \mathbb{F}_q^2 \)-linear, we can conveniently replace the trace alternating inner product by the Hermitian inner product, which is easier to compute.

If \( C \) is \( \mathbb{F}_q \)-additive and is even, i.e., \( w_H(c) \) is even for all \( c \in C \), then \( C \) is trace Hermitian self-orthogonal. If \( C \) is trace Hermitian self-orthogonal and \( C \) is \( \mathbb{F}_4 \)-linear, then \( C \) is an even code.

The quantum codes in Theorem 1.4.1 are modeled after classical codes with an additive structure, but the error operators are in fact multiplicative. An error \( E \) may have the same effect as \( ES \) where \( S \neq I \) is an element of the stabilizer group. A QEC is degenerate or impure if the set of correctable errors contains degenerate errors. Studies on impure codes has been rather scarce. The existence of two inequivalent \([6,1,3]\) impure qubit codes was shown in [3] Section IV. Remarkably, there is no \([6,1,3]\) pure qubit code. A systematic construction based on duadic codes and further discussion on the advantages of degenerate quantum codes are supplied in [3].

A very popular construction is based on nested classical codes. We denote the Euclidean dual of \( C \) by \( C^{⊥E} \).

Theorem 1.4.2 (Calderbank-Shor and Steane (CSS) Construction). Let \( C_j \) be an \([n,k_j,d_j]_q\)-code for \( j \in \{1,2\} \) with \( C_1^{⊥E} \subseteq C_2 \). Then there is an \([n,k_1 + k_2 - n, \min\{w_H(C_2 \setminus C_1^{⊥E}), w_H(C_1 \setminus C_2^{⊥E})\}]_q\)-code \( Q \).

The code is pure whenever \( \min\{w_H(C_2 \setminus C_1^{⊥E}), w_H(C_1 \setminus C_2^{⊥E})\} = \min\{d_1,d_2\} \).
Proof. Let $G_j$ and $H_j$ be the generator and parity-check matrices of $C_j$. Consider the linear code $C \subseteq \mathbb{F}_{q}^{2n}$ with generator matrix $\begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix}$. Since $C^\perp \subseteq C_2$, we have $H_1H_2^\top = 0$. Similarly, from $C_2^\perp \subseteq C_1$, we know $H_2H_1^\top = 0$. Define $C^\perp$ to be the code with parity check and generator matrices, respectively, $\begin{bmatrix} H_2 & 0 \\ 0 & H_1 \end{bmatrix}$ and $\begin{bmatrix} G_2 & 0 \\ 0 & G_1 \end{bmatrix}$. We verify that $C \subseteq C^\perp$, with $\dim_q C = 2n - (k_1 + k_2)$. By Theorem 1.3.4, $\dim_C Q = n - \dim_q C = k_1 + k_2 - n$. The distance computation is clear.

A special case of the CSS construction comes via a Euclidean dual-containing code $C^\perp \subseteq C$. From such an $[n, k, d]_q$-code $C$, one obtains an $[n, 2k - n, \geq d]_q$-code $Q$. The next method allows for most qubit CSS codes to be enlarged while avoiding a significant drop in the distance. The choice of the extra vectors in the generator matrix of $C'$ is detailed in [38, Section III].

**Theorem 1.4.3 (Steane Enlargement of CSS Codes).** Let $C$ be an $[n, k, d]_q$-code that contains its Euclidean dual $C^\perp \subseteq C$. Suppose that $C$ can be enlarged to $C' = [n, k', \geq k + 1, d]_q$. Then there exists a pure qubit code of parameters $[n, k + k' - n, \min\{d, [3d'/2]\}]$.

A generalization to the qudit case was subsequently given in [38], where the distance is $\min\{d, \lfloor 2d'/q \rfloor \}$. Comparing the minimum distances in the resulting codes, the enlargement offers a better chance of relative gain in the qubit case as compared with the $q > 2$ cases.

Lisoněk and Singh, inspired by the classical **Construction X**, proposed a modification to qubit stabilizer codes in [39]. The construction generalizes naturally to qudit codes.

**Theorem 1.4.4 (Quantum Construction X).** For an $[n, k]_q^2$-linear code $C$, let $e := k - \dim(C \cap C^\perp)$. Then there exists an $[n + e, n - 2k + e, d]_q$-code $Q$, with $d := d(Q) \geq \min\{d(C^\perp), d(C + C^\perp) + 1\}$, where $C + C^\perp := \{u + v : u \in C, v \in C^\perp\}$.

The case $e = 0$ is the usual stabilizer construction. To prevent a sharp drop in $d$, we want small $e$, i.e., large Hermitian hull $C \subseteq C^\perp$.

We shift our attention now to **propagation rules** and **bounds**. Most of them are direct consequences of the propagation rules and bounds on the classical rules used as ingredients in the above constructions.

**Proposition 1.4.5 (see [38, Theorem 6] for the binary case).** From an $[n, k, d]_q$-code, the following codes can be derived: an $[n, k - 1, \geq d]_q$-code by subcode construction, an $[n + 1, k, \geq d]_q$-code by lengthening, and an $[n - 1, k, \geq d - 1]_q$-code by puncturing.

The analogue of **shortening** is less straightforward. It requires the construction of an auxiliary code and, then, a check on whether this code has...
codewords of a given length. The details on how to shorten quantum codes are available in [26, Section 4], building upon the initial idea of Rains in [43].

How can we measure the goodness of a QEC? For stabilizer codes, given their classical ingredients and constructions, there are numerous bounds.

Theorem 1.4.6 (Quantum Hamming Bound, see [8] for the binary case). Let $Q$ be a pure $[[n, k, d]]_q$-code with $d \geq 2\ell + 1$ and $k > 0$. Then

$$q^{n-k} \geq \sum_{j=0}^{\ell} (q^2 - 1)^j \binom{n}{j}.$$  \hfill (1.7)

$Q$ is perfect if it meets the bound.

The proof comes from the observation that

$$q^n \geq \sum_{E \in \mathcal{F}_n(\ell)} \dim_{\mathbb{C}}(\overline{E} Q) = \dim_{\mathbb{C}} Q \cdot |\mathcal{F}_n(\ell)| = q^k \sum_{j=0}^{\ell} (q^2 - 1)^j \binom{n}{j}.$$  

The code in Example 3 is perfect. It has $2^{n-k} = 16 = \sum_{j=0}^{1} 3^j \binom{5}{j} = 1 + 15$.

Here is another bound which had been established as a necessary condition for pure stabilizer codes.

Theorem 1.4.7 (Quantum Gilbert-Varshamov Bound [18]). Let $n > k \geq 2$, $d \geq 2$, and $n \equiv k \pmod{2}$. A pure $[[n, k, d]]_q$-code exists if

$$q^{n-k+2} - 1 \geq \sum_{j=1}^{d-1} (q^2 - 1)^{j-1} \binom{n}{j}.$$  

An upper bound, which is well-suited for computer search, is the quantum Linear Programming (LP) bound. In the qubit case, the bound is explained in details in [8, Section VII]. The same routine adjusts immediately to the general qudit case, as was shown in [31, Section VI]. The main tools are the MacWilliams identities [40]. These are linear relations between the weight distribution of a classical code and its dual. They hold for all of the inner products we are concerned with here and have been very useful in ruling out the existence of quantum codes of certain ranges of parameters. Rains supplied a nice proof for the next bound, which is a corollary to the quantum LP bound, using the quantum weight enumerator in [43]. A quantum code that reaches the equality in the bound is said to be quantum MDS (QMDS).

Theorem 1.4.8 (Quantum Singleton Bound). An $[[n, k, d]]_q$-code with $k > 0$ satisfies $k \leq n - 2d + 2$.

Nearly all known families of classical codes over finite fields, especially those with well-studied algebraic and combinatorial structures, have been used
in each of the constructions above. A partial list, compiled in mid 2005 as [31 Table II], already showed a remarkable breadth. The large family of cyclic-type codes, whose corresponding structures in the rings of polynomials are ideals, has been a rich source of ingredients for QECs with excellent parameters. This includes the BCH, cyclic, constacyclic, quasi-cyclic, and quasi-twisted codes. In the family, the nestedness property, very useful in the CSS construction, comes for free. A great deal is known about their dual codes under numerous applicable inner products. For small \( q \), the structures allow for extensive computer algebra searches for reasonable lengths, aided by their minimum distance bounds.

The most comprehensive record for best-known qubit codes is Grassl’s online table [24]. Numerous entries have been certified optimal, while still more entries indicate gaps between the best-possible and the best-known. It is a two-fold challenge to contribute meaningfully to the table. First, for \( n \leq 100 \), many researchers have attempted exhaustive searches. Better codes are unlikely to be found without additional clever strategies. Second, for \( n > 100 \), computing the actual distance \( d(Q) \) tends to be prohibitive. As the length and dimension grow, computing the minimum distances of the relevant classical codes to derive the quantum distance is hard [50]. Improvements remain possible, with targeted searches. Recent examples include the works of Galindo et al. on quasi-cyclic constructions of quantum codes [22], where Steane enlargement is deployed, the search reported in [39] on cyclic codes over \( F_4 \), where Construction X is used with \( e \in \{1, 2, 3\} \), and similar random searches on quasi-cyclic codes done in [10] for qubit and qutrit codes.

Less attention has been given to record-holding qutrit codes, for which there is no publicly available database of comparative extent. A table listing numerous qutrit codes is kept by Y. Edel in [13] based on their explicit construction as quantum twisted codes in [4]. Better codes than many of those in the table have since been found.

Attempts to derive new quantum codes by shortening good stabilizer codes motivate closer studies on the weight distribution of the classical auxiliary codes, in particular when the stabilizer codes are QMDS. Shortening is very effective in constructing qudit MDS codes of lengths up to \( q^2 + 2 \) and minimum distances up to \( q + 1 \).

There has been a large literature on QMDS. All of the above constructions via classical codes as well as the propagation rules have been applied to families of classical MDS codes, particularly the Generalized Reed-Solomon and the constacyclic MDS codes. Since the dual of an MDS code is MDS, the dual distance is evident, leaving only the orthogonality property to investigate. While the theoretical advantages are clearly abundant, there are practical limitations. The length of such codes is bounded above by \( q^2 + 2 \), when \( q \) is even, and by \( q^2 + 1 \), when \( q \) is odd, assuming the MDS conjecture.

For qubit codes, the only nontrivial QMDS are those with parameters \([5, 1, 3]\), \([6, 0, 4]\), and \([2m, 2m - 2, 2]\). As \( q \) grows larger, the practical value of QMDS codes quickly diminishes, since controlling qudit systems with \( q > 3 \)
Quantum Error-Control Codes

is currently prohibitive. A list for $q$-ary QMDS codes, with $2 \leq q \leq 17$, is available in [27]. Another list that covers families of QMDS codes and their references can be found in [10] Table V. More works in QMDS continue to appear, with detailed analysis on the self-orthogonality conditions supplied from number theoretic and combinatorial tools.

Taking algebraic geometry (AG) codes as the classical ingredients is another route. A wealth of favourable results had already been available prior to the emergence of QECs. Curves with many rational points often lead, via the Goppa construction, to codes with good parameters. Their duals are well-understood, via the residue formula. Their designed distances can be computed from the Riemann-Roch theorem. Chen et al. showed how to combine Steane enlargement and concatenated AG codes to derive excellent qubit codes in [11]. A quantum asymptotic bound was established in [17]. Construction of QECs from AG codes was initially a very active line of inquiry. It had somewhat lessened in the last decade, mostly due to lack of practical values to add to the quest as $q$ grows.

Using codes over rings to construct QECs have also been tried. This route, however, does not usually lead to parameter improvements over QECs constructed from codes over fields. The absence of a direct connection from codes over rings to QECs necessitates the use of the Gray mapping, which often causes a significant drop in the minimum distance.

1.5 Going Asymmetric

So far we have been working on the assumption that the bit-flips and the phase-flips are equiprobable. Quantum systems, however, have noise properties that are dynamic and asymmetric. The fault-tolerant threshold is improved when asymmetry is considered [2]. It was Steane who first hinted at the idea of adjusting error-correction to the particular characteristics of the channel in [47]. Designing error control methods to suit the noise profile, which can be hardware-specific, is crucial. The study of asymmetric quantum codes (AQCs) gained traction when the ratios of how often $\sigma_z$ occurs over the occurrence of $\sigma_x$ were discussed in [30], with follow-up constructions offered soon after in [44]. Wang et al. established a mathematical model of AQC in the general qudit system in [51].

As in the symmetric case, $\mathcal{E}_n := \{\omega^\beta X(a)Z(b) : a, b \in \mathbb{F}_q^n, \beta \in \mathbb{F}_p\}$. An error $E := \omega^\beta X(a)Z(b) \in \mathcal{E}_n$ has $\operatorname{wt}_X(E) := \operatorname{wt}(a)$ and $\operatorname{wt}_Z(E) := \operatorname{wt}(b)$.

**Definition 4 (Asymmetric Quantum Codes).** Let $d_x$ and $d_z$ be positive integers. A qudit code $Q$ with dimension $K \geq q$ is called an asymmetric quantum code (AQC) with parameters $((n, K, d_z, d_x))_q$ or $[[n, k, d_z, d_x]]_q$, where $k = \log_q K$, if $Q$ detects $d_x - 1$ qudits of $X$-errors and, at the same time,
that pure CSS AQMDS codes can have were established in [14].

Assuming the validity of the MDS conjecture, all possible parameters and only if one of the followings holds:

1. \(q \) is arbitrary, \( n \geq 2, k \in \{1, n - 1\} \), and \( j \in \{0, n - k\} \).
2. \( q = 2, n \) is even, \( k = 1 \), and \( j = n - 2 \).
3. \( q \geq 3, n \geq 2, k = 1 \), and \( j = n - 2 \).

Theorem 1.5.2. Assuming the MDS conjecture, there is a pure CSS AQMDS code with parameters \([n, j, d_z, d_x]_q\), where \(\{d_z, d_x\} = \{n - k - j + 1, k + 1\} \) if and only if one of the followings holds:

...
4. \( q \geq 3, \ 2 \leq n \leq q, \ k \leq n - 1, \ \text{and} \ 0 \leq j \leq n - k \).
5. \( q \geq 3, \ n = q + 1, \ k \leq n - 1, \ \text{and} \ j \in \{0, 2, \ldots, n - k\} \).
6. \( q = 2^m, \ n = q + 1, \ j = 1, \ \text{and} \ k \in \{2, 2^m - 2\} \).
7. \( q = 2^m \) where \( m \geq 2, \ n = q + 2 \),
   \[
   \begin{cases} 
   k = 1, \ \text{and} \ j \in \{2, 2^m - 2\}, \\
   k = 3, \ \text{and} \ j \in \{0, 2^m - 4, 2^m - 1\}, \ \text{or}, \\
   k = 2^m - 1, \ \text{and} \ j \in \{0, 3\}.
   \end{cases}
   \]

Going forward, three general challenges can be identified. First, find better AQCs, particularly in qubit systems, than the currently best-known. More tools to determine or to lower bound \( d_z \) and \( d_x \) remain to be explored if we are to improve on the parameters. Second, construct codes with very high \( d_z \) to \( d_x \) ratio, since experimental results suggest that this is typical in qubit channels. Third, find conditions on the nested classical codes that yield impure codes.

1.6 Other Approaches and a Conclusion

We briefly mention other approaches to quantum error control before concluding.

Successful small-scale hardware implementations often rely on topological codes, first put forward by Kitaev \[32\]. This family of codes includes surface codes \[6\] and color codes \[5\]. Topological codes encode information in, mostly 2-dimensional, lattices. They are CSS codes with a clever design. The lattice, on which the stabilizer generators act locally, has a bounded weight. The extra restrictions make the error syndrome easier to infer.

Instead of block quantum codes, studies have been done on convolutional qubit codes, see, e.g., \[20\] and subsequent works that cited it. The logical qubits are encoded and transmitted as soon as they arrive in a steady stream. The rate \( k \) over \( n \) is fixed, but the length is not. This type of codes, like their classical counterparts, may be useful in quantum communication.

An approach, that does not require self-orthogonality, constructs entanglement-assisted quantum codes (EA-QECs) \[7\]. The price to pay is the need for a number of maximally entangled states, called ebits for entangled qubits, to increase either the rate or the ability to handle errors. Producing and maintaining ebits, however, tend to be costly, which offset their efficacy. Pairs of classical codes, whose intersections have some prescribed dimensions, were shown to result in EA-QECs in \[29\] Section 4. A formula on the optimal number of ebits that an EA-QEC requires is given in \[72\].

**Theorem 1.6.1.** Given a linear \([n, k, d]_q\)-code \( C \) with parity check matrix \( H \), the code \( C^{1-H} \) stabilizes an EA-QEC with parameters \([n, 2k - n + c, d; c]_q\), where \( c := \text{rank}(HH^\dagger) \) is the number of ebits required.
A larger class of QECs that includes all stabilizer codes is the **codeword stabilized (CWS) codes**. The framework was proposed by Cross et al. in [12] to unify stabilizer (additive) codes and known examples of good nonadditive codes. General constructions for large CWS codes are yet to be devised. Also currently unavailable are efficient encoding and decoding algorithms.

The bridge between classical coding theory and quantum error control was firmly put in place via the stabilizer formalism. Various generalizations and modifications have been studied since, benefitting both the classical and quantum sides of the error-control theory. Well-researched tools and the wealth of results in classical coding theory translate almost effortlessly to the design of good quantum codes, moving far beyond what is currently practical to implement in actual quantum devices. Research problems triggered by error-control issues in the quantum setup revive and expand studies on specific aspects of classical codes, which were previously overlooked or deemed not so interesting. This fruitful cross-pollination between the classical and the quantum, in terms of error control, is set to continue.
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