Quantum Search with Two-atom Collisions in Cavity QED

F. Yamaguchi$^1$, P. Milman$^2$, M. Brune$^2$, J. M. Raimond$^2$, S. Haroche$^{2,3}$

$^1$ E. L. Ginzton Laboratory, Stanford University, Stanford, CA 94305, USA
$^2$ Laboratoire Kastler Brossel, Département de Physique de l’Ecole Normale Supérieure,
24 rue Lhomond, F-75231 Paris Cedex 05 France
$^3$ Collège de France, 11 place Marcelin-Berthelot, F-75005, Paris France

We propose a scheme to implement two-qubit Grover’s quantum search algorithm using Cavity Quantum Electrodynamics. Circular Rydberg atoms are used as quantum bits (qubits). They interact with the electromagnetic field of a non-resonant cavity. The quantum gate dynamics is provided by a cavity-assisted collision, robust against decoherence processes. We present the detailed procedure and analyze the experimental feasibility.

Quantum mechanics makes it possible in principle to realize new information processing functions [1], ranging from the relatively simple quantum cryptography protocols [2] to complex quantum calculation algorithms [3,4]. They are based on manipulations of quantum entanglement between two-level quantum systems or qubits.

The practical implementation of quantum information processing, requiring an excellent isolation of the qubits from the environment, puts very severe constraints on the experimental systems. Many efforts have been recently devoted to the evaluation of various experimental approaches to quantum bits and quantum gates: trapped ions [2], cavity quantum electrodynamics [1], liquid state NMR [5], superconducting mesocircuits [6]. They culminated in the implementation of simple quantum algorithms [1].

In this context, cavity QED with circular Rydberg atoms and superconducting cavities presents a peculiar interest. The qubits are carried by long lived atomic levels or cavity states. Both the initial and final states of these qubits can be determined precisely. The resonant atom-cavity interaction, resulting in an energy exchange between the atom and the field, provides a direct mechanism to entangle the atomic and the cavity states [6] or to realize an atom-cavity quantum gate [1]. Using successive interactions of a series of atoms with the same cavity mode, we have tailored various entangled states such as EPR pairs [10] and GHZ triplets of entangled particles [11]. In these experiments, the quantum information is transiently stored in the cavity mode. The final fidelity is thus limited by the cavity losses, which are the main cause of decoherence.

We have recently demonstrated an alternative approach to quantum entanglement generation in cavity QED [2,13]. Two atoms directly interact with each other through a van der Waals interaction, assisted by the cavity mode. The entanglement dynamics only involves the virtual exchange of a photon with the field. To first order, the scheme is insensitive to cavity losses or to the presence of a stray thermal field in the mode. This new type of quantum gate opens interesting perspectives for quantum information processing in the cavity QED context. We show here that a simple extension of this experiment can be used to implement the two-qubit Grover search algorithm, with a high fidelity. This is the first proposal, to our knowledge, of implementation of this search algorithm in CQED. Note that cavity QED implementation of another algorithm has been independently proposed [6].

Let us recall briefly the main features of Grover’s search algorithm [4]. The goal is to find one item among $N$, which are stored in an unsorted database. The database can be accessed by an “oracle”, a “Black box” comparing any item with the searched one. It gives the “Yes” answer when the items match, “No” otherwise. The most efficient classical algorithm is to examine items one by one until the blackbox returns “Yes”. On average, $N/2$ inquires are necessary. In the quantum search algorithm [4], multiple items are simultaneously examined using a superposition of the corresponding states. The quantum search for the marked item requires only $O(\sqrt{N})$ inquires.

More precisely, the database items are represented by a quantum register with $n$ qubits, having $N = 2^n$ possible states, $|0\rangle = |00 \cdots 0\rangle$, $|1\rangle = |00 \cdots 1\rangle$, $|2^n - 1\rangle = |111 \cdots 1\rangle$. Let us assume that the marked item corresponds to state $|\tau\rangle$ and that the “Yes”/“No” answer from this blackbox is coded in a $\pi/0$ phase shift. The transformation performed by the oracle is thus $I_\tau = I - 2 |\tau\rangle \langle \tau|$, where $I$ is the $2^n \times 2^n$ identity matrix on the quantum register. Note that $I_\tau$ amounts to a conditional phase operation. The algorithm consists in a repetition of the

*Electronic address: yamaguchi@stanford.edu
transformation:

\[ Q = HI_0 HI_\tau , \]  

(time proceeds from right to left), where \( I_0 = I - 2 |0\rangle \langle 0 | \) and \( H = \prod_i H_i \) is the product of Hadamard gates acting on the \( i \)-th qubit. \( H_i \) transforms each qubit as

\[ \begin{align*}
H_i : & \begin{cases} 
|0\rangle_i \rightarrow \frac{1}{\sqrt{2}} (|0\rangle_i + |1\rangle_i) \\
|1\rangle_i \rightarrow \frac{1}{\sqrt{2}} (|0\rangle_i - |1\rangle_i)
\end{cases}
\]  

The sequence of \( Q \) operations acts on a state prepared initially in \( |\Psi\rangle = \prod_i \psi_i \rangle \), where \( \psi_i = H_i |0\rangle_i \). The initial state \(|\Psi\rangle\) is a superposition of all computational states with equal amplitudes, \( \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |i\rangle \). All states are then examined simultaneously by the blackbox, \( I_\tau \). Only state \(|\tau\rangle\) gains minus sign and the resulting register state is \( \frac{1}{\sqrt{N}} \left( \sum_{i \neq \tau} |i\rangle - |\tau\rangle \right) \). The operators \( HI_0 H \) finally perform an “inversion about the average” operation, which increases the probability amplitude of \(|\tau\rangle\). After \( O(\sqrt{N}) \) iterations of this elementary transformation, the probability to get the register in state \(|\tau\rangle\) is maximum and of the order of unity. A simple read-out of the register provides thus the searched item.

In the simple case case of two qubits \((n = 2)\), on which we will focus from now on, there are only 4 items, \(|00\rangle, |01\rangle, |10\rangle\) and \(|11\rangle\). After the first oracle operation, the average of the amplitudes of the two states, \( \frac{1}{2} \) for \( i \neq \tau \) and \( -\frac{1}{2} \) for \( i = \tau \), is \( \frac{1}{2} \), and the inversion about \( \frac{1}{2} \) leads to the amplitudes 0 for \( i \neq \tau \) and 1 for \( i = \tau \). Grover’s search thus requires only one transformation \( Q \) and one inquiry of the blackbox (note that the classical search requires in this case two inquiries on the average).

The \( H_i \) transformations are single qubits gates, easily performed in any physical implementation. The most critical part of the algorithm is the realization of the \( I_\tau \) and \( I_0 \) transformations, which produce an entangled state of the two qubits. We show now that the Grover operation \( Q \) reduces to single qubit gates and to two applications of the quantum phase gate \( I_{QPG} \); defined by the unitary matrix:

\[ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = I_{QPG} \]  

The operators \( I_\tau \) can be obtained by adding rotations of qubits 1 and 2 about \( z \)-axis of \( \theta_1 \) and \( \theta_2 \), respectively, to \( I_{QPG} \):

\[ Z_1(\theta_1)Z_2(\theta_2)I_{QPG} = I_{QPG}Z_1(\theta_1)Z_2(\theta_2) \]

\[ = \begin{pmatrix}
\cos \frac{\theta_1 + \theta_2}{2} & 0 & 0 & 0 \\
0 & \cos \frac{\theta_1 - \theta_2}{2} & 0 & 0 \\
0 & 0 & \cos \frac{\theta_1 - \theta_2}{2} & 0 \\
0 & 0 & 0 & -e^{\frac{i}{2}(\theta_1 + \theta_2)}
\end{pmatrix} \]

The relevant rotation angles \( \theta_1 \) and \( \theta_2 \) are \((\pi, \pi), (0, \pi), (\pi, 0) \) and \((0, 0)\) for implementing \( I_\tau \) up to a global phase \(|\tau\rangle = |00\rangle, |01\rangle, |10\rangle, |11\rangle\), respectively.

The sequence described by Eq.(1) for 2-bit Grover’s algorithm can be decomposed as

\[ [H_1 Z_1(\pi)] [H_2 Z_2(\pi)] I_{QPG} H I_{QPG} [Z_1(\theta_1)H_1] [Z_2(\theta_2)H_2] . \]  

It can be further simplified by the relationship,

\[ Z_i(\pm \theta) = H_i X_i(\mp \theta) H_i , \]  

where \( X_i(\theta) \) corresponds to the unitary transformation,

\[ |0\rangle \rightarrow \cos \frac{\theta}{2} |0\rangle + i \sin \frac{\theta}{2} |1\rangle , \]

\[ |1\rangle \rightarrow i \sin \frac{\theta}{2} |0\rangle + \cos \frac{\theta}{2} |1\rangle . \]

Using \( H_i^2 = 1 \), the final transformation performing the whole Grover search writes thus:

\[ SI_{QPG} HI_{QPG} P. \]  

In Eq.(6), \( S \) and \( P \) are abbreviations for the operations \( S_1 S_2 \) and \( P_1(\theta_1)P_2(\theta_2) \), respectively. The operation \( S_j \) is defined as \( X_j(-\pi)H_j \) and writes explicitly,

\[ S_j : \begin{cases} |0\rangle \rightarrow \frac{1}{\sqrt{2}} (-|0\rangle - |1\rangle) \\
|1\rangle \rightarrow \frac{1}{\sqrt{2}} (-|0\rangle - |1\rangle)
\end{cases} \]

The transformations \( P_j(\theta) = H_j X_j(-\theta) \) can be written also in a compact form as

\[ P_j(\theta) : \begin{cases} |0\rangle \rightarrow \frac{1}{\sqrt{2}} (e^{-\frac{i\theta}{2}} |0\rangle + e^{\frac{i\theta}{2}} |1\rangle) \\
|1\rangle \rightarrow \frac{1}{\sqrt{2}} (e^{-\frac{i\theta}{2}} |0\rangle - e^{\frac{i\theta}{2}} |1\rangle)
\end{cases} \]

We have thus finally expressed the Grover search as a simple sequence of single qubits rotations interrupted by two quantum phase gate operations only.

The scheme of the proposed implementation is displayed on figure 3. The cavity \( C \) is a Fabry-Perot resonator sustaining a resonant mode with a gaussian transverse geometry and a standing wave pattern along the cavity axis. Two atomic beams effusing from the same oven \( O \) cross the superconducting millimeter-wave cavity \( C \) at two separate antinodes in the mode standing wave. Two atoms \( A_1 \) and \( A_2 \), carrying qubits 1 and 2, are simultaneously prepared in box \( B \) in each atomic beam into a high lying circular Rydberg state. The relevant atomic levels \(|e_1\rangle, |g_1\rangle, |i_1\rangle\) are shown in Fig.3. The logical states 1 and 0 of qubit 1 are represented by \(|e_1\rangle\) and \(|g_1\rangle\) states of \( A_1 \) respectively. Qubit 2 uses instead as logical levels 1 and 0 states \(|i_2\rangle\) and \(|g_2\rangle\) of \( A_2 \) respectively (this choice is imposed by the quantum phase gate operation). Both atoms have the same velocity \( v \). They interact together with the cavity mode and are finally detected separately in the state-selective field ionization detectors \( D_1 \) and \( D_2 \).
While they cross the cavity mode, the atoms undergo single qubits rotations (transformations $P$, $H$ and $S$). They are produced by external classical microwave sources resonant on the $e \rightarrow g$ transition for $A_1$ and $g \rightarrow i$ transition for $A_2$. The amplitude and phase of these sources are carefully tuned to produce the required transformations. It is important that the microwave used for atom $A_1$ does not affect atom $A_2$ by mixing the $g$ and $e$ levels. In the same way, the microwave used for $A_2$ must not interact with $A_1$. It is thus essential that the atomic transitions have slightly different frequencies for the two atoms. We plan to use a set of electrodes creating in the cavity an inhomogeneous electric field, used to tune the atomic transitions through the Stark effect. Since the two atoms experience different fields, their frequencies can be controlled independently. Separate interactions with the two classical microwave sources can then be tailored.

In between these single qubit rotations, the atoms experience two dispersive interactions with the cavity mode providing, through the cavity-assisted van der Waals collision, the quantum phase gate (QPG) between the two qubits.

Let us now describe in more details the cavity assisted van der Waals collision, the quantum phase gate $I_{QPG}$ dynamics. For these interactions, the inhomogeneous component of the cavity field is suppressed. A voltage applied across the cavity mirrors provide an homogeneous field used to tune the common atomic frequencies at the proper value.

Let us now describe in more details the cavity assisted van der Waals collision between $A_1$ and $A_2$. The cavity field is assumed to be in its vacuum state at the beginning of the two-atom interaction. In a dispersive regime, where the detuning $\delta$ between the $e \rightarrow g$ atomic transition frequency $\omega_0$ and the cavity frequency $\omega$ is much greater than the atom-cavity coupling $\Omega$ ($\delta \gg \Omega$), the Hamiltonian for the two-atom system can be approximated by the effective expression

$$H_{eff} = \lambda \left[ \sum_{j=1,2} |e_j\rangle \langle e_j| + (S_1^+ S_2^- + S_1^- S_2^+) \right],$$

where $\lambda = \Omega^2/4\delta$, $S_j^+ = |e_j\rangle \langle g_j|$ and $S_j^- = |g_j\rangle \langle e_j|$. The first sum in $H_{eff}$ describes the cavity Lamb shift (or vacuum light shift) experienced by the two atoms. The second term describes the energy exchange between the atoms mediated by the cavity field.

When the interaction time $t$ is chosen so that $\lambda t = \pi$, the two-atom system undergoes the transition,

$$|g_1\rangle |g_2\rangle \rightarrow |g_1\rangle |g_2\rangle,$$
$$|g_1\rangle |i_2\rangle \rightarrow |g_1\rangle |i_2\rangle,$$
$$|e_1\rangle |g_2\rangle \rightarrow |e_1\rangle |g_2\rangle,$$
$$|e_1\rangle |i_2\rangle \rightarrow - |e_1\rangle |i_2\rangle.$$

This transformation corresponds to a conditional quantum phase gate (QPG) between the two qubits.

Let us discuss now the practical feasibility of this experiment. The coupling of the atoms to the cavity field is $\Omega/2\pi = 50$ kHz [13]. In order to get a good entanglement in the cavity enhanced collision, the detuning $\delta$ should be much larger than $\Omega$. We choose here $\delta/2\pi = 4\Omega$. This setting matches the dispersive regime requirement and provides a gate operation well described by $H_{eff}$. With this setting, the atom-cavity interaction time should be $2.5 \times 10^{-4}$ s to allow for two QPG gates operation. The time needed for the single qubit rotations is negligible at this scale. The velocity of the atoms should thus be of the order of $40$ m/s. This value is within reach of simple atomic beam techniques with transverse laser cooling to increase the density of slow atoms. The total interaction time with the mode is thus $120$ $\mu$s, short compared to the photon lifetime, $1$ ms in the present cavity [13].

We have performed a numerical simulation of the experiment to estimate the achievable fidelity. We used for that the exact hamiltonian describing the non resonant atom field coupling, in order to test the validity of the effective hamiltonian (11) approximation. Since the field in the cavity is only virtually populated, dissipation effects were not considered.

As is shown in Fig. 3(a) an efficiency of $\approx 94\%$ is achieved. The limited fidelity is due to that the real hamiltonian does not exactly produce the $I_{QPG}$ realized by its approximation $H_{eff}$. We also considered in the simulation the possibility of imperfection in the classical pulses duration. Their role is to decrease the probability of obtaining the searched state at the end of the process. By considering pulse imperfections of the order of $5\%$ in the simulations, we observed that the efficiency of our search decreased to $85\%$, as shown in Fig. 3(b). The dependency of the efficiency of our experimental proposal on pulse imperfections can be seen in Fig. 3(c), where the fidelity of the final state is plotted against the error in the classical pulses applied in the atoms.

We proposed here a simple implementation of the two qubit Grover search algorithm. It can be realized with minor amendments of our Rydberg atom-cavity setup. It explicitly makes use of qubit entanglement. Its experimental implementation will be an illustration of the power of cavity QED to manipulate complex entangled states for quantum information processing.

This work was supported by Japan Science and Technology Corporation and European Community (International Cooperative Research Project, “Quantum Entanglement”). Laboratoire Kastler Brossel, Université Pierre et Marie Curie and ENS, is associated with CNRS (UMR 8552).

[1] D. Deutsch, Proc. R. Soc. Lond. A 400, 97 (1985); D. Deutsch, and R. Jozsa, Proc. R. Soc. Lond. A 439, 553 (1992).
[2] C. H. Bennett, and G. Brassard, in Proceedings of IEEE International Conference on Computers, Systems, and Signal Processing, Bangalore, India (IEEE, New York,
FIG. 1. Experimental apparatus. Atoms $A_1$ and $A_2$ cross the cavity with same velocity $v$ but at different positions, allowing for individual manipulation of each one in the regions corresponding to pulses $H$, $S$ and $P$. The phase gates $(\mathcal{I}_{QP\mathcal{G}})$ are applied between each pair of classical pulse as shown in the figure. In the inset, the atomic level scheme with the corresponding frequencies.

FIG. 2. Results of the numerical simulation of the proposed experiment (a) Fidelity for the case where the item searched corresponds to state $|ei\rangle$. The state is obtained with a probability of $\approx 94\%$. (b) Fidelity when imperfections in the classical pulse are considered. The resulting state, for the case in which there is a 5% error in the classical pulses duration, coincides with the desired one with an $\approx 85\%$ probability. (c) Dependance of the fidelity on the pulse imperfections. In real experiments this imperfection can be of the order of 3\%.
(a) Probability distribution for different states: eg, ei, gg, gi.

(b) 5% pulse error distribution across the same states.

(c) Fidelity decrease with increasing pulse imperfections.