Regularization of the Hamiltonian constraint and the closure of the constraint algebra

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March 24, 2022

Abstract

In the paper we discuss the process of regularization of the Hamiltonian constraint in the Ashtekar approach to quantizing gravity. We show in detail the calculation of the action of the regulated Hamiltonian constraint on Wilson loops. An important issue considered in the paper is the closure of the constraint algebra. The main result we obtain is that the Poisson bracket between the regulated Hamiltonian constraint and the Diffeomorphism constraint is equal to a sum of regulated Hamiltonian constraints with appropriately redefined regulating functions.

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1 Introduction

The recent progress, made toward quantizing gravity has been mostly due to
the introduction of a new set of canonically conjugate variables by Ashtekar
[1]. In terms of the Ashtekar variables the gravitational constraints in the
canonical approach have acquired simple polynomial form. It has allowed
for different solutions to the quantum gravitational constraints to be inves-
tigated in the connection representation [2], and in the loop representation
[3]. Successful steps have been performed also toward incorporating scalar
[4] and fermion [5] matter fields in the theory. Regardless of the success of
the Ashtekar approach there are still a lot of difficulties in giving precise
physical meaning to the obtained results. An important problem in the the-
ory is the regularization of the products of distributional quantities involved
in the calculations. The need for regularization in the Ashtekar approach
was encountered by Jacobson and Smolin in their paper on the connection
representation [2]. Since then it has attracted a lot of attention. Thorough
discussions of the regularization procedure is given in [6] and [7]. The regu-
larization has been applied in [8], [9], [10] when the action of the Hamiltonian
constraint on the wave functions of the theory has been investigated. In [11]
detailed discussion has been given to the result from regularization for the
action of the local operator representing the metric. In [4] the regularization
has played major role for the successful inclusion of the matter fields in the
theory. Recently in [12] the authors have spelled in details the calculations
of the constraints algebra purely in the loop representation.

In the present paper we are concerned mostly with a precise definiton
of the regularization procedure and with consequences of regularization for
the coordinate invariance and the constraints closure. To avoid unnecessary
complications we work in the more conventional connection representation.
The content of the paper is as follows: In section 2. we justify the necessity
for regularization considering the action of the unregularized gravitational
constraints on the wave functions of the theory. After that in section 3. we
introduce the regularization procedure and calculate in detail the action of
the regulated Hamiltonian constraint. Despite the fact that such type of
calculations have been performed by Blencowe [8] in the loop representation,
we have been encouraged by the results obtained in [4] to investigate again the
apparent background dependance. As we will see an undesired imprint does
indeed survive in the contribution from smooth loops and loops with kinks
but, confirming the result from [4], we show that the background dependence drops completely from the contribution from self-intersections. In section 4, we investigate the constraints algebra at classical and quantum level, paying close attention to the problem for the algebra closure. On contrary to some previous results [9] we show that the Poisson bracket between the regulated Hamiltonian and the Diffeomorphism constraints equals a sum of regulated Hamiltonian constraints. We conclude with discussion of some open questions. Some technical details of the computations are included in appendices.

2 Ashtekar variables, Wilson loops, and necessity for regularization

The classical phase space of canonical quantum gravity a la Ashtekar consists of a configuration variable \( A_i^a \), which is a complex \( SU(2) \) connection on the spatial manifold, and its conjugate momentum \( \tilde{E}_i^a \), a triad with density weight one. (As usual: \( a, b, \ldots \) are spatial indices; \( i, j, \ldots \) are internal indices; each tilde denotes density weight one.) In terms of Ashtekar variables the constraints of the theory have the form:

\[
\tilde{G}_i = D_a \tilde{E}_i^a \simeq 0 \\
\tilde{V}_a = F_{ab}^i \tilde{E}_i^b \simeq 0 \\
\tilde{H} = \frac{1}{2} \epsilon_{ijk} F_{ab}^k \tilde{E}_a^{ai} \tilde{E}_b^{bj} \simeq 0,
\]

where \( D_a \) and \( F_{ab}^i \) are correspondingly the covariant derivative and the curvature defined with the Ashtekar connection \( A_i^a \). The constraints are the so called Gauge constraint, Vector constraint and Hamiltonian constraint. The first constraint generates rotations in the internal space, reflecting the freedom of choosing different \( \tilde{E}_i^a \)-s representing the same 3-metric. The second constraint modulo the first one generates spatial diffeomorphism transformations. The Hamiltonian constraint governs the evolution of the system under consideration from one space slice into another. As usual we will smear out the constraints with some arbitrary appropriately densitized fields:

\[
\mathcal{G}(\tilde{N}) = \int d^3x \tilde{N}_i(x) D_a \tilde{E}_i^a \simeq 0 \tag{1}
\]
\[ \mathcal{V}(\vec{N}) = \int d^3x N^a(x) F_{ab}^i(x) \bar{E}^b_i(x) \cong 0 \quad (2) \]

\[ \mathcal{H}(\vec{N}) = \int d^3x \vec{N}(x) \epsilon_{ijk} F_{ab}^k(x) \bar{E}^{ai}(x) \bar{E}^{bj}(x) \cong 0. \quad (3) \]

We also will write in explicit form the constraint \( \mathcal{C}(\vec{N}) \) which generates spatial diffeomorphism transformations:

\[ \mathcal{C}(\vec{N}) = \mathcal{V}(\vec{N}) - G(N^a \vec{A}_a) = \int d^3x N^a(x) [F_{ab}^i(x) \bar{E}^b_i(x) - A^i_a \partial_b \bar{E}^b_i] \cong 0. \]

The Poisson bracket of this constraint with any function \( f(A, E) \) of the canonical variables gives:

\[ \{ \mathcal{C}(\vec{N}), f(A, E) \} = \mathcal{L}_{\vec{N}} f(A, E), \quad (4) \]

where \( \mathcal{L}_{\vec{N}} \) is the Lie derivative along the field \( \vec{N} \).

The classical constraint algebra is closed, that is the Poisson brackets between constraints give as a result combinations of the constraints themselves. The exact expressions are:

\[ \{ \mathcal{G}(\vec{M}), \mathcal{G}(\vec{N}) \} = i \mathcal{G}(\vec{M} \times \vec{N}), \]

\[ \{ \mathcal{C}(\vec{N}), \mathcal{G}(\vec{M}) \} = \mathcal{G}(\mathcal{L}_{\vec{N}} \vec{M}), \]

\[ \{ \mathcal{G}(\vec{M}), \mathcal{H}(\vec{N}) \} = 0, \]

\[ \{ \mathcal{C}(\vec{N}), \mathcal{C}(\vec{M}) \} = \mathcal{C}(\mathcal{L}_{\vec{N}} \vec{M}), \]

\[ \{ \mathcal{H}(\vec{N}), \mathcal{C}(\vec{M}) \} = \mathcal{C}(\mathcal{L}_{\vec{N}} \vec{M}), \]

\[ \{ \mathcal{H}(\vec{M}), \mathcal{C}(\vec{N}) \} = -\mathcal{H}(\mathcal{L}_{\vec{N}} \vec{N}), \quad (5) \]

and

\[ \{ \mathcal{H}(\vec{N}), \mathcal{H}(\vec{M}) \} = \mathcal{C}(\vec{K}) + \mathcal{G}(K^a \vec{A}_a) = \mathcal{V}(\vec{K}) \quad (6) \]

with

\[ K^a(x) = \bar{E}^{ai}(x) \bar{E}^c_j(x) [\vec{M}(x) \partial_c \vec{N}(x) - \vec{N}(x) \partial_c \vec{M}(x)]. \quad (7) \]
In the process of quantization, the canonical variables become operators of multiplication and differentiation and the Poisson brackets become commutators. We also have to choose a particular factor ordering for expressions containing operator products (see [8] for discussion of the problem). In the constraints we will put all the “E”-s to the right of the “A”-s. The reason we choose such a factor ordering is the fact that with this ordering a set of solutions to all quantum constraints has been found. The solution obtained is given by regular functionals of knot and link classes of smooth, non-intersecting loops. However there is a problem with the chosen factor ordering: The Poisson bracket of two Hamiltonian constraints is non-trivial - on the right hand side we still have a combination of the constraints but the coefficients are not constants. They are functions of the basic variables. With the chosen factor ordering both “E”-s from (7) will appear to the right of the Vector constraint in (6) and this prevents the quantum constraint algebra from closing.

In the proposed paper we will work in the more conventional connection representation in which the wave functionals \( \Psi(A) \) depend on the Ashtekar connection. The loop and the connection representations are connected via the (not yet fully justified, see [13]) loop transform:

\[
\Psi(\gamma) = \int \mathrm{“}d\mu[A]\mathrm{”} W_\gamma(A) \Psi(A).
\]

In this Fourier-like transformation \( \mathrm{“}d\mu[A]\mathrm{”} \) is an unknown measure on the space of connections modulo gauge transformations \( A/\mathcal{G} \). The kernel of the transformation \( W_\gamma(A) \) is an Wilson loop. The Wilson loops form an infinite basis of gauge invariant functionals, parametrized by a loop \( \gamma \). They are defined by the trace of the path ordered exponential of the line integral of the connection along the loop \( \gamma \):

\[
W_\gamma(A) = \text{Tr} U(0,1) = \text{Tr} \left( \mathcal{P} \exp \oint ds \gamma^a(s) A^i_a(\gamma(s)) \tau_i \right),
\]

where

\[
U(s_1, s_2) = \text{Tr} \left( \mathcal{P} \exp \int_{s_1}^{s_2} ds \gamma^a(s) A_a(\gamma(s)) \right)
\]

is the holonomy of \( A \) along the loop \( \gamma \) and \( \tau_i = -\frac{i}{2} \sigma_i \); \( \sigma_i \) are the Pauli matrices. Because of the loop transform the investigation of the action of
constraints on the wave functionals in the loop representation is equivalent
to considering the corresponding action in the connection representation on
the Wilson loops.

By their construction as a trace of holonomy, the Wilson loops are auto-
matically gauge invariant. The action of the Diffeomorphism constraint on
the Wilson loop is given by:

\[
\hat{C}(\vec{N})W_\gamma(A) = \int d^3x N^a(x)F^i_{ab}(x) \oint ds \delta^3(\vec{x}, \gamma(s)) \dot{\gamma}^b(s) \text{Tr}[U(0, s)\tau_i U(s, 1)] = \\
= \oint ds \dot{\gamma}^b(s) N^a(\gamma(s)) F^i_{ab}(\gamma(s)) \text{Tr}[U(0, s)\tau_i U(s, 1)].
\]

Similarly we will get for the action of the Hamiltonian constraint:

\[
\hat{H}(\mathcal{M})W_\gamma(A) = \\
= \int d^3x \oint ds \oint dt \dot{\gamma}^a(s) \dot{\gamma}^b(t) M(x) \epsilon_{ijk} F^b_{ab}(x) \delta^3(\vec{x}, \gamma(s)) \delta^3(\vec{x}, \gamma(t)) L^{ij}(s, t; \gamma),
\]

where with \(L^{ij}(s, t; \gamma)\) we have denoted the “loop deformation” - the result
of the action of the Hamiltonian constraint on the loop itself. This action
amounts to breaking the loop and inserting Pauli matrices at certain points.

The analytic expression for \(L^{ij}(s, t; \gamma)\) is:

\[
L^{ij}(s, t; \gamma) = \text{Tr}[\partial(s - t)U(0, t)\tau^i U(t, s)\tau^j U(s, 1) + \\
+ \partial(t - s)U(0, s)\tau^i U(s, t)\tau^j U(t, 1)].
\]

Expression (10) above is ill defined and requires regularization. The problem arises from the fact that it contains two spatial \(\delta\)-functions integrated
in a 5-fold integral. In the next section we will consider the point-splitting
regularization as a possible way for solving the problem. This procedure will
make the action of \(\hat{H}(\mathcal{M})\) well defined and after appropriate renormaliza-
tion - finite. Also we will investigate the consequences of the regularization
procedure for the closure of the constraint algebra.
3 Regulated Hamiltonian constraint and its action on Wilson loops

3.1 How the regulator should look like?

In the process of regularization we use the following expression for \( H(\sim M) \):

\[
H_\epsilon(\sim M) = \int d^3 x \epsilon_{ijk} \sim M(x) F_{ab}^k(x) \int d^3 y f_\epsilon(\bar{x}, y) \bar{E}^{ai}(y) \int d^3 z f_\epsilon(\bar{x}, z) \bar{E}^{bj}(z).
\] (12)

In this expression \( f_\epsilon(\bar{x}, y) \) is a regulating function depending on a continuous parameter \( \epsilon \). Here the tilde over \( x \) means that \( f_\epsilon(\bar{x}, y) \) is a density with respect to its first argument. In [4] Smolin and Rovelli have used regularization with such symmetric point-splitting in the loop representation and their result of the action of the regulated Hamiltonian constraint is background independent.

The regulating function satisfies the requirement that for any smooth function \( \phi(x) \):

\[
\lim_{\epsilon \to 0} \int d^3 x \phi(x) f_\epsilon(\bar{x}, y) = \phi(y).
\] (13)

Particular examples of such functions are a normalized, weighted \( \vartheta \)-function:

\[
f_\epsilon(\bar{x}, y) = \sqrt{h(x)} f_\epsilon(x, y) = (3/4\pi\epsilon^3)^{\epsilon} \sqrt{h(x)} \vartheta[\epsilon - |\bar{x} - \bar{y}|]
\] (14)

or Gaussian function

\[
f_\epsilon(\bar{x}, y) = (\epsilon \sqrt{\pi})^{-3}\sqrt{h(x)} \exp[-\frac{|\bar{x} - \bar{y}|^2}{\epsilon^2}].
\]

In the above expressions \( h(x) \) is the determinant of the (arbitrary, i.e. Euclidian) background metric \( h^{ab}(x) \). This metric is used also in the definition of the distance \( |\bar{x} - \bar{y}| \). The process of regularization amounts to performing all of the calculations in the action of \( H_\epsilon(\sim M) \) on a Wilson loop and then taking the limit \( \epsilon \to 0 \). As we will see the problem of multiplying distributional quantities reduces to the emergence of a single pole in \( \epsilon \). Thus by renormalizing this last expression we will get a well defined, finite result.
Using (12) we get for the action of the regulated Hamiltonian constraint on the Wilson loop:

\[
\hat{H}_\epsilon^{\mathcal{M}} W_\gamma(A) = \int d^3x \oint ds \oint dt \dot{\gamma}^a(s) \dot{\gamma}^b(t) M(x) \epsilon_{ijk} F_{ab}^k(x) \\
\int d^3y \int d^3z \delta^3(\bar{y}, \gamma(s)) \delta^3(\bar{z}, \gamma(t)) f_\epsilon(\bar{x}, y) f_\epsilon(\bar{x}, z) L^{ij}(s, t; \gamma)
\]

\[
= \int d^3x \oint ds \oint dt \dot{\gamma}^a(s) \dot{\gamma}^b(t) M(x) \epsilon_{ijk} F_{ab}^k(x) f_\epsilon(\bar{x}, \gamma(s)) f_\epsilon(\bar{x}, \gamma(t)) L^{ij}(s, t; \gamma),
\]

where \( L^{ij}(s, t; \gamma) \) is defined with (11). We will use the \( \vartheta \)-function as a regulator in our calculations. Combining (14) with (15) we get:

\[
\hat{H}_\epsilon^{\mathcal{M}} W_\gamma(A) = \frac{9}{16\pi^2 \epsilon^6} \int ds \oint dt \dot{\gamma}^a(s) \dot{\gamma}^b(t) L^{ij}(s, t; \gamma) \\
\int d^3x M(x) \epsilon_{ijk} F_{ab}^k(x) h(x) \vartheta[|\epsilon - |\bar{x} - \gamma(s)||] \vartheta[|\epsilon - |\bar{x} - \gamma(t)||].
\]

Because of the \( \vartheta \)-functions the last integral is non-vanishing only for values of \( x \) in the region which is the intersection of the spheres with radii \( \epsilon \) and centers \( \gamma(s) \) and \( \gamma(t) \) correspondingly. The calculations from now on will depend on the type of the loop which parametrizes the functional \( W_\gamma(A) \). In this paper we will consider single smooth loops, loops with self-intersections and loops with kinks.

### 3.2 Smooth portions of loops

We will start with the calculation of the contribution from smooth portions of the loop \( \gamma \). On Figure 1 we have shown single smooth loop but our calculations will also be valid for cases when the loop \( \gamma \) has kinks or self-intersection.

From Figure 1 we can determine that the distance between the centers of the spheres \( \delta(s, t) \) is at most twice the (small) parameter \( \epsilon \), which enables us to make some approximations. Let us first consider the expansion:

\[
\delta(s, t) = \sum_{n=1}^{\infty} \gamma^{(n)}(s) \frac{(t-s)^n}{n!} \leq |\dot{\gamma}| |t-s| + \sum_{n=2}^{\infty} \gamma^{(n)}(s) \frac{(t-s)^n}{n!} \leq 2\epsilon.
\]
Figure 1: Smooth loop.

From here we can determine the range of one of the parameters along the loop, say $t$ with respect to $s$:

$$t \in \left[ s - \frac{2\epsilon \delta^-}{|\vec{\gamma}|}, s + \frac{2\epsilon \delta^+}{|\vec{\gamma}|} \right],$$

where $\delta^- = 1 + O(\epsilon)$ and $\delta^+ = 1 + O(\epsilon)$. This means that we can fix $s$ in (16) and expand all functions of $t$ about $s$ in powers of $\epsilon$. The first term in the expansion of $\dot{\gamma}^b(t)$ will give us the product:

$$\dot{\gamma}^a(s)\dot{\gamma}^b(s)F_{ab}^k,$$

which, because of the antisymmetry of $F_{ab}^k$, will make the whole integral vanishing. Thus the first non-vanishing term will come from the expansion of:

$$\frac{9}{16\pi^2\epsilon^6} \oint ds \oint dt \dot{\gamma}^a(s)\dot{\gamma}^b(s)(t-s)L^{ij}(s,t;\gamma)$$

$$\int d^3x M(x)\epsilon_{ijk}F_{ab}^k(x)h(x)\vartheta[\epsilon - |\vec{x} - \gamma(s)||\vartheta[\epsilon - |\vec{x} - \gamma(t)||. \quad (17)$$

In the expansion of the holonomies in $L^{ij}(s,t;\gamma)$ we have to keep only terms of zeroth order. Also, because the last integral in (17) is different from zero
only in a region of linear size $\epsilon$ we can replace this integral with its mean value:

$$M(x_0)F_{ab}^k(x_0)\sqrt{h(x_0)} \int d^3x \sqrt{h(x)}\vartheta[\epsilon - |\vec{x} - \gamma(s)||\vartheta[\epsilon - |\vec{x} - \gamma(t)|| =$$

$$M(x_0)F_{ab}^k(x_0)\sqrt{h(x_0)}V(\delta(s, t)),$$

where $V(\delta(s, t))$ is the volume of the intersection region, $\delta(s, t) = |\gamma(s) - \gamma(t)|$ and $x_0$ is a point in close vicinity to $\gamma(s)$ and $\gamma(t)$. The volume of the intersection of two spheres with radii $\epsilon$ and distance between their centers $\delta$ is equal to twice the volume cut from a sphere by a plane passing at a distance $\delta/2$ from the center of the sphere. The volume of the intersection is given by the expression:

$$V(\delta) = \frac{2}{3}\pi \epsilon^3 \left(2 - \frac{3\delta}{2\epsilon} + \left(\frac{\delta}{2\epsilon}\right)^3\right). \tag{18}$$

Because of the reparametrization invariance of the integrals in $s$ and $t$ we can use a parametrization in which $|\dot{\gamma}| = 1$. After expanding and performing the integration with respect to $t$ we get to the lowest order in $\epsilon$ (see Appendix 1):

$$\hat{H}_\epsilon(M)W_\gamma(A) =$$

$$\frac{3}{2\pi\epsilon}((\delta^+)^2 + (\delta^-)^2) \int ds \gamma^a(s)\dot{\gamma}^b(s)M(\gamma(s))F_{ab}^k(\gamma(s))\text{Tr}[U(0, s)]\tau^kU(s, 1)],$$

where we have written $M(\gamma(s))\sqrt{\gamma(s)}$ as a scalar function $M(\gamma(s))$. Here we see that simply by multiplying with $\epsilon$ and performing the limit $\epsilon \to 0$ we get a finite, well defined result:

$$\lim_{\epsilon \to 0} \epsilon \hat{H}_\epsilon(M)W_\gamma^{\text{smooth}}(A) =$$

$$\frac{3Z}{2\pi} \int ds \gamma^a(s)\dot{\gamma}^b(s)M(\gamma(s))F_{ab}^k(\gamma(s))\text{Tr}[U(0, s)]\tau^kU(s, 1)], \tag{19}$$
where $Z$ is an arbitrary renormalization constant. This result can be easily generalized for the case of a loop $\gamma$ with kinks and/or self-intersections - the closed integral in (19) should be replaced by a sum of integrals along smooth portions of the loop.

The result we obtained is different in its detail from the results obtained previously ([2], [9]) because of the different regularization schemes used, but in its general features it is similar. Unfortunately our result faces the same problem encountered before - it is background dependent because of the presence of the “acceleration” term $\ddot{\gamma}(s)$ (see [8] and [11] for discussion).

### 3.3 Contribution from self-intersections

The second contribution in the action of the Hamiltonian constraint on an Wilson loop will come from intersections. Let us consider a single self-intersecting loop. Let $\vec{\eta}^+$ and $\vec{\eta}^-$ be the unit tangent vectors to the loop at the intersection:

Here we will have two separate cases - one with $\dot{\gamma}(s_0) \equiv \vec{\eta}^+$ and $\dot{\gamma}(t_0) \equiv \vec{\eta}^-$, and another one with $s$ and $t$ replaced. In the above relations $s_0$ and $t_0$ are the values of the parameters along the loop, corresponding to the intersecting point. Again we use a parametrization in which $|\vec{\gamma}(s_0)| = \dot{\gamma}(t_0) = 1$. Keeping in mind the limiting procedure we are performing, we can write the result from the action of the Hamiltonian constraint on the intersecting portion of the loop as:
\[ \hat{H}_e(\mathcal{M})W^{\text{int}}_{\gamma}(A) = \frac{9}{16\pi^2}e^{\theta}M(\gamma(s_0))\epsilon_{ijk}F_{ab}^k(\gamma(s_0))\sqrt{h(\gamma(s_0))} \]

\[ \oint ds \oint dt \hat{\gamma}^a(s)\hat{\gamma}^b(t)L_{ij}(s,t;\gamma) \int d^3x \sqrt{h(x)}\vartheta[\epsilon - |\vec{x} - \gamma(s)|]\vartheta[\epsilon - |\vec{x} - \gamma(t)|]. \]

This last expression can be further simplified by replacing the arguments of the holonomies in \( L_{ij}(s,t;\gamma) \) by the values of the parameters \( s \) and \( t \) corresponding to the intersection.

\[ \hat{H}_e(\mathcal{M})W^{\text{int}}_{\gamma}(A) = \frac{9}{16\pi^2}e^{\theta}M(\gamma(s_0))F_{ab}^k(\gamma(s_0))\sqrt{h(\gamma(s_0))} \]

\[ \{ (\eta^+)^a(\eta^-)^b\text{Tr}[U(0,t_0)\tau^jU(t_0,s_0)\tau^iU(s_0,1)] + \\
+ (\eta^+)^b(\eta^-)^a\text{Tr}[U(0,s_0)\tau^iU(s_0,t_0)\tau^jU(t_0,1)] \} \]

\[ \oint ds \oint dt \int d^3x \sqrt{h(x)}\vartheta[\epsilon - |\vec{x} - \gamma(s)|]\vartheta[\epsilon - |\vec{x} - \gamma(t)|]. \quad (20) \]

To perform the integration we can again make use of (18) and write the last 5-fold integral as:

\[ I(\epsilon, \theta) = \int ds \int dt V(\delta(s,t)), \]

where the limits of integration are to be determined from Figure 3.

After tedious but straightforward calculations (see Appendix 2) we get the following result:

\[ \hat{H}_e(\mathcal{M})W^{\text{int}}_{\gamma}(A) = \frac{69}{20\epsilon \sin \theta}M(\gamma(s_0))F_{ab}^k(\gamma(s_0))\sqrt{h(\gamma(s_0))} \]

\[ (\eta^+)^a(\eta^-)^b\text{Tr}[U(t_0,s_0)\tau^iU(s_0,t_0)\tau^j] + \mathcal{O}(1), \]

where \( \theta \) is the angle between the vectors \( \vec{\eta}^+ \) and \( \vec{\eta}^- \). Again after performing multiplicative renormalization we get:
\[ \lim_{\epsilon \to 0} \epsilon \hat{\mathcal{H}}_\epsilon (M) W^\text{int}_\gamma (A) = \]
\[ = \frac{69 \epsilon_{ijk}}{20 \sin \theta} M^{\text{int}} (F^k_{ab})^{\text{int}} (\eta^+)^a (\eta^-)^b \text{Tr} [U(t_0, s_0) \tau^i U(s_0, t_0) \tau^j], \]  
\text{(21)}

where \( M^{\text{int}} \) and \( (F^k_{ab})^{\text{int}} \) are the values of \( M(x) \) and \( F^k_{ab}(x) \) at the intersection point. To make this result easier to understand let us write:
\[ F^k_{ab} = \frac{1}{2} \epsilon_{abc} B^{ck}, \]

where \( B^{ck} \) is the “magnetic field”. Then we will have:
\[ (\eta^+)^a (\eta^-)^b \epsilon_{abc} B^{ck} = (\vec{\eta}^+ \times \vec{\eta}^-)_c B^{ck} = \hat{n}_c B^{ck} \sin \theta, \]
where \( \hat{n}_c \) is the unit vector, normal to the plane defined by the loop at the intersection. Thus finally we get:
\[ \lim_{\epsilon \to 0} \epsilon \hat{\mathcal{H}}_\epsilon (M) W^\text{int}_\gamma (A) = \frac{69 \epsilon_{ijk} Z}{40} M^{\text{int}} (\hat{n}_c B^{ck})^{\text{intr}} \text{Tr} [U(t_0, s_0) \tau^i U(s_0, t_0) \tau^j]. \]  
\text{(22)}
In this form it is clear that the result is independent from the metric used in the regularization. Thus (up to a numerical factor) we recover the same result like the one obtained in [4] in the loop representation.

### 3.4 Contributions from kinks

The case of a loop with a kink is similar to the one with an intersection. The final result after performing the integration and renormalization is:

$$
\lim_{\epsilon \to 0} \epsilon \hat{H}_\epsilon (\mathcal{M}) W_\gamma^{\text{kink}}(A) = \frac{f(\theta)}{\sin \theta} M^{\text{kink}}(F_\gamma^{\text{kink}}(\eta^+)^a(\eta^-)^b \text{Tr}[U(0, s_0)\tau^k U(s_0, 1)]
$$

where

$$
f(\theta) = \frac{3}{40} \left( 23 \left( 1 - \frac{\theta}{\pi} \right) - \frac{9}{\pi} \sin \theta \cos \theta \right)
$$

Thus when there is a kink on the loop the result is again background dependent. The dependence shows up in the presence of $f(\theta)$ - a function of the angle between the tangent vectors at the kink, for the definition of which we need a background metric.

### 4 The algebra of the constraints

In this section we will present the calculations of the constraint algebra with a Regularized Hamiltonian constraint. Tsamis and Woodard suggest in [3] that a regulating procedure, which is not coordinate invariant should destroy the algebra closure. Surprisingly this is not exactly the case. After appropriate redefinition of the regulators the Hamiltonian and Diffeomorphism constraints do close, which means that the evolution generated by the regulated Hamiltonian constraint is consistent with the requirement for coordinate invariance.

#### 4.1 Hamiltonian with Gauss constraint

We start with the Poisson bracket between the Regularized Hamiltonian and the Gauge constraints:
\[ \{ \mathcal{G}(\vec{N}), \mathcal{H}_\epsilon(M) \} = \]
\[ \frac{1}{2} \int d^3x \epsilon_{ijk} M(x) \int d^3y f_\epsilon(\vec{x}, y) \int d^3z f_\epsilon(\vec{x}, z) \{ \mathcal{G}(\vec{N}), F_{ab}^k(x) \tilde{E}^{ai}(y) \tilde{E}^{bj}(z) \} = \]
\[ \int d^3x M(x) \int d^3y f_\epsilon(\vec{x}, y) \int d^3z f_\epsilon(\vec{x}, z) F_{ab}^k(x) \tilde{E}^{bj}(z) \]
\[ [(N_j(x) - N_j(y)) E^a_k(y) + (N_k(y) - N_k(z)) E^a_j(y)] . \quad (24) \]

In the unregulated case this bracket gives zero. Before considering the obtained result as a problem let us remember that the above calculation is a part of a limiting procedure. Because of the terms \((N_j(x) - N_j(y))\) and \((N_k(y) - N_k(z))\) if we perform the limit \(\epsilon \to 0\) at a classical level, the expression (24) will be of order \(\epsilon\) and will vanish. In the quantum case, which is interesting for us, the result is similar. Both terms in (24) have space indices like in a Hamiltonian constraint but rearranged internal indices. This means that the final result of the action of \(\{ \mathcal{G}(\vec{N}), \mathcal{H}_\epsilon(M) \}\) on Wilson loops will contain different types of braking of the loop \(\gamma\) but it will be \(\mathcal{O}(1)\). Thus we will have:

\[ \lim_{\epsilon \to 0} \epsilon \{ \mathcal{G}(\vec{N}), \mathcal{H}_\epsilon(M) \} W_\gamma(A) = 0. \]

### 4.2 Hamiltonian with Diffeomorphism constraint

To calculate the Poisson bracket between the Hamiltonian and the Diffeomorphism constraints we will use (1) to get:

\[ \{ \mathcal{H}_\epsilon^{IJ}(M), \mathcal{C}(\vec{N}) \} = \]
\[ = \frac{1}{2} \int d^3x \epsilon_{ijk} M(x) \int d^3y f_\epsilon(\vec{x}, y) \int d^3z f_\epsilon(\vec{x}, z) \{ F_{ab}^k(x) \tilde{E}^{ai}(y) \tilde{E}^{bj}(z), \mathcal{C}(\vec{N}) \} = \]
\[ = \frac{1}{2} \int d^3x \epsilon_{ijk} M(x) \int d^3y f_\epsilon(\vec{x}, y) \int d^3z f_\epsilon(\vec{x}, z) [\mathcal{L}_N F_{ab}^k(x) \tilde{E}^{ai}(y) \tilde{E}^{bj}(z) + \]

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\[ + F_{ab}^k(x) \mathcal{L}_N \tilde{E}^{ai}(y) \tilde{E}^{bj}(z) + F_{ab}^k(x) \tilde{E}^{ai}(y) \mathcal{L}_N \tilde{E}^{bj}(z) \]. \tag{25}

In this case it is important for us to use more explicit notation - in \( \mathcal{H}_e^{ff} \) we have shown explicitly the type of regulating functions we have used. As usual we are working on a compact manifold which will allow us to do integration by parts disregarding all boundary terms. Performing such an integration in (25), we get:

\[
\{ \mathcal{H}_e^{ff}(\mathcal{M}, \mathcal{C}(\tilde{N})) \} =

= -\frac{1}{2} \int d^3x \int d^3y \int d^3z \varepsilon_{ijk} F_{ab}^k(x) \tilde{E}^{ai}(y) \tilde{E}^{bj}(z) \{ \mathcal{L}(x) \tilde{N} (\mathcal{M}(x) f_e(\tilde{x}, y) f_e(\tilde{x}, z)) + \mathcal{M}(x) (\mathcal{L}(y) f_e(\tilde{x}, y)) f_e(\tilde{x}, z) + \mathcal{M}(x) f_e(\tilde{x}, y) (\mathcal{L}(z) f_e(\tilde{x}, z)) \}.
\]

After rearranging the terms in the last expression we will get:

\[
\{ \mathcal{H}_e^{ff}(\mathcal{M}, \mathcal{C}(\tilde{N})) \} =

= -\mathcal{H}_e^{ff}(\mathcal{L}_N \mathcal{M}) - \frac{1}{2} \int d^3x \int d^3y \int d^3z \varepsilon_{ijk} \mathcal{M}(x) \partial_e \tilde{N} c(x) F_{ab}^k(x) \tilde{E}^{ai}(y) \tilde{E}^{bj}(z)

f_e(\tilde{x}, z) 2 \left\{ f_e(\tilde{x}, y) + \frac{h(x)}{\partial_a N^a(x)} \left( N^c(x) \frac{\partial}{\partial x^c} f_e(\tilde{x}, y) + N^c(y) \frac{\partial}{\partial y^c} f_e(\tilde{x}, y) \right) \right\}.
\]

This final expression can be written in the form:

\[
\{ \mathcal{H}_e^{ff}(\mathcal{M}, \mathcal{C}(\tilde{N})) \} = -\mathcal{H}_e^{ff}(\mathcal{L}_N \mathcal{M}) + \mathcal{H}_e^{fg}(2 \mathcal{M} \partial_c \tilde{N}^c) - \mathcal{H}_e^{ff}(2 \mathcal{M} \partial_c \tilde{N}^c) \tag{26}
\]

In the second term in the right hand side of (26) the superscript \( g \) stands for the expression:

\[
g_e(\tilde{x}, y) = -\frac{h(x)}{\partial_a N^a(x)} \left( N^c(x) \frac{\partial}{\partial x^c} f_e(\tilde{x}, y) + N^c(y) \frac{\partial}{\partial y^c} f_e(\tilde{x}, y) \right).
\]
It can be shown that the function $g_\epsilon(\tilde{x}, y)$ satisfies the requirement
\[
\lim_{\epsilon \to 0} \int d^3 x \phi(x) g_\epsilon(\tilde{x}, y) = \phi(y).
\] (27)
for any smooth function $\phi(x)$ so it can be used as a regulating function. This means that the Poisson bracket between Hamiltonian and Diffeomorphism constraints gives as a result a sum of three regulated Hamiltonian constraints. Thus the process of regularization changes the unregulated expression (5) in such a way so this change does not destroy the constraints closure. Obviously if we perform in (26) the limit $\epsilon \to 0$ the last two terms we cancel each other and thus we will recover the unregulated result (5). If we proceed and quantize, the expression (26) will transform into its quantum version but the closure will not be affected in the process of quantization and it still will hold.

4.3 Two regulated Hamiltonian constraints

When calculating the Poisson bracket of two regulated Hamiltonian constraints we have to keep in mind the fact that the quantum commutator of two unregulated Hamiltonian constraints does not give combination of constraints. This means that it makes sense for us to work only at the classical level of the theory. Classically the Poisson brackets of two regulated Hamiltonian constraints can be shown to give:

\[
\{\mathcal{H}_\epsilon(\mathcal{M}), \mathcal{H}_\epsilon(\mathcal{N})\} =
\]
\[
\int d^3 x \int d^3 y \int d^3 z \int d^3 x' \int d^3 z' f_\epsilon(\tilde{x}', x) f_\epsilon(\tilde{x}', z') f_\epsilon(\tilde{x}, y) f_\epsilon(\tilde{x}, z)
\]
\[
[M(x')\partial_a N(x) - N(x')\partial_a M(x)]\tilde{E}_n^a(y) \tilde{E}_{dn}^b(z') \tilde{E}_k^b(z) F_{bd}^k(x') +
\]
\[
+ \int d^3 x \int d^3 y \int d^3 z \int d^3 x' \int d^3 z' f_\epsilon(\tilde{x}', x) f_\epsilon(\tilde{x}', z') F_{bd}^k(x') \tilde{E}_{dn}^b(z')
\]
\[
\{[N(x')\partial_a M(x) - M(x')\partial_a N(x)]f_\epsilon(\tilde{x}, y) f_\epsilon(\tilde{x}, z) \tilde{E}_n^a(y) \tilde{E}_k^b(z) +
\]

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\[
+ [\mathcal{N}(x') \mathcal{M}(x) - \mathcal{M}(x') \mathcal{N}(x)] \tilde{E}^a_b(y) \tilde{E}^b_a(z) \frac{\partial}{\partial x^a} (f_{\epsilon}(\bar{x}, y) f_{\epsilon}(\bar{x}, z) + \\
i \epsilon_{kmn} [\mathcal{N}(x') \mathcal{M}(x) - \mathcal{M}(x') \mathcal{N}(x)] f_{\epsilon}(\bar{x}, y) f_{\epsilon}(\bar{x}, z) A_{ap}(x) \tilde{E}^{a[p}_b(y) \tilde{E}^{b]n}(z)].
\]

In the last messy expression the first 5-fold integral can be rearranged in the form:

\[
\int d^3 x \int d^3 x' K^d_{\epsilon}(x', x) \int d^3 z f_{\epsilon}(\bar{x}, z) \tilde{E}^b_k(z) F_{bd}^k(x')
\]

where \( K^d_{\epsilon}(x', x) \) is the smeared version of (7):

\[
K^d_{\epsilon}(x', x) = f_{\epsilon}(\bar{x}', x) [\mathcal{M}(x') \partial_a \mathcal{N}(x) - \mathcal{N}(x') \partial_a \mathcal{M}(x)]
\]

\[
\int d^3 y f_{\epsilon}(\bar{x}, y) \tilde{E}^a_n(y) \int d^3 z' f_{\epsilon}(\bar{x}', z') \tilde{E}^{dn}(z').
\]

It is straightforward to check that classically for any smooth function \( \phi(x) \) \( K^d_{\epsilon}(x', x) \) behaves like a regulating function:

\[
\lim_{\epsilon \to 0} \int d^3 x' K^d_{\epsilon}(x', x) \phi(x') = K^d(x) \phi(x).
\]

If we classically perform the limit \( \epsilon \to 0 \) in (28) as we should expect, it reduces to the unregulated expression (27). The terms in the second 5-fold integral in (28) vanish - the first one because of the antisymmetry of \( F_{bd}^k(x) \), the second and the third one - because of the expression \( [\mathcal{N}(x') \partial_a \mathcal{M}(x) - \mathcal{M}(x') \partial_a \mathcal{N}(x)] \).

But with the presence of \( \epsilon \) we are not able to write (28) as a combination of constraints. Thus in the process of regularization the constraint closure is lost because of the \( \{ \mathcal{H}_\epsilon(\mathcal{M}), \mathcal{H}_\epsilon(\mathcal{N}) \} \) bracket, but this most probably just reflects the fact that the corresponding unregulated expression also prevents the quantum algebra from closing.
5 Conclusions

In conclusion we would like to emphasize again on the main problems we have encountered in our work:

- The first set of problems arises in connection with the requirement for background independence of the action of the regulated Hamiltonian constraint. The result we obtained shows that the arbitrary metric we have used in the calculations survives when the Hamiltonian acts on smooth portions of loops and on loops with kinks. In the case of loops with self-intersections the background dependence drops completely. In the case of smooth portions of loops the troublesome term is the so called “acceleration term”. As pointed out in [8] the “acceleration term” $\dot{\gamma}^b(s)$ is not a tensor quantity and its presence in (19) means that the result depends on the arbitrary metric we have used. This problem has been discussed in [3] where it is shown that smearing of the loops on which the loop operators in the loop representation are based makes the “acceleration term” to vanish. However the corresponding solution in the connection representation requires further investigation.

On the other hand the problem with the loops with kinks probably requires appropriate redefinition of the regularization procedure so the background dependent factor gets absorbed in the process of regularization [14]. To summarize: Even if the smearing of the loop succeeds to remove the background dependence coming from the smooth portions of the loop, we have to face the problem of the imprint of the metric used in the case of a loop with a kink.

- Another, though similar problem is concerned with the constraints closure. The Poisson bracket between two regulated Hamiltonian constraints give as a result an expression which apparently can not be cast into the form of a sum of (regulated ) constraints. It could be the case that the quantum commutator will preserve the constraints closure but there is no obvious reason for this to happen. It is possible that the problem should be faced at an earlier stage - a factor ordering should be sought, which preserves both the constraints closure and the physical meaning of the constraints.
6 Acknowledgments

I am grateful to Jorge Pullin for giving me the idea for this work and for the discussions afterwards. I also thank Plamen Fiziev, Don Neville, Petko Nikolov, Lee Smolin, and Modhavan Varadarajan for comments, suggestions and criticisms.
7 Appendices

7.1 Appendix 1

Here we will show the basic steps in the calculation of (17). We start from:

$$\hat{H}_\epsilon(\mathcal{M}) W_\gamma^\text{smooth}(A) = \frac{9}{16\pi^2\epsilon^6} \int ds \int dt \dot{\gamma}^a(s) \dot{\gamma}^b(s)(t - s)L^{ij}(s, t; \gamma)$$

$$\mathcal{M}(x_0) F^k_{ab}(x_0) \sqrt{h(x_0)} V(\delta(s, t)).$$

First, from the expansion of $L^{ij}(s, t; \gamma)$ we will get:

$$L^{ij}(s, t; \gamma) = \text{Tr}[\vartheta(s - t)U(0, s)\tau^i\tau^jU(s, 1) + \vartheta(t - s)U(0, s)\tau^i\tau^jU(s, 1)] =$$

$$= \frac{1}{2} \epsilon^{ijk} \text{Tr}[\vartheta(s - t)U(0, s)\tau_k U(s, 1) - \vartheta(t - s)U(0, s)\tau_k U(s, 1)].$$

In the function $V(\delta(s, t))$ working to lowest order in $\epsilon$ we can replace $\delta$ with $|t - s|$. Thus the integral reduces to:

$$\hat{H}(\mathcal{M}) W_\gamma^\text{smooth}(A) =$$

$$= \frac{3}{16\pi^3} \int ds \dot{\gamma}^a(s) \dot{\gamma}^b(s) \text{Tr}[U(0, s)\tau_k U(s, 1)] M(\gamma(s)) F^k_{ab}(\gamma(s)) \sqrt{\gamma(s)}$$

$$\times \left\{ \int_0^{2\delta^+} (t - s) \left[ 2 - \frac{3(t - s)}{2\epsilon} + \left( \frac{t - s}{2\epsilon} \right)^3 \right] d(t - s) - \int_{-2\delta^-}^0 (t - s) \left[ 2 + \frac{3(t - s)}{2\epsilon} - \left( \frac{t - s}{2\epsilon} \right)^3 \right] d(t - s) \right\} =$$

$$= \frac{3}{4\pi\epsilon} \int ds \dot{\gamma}^a(s) \dot{\gamma}^b(s) \text{Tr}[U(0, s)\tau_k U(s, 1)] M(\gamma(s)) F^k_{ab}(\gamma(s)) \sqrt{\gamma(s)}$$

$$\times \left\{ (\delta^+)^2 - (\delta^-)^2 + \frac{1}{5}(\delta^+)^5 + (\delta^-)^2 + (\delta^-)^3 - \frac{1}{5}(\delta^-)^5 \right\}.$$  

Because $((\delta^+)^3 - (\delta^-)^3) = \mathcal{O}(\epsilon)$ and also $((\delta^+)^5 - (\delta^-)^5) = \mathcal{O}(\epsilon)$ we can write the action of the regulated Hamiltonian constraint on smooth portions in the lowest order of $\epsilon$ as:

$$\hat{H}(\mathcal{M}) W_\gamma^\text{smooth}(A) =$$

$$= \frac{3}{2\pi\epsilon} ((\delta^+)^2 + (\delta^-)^2) \int ds \dot{\gamma}^a(s) \dot{\gamma}^b(s) M(\gamma(s)) F^k_{ab}(\gamma(s)) \text{Tr}[U(0, s)\tau_k U(s, 1)].$$
7.2 Appendix 2

To obtain the contribution from self-intersections, we have to compute the integral:

\[ I(\epsilon, \theta) = \int ds \int dt V(\delta(s, t)) \]

where \( V(\delta(s, t)) \) is given by [18] and the limits of integration can be determine from Figure 2. First we will change the variable \( s \) into:

\[ \xi = \frac{s \sin \theta}{2\epsilon} \]

where \( \sin \theta \) is the angle between \( \vec{\eta}^+ \) and \( \vec{\eta}^- \). Also we will introduce another variable \( \varphi \) via the relation \( t = 2\epsilon \xi \tan \varphi \). Thus we will have:

\[ V(\xi, \varphi) = \frac{2\pi \epsilon^3}{3} \left[ 2 - \frac{3\xi}{\cos \varphi} + \frac{\xi^3}{\cos^3 \varphi} \right] \]

Keeping \( \xi \) fixed we can perform the integration with respect to \( \varphi \) to get:

\[
\frac{8\pi \epsilon^4 \xi}{3} \int_0^{\arccos \xi} d\varphi \frac{d\varphi}{\cos^2 \varphi} \left[ 2 - \frac{3\xi}{\cos \varphi} + \frac{\xi^3}{\cos^3 \varphi} \right] = \\
= \pi \epsilon^4 \left\{ (4 + \xi^2)\sqrt{1 - \xi^2} - \frac{1}{2} \xi^2 (4 - \xi^2) \ln \left[ \frac{1 + \sqrt{1 - \xi^2}}{1 - \sqrt{1 - \xi^2}} \right] \right\}. 
\]

To obtain the final result we have to perform the integration with respect to \( \xi \):

\[ I(\epsilon, \theta) = \frac{4\pi \epsilon^5}{\sin \theta} \int_0^1 d\xi \left\{ (4 + \xi^2)\sqrt{1 - \xi^2} - \frac{1}{2} \xi^2 (4 - \xi^2) \ln \left[ \frac{1 + \sqrt{1 - \xi^2}}{1 - \sqrt{1 - \xi^2}} \right] \right\}. 
\]

Thus finally after this exercise in calculus we get:

\[ I(\epsilon, \theta) = \frac{4\pi^2 \epsilon^5}{\sin \theta} \frac{23}{30}. \]
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