Long-wave magnons in a ferromagnetic film

Gang Li,1 Chen Sun,1 Thomas Nattermann,2 and Valery L. Pokrovsky1,3
1) Department of Physics, Texas A&M University, College Station, Texas 77843-4242, USA
2) Institute für Theoretische Physik, Universität zu Köln, Zülpicher Str. 77a, D-50937 Köln, Germany
3) Landau Institute for Theoretical Physics, Chernogolovka, Moscow District, 142432, Russia

An asymptotically exact theory of spectrum and transverse distribution of magnetization in long-wave magnons is presented. It is based on exact analytical solution of linearized Landau-Lifshitz equation in a film. The quantization of the transverse wave vector and role of evanescent waves at different values of parameters and wave vectors is studied.

Introduction. In this article we present asymptotically exact theory of the spectrum and transverse distribution of magnetization in long-wave length magnons propagating in a ferromagnetic film. The theory is based on exact solution of linearized Landau-Lifshitz equation (LLE). To avoid complications we assume the film to be isotropic in the film plane, and external magnetic field \( H \) and the spontaneous magnetization \( M \) to be oriented in plane. Their direction is accepted for \( z \)-axis, whereas the \( x \)-axis is directed perpendicular to the film that occupies the volume between parallel planes \( x = \pm \frac{d}{2} \). The Hamiltonian \( \mathcal{H} \) includes exchange, Zeeman and dipolar interaction of magnetization:

\[
\mathcal{H} = \int_V d^3r \left[ \frac{D}{2} (\nabla \cdot \mathbf{M})^2 - \mathbf{H} \cdot \mathbf{M} \right] + \frac{1}{2} (\mathbf{M} \cdot \nabla) \int_V d^3r' (\mathbf{M}' \cdot \nabla') \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \tag{1}
\]

Here \( \mathbf{M}(\mathbf{r}) \) is the local magnetization vector, \( D \) is the exchange energy divided by \( M^2a \), where \( a \) is the lattice constant. \( V \) stands for volume and prime denotes dependence on the coordinate \( \mathbf{r}' \). Note that the coefficient \( D \) has dimensionality of square of length. The value \( \ell = \sqrt{D} \) is called dipolar length. This is the scale of distance at which dipolar and exchange interactions become of the same order of magnitude. Typically it is about 10-30nm. Theory of magnons in ferromagnetic films has important applications to real magnets and rather long history. In this brief article we can cite only several most important articles and give our apologies to many contributors to the subject. The first exact result was obtained by Damon and Eshbach for purely dipolar interaction. Kalinikos and Kalinikos and Slavin incorporated exchange interaction together with dipolar one and obtained an integral-differential equation for magnetization in a spin wave. They solved it employing a plausible, but uncontrollable approximation. Rezende assumed that magnetization does not depend of transverse coordinate \( x \) and exactly diagonalized the resulting Hamiltonian. Though such assumption is qualitatively justified for the transverse mode with the lowest energy, it is obviously invalid for higher transverse modes. In a recent work Sonin has found the magnon spectrum and shape of transverse mode at zero \( y \)-component of the magnon wave vector \( k_y \). His solution allows an explicit analytical expression in the limit \( d \gg \ell \) and \( 1/d \ll k_z \ll 1/\ell \). Our work is an extension of Sonin’s method to the case of general \( k_y \).

We show that formally the spectrum of magnons in a film has the same analytical form as in the bulk, but quantization of the transverse wave vector and transverse distribution of magnetization depend on thickness of the film \( d \) and other variables in a highly non-trivial way. In contrast to standard semiclassical approximation that becomes valid when the number of transverse mode \( n \) is a large number, in the ferromagnetic film if \( d \gg \ell \) there exist two different asymptotics in the ranges \( 1 \ll n \ll d/\ell \) and \( n \gg d/\ell \). In both cases for each \( n \) the distribution of magnetization across the film consists of one oscillating mode and two evanescent modes. All they have the same frequency. We obtain analytical solution at \( d \gg \ell \) for any \( n \), not only large ones. Due to symmetry of the problem magnon spectrum is divided in two infinite series. In simplest situation they correspond to even and odd transverse distribution of magnetization, \( n \)-th mode oscillates \( n \) times between boundaries. Evanescent waves can be neglected in the exchange-dominated range of wave vectors \( k_\parallel \equiv \sqrt{k_y^2 + k_z^2} \gg 1/\ell \). Otherwise evanescent waves must be taken in account. Specifically, they play important role in the case of thin films \( d \leq \ell \). Theory for small linear sizes has a special interest for applications to devices employing magnons instead of electrons as carriers of information.2,7 Our method can be extended to include anisotropy (spin-orbit interaction), tilted external magnetic field and other shapes of the sample.

Equations of motion and magnon solutions. A weak excitation of the equilibrium state is described by the transverse components of magnetization \( \mathbf{M} \equiv (M_x, M_y) \). They obey the linearized LLE:

\[
\dot{\mathbf{M}} = \gamma [(H - MD\Delta)\mathbf{M} + \mathbf{M}\mathbf{h}] \times \hat{z}, \tag{2}
\]

\( \hat{z} \) is the unit vector in \( z \)-direction and \( \Delta \equiv \nabla^2 \); \( \mathbf{h} = \nabla \perp \phi \) denotes the dipolar field induced by magnetization inside.
and outside the film, with $\nabla_\perp \equiv (\partial_x, \partial_y)^T$ and
\[ \phi (r) = -\nabla_\perp \cdot \int d^3r' M'|r - r'|^{-1}. \tag{3} \]
The number of parameters of the present problem can be reduced by the scale transformations
\[ t \rightarrow \omega_H^{-1} t, \quad r \rightarrow \sqrt{\frac{\chi}{4\pi}} t, \quad M \rightarrow M M. \tag{4} \]
Here $\omega_H \equiv \gamma H$ denotes the Larmor frequency and $\chi \equiv 4\pi M/H$ the static magnetic susceptibility (we absorb a factor $4\pi$ in its definition to simplify final expressions). In rescaled units equation of motion simplifies to
\[ M = [(1 - \Delta) M + \sqrt{\frac{\chi}{4\pi}} h] \times \hat{z}. \tag{5} \]

The equations for $h$ and $\phi$ remain unchanged. The remaining parameters of the problem are susceptibility $\chi$ and half of the sample width $d/2$ in new units

Applying laplacian $\Delta$ to eq. (6) and using magneto-static equation
\[ \Delta \phi = 4\pi \nabla_\perp \cdot M, \tag{6} \]
one gets the desired equation for $M$:
\[ \Delta M = [(1 - \Delta) M + \chi \nabla_\perp (\nabla_\perp \cdot M)] \times \hat{z}. \tag{7} \]

It must be solved with standard boundary conditions (BC) for Maxwell equations that requires continuity of tangential components of magnetic field $h$ and normal component of magnetic induction $b = h + 4\pi M$ at two surfaces of the film. Another set of BC originates from variation of magnetization (spins) on surfaces if they are free. It leads to equations:
\[ \partial_z M|_{x = \pm d/2} = 0. \tag{8} \]
We call them exchange boundary conditions (EBC).

In a propagating wave with in-plane wave vector $k_\parallel = k_y \hat{y} + k_z \hat{z}$, the oscillating components of magnetization can be written as
\[ M = \begin{pmatrix} m_x(x) \cos(k_\parallel \cdot r - \omega t) \\ m_y(x) \sin(k_\parallel \cdot r - \omega t) \end{pmatrix}. \tag{9} \]
The Ansatz \[ \text{(2)} \] turns eq. \[ \text{(7)} \] into a system of ordinary linear homogeneous differential equations with constant coefficients for the vector field $m = (m_x, m_y)^T$ which describes the transverse distribution of magnetization. General solution of such a system is a superposition of basic exponential solutions $m(x) = m_0 e^{i k_\parallel x}$. After division by $k^2 = k_\parallel^2 + k_z^2$ equation for $m_0$ reads
\[ (\Omega - i B z_3) m_0 = 0, \quad \Omega = \begin{pmatrix} \omega & -A_x \\ -A_y & \omega \end{pmatrix}. \tag{10} \]
Here $A_\alpha = 1 + k^2 + \chi k^2_\alpha$, and $B = \chi k^2_\parallel k_y$. $k_\parallel = k_\alpha/k$ denotes the cosine of direction and $z_3$ is the Pauli matrix.

The solvability condition of eq. \[ \text{(10)} \], $\omega^2 + B^2 - A_x A_y = 0$, delivers the magnon dispersion relation:
\[ \omega^2 = (1 + k^2) \left(1 + \chi + k^2 - \chi k^2_\perp \right). \tag{11} \]

It does not depend on the sample thickness and has therefore the same form as in the bulk. Boundary conditions will however restrict possible $k$-vectors, as it will be shown below. The dispersion relation \[ \text{(11)} \] can be treated as a cubic equation for $k^2$, assuming that $\omega$ and $k_\parallel$ are given. Its three solutions can be written as $k^2_i = k^2_{x,i} + k^2_\perp$. Thus $k_{x,i}$ is a function of $\omega$ and $k_\parallel$. Close investigation shows that all 3 roots of cubic equation for $k^2$ are real, one of them $k^2_1$ is positive, two others $k^2_2$ and $k^2_3$ are negative in the entire physically available range of parameters. Positive root $k^2_1$ corresponds to oscillating transverse mode, two negative roots correspond to evanescent waves.

Equations \[ \text{(10)} \] and boundary conditions are invariant under operation $x \rightarrow -x, k_y \rightarrow -k_y$. It means that all eigenvalues $\omega$ are at least double degenerate and the eigenfunctions with the same $\omega$ and opposite signs of $k_y$ are connected with a simple relation:
\[ m_{x,y}(x; k_y) = m_{x,y}(-x; -k_y). \tag{12} \]

The value $k_z$ enters in equations only as $k^2_x$. Therefore, the solution does not change at transformation $k_z \rightarrow -k_z$. These properties can be obtained from invariance of the Hamiltonian with respect to two discrete transformations: reflection in the central plane of the film combined with time reversal and reflection in the $(x, z)$-plane combined with time reversal. Time reversal in addition to reflection is necessary to keep pseudo-vector of spontaneous magnetization invariant.

The transverse distribution of magnetization $m(x)$ must be a real vector field. Therefore for any mode it can be written as follows:
\[ m(x) = a \cos k_x x + b \sin k_x x, \tag{13} \]
where $a$ and $b$ are real constant vectors. According to \[ \text{(10)} \], the coefficients $a, b$ obey the relation $\Omega a - B z_3 b = 0$ which implies the amplitude relations
\[ b = \Delta(k, \omega) \cdot a, \quad \Lambda = \sigma_3 \Omega / B. \tag{14} \]
Symmetry discussed above retains invariant coefficients $a_i$ and changes sign of coefficients $b_i$ ($i = 1, 2, 3$).

**Boundary conditions and consistency requirement.** The exchange BC \[ \text{(3)} \] include 4 equations, two on each surface. They cannot be satisfied with a single-mode solution \[ \text{(13)} \] associated with one of three possible values of $k^2_x$. Indeed, according to eq. \[ \text{(13)} \] such a solution depends only on two independent parameters, for example $a_x, a_y$. Only a proper superposition of three solutions can satisfy exchange and electromagnetic BC simultaneously. Such a general solution of the equation
\[ \mathbf{m}(x) = \sum_{i=1}^{3} (a_i \cos k_{ix} x + b_i \sin k_{ix} x), \]  

(15)

where \( k_{ix} \) denotes the \( x \)-component of the wave vector corresponding to \( i \)-th solution of cubic equation (it is purely imaginary for evanescent waves) and \( a_i, b_i \) are the vector amplitudes of \( i \)-th mode. Using eq. (15), the EBC equations can be rewritten in terms of 12 independent coordinates of vectors \( a_i, b_i; \ i = 1, 2, 3 \):

\[ \sum_{i=1}^{3} a_{ix} k_{ix} \sin \alpha_i = 0; \quad \sum_{i=1}^{3} b_{ix} k_{ix} \cos \alpha_i = 0; \quad \alpha_i = \frac{k_{ix} d_i}{2}. \]  

(16)

The magnetostatic BC are satisfied automatically for any distribution of magnetization if magnetic potential obeys the integral relation (3). In particular, it will be satisfied for magnetization represented by superposition (10). We have proved that any solution of equation of motions must be such a superposition. However, the inverse statement that any such a superposition is solution of equations of motions (13) is wrong. It happens because equations of motion contain not only differential, but also integral terms. The choice of valid solutions is realized by condition of consistency. It requires magnetic potential \( \phi \) to be a superposition of exponents \( e^{\pm ik_{ix} x} \) where \( k_{ix}^2 \) are solutions of the cubic equation discussed earlier. We will see that integrals in \( \phi(x) \) eq. (3) generate extra exponents of the type \( e^{\pm ik_{ix} x} \) that are not allowed by cubic equation. Consistency requires coefficients at them to be zero. Below we display an explicit form of these consistency equations (CE).

Integration over longitudinal coordinates \( y, z \) in eq. (3) can be performed explicitly with the result:

\[ \phi = -4\pi (d_x \eta_x + k_y \eta_y); \quad \eta_j = \frac{1}{2k_{ix}} \int_0^1 e^{-k_{ix}|x-x'|} m_j(x') dx'. \]  

(17)

The basic integrals that enter \( \eta_j(x) \), \( j = x, y \) are:

\[ \int_0^1 dx' e^{-k_{ix}|x-x'|} \frac{\cos k_{ix} x'}{2k_{ix}} \sin k_{ix} x' = \frac{1}{k_{ix}^2} \left( \cos k_{ix} x - \frac{k_{ix} x}{k_{ix}^2} \sin k_{ix} x \right). \]  

(18)

Here \( k_{ix}^2 = k_{ix}^2 + k_{ix}^2 \) and we denotes \( f_{ic} = k_{ix} \cos \alpha_i - k_{ix} \sin \alpha_i, f_{is} = k_{ix} \sin \alpha_i + k_{ix} \cos \alpha_i \) with \( \alpha_i = k_{ix} d_i/2 \). Eq. (18) visibly demonstrates the appearance in magnetic potential of exponents \( \exp(\pm k_{ix} x) \) forbidden by secular cubic equations for \( k^2 \) since it corresponds to \( k^2 = 0 \). It vanishes in \( \phi \) only due to superposition. The consistency equations require coefficients at \( \cos k_{ix} x \) and \( \sin k_{ix} x \) to be zero. The corresponding equations can be written as follows:

\[ \sum_{i=1}^{3} \frac{1}{k_{ix}^2} \left( k_{ix} a_{ix} f_{ic} + k_y b_{iy} f_{is} \right) = 0 \]

\[ \sum_{i=1}^{3} \frac{1}{k_{ix}^2} \left( k_{ix} b_{ix} f_{ic} + k_{iy} a_{ix} f_{is} \right) = 0. \]  

(19)

In order to turn CE together with exchange boundary conditions (10) into closed system of 6 equations for 6 independent amplitudes \( a_{ix}, b_{ix} \) it is possible to use relations between \( a_{iy}, b_{iy} \) and \( a_{ix}, b_{ix} \) following from equations of motion in form (10):

\[ a_{iy} = \frac{\omega}{A_{iy}} a_{ix} - \frac{B_i}{A_{iy}} b_{ix}; \quad b_{iy} = \frac{B_i}{A_{iy}} a_{ix} + \frac{\omega}{A_{iy}} b_{ix}, \]  

(20)

where \( A_{iy} = 1 + k_{y}^2 + \frac{\chi k_{x}^2}{k_{y}^2} \) and \( B_i = \frac{\chi k_{x} k_y}{k_{y}^2} \). Besides of that it is necessary to eliminate values \( k_{y}^2 \) and \( k_{ix} \) with \( i = 2, 3 \). As it follows from cubic equation for \( z = k_{x}^2 \), if the first (positive) root \( z_1 = k_{x}^2 \) is fixed, two others can be found from equation:

\[ k_{x}^2, 3 = -1 - \frac{\chi}{2} \pm \sqrt{\left(1 + \frac{\chi}{2} + \frac{k_{y}^2}{k_{x}^2}\right)^2 - \frac{\chi k_{y}^2}{k_{x}^2}}. \]  

(21)

In this way all \( k_{x}^2 \) and \( k_{ix} \) with \( i = 2, 3 \) are determined through the single positive wave vector \( k_{ix} \).

The system of 4 EBC (10) and 2 EC equations considered as 6 linear homogeneous equations for \( a_{ix}, b_{ix} \) has solutions only if its determinant is equal to zero. This requirement determines discrete set of \( k_{ix} \), i.e. transverse quantization of wave vector. This equation is exact in the framework of considered model. In Fig. 1 we show results of numerical calculations of quantized spectra from requirements of zero determinant for \( d = 100 \), and \( \chi = 2 \) for direction of propagation perpendicular and parallel to magnetization and spectra of the first transverse modes for a few different directions of propagation.

To give a more visible idea of mathematical procedure leading to these results, we consider 6 EBC-CE equations in some detail. Only \( \cos k_{ix} d/2 \) and \( \sin k_{ix} d/2 \) are oscillating functions of their arguments. The functions \( f_{ic}, f_{is} \) are linear functions of \( \cos \alpha_i, \sin \alpha_i \) and therefore also oscillate. Other functions containing \( \cos \alpha_2, 3, \sin \alpha_2, 3 \) are hyperbolic functions and change monotonically with \( k_{ix} \). In the 6×6 matrix \( C \) of the EBC-CE equations the first two columns are linear combinations of \( \sin \alpha_1, \cos \alpha_1 \), the rest are monotonal functions. Therefore, the determinant has a form \( \det C = K \sin^2 \alpha_1 + 2L \sin \alpha_1 \cos \alpha_1 + M \cos^2 \alpha_1 \), where \( K, L, M \) are monotonal functions of \( k_{ix} \) or \( \alpha_i \). Then equation \( \det C = 0 \) can be rewritten as \( K \tan^2 \alpha_1 + 2L \tan \alpha_1 + M = 0 \) with formal solution:

\[ \tan \alpha_1 = \frac{-L \pm \sqrt{L^2 - KM}}{2K}. \]
This is an implicit equation for $k_{1\nu}$. It shows that
the consequent quantized values $k_{1\nu}$ are located between
points $n\pi/d$ and that quantized values form two series
corresponding to two signs in front of square root in previous equation. Thus, the quantized values of $k_{1\nu}$ can
be enumerated by an index $\nu$ taking two values $+\pi$ and $-\pi$ and by an integer number $n$ taking values from 0 to $\infty$. We will denote these quantized values as $k_{\nu n}$. For
large $n$ the main part of $k_{\nu n}$ is $2\pi n/d$. An approximate
formula for the quantized values reads:
$$k_{\nu n} = \frac{2\pi n}{d} + \frac{2}{d} \arctan \frac{-L \pm \sqrt{L^2 - KM}}{2K}.$$ (21)

In the argument of arctan $k_{1\nu}$ must be replaced by $2\pi n/d$.

**General properties of magnon spectra in thick films.** In conclusion we describe general properties of spectra and structure of transverse modes in thick films $d \gg 1$.

At fixed direction of propagation given by $\theta \equiv \arccos(k_z/k_\parallel) = \text{const}$, the frequency $\omega_{\nu n}$ of a mode $\nu n$ with $n \ll d/2\pi$ as function of $k_\parallel$ has a minimum at non-zero value

$$k_{\parallel 0} \approx \left(\frac{\chi \cos^2 \theta}{2 + \chi \sin^2 \theta}\right)^{1/4} k_{1\nu}^{\nu n/2}.$$ 

According to this equation, $k_{\parallel 0} \gg k_{\nu n}$, but it is much less than 1 that makes this explicit result available. The limitation to $n$ ensures that $k_{\nu n} \ll 1$ and as a consequence $k_{\parallel 0} \ll 1$. The frequency in minimum is $\omega_{\min} = 1 + O(k_{\parallel 0}^2)$. The position of minimum eventually shifts to larger $k_\parallel$ with the growth of $n$ or equivalently $k_{\nu n}$ and at $k_{\nu n} = \frac{1}{2} \left(\sqrt{2 + \chi \sin^2 \theta} + 8\chi \cos^2 \theta - 2 - \chi \sin^2 \theta\right)^{1/2}$ goes
to $+\infty$. At fixed $n$ and $\theta$ decreasing, $k_{\parallel 0}$ decreases.
At $\theta = 0$ minimum and maximum coalesce. The point $k_\parallel = 0$ is the only minimum of frequency in the spectrum of any magnon mode propagating perpendicularly to permanent magnetization ($\cos \theta = 0$).

Maximum of frequency for any fixed $\theta$ except of $\theta = \pi/2$ is located at $k_{\parallel} = 0$. The value of frequency in maximum is equal to $\omega_{\max} = 1 + \chi$, frequency of ferromagnetic resonance. For a mode with $n$ not large such a value of frequency is reached right of minimum at

**Acknowledgements.** We thank E.A. Sonin for sending us his manuscript prior publication and discussions. This work was supported by the University of Cologne Center of Excellence QM2.

1R.W. Damon and J.R. Eschbach, J. Phys. Chem. Solids 19, 308 (1961).
2B.A. Kalinikos, Proc IEEE H 127, 4 (1980).
3B.A. Kalinikos and A.N. Slavin, J. Phys. C: Solid State Phys. 19, 7013 (1986).
4S.M. Rezende, Phys. Rev. B 79, 174411, (2009).
5E.A. Sonin, Phys. Rev. B 95, 144432 (2017).
6. Y. Sun, Z. Wang and L. Lu, Ch. 5 in Metallic Spintronics Devices, ed. by X. Wang, CRS (2014).
7. R.A. Duine, A. Brataas, S.A. Bender and Y. Tserkovnyak, arXiv:1505.01329 [cond-mat.meshall] (2015).
8. C. Sun, T. Nattermann and V.L. Pokrovsky J. Phys. D: Appl. Phys. 50, 143002 (2017).