Veselago Lens for Electrons:
Focusing and Caustics in
Graphene $p$-$n$ Junctions.

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Abstract

The focusing of electric current by a single $p$-$n$ junction in graphene is predicted. We show that precise focusing can be achieved by fine-tuning the densities of carriers on the n- and p-sides of the junction to equal values, whereas the current distribution in junctions with different densities resembles caustics in optics. This finding can be utilized in the engineering of electronic lenses and focused beam-splitters using gate-controlled $n$-$p$-$n$ junctions in graphene-based transistors.

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A lot of similarity exists between optics and electronics. Rays in geometrical optics are analogous to classical trajectories of electrons, while electron de Broglie waves interfere akin light. The electron microscope is one example of the technological implementation of this similarity. The analogy with optics may also hold a significant potential for semiconductor electronics. In optics, transparent interfaces between materials are used in lenses and prisms to manipulate light beams. So far, interfaces have played a rather different role in semiconductor electronics, where the central place was, for a long time, occupied by the \textit{p-n} junction (PNJ). Due to a depletion region near the contact between two semiconductors with different types of charge carriers (and a large energy gap), conventional PNJs are not suitable for precision manipulation of electron beams, which, if realized, may lead to a new functionality in microelectronics. From this perspective, a lot of promise is offered by a recently discovered \cite{1} truly two-dimensional gapless semiconductor - graphene \cite{2}. Fine-tuning of the carrier density in graphene by means of gates \cite{3, 4, 5} or doping of the underlying substrate \cite{6} was demonstrated, thus, paving the way towards controllable ballistic PNJs. On the one hand, the PNJ in graphene is highly transparent for the charge carriers \cite{7, 8}. On the other, as we show below, the transmission of electrons through the \textit{p-n} interface resembles optical refraction \cite{9} at the surface of metamaterials with negative refractive index \cite{10, 11, 12}: the straight interface is able to focus electric current whereas a ballistic stripe of \textit{p}-type graphene separating two \textit{n}-type regions acts as a lens.

The unique feature of the band structure of graphene (monolayer of graphite \cite{2, 13}) is that its valence band ($\pi$) and conduction band ($\pi^*$) touch each other. In
the absence of doping the Fermi level in graphene is at the energy which belongs to the both bands and corresponds to the Bloch states in the corners of the hexagonal Brillouin zone of this two-dimensional honeycomb crystal. For the states with a small quasimomentum $\hbar \vec{k}$ counted from the corresponding corner of the Brillouin zone, the dispersion $\varepsilon(\vec{k})$ and group velocity $\vec{V} = d\varepsilon/d(\hbar \vec{k})$ of electrons are given by

\[
\varepsilon_c(\vec{k}) = \hbar v_c, \quad \vec{V}_c = v \vec{k}/k, \text{ in conduction band,}
\]
\[
\varepsilon_v(\vec{k}) = -\hbar v_v, \quad \vec{V}_v = -v \vec{k}/k, \text{ in valence band.}
\]

Figure 1 illustrates such a dispersion for electrons in $n$-type (on the left) and $p$-type graphene (on the right). In a split-gate structure sketched in Fig. 1, voltages $\pm U$ applied to the two gates shift the degeneracy point of the electron dispersion cones down, by $\hbar v_\text{f.c}$ on the left and up, by $\hbar v_\text{f.v}$ on the right and, thus, form a PNJ separating the $n$- region with the density of electrons $\rho_e = k_\text{c}^2/\pi$ and the $p$-region with the density of holes $\rho_h = k_\text{v}^2/\pi$. Here $k_{\text{c(v)}}$ is the radius of the Fermi circle in the conduction (valence) band.

The transmission of charge by the PNJ bears striking resemblance to the refraction of light by left-handed metamaterials [10, 11, 12] with refractive index equal to $-1$. As a wave enters such a material, the relative direction of its group velocity $\vec{V}$ and the wave vector $\vec{k}$ of the wave reverses, from parallel (in vacuum) to anti-parallel. Therefore, upon refraction, the sign of the tangential velocity component of the propagating wave inverts, while the normal component remains the same. As a result, rays which diverge in vacuum become convergent after entering the metamaterial [9].
A similar incident occurs with electrons in the PNJ where the Fermi momentum of the charge carriers plays the same role as the refractive index in geometrical optics, with the sign determined by the type of band: positive for the conduction band and negative for the valence band.

Indeed, let us consider [with reference to Figs. 2(A) and 3(A)] a de Broglie wave of an electron approaching the PNJ from the $n$-side with velocity $\vec{V}_c = (v \cos \theta_c, v \sin \theta_c)$ and $\vec{k}_c = (k_c \cos \theta_c, k_c \sin \theta_c)$. At the interface, this wave is partly reflected to the state with $\vec{k'}_c = (-k_c \cos \theta_c, k_c \sin \theta_c)$ and partly transmitted to the valence band state with $\vec{V}_v = (v \cos \theta_v, v \sin \theta_v)$ and $\vec{k}_v = (-k_v \cos \theta_v, -k_v \sin \theta_v)$ on the $p$-side. The probability of the transmission is $\cos^2 \theta_c / \cos^2 (\theta_c + \theta_v)$ \[7, 8\]. The component of the electron momentum along a straight interface should be conserved. Accordingly, $k_c \sin \theta_c = -k_v \sin \theta_v$, that is, the transmission of electrons is governed by Snell’s law:

$$\frac{\sin \theta_c}{\sin \theta_v} = -\frac{k_v}{k_c} \equiv n.$$ \hspace{1cm} (1)

The negative sign of $n$ in Eq. (1) implies that the $n-p$ interface transforms a divergent flow of electrons emitted by a source on the $n$-side into a convergent flow on the $p$-side. This results in focusing illustrated in Fig. 2(A) for a symmetric junction, $\rho_h = \rho_e$ corresponding to $n = -1$. Under the latter condition electrons injected at $(-a, 0)$ in the $n$-region at the Fermi energy meet again in a symmetric spot at $(a, 0)$.

In an asymmetric junction such as shown in Fig. 3(A) for $n = -0.82$ (which corresponds to $\rho_h/\rho_e = 0.67$) a sharp focus transforms into a pair of caustics which coalesce in a cusp - a singularity in the density of classical trajectories. Similar singularities in the density of rays, as well as the interference patterns formed in their
vicinity were investigated in optics and classified \[14\] using the general catastrophe theory \[15\]. Ballistic trajectories of electrons in the \(p\)-region of an asymmetric PNJ are rays \(y = a \tan \theta_c + x \tan \theta_v\), where \(\theta_v\) is related to \(\theta_c\) by Eq. \((1)\). The condition for a singularity, \(\partial y/\partial \theta_c = 0\) determines the form of caustics \(y_{caust}(x)\) as well as the position \(x_{cusp}\) of the cusp,

\[
y_{caust}(x) = \pm \sqrt{\frac{x^{2/3} - x_{cusp}^{2/3}}{n^2 - 1}}, \quad x_{cusp} = |n|a. \tag{2}
\]

To detect focusing by a single flat interface in graphene one can use a small electric contact as a source of electrons, while another local probe located on the \(p\)-side can play the role of a detector. Electric conductance between the two contacts would reflect the probability for a carrier to get from the source to the probe. When the concentration of carriers is low their de Broglie wavelength is big enough for it to be not impossible to make contacts smaller than the wavelength. To study electron transmission in a phase-coherent system between contacts of such a small size the above-described classical picture should be complemented with the analysis of quantum interference pattern of electron de Broglie waves. Figs. 2(B) and 3(B) visualize the result of full quantum mechanical calculations of the current of electrons emitted at \((-a,0)\) and detected by a point contact: near the focal point in the symmetric PNJ [Fig. 2(B)] and in the vicinity of a cusp [Eq. \((2)\)] which appears when the symmetry \(\rho_h = \rho_e\) is lifted off [Fig. 3(B)]. The calculation was performed by applying the Kubo formula to the single-particle Dirac-like Hamiltonian \[21\] of electrons in graphene. Around, but not too close to the focus \((k_v r \gg 1)\) the analytically calcu-
lated current is \( j \sim \frac{(x - a)^2}{r^3} \) \([r = \sqrt{(x-a)^2 + y^2}\) stands for the distance from the probe to the focus]. The anisotropy of the current distribution is caused by the dependence of the transmission coefficient on the incidence angle and is smeared at shorter distances \( k_v r < 1 \). The current map calculated in the vicinity of the cusp for \( \rho_h \neq \rho_e \) shows characteristic patterns described by the canonical diffraction function for this type of wave catastrophe \([14]\). The maximum of the current would be when the probe is at the tip of the cusp, \((|n|a, 0)\). The width \( y^* \) of the bright spot near the cusp [Fig. 3(B)] or the focus [Fig. 2(A)] in the \( y \) direction can be estimated as \( y^* k_v \sim \max \left\{ 1, \left( \frac{1}{2} a k_v |n^{-1} - n| \right)^{1/4} \right\} \). Note that for a junction with \( n > 1 \) \((\rho_h > \rho_e)\) the pattern near the cusp is mirror-reflected as compared to that shown in Fig. 3(B) for \( n < 1 \).

It has been discovered \([16]\) in the scanning tunneling microscopy (STM) studies of elliptically shaped corals on the surface of copper that the presence of an impurity at one focus of the ellipse is reflected by the STM map in the vicinity of the other focus. Therefore, oscillations of the local density of states of electrons formed around a static local perturbation \([17]\) can be replicated through focusing by a carefully engineered fence of atoms. Similarly, focusing of electrons by a PNJ in graphene could create a ‘mirage’, which mimics the effect of a perturbation on the opposite side of the \( n-p \) interface. Consider, \( e.g.\), a small island of a bilayer \([18\ 5]\), which locally distinguishes between two sublattices (A and B) of the honeycomb lattice for electrons in the surrounding sheet [due to Bernal stacking of two adjacent monolayers \([19\ 20]\)]. It induces a change in the local electron density of states (LDoS), which is different on sublattices A and B. The long-range oscillations of the alternating LDoS
can be detected using STM: as a difference $\delta j_{A-B} \sim j_{A-B}^{(0)} \sin(2k_c v r)/r$ between the tunneling current from the STM tip to the A and to B sites. Fig. 2(C) shows the results (obtained using the Green functions technique) of a quantum-mechanical analysis of oscillations of $\delta j_{A-B}$ around the mirage image of a bilayer island formed on the other side of symmetric PNJ in the monolayer sheet. Fig. 2(D) shows the calculated mirage image of a spike of electrostatic potential (smooth at the scale of the lattice constant in graphene), which induces LDoS oscillations equal on the two sublattices. The difference in the sharpness of these two images is caused by the lack of backscattering off A-B symmetric scatterers specific to graphene [21].

Unlike the ideal left-handed metamaterial [10], focusing in the PNJ is not perfect. In symmetric junctions it occurs only for electrons exactly at the Fermi level, and it is spread into caustics for electrons excited to higher energies. This also implies that the results in Figs. 2 and 3 are only valid for low enough temperature, $T < \hbar v/a$. For electrons with different energies the patterns in Fig. 2 smear into patterns characteristic for a cusp in an asymmetric PNJ, e.g., with $\rho_h < \rho_e$ shown for the same type of perturbations in Figs. 3(C) and (D). In this respect, a certain reciprocity also exists: electrons with energy $\delta \varepsilon = \hbar v(k_v - k_c)$ counted from the Fermi level would be focused in the asymmetric PNJ.

Focusing of electrons by a sharp p-n junction in graphene can be used to turn the n-p-n junction into a Veselago lens for electrons. In such a device, Fig. 4(A) the density of charge carriers in the p-region (with width $w$) can be controlled by the top gate. If the densities in the n- and p-regions are equal $\rho_h = \rho_e$, charge carriers injected into graphene from the contact $S$ shown in Fig. 4(A) would meet again
in the focus at the distance $2w$ from the source [contact $D_3$ in Fig. 4(A)]. Varying the gate voltage over the $p$-region changes the ratio $n^2 = \rho_h/\rho_e$. This enables one to transform the focus into a cusp displaced by about $2(|n| - 1)w$ along the $x$-axis and, thus, to shift the strong coupling from the pair of leads $SD_3$ to either $SD_1$ (for $\rho_h < \rho_e$) or $SD_5$ (for $\rho_h > \rho_e$). Figs. 4(B,C) illustrate another graphene-based device in which a prism-shaped top-gate may be used as a focusing beam-splitter. For example, electrons emitted from the contact $B$, Fig. 4(B), are distributed between the contacts $b$ and $\beta$, whereas the signal sent from the contact $A$, Fig. 4(C), is replicated into the pair of contacts $a$ and $\alpha$. To mention, graphene has recently been contacted with a superconducting metal and the Josephson proximity effect through graphene has been observed \[22\]. Consequently, a beam splitter Fig. 4(B,C) can be used to experiment with Einstein-Podolsky-Rosen \[23\] pairs of particles.

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**Fig. 1.** Graphene $p$-$n$ junction (PNJ): monolayer of graphite is placed over the split gate which is used to create $n$- (left) and $p$-doped (right) regions. The energy diagram shows the position of the Fermi level with respect to the touching point of the valence and the conduction bands.

**Fig. 2.** Focusing of electrons by symmetric PNJ, $\rho_h = \rho_e$ (A) Classical trajectories of electrons diverging from a source at distance $a$ from the junction become convergent after refraction. (B) Interference-induced pattern in the charge current near the focal image of the source-contact. (C,D) "Quantum mirage" in graphene: local density of states oscillations around the image of a perturbation applied on the other side of PNJ: (C) a small island of bilayer and (D) potential of a remote Coulomb charge.

**Fig. 3.** Wave singularities in an asymmetric PNJ, $\rho_h/\rho_e = 0.67$ (A) formation of caustics by refracted waves. (B) Characteristic interference pattern for the current near the cusp. (C,D) Local density of states oscillations (in the region between caustics) created by (C) a small island of bilayer and (D) a remote Coulomb charge on the other side of PNJ.

**Fig. 4.** (A) Electron Veselago lens and (B, C) prism-shaped focusing beam splitter in the $n$-$p$-$n$ junction in graphene-based transistor.
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