On the strength of a weak variant of the axiom of counting

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In this paper \(\text{NFU}^{-\text{AC}}\) is used to denote Jensen’s modification of Quine’s ‘new foundations’ set theory (\(\text{NF}\)) fortified with a type-level pairing function but without the axiom of choice. The axiom \(\text{AxCount}_{\geq}\) is the variant of the axiom of counting which asserts that no finite set is smaller than its own set of singletons. This paper shows that \(\text{NFU}^{-\text{AC}} + \text{AxCount}_{\geq}\) proves the consistency of the simple theory of types with infinity (TSTI). This result implies that \(\text{NF} + \text{AxCount}_{\geq}\) proves that consistency of TSTI, and that \(\text{NFU}^{-\text{AC}} + \text{AxCount}_{\geq}\) proves the consistency of \(\text{NFU}^{-\text{AC}}\).

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1 Introduction

The axiom of counting (\(\text{AxCount}\)) was introduced by Rosser in [14] and asserts that every finite set has the same cardinality as its own set of singletons. When added to Quine’s ‘new foundations’ set theory (\(\text{NF}\)) or Jensen’s variant of \(\text{NF}\) that allows urelements (\(\text{NFU}\)), this axiom proves the comprehension scheme for formulae which fail to be stratified only by virtue of the fact that well-defined types can not be assigned to variables which range only over natural numbers. In the nineteen seventies two natural variants of \(\text{AxCount}\) emerged: \(\text{AxCount}_{\leq}\) and \(\text{AxCount}_{\geq}\). The axiom \(\text{AxCount}_{\leq}\) asserts that a finite set is no bigger than its own set of singletons, while \(\text{AxCount}_{\geq}\) asserts that a finite set is no smaller than its own set of singletons. It quickly became apparent that many of the strong consequences of \(\text{AxCount}\) (over both \(\text{NF}\) and \(\text{NFU}\)) also follow from \(\text{AxCount}_{\leq}\) [4, 5]. In contrast \(\text{AxCount}_{\geq}\) appears to be a much weaker assumption. This paper investigates the strength of \(\text{AxCount}_{\geq}\) over a theory, \(\text{NFU}^{-\text{AC}}\), which can be viewed as both a subtheory of \(\text{NFU}\) (by which we mean Jensen’s system [8] supplemented both with an axiom asserting the existence of a type-level pairing function and with the axiom of choice) and \(\text{NF}\), and shows that this axiom proves the consistency of the simple theory of types with infinity (TSTI).

In the context of \(\text{NF}\) very little is known about the relative strengths of the theories obtained by adding \(\text{AxCount}\), \(\text{AxCount}_{\leq}\) and \(\text{AxCount}_{\geq}\). Steven Orey [11] shows that \(\text{NF} + \text{AxCount}\) proves the consistency of \(\text{NF}\). In [5], Hinnion develops techniques that yield lower bounds on the consistency strengths of the theories \(\text{NF} + \text{AxCount}_{\leq}\), \(\text{NF} + \text{AxCount}_{\geq}\) and \(\text{NF}\) relative to subsystems of ZFC. This paper provides a new lower bound on the consistency strength of \(\text{NF} + \text{AxCount}_{\geq}\) relative to a well-understood ZF-style theory. This lower bound is stronger than any known lower bound on the consistency strength of \(\text{NF}\).

There is much clearer picture of the relative strengths of the theories obtained by adding \(\text{AxCount}\), \(\text{AxCount}_{\leq}\) and \(\text{AxCount}_{\geq}\) to \(\text{NFU}\), largely thanks to Jensen’s consistency proof of \(\text{NFU}\) [8]. This consistency proof yields the exact strength of \(\text{NFU}\) relative to a subsystem of ZFC, and Solovay and Holmes (unpublished) have also computed the exact strength of the theory \(\text{NFU} + \text{AxCount}\) relative to a subsystem of ZFC. The paper [9] separates the consistency strengths of the theories \(\text{NFU} + \text{AxCount}\), \(\text{NFU} + \text{AxCount}_{\leq}\) and \(\text{NFU} + \text{AxCount}_{\geq}\) by showing that \(\text{NFU} + \text{AxCount}\) proves the consistency of \(\text{NFU} + \text{AxCount}_{\leq}\) and \(\text{NFU} + \text{AxCount}_{\geq}\) proves the consistency of \(\text{NFU} + \text{AxCount}_{\leq}\). Here it is shown that \(\text{NFU} + \text{AxCount}_{\geq}\) proves the consistency of \(\text{NFU}\) answering a question raised in [9].

This paper is only concerned with variants of Quine’s ‘new foundations’ that are fortified with the axiom of infinity. It is interesting to note, however, that in [1], Enayat investigates the strengths of extensions of the

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theory $\text{NFU}^-\infty$ that is obtained by adding the negation of the axiom of infinity to Jensen’s modification of NF. Solovay has shown (unpublished) that $\text{NFU}^-\infty$ is equiconsistent with the subsystem of arithmetic $\text{I} \Delta_0 + \exp$. Both $\text{AxCount}$ and $\text{AxCount}_{<\infty}$ are inconsistent with $\text{NFU}^-\infty$. The theory $\text{NFU}^-\infty + \text{AxCount}_{<\infty}$ is equivalent to the theory $\text{NFUV}^-\infty$ which Enayat [1] shows is equiconsistent with Peano arithmetic (PA).

2 Background

In this section we present the axioms of the theories $\text{NFU}^-\text{AC}$ and the simple theory of types (TSTI), and the extensions and subsystems of these theories that we shall refer to in the next section of the paper. We also present some necessary facts relating to the development of mathematics in $\text{NFU}^-\text{AC}$ and outline Hinnion’s technique of interpreting well-founded set theories in the set of equivalence classes of topped well-founded extensional relations in $\text{NFU}^-\text{AC}$. A detailed development of mathematics in an extension of $\text{NFU}^-\text{AC}$ can be found in the textbook [6]. We also refer the reader to the monograph [3] for a treatment of advanced topics in the study of stratified set theories including extensions of $\text{NFU}^-\text{AC}$ and TSTI.

Throughout this paper we shall use $\mathcal{L}$ to denote the language of set theory: first-order logic endowed with the binary relation $\in$. We use $\mathcal{L}_{\text{PA}}$ to denote the language of arithmetic: first-order logic endowed with binary function symbols $+$ and $\cdot$, and constant symbols $0$ and $1$. As usual, we shall write PA for the $\mathcal{L}_{\text{PA}}$-theory that consists of all of the axioms of Peano arithmetic. If $\mathcal{L}'$ is a recursive language and $T$ is a recursively axiomatised $\mathcal{L}'$-theory then we write $\text{Cons}(T)$ for the $\mathcal{L}_{\text{PA}}$-formula which asserts that $T$ is consistent.

The simple theory of types is the simplification of the underlying system of [15] independently discovered by Ramsey and Chistwick. Following [10] we use TSTI to denote the simple theory of types fortified with the axiom of infinity. This theory is naturally axiomatised in the many-sorted language with sorts for each $n \in \mathbb{N}$.

**Definition 2.1** We use $\mathcal{L}_{\text{TST}}$ to denote the $\mathbb{N}$-sorted language endowed with binary relation symbols $\in_n$ for each sort $n \in \mathbb{N}$. There are variables $x^n, y^n, z^n, \ldots$ for each sort $n \in \mathbb{N}$ and well-formed $\mathcal{L}_{\text{TST}}$-formulae are built-up inductively from atomic formulae in the form $x^n \in_n y^{n+1}$ and $x^n = y^n$ using the connectives and quantifiers of first-order logic.

An $\mathcal{L}_{\text{TST}}$-structure $\mathcal{M}$ consists of a function $M$ with domain $\mathbb{N}$ where $M(0), M(1), \ldots$ are the domains of the sorts, and a function $\in^M$ with domain $\mathbb{N}$ such that for all $n \in \mathbb{N}$, $\in^M(n) \subseteq M(n) \times M(n + 1)$; we write $\mathcal{M} = \langle M, \in^M \rangle$.

**Definition 2.2** We use TSTI to denote the $\mathcal{L}_{\text{TST}}$-theory with axioms

\[
\forall n \in \mathbb{N}, \forall x^{n+1} y^{n+1} (x^{n+1} = y^{n+1} \iff \forall z^n (z^n \in_n x^{n+1} \iff z^n \in_n y^{n+1})),
\]

(Extensionality)

\[
\forall x^n, \exists y^{n+1} \forall x^n (x^n \in_n y^{n+1} \iff \varphi(x^n, y^{n+1})),
\]

(Comprehension)

\[
\exists x^1 \forall f^3 (f^3 : x^1 \rightarrow x^1 \text{ is injective but not surjective}).
\]

(Infinity)

If $n$ is a natural number with $n \geq 4$ then we use $\text{TSTI}_n$ to denote the weakening of TSTI which only allows formulae to refer to objects with type $< n$.

**Definition 2.3** Let $n \in \mathbb{N}$ with $n \geq 4$. We use $\mathcal{L}_n$ to denote the $n$-sorted language endowed with binary relation symbols $\in_k$ for each sort $k < n - 1$. There are variables $x^k, y^k, z^k, \ldots$ for each sort $k < n$ and well-formed $\mathcal{L}_n$-formulae are built-up inductively from atomic formulae in the form $x^k \in_k y^{k+1}$ where $k < n - 1$, and $x^k = y^k$ where $k < n$, using the connectives and quantifiers of first-order logic.

If $n \in \mathbb{N}$ then we use $\lfloor n \rfloor$ to denote the set $\{i \in \mathbb{N} \mid i \leq n\}$ and we use $(n)$ to denote the set $\{i \in \mathbb{N} \mid i < n\}$. An $\mathcal{L}_n$-structure $\mathcal{M}$ consists of a function $M$ with domain $\lfloor n - 1 \rfloor$ where $M(0), \ldots, M(n - 1)$ are the domains of the sorts, and a function $\in^M$ with domain $(n - 1)$ such that for all $k < n - 1$, $\in^M(k) \subseteq M(k) \times M(k + 1)$; we write $\mathcal{M} = \langle M, \in^M \rangle$.
**Definition 2.4** Let $n \in \mathbb{N}$ with $n \geq 4$. We use $\mathcal{TSTI}_n$ to denote the $\mathcal{L}_n$-theory with axioms

for all $k < n - 1$, $\forall x^{k+1} \forall y^{k+1} (x^{k+1} = y^{k+1} \iff \forall z^k (z^k \in_k x^{k+1} \iff z^k \in_k y^{k+1}))$.

(Extensionality)

for all $k < n - 1$ and for all well-formed $\mathcal{L}_n$-formulae $\varphi(x^k, z)$,

$\forall \exists y^{k+1} \forall x^k (x^k \in_k y^{k+1} \iff \varphi(x^k, z))$.

(Comprehension)

$\exists x^1 \exists f^3 (f^3 : x^1 \rightarrow x^1$ is injective but not surjective).

(Infinity)

If $\sigma_0, \ldots, \sigma_m$ is a proof of a contradiction from $\mathcal{TSTI}$ then there exists an $n \in \mathbb{N}$ such that $\sigma_0, \ldots, \sigma_m$ is a proof of a contradiction from $\mathcal{TSTI}_n$. Combining this with the observation that there exists a binary Turing machine which on input $n \geq 4$ and $k$ decides whether the sentence with Gödel number $k$ is an axiom of $\mathcal{TSTI}_n$ yields

$$PA \vdash ((\forall n \geq 4) \text{Cons}(\mathcal{TSTI}_n) \iff \text{Cons}(\mathcal{TSTI})) \quad (1)$$

A formula in the language of set theory is said to be stratified if the formula can be turned into a well-formed formula of $\mathcal{L}_{\mathcal{TST}}$ by decorating variables and instances of $\in$ appearing in the formula with types. In 1937, Quine proposed an axiomatisation of set theory, now dubbed ‘new foundations’ (NF) after the title of [12], that appears to avoid the set theoretic paradoxes by restricting Cantor’s unrestricted comprehension scheme to stratified formulae. In [8], Jensen considers a weakening of NF that permits both sets and non-sets (urelements) in the domain of discourse. Jensen was able to show that this modification of NF is consistent relative to a weak subsystem of ZFC and, unlike NF (cf. [16]), is consistent with both the axiom of choice and the negation of the axiom of infinity.

In this paper, we shall stick to the convention of using $\mathcal{NFU}$ and, unlike $\mathcal{TSTI}$, we shall use $\mathcal{NFU}^\text{−AC}$ to denote $\mathcal{NFU}$ minus the axiom of choice.

**Definition 2.5** We use $\mathcal{L}_{\mathcal{NFU}}$ to denote the extension of $\mathcal{L}$ obtained adding a unary predicate $S$ and a binary function symbol $(\cdot, \cdot)$.

The unary predicate $S$ will be used to distinguish sets from urelements and $(\cdot, \cdot)$ will act as a type-level pairing function. Before presenting the axioms of $\mathcal{NFU}^\text{−AC}$ we first need to extend the notion of stratification to $\mathcal{L}_{\mathcal{NFU}}$-formulae.

**Definition 2.6** The terms of $\mathcal{L}_{\mathcal{NFU}}$ are built-up inductively from variables using the function $(\cdot, \cdot)$. Let $\varphi$ be an $\mathcal{L}_{\mathcal{NFU}}$-formula. We use $\text{Term}(\varphi)$ to denote the set of $\mathcal{L}_{\mathcal{NFU}}$-terms appearing in $\varphi$. We say that $\sigma : \text{Term}(\varphi) \rightarrow \mathbb{N}$ is a stratification of $\varphi$ if for all terms $s$ and $t$ appearing in $\varphi$,

(i) if $s$ is a term appearing in $t$ then $\sigma(\cdot^t) = \sigma(\cdot^s)$,

(ii) if $s \in t$ is a subformula of $\varphi$ then $\sigma(\cdot^t) = \sigma(\cdot^s) + 1$,

(iii) if $s = t$ is a subformula of $\varphi$ then $\sigma(\cdot^t) = \sigma(\cdot^s)$.

If there exists a stratification of $\varphi$ then we say that $\varphi$ is stratified.

**Definition 2.7** We use $\mathcal{NFU}^\text{−AC}$ to denote the $\mathcal{L}_{\mathcal{NFU}}$-theory with axioms

$\forall x \forall y (S(x) \land S(y) \Rightarrow (x = y \iff \forall z (z \in x \iff z \in y)))$.

(Weak extensionality)

for all stratified $\mathcal{L}_{\mathcal{NFU}}$-formulae $\varphi(x, \bar{z})$, $\forall \exists y (S(y) \land \forall x (x \in y \iff \varphi(x, \bar{z})))$.

(Strong stratified comprehension)

$\forall x \forall y (x = y \iff (w, z) \Rightarrow (x = w \land y = z))$.

(Pairing)

**Definition 2.8** We use $\mathcal{NFU}$ to denote the $\mathcal{L}_{\mathcal{NFU}}$-theory that obtained from $\mathcal{NFU}^\text{−AC}$ by adding the axiom of choice.
Following [6, 7] we have opted to include an axiom that asserts the existence of a type-level pairing function in our axiomatisation of NFU\(^{-\text{AC}}\). We shall indicate below how this pairing function implies that there exists a Dedekind infinite set. Without the axiom of choice, Jensen’s theory [8] fortified with an axiom asserting the existence of a Dedekind infinite set is not equivalent to NFU\(^{-\text{AC}}\), however they do have the same consistency strength. One way of seeing this is to use [8] combined with work done in [10] to see that the axiom of choice can be consistently added to Jensen’s theory [8] fortified with an axiom asserting the existence of a Dedekind infinite set. In the presence of both the axiom of choice and an axiom asserting the existence of a Dedekind infinite set, Jensen’s theory [8] is equipped with a type-level pairing function that can be used in instances of the comprehension scheme. This shows that NFU\(^{-\text{AC}}\) and TSTI have exactly the same consistency strength:

**Theorem 2.9** (Jensen) \(\text{Cons(NFU}^{-\text{AC}}) \iff \text{Cons(TSTI)}\).

The set theory NF can be obtained from NFU\(^{-\text{AC}}\) by adding an axiom which says that everything is a set.

**Definition 2.10** We use NF to denote the \(\mathcal{L}_{\text{NFU}}\)-theory obtained from NFU\(^{-\text{AC}}\) by adding the axiom

\[
\forall x (S(x)).
\]

(2)

It should be noted that we could have axiomatised NF, as is done in [12], in the language \(\mathcal{L}\). In the presence of (2) the symbol \(S\) becomes redundant and the weak extensionality axiom reduces to the usual extensionality axiom for set theory. It follows from [13, 16] that any model of NF can be expanded to a model with a pairing function \((\cdot, \cdot)\) that satisfies the pairing axiom and can be used in instances of the stratified comprehension scheme without raising types. It should also be noted that [16] shows that the axiom of choice is inconsistent with NFU\(^{-\text{AC}}\) plus (2).

Stratified comprehension in the theory NFU\(^{-\text{AC}}\) guarantees the existence of a universal set which we denote \(V\). The fact that the function \(x \mapsto \{x, x\}\) is injective but not surjective implies that \(V\) is Dedekind infinite. Cardinal and ordinal numbers in NFU\(^{-\text{AC}}\) are defined to be equivalence classes of equipollent sets and equivalence classes of isomorphic well-orderings respectively. If \(X\) is a set then we use \(|X|\) to denote the cardinal number such that \(X \in |X|\). Stratified comprehension ensures that both the set of all ordinals (NO) and the set of all cardinals (NC) exist. We use NCI to denote the set of infinite cardinals. The least cardinal number, denoted \(0\), is the set of all sets and urelements that have no members. We use \(\iota\) to denote the function \(x \mapsto \{x\}\). Equipped with a successor operation \((S)\) we are able to define the natural numbers (\(\mathbb{N}\)) as the smallest inductive set:

\[
S(x) = \{y \mid (\exists z \in y)(y \vDash \{1\} \in x)\}
\]

\[
\mathbb{N} = \bigcap \{x \mid \{0\} \in x \land (\forall y \in x)(S(y) \in x)\}.
\]

Define + and \(\cdot\) on \(\mathbb{N}\) by: for all \(k, m, n \in \mathbb{N}\),

\[
k + m = n \text{ if there exists } x \in k \text{ and } y \in m \text{ with } x \cap y = \emptyset \text{ and } n = |x \cup y|
\]

\[
k \cdot m = n \text{ if there exists } x \in k \text{ and } y \in m \text{ such that } |x \times y| = n.
\]

By letting \(1 = S(0)\) we obtain an \(\mathcal{L}_{\text{PA}}\)-structure \(\langle \mathbb{N}, +, \cdot, 0, 1 \rangle\) that is a model of PA. If \(n\) is a concrete natural number then by adjoining the sets \(\varphi(\mathbb{N}), \varphi^2(\mathbb{N}), \ldots, \varphi^{n-1}(\mathbb{N})\) to the structure \(\langle \mathbb{N}, +, \cdot, 0, 1 \rangle\) we obtain a structure that is a model of \(n\)th order arithmetic.

This interpretation of Peano arithmetic allows NFU\(^{-\text{AC}}\) to describe the syntax of recursive languages. If \(\mathcal{L}'\) is a recursive language then expressions in \(\mathcal{L}'\) can be coded as elements of \(\mathbb{N}\), called a Gödel coding, in such a way so as effective properties of \(\mathcal{L}'\) expressions are definable by arithmetic (and therefore stratified) \(\mathcal{L}_{\text{NFU}}\)-formulae. Given a recursive language \(\mathcal{L}'\) we assume that a Gödel coding of \(\mathcal{L}'\) has been fixed and we write \(\varphi^\gamma\) for the Gödel code of \(\varphi\). We shall often omit the corners and equate a formula \(\varphi\) with its Gödel code. In [5], Hinnion shows that if an \(\mathcal{L}\)-structure \(\mathcal{M}\) is a set then there is a stratified formula \(\text{Sat}_{\mathcal{L}}(\varphi, a, M)\) which says that \(\varphi\) is an \(\mathcal{L}\)-formula, \(a\) is sequence of elements of \(M\) that agrees with the arity of \(\varphi\) and \(M\) satisfies \(\varphi[a]\). If \(\mathcal{L}'\) is a recursive language then Hinnion’s definition of satisfaction for \(\mathcal{L}\)-structures can easily be extended to define a ternary stratified formula \(\text{Sat}_{\mathcal{L}}\) which expresses satisfaction in an \(\mathcal{L}'\)-structure. Using the stratified formulae \(\text{Sat}_{\mathcal{L}}\) one can see that NFU\(^{-\text{AC}}\) proves the single sentence which asserts that the structure \(\langle \mathbb{N}, +, \cdot, 0, 1 \rangle\) is a model of PA. And, moreover, for any concrete natural number \(n\), NFU\(^{-\text{AC}}\) proves the single sentence which asserts that
the structure \( \langle \phi^{n-1}(\mathbb{N}), \ldots, \phi(\mathbb{N}), \mathbb{N}, +, \cdot, 0, 1 \rangle \) is a model of \( n \)th order arithmetic. If \( \mathcal{M} \) is a set structure in a recursive language \( \mathcal{L} \), \( \phi \) is an \( \mathcal{L} \)-formula and \( a \) is a sequence of elements of \( \mathcal{M} \) then we shall write \( \mathcal{M} \models \varphi[a] \) instead of \( \text{Sat}_{\mathcal{L}}(\varphi, a, \mathcal{M}) \).

The following definition mirrors the definition of an initial ordinal in ZFC:

**Definition 2.11** We say that an ordinal \( \alpha \) is initial if \((\forall \delta < \alpha)([\{\beta \mid \beta < \delta\}] < [\{\beta \mid \beta < \alpha\}])\). We use \( \omega \) to denote the first initial ordinal, \( \omega_1 \) to denote the least initial ordinal \( > \omega \), \( \omega_2 \) to denote the least initial ordinal \( > \omega_1 \), etc.

In ZFC cardinals correspond to initial ordinals. It is important to note that in NFU\(^{-AC} \) this coincidence does not occur. If \( R \) is a binary relation then we shall write \( \text{Dom}(R) \) for \( \text{dom}(R) \cup \text{ran}(R) \).

**Definition 2.12** Let \( \alpha \) be an ordinal. Define \( \text{Card}(\alpha) = |\text{Dom}(R)| \) where \( R \in \alpha \). For all \( n \in \mathbb{N} \), define \( \aleph_n = \text{Card}(\omega_n) \).

One unorthodox feature of NFU\(^{-AC} \) is the fact that it proves that there are sets, for example \( V \), which are not the same size as their own set of singletons. This motivates the introduction of the \( T \) operation which is defined on cardinals, ordinals and equivalence classes of isomorphic well-founded relations, and the definition of Cantorian and strongly Cantorian sets. If \( R \) is a relation then we use \( [R] \) to denote the set of all relations isomorphic to \( R \). If \( F \) is a function and \( X \) is a set then we write \( F^\ast X \) for the set of all things that can be obtained by applying \( F \) to an element of \( X \).

**Definition 2.13** We say that a set \( X \) is Cantorian if \( |X| = |^\ast X| \). We say a set \( X \) is strongly Cantorian if the restriction of the map \( \iota \) to \( X \) witnesses the fact that \( |X| = |^\ast X| \).

**Definition 2.14** If \( X \) is a set then define \( T(|X|) = |^\ast X| \). If \( R \) is a well-founded relation then define \( T(|R|) = |\{(\langle x, y \rangle) \mid \langle x, y \rangle \in R \}| \).

The \( T \) operation commutes with the functions \( + \) and \( \cdot \) defined on \( \mathbb{N} \) and is the identity on \( 0 \) and \( 1 \). Since both \( T^{\ast n} \mathbb{N} \) and \( T^{-1} \mathbb{N} \) contain \( 0 \) and are closed under \( S \), it follows that \( T^{\ast n} \mathbb{N} = \mathbb{N} \). Therefore the \( T \) operation is an automorphism of the interpretation of arithmetic in a model of NFU\(^{-AC} \). The axiom of counting (AxCount) asserts that this automorphism is the identity:

\[
(\forall n \in \mathbb{N})(T(n) = n). \tag{AxCount}
\]

This axiom was first introduced by Rosser in [14] in order to facilitate induction in NF. Orey’s [11] shows that NF + AxCount proves Cons(NF). As part of [7], which also initiates the comparison of extensions of NFU with subsystems and extensions of ZFC, Holmes investigates the strength of the theory NFU + AxCount in terms of which infinite cardinals this theory proves exist. In [2], Forster identifies two natural weakenings of AxCount:

\[
(\forall n \in \mathbb{N})(n \leq T(n)), \tag{AxCount_{\leq}}
\]

\[
(\forall n \in \mathbb{N})(n \geq T(n)). \tag{AxCount_{\geq}}
\]

Many of the strong consequences of AxCount also follow from the weaker assumption AxCount\(_{\leq} \). For example, [5] shows that NF + AxCount\(_{\leq} \) proves the consistency of Zermelo set theory. And [4] shows that if NFU\(^{-AC} \) + AxCount\(_{\leq} \) is consistent then so is NFU\(^{-AC} \) + (the function on \( \mathbb{N} \) defined by \( n \mapsto V_n \) exists). In contrast it is not known if NF + AxCount\(_{\leq} \) proves the consistency of Zermelo set theory. And the assertion that the function on \( \mathbb{N} \) defined by \( n \mapsto V_n \) exists proves AxCount\(_{\leq} \). The relative strengths of AxCount, AxCount\(_{\leq} \) and AxCount\(_{\geq} \) over NFU is studied in [9]:

**Theorem 2.15** (I) NFU + AxCount \( \vdash \) Cons(NFU + AxCount\(_{\leq} \)) (II) NFU + AxCount\(_{\leq} \) \( \vdash \) Cons(NFU + AxCount\(_{\geq} \))

In [9], the author also provides evidence which appears to indicate that, over NFU, AxCount\(_{\geq} \) is weak.

**Theorem 2.16** There is a model of NFU + AxCount\(_{\leq} \) which believes that NCI is finite.

In [10], it is shown that TSTI is equiconsistent with the set theory MOST that is a subsystem of ZFC that includes the axiom of choice. The fact that equiconsistencies between MOST and TSTI, and TSTI and NFU\(^{-AC} \) show that these theories have the same arithmetic yields the following strong equiconsistency:
Theorem 2.17 (Jensen, Mathias) We have Cons(NFU−AC) ⇐⇒ Cons(NFU). Moreover, if φ is an $L_{PA}$-sentence then NFU−AC ⊢ φ if and only if NFU ⊢ φ.

It follows from Theorem 2.17 that any occurrence of “NFU” in Theorem 2.15 can be replaced by “NFU−AC”.

In [5], Hinnion shows that subsystems of ZFC can be interpreted in substructures of the set of equivalence classes of isomorphic topped well-founded extensional relations in NF. Hinnion’s techniques have since been established (cf. [6, 7, 17]) as the standard method for proving lower bounds on the consistency strength of extensions of NF−AC relative to subsystems and extensions of ZFC.

Definition 2.18 A structure $\langle A, R \rangle$, where $R$ is a binary relation, is a BFEXT if

(i) Dom($R$) = $A$,
(ii) $\forall B ((B \neq \emptyset \land B \subseteq A) \Rightarrow (\exists b \in B (\forall c \in B) \neg ((c, b) \in R))$,
(iii) $\forall a, b \in A (a = b \iff \forall c ((c, a) \in R \iff (c, b) \in R))$.

We say that a binary relation $R$ is a BFEXT if $\langle \text{Dom}(R), R \rangle$ is a BFEXT.

Definition 2.19 Let $\langle A, R \rangle$ be a BFEXT. If $a \in A$ then define

$\text{seg}_R(a) = R \cap \{B \subseteq A \mid (a \in B \land (\forall b \in B) ((\forall c \in B) ((c, b) \in R \Rightarrow c \in B))\}$.

Definition 2.20 $\Omega = \{ R \mid (R \text{ is a BFEXT}) \land (\exists a \in \text{Dom}(R)) (R = \text{seg}_R(a)) \}$.

The fact that $\Omega$ is defined by a stratified set abstract shows that NFU−AC proves that $\Omega$ is a set. If $R \in \Omega$ then the $a \in \text{Dom}(R)$ with $R = \text{seg}_R(a)$ is unique—we shall use $\mathbb{1}_R$ to denote this element. We shall sometimes call $[R]$ the type of $R$.

Definition 2.21 The structure $\langle \text{BF}, \mathcal{E} \rangle$ is defined by

$\text{BF} = \{ [R] \mid R \in \Omega \}$

$\mathcal{E} = \{ ([R], [S]) \in \text{BF} \times \text{BF} \mid (\exists a \in \text{Dom}(S)) (R \equiv \text{seg}_S(a) \land (a, \mathbb{1}_S) \in S) \}$.

Theorem 2.22 (Hinnion) The structure $\langle \text{BF}, \mathcal{E} \rangle$ is well-founded and extensional.

A consequence of this theorem is that if $a$ is an equivalence class of isomorphic BFEXTs then $\text{seg}_\mathcal{E}(a)$ is a BFEXT. The BFEXTs represented by $a$ are related to $\text{seg}_\mathcal{E}(a)$ by the $\mathbb{T}$ operation.

Lemma 2.23 (Hinnion) If $a \in \text{BF}$ then $[\text{seg}_\mathcal{E}(a)] = \mathbb{T}^1(a)$.

By considering rank initial segments of the structure $\langle \text{BF}, \mathcal{E} \rangle$ Hinnion builds models of subsystems of ZFC.

Definition 2.24 Let $S$ be a well-founded extensional relation. Define

$S^0 = \{ a \in \text{Dom}(S) \mid (\forall b \in \text{Dom}(S)) ((b, a) \in S) \}$,

$S^{\alpha+1} = \{ a \in \text{Dom}(S) \mid (\forall b ((b, a) \in S \Rightarrow b \in S^\alpha)) \}$,

$S^\lambda = \bigcup_{\alpha < \lambda} S^\alpha$ for limit $\lambda$.

Note that for well-founded extensional $S$, the formula $'x = S^\alpha'$ is stratified and admits a stratification which assigns the same type to the variables $'x'$ and $'\alpha'$.

Definition 2.25 For an ordinal $\alpha$, we use $M_\alpha$ to denote $\text{seg}_\mathcal{E}^{\alpha+\alpha}$.

Note that $M_0 = [\text{BF}]$ and $|M_0| < \aleph_0$ and $|M_0| = \aleph_0$.

One of the achievements of [5], which we mentioned above, was to show that if AxCount$_{<\omega}$ holds in NF then the single sentence asserting that $M_\omega$ is a model of Zermelo set theory is provable. Even though the setting of [5] is NF, Hinnion’s argument can also be carried out in the weaker theory NFU−AC (cf. [6, 17]). Combining this with the work in [10] on the consistency strength of TST1 we note the following weak version of Hinnion’s result which we shall use in the next section.

Theorem 2.26 (Hinnion) NFU−AC + AxCount$_{<\omega}$ ⊢ Cons(TST1).
3 NFU–AC + AxCount$_2$ proves the consistency of TSTI

In this section I will show that NFU–AC + AxCount$_2$ proves the consistency of TSTI. The main tool used to prove this result is the technique, developed in [5], of using the class of topped well-founded extensional relations in NFU–AC to interpret well-founded set theories. It follows from Theorem 2.26 that if AxCount holds in NFU–AC then Cons(TSTI) holds. In light of this, all we need to prove is that the theory NFU–AC + AxCount$_2$ + ¬AxCount proves the consistency of TSTI.

Let $\mathcal{M}$ be a model of NFU–AC + AxCount$_2$ + ¬AxCount. The proof will show that there is an elementary $\mathcal{L}_\text{PA}$-substructure $\mathcal{A}$ of $\langle \mathbb{N}^M, +^M, \cdot^M, 0^M, 1^M \rangle$ that satisfies Cons(TSTI). It then follows from the elementarity of $\mathcal{A}$ that $\mathcal{M}$ satisfies Cons(TSTI). The fact that $\mathcal{A}$ satisfies Cons(TSTI) will be obtained by showing that for every $n \in \mathcal{A}$, there is a set substructure of $\langle \mathcal{B}_F, \mathcal{E} \rangle$ that is a model of TSTI$_n$. The structure $\mathcal{A}$ is obtained by considering the fixed points of $T$ acting on $\mathbb{N}^M$. The following Lemma is proved in the theory NFU–AC + AxCount$_2$ + ¬AxCount:

**Lemma 3.1** If $n \in \mathbb{N}$ is such that $T(n) = n$ then for all $m \leq n$, $T(m) = m$.

**Proof.** Suppose that there are $n, m \in \mathbb{N}$ with $m < n$, $T(n) = n$ and $T(m) < m$. But then $n - m \in \mathbb{N}$ and $T(n - m) = T(n) - T(m) = n - T(m) > n - m$ which contradicts AxCount$_2$. \hfill $\square$

**Definition 3.2** Define $\mathcal{A} = \langle A, +^A, \cdot^A, 0^A, 1^A \rangle$ to be the $\mathcal{L}_\text{PA}$-substructure of $\langle \mathbb{N}^M, +^M, \cdot^M, 0^M, 1^M \rangle$ with domain $\bar{A} = \{n \in \mathbb{N}^M \mid \mathcal{M} \vDash (T(n) = n)\}$.

Note that since $A \subseteq \mathbb{N}^M$, $A \neq \mathbb{N}$, $0^M \in A$ and $A$ is closed under $S^M$ it follows that $A$ is not a set of $\mathcal{M}$.

**Lemma 3.3** The $\mathcal{L}_\text{PA}$-structure $\langle \mathbb{N}^M, +^M, \cdot^M, 0^M, 1^M \rangle$ is a proper elementary end-extension of $\mathcal{A}$.

**Proof.** Lemma 3.1 implies that $\langle \mathbb{N}^M, +^M, \cdot^M, 0^M, 1^M \rangle$ is an end-extension of $\mathcal{A}$. It follows from the fact that $\mathcal{M} \vDash \neg\text{AxCount}$ that $A \neq \mathbb{N}^M$. That $\mathcal{A} \not\subseteq \langle \mathbb{N}^M, +^M, \cdot^M, 0^M, 1^M \rangle$ follows since $T^M$ is an automorphism of the structure $\langle \mathbb{N}^M, +^M, \cdot^M, 0^M, 1^M \rangle$ and PA has definable Skolem functions. \hfill $\square$

We now turn to showing that for all $n \in \mathcal{A}$, $\mathcal{A}$ satisfies Cons(TSTI$_n$). This will be achieved by working in $\mathcal{M}$ and showing that there is an $n \in \mathbb{N}$ such that $T(n) < n$ and there exists a set model of TSTI$_n$ in the structure $\langle \mathcal{B}_F, \mathcal{E} \rangle$. From this point on we work inside $\mathcal{M}$.

**Definition 3.4** If $\kappa$ is a cardinal then define

$$2^\kappa = \begin{cases} |\phi(X)| & \text{if there exists a set } X \text{ with } |\iota^\kappa X| = \kappa \\ \emptyset & \text{otherwise.} \end{cases}$$

This modification of the usual definition of cardinal exponentiation has the property that the function $\kappa \mapsto 2^\kappa$ is definable by a stratified formula which admits a stratification that assigns the same type to the result and the argument of the function. The following result shows that this exponentiation operation possesses the strictly inflationary property that we intuitively associate with cardinal exponentiation.

**Lemma 3.5** Let $\kappa$ be a cardinal. If $2^\kappa \neq \emptyset$ then $\kappa < 2^\kappa$.

**Proof.** The usual proof of Cantor’s theorem yields for all $X$, $|\iota^\kappa X| \prec |\phi(X)|$ and this proof only appeals to stratified instances of comprehension. \hfill $\square$

**Definition 3.6** Define $\boxdot : \mathbb{N} \rightarrow \text{NC}$ by

$$\boxdot(0) = 8_0,$$

$$\boxdot(n + 1) = \begin{cases} 2^{\boxdot(n)} & \text{if } \boxdot(n) \in \text{NC}, \\ \emptyset & \text{if } \boxdot(n) = \emptyset. \end{cases}$$

Note that stratified comprehension ensures that $\boxdot$ is a set. The following results are proved or adapted from results proved in [5].

**Lemma 3.7** If $2^\kappa \neq \emptyset$ then $T(2^\kappa) = 2^{T(\kappa)}$.

**Proof.** This follows immediately from the fact that for all $X$, $|\iota^\kappa \phi(X)| = |\phi(\iota^\kappa X)|$. \hfill $\square$
Lemma 3.8 Let $n \in \mathbb{N}$. If $\emptyset(n) \neq \emptyset$ then $\emptyset(T(n)) = T(\emptyset(n))$.

Proof. We prove this by induction on $n$. It holds for $n = 0$. Suppose that the Lemma holds for $n$. Suppose that $\emptyset(n+1) \neq \emptyset$. Therefore $\emptyset(n) \neq \emptyset$ and so $\emptyset(T(n)) = T(\emptyset(n))$. So,

$$\emptyset(T(n+1)) = \emptyset(T(n) + 1) = 2^{\emptyset(T(n))} = 2^T(\emptyset(n)) = T(2^{\emptyset(n)}) = T(\emptyset(n+1)).$$

Lemma 3.9 If $\kappa \leq T(|V|)$ then $2^\kappa \neq \emptyset$.

Proof. Let $X \in \kappa$ and let $f : X \rightarrow \iota^*V$ be an injection. Let $B = \text{ran}(f)$ and let $A = \bigcup B$. Therefore $|\iota^*A| = \kappa$.

Lemma 3.10 There exists an $n \in \mathbb{N}$ with $T(n) < n$ such that $\emptyset(n) \neq \emptyset$ and $\emptyset(n) \leq T^4(|V|)$.

Proof. If $\emptyset(n) \neq \emptyset$ and $\emptyset(n) \leq T^4(|V|)$ for all $n \in \mathbb{N}$, then we are done since AxCount fails and AxCount$_\tau$ holds. Suppose that $n + 1$ is least such that $\emptyset(n+1) = \emptyset$ or $\emptyset(n+1) \not\subseteq T^4(|V|)$. If $T(n+1) = n+1$ then $T(n) = n$. And, $\emptyset(n) \neq \emptyset$ and $\emptyset(n) \leq T^4(|V|)$. But then, by Lemma 3.9, $\emptyset(n+1) \neq \emptyset$. And, by Lemma 3.8, $T(\emptyset(n+1)) = \emptyset(n+1)$. So, $\emptyset(n+1) \leq |V|$ implies that $\emptyset(n+1) \leq T^4(|V|)$, which contradicts our assumptions. Therefore $T(n+1) < n+1$ and $T(n) < n$. Since $n+1$ was least such that $\emptyset(n+1) = \emptyset$ or $\emptyset(n+1) \not\subseteq T^4(|V|)$, it follows that $\emptyset(n) \neq \emptyset$ and $\emptyset(n) \not\subseteq T^4(|V|)$.

Lemma 3.11 Let $n \in \mathbb{N}$. If $\emptyset(T(n)) \neq \emptyset$ then $|M_n| \leq T^4(|V|)$.

Proof. The assertion $\emptyset(T(n)) \neq \emptyset \Rightarrow |M_n| \leq T^4(|V|)$ is stratified, so we can prove it by induction. The base case holds because $|M_0| = \emptyset$. Suppose that the Lemma holds for some $n \in \mathbb{N}$ and assume that $\emptyset(T(n+1)) \neq \emptyset$. Note that if $a \in M_{n+1}$ then $\{b \mid \langle b, a \rangle \in E\} \subseteq M_n$. The map $g : \iota^*M_{n+1} \rightarrow \iota^*M_n$ defined by $\langle a \rangle \mapsto \{b \mid \langle b, a \rangle \in E\}$ is injective. Therefore $T(|M_{n+1}|) \leq |\iota^*M_n|$. Now, $|\iota^*M_n| = T(|M_n|)$, therefore $2^T(|M_n|) = |\iota^*M_n|$. We also know that $2^{\emptyset(T(n))} \neq \emptyset$. And so,

$$T(|M_{n+1}|) \leq 2^T(|M_n|) \leq 2^{|\emptyset(T(n))|} = T(2^{\emptyset(T(n))}) = T(\emptyset(n+1)).$$

Lemma 3.12 Let $n \in \mathbb{N}$ be such that $\emptyset(n) \neq \emptyset$ and $\emptyset(n) \leq T^4(|V|)$. If $a \in M_n$ then there is a $b \in BF$ such that $a = T^2(b)$.

Proof. Let $n \in \mathbb{N}$ be such that $\emptyset(n) \neq \emptyset$ and $\emptyset(n) \leq T^4(|V|)$. Let $a \in M_n$. Therefore $\text{dom}(\text{seg}_\iota(a)) \subseteq M_n$, and so $|\text{dom}(\text{seg}_\iota(a))| \leq |M_n|$. Since $T(n) \leq n$, we know that $\emptyset(T(n)) \neq \emptyset$. Therefore, by Lemma 3.11,

$$|\text{dom}(\text{seg}_\iota(a))| \leq T(n) \leq |\emptyset(n) \leq T^4(|V|)|.$$

Let $f : \text{dom}(\text{seg}_\iota(a)) \rightarrow \iota^*V$ be an injection. Let $A = \bigcup \text{ran}(f)$ and define $S \subseteq A \times A$ by

$$(x, y) \in S \text{ if and only if } (f^{-1}(x^0), f^{-1}(y^0)) \in \text{seg}_\iota(a).$$

Let $b = [S]$. By Lemma 2.23 we have $T^2(a) = [\text{seg}_\iota(a)] = T^4(b)$. Therefore $a = T^2(b)$.

Lemma 3.13 Let $n \in \mathbb{N}$ be such that $\emptyset(n) \neq \emptyset$ and $\emptyset(n) \leq T^4(|V|)$. If $B \subseteq M_n$ then there exists $\bar{a} \in M_{n+1}$ such that for all $b \in M_n$ we have $\langle b, \bar{a} \rangle \in E$ if and only if $b \in B$.

Proof. Let $B \subseteq M_n$. It follows from Lemma 3.12 that for all $b$, there exists $b' \in BF$ such that $b = T^2(b')$. Therefore, by Lemma 2.23, for all $b \in BF$, $b = [\text{seg}_\iota(b')]$. Define

$$S = \left( \bigcup \{\text{seg}_\iota(b') \mid T^2(b') \in B\} \right) \cup \{(b', V) \mid T^2(b') \in B\}.$$ 

Now, $S \in \Omega$ and so let $\bar{a} = [S]$. It is clear that $\bar{a} \in M_{n+1}$ and for all $b \in M_n$ we have $\langle b, \bar{a} \rangle \in E$ if and only if $b \in B$.

Equipped with these results we are now in a position to show that $\mathcal{M}$ satisfies Cons(TSTI).

Lemma 3.14 $\mathcal{M} \models \text{Cons(TSTI)}$.

Proof. Suppose that $\mathcal{M} \models \neg\text{Cons(TSTI)}$. Therefore $\mathcal{M} \models (\exists k \geq 4)\neg\text{Cons(TSTI)}$. Since the arithmetic of $\mathcal{M}$ is elementarily equivalent to $\mathcal{A}$ this implies $\mathcal{A} \models (\exists k \geq 4)\neg\text{Cons(TSTI)}$. Let $k \in A$ be such that $\mathcal{A} \models \neg\text{Cons(TSTI)}$. Since $\mathcal{A}$ is an elementary submodel of the arithmetic of $\mathcal{M}$, this means that $\mathcal{M} \models \neg\text{Cons(TSTI)}$.
Note that $\mathcal{M} \models (T(k) = k)$. Now, work inside $\mathcal{M}$. Let $n \in \mathbb{N}$ be such that $T(n) < n$, $\exists(n) \neq \emptyset$ and $\exists(n) \leq T^4(|V|)$. Lemma 3.10 ensures that there exists an $n \in \mathbb{N}$ with these properties. Note that for all $i \leq n$, $\exists(i) \neq \emptyset$ and $\exists(i) \leq T^4(|V|)$. We shall build a set model of TSTI$_{n+1}$. Since $k < n$ this will yield a contradiction. Let $D : [n] \rightarrow V$ be defined by $D(i) = M_i$ for all $i \leq n$. Let $\in_D^D : (n) \rightarrow V$ be defined by $\in_D^D (i) = E | M_i \times M_{i+1}$. Note that stratified comprehension ensures that the functions $D$ and $\in_D^D$ are sets. The structure $D = (D, \in_D^D)$ is an $\mathcal{L}_{n+1}$-structure. We need to show that $\mathcal{D} \models TSTI_{n+1}$. To see that $\mathcal{D} \models (Extensionality)$ observe that the structure $(\mathcal{B}, \mathcal{E})$ is an extensional (Theorem 2.22) and for each $i \leq n$, $(\mathcal{B}, \mathcal{E})$ is an end-extension of $(M_i, \mathcal{E})$. We now turn to showing that $\mathcal{D} \models (Comprehension)$. Let $i < n$. Let $\varphi(x^i, z_0, \ldots, z_{m-1})$ be an $\mathcal{L}_{n+1}$-formula (according to $\mathcal{M}$). We need to show that $\mathcal{D} \models \forall z \exists y \forall x^i (x^i \in_i y^{i+1} \iff \varphi(x^i, z))$.

Let $\tilde{a} : (m) \rightarrow \bigcup \text{ran}(D)$ be a sequence such that for all $0 \leq \ell < m$, $\tilde{a}(\ell) \in D(j_i)$. Let

$$B = \{ b \in M_i \mid \exists \tilde{c}(\tilde{m} : [m] \rightarrow V) \land (\tilde{c}(0) = b) \land (\forall \ell \in (m))(\tilde{c}(\ell + 1) = \tilde{a}(\ell)) \land (D \models \varphi(\tilde{c})) \}. $$

Stratified comprehension ensures that $B$ is a set. Clearly $B \subseteq M_{i+1}$. By Lemma 3.13 there exists $d \in M_{i+1}$ such that for all $b \in M_i$, $D \models (b \in_i d)$ if and only if $(b, d) \in E$

$$\text{if and only if there exists } \tilde{c} : [m] \rightarrow V \text{ with } \tilde{c}(0) = b, \quad (\forall \ell \in (m))(\tilde{c}(\ell + 1) = \tilde{a}(\ell)), \quad \text{and } D \models \varphi(\tilde{c}).$$

This shows that Comprehension holds in $\mathcal{D}$.

Finally, we need to show that $\mathcal{D} \models (Infinity)$. Let $R = \{ (i, j) \mid i, j \in \mathbb{N} \land i < j \} \cup \{ (i, V) \mid i \in \mathbb{N} \}$. The relation $R$ is a BFEXT with $\mathcal{R} = \text{seg}_R(V)$, therefore $R \in \Omega$. Let $\mathcal{a} = [R]$. Note that $a \in M_1$. Let $\varphi(x, y)$ be the $\mathcal{L}$-formula $(x = (z_1, z_2)) \land (z_1, z_2 \in y) \land (z_2 = z_1 \cup \{z_1\})$. Let

$$B = \{ b \in M_2 \mid \exists \tilde{c}(\tilde{c} : (2) \rightarrow V) \land (\tilde{c}(0) = b) \land (\tilde{c}(1) = a) \land (\langle (M_1, \mathcal{E}) \models \varphi(\tilde{c}) \rangle) \}. $$

Stratified comprehension ensures that $B$ is a set. We also have that $B \subseteq M_3$. Using Lemma 3.13 we can find $d \in M_3$ such that for all $b \in M_2$ we have $(b, d) \in E$ if and only if $b \in B$. The point $d$ in $\mathcal{D}$ is an injective function that witnesses that $a$ is Dedekind infinite. This shows that $\mathcal{M} \models \text{Cons(TSTI}_{n+1})$. Since $k < n + 1$, $\mathcal{M} \models \text{Cons(TSTI}_{k})$, which is a contradiction.

Since $\mathcal{M}$ was an arbitrary model of NFU$^{-AC} + \text{AxCount}_{\geq} \iff \neg \text{AxCount}$ this proves:

**Theorem 3.15** NFU$^{-AC} + \text{AxCount}_{\geq} \models \text{Cons(TSTI)}$.

Combined with Theorem 2.9 this shows that the theory NFU$^{-AC} + \text{AxCount}_{\geq}$ has strictly stronger consistency strength than the theory NFU$^{-AC}$.

**Corollary 3.16** NFU$^{-AC} + \text{AxCount}_{\geq} \models \text{Cons(NFU}^{-AC})$.

Again, Theorem 2.17 allows any occurrence of “NFU$^{-AC}$” in Corollary 3.16 to be replaced with “NFU”. The following from [9] still remains open:

**Question 3.17** What is the exact consistency strength of NFU + AxCount$_{\geq}$ relative to a subsystem of ZFC?

Since the theory NF can be viewed as an extension of the theory NFU$^{-AC}$, Theorem 3.15 also yields:

**Corollary 3.18** NF + AxCount$_{\geq} \models \text{Cons(TSTI)}$.

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