SOLUTION TO A STOCHASTIC PURSUIT MODEL USING MOMENT EQUATIONS

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Abstract. The paper investigates the navigation problem of following a moving target, using a mathematical model described by a system of differential equations with random parameters. The differential equations, which employ controls for following the target, are solved by a new approach using moment equations. Simulations are presented to test effectiveness of the approach.

1. Introduction. Presently, stochastic differential equations can be employed as a mathematical model in many disciplines, including cybersecurity or physics. Most of these stochastic models, however, have remained inaccessible to applied scientists who are not experts in statistics and advanced mathematics. Inaccessibility in these and other fields might be because the main method for solving these equations is via finding the probability distribution function as a function of time using the equivalent Fokker-Planck equation, which tells us how the probability distribution function evolves in time. Moreover, solving these differential equations is not an easy task. Therefore, finding an explicit solution to a stochastic differential equation is possible only in some special cases. There is, thus, a great need for a simple and straightforward presentation of the methods for obtaining a solution to stochastic models.

Here we will consider a problem, whose basic mechanism is deterministic and fully understood, but at the same time incorporates elements of randomness, therefore it manifests itself as a stochastic process. Problems described by a differential
equation with random parameters appear in various areas of research and practice and constitute an important part of applications of mathematics.

Optimization of linear systems with random parameters are considered in many works, for example in monograph [12]. In particular, original results concerning stabilization of systems with random coefficients and random process are derived using moment equations and Lyapunov functions in monograph [4]. The results contained in the monograph offer a more convenient setting for applying the method in practice using suitable software, i.e. an engineering or economics application.

The aim of [10] is the expansion of achieved results to a new class of systems of linear differential equations with semi-Markov coefficients and random transformation of solutions performed simultaneously with jumps of a semi-Markov process. The work focuses on using the particular values of Lyapunov functions for the calculation of coefficients of the control vector which minimize the quality criterion, and also on establishing the necessary conditions of the optimal solution which enables the synthesis of the optimal control for the considered class of systems.

Reference [2] deals with systems of linear differential equations with coefficients depending on a Markov process, using the moment equations method.

Investigated in [3] are questions of solvability of initial value problems and stability of solutions. The mathematical model of foreign currency exchange market in the form of a stochastic linear differential equation with coefficients depending on a semi-Markov process is considered in [3]. The boundary of the unstable domain is determined using the moment equations method.

Paper [5] deals with a system of nonlinear differential equations under influence of white noise. This system can be used as a mathematical model of various real problems in finance, mathematical biology, climatology, signal theory and others. Necessary and sufficient conditions for the asymptotic mean square stability of the zero solution of this system are derived in the paper.

In many works which are dealing with stochastic systems of differential equations, methods are used which are different from the moment equations method. For instance, based on the linear matrix inequality technique, some criteria are obtained for exponential mean-square stability of nonlinear stochastic systems which model breast cancer stem cell dynamics with time-delays [8]. Three dimensional primitive equations with a small multiplicative noise are studied in [7]. The existence and uniqueness of solutions with small initial value in a fixed probability space are obtained. The proof is based on Galerkin approximation, Itô’s formula and weak convergence methods.

Specifically, we deal with the problem of navigation to a target described by differential equation with random parameters. Therefore, we introduce stochastic components into a mathematical model of this problem, so that it will be possible to take account of imperfections (approximations) in the present specification of the model. We offer a new approach to solve the problem by using moment equations and their quantification to derive controls for following the target. The origin of the theory of moment equations and their use in the examination of the properties of solutions can be found in the works by K. Valeev and his scientific school, see, for instance, [12]. The moment equations method was used also in works [2, 3, 5, 10] for studying stability of solutions to various kinds of systems with random structure.

2. Statement of the problem - the stochastic model. Firstly, we formulate the necessary assumptions for modeling with stochastic differential equations. Let
Let us assume given a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and let \(\xi(t)\) be a Markov process defined on this probability space (see [9]).

Let us consider a problem of pursuing a target. We will assume that the pursuer and his target are moving in a common vertical plane. We will also assume that the target moves at a constant speed on the ground along a line such that his course can randomly vary positively or negatively. The pursuer observes the target from above, i.e. from the air, trying to hit it. The status and speed of the target is known the whole time in the pursuer’s control system. The questions that interest us are whether the pursuer hit his target and, if he did not, what was the distance between the hit point and the target.

Let \(x, y\) be the coordinates of the pursuer and the target in the plane of movement. We will lay a coordinate system such that the direction of motion of the target is the same as the \(x\)-axis direction. We will assume that the target can change its direction of movement so quickly that its speed remains constant, but it cannot move faster than a limited value of \(v_{\text{lim}}\), for instance \(v_{\text{lim}} = 10\) m/s. Assuming these conditions, then the movement of the target can be described by a stochastic differential equation

\[
y' = b(\xi(t)), \quad |b(\xi(t))| \leq v_{\text{lim}}, \quad v_{\text{lim}} = 10\, ms^{-1}
\]  

where \(\xi(t)\) is a Markov process with two states

1) \(\xi(t) = \theta_1\), if the movement of the target is towards the positive direction, i.e. \(b(\xi(t) = \theta_1) = v > 0\);
2) \(\xi(t) = \theta_2\), if the movement of the target is towards the negative direction, i.e. \(b(\xi(t) = \theta_2) = -v < 0\).

The probability, that the Markov process is in state \(\theta_1\) or \(\theta_2\) will be denoted by \(p_1(t) = P\{\xi(t) = \theta_1\}\) or \(p_2(t) = P\{\xi(t) = \theta_2\}\), respectively.

We now define a system of differential equations that will describe the movement of the pursuer and the target in the \(xy\)-plane simultaneously. We will assume that the pursuer (or missile, for simplicity) has known descent speed \(v_g\), \(v_g \approx 100\) \(ms^{-1}\).

Moreover, it is known that the missile has a controlled stabilizer that can rotate the missile in a predicted direction to the angle \(\alpha\) relative to the vector of the missile velocity \(\mathbf{v}\). The angle between the vector \(\mathbf{v}\) and vertical direction we will denote as \(\varphi\). We assume that the deflection angle \(\varphi\) is not bigger than angle \(\varphi_0 \approx 15^\circ\).
There exist forces acting on the missile, the distribution of the forces is shown on Figure 1. As the missile velocity increases, it encounters air resistance, which opposes the motion. The force generated by air resistance is given by Newton’s law of air resistance, it is

\[ F_x = C_x(\alpha)S \rho \frac{v^2}{2} \]

where \( \rho \) is air density, \( S \) is the surface area of the axial section, and \( C_x(\alpha) \) is the coefficient determined by the shape of the missile.

In a direction perpendicular to the vector \( v \) exists a force

\[ F_y = C_y(\alpha)S \rho \frac{v^2}{2} \]

Angle \( \alpha \) can be changed by the guidance system of the pursuer and the force \( F_y \) could be understood as a control mechanism. Values \( S, C_x, C_y \) are unknown, therefore the equations of motion will be based on other facts. We will neglect changes in the value of \( C_x \) if the changes of the attack angle \( \alpha \) are small. Therefore we can say

\[ mg \cos \varphi \approx C_x(\alpha)S \rho \frac{v^2_g}{2}, \]

whence, under the assumption that \( \cos \varphi \approx 1 \), we can find the approximate value

\[ v_g = \sqrt{\frac{2mg}{C_x(\alpha)S \rho}}. \]

We can find projection forces on the horizontal axis \( O_x \), see Figure 1,

\[ (-mg \sin \varphi + F_y) \cos \varphi = -mg \frac{x}{v_g} + U \]

where \( U = C_y(\alpha)S \rho \frac{v^2}{2} \cos \varphi \). Define control function \( W = \frac{U}{m} \). Then the movement along the horizontal axis can be described by the differential equation

\[ \ddot{x} = -\delta \dot{x} + W \]  

(2)

where \( \delta = \frac{g}{v_g} \). Let \( y = x + x_1 \), \( \dot{x} = x_2 \). Then, in view of (1) and (2) we get

\[ \dot{x}_1 = b(\xi(t)) - x_2, \]
\[ \dot{x}_2 = -\delta x_2 + W. \]  

(3)

We will assume that the control function regulates the deviations of the missile trajectory \( x \) and velocity \( \dot{x} \) from the targets trajectory \( y \) and velocity \( \dot{y} \) respectively. Moreover, we will assume that the control function \( W \) will be linear, \( W_L = \alpha(y - x) + \beta(\dot{y} - \dot{x}) \), or

\[ W_L = \alpha x_1 + \beta(b(\xi(t)) - x_2) \]

where \( \alpha \) and \( \beta \) are coefficients of the control function \( W \), \( |W| \leq W_{\max} \),

\[ W = \left\{ \begin{array}{ll}
W_{\max}, & \text{if } W_L > W_{\max}, \\
W_L, & \text{if } |W_L| \leq W_{\max}, \\
-W_{\max}, & \text{if } W_L < -W_{\max}.
\end{array} \right. \]
3. Preliminary results. To derive the main result the method of moment equations is used. Such systems of moment equations for a non-homogenous linear system of stochastic equations was derived in [2]. Therefore we give a short summary of it.

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. On the probability space we consider the initial value problem formulated for stochastic system

\[
\frac{dx(t)}{dt} = A(t, \xi(t))x(t) + B(t, \xi(t)),
\]

\(x(0) = \varphi(\omega),\)

where \(A\) is an \(m \times m\) matrix with random elements, \(B\) is an \(m\)-dimensional column vector function whose elements are random variables, \(\varphi: \Omega \rightarrow \mathbb{R}^m\), \(\varphi \in C(\Omega)\), \(\xi(t)\) is random Markov process with a finite number of states \(\theta_k, k = 1, 2, \ldots, q\), with probabilities

\[
p_k(t) = P\{\xi(t) = \theta_k\}, \quad k = 1, 2, \ldots, q,
\]

and satisfy the system of linear differential equations

\[
\frac{dp_k(t)}{dt} = \sum_{s=1}^{q} \pi_{ks}(t)p_s(t)
\]

with transition matrix \((\pi_{ks}(t))^{q}_{k,s=1}\) satisfying

\[
\sum_{k=1}^{q} \pi_{ks}(t) \equiv 0, \quad \pi_{ks}(t) \begin{cases} \geq 0, & k \neq s, \\ \leq 0, & k = s. \end{cases}
\]

Moreover, we will use abbreviations

\[
A(t, \theta_k) = A_k(t), \quad B(t, \theta_k) = B_k(t), \quad k = 1, 2, \ldots, q.
\]

Definition 3.1. The \(m\)-dimensional random vector function \(x(t)\), whose components are random variables, is called a solution of the initial value problem (4), (5) if \(x(t)\) satisfies (4) and the initial condition (5) in the sense of strong solution of the initial Cauchy problem.

In the sequel, \(E_m\) denotes an \(m\)-dimensional Euclidean space and the \(m\)-dimensional row vector-functions \(f_k(t, x), n = 1, 2, \ldots, k = 1, 2, \ldots, q, x \in E_m\) are the particular probability density functions of \(x\).

Definition 3.2. Let \(x \in \mathbb{R}^m\) be a continuous random variable depending on a random Markov process \(\xi(t)\) with \(q\) possible states \(\theta_k, k = 1, 2, \ldots, q\). The vector function

\[
E^{(1)}\{t\} = \sum_{k=1}^{q} E_k^{(1)}\{t\}
\]

where

\[
E_k^{(1)}\{t\} = \int_{E_m} x f_k(t, x) \, dx, \quad k = 1, 2, \ldots, q,
\]

is called a moment of the first order of the random variable \(x(\xi(t))\). The values \(E_k^{(1)}\{x\}, k = 1, 2, \ldots, q\) are called particular moments of the first order.
Definition 3.3. Let \( x \in \mathbb{R}^m \) be a continuous random variable depending on a random Markov process \( \xi(t) \) with \( q \) possible states \( \theta_k, k = 1, 2, \ldots, q \). The matrix function

\[
E^{(2)}(t) = \sum_{k=1}^{q} E_k^{(2)}(t)
\]

where

\[
E_k^{(2)}(t) = \int_{\mathbb{R}^m} xx^* f_k(t, x) \, dx, \quad k = 1, 2, \ldots, q
\]

is called a moment of the second order of the random variable \( x(\xi(t)) \). The values \( E_k^{(2)}(t), k = 1, 2, \ldots, q \) are called particular moments of the second order.

Theorem 3.4. [2] The moment equations of the first and the second order respectively for the system (4) are

\[
\frac{dE_k^{(1)}(t)}{dt} = A_k(t) E_k^{(1)}(t) + B_k(t) p_k(t) + \sum_{j=1}^{q} \pi_{kj}(t) E_j^{(1)}(t),
\]

\[
\frac{dE_k^{(2)}(t)}{dt} = A_k(t) E_k^{(2)}(t) + E_k^{(2)}(t) A_k^*(t) + B_k(t) \left( E_k^{(1)}(t) \right)^* + E_k^{(1)}(t) B_k^*(t) + \sum_{j=1}^{q} \pi_{kj}(t) E_j^{(2)}(t),
\]

\( k = 1, 2, \ldots, q \).

4. Main results. The problem discussed in the first section, which was modeled on a probability space \( (\Omega, \mathcal{F}, P) \) by system of stochastic differential equations (3), can be rewritten as

\[
\dot{x}_1(t) = -x_2(t) + b(\xi(t)), \\
\dot{x}_2(t) = \alpha x_1(t) - (\delta + \beta) x_2(t) + \beta b(\xi(t)),
\]

where \( \xi(t) \) is above-mentioned Markov process. Or, if we denote

\[
x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -1 \\ \alpha & - (\delta + \beta) \end{pmatrix}, \quad b(\xi(t)) = b(\xi(t)) \begin{pmatrix} 1 \\ \beta \end{pmatrix}
\]

then system (10) takes the form

\[
\frac{dx}{dt} = Ax(t) + b(\xi(t)),
\]

which is already in the same form as system (4) with two possible states of the Markov process.

4.1. Moment equations for the process. Now, using the preliminary result formulated in Theorem 3.4, we will derive a system of ordinary differential equations whose solution behaves like the mean value of the solution to stochastic system (10).

Theorem 4.1. The mean value \( E^{(1)} = (m_1, m_2)^T \) of the solution to stochastic system (11) is the solution to the system of ordinary linear differential equations

\[
\frac{dm_1(t)}{dt} = -m_2(t) + a e^{-ct} \\
\frac{dm_2(t)}{dt} = \alpha m_1(t) - (\delta + \beta) m_2(t) + \beta a + \beta be^{-ct}
\]
where
\[ c \equiv \lambda + \nu, \quad a \equiv \frac{\lambda v_2 + \nu v_1}{c}, \quad b \equiv \left( p - \frac{\nu}{c} \right) (v_1 - v_2), \quad p \equiv p_1(0). \quad (13) \]

**Proof.** In view of (6), (7), (8) and due to the fact that both states are equally probable we get the system of differential equations
\[
\begin{align*}
\dot{p}_1(t) &= -\lambda p_1(t) + \nu p_2(t), \\
\dot{p}_2(t) &= \lambda p_1(t) - \nu p_2(t),
\end{align*}
\]
for determining the probability that there occurs one of the states \( \theta_1 \) or \( \theta_2 \) of the Markov process \( \xi(t) \). The parameters \( \lambda \equiv \pi_{11} \equiv \pi_{21}, \nu \equiv \pi_{12} \equiv \pi_{22} \) determine intensity of state changes of the process.

Because we need to know \( p_1(t) \), we solve the system (14). Using relationship \( p_2(t) = 1 - p_1(t) \) from the first equation of system (14), we get
\[ \frac{dp_1(t)}{dt} + (\lambda + \nu)p_1(t) = \nu, \]
or
\[ \frac{d}{dt} (p_1 e^{(\lambda + \nu)t}) = \nu e^{(\lambda + \nu)t}. \]
Then initial condition \( p_1(0) = p \) applies to obtain solution \( p_1(t) \),
\[ p_1(t) = \frac{\nu}{\lambda + \nu} + \left( p - \frac{\nu}{\lambda + \nu} \right) \frac{1}{e^{(\lambda + \nu)t}}. \quad (15) \]
The moment equations for system (11), using equation (9) in Theorem 3.4 are:
\[
\begin{align*}
\frac{dE^{(1)}_1}{dt} &= A_1 E^{(1)}_1 + b_1 p_1 - \lambda E^{(1)}_1 + \nu E^{(1)}_2, \\
\frac{dE^{(1)}_2}{dt} &= A_2 E^{(1)}_2 + b_2 p_2 + \lambda E^{(1)}_1 - \nu E^{(1)}_2,
\end{align*}
\]
where \( A_1 \equiv A_2, \quad b_1 = b (\xi(t) = \theta_1) = v_1 (1, \beta)^T, \quad b_2 = b (\xi(t) = \theta_2) = v_2 (1, \beta)^T. \)

Adding the two previous equations using \( E^{(1)} = E^{(1)}_1 + E^{(1)}_2 \), we get
\[ \frac{dE^{(1)}}{dt} = A E^{(1)} + b_1 p_1 + b_2 p_2. \]
In view of \( p_1(t) + p_2(t) = 1 \) this system can be written as
\[ \frac{dE^{(1)}}{dt} = A E^{(1)} + b_2 + (b_1 - b_2) p_1, \quad (16) \]
or
\[ \frac{dE^{(1)}}{dt} = \left( \begin{array}{c} \dot{m}_1 \\ \dot{m}_2 \end{array} \right) = \left( \begin{array}{cc} 0 & -1 \\ \alpha & - (\delta + \beta) \end{array} \right) \left( \begin{array}{c} m_1 \\ m_2 \end{array} \right) + v_2 \left( \begin{array}{c} 1 \\ \beta \end{array} \right) + (v_1 - v_2) \left( \begin{array}{c} 1 \\ \beta \end{array} \right) p_1, \]
Finally, using the notation of (13) and solution (15), system (16) can be rewritten into the desired form (12) with the initial condition \( E^{(1)}(0) = \langle x(0) \rangle = x(0) = (x_1(0), x_2(0))^T = (s(0), 0)^T. \)

**Theorem 4.1** provides us with a guarantee that a solution of ordinary linear differential equations (12) behaves as the mean value of a solution to the stochastic model of pursuit of a target. Therefore, instead of a solution to a complicated system of differential equations with random parameters, we can solve the system of
ordinary differential equations by known methods. The following results of simulations of the random process, and their comparison with the solution to system (12), support our claim.

4.2. Simulation results for the stochastic process. In this section we will monitor the mean value \( E^{(1)}(s(t)) \) of the stochastic process at time \( t = 10 \) depending on parameters \( \lambda, p \) and \( s(0) \). After this we compare the mean value \( E^{(1)}(s(t)) \) of the stochastic process \( s(t) \) as a solution to (12) with the mean value of several simulations of the stochastic process.

First, we need to calculate the value \( W_{\max} \). If we assume that the missile was launched from height 1000 m, then the missile is falling 10 s. The value \( W_{\max} \) can be calculated assuming that the initial velocity is \( x_2(0) = 0 \), the angle of deflection close to the ground is \( \varphi = 15^\circ \) and \( \delta = \frac{g}{v_g} \approx 0.1 \). If \( W = W_{\max} \), then from (3) we get equation

\[
\dot{x}_2 = -\delta x_2 + W_{\max},
\]

\( x_2(0) = 0 \).

By integrating this equation, in view of the initial condition, we get

\[
x_2(t) = -\frac{W_{\max}}{\delta} e^{-\delta t} + \frac{W_{\max}}{\delta},
\]

whence

\[
x_2(10) = \frac{W_{\max}}{\delta} (1 - e^{-10\delta}).
\]

On the other hand, given that the missile velocity in the horizontal direction satisfies

\[
\dot{x} = x_2(t) = gt \tan \varphi, \quad x_2(10) \approx 100 \tan 15^\circ = 26.8 \text{ m/s},
\]

we have \( W_{\max} \approx 4.24 \text{ m/s}^2 \). Let \( W_{\max} = 5, \alpha = 0.1146, \beta = 0.5 \), then the discriminant \( D = (\delta + \beta)^2 - 4\alpha = (0.1 + 0.5)^2 - 4 \cdot 0.1146 < 0 \), the roots of the characteristic equation of associated homogeneous system to system (12) are complex eigenvalues and the general solution to system (12) can be written in the form

\[
E^{(1)} = \begin{pmatrix} \omega \cos \varphi t + \varphi \sin \varphi t, & \varphi \cos \varphi t - \omega \sin \varphi t \\ -\alpha \cos \varphi t, & \alpha \sin \varphi t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{\omega t} + \begin{pmatrix} A_{11} & A_{12} \\ \alpha & A_{22} \end{pmatrix} \begin{pmatrix} 1 \\ e^{\varphi t} \end{pmatrix}
\]

where

\[
\omega = -\frac{\delta - \beta}{2}, \quad \varphi = \frac{\sqrt{4\alpha - (\delta + \beta)^2}}{2},
\]

\[
A_{11} = \frac{a \delta}{\alpha}, \quad A_{12} = \frac{b - A_{22}}{c}, \quad A_{22} = \frac{b \alpha + \beta b c}{c^2 + \alpha + c(\delta + \beta)}.
\]

Using the initial condition \( E^{(1)}(0) = (s, 0)^T \) we get constants \( c_1, c_2 \),

\[
c_1 = \frac{a + A_{22}}{\alpha}, \quad c_2 = \frac{s - c_1 \omega - A_{11} - A_{12}}{\varphi}.
\]

We are interested mainly in \( E^{(1)}(10) = \langle x_1(10) \rangle = \langle s(10) \rangle \), which is the mean value of the process \( s(t) \) at time \( t = 10 \), which represents the amount by which the missile missed the target.
Figure 2. The mean value of the process \( s(t) \) with parameters \( \lambda \) and \( p \): \( \lambda = 0.01; \ p = 0.1, 0.2, \ldots, 1; \ s(0) = 0, 4, 8, \ldots, 200. \)

Figure 3. The mean value of the process \( s(t) \) with parameters \( \lambda \) and \( p \): \( \lambda = 0.01, 0.02, \ldots, 0.2; \ p = 0.1, 0.2, \ldots, 1; \ s(0) = 0, 4, 8, \ldots, 200. \)

We will monitor the mean value of the process \( s(t) \) depending on intensity of state changes \( \lambda \), on the initial probability \( p = p_1(0) \) that the process will be in state \( \theta_1 \) and on the initial value \( s(0) \), as follows:

\[
\begin{align*}
\lambda &= 0.01; \ p = 0.1, 0.2, \ldots, 1; \ s(0) = 0, 4, 8, \ldots, 200; \\
\lambda &= 0.01, 0.02, \ldots, 0.2; \ p = 0.1, 0.2, \ldots, 1; \ s(0) = 0, 4, 8, \ldots, 200; \\
\lambda &= 0.01, 0.011, 0.012, \ldots, 0.21; \ p = 0, 0.01, \ldots, 1; \ s(0) = 0; \\
\lambda &= 0.01, 0.011, 0.012, \ldots, 0.21; \ p = 0, 0.01, \ldots, 1; \ s(0) = 0, 20, \ldots, 100.
\end{align*}
\]
Figure 4. The mean value of the process \( s(t) \) with parameters \( \lambda \) and \( p \): \( \lambda = 0.01, 0.011, 0.012, \ldots, 0.21; \, p = 0, 0.01, \ldots, 1; \, s(0) = 0 \).

The results of monitoring are recorded graphically on Figures 2 - 5. On the basis of these results, it can be concluded:

1) The largest mean value can be achieved if \( p = 1 \), \( s(0) \) is as large as possible, and in the opposite direction: the mean value is the smallest, if \( p = 0 \), \( s(0) < 0 \) is as small as possible. Therefore, the mean of the deflection of the target from the point of impact is greatest if it starts to move from a location that is as far away from the orthogonal projection of the missile’s starting point while at the same time begins to move away from the orthogonal deflection of the missile. (Figure 2)

2) The value \( \langle s(10) \rangle \) almost does not depend on \( \lambda \) or \( p \) and, from a certain value of \( \lambda \), depends only on \( s(0) \), for instance from \( \lambda > 0.1 \). Interestingly, when \( p = 0.5 \), then the mean value \( \langle s(10) \rangle \) does not depend on \( \lambda \) at all, moreover, for any \( \lambda \) and a fixed value \( s(0) \), it is the same. (Figure 3)

3) If \( \lambda > 0.1 \), the value \( \langle s(10) \rangle \) almost does not depend on \( p \) and is near zero.

The value \( |\langle s(10) \rangle| \) reaches its maximum when \( \lambda \) is as small as possible and the direction of the movement at the beginning is \( p = 0 \) or \( p = 1 \). (Figure 4, Figure 5)

Some simulations of the stochastic processes related to this work, which are not of the Itô type, can be found in [11]. Here two simulations of the stochastic process using the values \( \lambda = \nu = 1, \, p = 0.5, \, s(0) = 25 \) are performed and their mean value is compared to the mean value \( E^{(1)}_1 \{ s(t) \} \) of the stochastic process as the solution to (12), see Figure 6. The mean value of the two simulations (simulated \( E^{(1)}_1 \{ t \} \)) appears to explain nothing, but in this case, for certain values of \( t \), it is close to the solution of (12) \( (E^{(1)}_1 \{ t \}) \). Using a greater number of simulations it can be confirmed that the solution to system of ordinary linear differential equations (12) behaves as the mean value of the considered stochastic process. One hundred simulations of the stochastic process using the values \( \lambda = \nu = 0.3, \, p = 0.5, \, s(0) = 25 \) are shown in Figure 7. Their mean value (simulated \( E^{(1)}_1 \{ t \} \)) is very good approximation of
Figure 5. The mean value of the process $s(t)$ with parameters $\lambda$ and $p$: $\lambda = 0.01, 0.011, 0.012, \ldots, 0.21$; $p = 0, 0.01, \ldots, 1$; $s(0) = 0, 20, \ldots, 100$.

Figure 6. The mean value $E_1^{(1)} \{s(t)\}$ of the stochastic process as the solution to (12) and the mean value of two simulations of the stochastic process.

solution to (12) ($E_1^{(1)} \{t\}$). By increasing the number of simulations to a thousand we get full agreement of the mean values.

5. Conclusion. This work develops and solves the pursuit problem in a stochastic setting. Problems of this kind belong to a group of single-channel local control problems. The simple model determines the parameters of a random process so that the deviation of the missile impact point from the target reaches its maximum value. The target changes its direction randomly, which causes deviation of the missile from the target, while it is supposed that the target and also the missile are controlled optimally. Future research might study more complex situations that are associated with multi-channel control in a stochastic setting, with different conditions on the target and pursuer.
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