Generating Functions for Spherical Harmonics and Spherical Monogenics

P. Cerejeiras, U. Kähler & R. Lávička

Advances in Applied Clifford Algebras

ISSN 0188-7009
Volume 24
Number 4

Adv. Appl. Clifford Algebras (2014)
24:995-1004
DOI 10.1007/s00006-014-0495-8
Generating Functions for Spherical Harmonics and Spherical Monogenics

P. Cerejeiras, U. Kährler and R. Lávička*

To K. Gürlebeck

Abstract. In this paper, we study generating functions for the standard orthogonal bases of spherical harmonics and spherical monogenics in \( \mathbb{R}^m \). Here spherical monogenics are polynomial solutions of the Dirac equation in \( \mathbb{R}^m \). In particular, we obtain the recurrence formula which expresses the generating function in dimension \( m \) in terms of that in dimension \( m - 1 \). Hence we can find closed formulæ of generating functions in \( \mathbb{R}^m \) by induction on the dimension \( m \).

Keywords. Spherical harmonics, spherical monogenics, Gelfand-Tsetlin basis, orthogonal basis, generating function.

1. Introduction

It is well-known that classical orthogonal polynomials can be defined by generating functions. This close relationship allows for an indirect study of a given family of orthogonal polynomials by means of formal manipulations of its generating function. A classical example is the shifted Newtonian potential which is the generating function of spherical harmonics (see [30]). Not only one obtains several properties and recursion formulas of spherical harmonics by manipulation of the generating function but it also allows to establish new relationships with other families of orthogonal polynomials (see, for instances, [2], [25]). For example, the Gegenbauer polynomials

\[
C'_\nu(x) = \sum_{k=0}^{\infty} C'_{\nu}^k(x) h^k
\]

where \( \nu > 0 \), \( x, h \in \mathbb{R}, |x| \leq 1 \) and \( |h| < 1 \) (see e.g. [16, p. 18] or [27, p.173]). The classic approach to generating functions for spherical harmonics is to separate the angular part and construct the generating function of the

*Corresponding author.
associated Legendre polynomials [32, 1]. As will be clear in the sequel this approach to the generating function is not suitable for our purposes and we will present a different construction.

In [27], a general framework is developed for a study of properties of polynomial sequences, including the Appell property and generating functions. One of the principal advantages of a generating function is that instead of studying the action of an operator on each basis function one only needs to study the action of said operator on the generating function. This was used to great effect in many areas, such as Umbral calculus, quantum mechanics, and others [27, 30, 28]. Furthermore, generating functions form a bridge between analysis and discrete mathematics, by providing a really efficient tool for solving difference equations [31].

In this paper, we deal with generating functions for the standard orthogonal bases of spherical harmonics and spherical monogenics in $\mathbb{R}^m$. Possible applications of this theory lie in the study of monogenic operators, non-commutative combinatorics, and structural mechanics. Also, studying difference equations over the set of monogenic functions or their boundary values is closely linked with the study of problems in image processing by means of the so-called monogenic signal.

Orthogonal bases of spherical harmonics are well-known and have been studied for a long time. Spherical harmonics are useful in many theoretical areas and on applications such as structural mechanics, etc. In Clifford analysis, a similar role is played by spherical monogenics. Monogenic functions are defined as Clifford algebra valued solutions $f$ of the equation $\partial f = 0$ where $\partial$ is the Dirac operator on $\mathbb{R}^m$. Spherical monogenics are polynomial solutions of the Dirac equation. Since the Dirac operator $\partial$ factorizes the Laplace operator $\Delta$ in the sense that $\Delta = -\partial^2$ Clifford analysis can be understood as a refinement of harmonic analysis. On the other hand, monogenic functions are at the same time a higher dimensional analogue of holomorphic functions of one complex variable. See [5, 15, 19, 18] for an account of Clifford analysis.

The first construction of orthogonal bases of spherical monogenics valid for any dimension was given by F. Sommen, see [29, 15]. In dimension 3, explicit constructions using the standard bases of spherical harmonics were done also by K. Gürlebeck, H. Malonek, I. Caąao and S. Bock (see e.g. [3, 8, 9, 10, 11, 12, 13]). From the point of view of representation theory, the standard bases of spherical harmonics are nothing else than examples of the so-called Gelfand-Tsetlin bases, see [26]. V. Šouček proposed studying these bases in Clifford analysis. In particular, in [4], it is observed that the complete orthogonal system in $\mathbb{R}^3$ of [3] and F. Sommen’s bases [29, 15] can be both considered as Gelfand-Tsetlin bases. Actually, it turns out that Gelfand-Tsetlin bases in all cases so far studied in Clifford analysis are, by construction, uniquely determined and orthogonal and, in addition, they possess the so-called Appell property, see [24] for a recent survey, [21, 22] for the classical Clifford analysis, [14, 23] for Hodge-de Rham systems and [6, 7] for Hermitian Clifford analysis. Therefore we call them the standard orthogonal
bases in the sequel. For a detailed historical account of this topic, we refer to [4].

In this paper, we study generating functions for the standard orthogonal bases of spherical harmonics and spherical monogenics in $\mathbb{R}^m$. We obtain the recurrence formula which expresses the generating function in dimension $m$ in terms of that in dimension $m-1$, see below Theorem 1 for spherical harmonics and Theorem 2 for spherical monogenics. Using the recurrence formula, we can obtain closed formulæ of generating functions in $\mathbb{R}^m$ by induction on the dimension $m$. This is based on the generating function (1) for the Gegenbauer polynomials. It seems that analogous results can be obtained also for Hodge-de Rham systems [23] and even in Hermitian Clifford analysis [7]. But, in the hermitian case, the generating function for the Jacobi polynomials should be used instead of (1).

2. Spherical Harmonics

In this section, we study generating functions for spherical harmonics in $\mathbb{R}^m$. Denote by $\mathbb{B}_m$ the unit ball in $\mathbb{R}^m$. Let us recall the standard construction of an orthogonal basis in the complex Hilbert space $L^2(\mathbb{B}_m, \mathbb{C}) \cap \text{Ker } \Delta$ of $L^2$-integrable harmonic functions $g : \mathbb{B}_m \to \mathbb{C}$.

One proceeds by induction on the dimension $m$. Of course, in $\mathbb{R}^2$ the polynomials $\text{harm}_k^{\pm}, k \in \mathbb{N}_0^1$, give by

$$\text{harm}_k^{\pm}(x) = (x_1 \pm ix_2)^k/k!, \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad (2)$$

form an orthogonal basis of the space $L^2(\mathbb{B}_2, \mathbb{C}) \cap \text{Ker } \Delta$.

Now let $m \geq 3$. To construct the bases in higher dimensions, we need to introduce the following embedding factors. For $k, j \in \mathbb{N}_0^1$, we define

$$F_{m,j}^{(k)}(x) = |x|^k m^{|2+j-1|} C^m_{m/2+j-1}(x_m/|x|), \quad x = (x_1, \ldots, x_m) \in \mathbb{R}^m \quad (3)$$

where $|x|_m = \sqrt{x_1^2 + \cdots + x_m^2}$. Then, it is well-known that in $\mathbb{R}^m$ an orthogonal basis of the space $L^2(\mathbb{B}_m, \mathbb{C}) \cap \text{Ker } \Delta$ is formed by the polynomials $\text{harm}_k^{\pm}, k = (k_2, \ldots, k_m) \in \mathbb{N}_0^{m-1}$, given by

$$\text{harm}_k^{\pm}(x) = \text{harm}_{k_2}^{\pm}(x_1, x_2) \prod_{r=3}^m F_{r,k_r}^{(k_r)}(x_1, \ldots, x_r) \quad (4)$$

where $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$ and $k_*^{r} = k_2 + \cdots + k_r$. See e.g. [16, p. 35] or [22]. In difference to [22], we use another normalization of the embedding factors $F_{m,j}^{(k_m)}$ and we also change the notation for indices which in turns provides a more elegant expression for generating functions.

**Definition 1.** We define the generating function $H_m^{\pm}(x, h)$ of the orthogonal basis $\text{harm}_k^{\pm}, k \in \mathbb{N}_0^{m-1}$ of spherical harmonics in $\mathbb{R}^m$ by

$$H_m^{\pm}(x, h) = \sum_{k \in \mathbb{N}_0^{m-1}} \text{harm}_k^{\pm}(x) h^k$$
whenever the series on the right-hand side converges absolutely. Here \( x \in \mathbb{R}^m \), \( h = (h_2, \ldots, h_m) \in \mathbb{R}^{m-1} \) and \( h^k = h_2^k \cdots h_m^k \).

Obviously, the following result follows easily from (1).

**Lemma 1.** For \( x \in \mathbb{R}^m \) and \( h_m \in \mathbb{R} \), we have that
\[
\sum_{k_m=0}^{\infty} F_{m,j}(x)^{h_m} = \frac{1}{(1 - 2x_m h_m + h_m^2 |x_m|^2)^{\frac{m}{2}+j}}
\]
whenever \( |x_m| \leq 1 \), \( |h_m| < 1 \) and \( j \in \mathbb{N}_0 \).

Now we prove basic properties of the generating functions \( H_m^\pm \).

**Theorem 1.** For each \( m \geq 2 \) there is a neighborhood \( U_m \) of 0 in \( \mathbb{R}^{m-1} \) such that the following statements hold true.

(i) The generating functions \( H_m^\pm(x, h) \) are defined if \( |x_m| \leq 1 \) and \( h \in U_m \). Here \( x = (x_1, \ldots, x_m) \in \mathbb{R}^m \) and \( h = (h_2, \ldots, h_m) \in \mathbb{R}^{m-1} \).

(ii) For each \( k = (k_2, \ldots, k_m) \in \mathbb{N}_0^{m-1} \), we have that
\[
\text{harm}^\pm_k(x) = \frac{1}{k!} \partial^k H_m^\pm(x, h)|_{h=0}, \quad |x| \leq 1
\]
where \( k! = (k_2)! \cdots (k_m)! \) and \( \partial^k = \partial_{h_2}^{k_2} \cdots \partial_{h_m}^{k_m} \).

(iii) For \( m \geq 3 \), \( |x| \leq 1 \) and \( h \in U_m \), we have that
\[
H_m^\pm(x, h) = d_m^{1-\frac{m}{2}} H_{m-1}^\pm(x, h/d_m)
\]
where \( d_m = 1 - 2x_m h_m + h_m^2 |x_m|^2 \), \( x = (x_1, \ldots, x_{m-1}) \) and \( h/d_m = (h_2/d_m, \ldots, h_{m-1}/d_m) \).

**Proof.** We prove this theorem by induction on the dimension \( m \). It is easily seen that the theorem is true for \( m = 2 \). Indeed, we have that
\[
H_2^\pm(x_1, x_2, h_2) = \sum_{k_2=0}^{\infty} \frac{(x_1 \pm ix_2)^{k_2}}{k_2!} h_2^{k_2} = \exp((x_1 \pm ix_2)h_2).
\]

Now assume that the theorem is true for \( m - 1 \). Let \( H_{m-1}^\pm(x, h) \) be defined for \( h \in U_{m-1} = (-\delta_2, \delta_2) \times \cdots \times (-\delta_{m-1}, \delta_{m-1}) \) and \( |x|_{m-1} \leq 1 \) and let \( |x| \leq 1 \). It is easy to see that
\[
H_m^\pm(x, h) = \sum_k \left( \sum_{k_m=0}^{\infty} F_{m,k_m}^{(k_m)}(x) h_m^{k_m} \right) \text{harm}^\pm_k(x) h_m^k
\]
where the first sum is taken over all \( k = (k_2, \ldots, k_{m-1}) \in \mathbb{N}_0^{m-2} \). By Lemma 1, we have that
\[
\sum_{k_m=0}^{\infty} F_{m,k_m}^{(k_m)}(x) h_m^{k_m} = d_m^{1-\frac{m}{2}}(k_2 \cdots k_{m-1})
\]
if \( |h_m| < 1 \). Using this formula and (5), we have that
\[
H_m^\pm(x, h) = d_m^{1-\frac{m}{2}} \sum_k \text{harm}^\pm_k(x) (h/d_m)^k = d_m^{1-\frac{m}{2}} H_{m-1}^\pm(x, h/d_m)
\]
whenever \( h \in U_m = (-\delta_2/4, \delta_2/4) \times \cdots \times (-\delta_m-1/4, \delta_m-1/4) \times (-1/2, 1/2) \). Indeed, \( d_m \geq (1 - h_m |x_m|)^2 > 1/4 \) if \(|h_m| < 1/2\). Hence, if \( h \in U_m \) we have that \( h/d_m \in U_{m-1} \) and, by (5), we can easily see that some rearrangement of the power series defining \( H^\pm_m(x, h) \) converges at \( h \). Then Abel’s Lemma [20, Proposition 1.5.5, p. 23] proves that this power series converges absolutely on the whole \( U_m \), which finishes the proof of the theorem.

Using the recurrence formula (iii) of Theorem 1, we can find closed formulæ of generating functions for spherical harmonics in \( \mathbb{R}^m \) by induction on the dimension \( m \).

**Corollary 1.** In particular, we have the following formula

\[
H^\pm_3(x, h) = \frac{1}{(1 - 2x_3h_3 + h_3^2|x_3|^2)^{1/2}} \exp \left( \frac{(x_1 \pm ix_2)h_2}{1 - 2x_3h_3 + h_3^2|x_3|^2} \right).
\]

Here \( x = (x_1, x_2, x_3) \in \mathbb{R}^3 \) and \( h = (h_2, h_3) \in \mathbb{R}^2 \).

**Remark 1.** It is well-known that an orthogonal basis of real valued spherical harmonics in \( \mathbb{R}^m \) is formed by the polynomials \( \Re \text{harm}^+_k, \Im \text{harm}^+_k, k \in \mathbb{N}_0^{m-1} \). Here \( \Re z \) and \( \Im z \) are the real and imaginary part of the complex number \( z \). Hence the corresponding generating functions are \( \Re H^+_m, \Im H^+_m \).

**Remark 2.** If one replaces in the definition of the orthogonal basis (4) the polynomials \( \text{harm}^+_k (x_1, x_2) = (x_1 \pm ix_2)^{k_2}/(k_2!) \) with

\[
\overline{\text{harm}}^+_k (x_1, x_2) = (x_1 \pm ix_2)^{k_2},
\]

the corresponding generating functions \( \overline{H}^+_m \) are definitely different from \( H^+_m \) but they obviously satisfy again Theorem 1. In particular, we have that

\[
\overline{H}_2^\pm(x, h) = \sum_{k_2=1}^{\infty} (x_1 \pm ix_2)^{k_2}h_2^{k_2} = \frac{1 - (x_1 \mp x_2)i h_2}{1 - 2x_1h_2 + h_2^2|x_2|^2}.
\]

Here \( x = (x_1, x_2) \in \mathbb{R}^2 \) and \( h_2 \in \mathbb{R} \).

3. Spherical Monogenics

In this section, we introduce and investigate generating functions for spherical monogenics. For an account of Clifford analysis, we refer to [5, 15, 19, 18]. Denote by \( \mathcal{C} \) either the real Clifford algebra \( \mathbb{R}_{0,m} \) or the complex one \( \mathbb{C}_m \), generated by the elements \( e_1, \ldots, e_m \) such that \( e_j^2 = -1 \) for \( j = 1, \ldots, m \). As usual, a vector \( x = (x_1, \ldots, x_m) \in \mathbb{R}^m \) corresponds to the element \( \mathbf{x} = x_1e_1 + \cdots + x_me_m \) of the Clifford algebra \( \mathcal{C} \). Let \( G \subset \mathbb{R}^m \) be open. Then a continuously differentiable function \( f: G \to \mathcal{C} \) is called monogenic if it satisfies the equation \( \partial f = 0 \) on \( G \) where the Dirac operator \( \partial \) is defined as

\[
\partial = e_1\partial_{x_1} + \cdots + e_m\partial_{x_m}.
\]

Denote by \( L^2(\mathbb{B}_m, \mathcal{C}) \cap \text{Ker} \partial \) the space of \( L^2 \)-integrable monogenic functions \( g: \mathbb{B}_m \to \mathcal{C} \). It is well-known that \( L^2(\mathbb{B}_m, \mathcal{C}) \cap \text{Ker} \partial \) forms the
right $C\ell_m$-linear Hilbert space. Let us recall a construction of an orthogonal basis in this space which is quite analoguous to the harmonic case described in the previous section, see [22] for more details.

It is easy to see that in $\mathbb{R}^2$ the polynomials $\text{mon}_{k_2}, k_2 \in \mathbb{N}_0$, given by

$$
\text{mon}_{k_2}(x) = (x_1 - e_{12}x_2)^{k_2}/(k_2!), \quad x = (x_1, x_2) \in \mathbb{R}^2,
$$

form an orthogonal basis of the space $L^2(\mathbb{B}_2, C\ell_2) \cap \text{Ker}\, \partial$. Here we write $e_{12} = e_1e_2$ as usual. Now let $m \geq 3$. To construct the bases in higher dimensions, we need to introduce the embedding factors $X^{(k)}_{m,j}$ for $k, j \in \mathbb{N}_0$, defined as

$$
X^{(k)}_{m,j}(x) = \frac{m - 2 + k + 2j}{m - 2 + 2j} F^{(k)}_{m,j}(x) + F^{(k-1)}_{m,j+1}(x) \mathbf{x}_m
$$

where $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$ and $\mathbf{x} = x_1e_1 + \cdots + x_{m-1}e_{m-1}$. Here $F^{(k)}_{m,j}$ are given in (3) and we put $F^{(-1)}_{m,j+1} = 0$. Then it is well-known that an orthogonal basis of the space $L^2(\mathbb{B}_m, C\ell_m) \cap \text{Ker}\, \partial$ is formed by the polynomials

$$
\text{mon}_{k}(x) = X^{(k_m)}_{m,k_{m-1}} X^{(k_{m-1})}_{m-1,k_{m-2}} \cdots X^{(k_3)}_{3,k_2} \text{mon}_{k_2}(x_1, x_2), \quad x \in \mathbb{R}^m
$$

where $k = (k_2, \ldots, k_m) \in \mathbb{N}_0^{m-1}$ and $k_r = k_2 + \cdots + k_r$. Here

$$
X^{(k_r)}_{r,k_{r-1}} = X^{(k_r)}_{r,k_{r-1}}(x_1, \ldots, x_r)
$$

for $r = 3, \ldots, m$. Let us remark that due to non-commutativity of the Clifford multiplication the order of factors in the product (10) is important. See [22] for more details. In comparison with [22], we use another normalization of the embedding factors $X^{(k_m)}_{m,j}$ and we also change the notation for indices to get a nice expression for generating functions.

**Definition 2.** We define the generating function $M_m$ of the orthogonal basis $\text{mon}_k$, $k \in \mathbb{N}_0^{m-1}$ of spherical monogenicics in $\mathbb{R}^m$ by

$$
M_m(x, h) = \sum_{k \in \mathbb{N}_0^{m-1}} \text{mon}_k(x) \, h^k
$$

whenever the series on the right-hand side converges absolutely. Here $x \in \mathbb{R}^m$ and $h = (h_2, \ldots, h_m) \in \mathbb{R}^{m-1}$.

In particular, it is easily seen that

$$
M_2(x_1, x_2, h_2) = \sum_{k_2=0}^{\infty} \frac{(x_1 - e_{12}x_2)^{k_2}}{k_2!} h_2^{k_2} = \exp((x_1 - e_{12}x_2)h_2).
$$

Here $\exp((x_1 - e_{12}x_2)h_2) = \exp(x_1h_2)(\cos(x_2h_2) - e_{12}\sin(x_2h_2))$. To study the generating functions in higher dimensions we need to know the generating function of the embedding factors $X^{(k_m)}_{m,j}$.

**Lemma 2.** For $x \in \mathbb{R}^m$ and $h_m \in \mathbb{R}$, we have that

$$
\sum_{k_m=0}^{\infty} X^{(k_m)}_{m,j}(x) h_m^{k_m} = \frac{1 + h_m \mathbf{x}_m}{(1 - 2x_m h_m + h_m^2 \, |x|^2 m/2 + j)}
$$
whenever $|x|_m \leq 1$, $|h_m| < 1$ and $j \in \mathbb{N}_0$. Here $\mathbf{x} = x_1 \mathbf{e}_1 + \cdots + x_m \mathbf{e}_m$.

**Proof.** Put $\nu = m/2 - 1 + j$. By (9), the series we want to sum up is equal to

$$
\sum_{k_m=0}^{\infty} \frac{k_m + 2\nu}{2\nu} F_{m,j}^{(k_m)}(x) h_m^{k_m} + \sum_{k_m=1}^{\infty} F_{m,j+1}^{(k_m-1)}(x) h_m^{k_m} \mathbf{x} e_m = \Sigma_1 + \Sigma_2.
$$

Obviously, by Lemma 1, we get that

$$
\Sigma_2 = \frac{h_m \mathbf{x} e_m}{(1 - 2x_m h_m + h_m^2 |x|_m^2)\nu + 1}.
$$

Moreover, using Lemma 1 again, we have that

$$
\Sigma_1 = \frac{h_m}{2\nu} \frac{d}{dh_m} (1 - 2x_m h_m + h_m^2 |x|_m^2)\nu + 1 + \frac{1}{(1 - 2x_m h_m + h_m^2 |x|_m^2)\nu + 1},
$$

and hence

$$
\Sigma_1 = \frac{1 - x_m h_m}{(1 - 2x_m h_m + h_m^2 |x|_m^2)\nu + 1}.
$$

Finally, using $\mathbf{x} = \mathbf{x} + x_m \mathbf{e}_m$ we conclude that

$$
\Sigma_1 + \Sigma_2 = \frac{1 + h_m \mathbf{x} e_m}{(1 - 2x_m h_m + h_m^2 |x|_m^2)^{m/2+j}},
$$

which finishes the proof. \qed

Now we can prove basic properties of the generating functions $M_m$ quite similarly as in the harmonic case if, in this case, we use Lemma 2 instead of Lemma 1. Then we obtain the following result.

**Theorem 2.** For each $m \geq 2$ there is a neighborhood $U_m$ of 0 in $\mathbb{R}^{m-1}$ such that the following statements hold true.

(i) The generating functions $M_m(x, h)$ are defined if $|x|_m \leq 1$ and $h \in U_m$. Here $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$ and $h = (h_2, \ldots, h_m) \in \mathbb{R}^{m-1}$.

(ii) For each $k \in \mathbb{N}_0^{m-1}$, we have that

$$
\text{mon}_k(x) = \frac{1}{k!} \partial^k M_m(x, h)|_{h=0}, \quad |x|_m \leq 1
$$

where $k! = (k_2!) \cdots (k_m!)$ and $\partial^k = \partial_{h_2}^{k_2} \cdots \partial_{h_m}^{k_m}$.

(iii) For $m \geq 3$, $|x|_m \leq 1$ and $h \in U_m$, we have that

$$
M_m(x, h) = (1 + h_m \mathbf{x} e_m) d_m^{-\nu} M_{m-1}(x, h/d_m)
$$

where $d_m = 1 - 2x_m h_m + h_m^2 |x|_m^2$, $x = (x_1, \ldots, x_{m-1})$ and $h/d_m = (h_2/d_m, \ldots, h_{m-1}/d_m)$.

Using the recurrence formula (iii) of Theorem 2, we can find closed formulæ of generating functions for spherical monogenics in $\mathbb{R}^m$ by induction on the dimension $m$. 

Corollary 2. In particular, we have the following formula

\[ M_3(x, h) = \frac{1 + h_3 x e_3}{(1 - 2x_3 h_3 + h_3^2 |x|^2)^{3/2}} \exp \left( \frac{(x_1 - e_{12} x_2) h_2}{1 - 2x_3 h_3 + h_3^2 |x|^2} \right). \]

Here \( x = (x_1, x_2, x_3) \in \mathbb{R}^3 \) and \( h = (h_2, h_3) \in \mathbb{R}^2 \).

Remark 3. If one replaces in the definition of the orthogonal basis (10) the polynomials \( \text{mon}_{k_2}(x_1, x_2) = (x_1 - e_{12} x_2)^{k_2} / (k_2!) \) with \( \overline{\text{mon}}_{k_2}(x_1, x_2) = (x_1 - e_{12} x_2)^{k_2} \),

the corresponding generating functions \( \overline{M}_m \) are different from \( M_m \) but they obviously satisfy again Theorem 2. In particular, we have that

\[ \overline{M}_2(x, h_2) = \sum_{k_2=0}^{\infty} (x_1 - e_{12} x_2)^{k_2} h_2^{k_2} = \frac{1 - (x_1 + e_{12} x_2) h_2^{k_2}}{1 - 2x_1 h_2 + h_2^2 |x|^2}. \]

Here \( x = (x_1, x_2) \in \mathbb{R}^2 \) and \( h_2 \in \mathbb{R} \).

Acknowledgements

We would like to thank V. Souček for useful discussions on this topic. The work of the first and second authors was supported by Portuguese funds through the CIDMA - Center for Research and Development in Mathematics and Applications, and the Portuguese Foundation for Science and Technology (“FCT - Fundação para a Ciência e a Tecnologia”), within project PEst-OE/MAT/UI4106/2014. We would like to thank the anonymous referee for his/her helpful comments.

References

[1] K. Atkinson and W. Han, Spherical Harmonics and Approximations on the Unit Sphere: an Introduction. Springer, 2012.
[2] N. Asai, I. Kubo, H-H. Kuo, Generating functions of orthogonal polynomials and Szegő-Jacobi parameters. Probability and mathematical statistics, Vol. 23, Fasc. 2, (2003), 273-29.
[3] S. Bock and K. Gürlebeck, On a generalized Appell system and monogenic power series. Math. Methods Appl. Sci. 33 (2010), 394–411.
[4] S. Bock, K. Gürlebeck, R. Lávička, V. Souček, The Gel’fand-Tsetlin bases for spherical monogenics in dimension 3. Rev. Mat. Iberoamericana 28 (4) (2012), 1165-1192.
[5] F. Brackx, R. Delanghe, F. Sommen, Clifford analysis. Pitman, London, 1982.
[6] F. Brackx, H. De Schepper, R. Lávička, V. Souček, Gelfand-Tsetlin Bases of Orthogonal Polynomials in Hermitean Clifford Analysis. Math. Methods Appl. Sci. 34 (2011), 2167-2180.
[7] F. Brackx, H. De Schepper, R. Lávička, V. Souček, Embedding Factors for Branching in Hermitean Clifford Analysis. To appear in Complex Anal. Oper. Theory.
[8] I. Caçao, Constructive approximation by monogenic polynomials. Ph.D thesis, Univ. Aveiro, 2004.
[9] I. Caçao, K. Gürlebeck, S. Bock, *On derivatives of spherical monogenics*. Complex Var. Elliptic Equ. **51** (811) (2006), 847–869.

[10] I. Caçao, K. Gürlebeck, S. Bock, *Complete orthonormal systems of spherical monogenics - a constructive approach*. In: L.H. Son, W. Tutschke, S. Jain (Eds.), Methods of Complex and Clifford Analysis, Proceedings of ICAM, Hanoi, SAS International Publications, 2004.

[11] I. Caçao, K. Gürlebeck, H.R. Malonek, *Special monogenic polynomials and $L_2$-approximation*. Adv. appl. Clifford alg. **11** (S2) (2001), 47–60.

[12] I. Caçao and H. R. Malonek, *Remarks on some properties of monogenic polynomials*. ICNAAM 2006. International conference on numerical analysis and applied mathematics 2006 (T.E. Simos, G. Psihoyios, and Ch. Tsitouras, eds.), Wiley-VCH, Weinheim, 2006, pp. 596-599.

[13] I. Caçao and H. R. Malonek, *On a complete set of hypercomplex Appell polynomials*. Proc. ICNAAM 2008, (T. E. Timos, G. Psihoyios, Ch. Tsitouras, Eds.), AIP Conference Proceedings 1048, 647-650.

[14] R. Delanghe, R. Lávička, V. Souček, *The Gelfand-Tsetlin bases for Hodge-de Rham systems in Euclidean spaces*. Math. Meth. Appl. Sci. **35** (7) (2012), 745-757.

[15] R. Delanghe, F. Sommen, V. Souček, *Clifford algebra and spinor-valued functions*. Kluwer Academic Publishers, Dordrecht, 1992.

[16] C.F. Dunkl, Y. Xu, *Orthogonal Polynomials of Several Variables*. Cambridge University Press, Cambridge, 2001.

[17] J. E. Gilbert, M. A. M. Murray, *Clifford Algebras and Dirac Operators in Harmonic Analysis*. Cambridge University Press, Cambridge, 1991.

[18] K. Gürlebeck, K. Habetha, W. Sprößig, *Holomorphic functions in the plane and n-dimensional space*. Translated from the 2006 German original, with cd-rom (Windows and UNIX), Birkhäuser Verlag (Basel, 2008).

[19] K. Gürlebeck, W. Sprößig, *Quaternionic and Clifford Calculus for Physicists and Engineers*. J. Wiley & Sons, Chichester, 1997.

[20] S.G. Krantz, H.R. Parks, *A Primer of Real Analytic Functions*. Birkhäuser, Basel, 1992.

[21] R. Lávička, *Canonical bases for sl(2,C)-modules of spherical monogenics in dimension 3*. Arch. Math.(Brno) **46** (5) (2010), 339-349.

[22] R. Lávička, *Complete orthogonal Appell systems for spherical monogenics*. Complex Anal. Oper. Theory **6** (2) (2012), 477489.

[23] R. Lávička, *Orthogonal Appell bases for Hodge-de Rham systems in Euclidean spaces*. Adv. appl. Clifford alg. **23** (1) (2013), 113-124.

[24] R. Lávička, *Hypercomplex Analysis - Selected Topics*. Habilitation thesis, Faculty of Mathematics and Physics, Charles University, Prague, 2011.

[25] P. Maroni, J. Van Iseghem, *Generating functions and semi-classical orthogonal polynomials*. Proceedings of the Royal Society of Edinburgh: Section A Mathematics, Volume 124, Issue 05, January 1994, pp. 1003-1011.

[26] A. I. Molev, *Gelfand-Tsetlin bases for classical Lie algebras*. in: M. Hazewinkel (Ed.), Handbook of Algebra, Vol. 4, Elsevier, 2006, pp. 109-170.

[27] S. Roman, *The Umbral Calculus*. Academic Press Inc., 1984.
[28] G.-C. Rota, *Combinatorics, representation theory and invariant theory: the story of a ménage à trois*. Discrete Math. **193** (1998), 5-16.

[29] F. Sommen, *Spin groups and spherical means III*. Rend. Circ. Mat. Palermo (2) Suppl. No 1 (1989), 295-323.

[30] A. Sommerfeld, *Partial Differential Equations in Physics*. Academic Press Inc. Publishers, New York, 1949.

[31] H. S. Wilf, *Generatingfunctionology*. Academic Press, Inc., 1990.

[32] E. T. Whittaker and G.N. Watson, *A Course of Modern Analysis*. 4th ed., Cambridge University Press, Cambridge, 1996.

P. Cerejeiras and U. Kähler
CIDMA – Center for Research and Development in Mathematics and Applications
Department of Mathematics
University of Aveiro
Campus de Santiago
3810-193 Aveiro
Portugal

e-mail: pceres@ua.pt
         ukaehler@ua.pt

R. Lávička
Faculty of Mathematics and Physics
Charles University in Prague
Sokolovská 83
186 75 Praha 8
Czech Republic

Received: April 15, 2014.
Accepted: July 23, 2014.