**Locational Qubit Realization with One-dimensional Contact Interactions**

Taksu Cheon

*Laboratory of Physics, Kochi University of Technology, Tosa Yamada, Kochi 782-8502, Japan*

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We show that the $U(2)$ group structure of thin barriers can be adopted for quantum information processing when used in combination with environmental potential whose bouncing modes are profile preserving. Qubits are realized as wave functions localized in either side of the barrier which divides the one-dimensional system into two regions. It is argued that this model is a theoretical prototype of a robust and scalable quantum computing device.

Keywords: Quantum Computation, Quantum Wire, Quantum Contact Interaction

I. INTRODUCTION

Quantum computation has emerged as one of the prime source of inspiration for mathematical quantum mechanics \[1\]. It is a information processing based on quantum state belonging to a $U(2)$ group. Typically, spin one-half is considered as a natural playground. But any two level system can be utilized. In a separate development, $U(2)$ structure has been uncovered \[2, 3, 4, 5, 6\] in one dimensional system with generalized point interaction \[7, 8, 9\].

A natural question is whether it can be used for quantum information processing.

The purpose of this article is to show that that is indeed possible with the help of background potential, which move the particle back and forth while keeping the wave function profile. We consider controlling that motion by manipulating the properties of the barrier which separate the two spacial regions of the system. This quantum barrier is nothing but the point interaction placed at the origin as a barrier separating right and left regions.

The resulting model system takes the appearance of the quantum version of that ancient eastern calculational device of abacus. We argue that this model could be a prototype of a robust quantum qubit device which excels in stability, controllability and scalability.

II. THE MODEL

We consider a one dimensional system of quantum particle moving on $x$ axis subjected to a harmonic oscillator potential of frequency $\omega = 2\pi/T$ and the inverse square potential with the strength $g$:

$$V(x) = \frac{\omega^2}{2} x^2 + g \frac{1}{x^2}.$$  \hspace{1cm} (1)

With $g = 0$, the background potential is reduced to the elementary harmonic oscillator. In this article, we shall mainly work with this simple limit. Then, we further add the generalized point interaction placed at the origin whose characteristics are controllable. We define boundary vector at $x \to +0$ and $x \to -0$ as

$$\Psi = \begin{pmatrix} \psi(0_+) \\ \psi(0_-) \end{pmatrix}, \quad \Psi' = \begin{pmatrix} \psi'(0_+) \\ -\psi'(0_-) \end{pmatrix}. \hspace{1cm} (2)$$

The point interaction is described by an element $U$ of unitary group $U(2)$, that specifies the value of the boundary vector such that

$$(U - I)\Psi + i(U + I)\Psi' = 0. \hspace{1cm} (3)$$

In other word, all point interactions allowable in quantum mechanics form a family described by the set of four parameters, whose manifold structure is given by $U(2) \simeq S^1 \times S^3 \hspace{0.5cm} \[10, 11\].$

The elementary example of $\delta$-interaction is but a very special one parameter family within this wider class of interactions. In fact, generalized point interaction $U(2)$ comprises such exotic interactions that cause discontinuity in the wave function itself, and also the ones that have constant transmission probability which is independent of the particle energy.

Explicit construction of these highly singular point interactions has been achieved in terms of singular short-range limits of known interactions \[12, 13\]. Here, we only illustrate some of the prominent examples. The identity matrix $U = I$ results in $\psi'(0_+) = \psi'(0_-) = 0$, signifying the inpenetrable barrier at $x = 0$ with Neumann boundary at its both side. Similarly, the negative of identity matrix $U = -I$ results in $\psi(0_+) = \psi(0_-) = 0$, another
The eigenvalues of the system are unchanged from the free harmonic oscillator’s \( \{ \omega_n \} \), and its eigenfunctions of the system \( \{ \chi_n^\lambda \} \) are also analogous to the free one having discontinuity at the origin and being expanded/shrunk at one side;

\[
\begin{align*}
\chi_n^\lambda(x) &= N [\lambda \chi_n(|x|)\Theta(x) + \chi_n(|x|)\Theta(-x)] & n = 0, 2, 4, \cdots, \\
\chi_n^\lambda(x) &= N [\lambda \chi_n(|x|)\Theta(x) - \lambda^* \chi_n(|x|)\Theta(-x)] & n = 1, 3, 5, \cdots,
\end{align*}
\]

where the normalization is given by \( N = \sqrt{2/(|\lambda|^2 + 1)} \). Arbitrary state \( \psi(x, t) \) is represented as

\[
\psi(x, t) = \sum_n A_n \chi_n^\lambda(x) e^{i\omega t}
\]

\[
= \frac{\lambda}{|\lambda|^2 + 1} [\lambda^* S(x) - M(x)e^{i\omega t}] \Theta(x)e^{2i\omega t}
\]

\[
+ \frac{\lambda^*}{|\lambda|^2 + 1} [S(x) + \lambda M(x)e^{i\omega t}] \Theta(-x)e^{2i\omega t}
\]

FIG. 2: Illustrative representation of the construction of some of \( U(2) \) barriers in terms of multiple \( \delta \) functions placed in disappearing distances with diverging strength.
where we define
\[
S(x) = \frac{2}{\lambda^* N} \sum_m A_{2m} \chi_{2m}(|x|),
\]
\[
M(x) = -\frac{2}{\lambda N} \sum_m A_{2m+1} \chi_{2m+1}(|x|).
\]

Let’s assume that at \( t = 0 \), the wave function is localized in the region \( x > 0 \). That is possible only if we have \( M(x) = -S(x)/\lambda \). We then have
\[
\psi(x, 0) = \Theta(x) \ S(x)
\]
\[
\psi(x, T/2) = \left[ \cos \frac{\mu}{2} \Theta(x) + e^{i\nu} \sin \frac{\mu}{2} \Theta(-x) \right] S(x).
\]

Similarly, with \( S(x) = M(x)/\lambda^* \), we have
\[
\psi(x, 0) = \Theta(-x) \ M(x)
\]
\[
\psi(x, T/2) = \left[ e^{-i\nu} \sin \frac{\mu}{2} \Theta(x) - \cos \frac{\mu}{2} \Theta(-x) \right] M(x).
\]

Thus, our assertion is proven for \( \sigma \). We note that the parameter values \((\mu = 0, \nu = 0), (\mu = \pi, \nu = 0)\) and \((\mu = \pi/2, \nu = 0)\) respectively correspond to Identity, Not and Hadamard operations in quantum computation languages.

IV. HARD BARRIER

Now, we look at \( D \) which is the point interaction belonging to the torus \( \{\theta_+, \theta_-\} \). The effect of \( D \) on a qubit is applying a conditional phase whose value depends on whether the qubit is \(|0\rangle\) or \(|1\rangle\).
\[
\sin \frac{\theta_+}{2} \psi(0_, t) + \cos \frac{\theta_+}{2} \psi(0_+, t) = 0
\]
\[
\sin \frac{\theta_-}{2} \psi(0_-, t) - \cos \frac{\theta_-}{2} \psi(0_+, t) = 0.
\]

This is nothing but two non-communicating sub-systems separated by impenetrable barrier. It is sufficient to consider only one side, say, \( x > 0 \) side, since the structure of the problem is the same. We take out the index +.
\[
\sin \frac{\theta}{2} \psi(0, t) + \cos \frac{\theta}{2} \psi(0, t) = 0.
\]

For \( \theta = 0 \), one has Neumann boundary, and \( \psi(x, t) = \psi(x, 0) \) so the wave function stays the same (modulo ground state oscillation \( \exp(\omega/2t) \)). For \( \theta = \pi \), one has Dirichlet boundary, and \( \psi(x, t) = \psi(x, 0) \exp(i\omega t) \). so the wave function obtain phase \( \pi \) after half period \( T/2 = \pi/\omega \). For generic \( \theta \) between 0 and \( \pi \), one has some “in between” boundary condition which results in some mixture of \( \psi(x) \) with energy \( \varepsilon(\theta) \) which should be
\[
0 < \varepsilon < \omega
\]

So, after the half period, the wave function obtain phase
\[
\psi(x, T/2) = \exp(i\eta(\theta)\pi)\psi(x, 0),
\]

where we define \( \eta(\theta) = \varepsilon(\theta)/\omega \), provided that the energy spectra of the system is equally spaced. The phase \( \eta \) is a quantity satisfying \( 0 < \eta < 1 \), and \( \eta(\theta) \) should be some monotonous increasing function. Therefore, having \( D \) applied for half-period, desired phase would be added.

The problem is, that the system have only approximately equally spaced energy levels except for the case of large negative \( g \) which is the strong attractive limit for the inverse square part of the background potential. So the strict phase operation is not always possible with the application of corresponding \( U(2) \) point interaction, except for the special cases. The result of \( D \) operation, in general, is therefore corrupted by the mixing of undesired higher harmonics with random phases.

However, there are simple workaround of this problem. Instead of changing the wall property, we can apply additional constant potential \( V_{ad}(x) = \eta_\omega \) to obtain the desired phase after half period \( T/2 \). Another possibility is changing the harmonic oscillator frequency itself. The former method seems to be simpler and practical.

As in any qubit realization, the choice of the basis is arbitrary, and its any unitary transformation is a legi-
mate basis. In quantum cryptography, for example, two bases connected by the Hadamard transformation is utilized. Typically, spin up/down basis and right/left basis are used. In our case, that corresponds to the left/right localized basis and symmetric/antisymmetric basis.

V. PROSPECTUS

In mathematical term, the two-qubit operation comprises $U(4)$ group. This is a natural extension to the $U(2)$, whose quantum wire realization has been the subject of this work. Analogous realization of this $U(4)$ exists in the form of "quantum $X$-junction", a graph of four half lines whose edges are connected at single point \[14\]. Thus, the study of quantum graphs now carries even more urgency.

A practical question in the experimental realization of our scheme is how to change the characteristics of the thin barrier representing the generalized point interaction. Preferably, it is to be a quantally operating device with triggering mechanism utilizing particles of far smaller mass or energy scale compared to the particle used as the qubit carrier. Once such device is constructed (no doubt that will be done, in time), and if that trigger is coupled to the presence absence of the qubit carrier in neighboring device, we will have a realization of two-qubit operations such as control-not in place. This type of setup would be quite a bit more advantageous in term of scalability compared to the qubit manipulation utilizing the forced transition by external laser beam.

One of the advantage of our implementation is the robustness of the qubit due to the simplicity of the setup, which is matched only by the spin implementations. In contrast, most solid-state based approaches use particle states which could easily be lost in temperature fluctuations, whose suppression can be costly and potentially inhibiting in the setup requiring large number of qubits. We hope that the model considered here could offer a basis for an alternative location based quantum device which is simple, robust and truly scalable.

The content of the current work will be reported elsewhere in full \[15\].

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