RANDOM LIFTS OF $K_5 \setminus e$ ARE 3-COLOURABLE

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ABSTRACT. Amit, Linial, and Matoušek (Random lifts of graphs III: independence and chromatic number, Random Struct. Algorithms, 2001) have raised the following question: Is the chromatic number of random $h$-lifts of $K_5$ asymptotically (for $h \to \infty$) almost surely (a.a.s.) equal to a single number? In this paper, we offer the following partial result: The chromatic number of a random lift of $K_5 \setminus e$ is a.a.s. three.

1. INTRODUCTION

Let $G$ be a graph, and $h$ a positive integer. An $h$-lift of $G$ is a graph $\tilde{G}$ which is an $h$-fold covering of $G$ in the topological sense. Equivalently, there is a graph homomorphism $\phi: \tilde{G} \to G$ which maps the neighbourhood of any vertex $v$ in $\tilde{G}$ one-to-one onto the neighbourhood of the vertex $\phi(v)$ of $G$. The graph $G$ is called the base graph of the lift.

More concretely, we may say that an $h$-lift of $G$ has vertex set $V(G) \times [h]$ (where we let $[h] := \{1, \ldots, h\}$ as usual). The set $\{v\} \times [h]$ is called the fibre over $v$. Fixing an orientation of the edges of $G$, the edge set of an $h$-lift is of the following form: There exist permutations $\sigma_e$ of $[h]$, $e \in E(G)$, such that for every two adjacent vertices $u$ and $v$ of $G$, if the edge $uv$ is oriented $u \to v$, the edges between the fibres $\{v\} \times [h]$ and $\{u\} \times [h]$ are $(u, j)(v, \sigma_{uv}(j))$, $j \in [h]$. Changing the orientation of the edges in the graph does not change the lift, provided that permutations on edges on which the orientation is changed are replaced by their respective inverses. In this spirit, for an edge $uv$ in $G$, regardless of its orientation, we denote by $\sigma_{uv}$ the permutation for which the edges between the fibres are $\{(u, j)(v, \sigma_{uv}(j)) \mid j \in [h]\}$.

By a random $h$-lift we mean a graph chosen uniformly at random from the graphs just described, which amounts to choosing a permutation, uniformly at random, independently for every edge of $G$.

Random lifts of graphs have been proposed in a seminal paper by Amit, Linial, Matoušek, and Rozenman [4]. Their paper sketched results on connectivity, independence number, chromatic number, perfect matchings, and expansion of random lifts, and was followed by a series of papers containing broader and more detailed results by the same and other authors [1][2][3][8], and e.g. [5][7][6].

Key words and phrases. Graph theory, random graphs, random lifts of graphs, colouring, chromatic number.

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In [3] Amit, Linial, and Matoušek focused on independence and chromatic numbers of random lifts of graphs. They asked the following question.

Is there a zero-one law for the chromatic number of random lifts?

In particular, is the chromatic number of a random lift of $K_5$ a.a.s. (for $h \to \infty$) equal to a single number (which may be either 3 or 4)?

A random $h$-lift $\tilde{G}$ of $K_5$ a.a.s. has an odd cycle, whence a.a.s. we have $\chi(\tilde{G}) \geq 3$. Moreover, $\tilde{G}$ a.a.s. does not contain a 5-clique. Brooks’ theorem implies that a.a.s. $\chi(\tilde{G}) \leq 4$. So, a.a.s. $\chi(\tilde{G}) \in \{3, 4\}$.

In their paper, Amit, Linial, and Matoušek [3] conjectured that the chromatic number of random lifts of any fixed base graph obeys a zero-one law, i.e., it is asymptotically almost surely equal to a fixed number (depending only on the base graph). In the case when the base graph is $K_n$, they prove that $\chi(\tilde{G}) = \Theta(n/\log n)$ a.a.s. (the constant in the $\Theta$ notation may depend neither on $h$ nor on $n$). Five is the smallest value for $n$, for which this is not trivial.

In this paper, we contribute the following to this problem.

**Theorem 1.** A random lift of $K_5 \setminus e$ is a.a.s. 3-colorable.

2. Notation and Terminology

Let $G := K_5 \setminus e$. Clearly, $G$ is obtained by joining a cycle $C := [x_1, x_2, x_3]$ to a stable set $S := \{y_1, y_2\}$. Here, by join we mean that every vertex of $C$ is made adjacent to every vertex of $S$. From now on, $\tilde{G}$ will be a random $h$-lift of $G$. Let $\tilde{G}_C$ and $\tilde{G}_S$ denote the subgraphs of $\tilde{G}$ induced by the fibres over the vertices of $C$ and those over vertices of $S$, respectively. Moreover, for $x \in V(G)$, we denote by $V_x := \{x\} \times [h]$ the set of vertices of $\tilde{G}$ over $x$. Similarly, for any set $U$ of vertices of $\tilde{G}$ and $x \in V(G)$, we let $U_x := U \cap V_x$.

As an hors d’œuvre intended to familiarise the reader with the most basic random lift arguments, we serve the following easy lemma.

**Lemma 2.** The graph $\tilde{G}_C$ is a union of cycles, each of which is divisible by three. A.a.s., the number of cycles in $\tilde{G}_C$ is at most $\log^2 h$.

**Proof.** The cycles with length $3\ell$ of $\tilde{G}_C$ correspond to the cycles with length $\ell$ of the permutation $\sigma_{x_1 x_2} \circ \sigma_{x_2 x_3} \circ \sigma_{x_3 x_1}$. The latter is a uniformly distributed random permutation of $[h]$. It is a folklore fact (e.g., [9]) that the average number of cycles of a random permutation of $[h]$ is $\log h + o(1)$. The statement of the lemma now follows from Markov’s inequality. □

Lemma 2 allows us to assume that $\tilde{G}_C$ has at most $\log^2 h$ cycles. As a matter of fact, this is the only statement about $\tilde{G}_C$ which we need.

3. The 3-colouring algorithm

Our colouring algorithm is detailed in the box Algorithm 1. We use the colours red, black, and white, where the colour red will have a special significance. We
Algorithm 1 Three-Colour $\tilde{G}$

**Phase I:**
(1) The algorithm starts with all edges in $\tilde{G}_{C}$ exposed, but no edge in between $\tilde{G}_{C}$ and $\tilde{G}_{S}$ exposed. If $\tilde{G}_{C}$ has more than $\log^{2} h$ cycles, fail.
(2) Choose exactly one red vertex in each cycle of $\tilde{G}_{C}$.

**Phase II:**
(3) Expose all edges incident to red vertices. If there exists a vertex in $\tilde{G}_{S}$ which has two or more red neighbours, fail. Otherwise, let $P(0)$ be the set of pale vertices before the first iteration.
(4) For $t = 1, \ldots, \lceil h/3 \rceil$:
   (4.1) Let $v$ be chosen arbitrarily from the set $P(t - 1)$.
   (4.2) From the two non-exposed edges incident to $v$, expose one arbitrarily (the other edge remains unexposed). Let $u$ be the end-vertex in $\tilde{G}_{C}$ of the exposed edge.
   (4.3) Expose the other edge incident to $u$, and let $v'$ be the corresponding neighbour of $u$ in $\tilde{G}_{S}$. If $v' \in \bigcup_{s=0}^{t-1} P(s)$, fail. Otherwise $P(t) = P(t - 1) \cup \{v'\} \setminus \{v\}$ (this is now the new set of pale vertices).
   (4.4) Colour $u$ red.

**Phase III:**
(5) Expose all remaining edges.
(6) Colour every vertex red which is in $\tilde{G}_{S}$ and does not have a red neighbour.
(7) If the graph induced by the non-red vertices is acyclic, colour it black and white, otherwise fail.

Point the reader to the fact that, once Algorithm 1 has coloured a vertex, the vertex never changes its colour or becomes uncoloured again. A vertex of $\tilde{G}_{S}$ which is adjacent to precisely one red vertex is called pale (this is not a colour).

The algorithm works in three phases. In phase I, Steps (1–2), we destroy the uncoloured cycles of $\tilde{G}_{C}$ by colouring one vertex per cycle red. By Lemma 2 a.a.s., we colour at most $\log^{2} h$ vertices red in Phase I, i.e., Phase I fails with probability $o(1)$.

In Phase II, more accurately in the loop (4), the algorithm successively chooses uncoloured vertices of $\tilde{G}_{C}$ and colours them red. This is done by maintaining the set $P(\cdot)$ of pale vertices (i.e., those vertices of $\tilde{G}_{S}$ which are adjacent to precisely one red vertex).

In Phase III, Steps (5–7), the remaining vertices are coloured in a straightforward way.

The rationale behind the algorithm is as follows:

At any fixed time between Steps (3) and (5), consider the connected components of $\tilde{G}_{C}$ after deleting all red vertices. These are uncoloured paths of different lengths in $\tilde{G}_{C}$, separated by red vertices. We call them chunks. These chunks can be thought of as the vertices of a multi-graph, which we call the chunk-graph, whose
edges are the pale vertices in $\tilde{G}_C$: Every pale vertex has precisely two uncoloured neighbours in $\tilde{G}_C$, thus connecting the corresponding chunks. We refer to such a connection between chunks via a pale vertex as a chunk-edge. A chunk-edge may be a loop, which happens when a pale vertex have both uncoloured neighbours in the same chunk. Furthermore, there may be parallel chunk-edges in the chunk-graph, which happens when two pale vertices connect the same pair of chunks. The reason why, in Step (3) of the algorithm, we abort if a vertex has two or more red neighbors, is only because such vertices would not correspond to edges of the chunk-graph. Indeed, at the end of Phase II, there are only two kinds of uncolored vertices left: Those making up the chunk graph, and those being colored red in Step (6).

The chunk-graph is a random multi-graph. At Step (3), it has as many vertices as there are cycles in $\tilde{G}_C$ (at most $\log^2 h$ by Lemma 2), and as many edges as there are pale vertices. If the algorithm does not fail in Step (3), then to every red vertex there are two pale vertices, and they are all distinct. Hence, at this time, there are twice as many chunk-edges as there are chunks.

When the algorithm proceeds through loop (4), the number of chunks is increased as we colour more vertices of $\tilde{G}_C$ red. However, the number of pale vertices stays constant, and hence so does the number of chunk-edges.

The reasoning at this point is a heuristic analogy with the random (simple) graph model $G(n, m)$, where a set of $m$ edges is drawn uniformly at random from the set of all possible $m$-sets of edges between $n$ vertices. For us, $n$ is the number of chunks and $m$ is the number of chunk-edges. At Step (3), where $m = 2n$, we expect the chunk-graph to contain lots of cycles (including loops and parallel edges), which makes it unlikely that it can be coloured with just the two remaining colours. However, when $n$ grows and $m$ stays constant, a random graph $G(n, m)$ will be acyclic as soon as $m \ll n$, and we expect the same to be true for the chunk-graph.

There are complications in making this heuristic analogy work rigorously, the foremost being that the distribution of the edges in the chunk-graph is not uniform but instead depends on the sizes of the chunks. We will address these issues in the next section.

4. PROOF OF CORRECTNESS OF THE 3-COLOURING ALGORITHM

We prove that a.a.s. Algorithm 1 properly 3-colours $\tilde{G}$.

Lemma 3. A.a.s., Algorithm 1 does not fail in Steps (1), (3), or (4.3).

Proof. Lemma 2 implies that, a.a.s., the algorithm does not fail in Step (1).

For Step (3), note that, at this point in the algorithm, the probability that a fixed vertex in $\tilde{G}_S$ has two or more red neighbours is $O((\log^4 h)/h^2)$. Hence, the probability that there exists such a vertex having two or more red neighbours is $O((\log^4 h)/h) = o(1)$.

For Step (4.3), we see that for each fixed $t$, the probability that $v' \in \bigcup_{s=0}^{t-1} P(s)$ is $O(h^{-2/3})$. Thus, the probability that the algorithm fails after at most $t$ iterations
is $O(th^{-2/3})$. Consequently, the probability that the algorithm fails at Step (4.3) before completing $t := \lfloor h^{1/3} \rfloor$ iterations is $o(1)$. □

Denote by $T$ the last iteration (value of $t$) of the loop (4) which is completed (without failing). We let $R(t), t = 0, 1, \ldots, T$ be the set of vertices which are red after $t$ iterations of the loop (4). In particular, $R(0)$ is the set of vertices coloured red in Step (2). Let $R^+(t) := R(t) \setminus R(0)$. Recall that adding an index to a letter denoting a set refers to taking its intersection with the corresponding fibre, for example $R_x(t)$ refers to $V_x \cap R(t)$. Moreover, we use the following notation to refer to the cardinalities of each of these sets: If a set is denoted by an upper-case letter (possibly with sub- or superscript or followed by parentheses), the corresponding lower-case letter (with the same sub- or superscripts or parentheses) denotes its cardinality. For example $r_x(t) = |R_x(t)|$. We have the following.

**Lemma 4.** For each $x \in C$ and $t = 1, \ldots, T$, set $R^+_x(t)$ is uniformly distributed in the set of all $(r^+_x(t))$-element subsets of $V_x \setminus R_x(0)$.

**Proof.** Fix an $x \in C$. In every iteration of the loop (4) in which the fibre over $x \in C$ is selected in Step (4.3), when exposing the edge in Step (4.3), the vertex $u$ is selected uniformly at random from the set of all previously uncoloured vertices in $V_x$. In other words, for every fixed value of $R^+_x(t - 1)$, the distribution of $u$ is uniform. By induction, $R^+_x(t)$ is uniformly distributed. □

**Lemma 5.** In the loop (4) of Algorithm 1 a.a.s. no two adjacent vertices are coloured red.

**Proof.** Let $x_1, x_2 \in C$, and consider the situation after $T$ iterations, i.e., when the algorithm leaves the loop (4). By Lemma 4 at this time, the expected number of edges between $V_{x_1}$ and $V_{x_2}$ both of whose end vertices are red is at most

$$h \cdot \frac{T}{h - r_{x_1}(0)} \cdot \frac{T}{h - r_{x_2}(0)} = O\left(\frac{h^{5/3}}{(h - \log^2 h)^2}\right) = o(1).$$

□

Now, it only remains to show that when Step (7) of Algorithm 1 is reached, the graph consisting of the yet uncoloured vertices is a.a.s. acyclic.

Now, suppose that the algorithm has completed Phase II without failing, i.e., we find ourselves just before Step 5. Let $H$ denote the chunk graph as we defined in Section 3. Thus $H$ is a random multi-graph with $n \leq r(T) = T + r(0) = \Theta(h^{1/3})$ vertices and $m := p(T) = 2r(0) = O(\log^2 h)$ edges. In fact, if no two red vertices are adjacent, the first inequality becomes an equation, cf. Lemma 5. The distribution of $H$ can be described in terms of random permutations taking into account the edges which have already been exposed, and the sizes of the chunks. It appears sensible to guess that $H$ has no cycles. That is in fact correct.

Sizes of the chunks. The first thing we require to turn this analogy into a rigorous proof is an upper bound on the sizes of the chunks. We find it convenient to reduce the question to the distribution of the gaps between $n$ points drawn uniformly at
random from the interval $[0, 1]$. There, the probability that two consecutive points
eclose a gap of size $a$ is $(1 - a)^n$, which yields an upper bound of, say, $(2^{h \log n}) / n$
for the largest gap, a.a.s. In the following lemmas, we put this plan into action.

Let $n$ numbers $Y_1, \ldots, Y_n$ be drawn independently uniformly at random from
$[N]$, where $N$ is a function of $n$. Let $S_k$ be the $k$-th order statistics (i.e., $0 \leq S_1 \leq \cdots \leq S_n \leq 1$, and $\{S_1, \ldots, S_n\} = \{Y_1, \ldots, Y_n\}$) and set $S_0 := 0$ and
$S_{n+1} := N$.

We determine the distribution of $S_{k+1} - S_k$. This can be done directly, but it
also easily be derived from the Bapat-Beg theorem, of which the following is
a special case (see the appendix for a proof).

**Lemma 6.** Let $X_1, \ldots, X_n$ be points drawn independently uniformly at random in
$[0, 1]$ and denote by $S_k'$ the $k$-th order statistics. With $S'_0 := 0$ and $S'_{n+1} := 1$, for
each $k = 0, \ldots, n$, the distribution of $S'_{k+1} - S'_k$ is as follows: $P[S'_{k+1} - S'_k > a] = (1 - a)^n$.

For the discrete version we obtain the following.

**Lemma 7.** For every $a > 0$, we have

$$P[S_{k+1} - S_k > \frac{aN}{n}] \leq e^{-a+O(n/N)},$$

(with an absolute constant in the $O(\cdot)$).

**Proof.** Let $X_1, \ldots, X_n$ be drawn independently uniformly at random from $[0, 1]$. We
can assume that the $Y$'s are the $X$'s multiplied by $N$ and then rounded up:
$Y_j = \lceil NX_j \rceil$. We also assume that the permutation taking the $X$'s to the $Y$'s is
equal to the permutation taking the $S$'s to the $S'$'s (this condition makes sense when
two $Y$'s coincide). By Lemma 6 we conclude that

$$P[S_{k+1} - S_k > \frac{aN}{n}] \leq P[S'_{k+1} - S'_k > (\frac{aN}{n} - 2)/N]$$

$$= (1 - (a/n - 2/N))^n \leq e^{-a+2n/N}.$$ 

□

From this, we conclude the following.

**Lemma 8.** Let an $n$-subset $R$ be drawn uniformly at random from all the $n$-subsets
of $[N]$, and $a > 0$. The probability that there are $\lfloor aN/n \rfloor$ consecutive numbers not
in $R$ is at most $(n + 1)e^{-a+O(n/N)}$.

**Proof.** Let $b := \lfloor aN/n \rfloor$, and let $Y_1, \ldots, Y_b$ be drawn independently uniformly at
random from $[N]$. Let $A$ be the event that the $Y_j$'s are all distinct, $\bar{A}$ its comple-
ment, and let $B$ be the event that there are $b$ consecutive numbers not containing
any of the $Y_j$'s. Since $P(B)$ is a convex combination of $P(B|A)$ and $P(B|\bar{A})$, and
$P(B) \leq (n + 1)e^{-a+O(n/N)}$ by Lemma 7 this upper bound must also be true for the
smaller of the two conditional probabilities. But, clearly $P(B|A) \leq P(B|\bar{A})$. □

We can now prove the upper bound on the sizes of the chunks.
Lemma 9. Let $\omega \xrightarrow{h} \infty$ arbitrarily slowly. If $n$ is the number of red vertices in $\tilde{G}_C$ at the completion of Phase II of the algorithm, a.a.s. as $h \to \infty$, there is no chunk with size larger than $6(\omega + \log n) h/n$.

Proof. Choose an arbitrary $x \in C$. By Lemma 8, the conditions of Lemma 8 are satisfied if we let $n := r_x^+(T)$ and $N := |V_x \setminus R_x(0)|$. The vertices in $V_x \setminus R_x(0)$ are numbered in the following way.

For each cycle of $\tilde{G}_C$, choose an orientation. The numbers associated to the vertices in the intersection of $V_x \setminus R_x(0)$ and this cycle are then taken consecutively: starting with the vertex in $V_x \setminus R_x(0)$ which, in positive orientation, is next to the $R(0)$-vertex of the cycle, and continuing to number in positive orientation.

If there is a path in $\tilde{G}_C$ of length greater than $6(\omega + \log n) h/n$ not containing a red vertex, then there is a gap in $[N]$ larger than $(\omega + \log n) N/n$. (Notice that every third vertex of the path belongs to $V_x$. The factor 2 comes from the left and right end strips, i.e., the vertices which are close to the $R(0)$-vertex on a cycle but which do not have consecutive numbers.) By Lemma 8, the probability of this happening is at most

\[ (n + 1) e^{-\omega - \log n + O(n/N)} = \frac{n+1}{n} e^{-\omega + O(1)} = o(1). \]

Bounding the expected number of cycles in $H$. We now come to the classical first-moment argument which shows that, a.a.s., our random multi-graph $H$ has no cycles. For the remainder of this section, we condition on the event that the algorithm does not fail before Step 5, and that no two adjacent vertices have been coloured red (cf. Lemmas 3 and 5 respectively).

Lemma 10. The probability that the edge set of $H$ contains a fixed set $F$ of edges with $|F| = \ell$ is at most

\[ O\left( \ell! \left( \frac{m}{\ell} \right) \frac{\log^2 \ell n}{n^{2\ell}} \right). \]

Proof. Recall that $n$ denotes the number of vertices of $H$, which is equal to the number of chunks in $\tilde{G}_C$. This is equal to the number of red vertices at the end of Phase II, which is $\Theta(h^{1/3})$. The number $m$ of edges of $H$ is equal to the number $p(T)$ of pale vertices after termination of Phase II, which is $O(\log^2 h)$. The edges come in six different types, depending on which fibre $V_y$, $y \in S$, contains the corresponding pale vertex, and also which fibres contain the end-vertices of the two non-exposed edges adjacent to the pale vertex.

For each edge of $H$, one by one, we draw the two end-vertices one by one. An edge corresponding to a pale vertex $v$ of $\tilde{G}$ connects two fixed vertices of $H$ if the two yet unexposed edges incident to $v$ end turn out to be contained in the chunks corresponding to the fixed vertices of $H$. Since the sizes of the chunks are a.a.s. $O(h \log^2 n/n)$ by Lemma 9 and the number of possible neighbors of $v$ is between $h$ and $h - n - m + O(1) = \Theta(h)$, the probability that the edge of $H$ connects the two fixed vertices is $O\left( \frac{\log^2 n}{m} \right)$.

From this, the statement of the lemma follows. \qed
Now we adapt the classical first-moment calculation to prove that there are no cycles in $H$, and therefore, no cycles in the graph induced on uncoloured vertices in Step (7).

**Lemma 11.** A.a.s. $H$ contains no cycles.

**Proof.** By Lemma 10 the expected number of cycles of length $\ell \geq 1$ is

$$\sum_{C \text{cycle} \subseteq H} \mathbb{P}[C] = O\left(\binom{n}{\ell} \ell! \left(\frac{m}{\ell}\right) \frac{\log^{2\ell} n}{n^{2\ell}}\right).$$

Summing over all possible values of $\ell$, we obtain an upper bound for the expected number of cycles in $H$: With $t := (C \log^2 n)/n$ for a suitable constant $C$, we have

$$m \sum_{\ell=1}^{m} \frac{\binom{n}{\ell} \ell! \left(\frac{m}{\ell}\right) \frac{\log^{2\ell} n}{n^{2\ell}}}{t^{\ell}} = -1 + (1 + t)^m \leq -1 + e^{mt} = -1 + e^{(C \log^2 n)/n} = o(1).$$

$$\square$$

5. Conclusions

The argument for 3-colourability of random lifts of $K_5 \setminus e$ in this manuscript can be extended to a more general class of base graphs. Let $G := G_{k,s}$ be a graph obtained by joining a stable set $S$ of size $s$ to a cycle $C$ of size $k$, where $k \geq 3$ and $s \geq 1$. For $k = 3$ and $s = 2$ we recover $K_5 \setminus e$. The proof of Theorem 11 extends with hardly any changes to the following.

**Theorem 12.** The chromatic number of a random lift of $G_{k,s}$ is a.a.s. three.

It is known that the chromatic number of random 4-regular graphs (with uniform distribution) is three [10]. Even though random lifts of $K_{d+1}$ have some similarity to random $d$-regular graphs, adapting the methods of the latter to obtain results for random lifts of $K_{d+1}$ appears to be a challenging task.

**Appendix: Distribution of the Gaps Between $n$ Points Drawn in $[0,1]$**

As mentioned above, Lemma 6 is a special case of the Bapat-Beg theorem. For the sake of completeness, we give an elementary proof.

**Proof of Lemma 6** Clearly, $\min(X_1, \ldots, X_n)$ has cumulative distribution function $t \mapsto 1 - (1 - t)^n$. This settles the easy cases when $k = 0$ or, $k = n$.

Partitioning $\otimes_{j=1}^{n} [0,1]$ into $n!$ sets we need to compute

$$\mathbb{P}[S'_{k+1} - S'_k \leq a] = n! \int_{\mathbb{R}^n} 1_{\{0 \leq p_1 \leq \cdots \leq p_n \leq 1\}} 1_{\{pr_k \leq pr_{k+1} \leq pr_{k+a}\}} \, d\lambda^n. \quad (1)$$
Denoting
\[ v(\ell, t) := \int_{\mathbb{R}^t} 1_{\{0 \leq \ell_1 \leq \cdots \leq \ell_t \leq t\}} d\lambda^n = \frac{t!}{\ell!} \]
we have that (1) is equal to
\[
\int_0^1 \int_0^1 v(s, k-1)v(1-t, n-k-1)1_{s \leq t \leq s+a} \, dt \, ds =
\]
\[
= \int_0^1 v(s, k-1) \int_s^{\min(1,s+a)} v(1-t, n-k-1) \, dt \, ds =
\]
\[
= \frac{1}{(k-1)!(n-k-1)!} \int_0^1 s^{k-1} \int_s^{\min(1,s+a)} (1-t)^{n-k-1} \, dt \, ds \quad (2)
\]

We evaluate the inner integral
\[
\int_s^{\min(1,s+a)} (1-t)^{n-k-1} \, dt =
\]
\[
= \int_s^{\min(1,s+a)} (1-t)^{n-k-1} \, dt =
\]
\[
= \begin{cases}
\frac{1}{n-k} (1-s)^{n-k} & \text{if } s \leq 1-a \\
\frac{1}{n-k} (1-s)^{n-k} - \frac{1}{n-k} (1-a-s)^{n-k} & \text{if } s \geq 1-a.
\end{cases}
\]

Then the integral in (2) (without the factorial factor) becomes
\[
\frac{1}{n-k} \int_0^1 s^{k-1} (1-s)^{n-k} \, ds - \frac{1}{n-k} \int_0^{1-a} s^{k-1} (1-a-s)^{n-k} =
\]
\[
= -\frac{(k-1)!(n-k-1)!}{n!} (0-1) + \frac{(k-1)!(n-k-1)!}{n!} (0-(1-a)^n) =
\]
\[
= \frac{(k-1)!(n-k-1)!}{n!} (1-(1-a)^n).
\]

Hence, (1) is equal to
\[
n! \frac{1}{(k-1)!(n-k-1)!} \frac{(k-1)!(n-k-1)!}{n!} (1-(1-a)^n) = 1-(1-a)^n.
\]

\[ \square \]

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