Some remarks on smooth mappings of Hilbert and Banach spaces and their local convexity property

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Abstract: We analyze smooth nonlinear mappings for Hilbert and Banach spaces that carry small balls to convex sets, provided that the radius of the balls is small enough. Being focused on the study of new and mild sufficient conditions for a nonlinear mapping of Hilbert and Banach spaces to be locally convex, we address a suitably reformulated local convexity problem analyzed within the Leray-Schauder homotopy method approach for Hilbert spaces, and within the Lipschitz smoothness condition both for Hilbert and Banach spaces. Some of the results presented in the work prove to be interesting and novel even for finite-dimensional problems. Open problems related to the local convexity property for nonlinear mapping of Banach spaces are also formulated.

Keywords: nonlinear mapping; Hilbert space; Banach space; convex set

1. The local convexity mapping property

1.1. Introductory setting

The importance of local convexity of nonlinear mappings of Hilbert and Banach spaces in applied mathematics is well known [1,2,10,16,21,25,33,34–42,43,45,47]. It is especially true of the theory of nonlinear differential-operator equations [6,31], control theory and optimization theory[29], etc. Some of interesting and important for applications local convexity properties, related with mappings of Hilbert spaces, were first studied in [39], and related with mappings of Hilbert and Banach spaces were later generalized and studied both in [4,5,7,15,26,28,30,44] and in [12,18–20,22,24,33–35,37,38]. It is worth reminding that a nonlinear continuous mapping \( f : X \to Y \) of Banach spaces \( X \) and \( Y \) is called to be locally convex, if for any point \( a \in X \) there exists a ball \( B_r(a) \subset X \) of radius \( r > 0 \), such that its image \( f(B_r(a)) \subset Y \) is convex.

The property of local convexity holds for the special case of a differentiable mapping \( f : X \to Y \) of Hilbert spaces if the Frechét derivative \( f'(x) : X \to Y \) is Lipschitzian in a closed ball \( B_r(a) \subset X \) with radius \( r > 0 \) which is centered at point \( a \in X \) and the linear mapping \( f'(a) : X \to Y \), defined on the whole space \( X \), closed and surjective. We need to mention here that this notion differs from that introduced before in \[25\]. Concerning a special case of a differentiable mapping \( f : X \to Y \) of Hilbert spaces, the property of local convexity, as it was first stated in [39], holds and is based on the strong convexity of the ball \( B_r(a) \subset X \), if, in addition, the Frechet derivative \( f'_x : X \to Y, x \in X \) is Lipschitzian in a closed ball \( B_r(a) \subset X \) of radius \( r > 0 \), centered at point \( a \in X \), and the linear mapping
\(f'_a : X \to Y\) is surjective. This statement, is below proved making use of slightly different arguments from those done before in [3,39], giving rise to an improved estimation for the radius of the ball \(B_p(a) \subset X\), whose image \(f(B_p(a)) \subset Y\) proves to be convex.

More subtle techniques are required [12,22,35] for the local convexity problem for a nonlinear differentiable mapping \(f : X \to Y\) of Banach spaces. Thus, its analysis is carried out only for the case of Banach spaces with special properties. In particular, the locally convex functions between Banach spaces are analyzed and the conditions guaranteeing that a given function \(f : U \to Y\) is locally convex are stated. As in the Banach space case the local convexity property is based on much more subtle properties both of a mapping and a Banach space under regard, special attention is paid to the locally convex functions between Banach spaces. We suitably reformulated local convexity problem for Banach spaces, taking into account of the interplay between the modulus of convexity of a Banach space and the modulus of smoothness of a function \(f : X \to Y\), before used in the work [3] within a more general framework. It has been noticed that some of the results presented in the present work prove to be interesting themselves and novel even for finite-dimensional problems. Open problems related with the local convexity property for nonlinear mapping of Banach spaces are also formulated.

Let \(X, Y\) be Hilbert or Banach spaces and \(U \subset X\) be an open set. A mapping \(f : U \to Y\) is said to be Gâteaux differentiable [36,46] at a point \(a \in U\), if there exists a continuous linear mapping \(f'(a) : X \to Y\), such that for any \(h \in X\), there exists the limit

\[
\lim_{t \to 0} \frac{f(a + th) - f(a)}{t} = f'(a)h.
\]

The mapping \(f'(a) : X \to Y\) is called the Gâteaux derivative of the mapping \(f : U \to Y\) at the point \(a \in U\).

A mapping \(f : U \to Y\) is said to be Frechet differentiable at a point \(a \in U\), if there exists a continuous linear mapping \(f'(a) : X \to Y\), such that there exists the limit

\[
\lim_{||h|| \to 0} \frac{||f(a + h) - f(a) - f'(a)h||}{||h||} = 0.
\]

The mapping \(f'(a) : X \to Y\) is called the Frechet derivative of the mapping \(f : U \to Y\) at the point \(a \in U\). The mapping \(f' : X \to Y\) at the point \(a \in U\) is Frechet differentiable, it is also continuous and Gâteaux differentiable. A mapping \(f : U \to Y\) is called continuously Frechet differentiable, if the mapping \(X \ni a \to f'(a) \in L(X,Y)\) is continuous with respect to the corresponding topologies. Respectively, a mapping \(f : U \to Y\) is called continuously Gâteaux differentiable, if the mapping \(X \ni a \to f'(a) \in L(X,Y)\) is continuous with respect to the corresponding topologies. If a mapping \(f : U \to Y\) is continuously Gâteaux differentiable, then it is also continuously Frechet differentiable, thus being of the \(C^1\)-class. The latter means that the simplest way to state the continuous Frechet differentiability of a mapping \(f : U \to Y\) is to state its continuous Frechet differentiability. In general, a mapping \(f : U \to Y\) is assumed to be either continuously Frechet differentiable or locally Lipschitz. If a continuously differentiable mapping \(f : X \to Y\) is a bijection and its inverse \(f^{-1} : Y \to X\) is also continuously differentiable, it is called a Frechet diffeomorphism. Taking into account the classical Inverse Function Theorem [36,46], a continuously Frechet differentiable mapping \(f : X \to Y\), such that for any \(x \in X\) the derivative \(f'(x) : X \to Y\) is surjective, possesses the inverse mapping \(f'(x)^{-1} : Y \to X\), for which there exists a positive constant \(\alpha_x > 0\), such that \(||f'(x)u|| \geq \alpha_x ||u||\) for any \(u \in X\), thus defining a local Frechet diffeomorphism of Banach spaces \(X\) and \(Y\). In particular, this means that for each point \(x \in X\) there exists an open set \(U(x) \subset X\), such that \(f(U(x)) \subset Y\) is open in \(Y\) and \(f|_{U(x)} : U(x) \to f(U(x))\) is a local diffeomorphism. The general and often important for many applications problem of describing conditions on a locally Frechet diffeomorphism mapping \(f : X \to Y\) to be globally diffeomorphic is enough complicated, yet we are here interested only in the related and also important for applications local
convexity property of the constructed above mapping \( f|_{U(x)} : U(x) \to f(U(x)) \), namely, in description of conditions on the mapping \( f : X \to Y \), under which for any \( x \in X \) and some small enough \( \varepsilon > 0 \) there exists a ball \( B_\varepsilon(a_x) \subset U(x) \), being a convex set centered at some point \( a_x \in U(x) \), and such that the image \( f(B_\varepsilon(a_x)) \subset Y \) is convex too. The local convexity problem states as follows:

**Problem 1.** To construct at least sufficient conditions for a nonlinear smooth mapping \( f : X \to Y \) of a Hilbert space \( X \) into a Hilbert space \( Y \) to be locally convex.

Let \( (X, (\cdot, \cdot)) \) be a Hilbert space and a self-mapping \( f : X \to X \) is Frechet differentiable. The following proposition demonstrates some typical geometrical conditions, which a priori can be imposed on the self-mapping \( f : X \to X \) to guarantee it to satisfy the local convexity property.

**Proposition 1.** Let \( (X, (\cdot, \cdot)) \) be a Hilbert space and a Frechet differentiable mapping \( f : X \to X \) is such that the mapping \( h := f - cl : X \to X, c \in \mathbb{R}\setminus\{0\} \), is proper, that is, the preimage of a compact set is compact too. Assume also, in addition that if \( U \subset X \) is open and convex subset, then the image \( h(U) \subset X \) is simply connected and for any \( \tilde{\xi} \in X \) \( \sup_{u \in U} (\tilde{\xi}, u) < \infty \), then \( \sup_{u \in U} (\tilde{\xi}, h(u)) < \infty \). Then the mapping \( f : X \to X \) is locally convex.

**Proof.** To sketch a proof of this proposition, we take any \( \theta \in [0,1] \) and let \( f_\theta := \theta f + (1 - \theta)I : X \to X \) and remark that this mapping satisfies conditions assumed above too. Let \( v_0 \in f(U) \) and \( u_0 \in f^{-1}(v_0) \) and define the mapping \( g_\theta(u) := f_\theta(u) - f(u_0), u \in U \), satisfying the condition \( g_\theta(\partial U) \neq 0 \), where, obviously, \( f_\theta(\partial U) = \partial f_\theta(U), \theta \in [0,1] \). If now easy to check that the Leray-Schauder [36,43,46] topological degree \( \text{deg}(g_\theta, U; 0) \) is well defined, the preimage \( g_\theta^{-1}(0) \subset X \) is finite, coinciding with the finite number of elements, as the Frechet derivative \( g_\theta'(u) : X \to X \) is compact for any \( u \in U \). Taking now into account that the mapping \( [0,1] \ni \theta \to g_\theta \in C^1(X) \) is a homotopy with \( g_0 = I - u_0 \) and \( g_1 = f - v_0 \), one obtains easily that \( \text{deg}(f - v_0, U; 0) = 1 \). Since the latter holds for every \( v_0 \in f(U) \), it holds for all \( \theta \in [0,1] \), confirming that the mapping \( f|_U : U \to f(U) \) is a diffeomorphism. To state that the image \( f(U) \subset X \) is convex, we assume that \( v_0 \in f(S) \), where \( S \subset U \) is chosen in such a way that \( f(S) \subset X \) is not a strongly locally convex set. Let \( \varepsilon > 0 \) be small enough positive number, such that \( \text{inf}\{\lambda \in \text{Spec}(f'(u)) : u \in B_\varepsilon(u_0)\} = \varepsilon > 0 \). Then the mapping \( h(u) := u_0 + \varepsilon^{-1}f(u) - f(u_0), u \in U(u_0) \), is invertible and satisfies the inequalities

\[
\left( h^{-1} \right)'(v_0)\alpha|\alpha| > \varepsilon_0 M_0^{-1}(\alpha|\alpha|), \left( h^{-1} \right)'(v_0)\alpha|\alpha| \leq |\alpha||
\]

for any \( \alpha \in X \), where, by definition, \( M_0 := \sup\{\lambda \in \text{Spec}(f'(u_0)) : u_0 \in U \subset X \} \). From inequality (3) one derives that the vector \( (h^{-1})'(v_0)\alpha \in X \) lies in the interior of a strictly convex cone symmetric around the axis passing through \( v_0 \in f(S) \) in the direction of a vector \( \alpha \in X \) with respect the geometry induced on \( X \) by the inner product \((\cdot, \cdot)\). Replacing the vector \( \alpha \in X \) by \(-\alpha \in X \), we get that \(- (h^{-1})'(v_0)\alpha \in X \) lies in the interior of the strictly convex cone antipodal to that described above. It means that for \( \delta > 0 \) sufficiently small the points \( h^{-1}(p) \subset S \) and \( p \in f(S) \) lie together in the interior of one of these strictly convex cones, as well as the points \( h^{-1}(q) \subset S \) and \( q \in f(S) \) lie together in the interior of one of these strictly convex antipodal cones. Based on these geometric properties followed by analogical reasonings from [17], we finally state the convexity of the image \( f(U) \subset X \).  

### 1.2. The local convexity property: the Lipschitz smoothness analysis

To enter now into the problem, we begin with a novel proof of the local convexity result for a nonlinear smooth mapping \( f : X \to Y \) of Hilbert spaces, before formulated in [3,39] and stated there under different conditions. The following proposition holds.
Proposition 2. Let \( f : X \rightarrow Y \) be a nonlinear differentiable mapping of Hilbert spaces whose Frechet derivative \( f'_x : X \rightarrow Y, x \in B_r(a) \), in a ball \( B_r(a) \subset X \) centered at point \( a \in X \), is Lipschitzian with a constant \( L > 0 \), the linear mapping \( f'_x : X \rightarrow Y, a \in X \), is surjective and the adjoint mapping \( f''_a^*: Y \rightarrow X \) satisfies the condition \( \| f''_a^* \| \geq v \) for some positive constant \( v > 0 \). Then for any \( \varepsilon < \min \{ r, v / (4L) \} \) the image \( f_\varepsilon(a) := f(B_\varepsilon(a)) \subset Y \) is convex.

To prove Proposition 2, it is useful to state the following simple enough lemmas, based both on the Taylor expansion [1,15,33,36,46] of the differentiable mapping \( f : X \rightarrow Y \) at point \( x_0 \in B_r(a) \subset B_r(a) \) and on the triangle and parallelogram properties of the norm \( \| \cdot \| \) in a Hilbert space.

Lemma 1. Let a mapping \( f'_x : X \rightarrow Y, x \in X \), be \( L \)-Lipschitzian in a ball \( B_\rho(x_0) \subset B_r(a) \) of radius \( \rho > 0 \), centered at point \( x_0 := (x_1 + x_2)/2 \in B_r(a) \) for arbitrarily chosen points \( x_1, x_2 \in B_r(a) \). Then there exists such a positive constant \( \mu > 0 \) that the norm \( \| f''_x^*(y) \| \geq \mu \| y \| \) in the ball \( B_\rho(x_0) \subset X \) for all \( y \in Y \), there holds the estimation \( \| f(x_0) - y_0 \| \leq \mu \rho \) for \( y_0 := (y_1 + y_2)/2, y_1 := f(x_1), y_2 := f(x_2) \) and the equation \( f(x) = y_0 \) possesses a solution \( x \in B_\rho(x_0) \), such that \( \| x - x_0 \| \leq \mu^{-1} \| f(x_0) - y_0 \| \).

Proof. Really, the following Taylor expansions at point \( x_0 \in B_r(a) \) hold:

\[
y_1 = f(x_1) - f(x_0) = f'_x(x_1 - x_0) + \epsilon_1, \quad (4)
\]

\[
y_2 = f(x_2) - f(x_0) = f'_x(x_2 - x_0) + \epsilon_2,
\]

where \( \| \epsilon_j \| \leq \frac{L}{2} \| x_1 - x_0 \|^2 = \frac{L}{2} \| x_1 - x_2 \|^2, j = 1,2 \), as the mapping \( f'_x : X \rightarrow Y, x \in X \), is \( L \)-Lipschitzian. From (4) one obtains easily that

\[
y_0 = f(x_0) + \epsilon_0, \quad (5)
\]

where, evidently, \( \| \epsilon_0 \| \leq \frac{L}{2} \| x_1 - x_0 \|^2 \). Moreover, owing to the Lipschitzian property of the Frechet derivative \( f'_x(x) : X \rightarrow Y \), one can obtain the following inequality:

\[
\| f''_x^*(y) \| = \| f''_x^*(y) - f''_x^*(y) + f''_x^*(y) \| \geq \| f''_x^*(y) \| - \| f''_x^*(y) - f''_x^*(y) \| \geq \nu \| y \| - L \| x - a \| \| y \| \geq (v - L) \| y \| := \mu \| y \|
\]

for \( \mu_0 = \nu - L > 0 \), as the norm \( \| x - a \| \leq \varepsilon \). This, in particular, means that the adjoint mapping \( f''_x^*: Y \rightarrow X \) at \( x \in B_r(a) \) is invertible, defined on the whole Hilbert space \( Y \) and the norm of its inverse mapping \( (f''_x^*)^{-1} : X \rightarrow Y \) is bounded on the ball \( B_r(a) \subset X \) by the value \( 1/\mu_0 \). First observe that for \( \rho := \frac{\rho_0}{\| a \|} \| x_1 - x_2 \|^2 \) and \( \mu := v - 2L \varepsilon > 0 \) the following inequality

\[
\| f(x_0) - y_0 \| = \| \epsilon_0 \| \leq \frac{L}{2} \| x_1 - x_2 \|^2 = \rho \mu \mu_0^{-1} \rho \mu \mu_0^{-1} \| v - L \| \mu \mu_0^{-1} \| v - 2L \| \leq \rho \mu \mu_0^{-1} \| v - L \| \rho \mu \mu_0^{-1} \| v - 2L \| \leq \rho \mu \mu_0^{-1} \| v - L \| \rho \mu, \quad (7)
\]

based on expression (5), holds. Denote now by \( \bar{\varepsilon} \in B_\varepsilon(a) \) an arbitrary point satisfying the condition \( y_0 = f(\bar{\varepsilon}) \), whose existence is guaranteed by the standard implicit function
theorem [36,46], and denote $\bar{f} := \left(\int_0^1 f'_{x(t)} \, dt\right)^{-1}(\bar{x} - x_0) \in Y$, where the linear mapping 
\[
\left(\int_0^1 f'_{x(t)} \, dt\right)^{-1} : X \to Y
\]
holds. Really, for any \( \bar{y} \) holds.

\[ \left(\int_0^1 f'_{x(t)} \, dt\right)^{-1}(\bar{x} - x_0) = \int_0^1 f'_{x(t)}(\bar{x} - x_0) \, dt = \left(\int_0^1 f'_{x(t)} \, dt\right)(\bar{x} - x_0), \tag{8} \]

which holds owing to the continuation \( x(t) := x_0 + t(\bar{x} - x_0) \in B_{2\varepsilon}(x_0) \) for \( t \in [0,1] \).

Moreover, the following estimation
\[ \left\| \left(\int_0^1 f'_{x(t)} \, dt\right)^{-1} \right\| \leq \mu^{-1} \tag{9} \]
holds. Really, for any \( y \in Y \)
\[
\left\| \left(\int_0^1 f'_{x(t)} \, dt\right)(y) \right\| = \left\| \left(\int_0^1 f'_{x_0} \, dt\right)(y) + \left(\int_0^1 \left( f'_{x(t)} - f'_{x_0} \right) \, dt\right)(y) \right\| \geq \\
\geq \left\| \left(\int_0^1 f'_{x_0} \, dt\right)(y) \right\| - \left\| \left(\int_0^1 \left( f'_{x(t)} - f'_{x_0} \right) \, dt\right)(y) \right\| \geq \\
\geq \mu_0 \| y \| - \frac{\varepsilon}{2} \| x - x_0 \| \| y \| \geq (\mu_0 - L\varepsilon) \| y \| = (\nu - 2L\varepsilon) \| y \| = \mu \| y \|,
\]
as the norm \( \| x - x_0 \| = \| (\bar{x} - a) + (a - x_0) \| \leq \| (\bar{x} - a) \| + \| (a - x_0) \| \leq 2\varepsilon \).

Remark. The integral expressions, considered above with respect to the parameter \( t \in [0,1] \), are well defined in a Hilbert, or in general, in a Banach space \( Y \), as the related mapping \( f'_{x(t)}(\bar{x} - x_0) : [0,1] \to Y \), being continuous and of bounded variation, is a priori Riemann-Birkhoff type integrable \([8,9,11,13,14,27,32]\).

Denote by \( \langle \cdot \rangle \) the scalar product both on the Hilbert space \( X \) and the Hilbert space \( Y \). Then one easily obtains that
\[
\| x - x_0 \|^2 = \| x - x_0 \| \left(\int_0^1 f'_{x(t)} \, dt\right)(\bar{g}) = \| (\int_0^1 f'_{x(t)} \, dt)(\bar{x} - x_0) \| = \\
= \| (f(\bar{x}) - f(x_0)) \| = \| y_0 - f(x_0) \| \left(\int_0^1 f'_{x(t)} \, dt\right)^{-1}(\bar{x} - x_0) \| \leq \\
\leq \| y_0 - f(x_0) \| \| \bar{x} - x_0 \| \mu_n \leq \\
\leq \| y_0 - f(x_0) \| \| x - x_0 \| / \mu_n,
\]
yielding the searched for inequality
\[ \| \bar{x} - x_0 \| \leq \mu^{-1} \| f(x_0) - y_0 \| , \tag{11} \]
and proving the Lemma.

**Lemma 2.** For arbitrarily chosen points \( x_1, x_2 \in B_{\varepsilon}(a) \) the whole ball \( B_{\rho}(x_0) \) of radius \( \rho = \frac{\mu_0}{\varepsilon} \| x_1 - x_2 \|^2 \leq \varepsilon \), centered at point \( x_0 := (x_1 + x_2)/2 \in B_{\varepsilon}(a) \), belongs to the ball \( B_{\varepsilon}(a) \).
\[ \square \]
Proof. Consider for this the following triangle inequality and the related parallelogram identity on the Hilbert space $X$ for any point $x \in B_{\rho}(x_0)$:

$$||x-a|| = ||(x-x_0) + (x_0-a)|| \leq ||x-x_0|| + ||x_0-a|| =$$

$$= ||x-x_0|| + ||(x_1-a)/2 + (x_2-a)/2|| =$$

$$= ||x-x_0|| + [(||x_1-a||^2/2 + ||x_2-a||^2/2) - ||x_1-x_2||^2/4]^{1/2} \leq$$

$$\leq \rho + (\varepsilon^2 - ||x_1-x_2||^2/4)^{1/2}.$$  

For the right-hand side of (12) to be equal or less of $\varepsilon > 0$, it is enough to take such a positive number $\rho \leq \varepsilon$ that

$$\rho + (\varepsilon^2 - ||x_1-x_2||^2/4)^{1/2} \leq \varepsilon.$$  

This means that the following inequality should be satisfied:

$$\rho^2 \geq 2\rho - ||x_1-x_2||^2/4.$$  

The choice $\rho = \frac{\mu_0}{8C}||x_1-x_2||^2 \geq 2\varepsilon^2/(v - \varepsilon L)$ a priori satisfies the above condition (14) if $\varepsilon \leq \nu/(3L)$, thereby proving the Lemma. □

Proof. (Proof of Proposition 2). Now, based on Lemmas 1 and 2, it is easy to observe from (7) and (11) that a point $\bar{x} \in B_1(a)$, satisfying the equation $y_0 = f(\bar{x})$, belongs to the ball $B_\rho(x_0) \subset X$:

$$||\bar{x} - x_0|| \leq ||y_0 - f(x_0)||/\mu \leq \rho \mu / \mu = \rho,$$  

thereby proving our Proposition 2 and solving Problem (1). □

It is worth to mention here that our local convexity proof for a nonlinear smooth mapping $f : X \to Y$ of Hilbert spaces is slightly different from that presented before in [39] and gives a bit improved estimation of the radius of the ball $B_\rho(x_0) \subset X$, whose image $f(B_\rho(x_0)) \subset Y$ proves to be convex.

2. The local convexity mapping property: Banach spaces case

2.1. Banach spaces of the modulus of convexity of degree 2

Let now $X, Y$ be Banach spaces. A mapping $f : U \to Y$ defined on an open subset $U \subset X$ we will call locally convex if for each point $x \in U$ and its neighborhood $O_x \subset U$ there is a convex open neighborhood $U_x \subset O_x$ with convex image $f(U_x) \subset Y$. In this part we address the following problem.

Problem 2. Find at least sufficient conditions guaranteeing that a given function $f : U \to Y$ is locally convex.

We start from the modulus convexity definition following [15] (see also [35]).

Definition 1. Let $(X, || \cdot ||)$ be a Banach space and $B_X := \{ x \in X : ||x|| \leq 1 \}$ be a unit ball. For every $\varepsilon \in (0,2]$ we define the modulus of convexity (or rotundity) of $|| \cdot ||$ by

$$\delta_X(\varepsilon) = \inf_{x,y \in B_X} \left\{ 1 - \frac{||x+y||}{2} : ||x-y|| \geq \varepsilon \right\}.$$  

(16)

The norm $|| \cdot ||$ is called uniformly convex (or uniformly rotund), if $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$. The space $(X, || \cdot ||)$ is then called a uniformly convex space. Note also that $\delta_X(\varepsilon) = \inf_{Y \subset X} \{ \delta_Y(\varepsilon) : \dim Y = 2 \}$.
It is easy to observe that \( \delta_X(\varepsilon) \leq \varepsilon/2 \) for all \( \varepsilon \in (0, 2] \). The definition above can be equivalently reformulated owing to the following [15] lemma.

**Lemma 3.** \((X, \| \cdot \|)\) be a Banach space, \( S_X := \partial B_X \) be the boundary of the unit ball \( B_X \subset X \) and let \( \delta_X(\varepsilon), \varepsilon \in (0, 2], \) be the modulus of convexity of \( \| \cdot \| \). Then

\[
\delta_X(\varepsilon) = \inf_{x,y \in S_X} \left\{ 1 - \frac{\| x + y \|}{2} : \| x - y \| = \varepsilon \right\}.
\] (17)

Remark also, that by [35], each Banach space with norm, having modulus of convexity of power type 2, is superreflexive. In addition, based on the definition (16) one can be derived [15] the following useful lemma.

**Lemma 4.** The norm of a Banach space \( X \) has modulus of convexity of power \( p > 1 \), if and only if there is a positive constant \( C > 0 \) such that

\[
\frac{\| x + y \|}{2} \leq 1 - C\| x - y \|^p
\] (18)

for any points \( x, y \in X \) with \( \max\{\| x \|, \| y \|\} \leq 1 \).

2.2. The Banach space case: main result

To answer the Problem above, we need to recall some notions related to the differentiability and the Lipschitz property.

Let \( X, Y \) be Banach spaces and \( U \subset X \) be an open subset in \( X \). A function \( f : U \to Y \) is called

- **differentiable** at a point \( x_0 \in U \) if there is a linear continuous operator \( f'_x : X \to Y \) (called the Frechet derivative of \( f \) at \( x_0 \in U \)) such that

\[
\lim_{x \to x_0} \frac{\| f(x) - (f(x_0) + f'_x(x - x_0)) \|}{\| x - x_0 \|} = 0;
\]

- **Lip-differentiable** at a point \( x_0 \in U \), if there is a neighborhood \( W \subset U \) of \( x_0 \in U \), such that \( f : U \to Y \) is differentiable at each point \( x \in W \) and

\[
\sup_{x,y \in W, x \neq y} \frac{\| f(y) - (f(x) + f'_x(y - x)) \|}{\| y - x \|^2} < \infty.
\]

- **locally Lipschitz** at a point \( x_0 \in U \), if there is a neighborhood \( W \subset U \) of \( x_0 \in U \), such that

\[
\sup_{x,y \in W, x \neq y} \frac{\| f(x) - f(y) \|}{\| x - y \|} < \infty.
\]

**Theorem 1.** Let \( X \) be a Banach spaces whose norm has modulus of convexity of power type 2. A homeomorphism \( f : U \to V \) between two open subsets \( U, V \subset X \) is locally convex if

- the function \( f : U \to V \) is Lip-differentiable at each point \( x_0 \in U \) and
- the function \( f^{-1} : V \to U \) is locally Lipschitz at each point \( y_0 \in V \).

**Proof.** Fix any point \( x_0 \in U \). Given a neighborhood \( O(x_0) \subset U \) of \( x_0 \in U \) we should construct a convex neighborhood \( U(x_0) \subset O(x_0) \) with convex image \( f(U(x_0)) \). We lose no generality assuming that \( y_0 = f(x_0) = 0 \).

Using the Lip-differentiability of \( f : U \to V \) at \( x_0 \in U \), find a neighborhood \( W \subset O(x_0) \) of \( x_0 \in U \) and a real number \( L \) such that

\[
\| f(y) - f(x) - f'_x(y - x) \| \leq L\| y - x \|^2
\]
for all points \( x, y \in W \). Moreover, since the homeomorphism \( f^{-1} : V \to U \) is locally Lipschitz at the point \( y_0 = f(x_0) \), we can assume that
\[
\|x - y\| = \|f^{-1}(f(x) - f(y))\| \leq L\|f(y) - f(x)\|
\]
for any points \( x, y \in W \). By our assumption, the norm of the Banach space \( X \) has modulus of convexity of power type 2. Then, owing to the result (18) of Lemma 4, there is a positive constant \( C < 1 \) such that
\[
\frac{1}{2} \|x + y\| \leq 1 - C\|x - y\|^2
\]
for any points \( x, y \in B_X \). Take any \( \varepsilon > 0 \), such that
\[
\frac{1}{4}L^2 \leq \frac{C}{2}; \\
B_{\varepsilon}(z_0) = \{z \in X : \|z - z_0\| \leq \varepsilon\} \subset f(W);
\]
The choice of \( \varepsilon > 0 \) guarantees that the preimage \( A_{\varepsilon} = f^{-1}(B_{\varepsilon}(z_0)) \) of the \( \varepsilon \)-ball \( B_{\varepsilon}(z_0) = \{z \in V : \|z_0 - z\| \leq \varepsilon\} = \{z \in X : \|z\| \leq \varepsilon\} \), centered at \( z_0 = 0 \), lies in the neighborhood \( W \) on the point \( x_0 \in U \). Now the proof of the theorem will be complete as soon as we check that the closed neighborhood \( A_{\varepsilon} = f_{\varepsilon}(B_{\varepsilon}(z_0)) \subset W \) of \( x_0 \in U \) is convex. It suffices to check that for any points \( x, y \in A_{\varepsilon} \) the point \( \bar{x} = (x + y)/2 \) belongs to \( A_{\varepsilon} \subset W \), which happens if and only if its image \( f(\bar{x}) \in Y \) belongs to the ball \( B_{\varepsilon}(y_0) \subset Y \). The choice of \( f : U \to V \) guarantees that
\[
\|f(x) - f(\bar{x}) - f_\varepsilon'(x - \bar{x})\| \leq L\|x - \bar{x}\|^2
\]
and
\[
\|f(y) - f(\bar{x}) - f_\varepsilon'(y - \bar{x})\| \leq L\|y - \bar{x}\|^2.
\]
Adding these inequalities and taking into account that \( x - \bar{x} = -(y - \bar{x}) \), we get
\[
\|(f(x) + f(y)) - 2f(\bar{x})\| \leq 2L\|y - \bar{x}\|^2 = \frac{1}{2}L\|y - x\|^2 \leq \frac{1}{2}L^2\|f(y) - f(x)\|^2
\]
and hence
\[
\|\frac{f(x) + f(y)}{2} - f(\bar{x})\| \leq \frac{1}{2}L^2\|f(x) - f(y)\|^2 \leq \frac{C}{2}\|f(x) - f(y)\|^2.
\]
Since \( f(x), f(y) \in B_{\varepsilon}(z_0) \), we get
\[
\left\|\frac{f(x) + f(y)}{2}\right\| \leq \varepsilon - \frac{C}{2}\|f(y) - f(x)\|^2
\]
and
\[
\|f(z)\| \leq \left\|\frac{f(x) + f(y)}{2}\right\| + \left\|\frac{f(x) + f(y)}{2} - f(z)\right\| \leq \varepsilon - \frac{C}{2}\|f(y) - f(x)\|^2 + \frac{C}{2}\|f(y) - f(x)\|^2 = \varepsilon,
\]
which means that \( f(z) \in B_{\varepsilon}(y_0) \) and \( z \in A_{\varepsilon} \subset W \). \( \square \)

Remark. As follows from the proof of Theorem 1, when the spaces \( X \) and \( Y \) are Hilbert ones, the obtained above result reduces to that of Proposition 2, yet in its slightly weakened form.

2.3. The locally convex functions between Banach spaces

As above, let \( X, Y \) be Banach spaces and \( f : U \to Y \) be defined on an open subset \( U \subset X \), which is called locally convex, if each point \( x \in U \) has a neighborhood base consisting of open convex subsets \( U_x \subset X \) with convex images \( f(U_x) \subset Y \). In this part of paper we address the local convexity problem problem (2) formulated before. The answer
to this problem will be given in terms of the interplay between the modulus of smoothness of the function $f : U \to Y$ and the modulus of convexity of the Banach space $Y$.

Any Hilbert space $E$ of dimension $\dim(E) > 1$ has modulus of convexity $\delta_E(t) = 1 - \sqrt{1 - t^2} \leq \frac{t^2}{2}$. By [22], $\delta_X(t) \leq \delta_E(t) \leq \frac{t^2}{2}$ for each Banach space $X$. We shall say that the Banach space $X$ has modulus of convexity of degree $p$ if there is a constant $L > 0$ such that $\delta_X(t) \geq L t^p$ for all $t \in [0,1]$ that follows from the inequalities $L t^p \leq \delta_X(t) \leq \frac{t^2}{2}$ that $p \geq 2$. So, the Hilbert spaces have modulus of convexity of degree 2. Many examples of Banach spaces with modulus of convexity of degree 2 can be found in [23]. In particular, so is the Banach space $X = (\sum_{n=1}^{\infty} I_n(n))_2$, which is not isomorphic to a Hilbert space.

Next, we recall [22,23] the definition of the moduli of smoothness $\omega_n(f;t)$, $t \geq 0$, of a function $f : U \to Y$ defined on a subset $U \subset X$ of a Banach space $X$. By definition,

$$\omega_n(f;t) = \sup \left\{ \| \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x + \left(\frac{x}{2} - k)h\right)) : \|h\| \leq t, \ [x - nh/2, x + nh/2] \subset U \right\}.$$  \hspace{1cm} (20)

In particular,

$$\omega_1(f;t) = \sup \{ \| f(x) - f(y) \| : \| x - y \| \leq t, \ [x,y] \subset U \}

and

$$\omega_2(f;t) = \sup \{ \| f(x + h) - 2f(x) + f(x - h) \| : \|h\| \leq t, \ [x - h, x + h] \in U \},

where $[x,y] = \{sx + (1-s)y : s \in [0,1]\}$ stands for the segment connecting the points $x,y \in X$. Moreover, it is true [22] that $\omega_n(f;\frac{1}{m}t) \geq \frac{1}{nm} \omega_n(f;t)$ for each $m \in \mathbb{N}, t \geq 0$. Below we will formulate the following definition.

**Definition 2.** We shall say that a function $f : U \to Y, U \subset X$, is

- Lipschitz if there is a constant $L$ such that $\omega_1(f;t) \leq Lt$ for all $t \geq 0$;
- second order Lipschitz if there is a constant $L$ such that $\omega_2(f;t) \leq Lt^2$ for all $t \geq 0$;
- locally (second order) Lipschitz if each point $x \in U$ has a neighborhood $W \subset U$, such that the restriction $f|W : W \to Y$ is (second order) Lipschitz.

**Theorem 2.** Let $X$ be a Banach space and $Y$ be a Banach space with modulus of convexity of power 2. A homeomorphism $f : U \to V$ between two open subsets $U \subset X, V \subset Y$ of a Banach space $X$ is locally convex if

- the function $f : U \to V$ is locally second order Lipschitz;
- the function $f^{-1} : V \to U$ is locally Lipschitz.

**Proof.** Fix any point $x_0 \in U$. Given a neighborhood $O(x_0) \subset U$ of $x_0 \in U$ we should construct a convex neighborhood $U(x_0) \subset O(x_0)$ with convex image $f(U(x_0)) \subset V$. We lose no generality assuming that $y_0 = f(x_0) = 0$. Since $f$ is locally second order Lipschitz, the point $x_0 \in U$ has a neighborhood $W \subset O(x_0)$ such $\omega_2(f|W;t) \leq Lt^2$ for some real number $L$ and all positive $t \leq 1$. Moreover, since the homeomorphism $f^{-1} : V \to U$ is locally Lipschitz at the point $y_0 = f(x_0)$, we can assume that $\omega_1(f^{-1}|f(W);t) \leq L t$ for all $t \geq 0$. We can, in addition, also assume that max $\{ \text{diam}(W), \text{diam}(f(W)) \} \leq 1$.

By our assumption, the norm of the Banach space $X$ has modulus of convexity of power type 2 and, owing to the relationship (19) of Lemma 4, there is a positive constant $C < 1$, such that

$$\left\| \frac{x + y}{2} \right\| \leq 1 - C \| x - y \|^2$$
for any points \(x, y \in X\) with \(\max\{|x|, |y|\} \leq 1\). Take any positive \(\varepsilon < 1\) such that

- \(\frac{1}{\varepsilon} L^2 \leq \frac{C}{\varepsilon}\);
- \(B_{\varepsilon}(z_0) = \{z \in Y : |z - z_0| \leq \varepsilon\} \subset f(W)\).

The choice of \(\varepsilon > 0\) guarantees that the preimage \(A_\varepsilon = f^{-1}(B_\varepsilon(z_0))\) of the \(\varepsilon\)-ball \(B_\varepsilon(z_0) = \{z \in Y : |z - z_0| \leq \varepsilon\} = \{z \in Y : |z| \leq \varepsilon\}\) centered at \(z_0 = 0\) lies in the neighborhood \(W\) on the point \(x_0 \in U\). The proof of the theorem will be complete as soon as we check that the closed neighborhood \(A_\varepsilon = f^{-1}(B_\varepsilon(z_0)) \subset W\) of \(x_0 \in U\) is convex. It suffices to check that for any points \(x, y \in A_\varepsilon\) the point \(\bar{x} = \frac{x + y}{2}\) belongs to \(A_{\varepsilon} \subset W\), which happens if and only if its image \(f(\bar{x}) \in Y\) belongs to the ball \(B_{\varepsilon}(0)\). Let \(h = x - \bar{x} \in X\); the choice of \(f : U \to V\) guarantees that

\[
\|x - y\| = \|f^{-1}(f(x)) - f^{-1}(f(y))\| \leq \omega_1(f^{-1}(f(W)); f(x) - f(y)) \leq L\|f(x) - f(y)\|
\]

and

\[
\left\| \frac{f(x) + f(y)}{2} - f(z) \right\| = \frac{1}{2}\|f(z + h) - 2f(z) + f(x) - h\| \leq \frac{1}{2}\|f(x) - f(y)\| \leq \frac{1}{2}\|f(x) - f(y)\| \leq \frac{C}{\varepsilon}\|f(x) - f(y)\|.
\]

Since \(f(x), f(y) \in B_{\varepsilon}(z_0)\), we get

\[
\left\| \frac{f(x) + f(y)}{2} \right\| \leq 1 - \frac{C}{\varepsilon}\|f(y) - f(x)\|^2
\]

and

\[
\|f(\bar{x})\| \leq \left\| \frac{f(x) + f(y)}{2} \right\| + \left\| \frac{f(x) + f(y)}{2} - f(\bar{x}) \right\| \leq \left(\varepsilon - \frac{C}{\varepsilon}\|f(y) - f(x)\|^2\right) + \frac{C}{\varepsilon}\|f(y) - f(x)\|^2 = \varepsilon,
\]

which means that \(f(\bar{x}) \in B_{\varepsilon}(z_0)\) and \(\bar{x} \in A_{\varepsilon} \subset f^{-1}(B_{\varepsilon}(z_0)) \subset W\).

We say, following [35], that a function \(f : U \to Y\) defined on an open subset \(U \subset X\) of a Banach space \(X\) with values in a Banach space \(Y\) is Gâteaux differentiable at a point \(x_0 \in U\) if there is a linear operator \(f_{x_0}' : X \to Y\) (called the Gâteaux derivative of \(f : U \to Y\) at \(x_0 \in U\)) such that for every \(h \in X\)

\[
\lim_{t \to 0} \frac{f(x_0 + th) - f(x_0)}{t} = f_{x_0}'(h).
\]

By \(L(X, Y)\) we denote the Banach space of all bounded linear operators \(T : X \to Y\) from \(X\) to \(Y\), endowed with the standard operator norm \(\|T\| = \sup_{\|x\| \leq 1} \|T(x)\|\). The following proposition holds.

**Proposition 3.** Let \(X, Y\) be Banach spaces and \(U \subset X\) be an open subset. A function \(f : U \to Y\) is second order Lipschitz if it is Gâteaux differentiable at each point of \(U\) and the derivative map \(f' : U \to L(X, Y)\), \(f' : x \mapsto f_{x}'\), \(x \in X\), is Lipschitz.

**Proof.** Since the derivative map \(f' : U \to L(X, Y)\) is Lipschitz, there is a constant \(L\) such that

\[
\|f'_{x} - f'_{y}\| \leq L\|x - y\|
\]

for each \(x, y \in U\). The second order Lipschitz property of the map \(f\) will follow as soon as we check that

\[
\|f(x + h) - 2f(x) + f(x - h)\| \leq L\|h\|^2
\]
for each \( x \in U \) and \( h \in X \) with \([x - h, x + h] \subset U\). The Gâteaux differentiability of the function \( f : U \rightarrow Y \) implies the differentiability of the function
\[
g : [0,1] \rightarrow Y, \quad g(t) = f(x + th) - 2f(x) + f(x - th).
\]
Moreover,
\[
g'(t) = f'_{x + th}(h) - f'_{x - th}(h)
\]
and hence
\[
\|g'(t)\| = \|f'_{x + th} - f'_{x - th}\| \cdot \|h\| \leq L\|2th\| \cdot \|h\| = 2Lt\|h\|^2.
\]
Then
\[
\|f(x + h) - 2f(x) + f(x - h)\| = \|g(1) - g(0)\| = 2L \|h\| \int_0^1 \|g'(t)\| \, dt = 2L \|h\|^2 \int_0^1 t \, dt = L \|h\|^2.
\]

This proposition combined with Theorem 2 implies

**Corollary 1.** Let \( X \) be a Banach space and \( Y \) be a Banach space with modulus of convexity of power 2. A homeomorphism \( f : U \rightarrow V \) between two open subsets \( U \subset X, V \subset Y \) of a Banach space \( X \) is locally convex if

- the function \( f : U \rightarrow V \) is Gâteaux differentiable at each point of \( U \);
- the derivative map \( f' : U \rightarrow L(X,Y) \) is locally Lipschitz;
- the function \( f^{-1} : V \rightarrow U \) is locally Lipschitz.

The statement of Theorem 2 is eventually only sufficient. We also at present do not know if the requirement on the convexity modulus of \( Y \) is essential in Theorem 2 and Corollary 1.

**Remark.** Similar to the Hilbert space case, we need to mention here that the integral expressions, considered above with respect to the parameter \( t \in [0,1] \), are well defined in the Banach space \( Y \), as the related mapping \( f'_{\phi(t)}(x - x_0) : [0,1] \rightarrow Y \), being continuous and of bounded variation, is a priori Riemann-Birkhoff type integrable \([8,9,11,13,14,27,32]\).

**Problem 3.** Assume that \( X \) is a Banach space such that any locally second order Lipschitz homeomorphism \( f : X \rightarrow X \) with locally Lipschitz inverse \( f^{-1} : X \rightarrow X \) is locally convex. Is \( X \) (super)reflexive?

**Problem 4.** For every \( n \in \mathbb{N} \) let \( F_n \) be the set of all functions \( f_n : l_\infty(n) \rightarrow [0,1] \) on the \( n \)-dimensional Banach space \( l_\infty(n) = (\mathbb{R}^n, \| \cdot \|_\infty) \) such that

- \( f_n^{-1}(0,1] \subset (-1,1)^n\);
- \( \omega(f; t) \leq t \) for all \( t \geq 0 \);
- \( \omega(f; t) \leq t^2 \) for all \( t \geq 0 \).

Let also \( \epsilon_n = \sup \{ \|f_n\|_\infty : f_n \in F_n \} \). Is \( \lim_{n \rightarrow \infty} (1 + \epsilon_n)^n = \infty \)?

If the Problem 4 has an affirmative answer, then Problem 3 has negative answer. Namely, on the reflexive Banach space \( X = (\sum_{n=1}^{\infty} l_\infty(n))_2 \) there is a homeomorphism \( f : X \rightarrow X \) which is not locally convex but \( f : X \rightarrow X \) is second order Lipschitz and \( f^{-1} : X \rightarrow X \) is locally Lipschitz.

3. Conclusion

The paper analyses of smooth nonlinear mappings for Hilbert and Banach spaces that carry small balls to convex sets, provided that the radius of the balls is sufficiently small. The main goal is to establish new and mild sufficient conditions for a nonlinear mapping to
be locally convex. The analysis involves both Hilbert and Banach spaces, and some of the results are found to be interesting and novel even for finite-dimensional problems.

We specifically address a suitably reformulated local convexity problem for both Hilbert and Banach spaces. The local convexity property holds for differentiable mappings of Hilbert spaces if the Fréchet derivative is Lipschitzian in a closed ball with certain properties. We provide an improved estimation for the radius of the ball, ensuring that its image is convex. This result is established with arguments different from previous works, leading to a more refined analysis.

For Banach spaces, the local convexity problem is more intricate and requires subtle techniques. We analyze locally convex functions between Banach spaces, considering the interplay between the modulus of convexity of a Banach space and the modulus of smoothness of a function. This generalization allows for a deeper understanding of the local convexity property in the Banach space setting. Some of the presented results are novel even in the finite-dimensional case.

Moreover, we formulate open problems related to the local convexity property for nonlinear mappings of Banach spaces, highlighting areas where further research is needed.

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Contributions

Conceptualization, Y.P., P.P.; methodology M.V., M.G.; validation, Y.P., P.P.; investigation, Y.P., P.P.; writing original draft preparation, M.V., M.G.; writing-review and editing, M.V., M.G.; project administration, Y.P., P.P.; funding acquisition, Y.P., P.P. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

Authors report no potential conflict of interest.

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