THE WORD PROBLEM FOR
FREE ADEQUATE SEMIGROUPS

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Abstract. We study the complexity of computation in finitely generated free left, right and two-sided adequate semigroups and monoids. We present polynomial time (quadratic in the RAM model of computation) algorithms to solve the word problem and compute normal forms in each of these, and hence also to test whether any given identity holds in the classes of left, right and/or two-sided adequate semigroups.

1. Introduction

Adequate semigroups form a class of semigroups in which the cancellation properties of elements are reflected in the cancellation properties of idempotents. They form a natural common generalisation of inverse semigroups and cancellative monoids. Their importance was first recognised by Fountain in the 1970’s [6], but for many years their study was restricted by a lack of applicable methods. In the last few years, interest has been reawakened by the development of several new techniques and results (see for example [1] [2] [7] [8] [9]).

Free algebras form a natural focus of attention when studying any class of algebras in which they exist; indeed, an understanding of the free objects in a class of algebras usually yields considerable information about the class as a whole. In the case of adequate semigroups, the fact that free adequate semigroups of every rank exist follows from elementary principles of universal algebra (see for example [3] Proposition VI.4.5), but an explicit description proved elusive until recently. In [8], the first author gave a concrete geometric realisation of the free adequate semigroups (and monoids), inspired by Munn’s celebrated representation of the free inverse semigroups, in terms of directed, labelled, birooted trees under a natural combinatorial multiplication operation. In [9] he showed further that the certain natural subsemigroups are the free objects in the related categories of left adequate and right adequate semigroups (and monoids). The free left, right and two-sided adequate semigroups also turn out to be free objects in the larger classes of left, right and two-sided Ehresmann semigroups [2] [7] [10].

This representation immediately gave rise to a non-deterministic polynomial-time algorithm for the word problem in finite rank free adequate semigroups and monoids (and hence also in finite rank free left adequate and right adequate semigroups and monoids). Since a relation holds in a free algebra
in a category exactly if the corresponding identity holds in all algebras in the category, this also yields an algorithm to check whether a given identity holds in all adequate (or left adequate or right adequate) semigroups or monoids. This algorithm has proved surprisingly practical for human application to short words, with the intuitive geometric nature of the representation often allowing an effective use of guesswork to circumvent the issue of non-determinism. However, a non-deterministic algorithm is clearly not well-suited to computer implementation for larger words, and it would also be more satisfactory for theoretical reasons to know the precise asymptotic complexity of the problem.

In this paper, we apply some ideas from constraint satisfaction theory to refine the algorithm into a deterministic form, thus showing that the word problems for free adequate, free left adequate and free adequate semigroups and monoids, and hence also the problem of checking whether identities hold for all adequate semigroups or monoids, are decidable in quadratic (in the RAM model of computation) time. Moreover, we show how to efficiently (again, in quadratic time in the RAM model) compute normal forms (either trees or words) for elements of the free adequate semigroup or monoid.

2. Preliminaries

In this section we very briefly recall the definitions of (left, right and two-sided) adequate semigroups, and the first author’s characterisation of the free (left, right and two-sided) adequate semigroups monoids. The reader seeking a more complete introduction with examples is referred to [6] for adequate semigroups in general and [8] for free adequate semigroups and monoids.

Let $S$ be a semigroup whose idempotent elements commute. Denote by $S^1$ the monoid consisting of $S$ with a new identity element $1$ adjoined. Then $S$ is called left adequate if for every element $x \in S$ there is an idempotent element $x^+ \in S$ such that $ax = bx \iff ax^+ = bx^+$ for all $a, b \in S^1$. If $S$ is left adequate then the choice of $x^+$ is uniquely determined by $x$, and it is usual to consider $S$ as a $(2,1)$-algebra with the binary operation of multiplication and the unary operation $x \mapsto x^+$. In particular, we restrict attention to morphisms respecting both operations. Dually, $S$ is right adequate if for every $x \in S$ there is an idempotent $x^*$ with $xa = xb \iff x^*a = x^*b$ for all $a, b \in S^1$; right adequate semigroups are also $(2,1)$-algebras. The semigroup $S$ is called (two-sided) adequate if it is both left adequate and right adequate; the two maps $x \mapsto x^+$ and $x \mapsto x^*$, which in general will be different, make $S$ into a $(2,1,1)$-algebra.

Now let $\Sigma$ be an alphabet. A $\Sigma$-tree (or just a tree if the alphabet $\Sigma$ is clear) is a finite directed graph with edges labelled by letters from $\Sigma$, whose underlying undirected graph is a tree, together with two distinguished vertices (the start vertex and the end vertex) such that there is a (possibly empty) directed path from the start vertex to the end vertex. The (unique) simple path from the start vertex to the end vertex is termed the trunk of the tree; vertices and edges lying on it are called trunk vertices and trunk edges respectively. If $e$ is an edge in such a tree, we denote by $\alpha(e)$, $\omega(e)$ and $\lambda(e)$ respectively the source vertex, target vertex and label of $e$. We say
that a vertex \( v \) is a descendant of a vertex \( u \) if the unique simple undirected path between \( v \) and the start vertex passes through \( u \).

As a notational convenience, we let \( \Sigma' = \{ j' \mid j \in \Sigma \} \) be an alphabet disjoint from and in bijective correspondence with \( \Sigma \), and say that a \( \Sigma \)-tree \( X \) has an edge from \( u \) to \( v \) labelled \( j' \) to mean that it has an edge from \( v \) to \( u \) labelled \( j \). (Intuitively, the elements of \( \Sigma' \) can be thought of as labelling directed edges when read “in the wrong direction”. This notation will allow a unified consideration of labels and directions of edges; since label and direction play similar roles as obstructions to a morphism mapping one edge to another, considering them together simplifies our arguments in several places.)

A morphism \( \rho : X \to Y \) of \( \Sigma \)-trees \( X \) and \( Y \) is a map taking edges to edges and vertices to vertices which commutes with \( \alpha, \lambda \) and \( \omega \) and maps the start and end vertex of \( X \) to the start and end vertex of \( Y \) respectively. An isomorphism is a morphism which is bijective on both edges and vertices. A retraction is an idempotent morphism from a \( \Sigma \)-tree to itself; its image is called a retract. A tree is called pruned if it does not admit a non-identity retraction. (Structures without retractions are often called cores in graph theory.)

The \( \Sigma \)-tree with a single vertex and no edges is called trivial. The set of all isomorphism types of \( \Sigma \)-trees (including the trivial \( \Sigma \)-tree) is denoted \( UT^1(\Sigma) \) while the set of isomorphism types of non-trivial \( \Sigma \)-trees is denoted \( UT(\Sigma) \). The set of all isomorphism types of pruned trees [respectively, non-trivial pruned trees] is denoted \( T^1(\Sigma) \) [respectively, \( T(\Sigma) \)]. For any \( X \in UT^1(\Sigma) \) there is a unique \( Y \in T^1(\Sigma) \) which is isomorphic to a retract of \( X \) [8, Proposition 3.5]; we denote this pruned tree \( \overline{X} \) and call it the pruning of \( X \).

If \( X, Y \in UT^1(\Sigma) \) then the unpruned product \( X \times Y \) is (the isomorphism type of) the tree obtained by glueing together \( X \) and \( Y \), identifying the end vertex of \( X \) with the start vertex of \( Y \) and keeping all other vertices and all edges distinct; this is a well-defined, associative binary operation [8, Proposition 4.2]. If \( X \in UT^1(\Sigma) \) then \( X^{(+)} \) is (the isomorphism type of) the tree with the same labelled graph and start vertex of \( X \), but with end vertex of \( X^{(+)} \) the start vertex of \( X \). Dually, \( X^{(*)} \) is the isomorphism type of the idempotent tree with the same underlying graph and end vertex as \( X \), but with start vertex the end vertex of \( X \). We define corresponding pruned operations on \( T^1(\Sigma) \) by \( XY = \overline{X \times Y} \), \( X^{(*)} = \overline{X^{(*)}} \) and \( X^{(+)} = \overline{X^{(+)}} \).

A tree with a single edge and distinct start and end vertices is called a base tree; we identify each base tree with the label of its edge, thus viewing \( \Sigma \) itself as a set of \( \Sigma \)-trees. The main result of [8] is that \( T^1(\Sigma) \) is the free adequate monoid on \( \Sigma \), being freely generated under pruned multiplication, \( * \) and \( + \) by the base \( \Sigma \)-trees [8, Theorem 5.16]. The map \( X \to \overline{X} \) is a \((2,1,1,0)\)-morphism from \( UT^1(\Sigma) \) onto \( T^1(\Sigma) \) [8, Theorem 4.5]. Moreover, the submonoid of \( T^1(\Sigma) \) generated by the base trees under pruned multiplication and \( * \) [respectively, \( + \)] is the free left adequate [respectively, right adequate] monoid on \( \Sigma \) [9, Theorem 3.18]. Free adequate, left adequate or right adequate semigroups can all be obtained by discarding the trivial
tree (which is the identity element) in the corresponding monoids (see [8, Proposition 2.2] and [9, Proposition 2.6]).

3. Computing with Formulas and Trees

In this section we study the computational complexity of converting between well-formed formulas, over a generating set $\Sigma$ and the binary and unary operations in an adequate semigroup, and $\Sigma$-trees. This will allow us, in later sections, to use algorithms operating on $\Sigma$-trees to solve computational problems involving formulas.

For our complexity analysis throughout this paper, we shall work in the RAM model of computation, in which integer operations and indirection (finding a value stored at a known position in an array) take unit time. For simplicity we will analyse the complexity of problems for a fixed rank semigroup, say on an alphabet $\Sigma$, rather than the uniform complexity as the rank grows. In places where it is necessary to be formal, we shall regard formulas as words over the alphabet $\Omega$ consisting of generators from $\Sigma$ plus the symbols (, ), $*$ and + with the obvious meaning. We denote by $\Omega^*$ the set of all words over the alphabet $\Omega$, including the empty word which we denote $\epsilon$. Our measure of the size of an expression will be its length as a word over $\Omega$.

We assume $\Sigma$-trees are by default stored as a natural number representing the start vertex, a natural number representing the end vertex and a list of edges (in no particular order), each being a triple consisting of a label from $\Sigma$ and two natural numbers encoding its start vertex and its end vertex. Sometimes it will be expedient to convert trees to an alternative representation. Note that the same abstract $\Sigma$-tree can admit multiple representations, by numbering the vertices and ordering the edges differently. Our measure of the size of a $\Sigma$-tree will be the number of edges.

**Proposition 3.1.** Given a well-formed formula $\omega$, one can compute in quadratic time the (unpruned) $\Sigma$-tree which is its evaluation in $UT^1(\Sigma)$. Moreover, this tree has size linear in the length of $\omega$.

**Proof.** A formula of length $n$ can be evaluated by a depth-first traversal of a parse tree; this will clearly involve performing at most $n$ unpruned operations with trees whose size is $O(n)$. Clearly the unpruned (+) and (\*) operations on trees can be performed in constant time. Unpruned multiplication of trees can be performed in time linear in the number of edges in the trees, by first relabelling the vertices in the second tree (so that all references to its start vertex become the end vertex of the first tree, and all its other vertices are distinct from those in the first tree) and then concatenating the edge lists and setting start and end vertices appropriately.

Thus, the $O(n)$ unpruned operations can each be performed in $O(n)$ time, and the evaluation of the expression takes time $O(n^2)$. Moreover, the resulting tree clearly has exactly one edge for each occurrence of a generator in the expression, and hence has size linear in the size of the expression. $\square$

For our present purpose, the computations we wish to perform with trees will all take quadratic time, so there is no particular benefit in being able to compute the trees in faster than quadratic time. However, we remark that
the complexity of the algorithm given above can be improved by a more sophisticated approach, using what is known in the computer science literature as a “Union Find” algorithm. Under this approach, when performing multiplication, instead of merging the end vertex of one tree with the start vertex of another, we keep them separate (allowing the data structure to become a forest, rather than a tree) and maintain another data structure recording which vertices are to be merged at the end. An efficient implementation of this algorithm is extremely close to being linear time; see [4, Section 21.3] for more details.

Next, we shall show how a (not necessarily pruned) $\Sigma$-tree can be efficiently converted into a well-formed formula. We will define a function $\sigma : UT^1(\Sigma) \rightarrow \Omega^*$ such that for each tree $X$, $\sigma(X)$ is a well-formed formula which evaluates to $X$ in $UT^1(X)$, and then show that this function can be computed in quadratic time. Note that since $\sigma$ is a function defined on abstract trees, the algorithm produces a formula depending only on the abstract tree, and not on its representation. We shall exploit this in Section 4 below to compute normal forms (as formulas) for elements of the free adequate semigroup. To do this, we shall need a linear order on the set of all formulas; for now, we will assume that we have such an order fixed. We will discuss the choice and implementation of this order when we come to analyse the complexity of the algorithm.

Let $X$ be a tree. We begin by defining a function $\rho$ from the vertex set of $X$ to $\Omega^*$; this is done inductively by downwards induction on the distance of the vertex from the trunk. Let $v$ be a vertex, and suppose $\rho$ is already defined on all vertices strictly further from the trunk than $v$. Let $v_1,\ldots,v_p$ be the vertices adjacent to $v$ and strictly further from the trunk, noting that $\rho(v_i)$ is already defined for each $i$. For each $i$, let $e_i$ be the edge connecting $v$ to $v_i$, and let $a_i \in \Sigma$ be its label. Define a formula $\tau_i \in \Omega^*$ by:

$$\tau_i = \begin{cases} (a_i \rho(v_i))+, & \text{if } e_i \text{ is orientated away from } v \\ (\rho(v_i)a_i)^*, & \text{if } e_i \text{ is orientated towards } v \end{cases}$$

Now we define $\rho(v) \in \Omega^*$ to be the word obtained by sorting the words $\tau_i$ according to our ordering of formulas, and then concatenating. (If $p = 0$, that is, if $v$ is a “leaf”, this means $\rho(v) = e$.)

Now let $t_0,\ldots,t_q$ be the trunk vertices of $X$ and $b_1,\ldots,b_q$ the labels of the edges between them, both in the obvious order. We define

$$\sigma(X) = \rho(t_0)b_1\rho(t_1)b_2\ldots\rho(t_{q-1})b_q\rho(t_q).$$

A simple but tedious inductive argument, akin to those in [8], shows that $\sigma(X)$ evaluates to the tree $X$ in $UT^1(\Sigma)$, and that the number of characters in $\sigma(X)$ is at most four times the number of edges in $X$.

To compute $\sigma(X)$, we start by precomputing adjacency matrices for $X$ corresponding to each possible edge label and direction; it is easily seen that this can be done in $O(n^2)$ time where $n$ is the number of edges in $X$. It is immediate from the inductive method of definition how to compute $\sigma(X)$ by a simple depth first traversal (following non-trunk edges) from each of the trunk vertices; this involves considering each of $O(n)$ vertices once.
At each vertex, the only non-trivial operation is to sort the words \( \tau_i \) into order and then concatenate; the complexity of this of course depends on the choice of order. The sum length of all the words \( \tau_i \) is clearly \( O(n) \). If we choose the order to be lexicographic order (with respect to some arbitrary linear order on \( \Omega \)), then a careful implementation of radix sort gives us a lexicographically sorted list of formulas in \( O(n) \) time, and concatenation is clearly also \( O(n) \).

Thus, the total time required for the algorithm is \( O(n^2) \), and we have established:

**Proposition 3.2.** Given an unpruned \( \Sigma \)-tree \( X \), we can in quadratic time compute a well-formed formula which evaluates to \( X \) in \( UT^1(\Sigma) \). Moreover, the formula has size linear in the size of \( X \), and depends only on the isomorphism type of \( X \) and not on its representation.

### 4. The Word Problem

Recall that the word problem for an algebra \( A \) with a given generating set is the algorithmic problem of determining, given as input two well-formed formulas over the generating set and the operations of the algebra, whether the formulas represent the same element of the algebra. The word problem for free objects in a variety of algebras is of particular importance, since it is trivially equivalent to the problem of testing whether a given identity holds in all algebras of the variety.

In this section, we shall exhibit a quadratic time algorithm to solve the word problem in a free adequate monoid \( T^1(\Sigma) \). In fact in Section 5 below, we shall see that it is also possible to compute normal forms of elements of \( T^1(\Sigma) \) in quadratic time; this automatically yields another algorithm for the word problem (by computing normal forms and comparing), of the same asymptotic complexity. However, we present an explicit word problem algorithm first since this is simpler, potentially easier to implement, and illustrates in a simple context some of the ideas we will need in Section 5.

By Proposition 3.1 we can efficiently convert well-formed formulas in the free adequate monoid into unpruned \( \Sigma \)-trees of comparable size. It follows that to test (efficiently) whether two given expression \( x \) and \( y \) represent the same element of the free adequate monoid, that is, to solve the word problem, it suffices to compute corresponding \( \Sigma \)-trees \( X, Y \in UT^1(\Sigma) \), and then check (efficiently) if \( \overline{X} = \overline{Y} \) in \( T^1(\Sigma) \).

To solve this latter problem, we begin with an elementary proposition, which reduces it to a constraint satisfaction problem (formulated in terms of morphisms between structures, in the manner usual in the literature of areas such as graph theory and universal algebra—see for example [12]).

**Proposition 4.1.** Let \( X \) and \( Y \) be \( \Sigma \)-trees. Then the following are equivalent:

(i) \( \overline{X} = \overline{Y} \);

(ii) \( X \) and \( Y \) admit isomorphic retracts;

(iii) there is a morphism from \( X \) to \( Y \) and a morphism from \( Y \) to \( X \).

**Proof.** The equivalence of (i) and (ii) follows from [8, Proposition 3.5], so it suffices to establish the equivalence of (ii) and (iii).
If (ii) holds then, in particular, some retract of $X$ is isomorphic to a substructure of $Y$; composing the retraction of $X$ with the isomorphism yields a morphism from $X$ to $Y$. By symmetry of assumption there is also a morphism from $Y$ to $X$, so (iii) holds.

Now suppose (iii) holds, say $\sigma : X \to Y$ and $\tau : Y \to X$ are morphisms. Then the compositions $\tau \circ \sigma : X \to X$ and $\sigma \circ \tau : Y \to Y$ are maps on finite sets, and it follows that we may choose $n$ such that both $(\tau \circ \sigma)^n : X \to X$ and $(\sigma \circ \tau)^n : Y \to Y$ are idempotent, that is, are retractions of $X$ and $Y$ respectively. Let $X'$ and $Y'$ be the retracts which are the respective images of these retractions. Now it is easily verified that $\sigma$ and $\tau \circ (\sigma \circ \tau)^{n-1}$ restrict to mutually inverse isomorphisms between the retracts $X'$ and $Y'$, showing that (ii) holds.

□

Proposition 4.1 implies that to check if two $\Sigma$-trees are equivalent, and hence by the preceding arguments to solve the word problem for the free adequate semigroup on $\Sigma$, it suffices to check whether each $\Sigma$-tree admits a morphism to the other. Our main goal in the rest of this section, then, is an efficient algorithm to test, given an ordered pair of $\Sigma$-trees, whether there is a morphism from the first to the second. Our approach is essentially a constraint propagation algorithm, with the correctness of the result being shown by an arc consistency argument utilising the tree-like nature of our geometric representatives for elements. The ideas behind the proof are well known in the fields of constraint satisfaction and artificial intelligence (see for example [5]), but for the benefit of semigroup theorists who may not be familiar with these fields we present the algorithm in an elementary form:

**Algorithm 4.2.**

*Input:* Two $\Sigma$-trees $T_1$ and $T_2$ on $n$ and $m$ vertices respectively.  
*Output:* “Yes” if there exists a homomorphism from $T_1$ to $T_2$, “No” otherwise.

1. Consider the start vertex of $T_1$, label this vertex 1, and then use a depth-first traversal (ignoring direction of edges) to label the remaining vertices from 2 to $n$ in ascending order.
2. For each $i$ in \{1, …, $n$\}, let $B_i$ be the set of vertices in $T_2$.
3. For the start [end] vertex $i$ set $B_i$ to be the singleton set containing the start [end] vertex of $T_2$.
4. For $i$ descending from $n$ to 1, and each vertex $j > i$ adjacent to $i$, do the following:
   
   (i) Let $a \in \Sigma \cup \Sigma'$ be the label of the edge from $i$ to $j$ in $T_1$;
   
   (ii) Let $B_i := B_i \cap B_j^*$ where $B_j^* = \{ x \mid T_2 \text{ has an edge labelled } a \text{ from } x \text{ to some } y \in B_j \}$.

5. If $B_1 = \emptyset$, output “No”; otherwise output “Yes.”

**Proposition 4.3.** Algorithm 4.2 is correct, that is, $B_1$ is non-empty on completion of the algorithm if and only if there is a morphism from $T_1$ to $T_2$. 
Proof. For brevity, we identify the vertices with the labels from 1 to \( n \) assigned in the algorithm. Suppose first that there is a morphism \( \sigma : T_1 \to T_2 \). We claim that \( B_1 \) contains \( \sigma(i) \) for all \( i \), from which it follows in particular that \( B_1 \) contains \( \sigma(1) \) so that \( B_1 \) is non-empty as required. Indeed, if not, choose \( i \) maximal such that \( \sigma(i) \notin B_1 \). Clearly \( \sigma(i) \) was in \( B_i \) after Step 2 of the algorithm and, because \( \sigma \) preserves start and end vertices, also after Step 3; therefore, it must have been removed during Step 4. For this to have happened, there must have been a \( j > i \) and an edge from \( i \) to \( j \) (labelled \( a \in \Sigma \cup \Sigma' \), say) such that \( \sigma(i) \notin B_j^* \). By the definition of \( B_j^* \), this means there was (at the time of removal) no edge labelled \( a \) from \( \sigma(i) \) to any \( y \in B_j \). But because \( \sigma \) is a morphism, \( \sigma(j) \in B_j \) is connected to \( \sigma(i) \) by such an edge, so it must be that \( \sigma(j) \) was not in \( B_j \) at the time \( \sigma(i) \) was removed from \( B_i \). Now since \( B_j \) only gets smaller, \( \sigma(j) \) is not in \( B_j \) at the end of the algorithm. But \( j > i \), so this contradicts the maximality of \( i \).

Conversely, suppose \( B_1 \) is non-empty at the end of the algorithm. We define a morphism \( \sigma : T_1 \to T_2 \) inductively as follows. First, choose \( \sigma(1) \in B_1 \) arbitrarily. Now assume \( 1 < i < n \) and we have defined \( \sigma \) on the vertices \( 1, \ldots, i-1 \) and all edges between them, in such a way as to preserve adjacency, labels and directions of edges, and the start and end vertices if appropriate, and such that \( \sigma(p) \in B_p \) for \( 1 \leq p \leq i-1 \).

Since \( T_1 \) is a tree and the edges were numbered by a depth-first traversal, it follows that vertex \( i \) is connected to vertex \( k \) for some unique \( k < i \); suppose \( T_1 \) has an edge from \( k \) to \( i \) labelled \( a \in \Sigma \cup \Sigma' \).

Considering the way \( B_k \) is constructed, we see that every vertex in \( B_k \), including \( \sigma(k) \), is connected to some \( v \in B_i \). Moreover, if \( i \) happens to be the start [respectively, end] vertex of \( T_1 \), then \( B_i \) was originally set to contain only the start [end] vertex of \( T_2 \), so it must be that \( v \in B_i \) is the start [end] vertex of \( T_2 \). Thus, by defining \( \sigma(i) = v \) and \( \sigma(e) \) to be the appropriate edge, we extend \( \sigma \) to be defined on the vertices \( 1, \ldots, i \) and all edges between, with the appropriate properties.

We now analyse the complexity of Algorithm 4.2. At the start of the algorithm, we can precompute for each vertex in \( T_1 \) a list of edges adjacent to that vertex; this can be done in \( O(n) \) time.

Having done this, Step 1 of the algorithm (a simple depth first traversal of the tree \( T_1 \)) has complexity \( O(n) \). If we store the lists \( B_i \) as arrays of \( m \) boolean flags then Step 2 has complexity \( O(mn) \) since we need to initialise \( mn \) values. Step 3 has complexity \( O(m) \), since we must reset \( m-1 \) values for each of the start and end vertices.

The most interesting part is the complexity of Step 4. The number of iterations of the outer loop is clearly bounded by the number of edges in \( T_1 \), so it is \( O(n) \) and the precomputed lists of edges mean there is no extra overhead in finding the edges in the correct order. In each iteration, the fact that the corresponding edge has been found means Step 4(i) takes constant time. In Step 4(ii), computing \( B_j^* \) involves passing through the list of all \( O(m) \) edges of \( T_2 \) and for each edge checking (in constant time) if one of the ends lies in \( B_j \) and if the label is correct; this takes \( O(m) \) time. Computing the intersection is simply a boolean “and” operation on two arrays of length
m, and so also takes $O(m)$ time. Thus, Step 4 takes time $O(mn)$, and the total complexity of the algorithm is $O(mn)$.

Combining the above arguments with Proposition 3.1 we have established the following main result:

**Theorem 4.4.** The word problem for any finite rank free left adequate, free right adequate or free adequate semigroup is decidable in time polynomial (quadratic, in the RAM model of computation) in the combined length of the two formulas.

5. Pruned Trees and Normal Forms

In this section, we show how to efficiently compute the minimal retract of a $\Sigma$-tree. Combined with the results of Section 3, this will allow us to compute normal forms (as formulas) for elements of free adequate monoids. Our main algorithm is the following, the first four steps of which are essentially the same as in Algorithm 4.2:

**Algorithm 5.1.**

**Input:** A $\Sigma$-tree $T$ on $n$ vertices.

**Output:** The vertex set of a pruned subtree of $T$, isomorphic to the $T$.

1. Consider the start vertex of $T$, label this vertex 1, and then use a depth-first traversal (ignoring direction of edges) to label the remaining vertices from 2 to $n$ in ascending order.
2. For each $i$ in $\{1, \ldots, n\}$, set $B_i = \{1, \ldots, n\}$.
3. For the start [end] vertex $i$ set $B_i = \{i\}$.
4. For $i$ descending from $n$ to 1 and each $j$ with $j > i$ and $i$ connected to $j$, do the following:
   (i) Let $a \in \Sigma \cup \Sigma'$ be the label of the edge in $T$ from $i$ to $j$.
   (ii) Let $B_i := B_i \cap B^*_j$ where 

   $B^*_j = \{x \mid T \text{ has an edge labelled } a \text{ from } x \text{ to some } y \in B_j\}$.

5. Set $X = \{1, \ldots, n\}$.
6. For $w$ ascending from 1 to $n$ and $a \in \Sigma \cup \Sigma'$, do the following:
   (i) If $w \notin X$ then go to the next $w$.
   (ii) Otherwise, find all vertices $u$ such that $a$ labels an edge from $w$ to $u$ and put them in a list $K$.
   (iii) For each $u \in K$ such that $u > w$:
      (a) Check if $K \cap B_u = \{u\}$.
      (b) If not, then remove $u$ from $K$, and traverse the tree below $u$, removing $u$ and all its descendant vertices from $X$.
7. Output $X$.

Our next aim is to prove the correctness of this algorithm.

**Lemma 5.2.** The subtree $X$, as computed at the end of Algorithm 5.1, is a retract of $T$.

**Proof.** We shall show that each time a vertex and its descendants are removed from $X$ at Step 6(iii)(b), there is a retraction from the tree $X$ prior to
the removal, onto the tree $X$ after the removal. Since the successive subtrees
$X$ form a chain under inclusion, it is clear that composing these retractions
in the appropriate order yields a retraction from $T$ onto the final tree $X$, as
required.

Indeed, suppose $u$ and its descendants are removed from $X$ at some point.
Let $w$, $a$ and $K$ be as in the algorithm at that point, and let $X_1$ and $X_2$ be
the values of $X$ immediately before and after the deletion, respectively.

Note that, since the identity map is a morphism, it is easily verified that
$i \in B_i$ for all vertices $i$ of $T$. The fact that $u$ was removed means that
$K \cap B_u \neq \{u\}$, and we know $u \in K \cap B_u$, so we may choose some vertex
$v \in K \cap B_u$ with $v \neq u$.

First, we follow the procedure from the proof of Proposition 4.3 to inductively define a morphism $\sigma : T \to T$, but being more careful about our
choices in order to ensure that $\sigma(u) = \sigma(v) = v$. We start by setting $\sigma(i) = i$
for all $i < u$; since $i \in B_i$ for all $i$ it is easily verified that this is consistent
with the procedure in Proposition 4.3. Note in particular that $w < u$, so this
means $\sigma(w) = w$. Now since $u, v \in K$, there are edges from $w = \sigma(w)$ to $u$
and $v$ both labelled $a$, so in following the procedure of Proposition 4.3 we
may choose to set $\sigma(u) = v$. We now continue the process from the proof of
Proposition 4.3. For each vertex $r$ in turn, if $r$ is a descendant of $u$, then we
define $\sigma(r)$ as in the proof of Proposition 4.3 making any choices arbitrarily.
If $r$ is not a descendant of $u$ then the unique vertex $k < r$ adjacent to $r$ is
also not a descendant of $u$; thus, we have already defined $\sigma(k) = k$ and we
may set $\sigma(r) = r$.

Now $\sigma$ is a map on a finite set, and so has an idempotent power, say
$\sigma^i$. Since $v$ is not a descendant of $u$, we have $\sigma(v) = v$, and hence $\sigma^i(u) = 
\sigma^{i-1}(u) = v$, so $u$ is not in the image of $\sigma^i$. Since the image of $\sigma^i$ is a $\Sigma$-tree,
it must contain the start vertex and be connected, so we deduce that no
descendants of $u$ are in the image of $\sigma^i$. It follows that $\sigma^i$ maps $X_1$ to $X_2$.
Moreover, $\sigma$ fixes $X_2$, so restricting $\sigma^i$ to $X_1$ gives the required retraction
of $X_1$ onto $X_2$. \hfill \Box

**Lemma 5.3.** The retract $X$, as computed at the end of Algorithm 5.1, is
pruned.

**Proof.** Suppose not, say $X$ admits a proper retraction $\sigma : X \to X$. Let $u$
be a vertex in $X$ but not in the image of $\sigma$, and suppose $u$ is minimal with
respect to this condition. Then $u \neq 1$, since 1 labels the start vertex which
is fixed by every retraction. Thus, we may let $w$ be the unique vertex with
$w < u$ and $w$ adjacent to $u$.

Since $w < u$, by the minimality of the choice of $u$, we have $\sigma(w) = w$.
It follows that $\sigma(u)$ is connected to $w$ by an edge of the same label and
orientation as that connecting $u$ to $w$. This means that, when considering $w$
at step 6(iii), we would initially have had $\sigma(u) \in K$. Since $\sigma(u)$ is in the final
tree $X$, it was never removed from $K$. Moreover, composing the retraction
of $T$ onto $X$ (given by Lemma 5.2) with $\sigma$ gives a morphism of $T$ mapping
$u$ to $\sigma(u)$; it follows from the argument in the proof of Proposition 4.3 that
$\sigma(u) \in B_u$. 

This means that at the time $u$ was considered in Step 6(iii)(a) we had $\sigma(u) \in K \cap B_u$. But then $K \cap B_u \neq \{u\}$, so $u$ would have been removed from $X$, giving a contradiction. □

Turning to the complexity of the algorithm, Steps (1)-(4) are exactly as in Algorithm 4.2 (except that the source and target trees for the morphism are the same, so $m = n$), and by the same analysis as in Section 4, take time $O(n^2)$.

For efficiency, we store the set $X$ as an array of boolean flags. The time requirement for Step (5) is clearly $O(n)$. The loop in Step 6 is iterated at most $O(n)$ times. In each such iteration, step (i) takes constant time. Step (ii) cannot involve checking more than $O(n)$ vertices, so the total contribution to the time required will be $O(n^2)$. In step (iii), note that each element of $L$ is uniquely determined (across the entire algorithm) by the ordered pair $(w, u)$ where there is always an edge between $w$ and $u$; thus, the number of iterations of this step across the whole algorithm is at most twice the number of edges in the tree, which is $O(n)$. Within each iteration, each step takes $O(n)$ time, so the total contribution is $O(n^2)$.

Thus, we have established:

**Theorem 5.4.** Given a $\Sigma$-tree $T$, one can compute in polynomial time (quadratic time in the RAM model of computation) the pruned $\Sigma$-tree $\overline{T}$.

Combining with the results of Section 3, Theorem 5.4 allows us to compute normal forms (as formulas) in the free adequate monoid. Indeed, given a formula $w$, by Proposition 3.1 we may convert it in quadratic time to a corresponding unpruned $\Sigma$-tree $T$ of comparable size. By Theorem 5.4 we may then compute the pruned tree $\overline{T}$ in time quadratic in the size of $T$ and hence in the size of $w$. Finally, by Proposition 5.2 we can convert $\overline{T}$ into the uniquely defined formula $\sigma(\overline{T})$ in time quadratic in the size of $\overline{T}$; since $\overline{T}$ is no larger than $T$, this is also quadratic in the size of $T$, and hence in the size of $w$.

**Theorem 5.5.** Given a formula in the free adequate, left adequate or right adequate semigroup or monoid, one can compute a normal form in polynomial (quadratic in the RAM model of computation) time.

We note that the resulting language of normal forms for elements, which by definition is the set

$$\{\sigma(\overline{T}) \mid T \text{ is a pruned } \Sigma\text{-tree}\},$$

does not appear to have a completely elementary description without reference to trees. Of course one may check (in quadratic time) whether a given formula $w$ is a normal form by following the above procedure to convert $w$ to a normal form and then comparing with $w$; we do not know of a fundamentally easier method.

We also note that in the case of free inverse monoids (and semigroups), it is known [11, Theorem 11] that the word problem is decidable in (RAM) linear time. In the inverse case computations appear to be inherently simpler, as the operation corresponding to computing a minimal retract (namely, computing a minimal morphic image) can be performed by an iterative process of identifying vertices, where the fact a pair of vertices can be identified
is determined “locally”, by looking only in the immediate neighbourhood of
the vertices. It seems unlikely that quite such a fast algorithm can be ob-
tained in the adequate case, but one might still ask whether our algorithms
can be significantly improved upon. Also shown in [11, Theorem 11] is that
the word problem for a free inverse monoid is decidable (using a different
algorithm to the linear time one) in logarithmic space: the space complexity
of the word problem for free adequate monoids and semigroups is a natural
topic for future research.

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[11].

References

[1] J. Araújo, M. Kinyon, and A. Malheiro. A characterization of adequate semigroups
by forbidden subsemigroups. http://arxiv.org/abs/1111.4512v1 [math.GR], 2011.
[2] M. J. J. Branco, G. M. S. Gomes, and V. A. R. Gould. Left adequate and left
Ehresmann monoids. Internat. J. Algebra and Computation, 21:1259–1284, 2011.
[3] P. M. Cohn. Universal algebra, volume 6 of Mathematics and its Applications. D.
Reidel Publishing Co., Dordrecht, second edition, 1981.
[4] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein. Introduction to Algorithms.
The MIT Press, second edition, 2001.
[5] R Dechter and J. Pearl. Network-based heuristics for constraint-satisfaction problems.
Artificial Intelligence, 34:1–34, 1987.
[6] J. B. Fountain. Adequate semigroups. Proc. Edinburgh Math. Soc. (2), 22(2):113–125,
1979.
[7] G. M. S. Gomes and V. A. R. Gould. Left adequate and left Ehresmann monoids II.
J. Algebra, 348:171–195, 2011.
[8] M. Kambites. Free adequate semigroups. J. Australian Math. Soc., 91:365–390, 2011.
[9] M. Kambites. Retracts of trees and free left adequate semigroups. Proc. Edinburgh
Math. Soc., 54:731–747, 2011.
[10] M. V. Lawson. Semigroups and ordered categories I: the reduced case. J. Algebra,
141:422–462, 1991.
[11] M. Lohrey and N. Ondrusch. Inverse monoids: decidability and complexity of algebra-
ic questions. Inform. and Comput., 205(8):1212–1234, 2007.
[12] J. Nešetřil and P. Hell. Graphs and Homomorphisms. Oxford University Press, New
York, 2004.