Universality of single qudit gates

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Abstract

We consider the problem of deciding if a set of quantum one-qudit gates $S = \{g_1, \ldots, g_n\} \subset G$ is universal, i.e. if the closure $\langle S \rangle$ is equal to $G$. To every gate in $S$ we assign its image under the adjoint representation, i.e. $Ad : G \rightarrow SO(g)$, where $g$ is the Lie algebra of $G$. Matrices $Ad_g$ play the major role in deciding universality of the set $S$. We prove that $\langle S \rangle$ is infinite if the spectra of $Ad_g$’s satisfy a particular finite set of conditions. On the other hand, the infinite $\langle S \rangle$ is the whole $G$ if the simultaneous commutant of all $Ad_g$ is equal to the simultaneous commutant of all $Ad_g$, $g \in G$. We relate these conditions with the conditions for universal Hamiltonians. Our approach is illustrated by concrete problems concerning $SU(2)$, $SO(3)$ and $SU(3)$ universal gates.

1 Introduction

Quantum computer is a device that operates on a finite dimensional quantum system $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ consisting of $n$ qudits \textsuperscript{16} \textsuperscript{1} \textsuperscript{11} that are described by a $d$-dimensional Hilbert spaces, $\mathcal{H}_i \simeq \mathbb{C}^d$ \textsuperscript{18}. When $d = 2$ qudits are typically called qubits. The ability to effectively manufacture optical gates operating on many modes, using for example optical networks that couple modes of light \textsuperscript{1} \textsuperscript{19} \textsuperscript{20}, is a natural motivation to consider not only qubits but also higher dimensional systems in quantum computation setting. One of the necessary ingredients for a quantum computer to work properly is the ability to perform arbitrary unitary operation on the system $\mathcal{H}$. We distinguish two types of operations. First are one-qudit operations (one-qudit gates) that belong to $SU(\mathcal{H}_i) \simeq SU(d)$ and act on a single qudit. The second are $k$-qudit operations ($k$-qudit gates), $k \geq 2$, that belong to $SU(\mathcal{H}_i \otimes \cdots \otimes \mathcal{H}_i) \simeq SU(d^k)$ and act on the chosen $k$ qudits. A $k$-qudit gate is nontrivial if it is not a tensor product of $k$ single qudit gates. We say that one-qudit gates $S = \{g_1, \ldots, g_n\}$ are universal if any gate from $SU(d)$ can be built, with an arbitrary precision, using gates from $S$. Mathematically this means that the set $\langle S \rangle$ generated by elements from $S$ is dense in $SU(d)$ and its closure is the whole $SU(d)$, i.e. $\langle S \rangle = SU(d)$. It is known that once we have access to a universal set of one-qudit gates together with one additional two-qudit gate that does not map separable states onto separable states we can build, within a given precision, an arbitrary unitary gate belonging to $SU(\mathcal{H})$ \textsuperscript{3}. Thus in order to characterise universal sets of gates for quantum computing with qudits one needs to characterise sets that are universal for one qudit.

Although there are some qualitative characterisations of universal one-qudit gates the full understanding is far from complete. It is known, for example, that almost all sets of qudit gates are universal, i.e universal sets $S$ of the given cardinality $c$ form a Zariski open set in $SU(d)^c$. By the definition of a Zariski open set we can therefore deduce that non-universal gates can be characterised by vanishing of a finite number of polynomials in the matrices entries and their conjugates \textsuperscript{13} \textsuperscript{15}. These polynomials are, however, not known and it is hard to find operationally simple criteria that decide one-qudit gates universality. Some special cases of two and three dimensional gates have been studied in \textsuperscript{3} \textsuperscript{21}. The main obstruction in these approaches is the lack of classification of finite and infinite disconnected subgroups of $SU(d)$ for $d > 4$.

The goal of this paper is to provide some reasonable criteria for universality of one-qudit gates that can be applied even if one does not know classification of finite/infinite disconnected subgroups of $SU(d)$. To achieve this we divide the problem into two. First, using the fact that considered gates $S = \{g_1, \ldots, g_n\}$ belong to groups that are compact simple Lie groups $G$, we provide a criterion which allows to decide if an infinite subgroup is the whole group $G$. It is formulated in terms of adjoint representation matrices $Ad_g$, $g \in S$ and boils down to finding a dimension of the kernel of a matrix, whose coefficients are polynomial in the entries of gates and their complex...
conjugates. As the considered groups are compact and connected, any gate \( g \) can be written as \( g = \exp^X \), where \( X \) is an element of the Lie algebra of the group. Thus for a given set of gates \( S \) we also have a corresponding set of Lie algebra elements \( \mathcal{X} \). These elements can be treated as Hamiltonians. A set of Hamiltonians is universal iff the Lie algebra generated by its elements is the whole Lie algebra \([8, 22]\). Using the adjoint representation, this time in the setting of Lie algebras, we provide criteria for the universality of \( \mathcal{X} \) and show when they overlap with the criteria for the universality of \( S \). Next, we give sufficient conditions for a set generated by \( S \) to be infinite. They stem from inequalities that relate the distances of group elements and their commutators from the identity \([9, 2]\). In particular we show that for a pair of gates \( g_1 \) and \( g_2 \), for which the Hilbert–Schmidt distances from the centre \( Z(G) \) of \( G \) are less than \( \frac{1}{\sqrt{2}} \) and such that \([g_1, g_2] := g_1g_2g_1^{-1}g_2^{-1} \notin Z(G)\), deciding universality boils down to checking if the corresponding Lie algebra elements generate the whole Lie algebra. Next we show that for a gate whose distance from \( Z(G) \) is larger than \( \frac{1}{\sqrt{2}} \), \( \text{dist}(g, Z(G)) \geq \frac{1}{\sqrt{2}} \), there is always \( n \in \mathbb{N} \) such that \( \text{dist}(g^n, Z(G)) < \frac{1}{2^n} \). Moreover, using Dirichlet approximation theorems (and their modifications) we give an upper bound for the maximal \( N_G \) such that for every \( g \in G \) we have \( \text{dist}(g^n, Z(G)) < \frac{1}{\sqrt{n}} \) for some \( 1 \leq n \leq N_G \). We note, however, that the commutator of \([g_1^n, g_2^n] \) might belong to \( Z(G) \) even if \([g_1, g_2] \notin Z(G)\). Thus we next show that if the spectra of the adjoint representation matrices \( \text{Ad}_{g_1} \) and \( \text{Ad}_{g_2} \) satisfy certain conditions, then \([g_1^n, g_2^n] \notin Z(G)\). Gates that do not satisfy these conditions are called exceptional and we outline the procedure which leads to deciding their universality.

The last part of the paper concerns applications of the above ideas to \( SU(2) \), \( SO(3) \) and \( SU(3) \). In particular we give a full characterisation of the universal pairs of single qubit gates. As we show, in this case exceptional spectra are in direct correspondence with the characters of the finite subgroups of \( SU(2) \). We also characterise real and complex 2-mode beamsplitters that are universal when acting on an even number of qubits. Our approach allows to reprove the results of \([3, 24]\) without the knowledge of disconnected infinite or finite subgroups of \( SO(3) \) and \( SU(3) \).

## 2 Preliminaries

### 2.1 Compact semisimple Lie algebras

A real Lie algebra is a finite dimensional vector space \( g \) over \( \mathbb{R} \) together with a commutator \([\cdot, \cdot] : g \times g \to g\) that is: (1) bilinear (2) antisymmetric and (3) satisfies Jacobi identity \([[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0\). In this paper we will often skip ‘real’ as we will consider only real Lie algebras. A Lie algebra \( g \) is nonabelian if there is a pair \( X, Y \in g \) such that \([X, Y] \neq 0\). A subspace \( h \subset g \) is a subalgebra of \( g \) if and only if for any \( X, Y \in h \) we have \([X, Y] \in h\), i.e. \( h \) is closed under taking commutators. An important class of subalgebras are ideals. A subalgebra \( h \subset g \) is an ideal of \( g \) if for any \( X \in g \) and any \( Y \in h \) we have \([X, Y] \in h\). One easily checks that an intersection of ideals is an ideal.

**Definition 1.** A nonabelian Lie algebra \( g \) is simple if \( g \) has no ideals other than 0 and \( g \).

We say that a Lie algebra \( g \) is a direct sum of Lie algebras, \( g = \oplus_{i=1}^n g_i \), if and only if it is a direct sum of vector spaces \( \{g_i\}_{i=1}^n \) and \([g_i, g_j] = 0 \) for all \( i \neq j \). In this case \( g_i \)'s are ideals of \( g \). The algebras we will be interested in belong to a special class of either simple Lie algebras or their direct sums. In the following we briefly discuss their properties.

A representation of a real Lie algebra on a real vector space is a linear map \( \phi : g \to \text{End}_\mathbb{R}(V) \) that satisfies \( \phi([X, Y]) = [\phi(X), \phi(Y)] \). A representation is called irreducible if \( V \) has no \( \phi(g) \)-invariant subspace \( W \subset V \), i.e. a subspace for which \( \phi(X)W \subset W \), for all \( X \in g \). Irreducible representations are characterised by the Schur lemma that says a representation \( \phi : g \to \text{End}_\mathbb{R}(V) \) is irreducible if and only if the only endomorphism that commutes with all \( \phi(X) \) is proportional to the identity. As a Lie algebra itself one can consider representation of \( g \) on \( g \). In fact, there exists a canonical representation of this type that is called the adjoint representation:

\[
\text{ad} : g \to \text{End}(g), \quad \text{ad}_X(Y) := [X, Y].
\]

Note that invariant spaces of the adjoint representation are ideals and therefore the adjoint representation of a simple algebra is irreducible. Using the adjoint representation we define a bilinear form on \( g \), called the Killing form given by \( B(X, Y) = \text{tr}(\text{ad}_X \circ \text{ad}_Y) \)[4]. The Killing form satisfies

\[
B(\text{ad}_X(Y), Z) + B(X, \text{ad}_X(Z)) = 0.
\]

**Definition 2.** A real Lie algebra \( g \) is a compact semisimple Lie algebra if its Killing form is negative definite.

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1 Upon a choice of basis in \( g \) endomorphisms \( \text{ad}_X \) and \( \text{ad}_Y \) are matrices and hence we can compute the trace.
Assume now that \( \mathfrak{g} \) is a compact semisimple Lie algebra and let \( \mathfrak{a} \subseteq \mathfrak{g} \) be an ideal. Let \( \mathfrak{a}^\perp \) be the orthogonal complement of \( \mathfrak{a} \) with respect to the Killing form. For any \( X \in \mathfrak{g}, Y \in \mathfrak{a}^\perp \), and \( Z \in \mathfrak{a} \) we have
\[
B([X,Y], Z) = -B(Y, [X,Z]) = 0.
\]
Hence \( [X,Y] \in \mathfrak{a}^\perp \). Therefore \( \mathfrak{a}^\perp \) is also an ideal. Note next that \( [\mathfrak{a}, \mathfrak{a}^\perp] \subseteq \mathfrak{a} \cap \mathfrak{a}^\perp \). The restriction of \( B \) to the ideal \( \mathfrak{a} \cap \mathfrak{a}^\perp \) is obviously zero. But \( B \) is negative definite, hence \( \mathfrak{a} \cap \mathfrak{a}^\perp = 0 \). As a result \( \mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp \) is a direct sum of ideals. We can repeat this procedure for \( \mathfrak{a} \) and \( \mathfrak{a}^\perp \) and after several steps finally we get

**Fact 3.** A real compact semisimple Lie algebra is a direct sum of real compact simple Lie algebras.

Let us next choose a basis \( \{X_i\}_{i=1}^{\dim \mathfrak{g}} \) in \( \mathfrak{g} \) that satisfies \( B(X_i, X_j) = -\delta_{ij} \). In this basis \( \text{ad}_X \) is an antisymmetric trace zero real matrix, hence an element of the special orthogonal Lie algebra \( \mathfrak{so}(\dim \mathfrak{g}) \). Finally we remark that the subalgebra of a simple or semisimple Lie algebra need not to be simple/semisimple.

### 2.2 Compact semisimple Lie groups

A Lie group \( G \) is a group that has a structure of a differential manifold and the group operation is smooth. We say \( G \) is compact if it is a compact manifold, i.e. any open covering of \( G \) has a finite subcovering. It is well known that a closed subgroup of a Lie group is a Lie group [17, 7]. In this page we will always consider closed subgroups. An important class of subgroups are normal subgroups. \( H \subseteq G \) is a normal subgroup if for each \( g \in G \) we have \( gHg^{-1} \subseteq H \). We denote it by \( H \triangleleft G \). In this case the quotient \( G/H \) is a group. A disconnected \( G \) consists of connected components. Connected components of a Lie group are open and their number is finite if \( G \) is compact, as otherwise they would constitute an open covering of \( G \) in the identity component into components. But \( G \) is easily seen as the maps \( X \in G \) determine by their value at \( e \) continuous for every \( X \in G \), hence they map components into component. But \( e \in \text{ad}_X(G_e) \) for all \( g \in G \), hence \( \text{ad}_X(G_e) = G_e \). The quotient \( G/G_e \) is a group (because \( G_e \) is normal) which for compact \( G \) is a finite group called components group.

The connection between Lie groups and Lie algebras is established in the following way. Left invariant vector fields on \( G \) together with vector fields commutator form the Lie algebra \( \mathfrak{g} \) of a Lie group \( G \). Note that these fields are determined by their value at \( e \) and therefore \( \mathfrak{g} \) can be identified with the tangent space to \( G \) at \( e \), i.e. \( \mathfrak{g} = T_eG \). For every \( X \in \mathfrak{g} \) there is a unique one parameter subgroup \( \gamma(t) \) whose tangent vector \( e \) is \( X \). We define the exponential map \( \exp : \mathfrak{g} \to G \) to be: \( \exp(X) := \gamma(1) \). For any Lie group the image of the exponential map, \( \exp(\mathfrak{g}) \), is contained in the identity component \( G_e \) and when \( G \) is compact \( \exp(\mathfrak{g}) = G_e \). Therefore for a compact and connected group every element \( g \in G \) is of the form \( \exp(X) \) for some \( X \in \mathfrak{g} \). For matrix Lie groups \( G \subseteq \text{GL}(n, \mathbb{C}) \) these definitions simplify as the exponential map is the matrix exponential that is defined by \( e^X = \sum_{i=0}^{\infty} \frac{X^i}{i!} \) and the Lie algebra is defined as \( \mathfrak{g} = \{X : e^tX \in G, \forall t \in \mathbb{R}\} \).

**Definition 4.** A compact connected Lie group is simple/semisimple if its Lie algebra is compact and simple/semisimple.

Recall that the Lie algebra \( \mathfrak{h} \) of the identity component of \( H \triangleleft G \) is an ideal of the Lie algebra \( \mathfrak{g} \). We can also use equivalent definition that says a compact connected group \( G \) is simple if it has no connected normal subgroups. Similarly as for Lie algebras, compact semisimple Lie groups have a particularly nice structure.

**Fact 5.** Let \( G \) be a compact connected semisimple group. Then \( G = (G_1 \times \ldots \times G_k)/Z \), where each \( G_i \) is a simple compact group and \( Z \) is contained in the centre of \( G_1 \times \ldots \times G_k \).

A representation of a Lie group on a real vector space is a homomorphism \( \Phi : G \to \text{GL}_R(V) \), i.e. \( \Phi \) satisfies \( \Phi(g_1g_2) = \Phi(g_1)\Phi(g_2) \). A particularly important example is the adjoint representation of \( G \) on \( \mathfrak{g} \).

\[
\text{Ad} : G \to \text{Aut}(\mathfrak{g}), \quad \text{Ad}_g(X) := gXg^{-1}.
\]

The image of \( \text{Ad}_G \) is \( \text{Ad}_G = G/Z(G) \), where \( Z(G) \) is the centre of \( G \). For a semisimple compact Lie group \( Z(G) \) is finite by definition and therefore \( \text{Ad} \) is a finite covering homomorphism onto \( G/Z(G) \). For a compact connected simple Lie groups the adjoint representation is irreducible.

The relation between the adjoint representations of a compact connected semisimple Lie group and its Lie algebra, \( \text{Ad} \) and \( \text{ad} \), follows from the fact that \( \text{Ad} \) is a smooth homomorphism. For any \( X \in \mathfrak{g} \) and all \( t \in \mathbb{R} \) elements \( \text{Ad}_{e^{tX}} \) form a one-parameter subgroup in \( \text{Aut}(\mathfrak{g}) \) whose tangent vector at \( t = 0 \) is \( \text{ad}_X \). As this group is uniquely determined by its tangent vector we have \( \text{Ad}_{e^{tX}} = e^{\text{ad}_{tX}} \). Using this relation we easily see that the Killing form on \( \mathfrak{g} \) is invariant with respect to the adjoint action, i.e. \( B(\text{Ad}_gX, \text{Ad}_gY) = B(X, Y) \). Recall that for a compact \( G \) the Killing form is an inner product (negative definite) and therefore \( \text{Ad}_g \) is an orthogonal matrix belonging to \( \text{SO}(\mathfrak{g}) \). After a choice of orthonormal basis in \( \mathfrak{g} \), we can identify \( \text{Ad}_g \) with a matrix from \( \text{SO}(\dim \mathfrak{g}) \).
2.3 Subgroups of a compact semisimple Lie group

Let \( G \) be a Lie group. We say that \( H \subset G \) is a discrete subgroup of \( G \) if there is an open cover of \( H \) such that every open set in this cover contains exactly one element from \( H \) - we will call it a discrete open cover of \( H \). If \( G \) is compact every discrete subgroup is finite. To see this, assume that there is an infinite discrete subgroup \( H \) in a compact \( G \) and take the open cover of \( G \) that is a union of discrete open cover of \( H \) and the open set which consists of elements not in this discrete cover. Then this cover is infinite and has no finite subcover, hence we get contradiction. By similar argument any closed disconnected subgroup \( H \) of a compact \( G \) has finitely many connected components. The Lie algebra \( \mathfrak{h} \) of the identity component \( H_e \), is a subalgebra of \( \mathfrak{g} \) and the exponential map is surjective onto \( H_e \), however \( \mathfrak{h} \) needs not to be semisimple. We distinguish three possible types of closed subgroups of the compact Lie group \( G \): (1) finite discreet subgroups, (2) disconnected subgroups with a finite number of connected components, (3) connected subgroups.

In this paper we consider groups that are generated by finite number of elements from some compact semisimple Lie group \( G \). More precisely for \( S = \{g_1, \ldots, g_k\} \subset G \) we consider the closure of

\[
\langle S \rangle := \left\{ U_{i_1}^{k_1} \cdot \ldots \cdot U_{i_m}^{k_m} : U_{i_j} \in S, k_j \in \mathbb{N}, i_j \in \{1, \ldots, n\} \right\},
\]

which is a Lie subgroup of \( G \). In particular we want to know when \( \langle S \rangle = G \). It is known that almost any two elements generate a compact semisimple \( G \). Moreover, as was shown by Kuranishi [15] elements that are in a sufficiently small neighbourhood of \( e \) generate \( G \) if and only if their corresponding Lie algebra elements generate \( \mathfrak{g} \). The proof is, however, not contractive. The author of [13] shows that pairs generating \( G \) form a Zariski open subset of \( G \times G \). In our work we adopt and develop some of the ideas contained in [15] and [13] and this way obtain characterisation of sets \( S \) that generate groups \( SU(d) \) or \( SO(d) \).

**Fact 6.** The closure of \( S \) is a Lie group.

**Proof.** By theorem of Cartan [7, 17] we know that a closed subgroup of a Lie group is a Lie group. The set \( \overline{\langle S \rangle} \) is obviously closed and hence we are left with showing that it is has a group structure. By the construction \( S \) is invariant under multiplication and therefore \( \overline{\langle S \rangle} \) has this property too. As a direct implication of Dirichlet approximation theorem (see theorem [25]), for every element \( g \in S \) there is a sequence \( \{g^n\} \), such that \( \lim_{n \to \infty} g^n = I \) when \( \lim_{n \to \infty} g^{-n} = I \). Thus \( I \in \langle S \rangle \). Note, however, that by the same argument the sequence \( \{g^{n-1}\} \subset S \) converges to \( g^{-1} \). Thus \( S \) has a group structure. The result follows.

\[\square\]

3 Generating sets for compact semisimple Lie algebras and Lie groups

3.1 Generating sets for compact semisimple Lie algebras

In this section \( \mathfrak{g} \) will denote a compact semisimple Lie algebra. Let \( \mathcal{X} = \{X_1, \ldots, X_n\} \subset \mathfrak{g} \). We say that \( \mathcal{X} \) generates \( \mathfrak{g} \) if any element of \( \mathfrak{g} \) can be written as a linear combination of \( X_i \)'s and finitely nested commutators of \( X_i \)'s:

\[
\sum_i \alpha_i X_i + \sum_{i,j} \alpha_{i,j} [X_i, X_j] + \ldots
\]

Our aim is to provide a general criterion for compact semisimple Lie algebras that verifies when \( \mathcal{X} \subset \mathfrak{g} \) generates \( \mathfrak{g} \). To this end we use adjoint representation. Let \( \mathcal{C}(\text{ad}_\mathfrak{g}) = \{L \in \text{End}(\mathfrak{g}) : \forall X \in \mathfrak{g} \ [\text{ad}_X, L] = 0\} \) denotes the space of endomorphisms of \( \mathfrak{g} \) that commute with all \( \text{ad}_X \). \( \mathcal{X} \in \mathfrak{g} \). By the Jacobi identity \( \mathcal{C}(\text{ad}_\mathfrak{g}) \) is a Lie subalgebra of \( \text{End}(\mathfrak{g}) \). Moreover, also by Jacobi identity, if \( L \in \text{End}(\mathfrak{g}) \) commutes with \( \text{ad}_X \) and \( \text{ad}_Y \) then it also commutes with \( \text{ad}_{\alpha X + \beta Y} \) and \( \text{ad}_{[X,Y]} \). Let us denote by \( \mathcal{C}(\text{ad}_\mathfrak{g}) \) the solution set of

\[
[\text{ad}_X, \cdot] = 0, \ldots, [\text{ad}_{X_n}, \cdot] = 0.
\]

It is clear that if \( \mathcal{X} \) generates \( \mathfrak{g} \), then \( \mathcal{C}(\text{ad}_\mathfrak{g}) = \mathcal{C}(\text{ad}_\mathcal{X}) \). It happens that the converse is true for simple Lie algebras and with small modification also for semisimple.

**Lemma 7.** Let \( \mathfrak{g} \) be a compact simple Lie algebra and \( \mathcal{X} = \{X_1, \ldots, X_n\} \subset \mathfrak{g} \) be its finite subset. \( \mathcal{X} \) generates \( \mathfrak{g} \) if and only if \( \mathcal{C}(\text{ad}_\mathfrak{g}) = \{\lambda I : \lambda \in \mathbb{R}\} = \mathcal{C}(\text{ad}_\mathfrak{g}) \).
Proof. Note that since for a simple $\mathfrak{g}$ the adjoint representation is irreducible we have $C(\text{ad}_\mathfrak{g}) = \{ \lambda I : \lambda \in \mathbb{R} \}$ by the Schur lemma. Let us denote by $\mathfrak{h} \subset \mathfrak{g}$ the Lie algebra generated by $X$. Assume that $\mathfrak{h} \neq \mathfrak{g}$ but $C(\text{ad}_\mathfrak{g}) = C(\text{ad}_X)$. Using the Killing form we can decompose $\mathfrak{g}$ into a direct product of vector spaces (not necessarily Lie algebras), $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$. For any $X \in \mathfrak{h}$, $Y \in \mathfrak{g}$ and $Z \in \mathfrak{h}^\perp$ we have $\text{ad}_X Y \in \mathfrak{h}$ and $\text{ad}_X Z \in \mathfrak{h}^\perp$. The later is true as $B(\text{ad}_X Z, Y) = -B(Z, \text{ad}_X Y) = 0$, for any $Y \in \mathfrak{h}$. Therefore, for $X \in \mathfrak{h}$ operators $\text{ad}_X$ respect the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$ and have a block diagonal structure:

$$\text{ad}_X = \begin{pmatrix} \text{ad}_X|_\mathfrak{h} & 0 \\ 0 & \text{ad}_X|_{\mathfrak{h}^\perp} \end{pmatrix}.$$  \hfill (5)

Let $P : \mathfrak{g} \to \mathfrak{h}$ be the orthogonal, with respect to the Killing form, projection operator onto $\mathfrak{h}$. Then obviously $[P, \text{ad}_X] = 0$ for any $X \in \mathfrak{h}$. Note, however, that $P$ cannot belong to $C(\text{ad}_\mathfrak{g})$ as this would mean $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ which is a contradiction as $\mathfrak{g}$ is simple. \hfill $\square$

Let next $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$ be a decomposition of a semisimple $\mathfrak{g}$ into simple ideals. Let $\mathcal{X} = \{X_1, \ldots, X_n\} \subset \mathfrak{g}$. Every $X_i \in \mathcal{X}$ has a unique decomposition:

$$X_i = X_{i,1} + \ldots + X_{i,k}, \text{ where } X_{i,j} \in \mathfrak{g}_j.$$ 

Therefore $\mathcal{X}$ generates $\mathfrak{g}$ if every set $X_i = \{X_{i,1}, \ldots, X_{i,k}\}$ generates $\mathfrak{g}_i, i \in \{1, \ldots, k\}$.

Lemma 8. Let $\mathfrak{g}$ be a compact semisimple Lie algebra and $\mathcal{X} = \{X_1, \ldots, X_n\} \subset \mathfrak{g}$ its finite subset such that the projection of $\mathcal{X}$ onto every simple component of $\mathfrak{g}$ is nonzero. $\mathcal{X}$ generates $\mathfrak{g}$ if and only if $C(\text{ad}_\mathfrak{g}) = C(\text{ad}_\mathcal{X})$.

Finally let us remark that it is very important to consider not a defining but the adjoint representation. To see this let $X_1, X_2$ be two matrices that generate $\mathfrak{su}(2)$ and consider set $\mathcal{X} = \{I \otimes X_1, I \otimes X_2, X_1 \otimes I, X_2 \otimes I\} \subset \mathfrak{su}(4)$. Note that the Lie algebra generated by $\mathcal{X}$ is $\mathfrak{su}(2) \oplus \mathfrak{su}(2) \subset \mathfrak{su}(4)$. One checks by direct calculations that the only $4 \times 4$ matrix commuting with $\mathcal{X}$ is proportional to the identity. This is, however, not the case for matrices $\text{ad}_X, X \in \mathcal{X}$. Hence changing the adjoint representation in Lemma 7 into the defining one would result in the equality between $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and $\mathfrak{su}(4)$ which is of course not true.

3.2 Generating sets for compact semisimple Lie groups

We are interested in the following problem. Let $G$ be a compact connected semisimple Lie group and let $\mathcal{S} = \{g_1, \ldots, g_n\} \subset G$. We want to know when $<\mathcal{S}> = G$. As $G$ is compact and connected we have $g_i = e^{X_i}, X_i \in \mathfrak{g}$. Note that if we were promised that $<\mathcal{S}>$ is a connected subgroup of $G$ then using the correspondence between Lie groups and Lie algebras, $<\mathcal{S}> = G$ if and only if $\mathcal{X} = \{X_1, \ldots, X_n\}$ generates $\mathfrak{g}$. The difficulty of our problem is that we typically do not know if $<\mathcal{S}>$ is connected. Therefore we need criteria that could detect if the Lie subgroup $<\mathcal{S}>$ is or is not connected. As we already pointed out in Section 2.3 disconnected subgroups of $G$ can be either infinite or finite. We first explain how to detect infinite disconnected subgroups of $G$. To this end we use adjoint representation.

Let $C(\text{Ad}_G) = \{L \in \text{End}(\mathfrak{g}) : \forall g \in G \ [\text{Ad}_g, L] = 0 \}$ denote the space of endomorphisms of $\mathfrak{g}$ that commute with all $\text{Ad}_g, g \in G$. By the Jacobi identity $C(\text{Ad}_G)$ is a Lie subalgebra of $\text{End}(\mathfrak{g})$. Moreover, if $L \in \text{End}(\mathfrak{g})$ commutes with $\text{Ad}_g$ and $\text{Ad}_h$ then it also commutes with $\text{Ad}_{gh}$. Let us denote by $C(\text{Ad}_\mathfrak{g})$ the solution set of

$$[\text{Ad}_{g_1}, \cdot] = 0, \ldots, [\text{Ad}_{g_n}, \cdot] = 0.$$ 

It is clear that if $\mathcal{S}$ generates $G$ then $C(\text{Ad}_G) = C(\text{Ad}_\mathfrak{g})$. It happens that the converse if true for groups Lie algebra and with small modification also for semisimple, provided $<\mathcal{S}>$ is infinite.

Lemma 9. Let $G$ be a compact connected simple Lie group and $\mathcal{S} = \{g_1, \ldots, g_n\} \subset G$ its finite subset. Assume $<\mathcal{S}>$ is infinite. The set $\mathcal{S}$ generates $G$ if and only if $C(\text{Ad}_G) = \{\lambda I : \lambda \in \mathbb{R}\} = C(\text{Ad}_\mathfrak{g})$.

Proof. Note that since $G$ is connected and simple, the adjoint representation is irreducible and by the Schur lemma we have $C(\text{Ad}_G) = \{\lambda I : \lambda \in \mathbb{R}\}$. Let us denote by $H$ the closure of the group generated by $\mathcal{S}$. $H$ is a compact Lie group. Let $H_e$ be the identity component of $H$. As we know $H_e$ is a normal subgroup of $H$. Let $\mathfrak{h} \subset \mathfrak{g}$ be the Lie algebra of $H_e$. Assume that $\mathfrak{h} \neq \mathfrak{g}$ but $C(\text{Ad}_G) = C(\text{Ad}_\mathfrak{g})$. Using the Killing form we can decompose $\mathfrak{g}$ into a direct product of vector spaces (not necessarily Lie algebras), $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$. For any $g \in H, X \in \mathfrak{h}$ and $Y \in \mathfrak{h}^\perp$ we have...
Ad_\eta Y \in \mathfrak{h} \text{ and } Ad_\eta Y \in \mathfrak{h}^\perp. \text{ The later is true as } B(Ad_\eta Y, X) = B(Y, Ad_\eta^* X) = 0, \text{ for any } X \in \mathfrak{h}. \text{ Therefore, for } h \in H \text{ operators } Ad_h \text{ respect the decomposition } \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp \text{ and have a block diagonal structure: }

\begin{equation}
Ad_h = \begin{pmatrix}
Ad_h|_\mathfrak{h} & 0 \\
0 & Ad_h|_{\mathfrak{h}^\perp}
\end{pmatrix}.
\end{equation}

Let \( P : \mathfrak{g} \to \mathfrak{h} \) be the orthogonal, with respect to the Killing form, projection operator onto \( \mathfrak{h} \). Then obviously \( [P, Ad_h] = 0 \) for any \( h \in H \). Note, however, that \( P \) cannot belong to \( \mathcal{C}(Ad_G) \) as this would mean \( \mathfrak{h} \) is an \( Ad_G \) invariant subspace of \( \mathfrak{g} \) (or \( H_e \) is a normal subgroup of \( G \)) which is a contradiction as \( G \) is simple and connected. \( \blacksquare \)

Recall that up to a finite covering, any compact connected semisimple Lie group is a product of simple groups. By the similar argument as for simple Lie algebras we get:

**Lemma 10.** Let \( G \) be a compact connected semisimple Lie group and \( \mathcal{S} = \{g_1, \ldots, g_n\} \subset G \) its finite subset such that \( \langle \mathcal{S} \rangle \) is infinite and the projection of \( \mathcal{S} \) onto every simple component of \( G \) is also infinite. \( \mathcal{S} \) generates \( G \) if and only if \( \mathcal{C}(Ad_G) = \mathcal{C}(Ad_{\mathcal{S}}) \).

Note that the difference between \( \mathcal{C}(Ad_G) \) and \( \mathcal{C}(ad_G) \), where \( \mathcal{S} = \{g_1, \ldots, g_n\} \subset G \), \( g_i = e^{X_i} \) and \( \mathcal{X} = \{X_1, \ldots, X_n\} \subset \mathfrak{g} \) is possible only if \( \langle \mathcal{S} \rangle \) is disconnected. Therefore, even if \( \mathcal{X} \) generates \( \mathfrak{g} \), the group generated by \( \mathcal{S} \) can be smaller than \( G \). The adjoint representation is able to detect this kind of situation provided \( \langle \mathcal{S} \rangle \) is infinite. Note that \( \overline{\mathcal{S}} \) is infinite in particular when at least one of \( g_i \)'s is of infinite order. Hence

**Corollary 11.** Let \( G \) be a compact connected simple Lie group and \( \mathcal{S} = \{g_1, \ldots, g_n\} \subset G \) its finite subset such that at least one of \( g_i \)'s is of infinite order. \( \mathcal{S} \) generates \( G \) if and only if \( \mathcal{C}(Ad_{\mathcal{S}}) = \{\lambda I : \lambda \in \mathbb{R}\} \).

In the next section we characterise when \( \langle \mathcal{S} \rangle \) is infinite and when \( \mathcal{C}(Ad_{\mathcal{S}}) \) can be different form \( \mathcal{C}(ad_{\mathcal{X}}) \) for semisimple groups of our interest, i.e. for \( G = SU(d) \) and \( G = SO(d) \).

### 4 Groups \( SU(d) \) and \( SO(d) \)

In this section we focus on two groups \( G \) that are particularly important from the perspective of quantum computation and linear quantum optics, i.e. \( G = SO(d) \) or \( G = SU(d) \).

\( SO(d) = \{O \in \text{Gl}_d(\mathbb{R}) : O^t O = I, \det O = 1\} \), \( SU(d) = \{U \in \text{Gl}_d(\mathbb{C}) : U^t U = I, \det X = 1\} \).

Their Lie algebras \( \mathfrak{g} \) are:

\( \mathfrak{so}(d) = \{X \in \text{Mat}_d(\mathbb{R}) : X^t = -X, \text{tr} X = 0\} \), \( \mathfrak{su}(d) = \{X \in \text{Mat}_d(\mathbb{C}) : X^t = -X, \text{tr} X = 0\} \).

The centres of \( G \) are finite and given by \( Z(SU(d)) = \{\alpha I : \alpha \in \mathbb{C}, \alpha^d = 1\} \), \( Z(SO(2d)) = \{\pm I\} \) and \( Z(SO(2d+1)) = I \). Groups \( SU(d) \) for \( d \geq 2 \) and groups \( SO(d) \) for \( d \geq 3 \) and \( d \neq 4 \) are compact connected simple Lie groups. On the other hand \( SO(4) \) is still compact and connected but it is not simple as its Lie algebra is a direct sum of Lie algebras \( \mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3) \), hence \( SO(4) \) is semisimple. The Killing form on both \( \mathfrak{su}(d) \) and \( \mathfrak{so}(d) \), up to a constant factor, is given by \( B(X,Y) = \text{tr} XY \). We next introduce an orthonormal basis in \( \mathfrak{su}(d) \) and \( \mathfrak{so}(d) \). Let \( E_{kl} = |k\rangle\langle l| \) be a \( d \times d \) matrix whose only nonzero (and equal to 1) entry is \( (k,l) \). The commutation relations are \( [E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{kj} \).

One easily checks that for \( i, j \in \{1, \ldots, d\}, i < j \) matrices \( \{X_{ij}, Y_{ij}, Z_{i,i+1}\} \) form an orthogonal basis of \( \mathfrak{su}(d) \) and matrices \( \{X_{ij}\} \) of \( \mathfrak{so}(d) \). We will call these two bases the standard basis of \( \mathfrak{su}(d) \) and \( \mathfrak{so}(d) \) respectively.

Finally we recall that for a unitary matrix \( U \in SU(d) \) there is a unitary matrix \( V \) such that \( D = V^t UV = \text{diag}\{e^{i\phi_1}, \ldots, e^{i\phi_d}\} \). The diagonal entries of \( D \) constitute the spectrum of \( U \). As \( U = e^{X} \) for some \( X \in \mathfrak{su}(d) \) we get \( V^t VX = D = \text{diag}\{i\phi_1, \ldots, i\phi_d\} \). Matrices in \( SO(d) \) typically cannot be diagonalised by the orthogonal group. Nevertheless for a matrix \( O \in SO(d) \) there is an orthogonal matrix \( V \) such that \( R = V^t OV \) is block diagonal with two types of blocks: (1) one identity matrix \( I_k \) of dimension \( k \leq d \), (2) \( 2 \times 2 \) rotations by angles \( \phi_i \), i.e. matrices \( O(\phi_i) \) from \( SO(2) \). We again have \( O = e^{X} \) for some \( X \in \mathfrak{so}(d) \) and \( R = V^t XV \) is block diagonal and there are two types of blocks: (1) zero matrix \( 0_k \) of dimension \( k \leq d \), (2) Lie algebra elements corresponding to rotations by angle \( \phi_i \), i.e. matrices from \( \mathfrak{so}(2) \). We will call \( R \) and \( \tilde{R} \) normal forms of \( O \in SO(d) \) and \( X \in \mathfrak{so}(d) \) respectively and angles \( \phi_i \)'s the spectral angles.
4.1 The difference between $C(\text{Ad}_S)$ and $C(\text{ad}_X)$

4.1.1 The case of $SU(d)$

Let $S = \{U_1, \ldots, U_n\} \subset SU(d)$ and let $X = \{X_1, \ldots, X_n\}$ be such that $U_i = e^{X_i}$. In this section we study when the spaces $C(\text{Ad}_S)$ and $C(\text{ad}_X)$ are different. Note first that using $\text{Ad}_{X_i} = e^{\text{ad}X_i}$ we have $C(\text{Ad}_X) \subset C(\text{Ad}_S)$. Hence we are particularly interested in the situation when $C(\text{Ad}_S)$ is strictly larger then $C(\text{ad}_X)$. Matrices $U_i$ can be put into diagonal form $U_i = V_i \hat{D}_i V_i^\dagger$, where $V_i \in SU(d)$ and $\hat{D}_i = \{e^{i\phi_i}, \ldots, e^{i\phi_d}\}$, $\phi_j \in [0, 2\pi)$. Note now that $\text{Ad}_{U_i} = \text{Ad} V_i \hat{D}_i V_i^\dagger = O \text{Ad} \hat{D}_i O_i^\dagger$, where $O = \text{Ad} \hat{v}_i \in SO(d^2 - 1)$. Let us order the standard basis of $\mathfrak{su}(d)$ as follows $\{X_{12}, Y_{12}, \ldots, X_{d-1,d}, Y_{d-1,d}, Z_{12}, \ldots, Z_{d-1,d}\}$. The matrix $\text{Ad}_{D_i}$ in this basis has a block diagonal form:

$$
\text{Ad}_{D_i} = \begin{pmatrix}
O(\phi_{1,2}) & \cdots & & \\
& O(\phi_{1,d}) & \cdots & \\
& & \ddots & \cdots \\
& & & O(\phi_{d-1,d}) \\
& & & & I_{d-1}
\end{pmatrix},
$$

where

$$
O(\phi_{k,l}) = \begin{pmatrix}
\cos(\phi_{k,l}) & \sin(\phi_{k,l}) \\
-\sin(\phi_{k,l}) & \cos(\phi_{k,l})
\end{pmatrix},
$$

and $I_{d-1}$ is $(d-1) \times (d-1)$ identity matrix. Matrices from $X$ are diagonalised by the same operators $\{V_i\}_{i=1}^n$ as matrices form $S$, i.e. $X_i = V_i \hat{D}_i V_i^\dagger$ and $\hat{D}_i = i(\phi_i^1, \phi_i^2, \ldots, \phi_i^d)$. Hence $\text{ad}_{X_i} = \text{ad} V_i \hat{D}_i V_i^\dagger = O \text{ad} \hat{D}_i O_i^\dagger$, and we have (in the standard basis of $\mathfrak{su}(d)$ ordered as previously):

$$
\text{ad}_{D_i} = \begin{pmatrix}
X(\phi_{1,2}) & \cdots & & \\
& X(\phi_{1,d}) & \cdots & \\
& & \ddots & \cdots \\
& & & X(\phi_{d-1,d}) \\
& & & & 0_{d-1}
\end{pmatrix},
$$

where

$$
X(\phi_{k,l}) = \begin{pmatrix}
0 & \phi_{k,l} \\
-\phi_{k,l} & 0
\end{pmatrix},
$$

and $0_{d-1}$ is $(d-1) \times (d-1)$ zero matrix. Note that $\phi_{k,l} \in (-2\pi, 2\pi)$. Comparing structures of matrices $\text{Ad}_{D_i}$ and $\text{ad}_{D_i}$ we deduce that if all $\phi_{k,l} \neq \pm \pi$ then $C(\text{Ad}_S) = C(\text{ad}_X)$. The situation is different when $\phi_{k,l} = \pm \pi$. In this case $\text{Ad}_{D_i}$ has additional degeneracies as compared to $\text{ad}_{D_i}$ as $O(\phi_{k,l}) = O(\pm \pi) = -I_2$. Let $P$ be the rotation plane corresponding to the angle $\phi_{k,l} = \pm \pi$. One can then construct a rotation $O' \in SO(d^2 - 1)$ whose elementary rotation planes are exactly as in $\text{ad}_{D_i}$ except $P$ which is replaced by a plane $P'$, $P \bot P'$. This can be achieved using available $d - 1$ directions corresponding to $I_{d-1}$. If the rotation angle along $P'$ is also $\pi$ then $[\text{Ad}_{U_i}, O'] = 0$ and $[\text{ad}_{X_i}, O'] \neq 0$. Hence the space $C(\text{Ad}_{U_i})$ is larger than $C(\text{ad}_{X_i})$ and there is possibility that it might be true also for sets $C(\text{Ad}_S)$ and $C(\text{ad}_X)$. As a conclusion we get

**Fact 12.** The space $C(\text{Ad}_S)$ can be larger than $C(\text{ad}_X)$ if and only if the differences between spectral angles $\phi_{k,l}$ for at least one of the matrices $U_i \in S$ is equal to $\pm \pi$. 


4.1.2 The case of $SO(d)$

Similarly as for $SU(d)$ we consider $S = \{O_1, \ldots, O_n\} \subset SO(d)$ and $X = \{X_1, \ldots, X_n\}$ such that $O_i = e^{X_i}$. We have $C(\text{ad}_X) \subset C(\text{ad}_S)$ and our goal is to characterise cases when the space $C(\text{ad}_S)$ can be strictly larger than $C(\text{ad}_X)$. Matrices $O_i$ can be put into a standard form $O_i = V_i R_i V_i^\dagger$, where $V_i \in SO(d)$ and $R_i$ is a block diagonal matrix consisting of $k \leq \left\lfloor \frac{d}{2} \right\rfloor$ two dimensional blocks representing rotations by angles $\{\phi^1_1, \ldots, \phi^1_k\}$, $\phi^2_j \in (0, 2\pi)$ and one $(d - 2k)$-dimensional block that is identity matrix. Note now that $\text{Ad}_{O_i} = \text{Ad}_V \text{Ad}_{R_i} \text{Ad}_V$. Each matrix $\text{Ad}_{R_i}$ can be brought to the standard block diagonal form containing the following blocks:

1. $O(\phi^1_{a,b})$ and $O(\psi^1_{a,b})$, where $\phi^1_{a,b} = \phi^1_a - \phi^1_b$, $\psi^1_{a,b} = \psi^1_a + \psi^1_b$, $a < b$. The number of these blocks is $k(k - 1)$.

2. Identity block of dimension $k + \frac{(d - 2k)(d - 2k - 1)}{2}$.

3. Blocks $O(\phi^2_j)$, where $j \in \{1, \ldots, k\}$. Each block $O(\phi^2_j)$ appears $(d - 2k)$ times. Hence we have $k(d - 2k)$ blocks like this.

Matrices $\text{ad}_X$ have the same structure as matrices $\text{Ad}_{O_i}$ albeit the identity block is replaced by the 0-block of the same dimension and the rotational blocks $O(\phi^1_{a,b})$, $O(\psi^1_{a,b})$ and $O(\phi^2_j)$ are replaced by their Lie algebra elements. Repeating the reasoning for $SU(d)$ we get:

**Fact 13.** Let $G = SO(d)$. The space $C(\text{ad}_S)$ can be bigger than $C(\text{ad}_X)$ if and only if the difference or the sum of spectral angles $\phi^1_a$ and $\phi^1_b$ for at least one of the matrices $U_i \in S$ is equal to odd multiple of $\pi$.

4.2 Pairs generating infinite subgroups of $G$

Our first aim is to show that elements that are close to elements belonging to $Z(G)$ generate $G$ if the corresponding Lie algebra elements generate $g$. To this end we define a norm of $A \in \text{Mat}_d(\mathbb{C})$ by $\|A\| = \sqrt{\text{tr}(AA^\dagger)}$. Next we recall that the group commutator of two invertible matrices (with respect to matrix multiplication) is defined as $[A, B] = ABA^{-1}B^{-1}$. Naturally, if matrices commute in a usual sense then $[A, B] = I$. The following lemma relates the distance between $[A, B]$, and $I$ with the distances of $A$ and $B$ form the identity.

**Lemma 14.** Let $A, B \in G$ where $G = SU(d)$ or $G = SO(d)$ and let $C = [A, B]$. We have the following:

\[
\|C - I\| \leq \sqrt{2}\|A - I\|\|B - I\|.
\]

If $[A, C] = I$ and $\|B - I\| < 2$, then $[A, B] = I$.

**Proof.** Can be found in Lemmas 36.15 and 36.16 of [9].

We next define open balls in $G = SO(d)$ or $SU(d)$ centred around elements from $Z(G)$ and of radius $1/\sqrt{2}$,

\[
B^{1/\sqrt{2}} = \{g \in G : \|g - I\| < 1/\sqrt{2}\}.
\]

**Lemma 15.** Let $g, h \in B^{1/\sqrt{2}}$ and assume $[g, h] \neq I$. The group $< g, h >$ generated by $g, h$ is infinite.

**Proof.** Define the sequence $g_0 = g$, $g_1 = [g_0, h]$, $g_n = [g_{n-1}, h]$. By our assumptions $\|h - I\| = d \leq 1/\sqrt{2}$. Therefore using Lemma 14

\[
\|g_n - I\| \leq \sqrt{2}\|g_{n-1} - I\|.
\]

Thus $\|g_n - I\| \leq (\sqrt{2})^n \|g - I\|$ and $g_n \to I$, when $n \to \infty$. Assume that the sequence is finite, i.e. for some $N$ we have $g_N = I$. That means $[g_{N-1}, h] = I$. But $g_{N-1} = [g_{N-2}, h]$ and clearly $\|g_N - I\| < 2$ and by Lemma 1, $[g_{N-2}, h] = I$. Repeating this argument we get $[g, h] = I$ which is a contradiction. Therefore $< g, h >$ is infinite.

**Corollary 16.** Let $g \in B^{1/\sqrt{2}}$ and $h \in B^{1/\sqrt{2}}$, where $\alpha_1$ and $\alpha_2$ are such that $\alpha_1 I, \alpha_2 I \in Z(G)$ and assume $[g, h] \notin Z(G)$. Then the group $< g, h >$ is infinite.

**Proof.** If $\alpha_1 = \alpha_2 = 1$ the result follows from Lemma 15. For all other $\alpha_i$’s let $g' = \alpha_1^{-1} g$ and $h' = \alpha_2^{-1} h$. Then $h', g' \in B^{1/\sqrt{2}}$ and $[g', h'] \neq I$. Thus by Lemma 15 $< g', h' >$ is infinite. Note that $< g, h >$ is equal to $< g', h' >$ up to the finite covering and therefore is infinite too.
Finally \( g \) generates \( \mathbb{Z} \) of \( O \). The conditions for the spectral angles are as follows

\[
\sin i\phi_{1}, e^{i\phi_{2}}, \ldots, e^{i\phi_{d}} \text{ be the spectrum of } U_{d} \in SU(d). \quad \text{The conditions for } U_{d} \in SU(d) \text{ to belong to the ball } B_{\alpha_{m}I} \text{ read:}
\]

\[
U_{d} \in B_{\alpha_{m}I}^{\frac{1}{\sqrt{2}}} \iff \sum_{i=1}^{d} \sin^{2} \phi_{i} - \frac{\theta_{m}}{2} < \frac{1}{8}, \quad \sum_{i=1}^{d} \phi_{i} = 0 \text{ mod } 2\pi.
\]  

For \( SO(2k + 1) \) the centre is trivial and we have only one ball \( B_{I}^{\frac{1}{\sqrt{2}}} \). Let \( \{1, e^{i\phi_{1}}, e^{-i\phi_{1}}, \ldots, e^{i\phi_{k}}, e^{-i\phi_{k}}\} \) be the spectrum of \( O_{2k+1} \in SO(2k + 1) \). We have

\[
O_{2k+1} \in B_{I}^{\frac{1}{\sqrt{2}}} \iff \sum_{i=1}^{k} \sin^{2} \frac{\phi_{i}}{2} < \frac{1}{16}.
\]  

Finally \( Z(SO(2k)) = \{I, -I\} \) and we have two balls \( B_{I}^{\frac{1}{\sqrt{2}}}, B_{-I}^{\frac{1}{\sqrt{2}}} \). Let \( \{e^{i\phi_{1}}, e^{-i\phi_{1}}, \ldots, e^{i\phi_{k}}, e^{-i\phi_{k}}\} \) be the spectrum of \( O_{2k} \). The conditions for the spectral angles are as follows

\[
O_{2k} \in B_{I}^{\frac{1}{\sqrt{2}}} \iff \sum_{i=1}^{k} \sin^{2} \frac{\phi_{i}}{2} < \frac{1}{16},
\]

\[
O_{2k} \in B_{-I}^{\frac{1}{\sqrt{2}}} \iff \sum_{i=1}^{k} \sin^{2} \frac{\phi_{i} - \pi}{2} < \frac{1}{16}.
\]  

**Lemma 17.** Let \( G = SO(d) \) or \( G = SU(d) \). Let \( S = \{g_{1}, \ldots, g_{n}\} \subset G \) be such that \( g_{i} \in B_{\alpha_{d}I}^{\frac{1}{\sqrt{2}}} \), where \( \alpha I \in Z(G) \) and let \( X = \{X_{1}, \ldots, X_{n}\} \subset \mathfrak{g} \) be Lie algebra elements that satisfy \( e^{X_{i}} = g_{i} \). \( S \) generates \( G \) if and only if \( X \) generates \( \mathfrak{g} \).

**Proof.** By Lemma 10 matrices \( S \) generate \( G \) if they generate infinite subgroup and \( C(AdS) = C(AdG) \). The cases when spaces \( C(AdS) \) and \( C(AdX) \) can differ are characterised by Facts 12 and 13. Assume that \( S \subset SU(d) \). The spaces \( C(AdS) \) and \( C(AdX) \) can differ if and only if for one of the matrices \( g_{i} \in S \) we have \( \phi_{i}^{a,b} = k\pi \), where \( k \) is odd. But then \( \phi_{a}^{b} = \phi_{a}^{b} \pm \pi \) and

\[
\sin^{2} \frac{\phi_{a}^{b} \pm \pi - \theta_{m}}{2} + \sin^{2} \frac{\phi_{a}^{b} - \theta_{m}}{2} = 1,
\]
which means $g_i$ does not satisfies  \([12]\). Assume next that $S \subset SO(d)$. The spaces $C(\text{Ad}_g)$ and $C(\text{ad}_g \chi)$ can differ iff the difference or the sum of spectral angles $\phi_a^i$ and $\phi_b^i$ is equal to an odd multiple of $\pi$. For odd $d$ we arrive at
\[
\sin^2 \frac{\phi_b^i + \pi}{2} + \sin^2 \frac{\phi_a^i}{2} = 1,
\]
and for even $d$ we additionally have
\[
\sin^2 \frac{\phi_b^i + \pi - \pi}{2} + \sin^2 \frac{\phi_b^i - \pi}{2} = 1,
\]
which means $g_i$ does not satisfy \([13]\), \([14]\) or \([15]\).

\[\text{Fact 18.}\] For groups $G = SU(d)$ and $G = SO(d)$ there is $N_G \in \mathbb{N}$ such that for every $g \in G$, $g^n \in B^{1/\sqrt{d}}_{\alpha I}$ for some $\alpha I \in Z(G)$ and $1 \leq n \leq N_G$.

Proof. Let us first recall that by Dirichlet theorem (see Theorem 201 in \([14]\)), for given real numbers $x_1, x_2, \ldots, x_k$ we can find $n \in \mathbb{N}$ so that $nx_1, \ldots, nx_k$ all differ from integers by as little as we want. Let $\{\phi_1, \ldots, \phi_k\}$ be the spectral angles of $g \in G$ and let $\phi_i = 2\pi x_i$. By Dirichlet theorem we can always find $n$ such that $nx_i$‘s are close enough to integers to make $g^n$ to belong to $B^{1/\sqrt{d}}_{\alpha I}$. For $g \in G$ let $n_g$ be the smallest positive integer such that $g^{n_g} \in B^{1/\sqrt{d}}_{\alpha I}$ for some $\alpha I \in Z(G)$ (by Dirichlet theorem we know that $n_g < \infty$). Let $O_{n_g}^\alpha$ be an open neighbourhood of $g$ such that for any $h \in O_{n_g}^\alpha$ we have $h^{n_g} \in B^{1/\sqrt{d}}_{\alpha I}$. Note that there might be some $h \in O_{n_g}^\alpha$ for which $n_g$ is not optimal but this will not play any role. Let $\{O_{n_g}^\alpha\}_{\alpha I \in G}$ be the resulting open cover of $G$. As $G$ is compact there is a finite subcover \{$O_{n_g}^\alpha$\} and hence $N_G = sup_{n_g} n_g$, is well defined and finite.

By taking powers $1 \leq n \leq N_G$ we can move every element of $G$ into $B^{1/\sqrt{d}}_{\alpha I}$ for some $\alpha I \in Z(G)$. For $g_1, g_2 \in G$ such that $[g_1, g_2] \notin Z(G)$ let $g^n_1 \in B^{1/\sqrt{d}}_{\alpha I}$. If $[g_1^n, g_2^n] \notin Z(G)$ then $g_1, g_2$ is infinite. The sufficient condition for $[g_1^n, g_2^n] \notin Z(G)$ can be formulated in terms of spectral angles of matrices $\text{Ad}_{g_1}$ and $\text{Ad}_{g_2}$.

\[\text{Fact 19.}\] Let $g_1, g_2 \in G$ be such that $[g_1, g_2] \notin B^{1/\sqrt{d}}_{\alpha I}$ and $[g_1, g_2] \notin Z(G)$. Assume that $n_1, n_2 \in \{2, \ldots, N_G\}$ are such that $g_1^{n_1} \in B^{1/\sqrt{d}}_{\alpha I}$, where $\alpha I \in Z(G)$. Let $\phi_{a,b}^i, \psi_{a,b}^i$ be spectral angles of matrices $\text{Ad}_{g_i}$ (defined as in sections 4.1.1 and 4.1.2).

1. Let $G = SU(d)$. If a) for every $\phi_{a,b}^i \neq 0 \mod \pi$ we have $n_i \phi_{a,b}^i \neq 0 \mod \pi$ and $b)$ for every $\phi_{a,b}^i = (2k + 1)\pi$ we have $n_i \phi_{a,b}^i \neq 0 \mod 2\pi$ then $[g_1^{n_1}, g_2^{n_2}] \notin Z(G)$.

2. Let $G = SO(d)$. If a) for every $\phi_{a,b}^i \neq 0 \mod \pi$ and $\psi_{a,b}^i \neq 0 \mod \pi$ we have $n_i \phi_{a,b}^i \neq 0 \mod \pi$ and $n_i \psi_{a,b}^i \neq 0 \mod \pi$ and b) for every $\phi_{a,b}^i = (2k + 1)\pi$ and $\psi_{a,b}^i = (2k_2 + 1)\pi$ we have $n_i \phi_{a,b}^i \neq 0 \mod 2\pi$ and $n_i \psi_{a,b}^i \neq 0 \mod 2\pi$ then $[g_1^{n_1}, g_2^{n_2}] \notin Z(G)$.

Proof. Note that $[g_1, g_2] \notin Z(G)$ if $[\text{Ad}_{g_1}, \text{Ad}_{g_2}] \neq I$. Similarly $[g_1^{n_1}, g_2^{n_2}] \notin Z(G)$ iff $[\text{Ad}_{g_1^{n_1}}, \text{Ad}_{g_2^{n_2}}] \neq I$.

Let $\text{Ad}_{g_i} = O_i R_i O_i$, where $R_i$ is the normal form of $\text{Ad}_{g_i}$. As matrices $g^{n_i} \in B^{1/\sqrt{d}}_{\alpha I}$ for some $\alpha I \in Z(G)$ and $\text{Ad}_{g_i}^{n_i} = \text{Ad}_{g_i^{n_i}} = O_i R_i O_i$, the space $\mathcal{C} (\text{Ad}_{g_i^{n_i}})$ can be bigger than the space $\mathcal{C} (\text{Ad}_{g_i})$ if the implications 1. or 2. of Fact \([19]\) are not satisfied. The result follows.

Recall that since $\text{Ad} : G \rightarrow G/Z(G)$ is a finite covering homomorphism, we have that $S = <g_1, g_2>$ is infinite if and only if $\text{Ad}_S = <\text{Ad}_{g_1}, \text{Ad}_{g_2}>$ is infinite. Next, if at least one spectral angle of some $\text{Ad}_{g_i}$ is irrational multiple of $\pi$ then $\text{Ad}_S$ is infinite.

\[\text{Definition 20.}\] Assume $g \notin B^{1/\sqrt{d}}_{\alpha I}$ for any $\alpha I \in Z(G)$. A spectral angle $\phi$ of $\text{Ad}_g$ is called an exceptional angle iff

1. $\phi \neq 0 \mod \pi$ and there is $n \in \{2, \ldots, N_G\}$ such that $n\phi \equiv 0 \mod \pi$ and $g^n \in B^{1/\sqrt{d}}_{\alpha I}$, or

2. $\phi = (2k + 1)\pi$ and there is $n \in \{2, \ldots, N_G\}$ such that $n\phi \equiv 0 \mod 2\pi$ and $g^n \in B^{1/\sqrt{d}}_{\alpha I}$.

Next we define when a matrix $\text{Ad}_g$ is exceptional.
Definition 21. Matrices \( g \in G \) and \( \text{Ad}_g \in SO(g) \) will be called exceptional iff all spectral angles of \( \text{Ad}_g \) are rational multiples of \( \pi \) and at least one of them if an exceptional angle. The spectrum of such \( g \) or \( \text{Ad}_g \) is called an exceptional spectrum.

Corollary 22. If \( g_1, g_2 \) are not exceptional matrices and \( [g_1, g_2] \notin Z(G) \) then \( < g_1, g_2 > \) is infinite.

One can easily deduce from Definition 21 and properties of groups \( SU(d) \) and \( SO(d) \) that if the spectrum of \( \text{Ad}_g \) is exceptional, then all spectral angles of \( g \) must be rational multiples of \( \pi \). Next we ask how many spectra of \( g \) correspond to a given spectrum of \( \text{Ad}_g \). To answer this question we treat groups \( G = SO(d) \) and \( G = SU(d) \) separately. For \( G = SO(d) \) let \( k = \lfloor \frac{d}{2} \rfloor \) and for \( G = SU(d) \) let \( k = d \). We can define maps \( \Phi_{SO(d)} : \mathbb{R}^k \to \mathbb{R}^{k(k-1)/2} \times \mathbb{R}^{k(k-1)/2} \) and \( \Phi_{SU(d)} : \mathbb{R}^k \to \mathbb{R}^{k(k-1)/2} \) corresponding to systems of equations relating \( \phi_j \) and \( \phi_{a,b}, \psi_{a,b} \). For \( \Phi_{SO(d)} \) one easily checks that \( \text{Ker}(\Phi_{SO(d)}) = 0 \) and therefore for \( x \in \Phi_{SO(d)}(\mathbb{R}^k) \) there is only one \( y \in \mathbb{R}^k \) for which \( \Phi_{SO(d)}(y) = x \). In other words, each exceptional spectrum corresponds to the unique spectrum of an element \( g \in SO(d) \). The situation is different for \( SU(d) \). In this case there is an additional condition stemming form determinant that is given by the equation \( \sum_{i=1}^{d} \phi_i = 2m\pi \) and, under assumption that \( \phi_i \in [0, 2\pi) \), \( m \in \{1, \ldots, d-2\} \). Therefore there are \( d-2 \) spectra of \( g \in SU(d) \) that correspond to a given exceptional spectrum of \( \text{Ad}_g \).

Exceptional matrices need a separate treatment which we discuss in this work only for some low dimensional examples. In particular we show that exceptional spectra lead to finite subgroups of \( SU(2) \). The general case is beyond the scope of this paper. The following theorem summarises results of this section.

Theorem 23. Let \( S = \{g_1, g_2, \ldots, g_k\} \subset G \), where \( G = SO(d) \) and \( d \neq 4 \) or \( G = SU(d) \). Assume that there is at least one pair of matrices in \( S \) for which the spectra are not exceptional. Then \( < S > = G \) iff \( C(\text{Ad}_G) = C(\text{Ad}_S) \).

5 Computing \( N_G \)

In this section we find upper bounds for \( N_{SU(d)} \) and \( N_{SO(d)} \) using Dirichlet’s approximation theorem [10][13]:

Theorem 24. For a given real number \( a \) and a positive integer \( N \) there exist integers \( 1 \leq n \leq N \) and \( p \) such that no differs from \( p \) by at most \( \frac{1}{N+1} \), i.e.

\[
|na - p| \leq \frac{1}{N+1}.
\]  

(16)

In Section 5.1 we use Theorem 24 in calculation of \( N_G \) for \( G = SO(3) \) and \( G = SU(2) \) - these are two cases when \( g \in G \) has a one spectral angle. The simultaneous version of Dirichlet’s theorem gives a similar approximation for a collection of real numbers \( \phi_1, \ldots, \phi_k \). We will use it for \( SO(2k+1) \).

Theorem 25. For given real numbers \( a_1, \ldots, a_d \) and a positive integer \( N \) there exist integer \( 1 \leq n \leq N \) and integers \( p_1, \ldots, p_k \) such that

\[
|na_i - p_i| \leq \frac{1}{(N+1)^{1/d}}.
\]  

(17)

For groups \( SO(2k) \) and \( SU(d) \) we need to prove a modified version of Dirichlet’s theorem. To this end for any real number \( x \) and a positive integer \( d \) we define \( \{x\}_k \) to be the difference between \( x \) and the largest \( p + \frac{k}{d} \) that is smaller or equal to \( x \), where \( p \in \mathbb{Z} \), \( k \in \{0, 1, \ldots, d-1\} \). Clearly \( \{x\}_k \in [0, 1) \). For \( x = (x_1, \ldots, x_m) \in \mathbb{R}^m \) we define \( \{x\}_k = (\{x_1\}_k, \ldots, \{x_m\}_k) \). Let \( \mathcal{L}_{m,d} \) be the lattice in \( \mathbb{R}^m \) given by points

\[
(q_1, \ldots, q_m), \left(q_1 + \frac{1}{d}, \ldots, q_m + \frac{1}{d}\right), \ldots, \left(q_1 + \frac{d-1}{d}, \ldots, q_m + \frac{d-1}{d}\right),
\]

where \( q_1, \ldots, q_m \in \mathbb{Z} \). An important property of the lattice \( \mathcal{L}_{m,d} \) is that for any \( p, q \in \mathcal{L}_{m,d} \) we have \( p \pm q \in \mathcal{L}_{m,d} \). As a direct consequence of this property we get the following theorem.

Theorem 26. For \( a = (a_1, \ldots, a_m) \) and positive \( \epsilon < \frac{1}{md} \) there exist: integer \( 1 \leq n \leq \left[ \frac{1}{n_{\text{eu}} \epsilon} \right] \) and a point \( p = (p_1, \ldots, p_m) \in \mathcal{L}_{m,d} \) such that \( \forall i \in \{1, \ldots, m\} \):

\[
|na_i - p_i| < \epsilon.
\]  

(18)
Proof. For a given point \( a = (a_1, \ldots, a_m) \in \mathbb{R}^m \) consider \( dQ^m + 1 \) points:

\[
\{na_0, \{na\}_1, \ldots, \{na\}_{d-1}, \ n \in \{0, \ldots, Q^m\}\}
\] (19)

Next take an \( m \)-dimensional cube \([0, 1)^m\) and divide it into \( dQ^m\) boxes by drawing planes parallel to its faces at distances \( \frac{1}{\sqrt{dQ^m}} \). By Dirichlet’s pigeon hole principle, at least two points from (19) fall to the same box. Let these points be \( \{qa_i\}_1 \) and \( \{qa_j\}_j \), where \( i, j \in \{1, \ldots, d - 1\} \) and \( q_1 < q_2 \). Note that \( q_1 \) cannot be equal to \( q_2 \) as in this case \( \varepsilon > \frac{1}{\sqrt{dQ^m}} \). As the lattice \( L_{m,d} \) is invariant with respect to addition and subtraction of its points we have max\(|\{(q_1 - q_2) a_1\}_k| < \frac{1}{\sqrt{dQ^m}}\) where \( k = j - i \) if \( i < j \) or \( k = d + j - i \) when \( i > j \). The result follows. \[\square\]

5.1 Case of \( SU(2) \) and \( SO(3) \)

Fact 27. \( N_{SO(3)} = 12 \) and \( N_{SU(2)} = 6 \).

Proof. Let \( O \in SO(3) \) and let \([0, 2\pi) \ni \phi = 2\pi \) be its spectral angle. By Theorem 24 for a given \( n \) there are integers \( p \) and \( 1 \leq n \leq N \) such that \( |na - p| \leq \frac{1}{N+1} \). Multiplying this inequality by \( \pi \) yields \( |n\pi/2 - p\pi| \leq \frac{\pi}{N+1} \). Note that (13) simplifies to \( |\sin \frac{\phi}{2}| < \frac{1}{4} \), i.e. for a given \( \phi \) we look for \( n \) such that \( |n\phi/2 - p\phi/2| < \arcsin 1/4 \). Combining these two observations we need to find the smallest \( N \) such that \( \frac{\pi}{N+1} < \arcsin 1/4 \). It is

\[
N = \left\lceil \frac{\pi - \arcsin \frac{1}{4}}{\arcsin \frac{1}{4}} \right\rceil = 12.
\] (20)

Formula (20) gives an upper bound for \( N_{SO(3)} \). Note however that for \( \frac{\phi}{2} = \arcsin \frac{1}{4} \) the smallest \( n \) such that \( |n\arcsin \frac{1}{4} - \pi| < \arcsin \frac{1}{4} \) is exactly 12 (see figure 2(a)), hence \( N_{SO(3)} = 12 \).

![Figure 2: (a) Condition (13) for \( SO(3) \). Black dots corresponds to \( n\arcsin \frac{1}{4} \) and dashed segments are determined by \( |\sin \frac{\phi}{2}| < \frac{1}{4} \), (b) Conditions (12) for \( U \in SU(2) \). Black dots corresponds to \( n\arcsin \frac{1}{4} \) and dashed segments are determined by \( |\sin \frac{\phi}{2}| < \frac{1}{4} \) or \( |\sin \frac{\pi - \phi}{2}| < \frac{1}{4} \).](image)

Let next \( U \in SU(2) \) and \([0, 2\pi) \ni \phi = \alpha \pi \) be its spectral angle. By Theorem 24 for a given \( N \) there are integers \( p \) and \( 1 \leq n \leq N \) such that \( |na - p| \leq \frac{1}{N+1} \). Multiplying this inequality by \( \pi \) yields \( |n\pi/2 - p\pi/2| \leq \frac{\pi}{2(N+1)} \). Note that (13) simplifies to \( |\sin \frac{\phi}{2}| < \frac{1}{4} \) or \( |\sin \frac{\pi - \phi}{2}| < \frac{1}{4} \), i.e. for a given \( \phi \) we look for \( n \) such that \( |n\phi/2 - p\phi/2| < \arcsin 1/4 \). Combining these two observations we need to find the smallest \( N \) such that \( \frac{\pi}{2(N+1)} < \arcsin 1/4 \). It is

\[
N = \left\lceil \frac{\pi}{2} - \arcsin \frac{1}{4} \right\rceil = 6.
\] (21)

Formula (21) gives an upper bound for \( N_{SU(2)} \). Note however that for \( \frac{\phi}{2} = \arcsin \frac{1}{4} \) the smallest \( n \) such that \( |n\arcsin \frac{1}{4} - \frac{\phi}{2}| < \arcsin \frac{1}{4} \) is exactly 6 (see figure 2(b)). Hence \( N_{SU(2)} = 6 \). \[\square\]
**Fact 28.** The values of \(N_{SO(2k+1)}\) and \(N_{SO(2k)}\) are bounded from above in the following way:

\[
N_{SO(2k+1)} < \left(\frac{\pi}{\arcsin \frac{1}{4\sqrt{k}}}\right)^k, \tag{22}
\]

\[
N_{SO(2k)} < \left(\frac{1}{2} \frac{\pi}{\arcsin \frac{1}{4\sqrt{k}}}\right)^k. \tag{23}
\]

**Proof.** The spectral angles of \(O \in SO(d)\) are \(\{\phi_1, -\phi_1, \ldots, \phi_k, -\phi_k\}\) if \(d = 2k\) or \(\{\phi_1, -\phi_1, \ldots, \phi_k, -\phi_k, 0\}\) if \(d = 2k + 1\). We first address the case of \(SO(2k)\). Assume that \(\phi_i = a_i \pi\) for all \(i \in \{1, \ldots, k\}\). The lattice \(\pi \cdot \mathcal{L}_{k,2}\) corresponds exactly to points \(\left(\frac{\phi_1}{2}, \ldots, \frac{\phi_k}{2}\right)\) at which balls \(B_I\) and \(B_{-I}\) given by conditions (14) and (15) are centred. Let us next find the smallest hypercube \([-\frac{\phi_i}{2}, \frac{\phi_i}{2}]\times k\) contained in the ball \(B_I\). By symmetry, its edge length will be the same for balls \(B_{-I}\). To this end one needs minimise \(\sum_i \phi_i^2\) under the condition \(\sum_i \sin^2 \phi_i = \frac{1}{16}\). Calculations with the use of Lagrange multipliers show that the coordinates of the minimizing point are all equal and hence \(k \sin^2 \frac{\phi_i}{2} = \arcsin \frac{1}{16}\). That means \(\frac{\phi_i}{2} = \arcsin \frac{1}{4\sqrt{k}}\) is half of the edge length of the largest hypercube contained in a ball \(B_{\pm I}\). We next apply Theorem 26 to the lattice \(\mathcal{L}_{k,2}\) and the point \(a = (a_1, \ldots, a_k)\) with \(\epsilon = \frac{\arcsin \frac{1}{4\sqrt{k}}}{\pi} < \frac{1}{8}\). As a result we obtain point \(p \in \mathcal{L}_{k,2}\) such that:

\[
|na_i - p_i| < \arcsin \frac{1}{4\sqrt{k}}. \tag{24}
\]

where

\[
n < \left(\frac{\pi}{2(\arcsin \frac{1}{4\sqrt{k}})}\right)^k.
\]

For \(SO(2k+1)\) we can directly apply Theorem 25. Looking at the hypercube that is contained in one of the balls given by conditions (14) and (15) we get the desired result. \(\square\)

**Fact 29.** For \(d \geq 3\) the value of \(N_{SU(d)}\) is bounded from above by

\[
N_{SU(d)} < \left(\frac{1}{d} \frac{2\pi}{\beta_d}\right)^{d-1},
\]

where \(\beta_d\) is such that \((d - 1) \sin^2 \frac{\beta_d}{2} + \sin^2 \frac{(d-1)\beta_d}{2} = \frac{1}{8}\).

**Proof.** For \(U \in SU(d)\) let \(\{\phi_1, \ldots, \phi_d\}\) be the spectral angles of \(U\). Assume that for every \(i \in \{1, \ldots, d-1\}\) we have \(0, 2\pi) \ni \phi_i = a_i \pi\). As \(\sum_i \phi_i = 0 \mod 2\pi\) we can always put \(\phi_d = -\sum_{i=1}^{d-1} \phi_i\). We need to first find the edge length of the largest hypercube \([-\frac{\phi_i}{2}, \frac{\phi_i}{2}]^{(d-1)}\) contained in the ball \(B_I\). By symmetry of condition (13), this length will be the same for other balls. We need to minimise \(\sum_i \phi_i^2\) under the condition \(\sum_{i=1}^{d-1} \sin^2 \phi_i + \sin^2 (\sum_{i=1}^{d-1} \phi_i) = \frac{1}{8}\). Calculations with the use of Lagrange multipliers show that the coordinates of the minimizing point are all equal and hence \(\beta_d\) satisfies:

\[
(d - 1) \sin^2 \frac{\beta_d}{2} + \sin^2 \frac{(d-1)\beta_d}{2} = \frac{1}{8}. \tag{25}
\]

In order to apply Theorem 26 we need to check if \(\frac{\beta_d}{2\pi} < \frac{1}{2d}\). By equation (25) \(\beta_d\) is clearly close to zero and therefore we can assume that \(\sin \frac{\beta_d}{2}\) approximately equals to \(\frac{\beta_d}{2}\). Then it follows that \(\frac{\beta_d}{2\pi} = \frac{1}{2\pi \sqrt{2(d-1)}}\) which is clearly smaller than \(\frac{1}{2d}\). Thus we can apply Theorem 26 to the lattice \(\mathcal{L}_{d-1,d}\) and the point \(a = (a_1, \ldots, a_{d-1})\) with \(\epsilon = \frac{\beta_d}{2\pi} < \frac{1}{2d}\). As a result we obtain point \(p \in \mathcal{L}_{d-1,d}\) such that:

\[
|na_i - p_i| < \frac{\beta_d}{2\pi}, \tag{26}
\]

where

\[
n < \left(\frac{1}{d} \frac{2\pi}{\beta_d}\right)^{d-1}.
\]

The result follows. \(\square\)
For $d = 3$ we obtain $\frac{\phi_3}{2} = \arctan \sqrt{\frac{\phi_1 \phi_2}{\sqrt{1 + \phi_1^2 \phi_2^2}}}$. and $N_{SU(3)} < 154$. On the other hand numerical calculations yield $N_{SU(3)} = 49$. For orthogonal groups we have that numerical calculations yield $N_{SO(5)} = 172$ and $N_{SO(4)} = 86$, where the bounds given by (22) and (24) are $N_{SO(5)} < 312$ and $N_{SO(4)} < 151$ respectively. The difference between the bounds and values calculated numerically reflects the obvious fact that the considered hypercubes are rather brutal approximations of the balls $B_{\alpha I}$. However, we stress that the choice of hypercubes we made is the most optimal from the perspective of Dirichlet’s theorems. Let us also note that the upper bound for $N_G$ seems to be more accurate for $SO(4)$ than for $SU(3)$. We believe this stems from the fact that the ‘square-ball’ area ratio is smaller for $SU(3)$ than for $SO(4)$ (see figure 3). The way how these ratios should be incorporated into formulas for the upper bound on $N_G$ is left as an open problem. We suppose this should be done by introducing some additional factor that depends on the square-ball ratio.

Figure 3: The smallest hypercubes contained in the balls $B_{\alpha I}^{1/\sqrt{2}}$ for $SO(4)$ and $SU(3)$ respectively.

6 Universality for $SU(2)$ and $SO(3)$

In this section we discuss universality of gates in case when $G = SU(2)$ or $G = SO(3)$. For these groups adjoint matrices $Ad_g$ have only one spectral angle which means that the set of exceptional spectra is finite. As we show, exceptional spectra lead to finite subgroups of $SU(2)$ and $SO(3)$.

6.1 $SU(2)$ and $SO(3)$ - review of useful properties

In the following we recall useful facts about groups $SO(3)$ and $SU(2)$. In particular we introduce their parameterizations and briefly discuss the covering homomorphism given by the adjoint representation.

Commutation relations for Lie algebras of the considered groups are as follows:

\[\begin{align*}
su(2) & : [X, Y] = 2Z, \quad [X, Z] = -2Y, \quad [Y, Z] = 2X, \\
so(3) & : [X_{23}, X_{13}] = -X_{12}, \quad [X_{23}, X_{12}] = X_{13}, \quad [X_{13}, X_{12}] = X_{23}.
\end{align*}\] (27) (28)

Lie algebras $su(2)$ and $so(3)$ are isomorphic through the adjoint representation $ad : su(2) \to so(3)$. The isomorphism is established by $X \mapsto ad_X = -2X_{23}$, $Y \mapsto ad_Y = 2X_{13}$, $Z \mapsto ad_Z = -2X_{12}$.

Elements of groups $SU(2)$ and $SO(3)$ can be expressed using exponential map. By Cayley-Hamilton theorem we have:

\[\begin{align*}
SU(2) & : U(\phi, \vec{k}) = e^{\phi \cdot ad(\vec{k})} = e^{\phi(k_x X + k_y Y + k_z Z)} = \cos \phi I + \sin \phi (k_x X + k_y Y + k_z Z), \\
SO(3) & : O(\phi, \vec{k}) = e^{\phi \cdot ad(\vec{k})} = e^{\phi(-k_x X_{23} + k_y X_{13} - k_z X_{12})} = I + \sin \phi (-k_x X_{23} + k_y X_{13} - k_z X_{12}) + 2 \sin^2 \frac{\phi}{2} (-k_x X_{23} + k_y X_{13} - k_z X_{12})^2 ,
\end{align*}\] (29) (30)

\footnote{To simplify notation for Lie algebra $su(2)$ we skip lower indices for basis elements, e.g. $X := X_{12} \in su(2)$.}
where $\vec{k} = [k_x, k_y, k_z] \in \mathbb{R}^3$ is a rotation axis and $k_x^2 + k_y^2 + k_z^2 = 1$. Groups $SU(2)$ and $SO(3)$ are related by the covering homomorphism $\text{Ad} : SU(2) \to SO(3)$ given by $\text{Ad}_A = e^{ad_A}$, where $A \in \mathfrak{su}(2)$ and $\text{Ad} : U(\phi, \vec{k}) \mapsto O(2\phi, \vec{k})$. $\text{Ad}$ is in this case double covering. Using (29) we can easily calculate the product $U(\gamma, \vec{k}_{12}) = U(\phi_1, \vec{k}_1)U(\phi_2, \vec{k}_2)$, where:

$$\cos \gamma = \cos \phi_1 \cos \phi_2 - \sin \phi_1 \sin \phi_2 \vec{k}_1 \cdot \vec{k}_2,$$

$$\vec{k}_{12} = \frac{1}{\sin \gamma} \left( \vec{k}_1 \sin \phi_1 \cos \phi_2 + \vec{k}_2 \sin \phi_2 \cos \phi_1 + \vec{k}_1 \times \vec{k}_2 \sin \phi_1 \sin \phi_2 \right).$$

Making use of (31) one checks that two $SU(2)$ matrices $U(\phi_1, \vec{k}_1), U(\phi_2, \vec{k}_2) \notin \{I, -I\}$ commute iff the axes $\vec{k}_1$ and $\vec{k}_2$ are parallel, that is $[u(\vec{k}_1), u(\vec{k}_2)] = 0$. Similarly, they anticommute iff the axes $\vec{k}_1$ and $\vec{k}_2$ are orthogonal and rotation angles are $\phi_1 = \pm \pi/2 = \phi_2$. As for matrices from $SO(3)$, recall that they cannot anticommute. In order to check when they commute we note, that commuting and anticommuting $SU(2)$ matrices satisfy the identity $U_1 U_2 U_1^{-1} U_2^{-1} = \pm I$. But $\text{Ad}_\pm I = I$ and therefore $O(\phi_1, \vec{k}_1)$ commutes with $O(\phi_2, \vec{k}_2)$ iff either axes $\vec{k}_1$ and $\vec{k}_2$ are parallel or $\vec{k}_1 \perp \vec{k}_2$ and $\phi_1 = \pm \pi = \phi_2$.

**Fact 30.** Assume that $U(\phi_1, \vec{k}_1), U(\phi_2, \vec{k}_2) \in SU(2)$ and $O(\phi_1, \vec{k}_1), O(\phi_2, \vec{k}_2) \in SO(3)$ are two pairs of noncommuting matrices. Then for any $k, l \in \mathbb{N}$ if $k\phi_1$ and $l\phi_2$ are not integer or half integer multiples of $\pi$, we have $[U^k(\phi_1, \vec{k}_1), U^l(\phi_2, \vec{k}_2)] \notin Z(SU(2))$. Similarly, for any $k, l \in \mathbb{N}$ if $k\phi_1$ and $l\phi_2$ are not integer multiples of $\pi$ then $[O^k(\phi_1, \vec{k}_1), O^l(\phi_2, \vec{k}_2)] \notin Z(SO(3))$.

**Proof.** Follows directly from the conditions for commuting and anticommuting matrices in $SU(2)$ and $SO(3)$ combined with $U^n(\phi, \vec{k}) = U(n\phi, \vec{k}), O^n(\phi, \vec{k}) = O(n\phi, \vec{k})$.

### 6.2 Exceptional spectra and spaces $\mathcal{C}(\text{Ad}_G)$ for $SU(2)$ and $SO(3)$

Recall that adjoint representation maps $U(\phi, \vec{k}) \in SU(2)$ to $O(2\phi, \vec{k}_1) \in SO(3)$, i.e. $\text{Ad}_{U(\phi, \vec{k})} = O(2\phi, \vec{k})$. Matrix $\text{Ad}_{U(\phi, \vec{k})}$ has only one spectral angle, that is $2\phi$. By Definitions 20 and 21 the spectrum of $\text{Ad}_{U(\phi, \vec{k})}$ is exceptional if $2n\phi = 0 \mod \pi$ for some $n \leq N_{SU(2)}$, and $O(n\phi, \vec{k}) \in B^{1/\sqrt{2}}_I$. One checks that it happens exactly when $e^{i\phi}$ is a root of $1$ or $-1$ of order $n \in \{1, \ldots, N_{SU(2)}\}$.

For $O(\phi, \vec{k}) \in SO(3)$ the image of the adjoint representation is $SO(3)$ and the spectral angle of $\text{Ad}_{O(\phi, \vec{k})}$ is equal to $\phi$. Therefore $\phi$ is an exceptional angle if $n\phi = 0 \mod \pi$ for some $n \leq N_{SO(3)}$ and $O(n\phi, \vec{k}) \in B^{1/\sqrt{2}}_I$. One checks that it happens exactly when $e^{i\phi}$ is a root of unity of order $1 \leq n \leq N_{SO(3)}$. We can easily compute the number of exceptional spectra for $SU(2)$ and $SO(3)$ using the Euler totient function $\varphi(n)$. Note that the roots of $-1$ of order $n$ are the roots of unity of order $2n$.

Let us denote the sets of exceptional angles for $SU(2)$ and $SO(3)$ by $\mathcal{L}_{SU(2)}$ and $\mathcal{L}_{SO(3)}$ respectively. We have:

$$|\mathcal{L}_{SU(2)}| = 6 \varphi(n) + 6 \varphi(2n) = 24,$$

$$|\mathcal{L}_{SO(3)}| = 12 \varphi(n) = 46.$$  

The elements of sets $\mathcal{L}_G$ are of the form $\alpha \in \mathbb{C}_G$, where

$$\mathcal{L}_{SU(2)} = \left\{ \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{1}{4}, \frac{3}{4}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \frac{1}{4}, \frac{3}{4}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \frac{1}{4}, \frac{3}{4}, \frac{1}{2}, \frac{1}{2} \right\},$$

$$\mathcal{L}_{SO(3)} = \mathcal{L}_{SU(2)} \cup \left\{ \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \frac{1}{2}, \frac{1}{2} \right\}.$$  

We next discuss the conditions when the space $\mathcal{C}(\text{Ad}_{U(\phi_1, \vec{k}_1)}, \text{Ad}_{U(\phi_2, \vec{k}_2)})$ is different than $\mathcal{C}(\text{ad}_{u(\vec{k}_1)}, \text{ad}_{u(\vec{k}_2)})$. First, we note that elements $\{u(\vec{k}_1), u(\vec{k}_2)\}$ generate Lie algebra $\mathfrak{su}(2)$ iff $[u(\vec{k}_1), u(\vec{k}_2)] \neq 0$. In this case by Lemma 7 the solution set $\mathcal{C}(\text{ad}_{u(\vec{k}_1)}, \text{ad}_{u(\vec{k}_2)}) = \{\lambda I\}$. By Fact 31 the space $\mathcal{C}(\text{Ad}_{U(\phi_1, \vec{k}_1)}, \text{Ad}_{U(\phi_2, \vec{k}_2)})$ can be different than $\mathcal{C}(\text{ad}_{u(\vec{k}_1)}, \text{ad}_{u(\vec{k}_2)})$ if at least one $\phi_i$ is equal to $\frac{\kappa \pi}{2}$. In the following we give exact conditions when it happens.

**Fact 31.** Assume that $[u(\vec{k}_1), u(\vec{k}_2)] \neq 0$. The space $\mathcal{C}(\text{Ad}_{U(\phi_1, \vec{k}_1)}, \text{Ad}_{U(\phi_2, \vec{k}_2)})$ is larger than $\{\lambda I : \lambda \in \mathbb{R}\}$ if and only if: (1) $\phi_1, \phi_2 = \frac{k \pi}{2}$, (2) one of $\phi_i$’s is equal to $\frac{k \pi}{2}$ and $\vec{k}_1 \perp \vec{k}_2$, where $k$ is an odd integer.
Lemma 32. Assume that \( SU \) with the rotations \( \text{Ad} \) restriction to arbitrary endomorphism gives only three cases: when angles \( \phi \) and about the axis \( \vec{k} \) commutes with the rotations \( O(\pm \pi, \vec{k}_1) \) and \( O(\pm \pi, \vec{k}_2) \) are commutative.

6.3 Universal \( SU(2) \) gates

In this section we consider two matrices \( U(\phi_1, \vec{k}_1) \), \( U(\phi_2, \vec{k}_2) \) and ask when they generate \( SU(2) \). We treat separately three cases: when angles \( \phi_1 \) and \( \phi_2 \) are both non-exceptional, exactly one is exceptional and both are exceptional.

6.3.1 Two non-exceptional angles

We assume that \( \phi_1, \phi_2 \notin L_{SU(2)} \) and \( U(\phi_1, \vec{k}_1), U(\phi_2, \vec{k}_2) \) do not commute. Comparing the list of exceptional angles \( \vec{k} \) with Fact 31 we arrive at \( C(\text{Ad}_{U(\phi_1, \vec{k}_1)}, \text{Ad}_{U(\phi_2, \vec{k}_2)}) = \{ \lambda I : \lambda \in \mathbb{R} \} \). Since angles are non exceptional, we also know that \( < U(\phi_1, \vec{k}_1), U(\phi_2, \vec{k}_2) > \) is an infinite subgroup of \( SU(2) \). Therefore by Lemma 7

Lemma 32. Assume that \( U(\phi_1, \vec{k}_1) \) and \( U(\phi_2, \vec{k}_2) \) do not commute and \( \phi_1, \phi_2 \notin L_{SU(2)} \). Then \( < U(\phi_1, \vec{k}_1), U(\phi_2, \vec{k}_2) > = SU(2) \).
6.3.2 One non-exceptional angle

We assume that $\phi_1 \notin \mathcal{L}_{SU(2)}$, $\phi_2 \in \mathcal{L}_{SU(2)}$ and $U(\phi_1, \vec{k}_1)$, $U(\phi_2, \vec{k}_2)$ do not commute. By Fact 51 the space

$$\mathcal{C}(\text{Ad}_{U(\phi_1, \vec{k}_1)}, \text{Ad}_{U(\phi_2, \vec{k}_2)}) \neq \{1\},$$

if and only if $\phi_2 = \frac{m\pi}{2}$, where $m$ is an odd integer and $\vec{k}_1 \perp \vec{k}_2$. In the following we show that this is the only case when $U(\phi_1, \vec{k}_1), U(\phi_2, \vec{k}_2) \neq SU(2)$. Moreover we identify the resulting groups.

The situation when $\phi_1 \notin \mathcal{L}_{SU(2)}$, $\phi_2 \in \mathcal{L}_{SU(2)}$ reduces to the one considered in Section 6.3.1. To prove this consider two matrices $U(\phi_1, \vec{k}_1)$ and $U(\gamma, \vec{k}) = U(\phi_2, \vec{k}_2)U(\phi_1, \vec{k}_1)^{-1}(\phi_2, \vec{k}_2)$. Note that $\gamma = \pm \phi_1 \notin \mathcal{L}_{SU(2)}$. We need to only show that $\vec{k} \neq \vec{k}_1$. Assume on the contrary that $\vec{k} = \vec{k}_1$. Then $U(\phi_2, \vec{k}_2)U(\phi_1, \vec{k}_1)^{-1}(\phi_2, \vec{k}_2) = U(\pm \phi_1, \vec{k}_1)$. For $\gamma = \phi_1$ the matrices $U(\phi_1, \vec{k}_1)$, $U(\phi_2, \vec{k}_2)$ commute, which is a contradiction. When $\gamma = -\phi_1$ we have $U(\phi_2, \vec{k}_2)U(\phi_1, \vec{k}_1) = U(\phi_1, \vec{k}_1)U(\phi_2, \vec{k}_2)$. Let $U(\gamma_{21}, \vec{k}_{21}) = U(\phi_2, \vec{k}_2)U(\phi_1, \vec{k}_1)$ and $U(\gamma_{12}, \vec{k}_{12}) = U(-\phi_1, \vec{k}_1)U(\phi_2, \vec{k}_2)$.

$$U(\gamma_{21}, \vec{k}_{21}) = I \cos \gamma_{21} + \left(\vec{k}_2 \sin \phi_2 \cos \phi_1 + \vec{k}_1 \sin \phi_1 \cos \phi_2 - \vec{k}_2 \times \vec{k}_1 \sin \phi_2 \sin \phi_1 \right) \vec{X},$$

$$U(\gamma_{12}, \vec{k}_{12}) = I \cos \gamma_{12} + \left(-\vec{k}_1 \sin \phi_1 \cos \phi_2 + \vec{k}_2 \sin \phi_2 \cos \phi_1 + \vec{k}_1 \times \vec{k}_2 \sin \phi_1 \sin \phi_2 \right) \vec{X},$$

where

$$\cos \gamma_{21} = \cos \phi_2 \cos \phi_1 - \sin \phi_2 \sin \phi_1 \vec{k}_1 \cdot \vec{k}_2, \; \cos \gamma_{12} = \cos \phi_1 \cos \phi_2 + \sin \phi_1 \sin \phi_2 \vec{k}_1 \cdot \vec{k}_2.$$

We obtain the following two conditions for equality $U(\gamma_{21}, \vec{k}_{21}) = U(\gamma_{12}, \vec{k}_{12})$

$$\sin \phi_2 \sin \phi_1 \vec{k}_1 \cdot \vec{k}_2 = 0, \text{ and } \sin \phi_1 \cos \phi_2 = 0.$$ (42)

This is possible iff $\phi_2 = \pm \frac{\pi}{2}$ and $\vec{k}_1 \cdot \vec{k}_2 = 0$ (or $\phi_1 = k\pi$ but this is excluded). Therefore $\cos \gamma_{21} = \cos \gamma_{12} = 0$. Note that under assumptions $\phi_2 = \pm \frac{\pi}{2}$ and $\vec{k}_1, \vec{k}_2 = 0$ and identification: $b := U(\phi_1, \vec{k}_1)$, $x := U(\frac{\pi}{2}, \vec{k}_2)$, where $b$ is of finite order we get

$$H = \langle b, x | x^4 = I, b^n = I, xbx^{-1} = b^{-1} \rangle.$$ (43)

As $H$ contains $-I$ we have $(-b)^n = -I$ for $n$ odd. Let $a = -b$ then

$$H = \langle a, x | x^4 = I, a^{2n} = I, xax^{-1} = a^{-1} \rangle,$$ (44)

which is a dicyclic group of order $4n$ (it is a central extension of the dihedral group of order $2n$). In case when $a$ is of infinite order, after closure, we obtain a group consisting of two connected components. The first one is a one parameter group $U(t, \vec{k}_1)$ generated by $U(\phi_1, \vec{k}_1)$ and the second one is $U(\frac{\pi}{2}, \vec{k}_2)U(t, \vec{k}_1)$. In fact $U(\frac{\pi}{2}, \vec{k}_2)U(t, \vec{k}_1)$ is a normalizer of the group $U(t, \vec{k}_1)$ in $SU(2)$.

Lemma 33. Assume that $U(\phi_1, \vec{k}_1)$ and $U(\phi_2, \vec{k}_2)$ do not commute, $\phi_1 \notin \mathcal{L}_{SU(2)}$, $\vec{k}_1 \cdot \vec{k}_2 \neq 0$ or $\phi_2 \neq \pm \frac{\pi}{2}$, then they generate $SU(2)$. If $\vec{k}_1 \cdot \vec{k}_2 = 0$ and $\phi_2 = \pm \frac{\pi}{2}$ they generate 1) dicyclic group of order $4n$

$$n = \text{max(order}U(\phi_1, \vec{k}_1), \text{order}U(\phi_1 + \pi, \vec{k}_1)),$$ (45)

when order$U(\phi_1, \vec{k}_1) < \infty$ and 2) the normalizer of the group generated by $U(\phi_1, \vec{k}_1)$ in $SU(2)$ if order$U(\phi_1, \vec{k}_1) = \infty$.

6.3.3 Finite subgroups of $SU(2)$

In this section we briefly describe finite subgroups of $SU(2)$. We will derive them using the approach presented in the next section. In particular we will show that every finite subgroup of $SU(2)$ can be generated by $U(\phi_1, \vec{k}_1), U(\phi_2, \vec{k}_2)$ with some special $\phi_1, \phi_2 \in \mathcal{L}_{SU(2)}$ and $\vec{k}_1 \cdot \vec{k}_2$.

By the covering homomorphism $SU(2) \to SO(3)$ the finite nonabelian subgroups of $SU(2)$ can be regarded as central extensions of the finite nonabelian subgroups of $SO(3)$. The later are defined in terms of so called von Dyck groups.
Definition 34. The von Dyck group \((l,m,n)\) is a finite group with the following presentation:

\[
(l,m,n) = \{a,b,c|a^l = b^m = c^n = abc = e\},
\]

(46)

where \(e\) is the identity element of the group.

Nonabelian finite subgroups of \(SO(3)\) are given by \((2,2,n)\), \((2,3,3)\), \((2,3,4)\) and \((2,3,5)\), where \(n \geq 3\). On the other hand finite nonabelian subgroups of \(SU(2)\), denoted by \((l,m,n)\) are central extensions of these groups by an element of the order two. The extension has the structure of the Cartesian product

\[(l,m,n) = (l,m,n) \times \mathbb{Z}_2.\]

(47)

The following list contains all finite nonabelian subgroups of \(SU(2)\):

- Dicyclic group \(\langle 2,2,n \rangle = (2,2,n) \times \mathbb{Z}_2\) is the central extension of the dihedral group \((2,2,n)\). The group \((2,2,n)\) is generated by two rotations by \(\pm \pi\) about axes \(\vec{k}_1\) and \(\vec{k}_2\) separated by an angle \(\frac{\pi}{n}\). Their product is a rotation along \(\vec{k}_1 \times \vec{k}_2\) by \(\frac{2\pi}{n}\). The spectral angles for elements in \((2,2,n) \subset SU(2)\) are therefore \(\frac{2\pi}{m}\) and \(\frac{2\pi}{n}\) whereas the possible angles between axes are equal \(\{\frac{\pi}{2}, \frac{\pi}{2}\}\).

- Binary tetrahedral group \(\langle 2,3,3 \rangle = (2,3,3) \times \mathbb{Z}_2\) is the central extension of the tetrahedral group \((2,3,3) \simeq A_4\). The group \((2,3,3)\) is a symmetry group of the regular tetrahedron and consists of rotations by \(\frac{2k\pi}{3}\) about axes \(\vec{k}_1, \vec{k}_2, \vec{k}_3\) and \(\vec{k}_4\) such that \(\vec{k}_i \cdot \vec{k}_j = -\frac{1}{3}\) and of rotations by \(k\pi\) about axes \(\vec{l}_1, \vec{l}_2\) and \(\vec{l}_3\) such that \(\vec{l}_i \cdot \vec{l}_j = 0\) and \(\vec{k}_i \cdot \vec{l}_j = \pm \frac{\sqrt{3}}{3}\). The spectral angles for elements in \((2,3,3)\) are therefore: \(\frac{2\pi}{3}\) and \(\frac{2\pi}{3}\). The axes and angles between them are as in \((2,3,3)\) albeit \(\vec{k}_i \cdot \vec{k}_j = \pm \frac{1}{3}\).

- Binary octahedral group \(\langle 2,3,4 \rangle = (2,3,4) \times \mathbb{Z}_2\) is a central extension of the octahedral group \((2,3,4) \simeq S_4\). The group \((2,3,4)\) is a symmetry group of the regular octahedron (or, equivalently, of a cube) and consists of rotations by \(k\pi\) and \(\frac{2\pi}{3}\) about axes \(\vec{k}_1, \vec{k}_2, \vec{k}_3\) and \(\vec{k}_4\) such that \(\vec{k}_i \cdot \vec{k}_j = 0\), of rotations by \(k\pi\) about the axes \(\vec{l}_1, \ldots, \vec{l}_6\) for which \(\vec{l}_j \cdot \vec{l}_j = 0\) and of rotations by \(\frac{2k\pi}{3}\) about axes \(\vec{v}_1, \vec{v}_2, \vec{v}_3\) and \(\vec{v}_4\) such that \(\vec{v}_i \cdot \vec{v}_j = \pm \frac{1}{3}\). The angles between the axes corresponding to different rotations are the following: \(\vec{k}_i \cdot \vec{l}_j = \pm \frac{1}{\sqrt{3}}, \vec{k}_i \vec{v}_j = \pm \frac{2}{\sqrt{3}}\) and \(\vec{l}_i \vec{v}_j = \pm \frac{1}{\sqrt{3}}\). The spectral angles for elements in \((2,3,2)\) are therefore: \(\frac{\pi}{3}, \frac{2\pi}{3}\) and \(\frac{2\pi}{3}\). The axes and angles between them are as in \((2,3,4)\).

- Binary icosahedral group \(\langle 2,3,5 \rangle = (2,3,5) \times \mathbb{Z}_2\) is a central extension of the symmetry group of a regular icosahedron (or, equivalently, a regular dodecahedron) \((2,3,5) \simeq A_5\). The group \((2,3,2)\) consists of rotations by \(\frac{2\pi}{5}\) with the angles between rotation axes \(\vec{k}_i \cdot \vec{k}_j \in \{\pm \frac{1}{5}, \pm \frac{2\pi}{5}\}\), of rotations by \(\frac{2\pi}{5}\) with the angles between rotation axes \(\vec{l}_i, \ldots, \vec{l}_{10}\) takes values \(\vec{l}_i \cdot \vec{l}_j = \pm \frac{2\pi}{5}\) and of rotations by \(\frac{2\pi}{5}\) and \(\frac{2\pi}{5}\) where \(k\) is an odd number and the angle between rotation axes \(\vec{v}_i, \vec{v}_j\) is equal \(\vec{v}_i \cdot \vec{v}_j = \pm \frac{1}{\sqrt{5}}\). The angles between the axes \(\vec{l}_i, \vec{l}_j\) and \(\vec{k}_i, \vec{k}_j\) takes values \(\vec{l}_i \cdot \vec{l}_j \in \{\pm 0.795, \pm 0.188\}\) and \(\vec{k}_i \cdot \vec{k}_j \in \{\pm 0.525, \pm 0.851\}\) respectively.

### 6.3.4 Two exceptional angles

Let \(\phi_1, \phi_2 \in L_{SU(2)}\) and \(S = \{U(\phi_1, \vec{k}_1), U(\phi_2, \vec{k}_2)\} \subset SU(2)\). We first note that if all products of any length of matrices \(U(\phi_1, \vec{k}_1)\) and \(U(\phi_2, \vec{k}_2)\) have spectra in \(L_{SU(2)}\), the group \(S\) must be finite. This follows from the Schur’s solution of the Burnside problem (see Lemma 36.2 of [9]) which says that a finitely generated matrix group \(\Gamma\) over \(\mathbb{C}\) that is periodic (for every element in \(g \in \Gamma\) there is integer \(n \in \mathbb{N}\) such that \(g^n = I\)) must be finite. On the other hand, if some product of matrices \(U(\phi_1, \vec{k}_1)\) and \(U(\phi_2, \vec{k}_2)\) has non-exceptional spectrum and \(C(Ad_{U(\phi_1, \vec{k}_1)}, Ad_{U(\phi_2, \vec{k}_2)}) = \{\lambda I\}\) the resulting group must be \(SU(2)\). Thus in order to find the matrices that do not generate \(SU(2)\) we use the following algorithm:

1. Take all possible pairs of angles \(\phi_1, \phi_2 \in L_{SU(2)} \setminus \{\frac{\pi}{2}, \frac{3\pi}{2}\}\). Then compute \(\vec{k}_1 \cdot \vec{k}_2\) as

\[
\vec{k}_1 \cdot \vec{k}_2 = \frac{\cos \phi_1 \cos \phi_2 - \cos \gamma}{\sin \phi_1 \sin \phi_2}
\]

for each \(\gamma \in L_{SU(2)}\).

\(^{4}\)The case when both \(\phi_i\)'s are odd multiples of \(\frac{\pi}{2}\) was treated in lemma [9].
2. If $|\vec{k}_1 \cdot \vec{k}_2| > 1$, $\gamma$ cannot be a spectral angle for given $\phi_1, \phi_2, \vec{k}_1 \cdot \vec{k}_2$ and the generated group is $SU(2)$. If $|\vec{k}_1 \cdot \vec{k}_2| = 1$ the matrices commute. If $\vec{k}_1 \cdot \vec{k}_2 = 0$ and at least one of $\phi_i$’s is $\frac{\pi}{4}$ or $\frac{3\pi}{4}$ then by lemma [33] we cannot get $SU(2)$ and we obtain the group $<2, 2, n>$, where $n \in \{2, 3, 4, 5, 6\}$. The only remaining possibility for $\vec{k}_1 \cdot \vec{k}_2 = 0$ is both $\phi_i$’s belong to $\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\}$. The group generated in this case is $<2, 3, 4>$.

3. If $|\vec{k}_1 \cdot \vec{k}_2| < 1$, compute $\gamma_{nm}$ for all possible combinations of the form

$$U(n\phi_1, \vec{k}_1)U(m\phi_2, \vec{k}_2) = U(\gamma_{nm}, \vec{k}_{12}), \quad n, m \in \{1, \ldots, 6\}.$$  

If for some $n, m$, $\gamma_{nm} \not\in L_{SU(2)}$, then by lemma matrices $SU(\gamma, \vec{k}_{12})$ and $U(\phi_1, \vec{k}_1)$ or $U(\phi_2, \vec{k}_2)$ generate $SU(2)$.

4. If for all possible $m, n = \{1, \ldots, 6\}$ an exceptional matrix $U(\gamma_{nm}, \vec{k}_{12})$ is obtained, then consider the following compositions

$$\forall_{n_1, m_1, n_2, m_2} U(\gamma_{n_1 m_1}, \vec{k}_{12})U(\gamma_{n_2 m_2}, \vec{k}_{12}^2) = U(\gamma_{n_1 n_2, m_1 m_2}, \vec{k}_{12}^2), \quad (49)$$

where $\gamma_{n_1 n_2, m_1 m_2}, \vec{k}_{12}^2$ are given by [31]. Again the pairs $\{U(\phi_1, \vec{k}_1), U(\phi_2, \vec{k}_2)\}$ giving a non-exceptional matrix $U(\gamma_{n_1 n_2, m_1 m_2}, \vec{k}_{12}^2)$ generate $SU(2)$.

5. It turns to that after the step 4 the remaining pairs of matrices generate finite subgroups of $SU(2)$.

To show the efficiency of this algorithm we give a number of pairs of generators rejected after each step. One easily finds that the total number of possible pairs $\{U(\phi_1, \vec{k}_1), U(\phi_2, \vec{k}_2)\}$ is $(|L_{SU(2)}| - 4)^2 \cdot |L_{SU(2)}| = 96000$, where $|L_{SU(2)}| = 24$ and $\phi_1, \phi_2 \notin \{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\}$. Numerical computations show that 54.39% of initial pairs $\{U(\phi_1, \vec{k}_1), U(\phi_2, \vec{k}_2)\}$ do not satisfy the condition $|\vec{k}_1 \cdot \vec{k}_2| < 1$ and are rejected after the first step. Next, approximately 70.43% of the remaining $\{U(\phi_1, \vec{k}_1), U(\phi_2, \vec{k}_2)\}$ is rejected in the second step, i.e. their $U(\gamma_{nm}, \vec{k}_{12})$ is not an exceptional matrix for some $n, m \in \{1, \ldots, 6\}$. Finally we have obtained that 96.2% of all considered pairs $\{U(\phi_1, \vec{k}_1), U(\phi_2, \vec{k}_2)\}$ give a matrix with an non-exceptional spectrum. The products of the remaining pairs $\{U(\phi_1, \vec{k}_1), U(\phi_2, \vec{k}_2)\}$ give all matrices with exceptional spectral angles and generate a finite subgroup of $SU(2)$. We have listed all the generators of such subgroups in Tables 1 and 2. The following theorem summarises the obtained results.

**Theorem 35.** 2-mode gates $U(\phi_1, \vec{k}_1)$ and $U(\phi_2, \vec{k}_2)$ are universal on 2-modes if they do not commute and do not satisfy one of the following conditions:

1. $\phi_1, \phi_2 = \frac{k\pi}{2}$. In this case $U(\phi_1, \vec{k}_1)$ and $U(\phi_2, \vec{k}_2)$ generate a) the dihedral group $<2, 2, n>$ if the angle between $\vec{k}_1$ and $\vec{k}_2$ is equal to a rational multiple of $\pi$ and b) an infinite disconnected group if the angle between $\vec{k}_1$ and $\vec{k}_2$ is an irrational multiple of $\pi$.

2. If $\phi_1 = \frac{\pi}{4}$ or $\phi_2 = \frac{\pi}{2}$ and $\vec{k}_1 \perp \vec{k}_2$. In this case either dihedral or infinite disconnected group is generated.

3. $U(\phi_1, \vec{k}_1)$ and $U(\phi_2, \vec{k}_2)$ belong to Tables 1 or 2. In this case they generate $<2, 3, 3>, <2, 3, 4>, <2, 3, 5>$ which are finite nonabelian subgroups of $SU(2)$.

## 7 Universality of 2-mode beamsplitters

In this section we use our approach in the problem of the universality of a single gate that belong to $SO(2)$ or $SU(2)$ and acts on a $d$-dimensional space, where $d > 2$. More precisely, we consider Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_d$, where $\mathcal{H}_k \simeq \mathbb{C}$, $d > 2$. Next we take a matrix $B \in SU(2)$ or $B \in SU(2)$. This matrix will be referred to as a 2-mode beamsplitter. We assume that we can permute modes and therefore we have access to matrices $B$ and $B^\sigma = \sigma^i B \sigma_j$, where $\sigma = X$ is the permutation matrix. Next, we define matrices $B_{ij}$ or $B^\sigma_{ij}$ to be the matrices that act on a 2-dimensional subspace $\mathcal{H}_i \oplus \mathcal{H}_j \subset \mathcal{H}$ as $B$ or $B^\sigma$ respectively and on the other components of $\mathcal{H}$ as identity. This way we obtain the set of $2^d$ matrices $\mathcal{S}_d = \{B_{ij}, B^\sigma_{ij} : i < j, i, j \in \{1, \ldots, d\}\}$ in $SU(d)$ or $SO(d)$ respectively. Let us denote by $\mathcal{X}_d = \{b_{ij}, b^\sigma_{ij} \}$ the set of corresponding Lie algebra elements $B_{ij} = e^{b_{ij}}$, $B^\sigma_{ij} = e^{b^\sigma_{ij}}$. Our goal is to find out when $\mathcal{S}_d$ is universal, i.e. when $<\mathcal{S}_d >= SO(d)$ or $<\mathcal{S}_d >= SU(d)$. In particular we focus on showing, for which $B$ the set $\mathcal{S}_3$ is universal. It is known that for such $B$ also any set $\mathcal{S}_d$ with $d > 3$ will be universal (see [20] for two alternative proofs).
| No. | φ₁ | φ₂ | k₁ ⋅ k₂ | γ | (l, m, n) | No. | φ₁ | φ₂ | k₁ ⋅ k₂ | γ | (l, m, n) |
|-----|----|----|---------|---|----------|-----|----|----|---------|---|----------|
| 1   | ±π | ±π | (-1, 1) | ±π | (2, 2, n) | 2   | ±π | ±π | 0 | ±π | (2, 2, n) |
| 3   | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) | 4   | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) |
| 5   | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) | 6   | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) |
| 3   | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) | 8   | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) |
| 5   | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) | 6   | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) |
| 7   | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) | 8   | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) |
| 9   | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) | 10  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) |
| 11  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) | 12  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) |
| 13  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) | 14  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) |
| 15  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) | 16  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) |
| 17  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) | 18  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) |
| 19  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) | 20  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) |
| 21  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) | 22  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) |
| 23  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) | 24  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) |
| 25  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) | 26  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) |
| 27  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) | 28  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) |
| 29  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) | 30  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) |
| 31  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) | 32  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) |
| 33  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) | 34  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) |
| 35  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) | 36  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) |
| 37  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) | 38  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) |
| 39  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) | 40  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) |
| 41  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) | 42  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) |
| 43  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) | 44  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) |
| 45  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) | 46  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) |
| 47  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) | 48  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) |
| 49  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) | 50  | ±π/3 | ±π/3 | ±π/3 | ±π/3 | (2, 3, 3) |

Table 1: Generators of finite subgroups of SU(2): (2, 2, n), (2, 3, 3), (2, 3, 4), (2, 3, 5).

### 7.1 Spaces \( \mathcal{C}(\text{Ad}_{S_3}) \) and \( \mathcal{C}(\text{ad}_{X_3}) \)

In this section we characterise when \( \mathcal{C}(\text{Ad}_{S_3}) = \{\lambda I\} \) for both orthogonal and unitary beamsplitters. Our strategy here is to first check when \( \mathcal{C}(\text{ad}_{X_3}) = \lambda I \). This can be done relatively easy. Then we use Facts 12 and 13 to find \( \mathcal{C}(\text{Ad}_{S_3}) \).

#### 7.1.1 The case of orthogonal group

Let \( B \in SO(2) \) be a rotation matrix by an angle \( \phi \neq 0 \mod 2\pi \). Making use of the notation introduced in Section 6 we have

\[
S_3 = \{O(\pm \phi, \vec{k}_1), O(\pm \phi, \vec{k}_2), O(\pm \phi, \vec{k}_3)\},
\]

\[
X_3 = \{\pm \phi X_{23}, \pm \phi X_{13}, \pm \phi X_{12}\},
\]

(50)

where \( \vec{k}_1 = [1, 0, 0], \vec{k}_2 = [0, 1, 0], \vec{k}_3 = [0, 0, 1] \). Note that matrices belonging to \( X \) form a basis of the Lie algebra \( so(3) \) iff \( \phi \neq 0 \). Therefore by Lemma 7 we know that \( \mathcal{C}(\text{ad}_{X_3}) = \{\lambda I\} \). The adjoint matrices \( \text{Ad}_{O(\pm \phi, \vec{k}_i)} \) are again rotation matrices by angles \( \pm \phi \) along axes \( \vec{k}_i \). On the other hand, by Fact 13 we know that \( \mathcal{C}(\text{Ad}_{S_3}) \) can be different than \( \mathcal{C}(\text{ad}_{X_3}) \) only if \( \phi = \pm \pi \). Indeed in this case the adjoint matrices \( \text{Ad}_{O(\pm \phi, \vec{k}_i)} \) commute. Summing up we have

**Fact 36.** For a 2-mode orthogonal beamsplitter with \( \phi \neq 0 \mod 2\pi \) we have \( \mathcal{C}(\text{ad}_{X_3}) = \{\lambda I\} \). On the other hand \( \mathcal{C}(\text{Ad}_{S_3}) = \{\lambda I\} \) iff \( \phi \neq 0 \mod \pi \).
7.1.2 The case of unitary group

Let $B \in SU(2)$. Making use of the notation introduced in Section 4, we assume $B = U(\phi, \bar{k})$, $\phi \neq 0 \bmod \pi$, $\bar{k} = [k_x, k_y, k_z]$ and $k_x^2 + k_y^2 + k_z^2 = 1$. Therefore we have

$$\mathcal{X}_3 = \{b_{ij}, b_{ij}^\dagger\} = \phi \cdot \{k_x X_{ij} + k_y Y_{ij} + k_z Z_{ij}, -k_x X_{ij} + k_y Y_{ij} - k_z Z_{ij} : 1 \leq i < j \leq 3\},$$

$$(52)$$

$$\mathcal{S}_3 \{b_{ij}, b_{ij}^\dagger\} = \{I_{ij}(\phi) + \sin \phi(k_x X_{ij} + k_y Y_{ij} + k_z Z_{ij}), I_{ij}(\phi) + \sin \phi(-k_x X_{ij} + k_y Y_{ij} - k_z Z_{ij}) : 1 \leq i < j \leq 3\},$$

$$(53)$$

where $I_{ij}(\phi) = \cos \phi(E_{ii} + E_{jj}) + E_{ij}$, where $l \in \{1, 2, 3\} \setminus \{i, j\}$. We start from finding $C(\text{ad}_{\mathcal{X}_3})$. To this end note that $[b_{ij}, b_{ij}^\dagger] = 4k_y(k_x Z_{ij} - k_z X_{ij})$. If $[b_{ij}, b_{ij}^\dagger] \neq 0$ then $b_{ij}$ and $b_{ij}^\dagger$ generate $\mathfrak{su}(2)_{ij}$. Thus we have access to all elements $X_{ij}$, $Y_{ij}$ and $Z_{ij}$ $1 \leq i < j \leq 3$. Hence $\mathcal{X}_3$ generates $\mathfrak{su}(3)$ and $C(\text{ad}_{\mathcal{X}_3}) = \{\lambda I\}$. If in turn $[b_{ij}, b_{ij}^\dagger] = 0$ then we need to consider four cases: (1) $k_y \neq 0$ and $k_x = k_z = 0$, (2) $k_y = 0$ and $k_x \neq 0$ and $k_z \neq 0$, (3) $k_y = 0 = k_z$ and $k_x \neq 0$, (4) $k_y = 0 = k_z$ and $k_x \neq 0$.

1. In this case $b_{ij} = k_y Y_{ij} = b_{ij}^\dagger$, therefore we have access to all $\{Y_{ij}\}_{i<j}$, $i, j \in \{1, 2, 3\}$. But by commutation relations $[Y_{ij}, Y_{kj}] = -X_{jk}$, $[\hat{Y}_{ij}, Y_{jk}] = -X_{kj}$, $[\hat{Y}_{ij}, Y_{kj}] = -X_{ik}$ and $[X_{ij}, Y_{ij}] = 2Z_{ij}$. Thus we can generate all basis elements of $\mathfrak{su}(3)$ starting from $Y_{ij}$'s. This means $C(\text{ad}_{\mathcal{X}_3}) = \{\lambda I\}$.

2. In this case $b_{ij} = -b_{ij}^\dagger$. Direct calculations show that elements:

$$[b_{12}, [b_{12}, b_{13}]], [b_{12}, [b_{12}, b_{23}]], [b_{13}, [b_{13}, b_{12}]],$$

$$[b_{13}, [b_{13}, b_{23}]], [b_{23}, [b_{23}, b_{12}]], [b_{12}, [b_{12}, b_{13}, b_{23}]],$$

form a basis of $\mathfrak{su}(3)$. Thus $C(\text{ad}_{\mathcal{X}_3}) = \{\lambda I\}$.

3. In this case the algebra generated by $\mathcal{X}_3$ is clearly $\mathfrak{so}(3)$. Hence $C(\text{ad}_{\mathcal{X}_3}) \neq \{\lambda I\}$.
4. In this case the algebra generated by $\lambda_3$ abelian. Hence $C(\text{ad}\lambda_3) \neq \{\lambda I\}$.

We have just shown

**Fact 37.** For a 2-mode unitary beamsplitter $B = I \cos \phi + \sin \phi (k_x X + k_y Y + k_z Z)$, where $k_x^2 + k_y^2 + k_z^2 = 1$ we have $C(\text{ad}\lambda_3) = \{\lambda I\}$ unless (a) $k_y = 0 = k_z$ and $k_x = 1$, (b) $k_y = 0 = k_x$ and $k_z = 1$.

Next we characterise $C(\text{Ad}_{S_3})$. The adjoint matrices $\text{Ad}_{B_{ij}}$ and $\text{Ad}_{B_{ij}^*}$ are elements of $SO(\text{su}(3)) \simeq SO(8)$. The rotation angles of both $\text{Ad}_{B_{ij}}$ and $\text{Ad}_{B_{ij}^*}$ are $\pm \phi$, $2\phi$ and 0. On the other hand, by Fact 12 we know that $C(\text{Ad}_{S_3})$ can be different than $C(\text{ad}\lambda_3)$ only if the rotation angle is $\pm \pi$. This corresponds to situations when either $\phi = \pm \pi$ or $\phi = \pm \frac{\pi}{2}$. In the first case $B = -I$, thus obviously $C(\text{Ad}_{S_3}) \neq \{\lambda I\}$. The case $\phi = \pm \frac{\pi}{2}$ corresponds to $\frac{\pi}{2}, S_3 = \lambda_3$.

**Fact 38.** For a 2-mode unitary beamsplitter $B = I \cos \phi + \sin \phi (k_x X + k_y Y + k_z Z)$ we have $C(\text{Ad}_{S_3}) = \{\lambda I\}$ unless (a) $k_y = 0 = k_z$ and $k_x = 1$, (b) $k_y = 0 = k_x$ and $k_z = 1$, (c) $\phi = \pm \frac{\pi}{2}$ and $k_x = 0$.

**Proof.** Recall that $C(\text{ad}\lambda_3) \subset C(\text{Ad}_{S_3})$. Cases (a) and (b) correspond to situations when $C(\text{ad}\lambda_3) \neq \{\lambda I\}$. Case (c) follows from direct calculations for six $\text{Ad}_{g}$ matrices with $\phi = \pm \frac{\pi}{2}$ and $g \in S_3$. They were done with the help of symbolic calculation software. We only verify that when $\phi = \pm \frac{\pi}{2}$ and $k_z = 0$ indeed $C(\text{Ad}_{S_3}) \neq \{\lambda I\}$. Therefor we define $\mathfrak{h} = \text{Span}_{\mathbb{R}}\{Z_{12}, Z_{23}\}$, $\dim \mathfrak{h} = 2$ and show that for $\phi = \pm \frac{\pi}{2}$ and $k_x = 0$ the space $\mathfrak{h}$ is an invariant subspace for matrices $\text{Ad}_{B_{ij}}$ and $\text{Ad}_{B_{ij}^*}$, i.e. of $S_3$. To this end we calculate

$$\text{Ad}_{B_{12}}Z_{12} = -Z_{12}, \quad \text{Ad}_{B_{13}}Z_{12} = -Z_{23}, \quad \text{Ad}_{B_{23}}Z_{12} = Z_{12} + Z_{23},$$

$$\text{Ad}_{B_{12}}Z_{23} = Z_{23} + Z_{12}, \quad \text{Ad}_{B_{13}}Z_{23} = -Z_{12}, \quad \text{Ad}_{B_{23}}Z_{23} = -Z_{23}. \quad (54)$$

and $\text{Ad}_{B_{ij}^*}Z_{kl} = \text{Ad}_{B_{ij}}Z_{kl}$. Therefore the projection operator $P : \text{su}(3) \rightarrow \mathfrak{h}$ commutes with matrices from $S_3$ and thus it belongs to $C(\text{Ad}_{S_3})$. \hfill \square

It is interesting to look at the structure of the group $< S_3 >$ when $k_z = 0$ and $\phi = \frac{\pi}{2}$. Matrices are of the form $B_{ij} = e^{i\psi}E_{ij} - e^{-i\psi}E_{ji} + E_{kk}$ and $B_{ij}^* = -e^{-i\psi}E_{ij} + e^{i\psi}E_{ji} + E_{kk}$, where $1 \leq i < j \leq 3$, $k \neq i, j$ and $\psi \in [0, 2\pi)$. If $\psi$ is a rational multiple of $\pi$, then it is easy to see that $< S_3 >$ is a finite group and when $\psi$ is an irrational multiple of $\pi$ the group $< S_3 >$ is infinite and disconnected. In fact these are groups isomorphic to $\Delta(6n^2)$ and $\Delta(6\infty^2)$ given in $[12]$.  

**7.2 When $S_3$ is universal?**

Having characterised when $C(\text{Ad}_{S_3}) = \{\lambda I\}$ we check in this section when the group $< S_3 >$ is infinite and this way we get full classification of universal 2-mode beamsplitters.

**7.2.1 The case of orthogonal group**

Combining Theorem 23 with Fact 38 for $\phi \notin \mathcal{L}_{SO(3)}$ we obtain that the group generated by $S_3$ is exactly $SO(3)$. When $\phi \in \mathcal{L}_{SO(3)}$ we consider two matrices: $O(\gamma, \vec{k}_{13}) = O(\phi, \vec{k}_1)O(\phi, \vec{k}_3)$ and $O(\gamma, \vec{k}_{31}) = O(\phi, \vec{k}_3)O(\phi, \vec{k}_1)$. Their traces are the same and they yield the following equation that relates $\gamma$ and $\phi$

$$\cos \gamma = \frac{\cos^2 \phi + 2 \cos \phi - 1}{2}. \quad (56)$$

Note that $O(\gamma, \vec{k}_{13})$ and $O(\gamma, \vec{k}_{31})$ do not commute unless $\phi = 0 \mod \pi$. Moreover, if $\phi = \frac{(2k+1)\pi}{2}$, where $k \in \mathbb{Z}$, then matrices $O(\phi, \vec{k}_1)$, $O(\phi, \vec{k}_2)$ and $O(\phi, \vec{k}_3)$ are permutation matrices and they form 3-dimensional representation of $S_3$. For all remaining $\phi \in \mathcal{L}_{SO(3)}$ we calculate $\cos \gamma$ using (56) and compare it with the values of $\cos \alpha$ for all $\alpha \in \mathcal{L}_{SO(3)}$. We find out they never agree. Therefore $\gamma \notin \mathcal{L}$ and we can apply Theorem 23 and Fact 38 to $U(\gamma, \vec{k}_{13})$ and $U(\gamma, \vec{k}_{31})$. Summing up:

**Theorem 39.** Any 2-mode orthogonal beamsplitter with $\phi \notin \{\frac{k\pi}{2} : k \in \mathbb{Z}\}$ is universal on 3 and hence $n > 3$ modes.
7.2.2 The case of unitary group

Recall that by Fact 38 the space \( \mathcal{C}(\text{Ad}S_3) = \{ \lambda I \} \) if and only if all entries of a matrix \( B \in SU(2) \) are nonzero and at least one of them belongs to \( \mathcal{C} \). So we are left with checking if under these assumptions \( < S_1 > \) is infinite. Let \( \{ e^{i\phi}, e^{-i\phi} \} \) be the spectrum of \( B \). Matrices \( B_{ij} \) and \( B_{ij}^* \) have the same spectra \( \{ e^{i\phi}, e^{-i\phi}, 1 \} \). Looking at the definitions of the open balls \( B_{ij}^{1/\alpha} \), \( \alpha^3 = 1 \) we see that a matrix from \( SU(3) \) with one spectral element equal to one can be introduced (by taking powers) only to the ball with \( \alpha = 1 \). Moreover, the maximal \( n \) that is needed is exactly the same as for \( SO(3) \) and the exceptional angles belong to the set \( \mathcal{L}_{SO(3)} \). Therefore by Corollary 22, \( \phi \notin \mathcal{L}_{SO(3)} \) implies that the group generated by, for example, \( B_{12} \) and \( B_{23} \) is infinite. In the following we show that \( < S_1 > \) is infinite also for \( \phi \in \mathcal{L}_{SO(3)} \).

Let us consider \( < R > = < B_{12}(\phi), B_{23}(\phi) > \) with \( \phi \in \mathcal{L}_{SO(3)} \). Our goal is to show that \( R \subset S_3 \) generates infinite group. To this end we use the following procedure:

1. We calculate trace of the product \( B_{12}(\phi)B_{23}(\phi) \) and note it is real. Therefore spectrum of \( B_{12}(\phi)B_{23}(\phi) \) is of the form \( \{ e^{i\gamma}, e^{-i\gamma}, 1 \} \), where the relation between \( \phi \) and \( \gamma \) is given by
   \[
   \text{tr}B_{12}(\phi)B_{23}(\phi) = 2 \cos \phi + \cos^2 \phi + z^2 \sin^2 \phi = 2 \cos \gamma + 1. \tag{57}
   \]

2. Using 57, for each \( \gamma \in \mathcal{L}_{SO(3)} \) we compute
   \[
   z^2 = \frac{2 \cos \gamma + 1 - 2 \cos \phi - \cos^2 \phi}{\sin^2 \phi}, \tag{58}
   \]
   and check whether \( 0 < z^2 < 1 \). The pairs \( (\phi, \gamma) \) that fails these test are excluded form the further considerations. Note that \( z^2 = 1 \) corresponds to diagonal matrices \( B_{12}(\phi), B_{23}(\phi) \) and \( z^2 = 0 \) corresponds the situation when \( \mathcal{C}(S_3) \neq \{ \lambda I \} \).

3. For the pairs \( (\phi, \gamma) \) that give \( 0 < z^2 < 1 \) we consider the matrix \( U(\gamma') = B_{12}(2\phi)B_{23}(2\phi) \). Its trace is again real and we get
   \[
   \text{tr}B_{12}(2\phi)B_{23}(2\phi) = \frac{1}{2}(2 + 4 \cos(2\phi) + (1 - z^2)(\cos(4\phi) - 1)) = 2 \cos \gamma' + 1, \tag{59}
   \]
   where \( z^2 \) is determined by \( \phi \) and \( \gamma \). Direct computations show that \( \gamma' \notin \mathcal{L}_{SO(3)} \) if \( \phi \notin \{ \pm \frac{\pi}{2}, \pm \frac{3\pi}{4} \} \). We treat both of these cases separately.

4. For \( \phi = \pm \frac{2\pi}{3} \) and the fixed \( z^2 \) we consider yet another product of matrices \( U(\gamma') = U_{23}^2(\phi)U_{12}^2(\phi)B_{23}(\phi)U_{12}(\phi) \) with a real trace:
   \[
   \text{tr}B_{12}^2(\phi)B_{23}(\phi)B_{12}(\phi) = \frac{8 \cos \phi + 3 \cos(2\phi) + 4 \cos(3\phi) + 6 \cos(4\phi) + 4 \cos(5\phi) + \cos(6\phi) - 2}{8} + 32z^8 \sin^4 \phi \cos^2 \phi + 8z^2 \sin^2 \phi \cos \phi + 4 \cos(2\phi) + 2 \cos(3\phi) + \cos(4\phi) + 4 = 2 \cos \gamma''. \tag{60}
   \]

Direct computations show that \( \gamma'' \notin \mathcal{L}_{SO(3)} \), thus we are done for \( \phi \in \mathcal{L}_{SO(3)} \setminus \{ \pm \frac{\pi}{2}, \pm \frac{3\pi}{4} \} \). The same composition for \( U_{23} \left( \frac{\pi}{2} \right), U_{12} \left( \frac{\pi}{2} \right) \) may give a matrix of the spectral angle \( \gamma = \pm \frac{2\pi}{3} \).

For \( \phi = \pm \frac{\pi}{2} \) an additional treatment is needed. It basically consists of two main ingredients.

1. Assume \( B_{ij} \left( \frac{\pi}{2} \right) \) does not commute with its permutations \( B_{ij}^* \left( \frac{\pi}{2} \right) \) for \( 1 \leq i \leq j \leq 3 \). In this case we can use \( B_{ij}(\gamma) = B_{ij} \left( \frac{\pi}{2} \right) B_{ij}^* \left( \frac{\pi}{2} \right) \), \( 1 \leq i \leq j \leq 3 \) as the new set of generators. Note that the angle \( \gamma \) depends on the trace of \( B_{ij} \left( \frac{\pi}{2} \right) B_{ij}^* \left( \frac{\pi}{2} \right) \) as \( \cos \gamma = 1 - 2y^2 \). Thus \( \gamma \neq \pm \frac{\pi}{2} \) if \( y^2 \neq \frac{1}{4} \) and then we can apply the previous procedure to show that \( < B_{12}(\gamma), B_{23}(\gamma) > \) is infinite.

2. For \( \phi = \pm \frac{\pi}{2} \) and \( y^2 = \frac{1}{4}, z^2 = \frac{1}{4} \) we consider yet another product
   \[
   \text{tr}B_{12}^2 \left( \frac{\pi}{2} \right) B_{13} \left( \frac{\pi}{2} \right) B_{23} \left( \frac{\pi}{2} \right) B_{13}^* \left( \frac{\pi}{2} \right) = z^2 = 2 \cos \gamma'''.
   \]

We find out that the only \( \gamma \in \mathcal{L}_{SO(3)} \) satisfying \( 2 \cos \gamma = z^2 - 1 \) for 0 \( \leq z^2 \leq \frac{1}{4} \) are \( \gamma = \pm \frac{\pi}{3} \). But then \( z^2 = 0 \). Thus by Fact 38 the space \( \mathcal{C}(\text{Ad}S_3) \) is larger than \( \{ \lambda I \} \).
3. Finally we assume that matrices $B_{ij} \left( \frac{\pi}{2} \right)$ commute with their permutations. Recall that it happens if either $y = \pm 1$ and $x = z = 0$ or $y = 0$ and $x, z \neq 0$. The group generated for $y = \pm 1$ is of course finite. Therefore we need to consider only the case when $y = 0$ and $x, z \neq 0$. But in this case step 2 of the previous procedure is never satisfied (from equation (58) one can only obtain $z^2 = 0$ for $\gamma = \pm \frac{2\pi}{3}$).

Summing up

**Theorem 40.** Any 2-mode unitary gate whose all entries are nonzero and at least one of them is a complex number is universal on 3 and hence $n > 3$ modes.

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