ON ASYMPTOTIC EXPANSIONS AND SCALES OF SPECTRAL UNIVERSALITY IN BAND RANDOM MATRIX ENSEMBLES

A.KHORUNZHY
Institute for Low Temperature Physics, Kharkov, UKRAINE and University Paris-7, FRANCE

W.KIRSCH
Institute of Mathematics, Ruhr-University Bochum, Bochum, GERMANY

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Abstract

We consider real random symmetric $N \times N$ matrices $H$ of the band-type form with characteristic length $b$. The matrix entries $H(x,y), x \leq y$ are independent Gaussian random variables and have the variance proportional to $u\left(\frac{x-y}{b}\right)$, where $u(t)$ vanishes at infinity. We study the resolvent $G(z) = (H-z)^{-1}$, $\text{Im} z \neq 0$ in the limit $1 \ll b \ll N$ and obtain explicit expression $S(z_1, z_2)$ for the leading term of the first correlation function of the normalized trace $\langle G(z) \rangle = N^{-1} \text{Tr} G(z)$.

We examine $S(\lambda_1+i0, \lambda_2-i0)$ on the local scale $\lambda_1 - \lambda_2 = \frac{r}{N}$ and show that its asymptotic behavior is determined by the rate of decay of $u(t)$. In particular, if $u(t)$ decays exponentially, then $S(r) \sim -C b^2 N^{-1} r^{-3/2}$. This expression is universal in the sense that the particular form of $u$ determines the value of $C > 0$ only. Our results agree with those detected in both numerical and theoretical physics studies of spectra of band random matrices.

1 Problem, motivation and results

Random matrices play an important role in various fields of mathematics and physics. The eigenvalue distribution of large matrices was initially considered
by E.Wigner to model the statistical properties of energy spectrum of heavy
nuclei (see e.g. the collection of early papers \[29\]). Further investigations have
led to numerous applications of random matrices of infinite dimensions in such
branches of theoretical physics as statistical mechanics of disordered spin sys-
tems, solid state physics, quantum chaos theory, quantum field theory and others
(see monographs and reviews \[2, 11, 22, 25\]). In mathematics, the spectral the-
ory of random matrices has revealed deep links with the orthogonal polynomi-
als, integrable systems, representation theory, combinatorics, non-commutative
probability theory and other theories \[3, 11, 32, 35\].

In present paper we deal with the family of real symmetric random matri-
ces that can be referred to as the band-type one. In the simplest cas-
e the matrices
have zeros outside of a band around the principal diagonal. Inside of this band
they are assumed to be jointly independent random variables. The limiting
transition considered is that the band width \( b \) increases at the same time as the
dimension of the matrix \( n \) does.

There is a large number of papers devoted to the use of random matrices of
this type in models of quantum chaotical systems (see, e.g. \[30\] and references
therein). In these studies, one of the central topic is related with the transition
between fully developed chaos and complete integrability. The crucial observa-
tion made numerically \[9\] and then supported in the welth of theoretic al physics
papers (see, for example \[15, 33\]) is that the ratio \( b^2/n \) is the critical one for
the corresponding transition in spectral properties of band random matrices.

On the rigorous level, the eigenvalue distribution of \( H^{(n,b)} \) has been studied
\[4, 8, 26\]. It is shown that the limiting eigenvalue distribution exists, is non-
random and depends on the parameter \( \alpha = \lim_{n \to \infty} b/n \). However, the role
of the ratio \( b^2/n \) has not been revealed there. Recently, a series of papers
appeared where the band random matrices are studied in the context of the
non-commutative probability theory \[17, 31\]. These studies also deal with the
limit \( n, b \to \infty \) such that \( \alpha > 0 \).

In present paper we are concentrated on the case of \( \alpha = 0 \) represented by
the limit
\[
1 \ll b \ll n
\]
and study the first correlation function of the resolvent of band random matrices.
We show that the ratio \( \beta = \lim_{n \to \infty} b^2/n \) naturally arises when one considers
the leading term of this correlation function on the local scale. This can be
regarded as the support of the conjecture that the local properties of spectra of
band random matrices depend on the value of \( \beta \).

Let us describe our results in more details. We consider the ensemble
\( \{H^{(n,b)}\} \) of random \( N \times N \) matrices, \( N = 2n + 1 \) whose entries
\( H^{(n,b)}(x,y) \) have the variance proportional to \( u(x - y) \), where \( u(t) \geq 0 \) vanishes at infinity.
We consider the resolvent \( G^{(n,b)}(z) = (H^{(n,b)} - z)^{-1} \) and study the asymptotic
expansion of the correlation function
\[
C_{n,b}(z_1, z_2) = \mathbf{E}f_{n,b}(z_1)f_{n,b}(z_2) - \mathbf{E}f_{n,b}(z_1)\mathbf{E}f_{n,b}(z_2),
\]
where we denoted $f_{n,b}(z) = N^{-1} \operatorname{Tr} G^{(n,b)}(z)$. Keeping $z_i$ far from the real axis, we consider the leading term $S(z_1, z_2)$ of this expansion and find explicit expression for it. This term $S(r_1 + i0, r_2 - i0)$ regarded on the local scale $r_1 - r_2 = r/N$ exhibits different behavior depending on the rate of decay of the profile function $u(t)$.

Our main conclusion is that if $u(t) \sim |t|^{-1-\nu}$ as $t \to \infty$, then the value $\nu = 2$ separates two major cases. If $\nu \in (1, 2)$, then the limit of $S(r)$ depends on $\nu$. If $\nu \in (2, +\infty)$, then

$$
\frac{1}{Nb} S(r) = -\text{const} \cdot \sqrt{\frac{N}{b}} \cdot \frac{1}{|r|^{3/2}} (1 + o(1)).
$$

These results are in agreement with those predicted in theoretical physics studies. In particular, the last expression for $S$ coincides with the Altshuler-Shklovski asymptotics of the spectral correlation function (see e.g. [27]).

The paper is organized as follows. In Section 2 we determine the family of ensembles and present several already known results that will be needed. In Section 3 we formulate our main propositions and describe the scheme of their proofs. To illustrate this scheme, we present a short proof of the Wigner semicircle law for Gaussian Orthogonal Ensemble of random matrices. In Section 4 we study the correlation function $C_{n,b}(z_1, z_2)$ and obtain the explicit expression $S(z_1, z_2)$ for its leading term. In Section 5 we study the self-averaging property of $G(z)$ and prove auxiliary facts used in Section 4. Expressions derived in Section 4 are analyzed in Section 6, where the asymptotic behavior of $S(z_1, z_2)$ is studied. In Section 7 we give a summary of our observations.

## 2 Band Random Matrices and Wigner Law

### 2.1 The ensemble

Let us consider the family $A = \{a(x, y), x \leq y, x, y \in \mathbb{Z}\}$ of jointly independent random variables determined on the same probability space. We assume that they have joint Gaussian (normal) distribution with properties

$$
\mathbb{E} a(x, y) = 0, \quad \mathbb{E} [a(x, y)]^2 = v(1 + \delta_{xy}),
$$

where we denote by $\delta$ the Kronecker symbol;

$$
\delta_{xy} = \begin{cases} 
0, & \text{if } x \neq y, \\
1, & \text{if } x = y.
\end{cases}
$$

Here and below $\mathbb{E}$ denotes the mathematical expectation with respect to the measure generated by the family $A$.

Let $u(t), t \in \mathbb{R}$ be a piece-wise continuous function $u(t) = u(-t) \geq 0$ satisfying conditions

$$
\sup_{t \in \mathbb{R}} |u(t)| = \bar{u} < \infty \quad (2.2)
$$
\[ \int_{\mathbb{R}} u(t) dt = 1. \]  \hspace{1cm} (2.3)

For simplicity, we assume \( u(t) \) to be monotone for \( t \geq 0 \).

Given real parameter \( b > 0 \), we introduce an infinite matrix \( U^{(b)} \)

\[ U^{(b)}(x, y) = \frac{1}{b} u \left( \frac{x - y}{b} \right), \quad x, y \in \mathbb{Z}. \]

and determine the ensemble \( \{ H^{(n, b)} \} \) as the family of real symmetric matrices of the form

\[ H^{(n, b)}(x, y) = a(x, y) \sqrt{U^{(b)}(x, y)}, \quad x \leq y, \quad |x|, |y| \leq n, \]  \hspace{1cm} (2.4)

where \( b \leq N, \ N = 2n + 1 \) and the square root is assumed to be positive.

Let us note that the matrix (2.1) has the really band form when \( U^{(b)} \) is constructed with the help of a function \( u \) having a finite support, say

\[ u(t) = \begin{cases} 1, & \text{if } t \in (-\frac{1}{2}, \frac{1}{2}), \\ 0, & \text{otherwise.} \end{cases} \]

In this case the band width is less than or equal to \( 2b + 1 \). If \( b = N \), then matrices (2.4) coincide with those belonging to Gaussian Orthogonal Ensemble (GOE) \( 25 \). This random matrix ensemble is determined as the family \( \{ A_N \} \) of real symmetric matrices

\[ A_N(x, y) = \frac{1}{\sqrt{N}} a(x, y), \quad x, y = 1, \ldots, N, \]  \hspace{1cm} (2.5)

with \( \{ a(x, y) \} \) belonging to \( A \) (2.1). GOE together with its Hermitian and quaternion versions plays the fundamental role in the spectral theory of random matrices.

Random symmetric matrices (2.5) with independent arbitrary distributed random variables \( a(x, y) \) satisfying (2.1) is referred to as the Wigner ensemble of random matrices. This random matrix ensemble considered first by E. Wigner \( 30 \) is extensively studied in a series of papers (see e.g. \( 32 \) and references therein). In particular, in paper \( 24 \) the resolvent technique is developed to study the spectral properties of the Wigner ensemble. Actually, we follow a version of this technique, but restrict ourself with more simple case of Gaussian random variables. More general case of arbitrary distributed random variables would make the computations much more cumbersome. Let us repeat that the main task of this paper is to study the role of the ratio between \( b \) and \( N \) with respect to the spectral properties of random matrices.

Finally, it should be noted that we restrict ourself with the ensemble of real symmetric matrices for the sake of simplicity also. All results can be obtained by using essentially the same technique for the Hermitian analogue of \( H^{(n, b)} \).


2.2 Limiting eigenvalue distribution

Eigenvalue distribution of matrices \(H^{(n,b)}\) is described by the normalized eigenvalue counting function

\[
\sigma(\lambda; H^{(n,b)}) = \#\{\lambda_j^{(n,b)} \leq \lambda\} N^{-1},
\]

where \(\lambda_1^{(n,b)} \leq \ldots \leq \lambda_N^{(n,b)}\) are eigenvalues of \(H^{(n,b)}\). We denote by \(f_{n,b}(z), z \in \mathbb{C}\) the Stieltjes transform of the measure given by (2.6);

\[
f_{n,b}(z) = \int_{-\infty}^{\infty} \frac{d\sigma_{n,b}(\lambda)}{\lambda - z}, \quad \text{Im} \ z \neq 0.
\]

Given a Stieltjes transform \(f(z)\), one can restore corresponding measure \(d\sigma(\lambda)\) with the help of the inversion formula (see e.g. [12]).

The limiting behavior of (2.7) as \(n, b \to \infty\) was studied in a series of papers [1, 3, 23, 26]. It was proved in [26] that \(f_{n,b}(z)\) converges as \(n, b \to \infty\) in probability to a nonrandom function that depends on the ratio \(\alpha = \lim b/N\);

\[
p - \lim_{n,b \to \infty} f_{n,b}(z) = w_\alpha(z).
\]

In particular, if \(\alpha = 0\), then the function \(w_0(z) \equiv w(z)\) satisfies equation

\[
w(z) = \frac{1}{-z - vw(z)}.
\]

The solution of this equation is unique in the class of functions satisfying condition

\[\text{Im} \ w(z)\text{Im} \ z \geq 0\]

and can be represented in the form \(w(z) = \int (\lambda - z)^{-1} d\sigma_w(\lambda), \) where \(\sigma_w(\lambda)\) is the famous semicircle (or Wigner) distribution [36] with the density

\[
\rho_w(\lambda) = \sigma'_w(\lambda) = \frac{1}{2\pi v} \begin{cases} \sqrt{4v - \lambda^2}, & \text{if } |\lambda|^2 \leq 4v, \\ 0, & \text{if } |\lambda|^2 \geq 4v. \end{cases}
\]

This density was obtained first by E. Wigner [36] for eigenvalues of random matrices of the "full" form (2.5) and can be also obtained as the limit (2.8) with \(\alpha = 1\)

\[
\sigma_w(\lambda) = \lim_{N \to \infty} \sigma(\lambda; A_N)
\]

Thus, one gets the same eigenvalue distribution in the opposite limiting transitions of narrow \(\alpha = 0\) and wide \(\alpha = 1\) band widths. It is known that in the intermediate regime \(0 < \alpha < 1\) the limiting distribution differs from the semicircle (2.11) [26]. In present paper we concentrate ourself on the most interesting case \(\alpha = 0\).
In present paper we always consider the case of \( 1 \ll b \ll n \). As we have noted, the paper is aimed to detect the role of the parameter \( \beta = \lim_{N \to \infty} b^2/N \). To avoid technical problems, we restrict ourselves with the range

\[ b = n^\chi \quad 1/3 < \chi < 1. \quad (2.12) \]

We are convinced that our results are valid on the whole range \( 0 < \chi < 1 \).

### 3 Main Propositions and Scheme of the Proof

The resolvent

\[ G^{(n,b)}(z) = \left( H^{(n,b)} - zI \right)^{-1}, \quad \text{Im} \, z \neq 0 \]

is widely exploited in the spectral theory of operators. Its normalized trace

\[ \langle G^{(n,b)}(z) \rangle = \frac{1}{N} \text{Tr} \left( H^{(n,b)} - zI \right)^{-1} = \frac{1}{N} \sum_{j=1}^{N} \frac{1}{\lambda_j^{(n,b)} - z}. \]

The results of this section are related with the asymptotic behavior of \( \langle G^{(n,b)}(z) \rangle \) in the limit (2.12), with \( z \in \Lambda_\eta \),

\[ \Lambda_\eta = \{ z \in C : |\text{Im} \, z| \geq \eta \} \quad \text{with} \quad \eta = 2\sqrt{v} + 1. \quad (3.1) \]

#### 3.1 Main technical results

Our first statement concerns the pointwise convergence of the diagonal entries \( G^{(n,b)}(x,x;z), |x| \leq n \) of the resolvent. Let us determine the set

\[ B_L \equiv B_L(n,b) = \{ x \in \mathbb{Z} : |x| \leq n - bL \}. \quad (3.2) \]

**Theorem 3.1**

*Given \( \varepsilon > 0 \), there exists a natural \( L \) such that

\[ \sup_{x \in B_L} |G^{(n,b)}(x,x;z) - w(z)| \leq \varepsilon, \quad \forall z \in \Lambda_\eta, \quad (3.3) \]

for sufficiently large \( b, n \).*

The result of Theorem 3.1 is interesting by itself. We shall use it hardly in the proof of the following statement concerning the correlation function

\[ C_{n,b}(z_1, z_2) = E \langle G(z_1) \rangle \langle G(z_2) \rangle - E \langle G(z_1) \rangle \ E \langle G(z_2) \rangle. \quad (3.4) \]
**Theorem 3.2.**
If \( z_i \in \Lambda_0, i = 1, 2 \), then in the limit \( n, b \to \infty \) (2.12)

\[
C_{n,b}(z_1, z_2) = \frac{1}{Nb} S(z_1, z_2) + o\left(\frac{1}{Nb}\right). \tag{3.5}
\]

The explicit term of \( S(z_1, z_2) \) is given by relation

\[
S(z_1, z_2) = \frac{2v}{(1 - vw_1^2)(1 - vw_2^2)} Q(z_1, z_2), \tag{3.6}
\]
where \( w_j \equiv w(z_j), j = 1, 2 \) and \( Q(z_1, z_2) \) is given by the formula

\[
Q(z_1, z_2) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{w_1^2 w_2^2 \hat{u}_F(p)}{[1 - vw_1 w_2 \hat{u}_F(p)]^2} dp,
\]
where we denote by \( \hat{u}_F(p) \) the Fourier transform of \( u \)

\[
\hat{u}_F(p) = \int_{\mathbb{R}} u(t)e^{ipt} dt.
\]

It should be noted that in the case of GOE (2.6) relation (3.5) is valid with \( b \) replaced by \( N \) and expression (3.6) takes the following form (see e.g. [14, 20])

\[
S_{\text{GOE}}(z_1, z_2) = \frac{2v}{(1 - vw_1^2)(1 - vw_2^2)} \frac{w_1^2 w_2^2}{[1 - vw_1 w_2]^2}. \tag{3.7}
\]

Let us briefly explain why (3.6) and (3.7) lead to different asymptotic expressions on the local scale determined as

\[
z_1^{(N)} = \lambda + \frac{r_1}{N} + i0, \quad z_2^{(N)} = \lambda + \frac{r_2}{N} - i0 \tag{3.8}
\]
with \( \lambda \in \text{supp } d\sigma_w \) (2.10). It follows from equality (2.9) that

\[
\frac{w_1^2 w_2^2}{[1 - vw_1 w_2]^2} = \left( \frac{w_1 - w_2}{z_1 - z_2} \right)^2. \tag{3.9}
\]
This expression tends to infinity in the limit (3.8) and \( vw(z_1)w(z_2) \to 1 \) as well. But after dividing by \( N^2 \), one obtains from (3.7) and (3.9) that

\[
\frac{1}{N^2} S_{\text{GOE}}(z_1, z_2) = -\frac{1}{(r_1 - r_2)^2}(1 + o(1)). \tag{3.10}
\]
The left-hand side of (3.1) is usually called the wide (or smoothed) version of the eigenvalue density correlation function and the expression in the right-hand side of (3.10) is derived by various methods for different random matrix ensembles [13, 4, 8, 24].
In Section 6 we study $S(z_1, z_2)$ with the spectral parameters $z_1, z_2$ given by (3.8). Now the singularity of $Q(z_1, z_2)$ is determined by convergence of $1 - \nu w_1 w_2 \tilde{u}_F(p)$ to zero. This convergence depends on the behavior of $\tilde{u}_F(p)$ around the origin $p = 0$; that is why the rate of decay of $u(t)$ at infinity dictates the form of the limiting expression for $S$ in the local scale.

3.2 The method and short proof of semicircle law

We prove Theorem 3.1 in Sections 4 and 5. We are based on the moment relations approach for resolvents of random matrices proposed and developed in [21, 22, 28]. This technique is proved to be rather general, powerful and applicable to various random matrix ensembles. We use a modified version of this approach needed to study rather complex case of band random matrices. To make the subsequent exposition more transparent, let us describe the principal points of this method in application to the simplest case represented by GOE (2.6).

3.2.1 Families of averaged moments

In the early 70s F.Berezin observed [1] that regarding correlation functions of the formal density of states $\rho_N(\theta) = \sigma_N'(\theta)$

$$P_k^{(N)}(\Theta_k) = \mathbf{E} \rho_N(\theta_1) \cdots \rho_N(\theta_k),$$

$\Theta_k = (\theta_1, \ldots, \theta_k)$, one can derive for them a system of relations that resembles equalities for correlation functions of statistical mechanics. In this system $P_k^{(N)}$ is expressed via sum of $P_{k-1}^{(N)}$, $P_{k+1}^{(N)}$ and some terms that vanish in the limit $N \to \infty$. This can be rewritten in the vector form

$$\vec{P}^{(N)} = \vec{P}_0 + B \vec{P}^{(N)} + \vec{\varphi}^{(N)},$$

with certain operator $B$ and vector $\vec{\varphi}$ such that $\|B\| < 1$ and $\|\Phi^{(N)}\| = o(1)$ in appropriate Banach space. These properties prove existence of $\lim_{N \to \infty} \vec{P}^{(N)} = \vec{P}$; the special form of $B$ implies that the limiting $\vec{P}$ is nonrandom with the components $\prod \rho(\theta_j)$.

This approach has got its rigorous formulation on the base of the resolvent approach used first in the random matrix theory in the pioneering work [24]. Regarding the resolvent $G_N$, the main subject is given by the infinite system of moments

$$L_k^{(N)}(X_k, Y_k; Z_k) = \mathbf{E} \prod_{j=1}^{k} G_N(x_j, y_j; z_j), \quad k \in \mathbb{N}. \quad (3.11)$$

The technique proposed in [21, 28] and developed in [22] has been employed in the study of eigenvalue distribution of various ensembles of random operators and random matrices [20, 26].
In present paper we use the moment relations approach in its modified version. The main observation here is that often it is sufficient to study asymptotic behavior of \(L_1^{(N)}\) and \(L_2^{(N)}\) instead of the whole infinite family of the moments (3.11). This considerably reduces amount of computations and makes the proofs more transparent. To explain the principal steps of the proofs of Theorems 3.1 and 3.2, let us present here the short proof of the semicircle law for GOE.

### 3.2.2 Derivation of system of relations

The main ingredients in the derivation of moment relations are the resolvent identity
\[
G'(z) - G(z) = -G(z) (H' - H) G'(z),
\]
where \(G'(z) = (H' - z)^{-1}\), \(G(z) = (H - z)^{-1}\) and equality
\[
\mathbf{E} \gamma F(\gamma) = \mathbf{E} \gamma^2 \mathbf{E} F'(\gamma),
\]
where \(\gamma\) is the Gaussian random variable with zero mathematical expectation and \(F(t), t \in \mathbf{R}\) is a nonrandom function such that all integrals in (3.13) exist. Equality (3.13) is a simple consequence of the integration by parts formula.

Let us consider (3.12) with \(H' = A_N\) (2.6) and \(H = 0\). We obtain relation
\[
G_N(x, y) = \zeta \delta_{xy} - \zeta \sum_{s=1}^{N} G_N(x, s) A_N(s, y), \quad \zeta \equiv -z^{-1}.
\]

Regarding the normalized trace
\[
f_N(z) = N^{-1} \sum G_N(x, x) \equiv \langle G_N \rangle
\]
and using (3.13), we obtain relation
\[
\mathbf{E} f_N = \zeta - \zeta \frac{v}{N^2} \sum_{x,s=1}^{N} (1 + \delta_{xs}) \mathbf{E} \frac{\partial G_N(x, s)}{\partial A_N(s, x)}.
\]

One can easily find the partial derivatives with the help of (3.12). Remembering that \(H\) are real symmetric matrices, we have
\[
\frac{\partial G(x, y)}{\partial H(s, t)} = - [G(x, s)G(t, y) + G(x, t)G(s, y)] (1 + \delta_{st})^{-1}.
\]

Substituting (3.16) into (3.15), we obtain the first main relation for \(L_1^{(N)}\)
\[
\mathbf{E} f_N = \zeta + \zeta v \mathbf{E} f_N^2 + \phi_1^{(N)} + \psi_1^{(N)},
\]
where
\[
\phi_1^{(N)} = \zeta v N^{-1} \mathbf{E} \langle G_N^2 \rangle, \quad \text{and} \quad \psi_1^{(N)} = \zeta v \mathbf{E} \{f_N f_N^\gamma\}.
\]
and we denoted by $\xi^o$ the centered random variable

$$\xi^o = \xi - E\xi.$$  

Clearly, $\langle G \rangle^o = \langle G^o \rangle$ (here and till the end of the subsection we omit the subscript $N$ in $G_N$). If one can show that two last terms in (3.17) vanish as $N \to \infty$, then convergence $E f_N(z) \to w(z)$ will be proved.

We estimate the term $\phi_1$ with the help of two elementary inequalities that hold for the resolvent of a real symmetric matrix:

$$|f_N(z)| \leq \|G(z)| \leq |\text{Im} z|^{-1},$$

and

$$\|G^2(z)\| \leq |\text{Im} z|^{-2}.$$  

The last estimate implies that

$$\sum_s |G(x,s)|^2 = \|G\|_2^2 \leq |\text{Im} z|^{-2}.$$  

(3.18)

Inequality (3.18) means that if $z \in \Lambda_\eta$, then

$$\left| \phi^{(N)}_1 \right| \leq v\eta^{-3} N^{-1}.$$  

3.2.3 Selfaveraging property

To show that $\lim_{N \to \infty} \psi^{(N)}_1 = 0$, we prove that the variance of $f_N$ vanishes

$$\text{Var} f_N = E |f_N|^2 = O(N^{-2}).$$  

(3.19)

It is clear that

$$\text{Var} f_N = E \tilde{f}_N \bar{f}_N = E \tilde{f}_N f_N,$$

where we denoted $\tilde{f}_N = f_N(z)$. Applying (3.14) to the last factor $f_N$, we see that

$$E \tilde{f}_N f_N = -\frac{\zeta}{N} \sum_{s,t=1}^N E \{ f_N^o G(x,s)A_N(s,x) \}. $$

The using (3.13) and (3.16), we derive relation

$$E \tilde{f}_N f_N = \zeta v \bar{f}_N f_N f_N + \phi_2^{(N)} + \psi^{(N)}_2,$$  

where $\phi_2^{(N)} = \zeta v N^{-1} E \tilde{f}_N \langle G^2 \rangle$ and

$$\psi^{(N)}_2 = 2 \zeta v N^{-2} E \langle \bar{G}^2 G \rangle.$$  

The useful observation is that

$$E \tilde{f}_N f_N f_N = E \tilde{f}_N f_N E f_N + E \tilde{f}_N f_N f_N.$$  

(3.21)
Using this identity and taking into account estimates (3.19), we derive from (3.18) that
\[ E\bar{f}_N^c f_N \leq v\eta^{-1} \left\{ E\bar{f}_N^c f_N \cdot E|f_N| + E\bar{f}_N^c f_N \cdot |f_N| \right\} + \]
\[ v\eta^{-3}N^{-1}E|f_N^c| + 2v\eta^{-4}N^{-2}. \]
Taking into account that \(|f_N^c|^2 = \bar{f}_N^c f_N^c f_N^c f_N^c|\), we finally obtain
\[ E|f_N^c|^2 = E\bar{f}_N^c f_N \leq \]
\[ 2v\eta^{-2}E|f_N^c|^2 + v\eta^{-3}N^{-1} \left( E|f_N^c|^2 \right)^{1/2} + 2v\eta^{-4}N^{-2}. \]  
(3.22)
This immediately implies (3.19) provided \( z \in \Lambda_\eta \). Obviously, \( \psi_1^{(N)} \) admits the same estimate.

3.2.4 The semicircle law and further corrections
Returning back to (3.17) and gathering estimates for \( \phi_1^{(N)} \) and \( \psi_1^{(N)} \), one can easily derive that if \( z \in \Lambda_\eta \), then \( \lim_{N \to \infty} g_N(z) = w(z) \), with \( w(z) \) given by (2.10). Convergence of the Stieltjes transforms implies convergence of the corresponding measures. Thus, the semicircle law is proved.

It should be noted that relation (3.21) can be transformed into
\[ E\bar{f}_N^c f_N f_N = 2E\bar{f}_N^c f_N E f_N + E\bar{f}_N^c f_N E^c. \]
Substituting this into (3.18), we see that
\[ \text{Var} f_N = \frac{1}{N^2} \frac{2\zeta v}{1 - 2\zeta vE f_N} E\langle G^2 G \rangle + \frac{1}{1 - 2\zeta vE f_N} \left( \phi_2^{(N)} + E \left\{ \bar{f}_N^c f_N^c f_N^c f_N^c \right\} \right), \]
(3.23)
Using the resolvent identity
\[ G(z_1)G(z_2) = -\frac{G(z_1) - G(z_2)}{z_1 - z_2}, \quad G(z_i) = (H - z_i)^{-1} \]
(3.24)
and convergence of \( E f_N(z) \), one can easily find the limiting expression for \( E\langle G^2 G \rangle \). If one assumes that two last terms in (3.23) are values of the order \( o(N^{-2}) \), then one arrives at (3.7) (see e.g. [20] for more details).

4 Correlation Function of the Resolvent
Our approach is to apply systematically the scheme of subsection 3.2.2 to get the leading term of the correlation function \( C^{(n,b)}(z_1, z_2) \) (3.4). This term is expressed via the limit of the \( \lim E\langle G^{(n,b)}(z) \rangle = w(z) \) but we have to prove the
pointwise version of this convergence given by Theorem 3.1. This and other auxiliary propositions are addressed in subsections 4.1 and 4.2. In subsection 4.3 we give the proof of Theorem 3.2 on the base of these statements. In what follows, we omit super- and subscripts \( n \) and \( b \) and do not indicate the limits of summation when no confusion can arise.

4.1 Proof of Theorem 3.1

Using relations (3.12)-(3.14) with obvious changes and repeating computations of subsection 3.2.2, we obtain relation

\[
E_G(x, x) = \zeta + \zeta v E_G(x) U_G(x) + \zeta v \sum_{|s| \leq n} E[G(x, s)]^2 U(s, x),
\]

(4.1)

where

\[
U_G(x) = \sum_{|s| \leq n} G(s, s) U(s, x) = \frac{1}{b} \sum_{|s| \leq n} G(s, s) u\left(\frac{s - x}{b}\right).
\]

Let us denote the average \( E_G(x, x) \) by \( g(x) \) and rewrite (4.1) in the following form

\[
g(x) = \zeta + \zeta v^2 g(x) U_r(x) + \frac{1}{b} \Phi(x) + \Psi(x),
\]

(4.2)

where we denoted (cf. (3.17))

\[
\Phi(x) = \zeta v \sum_{|s| \leq n} E[G(x, s)]^2 u\left(\frac{s - x}{b}\right)
\]

(4.3)

and

\[
\Psi(x) = \zeta v E G^o(x, x) U_r^o(x).
\]

(4.4)

Let us consider the solution \( \{r(x), |x| \leq n\} \) of equation

\[
r(x) = \zeta + \zeta v r(x) U_r(x), \quad |x| \leq n.
\]

(4.5)

Given \( z \in \Lambda_\eta \) (3.1), one can prove that the system of equations (4.5) is uniquely solvable in the set of \( N \)-dimensional vectors \( \{\bar{r}\} \) such that

\[
\|\bar{r}\|_1 = \sup_{|x| \leq n} |r(x)| \leq 2\eta^{-1}, \quad \eta = |\text{Im } z|
\]

(4.6)

(see Lemma 4.1 at the end of this section). Certainly, \( r(x) \) depends on particular values of \( z, n \) and \( b \), so in fact we use denotation \( r(x) = r_{n, b}(x; z) \).

The following statements concern the differences

\[
D_{n, b}(x; z) = g_{n, b}(x; z) - r_{n, b}(x; z), \quad d_{n, b}(x; z) = r_{n, b}(x; z) - w(z),
\]

where \( w(z) \) is given as a solution of (2.9).
Proposition 4.1.  
Given $\varepsilon > 0$, there exists a number $L = L(\varepsilon)$ such that for all sufficiently large $b$ and $n$ satisfying (2.13)
\[
\sup_{x \in B_L} |d_{n,b}(x; z)| \leq \varepsilon, \quad z \in \Lambda_\eta,
\] (4.7)
with $B_L$ given by (3.2).

Proposition 4.2.  
If $z \in \Lambda_\eta$ (3.1) and (2.13) holds, then
\[
\sup_{|x| \leq n} |D_{n,b}(x; z)| = o(1), \quad n, b \to \infty.
\] (4.8)

Theorem 3.1 follows from (4.7) and (4.8). Under the same conditions one can find $L' \geq L$ such that
\[
\sup_{x \in B_{L'}} \left| \frac{\zeta}{1 - \zeta \nu U_{\nu}(x)} - w(z) \right| \leq 2\varepsilon.
\] (4.9)
Relation (4.9) follows from (3.3) added by (4.6), a priori estimate
\[
\sup_{|x| \leq n} |g(x)| \leq \frac{1}{|\text{Im} z|},
\] (4.10)
and observation that $L'$ has to satisfy condition $u(L - L') \leq \varepsilon$.

Proof of Proposition 4.1  
Let us consider the constant function $w_x(z) \equiv w(z)$ satisfying (2.10) that we rewrite in the following form similar to (4.5)
\[
w_x(z) = \zeta + \zeta \nu w_x(z) \frac{1}{b} \sum_{|t| \leq n} b \delta_x w_t(z), \quad |x| \leq n.
\]
Subtracting this equality from (4.5), we derive that $d(x) \equiv d_{n,b}(x; z)$ verifies equality
\[
d(x) = \zeta v d(x) U_r(x) + \zeta \nu w(z) U_{\nu}(x) + \zeta \nu w^2(z) [P_b + T(x)],
\]
where
\[
P_b = \frac{1}{b} \sum_{t \in \mathbb{Z}} u \left( \frac{t}{b} \right) - \int_{-\infty}^{\infty} u(s)ds
\] (4.11)
and
\[
T_{n,b}(x) = \frac{1}{b} \sum_{|t| \leq n} u \left( \frac{t - x}{b} \right) - \frac{1}{b} \sum_{t \in \mathbb{Z}} u \left( \frac{t}{b} \right).
\] (4.12)
It is clear that $|P_b| = o(1)$ as $b \to \infty$. Indeed, one can determine an even step-like function $u_d(t), t \in \mathbb{R}$, such that

$$u_d(t) = \sum_{k \in \mathbb{N}} u\left(\frac{k}{b}\right)I_{(k-1/b, k/b)}(t), \quad t \geq 0.$$ 

Then $u_d(t) \leq u(t)$ and $u_d(t) \to u(t)$ as $b \to \infty$ and the Beppo-Lévy theorem implies convergence of the corresponding integrals of (4.10).

Taking into account equality $r(x) = \zeta_1 - v\zeta U r(x)$, we can write that

$$d(x) = vwr(x)U_d(x) + vw^2 r(x) \left[P_b + T_{n,b}(x)\right],$$

where we denoted $w \equiv w(z)$. This relation, together with estimates (4.6) and $|w(z)| \leq |\text{Im} \, z|^{-1}$, implies inequality

$$\sup_{x \in B_L} |d(x)| \leq \tau \left(\sup_{x \in B_{L-1}} |d(x)| + \sup_{x \in B_L} |T_{n,b}(x)| + P_b\right) \leq$$

$$\tau \sum_{j=0}^{L} \tau^j \left(\sup_{x \in B_{L-j}} |T_{n,b}(x)| + P_b\right) + \tau^L \sup_{|x| \leq n} |d(x)|, \quad (4.13)$$

where $\tau \leq v\eta^{-2} < 1$. It is clear that due to monotonicity of $u(t)$, one gets

$$\sup_{x \in B_{L+1}} |T_{n,b}(x)| \leq \sup_{x \in B_L} |T_{n,b}(x)| \leq \frac{2}{b} \sum_{t=n-Lb}^{\infty} u\left(\frac{t}{b}\right) \leq 2 \int_{L}^{\infty} u(s)ds.$$ 

Given $\epsilon$, one can find such a number $k$ that $\tau^k < \epsilon/4$. Then we derive from (4.13) that

$$\sup_{x \in B_L} |d(x)| \leq \tau \sum_{j=0}^{k} \tau^j \sup_{x \in B_{L-j}} |T_{n,b}(x)| + 2\tau P_b + \epsilon/4 \leq$$

$$\tau \left(k+1\right) \sup_{x \in B_{L-k}} |T_{n,b}(x)| + 2\tau P_b + \epsilon/4.$$ 

Now it is clear that (4.9) holds for sufficiently large $L$ and $b$.

Proposition 4.1 is proved.

*Proof of Proposition 4.2.*
Subtracting (4.5) from (4.2), we obtain relation for $D(x) = D_{n,b}(x)$

$$D(x) = \zeta v D(x) U_r(x) + \zeta v g(x) U_D(x) + \zeta v \left[ \frac{1}{b} \Phi(x) + \Psi(x) \right]$$

that can be rewritten in the form

$$D(x) = v g(x) r(x) U_D(x) + v r(x) \left[ \frac{1}{b} \Phi(x) + \Psi(x) \right]$$

Regarding this relation as the coordinate form of a vector equality, one can write that

$$\vec{D} = v \left( I - W^{(g,r)} \right)^{-1} \left[ \frac{1}{b} \vec{\Phi}^{(r)} + \vec{\Psi}^{(r)} \right],$$

where we denote by $W^{(g,r)}$ a linear operator acting on vectors $e$ with components $e(x)$ as

$$\left[ W^{(g,r)} e \right](x) = v g(x) r(x) \sum_{|s| \leq n} e(s) U(s,x)$$

and vectors

$$\vec{\Phi}^{(r)}_{n,b}(x) = r(x) \phi_{n,b}(x), \quad \vec{\Psi}^{(r)}(x) = r(x) \Psi(x).$$

It is easy to see that if $z \in \Lambda_{\eta}$, then the estimates (4.6) and (4.10) imply inequality

$$\left\| W^{(g,r)} \right\| \leq \frac{v}{\eta^2} < 1/2. \quad (4.14)$$

Thus, to prove Proposition 4.2, it is sufficient to show that

$$\sup_x \left| \sum_s \mathbb{E} \left[ G(x,s) \right] u \left( \frac{s - x}{b} \right) \right| = O(1), \quad z \in \Lambda_{\eta} \quad (4.15)$$

and

$$\sup_x \mathbb{E} \left| G^o(x,x) U_D^o(x,x) \right| = o(1), \quad z \in \Lambda_{\eta}. \quad (4.16)$$

Relation (4.15) is a consequence of the bound (2.2) and inequality (3.18). Relation (4.16) reflects the selfaveraging property of $G^{(n,b)}$. This question is addressed in the next subsection. It should be noted that (4.16) will be proved independently from computations of this subsection. Assuming that this is done, we can say that Theorem 3.1 is proved. We complete this subsection with the proof of the following auxiliary statement.

**Lemma 4.1.**

_Equation (4.5) has a unique solution in the class of vectors satisfying condition (4.6)._
Let us consider the sequence of $N$-dimensional vectors $\{\vec{r}(k), k \in \mathbb{N}\}$ determined by relations for their components

$$r^{(k+1)}(x) = \zeta + \zeta vr^{(k)}(x)U_{r^{(k)}}(x), \quad r^{(1)}(x) = \zeta, \quad |x| \leq n.$$  

Then it is easy to derive that if $\vec{r}^{(k)}$ satisfies (4.6) and $z \in \Lambda_\eta$ (3.1), then $\vec{r}^{(k+1)}$ also satisfies (4.6). The difference $\chi_{k+1}(x) = r^{(k+1)}(x) - r^{(k)}(x)$ satisfies relations

$$\chi_{k+1}(x) = \zeta v \chi_k(x) U_{r^{(k)}}(x) + \zeta vr^{(k-1)}(x) U_{\chi_k}(x).$$

Obviously, $\|\chi_{k+1}\|_1 \leq \alpha \|\chi_k\|_1$ with $\alpha < 1$ provided $z \in \Lambda_\eta$. Lemma is proved. $\square$

### 4.2 The variance and selfaveraging property

The asymptotic relation (4.15) is a consequence of the fact that the variance of $G(x,x)$

$$\text{Var}(G^{(n,b)}) = E \left|\langle G^{(n,b)} \rangle^2 \right| = E \left\{ \langle G^{(n,b)}(z) \rangle^2 \langle G^{(n,b)}(\bar{z}) \rangle^2 \right\}$$

vanishes as $n,b \to \infty$. Instead of the direct proof of (4.15), we prefer to present the whole list of more general statements needed in studies of the correlation function of $G$. All of them can be proved independently of the Theorem 3.1 without use of its statement.

We start the list with the following three relations that concern the moments of diagonal elements of $G$.

**Proposition 4.3.**

If $z \in \Lambda_\eta$ (3.1), then the estimates

$$\sup_{|x| \leq n} E |G^0(x,x;z)|^2 = O(b^{-1}), \quad (4.17)$$

$$\sup_{|x| \leq n} E |U^0_{G^0}(x)|^2 = O(b^{-2}), \quad (4.18)$$

and

$$\sup_{|x| \leq n} E |U^4_{G^0}(x)|^4 = O(b^{-4}), \quad (4.19)$$

hold.

The following statement concerns the mixed moments of variables $G^0(x,x;z)$ and their sums.

**Proposition 4.4.**
If $z \in \Lambda_{\eta}$, then relations

$$
\sup_{|x|,|y| \leq n} |E G^\circ(x, x) U_G^\circ(y)| = O\left(b^{-2}\right), \quad (4.20)
$$

$$
\sup_{|x| \leq n} |E (G^\circ) G^\circ(x, x)| = O\left(n^{-1} b^{-1} + b^{-1} \text{Var} \langle G \rangle^{1/2}\right), \quad (4.21)
$$

and

$$
\sup_{|x|,|y| \leq n} |E \langle G^\circ \rangle G^\circ(x, x) U_G^\circ(y)| = O\left(n^{-1} b^{-2} + b^{-2} \text{Var} \langle G \rangle^{1/2}\right) \quad (4.22)
$$

are true in the limit $n, b \to \infty$.

Finally, we formulate

**Proposition 4.5.**

If $z \in \Lambda_{\eta}$, then relation

$$
\sup_{|x| \leq n} \left| E \left\{ \langle G^\circ \rangle \sum_s [G_2(x, s)]^2 u_b^2(s, x) \right\} \right| = O\left(n^{-1} b^{-2} + b^{-2} \text{Var} \langle G \rangle^{1/2}\right) \quad (4.23)
$$

is true in the limit $n, b \to \infty$.

Let us not that the estimates (4.21)-(4.23) admit also the estimates in terms of $n$ and $b$ that do not involve the variance of $\langle G \rangle$. However, derivation of the estimates would take more place and time and we restrict ourselves with the forms presented. It will be shown later that $\text{Var} \langle G \rangle = O(n^{-1} b^{-1})$. This fact together with the restriction (2.12) implies for (4.22) and (4.23) that

$$
\frac{1}{b^2} \frac{1}{\sqrt{nb}} \ll \frac{1}{nb}
$$

that is sufficient for us. We prove Propositions 4.3-4.5 in Section 5.

**4.3 Toward the correlation function**

Let us have a more close look at the correlation function

$$
C_{n,b}(z_1, z_2) = E \left\{ \langle G^{(n,b)}(z_1) \rangle^\circ \langle G^{(n,b)}(z_2) \rangle^\circ \right\}
$$

We follow the scheme described at the end of subsection 3.2 and introduce variables $G_j(x, y) = G^{(n,b)}(x, y; z_j), j = 1, 2$. To study the average

$$
E \langle G_1^\circ \rangle G_2(x, x) = R_{12}(x),
$$
we apply to $G_2(x, x)$ the resolvent identity (3.12) and obtain relation

$$R_{12}(x) = -\zeta_2 \sum_{|s| \leq n} E \{ \langle G_1^0 \rangle G_2(x, s) a(s, x) \} \sqrt{U(s, x)},$$

where $\zeta_2 = -z_2^{-1}$. We compute the last mathematical expectation with the help of formulas (3.13) and (3.16) and obtain equality (cf. (4.1))

$$R_{12}(x) = \zeta_2 v R_{12}(x) U_{g_2}(x) + \sum_s E G_2^2(x, s) G_2(x, s) U(s, x) + \zeta_2 v \left[ \Theta_{12}(x) + \Upsilon_{12}(x) \right],$$

where we denoted

$g_2(x) = E G(x, x; z_2),$

$U_{g_2}(x) = \sum_{|s| \leq n} g_2(s) U(s, x),$

$U_{R_{12}}(x) = \sum_{|s| \leq n} R_{12}(s) U(s, x),$

$$\Theta_{12}(x) = E \left\{ \langle G_1^0 \rangle \sum_{|s| \leq n} [G_2(x, s)]^2 U(s, x) \right\},$$

and

$$\Upsilon_{12}(x) = E \left\{ \langle G_1^0 \rangle U_{G_2}(x) G_2^2(x) \right\}.$$

Using denotation

$$q_2(x) = \frac{\zeta}{1 - \zeta v U_{g_2}(x)},$$

we obtain the following relation for $R_{12}$

$$R_{12}(x) = v q_2(x) g_2(x) U_{R_{12}}(x) + \frac{2vq_2(x)}{N} \sum_s F_{12}(x, s) U(s, x) + v q_2(x) [\Theta_{12}(x) + \Upsilon_{12}(x)],$$

where we denoted

$$F_{12}(x, s) = E G_2^2(x, s) G_2(x, s).$$

The terms $\Theta$ and $\Upsilon$ can be estimated with the help of Propositions 4.3-4.5. As we shall see in the next subsection, they do not contribute to the leading term of $R_{12}$. To obtain the explicit expression for the this leading term, it is necessary to study in detail the variable $F_{12}$. Now let us formulate corresponding statement and the auxiliary relations needed.

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Proposition 4.6.
If \( z \in \Lambda_\eta \), (3.1), then for arbitrary positive \( \varepsilon \) and large enough values of \( b \) and \( n \) (2.19) there exists the set \( B_L \) (3.2) with \( L \) such that

\[
\sup_{x \in B_L} b[F_{12}U](x, x) - \frac{1}{2\pi} \frac{w_2^2 w_1}{1 - v w_1 w_2 u_F(p)} \int_{\mathbb{R}} \frac{\tilde{u}_F(p)}{(1 - v w_1 w_2 u_F(p))^2} dp \leq \varepsilon. \tag{4.26}
\]

The proof of Proposition 4.6 is based on the similar statement formulated for the product \( G_1 G_2 \).

Proposition 4.7.
Given positive \( \varepsilon \), there exists such \( L \) that relations

\[
\sup_{x \in B_L} b \sum_{|s| \leq n} E G_1(x, s) G_2(x, s) U^k(s, x) - \frac{1}{2\pi} \int_{\mathbb{R}} \frac{w_1 w_2 \tilde{u}_F(p)}{(1 - v w_1 w_2 u_F(p))^2} dp \leq \varepsilon \tag{4.27}
\]
and

\[
\sup_{x \in B_L} \sum_{|s| \leq n} E G_1(x, s) G_2(x, s) - \frac{w_1 w_2}{1 - v w_1 w_2} \leq \varepsilon \tag{4.28}
\]
hold for all \( k \geq 1 \), all \( z_i \in \Lambda_\eta \) and large enough values of \( b \).

Remark. In the case when \( z_1 \neq z_2 \), relation (4.28) can be derived from the resolvent identity (3.24) with the help of the convergence (3.3) and the explicit form of \( w(z) \) (2.9).

We prove Proposition 4.6 in the next subsection. Relations (4.27) and (4.28) will be proved in Section 5.

4.4 Proof of Proposition 4.6 and Theorem 3.2

Let us assume that relations (4.27) and (4.28) are true and show that under conditions of Theorem 3.2 the leading term of \( R_{12} \) is of the order \( O(n^{-1}b^{-1}) \) and terms \( \Theta_{12} \) and \( \Upsilon_{12} \) of (5.2) do not contribute to it. We rewrite (4.25) in the form

\[
R_{12}(x) = v q_2(x) q_2(x) U_{R_{12}}(x) + 2 v q_2(x) N^{-1} [F_{12}U](x, x) + v q_2(x) [\Theta_{12}(x) + \Upsilon_{12}(x)]. \tag{4.29}
\]

Let us denote \( r_{12} = \sup_{|x| \leq n} |R_{12}(x)| \).
Taking into account \( U(x, y) \leq \bar{u}/b \) (2.2) and using inequalities of the form (3.19), it is easy to see that if \( z_i \in \Lambda_\eta \), then

\[
\frac{1}{N} [F_{12}U](x) \leq \frac{1}{Nb} E \left( \sum_s |G_1^2(x, s)|^2 \right)^{1/2} \left( \sum_s |G_2(s, x)|^2 \right)^{1/2} = O \left( \frac{1}{nb} \right).
\]

Regarding this estimate and relations (4.22), (4.23) we easily derive from (4.29) inequality (cf. (3.22))

\[
r_{12} \leq \frac{v}{n^2} r_{12} + \frac{C}{bn} + \frac{1}{b^2} \sqrt{r_{12}}
\]

with some constant \( C \). Since \( r_{12} \) is bounded for all \( z \in \Lambda_\eta \), then

\[
r_{12} = O \left( \frac{1}{nb} + \frac{1}{b^2} \right).
\]

Now condition (2.12) implies that \( r_{12} = O(1/nb) \) and therefore the general form of (3.5) is demonstrated.

Substituting (3.5) into the estimates (4.22) and (4.23), we obtain that

\[
\|\Theta_{12}\|_1 = o \left( \frac{1}{nb} \right) \quad \text{and} \quad \|\Upsilon_{12}\|_1 = o \left( \frac{1}{nb} \right).
\]

Thus, these terms of (4.29) do not contribute to the leading term of \( R_{12} \). To find this term in explicit form, we need the result of Proposition 4.6.

**Proof of Proposition 4.6.**

Regarding \( F_{12}(x, y) = E G_1^2(x, y) G_2(x, y) \), we apply to \( G_2 \) the resolvent identity (3.12). Computations similar to those of subsection 3.2.2 lead us to equality

\[
F_{12}(x, y) = \zeta_2 \delta_{xy} E G_2^2(x, x) + \zeta_2 v [t_{12}U](x, y) E G_2^2(y, y) + \zeta_2 v \left\{ [F_1 U](x, y) g_1(y) + F_{12}(x, y) U_{[g_2]}(y) + \Gamma(x, y) \right\},
\]

(4.30)

where

\[
t_{12}(x, y) = E T_{12}(x, y) = E G_1(x, y) G_2(x, y),
\]

and the vanishing terms are denoted by \( \Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 \):

\[
\Gamma_1(x, y) = \sum_s E \left\{ \Gamma_1(x, y) G_2^2(s, y) G_2(x, s) + 2 G_1^2(x, y) G_2(s, y) G_2(x, s) \right\} U(s, y),
\]

\[
\Gamma_2(x, y) = E \left\{ [T_{12}U](x, y) \left[ G_1^2(y, y) \right]^0 \right\} + E \left\{ [F_{12}U](x, y) G_2^2(y, y) \right\},
\]

and

\[
\Gamma_3(x, y) = E F_{12}(x, y) U_{[G_2]}(y).
\]

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Indeed, it is easy to show that
\[
\sup_{x,y} |\Gamma_j(x,y)| = O(b^{-1}), \quad z_1, z_2 \in \Lambda_{\eta}. \tag{4.31}
\]
This can be done with the help of inequality (3.19) and relations (4.17), (4.18), and (4.23).

Using definition of \(q_2(x)\) (4.24), we rewrite (4.30) as
\[
F_{12}(x,y) = v g_1(x)q_2(y) [F_{12}U](x,y) + R^{(1)}(x,y) + R^{(2)}(x,y) + v \tilde{\Gamma}(x,y), \tag{4.32}
\]
where we denoted
\[
R^{(1)}(x,y) = q_2(x) EG_1^2(x,x) \delta_{xy}, \tag{4.33a}
\]
and \(\tilde{\Gamma}(x,y) = \Gamma(x,y)q_2(y)\). Let us note that \(|R^{(1)}| \leq \eta^{-3}\) and \(|R^{(2)}| \leq v \eta^{-5}\) for \(z_i \in \Lambda_{\eta}\).

Let us determine the linear operator \(W\) that acts on \(N \times N\) matrices \(F\) according to the formula
\[
[W F](x,y) = v g_1(x) \left[ \sum_{|s| \leq n} F(x,s) U(s,y) \right] q_2(y).
\]
The a priori estimates \(|g_1(x)| \leq |\text{Im} z_1|^{-1}\), and \(|q_2(x)| \leq |\text{Im} z_2|^{-1}\) imply inequality (cf. (4.14))
\[
\|W\|_{(1,1)} \leq \frac{v}{\eta^2} < \frac{1}{2}, \quad z_i \in \Lambda_{\eta}, \tag{4.34}
\]
where the norm of \(N \times N\) matrix \(A\) is determined as \(\|A\|_{(1,1)} = \sup_{x,y} |A(x,y)|\).

This estimate is verified by direct computation of \(\|W A\|_{(1,1)}\) with \(\|A\|_{(1,1)} = 1\).

Then (4.32) can be rewritten as
\[
F_{12}(x,y) = v \sum_{m=0}^{\infty} \left[ W^m \left( R^{(1)} + R^{(2)} + v \tilde{\Gamma} \right) \right](x,y). \tag{4.35}
\]

The next steps of the proof of (4.26) are very elementary. We consider the first \(M\) terms of the infinite series and use the decay of the matrix elements \(U(x,y) = U^{(b)}(x,y)\). Indeed, if one considers (4.33) with \(x\) and \(y\) taken far enough from the endpoints \(-n, n\), then the variables \(g_1(s), q_2(t)\) enters into the finite series with \(s\) and \(t\) also far from the endpoints. Then one can use relations (3.3) and (4.9) and replace \(g_1\) and \(q_2\) by the constant values \(w_1\) and \(w_2\), respectively. This substitution leads simplifies expressions with the error terms that vanish as \(n, b \to \infty\). The second step is similar. It is to show that we
can use Proposition 4.7 and replace the terms \( R^{(1)} \) and \( R^{(2)} \) of the finite series of (4.33) by corresponding expressions given by formulas (4.27) and (4.28).

Let us start to perform this program. Taking into account the estimate of \( \Gamma \) and using boundedness of terms \( R^{(1)} \) and \( R^{(2)} \), we can deduce from (4.35) that

\[
b \sum_s F_{12}(x, s)U(s, x) = bv \sum_{m=0}^M \left[ W^m(R^{(1)} + R^{(2)})U \right](x, x) + \Delta^{(1)}(x, x),
\]

(4.36)

where \( M \) is such that given \( \varepsilon > 0, |\Delta^{(1)}(x, x)| < \varepsilon \) for large enough \( b \) and \( n \).

Now let us find such \( h \) that the following holds

\[
u(t) \leq \varepsilon, \forall |t| \geq h, \quad \text{and} \quad \int_{|t| \geq h} u(t) dt \leq \varepsilon.
\]

We determine matrix

\[
\hat{U}(x, y) = \begin{cases} 
U(x, y), & \text{if } |x - y| \leq bh; \\
0, & \text{if } |x - y| > bh
\end{cases}
\]

and denote by \( \hat{W} \) corresponding linear operator

\[
[\hat{WF}](x, y) = vg_1(x) \left[ \sum_{|s| \leq n} F(x, s)\hat{U}(s, y) \right] q_2(y).
\]

Certainly, \( \hat{W} \) admits the same estimate as (4.34).

Given \( \varepsilon > 0 \), let \( L \) the largest number among those required by conditions of Propositions 4.1 and 4.7. Let us denote by \( Q \) the first natural greater than \( (M + k)h \). Then one can write that

\[
bv \sum_{m=0}^M \left[ W^m(R^{(1)} + R^{(2)})U \right](x, x) = 
\]

\[
bv \sum_{m=0}^M \left[ (v\hat{W})^m(R^{(1)} + R^{(2)})\hat{U} \right](x, x) + \Delta^{(2)}(x, x),
\]

where

\[
\sup_{x \in B_L+Q} |\Delta^{(2)}(x, x)| \leq \varepsilon, \quad \text{as } n, b \to \infty.
\]

(4.37)

The proof of (4.37) uses elementary computations. Indeed, \( \Delta^{(2)}(x, x) \) is represented as the sum of \( M + 1 \) terms of the form

\[
bv^{m+1} \sum_{|x_1| \leq n} [g_1(x)]^m F(x, x_1)U(x, x_1)q_2(x_1) \cdots U(x_{m-2}, x_{m-1})q_2(x_{m-1}) \times
\]

\[
\]
\[ [R^{(1)} + R^{(2)}](x_{m-1}, x_m)U(x_m, x), \]

where the sum is taken over the values of \( x_i \) such that \( |x_j - x_{j+1}| > bh \) at least for one of the numbers \( j \leq m \).

Now remembering the a priori bounds for \( R^{(1)} \) and \( R^{(2)} \), estimates like (4.13) and taking into account the diagonal form of \( R^{(1)} \), one obtains the following estimate of \( \Delta^{(2)} \) by two terms

\[
\sup_{|x| \leq n} |\Delta^{(2)}(x, x)| \leq \frac{M}{\eta^{2m+3}} \sum_{|x_i| \leq n} U(x, x_1)U(x_1, x_2) \cdots U(x_m, x) + \frac{M}{\eta^{2m+5}} \sum_{|x_i| \leq n} U(x, x_1)U(x, x_m - 2, x_m - 1)U(x_m, x). \tag{4.38} \]

Regarding the first term in the right-hand side of (4.38) and assuming that \( |x_j - x_{j+1}| > bh \), one can observe that for large enough \( b, n \)

\[
\sum_{|x_j| \leq n} U^j(x, x_j)U^{m-j}(x_{j+1}, x) \leq \varepsilon.
\]

Indeed,

\[
\sum_{|x_i| \leq n} U(x, x_1)U(x_1, x_2) \cdots U(x_{j-1}, x_j) \leq \sum_{x_i \in \mathbb{Z}} U(x_1, x_2) \cdots U(x_{j-1}, x_j) \leq \left[ \int_{-\infty}^{\infty} u(t)dt + \frac{u(0)}{b} \right]^j \leq (1 + \bar{u}/b)^j.
\]

Let us also mention here that given \( \varepsilon > 0 \), one has for large enough \( n, b \) that

\[
\sup_{x \in B_{L+Q}} \left| \sum_{|s| \leq n} U^j(x, s) - 1 \right| \leq \varepsilon, \tag{4.39}
\]

where \( j \leq M \). This follows from elementary computations related with the differences (4.11) and (4.12) that vanish in the limit \( 1 \ll b \ll n \).

Similar but a little more modified reasoning can be used to estimate the second term in the right-hand side of (4.38). Now one can write that

\[
\sup_{|x| \leq n} |\Delta^{(2)}(x, x)| \leq 2\varepsilon \sum_{m=0}^{M} \frac{M\eta^{m+1}}{\eta^{2m+2}} \leq \varepsilon.
\]

Regarding the right-hand side of (4.37) with \( x \in B_{L+Q} \), one observes that the summations run over such values of \( x_j \) that \( |x - x_1| \leq bh, |x_i - x_{i+1}| \leq bh \), and thus \( x_j \in B_L \) for all \( j \leq k + m - 1 \). This means that we can apply relations
(3.3) and (4.9) to the right-hand side of (4.37) and replace \( g_1(x) \) by \( w(z_1), q_2(x) \) by \( w(z_2) \). We derive from (4.36) that

\[
(F_{12}U^k)(x, x) = bw_2 \sum_{m=0}^{M} (vw_1w_2)^m \hat{U}^m(x, s) \left[ R^{(1)} + R^{(2)}(s, t)\hat{U}(t, x) + \Delta^{(3)}(x, x) \right]
\]

with

\[
\sup_{x \in B_{L+Q}} |\Delta^{(3)}(x, x)| \leq 4\varepsilon.
\]

Finally, applying Proposition (4.7) to the expressions involved in \( R \) and taking into account that

\[
\sup_{x \in B_{L+Q}} |bU^{m+1}(x, x) - \frac{1}{2\pi} \int \hat{u}_F^{m+1}(p)dp| \leq \varepsilon,
\]

we obtain equality

\[
(F_{12}U)(x, x) = \frac{v}{2\pi} w_1^2w_2 \sum_{m=0}^{M} (vv_1v_2)^m \int \frac{\hat{u}_F^{m+1}(p)}{1 - vv_1v_2\hat{u}_F}dp + \Delta^{(5)}(x, x)
\]

with

\[
\sup_{x \in B_{L+Q}} |\Delta^{(5)}(x, x)| \leq \varepsilon \quad b, n \to \infty.
\]

Passing back in (4.41) to the infinite series and simplifying them, we arrive at the expression standing in the right-hand side of (4.26). Proposition is proved. \( \square \)

Let us complete the proof of Theorem 3.2. Remembering estimate (4.14), we can iterate relation (4.29) and obtain that

\[
R_{12}(x) = \frac{2vq_2(x)}{Nb} \sum_{m=0}^{\infty} \left[ (W^{(q_2-q_2)})^m f_{12}(x) + o(1/nb) \right]
\]

where we denoted \( f_{12}(x) = bq_2(x)[F_{12}U](x, x) \). Regarding the trace

\[
\frac{1}{N} \sum_{|x| \leq n} R_{12}(x) = \frac{1}{N} \sum_{x \in B_L} R_{12}(x)(1 + o(1))
\]

and repeating the arguments of the proof of Proposition 4.6 presented above, we can write that

\[
R_{12}(x) = \frac{2vw_2}{Nb} \sum_{m=0}^{M} \sum_{t} (bF_{12}U)(t, t)(vw_2^2U)^m(t, x) + \Delta^{(6)}(x, x)
\]
with \( \sup_{x \in B_L} |\Delta(6)(x, x)| \leq \text{vep}' \) provided \( n, b \to \infty \) (2.12). Finally, observing that \((bF_{12}U)(t, t)\) asymptotically does not depend on \( t \) (4.26), we arrive, with the help of (4.39), at the expression (3.6). Theorem 3.2 is proved.

5 Proof of auxiliary statements

Proof of Proposition 4.3

Let us consider the average \( \mathbb{E}G^0_1(x, x)G_2(y, y) \) and derive for it, with the help of formulas (3.12), (3.13) and (3.16) equality

\[
\mathbb{E}G^0_1(x, x)G_2(y, y) = \zeta_2 v \mathbb{E}G^0_1(x, x)G_2(y, y)U_{G_2}(y) + \\
\zeta_2 v \sum_s \mathbb{E}G^0_1(x, x) [G_2(y, s)]^2 U(s, y) + \\
2 \zeta_2 v \sum_s \mathbb{E}G_1(x, s)G_1(y, x)G_2(y, s)U(s, y).
\]

Applying to the first term of this equality the analogue of identity (3.21) and using \( q_2(x) \) (4.24), we obtain that

\[
\mathbb{E}G^0_1(x, x)G_2(y, y) = vq_2(y) \mathbb{E}G^0_1(x, x)G_2(y, y)U_{G_2}(y) + \\
vq_2(y) \sum_s \mathbb{E}G^0_1(x, x) [G_2(y, s)]^2 U(s, y) + \\
2vq_2(y) \sum_s \mathbb{E}G_1(x, s)G_1(y, x)G_2(y, s)U(s, y).
\]

(5.1)

We multiply both sides of this relation by \( U(x, t) \) and sum it over \( x \); then we get

\[
\mathbb{E}U^0_{G_1}(t, t)G_2(y, y) = vq_2(y) \mathbb{E}U^0_{G_1}(t)G_2(y, y)U_{G_2}(y) + \\
vq_2(y) \sum_s \mathbb{E}U^0_{G_1}(t) [G_2(y, s)]^2 U(s, y) + \\
2vq_2(y) \sum_s \mathbb{E}G_1(x, s)G_1(y, x)G_2(y, s)U(s, y)U(x, t).
\]

(5.2)

Regarding \( G_1(y, \cdot)U(\cdot, t) \) and \( G_2(y, \cdot)U(\cdot, y) \) in the last term as vectors in \( N \)-dimensional space, we derive from estimate (3.19) that

\[
\left| \sum_{s, x} \mathbb{E}G_1(x, s)G_1(y, x)G_2(y, s)U(s, y)U(x, t) \right| \leq \|G_1\| \left( \sum_x |G_1(y, x)U(x, t)|^2 \right)^{1/2} \left( \sum_s |G_2(y, s)U(s, y)|^2 \right)^{1/2}.
\]

(5.3)
Inequality (3.18) implies that the right-hand side of (5.3) is bounded by $b^{-2}\eta^{-3}$.

Let us multiply both sides of (5.2) by $U(y, r)$ and sum them over $y$. Then one obtains a relation that together with (3.18) and (5.3) implies the following estimate for variable $M_{12} = \sup_x \left( E \left| U_{G_1}^o(x) \right|^2 \right)^{1/2}$:

$$M_{12} \leq v\eta^{-2}M_{12} + v\eta^{-3}b^{-1}\sqrt{M_{12}} + 2v\eta^{-4}b^{-2}.$$

This proves (4.18).

Now (4.17) follows from (4.18) and relation (5.1).

To derive estimate (4.19), let us consider the variable

$$E U_{G_1}^o(x_1)U_{G_2}^o(x_2)U_{G_3}^o(x_3)U_{G_4}^o(x_4) = E \left[ U_{G_1}^o(x)U_{G_2}^o(x)U_{G_3}^o(x_3) \right]^o U_{G_4}^o(x_4).$$

Let us denote $T = U_{G_1}^o, U_{G_2}^o, U_{G_3}^o$ and $M(x_1, x_2, x_3, t) = ET^oG_4(t, t)$. We apply to $G_4(t, t)$ resolvent identity (3.14) and obtain relation

$$ET^oG_4(t, t) = v\zeta_4 ET^oG_4(t, t)U_{G_4}(t) +$$

$$v\zeta_4 ET^o \sum_s [G_4(s, t)]^2 U(s, t) +$$

$$v\zeta_4 \sum_{(i, j, k)} E U_{G_1}^o(x_i)U_{G_j}^o(x_j) \sum_{x, s, t} G_k(y, s)G_k(t, y)U(y, x_k)G_4(t, s)U(s, t).$$

(5.4)

Repeating previous computations and applying similar estimates, we obtain inequality

$$\left| \sum_t M(x_1, x_2, x_3, t)U(t, x_4) \right| \leq \frac{v}{\eta} E|TU_{G_1}^o(x_4)| + \frac{v}{\eta} E|T| E|U_{G_4}^o(x_4)| +$$

$$\frac{v}{\eta^3 b^2} E|T| + \frac{v}{\eta b^2} E|U_{G_1}^o(x_i)U_{G_j}^o(x_j)|.$$

(5.5)

Here we have applied inequalities (3.18) and (5.3) to the last two terms of relation (5.4). Now it is clear that (5.5) implies (4.19). Proposition 4.3 is proved.$\square$

**Proof of Proposition 4.4**

Estimate (4.20) follows from relation (5.2) and estimate (4.18). Regarding (5.1) and summing it over $x$, one can easily derive (4.21) with the help of the arguments used to prove (4.18).

Let us turn to the proof of (4.22). To do this, let us consider the variable

$$K(x, y) = E \langle G^o \rangle G^o(x, x)U_{G_1}^o(y) = E \left[ (G^o) U_{G_1}^o(y) \right]^o G(x, x$$
and apply to the last expression resolvent identity (3.12) and formulas (3.13) and (3.16). We obtain equality that can be written in the following form with denotation 
\[ R = \langle G^o \rangle U_G^o(y) \]

\[ E R^o G(x, x) = \zeta v E R^o G(x, x) U_G(x) + \sum_{i=1,2,3} \kappa_i(x, y), \quad (5.6) \]

where 
\[ \kappa_1(x, y) = \zeta v \sum_s E R^o G(x, s) G(x, s) U(s, x), \]
\[ \kappa_2(x, y) = 2\zeta v \sum_{s,t} E \langle G^o \rangle G(t, s) G(x, t) u^2_\kappa(t, y) G(x, s) U(s, x), \]
and 
\[ \kappa_3(x, y) = 2\zeta v N^{-1} \sum_{s,t} E G(t, s) G(x, t) U^c(y) G(x, s) u^2_\kappa(s, x). \]

Let us use identity 
\[ E R^o XY = E RX^o E Y + E RY^o E X + E R X^o Y^o - E R E X^o Y^o. \]
and can rewrite (5.6) in the form

\[ E R^o G(x, x) = \frac{vq(x)}{1 - vq(x)g(x)} [E U^c_G(x) G^o(x, x) - E \langle G^o \rangle U^c_G(y) E G^o(x) U^c_G(x)] + \]
\[ \frac{vq(x)}{1 - vq(x)g(x)} \sum_{i=1,2,3} \kappa_i(x, y). \quad (5.7) \]

Taking into account relation (4.18), inequalities (3.18) and (5.3), we obtain that
\[ |\kappa_i(x, y)| \leq 2\eta^{-2} b^{-2} (\text{Var} \langle G \rangle)^{1/2} \text{ for } i = 1, 2 \]
and
\[ |\kappa_3(x, y)| \leq 2\eta^{-3} b^{-2} N^{-1}. \]
Using them, we derive from (5.7) inequality
\[ |K(x, y)| \leq 2\eta^{-1} (\text{Var} \langle G \rangle)^{1/2} \left\{ \left( E |U^o_G(x)|^4 \right)^{1/2} + b^{-2} \left( E |U^o_G(x)|^2 \right)^{1/2} \right\} + \]
\[ 2\eta^{-1} b^{-2} (\text{Var} \langle G \rangle)^{1/2} + 2\eta^{-2} b^{-2} N^{-1}. \]
This leads to estimate (4.22). Proposition 4.4 is proved. \(\square\)

Proof of Proposition 4.5.

This proof of the estimate (4.23) is the most cumbersome. Here we have to use the resolvent identity (3.12) twice. However, the computations are based
on the same inequalities as those of the proofs of Propositions 4.3 and 4.4. Therefore we just indicate the principal lines of the proof and do not present the derivations of estimates.

To compute the mathematical expectation

\[ EM(x, s) = \mathbb{E} \langle G_1^0 \rangle [G_2(x, s)]^2, \]

let us apply to \( G_2(x, s) \) the resolvent identity (3.12). We obtain equality

\[ EM(x, s) = \zeta_2 \frac{u(0)}{b} \mathbb{E} \langle G_1^0 \rangle G_2(x, x) - \zeta_2 \mathbb{E} \langle G_1^0 \rangle \sum_t G_2(x, s) G_2(x, t) a(t, s) \sqrt{U(t, s)}. \]  

Relation (4.21) implies that the first term of the right-hand side of (5.8) is the value of the order indicated in (4.23). Let us consider the second term of (5.8). We compute mathematical expectation with the help of relations (3.13) and (3.16) and obtain expression

\[ \zeta_2 \mathbb{E} \langle G_1^0 \rangle \sum_t G_2(x, s) G_2(x, t) a(t, s) \sqrt{U(t, s)} = \sum_{i=1}^5 \Theta_i(x, s), \]  

where

\[ \Theta_1(x, s) = v\zeta_2 \mathbb{E} \langle G_1^0 \rangle G_2(x, s) G_2(x, s) \mathbb{E} U_G(s), \]

\[ \Theta_2(x, s) = v\zeta_2 \mathbb{E} \langle G_1^0 \rangle G_2(x, s) G_2(x, s) U_G^2(s), \]

\[ \Theta_3(x, s) = \frac{2v\zeta_2}{N} \mathbb{E} \sum_t G_2^2(s, t) U(t, s) G_2(x, s) G_2(x, t), \]

\[ \Theta_4(x, s) = v\zeta_2 \mathbb{E} \langle G_1^0 \rangle \sum_t [G_2(x, t)]^2 U(t, s) G_2(s, s), \]

and

\[ \Theta_5(x, s) = 2v\zeta_2 \mathbb{E} \langle G_1^0 \rangle \sum_t G_2(x, s) G_2(x, t) G_2(s, t) U(t, s). \]

\( \Theta_1 \) is of the form \( v\zeta_2 \mathbb{E} M(x, s) \mathbb{E} U_G(s) \) and can be put to the right-hand side of (5.9). The terms \( \Theta_2 \) and \( \Theta_3 \) are of the order indicated in the right-hand side of (4.23). This can be shown with the help of estimates of the form (5.3).

Regarding \( \Theta_4 \), we apply the resolvent identity (3.12) to factor \( G_2(s, s) \). Repeating the usual computations based on (3.13) and (3.16), we obtain that

\[ \Theta_4(x, s) = v\zeta_2^2 \sum_t EM(x, t) U(t, s) + v\zeta_2 \Theta_4(x, s) \mathbb{E} U_{G_2}(s) + \Omega(x, s), \]

where \( \Omega \) gathers the terms that are all of the order indicated in (4.23). This can be verified by direct computation with the use of estimates (4.18), (4.21), and
Not to overload this paper, we do not write down the terms constituting \( \Omega \) and do not present their estimates as well. Relation (5.10) is of the from that leads to the estimates needed for \( \sum E M(x,s)U(s,x) \).

Regarding \( \Theta_5(x,s) \), we apply (3.12) to \( G_2(s,t) \) and obtain, after the use of (3.13) and (3.16) that

\[
\Theta_5(x,s) = 2v\zeta^2 \frac{u(0)}{b} EM(x,s) + v\zeta \Theta_5(x,s) E U_{G_2}(s) + \Omega'(x,s),
\]

where \( \Omega'(x,s) \) consists of the terms that are of the order indicated in (4.23).

The form of (5.11) is also such that, being substituted into (5.9) and then into (5.8), it leads to the estimates needed. This observation shows that (4.23) is true.

**Proof of Proposition 4.7.**

We prove relation (4.27) with \( k = 1 \) because the general case does not differ from this one. To derive relations for the average value of variable \( t_{12}(x,y) = E G_1(x,y) G_2(x,y) \), we use identities (3.12)-(3.14) and repeat the proof of Proposition 4.6. Simple computations lead us to equality

\[
t_{12}(x,y) = g_1(x) q_2(x) \delta_{xy} + v g_1(y) q_2(y) [t_{12}] (x,y) + \Delta(x,y),
\]

(5.13)
where
\[
\sup_{x,y} |\Delta(x,y)| = o(1) \quad \text{and} \quad \sup_{x} \left| \sum_{y} \Delta(x,y) \right| = o(1) \quad (5.14)
\]
in the limit \(n, b \to \infty\) (2.12).

We rewrite relation (5.13) in the matrix form (cf. (4.35))
\[
t_{12} = \left( I - W^{(g,q)} \right)^{-1} \left[ \text{Diag}(g_1q_2) + \Delta \right] = \sum_{m=0}^{\infty} [W^{(g,q)}]^m (\text{Diag}(g_1q_2) + \Delta).
\]
Now we can apply to (5.15) the same arguments as to (4.35). Replacing \(g_1(x)\) and \(q_2(x)\) by \(w_1\) and \(w_2\), respectively, we derive from (5.14) that for \(x \in B_{L+Q}\)
\[
t_{12}(x,s) = \sum_{m=0}^{M} (w_1w_2)^{m+1} [U^{m}] (x,s) + o(1), \quad n, b \to \infty. \quad (5.16)
\]
Multiplying both sides of (5.16) by \(U(s,x)\) and summing it over \(s\), we obtain relation
\[
\sum_{|s| \leq n} t_{12}(x,s)U(s,x) = \sum_{m=0}^{M} (w_1w_2)^{m+1} \sum_{s} [U^{m}] (x,s) + o(1), \quad n, b \to \infty. \quad (5.17)
\]
Now convergence (4.40) implies relation that leads, with \(M\) replaced by \(\infty\), to (4.27).

To prove (4.28), let us sum (5.16) over \(s\). The second part of (5.14) tells us that the terms \(\Delta\) remains small when summed over \(s\). Thus we can write relations
\[
\sum_{s} t_{12}(x,s) = \sum_{m=0}^{M} (w_1w_2)^{m+1} \sum_{s} [U^{m}] (x,s) + o(1), \quad n, b \to \infty. \quad (5.18)
\]
Taking into account estimates for terms (4.11) and (4.12), it is easy to observe that convergence (4.39) together with (5.18) implies (4.28). \(\square\)

6  Asymptotic properties of \(S(z_1, z_2)\)

In the last decade, the main focus of the spectral theory of random matrices is related with the universality conjecture of local spectral statistics put forward first by F. Dyson \[13\]. This problem is addressed in a large number of papers where various random matrix ensembles are studied using different approaches (see e.g. the review \[16\]). The best understood are the Gaussian Unitary Ensemble (GUE) and its real symmetric analogue GOE (see (2.5)). The probability distribution of these ensembles are invariant with respect to the unitary
(orthogonal) transformations. This leads to the fact that the joint probability distribution of eigenvalues of these ensembles does not depend on the distribution of eigenvectors and is given in explicit form [2]. This allows one to use the powerful technique of the orthogonal polynomials that provides a detailed information of the spectral properties of GUE and GOE and related ensembles on the local scale (see [3, 14] for the initial results for Gaussian ensembles and [3, 11] for their generalizations).

The case of band random matrices is different because the probability distribution of the ensemble $H^{(n,b)}$ (2.4) is no more invariant under transformations of the coordinates. One of the possible ways to study the spectral properties of $H^{(n,b)}$ is to follow the resolvent expansions approach well-known in theoretical physics (see, for example [14]). A rigorous version of it has been developed in a series of papers [21, 22, 20].

In frameworks of the resolvent approach (see [20] for details), one considers the correlation function $C_{n,b}(z_1, z_2), \text{Im} z_j \neq 0$ (3.4) in the limit when the dimension of the matrix $N$ infinitely increases. Asymptotic expression for $S(z_1, z_2)$ regarded in the limit $z_1 = \lambda_1 + i0$, $z_2 = \lambda_2 - i0$ supplies one with the information about the local properties of eigenvalue distribution provided $\lambda_1 - \lambda_2 = O(N^{-1})$. Indeed, according to (2.7), the formal definition of the eigenvalue density $\rho_{n,b}(\lambda) = \sigma_{n,b}'(\lambda)$ is

$$\rho_{n,b}(\lambda) = \frac{1}{2i} [f_{n,b}(\lambda + i0) - f_{n,b}(\lambda - i0)].$$

Then one can consider expression

$$R_{n,b}(\lambda_1, \lambda_2) = -\frac{1}{4} \sum_{\delta_1, \delta_2 = -1, +1} \delta_1 \delta_2 C_{n,b}(\lambda_1 + i\delta_1 0, \lambda_2 + i\delta_2 0)$$

as the correlation function of $\rho_{n,b}$. In general, even if $R_{n,b}$ can be rigorously determined, it is difficult to carry out the direct study of it. Taking into account relation (3.5), one can pass to more simple expression

$$\Sigma_{n,b}(\lambda_1, \lambda_2) = -\frac{1}{4N b} \sum_{\delta_1, \delta_2 = -1, +1} \delta_1 \delta_2 S(\lambda_1 + i\delta_1 0, \lambda_2 + i\delta_2 0)$$

(6.1)

and assume that it corresponds to the leading term of $R_{n,b}(\lambda_1, \lambda_2)$ in the limit $n, b \rightarrow \infty$.

In present section we follow the same heuristic scheme. It should be noted that for Wigner random matrices this approach is justified by the study of the simultaneous limiting transition $N \rightarrow \infty, \text{Im} z_j \rightarrow 0$ in the studies of $C_N(z_1, z_2)$ [5, 19].
Theorem 6.1.
Let $S(z_1, z_2)$ is given by (3.6). Assume that function $\tilde{u}_F(p)$ is such that there exist positive constants $c_1, \delta$ and $\nu > 1$ that

$$\tilde{u}_F(p) = \tilde{u}_F(0) - c_1 |p|^{\nu} + o(|p|^{\nu})$$

for all $p$ such that $|p| \leq \delta$, $\delta \to 0$. Then

$$\Sigma_{n, b}(\lambda_1, \lambda_2) = \frac{1}{Nb} \frac{c_2}{|\lambda_1 - \lambda_2|^{2-1/\nu}} (1 + o(1))$$

for all $\lambda_j$, $j = 1, 2$ satisfying

$$\lambda_1, \lambda_2 \to \lambda \in (-2\sqrt{v}, 2\sqrt{v}).$$

Proof of Theorem 6.1.
Let us start with the terms of (6.1) that correspond to $\delta_1 \delta_2 = -1$. It follows from (2.9) that

$$1 - vw_1 w_2 = \frac{z_1 - z_2}{w_1 - w_2}.$$  

(6.5)

Also for the real and imaginary parts of $w(\lambda + i0) = \tau(\lambda) + i\rho(\lambda)$, we have

$$\tau^2 = \frac{\lambda^2}{4v^2}, \quad \rho^2 = \frac{4v - \lambda^2}{4v^2}$$

(6.6)

(here and below we omit variable $\lambda$). This implies existence of the limits $w(z_1) = w(z_2)$ for (6.4). One can easily deduce from (6.5) that in the limit (6.4)

$$1 - vw_1 w_2 = \frac{\lambda_1 - \lambda_2}{2i\rho(\lambda)} = o(1).$$

(6.7)

Also we have that

$$(1 - vw_1^2)(1 - vw_2^2) = 2 - 2v(\tau^2 - \rho^2) = 4v\rho^2.$$  

(6.8)

Now let us consider $Q(z_1, z_2)$ (3.8) and write that

$$Q(z_1, z_2) = \frac{1}{2\pi} \left( \int_{-\delta}^{\delta} \int_{\mathbb{R}\setminus(-\delta, \delta)} \frac{w_1^2 w_2^2 \tilde{u}_F(p)}{[1 - vw_1 w_2 \tilde{u}_F(p)]^2} dp \right) = I_1 + I_2.$$  

Relations (6.5) and (6.7) imply equality (cf. (3.9))

$$[1 - vw_1 w_2 \tilde{u}_F(p)]^2 = [\tilde{u}_F(p) - 1]^2 (1 + o(1)).$$

(6.9)

Since $u(t)$ is monotone, then

$$\liminf_{p \in \mathbb{R}\setminus(-\delta, \delta)} [\tilde{u}_F(p) - 1]^2 > 0.$$
This means that \( I_2 < \infty \) in the limit (6.4).

Regarding (6.7), we can write that in the limit (6.4)

\[
I_1 = \int_{-\delta}^{\delta} \frac{(2\pi)^{-1}w_1^2w_2^2 \tilde{u}_F(p)}{(1 - vv_1w_2 + vv_1w_2 [\tilde{u}_F(p) - 1])^2} dp = \int_{-\delta}^{\delta} \frac{(2\pi v)^{-1} \tilde{u}_F(p)(1 + o(1))}{(\frac{z_1 - z_2}{w_1 - w_2} + [\tilde{u}_F(p) - 1])^2} dp.
\]

Then we derive relation

\[
I_1(\lambda_1 + i0, \lambda_2 - i0) + I_1(\lambda_1 - i0, \lambda_2 + i0) = \frac{1}{\pi} \int_{-\delta}^{\delta} \frac{[\tilde{u}_F(p) - 1]^2 - \left(\frac{\lambda_1 - \lambda_2}{2\rho}\right)^2}{[\tilde{u}_F(p) - 1]^2 + \left(\frac{\lambda_1 - \lambda_2}{2\rho}\right)^2} \tilde{u}_F(p)(1 + o(1)) dp,
\]

where \( o(1) \) corresponds to (6.9) regarded in the limit (6.4).

Now let us use condition (6.2) and observe that

\[
\frac{1}{\pi} \int_{-\delta}^{\delta} \frac{c_1^2 p^{2\nu} + o(p^{2\nu}) - D^2}{[c_1^2 p^{2\nu} + o(p^{2\nu}) + D^2]^2} dp = \frac{2}{\pi D^{2-1/\nu}} \int_0^{\delta D^{-1/\nu}} \frac{c_1^2 s^{2\nu} + o(s^{2\nu}) - 1}{c_1^2 s^{2\nu} + o(s^{2\nu}) + 1} ds,
\]

where we denoted \( D = |\lambda_1 - \lambda_2|/(2\rho) \) and \( o(p^{2\nu}) \) corresponds to the limit \( \delta \to 0 \) (6.2). Now it is clear that if we take \( \delta \) such that \( \delta|\lambda_1 - \lambda_2|^{-1/\nu} \to \infty \), we obtain asymptotically

\[
I_1 + \bar{I}_1 = 4B_\nu(c_1) \frac{(2\rho)^{2-1/\nu}}{|\lambda_1 - \lambda_2|^{2-1/\nu}}
\]

where

\[
B_\nu(c_1) = \frac{1}{2\pi c_1^{1/\nu}} \left[ \int_0^\infty \frac{ds}{1 + s^{2\nu}} - 2 \int_0^\infty \frac{ds}{(1 + s^{2\nu})^2} \right].
\]

To prove relation (6.3), it remains to consider the sum

\[ I(\lambda_1 + i0, \lambda_2 - i0) + I(\lambda_1 - i0, \lambda_2 + i0). \]

It is easy to observe that relations of the form (6.8) imply boundedness of this sum in the limit (6.4)

Now gathering relations (6.8) and (6.11), we derive that

\[
\Sigma_{n,b}(\lambda_1, \lambda_2)s = \frac{1}{Nb(2\rho)^{1/\nu}} \frac{B_\nu(c_1)}{|\lambda_1 - \lambda_2|^{2-1/\nu}} (1 + o(1))
\]

This proves (6.3). □

Let us discuss two consequences of Theorem 6.1. Let us assume first that \( u(t) \) is such that

\[
u_2 \equiv \int t^2 u(t) dt < \infty.
\]
Then (6.2) holds with $\nu = 2$ and $c_1 = u_2$. Regarding the right-hand side of (3.5) in the limit (6.4) with $\lambda_j = \lambda + r_j/N$, $j = 1, 2$, we obtain asymptotic relation

$$\Sigma(\lambda_1, \lambda_2) = \frac{\sqrt{N}}{b} \frac{B_2(u_2)}{2(2\rho)^{1/2}} \frac{1}{|r_1 - r_2|^{3/2}} (1 + o(1)),$$

(6.15a)

where

$$B_2(u_2) = -\frac{1}{4\pi \sqrt{u_2}} \int_0^\infty \frac{ds}{1 + s^2} = -\frac{1}{4\pi \sqrt{u_2}} \Gamma \left( \frac{5}{4} \right) \Gamma \left( \frac{3}{4} \right).$$

(6.15b)

Now let us assume that (6.14) is not true. Suppose that there exists such $1 < \nu' < 2$ that

$$u(t) = O(|t|^{-1-\nu'}) \quad \text{as} \quad t \to \infty.$$

(6.16)

Then one can easily derive that (6.2) holds with $\nu = \nu'$. This follows from elementary computations based on equalities

$$\tilde{u}_F(p) = \tilde{u}_F(0) - \int_{-\infty}^\infty (1 - \cos pt) u(t) dt$$

and

$$\frac{1}{p} \int_{-\infty}^{\infty} (1 - \cos y) u(yp^{-1}) dy = |p|^{\nu'} \int_{-\infty}^{\infty} \frac{1 - \cos y}{|y|^{1+\nu'}} dy + o(|p|^{\nu'}), \quad p \to 0.$$

Therefore, if (6.16) holds, then

$$\Sigma(\lambda_1, \lambda_2) = \frac{N^{1-1/\nu}}{b} \frac{B_\nu(c_1)}{(2\rho)^{1/\nu}} \frac{1}{|r_1 - r_2|^{2-1/\nu}} (1 + o(1)).$$

(6.17)

The form of asymptotic expressions (6.15a) and (6.17) coincides with that determined by Altshuler and Shklovski for the spectral correlation function of band random matrices (see [27] for this and similar results). In these works, the factor $|r_1 - r_2|^{-3/2}$ appeared instead of usual for random matrices expression $|r_1 - r_2|^{-2}$ (see (3.10)). This has been interpreted as the evidence of (relatively) localized eigenvectors of $H^{(n,b)}$ in the limit $1 \ll b \ll n$ with the localization length $b^2/n$. Let us note that the asymptotic expressions similar to (6.15) have also appeared in the recent work [33], where the band random matrix ensemble $H^{(n,b)}$ was considered under condition (6.14). However it should be stressed that no explicit expressions like (3.6) and (6.15) were obtained neither in [27] nor in [33].
7 Summary

We consider a family of random matrix ensembles \( \{ H^{(n,b)} \} \) of the band-type form. More precisely, we are related with real symmetric \( N \times N \) matrices, \( N = 2n + 1 \), whose entries are jointly independent Gaussian random variables with zero mean value. The band-type form means that the variance of the matrix entries \( H^{(n,b)}(x,y) \) is proportional to \( u(x-y) \geq 0 \).

We study asymptotic behavior of the correlation function

\[
C_{n,b}(z_1,z_2) = \mathbb{E} f_{n,b}(z_1)f_{n,b}(z_2) - \mathbb{E} f_{n,b}(z_1)\mathbb{E} f_{n,b}(z_2),
\]

where \( f_{n,b}(z) \) is the normalized trace \( \langle G^{(n,b)}(z) \rangle \) of the resolvent of \( H^{(n,b)} \).

We have proved that if \( \text{Im } z_j \) is large enough, then in the limit \( 1 \ll b \ll N^{1/3} \)

\[
C_{n,b}(z_1,z_2) = \frac{1}{Nb} S(z_1,z_2) + o \left( \frac{1}{Nb} \right).
\]

We have found explicit form of the leading term \( S(z_1,z_2) \) in this limit. Assuming that expression \( \Sigma_{n,b}(\lambda_1,\lambda_2) \) (6.1) is closely related with the correlation function of the eigenvalue density, we have studied it in the limit \( N,b \to \infty \) and \( \lambda_1 - \lambda_2 = (r_1 - r_2)/N \).

Our main conclusion is that the limiting expression for \( \Sigma_{n,b} \) exhibits different behavior depending on the rate of decay of \( u(t) \) at infinity.

If \( \int t^2 u(t) dt < \infty \), then (6.1) is given by

\[
-\frac{C \sqrt{N}}{b} \frac{1}{|r_1 - r_2|^{3/2}} (1 + o(1)), \quad C > 0.
\]

If \( \hat{u}(t) = O(|t|^{-1-\nu}) \) with \( 1 < \nu < 2 \), then the asymptotic expression for (7.1) is proportional to

\[
\frac{t^{1-1/\nu}}{b} \frac{1}{|r_1 - r_2|^{2-1/\nu}}.
\]

In both cases the exponents do not depend on the particular form of the function \( u(t) \). Moreover, in the first case the exponents do not depend on \( u \) at all. This can be regarded as a kind of spectral universality for band random matrix ensembles. On can conject that these characteristics also do not depend on \( u \) at all.

Our results show that \( S(z_1,z_2) \) determines at least two scales of universality in the local spectral properties of band-type random matrices. These scales coincide with those detected in theoretical physics for the (relative) localization length and density-density correlation function for these ensembles \[27\]. In the papers cited also the third scale when \( u(t) = O(|t|^{-\gamma}) \) with \( \gamma \in (1/2,1) \) has been observed. It been shown to produce the asymptotics \( N^{-2}|r_1 - r_2|^{-2} \) which is typical for "full" random matrices like GOE \[6,12,24\]. Unfortunately, this asymptotic regime for band random matrices is out of reach of our technique.
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