Heisenberg’s wave packet reconsidered

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This note shows that Heisenberg’s choice for a wave function in his original paper on the uncertainty principle is simply a renormalized characteristic function of a stable distribution with certain restrictions on the parameters. Relaxing Heisenberg’s restrictions leads to a more general formulation of the uncertainty principle. This reformulation shows quantum uncertainty can exist at a macroscopic level. These modifications also give rise to a new form of Schrödinger’s wave equation as the equation of a vibrating string. Although a heat equation version can also be given, the latter shows the traditional formulation of Schrödinger’s equation involves a hidden Cauchy amplitude assumption.

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A generalized wave packet

We begin by showing that Heisenberg’s choice for a wave function in his original paper on the uncertainty principle is simply a renormalized characteristic function of a stable distribution, $S_{\alpha,\beta}(x; m, c)$ with $\alpha = 2$ and $\beta = 0$, and location and scale parameters $m$ and $c$. Relaxing the assumptions on $\alpha, \beta$ so that $0 < \alpha \leq 2, \beta \neq 0$, leads to a more general formulation of the uncertainty principle. These modifications also give rise to a new form of Schrödinger’s partial differential equation.

Consider the following wave packet $\psi(x, t)$, where at time $t = 0$, $\psi(x, 0)$ has the form

$$\psi(x, 0) = A_o \exp[imx - c|x|^\alpha],$$

where

$$A_o = \left[\frac{\alpha(2c)^{\frac{1}{\alpha}}}{2\Gamma(\frac{1}{\alpha})}\right]^{\frac{1}{2}}.$$  

(2)

It is easy to see that $\psi(x, 0)$ is normalized to unity:

$$\int_{-\infty}^{\infty} \psi(x, 0)^* \psi(x, 0) dx = A_o^2 \int_{-\infty}^{\infty} \exp[-2c|x|^\alpha] dx = 2A_o^2 \int_0^{\infty} \exp[-2cx^\alpha] dx.$$  

(3)
Using the relation
\[ \int_0^\infty y^k e^{-y^\alpha} dy = \frac{1}{\alpha} \Gamma\left(\frac{k + 1}{\alpha}\right) \]
and making the substitution \( u = (2c)^{\frac{1}{\alpha}} x \), we obtain
\[ 2A_o^2 \int_0^\infty \exp[-2cx^\alpha] dx = \frac{2A_o^2}{(2c)^{\frac{1}{\alpha}}} \int_0^\infty e^{-u^\alpha} du = \frac{2A_o^2}{(2c)^{\frac{1}{\alpha}}} \frac{1}{\alpha} \Gamma\left(\frac{1}{\alpha}\right) = 1. \]

Now, the form of the wave packet in Eq.(1) can be compared to Heisenberg’s original wave packet, denoted here \( H(x, 0) \):
\[ H(x, 0) = (2\tau)^{\frac{1}{4}} \exp[2\pi i\sigma_o x - \pi\tau x^2]. \]

If we make the substitutions
\[ 2\pi\sigma_o = m \]
\[ \pi\tau = c \]
\[ \alpha = 2 \]
in \( \psi(x, 0) \), the wave packet of Eq.(1), we obtain \( H(x, 0) \). (Note that with \( \alpha = 2 \), \( A_o = \left[\frac{2(2c)^{\frac{1}{2}}}{2\pi(\frac{1}{2})}\right]^{\frac{1}{2}} = \left(\frac{2c}{\pi}\right)^{\frac{1}{4}} = (2\tau)^{\frac{1}{4}} \).

Now let’s derive the amplitude function of \( \psi(x, 0) \), which will necessarily also give us the amplitude function of \( H(x, 0) \). First note that the log characteristic function of a stable distribution is
\[ \log \varphi(z) = \log \int_{-\infty}^\infty \exp[ixz]dF\left(\frac{x - m}{c'}\right) \]
\[ = imz - |c'|^\alpha |z|^\alpha[1 + i\beta(z/|z|)\tan(\pi\alpha/2)], \text{ if } \alpha \neq 1 \]
\[ = imz - |c'|^\alpha |z|^\alpha[1 + i\beta(z/|z|)(2/\pi)\log|z|], \text{ if } \alpha = 1 \]
where \( m \) is a real number, \( c' \geq 0, 0 < \alpha \leq 2, |\beta| \leq 1 \). Proof of this theorem, due to Khintchine and Lévy in 1936, may be found in [5] or [3]. Here \( \alpha \), the characteristic exponent, is essentially an index of peakedness (\( \alpha = 2 \) for the normal or Gaussian distribution, \( \alpha = 1 \) for the Cauchy distribution). The parameter \( \beta \) is an index of skewedness (\( \beta = 0 \) for symmetric distributions). The parameter \( c' = c^{\frac{1}{\alpha}} \) is a scale parameter (the standard deviation when \( \alpha = 2 \)). Finally, \( m \) is a location parameter (the mean if \( \alpha > 1 \); it is also the median or modal value of the distribution if \( \beta = 0 \)).
For $\beta = 0$ we obtain the characteristic function of a symmetric stable distribution, which is identical to Eq.(1), if we omit the normalizing constant $A_o$. Therefore, for the amplitude function of our wave packet, we take the Fourier transform, $A(z)$, of Eq.(1) to obtain

$$A(z) = \int_{-\infty}^{\infty} \psi(x,0) \exp[-ixz]dx = A_o \int_{-\infty}^{\infty} \exp[imx - c|x|^\alpha] \exp[-ixz]dx = A_0 s_{\alpha,0}(z;m,c).$$

(13)

In other words, we obtain a symmetric stable density function $s_{\alpha,0}(z;m,c) = dS_{\alpha,0}(z;m,c)$ with the normalization constant $A_o$ for the amplitude function. The symmetric stable density has $0 < \alpha \leq 2$, $\beta = 0$, and location and scale parameters $m$ and $c$, respectively. Note that the amplitude function is normalized so that the integral of its square is equal to 1. This involves the square of the probability density function $s_{\alpha,0}(z;m,c)$.

For Heisenberg’s case where $\alpha = 2$, we may explicitly solve for $A(z) = A(\sigma)$, which will be necessarily the Gaussian density multiplied by a normalizing constant. We reintroduce a factor of $2\pi$ to obtain

$$A(\sigma) = \int_{-\infty}^{\infty} (2\pi)^{\frac{1}{4}} \exp[2\pi i \sigma x - \pi \tau x^2] \exp[-2\pi i x \sigma] dx = \left(\frac{2}{\tau}\right)^{\frac{1}{4}} \exp\left[-\frac{\pi (\sigma - \sigma_o)^2}{\tau}\right].$$

(14)

This is Heisenberg’s amplitude function. That the integral of its square is 1 follows from:

$$\int_{-\infty}^{\infty} [A(\sigma)]^2 d\sigma = \left(\frac{2}{\tau}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \exp\left[-\frac{2\pi (\sigma - \sigma_o)^2}{\tau}\right] d\sigma = \left(\frac{2}{\tau}\right)^{\frac{1}{2}} \left(\frac{\tau}{2\pi}\right)^{\frac{1}{2}} 2 \int_{0}^{\infty} e^{-u^2} du = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1,$$

(15)

where we have used the substitution $u = \sqrt{\frac{2\pi}{\tau}} (\sigma - \sigma_o)$ . Note that the usual normalizing constant $\frac{1}{d\sqrt{2\pi}}$ for the Gaussian distribution (where $d$ is the standard deviation) has been absorbed into $A_o$. So above and below, when we write the stable density $s_{\alpha,\beta}(z;m,c)$, we will understand the omission of the usual normalizing constant, and will consider only the normalizing $A_o$ in the product $A_o s_{\alpha,\beta}(z;m,c)$. This will ensure that the square of the amplitude function is a probability distribution.

**Alternative amplitude functions**

For $\alpha = 1$, which corresponds to the Cauchy distribution, the normalizing constant $A_o = c^{\frac{1}{2}} = (c')^{\frac{1}{2}} = (c')^{\frac{1}{2}}$, so the amplitude function is

$$A(z) = A_o s_{1,0}(z;m,c) = c^{\frac{1}{2}} \sqrt{\frac{2}{\pi}} \frac{c}{c^2 + (z - m)^2}.$$

(16)
where we have removed a division by $\sqrt{2\pi}$ in the usual statement of the Cauchy. That this is the correct normalization for the amplitude function in Eq.(16) follows from the integral:

$$\int_{-\infty}^{\infty} [A(z)^2] dz = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{c^3}{[c^2 + (z - m)^2]^2} dz = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{1}{[1 + y^2]^2} dy,$$

(17)

where we have used the substitution $y = \frac{z - m}{c}$. We may now appeal to the relations, for $a, c > 0$ and $n$ a positive integer:

$$\int \frac{dx}{(ax^2 + c)^n} = \frac{1}{2(n-1)c} \frac{x}{(ax^2 + c)^{n-1}} + \frac{2n-3}{2(n-1)c} \int \frac{dx}{(ax^2 + c)^{n-1}}$$

(18)

and

$$\int \frac{dx}{ax^2 + c} = \frac{1}{\sqrt{ac}} \tan^{-1}\left[ \frac{x}{\sqrt{c}} \right].$$

(19)

Thus we get

$$\frac{2}{\pi} \int_{-\infty}^{\infty} \frac{dy}{[1 + y^2]^2} = \frac{2}{\pi^2} \int_{-\infty}^{\infty} \frac{dy}{1 + y^2} = \frac{2}{\pi^2} \int_{0}^{\infty} \frac{dy}{1 + y^2} = \frac{2}{\pi} \tan^{-1}[\infty] = 1.$$  

(20)

If we further generalize Eq.(1) by relaxing the constraint on $\beta$, we obtain the wave function

$$\psi(x, 0) = A_o \exp\{[imx - c|x|^\alpha][1 + i\beta(z/|z|)\tan(\pi\alpha/2)]\}.$$  

(21)

Note for the wave function in Eq.(21) that since $i$ multiplies $\beta$, the normalizing constant $A_o$ given in Eq.(2) is unchanged in terms of $\alpha$. For $\alpha = \frac{1}{2}$, which we will now consider, $A_o = c = (c')^{\frac{1}{2}}$. Thus for $\alpha = \frac{1}{2}$ and $\beta = -1$, we obtain for the amplitude function the completely positive stable distribution (sometimes called Pearson V), multiplied by the normalizing constant $c$:

$$A(z) = A_o s_{\frac{1}{2}, -1}(z; m, c) = c \frac{c}{\sqrt{(z - m)^3}} \exp\left[-\frac{c^2}{2(z - m)}\right].$$  

(22)

As a check, we integrate the probability function $P(z) = [A(z)]^2$ corresponding to the amplitude function in Eq.(22):

$$\int_{-\infty}^{\infty} [A(z)^2] dz = \int_{m}^{\infty} \frac{c^4}{(z - m)^3} \exp[-\frac{c^2}{(z - m)^3}] dz = \int_{0}^{\infty} \frac{1}{c^2} u^6 \frac{2c^2}{u^3} e^{-u^2} du$$

$$= 2 \int_{0}^{\infty} u^3 e^{-u^2} = 2 \frac{1}{2} \Gamma\left(\frac{4}{2}\right) = 1,$$

(23)

(24)

where we have used the substitution $u = \frac{c}{(z-m)^{\frac{1}{2}}}$. 
Finally, for the general case, we may express the amplitude function as a renormalized stable density, which is in turn represented by a Taylor expansion in the form of gamma functions \[2\] (alternative expansions may be found in \[1\]):

\[
A(z) = A_0 s_{\alpha,\beta}(z; 0, 1) = A_0 \frac{1}{z} \sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} \frac{\Gamma(1+k/\alpha)}{k!} (-z)^k \sin \left( \frac{k\pi}{2\alpha} (\beta - \alpha) \right),
\]

for \(z > 0\) and \(1 < \alpha < 2\). For \(z < 0\) we have the general relation \(s_{\alpha,\beta}(-z;m,c) = s_{\alpha,-\beta}(z;m,c)\). For \(0 < \alpha < 1\) we have the similar expansion, for \(z > 0\),

\[
A(z) = A_0 s_{\alpha,\beta}(z; 0, 1) = A_0 \frac{1}{z} \sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} \frac{\Gamma(1+k\alpha)}{k!} (-z^{-\alpha})^k \sin \left( \frac{k\pi}{2} (\beta - \alpha) \right).
\]

We may recover \(m\) and \(c\) in Eqs.(25,26) by the substitution \(z = \frac{u-m}{c\alpha}\).

The uncertainty relation

Now let’s consider the uncertainty relation. From Eq.(1), where the distribution is symmetric, we get the value for \((\triangle x)^2\) as:

\[
(\triangle x)^2 = \int_{-\infty}^{\infty} \psi^*(x,0)x^2\psi(x,0)dx.
\]

Inserting a factor of \(u^2 = (2\zeta)^2 x^2\) into the calculation of Eq.(5), we obtain

\[
(\triangle x)^2 = \frac{1}{(2\zeta)^2} \frac{\Gamma(\frac{3}{\alpha})}{\Gamma(\frac{1}{\alpha})}.
\]

For \(\alpha = 2\) this yields \((\triangle x)^2 = \frac{1}{4\zeta}\), or in Heisenberg’s formulation \(\frac{1}{4\pi\tau}\).

Next consider the uncertainty in \(z\) (or \(\sigma\)). First consider the case \(\alpha = 2\). From Eq.(15) we have

\[
(\triangle \sigma)^2 = \int_{-\infty}^{\infty} (\sigma - \sigma_o)^2 |A(\sigma)|^2 d\sigma = \left(\frac{2}{\tau}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} (\sigma - \sigma_o)^2 \exp \left[ \frac{-2\pi(\sigma - \sigma_o)^2}{\tau} \right] d\sigma = \frac{\tau}{4\pi}.
\]

Thus we obtain the uncertainty relation

\[
\triangle x \triangle \sigma = \frac{1}{4\pi}.
\]

From the de Broglie relation \(\triangle p = h \triangle \sigma\), where \(h\) is Planck’s constant, this becomes

\[
\triangle x \triangle p = \frac{h}{2}.
\]
However, for comparison with the results below, we will use for the (renormalized) Gaussian amplitude, the uncertainty relation in the form

\[ \Delta x \Delta z = \frac{1}{2}. \]  

(32)

Note that for the Cauchy density, where \( \alpha = 1, \beta = 0 \), the mean and variance don’t exist (“are infinite”). But we are considering a Cauchy amplitude, and hence the square of the Cauchy density (renormalized) for the probability density. For this density the second moment exists, as we will now demonstrate. From Eqs.(16,17), we calculate \((\Delta z)^2\) as:

\[
(\Delta z)^2 = \int_{-\infty}^{\infty} (z - m)^2[A(z)]^2 dz = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{c^3(z - m)^2}{c^2 + (z - m)^2} dz
\]

(33)

\[
= \frac{2c^2}{\pi} \int_{-\infty}^{\infty} \frac{y^2}{1 + y^2} dy = \frac{2c^2}{\pi} \frac{1}{2} \int_{0}^{\infty} \frac{dy}{1 + y^2} = c^2,
\]

(34)

where we have used the relation

\[
\int \frac{x^2 dx}{(ax^2 + c)^n} = \frac{1}{2(n-1)a} \frac{x}{(ax^2 + c)^{n-1}} + \frac{1}{2(n-1)a} \int \frac{dx}{(ax^2 + c)^{n-1}}.
\]

(35)

Thus we obtain the uncertainty relation, from Eqs.(28,34),

\[ \Delta x \Delta z = \frac{1}{\sqrt{2}}. \]  

(36)

For the Pearson V amplitude, we have from Eqs. (22,23)

\[
(\Delta z)^2 = \int_{-\infty}^{\infty} (z - m)^2[A(z)]^2 dz = c^2 \int_{m}^{\infty} \frac{c^2}{(z - m)^{2}} e^{-\frac{c^2}{(z - m)^2}} dz = c^4 \int_{0}^{\infty} \frac{1}{y} e^{-y} dy,
\]

(37)

where we have used the substitution \( y = \frac{c^2}{(z - m)} \). This integral is divergent. So instead we calculate

\[
\Delta z = \int_{-\infty}^{\infty} |z - m|[A(z)]^2 dz = \int_{m}^{\infty} \frac{c^4}{(z - m)^2} e^{-\frac{c^2}{(z - m)^2}} dz = c^2 \int_{0}^{\infty} e^{-y} dy = c^2.
\]

(38)

This yields, from Eqs.(28,38) the uncertainty relation

\[ \Delta x \Delta z = \sqrt{\frac{15}{2}}. \]  

(39)

It is easy to see from Eq.(28) that the general uncertainty relation, as a function of \( \alpha \), is

\[ \Delta x \Delta z = \frac{1}{\sqrt{2\pi}} \frac{\Gamma\left(\frac{3}{2}\alpha\right)}{\Gamma\left(\frac{1}{2}\alpha\right)}. \]  

(40)

This, then, is the reformulation of Heisenberg’s uncertainty relation. The uncertainty is a function of the characteristic exponent \( \alpha \) of the (renormalized) stable amplitude. As \( \alpha \to 0 \), the uncertainty becomes unbounded.
The time-dependent wave function and the dispersion relation

We can write the time-dependent wave equation corresponding to Eq.(1) as a superposition of plane waves:

$$\psi(x,t) = \int_{-\infty}^{\infty} A(z) \exp[i(zx - \nu(z)t)]dz,$$

where $A(z)$ is the stable amplitude—a renormalized stable density, and $\nu(z)$ is the frequency. A dispersion relation connects $\nu(z)$ to $z$.

From the de Broglie relations

$$E = h\nu$$

$$p = hz$$

we obtain the relation

$$\nu = \frac{zE}{p}$$

which gives as the time-dependent wave equation

$$\psi(x,t) = \int_{-\infty}^{\infty} A(z) \exp[i(z(x - \frac{E}{p}t))]dz.$$ 

(Note that we do not insert the classical relation $E = \frac{p^2}{2M}$, where $M$ is mass, at this point, because doing so does not yield a proper inverse Fourier transform.) Each plane wave equation $g(x,t) = \exp[iz(x - \frac{E}{p}t)]$ has differential operators $\frac{\partial^2}{\partial x^2}$ and $\frac{\partial^2}{\partial t^2}$ with eigenvalues $-z^2$ and $-z^2\frac{E^2}{p^2}$ respectively:

$$\frac{\partial^2 g}{\partial x^2} = -z^2 g$$

$$\frac{\partial^2 g}{\partial t^2} = -z^2\frac{E^2}{p^2}g.$$ 

These relations give rise to the partial differential equation

$$\frac{\partial^2 g}{\partial t^2} = \frac{E^2}{p^2} \frac{\partial^2 g}{\partial x^2}.$$ 

The time-dependent wave equation in Eq.(45) may be rewritten more fully (for $\alpha \neq 1$) as

$$\psi(x,t) = A_o \exp\{\left[i\alpha(x - \frac{E}{p}t) - c|(x - \frac{E}{p}t)|^{\alpha}\left[1 + i\beta((x - \frac{E}{p}t)/(|x - \frac{E}{p}t|))\tan(\pi\alpha/2)\right]\right]\}.$$ 

(49)
For symmetric distributions ($\beta = 0$), the probability density function corresponding to $\psi(x,t)$ is

$$P(x,t) = \psi^\ast (x,t)\psi(x,t) = A_0^2 \exp[-2c|\frac{E}{p}t|^\alpha], \quad (50)$$

which is the characteristic function of a stable density. For $\alpha = 2$, this is the Heisenberg time-dependent density.

**Schrödinger’s equation revisited**

Schrödinger’s equation may be viewed as a simple consequence of the Heisenberg uncertainty relations. Eq.(49) is a solution of the partial differential equation Eq.(48), so we have, as replacement for the Schrödinger equation, the partial differential equation

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{E^2}{p^2} \frac{\partial^2 \psi}{\partial x^2}, \quad (51)$$

which may be rewritten in the form

$$\nabla^2 \psi = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}. \quad (52)$$

This, of course, is the equation of a vibrating string, where $v = \frac{E}{p}$ is the speed of propagation of the waves. It is a true wave equation, by contrast to Schrödinger’s heat equation formalism, which relates $\frac{\partial \psi}{\partial t}$ to $\frac{\partial^2 \psi}{\partial x^2}$. In fact, noting from Eq.(49), letting $\beta$ equal zero for simplicity, and letting $\text{sgn } y$ denote $\text{sgn } (x - \frac{E}{p}t)$, that

$$\frac{\partial \psi}{\partial x} = (im - c\alpha|x - \frac{E}{p}t|^{\alpha-1} \text{sgn } y)\psi$$

$$\frac{\partial^2 \psi}{\partial x^2} = ((im - c\alpha|x - \frac{E}{p}t|^{\alpha-1} \text{sgn } y)^2 - c\alpha(\alpha - 1)|x - \frac{E}{p}t|^{\alpha-2})\psi \quad (53)$$

$$\frac{\partial \psi}{\partial t} = -\frac{E}{p} \frac{\partial \psi}{\partial x} \quad (54)$$

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{E^2}{p^2} \frac{\partial^2 \psi}{\partial x^2} \quad (55)$$

it does not appear to be particularly useful to relate $\frac{\partial \psi}{\partial t}$ to $\frac{\partial^2 \psi}{\partial x^2}$, although this can be done. In fact,

$$\frac{\partial \psi}{\partial t} = -\frac{E}{p} \frac{(im - c\alpha|x - \frac{E}{p}t|^{\alpha-1} \text{sgn } y)}{((im - c\alpha|x - \frac{E}{p}t|^{\alpha-1} \text{sgn } y)^2 - c\alpha(\alpha - 1)|x - \frac{E}{p}t|^{\alpha-2})} \frac{\partial^2 \psi}{\partial x^2}. \quad (56)$$
Only in the case of the Cauchy amplitude $\alpha = 1$ do we find this latter formulation in a simplified form:

$$\frac{\partial \psi}{\partial t} = -\frac{E}{p} \frac{1}{(im - c)} \frac{\partial^2 \psi}{\partial x^2}. \tag{58}$$

If we now make the substitutions $E = \frac{p^2}{2M}$, $p = \hbar \sigma$ we obtain

$$\frac{\partial \psi}{\partial t} = \frac{\hbar \sigma}{2M (m^2 + c^2)} \frac{im + c}{\partial x^2} \frac{\partial^2 \psi}{\partial x^2} \tag{59}$$

which may be rewritten

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2 \sigma}{2M (m^2 + c^2)} \frac{m - ic}{\partial x^2} \frac{\partial^2 \psi}{\partial x^2}. \tag{60}$$

It would appear that the traditional Schrödinger equation involves a hidden Cauchy amplitude assumption. The latter equation can be divided into two equations, one involving $m$ and the other involving $-ic$.

**Conclusion**

Stable distributions are the only distributions that exist as limit distributions of sums of random variables, thus giving rise to central limit theorems. Therefore they play a paramount role in the physical world. We have shown that Heisenberg’s original choice for a wave packet to illustrate his uncertainty principle is simply the characteristic function (the inverse Fourier transform) of a Gaussian distribution, leading to a Gaussian amplitude function with $\alpha = 2$ and $\beta = 0$. Relaxing Heisenberg’s assumptions to the general case $0 < \alpha \leq 2$, $|\beta| \leq 1$, leads to stable amplitudes renormalized so that the integral of their squares are probability distributions. The renormalization constant gives rise to a new form of Heisenberg’s uncertainty relation, expressed in terms of the characteristic exponent $\alpha$ of the underlying stable amplitude: $\Delta x \Delta z = \sqrt{\frac{\Gamma(\alpha)}{\Gamma\left(\frac{\alpha}{2}\right)}} \Gamma\left(\frac{\alpha}{2}\right)$. This relationship was illustrated by explicit calculation for the Gaussian ($\alpha = 2$), the Cauchy ($\alpha = 1$), and the Pearson V ($\alpha = \frac{1}{2}$, $\beta = -1$). As $\alpha \to 0$, the uncertainty $\Delta x \Delta z$ becomes unbounded. This means that, depending on the underlying stable amplitude, quantum uncertainty can arise at a macroscopic level.

By eschewing the ad hoc classical insertion $E = \frac{p^2}{2M}$, we were able to solve for the time-dependent wave equation as a superposition of plane waves, by taking the inverse Fourier transform of the stable amplitude function. For $\alpha = 2$, this recovers Heisenberg’s case. The
wave function follows the partial differential equation \( \frac{\partial \psi}{\partial x} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} \), which is the equation for a vibrating string. This is a proper wave equation, differing from Schrödinger’s equation, which is really a heat equation as it relates \( \frac{\partial}{\partial t} \), instead of \( \frac{\partial^2}{\partial t^2} \), to \( \frac{\partial^2}{\partial x^2} \). The traditional form of the Schrödinger equation can be recovered, but only in the case \( \alpha = 1 \). Thus it would appear that Schrödinger’s equation involves a hidden Cauchy amplitude assumption. This is not fatal, but is limiting. The more general heat equation relationship is given by Eq.(57).

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