A FORMAL SYSTEM OF MATHEMATICS BASED ON DEFINITIONS

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1. Introduction

It has been understood for more than a century that mathematical reasoning can in principle be formalized to the extend that it becomes machine readable and verifiable. The implementation of this program has gained momentum in the age of efficient computers. Many theorems have been translated into machine readable code in recent decades. The challenges of the implementation raised the desire for code that is convenient both for computer and humans. For example, the language ForTheL [2] uses English phrases ubiquitous in mathematics to assemble English sentences in its code.

The goal of this paper is to assemble a formal system out of a minimal collection of typical ingredients in a mathematics paper. Instead of English text, we write a stenographic notation that follows closely such ingredients but resembles more classical mathematical formulas. We then use this formal system for some first exploration in mathematics in Sections 4 and 5. We establish Russell’s paradox, construct the natural numbers and prove the Peano axioms.

To compare our system with standard axiomatizations in mathematics, recall that working mathematicians generally accept a standard of mathematical axiomatization based on propositional and predicate logic. These types of logic need an underlying content. The most commonly accepted universal content is Zermelo-Fraenkel set theory. A second example is Peano arithmetic. Basic statements of set theory concern sets being elements of other set, while basic statements in Peano arithmetic concern algebraic equations. Our system introduces a form of propositional and predicate logic and links it from the beginning to a content about objects and their equality. Further content arises in the process of doing mathematics. The identity of an object may not be entirely determined at the beginning of its use, but evolve with choices made throughout the mathematical process. This allows to build a system sophisticated enough to match Zermelo-Fraenkel set theory. We also develop a similar logical structure around attributes that are constructed from the basic attribute of equality. This structure

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is more basic and more constructive than propositional and predicate logic but equally important to describe the growing material developed in mathematics.

In Section 3 we give a concise formal description of our system. As an informal introduction, we discuss here the elements of the system and compare them with typical mathematical concepts. Basic statements are built from an attribute and a number of objects. The link between attribute and objects is typically expressed by the words “is” or “are”. An example of a basic statement is

\[ N x \]

where the capitalized letter stands for an attribute and the lower case letter stands for an object. For example, the letter \( N \) may have an interpretation as the attribute of being a natural number, then the formula means that \( x \) is a natural number. More precisely, there is a temporal component in our code. Reading the above as “is” assumes that the object \( x \) has been introduced earlier in the argument. If \( x \) does not occur earlier than the above line, then this line introduces the object \( x \) and can be interpreted as saying let \( x \) be a natural number.

Attributes may also take more than one object, the fundamental example is

\[ x = y \]

meaning that \( x \) and \( y \) are equal, or \( x \) is equal to \( y \). The latter formulation stresses more the order of the objects, but in case of the symmetric equality this emphasis is not necessary. The equality sign is the basic attribute within our formal system. It has the structural effect that it allows one of the equal objects to be substituted by the other in any further statement, Rule 9. The link between equality and substitution appears as early as in Freiherr von Wolff’s [6, §17], “Wenn ich ein Ding B für das Ding A setzen kann und es bleibt alles wie vorhin, so ist A und B einerlei” and “Wenn ich aber B für A setze kann und es bleibt nicht alles wie vorhin, so sind A und B unterschieden oder verschiedene Dinge.” A void substitution is caused by having the same variable on both sides of an equality. Such reflexive equalities are therefore true by default.

The negation “not” comes with its own symbol, a prime symbol behind an attribute. We write

\[ x = ' y \]

for \( x \) and \( y \) are not equal. Indeed, all attributes come in pairs, one with and one without the prime, which are classically dual in the sense of negation. We adopt the principle of the excluded middle, so we allow arguments by case distinction between a statement and its dual statement, Rule 3. We also adopt the rule of explosion, that is, a statement is proved by bringing its negation to a contradiction, Rule 4.
We represent the phrase “there exists” by brackets as in

\[ [N\xi] \xi = o \]

meaning that there exists a natural number \( \xi \) such that \( \xi \) is equal to \( o \). Similarly, we represent the phrase “for all” as

\[ \{N\xi\} \xi \neq o \]

meaning that for all natural numbers \( \xi \) we have that \( \xi \) is not equal to \( o \). Typographically, we have chosen brackets common on standard keyboards. The existential brackets are reminiscent of the letter “E” and the universal brackets are reminiscent of the letter “A”.

It appears that exactly one of the above bracketed statements is true. Indeed, these statements are formally dual to each other. Truth depends on what the object \( o \) might be. If for example it is zero, then the former statement is true and the latter is false, while if \( o \) is not a natural number, the latter is true and the former is false.

The grammar explicitly suggests that \( \xi \) is an object that we maintain some liberty to more narrowly specify later. We use greek letters for such deferred objects, and latin letters for objects that we deem determined already or that we determine at the time of first use. Such distinction of variables occurs in Frege’s Begriffsschrift [4]: “Alle Zeichen, die ich verwende, theile ich daher in solche, unter denen man sich Verschiedenes vorstellen kann, und in solche, die einen ganz bestimmten Sinn haben.”

Replacing greek by latin letters, we have similar statements to the above. The statement

\[ [Nx] x = o \]

means that \( x \) is a natural number and \( x \) is equal to \( o \), while

\[ \{Nx\} x \neq o \]

means that if \( x \) is a natural number, then \( x \) is not equal to \( o \). The type of letter switches therefore between the quantifier phrases “there exists” and “for all” on the one hand and the logical connectives “and” and “if, then” on the other hand.

These quantifier brackets are formally a process of concatenating statements. Rules 6 and 5 regulate this concatenation process for these brackets. In the case of existential brackets, we can simply deduce both parts of the concatenation from the concatenated statement. If a greek letter occurs at suitable place we use the moment of splitting the existential statement to introduce an object substituting a greek by a latin letter. In the case of universal brackets, we need that the first of the concatenated statements appears in the past, again with a latin letter in place of a greek letter when necessary, and we are then allowed to deduce the second statement, possibly with the same modification of the letters.
The deferment of determination of an object leads very naturally to
dependence of objects of each other. Consider the statement
\[\{N\xi\}[N\eta]\eta ='\xi\]
meaning that for all natural numbers there is another natural number
that is not equal to the former. This is a statement that appears true.
For example, if the first number is zero, we pick the second number to
be one, and if the first number is not zero, we pick the second number
to be zero. Note the importance of the order of determining the two
numbers. We need to know what the first number is, before we can
safely specify the second. It is therefore problematic, to replace in the
above formula only the second greek letter by a latin letter,
\[\{N\xi\}[Ny]\eta ='\xi\]
This assumes we have fixed the latin object already and it means that
for all natural numbers we have that the fixed object is a natural nu-
mer and the natural number is not equal to the fixed object. If the first
conclusion, namely that the fixed object is a natural number, is correct,
then the second conclusion is incorrect if we choose the natural number
to be the same as the fixed object. We adopt instead the depen-
dence notation
\[\{N\xi\}[Ny(\xi)]y(\xi) ='\xi\]
where \(y(\xi)\) may be read as \(y\ of\ \xi\). So our system has a functional
bracket for concatenating objects into new objects. Traditionally, the
first object in the concatenation is interpreted as a function that as-
sociates to each possible argument, that is the second object in the
concatenation, a new object. The function may well be determined
prior to the determination of the argument. The function in the last
displayed statement is called a choice for the earlier displayed state-
ment with corresponding greek variable, which is a guarantor for the
possibility of a suitable choice. The passage from the statement with a
greek letter to the statement with a concatenated object is close to the
classical axiom of choice in Zermelo-Fraenkel set theory. In our sys-

tem, choice lies at the heart of the formal system, Theorem \[12\]. However,
the replacement by a concatenated object has to be done with some
care. Consider the true statement
\[\{\xi = \xi\} [\eta = \eta] [\{\xi(\xi) = \xi\} \eta ='\xi] \{\xi(\xi) ='\xi\} \eta =\xi\]
which states that for every object there is another object that is either
equal or not equal to first object depending on a certain condition on
the first object. Substituting the second greek letter by a concatenated
object, we obtain
\[\{\xi = \xi\} [y(\xi) = y(\xi)] [\{\xi(\xi) = \xi\} y(\xi) ='\xi] \{\xi(\xi) ='\xi\} y(\xi) =\xi\]
This turns out to be a false statement. It makes a claim for an arbitrary
object, but the last part of the statement fails if we let this object be
the newly introduced object \( y \). Namely, we obtain
\[
\left[ \{ y(y) = y \} \ y(y) = \prime \ y \right] \ \{ y(y) = \prime \ y \} \ y(y) = y
\]
which states that certain two objects should be equal precisely if they are not equal. The problem is that the statement prior to applying choice works for all objects known prior to applying the choice. The choice then creates a new object, and this new object refuses to abide by the universal statement used to create itself. The example is a close variant of Russell’s paradox in \( \cite{3} \), which revealed a fault in this context in Frege’s Begriffsschrift \( \cite{1} \). The example is also somewhat in the spirit of the colloquial paradox ”This sentence is a lie”. Theorem 12 describes a more careful choice, avoiding that the new object will be forced to satisfy the statement used to create itself. Within our system, we interpret objects and concatenated object at the same level and are in particular free to equate and potentially substitute any such objects. Statements that play a particular role in our system are the statements
\[
y(x) = y \\
y(x) = \prime \ y
\]
They appear somewhat odd in classical mathematics due to the type mismatch between function and value of the function. In our system, the statements encode what is classically considered the domain of a function. The second statement says that the argument of the concatenation is in the domain of the function and the first says that the argument is not in the domain of the function. Note how typographically it appears in the first statement that the function refuses to accept the argument and just remains itself. The concept of domain is used in the formalism of choice to avoid Russell’s paradox. Validity of universal statements is only required for objects in the domain of some other object. To have an interesting theory, one then needs to guarantee objects with somewhat large domains. This is established by Theorems 14 and 16 which corresponds to the classical Zermelo-Fraenkel axioms of replacement, union, and power set on the one hand and the axiom of infinity on the other hand. The deferment of the identity of an object until certain other objects are identified allows to build a rich structure of interdependent objects. Functions are used in lieu of sets at the center of attention by von Neumann \( \cite{5} \), who justifies his choice by practicality: “Die technische Durchführung gestaltet sich jedoch beim Zugrundelegen des Funktionsbegriffes wesentlich einfacher, allein aus diesem Grunde haben wir uns für denselben entschieden.” The rich structure of functions matches the richness of the classical von Neumann universe in set theory. For convenience, and in analogy with the axiom of extension in the Zermelo-Fraenkel set theory, Theorem 10 explicitly restricts attention to how
objects react with the choice of other objects. Objects do not have additional characteristics of their own.

We turn our attention to attributes, which are represented by the equality symbol or by capital letters. Unlike lower case letters, behind which there is a universe of objects that can be disacovered with the full arsenal of non-constructive mathematical reasoning such as proof by cases, by contradiction, and by choice, the capital letters represent very concrete and explicitly constructed attributes. One should think of attributes as abbreviations for more complicated statements build from the basic attribute of equality.

To introduce a new attribute, one has to precisely write the abbreviated statement following Rule 7. We introduce for example the attribute of being a natural number by the string

\[ N_\xi : \{S_\phi\} \phi(\xi) =^* \phi \]

Here the statement to the left of the construction symbol \(\;\) may represent the phrase “define”, is an abbreviation for the statement to the right of the symbol. The right hand side uses more basic attributes such as the equality sign and capital letters that are defined earlier. The construction symbol is graphically a shorter variant of an equality symbol. Indeed, it plays a similar role in that it allows to substitute the statement on one side of it by the statement on the other side of it in any further statement. Rule 8 governs this process, which is subject to some minor syntactical rules concerning brackets. The construction symbol is used solely as part of definitions, in particular it does not come with a negated form of itself.

Despite being explicit, the attribute letters also come as latin and greek letters with similar effect on deferment. Latin attributes are assumed fixed already at the time of their use, while greek letters may be used before they are replaced by some precisely defined latin attribute. Greek attribute letters appear in documents called definitions and in claims of theorems, that are recorded for use in later documents. Definitions are governed by Rule 1 defining objects based on earlier existence proofs and Rule 0 defining attributes by construction. Claims are governed by Rule 2 these claims have to be followed by proofs except in the small number of exceptional cases of Theorems 10, 12, 14, and 16 which have the status of rules or axioms of the system. Objects or attributes recorded in definition documents may depend on greek attributes which will be specified later. This leads to a functional notation with attributes as arguments. For example,

\[ m(\Gamma) \]

could express the minimum of all natural numbers which satisfy the attribute \(\Gamma\), and

\[ A(\Gamma)\xi \]
could denote the attribute of $\xi$ being a natural number larger than all natural natural numbers satisfying attribute $\Gamma$. The functional notation is purely a way of organizing deferred explicit constructions, there is no rule similar to a choice axiom leading to functional expressions with attributes as arguments.

Unlike in the case of functional dependence of objects, where we have the liberty to change the number of arguments by concatenating further functional brackets, there is a rigidity around attributes in terms of number of arguments owed to the explicitness of the construction. The number of arguments in the context of attributes is regulated by arities associated with objects or attributes.

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3. FORMAL RULES

3.1. Documents. Our system of mathematics evolves as a growing number of documents of two types, definitions and theorems. We label the documents with natural numbers greater than nine, with larger numbers referring to later documents. The numbers zero to nine enumerate rules for the formal system.

Each document has a preamble, containing the type and label of the document, an informal title for mnemonic purpose but without direct relevance to the formal rules, and finally a possibly empty list of further labels of selected earlier definitions.

The main body of the document is a collection of strings. A definition contains a number of strings linearly ordered in time. We write these strings into a chain of rectangular boxes from top to bottom. A theorem consists of one earliest string called the claim, and in most cases a proof. The proof is a collection of strings put into rectangular boxes and arranged in a tree structure. Some boxes may be empty. Two boxes are connected by at most one edge. Each edge connects an earlier box with a later box called parent and child of each other. We draw the parent higher than the child. There is an earliest box of the proof called the root. Each box other than the root has exactly one parent. Each box has zero, one, or two children. A box without children is called a leaf. A box in a proof is earlier than another box, if it can be reached from the latter by a path with each step going from a box to its parent. An ancestor of a string in a given document is the string itself, or a string in the same document that is earlier than the given string, or a string in a definition that is earlier than the given document, or a claim in a theorem that is earlier than the given document.
Next to an edge we may write a justification for the edge. To the left of it we may write the number of one or several rules of the system, or labels of prior definitions or theorems containing relevant ancestors. To the right of an edge in a proof we may write relevant ancestors within the same proof, where each such ancestor is identified by the number of steps from box to parent needed to reach the ancestor from the leaf.

3.2. Symbols. Strings are finite chains of symbols. Some possible symbols are the opening and closing functional brackets,

\[(\ )\]

the opening and closing existential quantifier or existential brackets,

\[\lbrack \rbrack\]

the opening and closing universal quantifier or universal brackets,

\[\lbrace \rbrace\]

the duality symbol, and the construction symbol.

All other symbols are called letters. There are object letters and attribute letters. Each of these come in two forms, definite letters and indefinite letters.

We use characters of the greek alphabet for indefinite letters and possibly add a natural number as subscript. Object letters are lowercase and attribute letters are uppercase. Examples of indefinite letters are

\[\xi \eta \xi_0 \xi_1 \Gamma \Delta \Gamma_0 \Gamma_1\]

One particular definite attribute letter is the equality sign,

\[=\]

We use the latin alphabet for all other definite letters and add a subscript containing the label of a document. Object letters are lowercase and attribute letters are uppercase. Examples of definite letters are

\[x_{17} y_{10} A_{17} A_{10} B_{10}\]

A new combination of latin character and subscript in some document has the label of the present document as subscript. Subscripts of latin characters therefore refer to the earliest document in which the combination of character and subscript occurs. We may omit the subscript inside a document, if the subscript is the earliest label among the present document’s label and the labels listed in the preamble of the present document such that this combination of character and subscript appears in the document with that label. This is the purpose of the list of labels in the preamble.

With each definite letter, we associate a natural number called the attribute arity of the letter. With each definite attribute letter we additionally associate a pair of natural numbers called the object arity.
of the letter. If the attribute arity of a definite letter is not zero, we further associate with it for each number from one to the attribute arity a pair of numbers called the *implicit arity* of the letter and the number. The attribute arity of the equality sign is zero, its object arity is one and one, and it has no implicit arity.

Given a string in a document, we associate to each indefinite attribute letter occurring in the string an object arity consisting of two natural numbers. The object arity of an indefinite attribute letter may change between different strings in a document.

We say that a letter is *activated* at a string of a document, if it occurs in the string but not in any other ancestor of the string.

3.3. **Terms.** We first concatenate symbols into *attributes* and *terms*. An attribute is an attribute letter or a concatenation from left to right of an attribute letter and the duality symbol. The two possible attributes containing a particular attribute letter are called dual to each other. We call an attribute definite or indefinite following the type of its attribute letter. We associate with an attribute the same arities as with the attribute letter in it.

A chain of consecutive symbols inside a string we call a substring, if it is not followed to the right by a duality symbol inside the larger string. As duality symbols are only used directly following an attribute letter, each attribute letter inside a string is part of a unique substring which is an attribute.

An indefinite attribute or a definite attribute with zero attribute arity is an *attribute term*. Given a definite attribute of non-zero attribute arity, we obtain further attribute terms by concatenating from left to right first the given attribute, then the opening functional bracket, then in succession as many attribute terms as the attribute arity of the given attribute, and then the closing functional bracket. We require that the object arity of each of these attribute terms is equal to the implicit arity of the given attribute, which is the left most attribute in the concatenation, and the number of the attribute term in the succession. The object arity of the concatenated attribute term is the object arity of its left most attribute. Two concatenated attribute terms are dual, if they are obtained form each other by replacing the left most attribute by its dual.

A *basic object term* is an indefinite object letter, a definite object letter with zero attribute arity, or the concatenation from left to right of a definite object letter with non-zero attribute arity, then the opening functional bracket, then in succession as many attribute terms as the attribute arity of the object letter, and then the closing functional bracket. We require that the object arity of each of these attribute terms is equal to the implicit arity of the object letter and the number of the attribute term in the succession.
A basic object term is an object term. Given two not necessarily different object terms, we may obtain a further object term by concatenating from left to right first one object term, then the opening functional bracket, then the other object term, and finally the closing functional bracket.

Given any of the above terms, one may detect the most recent concatenation step of the term. The explicit brackets of the concatenation step are the right most closing bracket and the right most opening bracket that has the same number of opening and closing brackets between itself and the final closing bracket. If the most recent concatenation used a succession of several attribute terms, one may detect them as those attribute terms between the explicit concatenation brackets that do not lie between any further pair of opening and closing brackets.

3.4. **Statements.** Given an attribute term, we may concatenate an unquantified statement by concatenating in succession from left to right first as many object terms as the first number of the object arity of the attribute term, then the attribute term, and then in succession as many object terms as the second number of the object arity of the attribute term. Given an unquantified statement, one may detect the components of this concatenation, they are all the object and attribute terms which do not lie between any functional brackets and are not followed to the right by an opening functional bracket.

Two unquantified statements are dual to each other, if they are obtained from each other by replacing the attribute term used in the concatenation by its dual. An unquantified statement is admissible, if it contains no indefinite object letters.

We call an unquantified statement equality, if the attribute used in its concatenation is the equality symbol. We call the dual of an equality an inequality. We call the equality or inequality reflexive, if the two object terms used in the concatenation are the same.

Given two statements, we may concatenate a further statement by writing first an opening quantifier bracket, then one statement, then the matching closing quantifier bracket, and then the other statement. We call such a concatenated statement quantified. One can reconstruct the constituents of the most recent concatenation by identifying the closing quantifier bracket in the concatenation as the left most closing quantifier bracket that has the same number of opening and closing quantifier brackets between itself and the initial opening bracket. We call the first constituent term the hypothesis and the second constituent term the conclusion of the quantified statement. We call a quantified statement existential, if the matching pair of brackets used in the most recent concatenation are existential, otherwise we call it universal.
Two quantified statements are dual to each other, if precisely one is universal and they have the same hypothesis but mutually dual conclusions. A quantified statement is admissible if either both hypothesis and conclusion are admissible or the hypothesis is unquantified and contains precisely one indefinite object letter, possibly at several positions, and the conclusion becomes admissible if this indefinite letter is replaced by a definite object letter with zero attribute arity in all positions where the former occurs. We say an unquantified statement is contained in a statement, if it is a substring inside the statement that is not followed to the right by an opening functional bracket. Consider for each indefinite attribute in a statement an attribute term with the same object arity as the indefinite attribute, so that mutually dual indefinite attributes correspond to mutually dual attribute terms and the attribute terms contain only definite letters. We obtain a subordinate of the statement by replacing each of the indefinite attributes in the statement by the corresponding attribute term in all positions where the former occurs. Given a subordinate of a statement, and two indefinite object letters, one occurring in the subordinate and the other not occurring in the subordinate, we may obtain a further subordinate of the statement by replacing the former indefinite letter by the latter in all positions where the former occurs.

3.5. Constructions. A construction is concatenated from two statements by writing from left to right one statement, then the construction symbol, and then the other statement. We require that the left statement is unquantified and it contains precisely one definite letter, namely the left most letter in the attribute term used in the concatenation of the unquantified statement. We require that the left statement does not contain the duality symbol, and no letter occurs in more than one position in the left statement. We require that all indefinite letters in the left statement also occur in the right statement, but the definite letter in the left statement does not occur in the right statement. We require that the right statement does not contain any indefinite attribute letters other than those that also occur in the left statement. We call the construction admissible if the right statement becomes admissible when all indefinite object letters that appear in the left statement are replaced by a definite object letter of zero attribute arity in all positions of the right statement where they occur. Consider an admissible construction. Consider for each indefinite attribute in the construction an attribute term with the same object arity as the indefinite attribute, so that mutually dual indefinite attributes correspond to mutually dual attribute terms and the attribute terms contain only definite letters. Consider for each indefinite object letter
in the left statement an object term containing no indefinite attributes. If the right statement is quantified, we also require the object term to not contain any indefinite object letters. We obtain a subordinate of the construction by replacing each of the indefinite attributes in the statement by the corresponding attribute term in all positions where the former occurs and replacing each indefinite letter occurring in the left statement by the corresponding object term in all positions of the construction where the former occurs.

Given a subordinate of a construction, we can obtain a further subordinate by replacing both left and right statements by the respective dual statements.

Given a subordinate of a construction, and two indefinite object letters, one occurring in the subordinate and the other not occurring in the subordinate, we may obtain a further subordinate of the construction by replacing the former indefinite letter by the latter in all positions where the former occurs.

A subordinate still has one construction symbol and statements on both sides of the construction symbol. We call these statements the two sides of the subordinate.

3.6. Rules of adding strings. The process of formal mathematics is to add strings to the most recent document, until it is complete, and to start a new document with a preamble when the previous document is complete.

Assume the most recent document is a definition. We can decide it is complete, or add a box with a string at the bottom of the chain by one of the following two rules.

**Rule 0** (Construction in definition). We may add an admissible construction which activates the definite letter in the left statement.

**Rule 1** (Designation in definition). We may add a modified copy of the hypothesis of an existential claim of a prior theorem, assuming this hypothesis is unquantified and contains a unique indefinite letter. Here modification means that this indefinite letter is replaced by one and the same object term in all positions where the former occurs.

We require that this object term has only one object letter, only indefinite attribute letters, and no duality symbol. The object letter is definite, activated at the new string, and its attribute arity is equal to the number of different indefinite attribute letters in the claim. Each indefinite attribute letter in the claim appears in the object term and has the same object arity in the new string as in the claim.

We write the label of the theorem at the new edge.

Now assume the most recent document is a theorem. If it has no claim yet, we proceed with the following rule.
Rule 2 (Claim and root). We add a claim that is an admissible statement and does not activate any definite letter. If the theorem is one of Theorems 10 or 12 or 14 or 16 below, the theorem is complete with the claim. Otherwise, we start a proof with a root that contains a subordinate of the dual of the claim, where we require that each indefinite attribute letter in the claim is replaced by a definite attribute letter that is activated at the root and has the same object arity as the indefinite letter. We require that identical indefinite attribute letters are replaced by identical letters and different indefinite attribute letters are replaced by different letters.

Now assume the most recent document is a theorem that already has a claim and a root. If all leaves of the proof are empty, the theorem is complete. If the proof has leaves that are not empty, we consider one such leaf and add one or several children to this leaf using one of the next seven rules.

Rule 3 (Cases). We add a pair of children to this leaf with admissible statements that are mutually dual, contain no indefinite attribute letter, and do not activate any definite letter.

Rule 4 (Contradiction). If a subordinate of an ancestors of the leaf is dual to the statement in the leaf, we may add an empty child to this leaf and write the label of the ancestor at the new edge. If the leaf has a reflexive inequality, we may add an empty child to the leaf.

We may add a child to the leaf with an admissible reflexive equality, which does not activate any attribute letter and activates at most one object letter. We write the number of the present rule at the new edge. If we intend to immediately after adding this child use the reflexive equality for adding a child to this child using some other rule, we may omit adding the first child and proceed to the next child directly and write the present rule as part of the justification for the child at the edge.

Rule 5 (Deduction). Consider a subordinate for each of two ancestors of the leaf. If one subordinate is universal and the other subordinate is the hypothesis of the former, we may add a child to the leaf with the conclusion of the former. If one subordinate is universal with unquantified hypothesis containing an indefinite object letter, and the other subordinate is the modified hypothesis of the former, we may add a child to the leaf with the modified conclusion of the former. Here modification means the said indefinite object letter is replaced by one and the same object term in all positions where it occurs.

Instead of adding the child as above, we may combine with a further application of this Rule or Rule 6 to the quantified statement of the child, and directly add a child according to the further application.
We write this rule and the relevant ancestors at the first new edge of the succession of applications of Rules 5 and 6.

**Rule 6** (Designation in proof). Consider an object term containing no indefinite letters and a subordinate of an ancestor of the leaf that is an existential statement. Consider the modification of the subordinate obtained by doing nothing if the hypothesis of the subordinate is admissible and otherwise replacing the unique indefinite object letter occurring in the hypothesis by the considered object term in all positions of the subordinate where the former occurs.

We may add a child to the leaf with the hypothesis of the modified subordinate. If we have done this, we may add a child to this child with the conclusion of the modified subordinate.

We may omit the first child and directly add a child with the conclusion of the instantiation to the original leaf, provided the hypothesis of the subordinate is admissible.

Instead of adding a child with the conclusion as above, we may combine with a further application of this Rule or Rule 5 to the quantified conclusion, and directly add a child according to the further application.

We write this rule and the relevant ancestor at the first new edge of the succession of applications of Rules 5 and 6.

**Rule 7** (Construction in proof). We may add a child with an admissible construction that activates the definite letter in the left statement and does not contain any indefinite attributes.

We write this rule at the new edge.

**Rule 8** (Statement substitution). Consider a subordinate of an ancestor that is a construction, and consider a statement, itself a subordinate of some ancestor of the leaf.

Assume one side of the subordinate appears as the right most substring of the statement. We may add a child with a modification of the statement, meaning this rightmost substring is replaced by the other side of the subordinate.

Assume both sides of the subordinate are unquantified, and assume the statement contains one side of the subordinate at one or several positions. We may add a child with a modification of the statement, where modification means that this side of the subordinate is replaced by the other side of the subordinate in one or several positions where it occurs in the statement.

Instead of adding a child as above, we may combine with a subsequent applications of Rule 5 or Rule 6 breaking up the quantified statement of the child, and directly add a child following the further applications.

We refer to this rule and the relevant ancestors in the first new edge of the

**Rule 9** (Object substitution). Consider a statement of the leaf and an indefinite object letter.
Consider two object terms which are the two object terms in the concatenation of an equality that is an ancestor of the leaf.

Assume a modification of the statement, where the indefinite letter is replaced by one of the object terms in all positions, where it occurs, is a subordinate of an ancestor of the leaf. We may add a child with another modification of the statement, where the indefinite letter is replaced by the other object term in all positions where the former occurs.

We write this rule and the relevant ancestors at the new edge. The modification may be void if the equality is reflexive, in which case we copy the subordinate to the child and only write the relevant ancestor at the edge.

3.7. **First definitions and theorems.** We begin the process of producing documents until the stage that the four theorems explicitly addressed in Rule 2 appear.

**Theorem 10** (Object uniqueness).

\[
\{ \phi = \phi \} \{ \phi \neq \psi \} \{ \phi(\xi) = \phi(\xi) \} \{ \phi(\xi) = \phi \psi(\xi) \} = \psi
\]

**Definition 11** (Domain by attribute).

\[
M(\Gamma) : [ \phi = \phi ] \{ \Gamma \xi \} \phi(\xi) \neq \phi
D(\Gamma) \phi : [ \{ \Gamma \xi \} \phi(\xi) \neq \phi ] \{ \phi(\xi) \neq \phi \} \Gamma \xi
\]

**Theorem 12** (Choice, 11).

\[
\{ M(\Gamma) \} \{ \{ \Gamma \xi \} [ \Delta \eta ] \eta \Sigma \xi \} \{ D(\Gamma) \eta ] \{ \Gamma \xi \} [ \Delta \eta(\xi) ] \eta(\xi) \Sigma \xi
\]

**Definition 13** (Range, union, power).

\[
R \phi \psi : \{ \phi(\xi) \neq \phi \} \psi(\phi(\xi)) \neq \psi
U \phi \psi : \{ \phi(\xi) \neq \phi \} \{ \xi(\eta) \neq \xi \} \psi(\eta) \neq \psi
P \phi \psi : \{ \psi(\xi) = \psi \} \{ \xi(\eta) = \xi \xi \} \{ \phi(\eta) = \xi \phi \} \xi(\eta) = \xi \eta
\]

**Theorem 14** (Universe, 13).

\[
\{ \phi = \phi \} \{ \psi(\phi) \neq \psi \} [ R \phi \psi ] [ U \phi \psi ] P \phi \psi
\]

**Definition 15** (Identity object, extension).

\[
C \phi : \{ \phi(\xi) \neq \phi \} \phi(\xi) = \xi
E \phi \xi : \{ \phi(\xi) = \phi \} \phi = \xi
H \psi \phi : \{ \{ E \phi \xi \} \psi(\xi) \neq \psi \} \{ \psi(\xi) \neq \psi \} E \phi \xi \psi
\]
Theorem 16 (Infinity, 15).

\[
\{ \phi = \phi \} \left[ \psi(\phi) =' \psi \right] \{ \psi(\xi) =' \psi \} \{ \psi(\eta) =' \psi \} H_{\eta\xi}
\]

4. Basic theorems and definitions

Equality is a binary attribute. In case of a reflexive equality, one may want to replace this by a unary attribute. This is done in the following definition.

Definition 17 (Reflexive equality).

\[
I_{\xi} \colon \xi = \xi
\]

A number of rules of the system can be to similar effect be expressed in the language of the system. We present some examples. We first express that there is some object. It satisfies the attribute in Definition 17. As a consequence of Theorem 18 a definite object letter may be activated in a proof with this unary attribute, analogous to Rule 4 for a reflexive equality, which is used in the proof of the theorem.

Theorem 18 (Existence of object, 17).

\[
[I_{\xi}] I_{\xi}
\]

The next theorem allows to replace a reflexive equality by the unary attribute of Definition 17. This expresses a particular case of Rule 8 which is used in the proof of the theorem. In combination with Rule 4 the next theorem allows to add the unary attribute for every object.

Theorem 19 (Abbreviation reflexive equality, 17).

\[
\{ \xi = \xi \} I_{\xi}
\]
Every object is equal to itself as a consequence of Rule 4. The following theorem expresses this fact. It states that if an object is activated with any unary attribute, then it is equal to itself.

**Theorem 20** (Reflexive equality of arbitrary object).

\[
\{ \Gamma \xi \} \xi = \xi
\]

Given an object letter, that we assume activated with some unary attribute, we may assign it a new object letter by means of an equality. This is expressed by the following theorem.

**Theorem 21** (New equal object, 17).

The attribute in the following definition states that the object in question has an empty domain.

**Definition 22** (Attribute of empty domain).

\[
O \xi : \{ \eta = \eta \} \xi(\eta) = \xi
\]
Theorem 23 shows existence of an object with empty domain. It is constructed by applying the rule of choice with a condition that is impossible to satisfy.

**Theorem 23** (Existence of empty domain, [11][22]).

\[
\begin{align*}
\{O_{\xi}\} & \equiv \{O_{\xi}\} \times \\
\{I_{\xi}\} \times \{I_{\eta}\} & = \xi \\
[I_{\xi}'] & \equiv \{I_{\eta}\} \eta = \xi \\
[x = x] & = \{I_{\xi}'\} \times \{I_{\eta}'\} \eta \neq \xi
\end{align*}
\]

We use the above existence result to define an object with empty domain. We repeat the definition of the attribute of being empty, so that in future references we do not have to refer to both Definitions 22 and 24 in the same preamble.

**Definition 24** (Object with empty domain).

\[
O_{\xi} : O_{\xi} \equiv \{O_{\xi}\} \times \\
\{I_{\xi}\} \times \{I_{\eta}\} = \xi
\]

The following theorem expresses uniqueness of the object with an empty domain. The proof is an application of Theorem 10.

**Theorem 25** (Uniqueness of empty domain, [24]).

\[
\{O_{\xi}\} \xi = o
\]
Our next goal is to find an object with nonempty domain. To this end, the following theorem expresses existence of an object whose domain contains the object with empty domain. We use Theorem 14.

**Theorem 26** (Non-empty domain, 17, 24).

\[
\begin{align*}
[Iξ] & ξ(ο) = ξ \\
\{Iξ\} & ξ(ο) = ξ \\
14-4-5-6 & f(ο) = f \\
4-19 & If \\
5 & f(ο) = f
\end{align*}
\]

Having both an object with empty domain and an object with nonempty domain, we are ready to show that for any object one can find an object that is different. This theorem is somewhat parallel to Theorem 21.
The following two theorems together constitute Russell’s paradox. The first theorem states that for every object there is another object that is either equal or not equal to the first object depending on a certain condition on the first object. The proof prominently uses Theorem 27 to produce the different object when needed.

The second theorem states that this second object cannot be chosen prior to the first object, even if we allow it to float with the first object. If we have choose the second object before the first, we may then choose the first object be equal to the second object to obtain a contradiction.
Theorem 29 (Russell’s paradox, second part, [17]).

\[
\{I\eta\} [I\xi] \{\{\xi(\xi) = \xi\} \eta =' \xi\} \{\xi(\xi) =' \xi\} \eta =' \xi
\]

\[
[I\eta] \{\{\xi(\xi) = \xi\} \eta(\xi) =' \xi\} \{\xi(\xi) =' \xi\} \eta(\xi) =' \xi
\]

\[
[I\eta] [I\xi] \{\{\xi(\xi) = \xi\} \eta(\xi) =' \xi\} \{\xi(\xi) =' \xi\} \eta(\xi) = \xi
\]

\[
[I\xi]\{\{\xi(\xi) = \xi\} y(\xi) =' \xi\} \{\xi(\xi) =' \xi\} y(\xi) = \xi
\]

\[
\{y(y) = y\} y(y) =' y\} y(y) =' y\} y(y) = y
\]

\[
\{y(y) = y\} y(y) = y\}
\]

The term in \(M(\Gamma)\) in Definition 11 states that the attribute \(\Gamma\) is controlled by the domain of a function. The next theorem shows that this sub domain property is inherited by any more restrictive attribute.

Theorem 30 (Nested attributes, [11]).

\[
\{\{\Gamma\xi\} \Delta\xi\} \{M(\Delta)\} M(\Gamma)
\]
The abbreviation $C\phi$ in Definition 15 states that $\phi$ is an identity object, meaning for each object in its domain, it floats to become this very object.

**Theorem 31** (Empty object as identity object, 15, 24).

The following theorem states that for every sub domain attribute, there is an identity object with domain precisely described by this attribute.

**Theorem 32** (Existence identity function, 11, 15).
A successor of an object is an identity object whose domain contains precisely the original object as well as all objects in the domain of the original object. The attribute $E_{\phi \xi}$ states that $\xi$ is in this extended domain of $\phi$. The attribute $H_{\psi \phi}$ states that $\psi$ is the successor of $\phi$.

The purpose of the next two theorems is to show without using Theorem 16 that every identity object has a successor. The existence of such a successor is part of Theorem 16, but the main point of Theorem 16 is the existence of some object whose domain contains a successor of each of the objects in its domain.

We first show with the aid of Theorem 14 that for every identity object there is an object whose domain is at least the extended object of the original domain.

**Theorem 33** (Extension of domain, 13 15).

\[
\{C\phi\} [\psi = \psi] \{E_{\phi \xi}\} \psi(\xi) = ' \psi
\]
The next theorem uses the previous two theorems to show that every identity object has a successor.

**Theorem 34** (Existence of successor, [11] [15]).

\[
\{C\phi\} \{\psi = \psi\} \{E\phi\xi\} \psi(\xi) = \psi
\]

\[
\{\psi = \psi\} \{E\phi\xi\} \psi(\xi) = \psi
\]

\[
g(f) = f'
\]

\[
\{f(\xi) = f'\} \ g(f(\xi)) = f'
\]

\[
g(x) = g
\]

\[
\{f(\xi) = f'\} f(\xi) = x
\]
The next theorem states that an identity object cannot be in its own domain. In particular, for an identity object the extended domain is strictly larger than the domain of the object. Beginning with the empty object, which is an identity object, one may therefore use the previous theorem to construct larger and larger finite domains. As discussed more thoroughly in the next section, by Theorem 16 all these objects can be assumed in the domain of some object, which therefore has an infinite domain.

**Theorem 35** (Identity function not in its domain).  
\[
\{C\phi\} \phi(\phi) = \phi
\]
In this chapter we discuss the Peano axioms of the natural numbers. We will construct a model of the natural numbers and then prove that this model satisfies the Peano axioms. We use the object with empty domain as zero of the natural numbers. An inductive domain contains zero and with each of its objects also a successor of the object, with successor being the identity object on the extended domain as described in Definition 15. The attribute $S$ in the following definition states that $\phi$ has an inductive domain. An object that is in every inductive domain is called a natural number, this is expressed by the attribute $N$. The attribute $T$ describes an object with inductive domain consisting entirely of natural numbers.

**Definition 36** (Inductive domains, natural numbers, [11] [15] [24]).

\[
S\phi: [\phi(0) = \phi] \{\phi(\xi) = \phi, \phi(\eta) = \phi\} H\eta\xi \\
N\xi: \{S\phi\} \phi(\xi) = \phi \\
T\sigma: [D(N)\sigma] \{N\xi\} H\sigma(\xi)\xi
\]

The following theorem shows existence of an inductive domain. The proof uses the rule of infinity.

**Theorem 37** (Existence of inductive object, [15] [24] [36]).

\[
[\phi = \phi] S\phi
\]
The following theorem states that there is a domain containing every natural number.

**Theorem 38** (Subdomain property of natural numbers, [11][36].)

$$M(N)$$

$$M'(N)$$

$$\{\sigma = \sigma\} [N\xi] \sigma(\xi) = \sigma$$

$$s = s$$

$$Ss$$

$$Nx$$

$$s(x) = s$$

$$\{S\phi\} \phi(x) = 's$$

$$s(x) = 's$$

The next theorem gives an object with inductive domain containing precisely the natural numbers, floating with each natural number to the successor of the number.

**Theorem 39** (Minimal successor function, [11][15][17][36].)

$$[T\sigma] T\sigma$$
Definition 40 (Successor function).

\[
\{T\sigma\} T'\sigma
\]

\[
M(N)
\]

\[
\{N\xi\} \{I\eta\} H'\eta\xi
\]

\[
6 \ 0
\]

\[
N_x
\]

\[
\{I\eta\} H'\eta x
\]

\[
16 \ 4 \ 6
\]

\[
f(x) = ' f
\]

\[
\{f(\xi) = ' f\} \{f(\eta) = ' f\} H\eta\xi
\]

\[
5 \ 6 \ 0 \ 1
\]

\[
f(y) = ' f
\]

\[
H'yx
\]

\[
19 \ 4
\]

\[
I_y
\]

\[
5 \ 0 \ 5
\]

\[
H'yx
\]

\[
2
\]

\[
\{N\xi\} \{I\eta\} H\eta\xi
\]

\[
12 \ 6 \ 0 \ 1
\]

\[
D(N)s
\]

\[
\{N\xi\} [I(s(\xi))] Hs(\xi)\xi
\]

\[
T's
\]

\[
5 \ 0 \ 5
\]

\[
36 \ 8 \ 0
\]

\[
\{D(N)s\} \{N\xi\} H's(\xi)\xi
\]

\[
5 \ 6 \ 0 \ 3
\]

\[
N_x
\]

\[
H's(x)x
\]

\[
5 \ 6 \ 1 \ 4
\]

\[
Hs(x)x
\]

\[
1
\]

The object \(s\) from Definition 40 has been constructed so that it floats with every natural number to the successor of the number. The following theorem confirms that.

Theorem 41 (Auxiliary theorem, 15 36 40).

\[
\{N\xi\} Hs(\xi)\xi
\]
There are five Peano axioms. The first Peano axiom, Theorem 42, states that zero is a natural number.

**Theorem 42** (Peano Axiom I, 15 24 36).

\[
\begin{align*}
\text{No} & \\
\left[ N' \circ \right] & 36-8 \ 0 \\
\left[ S \phi \right] & 6 \ 0 \\
\left[ S r \right] & 6 \ 0 \\
\left[ r (o) = r \right] & 6 \ 0 \\
\left[ r (o) = r \right] & 2 \\
\end{align*}
\]

Induction is a proof method. If zero satisfies an attribute, and with every natural number that satisfies the attribute also its successor satisfies the attribute, then every natural number satisfies this attribute. This is stated in the following theorem. It is proved by producing an auxiliary inductive function.

**Theorem 43** (Induction, 11 15 36 40).

\[
\begin{align*}
\{ \text{Γ} \circ \} & \{ \text{Γ} \circ \} N \xi \} \{ \{ \text{Γ} \circ \} \Gamma s (\xi) \} \{ N \eta \} \text{Γ} \eta \\
\end{align*}
\]
The next theorem shows a uniqueness result for the successor. It is an application of the rule of function uniqueness.
Theorem 44 (Uniqueness of successor, [15]).

\[
\begin{align*}
\{\phi = \phi\} \quad & \{H\psi\phi\} \quad \{H\rho\phi\} \quad \psi = \rho \\
\{\phi = \phi\} \quad & [H\psi\phi] \quad [H\rho\phi] \quad \psi = \rho \\
\{\phi = \phi\} \quad & H\psi\phi \quad H\rho\phi \quad \psi = \rho \\
\end{align*}
\]

The second Peano axiom, Theorem 45, states that the successor of a natural number is again a natural number. This is inherited from the general inductive function, since natural numbers are precisely the objects in the intersection of all inductive domains.
Theorem 45 (Peano Axiom II, [18, 38, 40]).

\[
\{N\xi\} \; Ns(\xi)
\]

\[
\begin{array}{c}
[N\xi] \\
N' s(\xi)
\end{array}
\]

\[
6 \ 0 \\
N x
\]

\[
N' s(x)
\]

\[
36-8 \ 1 \\
\{S\phi\} \; \phi(x) = ' \phi
\]

\[
36-8 \ 1 \\
[S\phi] \; \phi(s(x)) = \phi
\]

\[
6 \ 0 \\
Sr
\]

\[
r(s(x)) = r
\]

\[
5 \ 1-3 \\
r(x) = ' r
\]

\[
36-8 \ 2 \\
[r(o) = ' r] \; \{r(\xi) = ' r\} \; [r(\eta) = ' r] \; H\eta\xi
\]

\[
6-5 \ 0-1 \\
r(y) = ' r
\]

\[
Hyx
\]

\[
41 \ 9 \\
Hs(x)x
\]

\[
44-4-5 \ 0-1 \\
y = s(x)
\]

\[
9 \ 0-6 \\
r(y) = r
\]

We postpone the discussion of the Peano axiom that is traditionally called the third. The fourth Peano axiom, Theorem 47, states that zero is not the successor of any natural number. Its proof uses the following theorem, that states that the successor of a natural number contains at least this natural number. The successor therefore can not have an empty domain.

Theorem 46 (Natural number in successor, [15, 38, 40]).

\[
\{N\xi\} \; s(\xi)(\xi) = ' s(\xi)
\]
[Nξ] s(x)(x) = s(x)

\[ \begin{align*}
\{Nξ \} s(\xi) &= 0 \\
N x \\
s(x) &= 0 \\
46-5 &= 1 \\
s(x)(x) &= s(x) \\
9-0 &= 1 \\
o(x) &= o \\
24 &= 0 \\
o(x) &= o \\
22-24 &= 5 \\
o(x) &= o \\
12 &= 0
\end{align*} \]

The fifth Peano axiom, Theorem 48, states the inductive principle. It is a variant of Theorem 43, which is prominently used in the proof.

Theorem 48 (Peano Axiom V, 36, 10).
To prove the third Peano axiom we need some preparation. Theorem 49 proves by induction that every natural number other than zero is the successor of some natural number. The theorem is then used to prove Theorem 50 which shows that every natural number is an identity object.

**Theorem 49** (Predecessor, 24 36 40).

\[
\{Nξ\} \{ξ = o\} [Nη] s(η) = ξ
\]
Theorem 50 (Natural number as identity object, \[13, 24, 36, 40\]).
The following three theorems together with Theorem 46 compare the domain of a natural number with that of its successor. While the natural number does not contain itself, Theorem 51 its successor does contain the natural number, Theorem 46. The domain of the successor is larger than the domain of the natural number, Theorem 52, but only by the natural number itself, Theorem 53.

Theorem 51 (Irreflexivity of containment, 15, 36).

\[
\{N\xi\} \xi(\xi) = \xi
\]

\[
[N\xi] \xi(\xi) = ^t \xi
\]

\[
6 \quad 0
\]

\[
N \times
\]

\[
x(x) = ^t x
\]

\[
50 \quad 1
\]

\[
C \times
\]

\[
x(x) = x
\]

\[
35 \quad 0
\]

\[
E \times
\]

\[
\{E x \xi\} s(x)(\xi) = ^t s(x)
\]

\[
E' x y
\]

\[
8 \quad 0
\]

\[
[x(y) = x] x = ^t y
\]

\[
6 \quad 0
\]

\[
x(y) = x
\]

\[
E x y
\]

\[
5 \quad 0 \quad 1
\]

\[
s(x)(y) = ^t s(x)
\]

\[
4 \quad 0
\]
Theorem 53 (Upper bound for successor domain, 15 36 40).

\[
\{N\xi\} \{s(\xi)(\eta) =' s(\xi)\} \{\xi(\eta) = \xi\} \xi=\eta
\]

\[
[N\xi] \{s(\xi)(\eta) =' s(\xi)\} \{\xi(\eta) = \xi\} \xi=\eta
\]

\[
6 \quad 0
\]

\[
N \xi
\]

\[
s(x)(y) =' s(x)
\]

\[
x(y) = x \quad x =' y
\]

\[
41 \quad 2
\]

\[
H s(x) x
\]

\[
15-8-6 \quad 0
\]

\[
\{s(x)(\xi) =' s(x)\} E x \xi
\]

\[
5 \quad 0-3
\]

\[
E x y
\]

\[
15-8 \quad 0
\]

\[
\{x(y) = x\} x = y
\]

\[
4
\]

We pause to observe by induction that zero in in the successor of every natural number.

Theorem 54 (Zero in successor, 24 36 40).

\[
\{N\xi\} s(\xi)(o) =' s(\xi)
\]

\[
[N\xi] s(\xi)(o) = s(\xi)
\]

\[
A \xi : s(\xi)(o) =' s(\xi)
\]

\[
[N\xi] \{A \xi\} A s(\xi)
\]

\[
6 \quad 0
\]

\[
N \xi
\]

\[
A x
\]

\[
A' s(x)
\]

\[
8 \quad 1-4
\]

\[
s(x)(o) =' s(x)
\]

\[
8 \quad 1-5
\]

\[
s(s(x))(o) = s(s(x))
\]

\[
52 \quad 1-4
\]

\[
A o
\]

\[
8 \quad 0-2
\]

\[
s(o)(o) = s(o)
\]

\[
42-46
\]

\[
s(o)(o) =' s(o)
\]

\[
48-5 \quad 1-2-3
\]

\[
s(s(x))(o) = s(s(x))
\]

\[
52 \quad 1-4
\]

\[
A o
\]

\[
8 \quad 0-5
\]

\[
s(x)(o) =' s(x)
\]

\[
2
\]

\[
s(s(x))(o) =' s(s(x))
\]

\[
1
\]
The next theorem establishes that containment is transitive across the natural numbers.

**Theorem 55** (Transitivity of containment, 24 36 40).

\[
\{N\eta\} \{N\zeta\} \{\eta(\xi) = \eta' \} \{\zeta(\eta) = \zeta' \} \zeta(\xi) = \zeta
\]

\[
[\eta(\xi) = \eta' \zeta(\eta) = \zeta' \zeta(\xi) = \zeta]
\]

\[
\text{y(x) = y}
\]

\[
[z(y) = z' \text{z}(x) = z]
\]

\[
A\zeta : \{\zeta(y) = \zeta' \} \zeta(x) = \zeta'
\]

\[
\{N\xi\} \{A\xi\} A's(\xi)
\]

\[
\{N\xi\} \{A\xi\} A's(\xi)
\]

Finally, we are in a position to prove the third Peano axiom. It states uniqueness for the predecessor of any natural number.
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Theorem 56 (Peano axiom III, $\text{[30]}$ $\text{[10]}$).

$\{N\xi\} \\{N\eta\} \{s(\xi) = s(\eta)\} \xi = \eta$

$$\begin{array}{c}
[N\xi] [N\eta] [s(\xi) = s(\eta)] \xi = \eta \\
\hline
\hline
\hline
\hline
6 \qquad \emptyset \\
\hline
N\theta \\
\hline
\hline
s(\xi) = s(\eta) \\
\hline
\hline
\xi = \eta \\
\hline
\hline
\hline
\hline
\hline
x = y \\
\hline
\hline
y = x \\
\hline
\hline
s(y)(x) = s(x) \\
\hline
\hline
s(y)(x) = s(x) \\
\hline
\hline
\end{array}$$

References

[1] Frege, G. Begriffsschrift, Eine der arithmetischen nachgebildete Formelsprache des reinen Denkens. Verlag L. Nebert, Halle (1879)

[2] Paskevich, A. The syntax and semantics of the ForTheL language. Excerpted from: Méthodes de formalisation des connaissances et des raisonnements mathématiques: aspects appliqués et théoriques. Doctoral thesis Paris XII University, 2007. http://nevidal.org/download/forthel.pdf (2007)

[3] Russell, B. The principles of mathematics. Cambridge University Press, Cambridge, 1903

[4] Zermelo, E. Über Grenzzahlen und Mengenbereiche. Fundamenta Mathematicae 16 (1930), 29–47

[5] von Neumann, J. Die Axiomatisierung der Mengenlehre. (German) Math. Z. 27 (1928), no. 1, 669–752.

[6] Freiherr von Wolff, C. Vernünftige Gedancken von Gott, der Welt und der Seele der Menschen, auch all Dingen überhaupt (1720)

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