ON THE TRANSVERSE KHOVANOV-ROZANSKY HOMOLOGIES: GRADED MODULE STRUCTURE AND STABILIZATION

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Abstract. In [9], the author proved that the Khovanov-Rozansky homology $H_N$ with potential $ax^{N+1}$ is an invariant for transverse links in the standard contact 3-sphere. In the current paper, we study the $\mathbb{Z}_2 \oplus \mathbb{Z}^3$-graded $\mathbb{Q}[a]$-module structure of $H_N$, which leads to better understanding of the effect of stabilization on $H_N$. As an application, we compute $H_N$ for all transverse unknots.

1. Introduction

1.1. The transverse Khovanov-Rozansky homology $H_N$. A contact structure $\xi$ on an oriented 3-manifold $M$ is an oriented tangent plane distribution such that there is a 1-form $\alpha$ on $M$ satisfying $\xi = \ker \alpha$, $d\alpha|_\xi > 0$ and $\alpha \wedge d\alpha > 0$. Such a 1-form is called a contact form for $\xi$. The standard contact structure $\xi_{st}$ on $S^3$ is given by the contact form $\alpha_{st} = dz - ydx + xdy = dz + r^2 d\theta$.

We say that an oriented smooth link $L$ in $S^3$ is transverse if $\alpha_{st}|_L > 0$. Two transverse links are said to be transverse isotopic if there is an isotopy from one to the other through transverse links.

Theorem 1.1. [1, 6, 7]

(1) Every transverse link is transverse isotopic to a counterclockwise transverse closed braid around the $z$-axis.
(2) Any smooth counterclockwise closed braid around the $z$-axis can be smoothly isotoped into a counterclockwise transverse closed braid around the $z$-axis without changing the braid word.
(3) Two counterclockwise transverse closed braids around the $z$-axis are transverse isotopic if and only if the braid word of one of them can be changed into that of the other by a finite sequence of transverse Markov moves. Here, by “transverse Markov moves”, we mean the following braid moves:
- Braid group relations generated by
  - $\sigma_i \sigma_i^{-1} = \sigma_i^{-1} \sigma_i = \emptyset$,
  - $\sigma_i \sigma_j = \sigma_j \sigma_i$, when $|i - j| > 1$,
  - $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_i$.
- Conjugation: $\mu \leftrightarrow \eta^{-1} \mu \eta$, where $\mu, \eta \in B_m$.
- Positive stabilization and destabilization: $\mu \in B_m$ $\leftrightarrow$ $\mu \sigma_m^{-1} \in (B_m)$.
In other words, all Markov moves are transverse Markov moves except the negative stabilization and destabilization $\mu \in B_m$ $\leftrightarrow$ $\mu \sigma_m \in (B_m)$.

Part (1) of Theorem 1.1 was established by Bennequin in [1], part (2) is a simple observation and part (3) was proved by Orevkov, Shevchishin in [6] and independently by Wrinkle in [7]. Theorem 1.1 means that there is a one-to-one correspondence

$\{\text{Transverse isotopy classes of transverse links}\} \longleftrightarrow \{\text{Closed braids modulo transverse Markov moves}\}$.

So, constructing invariants for transverse links is equivalent to constructing invariants for equivalence classes of closed braids modulo transverse Markov moves. For example, for a closed braid $B$ with writhe $w$ of $m$ strands, its self linking number $sl(B) = w_m$ is invariant under transverse Markov moves. So the self linking number is a transverse link invariant. See [1] for the original definition of the self linking number.

For more about transverse links, see, for example, [9].
Using the above correspondence, the author introduced in [9] a new homological invariant \( \mathcal{H}_N \) for transverse links. \( \mathcal{H}_N \) is a variant of the Khovanov-Rozansky homology defined in [4, 5]. We call \( \mathcal{H}_N \) the \( N \)th transverse Khovanov-Rozansky homology. The following is the main result of [9].

**Theorem 1.2.** [9] Theorem 1.2] Suppose \( N \geq 1 \). Let \( B \) be a closed braid and \( \mathcal{C}_N(B) \) the chain complex defined in Definition 2.13. Then the homotopy type of \( \mathcal{C}_N(B) \) does not change under transverse Markov moves. Moreover, the homotopy equivalences induced by transverse Markov moves preserve the \( \mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 3} \)-grading of \( \mathcal{C}_N(B) \), where the \( \mathbb{Z}_2 \)-grading is the \( \mathbb{Z}_2 \)-grading of the underlying matrix factorization and the three \( \mathbb{Z} \)-gradings are the homological, \( a \)- and \( x \)-gradings of \( \mathcal{C}_N(B) \).

Consequently, for the homology \( \mathcal{H}_N(B) = H(H(\mathcal{C}_N(B), d_{mf}), d_\chi) \) of \( \mathcal{C}_N(B) \) defined in Definition 2.13, every transverse Markov move on \( B \) induces an isomorphism of \( \mathcal{H}_N(B) \) preserving the \( \mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 3} \)-graded \( \mathbb{Q}[a] \)-module structure of \( \mathcal{H}_N(B) \).

1.2. Module structure of \( \mathcal{H}_N(B) \). The first part of the current paper is a more careful study of the \( \mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 3} \)-graded \( \mathbb{Q}[a] \)-module structure of \( \mathcal{H}_N(B) \), which refines [9] Theorem 1.11] and leads to Theorem 1.4 below.

Before stating Theorem 1.4, we introduce the following notations.

**Definition 1.3.** Let \( B \) be a closed braid. For \( (\varepsilon, i, j, k) \in \mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 3} \), denote by \( \mathcal{H}_N^{\varepsilon,i,j,k}(B) \) the subspace of \( \mathcal{H}_N(B) \) of homogeneous elements of \( \mathbb{Z}_2 \)-degree \( \varepsilon \), homological degree \( i \), \( a \)-degree \( j \) and \( x \)-degree \( k \). Replacing one of these indices by a “\(^*\)” means direct summing over all possible values of this index. For example:

\[
\mathcal{H}_N^{\varepsilon,i,*}(B) = \bigoplus_{j \in \mathbb{Z}} \mathcal{H}_N^{\varepsilon,i,j,k}(B),
\]

\[
\mathcal{H}_N^{\varepsilon,i,*,*}(B) = \bigoplus_{(j,k) \in \mathbb{Z}^{\oplus 2}} \mathcal{H}_N^{\varepsilon,i,j,k}(B).
\]

Similarly, for the \( \mathfrak{sl}(N) \) Khovanov-Rozansky homology \( H_N(B) \) defined in [4, 5], we denote by \( H_N^{\varepsilon,i,k}(B) \) the subspace of \( H_N(B) \) of homogeneous elements of \( \mathbb{Z}_2 \)-degree \( \varepsilon \), homological degree \( i \) and \( x \)-degree \( k \). Again, replacing one of these indices by a “\(^*\)” means direct summing over all possible values of this index.

**Theorem 1.4.** Let \( B \) be a closed braid, and \( (\varepsilon, i, k) \in \mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2} \). As a \( \mathbb{Z} \)-graded \( \mathbb{Q}[a] \)-module,

\[
\mathcal{H}_N^{\varepsilon,i,k}(B) \cong (\mathbb{Q}[a]\{ s(B) \})_a^{\oplus l} \oplus (\mathbb{Q}[a]\{ s(B) + 2 \})_a^{\oplus (\dim \mathcal{H}_N^{\varepsilon,i,k}(B) - l)} \oplus \bigoplus_{q=1}^{n} \mathbb{Q}[a]/(a)\{s_q\},
\]

where

- \( \{s\}_a \) means shifting the \( a \)-grading by \( s \),
- \( l \) and \( n \) are finite non-negative integers determined by \( B \) and the triple \( (\varepsilon, i, k) \),
- \( \{s_1, \ldots, s_n\} \subseteq \mathbb{Z} \) is a sequence determined up to permutation by \( B \) and the triple \( (\varepsilon, i, k) \),
- \( s(B) \leq s_q \leq c_+ - c_- - 1 \) and \( (N - 1)s_q \leq k - 2N + 2c_- \) for \( 1 \leq q \leq n \), where \( c_\pm \) is the number of \( \pm \) crossings in \( B \).

**Remark 1.5.** Note that \( s(B) \) and the number of components of \( B \) have the same parity. So, from [4], we know that \( H_N^{\varepsilon,i,k}(B) \cong 0 \) and, by Theorem 1.4, \( H_N^{\varepsilon,i,k}(B) \) is a torsion \( \mathbb{Q}[a] \)-module.

1.3. Stabilization. Applying a negative stabilization to a transverse closed braid \( B \), we get a new transverse closed braid \( B_- \). In contact geometry, this procedure is called a stabilization of the transverse link. In [9] Theorem 1.5], the author established that the chain complex \( \mathcal{C}_N(B_-) \) is isomorphic to \( \text{cone}(\pi_0)[-2,0] \), where

- \( \pi_0 : \mathcal{C}_N(B) \to \mathcal{C}_N(B)/a\mathcal{C}_N(B) \) is the standard quotient map,
- \( \text{cone}(\pi_0) \) is the mapping cone of \( \pi_0 \),
- \( \{j,k\} \) means shifting the \( a \)-grading by \( j \) and the \( x \)-grading by \( k \).

\(^2\)See Subsection 2.3 for our normalization of \( H_N(B) \).
 Therefore, there is a long exact sequence
\[ \cdots \to \mathcal{H}_N^{s-i,1,*,k}(B)[-2,0] \to \mathcal{H}_N^{s-i,1,*,k}(B)[-2,0] \to \mathcal{H}_N^{s-i,1,*,k}(B)[-2,0] \to \cdots \]

preserving the \(a\)- and \(x\)-gradings, where \( \mathcal{H}_N(B) := H(H(C_N(B)/aC_N(B), d_m), d_k) \).

Generally, it is not very easy to compute \( \mathcal{H}_N(B) \) even if \( \mathcal{H}_N(B) \) is known. So the above long exact sequence is not very useful when computing the homology of a stabilization of a transverse link. Using Theorem 1.4, we will take a closer look at the chain complex \( C \)

**Theorem 1.6.** Let \( B \) be a closed braid and \( B_- \) a stabilization of \( B \). Set \( s = sl(B) \). Then for any \((i, k) \in \mathbb{Z}^{\geq 2} \), there are a long exact sequence of \( \mathbb{Z} \)-graded \( Q[a]\)-modules

\[ (1.1) \cdots \to \mathcal{H}_N^{s-i,1,*,k}(B_-) \to \mathcal{H}_N^{s-i,1,*,k+N+1}(B_-)[1] \to \cdots \]

and a short exact sequence of \( \mathbb{Z} \)-graded \( Q[a]\)-modules

\[ (1.2) \quad 0 \to \mathcal{H}_N^{s-i,1,*,k}(B) \to \mathcal{H}_N^{s-i,1,*,k+N+1}(B_-) \to 0, \]

where \( \mathcal{H}_N(B) \) is the \( \mathfrak{sl}(N) \) Khovanov-Rozansky homology of \( B \) defined in [4].

In [2], Eliashberg and Fraser showed that two transverse unknots are transverse isotopic if and only if their self linking numbers are equal. Bennequin’s inequality [1] implies that the highest self linking number of a transverse unknot is \(-1\), which is attained by the 1-strand transverse closed braid. Denote by \( U_0 \) the transverse unknot with self linking \(-1\) and by \( U_m \) the transverse unknot obtained from \( U_0 \) by \( m \) stabilizations. Then every transverse unknot is transverse isotopic to \( U_m \) for some \( m \geq 0 \).

As an application of Theorem 1.6, we compute \( \mathcal{H}_N \) for all the transverse unknots. Before stating the result, let us recall that the \( \mathbb{Z} \)-grading of \( Q[a] \) is given by \( \deg_a = 2 \). We make \( Q[a] \) a \( \mathbb{Z}_2 \oplus \mathbb{Z}^{\geq 3} \)-graded \( Q[a]\)-module by making the \( \mathbb{Z}_2 \)-, homological and \( x \)-gradings all \( 0 \) on \( Q[a] \).

**Corollary 1.7.** Let \( \mathcal{F} \) and \( \mathcal{T} \) be the \( \mathbb{Z}_2 \oplus \mathbb{Z}^{\geq 3} \)-graded \( Q[a]\)-modules

\[ \mathcal{F} := \bigoplus_{l=0}^{\infty} Q[a]/(a)(1) \{ -1, N+1+2l \}, \]

\[ \mathcal{T} := \bigoplus_{l=0}^{\infty} Q[a]/(a)(1) \{ -1, N+1+2l \}, \]

where \( \langle \varepsilon \rangle \) means shifting the \( \mathbb{Z}_2 \)-grading by \( \varepsilon \) and \( \langle j,k \rangle \) means shifting the \( a \)-grading by \( j \) and the \( x \)-gradings by \( k \). Then,

\[ \mathcal{H}_N(U_0) \cong \mathcal{F} \oplus \mathcal{T}, \]

\[ \mathcal{H}_N(U_1) \cong \mathcal{F} \oplus \mathcal{T} \langle 1 \rangle \{ -1, N-1 \} \{ \| l \| \}, \]

and, for \( m \geq 2 \),

\[ \mathcal{H}_N(U_m) \cong \mathcal{F} \langle -2(m-1), 0 \rangle \oplus \mathcal{T} \langle m \rangle \{ -m, -m(N+1) \} \{ \| m \| \} \oplus \bigoplus_{l=1}^{m-1} \mathcal{F}/a \mathcal{F} \langle l \rangle \{ -2m+l, -(N+1) \} \{ l+1 \}, \]

where \( \| l \| \) means shifting the homological grading by \( l \).

1.4. **Organization of this paper.** In Section 2, we review the definition of \( \mathcal{H}_N \). Then we study the \( Q[a]\)-module structure of \( \mathcal{H}_N \) and prove Theorem 1.4 in Section 3. Finally, we prove Theorem 1.6 and Corollary 1.7 in Section 4.

This paper is self-contained for the most part. Of course, some prior knowledge of the Khovanov-Rozansky homology, especially of [1 2], will be helpful.

2. Definition of \( \mathcal{H}_N \)

In this section, we quickly review the definition of the transverse Khovanov-Rozansky homology \( \mathcal{H}_N \) in [2], which is every similar to the definition of the Khovanov-Rozansky homology in [3 4].
2.1. \( \mathbb{Z}_2 \oplus \mathbb{Z}_2^{\oplus 2} \)-graded matrix factorizations over \( \mathbb{Q}[a, x_1, \ldots, x_k] \).

**Definition 2.1.** We define a \( \mathbb{Z}_2^{\oplus 2} \)-grading on \( R = \mathbb{Q}[a, x_1, \ldots, x_k] \) by letting \( \deg a = (2, 0) \) and \( \deg x_i = (0, 2) \) for \( i = 1, \ldots, k \). We call the first component of this \( \mathbb{Z}_2^{\oplus 2} \)-grading the \( a \)-grading and denote its degree function by \( \deg a \). We call the second component of this \( \mathbb{Z}_2^{\oplus 2} \)-grading the \( x \)-grading and denote its degree function by \( \deg x \). An element of \( R \) is said to be homogeneous if it is homogeneous with respect to both the \( a \)-grading and the \( x \)-grading.

A \( \mathbb{Z}_2^{\oplus 2} \)-graded \( R \)-module \( M \) is a \( R \)-module \( M \) equipped with a \( \mathbb{Z}_2^{\oplus 2} \)-grading such that, for any homogeneous element \( m \) of \( M \), \( \deg(am) = \deg m + (2, 0) \) and \( \deg(x_i m) = \deg m + (0, 2) \) for \( i = 1, \ldots, k \). Again, we call the first component of this \( \mathbb{Z}_2^{\oplus 2} \)-grading of \( M \) the \( a \)-grading and denote its degree function by \( \deg a \). We call the second component of this \( \mathbb{Z}_2^{\oplus 2} \)-grading of \( M \) the \( x \)-grading and denote its degree function by \( \deg x \).

We say that the \( \mathbb{Z}_2^{\oplus 2} \)-grading on \( M \) is bounded below if both the \( a \)-grading and the \( x \)-grading are bounded below.

For a \( \mathbb{Z}_2^{\oplus 2} \)-graded \( R \)-module \( M \), we denote by \( M\{j,k\} \) the \( \mathbb{Z}_2^{\oplus 2} \)-graded \( R \)-module obtained by shifting the \( \mathbb{Z}_2^{\oplus 2} \)-grading of \( M \) by \( (j,k) \). That is, for any homogeneous element \( m \) of \( M \), \( \deg_M\{j,k\} m = \deg_M m + (j,k) \).

**Definition 2.2.** Let \( w \) be a homogeneous element with bidegree \( (2,2N+2) \) of \( R = \mathbb{Q}[a, x_1, \ldots, x_k] \). A \( \mathbb{Z}_2 \oplus \mathbb{Z}_2^{\oplus 2} \)-graded matrix factorization \( M \) of \( w \) over \( R \) is a collection of two \( \mathbb{Z}_2^{\oplus 2} \)-graded free \( R \)-modules \( M_0 \), \( M_1 \) and two homogeneous \( R \)-module maps \( d_0 : M_0 \to M_1 \), \( d_1 : M_1 \to M_0 \) of bidegree \( (1,N+1) \), called differential maps, such that
\[
d_1 \circ d_0 = w \cdot \text{id}_{M_0}, \quad d_0 \circ d_1 = w \cdot \text{id}_{M_1}.
\]
The \( \mathbb{Z}_2 \)-grading of \( M \) takes value \( \epsilon \) on \( M_\epsilon \). The \( a \)- and \( x \)-gradings of \( M \) are the \( a \)- and \( x \)-gradings of the underlying \( \mathbb{Z}_2^{\oplus 2} \)-graded \( R \)-module \( M_0 \oplus M_1 \).

We usually write \( M \) as \( M_0 \xrightarrow{d_0} M_1 \xrightarrow{d_1} M_0 \). Following [3], we denote by \( M\{1\} \) the matrix factorization \( M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} M_1 \) and write \( M\{j\} \)
\[
\text{the } j \text{ times}
\]
For any \( \mathbb{Z}_2 \oplus \mathbb{Z}_2^{\oplus 2} \)-graded matrix factorization \( M \) of \( w \) over \( R \) and \( j,k \in \mathbb{Z} \), \( M\{j,k\} \) is naturally a \( \mathbb{Z}_2 \oplus \mathbb{Z}_2^{\oplus 2} \)-graded matrix factorization of \( w \) over \( R \).

For any two \( \mathbb{Z}_2 \oplus \mathbb{Z}_2^{\oplus 2} \)-graded matrix factorizations \( M \) and \( M' \) of \( w \) over \( R \), \( M \oplus M' \) is naturally a \( \mathbb{Z}_2 \oplus \mathbb{Z}_2^{\oplus 2} \)-graded matrix factorization of \( w \) over \( R \).

Let \( w \) and \( w' \) be two homogeneous elements of \( R \) with bidegree \( (2,2N+2) \). For \( \mathbb{Z}_2 \oplus \mathbb{Z}_2^{\oplus 2} \)-graded matrix factorizations \( M \) of \( w \) and \( M' \) of \( w' \) over \( R \), the tensor product \( M \otimes_R M' \) is the \( \mathbb{Z}_2 \oplus \mathbb{Z}_2^{\oplus 2} \)-graded matrix factorization of \( w + w' \) over \( R \) such that:

- \( (M \otimes M')_0 = (M_0 \otimes M'_0) \oplus (M_1 \otimes M'_1) \), \( (M \otimes M')_1 = (M_1 \otimes M'_0) \oplus (M_1 \otimes M'_1) \);
- The differential is given by the signed Leibniz rule. That is, \( d(m \otimes m') = (dm) \otimes m' + (-1)^{\epsilon m} m \otimes (dm') \)

for \( m \in M_\epsilon \) and \( m' \in M'_\epsilon \).

**Definition 2.3.** Let \( w \) be a homogeneous element of \( R \) with bidegree \( (2,2N+2) \), and \( M \), \( M' \) any two \( \mathbb{Z}_2 \oplus \mathbb{Z}_2^{\oplus 2} \)-graded matrix factorizations of \( w \) over \( R \).

1. A morphism of \( \mathbb{Z}_2 \oplus \mathbb{Z}_2^{\oplus 2} \)-graded matrix factorizations from \( M \) to \( M' \) is a homogeneous \( R \)-module homomorphism \( f : M \to M' \) preserving the \( \mathbb{Z}_2 \oplus \mathbb{Z}_2^{\oplus 2} \)-grading satisfying \( d_M' f = f d_M \). We denote by \( \text{Hom}_{\text{mf}}(M,M') \) the \( \mathbb{Q} \)-space of all morphisms of \( \mathbb{Z}_2 \oplus \mathbb{Z}_2^{\oplus 2} \)-graded matrix factorizations from \( M \) to \( M' \).

2. An isomorphism of \( \mathbb{Z}_2 \oplus \mathbb{Z}_2^{\oplus 2} \)-graded matrix factorizations from \( M \) to \( M' \) is a morphism of \( \mathbb{Z}_2 \oplus \mathbb{Z}_2^{\oplus 2} \)-graded matrix factorizations that is also an isomorphism of the underlying \( R \)-modules. We say that \( M \) and \( M' \) are isomorphic, or \( M \cong M' \), if there is an isomorphism from \( M \) to \( M' \).

3. Two morphisms \( f \) and \( g \) of \( \mathbb{Z}_2 \oplus \mathbb{Z}_2^{\oplus 2} \)-graded matrix factorizations from \( M \) to \( M' \) are called homotopic if there is an \( R \)-module homomorphism \( h : M \to M' \) shifting the \( \mathbb{Z}_2 \)-grading by 1 such that \( f - g = d_M h + h d_M \). In this case, we write \( f \simeq g \). We denote by \( \text{Hom}_{\text{hmf}}(M,M') \) the \( \mathbb{Q} \)-space of all homotopy classes of morphisms of \( \mathbb{Z}_2 \oplus \mathbb{Z}_2^{\oplus 2} \)-graded matrix factorizations from \( M \) to \( M' \). That is, \( \text{Hom}_{\text{hmf}}(M,M') = \text{Hom}_{\text{mf}}(M,M')/ \simeq \).

\(^3\)An element of \( M \) is said to be homogeneous if it is homogeneous with respect to both \( \mathbb{Z} \)-gradings.
(4) $M$ and $M'$ are called homotopic, or $M \simeq M'$, if there are morphisms $f : M \to M'$ and $g : M' \to M$ such that $g \circ f \simeq \id_M$ and $f \circ g \simeq \id_{M'}$. $f$ and $g$ are called homotopy equivalences between $M$ and $M'$.

(5) We say that $M$ is homotopically finite if it is homotopic to a finitely generated graded matrix factorization of $w$ over $R$.

We define categories $\text{mf}^{\text{all}}_{R,w}$, $\text{mf}_{R,w}$, $\text{hmf}^{\text{all}}_{R,w}$, and $\text{hmf}_{R,w}$ by the following table.

| Category       | Objects                                                                 | Morphisms               |
|----------------|-------------------------------------------------------------------------|-------------------------|
| $\text{mf}^{\text{all}}_{R,w}$ | all $\mathbb{Z}_2 \oplus \mathbb{Z}^{\mathbb{Z}^2}$-graded matrix factorizations of $w$ over $R$ with the $\mathbb{Z}^{\mathbb{Z}^2}$-grading bounded below | $\text{Hom}_{\text{mf}}$ |
| $\text{mf}_{R,w}$   | all homotopically finite $\mathbb{Z}_2 \oplus \mathbb{Z}^{\mathbb{Z}^2}$-graded matrix factorizations of $w$ over $R$ with the $\mathbb{Z}^{\mathbb{Z}^2}$-grading bounded below | $\text{Hom}_{\text{mf}}$ |
| $\text{hmf}^{\text{all}}_{R,w}$ | all $\mathbb{Z}_2 \oplus \mathbb{Z}^{\mathbb{Z}^2}$-graded matrix factorizations of $w$ over $R$ with the $\mathbb{Z}^{\mathbb{Z}^2}$-grading bounded below | $\text{Hom}_{\text{hmf}}$ |
| $\text{hmf}_{R,w}$   | all homotopically finite $\mathbb{Z}_2 \oplus \mathbb{Z}^{\mathbb{Z}^2}$-graded matrix factorizations of $w$ over $R$ with the $\mathbb{Z}^{\mathbb{Z}^2}$-grading bounded below | $\text{Hom}_{\text{hmf}}$ |

**Definition 2.4.** If $a_0, a_1 \in R$ are homogeneous elements with $\deg a_0 + \deg a_1 = (2, 2N + 2)$, then denote by $(a_0, a_1)_R$ the $\mathbb{Z}_2 \oplus \mathbb{Z}^{\mathbb{Z}^2}$-graded matrix factorization $R \overset{a_0}{\twoheadrightarrow} R(1 - \deg a_0, N + 1 - \deg a_0) \overset{a_1}{\twoheadrightarrow} R$ of $a_0a_1$ over $R$. More generally, if $a_{1,0}, a_{1,1}, \ldots, a_{0,0}, a_{1,1} \in R$ are homogeneous with $\deg a_{j,0} + \deg a_{j,1} = (2, 2N + 2)$, then denote by

$$
\begin{pmatrix}
  a_{1,0} & a_{1,1} \\
  a_{2,0} & a_{2,1} \\
  \vdots & \vdots \\
  a_{0,0} & a_{1,1}
\end{pmatrix}_R
$$

the tensor product $(a_{1,0}, a_{1,1})_R \otimes_R (a_{2,0}, a_{2,1})_R \otimes_R \cdots \otimes_R (a_{0,0}, a_{1,1})_R$, which is a $\mathbb{Z}_2 \oplus \mathbb{Z}^{\mathbb{Z}^2}$-graded matrix factorization of $\sum_{j=1}^t a_{j,0}a_{j,1}$ over $R$, and is call the Koszul matrix factorization associated to the above matrix. We drop “$R$” from the notation when it is clear from the context.

Note that the above Koszul matrix factorization is finitely generated over $R$.

The following proposition from [1] is useful in computing the homology of some MOY graphs.

**Proposition 2.5.** [1] Proposition 10] Let $I$ be an ideal of $R$ generated by homogeneous elements. Assume $w$, $a_0$ and $a_1$ are homogeneous elements of $R$ such that $\deg w = \deg a_0 + \deg a_1 = (2, 2N + 2)$ and $w + a_0a_1 \in I$. Then $w \in I + (a_0)$ and $w \in I + (a_1)$.

Let $M$ be a $\mathbb{Z}_2 \oplus \mathbb{Z}^{\mathbb{Z}^2}$-graded matrix factorization of $w$ over $R$, and $\tilde{M} = M \otimes_R (a_0, a_1)_R$. Then $\tilde{M}/I\tilde{M}$, $M/(I + (a_0))M$ and $M/(I + (a_1))M$ are all $\mathbb{Z}_2 \oplus \mathbb{Z}^{\mathbb{Z}^2}$-graded chain complexes of $R$-modules.

(1) If $a_0$ is not a zero-divisor in $R/I$, then there is an $R$-linear quasi-isomorphism $f : \tilde{M}/I\tilde{M} \to (M/(I + (a_0))M) (1) \{1 - \deg a_0, N + 1 - \deg a_0\}$ that preserves the $\mathbb{Z}_2 \oplus \mathbb{Z}^{\mathbb{Z}^2}$-grading.

(2) If $a_1$ is not a zero-divisor in $R/I$, then there is an $R$-linear quasi-isomorphism $g : \tilde{M}/I\tilde{M} \to M/(I + (a_1))M$ that preserves the $\mathbb{Z}_2 \oplus \mathbb{Z}^{\mathbb{Z}^2}$-grading.

2.2. The matrix factorization associated to a MOY graph.

**Definition 2.6.** A MOY graph $\Gamma$ is an oriented graph embedded in the plane satisfying:

(1) Every edge of $\Gamma$ is colored by 1 or 2.
(2) Every vertex of $\Gamma$ is 1-, 2-, or 3-valent.
(3) Every 1-valent vertex of $\Gamma$ is either the initial point of a 1-colored edge or the terminal point of a 1-colored edge. We call 1-valent vertices of $\Gamma$ endpoints of $\Gamma$.
(4) Every 2-valent vertex of $\Gamma$ is the initial point of a 1-colored edge and the terminal point of a 1-colored edge.
(5) Every 3-valent vertex of $\Gamma$ is

- either the initial point of two 1-colored edges and the terminal point of a 2-colored edge,
- or the terminal point of two 1-colored edges and the initial point of a 2-colored edge.
In particular, Definition 2.6 means that every 2-colored edge of $\Gamma$ has a neighborhood that looks like the local configuration in Figure 1.

**Definition 2.7.** Let $\Gamma$ be a MOY graph. A marking of $\Gamma$ consists of:

1. A finite collection of marked points on $\Gamma$ such that
   - all endpoints are marked,
   - none of the 2- or 3-valent vertices are marked,
   - every 1-colored edge contains a marked point,
   - none of the 2-colored edges contain marked points.

2. An assignment that assigns to each marked point a single variable such that no two marked points are assigned the same variable.

Now suppose $\Gamma$ is a MOY graph with a marking. Let $x_1, \ldots, x_m$ be all the variables assigned to marked points on $\Gamma$ and $x_{i_1}, \ldots, x_{i_n}$ all the variables assigned to 1-valent vertices of $\Gamma$. We define $R$ to be the $\mathbb{Z}^2$-graded ring $R = \mathbb{Q}[a, x_1, \ldots, x_m]$ with the $\mathbb{Z}^2$-grading given by $\text{deg } a = (2, 0)$ and $\text{deg } x_i = (0, 2)$. Denote by $R_0$ the $\mathbb{Z}^2$-graded sub-ring $R_0 = \mathbb{Q}[a, x_{i_1}, \ldots, x_{i_n}]$ of $R$. We call $R_0$ the boundary ring of the marked MOY graph $\Gamma$.

Next, cut $\Gamma$ at all of its marked points. This breaks $\Gamma$ into simple marked MOY graphs $\Gamma_1, \cdots, \Gamma_p$, each of which is of one of the two types in Figure 2. Note that each $\Gamma_q$ is marked only at its endpoints. Denote by $R_q$ the $\mathbb{Z}^2$-graded polynomial ring over $\mathbb{Q}$ generated by $a$ and the variables marking $\Gamma_q$.

- If $\Gamma_q = \Gamma_{i:k}$ in Figure 2 then $R_q = \mathbb{Q}[a, x_i, x_k]$ and
  \[
  C_N(\Gamma_q) = (a \cdot \frac{x_{k}^{N+1} - x_i^{N+1}}{x_k - x_i}, x_k - x_i) R_q.
  \]

- If $\Gamma_q = \Gamma_{i,j:k,l}$ in Figure 2 then $R_q = \mathbb{Q}[a, x_i, x_j, x_k, x_l]$ and
  \[
  C_N(\Gamma_q) = \left( a \cdot \frac{g(x_k+x_i, x_kx_l)-g(x_i+x_j, x_kx_l)}{x_k+x_i, x_kx_l}, \frac{x_k + x_l - x_i - x_j}{x_kx_l - x_i, x_l} \right)_{R_q} \{0, -1\},
  \]
  where $g$ is the unique 2-variable polynomial satisfying $g(x + y, xy) = x^{N+1} + y^{N+1}$.

**Definition 2.8.**

\[
C_N(\Gamma) = \bigotimes_{q=1}^p (C_N(\Gamma_q) \otimes_{R_q} R),
\]

where the big tensor product “$\bigotimes_{q=1}^p$” is taken over the ring $R = \mathbb{Q}[a, x_1, \ldots, x_m]$.

Note that $C_N(\Gamma)$ is a $\mathbb{Z}_2 \oplus \mathbb{Z}^2$-graded matrix factorization of $w = \sum_{k=1}^n \pm a x_k^{N+1}$, where the sign is positive if $\Gamma$ points outward at the corresponding endpoint and negative if $\Gamma$ points inward at the corresponding endpoint.

---

4We consider the initial and terminal points of an edge part of that edge.
We view $C_N(\Gamma)$ as an object of the category $\text{hmf}^\text{all}_{R_0,w}$.

**Definition 2.9.** A MOY graph is called closed if it has no endpoints. If $\Gamma$ is a closed MOY graph, then $C_N(\Gamma)$ is a $\mathbb{Z}_2 \oplus \mathbb{Z}^2$-graded matrix factorization of $0$. So it is a homologically $\mathbb{Z}_2$-graded chain complex of $\mathbb{Z}^2$-graded $\mathbb{Q}[a]$-modules with a homogeneous differential map. We denote by $\mathcal{H}_N(\Gamma)$ the homology of this chain complex. Note that $\mathcal{H}_N(\Gamma)$ is a $\mathbb{Z}_2 \oplus \mathbb{Z}^2$-graded $\mathbb{Q}[a]$-module by inheriting the gradings of $C_N(\Gamma)$.

The following two lemmas are slight generalizations of the corresponding results in [4, 5].

**Lemma 2.10.** [9 Corollary 5.6, Lemma 3.11 and Proposition 7.1] As matrix factorizations over the respective boundary rings, we have:

\begin{equation}
(2.3) \quad C_N \begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1
\end{pmatrix} \simeq C_N \begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1
\end{pmatrix} \oplus C_N \begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1
\end{pmatrix} \oplus \{0, 1\} \oplus C_N \begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1
\end{pmatrix} \{0, 1\},
\end{equation}

\begin{equation}
(2.4) \quad C_N \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix} \simeq C_N \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix} \oplus C_N \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix} \{0, 1\},
\end{equation}

\begin{equation}
(2.5) \quad C_N \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix} \oplus C_N \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix} \oplus \{0, 1\} \oplus C_N \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix} \{0, 1\}.
\end{equation}

![Figure 3](image-url)
2.3. Definition of $H_N$. We first define the chain complex associated to a tangle diagram.

**Definition 2.12.** Let $T$ be an oriented tangle diagram. We call a segment of $T$ between two adjacent crossings/end points an arc. We color all arcs of $T$ by 1. A marking of $T$ consists of:

1. A collection of marked points on $T$ such that
   - none of the crossings of $T$ are marked,
   - all end points are marked,
   - every arc of $T$ contains at least one marked point,
2. An assignment of pairwise distinct homogeneous variables of bidegree $(0, 2)$ to the marked points such that every marked point is assigned a unique variable.

Let $T$ be an oriented tangle with a marking. Recall that $a$ is homogeneous of bidegree $(2, 0)$. Denote by

- $R$ the polynomial ring over $\mathbb{Q}$ generated by $a$ and all the variables associated to marked points of $T$,
- $R_\partial$ the polynomial ring over $\mathbb{Q}$ generated by $a$ and all the variables associated to end points of $T$.

Again, we call $R_\partial$ the boundary ring of $T$.

Cut $T$ at all of its marked points. This cuts $T$ into a collection \(\{T_1, \ldots, T_l\}\) of simple tangles, each of which is one of the three types in Figure 5 and is marked only at its end points. Denote by $R_i$ the polynomial ring over $\mathbb{Q}$ generated by $a$ and the variables marking end points of $T_i$.

![Tangle Diagram](image)

Figure 4.

If $T_i = A$, then $R_i = \mathbb{Q}[a, x_1, x_2]$ and $C_N(T_i)$ is the chain complex over $\text{hmf}_{R_i, a(x_1^{N+1} - x_2^{N+1})}$ given by

\[
C_N(T_i) = 0 \rightarrow C_N(A) \rightarrow 0,
\]

where the $C_N(A)$ on the right hand side is the matrix factorization associated to the MOY graph $A$, and the under-brace indicates the homological grading.

If $T_i = C_\pm$, then $R_i = \mathbb{Q}[a, x_1, x_2, y_1, y_2]$ and $C_N(T_i)$ is the chain complex over $\text{hmf}_{R_i, a(x_1^{N+1} + y_1^{N+1} - x_2^{N+1} - y_2^{N+1})}$ given by

\[
\begin{align*}
C_N(C_+) &= 0 \rightarrow C_N(\Gamma_1) \langle 1 \rangle \{1, N\} \xrightarrow{\chi^+} C_N(\Gamma_0) \langle 1 \rangle \{1, N - 1\} \rightarrow 0, \\
C_N(C_-) &= 0 \rightarrow C_N(\Gamma_0) \langle 1 \rangle \{-1, -N + 1\} \xrightarrow{\chi^0} C_N(\Gamma_1) \langle 1 \rangle \{-1, -N\} \rightarrow 0,
\end{align*}
\]

where $\Gamma_0$ and $\Gamma_1$ are the resolutions of $C_\pm$ given in Figure 5, the morphisms $\chi^0$ and $\chi^1$ are defined in Lemma 2.11 and the under-braces indicate the homological gradings.

Note that, in all three cases, the differential map of $C_N(T_i)$ consists of homogeneous morphisms of matrix factorizations preserving the $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$-grading. Of course, this differential map raises the homological grading by 1.

**Definition 2.13.** We define the chain complex $C_N(T)$ associated to $T$ to be

\[
C_N(T) := \bigotimes_{i=1}^l (C_N(T_i) \otimes_{R_i} R),
\]

where the big tensor product “\(\bigotimes_{i=1}^l\)” is taken over $R$. We view $C_N(T)$ as a chain complex of $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$-graded matrix factorizations over the ring $R_\partial$. 

8
$C_N(T)$ is equipped with a $\mathbb{Z}_2 \oplus \mathbb{Z}^{\mathbb{Z}^3}$-grading, where the $\mathbb{Z}_2 \oplus \mathbb{Z}^{\mathbb{Z}^2}$-grading comes from the underlying matrix factorization and the additional $\mathbb{Z}$-grading is the homological grading.

Note that, if $T$ is an oriented link diagram, then $C_N(T)$ is a chain complex over the category $\text{hmf}_{\mathbb{Q}[a],0}$.

**Lemma 2.14.** [3] Lemma 4.5, and Propositions 5.5, 6.1, 7.5] The homotopy type of $C_N(T)$ is independent of the marking of $T$ and invariant under positive Reidemeister move I and braid-like Reidemeister moves II and III.

Now let $L$ be a link diagram with a marking. Note $C_N(L)$ has two differential maps:

1. The differential $d_{mf}$ of the underlying matrix factorization structure of $C_N(L)$.
2. The differential $d_{\chi}$ from the crossing information given in equations (2.6), (2.7) and (2.8).

As a matrix factorization, $C_N(L)$ is a matrix factorization of $0$. So $d_{mf}^2 = 0$. Thus, the homology $H(C_N(L), d_{mf})$ is well defined. In fact, $H(C_N(L), d_{mf})$ inherits the $\mathbb{Z}_2 \oplus \mathbb{Z}^{\mathbb{Z}^3}$-grading of $C_N(L)$ and $(H(C_N(L), d_{mf}), d_{\chi})$ is a chain complex with a homological $\mathbb{Z}$-grading of $\mathbb{Z}_2 \oplus \mathbb{Z}^{\mathbb{Z}^2}$-graded $\mathbb{Q}[a]$-modules.

**Definition 2.15.** $\mathcal{H}_N(L) := H(H(C_N(L), d_{mf}), d_{\chi})$. It is a $\mathbb{Z}_2 \oplus \mathbb{Z}^{\mathbb{Z}^3}$-graded $\mathbb{Q}[a]$-module.

As a simple corollary of Lemma 2.14, we have:

**Corollary 2.16.** The $\mathbb{Z}_2 \oplus \mathbb{Z}^{\mathbb{Z}^3}$-graded $\mathbb{Q}[a]$-module $\mathcal{H}_N(L)$ is independent of the marking of $L$ and invariant under positive Reidemeister move I and braid-like Reidemeister moves II and III.

Clearly, Theorem 1.1 follows from Lemma 2.14 and Corollary 2.16.

2.4. **The $\mathfrak{sl}(N)$ Khovanov-Rozansky homology** $H_N$. If we set $a = 1$ in the above construction, then we get the $\mathfrak{sl}(N)$ Khovanov-Rozansky homology $H_N$ define in [4]. More precisely, for any tangle $T$, let

$$C_N(T) = C_N(T)/(a - 1)C_N(T).$$

Then $C_N(T)$ is the $\mathfrak{sl}(N)$ Khovanov-Rozansky chain complex defined in [4]. Note that $C_N(T)$ inherits the $\mathbb{Z}_2$, homological and $x$-gradings of $C_N(T)$. It also inherits the differentials $d_{mf}$ and $d_{\chi}$. For a link diagram $L$,

$$H_N(L) = H((C_N(L), d_{mf}), d_{\chi})$$

is the $\mathfrak{sl}(N)$ Khovanov-Rozansky homology defined in [4]. $H_N(L)$ inherits the gradings of $C_N(L)$ and is a $\mathbb{Z}_2 \oplus \mathbb{Z}^{\mathbb{Z}^2}$-graded $\mathbb{Q}$-linear space.
The $\mathbb{Z}_2 \oplus \mathbb{Z}$-graded matrix factorization $C_N(\Gamma) = C_N(\Gamma)/(a-1)C_N(\Gamma)$ of a MOY graph $\Gamma$ satisfies decompositions similar to those in Lemma 2.10.

Lemma 2.17. [4] As matrix factorizations over the respective boundary rings, we have:

\begin{align}
(2.11) \quad & C_N\begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix} \cong C_N\begin{pmatrix}
1 & 2 \\
1 & 1
\end{pmatrix} \{1\}_x \oplus C_N\begin{pmatrix}
1 \\
1
\end{pmatrix} \{1\} \{1-N\}_x, \\
(2.12) \quad & C_N\begin{pmatrix}
1 & 3 \\
1 & 1
\end{pmatrix} \cong C_N\begin{pmatrix}
1 & 2 \\
1 & 1
\end{pmatrix} \{-1\}_x \oplus C_N\begin{pmatrix}
2 \\
1
\end{pmatrix} \{1\}_x, \\
(2.13) \quad & C_N\begin{pmatrix}
1 & 1 & 1 \\
1 & 2 & 1 \\
1 & 1 & 1
\end{pmatrix} \oplus C_N\begin{pmatrix}
1 & 2 \\
1 & 1
\end{pmatrix} \cong C_N\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 2 \\
1 & 1 & 1
\end{pmatrix} \oplus C_N\begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}.
\end{align}

In the above, $\{\ast\}_x$ means shifting the $x$-grading by $\ast$.

The following invariance theorem for $H_N$ is established in [4].

Theorem 2.18. [4] The homotopy type of $C_N(T)$, including its $\mathbb{Z}_2 \oplus \mathbb{Z}^2$-grading, is independent of markings and invariant under all Reidemeister moves. Consequently, every Reidemeister move on $L$ induces an isomorphism of $H_N(L)$ preserving its $\mathbb{Z}_2 \oplus \mathbb{Z}^2$-graded $\mathbb{Q}$-linear space structure.

3. Graded Module Structure of $\mathcal{H}_N$

In this section, we study the $\mathbb{Z}_2 \oplus \mathbb{Z}^3$-graded $\mathbb{Q}[a]$-module of $\mathcal{H}_N$. The goal is to prove Theorem 1.4.

3.1. Resolved braids. In this subsection, we review some basic properties of resolved braids introduced in [8].

![Figure 6](image-url)

**Figure 6.**

**Definition 3.1.** For positive integers $b, i$ with $1 \leq i \leq b-1$, let $\tau_i$ be the MOY graph depicted in Figure 6. That is, from left to right, $\tau_i$ consists of $i-1$ downward 1-colored edges, then a downward 2-colored edge with two 1-colored edges entering through the top and two 1-colored edges exiting through the bottom, and then $b-i$ more downward 1-colored edges.

We use $(\tau_{i_1}, \ldots, \tau_{i_m})$ to represent the MOY graph formed by stacking the graphs $\tau_{i_1}, \ldots, \tau_{i_m}$ together vertically from top to bottom with the bottom end points of $\tau_{i_1}$ identified with the corresponding top end...
points of $\tau_{i_{i+1}}$. We call $(\tau_{i_1} \cdots \tau_{i_m})_b$ a resolved braid of $b$-strands. If the number of strands is clear from the context, then we drop the lower index $b$ and simply write $\tau_{i_1} \cdots \tau_{i_m}$.

Denote by $(\tau_{i_1} \cdots \tau_{i_m})_b$ the closed MOY graph obtained from $(\tau_{i_1} \cdots \tau_{i_m})_b$ by attaching a 1-colored edge from each end point at the bottom to the corresponding end point at the top. We call $(\tau_{i_1} \cdots \tau_{i_m})_b$ a closed resolved braid of $b$-strands. Again, if the number of strands is clear from the context, then we drop the lower index $b$ and simply write $\tau_{i_1} \cdots \tau_{i_m}$.

We use $(\emptyset)_b$ to represent $b$ vertical downward 1-colored edges, and, therefore, $(\emptyset)_b$ represents $b$ concentric 1-colored circles. Again, if the number of strands is clear from the context, then we drop the lower index $b$.

Remark 3.2.  
(1) Comparing Definition 3.1 to the resolutions in Figure 5, one can see that, if we choose a resolution for every crossing in a (closed) braid, then we get a (closed) resolved braid as defined in Definition 3.1.

(2) There are two obvious types of isotopies of resolved braids and closed resolved braids:
- $I_1$: If $|i - j| > 1$, then $\tau_i \tau_j$ is isotopic to $\tau_j \tau_i$;
- $I_2$: If $\mu$ and $\nu$ are two words in $\tau_1, \ldots, \tau_{b-1}$, then $\mu \nu$ is isotopic to $\nu \mu$.

Definition 3.3. We define the weight of the closed resolved braid $\tau_{i_1} \cdots \tau_{i_m}$ to be $w(\tau_{i_1} \cdots \tau_{i_m}) = i_1 + \cdots + i_m$.

In [8], the author introduced a scheme to perform inductive arguments on the weights of closed resolved braids using the decompositions in Lemma 2.17. The key to this scheme is Corollary 3.5 below, which is a simple consequence of Lemma 3.4.

Lemma 3.4. [8 Lemma 3.5] Let $\mu = \tau_{i_1} \cdots \tau_{i_m}$ be a resolved braid with $b$ strands satisfying:
- $m \geq 2$,
- $i_1 = i_m = i$,
- $i_l < i$ for $1 < l < m$.

Then, via a finite sequence of isotopies of type $I_1$, $\mu$ is isotopic to a resolved braid $\mu'$ that contains a segment of the form $\tau_j \tau_j$ or $\tau_j \tau_{j-1} \tau_j$ for some $j \leq i$.

Corollary 3.5. Let $\overline{\mu}$ be a closed resolved braids with $b$ strands. Then, via a finite sequence of isotopies of types $I_1$ and $I_2$, $\overline{\mu}$ is isotopic to a closed resolved braid of one of the following three types:

(a) $\tau_{i_1} \cdots \tau_{i_m} \tau_i$, where $i > i_1, \ldots, i_m$;
(b) $\tau_{i_1} \cdots \tau_{i_m} \tau_j \tau_j$;
(c) $\tau_{i_1} \cdots \tau_{i_m} \tau_j \tau_{j-1} \tau_j$. 

Figure 7.
3.2. Homology of closed resolve braids. In this subsection, we study the \(Q[a]\)-module structure of the homology of closed resolved braids. The goal is to establish Lemma 3.9 below.

Lemma 3.6. Let \(\emptyset_b\) be the closed resolved braid with \(b\)-strands corresponding to the empty word, that is, the MOY graph consisting of \(b\) concentric 1-colored circles. Define the \(\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}\)-graded \(Q[a]\)-modules \(\mathcal{M}_0\), \(\mathcal{M}_1\) and \(\mathcal{M}_\infty\) by

\[
\mathcal{M}_0 := \bigoplus_{N=1}^{\infty} Q[a] \langle 1 \rangle \{-1,1-N\} \oplus Q[a],
\]
\[
\mathcal{M}_1 := \bigoplus_{l=0}^{\infty} Q[a] \langle 1 \rangle \{-1,1-N+2l\},
\]
\[
\mathcal{M}_\infty := \bigoplus_{l=N}^{\infty} Q[a]/(a) \langle 1 \rangle \{-1,1-N+2l\}.
\]

Then, as a \(\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}\)-graded \(Q[a]\)-module,

\[
\mathcal{H}_N(\emptyset_b) \cong \mathcal{M}_1^{\oplus b} \oplus \left( \bigoplus_{j=0}^{b-1} \mathcal{M}_0^{\oplus j} \otimes \mathcal{M}_1^{\oplus (b-1-j)} \right) \otimes \mathcal{M}_\infty,
\]

where all the tensor products are over \(Q[a]\).

Proof. We prove this lemma by an induction on \(b\). Mark \((\emptyset)_1\) by a single variable \(x\). Then

\[
\mathcal{C}_N((\emptyset)_1) = ((N+1)ax_1^N,0)_{Q[a,x]} = Q[a,x] \rightarrow Q[a,x]\{−1,1−N\} \rightarrow Q[a,x].
\]

So \(\mathcal{H}_N((\emptyset)_1) \cong Q[a,x]/(ax^N) (1) \{−1,1−N\} \cong \mathcal{M}_1 \oplus \mathcal{M}_\infty\). This proves the lemma for \(b=1\).

Now assume the lemma is true for \((\emptyset)_{b−1}\). Consider \((\emptyset)_b\). Mark the \(j\)th circle in \((\emptyset)_b\) by a single variable \(x_j\). Then

\[
\mathcal{C}_N((\emptyset)_b) = \begin{pmatrix}
(N+1)ax_1^N & 0 \\
(N+1)ax_2^N & 0 \\
\vdots & \vdots \\
(N+1)ax_b^N & 0
\end{pmatrix} \otimes_{Q[a]} Q[a,x_b]/(ax_b^N) \langle 1 \rangle \{-1,1-N\}.
\]

Thus, by Proposition 2.25 \(\mathcal{C}_N((\emptyset)_b)\) is quasi-isomorphic to

\[
\left(\begin{array}{c}
(N+1)ax_1^N \\
(N+1)ax_2^N \\
\vdots \\
(N+1)ax_b^N
\end{array}\right)_{Q[a,x_1,x_2,\ldots,x_b-1]} \otimes_{Q[a]} Q[a,x_b]/(ax_b^N) \langle 1 \rangle \{-1,1-N\}.
\]

Note that:

1. \(\mathcal{C}_N((\emptyset)_{b-1}) \cong \begin{pmatrix}
(N+1)ax_1^N & 0 \\
(N+1)ax_2^N & 0 \\
\vdots & \vdots \\
(N+1)ax_{b-1}^N & 0
\end{pmatrix} \otimes_{Q[a]} Q[a,x_1,x_2,\ldots,x_{b-1}]\]
2. \(Q[a,x_b]/(ax_b^N) (1) \{-1,1−N\} \cong \mathcal{M}_1 \oplus \mathcal{M}_\infty\).
3. \(\mathcal{M}_1\) is a free \(Q[a]\)-module.
4. The homology of \(\begin{pmatrix}
(N+1)ax_1^N & 0 \\
(N+1)ax_2^N & 0 \\
\vdots & \vdots \\
(N+1)ax_{b-1}^N & 0
\end{pmatrix} \otimes_{Q[a]} \mathcal{M}_\infty\) is isomorphic to \(\mathcal{M}_0^{\oplus (b-1)} \otimes \mathcal{M}_\infty\).

Putting the above together, we get

\[
\mathcal{H}_N((\emptyset)_b) \cong \mathcal{H}_N((\emptyset)_{b-1}) \otimes_{Q[a]} \mathcal{M}_1 \oplus \mathcal{M}_0^{\oplus (b-1)} \otimes \mathcal{M}_\infty.
\]

This isomorphism and the assumption that the lemma is true for \((\emptyset)_{b-1}\) imply that the lemma is true for \((\emptyset)_b\). \(\Box\)
To discuss the homology of a general closed resolved braid, we need the following lemma, which is a slight refinement of the usual structure theorem of modules over a principal deal domain.

**Lemma 3.7.** [9] Lemma 9.2] Suppose that $M$ is a finitely generated $\mathbb{Z}$-graded $\mathbb{Q}[a]$-module. Then, as a $\mathbb{Z}$-graded $\mathbb{Q}[a]$-module, $M \cong \bigoplus_{i=1}^{\infty} \mathbb{Q}[a]\{s_i\}_a \oplus \bigoplus_{i=1}^{\infty} \mathbb{Q}[a]\{(a)^k\}_a \{t_k\}_a$, where $\{\ast\}_a$ means shifting the $a$-grading by $\ast$, and the sequences $\{s_i\} \subset \mathbb{Z}$, $\{(l_1, t_1), \ldots, (l_n, t_n)\} \subset \mathbb{Z}^{\geq 2}$ are uniquely determined by $M$ up to permutation. We call this decomposition the standard decomposition of $M$.

**Definition 3.8.** For a closed resolved braid $\tau$, we denote by $H_N^{\epsilon,k}(\tau)$ (resp. $C_N^{\epsilon,k}(\tau)$) the homogeneous component of $H_N(\tau)$ (resp. $C_N(\tau)$) of $\mathbb{Z}_2$-degree $\epsilon$, $a$-degree $j$ and $x$-degree $k$. If we replace one of these indices by $\ast$, it means we direct sum the components over all possible values of that index. For example, $H_N^{\epsilon,k}(\tau) = \bigoplus_{j \in \mathbb{Z}} H_N^{\epsilon,k}(\tau)$.

Similarly, we denote by $H_N^{\epsilon,k}(\tau)$ (resp. $C_N^{\epsilon,k}(\tau)$) the homogeneous component of $\mathbb{Z}_2$-degree $\epsilon$ and $x$-degree $k$ of the $\mathfrak{sl}(N)$ Khovanov-Rozansky homology $H_N(\tau)$ (resp. $C_N(\tau)$) of $\tau$.

**Lemma 3.9.** For a closed resolved braid $\tau_{i_1, \ldots, i_m}$ of $b$ strands, we have that, as a $\mathbb{Z}$-graded $\mathbb{Q}[a]$-module,

$$H_N^{\epsilon,k}(\tau_{i_1, \ldots, i_m}) \cong H_N^{\epsilon,k}(\tau_{i_1, \ldots, i_m}) \otimes_{\mathbb{Q}[a]} \mathbb{Q}[a]\{-b\}_a \oplus \bigoplus_{i=1}^{l} \mathbb{Q}[a]/(a)\{s_i\}_a,$$

where
- we give $H_N^{\epsilon,k}(\tau_{i_1, \ldots, i_m})$ the $a$-grading $0$, and $\{\ast\}_a$ means shifting the $a$-grading by $\ast$,
- up to permutation, the sequence $\{s_1, \ldots, s_l\}$ is uniquely determined by $(\tau_{i_1, \ldots, i_m})$, $N$, $k$ and $\epsilon$,
- $-b \leq s_i \leq -1$ and $(N-1)s_i \leq k - 2N + m$ for $i = 1, \ldots, l$.

**Proof.** From the construction of $C_N((\tau_{i_1, \ldots, i_m})$, one can see that $C_N^{\epsilon,k}(\tau_{i_1, \ldots, i_m})$ is a finitely generated free $\mathbb{Q}[a]$-module. This implies that $H_N^{\epsilon,k}(\tau_{i_1, \ldots, i_m})$ is a finitely generated $\mathbb{Z}$-graded $\mathbb{Q}[a]$-module. So, by Lemma 3.7, $H_N^{\epsilon,k}(\tau_{i_1, \ldots, i_m})$ has a unique standard decomposition. Now, to prove the lemma, we only need to verify that:

(I) The free part of $H_N^{\epsilon,k}(\tau_{i_1, \ldots, i_m})$ is isomorphic to $H_N^{\epsilon,k}(\tau_{i_1, \ldots, i_m}) \otimes_{\mathbb{Q}[a]} \mathbb{Q}[a]\{-b\}_a$.

(II) All torsion components of $H_N^{\epsilon,k}(\tau_{i_1, \ldots, i_m})$ are of the form $\mathbb{Q}[a]/(a)\{s\}_a$.

(III) If $H_N^{\epsilon,k}(\tau_{i_1, \ldots, i_m})$ contains a torsion component $\mathbb{Q}[a]/(a)\{s\}_a$, then $-b \leq s \leq -1$ and $(N-1)s \leq k - 2N + m$.

These three conclusions can be easily proved by an induction on the weight of $(\tau_{i_1, \ldots, i_m})$ using Lemmas 2.10 and 2.17 and Corollary 3.5.

If the weight of a closed resolved braid is 0, then it is $(0)_b$. By Lemma 3.6 (I-III) is true for $(0)_b$ for all $b \geq 0$.

Now assume that (I-III) is true for all closed resolved braids (on any number of strands) with weight less than the weight of $(\tau_{i_1, \ldots, i_m})$. By Corollary 3.5 via a finite sequence of isotopies of types $I_1$ and $I_2$, $(\tau_{i_1, \ldots, i_m})$ is isotopic to a closed resolved braid of one of the following three types:

(a) $(\tau_{j_1, \ldots, j_{m-1}, i})_b$, where $i > j_1, \ldots, j_m$;
(b) $(\tau_{j_1, \ldots, j_{m-1}, i, j})_b$;
(c) $(\tau_{j_1, \ldots, j_{m-1}, -1, i, j})_b$.

Of course, isotopies of types $I_1$ and $I_2$ do not change the weight of a closed resolved braid.

In Case (a), we have

\[
\begin{align*}
H_N((\tau_{i_1, \ldots, i_m}) & \cong H_N((\tau_{j_1, \ldots, j_{m-1}, i})_b), \\
H_N((\tau_{j_1, \ldots, j_{m-1}, i})_b & \cong H_N((\tau_{j_1, \ldots, j_{m-1}, i})_b)\{0, 1\} \oplus H_N((\tau_{j_1, \ldots, j_{m-1}, i})_b-1)\{1-N\}, \\
H_N((\tau_{j_1, \ldots, j_{m-1}, i})_b & \cong H_N((\tau_{j_1, \ldots, j_{m-1}, i})_b)\{1\} \oplus H_N((\tau_{j_1, \ldots, j_{m-1}, i})_b-1)\{1-N\}x,
\end{align*}
\]

where the second and third isomorphisms follow from Lemmas 2.10 and 2.17. The weights of both $(\tau_{j_1, \ldots, j_{m-1}, i})_b$ and $(\tau_{j_1, \ldots, j_{m-1}, i})_b-1$ are less than that of $(\tau_{i_1, \ldots, i_m})_b$. So (I-III) are true for $(\tau_{j_1, \ldots, j_{m-1}, i})_b$ and $(\tau_{j_1, \ldots, j_{m-1}, i})_b-1.$
Moreover, by Lemma 3.7, the standard decomposition of $H_{N}^{*,k}(⟨τ_{j_1}⋯τ_{j_{m-1}}⟩_b)$ is unique. It then follows from the above isomorphisms that (I-III) are true for $⟨τ_{i_1}⋯τ_{i_m}⟩_b$ too.

In Case (b), we have

\[
H_N(⟨τ_{i_1}⋯τ_{i_m}⟩) \cong H_N(⟨τ_{j_1}⋯τ_{j_{m-2}}τ_j⟩) \cong H_N(⟨τ_{j_1}⋯τ_{j_{m-2}}⟩_b) \oplus H_N(⟨0,1⟩_b) \oplus H_N(⟨τ_{j_1}⋯τ_{j_{m-2}}⟩_b) \{0,-1\},
\]

where the second and third isomorphisms follow from Lemmas 2.10 and 2.17. The weight of $⟨τ_{j_1}⋯τ_{j_{m-2}}⟩_b$ is less than that of $⟨τ_{i_1}⋯τ_{i_m}⟩_b$. So (I-III) are true for $⟨τ_{j_1}⋯τ_{j_{m-2}}⟩_b$. It then follows from the above isomorphisms that (I-III) are true for $⟨τ_{i_1}⋯τ_{i_m}⟩_b$ too.

In Case (c), we have

\[
H_N(⟨τ_{i_1}⋯τ_{i_m}⟩) \cong H_N(⟨τ_{j_1}⋯τ_{j_{m-3}}τ_j⟩) \cong H_N(⟨τ_{j_1}⋯τ_{j_{m-3}}⟩_b) \oplus H_N(⟨0,1⟩_b) \oplus H_N(⟨τ_{j_1}⋯τ_{j_{m-3}}⟩_b) \{0,-1\},
\]

where the second and third isomorphisms follow from Lemmas 2.10 and 2.17. The weights of $⟨τ_{j_1}⋯τ_{j_{m-3}}⟩_b$, $⟨τ_{j_1}⋯τ_{j_{m-3}}⟩_b$ and $⟨τ_{j_1}⋯τ_{j_{m-3}}⟩_b$ are less than that of $⟨τ_{i_1}⋯τ_{i_m}⟩_b$. So (I-III) are true for these three resolved closed braids. It then follows from the above isomorphisms that (I-III) are true for $⟨τ_{i_1}⋯τ_{i_m}⟩_b$ too.

**Corollary 3.10.** $H^{b+1,*}(⟨τ_{i_1}⋯τ_{i_m}⟩_b)$ is a direct sum of components of the form $Q[a]/(a)\{s,k\}$

**Proof.** From [4], we know that $H^{b+1,*}(⟨τ_{i_1}⋯τ_{i_m}⟩_b) = 0$. So the corollary follows from Lemma 3.9. □

### 3.3. Homology of a closed braid

We are now ready to prove Theorem 1.4.

Let $B$ be a closed braid of $b$ strands. Recall that $H_{N}(B) = H(H(C_{N}(B), d_{mf}), d_{a})$. Denote by $H_{\varepsilon,i,j,k}(C_{N}(B), d_{mf})$ the homogeneous component of $H(C_{N}(B), d_{mf})$ of $\mathbb{Z}_2$-degree $\varepsilon$, homological degree $i$, $a$-degree $j$ and $x$-degree $k$. We use the $\ast$-notation as introduced in Definition 1.3. Then, for every $(\varepsilon, k) \in \mathbb{Z}_2 \oplus \mathbb{Z}$, $(H_{\varepsilon,*,*}(C_{N}(B), d_{mf}), d_{a})$ is a bounded chain complex of finitely generated $\mathbb{Z}$-graded $Q[a]$-modules. Denote by $F_{\varepsilon,*,*}(C_{N}(B), d_{mf})$ and by $T_{\varepsilon,*,*}(C_{N}(B), d_{mf})$ the torsion part of $H_{\varepsilon,*,*}(C_{N}(B), d_{mf})$. Note that $s(l)(B) = c_+ - c_- - b$, where $c_{\pm}$ is the number of $\pm$ crossings in $B$. Then, by Lemma 3.9

- $F_{\varepsilon,*,*}(C_{N}(B), d_{mf}) = 0$ if and only if $s(l)(B) = c_+ - c_- - b$, where $c_{\pm}$ is the number of $\pm$ crossings in $B$. Then, by Lemma 3.9
- $T_{\varepsilon,*,*}(C_{N}(B), d_{mf})$ is a direct sum of finitely many components of the form $Q[a]/(a)\{s\}$. Then, differential map $H_{\varepsilon,*,*}(C_{N}(B), d_{mf}) \xrightarrow{d^\varepsilon} H_{\varepsilon+1,*,*}(C_{N}(B), d_{mf})$ takes the form

\[
\begin{pmatrix}
F_{\varepsilon,*,*} & 0 \\
T_{\varepsilon,*,*} & d^\varepsilon_{\varepsilon+1,*,*}
\end{pmatrix}_{\varepsilon+1,*,*},
\]

where $d^\varepsilon_{\varepsilon+1,*,*}$ are homogeneous homomorphisms of $\mathbb{Z}$-graded $Q[a]$-modules preserving the $a$-grading. This gives rise to two chain complexes $(F_{\varepsilon,*,*}, d_{\varepsilon,FF})$ and $(T_{\varepsilon,*,*}, d_{\varepsilon,TT})$. Moreover, $(H_{\varepsilon,*,*}(C_{N}(B), d_{mf}), d_{a})$ is isomorphic to the mapping cone of the chain map $F_{\varepsilon,*,*}||1|| \xrightarrow{d_{\varepsilon,TT}} T_{\varepsilon,*,*}$, where $||*|\|$ means shifting the homological grading up by $. Thus, we get the following lemma.

**Lemma 3.11.** There is a short exact sequence

\[
0 \rightarrow T_{\varepsilon,*,*} \rightarrow C_{N}^{\varepsilon,*,*}(B) \rightarrow F_{\varepsilon,*,*} \rightarrow 0,
\]
which induces a long exact sequence

$$\ldots \rightarrow H^i(T^{\varepsilon,*,k}) \rightarrow H_{\mathbb{N}}^{\varepsilon,i,k}(B) \rightarrow H^i(F^{\varepsilon,*,k}) \xrightarrow{d_{\varepsilon,F,T}} H^{i+1}(T^{\varepsilon,*,k}) \rightarrow \ldots$$

of $\mathbb{Z}$-graded $\mathbb{Q}[a]$-modules, where the arrows preserve the $\mathbb{Q}[a]$-grading.

**Proof.** This lemma follows from the standard construction of a long exact sequence from a mapping cone. $\square$

**Lemma 3.12.** $H^i(F^{\varepsilon,*,k}) \cong H_{\mathbb{N}}^{\varepsilon,i,k}(B) \otimes_{\mathbb{Q}} \mathbb{Q}[a]\{\text{sl}(B)\}_a$ for every $(\varepsilon, i, k) \in \mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$.

**Proof.** Recall that $C_N(B) := \mathcal{C}(B)/(a-1)\mathcal{C}(B)$ and $C_N(B)$ is a free $\mathbb{Q}[a]$-module. So there is a short exact sequence

$$0 \rightarrow C_N(B) \xrightarrow{a-1} C_N(B) \rightarrow C_N(B) \rightarrow 0.$$ 

This induces a long exact sequence

$$\ldots \rightarrow H^{\varepsilon,i,k}(C_N(B), d_{mf}) \xrightarrow{a-1} H^{\varepsilon,i,k}(C_N(B), d_{mf}) \rightarrow H^{\varepsilon,i,*}(C_N(B), d_{mf}) \rightarrow H^{\varepsilon+i,*}(C_N(B), d_{mf}) \xrightarrow{a-1} \ldots$$

preserving the $x$-grading. By [5], Lemma 9.1, the multiplication by $a-1$ is an injective endomorphism of $H^{\varepsilon,i,k}(C_N(B), d_{mf})$. So this long exact sequence breaks into a short exact sequence

$$0 \rightarrow (H^{\varepsilon,i,k}(C_N(B), d_{mf}), d_{\chi}) \xrightarrow{a-1} (H^{\varepsilon,i,k}(C_N(B), d_{mf}), d_{\chi}) \rightarrow (H^{\varepsilon,i,k}(C_N(B), d_{mf}), d_{\chi}) \rightarrow 0.$$ 

This shows that the chain complexes $(H(C_N(B), d_{mf}), d_{\chi})$ and $(H(C_N(B), d_{mf}),(a-1)H(C_N(B), d_{mf}), d_{\chi})$ are isomorphic to each other, and the isomorphism preserves the $\mathbb{Z}_2$, homological and $x$-gradings.

From the decomposition $H^{\varepsilon,i,*}(C_N(B), d_{mf}) = \bigoplus_{T^{\varepsilon,i,k}}$, it is clear that

$$(H^{\varepsilon,i,k}(C_N(B), d_{mf}))/((a-1)H^{\varepsilon,i,k}(C_N(B), d_{mf}), d_{\chi}) \cong (F^{\varepsilon,*,k}/(a-1)F^{\varepsilon,*,k}, d_{\chi,FF}).$$

So there is an isomorphism of chain complexes

$$(F^{\varepsilon,*,k}/(a-1)F^{\varepsilon,*,k}, d_{\chi,FF}) \cong (H^{\varepsilon,i,k}(C_N(B), d_{mf}), d_{\chi}).$$

Recall that $d_{\chi,FF}$ preserves the $a$-grading and $F^{\varepsilon,i,k} \cong H^{\varepsilon,i,k}(C_N(B), d_{mf}) \otimes_{\mathbb{Q}} \mathbb{Q}[a]\{\text{sl}(B)\}_a$, where the shift of the $a$-grading is independent of the homological grading $i$. Now let $n_i = \dim_{\mathbb{Q}} H^{\varepsilon,i,k}(C_N(B), d_{mf})$ and fix a basis for $H^{\varepsilon,i,k}(C_N(B), d_{mf})$. This basis induces a $\mathbb{Q}$-basis for $F^{\varepsilon,i,k}$ and allows us to identify $F^{\varepsilon,i,k}$ with $\mathbb{Q}[a]^{\oplus n_i}$. Thus, $(F^{\varepsilon,*,k}, d_{\chi,FF})$ is isomorphic to the chain complex

$$C = \ldots \rightarrow \mathbb{Q}[a]^{\oplus n_i} \{\text{sl}(B)\}_a \rightarrow \mathbb{Q}[a]^{\oplus n_{i+1}} \{\text{sl}(B)\}_a \rightarrow \ldots$$

where $D_i$ is the matrix of $d_{\chi,FF}$ relative to the bases of $F^{\varepsilon,i,k}$ and $F^{\varepsilon,i+1,k}$. Since $d_{\chi,FF}$ preserves the $a$-grading, all entries of $D_i$ are elements of $\mathbb{Q}$. Consider the chain complex

$$\mathcal{C} = \ldots \rightarrow \mathbb{Q}[a]^{\oplus n_i} \mathcal{D}_i \rightarrow \mathbb{Q}[a]^{\oplus n_{i+1}} \mathcal{D}_{i+1} \rightarrow \ldots$$

One can see that $(F^{\varepsilon,*,k}, d_{\chi,FF}) \cong C \cong \mathcal{C} \otimes_{\mathbb{Q}} \mathbb{Q}[a]\{\text{sl}(B)\}_a$ and, by isomorphism (3.1), $\mathcal{C} \cong C/(a-1)C \cong (H^{\varepsilon,i,k}(C_N(B), d_{mf}), d_{\chi})$. Combining these, we get

$$(F^{\varepsilon,*,k}, d_{\chi,FF}) \cong (H^{\varepsilon,i,k}(C_N(B), d_{mf}) \otimes_{\mathbb{Q}} \mathbb{Q}[a]\{\text{sl}(B)\}_a, d_{\chi}).$$

This implies that $H^i(F^{\varepsilon,*,k}) \cong H_{\mathbb{N}}^{\varepsilon,i,k}(B) \otimes_{\mathbb{Q}} \mathbb{Q}[a]\{\text{sl}(B)\}_a$. $\square$

**Proof of Theorem 4.4.** From Lemmas 3.11 and 3.12 we get a long exact sequence

$$\ldots \rightarrow H^i(T^{\varepsilon,*,k}) \rightarrow H_{\mathbb{N}}^{\varepsilon,i,k}(B) \rightarrow H^i(F^{\varepsilon,*,k}) \xrightarrow{d_{\varepsilon,F,T}} H^{i+1}(T^{\varepsilon,*,k}) \rightarrow \ldots$$

Denote by $F\mathcal{H}_{\mathbb{N}}^{\varepsilon,i,k}(B)$ the free part of the $\mathbb{Z}$-graded $\mathbb{Q}[a]$-module $\mathcal{H}_{\mathbb{N}}^{\varepsilon,i,k}(B)$ and by $T\mathcal{H}_{\mathbb{N}}^{\varepsilon,i,k}(B)$ the torsion part of $\mathcal{H}_{\mathbb{N}}^{\varepsilon,i,k}(B)$. Then the long exact sequence (3.2) splits into two exact sequences:

$$\ldots \rightarrow H^i(T^{\varepsilon,*,k}) \rightarrow T\mathcal{H}_{\mathbb{N}}^{\varepsilon,i,k}(B) \rightarrow 0,$$

$$0 \rightarrow F\mathcal{H}_{\mathbb{N}}^{\varepsilon,i,k}(B) \xrightarrow{J} H_{\mathbb{N}}^{\varepsilon,i,k}(B) \otimes_{\mathbb{Q}} \mathbb{Q}[a]\{\text{sl}(B)\}_a \xrightarrow{d_{\varepsilon,F,T}} H^{i+1}(T^{\varepsilon,*,k}) \rightarrow \ldots$$

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Since $T^{ε,i,*}k$ is a direct sum of finitely many components of the form $Q[a]/(a)\{s\}_a$, so is $H^i(T^{ε,*}k)$. From the exact sequence (3.3), one can see that $TH^{ε,i,*}k(B)$ is a quotient module of $H^i(T^{ε,*}k)$. Thus, $TH^{ε,i,*}k(B)$ is also a direct sum of finitely many components of the form $Q[a]/(a)\{s\}_a$. That is,

$$\text{(3.5)} \quad TH^{ε,i,*}k(N)_a \cong \bigoplus_{q=1}^{n} Q[a]/(a)\{s_q\}_a,$$

for some finite sequence $\{s_1, \ldots, s_n\}$ of integers.

Next we prove that the $Q$-linear map

$$FH^{ε,i,*}k(N)/a \to H^{ε,i,k}(B) \otimes Q[a]/(a)\{s\}_a$$

is an isomorphism. First, note that $H^{i+1}(T^{ε,*}k)$ is a direct sum of components of the form $Q[a]/(a)\{s\}_a$. So any multiple of $a$ in $H^{ε,i,k}(B) \otimes Q[a]\{s\}_a$ is in $\ker d_{N,e}^{FT} = \text{Im} f$. For any $u \in FH^{ε,i,*}k(B)$ such that $f(u) = (a - 1)v$ for some $v \in H^{ε,i,k}(B) \otimes Q[a]\{s\}_a$, there exist an $u' \in FH^{ε,i,*}k(B)$ satisfying $f(u') = av$. Thus,

$$f(-(a - 1)(u - u')) = -(a - 1)(f(u) - f(u')) = (a - 1)v = f(u).$$

But $FH^{ε,i,*}k(B) \to H^{ε,i,k}(B) \otimes Q[a]\{s\}_a$ is injective. So $u = -(a - 1)(u - u')$. This shows that the above $Q$-linear map is injective. Second, for every $v \in H^{ε,i,k}(B) \otimes Q[a]\{s\}_a$, there is a $u \in FH^{ε,i,*}k(B)$ such that $f(u) = av$. So $v = f(u) - (a - 1)v$. This shows that the above $Q$-linear map is surjective. Thus, it is an isomorphism.

The above $Q$-linear isomorphism implies that the rank of the $Z$-graded free $Q[a]$-module $FH^{ε,i,*}k(B)$ is equal to $\dim Q H^{ε,i,k}(B)$. Hence, by Lemma 3.7,

$$\text{(3.6)} \quad FH^{ε,i,*}k(N)_a \cong \bigoplus_{p=1}^{\dim Q H^{ε,i,k}(B)} Q[a]\{t_p\}_a.$$

From [1], we know that $H^{s(B)}_{a+1,i,k}(B) \cong 0$ for any $i, k$. So, for any $i, k$,

$$\text{(3.7)} \quad FH^{s(B)-1,i,*}k(N)_a \cong 0.$$

From the construction of $H(N)$, one can see that, when $\varepsilon = s(B)$, the parity of $t_p$ in (3.6) must be the same as that of $s(B)$. Since $FH^{s(B)}_{N,i,*}k(B) \to H^{s(B)}_{N,i,k}(B) \otimes Q[a]/(a)\{s\}_a$ is injective and preserves the $a$-grading, we know that $t_p \geq s(B)$ if $\varepsilon = s(B)$. Assume that $FH^{s(B)}_{N,i,*}k(B)$ contains a component $Q[a]\{t_p\}_a$ such that $t_p \geq s(B) + 4$. Denote by $1_p$ the 1 in $Q[a]\{t_p\}_a$. Then $f(1_p) = a^2v$ for some $v \in H^{s(B)}_{N,i,k}(B) \otimes Q[a]/(a)\{s\}_a$. Consider the exact sequence (3.4). Again, since $H^{i+1}(T^{s(B)},*)$ is a direct sum of finitely many components of the form $Q[a]/(a)\{s\}_a$, one can see that $av \in \ker d_{N,e}^{FT} = \text{Im} f$. So there exists a $u \in FH^{s(B)}_{N,i,*}k(B)$ such that $f(u) = av$. Therefore, $f(1_p) = f(au)$. But $f$ is injective. This means $1_p = au$, which is a contradiction. Thus, when $\varepsilon = s(B)$, we have $t_p = s(B)$ or $s(B) + 2$ for every $p$ and

$$\text{(3.8)} \quad FH^{s(B)-1,i,*}k(N)_a \cong (Q[a]/(a)\{s\}_a)^{\oplus l} \oplus (Q[a]/(a)\{s\}_a + 2)^{\oplus (\dim Q H^{s(B)}_{N,i,k}(B) - l)}$$

for some non-negative integer $l$.

By decompositions (3.5), (3.7) and (3.8), one can see that $H^{ε,i,*}k(N)_a$ admits a decomposition of the form given in Theorem 3.4. The uniqueness of this decomposition follows from Lemma 5.1. The only things left to prove are the bounds for $s_q$. In the remainder of this proof, we show that the bound for $s_q$ in Theorem 1.4 follows from the corresponding bounds in Lemma 5.1.

If we choose a resolution as in Figure 5 for each crossing of $B$, we get a closed resolved braid. We call such a closed resolved braid a resolution of $B$ and denote by $R(B)$ the set of all resolutions of $B$. As suggested in Figure 5, we call the resolution $C_0 \sim \Gamma_0 \sim 0$-resolution and $C_0 \sim \Gamma_1 \sim 1$ a $\pm 1$-resolution. For $\mathbf{w} \in R(B)$, assume it contains $m_{\mathbf{w},+} + m_{\mathbf{w},-}$ 2-colored edges, where $m_{\mathbf{w},\pm}$ is the number of 2-colored edges in $\mathbf{w}$ coming
from ±1-resolutions. From the construction of $C_N(B)$, especially local chain complexes \([2.7]\) and \([2.8]\), one can see that
\[
(3.9) \quad C_N(B) = \bigoplus_{\pi \in R(B)} C_N(\pi) \langle w \rangle \{ w, (N-1)w + m_{\pi,+} - m_{\pi,-} \} \| m_{\pi,-} - m_{\pi,+} \},
\]
where $w = c_+ - c_-$ is the writhe of $B$ and “$\| \|$” means shifting the homological grading by *. From Lemma \([3.9]\) we know that, if $H^{ε,ε,k}(N-1)w - \bar{m}_{\pi,+} + \bar{m}_{\pi,-}(\pi)\{w\}a$ contains a torsion component $\mathbb{Q}[a]/(a)\{s\}a$, then $w - b \leq s \leq w - 1$ and $(N-1)s \leq k - 2N + 2m_{\pi,-}$. Note that $w - b = s(B)$ and $m_{\pi,-} \leq c_-$. So, by decomposition \([3.9]\), we have that, if $T^{ε,ε,k}$ contains a component $\mathbb{Q}[a]/(a)\{s\}a$, then
\[
(3.10) \quad s(B) \leq s \leq w - 1 \text{ and } (N-1)s \leq k - 2N + 2c_-.
\]
Therefore, if $H^i(T^{ε,ε,k})$ contains a component $\mathbb{Q}[a]/(a)\{s\}a$, then $s$ satisfies the two bounds in \([3.10]\). Finally, by the exact sequence \([3.9]\), $T\mathcal{H}_N^{ε,ε,k}(B)$ is a quotient module of $H^i(T^{ε,ε,k})$. So, if $\mathcal{H}_N^{ε,ε,k}(B)$ contains a component $\mathbb{Q}[a]/(a)\{s\}a$, then $s$ satisfies the two bounds in \([3.10]\). This completes the proof of Theorem \([1.3]\). \(\square\)

4. Stabilization

In this section, we study how $\mathcal{H}_N$ changes under stabilization. The goal is to prove Theorem \([1.6]\).

4.1. Mapping cones. We now review some basic properties of mapping cones.

**Definition 4.1.** Let $A, B$ be two chain complexes of $\mathbb{Z}$-graded $\mathbb{Q}[a]$-modules and $f : A \rightarrow B$ a chain map preserving both the homological grading and the $a$-grading. Then the mapping cone $cone(f)$ is defined to be the chain complex given by:

- $cone^i(f) = \frac{A^i}{B^{i-1}}$, where $B^i = \bigoplus_{B^{i-1}}$
- the differential $cone^i(f) \xrightarrow{d} cone^{i+1}(f)$ is the map $\frac{A^i}{B^{i-1}} \xrightarrow{\begin{pmatrix} d_A & 0 \\ f & d_B \end{pmatrix}} \frac{A^{i+1}}{B^i}$, where $d_A$ and $d_B$ are the differential maps of $A$ and $B$.

**Lemma 4.2.** Suppose that $0 \rightarrow A \xrightarrow{\theta} B \xrightarrow{\varphi} C \rightarrow 0$ is a short exact sequence of chain complexes of $\mathbb{Z}$-graded $\mathbb{Q}[a]$-modules, where $f$ and $g$ preserve both the homological grading and the $a$-grading. Then, as $\mathbb{Z}$-graded $\mathbb{Q}[a]$-modules, $H^i(cone(f)) \cong H^{i-1}(C)$ and $H^i(cone(g)) \cong H^i(A)$.

**Proof.** Denote by $id_A$ the identity map from $A$ to itself. Define $\alpha : cone(id_A) \rightarrow cone(f)$ by $\frac{A^i}{A^{i-1}} \xrightarrow{\begin{pmatrix} id_A & 0 \\ 0 & f \end{pmatrix}} \frac{A^i}{A^{i-1}}$, and $\beta : cone(f) \rightarrow C \| 1 \|$ by $\frac{A^i}{B^{i-1}} \xrightarrow{(0, g)} C^{i-1}$. Then $\alpha, \beta$ are chain maps and

$0 \rightarrow cone(id_A) \xrightarrow{\alpha} cone(f) \xrightarrow{\beta} C \| 1 \| \rightarrow 0$

is a short exact sequence. It induces a long exact sequence

$\cdots \rightarrow H^i(cone(id_A)) \rightarrow H^i(cone(f)) \rightarrow H^{i-1}(C) \rightarrow H^{i+1}(cone(id_A)) \rightarrow \cdots$

Since $H(cone(id_A)) \cong 0$. This long exact sequence implies that $H^i(cone(f)) \cong H^{i-1}(C)$.\[17\]
Now define $\phi : A \to \text{cone}(g)$ by $A^i \left( \begin{array}{c} f \\ 0 \end{array} \right) \oplus B^i_{C^i-1}$ and $\psi : \text{cone}(g) \to \text{cone}(\text{id}_C)$ by $B^i_{C^i-1} \left( \begin{array}{c} g \\ 0 \end{array} \right) \oplus \text{id}_C$. Then $\phi, \psi$ are chain maps and

$$0 \to A \overset{\phi}{\to} \text{cone}(g) \overset{\psi}{\to} \text{cone}((\text{id}_C) \to 0$$

is a short exact sequence. It induces a long exact sequence

$$\cdots \to H^{i-1}(\text{cone}(\text{id}_C)) \to H^i(A) \to H^i(\text{cone}(g)) \to H^i(\text{cone}(\text{id}_C)) \to \cdots$$

Since $H(\text{cone}(\text{id}_C)) \cong 0$, this long exact sequence implies that $H^i(\text{cone}(g)) \cong H^i(A)$. □

**Lemma 4.43.** Suppose that $0 \to A \overset{f}{\to} B \overset{g}{\to} C \overset{h}{\to} D \to 0$ is an exact sequence of chain complexes of $\mathbb{Z}$-graded $\mathbb{Q}[a]$-modules, where $f, g$ and $h$ preserve both the homological grading and the $a$-grading. Then there is a long exact sequence of $\mathbb{Z}$-graded $\mathbb{Q}[a]$-modules

$$\cdots \to H^i(A) \to H^i(\text{cone}(g)) \to H^i(D) \to H^{i+1}(A) \to \cdots$$

**Proof.** Denote by $\pi : B \to B/f(A)$ the standard quotient map. Define $\alpha : \text{cone}(\pi) \to \text{cone}(g)$ by

$$B^i \oplus B^i/f(A^{i-1}) \to B^i \oplus C^i-1$$

which is well defined since $\text{ker} g = \text{Im} f$. Also, define $\beta : \text{cone}(g) \to D\|1\|$ by

$$D\|1\| \to B^i \oplus C^i-1 \to D^i-1.$$ Then $\alpha, \beta$ are chain maps and

$$0 \to \text{cone}(\pi) \overset{\alpha}{\to} \text{cone}(g) \overset{\beta}{\to} D\|1\| \to 0$$

is a short exact sequence. It induces a long exact sequence

$$\cdots \to H^i(\text{cone}(\pi)) \to H^i(\text{cone}(g)) \to H^i(D) \to H^{i+1}(\text{cone}(\pi)) \to \cdots$$

But $0 \to A \overset{f}{\to} B \overset{g}{\to} B/f(A) \to 0$ is a short exact sequence of complexes. So, by Lemma 4.2, we know that $H^i(\text{cone}(\pi)) \cong H^i(A)$. Thus, we have a long exact sequence

$$\cdots \to H^i(A) \to H^i(\text{cone}(g)) \to H^i(D) \to H^{i+1}(A) \to \cdots$$

□

4.2. **Stabilization and $\mathcal{H}_N$.** Next, we prove Theorem 1.6

**Proof of Theorem 1.6.** Let $B$ be a closed braid. Set $\mathcal{C}_N(B) = C_N(B)/a\mathcal{C}_N(B)$. Recall that $\pi_0$ is the standard quotient map $C_N(B) \overset{\pi_0}{\to} C_N(B)/a\mathcal{C}_N(B) = \mathcal{C}_N(B)$. Then there is a short exact sequence

$$0 \to C_N(B) \overset{\pi_0}{\to} \mathcal{C}_N(B) \{-2, 0\} \overset{\pi_0}{\to} \mathcal{C}_N(B) \{-2, 0\} \to 0.$$ Note that $d_{mf}$ is homogeneous with $\mathbb{Z}_2$-degree 1, homological degree 0, $a$-degree 1 and $x$-degree $N + 1$. Set $s = sl(B)$. Taking the homology with respect to $d_{mf}$, the above short exact sequence gives the following
long exact sequence.

\[
\cdots \rightarrow H^{s-1,i,*+N-1}(C(B), d_m)\{1\}_a \xrightarrow{\pi_0} H^{s-1,i,*+N-1}(\mathcal{E}(B), d_m)\{-1\}_a \xrightarrow{\pi_0} H^{s-1,i,*+N-1}(\mathcal{E}(B), d_m)\{-1\}_a \rightarrow \cdots
\]

Following the notations in Subsection 5.3, we denote by \(F^{*,i,*+k}\) the free part of \(H^{*,i,*+k}(C(B), d_m)\) and by \(T^{*,i,*+k}\) the torsion part of \(H^{*,i,*+k}(C(B), d_m)\). By Corollary 3.10 and the normalization of the local chain complexes (2.7) and (2.8), we know that \(F^{s-1,i,*+k}\) and \(T^{*,i,*+k}\) are direct sums of components of the form \(\mathbb{Q}[a]/(a)\{s\}_a\). So the above long exact sequence breaks into two exact sequences:

\[
\begin{align*}
\text{(4.1)} & \quad 0 \rightarrow H^{s-1,i,*+N-1}(C(B), d_m)\{-1\}_a \xrightarrow{\pi_0} H^{s-1,i,*+N-1}(\mathcal{E}(B), d_m)\{-1\}_a \rightarrow T^{*,i,*+k} \rightarrow 0 \\
\text{(4.2)} & \quad 0 \rightarrow F^{s-1,i,*+k}(C(B), d_m)\{-2\}_a \xrightarrow{\pi_0} H^{s-1,i,*+k}(\mathcal{E}(B), d_m)\{-2\}_a \rightarrow H^{s-1,i,*+N+1}(C(B), d_m)\{-1\}_a \rightarrow 0.
\end{align*}
\]

Applying Lemma 1.2 to the exact sequence (4.1), we get that

\[
H^{s-1,i,*+k}(cone(H(C(B), d_m) \xrightarrow{\pi_0} H(\mathcal{E}(B), d_m)), d_h)\{-1\}_a \cong H^{i-1}(T^{*,i,*+N+1}, d_h).
\]

By [9] Theorem 1.5, we have

\[
H^{s-1,i,*+k}(B_-) \cong H^{s-1,i,*+k}(cone(H(C(B), d_m) \xrightarrow{\pi_0} H(\mathcal{E}(B), d_m)), d_h)\{-2\}_a.
\]

So

\[
H^{s-1,i,*+k}(B_-) \cong H^{i-1}(T^{*,i,*+N+1}, d_h)\{-1\}_a.
\]

By Lemmas 3.11 and 3.12, there is a long exact sequence

\[
\cdots \rightarrow H^i(T^{*,i,*+k}) \rightarrow H_C^{*,i,*+k}(B) \rightarrow H_C^{*,i,*+k}(B) \otimes \mathbb{Q}[a]\{s\}_a \rightarrow H^{i+1}(T^{*,i,*+k}) \rightarrow \cdots
\]

Thus, we have a long exact sequence

\[
\cdots \rightarrow H^{s-1,i,*+k}(B_-) \rightarrow H^{s-1,i,*+k+1}(B)\{-1\}_a \rightarrow H^{s-1,i-1,k+1}(B)\otimes \mathbb{Q}[a]\{s-1\}_a \rightarrow H^{s-1,i+1,*+k}(B_-) \rightarrow \cdots
\]

This establishes the long exact sequence (4.1).

Now apply Lemma 1.2 to the exact sequence (4.2). Using also the fact that

\[
H^{s-1,i,*+k}(B_-) \cong H^{s-1,i,*+k}(cone(H(C(B), d_m) \xrightarrow{\pi_0} H(\mathcal{E}(B), d_m)), d_h)\{-2\}_a,
\]

we get a long exact sequence

\[
\cdots \rightarrow H^i(F^{*,i,*+k}) \rightarrow H^i(F^{*,i,*+k})(B_-) \rightarrow H_c^{s-1,i-1,k+1}(B)\{-1\}_a \rightarrow H^{i+1}(F^{*,i,*+k}) \rightarrow \cdots
\]

By Lemma 3.12 \(H^i(F^{*,i,*+k}) \cong H^{s-1,i,*+k}(B) \otimes \mathbb{Q}[a]\{s\}_a\), which is a free \(\mathbb{Q}[a]\)-module. From [4], we know that \(H^{s-1,i,*+k}(B) \cong 0\). So, by Theorem 1.4 \(H^{s-1,i-1,k+1}(B)\) is a torsion \(\mathbb{Q}[a]\)-module. Thus, the above long exact sequence breaks into the following short exact sequence:

\[
0 \rightarrow H^{s-1,i,*+k}(B) \otimes \mathbb{Q}[a]\{s\}_a \rightarrow H^{s-1,i,*+k}(B_-) \rightarrow H^{s-1,i-1,k+1}(B)\{-1\}_a \rightarrow 0.
\]

This establishes the short exact sequence (1.2).
4.3. Transverse unknots. We are now ready to prove Corollary 4.7. We start by a simple algebraic observation.

Lemma 4.4. Let \( \mathcal{F} = \bigoplus_{l=0}^{N-1} \mathbb{Q}[a] \langle 1 \rangle \{ -1, -N + 1 + 2l \} \) be as defined in Lemma 4.7.

(1) Assume \( f: \mathcal{F} \to \mathcal{F} \) is an injective homogeneous homomorphism of \( a \)-degree 2 and preserving other gradings. Then \( \text{coker} f \cong \mathcal{F}/a\mathcal{F} \).

(2) Assume \( g: \mathcal{F} \to \mathcal{F} \) is an injective homogeneous homomorphism preserving all gradings. Then \( g \) is an isomorphism.

Proof. The proofs for the two parts are very similar. We only include here the proof for Part (1) and leave Part (2) for the reader.

Denote by \( 1_t \) the “1” in \( \mathbb{Q}[a] \langle 1 \rangle \{ -1, -N + 1 + 2l \} \). Then, since \( f \) is an injective homogeneous homomorphism of \( a \)-degree 2 and preserves the \( x \)-grading, we know that \( f(1_t) = \lambda_t a 1_t \) for some \( \lambda_t \in \mathbb{Q} \setminus \{0\} \). The lemma follows from this. \( \square \)

Proof of Corollary 4.7. Setting \( b = 1 \) in Lemma 6.4, we get that \( \mathcal{H}_N(U_0) \cong \mathcal{F} \oplus \mathcal{T} \).

For \( m = 1 \), the exact sequences in Theorem 4.6 are non-vanishing at only two locations:

(4.3) \( 0 \to \mathcal{H}_N^{0,1,*}(U_1) \to \mathcal{H}_N^{1,0,*}(U_0) \{ -1, -N - 1 \} \to \mathcal{H}_N^{1,0,*}(U_1) \to 0 \),

(4.4) \( 0 \to \mathcal{H}_N^{1,0,*}(U_0) \otimes \mathbb{Q}[a] \{ -1 \} a \to \mathcal{H}_N^{1,0,*}(U_1) \to 0 \).

Recall that, from [1], we know that \( \mathcal{H}_N(U_m) \cong \mathcal{H}_N(U_0) \cong \bigoplus_{l=0}^{N-1} \mathbb{Q}[\langle 1 \rangle \{ -N + 1 + 2l \} \). So

(4.5) \( \mathcal{H}_N(U_m) \otimes \mathbb{Q}[a] \cong \mathcal{H}_N(U_0) \otimes \mathbb{Q}[a] \cong \bigoplus_{l=0}^{N-1} \mathbb{Q}[\langle 1 \rangle \{ 0, -N + 1 + 2l \} \cong \mathcal{F} \{ 1 \} a \).

Also, by Remark 1.5, \( \mathcal{H}_N^{0,1,*}(U_1) \) is a torsion \( \mathbb{Q}[a] \)-module. So exact sequence (4.3) breaks into

(4.6) \( 0 \to \mathcal{H}_N^{0,1,*}(U_1) \to \mathcal{T} \{ -1, -N - 1 \} \to 0 \),

(4.7) \( 0 \to \mathcal{F} \{ -1, -N - 1 \} \to \mathcal{F} \{ -1, -N - 1 \} \to \mathcal{H}_N^{0,2,*}(U_1) \to 0 \).

Thus, we have \( \mathcal{H}_N^{0,1,*}(U_1) \cong \mathcal{T} \{ -1, -N - 1 \} \) and, by Part (2) of Lemma 4.4, \( \mathcal{H}_N^{0,2,*}(U_1) \cong 0 \). Also, using exact sequence (4.3), we have \( \mathcal{H}_N^{1,0,*}(U_1) \cong \mathcal{F} \). Putting everything together, we have \( \mathcal{H}_N(U_1) \cong \mathcal{F} \oplus \mathcal{T} \{ 1 \} \{ -1, -N - 1 \} \).

Next, assume the corollary is true for \( U_m \) for some \( m \geq 1 \). We prove that the corollary is true for \( U_{m+1} \).

By (4.5), \( \mathcal{H}_N^{\varepsilon,i,*}(U_m) \otimes \mathbb{Q}[a] \cong \begin{cases} \mathcal{F} \{ 1 \} a & \text{if } \varepsilon = 1 \text{ and } i = 0, \\ 0 & \text{otherwise.} \end{cases} \)

So the exact sequences in Theorem 4.6 break into

(4.8) \( 0 \to \mathcal{F} \{ -2m \} a \to \mathcal{H}_N^{1,0,*}(U_{m+1}) \to 0 \),

(4.9) \( 0 \to \mathcal{H}_N^{0,1,*}(U_{m+1}) \to \mathcal{F} \{ -2m + 1, -N - 1 \} \to \mathcal{F} \{ -2m + 1, -N + 1 \} \to \mathcal{H}_N^{0,2,*}(U_{m+1}) \to 0 \),

(4.10) \( 0 \to \mathcal{H}_N^{1,1,2,*}(U_{m+1}) \to \mathcal{F} / a \mathcal{F} \{ -2m + l - 1, -(l + 1)(N + 1) \} \to 0 \),

(4.11) \( 0 \to \mathcal{H}_N^{m-1,m+1,*}(U_{m+1}) \to \mathcal{T} \{ -m - 1, -(m + 1)(N + 1) \} \to 0 \).

Exactness of (4.8) gives us

\( \mathcal{H}_N^{1,0,*}(U_{m+1}) \cong \mathcal{F} \{ -2m \} a. \)

Exactness of (4.10) and (4.11) give us

\( \mathcal{H}_N^{1,1,2,*}(U_{m+1}) \cong \mathcal{F} / a \mathcal{F} \{ -2m + l - 1, -(l + 1)(N + 1) \} \),

\( \mathcal{H}_N^{m-1,m+1,*}(U_{m+1}) \cong \mathcal{T} \{ -m - 1, -(m + 1)(N + 1) \}. \)

Finally, we look at exact sequence (4.9). By Remark 1.5, \( \mathcal{H}_N^{0,1,*}(U_{m+1}) \) is a torsion \( \mathbb{Q}[a] \)-module. This implies that \( \mathcal{H}_N^{0,1,*}(U_{m+1}) \cong 0 \) and we have a short exact sequence

\( 0 \to \mathcal{F} \{ -2m + 1, -N - 1 \} \to \mathcal{F} \{ -2m + 1, -N - 1 \} \to \mathcal{H}_N^{0,2,*}(U_{m+1}) \to 0. \)
Applying Part (1) of Lemma [4.4] to the above short exact sequence, we get
\[ H^{0,2,\ast,\ast}_N(U_{m+1}) \cong \mathcal{F}/a\mathcal{F}\{-2m - 1, -N - 1\}. \]

Now putting everything together, we have that
\[ H_N(U_{m+1}) \cong \mathcal{F}\{-2((m + 1) - 1), 0\} \oplus T\langle m + 1 \rangle\{-(m + 1), -(m + 1)(N + 1)\}\|m + 1\|
\oplus \bigoplus_{l=1}^{(m+1)-1} \mathcal{F}/a\mathcal{F}\langle l \rangle\{-2(m + 1) + l, -l(N + 1)\}\|l + 1\|. \]

This shows that the corollary is true for \( U_{m+1} \) too. \( \square \)

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