The Kähler-Ricci flow and K-polystability

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THE KÄHLER-RICCI FLOW AND K-POLYSTABILITY

By GÁBOR SZÉKELYHIDI

Abstract. We consider the Kähler-Ricci flow on a Fano manifold. We show that if the curvature remains uniformly bounded along the flow, the Mabuchi energy is bounded below, and the manifold is K-polystable, then the manifold admits a Kähler-Einstein metric. The main ingredient is a result that says that a sufficiently small perturbation of a cscK manifold admits a cscK metric if it is K-polystable.

1. Introduction. The question of existence of Kähler-Einstein metrics on Fano manifolds is a fundamental problem in Kähler geometry. The case of Kähler manifolds with zero or negative first Chern class was solved by Yau [23] and Aubin [1] showing that every such manifold admits a Kähler-Einstein metric. In the case of positive first Chern class the presence of holomorphic vector fields led to the first obstructions (Matsushima [14] and Futaki [11]). The problem can also be generalised to manifolds $M$ with an ample line bundle $L$, where we can ask whether there exists a constant scalar curvature (cscK) metric in the class $c_1(L)$. The Futaki invariant generalises to this context (see Calabi [2]) and since it enters later, we take this moment to define it. Let us choose a Kähler metric $\omega$ on $M$ in the class $c_1(L)$, and write $S(\omega)$ for the scalar curvature. Let the function $h$ satisfy

$$S(\omega) - \hat{S} = \Delta_\omega h,$$

where $\hat{S}$ is the average of $S(\omega)$ with respect to the volume form $\omega^n$. Given a holomorphic vector field $v$ on $M$ the Futaki invariant is defined to be

$$F(v) = \int_M v(h)\omega^n.$$

The main point is that $F(v)$ is independent of the metric we chose in $c_1(L)$, and if there exists a cscK metric in $c_1(L)$ then clearly $F(v) = 0$ for all holomorphic vector fields $v$.

Later, inspired by a conjecture due to Yau [24], a more subtle obstruction called K-polystability has been found which is related to stability in the sense
of geometric invariant theory. There are different notions of K-polystability, the original one in the context of Kähler-Einstein manifolds due to Tian [20], and a more algebraic one which applies to the more general problem of cscK metrics due to Donaldson [8]. The main ingredient in the definition is the notion of a test-configuration for a polarized variety \((M, L)\). This is a \(G^*\)-equivariant flat family \((\mathcal{M}, \mathcal{L})\) over \(\mathbb{C}\), such that the generic fibre is isomorphic to \((M, rL)\) for some \(r > 0\).

The central fibre \((M_0, L_0)\), which in Donaldson’s definition might be an arbitrary scheme, inherits a \(G^*\)-action. This allows one to define the Donaldson-Futaki invariant of the test-configuration, which coincides with the Futaki invariant of the vector field generating the \(G^*\)-action when \(M_0\) is smooth. We will only be interested in the case when the central fibre is smooth, so we will not give the detailed definition of the Donaldson-Futaki invariant. The polarized manifold \((M, L)\) is called K-polystable if the Donaldson-Futaki invariant is nonnegative for all test-configurations and is zero only for product configurations. A product configuration is a test-configuration where the central fibre is isomorphic to the general fibre.

The main conjecture is the following:

\textbf{Conjecture 1.} (Yau-Tian-Donaldson) A polarized manifold \((M, L)\) admits a cscK metric in \(c_1(L)\) if and only if it is K-polystable.

In one direction Donaldson has shown that if \((M, L)\) admits a cscK metric, then it is K-semistable. This is a weakening of K-polystability where the Donaldson-Futaki invariant is allowed to be zero as well. This was improved by Stoppa [19] who shows that existence of a cscK metric implies K-polystability (using Donaldson’s definition) when the manifold has no holomorphic vector fields. In the converse direction little progress has been made, except in the case of toric surfaces, where Donaldson [4] has proved Conjecture 1. In the Kähler-Einstein case Tian proved that the existence of a Kähler-Einstein metric implies his version of K-stability, where the central fibre of test-configurations is only allowed to have normal singularities. In the converse direction he showed that properness of a certain energy functional (the Mabuchi functional) implies the existence of a Kähler-Einstein metric, but it is still to be seen whether this can be related to an algebraic condition. In the case of toric Fano manifolds Conjecture 1 is also known thanks to the work of Wang-Zhu [22]. Our first result is the following.

\textbf{Theorem 2.} Suppose \((M, L)\) is cscK, and let \((M', L')\) be a sufficiently small deformation of the complex structures of \(M\) and \(L\). If \((M', L')\) is K-polystable then it admits a cscK metric.

It will be clear from the proof that we only need a weak version of K-polystability where we only consider test-configurations with smooth central fibres which are themselves small deformations of \((M, L)\). We have been informed that T. Brönnle has obtained some similar results on perturbing cscK metrics for
his Ph.D. thesis. When we are dealing with Fano varieties polarized by the anti-canonical bundle, then we do not need to keep track of the polarization since it is canonically defined. In particular the proof of the theorem shows that if $M$ is a Fano Kähler-Einstein manifold, and $M'$ is a sufficiently small deformation of $M$, then either $M'$ admits a Kähler-Einstein metric, or there is a test-configuration for $M'$ with smooth central fibre $M''$. Moreover $M''$ admits a Kähler-Einstein metric and it is itself a small deformation of $M$ (or it can also be equal to $M$).

We apply this result to the Kähler-Ricci flow on a Fano manifold. It was shown by Cao [3] that in the case of negative or zero first Chern class the flow converges to the Kähler-Einstein metric which is guaranteed to exist by Yau and Aubin’s theorems. In the Fano case Cao showed that the flow exists for all time, and the main problem is to find conditions under which it converges. In view of the Yau-Tian-Donaldson conjecture one would like to show that under the assumption of K-polystability the flow converges to a Kähler-Einstein metric. Without additional assumptions this seems out of reach at present. One interesting question is what we can say if we assume that the Riemannian curvature is uniformly bounded along the flow. The main result about the Kähler-Ricci flow that we use is the following, which is based on Perelman’s work [15].

**Theorem 3.** (see [17], [18]) Suppose that the Riemann curvature tensor is uniformly bounded along the Kähler-Ricci flow on a Fano manifold $M$. Write $J$ for the complex structure of $M$. We can then find a sequence of diffeomorphisms $\phi_k: M \to M$ such that $\phi_k^*(J)$ converges in $C^\infty$ to a smooth complex structure $J_0$, such that $(M, J_0)$ admits a Kähler-Ricci soliton. If in addition the Mabuchi functional of $M$ is bounded below, then $(M, J_0)$ admits a Kähler-Einstein metric.

The question is then to find conditions which ensure that the complex structure $J_0$ is isomorphic to $J$. Phong and Sturm [18] have introduced such a condition called Condition B. Let us briefly recall that a complex manifold $(M, J)$ satisfies Condition B if we cannot find a sequence of diffeomorphisms $\phi_k$ with $\phi_k^*(J)$ converging in $C^\infty$ to a complex structure $J_0$ which has a strictly higher dimensional space of holomorphic vector fields than $J$. The result in [18] was improved in [17] and it is the following:

**Theorem 4.** (Phong-Song-Sturm-Weinkove) Suppose that the Riemann curvature tensor is uniformly bounded along the Kähler-Ricci flow on a Fano manifold $M$. If the Futaki invariant of $M$ vanishes, and $M$ satisfies Condition B, then the flow converges exponentially fast in $C^\infty$ to a Kähler-Einstein metric.

In this direction we will prove the following related result:

**Theorem 5.** If the Riemann curvature tensor is uniformly bounded along the Kähler-Ricci flow on a Fano manifold $M$, and $M$ satisfies Condition B, then $M$ admits a Kähler-Ricci soliton.
The proof of this does not need Theorem 2. Note that if the Futaki invariant of $M$ with respect to the class $c_1(M)$ vanishes, then a Kähler-Ricci soliton on $M$ is necessarily a Kähler-Einstein metric. Our main result is that under the additional assumption that the Mabuchi functional is bounded from below, we can replace Condition B with K-polystability.

**THEOREM 6.** Suppose that the Riemann curvature tensor is uniformly bounded along the Kähler-Ricci flow on a Fano manifold $M$. Suppose in addition that the Mabuchi functional on $M$ is bounded from below and that $M$ is K-polystable. Then $M$ admits a Kähler-Einstein metric.

In the Fano case when saying that $M$ is K-polystable we mean that the pair $(M, -K_M)$ is K-polystable. Note that once the existence of a Kähler-Einstein metric is established, convergence of the flow follows from the work of Tian and Zhu [21]. Combined with the result of Phong-Song-Sturm-Weinkove [16] we obtain exponential convergence. We hope that the assumption that the Mabuchi functional is bounded from below can be removed in the future. Note that for toric varieties Donaldson [8] has shown that K-polystability implies that the Mabuchi functional is bounded from below.

We will see that instead of $M$ being K-polystable it is enough to require that there are no nonproduct test-configurations for $M$ with smooth central fibre which have zero Futaki invariant. The lower bound for the Mabuchi functional should be thought of as a semistability condition, which is strengthened to stability by excluding test-configurations with zero Futaki invariant. Bounding the curvature along the flow allows us to consider test-configurations with smooth central fibres. Also, the main difficulty in replacing Condition B with K-polystability is to obtain instead of just a sequence of diffeomorphisms $\phi_k$ with $\phi_k^* (J) \rightarrow J_0$, a test-configuration with central fibre $J_0$.

In the next section we give the proof of Theorem 5, and assuming Theorem 2 the proof of Theorem 6. The rest of the paper will then be devoted to the proof of Theorem 2.

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**2. Proofs of Theorems 5 and 6.**

**Proof of Theorem 5.** Under our assumptions Theorem 3 implies that we can find a sequence of diffeomorphisms $\phi_k$, such that the sequence of complex struc-
tures $J_k = \phi_k(J)$ converges to a complex structure $J_0$, and $(M, J_0)$ admits a Kähler-Ricci soliton. We want to show that Condition B implies that $J_0$ is isomorphic to $J$.

We have $c_1(M, J_k) \to c_1(M, J_0)$ and since these are integral cohomology classes we must have $c_1(M, J_k) = c_1(M, J_0)$ for sufficiently large $k$. This means that we can think of the canonical bundles $K_{J_k}$ of $(M, J_k)$ as a fixed complex line bundle over $M$ with complex structures varying with $k$. We can then take a basis of sections of $-lK_{J_k}$ for some large $l$ which gives an embedding of $(M, J_0)$ into projective space, and perturb it to a basis of sections of $-lK_{J_k}$ for large $k$. This will give a sequence of embeddings $V_k \subset \mathbb{P}^N$ of $(M, J_0)$ which converges to an embedding $V \subset \mathbb{P}^N$ of $(M, J_0)$.

The fact that all the $V_k$ are isomorphic to each other implies that they are all in the same orbit of $PGL(N+1, \mathbb{C})$ acting on the Hilbert scheme. The fact that the $V_k$ converge to $V$ means that $V$ represents a point in the closure of this orbit. If $J_0$ is not isomorphic to $J$, then $V$ represents a point in the boundary of the orbit of $V_k$, in which case its stabilizer subgroup must have dimension strictly greater than that of $V_k$. This is because the boundary of an orbit is a union of strictly lower dimensional orbits. The stabilizer in this case is just the group of holomorphic automorphisms, so this contradicts the assumption that $J$ satisfies Condition B. Therefore $J_0$ is isomorphic to $J$, and so $(M, J)$ admits a Kähler-Ricci soliton.

Assuming Theorem 2 we can give the proof of Theorem 6.

Proof of Theorem 6. In this case Theorem 3 implies that there exists a sequence of diffeomorphisms $\phi_k$ such that $J_k = \phi_k^*(J)$ converges to a complex structure $J_0$ which this time admits a Kähler-Einstein metric. As in the previous proof we have $c_1(M, J_k) = c_1(M, J_0)$ for sufficiently large $k$. This means that we can apply Theorem 2. Since by assumption $(M, J_k)$ is K-polystable, the theorem implies that $(M, J_k)$ admits a cscK metric in the class $c_1(M, J_k)$ for sufficiently large $k$. This is necessarily a Kähler-Einstein metric on $(M, J)$.

3. Perturbing cscK metrics. Suppose that the complex manifold $(M, J)$ admits a cscK metric $\omega$. In this section we study the problem of whether small deformations $J_t$ of the complex structure admit a cscK metric. We restrict attention to deformations of the complex structure which are compatible with $\omega$. When $(M, J)$ admits no holomorphic vector fields, then the implicit function theorem shows that every small such deformation admits a cscK metric. The case when $(M, J)$ has holomorphic vector fields is more subtle, and was also studied by LeBrun-Simanca [13] in relation with the Futaki invariant.

Since cscK metrics can be interpreted as zeros of a moment map, the deformation problem can be cast into a general framework. We first recall this moment map picture from Donaldson [6].
The moment map picture. Let us write $\mathcal{J}$ for the space of almost complex structures on $M$, compatible with $\omega$. The tangent space $T_J\mathcal{J}$ at a point $J$ can be identified with the space of 1-forms $\alpha \in \Omega^{0,1}(T^{1,0})$ which satisfy
\[
\omega(\alpha(X), Y) + \omega(X, \alpha(Y)) = 0.
\]
This space has a natural complex structure and also an $L^2$ inner product, which gives $\mathcal{J}$ the structure of an infinite dimensional Kähler manifold. Let us write $G$ for the group of exact symplectomorphisms of $(M, \omega)$. This acts naturally on $\mathcal{J}$ preserving the Kähler structure. The Lie algebra of $G$ can be identified with $C_0^\infty(M)$ via the Hamiltonian construction, and we identify it with its dual using the $L^2$ product induced by $\omega$. It was shown by Donaldson (also Fujiki [9]) that the map
\[
J \mapsto S(J, \omega) - \hat{S}
\]
is an equivariant moment map for this action. Here $S(J, \omega)$ is the “Hermitian scalar curvature” defined in [6], which when $J$ is integrable coincides with the usual scalar curvature of the Kähler metric defined by $(J, \omega)$ up to a constant factor. Moreover $\hat{S}$ is the average of $S(J, \omega)$, which is independent of $J$. We see therefore that if $J$ is integrable and is a zero of this moment map, then $(J, \omega)$ defines a cscK metric.

To explain what is meant by (1) being a moment map, define the following two operators at $J \in \mathcal{J}$. The infinitesimal action of $C_0^\infty(M)$ is given by
\[
P: C_0^\infty(M) \to T_J\mathcal{J},
\]
which we can also write as $P(H) = \overline{\partial}X_H$ where $X_H$ is the Hamiltonian vector field corresponding to $H$. The other operator is the derivative of $S(J, \omega)$,
\[
Q: T_J\mathcal{J} \to C_0^\infty(M).
\]
The fact that (1) gives a moment map can be expressed as
\[
\langle Q(\alpha), H \rangle_{L^2} = \Omega(\alpha, P(H)),
\]
where $\Omega$ is the symplectic form on $\mathcal{J}$.

The complex orbits. A key observation in [6] is that while the complexification $G^c$ of the group $G$ does not exist, one can still make sense of its orbits as leaves of a foliation, and a leaf containing an integrable complex structure can be interpreted as the space of Kähler metrics in a Kähler class. We briefly explain
how this works. We can complexify the action of $G$ on $J$ on the level of Lie algebras by extending the operator $P$ to

$$P: C^\infty_0(M, \mathbb{C}) \rightarrow T_J \mathcal{J}$$

in the natural way. We can then think of leaves of the resulting foliation on $\mathcal{J}$ as the orbits of $G^c$. With this in mind we will say that $J_0$ and $J_1$ are in the same $G^c$-orbit if we can find $\phi_t \in C^\infty_0(M, \mathbb{C})$ and a path $J_t \in \mathcal{J}$ for $t \in [0, 1]$ joining $J_0$ and $J_1$, which satisfies

$$\frac{d}{dt} J_t = P_t(\phi_t).$$

We write $P_t$ to emphasize that the operator $P$ depends on the complex structure.

When $J_0$ is integrable, then in fact there exists a diffeomorphism $f: M \rightarrow M$ and some $\psi \in C^\infty_0(M)$ such that $f^*(J_1) = J_0$ and $f^*(\omega) = \omega + i\partial \bar{\partial} \psi$. This means that up to the action of diffeomorphisms, integrable complex structures in the same $G^c$ orbit can be thought of as Kähler metrics in the same Kähler class on a fixed complex manifold.

We will later need to perturb complex structures in a $G^c$ orbit of an integrable $J$, so we give the relevant definition here. Let $J$ be an almost complex structure compatible with $\omega$ and let $U \subset L^2_k$ be a small ball around the origin for some sufficiently large $k$. For $\phi \in U$ we can define a complex structure $F_\phi(J)$ in the following way. For $t \in [0, 1]$ write

$$\omega_t = \omega - tdJd\phi.$$ 

Then

$$\frac{d}{dt} \omega_t = d\alpha$$

for a fixed one-form $\alpha$ with coefficients in $L^2_{k-1}$.

Write $X_t$ for the vector field dual to $-\alpha$ under the symplectic form $\omega_t$ (so $X_t$ has coefficients in $L^2_{k-2}$). Then

$$\frac{d}{dt} \omega_t = -d(tX_t, \omega_t) = -L_{X_t} \omega_t.$$ 

We can now define diffeomorphisms $f_t$ with coefficients in $L^2_{k-2}$ by integrating the family of vector fields $X_t$ for $t \in [0, 1]$.

$$\frac{d}{dt} f_t = X_t.$$
Then \( f_1^* (\omega_1) = \omega \), and we let \( F_\phi (J) = f_1^* J \) which has coefficients in \( L^2_{k-2} \). Note that when \( J \) is integrable then the two Kähler manifolds \((J, \omega - dJ \phi)\) and \((F_\phi (J), \omega)\) are isometric, so up to a diffeomorphism, we are just perturbing the metric in its Kähler class.

The map \( \phi \mapsto F_\phi (J) \) obtained this way is \( K \)-equivariant, where \( K \) is the stabilizer of \( J \) in \( G \). Here a diffeomorphism in \( K \) acts on \( U \in L^2_k \) linearly by pulling back the functions.

We will later need to know the derivative of the map \( \phi \mapsto F_\phi (J) \) at the origin. This is a map from \( U \in L^2_k \) to complex structures in \( L^2_{k-2} \). By the construction the derivative at the origin is given by

\[
(3) \quad DF_0(\phi) = J(L_X \phi) = JP(\phi),
\]

where \( X_\phi \) is the Hamiltonian vector field corresponding to \( \phi \).

**Construction of a local slice.** Suppose now that \((J_0, \omega)\) is a cscK metric on \( M \) (in particular \( J_0 \) is integrable). Following Kuranishi [12] we can construct a local slice for the action of \( G^c \) on \( J \) near \( J_0 \), which intersects the \( G^c \) orbit of every integrable complex structure near \( J_0 \). We will also allow some nonintegrable complex structures in the slice and this will allow the slice to be smooth.

Recall that the infinitesimal action of \( G^c \) is given by the complexification of \( P \),

\[
P: \quad C_0^\infty (M, \mathbb{C}) \to T_{J_0} \mathcal{J}.
\]

We also have an operator \( \bar{\partial} : T_{J_0} \mathcal{J} \to \Omega^{0,2}(T^{1,0}) \), and the two fit into an elliptic complex

\[
\begin{align*}
C_0^\infty (M, \mathbb{C}) & \xrightarrow{P} T_{J_0} \mathcal{J} \\
& \xrightarrow{\bar{\partial}} \Omega^{0,2}(T^{1,0}).
\end{align*}
\]

Let us write

\[
\hat{H}^1 = \{ \alpha \in T_{J_0} \mathcal{J} \mid P^* \alpha = \bar{\partial} \alpha = 0 \}.
\]

This is a finite dimensional vector space since it is the kernel of the elliptic operator \( PP^* + (\bar{\partial}^* \bar{\partial})^2 \) on \( T_{J_0} \mathcal{J} \) (for more details see [10]). We will write \( K \) for the stabilizer of \( J_0 \) in \( G \), i.e., the group of Hamiltonian isometries of \((J_0, \omega)\), and \( \mathfrak{k} \) for its Lie algebra. Note that \( \mathfrak{k} \) can be identified with the kernel of \( P \) in \( C_0^\infty (M, \mathbb{R}) \).

The group \( K \) acts naturally on \( \hat{H}^1 \), and we write \( K^c \) for the complexification of \( K \).

**Proposition 7.** There exists a ball \( B \subset \hat{H}^1 \) around the origin and a \( K \)-equivariant map

\[
\Phi : B \to \mathcal{J},
\]
such that the $\mathcal{G}^c$ orbit of every integrable $J$ near $J_0$ intersects the image of $\Phi$. Also if $x, x'$ are in the same $K^c$-orbit and $\Phi(x)$ is an integrable complex structure, then $\Phi(x), \Phi(x')$ are in the same $\mathcal{G}^c$-orbit. Moreover for all $x \in B$ we have $S(\Phi(x), \omega) \in \mathfrak{k}$.

We learned the idea of requiring this last condition in order to reduce the problem to a finite dimensional one from [5].

**Proof.** Following Kuranishi [12] we can construct a $K$-equivariant holomorphic map

$$\Phi_1: B_1 \to \mathcal{J}$$

from some ball $B_1$ in $\hat{H}^1$, such that the $\mathcal{G}^c$-orbit of every integrable complex structure near $J_0$ intersects the image of $\Phi_1$. The difference is that in Kuranishi’s situation instead of (4) the relevant elliptic complex is

$$\Gamma(T^{1,0}) \xrightarrow{\bar{\partial}} \Omega^{0,1}(T^{1,0}) \xrightarrow{\bar{\partial}} \Omega^{0,2}(T^{1,0}),$$

since he is constructing a slice for the action of the diffeomorphism group instead of $\mathcal{G}^c$. In addition we do not insist that all the complex structures in the image of $\Phi$ should be integrable; the integrable ones will correspond to an analytic subset of $B$ in the same way as in [12].

It only remains to show that we can perturb $\Phi_1$ in such a way as to satisfy the last statement in the proposition. We will perturb inside the $\mathcal{G}^c$ orbits using the map $F_\phi$ defined in the previous subsection. For $J \in \mathcal{J}$ sufficiently close to $J_0$ and $\phi \in L^2_\mathfrak{k}$ sufficiently small, we have a complex structure $F_\phi(J)$ in $L^{2}_{-2}$ if $l$ is large enough.

Let us write $\mathfrak{k}^\perp_l$ for the orthogonal complement to $\mathfrak{k}$ in the Sobolev space $L^2_l$, let $U_l \subset \mathfrak{k}^\perp_l$ be a small ball around the origin and consider the map

$$\begin{align*}
G: B_1 \times U_l &\to \mathfrak{k}^\perp_{l-4} \\
(x, \phi) &\mapsto \Pi_{l-4}^\perp S(F_\phi(\Phi_1(x)), \omega),
\end{align*}$$

where $\Pi$ is the $L^2$ orthogonal projection. It follows from (2) and (3) that the derivative of $G$ at the origin is given by

$$DG_{(0,0)}(\phi) = P^*P(\phi).$$

This is an isomorphism from $\mathfrak{k}^\perp_l \to \mathfrak{k}^\perp_{l-4}$, so by the implicit function theorem we can perturb $\Phi_1$ to

$$\Phi: B \to \mathcal{J},$$
where $B$ is a smaller ball than $B_1$, and so that for all $x \in B$ we have $S(\Phi(x), \omega) \in \mathfrak{k}$. Moreover $F$ and $\Phi_1$ are $K$-equivariant, therefore so are $G$ and $\Phi$.

Let us write $\Omega$ for the symplectic form on $B$ pulled back from $\mathcal{J}$ via $\Phi$. This form is preserved by the $K$-action on $B$, and a moment map for the action is given by

$$\mu(x) = S(\Phi(x), \omega) \in \mathfrak{k}.$$ 

Moreover points $x, x'$ in the same $K^c$ orbit correspond to complex structures in the same $G^c$-orbit if they represent integrable complex structures. Also note that the $K$ and $K^c$ actions on $B$ are just the linear ones induced by those on $\tilde{H}^1$. We have therefore reduced our problem to finding $K^c$-orbits which contain zeros of the moment map $\mu$.

**The finite dimensional problem.** We want to prove the following:

**Proposition 8.** After possibly shrinking $B$, suppose that $v \in B$ is polystable for the $K^c$-action on $\tilde{H}^1$. Then there is a $v_0 \in B$ in the $K^c$-orbit of $v$ such that $\mu(v_0) = 0$.

**Proof.** Let us identify $\tilde{H}^1$ with the tangent space to $B$ at the origin, and write $\Omega_0$ for the linear symplectic form induced on $\tilde{H}^1$ by $\Omega$. Also write

$$\nu: \tilde{H}^1 \to \mathfrak{k}$$

for the corresponding moment map, where we have identified $\mathfrak{k}$ with its dual using the inner product induced by the $L^2$ product on functions as before.

If $v \in B$ is polystable for the $K^c$-action then by the Kempf-Ness theorem there is a zero of the moment map $\nu$ in the $K^c$-orbit of $v$. In fact this is obtained by minimizing the norm over the $K^c$ orbit, so the zero of the moment map will still be in $B$. We can therefore assume without loss of generality that $\nu(v) = 0$.

We have the Taylor expansion

$$\mu(tv) = \mu(0) + t \frac{d\mu_0(v)}{dt} + \frac{t^2}{2} \frac{d^2}{dt^2} \bigg|_{t=0} \mu(tv) + O(t^3).$$

By assumption $\mu(0) = 0$ and since $0$ is a fixed point of the $K$-action we have $d\mu_0(v) = 0$. Also it is easy to check that

$$\frac{d^2}{dt^2} \bigg|_{t=0} \mu(tv) = \nu(v),$$

so in sum we have $\mu(tv) = O(t^3)$. 

For $x \in B$ let us write $K_x$ for the stabilizer of $x$ and $\mathfrak{k}_x$ for its Lie algebra. Note that $K_{tx} = K_x$ for all nonzero $t$. Then for all $\xi \in \mathfrak{k}_x$ we have
\[
\frac{d}{dt} \langle \mu(tx), \xi \rangle = \Omega_{tx}(x, \sigma_{tx}(\xi)) = 0,
\]
where $\sigma_x : \mathfrak{k} \to T_xB$ is the infinitesimal action and we have identified the tangent space to $B$ at every point with $\tilde{H}^1$. This implies that for all $x \in B$ we have
\[
\mu(x) \in \mathfrak{k}_{tx}^\perp.
\]

We now try to perturb $tv$ into a zero of $\mu$ in its $K_x$ orbit using Proposition 9 below for sufficiently small $t$. Write $Q_x$ for the operator
\[
\sigma_x^* \sigma_x : \mathfrak{k}_{tx}^\perp \to \mathfrak{k}_{tx}^\perp.
\]
We need to estimate the norm $\|Q_x^{-1}\|$ for $x = e^{i\xi} \cdot (tv)$ where $\|\xi\| < \delta$ for some fixed small $\delta > 0$. Clearly there is a constant $C$ such that
\[
\|\sigma_x(\eta)\|_{\Omega, 0}^2 \geq C\|\eta\|^2,
\]
for all $\eta \in \mathfrak{k}_x^\perp$ and $x = e^{i\xi} v$, where $\|\xi\| < \delta$.

Note that we have used the metric induced by $\Omega_0$. But $\sigma_{tx}(\eta) = t \sigma_x(\eta)$, and if the ball $B$ is sufficiently small then the metric on $B$ induced by $\Omega$ is bounded below by half of the metric induced by $\Omega_0$, so we have
\[
\|\sigma_{tx}(\eta)\|_{\Omega} \geq \frac{1}{4} t \|\sigma_x(\eta)\|_{\Omega_0}.
\]
It follows that for all $x = e^{i\xi} v$ with $\|\xi\| < \delta$ we have
\[
\langle \sigma_x^* \sigma_{tx}(\eta), \eta \rangle = \|\sigma_{tx}(\eta)\|_{\Omega}^2 \geq \frac{1}{4} t^2 C\|\eta\|^2,
\]
and so $\|Q_{tx}^{-1}\| < C_1 t^{-2}$.

Now Proposition 9 implies that if $C_1 t^{-2} \|\mu(tv)\| < \delta$, then there is a zero of $\mu$ in the $K^c$-orbit of $tv$. Since $\mu(tv) = O(t^3)$, this will be true for sufficiently small $t$, and so the proof of the proposition is complete.

We have used the following extension of a result in [7]. As in the previous proof, let us write $Q_x$ for the operator $\sigma_x^* \sigma_x$. This is an isomorphism
\[
Q_x : \mathfrak{k}_x^\perp \to \mathfrak{k}_x^\perp,
\]
and we write $\Lambda_x$ for the norm of $Q_x^{-1}$.
PROPOSITION 9. Suppose \( x_0 \in B \) satisfies \( \mu(x_0) \in \mathfrak{t}_{x_0} \). Given real numbers \( \lambda, \delta \) such that \( \Lambda_x \leq \lambda \) for all \( x = e^{i\xi} x_0 \) with \( \|\xi\| < \delta \), suppose that \( \lambda \|\mu(x_0)\| < \delta \). Then there is a point \( y = e^{i\eta} x_0 \) with \( \mu(y) = 0 \), where \( \|\eta\| \leq \lambda \|\mu(x_0)\| \).

In Donaldson’s statement it is assumed that \( \mathfrak{t}_{x_0} \) is trivial, but the proof of this slightly more general result is identical. We can now give the proof of Theorem 2.

Proof of Theorem 2. Let \((J, \omega)\) be a cscK metric on \( M \) in the class \( c_1(L) \). If \((M', L')\) is a small deformation of \((M, L)\) then necessarily \( c_1(L) = c_1(L') \) as cohomology classes on the underlying smooth manifold. In particular \( c_1(L) \) is a \((1,1)\)-class with respect to the complex structure of \( M' \) and by modifying \( J' \) and \( L' \) by a small diffeomorphism we can assume that \( J' \) is compatible with \( \omega \).

Any sufficiently small deformation \( J' \) of \( J \) which is integrable and compatible with \( \omega \) is represented by some \( v \in B \). If \( v \) is polystable for the \( K^c \)-action, then Propositions 7 and 8 imply that \( J' \) admits a cscK metric in the Kähler class \([\omega]\). What remains to be shown is that if \( v \) is not polystable then \( (M', L') \) is not K-polystable. By the Hilbert-Mumford criterion there exists a one-parameter subgroup \( \rho: \mathbb{C}^* \rightarrow K^c \) such that

\[
v_0 = \lim_{\lambda \to 0} \rho(\lambda) \cdot v
\]

is polystable (in fact it is a zero of the moment map for the linear symplectic form). Moreover \( v_0 \in B \), and it represents an integrable complex structure \( J_0 \) since integrability is a closed condition. Also, \( J_0 \) admits a cscK metric in \( c_1(L) \) and so \((M, J_0)\) has vanishing Futaki invariant (with respect to the polarization \( c_1(L) \)).

We can now construct a test-configuration for \((M, J')\) whose central fibre is \((M, J_0)\) as follows. First using \( \Phi_1 \) from the proof of Proposition 7 together with \( \rho \), we obtain an \( S^1 \)-equivariant holomorphic map

\[
F: \Delta \rightarrow J
\]

from a small disk \( \Delta \), such that \( F(t) \) is isomorphic to \( J' \) for nonzero \( t \), and \( F(0) \) is isomorphic to \( J_0 \). We let the total space of our test-configuration be \( \mathcal{M} = M \times \Delta \) as a smooth manifold, endowed with the almost complex structure which on the fibre \( M \times \{t\} \) is given by \( F(t) \). Since \( F \) is holomorphic, this gives an integrable complex structure on \( \mathcal{M} \). The \( S^1 \)-action on \( \mathcal{M} \) is the product action

\[
\lambda \cdot (x, t) = (\rho(\lambda) \cdot x, \lambda t),
\]

where we have identified elements of \( K \) with diffeomorphisms of \( M \) fixing \( J_0 \) (we can assume that \( \rho(S^1) \subset K \)). Let us write \( \mathcal{L} \) for the complex vector bundle underlying \( L \) and \( L' \). Let us also fix a connection \( \nabla \) on \( \mathcal{L} \) so that the \((0, 1)\)-part of
∇ with respect to the complex structure $J'$ gives $L$ the holomorphic structure $L'$. Note that $\rho$ induces an $S^1$-action on $L$ and we can assume that $\nabla$ is $S^1$-invariant. We now let $L$ be the pullback $\pi^*(L)$ under the projection $\pi: M \times \Delta \to M$, and endow it with the pullback connection and induced holomorphic structure (using the complex structure on $M$ that we have defined). Then $L$ has a natural $S^1$-action lifted from the action on $M$, which preserves the connection and hence is holomorphic. The restriction of $L$ to $M \times \{t\}$ for nonzero $t$ is just $L'$ and while the restriction to $M \times \{0\}$ might not be $L$ as a holomorphic bundle, it at least has the same first Chern class and is therefore ample. We have thus obtained a flat, polarized, $S^1$-equivariant family over $\Delta$ with general fibre $(M, J')$ polarized by $L'$, and central fibre $(M, J_0)$. The $S^1$-action extends to a $\mathbb{C}^*$-action, using which we can extend the family to a family over $\mathbb{C}$ so we have a test-configuration. Since the Futaki invariant of any vector field on $(M, J_0)$ vanishes, this test-configuration has zero Futaki invariant. Also $(M, J_0)$ has more holomorphic vector fields (since the dimension of the stabilizer $k_{v_0}$ is greater than that of $k_v$), so it is not isomorphic to $(M, J')$ and the test-configuration is not a product configuration. This shows that $(M, J')$ is not K-polystable.

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