COARSE ENTROPY

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Abstract. Coarse geometry studies metric spaces on the large scale. Our goal here is to study dynamics from a coarse point of view. To this end we introduce a coarse version of topological entropy, suitable for unbounded metric spaces, consistent with the coarse perspective on such spaces. As is the case with the usual topological entropy, the coarse entropy measures the divergence of orbits. Following Bowen’s ideas, we use \((n, \varepsilon)\)-separated or \((n, \varepsilon)\)-spanning sets. However, we have to let \(\varepsilon\) go to infinity rather than to zero.

1. Introduction

For a continuous self map of a compact metric space, viewed as a discrete time dynamical system via iteration, the topological entropy of Adler, Konheim, and McAndrew [AKM] can be seen as a measure of the divergence of orbits. Rufus Bowen [B] extended this definition to the noncompact case, as did later authors. While topological entropy is in some sense a global invariant, a map may have large or even infinite topological entropy even if it acts as the identity on all but a small portion of the space.

Coarse (or large scale, or asymptotic) geometry, as developed by Gromov [G] and many others in recent decades, considers properties of metric spaces which, roughly speaking, are visible to an observer at a vantage point receding to infinity. Since to a coarse geometer all bounded metric spaces are equivalent to a point, the focus is on unbounded spaces, for example the Cayley graph of a finitely generated infinite group. This example led to the success of coarse geometry in geometric group theory, where coarse properties of the Cayley graph (for instance the number of ends) give information about the group in question. For more on coarse geometry, see e.g. Roe [R].

Our goal here is to study dynamics from a coarse point of view. To this end we introduce a coarse version of topological entropy, suitable for unbounded metric spaces, consistent with the coarse perspective on such spaces. This entropy should be invariant under a notion of coarse conjugacy of dynamical systems, and so in particular should be insensitive to the behavior of the map on a bounded invariant subset. This is in stark contrast with the usual noncompact entropy. The theory we develop will apply most usefully to controlled maps (see Section 2) of finite dimensional spaces, as we will see.

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In Section 2 we introduce the notion of coarse conjugacy of maps on metric spaces. In Section 3 we introduce the coarse entropy $h_\infty$ of a map, show that it is an invariant of coarse conjugacy, and study its behavior. In Section 4 we compute the coarse entropy of linear maps on $\mathbb{R}^q$. We also compute $h_\infty$ for certain homotheties and relate it in this case to the box-counting dimension. In Section 5 we provide examples showing what can go wrong in infinite dimensional spaces.

2. **Coarse conjugacy**

If we want to investigate coarse dynamics, we need to define coarse conjugacy; this will play the same role as conjugacy in ordinary dynamics. It turns out that this is not completely trivial.

Let us start by fixing terminology and notation. This is important, since various authors use various terminology.

We will consider metric spaces, usually denoted $X, Y, Z$, with metric that we will denote $d$ (in all spaces). Then we will consider a map from the space to itself, and its iterations. To get the most general results, we do not assume anything about the map. However, if we restrict our attention to the class of controlled maps (see the definition below), we get some additional properties. Note that not all controlled maps are continuous and not all continuous maps are controlled.

We will call a map $\varphi : X \to Y$ controlled\footnote{Such maps are also called bornologous.} if there is an increasing function $L : [0, \infty) \to [0, \infty)$ such that for every $x, x' \in X$

$$d(\varphi(x), \varphi(x')) \leq L(d(x, x')).$$

If additionally for every $x, x' \in X$

$$d(x, x') \leq L(d(\varphi(x), \varphi(x'))),$$

then $\varphi$ is called a coarse embedding. If in addition to those two inequalities there exists a constant $M > 0$ such that for every $y \in Y$ there exists $x \in X$ such that $d(y, \varphi(x)) \leq M$, then $\varphi$ is called a coarse equivalence.

Clearly in the above definition we can replace $L$ by any increasing function larger than or equal to $L$. Observe that for any increasing function $L : [0, \infty) \to [0, \infty)$ there is a strictly increasing continuous function $\hat{L} : [0, \infty) \to [0, \infty)$ such that $L \leq \hat{L}$ and $\lim_{t \to \infty} \hat{L}(t) = \infty$ (we leave the proof of this simple fact to the reader as an entertainment). Therefore in the future we will always assume that $L$ is strictly increasing, continuous and $\lim_{t \to \infty} L(t) = \infty$.

If for two maps $\varphi, \varphi' : X \to Y$ there exists a constant $K > 0$ such that for every $x \in X$ we have $d(\varphi(x), \varphi'(x)) \leq K$, then we will say that $\varphi$ and $\psi$ are close. Clearly, closeness is an equivalence relation. A map $\psi : Y \to X$ is called a coarse inverse of $\varphi : X \to Y$ if $\psi \circ \varphi$ is close to the identity on $X$ and $\varphi \circ \psi$ is close to the identity on $Y$. The following facts are well-known.

**Lemma 2.1.** (a) The composition of controlled maps (respectively, coarse embeddings, coarse equivalences) is a controlled map (respectively, a coarse embedding, a coarse equivalence).

(b) Every coarse equivalence has a coarse inverse, and this coarse inverse is also a coarse equivalence.
(c) If a map is close to a controlled map (respectively, a coarse embedding, a coarse equivalence), then it is also a controlled map (respectively, a coarse embedding, a coarse equivalence).

(d) If maps $\zeta, \zeta'$ are close and a map $\xi$ is such that the compositions $\zeta \circ \xi, \zeta' \circ \xi$ make sense, then those compositions are also close.

(e) If maps $\zeta, \zeta'$ are close and a controlled map $\xi$ is such that the compositions $\xi \circ \zeta, \xi \circ \zeta'$ make sense, then those compositions are also close.

**Remark 2.2.** In view of (c), when applying (a), each time before we apply the next map in the composition, we can modify our map by a bounded amount.

In the rest of this section we will be using the map $L$ and the constant $K$ in the above sense. We will exploit the fact that $L$ can be replaced by a larger function and $K$ by a larger constant to use the same $L$ and $K$ for several maps under consideration.

The simplest idea for defining a coarse conjugacy between $f : X \to X$ and $g : Y \to Y$ would be to require that there exists a coarse equivalence $\varphi : X \to Y$ such that $\varphi \circ f$ is close to $g \circ \varphi$. However, in Example 2.7 we show that with this definition coarse conjugacy would not be a symmetric relation. Therefore we need a better definition.

**Definition 2.3.** Maps $f : X \to X$ and $g : Y \to Y$ are **coarsely conjugate** if there exists a coarse equivalence $\varphi : X \to Y$ with a coarse inverse $\psi : Y \to X$ such that $\varphi \circ f$ is close to $g \circ \varphi$ and $\psi \circ g$ is close to $f \circ \psi$.

**Conjecture 2.4.** If there exist coarse equivalences $\varphi : X \to Y$ and $\psi : Y \to X$ such that $\varphi \circ f$ is close to $g \circ \varphi$ and $\psi \circ g$ is close to $f \circ \psi$ then $f$ is coarsely conjugate to $g$.

**Example 2.5.** In Definition 2.3 we cannot, in general, choose an arbitrary coarse inverse $\psi$ of $\varphi$. For instance the maps $f, g : \mathbb{R} \to \mathbb{R}$, given by $f(x) = x^2$ and $g(x) = x^2 + 2x$ are coarsely conjugate via $\varphi(x) = x - 1$ and its coarse inverse (in fact, inverse) $\psi(x) = x + 1$, but not via $\varphi(x) = x - 1$ and its coarse inverse $\psi(x) = x$.

**Lemma 2.6.** Coarse conjugacy is an equivalence relation.

**Proof.** Clearly, coarse conjugacy is reflexive and symmetric. We will show that it is also transitive.

Let $f : X \to X$, $g : Y \to Y$ and $h : Z \to Z$ be three maps; let $f$ be coarsely conjugate to $g$ via $\varphi$ and $\psi$, and let $g$ be coarsely conjugate to $h$ via $\varphi'$ and $\psi'$. We want to show that $f$ is coarsely conjugate to $h$ by $\varphi' \circ \varphi$ and $\psi \circ \psi'$.

By Lemma 2.1, $\varphi' \circ \varphi$ and $\psi \circ \psi'$ are coarse equivalences. By Lemma 2.1 and Remark 2.2 the maps $\psi \circ \psi' \circ \varphi' \circ \varphi$ and $\varphi' \circ \varphi \circ \psi \circ \psi'$ are close to the corresponding identities, so $\psi \circ \psi'$ is a coarse inverse of $\varphi' \circ \varphi$.

By the assumption and Lemma 2.1 (d), $h \circ \varphi' \circ \varphi$ is close to $\varphi' \circ h \circ \varphi$. By the assumption and Lemma 2.1 (e), $\varphi' \circ h \circ \varphi$ is close to $\varphi' \circ \varphi \circ f$. Therefore $h \circ \varphi' \circ \varphi$ is close to $\varphi' \circ \varphi \circ f$. Similarly, $f \circ \psi \circ \psi'$ is close to $\psi \circ \psi' \circ h$.

Now let us return to the question whether we really need $\psi$ in the definition of coarse conjugacy.

**Example 2.7.** Let $X = \mathbb{Z}$ and $Y = \mathbb{R}$ and let $f : X \to X$ and $g : Y \to Y$ be defined by the same formula $x \mapsto x^2$. If $\varphi : X \to Y$ is the natural embedding, $\varphi(x) = x$,
then clearly $\phi$ is a coarse equivalence and $\phi \circ f = g \circ \phi$. However, $f$ is not coarsely conjugate to $g$, because there is no coarse equivalence $\psi : Y \to X$ for which $\psi \circ g$ is close to $f \circ \psi$.  

Indeed, suppose that such $\psi$ exists. Then for every $x > 0$ the set $\psi([x, x + 1])$ is contained in an interval of length $L(1)$, so it has at most $L(1) + 1$ elements. Therefore the set $f(\psi([x, x + 1]))$ has also at most $L(1) + 1$ elements, so the set $\psi(g([x, x + 1]))$ has at most $(2K + 1)(L(1) + 1)$ elements. However, the interval $g([x, x + 1])$ has length $2x + 1$, so there must be a point $n \in \mathbb{Z}$ whose preimage under $\psi$ has diameter at least $\frac{2x + 1}{(2K + 1)(L(1) + 1)}$. This means that there are $y, z \in \mathbb{R}$ with $\psi(y) = \psi(z)$ and $d(y, z) \geq \frac{2x + 1}{(2K + 1)(L(1) + 1)}$. Therefore, 

$$
\frac{2x + 1}{(2K + 1)(L(1) + 1)} \leq L(0),
$$

which is clearly not true if $x$ is sufficiently large.  

However, if we focus on controlled maps, we can dispense with $\psi$ in the definition of coarse conjugacy. In fact, we have

**Proposition 2.8.** Consider maps $f : X \to X$ and $g : Y \to Y$ for which there exists a coarse equivalence $\phi : X \to Y$ such that $\phi \circ f$ is close to $g \circ \phi$ and $g$ is controlled. Then $f$ is also controlled and for any coarse inverse $\psi$ of $\phi$ the maps $f$ and $g$ are coarsely conjugate via $\phi$ and $\psi$.

**Proof.** Let $\psi : Y \to X$ be a coarse inverse of $\phi$. In the proof we will be using all the time Lemma [2.1](#) and once Remark [2.2](#). The map $\psi \circ g \circ \phi$ is controlled. Since $\psi \circ \phi$ is close to the identity and $\phi \circ f$ is close to $g \circ \phi$, we see that $f$ is close to $\psi \circ \phi \circ f$, which is close to $\psi \circ g \circ \phi$. Thus, $f$ is controlled.

Further, we see that $f \circ \psi$ is close to $\psi \circ g \circ \phi \circ \psi$, which is close to $\psi \circ g$. Thus, $f \circ \psi$ is close to $\psi \circ g$, so $f$ and $g$ are coarsely conjugate via $\phi$ and $\psi$.  

The assumption that $g$ is controlled is important. The following example shows that it cannot even be replaced by the assumption that $f$ is controlled.

**Example 2.9.** Let $X = \mathbb{R}$, $Y = \{(x, y) \in \mathbb{R}^2 : y \in \{0, 1\}\}$, $f : X \to X$ be the identity, and $g(x, 0) = (x, 0)$, $g(x, 1) = (x^2, 1)$. If $\phi : X \to Y$ is given by $\phi(x) = (x, 0)$, then it is a coarse equivalence and $\phi \circ f = g \circ \phi$. However, there is no coarse equivalence $\psi : Y \to X$ such that $\psi \circ g$ is close to $f \circ \psi$. Indeed, if such $\psi$ exists, then by restricting it to $\mathbb{R} \times \{1\}$ and identifying this line with $\mathbb{R}$, we see that we can use Proposition [2.8](#) to deduce that the map $x \mapsto x^2$ is controlled. Since it is clearly not controlled, such $\psi$ cannot exist. 

A coarse conjugacy between maps need not work for their iterates if the maps are not controlled.

**Example 2.10.** Take $X = Y = [2, \infty)$, $f(x) = x^2$, $g(x) = x^2 + \frac{1}{x}$, and both $\phi$ and $\psi$ equal to the identity. Clearly, the pair $(\phi, \psi)$ is a coarse conjugacy between $f$ and $g$. However, it is not a coarse conjugacy between $f^2$ and $g^2$. Indeed, 

$$
g^2(x) - f^2(x) = -2x + \frac{1}{x^2} - \frac{x}{x^3 - 1}
$$

is not bounded.
However, it is easy to check that $f^2$ and $g^2$ are coarsely conjugate via $\varphi'(x) = x - \frac{1}{2x^2}$ and $\psi'(x) = x + \frac{1}{2x^2}$.

**Conjecture 2.11.** If $f$ and $g$ are coarsely conjugate then so are $f^n$ and $g^n$ for all natural $n$.

**Lemma 2.12.** Consider maps $f, f' : X \to X$, $g, g' : Y \to Y$ and a coarse equivalence $\varphi : X \to Y$ such that $g, g'$ are controlled, $\varphi \circ f$ is close to $g \circ \varphi$ and $\varphi \circ f'$ is close to $g' \circ \varphi$. Then $\varphi \circ f' \circ f$ is close to $g' \circ g \circ \varphi$.

**Proof.** By Proposition 2.8 $f$ and $f'$ are controlled. If $\psi$ is a coarse inverse of $\varphi$, then, as in the proof of Proposition 2.8, $f$ is close to $\psi \circ g \circ \varphi$. Now, by Lemma 2.1 $\varphi \circ f' \circ f$ is close to $\varphi \circ f' \circ \psi \circ g \circ \varphi$, which is close to $g' \circ \varphi \circ \psi \circ g \circ \varphi$, which is close to $g' \circ g \circ \varphi$. \hfill $\Box$

From Proposition 2.8 and Lemma 2.12 we get immediately the following corollary.

**Corollary 2.13.** If $f$ and $g$ are coarsely conjugate and $g$ is controlled, then for any natural $n$ the maps $f^n$ and $g^n$ are coarsely conjugate via the same coarse equivalences as $f$ and $g$.

### 3. Coarse entropy

Let $(X, d)$ be a metric space, and $f : X \to X$ a map. We want to define coarse entropy of $f$ using similar ideas as in the usual definitions of topological entropy. Mimicking the original definition of Adler, Konheim and McAndrew [AKM] can be difficult, since the space is not compact and the map is not necessarily continuous. Thus, we will try to mimic the definition of Bowen [B]. However, we have to incorporate the idea of closeness replacing equality. This means that instead of orbits we should use $\delta$-pseudoorbits. Fortunately, we know that $\delta$-pseudoorbits work well with Bowen’s definition (see [M]). Of course, we have to replace $\varepsilon$ going to 0 by $R$ going to infinity.

Thus, we define the **coarse entropy** of $f$ as

$$h_\infty(f) = \lim_{\delta \to \infty} \lim_{R \to \infty} \limsup_{n \to \infty} \frac{1}{n} \log s(f, n, \delta, x_0),$$

where $s(f, n, \delta, x_0)$ is the supremum of cardinalities of $R$-separated sets of $\delta$-pseudo-orbits of $f$ of length $n$ starting at $x_0$. As usual, a $\delta$-pseudoorbit of $f$ of length $n$ starting at $x_0$ is a sequence $(x_0, x_1, \ldots, x_n)$ such that $d(f(x_i), x_{i+1}) \leq \delta$ for $i = 0, 1, \ldots, n - 1$. The distance between the pseudoorbits $(x_0, x_1, \ldots, x_n)$ and $(y_0, y_1, \ldots, y_n)$ is the maximum of the distances $d(x_i, y_i)$ over $i = 0, 1, \ldots, n$. A set is $R$-separated if the distance between each two distinct elements of this set is at least $R$.

The value of $h_\infty(f)$ in the above definition does not depend on the choice of $x_0 \in X$. Indeed, if $y_0 \in X$ is another point, then a $\delta$-pseudoorbit starting at $x_0$ can have $y_0$ as the next element (and vice versa).

Given $f : X \to X$ and $g : Y \to Y$, we will say that the map $f$ is **coarsely embedded** in the map $g$ if there exists a coarse embedding of spaces $\varphi : X \to Y$ such that $\varphi \circ f$ is close to $g \circ \varphi$.

**Theorem 3.1.** If $f$ is coarsely embedded in $g$ then $h_\infty(f) \leq h_\infty(g)$. 
Proof. We will keep the same notation as in the definitions. Suppose that \((x_0, x_1, \ldots, x_m)\) is a \(\delta\)-pseudoorbit of \(f\) in \(X\). For \(i = 0, 1, \ldots, m - 1\) we have

\[
d \left( g(\varphi(x_i)), \varphi(x_{i+1}) \right) \leq d \left( g(\varphi(x_i)), \varphi(f(x_i)) \right) + d \left( \varphi(f(x_i)), \varphi(x_{i+1}) \right) \leq K + L(\delta).
\]

Thus, the image under \(\varphi\) of a \(\delta\)-pseudoorbit in \(X\) is a \((L(\delta) + K)\)-pseudoorbit of \(g\) in \(Y\). On the other hand, if two \(\delta\)-pseudoorbits in \(X\) are \(R\)-separated, then their images in \(Y\) are \(L^{-1}(R)\)-separated. Therefore,

\[
\limsup_{n \to \infty} \frac{1}{n} \log s(f, n, R, \delta, x_0) \geq \limsup_{n \to \infty} \frac{1}{n} \log s(g, n, L^{-1}(R), L(\delta) + K, \varphi(x_0)).
\]

The quantities \(R\) and \(L^{-1}(R)\) go to infinity simultaneously. Similarly, \(\delta\) and \(L(\delta) + K\) go to infinity simultaneously. In such a way we obtain \(h_{\infty}(f) \leq h_{\infty}(g)\). \qed

Corollary 3.2. If \(f\) is coarsely embedded in \(g\) and \(g\) is coarsely embedded in \(f\) then \(h_{\infty}(f) = h_{\infty}(g)\). Therefore, the coarse entropy is an invariant of coarse conjugacy. In particular, if we change the metric \(d\) to a metric that is bi-Lipschitz equivalent, or quasi-isometric, to \(d\), the coarse entropy will not change.

Remark 3.3. Maps \(f\) and \(g\) may each coarsely embed in the other without being coarsely conjugate. Let \(X\) be the binary tree with edges of unit length and \(Y\) be \(X\) with a ray attached at the root, each with the path metric (see Figure 1). If \(f\) and \(g\) are the identity maps on \(X\) and \(Y\) respectively, then \(g\) coarsely embeds in \(f\) as in the first diagram and \(f\) coarsely embeds in \(g\) via the inclusion. But \(f\) and \(g\) are not coarsely conjugate since \(X\) and \(Y\) are not coarsely equivalent, as their boundaries (a Cantor set, and the union of a Cantor set and an isolated point, respectively) are not homeomorphic.

Example 3.4. This is an example where \(f\) and \(g\) are homeomorphisms, they are conjugate via a Lipschitz (but not bi-Lipschitz) homeomorphism \(\varphi\) (that is, \(\varphi \circ f = g \circ \varphi\)), but \(h_{\infty}(g) > h_{\infty}(f)\).

Let \(X = Y\) be the half-plane \(\{(x, y) \in \mathbb{R}^2 : y \geq 0\}\). Let \(f : X \to X\) be given by the formula \(f(x, y) = (2x, y)\). The identity coarsely embeds \(f\) into the linear map of
\( \mathbb{R}^2 \) to itself given by the same formula, and, as we will see later, the coarse entropy of that map is \( \log 2 \). Therefore, by Theorem 3.1, we have \( h_\infty(f) \leq \log 2 \).

The map \( \varphi : X \to Y \) maps each horizontal line \( H_t = \{(x, y) \in \mathbb{R}^2 : y = t\} \) to itself by squeezing linearly the segment (in the variable \( x \)) \([-e^t, e^t]\) to the segment \([-1, 1]\) and translating the remaining two half-lines. Thus, if \( -e^y \leq x \leq e^y \), then \( \varphi(x, y) = (xe^{-y}, y) \); if \( x > e^y \) then \( \varphi(x, y) = (x - e^y + 1, y) \); and if \( x < -e^y \) then \( \varphi(x, y) = (x + e^y - 1, y) \). Clearly, \( \varphi \) is a homeomorphism.

Finally, we set \( g = \varphi \circ f \circ \varphi^{-1} \). Let us estimate the coarse entropy of \( g \). Take \( x_0 = (0, 0) \). If \( \delta > 1 \) then for every \( x \in [-1, 1] \) there is a \( \delta \)-pseudoorbit

\[
((0, 0), (0, \delta), (0, 2\delta), \ldots, (0, (n - 2)\delta), (x, (n - 2)\delta))
\]

of length \( n - 1 \). Therefore, there is a \( \delta \)-pseudoorbit of length \( n \) starting at \( x_0 \) and ending at \( z = g(x, (n - 2)\delta) \). The point \( z \) can be any chosen point of the image under \( g \) of \([-1, 1] \times \{(n - 2)\delta\}\). To find this image, we use the definition of \( g \). Its second coordinate is \( t := (n - 2)\delta \). For the first coordinate, we take the interval \([-e^t, e^t]\), multiply by 2 to get \([-2e^t, 2e^t]\), and shorten by \( e^t - 1 \) from both sides, to get \([-e^t - 1, e^t + 1]\). It follows that we can find an \( R \)-separated set of \( \delta \)-pseudoorbits of \( g \) of length \( n \) starting at \( x_0 \), which has cardinality \( (2e^t + 2)/R - 1 \), so

\[
s(g, n, R, \delta, x_0) \geq \frac{2}{R} e^{(n-2)\delta} - 1.
\]

Thus,

\[
\limsup_{n \to \infty} \frac{1}{n} \log s(g, n, R, \delta, x_0) \geq \delta,
\]

and hence, \( h_\infty(g) = \infty \).

A subset of a metric space is \( R \)-spanning if for every element of the space there is an element of the subset at distance less than \( R \). Let \( r(f, n, R, \delta, x_0) \) be the infimum of cardinalities of \( R \)-spanning sets of \( \delta \)-pseudoorbits of \( f \) of length \( n \) starting at \( x_0 \).

**Theorem 3.5.** We have

\[
h_\infty(f) = \lim_{\delta \to \infty} \lim_{R \to \infty} \limsup_{n \to \infty} \frac{1}{n} \log r(f, n, R, \delta, x_0).
\]

**Proof.** Any maximal \( R \)-separated set is also \( R \)-spanning. This proves that

\[
r(f, n, R, \delta, x_0) \leq s(f, n, R, \delta, x_0),
\]

so

\[
h_\infty(f) \geq \lim_{\delta \to \infty} \lim_{R \to \infty} \limsup_{n \to \infty} \frac{1}{n} \log r(f, n, R, \delta, x_0).
\]

On the other hand, in each ball of radius \( R \) centered in an element of an \( R \)-spanning set there may be at most one element of a \( 2R \)-separated set. This proves that

\[
r(f, n, R, \delta, x_0) \geq s(f, n, 2R, \delta, x_0),
\]

so

\[
h_\infty(f) \leq \lim_{\delta \to \infty} \lim_{R \to \infty} \limsup_{n \to \infty} \frac{1}{n} \log r(f, n, R, \delta, x_0).
\]

\( \square \)
Theorem 3.6. For any \( k \geq 1 \) we have \( h_\infty(f^k) \leq kh_\infty(f) \). If additionally \( f \) is controlled, then \( h_\infty(f^k) = kh_\infty(f) \).

Proof. Clearly, we have

\[
\text{Theorem 3.6.}
\]
then

\[
s(f, kn, R, \delta, x_0) \geq s(f^k, n, R, \delta, x_0).
\]

Therefore

\[
k \cdot \lim_{m \to \infty} \frac{1}{m} \log s(f, m, R, \delta, x_0) \geq k \cdot \lim_{n \to \infty} \frac{1}{kn} \log s(f, kn, R, \delta, x_0)
\]

\[
\geq \lim_{n \to \infty} \frac{1}{n} \log s(f^k, n, R, \delta, x_0),
\]

and thus, \( h_\infty(f^k) \leq kh_\infty(f) \).

Assume now that \( f \) is controlled with function \( L \). If \((x_0, x_1, \ldots, x_k)\) is a \( \delta \)-pseudo-orbit of \( f \), then by induction on \( k \) we get \( d(f^k(x_0), x_k) \leq \eta_k \), where

\[
\eta_k = \delta + L(\delta) + L^2(\delta) + \cdots + L^{k-1}(\delta).
\]

Thus, if \((x_0, x_1, \ldots, x_{nk})\) is a \( \delta \)-pseudo-orbit of \( f \), then \((x_0, x_k, \ldots, x_{nk})\) is an \( \eta_k \)-pseudo-orbit of \( f^k \).

Moreover, if \((x_0, x_1, \ldots, x_i)\) and \((y_0, y_1, \ldots, y_i)\) are \( \delta \)-pseudo-orbits of \( f \), then we get

\[
d(x_i, y_i) \leq d(f^i(x_0), f^i(y_0)) + d(f^i(x_i), y_i) \leq L^i(d(x_0, y_0)) + 2\eta_i
\]

for every \( i > 0 \). We may assume that \( L(t) \geq t \) for every \( t \), and then, if \( i \leq k \),

\[
d(x_i, y_i) \leq L^k(d(x_0, y_0)) + 2\eta_k.
\]

Therefore, if \( d(x_i, y_i) \geq L^k(0) + 2\eta_k \), then

\[
d(x_0, y_0) \geq L^{-k}(d(x_i, y_i) - 2\eta_k).
\]

Changing indices, we see that if \((x_0, x_1, \ldots, x_{nk})\) and \((y_0, y_1, \ldots, y_{nk})\) are \( \delta \)-pseudo-orbits of \( f \), then for \( m = 1, 2, \ldots, n - 1 \) and \( i = 1, 2, \ldots, k \), if \( d(x_{mk+i}, y_{mk+i}) \geq L^k(0) + 2\eta_k \), then

\[
d(x_{mk}, y_{mk}) \geq L^{-k}(d(x_{mk+i}, y_{mk+i}) - 2\eta_k).
\]

If the distance between those two pseudoorbits is at least \( R \), and \( R \geq L^k(0) + 2\eta_k \), then there are \( m \) and \( i \) such that \( d(x_{mk+i}, y_{mk+i}) \geq R \), and hence the distance between the \( \eta_k \)-pseudoorbits \((x_0, x_k, \ldots, x_{nk})\) and \((y_0, y_k, \ldots, y_{nk})\) of \( f^k \) is at least

\[
S_k = L^{-k}(R - 2\eta_k).
\]

This proves that

\[
s(f, kn, R, \delta, x_0) \leq s(f^k, n, S_k, \eta_k, x_0).
\]

If \( j = kn + i \) with \( i \leq k \), then \( s(f, j, R, \delta, x_0) \leq s(f, k(n + 1), R, \delta, x_0) \). Therefore

\[
\lim_{j \to \infty} \frac{1}{j} \log s(f, j, R, \delta, x_0) \leq \lim_{j \to \infty} \frac{1}{j} \log s(f, k[\lceil j/k \rceil], R, \delta, x_0)
\]

\[
= \lim_{n \to \infty} \frac{1}{kn} \log s(f, kn, R, \delta, x_0) \leq \frac{1}{k} \lim_{n \to \infty} \frac{1}{n} \log s(f^k, n, S_k, \eta_k, x_0).
\]

With \( \delta \) (and therefore also \( \eta_k \)) fixed, \( R \) and \( S_k \) go to infinity simultaneously, so

\[
k \cdot \lim_{R \to \infty} \limsup_{j \to \infty} \frac{1}{j} \log s(f, j, R, \delta, x_0) \leq \lim_{S \to \infty} \limsup_{n \to \infty} \frac{1}{n} \log s(f^k, n, S, \eta_k, x_0).
\]
Now, $\delta$ and $\eta_k$ go to infinity simultaneously, and thus, $h_\infty(f^k) \geq kh_\infty(f)$. \hfill \Box

**Example 3.7.** This example shows that in the above theorem, if we do not assume $f$ is controlled, then it can happen that $h_\infty(f^k) < kh_\infty(f)$.

Let $X$ be a disjoint union of rectangles $P_n$, $n = 0, 1, 2, \ldots$. Rectangle $P_{2m}$ has size $1 \times 2^m$ and rectangle $P_{2m+1}$ has size $2^m \times 1$. Let $c_n$ be the center of the rectangle $P_n$. On each rectangle the metric is the maximum of horizontal and vertical distances. If $x \in P_n$ and $y \in P_m$ for $n < m$, then

$$d(x, y) = d(x, c_n) + d(y, c_m) + (n + 1) + (n + 2) + \cdots + m$$

(that is, the distance between $P_n$ and $P_{n+1}$ is $n + 1$).

The map $f$ maps $P_n$ onto $P_{n+1}$ by a linear map that preserves the horizontal and vertical directions. Thus, as we apply $f$ repeatedly, the rectangles get alternately stretched horizontally while contracting vertically, and stretched vertically while contracting horizontally. However, $f^2$ only stretches each rectangle in one direction by the factor 2.

Assume that $\delta > 2$. For $m > 0$ we construct some special $\delta$-pseudoorbits of the length $2m + 2$. We set $x_0 = c_0$ and as $x_1$ we can take any point of $P_0$. For the next $2m + 1$ steps we just follow the orbit of $x_1$. If we choose locations of $x_1$ at the vertices of a square grid with vertical and rectangular distances of size $R/2^m$, then for two distinct points of this set the distance between the last or the last but one elements of the corresponding pseudoorbits will be at least $R$. There are $4^m/R$ of those vertices, so $s(f, 2m + 2, R, \delta, x_0) \geq 4^m/R$. Therefore,

$$\limsup_{n \to \infty} \frac{1}{n} \log s(f, n, R, \delta, x_0) \geq \log 2$$

so $h_\infty(f) \geq \log 2$.

On the other hand, when we look at $\delta$-pseudoorbits for $f^2$, then once we get into $P_n$ with $n > \delta$, we have to move in each step from $P_i$ to $P_{i+2}$. This means that up to a multiplicative and an additive constant, the maximal cardinality of an $R$-separated set of $\delta$-pseudoorbits will not be larger than for multiplication by 2 on the real line. We will see later that the coarse entropy of this multiplication is $\log 2$, and thus $h_\infty(f^2) \leq \log 2 < 2 \log 2 \leq 2h_\infty(f)$. \hfill \Box

**Theorem 3.8.** Let $f : X \to X$ and $g : Y \to Y$ be maps. Then

$$h_\infty(f \times g) \leq h_\infty(f) + h_\infty(g).$$

**Proof.** In $X \times Y$ we can take the max metric. If $(x_0, x_1, \ldots, x_n)$ is a $\delta$-pseudoorbit in $X$, and $(y_0, y_1, \ldots, y_n)$ is a $\delta$-pseudoorbit in $Y$, then $((x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n))$ is a $\delta$-pseudoorbit in $X \times Y$. Therefore, if $E_X$ is an $R$-spanning set of $\delta$-pseudoorbits of $f$ of length $n$ starting at $x_0$ and $E_Y$ is an $R$-spanning set of $\delta$-pseudoorbits of $g$ of length $n$ starting at $y_0$, then $E_X \times E_Y$ (understood in an obvious sense) is an $R$-spanning set of $\delta$-pseudoorbits of $f \times g$ of length $n$ starting at $(x_0, y_0)$. Hence,

$$r(f \times g, n, R, \delta, (x_0, y_0)) \leq r(f, n, R, \delta, x_0) \cdot r(g, n, R, \delta, y_0).$$

(1)
Therefore,
\[
\limsup_{n \to \infty} \frac{1}{n} \log r(f \times g, n, R, \delta, (x_0, y_0)) \\
\leq \limsup_{n \to \infty} \left( \frac{1}{n} \log r(f, n, R, \delta, x_0) + \frac{1}{n} \log r(g, n, R, \delta, y_0) \right) \\
\leq \limsup_{n \to \infty} \frac{1}{n} \log r(f, n, R, \delta, x_0) + \limsup_{n \to \infty} \frac{1}{n} \log r(g, n, R, \delta, y_0).
\]
By Theorem 3.5, we get \( h_\infty(f \times g) \leq h_\infty(f) + h_\infty(g) \).

\[\square\]

**Example 3.9.** This example shows that even if we assume that if \( f \) and \( g \) increase distances at most 2 times and do not decrease distances, we may not get equality in Theorem 3.8.

We define the spaces \( X \) and \( Y \) in a similar way as in Example 3.7 except that instead of rectangles, we take segments of the real line. The point \( c_n \) will be the left endpoint of the \( n \)-th segment, and the distance in the space is defined in a similar way as in Example 3.7. The length of the zeroth segment is 1. The lengths of the next segments will be determined by the maps \( f \) and \( g \). Both of them map the \( n \)-th segment onto the \((n+1)\)-st one in the linear way; it will be the multiplication by 1 or 2. If \( 2^{k^2} \leq n < 2^{(k+1)^2} \), then if \( k \) is even then \( f \) multiplies by 1 and \( g \) by 1; if \( k \) is odd then \( f \) multiplies by 2 and \( g \) by 1.

We may assume that \( \delta > 3 \). Then, if \( n = 2^{k^2} \) with \( k \) odd, the length of the \( n \)-th segment in \( X \) is at least \( 2^{2^{k^2} - 2^{(k-1)^2}} \), so

\[
\frac{1}{n} \log s(f, n, R, \delta, x_0) \geq \frac{1}{n} \log \frac{2^{2^{k^2} - 2^{(k-1)^2}}}{R} = \frac{2^{k^2} - 2^{(k-1)^2}}{2^{k^2}} \log 2 - \frac{1}{n} \log R,
\]
and therefore \( h_\infty(f) \geq \log 2 \). Similarly, \( h_\infty(g) \geq \log 2 \).

To obtain an upper estimate on \( h_\infty(f \times g) \), in view of the inequality (1), it is enough to construct for each \( n \) \( R \)-spanning sets of \( \delta \)-pseudo-orbits starting at \( x_0 \) for \( f \), and starting at \( y_0 \) for \( g \), with relatively small cardinalities. Let us do it for \( f \). Denote the \( n \)-th segment \( P_n \) and its length \( l_n \).

Concentrate first on the \( \delta \)-pseudo-orbits \((x_0, x_1, \ldots, x_n)\) for which \( x_i \in P_i \) (and \( x_0 = c_0 \)). We may assume that \( \delta \) is large and \( R \) is much larger. Set \( m = \lfloor \frac{R}{\delta} \rfloor \) and \( n = km \) for some integer \( k > 0 \). Partition each interval \( P_{jm} \) into subsegments of length \( R/3 \) (one of them may be shorter). Since \( f \) does not shorten the distances between points, if our \( \delta \)-pseudo-orbit has \( jm \)-th point in a given subsegment of the partition of \( P_{jm} \), then its \((j-1)m\)-th point is in some specific segment of length not larger than \( R/3 + 2m\delta \) of \( P_{(j-1)m} \). Since \( R/3 + 2m\delta \leq 2R/3 \), this segment can intersect at most 3 elements of the partition of \( P_{(j-1)m} \). Thus, if we code our \( \delta \)-pseudo-orbits by the elements of the partitions through which they pass, the number of the valid codes will be not larger than \( 3^k \left( 3l_n/R + 1 \right) \). On the other hand, if two \( \delta \)-pseudo-orbits have the same code, then their distance is at most \( 2R/3 < R \). Thus, there exists an \( R \)-spanning set of \( \delta \)-pseudo-orbits of length \( n \) of cardinality at most \( 3^k \left( 3l_n/R + 1 \right) \).

Now we have to deal with the fact that there are \( \delta \)-pseudo-orbits for which not necessarily \( x_i \in P_i \) for each \( i \). Once a \( \delta \)-pseudo-orbit gets to a segment \( P_i \) with \( i > \delta \), it has to move to the next segment with each application of \( f \). On the other hand, if \( R \) is large enough, if we distinguish between two points only if their distance is at
least $R$, the union of the segments $P_i$ with $i \leq \delta$ is seen as one point. Therefore our estimate of the cardinality of an $R$-spanning set has to be only multiplied by $n$.

The other thing we have to deal with is that we obtained our estimate only for $ns$ which are multiples of $m$. However, when taking a limit with respect to $n$, it does not matter whether we divide by $n$, or by $n - m$, or by anything in between.

If we use the estimate we obtained for $f$ and the analogous estimate for $g$ (where the length of the $n$th segment is $q_n$), we get

$$r(f \times g, n, R, \delta, (x_0, y_0)) \leq 3^k(3l_n/R + 1)n \cdot 3^k(3q_n/R + 1)n
\leq 3^{n-\delta/R}(3l_n/R + 1)(3q_n/R + 1)n^2.$$  

Taking into account that $\ln q_n = 2n$, we get

$$\limsup_{n \to \infty} \frac{1}{n} \log r(f \times g, n, R, \delta, (x_0, y_0)) \leq 12\frac{\delta}{R} \log 3 + \log 2.$$  

To compute the coarse entropy, we go to infinity with $R$ before we go to infinity with $\delta$, so $h_\infty(f \times g) \leq \log 2$.

Thus, in our example

$$h_\infty(f \times g) \leq \log 2 < 2 \log 2 = h_\infty(f) + h_\infty(g).$$

The idea of the above example is that since in the definition of coarse entropy we take the upper limit, for distinct maps those upper limits can be limits along different subsequences. However, if the maps are equal, we can take the same subsequences. Therefore we have the following result (suggested to us by Mariusz Lemańczyk).

**Proposition 3.10.** Let $f : X \to X$ be a map and $k \geq 2$ an integer. Then

$$h_\infty(F) = k h_\infty(f),$$

where $F = f \times f \times \cdots \times f$ ($k$ times).

**Proof.** In $X^k$ we take the max metric. If $(x^i_0, x^i_1, \ldots, x^i_n)$ are $\delta$-pseudoorbits of $f$ in $X$ for $i = 1, 2, \ldots, k$, then $((x^1_0, x^2_0, \ldots, x^k_0), \ldots, (x^1_n, x^2_n, \ldots, x^k_n))$ is a $\delta$-pseudoorbit of $F$ in $X^k$. Therefore, if $E$ is an $R$- separateset of $\delta$-pseudoorbits of $f$ of length $n$ starting at $x_0$, then $E^k$ is an $R$- separateset of $\delta$-pseudoorbits of $F$ of length $n$ starting at $(x_0, x_0, \ldots, x_0)$. Hence,

$$s(F, n, R, \delta, (x_0, x_0, \ldots, x_0)) \geq (s(f, n, R, \delta, x_0))^k,$$

and thus $h_\infty(F) \geq k h_\infty(f)$. Together with Theorem 3.8 applied inductively, we get $h_\infty(F) = k h_\infty(f)$. \qed

**4. Linear maps**

One of the basic tests whether our definition is good is whether the entropy of a linear map of a finite dimensional euclidean space is correct, that is, whether it is the sum of positive logarithms of the absolute values of eigenvalues. We will start with the expanding case.

**Lemma 4.1.** If $f : \mathbb{R}^q \to \mathbb{R}^q$ is a linear map with all eigenvalues of absolute value larger than 1 and the absolute value of the determinant of $f$ is $\Lambda$, then $h_\infty(f) = \log \Lambda$.
Proof. By changing the basis in $\mathbb{R}^q$ we may assume that for the Euclidean norm $\| \cdot \|$ there exists $\lambda > 1$ such that for every $x \in \mathbb{R}^q$ we have $\| f(x) \| \geq \lambda \| x \|$. Let $x_0$ be the origin of $\mathbb{R}^q$. Fix $\delta > 0$ and consider the set $\mathcal{O}_n$ of all $\delta$-pseudoorbits of $f$ of length $n$ starting at $x_0$. For such a $\delta$-pseudoorbit $(x_0, x_1, \ldots, x_n)$ we will call $x_n$ its final term. Let $K_n$ be the set of final terms of all elements of $\mathcal{O}_n$. In particular, $K_1 = B(\delta)$, where $B(t)$ denotes the closed ball centered at $x_0$ with radius $t$. Therefore, $K_n \supset f^{n-1}(B(\delta))$. It follows that if $E$ is an $R$-spanning set in $\mathcal{O}_n$, then the set of final terms of $E$ has to $R$-span $f^{n-1}(B(\delta))$. Thus, if $\text{Vol}$ denotes the $q$-dimensional volume, then the cardinality $|E|$ of $E$ satisfies

$$|E| \geq \frac{\text{Vol}(f^{n-1}(B(\delta)))}{\text{Vol}(B(R))} = \Lambda^{n-1} \frac{\text{Vol}(B(\delta))}{\text{Vol}(B(R))} = \Lambda^{n-1} \left(\frac{\delta}{R}\right)^q,$$

so $h_\infty(f) \geq \log \Lambda$.

We claim that the $\delta$-pseudoorbits of $f$ have the following shadowing property: if $(x_0, x_1, \ldots, x_n)$ is a $\delta$-pseudoorbit, then the orbit $(f^{-n}(x_n), f^{-n+1}(x_n), \ldots, x_n)$ is $\frac{\delta}{\lambda^n-1}$ close to it (remember that by our assumptions $f$ is a bijection). Indeed, by induction we get

$$\| x_k - f^{-n+k}(x_n) \| \leq \frac{\delta}{\lambda} + \frac{\delta}{\lambda^2} + \cdots + \frac{\delta}{\lambda^{n-k}} < \frac{\delta}{\lambda^{n-1}}.$$

In particular, we get $\| f^{-n}(x_n) \| < \frac{\delta}{\lambda^n-1}$, so $K_n \subset f^n(B(\frac{\delta}{\lambda^n-1}))$. The set $f^n(B(\frac{\delta}{\lambda^n-1}))$ is a $q$-dimensional ellipsoid of volume $\Lambda^n \text{Vol}(B(\frac{\delta}{\lambda^n-1}))$. This ellipsoid is contained in a $q$-dimensional box $A_n$ of volume $C_1 \Lambda^n$, where the constant $C_1$ does not depend on $n$. The thickness of $A_n$ (the minimal length of its edges) is at least $C_2 \lambda^n$ for some constant $C_2 > 0$ independent of $n$. Therefore for a constant $S > 0$ the set $A_n$ can be covered by $C_3 \Lambda^n$ subsets of diameter smaller than $S$, where $C_3$ does not depend on $n$. Consequently, we can find sets $E_n$ which are $S$-dense in $K_n$ and $|E_n| \leq C_3 \Lambda^n$.

If $(x_0, x_1, \ldots, x_n)$ and $(y_0, y_1, \ldots, y_n)$ are $\delta$-pseudoorbits and $\| x_0 - y_0 \| \leq S$, then the distances between $(x_0, x_1, \ldots, x_n)$ and $(f^{-n}(x_n), f^{-n+1}(x_n), \ldots, x_n)$, and between $(y_0, y_1, \ldots, y_n)$ and $(f^{-n}(y_n), f^{-n+1}(y_n), \ldots, y_n)$, are smaller than $\frac{\delta}{\lambda^n-1}$, while the distance between $(f^{-n}(x_n), f^{-n+1}(x_n), \ldots, x_n)$ and $(f^{-n}(y_n), f^{-n+1}(y_n), \ldots, y_n)$ is not larger than $S$. Thus, the distance between $(x_0, x_1, \ldots, x_n)$ and $(y_0, y_1, \ldots, y_n)$ is smaller than $\frac{\delta}{\lambda^n-1} + S$. Therefore, if for each element $x \in E_n$ we choose one $\delta$-pseudoorbit from $\mathcal{O}_n$ whose final term is $x$, we get an $R$-spanning subset in $\mathcal{O}_n$ of cardinality not larger than $C_3 \Lambda^n$, where $R = \frac{2\delta}{\lambda^n-1} + S$. This gives us the inequality $h_\infty(f) \leq \log \Lambda$. □

Lemma 4.2. If $f : \mathbb{R}^q \to \mathbb{R}^q$ is a Lipschitz continuous map with Lipschitz constant $\lambda > 1$ then $h_\infty(f) \leq q \log \lambda$.

Proof. Fix $\delta > 0$ and set $S = \frac{2\delta}{\lambda^n-1}$. Then fix a large integer $m$ and $R > 2S \lambda^m$. If $(x_0, x_1, \ldots, x_m)$ and $(y_0, y_1, \ldots, y_m)$ are $\delta$-pseudoorbits then by induction we see that for $i = 0, 1, \ldots, m$

$$\| x_i - y_i \| \leq \lambda^i \| x_0 - y_0 \| + 2\delta \frac{\lambda^i - 1}{\lambda - 1} < \lambda^m (\| x_0 - y_0 \| + S),$$

so in particular, if $\| x_0 - y_0 \| \leq S$, then $\| x_i - y_i \| \leq 2S \lambda^m$.

There is a constant $C$ such for every $\alpha > \beta > 0$ every subset of $\mathbb{R}^q$ of diameter less than $\alpha$ can be partitioned into less than $C(\alpha/\beta)^q$ subsets of diameter less than
β. Using this, we can define by induction for each \( j = 1, 2, \ldots, k \) a family \( A_j \) of sets of diameter less than \( S \), such that for every \( \delta \)-pseudoorbit \( (x_0, x_1, \ldots, x_{km}) \) of \( f \), where \( x_0 \) is the origin, \( x_{jm} \) belongs to exactly one element \( B \in A_j \), and then there are less than \( C(2S\lambda^m/S)^q = C2^q\lambda^{mq} \) elements of \( A_{j+1} \) to which \( x_{(j+1)m} \) can belong. Specifying the elements of \( A_1, A_2, \ldots, A_k \) to which the corresponding terms of our \( \delta \)-pseudoorbits belong, gives us a set of \( \delta \)-pseudoorbits of diameter less than \( 2S\lambda^m < R \). The number of such sets is at most \((C2^q\lambda^{mq})^k \). Thus,

\[
\frac{1}{km} \log r(f, km, R, \delta, x_0) \leq \frac{1}{km} \log(C2^q\lambda^{mq})^k = \frac{1}{m} \log(C2^q\lambda^{mq}) = \frac{1}{m} \log(C2^q) + q \log \lambda.
\]

By the same argument as in Example 3.9, we can replace \( km \) with any \( n \) and pass to the limit with \( n \), obtaining

\[
\limsup_{n \to \infty} \frac{1}{n} \log r(f, n, R, \delta, x_0) \leq \frac{1}{m} \log(C2^q) + q \log \lambda.
\]

As we take the limit of the left-hand side of the above inequality as \( R \to \infty \), we can assume that \( m \to \infty \), since the only condition for \( m \) is that \( R > 2S\lambda^m \). After taking the last limit, as \( \delta \to \infty \), we get \( h_\infty(f) \leq q \log \Lambda \).

**Theorem 4.3.** If \( f : \mathbb{R}^q \to \mathbb{R}^q \) is a linear map, then \( h_\infty(f) = \log \Lambda \), where \( \Lambda \) is the absolute value of the product of all eigenvalues of \( f \) that have absolute value larger than 1.

**Proof.** By changing the metric in \( \mathbb{R}^q \), we may consider \( \mathbb{R}^q \) as the product of two Euclidean spaces: \( X \) corresponding to the eigenvalues of \( f \) with absolute values larger than 1, and \( Y \) corresponding to the eigenvalues of \( f \) with absolute values less than or equal to 1. In this model, \( f = g \times h \), where \( g : X \to X \) is a linear map with all eigenvalues of absolute value larger than 1 and the absolute value of the determinant equal to \( \Lambda \), and \( h : Y \to Y \) is a linear map with all eigenvalues of absolute value smaller than or equal to 1.

By Lemma 4.1, \( h_\infty(g) = \log \Lambda \). To find the coarse entropy of \( h \), note that for every \( \varepsilon > 0 \) we can further change the metric in \( Y \) in such a way that \( h \) is Lipschitz continuous with Lipschitz constant \( 1 + \varepsilon \). Then, by Lemma 4.2, we get \( h_\infty(h) \leq q \log(1 + \varepsilon) \). Since \( \varepsilon \) is arbitrary, we get \( h_\infty(h) = 0 \).

Now, by Theorem 3.8 we get \( h_\infty(f) \leq \log \Lambda \), and by Theorem 3.1 we get \( h_\infty(f) \geq \log \Lambda \). Thus, \( h_\infty(f) = \log \Lambda \).

Let us consider another interesting example, where we can express the coarse entropy in terms of the properties of the map and the phase space. Let us recall the notion of the box-counting dimension (or rather ball-counting dimension, but in our case it will be the same) of a bounded space \( X \). It is equal to

\[
BCD(X) = \lim_{\varepsilon \to 0} \frac{\log r(X, \varepsilon)}{-\log \varepsilon}
\]

(if the limit exists), where \( r(X, \varepsilon) \) is the minimum cardinality of any \( \varepsilon \)-spanning subset of \( X \).

**Example 4.4.** Let \( S^{q-1} \) be the unit sphere in \( \mathbb{R}^q \). Let \( A \subset S^{q-1} \) be a set having box-counting dimension. Set

\[
X = \{ tx \in \mathbb{R}^q : t \geq 0, \ x \in A \}.
\]
Take \( \lambda > 1 \) and define \( f : X \to X \) by \( f(x) = \lambda x \). We will show that

\[
h_\infty(f) = (\text{BCD}(A) + 1) \log \lambda.
\]

Set \( \hat{A} = \{tx \in \mathbb{R}^q : 0 \leq t \leq 1, \ x \in A\} \). We will start by showing that \( \text{BCD}(\hat{A}) = \text{BCD}(A) + 1 \).

Let \( E \) be an \( \varepsilon \)-spanning set in \( A \) and \( D \) an \( \varepsilon \)-spanning set in \([0,1]\). Then \( \{tx : t \in D, \ x \in E\} \) is a \( 2\varepsilon \)-spanning set in \( \hat{A} \). Therefore

\[
\limsup_{\varepsilon \to 0} \frac{\log r(\hat{A}, 2\varepsilon)}{-\log \varepsilon} \leq \text{BCD}(A) + \text{BCD}([0,1]) = \text{BCD}(A) + 1.
\]

For \( t \in [1/2,1] \), let \( F_t \) be the projection to the sphere \( S_t \) of radius \( t \) centered at the origin: \( F_t(y) = t \frac{y}{\|y\|}. \) Set \( E_t = \{y \in E : t - \varepsilon < \|y\| < t + \varepsilon \}. \) If \( \varepsilon \) is sufficiently small and \( t \in [1/2,1] \), then whenever \( \|y\| - \|x\| < \varepsilon, \|x\| = t, \) and \( \|y - x\| < \varepsilon, \) then \( \|F_t(y) - x\| < 2\varepsilon \). Thus, \( |E| \geq r(S_t \cap A, 2\varepsilon) = r(A, 2\varepsilon/t) \geq r(A, 4\varepsilon) \). Dividing \([1/2,1]\) into \( m \) intervals of length larger than \( 2\varepsilon \) and considering as \( t \) the centers of those intervals, we see that \( |E| \geq mr(A, 4\varepsilon) \). We can take \( m > 1/(3\varepsilon) \), so

\[
\liminf_{\varepsilon \to 0} \frac{\log r(\hat{A}, \varepsilon)}{-\log \varepsilon} \geq \lim_{\varepsilon \to 0} \frac{\log r(A, 4\varepsilon) - \log(3\varepsilon)}{-\log \varepsilon} = \text{BCD}(A) + 1.
\]

Together with (2), we get \( \text{BCD}(\hat{A}) = \text{BCD}(A) + 1 \).

We have to prove that \( h_\infty(f) = \text{BCD}(\hat{A}) \). We will use the same method as in the proof of Lemma 4.1 and we will use terminology and some results from this proof.

If \( E \) is an \( R \)-spanning set in \( O_n \), then the set of final terms of \( E \) has to \( R \)-span \( f^{n-1}(B(\delta) \cap X) = B(\lambda^{n-1} \delta) \cap X \). However, covering \( B(\lambda^{n-1} \delta) \cap X \) with balls of radius \( R \) is the same as covering \( \hat{A} \) with balls of radius \( R/(\lambda^{n-1} \delta) \). Thus,

\[
r(f, n, R, \delta, x_0) \geq r \left( \hat{A}, \frac{R}{\lambda^{n-1} \delta} \right).
\]

We have

\[
\lim_{n \to \infty} \frac{1}{n} \log r \left( \hat{A}, \frac{R}{\lambda^{n-1} \delta} \right) = \lim_{n \to \infty} \frac{\log r \left( \hat{A}, \frac{R}{\lambda^{n-1} \delta} \right)}{-\log \frac{R}{\lambda^{n-1} \delta}} \left( \frac{1}{n} \log \frac{\delta}{R} + \frac{n - 1}{n} \log \lambda \right) = \text{BCD}(\hat{A}) \log \lambda.
\]

Therefore, \( h_\infty(f) \geq \text{BCD}(\hat{A}) \log \lambda \).

To get the opposite inequality, we use the fact that \( K_n \subset f^n \left( B\left(\frac{\delta}{\lambda^{n-1}}\right) \cap X \right) = B\left(\frac{\delta \lambda^n}{\lambda^{n-1}}\right) \cap X \). Covering \( B\left(\frac{\delta \lambda^n}{\lambda^{n-1}}\right) \cap X \) with balls of radius \( S \) is the same as covering \( \hat{A} \) with balls of radius \( S(\lambda - 1)/(\delta \lambda^n) \). Taking \( S = R - \frac{2\delta}{\lambda^{n-1}} \), we get an \( R \)-spanning subset in \( O_n \) of cardinality not larger than

\[
r \left( \hat{A}, \frac{S(\lambda - 1)}{\delta \lambda^n} \right) = r \left( \hat{A}, \frac{R(\lambda - 1) - 2\delta}{\delta \lambda^n} \right),
\]
Hence,
\[
\limsup_{n \to \infty} \frac{1}{n} \log r(f, n, R, \delta, x_0) \leq \limsup_{n \to \infty} \frac{1}{n} \log r(\hat{A}, R(\lambda - 1) - 2\delta) \frac{\delta \lambda^n}{R(\lambda - 1) - 2\delta} = \text{BCD}(\hat{A}) \lim_{n \to \infty} \frac{1}{n} \log \delta \lambda^n = \text{BCD}(\hat{A}) \log \lambda.
\]
Therefore, \(h_\infty(f) \leq \text{BCD}(\hat{A}) \log \lambda\), so \(h_\infty(f) = \text{BCD}(\hat{A}) \log \lambda\).

\[\diamondsuit\]

5. Entropy of the identity map

It seems unavoidable that whatever reasonable definition of the coarse entropy we try, if the space is large enough, then the entropy of the identity is positive (or even infinite). Here “large enough” basically means that the dimension is infinite.

**Example 5.1.** Let \(X\) be the space \(l_\infty\) of bounded real sequences, with the sup norm, and let \(f : X \to X\) be the identity map. Fix \(\delta, R > 0\). Let \(x_0\) be the zero sequence. If \(n \geq R/\delta\) then for every \(k\) there exists a \(\delta\)-pseudoorbit of length \(n\) starting at \(x_0\) and ending at the sequence whose only non-zero term is the \(k\)th one, and it is equal to \(R\). The set of those \(\delta\)-pseudoorbits is an \(R\)-separated set of cardinality infinity. This proves that \(h_\infty(f) = \infty\).

\[\diamondsuit\]

The above example and easily constructed similar ones are based on the property of the space \(X\) that for every \(R\) there are bounded sets with \(R\)-separated infinite subsets. However, there is an example of a space where the closure of every bounded set is compact, so every \(R\)-separated subset of a bounded set is finite, but nevertheless the identity has infinite coarse entropy.

**Example 5.2.** Let \(X\) be the half-line \([0, \infty)\) with the space \(\mathbb{R}^{2^k}\) attached at every integer \(k\) (with the origin on our half-line). The metric in \(X\) is “along the space”, so for example if \(x \in \mathbb{R}^k\) and \(y \in \mathbb{R}^l\) with \(k \neq l\), then \(d(x, y) = \|x\| + |l - k| + \|y\|\). Let \(f\) be the identity on \(X\).

Fix \(\delta\) and \(R\), and let \(x_0\) be the point 0 on our half-line. If \(n\) is large, look at the \(\delta\)-pseudoorbits from \(x_0\) that first go with step \(\delta\) along the half-line, and when they reach \(k = k(n) = \lfloor \delta n - 2R \rfloor\), they start spreading out in \(\mathbb{R}^{2^k}\). Their final distance from \(x_0\) is \(\delta n\) or anything less, so their final distance from the origin in \(\mathbb{R}^{2^k}\) is approximately \(2R\) or anything less. Thus, among the final points on those pseudoorbits are in particular all points in \(\mathbb{R}^{2^k}\) of the form \((0, \ldots, 0, R, 0, \ldots, 0)\). They form an \(R\)-separated set and there are \(2^k\) of them. This means that \(s(n, \delta, R, x_0) \geq 2^{k(n)}\). We get
\[
\limsup_{n \to \infty} \frac{1}{n} \log s(n, \delta, R, x_0) \geq \limsup_{n \to \infty} \frac{1}{n}(\delta n - 2R) \log 2 = \delta \log 2.
\]
Therefore, \(h_\infty(f) = \infty\).

\[\diamondsuit\]

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