Second-order optimality conditions for nonlinear programs and mathematical programs

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Abstract

It is well known that second-order information is a basic tool notably in optimality conditions and numerical algorithms. In this work, we present a generalization of optimality conditions to strongly convex functions of order $\gamma$ with the help of first- and second-order approximations derived from [Optimization 40(3):229-246, 2011] and we study their characterization. Further, we give an example of such a function that arises quite naturally in nonlinear analysis and optimization. An extension of Newton’s method is also given and proved to solve Euler equation with second-order approximation data.

Keywords: strong convexity of order $\gamma$; second-order approximation; $C^{1,1}$ functions; Newton’s method

1 Introduction

The concept of approximations of mappings was introduced by Thibault [2]. Sweetser [3] considered approximations by subsets of the space of continuous linear maps $L(X, Y)$, where $X$ and $Y$ are Banach spaces, and Ioffe [4] by the so-called fans. This approach was revised by Jourani and Thibault [5]. Another approach belongs to Allali and Amahroq [1]. Following the same ideas, Amahroq and Gadhi [6, 7] have established optimality conditions to some optimization problems under set-valued mapping constraints.

In this work, we explore the notion of strongly convex functions of order $\gamma$; see, for instance, [8–15] and references therein. Let $f$ be a mapping from a Banach space $X$ into $\mathbb{R}$, and let $C \subset X$ be a closed convex set. It is well known that the notion of strong convexity plays a central role. On the one hand, it ensures the existence and uniqueness of the optimal solution for the problem

$$\min_{x \in C} f(x).$$

On the other hand, if $f$ is twice differentiable, then the strong convexity of $f$ implies that its Hessian matrix is nonsingular, which is an important tool in numerical algorithms. Here we adopt the definition of a second-order approximation [1] to detect some equivalent properties of strongly convex functions of order $\gamma$ and to characterize the latter. Further-
more, for a $C^{1,1}$ function $f$ on a finite-dimensional setting, we show some simple facts. We also provide an extension of Newton’s method to solve an Euler equation with second-order approximation data.

The rest of the paper is written as follows. Section 2 contains basic definitions and preliminary results. Section 3 is devoted to main results. In Section 4, we point out an extension of Newton’s method and prove its local convergence.

2 Preliminaries
Let $X$ and $Y$ be two Banach spaces. We denote by $\mathcal{L}(X, Y)$ the set of all continuous linear mappings from $X$ into $Y$, by $\mathcal{B}(X \times X, Y)$ the set of all continuous bilinear mappings from $X \times X$ into $Y$, and by $B_Y$ the closed unit ball of $Y$ centered at the origin.

Throughout this paper, $X^*$ and $Y^*$ denote the continuous duals of $X$ and $Y$, respectively, and we write $\langle \cdot, \cdot \rangle$ for the canonical bilinear forms with respect to the dualities $(X^*, X)$ and $(Y^*, Y)$.

Definition 1 ([1]) Let $f$ be a mapping from $X$ into $Y$, $\bar{x} \in X$. A set of mappings $A_f(\bar{x}) \subset \mathcal{L}(X, Y)$ is said to be a first-order approximation of $f$ at $\bar{x}$ if there exist $\delta > 0$ and a function $r : X \to \mathbb{R}$ satisfying $\lim_{x \to \bar{x}} r(x) = 0$ such that

$$f(x) - f(\bar{x}) \in A_f(\bar{x})(x - \bar{x}) + \|x - \bar{x}\| r(x) B_Y$$  \hspace{1cm} (1)

for all $x \in \bar{x} + \delta B_X$.

It is easy to check that Definition 1 is equivalent to the following: for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$f(x) - f(\bar{x}) \in A_f(\bar{x})(x - \bar{x}) + \epsilon \|x - \bar{x}\| B_Y$$  \hspace{1cm} (2)

for all $x \in \bar{x} + \delta B_X$.

Remark 1 If $A_f(\bar{x})$ is a first-order approximation of $f$ at $\bar{x}$, then (2) means that for any $x \in \bar{x} + \delta B_X$, there exist $A(x) \in A_f(\bar{x})$ and $b \in B_Y$ such that

$$f(x) - f(\bar{x}) = A(x)(x - \bar{x}) + \epsilon \|x - \bar{x}\| b.$$

Hence, for any $x \in B(\bar{x}, \delta)$ and $A(x) \in A_f(\bar{x})$,

$$\|f(x) - f(\bar{x}) - A(x)(x - \bar{x})\| \leq \epsilon \|x - \bar{x}\|.$$  \hspace{1cm} (3)

If $A_f(\bar{x})$ is norm-bounded (resp. compact), then it is called a bounded (resp. compact) first-order approximation. Recall that $A_f(\bar{x})$ is a singleton if and only if $f$ is Fréchet differentiable at $\bar{x}$.

The following proposition proved by Allali and Amahroq [1] plays an important role in the sequel in a finite-dimensional setting.
**Proposition 1** ([1]) Let \( f : \mathbb{R}^p \rightarrow \mathbb{R} \) be a locally Lipschitz function at \( \bar{x} \). Then the Clarke subdifferential of \( f \) at \( \bar{x} \),

\[
\partial f(\bar{x}) := \text{co}\{\lim \nabla f(x_n) : x_n \in \text{dom} \nabla f \text{ and } x_n \rightarrow \bar{x}\},
\]

is a first-order approximation of \( f \) at \( \bar{x} \).

In [6], it is also shown that when \( f \) is a continuous function, it admits as an approximation the symmetric subdifferential defined and studied in [16].

The next proposition shows that Proposition 1 holds also when \( f \) is a vector-valued function. Let us first recall the definition of the generalized Jacobian for a vector-valued function (see [17, 18] for more details) and the definition of upper semicontinuity.

**Definition 2** The generalized Jacobian of a function \( g : \mathbb{R}^p \rightarrow \mathbb{R}^q \) at \( \bar{x} \), denoted \( \partial_x g(\bar{x}) \), is the convex hull of all matrices \( M \) of the form

\[
M = \lim_{n \rightarrow +\infty} Jg(x_n),
\]

where \( x_n \rightarrow \bar{x} \), \( g \) is differentiable at \( x_n \) for all \( n \), and \( Jg \) denotes the \( q \times p \) usual Jacobian matrix of partial derivatives.

**Definition 3** A set-valued mapping \( F : \mathbb{R}^p \rightrightarrows \mathbb{R}^q \) is said to be upper semicontinuous at a point \( \bar{x} \in \mathbb{R}^p \) if, for every \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that

\[
F(x) \subset F(\bar{x}) + \varepsilon B
\]

for every \( x \in \mathbb{R}^p \) such that \( \|x - \bar{x}\| < \delta \).

**Proposition 2** Let \( g : \mathbb{R}^p \rightarrow \mathbb{R}^q \) be a locally Lipschitz function at \( \bar{x} \). Then the generalized Jacobian \( \partial_x g(\bar{x}) \) of \( g \) at \( \bar{x} \) is a first-order approximation of \( g \) at \( \bar{x} \).

**Proof** Since the set-valued mapping \( \partial_x g(\cdot) \) is upper semicontinuous, for all \( \varepsilon > 0 \), there exists \( r_0 > 0 \) such that

\[
\partial_x g(x) \subset \partial_x g(\bar{x}) + \varepsilon B_{L(\mathbb{R}^p,\mathbb{R}^q)} \quad \text{for all } x \in \bar{x} + r_0 B_{\mathbb{R}^p}.
\]

We may assume that \( g \) is Lipschitzian in \( \bar{x} + r_0 B_{\mathbb{R}^p} \). Let \( x \in \bar{x} + r_0 B_{\mathbb{R}^p} \). We apply [17], Prop. 2.6.5, to derive that there exits \( c \in ]x,\bar{x}[ \) such that

\[
g(x) - g(\bar{x}) \in \partial_x g(c)(x - \bar{x}) \subset \partial_x g(\bar{x})(x - \bar{x}) + \varepsilon B_{L(\mathbb{R}^p,\mathbb{R}^q)}(x - \bar{x}).
\]

Since

\[
B_{L(\mathbb{R}^p,\mathbb{R}^q)}(x - \bar{x}) \subset \|x - \bar{x}\| B_{\mathbb{R}^q},
\]

we have

\[
g(x) - g(\bar{x}) \in \partial_x g(\bar{x})(x - \bar{x}) + \varepsilon \|x - \bar{x}\| B_{\mathbb{R}^q},
\]

which means that \( \partial_x g(\bar{x}) \) is a first-order approximation of \( g \) at \( \bar{x} \).
Recall that a mapping $f : X \to Y$ is said to be $C^{1,1}$ at $\bar{x}$ if it is Fréchet differentiable in neighborhood of $\bar{x}$ and if its Fréchet derivative $\nabla f(\cdot)$ is Lipschitz at $\bar{x}$.

Let $\bar{x} \in \mathbb{R}^p$, and let $f : \mathbb{R}^p \to \mathbb{R}$ be a $C^{1,1}$ function at $\bar{x}$. The generalized Hessian matrix of $f$ at $\bar{x}$ was introduced and studied by Hiriart-Urruty et al. [19] is the compact nonempty convex set

$$\partial^2_{\bar{x}}f(\bar{x}) := \co \{ \lim \nabla^2 f(x_n) : (x_n) \in \dom \nabla^2 f \text{ and } x_n \to \bar{x} \}, \quad (5)$$

where $\dom \nabla^2 f$ is the effective domain of $\nabla^2 f(\cdot)$.

**Corollary 1** Let $\bar{x} \in \mathbb{R}^p$, and $f : \mathbb{R}^p \to \mathbb{R}$ be a $C^{1,1}$ function at $\bar{x}$. Then, $\nabla f$ admits $\partial^2_{\bar{x}}f(\bar{x})$ as a first-order approximation at $\bar{x}$.

**Definition 4** ([1]) We say that $f : X \to Y$ admits a second-order approximation at $\bar{x}$ if there exist two sets $A_f(\bar{x}) \subset L(X, Y)$ and $B_f(\bar{x}) \subset B(X \times X, Y)$ such that

(i) $A_f(\bar{x})$ is a first-order approximation of $f$ at $\bar{x}$;
(ii) For all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$f(x) - f(\bar{x}) \in A_f(\bar{x})(x - \bar{x}) + B_f(\bar{x})(x - \bar{x})(x - \bar{x}) + \varepsilon \|x - \bar{x}\|^2 \mathbb{B}_Y$$

for all $x \in \bar{x} + \delta \mathbb{B}_X$.

In this case the pair $(A_f(\bar{x}), B_f(\bar{x}))$ is called a second-order approximation of $f$ at $\bar{x}$. It is called a compact second-order approximation if $A_f(\bar{x})$ and $B_f(\bar{x})$ are compacts.

Every $C^2$ mapping $f : X \to Y$ at $\bar{x}$ admits $(\nabla f(\bar{x}), \nabla^2 f(\bar{x}))$ as a second-order approximation, where $\nabla f(\bar{x})$ and $\nabla^2 f(\bar{x})$ are, respectively, the first- and second-order Fréchet derivatives of $f$ at $\bar{x}$.

**Proposition 3** ([1]) Let $f : \mathbb{R}^p \to \mathbb{R}$ be a $C^{1,1}$ function at $\bar{x}$. Then $f$ admits $(\nabla f(\bar{x}), \frac{1}{2} \partial^2_{\bar{x}}f(\bar{x}))$ as a second-order approximation at $\bar{x}$.

**Proposition 4** Let $f : X \to Y$ be a Fréchet-differentiable mapping. If $(\nabla f(\bar{x}), B_f(\bar{x}))$ is a bounded second-order approximation of $f$ at $\bar{x}$, then $\nabla f(\cdot)$ is stable at $\bar{x}$, that is, there exist $c, r > 0$ such that

$$\|\nabla f(x) - \nabla f(\bar{x})\| \leq c\|x - \bar{x}\|$$

for all $x \in \bar{x} + r \mathbb{B}_X$.

To derive some results for $\gamma$-strong convex functions, the following notions are needed.

**Definition 5** ([8]) Let $\gamma > 0$. We say that a map $f : X \to \mathbb{R} \cup \{+\infty\}$ is $\gamma$-strongly convex if there exist $c \geq 0$ and $g : [0, 1] \to \mathbb{R}^+$ satisfying

$$g(0) = g(1) = 0 \quad \text{and} \quad \lim_{\theta \to 0} \frac{g(\theta)}{\theta} = 1$$

(7)
and such that
\[ f(\theta x + (1 - \theta) y) \leq \theta f(x) + (1 - \theta) f(y) - c \|x - y\|^\gamma \tag{8} \]
for all \( \theta \in [0, 1] \) and \( x, y \in X \).

Of course, when \( c = 0, f \) is called a convex function. Otherwise, \( f \) is said \( \gamma \)-strongly convex. This class has been introduced by Polyak [11] when \( \gamma = 2 \) and \( g(\theta) = \theta (1 - \theta) \) and studied by many authors. Recently, a characterization of \( \gamma \)-strongly convex functions has been shown in [8]. For example, if \( f \) is \( C^1 \) and \( \gamma \geq 1 \), then (8) is equivalent to
\[ \langle \nabla f(x), y - x \rangle \leq f(y) - f(x) - c \|x - y\|^\gamma, \quad \forall x, y \in X. \tag{9} \]

Let \( f : X \to \mathbb{R} \cup \{+\infty\} \) and \( \bar{x} \in \text{dom} f := \{x \in X, f(x) < +\infty\} \) (the effective domain of \( f \)). The Fenchel-subdifferential of \( f \) at \( \bar{x} \) is the set
\[ \partial_{\text{Fen}} f(\bar{x}) = \{ x^* \in X^* : \langle x^*, y - \bar{x} \rangle \leq f(y) - f(\bar{x}), \forall y \in X \}. \tag{10} \]

Let \( \gamma > 0 \) and \( c > 0 \). The \((\gamma, c)\)-subdifferential of \( f \) at \( \bar{x} \) is the set
\[ \partial_{(\gamma, c)} f(\bar{x}) = \{ x^* \in X^* : \langle x^*, y - \bar{x} \rangle \leq f(y) - f(\bar{x}) - c \|\bar{x} - y\|^\gamma, \forall y \in X \}. \tag{11} \]

For more details on \((\gamma, c)\)-subdifferential, see [8]. Note that if \( x \notin \text{dom} f \), then \( \partial_{(\gamma, c)} f(\bar{x}) = \partial_{\text{Fen}} f(\bar{x}) = \emptyset \). Clearly, we have \( \partial_{(\gamma, c)} f(\bar{x}) \subset \partial_{\text{Fen}} f(\bar{x}) \). Note that the Fenchel-subdifferential defined by (10) coincides with the Clarke subdifferential of \( f \) at \( \bar{x} \) if the function \( f \) is convex. We also need to recall the following definitions.

**Definition 6** ([20]) We say that a map \( f : X \to \mathbb{R} \cup \{+\infty\} \) is 2-paraconvex if there exists \( c > 0 \) such that
\[ f(\theta x + (1 - \theta) y) \leq \theta f(x) + (1 - \theta) f(y) + c \min(\theta, 1 - \theta) \|x - y\|^2 \tag{12} \]
for all \( \theta \in [0, 1] \) and \( x, y \in X \).

It has been proved in [20] that if \( f \) is a \( C^1 \) mapping, then (12) is equivalent to
\[ \langle \nabla f(x), y - x \rangle \leq f(y) - f(x) + c \|x - y\|^2, \quad \forall x, y \in X. \tag{13} \]

### 3 Main results

In this section, we obtain the main results of the paper related to strongly convex functions of order \( \gamma \) defined by (7)-(8). We begin by showing some interesting facts of functions that admit a first-order approximation.

For any subset \( A \) of \( X^* \), we define the support function of \( A \) as
\[ s(A, x) = \sup \{ \langle x^*, x \rangle : x^* \in A \}. \tag{14} \]
It is well known that, for any convex function \( f : X \to \mathbb{R} \cup \{+\infty\} \), the 'right-hand' directional derivative at \( x \) in \( \text{dom} f \) (the domain of \( f \)) exists and, for each \( h \in X \), is

\[
d^+ f(x)(h) = \lim_{t \to 0^+} \frac{f(x + th) - f(x)}{t}.
\]

**Theorem 1** Let \( x \in X \). If \( f : X \to \mathbb{R} \cup \{+\infty\} \) is convex and continuous at \( \bar{x} \) and if \( A_f(\bar{x}) \subset X^* \) is a convex \( w(X^*, X) \)-closed approximation of \( f \) at \( \bar{x} \), then

\[
\partial_{\gamma} f(\bar{x}) \subset A_f(\bar{x}).
\]

**Proof** By the definition of \( A_f(\bar{x}) \), there exist \( \delta > 0 \) and \( r : X \to \mathbb{R} \) with \( \lim_{x \to \bar{x}} r(x) = 0 \) such that, for all \( x \in \bar{x} + \delta B_X \), \( t \in [0, \delta] \), and \( h \in X \), there exist \( A \in A_f(\bar{x}) \) and \( b \in [-1, 1] \) satisfying

\[
f(\bar{x} + th) - f(\bar{x}) - \|h\| r(\bar{x} + th)b = \langle A, h \rangle \leq s(\gamma; A_f(\bar{x}); h).
\]

By letting \( t \to 0^+ \), the directional derivative of \( f \) at \( \bar{x} \) satisfies

\[
d^+ f(\bar{x})(h) \leq s(\gamma; A_f(\bar{x}); h), \quad \forall h \in X.
\]

(15)

Using [21], Prop. 2.24, we get

\[
s(\partial_{\gamma} f(\bar{x}); h) \leq s(\gamma; A_f(\bar{x}); h).
\]

Since \( \partial_{\gamma} f(\bar{x}) \subset \partial_{\gamma} f(\bar{x}) \), we deduce that

\[
s(\partial_{\gamma} f(\bar{x}); h) \leq s(\gamma; A_f(\bar{x}); h).
\]

Hence we conclude that \( \partial_{\gamma} f(\bar{x}) \subset A_f(\bar{x}) \). \( \square \)

**Proposition 5** Let \( f : X \to \mathbb{R} \cup \{+\infty\} \) be a \( \gamma \)-strongly convex function. Assume that \( A_f(\bar{x}) \) is a compact approximation at \( \bar{x} \). Then \( A_f(\bar{x}) \cap \partial_{\gamma} f(\bar{x}) \neq \emptyset \).

**Proof** Let \( d \in X \) be fixed and define \( x_n := \bar{x} + \frac{1}{n} d \). Using Definition 1, we get, for \( n \) large enough, \( A_n \in A_f(\bar{x}) \) and \( b_n \in [-1, 1] \) such that

\[
\frac{1}{n} \langle A_n, d \rangle = f(\bar{x} + \frac{1}{n} d) - f(\bar{x}) - \frac{1}{n} \|d\| r(x_n)b_n.
\]

By \( \gamma \)-strong convexity we obtain

\[
\frac{1}{n} \langle A_n, d \rangle \leq \frac{1}{n} (f(\bar{x} + d) - f(\bar{x})) - c \|d\|^\gamma - \frac{1}{n} \|d\| r(x_n)b_n.
\]

By the compactness of \( A_f(\bar{x}) \), extracting a subsequence if necessary, we may assume that there exists \( A \in A_f(\bar{x}) \) such that \( \langle A_n, d \rangle \to \langle A, d \rangle \); and hence we obtain

\[
\langle A, d \rangle \leq f(\bar{x} + d) - f(\bar{x}) - c \|d\|^\gamma.
\]
Assume that $A \in \mathcal{A}_f(\bar{x}) \cap \partial_{\gamma,x} f(\bar{x})$. By the separation theorem there exists $h \in X$ with $\|h\| = 1$ such that

$$\min_{A \in \mathcal{A}_f(\bar{x})} \langle A, h \rangle > \sup_{x^* \in \partial_{\gamma,x} f(\bar{x})} \{ x^*, h \}.$$ 

Let $t > 0$ sufficiently small, so that

$$\min_{A \in \mathcal{A}_f(\bar{x})} \langle A, h \rangle > \frac{f(\bar{x} + th) - f(\bar{x})}{t},$$

in contradiction with relation (16) by taking $d = th$. \qed

Following a result by Rademacher, which states that a locally Lipschitzian function between finite-dimensional spaces is differentiable (Lebesgue) almost everywhere, we can prove the following result.

**Proposition 6** Let $\gamma \geq 1$, $\bar{x} \in \mathbb{R}^p$, and let $f : \mathbb{R}^p \to \mathbb{R}$ be continuous at $\bar{x}$. Assume that $f$ is a $\gamma$-strongly convex function. Then $\partial f(\bar{x}) = \partial_{\gamma,x} f(\bar{x})$.

**Proof** Obviously, we have $\partial_{\gamma,x} f(\bar{x}) \subset \partial f(\bar{x})$. Now let $A \in \partial f(\bar{x})$. For all $n$, there exists $x_n \in \text{dom} \nabla f$ such that $x_n \to \bar{x}$ and $\nabla f(x_n) \to A$. Since $f$ is $\gamma$-strongly convex and Fréchet differentiable at $x_n$ for all $n \in \mathbb{N}$, it follows by (9) that

$$\langle \nabla f(x_n), y - x_n \rangle \leq f(y) - f(x_n) - c\|y - x_n\|^\gamma, \quad \forall y \in \mathbb{R}^p, \forall n \in \mathbb{N}.$$ 

Letting $n \to +\infty$, we get

$$\langle A, y - \bar{x} \rangle \leq f(y) - f(\bar{x}) - c\|y - \bar{x}\|^\gamma, \quad \forall y \in \mathbb{R}^p,$$

which means that $\partial f(\bar{x}) \subset \partial_{\gamma,x} f(\bar{x})$. \qed

**Corollary 2** Let $\gamma \geq 1$, $\bar{x} \in \mathbb{R}^p$, and let $f : \mathbb{R}^p \to \mathbb{R}$ be continuous at $\bar{x}$. Assume that $f$ is a $\gamma$-strongly convex function. Then, for all $\varepsilon > 0$, there exists $r > 0$ such that

$$f(x) - f(\bar{x}) \in \partial_{\gamma,x} f(\bar{x})(x - \bar{x}) + \varepsilon \|x - \bar{x}\| \mathbb{B}$$

for all $x \in \bar{x} + r\mathbb{B}$, which means that $\partial_{\gamma,x} f(\bar{x})$ is a first-order approximation of $f$ at $\bar{x}$.

**Proof** It is clear that $\partial_{\gamma,x} f(\bar{x})$ is a first-order approximation of at $\bar{x}$. We end the proof by Propositions 1 and 6. \qed

The converse of Proposition 5 holds if (16) is valid for any $A \in \mathcal{A}_f(x)$ and $x \in X$.

**Proposition 7** Let $\gamma \geq 1$ and $f : X \to \mathbb{R} \cup \{+\infty\}$. Assume that, for each $x \in X$, $f$ admits a first-order approximation $\mathcal{A}_f(x)$ such that $\mathcal{A}_f(x) \subset \partial_{\gamma,x} f(x)$. Then $f$ is $\gamma$-strongly convex.
Proof Define $x_0 := \theta u + (1 - \theta)v$ for $\theta \in [0, 1]$ and $u, v \in X$. Let us take $A \in \mathcal{A}_f(x_0)$. Then

$$
(A, u - x_0) \leq f(u) - f(x_0) - c\|u - x_0\|^\gamma.
$$

Multiplying this inequality by $\theta$, we obtain

$$(a') \quad \theta(1 - \theta)(A, u - v) \leq \theta f(u) - \theta f(x_0) - c(1 - \theta)^\gamma \theta \|u - v\|^\gamma.
$$

In a similar way, since

$$
(A, v - x_0) \leq f(v) - f(x_0) - c\|v - x_0\|^\gamma,
$$

we get

$$(a'') \quad -(1 - \theta)(A, u - v) \leq (1 - \theta)f(v) - (1 - \theta)f(x_0) - c(1 - \theta)^\gamma \|u - v\|^\gamma.
$$

We deduce by addition of $(a')$ and $(a'')$ that

$$
f(x_0) \leq \theta f(u) + (1 - \theta)f(v) - cg(\theta)\|u - v\|^\gamma
$$

for all $u, v \in X$, where $g(\theta) = (1 - \theta)^\theta + (1 - \theta)^{\gamma\theta}$, so that $f$ is $\gamma$-strongly convex.

The next results are devoted to presenting some useful properties of the generalized Hessian matrix for a $C^{1,1}$ function in the finite-dimensional setting and a characterization of $\gamma$-strongly convex functions with the help of a second-order approximation.

**Proposition 8** Let $\bar{x} \in X$, and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be convex and Fréchet differentiable at $\bar{x}$. Suppose that $f$ admits $(\nabla f(\bar{x}), B_\gamma(\bar{x}))$ as a second-order approximation at $\bar{x}$ and that $B_\gamma(\bar{x})$ is compact. Then there exists $B \in B_\gamma(\bar{x})$ such that

$$
\sup_{B \in B_\gamma(\bar{x})} \langle Bd, d \rangle \geq 0, \quad \forall d \in X.
$$

If $f$ is 2-strongly convex, then we obtain

$$
\sup_{B \in B_\gamma(\bar{x})} \langle Bd, d \rangle \geq c\|d\|^2, \quad \forall d \in X,
$$

for some $c > 0$.

*Proof* We prove only the case where $f$ is convex. In a similar way, we can prove the other case. Let $d \in X$ and $\varepsilon > 0$ be fixed. We get for $n$ large enough $B_n \in B_\gamma(\bar{x})$ and $b_n \in [-1, 1]$ such that

$$
f\left(\bar{x} + \frac{1}{n}d\right) - f(\bar{x}) = \frac{1}{n} \langle \nabla f(\bar{x}), d \rangle + \frac{1}{n^2} \langle B_n d, d \rangle + \varepsilon \frac{1}{n^2} \|d\|^2 b_n.
$$

Since $f$ is convex, we obtain

$$
\langle B_n d, d \rangle + \varepsilon \|d\|^2 b_n \geq 0.
$$
By the compactness of $B_f(\bar{x})$, extracting a subsequence if necessary, we may assume that there exits $B \in B_f(\bar{x})$ such that $B_n$ converges to $B$; therefore

$$\langle Bd, d \rangle \geq 0,$$

and hence

$$\sup_{B \in B_f(\bar{x})} \langle Bd, d \rangle \geq 0, \quad \forall d \in X. \quad \square$$

When $X$ is a finite-dimensional space, we get the following essential result.

**Proposition 9** Let $f : \mathbb{R}^p \to \mathbb{R}$ be a $C^{1,1}$ function at $\bar{x}$. Assume that $f$ is $\gamma$-strongly convex. Then, for any $B \in \partial^2_f(\bar{x})$, we have the following inequality:

$$\langle Bd, d \rangle \geq c\|d\|^{\gamma}, \quad \forall d \in \mathbb{R}^p,$$

for some $c > 0$.

**Proof** It is clear that $(\nabla f(\bar{x}), \frac{1}{2} \partial^2 f(\bar{x}))$ is a second-order approximation of $f$ at $\bar{x}$. Now let $B \in \partial^2_f(\bar{x})$, so that there exists a sequence $(x_n) \in \text{dom} \nabla^2 f$ such that $x_n \to \bar{x}$ and $\nabla^2 f(x_n) \to B$. Since $f$ is $\gamma$-strongly convex, there exists $c > 0$ such that

$$\langle \nabla^2 f(x_n)d, d \rangle \geq c\|d\|^{\gamma}, \quad \forall d \in \mathbb{R}^p, \forall n \in \mathbb{N}.$$ 

Letting $n \to +\infty$, we have

$$\langle Bd, d \rangle \geq c\|d\|^{\gamma}, \quad \forall d \in \mathbb{R}^p. \quad \square$$

The preceding result shows that $\gamma$-strongly convex functions enjoy a very desirable property for generalized Hessian matrices. In fact, in this case, any matrix $B \in \partial^2_f(\bar{x})$ is invertible. The next result proves the converse of Proposition 9. Let us first recall the following characterization of l.s.c. $\gamma$-strongly convex functions.

**Theorem 2** (Amahroq et al. [8]) Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper and l.s.c. function. Then $f$ is $\gamma$-strongly convex iff $\partial f$ is $\gamma$-strongly monotone, that is, there exists a positive real number $c$ such that, for all $x, y \in X, x^* \in \partial f(x)$, and $y^* \in \partial f(y)$, we have

$$\langle x^* - y^*, x - y \rangle \geq c\|x - y\|^{\gamma}.$$ 

We are now in position to state our main second result.

**Theorem 3** Let $f : \mathbb{R}^p \to \mathbb{R}$ be a $C^{1,1}$ function. Assume that $\partial^2_f(\cdot)$ satisfies relation (20) at any $x \in \mathbb{R}^p$. Then $f$ is $\gamma$-strongly convex.

**Proof** Let $t \in [0,1]$ and $u, v \in \mathbb{R}^p$. Define $\varphi : \mathbb{R} \to \mathbb{R}$ as

$$\varphi(t) := f(u + t(v - u)),$$
so that \( \psi'(t) := \langle \nabla f(u + t(v - u)), v - u \rangle \). By the Lebourg mean value theorem \([22]\) there exists \( t_0 \in ]0,1[ \) such that

\[
\psi'(1) - \psi'(0) \in \partial_c \psi'(t_0).
\]

By using calculus rules it follows that

\[
\psi'(1) - \psi'(0) \in \partial_c \psi'(t_0) \subseteq \partial^2 f(u + t_0(v - u))(v - u).
\]

Hence, there exists \( B_{t_0} \in \partial^2 f(u + t_0(v - u)) \) such that \( \langle \nabla f(v) - \nabla f(u), v - u \rangle = \langle B_{t_0}(v - u), v - u \rangle \). The result follows from Theorem 2. \( \square \)

Hiriart-Urruty et al. [19] have presented many examples of \( C^{1,1} \) functions. The next proposition shows another example of a \( C^{1,1} \) function.

**Theorem 4** Let \( f : H \to \mathbb{R} \) be continuous on a Hilbert space \( H \). Suppose that \( f \) is convex (or 2-strongly convex) and that \(-f\) is 2-paraconvex. Then \( f \) is Fréchet differentiable on \( H \), and for some \( c > 0 \), we have that

\[
\| \nabla f(x) - \nabla f(y) \| \leq c \| x - y \| \quad \text{for all } x, y \in H.
\]

**Proof** Let \( x_0 \in X \). Clearly, \( f \) is locally Lipschitzian at \( x_0 \). Now let \( x_1^* \) and \( x_2^* \) be arbitrary elements of \( \partial f(x_0) \) and \( \partial (-f)(x_0) \), respectively. By [20], Thm. 3.4, there exists \( c > 0 \) such that \( \partial (-f)(x_0) = \partial^{(2,0)}(-f)(x_0) \), and for any \( y \in H \) and positive real \( \theta \), we have

\[
(a) \quad \theta \langle x_2^*, y \rangle \leq -f(x_0 + \theta y) + f(x_0) + c\theta^2 \| y \|^2
\]

and

\[
(a') \quad \theta \langle x_1^*, y \rangle \leq f(x_0 + \theta y) - f(x_0).
\]

Adding (a) and (a'), we get

\[
\theta \langle x_1^* + x_2^*, y \rangle \leq c\theta^2 \| y \|^2,
\]

and hence

\[
\langle x_1^* + x_2^*, y \rangle \leq c\theta \| y \|^2.
\]

Letting \( \theta \to 0 \), we have \( \langle x_1^* + x_2^*, y \rangle \leq 0 \), so that \( x_1^* = -x_2^* \). Since \( x_1^* \) and \( x_2^* \) are arbitrary in \( \partial f(x_0) \) and \( \partial (-f)(x_0) \), it follows that \( \partial f(x_0) \) is single-valued. Put \( \partial f(x_0) = \{ p(x_0) \} \). Since (a) and (a') hold for any \( \theta > 0 \) and \( y \in H \), we deduce that, for \( \theta = 1 \),

\[
\langle p(x_0), y \rangle \leq f(x_0 + y) - f(x_0)
\]

and

\[
f(x_0 + y) - f(x_0) - \langle p(x_0), y \rangle \leq c \| y \|^2.
\]
Hence, for all $y \neq 0$, we obtain
\[
\frac{|f(x_0 + y) - f(x_0) - \langle p(x_0), y \rangle|}{\|y\|} \leq c\|y\|. \tag{22}
\]

Letting $\|y\| \to 0$ in (22), we conclude that $f$ is Fréchet differentiable at $x_0$. Now since $-f$ is 2-paraconvex and $f$ is Fréchet differentiable, we may prove that there exists $c > 0$ such that
\[
-\langle \nabla f(x), y - x \rangle \leq -f(y) + f(x) + c\|x - y\|^2 \quad \text{for all } x, y \in H. \tag{23}
\]

For every $z \in H$, we have that
\[
-f(z) \geq -f(x) + \langle \nabla f(x), x \rangle - \langle \nabla f(x), z \rangle - c\|x - z\|^2.
\]

Thus
\[
-f(z) \geq f^\ast(\langle \nabla f(x), z \rangle - c\|x - z\|^2),
\]

so that
\[
f^\ast(\langle \nabla f(y), z \rangle - f(z),
\]
\[
f^\ast(\langle \nabla f(y), z \rangle + f^\ast(\langle \nabla f(x), z \rangle - c\|x - z\|^2),
\]

and hence
\[
f^\ast(\langle \nabla f(y), z \rangle - \langle \nabla f(x) - \nabla f(x), x \rangle)
\]
\[
\geq \langle \nabla f(y) - \nabla f(x), z - x \rangle - c\|x - z\|^2
\]
\[
\geq \sup_{z \in H} \{\langle \nabla f(y) - \nabla f(x), z - x \rangle - c\|x - z\|^2\}.
\]

This means that, for all $x, y \in H$,
\[
f^\ast(\langle \nabla f(y), z \rangle - \langle \nabla f(x) - \nabla f(x), x \rangle \geq \frac{1}{2c} \|\nabla f(y) - \nabla f(x)\|^2.
\]

Changing the roles of $x$ and $y$, we obtain
\[
f^\ast(\langle \nabla f(x), z \rangle - \langle \nabla f(y) - \nabla f(y), y \rangle \geq \frac{1}{2c} \|\nabla f(x) - \nabla f(y)\|^2.
\]

So by addition we get
\[
\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \frac{1}{c} \|\nabla f(x) - \nabla f(y)\|^2. \tag{24}
\]

Consequently, by the Cauchy-Schwarz inequality we obtain
\[
\|\nabla f(x) - \nabla f(y)\| \leq c\|x - y\| \quad \text{for all } x, y \in H. \quad \blacksquare
\]
4 Newton’s method

The aim of this section is to solve the Euler equation

\[ \nabla f(x) = 0 \quad (25) \]

by Newton’s method. The classic assumption is that \( f : \mathbb{R}^p \to \mathbb{R} \) a \( C^2 \) mapping and the Hessian matrix \( \nabla^2 f(x) \) of \( f \) at \( x \) is nonsingular. Here we prove the convergence of a natural extension of Newton’s method to solve (25) assuming that \( \nabla f(\cdot) \) admits \( \beta f(\cdot) \) as a first-order approximation. Clearly, if \( f : \mathbb{R}^p \to \mathbb{R} \) is a \( C^{1,1} \) mapping, then using Corollary 1, we obtain that \( \nabla f(\cdot) \) admits \( \partial^2 f(\cdot) \) as a first-order approximation.

This algorithm has been proposed by Cominetti et al. [23] with \( C^{1,1} \) data. Only some ideas were given, but it remains as an open question to state results on rate of convergence and local convergence of that algorithm. In the sequel, \( f : \mathbb{R}^p \to \mathbb{R} \) is a Fréchet-differentiable mapping such that its Fréchet derivative admits a first-order approximation, and \( \bar{x} \) is a solution of (25).

Algorithm (\( M \)) Starting from an arbitrary point \( x_0 \in B(\bar{x}, r) \), generate the sequences \((x_k)\) and \((h_k)\) as follows:

(i) \( h_k \in \mathbb{R}^p \) is a solution of \( 0 = \nabla f(x_k) + \beta_f(\bar{x})(h_k) \), and
(ii) \( x_{k+1} = x_k + h_k \).

Theorem 5 Let \( f : \mathbb{R}^p \to \mathbb{R} \) be a Fréchet-differentiable function, and \( \bar{x} \) be a solution of (25). Let \( \varepsilon, r, K > 0 \) be such that \( \nabla f(\cdot) \) admits \( \beta_f(\bar{x}) \) as a first-order approximation at \( \bar{x} \) such that, for each \( x \in B_{\mathbb{R}^p}(\bar{x}, r) \), there exists an invertible element \( B(x) \in B_f(x) \) satisfying \( \|B(x)^{-1}\| \leq K \) and \( \xi := \varepsilon K < 1 \). Then the sequence \((x_k)\) generated by Algorithm (\( M \)) is well defined for every \( x_0 \in B_{\mathbb{R}^p}(\bar{x}, r) \) and converges linearly to \( \bar{x} \) with rate \( \xi \).

Proof Since \( \nabla f(\bar{x}) = 0 \), we have

\[ x_{k+1} - \bar{x} = B(x_k)^{-1}(\nabla f(\bar{x}) - \nabla f(x_k) + B(x_k)(x_k - \bar{x})). \]

We inductively obtain that

\[ \|x_{k+1} - \bar{x}\| \leq K \|\nabla f(\bar{x}) - \nabla f(x_k) + B(x_k)(x_k - \bar{x})\|. \]

Thus

\[ \|x_{k+1} - \bar{x}\| \leq \xi \|x_k - \bar{x}\|, \]

which means that \( x_{k+1} \in B_{\mathbb{R}^p}(\bar{x}, r) \), and we have \( \|x_{k+1} - \bar{x}\| \leq \xi^k \|x_0 - \bar{x}\| \). Therefore the whole sequence \((x_k)\) is well defined and converges to \( \bar{x} \).

Now let us consider the following algorithm under less assumptions.

Algorithm (\( M' \)) Starting from an arbitrary point \( x_0 \in \mathbb{R}^p \), generate the sequences \((x_k)\) and \((h_k)\) as follows:

(i) \( h_k \in \mathbb{R}^p \) is a solution of \( 0 = \nabla f(x_k) + \beta_f(x_0)(h_k) \), and
(ii) \( x_{k+1} = x_k + h_k \).
Theorem 6 Let $U$ be an open set of $\mathbb{R}^p$, $x_0 \in U$, and $f : \mathbb{R}^p \to \mathbb{R}$ be a Fréchet-differentiable function on $U$. Let $\varepsilon, r, K > 0$ be such that $\nabla f(x)$ admits $\beta_f(x_0)$ as a strict first-order approximation at $x_0$ such that, for each $x \in B_{\mathbb{R}^p}(x_0, r)$, there exists a right inverse of $B(x) \in \beta_f(x_0)$, denoted by $\tilde{B}(x)$, satisfying $\|B(x)(\cdot)\| \leq K \|\cdot\|$ and $\xi := \varepsilon K < 1$.

If $\|\nabla f(x_0)\| \leq K^{-1}(1 - \varepsilon) r$ and $\nabla f$ is continuous, then the sequence $(x_k)$ generated by Algorithm $(M')$ is well defined and converges to a solution $\bar{x}$ of (25). Moreover, we have $\|x_k - \bar{x}\| \leq r \xi^k$ for all $k \in \mathbb{N}$ and $\|\bar{x} - x_0\| \leq \|\nabla f(x_0)\|K(1 - \xi)^{-1} < r$.

Proof We prove by induction that $x_k \in x_0 + r B_{\mathbb{R}^p}$, $\|x_k - x_k\| \leq K \xi^k \|\nabla f(x_0)\|$, and $\|\nabla f(x_k)\| \leq \xi^k \|\nabla f(x_0)\|$ for all $k \in \mathbb{N}$. For $k = 0$, these relations are obvious. Assuming that they are valid for $k < n$, we get

$$
\|x_n - x_0\| \leq \sum_{k=0}^{n-1} \|x_{k+1} - x_k\| \leq K \|\nabla f(x_0)\| \sum_{k=0}^{n-1} \xi^k
$$

$$
\leq K \|\nabla f(x_0)\| (1 - \xi)^{-1} < r.
$$

Thus $x_n \in x_0 + r B_{\mathbb{R}^p}$ and since $\nabla f(x_{n-1}) + B(x_{n-1})(x_n - x_{n-1}) = 0$, from Algorithm $(M')$ we have

$$
\|\nabla f(x_n)\| \leq \|\nabla f(x_n) - \nabla f(x_{n-1}) - B(x_{n-1})(x_n - x_{n-1})\| \leq \varepsilon \|x_n - x_{n-1}\|
$$

$$
\leq \varepsilon \|x_n\| \|\nabla f(x_0)\|
$$

and

$$
\|x_{n+1} - x_n\| \leq K \xi^n \|\nabla f(x_0)\|.
$$

Since $\xi < 1$, the sequence $(x_n)$ is a Cauchy sequence and hence converges to some $\bar{x} \in \mathbb{R}^p$ with $\|x_0 - \bar{x}\| < r$. Since $\nabla f$ is a continuous function, we get $\nabla f(\bar{x}) = 0$. \qed

5 Conclusions

In this paper, we investigate the concept of first- and second-order approximations to generalize some results such as optimality conditions for a subclass of convex functions called strongly convex functions of order $\gamma$. We also present an extension of Newton’s method to solve the Euler equation under weak assumptions.

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