SHARP ADAMS TYPE INEQUALITIES IN SOBOLEV SPACES \\
$W^{m,\frac{n}{m}}(\mathbb{R}^n)$ FOR ARBITRARY INTEGER $m$ \\

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Abstract. The main purpose of our paper is to prove sharp Adams-type inequalities in 
unbounded domains of $\mathbb{R}^n$ for the Sobolev space $W^{m,\frac{n}{m}}(\mathbb{R}^n)$ for any positive integer $m$ 
less than $n$. Our results complement those of Ruf and Sani [28] where such inequalities 
are only established for even integer $m$. Our inequalities are also a generalization of the 
Adams-type inequalities in the special case $n = 2$ and stronger than 
those in [28] when $n = 2m$ for all positive integer $m$ by using different Sobolev norms.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a bounded domain. The Sobolev embedding theorems say 
that $W^{k,p}_0(\Omega) \subset L^q(\Omega)$, $1 \leq q \leq \frac{np}{n-kp}$, $kp < n$ and that $W^{k,\frac{n}{k}}_0(\Omega) \subset L^q(\Omega)$, $1 \leq q < \infty$. However, we can show by easy examples that $W^{k,\frac{n}{k}}_0(\Omega) \not\subset L^\infty(\Omega)$. In this case, 
Yudovich [32], Pohozaev [26] and Trudinger [31] independently showed that $W^{1,n}_0(\Omega) \subset L_{\varphi_n}(\Omega)$ where $L_{\varphi_n}(\Omega)$ is the Orlicz space associated with the Young function 
$\varphi_n(t) = \exp\left(\frac{|t|^{n/(n-1)}}{n}\right) - 1$. In his 1971 paper [25], J. Moser finds the largest positive real number 
$\beta_n = n\omega_n^{\frac{1}{n-1}}$, where $\omega_n$ is the area of the surface of the unit $n-$ball, such that if $\Omega$ is a 
domain with finite $n-$measure in Euclidean $n-$space $\mathbb{R}^n$, $n \geq 2$, then there is a constant $c_0$ depending only on $n$ such that 
$$\frac{1}{|\Omega|} \int_{\Omega} \exp\left(\beta |u|^\frac{n}{n-1}\right) \, dx \leq c_0$$
for any $\beta \leq \beta_n$, any $u \in W^{1,n}_0(\Omega)$ with $\int_{\Omega} |\nabla u|^n \, dx \leq 1$. Moreover, this constant $\beta_n$ is sharp in the meaning that if $\beta > \beta_n$, then the above inequality can no longer hold with some $c_0$ independent of $u$. Such an inequality is nowadays known as Moser-Trudinger type inequality.

Moser’s result for first order derivatives was extended to high order derivatives by 
D. Adams [2]. Indeed, Adams found the sharp constants for higher order Moser’s type 
inequality. To state Adams’ result, we use the symbol $\nabla^m u$, $m$ is a positive integer, 
to denote the $m-$th order gradient for $u \in C^m$, the class of $m-$th order differentiable
functions:
\[
\nabla^m u = \begin{cases} 
\Delta^m u & \text{for } m \text{ even} \\
\nabla \Delta^{m-1} u & \text{for } m \text{ odd}
\end{cases}
\]
where \(\nabla\) is the usual gradient operator and \(\Delta\) is the Laplacian. We use \(||\nabla^m u||_p\) to denote the \(L^p\) norm (1 \(\leq p \leq \infty\)) of the function \(|\nabla^m u|\), the usual Euclidean length of the vector \(\nabla^m u\). We also use \(W^{k,p}_0(\Omega)\) to denote the Sobolev space which is a completion of \(C^\infty_0(\Omega)\) under the norm of \(||u||_{L^p(\Omega)} + \sum_{j=1}^k ||\nabla^j u||_{L^p(\Omega)}\). Then Adams proved the following:

**Theorem A.** Let \(\Omega\) be an open and bounded set in \(\mathbb{R}^n\). If \(m\) is a positive integer less than \(n\), then there exists a constant \(C_0 = C(n, m) > 0\) such that for any \(u \in W^{m, n/m}_0(\Omega)\) and \(||\nabla^m u||_{L^n(\Omega)} \leq 1\), then
\[
\frac{1}{|\Omega|} \int_{\Omega} \exp(\beta |u(x)|^{\frac{n}{n-m}}) dx \leq C_0
\]
for all \(\beta \leq \beta(n, m)\) where
\[
\beta(n, m) = \begin{cases} 
\frac{n}{m-1} \left[ \frac{n^2}{m(m+1)} \right]^{\frac{n}{n-m}} & \text{when } m \text{ is odd} \\
\frac{n}{m-1} \left[ \frac{n^2}{m^2} \right]^{\frac{n}{n-m}} & \text{when } m \text{ is even}
\end{cases}
\]
Furthermore, for any \(\beta > \beta(n, m)\), the integral can be made as large as possible.

Note that \(\beta(n, 1)\) coincides with Moser’s value of \(\beta_n\) and \(\beta(2m, m) = 2^{2m} \pi^m \Gamma(m+1)\) for both odd and even \(m\).

The Adams inequality was extended recently by Tarsi [29]. More precisely, Tarsi used the Sobolev space with Navier boundary conditions \(W^{m, n/m}_N(\Omega)\) which contains the Sobolev space \(W^{m, n/m}_0(\Omega)\) as a closed subspace:

**Theorem B.** Let \(n > 2\) and \(\Omega \subset \mathbb{R}^n\) be a bounded domain. Then there exists a constant \(C_0 = C(n, m) > 0\) such that for any \(u \in W^{m, n/m}_N(\Omega)\) with \(||\nabla^m u||_{L^n(\Omega)} \leq 1\)
\[
\frac{1}{|\Omega|} \int_{\Omega} \exp(\beta |u(x)|^{\frac{n}{n-m}}) dx \leq C_0
\]
for all \(\beta \leq \beta(n, m)\). Furthermore, the constant \(\beta(n, m)\) is sharp in the sense that if \(\beta > \beta(n, m)\) then the supremum is infinite.

The Adams inequality was also extended to compact Riemannian manifolds without boundary by Fontana [17]. Also, the singular Moser-Trudinger inequalities and the singular Adams inequalities which are the combinations of the Hardy inequalities, Moser-Trudinger inequalities and Adams inequalities are established in [122].

The Moser-Trudinger’s inequality and Adams inequality play an essential role in geometric analysis and in the study of the exponential growth partial differential equations where, roughly speaking, the nonlinearity behaves like \(e^{a|u|^{\frac{n}{n-m}}}\) as \(|u| \to \infty\). Here we mention Atkinson-Peletier [9], Carleson-Chang [12], Adimurthi et al. [3, 4, 5, 6, 7, 8],...
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We notice that when \( \Omega \) has infinite volume, the Moser-Trudinger’s inequality and Adams inequality don’t make sense since the left hand side is trivial. The sharp Moser-Trudinger type inequality for the first order derivatives in the case \(|\Omega| = +\infty\) was obtained by B. Ruf [27] in dimension two and Y.X. Li-Ruf [23] in general dimension. In fact, such an inequality at the subcritical case was derived earlier by Cao [11] in dimension two and by Adachi and Tanaka in high dimensions [1]. Recently, Ruf and Sani proved the Adams type inequality for higher derivatives of even orders when \( \Omega \) has infinite volume.

Indeed, Ruf and Sani proved the following Adams type inequality (see [28]):

**Theorem C.** Let \( m \) be an even integer less than \( n \). There exists a constant \( C_{m,n} > 0 \) such that for any domain \( \Omega \subseteq \mathbb{R}^n \)

\[
\sup_{u \in W_0^{m, \frac{n}{m}}(\Omega), \|u\|_{m,n} \leq 1} \int_\Omega \phi \left( \beta_0(n,m) \left| u \right|^{\frac{n}{n-m}} \right) dx \leq C_{m,n}
\]

where

\[
\beta_0(n,m) = \frac{n}{\omega_{n-1}} \left[ \frac{\pi^{\frac{n}{2}} 2^m \Gamma \left( \frac{m}{2} \right) \Gamma \left( \frac{n-m}{2} \right) }{n^{n-m}} \right]^{\frac{1}{n-m}},
\]

\[
\phi(t) = e^t - \sum_{j=0}^{\left\lfloor \frac{n-m}{2} \right\rfloor} \frac{j^2}{j!} t^j,
\]

\[
j_m = \min \left\{ j \in \mathbb{N} : j \geq n \right\} \geq \frac{n}{m}.
\]

This inequality is sharp in the sense that if we replace \( \beta_0(n,m) \) by any \( \beta > \beta_0(n,m) \), then the supremum is infinite.

We note that the norm \( \|u\|_{n,m} \) used in Theorem C is equivalent to the Sobolev norm

\[
\|u\|_{W^{m, \frac{n}{m}}(\Omega)} = \left( \|u\|_{\frac{n}{m}}^\frac{n}{m} + \sum_{j=1}^{m} \|\nabla^j u\|_{\frac{n}{m}}^\frac{n}{m} \right)^{\frac{m}{n}}.
\]

In particular, if \( u \in W_0^{m, \frac{n}{m}}(\Omega) \) or \( u \in W^{m, \frac{n}{m}}(\mathbb{R}^n) \), then \( \|u\|_{W^{m, \frac{n}{m}}} \leq \|u\|_{m,n} \).

The work of Ruf and Sani raised a good open question: Does Theorem C hold when \( m \) is odd?

One of the primary purposes of this paper is to answer the above question in an affirmative way. This is stated as follows:

**Theorem 1.1.** Let \( m \) be an odd integer less than \( n \): \( m = 2k + 1 \), \( k \in \mathbb{N} \) and let \( \beta(n,m) \) be as in Theorem A and the function \( \phi \) be as in Theorem C. Then there holds

\[
\sup_{u \in W^{m, \frac{n}{m}}(\mathbb{R}^n), \|\nabla (-\Delta + I)^k u\|_{\frac{n}{m}} + \|(-\Delta + I)^k u\|_{\frac{n}{m}} \leq 1} \int_{\mathbb{R}^n} \phi \left( \beta(n,m) \left| u \right|^{\frac{n}{n-m}} \right) dx < \infty.
\]
Moreover, the constant $\beta(n, m)$ is sharp in the sense that if we replace $\beta(n, m)$ by any $\beta > \beta(n, m)$, then the supremum is infinity.

In the special case $n = 2m$ and $m$ an arbitrary positive integer, we can prove the following stronger result which is the second main theorem of this paper:

**Theorem 1.2.** If $m = 2k + 1, \ k \in \mathbb{N}$, then for all $\tau > 0$, there holds

$$\sup_{u \in W^{m, 2}(\mathbb{R}^{2m}), \|\nabla (\Delta + \tau I)^{k} u\|_{2}^{2} + \tau \| (\Delta + \tau I)^{k} u\|_{2}^{2} \leq 1} \int_{\mathbb{R}^{2m}} \left( e^{\beta(m, m) u^{2}} - 1 \right) dx < \infty.$$  

If $m = 2k, \ k \in \mathbb{N}$, then for all $\tau > 0$, there holds

$$\sup_{u \in W^{m, 2}(\mathbb{R}^{2m}), \| (\Delta + \tau I)^{k} u\|_{2}^{2} \leq 1} \int_{\mathbb{R}^{2m}} \left( e^{\beta(m, m) u^{2}} - 1 \right) dx < \infty.$$  

Moreover, the constant $\beta(2m, m)$ is sharp in the above inequalities in the sense that if we replace $\beta(2m, m)$ by any $\beta > \beta(2m, m)$, then the supremums will be infinity.

We note that for $m = 2k + 1$ and any $a_{0} = 1, a_{2} > 0, \ldots, a_{m} > 0$, there is some $\tau > 0$ such that (see Lemma 2.2):

$$\left\| \nabla (\Delta + \tau I)^{k} u \right\|_{2}^{2} + \tau \left\| (\Delta + \tau I)^{k} u \right\|_{2}^{2} \leq \sum_{j=0}^{m} a_{m-j} \int_{\mathbb{R}^{n}} |\nabla^{j} u|^{2} dx$$

and for $m = 2k$ and any $a_{0} = 1, a_{2} > 0, \ldots, a_{m} > 0$, there is some $\tau > 0$ such that (see Lemma 2.1):

$$\left\| (\Delta + \tau I)^{k} u \right\|_{2}^{2} \leq \sum_{j=0}^{m} a_{m-j} \int_{\mathbb{R}^{n}} |\nabla^{j} u|^{2} dx.$$  

Thus, as a consequence, we will be able to establish the third main theorem of this paper. Namely, we will replace the norm $\| \cdot \|_{m, n}$ by $\| \cdot \|_{W^{m, \infty}}$ in the above Theorem C in the case $n = 2m$ for all positive integer $m$.

**Theorem 1.3.** Let $m \geq 1$ be an integer number. For all constants $a_{0} = 1, a_{1}, \ldots, a_{m} > 0$, there holds

$$\sup_{u \in W^{m, 2}(\mathbb{R}^{2m}), \int_{\mathbb{R}^{2m}} \left( \sum_{j=0}^{m} a_{m-j} |\nabla^{j} u|^{2} \right) dx \leq 1} \int_{\mathbb{R}^{2m}} \left[ \exp \left( \beta(2m, m) |u|^{2} \right) - 1 \right] dx < \infty.$$  

Furthermore this inequality is sharp, i.e., if $\beta(2m, m)$ is replaced by any $\beta > \beta(2m, m)$, then the supremum is infinite.

In the special case $n = 2m = 4k = 4$, the above theorem was proved by Yang in [33].

As a corollary of the above theorem, we have the following Adams type inequality with the standard Sobolev norm:

**Theorem 1.4.** Let $m \geq 1$ be an integer number. There holds

$$\sup_{u \in W^{m, 2}(\mathbb{R}^{2m}), \|u\|_{W^{m, 2}} \leq 1} \int_{\mathbb{R}^{2m}} \left[ \exp \left( \beta(2m, m) |u|^{2} \right) - 1 \right] dx < \infty.$$
Furthermore this inequality is sharp, i.e., if $\beta(2m, m)$ is replaced by any $\beta > \beta(2m, m)$, then the supremum is infinite.

Since the fact that if $u \in W^m_m(\Omega)$ or $u \in W^m_m(\mathbb{R}^n)$, then $\|u\|_{W^m_m} \leq \|u\|_{m,m}$, our result is stronger than the one in [28] in the case $m$ is even. Moreover, our theorems still hold when $m$ is odd.

We organize this paper as follows: In Section 2, we provide some preliminaries. We build an iterated comparison in Section 3 and use it to prove the Adams type inequalities (Theorem 1.2, Theorem 1.3 and Theorem 1.4) for the case $n = 2m = 4k$, $k \in \mathbb{N}$, namely when $m$ is even in Section 4. Section 5 is devoted to proving Theorems 1.2, 1.3 and 1.4 when $n = 2m = 4k + 2$, namely when $m$ is odd. In fact, we will first prove these theorems in the special case when $n = 2m = 6$. Then we will prove these theorems in the general case $n = 2m = 2(2k + 1)$. Finally, the Adams-type inequality when $m$ is odd in general (Theorem 1.1) is proved in Section 6.

2. Preliminaries

In this section, we provide some preliminaries. For $u \in W^{m,2}(\mathbb{R}^{2m})$ with $1 \leq p < \infty$, we will denote by $\nabla^j u$, $j \in \{1, 2, ..., m\}$, the $j$-th order gradient of $u$, namely

$$
\nabla^j u = \begin{cases} 
\Delta^{j/2} u & \text{for } j \text{ even} \\
\nabla \Delta^{(j-1)/2} u & \text{for } j \text{ odd}
\end{cases}.
$$

For $m = 2k$, $k \in \mathbb{N}$, $\tau > 0$, we have the following observations:

$$
(-\Delta + \tau I)^k u = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \tau^i \Delta^{k-i} u
$$

where

$$
\binom{k}{j} = \frac{k!}{j!(k-j)!}.
$$

Thus

$$
\int_{\mathbb{R}^{2m}} \left| (-\Delta + \tau I)^k u \right|^2 dx = \int_{\mathbb{R}^{2m}} \left| \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \tau^i \Delta^{k-i} u \right|^2 dx
$$

$$
= \int_{\mathbb{R}^{2m}} \sum_{0 \leq i,j \leq k} (-1)^{k-i} (-1)^{k-j} \binom{k}{i} \binom{k}{j} \tau^i \tau^j \Delta^{k-i} u \Delta^{k-j} u dx
$$

$$
= \sum_{s=0}^{2k} \sum_{i+j=s} (-1)^{k-i} (-1)^{k-j} \binom{k}{i} \binom{k}{j} \tau^i \tau^j \int_{\mathbb{R}^{2m}} \Delta^{k-i} u \Delta^{k-j} u dx
$$

$$
= \sum_{s=0}^{2k} \sum_{i+j=s} \binom{k}{i} \binom{k}{j} \tau^s \int_{\mathbb{R}^{2m}} |\nabla^{2k-s} u|^2 dx.
$$

From the coefficients of $x^s$ in the identity

$$(1 + x)^k (1 + x)^k = (1 + x)^{2k}$$


we have
\[ \sum_{i+j=s} \binom{k}{i} \binom{k}{j} = \binom{2k}{s} \]
and then
\[ (2.2) \quad \int_{\mathbb{R}^2_\infty} \left| (-\Delta + \tau I)^k u \right|^2 \, dx = \int_{\mathbb{R}^2_\infty} \left( \sum_{j=0}^{m} \binom{m}{j} \tau^{m-j} \left| \nabla^j u \right|^2 \right) \, dx. \]

From these observations, we have when \( m = 2k, \ k \in \mathbb{N} \):
\[ (2.3) \quad \left\| (-\Delta + \tau I)^k u \right\|_2 = \left[ \sum_{j=0}^{m} \binom{m}{j} \tau^{m-j} \left\| \nabla^j u \right\|_2 \right]^{1/2}. \]

From (2.1), (2.2) and (2.3), we have

**Lemma 2.1.** Assume \( m = 2k, \ k \in \mathbb{N} \). Let \( a_0 = 1, a_1, ..., a_m > 0 \). There exists a real number \( \tau > 0 \) such that for all \( u \in W^{m,2}(\mathbb{R}^{2m}) \):
\[ \left\| (-\Delta + \tau I)^k u \right\|_2^2 \leq \sum_{j=0}^{m} a_{m-j} \left\| \nabla^j u \right\|_2^2. \]

**Proof.** We just need to choose \( \tau > 0 \) such that
\[ \binom{m}{j} \tau^{m-j} \leq a_{m-j}, \ j = 0, 1, ..., m. \]

When \( m = 2k + 1, \ k \in \mathbb{N} \), we have
\[ (2.4) \quad \nabla (-\Delta + \tau I)^k u = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \tau^i \nabla^{k-i} u \]
where
\[ \binom{k}{j} = \frac{k!}{j!(k-j)!}. \]

Similarly, we can prove that
\[ (2.5) \quad \int_{\mathbb{R}^2_\infty} \left| \nabla (-\Delta + \tau I)^k u \right|^2 \, dx = \int_{\mathbb{R}^2_\infty} \left( \sum_{j=1}^{m} \binom{m-1}{j-1} \tau^{m-j} \left| \nabla^j u \right|^2 \right) \, dx. \]

Thus, we have for \( m = 2k + 1, \ k \in \mathbb{N} \):
\[ (2.6) \quad \left\| \nabla (-\Delta + \tau I)^k u \right\|_2^2 = \left[ \sum_{j=1}^{m} \binom{m-1}{j-1} \tau^{m-j} \left\| \nabla^j u \right\|_2 \right]^{1/2}. \]

By (2.3) and (2.6), we get
\[ (2.7) \quad \left\| \nabla (-\Delta + \tau I)^k u \right\|_2^2 + \tau \left\| (-\Delta + \tau I)^k u \right\|_2^2 = \sum_{j=0}^{m} \binom{m}{j} \tau^{m-j} \left\| \nabla^j u \right\|_2^2. \]
Lemma 2.2. Assume $m = 2k + 1$, $k \in \mathbb{N}$. Let $a_0 = 1, a_1, \ldots, a_m > 0$. There exists a real number $\tau > 0$ such that for all $u \in W^{m,2}(\mathbb{R}^{2m})$:

$$\left\| \nabla (-\Delta + \tau I)^k u \right\|_2^2 + \tau \left\| (-\Delta + \tau I)^k u \right\|_2^2 \leq \sum_{j=0}^{m} a_{m-j} \left\| \nabla^j u \right\|_2^2$$

Proof. Again, we just need to choose $\tau > 0$ such that

$$\binom{m}{j} \tau^{m-j} \leq a_{m-j}.$$

□

In the general case, we have the following result

Lemma 2.3. Assume that $m$ is an odd integer less than $n$: $m = 2k + 1$. There exists a real number $C > 0$ such that for all $u \in W^{m,2}(\mathbb{R}^n)$:

$$\left\| \nabla^m u \right\|_{\frac{n}{m}} + \frac{1}{C} \sum_{j=0}^{m-1} \left\| \nabla^j u \right\|_{\frac{n}{m}} \leq \left\| \nabla (-\Delta + I)^k u \right\|_{\frac{n}{m}} + \left\| (-\Delta + I)^k u \right\|_{\frac{n}{m}}.$$

We now introduce the Sobolev space of functions with homogeneous Navier boundary conditions:

$$W^{m,2}_N(B_R) := \left\{ u \in W^{m,2}(\mathbb{R}^n) : \Delta^j u = 0 \text{ on } \partial B_R \text{ for } 0 \leq j \leq \left\lfloor \frac{m-1}{2} \right\rfloor \right\}$$

where $B_R = \{ x \in \mathbb{R}^{2m} : |x| < R \}$. It is easy to see that $W^{m,2}_N(B_R)$ contains $W^{m,2}_0(B_R)$ as a closed subspace. Also, we define

$$W^{m,2}_{rad}(B_R) := \left\{ u \in W^{m,2}(\mathbb{R}^n) : u(x) = u(|x|) \text{ a.e. in } B_R \right\},$$

$$W^{m,2}_{N,rad}(B_R) = W^{m,2}_N(B_R) \cap W^{m,2}_{rad}(B_R).$$

Finally, we give some radial lemmas which will be used in our proofs (see [10, 18, 28]):

Lemma 2.4. If $u \in W^{1,\frac{n}{m}}(\mathbb{R}^n)$ then

$$|u(x)| \leq \left( \frac{1}{m\sigma_n} \right)^{\frac{m}{n}} \frac{1}{|x|^{\frac{n-1}{m}}} \|u\|_{W^{1,\frac{n}{m}}(\mathbb{R}^n)}$$

for a.e. $x \in \mathbb{R}^n$, where $\sigma_n$ is the volume of the unit ball in $\mathbb{R}^n$.

Lemma 2.5. If $u \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, is a radial non-increasing function, then

$$|u(x)| \leq \left( \frac{n}{\omega_{n-1}} \right)^{\frac{1}{p}} \frac{1}{|x|^p} \|u\|_{L^p(\mathbb{R}^n)}$$

for a.e. $x \in \mathbb{R}^n$. 
3. An iterated comparison principle

In this section, we still denote by $B_R$ the set $\{x \in \mathbb{R}^n : |x| < R\}$ and $|B_R|$ the Lebesgue measure of $B_R$, namely $|B_R| = \sigma_n R^n$ where $\sigma_n$ is the volume of the unit ball in $\mathbb{R}^n$. Let $u : B_R \to \mathbb{R}$ be a measurable function. The distribution function of $u$ is defined by

$$\mu_u(t) = |\{x \in B_R \mid |u(x)| > t\}|, \forall t \geq 0.$$ 

The decreasing rearrangement of $u$ is defined by

$$u^*(s) = \inf \{t \geq 0 : \mu_u(t) < s\}, \forall s \in [0, |B_R|],$$

and the spherically symmetric decreasing rearrangement of $u$ by

$$u^#(x) = u^*(\sigma_n |x|^n) \forall x \in B_R.$$ 

We have that $u^#$ is the unique nonnegative integrable function which is radially symmetric, nonincreasing and has the same distribution function as $|u|$.

Let $\tau > 0$ and $u$ be a weak solution of (3.1)

$$\begin{cases} 
-\Delta u + \tau u &= f \text{ in } B_R \\
 u &\in W^{1,2}_0(B_R)
\end{cases}$$

where $f \in L^{n/2}(B_R)$. We have the following result that can be found in [30]:

**Proposition 3.1.** If $u$ is a nonnegative weak solution of (3.1) then

$$-\frac{du^*}{ds}(s) \leq \frac{s^{n-2}}{n^2\sigma_n^{2/n}} \int_0^s (f^* - \tau u^*) \, d\tau, \forall s \in (0, |B_R|).$$

Now, we consider the problem

(3.3)

$$\begin{cases} 
-\Delta v + \tau v &= f^# \text{ in } B_R \\
v &\in W^{1,2}_0(B_R)
\end{cases}$$

Due to the radial symmetry of the equation, the unique solution $v$ of (3.3) is radially symmetric and we have

$$-\frac{d\hat{v}}{ds}(s) = \frac{s^{n-2}}{n^2\sigma_n^{2/n}} \int_0^s (f^* - \tau \hat{v}) \, d\tau, \forall s \in (0, |B_R|)$$

where $\hat{v}(\sigma_n |x|^n) := v(x)$. We have the following comparison of integrals in balls that again can be found in [30]:

**Proposition 3.2.** Let $u$, $v$ be weak solutions of (3.1) and (3.3) respectively. For every $r \in (0, R)$ we have

$$\int_{B_r} u^# \, dx \leq \int_{B_r} v \, dx.$$ 

We now apply the comparison principle for the polyharmonic operator. Let $u \in W^{2k,2}(B_R)$ be a weak solution of

(3.4)

$$\begin{cases} 
(-\Delta + I)^k u &= f \text{ in } B_R \\
u &\in W^{2k,2}_N(B_R)
\end{cases}$$
where \( f \in L^{\frac{2n}{n+2}}(B_R) \). If we consider the problem

\[
\begin{aligned}
(-\Delta + \tau I)^k v &= f^# \quad \text{in } B_R \\
v &\in W^{2k,2}_N(B_R)
\end{aligned}
\]

then we have the following comparison of integrals in balls:

**Proposition 3.3.** Let \( u, v \) be weak solutions of the polyharmonic problems \((3.4)\) and \((3.5)\) respectively. For every \( r \in (0, R) \) we have

\[
\int_{B_r} u^# \, dx \leq \int_{B_r} v \, dx.
\]

**Proof.** Since equations in \((3.4)\) and \((3.5)\) are considered with homogeneous Navier boundary conditions, they may be rewritten as second order systems:

\[
\begin{aligned}
(P1) \quad \begin{cases}
-\Delta u_1 + \tau u_1 = f \quad \text{in } B_R \\
u_1 &\in W^{1,2}_0(B_R)
\end{cases} &\quad (P_i) \quad \begin{cases}
-\Delta u_i + \tau u_i = u_{i-1} \quad \text{in } B_R \\
u_i &\in W^{1,2}_0(B_R) \quad i \in \{2, 3, ..., k\}
\end{cases} \\
(Q1) \quad \begin{cases}
-\Delta v_1 + \tau v_1 = f^# \quad \text{in } B_R \\
v_1 &\in W^{1,2}_0(B_R)
\end{cases} &\quad (Qi) \quad \begin{cases}
-\Delta v_i + \tau v_i = v_{i-1} \quad \text{in } B_R \\
v_i &\in W^{1,2}_0(B_R) \quad i \in \{2, 3, ..., k\}
\end{cases}
\end{aligned}
\]

where \( u_k = u \) and \( v_k = v \). Thus we have to prove that for every \( r \in (0, R) \)

\[
\int_{B_r} u^# \, dx \leq \int_{B_r} v^# \, dx.
\]

By the above proposition (Proposition 3.2), we have

\[
\int_{B_r} u^#_1 \, dx \leq \int_{B_r} v^#_1 \, dx.
\]

Now, if we have

\[
\int_{B_r} u^#_j \, dx \leq \int_{B_r} v^#_j \, dx \quad \text{for all } j = 1, ..., i,
\]

we will prove that

\[
\int_{B_r} u^#_{i+1} \, dx \leq \int_{B_r} v^#_{i+1} \, dx.
\]

Without loss of generality, we may assume that \( u_{i+1} \geq 0 \). In fact, let \( \overline{u}_{i+1} \) be a weak solution of

\[
\begin{aligned}
-\Delta \overline{u}_{i+1} + \tau \overline{u}_{i+1} &= |u_i| \quad \text{in } B_R \\
\overline{u}_{i+1} &\in W^{1,2}_0(B_R)
\end{aligned}
\]

then the maximum principle implies that \( \overline{u}_{i+1} \geq 0 \) and \( \overline{u}_{i+1} \geq |u_{i+1}| \) in \( B_R \).

Since \( u_{i+1} \) is a nonnegative weak solution of \((P \ (i+1))\) and \( v_{i+1} \) is a nonnegative weak solution of \((Q \ (i+1))\) then by Proposition 3.1 we have

\[
-\frac{du^*_{i+1}}{ds}(s) \leq \frac{s^{\frac{2n}{n+2}}}{n^2 \sigma_n^{2/n}} \int_0^s (u^*_i - \tau u^*_{i+1}) \, d\tau, \ \forall s \in (0, |B_R|),
\]

\[
-\frac{d\hat{v}_{i+1}}{ds}(s) = \frac{s^{\frac{2n}{n+2}}}{n^2 \sigma_n^{2/n}} \int_0^s (\hat{v}_i - \tau \hat{v}_{i+1}) \, d\tau, \ \forall s \in (0, |B_R|).
\]
Thus for all $s \in (0, |B_R|)$

$$\frac{d\hat{v}_{i+1}}{ds}(s) - \frac{du^*_i}{ds}(s) - \frac{s^{\frac{2}{n} - 2}}{n^2 \sigma_n^{2/n}} \int_0^s (\tau \hat{v}_{i+1} - \tau u^*_i) \, d\tau \leq \frac{s^{\frac{2}{n} - 2}}{n^2 \sigma_n^{2/n}} \int_0^s (u^*_i - \hat{v}_i) \, d\tau.$$  

Using the induction hypotheses, we get that

$$\int_0^s (u^*_i - \hat{v}_i) \, d\tau \leq 0 \quad \forall s \in (0, |B_R|)$$

and then

$$\frac{d\hat{v}_{i+1}}{ds}(s) - \frac{du^*_i}{ds}(s) - \frac{s^{\frac{2}{n} - 2}}{n^2 \sigma_n^{2/n}} \int_0^s (\tau \hat{v}_{i+1} - \tau u^*_i) \, d\tau \leq 0.$$  

Setting

$$y(s) = \int_0^s (\hat{v}_{i+1} - u^*_i) \, d\tau \quad \forall s \in (0, |B_R|)$$

we get

$$\begin{cases} y'' - \frac{s^{\frac{2}{n} - 2}}{n^2 \sigma_n^{2/n}} y \leq 0, \forall s \in (0, |B_R|) \\ y(0) = y'(|B_R|) = 0 \end{cases}.$$  

By maximum principle, we have that $y \geq 0$ which is the desired result. □  

From the above proposition, we have the following corollary:

**Corollary 3.1.** Let $u, v$ be weak solutions of the polyharmonic problems (3.4) and (3.5) respectively. For every convex nondecreasing function $\phi : [0, +\infty) \to [0, +\infty)$ we have

$$\int_{B_r} \phi(|u|) \, dx \leq \int_{B_r} \phi(|v|) \, dx.$$  

**Remark 3.1.** If $f \in C^\infty_0(\mathbb{R}^n)$, $\text{supp} f \subset B_R$, then we can conclude that $u$ and $v$ in Proposition 3.3 belong to $W^{m, \frac{m}{n}}(B_r)$ with $m = 2k$ or $2k + 1$.

### 4. Proofs of Theorems 1.2, 1.3 and 1.4 when $m$ is even

In this section, we will prove Theorem 1.2 in the case when $m$ is even, namely, $m = 2k, k \in \mathbb{N}$.

**Theorem 4.1.** Let $m = 2k$, $k \in \mathbb{N}$. For all $\tau > 0$, there holds

$$\sup_{u \in W^{m, 2}(|B_r|), \|(-\Delta + \tau I)^k u\|_2 \leq 1} \int_{\mathbb{R}^{2m}} \left(e^{\beta(2m,m)u^2} - 1\right) \, dx < \infty.$$  

Furthermore this inequality is sharp, i.e., if $\beta(2m,m)$ is replaced by any $\beta > \beta(2m,m)$, then the supremum is infinite.
Proof. Let \( u \in W^{m,2}(\mathbb{R}^2) \), \( \|(-\Delta + \tau I)^k u\|_2 \leq 1 \), by the fact that \( C_0^\infty(\mathbb{R}^2) \) is dense in \( W^{m,2}(\mathbb{R}^2) \), without loss of generality, we can find a sequence of functions \( u_i \in C_0^\infty(\mathbb{R}^2) \) such that \( u_i \to u \) in \( W^{m,2}(\mathbb{R}^2) \) and \( \int_{\mathbb{R}^2_m} \|(-\Delta + \tau I)^k u_i\|^2 \, dx \leq 1 \) and suppose that \( \text{supp} u_i \subset B_{R_i} \) for any fixed \( i \). Let \( f_i := (-\Delta + \tau I)^k u_i \). Consider the problem

\[
\begin{aligned}
\left\{ (-\Delta + \tau I)^k v_i = f_i^# \\
v_i \in W_N^{m,2}(B_{R_i})
\right.
\end{aligned}
\]

By the property of rearrangement, we have

\[
\int_{B_{R_i}} \|(-\Delta + \tau I)^k v_i\|^2 \, dx = \int_{B_{R_i}} \|(-\Delta + \tau I)^k u_i\|^2 \, dx \leq 1
\]

and by Corollary 3.1, we get

\[
\int_{B_{R_i}} (e^{\beta_0 v_i^2} - 1) \, dx = \int_{B_{R_i}} (e^{\beta_0 v_i^2} - 1) \, dx \leq \int_{B_{R_i}} (e^{\beta_0 v_i^2} - 1) \, dx.
\]

Also, from (4.1) and (2.3), we have

\[
\|v_i\|_{W^{1,2}} = \left[ \int_{B_{R_i}} (|v_i|^2 + |\nabla v_i|^2) \right]^{1/2} \leq \sqrt{\frac{1}{\tau^m} + \frac{1}{m\tau^{m-1}}}
\]

Now, writing

\[
\int_{B_{R_i}} (e^{\beta_0 v_i^2} - 1) \, dx \leq \int_{B_{R_0}} (e^{\beta_0 v_i^2} - 1) \, dx + \int_{B_{R_i} \setminus B_{R_0}} (e^{\beta_0 v_i^2} - 1) \, dx = I_1 + I_2
\]

where \( R_0 \) depends only on \( \tau \) and will be chosen later, we will prove that both \( I_1 \) and \( I_2 \) are bounded uniformly by a constant that depends only on \( \tau \).

Using Theorem B, we can estimate \( I_1 \). Indeed, we just need to construct an auxiliary radial function \( w_i \in W_N^{m,2}(B_{R_0}) \) with \( \|\nabla w_i\|_2 \leq 1 \) which increases the integral we are interested in. Such a function was constructed in [28]. For the completeness, we give the detail here. For each \( i \in \{1, 2, \ldots, k - 1\} \) we define

\[
g_i(|x|) := |x|^{m-2i}, \quad \forall x \in B_{R_0}
\]

so \( g_i \in W_{rad}^{m,2}(B_{R_0}) \). Moreover,

\[
\Delta^j g_i(|x|) = \left\{ \begin{array}{ll}
c_i^j |x|^{m-2(i+j)} & \text{for } j \in \{1, 2, \ldots, k - i\} \\
0 & \text{for } j \in \{k - i + 1, \ldots, k\}
\end{array} \right. \quad \forall x \in B_{R_0}
\]

where

\[
c_i^j = \prod_{h=1}^{j} [n + m - 2(h + i)] [m - 2(i + h - 1)], \quad \forall j \in \{1, 2, \ldots, k - i\}.
\]
Let
define 
\[ z_l (|x|) := v_l (|x|) - \sum_{i=1}^{k-1} a_ig_i (|x|) - a_k \forall x \in B_{R_0} \]
where
\[ a_i := \frac{\Delta^{k-i} v_l (R_0) - \sum_{j=1}^{i-1} a_j \Delta^{k-i} g_j (R_0)}{\Delta^{k-i} g_i (R_0)}, \forall i \in \{1, 2, ... k - 1 \}, \]
\[ a_k := v_l (R_0) - \sum_{i=1}^{k-1} a_i g_i (R_0). \]

We can check that (see [28])
\[ z_l \in W_{N, \text{rad}}^{m, 2} (B_{R_0}), \]
\[ \nabla^m v_l = \nabla^m z_l \text{ in } B_{R_0}. \]

We have the following lemma whose proof can be found in [28]:

**Lemma 4.1.** For \(0 < |x| \leq R_0\) we have for some \(d(m, R_0)\) only depending on \(m\) and \(R_0\) such that
\[
|v_l (|x|)|^2 \leq |z_l (|x|)|^2 \left( 1 + c_m \sum_{j=1}^{k-1} \frac{1}{R_0^j} \left\| \Delta^{k-j} v_l \right\|_{W^{1,2}}^2 + \frac{c_m}{R_0^{2m-1}} \left\| v_l \right\|_{W^{1,2}}^2 \right)^2 + d(m, R_0).
\]

Now, setting
\[ w_l (|x|) := z_l (|x|) \left( 1 + c_m \sum_{j=1}^{k-1} \frac{1}{R_0^j} \left\| \Delta^{k-j} v_l \right\|_{W^{1,2}}^2 + \frac{c_m}{R_0^{2m-1}} \left\| v_l \right\|_{W^{1,2}}^2 \right). \]
Since
\[ z_l \in W_{N, \text{rad}}^{m, 2} (B_{R_0}), \]
\[ \nabla^m v_l = \nabla^m z_l \text{ in } B_{R_0}. \]

we have
\[ w_l \in W_{N, \text{rad}}^{m, 2} (B_{R_0}) \]
and
\[
\left\| \nabla^m w_l \right\|_2 = \left\| \nabla^m z_l \right\|_2 \left( 1 + c_m \sum_{j=1}^{k-1} \frac{1}{R_0^j} \left\| \Delta^{k-j} v_l \right\|_{W^{1,2}}^2 + \frac{c_m}{R_0^{2m-1}} \left\| v_l \right\|_{W^{1,2}}^2 \right). \]
Note that
\[
\|\nabla^m z_l\|_2 = \|\nabla^m v_l\|_2 \\
\leq \left( 1 - \lambda \sum_{j=1}^{k-1} \|\Delta_j v_l\|_{W^{1,2}}^2 - \lambda \|v_l\|_{W^{1,2}}^2 \right)^{1/2} \\
\leq 1 - \frac{\lambda}{2} \sum_{j=1}^{k-1} \|\Delta_j v_l\|_{W^{1,2}}^2 - \frac{\lambda}{2} \|v_l\|_{W^{1,2}}^2
\]
where
\[
\lambda = \min \left\{ \left( \frac{m}{j} \right)^{m-j} : j = 0, 1, ..., m - 1 \right\}
\]
we have
\[
\|\nabla^m w_l\|_2 \leq \left( 1 - \frac{\lambda}{2} \sum_{j=1}^{k-1} \|\Delta_j v_l\|_{W^{1,2}}^2 - \frac{\lambda}{2} \|v_l\|_{W^{1,2}}^2 \right) \times \\
\times \left( 1 + c_m \sum_{j=1}^{k-1} \frac{1}{R_0^{d_j - 1}} \|\Delta_j v_l\|_{W^{1,2}}^2 + \frac{c_m}{R_0^{2m-1}} \|v_l\|_{W^{1,2}}^2 \right) \\
\leq 1 + \sum_{j=1}^{k-1} \left( \frac{c_m}{R_0^{d_j - 1}} - \frac{\lambda}{2} \right) \|\Delta_j v_l\|_{W^{1,2}}^2 + \left( \frac{c_m}{R_0^{2m-1}} - \frac{\lambda}{2} \right) \|v_l\|_{W^{1,2}}^2 \\
\leq 1
\]
if we choose \( R_0 = R_0(\tau) \) sufficiently large.

Finally, note that
\[
I_1 \leq e^{\beta_0 d(m,R_0)} \int_{B_{R_0}} e^{\beta_0 w_l^2} dx,
\]
using Theorem B, we can conclude that \( I_1 \) is bounded by a constant depending only on \( \tau \) since \( \|\nabla^m w_l\|_2 \leq 1 \) and \( w_l \in W_{n,rad}^{m,2}(B_{R_0}) \).

Now, we will estimate \( I_2 \). We choose \( R_0 \geq \left[ \frac{1}{m^{\sigma_n}} \left( \frac{1}{\tau^m} + \frac{1}{m^{2m-1}} \right) \right]^{-1/\tau} \) then from the Radial Lemma 2.4 we get that \( |v_l(x)| \leq 1 \) when \( |x| \geq R_0 \). Thus we have
\[
I_2 = \int_{B_{R_1} \setminus B_{R_0}} (e^{\beta_0 v_l^2} - 1) dx \\
\leq \sum_{j=1}^{\infty} j^{\beta_0} \int_{B_{R_1}} v_l^2 \\
\leq \frac{1}{\tau^m} \sum_{j=1}^{\infty} j^{\beta_0}.
\]
Thus we have that \( \int_{B_{R_1}} (e^{\beta_0 v_l^2} - 1) dx \) is bounded by a constant depending only on \( \tau \).
Combining the above estimates and using Fatou’s lemma, we can conclude that

$$\sup_{\|(-\Delta + \tau I)^k u\|_2 \leq 1} \int_{\mathbb{R}^{2m}} (e^{\beta_0 u^2} - 1) \, dx < \infty.$$ 

When $\beta > \beta_0$, it’s easy to check that the sequence given by Ruf and Sani (see Proposition 6.2. in [28]) will make our supremum blow up and we then complete the proof of Theorem 4.1. □

**Proof of Theorem 1.3 when $m$ is even:** Choose $\tau > 0$ as in Lemma 2.1, we have

$$\int_{\mathbb{R}^{2m}} \left( \sum_{j=0}^{m} a_{m-j} |\nabla^j u|^2 \right) \, dx \geq \left\| (-\Delta + \tau I)^k u \right\|_2^2$$

and then

$$\sup_{f_{2m}} \int_{\mathbb{R}^{2m}} \left( \sum_{j=0}^{m} a_{m-j} |\nabla^j u|^2 \right) \, dx \leq \sup_{\|(-\Delta + \tau I)^k u\|_2 \leq 1} \int_{\mathbb{R}^{2m}} (e^{\beta_0 u^2} - 1) \, dx.$$

Furthermore, we can check that the sequence given by Ruf and Sani (see Proposition 6.2. in [28]) will make the supremum in Theorem 1.3 becomes infinite and we complete the proof of Theorem 1.3.

**Proof of Theorem 1.4 when $m$ is even:** If we choose $a_i = 1, \ i = 0, 1, \ldots, m$, then by Lemma 2.1 we have proved Theorem 1.4 in the case $m = 2k$.

5. **Proofs of Theorems 1.2, 1.3 and 1.4 when $m$ is odd**

5.1. **Proofs of Theorems 1.2, 1.3 and 1.4 when $n = 2m = 6$**. For the convenience, first, we will prove Theorem 1.2 in the special case $k = 1$, i.e., we will prove that all $\tau > 0$, there holds

$$\sup_{u \in W^{3,2}(\mathbb{R}^6), \|\nabla (-\Delta + \tau I) u\|_2^2 + \tau \|(-\Delta + \tau I) u\|_2^2 \leq 1} \int_{\mathbb{R}^6} (e^{\beta_0 u^2} - 1) \, dx < \infty.$$ 

where $\beta_0 = \beta(6, 3)$.

**Proof.** Let $u \in W^{3,2}(\mathbb{R}^6)$ be such that

$$\|\nabla (-\Delta + \tau I) u\|_2^2 + \tau \|(-\Delta + \tau I) u\|_2^2 \leq 1.$$ 

Again, by the density of $C_0^\infty(\mathbb{R}^6)$ in $W^{3,2}(\mathbb{R}^6)$, there exists a sequence of functions $u_l \in C_0^\infty(\mathbb{R}^6)$: $u_l \rightarrow u$ in $W^{3,2}(\mathbb{R}^6)$,

$$\int_{\mathbb{R}^6} |\nabla (-\Delta + \tau I) u_l|^2 + \tau \|(-\Delta + \tau I) u_l\|_2^2 \, dx \leq 1$$

and $\text{supp} \, u_l \subset B_{R_l}$ for any fixed $l$. Set $f_l := (-\Delta + \tau I) u_l$ and consider the problem

$$\begin{cases} (-\Delta + \tau I) v_l = f_l^# \\ v_l \in W^{3,2}_N(B_{R_l}) \end{cases}.$$
By the properties of rearrangement, we have
\[
\int_{B_{R_l}} |f_l|^2 \, dx = \int_{B_{R_l}} |f^\#_l|^2 \, dx
\]
\[
\int_{B_{R_l}} |\nabla f^\#_l|^2 \, dx \leq \int_{B_{R_l}} |\nabla f_l|^2 \, dx
\]
which thus
\[
\int_{B_{R_l}} |(-\Delta + \tau I) v_l|^2 \, dx = \int_{B_{R_l}} |(-\Delta + \tau I) u_l|^2 \, dx
\]
\[
\int_{B_{R_l}} |\nabla (-\Delta + \tau I) v_l|^2 \, dx \leq \int_{B_{R_l}} |(-\Delta + \tau I) u_l|^2 \, dx
\]
So, we have
\[
\int_{\mathbb{R}^d} |\nabla (-\Delta + \tau I) v_l|^2 + \tau |(-\Delta + \tau I) v_l|^2 \, dx
\]
\[
\leq \int_{\mathbb{R}^d} |\nabla (-\Delta + \tau I) u_l|^2 + \tau |(-\Delta + \tau I) u_l|^2 \, dx
\]
\[
\leq 1.
\]
By the comparison argument (Corollary 3.1), we have
\[
\int_{B_{R_l}} \left( e^{\beta_0 v_l^2} - 1 \right) \, dx = \int_{B_{R_l}} \left( e^{\beta_0 v^\#_l} - 1 \right) \, dx \leq \int_{B_{R_l}} \left( e^{\beta_0 v_l^2} - 1 \right) \, dx
\]
Recall
\[
\int_{\mathbb{R}^d} |\nabla (-\Delta + \tau I) v_l|^2 + \tau |(-\Delta + \tau I) v_l|^2 \, dx
\]
\[
= \| \nabla^3 v_l \|^2_2 + 3\tau \| \Delta v_l \|^2_2 + 3\tau^2 \| \nabla v_l \|^2_2 + \tau^3 \| v_l \|^2_2.
\]
From (5.1) and (5.2), we have
\[
\| v_l \|_{W^{1,2}} = \left[ \int_{B_{R_l}} \left( |v_l|^2 + |\nabla v_l|^2 \right) \right]^{1/2}
\]
\[
\leq \sqrt{\frac{1}{\tau^3} + \frac{1}{3\tau^2}}.
\]
Now, write
\[
\int_{B_{R_l}} \left( e^{\beta_0 v_l^2} - 1 \right) \, dx \leq \int_{B_{R_0}} \left( e^{\beta_0 v_l^2} - 1 \right) \, dx + \int_{B_{R_1} \setminus B_{R_0}} \left( e^{\beta_0 v_l^2} - 1 \right) \, dx
\]
\[
= I_1 + I_2
\]
where \( R_0 \) depends only on \( \tau \) and will be chosen later. We will prove that both \( I_1 \) and \( I_2 \) are bounded uniformly by a constant that depends only on \( \tau \).
First, we will prove that $I_1$ is bounded by a constant depending only on $\tau$ using Theorem B. In order to do that, we will construct an auxiliary radial function $w_l$ such that $w_l \in W^{3,2}_N(B_{R_0})$, $\|\nabla^3 w_l\|_2 \leq 1$ and

$$\int_{B_{R_0}} \left(e^{30w_l^2} - 1\right) dx \leq C(R_0) \int_{B_{R_0}} e^{30w_l^2} dx.$$ 

The way to construct this radial function $w_l$ is very similar to the case when $m$ is even. Let

$$z_l(|x|) = v_l(|x|) - \frac{\Delta v_l(R_0)}{12} |x|^2 + \frac{R_0^2 \Delta v_l(R_0)}{12} - v_l(R_0), \forall x \in B_{R_0},$$

then $z_l \in W^{3,2}_{N,rad}(B_{R_0})$. Similar to that in the proof of Lemma 4.1, and by a combination of Radial Lemmas 2.4 and 2.5, we can prove that for $0 < |x| \leq R_0$ ($R_0 > 1$), there exists a universal constant $c > 0$ and a positive constant $d(R_0)$ depending only on $R_0$ such that

$$|v_l(|x|)|^2 \leq |z_l(|x|)|^2 \left(1 + c \frac{1}{R_0} \|\Delta v_l\|_2^2 + c \frac{1}{R_0} \|v_l\|_{W^{1,2}}^2\right)^2 + d(R_0).$$

Indeed, we have

$$v_l(|x|) = z_l(|x|) + \frac{\Delta v_l(R_0)}{12} |x|^2 - \frac{R_0^2 \Delta v_l(R_0)}{12} + v_l(R_0)$$

where

$$g(|x|) = \frac{\Delta v_l(R_0)}{12} |x|^2 - \frac{R_0^2 \Delta v_l(R_0)}{12} + v_l(R_0).$$

Then

$$|v_l(|x|)|^2 = |z_l(|x|) + g(|x|)|^2$$

$$= |z_l(|x|)|^2 + 2z_l(|x|) g(|x|) + |g(|x|)|^2$$

$$\leq |z_l(|x|)|^2 + |z_l(|x|)|^2 |g(|x|)|^2 + 1 + |g(|x|)|^2$$

$$= |z_l(|x|)|^2 \left(1 + |g(|x|)|^2\right) + 1 + |g(|x|)|^2.$$

Note that for $0 < |x| \leq R_0$ ($R_0 > 1$), we have by Radial lemmas 2.4 and 2.5:

$$|g(|x|)| = \frac{\Delta v_l(R_0)}{12} |x|^2 - \frac{R_0^2 \Delta v_l(R_0)}{12} + v_l(R_0)$$

$$\leq \left|\frac{\Delta v_l(R_0)}{12}\right| |x|^2 + \left|\frac{R_0^2 \Delta v_l(R_0)}{12}\right| + |v_l(R_0)|$$

$$\leq cR_0^2 \frac{1}{R_0^5} \|\Delta v_l\|_2 + c \frac{1}{R_0^5} \|v_l\|_{W^{1,2}}^2.$$

Thus (5.3) follows.

Setting

$$w_l(|x|) := z_l(|x|) \left(1 + c \frac{1}{R_0} \|\Delta v_l\|_2^2 + c \frac{1}{R_0} \|v_l\|_{W^{1,2}}^2\right), \forall x \in B_{R_0}$$
then it is clear that \( w_l \in W^{2,2}_{N,rad}(B_{R_0}) \). Moreover, we have the following inequalities
\[
\| \nabla^3 w_l \|_2 = \| \nabla^3 z_l \|_2 \left( 1 + c \frac{1}{R_0} \| \Delta v_l \|_2^2 + \frac{c}{R_0} \| v_l \|_{W^{1,2}}^2 \right)
\]
\[
= \| \nabla^3 v_l \|_2 \left( 1 + c \frac{1}{R_0} \| \Delta v_l \|_2^2 + \frac{c}{R_0} \| v_l \|_{W^{1,2}}^2 \right)
\]
\[
\leq (1 - 3\tau \| \Delta v_l \|_2^2 - 3\tau^2 \| \nabla v_l \|_2^2 - \tau^3 \| v_l \|_2^2)^{1/2} \left( 1 + c \frac{1}{R_0} \| \Delta v_l \|_2^2 + \frac{c}{R_0} \| v_l \|_{W^{1,2}}^2 \right)
\]
\[
\leq \left( 1 - \frac{3\tau}{2} \| \Delta v_l \|_2^2 - \frac{3\tau^2}{2} \| \nabla v_l \|_2^2 - \frac{\tau^3}{2} \| v_l \|_2^2 \right) \left( 1 + c \frac{1}{R_0} \| \Delta v_l \|_2^2 + \frac{c}{R_0} \| v_l \|_{W^{1,2}}^2 \right)
\]
\[
\leq 1
\]
if we choose \( R_0 \) sufficiently large. Furthermore,
\[
\int_{B_{R_0}} \left( e^{\beta_0 v_l^2} - 1 \right) \, dx \leq C(R_0) \int_{B_{R_0}} e^{\beta_0 v_l^2} \, dx.
\]
Thus by Theorem B, we have that \( I_1 \) is bounded by a constant depending only on \( \tau \).

Now, we will estimate \( I_2 \). We choose \( R_0 \geq \left[ \frac{1}{3\sigma_6} \left( \frac{1}{\tau^3} + \frac{1}{3\tau^2} \right) \right]^{1/4} \) then from the Radial lemma 2.4, we get that \( |v_l(x)| \leq 1 \) when \( |x| \geq R_0 \). Thus we have
\[
I_2 = \int_{B_{R_l} \setminus B_{R_0}} \left( e^{\beta_0 v_l^2} - 1 \right) \, dx
\]
\[
\leq \sum_{j=1}^{\infty} \frac{\beta_0^j}{j!} \int_{B_{R_l}} v_l^{2j}
\]
\[
\leq \frac{1}{\tau^3} \sum_{j=1}^{\infty} \frac{\beta_0^j}{j!}.
\]
Thus we have that \( \int_{B_{R_l}} \left( e^{\beta_0 v_l^2} - 1 \right) \, dx \) is bounded by a constant depending only on \( \tau \).

Combining the above estimates and using Fatou’s lemma, we can conclude that
\[
\sup_{\| \nabla (-\Delta + \tau I) u \|_2^2 + \tau \| (-\Delta + \tau I) u \|_2^2 \leq 1} \int_{\mathbb{R}^6} \left( e^{\beta_0 u^2} - 1 \right) \, dx < \infty.
\]
This completes the proof of Theorem 1.2. \( \square \)

**Proofs of Theorem 1.3 and Theorem 1.4 when \( n = 2m = 6 \):**

To prove Theorem 1.3 when \( m \) is odd, it suffices to choose \( \tau > 0 \) as in Lemma 2.2. Then we have
\[
\int_{\mathbb{R}^6} \left( \sum_{j=0}^{3} a_{m-j} |\nabla^j u|^2 \right) \, dx \geq \| \nabla (-\Delta + \tau I) u \|_2^2 + \tau \| (-\Delta + \tau I) u \|_2^2
\]
and we get
\[
\sup_{f_{R^6}} \left( \sum_{j=0}^{3} a_{m-j} |\nabla u|^2 \right) \leq \sup_{\|\nabla(-\Delta + \tau I)u\|^2 + \tau \|(-\Delta + \tau I)u\|^2} \int_{R^6} \left( e^{\beta_0 u^2} - 1 \right) dx \leq 1.
\]

When \( \beta > \beta_0 \), it is showed by Kozono, Sato and Wadade [19] and Proposition 6.2 in [28] that the supremum in Theorem 1.3 is infinite. In fact, the sequence of test functions which gives the sharpness of Adams’ inequality in bounded domains in [2] gives also the sharpness of Adams’ inequality in unbounded domains. This completes the proof of Theorem 1.3.

Moreover, we can choose \( a_0 = a_1 = a_2 = a_3 = 1 \) to get Theorem 1.4.

5.2. Proof of Theorem 1.2 when \( m = 2k + 1 \), \( k \in \mathbb{N} \). The idea to prove the Adams type inequality in this case is a combination of ideas in the previous subsection and ideas in Section 4.

Proof. Let \( u \in W^{m,2}(R^{2m}) \) be such that
\[
\left\| \nabla (-\Delta + \tau I) u \right\|_2^2 + \tau \left( -\Delta + \tau I \right) u \right\|_2^2 \leq 1.
\]

By density arguments, we can find a sequence of functions \( u_l \in C_0^\infty (\mathbb{R}^{2m}) \) such that \( u_l \to u \) in \( W^{m,2}(\mathbb{R}^{2m}) \), \( \int_{\mathbb{R}^{2m}} \left( |\nabla (-\Delta + \tau I) u_l|^2 + \tau |(-\Delta + \tau I) u_l|^2 \right) dx \leq 1 \) and \( \text{supp} u_l \subset B_{R_l} \) for any fixed \( l \). Let \( f_l := (-\Delta + \tau I)^k u_l \). Consider the problem
\[
\begin{cases}
(-\Delta + \tau I)^k v_l = f_l^# \\
v_l \in W^{m,2}_N(B_{R_l})
\end{cases}
\]

Such a \( v_l \) does exist by Section 3 and Remark 3.1. Moreover, by the properties of re-arrangement, we have
\[
\int_{B_{R_l}} \left| (-\Delta + \tau I)^k v_l \right|^2 dx = \int_{B_{R_l}} \left| (-\Delta + \tau I)^k u_l \right|^2 dx
\]
\[
\int_{B_{R_l}} \left| \nabla (-\Delta + \tau I)^k v_l \right|^2 dx \leq \int_{B_{R_l}} \left| \nabla (-\Delta + \tau I)^k u_l \right|^2 dx
\]
which leads to
\[
\int_{\mathbb{R}^{2m}} \left( |\nabla (-\Delta + \tau I)^k v_l|^2 + \tau |(-\Delta + \tau I)^k v_l|^2 \right) dx \leq 1
\]

Note that from (5.4) and the formula (2.7), we have
\[
\|v_l\|_{W^{1,2}} = \left[ \int_{B_{R_l}} \left( |v_l|^2 + |\nabla v_l|^2 \right) \right]^{1/2} \leq \sqrt{\frac{1}{\tau^m} + \frac{1}{m^2 \tau^{m-1}}}.
\]
By Corollary 3.1, we get
\[ \int_{B_{Rl}} \left( e^{\beta_0 u^2} - 1 \right) \, dx = \int_{B_{Rl}} \left( e^{\beta_0 u^2} - 1 \right) \, dx \leq \int_{B_{Rl}} \left( e^{\beta_0 v^2} - 1 \right) \, dx. \]

Here, \( \beta_0 = \beta(2m, m) \).

Again, we write
\[ \int_{B_{Rl}} \left( e^{\beta_0 v^2} - 1 \right) \, dx \leq \int_{B_{R0}} \left( e^{\beta_0 v^2} - 1 \right) \, dx + \int_{B_{Rl} \setminus B_{R0}} \left( e^{\beta_0 v^2} - 1 \right) \, dx \]

where \( R_0 \) depends only on \( \tau \) and will be chosen later. We will prove that both \( I_1 \) and \( I_2 \) are bounded uniformly by a constant that depends only on \( \tau \).

First, we will estimate \( I_2 \). We choose \( R_0 \geq \left[ \frac{1}{m\sigma_n} \left( \frac{1}{\tau m} + \frac{1}{m\tau^m - 1} \right) \right] ^{\frac{1}{nm}} \) then from the Radial lemma 2.4, we get that \( |v_l(x)| \leq 1 \) when \( |x| \geq R_0 \). Thus we have
\[ I_2 = \int_{B_{Rl} \setminus B_{R0}} \left( e^{\beta_0 v_l^2} - 1 \right) \, dx \]
\[ \leq \sum_{j=1}^{\infty} \frac{\beta_0^j}{j!} \int_{B_{Rl}} v_l^2 \]
\[ \leq \frac{1}{\tau m} \sum_{j=1}^{\infty} \frac{\beta_0^j}{j!} \]

Thus we have that \( \int_{B_{Rl}} \left( e^{\beta_0 v_l^2} - 1 \right) \, dx \) is bounded by a constant depending only on \( \tau \).

To estimate \( I_1 \), again, we need to construct an auxiliary radial function \( w_l \in W^{m,2}_N (B_{R_0}) \) with \( \| \nabla^m w_l \|_2 \leq 1 \) which increases the integral we are interested in. We will construct such the function by the very similar way as in the case \( m \) is even [28] and the case \( m = 3 \).

For each \( i \in \{0, 1, 2, \ldots, k-1\} \) we define
\[ g_i(|x|) := |x|^{m-1-2i}, \forall x \in B_{R_0} \]
so \( g_i \in W^{m,2}_{rad} (B_{R_0}) \). Moreover,
\[ \Delta^j g_i (|x|) = \begin{cases} c_i^j |x|^{m-1-2(i+j)} & \text{for } j \in \{1, 2, \ldots, k-i\} \setminus \{k-i+1, \ldots, k\} \\ 0 & \text{for } j \in \{k-i+1, \ldots, k\} \end{cases} \quad \forall x \in B_{R_0} \]
where
\[ c_i^j = \prod_{h=1}^{j} \left[ 6k - 2(i + h - 1) \right] \left[ 2k - 2(i + h - 1) \right], \forall j \in \{1, 2, \ldots, k-i\} \]

Let
\[ z_l(|x|) := v_l(|x|) - \sum_{i=0}^{k-1} a_i g_i(|x|) - a_k, \forall x \in B_{R_0} \]
where

\[ a_0 := \frac{\Delta^k v_l (R_0)}{\Delta^k g (R_0)} \]

\[ a_i := \frac{\Delta^{k-i} v_l (R_0) - \sum_{j=0}^{i-1} a_j \Delta^{k-i} g_j (R_0)}{\Delta^{k-i} g_i (R_0)}, \quad \forall i \in \{1, 2, \ldots, k-1\}, \]

\[ a_k := v_l (R_0) - \sum_{i=0}^{k-1} \alpha_i g_i (R_0). \]

We can check that

\[ z_l \in W_{N,rad}^{m,2} (B_{R_0}), \]

\[ \nabla^m v_l = \nabla^m z_l \text{ in } B_{R_0}. \]

Combining the proofs when \( m \) is even in \[28\] and when \( m = 3 \), the Radial Lemma 2.4 and 2.5, we have for \( R_0 \geq 1 \)

**Lemma 5.1.** For \( 0 < |x| \leq R_0 \), there exists some positive constant \( d(m, R_0) \) depending only on \( m \) and \( R_0 \) such that

\[ |v_l (|x|)|^2 \leq |z_l (|x|)|^2 \left( 1 + c_m \frac{1}{R_0} \|\Delta^k v_l\|_2^2 + c_m \sum_{j=1}^{k-1} \frac{1}{R_0} \|\Delta^{k-j} v_l\|_{W^{1,2}}^2 + \frac{c_m}{R_0} \|v_l\|_{W^{1,2}}^2 \right)^2 \]

\[ + d(m, R_0). \]

Now, setting

\[ w_l (|x|) := z_l (|x|) \left( 1 + c_m \frac{1}{R_0} \|\Delta^k v_l\|_2^2 + c_m \sum_{j=1}^{k-1} \frac{1}{R_0} \|\Delta^{k-j} v_l\|_{W^{1,2}}^2 + \frac{c_m}{R_0} \|v_l\|_{W^{1,2}}^2 \right). \]

Since

\[ z_l \in W_{N,rad}^{m,2} (B_{R_0}), \]

\[ \nabla^m v_l = \nabla^m z_l \text{ in } B_{R_0}. \]

we have

\[ w_l \in W_{N,rad}^{m,2} (B_{R_0}) \]

and

\[ \|\nabla^m w_l\|_2 = \|\nabla^m z_l\|_2 \left( 1 + c_m \frac{1}{R_0} \|\Delta^k v_l\|_2^2 + c_m \sum_{j=1}^{k-1} \frac{1}{R_0} \|\Delta^{k-j} v_l\|_{W^{1,2}}^2 + \frac{c_m}{R_0} \|v_l\|_{W^{1,2}}^2 \right). \]
Note that
\[ \left\| \nabla^m z_l \right\|_2 = \left\| \nabla^m v_l \right\|_2 \]
\[ \leq \left( 1 - \lambda \left\| \Delta^k v_l \right\|_2^2 - \lambda \sum_{j=1}^{k-1} \left\| \Delta^{k-j} v_l \right\|_{W^{1,2}}^2 - \lambda \left\| v_l \right\|_{W^{1,2}}^2 \right)^{1/2} \]
\[ \leq 1 - \frac{\lambda}{2} \left\| \Delta^k v_l \right\|_2^2 - \frac{\lambda}{2} \sum_{j=1}^{k-1} \left\| \Delta^{k-j} v_l \right\|_{W^{1,2}}^2 - \frac{\lambda}{2} \left\| v_l \right\|_{W^{1,2}}^2 \]
where
\[ \lambda = \min \left\{ \left( \begin{array}{c} m \\ j \end{array} \right)^{m-j} : j = 0, 1, ..., m. \right\} \]
we have
\[ \left\| \nabla^m w_l \right\|_2 \leq \left( 1 - \frac{\lambda}{2} \left\| \Delta^k v_l \right\|_2^2 - \frac{\lambda}{2} \sum_{j=1}^{k-1} \left\| \Delta^{k-j} v_l \right\|_{W^{1,2}}^2 - \frac{\lambda}{2} \left\| v_l \right\|_{W^{1,2}}^2 \right) \times \]
\[ \times \left( 1 + c_m \frac{1}{R_0} \left\| \Delta^k v_l \right\|_2^2 + c_m \sum_{j=1}^{k-1} \frac{1}{R_0} \left\| \Delta^{k-j} v_l \right\|_{W^{1,2}}^2 + \frac{c_m}{R_0} \left\| v_l \right\|_{W^{1,2}}^2 \right) \]
\[ \leq 1 \]
if we choose \( R_0 = R_0(\tau) \) sufficiently large.

Finally, note that
\[ I_1 \leq e^{\beta_0 d(m,R_0)} \int_{B_{R_0}} e^{\beta_0 u_l^2} dx, \]
by using Theorem B, we can conclude that \( I_1 \) is bounded by a constant depending only on \( \tau \) since \( \left\| \nabla^m w_l \right\|_2 \leq 1 \).

Combining the above estimates and applying Fatou’s lemma, we can conclude that
\[ \sup_{u \in W^{m,2}(\mathbb{R}^n), \left\| \nabla (-\Delta + I)^{k} u \right\|_{L^2}^2 + \tau \left\| (-\Delta + I)^{k} u \right\|_{L^2}^2 \leq 1} \int_{\mathbb{R}^n} \left( e^{\beta_0 u^2} - 1 \right) dx < \infty. \]

\[ \square \]

**Proofs of Theorem 1.3 and Theorem 1.4 when \( m \) is odd:** From Lemma 2.2, we have the conclusion of Theorem 1.3 when \( m = 2k + 1, \ k \in \mathbb{N}. \)

Again, when \( \beta > \beta_0 \), we can check that the sequence of test functions which gives the sharpness of Adams’ inequality in bounded domains in [2] gives also the sharpness of Adams’ inequalities in unbounded domains. See Proposition 6.2 in [28].

Moreover, we can choose \( a_j = 1, \ j = 0, ..., m \) to get the Theorem 1.4.

6. **Proof of Theorem 1.1**

**Proof.** Let \( u \in W^{m,\frac{m}{\beta}}(\mathbb{R}^n) \) be such that
\[ \left\| \nabla (-\Delta + I)^{k} u \right\|_{m}^{\frac{m}{\beta}} + \left\| (-\Delta + I)^{k} u \right\|_{m}^{\frac{m}{\beta}} \leq 1, \]
by density arguments, we can find a sequence of functions \( u_l \in C_0^\infty (\mathbb{R}^n) \) such that \( u_l \to u \) in \( W^{m, \frac{m}{n}} (\mathbb{R}^n) \), \( \int_{\mathbb{R}^n} \left( |\nabla (-\Delta + I)^k u_l|^\frac{n}{m} + |(-\Delta + I)^k u_l|^\frac{m}{n} \right) dx \leq 1 \) and supp \( u_l \subset B_{R_l} \) for any fixed \( l \). Let \( f_l := (-\Delta + I)^k u_l \) and consider the problem

\[
\begin{aligned}
&\left\{ \begin{array}{l}
(-\Delta + I)^k v_l = f_l^# \\
v_l \in W_N^{m,2} (B_{R_l})
\end{array} \right.
\end{aligned}
\]

By the properties of rearrangement, we have

\[
\begin{align*}
\int_{B_{R_l}} \left| (-\Delta + I)^k v_l \right|^{\frac{n}{m}} dx &= \int_{B_{R_l}} \left| (-\Delta + I)^k u_l \right|^{\frac{n}{m}} dx \\
\int_{B_{R_l}} \left| \nabla (-\Delta + I)^k v_l \right|^{\frac{n}{m}} dx &\leq \int_{B_{R_l}} \left| \nabla (-\Delta + I)^k u_l \right|^{\frac{n}{m}} dx
\end{align*}
\]

Therefore, we have

\[
(6.1) \quad \int_{\mathbb{R}^n} \left( \left| \nabla (-\Delta + I)^k v_l \right|^{\frac{n}{m}} + \left| (-\Delta + I)^k v_l \right|^{\frac{m}{n}} \right) dx \leq 1
\]

By Corollary 2.1, we get

\[
\int_{B_{R_l}} \phi \left( \beta_0 |u_l|^{\frac{n}{n-m}} \right) dx = \int_{B_{R_l}} \phi \left( \beta_0 |u_l^#|^{\frac{n}{n-m}} \right) dx \leq \int_{B_{R_l}} \phi \left( \beta_0 |v_l|^{\frac{n}{n-m}} \right) dx
\]

Here, \( \beta_0 = \beta(n, m) \).

Again, write

\[
\int_{B_{R_l}} \phi \left( \beta_0 |v_l|^{\frac{n}{n-m}} \right) dx \leq \int_{B_{R_0}} \phi \left( \beta_0 |v_l|^{\frac{n}{n-m}} \right) dx + \int_{B_{R_l}\setminus B_{R_0}} \phi \left( \beta_0 |v_l|^{\frac{n}{n-m}} \right) dx = I_1 + I_2
\]

where \( R_0 \) is a positive constant and will be chosen later. We will prove that both \( I_1 \) and \( I_2 \) are bounded uniformly.

To do that, again, first, we need to construct an auxiliary radial function \( w_l \in W_N^{m, \frac{n}{m}} (B_{R_0}) \) with \( \| \nabla^m w_l \|_{\frac{n}{m}} \leq 1 \) which increases the integral \( I_1 \). For each \( i \in \{0, 1, 2, \ldots, k-1\} \) we define

\[
g_i (|x|) := |x|^{m-1-2i}, \quad \forall x \in B_{R_0}
\]

so \( g_i \in W^{m, \frac{m}{m}} (B_{R_0}) \). Moreover,

\[
\Delta^j g_i (|x|) = \begin{cases} 
c_i^j |x|^{m-1-2(i+j)} & \text{for } j \in \{1, 2, \ldots, k-i\} \\
0 & \text{for } j \in \{k-i+1, \ldots, k\}
\end{cases} \quad \forall x \in B_{R_0}
\]

where

\[
c_i^j = \prod_{h=1}^{j} \left[ 6k - 2(i + h - 1) \right] \left[ 2k - 2(i + h - 1) \right], \quad \forall j \in \{1, 2, \ldots, k-i\}.
\]

Let

\[
z_l (|x|) := v_l (|x|) - \sum_{i=0}^{k-1} a_i g_i (|x|) - a_k, \quad \forall x \in B_{R_0}
\]
where

\[
a_0 := \frac{\Delta^k v_l (R_0)}{\Delta^k g (R_0)}
\]

\[
a_i := \frac{\Delta^{k-i} v_l (R_0) - \sum_{j=0}^{i-1} a_j \Delta^{k-i} g_j (R_0)}{\Delta^{k-i} g_i (R_0)}, \quad \forall i \in \{1, 2, \ldots k - 1\},
\]

\[
a_k := v_l (R_0) - \sum_{i=0}^{k-1} a_i g_i (R_0).
\]

We can check that

\[
z_l \in W^{m, \frac{n}{m}}_{N, \text{rad}} (B_{R_0}),
\]

\[
\nabla^m v_l = \nabla^m z_l \text{ in } B_{R_0}.
\]

By a similar argument to that in [28], and a combination of Radial Lemmas 2.4 and 2.5, we can prove that for \( R_0 \geq 1 \)

**Lemma 6.1.** For \( 0 < |x| \leq R_0 \) we have for some constant \( d(m, n, R_0) \) such that

\[
\left| v_l (|x|) \right|^{\frac{n}{m}} \leq \left| z_l (|x|) \right|^{\frac{n}{m}} \left( 1 + c_{m,n} \frac{1}{R_0} \left\| \Delta^k v_l \right\|^{\frac{n}{m}} + c_{m,n} \sum_{j=1}^{k-1} \frac{1}{R_0} \left\| \Delta^{k-j} v_l \right\|_{W^{1, \frac{n}{m}}}^{\frac{n}{m}} + \frac{c_{m,n}}{R_0} \left\| v_l \right\|_{W^{1, \frac{n}{m}}}^{\frac{n}{m}} \right) + d(m, n, R_0).
\]

Now, setting

\[
w_l (|x|) := z_l (|x|) \left( 1 + c_{m,n} \frac{1}{R_0} \left\| \Delta^k v_l \right\|^{\frac{n}{m}} + c_{m,n} \sum_{j=1}^{k-1} \frac{1}{R_0} \left\| \Delta^{k-j} v_l \right\|_{W^{1, \frac{n}{m}}}^{\frac{n}{m}} + \frac{c_{m,n}}{R_0} \left\| v_l \right\|_{W^{1, \frac{n}{m}}}^{\frac{n}{m}} \right).
\]

Since

\[
z_l \in W^{m, \frac{n}{m}}_{N, \text{rad}} (B_{R_0}),
\]

\[
\nabla^m v_l = \nabla^m z_l \text{ in } B_{R_0}
\]

we have

\[
w_l \in W^{m, \frac{n}{m}}_{N, \text{rad}} (B_{R_0})
\]

and

\[
\left\| \nabla^m w_l \right\|_{\frac{n}{m}} = \left\| \nabla^m z_l \right\|_{\frac{n}{m}} \left( 1 + \frac{c_{m,n}}{R_0} \left\| \Delta^k v_l \right\|^{\frac{n}{m}} + \sum_{j=1}^{k-1} \frac{c_{m,n}}{R_0} \left\| \Delta^{k-j} v_l \right\|_{W^{1, \frac{n}{m}}}^{\frac{n}{m}} + \frac{c_{m,n}}{R_0} \left\| v_l \right\|_{W^{1, \frac{n}{m}}}^{\frac{n}{m}} \right).
\]
Note that from Lemma 2.3:
\[ \|\nabla^m z_l\|_{\frac{n}{m}} = \|\nabla^m v_l\|_{\frac{n}{m}} \]
\[ \leq \left( 1 - \frac{1}{C} \|\Delta^k v_l\|_{\frac{n}{m}} - \frac{1}{C} \sum_{j=1}^{k-1} \|\Delta^{k-j} v_l\|_{\frac{n}{m}} - \frac{1}{C} \|v_l\|_{\frac{n}{m}} \right)^{m/n} \]
\[ \leq 1 - \frac{m}{nC} \|\Delta^k v_l\|_{\frac{n}{m}} - \frac{m}{nC} \sum_{j=1}^{k-1} \|\Delta^{k-j} v_l\|_{\frac{n}{m}} - \frac{m}{nC} \|v_l\|_{\frac{n}{m}}, \]

we have
\[ \|\nabla^m w_l\|_{\frac{n}{m}} \leq \left( 1 - \frac{m}{nC} \|\Delta^k v_l\|_{\frac{n}{m}} - \frac{m}{nC} \sum_{j=1}^{k-1} \|\Delta^{k-j} v_l\|_{\frac{n}{m}} - \frac{m}{nC} \|v_l\|_{\frac{n}{m}} \right) \times \]
\[ \times \left( 1 + c_{m,n} \frac{1}{R_0} \|\Delta^k v_l\|_{\frac{n}{m}} + c_{m,n} \sum_{j=1}^{k-1} \frac{1}{R_0} \|\Delta^{k-j} v_l\|_{\frac{n}{m}} + c_{m,n} \frac{R_0}{R_0} \|v_l\|_{\frac{n}{m}} \right) \]
\[ \leq 1 \]
if we choose \( R_0 \) sufficiently large.

Finally, note that
\[ I_1 \leq e^{\beta_0 d(m,n,R_0)} \int_{B_{R_0}} e^{\beta_0 w^2} dx, \]

by using Theorem B, we can conclude that \( I_1 \) is bounded by a constant depending only on \( n \) and \( m \).

Now, by the same argument as in [28] and noting that from (6.1) and Lemma 2.3, we have \( \|v_l\|_{W^{1,\frac{n}{m}}} \leq D \) for some constant \( D > 0 \), we can conclude that \( I_2 \) is also bounded by a constant depending only on \( n \) and \( m \).

Combining the above estimates and employing Fatou’s lemma, we can conclude that
\[ \sup_{u \in W^{m,\frac{n}{m}}(\mathbb{R}^n), \|\nabla(-\Delta+I)^k u\|_{\frac{n}{m}} + \|(-\Delta+I)^k u\|_{\frac{n}{m}} \leq 1} \int_{\mathbb{R}^n} \phi \left( \beta(n,m) |u|^{\frac{n}{n-m}} \right) dx < \infty. \]

Again, when \( \beta > \beta(n,m) \), it is showed by Kozono, Sato and Wadade [19] that the supremum is infinite. In fact, the sequence of test functions which gives the sharpness of Adams’ inequality in bounded domains in [2] gives also the sharpness of Adams’ inequality in unbounded domains (see Proposition 6.2 in [28]).

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