Gamma function: exact formulas and approximations

Definition and exact formulas

The incomplete Gamma function is defined by

\[\gamma(r, z) = \int_0^z e^{-t} t^{r-1} dt\]  

(1)

For non-negative integers \(k\), it may be expressed in terms of elementary functions as

\[\frac{\gamma(k+1, z)}{k!} = 1 - e^{-z} \sum_{n=0}^{k} \frac{z^n}{n!}\]

(2)

\[= 1 - \frac{z^k e^{-z}}{k!} \sum_{m=0}^{k} \frac{k!}{(k-m)!} \frac{k!}{z^m}\]

(3)

Useful approximations

For \(k \ll z\), it is a good approximation to keep only the first two terms in the sum of (2):

\[\frac{\gamma(k+1, z)}{k!} \approx 1 - \frac{z^k e^{-z}}{k!} \left(1 + \frac{k}{z}\right),\]

(4)

a result that is exact for \(k = 0\) and 1. For \(k = z \gg 1\), the above approximation needs to be replaced by

\[\frac{\gamma(z+1, z)}{z!} \approx \frac{1}{2} \left(1 - \frac{4}{3\sqrt{2\pi z}}\right).\]

(5)