A CRITERIA OF STRONG H-DIFFERENTIABILITY

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Abstract: We give a criteria for a Malliavin differentiable function to be strongly H-differentiable.

Keywords: Wiener space, H-$C^1$, strong differentiability, Malliavin derivative

1. Introduction

Let $W$ be the classical Wiener space and $H$ the associated Cameron-Martin space. A theory of a weak derivative over Wiener functional with respect to $H$ directions has long been developed (see [3], [5]). More recently, Üstünel and Zakai, in [2], or Kusuoka, in [1] have studied a strong derivative for Wiener functional, using the Fréchet differentiability on $H$. A Wiener functional $f$ is $H$-continuous, or $H$-$C$, if $h \mapsto f(w + h)$ is a.s. continuous on $H$, $H$-$C^1$ if $h \mapsto f(w + h)$ is a.s. Fréchet differentiable on $H$ with $H$-continuous derivative. $H$-$C^1$ functions are very useful in the study of invertibility of perturbations of the identity of Wiener space. Indeed if $u$ is a measurable $H$-$C^1$ function from $W$ to $H$, $I_W + u$ is invertible. This has been used by Üstünel to establish the following variational representation, where $B$ is a Brownian motion

$$- \log \mathbb{E} [e^{- f \circ B}] = \inf_u \mathbb{E} \left[ f \circ (B + u) + \frac{1}{2} \int_0^1 |\dot{u}(s)|^2 ds \right]$$

for some unbounded functions $f$.

Of course it is way more difficult to establish that a function is $H$-$C^1$ than it is to establish it is weakly $H$-differentiable. In this paper, we give a criteria for a weakly $H$-differentiable function to be $H$-$C^1$, namely the weak $H$-derivative has to be a.s. uniformly continuous on every zero-centered ball of $H$.

First we recall the formal setting of weak and strong $H$-derivative, then we establish the criteria. Finally, we expand the criteria to higher order derivatives.

2. Framework

Set $n \in \mathbb{N}$ and let $W$ be the canonical Wiener space $C([0, 1], \mathbb{R}^n)$. Let $H$ be the associated Cameron-Martin space

$$H = \left\{ \int_0^1 \dot{h}(s) ds, \dot{h} \in L^2([0, 1]) \right\}$$

and for $m \in \mathbb{N}^*$, $B_m = \{ h \in H, |h|_H \leq m \}$. Denote $\mu$ the Wiener measure and $W$ the coordinate process. $W$ is a Brownian motion under $\mu$ and we denote $(\mathcal{F}_t)$ the canonical filtration of $W$ completed with respect to $\mu$. Set $\text{Cyl}$ the set of cylindrical functions

$$\text{Cyl} = \left\{ F(W_{t_1}, ..., W_{t_p}), p \in \mathbb{N}^*, F \in \mathcal{S}(\mathbb{R}^n), 0 \leq t_1 < ... < t_p \leq 1 \right\}$$
where $\mathcal{S}(\mathbb{R}^n)$ denotes the set of Schwartz functions on $\mathbb{R}^n$.

For $f \in Cyl, w \in W$ and $h \in H$, we define
\[
\nabla_h f(w) = \left. \frac{d}{d\lambda} f(w + \lambda h) \right|_{\lambda=0}
\]

Riesz theorem enables us to consider $\nabla f$ as an element of $H$. For $1 < p < \infty$, we define
\[
|.|_{p,1} : f \in Cyl \mapsto |f|_{L^p(\mu)} + |\nabla f|_{L^p(\nu,H)}
\]

$\nabla f$ is a closable operator and we define $\mathbb{D}_{p,1}$ the closure of $Cyl$ for $|.|_{p,1}$.

Let $\delta$ be the adjoint operator of $\nabla$ and $L^0_0(\mu,H)$ be the set of the element of $L^0(\mu,H)$ whose density are adapted to $(\mathcal{F}_t)$. $L^0_0(\mu,H)$ is a subset of the domain of $\delta$, and for any $u \in L^0_0(\mu,H)$
\[
\delta u = \int_0^1 \dot{u}(s)dW(s)
\]

From now on for $u \in L^0_0(\mu,H)$, we will denote
\[
\rho(\delta u) = \exp \left( \delta u - \frac{1}{2} \int_0^1 |\dot{u}(s)|^2 ds \right)
\]

Now set $X$ a separable Hilbert space and $(e_i)_{i \in \mathbb{N}}$ an Hilbert base of $X$, define
\[
Cyl(X) = \left\{ \sum_{k=1}^p f_i e_{i_k}, p \in \mathbb{N}^*, (i_k) \in \mathbb{N}^p, (f_i) \in Cyl^p \right\}
\]

If $f = \sum_{k=1}^p f_i e_{i_k} \in Cyl(X)$, we define
\[
\nabla f[w][h] = \sum_{k=1}^p \nabla_h f_i e_{i_k}
\]

and $\nabla f$ is an element of $X \otimes H$.

We define $|.|_{p,1}$, similarly as before and $\nabla$ is once again a closable operator, we define $\mathbb{D}_{p,1}(X)$ the closure of $Cyl(X)$ for $|.|_{p,1}$. This enables us to define $\nabla^p$ for $p \geq 1$ by recurrence, we denote
\[
|.|_{p,k} : f \mapsto |f|_{L^p(\mu,X)} + \sum_{k=1}^p |\nabla^k f|_{L^p(\mu,X \otimes H \otimes \cdots \otimes H)}
\]

and $\mathbb{D}_{p,k}(X)$ the completion of $Cyl(X)$ for $|.|_{p,k}$.

Finally, we define the Ornstein-Uhlebeck semigroup $(P_t)$ as follow: set $t > 0$ and $f \in L^p(\mu,X)$ for some $p \geq 1$
\[
P_t f : w \in \mathcal{W} \mapsto \int_{\mathcal{W}} f \left( e^{-t}w + \sqrt{1 - e^{-2t}}y \right) \mu(dy)
\]

We will need the following technical results concerning $P_t$:

**Proposition 1.** Set $t > 0$ and $f \in L^1(\mu,X)$. For $h \in H$, we have $\mu$-a.s.
\[
P_t f(w + h) = P_t \left( (f(\cdot + e^{-t}h))(w) \right)
\]

If $f$ belongs to some $\mathbb{D}_{p,1}(X)$, we have $\mu$-a.s.
\[
P_t \nabla f = e^{t} \nabla P_t f
\]
**Proof:** For the sake of simplicity we address the case $X = \mathbb{R}$. The first assertion is an easy calculation.
For the second one, set $h \in H$, we have
\[
\nabla \rho(\delta h) = h \rho(\delta h)
\]
and
\[
P_t \rho(\delta h) = \rho(\delta(e^{-t}h))
\]
so
\[
P_t \nabla \rho(\delta h) = h \rho(\delta(e^{-t}h))
\]
\[
= e^t e^{-t} h \rho(\delta(e^{-t}h))
\]
\[
= e^t \nabla \rho(\delta(e^{-t}h))
\]
\[
= e^t \nabla P_t \rho(\delta h)
\]
and we conclude with density of the vector space generated by $\{\rho(\delta h), h \in H\}$ in $D_{p,1}$.

For more details on this setting see [3] or [5].

Now we give the definitions of strongly H-differentiable functions.

**Definition 1.** Set $u : \mathbb{W} \to X$ a measurable function. We say that
(i) $u$ is H-continuous (or H-C) if the map $h \mapsto u(w + h)$ is $\mu$-a.s. continuous on $H$.
(ii) $u$ is H-C$^1$ if the map $h \mapsto u(w + h)$ is $\mu$-a.s. Fréchet-differentiable and its Fréchet derivative $\nabla f$ is an H-continuous map from $\mathbb{W}$ to $X \otimes H$.
(iii) Set $p \in \mathbb{N}$, by recurrence, $u$ is H-C$^p$ if it is H-C$^{p-1}$ and its derivative of order $p-1$ is an H-C$^1$ map from $\mathbb{W}$ to $X \otimes H^{\otimes p-1}$.

We will need the following results concerning strong H-regularity for our main theorem, see [2] for their proof:

**Proposition 2.** Set $u : \mathbb{W} \to X$ such that $\nabla^k u$ is well-defined for every $k \in \mathbb{N}^*$. Assume there exists $p \in \mathbb{N}^*$ such that for every $\lambda \in \mathbb{R}_+$
\[
\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \left| \nabla^k u \right|_{L^p(\mu, X \otimes H^{\otimes k})} < \infty
\]
Then $\mu$-a.s. for every $h \in H$
\[
u(w + h) = \sum_{k=0}^{\infty} \frac{1}{k!} \nabla^k u(w) \left[ h^{\otimes k} \right]
\]

**Proposition 3.** Set $f \in L^p(\mu, X)$ for some $p > 1$, for every $t > 0$ and $\lambda \in \mathbb{R}_+$, we have
\[
\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \left| \nabla^k P_t f \right|_{L^p(\mu, X \otimes H^{\otimes k})} < \infty
\]
3. Main theorem

**Theorem 1.** Assume that \( f : \mathbb{W} \to X \) is in \( D_{p,1}(X) \) for some \( p > 1 \). Assume that \( h \mapsto \nabla f(w + h) \) uniformly continuous on every \( n \)-ball of \( H \). Then \( f \) is \( H - C^1 \) and its \( H \)-derivative is \( \nabla f \).

**Proof:** The hypothesis implies that \( h \mapsto \nabla f(w + h) \) is separable so the uniform continuity hypothesis can be written:

\[
\lim_{\epsilon \to 0} \sup_{h, k \in B_n, \|h - k\|_H \leq \epsilon} |\nabla f(w + h) - \nabla f(w + k)|_{X \otimes H} = 0 \quad a.s.
\]

As we just stated we can set \( A \subset \mathbb{W} \) of full measure such that for every \( w \in \mathbb{W} \) \( h \mapsto \nabla f(w + h) \) is continuous.

Set \( s > 0 \) and \( h \in H \). We know the action of \( P_s \) over the weak derivative:

\[
P_s \nabla f(w + h) = e^s \nabla P_s f(w + h) \quad a.s.
\]

We also have:

\[
P_s \nabla f(w + h) = P_s(\nabla f(\cdot + e^{-s}h))(w) \quad a.s.
\]

Since both terms are analytic, the set on which these equalities hold does not depend on \( h \). Now we denote, for \( m, n \in \mathbb{N}^* \):

\[
\theta_{nm}(w) = \sup_{h, k \in B_n, \|h - k\|_H \leq \frac{1}{m}} |\nabla f(w + h) - \nabla f(w + k)|_{X \otimes H}
\]

Observe that for \( h, k \in B_n \) verifying \( \|h - k\|_H < \frac{1}{m} \), we have:

\[
|P_s(\nabla f(\cdot + e^{-s}h))(w) - P_s(\nabla f(\cdot + e^{-s}k))(w)|_{X \otimes H} \leq P_s \theta_{nm}(w) \quad a.s.
\]

Since both terms have analytic modifications, the set of \( w \) on which this inequality stands is independent of \( h \) and \( k \).

Set \( (s_i) \) a sequence decreasing towards 0 and \( H_0 \) a countable dense subset of \( H \). We define:

\[
A' = A \cap \mathbb{W} : P_s \nabla f(w + h) = e^{s_i} \nabla P_{s_i} f(w + h), \forall h \in H, \forall i \in \mathbb{N}
\]
Observe that we know from \[2\] that \(\left\{w \in \mathbb{W} : \sum_{k=1}^{\infty} \frac{1}{(k+1)!} |\nabla^{k+1} P_s f(w)|_{H^k} < \infty, \forall x \in \mathbb{R}_+\right\}\) and \(\left\{w \in \mathbb{W} : P_s \nabla f(w + h) = P_s \nabla f(w) + \sum_{k=1}^{\infty} \frac{1}{k!} \nabla^k P_s \nabla f(w) \left[h^k\right], \forall h \in H\right\}\) are of full measure and \(H\)-invariant.

Set \(w \in A', i \in \mathbb{N}, h \in H \) and \(h' \in H_0\) such that \(|h-h'| \leq \frac{1}{m}\) and \(n \in \mathbb{N}\) such that \(B \left(h, \frac{1}{m}\right) \subset B_n\), we have:

\[
|P_s \nabla f(w + h) - \nabla f(w + h)|_{X \otimes H} \leq |P_s \nabla f(w + h) - P_s \nabla f(w + h')|_{X \otimes H} + |\nabla f(w + h') - \nabla f(w + h)|_{X \otimes H}
\]

This proves that:

\[
\lim_{i \to \infty} P_s \nabla f(w + h) = \nabla f(w + h) \quad \forall h \in H
\]

Now observe that for \(w \in \mathbb{W}\) such that \((\theta_{n,m}(w))_{m \in \mathbb{N}}\) converges toward 0, \(h \mapsto \nabla f(w + h)\) is uniformly continuous on \(B_n\) hence bounded. So for \(h, k \in B_n:\)

\[
|f(w + h) - f(w + k)|_X = \left|\int_0^1 \nabla f(w + \lambda h + (1 - \lambda) k)|k - h|d\lambda\right| \\
\leq \sup_{h' \in B_n} |\nabla f(w + h')|_{X \otimes H} |h - k|_H
\]

So the hypothesis imply that

\[
\lim_{\varepsilon \to 0} \sup_{h, k \in B_n, |h - k|_H \leq \varepsilon} |f(w + h) - f(w + k)|_X = 0 \quad a.s.
\]

where this supremum is a measurable random variable since \(f \in \mathbb{D}_{p,1}(X)\) and we can construct a full measure \(A'' \subset \mathbb{W}\) similar to \(A'\) where \(f\) takes the role of \(\nabla f\). We denote \(\hat{A} = A' \cap A''\). We have \(\mu(\hat{A}) = 1\) and for every \(w \in \hat{A}\) and \(h \in H:\)

\[
\lim_{i \to \infty} P_s f(w + h) = f(w + h) \\
\lim_{i \to \infty} P_s \nabla f(w + h) = \nabla f(w + h)
\]

Set \(i, j \geq \max(i_0, i_1, i_2)\), we have:

Now we can prove the differentiability of \(h \mapsto f(w + h)\). Set \(w \in A\) and \(h \in H\), we aim to prove that:

\[
\lim_{h' \to 0} \frac{1}{|h'|_H} |f(w + h + h') - f(w + h) - \nabla f(w + h)[h']|_X = 0
\]

Set \(h' \in H\), we have:
\[
\frac{1}{|h'|_H} \left| f(w + h + h') - f(w + h) - \nabla f(w + h)[h'] \right|_X \\
\leq \frac{1}{|h'|_H} \left| f(w + h + h') - f(w + h) - (P_s f(w + h + h') - P_s f(w + h)) \right|_X \\
+ \frac{1}{|h'|_H} \left| P_s f(w + h + h') - P_s f(w + h) - \nabla P_s f(w + h)[h'] \right|_X \\
+ \frac{1}{|h'|_H} \left| \nabla P_s f(w + h)[h'] - \nabla f(w + h)[h'] \right|_X
\]

We denote these three terms \( A_{h'}, B_{h'} \) and \( C_{h'} \) and we deal with each one of them separately.

\[
C_{h'} \leq \left| \nabla P_s f(w + h) - \nabla f(w + h) \right|_{X \otimes H} \\
\leq \left| e^{-s} P_s \nabla f(w + h) - \nabla f(w + h) \right|_{X \otimes H} \\
\rightarrow 0
\]

now the second term:

\[
B_{h'} = \frac{1}{|h'|_H} \sum_{k=2}^{\infty} \frac{1}{k!} \left| \nabla^k P_s f(w + h) \left[ h'^k \right] \right|_X \\
\leq \frac{1}{|h'|_H} \sum_{k=2}^{\infty} \frac{1}{k!} \left| \nabla^k P_s f(w + h) \right|_{X \otimes H^\otimes k} \left| h'^k \right|_H \\
= \sum_{k=1}^{\infty} \frac{1}{(k+1)!} \left| \nabla^{k+1} P_s f(w + h) \right|_{X \otimes H^\otimes k+1} \left| h'^k \right|_H \\
\rightarrow 0
\]

since \( \sum_{k=1}^{\infty} \frac{1}{(k+1)!} \left| \nabla^{k+1} P_s f(w + h) \right|_{X \otimes H^\otimes k+1} < \infty. \)

\[
A_{h'} = \lim_{j \to \infty} \frac{1}{|h'|_H} \left| P_s f(w + h + h') - P_s f(w + h) - (P_s f(w + h + h') - P_s f(w + h)) \right|_X
\]

We have

\[
\left| P_s f(w + h + h') - P_s f(w + h) - (P_s f(w + h + h') - P_s f(w + h)) \right|_X \\
\leq \sup_{\lambda \in [0,1]} \left| \nabla P_s f(w + h + \lambda h') - \nabla P_s f(w + \lambda h + \lambda h') \right|_{X \otimes H} \left| h' \right|_H
\]
Now set $\lambda \in [0,1]$, $\epsilon > 0$. We can assume that $|h - h'|_H \leq \frac{1}{m}$, set $n \in \mathbb{N}$ such that $B \left( h, \frac{1}{m} \right) \subset B_n$. We have:

$$
\begin{align*}
|\nabla P_{s_j} f(w + h + \lambda h') - \nabla P_{s_j} f(w + h + \lambda h')|_{X \otimes H} \\
\leq |\nabla P_{s_j} f(w + h + \lambda h') - \nabla P_{s_j} f(w + h)|_{X \otimes H} \\
+ |\nabla P_{s_j} f(w + h) - \nabla P_{s_j} f(w + h)|_{X \otimes H} \\
+ |\nabla P_{s_j} f(w + h) - \nabla P_{s_j} f(w + \lambda h + (1 - \lambda)h')|_{X \otimes H}
\end{align*}
$$

which is smaller than $\epsilon$ for $i$ and $j$ large enough. It ensures that $A_{h'}$ tends toward 0 when $h'$ converges toward 0, which concludes the proof. 

\section{4. Extension to higher order derivatives}

\textbf{Corollary 1.} Assume that $f : \mathcal{W} \to X$ is in $\mathbb{D}_{p,r}(X)$ for some $p > 1$. Assume that $h \mapsto \nabla^k f(w + h)$ is $\mu$-a.s. uniformly continuous on every $B_n$. Then $f$ is $H - C^r$ and its $H$-derivatives up to order $n$ are equal to its weak derivatives of the same order.

\textbf{Proof:} We prove this with a recurrence over $n$. The case $r = 1$ is theorem $\mathbb{H}$. Now set $r \leq 2$ and assume that the result is proven for every integer up to $k-1$. Set $n \in \mathbb{N}$ and $A$ a measurable subset of $\mathcal{W}$ such that $\mu(A) = 1$ and for every $w \in A$ $h \mapsto \nabla^r f(w + h)$ is uniformly continuous on $B_n$. Set $w \in A$, $B_n$ being closed, $h \mapsto \nabla^r f(w + h)$ is bounded is bounded on $B_n$. Consequently, $h \mapsto \nabla^{r-1} f(w + h)$ is lipschitz on $B_n$ and so is uniformly continuous on $B_n$. The recurrence hypothesis ensures that $f$ is $H-C^{r-1}$ and that its $H$ derivatives up to order $r-1$ are equals to its weak derivatives of the same order. Applying theorem $\mathbb{H}$ to $\nabla^{r-1} f$, we get that it is $H-C^1$ and that its $H$-derivative is $\nabla^r f$, which conclude the proof. 

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