BOUNDED DISTANCE GEODESIC FOLIATIONS IN RIEMANNIAN PLANES

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ABSTRACT. A conjecture of Burns-Knieper [BK91] asks whether a 2-plane with a metric without conjugate points, and with a geodesic foliation whose lines are at bounded Hausdorff distance, is necessarily flat. We prove this conjecture in two cases: under the hypothesis that the plane admits total curvature, and under the hypothesis of visibility at some point. Along the way, we show that all geodesic line foliations on a Riemannian 2-plane must be homeomorphic to the standard one.

1. INTRODUCTION

A complete Riemannian manifold $(M, g)$ is called without conjugate points if no geodesic on $M$ contains a pair of mutually conjugate points. The class of Riemannian manifolds without conjugate points contains the set of Riemannian manifolds with nonpositive curvature as a proper subset. E. Hopf [Hop48] proved that any metric without conjugate points on the 2-dimensional torus $T^2$ is necessarily flat. In fact, E. Hopf proved that the total curvature of such manifold must be nonpositive, hence by the Gauss Bonnet Theorem, the metric must be flat. In [Gre58], Green generalized Hopf’s idea to higher dimensions by showing that in any closed Riemannian manifold without conjugate points, the integral of its scalar curvature is nonpositive.

Although there are examples of manifolds without conjugate points having positive sectional curvatures at some point, there are many properties holding for nonpositively curved metrics that also hold for metrics without conjugate points. However this is not always the case. Recall that a smooth curve $\gamma : (-\infty, \infty) \to M$ in a Riemannian manifold $M$ is called a line if $\text{dist}(\gamma(t), \gamma(s)) = |t - s|$ for all $t, s \in \mathbb{R}$. The Flat Strip Theorem for a simply connected surface $M$ of nonpositive curvature states that any two lines $\sigma$ and $\gamma$ in $M$ with bounded Hausdorff distance must bound a flat strip, i.e., there exists an isometric embedding from $I \times \mathbb{R}$ to $M$ for some closed interval $I \subset \mathbb{R}$ such that $\sigma$ and $\gamma$ are the boundaries of the

Date: December 21, 2020.

2000 Mathematics Subject Classification. Primary: 53C23; Secondary: 53C20, 57N10.

Key words and phrases. Riemannian plane, geodesic foliation, conjugate points, Burns-Knieper conjecture.

*Partially supported by NSFC.

**Supported by research grants MTM2017-85934-C3-2-P from the MINECO, by ICMAT Severo Ochoa Project SEV-2015-0554 (MINECO) and by the Severo Ochoa Programme for Centers of Excellence in R&D, CEX2019-000904-S (MICINN).
image. However in [Bur92], Burns constructed an example showing that the flat strip theorem does not hold for metric without conjugate points. Nonetheless, his construction does not violate the following conjecture proposed in [BK91]:

**Conjecture 1.1.** Let $M$ be a simply connected surface with a complete Riemannian metric with no conjugate points. Suppose that $\mathcal{F}$ is a foliation on $M$ whose leaves are all geodesic lines, and any two of which are at finite Hausdorff distance. Then $M$ is flat.

In [BK91], the authors proved the flatness of $M$ under the extra assumption that for any given line $\gamma$ there exists a bounded Hausdorff distance line foliation $\mathcal{F}$ containing $\gamma$.

In this paper, we confirm the above conjecture in two different cases.

**Theorem 1.1.** If $M$ admits total curvature, then Conjecture 1.1 holds.

**Theorem 1.2.** If $M$ satisfies the visibility axiom from a given point $p$, then $M$ does not admit a line foliation with bounded Hausdorff distance. In particular if $M$ is the universal covering of a closed surface, then Conjecture 1.1 holds.

Our approach to the conjecture can be divided into two parts. In the first part, we study the topology of the foliation $\mathcal{F}$. General foliations on Riemannian surfaces by curves could be very complicated, cf [Kap40, Kap41, HR57], even after assuming that the leaves are geodesics. However, if we assume the leaves are lines, we have

**Theorem 1.3.** Let $(M^2, g)$ be a complete Riemannian surface homeomorphic to $\mathbb{R}^2$, and let $\mathcal{F}$ be a geodesic line foliation on $M$. Then $\mathcal{F}$ is homeomorphic to the standard straight line foliation on the Euclidean plane $\mathbb{R}^2$.

Note that Theorem 1.3 does not require a metric without conjugate points on $M$, so it might have an independent interest and could be applied to other situations.

The second part of our approach consists of a close study of compactifications of $M$ by the introduction of equivalence relations on the set of rays, which is an effective tool to control the global geometry of $M$. Such a tool has already proved to be useful in [GGS20]. In this paper, we pursue this direction further. We need the following pair of definitions.

**Definition 1.1.** Let $M$ be a complete Riemannian surface homeomorphic to $\mathbb{R}^2$. Let $\gamma_i : \mathbb{R} \to M$, $i = 1, 2$ be two non-intersecting lines in $M$. Then $\gamma_1$ and $\gamma_2$ bounds a region homeomorphic to the band $\mathbb{R} \times [0, 1]$, which has two ends. By changing the orientation of $\gamma_2$ if necessary, we assume $\gamma_1(t)$ and $\gamma_2(t)$ belong to the same end of the band as $t \to +\infty$.

1. We call $(\gamma_1, \gamma_2)$ a hyperbolic pair if
   \[
   \lim_{t \to +\infty} d(\gamma_1(t), \gamma_2(t)) = \infty, \quad \lim_{t \to -\infty} d(\gamma_1(t), \gamma_2(t)) = \infty.
   \]

2. We call $(\gamma_1, \gamma_2)$ a weak hyperbolic pair if the above inequalities hold for some sequence $t_k \to \infty$. 


It is clear that a hyperbolic pair is a weak hyperbolic pair, but the reciprocal does not need to hold. The importance of this definition lies in the following Propositions.

**Proposition 1.2.** Let $M$ be a complete Riemannian surface homeomorphic to $\mathbb{R}^2$ without conjugate points. Suppose the total curvature of $M$ exists. If $c(M) < 0$, then for any line $\gamma$ in $M$, there exists a line $\sigma$ such that $(\gamma, \sigma)$ is a hyperbolic pair.

**Proposition 1.3.** Let $M$ be a complete Riemannian surface homeomorphic to $\mathbb{R}^2$ without conjugate points satisfying the visibility axiom from some point $p$. Then for any line $\gamma$ in $M$, there exists a line $\sigma$ such that $(\gamma, \sigma)$ is a weak hyperbolic pair.

Once we have the existence of (weak) hyperbolic pairs, its combination with Theorem [1.2] will give relatively simple proofs of the conjecture.

### 1.1. Some properties of planes without conjugate points

We will collect here a few properties of Riemannian planes without conjugate points that, although elementary, will be used repeatedly in the paper. Recall that a line is a geodesic $\gamma : \mathbb{R} \to M$ such that it minimizes the distance between any two of its points.

**Lemma 1.4.** Let $M$ be a complete Riemannian plane without conjugate points. Then

1. Any geodesic $\gamma : \mathbb{R} \to M$ is a line;
2. any pair of lines $\gamma_1, \gamma_2 : \mathbb{R} \to M$ are disjoint or intersect at most in a single point;
3. each line $\gamma : \mathbb{R} \to M$ divides $M$ into two closed geodesically convex sets.

**Proof.** Using standard arguments, one shows that the lack of conjugate points implies that the exponential map is a diffeomorphism at every point. This shows the first and second points. The third follows immediately from the second. $\square$

### 2. Line foliations on a Riemannian plane

In this section we assume that $M$ is a Riemannian plane, and with a given foliation $\mathcal{F}$ by geodesic lines. For any leaf $\sigma \in \mathcal{F}$, $\sigma$ separates $M$ into two closed half spaces, that we denote as $H^\pm_\sigma$. For any two distinct leaves $\sigma, \gamma \in \mathcal{F}$, there is a half space defined by $\sigma$ containing $\gamma$, say $H^-_\sigma$. On the other hand there is a half space defined by $\gamma$ containing $\sigma$, say $H^+_\gamma$.

**Definition 2.1.** The region bounded by $\sigma$ and $\gamma$ is called a *strip*, denoted by

$$S^\sigma_\gamma = H^-_\sigma \cap H^+_\gamma.$$

We need the following lemma to show all those strips between two leaves behave well.

**Lemma 2.2.** Let $\sigma_i, i = 1, 3$ be two leaves in $\mathcal{F}$, $p$ be an interior point of the strip $S^{\sigma_3}_{\sigma_1}$. Let $\sigma_2$ be the leaf passing through $p$. Then $\sigma_2$ separates $S^{\sigma_3}_{\sigma_1}$, i.e.

$$S^{\sigma_2}_{\sigma_1} \cap S^{\sigma_3}_{\sigma_2} = \sigma_2,$$

and

$$S^{\sigma_3}_{\sigma_1} = S^{\sigma_2}_{\sigma_1} \cup S^{\sigma_3}_{\sigma_2}.$$
Proof. Suppose the contrary holds, i.e. \( \sigma_2 \) does not separate \( S_{\sigma_1}^{\sigma_3} \). Schematically, it looks like Figure 1.

Then we can choose six points \( P_i, i = 1, \ldots, 6 \) on the leaves as shown in the picture. Here is the detailed construction of choosing such points. For arbitrary chosen \( P_5 \in \sigma_3 \) and \( P_6 \in \sigma_1 \), we let \( P_5P_6 \) be a shortest geodesic connecting them. Clearly \( P_5P_6 \cap \sigma_2 = \emptyset \) because geodesic segments do not intersect within their interiors.

The \( P_5P_6 \) separates \( S_{\sigma_1}^{\sigma_3} \) into two parts:
\[
S_{\sigma_1}^{\sigma_3} = S_{\sigma_1}^{\sigma_3}(-) \cup S_{\sigma_1}^{\sigma_3}(+),
\]
(2.1)

Under the assumption that \( \sigma_2 \) does not separate \( S_{\sigma_1}^{\sigma_3} \), one of the regions in (2.1), say \( S_{\sigma_1}^{\sigma_3}(+) \), must contain \( \sigma_2 \). We choose any \( P_1 \in \sigma_1 \cap S_{\sigma_1}^{\sigma_3}(+) \) which differs from \( P_6 \). We then choose an arbitrary \( P_2 \in \sigma_2 \). By the same argument, the segment \( P_1P_2 \) separates the strip \( S_{\sigma_1}^{\sigma_2} \) into two parts, one of which contains \( \sigma_3 \), say \( S_{\sigma_1}^{\sigma_3}(-) \). We then choose arbitrary
\[
P_3 \in \sigma_2 \cap S_{\sigma_1}^{\sigma_3}(-) \cap S_{\sigma_1}^{\sigma_3}(+),
\]
\[
P_4 \in \sigma_3 \cap S_{\sigma_1}^{\sigma_3}(-) \cap S_{\sigma_1}^{\sigma_3}(+),
\]
which differs from \( P_2 \) and \( P_5 \). Since geodesic segments do not intersect more than once in their interior, the geodesic segments \( P_1P_2, P_3P_4 \) and \( P_5P_6 \) do not intersect. Then we denote the compact region in \( M \), bounded by the broken geodesic loop \( P_1P_2P_3P_4P_5P_6P_1 \), by \( \Omega \). Clearly \( \Omega \) is homeomorphic to the 2-dimensional disk.

By the definition of foliation and non-tangency of geodesic in Riemannian manifolds, through each point on \( P_1P_2, P_3P_4 \) and \( P_5P_6 \), there is a unique leaf passing through it. Therefore, there is a well defined nowhere singular line field \( \xi \) given by the foliation \( \mathcal{F} \) on \( \Omega \). Let \( \text{dbl}(\Omega) \) be the double of \( \Omega \), i.e. the gluing of two copies of \( \Omega \) along the common boundary. Clearly \( \text{dbl}(\Omega) \) is homeomorphic to \( S^2 \). We can glue the two foliations together along the boundary to get a nowhere singular line field on \( \text{dbl}(\Omega) \setminus \{P_1, P_2, P_3, P_4, P_5, P_6\} \). Denote the line field on \( \text{dbl}(\Omega) \) by \( \xi \). The local picture of \( \xi \) looks like Figure 2 at the corner point \( P_i \).
By the Hopf Index Theorem, we have
\[ \sum_{i=1}^{6} \text{ind}(\bar{\xi}(P_i)) = \chi(\text{dbl}(\Omega)), \]
where \( \text{ind}(\bar{\xi}(P_i)) \) is the Hopf index of the line field \( \bar{\xi} \) at the singularity \( P_i \) and \( \chi(\text{dbl}(\Omega)) \) is the Euler characteristic number of \( \text{dbl}(\Omega) \). The Euler number is clearly equals to 2 since the double is homeomorphic to the 2-dimensional sphere. On the other hand by [Hop89] page 109, the index of \( \bar{\xi} \) at \( P_i \) is \( \frac{1}{2} \). Therefore \( \sum_{i=1}^{6} \text{ind}(\bar{\xi}(P_i)) = 3 \), we get a contradiction. \( \square \)

**Proof of Theorem 1.3** In [HR57], it is seen that by passing to their quotient spaces, isomorphism classes of foliations in \( \mathbb{R}^2 \) with non-compact leaves are in one-to-one correspondence with the set of simply connected (not necessarily Hausdorff) one-dimensional manifolds without boundary with a countable basis. Thus, to prove our theorem, we only need to show that the leaf space of our line foliation \( F \) is homeomorphic to the real line \( \mathbb{R} \). We will do this by showing that \( M/F \) is Hausdorff.

First, for any two leaves \( \sigma_{-1} \) and \( \sigma_1 \), we can pick a geodesic segment (not necessarily unit speed) \( \gamma : [-2, 2] \) connecting them, i.e. \( \gamma(t) \in \sigma_i \) for \( i = -1, 1 \). By Lemma 2.2, for each \( t \in (-2, 2) \) there is a leaf \( \sigma_t \) in \( F \) passing through \( \gamma(t) \). Once again, Lemma 1.4 gives that \( \sigma_t \) and \( \gamma \) will intersect in a single point.

Therefore the sets \( \{\sigma_t : t \in (-2, 0)\} \) and \( \{\sigma_t : t \in (0, 2)\} \) serve respectively as separating neighborhoods for the two leaves \( \sigma_{-1} \) and \( \sigma_1 \). This proves that the leave space is Hausdorff; by [HR57], the leave space is homeomorphic to \( \mathbb{R} \) and our foliation is equivalent to the standard foliation of \( \mathbb{R}^2 \). \( \square \)

### 3. Total curvature and hyperbolic pairs

In this section we review basic properties of the ideal boundary of a surface admitting total curvature, and prove Proposition 1.2.

#### 3.1. Total curvature of surfaces

For an open complete surface, let us recall the following definition of total curvature. Let \( M \) be a 2-dimensional complete non-compact surface possibly with nonempty boundary, with a Riemannian metric \( g \). Denote by \( K : M \to \mathbb{R} \) the Gaussian curvature of \( M \), and let \( d\mu \) be its associated Riemannian density. Define \( K^+ = \max\{K, 0\} \) and \( K^- = \min\{K, 0\} \). Thus we
can define the total positive curvature $c_+(M)$ and total negative curvature $c_-(M)$ by

$$ c_+(M) := \int_M K^+ \, d\mu, \quad c_-(M) := \int_M K^- \, d\mu. $$

We will say that the total curvature $c(M)$ of $M$ exists, or that $M$ admits total curvature, if at least one of $c_+(M)$ or $c_-(M)$ is finite, and we write in that case

$$ c(M) := c_+(M) + c_-(M). $$

By theorems of [CV35] and [Hub66], the total curvature of $M$ exists if and only if $c_+(M)$ is finite.

### 3.2. The ideal boundary.

Since the general case is rather involved (cf. [Shi91, Shi96, SST03] for general discussion on the ideal boundary of surfaces admitting total curvature), we will focus our study on the special case when $(S, g)$ is homeomorphic to plane $\mathbb{R}^2$, with a complete Riemannian metric $g$ without conjugate points.

We fix a base point $O \in S$. The unit tangent sphere at $O$ is isometric to a unit circle; we will use the closed interval $[0, 2\pi]$ to parametrize it counterclockwise, with $\theta = 0$ and $\theta = 2\pi$ corresponding to some given ray $\gamma$ to be fixed later. For any $\theta \in [0, 2\pi]$, there exists a unique ray $\gamma_\theta$ emanating from $O$ along the direction $\theta$. We denote the set of rays starting at $O$ by $\Gamma_O$.

For $0 \leq \alpha < \beta < 2\pi$, the plane is divided into two parts by the rays $\gamma_\alpha$ and $\gamma_\beta$: the closed sector $T_{\alpha,\beta}$ containing every point in the rays $\gamma_\theta$ with $\alpha \leq \theta \leq \beta$, and its closed complement that we denote as $T_{\beta,\alpha}$.

To each one of these sectors, we associate the numbers

$$ \lambda_\infty(T_{\alpha,\beta}) := \beta - \alpha - c(T_{\alpha,\beta}), $$

and

$$ \lambda_\infty(T_{\beta,\alpha}) := 2\pi - (\beta - \alpha) - c(T_{\beta,\alpha}), $$

the rule of thumb being that $\lambda_\infty$ of a sector is its interior angle minus its total curvature.

One can define the pseudo-distance

$$ d_\infty(\gamma_\alpha, \gamma_\beta) = \min\{\lambda_\infty(T_{\alpha,\beta}), \lambda_\infty(T_{\beta,\alpha})\}, $$

between rays in $\Gamma_O$;

then the ideal boundary $S(\infty)$ is defined to be the quotient of the set $\Gamma_O$ by the equivalent relation $\sim$ given by $\gamma_\alpha \sim \gamma_\beta$ if and only if $d_\infty(\gamma_\alpha, \gamma_\beta) = 0$. Then $d_\infty$ can be extended to a metric on $S(\infty)$, which is called the Tits metric. The topology induced by $d_\infty$ is called the Tits topology.

Now, we need the following characterization of $S(\infty)$ by Shioya

**Proposition 3.1 ([Shi91]).** Let $S^2$ be a simply connected complete Riemannian surface admits total curvature with one end, then

1. If $\lambda_\infty(S) < \infty$, then $(S(\infty), d_\infty)$ is isometric to the circle of length $\lambda_\infty(S)$. 

(2) If $\lambda_\infty(S) = \infty$, then $(S(\infty), d_\infty)$ is at most continuum union of closed line segments.

**Proposition 3.2** ([Shi96]). Let $M$ be a Riemannian surface with total curvature. Then for any line $\gamma$, $d_\infty(\gamma(-\infty), \gamma(\infty)) \geq \pi$. Moreover, for any $x, y \in M(\infty)$, with $d_\infty(x, y) > \pi$, there exists a line $\sigma$ with $\sigma(-\infty) = x$ and $\sigma(\infty) = y$. For simplicity, we will say that such line $\sigma$ connects $x$ to $y$.

**Remark 3.3.** Shioya’s proof of Proposition 3.2 uses Gauss-Bonnet in the region bounded by two rays asymptotic to $x$ and $y$ and a segment connecting two points on these two rays. Therefore, it will also work in surfaces with boundary. For our application, we will apply 3.2 to a half plane without conjugate points.

The following result is already known. As mentioned in the introduction, it was proven by Green for compact Riemannian manifolds [Gre58]; Guimaraes extended it later to the complete case [Ga92]. We include here a new proof for surfaces that takes advantage of the structure of the ideal boundary.

**Proposition 3.4.** Let $M^2$ be an open complete Riemannian surface without conjugate points. Suppose the total curvature of $M$ exists. Then $c(M) := \int_M K d\mu \leq 0$, and equality holds if and only if the Riemannian universal cover of $M$ is isometric to the flat $\mathbb{R}^2$.

**Proof.** Let’s consider the Riemannian universal cover of $M$, denoted it by $S$. We first note that by Huber’s result $c(S) \leq 2\pi = 2\pi \chi(S)$. Therefore if $c(S) > 0$, then by 3.1 $S(\infty)$ is isometric to a circle of length

$$\lambda_\infty(S) = 2\pi \chi(S) - c(S) < 2\pi.$$

It follows that the diameter of $S(\infty)$ is $< \pi$. This contracts to the fact that any line $\gamma$ in $S$ must have $d_\infty(\gamma(+\infty), \gamma(-\infty)) \geq \pi$. Therefore $c(S) \leq 0$. Suppose that if $c(S) = 0$, then the ideal boundary of $S$ is isometric to the circle with length $2\pi$. It follows from Theorem 5.2.1 in [SST03] that

$$\lim_{r \to \infty} \frac{2A(r)}{r^2} = \lambda_\infty(S) = 2\pi,$$

where $A(r)$ is the length of the distance ball $B(O, r) = \{x \in S||x, O| \leq r\}$ for a fixed base point $O \in S$. Applying the rigidity theorem of [BE13], we know $S$ is isometric to the flat $\mathbb{R}^2$. □

3.3. **Ray convergence.** The following property is essentially proved in [SST03], where all rays are assumed emanating perpendicularly from a core of $M$. For completeness we include a simple proof in our special case. If the surface is a half plane, then there is only one sector between two rays, we called it $T_{\alpha,\beta}$ without referring to any orientation. The distance between $\gamma_\alpha$ and $\gamma_\beta$ is defined by:

$$d_\infty(\gamma_\alpha, \gamma_\beta) = \lambda_\infty(T_{\alpha,\beta}).$$

Therefore the ideal boundary of the half plane $M$ can be defined in a similar way as the surface $S$. 
Lemma 3.5 (Lemma 3.5.1 in [SST03]). Let $M$ be a Riemannian surface without conjugate points, homeomorphic to the upper half plane with totally geodesic boundary. Fix a base point $O \in \partial M$. Let $\{\gamma_i\}$ be a sequence of rays issued from $O$ which converges to a ray $\gamma_{\theta_\infty}$ with $\theta_i$ monotonic decreased to $\theta_\infty$. Suppose that
$$\sup_i d_\infty(\gamma_{\theta_1}, \gamma_i) < +\infty.$$ Then we have
$$d_\infty(\gamma_{\theta_1}, \gamma_{\theta_\infty}) < \infty.$$

**Proof.** For simplicity, we write $\gamma_{\theta_i}$ as $\gamma_i$ and $\gamma_{\theta_\infty}$ as $\gamma_\infty$. The region bounded by $\gamma_1$ and $\gamma_i$ in the upper half plane is denoted by $T_{1,i}$ since $\theta_i$ is decreasing. Note that in this case, there is only one sector between two rays, we called it $T_{1,i}$ without referring to any orientation. The distance is defined by:
$$d_\infty(\gamma_1, \gamma_i) = \lambda_\infty(T_{1,i})$$
for all $i \in \mathbb{N}$. By the monotonicity of $\{\theta_i\}$, we know $T_{1,\infty}$ contains every $\gamma_i$. Denote the open metric ball centered at $O$ with radius $r$ by $B(r)$, and the angle between $\gamma_i$ and $\gamma_j$ by $\theta_{i,j} = \theta_i - \theta_j$ for $i < j$, depicted in Figure 3.3.

By the definition of $d_\infty$, for each $i$ and all $r > 0$, we have
$$d_\infty(\gamma_1, \gamma_i) = \theta_{1,i} - c(T_{1,i} \cap B(r)) - c(T_{1,i} \setminus B(r)). \quad (3.1)$$
Since
$$c(T_{1,i} \setminus B(r)) \leq \int_{T_{1,i} \setminus B(r)} K^+ d\mu \leq \int_{T_{1,\infty} \setminus B(r)} K^+ d\mu.$$ Plugging it into (3.1), we have
$$d_\infty(\gamma_1, \gamma_i) \geq \theta_{1,i} - c(T_{1,i} \cap B(r)) - \int_{T_{1,\infty} \setminus B(r)} K^+ d\mu. \quad (3.2)$$
Since $B(r)$ is compact, we have
$$\lim_{i \to \infty} c(T_{1,i} \cap B(r)) = c(T_{1,\infty} \cap B(r)).$$
and clearly
\[ \lim_{i \to \infty} \theta_{1,i} = \theta_{1,\infty}. \]
Taking lower limit as \( i \to \infty \) of the (3.2), we get
\[ \liminf_{i \to \infty} d_{\infty}(\gamma_1, \gamma_i) \geq \theta_{1,\infty} - c(T_{1,\infty}) \geq \theta_{1,\infty} - \lambda_{\infty}(T_{1,\infty}). \tag{3.3} \]
By assumption \( S \) admits total curvature. It follows that \( \int_{T_{1,\infty}} K^+ d\mu < \infty \). Therefore
\[ \lim_{r \to \infty} \int_{T_{1,\infty}} K^+ d\mu = 0. \]
Letting \( r \to \infty \) on the right hand side of (3.3), we have
\[ \liminf_{i \to \infty} d_{\infty}(\gamma_1, \gamma_i) \geq \theta_{1,\infty} - \lambda_{\infty}(T_{1,\infty}) = \lambda_{\infty}(T_{1,\infty}). \tag{3.4} \]
The left hand side of (3.4) is finite by the assumption, therefore
\[ d_{\infty}(\gamma_1, \gamma_{\infty}) \leq \lambda_{\infty}(T_{1,\infty}) < \infty. \]
This finishes the proof. \( \Box \)

3.4. Existence of hyperbolic pairs.

*Proof of Proposition 1.2.\* Now we assume \( c(S) < 0 \), and we will provide a hyperbolic pair for a given \( \gamma \).

The first case is \( c(S) > -\infty \), then \( S(\infty) \) is isometric to a circle of length \( \ell > 2\pi \) by Proposition 3.1. In particular any line \( \gamma \) defines two ideal boundary points \( \gamma(-\infty) \) and \( \gamma(+\infty) \), which are at least \( \pi \) apart. Since \( \ell > 2\pi \), there exists a connected component of \( S \setminus \gamma(\mathbb{R}) \), denoted by \( M \) such that \( M(\infty) \) is an interval of length larger than \( \pi \). Clearly one can choose \( x, y \in M(\infty) \) different from \( \gamma(-\infty) \) and \( \gamma(+\infty) \) and with \( d_{\infty}(x, y) > \pi \). Connecting them yields a line \( \sigma \). Since our surface \( M \) has no conjugate points, \( \sigma \) does not intersects \( \gamma \). Moreover,
\[ \infty = \lim_{t \to \infty} d(\gamma_0(t), \gamma_x(t)) \leq \lim_{t \to \infty} d(\gamma(t), \sigma(t)), \]
and
\[ \infty = \lim_{t \to \infty} d(\gamma_{x}(t), \gamma_y(t)) \leq \lim_{t \to -\infty} d(\gamma(t), \sigma(t)), \]
imply that \( (\gamma, \sigma) \) is a hyperbolic pair.

The second case is \( c(M) = -\infty \), we have \( \lambda_{\infty}(S) = \infty \). Let \( \gamma : \mathbb{R} \to S \) be the given line. We will construct another line \( \sigma \) so that the pair \( (\gamma, \sigma) \) is a hyperbolic
pair. Clearly $\gamma$ divides $S$ into two parts, let consider one of them called $M$ such that $\lambda_\infty(M) = \infty$. Now we consider $M$ itself as a Riemannian surface without conjugate points, i.e. we do not view $M$ as a subset of $S$. We can define its ideal boundary $M(\infty)$ as we recalled the definition right before the Lemma 3.5. As before, we fix a base point of $\gamma(0) \in \partial M = \gamma(\mathbb{R})$. Let $\gamma_0(t) = \gamma(-t)$ and $\gamma_\pi(t) = \gamma(t)$ under the canonical identification of the unit tangent sphere at $O$ and $[0, \pi]$. Consider the set

$$I_0 = \{ \theta \in [0, \pi] \mid d_\infty(\gamma_0, \gamma_\theta) < \infty \}.$$ 

It is easy to see that $I_0$ is an interval: if $\theta \in I_0$, then for any $\theta'$ with $0 \leq \theta' \leq \theta$, the sector $T_{\theta,\theta'}$ contains the sector $T_{0,\theta'}$, and thus from $c(T_{0,\theta'}) > -\infty$, we get $c(T_{\theta,\theta'}) > -\infty$.

Clearly $0 \in I_0$, and $\pi \notin I_0$. There are two sub-cases; the first is when $I_0$ is a relatively open interval in $[0, \pi]$, i.e.

$$I_0 = (0, a), \quad a < \pi.$$ 

This means

$$\lim_{\theta \to a^-} d_\infty(\gamma_0, \gamma_\theta) = \infty$$

That is there exists $[x, y] \subset (0, a)$ such that

$$d_\infty(\gamma_0, \gamma_x) > 0, \quad \text{and} \quad d_\infty(\gamma_x, \gamma_y) > \pi$$

Then by Proposition 3.2, we can connect $\gamma_x$ to $\gamma_y$ to get a line $\sigma$. Similarly, we can define

$$I_\pi = \{ \theta \in [0, \pi] \mid d_\infty(\gamma_\theta, \gamma_\pi) < \infty \}.$$ 

The very same argument shows that if $I_\pi = (b, \pi]$ then we can find a hyperbolic pair.

So we only need to consider the second sub-case, i.e. $I_0 = [0, a]$ and $I_\pi = [b, \pi]$ with $a < b$. By the construction of $I_0$ and $I_\pi$ we know

$$d_\infty(\gamma_0, \gamma_c) = \infty = d_\infty(\gamma_\pi, \gamma_c),$$

where $c = \frac{a + b}{2}$. We consider

$$I_c = \{ \theta \in [0, \pi] \mid d_\infty(\gamma_\theta, \gamma_c) < \infty \}.$$ 

Clearly by Lemma 3.5, $I_c$ is a sub interval of $(a, b)$, with end points $a' \leq c \leq b'$. If one of the end point of $I_c$ is open, then the same construction as in the sub-case $I_0 = [0, a]$ will give two points in the ideal boundary of distance at least $1.5\pi$ apart in $M(\infty)$, therefore connecting them will give us a desired $\sigma$. If both end points are closed, i.e.

$$I_c = [a', b'],$$

with $a < a'$ and $b' < b$. Then we let $x = \frac{a + a'}{2}$ and $y = \frac{b' + b}{2}$. Then

$$d_\infty(\gamma_0, \gamma_x) = d_\infty(\gamma_x, \gamma_y) = d_\infty(\gamma_y, \gamma_\pi) = \infty.$$ 

Connecting $x$ to $y$ using 3.2 yields the desired $\sigma$. \qed
4. Visibility planes and weak hyperbolic pairs

**Definition 4.1.** A Riemannian manifold $M$ without conjugate points satisfies the visibility axiom at $p$ if for each $\varepsilon > 0$ there exists a constant $R = R(\varepsilon, p) > 0$ such that for any geodesic segment $\sigma : [a, b] \to M$ satisfying $d(p, \sigma) > R$, the angle $\angle_p(\sigma(a), \sigma(b)) \leq \varepsilon$.

If $p$ is arbitrary, such a manifold is called a visibility manifold; the reader can consult [Ebe72] or [EO73] for several of their properties.

**Definition 4.2.** Two geodesic rays $\gamma, \sigma : [0, \infty) \to M$ are asymptotes if there exists a constant $C > 0$ such that $d(\gamma(t), \sigma(t)) \leq C$ for every $t > 0$.

We will write $\gamma \sim \sigma$ to indicate that two rays are asymptotes. Due to the triangle inequality, $\sim$ is an equivalence relation on the set of rays in $M$.

**Definition 4.3.** Let $\Gamma_p$ be the set of all geodesic rays in $M$ emanating from $p$. We will use $M_a(\infty)$ to denote the set of equivalence classes $\Gamma_p / \sim$. If $\gamma \in \Gamma_p$, we will denote by $\gamma(\infty)$ its equivalence class in $M_a(\infty)$.

**Proposition 4.4.** Let $M$ be a open manifold satisfying the visibility axiom at some point $p$. Then for any two different rays $\gamma$ and $\sigma$ emanating from $p$, we have that $\gamma(\infty) \neq \sigma(\infty)$.

**Proof.** Suppose the contrary that there exists some $c > 0$ such that $d(\gamma, \sigma) < c$. Then $\angle_p(\gamma(t), \sigma(t)) \to 0$ since the segments connecting $\gamma(t)$ and $\sigma(t)$ clearly drifts to infinity as $t \to \infty$. On the other hand $\angle_p(\gamma(t), \sigma(t)) = \angle(\gamma'(0), \sigma'(0))$, a contradiction. □

**Proposition 4.5.** Let $x, y \in M_a(\infty)$ be two different points, and denote by $\gamma_x$ and $\gamma_y$ respectively the rays emanating from $p$ with $\gamma_x(\infty) = x$ and $\gamma_y(\infty) = y$. Then for any sequence $t_k \to \infty$, the sequence of the segments $\alpha_k$ connecting $\gamma_x(t_k)$ and $\gamma_y(t_k)$ has a subsequence that converges to a line in $M$.

**Proof.** Since $\angle_p(\gamma_x(t_k), \gamma_y(t_k))$ equals to $\angle(\gamma_x'(0), \gamma_y'(0)) = \varepsilon_0 > 0$, it follows from the definition of visibility at $p$ that there exists an $R = R(p, \varepsilon_0)$ such that $d(p, \alpha_k) \leq R$.

for $k \in \mathbb{N}$. Since closed balls of bounded radius are compact, the sequence of segments $\{\alpha_k\}_{k=1}^\infty$ subconverges to a line. □

If $M$ is a plane, then the limiting line $\alpha$ stays inside the open convex region bounded by the rays $\gamma_x$ and $\gamma_y$. Since this is the 2-dimensional case, we will switch from $M$ to $S$ to denote it in what follows.

**Proof of Proposition 4.3** We will see that, given $\gamma \in \mathcal{F}$, there is some line $\sigma$ such that $(\gamma, \sigma)$ is a weak hyperbolic pair. $\gamma$ separates $S$ into two open half planes. Let $\alpha_+$ and $\alpha_-$ be two different rays emanating from $p = \gamma(0)$ and lying within one component. By Proposition 4.5, we can construct a line $\sigma$ which lies within the
sector bounded by $\alpha_\pm$. On the other hand, from Proposition 4.4, $\gamma : [0, \infty) \to S$ and $\alpha_+$ are not asymptotes. Thus, there is a sequence $t_k \to \infty$ such that

$$d(\gamma(t_k), \alpha_+(t_k)) \to \infty$$

and therefore

$$d(\sigma(t_k), \gamma(t_k)) > d(\alpha_+(t_k), \gamma(t_k)) \to \infty.$$ 

A similar argument for $\sigma$ and $\gamma$ at $-\infty$ proves that $(\gamma, \sigma)$ is a weak hyperbolic pair. 

\section{Proofs of Theorems 1.1 and 1.2}

\textbf{Proof of Theorem 1.1.} Let $\gamma \in \mathcal{F}$ be a line. Suppose $c(S) < 0$; then by Proposition 1.2, there exists a line $\sigma$ such that $(\gamma, \sigma)$ is a hyperbolic pair. However this is impossible. In fact for any point $x_0 \not\in S^\sigma$ and $x \in H^+_\sigma$, there must be a leave $\bar{\gamma}$ passing through it and with bounded distance to $\gamma$, see Figure 5. Let

$$C = \text{dist}_{H}(\bar{\gamma}, \gamma),$$

be the Hausdorff distance between the two leaves, and parametrize $\gamma$ and $\bar{\gamma}$ such that $\text{dist}(\gamma(0), \bar{\gamma}(0)) \leq C$. Then after possible reversing the parameter of one of the lines, we have

$$\lim_{t \to \infty} \text{dist}(\gamma(t), \bar{\gamma}(t)) \leq 3C.$$ 

In fact, for any $t$, there exists $s$ such that $\text{dist}(\gamma(t), \bar{\gamma}(s)) \leq C$ by the definition of Hausdorff distance. Note that the $\gamma$ and $\bar{\gamma}$ are lines, therefore any two points on one line must realize the distance between them, it follows from the triangle inequality that $t \leq s + 2C$, and $s \leq t + 2C$, thus $|s - t| \leq 2C$. Therefore

$$\text{dist}(\gamma(t), \bar{\gamma}(t)) \leq \text{dist}(\gamma(t), \bar{\gamma}(s)) + |t - s| \leq 3C.$$ 

By the definition of hyperbolic pair, $\bar{\gamma}$ must exit $H^+_\sigma$, so $\bar{\gamma}$ must intersect $\sigma$ twice, this contradicts to the fact that geodesics do not bifurcate in Riemannian manifold. Therefore it follows that $c(S) = 0$ and by Proposition 3.4, $S$ is isometric to the flat $\mathbb{R}^2$. 

\textbf{Proof of Theorem 1.2.} The argument for the first part of the Theorem mimics the previous proof, but using Proposition 1.3 to replace the hyperbolic pair by a weak hyperbolic pair, and working with a sequence $t_k \to \infty$ instead of arbitrary $t \in \mathbb{R}$.

For the second part of the Theorem, suppose $\tilde{M}$ is the universal cover of a closed surface $\Sigma_g$ with genus $g \geq 1$ and without conjugate points. If $g = 1$, then by E.
Hopf’s theorem for torus without conjugate points, $\Sigma_g$ is a flat torus, and $M$ is the flat plane. If $g > 1$, then $M$ satisfies the visibility axiom, cf. [Ebe72]. Therefore from the first part of the Theorem, it cannot admit a line foliation with bounded Hausdorff distance. □

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