LOCALLY LIPSCHITZ GRAPH PROPERTY FOR LINES

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Abstract. On a non-compact, smooth, connected, boundaryless, complete Riemannian manifold \((M, g)\), some ideal boundary elements could be defined by rays (or equivalently, by Busemann functions). From the viewpoint of Aubry-Mather theory, these boundary elements could be regarded as an analogue to the static classes of Aubry sets, and thus lines should be thought of as the counterpart of the semi-static curves connecting different static classes. In Aubry-Mather theory, a core property is the Lipschitz graph property for Aubry sets and for some kind of semi-static curves. In this note, we prove such a result for a set of lines which connect the same pair of boundary elements. We also discuss an initial relation with ends (in the sense of Freudenthal).

Let \(M\) be a smooth, non-compact, complete, boundaryless, connected Riemannian manifold with Riemannian metric \(g\). Let \(TM\) be the tangent bundle and \(\pi\) be the canonical projection of \(TM\) onto \(M\). The distance \(d\) on \(M\) is induced by the Riemannian metric \(g\) and on \(TM\) it is induced by the Sasaki metric \(g_S\). We also use \(l\) to denote the length of a curve with respect to the Riemannian metric \(g\). Throughout this paper, all geodesic segments are always parameterized to be unit-speed. By a ray, we mean a geodesic segment \(\gamma: [0, +\infty) \to M\) such that \(d(\gamma(t_1), \gamma(t_2)) = |t_2 - t_1|\) for any \(t_1, t_2 \geq 0\). Throughout this paper, \(|\cdot|\) means Euclidean norm. By our assumptions on \(M\), for any point \(x \in M\), there always exists at least one ray emanating from \(x\). By definition, the Busemann function associated to a ray \(\gamma\) is defined as

\[b_\gamma(x) := \lim_{t \to +\infty} [d(x, \gamma(t)) - t].\]

Clearly, \(b_\gamma\) is a Lipschitz function with Lipschitz constant 1, i.e.

\[|b_\gamma(x) - b_\gamma(y)| \leq d(x, y).\]

Moreover, it is proved in [13] that \(b_\gamma\) is locally semi-concave with linear modulus (for the definition, see [9]).

By rays, or their Busemann functions, one could define a set \(M(\infty)\) of ideal boundary elements. By definition, \(M(\infty)\) is the set of equivalent classes of rays (or equivalently, Busemann functions), where two rays \(\gamma_1\) and \(\gamma_2\) (or equivalently, two

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Busemann functions $b_{\gamma_1}$ and $b_{\gamma_2}$) are equivalent if and only if $b_{\gamma_1} = b_{\gamma_2} + \text{const.}$ For more concrete information on $M(\infty)$, we refer to [3, 27]. It should be noted that one could also use horofunctions or $dl$ (i.e. distance like)-functions, instead of rays (or equivalently, Busemann functions), to define other kinds of ideal boundaries, for details, see [20, 27, 31]. A deeper study on the relationship among these boundaries and their influence on the geometry of the Riemannian metric and the dynamics of geodesic flow in the general setting should be very interesting.

Now we introduce a more restrictive notation than ray. A geodesic $\gamma : \mathbb{R} \to M$ is said to be a line if

$$d(\gamma(t_1), \gamma(t_2)) = |t_2 - t_1|$$

for any $t_1, t_2 \in \mathbb{R}$. Comparing with rays, lines are much rarer. Given a line $\gamma : \mathbb{R} \to M$, we can define the barrier function $B_\gamma$, under the motivation of Aubry-Mather theory, to be

$$B_\gamma(x) = b_{\gamma^+}(x) + b_{\gamma^-}(x),$$

where $\gamma^+, \gamma^-$ are two rays defined by $\gamma^\pm : [0, +\infty) \to M, \gamma^\pm(t) = \gamma(\pm t)$. It should be mentioned that this kind of barrier function has appeared in the literature (e.g. [28] Page 287, Line -15–Line -10), [30] Page 218, Line -3) in the context of non-negative Ricci curvature. Here, we follow the ideas from [13].

For any line $\gamma : \mathbb{R} \to M$, $-\gamma$ is defined by $(-\gamma)(t) := \gamma(-t)$ for any $t \in \mathbb{R}$. For any $\tau \in \mathbb{R}$ (resp. $\tau > 0$), and for any line $\gamma : \mathbb{R} \to M$ (resp. ray $\gamma : [0, \infty) \to M$), we can define the time translation $\gamma_\tau$ by $\gamma_\tau(t) := \gamma(t + \tau)$. Clearly for any line $\gamma$, $B_\gamma \geq 0$, $B_{-\gamma} = B_\gamma$ and $B_{\gamma_\tau} = B_\gamma$ for any $\tau \in \mathbb{R}$.

Rays, lines, Busemann functions and barrier functions are important objects and tools in the field of Riemannian geometry. They were widely studied and applied in the literature, for example, [3, 4, 5, 6, 7, 10, 12, 13, 18, 20, 21].

Since the elements in $M(\infty)$ are represented by rays, thus for any line $\gamma$, we could think $\gamma$ as a geodesic connecting two elements in $M(\infty)$: $(-\gamma)|[\tau_1, \infty)$ and $\gamma|[\tau_2, \infty)$ for $\tau_1, \tau_2 \in \mathbb{R}$. Since $\gamma$ is a line, $\gamma|[\tau_1, \infty)$ and $\gamma|[\tau_2, \infty)$ determine the same element in $M(\infty)$ for any $\tau_1, \tau_2 \in \mathbb{R}$, the same is true for $-\gamma$. So the choice of $\tau_1$ and $\tau_2$ is not crucial at all, and we may choose $\gamma^- := (-\gamma)|[0, \infty)$ and $\gamma^+ := \gamma|[0, \infty)$ as the representations of the two elements in $M(\infty)$ which are connected by the line $\gamma$.

Now we define

\((*)\) \quad $\hat{N}_{\gamma^-} := \{(\xi(t), \dot{\xi}(t)) \in TM : t \in \mathbb{R} \text{ and } \xi \text{ is a line in } M \text{ connecting } \gamma^- \text{ and } \gamma^+\}$,

and let $N_{\gamma^-}, N_{\gamma^+}$ be the projection of $\hat{N}_{\gamma^-}$ into $M$. In other words, the flow lines (with respect to the geodesic flow determined by $g$) $(\xi, \dot{\xi})$ in $\hat{N}_{\gamma^-}, \gamma^+$ must satisfy $b_\xi = b_{\gamma^\pm} + \text{const.}$

Our main result in this note is

**Theorem 1.** $\pi : \hat{N}_{\gamma^-} \to N_{\gamma^-}$ is a locally bi-Lipschitz map.

**Remark 2.** In this long remark, we discuss our motivation for Theorem 1. Readers who want to go into the proof of Theorem 1 directly may skip this remark. The motivation comes from Aubry-Mather theory. Aubry-Mather theory is a great progress in the study of convex Hamiltonian dynamical systems, which concentrates on the study of minimal invariant sets (e.g. Mather sets, Aubry sets and Mañé sets), connecting orbits among them and diffusion orbits along them, by minimization method. In Aubry-Mather theory, a core property is the Lipschitz graph property of Aubry sets. (For simplicity, here we only consider the autonomous version.)
In fact, it is quite easy to generalize the Lipschitz graph property by Mather’s curve shortening lemma \[24\] Page 186, Lemma] to the set:

\[
\mathcal{N}_{A_1,A_2} := \left\{ (\gamma(t),\dot{\gamma}(t)) \in TM : t \in \mathbb{R} \text{ and } (\gamma,\dot{\gamma}) \text{ is a semi-static orbit} \right\}
\]

where \(A_1\) and \(A_2\) are any two fixed static classes. Here, the definition of limit sets could be found in any textbook on dynamical systems (e.g. \[29\] Page 148); for semi-static orbits and static class, roughly speaking, semi-static orbits are a kind of minimal orbits which realize the so-called action-potential, and a static class is a subset of Aubry set, on which any two points can be embedded into a loop with arbitrarily small (non-negative) action with respect to the suitable modified Lagrangian. For concrete definitions or more details on Aubry-Mather theory, we refer to \[24\], \[25\], \[16\], especially to \[26\] Page 1670- Page 1671 (where static classes are nothing but elements of quotient Aubry set) and \[12\] Page 794 for readable versions. In \[11\] Proposition 0.3 and \[23\] Section 4, the study of viscosity solutions of a Hamilton-Jacobi equation with respect to a Tonelli Hamiltonian on a non-compact manifold was inspired by the method of constructing Busemann functions. Some connections between viscosity solutions and ideal boundaries were also realized by these authors. This viewpoint also appeared in a more general setting \[2\]. In \[13\], the authors studied the dynamics of lines and rays from the viewpoint of Aubry-Mather theory and tried to look for more analogies between the elements in \(M(\infty)\) and the static classes of Aubry set in Aubry-Mather theory. Clearly, there is some obvious evidence supporting the analogue. For example, by weak KAM theory, every static class could determine a (globally defined) viscosity solution up to a constant; for elements in \(M(\infty)\), it is also true. Thus, one would ask whether there is still some Lipschitz graph property for some invariant (with respect to geodesic flow) sets. This is the main motivation of this note and Theorem 1 could answer this problem in a confirmed way. We hope these viewpoints from dynamical systems would provide more geometric information of the Riemannain metric \(g\) in the future.

The main tool for the proof of this theorem is still the curve-shortening lemma \[24\] Page 186, Lemma].

**Proof.** In \[13\] Proposition 5.1, the authors proved that a flow line \((\xi, \dot{\xi})\) lies in \(\tilde{N}_{\gamma^-,\gamma^+}\) if and only if \(\xi \sim \gamma\). Here, \(\sim\) is an equivalence relation defined by \(\xi \sim \gamma\) if and only if \(\xi \prec \gamma\) and \(\gamma \prec \xi\), and the relation \(\prec\) is defined further as follows.

We say \(\xi \prec \gamma\) if:

- \(B_\gamma(\xi(t)) \equiv 0\);
- For any \(\tau \in \mathbb{R}\), \(\xi|[\tau, \infty)\) is a coray to \(\gamma^+\) and \((-\xi)|[\tau, \infty)\) is a coray to \(\gamma^-\). The definition of the coray (e.g. \[8\]) is postponed until we encounter Theorem 9, here we only need to mention a useful property (e.g. \[7\] Proposition 2.7): A ray \(\gamma_1\) is a coray to the ray \(\gamma\) if and only if \(b_\gamma(\gamma_1(t_1)) - b_\gamma(\gamma_1(t_2)) = t_2 - t_1\) for any \(t_1, t_2 \geq 0\).

We will prove that for any compact subset \(K\) of \(M\), there exists a constant \(C\) (depends on the compact subset \(K\)) such that

\[
d(\pi^{-1}(x_1) \cap \tilde{N}_{\gamma^-,\gamma^+}, \pi^{-1}(x_2) \cap \tilde{N}_{\gamma^-,\gamma^+}) \leq Cd(x_1, x_2)
\]

for any \(x_1, x_2 \in K \cap N_{\gamma^-,\gamma^+}\). First, we choose two connected open sets \(U_1, U_2\) with compact closures and such that \(D_1(K) \subset U_1\) and \(D_1(U_1) \subset U_2\), here, for any compact subset \(S\), \(D_1(S)\) means the closed ball \(D_1(S) := \{x \in M : d(x, S) \leq 1\}\),
and $\bar{U}_1$ means the closure of $U_1$. Then the Riemannian metric $g$, restricted on the open set $U_2$, is uniformly positive definite.

Now we reformulate Mather’s curve shortening lemma \cite{24} Page 186, Lemma] (see also \cite{25} Page 1361–Page 1362 for further explanation) as follows:

**Lemma 3.** There exist $\epsilon, \delta, C > 0$ such that if $\alpha, \beta : [t_0 - \epsilon, t_0 + \epsilon] \to M$ are two distinct geodesic segments in $U_1$ with $d(\alpha(t_0), \beta(t_0)) \leq \delta$ and

$$d((\alpha(t_0), \dot{\alpha}(t_0)), (\beta(t_0), \dot{\beta}(t_0))) \geq C d(\alpha(t_0), \beta(t_0)), $$

then there exist $C^1$ curves $c_1, c_2 : [t_0 - \epsilon, t_0 + \epsilon] \to U_2$ such that $c_1(t_0 - \epsilon) = \alpha(t_0 - \epsilon), c_1(t_0 + \epsilon) = \beta(t_0 + \epsilon), c_2(t_0 - \epsilon) = \beta(t_0 - \epsilon), c_2(t_0 + \epsilon) = \alpha(t_0 + \epsilon)$, and

$$4 \epsilon > l(c_1) + l(c_2).$$

For the proof of this lemma, we refer to \cite{24}. Despite some slight modifications (just restriction of Mather’s result to the setting of geodesic flow), the original proof still goes through. We remark that $\epsilon, \delta$ in the lemma could be taken arbitrarily small.

Based on this lemma, we could go on to prove our main result as follows. Assume that our theorem is not true. Then for any $\delta > 0, C > 0$, there exist two different lines $\xi$ and $\eta$ ($\xi$ and $\eta$ either are geometrically distinct (i.e. they differ as subsets of $M$) or differ by a parameterization) with $(\xi, \dot{\xi}), (\eta, \dot{\eta}) \in \hat{N}_{\gamma^+, \gamma^-}$, a real number $t_0$ (if necessary, we operate a time translation on $\xi$ or on $\eta$), such that $\xi(t_0) \in K, \eta(t_0) \in K$, $d(\xi(t_0), \eta(t_0)) \leq \delta$ and $d((\xi(t_0), \dot{\xi}(t_0)), (\eta(t_0), \dot{\eta}(t_0))) \geq C d(\xi(t_0), \eta(t_0))$. Choosing $\epsilon$ sufficiently small, we could assume that $\xi|[t_0 - \epsilon, t_0 + \epsilon] \subset U_1, \eta|[t_0 - \epsilon, t_0 + \epsilon] \subset U_1$. Since $\xi$ and $\eta$ are lines, $\xi|[t_0 - \epsilon, t_0 + \epsilon]$ and $\eta|[t_0 - \epsilon, t_0 + \epsilon]$ must be distinct geodesic segments. (If $\xi$ and $\eta$ are geodesically distinct, this claim is clear; if $\xi$ and $\eta$ only differ by a parameterization and moreover the claim is not true, $\xi$ and $\eta$ will be closed geodesics and this will be absurd since they are lines.) Choose $\delta, C$ suitably, such that the conditions of curve shortening lemma (Lemma 3) is satisfied. By the curve shortening lemma, there exist two $C^1$ curves $c_1, c_2 : [t_0 - \epsilon, t_0 + \epsilon] \to U_2$ such that $c_1(t_0 - \epsilon) = \xi(t_0 - \epsilon), c_1(t_0 + \epsilon) = \eta(t_0 + \epsilon), c_2(t_0 - \epsilon) = \eta(t_0 - \epsilon), c_2(t_0 + \epsilon) = \xi(t_0 + \epsilon)$, and

$$4 \epsilon > l(c_1) + l(c_2).$$

Thus,

$$B_\gamma(c_1(t_0))$$

$$= b_\gamma(c_1(t_0)) + b_\gamma(c_2(t_0))$$

$$\leq b_\gamma(\eta(t_0 + \epsilon)) + d(c_1(t_0), \eta(t_0 + \epsilon)) + b_\gamma(\xi(t_0 - \epsilon) + d(c_1(t_0), \xi(t_0 - \epsilon))$$

$$\leq b_\gamma(u(t_0 + \epsilon)) + b_\gamma(\xi(t_0 - \epsilon)) + l(c_1)$$

$$= b_\gamma(u(t_0)) - \epsilon + b_\gamma(\xi(t_0)) - \epsilon + l(c_1)$$

$$= b_\gamma(u(t_0)) + b_\gamma(\xi(t_0)) - 2 \epsilon + l(c_1),$$

where the first inequality follows from the fact that Busemann functions are Lipschitz with Lipschitz constant 1; the second equality holds because for any $\tau \in \mathbb{R}$, $\xi|\tau, \infty), \eta|\tau, \infty)$ are corays to $\gamma^+$ and $(-\xi)|\tau, \infty), (-\eta)|\tau, \infty)$ are corays to $\gamma^-$. 
Analogously, we could get
\[(\#)\quad B_\gamma(c_2(t_0)) \leq b_\gamma(\xi(t_0)) + b_{\gamma^-}(\eta(t_0)) - 2\epsilon + l(c_2).\]

Combining inequalities (\#) and (\#), we obtain
\[B_\gamma(c_1(t_0)) + B_\gamma(c_2(t_0)) \leq b_{\gamma^-}(\eta(t_0)) - 2\epsilon + l(c_1) + l(c_2) \leq 0,
\]
where the last inequality holds because \(B_\gamma \xi \equiv 0, B_\gamma \eta \equiv 0\) and \(l(c_1) + l(c_2) < 4\epsilon\). But it contradicts the fact that \(B_\gamma\) is a non-negative function. The contradiction proves Theorem 1.

By the procedure of the proof of Theorem 1, in fact we could get a stronger result as follows. For any line \(\gamma\), let
\[\dot{G}_\gamma := \{(\xi(t), \xi'(t)) \in TM : t \in \mathbb{R}\text{ and } \xi\text{ is a line with } \xi < \gamma\},\]
and \(G_\gamma\) be the projection of \(\dot{G}_\gamma\) into \(M\). In the proof of Theorem 1, only this fact is used: For any line \(\xi \in N_{\gamma^-}, \gamma^+\), \(\xi < \gamma\). So we get

**Corollary 4.** \(\pi : \dot{G}_\gamma \to G_\gamma\) is a locally bi-Lipschitz map.

**Remark 5.** Clearly, \(\dot{G}_\gamma\) is formally larger than \(\dot{N}_\gamma\). We believe that \(\dot{G}_\gamma\) should be strictly larger than \(\dot{N}_\gamma\) in some cases, where the confidence comes from the non-symmetric examples of parallel relation for lines [8, (23.6)] in a two-dimensional straight space and the discussion of order property on lines [13, Section 4], but we still lack of concrete examples so far. The construction of such an example seems, at least to the author, rather subtle. Even so, every line \(\xi\) in \(G_\gamma\) (namely, \(\xi < \gamma\)) connects the same pair of ends (in the sense of Freudenthal) as \(\gamma\) does; we will make this point clearer in Theorem 9. Also note that for a line \(\gamma, \gamma^+, \gamma^-\) always represent distinct elements in \(M(\infty)\), but they may represent the same end (considering Euclidean \(n\)-space \(\mathbb{R}^n\) with \(n \geq 2\), it has only one end and every geodesic is a line).

**Remark 6.** Both Theorem 1 and Corollary 4 could be regarded as analogies of Mather’s Lipschitz graph property. One may ask whether we could simply deduce them from Mather’s work. It is impossible at least by the following reasons:

1. Aubry sets are characterized by cohomologies (or equivalently, homologies) in classical Aubry-Mather theory. However, in our setting cohomologies and homologies are not involved due to the lack of recurrence of lines. This difference leads to the formulations and the proofs of the results in different settings being quite different.

2. In our setting, the set \(M(\infty)\), regarded as the analogue of Aubry sets, lies at infinity and is “imaginary” in some sense and we have to deal with it more or less in a different way.

3. The statement of Corollary 4 is much more general and we could not find a totally exact counterpart in the classical Aubry-Mather theory.
Remark 7. The injectivity of $\pi|\bar{N}_{\gamma^-}^{\gamma^+}$ or $\pi|\bar{G}_{\gamma}$ also follows from the differentiability of the barrier function $B$, on $G_{\gamma}[13$ Theorem 3.2].

Corollary 8. If $G_{\gamma} = M$, then by Corollary 4, $M$ will be foliated by the lines $\xi$ where $\xi < \gamma$. Moreover, the foliation is locally Lipschitz. In this case, $b_{\gamma^+}$ (also, $b_{\gamma^-}$) is a locally $C^{1,1}$ viscosity solution (in fact, classical solution) to the Hamilton-Jacobi equation

$$|\nabla u|_{g} = 1,$$

where $\nabla$ is the gradient, and $|\cdot|_{g}$ is the norm on $TM$, both determined by the Riemannian metric $g$.

In the rest of this note, motivated by the statements in Remark 5, we propose an initial relation of rays with the ends introduced by Freudenthal. For a non-compact, locally compact, Hausdorff topological space, following Freudenthal [17] we could define its (topological) ends. In our special setting of non-compact, complete Riemannian manifold, we could define them by rays [1, III 2]. An equivalence class of cofinal rays (here two rays $\gamma_1$ and $\gamma_2$ are said to be cofinal if for any compact subset $K$, there exists $t_K > 0$ such that $\gamma_1(t_1)$ and $\gamma_2(t_2)$ lie in the same connected component of $M \setminus K$ for all $t_1, t_2 \geq t_K$) is called an end of $M$. Let $\mathcal{E}(M)$ be the set of ends, equipped with the natural topology. It is known that $\mathcal{E}(M)$ is a compact, totally disconnected, Hausdorff topological space, and it is exactly the (topological) boundary of the Freudenthal compactification (here the terminology “boundary” is called to be “remainder” by topologists working on the compactification theory). For more details on end theory, we refer to [17], [22], [1], [21], [14], [22].

For a ray, it could represent either an element of the (metric) ideal boundary $M(\infty)$ or an element of the (topological) space $\mathcal{E}(M)$, thus rays connect these two objects. To go further, we shall introduce the notion of coray as follows.

For any ray $\gamma$, a ray $\tilde{\gamma} : [0, \infty) \to M$ is called to be a coray to $\gamma$ if there exist a sequence $x_k \to \tilde{\gamma}(0)$, a sequence $t_k \to \infty$ and a sequence of minimal geodesic segments $\gamma_k : [0, d(x_k, \gamma(t_k))] \to M$ connecting $x_k$ and $\gamma(t_k)$ such that $\gamma_k$ converge to $\tilde{\gamma}$ uniformly on any compact interval of $[0, \infty)$. To make Remark 5 clearer, we only need to prove

Theorem 9. For any ray $\gamma$, all corays to $\gamma$ are cofinal to $\gamma$.

Proof. Assume that $\tilde{\gamma}$ is a coray to $\gamma$, we shall prove that $\tilde{\gamma}$ and $\gamma$ are cofinal. Otherwise, there exists a compact subset $K$ such that for any $T > 0$, there exist $s_1, s_2 \geq T$ such that $\tilde{\gamma}(s_1)$ and $\gamma(s_2)$ lie in different connected components of $M \setminus K$. Since $\tilde{\gamma}$ and $\gamma$ are rays, if we choose $T$ large enough, $\tilde{\gamma}[T, \infty)$ and $\gamma[T, \infty)$ will lie in $M \setminus K$, and thus $\tilde{\gamma}[T, \infty)$ (respectively, $\gamma[T, \infty)$) will stay in a connected component of $M \setminus K$. Moreover, if $T$ is chosen larger, we can assume that $d(\tilde{\gamma}(t), K) \geq 1$ and $d(\gamma(t), K) \geq 1$ whenever $t \geq T$. Since $\tilde{\gamma}[T, \infty)$ and $\gamma[T, \infty)$ lie in different connected components, there exists $k_0 > 0$ such that for any $k \geq k_0$, the geodesic segment $\gamma_k[T, d(x_k, \gamma(t_k))]$ has a non-empty intersection with $K$; here the geodesic segments $\gamma_k$ come from the definition of coray. By the fact $\gamma_k[T, d(x_k, \gamma(t_k))]$ converge to $\tilde{\gamma}[T, \infty)$ uniformly on any compact interval of $[T, \infty)$, together with the fact $d(\tilde{\gamma}(t), K) \geq 1$ for any $t \geq T$, we will get a contradiction to the fact that $\gamma_k$ are minimal geodesic segments for all $k$ as follows.
Denote the diameter of $K$ by $r_1$ and $\max_{y \in K} d(\tilde{\gamma}(0), y)$ by $r_2$. Choose $S > r_1 + r_2 + 2$, then there exists $k_1 > k_0$ such that for any $k > k_1$, $d(\gamma_k(t), \tilde{\gamma}(t)) < \frac{1}{2}$ for any $t \in [0, S + T]$ and $\gamma_k(T + S)$ lies in the connected component of $M \setminus K$ where $\tilde{\gamma}([T, \infty))$ lies. Then for any $k > k_1$,

$$l(\gamma_k) \geq T + S - 1 + d(\gamma(t_k), K).$$

Moreover by the discussions in the previous paragraph, for any $k > k_1$, there exists a $t_k^*$ such that $\gamma_k(t_k^*) \in K$ and $\gamma_k(t) \notin K$ for $t_k^* < t \leq d(x_k, \gamma(t_k))$. Clearly $t_k^* > T + S$. For any $k \geq k_1$, we could construct $\xi_k$ such that $\xi_k$ is the conjunction of two minimal geodesic segments which connect $\gamma_k(0)$ and $\gamma_k(t_k^*), \gamma_k(t_k^*)$ and $\gamma(t_k)$, respectively. Then clearly

$$l(\xi_k) \leq r_2 + \frac{1}{2} + d(\gamma_k(t_k^*), \gamma(t_k)) \leq r_2 + \frac{1}{2} + r_1 + d(\gamma(t_k), K).$$

Hence $l(\xi_k) < l(\gamma_k)$ for all $k > k_1$, it contradicts the fact that $\gamma_k$ are minimal geodesic segments for all $k$.

For two rays $\gamma_1$ and $\gamma_2$, if $b_{\gamma_1} = b_{\gamma_2} + \text{const.}$, then $\gamma_1$ is a coray to $\gamma_2$ and $\gamma_2$ is a coray to $\gamma_1$ simultaneously. So we get

**Corollary 10.** Given two rays $\gamma_1$ and $\gamma_2$, if $b_{\gamma_1} = b_{\gamma_2} + \text{const.}$, then $\gamma_1$ and $\gamma_2$ represent the same end. In other words, if $\gamma_1$ and $\gamma_2$ represent the same element in $M(\infty)$, then they represent the same element in $\mathcal{E}(M)$. Of course, the converse is not true in general. For an easy counterexample, consider the Euclidean plane $\mathbb{R}^2$, where $\mathcal{E}(M)$ is one-point set, but $M(\infty)$, equipped with the quotient compact-open topology (here we choose Busemann functions as representatives of the elements in $M(\infty)$) is holomorphic to the circle $S^1$.

It is known [19], Theorem 1] that every paracompact, non-compact and boundaryless manifold admits a complete Riemannian metric. As a partial converse to Corollary 10, one would pose the following problem. This problem should be very fundamental and interesting and we are not sure whether it is definitely new or open. But, to our best, we could not find literature where such a problem was mentioned, posed or resolved.

**Problem 11.** On a smooth, non-compact, boundaryless, connected, paracompact manifold $M$, is there always a compete Riemannian metric $g$ on $M$ such that for any two rays representing the same end will represent the same ideal boundary element (with respect to the Riemannian metric $g$)?

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