Vacuum Energy Density and Cosmological Constant in dS Brane-World

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Abstract

We discuss the vacuum energy density and the cosmological constant of dS$_5$ brane world with a dilaton field. It is shown that a stable AdS$_4$ brane can be constructed and gravity localization can be realized. An explicit relation between the dS bulk cosmological constant and the brane cosmological constant is obtained. The discrete mass spectrum of the massive scalar field in the AdS$_4$ brane is used to acquire the relationship between the brane cosmological constant and the vacuum energy density. The vacuum energy density in the brane gotten by this method is in agreement with astronomical observations.

Keywords: Brane; Cosmological constant; Vacuum energy density; Gravity localization

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1 Introduction

In recent years, the old idea that placing our world on a domain wall in a higher-dimensional bulk space has been used to explain the hierarchy between the Planck scale and the electro-weak scale in the four-dimensional effective field theory [1-3]. After the notable papers of Randall and Sundrum [1, 4], physicists have made progress on using the idea of brane world to explain the cosmological problems [5, 6].

In the original Randall-Sundrum model, a flat 3-brane perpendicular to the fifth coordinate of the AdS$_5$ spacetime was embedded. All matter and interactions except for gravity are confined to the brane. Soon, the generalization of an AdS, flat or dS brane in the AdS bulk [7], and a flat or dS brane in the dS bulk were studied carefully[8]. The localization of gravity in these models has been discussed also [4,9-12]. For the investigation of the brane world cosmology, the radion field was introduced to stabilize the distance of branes [13, 14]. In some cases, the dilaton field was also needed to get more predictions on cosmology [7,15-17].

The recent astronomical observations on Type Ia supernovae [18, 19] and the cosmological microwave background [20, 21] give more believable answers to several long-existing problems, such as the cosmological constant, flatness of space and existence of inflation [22-27]. However, the vacuum energy density calculated by quantum field theory is much more larger than the possible observed value [28, 29]. Various attempts have been made in trying to solve the cosmological constant problem. Up to now, there still hasn’t a theory that can give a cosmological constant whose order is the same as that of the observed value. What is more puzzled, from the theoretical point of view, is that why the observed vacuum energy density is such a tiny value but non-zero [18-20,23-25].

In this paper, we discuss the vacuum energy density and the cosmological constant of the dS$_5$ brane world with dilaton field in a specially selected background metric. It is found that there is not an AdS$_4$ brane solution in a dS$_5$ bulk without other fields.
Furthermore, one can not find an AdS$_4$ brane structure by just introducing a dilaton field in the dS$_5$ bulk [7, 15, 17]. It is shown that both the dilaton field and the special background metric are necessary to get a stable AdS$_4$ brane in the dS$_5$ brane world. We study the localization of gravity by expanding the traceless transverse component of the perturbation of gravity in terms of the mass eigenstates of scalar field in the brane. An explicit relation between the dS bulk cosmological constant and the brane cosmological constant is obtained. We solve exactly the massive scalar field equation in the AdS$_4$ brane. The discrete mass spectrum of the massive scalar field in the brane is used to acquire the relationship between the brane cosmological constant and the vacuum energy density. The vacuum energy density in the brane world gotten by this method is in the same order with that of the observed value.

The paper is organized as follows. In section 2, we present the general framework of the dS brane world with dilaton field. Equations of motion for the gravity and dilaton field are derived. Section 3 is devoted to finding a stable AdS$_4$ brane solution in the dS brane world. The matching conditions on two sides of the brane give a useful relation between two cosmological constants. In section 4, localization of gravity is discussed by expanding the traceless transverse component of the fluctuation of gravity in terms of the mass eigenstates in the brane. We demonstrate that the deviation of classical gravity from the Newton’s law in the brane is too small to be observed. In section 5, we calculate the vacuum energy density of the brane by making use of exact solutions of massive scalar fields. The obtained vacuum energy density in the brane is in agreement with astronomical observations. In section 6, we give conclusion and some remarks.

2 Basic Setup and the Equations of Motion

We consider a dS$_5$ spacetime with an AdS$_4$ brane embedding at $y = 0$. It is assumed first that there is only gravity in the bulk. The action of the dS brane world can be
written as

\[ S = \int d^5 x \left[ \sqrt{-G} \left( \frac{M^3}{2} R - \Lambda \right) - \sqrt{-g} V \delta(y) \right], \tag{1} \]

where \( \Lambda \) (> 0) and \( M \) are the bulk cosmological constant and five-dimensional fundamental scale, respectively. Following Randall and Sundrum, we suppose the metric on the brane world is as follows

\[ ds^2 = e^{2A(y)} g_{\mu\nu} dx^\mu dx^\nu + dy^2, \tag{2} \]

where \( g_{\mu\nu} \) is the metric on the brane and takes the form as\(^4\)

\[ g_{\mu\nu} = \text{diag} \left( -1, \frac{\rho^2 \cos^2 \sqrt{-\lambda t}}{1 + \lambda \rho^2}, \frac{\rho^2 \sin^2 \theta \cos^2 \sqrt{-\lambda t}}{1 + \lambda \rho^2} \right). \tag{3} \]

Here \( \sqrt{-\lambda} \) (\( \lambda < 0 \)) is the curvature of the AdS brane.

The five dimensional Einstein equations for the above action read

\[ \sqrt{-G} \left( R_{MN} - \frac{1}{2} G_{MN} R \right) = -\frac{1}{M^3} \left[ \Lambda \sqrt{-G} G_{MN} + \sqrt{-g} V \delta_{MN} \delta^\mu_\mu \delta(y) \right]. \tag{4} \]

By making use of the ansatz (2), we transform Eq.(4) into the form

\[ 6(A')^2 + \frac{\Lambda}{M^3} = 6 \lambda e^{-2A}, \tag{5} \]

\[ 3 \lambda e^{-2A} + 3A'' = -\frac{V}{M^3} \delta(y), \]

where \( ' \) denotes derivative with respect to \( y \). It is obvious that one can not find a nontrivial solution of the above equations. Thus, we know there doesn’t exist an AdS\(_4\) brane structure in this dS\(_5\) spacetime with only gravity living in the bulk.

To obtain a stable AdS\(_4\) brane solution, we add a dilaton field \( \phi(y) \) in the dS\(_5\) spacetime. The action of the system is of the form

\[ S = \int d^5 x \left( \sqrt{-G} \left[ R - \frac{4}{3} (\nabla \phi)^2 - 2 \Lambda e^{\phi} \right] - 2 \sqrt{-g} \delta(y) V e^{\phi} \right). \tag{6} \]

We suppose the spacetime metric as follows

\[ ds^2 = e^{2A(y)} g_{\mu\nu} dx^\mu dx^\nu + e^{2B(y)} dy^2, \tag{7} \]

\(^4\)Here, for simplicity we define \( t \equiv x^0, \rho \equiv x^1, \theta \equiv x^2, \omega \equiv x^3 \). Five-dimensional suffices are denoted by capital Latin and the four-dimensional one by the Greek ones.
where \( B(y) \) is only a function of the fifth coordinate and \( g_{\mu\nu} \) takes the same form as Eq.(3).

Equation of motion of the dilaton field reads

\[
\sqrt{-G} \left( \frac{8}{3} \nabla^2 \phi - 2a\Lambda e^{a\phi} \right) - 2b\sqrt{-g} V \delta(y) e^{b\phi} = 0. \tag{8}
\]

The Einstein equation coupled with the dilaton field is of the form

\[
\sqrt{-G} \left( R_{MN} - \frac{1}{2} G_{MN} R \right) = \frac{4}{3} \sqrt{-G} \left[ \nabla_M \phi \nabla_N \phi - \frac{1}{2} G_{MN} (\nabla \phi)^2 \right] + \Lambda e^{a\phi} \sqrt{-G} G_{MN} + \sqrt{-g} V e^{b\phi} g_{\mu\nu} \delta_M^\mu \delta_N^\nu \delta(y) = 0. \tag{9}
\]

By making use of the ansatz for the metric (7), we can transform the equations of motion into the form

\[
\frac{32}{3} A' \phi'e^{-2B} - \frac{8}{3} B' \phi'e^{-2B} + \frac{8}{3} \phi'' e^{-2B} - 2a\Lambda e^{a\phi} - 2bV \delta(y) e^{b\phi} = 0, \tag{10}
\]

\[
-3\lambda e^{-2A+2B} + 6(A')^2 - 3A'B' + 3A'' + \frac{2}{3} (\phi')^2 + \Lambda e^{a\phi+2B} + V e^{b\phi} \delta(y) = 0, \tag{11}
\]

\[
-6\lambda e^{-2A+2B} + 6(A')^2 - \frac{4}{3} (\phi')^2 e^{-2B} + \frac{2}{3} (\phi')^2 + \Lambda e^{a\phi+2B} = 0. \tag{12}
\]

In the bulk, Eq.(10) and Eq.(11) reduce to

\[
\frac{32}{3} A' \phi'e^{-2B} - \frac{8}{3} B' \phi'e^{-2B} + \frac{8}{3} \phi'' e^{-2B} - 2a\Lambda e^{a\phi} = 0, \tag{13}
\]

\[
-3\lambda e^{-2A+2B} + 6(A')^2 - 3A'B' + 3A'' + \frac{2}{3} (\phi')^2 + \Lambda e^{a\phi+2B} = 0. \tag{14}
\]

### 3 Solution and Relationship between Cosmological Constants

To get an analytic solution of equations of motion (12), (13) and (14), we assume that the following relations are satisfied by fields in the system

\[
A = \alpha \phi, \quad B = \phi, \quad a = -2. \tag{15}
\]

Thus, Eq.(13) becomes

\[
\frac{32}{3} \alpha (\phi')^2 - \frac{8}{3} (\phi')^2 + \frac{8}{3} \phi'' + 4\Lambda = 0. \tag{16}
\]
It is easy to solve the above equation for the dilaton field $\phi(y)$. In the case of $\alpha < 1/4$, we acquire $\phi'(y)$ as follows

$$
\phi'(y) = \begin{cases} 
-D \tanh [(1 - 4\alpha)D(y + d_1)] , & \text{for } y > 0 \\
-D \tanh [(1 - 4\alpha)D(y + d_2)] , & \text{for } y < 0 
\end{cases}
$$

where $D = \sqrt{\frac{3\Lambda}{2(1 - 4\alpha)}}$. We present here a solution of the form

$$
\phi = \begin{cases} 
-\frac{1}{1 - 4\alpha} \ln \{\cosh [(1 - 4\alpha)D(y + d_1)]\} + f_1 , & \text{for } y > 0 \\
-\frac{1}{1 - 4\alpha} \ln \{\cosh [(1 - 4\alpha)D(y + d_2)]\} + f_2 , & \text{for } y < 0 
\end{cases}
$$

where $d_1$, $d_2$, $f_1$ and $f_2$ are constants which will be determined by self-tuning.

Imposing the discontinuity of $\phi'(y)$ at $y = 0$ on (10) and (11), we get the matching conditions as follows

$$
\phi'(0^+) - \phi'(0^-) = \frac{3}{4} bV ,
$$

$$
\phi'(0^+) - \phi'(0^-) = -\frac{1}{3\alpha} V .
$$

Thus, we have

$$
b = -\frac{4}{9\alpha} .
$$

The continuity of $(\phi')^2$ and discontinuity of $\phi'(y)$ at $y = 0$ give that

$$
\phi'(0^+) = -\phi'(0^-) .
$$

From equations (22) and (18), one can obtain

$$
d_1 = -d_2 \equiv d .
$$

By making use of equation (20), we can acquire $d$ as

$$
d = \frac{1}{\sqrt{6\Lambda(1 - 4\alpha)}} \ln \left| \frac{1 + F}{1 - F} \right| ,
$$

where $F = \frac{V}{3\alpha} \sqrt{\frac{1 - 4\alpha}{6\Lambda}}$.

Furthermore, $f_1$ and $f_2$ can be determined by $\phi(0^+) = \phi(0^-) = 0,$

$$
f_1 = f_2 = -\frac{1}{2(1 - 4\alpha)} \ln \left( 1 - \frac{V^2 (1 - 4\alpha)}{9\alpha^2 6\Lambda} \right) .
$$
Figure 1: The behavior of the function $K_1$ (unit of the transverse axis is $10^{24}$ and of the vertical axis is $10^{-51}$).

Inserting relations (15), (24) and (25) into the equations of motion (12) and (14), we get

$$-3\lambda e^{(-2\alpha+2)\phi} + \left(\frac{2}{3} - 6\alpha^2\right)(\phi')^2 + \left(1 - \frac{9}{2}\alpha\right)\Lambda = 0 ,$$  \hspace{1cm} (26)

$$-6\lambda e^{(-2\alpha+2)\phi} + \left(6\alpha^2 - \frac{4}{3}e^{-2\phi} + \frac{2}{3}\right)(\phi')^2 + \Lambda = 0 .$$  \hspace{1cm} (27)

The values of $\phi'(0)^2$ and $\phi(0)$ as well as the equation (27) give an explicit relation between the bulk cosmological constant $\Lambda$ and the brane cosmological constant $\lambda$,

$$\Lambda = 6\lambda - \frac{(6\alpha^2 - \frac{2}{3})}{36\alpha^2}V^2 .$$  \hspace{1cm} (28)

At this stage, we can say that it is really possible to get an AdS brane structure in dS spacetime by self-tuning the parameters. We show in the following by a numerical method that (26) and (27) are satisfied when the cosmological constant $\Lambda$, factor $\alpha$ and tension $V$ in the brane take some specially selected values. For the purpose, one can introduce functions $K_1$ and $K_2$ as follows

$$K_1 \equiv -3\lambda e^{(-2\alpha+2)\phi} + \left(\frac{2}{3} - 6\alpha^2\right)(\phi')^2 + \left(1 - \frac{9}{2}\alpha\right)\Lambda ,$$  \hspace{1cm} (29)

$$K_2 \equiv -6\lambda e^{(-2\alpha+2)\phi} + \left(6\alpha^2 - \frac{4}{3}e^{-2\phi} + \frac{2}{3}\right)(\phi')^2 + \Lambda .$$  \hspace{1cm} (30)
Recent observations of Type Ia supernovae and the cosmic microwave background indicate that our universe is dominated by a positive cosmological constant [18-21,25,26]. So it is reasonable to assume that the current age of Universe times the light speed equals the radius of the dS$_5$ spacetime. Therefore, we have

$$\frac{1}{\sqrt{\Lambda}} = \frac{c}{H_0}$$  \hspace{1cm} (31)$$

where $H_0$ is present-day value of the Hubble expansion rate. Namely, $\Lambda \simeq 5.89 \times 10^{-53} \text{m}^{-2}$. We make the computer program to draw different plots of $K_1$ and $K_2$ by automatically selecting different values of the parameters $\alpha$, $V$ and $d$. Our program give the following best fit values of $\alpha$, $V$ and $d$

$$\alpha = 0.249, \quad V = 2.99 \times 10^{-25}, \quad d = 4.94 \times 10^{27} \text{m}.$$  \hspace{1cm} (32)$$

In figure 1 and figure 2, we show plots of $K_1$ and $K_2$ as functions of $y$. It is clear that $K_1$ and $K_2$ are almost zero in the whole area of $|y| < \frac{1}{\sqrt{\Lambda}}$. We note that $K_1$ and $K_2$ deviate from zero very fast when $|y| \rightarrow d$. Possible reason of the fact is that the radius of five dimensional dS spacetime, the cosmological constant and parameter $d$ satisfy
the relation
\[ \frac{1}{\sqrt{\Lambda}} \sim \frac{1}{\sqrt{-\lambda}} \sim \frac{d}{10}. \] (33)

So \( |y| \to d \) is out of the dS spacetime. We can draw a conclusion that the equations (26) and (27) can be satisfied very well within the interval \( |y| < \frac{1}{\sqrt{\Lambda}} \) with the fine-tuned parameters.

### 4 Localization of Gravity

To demonstrate the localization of gravity of AdS\(_4\) brane in dS\(_5\) bulk with dilaton field, we rewrite the spacetime metric as follows

\[ ds^2 = e^{2A(y)}(-dt^2 + \gamma_i(t)^2 \delta_{ij} dx^i dx^j) + e^{2B(y)}dy^2, \quad i, j = 1, 2, 3, \] (34)

where
\[ \gamma_1 = \frac{\cos \sqrt{-\lambda} t}{(1 + \lambda \rho^2)} , \quad \gamma_2 = \frac{\rho \cos \sqrt{-\lambda} t}{\sqrt{1 + \lambda \rho^2}} , \quad \gamma_3 = \frac{\rho \sin \theta \cos \sqrt{-\lambda} t}{\sqrt{1 + \lambda \rho^2}} . \] (35)

Consider the perturbed metric \( h_{ij} \) in the form[11]

\[ ds^2 = e^{2A(y)} \left[ -dt^2 + \gamma_i(t)^2 (\delta_{ij} + h_{ij}(x^i)) dx^i dx^j \right] + e^{2B(y)}dy^2 . \] (36)

We are interested in the localization of the traceless transverse component of \( h_{ij} \), which corresponds to the graviton of the perturbation in the brane. It satisfies \( D_i h^{ij} = 0 \) and \( h^i_i = 0 \). Here \( D_i \) denotes the covariant differential with respect to the space metric \( \gamma_i^2(t) \delta_{ij} \). The traceless transverse component of \( h_{ij} \) is denoted by \( h \) for simplicity. \( h \) satisfies the following equation[11, 30]

\[ \frac{1}{\sqrt{-G}} \frac{\partial}{\partial x^M} \left( \sqrt{-G} G^{MN} \frac{\partial}{\partial x^N} h \right) = 0 . \] (37)

It should be noticed that this is equivalent to a five dimensional free scalar field equation. In our framework, the metric is diagonal and Eq.(37) can be reduced by expanding \( h \) in terms of four dimensional continuous mass eigen states

\[ h = \int dm \ \Phi_m(t, \vec{x}) \ \Psi(m, y) , \] (38)
where $\Phi_m(t, \vec{x})$ is the mass eigenstates of the four-dimensional scalar field\footnote{Here $\vec{x}$ denotes $(x^1, x^2, x^3) = (\rho, \theta, \omega)$.}

\[
\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left( \sqrt{-g} g^{\mu\nu} \frac{\partial}{\partial x^\nu} \Phi_m(t, \vec{x}) \right) = m^2 \Phi_m(t, \vec{x}) .
\]  

(39)

It is not difficult to obtain the equation for $\Psi(m, y)$ from Eq.(37) and Eq.(39)

\[
\Psi'' + (4A' - B')\Psi' + m^2 e^{2B-2A}\Psi = 0 .
\]  

(40)

Before considering the solution of Eq.(40), we find that the equation can be rewritten in a "supersymmetric" form by introducing transformations $\Psi = e^{-\frac{3}{2}\alpha B} u(z)$ and $b \equiv \frac{\partial}{\partial y} = e^{(1-\alpha)B}$. Thus, Eq.(40) becomes

\[
Q^\dagger Q u(z) = (-\partial_z - \frac{3}{2} \alpha \frac{\partial B}{\partial z})(\partial_z - \frac{3}{2} \alpha \frac{\partial B}{\partial z}) u(z) = m^2 u(z) .
\]  

(41)

This equation demonstrates that the eigenvalue of mass should be non-negative, i.e. no tachyon in the brane. Then the zero mode $m = 0$ is the lowest state which would be localized on the brane. Furthermore, we can rewrite the equation for $u(z)$ as the following one-dimensional Schrödinger-like equation in the $y$-direction with the eigenvalue $m^2$

\[
[-\partial_z^2 + V(z)] u(z) = m^2 u(z) ,
\]  

(42)
Figure 4: The behavior of the coordinate transformation $\frac{\partial z}{\partial y}$ (unit of the transverse axis is $10^{24}$ and of the vertical axis is $10^1$).

where the potential $V(z)$ is determined by $B(z)$ as

$$V(z) = \frac{3}{2} \alpha \left[ \frac{3}{2} \alpha \left( \frac{\partial B}{\partial z} \right)^2 + \frac{\partial^2 B}{\partial z^2} \right].$$ (43)

The localization is seen by solving Eq.(42). The localization of gravity can be concluded by the following two conditions: First, the potential $V(z)$ is volcano-like and contains a $\delta$-function attractive force at the brane position to trap the zero-mode of the bulk graviton; Second, There exist a normalizable state for the wave function $u_0(z)$ of $m = 0$ eigenvalue.

Figure 4 indicates that $b(y)$ could be treated as a linear function in physicist’s interested area. By the approximation, the potential $V(z)$ is of the form

$$\begin{align*}
V(z) &= \frac{3}{2} \alpha \left( \frac{5}{2} \alpha - 1 \right) D^2 e^{-2\gamma B} \tanh^2 [(1 - 4\alpha) D(z + \text{sgn}(z)d)] \\
&\quad - \frac{3}{2} \alpha e^{-2\gamma B} \left\{ \frac{3\Lambda}{2} \text{sech}^2 [(1 - 4\alpha) D(z + \text{sgn}(z)d)] \right\} - \frac{V}{4} e^{-2\gamma B} \delta(z).
\end{align*}$$ (44)

The $\delta$ function term in $V(z)$ guarantees that there exists an independent unitary bound state solution

$$u_0(z) = \frac{1}{\sqrt{L}} e^{-|z|/L},$$ (45)

where $L = \frac{4}{V}$ is the normalization constant. The figures 3 and 4 show that the potential (44) is a Volcano type. The perturbation of gravity with Volcano potential
has been studied extensively [11, 31, 32]. Therefore, we can say that the gravity of the dS$_5$ brane world with dilaton field in a specially selected background metric is well localized. That is to say, the deviation of the classical gravitation from the Newton’s law in the brane can be omitted.

5 Vacuum Energy Density and Cosmological Constants

The vacuum energy density in the AdS$_4$ brane can be calculated by summing up the zero point energy of different massive scalar fields living in the brane. The equation of motion for a massive scalar field in AdS$_4$ brane is of the form [33, 34]

$$\left[\frac{1}{\cos^3 \sqrt{-\lambda t}} \frac{\partial}{\partial t} \left( \cos^3 \sqrt{-\lambda t} \frac{\partial}{\partial t} \right) - \frac{(1 + \lambda \rho^2)^2}{\rho^2 \cos^2 \sqrt{-\lambda t} \partial \rho} \left( \rho^2 \frac{\partial}{\partial \rho} \right) - \frac{1 + \lambda \rho^2}{\rho^2 \cos^2 \sqrt{-\lambda t}} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) - \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right) + m_0^2 \right] \Phi(t; \rho, \theta, \phi) = 0.$$  \(\text{(46)}\)

The equation of motion can be solved exactly by the method of variables separation. Solutions can be written as follows

$$\Phi_{NIIm}(t; \rho, \theta, \phi) \propto U_{NI}(\rho) \left( \cos \sqrt{-\lambda t} \right)^{-1} P_I^N \left( \sin \sqrt{-\lambda t} \right) Y_{Im}(\theta, \phi),$$  \(\text{(47)}\)

where $P_I^N(\sin \sqrt{-\lambda t})$, with $I = -\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{m^2}{\lambda} + 2}, N = \sqrt{1 + \frac{k^2}{\lambda}}$, are associated Legendre functions, and $U(\rho)$, the radial part of the wave function,

$$U(\rho) = C \left( \sqrt{-\lambda \rho} \right)^l \left( 1 + \lambda \rho^2 \right)^{\frac{l}{2} + \frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{k^2}{\lambda}}} \times _2F_1 \left( \frac{1}{2}(l + \sqrt{1 + \frac{k^2}{\lambda}} + 2), \frac{1}{2}(l + \sqrt{1 + \frac{k^2}{\lambda} + 1}), l + \frac{3}{2}; -\lambda \rho^2 \right),$$  \(\text{(48)}\)

here $C$ is the normalization constant.

The natural boundary condition for $P_I^N(\sin(\sqrt{-\lambda t}))$ on $\sin(\sqrt{-\lambda t}) = \pm 1$ requires that $I$, $N$ to be integers. This gives the discrete mass spectrum of the scalar fields in
AdS$_4$ brane [34],
\[-\frac{m_0^2}{\lambda} + 2 = I(I + 1),
\]
\[1 + \frac{k^2}{\lambda} = N^2, \quad |N| \leq I. \quad (49)\]

The vacuum energy density on AdS$_4$ brane is obtained [33]
\[
\langle \rho \rangle = \frac{1}{8\pi^2} |\lambda| \left(\frac{E_{\text{Planck}}}{hc}\right)^2,
\]
where $E_{\text{Planck}}$ is the Planck energy.

The relation between the vacuum energy density and the cosmological constant in
the 5-dimensional spacetime can be expressed as
\[
\langle \rho \rangle = \frac{1}{8\pi^2} \left(\frac{E_{\text{Planck}}}{hc}\right)^2 \left(\frac{1}{324\alpha^2} - \frac{1}{36}\right)V^2 - \Lambda \right]. \quad (51)
\]

The vacuum energy density takes value
\[
\langle \rho \rangle = 7.66 \times 10^{-10} \text{erg} \cdot \text{cm}^{-3}. \quad (52)
\]

The vacuum energy density gotten here is consistent with the astronomical observations [20, 23, 28, 35, 36].

6 Conclusion and Remarks

The tiny vacuum energy density has puzzled physicists for nearly a century. The recent
astronomical observations on supernovae [18, 19] and CMB [20, 21] show that about two
third of the world energy is contributed by a small positive cosmological constant. The
most simple cosmology model is an asymptotic dS spacetime which has been discussed
widely in the literature [37-39]. In this paper, we have interpreted the tiny positive
cosmological constant as the curvature of a dS$_5$ brane world.

It is well known that the calculated vacuum energy density by quantum field theory
is much more larger than the possible observed value [28, 29]. Various attempts have
been made in trying to solve the cosmological constant problem. To the end of getting
a comparable value of vacuum energy density with the astronomical observations, we
have tried to construct a AdS\textsubscript{4} brane in the dS\textsubscript{5} spacetime. It was shown that a dilaton field is needed to get a stable AdS brane solution in the dS\textsubscript{5} spacetime. An explicit relation between the dS bulk cosmological constant and the AdS brane cosmological constant has been obtained. The discrete mass spectrum in AdS\textsubscript{4} brane was used to acquire vacuum energy density. The cosmological constant in the brane world gotten by this way is in the same order with the astronomical observations.

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