Morse–Floer theory for superquadratic Dirac-geodesics

Takeshi Isobe¹ · Ali Maalaoui²

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Abstract
In this paper we present the full details of the construction of a Morse–Floer type homology related to the superquadratic perturbation of the Dirac-geodesic model. This homology is computed explicitly using a Leray–Serre type spectral sequence and this computation leads us to several existence results of Dirac-geodesics.

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1 Introduction
The nonlinear sigma model in quantum field theory, consists of two Riemannian manifolds $M$ and $N$ and a Lagrangian $L$ defined by

$$L(f) = \frac{1}{2} \int_M |df|^2 \, dvol_M,$$

where $f : M \to N$ is a map chosen in an adequate space. The critical points of this Lagrangian are solutions to the nonlinear sigma model and are the well studied harmonic maps. Now, if we set $M = S^1$, then $L$ becomes the energy functional of loops which was still deeply investigated and whose critical points are geodesics on $N$ (see [30] for more details and results regarding the geodesic problem). Furthermore, through the addition of a supersymmetric structure we obtain an extended model called the supersymmetric sigma model see [14, 17, 25].

1 Graduate School of Economics, Hitotsubashi University, 2-1 Naka, Kunitachi, Tokyo 186-8601, Japan
2 Department of Mathematics, Clark University, 950 Main Street, Worcester, MA 01610, USA

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✉ Ali Maalaoui
amaalaoui@clarku.edu
Takeshi Isobe
t.isobe@r.hit-u.ac.jp

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Specifically, this model is obtained by adding a fermionic action, that is, by assuming that $M$ is a spin manifold, we consider the Lagrangian

$$L(f, \psi) = \frac{1}{2} \int_M |df|^2 d\text{vol}_M + \frac{1}{2} \int_M (D_f \psi, \psi) d\text{vol}_M,$$

where $D_f$ is the Dirac operator associated with the canonical connection $\nabla^{S(M)} \otimes f^*\nabla^{TN}$ on $\mathbb{S}(M) \otimes f^*TN$, where $\nabla^{S(M)}$ is the canonical lift of the Levi–Civita connection on $TM$ to the spinor bundle $\mathbb{S}(M) \to M$ and $\nabla^{TN}$ is the Levi–Civita connection on $TN$. There are extensive works on the existence of non-trivial critical points of this functional, called Dirac-Harmonic Maps. One can see for instance [6, 12, 29] and the references therein.

Dirac-geodesics on a compact manifold $N$ that we are concerned in this paper are the spinorial analogue of supersymmetric mechanical particles moving on $N$. They are 1-dimensional version of the Dirac-harmonic maps.

Dirac-geodesic is obtained as a critical point of the Dirac-geodesic action functional. It is defined on a configuration space $\mathcal{F}^{1,1/2}(S^1, N)$ defined by

$$\mathcal{F}^{1,1/2}(S^1, N) = \{ (\phi, \psi) : \phi \in H^1(S^1, N), \psi \in H^{1/2}(S^1, \mathbb{S}(S^1) \otimes \phi^*TN) \},$$

where $H^1(S^1, N) = \{ \phi \in H^1(S^1, \mathbb{R}^k) : \phi(s) \in N \text{ a.e. } s \in S^1 \}$ is the set of $H^1$-loops on $N$ (we may assume without loss of generality that $N \subset \mathbb{R}^k$ for some $k > 1$) and $H^{1/2}(S^1, \mathbb{S}(S^1) \otimes \phi^*TN) = \{ \psi \in H^{1/2}(S^1, \mathbb{S}(S^1) \otimes \mathbb{R}^k) : (\psi(s) \in \mathbb{S} \otimes T_{\phi(s)}N \text{ a.e. } s \in S^1 \}$ is the set of $H^{1/2}$-spinors along the loop $\phi$. Here we denote by $\mathbb{S} \cong \mathbb{C}$ the 1-dimensional spinor module, $\mathbb{S}(S^1) \to S^1$ is a spinor bundle on $S^1$ and $\mathbb{R}^k \to S^1$ is the trivial $\mathbb{R}^k$ bundle over $S^1$. Throughout this paper, we assume as in [20] that the metric structure on $\mathcal{F}^{1,1/2}(S^1, N)$ is induced from the canonical embedding $\mathcal{F}^{1,1/2}(S^1, N) \subset H^1(S^1, \mathbb{R}^k) \times H^{1/2}(S^1, \mathbb{S}(S^1) \otimes \mathbb{R}^k)$. See [20] and Sect. 3 for details. Recall that there are two spin structures on $S^1$. Throughout this paper, we fix one of them. The connected components of $\mathcal{F}^{1,1/2}(S^1, N)$ are classified by the free homotopy class $[S^1, N]$. For $\alpha \in \pi_1(N)$, we denote by $\mathcal{F}^{1,1/2}_\alpha(S^1, N)$ the component associated with the conjugacy class $\Lambda_\alpha N$ of $\alpha$.

For any loop $\phi \in H^1(S^1, N)$, there is a canonical connection $\nabla^\phi = \nabla^{\mathbb{S}(S^1)} \otimes \phi^*\nabla^{TN}$ on $\mathbb{S}(S^1) \otimes \phi^*TN$. Associated to this is a Dirac operator $D_\phi = \frac{\partial}{\partial s} \nabla^\phi \frac{\partial}{\partial s}$, where $\frac{\partial}{\partial s}$ is the Clifford multiplication by the angular vector field $\frac{\partial}{\partial s} \in TS^1$. Under the identification $\mathbb{S} \cong \mathbb{C}$, it is simply given by multiplication by $\pm i$. (The choice of the sign is not relevant in this paper). For more details, see [20, §2]. The Dirac-geodesic action functional $\mathcal{L}$ is defined by

$$\mathcal{L}(\phi, \psi) = \frac{1}{2} \int_{S^1} \left| \frac{d\phi}{ds} \right|^2 ds + \frac{1}{2} \int_{S^1} \langle \psi, D_\phi \psi \rangle ds,$$

where $\langle \cdot, \cdot \rangle$ is the canonical metric on $\mathbb{S}(S^1) \otimes \phi^*TN$.

Observe that $\mathcal{L}$ restricted to $H^1(S^1, N) \times \{ 0 \}$ is again the geodesic energy functional $E(\phi) = \frac{1}{2} \int_{S^1} \left| \frac{d\phi}{ds} \right|^2 ds$ whose critical points are closed geodesics on $N$, while for a fixed loop $\phi$, the second term of $\mathcal{L}$ is the pure Dirac fermion action and its critical points are harmonic spinors. In this paper, we focus our study on considering a perturbed Dirac-geodesic action functional $\mathcal{L}_H$ of the following form

$$\mathcal{L}_H(\phi, \psi) = \frac{1}{2} \int_{S^1} \left| \frac{d\phi}{ds} \right|^2 ds + \frac{1}{2} \int_{S^1} \langle \psi, D_\phi \psi \rangle ds - \int_{S^1} H(s, \phi(s), \psi(s)) ds,$$

where the perturbation $H : \mathbb{S}(S^1) \otimes TN \to \mathbb{R}$ is a smooth function, where $\mathbb{S}(S^1) \otimes TN \to S^1 \times N$ is the external tensor product and we write $H = H(s, \phi, \psi)$, where $s \in S^1, \phi \in N$.
and $\psi \in \Sigma(S^1) \otimes T\phi N$. We call critical points of $\mathcal{L}_H$ perturbed Dirac-geodesics. Perturbed Dirac-geodesic $(\phi, \psi)$ satisfies the following equations

$$\nabla_s \partial_s \phi - \frac{1}{2} R(\phi, \partial_s \phi \cdot \psi) + \nabla_\phi H(s, \phi, \psi) = 0,$$

$$D_\phi - \nabla_\psi H(s, \phi, \psi) = 0,$$

where $\nabla_s = \nabla_{\partial_s}$ and $\partial_s = \frac{\partial}{\partial s}$. See [20] for the derivation of these equations. We call these equations perturbed Dirac-geodesic equations.

The extended supersymmetric model in quantum mechanics consists of finding supersymmetric Dirac-geodesics $L_H$, where we take $H(s, \phi, \psi) = \frac{1}{12} R_{ijkl}(\phi) \langle \psi^i, \psi^j \rangle \langle \psi^k, \psi^l \rangle$, with $R_{ijkl}$ the curvature tensor of $N$ and $\psi = \psi^k \otimes \frac{\partial}{\partial y^k}$ ($y = (y^k)$ is a local coordinate on $N$). See [14, 17, 25].

As far as we know, the only existence results of (perturbed) Dirac-geodesics in the literature was studied in [20] for the case of $N = \mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$ a flat torus and the case of a compact manifold $N$ with a bumpy metric and “big” superquadratic perturbation. We also mention the extension of the previously stated work to the case of a torus target space as in [23, 24]. The pure unperturbed case was studied in [13], in the case of surfaces via a heat flow approach and explicit formula for Dirac-geodesics was given in the particular cases of the two sphere and the hyperbolic plane. See also [9] for a related problem. Other than these two cases, the existence of a non-trivial perturbed Dirac-geodesic is widely open. We also want to point out a different approach that was used in the case of surfaces in [26, 28] for $\alpha$-Dirac-harmonic maps. Also, in [27], the authors provide existence results for Dirac-harmonic maps from degenerate surfaces. In this paper, we aim to settle this question for some class of manifolds using a Morse–Floer theoretical approach for superquadratic Dirac-geodesics. Since the spectrum of the Dirac operator $D_{\phi}$ at any loop $\phi$ is unbounded from below and above, the functional $L_H$ is strongly indefinite in the sense that the Morse index and co-index at any critical point are infinite. Thus, we shall construct a Floer type homology theory for the functional $L_H$ leading us many existence results. In our setting, we will consider perturbations $H \in C^2$ satisfying

$$|\nabla^2_{\psi,\psi} H(s, \phi, \psi)| \leq C_1 (1 + |\psi|^{p-1}),$$

$$C_2 |\psi|^{p+1} + 2H(s, \phi, \psi) - C_3 \leq (\nabla_\psi H(s, \phi, \psi), \psi),$$

$$|\nabla^3_{\psi,\psi,\psi} H(s, \phi, \psi)| \leq C_4 (1 + |\psi|^{q_1}),$$

$$|\nabla^2_{\phi,\phi} H(s, \phi, \psi)| \leq C_4 (1 + |\psi|^{q_2}),$$

$$|\nabla^3 H(s, \phi, \psi)| \leq C(1 + |\psi|^{q_3}).$$

where $p > 1$, $0 \leq q_i < +\infty$ for $i = 1, 2, 3$.

We note that, by integrating (1.1) and (1.3), we have

$$|H(s, \phi, \psi)| \leq C (1 + |\psi|^{p+1}),$$

and

$$|\nabla_{\phi} H(s, \phi, \psi)| \leq C (1 + |\psi|^{q_1+1}).$$

for some constant $C > 0$.

We call conditions (1.1) and (1.2) superquadratic conditions since they imply that $H(s, \phi, \psi)$ behaves like $|\psi|^{p+1}$ as $|\psi| \to \infty$. We denote by $H_{p+1}^1$ the class of perturbations $H \in C^1(S(S^1) \otimes T\phi N)$ which satisfies (1.1)–(1.4) above. Notice that conditions (1.3) and (1.4) are not too restrictive. In fact, we only need them to ensure the higher regularity of
the functional $\mathcal{L}_H$. Conditions (1.1) and (1.2), are very similar to (F1) and (F3) in [20]. Yet, in order to state our results, the conditions that we are considering allow for more flexibility in picking the non-linearity $H$. Indeed, for the existence results in [20], extra conditions are imposed, like the non-negativity of the perturbation or the growth rate of the perturbation around zero (conditions (F5) and (F6)). These assumptions are natural in the setting of [20], because of the min-max procedure used to produce a solution. Since here we are following a different path, it is natural to relax the conditions.

Our main result is summarized as follows:

**Theorem 1.1** Let us assume $p \geq 3$. For any given $\alpha \in \pi_1(N)$, we have:

1. For $H \in \mathbb{H}^{3}_{p+1}$, the Morse–Floer homology $HF_*(\mathcal{L}_H, \mathcal{F}_\alpha^{1/2}(S^1, N); \mathbb{Z}_2)$ is well-defined.
2. For any $H, H' \in \mathbb{H}^{3}_{p+1}$, there is a natural isomorphism
   \[
   \Phi_{HH'} : HF_*(\mathcal{L}_H, \mathcal{F}_\alpha^{1/2}(S^1, N); \mathbb{Z}_2) \rightarrow HF_*(\mathcal{L}_{H'}, \mathcal{F}_\alpha^{1/2}(S^1, N); \mathbb{Z}_2).
   \]
   Thus the isomorphism class of $HF_*(\mathcal{L}_H, \mathcal{F}_\alpha^{1/2}(S^1, N); \mathbb{Z}_2)$ is independent of $H \in \mathbb{H}^{3}_{p+1}$. We define the $(p+1)$-Dirac–geodesic homology of $N$ with $\mathbb{Z}_2$-coefficients in the class $\alpha$, denoted by $DG^{p+1}H_*(\alpha; \mathbb{Z}_2)$, as the isomorphism class of $HF_*(\mathcal{L}_H; \mathcal{F}_\alpha^{1/2}(S^1, N); \mathbb{Z}_2)$ for any one of $H \in \mathbb{H}^{3}_{p+1}$.
3. If $N$ has the property that the length spectrum $L_S$ of the class $\alpha \in \pi_1(N)$ is bounded from above or the number of perturbed geodesics in the class $\alpha$ is finite, then we have a vanishing result: $DG^{p+1}H_*(\alpha; \mathbb{Z}_2) = 0$.
4. If $N$ is a flat manifold, all the above hold under the assumption $p > 1$.

In the following, we fix any one of $\alpha \in \pi_1(N)$ and denote $\mathcal{F}_{\alpha}^{1/2}(S^1, N)$ as $\mathcal{F}_{\alpha}^{1/2}(S^1, N)$ for simplicity. In the course of our proof, we will prove much more structural results about the homology of $HF_*(\mathcal{L}_H, \mathcal{F}_\alpha^{1/2}(S^1, N); \mathbb{Z}_2)$ defined for our functional $\mathcal{L}_H$, see Sect. 8 for details. As observed in [20], the configuration space $\mathcal{F}_{\alpha}^{1/2}(S^1, N)$ has a natural Hilbert bundle structure over the loop space $H^1(S^1, N)$ with projection map $\pi : \mathcal{F}_{\alpha}^{1/2}(S^1, N) \rightarrow H^1(S^1, N)$ defined by $\pi(\phi, \psi) = \phi$. The fiber over $\phi \in H^1(S^1, N)$ is then $\pi^{-1}(\phi) = H^{1/2}(S^1, S(S^1) \otimes \phi^*TN)$. Note that the loop space $H^1(S^1, N)$ is canonically embedded as the zero section in $\mathcal{F}_{\alpha}^{1/2}(S^1, N)$, $H^1(S^1, N) \ni \phi \sim (\phi, 0) \in \mathcal{F}_{\alpha}^{1/2}(S^1, N)$. By this identification, we can naturally consider $H^1(S^1, N)$ as a submanifold of $\mathcal{F}_{\alpha}^{1/2}(S^1, N)$. Observe that $\mathcal{L}_H$ restricted to $H^1(S^1, N)$ is a perturbed geodesic action

\[
E_{1,H}(\phi) = \frac{1}{2} \int_{S^1} |\dot{\phi}|^2 \, ds \quad \text{while its restriction to the fiber over } \phi \in H^1(S^1, N) \text{ is a perturbed Dirac fermion action}
\]

\[
E_{2,H,\phi}(\psi) = \frac{1}{2} \int_{S^1} \langle D_{\phi} \psi, \psi \rangle \, ds \quad \text{while its restriction to the fiber over } \phi \in H^1(S^1, N) \text{ is a perturbed Dirac fermion action}
\]

These two functionals were investigated from a Morse theoretical point of view and they are relatively well understood. Indeed, Morse theory for $E_{1,H}$ is a classical study, see [30, 33] for details and more recently [3]. For $E_{2,H,\phi}$, in [21, 22] Morse–Floer homology is constructed and computed for the superquadratic Dirac fermion action. The same problem was also investigated using a Rabinowitz–Floer homology approach in [32]. See also [15, 16] for variational study of nonlinear Dirac equations arising from quantum physics.
Because of the fiber bundle structure of $\mathcal{F}^{1,1/2}(S^1, N)$ and the splitting of the functional $\mathcal{L}_H$ into two parts, one geodesic and one fermionic, for a complex generated by critical points of $\mathcal{L}_H$, one can provide a filtration structure induced by the geodesic part via Morse index. From this, it natural to conjecture that there is a Leray–Serre type spectral sequence converging to the Morse–Floer homology $HF_*(\mathcal{L}_H, \mathcal{F}^{1,1/2}(S^1, N); \mathbb{Z}_2)$.

The main difference from a regular spectral sequence is the lack of topological objects obtained from the functionals in the underlying space of variations and this comes from the absence of the classical Morse index and the cell attaching procedure. In this work, in order to compute the homology $HF_*(\mathcal{L}_H, \mathcal{F}^{1,1/2}(S^1, N); \mathbb{Z}_2)$, we construct a Leray–Serre type spectral sequence whose 2nd page is given by

$$MH_*(E_{1,H}, H^1(S^1, N); \mathcal{H}(\mathcal{F}_*(E_{2,H}, H^{1/2}(S^1, S(S^1) \otimes TN); \mathbb{Z}_2)),$$

where $\mathcal{H}(\mathcal{F}_*(E_{2,H}, H^{1/2}(S^1, S(S^1) \otimes \phi^*TN); \mathbb{Z}_2))$ is a local coefficient system

$$\left\{ HF_*(E_{2,H}; \phi, H^{1/2}(S^1, S(S^1) \otimes \phi^*TN); \mathbb{Z}_2) \right\}_{\phi \in H^1(S^1, N)}$$
on $H^1(S^1, N)$ and $HF_*(E_{2,H}; \phi, H^{1/2}(S^1, S(S^1) \otimes \phi^*TN); \mathbb{Z}_2)$ is the Morse–Floer homology of $E_{2,H}; \phi$ on $H^{1/2}(S^1, S(S^1) \otimes \phi^*TN)$ with $\mathbb{Z}_2$-coefficient and

$$MH_*(E_{1,H}, H^1(S^1, N); \mathcal{H}(\mathcal{F}_*(E_{2,H}, H^{1/2}(S^1, S(S^1) \otimes TN); \mathbb{Z}_2))$$
is the Morse homology of $E_{1,H}$ on $H^1(S^1, N)$ with the above local coefficient system.

As we mentioned earlier, in [22] it has been proved that $HF_*(E_{2,H}; \phi, H^{1/2}(S^1, S(S^1) \otimes \phi^*TN); \mathbb{Z}_2)$ is well-defined and does not depend on the particular choices of $\phi$ and $H \in \mathbb{H}_{p+1}$ (see Sect. 8). Thus, the local coefficient system is in fact trivial. Moreover, it vanishes identically. Thus the spectral sequence collapses at the 2nd page and the vanishing of the homology of $\mathcal{L}_H$ follows. In fact, in Sect. 8, we will consider a different perturbed geodesic energy functional $E_{1,H}$ instead of $E_{1,H}$ to prove Theorem 1.1.

Since in the classical use of Morse theory, the non-vanishing of the homology is used to extract existence and multiplicity results, the vanishing of our homology might lead the reader to think that this approach is not adequate for proving existence of critical points. However, for some cases, the vanishing is seen as a topological obstruction, and leads to some existence results for the perturbed Dirac-geodesics for some interesting cases. Let us consider the case $\nabla_\psi H(s, \phi, 0) = 0$ for $s \in S^1$ and $\phi \in N$. In this case, there exist perturbed Dirac-geodesics of the form $(\phi, 0)$, where $\phi$ is a perturbed geodesic, that is, critical point of $E_{1,H}(\phi) = \frac{1}{2} \int_{S^1} |\dot{\phi}|^2 ds - \int_{S^1} H(s, \phi, 0) ds$. This kind of solution is easy to find from classical Morse theory for $E_{1,H}$ on $H^1(S^1, N)$ if the free loop space $H^1(S^1, N)$ has a rich topology. In such a case, we are interested in non-trivial solutions, i.e., solutions $(\phi, \psi)$ with $\psi \neq 0$. As one of applications of our vanishing result, we show that even when the topology of the free loop space is not so rich, the non-existence of non-trivial perturbed Dirac-geodesic forces the existence of infinitely many perturbed geodesics. In other words, there exists a non-trivial Dirac-geodesic if we know that there are at most finitely many perturbed geodesics. We say that $N$ satisfies condition $(\ast)$ if

$$(\ast) \quad H_*(\Lambda_\alpha N; \mathbb{Z}) \text{ vanishes in large degrees and is a finitely generated abelian group in each degree.}$$

Note that, under the assumption $(\ast)$, the Euler characteristic $\chi(\Lambda_\alpha N)$ of $\Lambda_\alpha N$ is well-defined. Then we have the following:
Theorem 1.2 Let us consider a compact manifold satisfying \((\ast)\) and such that the Euler characteristic \(\chi(\Lambda_\alpha N)\) is odd. Assume that \(H \in \mathbb{H}^{3}_{p+1} (p \geq 3)\) satisfies \(\nabla_{\psi} H(s, \phi, 0) = 0\) for all \(s \in S^1\) and \(\phi \in N\), then for a generic perturbation \(\tilde{H}\) of \(H\) satisfying the same conditions as \(H\) one of the following holds:

1. there exist infinitely many perturbed geodesics in \(\Lambda_\alpha N\), or
2. there exists at least one perturbed Dirac-geodesic \((\phi, \psi)\) with \(\phi \in \Lambda_\alpha N\) and \(\psi \neq 0\).

Note that under the assumption \((\ast)\), the existence of infinitely many perturbed geodesics as asserted in (1) is not a trivial fact.

If one also assumes an extra symmetry on \(H\), then we have:

Theorem 1.3 Assume that \(N\) satisfies \((\ast)\) and \(\chi(\Lambda_\alpha N)\) is odd. Assume also that \(H \in \mathbb{H}^{3}_{p+1} (p \geq 3)\) is even with respect to \(\psi\). Then for a generic perturbation \(\tilde{H}\) of \(H\) satisfying the same conditions as \(H\) one of the following holds:

1. there exist infinitely many perturbed geodesics in \(\Lambda_\alpha N\), or
2. there exist infinitely many perturbed Dirac-geodesics \((\phi, \psi)\) with \(\phi \in \Lambda_\alpha N\) and \(\psi \neq 0\).

By the evenness of \(H\), we have \(\nabla_{\psi} H(s, \phi, 0) = 0\) for all \(s \in S^1\) and \(\phi \in N\). Thus perturbed geodesics are perturbed Dirac-geodesics and the above theorem implies the existence of infinitely many perturbed Dirac-geodesics when \(H\) is even with respect to \(\psi\).

In Sect. 9, we provide some class of manifolds for which this condition is satisfied, namely aspherical manifolds.

Corollary 1.1 Assume that \(N\) is a compact aspherical manifold with odd Euler characteristic \(\chi(N)\) and let \(\Lambda_\alpha N\) be an arbitrary component of \(\Lambda N\) \((\alpha \in \pi_1(N))\). Then, for \(H \in \mathbb{H}^{3}_{p+1} (p \geq 3)\) satisfying \(\nabla_{\psi} H(s, \phi, 0) = 0\) \((s \in S^1, \phi \in N)\), one of the following assertions holds: For a generic perturbation \(\tilde{H}\) of \(H\) satisfying the same conditions as \(H\),

1. there exist infinitely many perturbed geodesics in \(\Lambda_\alpha N\), or
2. there exist perturbed Dirac-geodesics \((\phi, \psi)\) with \(\phi \in \Lambda_\alpha N\) and \(\psi \neq 0\).

If we further assume that \(H\) is even with respect to \(\psi\), then we have one of the following: For a generic perturbation \(\tilde{H}\) satisfying the same conditions as \(H\),

(i) there exist infinitely many perturbed geodesics in \(\Lambda_\alpha N\), or
(ii) there exist infinitely many perturbed Dirac-geodesics \((\phi, \psi)\) with \(\phi \in \Lambda_\alpha N\) and \(\psi \neq 0\).

One can also combine the vanishing of the homology to other existence results, such as the ones proved in [20], to deal with the case when the Euler characteristic is even. We state for instance the case of the flat torus \(\mathbb{T}^k = \mathbb{R}^k / \mathbb{Z}^k\):

Theorem 1.4 For the special case \(N = \mathbb{T}^k\), where \(\mathbb{T}^k\) is a flat torus, under the assumption that \(H \in \mathbb{H}^{3}_{p+1} (p > 1)\) and \(\nabla_{\psi} H(s, \phi, 0) = 0\) for all \(s \in S^1\) and \(\phi \in N\), we have one of the following: For a generic perturbation \(\tilde{H}\) of \(H\) satisfying the same conditions as \(H\),

1. there exist infinitely many perturbed geodesics in \(\Lambda_\alpha \mathbb{T}^k\), or
2. there exist at least two non-trivial perturbed Dirac-geodesics \((\phi, \psi)\) with \(\phi \in \Lambda_\alpha \mathbb{T}^k\).

Remark 1.1 (i) Actually, what we required for \(\tilde{H}\) in the above results are as follows: For Theorem 1.2 and Corollary 1.1, \((\phi, 0)\) is a non-degenerate critical point of \(L_{\tilde{H}}\) for all perturbed geodesic \(\phi\), while for Theorems 1.3 and 1.4, \(L_{\tilde{H}}\) is a Morse function on \(\mathbb{T}^{1.1/2}(S^1, N)\). We will see that these conditions are satisfied for a generic perturbation of a given function \(H\).
By considering the recent result proved in [23], the second alternative in Theorem 1.4 can be improved to the existence of at least $k + 2$ solutions if $k$ is even.

A more thorough discussion of the assumptions on the manifold and further examples are provided in Sect. 9.

Our paper is structured into four main parts:

- The first four Sections are devoted to the construction of the Floer homology while assuming transversality. This is done by introducing an adequate grading to replace the classical Morse index and then by studying the negative gradient flow lines.
- In Sects. 5, 6 and 7, we investigate the transversality conditions and we show that the homology can be defined for a generic set of perturbations. Then, we show that the homology is stable under perturbations by exhibiting a natural isomorphism between the different perturbed Floer homologies.
- In Sect. 8, we focus on computing the homology by introducing a Leray–Serre type spectral sequence as described above.
- In Sect. 9, we use our homology to prove some existence results for perturbed Dirac–geodesics.

## 2 Relative Morse index

We define a relative Morse index of a critical point $(\phi, \psi) \in \mathcal{F}^{1,1/2}(S^1, N)$ via the spectral flow of the linearization of the Euler–Lagrange equation. So, let us write $A_{(\phi, \psi), H}$ the bounded self-adjoint realization of the Hessian $d^2L_H(\phi, \psi)$ with respect to the $H^1 \times H^{1/2}$ metric on $\mathcal{F}^{1,1/2}(S^1, N)$:

$$d^2L_H(\phi, \psi) \left[ \left( \frac{X}{\xi} , \frac{Y}{\xi} \right) \right] = \left( \left( \frac{X}{\xi} , \frac{Y}{\xi} \right) \right) \left|_{T(\phi, \psi) \mathcal{F}^{1,1/2}(S^1, N)}, \right.$$  

where $(X, \xi), (Y, \xi) \in T(\phi, \psi) \mathcal{F}^{1,1/2}(S^1, N) = H^1(S^1, \phi^*TN) \times H^{1/2}(S^1, \mathbb{S}(S^1) \otimes \phi^*TN)$ and the metric on $\mathcal{F}^{1,1/2}(S^1, N)$ is defined in (3.4). Thus $A_{(\phi, \psi), H} : T(\phi, \psi) \mathcal{F}^{1,1/2}(S^1, N) \to T(\phi, \psi) \mathcal{F}^{1,1/2}(S^1, N)$ is a bounded self-adjoint operator.

Let us assume that $(\phi, \psi) \in \mathcal{F}^{1,1/2}(S^1, N)$ is a non-degenerate critical point. We fix a smooth pair $(\phi_0, \psi_0) \in \mathcal{F}^{1,1/2}(S^1, N)$ in the component containing $(\phi, \psi)$ and take a smooth path $((\phi_t, \psi_t))_{t \in [0,1]}$ connecting $(\phi_0, \psi_0)$ to $(\phi, \psi)$ in $\mathcal{F}^{1,1/2}(S^1, N)$, $(\phi_{t=0}, \psi_{t=0}) = (\phi_0, \psi_0)$, $((\phi_{t=1}, \psi_{t=1})) = (\phi, \psi)$. By the regularity theory, $(\phi, \psi)$ is smooth and we can always assume that $(\phi_t, \psi_t)$ is also a smooth pair for all $t$. We then define:

**Definition 2.1** We define relative Morse index $\mu_H(\phi, \psi)$ by

$$\mu_H(\phi, \psi) = -\text{sf}(A_{(\phi_t, \psi_t), H})_{0 \leq t \leq 1},$$

where $\text{sf}(A_{(\phi_t, \psi_t), H})_{0 \leq t \leq 1}$ is the spectral flow of a path of operators $A_{(\phi_t, \psi_t), H}$.

We need to prove that the quantity above is well-defined, i.e., $\mu_H(\phi, \psi) \in \mathbb{Z}$ and does not depend on the choice of the path $((\phi_t, \psi_t))$. In fact, we have the following:

**Lemma 2.1** The spectral flow $\text{sf}(A_{(\phi_t, \psi_t), H})_{0 \leq t \leq 1}$ is well-defined and does not depend on the choice of the path $((\phi_t, \psi_t))_{0 \leq t \leq 1}$.
**Proof** Well-definedness follows since, as we will see shortly below, $[A(\phi_t,\psi_t),H]_{0 \leq t \leq 1}$ is a Fredholm family. See [11] for general results about the spectral flow for Fredholm family. As for the independence of the path, the assertion follows if we show that the spectral flow $\text{sf}[A(\phi_t,\psi_t),H]_{t \in S^1}$ is 0 for any closed path $\{[\phi_t,\psi_t]\} \in S^1$, where $H_t$ is a loop of perturbations which satisfy (1.1)–(1.3) uniformly with respect to $t$. (This general case is not necessary for the proof of our assertion. However, for later purpose we treat a slightly general case.)

For simplicity, we write $A_t = A(\phi_t,\psi_t),H_t$.

Let us denote by $P_t(s) : T_{\phi(0)}N \rightarrow T_{\phi(s)}N$ the parallel translation along the path $[0, t] \ni \tau \mapsto \phi_t(s) \in N$. We define the operators $B_t = P_t^{-1} \circ A_t \circ P_t$ parametrized by $t \in S^1$. Note that

$$B_t : H^1(S^1, \phi_0^*TN) \times H^{1/2}(S^1, S(S^1) \otimes \phi_0^*TN) \rightarrow H^1(S^1, \phi_0^*TN) \times H^{1/2}(S^1, S(S^1) \otimes \phi_0^*TN),$$

for all $t \in S^1$ and $\text{sf}\{B_t\}_{t \in S^1} = \text{sf}\{A_t\}_{t \in S^1}$. We thus consider $\text{sf}\{B_t\}_{t \in S^1}$ instead of considering $\text{sf}\{A_t\}_{t \in S^1}$. Note that $A_t$ takes the following block form with respect to the canonical decomposition $T(\phi,\psi)^{S^{1/2}}(S^1, N) = H^1(S^1, \phi^*TN) \times H^{1/2}(S^1, S(S^1) \otimes \phi^*TN)$ (see Sect. 3.1):

$$A_t = \begin{pmatrix}
A_{t,11} & A_{t,12} \\
A_{t,21} & A_{t,22}
\end{pmatrix}.$$

The components $A_{ij,t}$ are given as follows:

$$A_{t,11}X = (\Delta + 1)^{-1} \left( -\nabla \nabla X - R(X, \partial_t \phi_t) \partial_t \phi_t + \frac{1}{2} \nabla R(\phi_t)(\psi_t, \partial_t \phi_t \cdot \psi_t) \\
+ \frac{1}{2} R(\phi_t)(\psi_t, \nabla X \cdot \psi_t) - H_t, \phi(\cdot, \phi_t, \psi_t)[X] \right),$$

$$A_{t,12} \xi = (\Delta + 1)^{-1} \left( \frac{1}{2} R(\phi_t)(\xi, \partial_t \phi_t \cdot \psi_t) + \frac{1}{2} R(\phi_t)(\psi_t, \partial_t \phi_t \cdot \xi) - H_t, \psi(\cdot, \phi_t, \psi_t)[\xi] \right),$$

$$A_{t,21}X = (1 + |D|)^{-1} \left( \partial_x \cdot \partial_y \psi^i X^m \Gamma^k_{mj}(\phi_t) \otimes \frac{\partial}{\partial y^k}(\phi_t) + \nabla \partial_y \Gamma^i_j(\phi_t) \partial_t \phi_t \partial_t \cdot \psi^k \otimes \frac{\partial}{\partial y^i}(\phi_t) \\
+ \Gamma^i_j(\phi_t) \partial_t X^j \partial_t \cdot \psi^k \otimes \frac{\partial}{\partial y^i}(\phi_t) \right),$$

$$A_{t,22} \xi = (1 + |D|)^{-1} (D \xi - H_t, \psi(\cdot, \psi_t)[\xi]),$$

where $\Delta = \frac{\partial^2}{\partial s^2}, D = D^{S(S^1)} \otimes \mathbb{R}^k$, $\nabla = \nabla_{\phi_t}, R(X, Y)Z = R^i_{jkl} X^j Y^k Z^l$, $R(\phi) : X, Y, \psi \mapsto \left\{ \xi, \partial_x \cdot \psi^i \otimes \frac{\partial}{\partial y^i}(\phi) \right\}$, $R^i_{jkl}(\phi) \otimes \frac{\partial}{\partial y^i}(\phi)$ for $\xi, \psi \in H^{1/2}(S^1, S(S^1) \otimes \phi^*TN)$ and $Y \in H^1(S^1, \phi^*TN)$ and $\Gamma^i_{jk}$ is the Christoffel symbol of the metric on $N$. We also use abbreviation $\partial_s = \frac{\partial}{\partial s}, \partial_t = \frac{\partial}{\partial t}$.

By the above form of $A_{ij,t}$, it is easy to see that $A_t$ can be written as

$$A_t = \begin{pmatrix}
A^0_{t,11} & O \\
O & A^0_{t,22}
\end{pmatrix} + \mathcal{K}_t,$$

where

$$A^0_{t,11} = (\Delta + 1)^{-1} (-\nabla \nabla s).$$
\[ A_{t,22}^0 = (1 + |D|)^{-1} D_{\phi_t} \]

and \( K_t \) is compact.

Now, we consider \( \{ B_t \}_{t \in \mathbb{S}^1} \). Observe that \( B_t \) is a 0th order elliptic operator acting on sections of the bundle \( \phi_0^* T N \oplus \mathbb{S}(S^1) \otimes \phi_0^* T N \to S^1 \) and its principal symbol is independent of \( t \). Considering \( S^1 \) as the quotient \( S^1 = [0, 1]/\{0\} \sim \{1\} \), we also have \( B_1 = P_1^{-1} \circ B_0 \circ P_1 \), where \( P_1 \) is considered as an automorphism of the bundle \( \phi_0^* T N \oplus \mathbb{S}(S^1) \otimes \phi_0^* T N \to S^1 \) (acting \( \phi_0^* T N \) parts only). In such a case, the spectral flow \( \text{sf}(B_t)_{t \in S^1} \) is given by the Booss–Wojciechowski’s desuspension formula [11, Theorem 17.17] as follows:

\[
\text{sf}(B_t)_{t \in S^1} = \text{ind}(P_{>0}(B_0) - P_1 P_{<0}(B_0)),
\]

where \( P_{>0}(B_0) \) and \( P_{<0}(B_0) \) are spectral projections of \( B_0 \) to the spectral set \([0, \infty)\) and \((-\infty, 0)\), respectively; \( P_{>0}(B_0) = 1_{[0, \infty)}(B_0) \), \( P_{<0}(B_0) = 1_{(-\infty, 0)}(B_0) \).

Define \( A_t^0 = \begin{pmatrix} A_{t,11}^0 & O \\ O & A_{t,22}^0 \end{pmatrix} \) and \( \mathcal{B}_t = \begin{pmatrix} B_{t,11}^0 & O \\ O & B_{t,22}^0 \end{pmatrix} \). Since the difference \( B_{t,0} - B_0 \) is a compact operator, \( P_{>0}(B_0) - P_{>0}(B_0) \) and \( P_{<0}(B_0) - P_{<0}(B_0) \) are compact. Thus by the index formula (2.6) and (2.7) we have

\[
\begin{align*}
\text{sf}(B_t)_{t \in S^1} &= \text{ind}(P_{>0}(B_0) - P_1 P_{<0}(B_0)) \\
&= \text{ind}(P_{>0}(B_{0,11}) - P_1 P_{<0}(B_{0,11})) + \text{ind}(P_{>0}(B_{0,22}) - P_1 P_{<0}(B_{0,22})) \\
&= \text{ind}(P_{>0}(B_{0,22}) - P_1 P_{<0}(B_{0,22})) \\
&= \text{ind}(P_{>0}(D_{\phi_0}) - P_1 P_{<0}(D_{\phi_0})) \\
&= \text{sf}(P_t^{-1} \circ D_{\phi_t} \circ P_1)_{t \in S^1} \\
&= \text{sf}(D_{\phi_t})_{t \in S^1},
\end{align*}
\]

where we have used in the third line the fact that \( B_{0,11}^0 = A_{0,11} \) is a non-negative operator so that \( P_{>0}(B_{0,11}) = 1 \) and \( P_{<0}(B_{0,11}) = 0 \).

On the other hand, by the Atiyah–Patodi–Singer index theorem [8, 11], the spectral flow \( \text{sf}(D_{\phi_t})_{t \in S^1} \) is given by the Fredholm index of the operator \( \frac{\partial}{\partial t} + D_{\phi_t} \) defined on \( S^1 \times S^1 \),

\[
\text{sf}(D_{\phi_t})_{t \in S^1} = \text{ind}\left( \frac{\partial}{\partial t} + D_{\phi_t} \right),
\]

where

\[
\frac{\partial}{\partial t} + D_{\phi_t} : H^1(S^1 \times S^1, \mathbb{S}(S^1 \times S^1)^\pm \otimes \phi^* T N) \to L^2(S^1 \times S^1, \mathbb{S}(S^1 \times S^1)^{-} \otimes \phi^* T N)
\]

is the Dirac operator and \( \mathbb{S}(S^1 \times S^1)^\pm \) is the \( \pm \)-spinor bundle associated with a spin structure on \( S^1 \times S^1 \) whose restriction to the first summand gives the trivial spin structure on \( S^1 \), while the restriction to the second summand gives the one of \( \mathbb{S}(S^1) \to S^1 \). By the Atiyah–Singer index theorem, we have

\[
\text{ind}\left( \frac{\partial}{\partial t} + D_{\phi_t} \right) = \int_{S^1 \times S^1} \text{ch}(\phi^* T N \otimes \mathbb{C}) \hat{\Lambda}(S^1 \times S^1)
\]

\[
= \int_{S^1 \times S^1} c_1(\phi^* T N \otimes \mathbb{C}) = \int_{S^1 \times S^1} \phi^* c_1(T N \otimes \mathbb{C}) = 0,
\]

where \( \text{ch} \) is the Chern character and \( \hat{\Lambda} \) is the \( \hat{A} \)-class. The last equation follows from \( c_1(T N \otimes \mathbb{C}) = 0 \) as the first Chern class of the complexification of the real bundle \( T N \). Thus, by (2.7) we have \( \text{sf}(A_t)_{t \in S^1} = \text{sf}(B_t)_{t \in S^1} = 0 \). This completes the proof. \( \square \)
3 The negative gradient flow of $\mathcal{L}_H$ on $\mathcal{F}^{1,1/2}(S^1, N)$

3.1 The gradient flow equation

In the following, we assume that $H$ satisfies (1.1)–(1.3) for some $p \geq 3$. For the case of flat manifolds $N$ as in the statement of Theorem 1.1 (4), we only need to assume $p > 1$. The reason is that, in this case, various curvature terms vanish and the equations take much simpler from. Since all proof for the flat case are much simpler than the general case, in the rest of this paper (except for Sect. 9) we only consider the general case. Proofs of similar results under the assumption $p > 1$ in the case of flat manifolds is left to the interested reader.

We assume that $N$ is isometrically embedded in $\mathbb{R}^k$. Thus, we regard $H^1(S^1, \mathbb{R})$ and $H^{1/2}(S^1, \mathbb{S}(S^1) \otimes \phi^*TN) \subset H^{1/2}(S^1, \mathbb{S}(S^1) \otimes \mathbb{R}^k)$, where $\mathbb{R}^k = \phi^*(N \times \mathbb{R}^k) \to S^1$ is the trivial bundle over $S^1$. The metric on $\mathcal{F}^{1,1/2}(S^1, N)$ is induced from these embeddings. To be more precise, recall that there is a canonical identification

$$T_{(\phi, \psi)}\mathcal{F}^{1,1/2}(S^1, N) = H^1(S^1, \phi^*TN) \times H^{1/2}(S^1, \mathbb{S}(S^1) \otimes \phi^*TN).$$

(3.1)

It is described as follows (see [20, §3.3] for more details): Let $(-1, 1) \ni t \mapsto (\phi_t, \psi_t) \in \mathcal{F}^{1,1/2}(S^1, N)$ be a smooth path passing through $(\phi, \psi) \in \mathcal{F}^{1,1/2}(S^1, N)$ at $t = 0$. That is, $(\phi_t, \psi_t)|_{t=0} = (\phi, \psi)$. We set $X := \frac{d}{dt} \bigg|_{t=0} \phi_t \in H^1(S^1, \phi^*TN)$. Using local coordinate $y = (y^k)$ on $N$, we write $\psi_t = \psi_t^i \otimes \frac{\partial}{\partial y^i}(\phi_t)$, where $\psi_t^i \in H^{1/2}(S^1, \mathbb{S}(S^1))$. Then the variation vector field $\frac{d}{dt} \bigg|_{t=0} \psi_t$ is given by

$$\frac{d}{dt} \bigg|_{t=0} \psi_t = \frac{d}{dt} \bigg|_{t=0} \psi_t^i \otimes \frac{\partial}{\partial y^i} + \psi_t^i \otimes \nabla \frac{\partial}{\partial y^i}(\phi_t) = \left( \frac{d}{dt} \bigg|_{t=0} \psi_t^i + X^j \Gamma^k_{ji}(\phi_t) \psi_t^j \right) \otimes \frac{\partial}{\partial y^k}(\phi),$$

(3.2)

where $\Gamma^k_{ij}$ is the Christopher symbol of the metric on $N$. We set $\xi^k = \frac{d}{dt} \bigg|_{t=0} \psi_t^k$ and $\xi = \xi^k \otimes \frac{\partial}{\partial y^k}(\phi)$. Then the identification (3.1) is given by

$$\frac{d}{dt} \bigg|_{t=0} (\phi_t, \psi_t) \mapsto (X, \xi).$$

(3.3)

Using the identification (3.1), the metric on $\mathcal{F}^{1,1/2}(S^1, N)$ is defined by

$$\left( \begin{array}{c} X \\ \xi \end{array} \right)_{T_{(\phi, \psi)}\mathcal{F}^{1,1/2}} = (X, Y)_{H^1} + \langle \xi, \zeta \rangle_{H^{1/2}(S^1)},$$

(3.4)

where $X, Y \in H^1(S^1, \phi^*TN) \subset H^1(S^1, \mathbb{R}^k)$, $(X, Y)_{H^1(S^1)} = ((\Delta + 1)X, Y)_{L^2(S^1)}$ ($\Delta = \nabla^2$) and $\xi, \zeta \in H^{1/2}(S^1, \mathbb{S}(S^1) \otimes \phi^*TN) \subset H^{1/2}(S^1, \mathbb{S}(S^1) \otimes \mathbb{R}^k)$, $(\xi, \zeta)_{H^{1/2}(S^1)} = ((1 + |D|)\xi, \zeta)_{L^2(S^1)}$ for $S^1 \subset \mathbb{S}^D(S^1) \boxtimes 1_{\mathbb{R}^k}$). With this metric, $\mathcal{F}^{1,1/2}(S^1, N)$ becomes a complete Hilbert manifold. The gradient of $\mathcal{L}_H$ with respect to this metric is denoted by $\nabla_{1,1/2} \mathcal{L}_H$. According to the decomposition (3.1), $\nabla_{1,1/2} \mathcal{L}_H$ reads as

$$\nabla_{1,1/2} \mathcal{L}_H(\phi, \psi) = \nabla_{\phi} \mathcal{L}_H(\phi, \psi) \oplus \nabla_{\psi} \mathcal{L}_H(\phi, \psi).$$

The components are given by the following proposition:

**Proposition 3.1** [20] For $(\phi, \psi) \in \mathcal{F}^{1,1/2}(S^1, N)$, we have the following:

$$\nabla_{\phi} \mathcal{L}_H(\phi, \psi) = (-\Delta + 1)^{-1} \left( -\nabla \partial_s \phi + \frac{1}{2} R(\phi)(\psi, \partial_s \phi \cdot \psi) - \nabla_{\phi} H(s, \phi, \psi) \right),$$

(3.5)
\[ \nabla_\psi \mathcal{L}_H(\phi, \psi) = \left(1 + |D|\right)^{-1}(D_\phi \psi - \nabla_\psi H(s, \phi, \psi)), \quad (3.6) \]

where

\[ R(\phi)(\psi, \partial_s \phi \cdot \psi) = \left(\psi, \partial_s \psi \ast \frac{\partial}{\partial y^j}(\phi)\right) \partial_s \phi \cdot R^i_{i\mu j}(\phi) g^{\mu s}(\phi) \frac{\partial}{\partial y^s}(\phi) \]

and \( R^i_{i\mu j} \) is the curvature tensor of the metric on \( N \).

See [20] for the details of the derivation of the above formula.

Thus the negative gradient flow equation \( \frac{d}{dt} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = -\nabla_{1,1/2} \mathcal{L}_H(\phi, \psi) \) takes the following form:

\[ \frac{\partial \psi}{\partial t} = (\Delta + 1)^{-1} \left( \nabla_\partial \psi \phi - \frac{1}{2} R(\phi)(\psi, \partial_s \phi \cdot \psi) + \nabla_\phi H(s, \phi, \psi) \right), \quad (3.7) \]

\[ \frac{\partial \psi}{\partial t} = -(1 + |D|)^{-1}(D_\phi \psi - \nabla_\psi H(s, \phi, \psi)). \quad (3.8) \]

### 3.2 Regularity and estimates for the negative gradient flow

In this section, we establish regularity properties of solutions to the negative gradient flow equation (3.7), (3.8). We first prove the following:

**Proposition 3.2** Let \( (\phi(t), \psi(t)) \in C^1(\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N)) \) be a solution to the negative gradient flow equation

\[ \frac{d}{dt} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = -\nabla_{1,1/2} \mathcal{L}_H(\phi, \psi) \quad (3.9) \]

which satisfies the condition \( \sup_{t \in \mathbb{R}} |\mathcal{L}_H(\phi(t), \psi(t))| = C_0 < +\infty \). Then there exists \( C(C_0) > 0 \) such that

\[ \sup_{t \in \mathbb{R}} \| \partial_s \psi(t) \|_{L^2(S^1)} + \sup_{t \in \mathbb{R}} \| \psi(t) \|_{H^{1/2}(S^1)} \leq C(C_0). \]

To prove the above proposition, we prepare some preliminary materials. We denote by \( H_0^-, H_0^0 \) and \( H_0^+ \) the negative, zero and positive subspaces of \( H^{1/2}(S^1, \mathbb{S}(S^1) \otimes \mathbb{R}^k) \) with respect to the Dirac operator \( D = D^0(S^1) \otimes I_{\mathbb{R}^k} \). Thus we have

\[ H^{1/2}(S^1, \mathbb{S}(S^1) \otimes \mathbb{R}^k) = H_0^- \oplus H_0^0 \oplus H_0^+. \quad (3.10) \]

We denote by \( P_0^-, P_0^0 \) and \( P_0^+ \) the spectral projections onto \( H_0^-, H_0^0 \) and \( H_0^+ \), respectively. For a spinor \( \psi \in H^{1/2}(S^1, \mathbb{S}(S^1) \otimes \mathbb{R}^k) \), we denote \( \psi^- = P_0^- \psi, \psi^0 = P_0^0 \psi \) and \( \psi^+ = P_0^+ \psi \). Thus, we have

\[ \psi = \psi^- + \psi^0 + \psi^+. \]

By the embedding \( \iota : N \leftrightarrow \mathbb{R}^k \), for \( \psi = \psi^k \otimes \frac{\partial}{\partial y^k}(\phi) \in H^{1/2}(S^1, \mathbb{S}(S^1) \otimes \phi^* TN) \) we have

\[ \tilde{\psi} := \iota_* \psi = \psi^k \otimes \iota_*(\frac{\partial}{\partial y^k}(\phi)) \in H^{1/2}(S^1, \mathbb{S}(S^1) \otimes \mathbb{R}^k). \]

Note that, we have

\[ D\tilde{\psi} = \frac{\partial}{\partial s} \cdot D_{\frac{\partial}{\partial y^k}} \tilde{\psi} = \frac{\partial}{\partial s} \cdot (\nabla_{\frac{\partial}{\partial y^k}} \psi^k \otimes \iota_*(\frac{\partial}{\partial y^k}(\phi))) + \psi^k \otimes (\nabla_{\frac{\partial}{\partial y^k}} \iota_*(\frac{\partial}{\partial y^k}(\phi))) \]

\[ + \psi^0 \otimes \iota_*(\nabla_{\frac{\partial}{\partial y^k}} \frac{\partial}{\partial y^k}(\phi))). \]
Lemma 3.1 Under the assumption of Proposition 3.2, there exists a positive constant $C$ depending only on $C_0$ such that the following holds for all $\tau \in \mathbb{R}$:

\[
\int_{\tau}^{\tau+1} \|\phi(t)\|_{H^1(S^1)}^2 \, dt \leq C(0),
\]

\[
\int_{\tau}^{\tau+1} \|\psi(t)\|_{H^{1/2}(S^1)}^2 \, dt \leq C(0),
\]

\[
\int_{\tau}^{\tau+1} \|\psi(t)\|_{L^{p+1}(S^1)}^{p+1} \, dt \leq C(0).
\]

Proof Let $-\infty < a < b < +\infty$ be arbitrary. Since $(\phi(t), \psi(t))$ is a solution to the negative gradient flow equation (3.9), we have

\[
\int_{a}^{b} \|\partial_t \phi(t)\|_{H^1(S^1)}^2 + \|\partial_t \psi(t)\|_{H^{1/2}(S^1)}^2 \, dt
\]

\[
= - \int_{a}^{b} \left\langle \nabla_{1/2} L_H(\phi(t), \psi(t)), \left(\begin{array}{c} \partial_t \phi(t) \\ \partial_t \psi(t) \end{array} \right) \right\rangle_{T(\phi(t), \psi(t)), L^{1,1/2}} \, dt
\]

\[
= - \int_{a}^{b} \frac{d}{dt} L_H(\phi(t), \psi(t)) \, dt
\]

\[
= L_H(\phi(a), \psi(a)) - L_H(\phi(b), \psi(b)) \leq 2C_0
\]

by our assumption $\sup_{t \in \mathbb{R}} |L_H(\phi(t), \psi(t))| =: C_0$. Since $-\infty < a < b < +\infty$ were arbitrary, we have

\[
\int_{-\infty}^{\infty} \|\partial_t \phi(t)\|_{H^1(S^1)}^2 \, dt + \int_{-\infty}^{\infty} \|\partial_t \psi(t)\|_{H^{1/2}(S^1)}^2 \, dt \leq 2C_0.
\]

Under the identification \(T_{(\phi(t), \psi(t))} : L^{1,1/2} \rightarrow H^1(S^1) \times H^{1/2}(S^1)\), taking the inner product with (3.9), we have

\[
(\partial_t \psi(t), \psi(t))_{H^{1/2}(S^1)} = -\left\langle \nabla_{1/2} L_H(\phi(t), \psi(t)), \left(\begin{array}{c} 0 \\ \psi(t) \end{array} \right) \right\rangle_{T(\phi(t), \psi(t)), L^{1,1/2}}
\]

\[
= - \int_{S^1} \langle \psi(t), D_\phi \psi(t) \rangle \, ds + \int_{S^1} \langle \nabla \psi H(s, \phi(t), \psi(t)), \psi(t) \rangle \, ds.
\]
By (1.2), we thus have
\[
(\partial_t \psi(t), \psi(t))_{H^{1/2}(S^1)} + 2L_H(\phi(t), \psi(t)) = \int_{S^1} |\partial_s \phi(t)|^2 ds - 2 \int_{S^1} H(s, \phi(t), \psi(t)) ds + \int_{S^1} \notag \langle \nabla_{\phi} H(s, \phi(t), \psi(t)), \psi(t) \rangle ds \geq \int_{S^1} |\partial_s \phi(t)|^2 ds + C_2 \int_{S^1} |\psi(t)|^{p+1} ds - C.
\]
(3.16)
Combining (3.16) with the assumption \(\sup_{t \in \mathbb{R}} |L_H(\phi(t), \psi(t))| = C_0 < +\infty\), we have
\[
\int_{S^1} |\partial_s \phi(t)|^2 ds + \int_{S^1} |\psi(t)|^{p+1} ds \leq 2C_0 + C \|\psi(t)\|_{H^{1/2}(S^1)} \|\partial_t \psi(t)\|_{H^{1/2}(S^1)}. \tag{3.17}
\]
On the other hand, there exists a positive constant \(\lambda_+ > 0\) (depending only on the smallest positive eigenvalue of \(D\)) such that
\[
\|\psi_0^+(t)\|_{H^{1/2}(S^1)}^2 \leq \lambda_+ \int_{S^1} \langle \psi_0^+, D\psi \rangle ds \leq \lambda_+ \int_{S^1} \langle \psi_0^+, D\phi \psi + A(\partial_s \phi, \partial_s \psi) \rangle ds \leq \lambda_+ \int_{S^1} \langle \psi_0^+, D\phi \psi \rangle ds + C \int_{S^1} |\partial_s \phi| \|\psi_0^+\| |\psi| ds \leq \lambda_+ \int_{S^1} \langle P(\phi(t)) \psi_0^+(t), D\phi(t) \psi(t) \rangle ds + C \int_{S^1} |\partial_s \phi| \|\psi_0^+\| |\psi| ds, \tag{3.18}
\]
where we have used (3.11) in the second inequality. Note that, since \((D\phi(t) \psi(t))(s) \in \mathbb{R}^k \otimes T_{\phi(t)(s)} N\), the last inequality of (3.18) follows, where \(P_y : \mathbb{R}^k \to T_y N\) is the orthogonal projection. We also note that, by Lemma 10.2 in the Appendix, \(P(\phi(t) \psi_0^+(t)) \in H^{1/2}(S^1, \mathbb{R}^k \otimes \phi(t)^* T N)\) so that the integral \(\int_{S^1} \langle P(\phi(t) \psi_0^+(t), D\phi(t) \psi(t) \rangle ds\) makes sense.

To estimate the integral \(\int_{S^1} \langle P(\phi(t) \psi_0^+(t), D\phi(t) \psi(t) \rangle ds\), we observe that \(\begin{pmatrix} 0 \\ P(\phi(t) \psi_0^+(t)) \end{pmatrix} \in T_{(\phi(t), \psi(t))} F^{1,1/2}\) so that we have
\[
\left| \int_{S^1} \langle P(\phi(t) \psi_0^+(t), D\phi(t) \psi(t) \rangle ds \right| = \left| \int_{S^1} \langle P(\phi(t) \psi_0^+(t), D\phi(t) \psi(t) \rangle ds - \int_{S^1} \langle \nabla \phi H(s, \phi(t), \psi(t)), P(\phi(t) \psi_0^+(t)) \rangle ds \right|
\leq \int_{S^1} \|\partial_t \phi(t)\|_{H^{1/2}(S^1)} \|P(\phi(t) \psi_0^+(t))\|_{H^{1/2}(S^1)}
\leq C(\|\partial_t \phi(t)\|_{H^{1/2}(S^1)} \|\psi(t)\|_{H^{1/2}(S^1)} + \|\partial_t \psi(t)\|_{H^{1/2}(S^1)} \|\partial_s \phi(t)\|_{L^2(S^1)} \|\psi(t)\|_{L^4(S^1)}), \tag{3.19}
\]
where in the last inequality, we have used (10.8) in the Appendix.

Combining (3.18) and (3.19), we obtain
\[
\|\psi_0^+(t)\|_{H^{1/2}(S^1)}^2 \leq C \left( \|\partial_t \psi(t)\|_{H^{1/2}(S^1)} \|\psi(t)\|_{H^{1/2}(S^1)} \right).
\]
where in the last inequality, we have used \( |\nabla_\psi H(s, \phi, \psi)| \leq C(1 + |\psi|^p) \) which follows from (1.5).

On the other hand, by (3.17), we have
\[
\|\partial_3 \phi(t)\|_{L^2(S^1)} \leq C(1 + \|\psi(t)\|_{H^{1/2}(S^1)})^{1/2}
\]  
(3.21)
and
\[
\|\psi(t)\|_{L^4(S^1)} \leq C\|\psi(t)\|_{L^{p+1}(S^1)} \leq C(1 + \|\psi(t)\|_{H^{1/2}(S^1)})^{1/p+1},
\]  
(3.22)
where by our assumption \( p \geq 3 \), we have used \( \|\psi(t)\|_{L^4(S^1)} \leq C\|\psi(t)\|_{L^{p+1}(S^1)} \) which follows from Hölder’s inequality. Plugging (3.21) and (3.22) into (3.20) and noting the inequality \( \|\psi^+_0\|_{L^q(S^1)} \leq C\|\psi\|_{L^q(S^1)} \) which holds for \( 1 < q < \infty \) (since \( P^+_0 \) is a pseudo differential operator of order 0), we obtain
\[
\|\psi^+_0(t)\|_{H^{1/2}(S^1)}^2 \leq C\|\partial_3 \psi(t)\|_{H^{1/2}(S^1)}\|\psi(t)\|_{H^{1/2}(S^1)}
+ C(1 + \|\partial_3 \psi(t)\|_{H^{1/2}(S^1)}\|\psi(t)\|_{H^{1/2}(S^1)})^{1/2 + \frac{1}{p+1}} \|\partial_2 \psi(t)\|_{H^{1/2}(S^1)}
+ C\|\psi(t)\|_{L^{p+1}(S^1)} + C\|\psi(t)\|_{L^{p+1}(S^1)}^{p+1}
+ C(1 + \|\partial_3 \psi(t)\|_{H^{1/2}(S^1)}\|\psi(t)\|_{H^{1/2}(S^1)})^{1/2 + \frac{2}{p+1}}.
\]  
(3.23)
Here, \( \frac{1}{2} + \frac{2}{p+1} \leq 1 \) for \( p \geq 3 \) implies that
\[
(1 + \|\partial_3 \psi(t)\|_{H^{1/2}(S^1)}\|\psi(t)\|_{H^{1/2}(S^1)})^{1/2 + \frac{2}{p+1}} \leq 1 + \|\partial_3 \psi(t)\|_{H^{1/2}(S^1)}\|\psi(t)\|_{H^{1/2}(S^1)}.
\]

Also, by (3.17), we have \( \|\psi(t)\|_{L^{p+1}(S^1)} \leq C(1 + \|\partial_3 \psi(t)\|_{H^{1/2}(S^1)}\|\psi(t)\|_{H^{1/2}(S^1)}) \). Thus, we have
\[
\|\psi^+_0(t)\|_{H^{1/2}(S^1)}^2 \leq C\|\partial_3 \psi(t)\|_{H^{1/2}(S^1)}\|\psi(t)\|_{H^{1/2}(S^1)} + C\|\psi(t)\|_{H^{1/2}(S^1)} + C
+ C(1 + \|\partial_3 \psi(t)\|_{H^{1/2}(S^1)}\|\psi(t)\|_{H^{1/2}(S^1)})^{1/2 + \frac{1}{p+1}} \|\partial_2 \psi(t)\|_{H^{1/2}(S^1)}
\leq C\|\partial_3 \psi(t)\|_{H^{1/2}(S^1)}\|\psi(t)\|_{H^{1/2}(S^1)} + C\|\psi(t)\|_{H^{1/2}(S^1)} + C
+ C(1 + \|\partial_3 \psi(t)\|_{H^{1/2}(S^1)}\|\psi(t)\|_{H^{1/2}(S^1)})^{1/2} \|\partial_2 \psi(t)\|_{H^{1/2}(S^1)},
\]  
(3.24)
where in the last inequality, we have used \( \frac{1}{2} + \frac{1}{p+1} \leq \frac{1}{2} \) for \( p \geq 3 \).

Integrating (3.24) over \([\tau, \tau + 1]\) with respect to \( t \), we have, by the Hölder’s inequality
\[
\int_\tau^{\tau+1} \|\psi^+_0(t)\|_{H^{1/2}(S^1)}^2 \, dt \leq C \left( \int_\tau^{\tau+1} \|\partial_3 \psi(t)\|_{H^{1/2}(S^1)}^2 \, dt \right)^{1/2} \left( \int_\tau^{\tau+1} \|\psi(t)\|_{H^{1/2}(S^1)}^2 \, dt \right)^{1/2}
\]
\[ C \left( \int_{\tau}^{\tau+1} \| \psi(t) \|_{H^{1/2}(S^1)}^2 \, dt \right)^{1/2} + C \]
\[ + C \left( \int_{\tau}^{\tau+1} \left( 1 + \| \partial_t \psi(t) \|_{H^{1/2}(S^1)} \| \psi(t) \|_{H^{1/2}(S^1)} \right) dt \right)^{1/2} \left( \int_{\tau}^{\tau+1} \| \partial_t \psi(t) \|^2_{H^{1/2}(S^1)} \, dt \right)^{1/2} \]
\[ \leq C + C \left( \int_{\tau}^{\tau+1} \| \psi(t) \|^2_{H^{1/2}(S^1)} \, dt \right)^{1/2} + C \left( \int_{\tau}^{\tau+1} \| \psi(t) \|^2_{H^{1/2}(S^1)} \, dt \right)^{1/4} \]
\[ \leq C + C \left( \int_{\tau}^{\tau+1} \| \psi(t) \|^2_{H^{1/2}(S^1)} \, dt \right)^{1/2}, \quad (3.25) \]

where in the second inequality, we have used \( \int_{-\infty}^{\infty} \| \partial_t \psi(t) \|^2_{H^{1/2}(S^1)} \, dt \leq 2C_0 \) (see (3.15)) and Hölder’s inequality once again.

By the same argument, we have
\[ \int_{\tau}^{\tau+1} \| \psi_0(t) \|^2_{H^{1/2}(S^1)} \, dt \leq C \left( 1 + \left( \int_{\tau}^{\tau+1} \| \psi(t) \|^2_{H^{1/2}(S^1)} \, dt \right)^{1/2} \right). \quad (3.26) \]

As for the component \( \psi_0^0 \), since the \( H^{1/2} \)-norm is equivalent to the \( L^{p+1} \)-norm on the finite dimensional space \( H_0^1 \), we have
\[ \| \psi_0^0(t) \|^2_{H^{1/2}(S^1)} \leq C \| \psi_0^0(t) \|^2_{L^{p+1}(S^1)} \]
\[ \leq C \left( 1 + \| \partial_t \psi(t) \|_{H^{1/2}(S^1)} \| \psi(t) \|_{H^{1/2}(S^1)} \right)^{2/p+1} \]
\[ \leq C \left( 1 + \| \partial_t \psi(t) \|_{H^{1/2}(S^1)} \| \psi(t) \|_{H^{1/2}(S^1)} \right), \quad (3.27) \]

where we have used (3.17) and \( \frac{2}{p+1} < 1 \).

Integrating (3.27) over \([\tau, \tau+1]\) and using Hölder’s inequality and (3.15), we have
\[ \int_{\tau}^{\tau+1} \| \psi_0^0(t) \|^2_{H^{1/2}(S^1)} \, dt \leq C \left( 1 + \left( \int_{\tau}^{\tau+1} \| \psi(t) \|^2_{H^{1/2}(S^1)} \, dt \right)^{1/2} \right). \quad (3.28) \]

By (3.25), (3.26) and (3.28), we have
\[ \int_{\tau}^{\tau+1} \| \psi(t) \|^2_{H^{1/2}(S^1)} \, dt \leq C \left( 1 + \left( \int_{\tau}^{\tau+1} \| \psi(t) \|^2_{H^{1/2}(S^1)} \, dt \right)^{1/2} \right). \quad (3.29) \]

(3.29) implies that
\[ \int_{\tau}^{\tau+1} \| \psi(t) \|^2_{H^{1/2}(S^1)} \, dt \leq C \quad (3.30) \]

for some \( C > 0 \) depending only on \( C_0 \).

On the other hand, integrating (3.17) over \([\tau, \tau+1]\) and using (3.15), (3.30) and Hölder’s inequality, we obtain
\[ \int_{\tau}^{\tau+1} \| \psi(t) \|^2_{H^{1/2}(S^1)} \, dt + \int_{\tau}^{\tau+1} \| \psi(t) \|_{L^{p+1}(S^1)} \, dt \]
\[ \leq C + C \left( \int_{\tau}^{\tau+1} \| \partial_t \psi(t) \|^2_{H^{1/2}(S^1)} \, dt \right)^{1/2} \left( \int_{\tau}^{\tau+1} \| \psi(t) \|^2_{H^{1/2}(S^1)} \, dt \right)^{1/2} \]
\[ \leq C, \quad (3.31) \]

where \( C > 0 \) depends only on \( C_0 \). This completes the proof. From (3.30) and (3.31), we complete the proof. \( \square \)
We now prove Proposition 3.2.

**Proof of Proposition 3.2.** Recall that the negative gradient flow equations take the form (3.7), (3.8). As before, we identify \( \phi \) and \( \psi \) with \( \delta \circ \phi \) and \( \delta \circ \psi \) and denote them by \( \phi \) and \( \psi \) for simplicity. Thus, we think of \( \phi \) as taking its values in \( \mathbb{R}^k \) and \( \psi \) takes its values in \( S(S^1) \otimes \mathbb{R}^k \). We first treat the spinor part of the Eq. (3.8). Recall that, from (3.11), the second equation, (3.8), is written as follows

\[
\partial_t \psi(t) = -(1 + |D|)^{-1}(D \psi - A(\partial_s \phi(t), \partial_s \psi(t)) - \nabla \psi H(s, \phi(t), \psi(t)))
\]

\[
= -(1 + |D|)^{-1}((D + \lambda) \psi(t) - \lambda(\psi(t) + A(\partial_s \phi(t), \partial_s \psi(t)) - \nabla \psi H(s, \phi(t), \psi(t)))
\]

\[
= -L_\lambda \psi(t) + (1 + |D|)^{-1}(\lambda(\psi(t) + A(\partial_s \phi(t), \partial_s \psi(t)) + \nabla \psi H(s, \phi(t), \psi(t)))
\]

(3.32)

where for \( \lambda \in \mathbb{R}, L_\lambda = (1 + |D|)^{-1}D_{-\lambda} \) and \( D_{-\lambda} = D + \lambda \).

We take \( \lambda \in \mathbb{R} \) such that \( -\lambda \notin \text{Spec}(D) \). The fundamental solution of the differential operator \( \frac{d}{dt} + L_\lambda : C^1(\mathbb{R}, H^{1/2}(S^1, S(S^1) \otimes \mathbb{R}^k)) \rightarrow C^0(\mathbb{R}, H^{1/2}(S^1, S(S^1) \otimes \mathbb{R}^k)) \) is then given by

\[
G_\lambda(t) = e^{-r_\lambda^+} 1_{(-\infty,0]}(t)P_\lambda^- + e^{-r_\lambda^-} 1_{[0,\infty)}(t)P_\lambda^+,
\]

(3.33)

where as in (3.10), \( P_\lambda^\pm : H^{1/2}(S^1, S(S^1) \otimes \mathbb{R}^k) \rightarrow H_\lambda^\pm \) are spectral projections with respect to the operator \( D_\lambda \) onto its positive/negative subspaces of \( H^{1/2}(S^1, S(S^1) \otimes \mathbb{R}^k) = H_\lambda^- \oplus H_\lambda^+ \) and \( L_\lambda = L_\lambda|_{H_\lambda^\pm} \). Since \(-\lambda \notin \text{Spec}(D)\), there exists \( \kappa > 0 \) such that the following holds:

\[
\|G_\lambda(t)\|_{\text{op}(H^{1/2}(S^1, S(S^1) \otimes \mathbb{R}^k))} \leq e^{-\kappa |t|},
\]

(3.34)

where for \( T : H^{1/2}(S^1, S(S^1) \otimes \mathbb{R}^k) \rightarrow H^{1/2}(S^1, S(S^1) \otimes \mathbb{R}^k), \|T\|_{\text{op}(H^{1/2}(S^1, S(S^1) \otimes \mathbb{R}^k))} \) denotes the operator norm of \( T \).

Using the fundamental solution \( G_\lambda \), the Eq. (3.32) is equivalent to the following integral equation

\[
\psi(t) = \int_{\mathbb{R}} G_\lambda(\tau)(1+|D|)^{-1}(\lambda(\psi(\tau) + A(\partial_s \phi(\tau), \partial_s \psi(\tau)) + \nabla \psi H(s, \phi(\tau), \psi(\tau)))) d\tau.
\]

(3.35)

From this and (3.34), we have

\[
\|\psi(t)\|_{H^{1/2}(S^1)} \leq \int_{\mathbb{R}} e^{-\kappa |\tau|} \|(1 + |D|)^{-1}(\lambda(\psi(\tau) + A(\partial_s \phi(\tau), \partial_s \psi(\tau)) + \nabla \psi H(s, \phi(\tau), \psi(\tau)))) \|_{H^{1/2}(S^1)} d\tau
\]

\[
= \sum_{n \in \mathbb{Z}} \int_{n}^{n+1} e^{-\kappa |\tau|} \|(1 + |D|)^{-1}(\lambda(\psi(\tau) + A(\partial_s \phi(\tau), \partial_s \psi(\tau)) + \nabla \psi H(s, \phi(\tau), \psi(\tau)))) \|_{H^{1/2}(S^1)} d\tau
\]

\[
\leq \left( \sum_{n \in \mathbb{Z}} e^{-\kappa |n|} \right) \sup_{\eta \in \mathbb{R}} \int_{\eta}^{\eta+1} \|(1 + |D|)^{-1}(\lambda(\psi(\tau) + A(\partial_s \phi(\tau), \partial_s \psi(\tau)) + \nabla \psi H(s, \phi(\tau), \psi(\tau)))) \|_{H^{1/2}(S^1)} d\tau
\]

\[
\leq C \sup_{\eta \in \mathbb{R}} \int_{\eta}^{\eta+1} \|(1 + |D|)^{-1}(\lambda(\psi(\tau) + A(\partial_s \phi(\tau), \partial_s \psi(\tau)) + \nabla \psi H(s, \phi(\tau), \psi(\tau)))) \|_{H^{1/2}(S^1)} d\tau.
\]

(3.36)

Here, we note that the following hold:

\[
|A(\partial_s \phi(t), \partial_s \psi(t))| \leq C|\partial_s \phi(t)||\psi(t)|,
\]

(3.37)
\[ |\nabla_\psi H(s, \phi(t), \psi(t))| \leq C(1 + |\psi(t)|^p). \] (3.38)

From (3.37) and the Sobolev embedding \( H^{1/2}(S^1) \subset L^q(S^1) \) for any \( 1 < q < +\infty \), we have \( A(\partial_s \phi(t), \partial_s \cdot \psi(t)) \in L^r(S^1) \) for any \( 1 < r < 2 \) and \( t \in \mathbb{R} \) and
\[
(1 + |D|)^{-1} A(\partial_s \phi(t), \partial_s \cdot \psi(t)) \in W^{1,r}(S^1)
\] (3.39)
for any \( t \in \mathbb{R} \) by the elliptic regularity. Again, by the Sobolev embeddings \( W^{1,r}(S^1) \subset H^{1/2}(S^1) \) for \( r \geq 1 \) and \( H^{1/2}(S^1) \subset L^{2r}/r(S^1) \) and Hölder’s inequality, we therefore have
\[
\|(1 + |D|)^{-1} A(\partial_s \phi(t), \partial_s \cdot \psi(t))\|_{H^{1/2}(S^1)} \leq C\|(1 + |D|)^{-1} A(\partial_s \phi(t), \partial_s \cdot \psi(t))\|_{W^{1,r}(S^1)}
\]
\[
\leq C\|A(\partial_s \phi(t), \partial_s \cdot \psi(t))\|_{L^{r}(S^1)}
\]
\[
\leq C\|\partial_s \phi(t)\|_{L^{2r}(S^1)}\|\psi(t)\|_{H^{1/2}(S^1)},
\] (3.40)

Similarly, by (3.38), we have \( \nabla_\psi H(s, \phi(t), \psi(t)) \in L^{p+1}(S^1) \) and
\[
\|(1 + |D|)^{-1} \nabla_\psi H(s, \phi(t), \psi(t))\|_{H^{1/2}(S^1)} \leq C\|(1 + |D|)^{-1} \nabla_\psi H(s, \phi(t), \psi(t))\|_{W^{1,p+1}(S^1)}
\]
\[
\leq C\|\nabla_\psi H(s, \phi(t), \psi(t))\|_{L^{p+1}(S^1)}
\]
\[
\leq C(1 + \|\psi(t)\|_{L^{p+1}(S^1)}^p).
\] (3.41)

Combining (3.36), (3.40) and (3.41), we obtain, by the Hölder’s inequality
\[
\|\psi(t)\|_{H^{1/2}(S^1)}^2 \leq C \sup_{\tau \in \mathbb{R}} \int_{\eta}^{\eta+1} (1 + \|\psi(\tau)\|_{H^{1/2}(S^1)} + \|\partial_s \phi(\tau)\|_{L^{2r}(S^1)}\|\psi(\tau)\|_{H^{1/2}(S^1)} + \|\psi(\tau)\|_{L^{p+1}(S^1)}^p) d\tau
\]
\[
\leq C \sup_{\eta \in \mathbb{R}} \left(1 + \left(\int_{\eta}^{\eta+1} \|\psi(\tau)\|_{H^{1/2}(S^1)}^2 d\tau \right)^{\frac{1}{2}}
\right.\]
\[
+ \left. \left(\int_{\eta}^{\eta+1} \|\partial_s \phi(\tau)\|_{L^{2r}(S^1)}^2 d\tau \right)^{\frac{1}{2}} \left(\int_{\eta}^{\eta+1} \|\psi(\tau)\|_{H^{1/2}(S^1)}^2 d\tau \right)^{\frac{1}{2}}
\right.
\[
+ \left. \left(\int_{\eta}^{\eta+1} \|\psi(\tau)\|_{L^{p+1}(S^1)}^{p+1} d\tau \right)^{\frac{1}{p+1}} \right)\). \] (3.42)

Hence, using (3.12)–(3.14), the Eq. (3.42) implies that
\[
\sup_{\tau \in \mathbb{R}} \|\psi(t)\|_{H^{1/2}(S^1)} \leq C(C_0)
\] (3.43)
for some \( C(C_0) > 0 \) depending only on \( C_0 \). This gives the desired estimate for \( \psi(t) \). To estimate \( \sup_{\tau \in \mathbb{R}} \|\partial_s \phi(\tau)\|_{L^{2r}(S^1)} \), using (3.17) and (3.43), we first have
\[
\|\partial_s \phi(\tau)\|_{L^{2r}(S^1)}^2 \leq C(1 + \|\partial_s \psi(\tau)\|_{H^{1/2}(S^1)}).
\] (3.44)

On the other hand, using Eq. (3.32), we have
\[
\|\partial_s \psi(\tau)\|_{H^{1/2}(S^1)} \leq C(\|\psi(t)\|_{H^{1/2}(S^1)} + (1 + |D|)^{-1} A(\partial_s \phi(t), \partial_s \cdot \psi(t))\|_{H^{1/2}(S^1)}
\]
\[
+ \|(1 + |D|)^{-1} \nabla_\psi H(s, \phi(t), \psi(t))\|_{H^{1/2}(S^1)}
\]
\[
\leq C(1 + \|\psi(t)\|_{H^{1/2}(S^1)} + \|\partial_s \phi(t)\|_{L^{2r}(S^1)}\|\psi(t)\|_{H^{1/2}(S^1)} + \|\psi(t)\|_{L^{p+1}(S^1)}^p),
\] (3.45)
where we have used (3.40) and (3.41).

By the embedding $H^{1/2}(S^1) \subset L^{p+1}(S^1)$ and (3.43), we have $\sup_{t \in \mathbb{R}} \| \psi(t) \|_{L^{p+1}(S^1)} \leq C(C_0)$. Combining this with (3.45), we obtain

$$\| \partial_t \psi(t) \|_{H^{1/2}(S^1)} \leq C(1 + \| \partial_t \phi(t) \|_{L^2(S^1)}).$$

(3.46)

By (3.44) and (3.46), we obtain

$$\| \partial_t \phi(t) \|_{L^2(S^1)}^2 \leq C(1 + \| \partial_t \phi(t) \|_{L^2(S^1)}).$$

(3.47)

Which implies that

$$\| \partial_t \phi(t) \|_{L^2(S^1)}^2 \leq C(C_0)$$

(3.48)

for all $t \in \mathbb{R}$, where $C(C_0) > 0$ is a constant depending only on $C_0$. This gives the desired estimate for $\phi(t)$ and completes the proof of Proposition 3.2.

Under the same assumptions of Proposition 3.2, we can further refine the result of Proposition 3.2 by using elliptic regularity theory. For this purpose, it is convenient to use $L_\lambda$-adapted norms on various function spaces. First of all, we notice that, since $L_1$ is a 0-th order (elliptic) pseudo-differential operator, $L_\lambda$ preserves Lebesgue spaces $L^p(S^1, \mathbb{S}(S^1) \otimes \mathbb{R}^k)$ ($1 < p < +\infty$), Sobolev spaces $W^{s,p}(S^1, \mathbb{S}(S^1) \otimes \mathbb{R}^k)$ ($0 < s < +\infty$, $1 < p < +\infty$) and Hölder spaces $C^{k,\alpha}(S^1, \mathbb{S}(S^1) \otimes \mathbb{R}^k)$ ($k \in \{0\} \cup \mathbb{N}$, $0 < \alpha < 1$), see [36, Chapter 13] for fundamentals of pseudo-differential calculus. We denote one such space by $E$.

We also notice that the spectrum of $L_\lambda$, considered as $L_\lambda : E \rightarrow E$ is the same as the spectrum of $L_\lambda : H^{1/2}(S^1, \mathbb{S}(S^1) \otimes \mathbb{R}^k) \rightarrow H^{1/2}(S^1, \mathbb{S}(S^1) \otimes \mathbb{R}^k)$ and they are given by $\text{Spec}(L_\lambda) = \{(1 + |\mu|)^{-1}(\mu + \lambda) : \mu \in \text{Spec}(D) \} \cup \{\pm 1\}$, where $\pm 1$ are in the essential spectrum. By our assumption $-\lambda \notin \text{Spec}(D)$, we have $\text{Spec}(L_\lambda) \cap \mathbb{R} = \emptyset$ and $L_\lambda : E \rightarrow E$ is hyperbolic in this sense.

By definition, a norm $\| \cdot \|$ on $E$ is called $L_\lambda$-adapted if it defines a norm which is equivalent to the original one on $E$ and satisfies the following properties:

$$\| \psi \| = \max\{\| P^-_\lambda(\psi) \|, \| P^+_\lambda(\psi) \|\} \quad \text{for all } \psi \in E$$

and there exists $\kappa > 0$ such that for all $t \geq 0$

$$\| e^{tL_\lambda} \psi^- \| \leq e^{-\kappa t} \| \psi^- \| \quad (\forall \psi \in E^-), \quad \| e^{-tL_\lambda} \varphi^+ \| \leq e^{-\kappa t} \| \varphi^+ \| \quad (\forall \varphi^+ \in E^+),$$

where $P^\pm_\lambda$ is the spectral projections as in the proof of Proposition 3.2 and $E^\pm_\lambda = P^\pm_\lambda(E)$.

For hyperbolic operators, the existence of such adapted norm is proved in [3, Lemma 1.1]. The proof is based on the spectral radius theorem and spectral mapping theorem. Because of the above estimate for $e^{\pm tL_\lambda}$ for $t \geq 0$, from now on, we will use adapted norms on various Lebesgue, Sobolev and Hölder spaces without mentioning it explicitly. In the following, for $T : E \rightarrow E$ a bounded linear operator between a Banach space $E$, we denote by $\| T \|_{\text{op}(E)}$ the operator norm of $T$.

Let us return to the regularity estimate. We have the following refinement of Proposition 3.2:

**Proposition 3.3** Let $(\phi(t), \psi(t)) \in C^1(\mathbb{R}, \mathbb{S}^{1,1/2}(S^1, N))$ be a solution to the negative gradient flow equation (3.9) which satisfies the same condition as in Proposition 3.2, $C_0 = \sup_{t \in \mathbb{R}} |\mathcal{L}_H(\phi(t), \psi(t))| < +\infty$. Then there exists $C(C_0) > 0$ such that the following holds

$$\sup_{t \in \mathbb{R}} \| \phi(t) \|_{C^{2,3/2}(S^1)} + \sup_{t \in \mathbb{R}} \| \psi(t) \|_{C^{1,3/2}(S^1)} \leq C(C_0).$$

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Proof In the course of the proof, $C(C_0)$ will denote constants which will depend only on $C_0$, but may change from line to line. We first consider the spinorial part of the flow equation. By the Sobolev embedding $H^{1/2}(S^1) \subset L^r(S^1)$ for all $r > 1$ and by the result of Proposition 3.2, we have

$$\sup_{t \in \mathbb{R}} \|\lambda \psi(t) + A(\partial_s \phi(t), \partial_s \cdot \psi(t)) + \nabla_\psi H(s, \phi(t), \psi(t))\|_{L^r(S^1)} \leq C(C_0)$$ (3.49)

for all $1 < r < 2$.

Thus, using elliptic regularity theory, we have

$$\sup_{t \in \mathbb{R}} \| (1 + |D|)^{-1}(\lambda \psi(t) + A(\partial_s \phi(t), \partial_s \cdot \psi(t)) + \nabla_\psi H(s, \phi(t), \psi(t)))\|_{W^{1,r}(S^1)} \leq C(C_0).$$ (3.50)

Since $1 < r < 2$, the Sobolev embedding theorem yields $W^{1,r}(S^1) \subset C^{0,\alpha}(S^1)$ for some $0 < \alpha < 1$ and

$$\sup_{t \in \mathbb{R}} \| (1 + |D|)^{-1}(\lambda \psi(t) + A(\partial_s \phi(t), \partial_s \cdot \psi(t)) + \nabla_\psi H(s, \phi(t), \psi(t)))\|_{C^{0,\alpha}(S^1)} \leq C(C_0)$$ (3.51)

following from (3.50).

Keeping in mind the formula (3.35), the mapping property $G_\lambda(t) : C^{0,\alpha}(S^1) \to C^{0,\alpha}(S^1)$ and the estimate $\|G_\lambda(t)\|_{C^{0,\alpha}(S^1)} \leq e^{-k|t|}$ for all $t \in \mathbb{R}$, where $k > 0$, we have

$$\|\psi(t)\|_{C^{0,\alpha}(S^1)}$$

$$\leq \int_{\mathbb{R}} e^{-k|\tau|} \| (1 + |D|)^{-1}(\lambda \psi(t) + A(\partial_s \phi(t), \partial_s \cdot \psi(t)) + \nabla_\psi H(s, \phi(t), \psi(t)))\|_{C^{0,\alpha}(S^1)} d\tau \leq C(C_0)$$ (3.52)

for all $t \in \mathbb{R}$.

We next look at the $\phi$ part of the flow equation (3.7). We first observe that, in terms of the local coordinate of $N$, $\nabla \partial_s \partial_s \phi = \partial^2_s \phi + \partial_s \phi_i \partial_s \phi^j \Gamma^k_{ij}(\phi) \frac{\partial}{\partial \phi^k}$. For simplicity, we write $\Gamma(\phi, \partial_s \phi, \partial_s \phi) = \partial_s \phi^i \partial_s \phi^j \Gamma^k_{ij}(\phi) \frac{\partial}{\partial \phi^k}$. Thus Eq. (3.7) takes the form

$$\partial_s \phi(t) = (-\Delta + 1)^{-1}\left(\Delta \phi(t) + \Gamma(\phi(t), \partial_s \phi(t), \partial_s \phi(t)) - \frac{1}{2} R(\phi(\psi, \partial_s \phi) \cdot \psi) + \nabla_\phi H(s, \phi, \psi)\right)$$

$$= -\phi(t) + (-\Delta + 1)^{-1}\left(\phi(t) + \Gamma(\phi(t), \partial_s \phi(t), \partial_s \phi(t)) - \frac{1}{2} R(\phi(\psi, \partial_s \phi) \cdot \psi) + \nabla_\phi H(s, \phi, \psi)\right).$$ (3.53)

Since the fundamental solution of the operator $\frac{d}{dt} + 1$ is $G(t) = e^{-t} \mathbf{1}_{[0,\infty)}(t)$, (3.53) is equivalent to

$$\phi(t) = \int_0^\infty e^{-\tau} (-\Delta + 1)^{-1}\left(\phi(t - \tau) + \Gamma(\phi(t - \tau), \partial_s \phi(t - \tau), \partial_s \phi(t - \tau)) - \frac{1}{2} R(\phi(t - \tau)(\psi(t - \tau), \partial_s \phi(t - \tau) \cdot \psi(t - \tau) + \nabla_\phi H(s, \phi(t - \tau), \psi(t - \tau))\right) d\tau.$$ (3.54)
Notice that \(|\Gamma(\phi, \partial_s \phi, \partial_x \phi)| \leq C|\partial_s \phi|^2\) for some constant \(C > 0\) depending only on \(N\). By the embedding \(L^1(S^1) \subset H^{-2}(S^1)\) for any \(s > \frac{1}{2}\) and Proposition 3.2, we thus have

\[
\|(-\Delta + 1)^{-1} \Gamma(\phi, \partial_s \phi, \partial_x \phi)\|_{W^{2,s,2}(S^1)} \leq C \|\Gamma(\phi, \partial_s \phi, \partial_x \phi)\|_{L^1(S^1)} \leq C(C_0)
\]

(3.55)

for all \(t \in \mathbb{R}\). We recall that the term \(R(\phi)\langle \psi, \partial_s \phi \cdot \psi \rangle\) is given by \(R(\phi)\langle \psi, \partial_s \phi \cdot \psi \rangle = \langle \psi, \partial_s \psi \otimes \frac{\partial}{\partial x} R_{il} L_1^j(\phi)^{gms} (\phi) \frac{\partial}{\partial y} (\phi) \rangle\) and it belongs to \(L^r(S^1)\) for any \(1 < r < 2\). Thus, by Proposition 3.2, we have the following bound for any \(1 < r < 2\) and for any \(t \in \mathbb{R}\):

\[
\|(-\Delta + 1)^{-1} R(\phi(\psi, \partial_s \phi \cdot \psi))\|_{W^{2,r}(S^1)} \leq C(C_0).
\]

(3.56)

By (1.6) and the Sobolev embedding \(H^{1/2}(S^1) \subset L^r(S^1)\) (for any \(1 < r < \infty\)), we also have \(\nabla \phi H(\cdot, \phi, \psi) \in L^r(S^1)\) for any \(1 < r < \infty\) and

\[
\|(-\Delta + 1)^{-1} \nabla \phi H(\cdot, \phi, \psi)\|_{W^{2,r}(S^1)} \leq C(C_0)
\]

(3.57)

for all \(t \in \mathbb{R}\). From (3.54)–(3.57), we obtain

\[
\sup_{t \in \mathbb{R}} \|\phi(t)\|_{W^{2,s,2}(S^1)} \leq C(C_0),
\]

(3.58)

where \(\frac{1}{2} < s < 1\) is arbitrary.

If we take \(s = \frac{3}{4}\) in (3.58), we obtain \(\sup_{t \in \mathbb{R}} \|\partial_s \phi(t)\|_{W^{1,2}(S^1)} \leq C(C_0)\). Combining this with the embedding \(W^{1,2}(S^1) \subset L^6(S^1)\), we have

\[
\sup_{t \in \mathbb{R}} \|\partial_s \phi(t)\|_{L^6(S^1)} \leq C(C_0).
\]

(3.59)

Using (3.52) and (3.59) the bounds (3.49) and (3.50) can be improved as follows:

\[
\sup_{t \in \mathbb{R}} \|\lambda \psi(t) + A(\partial_s \phi(t), \partial_s \cdot \psi(t)) + \nabla \phi H(s, \phi(t), \psi(t))\|_{L^6(S^1)} \leq C(C_0),
\]

(3.60)

\[
\sup_{t \in \mathbb{R}} \|(1 + |D|)^{-1} (\lambda \psi(t) + A(\partial_s \phi(t), \partial_s \cdot \psi(t)) + \nabla \phi H(s, \phi(t), \psi(t)))\|_{W^{1,6}(S^1)} \leq C(C_0).
\]

(3.61)

From the representation (3.35), (3.61) and the mapping property \(G_\lambda : W^{1,6}(S^1) \rightarrow W^{1,6}(S^1)\) with the bound \(\|G_\lambda(t)\|_{op(W^{1,6}(S^1))} \leq C e^{-|s||t|}\), we have, as in (3.52)

\[
\sup_{t \in \mathbb{R}} \|\psi(t)\|_{W^{1,6}(S^1)} \leq C(C_0).
\]

(3.62)

By (3.59) and (3.62), we have the following improvement of the estimates (3.55)–(3.57):

\[
\|(-\Delta + 1)^{-1} \Gamma(\phi(t), \partial_s \phi(t), \partial_x \phi(t))\|_{W^{2,3}(S^1)} \leq C(C_0),
\]

(3.63)

\[
\|(-\Delta + 1)^{-1} R(\phi(t))\langle \psi, \partial_s \phi \cdot \psi(t)\rangle\|_{W^{2,6}(S^1)} \leq C(C_0)
\]

(3.64)

and

\[
\|(-\Delta + 1)^{-1} \nabla \phi H(\cdot, \phi(t), \psi(t))\|_{W^{2,r}(S^1)} \leq C(C_0)
\]

(3.65)

for any \(1 < r < \infty\). Thus, by (3.54), (3.63)–(3.65), we have the following improvement of (3.58):

\[
\sup_{t \in \mathbb{R}} \|\phi(t)\|_{W^{2,3}(S^1)} \leq C(C_0).
\]

(3.66)
Using the Sobolev embedding $W^{2,3}(S^1) \subset C^{1, \frac{5}{2}}(S^1)$, we have
\[ \sup_{t \in \mathbb{R}} \| \partial_s \phi(t) \|_{C^{0, \frac{3}{2}}(S^1)} \leq C(C_0). \] (3.67)

Once again, we return to the equation of $\psi$, (3.32). Equations (3.62) and (3.67) and the Sobolev embedding $W^{1,6}(S^1) \subset C^{0, \frac{3}{2}}(S^1)$, yield
\[ \sup_{t \in \mathbb{R}} \| \lambda \psi(t) + A(\partial_s \phi(t), \partial_s \cdot \psi(t)) + \nabla_\psi H(s, \phi(t), \psi(t)) \|_{C^{0, \frac{3}{2}}(S^1)} \leq C(C_0) \]
and
\[ \sup_{t \in \mathbb{R}} \| (1 + |D|)^{-1}(\lambda \psi(t) + A(\partial_s \phi(t), \partial_s \cdot \psi(t)) + \nabla_\psi H(s, \phi(t), \psi(t))) \|_{C^{1, \frac{5}{2}}(S^1)} \leq C(C_0). \] (3.68)

Once more, using (3.35), (3.68) and the mapping property $G_\lambda(t) : C^{1, \frac{5}{2}}(S^1) \to C^{1, \frac{5}{2}}(S^1)$ with the bound $\|G_\lambda(t)\|_{\text{op}(C^{1, \frac{5}{2}}(S^1))} \leq e^{-|\kappa|t}$ for any $t \in \mathbb{R}$, where $\kappa > 0$, we have as in (3.52)
\[ \sup_{t \in \mathbb{R}} \| \psi(t) \|_{C^{1, \frac{5}{2}}(S^1)} \leq C(C_0). \] (3.69)

This gives the desired estimate for $\psi$.

Now, we again return to the equation of $\phi$, (3.7). By (3.67) and (3.69), we have
\[ \| (-\Delta + 1)^{-1} \Gamma(\phi(t), \partial_s \phi(t), \partial_s \cdot \phi(t)) \|_{C^{2, \frac{5}{2}}(S^1)} \leq C(C_0), \] (3.70)
\[ \| (-\Delta + 1)^{-1} R(\phi(t)) \langle \psi(t), \partial_s \phi(t) \cdot \psi(t) \rangle \|_{C^{2, \frac{5}{2}}(S^1)} \leq C(C_0), \] (3.71)
\[ \| (-\Delta + 1)^{-1} \nabla_\phi H(\cdot, \phi(t), \psi(t)) \|_{C^{2, \frac{5}{2}}(S^1)} \leq C(C_0). \] (3.72)

From the representation (3.54), (3.70)–(3.72), we finally obtained the desired estimate
\[ \sup_{t \in \mathbb{R}} \| \phi(t) \|_{C^{2, \frac{5}{2}}(S^1)} \leq C(C_0). \] (3.73)

This completes the proof. □

### 3.3 Fredholm property

In this section, we first give a functional analytic set up for the moduli problem of gradient flow lines. We then study its Fredholm property.

Let $x_- = (\phi_-, \psi_-, \phi_+ \cdot \psi_+) \in \text{crit}(\mathcal{L}_H)$. We define the $W^{1,2}$-Sobolev space of paths connecting $x_- \text{ and } x_+$ denoted by $W^{1,2}_{x_-, x_+}(\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N))$ as follows: Let us denote by $\iota(N)$ the injectivity radius of $N$. We define $(\phi, \psi) \in W^{1,2}_{x_-, x_+}(\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N))$ if and only if the following (i) and (ii) hold:

(i) $\phi \in W^{1,2}_{\text{loc}}(\mathbb{R}, H^1(S^1, N))$ and there exists $T > 0$ such that
\[ \phi(t) = \exp_{\phi_-}(X_-(t)) \] (3.74)
for $t \leq -T$ for some $X_- \in W^{1,2}((-\infty, -T], H^1(S^1, \phi_-^*TN))$ with $|X_-(t)(s)|_{T_{\phi_-(s)}N} < \iota(N)$ for all $(t, s) \in (-\infty, -T] \times S^1$,
\[ \phi(t) = \exp_{\phi_+}(X_+(t)) \] (3.75)
for \( t \geq T \) for some \( X_+ \in W^{1,2}([T, +\infty), H^1(S^1, \phi^*_+ TN)) \) with \( |X_+(t)(s)|T_{\phi_+(t)(s)}N < t(N) \) for all \((t, s) \in [T, +\infty) \times S^1 \).

(ii) \( \psi \in W^{1,2}_{\text{loc}}(\mathbb{R}, H^{1/2}(S^1, S(S^1) \otimes \phi^*_+ TN)) \) and

\[
\psi(t) = S_{-,t}(\psi_- + \xi_-(t)) \tag{3.76}
\]

for \( t \leq -T \) for some \( \xi_- \in W^{1,2}((-\infty, -T], H^{1/2}(S^1, S(S^1) \otimes \phi^*_+ TN)) \) (where \( T > 0 \) is as in (i)),

\[
\psi(t) = S_{+,t}(\psi_+ + \xi_+(t)) \tag{3.77}
\]

for \( t \geq T \) for some \( \xi_+ \in W^{1,2}([T, +\infty), H^{1/2}(S^1, S(S^1) \otimes \phi^*_+ TN)) \).

In (3.76) and (3.77), \( S_{\pm,t} : \phi^*_\pm TN \to \phi(t)^* TN \) denote the parallel translation along the path \([0, 1] \ni \tau \mapsto \exp_{\phi_\pm}(\tau X_\pm(t)) \in N \) for \( \pm t \geq T \).

We then define \( \mathcal{F}_{x_-,x_+} : W^{1,2}_{x_-,x_+}(\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N)) \to L^2(\mathbb{R}, T\mathcal{F}^{1,1/2}(S^1, N)) \) by

\[
\mathcal{F}_{x_-,x_+}(\ell) = \frac{d\ell}{dt} + \nabla_{1,1/2}\mathcal{L}_H(\ell) \tag{3.78}
\]

for \( \ell \in W^{1,2}_{x_-,x_+}(\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N)) \), where \( L^2(\mathbb{R}, T\mathcal{F}^{1,1/2}(S^1, N)) \) is a fiber bundle defined over \( W^{1,2}_{x_-,x_+}(\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N)) \) whose fiber at \( \ell \in W^{1,2}_{x_-,x_+}(\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N)) \) is \( L^2(\mathbb{R}, \ell^* T\mathcal{F}^{1,1/2}(S^1, N)) \), that is,

\[
L^2(\mathbb{R}, T\mathcal{F}^{1,1/2}(S^1, N))_\ell = \left\{ V(t) \in T_{\ell(t)}\mathcal{F}^{1,1/2}(S^1, N), \int_{\mathbb{R}} \|V(t)\|^2_{T_{\ell(t)}\mathcal{F}^{1,1/2}(S^1, N)} dt < +\infty \right\}. \tag{3.79}
\]

By our definition of the space \( W^{1,2}_{x_-,x_+}(\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N)) \), under the assumption (1.1)–(1.4) in the introduction, (3.78) is well-defined, i.e., \( \frac{d\ell}{dt} + \nabla_{1,1/2}\mathcal{L}_H(\ell) \in L^2(\mathbb{R}, \ell^* T\mathcal{F}^{1,1/2}(S^1, N)) \) for all \( \ell \in W^{1,2}_{x_-,x_+}(\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N)) \). Moreover, \( \mathcal{F}_{x_-,x_+} \) defines a \( C^2 \)-map from \( W^{1,2}_{x_-,x_+}(\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N)) \) to \( L^2(\mathbb{R}, T\mathcal{F}^{1,1/2}(S^1, N)) \). The proof of these regularity properties are proved in the Appendix, Sect. 10.2.

We shall prove the following:

**Proposition 3.4** Assume that \( x_-, x_+ \in \text{crit}(\mathcal{L}_H) \) are non-degenerate. Then \( \mathcal{F}_{x_-,x_+} \) defined by (3.78) is Fredholm. Its index at any \( \ell \in W^{1,2}_{x_-,x_+}(\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N)) \) is given by

\[
\text{ind}(d\mathcal{F}_{x_-,x_+}(\ell)) = \mu_H(x_-) - \mu_H(x_+).
\]

**Proof** We decompose the proof into two steps:

**Step 1. Reduction to the case of \( \ell \) which is constant near \( t = \pm \infty \).**

We first reduce to the case where \( \ell \in W^{1,2}_{x_-,x_+}(\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N)) \) is equal to \( x_\pm \) for \( \pm t \) large enough. That is, we shall show that it suffices to prove the assertion for the case where \( \ell \) satisfies \( \ell(t) = x_- \) for \( t \leq -R \) and \( \ell(t) = x_+ \) for \( t \geq R \) for some \( R > 0 \).

To prove this, let \( \ell(t) = (\phi(t), \psi(t)) \in W^{1,2}_{x_-,x_+}(\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N)) \) be arbitrary. By definition, we have the representation (3.74)–(3.77). For \( R > T \), we define \( \phi_R \) and \( \psi_R \) as follows:

\[
\phi_R(t) = \begin{cases} 
\phi_- & (t \leq -R - 1) \\
\exp_{\phi_-}((t + R + 1)x_-(-R)) & (-R - 1 \leq t \leq -R) \\
\phi(t) & (-R \leq t \leq R) \\
\exp_{\phi_+}((R + 1 - t)x_+(R)) & (R \leq t \leq R + 1) \\
\phi_+ & (R + 1 \leq t),
\end{cases}
\]
where \( S_{-; t; R} \) are parallel translations along paths \([0, 1] \ni \tau \mapsto \exp_{\phi_\pm}(\tau(t + R + 1)X_\pm(\pm R))\).

It can be easily checked that \( \ell_R := (\phi_R, \psi_R) \in W^{1,2}_{X_{-}, X_+}(\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N)) \) (see Appendix Sect. 10.2) and \( \ell_R \) converges to \( \ell = (\phi, \psi) \) as \( R \to \infty \) in the sense that the following holds

\[
\|\phi_R - \phi\|_{W^{1,2}(\mathbb{R}, H^1(S^1, \mathbb{R}^k))} \to 0, \quad (3.79) \\
\|\psi_R - \psi\|_{W^{1,2}(\mathbb{R}, H^{1/2}(S^1, \mathcal{S}(S^1) \otimes \mathbb{R}^k))} \to 0, \quad (3.80)
\]

as \( R \to \infty \).

We next show that \( d\mathcal{F}_{X_{-}, X_+}(\ell_R) \) converges (in the sense of the operator norm) to \( d\mathcal{F}_{X_{-}, X_+}(\ell) \) as \( R \to \infty \). For this purpose, we observe that by the condition \( X_\pm \in W^{1,2}_{-\infty, -T}(\mathbb{R}, H^1(S^1, \phi^*TN)) \), we have \( \|X_-(t)\|_{H^1(S^1)} \to 0 \) as \( t \to -\infty \). Similarly, we have \( \|X_+(t)\|_{H^1(S^1)} \to 0 \) as \( t \to \infty \). Combining these with the definition of \( \phi_R \), it follows that \( \|\phi_R - \phi\|_{L^\infty(\mathbb{R} \times S^1)} \to 0 \) as \( R \to \infty \). We take \( R > 0 \) so large that \( \|\phi_R - \phi\|_{L^\infty(\mathbb{R} \times S^1)} < t(N) \). We denote by \( S_R(t) : T_{\phi(t)}N \to T_{\phi_R(t)}N \) the parallel translation along the shortest geodesic from \( \phi(t) \) to \( \phi_R(t) \). By [20, Lemma 7.4], \( S_R(t) \) defines naturally a bounded linear operator which we also denote by \( S_R(t) \)

\[
S_R(t) : H^1(S^1, \phi(t)^*TN) \to H^1(S^1, \phi_R(t)^*TN), \quad (3.81) \\
S_R(t) = 1_{\mathcal{S}(S^1)} \otimes S_R(t) : H^{1/2}(S^1, \mathcal{S}(S^1) \otimes \phi(t)^*TN) \to H^{1/2}(S^1, \mathcal{S}(S^1) \otimes \phi_R(t)^*TN). \quad (3.82)
\]

We then have:

**Lemma 3.2** For \( \Theta = (\xi, \zeta) \in W^{1,2}(\mathbb{R}, \ell^*T\mathcal{F}^{1,1/2}(S^1, N)) \), we define

\[
(S_R(\Theta))(t) = (S_R(t)\xi(t), S_R(t)\zeta(t)).
\]

Then \( S_R : W^{1,2}(\mathbb{R}, \ell^*T\mathcal{F}^{1,1/2}(S^1, N)) \to W^{1,2}(\mathbb{R}, \ell_R^*T\mathcal{F}^{1,1/2}(S^1, N)) \) is an isomorphism. Moreover, we have the estimate

\[
\|S_R\Theta - \Theta\|_{W^{1,2}(\mathbb{R}, H^1(S^1, \mathbb{R}^k) \times H^{1/2}(S^1, \mathcal{S}(S^1) \otimes \mathbb{R}^k))} \leq \delta(R)\|\Theta\|_{W^{1,2}(\mathbb{R}, \ell^*T\mathcal{F}^{1,1/2}(S^1, N))}
\]

for some \( \delta(R) > 0 \) with \( \delta(R) \to 0 \) as \( R \to \infty \).

**Proof** By [20, Lemma 7.4], the operator norms \( \|S_R(t)\|_{\text{op}(H^1(S^1, \phi(t)^*TN), H^1(S^1, \phi_R(t)^*TN))} \) and \( \|S_R(t)\|_{\text{op}(H^{1/2}(S^1, \mathcal{S}(S^1) \otimes \phi(t)^*TN), H^{1/2}(S^1, \mathcal{S}(S^1) \otimes \phi_R(t)^*TN))} \) depends only on \( \|\phi(t)\|_{H^1(S^1)} \) and \( \|\phi_R(t)\|_{H^1(S^1)} \). Since \( \phi \in W^{1,2}(\mathbb{R}, H^1(S^1, N)) \), it follows that \( \sup_{t \in \mathbb{R}} \|\phi(t)\|_{H^1(S^1)} < +\infty \). Combining this with the definition of \( \phi_R \), it is easy to see that we have a uniform bound

\[
\|\phi_R(t)\|_{H^1(S^1)} \leq C
\]

for some \( C > 0 \) independent of \( t \in \mathbb{R} \) and \( R > T \). We thus have uniform bounds of the operator norms

\[
\|S_R(t)\|_{\text{op}(H^1(S^1, \phi(t)^*TN), H^1(S^1, \phi_R(t)^*TN))} \leq C, \quad (3.83) \\
\|S_R(t)\|_{\text{op}(H^{1/2}(S^1, \mathcal{S}(S^1) \otimes \phi(t)^*TN), H^{1/2}(S^1, \mathcal{S}(S^1) \otimes \phi_R(t)^*TN))} \leq C, \quad (3.84)
\]
where \( C > 0 \) does not depend on \( t \in \mathbb{R} \) and \( R > T \).

From (3.83) and (3.84), we see that
\[
S_R : L^2(\mathbb{R}, \ell^s T^1 S^1, N)) \to L^2(\mathbb{R}, \ell^s T^1 S^1, N)) \tag{3.85}
\]
defined by \((S_R \Theta)(t) = (S_R(t)\xi(t), S_R(t)\zeta(t))\) is a bounded linear map.

We next estimate the \( H^1(\mathcal{S}^1) \)-norm of the derivative \( \nabla_t(S_R(t)\xi(t)) = \nabla_tS_R(t)\xi(t) = S_R(t)(\nabla_t\xi(t)) \). For this purpose, recall that, in the notation of [20], \( S_R(t) \) is defined by
\[
S_R(t) = P_{\phi_R(t),\phi(t)}, \text{ where for } x, y \in N \text{ with } d(x, y) < t(N), \ P_{x,y} : T_xN \to T_yN \text{ denotes the parallel translation along the shortest geodesic from } x \text{ to } y. \tag{3.89}
\]

Thus
\[
\nabla_t S_R(t) = \nabla_x P_{\phi_R(t),\phi(t)}[\partial_t\phi_R(t)] + \nabla_y P_{\phi_R(t),\phi(t)}[\partial_t\phi(t)]. \tag{3.90}
\]

By (3.79) and (3.86), we have
\[
\nabla_t S_R(t) \to \nabla_x P_{\phi(t),\phi(t)}[\partial_t\phi(t)] + \nabla_y P_{\phi(t),\phi(t)}[\partial_t\phi(t)] = 0 \tag{3.91}
\]
in \( H^1(\mathcal{S}^1) \) as \( R \to \infty \) since \( \nabla_x P_{x,y} + \nabla_y P_{x,y} = 0 \). Moreover, the convergence is uniform with respect to \( t \).

We define \( \epsilon(R) = \sup_{t \in \mathbb{R}} \|\nabla_t S_R(t)\|_{H^1(\mathcal{S}^1)} \). Note that \( \epsilon(R) \to 0 \) as \( R \to \infty \) and
\[
\|\nabla_t S_R(t)\|_{H^1(\mathcal{S}^1)} \le C\|\nabla_t S_R\|_{H^1(\mathcal{S}^1)}\|\xi(t)\|_{H^1(\mathcal{S}^1)} \le C\epsilon(R)\|\xi(t)\|_{H^1(\mathcal{S}^1)}, \tag{3.92}
\]
where \( C > 0 \) depends only on the constant of the Sobolev embedding \( H^1(\mathcal{S}^1) \subset L^\infty(\mathcal{S}^1) \). (Thus, \( H^1(\mathcal{S}^1) \) becomes an algebra with respect to the point wise multiplication. This fact has been used in the first inequality of (3.92)). Obviously, the similar estimate holds for \( \xi(t) \in H^1(\mathcal{S}^1, \mathbb{S}(\mathcal{S}^1) \otimes \phi^*(t)T N) \),
\[
\|\nabla_t S_R(t)\|_{H^1(\mathcal{S}^1)} \le C\|\nabla_t S_R\|_{H^1(\mathcal{S}^1)}\|\xi(t)\|_{H^1(\mathcal{S}^1)} \le C\epsilon(R)\|\xi(t)\|_{H^1(\mathcal{S}^1)} \tag{3.93}
\]
On the other hand, for \( \xi \in L^2(\mathcal{S}^1, \mathbb{S}(\mathcal{S}^1) \otimes \phi^*(t)T N) \), we have
\[
\|\nabla_t S_R(t)\|_{H^1(\mathcal{S}^1)} \le C\|\nabla_x S_R(t)\|_{L^\infty(\mathcal{S}^1)}\|\xi(t)\|_{L^2(\mathcal{S}^1)} \le C\epsilon(R)\|\xi(t)\|_{L^2(\mathcal{S}^1)}, \tag{3.94}
\]
where we have used the Sobolev embedding \( H^1(\mathcal{S}^1) \subset L^\infty(\mathcal{S}^1) \) again. By interpolating (3.90) with (3.91), we have a similar estimate for the \( H^{1/2}(\mathcal{S}^1) \)-norm of \( \nabla_t S_R(t)\xi(t) \):
\[
\|\nabla_t S_R(t)\xi(t)\|_{H^{1/2}(\mathcal{S}^1)} \le C\epsilon(R)\|\xi(t)\|_{H^{1/2}(\mathcal{S}^1)} \tag{3.95}
\]
Combining (3.92) with the boundedness of (3.85), we see that
\[
S_R : W^{1,2}(\mathbb{R}, \ell^s T^1 S^1, N)) \to W^{1,2}(\mathbb{R}, \ell^s T^1 S^1, N)) \tag{3.96}
\]
defines a bounded linear map. Since the inverse of \( S_R \) is defined by the parallel translation along the shortest geodesic from \( \phi_R(t) \) to \( \phi(t) \), by the same reasoning as above it is bounded. The first assertion of the lemma is thus proved.

To prove the second assertion, we recall the estimates of [20, Lemma 7.4]:
\[
\|S_R\xi(t) - \xi(t)\|_{H^1(\mathcal{S}^1)} \le C(||\phi_R(t)\|_{H^1(\mathcal{S}^1)}, ||\phi(t)\|_{H^1(\mathcal{S}^1)})||\phi_R(t) - \phi(t)\|_{H^1(\mathcal{S}^1, \mathbb{R}^s)}\|\xi(t)\|_{H^1(\mathcal{S}^1)} \tag{3.97}
\]
\[
\|S_R\xi(t) - \xi(t)\|_{H^{1/2}(\mathcal{S}^1)} \le C(||\phi_R(t)\|_{H^1(\mathcal{S}^1)}, ||\phi(t)\|_{H^1(\mathcal{S}^1)})||\phi_R(t) - \phi(t)\|_{H^{1/2}(\mathcal{S}^1, \mathbb{R}^s)}\|\xi(t)\|_{H^{1/2}(\mathcal{S}^1)} \tag{3.98}
\]
where $C(\|\phi_R(t)\|_{H^1(S^1)}, \|\phi(t)\|_{H^1(S^1)})$ is a constant depending only on $\|\phi_R(t)\|_{H^1(S^1)}$ and $\|\phi(t)\|_{H^1(S^1)}$. By (3.92) and (3.88), we have

$$
\|\nabla_t(S_R(t)\xi(t)) - \nabla_t\xi(t)\|_{H^1(S^1)} \leq \|\nabla_t S_R(t)\xi(t)\|_{H^1(S^1)} + \|S_R(t)\nabla_t \xi(t) - \nabla_t \xi(t)\|_{H^1(S^1)} \\
\leq C \epsilon(R) \|\xi(t)\|_{H^1(S^1)} + C(\|\phi_R(t)\|_{H^1(S^1)}, \|\phi(t)\|_{H^1(S^1)}) \|\phi_R(t) - \phi(t)\|_{H^1(S^1)} \|\nabla_t \xi(t)\|_{H^1(S^1)}.
$$

(3.94)

Similarly, by (3.93) and (3.91) we have

$$
\|\nabla_t(S_R(t)\xi(t)) - \nabla_t\xi(t)\|_{H^{1/2}(S^1)} \leq \|\nabla_t S_R(t)\xi(t)\|_{H^{1/2}(S^1)} + \|S_R(t)\nabla_t \xi(t) - \nabla_t \xi(t)\|_{H^{1/2}(S^1)} \\
\leq C \epsilon(R) \|\xi(t)\|_{H^{1/2}(S^1)} + C(\|\phi_R(t)\|_{H^{1/2}(S^1)}, \|\phi(t)\|_{H^{1/2}(S^1)}) \|\phi_R(t) - \phi(t)\|_{H^{1/2}(S^1)} \|\nabla_t \xi(t)\|_{H^{1/2}(S^1)}.
$$

(3.95)

Since $\sup_{t \in \mathbb{R}} \|\phi_R(t) - \phi(t)\|_{H^1(S^1)} \to 0$ as $R \to \infty$, the asserted estimate follows from (3.92)–(3.95). This completes the proof. 

\[ \square \]

Lemma 3.3 We have

$$
\|S_R^{-1} \circ d\mathcal{F}_{x_+,x_+}(\ell_R) \circ S_R - d\mathcal{F}_{x_-,x_+}(\ell)\|_{op(W^{1,2}(\mathbb{R}, \ell^*T^*F^{1,1/2}(S^1, N)), L^2(\mathbb{R}, \ell^*T^*F^{1,1/2}(S^1, N)))} \to 0
$$
as $R \to \infty$.

\textbf{Proof} For $\Theta = (\xi, \xi) \in W^{1,2}(\mathbb{R}, \ell^*T^*F^{1,1/2}(S^1, N))$, we have

$$
S_R^{-1} \circ d\mathcal{F}_{x_-,x_+}(\ell_R) \circ S_R[\Theta](t) = \nabla_t \Theta(t) + S_R^{-1}(t) \nabla_t S_R(t) \Theta(t) \\
+ S_R^{-1}(t) \circ d\nabla_{1,1/2}L_H(\ell_R(t)) \circ S_R(t)[\Theta(t)].
$$

(3.96)

By (3.88), (3.91) and the uniform bound $\sup_{t \in \mathbb{R}} \|S_R^{-1}(t)\|_{H^1(S^1)} \leq C$ (which follows from [20, Lemma 7.4] as in the proof of Lemma 3.2), we have

$$
\|S_R^{-1}(t) \nabla_t S_R(t) \Theta(t)\|_{T_{\ell}(\mathcal{F}^{1,1/2}(S^1, N))} \leq \epsilon(R) \|\Theta(t)\|_{T_{\ell}(\mathcal{F}^{1,1/2}(S^1, N))}
$$

for some $\epsilon(R) > 0$ with $\epsilon(R) \to 0$ as $R \to \infty$.

On the other hand, by the estimate of Lemma 3.2 and $\ell_R \to \ell$ as $R \to \infty$ (in the sense that (3.79), (3.80) hold), we easily see that

$$
\|S_R^{-1}(t) \circ d\nabla_{1,1/2}L_H(\ell_R(t)) \circ S_R(t)[\Theta(t)] - d\nabla_{1,1/2}L_H(\ell(t))[\Theta(t)]\|_{T_{\ell}(\mathcal{F}^{1,1/2}(S^1, N))} \\
\leq \epsilon'(R) \|\Theta(t)\|_{T_{\ell}(\mathcal{F}^{1,1/2}(S^1, N))}
$$

for some $\epsilon'(R) > 0$ with $\epsilon'(R) \to 0$ as $R \to \infty$. From (3.96)–(3.98), we have the assertion of the lemma.

Since the space of Fredholm operator is open in the space of bounded linear operators with respect to the operator norm, in view of Lemma 3.3, if suffices to prove that $S_R^{-1} \circ d\mathcal{F}_{x_-,x_+}(\ell_R) \circ S_R$ is Fredholm for all large $R > 0$. By Lemma 3.2, this is equivalent to the assertion that $d\mathcal{F}_{x_-,x_+}(\ell_R)$ is Fredholm for all large $R > 0$. Notice also that the Fredholm index of $d\mathcal{F}_{x_-,x_+}(\ell)$ is equal to that of $d\mathcal{F}_{x_-,x_+}(\ell_R)$ for all large $R > 0$. Since $\ell_R = x_+$ when $\pm t \geq R + 1$, we have reduced the problem to the case where $\ell$ is constant near $t = \pm \infty$ as asserted.

\textbf{Step 2. Proof of Proposition 3.4 for $\ell$ constant near $t = \pm \infty$.}

Let us assume that $\ell \in W^{1,2}_{x_-,x_+}(\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N))$ satisfies $\ell(t) = x_-$ for $t \leq -T$ and $\ell(t) = x_+$ for $t \geq T$ for some $T > 0$. We set $\ell(t) = (\phi(t), \psi(t))$ and define the parallel
translation along the path $\phi(t) (-T \leq t \leq t)$ by $P_t : T_{\phi(t)}N \rightarrow T_{\phi(t)}N$. Note that, by our assumption, $P_t = P_{-T}$ for $t \leq -T$ and $P_t = P_T$ for $t \geq T$. By [20, Lemma 7.2], $\{P_t\}_{t \in \mathbb{R}}$ defines families of bounded linear operators

$$P_t : H^1(S^1, \phi^* TN) \rightarrow H^1(S^1, \phi(t)^* TN) \subset H^1(S^1, \mathbb{R}^k)$$

and

$$P_t = 1 \otimes P_t : H^{1/2}(S^1, S(S^1) \otimes \phi^* TN) \rightarrow H^{1/2}(S^1, S(S^1) \otimes \phi^*(t)TN) \subset H^{1/2}(S^1, S(S^1) \otimes \mathbb{R}^k),$$

which are continuous with respect to the respective operator norms. Since $P_t$ is independent of $t$ outside of a compact set $[-T, T]$, there exists $C_T > 0$ such that

$$\|P_t\|_{op(H^1(S^1, \phi^* TN), H^1(S^1, \mathbb{R}^k))} \leq C_T \quad (3.99)$$

and

$$\|P_t\|_{op(H^{1/2}(S^1, S(S^1) \otimes \phi^* TN), H^{1/2}(S^1, S(S^1) \otimes \mathbb{R}^k))} \leq C_T \quad (3.100)$$

for all $t \in \mathbb{R}$. For $\Theta = (\xi, \zeta) \in W^{1,2}(\mathbb{R}, x^* T^* F^{1,1/2}(S^1, N))$ (i.e., $\xi \in W^{1,2}(\mathbb{R}, H^1(S^1, \phi^* TN))$, $\zeta \in W^{1,2}(\mathbb{R}, H^{1/2}(S^1, S(S^1) \otimes \phi^* TN))$, we define $P(\Theta)(t) = P_t(\Theta(t))$. By the definition, we have $P(\Theta)(t) \in \ell^* (t) T^* F^{1,1/2}(S^1, N)$ for all $t \in \mathbb{R}$. Moreover, by (3.99) and (3.100), $P$ defines a bounded linear map $P : W^{1,2}(\mathbb{R}, T_x F^{1,1/2}(S^1, N)) \rightarrow W^{1,2}(\mathbb{R}, \ell^* T^* F^{1,1/2}(S^1, N))$. To see this, for $\Theta = (\xi, \zeta) \in W^{1,2}(\mathbb{R}, T_x F^{1,1/2}(S^1, N))$ we have $P(\Theta)(t) = (P_t(\xi)(t), P_t(\zeta)(t))$ and

$$\|P_t \xi(t)\|_{H^1(S^1)} \leq C_T \|\xi(t)\|_{H^1(S^1)},$$

$$\|\nabla_t(P_t \xi(t))\|_{H^1(S^1)} \leq \|P_t \nabla_t \xi(t)\|_{H^1(S^1)} \leq C_T \|\nabla_t \xi(t)\|_{H^1(S^1)},$$

$$\|P_t \zeta(t)\|_{H^{1/2}(S^1)} \leq C_T \|\zeta(t)\|_{H^{1/2}(S^1)},$$

$$\|\nabla_t(P_t \zeta(t))\|_{H^{1/2}(S^1)} = \|P_t \nabla_t \zeta(t)\|_{H^{1/2}(S^1)} \leq C_T \|\nabla_t \zeta(t)\|_{H^{1/2}(S^1)}$$

for all $t \in \mathbb{R}$. By (3.101)–(3.104), integrating over $\mathbb{R}$ we have

$$\int_{-\infty}^{\infty} \|P_t \xi(t)\|_{H^1(S^1)}^2 + \|\nabla_t(P_t \xi(t))\|_{H^1(S^1)}^2 \, dt \leq C_T^2 \int_{-\infty}^{\infty} \|\xi(t)\|_{H^1(S^1)}^2 + \|\nabla_t \xi(t)\|_{H^1(S^1)}^2 \, dt$$

and

$$\int_{-\infty}^{\infty} \|P_t \zeta(t)\|_{H^{1/2}(S^1)}^2 + \|\nabla_t(P_t \zeta(t))\|_{H^{1/2}(S^1)}^2 \, dt \leq C_T^2 \int_{-\infty}^{\infty} \|\zeta(t)\|_{H^{1/2}(S^1)}^2 + \|\nabla_t \zeta(t)\|_{H^{1/2}(S^1)}^2 \, dt$$

and the claim is proved.

Since $P$ is an isomorphism (the inverse is given by the parallel translation along the curve $\phi(-t) (-t \leq t \leq T)$), to complete the proof it suffices to prove the Fredholm property and the index formula for the operator

$$P^{-1} \circ dF_{x_- x_+} (\ell) \circ P : W^{1,2}(\mathbb{R}, T_x T^* F^{1,1/2}(S^1, N)) \rightarrow L^2(\mathbb{R}, T_x T^* F^{1,1/2}(S^1, N)),$$  \hspace{1cm} (3.105)

Since $P_t$ is the parallel transport, we have

$$P^{-1} \circ dF_{x_- x_+} (\ell) \circ P = \nabla_t + P^{-1} \circ d\nabla_{1/2} L_{\mathcal{H}}(\ell) \circ P.$$  \hspace{1cm} (3.106)

We set $A(t) := P^{-1} \circ d\nabla_{1/2} L_{\mathcal{H}}(\ell(t)) \circ P_t$. We observe that

$$A(-\infty) = A(-T) = P_{-T}^{-1} \circ d\nabla_{1/2} L_{\mathcal{H}}(\ell(-T)) \circ P_{-T} = d\nabla_{1/2} L_{\mathcal{H}}(x_-)$$

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and
\[ A(+\infty) = A(T) = P_T^{-1} \circ d\nabla_{1,1/2}L_H(\ell(T)) \circ P_T = P_T^{-1} \circ d\nabla_{1,1/2}L_H(x_+) \circ P_T \]
are invertible hyperbolic operators since we have assumed that \( x_-, x_+ \in \text{crit}(\mathcal{L}_H) \) are non-degenerate. In this sense, the family of operators \( \{A(t)\}_{t \in \mathbb{R}} \) is asymptotically hyperbolic. Moreover, we have:

**Lemma 3.4**  \( A(t) \) takes the following form

\[ A(t) = d\nabla_{1,1/2}L_H(x_-) + K(t), \]

where \( K(t) : T_{x_-}\mathcal{F}^{1,1/2}(S^1, N) \to T_{x_-}\mathcal{F}^{1,1/2}(S^1, N) \) is compact for all \( t \in \mathbb{R} \).

**Proof** By an approximation argument as in [20, Lemma 7.6], we see that \( (-\nabla_s^2 + 1)^{-1/2}\nabla_s : H^1(S^1, \phi^*TN) \to H^1(S^1, \phi^*TN) \subset H^1(S^1, \mathbb{R}^k) \) is a compact perturbation of \( (-\Delta + 1)^{-1/2}D : H^1(S^1, \phi^*TN) \to H^1(S^1, \mathbb{R}^k) \), where the latter operator acts on \( H^1(S^1, \phi^*TN) \) by composition with the canonical embedding \( H^1(S^1, \phi^*TN) \subset H^1(S^1, \mathbb{R}^k) \). Thus, the difference of their squares \( (-\nabla_s^2 + 1)^{-1/2}\nabla_s - (-\Delta + 1)^{-1}\Delta \) is also compact. Since \( \nabla_s \) is a compact perturbation of \( D \) (the later acts on vector fields along \( \phi \) after composing with the canonical embedding \( \phi^*TN \subset \mathbb{R}^k \) as above), \( (-\nabla_s^2 + 1)^{-1/2}\nabla_s \) is a compact perturbation of \( (-\Delta + 1)^{-1}\nabla_s^2 \). By a similar reasoning, \( (1 + |D\phi|)^{-1}D\phi \) is a compact perturbation of \( (1 + |D|)^{-1}D\phi \) again, \( (1 + |D|)^{-1} \) acts on spinors along \( \phi \) after composing with the canonical embedding \( \mathbb{S}(S^1) \otimes \phi^*TN \subset \mathbb{S}(S^1) \otimes \mathbb{R}^k \). Thus, by the formula (2.5) in Sect. 2, we see that \( d\nabla_{1,1/2}L_H(\ell(t)) \) takes the following form

\[ d\nabla_{1,1/2}L_H(\ell(t)) = \left( \begin{array}{c} (-\nabla_s^2 + 1)^{-1}\nabla_s^2 \\ O \end{array} \right) \left( \begin{array}{c} O \\ (1 + |D\phi(t)|)^{-1}D\phi(t) \end{array} \right) + \mathcal{K}_t \]

(3.106)

for some compact operator \( \mathcal{K}_t : T_{\ell(t)}\mathcal{F}^{1,1/2}(S^1, N) \to T_{\ell(t)}\mathcal{F}^{1,1/2}(S^1, N) \). Similarly, we have

\[ d\nabla_{1,1/2}L_H(x_-) = \left( \begin{array}{c} (-\nabla_s^2 + 1)^{-1}\nabla_s^2 \\ O \end{array} \right) \left( \begin{array}{c} O \\ (1 + |Dx_-|)^{-1}Dx_- \end{array} \right) + \mathcal{K} \]

(3.107)

for some compact operator \( \mathcal{K} : T_{x_-}\mathcal{F}^{1,1/2}(S^1, N) \to T_{x_-}\mathcal{F}^{1,1/2}(S^1, N) \).

Defining \( \tilde{\nabla}_s = P^{-1} \circ \nabla_s \circ P \) and \( \tilde{D}\phi = P^{-1} \circ D\phi \circ P \), by (3.106) we have

\[ P_t^{-1} \circ d\nabla_{1,1/2}L_H(\ell(t)) \circ P_t = \left( \begin{array}{c} (-\tilde{\nabla}_s^2 + 1)^{-1}\tilde{\nabla}_s^2 \\ O \end{array} \right) \left( \begin{array}{c} O \\ (1 + |D\phi(t)|)^{-1}D\phi(t) \end{array} \right) + \tilde{\mathcal{K}}_t, \]

(3.108)

where \( \tilde{\mathcal{K}}_t = P_t^{-1} \circ \mathcal{K}_t \circ P_t : T_{\ell(t)}\mathcal{F}^{1,1/2}(S^1, N) \to T_{\ell(t)}\mathcal{F}^{1,1/2}(S^1, N) \) is compact. By [20, Lemma 7.6], \( \{ (1 + |D\phi(t)|)^{-1}D\phi(t) - (1 + |Dx_-|)^{-1}Dx_- \}_{t \in \mathbb{R}} \) defines a continuous family of compact operators. By the same reasoning, \( \{ (-\tilde{\nabla}_s^2 + 1)^{-1/2}\tilde{\nabla}_s - (-\nabla_s^2 + 1)^{-1/2}\nabla_s \}_{t \in \mathbb{R}} \) is a continuous family of compact operators and therefore the difference of their squares \( \{ (-\tilde{\nabla}_s^2 + 1)^{-1/2}\tilde{\nabla}_s - (-\nabla_s^2 + 1)^{-1/2}\nabla_s \}_{t \in \mathbb{R}} \) also defines a continuous family of compact operators. From this observation, comparing (3.107) and (3.108), the assertion of the lemma follows. □

We now complete the proof of Step 2. By Lemma 3.4, we are now in a position to apply the result of Abbondandolo–Majer [1, Theorem 3.4], [2, Theorem B]. In fact, by the non-degeneracy of \( x_- \in \text{crit}(\mathcal{L}_H) \), \( d\nabla_{1,1/2}L_H(x_-) \) is a hyperbolic operator. Combining this with Lemma 3.4, we can apply Theorem 3.4 of [1] (see also Theorem B of [1]) and conclude that \( P^{-1} \circ d\mathcal{F}_{x_-}(\ell) \circ P = \nabla + A(t) : W^{1,2}(\mathbb{R}, T_{x_-}\mathcal{F}^{1,1/2}(S^1, N)) \to W^{1,2}(\mathbb{R}, T_{x_-}\mathcal{F}^{1,1/2}(S^1, N)) \).
Proof solutions to the negative gradient flow equations starting from $x \in \operatorname{specflow}$ of the family of operators $\{A(t)\}_{t \in \mathbb{R}}$:

$$\text{index}(d\mathcal{F}_{x_-,x_+}(\ell)) = \text{index}(P^{-1} \circ d\mathcal{F}_{x_-,x_+}(\ell) \circ P) = \operatorname{sf}(A(t))_{-\infty \leq t \leq +\infty}. \quad (3.109)$$

By our definition of the relative Morse index in Definition 2.1 in Sect. 2 (we can choose the base point of the path as $(\phi_0, \psi_0) = (\phi(0), \psi(0))$, we have

$$\operatorname{sf}(A(t))_{-\infty \leq t \leq +\infty} = \operatorname{sf}(A(t))_{-t \leq t} = \operatorname{sf}(A(t))_{-t \leq 0} + \operatorname{sf}(A(t))_{0 \leq t} = \mu_H(x_-) - \mu_H(x_+). \quad (3.110)$$

By (3.109) and (3.110), we have the desired index formula. □

For $x_- = (\phi_-, \psi_-), x_+ = (\phi_+, \psi_+) \in \text{crit}(\mathcal{L}_H)$, we denote by $\mathcal{M}(x_-, x_+)$ the space of solutions to the negative gradient flow equations starting from $x_-$ and ending at $x_+$,

$$\mathcal{M}(x_-, x_+) = \{ \ell \in W^{1,2}_{x_-,x_+}(\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N)) : \mathcal{F}_{x_-,x_+}(\ell) = 0 \}.$$

By Proposition 3.4 and the implicit function theorem, if $0 \in L^2(\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N))$ is a regular value of $\mathcal{F}_{x_-,x_+}$, $\mathcal{M}(x_-, x_+)$ is a manifold of dimension $\mu_H(x_-) - \mu_H(x_+)$. When $x_- \neq x_+$, $\mathcal{M}$ acts freely on $\mathcal{M}(x_-, x_+)$ by time shift: $\mathbb{R} \times \mathcal{M}(x_-, x_+) \ni (a, \ell) \mapsto \ell (\cdot + a) \in \mathcal{M}(x_-, x_+)$. The quotient of $\mathcal{M}(x_-, x_+)$ by this action is denoted by $\tilde{\mathcal{M}}(x_-, x_+) := \mathcal{M}(x_-, x_+)/\mathbb{R}$. This is the set of negative gradient flow lines and if $0 \in L^2(\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N))$ is a regular value of $\mathcal{F}_{x_-,x_+}$, $\tilde{\mathcal{M}}(x_-, x_+)$ is a manifold of dimension $\mu_H(x_-) - \mu_H(x_+) - 1$.

### 3.4 Compactness of flow lines

In this section, we prove compactness properties of the spaces $\mathcal{M}(x_-, x_+)$ and $\tilde{\mathcal{M}}(x_-, x_+)$ for $x_- = (\phi_-, \psi_-), x_+ = (\phi_+, \psi_+) \in \text{crit}(\mathcal{L}_H)$. The first result is a consequence of Proposition 3.3.

**Proposition 3.5** $\tilde{\mathcal{M}}(x_-, x_+) \subset C^0_{\text{loc}}(\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N))$ is relatively compact.

**Proof** Let $\ell = (\psi, \phi) \in \mathcal{M}(x_-, x_+)$ be arbitrary. Since $\mathcal{L}_H(\ell(t))$ is non-increasing along the negative flow line $\ell$, we have

$$\mathcal{L}_H(x_-) \leq \mathcal{L}_H(\ell(t)) \leq \mathcal{L}_H(x_+)$$

for all $t \in \mathbb{R}$. By Proposition 3.3, this implies the following: for some $C = C(x_-, x_+)$ depending only on $x_-$ and $x_+$,

$$\sup_{t \in \mathbb{R}} \| \phi(t) \|_{C^{0, \frac{5}{2}}(S^1)} + \sup_{t \in \mathbb{R}} \| \psi(t) \|_{C^{1, \frac{5}{2}}(S^1)} \leq C(x_-, x_+). \quad (3.111)$$

By (3.111) and the compactness of the embedding $C^{2, \frac{5}{2}}(S^1) \times C^{1, \frac{5}{2}}(S^1) \subset H^1(S^1) \times H^{1/2}(S^1)$, $\ell(t) : t \in \mathbb{R} \mapsto \mathcal{M}(x_-, x_+)$, $t \in \mathbb{R} \subset \mathcal{F}^{1,1/2}(S^1, N)$ is relatively compact. By the gradient flow equation (3.7), (3.8) and the elliptic regularity for the elliptic operators $-\Delta + 1$ and $1 + |D|$, (3.111) implies that there exists another constant $C = C(x_-, x_+)$ depending only on $x_-$ and $x_+$ such that

$$\sup_{t \in \mathbb{R}} \| \partial_t \phi(t) \|_{C^{0, \frac{5}{2}}(S^1)} + \sup_{t \in \mathbb{R}} \| \partial_t \psi(t) \|_{C^{1, \frac{5}{2}}(S^1)} \leq C(x_-, x_+). \quad (3.112)$$
Using the mean value theorem and the compactness of the embedding \( C^{2,\frac{3}{2}}(S^1) \times C^{1,\frac{5}{2}}(S^1) \subset H^1(S^1) \times H^{1/2}(S^1), \) (3.112) implies that \( M(x_-, x_+) \subset C^0(\mathbb{R}, F^{1,1/2}(S^1, N)) \) is an equicontinuous family. Therefore, by the Ascoli–Arzelà theorem, \( M(x_-, x_+) \subset C^0_{\text{loc}}(\mathbb{R}, F^{1,1/2}(S^1, N)) \) is relatively compact as asserted. This completes the proof. \( \square \)

Under the conditions (1.2) and (1.5), \( \mathcal{L}_H \) satisfies the Palais–Smale condition:

**Proposition 3.6** Assume that \( H \in C^1(S^1 \times \mathbb{S}(S^1) \otimes TN) \) satisfies (1.2) and (1.5). Then \( \mathcal{L}_H \) satisfies the Palais–Smale condition in the following sense:

Let us assume that \( \{ \ell_n \} = \{(\phi_n, \psi_n)\} \in F^{1,1/2}(S^1, N) \) satisfies \( \sup_{n \geq 1} \mathcal{L}_H(\ell_n) < +\infty \) and \( \| \nabla_{1/2} \mathcal{L}_H(\ell_n) \|_{T_{a_n}F^{1,1/2}(S^1, N)} \to 0 \) as \( n \to \infty \). Then there exist a subsequence \( \{ \ell_{n_k} \} \subset \{ \ell_n \} \) and \( \ell_\infty = (\phi_\infty, \psi_\infty) \in \text{crit}(\mathcal{L}_H) \) such that \( \| \ell_n - \ell_\infty \|_{H^1(S^1, \mathbb{R}^k)} \to 0 \) as \( n \to \infty \).

**Proof** Note that, in [20, Lemma 5.1], the Palais–Smale condition for \( \mathcal{L}_H \) has been verified under slightly different conditions on \( H \). In the present case, however, essentially the same argument applies and we omit the details here. \( \square \)

We next turn to the compactness issue of the set of flow lines. Recall from the last subsection, that for \( x_-(\phi_-, \psi_-), x_+(\phi_+, \psi_+) \in \text{crit}(\mathcal{L}_H), \) \( \hat{M}(x_-, x_+) \) denotes the set of negative gradient flow lines connecting \( x_- \) and \( x_+ \). For \( \ell \in \hat{M}(x_-, x_+) \), the associated flow line (compactified at both ends) \( \hat{\ell}(\mathbb{R}) := \{x_-\} \cup \ell(\mathbb{R}) \cup \{x_+\} \subset F^{1,1/2}(S^1, N) \) is compact. We define on \( \hat{M}(x_-, x_+) \) a metric \( d_H \) via

\[
d_H(\hat{\ell}_1, \hat{\ell}_2) = d_{\text{Hausdorff}}(\hat{\ell}_1(\mathbb{R}), \hat{\ell}_2(\mathbb{R})),
\]

where \( \hat{\ell}_i \in \hat{M}(x_-, x_+) \) \( (i = 1, 2) \) are flow lines corresponding to \( \ell_i \in M(x_-, x_+) \) and \( d_{\text{Hausdorff}} \) is the Hausdorff distance defined on compact sets in \( F^{1,1/2}(S^1, N) \). In the next proposition, we show that the metric space \( (\hat{M}(x_-, x_+), d_H) \) is relatively compact. We also give a characterization of its closure \( \overline{\hat{M}}(x_-, x_+) \).

**Proposition 3.7** The metric space \( (\hat{M}(x_-, x_+), d_H) \) is relatively compact, i.e., for any sequence \( \{\hat{\ell}_n\} \subset \hat{M}(x_-, x_+) \), there exists a subsequence \( \{\hat{\ell}_{n_k}\} \) and a compact subset \( K_\infty \in F^{1,1/2}(S^1, N) \) such that \( d_{\text{Hausdorff}}(\hat{\ell}_{n_k}(\mathbb{R}), K_\infty) \to 0 \) as \( n_k \to \infty \). The limit \( K_\infty \) has the following properties:

1. \( K_\infty \) is invariant under the negative gradient flow of \( \mathcal{L}_H \).
2. If \( \mathcal{L}_H \) is a Morse function on \( F^{1,1/2}(S^1, N) \), there exist finitely many critical points \( x_-(x_0, x_1, \ldots, x_{k-1}, x_k = x_+ \in \text{crit}(\mathcal{L}_H) \) and \( \ell_i \in M(x_i, x_{i+1}) \) for \( 0 \leq i \leq k - 1 \) such that \( K_\infty = \bigcup_{i=0}^{k-1} \ell_i(\mathbb{R}) \).

**Proof** Since \( \mathcal{L}_H \) is non-increasing along the negative gradient flow, we have \( \mathcal{L}_H(x_+) \leq \mathcal{L}_H(\ell_n(t)) \leq \mathcal{L}_H(x_-) \) for all \( n \geq 1 \) and \( t \in \mathbb{R} \). By Proposition 3.3, this implies the following uniform estimate

\[
\sup_{t \in \mathbb{R}} \| \phi_n(t) \|_{C^{2,2/3}(S^1)} + \sup_{t \in \mathbb{R}} \| \psi_n(t) \|_{C^{1,2/3}(S^1)} \leq C \tag{3.113}
\]

for some \( C > 0 \) independent of \( n \), where \( \ell_n = (\phi_n, \psi_n) \). Thus, there exists \( R > 0 \) such that

\[
\ell_n(\mathbb{R}) \subset B_R(C^{2,2/3}(S^1) \times C^{1,2/3}(S^1)) \cap F^{1,1/2}(S^1, N),
\]

where \( B_R(C^{2,2/3}(S^1) \times C^{1,2/3}(S^1)) \) is the closed ball of radius \( R \) with center at \( (0,0) \) in \( C^{2,2/3}(S^1, \mathbb{R}^k) \times C^{1,2/3}(S^1, \mathbb{S}(S^1) \otimes \mathbb{R}^k) \). Since the embedding \( C^{2,2/3}(S^1) \times C^{1,2/3}(S^1) \subset \)
For any $c$ this fact, we assume by contradiction that there are two distinct points $x, y \in K \subset \mathbb{R}$. Therefore, we also have $d_{\text{Hausdorff}}(\ell_{n_k}(\mathbb{R}), K) \to 0$ as $n_k \to \infty$.

Because all the flow lines $\ell_{n_k}(\mathbb{R})$ are invariant under the negative gradient flow, by the Hausdorff convergence, $K_\infty$ is also invariant under the negative gradient flow. Here, we used the following property of the Hausdorff convergence: Let $f : (X, d_X) \to (Y, d_Y)$ be a continuous map between two metric spaces. If $d_{H}(K_n, K_\infty) \to 0$ (as $n \to \infty$) for compact subsets $K_n, K_\infty \subset X$, then $d_{H}(f(K_n), f(K_\infty)) \to 0$ as $n \to \infty$. Applying this to the time $t$-flow map for any $t \in \mathbb{R}$ and compact sets $\ell_{n_k}(\mathbb{R})$ and $K_\infty$, we deduce the flow invariance of $K_\infty$.

Since we cannot find a suitable reference for this elementary property, for completeness, we give a proof. Let us recall $d_{H}(K_n, K_\infty) = \max[\sup_{x \in K_n} d_X(x, K_\infty), \sup_{y \in K_\infty} d_X(y, K_n)]$.

Assume contrary that $d_{H}(f(K_n), f(K_\infty))$ does not converge to 0. Then for some subsequence (we still denote by the full sequence), at least one of the following holds: (i) there exist $\epsilon > 0$ and $x_n \in K_n$ $(n \geq 1)$ such that $d_Y(f(x_n), f(K_\infty)) \geq \epsilon$ for all $n \geq 1$, or (ii) there exist $\epsilon > 0$ and $y_n \in K_\infty$ such that $d_Y(f(y_n), f(K_n)) \geq \epsilon$ for all $n$. If (i) holds, by the compactness of $K_\infty$ and $d_X(x_n, K_\infty) \leq d_{H}(K_n, K_\infty) \to 0$, taking a subsequence if necessary, we may assume that there exists $x_\infty \in K_\infty$ such that $d_X(x_n, x_\infty) \to 0$. But then, we arrive at a contradiction:

$$\epsilon \leq d_Y(f(x_n), f(K_\infty)) \leq d_Y(f(x_n), f(x_\infty)) \to 0.$$  

On the other hand, if (ii) holds, by the compactness of $K_\infty$, passing to a subsequence if necessary, we may assume that $d_X(y_n, y_\infty) \to 0$ for some $y_\infty \in K_\infty$. Then, by $d_X(y_\infty, K_n) \leq d_H(K_\infty, K_n) \to 0$, there exist $x_n \in K_n$ such that $d_X(x_n, y_\infty) \to 0$. We then arrive at a contradiction:

$$\epsilon \leq d_Y(f(y_n), f(K_n)) \leq d_Y(f(y_n), f(x_n)) \to 0.$$  

This finishes the proof of (1).

To prove the second assertion, assume that $\mathcal{L}_H$ is Morse. Since $\mathcal{L}_H(x_-) \leq \mathcal{L}_H \leq \mathcal{L}_H(x_+)$ on $K_\infty$, by the Hausdorff convergence, the Palais–Smale condition (Proposition 3.6) and the Morse property, there are at most finitely many critical points on $K_\infty$. Since $x_{\pm} \in \ell_{n}(\mathbb{R})$ for all $n$, we also have $x_{\pm} \in K_\infty$ by the Hausdorff convergence. To complete the proof, we need to show the following:

(i) For any $c \in \mathbb{R}$ with $\mathcal{L}_H(x_+) \leq c \leq \mathcal{L}_H(x_-)$, the set $K_\infty \cap \mathcal{L}_H^{-1}(c)$ consists of exactly one point.

(ii) For any $c \in \mathbb{R}$ with $c < \mathcal{L}_H(x_+)$ or $c > \mathcal{L}_H(x_-)$, we have $K_\infty \cap \mathcal{L}_H^{-1}(c) = \emptyset$.

As for (ii), observe that for any $n$, $\ell_{n}(\mathbb{R}) \cap \mathcal{L}_H^{-1}(c) \neq \emptyset$ if and only if $\mathcal{L}_H(x_+) \leq c \leq \mathcal{L}_H(x_-)$.

Thus, the assertion (ii) follows form the Hausdorff convergence $d_{\text{Hausdorff}}(\ell_{n_k}(\mathbb{R}), K_\infty) \to 0$.

Therefore, we also have $K_\infty \cap \mathcal{L}_H^{-1}(c) \neq \emptyset$ if $\mathcal{L}_H(x_+) \leq c \leq \mathcal{L}_H(x_-)$. Thus, it remains to prove $\#(K_\infty \cap \mathcal{L}_H^{-1}(c)) = 1$ for any $c \in \mathbb{R}$ with $\mathcal{L}_H(x_+) \leq c \leq \mathcal{L}_H(x_-)$. In order to prove this fact, we assume by contradiction that there are two distinct points $y, z \in K_\infty \cap \mathcal{L}_H^{-1}(c)$ for some $c \in \mathbb{R}$ with $\mathcal{L}_H(x_+) \leq c \leq \mathcal{L}_H(x_-)$. Again, from the Hausdorff convergence $\ell_{n_k}(\mathbb{R}) \to K_\infty$ and the fact that $\ell_{n_k}(\mathbb{R})$ is a flow line, there exists $y_k \in \ell_{n_k}(\mathbb{R})$ and $t_k \in \mathbb{R}$ such that $y_k \to y$ and $z_k := \ell_{n_k}(t_k, y_k) \to z$ as $k \to \infty$, where $\ell_{n_k}(t, y)$ is the solution to the
negative gradient flow equation (3.9) with $\ell_{n_k}(0, y) = y$. We may assume $t_k \geq 0$, otherwise we simply reverse the roles of $z_k$ and $y_k$.

Under this assumption, we have

$$
\int_0^{t_k} \| \nabla_{1/2} L_{H}(\ell_{n_k}(t, y_k)) \|^2_{T^{1/2}(S^1, N)} dt = L_H(y_k) - L_H(\ell_{n_k}(t_k, y_k)) \rightarrow L_H(y) - L_H(z) = 0. \tag{3.114}
$$

Knowing that $L_H$ is Morse and satisfies the Palais–Smale condition (see Proposition 3.6), the set $\text{crit}(L_H) \cup L_H^{-1}(\{L_H(z), L_H(x)\})$ is finite and (3.114) implies that $t_k \rightarrow 0$ or $\ell_{n_k}(0, t_k), y_k$ converges to some critical point of $L_H$. In any case, we arrive at a contradiction $y = z$. This completes the proof. \qed

We assume that $L_H$ is a Morse function on $\mathcal{F}^{1,1/2}(S^1, N)$ and

$$
\mathcal{F}_{x_-, x_+} : W^{1,2}_{x_-, x_+}(\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N)) \rightarrow L^2(\mathbb{R}, T^{1,1/2}(S^1, N))
$$

defined by (3.78) has $0 \in L^2(\mathbb{R}, T^{1,1/2}(S^1, N))$ as a regular value. Under this assumption, if $x_\neq x_+, \overline{M}(x_, x_+)$ is a manifold of dimension $\mu_H(x_\neq) - \mu_H(x_\neq) = 1$ as remarked at the end of the last subsection. Under these assumptions, as a corollary of Proposition 3.7, we have:

**Corollary 3.1** Under the above assumption, we have

1. If $\mu_H(x_\neq) - \mu_H(x_\neq) = 1$, $\overline{M}(x_, x_+)$ is a compact 0-dimensional manifold, hence it consists of at most finitely many points.

2. If $\mu_H(x_\neq) - \mu_H(x_\neq) = 2$, $\overline{M}(x_, x_+) is a 1-dimensional manifold and has a compactification $\overline{\mathcal{M}}(x_-, x_+)$. The boundary $\partial \overline{\mathcal{M}}(x_-, x_+)$ of this manifold (if non-empty) consists of broken flow lines of the form $\ell_1(\mathbb{R}) \cup \ell_2(\mathbb{R})$ for some $\ell_1 \in \mathcal{M}(x_-, y)$ and $\ell_2 \in \mathcal{M}(y, x_+)$, where $y \in \text{crit}(L_H)$ with $\mu_H(x_-) - \mu_H(y) = 1$.

### 3.5 Gluing negative gradient flow lines

We shall show that the converse of the compactness result (Corollary 3.1 (2)) holds under a certain transversality assumption.

Let $x,y,z \in \text{crit}(L_H)$ be such that $\mu_H(x) = \mu_H(z) + 1 = \mu_H(y) + 2$ and consider $\ell_1 \in \mathcal{M}(x, z)$ and $\ell_2 \in \mathcal{M}(z, y)$. We assume that $d\mathcal{F}_{x,z}(\ell_i) : W^{1,2}(\mathbb{R}, \ell_i^* T^{1,1/2}(S^1, N)) \rightarrow L^2(\mathbb{R}, \ell_i^* T^{1,1/2}(S^1, N))$ are surjective for $i = 1, 2$. We shall prove in what follows that under the above assumption, there exists a family of solutions $\{\ell_{1,2,R}\}$ to the negative gradient flow equation $\frac{d\ell_{1,2,R}}{dt} + \nabla_{1/2} L_H(\ell_{1,2,R}) = 0$ parametrized by large $R >> 1$ such that the flow line $\ell_{1,2,R}(\mathbb{R})$ converges to $\ell_1(\mathbb{R}) \cup \ell_2(\mathbb{R})$ in the Hausdorff metric as $R \rightarrow \infty$.

To construct such solutions, we first construct pregluings $\ell_1 \# R \ell_2$ and $\ell_2$ for all large $R >> 1$ which are good approximate solutions to the negative gradient flow equation.

We set $\ell_i = (\phi_i, \psi_i)$ for $i = 1, 2$ and $z = (\phi_2, \psi_2)$.

Construction of pregluing $\phi_1 \# R \phi_2$:

Let $\alpha \in C^\infty(\mathbb{R})$ be such that $\alpha(t) = 0$ for $t \leq -1$, $\alpha(t) = 1$ for $t \geq 1$ and $0 \leq \alpha(t) \leq 1$ for $-1 \leq t \leq 1$. For $R > 0$, we set $\alpha_R(t) = \alpha(R^{-1}t)$. By definition (see (3.74), (3.75)), $\phi_1$ and $\phi_2$ are expressed as

$$
\phi_1(t) = \exp_{\phi_2}(X_{1,+}(t)) \quad \text{for} \quad t \geq T_1,
$$

$$
\phi_2(t) = \exp_{\phi_2}(X_{2,-}(t)) \quad \text{for} \quad t \leq -T_2,
$$
where $T_1, T_2 > 0$,

$$X_{1,+} \in W^{1,2}([T_1, \infty), H^1(S^1, \phi_z^*TN))$$

and

$$X_{2,-} \in W^{1,2}((-\infty, -T_2], H^1(S^1, \phi_z^*TN)).$$

For $R > \max\{T_1, T_2\}$, we define pregluing $\phi_1#_R\phi_2$ by

$$\phi_1#_R\phi_2 = \begin{cases}
\phi_1(t + 2R) & (t \leq -R) \\
\exp_{\phi_2}((1 - \alpha_R(t))X_{1,+}(t + 2R) + \alpha_R(t)X_{2,-}(t - 2R)) & (-R \leq t \leq R) \\
\phi_2(t - 2R) & (t \geq R),
\end{cases}$$

Construction of pregluing $\psi_1#_R\psi_2$:

By definition (see (3.76), (3.77)), $\psi_1$ and $\psi_2$ are expressed as

$$\begin{align*}
\psi_1(t) &= S_{1,+}(\psi_z + \xi_{1,+}(t)) \quad \text{for} \quad t \geq T_1, \\
\psi_2(t) &= S_{2,-}(\psi_z + \xi_{2,-}(t)) \quad \text{for} \quad t \leq -T_2,
\end{align*}$$

where

$$\begin{cases}
\xi_{1,+} \in W^{1,2}([T_1, \infty), H^{1/2}(S^1, \mathbb{S}(S^1) \otimes \phi_z^*TN)), \\
\xi_{2,-} \in W^{1,2}((-\infty, -T_2], H^{1/2}(S^1, \mathbb{S}(S^1) \otimes \phi_z^*TN)),
\end{cases}$$

$S_{1,+}: \phi_z(t)^*TN \to \phi_1(t)^*TN$ is the parallel translation along path

$$[0, 1] \ni \tau \mapsto \exp_{\phi_2}(\tau X_{1,+}(t)) \in N$$

for $t \geq T_1$ and $S_{2,-}: \phi(z)^*TN \to \phi_2(t)^*TN$ is the parallel translation along path

$$[0, 1] \ni \tau \mapsto \exp_{\phi_2}(\tau X_{2,-}(t))$$

for $t \leq -T_2$.

For $R > \max\{T_1, T_2\}$, we define pregluing $\psi_1#_R\psi_2$ by

$$\psi_1#_R\psi_2 = \begin{cases}
\psi_1(t + 2R) & (t \leq -R) \\
S_{1,2,R;1}(\psi_z + (1 - \alpha_R(t))\xi_{1,+}(t + 2R) + \alpha_R(t)\xi_{2,-}(t - 2R)) & (-R \leq t \leq R) \\
\psi_2(t - 2R) & (t \geq R),
\end{cases}$$

where $S_{1,2,R;1}: \phi_z(t)^*TN \to (\phi_1#_R\phi_2)(t)^*TN$ is the parallel translation along path $[0, 1] \ni \tau \mapsto \exp_{\phi_2}(\tau X_{1,2,R;1}(t))$, where we set $X_{1,2,R;1} = (1 - \alpha_R(t))X_{1,+}(t + 2R) + \alpha_R(t)X_{2,-}(t - 2R)$.

We then define pregluing $\ell_1#_R\ell_2$ by $\ell_1#_R\ell_2 = (\phi_1#_R\phi_2, \psi_1#_R\psi_2)$. The following properties of the pregluing can be checked from the construction:

- The flow line $\ell_1#_R\ell_2(\mathbb{R})$ converges to the broken flow line $\ell_1(\mathbb{R}) \cup \ell_2(\mathbb{R})$ in the Hausdorff metric as $R \to \infty$.
- $\mathcal{F}_{xy}(\ell_1#_R\ell_2) \to 0$ as $R \to \infty$. Thus, for large $R > 0$, $\ell_1#_R\ell_2$ is a good approximate solution to the negative gradient flow equation.

A genuine solution to the equation $\mathcal{F}_{xy}(\ell) = 0$ can be constructed as a small perturbation of $\ell_1#_R\ell_2$.

**Proposition 3.8** Let us assume that $x, y, z \in \text{crit}(\mathcal{L}_H)$ with $\mu_H(x) = \mu_H(z) + 1 = \mu_H(y) + 2$. Let $\ell_1 \in \mathcal{M}(x, z)$ and $\ell_2 \in \mathcal{M}(z, y)$ be regular solutions to the negative gradient flow equation in the sense that $d\mathcal{F}_{xy}(\ell_1): W^{1,2}(\mathbb{R}, \epsilon_z^*T\mathcal{F}^{1,1/2}(S^1, N)) \to L^2(\mathbb{R}, \epsilon_z^*T\mathcal{F}^{1,1/2}(S^1, N))$ and $d\mathcal{F}_{zy}(\ell_2): W^{1,2}(\mathbb{R}, \epsilon_z^*T\mathcal{F}^{1,1/2}(S^1, N)) \to L^2(\mathbb{R}, \epsilon_z^*T\mathcal{F}^{1,1/2}(S^1, N))$ are surjective. Then there exists $R_0 > 0$ such that the following holds:
For any \( R \geq R_0 \), there exists a unique small \((X_{1,2;R}, \xi_{1,2;R})\) with
\[
' (X_{1,2;R}, \xi_{1,2;R}) \in \left( \begin{array}{c} 1 \\ -\Gamma (\ell_{1#R} \ell_2) \end{array} \right) \begin{pmatrix} 0 \\ \left( \ker (dF_{xy}(\ell_{1#R} \ell_2)) \right) \end{pmatrix},
\]
such that \( \ell_{1,2;R} := (\exp_{\phi_{1#R} \phi_2}(X_{1,2;R}), S(X_{1,2;R}(\psi_{1#R} \psi_2 + \xi_{1,2;R})) \in W^{1,2}(\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N)) \) solves the equation \( F_{xy}(\ell_{1,2;R}) = 0 \), where
\[
\Gamma (\ell_{1#R} \ell_2)[X] = -(\psi_{1#R} \psi_2)^{X_i}X^j\Gamma_{i,jk}(\phi_{1#R} \phi_2) \otimes \frac{\partial}{\partial y^j}(\phi_{1#R} \phi_2)
\]
for \( X = X^j(t) \frac{\partial}{\partial y^j}(\phi_{1#R} \phi_2)(t) \in W^{1,2}(\mathbb{R}, H^1(S^1, (\phi_{1#R} \phi_2)^*TN)) \), \( \ker (dF_{xy}(\ell_{1#R} \ell_2)) \) is the orthogonal complement of \( \ker (dF_{xy}(\ell_{1#R} \ell_2)) \subset W^{1,2}(\mathbb{R}, (\ell_{1#R} \ell_2)^*T\mathcal{F}^{1,1/2}(S^1, N)) \) and \( S(X_{1,2;R}) : T_{\phi_{1#R} \phi_2}(X_{1,2;R})N \rightarrow T_{\exp_{\phi_{1#R} \phi_2}(X_{1,2;R})}N \) is the parallel translation along path \([0,1] \ni t \mapsto \exp_{\phi_{1#R} \phi_2}(\tau X_{1,2;R}) \in N \).

**Proof** By the regularity of \( \ell_1 \) and \( \ell_2 \), we first observe that for large \( R > 0 \),
\[
dF_{xy}(\ell_{1#R} \ell_2) : W^{1,2}(\mathbb{R}, (\ell_{1#R} \ell_2)^*T\mathcal{F}^{1,1/2}(S^1, N)) \rightarrow L^2(\mathbb{R}, (\ell_{1#R} \ell_2)^*T\mathcal{F}^{1,1/2}(S^1, N))
\]
is surjective. To see this, we notice that Proposition 3.4 and the surjectivity of \( dF_{xz}(\ell_1) \) and \( dF_{zy}(\ell_2) \) imply the existence of a bounded right inverses
\[
R_{xz;\ell_1} : L^2(\mathbb{R}, \ell_{1#}^{x}T\mathcal{F}^{1,1/2}(S^1, N)) \rightarrow W^{1,2}(\mathbb{R}, \ell_{1#}^{x}T\mathcal{F}^{1,1/2}(S^1, N))
\]
and
\[
R_{zy;\ell_2} : L^2(\mathbb{R}, \ell_{2#}^{y}T\mathcal{F}^{1,1/2}(S^1, N)) \rightarrow W^{1,2}(\mathbb{R}, \ell_{2#}^{y}T\mathcal{F}^{1,1/2}(S^1, N))
\]
of \( dF_{xz}(\ell_1) \) and \( dF_{zy}(\ell_2) \), respectively. By gluing these right inverses appropriately, we obtain an approximate right inverse of \( dF_{xy}(\ell_{1#R} \ell_2) \) as follows. Let \( \beta, \gamma \in C^\infty(\mathbb{R}) \) be such that \( \beta(t) = 1 \) for \( t \leq -1 \), \( \beta(t) = 0 \) for \( t \geq 0 \), \( 0 \leq \beta(t) \leq 1 \) for \( -1 \leq t \leq 1 \), \( \gamma(t) = \beta(-t) \) for all \( t \in \mathbb{R} \) and \( \beta(t)^2 + \gamma(t)^2 = 1 \) for all \( t \in \mathbb{R} \). For \( R > 0 \), we set \( \beta_R(t) = \beta(R^{-1}t) \) and \( \gamma_R(t) = \gamma(R^{-1}t) \). We define
\[
R_{xz;\ell_1,\ell_2} = \begin{cases} \tau_{2R}R_{xz;\ell_1} \tau_{-2R} & (t \leq -R) \\ \beta_R^{-1} \tau_{2R}R_{xz;\ell_1} \tau_{-2R}S_1 \beta_R + \gamma_R S_2^{-1} \tau_{-2R}R_{xy;\ell_2} \tau_{2R}S_2 \gamma_R & (-R \leq t \leq R) \\ \tau_{-2R}R_{xy;\ell_2} \tau_{2R} & (t \geq R), \end{cases}
\]
where for \( a \in \mathbb{R} \), \( \tau_a \) is the time shifting operator defined by \( \tau_a(t) = \ell(t + a) \), \( S_1 \) and \( S_2 \) are parallel translations along paths \([0,1] \ni t \mapsto \exp_{\phi_2}(\tau X_{1,2;R}(t + 2R)) + (1 - \tau)X_{1,2;R,R} \) and \([0,1] \ni t \mapsto \exp_{\phi_2}(\tau X_{2,2;R}(t - 2R) + (1 - \tau)X_{1,2;R,R}) \), respectively. Observe that \( R_{xz;\ell_1,\ell_2} \) defines a bounded linear operator between \( L^2(\mathbb{R}, (\ell_{1#R} \ell_2)^*T\mathcal{F}^{1,1/2}(S^1, N)) \) and \( W^{1,2}(\mathbb{R}, (\ell_{1#R} \ell_2)^*T\mathcal{F}^{1,1/2}(S^1, N)) \). Moreover, from the definition of \( R_{xz;\ell_1,\ell_2} \) it can be easily checked that the operator norm
\[
\| R_{xz;\ell_1,\ell_2} \|_{\text{op}(L^2(\mathbb{R}, (\ell_{1#R} \ell_2)^*T\mathcal{F}^{1,1/2}), W^{1,2}(\mathbb{R}, (\ell_{1#R} \ell_2)^*T\mathcal{F}^{1,1/2}))}
\]
is uniformly bounded for large \( R > 0 \) and
\[
\| dF_{xy}(\ell_{1#R} \ell_2) \circ R_{xz;\ell_1,\ell_2} - \mathbf{1}_{L^2(\mathbb{R}, (\ell_{1#R} \ell_2)^*T\mathcal{F}^{1,1/2})} \|_{\text{op}(L^2(\mathbb{R}, (\ell_{1#R} \ell_2)^*T\mathcal{F}^{1,1/2}))} \rightarrow 0 \quad (3.115)
\]
as \( R \rightarrow \infty \).

By (3.115),
\[
dF_{xy}(\ell_{1#R} \ell_2) : W^{1,2}(\mathbb{R}, (\ell_{1#R} \ell_2)^*T\mathcal{F}^{1,1/2}(S^1, N)) \rightarrow L^2(\mathbb{R}, (\ell_{1#R} \ell_2)^*T\mathcal{F}^{1,1/2}(S^1, N))
\]
is onto for all large \( R > 0 \) and there exists a right inverse
\[
\mathcal{R}_{\ell_1, \ell_2; R} : L^2(\mathbb{R}, (\ell_1#_R \ell_2)^s T \mathcal{F}^{1,1/2}(S^1, N)) \to W^{1,2}(\mathbb{R}, (\ell_1#_R \ell_2)^s T \mathcal{F}^{1,1/2}(S^1, N))
\]
whose operator norm is uniformly bounded for all large \( R > 0 \).

We next observe that the differential of the map
\[
H^1(S^1, (\phi_1#_R \phi_2)^* T N) \times H^{1/2}(S^1, \mathbb{S}(S^1) \otimes (\phi_1#_R \phi_2)^* T N) \ni (X, \xi)
\]
\[
\mapsto (\exp_{\phi_1#_R \phi_2}(X), S(X)(\psi_1#_R \psi_2 + \xi)) \in \mathcal{F}^{1,1/2}(S^1, N)
\]
at \((X, \xi) = (0, 0)\) is given by
\[
H^1(S^1, (\phi_1#_R \phi_2)^* T N) \times H^{1/2}(S^1, \mathbb{S}(S^1) \otimes (\phi_1#_R \phi_2)^* T N) \ni \begin{pmatrix} X \\ \xi \end{pmatrix}
\]
\[
\mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} \Gamma(\ell_1#_R \ell_2) 0 \\ \xi \\ X \end{pmatrix} \in H^1(S^1, (\phi_1#_R \phi_2)^* T N) \times H^{1/2}(S^1, \mathbb{S}(S^1) \otimes (\phi_1#_R \phi_2)^* T N)
\]
\[
\cong T_{\ell_1#_R \ell_2} \mathcal{F}^{1,1/2}(S^1, N), \tag{3.116}
\]
where \( \Gamma(\ell_1#_R \ell_2) : H^1(S^1, (\phi_1#_R \phi_2)^* T N) \to H^{1/2}(S^1, \mathbb{S}(S^1) \otimes (\phi_1#_R \phi_2)^* T N) \) is given by
\[
\Gamma(\ell_1#_R \ell_2)[X] = -(\psi_1#_R \psi_2)^k X^j \Gamma^i_{jk}(\phi_1#_R \phi_2) \otimes \frac{\partial}{\partial y^i}(\phi_1#_R \phi_2)
\]
and the identification \( T_{\ell_1#_R \ell_2} \mathcal{F}^{1,1/2}(S^1, N) \cong H^1(S^1, (\phi_1#_R \phi_2)^* T N) \times H^{1/2}(S^1, \mathbb{S}(S^1) \otimes (\phi_1#_R \phi_2)^* T N) \) is as in (3.1) (see also \([20, (3.17), (3.19)]\)). In fact, (3.116) is the differential of the local coordinate map of \( \mathcal{F}^{1,1/2}(S^1, N) \) at \( \ell_1#_R \ell_2 \).

We then find a genuine solution to the equation \( \mathcal{F}_{xy}(\ell) = 0 \) in the following form,
\[
\ell = \ell_{1,2; R} := (\exp_{\phi_1#_R \phi_2}(X), S(X)(\psi_1#_R \psi_2 + \xi)),
\]
where \((X, \xi) \in W^{1,2}(\mathbb{R}, H^1(S^1, (\phi_1#_R \phi_2)^* T N) \times H^{1/2}(S^1, \mathbb{S}(S^1) \otimes (\phi_1#_R \phi_2)^* T N)) \) takes the form
\[
\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} \Gamma(\ell_1#_R \ell_2) 0 \\ \xi \\ X \end{pmatrix} = \mathcal{R}_{\ell_1, \ell_2; R}(\eta) \tag{3.117}
\]
for some \( \eta \in L^2(\mathbb{R}, (\ell_1#_R \ell_2)^s T \mathcal{F}^{1,1/2}(S^1, N)) \).

Note that (3.117) implies that
\[
\|X\|_{W^{1,2}(\mathbb{R}, H^1(S^1, (\phi_1#_R \phi_2)^* T N))} + \|\xi\|_{H^{1/2}(S^1, \mathbb{S}(S^1) \otimes (\phi_1#_R \phi_2)^* T N)}
\]
\[
\leq C\|\eta\|_{L^2(\mathbb{R}, (\ell_1#_R \ell_2)^s T \mathcal{F}^{1,1/2})} \tag{3.118}
\]
for some \( C > 0 \) independent of large \( R > 0 \).

By the Taylor’s expansion, when \( X, \xi \) is given by (3.117) we have
\[
\mathcal{F}(X, \xi) := \mathcal{F}_{xy}(\exp_{\phi_1#_R \phi_2}(X), S(X)(\psi_1#_R \psi_2 + \xi))
\]
\[
= \mathcal{F}_{xy}(\ell_1#_R \ell_2) + d\mathcal{F}_{xy}(\ell_1#_R \ell_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} \Gamma(\ell_1#_R \ell_2) 0 \\ \xi \\ X \end{pmatrix} + r(X, \xi)
\]
\[
= \mathcal{F}_{xy}(\ell_1#_R \ell_2) + d\mathcal{F}_{xy}(\ell_1#_R \ell_2) \circ \mathcal{R}_{\ell_1, \ell_2; R}(\eta) + r(X, \xi)
\]
\[
= \mathcal{F}_{xy}(\ell_1#_R \ell_2) + \eta + r(X, \xi),
\]
where the remainder term \( r(X, \xi) \) satisfies \( r(X, \xi) = o(\|\eta\|_{L^2(\mathbb{R}, (\ell_1#_R \ell_2)^s T \mathcal{F}^{1,1/2})}) \) by (3.118).
Since $\mathcal{F}_{xy}(\ell_1 \#_R \ell_2) \to 0$ as $R \to \infty$, by the standard Banach fixed point argument, there is a unique small $\eta \in L^2(\mathbb{R}, (\ell_1 \#_R \ell_2)\pi T\mathbb{S}^{1,1/2}(S^1, N))$ such that the equation $\mathcal{F}(X, \xi) = 0$ holds for $X, \xi$ defined through (3.117). This completes the proof.

\section{Definition of the Morse–Floer homology under the transversality condition}

Throughout this section, we assume that the following conditions are satisfied for $\mathcal{L}_H$, where $H \in \mathbb{H}^{3,1}_{p+1}$:

(i) $\mathcal{L}_H$ is a Morse function on $\mathbb{S}^{1,1/2}(S^1, N)$.

(ii) The negative gradient flow system $\frac{d\ell}{dt} = -\nabla_{1,1/2} \mathcal{L}_H(\ell)$ defines a Morse–Smale system.

This means that $0 \in L^2(\mathbb{R}, C^0, \mathbb{R} \times C^0, \mathbb{R}(S^1))$ is a regular value of the map $\ell \mapsto \frac{d\ell}{dt} + \nabla_{1,1/2} \mathcal{L}_H(\ell)$.

Under the above assumption (i), (ii), for $p \in \mathbb{Z}$ we define

$$C_p(\mathcal{L}_H) := \bigoplus_{z \in \text{crit}_p(\mathcal{L}_H)} \mathbb{Z}(z)$$

(4.1)

and

$$C_p(\mathcal{L}_H, \mathbb{Z}_2) := C_p(\mathcal{L}_H) \otimes \mathbb{Z}_2,$$

(4.2)

where $\text{crit}_p(\mathcal{L}_H) = \{ x \in \text{crit}(\mathcal{L}_H) : d\mathcal{L}_H(x) = 0, \mu_H(x) = p \}$. Thus $\{ C_p(\mathcal{L}_H, \mathbb{Z}_2) \}_{p \in \mathbb{Z}}$ is a graded group with grading given by the relative Morse index $\mu_H$.

For $x, y \in \text{crit}(\mathcal{L}_H)$ with $\mu_H(x) - \mu_H(y) = 1$, under the above assumption (i), (ii), by Corollary 3.1, $\hat{M}(x, y)$ is a finite set. For $p \in \mathbb{Z}$, we define

$$\partial_p(\mathcal{L}_H)(x) = \sum_{y \in \text{crit}_{p-1}(\mathcal{L}_H)} n(x, y)(y)$$

(4.3)

for a generator $x \in \text{crit}(\mathcal{L}_H)$, where $n(x, y) = \#\hat{M}(x, y) \mod 2$, and extend by linearity to define $\partial_p(\mathcal{L}_H) : C_p(\mathcal{L}_H, \mathbb{Z}_2) \to C_{p-1}(\mathcal{L}_H, \mathbb{Z}_2)$.

We note that the sum in (4.3) is finite and (4.3) is well-defined. Indeed, for $y \in \text{crit}(\mathcal{L}_H)$ with $n(x, y) \neq 0$, there holds $\hat{M}(x, y) \neq \emptyset$ and there exists a flow line connecting $x$ and $y$. Since $\mathcal{L}_H$ is non-increasing along the negative gradient flow, we have $\mathcal{L}_H(y) \leq \mathcal{L}_H(x)$. Then by the condition (i) and the Palais–Smale condition (see Proposition 3.6), the set $\{ y \in \text{crit}(\mathcal{L}_H) : \mathcal{L}_H(y) \leq \mathcal{L}_H(x) \}$ is a finite set. Thus the set of points $y \in \text{crit}_{p-1}(\mathcal{L}_H)$ with $n(x, y) \neq 0$ is a finite set for any $x \in \text{crit}_p(\mathcal{L}_H)$ and the sum in (4.3) is finite. By the standard argument, we have

\begin{proposition}
\{(C_p(\mathcal{L}_H, \mathbb{Z}_2), \partial_p(\mathcal{L}_H))\}_{p \in \mathbb{Z}} is a chain complex, that is, the following holds
$$\partial_{p-1}(\mathcal{L}_H) \circ \partial_p(\mathcal{L}_H) = 0$$

for all $p \in \mathbb{Z}$.
\end{proposition}

\begin{proof}
Let $x \in \text{crit}_p(\mathcal{L}_H)$ be arbitrary. We have

$$\partial_{p-1} \circ \partial_p(x) = \sum_{y \in \text{crit}_{p-2}(\mathcal{L}_H)} \sum_{z \in \text{crit}_{p-1}(\mathcal{L}_H)} n(x, z)n(z, y)(y).$$

(4.4)

\end{proof}
For arbitrary $y \in \text{crit}_{p-2}(\mathcal{L}_H)$, as we have observed in Corollary 3.1, $\overline{\mathcal{M}}(x,y)$ is a compact 1-dimensional manifold with boundary if it is not empty. The components of its boundary consists of broken flow lines with exactly one breaking, i.e., any component is a broken flow line of the form $\ell_1(\mathbb{R}) \cup \ell_2(\mathbb{R})$, where $\ell_1 \in \mathcal{M}(x,z)$, $\ell_2 \in \mathcal{M}(z,y)$ for some $z \in \text{crit}_{p-1}(\mathcal{L}_H)$. The converse is also true by the gluing result Proposition 3.8. Thus the number $\sum_{z \in \text{crit}_{p-1}(\mathcal{L}_H)} n(x,z)n(z,y) \pmod{2}$ is the number (mod 2) of the connected components of the boundary of $\overline{\mathcal{M}}(x,y)$. Since the number of the components of 1-dimensional manifold is even, (4.4) is 0 (mod 2). This completes the proof.

**Definition 4.1** Let $p \geq 3$. For $H \in \mathbb{H}^3_{p+1}$ satisfying (i) and (ii) above, we denote by $HF_*(\mathcal{L}_H, \mathcal{F}^{1,1/2}(S^1, N); \mathbb{Z}_2)$ the homology of the chain complex $\{(C_p(\mathcal{L}_H), \partial_p(\mathcal{L}_H))\}_{p \in \mathbb{Z}}$:

$$HF_p(\mathcal{L}_H, \mathcal{F}^{1,1/2}(S^1, N); \mathbb{Z}_2) := \frac{\ker \partial_p(\mathcal{L}_H)}{\text{Im} \partial_{p+1}(\mathcal{L}_H)}.$$ 

**5 Transversality**

**5.1 The Morse property is generic**

In this section, we show that $\mathcal{L}_H$ becomes a Morse function by generically perturbing $H$. First, we need to construct a separable Banach space of perturbations. So we let $(p_i)_{i \in \mathbb{N}}$ be a dense family of points in $N$ and similarly, for each $p_i$ we consider a dense family of vectors $\{z^i_j\}_{j \in \mathbb{N}} \subset T_{p_i}N$. For a fixed $r > 0$ less than the injectivity radius of $N$, we can identify points in $B_r(p_i)$ with vectors in $T_{p_i}N$, that is $p \in B_r(p_i)$ is identified with a uniquely determined vector $v^i_p \in T_{p_i}N$ by the relation $p = \exp_{p_i}(v^i_p)$. So we define now the functions $V_{ij} : N \to \mathbb{R}$ by

$$V_{i,j}(p) = \begin{cases} \rho(|v^i_p|^2)g_{p_i}(v^i_p, z^i_j) & \text{if } |v^i_p| < \frac{r}{\sqrt{2}}, \\ 0 & \text{otherwise} \end{cases},$$

where $| \cdot | = g(\cdot, \cdot)^{1/2}$ and $\rho$ is a cut-off function supported in $[0, \frac{r^2}{2}]$ and $\rho = 1$ on $[0, \frac{r^2}{4}]$. Similarly, we choose a dense family of spinors $\{\varphi^j_i\}_{j \in \mathbb{N}} \in \mathbb{S}(S^1) \otimes T_{p_i}N$ and construct the functions $W_{ij}^k$ defined by

$$W_{ij}^k(p, \psi) = \begin{cases} \rho\left(\frac{\varphi^j_i}{R_k}\right)\rho(|v^i_p|^2)\langle \psi, \varphi^j_i \rangle & \text{if } |v^i_p| < \frac{r}{\sqrt{2}} \text{ and } |\psi| < \frac{rK_k}{\sqrt{2}}, \\ 0 & \text{otherwise} \end{cases},$$

where $\{R_k\}_{k \in \mathbb{N}}$ is an increasing dense sequence of positive rational numbers. We define the space of perturbations $\mathbb{H}^3_b$ by $U \in \mathbb{H}^3_b$ if and only if there exist two sequences of numbers $\{c_{ij}\}$ and $\{d_{ijk}\}$, a function $\beta \in C^3(S^1, \mathbb{R})$ so that

$$U(s, p, \psi) = \beta(s) \sum_{i,j,k} (c_{ij}V_{ij}(p) + d_{ijk}W_{ij}^k(p, \psi)),$$

with

$$\|U\|_{3,b} := \|\beta\|_{C^3} + \sum_{i,j,k} |c_{ij}|\|V_{ij}\|_{C^3} + |d_{ijk}|\|W_{ij}^k\|_{C^3} < \infty.$$
One can easily see that $\mathbb{H}_3^b$ is a separable Banach space with the norm $\| \cdot \|_3, b$ since it is isomorphic to $C^3(S^1) \times \ell^1$. Now we can state the following:

**Proposition 5.1** There exists a residual set $\mathbb{H}_{reg} \subset \mathbb{H}_3^b$ such that $\mathcal{L}_{H+h}$ is a Morse function on $\mathcal{F}^{1,1/2}(S^1, N)$ for every $h \in \mathbb{H}_{reg}$.

To prove this, we consider the map $\mathcal{G} : \mathcal{F}^{1,1/2}(S^1, N) \times \mathbb{H}_3^b \to T\mathcal{F}^{1,1/2}(S^1, N)$ defined by

$$\mathcal{G}(\phi, \psi, h) = \nabla_{1,1/2}\mathcal{L}_{H+h}(\phi, \psi).$$

The following lemma holds:

**Lemma 5.1** For any $(\phi_0, \psi_0, h) \in \mathcal{G}^{-1}(0)$, the map

$$d\mathcal{G}(\phi_0, \psi_0, h) : T(\phi_0, \psi_0)\mathcal{F}^{1,1/2}(S^1, N) \times \mathbb{H}_3^b \to T(\phi_0, \psi_0)\mathcal{F}^{1,1/2}(S^1, N)$$

is surjective.

**Proof** The proof follows closely the idea in [37]. In order to do this, we will show that $\text{Ran}(d\mathcal{G}(\phi_0, \psi_0, h))$ is dense, which is equivalent to showing that

$$\text{Ran}(d\mathcal{G}(\phi_0, \psi_0, h))^\perp = \{0\}$$

So we take $Z = (Z_1, Z_2) \in \text{Ran}(d\mathcal{G}(\phi_0, \psi_0, h))^\perp$, then

$$\langle Z, d(\phi, \psi)\mathcal{G}(\phi_0, \psi_0, h)\xi \rangle = 0$$

for all $\xi \in T(\phi_0, \psi_0)\mathcal{F}^{1,2}(S^1, N)$ and

$$\langle Z, d_h\mathcal{G}(\phi_0, \psi_0, h)V \rangle = 0$$

for all $V \in \mathbb{H}_3^b$. Now since $H \in \mathbb{H}_3^b$ and $h \in \mathbb{H}_3^b$, it is easy to see from the first equation that $Z$ is a solution to a linear system of differential equations with coefficients in $C^1$ and thus $Z$ is $C^2$. Now we have

$$d_h\mathcal{G}(\phi_0, \psi_0, h)V = \begin{bmatrix} -(-\Delta + 1)^{-1}\nabla_\phi V \\ -|D|^{-1}\nabla_\psi V \end{bmatrix}.$$

Assume first that there exists $s_0 \in S^1$ such that $Z_1(s_0) \neq 0$. We can then consider a small interval $(s_0 - \delta, s_0 + \delta)$ for $\delta > 0$ and small, and $r > 0$ less than the injectivity radius, so that $\phi_0(s) = \exp_{\phi_0(s_0)}(\xi(s))$ and $|\xi(s)| < r/2$ for $s \in (s_0 - \delta, s_0 + \delta)$. Also, for $p \in B_r(\phi_0(s_0))$, we have the existence of a unique vector $v_p \in T_{\phi(s_0)}N$ so that $p = \exp_{\phi_0(s_0)}(v_p)$. By taking $\delta$ even smaller, we can assume by continuity that

$$g(Z_1(s), Z_1(s_0)) > 0, \text{ for } s \in (s_0 - \delta, s_0 + \delta).$$

Therefore, we can define now the following function

$$V(s, p) = \begin{cases} \gamma(s)\rho(|v_p|^2)g(v_p, Z_1(s_0)) & \text{if } |v_p| < \frac{r}{\sqrt{2}} \\ 0 & \text{otherwise} \end{cases},$$

where $\gamma$ is a cut-off function supported in $(s_0 - \delta/2, s_0 + \delta/2)$. First, notice that if $X \in T_pN$, then

$$d_p V(s, p)X = 2\gamma(s)\rho'(|v_p|^2)g(d \exp_{\phi_0(s_0)}^{-1} X, v_p)g(v_p, Z_1(s_0)) + \gamma(s)\rho(|v_p|^2)g(d \exp_{\phi_0(s_0)}^{-1} X, Z_1(s_0)).$$

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Hence, taking $\delta$ even smaller so that $|v_{\phi_0(s)}| < \frac{r^2}{4}$ when $s \in (s_0 - \delta, s_0 + \delta)$ (making $\rho'(|v_{\phi_0(s)}|^2) = 0$), we have that
\[
\langle Z, d_h \mathcal{G}(\phi_0, \psi_0, h) W \rangle = -(Z_1, (\Delta + 1)^{-1}\nabla_\phi V)_{H^1} \\
= -(Z_1, \nabla_\phi V)_{L^2} \\
= -\int_{s_0 - \frac{\delta}{2}}^{s_0 + \frac{\delta}{2}} \gamma(s) g(d\exp_{\phi_0(s)}^{-1} Z_1(s), Z_1(s_0)) ds
\]
But recall that $d\exp|_0 = id$. Hence, we have, by continuity, that
\[
\langle Z, d_h \mathcal{G}(\phi_0, \psi_0, h) W \rangle < 0.
\]
This leads to a contradiction and therefore $Z_1 \equiv 0$. This would lead to the conclusion if $V$ was in $\mathbb{H}^3_b$, which might not be the case, so by taking $p_i$ arbitrarily close to $\phi_0(s_0)$ and $z_i^f$ arbitrarily close to $Z_1(s_0)$, the same conclusion would hold by replacing $V$ by $\gamma(s) V_{ij} \in \mathbb{H}_b^3$ with $V_{ij}$ defined by (5.1).

Now we do a similar construction for $Z_2$. Indeed, if $Z_2(s_0) \neq 0$, one considers the function
\[
W(s, p, \psi) = \begin{cases} 
\gamma(s) \rho \left(\frac{r^2 + |p|^2}{R^2}\right) \rho(|v_p|^2)(\psi, Z_2(t_0)) & \text{if } |\psi| < \frac{R}{\sqrt{2}} \text{ and } |v_p| < \frac{r}{\sqrt{2}}, \\
0 & \text{otherwise}
\end{cases}
\]
where $\rho$ and $\gamma$ are as above with small $\delta > 0$ and $R$ is fixed so that $|\psi_0(s)| < \frac{R}{\sqrt{2}}$ for $t \in (s_0 - \delta, s_0 + \delta)$. Since $Z_1 \equiv 0$ it is easy to see that
\[
\langle Z, d_h \mathcal{G}(\phi_0, \psi_0, h) W \rangle = -(Z_2, (|D| + 1)^{-1}\nabla_\psi W)_{H^{1/2}} \\
= -(Z_2, \nabla_\psi W)_{L^2} \\
= -\int_{s_0 - \frac{\delta}{2}}^{s_0 + \frac{\delta}{2}} \gamma(s) (Z_2(s), Z_2(s_0)) ds < 0,
\]
which is another contradiction. But again, $W$ might not be in $\mathbb{H}_b^3$, so we make similar choices as in the $Z_1$ case to get an arbitrarily close function $\gamma W_{ijk}$ and the conclusion still holds. Therefore, $Z = 0$ and $\text{Ran}(d\mathcal{G}(\phi_0, \psi_0, h))$ is dense. Since $d\mathcal{G}$ is a Fredholm operator, it has closed range and hence it is surjective.

**Proof of Proposition 5.1** Once Lemma 5.1 is proved, the proof follows from a standard argument. Define
\[
S = \{ (\phi, \psi, h) \in \mathcal{F}^{1,1/2}(S^1, N) \times \mathbb{H}_b^3 : \mathcal{G}(\phi, \psi, h) = 0 \}.
\]
By Lemma 5.1, $S \subset \mathcal{F}^{1,1/2}(S^1, N) \times \mathbb{H}_b^3$ is a separable $C^1$-submanifold by Lemma 5.1. We define a $C^1$-map $\pi_{\mathbb{H}_b^3} : S \to \mathbb{H}_b^3$ by $\pi(\phi, \psi, h) = h$. As we have already observed, the differential $D_{(\phi, \psi)} \mathcal{G}(\phi, \psi, h)$ at any $(\phi, \psi, h) \in S$ is a compact perturbation of a self-adjoint Fredholm map $\begin{pmatrix} (\Delta + 1)^{-1}(-\nabla_\phi^2) & 0 \\ 0 & (1 + |D|)^{-1}D_\phi \end{pmatrix}$ and therefore it is Fredholm of index 0. By the $C^1$-regularity of $\pi_{\mathbb{H}_b^3}$ and the Sard–Smale theorem, there exists a residual subset $\mathbb{H}_{\text{reg}} \subset \mathbb{H}_b^3$ such that any $h \in \mathbb{H}_{\text{reg}}$ is a regular value of $\pi_{\mathbb{H}_b^3}$. Then for any $h \in \mathbb{H}_{\text{reg}}$, $\mathcal{G}(\cdot, \cdot, h) : T\mathcal{F}^{1,1/2}(S^1, N) \to T\mathcal{F}^{1,1/2}(S^1, N)$ has 0 as a regular value and $\mathcal{L}_{H+h}$ is Morse. This completes the proof.

\(\square\)
Remark 5.1 Notice that we can actually enlarge the class of perturbation $\mathbb{H}^3_b$ to allow non local perturbation. For instance, consider the set of functions $\mathbb{H}^3_b$ defined by $U \in \mathbb{H}^3_b$ if $U : H^1(S^1, N) \times S^1 \times N \to \mathbb{R}$ and
\[
U(\phi, s, p, \psi) = R(\phi) \beta(s) \sum_{i,j,k} c_{ijk} V_{ij}(p) + d_{ijk} W^k_{ij}(p, \psi)
\]
with $\beta \in C^3(S^1, \mathbb{R})$ and $R \in C^3(H^1(S^1, N), [0, 1])$. It becomes a Banach space with the norm
\[
\|U\|_{3, b} = \|R\|_{C^3} + \sum_{i,j,k} |c_{ijk}| \|V_{ij}\|_{C^3} + |d_{ijk}| \|W^k_{ij}\|_{C^3} < \infty
\]
Notice that $\mathbb{H}^3_b$ can be identified with $C^3(H^1(S^1, N), [0, 1]) \times \mathbb{H}^3_b$. The perturbation of the function $L_H$, denoted by $L_{H+Rh}$, is then defined by
\[
L_{H+Rh}(\phi, \psi) = L_H(\phi, \psi) + R(\phi) \int_{S^1} h(s, \phi(s), \psi(s)) \, ds,
\]
where we used the identification mentioned above, that is,
\[
\mathbb{H}^3_b \ni Rh = (R, h) \in C^3(H^1(S^1, N), [0, 1]) \times \mathbb{H}^3_b.
\]

5.2 The Morse–Smale property is generic

In this part, we will prove that there is a generic set of perturbations of the metric in such a way that the gradient flow with respect to this perturbed metric satisfies the Morse–Smale property. We will closely follow the proof presented in [22] for the case of the non-linear Dirac equation and we will adapt it to our setting. We also refer the reader to a different global construction of a perturbation set in [3].

Since $\mathcal{F}^{1,1/2}(S^1, N)$ is a Hilbert manifold, then it is separable, so we consider a dense sequence of points $\{p_i\}_{i \in \mathbb{N}} \subset H^1(S^1, N)$ and for each $p_i$ we consider a dense family of pairs $\{(u^1_{ij}, u^2_{ij})\}_{i,j \in \mathbb{N}} \subset H^1(S^1, p_i^*TN) \times H^{1/2}(S^1, \mathbb{S}(S^1) \otimes p_i^*TN)$ and a family of linear forms on $H^1(S^1, p_i^*TN) \times H^{1/2}(S^1, \mathbb{S}(S^1) \otimes p_i^*TN), \{(A_{ij}, B_{ij})\}_{k \in \mathbb{N}}$. We fix $r > 0$ small enough so that the parallel translation along the shortest geodesic $P_{p_i} : T_{p_i}N \to T_{p_i}N$ is well defined for $p \in B_r(p_i) = \{p \in H^1(S^1, N) : \|p - p_i\|_{H^1(S^1)} < r\}$ and let $\rho : C^\infty_c([0, +\infty))$ be compactly supported in $[0, r]$ and equals $\rho = 1$ on $[0, \frac{r}{2}]$. Then we can define functions $K_{ij}(x)$ for $x = (p, \psi)$ by
\[
K_{ij}(x) = K_{ij}(p) = \begin{cases} \rho(\|p_i - p\|_{H^1(S^1)}) P_{p_i} \big( A_{ij}(P_{p_i} \cdot) u^1_{ij} + B_{ij}(P_{p_i} \cdot) u^2_{ij} \big) & \text{if } p \in B_r(p_i) \\ 0 & \text{otherwise} \end{cases}
\]
where $\|p_i - p\|_{H^1(S^1)}$ means the distance between $p_i$ and $p$ in $H^1(S^1, \mathbb{R}^k)$. Without loss of generality, we can assume that the $p_i$ and $u^1_{ij}, u^2_{ij}$ are in $C^\infty$. We consider then the space $\mathcal{K}$ defined by $K \in \mathcal{K}$ if and only if $K$ is symmetric and there exist a sequence of numbers $\{c_{ijk}\} \in \mathbb{R}$ such that $K = \sum_{i,j,k} c_{ijk} K_{ijk}$ and
\[
\|K\|_{C^2} := \sum_{i,j,k} |c_{ijk}| \|K_{i,j,k}\|_{C^2(\mathcal{F}^{1,1/2}, \text{Sym}(T\mathcal{F}^{1,1/2}, T\mathcal{F}^{1,1/2}))} < \infty.
\]
where $T\mathcal{F}^{1,1}(S^1, N) \to T\mathcal{F}^{1,1/2}(S^1, N)$ is the bundle over $T\mathcal{F}^{1,1/2}(S^1, N)$ whose fiber at $(\phi, \psi) \in T\mathcal{F}^{1,1/2}(S^1, N)$ is defined by

$$
T(\phi, \psi)\mathcal{F}^{1,1}(S^1, N) = H^1(S^1, \phi^*TN) \times H^1(S^1, S(S^1) \otimes \phi^*TN).
$$

Here, $\text{Sym}(T\mathcal{F}^{1,1/2}, T\mathcal{F}^{1,1}) \to \mathcal{F}^{1,1/2}$ is the bundle of symmetric homomorphisms between $T\mathcal{F}^{1,1/2}$ and $T\mathcal{F}^{1,1}$. “Symmetric” here means that

$$
\theta \in \mathcal{F}^{1,1/2}(S^1, N) \text{ where } \theta(\phi, \psi)
$$

for any $\theta$ given a continuous map $\theta : \mathcal{F}^{1,1/2}(S^1, N) \to [0, 1]$, the space of perturbation that will be considered is then

$$
\mathbb{K}_\theta^2 = \{ K \in \mathbb{K} : \text{there exists } C > 0; \| K(x) \|_{\text{op}(T_x\mathcal{F}^{1,1/2}, T_x\mathcal{F}^{1,1})} \leq C \theta(x) \text{ for all } x \in \mathcal{F}^{1,1/2}(S^1, N) \}.
$$

If we endow it with the norm

$$
\| K \|_{\mathbb{K}_\theta^2} = \| K \|_{C^2(\mathcal{F}^{1,1/2})} + \sup_{x \in \mathcal{F}^{1,1/2}(S^1, N)} \frac{\| K(x) \|_{\text{op}(T_x\mathcal{F}^{1,1/2}, T_x\mathcal{F}^{1,1})}}{\theta(x)},
$$

it can be easily checked that the space $\mathbb{K}_\theta^2$ is a Banach space with the norm $\| \cdot \|_{\mathbb{K}_\theta^2}$.

Now given $\rho > 0$, we consider the ball $\mathbb{K}_\theta^2, \rho = \{ K \in \mathbb{K}_\theta^2; \| K \|_{\mathbb{K}_\theta^2} < \rho \}$. For $K \in \mathbb{K}_\theta^2, \rho$, we define the metric $G^K$ on $\mathcal{F}^{1,1/2}(S^1, N)$ by

$$
G^K(\phi, \psi)(X, Y) = \langle (1 + K(\phi, \psi))^{-1}X, Y \rangle_{H^1 \times H^{1/2}}
$$

for $X, Y \in T(\phi, \psi)\mathcal{F}^{1,1/2}(S^1, N)$. This metric is well-defined for $\rho < \frac{1}{2S_0}$, where $S_0$ is the norm of the embedding $H^1 \times H^1$ in $H^1 \times H^{1/2}$.

The gradient of $\mathcal{L}_\mathcal{H}$ with respect to $G^K$ is given by

$$
\nabla_{G^K} \mathcal{L}_\mathcal{H}(\phi, \psi) = (1 + K(\phi, \psi))\nabla_{1,1/2}\mathcal{L}_\mathcal{H}(\phi, \psi).
$$

The following is the main result of this section.

**Proposition 5.2** Assume that $\mathcal{L}_\mathcal{H}$ is a Morse function on $\mathcal{F}^{1,1/2}(S^1, N)$. Assume in addition that $\theta$ satisfies the following conditions:

1) $\theta(x) = 0$ for all $x \in \text{crit}(\mathcal{L}_\mathcal{H})$.

2) The zero set of $\theta$ is the closure of an open set.

3) If $W^u_{\mathcal{L}_\mathcal{H}}(x)$ and $W^s_{\mathcal{L}_\mathcal{H}}(y)$ intersect non-transversally at a point $z$ for some $x, y \in \text{crit}(\mathcal{L}_\mathcal{H})$, then $\theta$ is not identically zero on the flow line through $z$.

Then there exist $0 < \rho_0 < \frac{1}{2S_0}$ and a residual set $\mathbb{K}_{\text{reg}} \in \mathbb{K}_{\theta, \rho_0}$ such that for all $K \in \mathbb{K}_{\text{reg}}$ the negative gradient flow with respect to the metric $G^K$ satisfies the Morse–Smale property up to order 2.

Where $W^u_{\mathcal{L}_\mathcal{H}}(x)$ and $W^s_{\mathcal{L}_\mathcal{H}}(y)$ in the above proposition are unstable and stable manifolds of critical points $x, y \in \text{crit}(\mathcal{L}_\mathcal{H})$ of the vector field $-\nabla_{1,1/2}\mathcal{L}_\mathcal{H}$, respectively:

$$
W^u_{\mathcal{L}_\mathcal{H}}(x) := \{ c \in \mathcal{F}^{1,1/2} : \ell(t, c) \text{ exists for } t \leq 0 \text{ and } \lim_{t \to -\infty} \ell(t, c) = x \}.
$$

\( \text{Springer} \)
Lemma 5.2 There exists 0 < \delta \leq \frac{1}{350} such that for any 0 < \rho_0 < \delta and any \( K \in \mathbb{R}^2_{\rho, \rho_0} \) we have: Let \((\phi(t), \psi(t)) \in C^1(\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N))\) be a solution to the \( G^K \)-negative gradient flow equation

\[
\frac{d}{dt} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = -(1 + K(\phi, \psi)) \nabla_{1,1/2} L(H, \psi)
\]

which satisfies the condition \( \sup_{t \in \mathbb{R}} |L_H(\phi(t), \psi(t))| =: C_0 < +\infty \). Then there exists \( C(C_0) > 0 \) such that

\[
\sup_{t \in \mathbb{R}} \|\partial_\psi \phi(t)\|_{L^2(S^1)} + \sup_{t \in \mathbb{R}} \|\psi(t)\|_{H^{1/2}(S^1)} \leq C.
\]

Proof Set \( \ell(t) = (\phi(t), \psi(t)) \). We write \( K(\ell(t)) = \begin{pmatrix} K_{11}(t) & K_{12}(t) \\ K_{21}(t) & K_{22}(t) \end{pmatrix} \) according to the canonical decomposition \( (3.1) \). We first notice that since the metric \( G^K \) is uniformly equivalent to the standard one, we have a similar result to Lemma 3.1. That is, there exists a positive constant \( C = C(C_0, \rho_0) \) depending only on \( C_0 \) such that the following holds for all \( \tau \in \mathbb{R} \):

\[
\begin{align*}
\int_{\tau}^{\tau+1} \|\phi(t)\|^2_{H^1(S^1)} dt & \leq C, \\
\int_{\tau}^{\tau+1} \|\psi(t)\|_{H^{1/2}(S^1)} dt & \leq C, \\
\int_{\tau}^{\tau+1} \|\psi(t)\|_{L^{p+1}(S^1)} dt & \leq C.
\end{align*}
\]

Now following the steps in Proposition 3.2, we have

\[
\partial_\tau \psi(t) = -(1 + K_{22})(t)(1 + |D|)^{-1}(D\psi - A(\partial_s \phi(t), \partial_s \psi(t)) - \nabla \psi H(s, \phi(t), \psi(t)))
\]

\[
- K_{21} \nabla_\phi L_H(\phi(t), \psi(t))
\]

\[
= -(1 + K_{22})(1 + |D|)^{-1}(D\psi + \lambda \psi(t) - \lambda \psi(t) + A(\partial_s \phi(t), \partial_s \psi(t))
\]

\[
+ \nabla \psi H(s, \phi(t), \psi(t))) - K_{21} \nabla_\phi L_H(\phi(t), \psi(t))
\]

\[
= -L_\lambda \psi(t) - K_{22}(s) \nabla_\psi \psi(t) + (1 + K_{22})(1 + |D|)^{-1}(- \lambda \psi(t) + A(\partial_s \phi(t), \partial_s \psi(t))
\]

\[
+ \nabla \psi H(s, \phi(t), \psi(t))) - K_{21} \nabla_\phi L_H(\phi(t), \psi(t)).
\]

We set

\[
\begin{align*}
r(t) & = -K_{22}(t) L_\lambda \psi(t) + (1 + K_{22}(t))(1 + |D|)^{-1}(\lambda \psi(t) + A(\partial_s \phi(t), \partial_s \psi(t))
\]

\[
+ \nabla \psi H(s, \phi(t), \psi(t))) - K_{21}(t) \nabla_\phi L_H(\phi(t), \psi(t))
\]

\[
= -K_{22}(t) L_\lambda \psi(t) + (1 + K_{22}(t))(1 + |D|)^{-1}(\lambda \psi(t) + A(\partial_s \phi(t), \partial_s \psi(t))
\]

\[
+ \nabla \psi H(s, \phi(t), \psi(t))) + K_{21}(t)(1 + K_{11}(t))^{-1} K_{12}(t) \nabla_\psi L_H(\phi(t), \psi(t))
\]

\[
+ K_{21}(t)(1 + K_{11}(t))^{-1} \partial_\tau \phi(t).
\]

By using the fundamental solution \( G_\lambda \) (see (3.33)), by (5.7), we have

\[
\psi(t) = \int_{\mathbb{R}} G_\lambda(t - \tau) r(\tau) d\tau.
\]
Therefore, we have as in (3.36)
\[ \|\psi\|_{H^{1/2}} \leq C \sup_{\eta \in \mathbb{R}} \int_{\eta}^{\eta+1} \|r(\tau)\|_{H^{1/2}} d\tau. \]

Hence, it remains to estimate \( \|r(\tau)\|_{H^{1/2}} \).

\[
r(t) = (K_{21}(1 + K_{11})^{-1}K_{12} - K_{22})(t)\nabla_{\psi} L_{H} + (1 + |D|)^{-1}(\lambda_{\psi}(t) + A(\partial_{s}\phi(t), \partial_{s} \cdot \psi(t)) + \nabla_{\psi} H(s, \phi(t), \psi(t))) + K_{21}(1 + K_{11})^{-1}(t)\partial_{t}\phi(t). \tag{5.10}
\]

Here
\[
\|(K_{21}(1 + K_{11})^{-1}K_{12} - K_{22})(t)\nabla_{\psi} L_{H}\|_{H^{1/2}} \\
\leq C \|(K_{21}(1 + K_{11})^{-1}K_{12} - K_{22})(t)\nabla_{\psi} L_{H}\|_{H^{1}} \\
\leq C \rho_{0}\theta(\phi(t), \psi(t)) \|\nabla_{\psi} L_{H}\|_{H^{1/2}} \\
\leq C(1 + \|\psi(t)\|_{H^{1/2}(S^{1})} + \|\partial_{s}\phi(t)\|_{L^{2}(S^{1})}) \|\psi(t)\|_{H^{1/2}(S^{1})} + \|\psi(t)\|_{L^{p+1}(S^{1})}. \tag{5.11}
\]

The second term in (5.10) is estimated as in (3.40) and (3.41) and the last term is estimated as
\[
\|K_{21}(1 + K_{11})^{-1}\partial_{t}\phi(t)\|_{H^{1/2}} \leq C \rho_{0}\theta \|\partial_{t}\phi(t)\|_{H^{1}} \leq C \|\partial_{t}\phi(t)\|_{H^{1}}.
\]

Therefore, we obtain, by the Hölder’s inequality and the fact that \( \int_{-\infty}^{+\infty} \|\partial_{t}\phi(t)\|_{H^{1}} dt \leq C \) which follows as in (3.15) and the equivalence of \( G^{K} \) and the standard metric on \( \mathcal{G}^{1,1/2}(S^{1}, N) \),
\[
\|\psi(t)\|_{H^{1/2}(S^{1})} \leq C \sup_{\eta \in \mathbb{R}} \int_{\eta}^{\eta+1} (1 + \|\psi(\tau)\|_{H^{1/2}(S^{1})}) + \|\partial_{s}\phi(\tau)\|_{L^{2}(S^{1})}) \|\psi(\tau)\|_{H^{1/2}(S^{1})} \\
+ \|\psi(\tau)\|_{L^{p+1}(S^{1})} + \|\partial_{s}\phi(\tau)\|_{H^{1}(S^{1})}) d\tau \\
\leq C \sup_{\eta \in \mathbb{R}} \left( 1 + \left( \int_{\eta}^{\eta+1} \|\psi(\tau)\|_{H^{1/2}(S^{1})}^{2} d\tau \right)^{1/2} + \\
+ \left( \int_{\eta}^{\eta+1} \|\partial_{s}\phi(\tau)\|_{L^{2}(S^{1})}^{2} d\tau \right)^{1/2} \left( \int_{\eta}^{\eta+1} \|\psi(\tau)\|_{H^{1/2}(S^{1})}^{2} d\tau \right)^{1/2} + \\
+ \left( \int_{\eta}^{\eta+1} \|\psi(\tau)\|_{L^{p+1}(S^{1})}^{p+1} d\tau \right)^{1/p} + \left( \int_{\eta}^{\eta+1} \|\partial_{s}\phi(\tau)\|_{H^{1}(S^{1})}^{2} d\tau \right)^{1/2} \right). \tag{5.12}
\]

By (5.4), (5.5) and (5.6) we have,
\[
\sup_{t \in \mathbb{R}} \|\psi(t)\|_{H^{1/2}(S^{1})} \leq C(C_{0}) \tag{5.13}
\]

This gives the estimate for \( \psi(t) \).

To estimate \( \sup_{t \in \mathbb{R}} \|\partial_{s}\phi(t)\|_{L^{2}(S^{1})} \), we first notice that
\[
\left\{ (1 + K)^{-\frac{\partial_{t}}{\partial_{x}}}, \left( \begin{array}{c} 0 \\ \psi(t) \end{array} \right) \right\}_{T(\phi(t), \psi(t)) \mathcal{G}^{1,1/2}} = -\int_{S^{1}} \langle \psi(t), D_{\phi(t)} \psi(t) \rangle ds \\
+ \int_{S^{1}} \langle \nabla_{\psi} H(s, \phi(t), \psi(t)), \psi(t) \rangle ds.
\]
But if we define $\tilde{K}$ by $(1 + K)^{-1} = 1 + \tilde{K}$, we have that
\[
\left(\begin{pmatrix} 1 + K \end{pmatrix}^{-1} \frac{\partial \ell}{\partial t}, \begin{pmatrix} 0 \\ \psi(t) \end{pmatrix} \right)_{T(\phi(t), \psi(t))}^{1,1/2} = \left( \partial_t \psi(t), \psi(t) \right)_{H^{1/2}(S^1)} + \left( \tilde{K}_2 \partial_t \psi(t) + \tilde{K}_2 \partial_t \psi(t), \psi(t) \right)_{H^{1/2}(S^1)}.
\]

Therefore, by (1.2), we have
\[
(\partial_t \psi(t), \psi(t))_{H^{1/2}(S^1)} + (\tilde{K}_2 \partial_t \psi(t) + \tilde{K}_2 \partial_t \psi(t), \psi(t))_{H^{1/2}(S^1)} + 2\mathcal{L}_H(\phi(t), \psi(t)) = \int_{S^1} |\partial_s \phi(t)|^2 \, ds - 2 \int_{S^1} H(s, \phi(t), \psi(t)) \, ds + \int_{S^1} \langle \nabla \psi H(s, \phi(t), \psi(t)), \psi(t) \rangle \, ds \\
\geq \int_{S^1} |\partial_s \phi(t)|^2 \, ds + C_2 \int_{S^1} |\psi(t)|^{p+1} \, ds - C. \tag{5.14}
\]

Combining (5.14) with the assumption $\sup_{t \in \mathbb{R}} |\mathcal{L}_H(\phi(t), \psi(t))| = C_0 < +\infty$, we have
\[
\int_{S^1} |\partial_s \phi(t)|^2 \, ds + \int_{S^1} |\psi(t)|^{p+1} \, ds \leq 2C_0 + \frac{C}{2} \sup_{t \in [0,T]} \mathcal{L}_H(\phi(t), \psi(t)) \left( \|\partial_t \psi(t)\|_{H^{1/2}(S^1)} + \rho_0 \|\partial_t \phi(t)\|_{H^1(S^1)} \right). \tag{5.15}
\]

Which is the substitute of (3.21) with an extra term that is a factor of $\rho_0$. Hence, from (5.13), we get
\[
\|\partial_s \phi(t)\|_{L^2(S^1)}^2 \leq C \left( 1 + \|\partial_t \psi(t)\|_{H^{1/2}(S^1)} + \rho_0 \|\partial_t \phi(t)\|_{H^1(S^1)} \right). \tag{5.16}
\]

On the other hand, we have
\[
\partial_t \psi(t) = -K_2(t) \nabla \psi \mathcal{L}_H(\phi(t), \psi(t)) - (1 + K_2(t)) \nabla \psi \mathcal{L}_H(\phi(t), \psi(t)) \tag{5.17}
\]
and
\[
\|\partial_t \psi(t)\|_{H^{1/2}(S^1)} \leq C \left( \rho_0 \|\nabla \psi \mathcal{L}_H(\phi(t), \psi(t))\|_{H^1} + \|\psi(t)\|_{H^{1/2}(S^1)} \right) \\
+ \|\partial_t \psi(t)\|_{H^{1/2}(S^1)} \leq C \left( \rho_0 \|\nabla \psi \mathcal{L}_H(\phi(t), \psi(t))\|_{H^1} + \|\psi(t)\|_{H^{1/2}(S^1)} \right) \\
+ \|\partial_t \phi(t)\|_{L^2(S^1)} \|\psi(t)\|_{H^{1/2}(S^1)} + \|\psi(t)\|_{L^{p+1}(S^1)}^p. \tag{5.18}
\]

Similarly, by $\partial_t \phi = -(1 + K_{11}) \nabla \phi \mathcal{L}_H - K_{12} \nabla \psi \mathcal{L}_H$, we have
\[
\|\partial_t \phi(t)\|_{H^1} \leq C \left( \|\nabla \phi \mathcal{L}_H(\phi(t), \psi(t))\|_{H^1(S^1)} + \rho_0 \|\nabla \psi \mathcal{L}_H(\phi(t), \psi(t))\|_{H^{1/2}(S^1)} \right). \tag{5.19}
\]

Here, $\|\nabla \mathcal{L}_H(\phi(t), \psi(t))\|_{H^1}$ is estimated as
\[
\|\nabla \mathcal{L}_H(\phi(t), \psi(t))\|_{H^1} \leq \left( (-\Delta + 1)^{-1} \nabla_s \partial_s \phi \right)_{H^1} + \left( (-\Delta + 1)^{-1} R(\phi) \langle \psi, \partial_s \phi \cdot \psi \rangle \right)_{H^1} \\
\leq C \left( \|\partial_s \phi(t)\|_{L^2} + \|\partial_s \phi(t)\|_{L^2}^2 + \|\psi(t)\|_{H^{1/2}}^2 \|\partial_s \phi(t)\|_{L^2} + \|\psi(t)\|_{H^{1/2}}^{q+1} + 1 \right),
\]
where we have used (1.6) in the last step and
\[
\|\nabla \mathcal{L}_H(\phi(t), \psi(t))\|_{H^{1/2}(S^1)} \leq C \left( \|\psi(t)\|_{H^{1/2}} + \|\psi(t)\|_{H^{1/2}}^p \right).
\]
Combining the last two inequalities with (5.13) and (5.16–17), we have
\[ \| \partial_s \phi(t) \|_{L^2}^2 \leq C \left( 1 + \| \partial_s \phi(t) \|_{L^2}^2 + \rho_0 \| \partial_s \phi(t) \|_{L^2}^2 \right). \]

Thus, initially choosing \( \rho_0 \) small such that \( C \rho_0 < \frac{1}{2} \), we have a uniform bound
\[ \sup_{t \in \mathbb{R}} \| \partial_s \phi(t) \|_{L^2} \leq C(C_0). \]

This completes the proof. \( \square \)

In the next proposition, we prove the relative compactness of the flow lines. The following is an analogue of Proposition 3.5. Note that, due to the presence of the perturbed term, the proof of Proposition 3.3 cannot be applied directly in the present case and also the proof of Proposition 3.5 since it is based on the regularity result of Proposition 3.3. In fact, with a little more work, it can be possible to prove such a regularity statement as in Proposition 3.3 for the present case. The proof will be outlined after the proof of Lemma 8.4, see Remark 8.1. In the next, however, we give an alternative argument which is based on an idea of \([1]\).

**Proposition 5.3** Let \( 0 < \rho_0 < \frac{1}{2 \Sigma_{ij}} \), then for every \( K \in \mathbb{R}^2_{\theta, \rho_0} \) and \( x_+, x_- \in \text{crit}(\mathcal{L}_H) \), the moduli space \( \hat{\mathcal{M}}(x_+, x_-) \) is relatively compact.

The proof consists again of several steps.

**Lemma 5.3** There exists \( C = C(x_-, x_+) > 0 \) such that for every \( \ell \in \mathcal{M}(x_+, x_-) \), the length \( L(\ell) \) of \( \ell \) is bounded by \( C \). That is
\[ L(\ell) = \int_{-\infty}^{+\infty} \| \ell'(t) \|_{G^k} \, dt \leq C. \]

**Proof** The proof is similar to the one in \([1]\), but for the sake of completeness, we provide it here. Starting from the fact that \( \mathcal{L}_H \) is Morse and satisfies the (PS) condition, we have the existence of finitely many critical points \( x_1, \ldots, x_k \) with energy between \( \mathcal{L}_H(x_+) \) and \( \mathcal{L}_H(x_-) \). We let \( J = [\mathcal{L}_H(x_-), \mathcal{L}_H(x_+)] \). Using the invertibility of \( d^2 \mathcal{L}_H(x_i) \), we have the existence of \( r_0 > 0 \) small enough and \( c \geq 1 \), such that for \( x \in \bigcup_{i=1}^{k} B_{r_0}(x_i) \),
\[ \| d^2 \mathcal{L}_H(x) \| \leq c \text{ and } \| d^2 \mathcal{L}_H(x)^{-1} \| \leq c. \]

Using a local frame around each \( x_i \) we can assume that \( \mathcal{L}_H \) is defined on a small ball centered at \( x_i \) on a Hilbert space. Hence, we can apply the inverse function theorem to have that the map \( (d^2 \mathcal{L}_H)^{-1} \) is well defined on a neighborhood of \( B_{r_0}(0) \). Hence again by the inverse function theorem one has that
\[ \| \nabla_{G^k} \mathcal{L}_H(x) \| \geq \frac{1}{c} \| x - x_i \| \text{ for } x \in B_{r_0/2}(x_i). \] (5.20)

Using a Taylor’s expansion for \( \mathcal{L}_H \), we get for \( x \in B_{r_0}(x_i) \),
\[ |\mathcal{L}_H(x) - \mathcal{L}_H(x_i)| \leq \| \nabla_{G^k} \mathcal{L}_H(x) \| \| x - x_i \| + \frac{1}{2} \| x - x_i \|^2 \] (5.21)

Combining (5.20) and (5.21), one has for \( c_1 = c + \frac{c^2}{2} \) and \( x \in B_{r_0/2}(x_i) \)
\[ |\mathcal{L}_H(x) - \mathcal{L}_H(x_i)| \leq c_1 \| \nabla_{G^k} \mathcal{L}_H(x) \|^2. \] (5.22)
The second estimate on the gradient that we will need comes from the (PS) condition. Indeed, there exists $\delta > 0$ such that for $x \in \mathcal{L}_{H}^{-1}(J) \setminus \bigcup_{i=1}^{k} B_{r_{0}/2}(x_{i})$, we have
\[\|\nabla_{G} K \mathcal{L}_{H}(x)\| \geq \delta. \tag{5.23}\]

We define now $\mathcal{T}: J \to \mathbb{R}$ by
\[\mathcal{T}(w) = \inf\{\|\nabla_{G} K \mathcal{L}_{H}(x)\|, \mathcal{L}_{H}(x) = w\}.\]

Then using (5.22) and (5.23), we have that
\[\mathcal{T}(w) \geq \min\left\{\frac{1}{\sqrt{c_{1}}} \min_{i=1,\ldots,k} \left|w - \mathcal{L}_{H}(x_{i})\right|^{1/2}, \delta\right\}\]

Thus,
\[\frac{1}{\mathcal{T}(w)} \leq \max\left\{\sqrt{c_{1}}, \max_{i=1,\ldots,k} \frac{1}{\left|w - \mathcal{L}_{H}(x_{i})\right|^{1/2}}, \frac{1}{\delta}\right\}. \tag{5.24}\]

We deduce from this, that $\frac{1}{\mathcal{T}} \in L^{1}(J)$ and
\[\int_{J} \frac{1}{\mathcal{T}(w)} dw \leq 4k\sqrt{c_{1}} \left|J\right|^{1/2} + \frac{|J|}{\delta} \leq C \left|\mathcal{L}_{H}(x_{+}) - \mathcal{L}_{H}(x_{-})\right|^{1/2},\]

where $C = 4k\sqrt{c_{1}} + \frac{1}{\delta}|J|^{1/2}$. Now by definition, we have that
\[\mathcal{T}(\mathcal{L}_{H}(\ell(t))) \leq \|\nabla_{G} K \mathcal{L}_{H}(\ell(t))\|,\]

Hence if $\ell \in \mathcal{M}(x_{+}, x_{-})$, we have
\[\|\ell'(t)\| = -\frac{\langle \nabla_{G} K \mathcal{L}_{H}(\ell(t)), \ell'(t) \rangle}{\|\nabla_{G} K \mathcal{L}_{H}(\ell(t))\|} \leq -\frac{\langle \nabla_{G} K \mathcal{L}_{H}(\ell(t)), \ell'(t) \rangle}{\mathcal{T}(\mathcal{L}_{H}(\ell(t)))}.\]

Using the substitution $w = \mathcal{L}_{H}(\ell(t))$, one has
\[L(\ell) = \int_{\mathbb{R}} \|\ell'(t)\| dt \leq \int_{J} \frac{1}{\mathcal{T}(w)} dw \leq C \left|\mathcal{L}_{H}(x_{+}) - \mathcal{L}_{H}(x_{-})\right|^{1/2}.\]

\[\square\]

**Lemma 5.4** The set $\{\ell(t); \ell \in \mathcal{M}(x_{+}, x_{-}), t \in \mathbb{R}\}$ is relatively compact in $\mathcal{T}^{1,1/2}(S^{1}, N)$.

**Proof** Let $\ell = (\phi, \psi) \in \mathcal{M}(x_{+}, x_{-})$. We consider first the $\psi$ part. As in the proof of Lemma 5.2 (5.8), (5.9), if we consider $r(t)$ defined by
\[r(t) = (K_{21}(1 + K_{11})^{-1} K_{12} - K_{22}) \nabla_{\psi} \mathcal{L}_{H} + (1 + |D|)^{-1}(\lambda \psi(t) + A(\partial_{s} \phi(t), \partial_{s} \cdot \psi(t)) + \nabla_{\psi} H(s, \phi(t), \psi(t))) + K_{21}(1 + K_{11})^{-1}(t) \partial_{t} \phi(t) = r_{1}(t) + r_{2}(t) \tag{5.25}\]

where $r_{1}(t) = (K_{21}(1 + K_{11})^{-1} K_{12} - K_{22}) \nabla_{\psi} \mathcal{L}_{H}$, we have
\[\psi(t) = \int_{\mathbb{R}} G_{\lambda}(t - \tau) r(s) d\tau = \psi_{1}(t) + \psi_{2}(t).\]
Recall that
\[
\frac{d\ell}{dt} = -(1 + K(\ell))\nabla_{1,1/2}L_H(\ell).
\]
Therefore
\[
\frac{d\phi}{dt} = -\nabla_\phi L_H - K_{11}\nabla_\phi L_H - K_{12}\nabla_\psi L_H.
\]

By Lemma 5.2, this implies that \(\sup_{t \in \mathbb{R}} \| t^{-\frac{1}{2}} (t) \|_{H^1(S^1)} \) < \(+\infty\) and
\[
\sup_{t \in \mathbb{R}} \| K_{21} (1 + K_{11})^{-1} (t) \partial t \phi (t) \|_{H^1(S^1)} < \(+\infty\).
\]

Combining this with (3.49), (3.50), arguing as in the proof of Proposition 3.3, we have
\[
\sup_{t \in \mathbb{R}} \| \Psi_2 (t) \|_{C^{0,\alpha}(S^1)} < \(+\infty\) for some \(0 < \alpha < 1\).
\]
Thus the estimate (3.49) holds with \(r = 2\) and the estimate (3.50) holds with \(r = 2\). Then we have \(\sup_{t \in \mathbb{R}} \| \Psi_2 (t) \|_{H^1(S^1)} < \(+\infty\) and since \(\Psi_2 (t) = \int_{\mathbb{R}} G_2 (\tau) r_2 (t - \tau) d\tau\), as in Propositions 3.2 and 3.3, this implies a uniform bound \(\sup_{t \in \mathbb{R}} \| \Psi_2 (t) \|_{H^1(S^1)} < \(+\infty\).

For \(\psi_1\), we have
\[
\| \psi_1 (t) \|_{H^1} \leq \int_{\mathbb{R}} \| G_2 (\tau) \|_{H^1(S^1)} \| K_{21} (1 + K_{11})^{-1} K_{12} \|_{\text{lip}(H^{1/2}, H^1)} \|
\]
\[
\nabla_\phi L_H (\phi, \psi) \|_{H^1(S^1)} \| \nabla_\phi L_H (\phi, \psi) \|_{H^1(S^1)} \|
\]
\[
\leq C \rho_0 \int_{\mathbb{R}} e^{-k|\tau|} \left(1 + \| \psi (t - \tau) \|_{H^{1/2}(S^1)} + \| \partial_t \phi (t - \tau) \|_{L^2(S^1)} \| \psi (t - \tau) \|_{H^{1/2}(S^1)} \right)
\]
\[
+ \| \psi (t - \tau) \|_{H^{1/2}(S^1)} \|
\]
\[
(5.26)
\]
and using Lemma 5.2 we have \(\| \psi_1 (t) \|_{H^1(S^1)} \leq C\). From the estimates of \(\psi_1\) and \(\psi_2\), we have \(\sup_{t \in \mathbb{R}} \| \psi (t) \|_{H^1(S^1)} < \(+\infty\).

Now we want to show that the \(\phi\) component is relatively compact in \(H^1(S^1)\). We write
\[
\frac{d\phi}{dt} = -\nabla_\phi L_H - K_{11}\nabla_\phi L_H - K_{12}\nabla_\psi L_H
\]
\[
= -\phi + U(t)
\]
where
\[
U(t) = (-\Delta + 1)^{-1} \left( \phi + \Gamma (\phi (t), \partial_s \phi (t), \partial_s \psi (t)) - \frac{1}{2} R (\phi, \partial_s \phi, \psi) \right)
\]
\[
- K_{11}\nabla_\phi L_H (\phi, \psi) - K_{12}\nabla_\psi L_H (\phi, \psi)
\]

First recall that \(\phi (t), \psi (t) \in H^1(S^1) \times H^{1/2}(S^1)\) are uniformly bounded by Lemma 5.2. Now consider the operator
\[
U_1 (\phi, \psi) = (-\Delta + 1)^{-1} \left( \phi + \Gamma (\phi, \partial_s \phi, \partial_s \phi) - \frac{1}{2} R (\phi, \partial_s \phi, \psi) + \nabla_\phi H (s, \phi, \psi) \right)
\]
Clearly, \(U_1\) is a compact operator from \(H^1(S^1) \times H^{1/2}(S^1)\) to \(H^1(S^1)\) (cf., the proof of Proposition 3.3, (3.55)–(3.57)) and since, \(\sup_{t \in \mathbb{R}} (\| \phi \|_{H^1(S^1)} + \| \psi \|_{H^{1/2}(S^1)}) \leq C < \(+\infty\),

we have
\[
\{ U_1 (\ell (t)) : \ell \in \mathcal{M} (x_+, x_-), t \in \mathbb{R} \}
\]
is relatively compact in \(H^1(S^1)\) since it is the image of a bounded set via the operator \(U_1\).
Also, by the construction of the class $\mathbb{K}^2_{\delta_0}$, $K_{11}, K_{12}$ are compact and we have uniform boundedness of $\ell \in \mathcal{M}(x_+, x_-)$. Therefore,

$$\{K_{11} \nabla_{\phi} \mathcal{L}_H(\ell(t)) - K_{12} \nabla_{\psi} \mathcal{L}_H(\ell(t)) : \ell \in \mathcal{M}(x_+, x_-), t \in \mathbb{R}\}$$

is relatively compact in $H^1(S^1)$. Combining these, we conclude that $\{U(t) : \ell \in \mathcal{M}(x_-, x_+), t \in \mathbb{R}\}$ is relatively compact in $H^1(S^1)$. Now if $G_1$ is the fundamental solution of the operator $T$ defined by $T(\phi) = \frac{d\phi}{dt} + \phi$, we have

$$\phi(t) = G_1 * U(t).$$

Then by [1, Proposition 3.2], we have that the set $\{\phi(t) : \phi(t) = G_1 * U(t), \ell \in \mathcal{M}(x_-, x_+), t \in \mathbb{R}\}$ is relatively compact in $H^1(S^1)$. Since the embedding $H^1(S^1) \subset H^{1/2}(S^1)$ is compact, $\psi$-component of $\mathcal{M}(x_-, x_+)$ is also relatively compact in $H^{1/2}(S^1)$ and we conclude that $\{\ell(t), \ell \in \mathcal{M}(x_+, x_-), t \in \mathbb{R}\}$ is relatively compact in $\mathcal{F}^{1,1/2}(S^1, N)$. This completes the proof.

\[\Box\]

**Proof of Proposition 5.3** We consider now the set $\tilde{\mathcal{M}}(x_+, x_-)$ consisting of elements $\ell \in \mathcal{M}(x_+, x_-)$ that we parametrize by constant velocity. Notice that we have a map $\Pi : \mathcal{M}(x_+, x_-) \to \tilde{\mathcal{M}}(x_+, x_-)$. That is our new set of curves consists of maps $\tilde{\ell} : (0, 1) \to \mathcal{F}^{1,1/2}$ such that $\tilde{\ell}$ is the re-parametrization by constant velocity of an element $\ell \in \mathcal{M}(x_+, x_-)$. Observe that we can also include the points $\{0, 1\}$ by adding their respective images $x_+$ and $x_-$. So we have $\tilde{\mathcal{M}}(x_+, x_-) \subset C([0, 1], \mathcal{F}^{1,1/2})$.

Notice now that $\tilde{\ell}([0, 1]) = \ell(\mathbb{R}) \cup \{x_+, x_-\}$, therefore

$$\{\tilde{\ell}(t), \tilde{\ell} \in \tilde{\mathcal{M}}(x_+, x_-), t \in [0, 1]\} = \{\ell(t), \ell \in \mathcal{M}(x_+, x_-), t \in \mathbb{R}\} \cup \{x_+, x_-\}.$$

Hence, the set $\{\tilde{\ell}(t), \tilde{\ell} \in \tilde{\mathcal{M}}(x_+, x_-), t \in [0, 1]\}$ is relatively compact. Also, since the maps in this set are parametrized by constant velocity and the length of curves in $\mathcal{M}(x_+, x_-)$ is uniformly bounded, then the maps $\tilde{\mathcal{M}}(x_+, x_-) \subset C([0, 1], \mathcal{F}^{1,1/2})$ are uniformly Lipschitz. From this and Lemma 5.4, it follows from the Arzelá-Ascoli theorem that $\tilde{\mathcal{M}}(x_+, x_-)$ is a relatively compact subset of $C([0, 1], \mathcal{F}^{1,1/2})$. In order to continue, we only need to notice that the map $\Pi : \mathcal{M}(x_+, x_-) \to \tilde{\mathcal{M}}(x_+, x_-)$ induces a homeomorphism $\tilde{\Pi} : \mathcal{M}(x_+, x_-)/\mathbb{R} \to \tilde{\mathcal{M}}(x_+, x_-)$.

In this part we consider two critical points of $\mathcal{L}_H, x_-, x_+ \in \text{crit}(\mathcal{L}_H)$ such that $\mu_H(x_-) - \mu_H(x_+) \leq 2$. For $0 < \rho_0 < \delta$ $(\delta > 0$ is as in Lemma 5.2), we define the map

$$\tilde{\mathcal{F}}_{x_-, x_+} : W^{1,2}_{x_-, x_+}(\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N)) \times \mathbb{R}_{1, \rho_0}^2 \to L^2(\mathbb{R}, T\mathcal{F}^{1,1/2}(S^1, N))$$

by

$$\tilde{\mathcal{F}}_{x_-, x_+}(\ell, K) := \frac{d\ell}{dt} + (1 + K(\ell(t))) \nabla_{1,1/2} \mathcal{L}_H(\ell(t)).$$

We set $\tilde{\mathcal{M}}(x_-, x_+) = \tilde{\mathcal{F}}_{x_-, x_+}^{-1}(0)$.

**Lemma 5.5** For $(\ell, K) \in \tilde{\mathcal{M}}(x_-, x_+)$ the map

$$d_\ell \tilde{\mathcal{F}}_{x_-, x_+} : T_\ell W^{1,2}_{x_-, x_+}(\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N)) \to L^2(\mathbb{R}, e^{\ell} T\mathcal{F}^{1,1/2}(S^1, N))$$

is a Fredholm operator with index$(d_\ell \tilde{\mathcal{F}}_{x_-, x_+}) = \mu_H(x_-) - \mu_H(x_+)$. 

\[\square\]
Proof First of all, since \( \theta = 0 \) in the neighborhood of \( x_+ \) and \( x_- \), we can assume as in the proof of Proposition 3.4, that \( \ell \) is constant near \( t = \pm \infty \) and that \( \ell(t) = x_- \) when \( t \leq -T \) and \( \ell(t) = x_+ \) when \( t \geq T \). Hence, using the same notations, we just need to prove the index formula for the operator

\[
P^{-1} \circ d_{\ell} \tilde{\mathcal{F}}_{x_-} (\ell, K) \circ P : W^{1,2}(\mathbb{R}, T_{x_-} T^1 T^{1,1/2}(S^1, N)) \to L^2(\mathbb{R}, T_{x_-} T^1 T^{1,1/2}(S^1, N)).
\]

(5.27)

Again we write

\[
P^{-1} \circ d_{\ell} \tilde{\mathcal{F}}_{x_-} (\ell, K) \circ P = \nabla \ell + P^{-1} \circ d\nabla G K \mathcal{L}_H(\ell) \circ P.
\]

We set \( A(t) := P^{-1}_t \circ d\nabla G K \mathcal{L}_H(\ell(t)) \circ P_t \). We observe that

\[
A(-\infty) = A(-T) = P^{-1}_{-T} \circ d\nabla_{1,1/2} \mathcal{L}_H(\ell(-T)) \circ P_{-T} - d\nabla_{1,1/2} \mathcal{L}_H(x_-)
\]

and

\[
A(+\infty) = A(T) = P^{-1}_T \circ d\nabla_{1,1/2} \mathcal{L}_H(\ell(T)) \circ P_T - P^{-1}_T \circ d\nabla_{1,1/2} \mathcal{L}_H(x_+) \circ P_T
\]

are invertible hyperbolic operators. We claim that

\[
A(t) = d\nabla_{1,1/2} \mathcal{L}_H(x_-) + k(t),
\]

where the \((2,2)\)-component \( k_{22}(t) \) of

\[
k(t) = \begin{pmatrix} k_{11}(t) & k_{12}(t) \\ k_{21}(t) & k_{22}(t) \end{pmatrix} : T_{x_-} L^1 L^{1/2}(S^1, N) = H^1(S^1, \phi^* T N) \times H^1(S^1, S(S^1) \otimes \phi^* T N)
\]

\[
\to T_{x_-} L^1 L^{1/2}(S^1, N) = H^1(S^1, \phi^* T N) \times H^1(S^1, S(S^1) \otimes \phi^* T N)
\]

is compact for all \( t \in \mathbb{R} \). Indeed, we first write

\[
d_{\ell} \nabla G K \mathcal{L}_H(\ell)[v] = (1 + K(\ell)) d_{\ell} \nabla_{1,1/2} \mathcal{L}_H(\ell)[v] + d_{\ell} K[v] \nabla_{1,1/2} \mathcal{L}_H(\ell)
\]

Thus, as in the proof of Lemma 3.4, we can write

\[
d_{\ell} \nabla G K \mathcal{L}_H(\ell) = (1 + K(\ell)) \left( \left( -\nabla^2_s + 1 \right)^{-1} \left( -\nabla^2_s \right) \right) \left( O \right) (1 + |D_{\phi_-}|)^{-1} D_{\phi_-} + \mathcal{K}(t)
\]

\[
+ d_{\ell} K[\cdot] \nabla_{1,1/2} \mathcal{L}_H(\ell),
\]

(5.28)

where \( \mathcal{K}(t) : T_{\ell(t)} L^{1,1/2}(S^1, N) \to T_{\ell(t)} L^{1,1/2}(S^1, N) \) is compact.

By the construction of the element of \( \mathbb{K}^2_{\phi_-} \), for \( K \in \mathbb{K}^2_{\phi_-} \),

\[
K(\ell(t)) \left( \left( -\nabla^2_s + 1 \right)^{-1} \left( -\nabla^2_s \right) \right) \left( O \right) (1 + |D_{\phi_-}|)^{-1} D_{\phi_-}
\]

is compact for all \( t \). On the other hand, the compactness of the embedding \( H^1(S^1) \subset H^{1/2}(S^1) \) implies that the \((2,2)\)-component of \( d_{\ell} K[\cdot] \nabla_{1,1/2} \mathcal{L}_H(\ell) \) is compact. Thus, we have the assertion as stated above. Under the condition, we claim that the operator (5.27) is Fredholm with index \( \mu_H(x_-) - \mu_H(x_+) \). Note that the present case is different from the one of Lemma 3.4, since \( k(t) \) may not be compact (we only require the compactness of the \((2,2)\)-component of \( k(t) \)). To prove the Fredholm property, we argue as follows.

We write \( A(t) = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix} + \begin{pmatrix} k_{11}(t) & k_{12}(t) \\ k_{21}(t) & k_{22}(t) \end{pmatrix} \), where \( L_1 = \left( -\nabla^2_s + 1 \right)^{-1} \left( -\nabla^2_s \right) \) and
$L_2 = (1 + |D_{\phi_-}|)^{-1}D_{\phi_-}$. We first prove that $\nabla_t + A(t)$ has closed range with finite dimensional cokernel. To see this, we have

$$(\nabla_t + A(t)) \left( \begin{array}{c} \xi \\ 0 \end{array} \right) = \left( \begin{array}{c} \nabla_t \xi + L_1 \xi + k_{11}(t) \xi \\ k_{21}(t) \xi \end{array} \right).$$

Note that $L_1 + k_{11}(\pm \infty)$ are essentially positive hyperbolic operator and in such a case, $\nabla_t + L_1 + k_{11} : W^{1,2}(\mathbb{R}, H^1(S^1, \phi^*_+ TN)) \to L^2(\mathbb{R}, \phi^*_+ TN)$ is Fredholm as was proved, for example in [2, Theorem 5.1]. In particular, the codimension of the range of $\nabla_t + L_1 + k_{11}$ is finite. On the other hand,

$$(\nabla_t + A(t)) \left( \begin{array}{c} 0 \\ X \end{array} \right) = \left( \begin{array}{c} k_{12}(t)X \\ \nabla_tX + L_2X + k_{22}(t)X \end{array} \right).$$

As before, $L_2 + k_{22}(\pm \infty)$ is hyperbolic. In addition, $k_{22}(t)$ is compact for all $t$ and its limit $k_{22}(+\infty)$ is also compact since the limit of compact operators is compact. (The last fact also can be directly seen by the same reasoning as in the proof of Lemma 3.4). Thus, [2, Theorem B] implies that $\nabla_t + L_2 + k_{22} : W^{1,2}(\mathbb{R}, H^{1/2}(S^1, S(S^1) \otimes \phi^*_+ TN)) \to L^2(\mathbb{R}, H^{1/2}(S^1, S(S^1) \otimes \phi^*_+ TN))$ which is written as $\nabla_t + L_2 + k_{22}(+\infty) + k_{22}(t) - k_{22}(+\infty)$ is Fredholm. In particular, the codimension of the range of $\nabla_t + L_2 + k_{22}$ is finite. Combining these, the range of $\nabla_t + A(t)$ is closed with finite dimensional cokernel. Because the adjoint $-\nabla_t + A(t)^*$ is a similar form, $-\nabla_t + A(t)^*$ has closed range with finite dimensional cokernel. But this means that $\nabla_t + A(t)$ has a finite dimensional kernel. This proves that $\nabla_t + A(t)$ is Fredholm. To prove the index formula, for $0 \leq \epsilon \leq 1$ we consider the operator $\nabla_t + A_\epsilon(t)$, where

$$A_\epsilon(t) = \left( \begin{array}{cc} L_1 & 0 \\ 0 & L_2 \end{array} \right) + \left( \begin{array}{cc} k_{11}(t) & \epsilon k_{12}(t) \\ \epsilon k_{21}(t) & k_{22}(t) \end{array} \right).$$

By the same reasoning as above,

$$\nabla_t + A_\epsilon : W^{1,2}(\mathbb{R}, T_{\epsilon\mathbb{R}} T^*\mathbb{R}^{1,1/2}(S^1, N)) \to L^2(\mathbb{R}, T_{\epsilon\mathbb{R}} T^*\mathbb{R}^{1,1/2}(S^1, N))$$

is Fredholm for $0 \leq \epsilon \leq 1$. Since the Fredholm index is invariant under continuous deformations, we have

$$\text{index}(\nabla_t + A) = \text{index}(\nabla_t + A_0) = \text{index}(\nabla_t + L_1 + k_{11}) + \text{index}(\nabla_t + L_2 + k_{22}).$$

Since $L_1 + k_{11}(\pm \infty)$ are essentially positive, the first index in the above formula is given by the spectral flow

$$\text{index}(\nabla_t + L_1 + k_{11}) = \text{sf}[L_1 + k_{11}(t)]_{-\infty \leq t \leq +\infty}.$$

The second index is also given by the spectral flow since $k_{22}(t)$ is compact for all $t$, see [2]:

$$\text{index}(\nabla_t + L_2 + k_{22}) = \text{sf}[L_2 + k_{22}(t)]_{-\infty \leq t \leq +\infty}.$$

Combining these, we have

$$\text{index}(\nabla_t + A) = \text{sf}[L_1 + k_{11}(t)]_{-\infty \leq t \leq +\infty} + \text{sf}[L_2 + k_{22}(t)]_{-\infty \leq t \leq +\infty} = \text{sf}[A_0(t)]_{-\infty \leq t \leq +\infty}.$$

Since the spectral flow is a homotopy invariant, we also have

$$\text{sf}[A_0(t)]_{-\infty \leq t \leq +\infty} = \text{sf}[A(t)]_{-\infty \leq t \leq +\infty}.$$
Since \( sf(A(t))_{-\infty \leq t \leq +\infty} = \mu_H(x_-) - \mu_H(x_+) \) as in (3.110), we finally have the index formula as stated in the lemma. 

**Lemma 5.6** Let \( K \in \mathbb{R}^2_{0, \rho_0} \). The map 

\[
d^\ell\tilde{\mathcal{F}}_{x_-, x_+} : T_{\ell} W^1_{x_-, x_+}(\mathbb{R}_+^N, \mathcal{F}^{1,1/2}(S^1, N)) \to L^2(\mathbb{R}_+, \ell^* T\mathcal{F}^{1,1/2}(S^1, N))
\]

is onto for any \( \ell \) such that \( (\ell, K) \in \tilde{M}_{x_-, x_+} \) if and only if the unstable manifold \( W^u_{-\infty} \mathcal{L}_H(x_-) \) and the stable manifold \( W^s_{\infty} \mathcal{L}_H(x_+) \) intersect transversally.

**Proof** Using the same notations as in Proposition 3.4, we consider the operator 

\[
\tilde{\mathcal{F}} = P^{-1} \circ d^\ell\tilde{\mathcal{F}}_{x_-, x_+}(\ell, K) \circ P : W^{1,2}(\mathbb{R}_+, T_{\ell} \mathcal{F}^{1,1/2}(S^1, N)) \to L^2(\mathbb{R}_+, T_{\ell} \mathcal{F}^{1,1/2}(S^1, N)).
\]

(5.29)

We have

\[
\tilde{\mathcal{F}} = P^{-1} \circ d^\ell\tilde{\mathcal{F}}_{x_-, x_+}(\ell, K) \circ P = \nabla_t + P^{-1} \circ d\nabla G_K \mathcal{L}_H(\ell) \circ P.
\]

Therefore \( \tilde{\mathcal{F}} \) is onto if and only if \( d^\ell\tilde{\mathcal{F}}_{x_-, x_+}(\ell) \) is onto. By the results from [3], this is the case if and only if \( W^u(P^{-1} \circ d\nabla G_K \mathcal{L}_H(\ell) \circ P) \) and \( W^u(P^{-1} \circ d\nabla G_K \mathcal{L}_H(\ell) \circ P) \) intersect transversally. But this holds if and only if \( W^u_{-\infty} \mathcal{L}_H(x_-) \) and \( W^s_{\infty} \mathcal{L}_H(x_+) \) intersect transversally. 

**Lemma 5.7** Let \( (\ell, K) \in \tilde{M}(x_-, x_+) \) and consider the operator 

\[
d^\ell\tilde{\mathcal{F}}^+_{x_-, x_+}(\ell, K) : T_{\ell} W^{1,2}_{x_-, x_+}(\mathbb{R}_+^N, \mathcal{F}^{1,1/2}(S^1, N)) \to L^2(\mathbb{R}^+, \ell^* T\mathcal{F}^{1,1/2}(S^1, N))
\]

defined by restricting \( d^\ell\tilde{\mathcal{F}}_{x_-, x_+}(\ell, K) \) to \( T_{\ell} W^{1,2}_{x_-, x_+}(\mathbb{R}_+^N, \mathcal{F}^{1,1/2}(S^1, N)) \). Then \( d^\ell\tilde{\mathcal{F}}^+_{x_-, x_+}(\ell, K) \) is a left inverse.

**Proof** Again by using the same trick of parallel transport, it is enough to show the property for the operator 

\[
\tilde{\mathcal{F}}^+ = P^{-1} \circ d^\ell\tilde{\mathcal{F}}^+_{x_-, x_+}(\ell, K) \circ P : W^{1,2}(\mathbb{R}_+, T_{\ell} \mathcal{F}^{1,1/2}(S^1, N)) \to L^2(\mathbb{R}^+, T_{\ell} \mathcal{F}^{1,1/2}(S^1, N)).
\]

(5.30)

Then since it is asymptotically hyperbolic, one have again from [2] that it is a left inverse. 

**Lemma 5.8** Let \( (\ell, K) \in \tilde{M}(x_+, x_-) \) and assume that there exists \( a, b \in \mathbb{R} \) with \( a < b \) such that \( \theta(\ell(t)) \neq 0 \) for all \( t \in [a, b] \). Then for any \( w \in C^2(\mathbb{R}, \ell^* T\mathcal{F}^{1,1}) \) with \( \text{supp } w \subset [a, b] \) and for \( \varepsilon > 0 \), there exists \( k \in \mathbb{R}^2_0 \) such that 

\[
\|d^\ell_{\mathcal{F}}(\tilde{\mathcal{F}}_{x_-, x_+}(\ell, K)[k] - w\|_{\ell_+ T\mathcal{F}^{1,1}} < \varepsilon.
\]

**Proof** First we notice that 

\[
d^\ell_{\mathcal{F}}(\tilde{\mathcal{F}}_{x_-, x_+}(\ell, K)[k] = k(\ell)\nabla_{1,1/2} \mathcal{L}_H(\ell)
\]

and clearly \( \ell : \mathbb{R} \to \mathcal{F}^{1,1/2}(S^1, N) \) is an embedding of class \( C_3 \) since it is a solution to the negative gradient flow equation with respect to the gradient \( \nabla G_K \mathcal{L}_H \) which is \( C_2 \) by Proposition 10.1 in the “Appendix”. We consider the \( C^2 \) curve defined as follow:

\[
\tilde{k}(t) = \frac{\langle \cdot, \nabla_{1,1/2} \mathcal{L}_H(\ell(t)) \rangle}{\|\nabla_{1,1/2} \mathcal{L}_H(\ell(t))\|} w(t) + \frac{\langle \cdot, w(t) \rangle}{\|\nabla_{1,1/2} \mathcal{L}_H(\ell(t))\|^2} \nabla_{1,1/2} \mathcal{L}_H(\ell(t))
\]

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In the proof of Lemma 5.4, we have proved that $\ell(t) \in \mathcal{F}^{1,1}(S^1, N)$ for any $t \in \mathbb{R}$. It follows from this that $\nabla_{1,1/2}L_{H}(\ell(t))$ is also in $T_{\ell(t)}\mathcal{F}^{1,1}(S^1, N)$. To see this, it suffices to prove the $\psi$-component of $\nabla_{1,1/2}L_{H}(\ell(t))$, $(1 + |D|)^{-1}(D\psi - \nabla\psi H(s, \phi, \psi))$ is in $H^1$, where $\ell = (\phi, \psi)$. By the Sobolev embedding $H^1(S^1) \subset C^0(S^1)$, (1.5) and $\ell \in H^1(S^1)$, we have $D\phi \psi - \nabla\psi H(s, \phi, \psi) \in L^2(S^1)$. Thus, $(1 + |D|)^{-1}(D\phi \psi - \nabla\psi H(s, \phi, \psi)) \in H^1$ by the elliptic regularity. Since $w(t) \in T_{\ell(t)}\mathcal{F}^{1,1}(S^1, N)$, we thus have
\[ \tilde{k}(t) \in \text{Sym}(T_{\ell(t)}\mathcal{F}^{1,1}(S^1, N), T_{\ell(t)}\mathcal{F}^{1,1}(S^1, N)) \]
and since $H$ is $C^3$, we have that $\tilde{k}$ is $C^2$ in the $t$ variable and $\tilde{\psi}, \tilde{k}$ take values in $H^1 \times H^1$.

Now since $\ell : \mathbb{R} \to \mathcal{F}^{1,1/2}(S^1, N)$ is an embedding, for a given $\delta > 0$, there exists a neighborhood $U \subset \mathcal{F}^{1,1/2}(S^1, N)$ such that $\ell((a - \delta, b + \delta)) \subset U$ and a submersion $\tau : U \to (a - \delta, b + \delta)$ such that $\tau(\ell(t)) = t$ for $t \in (a - \delta, b + \delta)$. But we already assumed that $\theta > 0$ on $\ell([a, b])$, thus taking $\delta > 0$ and $U$ smaller if necessary, we can assume that $\text{int}_U \theta > 0$. Therefore, we can consider a cut-off function $\rho \in C^2(\mathcal{F}^{1,1/2}(S^1, N), \mathbb{R})$ such that $\rho = 1$ on $\ell([a, b])$ and $\text{supp} \; \rho \subset U$. We then set
\[ k(x) = \rho(x)\tilde{k}(\tau(x)), \]
for $x \in \mathcal{F}^{1,1/2}(S^1, N)$.

We set $\tau(\tau(x)) = \tilde{\tau}(x)$ for $x \in U$ and write $\tilde{\tau}(x) = (\tilde{\phi}_x, \tilde{\psi}_x)$. We also write $x = (\phi_x, \psi_x)$ for $x \in U$. For $U$ small enough, $\|\phi_x - \tilde{\phi}_x\|_{H^1}$ is small and so is $\|\phi_x - \tilde{\phi}_x\|_{L^\infty}$ by the Sobolev embedding $H^1(S^1) \subset L^\infty(S^1)$. Thus the parallel translation $P_{\phi_x, \psi_x}(s) : T_{\tilde{\phi}_x(s)}N \to T_{\phi_x(s)}N$ is defined for all $s \in S^1$. By [20, Lemma 7.4], it induces a bounded linear operator $P_{\phi_x, \psi_x} : H^1(S^1, \tilde{\phi}_x^*TN) \to H^1(S^1, \phi_x^*TN)$ which we denote by the same symbol. We finally define
\[ \hat{k}(x) = P_{\phi_x, \psi_x}(k(x)). \]
By the construction, this $\hat{k}$ satisfies $d_{K}\mathcal{F}_{x-}x_+ (\ell, K)[\hat{k}] = w$. But, $\hat{k}$ may not belong to $K_2$. We thus approximate $\hat{k}$ by an element of $K_2$ to obtain $k \in K_2$ such that $d_{K}\mathcal{F}_{x-}x_+ (\ell, K)[k]$ is arbitrary close to $d_{K}\mathcal{F}_{x-}x_+ (\ell, K)[\hat{k}] = w$. This completes the proof of the lemma. \hfill $\square$

**Lemma 5.9** For any $(\ell, K) \in \tilde{\mathcal{M}}(x_-, x_+), d\mathcal{F}_{x-}x_+ (\ell, K) : T_{\ell}W_{x-}x_+^1(\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N)) \times K_2 \to L^2(\mathbb{R}, \ell^*T\mathcal{F}^{1,1/2}(S^1, N))$ is a left inverse.

**Proof** We fix $(\ell, K) \in \tilde{\mathcal{M}}(x_-, x_+)$. We already proved in Lemma 5.5 that $d\mathcal{F}_{x-}x_+ (\ell, K)$ is Fredholm. So it is enough to show that $d\mathcal{F}_{x-}x_+ (\ell, K) : T_{\ell}W_{x-}x_+^1(\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N)) \times K_2 \to L^2(\mathbb{R}, \ell^*T\mathcal{F}^{1,1/2}(S^1, N))$ is onto. But since we have $\text{ Ran}(d\mathcal{F}_{x-}x_+ (\ell, K)) \subset \text{ Ran}(d\mathcal{F}_{x-}x_+ (\ell, K))$, we have that codim $d\mathcal{F}_{x-}x_+ (\ell, K) < \infty$ and hence to prove our lemma we just need to show that $\text{Ran}(d\mathcal{F}_{x-}x_+ (\ell, K))$ is dense in $L^2(\mathbb{R}, \ell^*T\mathcal{F}^{1,1/2}(S^1, N))$. Again we distinguish two cases.

i) The case $\theta(\ell(t)) = 0$ for all $t$.

In that case, by construction of the space $K_2$ we have that $K(\ell(t)) = 0$ for all $t$ and hence $\nabla_{GK}L_H = \nabla_{1,1/2}L_H$. Thus the stable and unstable manifolds of $-\nabla_{GK}L_H$ coincides with the ones of $-\nabla_{1,1/2}L_H$ and by the assumption they intersect transversally. Therefore, from Lemma 5.6 we have that $d\mathcal{F}_{x-}x_+ (\ell, K)$ is onto.
ii) The case $\theta(\ell(t)) \neq 0$ for some $t$.
Let $w \in L^2(\mathbb{R}, \ell^*T^1,1/2)$ and $\varepsilon > 0$. We consider an interval $[a, b]$ such that $\theta(\ell(t)) \neq 0$. Now by using Lemma 5.7 we have the existence of $v \in T_\ell W_{x_-, x_+}^1(\mathbb{R}, T^1,1/2(S^1, N))$ such that

$$d_\ell \tilde{\mathcal{F}}_{x_-, x_+}(\ell, K)[v] = w$$

for $t \in (-\infty, a] \cup [b, +\infty)$. Then the support of $\tilde{w} = w - d_\ell \mathcal{F}_{x_+, x_-}(\ell, K)[v] \in L^2(\mathbb{R}, \ell^*T^1,1/2(S^1, N))$ is contained in $[a, b]$. Now we need to approximate $\tilde{w}$ by a function in $C^2(\mathbb{R}, \ell^*T^1,1/2)$. This can be done first by transporting the space to the point $\ell(a)$. Indeed, using the notations of the proof of Proposition 3.4, we consider the operator $P_t$ defining the parallel transport along the path $\phi(\tau)$ for $a \leq \tau \leq t$, here $\phi$ is so that $\ell(t) = (\phi(t), \psi(t))$. Then we have

$$P_t : T_{\phi(a)} N \to T_{\phi(t)} N.$$ 

Thus $P_t$ induces a bounded linear operator

$$P_t : H^1(S^1, \phi(a)^*TN) \to H^1(S^1, \phi(t)^*TN)$$

Similarly

$$P_t = 1 \otimes P_t : H^1/2(S^1, \mathbb{S}(S^1) \otimes (\phi(a)^*TN) \to H^1/2(S^1, \mathbb{S}(S^1) \otimes (\phi(t)^*TN)$$

Thus by setting $w = P_t^{-1} \tilde{w}$, we see that $w \in L^2([a, b], T_{\ell(a)}T^1,1/2)$ and since it is valued in a fixed vector space, it can be approximated by a smooth function $w_\varepsilon$ so that

$$\|w - w_\varepsilon\|_{L^2([a,b], T_{\ell(a)}T^1,1/2)} < \varepsilon$$

Then by taking $w_\varepsilon = P_t w_\varepsilon$ we have that $w_\varepsilon \in C^2(\mathbb{R}, \ell^*T^1,1/2)$ and

$$\|\tilde{w} - w_\varepsilon\|_{L^2(\mathbb{R}, \ell^*T^1,1/2)} < C\varepsilon$$

for a fixed constant $C$ depending on the norm of the parallel transport $P_t$. Now using Lemma 5.8, we have the existence of $k$ such that

$$\|d_\mathcal{F}^t_{x_+, x_-}(\ell, K)[k] - w_\varepsilon\|_{L^2(\mathbb{R}, \ell^*T^1,1/2)} < \varepsilon$$

Therefore,

$$\|d_\mathcal{F}^t_{x_+, x_-}(\ell, K)[v, k] - w\|_{L^2(\mathbb{R}, \ell^*T^1,1/2)} \leq C\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this proves that Ran$(d_\mathcal{F}^t_{x_+, x_-}(\ell, K)) \subset L^2(\mathbb{R}, \ell^*T^1,1/2(S^1, N))$ is dense and this completes the proof.

\[\square\]

**Proof of Proposition 5.2.** If $x_+ = x_-$ the transversality is satisfied since $\ell(t) = x_+$ is a stationary solution and $d_\mathcal{F}^t_{x_-, x_+}(x_+) = \frac{d}{dt} + d\nabla_{1,1/2}L_H(x_+)$ is invertible. Next, we assume that $x_+ \neq x_-$. Using the result of Lemma 5.9, we have that $\mathcal{M}(x_-, x_+) = C^2$-submanifold of $W_{x_-, x_+}^1(\mathbb{R}, T^1,1/2(S^1, N)) \times \mathbb{K}^2_{\theta, \mu_0}$. Let $\pi_{\mathbb{K}^2_{\theta, \mu_0}} : \mathcal{M}(x_-, x_+) \to \mathbb{K}^2_{\theta, \mu_0}$ be the projection onto the second component. It is a Fredholm map of class $C^2$ and its index at $(\ell, K) \in \mathcal{M}(x_-, x_+)$ is

$$\text{index } \pi_{\mathbb{K}^2_{\theta, \mu_0}}(\ell, K) = \text{index } d_\ell \tilde{\mathcal{F}}_{x_-, x_+}(\ell, K) = \mu_H(x_-) - \mu_H(x_+).$$
Since $\mathbb{R}$ acts freely on $\mathcal{M}(x_-, x_+)$, the quotient $\hat{\mathcal{M}}(x_-, x_+) = \mathcal{M}(x_-, x_+)/\mathbb{R}$ is a $C^2$-manifold of dimension

$$\dim \hat{\mathcal{M}}(x_-, x_+) = \mu_H(x_-) - \mu_H(x_+) - 1.$$ 

Therefore, for $x_-$ and $x_+ \in \text{crit}(\mathcal{L}_H)$ with $\mu_H(x_-) - \mu_H(x_+) \leq 2$ we have that $\dim \hat{\mathcal{M}}(x_-, x_+) \leq 1$. Since $W^{1,2}(\mathbb{R}, \mathcal{F}^{1,1/2}) \times \mathbb{R}^2_{\theta, \rho}$ is second countable, we have that $\hat{\mathcal{M}}(x_-, x_+)$ is also second countable and by restricting the map $\pi_{\theta, \rho}^2$ to $\tilde{\pi}_{\theta, \rho}^2 : \hat{\mathcal{M}}(x_-, x_+) \to \mathbb{R}^2_{\theta, \rho}$ which is $C^2$, one applies the Sard–Smale theorem to conclude that the regular values of $\tilde{\pi}_{\theta, \rho}^2$ that we denote by $\mathbb{K}_{\text{reg}}(x_-, x_+)$ is residual in $\mathbb{K}_{\theta, \rho}^2$. We set then

$$\mathbb{K}_{\text{reg}} = \left\{ \hat{\mathcal{M}}(x_-, x_+) \in \mathbb{K}_{\text{reg}}(x_-, x_+) \right\}.$$ 

Since the set $(x_-, x_+ \in \text{crit}(\mathcal{L}_H), x_+ \neq x_-, \mu_H(x_-) - \mu_H(x_+) \leq 2}$ is countable, we have that $\mathbb{K}_{\text{reg}} \subset \mathbb{K}_{\theta, \rho}^2$ is residual. Hence for any $K \in \mathbb{K}_{\text{reg}}$ the operator

$$d_\ell \mathcal{F}_{x_-, x_+} : T_{\ell} W^{1,2}_{x_-, x_+}(\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N)) \to L^2(\mathbb{R}, \ell^* T\mathcal{F}^{1,1/2}(S^1, N))$$

is onto. This completes the proof of Proposition 5.2. □

6 Invariance of the Morse–Floer homology, I: independence of the metric on $\mathcal{F}^{1,1/2}(S^1, N)$

By the results of the previous section, for $K \in \mathbb{K}_{\text{reg}}$, as in Sect. 4, chain complex

$$\{C_*(\mathcal{L}_H; \mathbb{Z}_2), \partial_*(\mathcal{L}_H, G^K)\}$$

is defined by counting $-\nabla^G H^K \mathcal{L}_H$-flow lines of index difference 1. The homology of this complex is denoted by $HF_*(\mathcal{L}_H, \mathcal{F}^{1,1/2}(S^1, N), G^K; \mathbb{Z}_2)$. The aim of this section is to prove the following:

**Proposition 6.1** Let $K_0, K_1 \in \mathbb{K}_{\text{reg}}$, then there exists a chain complex isomorphism

$$\Phi : \{C_*(\mathcal{L}_H; \mathbb{Z}_2), \partial_*(\mathcal{L}_H, G^{K_0})\} \to \{C_*(\mathcal{L}_H; \mathbb{Z}_2), \partial_*(\mathcal{L}_H, G^{K_1})\}.$$ 

In particular $HF_*(\mathcal{L}_H, \mathcal{F}^{1,1/2}(S^1, N), G^{K_0}; \mathbb{Z}_2)$ is isomorphic to $HF_*(\mathcal{L}_H, \mathcal{F}^{1,1/2}(S^1, N), G^{K_1}; \mathbb{Z}_2)$.

The proof of this proposition will be made through several steps. First we will define an alternative functional by adding an extra real variable. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(s) = 2s^3 - 3s^2 + 1$ and we define the space $\mathcal{F}^{1,1/2}(S^1, N) = \mathcal{F}^{1,1/2}(S^1, N) \times \mathbb{R}$. We extend our functional $\mathcal{L}_H$ by $\mathcal{L}_H$ defined in $\mathcal{F}^{1,1/2}(S^1, N)$ as follows:

$$\tilde{\mathcal{L}}_H(x, s) = \mathcal{L}_H(x) + f(s).$$

Observe that $f$ is a Morse function with two critical points $s = 0, 1$ with Morse indices 1 and 0, respectively. We will assume that $\mathcal{L}_H$ is Morse, therefore, $\tilde{\mathcal{L}}_H$ is also Morse and

$$\text{crit}(\tilde{\mathcal{L}}_H) = \{(x, 0), (x, 1) \mid x \in \text{crit}(\mathcal{L}_H)\}.$$ 

Recall that the relative Morse index for a critical point $x$ of $\mathcal{L}_H$ is defined as follows

$$\mu_H(x) = -\text{sign} \{A_{x,t} \mid 0 \leq t \leq 1\},$$
where $A_{x_t, H}$ is the Hessian of $L_H$ at $x_t$ and $x_t$ is a path from a fixed base point $x_0 \in \mathcal{F}_{1/2}^1(S^1, N)$ to the critical point $x$. In the same way we define the relative index for critical points of $\tilde{L}_H$ as

$$ \tilde{\mu}_H(x, i) = -s f(\tilde{A}_{x_{s}, i, H})_{0 \leq t \leq 1}, $$

for $i = 0, 1$, where

$$ \tilde{A}_{x_{s}, i, H} = \begin{pmatrix} A_{x_s, H} & 0 \\ 0 & 2s - 1 \end{pmatrix} $$

and $s^i(t) = (1 - t)2 + ti$. Therefore, from the properties of the spectral flow we have that

$$ sf(\tilde{A}_{x_{s}, i, H})_{0 \leq t \leq 1} = sf(A_{x_{s}, H})_{0 \leq t \leq 1} + sf(2s^i(t) - 1)_{0 \leq t \leq 1}. $$

Hence,

$$ \tilde{\mu}_H(x, 0) = \mu_H(x) + 1, \quad \tilde{\mu}_H(x, 1) = \mu_H(x). $$

This gives us a splitting of the set of critical points of $\tilde{L}_H$ of the form

$$ \text{crit}_k(\tilde{L}_H) = (\text{crit}_{k-1}(\tilde{L}_H) \times \{0\}) \cup (\text{crit}_k(\tilde{L}_H) \times \{1\}), $$

where $\text{crit}_k(\tilde{L}_H) = \{(x, i) \in \text{crit}(\tilde{L}_H) : \tilde{\mu}_H(x, i) = k\}$. This naturally descends to the chain level and giving us

$$ C_k(\tilde{L}_H) = C_{k-1}(\tilde{L}_H) \oplus C_k(\tilde{L}_H). $$

Now we consider a smooth function $\rho \in C_\infty(\mathbb{R}, [0, 1])$ such that $\rho(s) = 1$ for $s \leq \frac{1}{3}$ and $\rho(s) = 0$ for $s \geq \frac{2}{3}$. We define then the metric $G^{p, K_0, K_1}$ which interpolates $G^{K_0}$ and $G^{K_1}$, for $x \in \mathcal{F}_{1/2}^1(S^1, N)$, $X, Y \in T_x\mathcal{F}_{1/2}^1(S^1, N)$ and $a, b \in \mathbb{R}$, by

$$ G^{p, K_0, K_1}(x)((X, a), (Y, b)) = \rho(s)G^{K_0}(x)(X, Y) + (1 - \rho(s))G^{K_1}(x)(X, Y) + ab. $$

**Lemma 6.1** The functional $\tilde{L}_H$ satisfies the Palais–Smale condition on $\mathcal{F}_{1/2}^1(S^1, N)$ equipped with the metric $G^{p, K_0, K_1}$.

**Proof** Assume that $|\tilde{L}_H(x_n, s_n)| \leq C$ and $\nabla_{G^{p, K_0, K_1}}\tilde{L}_H(x_n, s_n) \to 0$. In particular, we have that $f'(s_n) \to 0$. Therefore $s_n \to i$ for $i = 0$ or $1$. We will assume for instance that it converges to $1$. Hence, $f(s_n)$ is also bounded and for $n$ big enough $s_n$ will be close to $1$. Therefore, we have that $|\tilde{L}_H(x_n)|$ is bounded and $\nabla_{G^{p, K_0, K_1}}\tilde{L}_H(x_n) = \nabla_{G^{K_1}}\tilde{L}_H(x_n) \to 0$. Since $G^{K_1}$ is equivalent to the standard metric $G_0$ that we fixed in the beginning and $\tilde{L}_H$ satisfies the Palais–Smale condition (see Proposition 3.6), up to a subsequence, $(x_n)$ converges in $\mathcal{F}_{1/2}^1(S^1, N)$. This completes the proof. \hfill \Box

We consider now the negative gradient flow of $\tilde{L}_H$ connecting two critical points of $\text{crit}(\tilde{L}_H)$ with respect to the metric $G^{p, K_0, K_1}$. Then one finds three kinds of flow lines.

**Type 1:** These flow lines connect critical points of the form $(x_1, 0)$ and $(x_2, 0)$ for $x_1, x_2 \in \text{crit}(\tilde{L}_H)$. These flow lines take the form $(t(t), 0)$ and $x(t)$ satisfies

$$ \frac{dF}{dt} = -\nabla_{G^{K_0}}\tilde{L}_H(t), \quad F(0) = x_2, \quad F(\infty) = x_1. $$

**Type 2:** Similarly to the type 1 flow lines, they connect critical points of the form $(x_1, 1)$ and $(x_2, 1)$ for $x_1, x_2 \in \text{crit}(\tilde{L}_H)$, therefore they take the form $(t(t), 1)$ and $x(t)$ satisfies

$$ \frac{dF}{dt} = -\nabla_{G^{K_1}}\tilde{L}_H(t), \quad F(0) = x_2, \quad F(\infty) = x_1. $$

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Type 3: These flow lines \((\ell(t), s(t))\) of this type connect critical points of the form \((x_1, 0)\) and \((x_2, 1)\) where \(x_1, x_2 \in \text{crit}(\mathcal{L}_H)\). First we restrict the metric \(G^{p,K_0,K_1}\) to \(\mathcal{F}^{1.1/2}(S^1, N)\) that we denote it by \(\tilde{G}^{p,K_0,K_1}\). We notice now that

\[
\frac{d}{dt} \mathcal{L}_H(\ell(t)) = -\|\nabla \tilde{G}^{p,K_0,K_1} \mathcal{L}_H\|^2_{\tilde{G}^{p,K_0,K_1}} \leq 0,
\]

therefore \(\mathcal{L}_H(x_1) \leq \mathcal{L}_H(x_2)\) and \(\mathcal{L}_H(x_2) = \mathcal{L}_H(x_1)\) if and only if \(x_2 = x_1\) and the flow is stationary. Hence, we can split these flow lines to two categories.

The first category consists of flow lines of the form \((x, s(t))\) where \(s(t)\) solves

\[
s'(t) = -f'(s(t)), \quad s(-\infty) = 0, \quad s(+\infty) = 1.
\]

The second category consists of flow lines of the form \((\ell(t), s(t))\) connecting \((x_1, 0)\) and \((x_2, 1)\), where \(x_1 \neq x_2\).

As for flow lines of type 3 of the first category, we have:

**Lemma 6.2** Along the flow lines of the third type and first category the unstable manifold \(W^{u}_{-\nabla \tilde{G}^{p,K_0,K_1}}(x, 0)\) and the stable manifold \(W^{s}_{-\nabla \tilde{G}^{p,K_0,K_1}}(x, 1)\) intersect transversally.

**Proof** We consider a smooth function \(s_0 : \mathbb{R} \to \mathbb{R}\) such that \(s_0(t) = 1\) if \(t \geq 1\) and \(s_0(t) = 0\) if \(t \leq -1\) and let \(W = s_0 + W^{1.2}(\mathbb{R})\). We define then the map \(\mathfrak{F} : W^{1.2}_{\mathbb{R}} \times (\mathcal{F}^{1.1/2}(S^1, N)) \times W \to L^2(\mathbb{R}, T\mathcal{F}^{1.1/2}(S^1, N)) \times L^2(\mathbb{R})\) by

\[
\mathfrak{F}(x, s) = \left( \frac{dx}{dt} + \nabla \tilde{G}^{p,K_0,K_1} \mathcal{L}_H(x), \dot{s} + f'(s) \right).
\]

Now

\[
d\mathfrak{F}(x, s) : T_xW^{1.2}(\mathbb{R}, \mathcal{F}^{1.2}) \times W^{1.2}(\mathbb{R}) \to L^2(\mathbb{R}, x^*T\mathcal{F}^{1.1/2}) \times L^2(\mathbb{R})
\]

and one sees that

\[
d\mathfrak{F}(x, s) = d\mathfrak{F}_{x,s}(x) \oplus \left( \frac{d}{dt} + f''(s) \right).
\]

Clearly the map \(\frac{d}{dt} + f''(s) : W^{1.2}(\mathbb{R}) \to L^2(\mathbb{R})\) is Fredholm of index 1. Because its kernel is generated by \(s'\), it is onto. Thus it remains to study the first operator \(d\mathfrak{F}_{x,s}(x) : T_xW^{1.2}(\mathbb{R}, \mathcal{F}^{1.2}) \times L^2(\mathbb{R}, x^*T\mathcal{F}^{1.1/2})\). Notice now that

\[
d\mathfrak{F}_{x,s}(x) = \frac{d}{dt} + d\nabla \tilde{G}^{p,K_0,K_1}(\mathcal{L}_H(x).
\]

But since \(K_0\) and \(K_1\) are in \(\mathbb{K}_{\text{reg}}\), we have that \(K_0(x) = K_1(x) = 0\), hence \(\tilde{G}^{p,K_0,K_1} = G_0\) the standard metric in \(\mathcal{F}^{1.1/2}(S^1, N)\). Therefore, the operator becomes \(d\mathfrak{F}_{x,s}(x) = \frac{d}{dt} + d\mathcal{L}^{1.1/2}_{1,2}(\mathcal{L}_H(x). Since \(\mathcal{L}_H\) is assumed to be Morse, the operator \(d\mathcal{L}^{1.1/2}_{1,2}\) is hyperbolic and the surjectivity follow as in Proposition 5.2.

Now transversality needs to be checked only for the type 3 orbits from the second category. In this case, we do again the same construction as in Proposition 5.2 to get the following result

**Lemma 6.3** There exists a metric \(\hat{G}\) on \(\mathcal{F}^{1.1/2}(S^1, N)\) arbitrarily close to \(G^{p,K_0,K_1}\) such that:

- \(\mathcal{L}_H\) satisfies the Palais–Smale condition on \((\mathcal{F}^{1.1/2}, \hat{G})\)
- for any arbitrarily small neighborhood \(U\) of \(\text{crit}(\mathcal{L}_H)\), \(\hat{G}\) coincides with \(G^{p,K_0,K_1}\) on \((\mathcal{F}^{1.1/2}(S^1, N) \times (-\infty, \frac{1}{2})) \cup (\mathcal{F}^{1.1/2}(S^1, N) \times (\frac{1}{2}, +\infty)) \cup (U \times \mathbb{R})\).
\begin{itemize}
  \item $dL_H(x)[−\nabla\tilde{G}\tilde{L}_H(x, s)] < 0$ if $x \notin \text{crit}(L_H)$ and $s \in \mathbb{R}$
  \item The negative gradient flow of $\tilde{L}_H$ with respect to $\tilde{G}$ satisfies the Morse–Smale property up to order 2.
\end{itemize}

The idea of the proof is again to consider a function $\tilde{\theta} : \tilde{G}^{1,1/2} \to [0, 1]$ with the following properties:

\begin{enumerate}
  \item $\tilde{\theta} = 0$ on the set $(\mathcal{G}^{1,1/2}(S^1, N) \times [−\infty, 1/3)) \cup (\mathcal{G}^{1,1/2}(S^1, N) \times (2/3, +\infty)) \cup (U \times \mathbb{R})$.
  \item The zero set of $\tilde{\theta}$ is the closure of an open set.
  \item If $W_u^{\tilde{G}} \cdot \Phi\tilde{L}_H(x, i)$ and $W_s^{\tilde{G}} \cdot \Phi\tilde{L}_H(y, j)$, for $(x, i), (y, j) \in \text{crit}(\tilde{L}_H)$, intersect non transversally at a point $(x, s)$, then $\tilde{\theta}$ is not identically zero on the orbit passing through $(x, s)$.
\end{enumerate}

The rest of the proof is exactly similar to the proof of Proposition 5.2, so we will omit it. \(\square\)

**Proof of Proposition 6.1** We consider now the negative gradient flow of $\tilde{L}_H$ with respect to the metric $\tilde{G}$. Then again we have the same decompositions of the flow lines. That is, flow lines of type 1, 2 and 3 and the latter ones decompose to first and second category. Moreover, we have that the intersections of stable and unstable manifolds of critical points are transversal by Lemma 6.3.

We consider now the boundary operator by counting flow lines connecting critical points of $\tilde{L}_H$ with index difference 1:

$$\partial_k(\tilde{L}_H, \tilde{G}) : C_k(\tilde{L}_H; \mathbb{Z}_2) = C_{k-1}(\mathcal{L}_H; \mathbb{Z}_2) \oplus C_k(\mathcal{L}_H; \mathbb{Z}_2) \rightarrow C_{k-1}(\tilde{L}_H; \mathbb{Z}_2) = C_{k-2}(\mathcal{L}_H; \mathbb{Z}_2) \oplus C_{k-1}(\mathcal{L}_H; \mathbb{Z}_2).$$

Then one has that $\partial_k(\tilde{L}_H, \tilde{G})$ takes the form

$$\partial_k(\tilde{L}_H, \tilde{G}) = \begin{pmatrix} \partial_{k-1}(\mathcal{L}_H, G^{K_0}) & 0 \\ \Phi_{k-1} & \partial_k(\mathcal{L}_H, G^{K_1}) \end{pmatrix},$$

where $\Phi_k : C_k(\mathcal{L}_H; \mathbb{Z}_2) \rightarrow C_k(\mathcal{L}_H; \mathbb{Z}_2)$ is defined by counting the flow lines of type 3 (mod 2). For $\tilde{G}$-negative gradient flow lines of $\mathcal{L}_H$, the proof for the compactness of the moduli spaces of flow lines goes exactly as in Sect.3.2 and 3.4 and Proposition 5.3. In addition, we already have the transversality and the gluing of flow lines is proved as in Sect. 5.3. Hence we have that $\partial_{k-1}(\tilde{L}_H, \tilde{G}) \circ \partial_k(\tilde{L}_H, \tilde{G}) = 0$. Writing this using matrix notation we get that

$$\Phi_{k-1} \circ \partial_k(\mathcal{L}_H, G^{K_0}) + \partial_k(\mathcal{L}_H, G^{K_0}) \circ \Phi_k = 0$$

Therefore, $\Phi_* : C_*(\mathcal{L}_H, \mathbb{Z}_2) \rightarrow C_*(\mathcal{L}_H, \mathbb{Z}_2)$ is a chain map. Notice now that there is only one flow line connecting $(x, 0)$ to $(x, 1)$ for all $x \in \text{crit}(L_H)$. Therefore, if we order the critical points of $\mathcal{L}_H$ by increasing critical values of their energy, one sees that the chain map $\Phi$ can be represented by an upper triangular matrix with 1 in the diagonal, thus it is invertible and it induces a chain isomorphism. This completes the proof of Proposition 6.1. \(\square\)

With this fact, given $H$ satisfying (1.1)–(1.4) and if we assume that $\mathcal{L}_H$ is Morse, then the Morse–Floer homology of the pair $(\mathcal{L}_H, \mathcal{G}^{1,1/2}(S^1, N))$ is defined by

$$HF_* (\mathcal{L}_H, \mathcal{G}^{1,1/2}(S^1, N) ; \mathbb{Z}_2) := H F_* (\mathcal{L}_H, \mathcal{G}^{1,1/2}(S^1, N), G_K ; \mathbb{Z}_2)$$

for $K \in \mathbb{K}_{\text{reg}}$ and its isomorphism class is independent of the choice of $K \in \mathbb{K}_{\text{reg}}$. 

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In this section, we prove that the isomorphism class of the homology \( HF_* (\mathcal{L}_H, S^{1/2}; \mathbb{Z}_2) \) is independent of \( H \in \mathbb{H}_{p+1}^3 \). We consider two function \( H_0 \) and \( H_1 \) in \( \mathbb{H}_{p+1}^3 \) such that \( \mathcal{L}_{H_0} \) and \( \mathcal{L}_{H_1} \) are Morse. Let \( \eta(t) \) be a smooth function such that \( \eta(t) = 1 \) for \( t \geq 1 \), \( \eta(t) = 0 \) for \( t \leq 0 \) and \( |\eta'(t)| \leq 2 \) for all \( t \in \mathbb{R} \). We set \( H_t = \eta(t) H_1 + (1 - \eta(t)) H_0 \). To prove our result, we will study in this section the non-autonomous gradient flow equation defined by

\[
\frac{d\ell}{dt}(t) = -\nabla_{G^K} \mathcal{L}_{H_t}(\ell(t)), \quad t \in \mathbb{R}.
\]  

**Definition 7.1** We define metric \( d_{p+1} \) on \( \mathbb{H}_{p+1}^3 \) as follows: For \( H \) and \( H' \) in \( \mathbb{H}_{p+1}^3 \),

\[
d_{p+1}(H, H') = \sup_{(s, \phi, \psi) \in S^1 \times S(S^1) \otimes TN} \frac{|H(s, \phi, \psi) - H'(s, \phi, \psi)|}{1 + |\psi|_{p+1}}.
\]

It is easy to see that \( d_{p+1} \) is well defined on \( \mathbb{H}_{p+1}^3 \) and that if \( d_{p+1}(H, H') < \varepsilon \) then

\[
|H(s, \phi, \psi) - H'(s, \phi, \psi)| < \varepsilon (1 + |\psi|_{p+1})
\]

for all \( (s, \phi, \psi) \in S^1 \times S(S^1) \otimes TN \). From Sect. 5.2, we can always choose \( K \in \mathbb{R}_{\theta, \rho}^2 \) such that the negative gradient flows of \( \mathcal{L}_{H_0} \) and \( \mathcal{L}_{H_1} \) with respect to \( G^K \) satisfy the Morse–Smale property up to order 2. We first prove the following:

**Proposition 7.1** Let \( H_0 \) and \( H_1 \) in \( \mathbb{H}_{p+1}^3 \), there exists a constant \( \varepsilon_0 \) depending on the constants \( C_i (1 \leq i \leq 4) \) of (1.1)–(1.3) for \( H_0 \) and \( H_1 \) such that if \( d_{p+1}(H_0, H_1) < \varepsilon_0 \) then for any solution of (7.1) which satisfies \( \sup_{t \leq 0} \mathcal{L}_{H_0}(\ell(t)) \leq A \), there holds

\[
\sup_{t \geq 1} \mathcal{L}_{H_1}(\ell(t)) \leq C
\]

for some \( C = C(A, C_i) \) depending only on \( A \) and \( C_i (1 \leq i \leq 4) \).

Again this will be proved through several Lemmata. In what follows, \( C_k \) denotes constant depending only on \( C_1, C_2, C_3, C_4 \) of (1.1)–(1.3) for \( H_0 \) and \( H_1 \) and \( C_k(A) \) denotes constant depending only on the same constants \( C_1, C_2, C_3, C_4 \) and \( A \).

**Lemma 7.1** Let \( \varepsilon > 0 \) and \( H_1, H_0 \) in \( \mathbb{H}_{p+1}^3 \) be such that \( d_{p+1}(H_0, H_1) < \varepsilon \). Then for every solution \( \ell(t) \) of (7.1) that satisfies \( \sup_{t \leq 0} \mathcal{L}_{H_0}(\ell(t)) \leq A \), we have

\[
\mathcal{L}_{H_t}(\ell(t)) \leq A + 4\pi \varepsilon + 2\varepsilon \int_0^1 \|\psi(r)\|_{L_{p+1}(S^1)}^2 \, dr
\]  

for all \( t \in \mathbb{R} \) and

\[
\mathcal{L}_{H_1}(\ell(1)) + \int_0^1 \|\ell(r)\|_{G^K}^2 \, dr \leq A + 4\pi \varepsilon + 2\varepsilon \int_0^1 \|\psi(r)\|_{L_{p+1}(S^1)}^2 \, dr.
\]  

**Proof** Notice first that for \( t \geq 1 \), \( \mathcal{L}_{H_t}(\ell(t)) \leq \mathcal{L}_{H_1}(\ell(1)) \), therefore it is enough to show the proof for \( 0 \leq t \leq 1 \). Now, we have

\[
\mathcal{L}_{H}(\ell(t)) = \mathcal{L}_{H_0}(\ell(0)) + \int_0^t \frac{d}{dr} \mathcal{L}_{H}(\ell(r)) \, dr = \mathcal{L}_{H_0}(\ell(0)) + \int_0^t - \|\frac{d\ell}{dt}(r)\|_{G^K}^2 \, dr + \int_0^t \eta'(r) \int_{S^1} (H_1(s, \ell(r)) - H_0(s, \ell(r))) \, ds \, dr
\]
\[
\leq A + 2\varepsilon \int_{0}^{1} \int_{S^1} (1 + |\psi(r,s)|^{p+1}) \, ds \, dr
\]
\[
\leq A + 4\pi \varepsilon + 2\varepsilon \int_{0}^{1} \|\psi(r)\|_{L^{p+1}(S^1)}^{p+1} \, dr.
\]  
(7.4)

which yields (7.2). The proof of (7.3) follows by setting \( t = 1 \) and keeping the negative term involving \( \ell \) in the second inequality.

\[\square\]

**Lemma 7.2** Under the assumptions of the previous lemma, we have

\[
\|\partial_t \phi(t)\|_{L^2(S^1)}^2 + \|\psi(t)\|_{L^{p+1}(S^1)}^{p+1} \leq C_1(A) + C_2 \varepsilon \int_{0}^{1} \|\psi(r)\|_{L^{p+1}(S^1)}^{p+1} \, dr
\]  
\[+ \ C_3(\|\partial_t \phi(t)\|_{H^1(S^1)} + \|\partial_t \psi(t)\|_{H^{1/2}(S^1)}) \|\psi(t)\|_{H^{1/2}(S^1)},\]  
(7.5)

**Proof** We have

\[
\left\langle \begin{pmatrix} \partial_t \phi(t) \\ \partial_t \psi(t) \end{pmatrix}, \begin{pmatrix} 0 \\ \psi \end{pmatrix} \right\rangle_{G^K} = -\nabla^G K \mathcal{L}_H(\phi, \psi, \begin{pmatrix} 0 \\ \psi \end{pmatrix})
\]
\[= -d \mathcal{L}_H(\psi) \begin{pmatrix} 0 \\ \psi \end{pmatrix}
\]
\[= -\int_{S^1} (\psi, D \phi \psi) \, ds + \int_{S^1} (\nabla \psi H_t(s, \phi, \psi, \psi) \, ds
\]

and

\[
\left\langle \begin{pmatrix} \partial_t \phi(t) \\ \partial_t \psi(t) \end{pmatrix}, \begin{pmatrix} 0 \\ \psi \end{pmatrix} \right\rangle_{G^K} + 2L \mathcal{L}_H(\ell(t)) \geq \|\partial_t \phi(t)\|_{L^2(S^1)}^2 + C_4 \|\psi(t)\|_{L^{p+1}(S^1)}^{p+1} - C_5
\]

by (1.2). Hence, we have

\[
\|\partial_t \phi(t)\|_{L^2(S^1)}^2 + \|\psi(t)\|_{L^{p+1}(S^1)}^{p+1} \leq \begin{align*}
C_8(\|\partial_t \phi(t)\|_{H^1(S^1)} + \|\partial_t \psi(t)\|_{H^{1/2}(S^1)}) \|\psi(t)\|_{H^{1/2}(S^1)} + C_6(A) + C_7 \mathcal{L}_H(\ell(t)).
\end{align*}
\]

Using (7.2), we have that

\[
\|\partial_t \phi(t)\|_{L^2(S^1)}^2 + \|\psi(t)\|_{L^{p+1}(S^1)}^{p+1} \leq \begin{align*}
C_9(A) + C_{10} \varepsilon \int_{0}^{1} \|\psi(r)\|_{L^{p+1}(S^1)}^{p+1} \, dr
\end{align*}
\]
\[+ \ C_8(\|\partial_t \phi(t)\|_{H^1(S^1)} + \|\partial_t \psi(t)\|_{H^{1/2}(S^1)}) \|\psi(t)\|_{H^{1/2}(S^1)},
\]

which finishes the proof of the Lemma 7.2.  

\[\square\]

It follows from the previous lemma that if \( \varepsilon < \frac{1}{2C_10} \), we have that

\[
\int_{0}^{1} \|\partial_t \phi(t)\|_{L^2(S^1)}^2 \, dt + \int_{0}^{1} \|\psi(t)\|_{L^{p+1}(S^1)}^{p+1} \, dt \leq C_{11}(A) + C_{12} \int_{0}^{1} (\|\partial_t \phi(t)\|_{H^1(S^1)}) \|\psi(t)\|_{H^{1/2}(S^1)} \, dt
\]  
(7.6)

and

\[
\|\partial_t \phi(t)\|_{L^2(S^1)}^2 + \|\psi(t)\|_{L^{p+1}(S^1)}^{p+1}
\]
\[\leq C_{13}(A) + C_{14} \varepsilon \int_{0}^{1} (\|\partial_t \phi(r)\|_{H^1(S^1)} + \|\partial_t \psi(r)\|_{H^{1/2}(S^1)}) \|\psi(r)\|_{H^{1/2}(S^1)} \, dr
\]  
\[+ \ C_7(\|\partial_t \phi(t)\|_{H^1(S^1)} + \|\partial_t \psi(t)\|_{H^{1/2}(S^1)}) \|\psi(t)\|_{H^{1/2}(S^1)}.
\]  
(7.7)
Lemma 7.3 Under the assumptions of Lemma 7.1, we have

\[ \int_0^1 \| \psi(t) \|_{H^{1/2}(S^1)}^2 \, dt \leq C_{15}(A) + C_{16} \left( \left( \int_0^1 \| \partial_t \phi(t) \|_{H^{1}(S^1)}^2 \, dt \right)^{1/2} \left( \int_0^1 \| \psi(t) \|_{H^{1/2}(S^1)}^2 \, dt \right)^{1/2} \right. \]

\[ + \left. \left( \int_0^1 \| \partial_t \psi(t) \|_{H^{1/2}(S^1)}^2 \, dt \right)^{1/2} \left( \int_0^1 \| \phi(t) \|_{H^{1}(S^1)}^2 \, dt \right)^{1/2} \right. \]

\[ + \left. \left( \int_0^1 \| \partial_t \psi(t) \|_{H^{1/2}(S^1)}^2 \, dt \right)^{1/2} \left( \int_0^1 \| \partial_t \phi(t) \|_{H^{1}(S^1)}^2 \, dt \right)^{1/2} \right. \]

\[ + \left. \left( \int_0^1 \| \partial_t \phi(t) \|_{H^{1/2}(S^1)}^2 \, dt \right)^{1/2} \right) \cdot \psi^{-1}(s) \, ds + C \| \partial_t \phi(t) \|_{L^2(S^1)} \| \psi^{-1}(t) \|_{L^4(S^1)} \| \psi(t) \|_{L^4(S^1)} \]

\[ + C_{20} \int_{S^1} \left( \| \psi(t) \|_{L^p(S^1)}^p \right) \| \psi^{-1}(t) \|_{L^2(S^1)} \| \psi(t) \|_{L^4(S^1)} \| \psi(t) \|_{L^4(S^1)} \]

\[ + C_{21} \left( \| \psi(t) \|_{L^{p+1}(S^1)} \right) \| \psi(t) \|_{L^p(S^1)} \| \psi(t) \|_{L^4(S^1)} \| \psi(t) \|_{L^4(S^1)} \]

\[ \leq C_{19}(\| \partial_t \phi(t) \|_{H^{1}(S^1)} + \| \partial_t \phi(t) \|_{H^{1/2}(S^1)}) \left( \| \psi(t) \|_{H^{1/2}(S^1)} + \| \partial_t \phi(t) \|_{L^2(S^1)} \| \psi(t) \|_{H^{1/2}(S^1)} + \| \partial_t \phi(t) \|_{L^2(S^1)} \| \psi(t) \|_{L^2(S^1)} \right) \]

\[ + C_{22} \left( \| \psi(t) \|_{H^{1/2}(S^1)} + \| \psi(t) \|_{L^{p+1}(S^1)} \right) \| \psi(t) \|_{L^2(S^1)} \| \psi(t) \|_{L^2(S^1)} \| \psi(t) \|_{L^2(S^1)} \| \psi(t) \|_{L^4(S^1)} \]

By \( p + 1 \geq 4 \), the last three terms are estimated by using (7.7) and obtain

\[ \| \psi^{-1}(t) \|_{H^{1/2}(S^1)} \]

\[ \leq C_{23}(A) + C_{24} \left( \| \partial_t \phi(t) \|_{H^{1}(S^1)} + \| \partial_t \psi(t) \|_{H^{1/2}(S^1)} \right) \left( \| \psi(t) \|_{H^{1/2}(S^1)} \right) \]

\[ + \left( \int_0^1 \| \partial_t \psi(t) \|_{H^{1/2}(S^1)}^2 \, dt \right)^{1/2} \left( \int_0^1 \| \partial_t \phi(t) \|_{H^{1}(S^1)}^2 \, dt \right)^{1/2} \]

\[ + \| \phi(t) \|_{H^{1/2}(S^1)} + \int_0^1 \| \partial_t \phi(t) \|_{H^{1}(S^1)} + \| \partial_t \psi(t) \|_{H^{1/2}(S^1)} \| \psi(t) \|_{H^{1/2}(S^1)} \, dt \right) \].

(7.10)
Similarly we have
\[
\|\psi_0^-(t)\|_{H^{1/2}(S^1)}^2
\leq C_{23}(A) + C_{24}\left(\|\partial_t\phi(t)\|_{H^1(S^1)} + \|\partial_t\psi(t)\|_{H^{1/2}(S^1)}\right)\|\psi(t)\|_{H^{1/2}(S^1)}
+ \|\partial_s\phi(t)\|_{L^2(S^1)}^{1/2}\|\psi(t)\|_{L^4(S^1)}
\]
\[
+ \|\psi(t)\|_{H^{1/2}(S^1)} + \int_0^1 \left(\|\partial_t\phi(t)\|_{H^1(S^1)} + \|\partial_t\psi(t)\|_{H^{1/2}(S^1)}\right)\|\psi(t)\|_{H^{1/2}(S^1)} dt
\]  
(7.11)

Notice that the term \(\|\psi_0^0\|_{H^{1/2}(S^1)}^2\) can be estimated as follows
\[
\|\psi_0^0\|_{H^{1/2}(S^1)}^2 \leq C_{25}\|\psi(t)\|_{L^{p+1}(S^1)}^2 \leq C_{26}\|\psi(t)\|_{L^{p+1}(S^1)}^{p+1} + C_{26}
\]
\[
\leq C_{27}(A) + C_{28}\left(\int_0^1 \left(\|\partial_t\psi(t)\|_{H^1(S^1)} + \|\partial_t\psi(t)\|_{H^{1/2}(S^1)}\right)\|\psi(t)\|_{H^{1/2}(S^1)} dt
\]
\[
+ \left(\|\partial_t\phi(t)\|_{H^1(S^1)} + \|\partial_t\psi(t)\|_{H^{1/2}(S^1)}\right)\|\psi(t)\|_{H^{1/2}(S^1)}\right).
\]  
(7.12)

where (7.7) is used.

Combining these last three inequalities we get
\[
\|\psi(t)\|_{H^{1/2}(S^1)}^2
\leq C_{29}(A) + C_{30}\left(\|\partial_t\phi(t)\|_{H^1(S^1)} + \|\partial_t\psi(t)\|_{H^{1/2}(S^1)}\right)\|\psi(t)\|_{H^{1/2}(S^1)}
\]
\[
+ \|\partial_s\phi(t)\|_{L^2(S^1)}^{1/2}\|\psi(t)\|_{L^4(S^1)}
\]
\[
+ \|\psi(t)\|_{H^{1/2}(S^1)} + \int_0^1 \left(\|\partial_t\phi(t)\|_{H^1(S^1)} + \|\partial_t\psi(t)\|_{H^{1/2}(S^1)}\right)\|\psi(t)\|_{H^{1/2}(S^1)} dt.
\]  
(7.13)

Here, using the Cauchy–Schwartz inequality twice, we have
\[
\left(\|\partial_t\phi(t)\|_{H^1(S^1)} + \|\partial_t\psi(t)\|_{H^{1/2}(S^1)}\right)\|\partial_s\phi(t)\|_{L^2(S^1)}^{1/2}\|\psi(t)\|_{L^4(S^1)}
\]
\[
\leq \|\partial_t\phi(t)\|_{H^1(S^1)}^2 + \|\partial_t\psi(t)\|_{H^{1/2}(S^1)}^2 + \frac{1}{2}\|\partial_s\phi(t)\|_{L^2(S^1)}\|\psi(t)\|_{H^{1/2}(S^1)}
\]
\[
\leq \|\partial_t\phi(t)\|_{H^1(S^1)}^2 + \|\partial_t\psi(t)\|_{H^{1/2}(S^1)}^2 + \|\partial_s\phi(t)\|_{L^2(S^1)}^2 + \|\psi(t)\|_{L^4(S^1)}^4.
\]

Using again \(4 \leq p + 1\) and estimate the last two terms in the above inequality by using (7.7), we obtain
\[
\left(\|\partial_t\phi(t)\|_{H^1(S^1)} + \|\partial_t\psi(t)\|_{H^{1/2}(S^1)}\right)\|\partial_s\phi(t)\|_{L^2(S^1)}^{1/2}\|\psi(t)\|_{L^4(S^1)}
\]
\[
\leq C_{31}(A) + \|\partial_t\phi(t)\|_{H^1(S^1)}^2 + \|\partial_t\psi(t)\|_{H^{1/2}(S^1)}^2
\]
\[
+ C_{32}\left(\int_0^1 \left(\|\partial_t\phi(t)\|_{H^1(S^1)} + \|\partial_t\psi(t)\|_{H^{1/2}(S^1)}\right)\|\psi(t)\|_{H^{1/2}(S^1)} dt
\]
\[
+ \left(\|\partial_t\phi(t)\|_{H^1(S^1)} + \|\partial_t\psi(t)\|_{H^{1/2}(S^1)}\right)\|\psi(t)\|_{H^{1/2}(S^1)}\right).
\]
Inserting this into (7.13), we obtain

\[ \| \psi(t) \|_{H^{1/2}(S^1)}^2 \leq C_{33}(A) + C_{34} \left( \| \partial_t \phi(t) \|_{H^1(S^1)}^2 + \| \partial_t H(t) \|_{H^{1/2}(S^1)} \| \psi(t) \|_{H^{1/2}(S^1)} \right) \]

\[ + \| \partial_t \phi(t) \|_{H^1(S^1)}^2 + \| \partial_t \psi(t) \|_{H^{1/2}(S^1)}^2 + \| \psi(t) \|_{H^{1/2}(S^1)} \]

\[ + \int_0^1 \left( \| \partial_t \phi(t) \|_{H^1(S^1)} + \| \partial_t \psi(t) \|_{H^{1/2}(S^1)} \| \psi(t) \|_{H^{1/2}(S^1)} \right) \, dt \]  \tag{7.14}

After integrating on [0, 1] and the use of Hölder’s inequality, we finally have

\[ \int_0^1 \| \psi(t) \|_{H^{1/2}(S^1)}^2 \, dt \leq C_{35} \left( \int_0^1 \| \partial_t \phi(t) \|_{H^1(S^1)}^2 \, dt \right)^{1/2} \left( \int_0^1 \| \psi(t) \|_{H^{1/2}(S^1)}^2 \, dt \right)^{1/2} \]

\[ + \left( \int_0^1 \| \partial_t \psi(t) \|_{H^{1/2}(S^1)}^2 \, dt \right)^{1/2} \left( \int_0^1 \| \psi(t) \|_{H^{1/2}(S^1)}^2 \, dt \right)^{1/2} \]

\[ + \int_0^1 \| \partial_t \phi(t) \|_{H^1(S^1)}^2 \, dt + \int_0^1 \| \partial_t \psi(t) \|_{H^{1/2}(S^1)}^2 \, dt + \left( \int_0^1 \| \psi(t) \|_{H^{1/2}(S^1)}^2 \, dt \right)^{1/2} \]  \tag{7.15}

**Corollary 7.1** Under the assumptions of Lemma 7.1, we have

\[ \int_0^1 \| \psi(t) \|_{H^{1/2}(S^1)}^2 \, dt \leq C_{36}(A) + C_{37} \left( \int_0^1 \| \partial_t \phi(t) \|_{H^1(S^1)}^2 \, dt + \int_0^1 \| \partial_t \psi(t) \|_{H^{1/2}(S^1)}^2 \, dt \right) \]  \tag{7.16}

**Proof** This corollary follows easily by applying Young’s inequality first in the form \( ab \leq C_e a^2 + \epsilon b^2 \). Then grouping the terms yields the inequality. \( \square \)

**Proof of Proposition 7.1** First, from (7.3) and the equivalence of \( G^K \) and the standard metric on \( \mathcal{F}^{1,1/2}(S^1, N) \) we have

\[ \int_0^1 \| \partial_t \phi(t) \|_{H^1(S^1)}^2 \, dt + \int_0^1 \| \partial_t \psi(t) \|_{H^{1/2}(S^1)}^2 \, dt \]

\[ \leq C_{38}(A) - 2 \mathcal{L}_{H^1}(\ell(1)) + 4 \epsilon \int_0^1 \| \psi(t) \|_{L^{p+1}(S^1)}^{p+1} \, dt \].

Then from (7.6), it follows that

\[ \int_0^1 \| \partial_t \phi(t) \|_{H^1(S^1)}^2 \, dt + \int_0^1 \| \partial_t \psi(t) \|_{H^{1/2}(S^1)}^2 \, dt \]

\[ \leq C_{39}(A) - 2 \mathcal{L}_{H^1}(\ell(1)) + \epsilon C_{40} \int_0^1 \left( \| \partial_t \phi(t) \|_{H^1(S^1)} + \| \partial_t \psi(t) \|_{H^{1/2}(S^1)} \| \psi(t) \|_{H^{1/2}(S^1)} \right) \, dt \]

Again by using Young’s inequality and taking \( \epsilon < \frac{1}{2C_{40}} \), we have

\[ \int_0^1 \| \partial_t \phi(t) \|_{H^1(S^1)}^2 \, dt + \int_0^1 \| \partial_t \psi(t) \|_{H^{1/2}(S^1)}^2 \, dt \]

\[ \leq C_{41}(A) - 4 \mathcal{L}_{H^1}(\ell(1)) + \epsilon C_{42} \int_0^1 \| \psi(t) \|_{H^{1/2}(S^1)}^2 \, dt \].  \tag{7.17}
Combining this inequality with (7.16) one has
\[
\int_0^1 \| \psi(t) \|_{H^{1/2}(S^1)}^2 dt \leq C_{43}(A) - C_{44} \mathcal{L}_{H_1}(\ell(1)) + \varepsilon C_{45} \int_0^1 \| \psi(t) \|_{H^{1/2}(S^1)}^2 dt.
\]
Hence, for \( \varepsilon < \frac{1}{2C_{45}} \) we get
\[
\int_0^1 \| \psi(t) \|_{H^{1/2}(S^1)}^2 dt \leq C_{46}(A) - C_{47} \mathcal{L}_{H_1}(\ell(1)).
\] (7.18)

Plugging it back in (7.17), we have
\[
\int_0^1 \| \partial_t \phi(t) \|_{H^{1}(S^1)}^2 dt + \int_0^1 \| \partial_t \psi(t) \|_{H^{1/2}(S^1)}^2 dt \leq C_{48}(A) - C_{49} \mathcal{L}_{H_1}(\ell(1)).
\] (7.19)

Using Hölder’s inequality in (7.6) and (7.18) and (7.19), we have
\[
\int_0^1 \| \psi(t) \|_{L^{p+1}(S^1)}^{p+1} dt \leq C_{50}(A) - C_{51} \mathcal{L}_{H_1}(\ell(1)).
\] (7.20)

Inserting it back in (7.2), we get
\[
\mathcal{L}_{H_1}(\ell(1)) \leq C_{52}(A) - 2\varepsilon C_{51} \mathcal{L}_{H_1}(\ell(1)).
\]
Hence if we take \( \varepsilon_0 \leq \min\{ \frac{1}{2C_{10}}, \frac{1}{2C_{40}}, \frac{1}{2C_{45}} \} \) we have that
\[
\mathcal{L}_{H_1}(\ell(1)) \leq C_{52}(A).
\]

This completes the proof of Proposition 7.1. \( \square \)

The next lemma give a bound of the length of solutions \( \ell \) to (7.1) under assuming bounds of \( \mathcal{L}_{H_1}(\ell(t)) \) in both ends \( t = \pm \infty \).

**Lemma 7.4** Let \( H_0 \) and \( H_1 \) in \( \mathbb{H}^{3+1} \) and consider \( \varepsilon_0 > 0 \) as in Proposition 7.1. Assume that \( \sup_{t<0} \mathcal{L}_{H_0}(\ell(t)) \leq A \) and \( \inf_{t \geq 1} \mathcal{L}_{H_1}(\ell(t)) \geq B \), then there exists a constant \( C = C(A, B, C_1, C_2, C_3, C_4) \) depending only on \( A, B \) and constants \( C_1, C_2, C_3, C_4 \) of (1.1)–(1.3) for \( H_0 \) and \( H_1 \) such that the length of \( \ell \), \( L(\ell) \) satisfies
\[
L(\ell) \leq C.
\]

**Proof** Similarly to (7.3) we have that
\[
\int_{\mathbb{R}} \left\| \frac{d\ell}{dt}(t) \right\|_{G^k}^2 dt \leq C(A, B) + 2\varepsilon \int_0^1 \| \psi(t) \|_{L^{p+1}}^{p+1} dt, \tag{7.21}
\]
where \( C(A, B) \) is a constant depending only on \( A, B \) and \( C_i \) (\( 1 \leq i \leq 4 \)) of (1.1)–(1.3) for \( H_0, H_1 \).

By (7.20) and the lower bound \( \mathcal{L}_{H_1}(\ell(1)) \geq B \), we obtain from (7.21) that
\[
\int_{\mathbb{R}} \left\| \frac{d\ell}{dt}(t) \right\|_{G^k}^2 dt \leq C(A, B).
\]

Now it is enough to write that
\[
L(\ell) = L(\ell|_{(-\infty, 0]}) + L(\ell|_{[1, +\infty)}) + L(\ell|_{[0, 1]}).
\]
The first two terms are uniformly bounded as in Lemma 5.3. For the last term we have
\[
\int_0^1 \left\| \frac{d\ell}{dt}(t) \right\|_{T_c^1,1/2} dt \leq \left( \int_0^1 \left\| \frac{d\ell}{dt}(t) \right\|^2_{T_c^1,1/2} dt \right)^{1/2} \leq C \left( \int \left\| \frac{d\ell}{dt}(t) \right\|^2_{G^k} dt \right)^{1/2} \leq C(A, B).
\]

We consider now two functions $H_0$ and $H_1$ in $\mathbb{H}^3_{p+1}$ such that $d_{p+1}(H_0, H_1) < \varepsilon_0$ and $\mathcal{L}_{H_i}$ is a Morse function for $i = 0, 1$. We want to construct a chain homomorphism between $\{C_{\ast}(\mathcal{L}_{H_0}; \mathbb{Z}_2), \partial_{\ast}(\mathcal{L}_{H_0}, G^K)\}$ and $\{C_{\ast}(\mathcal{L}_{H_1}; \mathbb{Z}_2), \partial_{\ast}(\mathcal{L}_{H_1}, G^K)\}$ for generic metric $G^K$. For this purpose, we will study the moduli spaces of flow lines of the non-autonomous gradient flow. That is, for $x \in \text{crit}_k(\mathcal{L}_{H_0})$ and $y \in \text{crit}_k(\mathcal{L}_{H_1})$, we consider the solutions $\ell(t)$ of
\[
\frac{d\ell}{dt}(t) = -\nabla_{G^K} \mathcal{L}_{H_i}(\ell(t)), \quad \text{and} \quad \ell(-\infty) = x, \quad \ell(+\infty) = y.
\]
(7.22)
We denote by the set of solutions to (7.22) as $\mathcal{M}(x, y; H_i)$.

**Proposition 7.2.** Under the assumptions of Lemma 7.4, we have that $\mathcal{M}(x, y; H_i)$ is relatively compact in $C^0([\tau, \tau+1], \mathcal{G}^{1,1/2}((S^1, N))$.

**Proof.** By (7.17), (7.20) and the lower bound $\mathcal{L}_{H_i}(\ell(1)) \geq B$, we have
\[
\int_0^1 \left\| \psi(t) \right\|_{H^{1/2}(S^1)}^2 dt \leq C(A, B), \quad \int_0^1 \left\| \psi \right\|_{L_{p+1}(S^1)}^{p+1} dt \leq C(A, B).
\]
(7.23)
In what follows, $C(A, B)$ denotes various constants depending only on $A, B$ and $C_i$ ($1 \leq i \leq 4$) of (1.1)–(1.3) for $H_0, H_1$.

Combining the second of (7.23) with (7.21), we obtain
\[
\int_\mathbb{R} \left\| \partial_t \phi(t) \right\|_{H^1(S^1)}^2 dt + \int_\mathbb{R} \left\| \partial_t \psi(t) \right\|_{H^{1/2}(S^1)}^2 dt \leq C(A, B).
\]
(7.24)
On the other hand, integrating (7.14) over $[\tau, \tau+1]$, by Hölder’s inequality, (7.20) and (7.24), we have
\[
\int_\tau^{\tau+1} \left\| \psi(t) \right\|_{H^{1/2}(S^1)}^2 dt \leq C(A, B) \left( 1 + \left( \int_\tau^{\tau+1} \left\| \psi(t) \right\|_{H^{1/2}(S^1)}^2 dt \right)^{1/2} \right).
\]
Therefore, we obtain
\[
\sup_{\tau \in \mathbb{R}} \int_\tau^{\tau+1} \left\| \psi(t) \right\|_{H^{1/2}(S^1)}^2 dt \leq C(A, B).
\]
(7.25)
By integrating (7.7) over $[\tau, \tau+1]$ and using (7.23), (7.24) and (7.25), we have
\[
\int_\tau^{\tau+1} \left\| \partial_t \phi(t) \right\|_{H^1(S^1)}^2 dt + \int_\tau^{\tau+1} \left\| \psi(t) \right\|_{L_{p+1}(S^1)}^{p+1} dt \leq C(A, B) + \left( \int_\tau^{\tau+1} \left\| \psi(t) \right\|_{H^{1/2}(S^1)}^2 dt \right)^{1/2} \leq C(A, B)
\]
and
\[
\sup_{\tau \in \mathbb{R}} \int_\tau^{\tau+1} \left\| \partial_t \phi(t) \right\|_{L^2(S^1)}^2 dt \leq C(A, B), \quad \sup_{\tau \in \mathbb{R}} \int_\tau^{\tau+1} \left\| \psi(t) \right\|_{L_{p+1}(S^1)}^{p+1} dt \leq C(A, B).
\]
(7.26)
By (7.25) and (7.26), we have equivalent results to the ones in Lemma 5.2 (5.4), (5.5) and (5.6) which has the main estimates to prove Lemma 5.2. Combining with the length estimate of Lemma 7.4, the same procedure in the proof of Proposition 5.3 can be repeated here to get the desired relative compactness of $\mathcal{M}(x,y; H_t)$. 

Again we want to perturb the metric as in the previous section in order to achieve transversality. The main trajectories here causing problems are coming from stationary paths. That is, $\ell(t) = x_0$ for all $t \in \mathbb{R}$, where $x_0 \in \text{crit}_k(\mathcal{L}_{H_0}) \cap \text{crit}_k(\mathcal{L}_{H_1})$. Therefore, we will assume for now that

$$\text{crit}_k(\mathcal{L}_{H_0}) \cap \text{crit}_k(\mathcal{L}_{H_1}) = \emptyset \quad \text{for all} \quad k \in \mathbb{Z}. \quad (A)$$

This last assumption is not as restrictive as it seems, in fact it is generic as we show in the following

**Lemma 7.5** Let $H_0$ and $H_1$ in $\mathbb{H}_3^{\rho+1}$ be such that $\mathcal{L}_{H_0}$ and $\mathcal{L}_{H_1}$ are Morse. Then there exists a residual set $\mathbb{H}_b(H_1, H_0) \subset \mathbb{H}_b^3$ such that $\text{crit}_k(\mathcal{L}_{H_0}) \cap \text{crit}_k(\mathcal{L}_{H_1+h}) = \emptyset$ for all $k \in \mathbb{Z}$ and $h \in \mathbb{H}_b(H_1, H_0)$.

**Proof** We show that if $x = (\phi, \psi) \in \text{crit}_k(\mathcal{L}_{H_0})$, then the set of perturbations $h \in \mathbb{H}_b^3$ such that $x \notin \text{crit}(\mathcal{L}_{H_1+h})$ is an open dense set. Indeed, assume that $x \in \text{crit}(\mathcal{L}_{H_1})$, then $x = (\phi, \psi)$ satisfies Dirac-geodesic equations

$$\nabla_x \partial_s \phi - \frac{1}{2} R(\phi)(\psi, \partial_s \phi \cdot \psi) + \nabla_{\phi} H(s, \phi, \psi) = 0,$$

$$D_{\phi} - \nabla_{\psi} H(s, \phi, \psi) = 0$$

for $H = H_0, H_1$. Thus, $\phi, \psi$ satisfy

$$\nabla_{\phi} H_0(s, \phi(s), \psi(s)) = \nabla_{\phi} H_1(s, \phi(s), \psi(s)) \quad (7.27)$$

and

$$\nabla_{\psi} H_0(s, \phi(s), \psi(s)) = \nabla_{\psi} H_1(s, \phi(s), \psi(s)) \quad (7.28)$$

for all $s \in S^1$. It is easy to see that there exists an arbitrary small perturbation $h \in \mathbb{H}_b^3$ such that (7.27) or (7.28) are not satisfied for $H_1$ replaced by $H_1 + h$. An example of such a choice is given as follows. Since $(\phi, \psi) \in C^2(S^1) \times C^1(S^1)$, the graph of $x = (\phi, \psi)$, $G_x = \{(s, \phi(s), \psi(s)) : s \in S^1\} \subset S^1 \times S(S^1) \otimes T N$ is compact. We take a cutoff function $\rho \in C^\infty(S^1 \times S(S^1) \otimes T N)$ such that $\rho = 1$ in a neighborhood of $G_x$. For $\xi_0 \in S(S^1) \otimes \mathbb{R}^k$, we define

$$h(s, p, \xi) = \rho(s, p, \xi_0)(\xi_0, \xi)$$

for $s \in S^1$, $p \in N$ and $\xi \in S(S^1) \otimes T_p N$. We then have $\nabla_{\psi} h(s, \phi(s), \psi(s)) = P_{\phi(s)}(\xi_0)$. Thus, if we choose $\xi_0$ such that $P_{\phi(s)}(\xi_0) \neq 0$ and small enough, then (7.28) is not satisfied for arbitrary small perturbation $h$. Thus, there exists an open dense subset $\mathbb{H}_b(x) \subset \mathbb{H}_b^3$ such that the condition (A) is satisfied for $H_0$ and $H_1 + h$ with $h \in \mathbb{H}_b(x)$. Then, since $\mathcal{L}_{H_0}$ is Morse, crit($\mathcal{L}_{H_0}$) is countable and there exists a generic subset $\mathbb{H}_b(H_1, H_0) \subset \mathbb{H}_b^3$ such that the condition (A) is satisfied for $H_0$ and $H_1 + h$ with $h \in \mathbb{H}_b(H_1, H_0)$. This completes the proof.

Regarding the transversality issue, it is similar to the previous case, that is up to a perturbation of the metric $G^K$, the stable and unstable manifolds intersect transversally. Indeed, one starts first by taking a function $\theta : \mathcal{F}_{1,1/2} \to [0, 1]$ such that $\theta = 0$ on crit($\mathcal{L}_{H_0}) \cup$ crit($\mathcal{L}_{H_1}$)
and $\theta > 0$ elsewhere. Since we do not have stationary orbits under the assumption (A), $\dot{\ell} \neq 0$, hence there exists an interval $[a, b]$ such that if $\ell(t)$ is a flow line, $\ell : [a, b] \to S^{1,1/2}$ is an embedding. From this, the rest of the argument can be carried out exactly like in Sect. 5.2.

From the transversality, in a similar way as in Sect. 3, we have that $\mathcal{M}(x; H) \subset W_{x,y}^{1,2}(\mathbb{R}, S^{1,1/2})$ is a manifold of dimension $\mu_{H_0}(x) - \mu_{H_1}(y) = 0$. In fact, by Proposition 7.2, we showed that $\mathcal{M}(x; H)$ is relatively compact and it can be compactified by adding by broken trajectories. These trajectories can be described as follows: there exist $x = x_0, \ldots, x_p \in \text{crit}(\mathcal{L}_{H_0})$, $\ell_{0,i} \in \mathcal{M}(x_i, x_{i+1}, H_0)$ for $0 \leq i \leq p - 1$, $y_1, \ldots, y_q = y \in \text{crit}(\mathcal{L}_{H_1})$, $\ell_{1,i} \in \mathcal{M}(y_i, y_{i+1}, H_1)$ for $1 \leq i \leq q - 1$ and $\ell_{01} \in \mathcal{M}(x_p, y_1; H_1)$ such that

$$\mu_{H_0}(x_0) > \mu_{H_0}(x_1) > \cdots > \mu_{H_0}(x_p) \geq \mu_{H_1}(y_1) > \mu_{H_1}(y_2) > \cdots > \mu_{H_1}(y_q).$$

Therefore, in our case, since $\mu_{H_0}(x) = \mu_{H_1}(y)$, there is no broken trajectories and the manifold is just a finite number of points. Hence we denote

$$n_{01}(xy) = \#\mathcal{M}(x; H_1) \pmod{2}.$$

We define therefore the homomorphism $\Phi_{01} : C_k(\mathcal{L}; \mathbb{Z}_2) \to C_k(\mathcal{L}; \mathbb{Z}_2)$ on the generators by

$$\Phi_{01}(x) = \sum_{y \in \text{crit}^k(\mathcal{L})} n_{01}(xy)y$$

(7.29)

and extend by linearity.

**Lemma 7.6** Let $H_0$ and $H_1$ in $\mathbb{R}^p$ be such that $\mathcal{L}_{H_1}$ and $\mathcal{L}_{H_0}$ are Morse and $d_{p+1}(H_0, H_1) < \varepsilon_0$. Then under the assumption (A), the homomorphism $\Phi_{01} : \{C_*(\mathcal{L}; \mathbb{Z}_2), \partial_*(\mathcal{L}; G^K)\} \to \{C_*(\mathcal{L}; \mathbb{Z}_2), \partial_*(\mathcal{L}; G^K)\}$ defines a chain homomorphism.

**Proof** We consider arbitrary critical points $x_{0,k} \in \text{crit}_k(\mathcal{L}_{H_0})$ and $x_{1,k-1} \in \text{crit}_{k-1}(\mathcal{L}_{H_1})$. By transversality (up to perturbation of the metric $G^K$) we have that $\mathcal{M}(x_{0,k}, x_{1,k-1}; H_1)$ is a one dimensional manifold that is precompact and the boundary of its compactification $\overline{\mathcal{M}(x_{0,k}, x_{1,k-1}; H_1)}$ consists of the following two types of orbits:

i) Broken trajectories connecting $x_{0,k}$ to $x_{0,k-1} \in \text{crit}_{k-1}(\mathcal{L}_{H_0})$, that is flow lines $\ell_0 \in \mathcal{M}(x_{0,k}, x_{0,k-1}, H_0)$ then $x_{0,k-1}$ to $x_{1,k-1}$, that is a flow line $\ell_{01} \in \mathcal{M}(x_{0,k-1}, x_{1,k-1}; H_1)$.

ii) Broken trajectories connecting $x_{0,k}$ to $x_{1,k} \in \text{crit}_k(\mathcal{L}_{H_1})$, that is flow lines $\ell_{01} \in \mathcal{M}(x_{0,k}, x_{1,k}; H_1)$ then $x_{1,k}$ to $x_{1,k-1}$, that is a flow line $\ell_1 \in \mathcal{M}(x_{1,k}, x_{1,k-1}; H_1)$.

This of course require a gluing statement with the compactness result. The compactness issue is proved in Proposition 7.2. As for gluing, it relies mainly on the Fredholm property of the linearized flow equation and transversality. As seen above, both of which also hold in the present case if the metric $G^K$ is chosen appropriately. Thus the argument of Sect. 3.5 applies to prove gluing in the present case as well. This yields, in particular, that since this boundary is a boundary of one dimensional manifold, its cardinal is zero mod 2. So in terms of counting, we have

$$\sum_{x_{0,k-1} \in \text{crit}_{k-1}(\mathcal{L}_{H_0})} n_{0}(x_{0,k}, x_{0,k-1})n_{01}(x_{0,k-1}, x_{1,k-1})$$

$$+ \sum_{x_{1,k} \in \text{crit}_k(\mathcal{L}_{H_1})} n_{01}(x_{0,k}, x_{1,k})n_{1}(x_{1,k}, x_{1,k-1}) = 0 \pmod{2}$$
and this is equivalent to
\[ \Phi_{01} \circ \partial_k(\mathcal{L}_{H_0}, G^K) = \partial_k(\mathcal{L}_{H_1}, G^K) \circ \Phi_{01}. \]

This completes the proof. \( \square \)

Remark 7.1 The chain homomorphism \( \Phi_{01} \) constructed above depends on the specific choice of the homotopy \( H_t \). As in [22, Remark 6.1], it can be shown that it is in fact independent of the choice of homotopy if a certain condition is satisfied. For \( H_0, H_1 \in \mathbb{H}^3_{p+1} \) with \( d_{p+1}(H_0, H_1) < \epsilon \), we say that a homotopy between \( H_0 \) and \( H_1 \) in \( \mathbb{H}^3_{p+1} \) is admissible with constant \( L > 0 \) if \( H_t \) satisfies (1.1)–(1.4) for all \( t \) and there exists \( R > 0 \) such that \( H_t = H_0 \) for \( t \leq -R \), \( H_t = H_1 \) for \( t \geq R \) and \( |\frac{\partial H_t}{\partial t}(s, \phi, \psi)| \leq L \epsilon (1 + |\psi|^{p+1}) \) for all \( t \in \mathbb{R} \) and \( (s, \phi, \psi) \in S^{1} \times S(S^{1}) \otimes TN \). It is not difficult to see that all the arguments given above apply for admissible homotopies provided \( L \epsilon \) is small enough, thus giving a chain homomorphism for arbitrary admissible homotopy. By considering a homotopy of homotopies, it can be shown that the resulting chain homomorphism is independent of the choice of admissible homotopy. Prototypical example of this construction will be given in the proof of Lemma 7.7 below. That construction is easily generalized to the general case and we omit the details here.

Now we will prove a naturality result for the homomorphism \( \Phi_{01} \) at the homology level.

Proposition 7.3 There exists \( 0 < \epsilon < \epsilon_0 \) such that if we consider three function \( H_0 \), \( H_1 \) and \( H_2 \) in \( \mathbb{H}^3_{p+1} \) with \( \mathcal{L}_{H_i} \) is Morse for \( 0 \leq i \leq 2 \) and \( d_{p+1}(H_i, H_j) < \epsilon \) for \( 0 \leq i, j \leq 2 \). We assume furthermore that (A) holds for the pair \( H_i, H_j \) with \( H_i \neq H_j \). Then for a generic \( K \in \mathbb{R}_{p, \theta}^2 \), with \( \theta : \mathbb{S}^{1}/2 \to [0, 1] \) is adequately chosen and adapted to the three functions \( H_i \), the chain homomorphism \( \Phi_{ij} : \{C_*(\mathcal{L}_{H_i}; \mathbb{Z}_2), \partial_*(\mathcal{L}_{H_i}, G^K)\} \to \{C_*(\mathcal{L}_{H_j}; \mathbb{Z}_2), \partial_*(\mathcal{L}_{H_j}, G^K)\} \) are well defined (Lemma 7.4) and satisfies the following:

1) \( \Phi_{00} = id \).
2) \( \Phi_{02} = \Phi_{12} \circ \Phi_{01} \) in the homology level.

Proof of statement 1) First, by suitably choosing \( \theta \), we can assume that for a generic \( K \in \mathbb{R}_{p, \theta}^2 \), the Morse–Smale property up to order 2 holds for the negative gradient flows of \( \mathcal{L}_{H_i} \) for \( i = 0, 1, 2 \). To prove 1) in the proposition, we construct \( \Phi_{00} \) by taking the constant homotopy \( H_t(s, x) = H_0(s, x) \). In this case the flow (7.1) becomes the negative gradient flow of \( \mathcal{L}_{H_0} \) which satisfies the Morse–Smale property up to order 2. Hence, for \( x, y \in \text{crit}_k(\mathcal{L}_{H_0}) \), \( \mathcal{M}(x, y ; H_t) = \emptyset \) unless \( x = y \). In the case \( x = y \), it only contains the constant flow line \( \ell(t) = x \), therefore \( n_0(x, x) = 1 \) and \( n_0(x, y) = 0 \) for \( x \neq y \). Hence, \( \Phi_{00} = id \).

Now we move to the proof of the second point, which is more involved and technical. We consider the function \( H_{t ij} \) defined by
\[ H_{t ij}(t, s, \phi, \psi) = \eta(t)H_j(s, \phi, \psi) + (1 - \eta(t))H_i(s, \phi, \psi) \]
for \( 0 \leq i, j \leq 2 \) and also for \( R > 1 \),
\[ H_{02,R}(t, s, \phi, \psi) = \eta(t)H_{12}(t-R, s, \phi, \psi) + (1 - \eta(t))H_{01}(t + R, s, \phi, \psi), \]
where \( \eta \) is the same function at the beginning of this section.

We consider now the flow
\[
\frac{d\ell}{dt} = -\nabla_{G^K} \mathcal{L}_{H_{02,R}}(\ell(t)), \quad t \in \mathbb{R}.
\]
Since the flow is autonomous except in the intervals $[-R, -R + 1]$ and $[R, R + 1]$, all the compactness estimates hold as in Propositions 7.1 and 7.2. Indeed it is easy to see that
\[ \left| \frac{\partial H_{02, R}}{\partial t} \right| \leq 4\epsilon(1 + |\psi|^{p+1}). \]

Therefore, if $\epsilon < \frac{\epsilon_0}{4}$ (where $\epsilon_0 > 0$ is as in Proposition 7.1) we have the results of the previously cited propositions by integrating between $[-R, -R + 1] \cup [R, R + 1]$ instead of $[0, 1]$ in the proofs. Moreover, the estimates are independent of $R$ since the $\epsilon$ that we picked does not depend in $R$. Furthermore, under the assumption of the lemma, we may assume, after perturbing the metric if necessary, the system (7.30) satisfies the Morse–Smale property up to order 2. In fact, this follows from the same argument as remarked after the proof of Lemma 7.5 when $H_i \neq H_j$ for $i \neq j$. Notice that we did not exclude the case $H_i = H_j$ for some $i$ and $j$. In this case, in view of (1), it suffices to consider the case $H_0 = H_2$ and $H_0 \neq H_1$ to prove our claim. In this case, we also have $\dot{\ell}(t) \neq 0$ for the flow $\ell$ of (7.30); otherwise, we have $\ell(t) = x$ for some $x \in \mathcal{F}^{1,1/2}(S^1, N)$ and $\nabla_{GK} \mathcal{L}_{H_{02, R}}(x) = 0$. But this implies $x \in \text{crit}(\mathcal{L}_{H_0})$ by considering $t \leq -R$ and $x \in \text{crit}(\mathcal{L}_{H_1})$ by considering $1 - R \leq t \leq 0$ and therefore $x \in \text{crit}(\mathcal{L}_{H_0}) \cap \text{crit}(\mathcal{L}_{H_1})$. But this contradicts the assumption (A). Thus the Morse–Smale property will be satisfied after perturbing the metric.

Therefore one can define the chain homomorphism $\Phi_{02, R} : \{C_\ast(\mathcal{L}_{H_0}, \mathcal{Z}_2), \partial_\ast(\mathcal{L}_{H_0} G^K)\} \rightarrow \{C_\ast(\mathcal{L}_{H_2}, \mathcal{Z}_2), \partial_\ast(\mathcal{L}_{H_2} G^K)\}$ by counting solutions of (7.30) as in (7.29).

The assertion 2) of Proposition 7.3 follows from the following lemmata:

**Lemma 7.7** The chain homomorphism $\Phi_{02, R}$ is chain homotopy equivalent to $\Phi_{02}$.

**Proof** We consider again the following homotopy defined for $0 \leq r \leq 1$ by
\[ H_r(t, s, \phi, \psi) = r H_{02, R}(t, s, \phi, \psi) + (1 - r) H_{02}(t, s, \phi, \psi). \]

For $x_0 \in \text{crit}(\mathcal{L}_{H_0})$ and $x_2 \in \text{crit}(\mathcal{L}_{H_2})$, we define the operator
\[ \tilde{\mathcal{F}}_{x_0, x_2} : [0, 1] \times W^{1,2}_{x_0, x_2}(\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N)) \rightarrow L^2(\mathbb{R}, T\mathcal{F}^{1,1/2}(S^1, N)) \]
by
\[ \tilde{\mathcal{F}}_{x_0, x_2}(r, \ell) = \frac{d\ell}{dt} + \nabla_{GK} \mathcal{L}_{H_r}(\ell(t)). \]

We may assume, by a perturbation again of $G^K$, that 0 is a regular value of $\mathcal{F}_{x_0, x_2}$. Also, notice that
\[ \left| \frac{\partial H_r}{\partial t} \right| \leq 6\epsilon(1 + |\psi|^{p+1}). \]

Therefore, for $\epsilon < \frac{\epsilon_0}{6}$ ($\epsilon_0 > 0$ is as in Proposition 7.1), the compactness estimates of Propositions 7.1 and 7.2 hold, since the flow is autonomous except on the intervals of length 1, $[-R, -R + 1]$, $[0, 1]$ and $[R, R + 1]$ and the argument given above for the flow (7.30) applies in this case also. It follows then that the set $\mathcal{M}(x_0, x_2; H_r) = \tilde{\mathcal{F}}_{x_0, x_2}^{-1}(0) \subset [0, 1] \times W^{1,2}_{x_0, x_2}(\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N))$ is a manifold and relatively compact in $[0, 1] \times C^0_{\text{loc}}(\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N))$. Moreover, by transversality, we have that
\[ \dim \mathcal{M}(x_0, x_2; H_r) = \mu H_{0}(x_0) - \mu H_{2}(x_2) + 1. \]

We focus now on the zero dimensional case. So we take $x_{0,k} \in \text{crit}(\mathcal{L}_{H_0})$ and $x_{2,k+1} \in \text{crit}_{k+1}(\mathcal{L}_{H_2})$. By compactness, we can define the number $\tilde{n}(x_{0,k}, x_{2,k+1})$ as the cardinal of
$\mathcal{M}(x_{0,k}, x_{2,k+1}; H_r) \mod 2$. That is,

$$\bar{n}(x_{0,k}, x_{2,k+1}) = \#\mathcal{M}(x_{0,k}, x_{2,k+1}; H_r) \mod 2.$$  

Then we define the morphism $\tilde{\Phi} : C_k(\mathcal{L}_{H_0}; \mathbb{Z}_2) \to C_{k+1}(\mathcal{L}_{H_2}; \mathbb{Z}_2)$ on the generators by

$$\tilde{\Phi}(x_{0,k}) = \sum_{x_{2,k+1} \in \text{crit}_k(\mathcal{L}_{H_2})} \bar{n}(x_{0,k}, x_{2,k+1})x_{2,k+1} \quad (7.31)$$  

and extend by linearity. Here we observe that the above sum is finite by the fact that an analogue of Proposition 7.1 holds for flow lines defining (7.31) by the same reasoning as before and the Palais–Smale condition for $\mathcal{L}_{H_2}$.

Again by the usual compactness and gluing argument as described above, we have that $\mathcal{M}(x_{0,k}, x_{2,k}; H_r)$ is a one dimensional manifold and has a natural compactification to $\overline{\mathcal{M}}(x_{0,k}, x_{2,k}; H_r)$. Note that the non-compactness of $\mathcal{M}(x_{0,k}, x_{2,k}; H_r)$ only comes from the breaking of flow lines. By transversality of the negative gradient flow of $\mathcal{L}_{H_r}$ for $r = 0, 1$, the breaking of flow line only occurs for $0 < r < 1$ by the index constraint. Thus the components of its boundary $\partial \overline{\mathcal{M}}(x_{0,k}, x_{2,k+1}; H_r)$ is one of the following four types of (broken) flow lines:

- broken flow line whose components are $(r, \ell_0)$ and $\ell_2$, where $(r, \ell_0)$ satisfies $0 < r < 1$ and $\tilde{\mathcal{F}}(x_{0,k}, x_{2,k+1}; (r, \ell_0)) = 0$ (thus $\ell_0$ is a flow line of the vector field $-\nabla^{G_K} \mathcal{L}_{H_r}$ with $\ell_0(-\infty) = x_{0,k}$ and $\ell_0(+\infty) = x_{2,k+1}$) for some $x_{2,k+1} \in \text{crit}_{k+1}(\mathcal{L}_{H_2})$ and $\ell_2$ is a flow line of the vector field $-\nabla^{G_K} \mathcal{L}_{H_2}$ connecting $x_{2,k+1}$ and $x_2$.
- broken flow line whose components are $\ell_0$ and $(r, \tilde{\ell}_2)$, where $\ell_0$ is a flow line of the vector field $-\nabla^{G_K} \mathcal{L}_{H_0}$ connecting $x_{0,k}$ and $x_{0,k-1}$ for some $x_{0,k-1} \in \text{crit}_k(\mathcal{L}_{H_0})$ and $(r, \tilde{\ell}_2)$ satisfies $0 < r < 1$ and $\tilde{\mathcal{F}}(x_{0,k-1}, x_{2,k}; (r, \tilde{\ell}_2)) = 0$ (thus $\tilde{\ell}_2$ is a flow line of the vector field $-\nabla^{G_K} \mathcal{L}_{H_r}$ with $\tilde{\ell}_2(-\infty) = x_{0,k-1}$ and $\tilde{\ell}_2(+\infty) = x_{2,k}$).
- flow line connecting $x_{0,k}$ to $x_{2,k}$ via the flow of the vector field $-\nabla^{G_K} \mathcal{L}_{H_0}$.
- flow line connecting $x_{0,k}$ to $x_{2,k}$ via the flow of the vector field $-\nabla^{G_K} \mathcal{L}_{H_0}$.

In the above list, the first two correspond to breaking of flow lines, i.e., compactified points while the last two correspond to the boundary of $\mathcal{M}(x_{0,k}, x_{2,k}; H_r)$. Note that, since the negative gradient flow of $\mathcal{L}_{H_r}$ is not necessary be Morse–Smale for $0 < r < 1$, breaking of the first two types arise as the boundary of the compactified manifold.

Therefore, the sum of the number of these components is 0 modulo 2:

$$\sum_{x_{2,k+1} \in \text{crit}_k(\mathcal{L}_{H_2})} \bar{n}(x_{0,k}, x_{2,k+1})n_2(x_{2,k+1}, x_{2,k})$$

$$+ \sum_{x_{0,k-1} \in \text{crit}_{k-1}(\mathcal{L}_{H_0})} n_0(x_{0,k}, x_{0,k-1})\bar{n}(x_{0,k-1}, x_{2,k})$$

$$+ n_0(x_{0,k}, x_{2,k}) + n_{02, R}(x_{0,k}, x_{2,k}) = 0 \quad (\text{mod } 2). \quad (7.32)$$

But notice that

$$\left( \partial_{k+1}(\mathcal{L}_{H_2}; G^K) \circ \tilde{\Phi} + \tilde{\Phi} \circ \partial_k(\mathcal{L}_{H_0}; G^K) + \Phi_{02} + \Phi_{02, R} \right)(x_{0,k}) =$$

$$\sum_{x_{2,k} \in \text{crit}_k(\mathcal{L}_{H_2})} \sum_{x_{2,k+1} \in \text{crit}_{k+1}(\mathcal{L}_{H_2})} \bar{n}(x_{0,k}, x_{2,k+1})n_2(x_{2,k+1}, x_{2,k})x_{2,k}$$

$$+ \sum_{x_{2,k} \in \text{crit}_k(\mathcal{L}_{H_2})} \sum_{x_{0,k-1} \in \text{crit}_{k-1}(\mathcal{L}_{H_0})} n_0(x_{0,k}, x_{0,k-1})\bar{n}(x_{0,k-1}, x_{2,k})x_{2,k}.$$
Therefore, from (7.32), we have
\[ \partial_{k+1}(L_{H_2}, G^K) \circ \Phi + \Phi \circ \partial_k(L_{H_0}, G^K) + \Phi_{02} + \Phi_{02, R} = 0. \]
Hence \( \Phi \) is a chain homotopy between \( \Phi_{02} \) and \( \Phi_{02, R} \).

**Lemma 7.8** Consider an arbitrary \( x_{0,k} \in \text{crit}(L_{H_0}) \), then for \( R > 1 \) large enough,
\[ \Phi_{02, R}(x_{0,k}) = \Phi_{12} \circ \Phi_{01}(x_{0,k}). \]

**Proof** The idea of the proof lies mainly in studying the moduli space \( M(x_{0,k}, x_{2,k}, H_{02,R}) \) for \( x_{0,k} \in \text{crit}(L_{H_0}) \) and \( x_{2,k} \in \text{crit}(L_{H_2}) \), when \( R \to \infty \). It is similar to the case of Floer homology in [7] and [34]. We will give a sketch of it in our case. First, using the same compactness estimates as in Propositions 7.1 and 7.2, we can show that sequence of flow lines \( \psi_{02,R_n} \in M(x_{0,k}, x_{2,k}, H_{02,R_n}) \) is relatively compact in the Hausdorff topology when \( R_n \to \infty \). Moreover, its limit is a broken flow line whose component consists of the following types of trajectories:

1. The gradient flow lines of \( L_{H_0} \)
2. The gradient flow lines of \( L_{H_{01}} \)
3. The gradient flow lines of \( L_{H_1} \)
4. The gradient flow lines of \( L_{H_{12}} \)
5. The gradient flow lines of \( L_{H_2} \)

Notice that in the cases i), iii) and v) the flow lines are constant because of the index restriction. So the important flow lines are in the cases ii) and iv). It can be shown as in [7, §11.5] that \( \ell_n(-R_n) \to \ell_{01} \) in \( C^0_{\text{loc}}(\mathbb{R}, S^{1.1/2}(S^1, N)) \) and \( \ell_n(+R_n) \to \ell_{12} \) in \( C^0_{\text{loc}}(\mathbb{R}, S^{1.1/2}(S^1, N)) \) as \( R_n \to \infty \), where \( \ell_{01} \in M(x_{0,k}, x_{1,k}, H_{01}) \) and \( \ell_{12} \in M(x_{1,k}, x_{2,k}, H_{12}) \) for some \( x_{1,k} \in \text{crit}(L_{H_1}) \). Conversely, given broken trajectory from \( x_{0,k} \) to \( x_{1,k} \) then from \( x_{1,k} \) to \( x_{2,k} \), by the compactness argument and transversality, we can do the same gluing construction as in Sect. 3.5 to associate for \( R > 1 \) big enough a unique element in \( M(x_{0,k}, x_{2,k}, H_{02,R}) \) which converges (in the sense of Hausdorff topology) to the broken trajectory. Therefore, we have the following identification for \( R \) large enough:
\[ \bigcup_{x_{1,k} \in \text{crit}(L_{H_1})} M(x_{0,k}, x_{1,k}, H_{01}) \times M(x_{1,k}, x_{2,k}, H_{12}) \to M(x_{0,k}, x_{2,k}, H_{02,R}). \]

It follows that
\[ n_{02,R}(x_{0,k}, x_{2,k}) = \sum_{x_{1,k} \in \text{crit}(L_{H_1})} n_{01}(x_{0,k}, x_{1,k})n_{12}(x_{1,k}, x_{2,k}) \]
and
\[ \Phi_{02,R}(x_{0,k}) = \sum_{x_{2,k} \in \text{crit}(L_{H_2})} \sum_{x_{1,k} \in \text{crit}(L_{H_1})} n_{01}(x_{0,k}, x_{1,k})n_{12}(x_{1,k}, x_{2,k})x_{2,k} \]
\[ = \Phi_{12} \circ \Phi_{01}(x_{0,k}). \]
Proposition 7.1 for (7.30) which holds independent of big enough \( R > 1 \) as remarked before. Thus the number of such \( x_{2,k} \) independent of \( R > 1 \) big enough. Thus the proof of the lemma is completed. \( \square \)

Now we go back to the proof of Proposition 7.3, 2). Consider a cycle \( c_{0,k} = \sum_i x_{0,k}^i \in C_k(L_{H_0}; \mathbb{Z}_2) \). From Lemma 7.7, we have that 
\[
\Phi_{02}([c_{0,k}]) = \Phi_{02,R}([c_{0,k}]).
\]
Now for \( R > 1 \) big enough, possibly depending on \( c_{0,k} \), we have from Lemma 7.8 
\[
\Phi_{02,R}([c_{0,k}]) = \Phi_{12} \circ \Phi_{01}([c_{0,k}]).
\]
Combining these equalities, we finish the proof of the second point of Proposition 7.3. \( \square \)

As a corollary from the previous construction, we can drop the assumption (A) to define a homomorphism \( \Phi_{01} : HF_*(L_{H_0}, \mathcal{F}^{1,1/2}(S^1, N); \mathbb{Z}_2) \rightarrow HF_*(L_{H_1}, \mathcal{F}^{1,1/2}(S^1, N); \mathbb{Z}_2) \). Indeed, we consider \( H_0 \) and \( H_1 \) such that \( d_{p+1}(H_0, H_1) < \varepsilon \). Then we can perturb \( H_1 \) by \( h \in \mathbb{H}^3 \) so that the pairs \((H_0, H)\) and \((H, H_1)\) satisfy (A) for \( H = H_1 + h \) and such that \( d_{p+1}(H_0, H) < \varepsilon, d_{p+1}(H_1, H) < \varepsilon \). Then by Proposition 7.3, we have that 
\[
\Phi_{H_0,H} : HF_*(L_{H_0}, \mathcal{F}^{1,1/2}(S^1, N); \mathbb{Z}_2) \rightarrow HF_*(L_{H_1}, \mathcal{F}^{1,1/2}(S^1, N); \mathbb{Z}_2)
\]
and 
\[
\Phi_{H,H_1} : HF_*(L_{H}, \mathcal{F}^{1,1/2}(S^1, N); \mathbb{Z}_2) \rightarrow HF_*(L_{H_1}, \mathcal{F}^{1,1/2}(S^1, N); \mathbb{Z}_2)
\]
are well defined and hence we can define \( \Phi_{01} = \Phi_{H,H_1} \circ \Phi_{H_0,H} \). We claim that \( \Phi_{01} \) defined this way is independent of the perturbation \( H \). Indeed, consider another function \( H' = H_1 + h' \) such that \( d_{p+1}(H_0, H') < \varepsilon \) and \( d_{p+1}(H', H_1) < \varepsilon \). Then by Lemma 7.5, there exists two generic sets \( \mathbb{H}(H) \) and \( \mathbb{H}(H') \) in \( \mathbb{H}^3 \) such that for \( k \in \mathbb{H}(H) \) and \( k' \in \mathbb{H}(H') \) the condition (A) is satisfied for pairs \((H_0, H_1 + k), (H_1, H_1 + k), (H, H_1 + k), (H_0, H_1 + k'), (H_1, H_1 + k')\) and \((H', H_1 + k')\). Then we have in the homology level 
\[
\Phi_{H,H_1} \circ \Phi_{H_0,H} = (\Phi_{H_1+k,H_1} \circ \Phi_{H,H_1+k}) \circ (\Phi_{H_1+k,H} \circ \Phi_{H_0,H_1+k})
\]
\[
= \Phi_{H_1+k,H_1} \circ (\Phi_{H,H_1+k} \circ \Phi_{H_1+k,H}) \circ \Phi_{H_0,H_1+k}
\]
\[
= \Phi_{H_1+k,H_1} \circ \Phi_{H_0,H_1+k}.
\]
Similarly we have 
\[
\Phi_{H',H_1} \circ \Phi_{H_0,H'} = \Phi_{H_1+k',H_1} \circ \Phi_{H_0,H_1+k'}.
\]

Now notice that since the sets \( \mathbb{H}(H) \) and \( \mathbb{H}(H') \) are generic, then \( \mathbb{H}(H) \cap \mathbb{H}(H') \) is also generic, hence we can pick \( k = k' \in \mathbb{H}(H) \cap \mathbb{H}(H') \) and this leads to 
\[
\Phi_{H,H_1} \circ \Phi_{H_0,H} = \Phi_{H',H_1} \circ \Phi_{H_0,H'}.
\]
In a similar way we can show that the naturality also holds, that is \( \Phi_{00} = \text{id} \) and \( \Phi_{02} = \Phi_{12} \circ \Phi_{01} \). Therefore, we obtain the following:

**Corollary 7.2** Let \( H_0, H_1 \) and \( H_2 \) in \( \mathbb{H}^3 \) be such that \( L_{H_i} \) is Morse for \( i = 0, 1, 2 \) and \( d_{p+1}(H_i, H_j) < \varepsilon \) for \( 0 \leq i, j \leq 2 \). Then we have a natural isomorphism 
\[
\Phi_{ij} : HF_*(L_{H_i}, \mathcal{F}^{1,1/2}(S^1, N); \mathbb{Z}_2) \rightarrow HF_*(L_{H_j}, \mathcal{F}^{1,1/2}(S^1, N); \mathbb{Z}_2).
\]
Moreover, 
\[
\Phi_{00} = \text{id} \text{ and } \Phi_{02} = \Phi_{12} \circ \Phi_{01}.
\]
This allows us to see that the homology $HF_*(\mathcal{L}_H, \mathcal{F}^{1,1/2}(S^1, N); \mathbb{Z}_2)$ can be defined even when $\mathcal{L}_H$ is not a Morse function. Indeed, if $H \in \mathbb{H}_{p+1}^3$, we know from Sect. 5.1 that there exists a generic $h \in \mathbb{H}_b^3$ such that $d_{p+1}(H, H + h) < \varepsilon$ and that $\mathcal{L}_{H+h}$ is Morse. Hence we define

$$HF_*(\mathcal{L}_H, \mathcal{F}^{1,1/2}(S^1, N); \mathbb{Z}_2) := HF_*(\mathcal{L}_{H+h}, \mathcal{F}^{1,1/2}(S^1, N); \mathbb{Z}_2).$$

From Corollary 7.2, this construction is independent from the generic perturbation $h$, therefore the identification is well defined for any function $H$ in $\mathbb{H}_{p+1}^3$. This proves the first point of Theorem 1.1.

Now to prove the second point of Theorem 1.1. We need the following lemma.

**Lemma 7.9** The metric space $(\mathbb{H}_{p+1}^3, d_{p+1})$ is contractible (in fact, it is convex).

**Proof** We fix $H_0 \in \mathbb{H}_{p+1}^3$ and we consider the map $\mathcal{H} : [0, 1] \times \mathbb{H}_{p+1}^3 \to \mathbb{H}_{p+1}^3$ defined by

$$\mathcal{H}(\lambda, H) = \lambda H_0 + (1 - \lambda) H.$$

Now this map is well defined since the assumptions (1.1)–(1.3) are stable under convex combination. In remains to show that $\mathcal{H}$ is continuous. Indeed,

$$\begin{align*}
|\mathcal{H}(\lambda, H) - \mathcal{H}(\lambda', H')| & \leq |\mathcal{H}(\lambda, H) - \mathcal{H}(\lambda', H)| + |\mathcal{H}(\lambda', H) - \mathcal{H}(\lambda', H')| \\
& \leq |\lambda - \lambda'| |H_0 - H| + |1 - \lambda'| |H - H'| \\
& \leq |\lambda - \lambda'| (1 + |\psi|^{p+1}) d_{p+1}(H_0, H) + (1 + |\psi|^{p+1}) d_{p+1}(H, H')
\end{align*}$$

(7.39)

This yields to

$$d_{p+1}(\mathcal{H}(\lambda, H), \mathcal{H}(\lambda', H')) \leq |\lambda - \lambda'| d_{p+1}(H_0, H) + d_{p+1}(H, H'),$$

which gives us the continuity and clearly $H$ is a contraction from $\mathbb{H}_{p+1}^3$ to $\{H_0\}$. □

**Proof of Theorem 1.1 part 2**: We consider two functions $H$ and $H'$ in $\mathbb{H}_{p+1}^3$. Since this last space is contractible, it is connected in particular, so we take a continuous path $(H_\lambda)_{0 \leq \lambda \leq 1}$ in $\mathbb{H}_{p+1}^3$ connection $H$ to $H'$. We fix $\varepsilon > 0$ as in Proposition 7.3. Then there exists a subdivision $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n = 1$ such that $d_{p+1}(H_{\lambda_{i-1}}, H_{\lambda_i}) < \varepsilon$ for $1 \leq i \leq n$. From Corollary 7.2 we know that there exist natural isomorphisms $\Phi_{i-1,i} : HF_*(\mathcal{L}_{H_{\lambda_{i-1}}}, \mathcal{F}^{1,1/2}(S^1, N); \mathbb{Z}_2) \to HF_*(\mathcal{L}_{H_{\lambda_i}}, \mathcal{F}^{1,1/2}(S^1, N); \mathbb{Z}_2)$. We define then the isomorphism $\Phi_{H, H'} = \Phi_{n-1,n} \circ \cdots \circ \Phi_{1,2} \circ \Phi_{0,1}$. The isomorphism $\Phi_{H, H'}$ only depends on the homotopy class of $(H_\lambda)_{0 \leq \lambda \leq 1}$, but since the space $\mathbb{H}_{p+1}^3$ is contractible, we have that this isomorphism is independent of the choice of homotopy $(H_\lambda)_{0 \leq \lambda \leq 1}$. This finishes the proof of the second point of Theorem 1.1. □

**Definition 7.2** In view of the above result, for $p \geq 3$, we define the $(p + 1)$-Dirac-geodesic homology of $N$ by

$$DG^{p+1}H_*(N; \mathbb{Z}_2) := HF_*(\mathcal{L}_H, \mathcal{F}^{1,1/2}(S^1, N); \mathbb{Z}_2),$$

where $H \in \mathbb{H}_{p+1}$ and the definition is independent of $H \in \mathbb{H}_{p+1}$. □
8 Computation of the homology

8.1 $V$-homology

We consider the following functionals $E_1 : H^1(S^1, N) \to \mathbb{R}$ and $E_2 : \mathcal{F}^{1,1/2}(S^1, N) \to \mathbb{R}$ defined by

$$E_1(\phi) = \frac{1}{2} \int_{S^1} |\dot{\phi}(s)|^2 \, ds - \int_{S^1} v(s, \phi(s)) \, ds,$$

and

$$E_2(\phi, \psi) = \frac{1}{2} \int_{S^1} (D\phi \psi(s), \psi(s)) ds - \int_{S^1} H(s, \phi(s), \psi(s)) ds,$$

where $v = v(s, p)$ $(s \in S^1, p \in N)$ is a smooth function on $S^1 \times N$ such that $E_1$ becomes a Morse function on $H^1(S^1, N)$ and $H \in \mathbb{H}^3_{p+1}$. Notice that, if $\pi : \mathcal{F}^{1,1/2}(S^1, N) \to H^1(S^1, N)$ is the canonical projection, then

$$E_1 \circ \pi + E_2 = \mathcal{L}_{H+v}.$$

For simplicity, we will write $E_1$ instead of $E_1 \circ \pi$. We consider then the vector field $V$ on $\mathcal{F}^{1,1/2}(S^1, N)$ defined by

$$V(x) = \nabla^H E_1(x) + \nabla^N E_2(x),$$

where $\nabla^H$ stands for the horizontal gradient, that is $\nabla_\phi$ and $\nabla^N$ is the vertical gradient, that is the gradient on the fibers, i.e. $\nabla_\psi$. So $V$ can be written as

$$V(x) = \begin{pmatrix} (-\Delta + 1)^{-1} \left( - \nabla_s \partial_s \phi - \nabla_\phi v(s, \phi) \right) \\ (1 + |D|)^{-1} \left( D\phi \psi - \nabla_\psi H(s, \phi, \psi) \right) \end{pmatrix}$$

(8.1)

for $x = (\phi, \psi) \in \mathcal{F}^{1,1/2}(S^1, N)$.

From this decomposition, we see that we have the following:

Lemma 8.1 The rest points of the vector field $V$ are of the form $(\phi, \psi)$ where $\phi$ is a perturbed geodesic satisfying

$$-\nabla_s \partial_s \phi = \nabla_\phi v(s, \phi),$$

and $\psi$ is a critical point of the functional $E_2$ restricted to the fiber over $\phi$ satisfying

$$D\phi \psi = \nabla_\psi H(s, \phi, \psi).$$

We will denote by $\text{rest}(V)$ the rest points of the vector field $V$. Now notice that if $(\phi, \psi) \in \text{rest}(V)$, then $\phi$ has a natural index as a critical point of the Morse function $E_1$. Similarly, for $\psi$, we can define a relative index for it. Indeed, by considering the differential $dV$ we see that we can write

$$dV(\phi, \psi) = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}$$

(8.2)

so that for all $u = (X, \xi) \in T_{(\phi, \psi)} \mathcal{F}^{1,1/2}(S^1, N)$,

$$A_{11}[X] = \nabla_\phi^2 E_1[X]$$

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\[
A_{21}[X] = (1 + |D|)^{-1} \left( \partial_s \cdot \psi^i X^m \gamma^j_{mk} \otimes \frac{\partial}{\partial y^k}(\phi) + \nabla_X \Gamma^i_{jk}(\phi) \partial_s \psi^j \partial_s \psi^k \otimes \frac{\partial}{\partial y^i}(\phi) \right. \\
+ \Gamma^i_{jk}(\phi) \partial_s X^m \partial_s \psi^k \otimes \frac{\partial}{\partial y^i}(\phi) + \left. \Gamma^i_{jk}(\phi) \partial_s \psi^j \partial_s \psi^k \Gamma^m_{li} X^l \otimes \frac{\partial}{\partial y^m}(\phi) \right) - H_{\psi}(\phi, \psi)[X],
\]
(8.4)

\[
A_{22}[\xi] = \nabla^2_{\psi, \psi} E_{21}[\xi] \\
= (1 + |D|)^{-1}(D_\phi \xi - H_{\psi}(\phi, \psi))\{\xi\}.
\]
(8.5)

Again we use the same indexing process as in Sect. 2, that is, we fix \( x_0 = (\phi_0, \psi_0) \in \mathcal{F}^{1,1/2}(S^1, N) \) and a path \( x_t = (\phi_t, \psi_t)_{0 \leq t \leq 1} \) connecting it to \( x = (\phi, \psi) \in \text{rest}(V) \). The relative Morse index of \( x \) is then defined by

\[
\mu_0(x) := -sf[dV(x_t)]_{0 \leq t \leq 1}.
\]
(8.6)

Notice then by the above formula given for \( dV \), that

\[
sf[dV(x_t)]_{0 \leq t \leq 1} = sf[A_{11}(x_t)]_{0 \leq t \leq 1} + sf[A_{22}(x_t)]_{0 \leq t \leq 1}
\]

Therefore, we have that

\[
\mu_0(x) = \mu_1(\phi) + \mu_2(\phi, \psi)
\]
(8.7)

where

\[
\mu_1(\phi) = -sf[A_{11}(x_t)]_{0 \leq t \leq 1} \quad \text{and} \quad \mu_2 = -sf[A_{22}(x_t)]_{0 \leq t \leq 1}.
\]

We note that if we choose \( \phi_0 \) to be a minimizer of \( E_1 \), then \( \mu_1 \) coincides with the Morse index of \( \phi \) as a critical point of \( E_1 \). Based on this splitting of the index, we can define then

**Definition 8.1** For \((r, q) \in (\mathbb{N} \cup \{0\}) \times \mathbb{Z}\) we define the set \( \text{rest}_{r,q}(V) \subset \text{rest}(V) \) by

\[
\text{rest}_{r,q}(V) = \{(\phi, \psi) \in \text{rest}(V); \mu_1(\phi) = r, \mu_2(\phi, \psi) = q\},
\]

and

\[
\text{rest}_k(V) = \{(\phi, \psi) \in \text{rest}(V); \mu_0(\phi, \psi) = k\} = \bigcup_{r+q=k} \text{rest}_{r,q}(V).
\]

The vector field \( V \) has a good compactness property.

**Lemma 8.2** The vector field \( V \) satisfies the generalized Palais–Smale condition in the following sense: If a sequence \( \{x_k\} \subset \mathcal{F}^{1,1/2}(S^1, N) \) satisfies \( V(x_k) \to 0 \), \(|E_1(\phi_k)| \leq C \) and \( E_2(x_k) \leq C \) for some \( C \in \mathbb{R} \), then \( \{x_k\} \) have a convergent subsequence.

**Proof** This proof is similar to the classical Palais–Smale condition for the functional \( L_H \). In fact it is even easier since we have the compactness and convergence for the \( \phi_k \) part of \( x_k \) so the part that needs to be checked is for the \( \psi \) part. But this follows from the local trivialization around \( \phi \) and it is similar in nature to proof of the Palais–Smale in [20] and [19].

It is a classical result that for a generic set of perturbations \( v \), \( E_1 \) is Morse (see for example [37]). From [22], we have that for every critical point \( \phi \) of \( E_1 \), the functional \( E_2(\phi, \cdot) \) is Morse for a generic set of perturbations \( H \in \mathbb{H}^{3}_{p+1} \). Since the critical points of
\[ E_1 \] are countable, we have the existence of a generic subset of \( \mathbb{R}^{p+1}_x \) such that \( E_2(\phi, \cdot) \) is Morse for every \( \phi \in \text{crit}(E_1) \). Notice that we can also perturb the vector field \( V \) in a slightly different way from the perturbation of the gradient flow of \( L_H \) to achieve transversality, as follow

\[
V_K(x) = (1 + K) \left( (-\Delta + 1)^{-1} \left( -\nabla_s \partial_s \phi - \nabla_{\phi} v(s, \phi) \right) \right),
\]

where \( K \in \bar{\mathbb{K}}^2_{\theta, \rho_0} \subset \mathbb{K}^2_{\theta, \rho_0} \) for \( \theta \) chosen in a suitable way and we add the fact that \( K \) is lower triangular. That is, \( K \in \bar{\mathbb{K}}^2_{\theta, \rho_0} \) if and only if \( K \in \mathbb{K}^2_{\theta, \rho_0} \) and

\[
K = \begin{pmatrix} K_{11} & 0 \\ K_{21} & K_{22} \end{pmatrix}.
\]

Moreover, we can take \( K_{11} \) independent of \( \psi \), i.e., \( K_{11} = K_{11}(\phi) \). This choice is made in such a way to preserve the decoupling of the first equation of the flow. We claim that with this choice, transversality can be achieved. Notice that the main issue one needs to worry about is the result of Lemma 5.8. First we start by two rest points \( x_+ = (\phi_+, \psi_+) \) and \( x_- = (\phi_-, \psi_-) \). Notice that we have two kinds of flows for \(-V\). The first kind is when \( \phi_+ = \phi_- \), hence \( \ell(t) = (\phi_+, \psi(t)) \). In this case the stable and unstable manifold depend mainly on the \( \psi \) component and transversality can be achieved via a perturbation \( K \) with \( K_{11} = K_{21} = 0 \) as in [22]. Indeed, define the map

\[
\tilde{\mathcal{F}}^V_{x_-, x_+} : W^{1,2}_{x_-, x_+}(\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N)) \times \bar{\mathbb{K}}^2_{\theta, \rho_0} \rightarrow L^2(\mathbb{R}, T\mathcal{F}^{1,1/2}(S^1, N))
\]

by

\[
\tilde{\mathcal{F}}^V_{x_-, x_+}(\ell, K) := \frac{d\ell}{dt} + (1 + K(\ell(t)))V(\ell(t)).
\]

We also set, for a given \( K \in \bar{\mathbb{K}}^2_{\theta, \rho_0} \),

\[
\tilde{\mathcal{F}}^V_{x_-, x_+}(\ell, K) = \tilde{\mathcal{F}}^V_{x_-, x_+}(\cdot, K).
\]

In the case \( \phi_- = \phi_+ \) we will let \( H_{\phi_-}^{1/2} \) denote the fiber above \( \phi_- \). That is,

\[
H_{\phi_-}^{1/2} = H^{1/2}(S^1, S(S^1) \otimes \phi^* T N).
\]

We also define the map \( T_{\psi_-, \psi_+}^{K_{22}} : \psi_+ + W^{1,2}(\mathbb{R}, H^{1/2}_{\psi_-}) \rightarrow L^2(\mathbb{R}, H^{1/2}_{\psi_-}) \), where \( \psi_+ \in C^\infty(\mathbb{R}, H^{1/2}_{\psi_-}) \) is such that \( \psi_+(t) = \psi_- \) if \( t \leq -1 \) and \( \psi_+(t) = \psi_+ \) if \( t \geq 1 \), defined by

\[
T_{\psi_-, \psi_+}^{K_{22}}(\psi(t)) = \frac{d\psi}{dt} + (1 + K_{22}(\phi_-, \psi(t)))\nabla_\psi L_H(\phi_-, \psi(t)).
\]

**Lemma 8.3** Assume that \( \phi_- = \phi_+ \) and let \( \ell \) be a flow line of \(-V_K\) connecting \( x_- \) and \( x_+ \), where \( K \in \bar{\mathbb{K}}^2_{\theta, \rho_0} \), then the map \( \tilde{\mathcal{F}}^V_{x_-, x_+} \) is regular along \( \ell(t) \), if and only if the map \( T_{\psi_-, \psi_+}^{K_{22}} \) is regular along \( \psi(t) \).

**Proof** Since \( E_1 \) is decreasing along \(-V_K\)-flow lines, one has that \( \phi(t) \) is constant along the flow of \(-V_K\) and hence the flow stays in the same fiber \( H_{\phi_-}^{1/2} \). We compute now

\[
d_{\ell(t)} \tilde{\mathcal{F}}^V_{x_-, x_+}[X, Y] = \begin{bmatrix} \dot{X} + d^2E_1(\phi_-)X \\ d_\phi, \psi E_2X + d_\psi T_{\psi_-, \psi_+}^{K_{22}} \end{bmatrix} = \begin{pmatrix} B_{11} & 0 \\ B_{12} & B_{22} \end{pmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}.
\]
Because of the triangular nature of \( d_{\ell(t)} \tilde{\mathcal{F}}^{V, K}_{x_-, x_+} \) and \( B_{11} \) is invertible (because \( E_1 \) is Morse), we have that \( d_{\ell(t)} \tilde{\mathcal{F}}^{V, K}_{x_-, x_+} \) is a left inverse if and only if \( B_{22} = d_{\psi(t)} T^{K_{22}}_{\psi_-, \psi_+} \) is a left inverse. □

The second kind of flows is when \( \phi_+ \neq \phi_- \). In this case we have always that \( \nabla \psi E_1 (\phi(t)) \neq 0 \) for \( t \in \mathbb{R} \). We consider \( w \in C^2 (\mathbb{R}, \ell^* T^1 S^1) \) with supp \((w) \subset [a, b] \). We will show as in Lemma 5.8, the existence of \( \nabla \) in Lemma 5.8, we can always find \( \psi \) not depend on \( k \) such that

\[
\| k(\ell) V(\ell) - w \|_{\ell^* T^1 S^1} < \epsilon,
\]

for a given \( \epsilon > 0 \). First, we set \( w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \). We have that

\[
d_K \tilde{\mathcal{F}}^{V}_{x_-, x_+}[k] = k(\ell) V(\ell) = \begin{bmatrix} k_{11} \nabla \phi E_1 (\phi) \\ k_{21} \nabla \phi E_1 (\phi) + k_{22} \nabla \phi E_2 (\ell) \end{bmatrix}.
\]

Now since \( \nabla \phi E_1 (\phi(t)) \neq 0 \) for all \( t \in \mathbb{R} \), as in Lemma 5.8 (we replace \( \nabla_{1,1/2} L_H \) by \( \nabla \phi E_1 \) and argue as in the proof of Lemma 5.8), given \( \epsilon > 0 \), we can always find a \( k_{11} \) (which does not depend on \( \psi \)) such that

\[
\| k_{11} \nabla \phi E_1 (\phi) - w_1 \|_{p_1(\ell^* T^1 S^1)} < \epsilon,
\]

where here \( p_1 \) is the projection on the loop components of the tangent space. A similar procedure can be done for the second term, by taking \( k_{22} = 0 \). That is, from the construction in Lemma 5.8, we can always find \( k_{21} \) such that

\[
\| k_{21} \nabla \phi E_1 (\phi) - w_2 \|_{p_2(\ell^* T^1 S^1)} < \epsilon,
\]

where \( p_2 \) is the projection on the spinorial component. Hence, there exists \( k \in \mathbb{R}^2 \) such that

\[
\| k(\ell) V(\ell) - w \|_{\ell^* T^1 S^1} < \epsilon.
\]

Now the rest of the proof stays unchanged as in the proof of Lemma 5.8 in order to achieve transversality. □

In the following, we fix generic \( K \in \mathbb{R}^2 \) so that the transversality holds for the \(-V_K\)-flow lines and simply write \( V = V_K \).

The next step is then to define the \( V \)-homology. This is the homology of a chain complex generated by rest points of \( V \) and graded by \( \mu_0 \). As in the Morse homology, the boundary operator is defined by counting \( V \)-flow lines connecting two rest points of index difference 1. For this purpose, for \( x_- = (\phi_-, \psi_-) \in \text{rest}(V) \) and \( x_+ = (\phi_+, \psi_+) \in \text{rest}(V) \) with \( \mu_0(x_-) - \mu_0(x_+) = 1 \), we consider

\[
\frac{d \ell}{dt}(t) + V(\ell(t)) = 0, \quad \ell(-\infty) = x_-, \quad \ell(+\infty) = x_+.
\]

(8.9)

More precisely, we define

\[
\mathcal{F}^V_{x_-, x_+} : W^{1,2}_{x_-, x_+} (\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N)) \to L^2 (\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N))
\]

by \( \mathcal{F}^V_{x_-, x_+}(\ell) = \frac{d \ell}{dt} + V(\ell) \).

As in Proposition 10.2, \( \mathcal{F}^V_{x_-, x_+} \) is \( C^1 \) and \( \ell \) satisfies (8.9) if and only if \( \mathcal{F}^V_{x_-, x_+}(\ell) = 0 \). In order to define the \( V \)-homology, we need the compactness of \( V \)-flow lines. So we start by proving a uniform estimate for the \( \phi \)-component of the flow, as in Proposition 3.3.
Lemma 8.4 Let \( \ell = (\phi, \psi) \) be a solution of (8.9). Then the \( \phi \)-component of \( \ell \) satisfies
\[
\sup_{t \in \mathbb{R}} \| \phi(t) \|_{C^{2,\bar{z}}(S^1)} \leq C(x_-, x_+)
\]
for some \( C(x_-, x_+) > 0 \) depending only on \( x_-, x_+ \).

**Proof** First, notice that the \( \phi \)-component of \( \ell \), namely \( \phi(t) \), satisfies
\[
\frac{d\phi}{dt}(t) = -(1 + K_{11}(\phi)) \nabla_\phi E_1(\phi), \quad \phi(-\infty) = \phi_-, \quad \phi(+\infty) = \phi_+.
\]
where \( \phi_\pm \) is the \( \phi \)-components of \( x_\pm \). Thus \( \phi \) is a flow of the gradient \(-\nabla^{K_{11}} E_1\) (the gradient of \( E_1 \) with respect to the \( g^{K_{11}} \)-metric on \( H^1(S^1, N) \), where \( g^{K_{11}}(\phi)(X, Y) = ((1 + K_{11}(\phi))^{-1}X, Y)_{H^1(S^1)} \)). Hence, we have
\[
E_1(\phi(t)) \leq E_1(\phi(\pm)) \leq E_1(\phi_+), \quad (8.10)
\]
for all \( t \in \mathbb{R} \).

If the perturbation \( K_{11} \equiv 0 \), the assertion of the lemma follows exactly as in the proof of Proposition 3.3 from the inequality by (8.10). In the general case, the perturbation term \( K_{11}(\phi) \) is less regular and the bootstrap argument, as in the proof of Proposition 3.3, does not apply directly. However, this technical point can be overcome with a little trick.

Using the notation of the proof of Proposition 3.3, \( \phi \) satisfies the equation
\[
\frac{\partial \phi}{\partial t} = (1 + K_{11}(\phi))(-\Delta + 1)^{-1}(\Delta \phi + \Gamma(\phi, \partial_\phi \phi, \partial_\phi \phi) + \nabla_\phi v(s, \phi))
\]
\[
= (-\Delta + 1)^{-1}(\Delta \phi + \Gamma(\phi, \partial_\phi \phi, \partial_\phi \phi) + \nabla_\phi v(s, \phi)) + r(\phi), \quad (8.11)
\]
where
\[
r(\phi) = K_{11}(\phi)(-\Delta + 1)^{-1}(\Delta \phi + \Gamma(\phi, \partial_\phi \phi, \partial_\phi \phi) + \nabla_\phi v(s, \phi)).
\]
Using (8.10) and (3.55), we have for \( s > \frac{1}{2} \)
\[
\|(-\Delta + 1)^{-1}(\Delta \phi + \Gamma(\phi, \partial_\phi \phi, \partial_\phi \phi) + \nabla_\phi v(s, \phi))\|_{W^{2,\bar{z}}(S^1)} \leq C(x_-, x_+), \quad (8.12)
\]
for all \( t \in \mathbb{R} \), where \( C(x_-, x_+) > 0 \) is a constant depending only on \( x_\pm \).

On the other hand, by our definition of the perturbation \( K_{11} \), \( r(\phi) \) has at most the regularity of \( \phi \) and we cannot gain regularity from (8.11) by the argument of the proof of Proposition 3.3. To overcome this, as in (3.54), we write the Eq. (8.11) as an integral equation of the form
\[
\phi - \mathcal{R}(\phi) = \int_0^\infty e^{-\tau}(-\Delta + 1)^{-1}(\Delta \phi(t - \tau) + \Gamma(\phi(t - \tau), \partial_\phi \phi(t - \tau), \partial_\phi \phi(t - \tau) + \nabla_\phi v(s, \phi(t - \tau))) d\tau, \quad (8.13)
\]
where
\[
\mathcal{R}(\phi) = \int_0^\infty e^{-\tau}r(\phi(t - \tau)) d\tau. \quad (8.14)
\]
Here we observe that, by our definition of the perturbation as noted above, we can take the perturbation \( K_{11} \) such that \( K_{11}(\phi) \) and hence \( \mathcal{R}(\phi) \) has the same regularity of \( \phi \) (because we have chosen \( u_{ij}^1, u_{ij}^2 \) to be smooth and arbitrary \( K \in \mathbb{K}_2^2 \) can be approximated by finite sum of operators \( K_{ijk} \)). Moreover, the following estimates can be checked easily:
\[
\|\mathcal{R}(\phi)\|_{H^1(S^1)} \leq C \rho_0\|\phi\|_{H^1(S^1)}(1 + \|\phi\|^2_{H^1(S^1)}) \quad \text{for} \quad \phi \in H^1(S^1, N), \quad (8.15)
\]
For simplicity, we denote the extended operator by the same nearest point projection as in the proof of Proposition 10.1. Let \( R \) denote \( f \) and define two contractions, where \( \phi \) is a fixed point of \( \hat{\phi} \). On the other hand, the estimates (8.15)–(8.18) continue to hold on \( H^1(S^1, \mathbb{R}^k) \) and from (8.14) using the contraction mapping principle, we need to extend \( R \) to \( H^1(S^1, \mathbb{R}^k) \) (recall that \( i : N \subset \mathbb{R}^k \) is the isometric embedding). This can be done in a similar manner as in the proof of Proposition 10.1 in the “Appendix”. Namely, let \( N \subset V \subset U \) be two tubular neighborhoods of \( N \) and \( \pi_N : U \to N \) be a smooth nearest point projection as in the proof of Proposition 10.1. Let \( \rho \in C^\infty(\mathbb{R}^k) \) be a cut-off function such that \( \rho = 1 \) on \( V \) and \( \rho = 0 \) outside of \( U \). For \( \tilde{\phi} \in H^1(S^1, \mathbb{R}^k) \), we then define \( \tilde{K}_{11}(\tilde{\phi}) = \rho(\tilde{\phi})K_{11}(\pi_N(\tilde{\phi})) \) and \( \tilde{\Gamma}(\tilde{\phi}) = \rho(\tilde{\phi})\Gamma(\pi_N(\tilde{\phi}), \partial_t \pi_N(\tilde{\phi}), \partial_t \pi_N(\tilde{\phi})) \). Moreover, we have a uniform bound on \( \rho(t) \).

The estimates (8.15)–(8.18) continue to hold on \( H^1(S^1, \mathbb{R}^k) \) with a different constant \( C > 0 \). For simplicity, we denote the extended operator by the same \( R \).

From (8.15)–(8.18), for \( R > 0 \), if \( \rho_0 > 0 \) is chosen such that \( C\rho_0(1 + R^2) < R/2 \) and \( C\rho_0(1 + 2R^2)^3 < 1 \), then the two maps

\[ R : B_R(H^1) \to B_{R/2}(H^1) \]

and

\[ R : B_R(W^{2-s,2}) \to B_{R/2}(W^{2-s,2}) \]

define two contractions, where \( B_R(H^1) \) and \( B_R(W^{2-s,2}) \) are \( R \)-balls in \( H^1(S^1, \mathbb{R}^k) \) and \( W^{2-s,2}(S^1, \mathbb{R}^k) \). Thus, if we choose \( R = R(x_-, x_+) > 0 \) such that \( w \in B_{R/2}(W^{2-s,2}) \), by the contraction mapping principle, there exists a unique \( \phi \in W^{2-s,2}(S^1, \mathbb{R}^k) \) such that Eq. (8.14) is satisfied:

\[ \phi - R(\phi) = w. \]

On the other hand, \( \phi \in H^1(S^1, N) \subset H^1(S^1, \mathbb{R}^k) \) also satisfies Eq. (8.14) and \( W^{2-s,2}(S^1, \mathbb{R}^k) \) in \( H^1(S^1) \) if we choose \( s > \frac{1}{2} \) close to \( \frac{1}{2} \). Therefore, by the uniqueness of the solution of (8.14) on \( B_R(H^1) \), we must have \( \phi = \tilde{\phi} \), i.e., \( \phi \in W^{2-s,2}(S^1) \). Moreover, we have a uniform estimate

\[ \|\phi(t)\|_{W^{2-s,2}(S^1)} \leq C(x_-, x_+) \]

as a fixed point of \( \phi \mapsto R(\phi) + w \) on \( B_R(W^{2-s,2}) \). In this way, we gain regularity of \( \phi \). Taking \( s = \frac{1}{2} \), and using the Sobolev embedding, we obtain sup_{t \in \mathbb{R}} \|\partial_t \phi(t)\|_{L^6(S^1)} \leq C(x_-, x_+) \) as in (3.59) and sup_{t \in \mathbb{R}} \|w(t)\|_{W^{2-\frac{1}{2},2}(S^1)} \leq C(x_-, x_+). \) We then iterate the same trick on \( W^{2-\frac{1}{2},2}(S^1) \) and \( W^{2,\frac{1}{2}}(S^1) \) and from a \( W^{2-s,2}\)-bound on \( \phi(t) \), we obtain sup_{t \in \mathbb{R}} \|\phi(t)\|_{W^{2,\frac{1}{2}}(S^1)} \leq C(x_-, x_+). \) Iterating this process, as in the proof of Proposition 3.3, we finally obtain a uniform bound

\[ \|\phi(t)\|_{C^{2,\frac{1}{2}}(S^1)} \leq C(x_-, x_+) \]  

(8.19)
for all $t \in \mathbb{R}$. This completes the proof. \hfill \square

**Remark 8.1** The argument of the proof of Lemma 8.4 above can be used to prove the $C^{2,2/3} \times C^{1,2/3}$-regularity of perturbed gradient flows $\ell(t) = (\phi(t), \psi(t))$ satisfying Eq. (5.3) and $\sup_{t \in \mathbb{R}} |\mathcal{L}_H(\ell(t))| < +\infty$. The main point is that, as in the proof of Lemma 8.4, the perturbation $K$ can be chosen such that it has the same regularity as $\phi$ and when rewriting the flow equation as an integral equation, then similarly to the proof of Lemma 8.4, it takes the form

$$
\begin{pmatrix}
\phi(t) \\
\psi(t)
\end{pmatrix} - \begin{pmatrix}
\mathcal{R}_1(\phi(t), \psi(t)) \\
\mathcal{R}_2(\phi(t), \psi(t))
\end{pmatrix} = \begin{pmatrix}
w_1 \\
w_2
\end{pmatrix},
$$

where $\mathcal{R}_1$ and $\mathcal{R}_2$ involve the perturbation term as in the proof of Lemma 8.4 and $w_1$ and $w_2$ are unperturbed terms represented by (3.54) and (3.35), respectively. By Lemma 5.2 and the proof of Proposition 3.3, we have the uniform estimate $\sup_{t \in \mathbb{R}} \|w_1(t)\|_{W^{2-s,r}(S^1)} < +\infty$ and $\sup_{t \in \mathbb{R}} \|w_2(t)\|_{W^{1,r}(S^1)} < +\infty$ for any $s > 1/2$ and $1 < r < 2$. If the perturbation $K$ is small, it can be seen, as above, that the map $\begin{pmatrix} \phi \\ \psi \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{R}_1(\phi, \psi) \\ \mathcal{R}_2(\phi, \psi) \end{pmatrix}$ is a contraction on $W^{2-s,2} \times W^{1,r}$. (This of course requires, as in the proof of Lemma 8.4, the extension of $\mathcal{R}_1$ and $\mathcal{R}_2$ to $W^{2-s,2}(S^1, \mathbb{R}^4) \times W^{1,r}(S^1, \mathbb{S}(S^1) \otimes \mathbb{R}^k)$.) Thus, by the contraction mapping principle, we improve the regularity estimates of Lemma 5.2 as $\sup_{t \in \mathbb{R}} \|\phi(t)\|_{W^{2-s,r}(S^1)} < +\infty$ and $\sup_{t \in \mathbb{R}} \|\psi(t)\|_{W^{1,r}(S^1)} < +\infty$. We then iterate the process, similarly as in the proof of Lemma 8.4, and obtain $C^{2,2/3} \times C^{1,2/3}$ estimates as in Proposition 3.3. This gives an alternative proof of the relative compactness of $\mathcal{M}(x_-, x_+)$ in $C^0_{\text{loc}}(\mathbb{R}, \mathcal{F}^{1/2}(S^1, N))$ along the line of the proof of Proposition 3.5.

From the equation $\frac{d\phi}{dt} = -(1 + K_{11}(\phi))\nabla_\phi E_1(\phi)$ and the estimate of Lemma 8.4, we also have

$$
\sup_{t \in \mathbb{R}} \|\partial_t \phi(t)\|_{C^{2,2/3}(S^1)} \leq C(x_-, x_+). \tag{8.20}
$$

In the computation of $DG^{p+1} H_s(N; \mathbb{Z}_2) \cong H F_s(\mathcal{L}_H, \mathcal{F}^{1/2}; \mathbb{Z}_2)$, we are free to choose $H$ satisfying (1.1)–(1.4). Throughout this section, we choose $H$ such that

$$
|\nabla_\phi H(s, \phi, \psi)| \leq C \tag{8.21}
$$

for all $s, \phi, \psi$. Note that such a choice is possible maintaining the Morse condition, by first choosing $H$ independent of $\phi$ and then perturb $H + h$ with $h \in \mathbb{H}_{\text{reg}} \subset \mathbb{H}_b^3$ as in Proposition 5.1.

Similar to the proof of the compactness of Sect. 7, the following proposition is the first step to obtain the compactness result for $V$-flow lines.

**Proposition 8.1** Let $\ell$ be a solution of (8.9). Then there exists a constant $C$ depending on $x_+, x_-$ such that

$$
\sup_{t \in \mathbb{R}} |\mathcal{L}_H(\ell(t))| \leq C. \tag{8.22}
$$

**Proof** Because $E_1$ is decreasing along flow lines and therefore bounded, we only need to investigate the boundedness of $E_2$ along the flow. This is similar to the invariance of the homology of the functional $E_2$ under homotopy that was investigated in [22]. But the main difference here is that the perturbation in [22] is compactly supported. So if for instance $\phi(t)$ was constant outside an interval $[-T, T]$ then the proof works exactly the same. But here we need to take care of the part outside a compact set.
Lemma 8.5 Let $\mathcal{K}(\phi_-, \phi_+; E_1) \subset \mathcal{M}(\phi_-, \phi_+; E_1)$ be such that $\hat{\mathcal{K}}(\phi_-, \phi_+; E_1) = \mathcal{K}(\phi_-, \phi_+; E_1)/\mathbb{R}$ is a compact subset of $\mathcal{M}(\phi_-, \phi_+; E_1) = \mathcal{M}(\phi_-, \phi_+; E_1)/\mathbb{R}$. There exist constants $C = C(x_\pm)$ and $T_\pm = T(x_\pm) > 0$ (which also depend on $\mathcal{K}(\phi_-, \phi_+; E_1)$ but do not indicate its dependence for simplicity of notations) such that for any $K \in \mathcal{K}(\phi_-, \phi_+; E_1)$ (here $K$ is a compact subset of $\mathcal{K}(\phi_-, \phi_+; E_1)$) with $\phi \in \mathcal{K}(\phi_-, \phi_+; E_1)$ (here $\phi$ is the $\phi$-component of $K$), we have, reparametrizing $\ell(t)$ by $\ell(t + a)$ for some $a \in \mathbb{R}$ if necessary, 

$$E_2(\ell(-t)) \leq C(x_-)$$

for $t \geq T_-$ and 

$$-C(x_+) \leq E_2(\ell(t)),$$

for all $t \geq T_+$. 

**Proof** We recall that the $V$-flow equation is

$$
\begin{align*}
\partial_t \phi &= -(1 + K_{11}) \nabla_\phi E_1(\phi), \\
\partial_t \psi &= -K_{21} \nabla_\phi E_1(\phi) - (1 + K_{22}) \nabla_\psi E_2(\phi, \psi).
\end{align*}
$$

(8.23)

Using (8.23), we get 

$$\partial_t \psi = -K_{21}(1 + K_{11})^{-1} \partial_t \phi - (1 + K_{22}) \nabla_\psi E_2(\phi, \psi)$$

(8.24)

and

$$\nabla_\psi E_2(\phi, \psi) = -(1 + K_{22})^{-1} (\partial_x \psi - K_{21}(1 + K_{11})^{-1} \partial_t \phi).$$

(8.25)

It is a well known result now (see [1, Appendix A]) that since $E_1$ is Morse, there exist $C > 0$ and $\alpha > 0$ depending only on the compact set $\mathcal{K}(\phi_-, \phi_+; E_1)$ such that up to reparametrization by time-translation,

$$\| \partial_t \phi(t) \|_{H^1} \leq C e^{-\alpha |t|}.$$

(8.26)

We have then

$$E_2(\ell(t)) - E_2(x_-) = \int_{-\infty}^{t} \frac{d}{dt} E_2(\ell(t)) dt$$

$$= \int_{-\infty}^{t} (\nabla_\psi E_2(\ell(t)), \partial_t \psi)_{H^{1/2}(S^1)} dt + \int_{-\infty}^{t} (\nabla_\phi E_2(\ell(t)), \partial_t \phi)_{H^{1}(S^1)} dt$$

$$= -\int_{-\infty}^{t} ((1 + K_{22})^{-1} (\partial_x \psi - K_{21}(1 + K_{11})^{-1} \partial_t \phi), \partial_t \psi)_{H^{1/2}(S^1)} dt$$

$$+ \int_{-\infty}^{t} \left( \frac{1}{2} R(\phi)(\psi, \partial_x \phi \cdot \psi), \partial_t \phi \right) dt - \int_{-\infty}^{t} \langle \nabla_\phi H(s, \phi, \psi), \partial_t \phi \rangle dt$$

$$= -\int_{-\infty}^{t} ((1 + K_{22})^{-1} \partial_t \psi, \partial_t \psi)_{H^{1/2}(S^1)} dt$$

$$+ \int_{-\infty}^{t} ((1 + K_{22})^{-1} K_{21}(1 + K_{11})^{-1} \partial_t \phi, \partial_t \psi)_{H^{1/2}(S^1)} dt$$

$$+ \int_{-\infty}^{t} \left( \frac{1}{2} R(\phi)(\psi, \partial_x \phi \cdot \psi), \partial_t \phi \right) dt - \int_{-\infty}^{t} \langle \nabla_\phi H(s, \phi, \psi), \partial_t \phi \rangle dt$$

$$\leq -\frac{1}{2} \int_{-\infty}^{t} \| \partial_t \psi \|_{H^{1/2}(S^1)}^2 dt + C \int_{-\infty}^{t} e^{-|t|} (\| \partial_t \psi \|_{H^{1/2}(S^1)} + \| \psi \|_{L^2(S^1)}^2 + 1) dt$$
\[ \leq -\frac{1}{4} \int_{-\infty}^{t} \| \partial_{t} \psi \|_{L^{p+1}(S^1)}^2 dt + C \int_{-\infty}^{t} e^{-\alpha|t|} (\| \psi \|_{L^{p+1}(S^1)}^{p+1} + 1) \, dt, \]  

(8.27)

where in the third line, we have used (8.25) and in the seventh line, the equivalence of norms \( \frac{1}{2} \| \cdot \|_{H^{1/2}(S^1)}^{2} \leq (1 + K_{22})^{-1} \cdot \| \cdot \|_{H^{1/2}(S^1)}^{2} \leq 2 \| \cdot \|_{H^{1/2}(S^1)}^{2} \) (if \( \rho_{0} \) is small) and Lemma 8.4 and (8.26) are used. The last line follows from the Cauchy–Schwartz inequality and an elementary inequality of the form \( a^{2} \leq C(a^{p+1} + 1) \) (note that \( p \geq 3 \)).

On the other hand, by (1.2), we have

\[ d_{\psi} E_{2}(\ell(t)) [\psi] + 2 E_{2}(\ell(t)) \geq C \| \psi \|_{L^{p+1}(S^1)}^{p+1} - C \]

and using (8.25), we obtain

\[ \| \psi \|_{L^{p+1}(S^1)}^{p+1} \leq -C ((1 + K_{22})^{-1} (\partial_{t} \psi - K_{21} (1 + K_{11})^{-1} \partial_{t} \phi), \psi)_{H^{1/2}(S^1)} + 2 E_{2}(\ell(t)) + C \]

\[ \leq C (\| \partial_{t} \psi \|_{H^{1/2}(S^1)} + \| \partial_{t} \phi \|_{H^{1/2}(S^1)}) \| \psi \|_{H^{1/2}(S^1)} + C E_{2}(\ell(t)) + C. \]  

(8.28)

Combining (8.27) and (8.28), we obtain

\[ \| \psi \|_{L^{p+1}(S^1)}^{p+1} \leq C (\| \partial_{t} \psi \|_{H^{1/2}(S^1)} + \| \partial_{t} \phi \|_{H^{1/2}(S^1)}) \| \psi \|_{H^{1/2}(S^1)} + C E_{2}(\ell(t)) + C \]

\[ - C \int_{-\infty}^{t} \| \partial_{t} \psi \|_{H^{1/2}(S^1)}^{2} dt + C \int_{-\infty}^{t} e^{-\alpha|t|} (\| \psi \|_{L^{p+1}(S^1)}^{p+1} + 1) \, dt + C \]

\[ \leq C (\| \partial_{t} \psi \|_{H^{1/2}(S^1)} + \| \partial_{t} \phi \|_{H^{1/2}(S^1)}) \| \psi \|_{H^{1/2}(S^1)} + C \int_{-\infty}^{t} e^{-\alpha|t|} \| \psi \|_{H^{1/2}(S^1)} \, dt + C \]

\[ + C e^{-\alpha|t|} \| \psi \|_{H^{1/2}(S^1)}. \]  

(8.29)

where \( C(\ell(t)) \) is a constant depending only on \( \ell(t) \). Therefore, if we multiply by \( e^{-\alpha|t|} \) and integrate, we have

\[ \int_{-\infty}^{t} e^{-\alpha|t|} \| \psi(t) \|_{L^{p+1}(S^1)}^{p+1} \, dt \leq C(\ell(t)) e^{-\alpha|t|} \int_{-\infty}^{t} e^{-\alpha|t|} \| \psi \|_{L^{p+1}(S^1)}^{p+1} \, dt \]

\[ + C \int_{-\infty}^{t} e^{-\alpha|t|} \| \partial_{t} \psi \|_{H^{1/2}(S^1)} \| \psi \|_{H^{1/2}(S^1)} \, dt + C \int_{-\infty}^{t} e^{-2\alpha|t|} \| \psi \|_{H^{1/2}(S^1)} \, dt \]

and for \( T \) big enough and \( t \leq -T \), the first integral on the right hand side is absorbed in the left hand side to yield

\[ \int_{-\infty}^{t} e^{-\alpha|t|} \| \psi \|_{L^{p+1}(S^1)}^{p+1} \, dt \leq C(\ell(t)) e^{-\alpha|t|} + C \int_{-\infty}^{t} e^{-\alpha|t|} \| \partial_{t} \psi \|_{H^{1/2}(S^1)} \| \psi \|_{H^{1/2}(S^1)} \, dt \]

\[ + C \int_{-\infty}^{t} e^{-2\alpha|t|} \| \psi \|_{H^{1/2}(S^1)} \, dt. \]  

(8.30)

Using (8.29) and (8.30), we have

\[ \| \psi \|_{L^{p+1}(S^1)}^{p+1} \leq C(\ell(t)) + C \int_{-\infty}^{t} e^{-\alpha|t|} \| \partial_{t} \psi \|_{H^{1/2}(S^1)} \| \psi \|_{H^{1/2}(S^1)} \, dt + C \int_{-\infty}^{t} e^{-\alpha|t|} \| \psi \|_{H^{1/2}(S^1)} \, dt \]

\[ + C \| \partial_{t} \psi \|_{H^{1/2}(S^1)} \| \psi \|_{H^{1/2}(S^1)} + C e^{-\alpha|t|} \| \psi \|_{H^{1/2}(S^1)}. \]  

(8.32)

Next, we estimate \( \| \psi \|_{H^{1/2}} \). We argue as in Sect. 3 and use the same notations as in the proof of Lemma 3.1. From (8.25), it follows that

\[ |(\nabla\psi E_{2}(\phi, \psi), P_{\phi} \psi^{\dagger})_{H^{1/2}(S^1)}| \]
\[
\begin{align*}
&\left| \int_{S^1} \langle D_\phi \psi - \nabla \psi H(s, \phi, \psi), P_\phi \psi_0^+ \rangle \, ds \right| \\
&= \big| (1 + K_{22}^{-1} \partial_t \psi - K_{21}(1 + K_{11}^{-1} \partial_t \phi, P_\phi \psi_0^+) \big|_{H^{1/2}(S^1)} \\
&\leq C \left( \| \partial_t \psi \|_{H^{1/2}(S^1)} + \| \partial_t \phi \|_{H^{1/2}(S^1)} \right) \| P_\phi \psi_0^+ \|_{H^{1/2}(S^1)} \\
&\leq C \left( \| \partial_t \psi \|_{H^{1/2}(S^1)} + e^{-\alpha |t|} \left( \| \psi \|_{H^{1/2}(S^1)} + \| \partial_t \phi \|_{L^2(S^1)} \right) \| \psi \|_{L^4(S^1)} \right) \\
&\leq C \left( \| \partial_t \psi \|_{H^{1/2}(S^1)} + e^{-\alpha |t|} \left( \| \psi \|_{H^{1/2}(S^1)} + \| \partial_t \phi \|_{L^2(S^1)} \right) \| \psi \|_{L^4(S^1)} \right), \quad (8.33)
\end{align*}
\]

where (8.26), Lemmas 8.4 and 10.2 were used. Combining with (3.18), we obtain
\[
\| \psi_0^+ \|_{H^{1/2}(S^1)}^2 \leq \lambda + \int_{S^1} \langle P_\phi \psi_0^+, D_\phi \psi \rangle \, ds + C \int_{S^1} |\partial_t \phi| |\psi_0^+| |\psi| \, ds \\
\leq C \left( \| \partial_t \psi \|_{H^{1/2}(S^1)} + e^{-\alpha |t|} \left( \| \psi \|_{H^{1/2}(S^1)} + \| \partial_t \phi \|_{L^2(S^1)} \right) \| \psi \|_{L^4(S^1)} \right) \\
+ C \int_{S^1} |\partial_t \phi| |\psi_0^+| |\psi| \, ds + C \int_{S^1} (1 + |\psi|^p) |\psi_0^+| \, ds \\
\leq C \left( \| \partial_t \psi \|_{H^{1/2}(S^1)} + e^{-\alpha |t|} \left( \| \psi \|_{H^{1/2}(S^1)} + \| \partial_t \phi \|_{L^2(S^1)} \right) \| \psi \|_{L^4(S^1)} \right) \\
+ C \| \psi \|_{L^2(S^1)}^2 + C \| \psi \|_{L^{p+1}(S^1)}^{p+1} + C. \quad (8.34)
\]

Combining (8.32) and (8.34), we obtain
\[
\| \psi_0^+ \|_{H^{1/2}(S^1)}^2 \leq C(x_-) + C \int_{-\infty}^t e^{-\alpha |t|} \| \partial_t \psi \|_{H^{1/2}(S^1)} \| \psi \|_{H^{1/2}(S^1)} \, dt + C e^{-\alpha |t|} \| \psi \|_{H^{1/2}(S^1)} \\
+ C \int_{-\infty}^t e^{-\alpha |t|} \| \psi \|_{H^{1/2}(S^1)} \, dt + C \| \partial_t \psi \|_{H^{1/2}(S^1)} \| \psi \|_{H^{1/2}(S^1)}. \quad (8.35)
\]

In the same way, we have
\[
\| \psi_0^- \|_{H^{1/2}(S^1)}^2 \leq C(x_-) + C \int_{-\infty}^t e^{-\alpha |t|} \| \partial_t \psi \|_{H^{1/2}(S^1)} \| \psi \|_{H^{1/2}(S^1)} \, dt + C e^{-\alpha |t|} \| \psi \|_{H^{1/2}(S^1)} \\
+ C \int_{-\infty}^t e^{-\alpha |t|} \| \psi \|_{H^{1/2}(S^1)} \, dt + C \| \partial_t \psi \|_{H^{1/2}(S^1)} \| \psi \|_{H^{1/2}(S^1)}. \quad (8.36)
\]

The last component, \( \psi_0^0 \), is estimated, by using (8.32), as
\[
\| \psi_0^0 \|_{H^{1/2}(S^1)}^2 \leq C \| \psi_0^0 \|_{L^{p+1}(S^1)}^2 \\
\leq C(x_-) + C \int_{-\infty}^t e^{-\alpha |t|} \| \partial_t \psi \|_{H^{1/2}(S^1)} \| \psi \|_{H^{1/2}(S^1)} \, dt + C e^{-\alpha |t|} \| \psi \|_{H^{1/2}(S^1)} \\
+ C \int_{-\infty}^t e^{-\alpha |t|} \| \psi \|_{H^{1/2}(S^1)} \, dt + C \| \partial_t \psi \|_{H^{1/2}(S^1)} \| \psi \|_{H^{1/2}(S^1)}. \quad (8.37)
\]

Using (8.35)–(8.37), we obtain
\[
\| \psi \|_{H^{1/2}(S^1)}^2 \leq C(x_-) + C \int_{-\infty}^t e^{-\alpha |t|} \| \partial_t \psi \|_{H^{1/2}(S^1)} \| \psi \|_{H^{1/2}(S^1)} \, dt + C e^{-\alpha |t|} \| \psi \|_{H^{1/2}(S^1)} \\
+ C \int_{-\infty}^t e^{-\alpha |t|} \| \psi \|_{H^{1/2}(S^1)} \, dt + C \| \partial_t \psi \|_{H^{1/2}(S^1)} \| \psi \|_{H^{1/2}(S^1)}. \quad (8.38)
\]
Using the Cauchy–Schwartz inequality and absorbing the \( \| \psi \|_{H^{1/2}(S^1)} \) term in the right hand side of (8.38) to the left hand side, we get

\[
\| \psi \|_{H^{1/2}(S^1)^2}^2 \leq C(\mathcal{X}_-) + C\int_{-\infty}^t e^{-\alpha|t|} \| \partial_t \psi \|_{H^{1/2}(S^1)} \| \psi \|_{H^{1/2}(S^1)} dt
\]

\[
+C \int_{-\infty}^t e^{-\alpha|t|} \| \psi \|_{H^{1/2}(S^1)} dt + C \| \partial_t \psi \|_{H^{1/2}(S^1)^2} + C e^{-2\alpha|t|}. \tag{8.39}
\]

Therefore, if we multiply by the exponential term \( e^{-\alpha|t|} \) and integrate over \((-\infty, t]\), we find

\[
\int_{-\infty}^t e^{-\alpha|t|} \| \psi \|_{H^{1/2}(S^1)}^2 dt \leq C(\mathcal{X}_-) + C e^{-\alpha|t|} \int_{-\infty}^t e^{-\alpha|t|} \| \partial_t \psi \|_{H^{1/2}(S^1)} \| \psi \|_{H^{1/2}} dt
\]

\[
+C e^{-\alpha|t|} \int_{-\infty}^t e^{-\alpha|t|} \| \psi \|_{H^{1/2}} dt + C \int_{-\infty}^t e^{-\alpha|t|} \| \partial_t \psi \|_{H^{1/2}}^2 dt.
\tag{8.40}
\]

Using again the Cauchy–Schwartz inequality in the first integral of the right hand side of (8.40), we obtain, for \( T \) big enough and \( t \leq -T \),

\[
\int_{-\infty}^t e^{-\alpha|t|} \| \psi \|_{H^{1/2}(S^1)}^2 dt \leq C e^{-\alpha|t|} + C \int_{-\infty}^t e^{-\alpha|t|} \| \partial_t \psi \|_{H^{1/2}(S^1)}^2 dt. \tag{8.41}
\]

On the other hand, by (8.27) and (8.30), we have

\[
\int_{-\infty}^t \| \partial_t \psi \|_{H^{1/2}(S^1)}^2 dt \leq 4E_2(\mathcal{X}_-) - 4E_2(\ell(t)) + C \int_{-\infty}^t e^{-\alpha|t|} (\| \psi \|_{L^{p+1}(S^1)}^{p+1} + 1) dt
\]

\[
\leq C(\mathcal{X}_-) - 4E_2(\ell(t)) + C \int_{-\infty}^t e^{-\alpha|t|} \| \partial_t \psi \|_{H^{1/2}(S^1)} \| \psi \|_{H^{1/2}(S^1)} dt
\]

\[
+C \int_{-\infty}^t e^{-2\alpha|t|} \| \psi \|_{H^{1/2}(S^1)} dt
\]

\[
\leq C(\mathcal{X}_-) - 4E_2(\ell(t)) + C \int_{-\infty}^t (e^{-\frac{\alpha}{2}|t|} \| \partial_t \psi \|_{H^{1/2}(S^1)}^2 + e^{-\frac{\alpha}{2}|t|} \| \psi \|_{H^{1/2}(S^1)}^2) dt
\]

\[
+C \int_{-\infty}^t e^{-2\alpha|t|} (\| \psi \|_{H^{1/2}(S^1)}^2 + 1) dt. \tag{8.42}
\]

Then absorbing the term involving \( \| \partial_t \psi \|_{H^{1/2}(S^1)} \) in the right hand side into the left side (if \( T \) is big enough), we obtain

\[
\int_{-\infty}^t \| \partial_t \psi \|_{H^{1/2}(S^1)}^2 dt \leq C(\mathcal{X}_-) - 4E_2(\ell(t)) + C \int_{-\infty}^t e^{-\alpha|t|} \| \psi \|_{H^{1/2}(S^1)}^2 dt. \tag{8.43}
\]

By (8.41) and (8.43), we have

\[
\int_{-\infty}^t \| \partial_t \psi \|_{H^{1/2}(S^1)}^2 dt \leq C(\mathcal{X}_-) - 4E_2(\ell(t)) + C \int_{-\infty}^t e^{-\alpha|t|} \| \partial_t \psi \|_{H^{1/2}(S^1)}^2 dt. \tag{8.44}
\]

Thus, if \( T \) is large enough and \( t \leq -T \), we obtain

\[
E_2(\ell(t)) \leq \frac{C(\mathcal{X}_-)}{4}.
\]

Using the same argument, we have an estimate of the form

\[
E_2(\ell(t)) \geq -C(\mathcal{X}_+).
\]
for $t \geq T$ and this completes the proof. \hfill \square

**Completion of the proof of Proposition 8.1.** Since $\hat{M}(\phi_-, \phi_+; E_1)$ is compact only when $\mu_1(\phi_-) - \mu_1(\phi_+) = 0$, or 1, we decompose the proof into cases. Note that, to define the $V$-homology, we only need to consider $M(x_-, x_+; V)$ with $\mu_0(x_-) - \mu_0(x_+) \leq 2$. We always have $\mu_2(x_-) - \mu_2(x_+) \geq 0$ by the transversality and the triangular form of the equation of the flow. Thus, for our proof, we only need the case $\mu_1(\phi_-) - \mu_1(\phi_+) \leq 2$ and we will be assume this in the following. The general case, can be proved inductively, using a similar argument.

(I) When $\mu_1(\phi_-) - \mu_1(\phi_+) = 0$. In this case, $\hat{M}(\phi_-, \phi_+; E_1)$ is compact and Lemma 8.5 holds for any $\ell \in M(x_-, x_+; V)$. We fix $T = \max\{T(x_-), T(x_+}\}$. By Lemma 8.5, after reparametrization by time-translation, it remains to estimate $E_2(\ell(t))$ on the interval $[-T, T]$. Using the same calculation as in the proof of Lemma 8.5 (see (8.23)–(8.27)), we have for any $\epsilon > 0$, the following estimate

$$E_2(\ell(t)) - E_2(\ell(-T)) \leq -\frac{1}{4} \int_{-T}^{t} \|\partial_t \psi\|_{H^{1/2}(S^1)}^2 + \epsilon \int_{-T}^{t} \|\psi\|_{L^{p+1}(S^1)}^{p+1} + C(\epsilon)$$

for some $C(\epsilon) > 0$. In the derivation of this inequality, we used an elementary inequality of the form $a^2 \leq \epsilon a^{p+1} + C(\epsilon)$ for $a \geq 0$. Then by the same argument as in the proof of Proposition 7.1, we obtain an upper bound of the form

$$E_2(\ell(t)) \leq C(E_2(\ell(-T))) = C(x_-)$$

for $t \in [-T, T]$. Similarly, we also obtain a lower bound of the form

$$E_2(\ell(t)) \geq -C(E_2(\ell(T))) = -C(x_+)$$

This completes the proof of Proposition 8.1 for this case.

(II) When $\mu_1(\phi_-) - \mu_1(\phi_+) = 2$. The set $M(\phi_-, \phi_+; E_1)$ has a natural compactification $\hat{M}(\phi_-, \phi_+; E_1)$ and its boundary consists of broken flow lines of the form $\hat{\phi}_{1,2} := \phi_1(\mathbb{R}) \cup \phi_2(\mathbb{R})$ for some $\phi_1 \in M(\phi_-, \phi_0; E_1)$ and $\phi_2 \in M(\phi_0, \phi_+; E_1)$, where $\phi_0 \in \text{crit}(E_1)$ with $\mu_1(\phi_-) - \mu_1(\phi_0) = 1$. For $\epsilon > 0$ small enough, we consider the $\epsilon$-neighborhood of $\hat{\phi}_{1,2}$ defined by

$$U_\epsilon(\hat{\phi}_{1,2}) = \{\phi \in M(\phi_-, \phi_+; E_1) : d_H(\phi(\mathbb{R}), \hat{\phi}_{1,2}) < \epsilon\}.$$
We consider the first case. The other case is similar. We decompose \( \mathbb{R} \) into five intervals, (1) \((-\infty,-T]\), (2) \([-T,T]\), (3) \([T,\,T(\phi)-T]\), (4) \([T(\phi)-T,\,T(\phi)+T]\), (5) \([T(\phi)+T,\infty)\), and estimate \( E_2(\ell(t)) \) for \( t \) belonging to these intervals in this order.

In the interval (1). In this interval, by the exponential decay (i), we have, using the same proof of Lemma 8.5, an estimate of the form

\[
E_2(\ell(t)) \leq C(x_\ell), \quad t \in (-\infty, -T]
\]

for some \( C(x_\ell) > 0 \) depending only on \( x_\ell \).

In the interval (2). In this compact interval, we can apply the proof of (I) given above and obtain the estimate

\[
E_2(\ell(t)) \leq C(E_2(\ell(-T))) \leq C(x_\ell), \quad t \in [-T, T]
\]

where \( E_2(\ell(-T)) \) is estimated above by a constant depending only on \( x_\ell \) by (1). So there exists \( C(x_\ell) > 0 \) depending only on \( x_\ell \) such that the above estimate holds.

In the interval (3). This interval becomes unbounded when \( d_H(\phi(\bar{R}), 1_2) \rightarrow 0 \), but \( \|\partial_{t}(\phi(t))\|_{H^1(S^1)} \) has a uniform exponential decay. Thus, by the same argument of the proof of Lemma 8.4, and the estimate (2) above, the following estimate holds:

\[
E_2(\ell(t)) \leq C(E_2(\ell(T))) \leq C(x_\ell), \quad t \in [T, T(\phi) - T]
\]

for some constant \( C(x_\ell) > 0 \) depending only on \( x_\ell \).

In the interval (4). In this compact interval, as in the case (2) above, we can apply the proof of (I) and obtain the estimate by (3)

\[
E_2(\ell(t)) \leq C(E_2(\ell(T(\phi) - T))) \leq C(x_\ell), \quad t \in [T(\phi) - T, T(\phi) + T]
\]

for some constant \( C(x_\ell) > 0 \) depending only on \( x_\ell \).

In the interval (5). In this unbounded interval, by the exponential decay of \( \|\partial_{t}(\phi(t))\|_{H^1(S^1)} \), as in (1) above, we have the following estimate as in the proof of Lemma 8.4:

\[
E_2(\ell(t)) \leq C(E_2(\ell(T(\phi) + T))) \leq C(x_\ell), \quad t \in [T(\phi) + T, \infty)
\]

for some constant \( C(x_\ell) > 0 \) depending only on \( x_\ell \).

From (1)–(5), we obtain \( E_2(\ell(t)) \leq C(x_\ell) \) for all \( t \in \mathbb{R} \). In a similar way, we have \( E_2(\ell(t)) \geq -C(x_\ell) \). We thus have \( |E_2(\ell(t))| \leq C(x_\ell, x_+) \) for \( \ell(t) = (\phi(t), \psi(t)) \) with \( \phi \in U_E(1_2) \). Applying this argument to each neighborhood of the boundary of \( \tilde{E}(\phi_-, \phi_+; E_1) \), we have a uniform bound for \( E_2(\ell(t)) \) belonging to a neighborhood of the boundary. The complement of this neighborhood is compact and for \( \ell \) belonging to this set, we can apply the argument of (I) to obtain a uniform estimate for \( E_2(\ell(t)) \). Combining these two cases, we have a uniform estimate \( |E_2(\ell(t))| \leq C(x_\ell, x_+) \) which holds for all \( t \in M(x_\ell, x_+; V) \) and this completes the proof.

From this lemma, we obtain

**Lemma 8.6** Let \( \ell(t) = (\phi(t), \psi(t)) \in M(x_\ell, x_+; V) \). There exists \( C(x_\ell, x_+) > 0 \) such that

\[
\sup_{t \in \mathbb{R}} \|\psi(t)\|_{H^{1/2}(S^1)} \leq C(x_\ell, x_+).
\]
Lemma 8.5. From the proof of Proposition 8.1 (for the estimate in unbounded intervals, see (8.39), (8.46), (8.47) and Hölder’s inequality, we have

\[ \int_{-\infty}^{T} |\psi(t)|^2_{H^{1/2}(S^1)} dt \leq C(x_-, x_+). \]

Therefore, we have

\[ \sup_{\tau \in \mathbb{R}} \int_{\tau}^{\tau + 1} \|\psi(t)\|^2_{H^{1/2}(S^1)} dt, \quad \sup_{\tau \in \mathbb{R}} \int_{\tau}^{\tau + 1} \|\psi(t)\|^{p+1}_{L^{p+1}(S^1)} dt. \]

In fact, since we already have an estimate of Lemma 8.4, once we have estimated these by a constant depending only on $x_-, x_+$, the desired estimate follows as in the proof of Lemma 5.2.

Indeed, following the proof of Proposition 8.1, we decompose the proof into two cases. The first case is the case when $\phi$ belongs to a set $\mathcal{K}(\phi_-, \phi_+; E_1) \subset \mathcal{M}(\phi_-, \phi_+; E_1)$ with $\mathcal{K}(\phi_-, \phi_+; E_1)$ compact. We use the same notations as in the proof of Proposition 8.1 and Lemma 8.5. From the proof of Proposition 8.1 (for the estimate in unbounded intervals, see (8.44) and for the estimate in compact intervals, we argue as in the proof of Proposition 7.1), we have the following

\[ \int_{-\infty}^{\infty} \|\partial_t \psi(t)\|^2_{H^{1/2}(S^1)} dt \leq C(x_-, x_+). \] (8.45)

On the other hand, in the unbounded interval $(-\infty, -T]$, we have by (8.41), (8.45) and (8.30)

\[ \int_{-\infty}^{T} e^{-\alpha|t|} \|\psi(t)\|^2_{H^1(S^1)} dt \leq C(x_-, x_+) e^{-\alpha|t|} \] (8.46)

and

\[ \int_{-\infty}^{T} e^{-\alpha|t|} \|\psi\|^p_{L^{p+1}(S^1)} dt \leq C(x_-, x_+) e^{-\alpha|t|}. \] (8.47)

Then by (8.39), (8.46), (8.47) and Hölder’s inequality, we have

\[ \|\psi(t)\|^2_{H^{1/2}(S^1)} \leq C(1 + e^{-\alpha|t|}) + C\|\partial_t \psi(t)\|_{L^{1/2}(S^1)} \|\psi(t)\|_{H^{1/2}(S^1)} + C e^{-\alpha|t|} \|\psi(t)\|_{H^{1/2}(S^1)}. \]

Integrating this inequality over $[\tau, \tau + 1]$ ($\tau + 1 \leq -T$) and using (8.45) and Hölder’s inequality, we have

\[ \int_{\tau}^{\tau + 1} \|\psi(t)\|^2_{H^{1/2}(S^1)} dt \leq C(x_-, x_+) + C \left( \int_{\tau}^{\tau + 1} \|\psi(t)\|^2_{H^{1/2}(S^1)} dt \right)^{1/2}. \]

Therefore, we have

\[ \sup_{\tau \leq -T-1} \int_{\tau}^{\tau + 1} \|\psi(t)\|^2_{H^{1/2}(S^1)} dt \leq C(x_-, x_+). \]

Similarly, using (8.32), (8.45) and (8.46), we have

\[ \sup_{\tau \leq -T-1} \int_{\tau}^{\tau + 1} \|\psi(t)\|^{p+1}_{L^{p+1}(S^1)} dt \leq C(x_-, x_+). \]

One also has the same estimate for $\tau \geq T$. For the estimate for $-T-1 \leq \tau \leq T$, we argue in the same way as in the proof of Proposition 7.1 with the help of (8.26) and the estimate of Lemma 8.4, cf. (7.17) and (7.20). This gives the desired estimates

\[ \sup_{\tau \in \mathbb{R}} \int_{\tau}^{\tau + 1} \|\psi(t)\|^2_{H^{1/2}(S^1)} dt \leq C(x_-, x_+), \quad \sup_{\tau \in \mathbb{R}} \int_{\tau}^{\tau + 1} \|\psi(t)\|^{p+1}_{L^{p+1}(S^1)} dt \leq C(x_-, x_+) \] (8.48)

for $\phi$ belonging to $\mathcal{K}(\phi_-, \phi_+; E_1)$.

In the general case, for our purpose it suffices to consider the case $\mu_1(\phi_-) - \mu_1(\phi_+) = 2$ as before and decompose $\mathbb{R}$ into five intervals as in the proof of Proposition 8.1. In the
unbounded intervals (1), (3) and (5) in the proof of Proposition 8.1, we can argue exactly as in the previous case. In the bounded intervals (2) and (4), we can also argue as in the proof of Proposition 7.1 and obtain the similar estimate as in (8.48).

Combining these two cases, we have a uniform estimate of the form (8.48) which holds for any \( \ell \in M(x_-, x_+; V) \). □

From Lemmas 8.4 and 8.6, we have

**Proposition 8.2** Let \( \ell(t) = (\phi(t), \psi(t)) \in M(x_-, x_+; V) \) be arbitrary. There exists a constant \( C(x_-, x_+) > 0 \) depending only on \( x_-, x_+ \) such that

\[
\sup_{t \in \mathbb{R}} \| \phi(t) \|_{C^{2/3}(S^1)} + \sup_{t \in \mathbb{R}} \| \psi(t) \|_{C^{1,2/3}(S^1)} \leq C(x_-, x_+) .
\]

**Proof** The estimate for \( \phi \) is already given in Lemma 8.4. To prove the estimate for \( \psi \), we recall that (see Remark 8.1) the perturbation of the metric \( K \) can be taken to have the same regularity of \( \phi \). Since we already have \( C^{2,2/3} \)-estimate for \( \phi(t) \), we can apply the argument of the proof of Proposition 3.3. Applying this argument, we have a uniform \( C^{1,2/3} \)-bound for \( \psi \) once we have the estimate of Lemma 8.6. This completes the proof. □

In particular, by this proposition, as in Proposition 3.5 we deduce that the moduli space \( \tilde{M}(x_-, x_+; V) \) is a manifold of dimension \( \dim \tilde{M}(x_-, x_+; V) = \mu_0(x_-) - \mu_0(x_+) \) and relatively compact in \( C^0_{\text{loc}}(\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N)) \). In the case \( x_\neq x_+ \), \( \mathbb{R} \) acts freely as time translation and we obtain \( \tilde{M}(x_-, x_+; V) = \tilde{M}(x_-, x_+; V)/\mathbb{R} \) a manifold of dimension \( \mu_0(x_-) - \mu_0(x_+) - 1 \). We have then the following description of its compactification \( \tilde{\tilde{M}}(x_-, x_+; V) \):

**Proposition 8.3** Assume \( x_\neq x_+ \). The metric space \( \tilde{\tilde{M}}(x_-, x_+; V), d_{H} \) is relatively compact, i.e., for any sequence \( \{\tilde{\ell}_n\} \subset \tilde{\tilde{M}}(x_-, x_+; V) \), there exists a subsequence \( \{\tilde{\ell}_{n_k}\} \) and a compact subset \( \tilde{K}_{\infty} \subset \mathcal{F}^{1,1/2}(S^1, N) \) such that \( d_{\text{Hausdorff}}(\tilde{\ell}_{n_k}(\mathbb{R}), \tilde{K}_{\infty}) \to 0 \) as \( n_k \to \infty \). The limit \( \tilde{K}_{\infty} \) has the following properties:

1. \( \tilde{K}_{\infty} \) is invariant under the flow of \( V \),
2. There exist finitely many rest points \( x_\text{rest} = x_0, x_1, \ldots, x_{k-1}, x_k = x_+ \in \text{rest}(V) \) and \( \ell_i \in \tilde{M}(x_i, x_{i+1}; V) \) for \( 0 \leq i \leq k - 1 \) such that \( \tilde{K}_{\infty} = \bigcup_{i=0}^{k-1} \ell_i(\mathbb{R}) \).
3. If \( x_i = (\phi_i, \psi_i) \) then for all \( 0 \leq i \leq k - 1 \), \( \mu_1(\phi_i) \geq \mu_1(\phi_{i+1}) \) and if for some \( i \), \( \mu_1(\phi_i) = \mu_1(\phi_{i+1}) \) then \( \phi_i = \phi_{i+1} \) and \( \mu_2(\phi_i, \psi_i) > \mu_2(\phi_{i+1}, \psi_{i+1}) \).

**Proof** As we already mentioned, the existence of \( \tilde{K}_{\infty} \) follows from Proposition 8.2. The proof of (1)–(3), is similar in nature to the one of Proposition 3.7 with a few fundamental changes. Notice that here, the flow of \( V \) does not strictly decrease the functional \( C_{H+V} \), so the argument of Proposition 3.7 does not fully hold. But instead, \( V \) satisfies the following decreasing properties: If \( \ell(t) = (\phi(t), \psi(t)) \in \tilde{M}(x_-, x_+; V) \), \( x_+ = (\phi_+, \psi_+) \) and \( x_- = (\phi_-, \psi_-) \), then either \( \phi_- = \phi_+ = \phi(t) \) for all \( t \in \mathbb{R} \) and \( \psi \) strictly decreases \( E_2(\phi_-, \cdot) \). Or, \( \phi_- \neq \phi_+ \) and \( \phi(t) \) strictly decreases \( E_1 \).

Now, if \( \phi_- = \phi_+ \) then the proposition follows from applying Proposition 3.7 to flow line \( (\phi_-, \psi(t)) \). This is a special case of a flow line (i.e., \( \phi(t) \equiv \phi_- \)) treated in Proposition 3.7. So we assume that \( \phi_- \neq \phi_+ \). Then by transversality one has that \( \mu_1(\phi_-) > \mu_1(\phi_+) \).

Moreover, if we let \( \tilde{\ell}_n = (\phi^n, \psi^n) \), then we have that \( \{\phi^n\} \subset \tilde{\tilde{M}}(\phi_-, \phi_+, E_1) \). Therefore we have that \( \phi^n(\mathbb{R}) \) converges up to a subsequence, in the Hausdorff distance to a compact set \( K^1_{\infty} = \pi(\tilde{K}_{\infty}) \) and there exists \( \phi_- = \phi_0, \ldots, \phi_{k-1}, \phi_k = \phi_+ \), critical points of \( E_1 \) and \( \phi^i \in \tilde{M}(\phi_i, \phi_{i+1}, E_1) \) such that \( K^1_{\infty} = \bigcup_{i=0}^{k-1} \phi^i(\mathbb{R}) \).
Notice that in the previous decomposition, the $\phi_i$ are not necessary distinct and in case of equality, the corresponding connecting flow line is just the constant one. So there are two cases: (i) $\phi_1 = \phi_{i+1}$, (ii) $\phi_i \neq \phi_{i+1}$. For the case (i), $E_2$ is bounded on a compact set $\pi^{-1}(\phi_1) \cap \tilde{K}_\infty$ and we let $E_{2,i}^+ = \sup_{\pi^{-1}(\phi_1) \cap \tilde{K}_\infty} E_2$ and $E_{2,i}^- = \inf_{\pi^{-1}(\phi_1) \cap \tilde{K}_\infty} E_2$. In this case, we claim that $\pi^{-1}(\phi_1) \cap \tilde{K}_\infty$ consists of broken or unbroken flow lines connecting rest points $(\phi_i, \psi_i)$ and $(\phi_j, \psi_{j+1})$ for some $\psi_i, \psi_{j+1}$. To prove this, we need to show that for each $c \in [E_{2,i}^-, E_{2,i}^+]$, $\pi^{-1}(\phi_1) \cap E_{2}^{-1}(c)$ consists of exactly one point. But, since the $V$-flow coincides with the gradient flow of $E_2(\phi, \cdot)$ on the set $\pi^{-1}(\phi_1)$, the claim then follows from the same argument of the proof of Proposition 3.7. For the case (ii), there exists a flow line $\ell_i = (\phi^i, \psi^i)$ connecting two rest points $x_i = (\phi_i, \psi_i)$ and $x_{i+1} = (\phi_{i+1}, \psi_{j+1})$. Namely $\ell_i \in M(x_i, x_{i+1}; V)$. To see this, we need to show that, for any point $\phi_{i,j+1}$ on $\phi^i(\mathbb{R})$ $(\phi^i \in \mathcal{M}(\phi_i, \phi_{i+1}, E_1)$ is as above), $\pi^{-1}(\phi_{i,j+1}) \cap \tilde{K}_\infty$ consists of one point. Assume for the sake of contradiction that there exist two distinct points $(\phi_{i,j+1}, \psi_1), (\phi_{i,j+1}, \psi_2) \in \pi^{-1}(\phi_{i,j+1}) \cap \tilde{K}_\infty$. By the Hausdorff convergence, after reparametrizing $\ell_{n_k}$, we may assume that there exists $t_k \geq 0$ such that $\ell_{n_k}(0) \to (\phi_{i,j+1}, \psi_1)$ and $\ell_{n_k}(t_k) \to (\phi_{i,j+1}, \psi_2)$ as $k \to \infty$. But, then $t_k \to 0$ (for some subsequence), since otherwise there exists $\epsilon > 0$ such that $t_k \geq \epsilon$ and by the $C^1_{loc}$-convergence of $\pi(\ell_{n_k})$ to $\phi^i$, we have $\phi^i|_{[0, \epsilon]} \equiv \phi_{i,j+1}$. This implies that $\phi^i \equiv \phi_{i,j+1}$ and $\psi_i = \psi_{j+1}$. This contradicts the assumption $\phi_i \neq \phi_{j+1}$. Thus $t_k \to 0$ (for some subsequence) and $\psi_1 = \psi_2$ and the assertion follows. The last point of the proposition follows easily from the transversality property of the flow. \hfill $\Box$

From this, as in Sect. 3.4, we have

**Corollary 8.1** Under the transversality assumption as above, we have

1. If $\mu_0(x_-) - \mu_0(x_+) = 1$, then $\mathcal{M}(x_-, x_+; V)$ is a compact 0-dimensional manifold, hence it consists of at most finitely many points.
2. If $\mu_0(x_-) - \mu_0(x_+) = 2$, then $\mathcal{M}(x_-, x_+; V)$ is a 1-dimensional manifold and has a compactification $\overline{\mathcal{M}}(x_-, x_+; V)$. The boundary $\partial \overline{\mathcal{M}}(x_-, x_+; V)$ of this manifold (if non-empty) consists of broken flow lines of the form $\ell_1(\mathbb{R}) \cup \ell_2(\mathbb{R})$ for some $\ell_1 \in \mathcal{M}(x_-, y; V)$ and $\ell_2 \in \mathcal{M}(y, x_+; V)$, where $y \in \text{rest}(V)$ with $\mu_0(x_-) - \mu_0(y) = 1$.

To complete the description of the boundary of the compactified moduli space $\overline{\mathcal{M}}(x_-, x_+; V)$, it remains to show that for any pair $\ell_1 \in \mathcal{M}(x_-, y; V)$ and $\ell_2 \in \mathcal{M}(y, x_+; V)$, where $y \in \text{rest}(V)$ with $\mu_0(x_-) - \mu_0(y) = 1$ as in Corollary 8.1 (2), broken flow line $\ell_1(\mathbb{R}) \cup \ell_2(\mathbb{R})$ is a component of the boundary $\partial \overline{\mathcal{M}}(x_-, x_+; V)$. This is deduced from the gluing result for $V$-flow lines. In the present case, as we have proved above, we already have transversality along flow lines. Thus, the proof of the gluing result presented in Sect. 3.5 also applies to the present case. Namely, we first construct approximate solutions $\ell_1 \#_{\ell_2}$ as in Sect. 3.5 and, by the transversality of $V$-flows, construct approximate inverse of the differential of the operator $\mathcal{T}^V(\ell) := \frac{d^\ell}{dt} + V(\ell)$ at $\ell_1 \#_{\ell_2}$ by appropriately gluing right inverses of $d\mathcal{T}^V(\ell_1)$ and $d\mathcal{T}^V(\ell_2)$ as in Sect. 3.4. Then a family of genuine $V$-flows $\ell_{12,R}$ converging to the broken flow line $\ell_1(\mathbb{R}) \cup \ell_2(\mathbb{R})$ as $R \to \infty$ is obtained as a small perturbation of $\ell_1 \#_{\ell_2}$ as proved in Sect. 3.4 by the Banach fixed point argument.

Therefore, by counting $V$-flow lines connecting rest points of $\mu_0$-index difference 1, the homology $HF_*(V; \mathbb{Z}_2)$ is defined as usual. Namely, for $p \in \mathbb{Z}$, we define

$$C_p(V; \mathbb{Z}_2) = \bigoplus_{x \in \text{rest}_p(V)} \mathbb{Z}_2(x),$$

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where \( \text{rest}_p(V) = \{ x \in \text{rest}(V) : \mu_0(x) = p \} \). We define \( \partial^V_p : C_p(V; \mathbb{Z}_2) \to C_{p-1}(V; \mathbb{Z}_2) \) by

\[
\partial^V_p(x) = \sum_{y \in \text{rest}_{p-1}(V)} n(x,y) y
\]

(8.49)

for a generator \( x \in \text{rest}_p(V) \), where \( n(x,y) = \# \tilde{\mathcal{M}}(x,y; V) \) (mod. 2), and then we extend it by linearity. To see the above definition is well-defined, i.e., the sum in (8.50) is finite, we first observe that along the \( V \)-flow, \( E_1 \) is decreasing. Thus, for \( x = (\phi_x, \psi_x) \in \text{rest}_p(V) \) and \( y = (\phi_y, \psi_y) \in \text{rest}_{p-1}(V) \) with \( n(x,y) \neq 0 \), we have

\[
E_1(\phi_y) \leq E_1(\phi_x).
\]

(8.50)

Since \( E_1 \) satisfies the Palais–Smale condition, there are only finitely many \( \phi_y \in \text{crit}(E_1) \) which satisfies (8.50). On the other hand, by Proposition 8.1 (more precisely, the proof of the upper bound of \( E_2(\ell(t)) \) in Lemma 8.5), we have a uniform bound of the form

\[
E_2(\phi_y, \psi_y) \leq C(\alpha)
\]

(8.51)

for some constant \( C(\alpha) > 0 \) depending only on \( x \). Note here, we observe that, in the proof of Lemma 8.5, the constants \( T_\pm \) depends only on the \( E_1 \)-flow line connecting \( \phi_x \) and \( \phi_y \). But as we just observed form (8.50), such a constant only depends on \( x \) and the upper bound of the form (8.51) holds. Since there are only finitely many \( \phi_y \) and for each \( \phi_y, E_2(\phi_y, \cdot) \) satisfies the Palais–Smale condition on \([-\infty, C(\alpha)] \) (see [19]), there are only finitely many \( y = (\phi_y, \psi_y) \in \text{rest}_{p-1}(V) \) with \( n(x,y) \neq 0 \). Thus the sum in (8.49) is finite.

We then have, by Corollary 8.1, as usual \( \partial^V \circ \partial^V = 0 \) and the homology of the complex \( \{ C_*(V; \mathbb{Z}_2), \partial^V_* \} \) is defined:

\[
HF_*(V; \mathbb{Z}_2) = \frac{\text{ker} \partial^V_*}{\text{Im} \partial^V_{*+1}}.
\]

Remark 8.2 Notice that this construction still holds if we take a non-local perturbation of \( E_0(\phi) = 2 \int_{S^1} |\phi(s)|^2 ds \). That is, we can replace \( E_1 \) by \( \bar{E}_1 = E_0 + Rv \) with \( R \in C_3^\infty(H^1(S^1, N), [0, 1]) \) and \( v = \beta(s) \sum c_{i,j} V_{ij} \) as defined in Remark 5.1. Then

\[
\bar{E}_1(\phi) = E_0(\phi) + R(\phi) \int_{S^1} v(s, \phi(s)) ds.
\]

Now we consider \( K = \nabla E_0^{-1}(0) \), the critical set of \( E_0 \) and let

\[
U_\delta = \{ \phi \in H^1(S^1, N) : \text{dist}_{H^1}(\phi, K) < \delta \}.
\]

Since \( H^1(S^1, N) \) is paracompact, we can take \( R \in C^3(H^1(S^1, N), [0, 1]) \) such that \( R \) vanishes outside \( U_\delta \) and \( R = 1 \) on \( U_{\delta'} \) for some \( 0 < \delta' < \delta \). It is easy to see that when \( v \) is small enough the critical points of \( \bar{E}_1 \) are in \( U_{\delta'} \). Further, by choosing \( v \) appropriately, we can assume that this new perturbed functional is Morse. This construction allows us to perturb the functional in order to make it Morse while staying close enough to the critical set of the unperturbed functional \( E_0 \).

Now we fix \( \alpha \in \pi_1(N) \) and define the length spectrum of \( \Lambda_\alpha, LS_\alpha \subset [0, +\infty) \), by

\[
LS_\alpha = \{ E_0(\phi) ; \phi \in K \cap \alpha \}.
\]

It is easy to see that if \( LS_\alpha \) is bounded above, then so is the energy of the critical points of \( \bar{E}_1 \). In particular, the set of critical points of \( \bar{E}_1 \) in the class \( \alpha \) is a finite set.
For this $V$-homology, we have:

**Proposition 8.4** If $LS_\alpha$ is bounded from above or the set of perturbed geodesics (i.e., $\text{crit}(E_1)$) is finite, then the homology $H_{F_\alpha}(V; \mathbb{Z}_2)$ is equal to the homology $DG^{p+1}H_{\alpha}(N, \mathbb{Z}_2)$.

**Proof** We set $\chi = 1_{(-\infty, 0]}$ and consider the vector field

$$V_t = \chi(t)V + (1 - \chi(t))\nabla L_H.$$ 

Now, given $x_- \in \text{rest}(V)$ and $x_+ \in \text{crit}(L_H)$, we consider the functional $\mathcal{F}_{x-, x_+}^{V, V, \nabla L_H}: W^{1, 2}_{x-, x_+}((\mathbb{R}, \mathcal{T}^{1, 1/2}(S^1, N)) \to L^2(\mathbb{R}, T\mathcal{T}^{1, 1/2}(S^1, N))$ defined by

$$\mathcal{F}_{x-, x_+}^{V, V, \nabla L_H}(\ell)(t) = \frac{d\ell}{dt} + V_t(\ell(t)).$$

We want to study the moduli space $\mathcal{M}(x_-, x_+; V, \nabla L_H) := (\mathcal{F}_{x-, x_+}^{V, V, \nabla L_H})^{-1}(0)$. First, we notice that $\mathcal{F}_{x-, x_+}^{V, V, \nabla L_H}$ is Fredholm of index $\mu_0(x_-) - \mu_+ (x_+)$. To see this, we may assume as in Proposition 3.4 that $\ell$ is constant near $t = \pm \infty$. Then the assertion follows, as in the proof of Proposition 3.4, from the fact that

$$P^{-1} \circ d\ell_{\mathcal{F}_{x-, x_+}^{V, V, \nabla L_H}}(\ell) \circ P = \nabla \ell + A(t),$$

where $P$ is as in the proof of Proposition 3.4 and $A(t)$ is asymptotically hyperbolic and regulated in the sense of [1, page 703], i.e., limits $\lim_{t \to 0^\pm} A(t)$ exist at any $t_0 \in \mathbb{R}$ (notice here that we do not need the continuity of $A$, see [1, Theorem 3.4]). Again, using the same trick of perturbing the metric (this corresponds to considering $V_K$ and $(1 + K')\nabla L_H$ for some perturbations $K, K' \in \mathbb{R}^2_{0, 0}$), we can achieve transversality, making the moduli space $\mathcal{M}(x_-, x_+; V, \nabla L_H)$ a manifold of dimension $\mu_0(x_-) - \mu_+ (x_+)$. In fact, the only obstruction to construct such a perturbation as in Sects. 7 and 8 is the existence of constant flow lines. This can be excluded if $\text{rest}_k(V) \cap \text{crit}_k(L_H) = \emptyset$. As in Lemma 7.5, for a generic perturbation of $H$, this condition is satisfied. For simplicity, we use the same notation $V$ and $\nabla L_H$ to denote these perturbed vector fields.

Notice that, in fact, this space corresponds to $W^u(x_-, -V) \cap W^s(x_+, -\nabla L_H)$, that is the intersection of the stable manifold of $x_+$ with respect to $-\nabla L_H$ and the unstable manifold of $x_-$ with respect to $-V$. This is what is referred to in the literature as the space of hybrid trajectories (see [4]), since it corresponds to the coupled system:

$$\begin{align*}
\frac{d}{dt} u(t) &= -V(u(t)) & \text{for } t \in (-\infty, 0] \\
\frac{d}{dt} v(t) &= -\nabla L_H(v(t)) & \text{for } t \in [0, +\infty) \\
u(0) &= v(0).
\end{align*}$$

We now have:

**Lemma 8.7** $\mathcal{M}(x_-, x_+; V, \nabla L_H)$ is relatively compact in $C^0_{\text{loc}}(\mathbb{R}, \mathcal{T}^{1, 1/2}(S^1, N))$.

**Proof** Let $\ell(t) = (\phi(t), \psi(t)) \in \mathcal{M}(x_-, x_+; V, \nabla L_H)$ be arbitrary. We first observe that the restriction of $\phi$ to $(-\infty, 0]$ extends as an $E_1$-negative gradient flow to the whole $\mathbb{R}$. This result is proved in [31, Theorem 2.15] for the case of unperturbed negative gradient flow of the geodesic energy functional. Since the perturbation term $v$ is bounded, the same proof applies to the negative gradient flow of $E_1$. Note that the perturbation of the metric on $H^1(S^1, N)$ is irrelevant here. Thus, the same regularity estimate as in Lemma 8.4 holds for $\phi(t)$ for $t \in (-\infty, 0]$. Using this estimate and the fact that $L_H(\ell(0)) \geq L_H(x_+)$ (since $\ell$}
is the negative gradient flow of $L_H$ on $[0, \infty)$ and the end point condition $\ell(-\infty) = x_-$, also applying a similar argument as in the proof of Proposition 8.1, we have the estimate $|L_H(\ell(t))| \leq C(x_-, x_+)$ for all $t \in \mathbb{R}$. It follows then, as in the proof of Lemma 8.6, that

$$\sup_{t \leq -1} \int_{\tau}^{\tau+1} \|\psi(t)\|_{L^{1/2}(S^1)}^2 \, dt \leq C(x_-, x_+),$$

and

$$\sup_{t \geq 0} \int_{\tau}^{\tau+1} \|\psi(t)\|_{L^{1/2}(S^1)}^2 \, dt \leq C(x_-, x_+).$$

Combining these with the previous ones, we have $\int_{\tau}^{\tau+1} \|\partial_s \phi(t)\|_{L^2(S^1)}^2 \, dt \leq C(x_-, x_+)$, $\int_{\tau}^{\tau+1} \|\psi(t)\|_{L^{1/2}(S^1)}^2 \, dt \leq C(x_-, x_+)$ and $\int_{\tau}^{\tau+1} \|\psi(t)\|_{L^{p+1}(S^1)}^2 \, dt \leq C(x_-, x_+)$ for all $\tau \in \mathbb{R}$. Hence, similarly to the proof of Lemma 5.2, we have uniform estimates:

$$\sup_{t \in \mathbb{R}} \|\psi(t)\|_{L^{1/2}(S^1)} \leq C(x_-, x_+).$$

and

$$\sup_{t \in \mathbb{R}} \|\partial_s \phi(t)\|_{L^2(S^1)} \leq C(x_-, x_+).$$

Proceeding with bootstrapping (see Remark 8.1), as usual, we have bounds

$$\sup_{t \in \mathbb{R}} \|\psi(t)\|_{C^{1/3}(S^1)} \leq C(x_-, x_+)$$

and

$$\sup_{t \in \mathbb{R}} \|\partial_s \phi(t)\|_{C^{2/3}(S^1)} \leq C(x_-, x_+).$$

Hence, the relative compactness of $\mathcal{M}(x_-, x_+; V, -\nabla L_H)$ in $C_{\text{loc}}^0(\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N))$ follows as before.

Therefore, if $\mu_0(x_-) = \mu(H(x_+))$ then $\mathcal{M}(x_-, x_+; V, -\nabla L_H)$ is a 0-dimensional manifold. Moreover, it is compact since the non-compactness arises from breaking of $V_t$-flow lines (more precisely, by local compactness of Lemma 8.7, part of $V$-flow lines or $L_H$-negative flow lines break if the non-compactness occurs) which does not happen under the index constraint and the transversality. Hence, one can define the map $\Phi : C_* (\mathcal{L}_H; \mathbb{Z}_2) \rightarrow C_* (\mathcal{L}_H; \mathbb{Z}_2)$ on the generators as follows: for $x \in \text{rest}_k(V)$, $k \in \mathbb{Z}$,

$$\Phi(x) = \sum_{y \in \text{crit}_k(\mathcal{L}_H)} n_{\nabla H}(xy)y,$$

where $n_{\nabla H}(xy) = \# \mathcal{M}(xy; V, -\nabla L_H)(\text{mod } 2)$.

Similarly, one can define the map $\Psi : C_* (\mathcal{L}_H; \mathbb{Z}_2) \rightarrow C_* (V; \mathbb{Z}_2)$ counting flow lines between critical points of $\mathcal{L}_H$ and rest points of $V$ of the same index. In this
case the roles of $V$ and $\nabla L_H$ are changed and we consider the functional $\mathcal{F}_{x_-, x_+}^{\nabla L_H, V} : W_{x_-, x_+}^{1,2}(\mathbb{R}, \mathfrak{F}^{1,1/2}(S^1, N)) \to L^2(\mathbb{R}, \mathfrak{F}^{1,1/2}(S^1, N))$ defined by

$$
\mathcal{F}_{x_-, x_+}^{\nabla L_H, V}(\ell) = \frac{d\ell}{dt} + \hat{V}_t(\ell),
$$

where $x_\in \text{crit}(L_H)$, $x_\in \text{rest}(V)$ and $\hat{V}_t = \chi(t)\nabla L_H + (1 - \chi(t))V$. In this case, the Fredholm property of $\mathcal{F}_{x_-, x_+}^{\nabla L_H, V}$ and the transversality is achieved after perturbation of the vector fields as before and we obtain a 0-dimensional manifold $\mathcal{M}(x_-, x_+ ; \nabla L_H, V) := (\mathcal{F}_{x_-, x_+}^{\nabla L_H, V})^{-1}(0)$. Notice that $\ell \in \mathcal{M}(x_-, x_+ ; \nabla L_H, V)$ satisfies the following coupled system of similar to (8.52):

$$
\begin{align*}
\frac{du(t)}{dt} &= -\nabla L_H(u(t)) \quad &\text{for } t \in (-\infty, 0) \\
\frac{dv(t)}{dt} &= -V(v(t)) \quad &\text{for } t \in [0, +\infty) \\
u(0) &= v(0).
\end{align*}
$$

Notice that here the relative compactness of this moduli space is not guarantied in general since the restriction to $[0, +\infty)$ of the $v$ part of the flow, if extended to $(-\infty, 0]$, might not converge at $-\infty$. To overcome this, we will consider a modified moduli space. Let $x_0 \in \text{rest}(V)$ and we consider the moduli space

$$
\tilde{\mathcal{M}}(x_-, x_0, x_+) = W_{x_0}^{-V}(x_0) \cap W_{x_+}^{-V}(x_+) \cap W_{x_+}^V(x_+)
$$

This identification is done through the map $ev_{x_+, x_-} : \mathcal{M}(x_-, x_+, \nabla L_H, V) \to W_{x_+}^{-V}(x_+) \cap W_{x_+}^V(x_-)$ defined by

$$
ev_{x_+, x_-}(\ell) = \ell(0).
$$

So $\tilde{\mathcal{M}}(x_-, x_0, x_+)$ can be identified with $ev_{x_-, x_+}^{-1}(W_{x_0}^{-V}(x_0)) = ev_{x_-, x_+}^{-1}(\mathcal{M}(x_0, x_-, V))$.

The moduli space $\tilde{\mathcal{M}}(x_-, x_0, x_+)$ is relatively compact in $C^0_{\text{loc}}(\mathbb{R}, \mathfrak{F}^{1,1/2}(S^1, N))$. The proof is similar to the one of Lemma 8.7. In this case, the restriction of the $\phi$-component of $v$ to $[0, +\infty)$ extends as an $E_1$-negative gradient flow $\phi$ on the whole $\mathbb{R}$ with $\phi(-\infty) = \phi_0$, where $\phi_0$ is the $\phi$-part of $x_0$. We then obtain the same regularity estimate as in Lemma 8.4 for $\phi(t)$, $t \in [0, \infty)$. On the other hand, we have $L_H(x_-) \geq L_H(\ell(0))$. This together with the estimate for $\phi(t)$ for $t \in [0, \infty)$ and the end point condition $\ell(+\infty) = x_+$, we see that the argument of Lemma 8.5 and Proposition 8.1 applies and, as in the proof of Lemma 8.7, we have a uniform bound of $|L_H(\ell(t))|$ for $t \in \mathbb{R}$ and $\ell \in \tilde{\mathcal{M}}(x_-, x_+ ; \nabla L_H, V)$.

We let

$$
\tilde{\mathcal{M}}(x_-, x_+, \nabla L_H, V) = \bigcup_{x_0 \in \text{rest}(V)} \tilde{\mathcal{M}}(x_-, x_0, x_+).
$$

Notice that in this union, only the sets having $x_0 = (\phi_0, \psi_0)$ with $\mu_1(\phi_0) \geq \mu_1(\phi_+)$ will survive, the remaining ones will be empty. This union is then compact since the rest($V$) is a finite set, see Remark 8.2.

To complete the proof of Proposition 8.4, it remains to show that these are chain maps and that $\Phi \circ \Psi = \text{id}$ and $\Psi \circ \Phi = \text{id}$ in the homology level.

To show that $\Phi$ is indeed a chain map, we look at the moduli space $\mathcal{M}(x_-, x_+ ; V, \nabla L_H)$ for $\mu_0(x_-) - \mu_1(x_+) = 1$. By Lemma 8.7, the non-compactness of this moduli space arises from broken flow lines of the following form:
- Flow lines of $-V$ connecting $x_-$ to $x \in \text{rest}_{\mu_0(x_-)-1}(V)$ then $x$ to $x_+$ via the flow lines of $-V_t$.
- Flow lines of $-V_t$ connecting $x_-$ to $y \in \text{crit}_{\mu_H(x_+)}(\mathcal{L}_H)$, then $y$ to $x_+$ via the flow lines of $-\nabla \mathcal{L}_H$.

Conversely, by the gluing results of $-\nabla \mathcal{L}_H$ and $-V$ flow lines (see Sect. 3.5 and the argument following Corollary 8.1), these broken flow lines correspond to the ends of the moduli space. Therefore, the moduli space $\mathcal{M}(x_-, x_+; V, \nabla \mathcal{L}_H)$ has a natural compactification whose boundary consists of broken flow lines of the above types. Counting the number of the boundary modulo 2, we have
\[
\sum_{\mu_0(x) = \mu(x_-)-1} n_{V}(x_-, x)n_{\mathcal{L}_H}(x,x_+)+\sum_{\mu_H(y) = \mu(x_-)} n_{\mathcal{L}_H}(x_-, y)n(y,x_+) = 0 \pmod{2}.
\]
That means that
\[
\Phi \circ \partial_V = \partial_{\nabla \mathcal{L}_H} \circ \Phi.
\]

A similar property holds for the map $\Psi$.

The next step is to show that $\Psi \circ \Phi = \text{id}$. This requires another moduli space of hybrid trajectories. Indeed, we let $\chi_1 = 1_{[0,1]}$ and $\chi_2 = 1_{[1,\infty)}$ and consider for $R > 0$ the vector field
\[
\tilde{V}_{t,R} = \chi(t)V + R\chi_1(t)\nabla \mathcal{L}_H + \chi_2(t)\nabla_{\mathcal{L}_H}.
\]

The parameter $R$ should be seen here as a time rescaling parameter for the trajectories of $-\nabla \mathcal{L}_H$. Given $x_-, x_+ \in \text{rest}_k(V)$, we define the map $\tilde{\mathcal{F}}_{x_-, x_+}^{V_0, R} : W_{x_-, x_+}^{1,2}([R, 2^{1/2}(S^1, N)]) \times (0, \infty) \to L^2([R, 2^{1/2}(S^1, N)])$ by
\[
\tilde{\mathcal{F}}_{x_-, x_+}^{V_0, R}(\ell, R) = \frac{d\ell}{dt} + V_{t,R}.
\]

Note that the zero section of $\tilde{\mathcal{F}}_{x_-, x_+}^{V_0, R}$ corresponds to the system:
\[
\begin{align*}
\frac{d}{dt} u(t) &= V(u(t)) \quad \text{for } t \in (-\infty, 0] \\
\frac{d}{dt} v(t) &= -R\nabla \mathcal{L}_H(v(t)) \quad \text{for } t \in [0, 1] \\
\frac{d}{dt} w(t) &= V(w(t)) \quad \text{for } t \in [0, +\infty) \\
u(0) &= v(0) \\
v(1) &= w(0),
\end{align*}
\]

where $u(t) = \ell(t)$ for $t \in (-\infty, 0]$, $v(t) = \ell(t)$ for $t \in [0, 1]$ and $w(t) = \ell(t + 1)$ for $t \in [0, \infty)$. By the same reasons as before, this map is Fredholm and by perturbing the metric again, the moduli space $\mathcal{M}(x_-, x_+; V, \nabla \mathcal{L}_H, V) := (\tilde{\mathcal{F}}_{x_-, x_+}^{V_0, R})^{-1}(0)$ has the structure of a one dimensional manifold. Again the relative compactness of this moduli space is not guaranteed in general. So instead, we look at modified moduli space $\tilde{\mathcal{M}}(x_-, x_+; V, \nabla \mathcal{L}_H, V)$ of flow lines of $V_{t,R}$ such that $\ell(1) = w(0) \in W_{-V}(x_0)$ for a certain $x_0 \in \text{rest}(V)$ as in the previous case. The relative compactness of this moduli space in $C^0_{\text{loc}}([R, 2^{1/2}(S^1, N)]) \times (0, \infty)$ is then proved as in Lemma 8.7. By the index constraint and transversality for $V$-flow lines, the non-compactness only arises from $R \to 0$ or $R \to \infty$.

When $R \to 0$ we reach the boundary $\mathcal{M}(x_-, x_+; V) = \{x_-\}$ if $x_- = x_+$ and $\mathcal{M}(x_-, x_+; V) = \emptyset$ if $x_- \neq x_+$. On the other hand, when $R \to \infty$, flow line $(\ell_R, R) \in \mathcal{M}(x_-, x_+; V, \nabla \mathcal{L}_H, V)$ converges to broken flow line with components in
\[ \mathcal{M}(x_-, y; V, \nabla L_H) \times \mathcal{M}(y, x_+; \nabla L_H, V) \] for some \( y \in \text{crit}_k(L_H) \). Note that \( V \)-flow parts (i.e., \( v \) and \( w \) parts in Eq. (8.54)) do not break due to index constraint and transversality. The converse is also true by a standard gluing argument as before. Thus counting the number of the boundary of this moduli space modulo 2, we obtain
\[ \Psi \circ \Phi = \text{id}. \]
A similar construction, interchanging the roles of \( V \) and \( \nabla L_H \) in the above argument, can be done to show \( \Phi \circ \Psi = \text{id} \). Therefore, we have
\[ HF_s(V; \mathbb{Z}_2) = HF_s(L_H, S^{1,1/2}; S^1, N; \mathbb{Z}_2) = DG^{p+1}H_s(N; \mathbb{Z}_2). \]
This completes the proof of Proposition 8.4.

### 8.2 Leray–Serre type Floer spectral sequence

We will construct now a spectral sequence converging to the \( V \)-homology and this will lead us to the computation of the Dirac-geodesic homology. One can define a filtration \( (F_r(C_s(V; \mathbb{Z}_2)))_{r \geq 0} \) on \( C_s(V; \mathbb{Z}_2) \) as follows:
\[ F_r C_s(V; \mathbb{Z}_2) = \text{span}_\mathbb{Z}_2 \{ x = (\phi, \psi) \in \text{rest}_q(V); \mu_1(\phi) \leq r \}. \]
Clearly, we have the following properties:

i) \( F_r C_s(V; \mathbb{Z}_2) = \{ 0 \} \) for \( r < 0 \),

ii) \( F_r C_s(V; \mathbb{Z}_2) \subset F_{r+1} C_s(V; \mathbb{Z}_2) \),

iii) \( \bigcup_{r \geq 0} F_r C_s(V; \mathbb{Z}_2) = C_s(V; \mathbb{Z}_2) \).

Moreover, since the flow of \( V \) projects to the negative gradient flow of \( E_1 \), the boundary operator \( \partial^V \) decreases the index \( \mu_1 \). Therefore, \( \partial^V (F_r C_s(V; \mathbb{Z}_2)) \subset F_{r+1} C_s(V; \mathbb{Z}_2) \). So the filtration we have is in fact a chain filtration. Thus, we can consider its associated spectral sequence. It follows then from i)–iii) and [35, Chap 9, Theorem 2] that there exists a spectral sequence with the first page
\[ E^1_{r,q} = HF_s(F_r C(V; \mathbb{Z}_2)/F_{r-1} C(V; \mathbb{Z}_2)) \]
and the boundary operator \( \partial_1 \) corresponding to the triplet
\[ (F_r C(V; \mathbb{Z}_2), F_{r-1} C(V; \mathbb{Z}_2), F_{r-2} C(V; \mathbb{Z}_2)). \]
That is, \( \partial_1 \) is the boundary operator coming from the short exact sequence
\[ 0 \rightarrow F_{r-1} C(V; \mathbb{Z}_2) \rightarrow F_r C(V; \mathbb{Z}_2) \rightarrow F_{r-1} C(V; \mathbb{Z}_2) \rightarrow 0. \]

**Lemma 8.8** We have
\[ E^1_{r,q} = \bigoplus_{\phi \in \text{crit}_r(E_1)} D^{p+1}HF_q(H^{1/2}(S^1, \mathbb{S}(S^1) \otimes \phi_0^*TN); \mathbb{Z}_2), \]
where \( \phi_0 \in H^1(S^1, N) \) is an (arbitrary chosen) base point of \( H^1(S^1, N) \) and
\[ D^{p+1}HF_q(H^{1/2}(S^1, \mathbb{S}(S^1) \otimes \phi_0^*TN); \mathbb{Z}_2) \]
is the \( (p+1) \)-Dirac–Morse–Floer homology of the twisted spinor bundle \( \mathbb{S}(S^1) \otimes \phi_0^*TN \rightarrow S^1 \) as defined in [22].
**Proof** We first notice that
\[ F_r C(V; \mathbb{Z}_2) / F_{r-1} C(V; \mathbb{Z}_2) = \text{span}_{\mathbb{Z}_2} \{(\phi, \psi) \in \text{rest}(V); \phi \in \text{crit}_r(E_1); \psi \in \text{crit}(E_2(\phi, \cdot))\} \]
with the boundary operator
\[ \partial : F_r C_{r+q}(V; \mathbb{Z}_2) / F_{r-1} C_{r+q}(V; \mathbb{Z}_2) \to F_r C_{r+q-1}(V; \mathbb{Z}_2) / F_{r-1} C_{r+q-1}(V; \mathbb{Z}_2). \]
Now consider two rest points \( x_- = (\phi_-, \psi_-), x_+ = (\phi_+, \psi_+) \in \{(\phi, \psi) \in \text{rest}(V); \phi \in \text{crit}_r(E_1); \psi \in \text{crit}(E_2(\phi, \cdot))\} \) and let \( \ell = (\phi(t), \psi(t)) \in \mathcal{M}(x_-, x_+; V) \). \( \ell \) projects down to a flow \( \phi(t) \) of the negative gradient of \( E_1 \). Since \( \mu_1(\phi_-) = \mu_1(\phi_+) = r \), we have by transversality that the path \( \phi(t) \) is constant and that \( \phi(t) = \phi_- = \phi_+ \). This implies that \( \psi_- \) and \( \psi_+ \) are both critical points of \( E_2 \) restricted to the same fiber, that is, \( \psi_- \), \( \psi_+ \in \text{crit}(E_2(\phi, \cdot)) \) (\( \phi = \phi_- = \phi_+ \)) and the component \( \psi(t) \) of \( \ell \) is then a flow line of the negative gradient vector of \( E_2(\phi, \cdot) \). It follows then that
\[ \mathcal{M}(x_-, x_+; V) \cong \mathcal{M}(\psi_-, \psi_+; E_2(\phi, \cdot)), \]
where \( \mathcal{M}(\psi_-, \psi_+; E_2(\phi, \cdot)) \) is the set of negative gradient flows of \( E(\phi, \cdot) \) connecting \( \psi_- \) and \( \psi_+ \). Therefore, the \( E_1 \)-term is give by
\[ E_{r,q}^1 = \sum_{\phi \in \text{crit}_r(E_1)} \mathbb{Z}_2(\phi) \otimes HF_q(E_2(\phi, \cdot), H_1/2(S^1, \mathbb{S}(S^1) \otimes \phi^*TN); \mathbb{Z}_2), \]
where \( HF_*(E_2(\phi, \cdot), H_1/2(S^1, \mathbb{S}(S^1) \otimes \phi^*TN); \mathbb{Z}_2) \) is the Morse–Floer homology of \( E_2(\phi, \cdot) \) on \( H_1/2(S^1, \mathbb{S}(S^1) \otimes \phi^*TN) \) constructed in [22].

The homology \( HF_*(E_2(\phi, \cdot), H_1/2(S^1, \mathbb{S}(S^1) \otimes \phi^*TN); \mathbb{Z}_2) \) is independent of the fiber \( \phi \in \text{crit}(E_1) \). Indeed, if \( \phi_1 \) and \( \phi_2 \) are critical points of \( E_1 \) of index \( r \), then we have
\[ HF_*(E_2(\phi_1, \cdot), H_1/2(S^1, \mathbb{S}(S^1) \otimes \phi_1^*TN), \mathbb{Z}_2) = HF_*(E_2(\phi_2, \cdot), H_1/2(S^1, \mathbb{S}(S^1) \otimes \phi_2^*TN), \mathbb{Z}_2) = D^{p+1}HF_*(E_2(\phi_1, \cdot), H_1/2(S^1, \mathbb{S}(S^1) \otimes \phi_0^*TN), \mathbb{Z}_2), \]
where \( \phi_0 \in H^1(S^1, N) \) is the base point of \( H^1(S^1, N) \) and \( D^{p+1}HF_*(H_1/2(S^1, \mathbb{S}(S^1) \otimes \phi_0^*TN), \mathbb{Z}_2) \) is the \((p + 1)\)-Dirac–Morse–Floer homology of the twisted bundle \( \mathbb{S}(S^1) \otimes \phi_0^*TN \).

The above isomorphism follows from the result in [22] of the invariance of the homology with respect to functions \( H \in \mathbb{H}_p^k \). This can be seen by taking a homotopy connecting \( \phi_i \) to \( \phi_0 \) for \( i = 1, 2 \). Then using the parallel transport, we pull back the problem to one single fiber above \( \phi_0 \) and thus we will have an equivalent problem with a homotopy between two perturbations in \( \mathbb{H}_p^k \), hence they have the same homology as shown in [22]. Note that, by the main result of [22], the isomorphism obtained in this manner is canonically defined, i.e., independent of the choice of homotopy between \( \phi_i \) and \( \phi_0 \). Thus the local coefficient system \( \{HF_*(E_2(\phi, \cdot), H_1/2(S^1, \mathbb{S}(S^1) \otimes \phi^*TN); \mathbb{Z}_2)\}_{\phi \in H^1(S^1, N)} \) is in fact trivial. This completes the proof of Lemma 8.8.

With all these ingredient now we can state the third claim in Theorem 1.

**Corollary 8.2** The homology \( DG^{p+1}H_*(\mathbb{S}^{1,1/2}(S^1, N); \mathbb{Z}_2) \) vanishes.

**Proof** So far we constructed a first quadrant spectral sequence, with first page \( E_{r,q}^1 \) defined in (8.55). Again, from [35], we have that this spectral sequence converges to \( GH_*(C(V; \mathbb{Z}_2)) \).
But from [22], $D^{p+1} H F_*(H^{1/2}(S^1, S^1 \otimes \phi_0^* T N); \mathbb{Z}_2) = 0$. Therefore, by Lemma 8.8, the spectral sequence collapses at the first page and

$$G H_*(C(V; \mathbb{Z}_2)) = 0.$$ 

Therefore one has

$$H F_*(V; \mathbb{Z}_2) = 0.$$ 

Then by Proposition 8.4, we have the desired result

$$D G^{p+1} H F_*(N; \mathbb{Z}_2) = H F_*(V; \mathbb{Z}_2) = 0.$$ 

\[\square\]

9 Applications to the existence of superquadratic Dirac-geodesics

Because our homology vanishes, the reader might think that the approach by More theory is not adequate for proving existence of critical points of $\mathcal{L}_H$. As we shall show below, however, the vanishing of the homology is seen as a topological obstruction for some cases and it leads to some non-trivial existence results for perturbed Dirac-geodesics.

To show this, in this section, we assume that $\nabla_\psi \bar{H}(\phi, 0) = 0$. Note that, in this case, $(\phi, 0)$ is a critical point of $\mathcal{L}_H$ for any perturbed geodesic $\phi \in \text{crit}(E_{1, H})$.

Before stating our main results, we first give the following remark. For a given $E_{1, H}$, one can generically perturb $H$ by $h = h(s, \phi)$ in a way that for $\bar{H} = H + h$ also satisfies $\nabla_\psi \bar{H}(\phi, 0) = 0$ and $E_{1, \bar{H}}: \phi \mapsto \mathcal{L}_{\bar{H}}(\phi, 0)$ is Morse, see [37]. Also notice that for a given perturbed geodesic $\phi$ (i.e., critical point of $E_{1, \bar{H}}$), we have that

$$\nabla^2_{\psi, \psi} \mathcal{L}_{\bar{H}}(\phi, 0) = (1 + |D|)^{-1}\left(D_\phi - \nabla^2_{\psi, \psi} \bar{H}(\phi, 0)\right).$$

Since the spectrum of this operator is discrete, for $\lambda$ belonging to an open dense set of reals, $0 \notin \text{Spec}(\nabla^2_{\psi, \psi} \mathcal{L}_{\bar{H}}(\phi, 0) + \lambda \text{Id})$. Since the set of critical points of $E_{1, \bar{H}}$ is also discrete, we have that for $\lambda \in \Omega$ we have that

$$0 \notin \text{Spec}(\nabla^2_{\psi, \psi} \mathcal{L}_{\bar{H}}(\phi, 0) + \lambda \text{Id})$$

for all $\phi \in \text{crit}(E_{1, \bar{H}})$. In particular, if we perturb $\bar{H}$ even more, to a function $\bar{H}_\lambda = \bar{H} + \lambda \rho(|\psi|^2)|\psi|^2$, where $\rho \in C^\infty(\mathbb{R})$ is a cut-off function such that $\rho(t) = 1$ for $|t| \leq 1$ and $\rho(t) = 0$ for $|t| \geq 2$, we have that for $\lambda \in \Omega$, the functional $\mathcal{L}_{\bar{H}_\lambda}$ is Morse. Moreover, $\nabla_\psi \bar{H}_\lambda(\phi, 0) = 0$ and if $H$ is even with respect to $\psi$ then so is $\bar{H}_\lambda$. These show the following: For a given $H \in \mathbb{H}^{3, 2}_{p+1}$, there exists an arbitrary small perturbation $h$ such that for $\bar{H} = H + h$, $E_{1, \bar{H}}$ and $\mathcal{L}_{\bar{H}}$ are Morse on $H^1(S^1, N)$ and $\mathcal{F}^{1, 1/2}(S^1, N)$, respectively. Moreover, if $H$ is even with respect to $\psi$, the perturbed $\bar{H}$ can be chosen to be even with respect to $\psi$.

Based on the above observation, in the following, we assume that $E_{1, H}$ and $\mathcal{L}_H$ are Morse functions on $H^1(S^1, N)$ and $\mathcal{F}^{1, 1/2}(S^1, N)$. We fix a critical point $(\phi_0, 0) \in \text{crit}(\mathcal{L}_H)$ with $\phi_0$ a critical point of $E_{1, H}$ of index zero and we will compute the relative index $\mu_H$ starting from it.

Let us consider a component of the free loop space $\Lambda N = C^0(S^1, N)$. Since it corresponds to a conjugacy class in $\pi_1(N)$, it is labeled by $\pi_1(N)$. We fix then $\alpha \in \pi_1(N)$ and let $\Lambda_\alpha N$
the corresponding component in $\Lambda N$. We assume that $N$ satisfies the following finiteness condition:

\[ (*) \quad H_*(\Lambda_\alpha N; \mathbb{Z}) \text{ vanishes in large degrees and is a finitely generated abelian group in each degree.} \]

Assumption (\( * \)) is satisfied if $\Lambda_\alpha N$ has a homotopy type of finite CW-complex and examples of such manifolds will be provided later on. Now, notice that for such manifolds, the Euler characteristic $\chi(\Lambda_\alpha N)$ of $\Lambda_\alpha N$ is well-defined. We then have the following:

**Theorem 9.1** Let $H \in \mathbb{H}^{3}_{p+1}$ be as above. Let us also assume that the compact manifold $N$ satisfies (\( * \)) and the Euler characteristic $\chi(\Lambda_\alpha N)$ is odd. Then one of the following holds:

1. there exist infinitely many perturbed geodesics in $\Lambda_\alpha N$, or
2. there exists at least one perturbed Dirac-geodesic $(\phi, \psi)$ with $\phi \in \Lambda_\alpha N$ and $\psi \neq 0$ for the functional $\mathcal{L}_H$.

**Proof** We assume that there are only finitely many perturbed geodesic in $\Lambda_\alpha N$ and show that in this case, the assertion (2) holds.

Let $\phi_1, \phi_2, \ldots, \phi_p \in \Lambda_\alpha N$ be all perturbed geodesics in $\Lambda_\alpha$ and assume, for the sake of contradiction that there are no perturbed Dirac-geodesics. Then all critical points of $\mathcal{L}_H$ are of the form $(\phi_1, 0), (\phi_2, 0), \ldots, (\phi_p, 0)$. By the finiteness assumption, the Euler characteristic of $\chi(\Lambda_\alpha N)$ is well-defined. Since $E_1$ is Morse on $H^1(S^1, N) \cap \Lambda_\alpha N$ (we can also assume that the flow lines satisfy the Morse–Smale condition after a generic perturbation of the metric), $\phi_i (1 \leq i \leq p)$ generate the Morse complex for $E_{1,H}$ on $H^1(S^1, N) \cap \Lambda_\alpha N$. Note that $H^1(S^1, N) \cap \Lambda_\alpha N$ is homotopy equivalent to $\Lambda_\alpha N$. Therefore, we have

\[ \sum_{i=1}^{p} (-1)^{\mu_{E_{1,H}}(\phi_i)} = \chi(\Lambda_\alpha N), \quad (9.1) \]

where $\mu_{E_{1,H}}(\phi_i)$ is the Morse index of $E_{1,H}$ at $\phi_i$.

On the other hand, since $(\phi_i, 0) (1 \leq i \leq p)$ generate the Morse–Floer complex of $\mathcal{L}_H$ on $T^{1,1/2}(S^1, N)$ and $DG^{p+1}H F_e(T^{1,1/2}(S^1, N)) = 0$, we also have

\[ \sum_{i=1}^{p} (-1)^{\mu_{\mathcal{L}_H}(\phi_i,0)} = 0. \quad (9.2) \]

By (9.1) and (9.2), we have $\chi(\Lambda_\alpha N) = 0$ (mod 2) and this leads to a contradiction. Thus, there exists at least one perturbed Dirac geodesic $(\phi, \psi)$ with $\psi \neq 0$. We point out here that the two indices $\mu_{E_{1,H}}$ and $\mu_{\mathcal{L}_H}$ might be different. What makes this argument work is the fact that the left hand sides of (9.1) and (9.2) are both equal to $p$ in mod 2. \( \square \)

When $H(s, \psi, \psi)$ is even with respect to $\psi$, $H(s, \phi, -\psi) = H(s, \phi, \psi)$, we have:

**Theorem 9.2** Let $H \in \mathbb{H}^{3}_{p+1}$ be as above and assume that $N$ satisfies (\( * \)) and the Euler characteristic $\chi(\Lambda_\alpha N)$ is odd. Assume moreover that $H(s, \phi, \psi)$ is even with respect to $\psi$. Then one of the following holds:

1. there exist infinitely many perturbed geodesics in $\Lambda_\alpha N$, or
2. there exist infinitely many perturbed Dirac-geodesics $(\phi, \psi)$ with $\phi \in \Lambda_\alpha N$ and $\psi \neq 0$ for the functional $\mathcal{L}_H$.  

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There exist infinitely many perturbed Dirac-geodesics \( (\phi, \psi) \) with \( \psi \neq 0 \) and \( \phi \in \Lambda_\alpha N \). Assume, for the sake of contradiction, that there exist only finitely many perturbed Dirac-geodesics \( (\tilde{\phi}_k, \psi_k) (1 \leq k \leq q) \) with \( \psi_k \neq 0 \) and \( \tilde{\phi}_k \in \Lambda_\alpha N \). By the evenness of \( H \) with respect to \( \psi \), if \( (\phi, \psi) \) is a perturbed Dirac-geodesic, then so is \( (\phi, -\psi) \), i.e., perturbed Dirac-geodesics exist in pairs \( (\phi, \pm \psi) \). Thus \( q \geq 1 \) is an even integer. Then again comparing the Euler characteristics, as in the previous theorem, we have

\[
0 = \sum_{(\phi, \psi) \in \text{crit}(L_H)} (-1)^{\mu_H(\phi, \psi)} = \sum_{i=1}^{p} (-1)^{\mu_H(\phi_i, 0)} + \sum_{k=1}^{q} (-1)^{\mu_H(\tilde{\phi}_k, \psi_k)}
\]

\[
\equiv \sum_{i=1}^{p} (-1)^{\mu_H(\phi_i, 0)} \pmod{2}
\]

\[
\equiv \sum_{i=1}^{p} (-1)^{\mu_E(\phi_i)} = \chi(\Lambda_\alpha N) \pmod{2}.
\]

This, again, contradicts the oddness of \( \chi(\Lambda_\alpha N) \). Therefore, there exist infinitely many non-trivial perturbed Dirac-geodesics.

**Remark 9.1** If the condition \((*)\) is not satisfied, by Morse theory for \( E_1 \) on \( \Lambda_\alpha N \), the existence of infinitely many perturbed geodesics in \( \Lambda_\alpha N \) is a well established result. On the other hand, under the condition \((*)\), the existence of infinitely many perturbed geodesics in the component \( \Lambda_\alpha N \) is not obvious at all. In fact, for some cases, there exist only finitely many perturbed geodesics in \( \Lambda_\alpha N \), as we will discuss further down.

The finiteness condition \((*)\) is satisfied for a class of aspherical manifolds. By definition, a manifold \( N \) is called aspherical if its universal covering space \( \tilde{N} \) is contractible. In this case, \( \Lambda_\alpha N \) has the homotopy type of \( N \) for each \( \alpha \in \pi_1(N) \). Indeed, consider the fibration

\[
\Omega_\alpha(N) \longrightarrow \Lambda_\alpha N \overset{\text{ev}_0}{\longrightarrow} N,
\]

where \( \Omega_\alpha(N) \subset \Lambda_\alpha N \) is the based loop space in the class \( \Lambda_\alpha N \) and \( \text{ev}_0 \) is the evaluation map at \( 0 \), \( \text{ev}_0(\gamma) = \gamma(0) \). Since \( \Omega_\alpha(N) \simeq \Omega(\tilde{N}) \) (homotopy equivalence) and \( \Omega(\tilde{N}) \simeq \ast \) by the contractibility of \( \tilde{N} \), the fiber of the above fibration \( \Omega_\alpha(N) \) is contractible and we have homotopy equivalence \( \Lambda_\alpha N \simeq \sim N \). Hence, in this case, we have:

**Corollary 9.1** Let \( H \in \mathbb{H}^{p+1}_{p+1} \) be as in Theorem 9.1. Assume that \( N \) is a compact aspherical manifold with odd Euler characteristic \( \chi(N) \). Let \( \alpha \in \pi_1(N) \) be arbitrary given and consider the associated component \( \Lambda_\alpha N \) of \( \Lambda N \). Then one of the following holds:

1. there exist infinitely many perturbed geodesics in \( \Lambda_\alpha N \), or
2. there exists perturbed Dirac-geodesic \( (\phi, \psi) \) with \( \phi \in \Lambda_\alpha N \) and \( \psi \neq 0 \).

If we further assume that \( H \) is even with respect to \( \psi \), then we have one of the following:

1. there exist infinitely many perturbed geodesics in \( \Lambda_\alpha N \), or
2. there exist infinitely many perturbed Dirac-geodesics \( (\phi, \psi) \) with \( \phi \in \Lambda_\alpha N \) and \( \psi \neq 0 \).

When \( \chi(\Lambda_\alpha N) \) is even, the above argument does not apply. However, if we already know the existence of at least one perturbed Dirac-geodesic \( (\phi, \psi) \) with \( \psi \neq 0 \), a similar argument implies the following multiplicity result:
Theorem 9.3 Assume that a compact manifold $N$ satisfies $\ast$ and its Euler characteristic $\chi(\Lambda_\alpha N)$ is even. Moreover, assume that there exists a perturbed Dirac geodesic $(\phi, \psi)$ with $\phi \in \Lambda_\alpha N$ and $\psi \neq 0$. Then one of the following holds:

1. there exist infinitely many perturbed geodesics in $\Lambda_\alpha N$, or
2. there exist at least two Dirac geodesics $(\phi, \psi)$ with $\phi \in \Lambda_\alpha N$ and $\psi \neq 0$.

Proof We argue as in Theorems 9.1 and 9.2. So assume that there exist only finitely many perturbed geodesics in $\Lambda_\alpha N$. Again, by comparison of the Euler characteristics of the Morse homology of $E_1, H$ and the Morse–Floer homology of $L_H$, we conclude that there exist infinitely many perturbed Dirac-geodesics $(\phi, \psi)$ with $\psi \neq 0$ or there exist even number of perturbed Dirac geodesics $(\phi, \psi)$ with $\psi \neq 0$. Since there exists at least one perturbed Dirac-geodesic $(\phi, \psi)$ with $\psi \neq 0$ by our assumption, in either case, we conclude that there are at least two perturbed Dirac geodesics $(\phi, \psi)$ with $\psi \neq 0$. This completes the proof. □

The existence of at least one perturbed Dirac-geodesic $(\phi, \psi)$ with $\psi \neq 0$ was proved for some cases in [20]. In particular, for the case $N = \mathbb{T}^k$, the assumption $\ast$ is satisfied with $\chi(\mathbb{T}^k) = 0$ even. Thus, Theorem 9.3 applies and we have in this case:

Corollary 9.2 In the case of $N = \mathbb{T}^k$, a flat torus, we have one of the following: For each $\alpha \in \pi_1(\mathbb{T}^k) \cong \mathbb{Z}^k$,

1. there exist infinitely many perturbed geodesics in $\Lambda_\alpha \mathbb{T}^k$, or
2. there exist at least two perturbed Dirac-geodesics $(\phi, \psi)$ with $\psi \neq 0$ and $\phi \in \Lambda_\alpha \mathbb{T}^k$.

By considering a recent result of [23], the second alternative in the above corollary is improved to the existence of at least $k + 2$ solutions if $k$ is even.

We provide here some examples of aspherical manifolds:

Example 1. For a 2-dimensional compact manifold $N_g$ of genus $g$, its universal cover $\tilde{N}_g$ is contractible if and only if $g \geq 1$ if $N_g$ is orientable, and $g \geq 2$ if $N_g$ is non-orientable. Thus, these manifolds are aspherical. For the orientable case, we have $\chi(N_g) = 2 - 2g$ is even. In this case, combining the result of [20] and Theorem 9.3, we have the multiplicity result of Theorem 9.3 for the class of $H$ and Riemannian metrics on $N$ satisfying the assumption in [20, Theorem 1.2]. On the other hand, for the non-orientable case, $\chi(N_g) = 2 - g$. Therefore, the assumption of Corollary 9.1 is satisfied for $g \geq 3$ and $g$ odd. (For the case of even $g$ with $g \geq 2$, one can apply Theorem 9.3 just as before). Thus, for this class of manifolds, we have the existence result as stated in Corollary 9.1.

Example 2. The class of non-positively curved manifolds provide another important class of aspherical manifolds. By the Hadamard–Cartan theorem, $N$ is aspherical when $N$ is a non-positively curved manifold. Thus the assumption of Corollary 9.1 is satisfied when $\chi(N)$ is odd and that of Theorem 9.3 is satisfied when $\chi(N)$ is even. In both cases (for the even case, for the class of $H$ satisfying the assumption of [20, Theorem 1.2]), we have existence results as stated in Corollary 9.1 and Theorem 9.3.

Finally, we discuss the possibility of the second alternative of the conclusion of the theorems and corollaries, i.e., the possibility of the existence of perturbed Dirac-geodesics $(\phi, \psi)$ with $\psi \neq 0$. For example, consider the case where $N$ has a unique (up to reparametrization) genuine geodesic. This is the case when $N$ is a negatively curved manifold and $\alpha$ is not the class of contractible loops (For the contractible case, however, can be treated in a similar way). In this case, if the perturbation of the geodesic energy functional is small enough and made in a neighborhood of the unique $S^1$-family of genuine geodesics as in Remark 8.2, then the $S^1$-family of genuine geodesics split into finitely many non-degenerate critical points.
of the perturbed energy functional in that neighborhood and these are only critical points of the perturbed energy functional. Thus, if the perturbation is chosen appropriately, there are only finitely many perturbed geodesics and the second alternative of these theorems and corollaries hold in such a case.

10 Appendix

10.1 Multiplicative property of Sobolev functions

In this section, we collect some technical results which were used in the proofs of the results in the previous sections. The first is concerned with multiplicative property of fractional order Sobolev functions.

**Lemma 10.1** [10, Lemma 6.1] Let $1 < p < \infty$, $0 < s < \infty$, $1 < r < \infty$, $0 < \theta < 1$, $1 < t < \infty$ be such that $\frac{1}{r} + \frac{\theta}{t} = \frac{1}{p}$. For $f \in W^{s,t}(S^1) \cap L^\infty(S^1)$, $g \in W^{\theta s,p}(S^1) \cap L^t(S^1)$, we have $fg \in W^{\theta s,p}(S^1)$ and

$$\|fg\|_{W^{\theta s,p}(S^1)} \leq \|f\|_{L^\infty(S^1)} \|g\|_{W^{\theta s,p}(S^1)} + \|g\|_{L^t(S^1)} \|f\|_{W^{s,t}(S^1)} \|fg\|_{L^\infty(S^1)}^{1-\theta}. \quad (10.1)$$

For the proof, see [10]. Note that the conclusion of the lemma holds without the dimension restriction. However, we state it only for the 1-dimensional case since we only use it for this special case.

In particular, for $p = 2$, $s = 1$, $r = 4$, $\theta = \frac{1}{2}$, $t = 2$, Lemma 10.1 implies the following: For $f \in W^{1,2}(S^1)$ and $g \in H^{1/2}(S^1)$, we have $fg \in H^{1/2}(S^1)$ and

$$\|fg\|_{H^{1/2}(S^1)} \leq C(\|f\|_{L^\infty(S^1)} \|g\|_{H^{1/2}(S^1)} + \|g\|_{L^4(S^1)} \|f\|_{H^{1/2}(S^1)} \|fg\|_{L^\infty(S^1)}^{1/2}). \quad (10.2)$$

Note that, in 1-dimension, we have $H^1(S^1) \subset L^\infty(S^1)$ and $H^{1/2}(S^1) \subset L^4(S^1)$ by the Sobolev embedding theorem.

We also have the following corollary:

**Corollary 10.1** For any $\frac{1}{2} < s < 1$, we have the continuous multiplication:

$$H^{-1/2}(S^1) \times H^{1/2}(S^1) \ni (g, h) \mapsto gh \in H^{-s}(S^1). \quad (10.3)$$

**Proof** For $f \in H^{-1/2}(S^1)$, $g \in H^{1/2}(S^1)$ and $h \in C^\infty(S^1)$, the distribution $fg$ is defined by

$$\langle fg, h \rangle = \langle f, gh \rangle.$$

Thus, to prove the assertion, we need to prove $gh \in H^{1/2}(S^1)$ for $g \in H^{1/2}(S^1)$ and $h \in H^s(S^1)$. This follows form Lemma 10.1 by taking $p = 2$, $s = s$, $r = \frac{4s}{2s-1}$, $\theta = \frac{1}{2s}$ and $t = 2$. Or, we can directly prove this as follows: We note that the continuity of

$$L^2(S^1) \times H^s(S^1) \ni (f, g) \mapsto fg \in L^2(S^1) \quad (10.4)$$

and

$$H^s(S^1) \times H^s(S^1) \ni (f, g) \mapsto fg \in H^s(S^1), \quad (10.5)$$

where the continuity of (10.5) follows from $H^s(S^1) \subset L^\infty(S^1)$ for $s > \frac{1}{2}$ and therefore $H^s(S^1)$ becomes an algebra. Thus, for any fixed $h \in H^s(S^1)$, the following maps are continuous:

$$L^2(S^1) \ni g \mapsto gh \in L^2(S^1), \quad (10.6)$$
\[ H^s(S^1) \ni g \mapsto gh \in H^s(S^1). \] (10.7)

Since \( H^{1/2}(S^1) \) is obtained as an interpolation space between \( L^2(S^1) \) and \( H^s(S^1) \), the continuity of the multiplication \( H^{1/2}(S^1) \times H^s(S^1) \ni (f, g) \mapsto fg \in H^{1/2}(S^1) \) follows by interpolating (10.6) and (10.7).

**Lemma 10.2** Let \( \phi \in H^1(S^1, N) \). Then \( P_\phi \) defined by \( P_\phi(\psi)(s) = (1 \otimes P_\phi(s))\psi(s) \) for \( \psi \in H^{1/2}(S^1, \mathbb{S}(S^1) \otimes \mathbb{R}^k) \) and \( s \in S^1 \) defines a map \( P_\phi : H^{1/2}(S^1, \mathbb{S}(S^1) \otimes \mathbb{R}^k) \rightarrow H^{1/2}(S^1, \mathbb{S}(S^1) \otimes \phi^*TN) \). Moreover, we have the following estimate:

\[
\| P_\phi \psi \|_{H^{1/2}(S^1)} \leq C(\| \psi \|_{H^{1/2}(S^1)} + \| \partial_s \phi \|_{L^2(S^1)}^1 \| \psi \|_{L^1(S^1)}^1).
\] (10.8)

**Proof** Let \( \{ U_\alpha \}_{\alpha=1}^k \) be a covering of \( N \) consisting of local coordinates such that \( TN|_{U_\alpha} \cong U_\alpha \times \mathbb{R}^n \). For each \( 1 \leq \alpha \leq k \), let \( \{ e_{\alpha,j} \}_{j=1}^n \) be an orthonormal frame fields of \( TN|_{U_\alpha} \). For \( s \in S^1 \) such that \( \phi(s) \in N \), \( P_\phi \) is given by

\[
P_\phi(\psi)(s) = \sum_{j=1}^n (\psi(s), e_{\alpha,j}(\phi(s)))_{TN|U_\alpha} \otimes e_{\alpha,j}(\phi(s)) \tag{10.9}
\]

and the expression is independent of the choices of \( \alpha \) and the frame \( \{ e_{\alpha,j} \}_{j=1}^n \).

Let \( \{ \rho_\alpha \}_{\alpha=1}^k \) be a partition of unity subordinate to the covering \( \{ U_\alpha \}_{\alpha=1}^k \). By (10.9), we have

\[
P_\phi(\psi)(s) = \sum_{\alpha=1}^k \sum_{j=1}^n \rho_\alpha(\phi(s))(\psi(s), e_{\alpha,j}(\phi(s)))_{TN|U_\alpha} \otimes e_{\alpha,j}(\phi(s)) \tag{10.10}
\]

for any \( s \in S^1 \).

Note that (10.10) is written as the sum of the form \( m(\phi) \psi \), where \( m(\cdot) \) is a smooth function on \( N \). Thus, to prove (10.8), it suffices to prove the inequality for multiplication operator of the form \( H^{1/2}(S^1) \ni \psi \mapsto m(\phi) \psi \), where \( m \) is a smooth function on \( N \). This follows form (10.2) as follows: Note that \( m(\phi) \in H^1(S^1) \) and \( \| m(\phi) \|_{H^1(S^1)} \leq \| m \|_{C^1} (1 + \| \partial_s \phi \|_{L^2(S^1)}) \). Thus by (10.2), we have \( m(\phi) \psi \in H^{1/2}(S^1) \) and

\[
\| m(\phi) \psi \|_{H^{1/2}(S^1)} \leq C(\| m(\phi) \|_{L^\infty(S^1)} \| \psi \|_{H^{1/2}(S^1)} + \| \psi \|_{L^1(S^1)} \| m(\phi) \|_{L^2(S^1)}^{1/2} \| m(\phi) \|_{L^\infty(S^1)}^{1/2})
\]

\[
\leq C(\| \psi \|_{H^{1/2}(S^1)} + \| \partial_s \phi \|_{L^2(S^1)} \| \psi \|_{L^1(S^1)})
\]

This completes the proof.

### 10.2 Regularity properties of \( \mathcal{L}_H \) and \( \mathcal{F}_{x_-} \)

In this section, we prove regularity properties of \( \mathcal{L}_H \) and \( \mathcal{F}_{x_-} \). The following proposition proves the regularity of \( \mathcal{L}_H \).

**Proposition 10.1** Let us assume that \( H \in C^3(S^1 \times \mathbb{S}(S^1) \otimes TN) \) satisfies (1.1) and (1.3). Then the functional \( \mathcal{L}_H \) is \( C^2 \) on \( \mathcal{F}^{1,1/2}(S^1, N) \). Furthermore, if \( H \) satisfies (1.4), then \( \mathcal{L}_H \) is \( C^3 \) on \( \mathcal{F}^{1,1/2}(S^1, N) \).

**Proof** The functional \( \mathcal{L}_H \) consists of three terms. The first term \( E(\phi) = \frac{1}{2} \int_{S^1} | \frac{d\phi}{ds} | ds \) is the energy functional on the free loop space \( H^1(S^1, N) \) and it is \( C^\infty \), see [30]. The other parts
are $D(\phi, \psi) = \frac{1}{2} \int_{S^1} \langle \psi, D_\phi \psi \rangle \, ds$ and $\mathcal{H}(\phi, \psi) = \int_{S^1} H(s, \phi(s), \psi(s)) \, ds$. We shall prove that $D$ is $C^\infty$ and $\mathcal{H}$ is $C^2$ under the condition (1.1) and (1.3).

We first prove the regularity of $\mathcal{H}$. The first order partial derivatives of $\mathcal{L}_H$ (in the sense of Gâteaux) are given by

$$
 d_\phi \mathcal{H}(\phi, \psi)[X] = \int_{S^1} \langle \nabla_\phi H(s, \phi, \psi), X \rangle \, ds,
$$

$$
 d_\psi \mathcal{H}(\phi, \psi)[\eta] = \int_{S^1} \langle \nabla_\psi H(s, \phi, \psi), \eta \rangle \, ds,
$$

where $X \in H^1(S^1, \phi^*TN)$ and $\eta \in H^{1/2}(S^1, S(S^1) \otimes \phi^*TN)$.

By (1.5), (1.6) and Sobolev embedding $H^{1/2}(S^1) \subset L'(S^1)$ for any $r < +\infty$, these integrals converge. Moreover, by the dominated convergence theorem and Sobolev embeddings $H^1(S^1) \subset C^0(S^1)$ and $H^{1/2}(S^1) \subset L'(S^1)$ (for any $r < +\infty$), these partial differentials are continuous on $\mathcal{T}^{1,1/2}(S^1, N)$. Thus, $\mathcal{H}$ is $C^1$ on $\mathcal{T}^{1,1/2}(S^1, N)$. In a similar way, under the assumptions (1.1) and (1.3), second partial derivatives of $\mathcal{H}$ are given by

$$
 d^2_{\phi, \phi} \mathcal{H}(\phi, \psi)[X, Y] = \int_{S^1} \langle \nabla^2_{\phi, \phi} H(s, \phi, \psi), X, Y \rangle \, ds,
$$

$$
 d^2_{\phi, \psi} \mathcal{H}(\phi, \psi)[X, \eta] = \int_{S^1} \langle \nabla^2_{\phi, \psi} H(s, \phi, \psi), X, \eta \rangle \, ds,
$$

$$
 d^2_{\psi, \psi} \mathcal{H}(\phi, \psi)[\eta, \zeta] = \int_{S^1} \langle \nabla_\psi H(s, \phi, \psi), \eta, \zeta \rangle \, ds,
$$

where $X, Y \in H^1(S^1, \phi^*TN)$ and $\eta, \zeta \in H^{1/2}(S^1, S(S^1) \otimes \phi^*TN)$ and they converge. The continuity of these derivatives follows from the Sobolev embedding $H^1(S^1) \subset C^0(S^1)$ and $H^{1/2}(S^1) \subset L'(S^1)$ (for any $r < +\infty$) again. Therefore, $\mathcal{H}$ is $C^2$ on $\mathcal{T}^{1,1/2}(S^1, N)$.

We turn to the regularity of $D$. It is convenient to set everything in the setting of Sect. 3. Thus, as in Sect. 3, we assume that $N$ is isometrically embedded in $\mathbb{R}^k$ and we have canonical embeddings $H^1(S^1, N) \subset H^1(S^1, \mathbb{R}^k)$ and $H^{1/2}(S^1, S(S^1) \otimes \phi^*TN) \subset H^{1/2}(S^1, S(S^1) \otimes \mathbb{R}^k)$. It is also convenient to extend $D$ to $H^1(S^1, \mathbb{R}^k) \times H^{1/2}(S^1, S(S^1) \otimes \mathbb{R}^k)$. For this, let $\mathbb{R}^k \supset U \supset N$ be a tubular neighborhood of $N$ and $\pi_N : U \to N$ a smooth fibration defined by $d(x, \pi(x)) = d(x, N)$ (d is the Euclidean distance on $\mathbb{R}^k$). Let $\rho \in C^\infty(\mathbb{R}^k)$ be such that $\rho = 1$ on another tubular neighborhood $V \supset N$ with $N \subset V \subset U$ and $\rho = 0$ out side the set $U$. We extend the second fundamental form $A$ of the embedding $\iota : N \to \mathbb{R}^k$ as follows:

$$
 \hat{A}(y)(X, Y) = \rho(y) A(\pi_N(y))(X, \pi_N(y))(Y),
$$

where $y \in U, X, Y \in T_y \mathbb{R}^k = \mathbb{R}^k$ and $P(x) : \mathbb{R}^k \to T_x N$ is the orthogonal projection onto $T_x N$ for $x \in N$ and $\hat{A}(y)(\cdot, \cdot) \equiv 0$ for $y \in \mathbb{R}^k \setminus U$. Using this and formula (3.11) of $D_\phi \psi$, we extend $D_\phi \psi$ as follows:

$$
 \hat{D}_\phi \hat{\psi} = D\hat{\psi} - \hat{A}(\phi)(\partial_s \phi, \partial_s \hat{\psi})
$$

for $\phi \in H^1(S^1, \mathbb{R}^k)$ and $\hat{\psi} \in H^{1/2}(S^1, S(S^1) \otimes \mathbb{R}^k)$. Note that $D_\phi \psi = \hat{D}_\phi \phi \iota_\phi (\iota_\phi \psi) - \hat{A}(\iota \circ \phi)(\iota_\phi (\partial_s \phi), \partial_s \iota_\phi (\psi))$ for $\phi \in H^1(S^1, N)$ and $\psi \in H^{1/2}(S^1, S(S^1) \otimes \phi^*TN)$. We define the associated extension $\hat{D}$ of $D$ as

$$
 \hat{D}(\hat{\phi}, \hat{\psi}) = \frac{1}{2} \int_{S^1} \langle \hat{\psi}, \hat{D}_\phi \hat{\psi} \rangle \, ds.
$$
By the construction, we have \( D = \hat{D} \circ (t \times t_+) \) as functionals on \( \mathcal{F}^{1,1/2}(S^1, N) \). Since the canonical embedding \( t \times t_+ : \mathcal{F}^{1,1/2}(S^1, N) \to H^1(S^1, \mathbb{R}^k) \times H^{1/2}(S^1, \mathbb{S}(S^1) \otimes \mathbb{R}^k) \) is smooth, in order to prove the regularity of \( D \), it suffices to prove the regularity of \( \hat{D} \) defined on \( H^1(S^1, \mathbb{R}^k) \times H^{1/2}(S^1, \mathbb{S}(S^1) \otimes \mathbb{R}^k) \). The first component of \( \hat{D}_1(\phi, \psi) = \frac{1}{2} \int_{S^1} (\hat{\psi}, D^2 \hat{\phi} (\partial_s \hat{\phi}, \partial_s \hat{\psi})) \, ds \) is a continuous bilinear form, thus it is smooth on \( H^1(S^1, \mathbb{R}^k) \times H^{1/2}(S^1, \mathbb{S}(S^1) \otimes \mathbb{R}^k) \). \( C^\infty \)-regularity of the second component, \( \hat{D}_2(\phi, \psi) = \frac{1}{2} \int_{S^1} (\hat{\psi}, \hat{A}(\hat{\phi}(\partial_s \hat{\phi}, \partial_s \hat{\psi})) \, ds \) is proved as follows. The first order partial derivatives (in the sense of Gâteaux) are

\[
\begin{align*}
    d_\phi \hat{D}_2(\hat{\phi}, \hat{\psi})[\hat{X}] &= \int_{S^1} \langle \hat{\psi}, D_\hat{X} \hat{A}(\hat{\phi}(\partial_s \hat{\phi}, \partial_s \hat{\psi})) \rangle \, ds + \int_{S^1} \langle \hat{\psi}, \hat{A}(\hat{\phi}(\partial_s \hat{\phi}, \partial_s \hat{\psi})) \rangle \, ds,
    \\
    d_\psi \hat{D}_2(\hat{\phi}, \hat{\psi})[\hat{\eta}] &= \int_{S^1} \langle \hat{\eta}, \hat{A}(\hat{\phi}(\partial_s \hat{\phi}, \partial_s \hat{\psi})) \rangle \, ds + \int_{S^1} \langle \hat{\psi}, \hat{A}(\hat{\phi}(\partial_s \hat{\phi}, \partial_s \hat{\psi})) \rangle \, ds,
\end{align*}
\]

where \( \hat{X} \in H^1(S^1, \mathbb{R}^k) \) and \( \hat{\eta} \in H^{1/2}(S^1, \mathbb{S}(S^1) \otimes \mathbb{R}^k) \). By the Sobolev embeddings \( H^1(S^1) \subset C^0(S^1) \) and \( H^{1/2}(S^1) \subset L^r(S^1) \) (for any \( r < +\infty \)), these integrals converge. The continuity of these partial derivatives is also a consequence of the Sobolev embeddings. In a similar way, higher order differentiability follows. For example, partial derivative \( d^2_\phi \hat{D}_2(\hat{\phi}, \hat{\psi}) \) is given by

\[
\begin{align*}
    d^2_\phi \hat{D}_2(\hat{\phi}, \hat{\psi})[\hat{X}, \hat{Y}] &= \int_{S^1} \langle \hat{\psi}, D_{\hat{X}} D_{\hat{Y}} \hat{A}(\hat{\phi}(\partial_s \hat{\phi}, \partial_s \hat{\psi})) \rangle \, ds + \int_{S^1} \langle \hat{\psi}, D_{\hat{X}} \hat{A}(\hat{\phi}(\partial_s \hat{\phi}, \partial_s \hat{\psi})) \rangle \, ds \\
    &\quad + \int_{S^1} \langle \hat{\psi}, D_{\hat{Y}} \hat{A}(\hat{\phi}(\partial_s \hat{\phi}, \partial_s \hat{\psi})) \rangle \, ds,
\end{align*}
\]

where \( \hat{X}, \hat{Y} \in H^1(S^1, \mathbb{R}^k) \). The integral converges by the Sobolev embeddings \( H^1(S^1) \subset C^0(S^1) \) and \( H^{1/2}(S^1) \subset L^r(S^1) \) (for any \( r < +\infty \)). The continuity of this derivative is also easy to see by the Sobolev embeddings. This process can be continued as many times as you like. Thus, \( \hat{D}_2 \) is \( C^\infty \) on \( \mathcal{F}^{1,1/2}(S^1, N) \). This completes the proof of Proposition 10.1. \( \square \)

In the next proposition, we prove the regularity of \( \mathcal{F}_{x_{-x^+}} \).

**Proposition 10.2** Under the assumption of (1.1) and (1.3) for \( H \), the map

\[
\mathcal{F}_{x_{-x^+}} : W^{1,2}_{x_{-x^+}}(\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N)) \to L^2(\mathbb{R}, T\mathcal{F}^{1,1/2}(S^1, N))
\]

defined by \( \mathcal{F}_{x_{-x^+}}(\ell) = \frac{d\ell}{dt} + \nabla_{1,1/2} L_H(\ell) \) is \( C^1 \). Furthermore, if \( H \) satisfies (1.4), then \( \mathcal{F}_{x_{-x^+}} \) is \( C^2 \).

**Proof** We first prove that \( \mathcal{F}_{x_{-x^+}} \) is well-defined, i.e., \( \mathcal{F}_{x_{-x^+}}(\ell) \in L^2(\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N)) \) for \( \ell \in W^{1,2}_{x_{-x^+}}(\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N)) \). For this, let us recall the definition of \( W^{1,2}_{x_{-x^+}}(\mathbb{R}, \mathcal{F}^{1,1/2}(S^1, N)) \) given at the beginning of Sect. 3.3. It suffices to prove that

\[
\frac{d\ell}{dt}, \nabla_{1,1/2} L_H(\ell) \in L^2((-\infty, -T], \mathcal{F}^{1,1/2}(S^1, N))
\]

and

\[
\frac{d\ell}{dt}, \nabla_{1,1/2} L_H(\ell) \in L^2([T, +\infty), \mathcal{F}^{1,1/2}(S^1, N))
\]

for large \( T > 0 \).
Let \( \ell \in W^{1,2}_{\infty - \infty} (\mathbb{R}, C^{1,1/2_+}(S^1, N)) \) and take \( T > 0 \) so large that \( \ell(t) = (\phi(t), \psi(t)) \) takes the form (3.74)-(3.77). Since \( \phi(t) = \exp_{\phi_-}(X_- (t)) \) for \( t \leq -T \), we have

\[
\frac{d\phi}{dt}(t) = D_2 \exp_{\phi_-}(X_- (t)) [\nabla_t X_- (t)],
\]

where \( D_2 \) is the derivative of \( \exp_{\phi}(X) \) with respect to the variable \( X \in T_x N \). We shall use \( D_1 \exp \) to denote the derivative with respect to the variable \( x \in N \). We first note that from \( X_- \in W^{1,2}_-((-\infty, -T], H^1(S^1, \phi_+^* T N)) \) and \( \xi_- \in W^{1,2}_-((-\infty, -T], H^{1/2}(S^1, S(S^1) \otimes \phi_+^* T N)) \), we have

\[
\sup_{-\infty < t \leq -T} \|X_- (t)\|_{H^1(S^1)} \leq +\infty, \quad \sup_{-\infty < t \leq -T} \|\xi_- (t)\|_{H^{1/2}(S^1)} < +\infty. \tag{10.12}
\]

By the Sobolev embedding \( H^1(S^1) \subset C^0(S^1) \), the first of (10.12) implies that

\[
\sup_{-\infty \leq t \leq -T} \|X_- (t)\|_{C^0(S^1)} < +\infty. \tag{10.13}
\]

In fact, we may assume \( \sup_{-\infty \leq t \leq -T} \|X_- (t)\|_{C^0(S^1)} < \iota (N) \), the injectivity radius of \( N \) by taking \( T \) large. By (10.11) and (10.13), we have a pointwise estimate \( \left| \frac{d\phi}{dt} (t) \right| \leq C |\nabla_t X_- (t)| \)

\[
\left\| \frac{d\phi}{dt} (t) \right\|_{L^2(S^1)} \leq C \left\| \nabla_t X_- (t) \right\|_{L^2(S^1)} \leq C \left\| \nabla_t X_- (t) \right\|_{H^1(S^1)} \tag{10.14}
\]

for \(-\infty < t \leq -T \). We also have

\[
\nabla_s \left( \frac{d\phi}{dt} \right) = D_1 D_2 \exp_{\phi_-}(X_- (t)) [\nabla_t X_- (t), \partial_s \phi_-] + D_2^2 \exp_{\phi_-}(X_- (t)) [\nabla_t X_- (t), \nabla_s X_- (t)]
\]

\[
+ D_2 \exp_{\phi_-}(X_- (t)) [\nabla_s \nabla_t X_- (t)].
\]

Therefore, by (10.13) and the Sobolev embedding \( H^1(S^1) \subset C^0(S^1) \), we have a pointwise estimate

\[
\left| \nabla_s \left( \frac{d\phi}{dt} \right) \right| \leq C \left| \nabla_t X_- (t) \right| |\partial_s \phi_-| + |\nabla_s X_- (t)| + C \left[ \nabla_s \nabla_t X_- (t) \right] \]

\[
\leq C \left[ \nabla_t X_- (t) \right]_{H^1(S^1)} |\partial_s \phi_-| + |\nabla_s X_- (t)| + C \left[ \nabla_s \nabla_t X_- (t) \right] \]

and

\[
\left\| \nabla_s \left( \frac{d\phi}{dt} \right) \right\|_{L^2(S^1)} \leq C \left[ \nabla_t X_- (t) \right]_{H^1(S^1)} \left( |\partial_s \phi_-|_{L^2(S^1)} + |X_- (t)|_{H^1(S^1)} + \left\| X_- (t) \right\|_{H^1(S^1)} \right)
\]

\[
\leq C \left[ \nabla_t X_- (t) \right]_{H^1(S^1)} \tag{10.15}
\]

for \(-\infty < t \leq -T \) by (10.12). By (10.14) and (10.15), we have

\[
\frac{d\phi}{dt} \in L^2((-\infty, -T], H^1(S^1, \phi_+^* T N)) \tag{10.16}
\]

since \( \nabla_t X_- \in L^2((-\infty, -T], H^1(S^1, \phi_+^* T N)) \).

As for \( \frac{d\psi}{dt} \), we have

\[
\frac{d\psi}{dt} = \nabla_t \xi_- + (\psi_- + \xi_- (t)) + S_{-t} (\nabla_t \xi_- (t)). \tag{10.17}
\]
By the definition of $S_{-t} = P_{\phi_{-t}}(t)$ (let us recall that $P_{x,y} : T_x \to T_y$ is the parallel translation along the shortest geodesic from $x$ to $y$), we have

$$
\nabla_t S_{-t} = \nabla_t P_{\phi_{-t}}(t)[\partial_t \phi(t)],
$$

$$
\nabla_t \nabla_t S_{-t} = \nabla_t \nabla_t P_{\phi_{-t}}(t)[\partial_t \phi_- \partial_t \phi(t)] + \nabla^2 \nabla_t P_{\phi_{-t}}(t)[\partial_t \phi_- \partial_t \phi(t)] + \nabla \nabla P_{\phi_{-t}}(t)[\nabla \partial_t \phi(t)].
$$

From these inequalities, we have

$$
\| \nabla_t S_{-t} \|_{H^1(S^1)} \leq C \left( \| \partial_t \phi_- \|_{L^2(S^1)} + \| \partial_t \phi_- \|_{L^2(S^1)} \| \partial_t \phi(t) \|_{L^\infty(S^1)} \right)
$$

$$
\quad + \| \partial_t \phi_- \|_{L^2(S^1)} \| \partial_t \phi(t) \|_{L^\infty(S^1)} + \| \nabla \partial_t \phi(t) \|_{L^2(S^1)} \right)
$$

$$
\leq C \left( \| \nabla_t X_- (t) \|_{H^1(S^1)} + \| \partial_t \phi_- \|_{L^2(S^1)} \| \nabla_t X_- (t) \|_{H^1(S^1)} \right)
$$

(10.18)

for $-\infty < t \leq -T$, where we have used (10.14), (10.15) and the Sobolev embedding $H^1(S^1) \subset L^\infty(S^1)$. Here,

$$
\partial_t \phi_- = D_1 \exp_{\phi_-}(X_- (t)) [\partial_t \phi_-] + D_2 \exp_{\phi_-}(X_- (t)) [\nabla_t X_- (t)]
$$

and a pointwise estimate

$$
| \partial_t \phi(t) | \leq C | \partial_t \phi_- | + C | \nabla_t X_- (t) |
$$

for $-\infty < t \leq -T$ as before by (10.13). We thus obtain

$$
\| \partial_t \phi_- \|_{L^2(S^1)} \leq C \left( \| \partial_t \phi_- \|_{L^2(S^1)} + \| \nabla_t X_- (t) \|_{L^2(S^1)} \right) \leq C
$$

(10.19)

for $-\infty < t \leq -T$ by (10.12). By (10.18) and (10.19), we have

$$
\| \nabla_t S_{-t} \|_{H^1(S^1)} \leq C \| \nabla_t X_- (t) \|_{H^1(S^1)}
$$

(10.20)

for $-\infty < t \leq -T$.

On the other hand, we have

$$
\nabla_t S_{-t} = \nabla_t P_{\phi_{-t}}(t)[\partial_t \phi_-] + \nabla \nabla P_{\phi_{-t}}(t)[\partial_t \phi_-]
$$

and

$$
\| S_{-t} \|_{H^1(S^1)} \leq C + C \| \partial_t \phi_- \|_{L^2(S^1)} + C \| \partial_t \phi_- \|_{L^2(S^1)} \leq C
$$

(10.21)

for $-\infty < t \leq -T$ by (10.19).

By (10.17), (10.20), (10.21) and the multiplicative inequality

$$
\| fg \|_{H^{1/2}(S^1)} \leq C \| f \|_{H^{1/2}(S^1)} \| g \|_{H^{1/2}(S^1)}
$$

(which follows from (10.2) and the Sobolev embedding $H^1(S^1) \subset L^\infty(S^1)$. See also the proof of Corollary 10.1), we have

$$
\left\| \frac{d \psi}{dt} \right\|_{H^{1/2}(S^1)} \leq C \| \nabla_t X_- (t) \|_{H^1(S^1)} \| \psi_- + \xi_- (t) \|_{H^{1/2}(S^1)} + C \| \nabla_t \xi_- (t) \|_{H^{1/2}(S^1)}
$$

$$
\leq C \| \nabla_t X_- (t) \|_{H^1(S^1)} + C \| \nabla_t \xi_- (t) \|_{H^{1/2}(S^1)}
$$

(10.22)

for $-\infty < t \leq -T$, where we have used (10.12). Since the left hand side of (10.22) is $L^2$-integrable over $(-\infty, -T)$, we conclude $\frac{d \psi}{dt} \in L^2(\mathbb{R}, H^{1/2}(S^1), \mathbb{S}(S^1) \otimes \phi^*TN)$. Combining this with (10.16), we obtain $\frac{d \xi}{dt} \in L^2((-\infty, -T), T^\mathfrak{F}_{1,1/2}(S^1, N))$. By the same argument, we can prove $\frac{d \xi}{dt} \in L^2([T, +\infty), T^\mathfrak{F}_{1,1/2}(S^1, N))$.  

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We next prove $\nabla_{1,1/2} L_H(\ell) \in L^2((-\infty, -T), T\mathcal{F}^{1,1/2}(S^1, N))$. We only prove $\nabla_{1,1/2} L_H(\ell) \in L^1((-\infty, -T), T\mathcal{F}^{1,1/2}(S^1, N))$. Since $\ell(t) \to x_- \in \text{crit}(L_H)$ as $t \to -\infty$ and $L_H$ is $C^2$ in $\mathcal{F}^{1,1/2}(S^1, N)$ by Proposition 10.1, there exists $C > 0$ such that

$$\|\nabla_{1,1/2} L_H(\ell(t))\|_{T\mathcal{F}^{1,1/2}} = \|\nabla_{1,1/2} L_H(\ell(t)) - \nabla_{1,1/2} L_H(x_-)\|_{T\mathcal{F}^{1,1/2}} \leq C\|\ell(t) - x_-\|_{\mathcal{F}^{1,1/2}}.$$  

(10.23)

Thus the proof is reduced to the estimate of $\|\ell(t) - x_-\|_{\mathcal{F}^{1,1/2}}$ for $t \leq -T$. As before, we estimate $\|\phi(t) - \phi_-\|_{H^1(S^1)}$ and $\|\psi(t) - \psi_-\|_{H^{1/2}(S^1)}$ separately. We first estimate $\|\phi(t) - \phi_-\|_{H^1(S^1)}$. Let us recall $\phi(t) = \exp_{\phi_-}(X_-(t))$. Thus, by the mean value inequality, we have a pointwise estimate

$$\|\phi(t) - \phi_-\| \leq C|X_-(t)|,$$  

(10.24)

where $C = \sup\{|D_2 \exp_x(\tau X_-(t))| : x \in N, 0 \leq \tau \leq 1, -\infty < t \leq -T\} < +\infty$ by (10.13).

On the other hand, we have

$$\|\phi(t) - \phi_-\|_{H^1(S^1)} \leq C\|X_-(t)\|_{H^1(S^1)} \in L^2((-\infty, -T)).$$  

(10.26)

We next estimate $\|\psi(t) - \psi_-\|_{H^{1/2}(S^1)}$. Let us recall that $\psi(t) = S_{-t}(\psi_- + \xi(t))$. By [20, Lemma 7.4], we have the following estimate (in the notation of [20, Lemma 7.4], $S_{-t} = T_{\phi_-, \phi(t)}$)

$$\|\psi(t) - \psi_- - \xi_-\|_{H^{1/2}(S^1)} = \|S_{-t}(\psi_- + \xi_-\|_{H^{1/2}(S^1)}$$

$$\leq C(\|\phi_-\|_{H^1(S^1)}, \|\phi(t)\|_{H^1(S^1)}, \|\phi_- - \phi(t)\|_{H^1(S^1)}, \|\psi_- + \xi_-\|_{H^{1/2}(S^1)})$$

$$\leq C\|\phi_- - \phi(t)\|_{H^1(S^1)} \leq C\|X_-(t)\|_{H^1(S^1)},$$

where we have used (10.12) and (10.26). Thus, we have

$$\|\psi(t) - \psi_-\|_{H^{1/2}(S^1)} \leq \|\psi(t) - \psi_- - \xi_-\|_{H^{1/2}(S^1)} + \|\xi_-\|_{H^{1/2}(S^1)}$$

$$\leq C\|X_-(t)\|_{H^1(S^1)} + \|\xi_-\|_{H^{1/2}(S^1)} \in L^2((-\infty, -T)).$$  

(10.27)

Combining (10.23), (10.26) and (10.27), we have

$$\nabla_{1,1/2} L_H(\ell) \in L^2((-\infty, -T), T\mathcal{F}^{1,1/2}(S^1, N)).$$

$\nabla_{1,1/2} L_H(\ell) \in L^2(T, +\infty), T\mathcal{F}^{1,1/2}(S^1, N))$ is proved in a similar way. Therefore, we have proved that $\mathcal{F}_{1,-x_+}(\ell) \in L^2(\mathbb{R}, T\mathcal{F}^{1,1/2}(S^1, N))$ for $\ell \in W^{1,2}(\mathbb{R}, T\mathcal{F}^{1,1/2}(S^1, N))$.

Finally, we prove $C^1$-regularity of $\mathcal{F}_{1,-x_+}$. To prove differentiability, we need to prove that

$$D\mathcal{F}_{1,-x_+}(\ell)[V] = \nabla_t V + d\nabla_{1,1/2} L_H(\ell)[V]$$
defines a continuous linear operator on \( V \in W^{1,2}(\mathbb{R}, T^1/2(S^1, N)) \). We have an obvious estimate \( \| \nabla_1 V \|_{L^2(\mathbb{R}, T^1/2(S^1, N))} \leq \| V \|_{W^{1,2}(\mathbb{R}, T^1/2(S^1, N))} \). To prove the continuity of \( V \mapsto d\nabla_{1/2}L_H(\ell)[V] \), we first observe that, by the proof of Proposition 10.1, \( \| d\nabla_{1/2}L_H(\cdot) \|_{\text{op}(T^1/2)} \) is bounded on bounded set in \( T^1/2(S^1, N) \). Thus by (10.12), \( \| d\nabla_{1/2}L_H(\ell(\cdot)) \|_{\text{op}(T^1/2)} \) is uniformly bounded on \(-\infty < t \leq -T \) and

\[
\int_{-\infty}^{-T} \| d\nabla_{1/2}L_H(\ell(t))[V(t)] \|_{T^1/2}^2 dt \leq C \int_{-\infty}^{-T} \| V(t) \|_{T^1/2}^2 dt, \tag{10.28}
\]

where \( C = (\sup_{-\infty < t \leq -T} \| d\nabla_{1/2}L_H(\ell(t)) \|_{T^1/2})^2 < +\infty \). Therefore, \( F_{x-x_+} \) is differentiable. The continuity of the differential \( dF_{x-x_+} \) follows from \( \| \ell_n(t) - \ell(t) \|_{T^1/2} \to 0 \) \( (n \to \infty) \) when \( \ell_n \to \ell \) in \( W^{1,2}(\mathbb{R}, T^1/2(S^1, N)) \) \( (n \to \infty) \) and a similar estimate as in (10.28). Therefore, \( F_{x-x_+} \) is \( C^1 \). If we further assume that \( H \) satisfies \( (1.4) \), then \( L_H \) is \( C^3 \) by Proposition 10.2 and a similar argument as above applies to conclude that \( F_{x-x_+} \) is \( C^2 \). This completes the proof of Proposition 10.2. □

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