Quantum turbulence in atomic Bose-Einstein condensates

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Abstract.
Weakly interacting, dilute atomic Bose-Einstein condensates (BECs) have proved to be an attractive context for the study of nonlinear dynamics and quantum effects at the macroscopic scale. Recently, weakly interacting, dilute atomic BECs have been used to investigate quantum turbulence both experimentally and theoretically, stimulated largely by the high degree of control which is available within these quantum gases. In this article we motivate the use of weakly interacting, dilute atomic BECs for the study of turbulence, discuss the characteristic regimes of turbulence which are accessible, and briefly review some selected investigations of quantum turbulence and recent results. We focus on three stages of turbulence - the generation of turbulence, its steady state and its decay - and highlight some fundamental questions regarding our understanding in each of these regimes.

1. Introduction
Turbulence in ordinary fluids such as air or water consists of rotational eddies of different sizes which we term vortices. Vortices therefore are a hallmark signature of a turbulent flow (Barenghi et al., 2001). In superfluids, quantum vortices differ from their classical counterparts because of the quantization of circulation. This means that the rotational motion of a superfluid is constrained to discrete vortices which all have the same core structure. Turbulence in superfluid helium has been the subject of many recent experimental and theoretical investigations (Skrbek & Sreenivasan, 2012). Recently, experimentalists have been able to visualise individual vortex lines and reconnection events using tracer particles (Fonda et al., 2014). Weakly interacting, dilute atomic Bose-Einstein condensates (henceforth referred to as BECs) present a distinct platform to view and probe quantum turbulence. A key feature here is the ability to directly resolve the structure of individual vortices and in turn the dynamics of a turbulent vortex tangle (Henn et al., 2009). As a result of the quantized nature of vorticity, quantum turbulence in superfluid helium and in BECs can be viewed as a simpler, idealized analog of turbulence in ordinary fluids, and opens the possibility of studying problems which may be relevant to our general understanding of turbulence.

2. Why atomic Bose-Einstein condensates?
Since their first generation in 1995 (Davis et al., 1995; Anderson et al., 1995), atomic BECs have been used to study a wide variety of nonlinear dynamics, for example, solitons, vortices...
and four-wave mixing (Kevrekidis & Carretero-Gonzalez, 2008). A merit of exploiting BECs as a testbed of nonlinear physics lies with the immense control and flexibility they offer. For example:

- **Trapping geometry, shape and dimensionality**
  Optical and magnetic fields can be employed to precisely create a potential landscape for the atoms in the BEC, which in turn enables control of the shape and effective dimensionality of the system (Görlitz et al., 2001). A basic requirement of these gases is confinement in space to prevent contact with hot surfaces. This is typically provided by magnetic traps which are harmonic in shape and have the form (Pethick & Smith, 2002)

  \[ V_{\text{ext}}(r) = \frac{1}{2} m \omega (x^2 + y^2 + z^2), \]  

  where \( \omega \) is the trapping frequency and \( m \) the mass of the atom. This type of trap results in an atomic cloud with a radial density profile which resembles an inverted parabola. If a harmonic trap is used which is very strongly confining in one direction, for example

  \[ V_{\text{ext}}(r) = \frac{1}{2} m \omega (x^2 + y^2 + \epsilon z^2), \]  

  where \( \epsilon \gg 1 \), the dynamics in that direction are inhibited and the system becomes effectively two-dimensional (2D). By changing \( \epsilon \), one can easily change the effective dimensionality, which is particularly important in turbulence (2D turbulence is very different from 3D turbulence). In the same way, if the trap is very tightly confining in two directions, the dynamics is mainly in the third direction, and the system is effectively one-dimensional (1D).

  More complicated trapping geometries can be realised, for example a toroidal ring or a periodic optical lattice. Traps can also be made time-dependent by rotating or shaking the trapping potential. Furthermore, one can create localized potentials using optical fields, which can mimic an obstacle and be moved through the system on demand.

- **Interaction strength**
  Typically, the dominant atomic interaction in a BEC is the short-range and isotropic \( s \)-wave interaction. Experimentalists can employ magnetic Feshbach resonances (Inouye et al., 1998) to change the strength of these interactions and even their nature, i.e. whether they are attractive or repulsive (Roberts et al., 1998). Furthermore, by using atoms with relatively large magnetic dipole moments, e.g. \( ^{52}\text{Cr} \), it is possible to create a BEC where the atoms also experience significant dipole-dipole interactions, which are long-range and anisotropic, and greatly modify the static and dynamical properties of the system (Lahaye et al., 2009).

- **Vortex core optical imaging**
  The healing length which characterizes the vortex core size is typically around \( 10^{-7}\text{m} \) in a BEC (c.f. \( 10^{-10}\text{m} \) in superfluid Helium). By expanding the BEC (following release from the trapping potential), the vortex can be directly imaged and resolved via optical absorption (Madison et al., 2000; Raman et al., 2001). Advanced real time imaging of condensates containing vortices has also recently been developed (Freilich et al., 2010), allowing the precession of vortices to be observed.

In the limit of zero temperature and weak interactions, the evolution equation for the macroscopic condensate wavefunction, \( \phi(r,t) \), is a form of nonlinear Schrödinger equation, commonly known as the Gross-Pitaevskii Equation (GPE) (Pethick & Smith, 2002):

\[ i \hbar \frac{\partial \phi(r,t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}}(r,t) + g|\phi(r,t)|^2 - \mu \phi(r,t), \]  

(3)
where \( r \) is the position in space, \( t \) is time and \( \hbar \) is Planck's constant divided by \( 2\pi \). The GPE provides a good quantitative description of the dynamics of a BEC over all lengthscales available, from the vortex core to the system size, up to temperatures of approximately \( T \approx 0.5T_c \) (where the critical temperature \( T_c \) is of the order of \( \mu \)K in typical experiments). At the right-hand side we recognise the kinetic energy term, \( (-\hbar^2/2m)\nabla^2 \), the trapping potential \( V_{\text{ext}}(r,t) \) (which in general may be time-dependent), the interaction term, \( g|\phi(r,t)|^2 \) where \( g = 4\pi \hbar^2 a_s/m \) and \( a_s \) is the 3D \( s - \)wave scattering length, and the chemical potential \( \mu \).

The GPE can be almost exactly mapped to the classical Euler equation; the small difference, namely the quantum stress, regularizes the solutions, preventing singularities which may arise in an Euler fluid (Barenghi, 2008). The GPE is a practically exact model in the limit of zero temperature, where essentially all of the atoms exist in the Bose-Einstein condensate phase. In many experiments the condensate exists at well below the BEC transition temperature such that this approximation is justified. Extensions of the GPE to include the effect of thermal atoms provide a more complete (albeit not exact) physical model of a real BEC (Jackson et al., 2009) (see e.g. Proukakis & Jackson (2008) for an in depth review of finite temperature models).

However BECs suffer an important limitation. The systems which can be currently created in the laboratory contain a small number of atoms, typically \( 10^3 \) to \( 10^9 \), hence do not sustain the number of quantum vortices present in helium experiments. For example, up to a few hundred vortices have been achieved in the largest quasi-2D BECs (Abo-Shaeer et al., 2002). This brings to light the issue of length scales. A defining feature of classical turbulence, besides nonlinearity, is the huge number of length scales which are excited. The range of length scales available in turbulent superfluid helium at very small temperatures is perhaps even larger, since short wavelength helical waves along the vortices can be generated by nonlinear interactions, producing a turbulent cascade called the Kelvin wave cascade (Vinen, 2006). A simple question arises: can a BEC, containing a limited number of vortices, really become turbulent? Our tentative answer is yes. Numerical results thus far (Nore et al., 1997; Berloff & Sivestono, 2002; Kobayashi & Tsubota, 2005b; Yepez et al., 2009) suggest that kinetic energy is distributed over the length scales in agreement with the \( k^{-5/3} \) Kolmogorov scaling which is observed in ordinary turbulence (where \( k \) is the wavenumber) even over this small range of length scales. Therefore, the study of turbulence in a BEC represents an exciting opportunity to probe a new regime residing somewhere between chaos and turbulence.

In the remainder of this paper, we aim to identify some important open questions about turbulence in BECs; where appropriate we will review some of the work which has been carried out to date.

### 3. Quantum Turbulence in atomic BECs, where are we?

The following is an extensive, but by no means exhaustive, list of aspects yet to be understood regarding turbulence in atomic BECs. To structure our discussion, we distinguish the evolution of turbulent flow into three stages, namely;

1. **The generation of the turbulence.** What are the most effective and efficient ways to generate turbulence? Does the way in which the turbulence is generated affect the ‘type’ of turbulence created?

2. **The statistical steady state.** Are there universal features of turbulence, for instance, is the Kolmogorov energy spectrum present? What are the statistics of the turbulence velocity field?

3. **The decay of the turbulence.** How does the turbulence decay? What is the best way to measure the decay?
(i) The generation of the turbulence
To understand the generation of a vortex tangle in a quantum gas, we must first understand how individual vortices are nucleated. The very first creation of such a vortex took place in a two-component condensate and was driven by the rotation of one component around the other. The subsequent removal of the inner component resulted in the formation of a hollow core of a singly quantized vortex (Anderson et al., 2000). Further techniques for generating vortex structures soon followed, including the creation of vortex rings following the “snake instability” decay of a dark soliton (Anderson et al., 2001), phase imprinting (Leanhardt et al., 2002) and by a rapid quench through the transition temperature for the onset of Bose-Einstein condensation, i.e. the Kibble-Zurek mechanism (Weiler et al., 2008; Freilich et al., 2010).

However, to generate a large number of vortices in the system at any one time, two other techniques have proved to be more effective:

(i) Rotation of an anisotropic BEC excites surface modes leading to the nucleation of vortices at the edge which then drift into the bulk of the BEC. If the rotation is performed about only one axis, a vortex lattice is created (Hodby et al., 2001; Abo-Shaeer et al., 2002, 2001; Madison et al., 2000, 2001). In 3D, if the rotation is performed about more than one axis, a vortex tangle has been predicted to form (Kobayashi & Tsubota, 2007).

(ii) A moving (cylindrical) obstacle, such as that created by the potential from a blue-detuned laser beam moving through a quantum fluid, generates pairs of vortices in its wake when its speed exceeds a critical value (Raman et al., 2001). Recently, this method has been used to generate and study a collection of vortex dipoles in a 2D BEC (Neely et al., 2010).

Both methods generate a large number of vortices, in 2D as well as in 3D. However, one can bypass the initial transient period and begin with a nonequilibrium state of vortices. Experimentally, this can be achieved via imprinting a phase profile onto the condensate via laser beams, as performed by Leanhardt et al. (2002) for generating single vortices of arbitrary charge. The use of such imprinting to generate a vortex tangle has been implemented theoretically (White et al., 2010), with the resulting tangle similar to that depicted in Fig 3.

Our preliminary results with method (ii) (laser stirring), suggest that it is possible to generate a large number of vortices; we have found qualitative evidence that, by moving the obstacle along different paths, we can change the isotropy of the resulting tangle of vortices (Allen et al., 2014).

Fig. 1 (top row) shows the density isosurface of a 3D spherical condensate at three instants in time, after the condensate has been stirred for a time \( t_{\text{stir}} \), along a circular path with a Gaussian-shaped laser stirrer oriented in the \( z \)-direction. These results are based on simulations of the GPE, Eq. (3). For a simple measure of the isotropy of the tangle, we plot the projected vortex lengths \( L_x \), \( L_y \) and \( L_z \) in each Cartesian direction (bottom part of Fig. 1). All projected lengths rapidly increase during the stirring period \( (t < t_{\text{stir}}) \); after the laser has been removed \( (t > t_{\text{stir}}) \), \( L_x \), \( L_y \) and \( L_z \) all decrease for a short period of time. Later, only the vortex lengths \( L_x \) and \( L_y \) in the transverse \( (x \text{ and } y) \) directions further decrease, whereas \( L_z \) remains approximately constant because the vortex tangle decays into a regular lattice, as it is apparent from the final density isosurface (Top row, c)). This is as expected: in stirring the condensate circularly we impart angular momentum about the \( z \)-axis, and it is well known that the ground state of a superfluid with sufficient angular momentum features a lattice of regularly-spaced, vortex lines aligned along the \( z \)-axis.

In Fig. 2 the vortex length is shown when the stirring takes place, for the same amount of time, along a Figure-eight path. Again, the vortex length increases over the duration of the laser stirring \( (t < t_{\text{stir}}) \); however, after the laser is removed \( (t > t_{\text{stir}}) \), the tangle decays isotropically, i.e. all projected lengths \( L_x \), \( L_y \) and \( L_z \) decay together. For the Figure-eight path, the laser also moves through the centre of the condensate, where the density is higher, the vortices which
Figure 1. Top row: Density isosurfaces of a 3D spherical BEC at times $t\omega = a)\ 60, b)\ 100$ and c) 240 after stirring the condensate along the $z$–direction in a circular path. We see here that the surface plot picks up the vortex cores as well as some of the condensate edge. The resulting vortex length in each direction is shown over time in the bottom part of this figure where it has been normalised by the peak vortex length.

are generated are longer than those generated at the edge of the condensate (Allen et al., 2014); therefore, this laser stirring path is more efficient at creating a dense, random vortex tangle than simply stirring the condensate in a circular fashion.

(ii) Statistical state
Once the turbulence state has been created (by whichever means), its steady state properties can be investigated. BECs offer the possibility to study 2D and 3D turbulence and the cross-over region between the two (Parker & Adams, 2005). We now review our current understanding of the properties of a turbulent tangle of vortices in a BEC, first in 2D and then in 3D.

Quantum turbulence in 2D
In 2D classical turbulence, the conservation of enstrophy dictates that energy must flow from small scales of energy injection to large scales forming, for example, large clusters of vortices (Kraichnan, 1967). This inverse cascade is thought to underly Jupiter’s giant Red Spot and has been experimentally examined in classical fluids (Sommeria et al., 1988; Marcus, 1988) (for a review see Kellay & Goldburg (2002)). Attempts to observe the inverse cascade effect in quantum gases have led to modelling vortex generation (Parker & Adams, 2005; White et al., 2012; Fujimoto & Tsubota, 2011; Reeves et al., 2013), and to experiments on the dynamics of vortex dipoles created by a moving potential (Neely et al., 2010, 2013).
Figure 2. Vortex length, normalised by the peak length, in the $x$, $y$ and $z$ directions for a stirrer along the $z$-direction, moving in a Figure-eight path.

Numasato et al. (2010), evolved the 2D GPE to a turbulent state by initially imposing a random phase on the wavefunction. They did not observe a reverse cascade but rather a direct cascade. They argued that since the total number of vortices, and therefore the enstrophy, is not conserved in simulations of the GPE because of vortex pair annihilation, the inverse energy cascade is irrelevant for 2D quantum turbulence.

Conversely, Reeves et al. (2013) reported the numerical observation of the inverse cascade. They solved the 2D damped GPE (DGPE) and generated a turbulent state by imposing the fluid to flow past 4 stationary potential barriers. The speed of the flow was sufficiently high ($v \approx 0.822c$, where $c$ is the sound speed of the quantum fluid) so as to create many vortices and thereby a turbulent flow. Using a statistical algorithm they measured how prone ‘like-winding’ vortices, i.e. vortices of the same charge, were to cluster together depending on the amount of dissipation imposed. They found that an intermediate level of damping led to small clusters of like-winding vortices being formed and inferred an inverse energy cascade from analysis of the incompressible energy spectrum (energy associated with vortices). However, this analysis was carried out in the wake of the obstacle where like-winding vortices are naturally in the vicinity of each other. To make this result more convincing, a larger box for analysis would be desirable.

Similar work by White et al. (2012) implemented a rotating elliptical paddle to generate large numbers of vortices. Again, clustering of like-winding vortices was observed, but no inverse cascade was reported.

A possible reason for the lack of definitive evidence for or against the inverse cascade in quantum gases is their relatively small size, i.e. the systems which were studied lacked enough length scales. However, this limitation has not prevented the observation of the direct Kolmogorov-3D cascade in systems of similar size (Nore et al., 1997; Kobayashi & Tsubota, 2005a,b; Yepez et al., 2009; Kobayashi & Tsubota, 2007).

Quantum turbulence in 3D

In their seminal work, Nore et al. (1997) used the GPE to investigate 3D quantum turbulence in a homogeneous box by evolving an initial, large scale Taylor Green vortex. By decomposing the velocity field into divergence-free and curl-free parts, they obtained incompressible (associated with vortex motion) and compressible (associated with acoustic excitations) kinetic energy
spectra respectively. They showed that the incompressible kinetic energy spectrum is similar to the classical $k^{-5/3}$ Kolmogorov energy spectrum at scales down to the intervortex spacing. Similarly, Berloff & Svistunov (2002) began with a non-equilibrium state, generated by imprinting a random phase on the equilibrium wavefunction of a homogeneous box. They then observed the evolution of the quantum turbulence by solving the GPE and further allowing the system to evolve to phase coherence. Similarly, Kobayashi & Tsubota (2005a,b) imprinted a random phase on the equilibrium wavefunction of a homogeneous box and evolved according to the GPE. They introduced a dissipative term which only acted on scales smaller than the healing length to represent thermal dissipation in the system. They also obtained a decaying incompressible energy spectrum which has the Kolmogorov power law over the inertial range. In order to clarify the extent of this range, statistical steady turbulence was created by a moving random potential which continuously injected energy into the system at large scales; a damping term removed energy at small length scales. They found that the inertial range was slightly narrower for the continuously forced turbulence because the moving potential sets the energy containing range. Further to these methods, White et al. (2010) imprinted a staggered vortex array onto a harmonically trapped BEC and evolved the system according to the GPE (see Fig. 3). In this work, they calculated the probability density function of the velocity components and found that (in both 2D and 3D) it is not a Gaussian like in ordinary turbulence, in agreement with experimental results obtained in superfluid helium (Paoletti et al., 2008). This is an important observation which distinguishes quantum from classical turbulence.

Figure 3. 3D turbulent state of a harmonically trapped BEC. Condensate edge shown by blue shading, turbulent vortex tangle shown in purple (White et al., 2010). Figure courtesy of Angela C. White.

Using a similar method of imprinting, (Yepez et al., 2009) performed impressively large simulations in a homogeneous box using a quantum lattice gas algorithm (up to $5760^3$ grid points) and resolved scales smaller than the vortex core radius.

Most of the methods of generating quantum turbulence discussed so far in this section have a common aim: the incompressible energy spectrum of quantum turbulence after an initial, turbulent state has been set up (with the exception of (Kobayashi & Tsubota, 2005a,b)). However, the group of Tsubota have also carried out simulations where they dynamically create a vortex tangle by solving the DGPE with combined rotation along two axis of a harmonically trapped BEC. They found that by changing the ratio between the rotation frequencies in both directions, they could generate a vortex lattice or a more disordered array of vortices which formed a vortex tangle in which individual vortices appear to be nucleated with no preferred direction. They measured the incompressible energy spectrum and found it to be consistent with Kolmogorov law.

Experimentally, a small vortex tangle has been created in a harmonically trapped BEC through the combination of rotation and an external oscillating perturbation by Bagnato’s
They noticed that upon expansion of the condensate the usual inversion of aspect ratio of the gas (Mewes et al., 1996; Castin & Dum, 1996) did not happen. This effect could be a possible signature of the creation of a tangle of vortices. A further observation from this experiment was the identification of different regimes corresponding to the strength and duration of the external oscillatory potential. For small amplitude excitations, the only effect was a bending of the main axis of the cloud, irrespective of the duration of the oscillation. Increasing the amplitude of the oscillation caused regular vortices to be nucleated, with the number of vortices increasing monotonically with the duration of the external excitation. Increasing the excitation time further led to a turbulent vortex regime, and at very long hold times, the condensate fragmented or granulated.

This brings us to the final stage of the evolution of turbulence, the decay.

(iii) The decay of the turbulence

In classical turbulence, the cascade of kinetic energy over the length scales terminates at some very short scale where viscosity dissipates kinetic energy into heat. The absence of viscosity in quantum fluids means there must exist other mechanisms of energy dissipation. The most likely is acoustic emission. When two vortices reconnect, some energy is lost in the form of sound. Reconnections are also thought to create high frequency Kelvin waves on vortices. It is thought that, in superfluid helium, Kelvin waves interact nonlinearly and create shorter and shorter waves, until sound waves are emitted at high frequency (Vinen, 2006). This energy transfer is called the Kelvin wave cascade.

Experiments in superfluid helium show that, depending on the scale at which energy is injected, the decay of the turbulence can be in one of two forms (Baggaley et al., 2012):

(i) ‘Semiclassical’ or ‘Kolmogorov’ turbulence: The vortex tangle seems polarised and structured over many length scales. This type of turbulence is generated when the forcing is at length scales larger than the average intervortex spacing. In this regime, the vortex length \( L \) decays as \( L \sim t^{-3/2} \), which is consistent with the decay of a Kolmogorov spectrum.

(ii) ‘Ultraquantum’ or ‘Vinen’ turbulence: The scale of the forcing is less than the intervortex spacing, the vortex tangle seems random and possesses a single length scale, and the vortex length decays as \( L \sim t^{-1} \) (Walmsley & Golov, 2008).

How to measure the decay in a BEC is an open question. White et al. (2010) studied the decay of the turbulent tangle by numerically monitoring the vortex length, \( L \), over time. They showed that the line length increases initially as reconnections take place before decaying over time. By further solving the dissipative GPE, they confirmed that thermal dissipation leads to a faster decay of line length but could not clearly distinguish between \( L \sim t^{-1} \) or \( L \sim e^{-\alpha t} \) behaviour (\( \alpha \) being some decay parameter). However, is vortex linelength the best measure for this decay? Or can we again look to the incompressible energy spectrum to visualise the decay of vortices and draw some conclusions from both of these quantities? Furthermore, there is the question of how this is best achieved experimentally. The images taken of condensate density are typically column-integrated over the imaging direction which means that depth information becomes lost and an extraction of the true 3D vortex line length is not possible. Just as the attenuation of second sound is used to measure vortex length in helium, what surrogate measures of vortex line length are accessible experimentally in BECs? Such questions, both fundamental and practical in nature, will provide a rich source of research in these systems in the future.
4. Summary
We have discussed weakly interacting, dilute atomic Bose-Einstein condensates as tools for understanding the nature of quantum turbulence, motivated largely by the huge degree of control they offer. Even though the range of lengthscales excited in these systems is much less than in superfluid helium, the direct Kolmogorov energy cascade has been predicted to exist and the regimes of turbulence accessible is vast and interesting in its own right. We have distinguished three distinct phases of quantum turbulence - the generation of turbulence, its steady state and its decay - and briefly reviewed work done in understanding these so far, whilst highlighting fundamental questions about each phase.

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