SMOOTH APPROXIMATION OF
PLURISUBHARMONIC FUNCTIONS
ON ALMOST COMPLEX MANIFOLDS

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Abstract

This note establishes smooth approximation from above for $J$-plurisubharmonic functions on an almost complex manifold $(X, J)$. The following theorem is proved. Suppose $X$ is $J$-pseudoconvex, i.e., $X$ admits a smooth strictly $J$-plurisubharmonic exhaustion function. Let $u$ be an (upper semi-continuous) $J$-plurisubharmonic function on $X$. Then there exists a sequence $u_j \in C^\infty(X)$ of smooth strictly $J$-plurisubharmonic functions point-wise decreasing down to $u$.

In any almost complex manifold $(X, J)$ each point has a fundamental neighborhood system of $J$-pseudoconvex domains, and so the theorem above establishes local smooth approximation on $X$.

This result was proved in complex dimension 2 by the third author, who also showed that the result would hold in general dimensions if a parallel result for continuous approximation were known. This paper establishes the required step by solving the obstacle problem.

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1. Introduction.

On any smooth almost complex manifold $(X, J)$ there is a well-defined notion of $J$-plurisubharmonic functions of class $C^2$, namely those $u \in C^2(X)$ which satisfy the condition $i\partial \bar{\partial} u \geq 0$. This notion extends directly to the space of distributions $\mathcal{D}'(X)$ by requiring the current $i\partial \bar{\partial} u$ to be positive. It also extends to the space $\text{USC}(X)$ of upper semi-continuous functions $u : X \to [-\infty, \infty)$ in several ways – using viscosity theory, or by requiring that the restrictions to $J$-holomorphic curves in $X$ be subharmonic. These different extensions have been shown to be, in a precise sense, equivalent (see [16], [12]), and the space of such functions is denoted by $\text{PSH}(X, J)$.

We say that a function $u \in C^2(X)$ is strictly $J$-plurisubharmonic if $i\partial \bar{\partial} u > 0$ at every point. The manifold $X$ is then said to be $J$-pseudoconvex if it admits a smooth (proper) exhaustion function $\rho : X \to \mathbb{R}$ which is strictly $J$-plurisubharmonic. (See Remark 3.7 for other equivalent definitions.)

The main point of this paper is to establish the following (in §4).

**THEOREM 4.1. ($C^\infty$ Strict Approximation).** Suppose $(X, J)$ is an almost complex manifold which is $J$-pseudoconvex, and let $u \in \text{PSH}(X, J)$ be a $J$-plurisubharmonic function. Then there exists a decreasing sequence $\{u_j\} \subset C^\infty(X)$ of smooth strictly $J$-plurisubharmonic functions such that $u_j(x) \downarrow u(x)$ at each $x \in X$.

Now on any almost complex manifold $X$ every point $x$ has a fundamental neighborhood system of $J$-pseudoconvex domains – namely, small balls about $x$ in appropriate local coordinates. Consequently, as a special case of Theorem 4.1 we have local $C^\infty$ strict approximation on $X$ (see Corollary 4.2).

By this local regularization result a current $i\partial \bar{\partial} u \wedge i\partial \bar{\partial} v$ defined in [18] is a positive current for plurisubharmonic $u, v$ in the Sobolev class $W^{1,2}_{\text{loc}}$, in particular for bounded plurisubharmonic $u, v$ (see Proposition 4.2 and Proposition 5.2 there and compare with Corollary 2 in [19]). For an application of our global regularization result see Corollary 4.3, which concerns hulls of sets.

We note that in the case of plurisubharmonic functions on domains in $\mathbb{C}^n$, smoothing as in Theorem 4.1 is possible on all pseudoconvex, Reinhardt, and tube domains (see [7]), but there are smooth domains where not all plurisubharmonic functions are limits of a decreasing sequence of smooth plurisubharmonic functions (see [6]).

Theorem 4.1 was proved in complex dimension 2 by the third author (in [19]), who pointed out that his work would establish the result in general.
dimensions provided one could prove a certain parallel \textit{continuous} approximation theorem. The required continuous approximation result can be deduced from work of the first two authors on the obstacle problem – more precisely the Dirichlet problem with an obstacle function.

The discussion of this obstacle problem in [10] and [13] and its exact implementation in the context of almost complex analysis is somewhat scattered, and so, for clarity, we give a coherent exposition of the needed results in the first two sections of this note. Nevertheless, this note draws heavily on the work in [10], [12], [13], [18] and [19].

It is interesting to note that the work in [18] and [19] also involves solving the Dirichlet problem for the (almost) complex Monge-Ampère operator. In this case, however, the solutions are taken in the smooth category using results in [17], where the techniques are quite different from the viscosity methods employed in [10], [12], [13]. The idea of using the Monge-Ampère equation to approximate $J$-plurisubharmonic functions is probably due to J.-P. Rosay.

\textbf{Remark.} The main proof in this paper consists of combining a Richberg-type theorem (cf. [18, Thm. 3.1], [11, Thm. 9.10]) with the continuous approximation theorem which follows from solving the obstacle problem. The method applies generally to give smooth approximation of $F$-subharmonic functions whenever these two components can be established. An example is given in Appendix B where smooth approximation is established for subsolutions of the complex Hessian equations on a Kähler manifold.
2. The Obstacle Problem and Continuous Approximation for General Potential Theories.

We refer the reader to [10] or [13] for the concepts and terminology employed in this section.

Let $J^2(X) \rightarrow X$ be the bundle of 2-jets of real-valued functions on a manifold $X$. There is a natural splitting $J^2(X) = \mathbb{R} \times J^2_{\text{red}}(X)$ where the first factor corresponds to the value of the function.

Consider a subequation of the form $F = \mathbb{R} \times F_0$ with $F_0 \subset J^2_{\text{red}}(X)$. For a domain $\Omega \subset X$, let $F(\overline{\Omega})$ denote the set of $u \in \text{USC}(\overline{\Omega})$ such that $u\mid_{\Omega}$ is $F$-subharmonic (i.e., $u\mid_{\Omega}$ is a viscosity $F$-subsolution, cf. [2], [3]).

**THEOREM 2.1. (The Obstacle Problem).** Suppose that:

1. $F_0$ is locally affinely jet-equivalent to a constant coefficient (reduced) subequation $\bar{F}_0$,
2. $F_0$ has a monotonicity cone $M_0$ and $X$ carries a $C^2$ strictly $M$-subharmonic function $\psi$ where $M = \mathbb{R} \times M_0$,
3. $g \in C(X)$, and
4. $\Omega \subset X$ is a domain with smooth boundary $\partial \Omega$ which is both $F$- and $\bar{F}$-strictly convex.

Then the function
\[
    h(x) \equiv \sup_{u \in F[g]} u(x),
\]
where $F[g] \equiv \{ u(x) : u \in F(\overline{\Omega}) \text{ and } u \leq g \text{ on } \overline{\Omega} \}$, satisfies:

1. $h \in C(\overline{\Omega}) \cap F(\overline{\Omega})$,
2. $h \leq g$ on $\overline{\Omega}$
3. $h\mid_{\partial \Omega} = g\mid_{\partial \Omega}$

Furthermore,

1. $h$ is the Perron function, and $F[g]$ is the Perron family, for the Dirichlet problem for the subequation $F^g \equiv (\mathbb{R}^- + g) \times F_0$ on $\Omega$ with boundary function $\varphi \equiv g\mid_{\partial \Omega}$.
2. Comparison holds for $F^g$ on $X$. 
COROLLARY 2.2. (Continuous Strict Approximation). Suppose \( u \in F(\Omega) \).

(a) Then there exists a sequence of functions \( u_j \in C(\Omega) \cap F(\Omega) \) decreasing down to \( u \) on \( \overline{\Omega} \). In fact, if \( \{g_j\} \subset C(\Omega) \) is any sequence of continuous functions decreasing down to \( u \), the \( \{u_j\} \subset C(\Omega) \cap F(\Omega) \) can be chosen so that
\[
    u \leq u_j \leq g_j \quad \forall j. \tag{2.2}
\]

(b) Moreover, given \( \varepsilon_j \downarrow 0 \), the sequence \( \{u_j + \varepsilon_j \psi\} \) also decreases down to \( u \) on \( \overline{\Omega} \), and on each compact subset of \( \Omega \), the functions \( \{u_j + \varepsilon_j \psi\} \) are \( c \)-strict for some \( c > 0 \).

See 2.3 below for a definition and discussion of \( c \)-strictness.

Proof of Corollary 2.2. Pick \( g_j \in C(\Omega) \) with \( g_j \downarrow u \). Let \( u_j \) denote the solution of the obstacle problem for \( g_j \). Then \( u_j \in C(\Omega) \cap F(\Omega) \) and \( u_j \leq g_j \).

Since \( u \) is in the Perron family \( F[g_j] \), we have (2.2). This proves Part (a).

Part (b) follows from (a) and hypothesis (2).

Proof of Theorem 2.1. The following is proved in [10] but not stated explicitly as a theorem. It is however stated explicitly as Theorem 8.1.2 in [13] and the proof is given there based on results in [10].

THEOREM 8.1.2 in [13]. Suppose \( F \) is a subequation on a manifold \( X \) which is locally affinely jet-equivalent to a constant coefficient subequation. Suppose there exists a \( C^2 \) strictly \( M \)-subharmonic function on \( X \) where \( M \) is a monotonicity cone for \( F \). Then for every domain \( \Omega \subset\subset X \) whose boundary is strictly \( F \)- and \( \bar{F} \)-convex, both existence and uniqueness hold for the Dirichlet problem. That is, for every \( \varphi \in C(\partial\Omega) \) there exists a unique \( F \)-harmonic function \( u \in C(\Omega) \) with \( u|_{\partial\Omega} = \varphi \).

The adaptation to the general Obstacle Problem is given in Section 8.6 of [13]. What follows is a more detailed version of that argument.

By assumption we know that \( F = \mathbb{R} \times F_0 \) is affinely jet equivalent to the constant coefficient equation \( \mathbb{R} \times F_0 \subset \mathbb{R} \times J^2_{\text{red}} \), with a jet equivalence which is the identity on the first factor. Hence the subequation
\[
    F^g \equiv \{r \leq g(x)\} \times F_0
\]
is locally affinely jet equivalent to the subequation
\[
    F^g \equiv \{r \leq g(x)\} \times F_0
\]
We now consider the affine jet equivalence
\[
    \Phi : \mathbb{R} \times J^2_{\text{red}} \longrightarrow \mathbb{R} \times J^2_{\text{red}}
\]
given by
\[
    \Phi(r, J) \equiv (r - g(x), J).
\]
Applying this gives the local equivalence
\[ \Phi : F^g \rightarrow \{ r \leq 0 \} \times F_0 \equiv \mathbb{R}_- \times F_0, \]
and so composing this with the first equivalence shows that \( F^g \) is locally affinely jet-equivalent to the constant coefficient subequation \( \mathbb{R}_- \times F_0 \).

Now observe that if \( M_0 \) is a monotonicity cone for \( F_0 \), then \( M_- \equiv \mathbb{R}_- \times M_0 \) is a monotonicity cone for \( F^g \).

Note also that if \( \psi \) is strictly \( M \)-subharmonic function, then so is \( \psi - c \) for any constant \( c \leq 0 \) because \( M \) satisfies the basic negativity condition (N). Given a domain \( \Omega \subset\subset X \), we may therefore assume that \( \psi < 0 \) on a neighborhood of \( \Omega \). In this case, \( \psi \) is also \( M_- \)-strictly subharmonic on \( \Omega \).

Since \( F^g \) is locally jet-equivalent\(^\text{1}\) to a constant coefficient subequation, local weak comparison holds for \( F^g \). This is Theorem 10.1 in [10] and follows from the Theorem on Sums. Local weak comparison implies weak comparison (Theorem 8.3 in [10]). Now using Theorems 9.5 and 9.2 we have that comparison holds for \( F^g \) on \( X \).

The Dirichlet Problem for \( F^g \)-harmonics would now be solvable for arbitrarily prescribed boundary data \( \varphi \in C(\partial \Omega) \), (by either Theorem 12.4 in [10] or Theorem 8.1.2 above) if one could prove that the boundary is strictly \( F^g \) and \( \tilde{F}^g \) convex.

However, this is not true in general, and in fact existence fails for a boundary function \( \varphi \in C(\partial \Omega) \) unless \( \varphi \leq g \mid_{\partial \Omega} \). Nevertheless, if \( \partial \Omega \) is both \( F \) and \( \tilde{F} \) strictly convex, then existence holds for each boundary function \( \varphi \leq g \mid_{\partial \Omega} \). Section 8.6 in [13] provides a proof of this.

Here we give a proof but with attention restricted to the case at hand where \( \varphi = g \mid_{\partial \Omega} \). The Perron family for \( F^g \) with this boundary data consists of those functions \( u \in \text{USC}(\Omega) \) which are \( F \)-subharmonic on \( \Omega \) and satisfy the additional constraint that \( u \leq g \) on \( \Omega \). The dual subequation to \( F^g \) is \( \tilde{F^g} = [(\mathbb{R}_- - g) \times J^2_{\text{red}}(X)] \cup \tilde{F} \). Since \( \tilde{F^g} \subset \tilde{F} \), the \( \partial \Omega \) is strictly \( \tilde{F^g} \)-convex if it is strictly \( \tilde{F} \)-convex. However, \( \partial \Omega \) can never be strictly \( F^g \)-convex, as defined in Definition 11.10 of [10], because \( (F^g_\lambda)_x = \emptyset \) for \( \lambda > g(x) \).

Nevertheless, the only place that this hypothesis is used in proving Theorem 8.1.2 for \( H \) is in the barrier construction which appears in the proof of Proposition \( F \) in [10]. With \( \varphi(x_0) = g(x_0) \), the barrier \( \beta(x) \) as defined in (12.1) in [10] is not only \( F \)-strict near \( x_0 \) but also automatically \( F^g \)-strict since \( \beta < g \) in a neighborhood of \( x_0 \).

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\(^1\)See Appendix A for a discussion of jet-equivalence.
Definition 2.3. (Strictness). Let $F \subset J^2(X)$ be a subequation. A function $u \in F(\Omega)$ is strictly $F$-subharmonic (or simply strict) if for any $\varphi \in C_0^\infty(\Omega)$, there exists $\epsilon > 0$ such that $u + \epsilon \varphi \in F(\Omega)$.

Note that a $C^2$-function $u \in F(\Omega)$ is strict iff $J^2_x u \in \text{Int} F \forall x \in \Omega$.

In [10] there is the following related concept of $c$-strictness for $c > 0$.

Equip $J^2(X)$ with a bundle metric (induced, say, from a riemannian metric on $X$), and for $x \in X$, define $F^c_x \equiv \{ J \in F_x : \text{dist}_x(J, \sim F) \geq c \}$ where dist$_x$ denotes the distance in the fibre. A function $u \in F(\Omega)$ is said to be $c$-strict on a compact set $K \subset \Omega$ if $u$ is $F^c$-subharmonic on a neighborhood of $K$. The constant $c$ depends on the choice of bundle metric, but the condition of being $c$-strict on $K$ for some $c > 0$ does not. Strictness, as defined above, is equivalent to being locally $c$-strict on $\Omega$. (This is proved, though not explicitly stated, in §7 of [10].)

Remark 2.4. The main conclusion of Theorem 2.1 above can be stated in more appealing and succinct terms. Let us call the function $h$, defined in (2.1), the largest $F$-subharmonic minorant of $g$. Then we have the following abbreviated version of Theorem 2.1 and Corollary 2.2.

**Theorem 2.5.** Suppose $X, F = \mathbb{R} \times F_0$ and $\Omega$ are as in Theorem 2.1. Then given $g \in C(\Omega)$, the largest $F$-subharmonic minorant of $g$ on $\Omega$ is continuous and equals $g$ on the boundary of $\Omega$.

Moreover, given $u \in F(\Omega)$ there exists a sequence $\{u_j\} \subset C(\Omega) \cap F(\Omega)$ decreasing down to $u$ (with each $u_j$ strict).

3. Strict Continuous Approximation of Plurisubharmonic Functions on Almost Complex Manifolds

Let $(X, J)$ be an almost complex manifold, and let $F(J) \subset J^2_{\text{red}}(X)$ be the subequation defining the upper semi-continuous $J$-plurisubharmonic functions on $X$. (It is shown in [12] that all the different basic definitions of these functions are, in a precise sense, equivalent).\(^2\)

Proposition 4.5 in the paper [12] proves that the subequation $F(J)$ is locally jet equivalent to a constant coefficient reduced subequation (in fact to the standard subequation $F(J_0) \cong \{ i\partial \bar{\partial} u \geq 0 \}$ determined by a standard parallel $J_0$).

Furthermore, $F(J)$ is a convex cone subequation and in particular it satisfies $F(J) + F(J) \subset F(J)$. Therefore, $F(J)$ is a monotonicity cone for itself. A $C^2$-function $\psi$ is strictly $J$-plurisubharmonic (i.e., strictly $F(J)$-subharmonic) if $i\partial \bar{\partial} \psi > 0$ on $X$.

\(^2\)It is also shown at the end of section 7 in [12] that the various notions of $F(J)$-harmonic (including the notion of being maximal and continuous) are equivalent.
Definition 3.1. A domain $\Omega \subset X$ is called strictly $J$-pseudoconvex if it has a global $C^2$ defining function $\psi$ which is strictly $J$-plurisubharmonic on a neighborhood of $\overline{\Omega}$. Let $\tilde{F}(J)$ denote the dual subequation. One checks that
\[
F(J) + F(J) \subset F(J) \Rightarrow \tilde{F}(J) + F(J) \subset \tilde{F}(J) \Rightarrow F(J) \subset \tilde{F}(J),
\]
so if $\partial \Omega$ is strictly $F(J)$-convex, it is automatically strictly $\tilde{F}(J)$-convex.

Thus, as a special case of Theorem 2.5 we have the following.

**Theorem 3.2.** Let $\Omega \subset X$ be a strictly $J$-pseudoconvex domain in an almost complex manifold $(X,J)$. Let $g \in C(\Omega)$. Then the largest $J$-plurisubharmonic minorant of $g$ is continuous. Moreover, given $u \in PSH(\Omega)$ there exists a sequence $\{u_j\} \subset C(\Omega) \cap PSH(\Omega)$ decreasing down to $u$ (with each $u_j$ strict).

We now address the global question of continuous approximation of $J$-plurisubharmonic functions on $X$.

**Definition 3.3.** An almost complex manifold $(X,J)$ is $J$-pseudoconvex if it has a global $C^2$ strictly $J$-plurisubharmonic exhaustion function. (See Remark 3.7 below for equivalent definitions.)

It is standard that a strictly $J$-pseudoconvex domain $\Omega$ is itself $J$-pseudoconvex.

**Theorem 3.4.** Suppose $X$ is a $J$-pseudoconvex manifold. Then for each $u \in PSH(X)$ there exists a sequence of continuous strictly $J$-plurisubharmonic functions $u_j \in C(X)$ decreasing down to $u$ on $X$.

**Proof.** We shall adapt a part of the proof of the Theorem 1 from [19]. Take a decreasing sequence of continuous functions $\{g_k\}$ converging down to $u$. We begin with a result in smooth topology.

**Claim 3.5.** Let $h$ be an arbitrary continuous function on $X$, and suppose that $\rho : X \to \mathbb{R}$ is a $C^2$ (proper) exhaustion function. Then there exists a convex function $\chi \in C^\infty(\mathbb{R})$ with $\chi' \geq 1$ so that
\[
\chi(\rho(x)) \geq h(x) \quad \text{for all} \quad x \in X.
\]

**Proof.** Set $\psi(t) \equiv \sup\{h(x) : \rho(x) \leq t\}$ and note that
\[
\chi(\rho(x)) \geq h(x) \quad \forall x \in X \quad \iff \quad \chi(t) \geq \psi(t) \quad \forall t \in \text{range}(\rho).
\]
This reduces the claim to a one-variable claim. To establish this, assume that $\text{range}(\rho) = [0, \infty)$ and replace $\psi$ by a smooth function which is larger. Then choose $\chi \in C^\infty([0, \infty))$ to have $\chi(0) = \psi(0)$, $\chi'(0) \geq \max\{\psi'(0), 1\}$ and $\chi'' \geq \max\{\psi'', 0\}$. 

Now let $\rho \in C^\infty(X)$ be a strictly $J$-plurisubharmonic exhaustion function. For any smooth convex, increasing function $\chi \in C^\infty(\mathbb{R})$, with $\chi' \geq 1$, the
composition \( \chi \circ \rho \) is also a smooth strictly \( J \)-plurisubharmonic exhaustion. Thus, by Claim 3.5, with \( h \) taken to be \( g_1 \) plus any exhaustion function for \( X \), we can assume \( \rho \) is chosen so that
\[
\lim_{z \to \infty} (\rho(z) - g_1(z)) = +\infty
\]  
(3.1)
where \( \lim_{z \to \infty} \) denotes the limit in the one-point compactification of \( X \).

By (3.1) the sets \( U_k \equiv \{ \rho > g_1 + k \} \) provide a fundamental neighborhood system for the point at infinity. Since \( \rho \) is an exhaustion, we have that \( \{ \rho - k \geq t \} \subset U_k \) if \( t \) is sufficiently large. By Sard’s Theorem we may choose such \( t \) to be a regular value \( t_k \) of \( \rho - k \). Then \( \Omega_k \equiv \{ \rho - k < t_k \} \) is a strictly \( J \)-pseudoconvex domain, and
\[
\rho - k > g_1 (\geq g_k) \quad \text{on a neighborhood of } \sim \Omega_k. 
\]
(3.2)
Hence,
\[
\widetilde{g}_k \overset{\text{def}}{=} \max\{g_k, \rho - k\} = \rho - k \quad \text{on a neighborhood of } \sim \Omega_k. 
\]
(3.3)

Now let \( u_k \) be the largest \( J \)-psh minorant of \( \widetilde{g}_k \) on \( \Omega_k \), and note that \( u_k \) is continuous by Theorem 3.2. By (3.3) we have \( \widetilde{g}_k = \rho - k \) on a neighborhood of \( \sim \Omega_k \). Since \( \rho - k \) is \( J \)-psh, and \( u_k \) is the largest \( J \)-psh minorant of \( \widetilde{g}_k \), we have \( u_k = \rho - k \) on a neighborhood of \( \sim \Omega_k \). Thus we can extend \( u_k \) as a \( J \)-psh function to all of \( X \) by setting \( u_k = \rho - k \) on \( \sim \Omega_k \).

Note that since \( \widetilde{g}_k \equiv \max\{g_k, \rho - k\}, g_{k+1} \leq g_k, \) and \( g_k \downarrow u \), one has
\[
g_{k+1} \leq \widetilde{g}_k \quad \text{and} \quad \widetilde{g}_k \downarrow u. 
\]
(3.4)
By definition
\[
u_k \leq \widetilde{g}_k \quad \text{and} \quad u_k = \widetilde{g}_k \quad \text{on } \sim \Omega_k. 
\]
(3.5)
Now since \( u_{k+1} \leq \widetilde{g}_{k+1} \), and since \( u_k \) is the largest \( J \)-psh minorant of \( \widetilde{g}_k \) on \( \overline{\Omega}_k \), we have by (3.4) that \( u_{k+1} \leq u_k \) on \( \overline{\Omega}_k \). On the complement \( \sim \Omega_k \), we have \( u_k = \widetilde{g}_k \) and so \( u_{k+1} \leq u_k \) again by (3.4) and (3.5). Hence,
\[
u_{k+1} \leq u_k \quad \text{on } X. 
\]
(3.6)
Since \( u \leq \widetilde{g}_k \) is \( J \)-psh and \( u_k \) is the largest such minorant on \( \overline{\Omega}_k \), we have that \( u \leq u_k \) on \( \overline{\Omega}_k \). On the complement \( \sim \Omega_k \), we have \( u_k = \widetilde{g}_k \) and so \( u \leq u_k \) there as well. Hence,
\[
u \leq u_k \quad \text{and} \quad u_k \downarrow u \quad \text{on } X.
\]
In other words \( \{u_k\} \) is a decreasing sequence of continuous \( J \)-psh functions decreasing down to \( u \) on \( X \), and we can replace \( u_k \) with \( u_k + \frac{1}{k^p} \rho \) to make \( u_k \) strict.

Remark 3.7. (Equivalent Definitions of \( J \)-Pseudoconvexity). In defining \( J \)-pseudoconvexity it is enough to assume the existence of a continuous strictly \( J \)-plurisubharmonic exhaustion function \( \rho : X \to \mathbb{R} \). This
follows from the extension of Richberg’s Theorem to almost complex mani-

folds (Theorem 3.1 in [18]). Such manifolds are called *almost Stein manifolds*
in [4].

*J*-Pseudoconvex manifolds \((X,J)\) can also be characterized in terms of

the hulls of compact sets (see (4.1) below) by requiring that:

(i) There exists some \(u \in \text{PSH}^\infty(X,J)\) which is strict, and

(ii) For every compact \(K \subset X\), the hull \(\hat{K}_{C^0}\) is compact.

By Theorem 3.1 in [18] we have that the hulls \(\hat{K}_{C^0} = \hat{K}_{C^\infty}\) agree (see

Corollary 4.3 below). Therefore, *J*-Pseudoconvex manifolds can also be

characterized by the requiring:

(i) There exists some \(u \in \text{PSH}^0(X,J)\) which is strict, and

(ii) For every compact \(K \subset X\), the hull \(\hat{K}_{C^0}\) is compact.

For the proof one applies standard arguments (cf. [11, §4] or [9, Prop. 9.3])

to show that (i) and (ii) imply the existence of a strict PSH-exhaustion (in

either case).

4. Strict Smooth Approximation of Plurisubharmonic Functions on Almost Complex Manifolds

**THEOREM 4.1. (**\(C^\infty\) **Strict Approximation)**). Suppose \((X,J)\) is an

almost complex manifold which is *J*-pseudoconvex, and let \(u \in \text{PSH}(X,J)\)

be a *J*-plurisubharmonic function. Then there exists a decreasing sequence

\(\{u_j\} \subset C^\infty(X)\) of smooth strictly *J*-plurisubharmonic functions such that

\(u_j(x) \downarrow u(x)\) at each \(x \in X\).

**Proof.** Apply Theorem 3.1 in [18] and Theorem 3.4 above.

This generalizes Theorem 1 in [19] to arbitrary dimensions.

**COROLLARY 4.2. (Local \(C^\infty\) **Strict Approximation)**. Let \((X,J)\) be an

arbitrary (smooth) almost complex manifold. Then every point \(x \in X\)

has a fundamental system of neighborhoods \(U\) with the property that for

every \(u \in \text{PSH}(U,J)\) there is a decreasing sequence \(\{u_j\} \subset C^\infty(U)\) of strictly

*J*-plurisubharmonic functions such that \(u_j \downarrow u\).

**Proof.** Fix local coordinates in \(\mathbb{C}^n\) for \(X\) near \(x\) so that \(J\) is \(C^1\)-close to

the standard \(J_0\) at the origin. Then \(\chi(z) = |z|^2\) is strictly \(J\)-psh on the ball

\(B_\epsilon(0) = \{|z| < \epsilon\}\) for all \(\epsilon > 0\) sufficiently small. It is standard that any
domain which admits a \(C^2\) strictly \(J\)-plurisubharmonic defining function, is

*J*-pseudoconvex.

One can also give a more direct proof of Corollary 4.2 based on Theorem

3.2 above and Theorem 3.1 in [18].

Another immediate consequence of the global approximation Theorem 4.1

is that all the various possible definitions of the hull of a set actually agree.
Given a compact set $K \subset X$ we define its $J$-plurisubharmonic hull to be the set
\[ \hat{K} \equiv \{ x \in X : u(x) \leq \sup_K u \ \forall u \in PSH(X, J) \} . \] (4.1)

One could also define $\hat{K}_{C^0}$ and $\hat{K}_{C^\infty}$ by replacing $PSH(X, J)$ in (3.4) with $PSH^0(X, J) \equiv PSH(X, J) \cap C(X)$ and $PSH^\infty(X, J) \equiv PSH(X, J) \cap C^\infty(X)$ respectively.

**Corollary 4.3.** Suppose $(X, J)$ is $J$-pseudoconvex. Then for any compact $K \subset X$, one has $\hat{K} = \hat{K}_{C^0} = \hat{K}_{C^\infty}$.

**Proof.** Clearly $\hat{K} \subset \hat{K}_{C^0} \subset \hat{K}_{C^\infty}$, so it suffices to show that $\hat{K}_{C^\infty} \subset \hat{K}$. Suppose that $x \notin \hat{K}$. Then there exists $u \in PSH(X, J)$ with $u \leq 0$ on $K$ and $u(x) = 1$. Replace $u$ with $\max\{u, 0\}$. Let $\{u_j\}$ be the sequence given in Theorem 4.1. Then $u_j$ converges uniformly to 0 on the compact set $K$ and $u_j(x) \geq 1$ for all $j$. Hence, $x \notin \hat{K}_{C^\infty}$.

**Appendix A. Affine Jet-Equivalence.** A local affine jet-equivalence is a local isomorphism of the 2-jet bundle $J(\mathbb{R}^n) = \mathbb{R} \times \mathbb{R}^n \times \text{Sym}^2(\mathbb{R}^n)$ which is of the form:
\[ r' = r + r_0(x), \quad p' = k(x)p + p_0(x), \quad A' = h(x)Ah(x)t + L_x(p) + A_0(x) \]
where
\[ r_0(x) \text{ takes values in } \mathbb{R}, \]
\[ p_0(x) \text{ takes values in } \mathbb{R}^n, \]
\[ A_0(x) \text{ takes values in } \text{Sym}^2(\mathbb{R}^n), \]
(i.e., $J_0(x) \equiv (r_0(x), p_0(x), A_0(x))$ is a section of $J(\mathbb{R}^n)$)

and
\[ k(x) \text{ and } h(x) \text{ take values in } \text{GL}_n(\mathbb{R}), \text{ while} \]
\[ L_x \text{ takes values in } \text{Hom}(\mathbb{R}^n, \text{Sym}^2(\mathbb{R}^n)) \]

The regularity conditions on the jet-equivalence required in the proof of Theorem 10.1 in [10] are:

1. $k, h$ and $L$ are Lipschitz continuous, and
2. $J_0$ is continuous.

For the second jet equivalence in our application to the Obstacle Problem, $g \equiv h \equiv Id$ and $J_0(x) = (r_0(x), 0, 0)$, so our obstacle function $g(x) = -r_0(x)$ need only be continuous.
Appendix B. $\Sigma_m$-Subharmonic Functions.

As noted in Remark 1.3, for any subequation $F$, smooth approximation for $F$-subharmonic functions can be proved whenever continuous approximation and a Richberg-type theorem can be established for $F$. In this appendix we give just such a result for the complex hessian subequations on a Kähler manifold.

Let $X$ be a complex manifold of dimension $n$ with a fixed Kähler form $\omega$. We say that a function $u \in C^2(\Omega)$ is $\Sigma_m$-subharmonic on a domain $\Omega \subset X$ if $(dd^cu)^k \wedge \omega^{n-k} \geq 0$ for $k = 1, \ldots, m$. We say that a locally integrable function $u : \Omega \to [-\infty, +\infty)$ is $\Sigma_m$-subharmonic ($u \in \Sigma_m(\Omega)$) if $u$ is upper semicontinuous and

$$dd^cu \wedge dd^cu_1 \wedge \ldots \wedge dd^cu_{m-1} \wedge \omega^{n-m} \geq 0,$$

for any $C^2 \Sigma_m$-subharmonic functions $u_1, \ldots, u_{m-1}$ (they are defined in [1] for $\omega = \omega_{st} = dd^c(|z|^2)$ in $\mathbb{C}^n$ and in [5] and [14] for general Kähler form). This is just the subequation $F \equiv \Sigma_m$ defined on $X$ by the condition that the first $m$ elementary symmetric functions of the complex hessian satisfy $\sigma_\ell(\text{Hess}_Cu) \geq 0$ for $\ell = 1, \ldots, m$ (compare Example 18.1 in [10] and Lemma 7 in [20]).

A Richberg-type theorem for $\Sigma_m$ was proved in [20] (Theorem 2). Lu and Nguyen proved in [15] that on compact Kähler manifolds any quasi-$\Sigma_m$-subharmonic function can be approximated from above by smooth quasi-$\Sigma_m$-subharmonic functions (a function $u$ is quasi-$\Sigma_m$-subharmonic if the function $u + \rho$ is $\Sigma_m$-subharmonic where $\rho$ is local potential for $\omega$). Actually their global result implies that locally it is possible to regularize $\Sigma_m$-subharmonic functions. However, in the same way as in Theorem 4.1, we can prove a slightly stronger result.

**Theorem B.1.** Let $X$ be a $\Sigma_m$-pseudoconvex Kähler manifold. Let $u$ be a $\Sigma_m$-subharmonic function on $X$. Then there exists a decreasing sequence $u_j \in C^\infty(X)$ of $\Sigma_m$-subharmonic functions such that $u_j \downarrow u$.

By $\Sigma_m$-pseudoconvex we mean that $X$ has a global $C^2$ strictly $\Sigma_m$-subharmonic exhaustion function. In particular Stein manifolds are $\Sigma_m$-pseudoconvex.
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