RESIDUES AND TOPOLOGICAL YANG-MILLS THEORY IN TWO DIMENSIONS

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Abstract

A residue formula which evaluates any correlation function of topological $SU_n$ Yang-Mills theory with arbitrary magnetic flux insertion in two dimensions are obtained. Deformations of the system by two form operators are investigated in some detail. The method of the diagonalization of a matrix valued field turns out to be useful to compute various physical quantities. As an application we find the operator that contracts a handle of a Riemann surface and a genus recursion relation.
1 Introduction

Two dimensional topological Yang-Mills theory is an example of topological field theories which is precisely the two dimensional analogue of the Donaldson theory [1].

The physical observables of this theory in the favorable cases are identified with the cohomology classes of the moduli space $\mathcal{M}$ of the flat gauge fields on a Riemann surface.

The cohomology ring of $\mathcal{M}$ [2, 3, 4, 5] recently draws much attention in connection with the Floer cohomology group [6]. The correlation functions of topological Yang-Mills theory, which determine the cohomology ring, have been completely solved in the form of a multiple infinite sum by Witten using the exact solution of two dimensional physical Yang-Mills theory [7, 8, 9] and the correspondence between physical and topological Yang-Mills theories [10].

Apart from the unsolved problem of the cases of non compact gauge groups which may be important to analyze topological $W$ gravities, there is still much to do in this theory.

First of all it seems difficult to compute the explicit value of a correlation function and to investigate the cohomology ring of topological Yang-Mills theory of gauge groups other than $SU_2$ using the infinite sum formula found in [10]. Thus it would be desirable to find another formula of correlation functions which directly gives their values as rational numbers and is more suitable to study the cohomology ring.

Moreover the formula in [10] for the correlators that contain general two form operators has remained totally implicit because we must perform inversion of matrix field variables to evaluate it.

In this paper we give a formula which expresses any correlation function of $SU_n$ theory with arbitrary magnetic flux as a residue evaluated at the origin of the Cartan subalgebra generalizing the previous results [4, 11].

In the process we develop systematically the method of inversion of variables using residues and the diagonalization of matrix valued field.

In addition to its practical value, this residue formula sheds light on the structure of the topological Yang-Mills theory. For example, topological Yang-Mills theory may be
regarded as a kind of matrix models because the evaluation of correlation functions by the residue can be written as an integration over eigenvalues of Hermite matrixes. We also expect that the residue formula will be useful to consider systematically the quantum deformation of the cohomology ring [4], the coupling of topological Yang-Mills theory to topological gravity and string interpretation of the large $N$ expansion of topological Yang-Mills theory [12]. Jeffrey and Kirwan have studied the non-Abelian localization in [13] and proved the residue formula for $SU_2$ [14]. Blau and Thompson have studied two dimensional topological Yang-Mills theory as well as other closely related gauge theories based on the path integral approach. They introduced the Abelianization, i.e., the diagonalization of the matrix valued fields in [15, 16], which naturally reproduces the infinite sum formula mentioned above. In [17], the Abelianization was related to the localization of the gauge field path integral to the reducible gauge fields using the equivariant supersymmetry. The relation between this localization and the non-Abelian localization to the Yang-Mills connections seems still mysterious. This paper is organized as follows. In Sect.2.1 the magnetic flux for gauge theory on a two dimensional surface is introduced. Sect.2.2 is devoted to the standard construction of the BRST observables. In Sect.3 we review the result about physical Yang-Mills theory and the infinite sum formula for correlation functions of topological Yang-Mills theory. In Sect.4, we give the residue formula for the correlation functions with arbitrary magnetic flux which contain no two form operators but the symplectic form $\omega$. We also describe some concrete examples of the correlation functions. In Sect.5 we treat the cases in which arbitrary two form operators are inserted in a correlator. This corresponds to considering deformations of the original topological theory. The most general residue formula for the correlation functions is obtained. Finally in Sect.6, as an application of the residue formula we find the observable that restricts the path integral to the subspace where the holonomy of any gauge field around a cycle is trivial and does nothing else.
2 Topological Yang-Mills on a Riemann Surface

2.1 ‘t Hooft Magnetic Flux

Here we describe the geometrical setting for $SU_n$ topological Yang-Mills theory on a genus $g$ Riemann surface $\Sigma_g$. We introduce ‘t Hooft magnetic flux $a$, which is the terminology originally used in four dimensional gauge theories [10, 18, 19], as follows.

Pick a point $P$ on $\Sigma_g$ and put the boundary condition on gauge fields that the holonomy around $P$ must be $X^d$, where

$$X = \text{diag} \left( e^{\frac{2\pi \sqrt{-1}}{n}}, ..., e^{\frac{2\pi \sqrt{-1}}{n}} \right)$$

(2.1)

is a generator of the center $\mathbb{Z}_n$ of $SU_n$.

This is called the $SU_n$ gauge theory with $d$ units of magnetic flux. Note that the magnetic flux $d$ is defined only modulo $n$ and the theory with $d$ units of magnetic flux are related to one with $(n - d)$ units of magnetic flux by the charge conjugation. Modulo the degrees of freedom of the ghost zero modes associated with the residual gauge symmetries for the cases of $(n, d) \neq 1$, path integral of the topological theory can be regarded as a integration over the moduli space of flat gauge fields on $\Sigma_g - P$ with the prescribed holonomy around $P$, which we denote as $M_g(n, d)$. $M_g(n, d)$ is also the absolute minima of the ordinary Yang-Mills action.

We list some properties of $M_g(n, d)$ relevant to the later discussion;

- $M_g(n, d)$ has the real dimension equal to the ghost number violation $2(n^2 - 1)(g - 1)$.
- $M_g(n, d) \cong M_g(n, n - d)$ due to the charge conjugation symmetry.
- When $(n, d) = 1$, $M_g(n, d)$ is smooth and has no reducible gauge fields.

In the case $(n, d) \neq 1$, $M_g(n, d)$ always has reducible gauge fields, which make the analysis of the theory difficult. In this paper we define correlation functions of topological Yang-Mills theory as expansion coefficients in coupling constant of the partition function $Z(\epsilon)$ of physical Yang-Mills theory [10]. This procedure indeed gives the correct answer for the case $(n, d) = 1$. For the other cases $(n, d) \neq 1$, in addition to a polynomial part, there also
exist non-local terms in $\epsilon$ in the expansion of $Z(\epsilon)$ due to the reducible flat gauge fields, which makes the identification of topological correlation functions with the polynomial part in $\epsilon$ of $Z(\epsilon)$ somewhat doubtful. However even in these cases the result obtained by this approach is consistent with the Riemann-Roch-Verlinde formula, as we will see later, with the reservation that the existence of the Riemann-Roch formula itself on a moduli space with singularities is also presumed. Thus we will not worry about the singularities associated with reducible flat gauge fields in the cases $(n, d) \neq 1$ as long as correlation functions are concerned. For the basic facts about the Lagrangian and BRST symmetry see [10, 16, 20].

2.2 Topological Observables

Here we review the standard construction of the BRST observables of the topological theory [20]. Mathematical treatments of this subject for general $SU_n$ gauge group can be found in [21, 22]. Let $(A_i, \psi_i, \phi)$ be the basic topological multiplet of the topological Yang-Mills theory, with the BRST symmetry $\delta A = \psi, \delta \psi = -D \phi, \delta \phi = 0$.

It is useful to redefine the fields as $(\bar{\phi}, \bar{\psi}, \bar{F}) = (\sqrt{\frac{-1}{2\pi}} \phi, \sqrt{\frac{-1}{2\pi}} \psi, \sqrt{\frac{-1}{2\pi}} F)$.

There are two ways of component expansions of the matrix-valued field $\bar{\phi}$. The first one is the expansion with respect to an orthonormal basis of Lie algebra, $\bar{\phi} = \sum_a \bar{\phi}^a J_a$.

The second one, which turns out to be more useful, is defined using a diagonalization

$$\bar{\phi} = \text{diag}(z_1, ..., z_n) = \sum_{i=1}^{n-1} x_i H_i, \quad x_i = z_i - z_{i+1} = \langle \alpha_i, \bar{\phi} \rangle,$$

where $\{H_i\}$ and $\{\alpha_i\}$ are the set of fundamental coweights and the simple roots of $SU_n$ respectively, and the Weyl group action is simply the permutations of $\{z_i\}$.

Now the zero form operator of ghost number $2m$ is defined by

$$\mathcal{O}_m = \frac{1}{m!} \text{Tr}(\bar{\phi}^m).$$

Next one form operator of ghost number $(2m - 1)$ is

$$V_m(a) = \oint_{C_a} \frac{1}{m!} m \text{Tr}(\bar{\phi}^{m-1} \bar{\psi}),$$

(2.4)
where \(C_a, 1 \leq a \leq 2g\) are the 1-homology basis of \(\Sigma_g\) such that \(C_a \cdot C_{b+g} = -\delta_{ab}\).

Note that \(\{V_m(a)\}\) transform among themselves under the mapping class group of \(\Sigma_g\) \[2, 4\] and only the modular invariant combination of those;

\[
\Xi_{lm} = \sum_{a=1}^{g}(V_l(a)V_m(a + g) - V_l(a + g)V_m(a)) \tag{2.5}
\]

are nonvanishing in the correlation functions.

Finally two form operator of ghost number \((2m - 2)\) is similarly constructed as

\[
\mathcal{O}^{(2)}_m = -\frac{1}{m!} \int_\Sigma \text{Tr}(m\varphi^{(m-1)}F + \frac{1}{2}m(m - 1)\varphi^{(m-2)}\psi^2). \tag{2.6}
\]

In particular, there always exist observables associated with the degree two Casimir invariant; \(O_2\) and \(\omega = O_2^{(2)}\) which is the standard symplectic two form\(^b\) on \(\mathcal{M}_g(n, d)\).

They play the special role in physical/topological Yang-Mills correspondence \[10\].

The correlation function of the form;

\[
\left\langle \prod_i O_i \prod_j V_{m_j}(a_j) \prod_k \mathcal{O}^{(2)}_{n_k}, [\mathcal{M}_g(n, d)] \right\rangle \tag{2.7}
\]

is non-vanishing only if the observables inside the correlator satisfy the ghost number selection rule,

\[
\sum_i 2l_i + \sum_j (2m_j - 1) + \sum_k (2n_k - 2) = 2(n^2 - 1)(g - 1). \tag{2.8}
\]

Hereafter we will frequently use the notation \(\overline{g} = (g - 1)\).
3 Physical Yang-Mills Theory

3.1 Infinite Sum Formula

In [7, 8, 10], a multiple infinite sum formula was obtained for the partition function of physical Yang-Mills theory.

Physical Yang-Mills theory can be described by the same field content as the basic BRST multiplet of the topological Yang-Mills so that the generalized Lagrangian of the model with a polynomial \( Q(x) = \epsilon \frac{x^2}{2!} + \sum_{m \geq 3} \frac{\delta_m x^m}{m!} \) with nilpotent \( \{\delta_m\} \) is given by

\[
L = -\text{Tr}Q(\phi) + \text{Tr}(\phi F + \frac{1}{2} \bar{\psi} \psi). \tag{3.1}
\]

Then the partition function is written as the following multiple infinite sum closely related to the multiple zeta value investigated in [23]

\[
Z(Q, \omega) = (-1)^{(n-1)d+|\Delta|} \gamma_{n}^{d} \sum_{l_{1}=1}^{\infty} \cdots \sum_{l_{n-1}=1}^{\infty} \prod_{\alpha \in \Delta^{+}} \langle \alpha, \Phi \rangle^{-\frac{\epsilon g}{2}} e^{\text{Tr}Q(\Phi)} e^{-d(\lambda_{1}, \Phi)}, \tag{3.2}
\]

where we set \( \Phi = 2\pi \sqrt{-1} \sum_{i=1}^{n-1} H_{i} \lambda_{i} \) and \( H_{i} \) is the \( i \)-th fundamental coweight.

The normalization of the partition function is determined so as to produce the corresponding topological correlation function by the expansion in the coupling constant \( \epsilon \).

3.2 Non-Abelian Localization

Due to the non-Abelian localization theorem [14, 13], see also [17], the path integral of physical Yang-Mills theory can be localized around the solution of the equation of motion which are the critical points of \( S_{YM} \). Noting that the flat gauge fields are the absolute minima which give the dominant contribution, it is seen that the partition function of the physical Yang-Mills theory has the following connection with the correlation function of the topological theory in the case of \( (n, d) = 1 \),

\[
Z(Q, \omega) = \left\langle e^{\omega} e^{\text{Tr}Q(\phi)} , [\mathfrak{M}_{g}(n, d)] \right\rangle + \{\text{contributions of non-flat solutions}\}. \tag{3.3}
\]

The first term of the right hand side of (3.3) is a polynomial in \( \epsilon \) and represents the corresponding correlation function of topological Yang-Mills theory, while the second term
means the contributions to the path integral from the solutions of the Yang-Mills equation with non-zero action \( S_{YM} \) which has the \( \epsilon \)-dependence \( \simeq \exp(-S_{YM}/(2\pi)^2\epsilon) \).
More precisely it was shown \([10]\) that the infinite sum of the form
\[
\sum_{l_1=1}^{\infty} \cdots \sum_{l_n=1}^{\infty} \prod_{\alpha \in \Delta_+} \langle \alpha, \Phi \rangle^{2\gamma} e^{i\text{Tr}\Phi^2/2}\text{Tr}B(\Phi)e^{-d(\lambda_1,\Phi)}
\]
vanishes exponentially for \( \epsilon \to 0 \) if \((n,d) = 1\) and the ghost number of \( B \) is greater than \( 4\gamma|\Delta_+| \). When \((n,d) \neq 1\), reducible flat gauge fields give additional terms which are non-local in the coupling constant. Even in these cases the polynomial part could be given the interpretation as topological correlation functions.
Thus we obtain the infinite sum formula for correlation functions of topological Yang-Mills theory which is somewhat conjectural for \((n,d) \neq 1\) cases;
\[
\left\langle e^{\omega \text{Tr}B(\bar{\phi})}, [\mathcal{M}_g(n,d)] \right\rangle = (-1)^{(n-1)d+|\Delta_+|\gamma n^2} \sum_{l_1=1}^{\infty} \cdots \sum_{l_n=1}^{\infty} \prod_{\alpha \in \Delta_+} \langle \alpha, \Phi \rangle^{2\gamma} \text{Tr}B(\Phi)e^{-d(\lambda_1,\Phi)}.
\]
(3.5)
The above formula is well-defined only when the right hand side converges. This in particular implies that the ghost number of \( B \) should be less than \( 4\gamma|\Delta_+| \) in (3.5).

4 Residue Formula

4.1 Residue Form with Magnetic Flux

Here we propose the residue formula for correlation functions of topological Yang-Mills theory with arbitrary magnetic flux. For the case of \((n,d) = (2,1)\) and the case of \((n,0)\) were treated by Thaddeus \([2]\) and Szenes \([11]\) respectively.
To present the formula we introduce the following symbol;
\[
\theta \left( \frac{d \cdot i}{n} \right) = \frac{d \cdot i}{n} - \left[ \frac{d \cdot i}{n} \right].
\]
(4.1)
Now define the multi-variable residue form $\Omega_g^{(k)}(n, d)$ for $SU_n$ theory with $d$ units of magnetic flux by

$$
\Omega_g^{(k)}(n, d) = (-1)^{(n-1)(d-1)+|\Delta_+|} (nk^{n-1})^{\frac{r}{2}} \prod_{i=1}^{n-1} \frac{dx_i}{x_i} \frac{kx_i}{(e^{kx_i} - 1)} \prod_{\theta(d \cdot i/n) = 0} \frac{1}{2} (e^{kx_i} + 1) \prod_{\theta(d \cdot i/n) \neq 0} e^{k\theta(d \cdot i/n)x_i} \prod_{\alpha \in \Delta_+} \langle \alpha, \Phi \rangle^{-2\gamma}. \tag{4.2}
$$

The residue formula of the correlation function is given by

$$
\langle e^{k\omega} \text{Tr} B(\Phi), [\mathcal{M}_g(n, d)] \rangle = \text{Res}_{\{x_i = 0\}} \left( \Omega_g^{(k)}(n, d) \text{Tr} B(\Phi) \right). \tag{4.3}
$$

The order of evaluations of residues above is $x_{n-1} < x_{n-2} < \cdots < x_1$.

The equivalence of the multiple infinite sum formula (3.5) and the residue formula (4.3) can be seen using the localization of infinite residues sum argument as follows.

First substituting the partial fraction expansion in the residue for

$$
\frac{e^{\theta(d \cdot i/n)x_i}}{e^{x_i} - 1} = \sum_{m \in \mathbb{Z}} e^{m \frac{2\pi \sqrt{-1}}{n} di} \frac{1}{x_i - 2\pi \sqrt{-1} m}, \quad \theta(d \cdot i/n) \neq 0 \tag{4.4}
$$

$$
\frac{1}{2} \frac{e^{x_i + 1}}{e^{x_i} - 1} = \sum_{m \in \mathbb{Z}} \frac{1}{x_i - 2\pi \sqrt{-1} m} \tag{4.5}
$$

we can see that the residue of the form in the right hand side of (4.3) evaluated at any dominant integral weight shifted the weight by $\rho$, $\{(l_1, \ldots, l_{n-1}) | l_i \geq 1\}$, coincides with a corresponding summand in (3.5) up to a constant;

$$
\text{Res}_{\{x_i = 2\pi \sqrt{-1} l_i\}} \left( \Omega_g^{(l)}(n, d) \text{Tr} B(\Phi) \right) = (-1)^{(n-1)} \frac{1}{n} \times (-1)^{(n-1)d + |\Delta_+|} n^g \prod_{\alpha \in \Delta_+} \langle \alpha, \Phi \rangle^{-2\gamma} \text{Tr} B(\Phi) e^{-d(l_1, \Phi)}. \tag{4.6}
$$

The set of the dominant integral weights shifted by $\rho$ constitute the lattice points set of one of the $n!$ Weyl chambers and (4.6) is invariant under the Weyl group action.

Thus we can sum over residues over the $n!$ Weyl chambers instead of the single chamber as in (3.5). Let $H_\alpha$ be the subset of the weight lattice perpendicular to the root $\alpha$,

$$
H_\alpha = \{(l_i) | l_i \in \mathbb{Z}, \langle \Phi, \alpha \rangle = 0\}. \quad \text{Then the union of the lattice points of the } n! \text{ Weyl chambers coincides with the complement in the weight lattice of the union of the hypersurfaces:}
$$
Let $L = \bigcup_{\alpha \in \Delta^+} H_{\alpha}$, and we have

$$\sum_{(l_i) \in L} \prod_{\{x_i = 2\pi \sqrt{-1} l_i\}} \prod_{\{x_{n-1} = 2\pi \sqrt{-1} l_{n-1}\}} \left( \Omega^{(1)}_g(n,d) \text{Tr} B(\bar{\phi}) \right)$$

$$= (-1)^{(n-1)}(n-1)! \left( e^{\omega \text{Tr} B(\bar{\phi})} \left[ \mathfrak{M}_g(n,d) \right] \right). \quad (4.7)$$

Now using the standard residue theorem which tells that the total sum of the residues for one variable with the remaining variables fixed is zero \[11\] repeatedly from $x_{n-1}$ to $x_1$, we can reduce the original residues sum (4.7) to that over sets of the form: $\bigcap_{\alpha \in I} H_{\alpha} \cap \bigcup_{\beta \notin I} H_{\beta}$, where each $I$, beginning with $\emptyset$, eventually becomes $\Delta^+$ in this process. At last (4.7) is expressed by the single residue evaluated at the origin $= \bigcap_{\alpha \in \Delta^+} H_{\alpha}$. Thus the equivalence of the infinite sum formula (3.5) and the residue formula (4.3) follows for $\text{Tr} B(\bar{\phi})$ such that (3.5) is convergent. We also conjecture that the residue formula (4.3) is valid for arbitrary $\text{Tr} B(\bar{\phi})$. One evidence for this conjecture is the fact that if we insert the $\hat{A}$-genus $\prod_{\alpha \in \Delta^+} \left( \frac{\langle \alpha/2, \bar{\phi} \rangle}{\sinh(\langle \alpha/2, \bar{\phi} \rangle)} \right)^{2g} \quad [11]$ as a gauge invariant zero form operator in (4.3),

$$\text{Res}_{\{x_i = 0\}} \left( \Omega^{(k)}_g(n,d) \prod_{\alpha \in \Delta^+} \left( \frac{\langle \alpha/2, \bar{\phi} \rangle}{\sinh(\langle \alpha/2, \bar{\phi} \rangle)} \right) \right)^{2g} \quad (4.8)$$

gives the twisted Verlinde dimension \[24\] of current algebra of level $(k - n)$ for any $k$ such that $k \equiv 0 \mod \frac{n}{(n,d)}$ in accord with the prediction of the Riemann-Roch formula, the existence of which is also conjectural for the cases of $(n,d) \neq 1$. 
4.2 Some examples

Here we will give some explicit form of the residue formulas.

First for $SU_2$ gauge group, the diagonalization of bosonic ghost becomes

$$\bar{\phi} = \frac{1}{2} \begin{pmatrix} x_1 & 0 \\ 0 & -x_1 \end{pmatrix}, \quad \mathcal{O}_2 = \frac{1}{4} x_1^2$$

The residue forms for $d = 0, 1$ read as

$$\Omega_g^{(k, 0)} = -(-2k)^{\frac{1}{2}} dx_1 \frac{e^{kx_1} + 1}{e^{kx_1} - 1} (x_1)^{-2\pi}.$$  \hspace{1cm} (4.9)

$$\Omega_g^{(k, 1)} = -(-2k)^{\frac{1}{2}} dx_1 k \frac{e^{kx_1/2}}{e^{kx_1} - 1} (x_1)^{-2\pi}.$$  \hspace{1cm} (4.10)

The correlation functions which have been completely solved in [2, 10] can be elegantly expressed as follows:

$$\sum_{g=0}^{\infty} \lambda e^{\omega e^{a\mathcal{O}_2}} \left\langle [\mathcal{M}_g(2, 0)] \right\rangle = -e^{-\frac{1}{2} a \lambda} \sqrt{\lambda/2} \cot \left( \sqrt{\lambda/2} \right).$$ \hspace{1cm} (4.11)

$$\sum_{g=0}^{\infty} \lambda e^{\omega e^{a\mathcal{O}_2}} \left\langle [\mathcal{M}_g(2, 1)] \right\rangle = e^{-\frac{1}{2} a \lambda} \sqrt{\lambda/2} \sin \left( \sqrt{\lambda/2} \right).$$ \hspace{1cm} (4.12)

The generating function of $SU_2$ correlators of all genera are also considered in [5].

Next for $SU_3$, the diagonalization of $\bar{\phi}$ becomes

$$\bar{\phi} = \frac{1}{3} \begin{pmatrix} 2x_1 + x_2 & 0 & 0 \\ 0 & -x_1 + x_2 & 0 \\ 0 & 0 & -x_1 - 2x_2 \end{pmatrix},$$

$$\mathcal{O}_2 = \frac{1}{3} (x_1^2 + x_1 x_2 + x_2^2), \quad \mathcal{O}_3 = \frac{1}{54} (2x_1 + x_2)(x_1 + 2x_2)(x_1 - x_2).$$

The residue forms for $SU_3$ are given by

$$\Omega_g^{(k, 0)} = (-3k^2)^{\frac{1}{2}} dx_1 dx_2 (k/2)^{2} e^{kx_1} \frac{e^{kx_2} + 1}{e^{kx_1} - 1} \frac{1}{e^{kx_2} - 1} (x_1 x_2 (x_1 + x_2))^{-2\pi}.$$ \hspace{1cm} (4.13)

$$\Omega_g^{(k, 1)} = (-3k^2)^{\frac{1}{2}} dx_1 dx_2 k^2 e^{kx_1/3} e^{2kx_2/3} \frac{e^{kx_2}}{e^{kx_1} - 1} \frac{1}{e^{kx_2} - 1} (x_1 x_2 (x_1 + x_2))^{-2\pi}.$$ \hspace{1cm} (4.14)
We give two simple examples of correlation functions computed by the residue formula;

\[
\langle e^{\omega}e^{a\phi_2}e^{b\phi_3}, [\mathcal{M}_3(3, 0)] \rangle = \frac{19}{4151347200} k^{16} - \frac{1}{53222400} a k^{14} + \frac{1}{2419200} a^2 k^{12} - \left( \frac{1}{120960} a^3 + \frac{1}{4354560} b^2 \right) k^{10} + \left( \frac{1}{17280} a^4 - \frac{1}{31104} a b^2 \right) k^8 + \left( \frac{1}{2592} a^5 - \frac{1}{46656} a^2 b^2 \right) k^6 + \left( \frac{1}{3888} a^6 - \frac{7}{34992} a^3 b^2 + \frac{107}{7558272} b^4 \right) k^4 \tag{4.15}
\]

\[
\langle e^{3m\omega}e^{a\phi_2}e^{b\phi_3}, [\mathcal{M}_3(3, 1)] \rangle = \frac{9708939}{512512000} m^{16} - \frac{160911}{1971200} a m^{14} + \frac{1}{896} b m^{13} + \frac{15363}{89600} a^2 m^{12} - \frac{3}{320} a b m^{11} + \left( \frac{1011}{4480} a^3 + \frac{83}{53760} b^2 \right) m^{10} + \frac{21}{640} a^2 b m^9 + \left( \frac{123}{640} a^4 - \frac{5}{384} a b^2 \right) m^8 - \left( \frac{5}{96} a^3 b - \frac{7}{10368} b^3 \right) m^7 + \left( -\frac{3}{32} a^5 + \frac{7}{192} a^2 b^2 \right) m^6 - \left( -\frac{1}{48} a^4 b + \frac{19}{2592} a b^3 \right) m^5 + \left( \frac{1}{48} a^6 - \frac{7}{432} a^3 b^2 + \frac{107}{93312} b^4 \right) m^4 \tag{4.16}
\]

Finally for $SU_4$, the diagonalization of $\bar{\phi}$ becomes

\[
\bar{\phi} = \frac{1}{4} \begin{pmatrix}
3x_1 + 2x_2 + x_3 & 0 & 0 & 0 \\
0 & -x_1 + 2x_2 + x_3 & 0 & 0 \\
0 & 0 & -x_1 - 2x_2 + x_3 & 0 \\
0 & 0 & 0 & -x_1 - 2x_2 - 3x_3
\end{pmatrix}
\]

In this case we have three different theories with magnetic flux units $d = 0, 1, 2$.

\[
\Omega^{(k)}_g(4, 0) = -\left(4k^3\right)^7 d x_1 d x_2 d x_3 (k/2)^3 \frac{e^{kx_1} + 1}{e^{kx_1} - 1} \frac{e^{kx_2} + 1}{e^{kx_2} - 1} \frac{e^{kx_3} + 1}{e^{kx_3} - 1} \left( x_1 x_2 x_3 (x_1 + x_2)(x_2 + x_3)(x_1 + x_2 + x_3)^{-2\sigma} \right) \tag{4.17}
\]

\[
\Omega^{(k)}_g(4, 1) = \left(4k^3\right)^7 d x_1 d x_2 d x_3 k^3 \frac{e^{kx_1/4}}{e^{kx_1} - 1} \frac{e^{2kx_2/4}}{e^{kx_2} - 1} \frac{e^{3kx_3/4}}{e^{kx_3} - 1} \left( x_1 x_2 x_3 (x_1 + x_2)(x_2 + x_3)(x_1 + x_2 + x_3)^{-2\sigma} \right) \tag{4.18}
\]

\[
\Omega^{(k)}_g(4, 2) = -\left(4k^3\right)^7 d x_1 d x_2 d x_3 (k/2)^2 k^2 \frac{e^{2kx_1/4}}{e^{kx_1} - 1} \frac{e^{kx_2} + 1}{e^{kx_2} - 1} \frac{e^{2kx_3/4}}{e^{kx_3} - 1} \left( x_1 x_2 x_3 (x_1 + x_2)(x_2 + x_3)(x_1 + x_2 + x_3)^{-2\sigma} \right) \tag{4.19}
\]
4.3 Bernoulli Expansions

In principle, by substituting in (4.3) the Fourier expansions [25];

\[
\frac{1}{2} e^x + \frac{1}{2} e^{-x} - 1 = \sum_{m=0}^{\infty} B_{2m} \frac{x^{2m}}{(2m)!}
\]

we can express any correlation functions by a finite sum of \((n-1)\)-ple products of Bernoulli polynomials. Here we will present the simplest ones.

To this end it is convenient to introduce the following notations;

\[
b_m(0) = \frac{B_m}{m!} \text{ for } m \neq 1, \text{ and } b_1(0) = 0,
\]

\[
b_m(\theta(d \cdot i/n)) = \frac{B_m(\theta(d \cdot i/n))}{m!}, \theta(d \cdot i/n) \neq 0
\]

Then the correlation function of \(SU_3\) theory can be written as a sum of double products of Bernoulli polynomials;

\[
\langle e^{\omega x_1 a_1 x_2 a_2} \, [\mathcal{M}_g(3, d)] \rangle = (-3)^7 \sum_{m_1+m_2=6} (-1)^{m_2} H_{m_2-m_1} b_{m_1-a_1}(\theta(d/3)) b_{m_2-a_2}(\theta(2d/3)).
\]

Similarly the correlation function of \(SU_4\) theory is given by a sum of triple products of Bernoulli polynomials;

\[
\langle e^{\omega x_1 a_1 x_2 a_2 x_3 a_3} \, [\mathcal{M}_g(4, d)] \rangle = (-1)^{d-1} (4)^7 \sum_{m_1+m_2+m_3=12} \sum_{l_1+l_2+l_3=6} (-1)^{l_1+l_2+l_3} H_{l_1} H_{l_2} H_{l_3} C_{l_3-m_3} \quad b_{m_1-a_1}(\theta(d/4)) b_{m_2-a_2}(\theta(2d/4)) b_{m_3-a_3}(\theta(3d/4)).
\]

In this way we can express any correlation function of \(\omega\) and \(\phi\) as a finite sum of known rational numbers. It would be interesting if we understand the relevance of the arithmetic properties of Bernoulli numbers [23, 25] to two dimensional gauge theories.
5 Deformations by Two Form Operators

5.1 Witten’s Formula

So far we have treated correlation functions which contain arbitrary zero operators but
do not contain any two form operator other than the standard symplectic form \( \exp(k\omega) \).
Now we describe the computation of correlators with arbitrary two form operator following
[10]. Here again the residue method will turn out to be useful.

Let \( \mathcal{O} \) be a gauge invariant polynomial of \( \{\phi^a\} \) of the form,
\[
\mathcal{O} = \mathcal{O}_2 + \sum_{m \geq 3} c_m \mathcal{O}_m, \quad (5.1)
\]
and \( \mathcal{O}^{(2)} \) be the associated two form operator.
\[
\mathcal{O}^{(2)} = -\int_{\Sigma} \left( \frac{1}{2} M_{ab} \overline{\psi} \gamma^a \psi + \frac{\partial \mathcal{O}}{\partial \phi^a} F^a \right), \quad M_{ab} = \frac{\partial^2 \mathcal{O}}{\partial \phi^a \partial \phi^b}. \quad (5.2)
\]
The insertion of \( \exp(\mathcal{O}^{(2)}) \) in the correlator corresponds to the deformation of the original
Lagrangian by the two form operator.

By computing the fermion determinant and the Jacobian of the change of bosonic vari-
ables, Witten gave the following formula \[10\]
\[
\langle e^{k\mathcal{O}^{(2)}} \text{Tr} \mathcal{B}(\phi), [\mathcal{M}_g(n, d)] \rangle = \langle e^{k\omega} \det M(Q(\phi)) \overline{\mathcal{B}}(Q(\phi)), [\mathcal{M}_g(n, d)] \rangle, \quad (5.3)
\]
where \( Q(\phi) \) is the power series defined by the change of variables;
\[
\hat{\phi}^a = \phi^a(\phi) \equiv \frac{\partial \mathcal{O}}{\partial \phi^a}, \quad \overline{\phi}^a = Q^a(\phi). \quad (5.4)
\]

5.2 Inversion of Variables and Residues

At first sight it might seem necessary to convert the original field variable \( \phi \) into a power
series of \( \hat{\phi} \) in order to evaluate the right hand side of (5.3). But it is sufficient to find only
the inversion of the gauge invariants \( \{\mathcal{O}_m\} \). The gauge invariants constructed by \( \{\hat{\phi}\} \) are
defined by \( \hat{\mathcal{O}}_m(\phi) = \mathcal{O}_m(\phi), \quad 2 \leq m \leq n \). The two sets of Casimir invariants \( \{\mathcal{O}_m\} \) and
\( \{\hat{\mathcal{O}}_m\} \) are related by certain polynomial equations: \( \hat{\mathcal{O}}_m = F_m(\mathcal{O}_2, ..., \mathcal{O}_n) \), and we have
only to convert them to evaluate the right hand side of (5.3) \( \mathcal{O}_m = G_m(\hat{\mathcal{O}}_2, \ldots, \hat{\mathcal{O}}_n) \).

Now the expansion coefficients of \( G_m \) defined by

\[
G_m(p_2, \ldots, p_n) = \sum_{l_2, \ldots, l_n \geq 0} G_m(l_2, \ldots, l_n) p_2^{l_2} \cdots p_n^{l_n},
\]

(5.5)
can be obtained using the Cauchy formula [26];

\[
G_m(l_2, \ldots, l_n) = \text{Res}_{\{q_i = 0\}} \left( \frac{G_m(p)}{(p_{l_2+1}^{l_2+1} \cdots p_{l_n+1}^{l_n+1})} dp_2 \cdots dp_n \right) \cdot
\]

(5.6)

Thus we get at least formally the following residue formula for the power series \( G_m \) of \( p_i \)

\[
G_m(p_2, \ldots, p_n) = \text{Res}_{\{q_i = 0\}} \left( q_m \det \left( \frac{\partial F_i}{\partial q_j} \right) \prod_{i=2}^n \frac{dq_i}{(F_i(q) - p_i)} \right).
\]

(5.7)

### 5.3 Diagonalization

To get the explicit polynomial relations \( \{ F_m \} \) of the previous subsection between the old and new Casimir invariants, it suffices to know only the change of variables for the diagonalization of the fields \( \phi \) because of gauge invariance;

\[
\hat{\phi} = \sum_{i=1}^{n-1} \hat{x}_i H_i = \text{diag}(\hat{z}_1, \ldots, \hat{z}_n), \quad \hat{\mathcal{O}}_l = \sum_{i=1}^n \frac{1}{l!} (\hat{z}_i)^l,
\]

\[
\hat{x}_i = y_i(x) = C_{ij} \frac{\partial \mathcal{O}}{\partial x_j} = (z_i - z_{i+1}) + \sum_{m \geq 2} \frac{c_{m+1}}{m!} (z_i^m - z_{i+1}^m),
\]

(5.8)

\[
\hat{z}_i = z_i + \sum_{m \geq 2} \frac{c_{m+1}}{m!} z_i^m - \frac{1}{n} \sum_{m \geq 2} c_{m+1} \mathcal{O}_m
\]

We also have the following determinant formula by the diagonalization;

\[
\det M(\phi) = n \det \left( \frac{\partial^2 \mathcal{O}}{\partial x_i \partial x_j} \right) \prod_{\alpha \in \Delta^+} \left( \frac{\langle \alpha, \hat{\phi} \rangle}{\langle \alpha, \phi \rangle} \right)^2.
\]

(5.9)

Now we can compute any correlation functions (5.3) using (5.7),(5.8) and (5.9).

The consistency of our formalism may be checked by considering deformations of \( SU_2 \) theory because in \( SU_2 \) theory any observable can be expressed by \( \omega, \mathcal{O}_2 \) and \( V_2(a) \).

For example the two form observable associated with \( \mathcal{O} = \mathcal{O}_2 - (2l - 1)! b_l \mathcal{O}_{2l} \) can be written by the observables associated with the Casimir invariant of second degree as

\[
\mathcal{O}^{(2)} = \omega - b_l \left( \mathcal{O}_2^{l-1} \omega - (l - 1) \mathcal{O}_2^{l-2} \sum_{a=1}^q V_2(a) V_2(a + g) \right).
\]

(5.10)
It can be seen that the use of the Witten’s formula and the direct substitution of (5.10) in the left hand side of (5.3) give the same answer.

Next take, for example, the $SU_3$ theory with the two form operator associated with $O = O_2 - 6mO_3$. The polynomial relation between old and new Casimir invariants is given by

\[
\begin{align*}
\hat{O}_2 &= O_2 - 18mO_3 + 3m^2O_2^2 \\
\hat{O}_3 &= O_3 - mO_2^2 + 9m^2O_2O_3 + m^3(O_2^3 - 54O_3^2) \tag{5.11}
\end{align*}
\]

From the residue formula (5.7), we get the inversion of the polynomial relation (5.11) as follows;

\[
\begin{align*}
O_2 &= \hat{O}_2 + 18m\hat{O}_3 + 15m^2\hat{O}_2^2 + 378m^3\hat{O}_3\hat{O}_2 + m^4(2916\hat{O}_3^2 + 270\hat{O}_2^3) + \cdots, \\
O_3 &= \hat{O}_3 + m\hat{O}_2^2 + 27m^2\hat{O}_2\hat{O}_3 + m^3(216\hat{O}_3^2 + 20\hat{O}_2^3) + 810m^4\hat{O}_2^2\hat{O}_3 + \cdots \tag{5.12}
\end{align*}
\]

The determinant that appears in Witten’s formula is given by

\[
\det M = (1 - 12m^2O_2)(1 - 9m^2O_2 + 54m^3O_3)^2. \tag{5.13}
\]

Then by the formula (5.3) we get the results

\[
\begin{align*}
\langle e^{3aO_2}, [\mathbb{M}_2(3, 1)] \rangle &= \frac{477}{2240}a^8 + \frac{189}{8}a^6m^2 - 27a^5m^3 - \frac{405}{4}a^4m^4 \\
\langle e^{3aO_2}, [\mathbb{M}_3(3, 1)] \rangle &= \frac{9708939}{512512000}a^{16} + \frac{482733}{98560}a^{14}m^2 - \frac{27}{28}a^{13}m^3 + \frac{1244403}{14480}a^{12}m^4 \\
&\quad - 243a^{11}m^5 - \frac{38151}{16}a^{10}m^6 - \frac{423549}{80}a^8m^8. \tag{5.14}
\end{align*}
\]

**5.4 Generalized Residue Formula**

One method to compute the correlators with a general two form operator was to expand the Casimir invariants $\{O_m\}$ into the series of $\{\hat{O}_m\}$ perturbatively and substitute them in the right hand side of (5.3). Here we give another method in the form of residue formula.

First we use the residue formula of the previous section in the right hand side of (5.3) to obtain

\[
\langle e^{kO_2} \text{Tr} B(\phi), [\mathbb{M}_g(n, d)] \rangle = \text{Res}_{\{g=0\}} \left( \Omega_g^{(k)}(n, d) \text{Tr} B(Q(y)) \det M^T(Q(y)) \right). \tag{5.16}
\]
Then if we change the integration variables from $y_i = \hat{x}_i$ to $Q_i(y) = x_i$, we obtain the generalized residue formula;

$$\left\langle e^{k\mathcal{O}^{(2)}} \text{Tr} B(\overline{\phi}), \left[ \mathfrak{M}_g(n, d) \right] \right\rangle = \text{Res}_{\{x_i=0\}} \left( \Omega_g^{(k)}(n, d; \mathcal{O}) \text{Tr} B(x) \right),$$

(5.17)

where

$$\Omega_g^{(k)}(n, d; \mathcal{O}) = (-1)^{(n-1)(d-1)+|\Delta_+|} \left( nk^{n-1} \right) \prod_{i=1}^{n-1} dx_i \frac{k}{(e^{kx_i} - 1)}$$

$$J^g \prod_{\theta(d+i/n)=0} \frac{1}{2(e^{ky_i} + 1)} \prod_{\theta(d+i/n)\neq0} e^{k\theta(d+i/n)y_i} \prod_{\alpha \in \Delta_+} \left( \alpha, \overline{\phi} \right)^{-\mathcal{F}}$$

$$= J^g \prod_{i=1}^{n-1} \left( \frac{e^{kx_i} - 1}{e^{ky_i} - 1} \right) \prod_{\theta(d+i/n)=0} \left( \frac{e^{ky_i} + 1}{e^{kx_i} + 1} \right) \prod_{\theta(d+i/n)\neq0} e^{k\theta(d+i/n)(y_i - x_i)} \Omega_g^{(k)}(n, d)$$

(5.18)

and the Jacobian is

$$J = \det \left( \frac{\partial y_i}{\partial x_i} \right) = n \det \left( \frac{\partial^2 \mathcal{O}}{\partial x_i \partial x_j} \right).$$

(5.19)

Thus we can say that the deformation of topological Yang-Mills theory by a two form operator is equivalent to the insertion of a certain zero form operator. Note that in this section the reduction of the gauge group to the abelian subgroup \cite{16, 14} was the powerful tool to compute explicitly the various physical quantities.

\section{Recursion Relation}

\subsection{Wick Contraction of One Form Operators}

Here as an application of the residue formula described above we will consider correlation functions containing one form observables. It is not difficult to compute them because we can use the physical Yang-Mills Lagrangian \cite{17} to integrate out all one form observables in correlators of the topological theory \cite{10} owing to the physical/topological Yang-Mills correspondence. Indeed using the gauge invariance and the diagonalization we get the following contraction formula in the presence of the two form operator $\exp(k\mathcal{O}^{(2)})$;

$$\langle V_m(a)V_i(b + g) \rangle = - \frac{1}{k} \delta_{ab} \sum_{ij} H_{ij}^{-1} \frac{\partial \mathcal{O}_m}{\partial x_i} \frac{\partial \mathcal{O}_t}{\partial x_j},$$

(6.1)

where $H_{ij}$ is the Hessian $H_{ij} = \frac{\partial^2 \mathcal{O}}{\partial x_i \partial x_j}$. 
6.2 Handle Contracting Operator; $SU_2$

In [2] Thaddeus rigorously proved that as a cohomology class $V_2(a)$ is the Poincare dual of the subspace $N_g(a)$ of $\mathcal{M}_g(2,1)$ where the holonomy around the cycle $C_a$ is trivial.

The physical meaning of it is that $V_2(a)$ has only the effect of reducing the path integral to the flat gauge fields that have the trivial holonomy around $C_a$. Thus $V_2(a)$ may be regarded as a operator which contracts the cycle $C_a$. The cup product of them $V_2(a)V_2(a+g)$ is the Poincare dual to $N_g(a) \cap N_g(a+g)$ where the holonomies around the both cycles $C_a$ and $C_{a+g}$ are trivial.

Thus we call here $H_2(a) \equiv V_2(a)V_2(a+g)$ the operator that contracts the $a$-th handle.

Noting that $N_g(a) \cap N_g(a+g)$ is diffeomorphic to $\mathcal{M}_{g-1}(2,1)$, we have the relation between correlators of genus $g$ and $g-1$,

$$\langle V_2(a)V_2(a+g)(\cdots, [\mathcal{M}_g(2,1)]) = \langle (\cdots, [\mathcal{M}_{g-1}(2,1)]) \rangle,$$  \hspace{1cm} (6.2)

where $(\cdots)$ means any operators. $H_2(a)$ is the inverse of the handle operator in the ordinary topological field theories in two dimensions.

The physical derivation of this effect [10] is as follows. Consider a correlation function of $SU_2$ theory ; $\langle V_2(a)V_2(a+g)\text{Tr}B(\vec{\phi})e^{k\omega}, [\mathcal{M}_g(2,d)] \rangle$. We can easily integrate out the one form operators if we use the physical Yang-Mills Lagrangian (3.1) which produces the trivial propagator for the fermions and then return to the topological Lagrangian to get

$$\langle V_2(a)V_m(a+g) \rangle = -\frac{2}{k}O_2.$$  \hspace{1cm} (6.3)

Then according to the residue formula (4.3) it is clear that the insertion of $-\frac{2}{k}O_2 = -\frac{x^2}{2k}$ reduce the genus of the surface by one.

Thus we have found the formula which is equivalent to (6.2),

$$\langle V_2(a)V_2(a+g)\text{Tr}B(\vec{\phi})e^{k\omega}, [\mathcal{M}_g(2,d)] \rangle$$

$$= \left\langle -\frac{2}{k}O_2\text{Tr}B(\vec{\phi})e^{k\omega}, [\mathcal{M}_g(2,d)] \right\rangle = \left\langle \text{Tr}B(\vec{\phi})e^{k\omega}, [\mathcal{M}_{g-1}(2,d)] \right\rangle$$  \hspace{1cm} (6.3)

6.3 Handle Contracting Operator; Generalization to $SU_n$

The identification of $V_2(a)$ with the $a$-th cycle contracting operator of $SU_2$, $d = 1$ theory was possible [2] because $V_2(a)$ is the only observable that satisfies the two requirements:
6 RECURSION RELATION

(1) It should have the ghost number \((n^2 - 1) = 3\).

(2) It must be fixed by the modular transformations that fix \(C_a\).

It seems impossible to extend this pure topological method to higher rank gauge groups \(SU_n, n > 3\). Nevertheless we can identify even for higher rank \(SU_n\) theories the operator which contracts the \(a\)-th cycle by using the generalized residue formula (3.17) and the contraction formula of fermions (6.1). We claim that for general \(SU_n\) the operator that contracts the \(a\)-th handle is the following:

\[
H_n(a) = \prod_{l=1}^{n-1} l! \ V_2(a) \cdots V_{n-1}(a) \prod_{l=1}^{n-1} l! \ V_2(a + g) \cdots V_{n-1}(a + g).
\]  

(6.4)

Indeed in the presence of the general two form operator \(\exp(kO^{(2)})\), the Wick contraction of fermions in the physical Yang-Mills theory (3.1) gives,

\[
\langle H_n(a) \rangle = \left( \prod_{l=1}^{n-1} l! \right)^2 \ det \left( \langle V_n(a)V_m(a + g) \rangle \right) \]

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It is clear that the above equation is also valid even when we insert any one form operators. Here we describe explicitly the Wick contraction for the case of $SU_4$ and $\mathcal{O} = \mathcal{O}_2$. The handle contracting operator for $SU_4$ is

$$H_4(a) = 2 \cdot 3! \cdot V_2(a)V_3(a)V_4(a) \cdot 2 \cdot 3! \cdot V_2(a + g)V_3(a + g)V_4(a + g),$$

and the Wick contraction of this becomes

$$\langle H_4(a) \rangle = \frac{(2!3!)^2}{k^3} \begin{vmatrix} -2\mathcal{O}_2 & -3\mathcal{O}_3 & -4\mathcal{O}_4 \\ -3\mathcal{O}_3 & 1/4\mathcal{O}_2^2 - 6\mathcal{O}_4 & -7/12\mathcal{O}_2\mathcal{O}_3 \\ -4\mathcal{O}_4 & -7/12\mathcal{O}_2\mathcal{O}_3 & 1/36\mathcal{O}_2^3 - 1/12\mathcal{O}_3^2 + \mathcal{O}_2\mathcal{O}_4 \end{vmatrix}$$

$$= \frac{144}{k^3} \left( \frac{1}{72}\mathcal{O}_2^6 - \frac{5}{6}\mathcal{O}_2^4\mathcal{O}_4 - \frac{17}{36}\mathcal{O}_2^3\mathcal{O}_3^2 + 16\mathcal{O}_2^2\mathcal{O}_4^2 + 6\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4 - 96\mathcal{O}_4^3 - \frac{3}{4}\mathcal{O}_3^4 \right)$$

$$= \frac{1}{4k^3} (x_1x_2x_3(x_1 + x_2)(x_2 + x_3)(x_1 + x_2 + x_3))^2. \quad (6.8)$$

7 Footnotes

(a) Magnetic flux here represents an element of $H^2(\Sigma_g, \mathbb{Z}_n) \cong \mathbb{Z}_n$ which classifies the topology of $SU_n/\mathbb{Z}_n$ bundles on $\Sigma_g$.

(b) The ample generator of the Picard group $\cong \mathbb{Z}$ is $\frac{n}{(n,d)}\omega$, and the first Chern class is $2n\omega$ in any case.

(c) If $k \not\equiv 0 \mod \frac{n}{(n,d)}$, then the twisted Verlinde dimension is precisely zero, while the residue (4.8) gives a rational number.
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