Quantum-classical correspondence via coherent state in integrable field theory

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We consider the problem of quantum-classical correspondence in integrable field theories. We propose a method to construct a field theoretical coherent state, in which the expectation value of the quantum field operator exactly coincides with the classical soliton. We also discuss the time evolution of this quantum state and the instability due to the nonlinearity.

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Introduction. The quantum-classical correspondence has been a fundamental problem since the foundation of quantum mechanics [1]. The problem of how to deduce the classical mechanics from the quantum theory has been discussed in various ways, for instance, the Ehrenfest’s theorem [2], the WKB analysis [3]. Physical quantities in the classical mechanics appears as the expectation values in the quantum mechanics. It follows that the essential problem in the quantum-classical correspondence is to find a quantum state in which the expectation value of the canonical variable of the system coincides with the classical counterpart.

The coherent states are known to be quantum states which behave classically [4]. In particular, in the case of the harmonic oscillator, a coherent state is a localized wave packet which oscillates exactly in the same frequency as the classical particle without changing its form [5]. Concerning the integrable system, coherent states play a prominent roll with respect to the quantum-classical correspondence. In [6], KdV soliton was constructed as a coherent state of the unharmonic oscillator. Moreover, the field theoretical coherent state was constructed in the sine-Gordon model [7].

In this letter, we consider the quantum-classical correspondence in integrable field theories. The most simple and fundamental example is the one-dimensional Bose gas with contact interactions, described by the Hamiltonian

\[ H = \sum_{j=1}^{N} \left( -\partial_{j}^{2} \right) + \sum_{1 \leq j < k \leq N} 2c\delta(x_j, x_k), \tag{1} \]

where \( N \) is the number of particle, \( c \) is the coupling constant and \( \partial_j := \frac{\partial}{\partial x_j} \). This is a quantum integrable system and exact eigenstates and eigenenergies are obtained via the Bethe ansatz method [5]. In the quantum field description, the time evolution of the Boson field operator obeys the quantum nonlinear Schrödinger (NLS) equation. In the classical limit where the quantum field operator is replaced by a commutative complex scalar field, the classical NLS equation is known to be classically integrable and has soliton solutions [8].

Identifying the quantum state corresponding to the classical soliton has been a long standing problem. In the attractive case \( c < 0 \), the classical NLS equation has the bright soliton solution and the corresponding quantum state is constructed in terms of the bound states associated with the complex Bethe roots called string [10, 11]. In the repulsive case \( c > 0 \), the classical solution is the dark soliton. It has been argued that the quantum wave packet constructed from the superposition of the hole-type excitations via the Bethe ansatz is corresponding to the classical dark soliton [12, 13].

In the attractive case, the bright soliton is obtained from the \( N \)-particle bound states in the limit \( N \to \infty \) and \( c \to 0 \) while keeping the product \( Nc \) finite. This can be regarded as a large quantum-number limit. Moreover, the time evolution of the quantum state does not obey the law of quantum mechanics. The time dependent quantum state is obtained from the Galilean transformation. In the repulsive case, the quantum wave packet corresponding to the classical dark soliton collapses due to the interference of the different energy eigenstates.

In this letter, we consider the quantum-classical correspondence in integrable field theories. The 1D Bose gas [14] can be described by the Bose field operators satisfying the canonical equal-time commutation relations

\[ [\hat{\psi}(x, t), \hat{\psi}^\dagger(y, t)] = \delta(x - y), \tag{2} \]

\[ [\hat{\psi}(x, t), \hat{\psi}(y, t)] = [\hat{\psi}^\dagger(x, t), \hat{\psi}^\dagger(y, t)] = 0. \tag{3} \]

The vacuum \( |0\rangle \) satisfies

\[ \hat{\psi}(x, t)|0\rangle = 0, \quad \langle 0|\hat{\psi}^\dagger(x, t) = 0, \quad \langle 0|0\rangle = 1. \tag{4} \]

Quantum field theory. The 1D Bose gas [14] can be described by the Bose field operators satisfying the canonical equal-time commutation relations

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The state space is generated by the successive actions of the creation operator \( \hat{\psi}^\dagger(x) \) on the vacuum as
\[
|\varphi_N\rangle = \int x_1 \cdots x_N \varphi_N(x_1, \ldots, x_N) \times \hat{\psi}^\dagger(x_1) \cdots \hat{\psi}^\dagger(x_N) |0\rangle,
\]
where \( \varphi_N(x_1, \ldots, x_N) \) is the corresponding \( N \)-body wave function. The Hamiltonian \( \hat{H} \) is written in terms of the field operators as
\[
\hat{H} = \int x \left[ -\hat{\psi}^\dagger \partial_x^2 \hat{\psi} + c \hat{\psi}^\dagger \hat{\psi} \right].
\]
In the Heisenberg picture, the time evolution of the field operator \( \hat{\psi}(x, t) \) is given by
\[
\partial_t \hat{\psi} = [\hat{\psi}, \hat{H}] = -\partial_x^2 \hat{\psi} + 2c\hat{\psi}^\dagger \hat{\psi},
\]
which we call the quantum nonlinear Schrödinger (NLS) equation. The formal solution is explicitly written as
\[
\hat{\psi}(x, t) = e^{it\hat{H}} \hat{\psi}(x)e^{-it\hat{H}}.
\]

Classical field theory. Replacing the quantum field operators \( \hat{\psi}(x, t) \) and \( \hat{\psi}^\dagger(x, t) \) by the commutative complex scalar field \( f(x, t) \) and \( f^\ast(x, t) \), we call “classicalization,” we obtain the classical NLS equation
\[
\partial_t f = -\partial_x^2 f + 2cf^\ast f f.
\]
This can be solved via the inverse scattering method and has a soliton solutions. The energy functional is obtained through the classicalization in the Hamiltonian as
\[
E[f, f^\ast] = \int x \left[ -f^\ast \partial_x^2 f + cf^\ast f f \right],
\]
in terms of which the classical NLS equation can be recast into the form
\[
\partial_t f = \{ f, E \}, \quad \partial_t f^\ast = \{ f^\ast, E \},
\]
where the Poisson bracket for two functionals is defined by
\[
\{ F, G \} := \frac{1}{i} \int x \left( \frac{\delta F}{\delta f} \frac{\delta G}{\delta f^\ast} - \frac{\delta G}{\delta f} \frac{\delta F}{\delta f^\ast} \right).
\]
Then it follows that the equal-time canonical relation
\[
\{ f(x, t), f^\ast(y, t) \} = \frac{1}{i} \delta(x - y)
\]
and the time evolution of the physical quantity \( F \)
\[
\partial_t F = \{ F, E \}.
\]

Coherent state. Let \( f(x, t) \) be the exact soliton solution of the classical NLS equation or \( (11) \). We construct a quantum state corresponding to the classical soliton at the initial time \( t = 0 \). The main object of this letter is the coherent state in the quantum field theory defined as
\[
|f\rangle := e^{A} |0\rangle, \quad A := \int x \left[ f(x)\hat{\psi}^\dagger(x) - f^\ast(x)\hat{\psi}(x) \right],
\]
where \( f(x) := f(x, t = 0) \). The normalization \( \langle f|f\rangle = 1 \) follows from \( A^\dagger = -A \). One can easily see that
\[
\{ \hat{\psi}(x), A \} = f(x), \quad \{ \hat{\psi}^\dagger(x), A \} = f^\ast(x),
\]
\[
\{ f, A \} = \hat{\psi}(x), \quad \{ f^\ast, A \} = \hat{\psi}^\dagger(x).
\]
It follows that the coherent state is an eigenvector of \( \hat{\psi}(x) \) with the eigenvalue \( f(x) \)
\[
\hat{\psi}(x)|f\rangle = f(x)|f\rangle.
\]
Moreover, as for the expectation values, we have the following relations
\[
\langle f|\hat{\psi}(x)|f\rangle = f(x), \quad \langle f|\hat{\psi}^\dagger(x)|f\rangle = f^\ast(x),
\]
which means that the coherent state classicallyize the field operators \( \hat{\psi}, \hat{\psi}^\dagger \) to the scalar fields \( f, f^\ast \), respectively.

Time evolution. Let us proceed to the time evolution of the coherent state. According to the principle of quantum mechanics, the coherent state \( |f\rangle \) is time-evolved as
\[
|f, t\rangle := e^{-i\hat{H}t} |f\rangle = e^{A(-t)} |0\rangle,
\]
where
\[
A(t) := \int x \left[ f(x)\hat{\psi}^\dagger(x, t) - f^\ast(x)\hat{\psi}(x, t) \right].
\]
The expectation value of the field operator at time \( t \) is given by
\[
\langle f, t|\hat{\psi}(x)|f, t\rangle = \langle f|\hat{\psi}(x, t)|f\rangle,
\]
which is not equal to the classical solution \( f(x, t) \). Here let us introduce the “classically” time evolved coherent state \( |\tilde{f}, t\rangle \) as
\[
|\tilde{f}, t\rangle := e^{\tilde{A}(t)} |0\rangle,
\]
\[
\tilde{A}(t) := \int x \left[ f(x, t)\hat{\psi}^\dagger(x) - f^\ast(x, t)\hat{\psi}(x) \right],
\]
whose expectation value describes the exact time evolution of the classical soliton
\[
\langle \tilde{f}, t|\hat{\psi}(x)|\tilde{f}, t\rangle = f(x, t).
\]
At the initial time \( t = 0 \), they are equal to each other
\[
A(0) = \tilde{A}(0) = A, \quad |f, t = 0\rangle = |\tilde{f}, t = 0\rangle = |f\rangle.
\]
They are evolved in time according to
\[
\partial_t A(t) = \frac{\hbar}{i}[A, \hbar], \quad \partial_t \tilde{A}(t) = \{A, E\}.
\] (27)

As a quantity representing the difference between quantum and classical states, we introduce a function \( r(t) \) as
\[
 r(t) := \langle f, t | \tilde{f}, t \rangle = \langle 0 | e^{-\tilde{A}(t)} e^{A(t)} | 0 \rangle,
\] (28)
which starts from 1 and decays to 0. At an infinitesimal time \( \Delta t \), we have
\[
 A(-\Delta t) = A - \Delta t 1^\hbar [A, \hbar] =: A - \Delta t F,
\] (29)
\[
 \tilde{A}(\Delta t) = A + \Delta t \{A, E\} =: A + \Delta t G.
\] (30)

Using Baker-Campbell-Hausdorff formula, we can explicitly evaluate \( r(t) \) in the form
\[
 r(\Delta t) = 1 - \frac{1}{i}c \Delta t \int |f|^4 x + cO(\Delta t^2). \tag{31}
\]

In the case of \( c = 0 \), the overlap \( r(t) \) remains 1 for all \( t \) and the time evolution of the coherent state exactly coincides with the classical soliton. The cancellation of the linear terms suggests that the instability of quantum soliton originates from the nonlinear term in the Hamiltonian (6).

**Conclusion.** In this letter, we constructed the field theoretical coherent state (15) and discussed the quantum-classical correspondence in the integrable field theory. We consider here the case of the one-dimensional Bose gas as an example. However, similar arguments are possible in the case of other integrable field theories.

The expectation values of the field operator with respect to the coherent state is identical to the classical soliton at initial time. However, the unitary time evolution of the coherent states breaks the coherent property due to the nonlinearity. Consequently the expectation values of the field operators with respect to this state do not coincide with the classical solutions anymore. We discussed the difference between quantum and classical time evolutions of coherent states by evaluating the overlaps between these states.

To elucidate the relationship between the coherent states and the previously constructed quantum bright and dark solitons are the next problem to be studied.

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