Superstring Theory on $AdS_2 \times S^2$
as a Coset Supermanifold

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Abstract
We quantize the superstring on the $AdS_2 \times S^2$ background with Ramond-Ramond flux using a $PSU(1,1|2)/U(1) \times U(1)$ sigma model with a WZ term. One-loop conformal invariance of the model is guaranteed by a general mechanism which holds for coset spaces $G/H$ where $G$ is Ricci-flat and $H$ is the invariant locus of a $Z_4$ automorphism of $G$. This mechanism gives conformal theories for the $PSU(1,1|2) \times PSU(2|2)/SU(2) \times SU(2)$ and $PSU(2,2|4)/SO(4,1) \times SO(5)$ coset spaces, suggesting our results might be useful for quantizing the superstring on $AdS_3 \times S^3$ and $AdS_5 \times S^5$ backgrounds.

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1. Introduction

Two dimensional sigma models with target space supermanifolds naturally appear in the quantization of superstring theory with Ramond-Ramond (RR) backgrounds. These sigma models possess manifest target space supersymmetry and have no worldsheet spinors. The fermionic fields are worldsheet scalars as in the usual Green-Schwarz (GS) formalism. It was recently shown that a sigma model on the supergroup manifold $PSU(1,1|2)$ can be used for quantizing superstring theory on the $AdS_3 \times S^3$ background with RR flux \[1,2\]. In this paper, we show that the sigma model defined on the coset supermanifold $PSU(1,1|2)/U(1) \times U(1)$ can be used to quantize superstring theory on the $AdS_2 \times S^2$ background with RR flux. Furthermore, we show that this sigma model is one-loop conformal invariant and we expect that it is conformal to all orders in perturbation theory. Our methods can also be used to produce one-loop conformal sigma models for $AdS_3 \times S^3$ and $AdS_5 \times S^5$ backgrounds. In the last case, incorporation of the model into a consistent superstring theory remains to be done.

There are two conventional approaches for describing the superstring, neither of which has been useful for quantizing the superstring in backgrounds with RR flux. One such approach is the covariant GS formalism where spacetime-supersymmetry is manifest and worldsheet supersymmetry is absent. The covariant GS superstring action can be defined classically in any background which satisfies the supergravity equations of motion. When the background fields satisfy these equations, the GS action is classically invariant under $\kappa$-symmetry which is necessary for removing unphysical fermionic degrees of freedom. Recently, this classical action was explicitly constructed for the case of the $AdS_5 \times S^5$ background \[3,4\]. Like the flat ten-dimensional GS action, the $AdS_5 \times S^5$ action has four-dimensional and six-dimensional classical analogs – the $AdS_2 \times S^2$ and the $AdS_3 \times S^3$ actions respectively \[5,6\]. Unfortunately, it is not known how to quantize any of these GS actions.

Another conventional approach to constructing superstring actions uses the RNS formalism where quantization is straightforward since the action is free in a flat background. The RNS formalism was successful for quantizing the superstring in an $AdS_3 \times S^3$ background with NS/NS flux \[7\]. However, the RNS approach has not been used to describe the superstring in RR backgrounds because of the complicated nature of the RR vertex operator.

Over the last five years, an alternative approach to constructing superstring actions has been developed which combines the advantages of the GS and RNS approaches \[8\].
Like the GS approach, this hybrid approach uses spacetime spinor variables as fundamental fields, allowing simple vertex operators for RR fields. And like the RNS approach, it reduces to a free action for a flat background, so quantization is straightforward. The worldsheet variables of the hybrid formalism are related to the RNS worldsheet variables by a field redefinition, and the action contains critical $N = 2$ worldsheet superconformal invariance which replaces the $\kappa$-symmetry of the GS action. This $N = 2$ worldsheet superconformal invariance is related to a twisted BRST invariance of the RNS formalism and is crucial for removing unphysical states.

The only disadvantage of this hybrid approach is that ten-dimensional Lorentz invariance cannot be kept manifest. The maximum amount of invariance which can be kept manifest (after Wick-rotating) is a $U(5)$ subgroup of the Lorentz group $[9]$. However, depending on the desired background, there are other ways of breaking the manifest Lorentz invariance which are more convenient. For example, one choice is to break the $SO(9,1)$ Lorentz invariance down to $SO(5,1) \times U(2)$ $[10]$. This choice is convenient for describing compactifications of the superstring to six dimensions and was used successfully in $[1]$ (see also $[2]$) for quantizing the superstring in an $AdS_3 \times S^3$ background with RR flux. The worldsheet description of strings propagating on $AdS_3 \times S^3$ is given by a certain modification of a sigma model on $PSU(1,1|2)$. Another choice is to break the $SO(9,1)$ Lorentz-invariance down to $SO(3,1) \times U(3)$ $[11,12]$. This choice is convenient for describing compactifications of the superstring to four dimensions and will be used in this paper for quantizing the superstring in an $AdS_2 \times S^2$ background with RR flux. The sigma model presented here is based on the coset supermanifold $PSU(1,1|2)/U(1) \times U(1)$.

As was shown in $[1]$ and $[2]$, the $PSU(1,1|2)$ sigma model is conformal invariant because of the Ricci flatness of the group supermanifold. To get an $AdS_2 \times S^2$ model, one has to form a coset space by dividing $PSU(1,1|2)$ by a $U(1) \times U(1)$ subgroup. Unfortunately, the resulting coset space turns out not to be Ricci flat. Indeed, on general grounds, only division by symmetric subgroups $[4]$ preserves Ricci flatness. Nevertheless, the $U(1) \times U(1)$ subgroup is quite special – it is the invariant locus of a $Z_4$ automorphism of $PSU(1,1|2)$. It is also a symmetric subgroup of the bosonic part of $PSU(1,1|2)$. Thanks to this $Z_4$ action, one can add a WZ term that can be used to restore one-loop conformal invariance. This WZ term is $d$-exact and the corresponding interaction can be written in terms of manifestly

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$^1$ $H$ is a symmetric subgroup of $G$ if it is the invariant locus of a $Z_2$ automorphism of $G$. 
\(PSU(1,1|2)\) left invariant currents. This mechanism for constructing a conformal field theory works for any coset space \(G/H\), provided that \(G\) is Ricci flat and \(H\) is the invariant locus of a \(\mathbb{Z}_4\) automorphism of \(G\). For example, superstring theory on \(AdS_3 \times S^3\) can be obtained from a conformal field theory based on the \(PSU(1,1|2) \times PSU(2|2)/SU(2) \times SU(2)\) coset. Our construction also works for the \(PSU(2,2|4)/SO(4,1) \times SO(5)\) coset manifold and leads to a conformal field theory that could be the starting point for quantizing superstring theory on \(AdS_5 \times S^5\).

The \(AdS_2 \times S^2\) background appears as the near horizon limit of a four-dimensional extremal black hole. To realize this background in type IIB string theory one has to consider compactification on a Calabi-Yau (CY) manifold \(X\) and wrap an appropriate number of 3-branes over 3-cycles of \(X\). The four-dimensional metric would be that of an extremal black hole. The ten dimensional geometry is not a direct product and the complex moduli of the CY vary as a function of the radial coordinate of the black hole. In the near horizon limit the CY moduli are fixed by the attractor equation \([14]\) and \(X \to X_{\text{attr}}\). At the attractor point the periods \((p^I,q_I)\) of the CY are proportional to D-brane charges

\[
\text{Im}(Cp^I) = Q^I, \quad \text{Im}(Cq_I) = \tilde{Q}_I, \quad (1.1)
\]

where \(Q^I\) and \(\tilde{Q}_I\) are electric and magnetic charges and \(C\) is a complex constant. The CY \(X_{\text{attr}}\) is uniquely fixed by the D-brane charges \([14]\), up to the complex constant \(C\). The radius of \(AdS_2 \times S^2\) is given by

\[
R = \frac{1}{4} \sqrt{Q^I Q^J N_{IJ}} \quad (1.2)
\]

where \(N_{IJ}\) is the vector superpotential \([15]\). The conformal theory describing the strings in the near horizon geometry (\(AdS_2 \times S^2\) background) factorizes into the product

\[
\left(AdS_2 \times S^2\right) \times X_{\text{attr}} \times H_\rho, \quad (1.3)
\]

where the first factor is the conformal field theory of \(AdS_2 \times S^2\) constructed in this paper, the second factor is a sigma model on the Calabi-Yau \(X_{\text{attr}}\) (which can in principle be replaced by any internal \(N = 2\) \(c = 9\) superconformal theory), and the last factor is the

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2 The details of this construction and its relation to \([1]\) will be discussed in \([13]\).

3 This complex constant will be related in section (2.3) to the vacuum value of the supergravity vector compensator.
conformal theory of a free chiral boson $\rho$. The product (1.3) is not exactly a direct product since changing the D-brane charges adjusts the CY moduli fixed by (1.1) and at the same time changes the radius of $AdS_2 \times S^2$. This adjustment is ensured by the presence of manifest four dimensional $N = 2$ supersymmetry in our model. The D-brane charges are quantized, but this is a non-perturbative effect that cannot be seen in the perturbative worldsheet theory.

The plan of this paper is as follows: In section 2, we shall review the four-dimensional version of the hybrid action in a flat \cite{1} and curved \cite{2} background, and then discuss this action in an $AdS_2 \times S^2$ background with RR flux. In sections 3-4, we shall show that this hybrid action is equivalent to a sigma model action for the coset supermanifold $PSU(1,1|2)/U(1) \times U(1)$ including a Wess-Zumino term. This action is similar to the GS action considered by Zhou \cite{3}(which was based on the $AdS_5 \times S^5$ action of Metsaev and Tseytlin \cite{4}), but has a crucial difference – it includes a metric for the fermionic currents. The $\kappa$-symmetry is explicitly broken by the kinetic term for fermions and is replaced by $N = 2$ worldsheet superconformal invariance. At the end of section 4, we compute the one-loop beta functions and show that they vanish for a certain coefficient in front of the WZ term. We demonstrate that the sigma model on the coset space $G/H$ with the WZ term is one-loop conformal invariant provided that the group $G$ is Ricci flat and $H$ is the fixed locus of a $Z_4$ automorphism of $G$.

In section 5, we present a geometrical proof of one-loop conformal invariance based on target-space considerations. The target space of our sigma model is the coset supermanifold $G/H$ with a $G$-invariant background metric $g_{AB}$ and an invariant antisymmetric tensor field strenght $H_{ABC}$. The choice of WZ term (i.e. antisymmetric field strength $H$) is a subtle modification of the usual one which is possible because of $Z_4$ symmetry. After computing the curvature of super coset spaces, we show that the non-vanishing Ricci curvature of the coset space is cancelled by the $H_{ABC}$ stress tensor in the appropriate Einstein’s equations. Moreover, $H_{ABC}$ satisfies its own field equation. In addition, we confirm that the dilaton expectation value is constant to two loops because the coset supermanifolds in question have vanishing scalar curvature.

Finally, in section 6 we speculate on the possible relation between conformal field theories on the coset spaces $PSU(1,1|2) \times PSU(2|2)/SU(2) \times SU(2)$ and $PSU(2,2|4)/SO(4,1) \times SO(5)$ and quantization of the superstring in $AdS_3 \times S^3$ and $AdS_5 \times S^5$ backgrounds.
2. Hybrid Superstring in Four Dimensions

We will start by describing the hybrid action based on the $SO(3, 1) \times U(3)$ splitting of the Lorentz group \[11,12\]. This action is very similar to the four-dimensional version of the GS action, but includes some crucial additional terms. As will be reviewed below, the Type II four-dimensional GS action contains four $\kappa$-symmetries which are replaced by $N = (2, 2)$ superconformal invariance in the hybrid action. There are three main differences, however, between the four-dimensional GS action and the action based on the hybrid approach. Firstly, the action in the hybrid approach is a free action in a flat background, so quantization is straightforward. Secondly, it includes compactification fields which cancel the conformal anomaly. And thirdly, the action contains a term coupling the dilaton zero mode with the worldsheet curvature, so scattering amplitudes have the expected dependence on the coupling constant.

2.1. Hybrid approach in a flat four-dimensional background

Before discussing the $AdS_2 \times S^2$ background, it will be useful to review the action in the hybrid approach for a flat four-dimensional background. In the $SO(3, 1) \times U(3)$ version there are ten bosonic spacetime variables which split into $X^m$ for $m = 0$ to 3, $Y^j$ and $\bar{Y}_j$ for $j = 1$ to 3. The $Y^j$ and $\bar{Y}_j$ variables describe the compactification manifold while the $X^m$ variables describe the four-dimensional spacetime. As in the RNS approach, the hybrid model has left-moving fermions $(\psi^j, \bar{\psi}_j)$ and right-moving fermions $(\hat{\psi}^j, \hat{\bar{\psi}}_j)$ associated with $Y^j$ and $\bar{Y}_j$. One also has sixteen fermionic worldsheet variables transforming as four-dimensional spinors which split into $(\theta^\alpha, p_\alpha)$, $(\bar{\theta}^{\dot{\alpha}}, \bar{p}_{\dot{\alpha}})$, $(\hat{\theta}^\alpha, \hat{p}_\alpha)$, $(\hat{\bar{\theta}}^{\dot{\alpha}}, \hat{\bar{p}}_{\dot{\alpha}})$ for $\alpha, \dot{\alpha} = (1, 2)$\footnote{The fermionic directions are labeled by $[\alpha, \dot{\alpha}, \hat{\alpha}, \hat{\dot{\alpha}}]$ but in order to simplify the notation (in those cases where there is no confusion), we will put a hat only on top of the fields $\theta, \hat{\theta}, p, \hat{p}, \ldots$.}. The $\theta$’s and $\hat{\theta}$’s correspond to the left-moving and right-moving fermionic variables of $N = 2$, $D = 4$ superspace and the $p$’s and $\hat{p}$’s are their conjugate momenta. Finally, one has a chiral and anti-chiral boson which will be called $\rho$ and $\hat{\rho}$. One can show that the fermions and chiral boson of the hybrid formalism are related by a field-redefinition to the ten $\psi$’s, two bosonic ghosts and two fermionic ghosts of the RNS formalism. The six RNS $\psi$’s from the compactification directions are not quite the same as the six $\psi$’s in the hybrid formalism, but are related by a factor involving the $\beta, \gamma$ ghosts.
In a flat or toroidal background, the worldsheet action for these fields in superconformal gauge is

\[ S = \frac{1}{\alpha'} \int dz \bar{z} \left[ \frac{1}{2} \partial X^m \partial X_m + p_\alpha \partial \theta^\alpha + \bar{p}_\alpha \partial \bar{\theta}^{\bar{\alpha}} + \bar{\rho} \partial \rho + \rho \partial \bar{\rho} \right] \]

where we have not tried to write the action for the chiral and anti-chiral boson \( \rho \) and \( \bar{\rho} \). The action of (2.1) is quadratic so all fields are free and it is completely straightforward to compute their OPE’s. We only consider the Type II superstring in this paper, so all worldsheet fields satisfy periodic boundary conditions.

The above action is manifestly conformal invariant, and it also contains a non-manifest \( N = (2,2) \) superconformal invariance. The left-moving \( N = 2 \) generators are given by

\[
T = \frac{1}{2} \partial X^m \partial X_m + p_\alpha \partial \theta^\alpha + \bar{p}_\alpha \partial \bar{\theta}^{\bar{\alpha}} + \frac{1}{2} \partial \rho \partial \rho \\
+ \partial Y^j \partial \bar{Y}_j + \frac{1}{2} (\psi^j \bar{\partial} \psi_j + \bar{\psi}_j \partial \psi^j), \tag{2.2}
\]

\[
J = i \partial \rho + \psi^j \bar{\psi}_j, \]

where

\[
d_\alpha = p_\alpha + i \sigma^m_{\alpha \bar{\alpha}} \bar{\theta}^{\bar{\alpha}} \partial X_m - \frac{1}{2} (\bar{\theta})^2 \partial \theta_\alpha - \frac{1}{4} \theta_\alpha \partial (\bar{\theta})^2, \tag{2.3}
\]

\[
\bar{d}_{\bar{\alpha}} = \bar{p}_{\bar{\alpha}} + i \sigma^m_{\alpha \bar{\alpha}} \theta^\alpha \partial X_m - \frac{1}{2} (\theta)^2 \partial \bar{\theta}_{\bar{\alpha}} + \frac{1}{4} \bar{\theta}_{\bar{\alpha}} \partial (\theta)^2,
\]

and \((d)^2\) means \(\frac{1}{2} e^{\alpha \beta} d_\alpha d_\beta\). The right-moving \( N = 2 \) generators are obtained from (2.2) by replacing \( \partial \) with \( \bar{\partial} \) and placing hats on the worldsheet variables. Note that \( (d_\alpha, \bar{d}_{\bar{\alpha}}, \bar{d}_\alpha, \bar{\bar{d}}_{\bar{\alpha}}) \) anti-commute with the spacetime-supersymmetry generators

\[
q_\alpha = \int dz \, Q_\alpha \quad \text{where} \quad Q_\alpha = p_\alpha - i \sigma^m_{\alpha \bar{\alpha}} \bar{\theta}^{\bar{\alpha}} \partial X_m - \frac{1}{4} (\bar{\theta})^2 \partial \theta_\alpha, \\
\bar{q}_{\bar{\alpha}} = \int dz \, \bar{Q}_{\bar{\alpha}} \quad \text{where} \quad \bar{Q}_{\bar{\alpha}} = \bar{p}_{\bar{\alpha}} - i \sigma^m_{\alpha \bar{\alpha}} \theta^\alpha \partial X_m - \frac{1}{4} (\theta)^2 \partial \bar{\theta}_{\bar{\alpha}}. \tag{2.4}
\]

To obtain \( \hat{q}_\alpha \) and \( \hat{\bar{q}}_{\bar{\alpha}} \), one just substitutes \( \theta \leftrightarrow \bar{\theta} \) and replaces \( \partial \) with \( \partial \).

The \( N = 2 \) generators of (2.2) split into two pieces, one piece depending on the ‘six-dimensional’ variables \( (Y^j, \psi^j) \), and the other piece depending on the remaining ‘four-dimensional’ variables \( (X^m, \theta^\alpha, \bar{\theta}^{\bar{\alpha}}, p_\alpha, \bar{p}_{\bar{\alpha}}, \rho) \). One can check that the six-dimensional contribution forms an \( N = 2 \ c = 9 \) algebra while the four-dimensional contribution forms an
\( N = 2 c = -3 \) algebra, summing to a critical \( N = 2 c = 6 \) algebra. The above action is easily generalized to the case when the six-dimensional compactification manifold is described by an \( N = 2 c = 9 \) superconformal field theory. In this case, one simply replaces the flat six-dimensional contribution to the action and \( N = 2 \) generators by their non-flat six-dimensional counterpart.

The action of (2.4) can be written in manifestly spacetime-supersymmetric notation using the supersymmetric combinations

\[
\Pi_j^m = \partial_j X^m + i \sigma_{\dot{a}}^m (\tilde{\theta}^\dot{a} \partial_j \theta^a + \theta^a \partial_j \tilde{\theta}^\dot{a} + \tilde{\theta}^\dot{a} \partial_j \theta^a + \tilde{\theta}^\dot{a} \partial_j \tilde{\theta}^\dot{a}).
\]

(2.5)

In terms of \( \Pi_j^m \), the action (2.4) can be written as a sum of the action describing an \( N = 2 \) \( c = 9 \) superconformal theory \( S_C \) and a four-dimensional contribution

\[
S = S_C + \frac{1}{\alpha'} \int d\bar{z} d\bar{z}' \left( \frac{1}{2} \Pi_{\bar{z}}^m \Pi_{\bar{z}'}^m + d_{\alpha} \Pi_{\bar{z}}^\alpha + \bar{\Pi}^\alpha_{\bar{z}} + \bar{\Pi}^{\dot{\alpha}}_{\bar{z}} + \bar{d}_{\dot{\alpha}} \bar{\Pi}^{\alpha}_{\bar{z}} + \bar{d}_{\dot{\alpha}} \bar{\Pi}^{\dot{\alpha}}_{\bar{z}} \right.

\[ + \epsilon^{j k} \left[ \Pi_{\bar{z}}^m i \sigma_{\dot{m}}^{m \dot{a}} (\theta_\dot{a} \partial_k \tilde{\theta}^\dot{a} + \tilde{\theta}^\dot{a} \partial_k \theta_\dot{a} - \tilde{\theta}^\dot{a} \partial_k \tilde{\theta}^\dot{a} - \tilde{\theta}^\dot{a} \partial_k \theta_\dot{a}) \right.

\[ + \left( \theta_\alpha \partial_j \tilde{\theta}^{\dot{a}} + \tilde{\theta}^{\dot{a}} \partial_j \theta_\alpha \right) \left( \tilde{\theta}^{\dot{a}} \partial_k \tilde{\theta}^{\dot{a}} + \tilde{\theta}^{\dot{a}} \partial_k \tilde{\theta}^{\dot{a}} \right) \right)
\]

where \( j, k = z, \bar{z} \) and \( d_{\alpha} \) differs from the definition of (2.3) by terms which vanish on-shell. Note that the last two lines of (2.6) represent the standard Wess-Zumino term of the four-dimensional Type II Green-Schwarz action in conformal gauge.

2.2. Type II action in a general curved background

Using the action of (2.1) in a flat background and the massless vertex operators described in [12], it is easy to guess the following action in a curved background:

\[
S = S_C + \frac{1}{\alpha'} \int d\bar{z} dz \left( \frac{1}{2} \Pi_{\bar{z}}^c \Pi_{z c} + B_{AB} \Pi_{\bar{z}}^A \Pi_{z B} \right.

\[ + d_{\alpha} \Pi_{\bar{z}}^\alpha + \bar{d}_{\dot{\alpha}} \bar{\Pi}_{\bar{z}}^{\dot{\alpha}} + \tilde{\Pi}_{\bar{z}}^\alpha + \bar{\tilde{\Pi}}_{\bar{z}}^{\dot{\alpha}} \]

\[ + d_{\alpha} P^{\alpha \dot{\beta}} \tilde{d}_{\dot{\beta}} + \bar{d}_{\dot{\alpha}} \bar{P}^{\dot{\alpha} \dot{\beta}} \tilde{\bar{d}}_{\dot{\beta}} + d_{\alpha} Q^{\alpha \dot{\beta}} \tilde{d}_{\dot{\beta}} + \bar{d}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha} \dot{\beta}} \tilde{\bar{d}}_{\dot{\beta}} \right),
\]

(2.7)

where \( \Pi_{\bar{z}}^A \) is written in terms of the supervierbein \( E^A_M \) as \( \Pi_{\bar{z}}^A = E^A_M \partial_j Z_M \) with \( Z^M = (X^m, \theta^\mu, \tilde{\theta}^{\dot{\mu}}, \tilde{\tilde{\theta}}^{\dot{\mu}}, \tilde{\tilde{\tilde{\theta}}^{\dot{\mu}}}) \), and the index \( A \) takes the tangent-superspace values \( [c, \alpha, \dot{a}, \dot{\alpha}, \dot{\dot{a}}, \dot{\dot{\alpha}}] \). Once again, to simplify notation we denote \( \tilde{\theta}^\dot{a} \) as \( \tilde{\theta}^\dot{a} \) and \( \tilde{\tilde{\theta}}^{\dot{\alpha}} \) as \( \tilde{\theta}^{\dot{a}} \). In other words, we put the hat over the symbol instead of the index. We will also use this rule for all other
worldsheet fields. But we will use the standard notations \([c, \alpha, \dot{\alpha}, \hat{\alpha}, \hat{\dot{\alpha}}]\) for the background fields such as \(Q^{\alpha\hat{\beta}}\).

Spinor indices \([\alpha, \dot{\alpha}]\) and \([\hat{\alpha}, \hat{\dot{\alpha}}]\) correspond to the left and right-moving degrees of freedom and \(c\) is a vector index. The lowest component of \(E^c_m\) is the vierbein and the lowest components of \(E^\alpha_m\) and \(E^{\hat{\alpha}}_m\) are the gravitini. The antisymmetric field \(B_{MN}\) is the two-form superfield where the lowest component of \(B_{mn}\) is the NS/NS two-form. The superfields \(P^{\alpha\hat{\beta}}\) and \(\bar{P}^{\dot{\alpha}\hat{\beta}}\) contain the self-dual and anti-self-dual parts of the RR graviphoton field-strength as their lowest components. Similarly, the superfields \(Q^{\alpha\hat{\beta}}\) and \(\bar{Q}^{\dot{\alpha}\hat{\beta}}\) contain the derivatives of the two RR scalars as their lowest components. The last term in (2.7), \(S_C\), is the action that describes the compactification manifold.

In a flat background, one can check that the action (2.7) reduces to (2.6). Furthermore, linear perturbations around the flat background reproduce the massless vertex operators of \([12]\).

Note that \(d\) and \(\hat{d}\) are fundamental fields in the action of (2.7), and in a flat background, they can be expressed in terms of the free fields \(p\) and \(\bar{p}\) using the definitions (2.3). If \(d\) and \(\hat{d}\) are set to zero and the compactification fields are ignored, the action of (2.7) is precisely the four-dimensional version of the covariant GS action in a curved background. When the background satisfies certain torsion constraints, the covariant GS action has \(\kappa\)-symmetries which allow half of the \(\theta\)’s to be gauge-fixed. However, it has only been possible to quantize this action in light-cone gauge, which is not accessible for arbitrary backgrounds \([14]\).

By including the \(d\) and \(\hat{d}\) fields, as well as the chiral boson \(\rho\) and the compactification fields, it is possible to covariantly quantize the action (2.7) in a manner analogous to the normal-coordinate expansion used in bosonic string or RNS superstring sigma models. In a curved background, one still has an \(N = 2\) superconformal invariance at the quantum level if the background fields are on-shell. The \(N = 2\) left-moving generators are given by

\[
T = \Pi^\bar{c}_z \Pi_{cz} + d_\alpha \Pi^\alpha_z + \bar{d}_{\dot{\alpha}} \Pi^{\dot{\alpha}}_z + \frac{1}{2} \partial \rho \partial \rho + T_C ,
\]

\[
G = e^{i\rho} (d)^2 + \bar{G}_C , \quad \bar{G} = e^{-i\rho} (\bar{d})^2 + G_C ,
\]

\[
J = i \partial \rho + J_C ,
\]

\text{As explained in reference \([12]\), the complete Type II action also contains a ‘Fradkin-Tseytlin’ term which couples the spacetime dilaton to the \(N = (2,2)\) worldsheet supercurvature. But since the dilaton is constant in the \(AdS_2 \times S^2\) background, this term will only give the contribution} \(\phi \int R + a \int F\) \text{where} \(\int R\) \text{is the worldsheet Euler number,} \(\int F\) \text{is the worldsheet U(1) instanton number (coming from the U(1) gauge field of the N=2 worldsheet), and} e^{-\phi - ia} \text{is the background value of the supergravity vector compensator} Z^0 \text{(which is related by supergravity equations of motion to the dilaton and NS/NS axion background values)} \([12]\).
where \([T_C, G_C, \bar{G}_C, J_C]\) are the \(N = 2\) \(c = 9\) generators describing the compactification manifold. One can check that the classical equations of motion of (2.7) imply that \((d)^2\) and \((\bar{d})^2\) are holomorphic, so the \(N = 2\) generators are holomorphic at least at the classical level \([12,16]\). Holomorphicity at the quantum level is expected to imply that the background superfields satisfy their low-energy equations of motion. This has been explicitly checked at one-loop for the heterotic superstring in a curved background \([16]\), but has not yet been checked for the Type II superstring in a curved background. So the action of (2.7) is still a conjecture at the quantum level but we will provide evidence for this conjecture by explicitly showing that the one-loop conformal anomaly vanishes when the background is chosen to describe \(AdS_2 \times S^2\).

2.3. Type IIA action in \(AdS_2 \times S^2\) background with RR flux

To describe the \(AdS_2 \times S^2\) background with RR flux, one simply needs to substitute the values for the superfields into (2.7). In this background, the field-strength of the RR graviphoton is equal to \(F_{01} = F_{23} = N\) where the integer \(N\) counts the number of branes. But in the presence of a Type IIA (or Type IIB) Calabi-Yau compactification with \(h\) Kahler moduli (or \(h\) complex moduli), there is some ambiguity which of the \(h + 1\) RR vector fields is called the graviphoton. There is an auxiliary two-form, \(T_{\mu \nu}\), in the supergravity multiplet whose on-shell self-dual part satisfies the equation of motion \([17]\)

\[
T_{\alpha \beta} = \frac{4N_{PQ} Z^Q}{N_{RST} Z^R Z^S} F_{\alpha \beta}^P ,
\]

where \(P, Q, R, S = 0, \ldots, h\), \(Z^0 = e^{-\phi - ia}\) is the supergravity vector compensator, \(Z^P / Z^0\) are the Kahler (or complex) moduli for \(P = 1, \ldots, h\), \(N_{PQ}\) is the vector superpotential, and \(F_{\alpha \beta}^P\) is the self-dual part of the \(P^{th}\) vector field strength. \(T_{\alpha \beta}\) is the lowest component of the superfield \(P_{\alpha \beta}\) and \(F_{\alpha \beta} = Z^0 T_{\alpha \beta}\) gives the linear combination of RR field-strengths which is turned on. So in the \(AdS_2 \times S^2\) background, \(P_{\alpha \beta}\) and \(\bar{P}_{\dot{\alpha} \dot{\beta}}\) take the values

\[
P_{\alpha \beta} = (Z^0)^{-1} F_{\alpha \beta} = (Z_0)^{-1} N(\sigma^{01})_{\alpha \beta} = (Z_0)^{-1} N \delta_{\alpha \beta} ,
\]

\[
\bar{P}_{\dot{\alpha} \dot{\beta}} = (\bar{Z}^0)^{-1} F_{\dot{\alpha} \dot{\beta}} = (\bar{Z}_0)^{-1} N(\sigma^{01})_{\dot{\alpha} \dot{\beta}} = (\bar{Z}_0)^{-1} N \delta_{\dot{\alpha} \dot{\beta}} .
\]

The dependence of \(P_{\alpha \beta}\) on the string coupling constant \(g = e^\phi\) can be understood by recalling that the \(e^{-2\phi}\) term is absent in the \(FF\) kinetic term for the RR field. So \(P_{\alpha \beta}\) is related to the RR field strength by a factor of \(g\). The dependence of \(P_{\alpha \beta}\) on the phase
factor $e^{ia}$ from $(Z_0)^{-1}$ comes from its non-zero $R$-charge where $e^{ia}$ is related to the phase of the constant $C$ in (1.1). In the rest of this paper, we shall assume that $a = 0$ although all our formulas can be generalized to non-zero $a$ by simply rotating all fields by a phase factor proportional to their $R$-charge.

Note that the $h$ linear combinations of field-strengths which are not turned on are given by

$$F_{\alpha \beta}^P - \frac{1}{4} Z^P T_{\alpha \beta}, \quad (2.11)$$

for $P = 1, ..., h$. This can be seen from $N = 2 D = 4$ spacetime-supersymmetry since the Calabi-Yau gauginos transform into (2.11) under supersymmetry [15].

Substituting (2.10) into (2.7), we obtain

$$S = \frac{1}{\alpha'} \int d\bar{z} dz \left( \frac{1}{2} \Pi_z^c \Pi_{zc} + B_{AB} \Pi_z^A \Pi_z^B + d_\alpha \Pi_z^\alpha + \hat{d}_\dot{\alpha} \hat{\Pi}_z^\dot{\alpha} + \hat{d}_\dot{\alpha} \hat{\Pi}_z^\dot{\alpha} + Ng d_\alpha \hat{d}_\dot{\beta} \delta^{\alpha \dot{\beta}} + Ng \hat{d}_\dot{\alpha} d_\dot{\beta} \delta^{\dot{\alpha} \dot{\beta}} \right) + S_C. \quad (2.12)$$

In the above action, $\Pi^A$ and $B_{AB}$ are defined in precisely the same manner as in the covariant GS $AdS_2 \times S^2$ action of Zhou [4], which was based on the $AdS_5 \times S^5$ action of Metsaev and Tseytlin [8]. However, the second and third lines of (2.12) are not present in the usual GS action and are crucial for quantization. As discussed in [4] and will be reviewed in the next section, the objects $\Pi^A$ can be constructed out of currents of the supergroup $PSL(1,1|2)/U(1) \times U(1)$. In the $AdS_2 \times S^2$ background, $B_{AB}$ can be written in the following simple form:

$$B_{\alpha \beta} = B_{\beta \alpha} = -\frac{1}{4Ng} \delta_{\alpha \beta}, \quad B_{\dot{\alpha} \dot{\beta}} = B_{\dot{\beta} \dot{\alpha}} = -\frac{1}{4Ng} \delta_{\dot{\alpha} \dot{\beta}}, \quad (2.13)$$

with all other components of $B_{AB}$ vanishing. A similarly simple form for $B_{AB}$ occurs in the $AdS_3 \times S^3$ and $AdS_5 \times S^5$ backgrounds. Equation (2.13) is easy to prove since $H_{ABC} = \nabla_{[AB} B_{BC]} + T_{[AB}^D B_{C]D}$, so the only non-zero components of $H_{ABC}$ are

$$H_{ca\dot{a}} = T_{ca\dot{a}} B_{\alpha \beta} + T_{ca\dot{a}} \hat{B}_{\alpha \beta} = Ng \sigma_c \alpha \gamma \delta^{\beta \gamma} \left( -\frac{1}{4Ng} \delta_{\alpha \beta} \right) + Ng \sigma_c \gamma \alpha \delta^{\beta \gamma} \left( -\frac{1}{4Ng} \delta_{\alpha \beta} \right) = -\frac{1}{2} (\sigma_c)_{\alpha \dot{a}}, \quad (2.14)$$
and $H_{c\dot{a}} = \frac{1}{2}(\sigma_c)_{c\dot{a}}$, which can be obtained from (2.14) by replacing the fermionic unhatted indices by hatted and vice versa. The precise normalizations are fixed by the supergravity equations. In (2.14), we have used the torsion constraints of [12] to relate the torsion components to the Ramond-Ramond field-strengths by the equations

$$T_{c\alpha}^\beta = (\sigma_c)^\alpha_{\alpha'\beta}, \quad T_{c\dot{a}}^\beta = (\sigma_c)^\alpha_{\dot{a}'\beta},$$

$$T_{c\alpha}^\dot{\alpha} = -(\sigma_c)^\alpha_{\alpha'\dot{\alpha}'}, \quad T_{c\dot{a}}^\dot{\alpha} = -(\sigma_c)^\alpha_{\dot{a}'\dot{\alpha}'}.$$

(2.15)

As in the hybrid action for $AdS_3 \times S^3$ [1], it is convenient to integrate out the $d$ and $\hat{d}$ fields. This is possible because of the quadratic term $d\hat{d}$ produced by the RR flux, which implies that $d$ and $\hat{d}$ are auxiliary fields. Substituting the equations of motion for $d$ and $\hat{d}$ into (2.12) and taking (2.13) into account, one obtains

$$S = S_C + \frac{1}{\alpha'} \int dxd\bar{x} \left[ \frac{1}{2} \Pi^{\alpha}_{z} \Pi_{z\alpha} + \frac{3}{4Ng} (\delta_{\alpha\beta} \hat{\Pi}^{\beta}_{z} \Pi^{\alpha}_{z} + \delta_{\dot{a}\dot{b}} \hat{\Pi}^{\dot{a}}_{z} \Pi^{\dot{b}}_{z}) + \frac{1}{4Ng} (\delta_{\alpha\beta} \hat{\Pi}^{\beta}_{z} \Pi^{\alpha}_{z} + \delta_{\dot{a}\dot{b}} \hat{\Pi}^{\dot{a}}_{z} \Pi^{\dot{b}}_{z}) \right].$$

(2.16)

The $Ng$ dependence can be simplified by rescaling $E^c_M \to (Ng)^{-1}E^c_M$, and $E^\alpha_M \to (Ng)^{-\frac{1}{2}}E^\alpha_M$ to obtain

$$S = S_C + \frac{1}{2\alpha' g^2 N^2} \int dxd\bar{x} \left[ \Pi^{\alpha}_{z} \Pi_{z\alpha} + \frac{3}{2} (\delta_{\alpha\beta} \hat{\Pi}^{\beta}_{z} \Pi^{\alpha}_{z} + \delta_{\dot{a}\dot{b}} \hat{\Pi}^{\dot{a}}_{z} \Pi^{\dot{b}}_{z}) + \frac{1}{2} (\delta_{\alpha\beta} \hat{\Pi}^{\beta}_{z} \Pi^{\alpha}_{z} + \delta_{\dot{a}\dot{b}} \hat{\Pi}^{\dot{a}}_{z} \Pi^{\dot{b}}_{z}) \right].$$

(2.17)

The $N = 2$ superconformal generators are the same as in (2.8) with $d$ replaced by its equation of motion, and the compactification-independent part of the stress tensor will be shown in section 3 to be the conformal generator for a sigma model based on the supergroup $PSU(1,1|2)/U(1) \times U(1)$. Note that the action and constraints are invariant under the transformation which takes the bosonic currents $\Pi^{c} \to -\Pi^{c}$, the fermionic currents $\Pi \to i\Pi$ and $\hat{\Pi} \to -i\hat{\Pi}$, and which shifts $\rho \to \rho + \pi$ and $\bar{\rho} \to \bar{\rho} + \pi$. The presence of this $\mathbb{Z}_4$ symmetry will be important later.

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7 This rescaling is chosen such that the torsion constraint for $T^c_{\alpha\dot{a}}$ remains independent of $Ng$. 

11
2.4. Perturbative derivation of $AdS_2 \times S^2$ action

In the hybrid action for $AdS_3 \times S^3$ of reference [1], the action was justified using a Ramond-Ramond vertex operator to perturb around a flat background. This justification was important in the $AdS_3 \times S^3$ case since there did not exist a hybrid action for the general six-dimensional Type II superstring background. Although the existence of (2.7) as a hybrid action for the general four-dimensional background [12] makes such a justification less necessary in the $AdS_2 \times S^2$ action, we shall show in this section that such a perturbation is possible and leads to the same result of (2.16).

The vertex operator in a flat background for a constant graviphoton field-strength $F^{01} = F^{23} = N$ is given by

$$V = Ng \int dzd\bar{z} (\delta_{\alpha\beta} Q^\alpha \hat{Q}^\beta + \delta_{\dot{\alpha}\dot{\beta}} \bar{Q}^{\dot{\alpha}} \bar{\hat{Q}}^{\dot{\beta}}),$$

(2.18)

where the $g$ dependence in $V$ comes for the same reason as in (2.10) and $Q$ is given in terms of $p$’s and $\theta$’s by (2.4). It is clear that $V$ is a physical operator since the supersymmetry currents $Q$ commute with the $N = 2$ constraints of (2.2). Adding $V$ to the flat action of (2.1) produces the action

$$S = \frac{1}{\alpha'} \int dzd\bar{z} \left( \frac{1}{2} \bar{\partial} X^m \partial X_m + p_{\alpha} \bar{\partial} \theta^\alpha + \bar{p}_{\dot{\alpha}} \bar{\partial} \bar{\theta}^{\dot{\alpha}} + \bar{p}_{\dot{\alpha}} \bar{\partial} \bar{\theta}^{\dot{\alpha}} + Ng(\delta_{\alpha\beta} Q^\alpha \hat{Q}^\beta + \delta_{\dot{\alpha}\dot{\beta}} \bar{Q}^{\dot{\alpha}} \bar{\hat{Q}}^{\dot{\beta}}) \right) + SC.$$  

(2.19)

Now one can integrate the $Q$’s out, and keeping terms only up to the cubic order in fields, one can just replace $Q = (Ng)^{-1} \bar{\partial} \bar{\theta}$ and $\hat{Q} = -(Ng)^{-1} \partial \theta$. Finally, integrating by parts and using the fact that terms proportional to $\bar{\partial} \partial X^m$ or $\partial \bar{\partial} \theta$ can be removed by redefining $X^m$ or $\theta$, one can write the action as

$$S = \frac{1}{\alpha'} \int dzd\bar{z} \left( \frac{1}{2} \bar{\partial} X^m \partial X_m + (Ng)^{-1}(\delta_{\alpha\beta} \bar{\partial} \theta^\alpha \partial \theta^\beta + \delta_{\dot{\alpha}\dot{\beta}} \bar{\partial} \bar{\theta}^{\dot{\alpha}} \bar{\partial} \bar{\theta}^{\dot{\beta}}) + i\sigma^m_{\alpha\dot{\alpha}} (\bar{\theta}^{\dot{\alpha}} \partial_j X_m \partial_k \theta^\alpha + \theta^\alpha \partial_j X_m \partial_k \bar{\theta}^{\dot{\alpha}} - \bar{\theta}^{\dot{\alpha}} \partial_j X_m \partial_k \theta^\alpha - \bar{\theta}^{\dot{\alpha}} \partial_j X_m \partial_k \bar{\theta}^{\dot{\alpha}}) \right) + SC.$$  

(2.20)

The action (2.20) supplemented by the total derivative term

$$\Delta L = -\epsilon^{jk} \frac{1}{2Ng\alpha'} (\delta_{\alpha\beta} \partial_j \bar{\theta}^{\alpha} \partial_k \theta^\beta + \delta_{\dot{\alpha}\dot{\beta}} \partial_j \bar{\theta}^{\dot{\alpha}} \partial_k \bar{\theta}^{\dot{\beta}}),$$

(2.21)

reproduces the action of (2.16) up to cubic order in the fields.
3. AdS$_2 \times S^2$ from the $PSU(1,1|2)/U(1) \times U(1)$ coset

From the work of [1] and [2], it is natural to expect that quantization of strings in the AdS$_2 \times S^2$ background requires a sigma model based on a quotient supermanifold $PSU(1,1|2)/U(1) \times U(1)$. In this section, we discuss this coset supermanifold and its higher dimensional analogs. We also introduce a $\mathbb{Z}_4$ symmetry that will play an important role in our considerations. Finally, we construct a sigma model action as a gauged principal chiral field. The model constructed in this section is not conformal and the necessary modification (a WZ term) will be discussed in the next section.

As we will confirm in this and in the next sections, the action (2.17) indeed combines the sigma model and the WZ terms of the $PSU(1,1|2)/U(1) \times U(1)$ coset. To make the identification, one notices that the objects $[\Pi^c, \Pi^\alpha, \Pi^{\dot{\alpha}}, \hat{\Pi}^\alpha, \hat{\Pi}^{\dot{\alpha}}]$ generate global $PSU(1,1|2)$ rotations and can be identified with the sigma model currents.

3.1. Coset spaces and $\mathbb{Z}_4$ symmetry

Let us start this section with a discussion of the signature of the coset space. There are two closely related cosets, $PSU(2|2)/U(1) \times U(1)$ and $PSU(1,1|2)/U(1) \times U(1)$, that differ by the choice of real structure on the group. The first coset is based on the $SU(2) \times SU(2) \subset PSU(2|2)$ and leads to the $S^2 \times S^2$ geometry with the signature being $(2,2)$. The other coset leads to the AdS$_2 \times S^2$ geometry with the signature $(1,3)$. Clearly, the physics of these two backgrounds is very different, but they share many common algebraic properties.

The super Lie algebras $psu(2|2)$, $psu(1,1|2)$ are the algebras of $4 \times 4$ matrices with bosonic diagonal blocks and fermionic off-diagonal blocks

$$M = \begin{pmatrix} A & X \\ Y & B \end{pmatrix}$$

where $trA = trB = 0$. (3.1)

The bracket is defined to be the commutator projected on the doubly-traceless subspace. The supertrace is defined as $Str(M) = TrA - TrB$ and the (super) antihermiticity condition for $psu(2|2)$ is [19]

$$M^\dagger = \begin{pmatrix} A^\dagger & -iY^\dagger \\ -iX^\dagger & B^\dagger \end{pmatrix} = -M \quad \rightarrow \quad A = -A^\dagger, \quad B = -B^\dagger, \quad X = iY^\dagger.$$ (3.2)
In other words, the Grassmann even matrices $A, B$ are traceless antihermitian, and the Grassmann odd $X, Y$ matrices are related to each other. For the case of $psu(1, 1|2)$, the anti-hermiticity condition is

$$M^\dagger \equiv \begin{pmatrix} \sigma_3 A^\dagger \sigma_3 & -i\sigma_3 Y^\dagger \\ -iX^\dagger \sigma_3 & B^\dagger \end{pmatrix} = -M \rightarrow A = -\sigma_3 A^\dagger \sigma_3^{-1}, \quad B = -B^\dagger, \quad X = i\sigma_3 Y^\dagger. \quad (3.3)$$

The bosonic geometry of the $PSU(1, 1|2)$ is $AdS_3 \times S^3$, so to construct a string theory on $AdS_2 \times S^2$, we have to quotient the group by (the right action of) $U(1) \times U(1)$. The first $U(1)$ is embedded into $SU(1, 1)$ to produce $AdS_2$, while the second is embedded into $SU(2)$ to give rise to $S^2$. Since $PSU(1, 1|2)$ has already the right number of fermions (8 real ones), we divide only by a bosonic subgroup. For a reason that will become clear later, we will think of this subgroup as an invariant locus of a $\mathbb{Z}_4$ automorphism. Both for $psu(1, 1|2)$ and $psu(2|2)$ the $\mathbb{Z}_4$ automorphism is generated by conjugation $M \rightarrow \Omega(M) \equiv \Omega^{-1} M \Omega$ with the matrix

$$\Omega = \begin{pmatrix} \sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix}. \quad (3.4)$$

This conjugation respects the anti-hermiticity conditions given above and manifestly gives an algebra automorphism. In addition, the invariant subalgebra $\Omega(M) = M$ is the desired bosonic $u(1) \oplus u(1)$ algebra. Finally, $\Omega^4(M) = M$.

Since this $\mathbb{Z}_4$ automorphism will play a key role in our construction, it is of interest to show that it is also present for other $AdS_d \times S^d$ spaces of relevance. For example, $AdS_5 \times S^5$ appears as the bosonic part of the super-quotient $PSU(2, 2|4)/SO(4, 1) \times SO(5)$. In general, the Lie superalgebra $psu(n, n|2n)$ has a $\mathbb{Z}_4$ automorphism whose invariant locus is $usp(n, n) \times usp(2n)$ [20]. The anti-hermiticity condition for $psu(n, n|2n)$ is as follows

$$M^\dagger \equiv \begin{pmatrix} \Sigma A^\dagger \Sigma & -i\Sigma Y^\dagger \\ -iX^\dagger \Sigma & B^\dagger \end{pmatrix} = -M \rightarrow A = -\Sigma A^\dagger \Sigma, \quad B = -B^\dagger, \quad X = i\Sigma Y^\dagger, \quad (3.5)$$

where $\Sigma = \sigma_3 \otimes I_n$ and $I_n$ is $(n \times n)$ identity matrix. Then the $\mathbb{Z}_4$ automorphism is generated by

$$M = \begin{pmatrix} A & X \\ Y & B \end{pmatrix} \rightarrow \Omega(M) \equiv \begin{pmatrix} JA^\dagger J & -JY^\dagger J \\ JX^\dagger J & JB^\dagger J \end{pmatrix}, \quad \text{where} \quad J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}. \quad (3.6)$$

---

8 Antihemitian (super)matrices form a superalgebra because $(MN)^\dagger = N^\dagger M^\dagger$. On block matrices dagger means transposition composed with ordinary complex conjugation ($\overline{\epsilon_1 \epsilon_2} = \bar{\epsilon}_1 \bar{\epsilon}_2$).
While it is not expressed as conjugation, one can verify that it is a Lie algebra automorphism compatible with the antihermiticity condition (3.3). Moreover, its fourth power is the identity and the $\mathbb{Z}_4$ invariant locus is precisely $\text{usp}(n,n) \times \text{usp}(2n)$. When $n = 2$, $\text{su}(2,2) \sim \text{so}(4,2)$, $\text{usp}(2,2) \sim \text{so}(4,1)$, $\text{usp}(4) \sim \text{so}(5)$ (see [21]) and we recover the quotient that leads to $\text{AdS}_5 \times S^5$ geometry. Other group signatures can be discussed similarly.

The $\mathbb{Z}_4$ action can be used to decompose the Lie algebra $\mathcal{G}$ as

$$
\mathcal{G} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \,,
$$

where the subspace $\mathcal{H}_k$ is the eigenspace of the $\mathbb{Z}_4$ generator $\Omega$ with eigenvalue $(i)^k$. The subspaces $\mathcal{H}_1$ and $\mathcal{H}_3$ contain all the fermionic generators of the algebra $\mathcal{G}$ and $\mathcal{H}_2$ contains the bosonic generators of $\mathcal{G}$ that are not in the subalgebra $\mathcal{H}_0$. The definition of $\mathbb{Z}_4$ implies that the hermitian conjugate of $\mathcal{H}_1$ is $\mathcal{H}_3$ [14]. Given that $\mathbb{Z}_4$ is an automorphism of the Lie algebra, the decomposition $\oplus_k \mathcal{H}_k$ satisfies

$$
[\mathcal{H}_m, \mathcal{H}_n] \subset \mathcal{H}_{m+n} \, (\text{mod} \, 4) \,. 
$$

The bilinear form is also $\mathbb{Z}_4$ invariant and hence

$$
\langle \mathcal{H}_m, \mathcal{H}_n \rangle = 0 \, \text{unless} \, \, n + m = 0 \, (\text{mod} \, 4) \,. 
$$

Let us also mention that there is an alternative way to impose a hermiticity condition based on a second way to define complex conjugation on Grassmann variables. This conjugation, denoted $\#$, satisfies [22]: $(c\epsilon)^\# = \bar{c} \epsilon^\#$, $(\epsilon^\#)^\# = (-)^{\text{deg}(\epsilon)} \epsilon$, and $(\epsilon_1 \epsilon_2)^\# = \epsilon_1^\# \epsilon_2^\#$. The alternative antihermiticity condition is

$$
M^+ \equiv \begin{pmatrix} A^+ & Y^+ \\ -X^+ & B^+ \end{pmatrix} = -M \quad \rightarrow \quad A = -A^+ \, , \, B = -B^+ \, , \, X = -Y^+ 
$$

where on block matrices $^+$ denotes transposition composed with $\#$-conjugation. Note that $(MN)^+ = N^+ M^+$. The above antihermiticity condition is also compatible with the $\mathbb{Z}_4$ action (3.6) and restricts $\text{psl}(2n|2n)$ (over the complex) to $\text{psu}(2n|2n)$ and the invariant locus to $\text{usp}(2n) \times \text{usp}(2n)$ ($\text{usp}(4) \sim \text{so}(5)$ for $n = 2$).

The eigenvectors in $\mathcal{H}_1$ and $\mathcal{H}_3$ thus give a complex basis for the real Lie algebra $\mathcal{G}$.
To illustrate these ideas concretely, we present the basis for the \( psu(2|2) \) algebra explicitly and the \( \mathbb{Z}_4 \) decomposition. The Lie brackets of the algebra are

\[
[K_{\mu\nu}, K_{\rho\sigma}] = \delta_{\mu\rho} K_{\nu\sigma} - \delta_{\mu\sigma} K_{\nu\rho} - \delta_{\nu\rho} K_{\mu\sigma} + \delta_{\nu\sigma} K_{\mu\rho},
\]

\[
[K_{\mu\nu}, S_{\rho\alpha}] = \delta_{\mu\rho} S_{\nu\alpha} - \delta_{\nu\rho} S_{\mu\alpha},
\]

\[
\{S_{\mu\alpha}, S_{\nu\beta}\} = \frac{1}{2} \epsilon_{\mu\rho\sigma\alpha} \epsilon_{\alpha\beta} K^{\rho\sigma},
\]

where the indices \( \mu, \nu, \rho, \sigma = (0, \ldots, 3) \) and \( \alpha, \beta = (1, 2) \). The invariant bilinear form in the algebra is

\[
\langle K_{\mu\nu}, K_{\rho\sigma} \rangle = \epsilon_{\mu\rho\sigma\alpha} \epsilon_{\alpha\beta}, \quad \langle S_{\mu\alpha}, S_{\nu\beta} \rangle = \delta_{\mu\nu} \epsilon_{\alpha\beta}.
\]

(3.12)

Denoting \( S_{\mu} \equiv S_{\mu 1}, \tilde{S}_{\mu} \equiv S_{\mu 2} \), one obtains for the \( \mathbb{Z}_4 \) invariant subspaces:

\[
\mathcal{H}_0 = \{K_{03}, K_{12}\}
\]

\[
\mathcal{H}_1 = \{S_{\mu} + i\tilde{S}_{\mu} \mid \mu = 0, \ldots, 3\}
\]

\[
\mathcal{H}_2 = \{K_{01}, K_{02}, K_{13}, K_{23}\}
\]

\[
\mathcal{H}_3 = \{S_{\mu} - i\tilde{S}_{\mu} \mid \mu = 0, \ldots, 3\}.
\]

(3.13)

The reader can explicitly check that this decomposition satisfies (3.9) and (3.10).

3.2. Sigma model action

The easiest way to construct a sigma model action on the coset space is by gauging the subgroup \( H \) whose Lie algebra is \( \mathcal{H}_0 \) (which will sometimes be simply called \( \mathcal{H} \)). Let \( g(x) \in G \) describe the map from the worldsheet into the group \( G \). The current \( J = g^{-1} dg \) is valued in the Lie algebra \( \mathcal{G} \). Introducing the gauge field \( A \) taking values in \( \mathcal{H}_0 \), one can define a gauged action

\[
S[G, A] = \frac{1}{4\pi\lambda^2} \int d^2x \ Str(J - A)^2.
\]

(3.14)

It is convenient to decompose the current \( J \) into two pieces \( J^{(0)} \in \mathcal{H}_0 \) and \( J' \in (\mathcal{G} \setminus \mathcal{H}_0) = \mathcal{H}' \). There is a natural metric on \( \mathcal{G} \), given by \( (A, B) = Str(AB) \) which allows us to make a canonical choice of \( J' \) such that \( Str(J'J^{(0)}) = 0 \). Now consider the gauge transformation \( g(x) \rightarrow g(x)h(x) \). Taking into account that \( [\mathcal{H}_0, \mathcal{H}'] \subset \mathcal{H}' \) (the subgroup \( H \) is reductive) we get that \( J' \) transforms by conjugation \( J' \rightarrow h^{-1}J'h \). Meanwhile, the current \( J^{(0)} \) transforms inhomogeneously as \( J^{(0)} \rightarrow h^{-1}J^{(0)}h + h^{-1}dh \). The inhomogeneous term in
this transformation cancels the inhomogeneous transformation for the gauge field $A$, such that the action (3.14) remains invariant.

Integrating out the gauge field $A$, we obtain an action for the sigma model on a coset space

$$S_{G/H}^{(0)} = \frac{1}{4\pi \lambda^2} \int d^2x \text{Str}(J'^2).$$ (3.15)

It is clear that this action is gauge invariant with respect to the gauge transformation $g(x) \to g(x)h(x)$ and therefore is defined on the coset space $G/H$. The group $G$ acts on the coset space by global left multiplication, namely for coset representatives $[g]$ and group element $g_0 \in G$ we have $g_0 : [g] \to [g_0g]$. It is clear that $J'$ (as well as $J$) is invariant under this transformation. Observe that the action (3.15) can be easily generalized by replacing the $\text{Str}()$ by any $\text{ad}(H)$ invariant metric $b(J',J')$. The invariance of the metric implies that the action would still possess the same gauge invariance.

The sigma model action (3.15), however, cannot be the right string worldsheet action since this sigma model is not conformal. At one loop, a counter term proportional to the Ricci tensor gets generated. As we will see in Section 5, the Ricci tensor for the quotient $PSU(1,1|2)/U(1) \times U(1)$ is non-zero.

4. WZ term and conformal invariance

In order to make the theory conformal, we modify the action by adding an extra term. This term will be a special version of a WZ term. Let us first remind the reader the structure of the WZ term for a supergroup. For a (super) group $G$, the WZ term arises from a closed 3-form. Indeed, the three form reads

$$\tilde{\Omega} = \text{Str} (J \wedge [J \wedge J]) = f_{MNK} J^M \wedge J^N \wedge J^K,$$ (4.1)

where $J^A$ are left-invariant one forms defined as $g^{-1}dg = \sum_M J^M T_M$, $f_{MNK} \equiv g_{MP} f_{N^K}$ are the totally (graded) antisymmetric structure constants, and $g_{MN}$ is a $G$ invariant bilinear form on $G$. The WZ term is simply the integral of the pullback of $\tilde{\Omega}$ over a three manifold whose boundary is the world sheet. One can readily verify that $d\tilde{\Omega} = 0$ making use of the Maurer Cartan identities

$$dJ^K = -\frac{1}{2} [J \wedge J]^K = -\frac{1}{2} f^K_{MN} J^M \wedge J^N,$$ (4.2)
and the Jacobi identity for $G$. The three form should be closed, so it is at least locally exact and thus can arise from a two-form $B$.

The possibility of writing the WZ term for the $G/H$ coset arises due to the very special nature of the subgroup $H \subset G$. The group $G$ admits a $\mathbb{Z}_4$ automorphism and the subgroup $H$ is the fixed locus of this action (see for example (3.4), (3.6)). This is a general statement which is valid for any coset $PSU(n, n|2n)/USp(n, n) \times USp(2n)$. In this paper, we will mainly deal with the $AdS_2 \times S^2$ case ($n = 1$) and make some comments on $n = 2$.

Let us denote the projections of the 1-form $J$ on the corresponding subspaces $\mathcal{H}_k$ of (3.8) as
$$J^{(i)} = J|_{\mathcal{H}_i}. \quad (4.3)$$

The naive restriction of the three-form $\mathcal{O}$ to $G/H$ (obtained by letting indices run only over coset values) is not even closed\(^{11}\). Still, one can write a WZ term using the $\mathbb{Z}_4$ decomposition of the algebra. This WZ term is precisely that of [5] (which is a straightforward generalization of the WZ terms in [3,4,6]) and can be written as
$$\Omega = Str\bigl(\left[ J^{(1)} \wedge J^{(3)} \right] \wedge J^{(2)} - \left[ J^{(3)} \wedge J^{(3)} \right] \wedge J^{(2)} \bigr) . \quad (4.4)$$

This 3-form is closed and its variation is exact. Let us introduce the indices $\{i, j, k, \cdots\}$ for $\mathcal{H}_0$, $\{a, b, c, \cdots\}$ for $\mathcal{H}_1$, $\{l, m, n, \cdots\}$ for $\mathcal{H}_2$, and $\{a', b', c', \cdots\}$ for $\mathcal{H}_3$. Then the 3-form $\Omega$ (4.4) can be rewritten as
$$\Omega = f_{mab} J^m \wedge J^a \wedge J^b - f_{ma'b'} J^m \wedge J^{a'} \wedge J^{b'} . \quad (4.5)$$

Using the Maurer-Cartan identities (4.2) for the one-forms $J^a$ and $J^{a'}$, it is easy to check that the WZ 3-form $\Omega$ given above is $d$-exact
$$\Omega = d \ Str(J^{(1)} \wedge J^{(3)}) = d (g_{aa'} J^a \wedge J^{a'}) \equiv d \Omega^{(2)}. \quad (4.6)$$

The full action can now be written as the sum of the kinetic term (3.15), and the (two-dimensional) integral of $\Omega^{(2)}$

$$S_{G/H} = S_{G/H}^{(0)} + \frac{ik}{2\pi \lambda^2} \int \Omega^{(2)} = \frac{1}{2\pi \lambda^2} \int d^2x Str\left( \frac{1}{2} J^{(2)}_{\mu} J^{(2)}_{\mu} + J^{(1)}_{\mu} J^{(3)}_{\mu} + ik \epsilon_{\mu\nu J^{(1)}_{\mu} J^{(3)}_{\nu}} \right) . \quad (4.7)$$

\(^{11}\) The rules for working with forms and exterior derivatives on homogeneous spaces $G/H$ were reviewed in [23].
where \( k \) will be determined later\(^{12} \). Let us stress that this action differs from that of \( B \) since it contains a kinetic term for fermions. This kinetic term breaks \( \kappa \)-symmetry, however, instead it allows \( N = 2 \) worldsheet superconformal invariance. To construct the \( N = 2 \) superconformal generators, it is useful to rewrite the action as (with \( \epsilon^{01} = +1 \))

\[
S_{G/H} = \frac{1}{\pi \lambda^2} \int d^2 x \; \text{Str} \left( J_z^{(2)} J_{\bar{z}}^{(2)} + (1 + k) J_z^{(1)} J_{\bar{z}}^{(3)} + (1 - k) J_z^{(1)} J_{\bar{z}}^{(3)} \right).
\] (4.8)

Although the coefficient \( k \) can be fixed by requiring one-loop conformal invariance, we can try to guess the appropriate value of \( k \) by examining the classical equations of motion. Taking into account that the variation of \( J \) satisfies the equation \( \delta J = d \delta X + [J, \delta X] \), we obtain

\[
D_z J_{\bar{z}}^{(3)} + D_{\bar{z}} J_z^{(3)} + k(\partial_z J_{\bar{z}}^{(3)} + [J_z^{(1)}, J_{\bar{z}}^{(2)}] + [J_z^{(0)}, J_{\bar{z}}^{(3)}] - (z \leftrightarrow \bar{z})) = 0
\]

\[
D_z J_{\bar{z}}^{(1)} + D_{\bar{z}} J_z^{(1)} + k(\partial_z J_{\bar{z}}^{(1)} + [J_z^{(3)}, J_{\bar{z}}^{(2)}] + [J_z^{(0)}, J_{\bar{z}}^{(1)}] - (z \leftrightarrow \bar{z})) = 0
\] (4.9)

\[
D_z J_{\bar{z}}^{(2)} + D_{\bar{z}} J_z^{(2)} + 2k([J_z^{(3)}, J_{\bar{z}}^{(3)}] - [J_z^{(1)}, J_{\bar{z}}^{(1)}]) = 0.
\]

For \( k = \pm 1/2 \), a significant simplification happens. Using a linear combinations of the above equations and Maurer-Cartan identities \( \partial_z J_{\bar{z}} - \partial_{\bar{z}} J_z + [J_z, J_{\bar{z}}] = 0 \) and setting \( k = 1/2 \) we obtain

\[
D_z J_{\bar{z}}^{(3)} = 0 , \quad D_z J_{\bar{z}}^{(1)} = 0 ,
\]

\[
D_{\bar{z}} J_z^{(2)} - [J_z^{(1)}, J_{\bar{z}}^{(1)}] = 0 ,
\]

\[
D_z J_z^{(2)} + [J_z^{(3)}, J_{\bar{z}}^{(3)}] = 0 ,
\] (4.10)

where the covariant derivative is defined as \( D_\mu = \partial_\mu + [J^{(0)}_\mu, ] \). So for \( k = 1/2 \), the current \( J^{(1)}_z \) is covariantly holomorphic and the current \( J^{(3)}_{\bar{z}} \) is covariantly antiholomorphic. (The other choice of \( k = -1/2 \) would flip the roles of \( J^{(1)} \) and \( J^{(3)} \).) This means that any \( H \)-invariant combination of \( J^{(1)} \) currents will be holomorphic when \( k = 1/2 \).

Note that when \( k = 1/2 \), the actions of (4.8) and (2.17) agree since \( \Pi^c \) is identified with \( J^{(2)} \), \( \Pi^\alpha \) and \( \Pi^\alpha \) are identified with \( J^{(1)} \), \( \Pi^\alpha \) and \( \Pi^\alpha \) are identified with \( J^{(3)} \). Furthermore, the four elements of \( J^{(1)} \) decompose under \( H = U(1) \times U(1) \) as \( (J^{(1)}_+, J^{(1)}_-, J^{(1)}_+, J^{(1)}_-) \) where

\[\text{The coefficient } i \text{ in (4.7) makes the contribution of the WZ term to the Euclidean worldsheet action real. This is a little unusual but is forced by conformal invariance (which will imply that } k \text{ is real). Moreover, the field } B \text{ has non zero components } B_{\alpha \alpha'} \text{ and } B_{\alpha' \alpha} \text{ only in fermionic directions and we are not aware of any quantization condition that would require this term to be imaginary.}\]
the first two are identified with $\Pi^\alpha$ and the second two are identified with $\Pi^{\dot{\alpha}}$. So we can define two classically holomorphic, $H$ invariant combinations of $J^{(1)}$ which are

$$ A^+ = J^{(1)}_{++} J^{(1)}_{--}, \quad A^- = J^{(1)}_{+-} J^{(1)}_{-+}, $$

(4.11)

where we dropped the $z, \bar{z}$ indices. We conjecture that the 2d QFT theory given by (4.8) is exactly conformal for $k = 1/2$ and that the composite fields $A^+(z)$ and $A^-(z)$ remain holomorphic at the quantum level. This could be checked by perturbative calculations. Furthermore, we conjecture that the OPE’s of $A^+$ and $A^-$ form some version of a $W$-algebra

$$ A^+(z) A^-(w) = -\frac{1}{(z-w)^4} + \frac{1}{(z-w)^2} T + \frac{1}{(z-w)} W_3, $$

(4.12)

where $T$ is the stress-tensor and $W_3$ is some new spin-3 current. The OPE’s of $A^+$ and $A^-$ with themselves should be regular. The conformal anomaly of this $W$-algebra coincides with the superdimension of the coset space which is $c = -4$. It follows from our calculations of the effective action that the $W$-algebra persist at one-loop level. Remarkably, it is closely related to an $N = 2$ superconformal algebra and can easily be converted into one by adding an extra chiral boson $\rho$. Let us fix the normalization of $\rho$ by requiring that $\langle \rho(z) \rho(w) \rangle = \log(z-w)$. Then the fields $e^{\pm i \rho}$ have dimension $\Delta = -1/2$ and the combinations $G^\pm = e^{\pm i \rho} A^\pm$ together with stress-energy tensor $T + \frac{i}{2} \partial \rho \partial \rho$ and $U(1)$ current $j = i \partial \rho$ generate a $c = -3$ $N = 2$ superconformal algebra. Adding any $N = 2$ $c = 9$ superconformal field theory for the CY manifold to this model produces a critical $N = 2$ $c = 6$ superconformal field theory. This superconformal field theory is precisely the one described in the previous section where $A^+ = (d)^2$ and $A^- = (\bar{d})^2$. It is related to the RNS version of the superstring by using the RNS stress tensor, BRST current, $b$ ghost, and ghost number current as twisted $N = 2$ generators, redefining the worldsheet variables, and then untwisting to get a critical $c = 6$ superconformal field theory.

4.1. One loop beta function

In this subsection we use the background field formalism to compute the one loop effective action and verify that there are no UV divergences for $k = \pm 1/2$. We write group elements as $g = \tilde{g} e^{\lambda X}$ where $\tilde{g}$ is the background field, $X \in \mathcal{G}$ parameterizes quantum fluctuations around $\tilde{g}$, and $\lambda$ is the coupling constant (i.e. the inverse radius of $AdS_2 \times S^2$) inserted here for convenience. Then the current $J_\mu$ can be written as

$$ J_\mu = g^{-1} \partial_\mu g = e^{-\lambda X} \bar{J}_\mu e^{\lambda X} + e^{-\lambda X} \partial_\mu e^{\lambda X}, $$

(4.13)

20
where $\tilde{J}_\mu = \tilde{g}^{-1} \partial_\mu \tilde{g}$ is the background current. The action for the coset space is given by (4.7) which in our parameterization becomes

$$S_{G/H} = \frac{1}{2\pi \lambda^2} \int \text{Str} \left( \frac{1}{2} \left( e^{-\lambda X} \tilde{J}_\mu e^{\lambda X} \big|_{G\setminus H_0} + e^{-\lambda X} \partial_\mu e^{\lambda X} \big|_{G\setminus H_0} \right)^2 + + i e^{\mu \nu} \left( e^{-\lambda X} \tilde{J}_\mu e^{\lambda X} + e^{-\lambda X} \partial_\mu e^{\lambda X} \big|_{H_1} \left( e^{-\lambda X} \tilde{J}_\nu e^{\lambda X} + e^{-\lambda X} \partial_\nu e^{\lambda X} \big|_{H_3} \right) \right).$$

(4.14)

The gauge invariance of the original action $g \to gh$ (where $h \in H$) allows us to choose a gauge for $X$ such that $X \in G \setminus H_0$. The gauge invariance of the effective action for the background field becomes manifest in this gauge for $X$. Under $\tilde{g} \to \tilde{g}h$ we simply change variables $X \to h^{-1}Xh$ in the functional integral. As the subgroup $H$ is reductive, $h^{-1}Xh \in G \setminus H_0$ and so we are still in the same gauge for $X$. The action (4.14) is invariant under such transformations. The integration measure is certainly invariant for $k = 0$ since there is no chirality. For all other $k$, we may use the same regulator as for $k = 0$ to see that the measure is invariant. The gauge invariance of the effective action guarantees that our theory makes sense on the coset space even quantum-mechanically.

To compute the effective action for our model, we first have to expand (4.14) in terms of $X$ and then evaluate all 1PI diagrams with external lines of the background currents $\tilde{J}$. To compute the beta function we need to renormalize UV divergent diagrams. The IR divergences can be dealt with in the standard fashion by adding a small mass term $\mu$ for $X$. By power counting, the UV primordially divergent diagrams can have no more than two external lines. Thus we need to evaluate those.

The expansion of (4.14) contains the zeroth order term in $X$ that is simply the action for the background field and the linear term in $X$ which does not contribute to 1PI diagrams. We are interested in terms of second order in $X$ which are the only ones we need to compute the one-loop beta function. There is the kinetic term for $X$ that is simply

$$\frac{1}{4\pi} \int d^2x \text{Str} \left( \partial_\mu X \right)^2,$n

(4.15)

and terms which include interactions between $\tilde{J}$ and $X$. These terms can be divided into three subsets: (i) terms containing $\tilde{J}^{(2)}$, (ii) terms containing $\tilde{J}_z^{(1)}$ and/or $\tilde{J}_z^{(3)}$ and (iii) terms containing $\tilde{J}_z^{(3)}$ and/or $\tilde{J}_z^{(1)}$.

13 One might also consider terms containing $\tilde{J}^{(0)}$, but the effective action should be independent of them since their appearance would indicate the breakdown of $H$ gauge invariance. Using methods similar to those described below, we explicitly checked that such terms were absent in the one-loop effective action.
Terms of type (i) are given by

\[ \frac{1}{\pi} \int d^2 x \ Str \left( \frac{1}{2} + k \partial X^{(1)} [\tilde{J}_z^{(2)}, X^{(1)}] + \left( \frac{1}{2} - k \right) \partial X^{(1)} [\tilde{J}_z^{(2)}, X^{(1)}] + \right. \\
\left. \left( \frac{1}{2} - k \right) \partial X^{(3)} [\tilde{J}_z^{(2)}, X^{(3)}] + \left( \frac{1}{2} + k \right) \partial X^{(3)} [\tilde{J}_z^{(2)}, X^{(3)}] + \\- k \tilde{J}_z^{(2)}[[\tilde{J}_z^{(2)}, X^{(1)}], X^{(3)}] + k \tilde{J}_z^{(2)}[[\tilde{J}_z^{(2)}, X^{(3)}], X^{(1)}] + \\+ \tilde{J}_z^{(2)}[[\tilde{J}_z^{(2)}, X^{(2)}], X^{(2)}] \right) . \]  

Equation (4.16)

The first four terms above contain a single background current and therefore give rise to fish-type divergent diagrams in second order of perturbation theory. Their combined contribution to the divergent piece is equal to

\[ l_m = \log(\Lambda/\mu) \tilde{J}_z^l \tilde{J}_z^m \left( \frac{1}{2} - k \right)^2 (f_{lab} f_{b'a'm} g^{bb'} g^{aa'}) + \left( \frac{1}{2} + k \right)^2 (f_{la'b'} f_{bam} g^{b'bg'a'}) \]  

Equation (4.17)

where \( \Lambda(\mu) \) denotes UV(IR) cutoff. Our convention for the indices and subspaces is the same as defined above equation (4.5). We put group-theory factors in parentheses to make comparison easier.

Similarly, the last three terms in (4.16) contain two background lines and also renormalize the propagator at one loop. Evaluating their contribution to the divergent piece, we obtain

\[ l \circ m = \log(\Lambda/\mu) \tilde{J}_z^l \tilde{J}_z^m \left( k (f_{lab} f_{b'a'm} g^{bb'} g^{aa'}) \right. \left. - k (f_{la'b'} f_{bam} g^{b'bg'a'}) + (f_{ik} f_{njm} g^{kn} g^{ij}) \right) , \]  

Equation (4.18)

The first two terms above have the same group factors as those in (4.17), so adding them makes the coefficients in front of both equal to \( 1/4 + k^2 \). Their sum can be written in terms of the group factor in the third term of (4.18) since the groups we choose for our coset constructions have vanishing dual Coxeter number, which implies, in particular that the \( (l, m) \) components of the Cartan-Killing form vanish:

\[ 0 = f_{lab} f_{b'a'm} g^{bb'} g^{aa'} + f_{la'b'} f_{bam} g^{b'bg'a'} + 2 f_{ik} f_{njm} g^{kn} g^{ij} . \]  

Equation (4.19)
Adding the contributions from both types of diagrams and using (4.13), we find that the sum is equal to
\[
\left( \frac{1}{2} - 2k^2 \right) \log(\Lambda/\mu) \tilde{J}_z^a \tilde{J}_z^{a'} (f_{lik} f_{njm} g^{kn} g^{ij}) .
\] (4.20)
Thus, there is no renormalization only when \( k = \pm 1/2 \), which are the same values of \( k \) that we guessed using the classical equations of motion (4.9).

Repeating the same steps for terms of type (ii) gives
\[
\frac{1}{\pi} \int d^2 x \ \text{Str} \left( \left( \frac{1}{2} - \frac{k}{2} \right) \partial X^{(2)} [\tilde{J}_z^{(3)}, X^{(3)}] + \left( \frac{1}{2} - \frac{3k}{2} \right) \partial X^{(3)} [\tilde{J}_z^{(3)}, X^{(2)}] + \left( \frac{1}{2} - \frac{k}{2} \right) \partial X^{(2)} [\tilde{J}_z^{(1)}, X^{(1)}] + (k + 1) \tilde{J}_z^{(1)} [\tilde{J}_z^{(3)}, X^{(1)}]\right).
\] (4.21)
The fish-type diagrams give the divergent contribution
\[
\frac{1}{\pi} \int d^2 x \ \text{Str} \left( \left( \frac{1}{2} - \frac{3k}{2} \right) \partial X^{(2)} [\tilde{J}_z^{(1)}, X^{(3)}] + \left( \frac{1}{2} - \frac{k}{2} \right) \partial X^{(3)} [\tilde{J}_z^{(1)}, X^{(2)}] + (k + 1) \tilde{J}_z^{(1)} [\tilde{J}_z^{(3)}, X^{(1)}]\right).
\] (4.22)
while the divergent contribution from the last three terms in (4.21) is
\[
\frac{1}{\pi} \int d^2 x \ \text{Str} \left( \left( \frac{1}{2} - \frac{3k}{2} \right) \partial X^{(2)} [\tilde{J}_z^{(1)}, X^{(3)}] + \left( \frac{1}{2} - \frac{k}{2} \right) \partial X^{(3)} [\tilde{J}_z^{(1)}, X^{(2)}] + (k + 1) \tilde{J}_z^{(1)} [\tilde{J}_z^{(3)}, X^{(1)}]\right).
\] (4.23)
Again, the first term above has the same group structure as in (4.22) while the second can be expressed through it using vanishing of the \( a, a' \) components of the Cartan-Killing form in a way similar to (4.19). Summing both types of diagrams in this case gives
\[
(2k^2 - \frac{1}{2}) \log(\Lambda/\mu) \tilde{J}_z^a \tilde{J}_z^{a'} (f_{amb} f_{b'n'a'} g^{bb'} g^{mn}) .
\] (4.24)
which vanishes for the same values of \( k = \pm 1/2 \). Terms of type (iii) in the expansion of the action can be dealt with in the same manner and with the same conclusion that \( k = \pm 1/2 \).

One can immediately deduce from the above results that the one loop beta function for \( 1/\lambda^2 \) is proportional to \( (2k^2 - \frac{1}{2}) \), while the renormalization of \( k \) is identically equal to zero independently of the values of \( \lambda \) and \( k \). We therefore conclude that the couplings \( \lambda \) and \( k \) are not renormalized at one loop. In fact, we believe that the theory given by (4.8) is exactly conformal for \( k = \pm 1/2 \) and we plan to address this issue in the near future.
5. Ricci Curvature, Field Equations and Conformal Invariance

Here we study the geometrical properties of coset supermanifolds. We first show how to compute the Ricci curvature of homogeneous supermanifolds \( G/H \) in terms of structure constants of the associated Lie algebras. In doing this, we will explain the role of the chosen Lie algebra metric. This will illuminate the metric interpretation of the work of [1] and [2]. Then we compute the curvature of \( G/H \) in terms of the curvature of \( G \) plus additional terms. These formulas will confirm that \( PSU(1,1|2)/U(1) \times U(1) \) and \( PSU(2,2|4)/SO(4,1) \times SO(5) \) are not flat and therefore the corresponding sigma models require some modification for conformal invariance.

In the second subsection, we present another proof of one-loop conformal invariance of our model. We show that the background fields implied by the kinetic term and WZ term satisfy the appropriate target space field equations that arise from the conditions of one-loop conformal invariance. This approach makes use of the geometrical properties of the coset in question. We also show that there is no two-loop correction to the central charge.

5.1. Ricci curvature and Lie algebra metrics

We consider here metrics on homogeneous supermanifolds obtained as coset spaces \( G/H \) where \( G \) is a Lie supergroup and \( H \) is a sub(super)group of \( G \). Our discussion is a generalization of the familiar results for bosonic cosets [24,25] to the super case. A useful reference is [26].

We focus throughout on reductive coset spaces, i.e. spaces where the Lie algebra \( G \) can be decomposed as a direct vector space sum of the Lie algebra \( H \) of \( H \) and an \( \text{ad}(H) \) invariant space \( G \setminus H \). We choose a basis \( \{ T_M \} \) of generators for \( G \), and use the letters \( \{ M, N, P, \ldots \} \) to index the generators of \( G \), the letters \( \{ I, J, K \ldots \} \) to index the generators of \( H \) and \( \{ A, B, C \ldots \} \) for the elements of \( G \setminus H \).

Assume that \( G \) has an \( \text{ad}(G) \) invariant bilinear form \( (T_M, T_N) = b_{MN} \) whose restriction to \( H \) is non-degenerate and thus can be used to produce a reductive orthogonal decomposition \( G = H \oplus (G \setminus H) \). The homogeneous space admits a \( G \)-invariant metric arising from the restriction of the bilinear form \( b \) to \( G \setminus H \). The following result then holds for the curvature of this metric ([24], vol. 2, p. 203):

\[
R_{ABDC} + (B \leftrightarrow D) = \frac{1}{4} f^E_{AB} b_{EF} f^F_{CD} + f^I_{AB} b_{IJ} f^J_{CD} + (B \leftrightarrow D) . \tag{5.1}
\]
It is clear that the Riemann curvature tensor does depend on the choice of bilinear $b$. To find an expression for the Ricci curvature, we contract with the inverse metric $b^{BD}$

$$R_{AC} = \left( \frac{1}{4} f_{AB}^E b_{EF} f_{CD}^F + f_{AB}^I b_{IJ} f_{CD}^J \right) b^{BD} = \frac{1}{4} f_{AD}^E f_{BE}^D - f_{AD}^I f_{CI}^D \quad (5.2)$$

The bilinear form drops out of the Ricci tensor as a consequence of its group invariance and the orthogonality of $G \setminus H$ and $H$. This is the desired result for the Ricci curvature of an ordinary homogeneous manifold $G/H$. We now claim that for a super coset manifold we simply need to replace one of the contractions by a supertrace:

$$R_{AB} (G/H) = -\frac{1}{4} f_{AD}^E f_{BE}^D (-)^E - f_{AD}^I f_{CI}^D (-)^I . \quad (5.3)$$

It should be noted that structure constants of a Lie superalgebra are commuting numbers and thus $f_{PQ}^M = 0$ unless $\epsilon(M) = \epsilon(P) + \epsilon(Q) (\text{mod } 2)$. It follows that $R_{AB}$ vanishes unless both $A$ and $B$ are commuting or both are anticommuting. As a consistency check of the above equation, one can verify the expected exchange symmetry $R_{AB} = (-)^{AB} R_{BA}$, and that (5.3) agrees with results derived in [26]. Note that as in the bosonic case, the Ricci curvature is independent of the bilinear form. When $H$ is the identity, the result (5.3) reduces to

$$R_{MN}(G) = -\frac{1}{4} f_{MQ}^P f_{NP}^Q (-)^P = -\frac{1}{4} \text{Str} (T_M T_N) = \frac{1}{4} \kappa_{MN} , \quad (5.4)$$

where $T_M$ are the generators of $G$ in the adjoint representation $^P (T_M)_N = f_{MN}^P$ and $\kappa$ is the Cartan-Killing metric on $G$ (positive definite for compact semisimple Lie algebras). This is an important fact: the Ricci curvature of a $G \times G$ invariant metric on $G$ depends only on the Cartan-Killing form $\kappa$ on $G$. This confirms the metric interpretation of the computation of [1,2]. Their $PSU(1,1|2)$ metric arises from the defining representation of $SU(1,1|2)$ but its Ricci curvature is still given by the Killing form which vanishes.

5.2. Further results and the curvature of $PSU(1,1|2)/U(1) \times U(1)$

The Ricci curvature of a group $G$, as given in (5.4), takes a useful form when $G$ has a subgroup $H$. A small calculation making use of exchange symmetries gives

$$R_{AB}(G) = -\frac{1}{4} f_{AN}^M f_{BM}^N (-)^M = -\frac{1}{4} f_{AD}^E f_{BE}^D (-)^E - \frac{1}{2} f_{AD}^I f_{BI}^D (-)^I , \quad (5.5)$$
where we use the index convention of the previous subsection. If $H$ defines a symmetric subgroup (i.e. $[G \setminus H, G \setminus H] \subset H$), then

$$\mathcal{R}_{AB}(G) = -\frac{1}{2} f^I_{AD} f^D_{BI} (-)^I .$$

(5.6)

Combining (5.3) with (5.5), we find the following two expressions for the Ricci curvature of a coset:

$$\mathcal{R}_{AB}(G/H) = \mathcal{R}_{AB}(G) - \frac{1}{2} f^I_{AD} f^D_{BI} (-)^I ,$$

(5.7)

$$\mathcal{R}_{AB}(G/H) = 2 \mathcal{R}_{AB}(G) + \frac{1}{4} f^E_{AD} f^D_{BE} (-)^E .$$

(5.8)

Note that the extra terms involve sums over $H$ indices in (5.7) and involve sums over $G \setminus H$ indices in (5.8).

It will also be useful to have an expression for the scalar curvature $\mathcal{R}(G/H)$ of the coset in terms of the scalar curvature $\mathcal{R}(G)$ and the scalar curvature $\mathcal{R}(H)$. Decomposing the scalar curvature of $G$ in terms of subgroup and coset indices, we find

$$\mathcal{R}(G) = \mathcal{R}_{IJ}(G) g^{JI} + \mathcal{R}_{AB}(G) g^{BA} ,$$

$$= -\frac{1}{4} f^E_{ID} f^D_{EI} (-)^E g^{JI} - \frac{1}{4} f^K_I f^L_K (-)^L g^{JI} + \mathcal{R}_{AB}(G) g^{BA} ,$$

(5.9)

$$= -\frac{1}{4} f^E_{ID} f^D_{EI} (-)^E g^{JI} + \mathcal{R}(H) + \mathcal{R}_{AB}(G) g^{BA} .$$

On the other hand, we also have

$$-\frac{1}{4} f^E_{ID} f^D_{EI} (-)^E g^{JI} = -\frac{1}{4} f^I_{AD} f^D_{BI} (-)^I g^{BA} ,$$

(5.10)

$$= \frac{1}{2} \mathcal{R}(G/H) - \frac{1}{2} \mathcal{R}_{AB}(G) g^{BA} ,$$

where the first step is proved by raising and lowering indices using $f_{FID} = g_{FEf} f^E_D$, $f^E_{ID} = (-)^F g^{EF} f_{FID}$, and the total graded antisymmetry of the structure constants with three lower indices. The second step made use of (5.7). Finally, using (5.9), we find that

$$\mathcal{R}(G) = \frac{1}{2} \mathcal{R}(G/H) + \frac{1}{2} \mathcal{R}_{AB}(G) g^{BA} + \mathcal{R}(H) .$$

(5.11)

Let us now consider the computation of the Ricci curvature for the special cases. When $G/H$ is a symmetric coset space, we find from (5.8) that

$$\mathcal{R}_{AB}(G/H) = 2 \mathcal{R}_{AB}(G) [G/H \text{ symmetric}] .$$

(5.12)
So if we take a Ricci flat supergroup and divide by a symmetric subgroup, we get a Ricci flat coset, thus a one-loop conformal sigma model. The coset space \( G/H = PSU(1, 1|2)/U(1) \times U(1) \), however, is not symmetric. Because \( H \) does not contain fermionic generators while there are bosonic generators in \( G \setminus H \). Indeed, the bracket of a fermionic generator and any bosonic generator in \( G \setminus H \) would be a fermionic generator in \( G \setminus H \). Thus, there is no reason to expect this coset space to be Ricci flat. To show that the curvature of \( G/H = PSU(1, 1|2)/U(1) \times U(1) \) is definitely non-vanishing, note that \( G_B/H \) is a symmetric space, where \( G_B = SU(2) \times SU(2) = S^3 \times S^3 \) denotes the bosonic subgroup of \( G \). When \( G \) is arbitrary, \( H \) is bosonic, and \( G_B/H \) is symmetric, (5.7) and (5.6) gives

\[
R_{AB}(G/H) = R_{AB}(G_B) = R_{AB}(S^3 \times S^3) \neq 0 ,
\]

(5.14)
since the Ricci curvature of round spheres is necessarily nonvanishing. A similar result holds for \( PSU(2, 2|4)/SO(4, 1) \times SO(5) \), so it also has non-vanishing Ricci curvature.

5.3. Field equations

We begin our analysis by recalling the familiar background field equations that arise from the conformal invariance conditions on two dimensional bosonic sigma models. When the target space is a manifold with metric \( G_{\mu\nu} \), antisymmetric field \( B_{\mu\nu} \) and dilaton \( \Phi \), the sigma model action takes the form \([27]\)

\[
S = \frac{1}{4\pi\alpha'} \int d\sigma d\tau \sqrt{g} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) \\
+ \frac{1}{4\pi\alpha'} \int d\sigma d\tau \sqrt{g} \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu B_{\mu\nu}(X) \\
+ \frac{1}{8\pi} \int d\sigma d\tau \sqrt{g} R(\gamma) \Phi(X) .
\]

(5.15)
The conditions of conformal invariance imply that background fields satisfy the equations

\[
\beta^G_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} H^2_{\mu\nu} + 2\nabla_\mu \nabla_\nu \Phi = 0 ,
\]

\[
\beta^B_{\mu\nu} = \frac{1}{2} \nabla^\lambda H_{\lambda\mu\nu} - \nabla^\lambda \Phi H_{\lambda\mu\nu} = 0 ,
\]

(5.16)

\[
\beta^\Phi = \frac{D - 26}{6} + \frac{\alpha'}{2} \left( -R + \frac{H^2}{12} + 4(\nabla \Phi)^2 - 4\nabla^2 \Phi \right) = 0 ,
\]

(5.17)
where $\beta^G$ and $\beta^B$ have been calculated to one loop, while the $\alpha'$ term in $\beta^\Phi$ arises at two loops.

In the case of the bosonic sigma model, the condition of vanishing beta function implies the target-space equations of motion. Here we want to use this statement when the target space is a supermanifold. Formally, one only needs to replace the bosonic indices $\mu, \nu \cdots$ by indices $A, B, \cdots$ running over bosonic and fermionic values\[14\]. Being a part of string theory, the $AdS_n \times S^n$ sigma model should be supplemented by a set of ghost CFT’s and an internal space CFT as described in the previous sections. This cancels the conformal anomaly, implying that the term $(D - 26)$ in $\beta^\Phi$ can be set to zero. Moreover, for the $AdS_d \times S^d$ model, we expect to have a constant dilaton so the equations of motion become

$$
\beta^G_{MN} = R_{MN} + \frac{1}{4} H^P_{MQ} H^Q_{NP} (-)^P = 0, \\
\beta^B_{NP} = (-)^M \nabla^M H_{MN} = 0, \\
\beta^\Phi = -\frac{\alpha'}{3} R = 0.
$$

(5.17)

The first two equations are conditions of one-loop conformal invariance while the third equation guarantees that the conformal anomaly is not renormalized at two loops. In fact, these equations are too restrictive and can be relaxed without breaking worldsheet conformal invariance. Namely, the r.h.s. of these equations can be replaced by the gauge transform of the corresponding fields (i.e. the transformation of the $G$ and $B$ field under diffeomorphisms plus the gauge transformation of the $B$ field).

Let us begin with the Einstein equation, the first of the above. For a WZW model for a group manifold, comparison with (5.4) shows that setting $H^M_{NP} = f^M_{NP}$ makes the equation work. In our case of a coset manifold $G/H$, taking into account (5.8) and $R_{AB}(G) = 0$, the Einstein equation become

$$
R_{AB}(G/H) = \frac{1}{4} f^E_{AD} f^D_{BE} (-)^E = -\frac{1}{4} H^E_{AD} H^D_{BE} (-)^E.
$$

(5.18)

We see that a naive attempt to identify $H_{ABC}$ with structure constants would fail. We can examine our proposal of section 4 to identify the $H$ field. The WZ 3-form is given by (4.5) and we will repeat this expression here

$$
\Omega = H_{ABC} J^A \wedge J^B \wedge J^C, \\
= f_{mab} J^m \wedge J^a \wedge J^b - f_{ma'b'} J^m \wedge J^a' \wedge J^{b'}.
$$

(5.19)

\[14\] Sign factors that would disappear in the bosonic case are sometimes necessary, and can generally be found by considerations of symmetries under exchanges of indices.
This implies that the only non zero components of $H$ are $H_{mab} = f_{mab}$, $H_{ma'b'} = -f_{ma'b'}$, and those with the indices $m, a, b$ or $m, a', b'$ being permuted. Indeed, the field strength $H$ equals the structure constants only up to some crucial sign factors. Raising indices we find

$$
\begin{align*}
H_{mb}^{a'} & = f_{mb}^{a'}, & H_{mb'}^{a} & = -f_{mb'}^{a}, \\
H_{ab}^{m} & = f_{ab}^{m}, & H_{a'b}^{m} & = -f_{a'b}^{m}.
\end{align*}
$$

(5.20)

It is easy to check that with the $H$’s listed above (5.18) is satisfied: With indices only running over the coset, and due to the $\mathbb{Z}_4$ grading, we can only have $(A, B) = (n, m)$, and $(A, B) = (a, b')$ (or vice versa). In both cases the products in the right hand side always involve an $H$ from the first column in (5.20) and an $H$ from the second column. This confirms that the Einstein equation (5.18) is satisfied.

We now verify the last equation in (5.17) – the scalar curvature $R$ of the coset supermanifold vanishes. This is readily done with the help of equation (5.11). Since the numerator group $G$ is Ricci flat, we have $R(G) = 0$ and $R_{AB}(G) = 0$. Moreover, the group $H$ is purely bosonic and corresponds to $AdS_2 \times S^2$ or their higher dimensional counterparts. While the Ricci curvature of $H$ does not vanish, its scalar curvature does. It follows from these facts that $R(G/H) = 0$, which is the desired result. The central charge of the coset model remains unchanged to two loops.

Finally, we turn to the second equation in (5.17); the field equation for the three-form. For the coset case it reads

$$
(-)^{A} \nabla^{A} H_{ABC} = g^{AF} \nabla_{F} H_{ABC} = g^{AF} (\partial_{F} H_{ABC} + \cdots),
$$

(5.21)

where the dotted terms denote contributions proportional to the Christoffel connection. Because of homogeneity, it is sufficient to check that (5.21) holds at the identity point of $G/H$. At this point, to be denoted as 0, the Christoffel coefficients vanish and we have

$$
(-)^{A} \nabla^{A} H_{ABC} \big|_{0} = g^{AF} (\partial_{F} H_{ABC})|_{0}.
$$

(5.22)

It will be useful to treat this left hand side as a two-form, introducing the notation

$$
\beta^{B} \equiv g^{AF} (\partial_{F} H_{ABC})|_{0} J^{B} \wedge J^{C}.
$$

(5.23)

We will now show that the $\mathbb{Z}_4$ automorphism of the group $G$ together with its Ricci flatness guarantees that $\beta^{B}$ is $d$-exact. This is sufficient for one-loop conformal invariance since it implies that the relevant counter term is a total derivative in the two-dimensional
effective action. Indeed, as it was shown in section 4, the $Z_4$ action and Ricci flatness are enough to ensure one-loop conformal invariance. Moreover, we will show that when $H$ is semisimple, $\beta^B$ vanishes identically. The cases of $AdS_n \times S^n$ for $n = 3, 5$ are of this type. The case $n = 2$ is different, but $\beta^B$ vanishes anyway.

To evaluate (5.23), we need the expansion of $H_{ABC}(X)$ to linear order in $X$. For this, one uses $J^A = dX^A - \frac{1}{2} f^A_{BC} X^B dX^C + \cdots$, and, $\Omega = H_{ABC}(X) dX^A \wedge dX^B \wedge dX^C$, which lead to

$$H_{ABC}(X) = H_{ABC} - \frac{1}{2} \left( H_{EBC} f^E_{FA} + (-)^{(A+B)E} H_{ABE} f^E_{FC} \right) X^F + O(X^2)$$

(5.24)

Replacing into (5.23) and noticing that $g^{FA} f_{AFE} = 0$, we find

$$\beta^B \sim g^{FA} \left( (-)^{(C+B)F} H_{ACE} f^E_{FB} + (-)^{(B+C)F} H_{ABE} f^E_{FC} \right) J^B \wedge J^C.$$  

(5.25)

For (5.25) to be non-vanishing, $Z_4$ symmetry implies that $(B, C) = (m, n)$, $(B, C) = (b', c)$ or $(B, C) = (b, c')$. For $(B, C) = (m, n)$, expanding out, substituting the values of $H$, and using the Jacobi identity, we get an answer proportional to $V_i f_{mn}^i J^m \wedge J^n$, where $V_i \equiv f_{ai}^a$. For the case $(C, B) = (c, b')$, using the Jacobi identity and the vanishing of $R_{cb'}(G)$, we find a single term proportional to $V_i f_{bc}^i J^b \wedge J^{c'}$. Evaluating the relative coefficients of the various terms, one finds

$$\beta^B \sim V_i \left( f_{mn}^i J^m \wedge J^n + f_{bc}^i J^b \wedge J^{c'} + f_{b'c}^i J^{b'} \wedge J^{c'} + f_{jk}^i J^j \wedge J^k \right),$$

(5.26)

where the last term has been added by hand using $V_i f_{jk}^i = f_{ai}^a f_{jk}^i = 0$, a fact that follows from the Jacobi identity of $G$. Now, using (4.2), one can rearrange (5.26) into the $d$-exact term:

$$\beta^B \sim V_i dJ^i = d \left( V_i J^i \right),$$

(5.27)

which is sufficient for one-loop conformal invariance since it implies that the worldsheet counter term is a total derivative, i.e. it can be removed by a gauge transformation of the $B$ field.

For the $AdS_2 \times S^2$ case, the r.h.s. of (5.27) vanishes since, using (3.13), one can check that $V_i = f_{ai}^a = 0$. For the higher dimensional $AdS$ cases the Lie algebra of the subgroup $H$ is semisimple and therefore every generator $T_i$ can be written as a commutator of other generators. As a result, the vanishing of $V_i$ is implied by the vanishing of $V_i f_{jk}^i$ discussed above. This concludes our verification of the target space equations of motion.
6. String theory in 6 and 10 dimensions

As we already mentioned in the previous section, our $AdS_2 \times S^2$ computations can be repeated for $AdS_3 \times S^3$ and $AdS_5 \times S^5$ backgrounds by replacing $PSU(1,1|2)/U(1) \times U(1)$ with $PSU(1,1|2) \times PSU(2|2)/SU(1,1) \times SU(2)$ or with $PSU(2,2|4)/SO(4,1) \times SO(5)$ and choosing the appropriate $Z_4$ action. The WZ term is given by the expression (4.4) in all of these cases. Although our results only prove one-loop conformal invariance, we conjecture that these actions are conformally invariant to all loops and therefore describe conformal field theories.

The remaining problem for the higher dimensional cases is to construct a consistent string theory from the conformal field theory of these coset spaces. In the six-dimensional case, the conformal anomaly of the coset model is equal to $c = -10$. As discussed in \[1,2\], there are two ghost-like chiral bosons which are necessary for $N = 2$ worldsheet superconformal invariance and which contribute $c = +2$ to the conformal anomaly. As will be shown in \[3\], the remaining $c = +8$ to cancel the anomaly comes from the ghosts for “harmonic” constraints which eliminate half of the fermionic currents and reduce the $PSU(1,1|2) \times PSU(2|2)/SU(2) \times SU(2)$ coset model to the $PSU(2|2)$ model discussed in \[1\] and \[2\]. In the ten-dimensional case, one would have to add a $c = +22$ ghost system to cancel the anomaly. Unfortunately, the lack of a covariant GS formalism in ten dimensions makes it unclear how to add this additional ghost system. To understand the string theory in $AdS_5 \times S^5$ background it may be important to analyze the $W$ algebra of the corresponding coset.

It might be interesting to try to formulate string theory on an arbitrary supermanifold. This idea is very simple but is not easy to implement. Sigma models on supermanifolds are generically non-unitary, so one needs to find a set of constraints (analogous to the $N = 2$ superconformal constraints of this paper) to remove the negative-norm states. Although such constraints are known for some special cases (e.g. those related to the RNS superstring by a field redefinition), they are unknown in general form.

We have given a formulation of string theory on an interesting four-dimensional background involving RR fields. We hope our results will pave the way to the construction of full string theories for the higher dimensional cases.

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33
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