Sharp hypotheses and bispatial inference

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Abstract: A fundamental class of inferential problems are those characterised by there having been a substantial degree of pre-data (or prior) belief that the value of a model parameter was equal or lay close to a specified value, which may, for example, be the value that indicates the absence of an effect. Standard ways of tackling problems of this type, including the Bayesian method, are often highly inadequate in practice. To address this issue, an inferential framework called bispatial inference is put forward, which can be viewed as both a generalisation and radical reinterpretation of existing approaches to inference that are based on P values. It is shown that to obtain a post-data density function for a given parameter, it is convenient to combine a special type of bispatial inference, which is constructed around one-sided P values, with a previously outlined form of fiducial inference. Finally, by using what are called post-data opinion curves, this bispatial-fiducial theory is naturally extended to deal with the general scenario in which any number of parameters may be unknown. The application of the theory is illustrated in various examples, which are especially relevant to the analysis of clinical trial data.

Keywords: foundational issues; Gibbs sampler; organic fiducial inference; parameter and sampling space hypotheses; post-data opinion curve; pre-data knowledge; relative risk.
1. Introduction

Let us begin with the following definition:

**Definition 1: Sharp and almost sharp hypotheses**

The hypothesis that a model parameter $\theta$ lies in an interval $[\theta_0, \theta_1]$, will be defined as a sharp hypothesis if $\theta_0 = \theta_1$, and as an almost sharp hypothesis if the difference $\theta_1 - \theta_0$ is very small in the context of our uncertainty about $\theta$.

Clearly, any importance attached to a hypothesis of either of these two types should not generally have a great effect on the way that, on the basis of data, we make inferences about $\theta$ if there was no exceptional reason to believe that it would be true or false before the data were observed. Taking this into account, it will be assumed that we are in the following scenario:

**Definition 2: Scenario of interest**

This scenario is characterised by there having been a substantial degree of belief, before the data were observed, that a given sharp or almost sharp hypothesis could have been true, but conditional on this hypothesis not being true, there was no or very little pre-data knowledge about the parameter $\theta$.

Perhaps some may try to dismiss the importance of this type of scenario, however it is one of the most fundamental problems of statistical inference that arise in practice. Let us consider the following examples.

**Example 1: Intervening in a system**

If $\theta$ is a parameter of one part of a system, and an intervention is made in a second part of the system that is arguably completely disconnected from the first part, then there will be a high degree of belief that the value of $\theta$ will not change as a result of the
intervention, i.e. there is a strong belief in a sharp hypothesis about $\theta$.

**Example 2: A randomised-controlled trial**

Let us imagine that a sample of patients is randomly divided into a group of $n_t$ patients, namely the treatment group, that receive a new drug $T$, and a group of $n_c$ patients, namely the control group, that receive a standard drug $C$. We will assume that $e_t$ patients in the treatment group experience a given adverse event, e.g. a heart attack, in a certain period of time following the start of treatment, and that $e_c$ patients in the control group experience the same type of event in the same time period. Using this sample information, it will be supposed that the aim is to make inferences about the relative risk $\pi_t/\pi_c$, where $\pi_t$ and $\pi_c$ are the population proportions of patients who would experience the adverse event when given drug $T$ and drug $C$ respectively. Now, if the action of drug $T$ on the body is considered to be very similar to the action of drug $C$, which is in fact often the case in practice in this type of clinical trial, then there may well have been a strong pre-data belief that this relative risk would be close to one, or in other words, that the almost sharp hypothesis that the relative risk would lie in a narrow interval containing the value one would be true.

It would appear that a common way of dealing with the strong pre-data belief that a sharp or almost sharp hypothesis is true is to simply ignore its inconvenient presence. However, doing so means that inferences based on the observed data will often not be even remotely honest. On the other hand, a formal method of addressing this issue that has received some attention is the Bayesian method. Let us take a quick look at how this method would work in a simple example.

**Example 3: Application of the Bayesian method**

Let us suppose that we are interested in making inferences about the mean $\mu$ of a normal density function that has a known variance $\sigma^2$, on the basis of a sample of values $x$ drawn
from the density function concerned. It will be assumed that we are in the scenario of Definition 2 with the sharp hypothesis concerned being the hypothesis that \( \mu = 0 \). Under the Bayesian paradigm, it would be natural to incorporate any degree of pre-data belief that \( \mu = 0 \) into the analysis of the data by assigning a positive prior probability to this hypothesis, i.e. the prior probability \( P_\ast(\mu = 0) > 0 \).

However, the only accepted way of expressing a lack of knowledge about a parameter under this paradigm is the controversial strategy of placing a diffuse proper or improper prior density over the parameter concerned. Taking this into account, let us assume, without a great loss of generality, that the prior density function of \( \mu \) conditional on \( \mu \neq 0 \) is a normal density function with a mean of zero and a large variance \( \sigma_0^2 \).

The inadequacy of the strategy in question is clearly apparent in the uncertainty there would be in choosing a value for the variance \( \sigma_0^2 \), and this issue becomes very hard to conceal after appreciating that the amount of posterior probability given to the hypothesis that \( \mu = 0 \) is highly sensitive to changes in this variance. For example, the natural desire to allow the variance \( \sigma_0^2 \) to tend to infinity results in this posterior probability tending to one for any given data set \( x \) and any given probability \( P_\ast(\mu = 0) > 0 \).

It can be easily argued, therefore, that the application of the Bayesian method in the case just outlined has an appalling result. Moreover, similar results occur in cases where the sampling density of the parameter of interest \( \theta \) is not normal and/or the prior density of this parameter has a more general form, and importantly, also in cases where the hypothesis in question is an almost sharp rather than a simple sharp hypothesis. This clearly gives us a strong motivation to look for an alternative method for making inferences about \( \theta \) in the scenario of interest. Following a similar path to that of Bowater and Guzmán (2019b), the aim of the present paper is to develop a satisfactory method of this type on the basis of classical ideas about statistical inference. This method of inference will be called bispatial inference.
Before going further, let us summarise the structure of the paper. In the next section, a general theory of bispatial inference is broadly outlined. A special formalisation of this theory is then developed in detail in Section 3. Given that all reasonable objectives for making inferences about the parameter $\theta$ can not be conveniently achieved by using this theory alone, a method of inference is put forward in Section 4 that is based on combining bispatial inference with a specific type of fiducial inference. In the final main section of the paper (Section 5), this combined theory is extended to cases where various model parameters are unknown.

2. General theory of bispatial inference

To clarify, let us now adjust our notation slightly by assuming that the data set to be analysed $x = \{x_1, x_2, \ldots, x_n\}$ was drawn from a joint density or mass function $g(x | \theta)$ that depends on a set of parameters $\theta = \{\theta_i : i = 1, 2, \ldots, k\}$, where each $\theta_i$ is a one-dimensional variable.

2.1. A note about probability

The concept of probability will be interpreted under the definition of generalised subjective probability that was comprehensively outlined in Bowater and Guzmán (2018b). Given that it will not be necessary to explicitly discuss this definition of probability in the present paper, the reader is referred to this earlier work for further information. Nevertheless, in relation to the general topic in question, there is a specific issue that should not be overlooked. In particular, we observe that when events are repeatable, the concept of the probability of an event and the concept of the proportion of times the event occurs in the long term are often used interchangeably. However, this is not always appropriate. The reason for this is that a population proportion is a fact about the physical world, while under the definition of probability that will be adopted, a probability is primarily
always a measure of an individual’s state of mind. Therefore, where necessary, we will denote the population proportion of times any given event \( A \) occurs by \( \rho(A) \), while the probability of the event \( A \) will, as usual, be denoted by \( P(A) \).

### 2.2. Parameter and sampling space hypotheses

The theory of inference that will be developed is based on a hypothesis \( H_P \) that concerns an event in the parameter space, and an equivalent hypothesis \( H_S \) that is stated in terms of the proportion of times an event in the sampling space will occur in the long run. The link that is made between the parameter and sampling spaces through the attention given to these two hypotheses is the reason that this type of inference will be called bispatial inference. More specifically, these two types of hypothesis will be assumed to have the following definitions.

**Definition 3: Parameter space hypothesis \( H_P \)**

Given that \( H : C \) denotes the hypothesis \( H \) that a given condition \( C \) is true, the parameter space hypothesis \( H_P \) is defined by:

\[
H_P : \theta \in \Theta_0
\]

where \( \Theta_0 \) is a given subset of the entire space \( \Theta \) over which the parameter \( \theta \) is defined.

**Definition 4: Sampling space hypothesis \( H_S \)**

The two conditions that the sampling space hypothesis \( H_S \) must satisfy are, first, that it must be equivalent to the hypothesis \( H_P \), and second, that it must have the following form:

\[
H_S : \rho(J(X^*) \in J_0(x)) \in A_0
\]

where \( J(X^*) \) is a statistic calculated on the basis of an as-yet-unobserved second sample \( X^* \) of values drawn from the density function \( g(x | \theta) \), which is possibly of a different size.
to the observed (first) sample $x$, the set $J_0(x)$ is a given subset of the entire space $J$ over which the statistic $J$ is defined, and the set $A_0$ is a given subset of the interval $[0, 1]$. Also, it should be clarified that the definition of the set $J_0(x)$ will depend, in general, on the data set $x$.

2.3. Inferential process

It will be assumed that inferences are made about the parameter $\theta$ by proceeding through the steps of the following algorithm:

Step 1: Formation of a suitable hypothesis $H_P$. The choice of this hypothesis should be made with the goal in mind of being able to make useful inferences about $\theta$.

Step 2: Assessment of the likeliness of $H_P$ being true using only pre-data knowledge about $\theta$. It is not necessary that this assessment is expressed in terms of a formal measure of uncertainty, e.g. a probability does not need to be assigned to this hypothesis.

Step 3: Formation of a suitable hypothesis $H_S$.

Step 4: Assessment of the likeliness of $H_S$ being true. This of course, in general, can only be done after the data $x$ have been observed. In carrying out this assessment, all relevant factors need to be taken into account including, in particular, the assessment made in Step 2 and the known equivalency between the hypotheses $H_P$ and $H_S$ after the data have been observed.

Step 5: Conclusion about the likeliness of $H_P$ being true on the basis of the data $x$. This is directly implied by the assessment made in Step 4.

2.4. First example: Two-sided P values

In the next three sections, we will apply the method outlined in the previous section to the problem of inference referred to in Example 3 of the Introduction, i.e. that of making
inferences about a normal mean $\mu$ when the population variance $\sigma^2$ is known.

To give a context to this problem, let us imagine that a patient is being constantly monitored with regard to the concentration of a certain chemical in his blood. We will assume that the measurements of this concentration are notably imprecise, and in particular, it will be assumed that any such measurement follows a normal density function with known variance $\sigma^2/2$ centred at the true concentration. Also, let us suppose that the data $x$ is simply the difference between two measurements of this concentration taken at time points immediately before and after the patient is subjected to some kind of intervention.

Now, if this intervention would not be expected to affect the concentration of the chemical of interest, there is likely to be a substantial degree of pre-data belief that the true difference in this concentration between these time points, namely the difference $\mu$, will be very small. In fact, to begin with, let us assume that the two time points in question are so close together that we find ourselves in the scenario of Definition 2 with the hypothesis concerned being the sharp hypothesis that $\mu = 0$. It can be seen therefore that we have effectively arrived at a specific form of Example 1 of the Introduction.

Under the assumptions that have been made, it is reasonable, as part of Steps 1 and 3 of the algorithm of Section 2.3, to define the hypotheses $H_P$ and $H_S$ as follows:

\[
H_P : \mu = 0 \quad \text{and} \quad H_S : \rho(\{ X^* < -|x| \} \cup \{ X^* > |x| \}) = 2\Phi(-|x|/\sigma)
\]

where $X^*$ is the unobserved difference between two additional measurements of the concentration in question taken at the time points concerned, and $\Phi(y)$ is the cumulative density of a standard normal distribution at the value $y$. These two hypotheses are clearly equivalent. Observe that the quantity on the right-hand side of equation (1) would be the standard two-sided P value if $H_P$ was regarded as being the null hypothesis.

Now, in Step 4 of the algorithm being considered, although a small value for this
two-sided P value would naturally disfavour the hypothesis \( H_S \), and in particular favour the left-hand side of the equality in equation (1) being greater than this P value, this would need to be balanced by how much the pre-data assessment in Step 2 favoured the hypothesis \( H_P \). Nevertheless, if this P value is very small then, even if this pre-data assessment heavily favoured the hypothesis \( H_P \), the hypothesis \( H_S \) could rationally be assessed to be quite unlikely to be true. As will always be the case, the evaluation of the likeliness of the hypothesis \( H_P \) in Step 5 of the algorithm in question should be the same as the evaluation of the likeliness of the hypothesis \( H_S \) in Step 4.

2.5. Second example: Q values

Let us now consider the more general case where the hypothesis on which the scenario of Definition 2 is based is the almost sharp hypothesis that \( \mu \) lies in the interval \([−\varepsilon, \varepsilon]\), where \( \varepsilon \) is a small positive constant.

In this case, it is reasonable to define the hypotheses \( H_P \) and \( H_S \) as follows:

\[
H_P : \mu \in [−\varepsilon, \varepsilon] \quad \text{and} \\
H_S : \rho(\{ X^* < −|x| \} \cup \{ X^* > |x| \}) \leq q(\varepsilon)
\]

where

\[
q(\varepsilon) = \Phi((−|x|−\varepsilon)/\sigma) + \Phi((−|x|+\varepsilon)/\sigma)
\]

It can easily be shown that these two hypotheses are equivalent. Notice that the value \( q(\varepsilon) \) as specified in equation [2] can be referred to as a Q value, since it falls within the general definition of a Q value as presented and discussed in Bowater and Guzmán (2019b).

Similar to the previous example, although we would naturally disfavour the hypothesis \( H_S \), and as a result, the hypothesis \( H_P \) if this Q value was small, this would need to be balanced by how much the hypothesis \( H_P \) was favoured before the value \( x \) was observed.
2.6. Third example: One-sided P values

To give another example, let us look at an alternative way of defining the hypotheses $H_P$ and $H_S$ in the context of the previous example. In particular, let us now assume that, if $x \leq 0$, then these hypotheses are defined as:

\[
H_P : \mu \geq -\varepsilon
\]
\[
H_S : \rho(X^* < x) \leq \Phi((x + \varepsilon)/\sigma)
\]

while if $x > 0$, they have the definitions:

\[
H_P : \mu \leq \varepsilon
\]
\[
H_S : \rho(X^* > x) \leq \Phi((-x + \varepsilon)/\sigma)
\]

Again, it can be seen that the hypotheses in each of these two pairs of hypotheses $H_P$ and $H_S$ are equivalent. Also, observe that the quantities on the right-hand sides of the inequalities in the two definitions of the hypothesis $H_S$ would be standard one-sided P values if the hypotheses $H_P$ that correspond to these definitions were regarded as being the null hypotheses.

Clearly, the substantial degree of pre-data belief that $\mu$ lies in the interval $[-\varepsilon, \varepsilon]$ should be reflected in the pre-data assessment of the likeliness of either of the hypotheses $H_P$ that have just been defined. Furthermore, similar to what was seen in the previous examples, a substantial degree of pre-data belief in whichever of these two hypotheses $H_P$ is applicable would need to be appropriately balanced by the information in the data that is summarised by the one-sided P value that appears in the corresponding hypothesis $H_S$, in order to obtain an adequate assessment of the likeliness of this latter hypothesis.

2.7. Discussion of examples

Although all the methods that have been outlined in the previous sections can be applied to many other problems of inference than the simple one that has been considered, the
method based on one-sided P values is much more widely applicable that the methods based on two-sided P values and Q values, in particular, it is able to cope better with sampling densities that are multimodal and/or non-symmetric.

Also, it can be argued that it is less easy to evaluate the likeliness of the hypotheses $H_S$ that are based on two-sided P values and Q values than those that are based on one-sided P values. With regard to the examples that have been presented, the observation that motivates the argument being referred to is that, for any given value of $x$, one of the two open intervals over which the two-sided P values and Q values are calculated will contain a proportion of the sampling density that always decreases in size as the mean $\mu$ moves away from zero, despite of course this change in $\mu$ always causing the total proportion of the sampling density contained in these two intervals to increase.

For the reasons that have just been given, from now on, we will not actively consider the methods based on two-sided P values and Q values. Instead, the method based on one-sided P values, and developments of this method, will constitute the main form of bispatial inference that will be explored. Although in the examples considered, a disadvantage of this method would appear to be that it does not allow us to directly assess the likeliness of the almost sharp hypothesis that $\mu$ lies in the interval $[-\varepsilon, \varepsilon]$ after the data have been observed, it will be shown later how this difficulty can be overcome.

3. Special form of bispatial inference

Let us now formalise the specific type of bispatial inference that has just been identified.

For the moment, it will be assumed that the only unknown parameter on which the sampling density $g(x | \theta)$ depends is the parameter $\theta_j$, either because there are no other parameters in the model, or because all the other parameters are known. To clarify, the scenario of interest will still be the scenario outlined in Definition 2, and therefore, given the previous assumption, the almost sharp hypotheses on which this scenario is based
becomes the hypothesis that $\theta_j$ lies in the narrow interval $[\theta_{j0}, \theta_{j1}]$.

3.1. Test statistic

It will be assumed that a test statistic $T(x)$ (which will be also denoted simply as $t$) satisfies the following two requirements:

1) Similar to what in Bowater (2019a) was defined as being a fiducial statistic, it is necessary that the test statistic $T(x)$ is a univariate statistic of the sample $x$ that can be regarded as efficiently summarising the information contained in this sample about the parameter $\theta_j$, given the values of other statistics that do not provide any information about this parameter.

2) If $F(t \mid \theta_j, u)$ is the cumulative distribution function of the unobserved test statistic $T(X)$ evaluated at its observed value $t$ conditional on a value for the parameter $\theta_j$, and $U(X)$ being equal to $u$, where $u$ are the observed values of an appropriate set of ancillary statistics $U(X)$, then it is necessary that, over the set of allowable values for $\theta_j$, the probability $F(t \mid \theta_j, u)$ strictly decreases as $\theta_j$ increases.

As far as the examples that will be considered in this paper are concerned, condition (1) will be satisfied, in a simple and clear-cut manner, by $T(x)$ being a univariate sufficient statistic for $\theta_j$. As a result, the set of ancillary statistics $U(X)$ in condition (2) will naturally be chosen to be empty in these examples, which in fact would usually be the case when the choice of $T(x)$ is more general.

3.2. Parameter and sampling space hypotheses

If the condition

$$F(t \mid \theta_j = \theta_{j0}, u) \leq F'(t \mid \theta_j = \theta_{j1}, u)$$

(4)
holds, where \( F'(t|\theta_j = \theta_{j1}, u) \) is the conditional probability \( P(T(X) \geq t|\theta_j = \theta_{j1}, u) \), then the hypotheses \( H_P \) and \( H_S \) will be defined as:

\[
H_P : \theta_j \geq \theta_{j0} \\
H_S : \rho(T(X^*) \leq t|u) \leq F(t|\theta_j = \theta_{j0}, u)
\]  (5)

where the as-yet-unobserved sample \( X^* \) is as defined in Section 2.2 except that now it will be assumed to be always of the same size as the sample \( x \), i.e. it must consist of \( n \) observations, and where \( \rho(T(X^*) \leq t|u) \) is the unknown population proportion of times that \( T(X^*) \leq t \) conditional on \( U(X^*) = u \). On the other hand, if the condition in equation [4] does not hold, then the hypotheses in question will be defined as:

\[
H_P : \theta_j \leq \theta_{j1} \\
H_S : \rho(T(X^*) \geq t|u) \leq F'(t|\theta_j = \theta_{j1}, u)
\]  (6)

It can easily be seen that the hypotheses in each of the two pairs of hypotheses \( H_P \) and \( H_S \) that have just been defined are equivalent.

We will assume that to make inferences about a parameter of interest, the same algorithm will be used as was outlined in Section 2.3. Clearly though, in Step 2 of this algorithm, some special attention will often need to be placed in assessing the likeliness of the almost sharp hypothesis that \( \theta_j \in [\theta_{j0}, \theta_{j1}] \) before the data were observed, since we can see that this hypothesis will always be included in the hypothesis \( H_P \), but will not generally be equivalent to \( H_P \). Also, it now will be assumed that in Step 4 of the algorithm in question, the goal is usually to assign a probability to the hypothesis \( H_S \).

With this goal in mind, it may be helpful to try to determine the minimum probability that could sensibly be assigned to the hypothesis \( H_S \). In particular, for a reason that should be obvious, it would not seem sensible to assign a probability to this hypothesis that is less than the probability that would be assigned to it if nothing or very little had been known about the parameter \( \theta_j \) before the data were observed. One way, but not as
yet a widely accepted way, of making inferences about $\theta_j$ under these circumstances is to use the fiducial method of inference (which has its origins in Fisher 1935, 1956) and, given the interpretation of the concept of probability being relied on in the present paper (see Section 2.1), it would seem appropriate to consider applying the form of this type of inference that has been called subjective, or more recently, organic fiducial inference, see for example Bowater (2017, 2018a, 2019a). Therefore, as a feature of the examples in the next two sections, organic fiducial inference will be applied to obtain what will be possible to regard as being a minimum value for the probability of the hypothesis $H_S$ being true. We will denote this minimum value as $P_f(H_S)$.

3.3. First example: Inference about a normal mean with variance known

Let us return to the example that was outlined in Section 2.6. We can see that this example fits within the special framework for bispatial inference that has just been outlined. In particular, the difference $x$ is clearly a suitable test statistic $T(x)$, since it is a sufficient statistic for $\mu$ that will satisfy condition (2) of Section 3.1 for any observed $x$. Also, the way that the hypotheses $H_P$ and $H_S$ were specified in Section 2.6 matches how these hypotheses would be specified by using the definitions in Section 3.2.

In this earlier example, let us now more specifically assume that $\sigma = 1$, $\varepsilon = 0.2$ and $x = 2.7$. Under these assumptions, the one-sided P value on the right-hand side of equation (3) is 0.0062. Since this P value is obviously small, but not very small, if a substantial probability of around 0.3 would have been placed on the hypothesis that $\mu \in [-0.2, 0.2]$ before the value $x$ was observed, it would seem possible to justify a probability in the range of say 0.03 to 0.08 being placed on the hypothesis $H_S: \rho(X^* > 2.7) \leq 0.0062$ being true, and as a result, on the hypothesis $H_P: \mu \leq 0.2$ being true after the value $x$ has been observed.

The probability that would be assigned to the hypothesis $H_P$ after the value $x$ has
been observed by applying the strong fiducial argument (see Bowater 2019a) as part of the method of organic fiducial inference would be equal to 0.0062, i.e. the one-sided P value of interest. Since this form of reasoning could be regarded as justifying the value of 0.0062 as being a minimum value for the probability in question, it is therefore appropriate that it is well below the range for this probability that has been proposed. To be more clear, this value is the probability $P_f(H_S)$ referred to in the last section.

3.4. Second example: Inference about a binomial proportion

Let us imagine that a random sample of patients are switched from being given a standard drug $C$ to being given a new drug $T$. After a period of time has passed, they are asked which out of the two drugs $C$ and $T$ they prefer. The proportion of patients who prefer drug $T$ to drug $C$, after patients who do not express a preference have been excluded, will be denoted by $q$. Given this sample proportion, it will be assumed that the aim is make inferences about its corresponding population proportion $\pi$. For a similar reason with regard to the nature of drugs $C$ and $T$ as that given in Example 2 of the Introduction, let us also suppose that the scenario of Definition 2 applies with the hypothesis concerned being the hypothesis that the proportion $\pi$ lies in a narrow interval centred at 0.5, which will be denoted as $[0.5 - \varepsilon, 0.5 + \varepsilon]$.

Observe that the sample proportion $q$ clearly satisfies the requirements of Section 3.1 to be a suitable test statistic $T(x)$. To give a more specific example, we will assume that there are twelve patients in the sample, of whom nine prefer drug $C$ to drug $T$, one prefers drug $T$ to drug $C$ and two do not express a preference, and therefore $q = 0.1$. Also, let the constant $\varepsilon$ be equal to 0.03. It now follows that, under the definitions of Section 3.2, the hypotheses $H_P$ and $H_S$ would be specified as:

$$H_P : \pi \geq 0.47$$

$$H_S : \rho(Q^* \leq 0.1) \leq 0.53^{10} + 10(0.47)(0.53^9) = 0.0173$$

(7)
where $Q^*$ is the proportion of patients who would prefer drug $T$ to drug $C$ in an as-yet-unobserved sample of ten patients who express a preference between the two drugs. We can see that again the one-sided $P$ value, i.e. the value 0.0173 in equation (7), is reasonably small. Therefore, if a pre-data probability of say 0.3 would have been placed on the hypothesis that $\pi \in [0.47, 0.53]$, it would seem possible to justify a probability in the range of say 0.03 to 0.08 being placed on the hypothesis $H_S$ being true, and as a result, on the hypothesis $H_P$ being true after the proportion $q$ has been observed.

The probability that would be assigned to the hypothesis $H_P$ after the value $q$ has been observed by using the strong fiducial argument, and a local pre-data (LPD) function for $\pi$ (see Bowater 2019a) defined by:

$$\omega_L(\pi) = b \quad \text{if} \quad 0 \leq \pi \leq 1 \quad \text{and} \quad 0 \quad \text{otherwise}$$

where $b > 0$, as part of the method of organic fiducial inference would be equal to 0.0070. Since this probability could be regarded as being a minimum value for the probability of the hypothesis $H_S$, in particular as the value $P_f(H_S)$ referred to in Section 3.2 it is again therefore appropriate that it is well below the range for this probability that has been proposed. We will return to this topic in Sections 5.4 and 5.5.

3.5. Foundational basis of the theory

In the examples considered in the previous section and Section 3.3, it was inherently assumed that the smaller the size of the one-sided P value that appears in the hypothesis $H_S$, the less inclined we should be to believe that this hypothesis is true. However, what is the foundational basis for this assumption? We will now try to offer some kind of answer to this question.

It can be seen that the two versions of the hypothesis $H_S$ in equations (5) and (6) can both be represented as:

$$H_S : \rho(A) \leq \beta$$

(9)
where $A$ is a given condition and $\beta$ is a given one-sided P value. Therefore, the population proportion of times condition $A$ is satisfied will be less than or equal to $\beta$ if the corresponding hypothesis $H_P$ is true, or in other words, if the parameter $\theta_j$ is restricted in some way. However, we could also calculate a probability for condition $A$ being satisfied without placing restrictions on the parameter $\theta_j$ by using the fiducial argument. In particular, the fiducial probability in question would be defined as:

$$P_f(A) = \int_A \int_{\theta_j} g(x^* \mid \theta_j, u) f(\theta_j \mid x) d\theta_j dx^*$$

(10)

where $f(\theta_j \mid x)$ is an appropriate fiducial density function for the parameter $\theta_j$.

It will be helpful if we now look at a specific example, and so let us again consider the example of Section 3.3. In this case, the fiducial density of the parameter of interest $\mu$, i.e. the density $f(\mu \mid x)$, obtained by using the strong fiducial argument is defined by the expression $\mu \sim N(x, \sigma^2)$. On the basis of this fiducial density for $\mu$, it is simple to show how, by using equation (10), we obtain the result that $P_f(A)$ equals 0.5, where as we know $A = \{X^* < x\}$ or $A = \{X^* > x\}$, which of course is a special result that in fact could have been derived by using more direct fiducial reasoning. We could interpret this result as meaning that the probability that we should assign to condition $A$ being true if we had known nothing or very little about $\theta_j$ before the sample $x$ was observed should be 0.5.

In this regard, if we were to propose assigning a large probability to the hypothesis $H_S$ being true when the P value $\beta$ in equation (9) was quite small, then it would seem fair if we were asked how we can justify doing this given the large difference between this P value and the probability $P_f(A)$. To give an answer to this question, it is reasonable to argue that the only circumstances in which we could possibly be would be where the importance attached to the probability $P_f(A)$ has been greatly diminished by there having been a high degree of belief that the hypothesis $H_P$ was true before the sample $x$ was observed, which in the context of the scenario of Definition 2, would mean a high
degree of pre-data belief that $\mu$ lay in the interval $[-\varepsilon, \varepsilon]$.

Furthermore, let us suppose that a given probability of $\gamma_0$ would be assigned to the hypothesis $H_S$ being true if the P value $\beta$ was equal to a given value $\beta_0$. Now, if we imagine a scenario in which the value of $\beta$ is smaller than $\beta_0$, and therefore further away from the probability $P_f(A)$, then it can be easily argued that the only way we could justify assigning the same probability $\gamma_0$ to the hypothesis $H_S$ would be if, for an unrelated reason, it was decided that our pre-data belief that $\mu \in [-\varepsilon, \varepsilon]$ should be increased. Also, it is fairly uncontroversial to argue that for any fixed sample $x$, the degree of pre-data belief that $\mu \in [-\varepsilon, \varepsilon]$ and the degree of post-data belief in the hypothesis $H_P$ should be positively correlated. As a logical consequence of these arguments, it follows that, if there is a fixed degree of pre-data belief that $\mu \in [-\varepsilon, \varepsilon]$, then the probability we should wish to assign to the hypothesis $H_S$ should decrease as the value that the P value $\beta$ is assumed to take is made smaller. Therefore, we hope that an adequate answer to the question posed at the start of this section has been provided.

Another foundational issue that no doubt some would try to raise centres on the argument that the probabilities that are assigned to the hypothesis $H_S$ as part of the method that has been outlined should be treated as posterior probabilities, each of which corresponding to the Bayesian update of some given prior distribution function for $\theta_j$. However, to be able to use the Bayesian method concerned some justification would need to be given as to why such a prior distribution function is a good representation of our pre-data beliefs about $\theta_j$. The fact that, in the context of the scenario of interest in Definition 2, this is in general going to be extremely difficult to do is consistent with the motivation for the method of bispatial inference given in the Introduction.
4. Bispatial-fiducial inference

The methodology of Sections 3.1 and 3.2 allows us to determine a post-data probability for the hypothesis \( H_P \) being true. Clearly though, it would be preferable to have a post-data distribution function for the parameter \( \theta_j \). For this reason, let us now consider generalising the methodology that has been proposed.

In particular, if

\[
F(t \mid \theta_j = \theta_j^*, u) \leq F'(t \mid \theta_j = \theta_j^*, u)
\]

where \( \theta_j^* \) is any given value of \( \theta_j \), then let us define the hypotheses \( H_P \) and \( H_S \) as:

\[
H_P : \theta_j \geq \theta_j^*
\]

\[
H_S : \rho(T(X^*) \leq t \mid u) \leq F(t \mid \theta_j = \theta_j^*, u),
\]

otherwise, we will define these hypotheses as:

\[
H_P : \theta_j \leq \theta_j^*
\]

\[
H_S : \rho(T(X^*) \geq t \mid u) \leq F'(t \mid \theta_j = \theta_j^*, u)
\]

We can observe that a post-data distribution function for \( \theta_j \) could be constructed if, for each value of \( \theta_j^* \) within the range of allowable values for \( \theta_j \), we were able to consistently evaluate the probability of the hypothesis \( H_S \) as defined by these equations. Obviously, it would be a little awkward to do this by directly assessing the likeliness of the hypothesis \( H_S \) being true for all the values of \( \theta_j^* \) concerned, however no assumption has been made regarding whether assessments of this type should be made directly or indirectly.

Therefore, we now will explore a way of making all but one of these assessments indirectly by using again the method of organic fiducial inference. The general method that results from doing this will be referred to as bispatial-fiducial inference.
4.1. First proposed method

Under the assumption that the hypothesis $H_S$ satisfies the more conventional definition of this type of hypothesis given in Section 3.2, let us assume that we directly weigh up, and then determine a value for the probability of this hypothesis being true. This probability will be denoted by the value $\gamma$.

Furthermore, it will be assumed that the method of organic fiducial inference is used to derive a fiducial density function for $\theta_j$ conditional on $\theta_j$ not lying in the interval $[\theta_{j0}, \theta_{j1}]$. In this approach to inference, the global pre-data (GPD) function $\omega_G(\kappa)$ (see Bowater 2019a) offers the principal, if not exclusive, means by which pre-data beliefs about a parameter of interest $\kappa$ can be expressed. Given that it is being assumed that, under the condition that $\theta_j$ does not lie in the interval $[\theta_{j0}, \theta_{j1}]$, nothing or very little would have been known about $\theta_j$ before the data were observed, it is appropriate to use a neutral GPD function for $\theta_j$ that has the following form:

$$\omega_G(\theta_j) = \begin{cases} 
0 & \text{if } \theta_j \in [\theta_{j0}, \theta_{j1}] \\
\frac{a}{\omega_G(\theta_j)} & \text{otherwise}
\end{cases} \quad (11)$$

where $a > 0$. On the basis of this GPD function, the fiducial density function of $\theta_j$ that is of interest can often be derived by directly applying what, in Bowater (2019a), was referred to as the moderate fiducial argument (when Principle 1 of this earlier paper can be applied), or more generally is defined by the following expression:

$$f(\theta_j | \theta_j \notin [\theta_{j0}, \theta_{j1}], x) = C_0 f_S(\theta_j | x) \quad (12)$$

where $f_S(\theta_j | x)$ is the only, or otherwise, a suitable fiducial density for $\theta_j$ that results from applying the strong fiducial argument (either under Principle 1 or Principle 2 of Bowater 2019a), and $C_0$ is a normalising constant.

Given the assumptions that have been made, if the condition in equation (4) holds, which implies that $H_P$ is the hypothesis that $\theta_j \geq \theta_{j0}$, then it can be deduced that the
post-data probability of the event \( \theta_j \in [\theta_{j0}, \theta_{j1}] \) is defined by:

\[
P(\theta_j \in [\theta_{j0}, \theta_{j1}] \mid x) = \gamma - \lambda(1 - \gamma)
\]  

(13)

where the probability \( \gamma \) is as defined at the start of this section, and \( \lambda \) is given by:

\[
\lambda = \frac{P_f(\theta_j > \theta_{j1} \mid \theta_j \notin [\theta_{j0}, \theta_{j1}], x)}{P_f(\theta_j < \theta_{j0} \mid \theta_j \notin [\theta_{j0}, \theta_{j1}], x)}
\]

where \( P_f(A \mid \theta_j \notin [\theta_{j0}, \theta_{j1}], x) \) denotes the fiducial probability of the event \( A \) conditional on \( \theta_j \notin [\theta_{j0}, \theta_{j1}] \) that is derived by integrating over the fiducial density of \( \theta_j \) given by equation (12). Under the condition in equation (4), it also follows that the post-data density function of \( \theta_j \), which will be denoted simply as \( p(\theta_j \mid x) \), is defined over all of its domain except for the interval \([\theta_{j0}, \theta_{j1}]\) by the expression:

\[
p(\theta_j \mid x) = \begin{cases} 
(1 - \gamma)f(\theta_j \mid \{\theta_j < \theta_{j0}\}, x) & \text{if } \theta_j < \theta_{j0} \\
\lambda(1 - \gamma)f(\theta_j \mid \{\theta_j > \theta_{j1}\}, x) & \text{if } \theta_j > \theta_{j1}
\end{cases}
\]  

(14)

where \( f(\theta_j \mid B, x) \) denotes the fiducial density in equation (12) conditioned on the event \( B \). It should be obvious how to modify the definitions in equations (13) and (14) when the condition in equation (4) does not hold. Notice that the assignment of a probability \( \gamma \) to the hypothesis \( H_S \) that is greater than or equal to the minimum value \( P_f(H_S) \) for this probability that was referred to at the end of Section 3.2, and discussed further in Sections 3.3 and 3.4, should logically be a sufficient (but not a necessary) requirement for ensuring that the probability \( P(\theta_j \in [\theta_{j0}, \theta_{j1}] \mid x) \) given in equation (13) is not negative.

Although, if \( \theta_{j1} > \theta_{j0} \), the definitions in equations (13) and (14) do not fully specify the form taken by the post-data density function of \( \theta_j \), this may not be a great problem if the aim is to only derive post-data probability intervals for \( \theta_j \), since the narrow interval \([\theta_{j0}, \theta_{j1}]\) over which this density is undefined may often lie wholly inside or outside of the intervals of this type that are of greatest interest. On the other hand, it will of course often be indispensible to have a full rather than a partial definition of this post-data
density function, e.g. for determining the post-data expectations of general functions of \( \theta_j \), and for simulating values from this kind of density function.

One way around this problem is to simply complete the definition of the post-data density function of \( \theta_j \) by assuming that, when \( \theta_j \) is conditioned to lie in the interval \([\theta_{j0}, \theta_{j1}]\), it has a uniform density function over this interval. Therefore, the full definition of this density function would consist of what is required by equation (14), and by the expression:

\[
p(\theta_j | x) = \frac{(\gamma - \lambda(1 - \gamma))}{(\theta_{j1} - \theta_{j0})} \quad \text{if } \theta_j \in [\theta_{j0}, \theta_{j1}]
\]

Again, since by definition the interval \([\theta_{j0}, \theta_{j1}]\) is narrow, this simple solution to the problem concerned may, in some circumstances, be considered adequate.

Nevertheless, it is a method that has two clear disadvantages. First, the post-data density function of \( \theta_j \) that it gives rise to will, in general, be discontinuous at the values \( \theta_{j0} \) and \( \theta_{j1} \). Second, the way in which the post-data density function of \( \theta_j \) conditional on the event \( \theta_j \in [\theta_{j0}, \theta_{j1}] \) is determined does not take into account our pre-data beliefs about \( \theta_j \), or the information contained in the data.

We will now try to enhance the methodology that has been considered in the present section with the aim of addressing these two issues.

4.2. A more sophisticated method

The method that has just been outlined is based on determining a fiducial density for \( \theta_j \) conditional on \( \theta_j \) lying outside of the interval \([\theta_{j0}, \theta_{j1}]\) using the neutral GPD function for \( \theta_j \) given in equation (11). We now will attempt to construct a fiducial density for \( \theta_j \) conditional on \( \theta_j \) lying inside this interval using a more general type of GPD function for \( \theta_j \).
In particular, it will be assumed that this GPD function has the following form:

\[
\omega_G(\theta_j) = \begin{cases} 
1 + \tau h(\theta_j) & \text{if } \theta_j \in [\theta_{j0}, \theta_{j1}] \\
0 & \text{otherwise}
\end{cases}
\] (15)

where \(\tau \geq 0\), and \(h(\theta_j)\) is a continuous unimodal density function on the interval \([\theta_{j0}, \theta_{j1}]\) that is equal to zero at the limits of this interval. On the basis of this GPD function, the fiducial density function of \(\theta_j\) that is of interest can often be derived by directly applying the weak fiducial argument (see Bowater 2019a), or more generally is defined by the following expression:

\[
f(\theta_j | \theta_j \in [\theta_{j0}, \theta_{j1}], x) = C_1 \omega_G(\theta_j) f_S(\theta_j | x)
\] (16)

where \(f_S(\theta_j | x)\) is the same fiducial density function that appeared in equation (12), and \(C_1\) is a normalising constant. Let us therefore make the assumption that this expression is combined with equations (13) and (14) to obtain a full definition of the post-data density function of \(\theta_j\) over all values of \(\theta_j\).

More specifically, though, it will be assumed that the value \(\tau\) in equation (15) is chosen such that this overall density function for \(\theta_j\) is made equivalent to a fiducial density function for \(\theta_j\) that is based on a continuous GPD function for \(\theta_j\) over all values of \(\theta_j\). If the hypothesis \(H_S\) is assigned a probability \(\gamma\) that is greater than the minimum value \(P_f(H_S)\) as defined in Section 3.2, then logically, all other things equal, such a value for \(\tau\) will always exist and be unique.

This criterion for choosing the value of \(\tau\) will, in general, ensure that the post-data density function for \(\theta_j\) will be continuous over all values of \(\theta_j\). Also, if this density was conditioned to lie in the interval \([\theta_{j0}, \theta_{j1}]\), then it still would be formed in a way that takes into account our pre-data beliefs about \(\theta_j\), and allows these beliefs to be modified on the basis of the data. Therefore, the difficulties are avoided that were identified as being associated with the method that was proposed in the previous section.

Furthermore, there are two reasons why the criterion in question regarding the choice
of $\tau$ can be viewed as not being a substantial restriction on the way we are allowed to express our pre-data knowledge about the parameter $\theta_j$. First, going against this rule may clearly have undesirable consequences, and therefore it can be regarded as being a useful guideline in choosing a suitable GPD function for $\theta_j$ when $\theta_j$ is conditioned to lie in the interval $[\theta_{j0}, \theta_{j1}]$. Second, any detrimental effect of this rule may not be that apparent given the great deal of imprecision there will usually be in the specification of this GPD function.

4.3. First example: Revisiting the simple normal case

To give an example of the application of the method proposed in the previous section, let us return to the problem of making inferences about the mean of a normal distribution that was considered in both Sections 2.6 and 3.3. All assumptions about the values of quantities of interest that were made in Section 3.3 will be maintained.

In this example, the fiducial density defined by equation (12), i.e. the density $f(\mu | \mu \notin [-0.2, 0.2], x)$, can be derived by using the moderate fiducial argument, which implies that it is simply the density $f_S(\mu | x)$, i.e. the density function for $\mu$ defined by the expression $\mu \sim N(2.7, 1)$, conditioned not to lie in the interval $[-0.2, 0.2]$.

Let us also make the assumption that the density function $h(\theta_j)$ that appears in equation (15) is defined as being a beta density function for $\mu$ on the interval $[-0.2, 0.2]$ with both its shape parameters equal to 4. This density function clearly satisfies the conditions on the function $h(\theta_j)$ that were given in Section 4.2. Furthermore, it is a reasonable choice for this function given that it is smooth, its mode is at $\mu = 0$ and it is symmetric.

If a sensible value $\gamma$ is assigned to the probability of the hypothesis $H_S$, then the assumptions that have been made would determine the post-data density $p(\mu | x)$ according to the expressions in equations (13), (14) and (16). On the other hand, of course, this
density function could have been defined according to the simple proposals for its full and partial specification outlined in Section 4.1 without the need for the assumption about the form of the density \( h(\theta_j) \).

The curves in Figure 1 represent the post-data density \( p(\mu \mid x) \) outside of the interval \([-0.2, 0.2]\) for all the definitions of this density function being considered, while, over the whole of the real line, they more specifically represent this function under its more sophisticated definition given in Section 4.2. The range for \( \gamma \), i.e. the probability \( P(H_S) \), that underlies this figure is equal to what was suggested for this example in Section 3.3, i.e. the range of 0.03 to 0.08. In particular, the solid curve in this figure depicts the post-data density for \( \mu \) when \( \gamma \) is 0.05, while the long-dashed and short-dashed curves in this figure depict this density when \( \gamma \) is 0.03 and 0.08 respectively. The accumulation of probability mass around the value of zero in these density functions would be anticipated given the strong pre-data belief that \( \mu \) would be close to zero, though as we know, the importance of this pre-data opinion about \( \mu \) is assessed in the context of having observed the data value \( x \) to obtain the density functions that are shown.
4.4. Second example: Revisiting the binomial case

To give a second example of the application of the method being considered, let us return to the problem of making inferences about a binomial proportion that was examined in Section 3.4, again with the same assumptions in place about the values of quantities of interest that were made earlier.

In using the method of organic fiducial inference in this example to derive the fiducial density \( f_S(\pi \mid x) \) that is required by equations (12) and (16), it would once more seem reasonable to define the LPD function \( \omega_L(\pi) \) as in equation (8). Also, as explained in Bowater (2019a), this fiducial density for \( \pi \) will be fairly insensitive to the choice made for this latter function. We will, in addition, assume that the density \( h(\theta_j) \) required by equation (15) is the same type of beta density that it was in the previous section, but this time defined on the interval \([0.47, 0.53]\). As a result, we now can specify the post-data density \( p(\pi \mid x) \) using equations (13), (14) and (16), assuming again, of course, that a sensible value \( \gamma \) has been assigned to the probability of \( H_S \).

The weighted histogram in Figure 2 represents this post-data density for the case where \( \gamma = 0.05 \). The numerical output on which this histogram is based was generated by the method of importance sampling, more specifically by appropriately weighting a sample of three million independent random values from the fiducial density \( f_S(\pi \mid x) \). On the other hand, the curves in Figure 2 represent approximations to the post-data density \( p(\pi \mid x) \) obtained by using the posterior density for \( \pi \) that is based on the Jeffreys prior for this problem in place of the density \( f_S(\pi \mid x) \). Additional simulations showed that this type of approximation was satisfactory. In particular, the solid curve in Figure 2 approximates the density \( p(\pi \mid x) \) for the case where \( \gamma = 0.05 \) and, as was expected, it closely approximates the histogram in this figure. Similarly, the long-dashed and short-dashed curves in this figure approximate the density in question in the cases where \( \gamma \) is 0.03 and 0.08 respectively. Again, the range for \( \gamma \) being considered is 0.03 to 0.08, which
is what was suggested for this example in Section 3.4.

The accumulation of probability mass around the value of 0.5 in the density functions in Figure 2 is clearly an artefact of the strong pre-data opinion that was held about \( \pi \).

5. Extending bispatial-fiducial inference to multiparameter problems

5.1. General assumptions

It was assumed at the start of Section 3 that the parameter \( \theta_j \) is the only unknown parameter in the sampling model. Let us now assume that all the parameters \( \theta_1, \theta_2, \ldots, \theta_k \) in the sampling model are unknown.

More specifically, we will define the subset of parameters \( \theta^A = \{\theta_1, \theta_2, \ldots, \theta_m\} \) such that what is known about each parameter \( \theta_j \) in this set, conditional on all other parameters in the model, i.e. \( \theta_{-j} = \{\theta_1, \ldots, \theta_{j-1}, \theta_{j+1}, \ldots, \theta_k\} \), being known, satisfies the requirements of the scenario of Definition 2. Furthermore, it will be assumed that the set of all the remaining parameters in the sampling model, i.e. the set \( \theta^B = \{\theta_{m+1}, \theta_{m+2}, \ldots, \theta_k\} \), is such that there was no or very little pre-data knowledge about each parameter \( \theta_j \) in
this set conditional on all other parameters in the model, i.e. the set $\theta_{-j}$, being known.

We can clearly justify deriving the post-data density of any given parameter $\theta_j$ in the set $\theta^A$ conditional on all the parameters in the set $\theta_{-j}$ using the method outlined in Section 4.2, meaning that this density function would be defined by equations (13), (14) and (16). The set of full conditional post-data densities that result from doing this for each parameter in the set $\theta^A$ could therefore be denoted as:

$$p(\theta_j | \theta_{-j}, x) \quad j = 1, 2, \ldots, m$$

(17)

It is also clearly defensible to specify the post-data density of any given parameter $\theta_j$ in the set $\theta^B$ conditional on all the parameters in the set $\theta_{-j}$ as being the fiducial density $f_S(\theta_j | x)$ that appears in equations (12) and (16). Doing this for each parameter in the set $\theta^B$ would therefore give rise to the following set of full conditional post-data densities:

$$f_S(\theta_j | \theta_{-j}, x) \quad j = m + 1, m + 2, \ldots, k$$

(18)

**Statement about incompatible full conditional densities**

It is not acknowledged in the following section or in Sections 5.4 and 5.5 that the stationary density of an ergodic Gibbs sampler is affected, in general, by the scanning order of the variables on which the sampler is based when the full conditional densities concerned are incompatible. These sections will be rewritten in due course to take this important issue into account. Nevertheless, doing so will not affect the relevance of the results that are currently presented in the examples in the latter two sections.

**5.2. Post-data densities of various parameters**

It is obviously reasonable to claim that if the set of densities that results from combining the sets of full conditional densities in equations (17) and (18) determine a joint density function for the parameters $\theta_1, \theta_2, \ldots, \theta_k$, then this density function should be regarded as
being the post-data density function of these parameters, i.e. the density \( p(\theta \mid x) \). Under the assumption that such a density function exists, it can be easily shown that it must be unique.

To corroborate that the combined set of full conditional densities in equations (17) and (18) actually determine a joint density function for the parameters concerned, it will be assumed that it is adequate to exclusively apply the computational method that was proposed for this type of purpose in Bowater (2018a), although it would be inappropriate to ignore the possibility that alternatively, or in addition, an analytical method could be used, e.g. the analytical method proposed in this earlier paper. The computational method in question is based on applying the Gibbs sampling algorithm (Geman and Geman 1984 and Gelfand and Smith 1990), and analysing the results that it produces using appropriate convergence diagnostics (Gelman and Rubin 1992 and Brooks and Roberts 1998). The use of this method will be explored in the examples that will be considered in Sections 5.4 and 5.5.

5.3. Post-data opinion curve

To construct any of the post-data densities \( p(\theta_j \mid \theta_{-j}, x) \) in equation (17), there is still one important issue that needs to be addressed, which is that the assessment of the likeliness of the relevant hypothesis \( H_S \) in equation (5) or (6) will generally depend on the values of the parameters in the set \( \theta_{-j} \). This of course will be partially due to the effect that the values of these parameters can have on the one-sided P value that appears in the definition of this hypothesis. Therefore, in general, we will not need to assign just one probability to the hypothesis \( H_S \), but various probabilities conditional on the values of the parameters in the set \( \theta_{-j} \).

Faced with the inconvenience that this can cause, it is possible to simplify matters greatly by assuming that, the probability that is assigned to any given hypothesis \( H_S \),
i.e. the probability $\gamma$, will be the same for any fixed value of the one-sided P value that appears in the definition of this hypothesis, no matter what values are actually taken by the parameters in the set $\theta_{-j}$. It can be argued that such an assumption would be reasonable in many situations. If this assumption is made then, the probability $\gamma$ will clearly be a mathematical function of the one-sided P-value that appears in the hypothesis $H_S$ concerned, i.e. the value $\beta$ in terms of the notation in equation (9). We will call this function the post-data opinion (PDO) curve for the parameter $\theta_j$ conditional on the parameters $\theta_{-j}$.

Notice that, when using the Gibbs sampling procedure mentioned in the last section, we will only need to define this curve over the range of values of $\beta$ that are likely to appear in the hypothesis $H_S$ having taken into account the data $x$ that have been observed. Also, it could be hoped that, in many cases, it would be possible to adequately specify this curve by first assessing the probability $\gamma$ for a small number of carefully selected values of $\beta$, and then fitting a suitable smooth curve through the resulting points.

Furthermore, it is clear that there are rules which can be employed to ensure that any given PDO curve is chosen in a sensible manner. Perhaps the most obvious rule of this kind is that a PDO curve, in general, must be chosen so that it is a monotonically increasing function. Other characteristics that we would expect this type of curve to have will be discussed in the context of the examples that will be considered in the next two sections.

5.4. First example: Normal case with variance unknown

To give an example of the application of the method of inference that has just been proposed, i.e. bispatial-fiducial inference for multiparameter problems, let us return to the example that was first considered in Section 3.4, however let us now assume that the difference in the performance of the two drugs of interest is measured by the difference
in the concentration of a certain chemical (e.g. cholesterol) in the blood, i.e. the level observed for drug T minus the level observed for drug C, rather than the preferences of the patients concerned. The set of these differences for all the patients in the sample will be the data set \( x \). This example therefore also has similarities with the example that was first considered in Section 2.4. As in this earlier example, it will be assumed in addition that the data values \( x_i \) follow a normal distribution with an unknown mean \( \mu \), although now its variance \( \sigma^2 \) will also be assumed to be unknown.

For a similar reason to that used in Example 2 in the Introduction, let us suppose furthermore that, for any given value of \( \sigma^2 \), the scenario of Definition 2 would apply with the hypothesis in question being the hypothesis that \( \mu \) lies in the narrow interval \([−\varepsilon, \varepsilon]\). On the other hand, it will be assumed, as could often be done in practice, that nothing or very little would have been known about the variance \( \sigma^2 \) before the data were observed given any value for \( \mu \). Therefore, in terms of the notation of Section 5.1, the set of parameters \( \theta^A \) will only contain \( \mu \), and the set \( \theta^B \) will only contain \( \sigma^2 \).

To determine the post-data density \( p(\mu | \sigma, x) \), the test statistic \( T(x) \) as defined in Section 3.1 will be assumed to be the sample mean \( \bar{x} \), which clearly satisfies what is required to be such a statistic. Under this assumption, the hypotheses \( H_P \) and \( H_S \) will be as given in Section 2.6, except that now the mean \( \bar{x} \) takes the place of the value \( x \), the standard error \( \sigma/\sqrt{n} \) takes the place of \( \sigma \), and the random variable \( X^* \) is substituted by \( \bar{X}^* \), i.e. by the mean of an as-yet-unobserved sample of \( n \) additional observations from the population concerned. If we more specifically assume, as we will do from now on, that \( n = 9, \bar{x} = 2.7 \) and \( \varepsilon = 0.2 \), then these hypotheses can be simply defined as

\[
H_P : \mu \leq 0.2 \\
H_S : \rho(\bar{X}^* > 2.7) \leq \Phi(-7.5/\sigma) \quad (= \beta)
\]

Clearly, the one-sided P value \( \beta \) in the hypothesis \( H_S \) depends on the variable \( \sigma \). Moreover, given that there is a one-to-one relationship between \( \sigma \) and this P value, the PDO
Figure 3: Post-data opinion (PDO) curve for $\mu$ conditional on $\sigma$

curve for $\mu$ conditional on $\sigma$ will completely describe the assessment of the probability of $H_S$, i.e. the probability $\gamma$, in all circumstances of interest. To give a simple example, we will assume that this PDO curve has the algebraic form: $\gamma = \beta^{0.6}$. In Figure 3, this PDO curve is represented by the solid curve.

Similar to what was already clarified in Section 3.5, the fiducial density function of $\mu$ conditional on $\sigma$ that is obtained by using the strong fiducial argument, i.e. the density $f_S(\mu|\sigma,x)$, is defined by the expression $\mu \sim N(2.7,\sigma^2/9)$. The lower dotted line in Figure 3 represents what the PDO curve under discussion would need to be so that the probability $\gamma$ equals the probability that would be assigned to the hypothesis $H_S$ by this fiducial density function. For the same reason as given in Section 3.2, we would expect any choice for the PDO curve concerned to be always higher than this dotted line, which is the case, as can be seen, for the PDO curve that has been proposed.

Using this proposed PDO curve and the fiducial density $f(\mu|\mu \notin [-0.2,0.2],\sigma,x)$ defined by equation (12) as inputs into the method described in Section 4.1 enables us to calculate, via equation (13), the post-data probability of $\mu$ lying in the interval $[-0.2,0.2]$ for any given value of the P value $\beta$. The dashed curve in Figure 3 was constructed by
plotting this post-data probability against different values for the P value in question. It can be seen that this curve is monotonically increasing, which is a desirable outcome. If this had not been the case, then it could have been quite reasonably concluded that the PDO curve on which this dashed curve is based does not represent logically structured beliefs about the parameters $\mu$ and $\sigma$ in the context of what has been assumed about these parameters.

Finally, the upper dotted curve in Figure 3 represents what the PDO curve of interest would need to be if, under the assumptions already made, we decided that independent of the size of the P value $\beta$, we would place a post-data probability on $\mu$ lying in the interval $[-0.2, 0.2]$ that was equal to the limiting value assigned to this probability by the dashed curve as $\beta$ tends to 0.5, i.e. a probability of 0.32. Clearly, any sensible PDO curve in the case being considered would need to lie below this dotted curve and, as can be seen, the proposed PDO curve also satisfies this constraint.

If in addition we choose the density function $h(\mu)$, which is required by equation (15), to be the same as it was in the example in Section 4.3 then the assumptions that have been made fully determine the post-data density $p(\mu | \sigma, x)$ according to the expressions given in equations (13), (14) and (16). Furthermore, if we more specifically assume that the sample variance $s^2 = 9$, then the fiducial density of $\sigma$ conditional on $\mu$ that is obtained by using the strong fiducial argument, i.e. the density $f_S(\sigma | \mu, x)$, is defined by the following expression:

$$\sigma^2 | \mu, x \sim \text{Scale-inv-}\chi^2(9, 8 + (\mu - 2.7)^2)$$

i.e. it is a scaled inverse $\chi^2$ density function with 9 degrees of freedom and scaling parameter equal to $8 + (\mu - 2.7)^2$.

Figure 4 shows some results from running a Gibbs sampler on the basis of the full conditional post-data densities that have just been defined, i.e. the density $p(\mu | \sigma, x)$ and the density $f_S(\sigma | \mu, x)$. In particular, the histograms in Figures 4(a) and 4(b) represent
the marginal post-data density functions of $\mu$ and $\sigma$ respectively. These histograms were formed on the basis of a single run of three million samples of $\mu$ and $\sigma$ generated by the Gibbs sampler after an initial 500 samples of its output were excluded due to the values concerned being classified as belonging to its burn-in phase. The sampling of the density $p(\mu \mid \sigma, x)$ was based on the Metropolis algorithm (Metropolis et al. 1953), while each value drawn from the density $f_S(\sigma \mid \mu, x)$ was independent from the preceding iterations. In addition to this analysis, the Gibbs sampler was run various times from different starting points, and a careful study of the output of these runs using appropriate diagnostics provided no evidence to suggest that the full conditional densities in question do not determine a joint density function for $\mu$ and $\sigma$. The curves overlaid on the histograms in Figures 4(a) and 4(b) are the marginal fiducial density functions for the corresponding parameters in the case where the full conditional fiducial densities for $\mu$ and $\sigma$ are the fiducial densities $f_S(\mu \mid \sigma, x)$ and $f_S(\sigma \mid \mu, x)$, which have already been specified in this section, i.e. in the case where both the parameters $\mu$ and $\sigma$ belong to the set $\theta^B$.

The accumulation of probability mass around the value of zero in the marginal post-
data density of $\mu$, as is apparent from Figure 4(a), and the fact that the upper tail of the marginal post-data density of $\sigma$ tapers down to zero slightly more slowly than the aforementioned marginal fiducial density for $\sigma$, as is apparent from Figure 4(b), are both clearly artefacts of the strong pre-data opinion that was held about $\mu$.

5.5. Second example: Inference about a relative risk

To go, in a certain sense, completely around the circle of examples considered in the present paper, let us now try to directly address the problem of inference posed in Example 2 of the Introduction, i.e. that of making inferences about a relative risk $\pi_t/\pi_c$.

For the same reason as given in the description of this example, let us assume that the scenario of Definition 2 would apply if $\pi_t$ was unknown and $\pi_c$ was known, and also if $\pi_c$ was unknown and $\pi_t$ was known. In particular, if we define $\text{odds}(a) = a/(1 - a)$ then, with respect to the parameter $\pi_t$, the hypothesis in this scenario will be assumed to be that $\text{odds}(\pi_t)$, i.e. the odds of $\pi_t$, lies in the narrow interval $I(\pi_c) = (\text{odds}(\pi_c)/(1 + \varepsilon), \text{odds}(\pi_c)(1 + \varepsilon))$, which is clearly an interval that contains $\text{odds}(\pi_c)$, i.e. the odds of $\pi_c$. In a symmetrical way, we will assume that, with respect to the parameter $\pi_c$, the hypothesis in the scenario of Definition 2 is that $\text{odds}(\pi_c)$ lies in the narrow interval $I(\pi_t) = (\text{odds}(\pi_t)/(1 + \varepsilon), \text{odds}(\pi_t)(1 + \varepsilon))$. Of course, having a high degree of pre-data belief that $\text{odds}(\pi_t) \in I(\pi_c)$ logically implies that one should have a high degree of pre-data belief that $\text{odds}(\pi_c) \in I(\pi_t)$. As a result of what has just been assumed, it is clear that, in terms of the notation of Section 5.1, the set of parameters $\theta^A$ will contain both the parameters $\pi_t$ and $\pi_c$, while the set $\theta^B$ will be empty.

Notice that, due to the independence of the data between the treatment and control groups, the fiducial density of $\pi_t$ conditional on $\pi_c$ and the fiducial density of $\pi_c$ conditional on $\pi_t$ derived by the strong fiducial argument will simply be the marginal fiducial densities of $\pi_t$ and $\pi_c$ derived by this argument respectively. Let us suppose that each of
these marginal fiducial densities, i.e. $f_S(\pi_t | x)$ and $f_S(\pi_c | x)$, is defined in the same way as the fiducial density $f_S(\pi | x)$ was defined in Section 4.4, i.e. on the basis of the LPD function $\omega_L(\pi)$ given in equation (8).

Clearly, these marginal fiducial densities would define a joint fiducial density function for $\pi_t$ and $\pi_c$ over all allowable values of $\pi_t$ and $\pi_c$. However, of course, the use of such a joint density function as a post-data density function of $\pi_t$ and $\pi_c$ in the present example would not be appropriate given what was known about $\pi_t$ and $\pi_c$ before the data were observed. In spite of this, it would be interesting to see what the marginal density of the relative risk $\pi_t/\pi_c$ over this joint fiducial density would look like. Therefore, the histogram in Figure 5 represents this marginal density function for the case where, in terms of the notation given in the Introduction, the observed counts are specified as follows: $e_t = 6$, $n_t = 20$, $e_c = 18$ and $n_c = 30$. This histogram was constructed on the basis of a sample of three million independent random values drawn from the marginal fiducial density concerned.

By contrast, it is standard practice to form (Neyman-Pearson) confidence intervals

Figure 5: Marginal fiducial density of the relative risk $\pi_t/\pi_c$
for any given relative risk by approximating the sampling density of the logarithm of
the relative risk by a normal density function, with the usual estimate of the variance
of this density function treated as though it is the true value of this variance (see for
example Altman 1991). For this reason, a curve has been overlaid on the histogram in
Figure 5 representing the confidence density function (see for example Efron 1993) of the
relative risk $\pi_t/\pi_c$ constructed on the basis of confidence intervals of this kind. It can be
clearly seen that this density function is very different from the density function of $\pi_t/\pi_c$
that is depicted by the histogram in this figure, which indicates the inadequacy of the
approximate confidence intervals in question, even if nothing or very little was known
about the parameters $\pi_t$ and $\pi_c$ before the data were observed.

To simplify proceedings, let us assume from now on that $\varepsilon = 0.08$, and the event
counts are equal to the values that have just been given. Under these assumptions, if for
a given value of $\pi_c$ the following condition holds

$$\beta_0 = \sum_{y=0}^{6} \binom{20}{y} (\pi_{t0})^y (1-\pi_{t0})^{20-y} \leq \sum_{y=6}^{20} \binom{20}{y} (\pi_{t1})^y (1-\pi_{t1})^{20-y} = \beta_1$$

where $\pi_{t0} = \text{odds}^{-1}(\text{odds}(\pi_c)/(1.08))$ and $\pi_{t1} = \text{odds}^{-1}(\text{odds}(\pi_c)(1.08))$, then in deter-
mining the post-data density $p(\pi_t | \pi_c, x)$, the hypotheses $H_P$ and $H_S$ would be defined
as:

$$H_P : \pi_t \geq \pi_{t0}$$
$$H_S : \rho(E_t^* \leq 6) \leq \beta_0$$

while otherwise, these hypotheses would have the definitions:

$$H_P : \pi_t \leq \pi_{t1}$$
$$H_S : \rho(E_t^* \geq 6) \leq \beta_1$$

where in both definitions of the hypothesis $H_S$, the random variable $E_t^*$ is assumed to be
the number of patients who would experience the adverse event concerned when receiving
drug $T$ in an additional as-yet-unrecruited group of $n_t = 20$ patients.
Clearly, the one-sided P values in the two versions of the hypothesis $H_S$, i.e. the P values $\beta_0$ and $\beta_1$, depend on the value of $\pi_c$. For this reason, to completely describe, in all circumstances of interest, what probability should be given to the hypothesis $H_S$, i.e. the probability $\gamma$, we would require the information that is conveyed by two separate PDO curves, where each one would correspond to one of the definitions of $H_S$ concerned. However, to give an example, it will be assumed that these two PDO curves are in fact the same, which actually is a reasonably justifiable assumption to make. In particular, we will assume that the single PDO curve in question has the simple algebraic form:

$$\gamma = ((0.92)^{\beta})^{0.6}$$

where $\beta$ is the P value of interest, i.e. the value $\beta_0$ or $\beta_1$. This PDO curve is represented by the solid curve in Figure 6(a).

The two highest short-dashed curves in this figure (which in fact appear to be almost a single long-dashed curve because they are so close together) represent what the PDO curve under discussion would need to be so that the probability $\gamma$ equals the probability that would be assigned to the hypothesis $H_S$ by the fiducial density $f_S(\pi_t | x)$ defined earlier. Each of these curves corresponds to one of the definitions of the hypothesis $H_S$. Of course, we would expect any choice for this PDO curve to be always higher than these two dashed curves, which is the case, as can be seen, for the PDO curve that has been proposed.

In determining the post-data density $p(\pi_c | \pi_t, x)$, the hypotheses $H_P$ and $H_S$ would be defined in a similar way to how they have just been defined. Even though it is now $\pi_c$ rather than $\pi_t$ that is the unknown parameter, it will be assumed that the PDO curves that are associated with the two definitions of $H_S$ that apply in this case are again equal to the single PDO curve defined in equation (19), which, taking into account especially the discrete nature of the data, is a slightly unsophisticated, but nonetheless, adequate assumption to make for the purposes of giving an example.
Figure 6: Graph (a) shows the PDO curve used when either $\pi_t$ or $\pi_c$ is treated as the only unknown parameter. Graphs (b) to (d) show the marginal post-data densities of $\pi_t$, $\pi_c$ and the relative risk $\pi_t/\pi_c$ respectively.

The two lowest dashed curves in Figure 6(a) represent what the PDO curve in this case would need to be so that the probability $\gamma$ equals the probability that would be assigned to $H_S$ by the fiducial density $f_S(\pi_c | x)$. As before, each curve corresponds to one of the definitions of $H_S$. These two curves are clearly quite close to the other two (almost overlaid) dashed curves in this figure, but reassuringly a long way below the
proposed PDO curve.

To fully determine the full conditional post-data densities $p(\pi_t \mid \pi_c, x)$ and $p(\pi_c \mid \pi_t, x)$ according to the expressions given in equations (13), (14) and (16), let us specify the density functions $h(\pi_t)$ and $h(\pi_c)$, which are required by equation (15), such that $\log(\text{odds}(\pi_t))$ and $\log(\text{odds}(\pi_c))$ have beta density functions on the intervals obtained by converting, respectively, the intervals $I(\pi_c)$ and $I(\pi_t)$ to the logarithmic scale, with both shape parameters of this density function equal to 4.

Figures 6(b) to 6(d) show some results from running a Gibbs sampler on the basis of approximations to the full conditional post-data density functions being considered, and in particular, the histograms in these three figures represent the marginal post-data density functions of $\pi_t$, $\pi_c$ and the relative risk $\pi_t/\pi_c$ respectively. These histograms were formed on the basis of a single run of three million samples of $\pi_t$ and $\pi_c$ generated by the Gibbs sampler after allowing for its burn-in phase by excluding an initial 1000 samples of its output. The sampling of both the densities $p(\pi_t \mid \pi_c, x)$ and $p(\pi_c \mid \pi_t, x)$ was based on the Metropolis algorithm. Furthermore, as in the previous example, the Gibbs sampler was also run various times from different starting points, and there was no suggestion from using appropriate diagnostics that these full conditional densities do not determine a joint density function for $\pi_t$ and $\pi_c$.

As already mentioned, there was an approximate aspect to these simulations, but this was simply due to the fiducial densities $f_S(\pi_t \mid x)$ and $f_S(\pi_c \mid x)$ being approximated, respectively, by the posterior densities of $\pi_t$ and $\pi_c$ that are based on the use of the corresponding Jeffreys prior for these parameters, which can be recalled is an approximation that was also used in Section 4.4. Similar to earlier, additional simulations showed that, for the data in question, using this method to approximate the two fiducial densities concerned was very adequate.

Under this type of approximation, the curves overlaid on the histograms in Figures 6(b)
and 6(c) represent the fiducial densities $f_S(\pi_t \mid x)$ and $f_S(\pi_c \mid x)$ respectively. It can be seen that relative to these fiducial densities, the marginal post-data density of $\pi_t$ could be described as being drawn towards the sample proportion in the control group, i.e. $e_c/n_c = 0.6$, which is especially apparent in the upper tail of this density, while the marginal post-data density of $\pi_c$ could be described as being drawn towards the sample proportion in the treatment group, i.e. $e_t/n_t = 0.3$, which is especially apparent in the lower tail of this density. These characteristics would of course be expected given the nature of the pre-data opinion about $\pi_t$ and $\pi_c$ that has been incorporated into the inferential process.

The curve overlaid on the histogram in Figure 6(d), by contrast, represents the confidence density function of the relative risk $\pi_t/\pi_c$ that was referred to earlier, and which also appears in Figure 5. As can be seen, it is very different from the marginal post-data density of this relative risk. Of course, the accumulation of probability mass around the value of one for $\pi_t/\pi_c$ in this latter density function is clearly an artefact of the strong pre-data opinion that was held about the parameters concerned.

6. A closing remark

Taking into account all that was discussed in the Introduction, observe that, even to attempt to construct adequate Bayesian solutions to problems of inference that are loosely similar to the type of problem that has been of interest in the present paper would require the elicitation of at least $m$ full conditional prior density functions, assuming that the set $\theta^A$, as defined in Section 5.1, contains $m$ parameters. Applying the bispatial-fiducial method that has been proposed to these problems would require, under the kind of assumptions made in Sections 5.4 and 5.5, the elicitation of the same number of post-data opinion (PDO) curves, i.e. $m$ such curves. In this sense, therefore, this latter method does not carry an extra burden in terms of the required quantification of opinions.
about the parameters in the model that are based on pre-data knowledge about these parameters. Moreover, the case can reasonably be made that, given their connection with the objective information contained in the data, these PDO curves will generally be easier to determine than the prior density functions in question, and above all, of course, by using this method, we can obtain a direct solution to the precise problem that we actually have been concerned with rather than to clearly distinct, and arguably, much less common types of problem.

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