Algorithm for solving problems related to the natural vibrations of electro-viscoelastic structures with shunt circuits using ANSYS data

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ABSTRACT
An algorithm for numerical realisation of a mathematical statement of the natural vibrations problem for electro-viscoelastic bodies with passive external electric circuits (i.e. shunting circuits) with an arbitrary configuration using the finite element method is proposed in the present paper. The proposed algorithm allows considering the viscoelastic properties of materials using the model of linear hereditary viscoelasticity with complex dynamic moduli and is used to solve 3D solid structure problems that are compatible for ANSYS package element types. This technique implies the usage of the global assembled matrices of stiffness and mass, formed in the ANSYS package. The basis of the algorithm is a novel approach that allows performing decomposition of the global assembled stiffness matrix formed in the ANSYS software package into constituents that are needed for calculation of the natural vibration frequencies of the objects under study. These matrix components are used in the program that was written in FORTRAN (Formula Translation) language. This problem could be efficiently applied for analysis of the dynamic processes in smart systems based on piezoelectric materials and could also form a basis for the development of numerical finite element algorithms for optimization of the dissipative characteristics of electromechanical systems with shunted piezoelectric elements.

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1. Introduction
Taking into account the manifold options for control of the dynamic characteristics of structures with piezoelectric elements and external electric circuits, the search for optimal solutions using mathematical modelling is still ongoing. Many studies to date have utilized a finite element method for the numerical modelling, as highlighted in a number of review papers [1–3].

It is a well-known fact that the development of a full program for realization of a numerical solution for the natural vibrations problem using the finite element method is a rather difficult and time-consuming process, beginning from building the geometry and ending with processing the results of the solution. Nowadays, the finite element
method has been realized in many different application software packages (ANSYS, Abaqus, NASTRAN, etc.), providing possibilities to find solutions to different continuous media mechanic problems (including coupled problems). Therefore, utilization of the existing software packages makes it possible for researchers to significantly accelerate their investigations.

The issues related to the dynamic behaviour of piezoelectric-based smart structures connected to an external electric circuit are mainly modelled with the aid of algorithms incorporated in commercial engineering software packages (such as ANSYS, Abaqus, MSC or NASTRAN) or using the original author’s calculation procedures realized in various computational packages, such as MatLab [4–26]. Unfortunately, the latter case as a rule considers only 2D structures, such as plates or beams.

Conventionally, when analyzing electromechanical systems by the finite element method with the aid of numerical engineering software packages, three kinds of problems are considered: the problem of natural vibrations (modal analysis), the problem of steady-state vibrations (harmonic analysis) and the problem of transient processes (transient analysis) [1]. The damping properties can be estimated by the magnitude of the amplitude in the resonant mode or by the damping rate of the transient processes. In the first case, the problem of steady-state vibrations is solved, while in the second case, the dynamic problem with the initial conditions is solved. Solving these problems alone though is rather inefficient for achieving optimization of the dynamic properties due to a number of reasons. For instance, to obtain amplitude values at resonant modes on the basis of the steady-state vibrations problem, it is necessary to repeatedly solve the problem at different external excitation frequencies. For this problem or the problem with the initial conditions, the found optimal solutions depend on the specific modelling options used for the system loading, and hence the optimal solution may change, if one changes these conditions.

Regarding this, the natural vibrations problem is a more convenient tool for analyzing the dynamic behaviour of electromechanical systems (e.g. structures with piezoelectric elements and external electric circuits). Also it is a more convenient tool due to the fact that the dynamic characteristics obtained during the process of finding the problem solution (e.g. complex natural vibration frequencies of the system) do not depend on the kind of external loading. Moreover in the context of solving the optimization problem, no repeated solutions are required for all the same values of the parameters one is searching for, as it is needed for the steady-state vibrations problem when calculating amplitude vs. frequency plots, for instance.

However despite the continuous updating process and the inclusion of new modules into finite element software packages, it is often necessary to solve new tasks that are beyond the current abilities of software.

To overcome this problem, it helps that the majority of modern software packages allow obtaining the information about the geometry and finite-element mesh. Using this data, it is possible to assemble global finite-element matrices and to build up the resolving matrix-like equation in the required form. With some software packages, it is also possible to receive global assembled matrices required for resolving the equation directly. The matrices obtained one way or another can be further used in special programs and in algorithms developed by researchers.
In the present work, an algorithm for numerical realization of a mathematical statement of the natural vibrations problem for electro-viscoelastic bodies with an external electric circuit, which allows using the stiffness and mass matrices formed in the ANSYS application software package (License Academic Research Mechanical and CFD № 1064623) is proposed. This algorithm also allows taking into account the viscoelastic properties of structures on the basis of a viscoelasticity model with complex dynamic moduli.

2. Mathematical statement of the problem of natural vibrations

A piecewise-homogeneous body of volume \( V = V_1 + V_2 \), where volume \( V_1 \) consists of \( N \) homogeneous elastic or viscoelastic parts, and volume \( V_2 \) consists of \( M \) electroelastic (piezoelectric) elements was established as the object under study. A part of the \( V_2 \) volume surface is covered with electrodes.

The variational equation of motion of the body, consisting of elastic/viscoelastic and piezoelectric elements was derived on the basis of the linear theory of elasticity relations and quasi-static Maxwell's equations [27–30]:

\[
\sum_{n=1}^{N} \left( \int_{V_1^n} (\sigma_{ij} \delta \varepsilon_{ij} + \rho_n \dddot{u}_i \delta u_i) \, dV \right) + \sum_{m=1}^{M} \left( \int_{V_2^m} (\sigma_{ij} \delta \varepsilon_{ij} - D_i \delta E_i + \rho_m \dddot{u}_i \delta u_i) \, dV \right) = \int_{S_\sigma} \rho_i \delta u_i \, dS + \int_{S_p} q_e \delta \varphi \, dS
\]

where, \( D_i \) and \( E_i \) are the components of the electric flux density vector and the electric field intensity vector, respectively; \( \sigma_{ij} \) represents the components of the symmetric Cauchy stress tensor; \( \varepsilon_{ij} \) represents the components of the linear strain tensor; \( u_i \) represents the components of the displacement vector; \( \rho_n \) is the mass density of the \( n \)-th constituent of the piecewise-homogeneous body of volume \( V_1^n \); \( \rho_m \) is the mass density of the piezoelectric material of the \( m \)-th constituent of the electroelastic body of \( V_2^m \); \( S_\sigma \) is part of the \( V \) volume surface, where the surface loads \( P_i \) are prescribed; \( S_p = S_{el} + S_0 \) is the surface of the piezoelectric body of the volume \( V_2 \), where part of the surface \( S_{el} \) is covered with electrodes and part of the surface \( S_0 \) is without electrodes; \( q_e \) is the electric charge surface density; \( \varphi \) is the electric potential and \( \delta \) is the variation of the corresponding variable.

For the electric field, the potentiality condition is fulfilled and can be written as:

\[
\varphi_{,i} = -E_i
\]

For the case of isothermal processes, the following physical relations are valid:

- for the elastic parts of the \( V_1 \) volume:

\[
\sigma_{ij} - \sigma_{\delta ij} = 2G^{(n)} \left( \varepsilon_{ij} - \frac{1}{3} \delta \varepsilon_{ii} \right), \quad \sigma = B^{(n)} i j
\]

- for the viscoelastic parts of the \( V_1 \) volume:
\[
\sigma_{ij} = 2G_m^0 \left( e_{ij} - \int_0^t R^{(m)}(t-\tau)e_{ij}(\tau)d\tau \right),
\]
\[
\sigma = B_m^0 \left( \vartheta - \int_0^t K^{(m)}(t-\tau)\vartheta(\tau)d\tau \right)
\]

- for the piezoelectric elements of the \( V_2 \) volume:
\[
\sigma_{ij} = C_{ijkl}^{m}e_{kl} - \beta_{ijk}^{m}E_k \\
D_k = \beta_{ijk}^{m}E_j + e_{ki}^{m}E_i
\]

where \( G^{(n)}, B^{(n)} \) are the elastic shear and bulk moduli; \( G_m^0, B_m^0 \) are the instant shear and bulk moduli; \( R^{(m)}, K^{(m)} \) are the relaxation kernels; \( \sigma \) is the mean stress; \( \vartheta \) is the volumetric strain; \( s_{ij}, e_{ij} \) are the components of the deviators of the stress and strain tensors; \( C_{ijkl} \) are the components of the piezoelectric material elastic constant tensor; \( \beta_{ijk} \) and \( e_{ki} \) are the components of the piezoelectric coefficients tensor and the dielectric coefficients tensor, \( i, j, k, l = 1,2,3 \).

The properties of a viscoelastic material are described by the model of linear hereditary viscoelasticity with complex dynamic moduli. This model is well-fitted to describing the mechanical behavior of a wide class of structural materials, including composite materials.

For the electro-viscoelastic problem under study, the boundary conditions can be divided into two types: mechanical and electrical.

The mechanical boundary conditions are written by analogy with problems in the theory of elasticity and viscoelasticity:
\[
S_\sigma: \quad \sigma_{ij}n_j = 0, \quad S_u: \quad u_i = 0
\]

where \( S_\sigma \) is the surface, where the stresses \( \sigma_{ij} \) are prescribed; \( S_u \) is the surface, where the components of the displacement vector \( u_i \) are given and \( n_j \) are the components of the unit normal vector for the \( S_\sigma \) surface.

For the non-electrode parts of the surface \( S_0 \) of a piezoelectric body of volume \( V_2 \) there are no electric boundary conditions due to the fact that these parts of the surface are non-conductive and hence there are no free electric charges on them. Taking into account the used form of Maxwell’s equations, this condition can be written as:
\[
\int_S (\vec{n} \cdot \vec{D})dS = 0 \quad \text{or} \quad \text{div} \vec{D} = 0
\]

The electric boundary conditions depend on the way electric energy is transferred to the piezoelectric body. The supplying and withdrawal of electrical energy for the piezoelectric body are performed with the aid of the electrode coverage on parts of the body’s surface. Further, it is implied that the electrode coverages are thin ideal conductors with negligible mass. The presence of electrode coverage at the surface \( S_{el} \) makes it an equipotential surface.

The specific form of boundary conditions for the electric component of the state vector depends on the type of energy sources. However, existing well-known models for
the energy sources from electrical engineering, such as the current generator and voltage generator models, can be used.

Supposing one of the electrode surfaces of the piezoelectric element is grounded, i.e. it has a zero-value electric potential, the electric boundary condition for this part of the surface has the following form:

$$\varphi|_{S_{el}^\phi} = 0$$  \hspace{1cm} (8)

where $S_{el}^\phi$ is the part of the electrode surface $S_{el}$ of volume $V_2$, where the electric potential is prescribed. If there is no an external power supply, the other parts of the electrode surface $S_{el}$ can be considered free (in this case, the open circuit mode is realized) or it can be considered to have zero-value electric potential (8) (here, the short circuit mode is realized).

By using the electrode surfaces, the external electric circuits can be attached with an arbitrary configuration to the system under consideration, which could include resistive elements ($R$), capacitive elements ($C$) or inductive elements ($L$). If these circuits are not supplied by an external power source, they are classed as internal elements of the system and then the following term should be added to Equation (1):

$$\sum \delta A = \delta A_L + \delta A_R + \delta A_C$$  \hspace{1cm} (9)

This term takes into account all the internal works $\sum \delta A$ of the electric field with a potential difference $\pm \varphi$ for transferring the arbitrary electric charge on the circuit elements $\delta A_L$, $\delta A_R$, $\delta A_C$:

$$\delta A = \varphi \delta q$$  \hspace{1cm} (10)

On the basis of some well-known relations from electrical engineering [31], the expressions for all of the terms in Equation (9) can be obtained. An external circuit made of a single element from among those presented in Figure 1 is considered here:

- for a resistive circuit:

$$\delta A_R = \frac{1}{R} \int (\varphi_1^R - \varphi_2^R) \delta \varphi dt = 0$$  \hspace{1cm} (11)

where $\varphi_1^R$, $\varphi_2^R$ are the electric potentials at points 1 and 2 (Figure 1(a)).

- for a capacitive circuit:

$$\delta A_C = C (\varphi_1^C - \varphi_2^C) \delta \varphi = 0$$  \hspace{1cm} (12)

where $\varphi_1^C$, $\varphi_2^C$ are the electric potentials at points 1 and 2 (Figure 1(b)).

Figure 1. Elements of external electric circuits: (a) resistance ($R$), (b) capacitance ($C$), (c) inductance ($L$).
- for an inductive circuit:

\[ \delta A_L = \frac{1}{L} \int \left( \varphi_L^1 - \varphi_L^2 \right) \delta \varphi \, dt \, dt = 0 \]  \hspace{1cm} (13)

where \( \varphi_L^1, \varphi_L^2 \) are the electric potentials at points 1 and 2 (Figure 1(c)).

Taking into account the form of the terms (11–13) in expression (9), the variational equation for the motion of an electro-viscoelastic body with an external electric circuit takes the following form:

\[
\sum_{n=1}^{N} \left( \int_{V_n} \left( \sigma_{ij} \delta \epsilon_{ij} + \rho_n \delta u_i \right) \, dV \right) + \sum_{m=1}^{M} \left( \int_{V_m} \left( \sigma_{ij} \delta \epsilon_{ij} - D_i \delta E_i + \rho_m \delta u_i \right) \, dV \right) - \\
- \int_{S_p} p_i \delta u_i \, dS - \int_{S_p} q_e \delta \varphi \, dS + \sum_{n=1}^{n_L} \frac{1}{L_p} \int_{S_p} \left( \varphi_L^1 - \varphi_L^2 \right) \delta \varphi \, dt \, dt + \\
+ \sum_{q=1}^{n_R} \frac{1}{R_q} \int_{S_p} \left( \varphi_R^1 - \varphi_R^2 \right) \delta \varphi \, dt + \sum_{r=1}^{n_C} \left( \varphi_C^1 - \varphi_C^2 \right) \delta \varphi = 0
\]  \hspace{1cm} (14)

where \( \varphi_L^{\text{cir}} - \varphi_L^{\text{cir}} \) is the potential difference in the corresponding circuit element \( \text{cir} = L_p, R_q, C_r, n_L, n_R, n_C \) are the quantities of the inductive, resistive and capacitive elements, respectively; \( L_p, R_q, C_r \) are the values of the inductance, resistance and capacitance of the corresponding circuit elements.

For the problem of natural vibrations under zero-value boundary conditions (6–8), eigensolutions are sought as:

\[
u_n(x, t) = \bar{u}_n(x) e^{i \omega t}, \quad \varphi(x, t) = \bar{\varphi}(x) e^{i \omega t}
\]  \hspace{1cm} (15)

where \( \omega = \omega_\text{Re} + i \omega_\text{Im} \) gives the complex natural vibration frequency, wherein \( \omega_\text{Re} \) corresponds to the circular natural vibration frequency; \( \omega_\text{Im} \) is the rate of its damping (the damping ratio of vibrations) and \( \bar{u}_n(x), \bar{\varphi}(x) \) are the natural vibration forms.

For the problems on natural vibrations, the following option of physical relations (4) is used [32]:

\[
\sigma_{ij} - \sigma \delta_{ij} \approx 2 \tilde{G}^{(n)} \left( \epsilon_{ij} - \frac{1}{3} \vartheta \delta_{ij} \right),
\]

\[
\sigma \approx \tilde{B}^{(n)} \vartheta
\]  \hspace{1cm} (16)

where \( \tilde{G}^{(n)} = G_\text{Re}(n) + i G_\text{Im}(n) \), \( \tilde{B}^{(n)} = B_\text{Re}(n) + i B_\text{Im}(n) \) are the complex dynamic shear and bulk moduli; \( G^{(n)}, B^{(n)} \) are the elastic shear and bulk moduli; \( \sigma \) is the mean stress; \( \vartheta \) is the volumetric strain and \( s_{ij}, e_{ij} \) are the components of the deviatoric parts of the stress and strain tensors, \( i, j, = 1,2,3 \).

Considering the form of solution (15), the variational equation for the problem of the natural vibrations of an electro-viscoelastic body with an external electric circuit in the case of an absence of external loads takes the following form:
$N \sum_{n=1}^{N} \left( \int_{V_n} (\sigma_{ij} \delta \epsilon_{ij} - \rho_n \omega^2 u_i \delta u_i) \, dV \right) +$

$+ M \sum_{m=1}^{M} \left( \int_{V_m} ((\sigma_{ij} \delta \epsilon_{ij} - D_i \delta E_i - \rho_m \omega^2 u_i \delta u_i) \, dV \right)$

$- \sum_{p=1}^{n_l} \frac{1}{\omega^2 L_p} \delta \phi \left( \varphi_1^{k_p} - \varphi_2^{k_p} \right) +$

$+ \sum_{q=1}^{n_e} \frac{1}{\omega R_q} \left( \varphi_1^{k_{pq}} - \varphi_2^{k_{pq}} \right) \delta \phi + \sum_{r=1}^{n_c} C_r (\varphi_1^{k_c} - \varphi_2^{k_c}) \delta \phi = 0 \tag{17}$

Typical procedures in the finite element method reduce the problem of the natural vibrations of electro-viscoelastic bodies with an external circuit, which is described by Equation (16), to the following algebraic eigenvalue problem:

$$([K] - \omega^2 [M] + [C(\omega)]) \{\delta\} = 0 \tag{18}$$

Equation (18) sufficiently differs from the generalized eigenvalue problem due to the presence of the matrix $[C(\omega)]$, which describes the external electric circuit. In view of relations (11–13), this matrix can be represented as:

$$[C(\omega)] = - \sum_{p=1}^{n_l} \frac{1}{\omega^2 L_p} [K_{lp}] + \sum_{q=1}^{n_e} \frac{1}{\omega R_q} [K_{pq}] + \sum_{r=1}^{n_c} C_r [K_{cr}] \tag{19}$$

where $n_l, n_e, n_c$ are the numbers of the inductive, resistive and capacitive elements respectively. As a result this leads to the necessity for development of an algorithm allowing one to obtain global stiffness matrices for each element of the electric circuit $[K_{lp}], [K_{pq}], [K_{cr}]$ from the modal analysis module in the ANSYS package.

For the case of a piecewise-homogeneous body consisting of $M$ electroelastic and $N$ viscoelastic parts, the global stiffness matrix $[K]$ can be represented as the sum of the corresponding piezoelectric and viscoelastic parts $[K^{n_p}_p]$ and $[K^{n}_{vis}]$:

$$[K] = \sum_{n=1}^{N} [K^{n}_{vis}] + \sum_{m=1}^{M} [K^{m}_p] \tag{20}$$

According to relations (16) the stiffness matrix of $n$-th viscoelastic part using the algorithm described in [32] can be described as:

$$[K^{n}_{vis}] = \hat{B}^{(n)} [K^{n}_B] + \hat{G}^{(n)} [K^{n}_B] \tag{21}$$

The numerical realization of the problem established using the finite element method reduces the main resolving equation to:

$$\left\{ \begin{array}{l}
\sum_{n=1}^{N} \left[ (G^{(n)}_{\text{Re}} [K^{n}_G] + B^{(n)}_{\text{Re}} [K^{n}_G]) + i (G^{(n)}_{\text{Im}} [K^{n}_G] + B^{(n)}_{\text{Im}} [K^{n}_G]) \right] + \\
+ \sum_{m=1}^{M} [K^{m}_p] - \omega^2 [M] - \\
- \sum_{p=1}^{n_l} \frac{1}{\omega^2 L_p} [K_{lp}] + \sum_{q=1}^{n_e} \frac{1}{\omega R_q} [K_{pq}] + \sum_{r=1}^{n_c} C_r [K_{cr}] 
\end{array} \right\} \{u\} = \{0\} \tag{22}$$
Here the first row of the equation represents the relations for the elastic or viscoelastic parts of the object under study, the second row is the relations for the piezoelectric elements and global mass matrix, and finally the third row describes the electric circuit connected to the piezoelectric element’s electrodes. The following notations are introduced: \( K_n^B \), \( K_n^G \) are the bulk and shear components of the stiffness matrix of the \( n \)-th viscoelastic part; \( K_m^p \) is the stiffness matrix of the \( m \)-th piezoelectric element; \( M \) is the global mass matrix of the whole structure; \( K_n^{lp} \) are the \( n_L \) matrices for the inductive elements of the circuit; \( K_n^{rq} \) are the \( n_R \) matrices for the resistive elements of the circuit; \( K_n^{cr} \) are the \( n_C \) matrices for the capacitive elements of the circuit; \( L_p \), \( R_q \), \( C_r \) are the magnitudes of the inductance, resistance and capacitance, respectively; and \( \omega = \omega_{Re} + i\omega_{Im} \) is the sought-for complex natural frequency.

The condition for the existence of a non-trivial solution of Equation (22) is equality with zero for the following determinant:

\[
D(\omega) = \det \left( \sum_{m=1}^{M} [K_n^p] + \sum_{n=1}^{N} \left( \left( C_{Re}^{(n)} [K_n^G] + B_{Re}^{(n)} [K_n^B] \right) + i \left( C_{Im}^{(n)} [K_n^G] + B_{Im}^{(n)} [K_n^B] \right) \right) - \omega^2 [M] - \sum_{p=1}^{n_L} \frac{1}{\omega^2 L_p} [K_n^{lp}] + \sum_{q=1}^{n_R} \frac{1}{i\omega R_q} [K_n^{rq}] + \sum_{r=1}^{n_C} C_r [K_n^{cr}] \right) = 0 \quad (23)
\]

The mathematical statement of the problem is presented in detail in [33].

3. Algorithm for numerical solution on the basis of the ANSYS package

ANSYS has no direct possibilities for obtaining a solution of the problem of the natural vibrations of electro-viscoelastic bodies with an external electric circuit corresponding to the mathematical statement presented above.

The first problem is that it is impossible to describe the properties of viscoelastic material by complex dynamic moduli according to formulation (16) when using the modal analysis module realized in ANSYS. This can be explained by the fact that the ANSYS package solver is only able to solve eigenvalue problems that are described by a matrix equation in the form of (24):

\[
(-\omega^2 [M] + i[D] + [K]) \{ \delta \} = 0 \quad (24)
\]

Here, \([D]\) is the damping matrix with the real coefficients linearly dependent on \( \omega \), as described in detail in [32].

However, this option for consideration of the damping properties has no possibility of describing the viscoelasticity on the basis of complex dynamic moduli. This fact leads to the necessity for development of an algorithm allowing representation of the stiffness matrix of the \( n \)-th viscoelastic part of structure \([K^n]\) considering (16) as follows:
\[
[K^n] = B[K^G_B] + G[K^G_G] = \\
= G^{(n)}_{\text{Re}}[K^G_G] + B^{(n)}_{\text{Re}}[K^G_B] + i\left\{ G^{(n)}_{\text{Im}}[K^G_G] + B^{(n)}_{\text{Im}}[K^G_B] \right\} = ,
\]

(25)

where \(G^{(n)}(\omega), G^{(n)}_{\text{Re}}(\omega), B^{(n)}_{\text{Re}}(\omega), B^{(n)}_{\text{Im}}(\omega)\) are the functions of frequency \(\omega\).

Regarding this the resolving system of Equation (18) takes the form (26)

\[
(-\omega^2[M] + [K_{\text{Re}}(\omega)] + i[K_{\text{Im}}(\omega)] + [C(\omega)]) \{\delta\} = 0,
\]

(26)

which significantly differs from Equation (24) due to the presence of the matrices \([C(\omega)], [K_{\text{Re}}(\omega)]\) and \([K_{\text{Im}}(\omega)]\), which have a complicated dependence on \(\omega\). So due to this, Equation (26) cannot be solved with just the ANSYS package.

This leads to the necessity for development of a special algorithm that would allow obtaining the stiffness matrix \([K^n_{\text{vis}}]\), as formed in the modal analysis module of ANSYS, in the form (21).

The second problem is the incapability of the ANSYS modal analysis solver to solve eigenvalue problems described by matrix equations different from (24). This aspect does not allow solving the above-stated natural vibration problem with an external electric circuit. Although the ANSYS package allows modelling elements of external electric circuits, such as resistors, inductors, capacitors and also current and voltage sources, these kinds of elements are available only for harmonic analysis or transient analysis. The lack of possibility of obtaining a solution to the natural vibration problem for electro-elastic bodies with external electric circuits is explained by the fact that relations describing the behaviour of external circuit elements (resistors and inductors, in particular) contain multipliers inversely proportional to the frequency (22).

Thus, the modal analysis module in the ANSYS software package is not able to provide a solution for the natural vibration problem for electro-viscoelastic bodies with an external electric circuit. Nevertheless, this module can allow obtaining the global stiffness and mass matrices in a form that allows one to receive all the matrices included in Equation (22) and to therefore achieve a solution to the problem.

Using the ANSYS software package to retrieve all the matrices included in Equation (22), only three kinds of matrices are available, namely the global stiffness matrix \([K]\), the global mass matrix \([M]\) and the global damping matrix \([D]\) (24–25). These matrices can be obtained only from the modal analysis module. For all other cases (e.g. static analysis, harmonic analysis and transient analysis), only one global matrix, \([\bar{K}]\) is formed, which includes both the mechanical and mass properties of the object under study. Furthermore, this matrix cannot be decomposed to the matrices required for solution of the problem. These two problems led to the need to develop additional algorithms. Specifically, these algorithms are needed to help receive all the required matrices for Equation (22) by decomposition of the global stiffness matrix \([K]\) obtained from ANSYS. For modelling external electrical circuits in ANSYS, there are two-node finite elements simulating resistive elements \((R)\), inductive elements \((L)\) and capacitive elements \((C)\).

These finite elements can be described by the following matrix relations:

\[
[K^e_R] = \frac{1}{i\omega R} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad [K^e_L] = -\frac{1}{\omega^2 L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad [K^e_C] = C \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}
\]

(27)
However, due to the presence of multipliers dependent on the frequency for the resistive and inductive elements, they cannot be used for the modal analysis. However, the possibility of using finite elements to describe the capacitive elements remains since the matrix relation for these elements does not contain a frequency-dependent multiplier.

Regarding this aspect, the necessity to develop an algorithm that would allow obtaining the matrices $[K_{lp}], [K_{Re}], [K_{C}]$ from the modal analysis module in the ANSYS software package thus became apparent. In the most common form, this algorithm can be represented as shown in Figure 2.

In order to overcome these difficulties, a special algorithm was developed for the matrix decomposition. This algorithm was realized in APDL (ANSYS Parametric Design Language) and allowed turning the decomposition of the global stiffness matrix, obtained from the modal analysis module in ANSYS, into the matrices $[K_{vis}^n], [K_{vis}^m]$ and $[C(\omega)]$. Here, $[K_{vis}^n]$ contains elements only for the $n$-th viscoelastic part of the structure, while $[K_{vis}^m]$ contains elements only for the $m$-th piezoelectric part of the structure and $[C(\omega)]$ contains elements only for the external electric circuit. The matrices $[K_{vis}^n]$ and $[C(\omega)]$ are decomposed as follows: $[K_{vis}^n]$ is decomposed into matrices $[K_{vis}^G], [K_{vis}^B]$, and $[C(\omega)]$ is decomposed into matrices $[K_{lp}], [K_{Re}], [K_{C}]$. The most common form of the algorithm for obtaining all of the matrices included in Equation (22) is presented in Figure 3. For dividing the global stiffness matrix $[K]$ into its constituents related to

![Diagram](image)

**Figure 2.** The common scheme for the natural vibration problem solution using the finite element method.
different parts of the object under study (e.g. the elastic elements, viscoelastic elements, piezoelectric elements and external circuit elements), the approach described in [19] was used. The concept of this approach is as follows: the matrix $[K]$ is decomposed into the required constitutive elements by varying the material constant values and the mathematical addition and subtraction operations.

4. Description of the algorithms for decomposition of the matrices

In the first stage, the full finite-element model of the structure under study is built in ANSYS: the geometry is set, the material characteristics are specified and the process of meshing the geometric model into finite elements is done.

The problem is solved in a 3D-statement. The finite elements with quadratic approximation of the nodal unknowns (SOLID 186, SOLID 226 [34]) are used while building the finite-element model. It should be noted that decomposition of the global stiffness matrix into bulk and shear parts in the form of (16) can be realized only for SOLID finite element types, and but is not possible for the SHELL-type elements. Depending on the required tolerance of approximation and the possible restrictions put on the number of nodal unknowns for the full-order finite-elemental model, any combination of the following element types can be used to realize the algorithm under study: for elastic/viscoelastic parts – SOLID45, SOLID92, SOLID95, SOLID185, SOLID186, SOLID187; for electroelastic parts – SOLID5, SOLID98, SOLID226, SOLID227.

Consider an algorithm of matrices’ decomposition for the example of a piecewise-homogeneous body made completely of a viscoelastic material (having only one viscoelastic element) with a single piezoelectric element attached to its’ surface. The piezoelectric element’s electrodes are connected to a single external electric circuit with an arbitrary configuration.
4.1. Extraction of stiffness matrices for the elements of the external electric circuit

For simulating electric circuits in ANSYS, the special two-nodal finite elements are used. For this, special two-nodal finite elements matrices of stiffness \([K^e_L]\), \([K^e_R]\), \([K^e_C]\) for the inductive, resistive and capacitive elements, respectively, take the form shown in (27). As can be seen from (27), the structures of the matrices \([K^e_L]\), \([K^e_R]\), \([K^e_C]\) are identical. The only difference between them is in the multipliers before the matrices. These multipliers are also presented in Equation (22) as multipliers for the global matrices \([K_L]\), \([K_R]\), \([K_C]\).

Due to the fact that these matrices contain no parameters characterizing the specific kind of circuit element, the type of external circuit’s finite element can be arbitrary selected to receive the required matrices for the demanded structure from ANSYS for each circuit element. This peculiarity allows further simulating all the circuit’s elements (resistors, inductors and capacitors) with the help of only capacitive-like finite elements for obtaining the stiffness matrix describing the external electric circuit.

The parameters of the external circuit’s elements, such as capacitance, inductance and resistance, in ANSYS are assigned by corresponding sets of the so-called real constants [34]. In order to obtain the stiffness matrix for each circuit element, it is necessary to specify as many sets of real constants for as many circuit elements as the external circuit contains. For instance, four finite elements will be used for the circuit’s simulation if the real circuit is composed of one resistor, two inductance coils and one capacitor. Consequently, the four sets of real constants should be specified. For further convenience let’s accept that the real constants for the external circuit’s finite elements may take only the value 1 or 2. Since only capacitive types of finite elements are used for obtaining the corresponding matrices, these real constants have a meaning of capacitance (in this case 1F or 2F, respectively).

In the first stage, the global stiffness matrix \([K^0_{cir}]\) is written, in which for all the elements of the external circuit, all the real constant’s values are set equal to 1. Furthermore, the set of \(Z\) global stiffness matrices \([K^i_{cir}]\) is written (where \(Z\) is the number of elements of the electric circuit) while changing the values of the real constants from 1 to 2, where \(i = 1, 2\) is the number of finite elements for which the real constant is set equal to 2. It is important to note that the real constant should take a value of 2 only for single \(i\)-th circuit’s element. All other real constants related to the other circuit elements should remain equal to 1. As a result, the stiffness matrix for each \(i\)-th circuit element \([K^{cir(i)}]\) (resistor, capacitor or inductor) can be obtained by subtracting the matrix \([K^0_{cir}]\) from the matrix \([K^i_{cir}]\):

\[
[K^{cir(i)}] = [K^i_{cir}] - [K^0_{cir}]
\] (28)

Since the parameters for the continual part of the structure (viscoelastic and piezoelectric) have not been changed during the process of writing the matrices \([K^i_{cir}]\), when performing the subtraction operations (28) all the components of the matrices related to the structure will be zero. After performing all the operations (28), the matrices \([K^{cir(i)}]\) with dimensions corresponding to the dimensions of the global resolving system of algebraic equations were obtained. At the same time, they only had the coefficients (1)
and \((-1)\) at positions related to the nodal unknowns of the corresponding electric circuit’s element.

As a result, the matrix with the required structure for each specific element of the external circuit could be retrieved. Depending on the kind of element in the real electric circuit, these matrices \([K_{\text{Cir}(i)}]\) when substituting in (22) should be multiplied by the corresponding coefficients: \(1/\omega^2 L\) for inductive element, \(-i/\omega R\) for resistive element and \(C\) for capacitive element. After these multiplications, the required matrices \([K_p]\), \([K_{R_q}]\), \([K_C]\) were received. Then, summation of these matrices leads to formation of the matrix \([C(\omega)]\) describing the external electric circuit.

### 4.2. Extraction of the stiffness matrix related to the piezoelectric constituent

The next step is extraction of the stiffness matrix \([K_p]\) related to the piezoelectric constituent of volume \(V_2\) from the global assembled stiffness matrix \([K]\). The process of extraction of the \([K_p]\) matrix in the form of multiplication of the real multiplier, which is a physical constant of the material, and the matrix containing the coefficients not depending on the material parameters is rather problematic due to the anisotropy of the material properties of the piezoelectric component.

Let’s introduce two assumptions:

1. The material parameters of a piezoelectric material remain the same during the process of decomposition of the global stiffness matrix \([K]\);
2. Since the form of physical relations for the elastic and viscoelastic parts is the same (3–4, 16) the viscoelastic components were considered as elastic ones in order to obtain stiffness matrices of the required structure. For this, the material parameters \(E, \nu\), related to the viscoelastic part, were varied.

Furthermore, two additional global stiffness matrices were obtained. In order to assemble the first one \([K_1]\), the following values of material parameters related to the viscoelastic part of the structure (on the basis of assumption 2): \(E_1 = 1\) Pa, \(\nu_1 = 0.1\) should be set. Alongside this, all the values of the real constants for the elements of external circuit should be equal to 1F. The values of the material parameters \(E_2 = 2\) Pa, \(\nu_2 = 0.1\) should be assigned for assembling the second global stiffness matrix \([K_2]\). The sets of material parameters must be chosen in such a way that the following conditions are satisfied: \(\nu_1 = \nu_2\), and \(E_2 - E_1 = 1\) Pa. This allows obtaining the stiffness matrix related only to the piezoelectric components and containing all the piezoelectric material characteristics within. After assembling and writing the matrices \([K_1]\) and \([K_2]\), the matrix for the piezoelectric constituent can be obtained using the following algorithm.

First, it is necessary to identify the parts in the global stiffness matrix containing elements related only to the viscoelastic component \([K_{2-1}]\) (here it plays only supplementary role and won’t be used further):

\[
[K_{2-1}] = [K_2] - [K_1]
\]  

(29)

After that, the stiffness matrix related to the piezoelectric part of the object under study can be obtained as follows:
\[
[K_p] = [K_1] - [K_{2-1}] - \sum_{i=1}^{Z} [K_{\text{Cir}(i)}]
\]  

(30)

It is important to note once more that the matrix \([K_p]\) should be extracted in such a way that it contains all the information about the physical and mechanical properties of the piezoelectric material, unlike the matrices \([K_{\text{Cir}(i)}]\) for the external circuit’s elements. Regarding this, there is no need for multiplication of this matrix for any additional coefficients.

### 4.3. Decomposition of the matrix related to the viscoelastic part into shear and bulk components

In order to have the possibility of using complex dynamic moduli for solving the natural vibration problem components \([K_8]\) and \([K_G]\) of the global stiffness matrix (21) related to the viscoelastic part proportional to the bulk and shear elastic moduli (assumption 2 of paragraph 4.2) should be obtained. Such a form of the stiffness matrix (21) allows using the viscoelasticity model under consideration (16) by substitution of the elastic constants \(G\) and \(B\) with the corresponding viscoelastic parameters. Since the ANSYS package does not allow representing the stiffness matrix as the sum of bulk and shear components directly, so the algorithm presented in [32] was used.

For an arbitrary isotropic elastic body, the material properties can be set up only with the aid of technical constants, such as Young’s modulus \(E\) and Poisson’s coefficient \(\nu\). To obtain matrices \([K_8]\) and \([K_G]\) contained in Equation (21), two additional stiffness matrices with different sets of constants \(E, \nu\) should be retrieved from ANSYS. Concerning the real constants for the electric circuit’s elements, let’s accept they are equal to 1F. The values of the material constants are set as follows: \(E_01 = 1\) Pa, \(\nu_01 = 0.1\) for obtaining the first additional matrix \([K_{01}^\text{vis}]\). For the second matrix \([K_{02}^\text{vis}]\), let’s set the values of the material constants as \(E_02 = 2\) Pa, \(\nu_02 = 0.2\). The material constant must be chosen in such way that the following conditions are satisfied: \(\nu_1 \neq \nu_2, E_2 \neq E_1\).

The next step is selection of the components related only to the viscoelastic material and containing no information about the electric circuit and piezoelectric part from the new matrices \([K_{01}^\text{vis}]\) and \([K_{02}^\text{vis}]\). Using the previously obtained matrices \([K_p]\) and \([K_{\text{Cir}(i)}]\), this can be performed in the following way:

\[
[K_{\text{vis}}^1] = [K_{\text{vis}}^{01}] - [K_p] - \sum_{i=1}^{Z} [K_{\text{Cir}(i)}];
\]

\[
[K_{\text{vis}}^2] = [K_{\text{vis}}^{02}] - [K_p] - \sum_{i=1}^{Z} [K_{\text{Cir}(i)}].
\]

(31)

Next, the following system of matrix equations can be composed on the basis of the matrices obtained using (30):

\[
\begin{cases}
[K_{\text{vis}}^1] = a_1[K_8] + b_1[K_G] \\
[K_{\text{vis}}^2] = a_2[K_8] + b_2[K_G]
\end{cases}
\]

(32)

where \(a_1 = \frac{E_01}{3 - 6\nu_01}; \quad a_2 = \frac{E_02}{3 - 6\nu_02}; \quad b_1 = \frac{E_01}{2 + 2\nu_01}; \quad b_2 = \frac{E_02}{2 + 2\nu_02},\) the unknowns are the matrices \([K_8]\) and \([K_G]\).
After the solution of system (32) the following relations for defining the matrices $[K_B]$ and $[K_G]$ can be obtained:

$$
\begin{align*}
[K_B] &= \frac{b_1[K_{B0}]-b_2[K_{B1}]}{a_1}\frac{b_1}{a_1b_1-a_1b_2} \\
[K_G] &= \frac{a_2[K_{G0}]-a_1[K_{G1}]}{a_2b_1-a_1b_2}
\end{align*}
$$

(33)

The matrices $[K_B]$ and $[K_G]$ obtained in such a way contain only «geometric» coefficients and no information about the material. These matrices should be multiplied for the corresponding components of complex dynamic moduli $\tilde{B}$ and $\tilde{G}$ when substituting them into Equation (22).

Once more, it is important to point out that the present algorithm of decomposition of the stiffness matrix into bulk and shear components is valid only for matrices assembled in ANSYS when using only PLANE or SOLID types of finite elements and is not valid for matrices assembled when using SHELL types of finite elements. In other words, this algorithm can be applied only for solving the problems having a 2D or 3D statement. The application of different shell theories for reducing the dimensions of the problem under study in this case is unacceptable. This fact can be explained by being due to differences in the physical relations for these types of elements.

### 4.4. Forming the final resolving equation

In sections 4.2–4.3 the algorithms of obtaining matrices $[K_P]$, $[K_B]$ and $[K_G]$ for a piecewise-homogeneous electro-viscoelastic body composed of one viscoelastic part and one piezoelectric part were described. If the piecewise-homogeneous electro-viscoelastic body consists of $N$ electroelastic and $M$ viscoelastic elements, then it is necessary to repeat the algorithm described in section 4.2 $N$ times and algorithm described in section 4.3 $M$ times, respectively, in order to obtain all the required matrices.

After getting all the required matrices, the equation in the form of (22), which is solved in relation to the unknown complex natural vibration frequencies of the system under study $\omega$, was composed. The condition of existence of a non-trivial solution of Equation (21) is the condition of equality to zero as the determinant (23).

It was necessary to take into account the large dimensions of the algebraic problem and also the possibility of obtaining a solution to the algebraic problem of complex eigenvalues when choosing the algorithm for searching for the eigenvalue problem solution. It should be noted that for the problems under consideration, it made sense to solve only a partial eigenvalue problem, i.e. to define not the full eigenvalues spectrum, but only some part of it.

The last circumstance produces a strict condition, in the fact that the eigenvalues must be guaranteed to be determined, either in some specified range or in the required sequence, for example, in ascending order of the real parts in the case of complex eigenvalues. Taking into account all the above-mentioned points, the method presented in [33] was chosen for solution of the problem. The program realizing this method was written in FORTRAN language and included different options for choosing the initial approximations.
5. Approbation of the algorithm

For demonstrating the reliability of the results obtained on the basis of the proposed algorithm, a series of numerical experiments was carried out. A cantilever plate with a piezoelectric element attached on its surface was chosen as the object under investigation (Figure 4).

The dimensions of the plate were: \( l_1 = 400 \text{mm}, b_1 = 100 \text{mm}, h_1 = 1 \text{mm} \). The dimensions of the piezoelectric element were: \( l_p = 50 \text{mm}, b_p = 20 \text{mm}, h_p = 0.36 \text{mm} \). The piezoelectric element’s centre of mass was located within 35 mm offset from the clamped edge and at the centre of the plate (50 mm offset from the edge having the length \( l_1 \)). The piezoelectric element was made of the piezoceramic ZTS-19 (lead zirconate-titanate), polarized along the z-axis, with the following physical and mechanical properties:

- \( C_{11} = C_{22} = 10.9 \times 10^{10} \text{Pa}, \ C_{13} = C_{23} = 5.4 \times 10^{10} \text{Pa}, \ C_{12} = 6.1 \times 10^{10} \text{Pa}, \ C_{33} = 9.3 \times 10^{10} \text{Pa}, \ C_{44} = C_{55} = C_{66} = 2.4 \times 10^{10} \text{Pa}, \ \beta_{13} = \beta_{23} = -4.9 \ C/\text{m}^2, \ \beta_{33} = 14.9 \ C/\text{m}^2, \ \beta_{51} = \beta_{42} = 10.6 \ C/\text{m}^2, \ \mu = 22 = 8.2 \times 10^{-9} \text{F/m}, \ \nu = 33 = 8.4 \times 10^{-9} \text{F/m}, \ \rho = 7500 \text{kg/m}^3. \)

The plate was made of an isotropic material with the following characteristics:

- \( E = 7 \times 10^{10} \text{Pa}, \ \nu = 0.34, \ \rho = 2700 \text{kg/m}^3. \)

As one of the options for assessing the reliability of the results obtained, a convergence analysis was performed on the object under study (Figure 4). For these tests, it was assumed that the material of the plate had no viscoelastic properties (was purely elastic). The piezoelectric element was connected to the external electric circuit comprising a series \( RL \)-circuit consisting of a series-connected resistor having resistance \( R \) and an inductance coil having inductance \( L \) (Figure 4(b)). The following values of resistance and inductance were chosen for the numerical calculations: \( L = 300 \text{H}, R = 8000 \text{Ohm}. \)

The calculations performed showed a convergence of the results. For the first complex natural vibration frequency, for instance, the difference between the results obtained using the finite-element mesh with 510 nodal unknowns and 5678 nodal unknowns for the real part was 0.8% and for the imaginary part was 20.5%. The difference between the results obtained using the finite-element mesh with 5678 nodal unknowns and 13323 nodal unknowns for the real part was 0.09% and for the imaginary part was 3.7%.
As mentioned earlier, the ANSYS package has the possibility for specifying damping proportional to the full stiffness matrix with the aid of the coefficients $g$ or $m_p$ (25). This damping can be considered as a particular case for the model of linear viscoelasticity, which is described by complex dynamic moduli, when the shear and bulk components of the complex dynamic modulus satisfy the following condition $G^{(e)}_{lm}/G^{(e)}_{Re} = B^{(e)}_{lm}/B^{(e)}_{Re}$.

In this case, the results obtained on the basis of the proposed algorithm, when $\delta = G^{(e)}_{lm}/G^{(e)}_{Re} = B^{(e)}_{lm}/B^{(e)}_{Re}$, can be compared to the results obtained on the basis of ANSYS package when $m_p = \delta/2$.

Tables 1 and 2 present the results of the calculations of the natural vibration frequencies for the first five eigenfrequencies of the structure with the piezoelectric element (Figure 4(a)), when it is operated in the open circuit mode (Table 1), and in the short circuit mode (Table 2), when the material of the plate has viscoelastic properties. For the calculations using the presented algorithm, the damping parameter was chosen as $\delta = 0.04$. The damping coefficient in ANSYS was specified as follows: $m_p = 0.02$.

The difference between the results obtained on the basis of these two techniques did not exceed 0.007%. This result confirmed the reliability of the results obtained on the basis of the proposed algorithm.

The other series of numerical calculations were performed for verifying the validity of the results obtained on the basis of the proposed algorithm. For these calculations, it was assumed that the material of the host structure had viscoelastic properties and a series $RL$-circuit was connected to the piezoelectric element electrodes (Figure 4(b)). If one sets both parameters of inductance and resistance to be negligibly small (in the order of about $10^{-15}$), this is equivalent to the case of when the piezoelectric element operates in the short circuit mode. Otherwise if one sets both parameters of inductance and resistance extremely high (in the order of about $10^{15}$), it is equivalent to the case when the piezoelectric element operates in the open circuit mode. This peculiarity

### Table 1. Complex natural vibration frequencies of the viscoelastic plate with the attached piezoelectric element, operating in the open circuit mode, when an external circuit is absent.

| Number of frequency | The results obtained on the basis of ANSYS | The results obtained on the basis of the proposed algorithm |
|---------------------|-------------------------------------------|-----------------------------------------------------------|
| 1                   | $5.527 - i 0.104$                         | $5.527 - i 0.104$                                         |
| 2                   | $33.841 - i 0.649$                        | $33.841 - i 0.649$                                        |
| 3                   | $42.153 - i 0.826$                        | $42.153 - i 0.826$                                        |
| 4                   | $93.643 - i 1.829$                        | $93.643 - i 1.829$                                        |
| 5                   | $130.201 - i 2.554$                       | $130.201 - i 2.554$                                       |

### Table 2. Complex natural vibration frequencies of the viscoelastic plate with the attached piezoelectric element, operating in the short circuit mode, when an external circuit is absent.

| Number of frequency | The results obtained on the basis of ANSYS | The results obtained on the basis of the proposed algorithm |
|---------------------|-------------------------------------------|-----------------------------------------------------------|
| 1                   | $5.507 - i 0.103$                         | $5.507 - i 0.103$                                         |
| 2                   | $33.758 - i 0.648$                        | $33.758 - i 0.648$                                        |
| 3                   | $42.152 - i 0.826$                        | $42.152 - i 0.826$                                        |
| 4                   | $93.520 - i 1.827$                        | $93.520 - i 1.827$                                        |
| 5                   | $130.200 - i 2.553$                       | $130.200 - i 2.553$                                       |
allows performing a comparison between the results obtained with the help of the proposed algorithm and the results obtained in the ANSYS package.

Tables 3 and 4 shows the results of the calculations of the first five complex natural vibration frequencies for the case of the viscoelastic host structure with the attached piezoelectric element connected to the series RL-circuit (Figure 4(b)). The parameters of the circuit elements correspond to the piezoelectric element open circuit mode ($L = 10^{15}$ H, $R = 10^{15}$ Ohm in Table 3) and short circuit mode ($L = 10^{-15}$ H, $R = 10^{-15}$ Ohm in Table 4). For the calculations on the basis of the present algorithm, the viscoelastic material properties were defined by the value of $\delta = 0.04$. For comparing the results in Tables 3 and 4, the results obtained on the basis of ANSYS with $m_p = 0.02$ were also presented.

The data presented in Tables 3 and 4 show that the results obtained on the basis of ANSYS and on the basis of the proposed algorithm differed by less than 0.02%.

6. Conclusions

An algorithm for numerical realization of the problem of the natural vibrations of electro-viscoelastic bodies with an external electric circuit was proposed herein. This algorithm utilizes some possibilities of the ANSYS commercial software package that allow using the global assembled finite-element matrices formed within it. Furthermore, these matrices were decomposed in order to retrieve the required components for construction of the resolving matrix-like equation.

The presented algorithm is applicable to piecewise-homogeneous electro-viscoelastic bodies of an arbitrary geometry with attached an external electric circuit of arbitrary configuration composed of resistive, inductive and capacitive elements.

This approach to solving the problem of natural vibrations for electro-viscoelastic bodies with an external electric circuit is an effective basis for verification of algorithms for the

### Table 3. Complex natural vibration frequencies of the viscoelastic plate with piezoelectric element connected to the series RL-circuit with the following values of parameters $L = 10^{15}$ H, $R = 10^{15}$ Ohm (simulation of the open circuit mode).

| Number of frequency | The results obtained on the basis of ANSYS (no external circuit, short-circuited piezoelectric element) | The results obtained on the basis of the proposed algorithm |
|---------------------|---------------------------------------------------------------------------------|--------------------------------------------------------|
| 1                   | 5.527 $\pm$ 0.104                                                             | 5.527 $\pm$ 0.104                                      |
| 2                   | 33.841 $\pm$ 0.649                                                            | 33.841 $\pm$ 0.649                                     |
| 3                   | 42.153 $\pm$ 0.826                                                            | 42.153 $\pm$ 0.826                                     |
| 4                   | 93.643 $\pm$ 1.829                                                            | 93.643 $\pm$ 1.829                                     |
| 5                   | 130.201 $\pm$ 2.553                                                           | 130.201 $\pm$ 2.553                                    |

### Table 4. Complex natural vibration frequencies of the viscoelastic plate with the piezoelectric element connected to the series RL-circuit with the following values of parameters $L = 10^{-15}$ H, $R = 10^{-15}$ Ohm (simulation of the short circuit mode).

| Number of frequency | The results obtained on the basis of ANSYS (no external circuit, open-circuited piezoelectric element) | The results obtained on the basis of the proposed algorithm |
|---------------------|-------------------------------------------------------------------------------------------------|--------------------------------------------------------|
| 1                   | 5.507 $\pm$ 0.103                                                                             | 5.507 $\pm$ 0.103                                      |
| 2                   | 33.758 $\pm$ 0.648                                                                            | 33.758 $\pm$ 0.648                                     |
| 3                   | 42.153 $\pm$ 0.826                                                                            | 42.153 $\pm$ 0.826                                     |
| 4                   | 93.520 $\pm$ 1.827                                                                            | 93.520 $\pm$ 1.827                                     |
| 5                   | 130.200 $\pm$ 2.553                                                                           | 130.200 $\pm$ 2.553                                    |
construction of electric analogues of smart-structures having the form of electric schemes [35] and for development of algorithms as strategies for realization of dynamic behaviour control for smart structures based on piezoelectric elements and an external electric circuit.

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