REGULARITY OF FULLY NON-LINEAR ELLIPTIC EQUATIONS ON HERMITIAN MANIFOLDS

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Abstract. In this paper we propose new insights and ideas to set up quantitative boundary estimates for solutions to Dirichlet problem of a class of fully non-linear elliptic equations on compact Hermitian manifolds with real analytic Levi flat boundary. With the quantitative boundary estimates at hand, we can establish the gradient estimate and give a unified approach to investigate the existence and regularity of solutions of Dirichlet problem with sufficiently smooth boundary data, which include the geodesic equation in the space of Kähler metrics as a special case. Our method can also be applied to Dirichlet problem for analogous fully non-linear elliptic equations on a compact Riemannian manifold with concave boundary.

1. Introduction

The most important fully non-linear elliptic equation is perhaps the complex Monge-Ampère equation, which is closely related to the volume form and the representation of Ricci form of Kähler manifolds. Calabi conjectured in [15] that any (1, 1)-form which represents the first Chern class of a closed Kähler manifold is the Ricci form of a Kähler metric with the same Kähler class as the original one. This fundamental problem can be reduced to solving non-degenerate and smooth complex Monge-Ampère equation on closed Kähler manifolds. In his pioneering paper, Yau [74] proved Calabi’s conjecture and showed that any closed Kähler manifold with zero or negative first Chern class admits a unique Kähler-Einstein metric in the given Kähler class. The existence of Kähler-Einstein metrics on closed Kähler manifolds of negative first Chern class was also proved by Aubin [1] independently. A heat equation proof of Yau’s theorem was obtained by Cao [17] in which he introduced the Kähler-Ricci flow. Yau’s work was partially extended by Tosatti-Weinkove [69] to closed Hermitian manifolds (see also [29] for a parabolic version). The other seminal works concerning the complex Monge-Ampère equation was done by Bedford-Taylor [3, 5] on generalized solutions in the sense of pluripotential theory, and by Caffarelli-Kohn-Nirenberg-Spruck [10] on the Dirichlet problem for the complex Monge-Ampère equation on a strictly pseudoconvex domain in $\mathbb{C}^n$.

This is the first part of a series of researches devoted to the study of Dirichlet problem for fully non-linear elliptic equations on complex manifolds, which include [arXiv:2001.09238], [arXiv:2106.14837], [Pure Appl. Math. Q. 16 (2020), 1585-1617; MR4221006] and [Calc. Var. PDE. 60 (2021), Paper No. 162, 20 pp.; MR4290375].
This paper is devoted to investigating the solvability and regularity of solutions to fully non-linear elliptic equations on Hermitian manifolds. Let \((M, J, \omega)\) be a compact Hermitian manifold of complex dimension \(n \geq 2\) with Levi flat boundary \(\partial M\), where \(\omega\) is the Kähler form locally given by
\[
\omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j,
\]
and \(J\) denotes the underlying complex structure. Throughout the paper, unless otherwise indicated, we assume in addition that the boundary is smooth and real analytic.

Let \(\chi = \sqrt{-1} \chi_{i\bar{j}} dz^i \wedge d\bar{z}^j\) be a smooth real \((1,1)\)-form on \(\bar{M} := M \cup \partial M\). We are concerned with fully non-linear elliptic equations for deformation of \(\chi\),
\[
F(g[u]) := f(\lambda(g[u])) = \psi \text{ in } M,
\]
with the boundary value condition \(u = \varphi\) on \(\partial M\), where \(\psi\) and \(\varphi\) are both two functions with suitable regularity, \(g[u] = \chi + \sqrt{-1} \partial \bar{\partial} u\), and \(\lambda(g[u]) = (\lambda_1, \cdots, \lambda_n)\) denote the eigenvalues of \(g[u]\) with respect to \(\omega\).

The study of fully non-linear elliptic equations analogous to (1.1) in the setting of real variables goes back to the work of Caffarelli-Nirenberg-Spruck [12] and Ivochkina [47] who dealt with some special cases. The function \(f\) is assumed to be a smooth symmetric function defined in an open symmetric and convex cone \(\Gamma \subset \mathbb{R}^n\) with vertex at the origin, and boundary \(\partial \Gamma \neq \emptyset\),
\[
\Gamma_n \subseteq \Gamma \subseteq \Gamma_1,
\]
where \(\Gamma_n = \{\lambda \in \mathbb{R}^n : \text{each } \lambda_i > 0\}\) and \(\Gamma_1 = \{\lambda \in \mathbb{R}^n : \sum_{i=1}^n \lambda_i > 0\}\). Moreover, \(f\) shall satisfy the following structure conditions:
\[
\text{(1.2)} \quad f_i := \frac{\partial f}{\partial \lambda_i} > 0 \text{ in } \Gamma, \quad 1 \leq i \leq n,
\]
\[
\text{(1.3)} \quad f \text{ is concave in } \Gamma,
\]
\[
\text{(1.4)} \quad \delta_{\psi,f} := \inf_{M} \psi - \sup_{\partial \Gamma} f > 0,
\]
where
\[
\sup_{\partial \Gamma} f := \sup_{\lambda_0 \in \partial \Gamma} \lim_{\lambda \to \lambda_0} f(\lambda).
\]
The constant \(\delta_{\psi,f}\) measures if the equation is degenerate. Namely, the equation is called non-degenerate (respectively, degenerate) if \(\delta_{\psi,f} > 0\) (respectively, if \(\delta_{\psi,f}\) vanishes).

In addition, we assume
\[
\text{(1.5)} \quad \text{For any } \sigma < \sup_{\Gamma} f \text{ and } \lambda \in \Gamma, \quad \lim_{t \to +\infty} f(t\lambda) > \sigma
\]
so that one can apply Székelyhidi’s [64] Liouville type theorem extending a result of Dinew-Kołodziej [25] to derive the gradient estimate via a blow-up argument. We shall emphasize that conditions (1.2), (1.3) and (1.5) allow many natural functions,
for instance if $f$ is homogeneous of degree one with $f > 0$ in $\Gamma$, or the function $f$ associated with $\Gamma = \Gamma_n$.

In order to study equation (1.1) within the framework of elliptic equations, we shall find the solutions in the class of $C^2$-admissible functions $u$ satisfying

$$\lambda(g[u]) \in \Gamma.$$ 

In the theory of fully non-linear elliptic equations, the standpoint of a subsolution $u \in C^2(\bar{M})$ satisfying

$$f(\lambda(g[u])) \geq \psi \text{ in } M,$$

with $\underline{u} = \varphi$ on $\partial M$, without geometric restriction to the boundary, plays important roles in deriving a priori estimates for Dirichlet problem and has a great advantage in applications to geometric problems (cf. [19, 35, 38, 41, 45] and references therein). The admissible subsolution was used by Hoffman-Rosenberg-Spruck [45] and further developed in [35, 30] to derive second order boundary estimates for real and complex Monge-Ampère equation on general bounded domains. In addition, Guan [32] recently used it to derive the global second order estimate for general fully non-linear elliptic equations on Riemannian manifolds. Influenced by the work of Guan [32], a flexible notion of a $C$-subsolution was recently introduced by Székelyhidi [65]. With the assumption of the existence of $C$-subsolutions, Székelyhidi established $C^{2,\alpha}$-estimate for admissible solutions of fully non-linear elliptic equations satisfying (1.2)-(1.5) on closed Hermitian manifolds. The concept of $C$-subsolution turns out to be applicable for fully non-linear elliptic equations in the setting of closed manifolds (cf. [66, 23]).

1.1. Statement of main results. We show that Dirichlet problem (1.1) is uniquely solvable in the class of admissible functions, provided that the function $\psi$ and boundary data $\varphi$ are sufficiently smooth, and the Dirichlet problem admits an admissible subsolution taking the same boundary data.

**Theorem 1.1.** Let $(M, J, \omega)$ be a compact Hermitian manifold with real analytic Levi flat boundary. Let $\psi \in C^{k,\alpha}(\bar{M})$ and $\varphi \in C^{k+2,\alpha}(\partial M)$ for an integer $k \geq 2$ and $0 < \alpha < 1$. In addition to (1.2)-(1.5), we assume that there exists an admissible subsolution $\underline{u} \in C^3(\bar{M})$ obeying (1.6). Then Dirichlet problem (1.1) has a unique admissible solution $u \in C^{k+2,\alpha}(M)$. Moreover,

$$|u|_{C^2(\bar{M})} \leq C,$$

where $C$ depends only on $|\varphi|_{C^{2,1}(\bar{M})}$, $|\psi|_{C^{1,1}(\bar{M})}$, $|\underline{u}|_{C^2(\bar{M})}$, $|\chi|_{C^2(\bar{M})}$ and other known data (but not on $(\delta_{\psi,f})^{-1}$).

**Remark 1.2.** Throughout this paper, we say a constant $C$ does not depend on $(\delta_{\psi,f})^{-1}$ if $C$ remains bounded as $\delta_{\psi,f}$ tends to zero. We say the constant $\kappa$ depends on $\delta_{\psi,f}$ if $\kappa \to 0$ when $\delta_{\psi,f} \to 0$. Moreover, in the theorems, we assume that the function $\varphi$ is extended to a $C^3$ function on $\bar{M}$, still denoted as $\varphi$. 
We can apply our results to investigate degenerate equations, as our \textit{a priori} estimates do not depend on $(\delta_{\psi,f})^{-1}$. The degenerate fully non-linear elliptic equations have been studied from different aspects, including \cite{37,40,48,52,67,68}. They are also closely connected with the research of certain objects in differential geometry and analysis. For instance, the homogeneous of complex Monge-Ampère equation
\begin{equation}
(\omega + \sqrt{-1} \partial \bar{\partial} u)^n = 0 \text{ in } M, \quad u = \varphi \text{ on } \partial M
\end{equation}
plays some important roles in complex geometry and complex analysis (cf. \cite{4,18,26,49,63}). See also \cite{42,58,62}. The following works should be mentioned:

- Guan’s \cite{38,39} proof of Chern-Levine-Nirenberg conjecture on intrinsic norms.
- The work concerning Donaldson’s conjecture in Kähler geometry due to Chen \cite{19}. For more related topics, please refer to \cite{8,16,20,21,22,27,44,56,59,60,61} and references therein.
- The regularity of pluricomplex Green function in the pluripotential theory (cf. \cite{6,24,30,31,54,55} and references therein).

Applying Theorem \ref{thm:main} and the method of approximation, we can solve degenerate fully non-linear equations on compact Hermitian manifolds with real analytic Levi flat boundary.

\textbf{Theorem 1.3.} In addition to \ref{eq:main}, \ref{eq:main2} and \ref{eq:main3}, we assume $f \in C^\infty(\Gamma) \cap C^0(\bar{\Gamma})$. Let $\varphi \in C^{2,1}(\partial M)$ and $\psi \in C^{1,1}(M)$ be a function satisfying $\delta_{\psi,f} = 0$. Suppose that there is a strictly admissible subsolution $\underline{u} \in C^{2,1}(M)$ with $\underline{u} = \varphi$ on $\partial M$ satisfying
\begin{equation}
f(\lambda(g[u])) \geq \psi + \delta_0 \text{ in } M
\end{equation}
for some $\delta_0 > 0$. Then Dirichlet problem \ref{eq:dirichlet} admits a weak solution $u \in C^{1,\alpha}(\bar{M})$, $\forall 0 < \alpha < 1$, with $\lambda(g[u]) \in \bar{\Gamma}$ and $\Delta u \in L^\infty(M)$.

Our method and results may enable us to attack various problems from complex geometry, for instance the geodesic equations in the space of Kähler metrics and the construction and regularity of geodesic rays associated to test configurations studied in the cited literature above.

The primary difficulty is deriving the gradient estimate. It is pretty hard to prove the gradient bound for general fully non-linear elliptic equations on curved complex manifolds. The blow-up argument is an alternative approach to deriving gradient estimate, as shown by Chen \cite{19} for Dirichlet problem of complex Monge-Ampère equation on $X \times A$ where $A = S^1 \times [0,1]$ and $X$ is a closed Kähler manifold, and by Dinew-Kołodziej \cite{25} for complex $k$-Hessian equation on closed Kähler manifolds using Hou-Ma-Wu’s \cite{46} second order estimate of the form
\begin{equation}
\sup_M |\partial \bar{\partial} u| \leq C(1 + \sup_M |\nabla u|^2).
\end{equation}
The results of Hou-Ma-Wu and Dinew-Kołodziej in \cite{46,25} have been extended extensively by Székelyhidi \cite{65} to fully non-linear elliptic equations satisfying \ref{eq:main}-\ref{eq:main3} on closed Hermitian manifolds. We also refer the reader to \cite{66,70,71,75} for the
second order estimate (1.10) of complex Monge-Ampère equation for \((n - 1)\)-PSH functions and complex \(k\)-Hessian equations on closed complex manifolds.

In this paper we derive the gradient estimate by a blow-up argument. To achieve this, a specific problem that we have in mind is to establish a quantitative version of second order boundary estimates, which claims that the complex Hessian on the boundary can be dominated by a quadratic term of the boundedness for the gradient of the unknown solution. In the following theorem, we derive such quantitative boundary estimates.

**Theorem 1.4.** Suppose that \((M, J, \omega)\) is a compact Hermitian manifold with real analytic Levi flat boundary. Let \(\psi \in C^1(\overline{M})\) and \(\varphi \in C^3(\partial M)\). Suppose, in addition to (1.2) - (1.5), that there exists an admissible subsolution \(u \in C^2(\overline{M})\) to Dirichlet problem (1.1). Then for any admissible function \(u \in C^3(\overline{M}) \cap C^2(\overline{M})\) solving the Dirichlet problem, we derive

\[
\sup_{\partial M} |\partial \overline{\partial} u| \leq C(1 + \sup_M |\nabla u|^2),
\]

where \(C\) is a uniform positive constant depending only on \(|\varphi|_{C^3(\overline{M})}\), \(|u|_{C^2(\overline{M})}\), \(|\psi|_{C^1(\overline{M})}\), \(|\chi|_{C^1(\overline{M})}\), \(\sup_{\partial M} |\nabla u|\) and other known data under control (but not on \((\delta \psi, f)^{-1}\)).

The quantitative boundary estimate (1.11) was established by Chen [19] for Dirichlet problem of complex Monge-Ampère equation on \(X \times A\), and further by Phong-Sturm [61] for Dirichlet problem of complex Monge-Ampère equation on compact Kähler manifolds with real analytic Levi flat boundary. See also Phong-Song-Sturm’s survey [58]. As in the statement of Lemma 7.17 in [9], (1.11) indeed holds for complex Monge-Ampère equation on compact Kähler manifolds without assuming the boundary to be real analytic Levi flat. To the best knowledge of the author, only the literature mentioned above has studied this topic.

By using Theorems 1.4 and 5.1 we can establish the desired second order estimate of the form (1.10) for Dirichlet problem (1.1). Combining it with Székelyhidi’s [64] Liouville type theorem, we can prove the gradient bound via a blow-up argument, and then obtain a uniform bound of the complex Hessian

\[
\sup_M |\partial \overline{\partial} u| \leq C,
\]

where \(C\) is a uniform positive constant which does not depend on the solution \(u\) and its derivatives. With the uniform estimate for complex Hessian at hand, conditions (1.2) - (1.4) guarantee equation (1.1) to be uniformly elliptic and concave for admissible solutions, then the Evans-Krylov theorem [28, 50, 51], adapted to complex setting (cf. [72]), yields the \(C^{2,\alpha}\) estimate. We can also follow the line of the proof in [33] to derive the estimates for the real Hessian so that one can directly apply Evans-Krylov theorem. The higher order regularity follows from the standard Schauder theory.

Finally, we remark that it is pretty hard to prove gradient bound directly, as Blocki [7], Hanani [43] and Guan-Li [33] did for complex Monge-Ampère equation, and Zhang...
did for $\omega$-plurisubharmonic solutions of complex $k$-Hessian equations. Also, the direct proof of gradient estimate of complex inverse $\sigma_k$ equation was obtained in [36].

1.2. **Sketch proof of Theorem 1.4.** It is interesting but challenging to derive such quantitative boundary estimates for Dirichlet problem of fully non-linear elliptic equations (1.1). We propose new insights and ideas to overcome the difficulties in this paper. The proof of the quantitative boundary estimate is based on two ingredients:

We set up Lemma 3.1 in an attempt to bound double normal derivative of the solution on the boundary. This lemma states that for a certain Hermitian matrix, if there is a diagonal element, $a$ say, satisfying a quadratic growth condition, then the eigenvalues concentrate near the corresponding diagonal elements. Lemma 3.1 allows us to prove that, under the assumptions of Theorem 1.4, the boundary estimates for double normal derivative can be dominated by the quadratic term of the boundedness for tangential-normal derivatives of the admissible solutions on the boundary (see Proposition 4.1). On the other hand, we prove in Proposition 4.3 that the boundedness of boundary estimates for tangential-normal derivatives depends linearly on the supremum of gradient term. Theorem 1.4 immediately follows from these two steps.

1.3. **Some phenomena on regularity assumptions.** It should be stressed that there are several notable phenomena on regularity assumptions on the boundary and boundary data, which are rather different from the results and counterexamples for real Monge-Ampère equations on domains in Euclidean spaces due to Caffarelli-Nirenberg-Spruck [13] and Wang [73].

Following is the explanation. The tangential operator on the boundary and the distance function $\sigma$ to the boundary play important roles in constructing the local barrier functions, as shown in the proof of boundary estimates in [10, 11, 12, 45, 35, 30]. In our case, we apply local coordinate (4.13) to construct a tangential operator (4.15) and a local barrier function $\tilde{\Psi}$ in (4.16). In the proof of Proposition 4.3 we only use $\sigma$, $\rho$, $\nu$ and their derivatives up to second order, $\varphi$ and its derivatives up to third order. It is worth stressing that when the boundary data is a constant, the constant in (4.12) of Proposition 4.3 depends only on $\partial M$ up to second derivatives and other known data (see Remark 4.4).

As a result, the regularity assumptions in Theorem 1.1 can be further weakened when $M = X \times S$ is a product of a closed complex manifold $X$ with a compact Riemann surface $S$ with boundary. More precisely, if $\psi \in C^2(M)$ as well as $\partial M \in C^3$, $\varphi \in C^3$, then the solution $u$ lies in $C^{2,\alpha}(M)$ with $0 < \alpha < 1$. At least on some convex domains $\Omega \subset \mathbb{R}^2$, such $C^3$ regularity assumptions on boundary and boundary data are optimal for Dirichlet problem of nondegenerate real Monge-Ampère equations as shown by Wang [73], in which he constructed counterexamples to show that if either $\partial \Omega$ or $\varphi$ is only $C^{2,1}$ smooth, then the solution $u$ to real Monge-Ampère equation on $\Omega \subset \mathbb{R}^2$ may fail to be $C^2$ smooth near the boundary. For the Dirichlet problem with homogeneous boundary data on products $X \times S$, the regularity assumption
on the boundary may be further relaxed to $C^{2,\beta}$ ($0 < \beta < 1$) in this special case. We shall remark here that the $C^{2,\alpha}$ boundary regularity may follow from a result of Silvestre-Sirakov [64].

**Theorem 1.5.** Assume $M = X \times S$, where $X$ is a closed complex manifold and $S$ is a compact Riemann surface with $C^{2,\beta}$ boundary ($0 < \beta < 1$). Suppose, in addition to (1.2)-(1.5), $\varphi = 0$ and $\psi \in C^2(\bar{M})$, that Dirichlet problem (1.1) has a $C^2$-smooth admissible subsolution. Then the Dirichlet problem supposes a unique $C^{2,\alpha}$ admissible solution, where $0 < \alpha < 1$ depends on $\beta$ and other known data.

For the Dirichlet problem of degenerate fully non-linear elliptic equations on $M = X \times S$, the regularity assumptions in Theorem 1.3 can also be further weakened to $\varphi \in C^{2,1}$ and $\partial M \in C^{2,1}$. Furthermore, as in Theorem 1.5 we have the corresponding result for degenerate equations when $M = X \times S$. Such regularity assumptions on boundary and boundary data are impossible for homogeneous real Monge-Ampère equation on certain bounded domains $\Omega \subset \mathbb{R}^n$ as shown by [13], in which the $C^{3,1}$-regularity assumptions on boundary and boundary data are optimal for the optimal $C^{1,1}$ global regularity of the weak solution to homogeneous real Monge-Ampère equation on $\Omega$.

Notice that the boundary of a $C^2$ bounded domain $\Omega \subset \mathbb{R}^n$ has at least one strictly convex point at which every principal curvature of $\partial \Omega$ is positive, to be compared with the Levi flatness of $M = X \times S$. These new features show that the assumption on the boundary in the corresponding theorem is essential for improving the regularity assumptions on the boundary and the boundary data.

**Organization.** The rest of this paper is organized as follows. In Section 2 we outline the notation and some useful results. In Section 3 we derive a quantitative lemma which is the crucial ingredient in the proof of quantitative boundary estimates. In Section 4 we establish quantitative boundary estimates. In Section 5 we solve the Dirichlet problem by using the method of continuity and approximation. In Section 6 we briefly investigate the solvability and regularity of Dirichlet problem for fully non-linear elliptic equations on Riemannian manifolds with concave boundary. In Appendix A we finally present a characterization of the level sets of $f$ satisfying (1.2), (1.3) and (1.5), which plays an important role in proving the boundary estimates for ‘tangential-normal’ derivatives.

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## 2. Preliminaries

In this section, we outline and set up some useful results. For convenience we first denote 

$$\bar{I} = (1, \cdots , 1), \quad \lambda[v] = \lambda(g[v]), \quad \lambda = \lambda[u], \quad \Lambda = \lambda[u].$$
Based on a characterization of level sets of \( f \) satisfying (1.2), (1.3) and (1.5), we give a new proof of Part (b) of Lemma 9 in [65]. We state it as follows.

**Lemma 2.1.** Suppose (1.2), (1.3) and (1.5) hold. Then for \( \sigma \in (\sup_{\partial \Gamma} f, \sup_{\Gamma} f) \) and \( t > 0 \), we have

\[
\sum_{i=1}^{n} f_{i}(\lambda) > \frac{f(t\bar{1}) - \sigma}{t} \quad \text{in} \quad \partial \Gamma^\sigma := \{ \lambda \in \Gamma : f(\lambda) = \sigma \}.
\]

In particular, if \( u \) is a \( C^2 \) admissible solution of equation (1.1) then

\[
\sum_{i=1}^{n} f_{i}(\lambda(g[u])) \geq \kappa > 0.
\]

where \( \kappa = \frac{f((1+c_0)\bar{1}) - \sup_{M}\psi}{1+c_0} \), which is independent of \( \delta_{\psi,f} \), and \( c_0 \) is the positive constant satisfying \( f(c_0\bar{1}) = \sup_{M}\psi \).

**Proof.** In Appendix A we obtain a characterization of level sets of \( f \) when it satisfies (1.2), (1.3) and (1.5). The proof of Lemma 2.1 is based on this characterization. Fix \( t > 0 \). By (1.3) and Lemma A.4, one has

\[
 t \sum_{i=1}^{n} f_{i}(\lambda) \geq \sum_{i=1}^{n} f_{i}(\lambda)\lambda_{i} + f(t\bar{1}) - f(\lambda) > f(t\bar{1}) - \sigma \quad \text{in} \quad \partial \Gamma^\sigma.
\]

Furthermore, (2.2) holds for \( \kappa = \frac{f((1+c_0)\bar{1}) - \sup_{M}\psi}{1+c_0} \) by setting \( t = 1 + c_0 \).

An important ingredient in the proof of a priori estimates for fully non-linear elliptic equations is the following lemma.

**Lemma 2.2** ([34], Lemma 2.2). Suppose that \( f \) satisfies (1.2) and (1.3). Let \( K \) be a compact subset of \( \Gamma \) and \( \beta > 0 \). There is a constant \( \varepsilon > 0 \) such that, for any \( \mu \in K \) and \( \lambda \in \Gamma \), when \( |\nu_{\mu} - \nu_{\lambda}| \geq \beta \),

\[
\sum_{i=1}^{n} f_{i}(\lambda) (\mu_{i} - \lambda_{i}) \geq f(\mu) - f(\lambda) + \varepsilon(1 + \sum_{i=1}^{n} f_{i}(\lambda)),
\]

where \( \nu_{\lambda} = Df(\lambda)/|Df(\lambda)| \) denotes the unit normal vector to the level surface of \( f \) through \( \lambda \).

For the given admissible subsolution \( u, \lambda \) falls in a compact subset of \( \Gamma \). We take

\[
\beta := \frac{1}{2} \min_{M} \text{dist}(\nu_{\lambda}, \partial \Gamma_{n}) > 0.
\]

It follows from Lemma 2.2 and Lemma 6.2 in [12] that when \( |\nu_{\lambda} - \nu_{\lambda}| \geq \beta \) we have

\[
\mathcal{L}(u - u) \geq \varepsilon(1 + \sum_{i=1}^{n} f_{i}(\lambda)).
\]
While $|\nu - \nu_\lambda| < \beta$ ensures $\nu_\lambda - \beta 1 \in \Gamma_n$ and

$$f_i(\lambda) \geq \frac{\beta}{\sqrt{n}} \sum_{j=1}^{n} f_j(\lambda)$$

for each $i = 1, \ldots, n$.

From the original proof of Lemma 2.2 in [34], we know that the constant $\varepsilon$ in this lemma depends only on $\lambda_\lambda$, $\beta$ and other known data. Please refer to [34] for the detail.

In analogy with Lemma 2.2, Székelyhidi also proved the following lemma.

**Lemma 2.3** ([65], Proposition 5). Let $f$ be the function satisfying (1.2) and (1.3), and let $\sup_{\Gamma} f < \sigma < \sup_{\Gamma} f$. Suppose that there exists a $C$-subsolution $u \in C^2(M)$. Then there exist two uniform positive constants $R_0$ and $\varepsilon_1$, such that if $\lambda \in \partial \Gamma_0$ and $|\lambda| \geq R_0$, then either

$$(2.5) \quad \sum_{i=1}^{n} f_i(\lambda)(\Delta_i - \lambda_i) \geq \varepsilon_1 \sum_{j=1}^{n} f_j(\lambda),$$

or $f_i(\lambda) \geq \varepsilon_1 \sum_{j=1}^{n} f_j(\lambda)$ for each $i = 1, \ldots, n$.

It shall be noted that any admissible subsolution satisfying (1.6) is clearly the $C$-subsolution introduced by Székelyhidi [65], so we can apply either Lemma 2.2 or Lemma 2.3 to derive the estimates established in this paper.

Throughout this paper we use derivatives with respect to Chern connection $\nabla$ associated with $\omega$, and in local coordinates $z = (z_1, \ldots, z_n)$ use notations such as

$$v_i = \nabla_{\partial} v, \quad v_{ij} = \nabla_{\partial_j} \nabla_{\partial_i} v, \quad v_{ij} = \nabla_{\bar{\partial}_j} \nabla_{\bar{\partial}_i} v, \ldots.$$  

We denote $g = g[u]$ and $\underline{g} = g[u]$ for the solution $u$ and the subsolution $\underline{u}$ respectively. Given a Hermitian matrix $A = \{a_{ij}\}$, we write $F^{ij}(A) = \frac{\partial F}{\partial a_{ij}}(A)$.

### 3. Quantitative Lemmas

In this section we set up a lemma which states that if the parameter $a$ satisfies a quadratic growth condition then the eigenvalues concentrate near the corresponding diagonal elements. Namely,

**Lemma 3.1.** Let $A$ be an $n \times n$ Hermitian matrix

$$
\begin{pmatrix}
    d_1 & a_1 \\
    d_2 & a_2 \\
    \vdots & \vdots \\
    d_{n-1} & a_{n-1} \\
    \bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_{n-1} & \bar{a}
\end{pmatrix}
$$

with $d_1, \ldots, d_{n-1}, a_1, \ldots, a_{n-1}$ fixed, and with $\bar{a}$ variable. Denote $\lambda = (\lambda_1, \ldots, \lambda_n)$ by the eigenvalues of $A$. Let $\epsilon > 0$ be a fixed constant. Suppose that the parameter $\bar{a}$ in
A satisfies the quadratic growth condition

\[(3.1) \quad a \geq \frac{2n-3}{\epsilon} \sum_{i=1}^{n-1} |a_i|^2 + (n-1) \sum_{i=1}^{n-1} |d_i| + \frac{(n-2)\epsilon}{2n-3},\]

where \(\epsilon\) is a positive constant. Then the eigenvalues behavior like

\[|d_\alpha - \lambda_\alpha| < \epsilon, \quad \forall 1 \leq \alpha \leq n - 1,\]
\[0 \leq \lambda_n - a < (n-1)\epsilon.\]

This lemma is in fact a quantitative version of the following lemma.

**Lemma 3.2** ([12], Lemma 1.2). Consider the \(n \times n\) symmetric matrix

\[A = \begin{pmatrix} d_1 & a_1 \\ d_2 & \ddots & \vdots \\ \vdots & \ddots & a_{n-1} \\ a_1 & a_2 & \cdots & a_{n-1} & a \end{pmatrix}\]

with \(d_1, \ldots, d_{n-1}\) fixed, \(|a|\) tends to infinity and

\[|a_i| \leq C, \quad i = 1, \ldots, n.\]

Then the eigenvalues \(\lambda_1, \ldots, \lambda_n\) behave like

\[\lambda_\alpha = d_\alpha + o(1), \quad 1 \leq \alpha \leq n - 1,\]
\[\lambda_n = a (1 + O(1/a)),\]

where the \(o(1)\) and \(O(1/a)\) are uniform—depending only on \(d_1, \ldots, d_{n-1}\) and \(C\).

In their seminal paper, Caffarelli-Nirenberg-Spruck [12] proved Lemma 3.2 and applied it to derive boundary estimates for double normal derivative of certain fully non-linear elliptic equations on \(\Omega \subset \mathbb{R}^n\). Since then Lemma 3.2 plays an important role in the boundary estimates for second derivatives in the study of Dirichlet problem for general fully non-linear elliptic equations (cf. [57, 68, 32] and references therein).

Our main estimate in this paper is the quantitative boundary estimate which states that the complex Hessian on the boundary can be bounded by a quadratic term of the boundedness for the gradient of unknown solutions. Except complex Monge-Ampère equation, such quantitative boundary estimates for general fully non-linear elliptic equations are poorly understood on this subject, because Lemma 3.2 does not figure out how do the eigenvalues concentrate explicitly near the corresponding diagonal elements of the matrix \(A\) when \(|a|\) is sufficiently large.

We make progress on the subject. Lemma 3.1 allows us to follow the track of the behavior of the eigenvalues as \(|a|\) tends to infinity.

We start with the case of \(n = 2\). In this case, we prove that if \(a \geq \frac{|a_1|^2}{\epsilon} + d_1\) then

\[0 \leq d_1 - \lambda_1 = \lambda_2 - a < \epsilon.\]
Let's briefly present the discussion as follows: For \( n = 2 \), the eigenvalues of \( A \) are
\[
\lambda_1 = \frac{a + d_1 - \sqrt{(a - d_1)^2 + 4|a_1|^2}}{2} \quad \text{and} \quad \lambda_2 = \frac{a + d_1 + \sqrt{(a - d_1)^2 + 4|a_1|^2}}{2}.
\]
We can assume \( a_1 \neq 0 \); otherwise we are done. If \( a \geq \frac{|a_1|^2}{\epsilon} + d_1 \) then one has
\[
0 \leq d_1 - \lambda_1 = \lambda_2 - a = \frac{2|a_1|^2}{\sqrt{(a - d_1)^2 + 4|a_1|^2} + (a - d_1)} < \frac{|a_1|^2}{a - d_1} \leq \epsilon.
\]
Here we use \( a_1 \neq 0 \) to verify that the strictly inequality in the above formula holds. We hence obtain Lemma 3.1 for \( n = 2 \).

The following lemma enables us to count the eigenvalues near the diagonal elements via a deformation argument. It is an essential ingredient in the proof of Lemma 3.1 for general \( n \).

**Lemma 3.3.** Let \( A \) be a Hermitian \( n \times n \) matrix
\[
\begin{pmatrix}
d_1 & a_1 \\
d_2 & a_2 \\
\vdots & \vdots \\
ad_{n-1} & a_{n-1}
\end{pmatrix}
\]
with \( d_1, \ldots, d_{n-1}, a_1, \ldots, a_{n-1} \) fixed, and with \( a \) variable. Denote \( \lambda = (\lambda_1, \ldots, \lambda_n) \) by the the eigenvalues of \( A \) with the order \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \). Fix a positive constant \( \epsilon \).

Suppose that the parameter \( a \) in the matrix \( A \) satisfies the following quadratic growth condition
\[
\tag{3.2} a \geq \frac{1}{\epsilon} \sum_{i=1}^{n-1} |a_i|^2 + \sum_{i=1}^{n-1} [d_i + (n - 2)|d_i|] + (n - 2)\epsilon.
\]
Then for any \( \lambda_\alpha \ (1 \leq \alpha \leq n - 1) \) there exists an \( d_{i_\alpha} \) with lower index \( 1 \leq i_\alpha \leq n - 1 \) such that
\[
\tag{3.3} |\lambda_\alpha - d_{i_\alpha}| < \epsilon,
\]
\[
\tag{3.4} 0 \leq \lambda_n - a < (n - 1)\epsilon + |\sum_{\alpha=1}^{n-1} (d_\alpha - d_{i_\alpha})|.
\]

**Proof.** Without loss of generality, we assume \( \sum_{i=1}^{n-1} |a_i|^2 > 0 \) and \( n \geq 3 \) (otherwise we are done, since \( A \) is diagonal or \( n = 2 \)). Note that in the assumption of the lemma the eigenvalues have the order \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \). It is well known that, for a Hermitian matrix, any diagonal element is less than or equals to the largest eigenvalue. In particular,
\[
\tag{3.5} \lambda_n \geq a.
\]
We only need to prove (3.3), since (3.4) is a consequence of (3.3), (3.5) and
\[ \sum_{i=1}^{n} \lambda_i = \text{tr}(A) = \sum_{\alpha=1}^{n-1} d_\alpha + a. \]

Let’s denote \( I = \{1, 2, \ldots, n - 1\} \). We divide the index set \( I \) into two subsets by
\[
B = \{ \alpha \in I : |\lambda_\alpha - d_i| \geq \epsilon, \forall i \in I \}
\]
and \( G = I \setminus B = \{ \alpha \in I : \text{There exists an } i \in I \text{ such that } |\lambda_\alpha - d_i| < \epsilon \} \).

To complete the proof we need to prove \( G = I \) or equivalently \( B = \emptyset \). It is easy to see that for any \( \alpha \in G \), one has
\[ |\lambda_\alpha| < \sum_{i=1}^{n-1} |d_i| + \epsilon. \]

Fix \( \alpha \in B \), we are going to give the estimate for \( \lambda_\alpha \). The eigenvalue \( \lambda_\alpha \) satisfies
\[ (\lambda_\alpha - a) \prod_{i=1}^{n-1} (\lambda_\alpha - d_i) = \sum_{i=1}^{n-1} (|a_i|^2 \prod_{j \neq i} (\lambda_\alpha - d_j)). \]

By the definition of \( B \), for \( \alpha \in B \), one then has \( |\lambda_\alpha - d_i| \geq \epsilon \) for any \( i \in I \). We therefore derive
\[ |\lambda_\alpha - a| \leq \sum_{i=1}^{n-1} \frac{|a_i|^2}{|\lambda_\alpha - d_i|} \leq \frac{1}{\epsilon} \sum_{i=1}^{n-1} |a_i|^2, \text{ if } \alpha \in B. \]

Hence, for \( \alpha \in B \), we obtain
\[ \lambda_\alpha \geq a - \frac{1}{\epsilon} \sum_{i=1}^{n-1} |a_i|^2. \]

For a set \( S \), we denote \( |S| \) the cardinality of \( S \). We shall use proof by contradiction to prove \( B = \emptyset \). Assume \( B \neq \emptyset \). Then \( |B| \geq 1 \), and so \( |G| = n - 1 - |B| \leq n - 2 \).

We compute the trace of the matrix \( A \) as follows:
\[ \text{tr}(A) = \lambda_n + \sum_{\alpha \in B} \lambda_\alpha + \sum_{\alpha \in G} \lambda_\alpha \]
\[ > \lambda_n + |B|(a - \frac{1}{\epsilon} \sum_{i=1}^{n-1} |a_i|^2) - |G|\left(\sum_{i=1}^{n-1} |d_i| + \epsilon\right) \]
\[ \geq 2a - \frac{1}{\epsilon} \sum_{i=1}^{n-1} |a_i|^2 - (n - 2)(\sum_{i=1}^{n-1} |d_i| + \epsilon) \]
\[ \geq \sum_{i=1}^{n} d_i + a = \text{tr}(A), \]
where we use (3.2), (3.3), (3.7) and (3.10). This is a contradiction.
We now prove $B = \emptyset$. Therefore, $G = I$ and the proof is complete. 

We consequently obtain

**Lemma 3.4.** Let $A(a)$ be an $n \times n$ Hermitian matrix

$$A(a) = \begin{pmatrix} d_1 & a_1 \\ d_2 & a_2 \\ \vdots \\ d_{n-1} & a_{n-1} \\ \bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_{n-1} \end{pmatrix}$$

with $d_1, \ldots, d_{n-1}, a_1, \ldots, a_{n-1}$ fixed, and with $a$ variable. Assume that $d_1, d_2, \ldots, d_{n-1}$ are distinct each other, i.e. $d_i \neq d_j, \forall i \neq j$. Denote $\lambda = (\lambda_1, \ldots, \lambda_n)$ by the the eigenvalues of $A(a)$. Given a positive constant $\epsilon$ with $0 < \epsilon \leq \frac{1}{2} \min \{|d_i - d_j| : \forall i \neq j\}$. If the parameter $a$ satisfies the quadratic growth condition

\begin{align}
\tag{3.12}
a \geq \frac{1}{\epsilon} \sum_{i=1}^{n-1} |a_i|^2 + (n-1) \sum_{i=1}^{n-1} |d_i| + (n-2) \epsilon,
\end{align}

then the eigenvalues behavior like

$|d_\alpha - \lambda_\alpha| < \epsilon, \forall 1 \leq \alpha \leq n-1$,

$0 \leq \lambda_n - a < (n-1) \epsilon$.

**Proof.** The proof is based on Lemma 3.3 and a deformation argument. Without loss of generality, we assume $n \geq 3$ and $\sum_{i=1}^{n-1} |a_i|^2 > 0$ (otherwise $n = 2$ or the matrix $A(a)$ is diagonal, and then we are done). Moreover, we assume in addition that $d_1 < d_2 \cdots < d_{n-1}$ and the eigenvalues have the order

$\lambda_1 \leq \lambda_2 \cdots \leq \lambda_{n-1} \leq \lambda_n$.

Fix $\epsilon \in (0, \mu_0]$, where $\mu_0 = \frac{1}{2} \min \{|d_i - d_j| : \forall i \neq j\}$. We denote

$$I_i = (d_i - \epsilon, d_i + \epsilon)$$

and

$$P_0 = \frac{1}{\epsilon} \sum_{i=1}^{n-1} |a_i|^2 + (n-1) \sum_{i=1}^{n-1} |d_i| + (n-2) \epsilon.$$

Since $0 < \epsilon \leq \mu_0$, the intervals disjoint each other

\begin{align}
\tag{3.13}
I_\alpha \bigcap I_\beta = \emptyset \text{ for } 1 \leq \alpha < \beta \leq n-1.
\end{align}

In what follows, we assume that the parameter $a$ satisfies (3.12) and the Greek letters $\alpha, \beta$ range from 1 to $n-1$. Let

$$\text{Card}_a : [P_0, +\infty) \to \mathbb{N}$$

...
be the function that counts the eigenvalues which lie in $I_\alpha$. (Note that when the eigenvalues are not distinct, the function $\text{Card}_\alpha$ means the summation of all the multiplicities of distinct eigenvalues which lie in $I_\alpha$). This function measures the number of the eigenvalues which lie in $I_\alpha$.

We are going to prove that $\text{Card}_\alpha$ is continuous on $[P_0, +\infty)$ in an attempt to complete the proof.

Firstly, Lemma 3.3 asserts that if $a \geq P_0$, then

$$\lambda_\alpha \in \bigcup_{i=1}^{n-1} I_i, \forall 1 \leq \alpha \leq n-1. \tag{3.14}$$

It is well known that the largest eigenvalue $\lambda_n \geq a$, while the smallest eigenvalue $\lambda_1 \leq d_1$. Combining it with (3.14) one has

$$\lambda_n \geq a > \sum_{i=1}^{n-1} |d_i| + \epsilon, \tag{3.15}$$

Thus $\lambda_n \in \mathbb{R} \setminus \left( \bigcup_{i=1}^{n-1} \overline{T_i} \right)$ where $\overline{T_i}$ denotes the closure of $I_i$. Therefore, the function $\text{Card}_\alpha$ is continuous (and so it is constant), since (3.15), (3.13), $\lambda_n \in \mathbb{R} \setminus \left( \bigcup_{i=1}^{n-1} I_i \right)$ and the eigenvalues of $A(a)$ depend on the parameter $a$ continuously.

The continuity of $\text{Card}_\alpha(a)$ plays a crucial role in this proof. Following the line of the proof Lemma 1.2 of Caffarelli-Nirenberg-Spruck [12] (i.e. Lemma 3.2 above), in the setting of Hermitian matrices, one can show that for $1 \leq \alpha \leq n-1$,

$$\lim_{a \to +\infty} \text{Card}_\alpha(a) \geq 1. \tag{3.16}$$

It follows from (3.15), (3.16) and the continuity of $\text{Card}_\alpha$ that

$$\text{Card}_\alpha(a) = 1, \forall a \in [P_0, +\infty), 1 \leq \alpha \leq n-1. \tag{3.17}$$

Together with (3.14), we prove that, for any $1 \leq \alpha \leq n-1$, the interval $I_\alpha = (d_\alpha - \epsilon, d_\alpha + \epsilon)$ contains the eigenvalue $\lambda_\alpha$. We thus complete the proof of Lemma 3.4.

Suppose that there are two distinct index $i_0, j_0$ ($i_0 \neq j_0$) such that $d_{i_0} = d_{j_0}$. Then the characteristic polynomial of $A$ can be rewritten as the following

$$(\lambda - d_{i_0}) \left[ (\lambda - a) \prod_{i \neq i_0} (\lambda - d_i) - |a_{i_0}|^2 \prod_{j \neq j_0, j \neq i_0} (\lambda - d_j) - \sum_{i \neq i_0} |a_i|^2 \prod_{j \neq i, j \neq i_0} (\lambda - d_j) \right].$$

So $\lambda_{i_0} = d_{i_0}$ is an eigenvalue of $A$ for any $a \in \mathbb{R}$. Noticing that the following polynomial

$$(\lambda - a) \prod_{i \neq i_0} (\lambda - d_i) - |a_{i_0}|^2 \prod_{j \neq j_0, j \neq i_0} (\lambda - d_j) - \sum_{i \neq i_0} |a_i|^2 \prod_{j \neq i, j \neq i_0} (\lambda - d_j)$$
is the characteristic polynomial of the \((n - 1) \times (n - 1)\) Hermitian matrix

\[
\begin{pmatrix}
    d_1 & a_1 \\
    \ddots & \ddots \\
    \hat{d}_{i_0} & a_{i_0} & \ddots \\
    \vdots & \ddots & \ddots & \ddots \\
    \hat{a}_1 & \cdots & \hat{a}_{i_0} & (|a_{j_0}|^2 + |a_{i_0}|^2)^{\frac{1}{2}} & \cdots & a
\end{pmatrix}
\]

where \(\hat{\ast}\) indicates deletion. Therefore, \((\lambda_1, \cdots, \hat{\lambda}_{i_0}, \cdots, \lambda_n)\) are the eigenvalues of the above \((n - 1) \times (n - 1)\) Hermitian matrix. Hence, we obtain

**Lemma 3.5.** Let \(A\) be an \(n \times n\) Hermitian matrix

\[
\begin{pmatrix}
    d_1 & a_1 \\
    \ddots & \ddots \\
    d_{i_0} & a_{i_0} & \ddots \\
    \vdots & \ddots & \ddots & \ddots \\
    \hat{a}_1 & \cdots & \hat{a}_{i_0} & (|a_{j_0}|^2 + |a_{i_0}|^2)^{\frac{1}{2}} & \cdots & a
\end{pmatrix}
\]

with \(d_1, \cdots, d_{n-1}, a_1, \cdots, a_{n-1}\) fixed, and with \(a\) variable. Let

\[
I = \begin{cases} 
\mathbb{R}^+ = (0, +\infty) & \text{if } d_i = d_1, \forall 2 \leq i \leq n - 1, \\
(0, \mu_0) & \text{for } \mu_0 = \frac{1}{2} \min\{|d_i - d_j| : d_i \neq d_j\} & \text{otherwise.}
\end{cases}
\]

Denote \(\lambda = (\lambda_1, \cdots, \lambda_n)\) by the the eigenvalues of \(A\). Fix \(\epsilon \in I\). Suppose that the parameter \(a\) in \(A\) satisfies (3.12). Then the eigenvalues behavior like

\[
|d_{\alpha} - \lambda_{\alpha}| < \epsilon, \ \forall 1 \leq \alpha \leq n - 1,
\]

\[
0 \leq \lambda_n - a < (n - 1)\epsilon.
\]

Applying Lemmas 3.3 and 3.5, we complete the proof of Lemma 3.1 without restriction to the applicable scope of \(\epsilon\).

**Proof of Lemma 3.7.** We follow the outline of the proof of Lemma 3.4. Without loss of generality, we may assume

\[
n \geq 3, \sum_{i=1}^{n-1} |a_i|^2 > 0, \ d_1 \leq d_2 \leq \cdots \leq d_{n-1} \text{ and } \lambda_1 \leq \lambda_2 \leq \cdots \lambda_{n-1} \leq \lambda_n.
\]

Fix \(\epsilon > 0\). Let \(I'_\alpha = (d_\alpha - \frac{\epsilon}{2n-3}, \hat{d}_\alpha + \frac{\epsilon}{2n-3})\) and

\[
P'_0 = \frac{2n-3}{\epsilon} \sum_{i=1}^{n-1} |a_i|^2 + (n - 1) \sum_{i=1}^{n-1} |d_i| + \frac{(n - 2)\epsilon}{2n - 3}.
\]
In what follows we assume (3.1) holds. The connected components of \( \bigcup_{\alpha=1}^{n-1} I'_\alpha \) are as in the following:

\[
J_1 = \bigcup_{\alpha=1}^{j_1} I'_\alpha, \quad J_2 = \bigcup_{\alpha=j_1+1}^{j_2} I'_\alpha, \ldots, \quad J_i = \bigcup_{\alpha=j_{i-1}+1}^{j_i} I'_\alpha, \ldots, \quad J_m = \bigcup_{\alpha=j_{m-1}+1}^{n-1} I'_\alpha.
\]

Moreover

\[ J_i \cap J_k = \emptyset, \quad \text{for } 1 \leq i < k \leq m. \]

It plays formally the role of (3.13) in the proof of Lemma 3.4.

As in the proof of Lemma 3.4 we let \( \widetilde{\text{Card}}_k : [P'_0, +\infty) \to \mathbb{N} \)

be the function that counts the eigenvalues which lie in \( J_k \). (Note that when the eigenvalues are not distinct, the function \( \widetilde{\text{Card}}_k \) denotes the summation of all the multiplicities of distinct eigenvalues which lie in \( J_k \)). By using Lemma 3.3 and

\[
\lambda_n \geq a \geq P'_0 > n - 1 \sum_{i=1}^{n-1} |d_i| + \frac{\epsilon}{2n - 3}
\]

we conclude that if the parameter \( a \) satisfies the quadratic growth condition (3.1) then

\[
\lambda_n \in \mathbb{R} \setminus (\bigcup_{k=1}^{m} I'_k) = \mathbb{R} \setminus (\bigcup_{i=1}^{m} J_i),
\]

(3.17)

\[
\lambda_\alpha \in \bigcup_{i=1}^{n-1} I'_i = \bigcup_{i=1}^{m} J_i \quad \text{for } 1 \leq \alpha \leq n - 1.
\]

Similarly, \( \widetilde{\text{Card}}_i(a) \) is a continuous function with respect to the variable \( a \). So it is a constant. Combining it with Lemma 3.5 we see that \( \widetilde{\text{Card}}_i(a) = j_i - j_{i-1} \) for sufficiently large \( a \). Here we denote \( j_0 = 0 \) and \( j_m = n - 1 \). The constant of \( \text{Card}_i \) therefore follows that

\[
\widetilde{\text{Card}}_i(a) = j_i - j_{i-1}.
\]

We thus know that the \( (j_i - j_{i-1}) \) eigenvalues

\[
\lambda_{j_{i-1}+1}, \lambda_{j_{i-1}+2}, \ldots, \lambda_{j_i}
\]

lie in the connected component \( J_i \). Thus, for any \( j_{i-1} + 1 \leq \gamma \leq j_i \), we have \( I'_\gamma \subset J_i \) and \( \lambda_\gamma \) lies in the connected component \( J_i \). Therefore,

\[
|\lambda_\gamma - d_\gamma| < \frac{(2(j_i - j_{i-1}) - 1)\epsilon}{2n - 3} \leq \epsilon.
\]

Here we also use the fact that \( d_\gamma \) is midpoint of \( I'_\gamma \) and every \( J_i \subset \mathbb{R} \) is an open subset. \( \square \)
4. Quantitative Boundary Estimates

Given a point \( p_0 \in \partial M \). Let \( \sigma \) be the distance function to \( \partial M \), and let \( \rho(z) = \text{dist}_{M}(z, p_0) \) be the distance function from \( z \) to \( p_0 \) with respect to \( \omega \), and

\[
\Omega_\delta = \{ z \in M : \rho(z) < \delta \}.
\]

First of all, we derive \( C^0 \)-estimate, gradient estimate on the boundary and the boundary estimates for double tangential derivatives. Let \( w \in C^2(\bar{M}) \) be the function solving

\[
\Delta w + g^{\bar{i}\bar{j}} \chi_{\bar{i}\bar{j}} = 0 \text{ in } M, \quad w = \varphi \text{ on } \partial M.
\]

The maximum principle and comparison principle imply that

\[
u \leq u \leq w \text{ in } M.
\]

Combining with \( u = u = w = \varphi \) on \( \partial M \), we know that there is a uniform constant \( C \) depending only on \( \sup_\bar{M} |w|, \sup_M |u|, \sup_{\partial M} |\nabla w| \) and \( \sup_{\partial M} |\nabla u| \) such that

\[
\sup_\bar{M} |u| + \sup_{\partial M} |\nabla u| \leq C.
\]

Moreover, we can write \( u - \bar{u} = h\sigma \) in \( \{ z \in M : \sigma < \delta \} \) for \( 0 < \delta \ll 1 \) and \( h \in C^2 \).

Thus, we have

\[
\sup_{\partial M} |\nabla^2 u(\xi_1, \xi_2)| \leq C, \quad \forall \xi_1, \xi_2 \in T_{\partial M}, |\xi_1| = |\xi_2| = 1
\]

where \( \nabla^2 u \) denotes the real Hessian of \( u \).

Applying Lemma 3.1 we first show that the boundary estimates for double normal derivatives can be dominated by a quadratic term of the boundary estimates for tangential-normal derivatives.

**Proposition 4.1.** Let \( (M, J, \omega) \) be a compact Hermitian manifold with \( C^2 \) Levi flat boundary, let \( \nu \) be the unit inner normal vector along the boundary. We denote \( \xi_n = \frac{1}{2}(\nu - \sqrt{-1}J\nu) \). Let \( u \in C^3(M) \cap C^2(\bar{M}) \) be an admissible solution to Dirichlet problem \((1.1)\). Fix a point \( p_0 \) at the boundary. Then for \( \xi_\alpha, \xi_\beta \in T_{\partial M} \cap JT_{\partial M} \) \((\alpha, \beta = 1, \ldots, n - 1)\) satisfying \( J\xi_\alpha = \sqrt{-1}\xi_\alpha, J\xi_\beta = \sqrt{-1}\xi_\beta \) and \( g(\xi_\alpha, \bar{\xi}_\beta) = \delta_{\alpha\beta} \) at \( p_0 \in \partial M \), we have

\[
g(\xi_n, J\bar{\xi}_n)(p_0) \leq C(1 + \sum_{\alpha=1}^{n-1} |g(\xi_\alpha, J\bar{\xi}_n)(p_0)|^2),
\]

where \( C \) is a uniform positive constant depending only on \( |u|_{C^0(\bar{M})}, |u|_{C^2(\bar{M})} \) and other known data (but neither on \( \sup_M |\nabla u| \) nor on \( (\delta_{\varphi, f})^{-1} \)).

The Levi flatness of the boundary implies that \( \partial M \) can be foliated by complex analytic hypersurfaces, see [2]. These leaves are complex local submanifolds and their complex tangent spaces are precisely the holomorphic tangent spaces \( T_{\partial M} \cap JT_{\partial M} \) to \( \partial M \). We shall remark here that Frobenius theorem guarantees that the Levi flat
real hypersurface $\Sigma$ of class $C^k$ can be foliated $C^{k-1}$ complex hypersurfaces, while Barrett-Fornaess [2] proved that the foliation is indeed of class $C^k$. Namely,

**Lemma 4.2** ([2]). Let $\Sigma$ be a $C^k$ ($k \geq 2$) Levi flat real hypersurface in a complex manifold of complex dimension $n$. The induced foliation is indeed of class $C^k$.

Let $\mathcal{L}$ be the leaves on $\partial M$, and let $\mathcal{L}_{p_0}$ be the leaf passing through $p_0 \in \partial M$. Such leaves allow one to construct special local holomorphic coordinates centered at the boundary points. To be precise, given a point $p_0 \in \partial M$, there is always a local holomorphic coordinate system $w = (w_1, \ldots, w_n)$ of $\bar{M}$ centered at $p_0$ such that $(w_1, \ldots, w_{n-1})$ is a local coordinate of $\mathcal{L}_{p_0}$ near $p_0$, $g(\frac{\partial}{\partial w_\alpha}, \frac{\partial}{\partial \bar{w}_\beta})(p_0) = \delta_{\alpha\beta}$ ($1 \leq \alpha, \beta \leq n-1$), and $\mathcal{L}_{p_0}$ is defined locally by $w_n = 0$. By changing the coordinates $w = (w_1, \ldots, w_n)$ if necessary, one can pick local holomorphic coordinates $z = (z_1, \ldots, z_n)$ ($z_i = x_i + \sqrt{-1}y_i$) such that

$$z_\alpha = w_\alpha, \ g_{ij}(p_0) = \delta_{ij},$$

where $1 \leq \alpha \leq n-1, 1 \leq i, j \leq n$.

**Proof of Proposition 4.1.** Given $p_0 \in \partial M$. In what follows the discussion is done at $p_0$, and the Greek letters $\alpha, \beta$ range from 1 to $n-1$. For the local coordinate $z = (z_1, \ldots, z_n)$ given by (4.5), we assume further that $\{g_{\alpha\beta}\}$ is diagonal at $p_0$. It follows from the Levi flatness of $\partial M$ and the boundary value condition that

$$g_{\alpha\beta} = g_{\alpha\beta} = g_{\alpha\beta}[\varphi]$$

at $p_0$.

Firstly, we claim that there exist two uniform positive constants $\varepsilon_0, R_0$ depending on $g$ and $f$, such that

$$f(\underline{\varphi}_1, \ldots, \underline{\varphi}_{(n-1)(n-1)} - \varepsilon_0, R_0) \geq \psi$$

and $(\underline{\varphi}_1 - \varepsilon_0, \ldots, \underline{\varphi}_{(n-1)(n-1)} - \varepsilon_0, R_0) \in \Gamma$.

We leave the proof of (4.7) at the end of the proof of this proposition. Next, we apply Lemma 3.1 together with (4.7) to establish the quantitative boundary estimates for double normal derivative. Let’s denote

$$A(R) = \begin{pmatrix}
g_{11} & g_{1n} 
g_{21} & g_{2n} 
\vdots & \vdots 
g_{n1} & g_{nn}
g_{n1} & g_{n(n-1)} 
\end{pmatrix}$$
and

\[ A(R) = \begin{pmatrix}
g_{11} & g_{1\bar{n}} \\
g_{21} & g_{2\bar{n}} \\
\vdots & \vdots \\
g_{n1} & g_{n\bar{n}}
g_{1\bar{1}} & g_{2\bar{1}} & \cdots & g_{(n-1)(n-1)\bar{n}} \\
\end{pmatrix}.\]

Let’s pick the parameter \( \epsilon \) in Lemma 3.1 as \( \epsilon = \frac{\epsilon_0}{128} \), and let

\[ R_c = \frac{128(2n-3)}{\epsilon_0} \sum_{\alpha=1}^{n-1} |g_{\alpha\bar{\alpha}}|^2 + (n-1) \sum_{\alpha=1}^{n-1} |g_{n\alpha\bar{n}}| + \frac{(n-2)\epsilon_0}{128(2n-3)} + R_0, \]

where \( \epsilon_0 \) and \( R_0 \) are the constants from (4.7). Lemma 3.1 applies to \( A(R_c) \) and the eigenvalues of \( A(R_c) \) shall behavior like

\[ \lambda(A(R_c)) \in (g_{1\bar{1}} - \frac{\epsilon_0}{128}, \cdots, g_{(n-1)(n-1)\bar{n}} - \frac{\epsilon_0}{128}, R_c) + \Gamma_n \subset \Gamma. \]

Applying (1.2), (4.6), (4.7) and (4.8), one hence has

\[ F(A(R_c)) = F(A(R_c)) \geq f(g_{1\bar{1}} - \frac{\epsilon_0}{128}, \cdots, g_{(n-1)(n-1)\bar{n}} - \frac{\epsilon_0}{128}, R_c) \geq \psi. \]

Therefore,

\[ g_{n\bar{n}}(p_0) \leq R_c. \]

We notice that there is a difference between Lemma 3.1 and Lemma 3.4. In order to apply Lemma 3.4 we shall make a perturbation so that one gets a matrix satisfying the assumptions of Lemma 3.4. However, the perturbation is not needed when one applies Lemma 3.1 since Lemma 3.1 is applicable without restriction to scope of the parameter \( \epsilon \).

To finish the proof of Proposition 4.1 what is left to prove is the key inequality (4.7).

We propose two proofs of (4.7) here. Writing

\[ B(R) = \begin{pmatrix}
g_{11} & g_{1\bar{n}} \\
g_{21} & g_{2\bar{n}} \\
\vdots & \vdots \\
g_{n1} & g_{n\bar{n}}
g_{1\bar{1}} & g_{2\bar{1}} & \cdots & g_{(n-1)(n-1)\bar{n}} \\
\end{pmatrix}.\]

The first proof is as in the following: For \( R > \sup_{\partial M} |g| \), one has

\[ f(\lambda(B(R))) > \psi \text{ on } \partial M. \]
It follows from (1.2), (1.3), (4.9) and the openness of $\Gamma$ that

$$f(\lambda(B(R_1)) - \frac{\epsilon_0}{2} I) > \psi \text{ and } (\lambda(B(R_1)) - \frac{\epsilon_0}{2} I) \in \Gamma,$$

where $\epsilon_0$ (small enough) and $R_1$ (large enough) are two uniform positive constants depending only on $g$ and other known data. Moreover, by applying Lemma 3.1 to the matrix $B(R)$ (by setting the parameter $\epsilon = \frac{\epsilon_0}{128}$ in Lemma 3.1), we know that the eigenvalues $\lambda(B(R_2)) = (\mu_1, \ldots, \mu_n)$ ($R_2 = \frac{128(2n-3)}{\epsilon_0} |g|^2 + (n-1)^2 |g| + \frac{(n-2)\epsilon_0}{128(2n-3)} + R_1$) behavior as

$$\frac{\epsilon_0}{128} \leq \mu_2 \leq \frac{\epsilon_0}{128}, \; R_2 \leq \mu_n < R_2 + \frac{(n-1)\epsilon_0}{128}.$$

Combining it with (4.10) we can prove (4.7) holds with $\epsilon_0 = \frac{63}{128} \epsilon_0$ and $R_0 = R_2 + \frac{(n-65)\epsilon_0}{128}$, where $\epsilon_0$ is the constant from (4.10).

We shall point out that in this proof condition (1.3) may be replaced by the convexity of the level sets of $f$. Moreover, condition (4.9) can be also derived from

$$\lim_{R \to +\infty} f(\lambda(B(R))) > \psi \text{ on } \partial M.$$

This condition can be achieved by the boundary data $\varphi$ according to Lemma 3.1 and (4.6). Also, this condition is satisfied by a $C$-subsolution $\underline{u}$ with the same boundary value condition $\underline{u}|_{\partial M} = \varphi$.

The second proof is the following: Applying Lemma 3.1 to $B(R)$ we can prove that there is a uniform positive constant $R_3$ depending on $g$ such that

$$\left(\underline{g}_{11}, \ldots, \underline{g}_{(n-1)(n-1)}, R_3\right) \in \Gamma.$$

Here we also use the fact that $\Gamma$ is an open set. The ellipticity and concavity of equation (1.1), couple with Lemma 6.2 in [12], therefore yield that

$$F(A) - F(B) \geq F^{ij}(A)(a_{ij} - b_{ij})$$

for the Hessian matrices $A = \{a_{ij}\}$ and $B = \{b_{ij}\}$ with $\lambda(A), \lambda(B) \in \Gamma$. Thus, there exists a uniform positive constant $R_4 \geq R_3$ depending only on $g$ such that

$$f(\underline{g}_{11}, \ldots, \underline{g}_{(n-1)(n-1)}, R_4) = F(\text{diag}(\underline{g}_{11}, \ldots, \underline{g}_{(n-1)(n-1)}), R_4)) > F(\underline{g}) \geq \psi.$$

Thus one can derive (4.7) holds. We thus complete the proof of Proposition 4.1.
Moreover, we shall point out that one can also apply Lemma 3.4 to prove Proposition 4.1. In order to apply Lemma 3.4 we shall make a perturbation:

\[ A(R, \varepsilon_0) = \begin{pmatrix}
  \tilde{g}_{11} & \tilde{g}_{12} & \cdots & \tilde{g}_{1n} \\
  \tilde{g}_{21} & \tilde{g}_{22} & \cdots & \tilde{g}_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  \tilde{g}_{n1} & \tilde{g}_{n2} & \cdots & \tilde{g}_{nn} \\
\end{pmatrix} \]

where \( \tilde{g}_{\alpha\beta} - \varepsilon_0 \leq \tilde{g}_{\alpha\beta} \leq \tilde{g}_{\alpha\beta} \) and \( |\tilde{g}_{\alpha\beta} - \tilde{g}_{\beta\beta}| \geq \frac{\varepsilon_0}{64(n-1)} \) for \( 1 \leq \alpha < \beta \leq n - 1 \), where \( \varepsilon_0 \) is the constant from (4.7).

Next, we are going to prove the following proposition when \( \partial M \) is real analytic Levi flat.

**Proposition 4.3.** Assume the boundary is real analytic Levi flat. Let \( u \in C^3(M) \cap C^2(\bar{M}) \) be an admissible solution of Dirichlet problem (1.1), \( \psi \in C^1(\bar{M}) \) and \( \varphi \in C^3(\partial M) \). Suppose, in addition to (1.2) - (1.5), that there is an admissible subsolution \( u \in C^2(\bar{M}) \) to the Dirichlet problem. Then for any \( T \in T_{\partial M} \cap JT_{\partial M} \) with \( |T| = 1 \), there exists a uniform constant \( C \) depending only on \( |\varphi|_{C^3(M)}, |u|_{C^2(\bar{M})}, \sup_{\partial M} |\nabla u|, |\psi|_{C^1(\bar{M})}, \) and other known data (but not on \( \sup_{M} |\nabla u| \)), such that

\[ |\nabla^2 u(T, \nu)| \leq C(1 + \sup_{M} |\nabla u|). \]

Moreover, \( C \) is independent of \( (\delta_{\varphi, f})^{-1} \).

**Remark 4.4.** When \( \varphi \) is a constant, the constant \( C \) in (4.12) depends on \( |\psi|_{C^1(\bar{M})}, |u|_{C^2(\bar{M})}, \sup_{\partial M} |\nabla u|, |\psi|_{C^1(\bar{M})}, \) and other known data.

**Remark 4.5.** Combining these two Propositions, we immediately derive Theorem 1.4.

According to a theorem in [14] (see also [53]), for each \( p_0 \in \partial M \), one can pick local holomorphic coordinates

\[ (z_1, \cdots, z_n), \ z_i = x_i + \sqrt{-1} y_i, \]

centered at \( p_0 \), such that \( g_{ij}(p_0) = \delta_{ij} \) and \( \partial M \) is locally given by

\[ \Re(z_n) = 0. \]

Under local holomorphic coordinates (4.13), we take the tangential operator on boundary as

\[ D = \pm \frac{\partial}{\partial x_\alpha}, \pm \frac{\partial}{\partial y_\alpha}, 1 \leq \alpha \leq n - 1. \]
(Notice \( D \) is just defined locally). In particular, when \( M = X \times S \) is a product of a closed complex manifold \( X \) with a compact Riemann surface with boundary \( S \), as in [19], we simply choose \( D \) as follows

\[
D = \pm \frac{\partial}{\partial x_\alpha}, \pm \frac{\partial}{\partial y_\alpha},
\]

where \( z' = (z_1, \cdots, z_{n-1}) \) is local holomorphic coordinate of \( X \). Thus our results extend the boundary estimates in [19, 61] from complex Monge-Ampère equation to more general equations.

Next, we outline the following lemma.

**Lemma 4.6.** There is a positive constant \( C \) depending only on \( |\varphi|_{C^1(M)} \), \( |\chi|_{C^1(M)} \), \( \psi_{C^1(M)} \) and other known data (but neither on \( \sup_M |\nabla u| \) nor on \( (\delta \psi, \psi) \)) such that

\[
|\mathcal{L}D(u - \varphi)| \leq C \left( 1 + \left( 1 + \sup_M |\nabla u| \right) \sum_{i=1}^{n} f_i + \sum_{i=1}^{n} f_i |\lambda_i| \right), \quad \text{in } \Omega_\delta
\]

for some small \( \delta > 0 \), where \( \mathcal{L} \) is the linearized operator of equation (1.1) which is given by

\[
\mathcal{L}v = F^{ij} v_{ij} \text{ for } v \in C^2(M).
\]

Here \( F^{ij} = F^{ij}(\varphi) \).

**Proof.** We follow the notation used in [33]. By straightforward computation, we have

\[
(u_{x_k})_{z_j} = u_{x_k} \bar{z}_j + \Gamma^l_{kj} u_{l}, \quad (u_{y_k})_{z_j} = u_{y_k} \bar{z}_j - \sqrt{-1} \Gamma^l_{kj} u_{l},
\]

\[
(u_{x_k})_{z_i z_j} = u_{x_k} z_i \bar{z}_j + \Gamma^l_{ik} u_{l} + \Gamma^l_{jk} u_{l} - g^{lm} R_{ijkm} u_{l},
\]

\[
(u_{y_k})_{z_i z_j} = u_{y_k} z_i \bar{z}_j + \sqrt{-1} \left( \Gamma^l_{ik} u_{l} - \Gamma^l_{jk} u_{l} \right) - \sqrt{-1} g^{lm} R_{ijkm} u_{l},
\]

\[
(u_{x_k})_{z_i z_j} = u_{z_i z_j} + g^{lm} R_{ijkm} u_{l} - T^l_{ik} u_{l} - T^l_{jk} u_{l},
\]

\[
(u_{y_k})_{z_i z_j} = u_{z_i z_j} + \sqrt{-1} g^{lm} R_{ijkm} u_{l} - \sqrt{-1} \left( T^l_{ik} u_{l} - T^l_{jk} u_{l} \right).
\]

On the other hand, differentiating equation (1.1), we get

\[
\mathcal{L}(u_{x_k}) = (\psi)_{x_k} - F^{ij} (\chi_{ij})_{x_k} + F^{ij} g^{lm} R_{ijkm} u_{l} - 2 \Re (F^{ij} T^l_{ik} u_{l}),
\]

\[
\mathcal{L}(u_{y_k}) = (\psi)_{y_k} - F^{ij} (\chi_{ij})_{y_k} + \sqrt{-1} F^{ij} g^{lm} R_{ijkm} u_{l} + 2 \Im (F^{ij} T^l_{ik} u_{l}).
\]

Together with \( u_{ij} = \varphi_{ij} - \chi_{ij} \), one has

\[
|\mathcal{L}((u - \varphi)_{x_k})|, \ |\mathcal{L}((u - \varphi)_{y_k})| \leq C \left( 1 + \left( 1 + \sup_M |\nabla u| \right) \sum_{i=1}^{n} f_i + \sum_{i=1}^{n} f_i |\lambda_i| \right).
\]

\[\square\]
Proof of Proposition 4.3. The proposition is proved by constructing barrier functions. The constructions of this type of barrier functions go back at least to the work \[35, 30\].

Let’s take

\[\tilde{\Psi} = A_1 \sqrt{b_1 (u - u)} - A_2 \sqrt{b_1 \rho^2} + A_3 \sqrt{b_1} (N \sigma^2 - t \sigma)\]

(4.16)

\[+ \frac{1}{\sqrt{b_1}} \sum_{\tau < n} |(u - \varphi)_{\tau}|^2 + D(u - \varphi),\]

where \(b_1 = 1 + \sup_M |\nabla (u - \varphi)|^2 + \sup_M |\nabla \varphi|^2\).

In what follows we denote by \(\tilde{u} = u - \varphi\). We shall point out that the constants appearing in the proof of quantitative boundary estimates, such as \(C, C_0, C_1, C_2, A_1, A_2, A_3, \cdots\), etc depend neither on \(|\nabla u|\) nor on \((\delta_{\psi, f})^{-1}\).

Let \(\delta > 0\) and \(t > 0\) be sufficiently small constants such that \(N \delta - t \leq 0\) (where \(N\) is a positive constant sufficiently large to be determined later), \(\sigma\) is \(C^2\) and

\[\frac{1}{4} \leq |\nabla \sigma| \leq 2, \quad |\mathcal{L} \sigma| \leq C_2 \sum_{i=1}^n f_i, \quad |\mathcal{L} \rho|^2 \leq C_2 \sum_{i=1}^n f_i, \quad \text{in } \Omega_\delta.\]

(4.17)

Furthermore, we can choose \(\delta\) and \(t\) small enough such that

\[\max\{|2N \delta - t|, t\} \leq \min\left\{\frac{\varepsilon}{2C_2}, \frac{\beta}{16\sqrt{nC_2}}\right\},\]

(4.18)

where \(\beta = \frac{1}{2} \min_M \text{dist}(\nu_{\lambda_{\tilde{u}}}, \partial \Gamma_n)\) is the constant in (2.4), \(\varepsilon\) is the constant in Lemma 2.2 and \(C_2\) is the constant in (4.17).

By straightforward calculation and an elementary inequality \(|a - b|^2 \geq \frac{1}{2} |a|^2 - |b|^2\), one has

\[\mathcal{L} \left(\sum_{\tau < n} |\tilde{u}_{\tau}|^2\right) = \sum_{\tau < n} \left\{F^{ij}_{\tau} (\tilde{u}_{\tau i} \tilde{u}_{\tau j} + \tilde{u}_{\tau i} \tilde{u}_{\tau j}) + \mathcal{L} (\tilde{u}_{\tau i}) \tilde{u}_{\tau r} + \mathcal{L} (\tilde{u}_{\tau r}) \tilde{u}_{\tau i}\right\}\]

(4.19)

\[\geq \frac{1}{2} \sum_{\tau < n} F^{ij}_{\tau} g_{\tau i} g_{\tau j} - C'_1 \sqrt{b_1} \sum_{i=1}^n f_i |\lambda_i| - C'_1 t \sqrt{b_1} - C'_1 b_1 \sum_{i=1}^n f_i.\]

By Proposition 2.19 in \[32\], there is an index \(r\) such that

\[\sum_{\tau < n} F^{ij}_{\tau} g_{\tau i} g_{\tau j} \geq \frac{1}{4} \sum_{i \neq r} f_i |\lambda_i|^2.\]

By (4.17), (4.19) and Lemma 4.6 we therefore arrive at the following key inequality

\[\mathcal{L} (\tilde{\Psi}) \geq A_1 \sqrt{b_1} \mathcal{L} (u - u) + \frac{1}{8 \sqrt{b_1}} \sum_{i \neq r} f_i |\lambda_i|^2 + A_3 \sqrt{b_1} \mathcal{L} (N \sigma^2 - t \sigma)\]

(4.20)

\[- C_1 - C_1 \sum_{i=1}^n f_i |\lambda_i| - (A_2 C_2 + C_1) \sqrt{b_1} \sum_{i=1}^n f_i.\]
Case I: If $|\nu_\lambda - \nu_{\bar{\lambda}}| \geq \beta$, then by Lemma 2.2 above and Lemma 6.2 in [12], we have

$$\mathcal{L}(u-u) \geq \varepsilon(1 + \sum_{i=1}^{n} f_i),$$

where $\varepsilon$ is the positive constant in Lemma 2.2. From the proof of Lemma 2.2 presented in [34], we can check that $\varepsilon$ is determined by $f_i, \beta$ and $\lambda$.

Lemma A.4 states that $\sum_{i=1}^{n} f_i\lambda_i > 0$ in $\Gamma$; while the concavity of the equation implies $\sum_{i=1}^{n} f_i(\lambda_i - \lambda_i) \geq f(\bar{\lambda}) - f(\lambda) \geq 0$. We can apply them to derive

$$\sum_{i=1}^{n} f_i|\lambda_i| = 2 \sum_{\lambda_i \geq 0} f_i \lambda_i - \sum_{i=1}^{n} f_i \lambda_i \leq \frac{\varepsilon}{16\sqrt{b_1}} \sum_{\lambda_i \geq 0} f_i\lambda_i^2 + \frac{16\sqrt{b_1}}{\varepsilon} \sum_{i=1}^{n} f_i,$$

$$\sum_{i=1}^{n} f_i|\lambda_i| = \sum_{i=1}^{n} f_i \lambda_i - 2 \sum_{\lambda_i < 0} f_i \lambda_i \leq \frac{\varepsilon}{16\sqrt{b_1}} \sum_{\lambda_i < 0} f_i\lambda_i^2 + \left(\frac{16\sqrt{b_1}}{\varepsilon} + |\lambda|\right) \sum_{i=1}^{n} f_i,$$

where $\varepsilon > 0$ is a constant to be determined later.

From (4.17) and (4.18) it is easy to see that

$$\mathcal{L}(N\sigma^2 - t\sigma) \geq -\frac{\varepsilon}{2} \sum_{i=1}^{n} f_i.$$

Let $\varepsilon = 1/C_1$ and

$$A_1 \geq A_3 + \frac{2}{\varepsilon} (16C_1^2 + A_2C_2 + C_1 \sup_{M} |\lambda| + C_1).$$

Then it follows from (4.20), (4.21) and (4.22) that

$$\mathcal{L}\bar{\Psi} \geq A_1 \sqrt{b_1} \mathcal{L}(u-u) + \frac{1}{16\sqrt{b_1}} \sum_{i \neq r} f_i\lambda_i^2 - C_1$$

$$- (C_1 + A_2C_2 + 16C_1^2 + \frac{A_3\varepsilon}{2} + \frac{C_1 \sup_{M} |\lambda|}{\sqrt{b_1}}) \sqrt{b_1} \sum_{i=1}^{n} f_i \geq 0, \text{ in } \Omega_\delta.$$

Case II: Suppose that $|\nu_\lambda - \nu_{\bar{\lambda}}| < \beta$ and therefore $\nu_\lambda - \beta\bar{\lambda} \in \Gamma_n$ and

$$f_i \geq \frac{\beta}{\sqrt{n}} \sum_{j=1}^{n} f_j, \text{ i.e., } F^{ij} \geq \frac{\beta}{\sqrt{n}} (F^{pq}g_{pq})g^{ij}.$$

Together with (4.18) one can derive

$$\mathcal{L}(N\sigma^2 - t\sigma) = (2N\sigma - t)\mathcal{L}\sigma + 2NF^{ij}\sigma_i\sigma_j \geq \frac{\beta N}{16\sqrt{n}} \sum_{i=1}^{n} f_i.$$
As in [34] we know there exist two uniform positive constants \(c_0\) and \(C_0\), such that

\[
\sum_{i \neq r} f_i \lambda_i^2 \geq c_0 \sum_{i=1}^{n} f_i \lambda_i^2 - C_0 \sum_{i=1}^{n} f_i.
\]

We can check that \(c_0\) depends only on \(\beta\) and \(n\), and \(C_0\) depends only on \(\beta\), \(n\) and \(\sup_{\Omega} |\lambda u|\). Therefore, by choosing \(N \geq 32(C_0 + C_1 + A_2 C_2)/(\beta A_3)\), one derives

\[
\mathcal{L} \tilde{\Psi} \geq \frac{c_0 \beta}{8 \sqrt{b_1 n}} |\lambda|^2 \sum_{i=1}^{n} f_i + \frac{\beta A_3 N}{32 \sqrt{n}} \sqrt{b_1} \sum_{i=1}^{n} f_i - C_1 |\lambda| \sum_{i=1}^{n} f_i - C_1,
\]

where we use \(\mathcal{L}(u - \bar{u}) \geq 0\) in \(M\). Moreover, Lemma 2.1 together with Cauchy-Schwarz inequality, implies that

\[
\mathcal{L} \tilde{\Psi} \geq \frac{\beta A_3 N}{64 \sqrt{n}} \sqrt{b_1} \sum_{i=1}^{n} f_i - C_1 + \left( \frac{c_0 \beta^2 A_3 N}{16 \sqrt{n}} - C_1 \right) |\lambda| \sum_{i=1}^{n} f_i, \text{ in } \Omega_\delta.
\]

If we choose \(A_3\) sufficiently large then

\[
\mathcal{L} \tilde{\Psi} \geq 0, \text{ in } \Omega_\delta.
\]

The boundary is locally given by \(\Re(z_n) = 0\) under local holomorphic coordinates \([4.13]\), thus \(D(u - \varphi) = 0\), \((u - \varphi)_{\tau} = 0\) on \(\partial M \cap \overline{\Omega}_\delta\) (\(\forall 1 \leq \tau < n\)), for some \(\delta > 0\), where \(D\) is defined in \([4.15]\). Thus we can assume \(0 < t, \delta \ll 1, N\delta - t \leq 0\) such that

\[
\tilde{\Psi} = A_1 \sqrt{b_1}(u - u) - A_2 \sqrt{b_1} \rho^2 + A_3 \sqrt{b_1}(N\sigma^2 - t\sigma)
\]

\[
+ \frac{1}{\sqrt{b_1}} \sum_{\tau < n} |(u - \varphi)_{\tau}|^2 + D(u - \varphi) \leq 0
\]

on \(\partial M \cap \overline{\Omega}_\delta\). On the other hand, \(\rho = \delta\) and \(u - \bar{u} \leq 0\) on \(M \cap \partial \Omega_\delta\). Hence, if \(A_2 \gg 1\) then \(\tilde{\Psi} \leq 0\) on \(M \cap \partial \Omega_\delta\), where we use again \(N\delta - t \leq 0\). Therefore, \(\tilde{\Psi} \leq 0\) in \(\Omega_\delta\) by applying maximum principle. Together with \(\tilde{\Psi}(0) = 0\), one has \(\tilde{\Psi}_\nu(0) \leq 0\). Thus

\[
\nabla_\nu D(u - \varphi)(0) \leq - A_1 \sqrt{b_1}(u - u)_\nu(0) + A_2 \sqrt{b_1}(\rho^2)_\nu(0)
\]

\[
- A_3 \sqrt{b_1}(N\sigma^2 - t\sigma)_\nu(0)
\]

\[
- \frac{1}{\sqrt{b_1}} \sum_{\tau < n} ((u - \varphi)_{\tau} \nabla_\nu (u - \varphi))_{\tau}(0)
\]

\[
- \frac{1}{\sqrt{b_1}} \sum_{\tau < n} ((u - \varphi)_{\tau} \nabla_\nu (u - \varphi))_{\tau}(0)
\]

\[
= - A_1 \sqrt{b_1}(u - u)_\nu(0) + A_2 \sqrt{b_1}(\rho^2)_\nu(0)
\]

\[
- A_3 \sqrt{b_1}(N\sigma^2 - t\sigma)_\nu(0)
\]

\[
\leq C \sqrt{b_1}(1 + \sup_{\Omega} |\nabla (u - \bar{u})|) \leq C'(1 + \sup_{M} |\nabla u|).
\]
Here we use (4.3) and \((u - \varphi)_\tau(0) = 0\) for \(1 \leq \tau \leq n - 1\). The above discussion also works if we take the operator as \(-D\). Therefore, we derive

\[(4.27) \quad |\nabla u| \leq C(1 + \sup_M |\nabla u|)\]

at \(p_0\), where \(C\) is a positive constant depends only on \(|\varphi|_{C^2(M)}, |u|_{C^2(M)}|\psi|_{C^1(M)}\) and other known data (but not on \(\sup_M |\nabla u|\)). Moreover, the constant \(C\) in (4.27) does not depend on \((\delta_{\psi,f})^{-1}\).

**Remark 4.7.** Together with the boundary value condition, Theorem 1.4 immediately gives the quantitative boundary estimate for real Hessian

\[\sup_{\partial M} |\nabla^2 u| \leq C(1 + \sup_M |\nabla u|^2).\]

### 5. Solving Equations

The following second order estimate is essentially due to Székelyhidi [65].

**Theorem 5.1.** Let \((M, J, \omega)\) be a compact Hermitian manifold of complex dimension \(n \geq 2\) with smooth boundary. Let \(\psi \in C^2(M) \cap C^{1,1}(\bar{M})\). Suppose, in addition to (1.2)-(1.4), that there is an admissible subsolution \(u \in C^2(\bar{M})\) obeying (1.6). Then for any admissible solution \(u \in C^4(M) \cap C^2(\bar{M})\) of Dirichlet problem (1.1), there exists a uniform positive constant \(C\) depending only on \(|u|_{C^0(M)}, |\psi|_{C^2(M)}, |u|_{C^2(M)}\), \(|\chi|_{C^2(M)}\) and other known data such that

\[(5.1) \quad \sup_M |\partial \bar{\partial} u| \leq C(1 + \sup_M |\nabla u|^2 + \sup_{\partial M} |\partial \bar{\partial} u|).\]

**Remark 5.2.** Following the outline of proof of Proposition 13 in [65], we can use Lemma 2.2 in place of Lemma 2.3 to check that Székelyhidi’s second order estimate still holds for the Dirichlet problem without assuming (1.5). One can further verify that the constant \(C\) in (5.1) does not depend on \((\delta_{\psi,f})^{-1}\).

The existence results follow from the standard continuity method and the above estimates. We assume \(\psi, u \in C^\infty(\bar{M})\). The general case of \(u \in C^2(\bar{M})\) and \(\psi \in C^{k,\alpha}(\bar{M})\) follows by approximation process. Let’s consider a family of Dirichlet problems:

\[(5.2) \quad F(\mathfrak{g}[u^t]) = (1 - t)F(\mathfrak{g}[u]) + t\psi \text{ in } M, \quad u^t = \varphi \text{ on } \partial M.\]

We set

\[I = \{t \in [0,1] : \text{there exists } u^t \in C^{4,\alpha}(\bar{M}) \text{ solving equation (5.2)}\}.\]

Clearly \(0 \in I\) by taking \(u^0 = u\). The openness of \(I\) is follows from the implicit function theorem and the estimates.
We can verify that $u$ is the admissible subsolution along the whole method of continuity. Combining Theorem 1.4 with Theorem 5.1, we can derive that there exists a uniform positive constant $C$ depending not on $|\nabla u|^2_{C^0(\overline{M})}$ such that

$$\sup_M |\Delta u^t| \leq C(1 + \sup_M |\nabla u|^2_t).$$

One thus applies the blow-up argument used in [65] (also previously appeared in [19, 25]) to derive the gradient estimate, and so $|\partial u|^t$ has a uniform bound. Applying Evans-Krylov theorem [28, 50, 51], adapted to the complex setting (cf. [72]), and Schauder theory, one obtains the required higher $C^{k,\alpha}$ regularities.

We now complete the proof of Theorem 1.1. Moreover, Theorem 1.3 can be derived by combining Theorem 1.1 with the method of approximation.

6. The Dirichlet problem on compact Riemannian manifolds with concave boundary

Let $(M, g)$ be a compact Riemannian manifold with smooth boundary $\partial M$ whose second fundamental form $\Pi$ is non-positive. We call such $\partial M$ a concave boundary. Throughout this section we use the Levi-Civita connection $\nabla$ of $(M, g)$. Under Levi-Civita connection, we denote the covariant derivatives as follows

$$\nabla_i = \nabla_{e_i}, \quad \nabla_{ij} = \nabla_i \nabla_j - \Gamma^k_{ij} \nabla_k,$$

where $\Gamma^k_{ij}$ are the Christoffel symbols.

In this section we briefly discuss the Dirichlet problem analogous to (1.1) on such a Riemannian manifold,

$$(6.1) \quad f(\lambda(g[u])) = \psi \text{ in } M$$

with $u = \varphi$ on $\partial M$, where we denote $g[u] = \chi + \nabla^2 u$, $\chi$ is a smooth and symmetric $(0,2)$-tensor on $M$, and $\lambda(g) = (\lambda_1, \cdots, \lambda_n)$ are the eigenvalues of $g$ with respect to the Riemannian metric $g$.

As in the complex setting, we only need to derive the quantitative boundary estimates. The following theorem states that the quantitative boundary estimates hold for Dirichlet problem (6.1) on the Riemannian manifold with concave boundary.

**Theorem 6.1.** With the above notations. Let $\psi \in C^1(M) \cap C^{0,1}(\overline{M})$ and $\varphi \in C^3(\partial M)$. Suppose, in addition to (1.2)-(1.5), that Dirichlet problem (6.1) admits an admissible subsolution $u \in C^2(\overline{M})$. Then for any admissible solution $u \in C^3(M) \cap C^2(\overline{M})$ of the Dirichlet problem, we have

$$(6.2) \quad \sup_{\partial M} |\nabla^2 u| \leq C(1 + \sup_M |\nabla u|^2),$$

where $C$ is a uniform constant depending only on $|u|_{C^0(\overline{M})}$, $\sup_{\partial M} |\nabla u|$, $|\varphi|_{C^{2,1}(\overline{M})}$, $|\nabla^2 u|_{C^0(\overline{M})}$, $|\chi|_{C^2(\overline{M})}$, $|\psi|_{C^{0,1}(\overline{M})}$ and other known data (but neither on $\sup_M |\nabla u|$ nor on $(\delta_{\psi,f})^{-1}$).
Proof. Here we sketch the proof. Let \((e_1, \ldots, e_n)\) be smooth orthonormal local frames around \(x \in \partial M\), \(e_n\) the interior normal to \(\partial M\) along the boundary when restricted to \(\partial M\). It is similar to the proof of Proposition 4.1, one may derive
\[
\sup_{\partial M} |\nabla_{\alpha n} u| \leq C(1 + \sup_M |\nabla u|), \quad 1 \leq \alpha < n.
\]
It is analogous to the proof of Proposition 4.1, we only need to verify that
\[
(\mathfrak{g} - \mathfrak{g})|_{\partial M}(e_{\alpha}, e_{\beta}) \geq 0, \quad \forall 1 \leq \alpha, \beta \leq n - 1.
\]
Next we check (6.4). Since \(u - u = 0\) on \(\partial M\), we have
\[
\nabla_{\alpha \beta}(u - u) = -\nabla_n(u - u)\Pi(e_{\alpha}, e_{\beta}) \quad \text{on} \ \partial M, \quad \forall 1 \leq \alpha, \beta \leq n - 1.
\]
From (4.2) we know
\[
\nabla_n(u - u) \geq 0 \quad \text{on} \ \partial M.
\]
Thus (6.4) holds when \(\partial M\) is concave. The proof is complete.

\(\square\)

**Theorem 6.2.** Let \((M, g)\) be a compact Riemannian manifold with smooth concave boundary. Suppose, in addition to (1.2)-(1.5), that there exists an admissible subsolution \(u \in C^3(\bar{M})\) of Dirichlet problem (6.1) with \(\psi \in C^\infty(\bar{M})\) and \(\varphi \in C^\infty(\partial M)\). Then there exists a unique smooth admissible function \(u \in C^\infty(\bar{M})\) to solve the Dirichlet problem.

**Theorem 6.3.** Let \((M, g)\) be a compact Riemannian manifold with smooth concave boundary. Suppose, in addition to (1.2), (1.3) and (1.5), that \(f \in C^\infty(\Gamma) \cap C^0(\Gamma)\). Assume \(\varphi \in C^{2,1}(\partial M)\), and the function \(\psi \in C^{1,1}(\bar{M})\) satisfies \(\delta_{\psi,f} = 0\). If Dirichlet problem (6.1) admits a strictly admissible subsolution \(u \in C^{2,1}(M)\), then it admits a weak solution \(u \in C^{1,1}(M)\) with \(\lambda(g[u]) \in \Gamma\) with \(\Delta u \in L^\infty(M)\).

**Appendix A. A characterization of level sets of \(f\)**

In this appendix we present a characterization of level sets of \(f\) which satisfies (1.2), (1.3) and (1.5). We prove

**Proposition A.1.** Suppose (1.2) and (1.3) hold, and \(\sigma \in (\sup_{\partial \Gamma} f, \sup_\Gamma f)\). Denote by \(l_\lambda = \{t\lambda : t > 0\}\). Then the following statements are equivalent.

1. Condition (1.5) holds.
2. For any \(\lambda \in \Gamma\), the ray \(l_\lambda\) intersects every \(\partial \Gamma^\sigma\).
3. For any \(\lambda \in \Gamma\), the ray \(l_\lambda\) intersects every \(\partial \Gamma^\sigma\) at a unique one point.

**Proposition A.2.** Suppose that \(f\) satisfies (1.2)-(1.3). Let \(\sigma \in (\sup_{\partial \Gamma} f, \sup_\Gamma f)\). If there is an \(\lambda'' \in \partial \Gamma^\sigma\) such that \(\lambda'' \cdot \nu_{\lambda''} < 0\), then we have \(\lambda' \cdot \nu_{\lambda'} = 0\) and the origin \(0 \in T_{\lambda'}\partial \Gamma^\sigma\).
Proof. Let \( g : \partial \Gamma^\sigma \to \mathbb{R} \) be a function defined by \( g(\lambda) = \sum_{i=1}^{n} f_i(\lambda) \lambda_i \). Let \( c_\sigma \) be the positive constant which satisfies \( f(c_\sigma \tilde{\lambda}) = \sigma \). Clearly, \( g \) is continuous, and
\[
g(c_\sigma \tilde{\lambda}) = c_\sigma \sum_{i=1}^{n} f_i(c_\sigma \tilde{\lambda}) > 0.
\]
Note that \( \lambda'' \cdot \nu_{\lambda''} < 0 \). So we have \( \lambda' \in \partial \Gamma^\sigma \) such that \( g(\lambda') = 0 \), i.e., \( \lambda' \cdot \nu_{\lambda'} = 0 \).

\[\Box\]

**Corollary A.3.** Given \( \sigma \in (\sup_{\partial \Gamma} f, \sup_{\Gamma} f) \), and we assume \( f \) satisfies (1.2) and (1.3). If there is an \( \lambda \in \partial \Gamma^\sigma \) such that \( \sum_{i=1}^{n} f_i(\lambda) \lambda_i \leq 0 \), then there exists an \( \mu \in \partial \Gamma^\sigma \) such that \( f(t\mu) < \sigma \) for any \( t > 0 \).

Geometrically, the ray \( \{t\mu : t > 0\} \) does not intersect \( \partial \Gamma^\sigma \).

Proof. Given \( \sigma \in (\sup_{\partial \Gamma} f, \sup_{\Gamma} f) \). The conditions (1.2) and (1.3) ensure that \( \partial \Gamma^\sigma \) is a smooth convex hypersurface. Let \( \lambda' \in \partial \Gamma^\sigma \) be the point which satisfies \( \lambda' \cdot \nu_{\lambda'} = 0 \). Then \( 0 \in T_{\lambda'} \partial \Gamma^\sigma \). For any \( \hat{\lambda} \in \Gamma^\sigma \) (i.e., \( f(\hat{\lambda}) \geq \sigma \)), one obtains \( (\hat{\lambda} - \lambda') \cdot \nu_{\lambda'} \geq 0 \), as the level set \( \partial \Gamma^\sigma \) is a smooth convex hypersurface. Let \( \epsilon > 0 \) be a sufficiently small constant such that \( \mu = \lambda' - \epsilon \nu_{\lambda'} \in \Gamma \). Then
\[
(t\mu - \lambda') \cdot \nu_{\lambda'} < 0 \text{ for any } t > 0.
\]
Hence, \( f(t\mu) < \sigma \) for any \( t > 0 \). \(\Box\)

The above corollary also yields the following:

**Lemma A.4.** Suppose (1.2), (1.3) and (1.5) holds. Then \( \sum_{i=1}^{n} f_i(\lambda) \lambda_i > 0 \) in \( \Gamma \).

**Remark A.5.** See also the inequality (8)' of Caffarelli-Nirenberg-Spruck [12].

**Proof of Proposition A.7.** Corollary A.3 and Lemma A.4 imply that if one of the above three conditions (1), (2), (3) holds then
\[
(A.1) \quad \sum_{i=1}^{n} f_i(\lambda) \lambda_i > 0.
\]
Hence, \( f(t\lambda) \) is strictly monotone increasing in \( t \in \mathbb{R}^+ \), and so (2) \(\iff\) (3).

(1) \(\Rightarrow\) (2): If there exist \( \sigma \in (\sup_{\partial \Gamma} f, \sup_{\Gamma} f) \) and \( \lambda \in \Gamma \), such that the ray \( \{t\lambda : t > 0\} \) does not intersect \( \partial \Gamma^\sigma \), then one easily derive \( f(t\lambda) < \sigma \) for any \( t > 0 \). It is a contradiction.

(2) \(\Rightarrow\) (1): For any \( \lambda \in \Gamma \) and \( \sigma \in (\sup_{\partial \Gamma} f, \sup_{\Gamma} f) \), we know that there is \( t_0 > 0 \) such that \( f(t_0\lambda) = \sigma \). So condition (1.5) holds as one has (A.1). \(\Box\)
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