Abstract: In this paper we consider the massless translation invariant Nelson model with ultraviolet cutoff. It is proven that the fiber operators have no ground state if there is no infrared cutoff.

1. Introduction

In this paper we study the translation invariant massless Nelson model. The model can (after a unitary transformation) be written as a direct integral of fiber operators \( \{H(\xi)\}_{\xi \in \mathbb{R}^3} \). The spectral properties of these operators were first investigated by J. Fr"olich in his PhD-thesis, which was published in the two papers [5] and [6]. Fr"olich showed, that if the field is massive or there is an infrared cut-off then \( H(\xi) \) has a ground state for \( \xi \) in an open ball around 0. He also proved, that if the field is massless, no infrared conditions are imposed and a ground state exists for sufficiently many of the \( H(\xi) \), then one can reach some physically unacceptable conclusions. The aim of this paper is to prove that \( H(\xi) \) does not have a ground state if the field is massless and no infrared conditions are assumed. We shall briefly review central results about existence of ground states in the massless Nelson model.

In the paper [10], it is proven that ground states exist in a non-equivalent Fock representation. A consequence of this result is that the usual "taking the massgap to 0" strategy for proving existence of ground states does not work. This strongly indicates that there should be no ground state.

A proof of absence of ground states in a similar model was given by I. Herbst and D. Hasler in the paper [8]. They consider the fiber operators of the massless and translation invariant Pauli-Fierz model \( \{\mathcal{H}(\xi)\}_{\xi \in \mathbb{R}^3} \). They prove that \( \mathcal{H}(\xi_0) \) has no ground state if \( \xi \mapsto \inf(\sigma(\mathcal{H}(\xi))) \) is differentiable at \( \xi_0 \) and has a non-zero derivative. One may easily work out the same problem for the Nelson model and obtain the same conclusions. However proving the existence of a non-zero
derivative is an extremely hard problem and such a result has only been achieved for weak coupling and small $\xi$ (see [1]). Furthermore, $\xi = 0$ is a global minimum for $\xi \mapsto \inf(\sigma(H(\xi)))$ and therefore the derivative is 0. However, $H(0)$ has no ground states shall prove below.

In fact we shall prove that $H(\xi)$ has no ground state for any non-zero coupling strength and $\xi \in \mathbb{R}^3$. Our proof is based on strategy used by I. Herbst and D. Hasler, but we remove the assumption regarding the existence of a non-zero derivative. Instead we use rotation invariance of the map $\xi \mapsto \inf(\sigma(H(\xi))),$ non degeneracy of ground states and the HVZ-theorem.

2. Notation and preliminaries

We start by fixing the measure theoretic notation. Let $(\mathcal{M}, \mathcal{F}, \mu)$ be a $\sigma$-finite measure space and $X$ be a separable Hilbert space. We will write $L^p(\mathcal{M}, \mathcal{F}, \mu, X)$ for the Hilbert space valued $L^p$-space. If $X = \mathbb{C}$ it will be omitted from the notation. In case $\mathcal{M}$ is a topological space we will write $B(\mathcal{M})$ for the Borel $\sigma$-algebra.

Let $\mathcal{H}$ denote a Hilbert space and $n \geq 1$. We write $\mathcal{H} \otimes n$ for the $n$-fold tensor product. Write $S_n$ for the set of permutations of $\{1, \ldots, n\}$ and let $\mathcal{H}$ be a Hilbert space. The symmetric projection is the unique bounded extension of the map $S_n(f_1 \otimes \cdots \otimes f_n) = \frac{1}{n!} \sum_{\sigma \in S_n} f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)}$

and $S_0$ is the identity on $\mathcal{H} \otimes 0 = \mathbb{C}$. In certain cases we can realise tensor produces as concrete spaces:

$$L^2(\mathcal{M}, \mathcal{F}, \mu, X) = L^2(\mathcal{M}, \mathcal{F}, \mu) \otimes X$$

$$(L^2(\mathcal{M}, \mathcal{F}, \mu))^\otimes n = L^2(\mathcal{M} \otimes n, \mathcal{F} \otimes n, \mu \otimes n).$$

with the tensor products $f \otimes x = k \mapsto f(k)x$ and $f_1 \otimes \cdots \otimes f_n = (k_1, \ldots, k_n) \mapsto f_1(k_1)\ldots f_n(k_n)$. In the case $\mathcal{H} = L^2(\mathcal{M}, \mathcal{F}, \mu)$ we have for $n \geq 1$

$$(S_n f)(k_1, \ldots, k_n) = \frac{1}{n!} \sum_{\sigma \in S_n} f(k_{\sigma(1)}, \ldots, k_{\sigma(n)}).$$

We note that $f \in S_n(L^2(\mathcal{M}, \mathcal{F}, \mu)^\otimes n)$ if and only if $f \in L^2(\mathcal{M}^\otimes n, \mathcal{F}^\otimes n, \mu^\otimes n)$ and $f(k_1, \ldots, k_n) = f(k_{\sigma(1)}, \ldots, k_{\sigma(n)})$ for any $\sigma \in S_n$. Write $\mathcal{H} \otimes n = S_n(\mathcal{H}^\otimes n)$.

The bosonic Fock space is defined by

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^\otimes n.$$

where $S_0 = 1$. We will write an element $\psi \in \mathcal{F}(\mathcal{H})$ in terms of its coordinates as $\psi = (\psi^{(n)})$ and define the vacuum $\Omega = (1, 0, 0, \ldots)$. Furthermore, for $\mathcal{D} \subset \mathcal{H}$
and $f_1, \ldots, f_n \in \mathcal{H}$ we introduce the notation

$$S_n(f_1 \otimes \cdots \otimes f_n) = f_1 \otimes_s \cdots \otimes_s f_n$$

$$\epsilon(f_i) = \sum_{n=0}^{\infty} \frac{f \otimes_n}{\sqrt{n!}}$$

$\mathcal{J}(D) = \{ \Omega \} \cup \{ f_1 \otimes_s \cdots \otimes_s f_n \mid f_i \in D, n \in \mathbb{N} \}$

$\mathcal{L}(D) = \{ \epsilon(f) \mid f \in D \}$

where $f \otimes_0 = \Omega$. One may prove that if $D \subset \mathcal{H}$ is dense then $\mathcal{L}(D)$ is a linearly independent total subset of $\mathcal{J}(\mathcal{H})$. From this one easily concludes $\mathcal{J}(\mathcal{D})$ is total.

For $g \in \mathcal{H}$ one defines the annihilation operator $a(g)$ and creation operator $a^\dagger(g)$ on symmetric tensors in $\mathcal{F}(\mathcal{H})$ using $a(g)\Omega = 0, a^\dagger(g)\Omega = g$ and

$$a(g)(f_1 \otimes_s \cdots \otimes_s f_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (g, f_i) f_1 \otimes_s \cdots \otimes_s \hat{f}_i \otimes_s \cdots \otimes_s f_n$$

$$a^\dagger(g)(f_1 \otimes_s \cdots \otimes_s f_n) = \sqrt{n + 1} g \otimes_s f_1 \otimes_s \cdots \otimes_s f_n$$

where $\hat{f}_i$ means that this element is omitted. One can show that these operators extends to closed operators on $\mathcal{F}(\mathcal{H})$ and that $(a(g))^* = a^\dagger(g)$. Furthermore, we have the canonical commutation relations which states

$$[a(f), a(g)] = 0 = [a^\dagger(f), a^\dagger(g)]$$

and $[a(f), a(g)] = (f, g)$.

One now introduces the selfadjoint field operators

$$\varphi(g) = a(g) + a^\dagger(g).$$

If $\omega$ is a selfadjoint operator on $\mathcal{H}$ with domain $\mathcal{D}(\omega)$ then we define the second quantisation of $\omega$ to be the selfadjoint operator

$$d\Gamma(\omega) = 0 \oplus \bigoplus_{n=1}^{\infty} \sum_{k=1}^{n} (1 \otimes)^{k-1} \omega(1 \otimes)^{n-k} |_{\mathcal{H}^\otimes_n}. \quad (2.1)$$

If $\omega$ is a multiplication operator on $\mathcal{H} = L^2(\mathcal{M}, \mathcal{F}, \mu)$ we define $\omega_n : \mathcal{M} \to \mathbb{R}$ by

$$\omega_0 = 0 \text{ and } \omega_n(k_1, \ldots, k_n) = \omega(k_1) + \cdots + \omega(k_n).$$

Then $d\Gamma(\omega)$ acts on elements in $\mathcal{H}^\otimes_n$ as multiplication by $\omega_n(k_1, \ldots, k_n) = \omega(k_1) + \cdots + \omega(k_n)$. The number operator is defined as $N = d\Gamma(1)$. Let $U$ be a unitary map from $\mathcal{H}$ to $\mathcal{K}$. Then we define the unitary map

$$\Gamma(U) = 1 \oplus \bigoplus_{n=1}^{\infty} U \otimes \cdots \otimes U |_{\mathcal{H}^\otimes_n}.$$
Lemma 2.1. Let $\omega \geq 0$ be selfadjoint and injective. If $g \in \mathcal{D}(\omega^{-1/2})$ then $\varphi(g)$, $a^!(g)$ and $a(g)$ are $d\Gamma(\omega)^{1/2}$ bounded. In particular $\varphi(g)$ is $N^{1/2}$ bounded. We have the following obvious lemma which is useful for calculations

$$
\|\varphi(g)\| \leq 2\|\omega^{-1/2} + 1\|\|d\Gamma(\omega) + 1\|^{1/2}\|\psi\|
$$

which holds on $\mathcal{D}(d\Gamma(\omega)^{1/2})$. In particular, $\varphi(g)$ is infinitesimally $d\Gamma(\omega)$ bounded. Furthermore, $\sigma(d\Gamma(\omega) + \varphi(g)) = -\|\omega^{-1/2}g\|^2 + \sigma(d\Gamma(\omega))$.

We have the following obvious lemma which is useful for calculations

Lemma 2.2. Let $f, g \in \mathcal{H}$. Then $\epsilon(g) \in \mathcal{D}(N^n)$ for all $n \geq 0$. Furthermore:

1. $a(g)\epsilon(f) = \langle g, f \rangle \epsilon(f)$ and $(\epsilon(g), \epsilon(f)) = \epsilon(g, f)$.
2. If $f \in \mathcal{D}(\omega)$ then $\epsilon(f) \in \mathcal{D}(d\Gamma(\omega))$ and $d\Gamma(\omega)\epsilon(f) = a^!(\omega f)\epsilon(f)$. In particular we find $\epsilon(g) = \epsilon(g, \omega f)\epsilon(g, f)$.

Let $A \in \mathcal{B}(\mathbb{R}^n)$. In this paper we shall mainly encounter spaces of the form

$$
\mathcal{H}_A = (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), 1_A \lambda_n)
$$

where $\lambda_n$ is the Lebesgue measure. Note $\mathcal{H}_A^{\otimes n} = L^2((\mathbb{R}^n)^n, \mathcal{B}(\mathbb{R}^n)\otimes_n, 1_A \lambda_n^{\otimes n})$. We also define

$$
\mathcal{C}S_A = \{ f \in \mathcal{H}_A \mid \exists R > 0 \text{ such that } 1_{B_R(0)}f = 1_A \lambda_n \text{ almost everywhere}\}.
$$

which is obviously a dense subspace inside $\mathcal{H}_A$. We will also need the contraction $P_A : \mathcal{H}_{\mathbb{R}^n} \to \mathcal{H}_A$ defined by

$$
P_A(v) = v
$$

$1_A \lambda_n$ almost everywhere. Let $\omega : \mathbb{R}^n \to \mathbb{R}$ be a measurable map. Then $\omega_A$ is defined to be multiplication by $\omega$ on the space $\mathcal{H}_A$. Define furthermore $d\Gamma(k_A) = (d\Gamma((k_1)_A), \ldots, d\Gamma((k_n)_A))$ where $k_i : \mathbb{R}^n \to \mathbb{R}$ is projection to the $i$th coordinate and let $g^{(n)} : (\mathbb{R}^n)^n \to \mathbb{R}^n$ be given by $g^{(0)}(0) = 0$ and $g^{(n)}(k) = k_1 + \cdots + k_n$ for $n \geq 1$. Then for $K : \mathbb{R}^n \to \mathbb{R}$ we have

$$
K(\xi - d\Gamma(k_A)) = \bigoplus_{n=0}^{\infty} K_A(\xi - g^{(n)})
$$

where $K_A(\xi - g^{(n)})$ is to be interpreted as the corresponding multiplication operator on $\mathcal{H}_A^{\otimes n}$. In case $A = \mathbb{R}^n$ we will omit $A$ from the notation.

We shall also encounter vectors of operators. Let $B_1, \ldots, B_n$ be operators on a Hilbert space $\mathcal{H}$ and define $B = (B_1, \ldots, B_n)$ from $\cap_{i=1}^{n} \mathcal{D}(B_i)$ into $\mathcal{H}^n$ by $B\psi = (B_1\psi, \ldots, B_n\psi)$. Note $\mathcal{H}^n = \bigoplus_{k=1}^{n} \mathcal{H}$ and is also a Hilbert space. For any $k \in \mathbb{R}^n$ we define

$$
k \cdot B = \sum_{i=1}^{n} k_i B_i.
$$

In particular we find for $\psi \in \mathcal{D}(B)$

$$
\|k \cdot B\psi\|^2 = \sum_{i,j=1}^{n} \langle k_i B_i \psi, k_j B_j \psi \rangle \leq \sum_{i,j=1}^{n} |k_i||k_j||B_i\psi|||B_j\psi||
$$

$$
\leq \sum_{i,j=1}^{n} \frac{1}{2}|k_i|^2||B_i\psi||^2 + \frac{1}{2}|k_j|^2||B_j\psi||^2 = \|k\|^2\|B\psi\|^2
$$

(2.2)
3. The operator - basic properties and the main result

Fix $K, \omega : \mathbb{R}^\nu \rightarrow [0, \infty)$ measurable and let $v \in \mathcal{H}$. Define for $A \in \mathcal{B}(\mathbb{R}^\nu)$ and $\xi \in \mathbb{R}^\nu$ the Hamiltonian

$$H_\mu(\xi, A) = K(\xi - d\Gamma(k_A)) + d\Gamma(\omega_A) + \mu \varphi(v_A)$$

where $v_A = P_A(v)$. We have

**Lemma 3.1.** Assume $\omega > 0 \lambda_\nu$ almost everywhere, $v \in \mathcal{D}(\omega^{-1/2})$ and $A \in \mathcal{B}(\mathbb{R}^\nu)$. Then $\omega_A \geq 0$ is injective and $v_A \in \mathcal{D}(\omega_A^{-1/2})$. Furthermore, $H_\mu(\xi, A)$ is selfadjoint on $\mathcal{D}(H_0(\xi, A)) = \mathcal{D}(d\Gamma(\omega)) \cap \mathcal{D}(d\Gamma(K(\xi - d\Gamma(k_A))))$ and essentially selfadjoint on any core for $H_0(\xi, A)$. Also, $H_\mu(\xi, A) \geq -\mu^2\|\omega^{-1/2}v\| \text{ independent of } A$ and $\xi$.

**Proof.** We know $\{\omega \leq 0\}$ is a $\lambda_\nu$ 0 set and therefore a $1_A\lambda_\nu$ 0 set. Hence $\omega_A \geq 0$ is injective. That $v_A \in \mathcal{D}(\omega_A^{-1/2})$ is obvious as $\omega^{-1/2}v$ is square integrable over $\mathbb{R}^\nu$. For each $n \in \mathbb{N}_0$ we define a map $G_\xi^{(n)} = K(\xi - g^{(n)}) + \omega_n$ and define the selfadjoint operator $B_\xi = \bigoplus_{n=0}^\infty G_\xi^{(n)}$ on $\mathcal{F}(\mathcal{H}_\xi)$. Using $\max\{K(\xi - g^{(n)}), \omega\} \leq G_\xi^{(n)} = K(\xi - g^{(n)}) + \omega_n$ we note

$$\mathcal{D}(B_\xi) = \mathcal{D}(K_A(\xi - d\Gamma(k))) \cap \mathcal{D}(d\Gamma(\omega)) \text{ and } H_0(\xi, A) = B_\xi.$$

In particular, $H_0(\xi, A)$ is selfadjoint. For $\psi \in \mathcal{D}(H_0(\xi, A))$ we have $\|d\Gamma(\omega)\psi\| \leq \|H_0(\xi, A)\psi\|$ and so we find via Lemma [2.1] and the Kato Rellich theorem that

$$H_\mu(\xi, A) := H_0(\xi, A) + \mu \varphi(v_A)$$

is selfadjoint on $\mathcal{D}(H_0(\xi, A))$ and any core for $H_0(\xi, A)$ is a core for $H_\mu$. Using Lemma [2.1] again we find $H_\mu(\xi, A) \geq 0 - \mu^2\|\omega^{-1/2}v\|^2 \geq -\mu^2\|\omega^{-1/2}v\|^2$. \hfill $\square$

**Hypothesis 1:**
1. $K \in C^2(\mathbb{R}^\nu, \mathbb{R})$ is non negative and there is $C_K > 0$ such that $\|\nabla K\|^2 \leq C_K(1 + K)$ and $\|D^2 K\|^2 \leq C_K$ where $D^2 K$ is the Hessian of $K$.
2. $\omega : \mathbb{R}^\nu \rightarrow [0, \infty)$ is continuous and $\omega > 0 \lambda_\nu$ almost everywhere.
3. $v \in \mathcal{D}(\omega^{-1/2})$.

Under these hypothesis we define maps

$$\nabla K(\xi - d\Gamma(k_A)) = (\partial_1 K(\xi - d\Gamma(k_A)), \ldots, \partial_\nu K(\xi - d\Gamma(k_A)))$$

$$\Sigma_A(\xi) = \inf(\sigma(H_\mu(\xi, A)))$$

We have the following lemma

**Lemma 3.2.** Assume Hypothesis 1. The following holds

1. $\mathcal{D}(K(\xi - d\Gamma(k_A))) \subset \mathcal{D}(\nabla K(\xi - d\Gamma(k_A)))$ and for $\psi \in \mathcal{D}(K(\xi - d\Gamma(k_A)))$ we have $\|\nabla K(\xi - d\Gamma(k_A))\psi\|^2 \leq C_K\|K(\xi - d\Gamma(k_A))\psi\|^2 + C_K\|\psi\|^2$
2. $\mathcal{D}(K(\xi - d\Gamma(k_A)))$ is independent of $\xi$. On $\mathcal{D}(K(\xi - d\Gamma(k_A)))$ we have

$$K(\xi + a - d\Gamma(k_A)) = K(\xi - d\Gamma(k_A)) + a \cdot \nabla K(\xi - d\Gamma(k_A)) + E_{\xi,A}(a)$$

where $\|E_{\xi,A}(a)\| \leq C_K\|a\|^2$. In particular, $\mathcal{D}(H_\mu(\xi, A))$ is independent of $\xi$. 

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Let $\psi \in D(K(\xi - d\Gamma(k_A)))$. Then
\begin{align}
\|K(\xi + a - d\Gamma(k_A)) - K(\xi - d\Gamma(k_A))\|_2^2 \\
\leq C_K^2\|a\|^2\|K(\xi - d\Gamma(k_A))\|_2^2 + (1 + \|a\|^2)C_K\|a\|_2^2\|\psi\|_2^2.
\end{align}
(3.2)

Furthermore, $\xi \mapsto H_\mu(\xi)\psi$ is continuous for any $\psi \in cD(H_\mu(0, A))$ and $\xi \mapsto H_\mu(\xi, A)$ is continuous in norm resolvent sense. In particular, the map $\xi \mapsto \Sigma_\mu(\xi)$ is continuous.

(4) Let $D \subset C\mathcal{S}_A$ be a dense subspace. Then $\mathcal{L}(D)$ and $\mathcal{J}(D)$ span cores for $H_\mu(\xi, A)$.

Proof. To prove (1) we calculate for $\psi \in D(K(\xi - d\Gamma(k_A)))$
\begin{align}
\sum_{i=1}^n \sum_{n=0}^\infty \int_{A^n} |\psi^{(n)}(k)\partial_t K(\xi - g^{(n)}(k))|^2 d\lambda^n_k
\leq \sum_{n=0}^\infty \int_{A^n} C_k|\psi^{(n)}(k)K(\xi - g^{(n)}(k))|^2 d\lambda^n_k + C_k\|\psi^{(n)}\|
= C_K\|K(\xi - d\Gamma(k))\|_2^2 + C_K
\end{align}
This proves (1). To prove (2) we use the fundamental theorem of calculus twice and arrive at
\begin{align}
K(\xi + a - k) = K(\xi - k) + a \cdot \nabla K(\xi - k) + a \cdot \int_0^1 \int_0^1 D^2 K(\xi + sta - k)adsdt
\end{align}
Define $G_{\xi, A}(k) = a \cdot \int_0^1 \int_0^1 D^2 K(k + sta)adsdt$, and note $|G_{\xi, A}(k)| \leq C_K\|a\|^2$ uniformly in $k$ and $\xi$. Thus if we define $E_{\xi, A}(a) = G_{\xi, A}(\xi - d\Gamma(k_A))$ we find that $E_{\xi, A}(a)$ is bounded with norm bound $C_K\|a\|^2$. Let $\psi \in D(K(\xi - d\Gamma(k_A)))$. Then $\psi \in D(K(\xi - d\Gamma(k_A))) + \{a, \nabla K(\xi - d\Gamma(k_A)) + E_{\xi, A}(a)\}$ by part 1. We have the point wise identity:
\begin{align}
(K(\xi - d\Gamma(k_A)) + a \cdot \nabla K(\xi - d\Gamma(k_A)) + E_{\xi, A}(a))\psi^{(n)} = K(\xi + a - g^{(n)})\psi^{(n)}
\end{align}
showing $K(\xi + a - g^{(n)})\psi^{(n)}$ is square integrable and the sum of squared norms is finite. Hence $\psi \in D(K(\xi + a - d\Gamma(k_A)))$ and equation (3.1) holds. We have thus proven $D(K(\xi + a - d\Gamma(k_A))) \subset D(K(\xi - d\Gamma(k_A)))$ for all $\xi \in \mathbb{R}^n$ however using $\xi = \xi - a$ we find the other inclusion. This proves (2).

To prove (3) we note that equation (3.2) is easily obtained from statements (1) and (2). Using
\begin{align}
(H_\mu(\xi + a, A) - H_\mu(\xi, A))\psi = (K(\xi + a - d\Gamma(k_A)) - K(\xi - d\Gamma(k_A)))\psi
\end{align}
for any $\psi \in D(H_\mu(\xi, A))$ and equation (3.2) we immediately obtain continuity for $\xi \mapsto H_\mu(\xi, A)\psi$. To prove the statement regarding norm resolvent convergence we calculate using equation (3.2)
\begin{align}
\|(H_\mu(\xi + a, A) + i)^{-1} - (H_\mu(\xi, A) + i)^{-1}\|^2
\leq C_K\|a\|^2\|K(\xi - d\Gamma(k))\|_2^2 + (1 + \|a\|^2)C_K\|a\|^2
\end{align}
which goes to 0 for a tending to 0. Continuity of $\xi \mapsto \inf(\sigma(H_0(\xi, A)))$ now follows from continuity of the spectral calculus and the existence of a $\xi$-independent lower bound by Lemma 3.3.

It only remains to prove statement (4). By Lemma 3.3 it is enough to check that $f(\mathcal{D})$ and $L(\mathcal{D})$ span a core for $H_0(\xi, A)$. Let $f_1, \ldots, f_n \in CS_A$. Pick $R > 0$ such that $1_{B_R(0)} f_i = f_i 1_{A^\lambda}$ almost everywhere for all $i \in \{1, \ldots, n\}$ and note that $1_{B_R(0)} f_1 \otimes_s \cdots \otimes_s f_n = f_1 \otimes_s \cdots \otimes_s f_n 1_{A^\lambda}$ almost everywhere. Let $C = \sup_{k \in B_R(\omega)} \omega(k)$. Using the fundamental theorem of calculus we find the following point wise inequality for $k \in B_R(0)^n$:

$$|K(\xi - g^n(k))| = K(\xi) + \|g(\xi)(k)\| \|\nabla K(\xi)\| + \|g^{(n)}(k)\|^2 C \lesssim (1 + n^2 R^2)$$

Where $\widetilde{C} = \max\{K(\xi) + \frac{1}{2}\|\nabla K(\xi)\|, (1 + C)\}$ and we used that $\|g^{(n)}(k)\| \leq n R$ for $k \in B_R(0)^n$. We therefore find the following point wise estimates on $B_R(0)^n$:

$$(K(\xi - g^n) + \omega_n)^2 |f_1 \otimes_s \cdots \otimes_s f_n| \leq (\widetilde{C}(1 + n^2 R^2) + n C)^2 |f_1 \otimes_s \cdots \otimes_s f_n|$$

Integrating yields $f_1 \otimes_s \cdots \otimes_s f_n \in \mathcal{D}(H_0(\xi, A)^p)$ and

$$\|H_0(\xi, A)^p f_1 \otimes_s \cdots \otimes_s f_n\| \leq (\widetilde{C}(1 + n^2 R^2) + n C)^p \|f_1 \otimes_s \cdots \otimes_s f_n\| \quad (3.3)$$

Multiplying by $\frac{1}{p!}$ and summing over $p$ yields a finite number so $f_1 \otimes_s \cdots \otimes_s f_n$ is analytic for $H_0(\xi)$. Now, $\mathcal{O}$ is an eigenvector for $H_0(\xi)$ and therefore analytic we see $f(\mathcal{D})$ is a total set of analytic vectors for $H_0(\xi, A)$ and therefore it spans a core for $H_0(\xi, A)$ by Nelson analytic vector theorem.

By equation (3.3) we see $f^{\otimes n} \in \mathcal{D}(H_0(\xi, A)^p)$ and

$$\|H_0(\xi, A)^p f^{\otimes n}\| \leq (\widetilde{C}(1 + n^2 R^2) + n C)^2 \|f^{\otimes n}\| \leq (\widetilde{C}^{1/2}(1 + n R) + \sqrt{n C})^{2p} \|f^{\otimes n}\| \leq (\widetilde{C}^{1/2})^{2p} \|f^{\otimes n}\|.$$ 

This also holds for $n = 0$ as we in this case obtain $\|H_0(\xi, A)^0 f^{\otimes n}\| = \sqrt{K(\xi)}^{4p} \leq (\widetilde{C}^{1/2})^{4p}$. Multiplying by $\frac{1}{p!}$ and summing over $n$ yields a finite number so $e(f_1) \in \mathcal{D}(H_0(\xi, A)^p)$ for all $p$. Now

$$\sum_{p=0}^{\infty} \frac{1}{(2p)!} \|H_0(\xi, A)^p e(1_{\mathcal{D}})\| \leq \sum_{p=0}^{\infty} \frac{1}{(2p)!} \sum_{n=0}^{\infty} \frac{1}{n!} \|H_0(\xi, A)^p f^{\otimes n}\|$$

$$\leq \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \sum_{p=0}^{\infty} \frac{1}{(2p)!} (\widetilde{C}^{1/2}(1 + n R) + \sqrt{n C})^{2p} \|f_1^{\otimes n}\|$$

$$\leq \sum_{n=0}^{\infty} \frac{\|f\| e(\sqrt{\widetilde{C}^{1/2}(1 + n R) + \sqrt{n C}})}{\sqrt{n!}} < \infty$$

Thus $e(f_1)$ is semi analytic for $H_0(\xi)$. This implies $\{e(f) \mid f \in \mathcal{D}\}$ spans a dense subspace of semi analytic vectors for $H_0(\xi, A)$, which is a core by the Masson- McClary theorem.

**Hypothesis 2:** We assume
Consider the map $K, \omega$ and $v$ are rotation invariant. Furthermore $k \mapsto e^{-tK(k)}$ is positive definite for all $t$.

(2) $\omega$ is sub-additive, $\omega(x_1) < \omega(x_2)$ of $|x_1| < |x_2|$. Also $C_\omega = \lim_{k \to 0} \|k\|^{-1} \omega(k)$ exists and is strictly positive.

(3) $v \notin D(\omega_{-1})$

The physical choices for the 3-dimensional Nelson model are $\omega(k) = |k|$, $K \in \{k \mapsto |k|^2, k \mapsto \sqrt{|k|^2 + m - m}\}$ and $v = \omega^{-1/2} \chi$ where $\chi : \mathbb{R}^\nu \to \mathbb{R}$ is a spherically symmetric ultra violet cutoff. It is well known that Hypothesis 1 and 2 are fulfilled in this case. We can now state the main theorem of this paper:

**Theorem 3.3.** Assume Hypothesis 1 and 2 along with $\nu \geq 3$. Then $H_\mu(\xi)$ has no ground states for any $\xi$ and $\mu \neq 0$.

4. Proof of Theorem [3,3]  

We start with proving series of lemmas which we shall need. We work under Hypothesis 1 and 2. The first Lemma is known and we only sketch the proof.

**Lemma 4.1.** The map $\xi \mapsto \Sigma(\xi)$ is rotation invariant.

*Proof.* Let $O$ denote any orthogonal matrix with dimensions $\nu$. Define the unitary map $\tilde{O} : \mathcal{H} \to \mathcal{H}$ by $(\tilde{O}f)(k) = f(Ok)\lambda_\nu$ almost everywhere. Let $f,g \in \mathcal{CS}$ and note $\tilde{O}f, \tilde{O}g \in \mathcal{CS}$. In particular, $\Gamma(\tilde{O})f = f(\tilde{O}) \in \mathcal{D}(H_\mu(\xi))$ for all $\xi$. One now easily calculates using Lemma 2.2

$$\langle \epsilon(g), \Gamma(\tilde{O})^*H_\mu(\xi)\Gamma(\tilde{O})\epsilon(f) \rangle = \langle \epsilon(g), H_\mu(O\xi)\epsilon(f) \rangle.$$  

Now $\mathcal{L}(\mathcal{CS})$ is total so we find $H_\mu(O\xi) = \Gamma(\tilde{O})^*H_\mu(\xi)\Gamma(\tilde{O})$ on $\mathcal{L}(\mathcal{CS})$ which is spans a core for $H_\mu(O\xi)$ and so $\Gamma(\tilde{O})^*H_\mu(\xi)\Gamma(\tilde{O}) = H_\mu(O\xi)$. □

For any $x \in \mathbb{R}^\nu \setminus \{0\}$ we write $\tilde{x} = ||x||^{-1}x$. The next small lemma is basically spherical coordinates.

**Lemma 4.2.** Let $U \subset \mathbb{R}^\nu$ be invariant under multiplication by elements in $(0, \infty)$. Then for any positive, rotation invariant, measurable map $f$ we have

$$\int_U f(x)\lambda_\nu(x) = n\lambda_\nu(U \cap B_1(0)) \int_0^\infty f(ke_1)k^{\nu-1}d\lambda_1(k)$$

where $e_1$ is the first standard basis vector. If $U$ is open then $\lambda_\nu(U \cap B_1(0)) \neq 0$.

*Proof.* Consider the map $g : \mathbb{R}^\nu \to [0, \infty)$ given by $g(x) = |x|$. Define the transformed measure on $([0, \infty), \mathcal{B}([0, \infty]))$ by

$$\mu = (1_U \lambda_\nu) \circ g^{-1}$$

The transformation theorem implies

$$\mu([0, a]) = \lambda_\nu(a(U \cap B_1(0))) = \nu \lambda_\nu(\tilde{U} \cap B_1(0)) \int_0^a k^{\nu-1}d\lambda_1(k)$$

The physical choices for the 3-dimensional Nelson model are $\omega(k) = |k|$, $K \in \{k \mapsto |k|^2, k \mapsto \sqrt{|k|^2 + m - m}\}$ and $v = \omega^{-1/2} \chi$ where $\chi : \mathbb{R}^\nu \to \mathbb{R}$ is a spherically symmetric ultra violet cutoff. It is well known that Hypothesis 1 and 2 are fulfilled in this case. We can now state the main theorem of this paper:

**Theorem 3.3.** Assume Hypothesis 1 and 2 along with $\nu \geq 3$. Then $H_\mu(\xi)$ has no ground states for any $\xi$ and $\mu \neq 0$.  

4. Proof of Theorem [3,3]  

We start with proving series of lemmas which we shall need. We work under Hypothesis 1 and 2. The first Lemma is known and we only sketch the proof.

**Lemma 4.1.** The map $\xi \mapsto \Sigma(\xi)$ is rotation invariant.

*Proof.* Let $O$ denote any orthogonal matrix with dimensions $\nu$. Define the unitary map $\tilde{O} : \mathcal{H} \to \mathcal{H}$ by $(\tilde{O}f)(k) = f(Ok)\lambda_\nu$ almost everywhere. Let $f,g \in \mathcal{CS}$ and note $\tilde{O}f, \tilde{O}g \in \mathcal{CS}$. In particular, $\Gamma(\tilde{O})f = f(\tilde{O}) \in \mathcal{D}(H_\mu(\xi))$ for all $\xi$. One now easily calculates using Lemma 2.2

$$\langle \epsilon(g), \Gamma(\tilde{O})^*H_\mu(\xi)\Gamma(\tilde{O})\epsilon(f) \rangle = \langle \epsilon(g), H_\mu(O\xi)\epsilon(f) \rangle.$$  

Now $\mathcal{L}(\mathcal{CS})$ is total so we find $H_\mu(O\xi) = \Gamma(\tilde{O})^*H_\mu(\xi)\Gamma(\tilde{O})$ on $\mathcal{L}(\mathcal{CS})$ which is spans a core for $H_\mu(O\xi)$ and so $\Gamma(\tilde{O})^*H_\mu(\xi)\Gamma(\tilde{O}) = H_\mu(O\xi)$. □

For any $x \in \mathbb{R}^\nu \setminus \{0\}$ we write $\tilde{x} = ||x||^{-1}x$. The next small lemma is basically spherical coordinates.

**Lemma 4.2.** Let $U \subset \mathbb{R}^\nu$ be invariant under multiplication by elements in $(0, \infty)$. Then for any positive, rotation invariant, measurable map $f$ we have

$$\int_U f(x)\lambda_\nu(x) = n\lambda_\nu(U \cap B_1(0)) \int_0^\infty f(ke_1)k^{\nu-1}d\lambda_1(k)$$

where $e_1$ is the first standard basis vector. If $U$ is open then $\lambda_\nu(U \cap B_1(0)) \neq 0$.

*Proof.* Consider the map $g : \mathbb{R}^\nu \to [0, \infty)$ given by $g(x) = |x|$. Define the transformed measure on $([0, \infty), \mathcal{B}([0, \infty]))$ by

$$\mu = (1_U \lambda_\nu) \circ g^{-1}$$

The transformation theorem implies

$$\mu([0, a]) = \lambda_\nu(a(U \cap B_1(0))) = \nu \lambda_\nu(\tilde{U} \cap B_1(0)) \int_0^a k^{\nu-1}d\lambda_1(k)$$
For any $\Sigma$, Lemma 4.3. $H$ implying this result was proven in the paper [7] under the extra assumption that $m > 0$ then $\exists k \in U \cap B_1(0)$ so $U \cap B_1(0) \neq \emptyset$. Hence if $U$ is open and non empty we find $U \cap B_1(0)$ is open and non empty so $\lambda_n(U \cap B_1(0)) \neq 0$. □

**Lemma 4.3.** $\Sigma$ has a global minimum at $\xi = 0$.

**Proof.** This result was proven in the paper [7] under the extra assumption that there is $m > 0$ such that $\omega \geq m$. The proof used in [7] does however generalise to our setting. Another way to derive it to consider $\omega_n = 1/n + \omega$ and let $H_n(\xi) = K(\xi - d\Gamma(k)) + d\Gamma(\omega_n) + \mu \varphi(v)$ Write $\Sigma_n(\xi) = \inf(\sigma(H_n(\xi)))$. Now $\text{Span}(\mathcal{F}(\mathcal{C}) \mathcal{S})$ is a common core for the $H_n(\xi)$ and $H(\xi)$ by Lemma 3.2 and for $\psi$ in this set we see

$$
\lim_{n \to \infty} (H_n(\xi) - H(\xi))\psi = \lim_{n \to \infty} \frac{1}{n} N \psi = 0
$$

implying $H_n(\xi)$ converges to $H(\xi)$ in strong resolvent sense by [11] Theorem VIII.25. For any $\varepsilon > 0$ we may pick $\psi \in \text{Span}(\mathcal{F}(\mathcal{C}) \mathcal{S})$ such that

$$
\Sigma_n(\xi) + \varepsilon \geq \langle \psi, H_n(\xi)\psi \rangle \geq \langle \psi, H(\xi)\psi \rangle \geq \Sigma(\xi)
$$

In particular, $\Sigma_n(\xi) \geq \Sigma(\xi)$ for all $n \in \mathbb{N}$. By [11] Theorem VIII.24 we find a sequence $\{\lambda_n\}_{n=1}^{\infty}$ converging to $\Sigma(\xi)$ with $\lambda_n \in \sigma(H_n(\xi))$.

Hence $0 \leq \Sigma_n(\xi) - \Sigma(\xi) \leq \lambda_n - \Sigma(\xi)$ so $\Sigma_n(\xi)$ converges to $\Sigma(\xi)$. Now $\Sigma_n$ has a global minimum at $\xi = 0$ and so

$$
\Sigma(0) = \lim_{n \to \infty} \Sigma_n(0) \leq \lim_{n \to \infty} \Sigma_n(\xi) = \Sigma(\xi)
$$

finishing the proof. □

For every $\xi \in \mathbb{R}^n$ and $0 < \varepsilon < 1$ we define

$$
S_{\varepsilon}(\xi) = \{k \in \mathbb{R}^n \setminus \{0\} \mid |\hat{k} \cdot \xi| < (1 - \varepsilon)||\xi||\}.
$$

where $\hat{k} = k/||k||$. The following Lemma is essential:

**Lemma 4.4.** Let $\xi \in \mathbb{R}^n$. Then

1. $\Sigma(\xi - k) + \omega(k) > \Sigma(\xi)$ if $k \notin \mathbb{R}\xi$.
2. For any $1 > \varepsilon > 0$ there exists $D := D(\varepsilon,\xi) < 1$ and $r := r(\varepsilon,\xi) > 0$ such that for all $k \in B_r(0) \cap S_{\varepsilon}(\xi)$ we have

$$
\Sigma(\xi - k) - \Sigma(\xi) \geq -D\omega(k)
$$
Proof. We start by proving (1). Assume \( \xi = 0 \) and \( k \neq 0 \). If \( \omega(k) = 0 \) then by Hypothesis 2 we have \( \omega(k') < 0 \) for all \( k' \in B_{|k|}(0) \) which contradicts Hypothesis 1. So if \( \xi = 0 \) the result is trivial since \( \Sigma(\xi - k) - \Sigma(\xi) > -\frac{1}{2}\omega(k) > -\omega(k) \) holds for all \( k \neq 0 \) by Lemma 4.3.

Assume now \( \xi \neq 0 \) and let \( k \notin R\xi \). By rotation invariance of \( \Sigma \) (Lemma 4.4) we may calculate

\[
\Sigma(\xi - k) - \Sigma(\xi) = \Sigma(\xi - k) - \Sigma \left( \frac{\|\xi\|}{\|\xi - k\|}(\xi - k) \right)
\]  

(4.1)

By Lemma 4.5 we have \( \Sigma(\xi - k) + \omega(k) \in \sigma(H(\xi)) \) and so \( \Sigma(\xi) \leq \Sigma(\xi - k) + \omega(k) \) implying

\[
\Sigma(\xi - k) - \Sigma \left( \frac{\|\xi\|}{\|\xi - k\|}(\xi - k) \right) \geq -\omega \left( \frac{\|\xi\|}{\|\xi - k\|}(\xi - k) - \xi - k \right)
\]

\[
= -\omega \left( \frac{\|\xi\|}{\|\xi - k\|}(\xi - k - \xi) \right)
\]  

(4.2)

Now \( \|\xi\| - \|\xi - k\| \leq \|k\| \) by the reverse triangle inequality. If equality holds we have either \( \|\xi\| = \|\xi - k\| + \|k\| \) or \( \|\xi - k\| = \|\xi\| + \|k\| \). By [14, Page 9] either \( k \) and \( \xi - k \) are linearly dependent or \( k \) and \( \xi \) are linearly dependent. In any case \( \xi \) and \( k \) are linearly independent which (as \( \xi \neq 0 \)) implies \( k = \alpha \xi \) for some \( \alpha \in \mathbb{R} \).

So since \( k \notin R\xi \) we find \( ||\xi|| - ||\xi - k|| < ||k|| \) and so

\[
\omega \left( \frac{\|\xi\|}{\|\xi - k\|}(\xi - k) \right) < \omega(k)
\]

by Hypothesis 2. Combining this and equations (4.1) and (4.2) we find statement (1). To prove statement (2) we continue to calculate for \( k \in S_{r}(\xi) \) (which is disjoint from \( R\xi \))

\[
||\xi - k|| - ||\xi|| = \frac{||\xi - k||^2 - ||\xi||^2}{||\xi - k|| + ||\xi||} = |k| \left| \frac{-\xi \cdot \hat{k} + ||k||}{||\xi - k|| + ||\xi||} \right| \leq |k| \left( 1 - \frac{\|k\|}{\|\xi\|} \right)
\]  

(4.3)

Pick \( n \) such that \( D := (1 + 1/n)(1 - 1/n)^{-1}(1 - \varepsilon/2) < 1 \) and \( R > 0 \) such that

\[
C_{\omega}(1 - 1/n)||k|| < \omega(k) \leq C_{\omega}(1 + 1/n)||k||
\]  

(4.4)

for all \( k \in B_{R}(0) \). Pick \( r = \min \{ \frac{\|\xi\|}{D\omega}, R \} \). Using equations (4.1), (4.2), (4.3) and (4.4) we find

\[
\Sigma(\xi - k) - \Sigma(\xi) \geq -C(1 + 1/n) \left( 1 - \varepsilon + \frac{|k|}{\|\xi\|} \right) |k| \geq -D\omega(k)
\]

for \( k \in B_{r}(0) \cap S_{r}(\xi) \).

The following lemma is well known see e.g. [4].

**Lemma 4.5.** Define \( A = \{ \nu \neq 0 \} \). Assume \( H_{\mu}(\xi, A) \) has a ground state for some \( \mu \neq 0 \) and \( \xi \in \mathbb{R}^{v} \). Then the corresponding eigenspace is non degenerate.

We will now sharpen this result.
Lemma 4.6. Assume $H_\mu(\xi)$ has a ground state for some $\mu \neq 0$ and $\xi \in \mathbb{R}^\nu$. Then the corresponding eigenspace is non degenerate if $\nu \geq 2$.

**Proof.** Define $A = \{v \neq 0\}$. By Lemma A.3 there is a unitary map

$$U : \mathcal{F}(\mathcal{H}) \rightarrow \mathcal{F}(\mathcal{H}_A) \oplus \bigoplus_{n=1}^{\infty} \mathcal{F}(\mathcal{H}_A) \otimes \mathcal{H}_{A^\nu}^\otimes,$$

such that

$$U H_\mu(\xi) U^* = H_\mu(\xi, A) \oplus \bigoplus_{n=1}^{\infty} H_{n,\mu}(\xi, A)|_{\mathcal{F}(\mathcal{H}_A) \otimes \mathcal{H}_{A^\nu}^\otimes} \quad (4.5)$$

for all $\xi \in \mathbb{R}^\nu$ where

$$H_{n,\mu}(\xi, A) = \int_{(A^\nu)^n} H_\mu(\xi - k_1 - \cdots - k_n, A) + \omega(k_1) + \cdots + \omega(k_n) d\lambda_\nu^\otimes(k).$$

Let $\psi$ be any ground state for $H_{\mathbb{R}^\nu}(\xi)$. We prove $U \psi = (\tilde{\psi}(0), 0, 0, \ldots)$. Write $U \psi = (\tilde{\psi}(n))$ and assume towards contradiction that $\tilde{\psi}(n) \neq 0$ for some $n \geq 1$. Then $\tilde{\psi}(n)$ is an eigenvector of $H_{n,A}(\xi)$ corresponding to the eigenvalue $\Sigma(\xi)$. The spectral projection of $H_{n,A}(\xi)$ onto $\Sigma(\xi)$ is given by

$$\int_{(A^\nu)^n} 1_{\Sigma(\xi)}(H(\xi - k_1 - \cdots - k_n) + \omega(k_1) + \cdots + \omega(k_n)) d\lambda_\nu^\otimes(k) \neq 0.$$

Hence $\Sigma(\xi)$ is an eigenvalue for $H_\mu(\xi - k_1 - \cdots - k_n, A) + \omega(k_1) + \cdots + \omega(k_n)$ on a set of positive $\lambda_\nu^\otimes$ measure. Sub-additivity of $\omega$ along with Lemmas A.3 and A.5 gives

$$\Sigma(\xi) \geq \Sigma_A(\xi - k_1 - \cdots - k_n) + \omega(k_1) + \cdots + \omega(k_n) \geq \Sigma(\xi - k_1 - \cdots - k_n) + \omega(k_1 + \cdots + k_n) \geq \Sigma(\xi)$$

most hold on a set of positive $\lambda_\nu^\otimes$ measure. By Lemma A.4 we see that this can only hold for $k \in (\mathbb{R}^\nu)^n$ with $k_1 + \cdots + k_n \in \text{Span}(\xi)$. But the rank theorem implies that the set of $k$ satisfying this is a subspace of $(\mathbb{R}^\nu)^n$ of dimension $\nu n - (\nu - 1) < \nu n$. However such a subspace must have $\lambda_\nu$ measure 0 which is a contradiction.

We now finish the proof as follows. Assume $\psi_1, \psi_2$ are orthogonal eigenvectors corresponding to the eigenvalue $\Sigma(\xi)$. Then $U \psi_i = (\tilde{\psi}_i, 0, 0, \ldots)$. Now $U$ preserves the inner product so $\tilde{\psi}_1$ and $\tilde{\psi}_2$ are orthogonal eigenvectors for $H_\mu(\xi, A)$ corresponding to the eigenvalue $\Sigma(\xi)$ so in particular $\Sigma(\xi) \geq \Sigma_A(\xi)$. By equation (4.5) we conclude that $\Sigma(\xi) = \Sigma_A(\xi)$ and therefore $H_\mu(\xi, A)$ has two orthogonal ground states. This is a contradiction with Lemma 4.5. $\square$

The next two Lemmas are an adapted version of the corresponding ones found in [8]. For $\xi \in \mathbb{R}^\nu$ and $k \neq 0$ we define

$$Q_0(k, \xi) = \omega(k)(H(\xi) - \Sigma(\xi) + \omega(k))^{-1} \quad P_0(\xi) = 1_{\text{null}}(H(\xi)).$$
Lemma 4.7. Fix $\xi \in \mathbb{R}^\nu$ and $R > 0$. Then $\hat{K} \cdot \nabla K(\xi - d\Gamma(k))Q_0(k, \xi)$ is uniformly bounded for $k \in B(0,R) \setminus \{0\}$. We also have
\[
\lim_{k \to 0} \hat{K} \cdot \nabla K(\xi - d\Gamma(k))Q_0(k, \xi)(1 - P_0(\xi)) = 0 \quad (4.6)
\]
Proof. Note $\hat{K} \cdot \nabla K(\xi - d\Gamma(k))Q_0(k)$ is bounded for $k \neq 0$ by the closed graph theorem and Lemma 3.2. For $\psi \in \mathcal{F}(\mathcal{H})$ we find by equation (2.2) that
\[
\|\hat{K} \cdot \nabla K(\xi - d\Gamma(k))Q_0(k, \xi)\| \leq \sum_{i=1}^\nu \|\partial_i K(\xi - d\Gamma(k))Q_0(k, \xi)\| \psi
\]
so it is enough to see $\partial_i K(\xi - d\Gamma(k))Q_0(k, \xi)$ is uniformly bounded on $B(0,R) \setminus \{0\}$ for any $R > 0$ and converges strongly to $\partial_i K(\xi - d\Gamma(k))P_0(\xi)$. We have
\[
\partial_i K(\xi - d\Gamma(k))Q_0(k, \xi) = \partial_i K(\xi - d\Gamma(k)) \frac{\omega(k)}{H(\xi) - \Sigma(\xi) + \omega(k) + 1} + \partial_i K(\xi - d\Gamma(k)) \frac{\omega(k)}{H(\xi) - \Sigma(\xi) + \omega(k) + 1}
\]
Now $\omega$ is continuous and goes to 0 as $k$ tends to 0 so $Q_0(k, \xi)$ goes strongly to $P_0(\xi)$. Hence it is enough to see $\partial_i K(\xi - d\Gamma(k))(H(\xi) - \Sigma(\xi) + \omega(k) + 1)^{-1}$ is uniformly bounded in $k$ and converges to $\partial_i K(\xi - d\Gamma(k))(H(\xi) - \Sigma(\xi) + 1)^{-1}$ in norm. But this is obvious from the equality
\[
\partial_i K(\xi - d\Gamma(k)) \frac{1}{H(\xi) - \Sigma(\xi) + \omega(k) + 1} = \partial_i K(\xi - d\Gamma(k)) \frac{1}{H_\mu(\xi) - \Sigma(\xi) + 1} + \partial_i K(\xi - d\Gamma(k)) \frac{\omega(k)}{H_\mu(\xi) - \Sigma(\xi) + 1 + \omega(k)}
\]
because the first term is constant and the other term is uniformly bounded and goes to 0. \hfill \Box

For $\xi \in \mathbb{R}^\nu$ and $k \notin \mathbb{R}^\xi$ we may by Lemma 4.4 define
\[
Q(k, \xi) = \omega(k)(H(\xi - k) - \Sigma(\xi) + \omega(k))^{-1}
\]
Lemma 4.8. Fix $\xi \in \mathbb{R}^\nu$. There is a vector $v(\xi) \in \mathbb{R}^\nu$ such that
\[
P_0(\xi)\hat{K} \cdot \nabla K(\xi - d\Gamma(k))P_0(\xi) = \hat{K} \cdot v(\xi)P_0(\xi)
\]
for any $k \in \mathbb{R}^\nu \setminus \{0\}$. Pick $0 < \varepsilon < 1$ such that $\hat{K} \cdot C_{\omega}v(\xi) < \frac{1}{2}$ for all $k \in S_\varepsilon(C_{\omega}v(\xi))$. Define
\[
\tilde{S}_\varepsilon(\xi) = S_\varepsilon(\xi) \cap S_\varepsilon(C_{\omega}v(\xi)).
\]
If $\nu \geq 3$ then $S_\varepsilon$ is open, non-empty and invariant under positive scalings. Furthermore,
\[
w = \lim_{k \to 0, k \in \tilde{S}_\varepsilon(\xi)} Q(k, \xi) - (1 - C_{\omega}\hat{K} \cdot v(\xi))^{-1}P_0(\xi) = 0. \quad (4.7)
\]
Proof. As \( \xi \) is fixed in this proof it will be omitted from the nation of \( Q, Q_0, B \) and \( P_0 \). If \( P_0 = 0 \) we can pick \( v(\xi) = 0 \). If \( P_0(\xi) = 0 \) then \( S(\xi) \) \( \parallel \) given dimension 1 by Lemma 3.1 and is spanned by a vector \( \psi \in D(H (\xi)) \). Using \( P_0 = |\psi\rangle\langle\psi| \) we find that \( v(\xi) = \langle\psi, \partial K(\xi - d)\psi\rangle \) does the trick. Furthermore, \( S_\epsilon \) is obviously open and invariant under positive scaling since this holds for \( S(\xi) \) and \( S_\epsilon(C_\nu S(\xi)) \). Furthermore any non-zero vector which is orthogonal to \( \xi \) and \( v(\xi) \) is in \( S_\epsilon \) and such vector will always exist if \( \nu \geq 3 \).

It remains only to prove equation 4.7. By Lemma 4.1 we may pick \( R(\xi, \epsilon) > 0 \) such that for \( k \in \tilde{S}_\epsilon(\xi) \cap B_{R(\xi, \epsilon)}(0) \) we have

\[
\|Q(k)\| \leq (1 - D(\xi, \epsilon))^{-1} \forall k \in \tilde{S}_\epsilon(\xi) \cap B_{R(\xi, \epsilon)}(0) \quad (4.8)
\]

Using Lemma 3.2 we may calculate for \( k \in \tilde{S}_\epsilon(\xi) \):

\[
Q(k) = Q_0(k) + \omega(k)^{-1}Q_0(k)(H(\xi) - H(\xi - k))Q(\xi) = \frac{|k|}{\omega(k)}Q_0(k)(\hat{k} \cdot \nabla K(\xi - d)\omega) - |k|^{-1}E_\epsilon k(\xi)Q(\xi) = Q_0(k) + \frac{|k|}{\omega(k)}Q_0(k)(\hat{k} \cdot \nabla K(\xi - d)\omega)Q(\xi) + \alpha_1(k) \quad (4.11)
\]

where \( \alpha_1(k) := -Q_0(k)\omega(k)^{-1}E_\epsilon k(\xi)Q(\xi) \). We also have

\[
Q(k) = Q_0(k) + \omega(k)^{-1}Q(\xi) - H(\xi - k)Q_0(k) = Q_0(k) + \frac{|k|}{\omega(k)}Q_0(k)(\hat{k} \cdot \nabla K(\xi - d)\omega)Q_0(k) + \alpha_2(k) \quad (4.13)
\]

where \( \alpha_2(k) := -Q_0(k)\omega(k)^{-1}E_\epsilon k(\xi)Q(k) \). Note \( \alpha_1(k) \) goes to 0 in norm for \( k \) tending to 0 in \( \tilde{S}_\epsilon(\xi) \) by equation (4.13), Lemma 3.2 and the uniform bound \( \|Q_0(k)\| \leq 1 \). Inserting equation (4.13) into equation (4.11) we find

\[
Q(k) = Q_0(k) + \frac{|k|}{\omega(k)}Q_0(k)(\hat{k} \cdot \nabla K(\xi - d)\omega)Q_0(k) + \frac{|k|^2}{\omega(k)^2}Q_0(k)(\hat{k} \cdot \nabla K(\xi - d)\omega)Q_0(k)(\hat{k} \cdot \nabla K(\xi - d)\omega)Q_0(k) + o(k)
\]

Where

\[
o(k) = Q_0(k)(\hat{k} \cdot \nabla K(\xi - d)\omega)\alpha_2(k) + \alpha_1(k) = -\frac{|k|}{\omega(k)}Q_0(k)(\hat{k} \cdot \nabla K(\xi - d)\omega)Q_0(k)|k|^{-1}E_\epsilon k(\xi)Q(k) + \alpha_1(k)
\]

Note \( o(k) \) goes to 0 in norm for \( k \) tending to 0 in \( \tilde{S}_\epsilon(\xi) \) by equation (4.13), Lemmas 3.2 and 4.1, the uniform bound \( \|Q_0(k)\| \leq 1 \) and the fact that \( |k|\omega(k)^{-1} \) has a limit for \( k \) tending to 0. Using equation (4.14) and appealing to the limit found in Lemma 4.1 along with the uniform bounds in Lemma 4.1 and equation (4.8).
we now see \((1 - P_0)Q(k)\) and \(Q(k)(1 - P_0)\) goes to 0 weakly for \(k\) tending to 0 inside \(\tilde{S}_\varepsilon(\xi)\). Hence we find

\[
w - \lim_{k \to 0, k \in S_\varepsilon(\xi)} Q(k) - P_0Q(k)P_0 = 0. \tag{4.15}
\]

From equation (4.11) we find

\[
P_0Q(k)P_0 = P_0Q_0(k)P_0 + \frac{|k|}{\omega(k)}P_0\hat{\nabla}K(\xi - d\Gamma(\omega))Q(k)P_0 + P_0\alpha_1(k)P_0
\]

\[
= P_0 + \frac{|k|}{\omega(k)}P_0(k \cdot \nabla K(\xi - d\Gamma(\omega)))(1 - P_0)Q(k)P_0
\]

\[
+ \left(\frac{|k|}{\omega(k)} - C_\omega\right)\hat{k} \cdot v(\xi)P_0Q(k)P_0 + C_\omega\hat{k} \cdot v(\xi)P_0Q(k)P_0 + P_0\alpha_1(k)P_0
\]

Write \(D_k = (1 - C_\omega \hat{k} \cdot v(\xi))^{-1}\) and that for \(k \in \tilde{S}_\varepsilon(\xi)\) we have \(|D_k| \leq 2\). A little algebra yields

\[
P_0Q(k)P_0 - D_kP_0 = D_k\frac{|k|}{\omega(k)}P_0(\hat{k} \cdot \nabla K(\xi - d\Gamma(\omega)))(1 - P_0)Q(k)P_0
\]

\[
+ D_k\left(\frac{|k|}{\omega(k)} - C_\omega\right)\hat{k} \cdot v(\xi)P_0Q(k)P_0 + D_kP_0\alpha_1(k)P_0
\]

The second and third term converges to 0 in norm since for \(k\) tending to 0 inside \(\tilde{S}_\varepsilon(\xi)\) since \(D_k\) and \(\hat{k} \cdot v(\xi)P_0Q(k)P_0\) are uniformly bounded by equation (4.8) and \(\alpha_1(k)\) converges to 0 in norm since for \(k\) tending to 0 inside \(\tilde{S}_\varepsilon(\xi)\). Sandwitching the first term with two vectors \(\phi, \psi \in \mathcal{F}(\mathcal{H})\) we find

\[
D_k\frac{|k|}{\omega(k)}\sum_{i=1}^n \hat{k}_i \langle \partial_i K(\xi - d\Gamma(k))P_0\psi, (1 - P_0)Q(k)P_0\phi \rangle
\]

Now \(\langle \partial_i K(\xi - d\Gamma(k))P_0\psi, (1 - P_0)Q(k)P_0\phi \rangle\) converges to 0 for \(k\) going to 0 inside \(\tilde{S}_\varepsilon(\xi)\) by equation (4.13) and \(\frac{|k|}{\omega(k)}\hat{k}_i\) remains bounded as \(k\) goes to 0. Therefore first term goes weakly to 0 for \(k\) going to 0 inside \(\tilde{S}_\varepsilon(\xi)\). \(\square\)

**Proof (Theorem 3.3).** Fix notation from Lemma 4.8. Assume that a ground state \(\psi\) exist and pick \(\eta \in \mathcal{D}(\mathcal{X}^{1/2})\) such that \(\langle \psi, \eta \rangle > \frac{1}{2}\). Then by Lemma B.14 in Appendix B we have the pull through formula

\[
\langle \eta, A_1\psi(k) \rangle = \mu \frac{v(k)}{\omega(k)}\langle \eta, Q(k)\psi \rangle.
\]

Now

\[
\lim_{k \to 0, k \in S_\varepsilon(\xi)} \langle \eta, Q(k)\psi \rangle - (1 - C_\omega \hat{k} \cdot v(\xi))^{-1}\langle \eta, \psi \rangle = 0
\]
and since \((1 - C_\omega \cdot v(\xi))^{-1}\langle \eta, \psi \rangle\) is uniformly bounded from below in \(\tilde{S}_\epsilon(\xi)\) by \(\frac{1}{2}\), we find that there is \(R > 0\) such that
\[
|\langle \eta, A_1 \psi(k) \rangle|^2 \geq \frac{\mu^2}{16} \frac{|v(k)|^2}{\omega(k)^2}
\]
for all \(k \in \tilde{S}_\epsilon(\xi) \cap B_R(0)\). Using Hypothesis 1 and 2 we see \(\omega(Re_1)^2 > 0\) because if that was not true then \(\omega \leq 0\) on \(B_R(0)\) which is a contradiction. Hence we find
\[
\infty = \int_{B_R(0)} \frac{|v(k)|^2}{\omega(k)^2} d\lambda_\nu \leq \frac{1}{\omega(Re_1)^2} \int_{B_R(0)^c} |v(k)|^2 d\lambda_\nu + \int_{B_R(0)} \frac{|v(k)|^2}{\omega(k)^2} d\lambda_\nu
\]
as \(v \in \mathcal{H}\) we find that the integral of \(\omega(k)^{-2}|v(k)|^2\) over \(B_R(0)\) must be infinite. Using Lemma 4.2 we find
\[
\infty = \int_{B_R(0)} \frac{|v(k)|^2}{\omega(k)^2} d\lambda_\nu = \lambda_\nu(B_1(0)) \int_0^\infty 1_{B_R(0)}(xe_1) \frac{|v(ke_1)|^2}{\omega(ke_1)^2} k^{\nu-1} d\lambda_1(ke_1)
\]
as \(\lambda_\nu(B_1(0))\) we see that the latter integral must be infinite. Furthermore since \(\tilde{S}_\epsilon(\xi)\) is open and not empty we have
\[
\int_{\tilde{S}_\epsilon(\xi) \cap B_R(0)} \frac{|v(k)|^2}{\omega(k)^2} d\lambda_\nu = \nu \lambda_\nu(\tilde{S}_\epsilon(\xi) \cap B_1(0)) \int_0^\infty 1_{B_R(0)}(xe_1) \frac{|v(ke_1)|^2}{\omega(ke_1)^2} k^{\nu-1} d\lambda_1(ke_1)
\]
by Lemma 4.2 so \(|\langle \eta, A_1 \psi(k) \rangle|^2\) is not integrable. On the other hand we find
\[
|\langle \eta, A_1 \psi(k) \rangle|^2 \leq \|(N + 1)^{1/2} \eta\|^2 \|(N + 1)^{-1/2} A_1 \psi(k)\|^2
\]
\[
= \|(N + 1)^{1/2} \eta\|^2 \sum_{i=1}^\infty \int_{\mathbb{R}^{(n-1)\nu}} |\psi^{(n)}(k, k_1, \ldots, k_{n-1})|^2 d\lambda_\nu \otimes \cdots \otimes |k_1, \ldots, k_{n-1}|
\]
which is integrable with integral \(\|(N + 1)^{1/2} \eta\|^2 \|\psi\|^2\) by definition of the Fock space norm. This is the desired contradiction. \(\square\)

### A. Partitions of unity and the essential spectrum.

In this section we prove a few technical ingredients. Hypothesis 1 will be assumed throughout this section. Define \(V_A : \mathcal{H} \to \mathcal{H}_A \oplus \mathcal{H}_{A^c}\) by
\[
V_A(f) = (P_A f, P_{A^c} f).
\]
Then \(V_A\) is unitary with \(V_A^*(f,g) = f 1_A + g 1_{A^c}\) almost everywhere. The following Lemma can be found in e.g. [9]:

**Lemma A.1.** There is a unique isomorphism \(U : \mathcal{F}(\mathcal{H}_A \oplus \mathcal{H}_{A^c}) \to \mathcal{F}(\mathcal{H}_A) \otimes \mathcal{F}(\mathcal{H}_{A^c})\) with the property that \(U(\epsilon(f_1 \oplus f_2)) = \epsilon(f_1) \otimes \epsilon(f_2)\).

The following Lemma is obvious
Lemma A.2. There is a unique isomorphism
\[ U : \mathcal{F}(\mathcal{H}_A) \otimes \mathcal{F}(\mathcal{H}_{A^*}) \to \mathcal{F}(\mathcal{H}_A) \oplus \bigoplus_{n=1}^{\infty} \mathcal{F}(\mathcal{H}_A) \otimes \mathcal{H}_{A^c}^\otimes \]
such that
\[ U(w \otimes \{\psi^{(n)}\}_{n=0}^{\infty}) = \psi^{(0)} w \oplus \bigoplus_{n=1}^{\infty} w \otimes \psi^{(n)}. \]
Note that we may identify
\[ \mathcal{F}(\mathcal{H}_A) \otimes \mathcal{H}_{A^c}^\otimes = (1 \otimes S) L^2(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), 1_{(A^c)^n}, \lambda_{\nu\nu}, \mathcal{F}(\mathcal{H}_A)) \]
where \(1 \otimes S\) acts on \( L^2(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda_{\nu\nu}, \mathcal{F}(\mathcal{H}_A)) \) like
\[ (S_n f)(k_1, \ldots, k_n) = \frac{1}{n!} \sum_{\sigma \in S_n} f(k_{\sigma(1)}, \ldots, k_{\sigma(n)}). \]
Now we define
\[ H_{\rho}^{(n)}(\xi, k_1, \ldots, k_n) = H_{\rho}(\xi - k_1 - \cdots - k_n, A) + \omega(k_1) + \cdots + \omega(k_n) \]
which is strongly resolvent measurable in \((k_1, \ldots, k_n) \in (A^c)^n\) since \(\xi \mapsto H(\xi)\) is strong resolvent measurable by Lemma 3.2. In particular
\[ H_{n,A}(\xi) = \int_{(A^c)^n} H_{\rho}(\xi, k_1, \ldots, k_n) d\lambda_{\nu\nu}(k_1, \ldots, k_n) \]
defines a selfadjoint operator on \(L^2(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda_{\nu\nu}, \mathcal{F}(\mathcal{H}_A))\) and it is reduced by the projection \(1 \otimes S\). To see this we note that \(1 \otimes S\) commutes with the unitary group of \(H_{n,A}(\xi)\) since \(H_{\rho}^{(n)}(\xi, k_1, \ldots, k_n)\) symmetric in the variables \(k_1, \ldots, k_n\). Combining the above observations one arrives at the following lemma.

Lemma A.3. Let \(A \in \mathcal{B}(\mathbb{R}^\nu)\) and assume \(1_A v = v \lambda_{\nu}\) almost everywhere. Define \(j_\lambda : \mathcal{H}_A \to \mathcal{H}_A \oplus \mathcal{H}_{A^c}\) by \(j_\lambda f = (f, 0)\) and \(j_{A^*} f = (0, f)\) and define \(Q_\lambda = V_\lambda j_\lambda\). There is a unitary map
\[ U : \mathcal{F}(\mathcal{H}) \to \mathcal{F}(\mathcal{H}_A) \oplus \bigoplus_{n=1}^{\infty} \mathcal{F}(\mathcal{H}_A) \otimes \mathcal{H}_{A^c}^\otimes \]
such that
\[ UH_{\rho}(\xi)U^* = H_{\rho}(\xi, A) \oplus \bigoplus_{n=1}^{\infty} H_{n,A}(\xi) |_{\mathcal{F}(\mathcal{H}_A) \otimes \mathcal{H}_{A^c}^\otimes} := G_{\rho}(\xi) \]  
(A.1)
for all \(\xi \in \mathbb{R}^\nu\). In particular \(\Sigma_A(\xi) \geq \Sigma(\xi)\) for all \(\xi \in \mathbb{R}^\nu\). Furthermore
\[ U |_{\mathcal{F}(\mathcal{H}_A)} = \Gamma(Q_\lambda). \]
Let \( g_1, \ldots, g_n \in \mathcal{H}_A \) and let \( K \subset CS_A \) be a subspace. Define
\[
D = \{ Q_A \cdot g_1 \otimes \cdots \otimes Q_A \cdot g_n \} \\
\cup \bigcup_{h=1}^{\infty} \{ h_1 \otimes \cdots \otimes s \in \mathcal{H}_A \}.
\]

If \( \psi \in \text{Span}(\mathcal{J}(K)) \) we have
\[
U^* (\psi \otimes g) \in \text{Span}(D).
\]
where \( \lambda \in \mathbb{C} \).

Proof. Define \( U = U_2 U_1 \Gamma(V_A) \). Let \( f, h \in CS \) and write for \( C \in \{ A, A^c \} f_C = P_{C}(f), h_C = P_{C}(h) \in CS_C \). Then
\[
U \varepsilon(f) = U_2 U_1 \varepsilon(f_A, f_{A^c}) = U_2 \varepsilon(f_A) \otimes \varepsilon(f_{A^c}) = \varepsilon(f_A) \oplus \sum_{n=1}^{\infty} \varepsilon(f_A) \otimes \frac{1}{\sqrt{n!}} f_{A^c}^{(n)}
\]
which one may check is in \( D(G(\xi)) \). A long but easy calculation using Lemma 2.2 yields
\[
\langle \varepsilon(h), U^* G(\xi) U \varepsilon(f) \rangle = \langle U \varepsilon(h), G(\xi) U \varepsilon(f) \rangle = \langle \varepsilon(h), H(\xi) \varepsilon(f) \rangle
\]
As \( L(CS) \) is total we find \( H_n(\xi) \) and \( U^* G(\xi) U = H(\xi) \) on \( L(CS) \) which spans a core for \( H_\mu(\xi) \). Hence \( U^* G(\xi) U = H_\mu(\xi) \) as both operators are self-adjoint. This proves the claim regarding the transformations. The remaining statements except equations (A.3) and (A.4) can be found in \( [3] \). However equations (A.3) and (A.4) follows from \( U \mid \mathcal{J}(H_\lambda) = \Gamma(Q_A) \) and equation (A.1). \( \square \)

We have the following Lemma

**Lemma A.4.** Let \( k_1, \ldots, k_\ell \in \mathbb{R}^n \) be different. If there is \( \varepsilon > 0 \) such that
\( (B_\varepsilon(k_1) \cup \cdots \cup B_\varepsilon(k_\ell)) \cap \{ v \neq 0 \} \) is a \( \lambda_v \) 0-set then \( \Sigma(\xi - k_1 - \cdots - k_\ell) + \omega_\varepsilon(k_1, \ldots, k_\ell) \subset \sigma_{\text{ess}} (H_\mu(\xi)) \).

Proof. Pick \( \varepsilon > 0 \) such that the balls \( B_\varepsilon(k_1), \ldots, B_\varepsilon(k_\ell) \) are pairwise disjoint and we have \( (B_\varepsilon(k_1) \cup \cdots \cup B_\varepsilon(k_\ell)) \cap \{ v \neq 0 \} \) is a \( \lambda_v \) 0-set. Let \( \varepsilon_n = \varepsilon, B_n^{(i)} = B_n(k_i), B_n = B_n^{(1)} \cap \cdots \cap B_n^{(\ell)}, k_0 = k_1 + \cdots + k_\ell, A_n = B_n^{(1)} \times \cdots \times B_n^{(\ell)} \) and let
\[
g_n^{(i)} = \lambda_v (B_n^{(i)} \setminus B_n^{(i+1)})^{-1/2} B_n^{(i)} B_n^{(i+1)}
\]
\[A_n = \{ f \in CS \mid fB_n^{(i)} = f \lambda_v \text{ almost everywhere for all } i \in \{1, \ldots, n\} \}
\]
\[A_\infty = \bigcup_{n=1}^{\infty} A_n.
\]
Note that \( CS \subset A_\infty \) so \( A_\infty \) is a dense subspace of \( \mathcal{H} \). In particular, \( \mathcal{J}(A_\infty) \) spans a core for \( H_\mu(\xi - k_0) \) by Lemma 5.2. For each \( p \in \mathbb{N} \) we may thus pick
\( \psi_p \in \mathcal{J}(A_{\infty}) \) such that \( \| (H_\mu(\xi - k_0) - \Sigma(\xi - k_0)) \psi_p \| \leq 1/p \). By Lemma 1 there is \( u_1(p) \) such that

\[
\sup_{x=(x_1, \ldots, x_\ell) \in A_n} \| (H(\xi - x_1 - \cdots - x_\ell) - H(\xi - k_0)) \psi_p \| \leq \frac{1}{p}
\]

for all \( n \geq u_1(p) \). Note now that \( \psi_p \) may we written as

\[
\psi_p = a(p)\Omega + \sum_{i=1}^{b} \sum_{j=1}^{c(p)} \alpha_{i,j}(p)f^i_j(p) \otimes \cdots \otimes f^i_1(p)
\]

for some \( a(p), b(p), c(p), \alpha_{i,j}(p) \) constants and \( f^i_j(p) \in A_{\infty} \). Note that each \( f^i_j(p) \) is in fact contained in some \( A_{(i,j,p)} \) by definition so defining \( u_2(p) = \max_{i,j\{1,2,3\}} \| (H(\xi - k_0)) \psi_p \| \leq 1/p \) and \( \ell \in \epsilon_{u_2} \leq \delta \)

To summarise we have found vectors \( \psi_p \in \mathcal{D}(H(\xi)) \) and a strictly increasing sequence of numbers \( \{ u_p \}_{p=1}^{\infty} \subset \mathbb{N} \) such that

1. \( \| (H(\xi - k_0) - \Sigma(\xi - k_0)) \psi_p \| \leq 1/p \).
2. \( \sup_{k \in B_{u_2}(k_1, \ldots, k_\ell)} \| (H(\xi - k) - H(\xi - k_0)) \psi_p \| \leq \frac{1}{p} \) and \( \ell \in \epsilon_{u_2} \leq \delta \).
3. \( \psi_p \in \text{Span}(\mathcal{J}(A_{u_2})) \).

For each \( n \in \mathbb{N} \) and \( A \in \{ B_{u_2}(k_1, \ldots, k_\ell) \} \) define \( V_n = V_{B_{u_2}(k_0)^n} \) and \( j_{n,A} : H_k \to H_{B_{u_2}(k_0)^n} \) by \( j_{n,A}(f) = (f,0) \) and \( j_{n,B_{u_2}^c} = (0,f) \). Furthermore we set \( Q_{n,A} = V_n^\ast j_{n,A} \) and let \( U_n \) be the unitary map from Lemma A.4 corresponding to \( B_{u_2}^c \).

Fix \( f \in H \). Then the following equalities holds \( \lambda_v \) almost everywhere:

\[
Q_{n,B_{u_2}^c}P_{B_{u_2}^c}(f) = V_n^\ast(0,P_{B_{u_2}^c}(f)) = 1_{B_{u_2}^c}P_{B_{u_2}^c}(f) = 0 \quad \text{(A.5)}
\]

\[
Q_{n,B_{u_2}^c}P_{B_{u_2}^c}(f) = V_n^\ast(P_{B_{u_2}^c}(f),0) = 1_{B_{u_2}^c}P_{B_{u_2}^c}(f) = 1_{B_{u_2}^c}f \quad \text{(A.6)}
\]

since \( P_{B_{u_2}^c}(f) = f 1_{B_{u_2}^c}\lambda_v \)-almost everywhere and \( P_{B_{u_2}^c}(f) = f 1_{B_{u_2}^c}\lambda_v \)-almost everywhere. For \( f \in A_n \) we have \( 1_{B_{u_2}^c}f = f \) and so we obtain the two equalities

\[
\Gamma(Q_{n,B_{u_2}^c})\Gamma(P_{B_{u_2}^c})\psi = \Gamma(1_{B_{u_2}^c})\psi = \psi \quad \forall \psi \in \text{Span}(\mathcal{J}(A_n)) \quad \text{(A.7)}
\]

\[
Q_{n,B_{u_2}^c}P_{B_{u_2}^c}g_n^{(i)} = 1_{B_{u_2}^c}g_n^{(i)} = g_n^{(i)} \quad \text{(A.8)}
\]

for all \( i \in \{1, \ldots, \ell\} \). We now define the Weyl sequence as follows:

\[
\phi_p = \sqrt{\bar{U}_{u_p}^\ast(\Gamma(P_{B_{u_p}^c})\psi_p \otimes P_{B_{u_p}^c}g_{u_p}^{(1)} \otimes \cdots \otimes P_{B_{u_p}^c}g_{u_p}^{(\ell)})}
\]

We will now prove

1. \( \phi_p \in \mathcal{D}(F_\mu) \).
2. \( \phi_p \) is orthogonal to \( \phi_r \) for \( p \neq r \).
3. \( \| \phi_p \| = 1 \) for all \( p \in \mathbb{N} \).
4. \( \| (H(\xi) - \Sigma(\xi - k_0) - \omega_n(k_1, \ldots, k_n)) \phi_p \| \) converges to 0.
(1): Define for all \( p \in \mathbb{N} \) the set
\[
C_p = \{ g_{u_p}^{(1)} \otimes_s \cdots \otimes_s g_{u_p}^{(\ell)} \} \\
\cup \bigcup_{q=1}^{\infty} \{ h_1 \otimes_s \cdots \otimes_s h_q \otimes_s g_{u_p}^{(1)} \otimes_s \cdots \otimes_s g_{u_p}^{(\ell)} | h_1 \in A_{u_p} \} \subset \mathcal{J}(\mathcal{C}\mathcal{S})
\]
and let \( K_p = P_{B_{u_p}} A_{u_p} \subset \mathcal{C}\mathcal{S}B_{u_p} \) since \( P_{B_{u_p}} \) maps \( \mathcal{C}\mathcal{S} \) into \( \mathcal{C}\mathcal{S}B_{u_p} \). Using equation (A.5) we find \( Q_{u_p, B_{u_p}} K_p = 1_{B_{u_p}} A_{u_p} = A_{u_p} \) and Lemma A.3 implies \( \psi_p \in C_p \subset \mathcal{J}(\mathcal{C}\mathcal{S}) \subset \mathcal{D}(\mathcal{H}_p(\xi)) \) as required.

(2): Let \( r < p \). Then \( \phi_r \in \text{Span}(C_r) \) and \( \phi_p \in \text{Span}(C_p) \), so we just need to see that every element in \( C_p \) and \( C_r \) are orthogonal. Let \( \psi_1 \in C_p \) and \( \psi_2 \in C_r \). Note every tensor in \( C_p \) has a factor \( g_{u_p}^{(1)} \) and that this factor is orthogonal to \( g_{u_r}^{(j)} \) for all \( i \) by construction. Furthermore for any \( h \in A_{u_r} \) we see that \( h \) is supported in \( B_{u_r} \subset B_{u_p} \) and hence \( g_{u_r}^{(1)} h = 0 \), so \( g_{u_r}^{(1)} \) is orthogonal to any element in \( A_{u_r} \). This implies \( \psi_1 \) contains a factor orthogonal to all factors in \( \psi_2 \) and thus \( \psi_1 \) is orthogonal to \( \psi_2 \).

(3): \( Q_{u_p, B_{u_p}} \) and \( Q_{u_p, B_{u_p}} \) are isometries and which implies \( \Gamma(Q_{u_p, B_{u_p}}) \) and \( \Gamma(Q_{u_p, B_{u_p}}) \) are isometries. Using equations (A.7) and (A.8) we calculate
\[
\| \phi_p \| = \sqrt{\ell!} \| \Gamma(P_{B_{u_p}}) \psi_p \| \| P_{B_{u_p}} g_{u_p}^{(1)} \otimes_s \cdots \otimes_s P_{B_{u_p}} g_{u_p}^{(\ell)} \| \\
= \sqrt{\ell!} \| \Gamma(Q_{u_p, B_{u_p}}) \psi_p \| \| \Gamma(Q_{u_p, B_{u_p}}) P_{B_{u_p}} g_{u_p}^{(1)} \otimes_s \cdots \otimes_s P_{B_{u_p}} g_{u_p}^{(\ell)} \| \\
= \sqrt{\ell!} \| \psi_p \| \| g_{u_p}^{(1)} \otimes_s \cdots \otimes_s g_{u_p}^{(\ell)} \| = 1
\]
where we used \( g_{u_p}^{(i)} \) and \( g_{u_p}^{(j)} \) are normalised and orthogonal if \( i \neq j \) and
\[
\| g_{u_p}^{(1)} \otimes_s \cdots \otimes_s g_{u_p}^{(\ell)} \|^2 = \frac{1}{\ell!} \sum_{\sigma \in S_{\ell}} \| g_{u_p}^{(1)} \otimes_s \cdots \otimes_s g_{u_p}^{(\ell)} g_{u_p}^{(\sigma(1))} \otimes_s \cdots \otimes_s g_{u_p}^{(\sigma(\ell))} \| = \frac{1}{\ell!}
\]

(4): Define the function \( g_{u_p} = g_{u_p}^{(1)} \otimes_s \cdots \otimes_s g_{u_p}^{(\ell)} \). Using Lemma A.3 we see that \( \| (H(\xi) - \Sigma(\xi - k_0) - \omega(\nu k_1, \ldots, k_n)) \phi_p \| \) is given by
\[
\sqrt{\ell!} \left( \int_{B_{u_p}} \| H_{B_{u_p}}(\xi - x_1 - \cdots - x_{\ell}) + \omega(\xi, 1, \ldots, x_{\ell}) - \Sigma(\xi - k_0) - \omega(\xi, k_1, \ldots, k_\ell) \| (P_{B_{u_p}}) \psi_p \|^2 |g_{u_p}(x)|^2 d\lambda_{\nu}(x) \right)^{1/2} := \sqrt{\ell!} \gamma
\]
Using the triangle inequality, \( \| \Gamma(P_{B_{n_p}^+}) \psi_p \| = 1 \), \( \Gamma(Q_{n_p,B_{n_p}^-}) \Gamma(P_{B_{n_p}^-}) \psi_p = \psi_p \) and Lemma A.3 we find \( \gamma \leq C_1 + C_2 + C_3 \) where

\[
C_1 = \left( \int_{B_{n_p}^+} \| (H(\xi - x_1 - \cdots - x_n) - H(\xi - k_0)) \psi_p \|^2 |g_{u_p}(x)|^2 d\lambda_\nu(x) \right)^{1/2}
\]

\[
C_2 = \left( \int_{B_{n_p}^-} |(\omega_\nu(x_1, \ldots, x_\ell) - \omega_\nu(k_1, \ldots, k_\ell))| |g_{u_p}(x)|^2 d\lambda_\nu(x) \right)^{1/2}
\]

\[
C_3 = \| (H(\xi - k_0) - \Sigma(\xi - k_0)) \psi_p \| \left( \int_{B_{n_p}^+} |g_{u_p}(x)|^2 d\lambda_\nu(x) \right)^{1/2}
\]

Let \( f : (\mathbb{R}^\nu)^n \to \mathbb{R} \) be non negative and symmetric. Using that the \( g_{u_p}^{(i)} \) have disjoint support one finds

\[
|g_{u_p}(x_1, \ldots, x_\ell)|^2 = \frac{1}{d!^2} \sum_{\sigma \in \mathcal{S}_n} \prod_{i=1}^\nu g_{u_p}^{(\sigma(i))}(x_i) g_{u_p}^{(\sigma(i))}(x_i) = \frac{1}{d!^2} \sum_{\sigma \in \mathcal{S}_n} \prod_{i=1}^\nu (g_{u_p}^{(i)}(x_i))^2
\]

Thus using permutation invariance of \( f \) we find

\[
\int_{B_{n_p}} f(x) |g_{u_p}(x)|^2 d\lambda_\nu(x) = \frac{1}{d!} \int_{A_{n_p}} f(x) \prod_{i=1}^\nu (g_{u_p}^{(i)}(x_i))^2 d\lambda_\nu(x)
\]

Thus \( \sqrt{\nu} C_3 = \| (H(\xi - k_0) - \Sigma(\xi - k_0)) \psi_p \| \leq p^{-1} \). Furthermore

\[
\sqrt{\nu} C_1 \leq \sup_{(x_1, \ldots, x_n) \in A_{n_p}} \| (H(\xi - x_1 - \cdots - x_\ell) - H(\xi - k_0)) \psi_p \| \leq p^{-1}
\]

\[
\sqrt{\nu} C_2 \leq \sup_{(x_1, \ldots, x_n) \in A_{n_p}} |\omega_\xi(x_1, \ldots, x_\ell) - \omega_\xi(k_1, \ldots, k_\ell)|
\]

By continuity of \( \omega \) we now see \( \sqrt{\nu} \gamma \) goes to 0 for \( p \) tending to \( \infty \). \( \square \)

**Lemma A.5.** Let \( k_1, \ldots, k_\ell \in \mathbb{R}^\nu \). Then \( \Sigma(\xi - k_1 - \cdots - k_\ell) + \omega_\nu(k_1, \ldots, k_\ell) \in \sigma_{ess}(H_{\mu}(\xi)) \).

**Proof.** Assume first \( k_1, \ldots, k_\ell \in \mathbb{R}^\nu \) are different elements and define \( A_n = B_{1/n}(k_1) \cup \cdots \cup B_{1/n}(k_\ell) \). Let \( v_n = 1_{A_n} v \) and note that \( v_n \in \mathcal{D}(\omega^{-1/2}) \) and

\[
\lim_{n \to \infty} \| (v_n - v)(\omega^{-1/2} + 1) \| = 0
\]

by dominated convergence. Define

\[
H^{(n)}(\xi) = \Omega(\xi - d\Gamma(k)) + d\Gamma(\omega) + \mu \varphi(v_n) \geq -\mu^2 \| \omega^{-1/2} v_n \|^2 \geq -\mu^2 \| \omega^{-1/2} v \|^2
\]

\[
\Sigma_n(\xi) = \inf(\sigma(H(\xi))
\]

Using Lemma 2.1 we find

\[
\| (H_{\mu}(\xi) + i)^{-1} - (H^{(n)}(\xi) + i)^{-1} \| \leq \| \varphi(v - v_n)(H_{\mu}(\xi) + i)^{-1} \|
\]

\[
\leq \| \varphi(v - v_n)(\omega^{-1/2} + 1) \| \| (d\Gamma(\omega) + 1)^{1/2} (H_{\mu}(\xi) + i)^{-1} \|
\]
so $H^{(n)}(\xi)$ converges to $H_{\mu}(\xi)$ in norm resolvent sense for all $\xi \in \mathbb{R}^\nu$. The uniform lower bound of $\Sigma_n(\xi)$ and norm resolvent convergence now implies $\Sigma_n(\xi)$ converges to $\Sigma(\xi)$ for all $\xi$.

By Lemma [A.3] we have $\Sigma_n(\xi - k_1 - \cdots - k_\ell) + \omega_n(k_1, \ldots, k_\ell) \in \sigma_{ess}(H^{(n)}(\xi))$. Now $\Sigma_n(\xi - k_1 - \cdots - k_\ell) + \omega_n(k_1, \ldots, k_\ell)$ converges to $\Sigma(\xi - k_1 - \cdots - k_\ell) + \omega(k_1, \ldots, k_\ell)$ and $H^{(n)}(\xi)$ converges to $H(\xi)$ in norm resolvent sense so we are done in the case where $k_1, \ldots, k_\ell$ are different. The conclusion now follows since $\Sigma$ and $\omega$ are continuous, $\{(k_1, \ldots, k_\ell) \mid k_i \neq k_j \forall i, j\}$ is dense and $\sigma_{ess}(H(\xi))$ is closed.

### B. Proof of pull through formula

This appendix is devoted to proving the pull through formula. The in case $K(k) = |k|^2$ one could compute everything directly using tools as in [3]. However the other possible choices of $K$ require a more sophisticated approach as we use the formalisation developed in [3] and the reader should consult this paper for the proofs. Let $\mathcal{H} = L^2(\mathcal{M}, \mathcal{E}, \mu)$, where $(\mathcal{M}, \mathcal{E}, \mu)$ is assumed to be $\sigma$-finite. We start by defining

$$
\mathcal{F}_+(\mathcal{H}) = \bigotimes_{n=0}^{\infty} \mathcal{H}^{\otimes,n}
$$

with coordinate projections $P_n$ and $\mathcal{H} = L^2(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda_\nu)$. For $(\psi^{(n)}), (\phi^{(n)}) \in \mathcal{F}_+(\mathcal{H})$ we define

$$
d((\psi^{(n)}), (\phi^{(n)})) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{\|\psi^{(n)} - \phi^{(n)}\|}{1 + \|\psi^{(n)} - \phi^{(n)}\|}
$$

where $\|\cdot\|$ is the Fock space norm. This makes sense since $P_n(\mathcal{F}_+(\mathcal{H})) \subset \mathcal{F}(\mathcal{H})$.

We now have

**Lemma B.1.** The map $d$ defines a metric on $\mathcal{F}_+(\mathcal{H})$ and turns this space into a complete separable metric space and a topological vector space. The topology and Borel $\sigma$-algebra is generated by the projections $P_n$. If a sequence $\{\psi_n\}_{n=1}^{\infty} \subset \mathcal{F}(\mathcal{H})$ is convergent/Cauchy then it is also convergent/Cauchy with respect to $d$. Also any total/dense set in $\mathcal{F}_b(\mathcal{H})$ will be total/dense in $\mathcal{F}_+(\mathcal{H})$ as well.

For each $a \in \mathbb{R}$ we define

$$
\|\cdot\|_{a,+} = \lim_{n \to \infty} \left( \sum_{k=0}^{n} (k+1)^{2a} \|P_k(\cdot)\|^2 \right)^{\frac{1}{2}}
$$

which is measurable from $\mathcal{F}_+(\mathcal{H})$ into $[0, \infty]$. Let

$$
\mathcal{F}_{a,+}(\mathcal{H}) = \{ \psi \in \mathcal{F}_+(\mathcal{H}) \mid \|\psi\|_{a,+} < \infty \}.
$$

Note $\|\cdot\|_{a,+}$ restricts to a norm on $\mathcal{F}_{a,+}(\mathcal{H})$ that comes from an inner product. In particular $\mathcal{F}_{a,+}(\mathcal{H})$ is a Hilbert space and for $a \geq 0$ we have $\mathcal{F}_{a,+}(\mathcal{H}) = \mathcal{D}((N+1)^a)$. We summarise as follows
Lemma B.2. \(|\cdot|_{a,+}\) defines measurable map from \(\mathcal{F}_+(\mathcal{H})\) to \([0, \infty]\), and restricts to a norm on the spaces \(\mathcal{F}_{a,+}(\mathcal{H})\) that comes from an inner product turning \(\mathcal{F}_{a,+}(\mathcal{H})\) into a Hilbert space.

The point of defining a metric on \(\mathcal{F}_+(\mathcal{H})\) and finding a dense set is that most of the operations we will encounter in this chapter are continuous on \(\mathcal{F}_+(\mathcal{H})\). Therefore many operator identities only needs to be proven on well behaved vectors. Fix now \(v \in \mathcal{H}\). We now define the following maps on \(\mathcal{F}_+(\mathcal{H})\)

\[
\begin{align*}
    a_+(v)(\psi^{(n)}) &= (a_n(v)\psi^{(n+1)}) \\
a^{\dagger}_+(v)(\psi^{(n)}) &= (0, a_0^+(v)\psi^{(0)}, a_1^+(v)\psi^{(1)}, \ldots) \\
\varphi_+(v) &= a_+(v) + a^{\dagger}_+(v)
\end{align*}
\]

Where \(a_n(v)\) is annihilation from \(\mathcal{H}^{\otimes(n+1)}\) to \(\mathcal{H}^{\otimes,n}\) and \(a^0_+(f)\) is creation from \(\mathcal{H}^{\otimes,n}\) to \(\mathcal{H}^{\otimes(n+1)}\).

Lemma B.3. The maps \(a_+(v), a^{\dagger}_+(v)\) and \(\varphi_+(v)\) are all continuous. For \(B \in \{a, a^{\dagger}, \varphi\}\) we have

\[
    B_+(v)\psi = B(v)\psi \text{ if } \psi \in \mathcal{D}(B(v)). \quad (B.1)
\]

Furthermore we have the commutation relations

\[
\begin{align*}
    [a_+(v), a^{\dagger}_+(g)] &= \langle v, g \rangle \\
    [\varphi_+(v), \varphi^{\dagger}_+(g)] &= 2i\text{Im}(\langle v, g \rangle)
\end{align*}
\]

We now move on to the second quantisation of unitaries and selfadjoint operators. Let \(U\) be unitary on \(\mathcal{H}\) and \(\omega = (\omega_1, \ldots, \omega_p)\) be a tuple of strongly commuting selfadjoint operators on \(\mathcal{H}\). We then define

\[
\begin{align*}
    d\Gamma_\omega &= (d\Gamma_1(\omega_1), \ldots, d\Gamma_p(\omega_p)) \\
    d\Gamma^{(n)}_\omega &= (d\Gamma_{(n)}(\omega_1), \ldots, d\Gamma_{(n)}(\omega_p))
\end{align*}
\]

which are now tuples of strongly commuting selfadjoint operators (this is easily checked using the unitary group). Let furthermore \(f : \mathbb{R}^p \to \mathbb{C}\) be a map. We then define

\[
    f(d\Gamma_\omega) = \prod_{n=0}^{\infty} f(d\Gamma^{(n)}_\omega) \mathcal{D}(f(d\Gamma_\omega)) = \prod_{n=0}^{\infty} \mathcal{D}(f(d\Gamma^{(n)}_\omega))
\]

\[
    \Gamma_+(U) = \prod_{n=0}^{\infty} \Gamma_{(n)}(U).
\]

If \(\omega : \mathcal{M} \to \mathbb{R}^p\) is measurable then we may identify \(\omega\) as such a tuple of commuting selfadjoint operators. In this case \(f(d\Gamma^{(n)}_\omega)\) is multiplication by the map \(f(\omega(k_1) + \cdots + \omega(k_n))\). The following lemma is now obvious.

Lemma B.4. The map \(\Gamma_+(U)\) is an isometry on \(\mathcal{F}_+(\mathcal{H})\) and is thus continuous. Furthermore we have

\[
\begin{align*}
    f(d\Gamma_+(\omega))\psi &= f(d\Gamma(\omega))\psi, \quad \psi \in \mathcal{D}(f(d\Gamma(\omega))) \\
    \Gamma_+(U)\psi &= \Gamma(U)\psi, \quad \psi \in \mathcal{F}_0(\mathcal{H})
\end{align*}
\]
We will now consider a class of linear functionals on $\mathcal{F}_+(\mathcal{H})$. For each $n \in \mathbb{N}$ we let $Q_n : \mathcal{F}_+(\mathcal{H}) \to \mathcal{N}$ denote the linear projection which preserves the first $n$ entries of $\langle \psi, \cdot \rangle$ and projects the rest of them to 0. For $\psi \in \mathcal{N}$ there is $K \in \mathbb{N}$ such that for $n \geq K$ we have $Q_n \psi = \psi$. For $\phi \in \mathcal{F}_+(\mathcal{H})$ we may thus define the pairing

$$\langle \psi, \phi \rangle_+ := \langle \psi, Q_n \phi \rangle = \sum_{i=0}^{K} \langle \psi^{(i)}, \phi^{(i)} \rangle,$$  \hspace{1cm} (B.2)

where $n \geq K$.

**Lemma B.5.** The map $Q_n$ above is linear and continuous into $\mathcal{F}(\mathcal{H})$. The paring $\langle \cdot, \cdot \rangle_+$ is sesquilinear, and continuous in the second entry. If $\phi \in \mathcal{F}_+(\mathcal{H})$ then $\psi \mapsto \langle \psi, \phi \rangle_+$ is continuous with respect to $\| \cdot \|_{a,+}$. Furthermore, the collection of maps of the form $\langle \psi, \cdot \rangle_+$ will separate points of $\mathcal{F}_+(\mathcal{H})$.

**Corollary B.6.** Let $\phi \in \mathcal{F}_+(\mathcal{H})$ for some $a \leq 0$, $\mathcal{D} \subset \mathcal{N}$ be dense in $\mathcal{F}(\mathcal{H})$ and assume $\langle \psi, \phi \rangle_+ = 0$ for all $\psi \in \mathcal{D}$. Then $\phi = 0$.

We also have the following formal adjoint relations

**Lemma B.7.** Let $\psi \in \mathcal{N}$, $\phi \in \mathcal{F}_+(\mathcal{H})$, $v \in \mathcal{H}$ and $U$ be unitary on $\mathcal{H}$. Then we have

$$\langle a^\dagger(v) \psi, \phi \rangle_+ = \langle \psi, a_+(v) \phi \rangle_+,$$

$$\langle a(v) \psi, \phi \rangle_+ = \langle \psi, a^\dagger(v) \phi \rangle_+,$$

$$\langle \varphi(v) \psi, \phi \rangle_+ = \langle \psi, \varphi_+(v) \phi \rangle_+,$$

$$\langle \Gamma(U) \psi, \phi \rangle_+ = \langle \psi, \Gamma_+(U^*) \phi \rangle_+.$$

Let $\omega = (\omega_1, \ldots, \omega_p)$ be a tuple of commuting selfadjoint operators, $f : \mathbb{R}^p \to \mathbb{C}$, $\psi \in \mathcal{N} \cap \mathcal{D}(f(d\Gamma(\omega)))$ and $\phi \in \mathcal{D}(\overline{\mathcal{F}}(d\Gamma_+(\omega)))$ we have

$$\langle f(d\Gamma(\omega)) \psi, \phi \rangle_+ = \langle \psi, \overline{\mathcal{F}}(d\Gamma_+(\omega)) \phi \rangle_+.$$

We now consider functions with values in $\mathcal{F}_+(\mathcal{H})$. Let $(X, \mathcal{X}, \nu)$ be a $\sigma$-finite and countably generated measure space. Define the quotient

$$\mathcal{M}(X, \mathcal{X}, \nu) = \{ f : X \to \mathcal{F}_+(\mathcal{H}) \mid f \text{ is } \mathcal{B}(\mathcal{F}_+(\mathcal{H})) \text{ measurable} \}/ \sim,$$

where we define $f \sim g \iff f = g$ almost everywhere. We are interested in the subspace

$$\mathcal{C}(X, \mathcal{X}, \nu) = \{ f \in \mathcal{M}(X, \mathcal{X}, \nu) \mid x \mapsto P_n f(x) \in L^2(X, \mathcal{X}, \nu, \mathcal{H}^{\otimes n}) \ \forall n \in \mathbb{N}_0 \}.$$

Lemma [B.2] shows that $x \mapsto \| f(x) \|_{a,+}$ is measurable for functions $f \in \mathcal{C}(X, \mathcal{X}, \nu)$ and so the integral

$$\int_X \| f(x) \|^2_{a,+} d\nu(x)$$

always makes sense. If $a = 0$ then it is finite if and only if $f \in L^2(X, \mathcal{X}, \nu, \mathcal{F}_b(\mathcal{H}))$. We write $f \in \mathcal{C}(X, \mathcal{X}, \nu)$ as $(f^{(n)})$ where $f^{(n)}(x) = x \mapsto P_n f(x)$. For $f, g \in \mathcal{C}(X, \mathcal{X}, \nu)$ we define

$$d(f, g) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{\| f^{(n)} - g^{(n)} \|^2_{L^2(X, \mathcal{X}, \nu, \mathcal{H}^{\otimes n})}}{1 + \| f^{(n)} - g^{(n)} \|^2_{L^2(X, \mathcal{X}, \nu, \mathcal{H}^{\otimes n})}}.$$

We can now summarise.
Lemma B.8. \( d \) is a complete metric on \( C(X, \mathcal{X}, \nu) \) such that \( C(X, \mathcal{X}, \nu) \) becomes separable topological vector space. The topology is generated by the maps \( f \mapsto (x \mapsto P_n f(x)) \). Furthermore \( L^2(X, \mathcal{X}, \nu, \mathcal{F}_0(H)) \subset C(X, \mathcal{X}, \nu) \) and convergence in \( L^2(X, \mathcal{X}, \nu, \mathcal{F}_0(H)) \) implies convergence in \( C(X, \mathcal{X}, \nu) \). Also the map \( x \mapsto \|f(x)\|_{a,+} \) is measurable for any \( f \in C(X, \mathcal{X}, \nu) \) and \( a \in \mathbb{R} \).

We now move on to discuss some actions on this space. This is strongly related to the direct integral and readers should look up the results in [12]. Let \( n \geq 1 \), \( v \in \mathcal{H}, U \) be unitary on \( \mathcal{H}, \omega = (\omega_1, \ldots, \omega_p) \) a tuple of selfadjoint multiplication operators on \( \mathcal{H} \), \( m : \mathcal{M}^n \rightarrow \mathbb{R}^p \) measurable and \( g : \mathbb{R}^p \rightarrow \mathbb{R} \) a measurable map. Then we wish to define operators on \( C(\mathcal{M}^\ell, \mathcal{E}^{\otimes \ell}, \mu^{\otimes \ell}) \) for \( \ell \geq 1 \) by

\[
(a_{\otimes,\ell}^1(v)f)(k) = a_{+}^1(v)f(k),
(a_{\otimes,\ell}(v)f)(k) = a_{+}(v)f(k),
(\varphi_{\otimes,\ell}(v)f)(k) = \varphi_{+}(v)f(k),
(I_{\otimes,\ell}(f))(k) = I_{+}(f)(k),
(g(dI_{\otimes,\ell}(\omega) + m)f)(k) = g(dI_{+}(\omega) + m(k))f(k).
\]

We further define \( C(\mathcal{M}, \mathcal{E}^{\otimes 0}, \mu^{\otimes 0}) = \mathcal{F}_+(\mathcal{H}) \) along with \( a_{\otimes,0}^1(v) = a_{+}^1(v), a_{\otimes,0}(v) = a_{+}(v), \varphi_{\otimes,0}(v) = \varphi_{+}(v) \) and \( I_{\otimes,0} = I_{+}(U) \). We have the following lemma.

**Lemma B.9.** The \( a_{\otimes,\ell}(v), a_{\otimes,\ell}(v), \varphi_{\otimes,\ell}(v) \) and \( I_{\otimes,\ell}(U) \) are well defined and continuous for all \( \ell \in \mathbb{N}_0 \). Let \( f \in C(\mathcal{M}^\ell, \mathcal{E}^{\otimes \ell}, \mu^{\otimes \ell}) \). If \( f(k) \in \mathcal{D}(g(dI_{+}(\omega) + m(k))) \) for all \( k \) then \( k \mapsto P_n(g(dI_{+}(\omega) + m(k))f(k)) \) is measurable. Thus as domain of \( g(dI_{+}(\omega) + m) \) we may choose

\[
\lim_{\ell \to 0} \left\{ f \in C(\mathcal{M}^\ell, \mathcal{E}^{\otimes \ell}, \mu^{\otimes \ell}) \left| f(k) \in \mathcal{D}(g(dI_{+}(\omega) + m(k))) \text{ for a.e. } k \in \mathcal{M}^\ell \right. \right\},
\]

\[
\int_{\mathcal{M}^\ell} \|P_n g(dI_{+}(\omega) + m(k))f(k)\|^2 d\mu^{\otimes \ell}(k) < \infty.
\]

We will now introduce the pointwise annihilation operators. For \( \psi = (\psi(n)) \in \mathcal{F}_+(\mathcal{H}) \) we define \( A_\ell \psi \in C(\mathcal{M}^\ell, \mathcal{E}^{\otimes \ell}, \mu^{\otimes \ell}) \) by

\[
P_n(A_\ell \psi)(k_1, \ldots, k_\ell) = \sqrt{(n + \ell)(n + \ell - 1) \cdots (n + 1)}\psi(n+\ell)(k_1, \ldots, k_\ell, \ldots, \cdot),
\]

which is easily seen to be well defined and take values in \( \mathcal{H}^{\otimes n} \). We can prove

**Lemma B.10.** \( A_\ell \) is a continuous linear map from \( \mathcal{F}_+(\mathcal{H}) \) to \( C(\mathcal{M}^\ell, \mathcal{E}^{\otimes \ell}, \mu^{\otimes \ell}) \) and from \( D(\mathcal{N}^\ell) \) into \( L^2(\mathcal{M}^\ell, \mathcal{E}^{\otimes \ell}, \mu^{\otimes \ell}, \mathcal{F}(\mathcal{H})) \). Furthermore \( \psi \in D(\mathcal{N}^{\ell/2}) \iff A_\ell \psi \in L^2(\mathcal{M}^\ell, \mathcal{E}^{\otimes \ell}, \mu^{\otimes \ell}, \mathcal{F}(\mathcal{H})) \) and if \( \psi \in \mathcal{F}(\mathcal{H}) \) we have \( A_\ell \psi \) is almost everywhere \( \mathcal{F}^{-\frac{\ell}{2},+}(\mathcal{H}) \) valued.

Fix \( v \in \mathcal{H} \) and \( \ell \in \mathbb{N}_0 \). We then define a map \( z_\ell(v) : C(\mathcal{M}^\ell, \mathcal{E}^{\otimes \ell}, \mu^{\otimes \ell}) \rightarrow C(\mathcal{M}^{\ell+1}, \mathcal{E}^{\otimes (\ell+1)}, \mu^{\otimes (\ell+1)}) \) by

\[
(z_0(v)\psi)(k) = v(k)\psi \text{ and } (z_\ell(v)\psi)(x, k) = v(x)\psi(k)
\]

when \( \ell \geq 1 \). One may prove
Lemma B.11. The map $z_1(v)$ introduced above is linear and continuous. Both as a map from $C(\mathcal{M}^\ell, \mathcal{E}^\otimes \ell, \mu^\otimes \ell)$ into the space $C(\mathcal{M}^{\ell+1}, \mathcal{E}^\otimes (\ell+1), \mu^\otimes (\ell+1))$ and from $L^2(\mathcal{M}^\ell, \mathcal{E}^\otimes \ell, \mu^\otimes \ell, \mathcal{F}(\mathcal{H}))$ into the space $L^2(\mathcal{M}^{\ell+1}, \mathcal{E}^\otimes (\ell+1), \mu^\otimes (\ell+1), \mathcal{F}(\mathcal{H}))$.

Lastly we look at permutation and symmetrisation operators. Let $\ell \geq 1$ and $\sigma \in S_\ell$ where $S_\ell$ is the set of permutations of $\{1, \ldots, \ell\}$. Defining $\hat{\sigma} : \mathcal{M}^\ell \to \mathcal{M}^\ell$ by $\hat{\sigma}(k_1, \ldots, k_\ell) = (k_{\sigma(1)}, \ldots, k_{\sigma(\ell)})$. Define $\hat{\sigma} : \mathcal{C}(\mathcal{M}^\ell, \mathcal{E}^\otimes \ell, \mu^\otimes \ell) \to \mathcal{C}(\mathcal{M}^\ell, \mathcal{E}^\otimes \ell, \mu^\otimes \ell)$ by

$$(\hat{\sigma} f)(k_1, \ldots, k_\ell) = f(k_{\sigma(1)}, \ldots, k_{\sigma(\ell)}) = (f \circ \hat{\sigma})(k_1, \ldots, k_\ell).$$

Define now

$$S_\ell := \frac{1}{(\ell - 1)!} \sum_{\sigma \in S_\ell} \hat{\sigma}.$$ 

One may prove:

Lemma B.12. Let $\ell \in \mathbb{N}$. For $\sigma \in S_\ell$ the map $\hat{\sigma}$ defines a linear bijective isometry from $\mathcal{C}(\mathcal{M}^\ell, \mathcal{E}^\otimes \ell, \mu^\otimes \ell)$ to $\mathcal{C}(\mathcal{M}^\ell, \mathcal{E}^\otimes \ell, \mu^\otimes \ell)$ and from $L^2(\mathcal{M}^\ell, \mathcal{E}^\otimes \ell, \mu^\otimes \ell, \mathcal{F}(\mathcal{H}))$ to $L^2(\mathcal{M}^\ell, \mathcal{E}^\otimes \ell, \mu^\otimes \ell, \mathcal{F}(\mathcal{H}))$. Also $\hat{\sigma} \Lambda \psi = \Lambda \psi$ and if $\pi \in S_\ell$ then $\hat{\pi} \hat{\sigma} = \hat{\pi} \circ \hat{\sigma}$.

Furthermore $S_\ell$ is continuous and linear from $\mathcal{C}(\mathcal{M}^\ell, \mathcal{E}^\otimes \ell, \mu^\otimes \ell)$ into the space $\mathcal{C}(\mathcal{M}^\ell, \mathcal{E}^\otimes \ell, \mu^\otimes \ell)$ and it satisfies relation $S_\ell^2 = \ell S_\ell$. Furthermore $S_\ell$ is also continuous from $L^2(\mathcal{M}^\ell, \mathcal{E}^\otimes \ell, \mu^\otimes \ell, \mathcal{F}(\mathcal{H}))$ into $L^2(\mathcal{M}^\ell, \mathcal{E}^\otimes \ell, \mu^\otimes \ell, \mathcal{F}(\mathcal{H}))$.

We can now calculate commutators (more commutation relations can be found in B.3 but we will only cite those used here)

Lemma B.13. Let $\omega : \mathcal{M} \to \mathbb{R}^p$ be measurable, $v \in \mathcal{H}$ and let $f : \mathbb{R}^p \to \mathbb{R}$ be measurable. Then

$$\varphi_\exists(v)A_1 = A_1 \varphi_+(v) - z_0(v) \quad \text{(B.3)}$$

Let $\ell \geq 1$. If $\psi \in D(f(d\Gamma_\omega(\omega)))$ then $A_\ell \psi \in D(f(d\Gamma_\exists(\omega_\ell)))$ where we define $\omega_\ell(k_1, \ldots, k_\ell) = \omega(k_1) + \cdots + \omega(k_\ell)$ and

$$f(d\Gamma_\exists(\omega_\ell)) = A_\ell f(d\Gamma_+((\omega_\ell) \psi).$$

We can now prove the pull-through formula.

Lemma B.14. Let $\mathcal{H} = L^2(\mathbb{R}^\nu, \mathcal{B}(\mathbb{R}^\nu), \lambda_\nu)$ and $\omega, v, K$ satisfy Hypothesis 1 and 2 and let $\mu \in \mathcal{M}, \xi \in \mathbb{R}^\nu, \nu \geq 2$. Assume $\psi$ is a ground state for $H_\mu(\xi)$. Then we have

$$(A_1 \psi)(k) = -\mu \nu(k)(H_\mu(\xi - k) + \omega(k) - \Sigma(\xi))^{-1} \psi$$

almost everywhere.

Proof. First we note $(H_\mu(\xi - k) + \omega(k) - \Sigma(\xi))^{-1}$ exists as a bounded operator away from the zero set $\mathbb{R} \xi$ by Lemma B.3. Define the lifted operators on $\mathcal{F}_+(\mathcal{H})$ and $\mathcal{C}(\mathcal{M}^\ell, \mathcal{E}^\otimes \ell, \mu^\otimes \ell)$ respectively

$$H_+(\xi) = K(\xi - d\Gamma_+(k)) + d\Gamma_+(\omega) + \mu \varphi_+(v)$$

$$H_\exists(\xi) = K(\xi - g - d\Gamma_{\exists,1}(k)) + d\Gamma_{\exists,1}(\omega) + \omega + \mu \varphi_{\exists,1}(\omega).$$
where \( g : \mathbb{R}^v \to \mathbb{R}^v \) is given by \( g(k) = k \). The domains are

\[
\mathcal{D}(H_{\phi}(\xi)) = \mathcal{D}(d\Gamma_{\omega}(\omega)) \cap \mathcal{D}(K(\xi - d\Gamma_{\omega}(g)))
\]
\[
\mathcal{D}(H_{\phi}(\xi)) = \mathcal{D}(d\Gamma_{\omega,1}(\omega') + \omega_1) \cap \mathcal{D}(K(\xi - g - d\Gamma_{\omega,1}(g)))
\]

By Lemma \[ \text{B.13} \] we have \( A_{\psi} \in \mathcal{D}(H_{\phi}(\xi)) \) since \( \psi \in \mathcal{D}(H(\xi)) \subset \mathcal{D}(H_{+}(\xi)) \).

Using Lemmas \[ \text{B.3} \] \[ \text{B.4} \] and \[ \text{B.13} \] we also obtain

\[
h := (H_{\phi}(\xi) - \Sigma(\xi))A_{\psi} = -\mu z_0(v)\psi + A_{1}(H_{+}(\xi) - \Sigma(\xi))\psi = -\mu z_0(v)\psi
\]

which is Fock space valued. Let \( M \) be a zeroset such that:

1. \( A_{\psi} \in \mathcal{F}_{-1/2,1}(\mathcal{H}) \) valued on \( M^c \) (see Lemma \[ \text{B.10} \]).
2. \( h(k) = (H_{+}(\xi - k) + \omega(k))(A_{\psi})(k) \) and \( h(k) \in \mathcal{F}(\mathcal{H}) \) for \( k \in M^c \).
3. \( (H_{\mu}(\xi - k) + \omega(k) - \Sigma(\xi))^{-1} \) exists on \( M^c \).

Fix \( k \in M^c \). For any vector \( \phi \) such that both \( (H_{\mu}(\xi - k) + \omega(k) - \Sigma(\xi))^{-1} \phi \) and \( \phi \) is in \( \mathcal{N} \) (this set is dense by Proposition \[ \text{B.2} \]) we find using Lemma \[ \text{B.7} \] that

\[
\langle \phi, A_{\psi}(\xi)(k) \rangle = \langle (H_{\mu}(\xi - k) + \omega(k) - \Sigma(\xi))(H_{\mu}(\xi - k) + \omega(k) - \Sigma(\xi))^{-1}\phi, A_{\psi}(k) \rangle
\]
\[
= \langle (H_{\mu}(\xi - k) + \omega(k) - \Sigma(\xi))^{-1}\phi, h(k) \rangle
\]
\[
= \langle \phi, (H_{\mu}(\xi - k) + \omega(k) - \Sigma(\xi))^{-1}h(k) \rangle.
\]

Corollary \[ \text{B.6} \] finishes the proof.

References

1. Abdesselam A., Hasler, D.: Analyticity of the Ground State Energy for Massless Nelson Models. Commun. Math. Phys. (2012) 310: 511-536.
2. Betz V., Hiroshima F. and Lorinczi J.: Feynman-Kac-Type Theorems and Gibbs Measures on Path Space, with applications to rigorous Quantum Field Theory. 2011 Walter De Gruyter GmbH and Co. KG, Berlin/Boston.
3. Dam, T. N.,Møller J. S.: Spin Boson Type Models Analysed Through Symmetries. In preparation.
4. Dam, T. N.,Møller J. S.: Large interaction asymptotics of spin-boson type models. In preparation.
5. Fröhlich J.: Existence of dressed one-electron states in a class of persistent models. Fortschr. Phys. 22 (1974), 159-198.
6. Fröhlich J.: On the infrared problem in a model of scalar electrons and massless scalar bosons. Ann. Inst. Henri Poincare 19 (1973), 1-103
7. Gross L.: Existence and uniqueness of physical ground states, J. Funct. Anal. 10 (1972) 321-309.
8. Hasler, D., Herbst I.: Absence of Ground States for a Class of Translation Invariant Models of Non-relativistic QED. Commun. Math. Phys. (2008) 279: 769-787.
9. Parthasarathy K.R.: An Introduction to Quantum Stochastic Calculus, Monographs in Mathematics, vol. 85, Birkhäuser, Basel, 1992.
10. Pizzo, A.: One-particle (improper) States in Nelsons Massless Model. Ann. Inst. Henri Poincaré 4 (2003): 439-486.
11. Reed M., Simon B.: Methods of Modern Mathematical Physics I. Functional Analysis Revised and enlarged edition. Academic Press 1980.
12. Reed M., Simon B.: Methods of Modern Mathematical Physics 4. Analysis of operators. Academic Press 1978.
13. Schilling, R.: Measures Integrals and Martingales second edition. 2017 Cambridge University Press, Cambridge.
14. Weidmann J.: Linear Operators in Hilbert Spaces, 1980 Springer-Verlag New York Inc.