Gauge theory of glass transition

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Abstract. A new analytical approach for the description of the glass transition in a frustrated system is suggested. The theory is based on the non-equilibrium dynamics technique, and takes into account the interaction of the local order field with the massive gauge field, which describes frustration-induced plastic deformation. The glass transition is regarded as a phase transition interrupted because of the premature critical slowing-down of one of the degrees of freedom caused by the frustrations. It is shown that freezing of the system appears when the correlation length and relaxation time of the gauge field diverge. The Vogel–Fulcher–Tammann relation for the transition kinetics and the critical exponent for the nonlinear susceptibility, $2.5 \lesssim \gamma \leq 3$, are derived in the framework of the suggested approach. An expression for the temperature dependence of the heat capacity near to the glass transition is derived. This dependence is qualitatively in good agreement with experimental data. The presented theory reproduces the characteristic form of the $\langle \phi \phi \rangle_t$ correlation function dependence on time, and explains the boson peak appearance on this curve. In addition, the function of the glass transition temperature value with cooling rate is derived; this dependence fully conforms with known experimental data.

Keywords: frustrated systems (theory), renormalization group, spin glasses (theory), gauge theories
1. Introduction

The formulation of the microscopic glass transition theory has remained one of the most intriguing but still unresolved problems of condensed matter physics [1]–[3] for a long time. Many systems which manifest this phenomenon regardless of their nature enable us to conclude that this phenomenon does not depend on any microscopic details, but is determined by the symmetry properties of the systems, as in the case of phase transitions, regardless of the scale. However, the search for an answer to the question of whether the glass transition is a phase transition, or its nature is purely dynamical, remains a fundamental problem of glass state physics. The glass transition is indeed attended by continuous change of the system’s volume and entropy and a sharp change of its temperature derivatives. Therefore, it is similar to a second order phase transition according to the Ehrenfest classification. However, this transition is not attended by the formation of a long-range order, or by a susceptibility divergence, and depends on the cooling rate, which is confusing and does not allow us to label it as an authentic phase transition. If the glass transition is a purely dynamical phenomenon, that is a full arrest of diffusion takes place thanks to the very rapid cooling, then why does the heat capacity jump near the glass transition temperature, and the nonlinear susceptibility diverge? Why does the critical slowing-down of relaxation processes, which appears as the non-Arrhenius behavior of the relaxation time temperature dependence, take place? On the other hand, if the glass transition is a thermodynamic phase transition, then why is this critical slowing-down described with the Vogel–Fulcher–Tammann equation, but not with a power function, attached for critical dynamics? Why does neither the correlation

\[ T \rightarrow T_g^+ \]

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radius nor the linear susceptibility diverge in $T_g$? Besides, why does the value of the glass transition temperature depend on the cooling rate?

It is considered that a clue is the fact that the relaxation time is extremely large, owing to which the system is non-equilibrium because its relaxation time becomes so large that it cannot reach the equilibrium state in the observation time, and the system’s physical properties depend on the kinetics of the processes. This drastically distinguishes this problem from the equilibrium one, where usually the static case is considered, and requires the use of new methods of non-equilibrium statistical physics, which go beyond the scope of the usual theoretical approaches. Hence, the way to solve the basic problem of the theoretical description of the glass transition lies in the development of fundamental concepts about the non-equilibrium dynamics of freezing of complex collective systems and an appropriate technique of analytical description.

The new theoretical approach to the glass transition description, introduced in this paper, is based on well known methods of description of the critical phenomena and non-equilibrium dynamics, and promotes the ideas mentioned in [4]–[11]. According to this theory the glass transition is accompanied by the formation of local ordering in the same way as the second order phase transition. We suppose that the key feature, which distinguishes one phenomenon from the other, is the presence of the disorder-induced frustrations. The frustrations distort the system’s space [8], make it heterogeneous, and lead to its plastic deformation. Below it will be shown that the true phase transition is thwarted and replaced by a glass type transition at higher temperatures due to these frustrations.

There are two causes of the frustrations. The first one is the geometric frustration incurred by the naturally preferred structure of the system [9, 10], e.g., the local icosahedral order in simple glass models that is experimentally observed in several metallic glass formers. Such locally preferred structures are incompatible with the periodical ordered structure that leads to the defects (frustrations) forming. In the supercooled liquid state these defects are movable. The second one is extremely important and corresponds to the introduction of unmovable quenched frustrations in a supercooled system. Such frustrations are characteristic, for example, for spin glass formers [8]. In the continuum approximation of the gauge theory these defects act as sources $J$ for the gauge field $A$.

In this paper we study only a special case of the glass-forming systems. It is an $SO(3)$ spin model that is frustrated by randomly quenched topological defects (disclinations)(see figure 1), which are sources of appropriate gauge fields [5, 6], [8]–[10], [12]. Of course this model is not specific for all varieties of glass-forming systems, but it allows us to check our basic ideas on the relatively simple example. We suppose that an ordering starts in the system, and the critical fluctuations of the local order parameter in the disordered phase increase as the temperature approaches the phase transition point, $T_c$. However, the system contains the frustrations, and the energy of an ordered domain containing these frustrations gets the additional contribution which grows with increasing of its volume. Therefore, the fluctuations grow while their size remains less than some characteristic value, $L_{\text{frust}}$, which depends on the concentration of these frustrations, $L_{\text{fluct}} < L_{\text{frust}}$. Since the fluctuations cannot grow larger than $L_{\text{frust}}$, a critical slowing-down occurs when they reach this size. Below we show that this critical behavior is attributed to vanishing of the effective mass of the gauge field, which occurs at the temperature $T_g > T_c$. As a result, the system freezes in a state with a disordered structure. Thus, the glass transition
Figure 1. Frustration can be introduced into the model with the help of a topological defect (disclination) curving the space so that the local equilibrium configuration of spins contains an entire turn of their direction while going around this disclination along a closed contour. The availability of such defects in the system degenerates its collinear ground state.

represents the critical slowing-down of fluctuation growth by disorder-induced frustrations, which complies with the ‘frustration-limited domain theory’ (FLDT) [10] and correlates with other approaches [8, 13].

The paper is organized as follows. First of all we put forward the gauge theory for description of the Heisenberg glass model, and discuss the method of introduction of frustrative disorder in the theory. Next we represent the partition function of the system using the functional technique of description of the non-equilibrium dynamics. This allows us to consider the non-equilibrium dynamics of the frustrated system in critical behavior. In the final part of the paper the offered theory is used for description of the characteristic features of the glass transition.

2. Model

In more detail, let us consider an example of the three-dimensional static isotropic Heisenberg model, whose order parameter, $s$, is considered to be a local magnetization vector with the $SO(3)$ continuous symmetry group. The pure model describes the system undergoing the paramagnetic–ferromagnetic phase transition, with no quenched disorder available. The static action of the model is well known and has the form of

$$ S = \int \left[ \frac{1}{2} (\partial_i s)^2 + U(s) \right] \, dz, $$

where $U(s) = \mu^2 s^2 + vs^4$. Note that at $\mu^2 < 0$ the action is invariant under the $SO(3)$ gauge transformations and $\langle s \rangle = 0$. However, at $\mu^2 < 0$ the symmetry is explicitly broken, since the system can arbitrarily ‘choose’ only one state from all equivalent states with the $U(s)$ potential minima situated on the $|s| = \frac{i\mu}{\sqrt{2v}}$ sphere. We fix the vacuum by means of fixing a point on the sphere. The system is no longer symmetrical with respect to the $SO(3)$ gauge group, but it is invariant under the $SO(2)$ group of the rotation around the chosen direction.

In a disordered spin system the equilibrium spin orientations at different points are not collinear and a suitable connection between orientations is made by introducing a gauge
field, \( A^\mu_a \), and replacing the ordinary derivative, \( \partial_i s \), by a covariant derivative, \( D_i s \) [8, 12]:

\[
S = \int \left[ \frac{1}{2}(D_is)^2 + U(s) + \frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + J^a_\mu A^a_\mu \right] \, \text{dr},
\]

where

\[
D_i s^a = \partial_i s^a + g\varepsilon^{abc} A^b_\mu s^c,
\]

\[
F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\varepsilon^{abc} A^b_\mu A^c_\nu,
\]

\( J^a_\mu \) is the source of the \( A^a_\mu \) field which is introduced to the action in addition to the general part. The reason for this is discussed below.

The method of introduction of disorder in the theory is very important and plays a key part. In order that a spin system possess glass properties we should inject into this system a disorder which gives rise to frustration of its structure. Unfortunately, earlier works did not pay attention to this fact properly. In [6, 7], for example, the gauge field is free, and any quenched randomness is absent. This is right when the dynamic soliton works did not pay attention to this fact properly. In [6, 7], for example, the gauge field in the system does not yet mean the presence of frustrations. The quenched frustrations can be represented by immovable disclinations in an arbitrary configuration with some \( P(A^a_\mu) \) distribution function. It is supposed that the quenched gauge field describes the frustrations. But it is not quite right either, since the presence of the quenched gauge field in the system does not yet mean the presence of frustrations. The quenched frustrations can be represented by immovable disclinations passing through the frustration planes [8, 11] (see figure 1). They are the static sources of the \( A^a_\mu \) gauge field, and should be injected into the model by means of the static sources field, \( J^a_\mu \). Therefore, in contrast to [5, 8], we believe that it is more correct to consider the source field, \( J^a_\mu \), but not the gauge field, \( A^a_\mu \), in the capacity of the quenched random field. In this case the \( A^a_\mu \) field remains as the dynamic one. For illustration the \( A^a_\mu \) field can be interpreted as a local relative rotation of neighboring spins, which corresponds to their local equilibrium. Spins can be movable, but their local equilibrium configuration around a static frustration in the \( \delta V \) volume, bounded by the \( \delta S \) sphere, should satisfy the following condition:

\[
\frac{1}{2} \int_{\delta S} F^a_{\mu\nu} \, \text{d}S_{\nu} = \int_{\delta V} J^a_\mu \, \text{dr} \neq 0,
\]

which follows from the principle of least action.

Expansion of the local magnetization field, \( s \), near one of the vacuum states, for instance \( (s) = (0, 0, i\mu/\sqrt{2v}) \), in small \( \phi = s - i\mu/\sqrt{2v} \) deviations, and of the gauge transformation properties allow to rewrite (1) in the form of the action of two massive vector bosons, \( A^\mu_\kappa \) (\( \kappa = \{1, 2\} \)), with the mass \( M_0 = ig\mu/\sqrt{2v} \), one massless vector boson, \( A^3_\mu \), and one scalar field, \( \phi \):

\[
S = \int \left[ \frac{1}{2}(\partial_\mu \phi)^2 + 2\mu^2 \phi^2 + \frac{g_2^2}{4v} A^\kappa_\mu A^\kappa_\mu + \frac{1}{4} F^a_\mu F^a_{\mu\nu} + v\phi^4 + \frac{g_2^2}{2} \phi^2 A^a_\mu A^a_\mu + J^a_\mu A^a_\mu \right] \, \text{dr}.
\]

Note, that this Lagrangian is not gauge-invariant in the presence of external sources, which is important below.

Let us consider a system with randomly quenched frustrations. The frustrations can be presented as point or linear defects which are continuously distributed over a
Thus, the frustrations lead to the renormalization of the gauge field mass. Let us suppose that the $J^a_\mu$ field is static and absolutely random. Therefore, for simplicity, we assume that $\langle J^a_\mu(r) J^a_\mu(r') \rangle = I_0 \delta(r - r')$, where $I_0$ is the intensity of the quenched disorder. It is shown below that this parameter is proportional to some ‘structural temperature’.

Averaging over $J^a_\mu$ leads to redefinition of the partition function:

$$Z = \int \left[ \int \exp \left( -S - \frac{1}{4} \int I_0^{-1} J^a_\mu^2 \, d\mathbf{r} \right) D\mathbf{J}_\mu^a \right] D\phi D\mathbf{A}_\mu^a,$$

where $\int \cdots D\mathbf{x}$ is the continual integral. It leads to an additional contribution to the $A_\mu^a$ ‘mass’, which takes the form

$$M^2 = -I_0 + \mu^2 g^2 / 4v.$$

Thus, the frustrations lead to the renormalization of the gauge field mass.

The renormalization of the gauge field mass affects the critical behavior of the system, since it shifts the $M^2 = 0$ singularity to the temperature range above the paramagnetic–ferromagnetic transition point, $T_c$. If we assume that $\mu^2 = \alpha k_B (T - T_c)$, where $\alpha$ is some constant, then from (6) we have the critical divergence of the $A_\mu^a$ field correlation radius at $T_k = T_c + 4I_0v / \alpha k_B g^2$ in the paramagnetic phase. One can suppose that this can lead to critical slowing-downs of the fluctuations. Thus, the disorder-induced frustrations inhibit the growth of the $\phi$ field correlation length, and the system freezes in a disordered state, which conforms with the glass transition description in FLDT.

### 3. Description of the non-equilibrium dynamics of the model

In order to examine the above assumption we should investigate the kinetics and susceptibility of the considered model near $T_g$. To do this, we will consider the non-equilibrium dynamics of the system near the critical point. This is a classical problem. Therefore, one can apply either the method of the dynamic generating functional [14], or the classical limit of the Keldysh technique [15], since both these methods coincide in the classical limit. We will choose units such that $k_B T_g = 1$. The use of the functional technique for the description of the non-equilibrium dynamics leads to the representation of the partition function of the system in the form of

$$Z = \int \exp(-S^*) D\bar{\phi} D\bar{A}_\mu^a;$$

where

$$S^* = \frac{1}{2} \int \left[ \bar{\phi}(t, \mathbf{r}) \hat{G}^{-1}(t - t', \mathbf{r} - \mathbf{r'}) \bar{\phi}(t', r') \right.$$

$$+ \bar{A}_\mu^a(t, \mathbf{r}) \Delta^{-1}_{\mu\nu}(t - t', \mathbf{r} - \mathbf{r'}) A_\nu^a(t', r') \Big] \, d\mathbf{r} \, dt \, d\mathbf{r}' \, dt'$$

$$+ \int \left[ g \varepsilon^{abc}(\partial_\mu A_\nu^a) A_\mu^b A_\nu^c + g \varepsilon^{abc}(\partial_\mu A_\nu^a) \bar{A}_\mu^b A_\nu^c 
+ g \varepsilon^{abc}(\partial_\mu A_\nu^a) A_\mu^b \bar{A}_\nu^c + g^2 \varepsilon^{abc} \varepsilon^{aij} \bar{A}_\mu^b A_\nu^c A_\mu^i A_\nu^j 
+ g^2 \bar{A}_\mu^a A_\mu^a \phi^2 + g^2 (A_\mu^a)^2 \bar{\phi} \phi + v 4 \bar{\phi} \phi^3 \right] \, d\mathbf{r} \, dt,$$

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\[ \vec{\phi} = \{ \bar{\phi}, \phi \}, \text{ and } \vec{A}_\mu^a = \{ A^a_\mu, A^a_\mu \} \text{ are vectors, the components of which are named 'quantum' and 'classical' respectively [15],} \]

\[ G^{-1} \text{ and } \Delta^{-1}_{\mu\nu} \text{ are matrices, inverse to the Green function matrices having the following form:} \]

\[ \hat{G} = \begin{pmatrix} G^K & G^A \\ G^R & 0 \end{pmatrix}, \quad \hat{\Delta}_{\mu\nu} = \begin{pmatrix} \Delta^K_{\mu\nu} & \Delta^A_{\mu\nu} \\ \Delta^R_{\mu\nu} & 0 \end{pmatrix}. \quad (9) \]

Note that unlike Yang–Mills theory here there is no necessity to introduce an additional ghost field in the theory because of the gauge symmetry breaking and the presence of the gauge field mass. All Green functions of the theory are explicitly determined for the same reason. The components of the Green function of the scalar field are

\[ G^{R(A)}(k, \omega) = \frac{1}{k^2 + \mu^2 \pm i\Gamma_\phi \omega}, \quad G^K(k, \omega) = \frac{2\Gamma_\phi}{(k^2 + \mu^2)^2 + \Gamma_\phi^2 \omega^2}, \quad (10) \]

where \( \Gamma_\phi \) is the kinetic coefficient, which corresponds to the local magnetization. The components of the Green function of the massive gauge field are

\[ \Delta^{R(A)}_{\mu\nu}(k, \omega) = \frac{\delta_{\mu\nu}}{k^2 + M^2 \pm i\Gamma_A \omega}, \quad \Delta^K_{\mu\nu}(k, \omega) = \frac{2\Gamma_A \delta_{\mu\nu}}{(k^2 + M^2)^2 + \Gamma_A^2 \omega^2}, \quad (11) \]

where \( \Gamma_A \) is the kinetic coefficient, which corresponds to the gauge field. Their graphic forms are presented in figure 2.

4. Renormalization procedure features

The critical behavior of the system close to the \( T_g \) can be considered within the critical dynamics technique [16,17]. The critical dynamics rests on the hypothesis of dynamical scaling, according to which the action should be invariant under scale transformations, which conformally expand the space and time coordinates (\( \omega \sim k^{d-}\)). In this case the summarized dimension, \( d = d_k + d_\omega \) (\( d_\omega = z \) is the dynamic exponent), has the same role as the conventional (momentum) dimension, \( d_k \), in the static case. The canonical dimensions of the fields and model parameters are determined from the condition of dimensionless action. The corresponding summarized canonical dimensions, \( d[F] \), of any values, \( F \), are defined as

\[ d[F] = d_k[F] + z \cdot d_\omega[F], \]

where \( d_\omega[F] \) is the frequency dimension [18, 19]. The canonical dimensions of the values of our theory are given in the following table.

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The renormalization procedure only refines these values, which leads to the replacement of the canonical dimensions by the critical ones $\Delta[F] = d[F] + O[F](\varepsilon)$.

The renormalization procedure is carried out with the standard method. It is assumed that the fields $\phi$, $\bar{\phi}$, $A$, and $\bar{A}$ are slow-varying ones, such that the Fourier-transformed fields have only long-wave components: $|k| < k_0$; $\omega < \omega_0$. In the first step of the RG transformations one integrates the partition function over the components of the fields in the limited wave band $\Lambda k_0 < k < k_0$, $\Lambda^2 \omega_0 < \omega < \omega_0$, where $\Lambda \ll 1$ is the regularization parameter (cutoff of the momentum). It is expected that under certain conditions this action has a structure similar to the original one, in this case a model is multiplicatively renormalizable. One can check that the formulated model satisfies this criterion.

As a result one gets an effective action $S_\Lambda$ with renormalized parameters ($Z_{\Gamma_A}$, $Z_{\Gamma_{\bar{A}}}$, $Z_v$, $Z_\mu$, $Z_M$, $Z_{\bar{A}}$, and $Z_{\bar{\phi}}^2$), which are named ‘constants of renormalization’ and depend on the cutoff $\Lambda$. In the second step, one makes the inverse scaling transformation of the fields ($Z_A$, $Z_{\bar{A}}$, $Z_{\bar{\phi}}$, $Z_{\bar{\phi}}$) and coordinates which is aiming to restore the original cutoff scale $k_0$ and $\omega_0$. Then the renormalized parameters have the following form:

$$
\begin{align*}
\Gamma^{(R)}_{\bar{\phi}} &= Z_{\Gamma_{\bar{\phi}}} Z_{\bar{\phi}} Z_{\bar{\phi}} Z_{\phi} \Lambda^{d+z} = \Lambda^{d+2-2(1+d/2)} Z_{\Gamma_{\phi}} = Z_{\Gamma_{\phi}}, \\
\mu^{(R)2} &= Z_{\mu^2} Z_{\bar{\phi}} Z_{\phi} \Lambda^{d+z} = \Lambda^{2} Z_{\mu^2}, \\
M^{(R)2} &= Z_{M^2} Z_{\phi} Z_{A} Z_{\phi} \Lambda^{d+z} = \Lambda^{-2} Z_{M^2}, \\
\Gamma^{(R)}_A &= Z_{\Gamma_A} Z_A Z_{\phi} Z_{\phi} \Lambda^{d+z} = \Lambda^{\varepsilon} Z_{\Gamma_A}, \\
g^{(R)} &= Z_{g^2} Z_{\phi} Z_{A} Z_{\phi} \Lambda^{2d+2z+1} = \Lambda^{-\varepsilon/2} Z_{g^2}, \\
g^{(R)2} &= Z_{g^2} Z_{\phi} Z_{A} Z_{\phi} \Lambda^{2d+3z} = Z_{g^2} Z_{\phi} Z_{A} Z_{\phi} \Lambda^{2d+3z} = \Lambda^{-\varepsilon} Z_{g^2}, \\
v^{(R)} &= Z_{v^2} Z_{\phi} Z_{A} Z_{\phi} \Lambda^{3d+3z} = \Lambda^{-\varepsilon} Z_{v^2}.
\end{align*}
$$

It is assumed that $S$ is invariant with respect to the above scale transformations in the critical point.

Close to $M^2 = 0$ ($T \simeq T_c$) the gauge field becomes massless, but the scalar field remains massive with $\mu^2 = 4I_0 v^2$. Therefore, the contribution to the renormalization is made only by the loops of the gauge field propagators. According to the separation of massive field theorem [20] the Feynman diagrams, containing the propagators of field, the mass of which is appreciably larger than the external momentum, are inversely proportional to the degree of this mass, and make a finite contribution to the
renormalization. In figure 3 some graphs giving logarithmically divergent contributions to the renormalized theory are presented.

Let us explain in detail the renormalization of $M^2$ as an example. We will limit ourselves to using the one-loop approximation, which is quite enough for the demonstration of all features of the theory. In this case the renormalization constant of $M^2$ has the form

$$Z_{M^2} \delta_{\mu\nu} \simeq M^2 \delta_{\mu\nu} - \frac{6M^2g^2}{(2\pi)^d_k+1} \int_{\Lambda_0}^{k_0} \Delta^{R}_{\mu\lambda}(k, \omega) \Delta^{K}_{\nu\omega}(k, \omega) \, dk \, d\omega$$

$$\simeq M^2 \delta_{\mu\nu} - \frac{12M^2g^2}{(2\pi)^d_k+1} \int_{\Lambda_0}^{k_0} \pi (d_k \pi d_k/2) \Gamma(1 + d_k/2) \delta_{\mu\nu} k^{d_k-1} \frac{1}{(k^4 + \omega^2)(k^2 + i\omega)} \, dk \, d\omega$$

$$= M^2 \delta_{\mu\nu} - \frac{3M^2g^2}{(2\pi)^d_k} \frac{\delta_{\mu\nu} \pi d_k/2}{\Gamma(1 + d_k/2)} \int_{\Lambda_0}^{k_0} \frac{k^{d_k-1}}{k^4} \, dk. \tag{13}$$

One can see that the integral in this expression introduces a logarithmically divergent contribution into the $M^2$ renormalization if the momentum dimension is $d_k = 4$. In this case we get the following expression for the renormalized value of the $M^2$:

$$M^{2(R)} = e^{2\xi} Z_{M^2} \simeq e^{2\xi} \left[ M^2 - 3 \frac{M^2g^2}{8\pi^2} \xi \right], \tag{14}$$

where $\xi = \ln(1/\Lambda)$ is the logarithmically divergent factor.

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Figure 4. The one-loop graph which gives the non-trivial contribution to the renormalization.

In the same way one can get other terms of the renormalized action:

\[
\begin{align*}
\mu^{2(R)} &= e^{2\varepsilon} Z_{\mu^2} \simeq e^{2\varepsilon} \left[ \mu^2 - \frac{M^2 g^2}{8\pi^2} \xi \right], \\
\Gamma_A^{(R)} &= e^{-\varepsilon} Z_{\Gamma_A} \simeq e^{-\varepsilon} \left[ \Gamma_A + 3 \frac{\Gamma_A g^4}{4\pi^2} \xi + \frac{\Gamma_A g^2}{8\pi^2} \xi \right], \\
g^{(R)} &= e^{\xi/2} Z_g \simeq e^{\xi/2} \left[ g - \frac{g^4}{8\pi^2} \xi \right], \\
g^{2(R)} &= e^{\varepsilon} Z_{g^2} \simeq e^{\varepsilon} \left[ g^2 - \frac{g^4}{4\pi^2} \xi \right], \\
v^{(R)} &= e^{\varepsilon} Z_v \simeq e^{\varepsilon} \left[ v - \frac{g^4}{8\pi^2} \xi \right],
\end{align*}
\]

where \( \varepsilon = 4 - d_k \).

One can see that the considered theory is renormalizable, since the renormalization procedure leads to the correction of only existing terms of the Lagrangian, and all renormalized parameters are logarithmically divergent when \( d_k = 4 \), which thereby is the critical dimension.

The interaction of the local magnetization fluctuations per gauge field (figure 4) plays the key part in the considered theory. This becomes clear when we consider in detail the contribution to the renormalization of the \( \Gamma_\phi \) node of the a diagram, which is shown in figure 3. This term is also interesting because the \( \Gamma_\phi \) node is proportional to the relaxation time of the local magnetization field, and determines the kinetics of the glass transition: the loop of the light field, \( A_{\mu}^a \), which is given in figure 4, makes the logarithmically divergent contribution \( \sim \ln(1/\Lambda)\delta(\omega) \). This term determines the divergence contribution of the a graph (figure 3) in which the massive field loop leads only to multiplying the logarithm by a factor of \( 4I_0 v/g^2 \):

\[
Z_{\Gamma_\phi} \approx \Gamma_\phi + \frac{4g^2 I_0 v \ln(1/\Lambda)}{\pi^2} \int_0^{t_0} e^{-4I_0 v|t|/\Gamma_\phi g^2} dt \\
= \Gamma_\phi + \frac{g^4 \ln(1/\Lambda)}{\pi^2} \left( 1 - e^{-4I_0 v|t_0|/\Gamma_\phi g^2} \right),
\]

where \( t_0 \) is the time of the observation of the system. One can see that for \( I_0 \to 0 \), or with a short observation time, \( \Gamma_\phi g^2 / 4I_0 v \gg t_0 \), this contribution becomes negligibly small, and the theory becomes nonrenormalizable. This relates to the symmetry properties of the Yang–Mills model. In this case the fluctuation-dissipation theorem (FDT) is always broken because of free energy transfer between the local magnetization modes and massless gauge field modes (Goldstone modes). However, there is also some problem in the presence
of the quenched disorder: this node becomes nonlocal in time. This formally makes the renormalization group equations non-autonomous for \( t_o \approx \Gamma_\phi g^2/4I_0v \), which violates the correctness of the renormalization procedure. The non-autonomous renormalization group has recently been considered in physical problems rather often (see, e.g., [21,22]). In order to avoid this problem we note that the renormalization group equations are autonomous in the two limit cases: \( t_o \ll \Gamma_\phi g^2/4I_0v \) and \( t_o \gg \Gamma_\phi g^2/4I_0v \); the physical solution is the result of matching these two limit cases.

At large time scales, \( t_o \gg \Gamma_\phi g^2/4I_0v \), the contribution to the \( \Gamma_\phi \) renormalization is logarithmically divergent. Hence, in the one-loop approximation the renormalization group of the model under study has the form

\[
\begin{align*}
\frac{\partial \ln(\Gamma_\phi)}{\partial \xi} &= g^4/\pi^2, \\
\frac{\partial \ln(\Gamma_A)}{\partial \xi} &= -\varepsilon + 3g^4/4\pi^2 + g^2/8\pi^2, \\
\frac{\partial \ln(M^2)}{\partial \varepsilon} &= 2 - 3g^2/8\pi^2, \\
\frac{\partial \ln(\mu^2)}{\partial \varepsilon} &= 2 - M^2g^2/8\mu^2\pi^2 \approx 2, \\
\frac{\partial \ln(g^2)}{\partial \varepsilon} &= \varepsilon - g^2/4\pi^2, \\
\frac{\partial \ln v}{\partial \varepsilon} &= \varepsilon - g^4/8\pi^2v^2.
\end{align*}
\]  

(17)

From the condition for the existence of a stable point, \( \partial \ln(g^2)/\partial \xi = 0 \), \( \partial \ln(v)/\partial \xi = 0 \), we get \( g^2 = 4\pi^2\varepsilon \), and \( v = g^2/2 \). In addition, one can see that

\[
M^2 = \frac{\alpha k_B g^2}{4v}(T - T_k) \approx e^{2z}.
\]  

(18)

At small time scales, \( t_o \ll \Gamma_\phi g^2/4I_0v \), the contribution to the \( \Gamma_\phi \) renormalization is negligibly small. In this case the RG equations have a form similar to (17) except for the first equation, which has the form

\[
\frac{\partial \ln(\Gamma_\phi)}{\partial \xi} = 0.
\]  

(19)

In order to match the solutions found in (17) and (19) it is necessary to analyze the renormalization in the \( t_o \sim \Gamma_\phi g^2/4I_0v \) region, and get a matching function, \( \Phi(\xi) \), as was done in [21,17]:

\[
\frac{\partial \ln(\Gamma_\phi)}{\partial \xi} = g^4\Phi(\xi)/\pi^2,
\]

where \( \xi \sim 1 \). The matching function, which is 1 in the case of large time scales, \( \xi \gg 1 \), and 0 in the case of small time scales, \( \xi \ll 1 \), can be chosen in the form

\[
\Phi(\xi) = 1 - \Lambda^z = 1 - \exp(-z\varepsilon),
\]

where \( z \approx 2 \) is the dynamical index [18,19]. As a result

\[
\tau_{rel} = \frac{\Gamma_\phi}{\Gamma_\phi} \propto \exp\left(\frac{2vg^2T_k}{\alpha\pi^2(T - T_k)}\right).
\]  

(20)

Hence, the critical slowing-down of all relaxation processes does occur in \( T_k \), and follows the Vogel–Fulcher–Tammann relation, which in our case was derived from microscopic reasons by means of the Keldysh technique and the critical dynamics method.

Note that if frustrations are absent, \( I_0 \to 0 \), the freezing temperature coincides with the phase transition temperature. Then the diagrams with the loops of \( \phi \) and \( \tilde{\phi} \) fields become divergent, and the system experiences the paramagnetic–ferromagnetic phase transition, which is described within the standard critical dynamics [14].

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5. Linear and nonlinear susceptibilities

It should be noted that the above form of the renormalization group leads to the FDT violation on small time scales: in the case of \( t_o \ll \Gamma \phi g^2/4I_0v \) the Keldysh part of the propagator becomes insignificant in the critical region, \( G^K \to 0 \), which violates the ratio

\[
G^K(\omega) = \frac{T}{\omega} \left\{ G^R(\omega) - G^A(\omega) \right\},
\]

(21)
corresponding to the fluctuation-dissipation theorem [15]. After the time \( t_o = \Gamma \phi g^2/4I_0v \) the system is for all practical (dynamical) purposes as if in equilibrium, and FDT holds for any time \( t > t_o \). Thus \( \Gamma \phi g^2/4I_0v \) is the characteristic scale of time homogeneity.

As the renormalization group implies dynamic self-similarity of the system in the transformation of scales, this time inhomogeneity exists in all scales, and the hierarchy of \( t_o \) is an ultrametric set. It leads to the breakdown of ergodicity in the phase space, which happens in \( T_g \): at temperatures close to \( T_g \) the geometry of the region of phase space available at a certain time becomes more and more complicated, and the system is slowed down and does not reach an equilibrium state in finite times. However, it does not remain permanently confined in any finite region within which it is in local equilibrium.

This makes it possible to contend that the freezing temperature \( T_g \) is the glass transition temperature. This subject needs more detailed investigation; below we will consider some arguments supporting this conclusion. (1) First of all, the picture of the breakdown of ergodicity stated above is in good agreement with the scenario of weak ergodicity breaking [23], which considers a phase space of a vitrificated system with ‘traps’ separated by finite barriers with a wide distribution of lifetimes. (2) In contrast to the majority of existing approaches to the theoretical description of the glass transition there was no need to introduce any specific order parameter into the given model. However, to check our conclusions it is easy to calculate, e.g., the Edwards–Anderson parameter within the suggested approach:

\[
q = \langle s(t) s(0) \rangle \sim \exp(-|t|/\Gamma_\phi(T)).
\]

(22)
As is shown above, the relaxation time, \( \Gamma_\phi \), diverges in \( T_g \). Consequently \( q \neq 0 \) at \( t \to \infty \), which tells us about the appearance of local magnetization and confirms the statement that \( T_g \) is the temperature of the glass transition. (3) It is evident from (10) that the linear susceptibility, \( \chi_L = \partial \langle \phi \rangle / \partial h \sim \mu^{-2} = g^2/4I_0v \) (\( h \) is an external source of the field \( \phi \)), is finite in \( T_g \). The correlation length, \( r_{cor} \sim \sqrt{g^2/4I_0v} \), is finite too. However, the nonlinear susceptibility, \( \chi_N = \partial^3 \langle \phi \rangle / \partial h^3 \) [24], diverges near \( T_g \): it is not difficult to check that the nonlinear susceptibility (see figure 5) encloses infinitely increasing contributions. The simplest one of them corresponds to the \( \hbar \) diagram in figure 3, and gives the divergent contribution proportional to \( (\hbar^{\xi})^{-3!/(2+3\varepsilon/8)} \). Hence in the one-loop approximation it is possible to estimate \( \chi_N \propto (T - T_g)^{-\gamma} \) for \( T \to T_g^+ \), where \( 2.5 \lesssim \gamma < 3 \) (figure 6). The lower limit is determined by the critical dimension, \( d = 4 \), the upper limit by the real dimension, \( d = 3 \). This is in good agreement with the experimental observations [24]. Thus, one can assert that the system freezes in the state with a disordered structure of the local magnetization field, and \( T_g \) is the glass transition temperature.
Figure 5. The graphic representation of the nonlinear susceptibility, $\chi_N = \langle \phi \phi \phi \phi \rangle_{k=0}$ (the external momenta are $k = 0$ in the momentum representation).

Figure 6. The qualitative presentation of the temperature function of the nonlinear susceptibility, $\chi_N \propto 1/(T - T_g)^\gamma$, and linear susceptibility, $\chi_L \propto 1/(T - T_c)$. The $\chi_L$ curve is not described by this theory at $T < T_g$, it is finite in this region.

6. Calculation of the temperature function of the heat capacity at $T \to T_g^+$

Use of the theory allows us to determine the temperature dependence of the system’s heat capacity at $T \to T_g^+$. It is convenient when carrying out the analysis to consider the correlation functions in the $(t, k)$-representation, and take into account that $\langle \bar{\phi} \bar{\phi} \rangle = \langle \bar{A} \bar{A} \rangle = 0$. As a result we obtain

$$c_p = \frac{dU}{dT} \approx -k_B \ln Z - k_B T \frac{dZ}{dT} = -k_B \ln Z$$

$$+ \frac{V k_B T \alpha g^2}{4v} \frac{\partial}{\partial T} \left[ (T - T_g) \int_p \langle \bar{A} \bar{A} \rangle_{t=0} \right]$$

$$+ V k_B T \alpha \frac{\partial}{\partial T} \left[ (T - T_c) \int_p \langle \bar{\phi} \bar{\phi} \rangle_{t=0} \right]$$

$$+ V k_B T \frac{\partial}{\partial T} \left[ \int_p \langle \bar{\phi} \bar{\phi} \rangle_{t=0} \int_p \langle \bar{A} \bar{A} \rangle_{t=0} \right]$$

$$+ \int_p \langle \bar{\phi} \bar{\phi} \rangle_{t=0} \int_p \langle \bar{A} \bar{A} \rangle_{t=0} + \int_p \langle \bar{\phi} \bar{\phi} \rangle_{t=0} \int_p \langle \phi \phi \rangle_{t=0} \right], \quad (23)$$

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Figure 7. The qualitative picture of the temperature dependence of the heat capacity near \( T_g \) which is obtained from the theory.

where the \( \int_p (\cdots) = \int_{-\infty}^{\infty} (\cdots) \ dp \) notation is introduced. Since the matter keeps disordered structure at freezing, we believe that \( Z \) weakly depends on the temperature at \( T_g \). Therefore, let us assume that first term is some constant. From (10) and (11) it is not difficult to obtain

\[
\langle \bar{\phi} \phi \rangle_t = \theta(t) \Gamma^{-1}(T) e^{-t \varepsilon_p(m)/\Gamma(T)}, \quad \langle \phi \phi \rangle_t = \frac{1}{\varepsilon_p(m)} e^{-|t| \varepsilon_p(m)/\Gamma(T)},
\]

\[
\langle \bar{A}A \rangle_t = \theta(t) \Gamma^{-1}(T) e^{-t \varepsilon_p(M)/\Gamma_A(T)}, \quad \langle AA \rangle_t = \frac{1}{\varepsilon_p(M)} e^{-|t| \varepsilon_p(M)/\Gamma_A(T)},
\]

whence

\[
\int_p \langle \bar{\phi} \phi \rangle_{t=0} = \frac{1}{\Gamma \varepsilon_p(T)}, \quad \int_p \langle \bar{A}A \rangle_{t=0} = \frac{1}{\Gamma_A(T)}.
\]

From (17) it follows that \( \Gamma_A(T) \) is weakly dependent on temperature in the critical region. Therefore, let us believe that \( \Gamma_A(T) \approx \Gamma_A = \text{const} \); consequently we obtain

\[
c_p = -k_B \ln Z + \frac{k_B T \alpha g^2}{4 \pi \Gamma_A} + k_B T \alpha \left( \frac{T - T_c}{\Gamma \varepsilon_p(m)} \right) \left( \frac{1}{\Gamma \varepsilon_p(M)} \right) + \frac{k_B T \alpha}{\Gamma \varepsilon_p(m)} \left( \frac{1}{\Gamma_A} \right) \left( \frac{1}{\Gamma \varepsilon_p(M)} \right) \left( \frac{1}{\Gamma \varepsilon_p(m)} \right).
\]

Since the system is close to \( T_g \) then \( \varepsilon_p(M) \ll \varepsilon_p(m) \). Therefore, the first integral in the last term gives the most considerable contribution to the heat capacity. Hence, one can get the qualitative form of the temperature dependence of the heat capacity close to the glass transition:

\[
c_p(T) \propto T \frac{\partial}{\partial T} \left[ e^{-C T_g/(T - T_g)} \right] = \frac{T e^{-C T_g/(T - T_g)}}{(T - T_g)^3} \left[ (C + 1) T_g - T \right],
\]

where \( C = 2 \alpha g^2 / \pi \). The graph of this expression is shown in figure 7. In this figure one can see the sharp growth of the heat capacity near the glass transition which then drops at \( T_g \), which is characteristic for vitreous systems. By using the suggested theory

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it is not difficult to give a simple qualitative explanation of this dependence: when the system’s temperature approaches $T_c$, the fluctuations are increased, and, consequently, the heat capacity increases too, like in the case of the second order phase transition; however, the quenched frustrations block this increase over scales which exceed some finite scale which is determined by these frustrations. As a result the heat capacity does not infinitely diverge in $T_g$, but has a finite maximum in $T_{\text{max}}>T_g$, and then falls. This physical picture resembles the second order phase transition in confined space.

7. Dependence of the glass transition temperature, $T_g^*$, on the cooling rate, $v$

The introduced theory makes it possible to account for the widely known and very important phenomenon of the glass transition temperature’s dependence on the cooling rate, $v$. Let us denote the effective value of the glass transition temperature, which corresponds to a cooling rate, $v$, as $T_g^*(v)$. It is known from experiments that this temperature increases when the cooling rate is increased, and this dependence has the form of a logarithmic function: $T_g^*(v) \propto 1/(\text{const} - \ln v)$.

Note, that the fluctuations have a characteristic life time, which increases at temperatures approaching $T_g$. Let us consider the fluctuation’s behavior when the system’s temperature is uniformly decreasing from some initial value, $T_o$, with a cooling rate, $v$ (K s$^{-1}$): $T(t) = T_o - vt$. The time of fluctuation relaxation uniformly increases in this case, and at some values of the temperature and ratio, for instance at $t = t^*$, it leads to the condition that in a small time period, $\delta t$, the system’s temperature decreases so that the relaxation time grows by the same quantity $\delta \tau = \delta t$. This is equivalent to transition from the damped in time fluctuations into the growing in time fluctuations. Hence, starting from this moment, the fluctuations rapidly grow with further cooling of the system, which leads to freezing of the system. Mathematically this condition can be expressed using the simple equation $d\tau/dt|_{t=t^*} = 1$. Since $\tau(T)$ and $T(t)$ have known values, it is not difficult to obtain the following expression:

$$v = \frac{x^2 T_g}{\tau_0} \exp\left(-\frac{C}{x}\right),$$

where $x = [(T_o - T_g) - vt^*]/T_K = (T_g^*(v) - T_g)/T_K$. If we find its logarithm we can get the functional dependence of the glass transition temperature on the system’s cooling rate:

$$T_g^*(v) = T_g + \frac{T_g C}{2 \ln x - \ln(v\tau_o/T_g^*)}.$$  

When the cooling rate is low, $v\tau_o \ll T_g$, and, accordingly, when $x \ll C$, we obtain

$$T_g^*(v) - T_g \approx \frac{T_g C}{\ln(T_g/v\tau_o)},$$

which is in good agreement with experiment results [25]. If the cooling rate is high, $v \to T_g/\tau_o$, then we come to

$$T_g^*(v) - T_g \approx (v\tau_o T_g)^{1/2}.$$
Figure 8. The graphic representation of $\langle \phi \phi \rangle_t$ in the ‘one-loop’ approximation: (a) is the term corresponding to the relaxation of the free field of magnetization, (b) is the term which contains the self-action of the magnetization field by means of the gauge field.

Figure 9. The dependence of $\langle \phi \phi \rangle_t$ on $\ln(t)$: (a) is the contribution of $\langle \phi \phi \rangle_t^0 = G^R(t)$, which is the Debye-relaxation; (b) is the contribution of the second term which is given by the cooperative effects; (c) is the sum of the first and second terms.

8. Time-dependence correlation function and boson peak

Let us derive the time-dependence correlation function $\langle \phi \phi \rangle_t$ near the glass transition function. We confine ourselves to the ‘one-loop’ approximation, since we believe that this is enough for demonstration of the basic properties of the dependence (figure 8). Using (16) we can write

$$\langle \phi \phi \rangle_{t,k=0} \simeq G^K(t) + \frac{4g^2I_0v\xi}{\pi^2} \int G^R(t_1)e^{-4I_0v|t_2-t_1|/\Gamma g^2}G^A(t-t_2)dt_1dt_2$$

$$\simeq G^K(t) + \frac{g^2I_0v\xi}{\Gamma g^2} \ln \left( \frac{4vT_g}{\alpha k_B g^2(T-T_g)} \right) \int_0^t G^K(t_1)e^{-4I_0v|t-t_1|/\Gamma g^2}dt_1. \quad (24)$$

In figure 9 the curve of the derived function is presented. One can see that the first peak in this curve, which is given by the first term of the expression, corresponds to the high-temperature system in which any collective effects are absent. The wide long-term peak is described by the second term, and corresponds to the cooperative processes, which arise because of self-action of the magnetization field by means of the gauge field induced by frustrations. It is natural that this peak is directly associated with the critical slowing-

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down induced by the collectivization of motion and corresponds to $\alpha$-relaxation. The superposition of these terms gives the resulting correlation function, which is characteristic for vitreous systems. Besides, this curve can contain a characteristic gap and a following jump between the basic peaks, which is commonly observed in vitreous systems and is termed the 'boson peak'.

9. Conclusions

The glass transition, according to the presented theory, can be regarded as a phase transition interrupted because of the premature critical slowing-down of one of the degrees of freedom caused by the frustrations. We have described the critical behavior of the frustrated 3D Heisenberg model in terms of the gauge theory and shown that this critical behavior is attributed to vanishing of the effective mass of the gauge field, which occurs at the temperature $T_g > T_c$. As a result, the system freezes in a state with a disordered structure. It is necessary to note once again that the gauge symmetry breakdown is not spontaneous in the considered model. It is induced by the random sources of the gauge field, which is connected with the geometrical frustration. The symmetry breakdown leads to the appearance of the gauge field mass. However, the gauge field mass becomes small when the temperature reaches $T_g$ ($T_g > T_c$). Close to $T_g$ the fluctuations of the local magnetization field are small, but the gauge field fluctuations dramatically increase, which leads to the critical slowing-down of the gauge field dynamics, as well as the dynamics of the local magnetization field. This slowing-down freezes the disordered structure of the local magnetization field with finite correlation length and susceptibility, and can be interpreted as a glass transition. Thus, the glass transition represents the critical slowing-down of fluctuation growth by disorder-induced frustrations, which complies with the 'frustration-limited domain theory' [10] and correlates with other approaches [8,13].

The simplest model system was described above, but the results can be generalized for another vitreous systems, since they are caused by the general dynamic properties of frustrated systems, and are not connected with their microscopic features. It should also be noted that it is not only quenched randomness that leads to the appearance of the gauge field mass. The disorder in a molecular system can also be induced by geometrical frustrations. The geometrical frustrations connected with the topological features of the condensed matter structure can produce a massive gap of $A_\mu$. For example, in the model of the defected states of the orientation order [9] the mass of the gauge field is determined by the summed topological charge, $Q$, which is connected with the curvature of the hypersphere, corresponding to the locally preferable structure. Besides, the theory describes all the general properties of the glass transition very well. The expression for the temperature dependence of the heat capacity near to the glass transition, which is derived in this paper, is qualitatively in good agreement with experimental data. The derived dependence of the glass transition temperature on the cooling rate also conforms with known experimental data for manifold vitreous systems. On top of this, this theory reproduces the characteristic form of the $\langle \phi \phi \rangle_t$ correlation function dependence on time, and explains the appearance of the boson peak on this curve. This allows us to hope that the suggested approach for the description of the glass transition will be applicable to the description of the glass transition in a wider range of vitreous systems.
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