NEW APPROACH TO LOCAL PARAMETRIX PROBLEMS IN RIEHMANN–HILBERT THEORY

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ABSTRACT. In this work we study local parametrix problems in the setting of the steepest descent method for Riemann–Hilbert problems. In particular, we focus on general conditions under which one can approximate the exact solution by an explicitly solvable model solution. Our findings show that, provided one can obtain certain a priori estimates on the exact solution, the existence of the local parametrix solution can be deduced without the need to write it out explicitly in terms of special functions, as it is usually done. The a priori estimates are also obtained in a concrete example, namely the Riemann–Hilbert problem associated with orthogonal polynomials on \([-1, 1]\).

1. INTRODUCTION

This paper deals with the analysis of local Riemann-Hilbert (R-H) problems (also called ‘parametrix problems’) in the setting of the nonlinear steepest descent method (\([3]\)). Such problems occur frequently in R-H theory and can either result in the leading asymptotics (\([7, 8]\)), or contribute to higher order corrections (\([2]\) Chapter 7, \([9, 13]\)). Interestingly, even in the second case one has to deal with such local problems to obtain rigorous leading asymptotics. Hence, the natural question arises, whether the construction of the explicit parametrix solution can be avoided completely in cases when higher order corrections are not needed. Before tackling this question, let us consider as an example the Airy parametrix problem.

1.1. Airy parametrix problem. The Airy parametrix problem occurs in the setting of inverse scattering and orthogonal polynomials (\([2]\) Chapter 7, \([13]\)). The general setting is as follows. One starts with an initial global R-H problem in \(\mathbb{C} \cup \{\infty\}\), meaning that the boundary conditions are fixed at infinity. A series of conjugation and deformation steps leads to an equivalent R-H problem with two types of jump matrices. Ones that have a simple explicit form and ones that converge exponentially fast to the identity matrix, except in the vicinity of finitely many points. Around these ‘oscillatory points’ one has to solve a local parametrix R-H problem, which can be transformed to a global R-H problem through a change of variables. To avoid confusion, we shall throughout this paper refer to the global R-H problem after the conjugation and deformation steps as the ‘global R-H problem’, while to the local (even in its global form) as the ‘parametrix R-H problem’. In the case of the Airy parametrix problem the contour is given by four rays \(\Sigma_i\), \(i = 1, \ldots, 4\) in the complex plane that meet at the origin as is shown below.
To the above contour we associate the following piecewise constant jump matrix \( v \):

\[
\begin{align*}
\begin{cases}
1 & -i \\
0 & 1 \\
1 & 0 \\
i & 1
\end{cases}, & \quad k \in \Sigma_1, \\
\begin{cases}
0 & -i \\
-1 & 0 \\
1 & 0 \\
i & 1
\end{cases}, & \quad k \in \Sigma_2, \\
\begin{cases}
0 & -i \\
-1 & 0 \\
1 & 0 \\
i & 1
\end{cases}, & \quad k \in \Sigma_3, \\
\begin{cases}
1 & 0 \\
i & 1
\end{cases}, & \quad k \in \Sigma_4.
\end{align*}
\]

The R-H problem with data \((v, \Sigma)\) is stated as follows:

- \( P(z) \) is holomorphic in \( \mathbb{C} \setminus \Sigma \) with \( \Sigma := \bigcup_{i=1}^{4} \Sigma_i \),
- \( P_+(k) = P_-(k)v(k), \quad k \in \Sigma \),
- \( P(z) \to \frac{1}{2\sqrt{\pi}} z^{-\sigma_3/4} e^{\pi i/12} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} e^{-2/3z^{3/2}\sigma_3}, \quad z \to \infty. \)

The above limit is assumed to be uniform in all directions, and all roots have a branch cut on \( \mathbb{R}_- \). The jump matrices satisfy the cyclic condition, i.e. their cyclic product when taking into account the contour orientation evaluates to the identity matrix:

\[
\begin{pmatrix}
1 & -i \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
i & 1
\end{pmatrix}^{-1}
\begin{pmatrix}
0 & -i \\
-1 & 0
\end{pmatrix}^{-1}
\begin{pmatrix}
1 & 0 \\
i & 1
\end{pmatrix}^{-1} = \mathbb{I}.
\]

This implies that solutions without the normalization at infinity correspond bijectively to entire matrix-valued functions (12). Hence, it is enough to specify the...
form of the solution in e.g. $\Omega_1$ through an entire matrix-valued function $E(z)$:

$$P(z) = \begin{cases} 
E(z), & z \in \Omega_1, \\
E(z) \begin{pmatrix} 1 & 0 \\
-i & 1 \end{pmatrix}, & z \in \Omega_2, \\
E(z) \begin{pmatrix} 0 & i \\
i & 1 \end{pmatrix}, & z \in \Omega_3, \\
E(z) \begin{pmatrix} 1 & i \\
0 & 1 \end{pmatrix}, & z \in \Omega_4. 
\end{cases}$$

Taking the second derivative of the asymptotic behaviour of $E(z)$ as $z \to \infty$ and then considering only the highest order term, we can expect that

$$E''(z) \sim zE(z), \ z \to \infty.$$  

This heuristics leads us to the Airy function, which is a special entire solution of the Schrödinger equation with a linear potential

$$-\psi'' + z\psi = 0$$

with asymptotics for $z \to \infty$ given by

$$\text{Ai}(z) = \frac{1}{2\sqrt{\pi}} \left( z^{-1/4} + O(z^{-7/4}) \right) e^{-2/3z^{3/2}}, \ |\arg(z)| < \pi,$$

where all roots have branch cuts on $\mathbb{R}_-$ and the convergence at infinity is uniform in any closed sector not containing $\mathbb{R}_-$. Similarly we need also the derivative of the Airy function given by

$$\text{Ai}'(z) = -\frac{1}{2\sqrt{\pi}} \left( z^{1/4} + O(z^{-5/4}) \right) e^{-2/3z^{3/2}}, \ |\arg(z)| < \pi,$$

with the same constraints on the convergence rate at infinity.

From the above asymptotic expansions of $\text{Ai}(z)$ and $\text{Ai}'(z)$ we can try to guess the explicit form of $E(z)$:

$$E(z) = \left( \begin{array}{c}
\text{Ai}(z) \\
\text{Ai}'(z) \\
\rho \text{Ai}(\rho z) \\
\rho^2 \text{Ai}'(\rho^2 z) 
\end{array} \right) e^{\pi i \gamma_3/12},$$

with $\rho := e^{2\pi i / 3}$. It turns out that while the asymptotics of the Airy functions hold uniformly only away from the negative real axis, the asymptotics of $P(z)$ given by (3) will be uniform in all directions and have the required form. This follows from the linear relation

$$\text{Ai}(z) + \rho \text{Ai}(\rho z) + \rho^2 \text{Ai}(\rho^2 z) = 0$$

between three different solution of the Schrödinger equation (3) which is of second order. Note that $\text{Ai}(e^{2\pi i \varphi} z)$ has an asymptotic formula at infinity which holds uniformly away from the ray with angle $\pi - \varphi$.  

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1.2. **Outline of this paper.** In the next section we show that the existence of the parametrix solution can in fact be proven without knowing its explicit form, provided certain a priori $L^2$-estimates of the exact solution to the global R-H problem and a regular enough model solution is known. Our method uses the connection between R-H problems and singular integral equations and relies on results from [13]. In section 3 an application of our approach is presented for the case of orthogonal polynomials on the interval $[-1, 1]$. The corresponding conjugation and deformation steps, as well as the explicit parametrix construction involving Bessel functions can be found in [9]. In the discussion section we elaborate on the advantages and limitations of our approach and mention future challenges related to obtaining the a priori $L^2$-estimates. The appendix contains a short technical argument which shows that a technical condition assumed in the main text is in fact no restriction in the applications of interest.

2. **Approximating solutions of R-H problems**

Consider a R-H problem with data $(v, \Sigma)$, i.e. we are looking for a $d \times d$ matrix-valued function $S$ such that

- $S(z)$ is analytic for $z \in \mathbb{C} \setminus \Sigma$,
- $S_+(k) = S_-(k)v(k), \quad k \in \Sigma$,
- $S(z) = I + O(z^{-1}), \quad z \to \infty$.

The general question we are trying to answer is the following: Given a solution $N$ of a model R-H problem with data $(v_N, \Sigma)$, is it a good approximation for $S$? We refer to $S$ as the 'exact solution' and to $N$ as the 'model solution'. First we need to specify what we mean by $N$ being a good approximation of $S$. In many mathematical problems, in particular those that involve scattering theory, we are interested in the first Taylor term of $S$ at infinity, i.e. the quantity $\psi$ given by

$$S(z) = I + \frac{\psi}{z} + O(z^{-2}), \quad z \to \infty.$$  

(10)

Hence we would like $\psi_N$ defined analogously, i.e.

$$N(z) = I + \frac{\psi_N}{z} + O(z^{-2}), \quad z \to \infty,$$

(11)

to be close to $\psi$. On the other hand, in the case of orthogonal polynomials pointwise estimates are more relevant. Hence we would like $S(z)$ to be close to $N(z)$ for $z \in \mathbb{C} \setminus \Sigma$.

Usually the R-H problem for $S$ and $N$ depends continuously on some auxiliary continuous or discrete parameter $t \in T$ which we call 'time' (later it is the degree $n$ of orthogonal polynomials). In the case of scattering theory we want to prove that

$$\psi(t) = \psi_N(t) + o(1), \quad t \to \infty,$$

(12)

while in the case of orthogonal polynomials that

$$S(z, t) = N(z, t) + o(1), \quad z \in \mathbb{C} \setminus \Sigma, \quad t \to \infty,$$

(13)

where the error term is a measure of the accuracy of the approximation.
2.1. **Singular integral formulation of R-H problems.** To approach the problems posed in the last section, we need to reformulate a R-H problem as an singular integral equation. To this end, let us define the Cauchy operator $C_\Sigma$ associated to an oriented contour $\Sigma$:

$$C_\Sigma : L^2(\Sigma) \to O(\mathbb{C} \setminus \Sigma), \quad f \to C_\Sigma(f)(z) := \frac{1}{2\pi i} \int_{\Sigma} \frac{f(k)}{k - z} dk.$$  

The only requirement needed for the above expression to be well defined, is that $(k - z)^{-1}$ is in $L^2(\Sigma)$ for every $z \in \mathbb{C} \setminus \Sigma$, or equivalently $k^{-1}$ is in $L^2(\Sigma)$. Given some further regularity assumptions on $\Sigma$ which are fulfilled in most applications ([10]), we can define two bounded operators given by:

$$C_{\Sigma}^{\pm} : L^2(\Sigma) \to L^2(\Sigma), \quad f \mapsto C_{\Sigma}^{\pm}(f)(k) := \lim_{z \to k^{\pm}} C_\Sigma(f)(z),$$

where the limit is assumed to be nontangential in which case it exists a.e. on $\Sigma$. Here the $+$($-$) sign corresponds to taking the left (right) limit to the oriented contour $\Sigma$.

We now turn to a bijection between solutions of R-H problems and solutions of certain singular integral equations. These results can be found in [15]. Define the operator $C_\Sigma^w$ associated to a R-H problem with data $(v, \Sigma)$ given by

$$C_\Sigma^w : L^2(\Sigma) \to L^2(\Sigma), \quad f \mapsto C_\Sigma^w(fw),$$

where $w := v - I \in L^2(\Sigma) \cap L^\infty(\Sigma)$. Here we abuse notation by denoting by $L^p(\Sigma)$ the space of matrix-valued functions with entries in $L^p(\Sigma)$. The operator $C_\Sigma^w$ is evaluated entrywise for matrix inputs. From its boundedness and the assumption $w \in L^\infty(\Sigma)$, it follows that $C_\Sigma^w$ is a bounded operator.

Let $M$ be a solution to the R-H problem with data $(v, \Sigma)$ (with normalization $M(z) \to I$ as $z \to \infty$). Then we can define

$$\Phi(k) := M_-(k) - I.$$  

If $\Phi \in L^2(\Sigma)$, then $\Phi$ satisfies the following singular integral equation:

$$(1 - C_\Sigma^w)\Phi = C_\Sigma^w(w).$$

Conversely a solution $\Phi$ of [15] gives rise to a solution of the R-H problem via

$$M := I + C_\Sigma^w(\Phi + I).$$

Hence, there is a bijective correspondence between solutions of the R-H problem with data $(v, \Sigma)$ such that $M_-(k) - I \in L^2(\Sigma)$ and solutions $\Phi$ of [15] (see also [13]). Moreover, as existence implies uniqueness for the class of R-H problems we are interested in ([2] Chapter 3, [4]), the same is true for the singular integral equation [15].

2.2. **Residual R-H problem.** In the following we make the additional assumption that $\det v = I$ which implies that the determinants of $S$ and $N$ are constant equal to 1 in $\mathbb{C}$, by Liouville’s Theorem. Furthermore, the jump contour $\Sigma$ should be time-independent. We denote by $\Gamma$ a smooth clockwise oriented Jordan curve such that $\Sigma \cap \Gamma$ consists of only finitely many intersection points.
As $\det N \equiv 1$, we can define a new matrix-valued function $R := SN^{-1}$. We call $R$ the 'residual solution' and it satisfies a R-H problem with data $(v_R, \Sigma)$ where

$$v_R := N_- vv_N^{-1} N_-^{-1}.$$  

From the last section we know that it can be also written in integral form:

$$R(z) = \mathbb{I} + \frac{1}{2\pi i} \int_{\Sigma} \frac{R_-(k)(v_R(k) - \mathbb{I})}{k - z} dk$$

$$= \mathbb{I} + \frac{1}{2\pi i} \int_{\Sigma} \frac{S_-(k)(v(k)v_N^{-1}(k) - \mathbb{I})N_-^{-1}(k)}{k - z} dk.$$  

The quantity of interest is the $L^1(\Sigma)$-norm of the integrand, i.e.

$$\|S_-(k)(v(k)v_N^{-1}(k) - \mathbb{I})N_-^{-1}(k)\|_{L^1(\Sigma)}$$

for $z \in \Gamma$ and the integral is taken with respect to $k$ which is denoted by the subscript. Let us assume that (21) is of order $\varepsilon(t)$ uniformly for $z \in \Gamma$, where $\varepsilon : T \to \mathbb{R}_+$ and $T$ is the parameter space (either $\mathbb{R}_+$ or $\mathbb{N}$). Observe that in this case we have

$$R^{\pm 1}(k) = \mathbb{I} + O(\varepsilon(t)), \quad k \in \Gamma.$$  

Define two matrix-valued function $M^I, M^{II}$ by

$$M^I(z) = \begin{cases} S(z), & z \in \Gamma_e \\ \mathbb{I}, & z \in \Gamma_i. \end{cases}$$

$$M^{II}(z) = \begin{cases} N(z), & z \in \Gamma_e \\ \mathbb{I}, & z \in \Gamma_i. \end{cases}$$

where $\Gamma_e, (\Gamma_i)$ denotes the exterior (interior) of the Jordan curve $\Gamma$. Again, $M^I, M^{II}$ satisfy R-H problems with the jump on $\Gamma$ taking the form:

$$M^I_+(k) = M^I_-(k)v^I(k)$$

with

$$v^I(k) = \begin{cases} S(k), & k \in \Gamma \\ v(k), & k \in \Sigma \cap \Gamma_e, \end{cases}$$

and

$$M^{II}_+(k) = M^{II}_-(k)v^{II}(k)$$

with

$$v^{II}(k) = \begin{cases} N(k), & k \in \Gamma \\ v_N(k), & k \in \Sigma \cap \Gamma_e. \end{cases}$$

Assuming for the moment that

$$\|v - v_N\|_{L^\infty(\Sigma \cap \Gamma_e)} \to 0,$$

for $t \to \infty$, we see that the jump matrices of $M^I$ and $M^{II}$ converge uniformly to each other, provided we can show that

$$\|S - N\|_{L^\infty(\Gamma)} \to 0$$
for $t \to \infty$. Note that $S - N = (R - I)N$ and we can therefore conclude
\begin{equation}
\|S - N\|_{L^\infty(\Gamma)} = O(\|N\|_{L^\infty(\Gamma)} \varepsilon(t)).
\end{equation}
In the next section we shall see that given (29), it is sufficient that
\begin{equation}
\|N\|_{L^\infty(\Gamma)}(t) \to 0, \quad t \to \infty,
\end{equation}
for $N$ to be a good approximation of $S$ in $\Gamma_c$.

2.3. Riemann–Hilbert problems in applications. We now consider specific conditions under which (32) can be achieved. In the case of scattering theory and orthogonal polynomials ([1], [9], [13]), the contour $\Sigma$ can be written as a union of $\Sigma_{\text{mod}}$ and $\Sigma_{\text{exp}}$. For $v_N$ we have
\begin{equation}
v_N(k) = \begin{cases} v(k), & k \in \Sigma_{\text{mod}}, \\ I, & k \in \Sigma_{\text{exp}}. \end{cases}
\end{equation}
Hence, $N$ is the exact solution of the global R-H problem provided the jump on $\Sigma_{\text{exp}}$ is ignored. On $\Sigma_{\text{exp}}$ the jump matrix $v$ converges uniformly exponentially fast to the identity matrix, except in the vicinity of a finite number of points $\kappa \in K \subset \Sigma_{\text{exp}}$ (we often have $K \subset \Sigma_{\text{exp}} \cap \Sigma_{\text{mod}}$ ([9], [4])).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Jump contour occurring in the R-H formulation of the KdV equation with steplike initial data. The dotted line is $\Sigma_{\text{exp}}$, while the solid is $\Sigma_{\text{mod}}$.}
\end{figure}

For each $\kappa \in K$ the local behaviour of the jump matrix is given by
\begin{equation}
|v(k) - I| = O(e^{-ct|k-\kappa|^{1/2}}),
\end{equation}
for some positive constant $c$. Meanwhile, $v_N$ and hence also the model solution $N$ is either independent or periodic in $t$ and is uniformly bounded for $z \in \mathbb{C} \setminus \Sigma$, except in the vicinities of the points $\kappa \in K$ where it can have fourth-root singularities, i.e.
\begin{equation}
|N(k)| = O(|k - \kappa|^{-1/4}),
\end{equation}
(the same is true for $N^{-1}$ as the determinant of $N$ is unity). In particular, we get
\begin{equation}
\|N\|_{L^\infty(\Gamma)} = O(1)
\end{equation}
provided $K$ and $\Gamma$ are disjoint. Before we turn to (21), we make the additional assumption that $\Sigma_{\text{exp}}$ and $\Sigma_{\text{mod}}$ are bounded and contained in $\Gamma_i$:
\begin{equation}
\Sigma = \Sigma_{\text{mod}} \cup \Sigma_{\text{exp}} \subset \Gamma_i.
\end{equation}
While this is indeed the case for orthogonal polynomials on $[-1,1]$, for the R-H formulation of inverse scattering transforms we have that $\Sigma^{exp}$ is in general unbounded \cite{1,13}. However, in the appendix we provide an argument that shows that the following methods can also be applied in this case.

Let us now apply the Cauchy–Schwarz inequality to (21):

$$\|S^{-}(k)(v(k) - I)N^{-1}(k)\|_{L^1(\Sigma^{exp})}$$

$$\leq \|S^{-}\|_{L^2(\Sigma^{exp})} \left\| \frac{(v(k) - I)N^{-1}(k)}{k - z} \right\|_{L^2(\Sigma^{exp})}.$$  

Note that we made use of $v_N(k) = I$ for $k \in \Sigma^{exp}$ and as $v(k)v_N(k)^{-1} = I$ for $k \in \Sigma^{mod}$ we only need to integrate over $\Sigma^{exp}$. As already mentioned, we require $\Sigma^{exp}$ to be fully contained in $\Gamma_i$, in fact we need

$$\text{dist}(\Sigma^{exp}, \Gamma) > 0.$$  

The last condition allows us to ignore the singular term $(k - z)^{-1}$ as $k \in \Sigma^{exp}$ and $z \in \Gamma$. Now from the assumptions \cite{34, 35} on $v$ and $N$ it follows that

$$\|(v - I)N^{-1}\|_{L^2(\Sigma^{exp})} = O(t^{-1/2}),$$

where the main contributions come from the points $\kappa \in K$. We see that in order to guarantee that (21) goes to 0, it is sufficient to make the assumption that

$$\|S^{-}\|_{L^2(\Sigma^{exp})} = O(1).$$

We call estimates of the form \cite{70} 'a priori estimates', as they have to be established before an approximation for $S$ is known. The last section is dedicated to proving these estimates in the case of the R-H problem associated with orthogonal polynomials on the interval $[-1,1]$. In such cases we get

$$\|S - N\|_{L^\infty(\Gamma)} = O(t^{-1/2}),$$
which implies
\[ \|S - N\|_{L^p(\Gamma)} = O(t^{-1/2}) \]
for any \( p \geq 1 \), as \( \Gamma \) has finite length.

Let us now reconsider our solutions \( M^I \) and \( M^{II} \) of the R-H problems with data \((S, \Gamma)\) and \((N, \Gamma)\) respectively. From the uniform boundedness of \( N \) away from \( \mathcal{K} \), it follows that \( w^{II} := N - \mathbb{I} \) and hence because of \( (42) \) \( w^{I} := S - \mathbb{I} \) are in \( L^\infty(\Gamma) \subseteq L^2(\Gamma) \). Thus, we can associate to \( M^I, M^{II} \) the corresponding unique solutions \( \Phi^I, \Phi^{II} \) of
\[ (44) \quad (1 - C_\Gamma^w w^{I}) \Psi = C_\Gamma^w (w^{I}) \]
and
\[ (45) \quad (1 - C_\Gamma^w w^{II}) \Psi = C_\Gamma^w (w^{II}) \]
respectively. The key observation comes from \cite{15} which tells us that the singular integral operators on the left-hand sides,
\[ (46) \quad 1 - C_\Gamma^w \quad \text{and} \quad 1 - C_\Gamma^w \]
are Fredholm of index 0 (also compare with \cite{13}). Hence, existence and uniqueness of the solution of \( (44) \) and \( (45) \) implies the invertibility of both operators. In particular, we can write
\[ (47) \quad \Phi^I = (1 - C_\Gamma^w)^{-1} C_\Gamma^w (w^{I}) \]
and
\[ (48) \quad \Phi^{II} = (1 - C_\Gamma^w)^{-1} C_\Gamma^w (w^{II}) \]
We now make use of the following Theorem taken from \cite{13}.

**Theorem 2.1.** Let \( \Gamma \) be a contour of finite length such that the Cauchy operators \( C_\pm^\Gamma \) are bounded from \( L^2(\Gamma) \) to itself. Define two R-H problems \((v_i, \Gamma), i = 1, 2\), where solutions are assumed to be normalized to the identity matrix as \( z \to \infty \). Furthermore, introduce a parameter \( t \in T \) where \( T \) is either \( \mathbb{R}_+ \) or \( \mathbb{N} \). The jump matrix \( v_1 \) can depend on \( t \), while \( v_2 \) is assumed to be stationary such that
\[ (49) \quad w_1(t), w_2 \in L^2(\Gamma) \cap L^\infty(\Gamma), \]
where \( w_1(t) := v_1(t) - \mathbb{I}, w_2 := v_2 - \mathbb{I} \). Moreover, assume that
\[ (50) \quad \|v_1(t) - v_2\|_{L^\infty(\Gamma)} = O(\varepsilon(t)) \]
where \( \varepsilon(t) \) is positive and tends to 0 as \( t \to \infty \).

If \( 1 - C_{w_2}^\Gamma \) is invertible, then so is \( 1 - C_{w_1(t)}^\Gamma \) for \( t \) large enough and the unique solutions \( \Phi_1(t) \) and \( \Phi_2 \) of
\[ (51) \quad (1 - C_{w_1(t)}^\Gamma) \Psi = C_{\Gamma}^w (w_1(t)) \]
and
\[ (52) \quad (1 - C_{w_2}^\Gamma) \Psi = C_{\Gamma}^w (w_2) \]
satisfy
\[ (53) \quad \|\Phi_1(t) - \Phi_2\|_{L^2(\Gamma)} = O(\varepsilon(t)). \]
Furthermore, the unique solutions $M_1$ and $M_2$ of the corresponding R-H problems satisfy for any $l \in \mathbb{N}$ the estimate

$$M_1(z) - M_2(z) = \sum_{i=1}^{l} \frac{\psi_i}{z} + O(z^{-l-1}), \quad z \to \infty,$$

where $\psi_i$ are matrices that depend only on $t$ but not on $z$, with

$$\|\psi_i\|_{\infty} = O(\varepsilon(t))$$

and

$$M_1(z) = M_2(z) + O(\varepsilon(t))$$

locally uniformly for $z \in \mathbb{C} \setminus \Gamma$.

**Remark 2.2.** The simplification of the above Theorem compared to the one in [13] comes from the finite length of $\Gamma$. This implies that the $L^\infty(\Gamma)$-norm of $f \in L^\infty(\Gamma)$ bounds all moments of $f$ in all $L^p(\Gamma)$-spaces for $p \in [1, \infty]$. Hence we only require one convergence rate $\varepsilon(t)$.

We are now in a position to state the following result:

**Theorem 2.3.** Suppose $S, N, v, v_N, \Sigma^{mod}, \Sigma^{exp}, \Gamma$ satisfy the assumptions stated in this section, i.e. (33), (34), (35), (36), (37), (39) and (41) hold. Then

$$S(z) - N(z) = \sum_{i=1}^{l} \frac{\psi_i}{z} + O(z^{-l-1}), \quad z \to \infty$$

where $\psi_i$ are matrices that depend only on $t$ but not on $z$ with

$$\|\psi_i\|_{\infty} = O(t^{-1/2}).$$

and

$$S(z) - N(z) = O(t^{-1/2})$$

locally uniformly for $z \in \mathbb{C} \cap \Gamma_e$.

The next lemma tells us that a priori $L^2$-estimates of solutions to a family of R-H problems can be uniformly extended to larger contours and are stable under contour deformations which are common in the nonlinear steepest descent method.

**Lemma 2.4.** Let $\Sigma$ and $\Gamma$ be oriented contours in $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, which are disjoint except for a finite number of points. Assume that the operators $C_{\Sigma, \Gamma}^{\pm}$ are well-defined and bounded on $L^2(\Sigma \cup \Gamma)$. Let $f \in O(\hat{\mathbb{C}} \setminus \Sigma)$ be given, such that $f$ is continuous at infinity in the case that $\infty \in \Sigma$ and the $\pm$ limits on $\Sigma$ exist in the usual sense satisfying $f_+ - f_- \in L^2(\Sigma)$. Then

$$\|f - f(\infty)\|_{L^2(\Gamma)} \leq C(\Gamma \cup \Sigma)\|f_+ - f_-\|_{L^2(\Sigma)}$$

for some positive constant $C(\Gamma \cup \Sigma)$ independent of $f$.

**Proof.** Note that because of the conditions on $f$, it follows from the properties of the Cauchy integral operator that

$$f - f(\infty) = C_{\Sigma}(f_+ - f_-) = C_{\Sigma, \Gamma}^{\pm}(f_+ - f_-).$$
where the last equality is true because \( f_+ = f_- \) on \( \Gamma \setminus \Sigma \), as \( f \in \mathcal{O}(\hat{\mathcal{C}} \setminus \Sigma) \). Hence we conclude
\[
\|f - f(\infty)\|_{L^2(\Gamma)} \leq \|\mathcal{C}_\pm\|_{L^2(\Sigma, \mathbb{R}^d)}\|f_+ - f_-\|_{L^2(\Sigma)} = \|\mathcal{C}_\pm\|_{L^2(\Sigma, \mathbb{R}^d)}\|f_+ - f_-\|_{L^2(\Sigma)}
\]
which shows that we can choose \( C(\Sigma \cup \Gamma) := \|\mathcal{C}_\pm\|_{L^2(\Sigma, \mathbb{R}^d)} \).

\[\]

2.4. The vector case. Similar arguments work in the case that the initial R-H problem is not stated for a matrix \( S \), but rather for a vector \( s \). Hence, we are looking for a vector-valued function \( s \), such that

- \( s(z) \) is analytic for \( z \in \mathbb{C} \setminus \Sigma \),
- \( s_+(k) = s_-(k) \) for \( k \in \Sigma \),
- \( s(z) = s_\infty + O(z^{-1}) \) as \( z \to \infty \).

where we assume that \( s_\infty \neq 0 \) is time-independent. However, we still need a matrix-valued model solution \( N \), which is normalized to the identity matrix at infinity. One has to be careful about uniqueness, which is automatically implied for matrix solutions with jump matrices having determinant 1 ([2] Chapter 3, [4]), but has to be proven separately in the vector case ([4]). For the following argument only uniqueness and uniform pointwise invertibility of the model matrix solution is needed.

As before we define a vector-valued function \( r := sN^{-1} \) which can be written in the integral form
\[
r(z) = s_\infty + \frac{1}{2\pi i} \int_{\Sigma} \frac{s_-(k)(v(k)\nu^{-1}_N(k) - \mathbb{I})N^{-1}_-(k)\nu_N(k)}{k - z} \, dk.
\]

Control of the integrand gives us as before an estimate
\[
r(k) - s_\infty = O(\varepsilon(t)), \quad k \in \Gamma,
\]
where \( \varepsilon(t) \) is positive and decays to 0. The matrices \( M^S, N^N \) are now substituted for vectors \( m^S, m^N \) given by:
\[
m^S(z) = \begin{cases} 
  s(z) & \text{for } z \in \Gamma_e, \\
  s_\infty & \text{for } z \in \Gamma_i.
\end{cases}
\]
\[
m^N(z) = \begin{cases} 
  s_\infty N(z) & \text{for } z \in \Gamma_e, \\
  s_\infty & \text{for } z \in \Gamma_i.
\end{cases}
\]

Note however that unlike the matrix case, the jump matrices of \( m^S, m^N \) are not yet uniquely specified. At least for \( m^N \) we have a canonical choice for the jump matrix given by
\[
m^N(k) = m^N(k)N(k), \quad \text{for } k \in \Gamma.
\]

The above R-H problem has the unique solution \( m^N \), which follows from Liouville’s Theorem after multiplying any given solution by \( N^{-1} \) (\( N \) is assumed to be invertible) in \( \Gamma_e \). We would like to find a jump matrix for \( m^S \) which is close in the \( L^\infty(\Gamma) \)-norm to \( N \). Define \( \delta := s - s_\infty N \). Then we know that \( \|\delta\|_{L^\infty(\Gamma)} = O(\|N\|_{L^\infty(\Gamma)} \varepsilon(t)) \).

As we assumed that \( s_\infty \neq 0 \), we can write down a matrix \( D(k) \) corresponding to the
linear map $f \mapsto \langle f, s_\infty \rangle \delta(k)/\|s_\infty\|_2$. Then we have $\|D\|_{L^\infty(\Gamma)} = O(\|N\|_{L^\infty(\Gamma)}\varepsilon(t))$ such that

$$\delta(k) = s_\infty D(k), \quad k \in \Gamma.$$  

Hence, $m^s$ satisfies the jump condition

$$m_+(k) = m^s(k)(N(k) + D(k)), \quad k \in \Gamma.$$  

Subsequently, for the jump matrices we have as before

$$\|N + D\| - N\|_{L^\infty(\Gamma)} = O(\|N\|_{L^\infty(\Gamma)}\varepsilon(t))$$

From this it follows that for $t$ large enough (i.e. $\varepsilon(t)$ small enough) the R-H problem for $m^s$ has indeed a unique solution, i.e. the corresponding singular integral operator is invertible (see Theorem [2]), provided $\|N\|_{L^\infty(\Gamma)}$ is bounded. The rest of the analysis follows analogously as in the matrix case.

3. Application to Orthogonal Polynomials on $[-1, 1]$

3.1. Riemann–Hilbert formulation of orthogonal polynomials. In this section we assume that the measure $d\mu$ on $[-1, 1]$, associated to a series of orthogonal polynomials, is absolutely continuous, i.e. can be written as

$$d\mu(x) = h(x)dx, \quad x \in [-1, 1]$$

for some real-valued function $h$. Furthermore we assume that $h$ satisfies the following inequality

$$0 < c < h(x) < C < +\infty, \quad x \in [-1, 1],$$

for some constants $c, C$. Consider now the following R-H problem:

For any $n \in \mathbb{N}_+$ find a $2 \times 2$ matrix-valued function $X = X^{(n)}$ on $\mathbb{C} \setminus [-1, 1]$, such that

- $X(z)$ is analytic in $\mathbb{C} \setminus [-1, 1]$,
- $X_+(x) = X_-(x)\begin{pmatrix} 1 & h(x) \\ 0 & 1 \end{pmatrix}, \quad x \in (-1, 1),$
- $X(z) = \begin{pmatrix} (2z)^n & 0 \\ 0 & (2z)^{-n} \end{pmatrix}(\mathbb{I} + O(z^{-1})), \quad z \to \infty.$

The usual argument shows that if a solution exists with at most square integrable singularities at $x = \pm 1$, it is necessary unique ([2], Chapter 3). The following result explains the connection between these R-H problems and orthogonal polynomials:

**Theorem 3.1.** (Fokas, Its, Kitaev ([5, [6])) The unique solution with square integrable singularities at $x = \pm 1$ of the above R-H problem is given by

$$X^{(n)}(z) = \begin{pmatrix} p_n(z) & C^{-1,1}(p_n h)(z) \\ \eta_{n-1} p_{n-1}(z) & \eta_{n-1} C^{-1,1}(p_{n-1} h)(z) \end{pmatrix}$$

where $p_n(z)$ is the $n$-th orthogonal polynomial with leading coefficient $2^n$ and

$$\eta_n := -\pi i \|p_n\|_{L^2([-1, 1]; d\mu(x))}^{-2}.$$  

Usually the R-H problem for orthogonal polynomials is normalized without the factors $2^{\pm n}$ at infinity, in which case the $p_n(z)$ would be just the monic orthogonal polynomials. The next theorem found explains this discrepancy ([13]):
\textbf{Theorem 3.2.} Assume the weight \( h(x) \) on \([-1, 1]\) satisfies the Szegö condition, i.e.
\begin{equation}
\int_{-1}^{1} \frac{\log h(x)}{\sqrt{1 - x^2}} \, dx > -\infty.
\end{equation}
Then
\begin{equation}
\lim_{n \to \infty} \| p_n \|_{L^2([-1,1],d\mu(z))} = \sqrt{\pi} \exp \left( \frac{1}{2\pi} \int_{-1}^{1} \frac{\log h(x)}{\sqrt{1 - x^2}} \, dx \right).
\end{equation}

Hence assuming the Szegö condition which follows from \((72)\), the \( L^2([-1,1], d\mu(x)) \)-norm of \( p_n(z) \) converges as \( n \) tends to infinity. In particular, the resulting uniform boundedness of the norm is similar to the a priori \( L^2 \)-estimates needed for our novel approach for solving local parametrix problems. We state the following corollary:

\textbf{Corollary 3.3.} Assume \( h \) satisfies \((72)\). Then the entries of the unique solution \( X^{(n)} \) are uniformly bounded in \( L^2([-1,1]) \) as \( n \) tends to infinity. Here \( L^2([-1,1]) \) denotes the \( L^2 \)-space on \([-1,1]\) with the Lebesgue measure.

\textit{Proof.} For the entries in the first column we have by \((76)\) uniform boundedness in \( L^2([-1,1], h(x)dx) \). From condition \((72)\) we obtain uniform boundedness of \( p_n \) and \( p_n h \) in \( L^2([-1,1]) \). The statement follows from the boundedness of the operator \( \mathcal{C}([-1,1]) \) on \( L^2([-1,1]) \) and the convergence of \( \eta_n \). \qed

3.2. \textbf{Steepest descent method for orthogonal polynomials on \([-1,1]\).} Let us apply the results from the last section to the strong asymptotics of orthogonal polynomials on the interval \([-1,1]\). In particular, we consider a special case found in \([9]\), namely \( h \) is real analytic in some neighbourhood of \([-1,1]\) where its magnitude satisfies constraint \((72)\).

In \([9]\) the authors perform jump matrix factorizations and contour deformations to arrive at the following R-H problem on a 'lense' \( \Sigma := \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \):

Find a \( 2 \times 2 \) matrix-valued function \( S \) such that
\begin{itemize}
\item \( S(z) \) is analytic in \( \mathbb{C} \setminus \Sigma \),
\item \( S_+(z) = S_-(z) v(z) \) for \( z \in \Sigma \) with:
\begin{equation}
v(z) = \begin{cases}
1 & \text{for } z \in \Sigma_1 \cup \Sigma_3, \\
\frac{1}{h(z)^{-1} \varphi(z)^{-2n}} & \text{for } z \in \Sigma_2 = [-1,1],
\end{cases}
\end{equation}
\item \( S(z) = \mathbb{I} + O(z^{-1}) \), as \( z \to \infty \),
\end{itemize}
with at most square integrable singularities at \( z = \pm 1 \). Here
\begin{equation}
\varphi(z) := z + \sqrt{z^2 - 1}
\end{equation}
maps \( \mathbb{C} \setminus [-1,1] \) biholomorphically to the exterior of the unit disc. In particular \( |\varphi(z)| > 1 \) and \( |\varphi(z)| \to 1 \) as \( z \) approaches \([-1,1]\). Hence, looking at the jump matrix \((77)\), we identify \( z = \pm 1 \) as the oscillatory points.
The solution $S$ is related to the solution $X$ from Theorem 3.1 via

\begin{equation}
S(z) = \begin{cases}
X(z)\varphi(z)^{-n\sigma_3}, & z \in \Omega_\infty, \\
X(z)\varphi(z)^{-n\sigma_3} \begin{pmatrix} 1 & 0 \\ -h(z)^{-1}\varphi(z)^{-2n} & 1 \end{pmatrix}, & z \in \Omega_u \cup \Omega_l.
\end{cases}
\end{equation}

We already know that the $L^2([-1,1])$-norm of the entries of $X_k$ are uniformly bounded as $n$ tends to infinity. As $|\varphi_k(x)| = 1$ for $x \in [-1,1]$, the same is true for $X\varphi_{-n\sigma_3}^{-1}$. Our extension Lemma 2.4 implies that the uniform $L^2$-boundedness of $X\varphi_{-n\sigma_3}$ remains true on the boundary of the lense. Finally, note that $|\varphi(z)| \geq 1$ on $\Sigma_{1,3}$ and so we get uniform $L^2(\Sigma)$-boundedness of the matrix entries of $S$. Hence we have shown the a priori estimates needed for our approach.

The corresponding model problem has only a jump on $\Sigma^2 = \Sigma^{mod} = [-1,1]$, i.e.

\begin{equation}
N_+(x) = N_-(x) \begin{pmatrix} 0 & h(x) \\ -h(x)^{-1} & 0 \end{pmatrix}, \quad x \in [-1,1].
\end{equation}

The explicit solution given in [9] has the form

\begin{equation}
N(z) = D_\infty^{\sigma_3} \begin{pmatrix} a(z) + a(z)^{-1} & a(z) - a(z)^{-1} \\ 2 & -2i \\
2i & a(z) + a(z)^{-1} \end{pmatrix} D(z)^{-\sigma_3},
\end{equation}

where $D(z)$ is the Szegö function associated with $h$:

\begin{equation}
D(z) := \exp \left( \frac{\sqrt{z^2 - 1}}{2\pi} \int_{-1}^{1} \frac{\log h(x)}{\sqrt{1 - x^2} (z - x)} \, dx \right), \quad z \in \mathbb{C} \setminus [-1,1],
\end{equation}

\begin{equation}
D_\infty := \lim_{z \to \infty} D(z) = \exp \left( \frac{1}{2\pi} \int_{-1}^{1} \frac{\log h(x)}{\sqrt{1 - x^2}} \, dx \right)
\end{equation}

and

\begin{equation}
a(z) = \left( \frac{z - 1}{z + 1} \right)^{1/4}
\end{equation}
with a branch cut on $[-1, 1]$ and $a(\infty) = 1$. It follows from ([11], Chapter 4) that $D(z)$ is bounded around $z = \pm 1$. Hence, we see that our model solution has the required behaviour around the oscillatory points $z = \pm 1$. Moreover,

$$|\varphi(z)| \leq 1 + |z^2 - 1|^{1/2} + O(z \mp 1) = e^{|z^2 - 1|^{1/2}} + O(z \mp 1)$$

around $z = \pm 1$. Hence also condition (34) is satisfied. For $\Gamma$ we can take any smooth Jordan curve encircling $\Sigma$. From Theorem 2.3 it follows that

$$S(z) - N(z) = \sum_{i=1}^{l} \frac{\psi_i}{z} + O(z^{-l-1}), \quad z \to \infty,$$

where $\psi_i$ are matrices that depend only on $n$ but not on $z$ with

$$\|\psi_i\|_\infty = O(n^{-1/2})$$

and

$$S(z) = N(z) + O(n^{-1/2})$$

locally uniformly in $\mathbb{C} \cap \Gamma_n$. As the lense and $\Gamma$ can be made arbitrary narrow, this result even holds locally uniformly in $\mathbb{C} \setminus [-1, 1]$.

4. Discussion

We have shown that an explicit construction of the parametrix solution can be avoided in the case that a priori estimates for the exact solution are known. However, the error terms obtained this way are in general worse than the actual error terms, which is the result of the a priori estimates containing very limited information on the local structure of the exact solution around the oscillatory point. In [9] the authors show

$$S(z) = N(z) + O(n^{-1})$$

uniformly away from $[-1, 1]$, while our approach only gives $O(n^{-1/2})$. Furthermore the explicit solution of the parametrix problem in terms of Bessel functions at infinity leads to an asymptotic expansion of the form

$$S(z) = N(z) \left( \mathbb{I} + \sum_{k=1}^{l} \frac{R_k(z)}{n^k} + O(n^{-l-1}) \right)$$

for $l \in \mathbb{N}_+$, which then translates into a similar asymptotic expansion of quantities related to orthogonal polynomials. While this part of the analysis is not performed in this paper, it should be clear that our method would result in asymptotic formulas for the same quantities with errors of order $O(n^{-1/2})$ instead of the full expansion in terms of $n^{-1}$. We nonetheless believe that our approach works faster and is less technical in cases where only leading asymptotics are of interest and a priori $L^2$-estimates can be obtained. A future challenge would be proving these estimates in different settings, e.g. long time asymptotics of nonlinear PDEs solvable via scattering theory or strong asymptotics of orthogonal polynomials on the real line.
APPENDIX A. CUTTING $\Sigma^{exp}$’S TIPS

We now consider the setting of section 2.3, except that $\Sigma^{exp}$ is unbounded. Choose a bounded domain $V \subseteq \mathbb{C}$ containing all the oscillatory points $\kappa \in K$ and define
\begin{equation}
\Sigma^{exp}_0 := \Sigma^{exp} \setminus V, \quad \Sigma^{exp}_\kappa := \Sigma^{exp} \cap V.
\end{equation}
The crucial assumption is that on $\Sigma^{exp}_0$ the matrix $w$ and its moments decay exponentially fast, i.e.
\begin{equation}
\|w(k)k^i\|_{L^p(\Sigma^{exp})} = O(e^{-ct})
\end{equation}
for $c$ positive, $i = 0, \ldots, l - 1$ and $p = [1, \infty]$.

Let us now introduce the following R-H problem with data $(v_K, \Sigma)$ given by
\begin{equation}
v_K(k) = \begin{cases}
    v(\kappa), & k \in \Sigma^{mod} \cup \Sigma^{exp}_0 \\
    \mathbb{I}, & k \in \Sigma^{exp}_\kappa
\end{cases}
\end{equation}
Then
\begin{equation}
\|v_K - v_N\|_{L^p(\Sigma)} = \|w_K\|_{L^p(\Sigma^{exp})} = O(e^{-ct})
\end{equation}
where $w_K := v_K - \mathbb{I}$ and $p = 2, \infty$. Hence, as before we can use Theorem B.1. in \textcircled{13} to conclude that for $t$ large enough a unique solution $K$ exists and
\begin{equation}
\|K - N\|_{L^2(\Sigma)} = O(e^{-ct}),
\end{equation}
\begin{equation}
K(z) - N(z) = \sum_{i=1}^l \frac{\psi_i}{z} + O(z^{-l-1}), \quad z \to \infty,
\end{equation}
where $\psi_i$ are matrices that depend only on $t$ but not on $z$ with
\begin{equation}
\|\psi_i\|_{\infty} = O(e^{-ct})
\end{equation}
and
\begin{equation}
K(z) = N(z) + O(e^{-ct})
\end{equation}
locally in $\mathbb{C} \setminus (\Sigma^{mod} \cup \Sigma^{exp}_0)$. Using the solution $K$ instead of $N$ as the model solution, we arrive as in \textcircled{38}, at the problem of estimating
\begin{equation}
\left\| \frac{(v(k) - \mathbb{I})K^{-1}(k)}{k - z} \right\|_{L^2(\Sigma^{exp})}
\end{equation}
for $z \in \Gamma$. As $\Sigma^{exp}$ is bounded, we can choose $\Gamma$ such that $\text{dist}(\Sigma^{exp}_\kappa, \Gamma) > 0$ and thus we can ignore the $(k - z)^{-1}$ term. We write
\begin{equation}
\left\| (v - \mathbb{I})K^{-1} \right\|_{L^2(\Sigma^{exp})} \leq \left\| (v - \mathbb{I})(K^{-1} - N^{-1}) \right\|_{L^2(\Sigma^{exp})} + \left\| (v - \mathbb{I})N^{-1} \right\|_{L^2(\Sigma^{exp})}.
\end{equation}
As $v$ is uniformly bounded on $\Sigma^{exp}_\kappa$ and the determinant of $K$ and $N$ is 1, we have that
\begin{equation}
\left\| (v - \mathbb{I})(K^{-1} - N^{-1}) \right\|_{L^2(\Sigma^{exp})} \leq C \left\| K - N \right\|_{L^2(\Sigma^{exp})} = O(e^{-ct}),
\end{equation}
for some positive constant $C$. The second term $\left\| (v - \mathbb{I})N^{-1} \right\|_{L^2(\Sigma^{exp})}$ can be approximated as before to be of order $O(t^{-1/2})$. Hence we can substitute $K$ for $N$ in Theorem (2.3). Together with \textcircled{93} and \textcircled{97} we arrive at Theorem 2.3 for $S$ and $N$. 

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References

[1] K. Andreiev, I. Egorova, T.-L. Lange, G. Teschl, Rarefaction waves of the Korteweg-de Vries equation via nonlinear steepest descent, J. Differential Equations 261, 5371–5410 (2016).

[2] P. Deift, Orthogonal Polynomials and Random Matrices: A Riemann–Hilbert Approach, Courant Lecture Notes 3, Amer. Math. Soc., Rhode Island 1998.

[3] P. Deift and X. Zhou, A steepest descent method for oscillatory Riemann–Hilbert problems. Asymptotics for the MKdV equation, Ann. of Math. 137, 295–368 (1993).

[4] I. Egorova, M. Porkowski, G. Teschl, On vector and matrix Riemann-Hilbert problems for KdV shock waves. arXiv:1907.09792

[5] A. S. Fokas, A. R. Its, A. V. Kitaev, Discrete Painlevé and their appearance in quantum gravity, Comm. Math. Phys. 142(2): 313–344 (1991).

[6] A. S. Fokas, A. R. Its, A. V. Kitaev, The isomonodromy approach in the theory of two-dimensional quantum gravitation, Uspekhi Mat. Nauk, 45:6(276), 135–136 (1990); Russian Math. Surveys, 45:6, 155–157 (1990).

[7] K. Grunert, G. Teschl, Long-time asymptotics for the Korteweg–de Vries equation via nonlinear steepest descent, Math. Phys. Anal. Geom. 12, 287–324 (2009).

[8] H. Krüger, G. Teschl, Long-time asymptotics of the Toda lattice for decaying initial data revisited, Rev. Math. Phys. 21:1, 61–109 (2009).

[9] A. B. J. Kuijlaars, K. T-R McLaughlin, W. Van Assche, M. Vanlessen, The Riemann–Hilbert approach to strong asymptotics for orthogonal polynomials on [-1, 1], Advances in Mathematics 188(2): 337–398 (2001).

[10] J. Lenells, Matrix Riemann-Hilbert problems with jumps across Carleson contours, Monatsh. Math. 186:1, 111-152 (2018).

[11] N. I. Muskhelishvili, Singular Integral Equations, P. Noordhoff Ltd., Groningen, 1953.

[12] W.-Y. Qiu, R. Wong, Asymptotic expansions for Riemann–Hilbert problems, Analysis and Applications, Vol 6, No 3, 269–298 (2009).

[13] M. Piorkowski, Parametrix problem for the Korteweg–de Vries equation with steplike initial data. arXiv:1908.11340

[14] G. Szegő, Orthogonal Polynomials, 4th Edition, Colloquium Publications, Vol. 23, Amer. Math. Soc., Providence, RI 1975.

[15] X. Zhou, The Riemann–Hilbert problem and inverse scattering, SIAM J. Math. Anal. 20–4, 966-986 (1989).

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