JACKSON THEOREM AND MODULUS OF CONTINUITY FOR
UNITARY REPRESENTATIONS OF LIE GROUPS.

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Abstract. For a strongly continuous unitary representation of a Lie group
$G$ in a Hilbert space $H$ we consider an analog of the Laplace operator $L$
and use it to define subspaces of Paley-Wiener vectors $PW_{\sigma}(L^{1/2})$. It allows to
introduce notion of the best approximation $E(\sigma, f)$ of a general vector in $H$
by Paley-Wiener vectors of a certain bandwidth $\sigma > 0$. The existence of a
group representation in $H$ is used to introduce a family of moduli of continuity
$\Omega^r(s, f)$, $r \in \mathbb{N}$, $s > 0$, of vectors in $H$. The main objective of the paper is to
prove the so-called Jackson-type estimate $E(\sigma, f) \leq C\Omega^r(\sigma^{-1}, f)$.

1. Introduction and Main Results

One of the main goals of the classical harmonic analysis is to describe relations
between frequency content of a function and its smoothness. A famous result in
this direction is the so-called Jackson Theorem for functions in $L_2(\mathbb{R})$:
(1.1) $\inf_{g \in PW_{\sigma}(\mathbb{R})} \| f - g \| = \| f - \mathcal{P}_\delta f \| = \mathcal{E}(\sigma, f) \leq C\omega^r(\sigma^{-1}, f)$, $\sigma > 0$, $r \in \mathbb{N},$
where the Paley-Wiener space $PW_{\sigma}(\mathbb{R})$, $\sigma > 0$, is the space of functions in $L_2(\mathbb{R})$
whose Fourier transform has support in $[-\sigma, \sigma]$, $\mathcal{P}$ is the orthogonal projection of
$L_2(\mathbb{R})$ onto $PW_{\sigma}(\mathbb{R})$, and
$\omega^r(s, f) = \sup_{0 \leq \tau \leq s} \| (T(\tau) - I)^r f \|_2$, $T(\tau)f(x) = f(x + \tau),$
is the modulus of continuity. Similar estimate also holds true in the case of one
dimensional torus $\mathbb{T}$ if one will replace $PW_{\sigma}$ by the space of trigonometric polynomials degree $\leq n$. In the case of $\mathbb{R}^d$ and $\mathbb{T}^d$ one defines corresponding modules of
continuity by using one-parameter translation groups along single coordinates
(1.2) $T_j(\tau)f(x_1, ..., x_j, ..., x_d) = f(x_1, ..., x_j + \tau, ..., x_d), \ f \in L_p$, $1 \leq p \leq \infty,$
whose infinitesimal operators are partial derivatives $\partial/\partial x_j$, $1 \leq j \leq d$. The corre-
sponding spaces $PW_{\sigma}$ and spaces on trigonometric polynomials can be introduced
in terms of the Laplace operator $\Delta = \partial^2_1 + ... + \partial^2_d$.

The goal of this paper is to develop a unified approach to Jackson-type estimates
in a space of a strongly continuous unitary representation of a Lie group. Namely,
we consider an appropriate notion of Paley-Wiener vectors and a modulus of con-
tinuity in a space of unitary representation of a Lie group $G$ in a Hilbert space $H$
and prove an analog of the Jackson inequality (1.1) in a such general setting. Note

1991 Mathematics Subject Classification. 43A85, 41A17;
Key words and phrases. Non-compact symmetric space, Laplace-Beltrami operator, entire
functions of exponential type, Bernstein and Nikol'skii inequalities.
that an approach to a generalization of the classical approximation theory to a Banach space in which a strongly continuous bounded representation of a Lie group is given was outlined without complete proofs in [14]-[20]. The problem of developing approximation theory in non-Euclidean settings attracted attention of many mathematicians and in particular was recently considered in [2, 3, 4, 5, 8, 11, 12, 13, 21].

In the present paper we show (see also [14]-[21]) that the existence of a strongly continuous unitary representation of a Lie group in a Hilbert space $H$ implies that $H$ is equipped with a set of operators $D_1, D_2, ..., D_d$ which generate strongly continuous one-parameter groups of unitary operators $T_1(t), T_2(t), ..., T_d(t), \ t \geq 0$ and the following properties hold:

1. The operator $L = -D_1^2 - ... - D_d^2$ is a non-negative self-adjoint operator in $H$ and the domain of $L^{r/2}$, $r \in \mathbb{N}$, with the graph norm $\|f\| + \|L^{r/2}f\|$ coincides with the space $H_r$ with the norm

$$\|f\|_r = \|f\|_H + \sum_{1 \leq i_1, ..., i_r \leq d} \|D_{i_1}...D_{i_r}f\|_H, \ r \in \mathbb{N}. \quad (1.3)$$

2. For every $f \in H^1$ the following formula holds true

$$D_jTf(t_1, ..., t_d) = \sum_{k=1}^{d} \zeta_{j}^k(t)\partial_k Tf(t_1, ..., t_d), \ f \in H^1, \quad (1.4)$$

where $t = (t_1, ..., t_d)$ is in the standard open unit ball $U$ in $\mathbb{R}^d$, a vector-valued function $Tf : \mathbb{R}^d \rightarrow H$ is defined as

$$Tf(t_1, t_2, ..., t_d) = T_1(t_1)T_2(t_2)...T_d(t_d)f,$$

functions $\zeta_{j}^k(t)$ belong to $C^\infty(U), \ \partial_k = \frac{\partial}{\partial t_k}$. \quad (1.5)

The fact that the formula (1.4) holds for any strongly continuous unitary representation is proved in Appendix. The first property on this list allows to introduce Paley-Wiener vectors which are used as the apparatus for approximation. The formula (1.4) allows for construction of an abstract version of the Hardy-Steklov smoothing operator. This construction is used to establish equivalence of a modulus of continuity (see below) and the K-functional for the pair $(H, H')$ (section 3).

**Definition 1.** $PW_\sigma(L) \subset H$ denote the image space of the projection operator $1_{[0, \sigma]}(L)$ to be understood in the sense of Borel functional calculus for self-adjoint operators.

It is obvious that the space $PW_\sigma(L)$ is a linear closed subspace in $H$ and the space $\bigcup_{\sigma > 0} PW_\sigma(L)$ is dense in $H$.

The following theorem contains generalizations of several results from classical harmonic analysis (in particular the Paley-Wiener theorem). It follows from our results in [16], [19], [20], [6].

**Theorem 1.1.** The following statements hold:

1. (Bernstein inequality) $f \in PW_\sigma(L)$ if and only if $f \in H^\infty = \bigcap_{k=1}^\infty H^k$, and the following Bernstein inequalities holds true

$$\|L^{s/2}f\|_H \leq \sigma^s \|f\|_H \quad \text{for all } s \in \mathbb{R}_+; \quad (1.6)$$
(2) (Paley-Wiener theorem) \( f \in PW_\sigma (L) \) if and only if for every \( g \in H \) the scalar-valued function of the real variable \( t \mapsto \langle e^{itL} f, g \rangle \) is bounded on the real line and has an extension to the complex plane as an entire function of the exponential type \( \sigma; \)

Next, we define the best approximation

\[
\mathcal{E} (\sigma, f) = \inf_{g \in PW_\sigma (L)} \| f - g \| = \| f - P_\sigma f \| ,
\]

where \( P_\sigma \) is the orthogonal projector of \( H \) onto \( PW_\sigma (L) \). We also use the Schrödinger group \( e^{itL} \) to introduce the modulus of continuity

\[
\omega_L^r (t, f) = \sup_{0 \leq \tau \leq t} \| (e^{itL} - I)^r f \| .
\]

In section 2 in Theorem 2.1 we prove the following Jackson-type estimate

\[
\mathcal{E} (\sigma, f) \leq C \omega_L^r (\sigma^{-1}, f).
\]

An analog of a Sobolev space is introduced as the space \( H^r \) of vectors in \( H \) for which the following norm is finite

\[
\| f \|_{H^r} = \| f \|_H + \sum_{k=1}^r \sum_{1 \leq j_1, \ldots, j_k \leq d} \| D_{j_1} \ldots D_{j_k} f \|_H,
\]

where \( r \in \mathbb{N} \), \( f \in H \). By using the closed graph theorem and the fact that each \( D_i \) is a closed operator in \( H \), one can show that this norm is equivalent to the norm

\[
\| f \|_r = \| f \|_H + \sum_{1 \leq j_1, \ldots, j_r \leq d} \| D_{j_1} \ldots D_{j_r} f \|_H, \quad r \in \mathbb{N}.
\]

Note, that for the \( K \)-functional

\[
K \left( s^r, f, H, D\left( L^{r/2} \right) \right) = \inf_{g \in D\left( L^{r/2} \right)} \left( \| f - g \|_H + s^r \| g \|_{D\left( L^{r/2} \right)} \right),
\]

one has \( 1 \)

\[
\omega_L^r (s^r, f) \leq CK \left( s^r, f, H, D\left( L^{r/2} \right) \right) \leq C \left( \omega_L (s^r, f) + s^r \| f \| \right)
\]

We show in section 3 that the space \( D\left( L^{r/2} \right) \) with the graph norm coincides with the space \( H^r \) and their norms are equivalent, i.e.

\[
\| L^{r/2} f \| \sim \| f \| + \| L^{r/2} f \| \sim \| f \| + \sum_{1 \leq j_1, \ldots, j_r \leq d} \| D_{j_1} \ldots D_{j_r} f \|,
\]

where \( \Lambda = I + L = I - D_1 - \ldots - D_d \).

Using the groups \( T_1, \ldots, T_d \), \( d \geq n = \text{dim} M \), we define the modulus of continuity by the formula

\[
\Omega^r (s, f) = \sum_{1 \leq j_1, \ldots, j_r \leq d} \sup_{0 \leq \tau_{j_1} \leq s} \ldots \sup_{0 \leq \tau_{j_r} \leq s} \| (T_{j_1} (\tau_{j_1}) - I) \ldots (T_{j_r} (\tau_{j_r}) - I) f \|_H,
\]

where \( f \in H \), \( r \in \mathbb{N} \), and \( I \) is the identity operator in \( H \). It is shown in Theorem 4.4 that for the \( K \)-functional

\[
K (s^r, f, H, H^r) = \inf_{g \in H^r} (\| f - g \|_H + s^r \| g \|_{H^r}),
\]
the following double inequality holds
\[ c\Omega(s, f) \leq K(s', f, H, H') \leq C (\Omega(s, f) + s\|f\|_{H'}) , \]
where positive constant \( c, C \) are independent on \( f \). Thus from (1.16) the main result of the paper follows
\[ \mathcal{E}(\sigma, f) \leq C \{ \Omega^-(\sigma^{-1}, f) + \sigma^{-\tau}\|f\| \} , \]
Remark 1.2. It is important to notice that since \( \Omega(f, \tau) \) cannot be of order \( o(\tau^+) \)
when \( \tau \to 0 \) (unless \( f \) is invariant), the behavior of the right-hand side in (1.17) is
determined by the first term when \( \sigma \to \infty \). In particular, if \( f \in H' \), then due to
the inequality (see below)
\[ \Omega(s, f) \leq s^{-k}\Omega(s, f) \leq C \|f\|_{H'} , \]
one has the best possible estimate
\[ \mathcal{E}(\sigma, f) \leq C \Omega(s, f) \leq C \sigma^{-\tau}\|f\|_{H'} . \]

Example 1. A compact homogeneous manifold. The situation on a unit
sphere is typical for at least all two-point homogeneous compact manifolds. Consider
the unit sphere
\[ S^n = \{ x = (x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \|x\|^2 = x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = 1 \} . \]
Let \( e_1, \ldots, e_{n+1} \) be the standard orthonormal basis in \( \mathbb{R}^{n+1} \). If \( SO(n+1) \) and \( SO(n) \)
are the groups of rotations of \( \mathbb{R}^{n+1} \) and \( \mathbb{R}^n \) respectively then \( S^n = SO(n+1)/SO(n) \).
On \( S^n \) we consider vector fields \( X_{ij} = x_i \partial_{x_j} - x_j \partial_{x_i}, i < j \), which are generators
of one-parameter groups of rotations \( \exp tX_{ij} \in SO(n+1) \) in the plane \( (x_i, x_j) \).
These groups are defined by the formulas for \( \tau \in \mathbb{R} \),
\[ \exp \tau X_{ij} \cdot (x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_i \cos \tau - x_j \sin \tau, \ldots, x_i \sin \tau + x_j \cos \tau, \ldots, x_{n+1}) . \]
Clearly, there are \( d = \frac{1}{2}n(n-1) \) such groups. Let \( T_{ij}(\tau) \) be a one-parameter group
which is a representation of \( \exp \tau X_{ij} \) in the space \( L_2(S) \). It acts on \( f \in L_2(S^n) \) by
the following formula
\[ T_{ij}(\tau)f(x_1, \ldots, x_{n+1}) = f(x_1, \ldots, x_i \cos \tau - x_j \sin \tau, \ldots, x_i \sin \tau + x_j \cos \tau, \ldots, x_{n+1}) . \]
The infinitesimal operator of this group will be denoted as \( D_{ij} \). The operator \( L = -\sum_{i,j} D_{ij}^2 \)
is the regular Laplace-Beltrami operator on \( S^n \) and spaces \( PW_\sigma(S^n) \)
are comprised of appropriate linear combinations of spherical harmonics.

Remark 1.3. This example explains reasons why \( d \) is typically greater than \( n = \dim M \).
In this case it happens because vector fields \( D_{ij} \) can vanish along low dimen-
sional submanifolds. For example on \( S^2 \subset \mathbb{R}^3 \) one needs three fields \( X_{12}, X_{13}, X_{23} \)
since they vanish at the poles \( (0, 0, \pm 1), (0, \pm 1, 0), (\pm 1, 0, 0) \) respectively.

Example 2. A non-compact homogeneous manifold.
Consider the upper half of the hyperboloid
\[ \mathbb{H}^+ = \{ x = (x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : -x_1^2 - x_2^2 - \ldots - x_n^2 + x_{n+1}^2 = 1, x_{n+1} > 0 \} . \]
Let \( e_1, \ldots, e_{n+1} \) be the standard orthonormal basis in \( \mathbb{R}^{n+1} \). If \( SH(n+1) \) is the
group of hyperbolic rotations which means it preserves the form
\[ [x, y] = -x_1 y_1 - \ldots - x_n y_n + x_{n+1} y_{n+1} \]
then \( \mathbb{H}^+ = SH(n+1)/SO(n) \). On \( \mathbb{H}^+ \) we consider the vector fields \( X_{ij} = x_j \partial_{x_i} - x_i \partial_{x_j}, i < j < n + 1 \), which generate euclidean rotation groups in the planes
(x_i, x_j), i < j < n + 1, and the fields X_{i,n+1} = x_{n+1} \partial_{x_i} + x_i \partial_{x_{n+1}} which are generators of the hyperbolic groups of rotations in the planes (x_i, x_{n+1}). These groups are defined by the formulas for $\tau \in \mathbb{R}$,

$$\exp \tau X_{i,j}(x_1, ..., x_{n+1}) = (x_1, ..., x_i \cosh \tau - x_j \sinh \tau, ..., x_i \sinh \tau + x_j \cosh \tau, ..., x_{n+1}),$$

$$\exp \tau X_{i,n+1} \cdot (x_1, ..., x_{n+1}) = (x_1, ..., x_i \cos \tau - x_{n+1} \sin \tau, ..., x_i \sin \tau + x_{n+1} \cos \tau).$$

Strictly continuous one-parameter groups of operators $T_{i,j}(\tau)$ which are representations of $\exp \tau X_{i,j}$ in the space $L_2(\mathbb{H}_n^+)$ can be used to construct corresponding modulus of continuity $\Omega(\sigma, f)$. Their infinitesimal operators $D_{i,j}$ are just operators $X_{i,j}$ in the space $L_2(\mathbb{H}_n^+)$ and $L = -\sum_{i,j=1}^{n+1} D_{i,j}^2$ is an elliptic self-adjoint non-negative operator in $L_2(\mathbb{H}_n^+)$ which has continuous spectrum. As well as we know, the spectral resolution of this operator is unknown. However, the abstract Definition 2 and notion of best approximation (1.4) still make sense.

**Example 3. Schrödinger representation of the Heisenberg group.**

The $(2n + 1)$-dimensional Heisenberg group $\mathbb{H}_{2n+1}$ has a unitary representation in the space $L_2(\mathbb{R}^n)$

$$T(p, q, x) = e^{i(t + (p, x))} f(x + p), \quad p, q, x \in \mathbb{R}^n, \quad t \in \mathbb{R}.$$

One can consider the following set of infinitesimal operators where $i = \sqrt{-1}$:

$$D_j = \partial_j, \quad 1 \leq j \leq n; \quad D_j = ix_j, \quad n + 1 \leq j \leq 2n; \quad D_{2n+1} = i.$$

In this case every $T_j(\tau), 1 \leq j \leq n$, is a translation (1.3) along variable $x_j$,

$$T_j(\tau)f(x_1, ..., x_n) = e^{i\tau x_j}f(x_1, ..., x_n), \quad n + 1 \leq j \leq 2n,$$

and $T_{2n+1}(\tau)f(x_1, ..., x_n) = e^{i\tau}f(x_1, ..., x_n)$. The operator $L$ is the shifted $n$-dimensional linear oscillator

$$L = -\Delta + |x|^2 + 1,$$

where

$$\Delta = \sum_{j=1}^{n} \partial_j^2, \quad |x|^2 = \sum_{j} x_j^2, \quad x = (x_1, ..., x_n).$$

It is known that the spectrum of this operator is discrete and its eigenfunctions are products of one-dimensional Hermite functions. One can easily describe corresponding Paley-Wiener spaces and to construct corresponding modulus of continuity by using groups of operators $T_j, 1 \leq j \leq 2n + 1.$

2. **Jackson inequality for the Schrödinger group of a self-adjoint operator**

Let $L$ be a self-adjoint operator in a Hilbert space $\mathbb{H}$ and $\omega_L^m$ is defined as in (1.8). For the modulus of continuity introduced in (1.8) the following inequalities hold:

$$\omega_L^m(s, f) \leq s^k \omega_L^{m-k}(s, L^k f)$$

and

$$\omega_L^m(as, f) \leq (1 + a)^m \omega_L^m(s, f), \quad a \in \mathbb{R}^+.$$

The first one follows from the identity

$$(e^{isL} - 1)^k f = \int_0^s ... \int_0^s e^{i(\tau_1 + ... + \tau_k)L} L^k f d\tau_1 ... d\tau_k,$$
Moreover, for any $E$ because

\[ E \leq (2.5) \]

\[ f \in D \]

Using this formula we can extend the abstract function \( e \) as

\[ \omega_1^1 (s_1 + s_2, f) \leq \omega_1^1 (s_1, f) + \omega_1^1 (s_2, f) \]

which is easy to verify.

**Theorem 2.1.** There exists a constant $c = c(m) > 0$ such that for all $\omega > 0$ and all $f$ in $H$

\[ (2.4) \]

\[ E(\sigma, f) \leq c \omega_1^m (1/\sigma, f). \]

Moreover, for any $1 \leq k \leq m$ there exists a $C = C(m, k) > 0$ such that for any $f \in D(L^k)$ one has

\[ (2.5) \]

\[ E(\sigma, f) \leq C \sigma^m \omega_1^{m-k} (1/\sigma, L^k f), \quad 0 \leq k \leq m. \]

**Proof.** First, we note that if $h \in L_1(\mathbb{R})$ is an entire function of exponential type $\sigma$ then for any $f \in L_2(M)$ the vector

\[ g = \int_{-\infty}^{\infty} h(t) e^{itL} f dt \]

belongs to $PW_\sigma(L)$. Indeed, for every real $\tau$ we have

\[ e^{i\tau L} g = \int_{-\infty}^{\infty} h(t) e^{i(t+\tau)L} f dt = \int_{-\infty}^{\infty} h(t - \tau) e^{itL} f dt. \]

Using this formula we can extend the abstract function $e^{i\tau L} g$ to the complex plane as

\[ e^{izL} g = \int_{-\infty}^{\infty} h(t - z) e^{itL} f dt. \]

One has

\[ \|e^{izL} g\| \leq \|f\| \int_{-\infty}^{\infty} |h(t - z)| dt. \]

Since by assumption $h \in L_1(\mathbb{R})$ is an entire function of exponential type $\sigma$ we have for $z = x + iy$ and $u = t - x$

\[ \int_{-\infty}^{\infty} |h(t - z)| dt = \int_{-\infty}^{\infty} |h(u - iy)| du \leq e^{\sigma|y|} \|h\|_{L_1(\mathbb{R})}, \]

because

\[ h(u - iy) = \sum_{k=0}^{\infty} \frac{(-iy)^k}{k!} h^{(k)}(u), \quad \|h^{(k)}\|_{L_1(\mathbb{R})} \leq \sigma^k \|h\|_{L_1(\mathbb{R})}. \]

Thus

\[ \|e^{izL} g\| \leq \|f\| \int_{-\infty}^{\infty} |h(t - z)| dt \leq \|f\| e^{\sigma|y|} \|h\|_{L_1}. \]

It shows that for every vector $g^* \in H$ the function $\langle e^{izL} g, g^* \rangle$ is an entire function and

\[ \left| \langle e^{izL} g, g^* \rangle \right| \leq \|g^*\| \|f\| e^{\sigma|y|} \|f\|_{L_1(\mathbb{R})}. \]

In other words the $\langle e^{izL} g, g^* \rangle$ is an entire function of the exponential type $\sigma$ which is bounded on the real line and an application of the classical Bernstein theorem gives the inequality

\[ \left| \left( \frac{d}{dt} \right)^k \langle e^{itL} g, g^* \rangle \right| \leq \sigma^k \sup_{t \in \mathbb{R}} \left| \langle e^{itL} g, g^* \rangle \right|. \]
Since
\[
\left( \frac{d}{dt} \right)^k \langle e^{itL}g, g^* \rangle = \langle e^{itL}(iL)^kg, g^* \rangle
\]
we obtain for \( t = 0 \)
\[
|\langle L^kg, g^* \rangle| \leq \sigma^k \|g^*\| \|f\| \int_{-\infty}^{\infty} |h(\tau)|d\tau.
\]
Choosing \( g^* \) such that \( \|g^*\| = 1 \) and \( \langle L^kg, g^* \rangle = \|L^kg\| \) we obtain the inequality
\[
\|L^kg\| \leq \sigma^k \|f\| \int_{-\infty}^{\infty} |h(\tau)|d\tau
\]
which implies that \( g \) belongs to \( PW_{\sigma}(L) \).

Let
\[
(2.6) \quad h(t) = a \left( \frac{\sin(t/n)}{t} \right)^n
\]
where \( n = 2(m + 3) \) and
\[
a = \left( \int_{-\infty}^{\infty} \left( \frac{\sin(t/n)}{t} \right)^n dt \right)^{-1}.
\]
With such choice of \( a \) and \( n \) function \( h \) will have the following properties:
(1) \( h \) is an even nonnegative entire function of exponential type one;
(2) \( h \) belongs to \( L^1(\mathbb{R}) \) and its \( L^1(\mathbb{R}) \)-norm is 1;
(3) the integral
\[
(2.7) \quad \int_{-\infty}^{\infty} h(t)|t|^m dt
\]
is finite.

Next, we observe the following formula
\[
(-1)^{m+1}(e^{i\sigma L} - I)^mf = (-1)^{m+1}\sum_{j=0}^{m}(-1)^{m-j}C_m^j e^{js(iL)}f = \sum_{j=1}^{m} b_j e^{js(iL)}f - f, \quad b_1 + b_2 + ... + b_m = 1.
\]
Consider the vector
\[
(2.9) \quad Q_{h}^{\sigma,m}(f) = \int_{-\infty}^{\infty} h(t) \left\{ (-1)^{m+1}(e^{i\sigma L} - I)^mf + f \right\} dt.
\]
According to (2) we have
\[
Q_{h}^{\sigma,m}(f) = \int_{-\infty}^{\infty} h(t) \sum_{j=1}^{m} b_j e^{j\sigma L}f dt = \int_{-\infty}^{\infty} \Phi(t)e^{t\sigma L}f dt.
\]
where
\[
\Phi(t) = \sum_{j=1}^{m} b_j \left( \frac{\sigma}{j} \right) \left( i\sigma \right) \left( t\frac{\sigma}{j} \right), \quad b_1 + b_2 + ... + b_m = 1.
\]
Since the function \( h(t) \) has exponential type one every function \( h(t\sigma/j) \) has the type \( \sigma/j \) and because of this the function \( \Phi(t) \) has exponential type \( \sigma \). It also belongs to \( L^1(\mathbb{R}) \) and as it was just shown it implies that the vector \( Q_{h}^{\sigma,m}(f) \) belongs to
operator. We introduce the self-adjoint non-negative operator means
\[ D \]

Let
\[ L \]

the formula
\[ P W_c(L) \]

since
\[ D \]

is finite by the choice of
\[ h \]

we obtain by using (2.2)
\[ \mathcal{E}(f, f) \leq \| f - Q^\sigma_m(f) \| \leq \int_{-\infty}^{\infty} h(t) \left\| (e^{iL} - I)^m f \right\| dt \leq \int_{-\infty}^{\infty} h(t) \omega^m_L(f, t/\sigma) dt \leq C \omega^m_L(f, 1/\sigma), \quad c = \int_{-\infty}^{\infty} h(t)(1 + |t|)^m dt. \]

If \( f \in D(L^k) \) then by using (2.1) we have
\[ \mathcal{E}(\sigma, f) \leq \int_{-\infty}^{\infty} h(t) \omega^m_L(t/\sigma, f) dt \leq \frac{\omega^m_L(1/\sigma, L^k f)}{\sigma^k} \int_{-\infty}^{\infty} h(t)|t|^k(1 + |t|)^{m-k} dt \leq \frac{C}{\sigma^k} \omega^m_L(1/\omega, L^k f), \]

where
\[ C = \int_{-\infty}^{\infty} h(t)|t|^k(1 + |t|)^{m-k} dt \]
is finite by the choice of \( h \). The inequalities (2.4) and (2.5) are proved. \( \square \)

3. Unitary representations. Equivalence of norms

A strongly continuous unitary representation of a Lie group \( G \) in a Hilbert space \( H \) is a homomorphism \( T : G \to U(H) \) where \( U(H) \) is the group of unitary operators of \( H \) such that \( T(g)f, \ g \in G, \) is continuous on \( G \) for any \( f \in H. \) The Garding space \( G \) is defined as the set of vectors \( h \) in \( H \) that have the representation \( h = \int_G \varphi(g)T(g)f dg, \) where \( f \in H, \ \varphi \in C_0^\infty(G), \) \( dg \) is a left-invariant measure on \( G. \) If \( X \in g \) is identified with a right-invariant vector field
\[ X \varphi(g) = \lim_{t \to 0} \frac{\varphi(\exp tX \cdot g) - \varphi(g)}{t}, \]

then one has a representation \( D(X) \) of \( g \) by operators which act on \( G \) by the formula
\[ D(X)h = -\int_G X \varphi(g)T(g)f dg. \] The Garding space \( G \) is invariant with respect to all operators \( D(X), \ X \in g, \) and dense in every \( H^r. \)

If \( X_1, ..., X_d \) is a basis in \( g \) we associate with every \( X_j, \ 1 \leq j \leq d, \) a strongly continuous one-parameter group of isometries \( t \mapsto T(\exp tX_j), \ t \in \mathbb{R}, \) whose generator is denoted as \( D_j, \ 1 \leq j \leq d. \) The Laplace operator \( 10 \) is defined on \( G \) by the formula
\[ L_G = -D_1^2 - D_2^2 - ... - D_d^2. \]

Since \( L_G \) is symmetric and the differential operator \( -\sum_{i=1}^d X_i^2 \) is elliptic on the group \( G \) the Theorem 2.2 in [10] implies that \( L_G \) is essentially self-adjoint, which means \( L_G = \overline{L_G}. \) In other words, the closure \( \overline{L_G} = L \) of \( L_G \) from \( G \) is a self-adjoint operator. We introduce the self-adjoint non-negative operator
\[ \Lambda = I + L = I - D_1^2 - ... - D_d^2. \]

Let \( D = \sqrt{\Lambda} \) be the non-negative square root from \( \Lambda. \) Our first result is the following.
Theorem 3.1. The space $\mathbf{H}^r$ with the norm $\| \cdot \|_{\mathbf{H}}$ is isomorphic to the domain of $\Lambda^{r/2}$ with the norm $\| \Lambda^{r/2} f \|_{\mathbf{H}}$.

Proof. In the case $r = 2k$, the inequality

$\| f \|_{\mathbf{H}^{2k}} \leq C(k) \| \Lambda^k f \|_{\mathbf{H}}$ (3.1)

is shown in [9], Lemma 6.3. The reverse inequality is obvious. We consider now the case $r = 2k + 1$. If $f \in \mathbf{H}^2 = \mathcal{D}(\Lambda)$, then since $\mathcal{D}(\Lambda) \subset \mathcal{D}(\Lambda^{1/2})$ we have

$$
\| f \|_{\mathbf{H}}^2 + \sum_j \| D_j f \|_{\mathbf{H}}^2 = \langle f, f \rangle + \sum_j \langle D_j f, D_j f \rangle = \langle f, f \rangle + \left( - \sum_j D_j^2 f, f \right) = \langle f - \sum_j D_j^2 f, f \rangle = \langle \Lambda f, f \rangle = \| \Lambda^{1/2} f \|_{\mathbf{H}}^2.
$$

These equalities imply that $\mathbf{H}^1$ is isomorphic to $\mathcal{D}(\Lambda^{1/2})$. Our goal is to prove existence of an isomorphism between $\mathbf{H}^{2k+1}$ and $\mathcal{D}(\Lambda^{k+1/2})$. It is enough to establish equivalence of the corresponding norms on the set $\mathbf{H}^{4k+2} = \mathcal{D}(\Lambda^{2k+1})$ since the latest is dense in $\mathbf{H}^{2k+1}$. If $f \in \mathbf{H}^{4k+2} \subset \mathbf{H}^{2k}$ then $D_j f \in \mathbf{H}^{2k+1} \subset \mathbf{H}^{2k}$ and $\Lambda^k f = \sum_{m \leq k} \sum D_{j_1}^2 \ldots D_{j_m}^2 f$. Thus if $f \in \mathbf{H}^{4k+2}$ then

$$
\| D_{j_1} \ldots D_{j_{2k+1}} f \|_{\mathbf{H}} \leq C \| \Lambda^k D_{j_{2k+1}} f \|_{\mathbf{H}} = \left\| \sum_{m \leq k} \sum D_{j_1}^2 \ldots D_{j_m}^2 D_{j_{2k+1}} f \right\|_{\mathbf{H}}.
$$

Multiple applications of the identity $D_i D_j - D_j D_i = \sum_k c_{i,j}^k D_k$ which holds on $\mathbf{H}^2$ lead to the inequality

$$
\| D_{j_1} \ldots D_{j_{2k+1}} f \|_{\mathbf{H}} \leq C \left( \| D_{j_{2k+1}} \Lambda^k f \|_{\mathbf{H}} + \| R f \|_{\mathbf{H}} \right),
$$

where $R$ is a polynomial in $D_1, \ldots, D_d$ whose degree $\leq 2k$. According to (3.1) and (3) we have that

$$
\| D_{j_{2k+1}} \Lambda^k f \|_{\mathbf{H}} \leq \| \Lambda^{1/2} \Lambda^k f \|_{\mathbf{H}} = \| \Lambda^{k+1/2} f \|_{\mathbf{H}}
$$

and also $\| R f \|_{\mathbf{H}} \leq \| f \|_{\mathbf{H}^{2k}} \leq C(k) \| \Lambda^k f \|_{\mathbf{H}}$. Since $\| \Lambda^k f \|_{\mathbf{H}}$ is not decreasing with $k$ we get the following estimate

$$
\| D_{j_1} \ldots D_{j_{2k+1}} f \|_{\mathbf{H}} \leq C(k) \| \Lambda^{k+1/2} f \|_{\mathbf{H}}, \quad f \in \mathbf{H}^{4k+2}.
$$

Now, since for $f \in \mathbf{H}^{4k+2}$ we have $D_{j_1} \ldots D_{j_{2k}} f \in \mathbf{H}^{2k+2} \subset \mathbf{H} = \mathcal{D}(\Lambda^{1/2})$, and the equality $\Lambda^k f = \sum_{m \leq k} \sum D_{j_1}^2 \ldots D_{j_m}^2 f$, holds we obtain, by using (3)

$$
\| \Lambda^{k+1/2} f \|_{\mathbf{H}} = \| \Lambda^{1/2} \sum_{m \leq k} \sum D_{j_1}^2 \ldots D_{j_m}^2 f \|_{\mathbf{H}} \leq C \| f \|_{\mathbf{H}^{2k+1}}, \quad C = C(k).
$$

Theorem is proved. \qed
4. The Hardy-Steklov operator and the \( K \)-functional

We introduce a generalization of the classical Hardy-Steklov operator. For a positive small \( s \), natural \( r \) and \( 1 \leq j \leq d \) we set

\[
H_{j,r}(s)f = (s/r)^{-r} \int_0^{s/r} \ldots \int_0^{s/r} \sum_{k=1}^{r} (-1)^k C_r^k \tau_j(k(\tau_{j,1} + \ldots + \tau_{j,r})f d\tau_{j,1} \ldots d\tau_{j,r},
\]

where \( C_r^k \) are the binomial coefficients and then define the Hardy-Steklov operator:

\[
H_r(s)f = \prod_{j=1}^{d} H_{j,r}(s)f = H_{1,r}(s)H_{2,r}(s) \ldots H_{d,r}(s)f, \quad f \in H.
\]

For every fixed \( f \in H \) the function \( H_r(s)f \) is an abstract valued function from \( \mathbb{R} \) to \( H \) and it is a linear combination of some abstract valued functions of the form

\[
(s/r)^{-rd} \int_0^{s/r} \ldots \int_0^{s/r} T f(\tau)d\tau_{1,1} \ldots d\tau_{d,r},
\]

where

\[
\tau = (k_1 \tau_1, k_2 \tau_2, \ldots, k_d \tau_d), \quad 1 \leq k_j \leq r,
\]

\[
\tau_j = (\tau_{j,1} + \tau_{j,2} + \ldots + \tau_{j,r}), \quad 1 \leq j \leq d,
\]

and

\[
T f(\tau) = T_1(k_1 \tau_1)T_2(k_2 \tau_2) \ldots T_d(k_d \tau_d)f.
\]

Let \( F(x_1, x_2, \ldots, x_N) \) be a function on \( \mathbb{R}^N \) that takes values in the Hilbert space \( H \)

\[
F : \mathbb{R}^N \mapsto H.
\]

For \( 1 \leq i \leq N \) we introduce the difference operator by the formula

\[
(\Delta_i(s)F)(x_1, x_2, \ldots, x_N) = F(x_1, x_2, \ldots, x_{i-1}, s, x_{i+1}, \ldots, x_N) - F(x_1, x_2, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_N).
\]

Our nearest goal is to prove the following theorem.

**Theorem 4.1.** The following holds:

1. For every \( f \in H \) and \( s > 0 \) the function \( H_r(s)f \) belongs to \( H^r \).
2. For every \( 0 \leq q < r \) the "mixed derivative" \( D_{j_1} \ldots D_{j_q} H_r(s)f \), \( 1 \leq j_k \leq d \) is another abstract valued function with values in \( H^{r-q} \) and it is a linear combination of abstract valued functions (with values in \( H^r \)) of the form

\[
(s/r)^{-rd} \int_0^{s/r} \ldots \int_0^{s/r} \mu_{j_1, \ldots, j_q}^{i_1, \ldots, i_m}(\cdot) \Delta_{i_1, k_1}(s/r) \ldots \Delta_{i_l, k_l}(s/r) T f(\cdot) d\cdot,
\]

where

\[
\max_{0 \leq \tau_{i,j} \leq s} |\partial^p \mu_{j_1, \ldots, j_q}^{i_1, \ldots, i_m}(\cdot)| \leq cs^{r-l}, \quad p \in \mathbb{N} \cup \{0\},
\]

and \( 0 < s < 1 \), \( 0 \leq m \leq rd \), \( 0 \leq l \leq m \), where \( l = 0 \) corresponds to the case when the set of indices \( \{i_1, \ldots, i_l\} \) is empty.
The main ingredient of the proof of this theorem is the next lemma which will be proved now. For the rest of the proof of Theorem 4.1 we refer to [21].

**Lemma 4.2.** For every $s > 0$ the operator $H_1(s)$ maps $H$ to $H^1$.

**Proof.** Let’s assume first that $f \in H^1$. Since every $D_j$, $1 \leq j \leq d$, is a closed operator to show that the term (4.1) belongs to $D(D_j)$ it is sufficient to show existence of the integral (see notations (4.2), (4.3))

\[(4.8) \quad \int_0^{s/r} ... \int_0^{s/r} D_j T f(\tau) d\tau_{1,1} ... d\tau_{d,r}, \]

According to (1.1) the last integral equals to

\[(4.9) \quad \sum_{i=1}^{d} \left( s/r \right)^{-r d} \int_0^{s/r} ... \int_0^{s/r} \frac{\partial}{\partial \tau_{1,1,i}} \left( s/r \right) \Delta_i(s/r) T f(\tau) d\tau_i + \right.

\[\left. \left( s/r \right)^{-r d} \int_0^{s/r} ... \int_0^{s/r} \partial_i \xi_j \left( s/r \right) T f(\tau) d\tau_i \right) = B_j(s)f, \]

where $\tau = (\tau_{1,1}, ..., \tau_{1,r}, ..., \tau_{n,1}, ..., \tau_{n,r})$, $\tau_{i}^{s/r} = (\tau_{1,1}, ..., \tau_{1,r}, ..., \tau_{i,1}, ..., \tau_{i,r}, ..., \tau_{n,1}, ..., \tau_{n,r})$, $\tau_{0}^{s/r} = (\tau_{1,1}, ..., \tau_{1,r}, ..., \tau_{i,1}, ..., \tau_{i,r} + 0, ..., \tau_{i,1} + ... + \tau_{i,r}, ..., \tau_{n,1} + ... + \tau_{n,r})$, $d\tau = d\tau_{1,1} ... d\tau_{n,r}$, and $(d\tau)_i = d\tau_{1,1} ... d\tau_{i-1,r} d\tau_{i,r} d\tau_{i+1,1} ... d\tau_{n,r}$, where the term $d\tau_{i,r}$ is missing.

Since integrand of each of the three integrals is bounded it implies existence of (4.8) which, in turn, shows that (1.1) is an element of $D(D_j)$ for every $1 \leq j \leq d$.

Thus if $f$ belongs to $H^1$ then $H(s)f$ takes values in $H^1$ and the formula $D_j H(s)f = B_j(s)f$ holds where the operator $B_m(s)$ is bounded. This fact along with the fact that $H^1$ is dense in $H$ implies the formula $D_j H(s)f = B_j(s)f$ for every $f \in H$. Thus we proved the first part of the Lemma for $q = 1$.

We will need the following Lemma which can be verified directly.

**Lemma 4.3.** For every bounded operators $B_1, B_2, ..., B_n$ the following formulas hold

\[(4.11) \quad B_1 B_2 ... B_n - I = B_1 (B_2 - I) + ... + B_1 B_2 ... B_{n-1} (B_n - I), \]
\( (B_1 - I)B_2...B_n = (B_1 - I) + B_1(B_2 - I) + ... + B_1B_2...B_{n-1}(B_n - I) \).

**Theorem 4.4.** For the \( K \)-functional

\[
K(s^r, f, H, H') = \inf_{g \in H'} (\| f - g \|_H + s^r \| g \|_{H'}) ,
\]

the following double inequality holds

\[
e^{-\Omega'}(s, f) \leq K(s^r, f, H, H') \leq C(\Omega'(s, f) + s^r \| f \|_H),
\]

where positive constant \( c, C \) are independent on \( f \).

**Proof.** First we prove the right-hand side of the inequality (4.14). The following identities which can be verified directly play important role in the roof. By using the formula (4.11) we obtain

\[
\left\| \left( -1 \right)^{n(r+1)} H_r(s)f - f \right\|_H = \left\| \prod_{j=1}^{n} \left( -1 \right)^{n+1} H_j, r(s)f - f \right\|_H \leq \left( \Omega'(s, f) \right) \leq C \Omega'(s, f).
\]

To estimate \( s^r \| H_r(s)f \|_H \) we note that \( \| H_r(s)f \|_H \leq C \| f \|_H \). According to Theorem 4.4 the estimate \( \| H_r(s)f \|_H \) is estimated for \( 0 \leq s \leq 1 \) by

\[
s^{-t} \sup_{0 \leq \tau \leq s} \| \Delta_{j_{1}, k_{1}}(s/r)\Delta_{j_{2}, k_{2}}(s/r)T(\cdot) \| \leq C \Omega^t(s, f),
\]

where \( T(\cdot) = \prod_{j=1}^{n} \prod_{k=1}^{r} \tau_j(\tau_{j,k}) \). By the definition of \( \Delta_{j,k}(s/r) \) the expression \( \Delta_{j,k}(s/r)T(\cdot) \) differs from \( T(\cdot) \) only in that in place of the factor \( T_j(\tau_{j,k}) \) the factor \( T_j(s/r) - I \) appears. Multiple applications of the identity (4.12) to the operator \( \Delta_{j_{1}, k_{1}}(s/r)\Delta_{j_{2}, k_{2}}(s/r)T(\cdot) \) allow its expansion into a sum of operators each of which is a product of not less than \( l \leq r \) of operators \( T_i(\sigma_i) - I, \sigma_i \in (0, s/r), 1 \leq i \leq n \). Consequently, (4.14) is dominated by a multiple of \( s^{-t} \Omega^t(s, f) \). By summing the estimates obtained above we arrive at the inequality

\[
K(s^r, f, H, H') \leq \left\| \left( -1 \right)^{n(r+1)} H_r(s)f - f \right\|_H + s^r \| H_r(s)f \|_H \leq \left( \sum_{l=1}^{n} s^{-l} \Omega^l(s, f) + s^r \| f \|_H \right), \quad 0 \leq s \leq 1.
\]

Note, that by repeating the known proof for the classical modulus of continuity one can prove the inequality

\[
\Omega^l(s, f) \leq C \left( s^l \| f \|_H + s^l \int_{l}^{1} \tau^{-1-i} \Omega^{k+r}(\tau, f) d\tau \right),
\]

which implies \( s^{-l} \Omega^l(s, f) \leq C \left( s^l \| f \|_H + \Omega^r(s, f) \right) \). By applying this inequality to (4.14) and taking into account the inequality \( K(s^r, f, H, H') \leq \| f \|_H \), we obtain the right-hand side of the estimate (4.14).

To prove the left-hand side of (4.14) we first notice that the following inequality holds

\[
\Omega^r(s, g) \leq C s^{k} \sum_{1 \leq j_1, \ldots, j_k \leq n} \Omega^{r-k}(s, D_{j_1} \ldots D_{j_k} g), g \in H^k, C = C(k, r), k \leq r,
\]
which is an easy consequence of the identity \( \mathbf{(1.12)} \) and the identity \( (T_j(t) - I) g = \int_0^t T_j(\tau) D_j g d\tau, \ g \in D(D_j). \) From here, for any \( f \in \mathbf{H}, \ g \in \mathbf{H^r} \) we obtain
\[
\Omega^r(s, f) \leq \Omega^r(s, f - g) + \Omega^r(s, g) \leq C \left( \| f - g \|_\mathbf{H} + s^r \| g \|_\mathbf{H^r} \right).
\]
Theorem is proven.

\[\square\]

5. Appendix A. The Campbell-Hausdorff Formula

Lie algebra \( \mathbf{g} \) of a Lie group \( G \) can be identified with the tangent space \( T_e(G) \) of \( G \) at the identity \( e \in G \). Let \( \exp(tX) : T_e(G) \to G, \ t \in \mathbb{R}, \ X \in T_e(G), \) be the exponential geodesic map i.e. \( \exp(tX) = \gamma(1) \), where \( \gamma(t) \) is a geodesic of a fixed left-invariant metric on \( G \) which starts at \( e \) with the initial vector \( tX \in T_e(G) \):
\[
\gamma(0) = e, \quad \frac{d\gamma(0)}{dt} = tX.
\]
It is known that \( \exp \) is an analytic homomorphism of \( \mathbb{R} \) onto one parameter subgroup \( \exp(tX) \) of \( G \): \( \exp((s + t)X) = \exp(sX) \exp(tX), \ s, t \in \mathbb{R} \).

Let \( X_1, ..., X_d, \ d = \text{dim} \ G \) form a basis in the Lie algebra of \( G \), then one can consider the following coordinate system in a neighborhood of identity \( e \)
\[
(t_1, ..., t_d) \mapsto \exp(t_1X_1 + ... + t_dX_d).
\]

If \( Y_1 = s_1X_1 + ... + s_dX_d \) and \( Y_2 = t_1X_1 + ... + t_dX_d \) then \( \exp Y_1 \exp Y_2 = \exp Z \),
where \( Z \) is given by the Campbell-Hausdorff formula
\[
Z = Y_1 + Y_2 + \frac{1}{2}[Y_1, Y_2] + \frac{1}{12}[Y_1, [Y_1, Y_2]] - \frac{1}{12}[Y_2, [Y_1, Y_2]] - \frac{1}{24}[Y_2, [Y_1, [Y_1, Y_2]]] + ... .
\]
It implies that \( Z = \zeta_1X_1 + ... + \zeta_dX_d \), where
\[
\zeta_j = s_j + t_j + O(\epsilon^2), \quad |t_j|, |s_j| \leq \epsilon, \quad 1 \leq j \leq d.
\]
One can also consider another local coordinate system around \( e \) which is given by the formula
\[
(t_1, ..., t_d) \mapsto \varphi \left( \sum_{j=1}^d t_j X_j \right) = \exp(t_1X_1) ... \exp(t_dX_d).
\]

**Theorem 5.1.** If \( T : G \to GL(\mathbf{H}) \) is a strongly continuous bounded representation of \( G \) in a Banach space \( \mathbf{H} \) and \( T_j(t) = T(\exp tX_j) \), where \( \{X_1, ..., X_d\} \) is a basis in \( \mathbf{g} \) then the formula \( \mathbf{(5.4)} \) is satisfied for groups \( T_j \) and their infinitesimal operators \( D_j \), \( 1 \leq j \leq d \).

**Proof.** The fact that \( \mathbf{H^1} = \bigcap_{j=1}^d D(D_j) \) is dense in \( \mathbf{H} \) and invariant with respect to \( T \) is well known \( \mathbf{[9, 10]} \). Since \( \exp \) and \( \varphi \) are diffeomorphisms in a neighborhood of zero in \( \mathbf{g} \) the map \( \exp^{-1} \circ \varphi : \mathbf{g} \to \mathbf{g} \) is also a diffeomorphism. The formulas \( \mathbf{(5.2)} \) and \( \mathbf{(5.3)} \) give connection between \( \mathbf{(5.1)} \) and \( \mathbf{(5.4)} \)
\[
\varphi \left( \sum_{j=1}^d t_j X_j \right) = \exp \left( \sum_{k=1}^d \alpha_k(t) X_k \right), \quad t = (t_1, ..., t_d),
\]
where
\[
\alpha_j(t) = t_j + O(\epsilon^2), \quad |t_j| \leq \epsilon, \quad 1 \leq j \leq d.
\]
In particular, (5.2) implies $\exp \tau X_j \exp \sum_{i=1}^d t_i X_i = \exp \sum_{k=1}^d \gamma_k^j(t, \tau) X_k, \ t = (t_1, ..., t_d)$, where

\begin{equation}
\gamma_k^j(t, \tau) = t_k + \tau \zeta_k^j(t) + \tau^2 R_k^j(t, \tau),
\end{equation}

and $\zeta_k^j(t) = \delta_k^j + Q_k^j(t)$, where $\delta_k^j$ is the Kronecker symbol and $Q_k^j(t)$ and $R_k^j(t, \tau)$ are convergent series in $t_1, ..., t_d$ and $t_1, ..., t_d, \tau$ respectively.

Since for $f \in \mathbf{H}^1$ one has $D_j T(g) f = \frac{d}{d\tau} \tau (\exp \tau X_j) T(g) f|_{\tau=0}$ we obtain for $f \in \mathbf{H}^1$ the following

\[
D_j T_1(t_1) ... T_d(t_d) f = D_j T \left( \varphi \left( \sum_{i=1}^d t_i X_i \right) \right) = D_j T \left( \exp \sum_{i=1}^d \alpha_i(t) X_i \right) f =
\]

\[
\frac{d}{d\tau} \tau (\exp \tau X_j) T \left( \varphi \left( \sum_{i=1}^d t_i X_i \right) \right) f|_{\tau=0} = \frac{d}{d\tau} \tau (\exp \sum_{i=1}^d \gamma_i^j(\alpha, \tau) X_i) f|_{\tau=0},
\]

where $\alpha = (\alpha_1(t), ..., \alpha_d(t))$ and according to (5.7) $\gamma_i^j(\alpha, \tau) = \alpha_i(t) + \tau \zeta_i^j(\alpha(t)) + \tau^2 R_i^j(\alpha, \tau)$. By using the Chain Rule and (5.5) we finally obtain the formula (1.4)

\[
D_j T_1(t_1) ... T_d(t_d) f = \frac{d}{d\tau} \tau (\exp \sum_{i=1}^d \gamma_i^j(\alpha, \tau) X_i) f|_{\tau=0} =
\]

\[
\sum_{k=1}^d \left( \frac{d}{d\tau} \gamma_k^j(\alpha, \tau) \right)|_{\tau=0} \partial_k T \left( \exp \sum_{i=1}^d \gamma_i^j(\alpha, \tau) X_i \right) f|_{\tau=0} = \sum_{k=1}^d \zeta_k^j(t) \partial_k T_1(t_1) ... T_d(t_d) f.
\]

Lemma is proved. 

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