Jacobi diagrams on surfaces and quantum invariants

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Abstract

This is a preliminary version; it will be completed shortly.

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1 Introduction

The relationship between quantum link invariants (which generalize the Jones polynomial), and the Kontsevich integral (universal finite type invariant of knots/links) is fairly well understood. The quantum link invariants were extended to 3-manifolds by Witten and Reshetikhin-Turaev (WRT). One associates to a compact oriented 3-manifold $M$, a root of unity $\xi$, and a semi-simple Lie algebra $\mathfrak{g}$, for example $\mathfrak{sl}_2$, a complex number $\tau_{M}(\xi)$.

The Kontsevich integral was extended to an invariant of 3-manifolds (LMO) by Le-Murakami-Ohtsuki. LMO takes values in a certain algebra $A(\emptyset)$ of Jacobi diagrams. Any finite-dimensional semi-simple metrized Lie algebra $\mathfrak{g}$ defines a linear map $A(\emptyset) \to \mathbb{Q}[[h]]$ called the weight system associated to $\mathfrak{g}$. For a Jacobi diagram $D \in A(\emptyset)$, $\hat{W}_{\mathfrak{sl}_2}(D) = \sum_{d \geq 0} W_{\mathfrak{sl}_2}(D) h^d$, where $W_{\mathfrak{sl}_2}(D) \in \mathbb{Q}$, $d$ = degree of $D$.

The relationship between WRT and LMO invariants is more complex, and is best described by the theory of the unified invariant $J_M(q)$, whose evaluation at any root of unity $q = \xi$ coincides with the value of the WRT invariant at that root: $J_M(\xi) = \tau_M(\xi)$. Habiro’s invariant takes values in the Habiro’s ring:

$$\hat{\mathbb{Z}}[q] := \lim_{\leftarrow n} \mathbb{Z}[q] / ((1 - q)(1 - q^2) \cdots (1 - q^n)),$$

the cyclotomic completion of $\mathbb{Z}[q]$. Every element $f(q) \in \hat{\mathbb{Z}}[q]$ can be written (not uniquely) as an infinite sum $f(q) = \sum_{k \geq 0} f_k(q)(1 - q)(1 - q^2) \cdots (1 - q^h)$, with $f_k(q) \in \mathbb{Z}[q]$. When $q = \xi$ a root of unity, only a finite number of these terms are not zero, hence the evaluation $ev_{\xi}(f(q))$ is well-defined. Moreover, it only depends on $f(q)$. Habiro’s ring has the property that the formal Taylor series expansion of $f(q) \in \hat{\mathbb{Z}}[q]$ at a root $\xi$ of 1

$$T_{\xi}(f) = \sum_{n=0}^{\infty} \frac{f(n)(\xi)}{n!}(q - \xi)^n$$

is well defined. Moreover, the map $T_{\xi} : \hat{\mathbb{Z}}[q] \to \hat{\mathbb{Z}}[\xi][[q - \xi]]$ is injective, i.e. a function in $\hat{\mathbb{Z}}[q]$ is determined by its Taylor expansion at a point in the domain $U$, the set of roots of 1. While the Taylor series $T_{\xi}f(\xi)$ has convergence radius zero, in the $p$-adic topology $T_{\xi}f(\xi)$ converges to $f(\xi)$.

Properties of the Habiro’s ring imply that the following diagram is commutative [3, 2]:

$\{ZHS\} \xrightarrow{J_M(q)} \hat{\mathbb{Z}}[q] \xrightarrow{T_{\xi}} \mathbb{Z}[1 - q]$

$\xrightarrow{\text{LMO}} \xrightarrow{\hat{A}(\emptyset)_{\text{weight}}} \mathbb{Q}[[h]]$

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where \( \{ZHS\} \) denotes the set of integral homology 3-spheres. This puts into perspective the well-known Ohtsuki’s result that for an integer homology 3-sphere \( M \), the perturbative \( PSU(2) = SO(3) \) invariant (see [7]) recovers from the LMO invariant as \( \tau^{SO(3)}(M) = \tilde{W}_{\mathrm{sl}_2}(Z^{\mathrm{LMO}}(M)) \), while also proving that \( \tau^{SO(3)}(M) \) has integer coefficients (a priori \( \tau^{SO(3)}(M) \in \mathbb{Q}[\langle q - 1 \rangle] \)).

Both WRT and LMO invariants extend to 3-manifolds with boundary in a nice way (TQFTs). However, the two types of functorialities have different flavor. The LMO functor is appropriate for homology cylinders, which are a generalization to cobordisms of integer homology 3-spheres. The TQFTs for the WRT invariants, by contrast, make use of the totality of 3-manifolds with boundary.

The weight system construction generalizes to a map \( W_g : \big( \mathcal{A}(H_Q), \star \big) \to (S(\mathfrak{g} \otimes H_Q)[[t]], \star) \) from the Hopf algebra of symplectic Jacobi diagrams, where \( H_Q = H \otimes \mathbb{Q} \), and \( H = H_1(\Sigma_{g,b}, \mathbb{Z}) \), to the deformation quantization of the symmetric algebra \( S(\mathfrak{g} \otimes H_Q) \), whose Poisson structure is induced by the symplectic form on \( H_Q \) (see [4]). The LMO homomorphism sends the monoid of homology cylinders \( \mathcal{I}C(\Sigma_{g,b}) \) (see Section 2 below) to the group-like elements \( GLike \mathcal{A}(H_Q) \), and composition of cobordisms is sent to the multiplication \( \star \) which can be described in purely algebraic-combinatorial terms (see [3]). (On a side note, the induced Lie bracket \([\cdot, \cdot]_\ast\) on the reduction of \( \mathcal{A}(H_Q) \) to connected and tree-like Jacobi diagrams \( \mathcal{A}^{\text{tree}}(H_Q) \) is the one considered independently by Kontsevich and Morita in 1993.) The map \( W_g \) sends \( \star \) to the Moyal-Weyl product on \( S(\mathfrak{g} \otimes H_Q)[[t]] \) (see [4]).

This provides a means by which some functoriality properties of WRT and LMO invariants can be compared. The aim of this paper is to introduce (adapt) another ingredient to this goal: Jacobi diagrams on surfaces. In Section 4 we introduce a new map modeled on the LMO homomorphism. Section 5 ...

## 2 Cobordisms over \( \Sigma_{g,b} \)

Let \( \Sigma_{g,b} \) denote a compact connected oriented surface of genus \( g \) with \( b \) boundary components. For our purposes, \( b \) will be 0 or 1. Let \( H_g \) denote a handlebody of genus \( g \) such that \( \partial H_g = \Sigma_g \). The mapping class group of \( \Sigma_{g,b} \) is the group of isotopy classes of homeomorphisms \( \Sigma_{g,b} \to \Sigma_{g,b} \) which preserve the orientation and fix the boundary pointwise. It is well-known that every compact connected oriented 3-manifold is homomorphic to \( H_g \cup_{f} H_g \) for some orientation preserving homeomorphism \( f \) of \( \Sigma_b \). The Torelli group of \( \Sigma_{g,b} \) is the subgroup of the mapping class group, whose elements are represented by homeomorphisms that induce the identity map on the first homology \( H_1(\Sigma_{g,b}) \). It is well-known that every (compact connected oriented) integer homology 3-sphere is homeomorphic to \( H_g \cup_{f} H_g \) for some orientation preserving homeomorphism \( f \) of \( \Sigma_b \), who isotopy class is in the Torelli group.

We shall call a pair \((M, m)\) a cobordism of \( \Sigma_{g,b} \) if \( M \) is a compact connected oriented 3-manifold and \( m : \partial(\Sigma_{g,b} \times [-1, 1]) \to \partial M \) is an orientation-preserving homeomorphism. \((M, m)\) and \((M', m')\) are equivalent if there is an orientation-preserving homeomorphism \( f : M \to M' \) such that \( f|_{\partial M} \circ m = m' \), \( m_{\pm} = m(\cdot, \pm 1) : \Sigma_{g,b} \to M \) determines top and bottom surfaces, and are used to compose cobordisms: \( M \circ M' = M \cup_{m_{\pm} \circ (m')^{-1}} M' \). Here \((\Sigma_{g,b} \times [-1, 1], \text{id})\) is the identity element of resulting monoid \( \text{Coh}(\Sigma_{g,b}) \).

Notice the submonoids of homology cobordisms and homology cylinders over \( \Sigma_{g,b} \), where a cobordism \((M, m)\) is called a homology cobordism if both \( m_{\pm} \) and \( m_{-} \) induce isomorphisms \( H_*(\Sigma_{g,b}) \to H_*(M) \), and a homology cylinder if in addition the two induced isomorphisms coincide.

The mapping cylinder construction associates to a surface homeomorphisms \( f \) the cobordism \((\Sigma_{g,b} \times [-1, 1], \text{id} \times \{-1\} \cup f \times \{1\})\). Thus, Torelli group is contained in the monoid of homology cylinders.
3 Jacobi diagrams on surfaces

4 The LMO map

5 References

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