Concentration properties of semi-vertex transitive graphs and random bi-coset graphs

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Abstract

It is well-known that concentrators are sparse graphs of high connectivity, which play a key role in the construction of switching networks; and any semi-vertex transitive graph is isomorphic to a bi-coset graph. In this paper, we prove that random bi-coset graphs are almost always concentrators, and construct some examples of semi-vertex transitive concentrators.

1 Introduction

Many problems on information transmission and complexity have shown the importance of constructing graphs that are highly connected yet sparse. On one hand, concentrators are a key building block for the construction of a class of graphs called superconcentrators which are useful in the study of algorithmic complexity. On the other hand, concentrators are the basis for another class of graphs called generalized connectors (20). For the importance of concentrator-like bipartite graphs in constructing low complexity error-correcting codes, refer to Tanner (19).

Recall a bipartite graph is an \((n, \theta, k, \alpha, c)\) bounded strong concentrator (bsc) if it is a bipartite graph with \(n\) inputs, \(\theta n\) outputs and at most \(kn\) edges such that \(\Gamma_X \geq c|X|\) for any set \(X\) of inputs with \(|X| \leq \alpha n\). Here \(\Gamma_X\) is the set of outputs connected to \(X\) and \(|\cdot|\) is the cardinality of a set.

An \((n, k)\)-superconcentrator is a directed graph with \(n\) inputs and \(n\) outputs, and at most \(kn\) edges satisfying that for any \(1 \leq r \leq n\) and any two sets of \(r\) inputs and \(r\) outputs, there are \(r\) vertex disjoint paths connecting the two sets. A family of linear superconcentrators of density \(k\) is a set of \((n_i, k + o(1))\) superconcentrators, with \(n_i \to \infty\), as \(i \to \infty\), which is most useful in theoretical computer science. Note an \((n, k, c) - \text{expander}\) is a bipartite graph with \(n\) inputs, \(n\) outputs and at most \(kn\) edges such that for any subset \(A\) of inputs,

\[|N(A)| \geq \left(1 + c \left(1 - \frac{|A|}{n}\right)\right)|A|,
\]

where \(N(A)\) is the set of all neighbors of \(A\).

A graph \(G = (V, E)\) on \(n\) vertices with maximal degree \(d\) is called an \((n, d, c) - \text{magnifier}\) if \(|N(X) - X| \geq c|X|\) for any vertex set \(|X| \leq \frac{n}{2}\). For a graph \(G = (V, E)\) with \(V = \{v_1, v_2, \ldots, v_n\}\), its (extended) double cover is the bipartite graph \(H\) on input set \(X =\)

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\{x_1, \cdots, x_n\} \text{ and output set } \ Y = \{y_1, \cdots, y_n\} \text{ such that } x_i \in X \text{ and } y_j \in Y \text{ are adjacent if and only if } i = j \text{ or } v_iv_j \in E.

**Lemma 1.1**. Let graph \( G = (V, E) \) be an \((n, d, c) - \text{magnifier}\) and \( H \) its extended double cover. Then \( H \) is an \((n, d + 1, c) - \text{expander}\).

By this lemma, to construct a family of linear expanders, it suffices to construct a family of linear magnifiers, i.e., to construct, for some fixed \( d \) and \( c > 0 \), a family of \((n_i, d, c) - \text{magnifiers}\) with \( n_i \to \infty\).

It is well-known that random \( d - \text{regular} \) graphs are magnifiers for \( d \geq 5 \). But an explicit construction is needed for application. However, such construction is much more difficult. Margulis gave the first explicit family of linear magnifiers of density 5 and proved it has expansion \( c \) for some \( c > 0 \) by several deep results from the theory of group representations. But he didn’t bound \( c \) strictly away from 0. Then Gaber and Galil modified Margulis’ construction to obtain a family of linear magnifiers with density 7 and expansion \( \frac{2 - \sqrt{3}}{2} \), and used this family to construct explicitly a family of linear superconcentrators of density 271.8. Schöning \[15\] constructed the smaller superconcentrators of density 28, which is the best density.

Note the Cayley diagram of any non-abelian finite simple group \( G \) is a magnifier. Kassabov \[9\] constructed explicitly generating \( \{k\} \) \( \mu \)-bounded degree magnifiers of \((2, 1, 1) - \text{magnifiers}\) with \( \mu \)-bounded degree magnifiers of \((2, 1, 1) - \text{magnifiers}\) with ignoring orientation but remaining multiple edges.

Recall many constructions of magnifier graphs are Cayley graphs, and many families of finite simple groups are magnifier families. Kassabov \[9\] constructed explicitly generating sets \( S_n \) and \( \tilde{S}_n \) of the alternating and the symmetric groups respectively to obtain two families \( \{X(\text{Alt}(n), S_n)\}_{n \geq 1} \) and \( \{X\left(\tilde{S}_n, \tilde{S}_n\right)\}_{n \geq 1} \) of bounded degree magnifiers of Cayley graphs. Kassabov et al \[10\] proved that there exist \( k \in \mathbb{N} \) and \( 0 < \varepsilon < \infty \) such that any non-abelian finite simple group \( G \), which is not a Suzuki group, has a set of \( k \) generators for which the Cayley graph \( X(G, S) \) is an \( \varepsilon - \text{expander} \) (here we call \( \varepsilon - \text{magnifier} \)). Notice we will use the result in \[9\] to construct concentrators in Section 6.

For any graph \( H \), let \( \mu_1(H) \) denote the second largest eigenvalue in absolute value of its adjacency matrix \( A_H \). When \( H \) is \( d \)-regular, the normalized adjacency matrix \( A_H^* = \frac{1}{d}A_H \) of \( H \) is doubly stochastic and \( \mu_1^*(H) = \frac{1}{d}\mu_1(H) \); where \( \mu_1^*(H) \) is the second largest eigenvalue in absolute value of \( A_H^* \).

**Theorem (Alon-Roichman)** 1.2[4]. For any \( \varepsilon > 0 \), there is a \( c(\varepsilon) > 0 \) depending only on \( \varepsilon \) such that the following holds. Let \( G \) be a group of order \( n \) and let \( S \) be a set of \( c(\varepsilon) \log n \) elements of \( G \) chosen uniformly and independently at random. Then

\[
\mu_1^*(X(G, S)) \text{ is concentrated around its mean and } E(\mu_1^*(X(G, S))) < \varepsilon.
\]

By Theorem 1.2, we have

**Corollary 1.3**[6]. For any \( \varepsilon > 0 \), there is a \( c^*(\varepsilon) > 0 \) depending only on \( \varepsilon \) satisfying that for any finite group \( G \) with \( n \) elements, the Cayley graph \( X(G, S) \) is an \((n, 2|S|, \varepsilon) - \text{expander} (\text{magnifier}) \) with high probability as \( n \to \infty \), where \( S \) is a multiset of \( c^*(\varepsilon) \log n \) random elements of \( G \).

Christofides and Markström[6] generalized Alon-Roichman theorem to random coset graphs. For any \( p \in (0, 1) \), let \( H_p \) be the weighted entropy function defined by

\[
H_p(x) = x \log \frac{x}{p} + (1 - x) \log \frac{1 - x}{1 - p}, \text{ } x \in [0, 1].
\]
Then by [6], the following theorem holds.

**Theorem 1.4** [6]. Let $S$ be a multiset of $k$ elements of a finite group $G$ chosen independently and uniformly at random, and $D$ the sum of the dimensions of the irreducible representations of the group $G$. Then for any $0 < \varepsilon < 1$,

$$P_r(\mu^*(X(G, S)) > \varepsilon) \leq 2D \exp \left\{ -kH_2 \left( \frac{1 + \varepsilon}{2} \right) \right\}.$$

For any subgroup $H$ of $G$, denote by $D(G, H)$ the sum of the dimensions of the irreducible representations of $G$ which do not contain the trivial representation of $H$ when decomposed into irreducible representations of $H$.

Let $S$ be a multiset of $G$ chosen independently and uniformly at random. The random coset graph $X(G, H; S)$ of $G$ with respect to $H$ and $S$ is defined as follows: its vertices are all right cosets of $H$ in $G$, and there is an edge between $Hg_1$ and $Hg_2$ if and only if $g_2g_1^{-1} \in HSH$.

**Theorem 1.5** [6]. For the random coset graph $X = X(G, H; S)$ and any $0 < \varepsilon < 1$,

$$P_r(\mu^*(X) > \varepsilon) \leq 2D(G, H) \exp \left\{ -|S|H_2 \left( \frac{1 + \varepsilon}{2} \right) \right\}.$$

Note the extended double cover is a superconcentrator. Moreover superconcentrators can be constructed by bounded concentrators through recursive construction of [21]. The Cayley graph is vertex transitive. Tanner [18] constructed several explicit concentrators by the generalized polygons and his technique for deciding the concentration properties of a graph by analysing its eigenvalues can be stated as follows:

Assume $I$ and $O$ are two disjointed sets of sizes $n$ and $m$ respectively. Let $H$ be a bipartite graph with $I$ as input vertex set and $O$ as output vertex set such that edges connect input vertices to output ones, and the degree of each input vertex is $k$ and that of each output vertex is $r = n/m$. Write $A = [a_{ij}]$ for the incidence matrix of $H : a_{ij} = 1$ if the $i$th input vertex $a_i$ is connected to the $j$th output vertex $b_j$ and $= 0$ otherwise. Note $AA^T$ is diagonalizable and has real nonnegative eigenvalues due to it is symmetric and nonnegative definite. If $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the ordered eigenvalues of $AA^T$, then the following holds.

**Theorem 1.6** [18]. If $\lambda_1 > \lambda_2$, then for any $0 < \alpha \leq \frac{m}{n}, H$ is an $(n, \frac{m}{n}, l, \alpha, c(\alpha))$ bsc with

$$c(\alpha) \geq \frac{l^2}{[\alpha(kr - \lambda_2) + \lambda_2]}.$$

**Definition 1.1.** For a bipartite graph $H$ with the vertex bipartition $U(H)$ and $W(H)$, $H$ is called semi-vertex transitive if $Aut(H)$ is transitive on $U(H)$ and $W(H)$ respectively.

**Definition 1.2.** Let $G$ be a group with $L$ and $N$ as its two subgroups. For a set $S$ of some bi-cosets $NgL$, define the bi-coset bipartite graph $X = C(G, L, N; S)$ of $G$ with respect to $L, N, S$ as follows: its vertex set $V(X)$ is $[G : L] \cup [G : N]$, and its edge set $E(X)$ is

$$\{ \{Lg, Nsg\} : g \in G, s \in S \}.$$

Particularly, $BC(G, S) := C(G, \{1\}, \{1\}; S)$ is called the bi-Cayley graph of $G$ with respect to $S$. In addition, if $S$ is a multiset of $G$ chosen independently and uniformly at random,
then we call $X = C(G, L, N; S)$ a random bi-coset graph of $G$ with respect to $L$, $N$ and $S$.

The following Proposition 1.7 on bi-coset bipartite graphs is well-known:

**Proposition 1.7.** (i) Let $X = C(G, L, N; S)$ be the bi-coset graph of $G$ with respect to $L$, $N$, and $S$. Then the following hold.

- The degrees of $Lg$ and $Ng$ are $|S : N|$ and $|S : L|$ respectively for any $g \in G$, and $X$ is regular if and only if $|L| = |N|$.
- $G \subseteq Aut(X)$ (acting on $[G : L]$ and $[G : N]$ by right multiplication), $X$ is semi-vertex transitive.
- $X$ is connected if and only if $G = (S^{-1}S)$.

(ii) Every semi-vertex transitive graph is isomorphic to some bi-coset graph.

For any bi-coset graph $X = C(G, L, N; S)$ of $G$ with respect to $L$, $N$ and $S$, let $A = [a_{ij}]$ be its incidence matrix and $M_L = \frac{1}{\sqrt{|G|}} A A^T$. Then one of our main results is stated as follows.

**Theorem 1.8.** Let $X = C(G, L, N; S)$ be a random bi-coset graph of $G$ with respect to $L$, $N$ and $S$ ($|S| = k$). Then for any $0 < \varepsilon < 1$,

$$P_r \left( \mu^* (M_L) > \varepsilon + \frac{1 - \varepsilon}{2k} \right) \leq 2D(G, L) \exp \left\{ -(k^2 - k) H_2 \left( \frac{1}{2} + \varepsilon \right) \right\} .$$

By Theorem 1.8 and Proposition 1.7(ii), we have

**Corollary 1.9.** The set of all semi-vertex transitive graphs isomorphic to some bi-coset graph $C \left( G, L, N; S \right)$ with $|S| = k$ is just the sample space for the random bi-coset graph $X = C(G, L, N; S)$ with $|S| = k$; and hence in this sense, almost always semi-vertex transitive graphs are concentrators.

Our other main results are constructions of semi-vertex transitive concentrators which are presented in Sections 4-6, which may be simple but we can not find them in the former papers. Moreover we give the theory explanation of the golay codes constructed by Mathieu groups are good.

2 Preliminaries from representation theory

A **representation** $\rho$ of a finite group $G$ is a homomorphism $\rho : G \rightarrow \bigcup(\mathbb{H})$, where $\mathbb{H}$ is a finite dimensional Hilbert space and $\bigcup(\mathbb{H})$ is the group of unitary operators on $\mathbb{H}$. The dimension $d_\rho$ of $\rho$ is the dimension of $\mathbb{H}$. Fix a basis for $\mathbb{H}$, then each $\rho(g)$ is associated with a unique unitary matrix $[\rho(g)]$ satisfying $[\rho(gh)] = [\rho(g)][\rho(h)]$ for any $g, h \in G$.

For a fixed representation $\rho : G \rightarrow \bigcup(\mathbb{H})$, a subspace $V \subseteq \mathbb{H}$ is invariant or a $G$-invariant subspace if $\rho(g) V \subseteq V$ for all $g \in G$. In this case the restriction $\rho_V : G \rightarrow V$ given by restricting $\rho(g)$ to $V$ is also a representation. If $\rho$ has no $G$-invariant subspace other than $\{0\}$ and $\mathbb{H}$, then $\rho$ is irreducible. Equip $\mathbb{H}$ with an inner product $\langle \cdot, \cdot \rangle$, and define a new inner product $\langle \cdot, \cdot \rangle^* $ which is preserved under the action of $\rho$ as follows:

$$\langle v, w \rangle^* = \frac{1}{|G|} \sum_{g \in G} \langle \rho(g)v, \rho(g)w \rangle, \; v, w \in \mathbb{H} .$$

When $\rho$ is not irreducible, there is a nontrivial invariant subspace $V \subseteq \mathbb{H}$; and as $\langle \cdot, \cdot \rangle^*$ is invariant under each unitary map $\rho(g)$, the subspace $V^\perp = \{ u | v \in V, \langle u, v \rangle^* = 0 \}$ is also
invariant. Corresponding to the decomposition $\mathbb{H} = V \bigoplus V^\perp$, $\rho(g)$ has a natural decomposition $\rho(g) = \rho_V(g) \bigoplus \rho_{V^\perp}(g)$ for any $g \in G$. Repeating this process, we see $\mathbb{H}$ has a direct-sum decomposition: $\mathbb{H} = V_1 \bigoplus \cdots \bigoplus V_k$, and the following theorem holds.

**Theorem (Complete reducibility)** 2.1. Any representation $\rho$ can be decomposed into irreducible representations: $\rho = \sigma_1 \oplus \cdots \oplus \sigma_k$, where $\sigma_i = \rho_{V_i}$.

**Definition 2.1.** Any two representations $\rho_1 : G \rightarrow \bigcup(\mathbb{H}_1)$ and $\rho_2 : G \rightarrow \bigcup(\mathbb{H}_2)$ are equivalent if there is an isomorphism $\theta : \mathbb{H}_1 \rightarrow \mathbb{H}_2$ such that

$$\theta(\rho_1(g)v) = \rho_2(g)\theta(v), \quad g \in G, \ v \in \mathbb{H}_1.$$  

**Theorem 2.2.** Any finite group $G$ has only a finite number of irreducible representations up to equivalence.

Let $G^*$ denote a set of representations containing exactly one from each equivalence class. There are two important representations in our analysis: one is the trivial representation $1 : G \rightarrow \bigcup(\mathbb{C}[G])$, i.e., $g \rightarrow id$, which is irreducible; the other is the regular representation $R : G \rightarrow \bigcup(\mathbb{C}[G])$, where $\mathbb{C}[G]$ is the vector space over the complex field $\mathbb{C}$ generated by $G$, that is

$$\mathbb{C}[G] = \left\{ \sum_{g \in G} \alpha_g g \bigg| \alpha_g \in \mathbb{C} \right\}.$$  

Notice $\left( Rg \right) (\sum \alpha_h h) = \Sigma \alpha_h Rgh$; and $R$ is not irreducible and

$$R = \bigoplus_{\rho \in G^*} \rho \bigoplus \cdots \bigoplus \rho.$$  

Since $\mathbb{C}[G]$ has dimension $|G|$, we have $|G| = \sum d_{\rho}^2$. Define $D = D(G)$ by $D = \sum d_{\rho}$. Then

$$\sqrt{|G|} < D(G) \leq |G|.$$  

### 3 Proof of Theorem 1.8

Firstly, we verify the conditions in the proof of [6] Theorem 5 are satisfied. Note [6] Theorem 5 is stated as Theorem 1.4 in our paper.

Let $s_1, \cdots, s_k$ be chosen independently and uniformly at random from $G$ and

$$S = \{s_1, \cdots, s_k\}.$$  

From theorem 1.6, we know that if $\lambda_1 > \lambda_2$, then for any $0 < \alpha \leq \frac{m}{\pi}$, $H$ is an $(n, \frac{m}{\pi}, l, \alpha, c(\alpha))$ bsc with

$$c(\alpha) \geq \frac{l^2}{[\alpha(l \nu - \lambda_2) + \lambda_2]}.$$  

Moreover the eigenvalue ordering of $X \left( (G, H; SS^{-1} - \{ke\}) \right)$ has same magnitude relation to the eigenvalue ordering of $X \left( (G, H; SS^{-1}) \right)$. Thus we consider $X \left( (G, H; SS^{-1} - \{ke\}) \right)$ following.

Let $B = SS^{-1}$ be the multiset as the product of $S$ and $S^{-1}$, and $B^* = SS^{-1} - \{ke\}$, where $e$ is the unit element of $G$,

$$s = \frac{1}{2(k^2 - k)} \sum_{a \in B^*} (a + a^{-1}).$$
Notice the matrix of the linear operator

\[ R(s) = \frac{1}{2(k^2 - k)} \sum_{a \in B^*} (R(a) + R(a^{-1})) \]

with respect to the standard basis of \( \mathbb{C}[G] \) is the normalized adjacency matrix of \( X(G, SS^{-1} - \{k e\}) \), and its eigenvalue 1 corresponds to the trivial representation. By the decomposition of \( R \), we have \( \mu^* = \max_{\rho \in G\setminus \{1\}} \| \rho(s) \| \), where \( \| \cdot \| \) is the operator norm.

For any non-trivial representation \( \rho \) of \( G \), let

\[ B_a = \frac{1}{2} [\rho(a) + \rho(a^{-1})], \ a \in B^*, \]

and \( \mu_1, \ldots, \mu_{d_\rho} \) be the eigenvalues of the \( \sum_{a \in B^*} B_a \) arranged in decreasing order of their absolute values, and \( \lambda \) be an eigenvalue of \( \sum_{a \in B^*} B_a \) chosen uniformly at random. Then

\[ \mathbb{E} [\lambda | s_1, \ldots, s_k] = \frac{1}{d_\rho} \sum_{a \in B^*} T_r(B_a), \ \mathbb{E} [\lambda] = \frac{1}{d_\rho} \mathbb{E} \left[ \sum_{a \in B^*} T_r(B_a) \right]. \]

By the decomposition of \( R \), \( \sum_{\rho \in G} \rho(g) \) is the zero operator, and further we have

\[
\mathbb{E} [\lambda] = \frac{1}{d_\rho} \sum_{1 \leq i < j \leq k} \mathbb{E} \left[ \text{Tr} \left( B_{s_i s_j^{-1}} \right) \right] = \frac{k^2 - k}{d_\rho |G|^2} \sum_{x, y \in G} \text{Tr} \left( B_{xy^{-1}} \right) \\
= \frac{k^2 - k}{2d_\rho |G|^2} \sum_{x, y \in G} \text{Tr} \left( \rho \left( xy^{-1} \right) + \rho \left( yx^{-1} \right) \right) \\
= \frac{k^2 - k}{2d_\rho |G|^2} \sum_{y \in G} \text{Tr} \left( \sum_{x \in G} \rho \left( xy^{-1} \right) + \rho \left( yx^{-1} \right) \right) \\
= \frac{k^2 - k}{d_\rho |G|^2} \sum_{y \in G} \text{Tr} \left( \sum_{g \in G} \rho(g) \right) = 0.
\]

Now let \( \lambda_i = \mathbb{E} (\lambda \mid B_{a_1}, \ldots, B_{a_k}) \), then \( \lambda_0, \ldots, \lambda_{k^2 - k} \) is a martingale with \( |\lambda_i - \lambda_{i-1}| \leq 1 \), since \( \lambda_i - \lambda_{i-1} = \frac{1}{d_\rho} \text{Tr} (B_{a_i}) \) and \( \rho(a_i), \rho(a_i^{-1}) \) are unitary operators, thus \( |\text{Tr}(B_{a_i})| \leq d_\rho \).

Applying Hoeffding-Azuma inequality, we conclude that

\[ \mathbb{P} \left( |\lambda| \geq \varepsilon (k^2 - k) \right) \leq 2 \exp \left\{ -k^2 \varepsilon / 2 \right\} \]

At last we have that

\[
\mathbb{P} \left( \| \rho(s) \| \geq \varepsilon \right) = \mathbb{P} \left( \left\| \frac{1}{k^2 - k} \left( \sum_{i=1}^{k^2} B_{a_i} \right) \right\| \geq \varepsilon \right) \\
= \mathbb{P} \left( |\lambda_1| \geq \varepsilon (k^2 - k) \right) \\
\leq \sum_{i=1}^{d_\rho} \mathbb{P} \left( |\lambda_i| \geq \varepsilon (k^2 - k) \right) \\
= d_\rho \mathbb{P} \left( |\lambda| \geq \varepsilon (k^2 - k) \right) \leq 2d_\rho \exp \left\{ - (k^2 - k) / 2 \right\} \]
Therefore, summing over all irreducible non-trivial representations of $G$, we have the following: For any $0 < \varepsilon < 1$,

$$P_r(\mu^*(X(G, A^*)) > \varepsilon) \leq 2D\exp \left\{ -(k^2 - k)H_\frac{1}{2} \left( \frac{1}{2} + \varepsilon \right) \right\}.$$ 

Notice $AA^T - kI$ is the adjacency matrix of $X(G, H; B^*)$, and the matrix

$$M'_L = \frac{1}{2(|S|^2 - k)}(AA^T - kI)$$

is the normalized adjacency matrix of $X(G, H; B^*)$. If $\lambda_1^* \geq \lambda_2^* \geq \cdots \geq \lambda_n^*$ is the ordered normalized eigenvalues of $AA^T - kI$, then $\frac{(2(|S|^2 - k))\lambda_1^*}{2|S|^2} \geq \frac{(2(|S|^2 - k))\lambda_2^*}{2|S|^2} \geq \cdots \geq \frac{(2(|S|^2 - k))\lambda_n^*}{2|S|^2}$ are the ordered normalized eigenvalues of $AA^T$. Thus we can obtain that if $\lambda_1^* > \lambda_2^*$, then $\frac{(2(|S|^2 - k))\lambda_1^*}{2|S|^2} > \frac{(2(|S|^2 - k))\lambda_2^*}{2|S|^2}$. Let $M_L = \frac{1}{2|S|^2} AA^T$, then by Theorem 1.5, we see that for any $0 < \varepsilon < 1$,

$$P_r \left( \mu^*(M_L) > \varepsilon + \frac{1 - \varepsilon}{2k} \right) \leq 2D(G, H)\exp \left\{ -(k^2 - k)H_\frac{1}{2} \left( \frac{1}{2} + \varepsilon \right) \right\}.$$ 

\[\square\]

**Corollary 3.1.** If $|S| = k$ is large enough, then the random bi-coset graph $X = C(G, L, N; S)$ of $G$ with respect to $L, N$ and $S$ is almost always a concentrator.

**Remark 3.1.** Let $S^* = S \cup \{1\}$. Then the random bi-coset graph $X = C(G, L, N; S^*)$ of $G$ with respect to $L, N$ and $S^*$ is almost always a concentrator if $|S| = k$ is large enough.

## 4 Bsc and semi-vertex transitive graph from generalized polygons

The generalized polygons are incidence structures consisting of points and lines of which the bipartite graphs have diameter $D$ and girth $2D$ for some integer $D$. Tanner\[13\] proved every bipartite graph of the generalized polygons whose any point is incident on $s + 1$ lines and any line is incident on $r + 1$ points, is a good bsc for $D - gons$ with $D = 3, 4, 6, 8$.

Note that for $D = 6$, the smallest thick generalized hexagon has order $(2,2)$, namely $s = 2, r = 2$. Let $X$ be the bipartite graph of the generalized hexagon of order $(2,2)$. It is known that any automorphism of $X$ does not act transitively on the vertices, and $X$ has two orbits that are the two halves of the bipartition, i.e. $X$ is semi-vertex transitive. For other $D$ and $(s, r)$, we do not know generally whether the related bipartite graphs are semi-vertex transitive; but there are some $D$ and $(s, r)$, the related bipartite graphs are not semi-vertex transitive.

**Proposition 4.1.** $X$ is a semi-vertex transitive bsc.
5 Bsc and semi-vertex transitive graph from designs of the Mathieu groups

Definition 5.0.1. Given any natural numbers \(v, k, \gamma, t\) with \(v > k > t \geq 2\). Let \(P\) be a set of elements called points and \(B\) a set of subsets of \(P\) called blocks such that

\[
(a) \ |P| = v, \ (b) \ |B_i| = k \text{ for all } B_i \in B,
(c) \text{ any } t - \text{tuple of points is contained in exactly } \gamma \text{ blocks.}
\]

Then the system \(D = (P, B)\) is called a \(t - (v, k, \gamma)\) design.

Proposition 5.0.1. Assume \(D = (P, B)\) is a \(t - (v, k, \gamma)\) design. Then for any natural number \(s \leq t\) and any subset \(S\) of \(P\) with \(|S| = s\), the total number of blocks incident with each element of \(S\) is given by

\[
\gamma_s = \frac{(v-s)(v-s-1) \cdots (v-t+1)}{(k-s)(k-s-1) \cdots (k-t+1)}.
\]

Particularly, a \(t - (v, k, \gamma)\) design is also an \(s - (v, k, \gamma_s)\) design.

Let \(r := \gamma_1\) be the total number of blocks incident with a given point.

Proposition 5.0.2. Let \(D = (P, B)\) be a \(t - (v, k, \gamma)\) design. Then

\[
bk = vr, \quad \binom{v}{t} \gamma = b \binom{k}{t}, \quad r(k-1) = \gamma_2(v-1).
\]

Definition 5.0.2. A balanced incomplete block design (BIBD) with parameters \((v, b, r, k, \lambda)\) is an arrangement of \(v\) distinct objects \(b\) blocks such that each block contains exactly \(k\) distinct objects, each object occurs in exactly \(r\) different blocks, and every pair of distinct objects \(a_i, a_j\) occur together in exactly \(\lambda\) blocks. Obviously a BIBD is a \(2 - (v, b, r, k, \gamma_2)\) design.

Definition 5.0.3. Assume \(P = \{a_1, \cdots, a_v\}\) is a set of \(v\) objects, and \(B = \{B_1, \cdots, B_b\}\) is a set of \(b\) blocks consisting of elements of \(P\), and \(D = (P, B)\) is a BIBD with parameters \((v, b, r, k, \lambda)\). Define \(X_D\) as the following bipartite graph: This bipartite graph has \(A\) as the set of left vertices and \(B\) as the set of right vertices such that there is an edge between an \(a_i\) and a \(B_j\) with \(a_i \in B_j\).

Let \(v \times b\) matrix \(A = (a_{i,j})\) be the incidence matrix of \(D\), where \(a_{i,j} = I\{a_i \in B_j\}\).

Lemma 5.0.1. The character and minimal polynomials of \(AA^T\) are respectively

\[
P(x) = (x - rk)(x - (r - \lambda))^{v-1}, \quad x \in \mathbb{R}^1,
\]

\[
m(x) = (x - rk)(x - (r - \lambda)), \quad x \in \mathbb{R}^1.
\]

Hence \(\lambda_1 = rk, \ \lambda_2 = r - \lambda\) and \(r - \lambda < rk\), and \(X_D\) is a bsc.

For any graph \(X\) with \(n\) vertices, its adjacency matrix has \(n\) eigenvalues which are denoted by \(\mu_0 \geq \mu_1 \geq \cdots \geq \mu_{n-1}\) with decreasing order.

Definition 5.0.4 (Bipartite Ramanujan Graphs). For any \((c, d)-\text{regular bipartite graph}\) \(X\), we call it a Ramanujan graph if \(\mu_1 \leq \sqrt{c-1} + \sqrt{d-1}\).

Recall Høholdt and Janwa\[7\] proved the bipartite graph of a \((v, b, r, k, \lambda)\) BIBD is a \((r, k)-\text{regular}\) bipartite Ramanujan graph with \(\mu_1 = \sqrt{r - \lambda}\), and it is the optimal expander graph with these parameters.
Let $D = (P, B)$ be a $t - (v, k, \gamma)$ design and $p$ a point of $D$. Define a new design $D_p$ depending on $p$ as follows: its point set is $P - \{p\}$, and its block set is

$$\{B - \{p\} \mid B \text{ is any block containing } p \text{ in } D\}.$$ 

Note $D_p$ is a $(t - 1) - (v - 1, k - 1, \gamma)$ design and we call it the contraction of $D$ at $p$.

Given a permutation $g$ of $P$, if for any block $B$ of $D$, $B^g = \{x^g : x \in B\}$ is also a block of $D$, then $g$ is called a automorphism of $D$. Clearly all automorphisms of $D$ forming a group, which is called the automorphism group of $D$ and denoted by $\text{Aut}(D)$.

In Subsections 5.1-5.2, we will construct some semi-vertex transitive $bscs$ from Mathieu groups.

### 5.1 The large Mathieu groups

The projective plane $\text{PG}(2, 4)$ can be extended 3 times leading to the unique designs with parameters $3 - (22, 6, 1)$, $4 - (23, 7, 1)$ and $5 - (24, 8, 1)$, denoted by $M_{22}, M_{23}$ and $M_{24}$ respectively. It is well-known that $\text{Aut}(M_i)$ is the Mathieu group $M_i$ for any $i \in \{22, 23, 24\}$.

Let $\Omega$ be a set with $|\Omega| = n$, and $H$ a subgroup of the symmetric group $\text{Sym}(n)$. We say $H$ is transitive on $\Omega$ if for every $a$ and $b$ in $\Omega$, there exists $\pi \in H$ such that $a^\pi = b$. Given a natural number $k \leq n$. If for every list of $k$ distinct points $a_1, \ldots, a_k$ and very list of $k$ distinct points $b_1 \cdots, b_k$, there exists a $\pi \in H$ such that $a_i^\pi = b_i$ for all $i$, then $H$ is said to be $k$-transitive. Clearly, if $H \neq \{1\}$ is $k$-transitive, then it is $m$-transitive for all $m \leq k$.

**Proposition 5.1.1.** The group $M_{24}$ is 5-transitive on the 24 points of $M_{24}$, and the group $M_{24-i}$ is $(5 - i)$-transitive on $24 - i$ points of $M_{24-i}$ for any $i \in \{1, 2\}$. Particularly, $M_j$ is transitive on the $j$ points for any $j \in \{22, 23, 24\}$.

**Theorem 5.1.1.** For any $i \in \{22, 23, 24\}$, let $X_{M_i}$ be the corresponding bipartite graph of the design $M_i$. Then each $X_{M_i}$ is a semi-vertex transitive $bsc$.

**Proof:** Fix an $i \in \{22, 23, 24\}$. Let $M_i = (P, B)$. Then for any $a$ and $b$ in $P$, there exists $g \in M_i$ with $a^g = b$. Assume $a \in B_j$ and $B_j \in B$, since $g$ is a automorphism of the design, we have that $B_j^g \in B$, and $g$ maps the neighbors of $a$ to the neighbors of $b$, and $X_{M_i}$ is semi-vertex transitive.

By Proposition 5.0.2, $3 - (22, 6, 1)$, $4 - (23, 7, 1)$ and $5 - (24, 8, 1)$ are $\{22, 23, 21, 6, 5\}$, $(23, 506, 77, 6, 21)$ and $(24, 759, 253, 8, 77) BIBDs$ respectively. From Lemma 5.0.1 and Theorem 5.1.1, we obtain that each $X_{M_i}$ is a $bsc$. Moreover, each $X_{M_i}$ is a bipartite Ramanujan graph by the main result of H	extcite{holdt1997ramanujan}.

### 5.2 The small Mathieu groups

Let $\Omega = \{1, 2, \cdots, 12\}$ and consider the following permutations on $\Omega$:

\[
\begin{align*}
\mu &= (1 \ 2 \ 3)(4 \ 5 \ 6)(7 \ 8 \ 9), \\
b &= (2 \ 5 \ 3 \ 9)(4 \ 8 \ 7 \ 6), \\
y &= (10 \ 11)(4 \ 7)(5 \ 8)(6 \ 9), \\
a &= (2 \ 4 \ 3 \ 7)(5 \ 6 \ 8 \ 9), \\
x &= (1 \ 10)(4 \ 5)(6 \ 8)(7 \ 9), \\
z &= (11 \ 12)(4 \ 9)(5 \ 7)(6 \ 8).
\end{align*}
\]

Then $M_{12} = \langle \mu, a, b, x, y, z \rangle$. Let $\Delta = \{1, 2, 3, 10, 11, 12\}$ and $D_{12} = (\Omega, B)$, where

$$B = \{\Delta^g : g \in M_{12}\}.$$ 

Note $M_{12}$ is the automorphism group of the design $D_{12}$, which is a $5 - (12, 6, 1)$ design. Let $p$ be any element of $\Omega$, say $p = 12$. Then the contraction $(D_{12})_p$ of $D_{12}$ at $p = 12$ is a $4 - (11, 5, 1)$ design and $M_{11} = \text{Aut}((D_{12})_p)$. Similarly we can construct a $3 - (10, 4, 1)$...
design and a $2 - (9, 3, 1)$ design, $M_{10}$ and $M_9$ are their automorphism groups respectively. So we can denote the designs $4 - (11, 5, 1), 3 - (10, 4, 1),$ $2 - (9, 3, 1)$ by $D_{11}, D_{10}, D_9$ respectively.

Notice $5 - (12, 6, 1), 4 - (11, 5, 1), 3 - (10, 4, 1), 2 - (9, 3, 1)$ are respectively $(12, 132, 66, 6, 30), (11, 66, 30, 5, 12), (10, 30, 12, 4, 4), (9, 36, 8, 3, 1)$ BIBDs. Similarly to Theorem 5.1.1, we can prove

**Theorem 5.2.1.** Let $X_{D_i}$ be the corresponding bipartite graph of the design $D_i$, then for any $i \in \{9, 10, 11, 12\}$, $X_{D_i}$ is a semi-vertex transitive $bsc$.

### 6 Symmetric group and a sequence of concentrators

Assume $X = (V, E)$ be a connected graph with $n$ vertices and $Q = Q_X = diag(d(v)) - A_X$ where $A_X$ is its adjacency matrix. Let $\lambda(X)$ be the second smallest eigenvalue of $Q$. When $X$ be a $d$ - regular graph, then $\lambda(X)$ is the difference between $d$ and the second largest eigenvalue of $X$. The following result holds.

**Theorem 6.1**. If a $d$-regular graph $X = (V, E)$ is an $(n, d, \varepsilon) - magnifier$, then

$$\lambda(G) \geq \frac{\varepsilon^2}{4 + 2\varepsilon^2}.$$ 

Recall the following result from Kassabov[9].

**Theorem 6.2**. For every natural number $n$, there is a generating set $S_n$ (of size at most $\ell$) of the alternating group $Alt(n)$ such that the Cayley graphs $X(Alt(n), S_n)$ form a family of $\varepsilon - expanders$ ($magnifiers$). Here $\ell$ and $\varepsilon > 0$ are some universal constants. Similarly there is a generating set $\tilde{S}_n$ of the symmetric group $Sym(n)$ with the same property.

Let $L$ be a subgroup of $Alt(n)$ or $Sym(n)$. Then the bi-coset graphs

$$X = C(G, L, \{1\}, (S_n \cup \{1\}) L) \text{ and } Y = C(G, L, \{1\}, \left(\tilde{S}_n \cup \{1\}\right) L)$$

are semi-vertex transitive graphs, and are respectively $\left(|(S_n \cup \{1\}) L|, |S_n \cup \{1\}\right) -$ bipartite graph and $\left(\left|\tilde{S}_n \cup \{1\}\right| L, \left|\tilde{S}_n \cup \{1\}\right|\right) -$ bipartite graph.

**Theorem 6.3.** The bi-coset graphs

$$X = C(G, L, \{1\}, (S_n \cup \{1\}) L) \text{ and } Y = C(G, L, \{1\}, \left(\tilde{S}_n \cup \{1\}\right) L)$$

are $bscs$.

**Proof:** We only prove the case for alternating groups. Let $M_L^T = AA^T$, where $A$ is the adjacency matrix of $X$. Then $M_L^T$ is a symmetric nonnegative definite matrix and has nonnegative eigenvalues, which can be considered as the adjacency matrix of the coset graph

$$D = (G/L, (S_n \cup \{1\})(S_n \cup \{1\})^{-1}).$$

Notice $D$ is a $2|\left|S_n \cup \{1\}\right|(S_n \cup \{1\})^{-1} - regular$ graph. Assume

$$G/L = \{L_{g_1}, \cdots, L_{g_m}\} \text{ and } A = \{L_{g_1}, \cdots, L_{g_r}\} \text{ with } r \leq \left\lfloor \frac{m}{2} \right\rfloor.$$

10
Then by $X(\text{Alt}(n), S_n)$ is an $\varepsilon$-expander (magnifier), we have

$$|A((S_n \cup \{1\})(S_n \cup \{1\})^{-1}) \setminus A| \geq |A(S_n \cup \{1\}) \setminus A| = |A_{S_n \setminus A| \geq \varepsilon |A|.$$  

Hence we obtain

$$|((A((S_n \cup \{1\})(S_n \cup \{1\})^{-1})/(L)) \setminus (A/L)|$$

$$\geq |((A(S_n \cup \{1\})/L) \setminus (A/L)) = |((A_{S_n}/L) \setminus (A/L))| \geq \varepsilon |A/L|.$$  

Therefore,

$$D = (G/L, (S_n \cup \{1\})(S_n \cup \{1\})^{-1})$$

is an $\varepsilon$-expander graph, namely an $(n, d, \varepsilon)$-magnifier. By Theorem 6.1,

$$X = C(G, L, \{1\}, (S_n \cup \{1\})L)$$

is a bsc. \qed

7 Concluding remarks

We prove the random bi-coset graphs are almost always concentrators, and construct some examples of semi-vertex transitive bscs. Because generalized $D-gons$ do not exist for arbitrary parameters $n$ and $k$, Tanner \cite{18} did not provide a complete solution to the problem of constructing concentrators. However, we can get a sequence of concentrators by symmetric groups or alternating groups with their appropriate subgroups for arbitrary parameter $n$ and some $k \leq \ell$ in Section 6.

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