The global attractors and their Hausdorff and fractal dimensions estimation for the higher-order nonlinear Kirchhoff-type equation*

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Abstract
We investigate the global well-posedness and the longtime dynamics of solutions for the higher-order Kirchhoff-type equation with nonlinear strongly dissipation:

\[ u_{tt} + (-\Delta)^m u_t + \phi((\frac{m}{2} \Delta) u^2)(-\Delta)^m u + g(u) = f(x), \quad x \in \Omega, \ t > 0, m > 1 \]

Under the proper assume, the main results are that existence and uniqueness of the solution is proved by using priori estimate and Galerkin method, the existence of the global attractor with finite-dimension, and estimation Hausdorff and fractal dimensions of the global attractor.

Key words: Higher order; Attractor; Kirchhoff; Hausdorff dimension; Fractal dimension

1 Introduction

We consider the problem

\[ u_{tt} + (-\Delta)^m u_t + \phi((\frac{m}{2} \Delta) u^2)(-\Delta)^m u + g(u) = f(x), \quad x \in \Omega, \ t > 0, m > 1 \] (1.1)

\[ u(x, t) = 0, \quad \frac{\partial u}{\partial n} = 0, \quad \ldots, \quad i = 1, 2, \ldots, n \Omega - 1 \] (1.2)

\[ u(x, 0) = u_0, \quad \frac{\partial u}{\partial t} (x, 0) = 0 \] (1.3)

Where \( \Omega \) is a bounded domain of \( \mathbb{R}^n \), with a smooth Dirichlet boundary \( \partial \Omega \) and initial value, the damping coefficient is function of the \( L^2 \)-norm of the gradient \( m \) power, \( g(u) \) is a nonlinear forcing, \( (-\Delta)^m u_t \) is a strongly dissipation.

There have been many researches on the global attractors existence of the Kirchhoff equation with strong dissipation, we can see [1,2,3]. There are lots of recent results on the global attractor of Kirchhoff equation, we can refer [4,5,6,7].

Zhijian Yang and Pengyan Ding [8] studied the longtime dynamics of the Kirchhoff equation with strong damping and critical nonlinearity on \( \mathbb{R}^n \):

\[ u_{tt} - \Delta u_t - M(\nabla u \Delta u^2)\Delta u + u_x + g(x, u) = f(x). \] (1.4)

They obtain the well-posedness, the existence of the global and exponential attractors in \( H = L^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \) in critical nonlinearity case. Their novelty is that it overcomes the essential difficulties that is the Sobolev embedding on \( \mathbb{R}^n \) and the critical growth of \( g \) cause the lack of compactness.
Recently, Zhijian Yang, Pengyan Ding and Lei Li [9] also studied longtime dynamics of the Kirchhoff equation with fractional damping and supercritical nonlinearity:

\[ u_{tt} - M (\nabla u \nabla u) \Delta u + (-\Delta)^{\alpha} u_t + f(u) = g(x), \quad x \in \Omega, \ t > 0, \]  

\[ u \big|_{t=0} = 0, \quad u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \]  

(1.5) (1.6)

Where \( \alpha \in (\frac{1}{2},1) \), \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with the smooth boundary, they show that (i) even if \( p \) (the growth exponent \( p \) of the nonlinearity \( f(u) \)), \( 1 \leq p \leq \frac{N+4\alpha}{(N-4\alpha)^+} \), the well-posedness and longtime behavior of the solutions of the equation are of the characters of the parabolic equation; (ii) when \( \frac{N+4\alpha}{(N-4\alpha)^+} \leq p < \frac{N+4}{(N-4)^+} \), the limit solutions exist and possesses a weak global attractor.

Chueshov [10] first studied the well-posedness and the global attractor for the IBVP of Kirchhoff wave models with strong nonlinear damping:

\[ u_{tt} - \sigma (\nabla u \nabla u) + \phi (\nabla u \nabla u) \Delta u + g(u) = h(x) \]  

(1.7)

He established a finite-dimensional global attractor in the sense of partially strong topology. In particular, in nonsupercritical case: (i) the partially strong topology becomes strong; (ii) an exponential attractor is obtained in natural energy space \( H(\Omega) = H^1(\Omega) \cap L^{p+1}(\Omega) \times L^2(\Omega) \).

Guigui Xu and Guoguang Lin [11] studied the global attractor and their dimensions estimation for the generalized Boussinesq equation:

\[ u_{tt} - \Delta u - \Delta u_{tt} + \alpha \Delta^2 u + \beta \Delta^2 u_{tt} - \Delta u_{tt} - \Delta |u|^p = f(x). \]  

(1.8)

Under the existence of the global solution, it is discussed that the global attractor and infinite Hausdorff dimension and fractional dimension.

The main details of this paper are arranged as follow:

In section 2, under the assume of Lemma 2.1 and Lemma 2.2, we get the existence and uniqueness of solution; in section 3, we obtain the global attractor of the problems (1.1)-(1.3); in section 4, we consider the finite Hausdorff dimension and fractal dimension of the global attractor.

2 Main results of the paper

For convenience, we denote the simple symbol, \( \| . \| \) represents norm, \( (., .) \) represents inner product.

and \( f = f(x), H^m_0(\Omega) = H^m(\Omega) \cap H^{1}_0(\Omega), H^{2m}(\Omega) = H^{2m}(\Omega), \ (-\Delta)^{\frac{m}{2}} = D^m, \ \| . \|_L^\infty, \ C_i \ (i=0,1\cdots 7) \) is a constant, \( m_i \ (i=0,1\cdots 4) \) is also a constant.

Lemma 2.1. Assume
(1) $\phi(\Box D^n u \Box^2) : R^+ \rightarrow R^+$ is a differentiable function;

(2) $\epsilon \phi(\Box D^n u \Box^2) \Box D^n u \Box^2 \geq \epsilon \Phi(\Box D^n u \Box^2) + \frac{1}{4} \epsilon^2 \Box D^n u \Box^2$ where $\Phi' = \phi$;

(3) $\Phi(\Box D^n u \Box^2) \geq \epsilon \Box D^n u \Box^2 + C_0$;

(4) $g(u) u \geq \epsilon G(u) + \frac{1}{2} u^2$, where $G'(u) = g(u)u$;

(5) $J(u) = \int G(u) dx$;

(6) $f(x) \in L^2(\Omega)$.

Then the solution $(u, v)$ of the problems (1.1) - (1.3) satisfies $(u, v) \in H^m(\Omega) \times L^2(\Omega)$, and satisfies:

$$
\Box (u, v) \Box^2_{H^m \times L^2} \equiv \Box D^n u \Box^2 + \Box v \Box^2 \leq W(0)e^{-\alpha t} + \frac{C}{\alpha}(1 - e^{-\alpha t}).
$$

(2.1)

Where $u = u_i + \epsilon u$, $W(0) = v_0^2 + \epsilon^2 u_0^2 + \epsilon \Box D^n u_0 \Box^2 + 2 \int J(u_0)$, there exist $t = t_i > 0$ and $R_0$, such that

$$
\lim_{t \rightarrow \infty} \Box (u, v) \Box^2 \leq \frac{C}{\alpha} = R_0.
$$

(2.2)

Proof: Let $v = u_i + \epsilon u$ we use $v$ multiply both sides of equation (1.1) and obtain

$$
(a_{ii} + (-\Delta)^m u_j + \phi(\Box D^n u \Box^2)(-\Delta)^m u + g(u), v) = (f(x), v).
$$

(2.3)

$$
(a_{ii}, v) = (v - \epsilon u, v) = (v, v) - \epsilon(v - \epsilon u, v)
= \frac{1}{2} \frac{d}{dt} \Box v \Box^2 + \frac{1}{2} \frac{d}{dt} \Box u \Box^2 - \Box v \Box^2 + \epsilon^2 \Box u \Box^2.
$$

(2.4)

$$
((-\Delta)^m u_j, v)
= (D^n v - \epsilon D^n u, D^n v)
= \Box D^n v \Box^2 - \epsilon(D^n u, D^n u) + \epsilon D^n u
$$

(5.5)
\[
\phi(\sum D^n u \Box^2) (-\Delta) u + \epsilon u
\]
\[
= \phi(\sum D^n u \Box^2) (-\Delta) u + \epsilon u + \epsilon \phi(\sum D^n u \Box^2) \sum D^n u \Box^2
\]
\[
= \frac{1}{2} d t \Phi(\sum D^n u \Box^2) + \epsilon \phi(\sum D^n u \Box^2) \sum D^n u \Box^2
\]
\[
\geq \frac{1}{2} d t \Phi(\sum D^n u \Box^2) + \epsilon \Phi(\sum D^n u \Box^2) + \frac{1}{4} \epsilon^2 \sum D^n u \Box^2
\]

\[
(\mathcal{g}(u), v)
\]
\[
= (\mathcal{g}(u), u) + \epsilon (\mathcal{g}(u), u)
\]
\[
= \frac{d}{dt} \int G(u) dx + \epsilon \mathcal{g}(u, u)
\]
\[
\geq \frac{d}{dt} \int G(u) dx + \epsilon^2 \int G(u) dx
\]
\[
\geq \frac{d}{dt} J(u) + J(u).
\]

\[
(f(x), v)
\]
\[
\leq \frac{1}{2\epsilon^2} \int f \Box^2 + \frac{2}{\epsilon} \int v \Box^2.
\]

From the above, we have
\[
\frac{d}{dt}[\sum v \Box^2 + \epsilon^2 \sum u \Box^2 + \epsilon \sum D^n u \Box^2 + 2J(u)]
\]
\[
+ (2m \lambda_1 - \epsilon^2 - 2\epsilon) \sum v \Box^2 + 2\epsilon \sum u \Box^2 + (2\epsilon - \epsilon^2) \sum D^n u \Box^2 + 2\epsilon^2 J(u)
\]
\[
\leq \frac{1}{2}\epsilon^2 \int f \Box^2 + C_0 := C.
\]

Where we take proper constant \( m_0 \) and \( \epsilon \), such that:
\[
\begin{cases}
\alpha_1 = 2m_0 \lambda_1 - \epsilon^2 - 2\epsilon \\
\alpha_2 = 2\epsilon - \epsilon^2 \\
\end{cases}
\]
\[
(2.10)
\]

Then we take \( \alpha = m \min \{ \alpha_1, 2\epsilon, \epsilon \frac{\lambda_1}{m_0}, \epsilon^2 \} \), we obtain:
\[
\frac{d}{dt} W(t) + \alpha W(t) \leq C,
\]
\[
(2.11)
\]

where
\[
W(t) = \int v \Box^2 + \epsilon^2 \sum u \Box^2 + \epsilon \sum D^n u \Box^2 + 2J(u).
\]
\[
(2.12)
\]

By using Gronwall inequality, we obtain:
\[
W(t) \leq W(0)e^{-\alpha t} + \frac{C}{\alpha}(1 - e^{-\alpha t}),
\]
\[
(2.13)
\]

where
\[ W(0) = v_0 \|^2 + \varepsilon \|^2 + u_0 \|^2 + \varepsilon D^\alpha u_0 \|^2 + 2J(u_0). \] (2.14)

So, we have:

\[ \Box (u,v) \|^2_{H^{2\alpha,2}} = \Box D^\alpha u \|^2 + \Box v \|^2 \leq W(0)e^{-\alpha t} + \frac{C}{\alpha}(1 - e^{-\alpha t}). \] (2.15)

And

\[ \lim_{t \to \infty} \Box (u,v) \|^2_{H^{2\alpha,2}} \leq \frac{C}{\alpha}. \] (2.16)

Thus there exist \( t = t_1(\Omega) \) and \( R_1 \), such that

\[ \Box (u,v) \|^2_{H^{2\alpha,2}} \leq \frac{C}{\alpha} = R_1(t > t_1) . \] (2.17)

**Lemma 2.2.** Assume

1. \( g(u) \leq C_1(1 + |u|^p) \), \( p \leq \frac{2n}{n - 2\alpha} \), \( n \geq 3 \);

2. \( \varepsilon_1 \leq \mu_0 \leq \mu \leq \mu_1 \), \( \mu \approx \left\{ \begin{array}{ll} \mu_0 : & \frac{d}{dt} \Box D^\alpha u \|^2 \geq 0 \\ \mu_1 : & \frac{d}{dt} \Box D^\alpha u \|^2 < 0 \end{array} \right. \);

3. \( 0 \leq \frac{d\phi(x)}{dx} \leq C_2 \);

4. \( f(x) \in L^2(\Omega) \).

Then the solution \( (u,v) \) of the problems (1.1) - (1.3) satisfies \( (u,v) \in H^{2\alpha}(\Omega) \times H^\alpha(\Omega) \), and satisfies

\[ \Box (u,v) \|^2_{H^{2\alpha,2}} + \Box M(0) \leq M(0)e^{-\beta t} + \frac{C}{\beta}(1 - e^{-\beta t}) . \] (2.18)

where \( v = u_0 + \varepsilon_1u_0 + M(0) = \Box D^\alpha v_0 \|^2 + \Box D^\alpha u_0 \|^2 + \Box (\mu - \varepsilon_1)D^\alpha u_0 \|^2 \), there exist \( t = t_2(\Omega) \) and \( R_2 \), such that

\[ \lim_{t \to \infty} \Box (u,v) \|^2_{H^{2\alpha,2}} \leq \frac{C}{\beta} = R_2 . \] (2.19)

**Proof** Let \( (\Delta)^\alpha u = (\Delta)^\alpha u_1 + \varepsilon_1(\Delta)^\alpha u \) we use \( (\Delta)^\alpha u \) multiply both sides of equation (1.1) and obtain

\[ (u_n + (\Delta)^\alpha u_1 + \phi(\Box D^\alpha u \|^2)(\Delta)^\alpha u + g(u), (\Delta)^\alpha v) = (f(x), (\Delta)^\alpha v) . \] (2.20)
\[ (u_{\mu}, (-\Delta)^{n} v) \]
\[ = (v_{\mu} - \varepsilon_{1} u_{\mu}, (-\Delta)^{n} v) \]
\[ = \frac{1}{2} \frac{d}{dt} \square D_{\mu}^{n} v \square - \frac{1}{2} \varepsilon_{1} (v_{\mu} - \varepsilon_{1} u_{\mu}, (-\Delta)^{n} v) \]
\[ = \frac{1}{2} \frac{d}{dt} \square D_{\mu}^{n} v \square - \frac{1}{2} \varepsilon_{1} \square D_{\mu}^{n} v \square + \frac{1}{2} \varepsilon_{1} (u_{\mu}, (-\Delta)^{n} u_{\mu} + \varepsilon_{1} (-\Delta)^{n} u) \]
\[ = \frac{1}{2} \frac{d}{dt} \square D_{\mu}^{n} v \square - \frac{1}{2} \varepsilon_{1} \square D_{\mu}^{n} v \square + \frac{1}{2} \varepsilon_{1} \frac{1}{2} \frac{d}{dt} \square D_{\mu}^{n} u \square + \frac{3}{2} \varepsilon_{1} D_{\mu}^{n} u \square . \]

\[ \langle (-\Delta)^{n} u, (-\Delta)^{n} v \rangle \]
\[ = \langle (-\Delta)^{n} u - \varepsilon_{1} (-\Delta)^{n} u, (-\Delta)^{n} v \rangle \]
\[ = \langle (-\Delta)^{n} v \square - \varepsilon_{1} (-\Delta)^{n} u \square, (-\Delta)^{n} v \rangle \]
\[ = \langle (-\Delta)^{n} v \square - \frac{1}{2} \varepsilon_{1} \frac{d}{dt} \square (-\Delta)^{n} u \square - \varepsilon_{1} \square (-\Delta)^{n} u \square \rangle \]
\[ \geq \frac{1}{2} m_{2} \lambda_{2} \square D^{n} v \square + \frac{1}{2} \\square (-\Delta)^{n} v \square + \frac{1}{4} \\square (-\Delta)^{n} v \square \]
\[ + \frac{1}{8} \square (-\Delta)^{n} u \square - \frac{1}{2} \varepsilon_{1} \frac{d}{dt} \\square (-\Delta)^{n} u \square - \varepsilon_{1} \square (-\Delta)^{n} u \square . \]

\[ (\phi \square D_{\mu}^{n} u \square) (-\Delta)^{n} u, (-\Delta)^{n} v \]
\[ = \phi (\square D_{\mu}^{n} u \square) \frac{1}{2} \frac{d}{dt} \square (-\Delta)^{n} u \square + \varepsilon_{1} \phi (\square D_{\mu}^{n} u \square) \square (-\Delta)^{n} u \square \]
\[ \geq \mu \frac{1}{2} \frac{d}{dt} \\square (-\Delta)^{n} u \square + \varepsilon_{1} \mu_{0} \\square (-\Delta)^{n} u \square . \]

\[ (g(u), (-\Delta)^{n} v) \]
\[ \geq - \frac{1}{2} \square g(u) \square - \frac{1}{2} \\square (-\Delta)^{n} v \square . \]

According to assume (1), we can get \( \square g(u) \square \leq C_{4} \square u \square^{p} + C_{4} \), and accord to Poincare inequality \( \square g(u) \square \leq C_{4} m_{2} \lambda_{3} \square D_{\mu}^{n} u \square^{p} + C_{4} \), then accord to Lemma2.1. \( \square D_{\mu}^{n} u \square^{p} < \infty \), so, we have \( \square g(u) \square \leq C_{4} \).

\[ (g(u), (-\Delta)^{n} v) \]
\[ \geq - C_{6} - \frac{1}{2} \\square (-\Delta)^{n} v \square . \]

\[ (f(x), (-\Delta)^{n} v) \leq \\square f \square + \frac{1}{4} \\square (-\Delta)^{n} v \square . \]

From the above , we have
\[ \frac{d}{dt} \square D_{\mu}^{n} v \square + \varepsilon_{1} \square D_{\mu}^{n} u \square + (\mu - \varepsilon_{1}) \square (-\Delta)^{n} u \square \]
\[ \geq \frac{1}{4} m_{2} \lambda_{3} - 2 \varepsilon_{1} \square D_{\mu}^{n} v \square + 2 \varepsilon_{1} \square D_{\mu}^{n} u \square + (-2 \varepsilon_{1}^{2} + 2 \varepsilon_{1} \mu_{0}) \square (-\Delta)^{n} u \square \]
\[ \geq \frac{1}{4} \\square (-\Delta)^{n} v \square \leq \\square f \square + C_{6} := C_{5} . \]

Next, accord to assume (2), we see \( \mu - \varepsilon_{1} \geq 0, -2 \varepsilon_{1}^{2} + 2 \varepsilon_{1} \mu_{0} \geq 0 \) and we take proper constant \( m_{2} \) such that \( \frac{1}{4} m_{2} \lambda_{3} - 2 \varepsilon_{1} \geq 0 \).
Then, we take \( \beta = \min \left\{ \frac{1}{4} m_x \lambda_x - 2 \epsilon_1, 2 \epsilon_1, \frac{-2 \epsilon_2 + 2 \epsilon_1 \mu_1}{\mu - \epsilon_1} \right\} \), we obtain

\[
\frac{d}{dt} M(t) + \beta M(t) \leq C_\delta,
\]

(2.28)

where

\[
M(t) = \Box \, D^n v \, \Box^2 + \epsilon_1^2 \Box \, D^n u \, \Box^2 + (\mu - \epsilon_1) \Box \, (-\Delta)^n u \, \Box^2.
\]

(2.29)

By using Gronwall inequality, we obtain

\[
\Box (u, v) \leq \Box (\Delta)^n u \leq M(0)e^{-\beta t} + \frac{C_1}{\beta}(1 - e^{-\beta t}).
\]

(2.30)

And

\[
\lim_{t \to \infty} \Box (u, v) \leq \frac{C_1}{\beta} = R_1.
\]

(2.31)

Thus there exist \( t = t_2(\Omega) \) and \( R_1 \), such that

\[
\Box (u, v) \leq \frac{C_1}{\beta} = R_1, \quad (t > t_2).
\]

(2.32)

**Theorem 2.1.** Lemma 2.1, Lemma 2.2 holds; the initial boundary value problem (1.1) with Dirichlet boundary exists unique smooth solution \((u, v) \in L^\infty ([0, +\infty); H^m \times H^m)\).

Proof. By Lemma 2.1-Lemma 2.2 and Glerkin method, we can easily obtain the existence of solution of equation \((u, v) \in L^\infty ([0, +\infty); H^m \times H^m)\), the procedure is omitted. Next, we prove the uniqueness of solution in detail.

Let \( u, v \) are two solutions of equation (1.1), we denote \( w = u - v \), then two equations subtract and obtain

\[
w_u + (-\Delta)^n w + \phi(\Box D^n u \, \Box^2)(-\Delta)^n u - \phi(\Box D^n v \, \Box^2)(-\Delta)^n v + g(u) - g(v) = 0.
\]

(2.33)

By using \( w \) to inner product of the equation (2.33), and we have

\[
(w_u + (-\Delta)^n w + \phi(\Box D^n u \, \Box^2)(-\Delta)^n u - \phi(\Box D^n v \, \Box^2)(-\Delta)^n v + g(u) - g(v), w) = 0
\]

(2.34)

\[
((-\Delta)^n w, w) = \Box \, D^n w \, \Box^2 \geq m_x \lambda_x \Box \, w^2.
\]

(2.35)
\( (\phi (\Box D^n u \Box^2) (\Box^2 D^n v - \phi (\Box D^n v \Box^2)) (\Box^2 D^n v, w_j) \)
\[
= (\phi (\Box D^n u \Box^2) (\Box^2 D^n v - \phi (\Box D^n v \Box^2)) (\Box^2 D^n v, w_j) + \phi (\Box D^n u \Box^2) (\Box^2 D^n u, w_j)
\]
\[
= \phi (\Box D^n u \Box^2) (\Box^2 D^n v, w_j) + \phi (\Box D^n u \Box^2) (\Box^2 D^n v, w_j)
\]
\[
= \frac{1}{2} \phi (\Box D^n u \Box^2) \frac{d}{dt} D^n w \Box^2 + \phi (\Box D^n u \Box^2) (\Box^2 D^n u, w_j) (\Box^2 D^n v, w_j).}
\]
\[
(2.36)
\]

where
\[
\phi (\Box D^n u \Box^2) (\Box^2 D^n v, w_j) \leq \phi (\Box D^n u \Box^2) (\Box^2 D^n u, w_j) (\Box^2 D^n v, w_j).
\]
\[
(2.37)
\]

According to Lemma 2.1, Lemma 2.2 and young inequality; so, exist a constant \( C_1 \) such that
\[
\phi (\Box D^n u \Box^2) (\Box^2 D^n v, w_j) \leq C_1 (\Box D^n u \Box^2 + \Box D^n v, w_j).
\]
\[
(2.38)
\]

According to (2.37) – (2.39), we have
\[
(\phi (\Box D^n u \Box^2) (\Box^2 D^n v - \phi (\Box D^n v \Box^2)) (\Box^2 D^n v, w_j)
\]
\[
\geq \frac{\mu}{2} \frac{d}{dt} \Box D^n w \Box^2 + \frac{C_1}{2} \Box D^n w \Box^2 + \Box (w_j \Box^2)
\]
\[
\geq \frac{\mu}{2} \frac{d}{dt} \Box D^n w \Box^2 - \frac{C_1}{2} \Box D^n w \Box^2 - \frac{C_2}{2} \Box w_j \Box^2.
\]
\[
(2.39)
\]

From the above, we obtain
\[
\frac{d}{dt} \Box w_j \Box^2 + \mu \Box D^n w \Box^2 + (2m_1 \lambda_4 - m_4 \lambda_5 - C_\gamma) \Box w_j \Box^2 - (C_\gamma + m_4 \lambda_5) \Box D^n w \Box^2 \leq 0
\]
\[
(2.40)
\]

Take \( \gamma = \min \{ \frac{-(C_\gamma + m_4 \lambda_5)}{\mu}, 2m_1 \lambda_4 - m_4 \lambda_5 - C_\gamma \} \), we have
\[
\frac{d}{dt} N(t) + \gamma N(t) \leq 0,
\]
\[
(2.41)
\]

where
\[
N(t) = \Box w_j \Box^2 + \mu \Box D^n w \Box^2.
\]
\[
(2.42)
\]

By using Gronwall inequality, we obtain
\[
N(t) \leq N(0) e^{\gamma t} = 0.
\]
\[
(2.43)
\]

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O c t o b e r  2 0 1 6  
w w w . c i r w o r l d . c o m
Therefore

\[ u = v. \]

(2.45)

So we prove the uniqueness of the solution.

### 3 Global attractor

**Theorem 3.1.** \[^{[12]}\] Let \( E_1 \) be a Banach space, and \( \{S(t)\}(t \geq 0) \) are the semigroup operator on \( E_1 \).

\[ S(t) : E_1 \rightarrow E_1, \quad S(t + s) = S(t)S(s) \quad (\forall t, s \geq 0), \quad S(0) = I, \]

where \( I \) is a unit operator. Set \( S(t) \) satisfy the follow conditions.

1) \( S(t) \) is uniformly bounded, namely \( \forall R > 0, \quad [u \leq R] \), it exists a constant \( C(R) \), so that

\[ \| S(t)u \|_{E_1} \leq C(R) \quad (t \in [0, +\infty)); \]

2) It exists a bounded absorbing set \( B_0 \subset E_1 \), namely, \( \forall B \subset E_1 \), it exists a constant \( t_0 \), so that \( S(t)B \subset B_0 \) \( (t \geq t_0); \)

Where \( B_0 \) and \( B \) are bounded sets.

3) When \( t > 0 \), \( S(t) \) is a completely continuous operator \( A \).

Therefore, the semigroup operator \( S(t) \) exists a compact global attractor.

**Theorem 3.2.** \[^{[12]}\] Under the assume of Theorem 2.1, equations have global attractor

\[ A = \omega(B_0) = \bigcap_{s \geq 0} \bigcup_{t \geq s} S(t)B_0; \]

Where \( B_0 = \{(u, v) \in H^m(\Omega) \times H^m(\Omega) \mid \|u\|_{H^m} + \|v\|_{H^m} \leq R_0 + R_1\} \), \( B_0 \) is the bounded absorbing set of \( H^m(\Omega) \times H^m(\Omega) \) and satisfies

1) \( S(t)A = A, \quad t > 0; \)

2) \( \lim_{t \to \infty} \text{dist}(S(t)B, A) = 0 \), here \( B \subset H^m \times H^m \) and it is a bounded set,

\[ \lim_{t \to \infty} \text{dist}(S(t)B, A) = \sup \{ \inf_{x \in B} \text{dist}(S(t)x, A) \} \to 0, \quad t \to \infty. \]

Proof. Under the conditions of Theorem 2.1, it exists the solution semigroup \( S(t) \), \( S(t) : E_1 \rightarrow E_1 \), here \( E_1 = H^m \times H^m \).

(1) from Lemma 2.1 to Lemma 2.2, we can get that \( \forall B \subset H^m \times H^m \) is a bounded set that includes in the ball \( \{u \leq R\} \),
This shows that $S(t)$ is uniformly bounded $H^{2m} \times H^m$.

(2) Furthermore, for any $(u_0, v_0) \in H^{2m} \times H^m$, when $t \geq \max \{t_1, t_2\}$, we have,

$$
\square S(t)(u_0, v_0) \leq u_0^2 u_{2m} + \|u\|_{H^{2m}}^2 \leq u_0^2 u_{2m} + \|v\|_{H^m}^2 \leq R_0 + R_1
$$

So we get $B_0$ is the bounded absorbing set.

(3) Since $H^{2m} \times H^m \rightarrow H^n \times L^2$ is compact embedded, which means that the bounded set in $H^{2m} \times H^m$ is the compact set in $H^n \times L^2$, so the semigroup operator $S(t)$ exists a compact global attractor $\Lambda$.

4 Hausdorff and fractal dimensions for the global attractor

Theorem 4.1. Under the conditions of Theorem 3.2, the global attractor $A$ of problem (1.1)-(1.3) has infinite Hausdorff dimension and fractal dimension, and $d_H(A) < \frac{1}{\varepsilon} n$, $d_F(A) < \frac{2}{\varepsilon} n$.

Proof. Problem (1.1) can be written

$$
u_{n} + A^m u_{2m} + \phi(\|A^{-1} u\|^2) A^m u + g(u) = f(x),
$$

where $-\Delta = A$.

Let $\psi = R_\varepsilon \varphi = (u, v), \varphi = (u, u_j), v = u_j + \varepsilon u_j, R_{\varepsilon} : (u, u_j) \rightarrow (u, u_j + \varepsilon u_j)$, is an isomorphic mapping, so the equation of (4.1) is

$$
\psi + \Lambda \psi + \bar{g}(\psi) = \bar{f}.
$$

Where $\psi = (u, u_j + \varepsilon u_j)^T$, $\bar{g}(\psi) = (0, g(u_j))^T$, $\bar{f}(x) = (0, f(x))^T$.

Let $F: E_1 \rightarrow E_1$ is Frechet differentiable, the linearized equation of (4.3) is

$$
P_{\varepsilon} + \Lambda \psi + \bar{g}(\psi) P = 0.
$$
where $P = (U_t, U_k + eU_k \cdot F_k)$, $F_k(\psi)U = (0, g_s(u)U)$. $U$ is solution of (4.2).

For a fixed $(u_0, v_0) \in E_1$, let $\gamma_1, \gamma_2, ..., \gamma_N$ are N elements of $E_1$, let $P_1(0), P_2(0), ..., P_N(0)$ are N solutions of linear equation (4.4) with initial value $P_1(0) = \gamma_1, P_2(0) = \gamma_2, ..., P_N(0) = \gamma_N$, so, we have

$$\square P(t)A \gamma_1 \wedge A \gamma_2 \wedge A \gamma_3 \wedge A \gamma_N \exp \left( \int_0^t tr^N F_q(\psi) \cdot Q_N(\tau) d\tau \right).$$

(4.5)

Where $\wedge$ represents the outer product, $tr$ represents the trace, $Q_N(\tau)$ is an orthogonal projection from the space $E_1$ to the subspace spanned by $\{ P_1(t), P_2(t), ..., P_N(t) \}$.

For a given time $\tau$, let $\theta_j(\tau) = (\xi_j(\tau), \eta_j(\tau))^T$, $j = 1, 2, ..., N$, is the standard orthogonal basis of the space $span\{ P_1(t), P_2(t), ..., P_N(t) \}$.

We denote inner product in $E_1$, $(\xi, \eta, (\xi, \eta)) = ((\xi, \xi) + (\eta, \eta))$.

From the above, we have

$$Tr F_j(\psi(\tau)) \cdot Q_N(\tau) = \sum_{j=1}^{N} (F_j(\psi(\tau)) \cdot Q_N(\tau) \theta_j(\tau), \theta_j(\tau))_{E_1}$$

$$= \sum_{j=1}^{N} (F_j(\psi(\tau)) \theta_j(\tau), \theta_j(\tau))_{E_1}$$

$$= N \cdot (F_j(\psi(\tau)) \theta_j(\tau), \theta_j(\tau))_{E_1}$$

(4.6)

$$= -(\Lambda x \theta_j, \theta_j) - (g, \theta_j, \theta_j).$$

(4.7)

$$= ((A^n \xi_j - \eta_j, (\phi(\square A^2 u \cdot \square^2) - e) A^n \xi_j - \eta_j, \xi_j + A^n \eta_j - e \eta_j, \eta_j), (\xi_j, \eta_j))$$

$$= (e \xi_j - \eta_j, (\phi(\square A^2 u \cdot \square^2) - e) A^n \xi_j - \eta_j, \xi_j + A^n \eta_j - e \eta_j, \eta_j)$$

$$= e \square \xi_j \|^2 - (1 + e^2)(\xi_j, \eta_j) + (\phi(\square A^2 u \cdot \square^2) - e)(A^n \xi_j, \eta_j) + A^n \eta_j - e \eta_j, \eta_j)$$

$$- (1 + e^2)(\xi_j, \eta_j) + (\phi(\square A^2 u \cdot \square^2) - e)(A^n \xi_j, \eta_j)$$

$$\geq I_1 + (\mu_1 - e)(\xi_j, \eta_j) - (1 + e^2)(\xi_j, \eta_j)$$

$$\geq (I_1(\mu_1 - e) - (1 + e^2))(\xi_j, \eta_j).$$

(4.8)

There exists a constant $I_1$, such that
\( l_{1}(\mu_{1} - \varepsilon) - (1 + \varepsilon^2) \geq 0. \) \hspace{1cm} (4.10)

So we have

\[
(A, \theta_{j}, \theta_{j}) = \varepsilon \int \xi_{j} \, d\xi \, d\eta_{j} - \varepsilon \int \eta_{j} \, d\eta_{j} \leq \varepsilon \int \xi_{j} \, d\xi \, d\eta_{j} + (l_{2} - \varepsilon) \int \eta_{j} \, d\eta_{j} \leq 0.
\] \hspace{1cm} (4.11)

There exists a constant \( l_{2} \), such that

\( l_{2} - \varepsilon \geq 0 \) \hspace{1cm} (4.12)

Take \( \delta = \min(\varepsilon, l_{2} - \varepsilon) \), so

\[
(A, \theta_{j}, \theta_{j}) \geq \varepsilon \int \xi_{j} \, d\xi \, d\eta_{j} + (l_{2} - \varepsilon) \int \eta_{j} \, d\eta_{j} \geq \delta \int \xi_{j} \, d\xi \, d\eta_{j} + \int \eta_{j} \, d\eta_{j}.
\] \hspace{1cm} (4.13)

\[
(g, (\psi) \theta_{j}, \theta_{j}) = (0, g, (\psi) \xi_{j}) \cdot (\xi_{j}, \eta_{j}) = (g, \xi_{j}, \eta_{j}) \geq - \delta \int \xi_{j} \, d\xi \, d\eta_{j} - \int \eta_{j} \, d\eta_{j}.
\] \hspace{1cm} (4.14)

Now, suppose that \((u_{0}, u_{1}) \in A, A \) is a bounded absorbing set in \( E_{1} \); \( \psi(t) = (u(t), u_{1}(t) + \varepsilon u(t)) \in \mathcal{E}_{1} \).

\[ u(t) \in D(A) . \] Then there exists a \( s \in [0, 1] \), we have mapping \( g_{s} : D(A) \rightarrow \sigma(V_{r}, H^{1}) \), such that

\[
\sup \| g_{s} \| \leq r < \infty.
\] \hspace{1cm} (4.15)

According to (4.7),(4.13),(4.14), we have

\[
(F_{r}(\psi(r))\theta_{j}(r), \theta_{j}(r))_{\mathcal{E}_{1}} \leq -\delta \int \xi_{j} \, d\xi \, d\eta_{j} + \int \eta_{j} \, d\eta_{j} + \frac{\delta}{2} \int \xi_{j} \, d\xi \, d\eta_{j} \leq \frac{\delta}{2} \int \xi_{j} \, d\xi \, d\eta_{j} + \frac{\delta}{2} \int \eta_{j} \, d\eta_{j}.
\] \hspace{1cm} (4.16)

Because of \( \theta_{j}(r) = (\xi_{j}(r), \eta_{j}(r)) \), is the standard orthogonal basis, so

\[
\int \xi_{j} \, d\xi \, d\eta_{j} + \int \eta_{j} \, d\eta_{j} = 1.
\] \hspace{1cm} (4.17)
\[
\sum_{j=1}^{N} (F_j(\psi(\tau))\theta_j(\tau), \theta_j(\tau))_{E_j} \leq -\frac{N\delta}{2} + \frac{r}{2} \sum_{j=1}^{N} \xi_j \dot{\xi}_j.
\]

Almost to all \(t\), making
\[
\sum_{j=1}^{N} \xi_j \dot{\xi}_j \leq \sum_{j=1}^{N} \lambda_j^{-1}.
\]

So we have
\[
TrF_j(\psi(\tau)) \cdot Q_j(\tau) \leq -\frac{N\delta}{2} + \frac{r}{2} \sum_{j=1}^{N} \lambda_j^{-1}. \tag{4.20}
\]

Let
\[
q_N(\tau) = \sup_{\psi} \sup_{\eta \in E_j} \left( \int_0^\tau \text{Tr}F_j(S(\tau)\psi_0) \cdot Q_j(\tau) d\tau \right). \tag{4.21}
\]

According to (4.20), we have
\[
q_N = \lim_{t \to \infty} q_N(\tau). \tag{4.22}
\]

Therefore, the Lyapunov exponent of \(A\) is uniformly bounded.
\[
\kappa_1 + \kappa_2 + \cdots + \kappa_N \leq -\frac{N\delta}{2} + \frac{r}{2} \sum_{j=1}^{N} \lambda_j^{-1}. \tag{4.24}
\]

Then, exist a \(s \in [0,1]\), such that
\[
(q_j)_s \leq -\frac{N\delta}{2} + \frac{r}{2} \sum_{j=1}^{N} \lambda_j^{-1} \leq \frac{r}{2} \sum_{j=1}^{N} \lambda_j^{-1} \leq \frac{N\delta}{7}. \tag{4.25}
\]

where \(\lambda_j\) is eigenvalue of \(A^s\), and \(\lambda_1 < \lambda_2 < \cdots < \lambda_n\).
\[ q_N \leq -\frac{N\delta}{2}(1 - \frac{1}{N}\sum_{j=1}^{N} \lambda_j^{-r-1}) \leq -\frac{\delta}{4} \frac{N\delta}{2}. \]  

(4.26)

So

\[ \max_{1 \leq j \leq N} \left( \frac{q_j}{\|u_j\|} \right) \leq \frac{2}{N}. \]  

(4.27)

So, we can acquire \( d_N(A) < \frac{2}{N} \) .

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