BIRATIONAL MODIFICATIONS OF-surfaces via
UNPROJECTIONS

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March 1, 2010

Abstract. We describe elementary transformations between minimal models of rational surfaces in terms of unprojections. These do not fit into the framework of Kustin–Miller unprojections as introduced by Papadakis and Reid, since we have to leave the world of projectively Gorenstein varieties. Also, our unprojections do not depend only on the choice of the unprojection locus but need extra data corresponding to the choice of a divisor on this unprojection locus.

Introduction

Unprojection, introduced by Miles Reid in [R2], is a technique to describe birational transformations in higher dimensional geometry explicitly in terms of commutative algebra. As explained in [PR, Section 2.3], the prototype and easiest example of an unprojection is the Castelnuovo blow-down of a rational \((-1)-\)curve lying on a smooth cubic surface in \(P^3\) as the inverse of a projection from a del Pezzo surface of degree 4 in \(P^4\), which also explains the name unprojection.

Since then, unprojections have been applied to many explicit constructions in birational geometry of surfaces and 3-folds, compare [ABM], [CPR], [RS],... Moreover, these techniques have also proved to be useful for the construction of key varieties, see, for example, [R2] or [NP]. Finally, the general theory of unprojection has been developed further by the second author and a general framework has been proposed in [P].

In this note we return to the two-dimensional case. Here, every birational map between two smooth projective surfaces can be factored into a sequence of blow-ups of points and Castelnuovo blow-downs of rational \((-1)-\)curves. A prominent class of birational transformations between smooth surfaces are the elementary transformations, which relate minimal models of surfaces of Kodaira dimension \(\kappa = -\infty\) to each other, compare [B, Chapter 12]. Their higher-dimensional generalisations are Sarkisov links, compare [CPR].

Question. Can one describe elementary transformations of surfaces in terms of unprojections?

For reasons of simplicity and since all relevant problems already occur within this class of surfaces, we will restrict ourselves to minimal rational surfaces. Already here we encounter new types of unprojections and new phenomena.

\textbf{2000 Mathematics Subject Classification.} 14M05, 14E05, 14J26, 13H10.
show up. This is related to the fact that an elementary transformation depends not only on the choice of a curve but needs also the choice of a point on it.

**Answer.** Yes, this can be done for minimal rational surfaces. However, the unprojection is no longer determined by the unprojection locus alone. Moreover, this cannot be done within the framework of projectively Gorenstein varieties, as in the classical case of Kustin–Miller unprojections.

Minimal rational surfaces consist of $\mathbb{P}^2$ and Hirzebruch surfaces. By definition, the Hirzebruch surface, sometimes also called Segre surface, $F_d$ is the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d)) \to \mathbb{P}^1$. An elementary transformation of $F_d$ is the following: we choose a point lying on a fibre of this projection and blow it up. The strict transform of this fibre on the blow-up is a rational $(-1)$-curve and blowing it down we obtain the desired elementary transformation of $F_d$. Depending on the position of the point we blew up to start with, the resulting surface is isomorphic to $F_{d+1}$ or $F_{d-1}$.

As embedding for the Hirzebruch surfaces we choose their realisations as surfaces of minimal degree, i.e., as scrolls. As unprojection locus $\Gamma$ we choose a line lying on this scroll, which corresponds to the fibre of the projection of this Hirzebruch surface onto $\mathbb{P}^1$. In terms of rings and ideals we have

$$V(I) = \Gamma \subset \text{Proj} S \subset \mathbb{P}^N.$$  

The point of departure for unprojections is the $S$-module $\text{Hom}_S(I, S)$, which in our situation turns out to be generated by two elements in degree zero. This implies that the associated unprojection ring is "not geometric", although somewhat similar rings have been considered in [R2] to describe 3-fold flips. Geometrically, this is related to the fact that unprojections correspond to contractions and that $\Gamma$ has not negative self-intersection, which would be necessary in order to contract it.

Instead, we will use natural submodules of $\text{Hom}_S(I, S)$ to construct our unprojections. Geometrically, these submodules correspond to choosing a divisor $D$ on $\Gamma = V(I)$. In case $D$ is a divisor of degree $k \geq 1$, whose support consists of $k$ distinct points, our results specialise to the following

**Theorem.** The unprojection of $X = \text{Proj} S \subset \mathbb{P}^N$ with respect to $D \subset \Gamma \subset X$ is a normal and projectively Cohen–Macaulay surface inside $\mathbb{P}(1^{N+1}, k)$, which arises from $X$ by first blowing up $D$ and then contracting the strict transform of $\Gamma$.

In this special case the unprojection is smooth outside a toric singularity of type $\frac{1}{k}(1, 1)$, which is induced from the singularity of the ambient weighted projective space. We refer to Section 2 for the general case.

The case $k = 1$ corresponds to elementary transformations of Hirzebruch surfaces. Moreover, if $\text{Proj} S \cong F_d$ and we vary the divisor $D$, which is just one point in this case, we obtain a 1-parameter family of unprojections, all of which are isomorphic to $F_{d-1}$ except one surface which is isomorphic to $F_{d+1}$. This fits nicely into the deformation and degeneration theory of Hirzebruch surfaces, confer [BHPV] Theorem VI.8.1].
Finally, we give an application to odd Horikawa surfaces, i.e., to minimal surfaces of general type with \( K^2 = 2p_g - 3 \). More precisely, for an odd Horikawa surface with \( p_g \geq 7 \) the canonical and the bicanonical image are rational surfaces. Moreover, the canonical image is a Hirzebruch surface realised as surface of minimal degree. Then the birational transformation relating canonical and bicanonical image corresponds to an unprojection of the type considered in this article with \( k = 2 \).

Acknowledgement. We thank Francesco Zucconi and the referee for remarks, comments and pointing out a couple of inaccuracies. Stavros Papadakis is a participant of the Project PTDC/MAT/099275/2008, a member of CAMSGD (IST/UTL), and gratefully acknowledges funding from the Portuguese Fundação para a Ciência e a Tecnologia (FCT) under research grant SFRH/BPD/22846/2005 of POCI2010/FEDER. Much of this work was done while he was visiting the university of Düsseldorf. We thank the Mathematisches Institut and the DFG-Forschergruppe Classification of Algebraic Surfaces and Compact Complex Manifolds for the kind hospitality and financial support.

1. Hirzebruch surfaces and scrolls

We fix once and for all an arbitrary field \( k \) over which all our schemes will be defined. Let \( d \geq 0 \) be a non-negative integer. Then the Hirzebruch surface, or, Segre surface, \( F_d \) is defined to be the \( \mathbb{P}^1 \)-bundle \( \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d)) \to \mathbb{P}^1 \).

We denote by \( \Gamma \) the class of a fibre of this projection. Moreover, there exists a section \( \Delta_0 \) with self-intersection \( -d \), which is unique if \( d \neq 0 \).

We remark that \( F_0 \) is isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \), and that \( F_1 \) is isomorphic to \( \mathbb{P}^2 \) blown-up in one point. Moreover, \( F_1 \) is the only Hirzebruch that is not minimal, and among Hirzebruch surfaces, the only del Pezzo surfaces are \( F_0 \) and \( F_1 \). Now, a projectively Cohen–Macaulay scheme \( X \) is projectively Gorenstein if and only if there exists a \( k \in \mathbb{Z} \) such that \( \omega_X \cong \mathcal{O}_X(k) \), see [Eis, Section 21.11]. Thus, apart from \( F_0 \) and \( F_1 \), Hirzebruch surfaces do not possess embeddings into projective space that are projectively Gorenstein. In particular, elementary transformations of minimal rational surfaces in terms of unprojections cannot be described within the framework of Kustin–Miller unprojections as in [PR].

However, Hirzebruch surfaces do possess nice embeddings into projective space. Namely, for integers \( m, n \) satisfying \( n \geq m \geq 1 \) we define \( F(m, n) \) to be the surface scroll in \( \mathbb{P}^{m+n+1} \) defined by the vanishing of the 2 \times 2-minors of

\[
\begin{pmatrix}
  x_{00} & \ldots & x_{0m-1} & x_{10} & \ldots & x_{1n-1} \\
  x_{01} & \ldots & x_{0m} & x_{11} & \ldots & x_{1n}
\end{pmatrix}.
\]

Abstractly, this scroll is isomorphic to \( F_{n-m} \). Moreover, \( F(m, n) \) corresponds to embedding \( F_d \) with \( d = n - m \) via the complete linear system \( |\Delta_0 + n\Gamma| \) into projective space. Under this embedding, the projection onto \( \mathbb{P}^1 \) is given by the ratios of the columns of (1):

\[
F(m, n) \rightarrow \mathbb{P}^1 \\
[ x_{00} : \ldots : x_{0m} : x_{10} : \ldots : x_{1n} ] \mapsto [ x_{00} : x_{01} ] = \ldots = [ x_{1n-1} : x_{1n} ].
\]
Let us fix the fibre $\Gamma$ over $[0 : 1]$, which is given by the vanishing of the first row in (1):

$$\Gamma = \{x_{00} = \ldots = x_{0m-1} = x_{10} = \ldots = x_{1n-1} = 0\} \cap \mathbb{F}(m, n).$$

Apart from the nice determinantal description there is another reason why these embeddings of the Hirzebruch surfaces are distinguished. Namely, a non-degenerate and integral surface in $\mathbb{P}^N$ has degree at least $N - 1$. If such a surface has degree equal to $N - 1$, then a theorem of del Pezzo states that it is precisely one of the $\mathbb{F}(m, n)$’s above, $\mathbb{P}^2$, the Veronese surface in $\mathbb{P}^5$, or the cone over a rational normal curve, confer [EH] for a modern account. The case of the cone over a rational normal curve corresponds to having only one block in the matrix (1) above.

We set $R = k[x_{0i}, x_{1j}]$ with $i = 0, \ldots, m$ and $j = 0, \ldots, n$, which is the homogeneous coordinate ring of $\mathbb{P}^{m+n+1}$. Let $Q$ be the ideal of $R$ corresponding to $\mathbb{F}(m, n)$, i.e., the homogeneous ideal generated by the $2 \times 2$ minors of (1). We set $S = R/Q$ and define $I$ to be the ideal of $S$ defining $\Gamma$, i.e., the ideal corresponding to (2)

$$I = (x_{00}, \ldots, x_{0m-1}, x_{10}, \ldots, x_{1n-1}) \subset S = R/Q.$$  

The ring $S$ is Cohen–Macaulay, which is related to the fact that the embedding of $\mathbb{F}(m, n)$ into $\mathbb{P}^{m+n+1}$ is given by a complete linear system, confer [Eis, Exercise 18.16]. Alternatively, it also follows from the determinantal description of $S$ by Eagon’s theorem, see [Eis, Theorem 18.18].

Since we want to unproject from $\Gamma = V(I)$, we need an analysis of the graded $S$-module $\text{Hom}_S(I, S)$, compare [P, Section 4]. From loc.cit. or [PR, Section 1] we recall the short exact sequence

$$0 \rightarrow S \rightarrow \text{Hom}_S(I, S) \xrightarrow{\text{res}} \text{Ext}_S^1(S/I, S) \rightarrow 0,$$

where res stands for Poincaré residue map. Obviously, the inclusion $i$ of $I$ into $S$ lies in $\text{Hom}_S(I, S)$. Moreover, the map

$$\phi : x_{0i} \mapsto x_{0i+1} \quad i = 0, \ldots, m - 1$$

$$x_{1j} \mapsto x_{1j+1} \quad j = 0, \ldots, n - 1$$

sending the first row of (1) to the second row defines a homomorphism of degree zero of graded $S$-modules from $I$ to $S$. This is best seen by considering the element $\tilde{s} = x_{01}/x_{00} = \ldots = x_{1n}/x_{1n-1}$ in the field of fractions $k(S)$ of the domain $S$. Then multiplication by $\tilde{s}$ induces a homomorphism of $S$-modules from $I$ to $k(S)$ which yields $\phi$.

**Proposition 1.1.** The $S$-module $\text{Hom}_S(I, S)$ is generated by the two elements $i$ and $\phi$, both of which are of degree zero.

Since this module is crucial for the construction and analysis of unprojections, we decided to give two proofs – one more geometric and one purely algebraic:

**First Proof.** Let $X = \text{Proj} S$ together with its very ample invertible sheaf $\mathcal{O}_X(1)$. From the determinantal description we infer that $X$ is projectively normal, which implies $S = \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{O}_X(n))$. Moreover, the sheafification
of $\text{Hom}_S(I, S)$ is $O_X(\Gamma)$, which implies that there is a natural injection of graded $S$-modules

$$\alpha : \text{Hom}_S(I, S) \to \bigoplus_{n \in \mathbb{Z}} H^0(X, O_X(\Gamma)(n)) =: M.$$  

Thus, there are no elements of negative degree in $\text{Hom}_S(I, S)$. In degree zero, we have $\iota$ and $\phi$ in $\text{Hom}_S(I, S)$ and $h^0(X, O_X(\Gamma)) = 2$, which implies that $\alpha$ is an isomorphism in degree zero.

Using the explicit description of global sections of invertible sheaves on scrolls in terms of bihomogeneous polynomials as in [R, Chapter 2], it follows easily that $M$ is generated as an $S$-module in degree zero. It follows that $\alpha$ is an isomorphism of graded $S$-modules and hence that $\text{Hom}_S(I, S)$ is generated by $\iota$ and $\phi$. 

**Second Proof.** Denote by $B$ the following subset of $R$

$$B = \{1\} \cup \{x_{0i} \cdot x_{0m}^a \cdot x_{1n}^b \mid 2 \leq i \leq m - 1, \ a, b \geq 0\} \cup \{x_{0m}^a \cdot x_{1n}^b \mid a, b \geq 0, (a, b) \neq (0, 0)\} \cup \{x_{10}^a \cdot x_{11}^b \mid 1 \leq i \leq n - 1, \ a, b \geq 0\}.$$  

**First Claim:** The set $B$ is a basis of the $k$-vector space $R/(Q + (x_{00}, x_{01}))$. Set $Q_1 = Q + (x_{00}, x_{01})$. Using the relations $x_{0i}x_{0j} = x_{0i-1}x_{0j+1}$, $x_{1i}x_{1j} = x_{1i-1}x_{1j+1}$ and $x_{0i}x_{1j} = x_{0i-1}x_{1j+1}$ of $Q_1$ it is not difficult to see that given a monomial $w \in R$ there exists another monomial $w' \in R$ with $w - w' \in Q_1$ such that $w' \in B$. This shows that $B$ spans $R/Q_1$ as $k$-vector space. To prove linear independence, we consider the $k$-algebra homomorphism $g : R \to k[z, s, t]$ defined by

$$g(x_{0i}) = z^{m-i}s^i, \ 0 \leq i \leq m \quad \text{and} \quad g(x_{1j}) = t^{n-j}s^j, \ 0 \leq j \leq n.$$  

Now, assume that we have an element

$$a = \sum_{b \in B} a_b \cdot b \in Q_1,$$  

where almost all $a_b = 0$.

Using $Q \subseteq \ker g$, we see that $g(a) = \sum_b a_bg(b)$ lies inside the ideal of $k[z, s, t]$ generated by $zt^{m-1}$. On the other hand, none of the monomials $g(b), b \in B$ is divisible by $zt^{m-1}$. Hence if $g(a) \neq 0$ we get a contradiction, so $g(a) = 0$. Since it is clear that the set $\{g(b) \mid b \in B\}$ is linearly independent, we get $a_b = 0$ for all $b \in B$ and we conclude linear independence of $B$. This proves the claim.

**Second Claim:** If $u \in R$ fulfills $ux_{1n-1} \in (x_{00}) + Q$ then $u \in Q_1$.

Changing by elements of $Q_1$ and using the first claim, we may assume that $u$ is of the form $u = \sum_{b \in B} a_b$ with almost all $a_b = 0$. By assumption we have $g(u)ts^{n-1} = g(ux_{1n-1}) \in (zt^{m})$, hence $g(u) \in (zt^{m-1})$. However, we have seen in the proof of the first claim that this implies $a_b = 0$ for all $b \in B$ and proves the second claim.

Finally, we prove our assertion about $\text{Hom}_S(I, S)$: Since $S = R/Q$ is a domain, the element $x_{00}$ is $S$-regular. Moreover, since $I$ is an ideal of $S$ and $x_{00}$ is $S$-regular, it follows that the $S$-module homomorphism $\text{Hom}_S(I, S) \to S$ given by $f \mapsto f(x_{00})$, is injective. Now, let $f \in \text{Hom}_S(I, S)$ and set $u =
We are done if we show that \( u \) lies inside the ideal generated by \( x_{00} \) and \( x_{01} \) of \( S \). However, this follows from the computation

\[
ux_{1n-1} = f(x_{00})x_{1n-1} = f(x_{00}) = x_{00}f(x_{1n-1}) \in (x_{00}) \subseteq S.
\]

together with the second claim above. \( \square \)

Remark 1.2. In fact, \( i \) and \( \phi \) are defined over the integers. Since they generate \( \text{Hom}_S(I, S) \) over any field, in particular over all prime fields, it follows that Proposition 1.1 holds in fact over the integers.

Since \( \text{Hom}_S(I, S) \) has two generators in degree zero, the Kustin–Miller unprojection with respect to the whole \( S \)-module \( \text{Hom}_S(I, S) \) yields a graded ring, whose component of degree zero is a vector space of dimension at least two, i.e., the unprojection ring is "not geometric". Although even negatively graded rings occur in the description of 3-fold flips \([R2, \text{Section 11}]\), we will use natural submodules of \( \text{Hom}_S(I, S) \) instead.

A geometric interpretation why the unprojection ring associated to the whole \( S \)-module \( \text{Hom}_S(I, S) \) does not give the "right" object is the following observation: the unprojection locus \( \Gamma = V(I) \) is a curve with self-intersection zero, whereas for the existence of a morphism contracting \( \Gamma \) we would need that \( \Gamma \) has negative self-intersection.

2. GENERALISED UNPROJECTIONS

We keep the notations introduced so far. As already noted before, taking the unprojection ring with respect to the whole \( S \)-module \( \text{Hom}_S(I, S) \) yields a graded ring, which is not "geometric", which is why we consider suitable submodules.

In view of the natural short exact sequence \( 3 \) we will consider submodules of \( \text{Hom}_S(I, S) \) of the form \( \text{res}^{-1}(N) \), where \( N \) is a submodule of \( \text{Ext}^1_S(S/I, S) \). Recall that \( \text{Hom}_S(I, S) \) is generated as \( S \)-module by two elements \( i, \phi \) in our setup by Proposition 1.1. Then, in case \( N \) is a cyclic \( S \)-module we are led to considering submodules of \( \text{Hom}_S(I, S) \) that are generated by \( i \) and another element \( f\phi \), where \( f \in S \) is a homogeneous element. Motivated by \([PR]\) and \([P]\) we define

**Definition 2.1.** Let \( S \) be the homogeneous coordinate ring of \( \mathbb{F}(m, n) \) inside \( \mathbb{P}^{m+n+1} \) and let \( f \in S \) be homogeneous of degree \( k \geq 1 \). The **generalised unprojection ring** of \( S \) with respect to the unprojection ideal \( I \) and to \( f \) is defined as

\[
S_{un}(f) = \frac{S[T]}{(Tu - f\phi(u), u \in I)},
\]

where \( T \) is a variable of degree \( k \).

**Lemma 2.2.** Let \( f_1, f_2 \) be homogeneous elements of \( S \) of the same degree with \( f_1 - f_2 \in I \). Then there exists an isomorphism of graded rings

\[
S_{un}(f_1) \cong S_{un}(f_2).
\]

In particular, if \( f_1 \in I \) then \( S_{un}(f_1) \) is not a domain.
Remark 2.4. If the ground field is algebraically closed then choosing a homogeneous element \( f \) to be a homogeneous polynomial in \( x_{0m} \) and \( x_{1n} \). Since \( x_{0m} \) and \( x_{1n} \) are homogeneous coordinates on \( \Gamma = V(I) \) we remark the following.

Theorem 2.5. Let \( f \neq 0 \) be homogeneous of degree \( k \geq 1 \) in \( x_{0m} \) and \( x_{1n} \). Then \( \text{Proj} \ S_{un}(f) \) is an integral, normal, and projectively Cohen–Macaulay surface inside weighted projective space \( \mathbb{P}(1^n + n^2, k) \). Its homogeneous ideal is generated by the \( 2 \times 2 \)-minors of the matrix

\[
\begin{pmatrix}
  x_{00} & \ldots & x_{0m-1} \\
  x_{01} & \ldots & x_{0m}
\end{pmatrix}
\begin{pmatrix}
  x_{10} & \ldots & x_{1n-1} \\
  x_{11} & \ldots & x_{1n}
\end{pmatrix}
\begin{pmatrix}
  f \\
  T
\end{pmatrix},
\]

where the \( x_{ij} \) are of degree one and \( T \) is of degree \( k \).

Proof. The description of the homogeneous ideal follows directly from the presentation of \( S \) as the vanishing of the \( 2 \times 2 \) minors of \([1]\) and the definition of \( S_{un}(f) \).

Let \( Q_2 \) be the ideal of \( R_2 = k[x_{ij}, T] \) generated by the \( 2 \times 2 \) minors of \([1]\). We want to show that if \( P \) is a minimal prime ideal over \( Q_2 \) then it has codimension equal to \( m + n \), which is the maximum possible by a result of Eagon, compare [Eis, Exercise 10.9].

Denote by \( I^e \) the ideal of \( R_2 \) generated by the subset \( I + Q_2 \). First, assume that \( I^e \subseteq P \), which implies \( \text{codim}(P) \geq \text{codim}(I^e) \). However, \( I^e \) contains \( x_{00}, \ldots, x_{0m-1}, x_{10}, \ldots, x_{1n-1}, f x_{0m} \), which form a regular sequence, which implies \( \text{codim}(I^e) \geq m + n + 1 \). Hence this case does not exist and we have \( I^e \not\subseteq P \), i.e., \( V(P) \cap (V(Q_2) - V(I^e)) \neq \emptyset \). By [NP2, Remark 2.5], the inclusion of rings induces an isomorphism \( \text{Spec} \ S_{un}(f) - V(I^e) \cong \text{Spec} \ S - V(I) \).

Since \( \text{Spec} \ S - V(I) \) is irreducible of dimension three, we see that \( P \) has codimension \( m + n \). Thus, every minimal prime over \( Q_2 \) has codimension \( m + n \), which means that \( S_{un}(f) \) is a determinantal ring and such rings are known to be Cohen–Macaulay by a result of Eagon, compare [Eis, Theorem 18.18].

By the above arguments, the irreducible open subset \( \text{Spec} \ S_{un}(f) - V(I^e) \) of \( \text{Spec} \ S_{un}(f) \) meets every irreducible component of \( \text{Spec} \ S_{un}(f) \). From this we conclude that \( \text{Spec} \ S_{un}(f) \) is irreducible.

From the isomorphism \( \text{Spec} \ S_{un}(f) - V(I^e) \cong \text{Spec} \ S - V(I) \) it follows that \( S_{un}(f) \) is generically reduced. In particular, being Cohen–Macaulay there are
no embedded components and it follows that $S_{\text{un}}(f)$ is reduced. Together with the irreducibility it follows that $S_{\text{un}}(f)$ is a domain.

Using the isomorphism $\text{Spec } S_{\text{un}}(f) - V(I^c) \cong \text{Spec } S - V(I)$ once more, we obtain normality outside $V(I^c)$. A straightforward calculation using the Jacobian criterion shows normality along $V(I^c)$. Thus, $S_{\text{un}}(f)$ is normal. \hfill \Box

**Theorem 2.6.** Let $\varpi : \hat{X} \to X = \text{Proj } S$ be the blow-up of the ideal $(I, f)$. Then there exists a factorisation

$$
\begin{array}{ccc}
\hat{X} & \xrightarrow{\text{cont}_\Gamma} & X_{\text{un}} = \text{Proj } S_{\text{un}}(f), \\
\varpi & & \\
\end{array}
$$

where $\text{cont}_\Gamma$ is contraction of the strict transform of $\Gamma = V(I)$ on $\hat{X}$.

**Proof.** Considered as an ideal of $R = k[x_{ij}]$, the ideal $(I, f)$ is generated by the regular sequence $x_{00}, ..., x_{0m-1}, x_{10}, ..., x_{1n-1}, f$. By [EH, Exercise IV-26], the Rees algebra $\tilde{R}$ of $R$ with respect to $(I, f)$ is isomorphic to

$$
R[T_{00}, ..., T_{0m-1}, T_{10}, ..., T_{1n-1}, T_f] / B,
$$

where the $T_{ij}$ and $T_f$ are indeterminants and where $B$ is the ideal generated by the $2 \times 2$ minors of

$$
\begin{pmatrix}
x_{00} & ... & x_{0m-1} & x_{10} & ... & x_{1n-1} & f \\
T_{00} & ... & T_{0m-1} & T_{10} & ... & T_{1n-1} & T_f
\end{pmatrix}.
$$

Taking Proj, we obtain the blow-up $\varpi : \hat{X} \to X$. We denote by $\hat{\Gamma}$ the strict transform of $\Gamma$, which is cut out by $x_{00} = ... = x_{0m-1} = 0$, $x_{10} = ... = x_{1n-1} = 0$, $T_{00} = ... = T_{0m-1} = 0$, and $T_{10} = ... = T_{1n-1} = 0$.

On the level of commutative algebra, $\varpi$ corresponds to eliminating the $T_{0i}$’s, the $T_{ij}$’s and $T_f$, whereas eliminating only the $T_{0i}$’s and the $T_{ij}$’s induces a map $\text{cont}_\Gamma$ from $\hat{X}$ onto $X_{\text{un}}$.

A straightforward calculation shows that $\text{cont}_\Gamma$ is in fact a morphism, that it is an isomorphism outside $\hat{\Gamma}$ and that it contracts $\hat{\Gamma}$ to the vertex of the weighted projective space in which $X_{\text{un}}$ lies. Since $X_{\text{un}}$ is normal by Theorem 2.5, Zariski’s main theorem shows that $\text{cont}_\Gamma$ is in fact the contraction of $\hat{\Gamma}$. \hfill \Box

Let us assume that $D = \sum_i k_i P_i$, where the $k_i$ are positive integers with $k = \sum_i k_i$ and where the $P_i$ are distinct points that are rational over the ground field. Note that this assumption on $D$ can always be fulfilled if the ground field is algebraically closed. Then calculations similar to those in [EH, Chapter IV.2.3] show that we obtain $X_{\text{un}} = \text{Proj } S_{\text{un}}(f)$ as follows:

1. For each $i$, blow up $\text{Proj } S$ at $P_i$. Then blow up the intersection point of the strict transform of $\Gamma$ with the resulting $(-1)$-curve of the blow-up etc. until, for every $i$ we get a chain $C_i$ of $(k_i - 1)$ rational $(-2)$-curves and a $(-1)$-curve. It is understood that $C_i$ is empty if $k_i = 1$.
2. The strict transform $\hat{\Gamma}$ is a rational curve with self-intersection $-k$. Contracting $\hat{\Gamma}$ and all the $C_i$’s we obtain $X_{\text{un}}$. 
From this description we can read off the singularities of $X_{\text{un}}$: it has a toric singularity of type $\frac{1}{k}(1,1)$ coming from the contraction of $\tilde{\Gamma}$, and every contracted chain $C_i$ contributes a cyclic quotient singularity of type $\frac{1}{k_i}(1,k_i-1)$, i.e., a Du Val singularity of type $A_{k_i-1}$.

3. Elementary transformations

As in the previous sections, let $S$ be the homogeneous coordinate ring of $F(m,n)$ inside $\mathbb{P}^{m+n+1}$ given by (1) and recall that we assumed $n \geq m \geq 1$. As unprojection divisor we take the line $\Gamma = V(I)$ lying on $F(m,n)$ as in (2). We note that $x_0^m$ and $x_1^n$ can be viewed as coordinates on $\Gamma$. Moreover, in our unprojection setting we choose $0 \neq f \in S$ of degree one, which for our unprojection purposes we may assume to be of the form $f_{a,b} = a x_0^m + b x_1^n$ with $[a:b] \in \mathbb{P}^1$, cf. Lemma 2.2. As already noted in Remark 2.4, our unprojection data consists of an unprojection locus, which is a line, and a point on this line.

Proposition 3.1. There exists an isomorphism

$$\text{Proj } S_{\text{un}}(f_{a,b}) \cong \begin{cases} F(m,n+1) & \text{if } [a:b] = [0:1] \\ F(m+1,n) & \text{else,} \end{cases}$$

which is induced by a projective linear transformation of the ambient $\mathbb{P}^{m+n+2}$.

Proof. If $a = 0$ or $b = 0$ this follows directly from comparing (4) with (1). We may thus assume $a \neq 0$. For all $i = 0,...,n-1$ we add $b$ times the $(n-i)$.th column of the middle block of (4) to the $(m-i)$.th column of the left block of (4). This is possible since we assumed $n \geq m$ and a linear change of variables yields the desired isomorphism. □

Remark 3.2. By Theorem 2.6 we have realised all elementary transformations of Hirzebruch surfaces in our setting.

Moreover, we obtain a family of unprojections parametrised by $\mathbb{P}^1$. One member of this family is isomorphic to $F_{n-m+1}$, whereas all the others are isomorphic to $F_{n-m-1}$. This fits nicely into the deformation and degeneration theory of Hirzebruch surfaces as explained in [BHPV, Theorem VI.8.1].

The inverse of the unprojection $\text{Proj } S \to \text{Proj } S_{\text{un}}(f_{a,b})$, corresponds to eliminating the new variable $T$ of $S_{\text{un}}(f_{a,b})$, and is induced by a projection from $\mathbb{P}^{m+n+2}$ onto $\mathbb{P}^{m+n+1}$, confer [Ha, Proposition 8.20].

4. (B)canonical images of Horikawa surfaces

In this final section we work over the complex numbers. If $S$ is a minimal surface of general type then Noether’s inequality $K^2 \geq 2p_g - 4$ holds true, confer [BHPV, Theorem VII.3.1]. In case of equality $K^2 = 2p_g - 4$, i.e., if $S$ is a so-called even Horikawa surface, then the canonical map is a generically finite morphism of degree 2 onto a surface of minimal degree in $\mathbb{P}^{p_g-1}$, which is the key to the classification of these surfaces, compare [BHPV, Chapter VII.9]. Also, it is not difficult to show that the bicanonical map is a morphism that coincides with the canonical map followed by the second Veronese embedding.
Now, let $S$ be an odd Horikawa surface, i.e., a minimal surface of general type with $K^2 = 2p_g - 3$. These have been classified in \cite{Ho}, and we will assume that we are in case A in Horikawa’s terminology with smooth canonical image. This is the generic case, and if $p_g \geq 7$ it is even automatically fulfilled, compare \cite[Theorem 1.3]{Ho}.

Then the image of the canonical map is a smooth surface $X$ of minimal degree in $\mathbb{P}^{p_g - 1}$. Hence there exist integers $n \geq m \geq 1$ with $p_g = m + n + 2$ such that the canonical image is $F(m, n)$, which is abstractly isomorphic to $F_{n-m}$.

The canonical system $|K_S|$ has a unique base point, whose indeterminacy is resolved by a single blow-up $\tilde{S} \to S$. The resulting $(-1)$-curve on $\tilde{S}$ maps to a line $\Gamma \subset F(m, n) \subset \mathbb{P}^{p_g - 1}$ and after an appropriate choice of coordinates we may assume that $\Gamma$ and $F(m, n)$ are as in Section 1. Then $S$ determines two points $x, y$ on $\Gamma$, possibly infinitely near, cf. \cite[Theorem 1.3]{Ho}. We denote by $\pi: \tilde{X} \to X$ their blow-up, by $E_x, E_y$ the corresponding exceptional divisors, and by $\tilde{\Gamma}$ the strict transform of $\Gamma$ on $\tilde{X}$. Moreover, we obtain a factorisation $\tilde{S} \to S^* \to \tilde{X} \to X$, where $\tilde{S} \to S^*$ is birational, $S^*$ has at worst Du Val singularities, and where $S^* \to X$ is finite and flat of degree 2. On $\tilde{X}$ we consider the line bundle

$$L = \mathcal{O}_{\tilde{X}}(\pi^*\Delta_0 + (n - 4)\pi^*\Gamma - 2E_x - 2E_y).$$

Then the canonical map of $\tilde{S}$ (and hence $S$) factors over the complete linear system $|L|$ on $\tilde{X}$ and we already noted that we can identify its image with $F(m, n)$. Moreover, the $(-1)$-curve on $\tilde{S}$ maps to $\tilde{\Gamma}$ on $\tilde{X}$ and thus maps to $\Gamma$ under the canonical map.

From \cite[Theorem 1.3]{Ho} it follows easily that the bicanonical map factors over $|L^{\otimes 2}|$ on $\tilde{X}$. This map contracts $\tilde{\Gamma}$ to an $A_1$-singularity. A straightforward computation counting the quadratic relations coming the $2 \times 2$ minors of $L$, we see that the map $H^0(L^{\otimes 2}) \to H^0(L^{\otimes 2})$ has 1-dimensional coimage. This implies that the canonical ring of $S$ has $p_g$ generators in degree one and precisely one new generator in degree two.

Let us denote by $X_1 \subset \mathbb{P}(1^{p_g})$ and by $X_2 \subset \mathbb{P}(1^{p_g}, 2)$ the projection from the canonical model of $S$ onto the weighted projective space corresponding to generators in degree 1 (canonical image) and generators in degree $\leq 2$ (weighted bicanonical image). Then, putting all observations above together we can interpret Horikawa’s results \cite[Section 1]{Ho} in our setting as follows:

**Proposition 4.1.** The odd Horikawa surface $S$ determines a line $\Gamma$ on $X_1$ and two points $\{x, y\}$ on this line, which are possibly infinitely near. Then the inverse of the natural projection $X_2 \to X_1,$

$$X_1 \dashrightarrow X_2$$

is a generalised unprojection with unprojection data $\Gamma$ and divisor $D = x + y$ on $\Gamma$.

Finally, we remark that we cannot realise the other birational modifications of Theorem 2.6 by images of pluricanonical maps of surfaces of general type: If the canonical map has two-dimensional image then $p_g \geq 3$ and in order to
get a scroll as canonical image we even need $p_g \geq 4$. For such surfaces we have $K^2 \geq 4$ by Noether’s inequality. However, for minimal surfaces of general type $X$ with $K^2_X \geq 3$ all pluricanonical images $\text{im}(\varphi_i(X))$ with $i \geq 3$ are birational to $X$ by Bombieri’s theorem (confer [BHPV, Theorem VII.5.1]) and thus no rational surfaces.

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