Scale Factor Duality for Domain Walls Holography

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November 20, 2019

Abstract

We describe a scale factor duality (SFD) for domain wall (DW) spacetimes in Einstein gravity with a single scalar field and self-interaction potential. The duality comprises a scale factor inversion mapping large scales to small scales, and a transformation of the field and its potential which preserves the form of the first-order BPS system for static and isotropic DWs. By construction, SFD maps the asymptotic AdS boundary to the vicinity of a naked singularity in the dual theory with a special Liouville (exponential) scalar field potential. The holographic implementation of SFD relates the dual pair of domain walls to a dual pair of the corresponding holographic renormalization group flows, with a map between their UV and IR regimes. We consider a few examples and show that SFD is a symmetry of the GPPZ flow. We also demonstrate how SFD can be extended to act on the tensor perturbation modes in general, and on the scalar perturbation modes in the particular case of exponential potentials.

Keywords: Scale factor duality, gauge/gravity correspondence, holographic RG flow.

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1 Introduction

At high energies, the UV asymptotic properties of relativistic field theories are determined by conformal field theories (CFTs). At lower energies, if (as usual) scaling and conformal symmetries are broken and in the absence of small parameters, no consistent perturbative methods are available in the IR limit. Then an efficient alternative description might exist when the small- and large-scale regimes of two theories are related by some UV/IR duality.\(^1\) An important example of such has been proposed long ago by Veneziano [2], observing that dilaton gravity coupled to scalar matter is invariant under inversion of the scale factor in cosmological space-times. Given the importance of understanding the big-bang, it comes as no surprise that such symmetries found many applications in early-universe cosmology [3–6].

\(^1\)The most representative examples are the T-duality of superstrings and the gauge/gravity holographic correspondence [1].
The purpose of the present work is to implement a scale factor duality (SFD) for domain-wall spacetimes, and to describe its holographic implications in the framework of gauge/gravity correspondence [1]. Domain walls (DWs) are a class of “vacuum solutions” of (super)gravity [9, 10], appearing in the D-brane solutions of superstring theory, in Randall-Sundrum inspired scenarios and, in particular, as the correspondents of the holographic renormalization group (RG) flows [11,12].

We consider static and flat DWs with $d$-dimensional Poincaré symmetry:

$$ds^2 = e^{2A(z)} \left[ dz^2 + \eta_{ab} dx^a dx^b \right]; \quad e^{A(z)} \equiv a(z); \quad \phi = \phi(z),$$

in conformal coordinates, where the scale factor $a(z)$ is a Weyl factor. The scalar field $\phi$ (which may be seen as the dilaton in the string-frame) possesses an interaction potential $V(\phi)$ in the Einstein-frame action:

$$S = \int d^{d+1}x \sqrt{-g} \left[ \frac{1}{\kappa^2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right].$$

Our main observation is that the scale factor duality transformations

$$a \mapsto \tilde{a} = \frac{c^2}{a} \quad \phi \mapsto \tilde{\phi}(\phi) \quad V(\phi) \mapsto \tilde{V}(\tilde{\phi})$$

where $c$ is a free parameter, preserve the form of the DW equations. Thus SFD maps two domain wall solutions in theories with different scalar potentials $V(\phi)$ and $\tilde{V}(\tilde{\phi})$. By construction, this map is such that the vicinity of a singularity, where $a \to 0$, is mapped to a “boundary”, where $\tilde{a} \to \infty$, and we can reconstruct explicitly one asymptotic geometry in terms of the other. Besides preserving the full second-order Einstein equations for the domain walls, (1.3) also preserves the usual (pseudo-)BPS first-order system when the scalar potential is written in terms of the usual (fake-)superpotential [13,14].

The transformation of the scalar field is very non-trivial, and has some interesting properties. The first part of this article is devoted to their description in two important cases: asymptotically AdS vacua and asymptotically Liouville (exponential) potentials. We show that SFD preserves the exponential form $V = V_0 e^{v \phi}$, only changing the parameters (in particular $v$), and maps AdS space into a special Liouville model with the exponent $v = 2/\sqrt{d-1}$. Combined with the scale-factor inversion, this particular transformation allows us to relate the AdS boundary to the naked singularity present in the Liouville domain wall. A similar relation is valid (we construct it explicitly) for domain walls with an asymptotically AdS vacuum.

The second part of this paper concerns the natural application of SFD to holography. In this context, SFD is a map between holographic RG flows. Information

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2The analogous cosmological duality has been described by the authors in [7,8].

3$\eta_{ab}$ is the Minkowski metric, where $a, b = 0, 1, 2, \cdots, d-1$ denote $d$-dimensional “transversal” coordinates and $\mu, \nu$ bulk coordinates. Newton’s constant in $(d+1)$ dimensions is (proportional to) $\kappa^2$.

4Not only for Poincaré symmetry but for curved DWs as well.
about the flow—fixed points, the central charge and the (anomalous) dimension of operators—can be read from the holographic beta-function \( \beta(\phi) = -d\phi/d\log a \), that is the evolution of the boundary coupling \( \phi(a) \) as a function of the running energy scale \( \log a \). The explicit SFD transformations of the beta-function and of the remaining RG data turns out to have a quite simple form.

Over the years, a fruitful line of work in the extensive literature devoted to holographic RG flows has been a bottom-up phenomenological approach: one classifies different bulk evolutions, so as to identify possible behaviors of RGs [15–18]. In these models, singular domain walls with an asymptotically exponential potential are very important examples of flows that do not end in a standard AdS fixed point. SFD is a special link between different pairs of such evolutions, which could reveal some interesting physics. For instance, we have found that SFD is an exact symmetry of at least one important solution, the GPPZ flow [19].

The third part of this work is about fluctuations around backgrounds related by SFD. We find that the equations for tensor modes around any domain wall present a symmetry under SFD relating the modes and their conjugate momenta; the equations for the scalar modes posses the same symmetry only in theories with exponential potentials. The symmetry can be interpreted in terms of the well known “SUSY quantum-mechanical” [20] description of domain wall fluctuations. Dual modes (around dual backgrounds) are superpartners, and each can be found from acting with the SUSY charge operator on the other. This relates the corresponding “wave functions” in such a way that fixing the boundary conditions of fluctuations in one model fixes them in the dual as well, even in cases where the extension of SFD is only asymptotic, such as in asymptotically Liouville/AdS backgrounds.

Holographically, fluctuations can be used to give VEVs of operators or the mass spectrum of composite particles in the QFT, depending on how one ascribes Dirichlet conditions in the UV. In the latter case, the spectrum is fixed by assigning Dirichlet conditions in the IR as well; SFD relates these to the conditions at the UV of the dual model. We give an example of how this can be used, in some cases, to solve ambiguities that have been noted to sometimes appear in the spectrum of singular domain walls. Our final result is an application of SFD to the method of holographic renormalization developed in [21], valid at spaces such as Liouville models in which there is no AdS UV boundary. As expected from the background interpretation of the scale factor as the holographic energy scale, we show explicitly that SFD maps the VEV of the deformation operator driving the RG flow to a dual VEV evaluated at the reciprocal energy scale \( \tilde{E} = 1/E \).

The structure of the paper is as follows. In Sect.2 we review the description of isotropic domain walls in terms of a system of first-order differential equations. In Sect.3 we introduce SFD, describe its properties and construct explicit examples of dual domain walls, focusing in asymptotically AdS/Liouville geometries. In Sect.4, we discuss the duality between domain walls as a duality between holographic RG flows. In Sect.5 we investigate the duality of fluctuations and the consequences for wave-functions and spectra of \( d \)-dimensional eigenstates. In Sect.6 we conclude by
making a summary of our results and discussing future prospects. Some secondary points and examples are left for the appendices.

2. Domain walls dictionary

Let us briefly review the basic features of domain walls in Einstein gravity coupled to one scalar field [9, 10]. The field equations obtained from the action (1.2) for the flat, isotropic and static ansatz (1.1) have the form

\[ 2(d-1)A'' + (d-1)(d-2)A^2 = -\kappa^2 \left[ \frac{1}{2} \phi'^2 + e^{2A} V(\phi) \right] \] (2.1a)

\[ \phi'' + (d-1)A'\phi' = e^{2A} \partial_\phi V(\phi) \] (2.1b)

\[ d(d-1)A^2 - \frac{1}{2} \kappa^2 \phi'^2 = -\kappa^2 e^{2A} V(\phi) \] (2.1c)

where \( \prime \equiv d/dz \). Eqs.(2.1) can be rewritten in an equivalent form as the first-order system

\[ A'(z) = -\frac{\kappa}{d-1} e^{A(z)} W(\phi) \] (2.2a)

\[ \phi'(z) = \frac{2}{\kappa} e^{A(z)} \frac{dW(\phi)}{d\phi} \] (2.2b)

by introducing an (auxiliary) function \( W(\phi) \) called the ‘superpotential’ due to its (fake or true) supersymmetric origin [13,14,22,23]. The constraint (2.1c) determines the potential \( V(\phi) \) in terms of the superpotential:

\[ V(\phi) = 2\kappa^2 \left[ \frac{dW(\phi)}{d\phi} \right]^2 - \frac{d}{d-1} W^2(\phi), \] (2.3)

or vice-versa—when \( V(\phi) \) is given, one should solve Eq.(2.3) for \( W(\phi) \) and then to use this (non-unique) solution for solving the first-order system. We are mostly interested in the proper supergravity DWs of BPS-type, where the superpotential is an important input and the first-order system is nothing but the integrability (consistency) conditions for the BPS Killing spinor equations. Notice that in the patches of spacetime where \( W(\phi) \) is monotonic, one can use \( \phi \) itself as a coordinate and find \( A = A(\phi) \) from the equation \( dA/d\phi = -1/\beta(\phi) \), where

\[ \beta(\phi) = -\frac{d}{dA} = \frac{2(d-1)}{\kappa^2} \frac{\partial_\phi W(\phi)}{W(\phi)}. \] (2.4)

Fixing the integration constant \( A_0 \) in this equation amounts to fixing the overall “amplitude” of the scale factor \( a = e^A \), which can always be rescaled into the coordinates \( x^a \). The notation for \( \beta(\phi) \) is motivated by its holographic interpretation as the beta-function for the running coupling \( \phi \) in the dual QFT (cf. Sect.4). It can often be used to conveniently parameterize the DW geometry, for example the Ricci scalar is

\[ R = -\frac{d(d+1)}{(d-1)^2} \left[ 1 - \frac{\kappa^2}{(d+1)(d-1)} \beta^2(\phi) \right] \kappa^2 W^2(\phi). \] (2.5)
In terms of $\phi$, the geometric properties of the DWs are completely determined by the superpotential and its derivative, i.e. $\beta(\phi)$—qualitatively, one must know whether $W(\phi)$ has extrema, zeros and its behavior at $\phi \to \pm \infty$. Indeed, given $W(\phi)$, the scale factor $a(\phi)$ can be easily obtained by integration of Eq.(2.4). But the derivation of the “proper” DW solution $(a(z), \phi(z))$ requires further integrations, and its exact form is available only when the inverse function $\phi(a)$ is known explicitly.

3. Scale factor duality for domain walls

The first-order system in conformal coordinates (2.2) is invariant under the ‘scale factor duality’ transformations, which consist of a scale-factor inversion

$$\tilde{a}(\tilde{z}) = \frac{c^2}{a(z)}; \quad \tilde{z} = \pm z + \text{constant} \quad (3.1a)$$

together with a specific transformation of the scalar field and its potential,

$$\tilde{\phi}^2 = -\phi^2 + \frac{\kappa^2}{d-1}a^2(\phi)W^2(\phi), \quad \tilde{V}(\tilde{\phi}) = -\frac{a^4(\phi)}{c^4} [V(\phi) + 2W^2(\phi)]. \quad (3.1b)$$

Thus if $a(z)$ is a solution for a given superpotential $W(\phi)$ (and a potential $V(\phi)$), then SFD generates another solution $\tilde{a}(\tilde{z})$ for a system with a different superpotential $\tilde{W}(\tilde{\phi})$ and potential $\tilde{V}(\tilde{\phi})$. The scale-factor inversion implies that the superpotential transforms as

$$\tilde{a}(\tilde{\phi})\tilde{W}(\tilde{\phi}) = \mp a(\phi)W(\phi) \quad \text{or equivalently} \quad \tilde{W}(\tilde{\phi}) = \mp \frac{a^2(\phi)}{c^2} W(\phi), \quad (3.2)$$

with the signs following those of (3.1a). We can further set $c = 1$ by choosing the normalization of the scale factors in the dual solutions.

Given the scale factor $a(\phi)$ as a function of the scalar field, together with Eq.(2.4), it is straightforward to derive the SFD transformation of the scalar field

$$\tilde{\phi}(\tilde{\phi}) = \tilde{\phi}_0 \pm \int_{\phi_0}^{\phi} d\phi \left[ \frac{4(d-1)}{\kappa^2 \beta^2(\phi)} - 1 \right]^{1/2}. \quad (3.3)$$

The condition that r.h.s. has to be real introduces the upper bound

$$\beta^2(\phi) \leq 4(d-1)/\kappa^2. \quad (3.4)$$

If this bound is violated in a given model, the kinetic energy of the dual field has the wrong sign, and the dual domain wall is unstable. In other words, the bound (3.4) preserves the null energy condition (NEC), which for the single homogeneous
scalar field reads $\phi'^2(z) \geq 0$. It is not difficult to derive the transformation of the beta-functions,

$$
\tilde{\beta}(\tilde{\phi}) = \pm \left[ \frac{4(d-1)}{\kappa^2 \beta^2(\phi)} - 1 \right]^{1/2} \beta(\phi)
$$

where the signs follow the ones in (3.3). This can be written in a symmetric way (that however hides the relation between signs) as

$$
\beta^2(\phi) + \tilde{\beta}^2(\tilde{\phi}) = 4(d-1)/\kappa^2,
$$

and from the symmetry of this equation one sees that if (3.4) holds for one model it holds for the dual pair as well.

The ambiguities in sign appearing in Eqs.(3.2), (3.3) and (3.5) correspond to different ambiguities in Eq.(2.3), which determines the superpotential from the “true” potential $V(\phi)$. To be definite, in what follows we adopt the following conventions:

i) We always choose $\tilde{z} = -z + \text{const. in (3.1a)}$. Thus both $\tilde{W} > 0$ and $W > 0$, i.e. the plus sign is chosen in (3.2). Choosing a definite sign of the superpotential has no physical consequence, it only stipulates that both $a$ and $\tilde{a}$ grow with their respective radial coordinates.

ii) We always choose the plus sign in Eqs.(3.3) and (3.5). This particular sign ambiguity stems from invariance of Eq.(2.3) under a change of sign of $d\tilde{W}/d\tilde{\phi}$. Once we have fixed $\tilde{W} > 0$ in item i), the sign of $\tilde{\beta}$ is fixed by $d\tilde{W}/d\tilde{\phi}$, cf. Eq.(2.4). A choice of the opposite sign would select a different $\tilde{W}(\tilde{\phi})$, but in a theory with the same $\tilde{V}(\tilde{\phi})$.

The first-order system gives a restricted class of solutions of the second-order equations (2.1), but SFD is a symmetry of the full second-order equations (2.1) as well. Moreover, although we are considering here only flat domain walls, SFD (as defined above in conformal coordinates) actually holds for domain walls with any constant curvature, i.e. not only for Minkowski but also for dS and AdS internal $d$-dimensional slices. This has been shown in [7,8] for cosmological spacetimes, but the translation to DW case is known to be straightforward.

Preservation of the first-order system is related to the way SFD acts on the effective action. First of all, SFD is a symmetry of the equations only: the transformations do not preserve the action. Dismissing a volume factor $\int d^d x$, the action for the ansatz (1.1) can be put in the well-known (pseudo)-BPS form

$$
S = \int_{z_1}^{z_2} dz \ a^d \left[ \frac{d(d-1)}{\kappa^2} \left( \frac{a'}{a} + \frac{\kappa a W}{d-1} \right)^2 - \frac{1}{\kappa} \left( \frac{x}{2a} \phi' - a \frac{dW}{d\phi} \right)^2 \right] - \left[ \frac{2d}{\kappa^2} a^{d-1} \left( \frac{a'}{a} + \frac{\kappa}{d} a W \right) \right]_{z_1}^{z_2}
$$

\footnote{See [17] for an extensive discussion of the solutions of the superpotential equation (2.3).}
The last (boundary) term comes from a total derivative. The squared terms in the bulk integral give the first order system (2.2), thus the on-shell action is simply the boundary term. Making the duality transformation (with $dz = \pm d\tilde{z}$), we get

$$S = \pm \int_{\tilde{z}_1}^{\tilde{z}_2} d\tilde{z} \frac{c^d}{\tilde{a}^d} \left[ \frac{d(d-1)}{\kappa^2} \left( \frac{\ddot{a}}{\tilde{a}} + \frac{\kappa \dot{a} W}{d-1} \right)^2 - \frac{1}{\kappa} \left( \frac{d\phi}{d\tilde{a}} \right)^2 \left( \frac{\kappa \ddot{\phi} - \dot{a} \dot{W}}{d\phi} \right)^2 \right]$$

$$(3.8)$$

The action is clearly not invariant but the separation of squares is preserved; hence solutions of the first-order system for $\tilde{W}(\tilde{\phi})$ extremize $S$ and on such solutions the on-shell action is again a “pure” boundary term.

Since the only geometrical d.o.f. of (1.1) is the scale factor, SFD completely determines the geometry of the dual domain wall. For example, the transformation of the Ricci scalar (2.5),

$$\frac{R}{\kappa^2 W^2} + \frac{\tilde{R}}{\kappa^2 \tilde{W}^2} = - \frac{2d}{d-1},$$

$$(3.9)$$

is quite useful in relating the asymptotic behavior (near AdS boundaries or near singularities) of dual geometries. More interestingly, SFD gives a simple map between causal structures—knowing the Penrose diagram of a domain wall solution, we can very easily find the diagram of its dual. This is explained in detail in App.A.

### 3.1 Liouville potentials and AdS space

We now want to give explicit examples of the construction of dual domain walls. The simplest one is found for Liouville, i.e. exponential, potentials

$$W(\phi) = \frac{2/v^2}{\kappa L} e^{v \kappa \phi / 2}$$

$$(3.10)$$

$$V(\phi) = - \frac{4/v^4}{(d-1)\kappa^2 L^2} \left[ d - (v/v_c)^2 \right] e^{v \kappa \phi}, \quad v_c = \sqrt{\frac{2}{d-1}}.$$  

$$(3.11)$$

Here $v > 0$ is a dimensionless parameter. The amplitude has been chosen appropriately to simplify the scale factor in terms of a length scale $L > 0$. We assume that $V \leq 0$, hence

$$0 < v < \sqrt{d} \ v_c.$$  

$$(3.12)$$

The beta-function given by (2.4) is a constant parameterized by $v$,

$$\beta = (d-1)v / \kappa$$

$$(3.13)$$

so integrating Eq.(2.4) we find

$$a(\phi) = \exp\left[ - \frac{\kappa \phi}{(d-1)v} \right],$$

$$(3.14)$$
while integration of the first-order system (2.1) yields

$$a(z) = \left[\frac{(v^2 - v_c^2)(z_0 - z)}{v^2 L}\right]^{\frac{v^2}{\sqrt{v^2 - v_c^2}}} \begin{cases} \text{if } v_c < v, & z \in (-\infty, z_0) \\ \text{if } v_c > v, & z \in (z_0, +\infty) \end{cases}$$

(3.15)

$$a(z) = \exp\left[\frac{z_0 - z}{L}\right] \begin{cases} \text{if } v_c = v, & z \in (-\infty, \infty) \end{cases}$$

(3.16)

where $z_0$ is an integration constant. In all cases there is a singularity when $a \to 0$, but with one crucial difference: for $v > v_c$ the singularity is reached at the finite radius $z_0$. For $v \leq v_c$ the point $z_0$ is where the scale factor diverges, and the singularity is at $z = +\infty$. The solutions for $\phi$ are

$$\kappa \phi(z) = -\frac{2}{v} \log \left[\frac{(v^2 - v_c^2)(z_0 - z)}{v^2 L}\right]^{\frac{v^2}{\sqrt{v^2 - v_c^2}}} \begin{cases} \text{if } v \neq v_c \end{cases}$$

(3.17)

$$\kappa \phi(z) = -\frac{2}{v} \left(\frac{z_0 - z}{L}\right) \begin{cases} \text{if } v = v_c \end{cases}$$

(3.18)

with ranges of $z$ again as in (3.15)-(3.16). Note that the singularity is always at $\phi \to +\infty$; cf. (3.14).

The SFD dual domain wall has $\tilde{a} = c^2/a$. For $v \neq v_c$,

$$\tilde{a}(\tilde{z}) = \left[\frac{(v^2 - v_c^2)(\tilde{z}_0 - \tilde{z})}{v^2 \tilde{L}}\right]^{-\frac{v^2}{\sqrt{v^2 - v_c^2}}} \tilde{z} - \tilde{z}_0 = -(z - z_0)$$

(3.19)

for some constant $\tilde{L}$. Eq.(3.19) has the same structure as (3.15), but with

$$v^2 - v_c^2 = v_c^2 - v^2.$$  (3.20)

This transformation leaves $v_c$ invariant, so the critical model with $v = v_c$ is dual to itself, with $\tilde{a}(\tilde{z}) = a(z)$. This can be seen directly from the form of the scale factor (3.15).

Since the scale factors have the same form, the potential and the superpotential must again be exponentials. The dual field $\tilde{\phi}$ is given by Eq.(3.3), whose integral is trivial,

$$\tilde{\phi}/\tilde{v} = -\phi/v.$$  (3.21)

We have chosen the plus sign in (3.3), and also set the integration constant to zero. With this choice, from Eq.(3.2), we find that the dual potential and superpotential are

$$\tilde{W}(\tilde{\phi}) = \tilde{W}_0 e^{\tilde{\phi}/\tilde{v}/2}, \quad \tilde{V}(\tilde{\phi}) = -\frac{\tilde{W}_0^2}{d-1} \left[\tilde{d} - (\tilde{v}/v_c)^2\right] e^{\tilde{\phi}/\tilde{v}}.$$  (3.22)

Note that we have the map $\{\phi \to -\infty\} \leftrightarrow \{\tilde{\phi} \to +\infty\}$, corresponding, from Eq.(3.14) to $\{a \to \infty\} \leftrightarrow \{\tilde{a} \to 0\} \text{ (and vice-versa).}$
Thus we have found that SFD is a map between exponential potentials:

\[ V(\phi) = V_0 e^{\nu \phi} \quad \leftrightarrow \quad \tilde{V}(\tilde{\phi}) = \tilde{V}_0 e^{\tilde{\nu} \tilde{\phi}} \]  

(3.23)

\[ 0 < v^2 < v_c^2 \quad \leftrightarrow \quad v_c^2 < \tilde{v}^2 < 2v_c^2 \]  

(3.24)

The duality reflects a value of \( v^2 \) across the critical \( v_c^2 \), as shown in Fig.1. Then \( v^2 = 0 \) is mapped to \( \tilde{v}^2 = 2v_c \), and any \( v^2 > 2v_c^2 \) is mapped to a \( \tilde{v}^2 < 0 \). So requiring that the pair \((v, \tilde{v})\) is real introduces the bound

\[ v^2 \leq 2v_c^2 \quad \text{or} \quad v^2 \leq 4/(d - 1). \]  

(3.25)

Eq.(3.20) is equivalent to the beta-function transformation (3.6) and the bound (3.25) corresponds to the bound (3.4). As discussed, models violating the bound have their duals with the wrong sign of the kinetic term.

A Liouville model with exponential parameter

\[ v = \sqrt{2}v_c = 2/\sqrt{d - 1}, \]  

(3.26)

for which (3.25) is saturated, will be called a ‘special Liouville’ solution. The special nomenclature is because the corresponding dual parameter is \( \tilde{v} = 0 \), which does not correspond to an exponential potential anymore but to a (negative) constant \( \tilde{V} \), hence the dual solution of special Liouville is AdS space. Note however that, as \( v \to 0 \) the solutions for \( a \) and \( \phi \) must be looked at carefully. In particular, with our chosen parameterization for the superpotential (3.10), the limit \( v \to 0 \) must be taken with the fixed product \( v^2L \to 2\ell/(d - 1) \), where \( \ell \) is the radius of AdS. For example, Eq.(3.14) does not have a limit, because in pure AdS there is no scalar field; but Eq.(3.15) does have the right limit, giving \( a = \ell/z \).

### 3.2 Asymptotically AdS solutions and their duals

We next consider an asymptotically AdS domain wall DW, and use SFD to find the dual solution \( \tilde{D}W \). The superpotential has the standard form corresponding the quadratic approximation of the potential near a maximum,

\[ W(\phi) = d - 1 \frac{\phi}{\kappa \ell} + \frac{s}{4\kappa \ell} (\kappa \phi)^2 + \cdots \]  

(3.27)

\[ V(\phi) = -\frac{d(d - 1)}{\kappa^2 \ell^2} + \frac{1}{2} m^2 \phi^2 + \cdots, \]  

(3.28)

\[ s = \frac{1}{2} \left( d \pm \sqrt{d^2 + 4m^2 \ell^2} \right) > 0, \quad -\frac{d^2}{4\kappa^2} < m^2 < 0 \]  

(3.29)
We are considering solutions with $0 < \kappa \phi \ll 1$, near the AdS vacuum with radius $\ell$ located at $\phi = 0$. The geometry will be near the AdS boundary. The function (2.4) is
\[ \beta(\phi) = s \phi + \cdots \] (3.30)
To find the dual field $\tilde{\phi}$, we must insert this into Eq.(3.3) and integrate. Keeping only the leading term we get
\[ \kappa \tilde{\phi} = -\frac{2\sqrt{d-1}}{s} \log(\kappa \phi) + \cdots \] (3.31)
For $0 < \kappa \phi \ll 1$, we have $\kappa \tilde{\phi} \gg 1$. Inverting (3.31) as $\phi = \phi(\tilde{\phi})$,
\[ \kappa \phi = \left(e^{-\frac{\sqrt{d-1}}{2s \sqrt{\kappa \tilde{\phi}}}}\right)^s \left[1 + O\left(e^{-2s \frac{\sqrt{d-1}}{\sqrt{\kappa \tilde{\phi}}}}\right)\right] \] where $e^{-\frac{\sqrt{d-1}}{2s \sqrt{\kappa \tilde{\phi}}}} \ll 1$. (3.32)
We now want to find the superpotential for $\tilde{D}W$. Inserting (3.30) into Eq.(3.5) and using (3.32) we get
\[ \tilde{\beta}(\tilde{\phi}) = \frac{2\sqrt{d-1}}{s} - \frac{s^2}{4\sqrt{d-1}} \left(e^{-\frac{\sqrt{d-1}}{2s \sqrt{\kappa \tilde{\phi}}}}\right)^2 + O\left(e^{-4s \frac{\sqrt{d-1}}{\sqrt{\kappa \tilde{\phi}}}}\right). \] (3.33)
Eq.(2.4) can be solved as a differential equation for $\tilde{W}(\tilde{\phi})$, resulting in an exponential superpotential, and in the asymptotic expression for the potential of the dual model:
\[ \tilde{W}(\tilde{\phi}) = \tilde{W}_0 e^{s \sqrt{\kappa \tilde{\phi}}} \left[1 + \frac{s}{8(d-1)} e^{-s \frac{\sqrt{d-1}}{\sqrt{\kappa \tilde{\phi}}}} + \cdots\right] \] (3.34)
\[ \tilde{V}(\tilde{\phi}) = -\left(\frac{d-2}{d-1}\right) \tilde{W}_0^2 e^{2s \sqrt{\kappa \tilde{\phi}}} \left[1 + \frac{s(2s + d - 2)}{4(d-1)(d-2)} e^{-s \frac{\sqrt{d-1}}{\sqrt{\kappa \tilde{\phi}}}} + \cdots\right] \] (3.35)
with $\kappa \tilde{\phi} \gg 1$. The first term of $\tilde{V}(\tilde{\phi})$ is the leading one, and it is just the special-Liouville potential; the other terms vanish for $\tilde{\phi} \to \infty$. We plot asymptotic regions of the dual potentials (3.28) and (3.35) in Fig.2.
Of course, if we include more (subleading) terms in the ellipsis in Eq.(3.27), then the formula (3.35) for $\tilde{V}(\tilde{\phi})$ will include more terms inside the brackets. But note that the leading term does not depend on $s$; so we could, for example, start with $s = 0$ and get the same potential at leading order. Say, if instead of (3.27) we start with a cubic superpotential,
\[ W(\phi) \approx \frac{d-1}{\kappa \ell} + \frac{D}{6\kappa \ell}(\kappa \phi)^3, \quad \kappa \phi \ll 1, \]
the domain wall structure is very different, but after calculating $\tilde{W}(\tilde{\phi})$ and $\tilde{V}(\tilde{\phi})$ we still find the same leading asymptotic exponential behavior. This was expected, since AdS is always mapped by SFD to the special Liouville solution.
4. SFD as a relation between holographic RG flows

The DW first-order system (2.2) has a standard holographic interpretation as RG equations of certain QFT$_d$ with an energy scale $E = a(z)$ and running coupling $g = \phi(z)$, driven by a scalar operator $\mathcal{O}$ [22, 24]. The RG flow is determined by the holographic beta-function $\beta \equiv -dg/d\log E$ given by Eq.(2.4). In this context, SFD defines a map between the RG flows of two holographic QFT$_d$s which, because of scale factor inversion, relates their ultraviolet and infrared limits. The transformation of the beta-functions given by Eq.(3.6), now should be seen as a statement about the pair of dual QFTs:

$$\beta^2(\phi) + \tilde{\beta}^2(\tilde{\phi}) = 4(d - 1)/\kappa^2,$$

(4.1)

and, together with (3.3), it allows us to construct the RG flow of $\tilde{\text{QFT}}$ from the one of QFT. Other features of the holographic RG flows are inherited by the domain wall solutions and the corresponding action of SFD. For example, the running of the anomalous scaling dimension of $\mathcal{O}$, given by $s(\phi) = -d\beta/d\phi$, transforms as:

$$\beta^2(\phi)s(\phi) = \beta^2(\tilde{\phi})s(\tilde{\phi}).$$

(4.2)

Another important characteristic of the QFTs is the central-charge function\footnote{In Einstein gravity, and in asymptotically AdS$_{d+1}$ backgrounds, the two central functions $c$ and $a$ coincide: $c = a$.} [25–27] whose SFD transformation is found readily from Eq.(3.2),

$$c(\phi) \equiv \frac{2(d - 1)d^{-1}\pi^{d/2}}{\kappa^{d+1}\Gamma(d/2)|-W(\phi)|^{d-1}}, \quad \frac{\tilde{c}(\tilde{\phi})}{\tilde{a}^{d-1}(\tilde{\phi})} = \frac{c(\phi)}{a^{d-1}(\phi)}.$$

(4.3)

At the UV fixed point of the RG flow (i.e. at the AdS$_d$ boundary of the DW bulk), $c(= a)$ is a natural generalization of the central charge(s) of the corresponding UV

---

Figure 2: Asymptotic regions of dual potentials. (a) The quadratic potential (3.28); (b) The Liouville potential (3.35)
CFTs. In RG flows between two fixed points, this function obeys the (generalized) Zamolodchikov’s $c$-theorem: $c_{\text{UV}} \geq c_{\text{IR}}$, assuring the decreasing of the central functions during the RG flows. The inequality follows if the null energy condition is satisfied along the domain wall evolution [26]; under the condition (3.4) SFD preserves the NEC, so it also guarantees the validity of the $c$-theorem, i.e. $\tilde{c}_{\text{UV}} \geq \tilde{c}_{\text{IR}}$.

Perhaps the most remarkable feature of the map (4.1) is that it does not preserve the RG fixed points, i.e. the positions of the zeros of $\beta(\phi)$ and $\tilde{\beta}(\tilde{\phi})$. Instead, as seen in §3.2, SFD maps a QFT near a UV (AdS) fixed point with small $\kappa \phi \ll 1$ to the IR regime of $\tilde{\text{QFT}}$ with a logarithmically diverging coupling $\kappa \tilde{\phi} \gg 1$; cf. Eq.(3.31). Such “flows to infinity” of the scalar field are relevant in phenomenological applications of holography to QCD [15,16,18,28], and in holography applied to condensed matter [29,30]. The Liouville asymptotics are particularly important when located at the IR limit, i.e. at a singularity, which is the case in our discussions. For example, using the methods of [16], we can show that the special-Liouville singularity is “confining”, in the sense that the holographic Wilson loops of the QFT obey the area-law in the IR. Flows that end in a naked singularity in the IR are an indication of a non-trivial IR structure of the holographic QFT. On this note, all our Liouville singularities are ‘good’ by the criterion of Gubser [31], since the (negative) potential $V(\phi)$ is bounded from above. Hence the singular domain walls can be obtained from a black hole solution whose horizon shrinks to zero, and the field theories have a well-defined finite-temperature limit. The advantage of the use of (asymptotic) scale factor duality between the UV and IR regimes of pairs of holographic QFTs is that one can describe the problematic IR behavior (near the Liouville-singularity) of one QFT with the well-defined, nearly UV data of its dual $\tilde{\text{QFT}}$.

As an illustration of what kinds of pairs of dual RG flows can be obtained as a result of the SFD transformations, we now consider three examples of flows with a UV fixed point, and their respective images shown in Fig.3.

I) In the standard RG flow between two CFTs the domain wall interpolates between two AdS vacua where $V(\phi)$ has a maximum (the UV point) and a minimum (the IR point), with $s(\phi_{\text{IR}}) < 0 < s(\phi_{\text{UV}})$. The image of this flow interpolates between two special-Liouville asymptotics.
II) The flow starts in the AdS UV and runs to a general Louville singularity in
the IR. Its image has two different Liouville asymptotics, with the special
potential in the IR limit (as image of the AdS boundary).

III) The third example is a flow starting at AdS and ending with a special Liouville
IR; its dual has, therefore, the same types of asymptotics.

In §3.2 we have given the explicit expression of $\hat{\beta}(\phi)$ and $\tilde{\phi}(\phi)$, dual to the vicinity
of a UV fixed point of $\beta(\phi)$ when $0 \ll \kappa \phi \ll 1$. We saw that $\hat{\phi} \approx -\frac{2\sqrt{d-1}}{\kappa \phi} \log(\kappa \phi)$,
so as $\phi \to 0^+$ we have $\hat{\phi} \to +\infty$. The dual beta-functions $\hat{\beta}(\phi)$ and $\tilde{\beta}(\phi)$, given in
Eqs.(3.30) and (3.33), are plotted in Fig.4(a).

In general, given a beta function $\beta(\phi)$, it is not possible to invert Eq.(3.3) to find
$\hat{\phi}(\phi)$, hence we cannot give a closed expression for $\hat{\beta}(\phi)$. However, it is possible
to give $\tilde{\beta}(\phi)$, so we can still see the behavior of the dual RG flow of QFT, even
without knowing the explicit transformation of the fields. Let us illustrate this
with examples of the three types above.

The simplest example of Type I flow is given by the potential
\[
W(\phi) = \frac{1}{\kappa \ell} \left[ 2 + \alpha^{-2} s \sin^2 \left( \frac{\alpha}{2} \kappa \phi \right) \right] , \quad \kappa \beta(\phi) = \frac{s \sin(\alpha \kappa \phi)}{1 + \frac{\alpha}{3\alpha} \sin^2 \left( \frac{\alpha}{2} \kappa \phi \right) } . \tag{4.4}
\]
We take $d = 4$. There is a UV fixed point at $\phi_{UV} = 0$ with anomalous dimension
$s(\phi_{UV}) > 0$, and a IR fixed point at $\phi_{IR} = \pi / \kappa \alpha$ with $s(\phi_{IR}) = -3\alpha^2 s / (3\alpha^2 + s) < 0$.
The dual $\tilde{\beta}$ as a function of the original field $\phi$ can be readily found from Eq. (3.5).
We plot both $\beta(\phi)$ and $\tilde{\beta}(\phi)$ in Fig.4(b).

A well-studied flow of Type II is the Coulomb branch of $\mathcal{N} = 4$ SYM in $d = 4$.
The superpotential is given by [24,32]
\[
W(\phi) = \frac{2}{\kappa} \left[ e^{-\kappa \phi / \sqrt{2}} + e^{2\kappa \phi / \sqrt{2}} \right] , \quad \beta(\phi) = \frac{2\sqrt{6}}{\kappa} \left( \frac{e^{\sqrt{3/2} \kappa \phi} - 1}{e^{\sqrt{3/2} \kappa \phi} + 2} \right) . \tag{4.5}
\]
The RG flows from an AdS UV fixed point at $\phi = 0$, driven by the VEV an operator
with anomalous dimension $s = 2$, up to a null Liouville singularity at $\phi = -\infty$
(produced by a disc of D3-branes in the “lifted” 10-dimensional SUGRA), where
$W \sim e^{-\nu_c \kappa \phi / 2}$. Here $\nu_c = \sqrt{3/3}$ is the critical parameter (3.11). We plot $\beta(\phi)$
and its dual $\tilde{\beta}(\phi)$ in Fig.4. Note that the critical-Liouville limit is invariant, as expected.
The behavior dual to the UV limit is $\tilde{\beta} \approx -2\sqrt{d-1} / \kappa$; the minus sign comes from
the fact that $\beta$ and $\phi$ are negative. Here we can also look at the map (3.3) between
fields $\phi$ and $\hat{\phi}$ asymptotically: for $-1 \gg \kappa \phi < 0$, we have $\kappa \phi \approx 3 \log |\phi| \to -\infty$,
and for $\phi \to -\infty$ we have $\hat{\phi} \approx -\phi \to +\infty$. (So the flow of $\hat{\beta}(\hat{\phi})$ should be read in
the opposite direction when as function of $\phi$.)

Another interesting example of Type II has been discussed in [18], with
\[
V(\phi) = \left[ \frac{s(s-d) + d(d-1)\nu^2}{2\kappa^2 \ell^2} \right] \phi^2 - \frac{d(d-1)}{\kappa^2 \ell^2} \cosh(\nu \kappa \phi) . \tag{4.6}
\]
Figure 4: (a) $\beta(\phi)$ near an (AdS) UV fixed point (left panel) and its dual $\tilde{\beta}(\tilde{\phi})$ near a Liouville singularity (right panel). (b) $\beta(\phi)$ for a standard RG flow, and its dual $\tilde{\beta}(\phi)$, as a function of the original field. (c) $\beta(\phi)$ of the CB flow and its dual $\tilde{\beta}(\phi)$, as a function of the original field. (d) $\beta(\phi)$ of the GPPZ flow and its dual $\tilde{\beta}(\phi)$, as a function of the original field.
This potential has rich QCD phenomenology for finite temperature solutions (i.e. black hole geometries); it has AdS asymptotics \((3.28)\) for \(\kappa \phi \ll 1\), and for \(\phi \to \infty\) it goes to \(V(\phi) \sim -e^{\kappa \phi} \). For \(v = \sqrt{2} v_c\), the flow becomes an example of Type III, with a special Liouville singularity. This is precisely the numeric example examined in [17], in connection with the dynamical instability of [18].

The flows of \(\beta(\phi)\) in Types I and II do not have UV fixed points: the corresponding domain wall solutions do not have an AdS boundary, they have Liouville asymptotics instead. Although one can define a kind of gauge/gravity correspondence between the bulk DW solutions and distinct QFTs, there are still no proofs or convincing (stringy or CFT) arguments in favor of the holographic nature of these relationships. Hence these models requires a proper definition of a new holographic renormalization procedure, different from the standard AdS/CFT one [22, 24, 33] which explicitly uses the AdS boundary to define the UV “bare action”. In Ref. [21], Kiritsis et al. show that a consistent renormalization can indeed be carried out in Liouville models with \(v\) within the bound \((3.12)\). In §5.4 we show how SFD gives, in this scenario, a relation between VEVs of the operators sourced by \(\phi\) and \(\tilde{\phi}\).

Finally, there are the flows of Type III, which are “asymptotically symmetric” under SFD, and both \(\beta(\phi)\) and \(\beta(\tilde{\phi})\) have a UV fixed point and a special-Liouville IR asymptotic. This symmetry may seem too restrictive, but we can give a remarkable example: the GPPZ flow [19], extensively examined (cf. [24,32,34]). The superpotential is

\[
W(\phi) = \frac{d - 1}{2 \kappa \ell} \left[ 1 + \cosh \left( \frac{\kappa \phi}{\sqrt{d - 1}} \right) \right] = \frac{d - 1}{\kappa \ell} \cosh^2 \left( \frac{\kappa \phi}{2 \sqrt{d - 1}} \right),
\]

usually considered for \(d = 4\). The AdS boundary is at \(\phi = 0\) and the special-Liouville singularity at \(\phi \to +\infty\), near which \(W \sim e^{\kappa \phi} \) with \(v = \sqrt{2} v_c\). This proves the asymptotic symmetry, but that is not all. Actually, the GPPZ flow is exactly self-dual under SFD. This can be easily shown. The solution in conformal coordinates is

\[
a(z) = \tan(-z/\ell), \quad \sinh \left[ \frac{\kappa \phi}{2 \sqrt{d - 1}} \right] = \cot(-z/\ell), \quad -\frac{\pi}{2} \ell < z < 0,
\]

and the beta-function is

\[
\beta(\phi) = \frac{2 \sqrt{d - 1}}{\kappa} \tanh \left( \frac{\kappa \phi}{2 \sqrt{d - 1}} \right).
\]

Scale factor inversion, with \(\tilde{z} = -z\) and \(c = 1\), gives \(\tilde{a}(\tilde{z}) = \cot(\tilde{z}/\ell)\), with \(0 < \tilde{z} < \frac{\pi}{2} \ell\). Integrating Eq.\((3.3)\), we get \(\cot \left( \frac{-\kappa \tilde{\phi}}{2 \sqrt{d - 1}} \right) = \cosh \left( \frac{\kappa \phi}{2 \sqrt{d - 1}} \right)\), with \(-\infty < \tilde{\phi} < 0\). Then with Eq.\((3.2)\) we find the dual superpotential

\[
\tilde{W}(\tilde{\phi}) = \frac{d - 1}{\kappa \ell} \cosh^2 \left( \frac{-\kappa \tilde{\phi}}{2 \sqrt{d - 1}} \right).
\]
which has exactly the same form as (4.7). In Fig.4(d) we plot the beta-functions \( \beta(\phi) \) and \( \tilde{\beta}(\phi) \) as we did for the other examples. The asymmetry between the functions is a result of plotting \( \tilde{\beta} \) as a function of \( \phi \) and not of \( \tilde{\phi} \). This illustrates how a similar “distortion” must be considered in the examples of Figs.4(b)-(c). Note how the asymptotics are swapped, as a result of \( \{ \phi = 0 \} \mapsto \{ \tilde{\phi} = \infty \} \) and vice-versa.

Let us mention that a more general family of SFD self-dual models can be constructed from the cosmological models considered in [7] after applying the “DW/FRW correspondence” rules of [35, 36].

5. SFD for fluctuations and their spectra

We now turn to the effect of SFD transformations on the fluctuations around the domain wall geometry. We will be interested in tensor and scalar modes, which are the most relevant for holography. The linearized metric is

\[
ds^2 = e^{2A(z)} \left[ (1 - (d - 2)B)dz^2 + [(1 + B)\eta_{ab} + h_{ab}] dx^a dx^b \right]
\]

\[
\partial_a h_{ab} = 0, \quad h^a_a = 0, \quad \phi(z, x^a) = \bar{\phi}(z) + \chi(z, x^a)
\]

Tensor modes \( h_{ab}(z, x^a) \) are transverse and traceless, and the scalar sector is written in Newtonian gauge. We mark the background field with an overbar. Since we are working with a single scalar field, there is only one true scalar degree of freedom. The ‘Bardeen potential’ \( B(z, x^a) \), and the perturbation of the scalar field \( \chi(z, x^a) \) combine into the gauge-invariant ‘curvature perturbation’ \[38\] \( \zeta = B - (A'/\bar{\phi})\chi; \) we use \( \zeta \) to describe the scalar d.o.f. The linearized field equations are

\[
h''_{ab} + (d - 1)A' h'_{ab} + \Box_d h_{ab} = 0
\]

\[
\zeta'' + [(d - 1)A'(z) + \partial_z \log \beta^2(z)] \zeta' + \Box_d \zeta = 0
\]

where \( \beta(\bar{\phi}) \) is given by Eq.(2.4).

5.1 SFD and S-duality

Eqs.(5.2) and (5.3) can be obtained from an effective quadratic action \( I \). Writing \( f \) for either the scalar mode \( \zeta \) or the tensor modes \( h \) (ignoring polarization indexes),

\[
I = \frac{1}{2} \int d^d x \int dz \ G(z) \left( f'^2 - \eta^{ab} \partial_a f \partial_b f \right) \begin{cases} G_h(z) = e^{(d-1)A(z)} \\ G_\zeta(z) = e^{(d-1)A(z)} \beta^2(z) \end{cases}
\]

and if we pass to phase space with the canonical momentum \( \Pi = \delta I/\delta f' = Gf' \) we get a Hamiltonian

\[
\mathcal{H} = \frac{1}{2} \int d^d x \int dz \left[ \frac{\Pi^2}{G} + G(\partial f)^2 \right] = \frac{1}{2} \int d^d k \int dz \left[ \frac{||\Pi||^2}{G} + Gk^2|f_k|^2 \right]
\]

\[8\]The description of fluctuations around domain walls is the same as in cosmology, see e.g. [37]. For domain wall conventions, see e.g. [17].
We have used Fourier modes in the transverse space, with $|f_k|^2 = f_k f_{-k}$, etc. and $k^2 = \eta_{ab} k^a k^b < 0$ is a timelike vector. In [5] it was noted that (5.5) is invariant under the transformation

$$ G \mapsto \tilde{G} = \frac{\lambda^2}{G}, \quad f_k \mapsto \tilde{f}_k = \frac{1}{\lambda k} \Pi_k, \quad \Pi_k \mapsto \tilde{\Pi}_k = -\lambda k f_k \quad (5.6) $$

where $\lambda$ is a parameter. This was called S-duality. It is a “canonical transformation”, exchanging $\Pi$ and $f$; it preserves the Hamilton equations and it swaps the two second-order equations for $f$ and $\Pi$, viz.

$$ f''_k + (G'/G) f'_k - k^2 f_k = 0, \quad \Pi''_k - (G'/G) \Pi'_k - k^2 \Pi_k = 0, \quad (5.7) $$

the first of which corresponds to Eqs.(5.2)/(5.3).

Now, for tensor modes we have $G_h(z) = [a(z)]^{d-1}$. Hence $G_h \mapsto \tilde{G}_h$ is an inversion of the scale factor in conformal coordinates: it is SFD. More precisely, SFD can be extended to tensor modes as

$$ a \mapsto \tilde{a} = c^2/a \quad h_k \mapsto \tilde{h}_k = \frac{1}{c^2} k^{-1} \Pi_k^{(h)} \quad \Pi_k \mapsto \tilde{\Pi}_k^{(h)} = -\beta d^{-1} k \ h_k \quad (5.8) $$

For scalar modes, the same is not true in general, because inversion of $G_\zeta(z) = [a(z)]^{d-1} \beta^2(z)$ does not correspond to an inversion of $a(z)$. However, in Liouville models, where $\beta$ is a constant, $G_\zeta \mapsto \tilde{G}_\zeta$ is again an inversion of the scale factor: again, it corresponds to SFD. Therefore, SFD can be extended to scalar modes in Liouville models as

$$ a \mapsto \tilde{a} = c^2/a \quad \zeta_k \mapsto \tilde{\zeta}_k = \frac{1}{\sqrt{c^2 z}} k^{-1} \Pi_k^{(c)} \quad \Pi_k \mapsto \tilde{\Pi}_k^{(c)} = -\beta^2 c d^{-1} k \ \zeta_k \quad (5.9) $$

with $\beta = \text{constant}$. When a domain wall is only asymptotically Liouville, as in the examples of Sect.4, the transformation for the scalar modes is valid asymptotically.

The name S-duality was given in [5] because (5.6) is a generalization of the usual strong-/weak-coupling duality of string theory. This is in fact also true for SFD in the examples we have discussed. As a function of the string-frame dilaton $\phi_S$, the Liouville solution (3.14) gives

$$ G \sim a^{d-1} = e^{-\kappa \phi/v} = e^{-2\kappa \phi_S/v}. $$

Hence $G \mapsto 1/G$ is equivalent to $\phi_S \mapsto -\phi_S$, i.e. it is indeed equivalent to an inversion of the string coupling $g_S^2 = e^{\phi_S}$.

---

9We write the effective string-frame action as

$$ S = \frac{1}{\kappa_S^2} \int d^{d+1}x \sqrt{-g_S} e^{-2\phi_S} [R_S + 4 g_S^{\mu\nu} \partial_\mu \phi_S \partial_\nu \phi_S + V_S(\phi_S)] $$

$V_S(\phi_S)$ is the effective potential found after truncating the SUGRA action; $\kappa_S^2$ is a normalization constant determined by the vacuum value $\Phi_0$ of the dilaton $\Phi = \phi_S + \Phi_0$. The “string metric” $g_S^{\mu\nu}$ is related to the Einstein metric by $g^{\mu\nu} = e^{-\frac{2\phi_S}{d-1}} g_S^{\mu\nu}$. The canonically normalized field $\phi$ appearing in the Einstein action is given by $\phi_S = \frac{1}{2} [1/(d-1)]^{1/2} \phi$. 

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5.2 Dual Schrödinger equations

It is well known that fluctuations around domain walls can be described by a one-dimensional Schrödinger problem with a potential dictated by the background dynamics. With a change of variables,

$$\psi_k(z) \equiv e^{-\int W(z) dz} f_k(z), \quad W(z) \equiv -\frac{1}{2} G'(z)/G(z),$$

Eq.(5.7) becomes a Schrödinger equation with energy $M_k^2 = -k^2 > 0$,

$$H_1 \psi_k = \left[-\frac{d^2}{dz^2} + V(z)\right] \psi_k = M_k^2 \psi_k,$$

where

$$V(z) = W^2(z) - W'(z).$$

This Hamiltonian is factorizable as

$$H_1 = Q^\dagger Q, \quad \text{for} \quad Q = d/dz + W(z), \quad Q^\dagger = d/dz - W(z),$$

and has a “superpartner” $H_2 = QQ^\dagger$ whose potential has a flipped sign,

$$H_2 \psi_k = QQ^\dagger \psi_k = -\psi_k'' + \left[W^2(z) + W'(z)\right] \psi_k.$$

The special factorization relates the eigenstates and the spectra of $H_1$ and $H_2$. If $\psi_k^{(A)}$ are the wave-functions of $H_A$,

$$Q^\dagger Q \psi_k^{(1)} = M_k^2 \psi_k^{(1)} \quad \psi_k^{(2)}(z) = M_k^{-1} Q \psi_k^{(1)}$$

$$QQ^\dagger \psi_k^{(2)} = M_k^2 \psi_k^{(2)} \quad \psi_k^{(1)}(z) = M_k^{-1} Q \psi_k^{(2)}$$

What we have here is known as ‘supersymmetric quantum mechanics’ (SUSY QM) [20]; the function $W(z)$ is (also) called ‘superpotential’.

We can rephrase the results of §5.1 in terms of SUSY QM. The duality of tensor modes written in (5.8) here manifests itself in the fact that SFD is SUSY for the tensor superpotential, which can be read from Eq.(5.2),

$$W_T(z) = -\frac{1}{2} (d - 1) A'(z).$$

Indeed, scale-factor inversion amounts to just flipping the sign of $W_T(z)$. Therefore SFD represents a crossed map between the SUSY partners of dual domain wall
solutions, as shown in Fig.5. If we solve the tensor wave functions around a domain wall, (5.15) gives us automatically the tensor wave functions around its scale-factor dual.

The SUSY QM representation is also valid for scalar modes in general, with the SUSY partner of the curvature perturbation \( \zeta \) being the Bardeen potential \( B \) [17]. Just as S-duality, the relation between SUSY QM and SFD, however, only holds for the scalar modes on Liouville backgrounds. We can find the scalar superpotential from Eq.(5.3),

\[
\mathcal{W}_S(z) = \frac{1}{2} (d - 1) A'(z) + \partial_z \log |\beta(z)|. \tag{5.17}
\]

In Liouville models \( \beta \) is a constant, hence (5.17) is equal to the superpotential (5.16) and SFD is equivalent to SUSY QM.

The most important fact about the extension of SFD to fluctuations is that, basically, it exchanges a mode with its derivative. In the S-duality context, this corresponds to an exchange between \( f_k \) and \( \Pi_k \); in the SUSY QM context, it corresponds to the operation \( M_k^{-1} Q \psi_k \) leading to a superpartner. Note that when we express \( \psi_k = e^{-\int W dz} f_k \), the dual fluctuation/superpartner, given by (5.15) as

\[
e^{\int W dz} \tilde{f}_k = \tilde{\psi}_k = M_k^{-1} (d/dz + \mathcal{W}) \psi_k = M_k^{-1} e^{-\int W dz} f'_k,
\]

is equivalent to (5.6), i.e. to

\[ \tilde{f}_k = \frac{1}{\kappa} \Pi_k = \frac{1}{\kappa} G f'_k \]

where by the definition (5.10) we have \( e^{-\int W dz} = G^{1/2} \). This duality between the fluctuation and its derivative has important consequences for the boundary conditions of the dual models, as we illustrate concretely below.

### 5.2.1 Wave-functions for (asymptotically) Liouville models

In Sect.4 we saw that many interesting domain walls are asymptotically Liouville (or/and AdS), and in all these pairs of models the potentials of the fluctuation equations (5.11) for both tensor and scalar modes are asymptotically related by the SFD transformations. The potential of interest is

\[
\mathcal{V}(z) = \frac{\delta(\delta - 1)}{(z_s - z)^2} + O \left( \frac{1}{|z_s - z|} \right), \quad \delta \equiv \frac{1}{v^2 - v_c^2}. \tag{5.18}
\]

According to (3.20) the parameter \( \delta \) transforms as

\[ \tilde{\delta} + \delta = 0, \tag{5.19} \]

and in Table 1 we show in each column the ranges of variables related by SFD. The point \( z_s \) may be the location of a timelike boundary or of a timelike singularity, depending on whether \( \delta < 0 \) or else \( \delta > 0 \).
Timelike Boundary | Timelike Singularity
---|---
$v^2 \in (0, v_c^2)$ | $v^2 \in (v_c^2, 2v_c^2)$
$\delta \in (-\infty, -1/v_c^2)$ | $\delta \in (1/v_c^2, +\infty)$
$z < z_*$ | $z > z_*$

Table 1: Ranges of $\delta$, $v$ and $z$ in (asymptotically) Liouville models.

The asymptotic solution near $z_*$ is
\[
\psi_1^k(z) = C_+(k)(z - z_*)^\delta [1 + b_+(k)(z - z_*)^2 + O(z - z_*)^3] + C_-(k)(z - z_*)^{1-\delta} [1 + O(z - z_*)^2] \tag{5.20}
\]
and the dual solution can be obtained by applying the $Q$ operator,
\[
\tilde{\psi}_k^2(\tilde{z}) = -\frac{(1 + 2\tilde{\delta})C_-(k)}{M_k}(\tilde{z}_* - \tilde{z})^{\tilde{\delta}} + \frac{2b_+(k)C_+(k)}{M_k}(\tilde{z}_* - \tilde{z})^{1-\tilde{\delta}} + \cdots \tag{5.21}
\]
where $\tilde{z}_* - \tilde{z} = z - z_*$. Of course, (5.21) has the same form as (5.20), but with the “dual integration constants” related as
\[
\tilde{C}_+ = -2\nu C_- / M_k \quad \tilde{C}_- = 2b_+ C_+/M_k. \tag{5.22}
\]
This simple observation makes it evident that, by fixing boundary conditions at $z_*$ in $\psi_1^k$, the SFD dual boundary condition at $\tilde{z}_*$ in $\tilde{\psi}_k^2$ are univocally determined. Thus, even when SFD holds only asymptotically, it gives information about the dual wave-functions near the dual boundary/singularity; in particular, by relating their Dirichlet to Neumann (or mixed) boundary conditions.

Going back to the original perturbation with Eq.(5.10), we find
\[
f_k(z) \approx \frac{L^{\delta + 1}}{1 - v_c^2/v^2} [C_+(k) + C_-(k) |z - z_*|^{-2\delta + 1}]. \tag{5.23}
\]
In the near-boundary limit we have $\delta < 0$. Within the standard holographic renormalization, there are two kinds of Dirichlet conditions to be imposed at the UV, corresponding to the following two independent solutions:

a) The finite solution, $f_k \sim C_+(k)$, for scalar modes, acts as a source for the operator $\mathcal{O}$ driving the RG flow. (For tensor modes, it couples to the QFT energy-momentum tensor.) We discuss this in detail in §5.4.

b) The vanishing solution $f_k \sim C_-(k) |z - z_*|^{-2\delta + 1}$ corresponds to composite bound states of the $d$-dimensional QFT, i.e. “glueballs” [15, 16, 39–41], with a mass spectrum $M_k$ given by the eigenvalues of the Schrödinger equation (5.11). We discuss them now in §5.3.

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Bound states of the $d$-dimensional QFT occur when $\psi_k$ is normalizable both in the UV and in the IR, i.e. when
\[
\int_{z_{\text{IR}}}^{z_{\text{UV}}} dz \ |\psi_k(z)|^2 < \infty,
\] (5.24)
equating that the kinetic term of the effective $d$-dimensional action for $f(z,x)$ is finite.

We are as usual interested in domain walls which are asymptotically Liouville or AdS. The first thing we prove is that the quantum-mechanical SUSY is broken for the potential (5.16). This means that neither the solution of $Q\psi_0^{(1)} = 0$ nor the solution of $Q^{\dagger}\psi_0^{(2)} = 0$ are square-integrable, which is easily verifiable, since the solutions are
\[
\psi_0^{(1)}(z) = C_{\text{UV}} \exp \left[ \frac{(d-1)}{2} A(z) \right], \quad \psi_0^{(2)}(z) = C_{\text{IR}} \exp \left[ -\frac{(d-1)}{2} A(z) \right],
\] (5.25)
and $a(z)$ is given by (3.15) and/or (3.16) in the UV and IR asymptotics. Hence $Q$ and $Q^{\dagger}$ do not change the “energy levels”, and the partner models have a completely degenerated spectrum [20]
\[
H_1\psi_n^{(1)} = M_{(1)n}^2 \psi_n^{(1)} \quad \text{and} \quad \tilde{H}_2\tilde{\psi}_n^{(2)} = \tilde{M}_{(2)n}^2 \tilde{\psi}_n^{(2)} = M_n^2
\] (5.26)
Therefore, given a domain wall with a tensor mass spectrum $\{M_n\}$, SFD yields a different domain wall which has the same spectrum.

5.3.1 Stability

The fluctuations are stable as long as $M_k^2 \geq 0$. Starting from the fact that
\[
\int_{z_{\text{IR}}}^{z_{\text{UV}}} dz \ [Q\psi_k(z)]^* [Q\psi_k(z)] \geq 0,
\]
the special factorization of the SUSY Hamiltonians implies that $\psi_k^*(z) Q \psi_k(z) |_{z_{\text{IR}}}^{z_{\text{UV}}} + M_k^2 \int_{z_{\text{IR}}}^{z_{\text{UV}}} dz |\psi_k(z)|^2 \geq 0$, hence a sufficient condition for $M_k^2 \geq 0$ is that the boundary term vanishes,
\[
\psi_k^*(z) Q \psi_k(z) |_{z_{\text{IR}}}^{z_{\text{UV}}} = 0.
\] (5.27)
Thus imposing Dirichlet or Neumann(-like) boundary conditions,
\[
\begin{align*}
\text{(Dirichlet)} \quad & \psi_k(z_{\text{IR}}) = \psi_k(z_{\text{UV}}) = 0 \\
\text{("Neumann")} \quad & Q\psi_k(z_{\text{IR}}) = Q\psi_k(z_{\text{UV}}) = 0
\end{align*}
\] (5.28, 5.29)
is sufficient to ensure stability of the fluctuations. (There are, of course, other “mixed” choices.) Note that the form of Eq. (5.27) is such that, taking into account (5.15), if DW is stable then $\tilde{DW}$ will be as well.

It is not guaranteed that (5.27) can indeed be imposed. Analyzing the asymptotic solutions of (5.11), it was shown in [17] that stability is ensured for domain walls which have: an AdS UV boundary within the BF unitarity bound and a regular (i.e. non-singular) IR, or a singular Liouville IR satisfying the condition

$$\delta \geq \frac{3}{2}, \quad \text{hence} \quad v^2 < v^2_c \leq v^2_c + \frac{2}{7}, \quad (5.30)$$

cf. Eq.(5.18). This condition ensures that the Hamiltonian with the Schrödinger potential (5.18) is self-adjoint. When (5.30) holds, we are forced to make $C_\gamma = 0$ in (5.20), otherwise the wave function is not square-integrable. Moreover, when (5.30) holds, normalizability forces the Dirichlet boundary condition $\psi(z_{IR}) = 0$, leaving only one integration constant to be fixed at $z_{UV}$. We thus have a well-defined spectrum imposed by the condition of normalizability of the wave function.

When (5.30) does not hold, i.e. when $\delta \in (0, 3/2)$, normalizability does not forcefully imply an IR Dirichlet condition. (Both terms in (5.20) are normalizable.) Then the Hamiltonian of the Schrödinger potential (5.18) is not self-adjoint, but there are techniques that allow the construction of families of self-adjoint extensions, described in terms of a continuous parameter [42–44]. It can be shown [45] that there exists only one negative eigenvalue $M^2$ as long as $\delta(\delta - 1) \geq -\frac{1}{4}$, and this mode can always be isolated and excluded, thus ensuring the stability of the spectrum.

Now, recall that for potentials with well-defined SFD duals, the parameter $v$ is bounded from above by the special Liouville value $0 < v^2 \leq 2v^2_c$ (thus for $\delta < 0$), the fluctuations are not integrable because then the singularity lies at $z \to \infty$, hence $V(z)$ is not bounded and the spectrum is a continuum. From the previous discussion, this implies that for $d = 3$ one must impose an IR boundary condition other than simple normalizability of $\psi(z)$. Now, since SFD gives a map between the UV and the IR of dual models, it can be used to translate such an extra IR condition into a UV condition of the dual model. Such a map has important implications. As already said, for $\delta \in (0, 3/2)$ there is no reason for imposing $C_\gamma = 0$ in (5.20). On the other hand, for SFD to relate both models it is necessary that the corresponding dual solution in (5.21) also be square-integrable and, if $C_\gamma \neq 0$, this only happens if $\delta = -\tilde{\delta} < 1/2$. Hence the consistency of SFD is more restrictive than the "normalizability condition", requiring that $C_\gamma = 0$ whenever $\delta$ lies not in (5.30) but in the (larger) interval $\delta > 1/2$. When $d = 3$ we have $\delta > 1$ and when $d = 2$ we have $\delta > 1/2$, therefore for bound states SFD always fixes univocally the IR boundary condition—a non-trivial result that...
completely solves possible ambiguities as the ones described in [16, 17, 40]. This point is illustrated in the example below.

5.3.2 Example: GPPZ flow in $d = 3$

Consider the GPPZ flow (4.7) in 3+1 dimensions. This is a useful example for two reasons. First, it has the peculiar property of being symmetrical under SFD. Second, since $d = 3$, it provides an example of the situation described above: the wave-functions are automatically normalizable in the IR, so one needs an extra criterion for fixing the IR boundary conditions. We will show how the invariance under SFD can be used to fix the IR behavior of the wave-function in terms of its UV properties.

The background geometry, given by Eq.(4.8), shows an AdS boundary at $z_{\text{UV}} = -\frac{\pi}{2}\ell$ and a Liouville singularity at $z_{\text{IR}} = 0$. Given $a(z)$, we find the SUSY QM superpotential (5.16), and the potential (5.12) for tensor modes:

$$W_T(z) = \frac{2}{\ell} \sin(-2z/\ell); \quad V_T(z) = \frac{2}{\ell^2} \cos^2(-z/\ell).$$ (5.31)

The Schrödinger equation (5.11) has the exact solution

$$\psi(z) = C_+ \left[ \sqrt{M^2} \sin(-\sqrt{M^2}z) + \frac{1}{\ell} \tan(z/\ell) \cos(\sqrt{M^2}z) \right] + C_- \left[ \sqrt{M^2} \cos(-\sqrt{M^2}z) - \frac{1}{\ell} \tan(z/\ell) \sin(\sqrt{M^2}z) \right]$$ (5.32)

where $C_\pm$ are arbitrary constants to be fixed by the boundary conditions. As it should be expected from our previous discussion, $\psi(z)$ is regular at the IR, with $\psi(z_{\text{IR}}) = C_- \sqrt{M^2}$. Thus normalizability at the singularity does not require neither $C_+$ nor $C_-$ to be zero.

We then turn to the UV limit, where we must impose Dirichlet conditions such that $\psi(z_{\text{UV}}) = 0$ (we are after bound states). This fixes the mass spectrum to be $M_k^2 = k^2/\ell^2$, with $k = 2, 3, 4, 5, \cdots$. The eigenfunctions split into two classes, depending on whether $k$ is even or odd. For odd $k = 2n + 3$ we must take $C_- = 0$ and the normalized functions are

$$\psi_n^+(z) = \frac{1}{\sqrt{\pi \ell (n+1)(n+2)}} \left[ (2n+3) \sin \left( \frac{2n+3}{\ell} z \right) - \cos \left( \frac{2n+3}{\ell} z \right) \tan \left( \frac{1}{\ell} z \right) \right]$$

$$M_n^2 = \frac{(2n+3)^2}{\ell^2}; \quad n = 0, 1, 2, 3, \cdots$$ (5.33)

For even $k = 2n$, we have to take $C_+ = 0$ instead, and the normalized solutions are

$$\psi_n^-(z) = \frac{2}{\sqrt{\pi \ell (4n^2-1)}} \left[ 2n \cos \left( \frac{2n}{\ell} z \right) + \tan \left( \frac{1}{\ell} z \right) \sin \left( \frac{2n}{\ell} z \right) \right], \quad M_n^2 = \frac{4n^2}{\ell^2}$$ (5.34)

\footnote{We describe tensor modes, but a similar discussion goes for the scalar ones. Note that the exact invariance under SFD extends to the scalar modes even if this is not a Liouville model.}
with \( n = 1, 2, 3 \cdots \). Thus we have two sets of eigenfunctions which are regular in the singularity and give discrete spectra in the UV. The functions \( \psi^+ \) vanish at \( z_{IR} = 0 \), which is usually the boundary condition imposed by regularity at the singularity. The functions \( \psi^- \) do not vanish at \( z_{IR} \) but they are finite and hence square-integrable, so they cannot be discarded.

Now we use SFD. The SUSY QM superpotential and potential are

\[
\tilde{W}_T(\tilde{z}) = \frac{2/\ell^2}{\sin(2\tilde{z}/\ell)}, \quad \tilde{V}_T(\tilde{z}) = \frac{2/\ell^2}{\sin^2(\tilde{z}/\ell)}
\]

with \( 0 < \tilde{z} < \frac{\pi}{2} \ell \), and \( \tilde{z}_{IR} = \frac{\pi}{2} \ell \) and \( \tilde{z}_{UV} = 0 \). Note that the SUSY partners (5.31) and (5.35) are actually identical, there is only a translation by \( \frac{\pi}{2} \ell \). This is a consequence of the symmetry under SFD. Using (5.15), i.e. applying the operator \( M^{-1}_nQ \) on \( \psi^+_n(z) \), we find

\[
\tilde{\psi}^+_n(\tilde{z}) = \frac{\ell}{2n+3} \left[ \frac{2/\ell}{\sin(-2\tilde{z}/\ell)} \right] \psi^+_n(z) \bigg|_{z=-\tilde{z}}
\]

\[
= \frac{1}{\sqrt{\pi\ell(n+1)(n+2)}} \left[ (2n+3) \cos\left(\frac{2n+3}{\ell}\tilde{z}\right) - \sin\left(\frac{2n+3}{\ell}\tilde{z}\right) \cot\left(\frac{\tilde{z}}{\ell}\right) \right]
\]

with \( \tilde{M}^2_n = (2n+3)^2/\ell^2 \) and \( n = 0, 1, 2, 3, \cdots \). These are the same as the original functions (5.33), which is also consistent with the self-duality of the model. But for the even functions (5.34), the dual solutions are

\[
\tilde{\psi}^-_n(\tilde{z}) = \frac{\ell}{2n} \left[ \frac{2/\ell}{\sin(-2\tilde{z}/\ell)} \right] \psi^-_n(z) \bigg|_{z=-\tilde{z}}
\]

\[
= \frac{2}{\sqrt{\pi\ell(4n^2-1)}} \left[ \cos\left(\frac{2n}{\ell}\tilde{z}\right) \cot\left(\frac{1}{\ell}\tilde{z}\right) + 2n \sin\left(\frac{2n}{\ell}\tilde{z}\right) \right].
\]

which are not the same as (5.34), an inconsistency with self-duality. The point, however, is that these solutions should be discarded. Indeed, they diverge at \( \tilde{z}_{UV} \), so they do not represent bound states. But to force \( \tilde{\psi}^- = 0 \), we must take \( \psi^- = 0 \), and hence choose \( C_- = 0 \) in (5.32), leaving only the functions (5.33) as eigenfunctions. Thus in the end, consistency with the symmetry of the model under SFD has discarded half of the spectrum allowed by normalizability, by ultimately fixing a specific IR Dirichlet condition.

5.4 Renormalized actions and dual VEVs

In §5.3 we have described the effect of SFD on the bound-states of the QFT. Now we discuss the fluctuations that act as sources in the holographic QFT. The usual procedure is to renormalize the on-shell action for these solutions, which can then be differentiated with respect to the Dirichlet boundary conditions to yield correlation

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functions. This amounts to leaving $C_+(k)$ arbitrary in Eq. (5.23). The image of this solution under SFD, given by Eq. (5.22), is irregular: a divergent fluctuation near a singularity. Therefore SFD is not a well-defined map of holographic sources. This is actually not so surprising: there should be no sources in the singularity.

It is nevertheless interesting to see how SFD fits with holographic RG descriptions that are valid not only in the UV but at arbitrary scales in the flow. Such a description was developed in [21]. The authors construct a renormalized action for Liouville models with “bare” potential $V(\phi) = L^{-2} e^{v \kappa \phi}$,

$$S^{(\text{ren})}[\phi, \gamma_{ab}] = \frac{1}{\kappa^2 L} \int d^d x \sqrt{-\gamma} \left[ D_0 \exp \left( \frac{d}{(d-1)v} \kappa \phi \right) + L^2 D_1 \exp \left( \frac{(d-2)}{v} \kappa \phi \right) \right]$$

where $D_0$ and $D_1$ are two arbitrary scheme-dependent constants. This action is the generating functional of connected correlators, obtained from differentiation with respect to the sources $\phi$ and $\gamma_{ab}$. Most importantly, it is valid at any energy scale along the flow. The bulk metric is a generalization of (1.1) for a not necessarily homogeneous $\gamma_{ab}$. In the homogeneous case, $\gamma_{ab} = e^{A(r)} \eta_{ab}$, and $e^A$ gives the energy scale of the QFT as usual. But in general one can separate from $\gamma_{ab}(x, r)$ a part that changes along $r$ with a local Weyl transformation, which defines the beta-function.

SFD, as it relates different Liouville models with parameters $v$ and $\tilde{v}$ according to Eq. (3.20), should work on the $S^{(\text{ren})}$ in the same way. This is indeed the case: the transformation of the fields under SFD, given by Eq. (3.21), is precisely the correct one to make the $\phi$ dependence of $S^{(\text{ren})}$ invariant, only changing $v \leftrightarrow \tilde{v}$; note that only the combination $\phi/v$ appears in $S^{(\text{ren})}$. Eq. (5.36) is written for general inhomogeneous slices, for which scale-factor inversion is not actually defined. But given that SFD is also a symmetry of linear fluctuations (these are pure Liouville models), we can expect it to preserve the form of the metric dependence of $S^{(\text{ren})}$. Scale-factor inversion simply changes the surface $\Sigma_r$ where $S^{(\text{ren})}$ is evaluated, i.e. it relates generators at different energy scales.

It is instructive to check this on homogeneous backgrounds, as follows. The renormalized action can be written as a function of the energy scale $E$ and of the coupling $\phi$ as [21]

$$S^{(\text{ren})}[E, \phi(E)] = C_R \int d^d x \exp \left[ d \left[ A(E) - \mathcal{A}(\phi(E)) \right] \right],$$

$$\mathcal{A}(\phi) \equiv -\frac{1}{2(d-1)} \int_{\phi_0}^{\phi} d\varphi \frac{W(\varphi)}{\partial_{\varphi} W(\varphi)}$$

where $\phi_0$ is an initial condition and $C_R$ a scheme-dependent constant. It is immediate to check, using Eq. (2.4), that $S^{(\text{ren})}[E, \phi(E)]$ is scale-invariant, i.e. that $dS^{(\text{ren})}/d\log E = 0$. This as an indication of the fact that $S^{(\text{ren})}$ is valid not only
in the UV but all along the flow. The renormalized action can also be written as a function of the energy scale $E$ and of the coupling $\phi$ independently, as

$$S^{(\text{ren})}[E,\phi] = C_R \int d^d x \exp \left[ d (A(E) - A(\phi)) \right]$$  \hspace{1cm} (5.39)

where it should be understood that $\phi$ varies while keeping $E$ fixed, so we can find the renormalized VEV of the operator $\mathcal{O}$ sourced by $\phi$, at energy $E$,

$$\langle \mathcal{O} \rangle_{\phi,E} = \frac{\delta S^{(\text{ren})}}{\delta \phi} = \frac{dC_R}{2(d-1)} \frac{W(\phi)}{\partial_\phi W(\phi)} e^{-dA(\phi)} e^{dA(E)}.$$

This allows one to find the quantum effective action by taking a Legendre transform of $S^{(\text{ren})}$, but here we are interested, instead, in finding the effect of SFD on the VEV. In Liouville models, the expression simplifies greatly, and we are left with

$$\langle \mathcal{O} \rangle_{\phi,E} = \frac{dC_R}{(d-1)v} \exp \left[ \frac{d(\phi - \phi_0)}{(d-1)v} \right] e^{dA(E)} \hspace{1cm} (5.41)$$

It should be noted that the dependency of the energy-scale with the scale factor is the usual, viz. $e^{A(E)} = E$ (modulo an arbitrary multiplicative constant). But writing it as a function of $A$ allows us to see clearly, now, the effect of SFD: taking into account the field transformation (3.21) and scale factor inversion,

$$\frac{dC_R}{(d-1)v} \exp \left[ \frac{d(\phi - \phi_0)}{(d-1)v} \right] e^{dA(E)} = \frac{dC_R}{(d-1)v} \exp \left[ -\frac{d(\phi - \phi_0)}{(d-1)v} \right] e^{-dA(E)}$$

This can be written as

$$\langle \hat{\mathcal{O}} \rangle_{\hat{\phi},\hat{E}} = \frac{1}{\sqrt{2q-1}} \langle \mathcal{O} \rangle_{-\hat{\phi},\frac{1}{\hat{E}}} \hspace{1cm} (5.42)$$

where $q \equiv (v_c/v)^2$. We see that dual VEVs depend on the fields in a strong-/weak-coupling relation, and are evaluated at reciprocal energy scales.

6. Conclusion and a summary of results

The results obtained in the present paper are based on the observation that the static domain wall equations for a variety of (super)gravity models possess a discrete $Z_2$ symmetry under inversion of the scale factor, which can be consistently implemented within the framework of gauge/gravity correspondence. The SFD transformations (3.1) provide a map between boundaries and singularities of the dual domain walls, and between UV and IR regimes of the holographic RG flows in the corresponding pair of dual QFTs. We have also shown that SFD appears to be an asymptotic (and exact in some cases) symmetry of the linear perturbations around these DWs backgrounds, relating the fluctuation modes of a given DW with the “conjugate momenta” of the modes around the dual DW. Let us summarize our main results, their holographic implications and eventual further applications in different areas of the holographic research.
SFD as a solution generating technique. In principle, SFD provides a map from any domain wall solution of a theory with a scalar potential $V(\phi)$ to a dual domain wall solution of a theory with a different scalar potential $\tilde{V}(\tilde{\phi})$. In practice, the transformation $\phi \mapsto \tilde{\phi}$, $V \mapsto \tilde{V}$ can be very complicated and difficult (or impossible) to be written down explicitly. Also, SFD may map a stable theory with $(d\phi/dz)^2 > 0$ to an unstable one with $(d\tilde{\phi}/d\tilde{\eta})^2 < 0$, i.e. it may violate the NEC. To avoid this, one must restrict oneself to solutions which satisfy an upper bound on the holographic beta-function, Eq.(3.4).

Liouville potentials, AdS and RG flows. There is a special class of scalar potentials for which SFD is particularly simple: exponential ones, $V(\phi) = V_0 e^{v\phi}$. In fact, SFD preserves the exponential form, while changing the exponent’s parameter $v$ by a reflection of $v^2$ about a critical value $v^2_c = 2/(d-1)$. In this class of models, the bound (3.4) corresponds to an upper bound on the possible values of $v^2$, which is saturated when $v^2 = 2v^2_c$. The corresponding ‘special Liouville model’, with $v = 2/\sqrt{d-1}$, is dual, by SFD, to AdS space. The dual of an asymptotically AdS domain wall with the usual expansion of $V(\phi)$ around the vacuum is a theory with the potential $\tilde{V}(\tilde{\phi})$ which is special-Liouville at leading order, with perturbative exponential corrections given by Eq.(3.35).

SFD thus maps flows with AdS fixed points to asymptotically Liouville theories. In general, SFD can be seen as a map between the RG flows of a pair of QFTs, the relation between the beta-functions being given by Eq.(4.1); the map is between UV and IR regimes, as a consequence of scale-factor inversion. We have given some examples in Sect.4 by describing the images under SFD of known flows, some of which have applications in holographic QCD. In particular, we have shown that the GPPZ flow is self-dual, i.e. SFD is an exact symmetry of this model. This implies a remarkable relation between its IR and UV regimes.

Extension for fluctuations. SFD can be extended to fluctuations around isotropic domain walls as a “canonical transformation” swapping a fluctuation mode and its conjugate momentum. The extension is always valid for tensor modes, while for scalar modes it holds in Liouville models. In §5.1 we interpret this extension in terms of the S-duality found in cosmological spacetimes by Brustein, Gasperini and Veneziano [5]. In §5.2 we show that it can also be interpreted as a “SUSY QM” of the effective Schrödinger equation.

By itself, the existence of a general duality for tensor modes implies that SFD is a “solution generating technique” for tensor fluctuations in a similar way as it is for background solutions. We illustrate this fact further in App.B by using SFD to give the exact solution of fluctuations around a complicated domain wall.

We then consider asymptotically Liouville and/or AdS backgrounds. We show that in these DWs the quantum-mechanical SUSY is broken, which combined with SFD implies that the tensor spectra of dual models is identical (and the scalar spectrum of dual Liouville models is identical as well), and has no massless mode.
Also, scalar modes are asymptotically related by SFD, as well as the tensor modes (which are always). The asymptotic dual solutions (5.20) present a relation between the boundary conditions at the boundary/singularity, given by Eq.(5.22). These domain walls are stable, and the specific relation between boundary conditions introduced by SUSY QM/SFD preserves the condition for stability, Eq.(5.27).

**Mass spectra.** The KK modes whose wave-functions vanish at the UV boundary correspond to composite particles in the $d$-dimensional QFT [39]. The derivation of their mass spectrum is of interest to phenomenological QCD, and domain walls with Liouville asymptotic impart their own characteristics on the spectrum. For example, the absence of a zero-mode discussed above corresponds to a mass gap. The exact form of the mass spectrum crucially depends on the boundary conditions to be imposed on fluctuations at the IR. Assigning these conditions can be somewhat arbitrary, clouding the physical interpretation, and singular domain walls have been regarded as a natural way to obtain an IR cutoff without the need of introducing by hand a “hard-wall” (see e.g. [41]). Indeed, in the most fortunate cases, the normalizability condition (5.24) can, by itself, provide a natural Dirichlet boundary condition at the IR. For Liouville singularities, this is the case when the parameters obey the bound (5.30). Then the natural IR condition is $\psi(z_{IR}) = 0$, or $C_- = 0$ in Eq.(5.20), because this is the only way $\psi(z)$ to be made square-integrable. We have shown that for $d = 4$ the so-called ‘computability bound’ [17] is identical with the restriction (3.25) that selects the Liouville models that possess good (NEC-preserving) dual pairs under SFD. For $d \geq 5$, the computability bound is less restrictive than (3.25) and all NEC-preserving models are “(spectrally) computable”.

When the ‘computability bound’ is not satisfied, one cannot call upon normalizability to give a natural IR boundary condition. One of the applications of SFD is to provide an appropriate (additional) boundary condition in some of these problematic cases. By its very definition SFD transforms the IR boundary condition into a UV one for the dual domain wall: according to (5.22), the condition for a bound state, $\tilde{\psi}(\tilde{z}_{UV}) = 0$, implies $\psi(z_{IR}) = 0$. This mechanism is only justifiable in some of the models however. Indeed, for $d \geq 4$, if we have a domain wall for which the computability bound is violated, applying SFD will lead us to a domain wall with the wrong sign of the kinetic energy. Nevertheless, in §5.3.2, we give an example of how SFD can be successfully used to fix the IR boundary conditions in the GPPZ flow in $d = 3$. The GPPZ flow is self-dual hence, although the special-Liouville singularity violates the computability bound in $d = 3$, imposing SFD-consistent boundary conditions at the IR implies that half of the mass spectrum has to be discarded.

**Open problems.** We conclude with a few comments concerning the origin of the proposed scale factor duality for DW geometries.

As stated in the Introduction, the SFD developed here is somewhat similar to
Veneziano’s extension of the abelian T-duality for cosmological spacetimes [2]. But here the duality is adapted for an open space-like “radial” direction (in conformal coordinates). Moreover, the two SFDs are not equivalent even when both are applied to cosmological models (see [7] for a comparison of their properties).

Our motivation to introduce SFD, in the form we have derived it in Sect.3, has been rather ad hoc, based on the fact that there exists a specific discrete transformation that preserve the form of the domain-wall equations under scale factor inversion. Yet, there are some hints about an eventual stringy nature of SFD based on the prominence of (asymptotically) Liouville potentials. The Liouville potentials in Einstein frame can be obtained from a linear-dilaton theory in the string frame. Exponential potentials are also typical in the effective action for Dp-branes in the near-horizon limit [46], and can be obtained from a higher dimensional AdS solution via a dimensional reduction [30, 47, 48]. In fact, as explained in [30], we can obtain $V(\phi) = V_0 e^{v_2 \phi}$ in two different ways. If $v^2 < v_c^2$, we start with with pure AdS gravity in $2\sigma + 1$ dimensions, make the compactification of an internal $p$-dimensional torus with $p = 2\sigma - d$ and get the $(d+1)$-dimensional Einstein-scalar theory with a Liouville potential proportional to $\Lambda (2\sigma + 1)$. Instead, if $v^2 > v_c^2$, we start with pure Einstein gravity (with $\Lambda = 0$) in $2\sigma + 1$ dimensions, compactify an internal $p$-dimensional sphere with curvature $R(p)$ and get $(d+1)$-dimensional Einstein-scalar theory with a Liouville potential proportional to $R(p)$. The curious thing is that SFD, since it reflects $v^2$ around the critical value $v_c^2$, swaps these two different constructions, but in a rather implicit way: dual parameters $v$ and $\tilde{v}$ must correspond not only to different constructions, but also to parent spaces with different dimensions, such that $\tilde{\sigma} - \sigma = \frac{1}{2}(d - 1)$. Similar phenomena take place in open string T-dualities as well, hinting at an intrinsic relation between SFD and the superstrings dualities.

A. Causal structure of SFD dual models

The causal structure of dual domain walls is related by SFD in a straightforward manner. Since it is defined in conformal coordinates, SFD preserves the causal nature of a surface. Thus if we generically call $\{a = 0\}$ a “singularity” and $\{a = \infty\}$ a “boundary”, it is not hard to see that SFD maps

- null boundaries $\leftrightarrow$ null singularities
- timelike boundaries $\leftrightarrow$ timelike singularities

This is shown in Fig.6.

The best examples are the Liouville models. A Liouville model with $v > v_c$ has the singularity at a finite distance which we can set to $z_0 = 0$, so it is timelike, while the boundary lies at $z = -\infty$ so it is null. The Penrose diagram is in Fig.6(a). On the other hand, the dual Liouville model, with $v < v_c$, will have null singularity and a time-like boundary as shown in Fig.6(b). The critical model with $v = v_c$ has
boundary at $z = \infty$ and singularity at $z = -\infty$, so both are null and the diagram is that of Fig.6(c).

The Penrose diagram of a self-dual model must be shape-invariant under SFD. This is illustrated by the Liouville model with $v = v_c$. Meanwhile, a self-dual model with an AdS boundary (e.g. the GPPZ geometry) has a diagram like in Fig.6(d) (which is also invariant because boundary and singularity are both timelike).

On the other hand, for a model which is not self-dual, the shape of the diagram is not preserved. Fig.6(d) is also the diagram of, say, a domain wall whose potential has two different Liouville asymptotics: with $v > v_c$ as $a \to 0$ and $v < v_c$ as $a \to \infty$. If these limits are reversed, we have a Fig.6(c) diagram, etc.

Note that a discrete spectrum occurs in models where the interval $[z_{IR}, z_{UV}]$ is finite (then the Schrödinger problem becomes a finite box). The casual structure of such domain walls is that of Fig.6(d) and, as we have shown, this structure is preserved by SFD. Continuous spectra, on the other hand, happen when there is an infinite, or semi-infinite range of the conformal coordinate $z$, such as in in diagrams (a), (b) and (c). Again, this structure is preserved by SFD.

B. RG flow from AdS to Critical Liouville, and its dual

Here we present another example of background for which the tensor fluctuations can be solved exactly. The interesting thing about this model is that the dual background under SFD is quite complicated, but the duality gives us the corresponding fluctuations nevertheless.

We start with a model given by

$$\begin{align*}
W(\phi) &= \frac{d-1}{2\tau} \cosh \left( \frac{\kappa \phi}{\sqrt{2(d-1)}} \right), \\
\beta(\phi) &= \sqrt{\frac{2(d-1)}{\tau}} \tanh \left( \frac{\kappa \phi}{\sqrt{2(d-1)}} \right).
\end{align*}$$

(B.1)

There is a UV AdS boundary with radius $\ell$, and with $s = 1$, while for $\kappa \phi \gg 1$ we have $W \sim e^{\kappa \phi}/\sqrt{2(d-1)}$, thus a critical Liouville singularity. The beta-function coincides with (4.9) near the UV fixed point. (Actually, the two functions coincide
after a rescaling of \( \phi \). This illustrates how the same flux may correspond to very different bulk geometries. Solving the domain wall profile, we find

\[
a(z) = \frac{1}{\sinh(z/\ell)}, \quad \propto \phi(z) = \sqrt{2(d-1)} \frac{z}{\ell}, \quad 0 < z < \infty. \tag{B.2}
\]

The AdS boundary is at \( z_{\text{UV}} = 0 \), and near the critical Liouville singularity at \( z_{\text{IR}} = \infty \) we have \( a(z) \sim e^{-z/\ell} \); cf. Eq. (3.16).

From (B.2) we find the SUSY QM potential and superpotential of the tensor fluctuations to be

\[
\mathcal{W}_T(z) = \frac{d-1}{2\ell} \coth(z/\ell), \quad \mathcal{V}_T(z) = \frac{(d-1)^2}{4\ell^2} + \frac{d^2-1}{4\ell^2} \frac{1}{\sinh^2(z/\ell)}. \tag{B.3}
\]

The UV-square-integrable solution of the Schrödinger equation, satisfying the Dirichlet boundary condition \( \psi(z_{\text{UV}}) = 0 \) at \( z_{\text{UV}} = 0 \), is

\[
\psi(z) = C \left[ \sinh \left( \frac{z}{\ell} \right) \right] \frac{d+1}{2} \Gamma \left( \frac{d+1}{4} - \frac{ig\ell}{2}, \frac{d+1}{4} + \frac{ig\ell}{2}; \frac{d+2}{2}; -\sinh^2 \left( \frac{z}{\ell} \right) \right),
\]

\[
g = \sqrt{M^2 - \frac{(d-1)^2}{4\ell^2}}. \tag{B.4}
\]

Near the boundary, this behaves as \( \psi(z) \approx C \left( \frac{z}{\ell} \right)^{(d+1)/2} \). Near the singularity,

\[
\psi(z) \approx C|\lambda_g| \cos \left[ g(z - \ell \log 2) + \frac{1}{2} \delta_g \right], \quad |\lambda_g| = \frac{2\Gamma \left( ig\ell \right) \Gamma \left( \frac{d+2}{2} \right) e^{-i\delta_g/2}}{\Gamma \left( \frac{d+1}{4} + \frac{ig\ell}{2} \right) \Gamma \left( \frac{d+3}{4} - \frac{ig\ell}{2} \right)}, \quad \delta_g \text{ is the phase-shift of the wave-function.} \tag{B.5}
\]

Now, with SFD we are going to construct a new domain wall, with a special-Liouville singularity (dual to the AdS boundary) and an asymptotic critical-Liouville boundary (dual to the critical Liouville singularity). Using \( c = 1 \) and \( \tilde{z} = -z \), we have the dual scale-factor

\[
\tilde{a}(\tilde{z}) = \sinh (-\tilde{z}/\ell), \quad -\infty < \tilde{z} < 0. \tag{B.6}
\]

The dual scalar field can be expressed implicitly as

\[
\propto \tilde{\phi} = \sqrt{2(d-1)} \log \left[ \sqrt{\left( \frac{2 + \sqrt{2}}{2 - \sqrt{2}} \right) \left( \frac{\gamma - 1}{\gamma + 1} \right) \left( \frac{2\gamma - \sqrt{2\gamma^2 - 1} + 1}{2\gamma + \sqrt{2\gamma^2 - 1} - 1} \right) \times \left( \frac{\gamma + \sqrt{\gamma^2 - 1/2}}{\sqrt{2}} \right)^{\sqrt{2}}} \right],
\]

\[
\gamma \equiv \coth \left( \frac{\frac{\propto \tilde{\phi}}{\sqrt{2(d-1)}}}{\sqrt{2}} \right) = \coth (-\tilde{z}/\ell). \tag{B.7}
\]
Although it is not possible to invert this function exactly, it is easy to verify that the asymptotics give the correct SFD results.\textsuperscript{12} The dual superpotential (and every bulk quantity) can be written as a function of $\tilde{\phi}$ only implicitly, through $\gamma$, viz.

$$\tilde{W} = \frac{(d - 1)}{\kappa \ell} \gamma \sqrt{\gamma^2 - 1}, \quad (B.8)$$

but with this we can plot the function $\tilde{W}(\tilde{\phi})$; we do so in Fig.B.

Here the duality between fluctuations is very useful. Although the background is a very complicated implicit function of the field $\tilde{\phi}$, the tensor fluctuations can easily be found explicitly. The SUSY QM superpotential and the Schrödinger potential, as always, are found from (B.6),

$$\tilde{W}_T(\tilde{z}) = \frac{(d - 1)}{2\ell} \coth (-\tilde{z}/\ell), \quad \tilde{V}_T(\tilde{z}) = \frac{(d - 1)^2}{4\ell^2} + \frac{(d - 1)(d - 3)}{4\ell^2 \sinh^2 (-\tilde{z}/\ell)}. \quad (B.9)$$

The Schrödinger equation can be solved as usual, but it is easier to use (5.15)

$$\tilde{\psi}(\tilde{z}) = \tilde{C} \left[ \sinh \left( \frac{\tilde{z}}{\ell} \right) \right]_{\tilde{z} = -\tilde{z}} = \frac{1}{\sqrt{M^2}} \left[ \partial_\tilde{z} + \frac{(d - 1)}{2\ell} \coth \left( \frac{\tilde{z}}{\ell} \right) \right] \psi(\tilde{z}) \bigg|_{\tilde{z} = -\tilde{z}}. \quad (B.10)$$

Using some properties of the hypergeometric function we can write the result

$$\tilde{\psi}(\tilde{z}) = \tilde{C} \left[ \sinh \left( \frac{|\tilde{z}|}{\ell} \right) \right]_{\tilde{z} = -\tilde{z}} = 2F_1 \left( \frac{d-1}{4} - \frac{i\gamma}{2}; \frac{d-1}{4} + \frac{i\gamma}{2}; \frac{d}{2}; -\sinh^2 \left( \frac{|\tilde{z}|}{\ell} \right) \right),$$

$$g = \sqrt{M^2 - \frac{(d-1)^2}{4\ell^2}}, \quad \tilde{C} = \frac{d}{\sqrt{M^2 \ell^2} C}.$$}

\textsuperscript{12}Near the the boundary, $|\tilde{z}|/\ell \gg 1$ ($\gamma \rightarrow 1$), we have $\tilde{\phi} \approx -\phi$, hence $\tilde{W}(\tilde{\phi}) \sim e^{\frac{\kappa \tilde{\phi}}{\sqrt{d-1}}} \ll$. This is indeed the critical Liouville behavior. Near the singularity, $\tilde{z} \rightarrow 0$ ($\gamma \rightarrow \infty$), Eq.(B.7) gives $\kappa \tilde{\phi}/\sqrt{2(d-1)} \approx e^{-\frac{\kappa \tilde{\phi}}{\sqrt{d-1}}} \ll 1$, hence $\tilde{W}(\tilde{\phi}) \sim e^{\kappa \tilde{\phi}/\sqrt{d-1}} \gg 1$, which is a special Liouville singularity.
Near the singularity ($\tilde{z} \to 0$), we have $\tilde{\psi}(\tilde{z}) \approx \tilde{C} \left( |\tilde{z}| / \ell \right)^{\frac{d-1}{2}}$, while at the boundary ($|\tilde{z}| \to \infty$)

$$
\tilde{\psi}(\tilde{z}) \approx \tilde{C} |\tilde{\lambda}| \cos \left[ g \left( |\tilde{z}| - \ell \log 2 \right) + \frac{\delta g}{2} \right], \quad |\tilde{\lambda}| = \frac{2 \Gamma (ig\ell)^{(\frac{d}{2})} e^{-\frac{ig\ell}{2}}}{\Gamma \left( \frac{d-1}{4} + \frac{ig\ell}{2} \right) \Gamma \left( \frac{d+1}{4} + \frac{ig\ell}{2} \right)}.
$$

(B.11)

Comparison with (B.5) gives the relation between the dual phase-shifts

$$
\delta_g = \tilde{\delta}_g - \arctan \left( \frac{2g\ell}{d-1} \right).
$$

(B.12)

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