BAUM-BOTT RESIDUE OF FLAGS OF HOLOMORPHIC DISTRIBUTIONS

ANTONIO M. FERREIRA AND FERNANDO LOURENÇO

Abstract. In this work we extend the residue theory from flag of holomorphic foliations to flag of holomorphic distributions and we provide an effective way to calculate this class in certain cases. As a consequence, we show that if we consider a flag $\mathcal{F} = (F_1, F_2)$ of holomorphic distributions on $\mathbb{P}^3$, we get a relation between the degrees of the distributions in the flag, the tangency order of distributions, the Euler characteristic and the degree of the curve $C$.

1. Introduction

The residue theory of holomorphic foliations was developed by several authors, see for example Baum and Bott [3, 4] and Suwa [18, 19]. Let $\mathcal{F}$ be a holomorphic foliation of dimension $k$ on a complex manifold $M$ and let $\varphi$ be a holomorphic symmetric polynomial of degree $d$ satisfying the vanishing condition $n - k < d \leq n$ and let $Z \subset S(\mathcal{F})$ be a compact connect component of singular set of $\mathcal{F}$, there exists a homology class $\text{Res}_\varphi(\mathcal{F}, Z) \in H_{2(n-d)}(Z; \mathbb{C})$ such that if $M$ is compact

$$\varphi(N_\mathcal{F}) \sim [M] = \sum_Z \text{Res}_\varphi(\mathcal{F}; Z),$$

where $N_\mathcal{F}$ denotes the normal sheaf of the foliation $\mathcal{F}$.

In general, there is no way to compute the residue class $\text{Res}_\varphi(\mathcal{F}; Z)$ and this is an open problem in foliation theory. Several authors have worked on this topic and have obtained interesting results, see [3, 6, 12, 20].

For instance in the case $k = 1$ and $S(\mathcal{F})$ consists only of isolated singularities we have an expression of the above residue, see [3, Theorem 1]:

$$\text{Res}_\varphi(\mathcal{F}; p) = \text{Res}_p \left[ \begin{array}{c} \varphi(Jv) \\ v_1, \ldots, v_n \end{array} \right],$$

where $v = (v_1, \ldots, v_n)$ is a germ of holomorphic vector field at $p$, local representative of $\mathcal{F}$, $Jv$ is its Jacobian matrix and $\text{Res}_p \left[ \begin{array}{c} \varphi(Jv) \\ v_1, \ldots, v_n \end{array} \right]$ is the Grothendieck residue.

Let $\mathcal{F}$ be a dimension $k$ holomorphic foliation on a compact complex manifold $M$ such that $\text{dim} S(\mathcal{F}) \leq k - 1$, and let $Z$ be an irreducible compact component of $S(\mathcal{F})$ of dimension $k - 1$. In this case, Baum and Bott in [3, Theorem 3, p. 285], and Corrêa and Lourenço in [10] give an effective way to compute the residue class $\text{Res}_\varphi(\mathcal{F}; Z)$, where $\varphi$ is a symmetric homogeneous polynomial of degree $n - k + 1$. To do this, take a generic point $p \in Z$ such that $p$ is a point where $Z$ is smooth and disjoint from the other singular components. Now, consider $D_p$ a ball centered at $p$, of dimension $n - k + 1$ sufficiently small and transversal to $Z$ in $p$. Thus
\[
\text{Res}_\varphi(F; Z) = \text{Res}_p \left[ \frac{\varphi(Jv|D_p)}{v_1, \ldots, v_{n-k+1}} \right] [Z]
\]

where \( \text{Res}_p \left[ \frac{\varphi(Jv|D_p)}{v_1, \ldots, v_{n-k+1}} \right] \) represents the Grothendieck residue of the foliation \( F \) restricted to \( D_p \). For other progress in residue theory we refer to \[6, 12\].

We define a 2-flag of holomorphic foliations by a sequence of 2 foliations \( F \) such that the leaves of the foliation \( F \) is contained in \( F_2 \) ones. We also define the singular set of \( F \), by union of the singular sets of the foliations, i.e., \( S(F) := S(F_1) \cup S(F_2) \). For an overview about flag theory we refer to \[3\].

There exist many works in flags theory. Feigin studied characteristic classes of flags in 1975, see \[13\], where the author investigates an obstruction for existence of flags integrably homotopic. Mol in \[16\] studied the behavior of singularities of flags and its polar varieties. In the same sense, Corrêa and Soares studied the Poincaré problem for flags in \[8\].

More recently in \[5\], Theorem 2] Brasselet, Corrêa and Lourenço studied residues of flags and they proved a residue theorem of Baum-Bott type for flags.

Although there is a residue theory for flag, it is not simple to calculate the residue of flag in general. The goal of this paper is to show a partial answer of this problem. We start the paper with an extension of the residue theorem of flag to distribution.

**Theorem 1.1.** Let \( F = (F_1, F_2) \) be a 2-flag of singular holomorphic distributions of codimension \( s_1, s_2 \) respectively on a compact complex manifold \( M \) of dimension \( n \). Let \( \varphi_1 = c_{p_1} \cdots c_{p_k} \) and \( \varphi_2 = c_{t_1} \cdots c_{t_q} \) be Chern monomials, of degrees \( d_1, d_2 \) respectively, such that \( p_i > s_1 - s_2 \) for some \( i \) or \( t_j > s_2 \) for some \( j \). Then for each compact connected component \( Z \) of \( S(F) \) there exists a class \( \text{Res}_{\varphi_1, \varphi_2}(F, N_F; Z) \in H_{2n-2(d_1+d_2)}(Z; \mathbb{C}) \) such that

\[
\sum_{\chi} (\iota_\chi)_* \text{Res}_{\varphi_1, \varphi_2}(F, N_F; Z_\chi) = \left( \varphi_1(N_{12}) \varphi_2(N_2) \right) \cap [M] \quad \text{in} \quad H_{2n-2(d_1+d_2)}(M; \mathbb{C})
\]

where \( \iota_\chi \) denotes the embedding of \( Z_\chi \) on \( M \).

We prove a result about residue of flag of distributions on isolated singularities.

**Theorem 1.2.** Let \( F = (F_1, F_2) \) be a 2-flag of holomorphic distributions on a compact complex manifold \( M \) of dimension \( n \), \( \varphi_1 \) and \( \varphi_2 \) be homogeneous symmetric polynomials, respectively of degrees \( d_1 > 0 \) and \( d_2 > 0 \) and \( p \) be an isolated point of \( S(F) \). Then

\[
\text{Res}_{\varphi_1, \varphi_2}(F, N_F; p) = 0.
\]

With this tool on the hand we consider a 2-flag on \( \mathbb{P}^3 \) and we prove the following.

**Theorem 1.3.** Let \( F = (F_1, F_2) \) be a 2-flag of holomorphic foliations on \( \mathbb{P}^3 \) with \( \deg(F_1) = d_1 \) thus

\[
(1 + d_1 - d_2) \sum_{Z \in S(F_2)} \deg(Z) \text{Res}_{\varphi_2}(F_2|p; Z) = \sum_{Z \in S(F)} \text{Res}_{\varphi_1 \varphi_2}(F, N_F; Z),
\]
where \( \deg(Z) \) is the degree of the irreducible component \( Z \), \( \text{Res}_{\varphi_2}(\mathcal{F}_2|B_p;p) \) represents the Grothendieck residue of the foliation \( \mathcal{F}_2|B_p \) at \{p\} = \( Z \cap B_p \) with \( B_p \) a transversal ball and either \( \varphi_2 = c_1^2 \) or \( \varphi_2 = c_2 \).

This previous result is a partial advance in goal of calculates the residue of flags and it is an effective way to calculate this residue when \( S(F) \) has only one irreducible component.

For the last result of this paper we need the following definition due to G. N. Costa, see [11]. Let \( \mathcal{F} \) be a holomorphic distribution on \( M \) with singular set \{\( p_1, \ldots, p_r \)\} \( \cup \) \( C \), where \{\( p_1, \ldots, p_r \)\} are isolated points and \( C \) is an irreducible smooth curve, and let \( \pi: M \to M \) be the blow up morphism along \( C \) with exceptional divisor \( \mathcal{E} = \pi^{-1}(C) \). We say that \( \mathcal{F} \) is a special holomorphic distribution if the pull-back distribution \( \tilde{\mathcal{F}} \) on \( \tilde{M} \) has only isolated singularities and \( \mathcal{E} \) is an invariant set. We show, for special distribution, a relation between the degrees of the distributions in the flag, the tangency order of the distributions, the Euler characteristic and the degree of the curve \( C \).

**Theorem 1.4.** Let \( \mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2) \) be a 2-flag of holomorphic distributions on \( \mathbb{P}^3 \) satisfying the conditions:

1. \( S(\mathcal{F}_1) = \{ \text{isolated points} \} \),
2. \( S(\mathcal{F}_2) = \{ \text{isolated points} \} \cup \{ C \} \), where \( C \) is an irreducible smooth curve,
3. \( \mathcal{F}_2 \) is special along \( C \).

With this information we have the following relation

\[
\deg(C) \left[ (1 + d_1 - d_2) \left( -l_2(2 + 3l_2) + 2 \right) + (2l_2 - l_1) \left( -3l_2(2 + 4l_2 - d_2) + 2 + d_2 \right) \right] =
-\chi(C) \left( -l_2(2 + 3l_2) + 2 \right) (2l_2 - l_1),
\]

where \( d_i = \deg(\mathcal{F}_i), \quad l_i = \text{tang}(\pi^*\mathcal{F}, \mathcal{E}) \) for blow up \( \pi \) along the curve \( C \) with exceptional divisor \( \mathcal{E} \) and \( \chi(C) \) the Euler characteristic of \( C \).

**Corollary 1.5.** If \( 2l_2 = l_1 \) we have

\[
d_2 = d_1 + 1.
\]

**Corollary 1.6.** If \( 2l_2 \neq l_1 \) we get an expression of the Euler characteristic of the curve \( C \).

\[
\chi(C) = \deg(C) \left[ \frac{1 + d_1 - d_2}{2l_2 - l_1} + \frac{2 + d_2 - 3l_2(2 + 4l_2 - d_2)}{2 - l_2(2 + 3l_2)} \right].
\]

2. **Flag of Holomorphic Distributions**

Let us begin by recall the basic material and results in singular holomorphic foliations and distributions. Let \( M \) be a complex manifold of dimension \( n \) and \( \Theta_M \) be the sheaf of germs of holomorphic vector fields. For this section we refer to [5] [10].

A singular holomorphic distribution of dimension \( k \) on \( M \) is a coherent subsheaf \( \mathcal{F} \) of \( \Theta_M \) of rank \( k \).

If \( \mathcal{F} \) satisfies the following integrability condition

\[
[\mathcal{F}_x, \mathcal{F}_x] \subset \mathcal{F}_x \quad \text{for all} \quad x \in M,
\]

we say that \( \mathcal{F} \) is a holomorphic foliation. The normal sheaf of \( \mathcal{F} \) is defined by \( \mathcal{N}_\mathcal{F} := \Theta_M/\mathcal{F} \), such that it is torsion free (it means that \( \mathcal{F} \) is saturated). With this definition we have the following exact sequence
We define the singular set of the distribution \( F \) by
\[
S(F) := \text{Sing}(N_F) = \{ p \in M; N_{F,p} \text{ is not locally free} \}.
\]
We assume that \( \text{codim}(S(F)) \geq 2 \).

In [3] P. Baum and R. Bott developed a general residue theory for singular holomorphic foliations on \( M \) using differential geometry. More precisely, let \( F \) be a holomorphic foliation of dimension \( k \) on \( M \) and \( \varphi \) be a homogeneous symmetric polynomial of degree \( d \) satisfying \( n - k < d \leq n \). Let \( Z \) be a compact connected component of the singular set \( S(F) \).

Then, there exists a homology class, called residue \( \text{Res}_\varphi(F;Z) \in H_{2(n-d)}(Z;\mathbb{C}) \) such that it depends only on \( \varphi \) and on the local behavior of the leaves of \( F \) near \( Z \), satisfying
\[
\varphi(N_F) \sim [M] = \sum_Z \text{Res}_\varphi(F;Z).
\]

In [13] Suwa developed a residue theory for holomorphic distributions using certain Chern polynomials, and the residues arise from the vanishing by rank reason, instead of foliations that use the Vanishing Theorem.

Now we can define flags of holomorphic distributions. For this consider \( M \) a complex manifold of dimension \( n \).

**Definition 2.1.** Let \( F_1, F_2 \) be two holomorphic distributions on \( M \) of dimensions \( q = (q_1, q_2) \). We say that \( F := (F_1, F_2) \) is a 2-flag of holomorphic distributions if \( F_1 \) is a coherent sub \( O_M \)-module of \( F_2 \). Furthermore if each \( F_i \) is integrable we say that \( F \) is a 2-flag of holomorphic foliations.

We note that, for \( x \in M \setminus \bigcup_{i=1}^2 S(F_i) \) the inclusion relation \( T_x F_1 \subset T_x F_2 \) holds, giving that the leaves of \( F_1 \) are contained in leaves of \( F_2 \), when we have integrability. Here \( TF_i \) is the tangent sheaf of the distribution \( F_i \), but throughout the text we will abuse of notation and denote it simply by \( F_i \). Now we observe that we have a diagram of short exact sequences of sheaves, called "turtle diagram".

We define the singular set \( S(F) \) of the flag \( F \) to be the analytic set \( S(F_1) \cup S(F_2) \) and \( N_F := N_{12} \oplus N_2 \) to be the normal sheaf of the flag, where \( N_{12} \) is the relative quotient sheaf \( F_2/F_1 \).
3. Chern-Weil Theory of Characteristic Class

In this section we present the basic tools for working with residue of flags. The residue theory was developed firstly by Baum-Bott by using differential geometry. Lehman and Suwa on the decade of 1980 and 1990 present a new approach of residue theory using Chern-Weil theory. We use this last approach, for more details see [19].

Definition 3.1. A connection for a complex vector bundle $E$ on $M$ is a $\mathbb{C}$-linear map

$$\nabla : A^0(M, E) \rightarrow A^1(M, E)$$

that satisfies

$$\nabla(f.s) = df \otimes s + f.\nabla(s) \quad \text{for} \quad f \in A^0(M) \quad \text{and} \quad s \in A^0(M, E).$$

If $\nabla$ is a connection for $E$, then it induces a $\mathbb{C}$-linear map

$$\nabla := \nabla^2 : A^1(M, E) \rightarrow A^2(M, E)$$

satisfying

$$\nabla(\omega \otimes s) = d\omega \otimes s - \omega \wedge \nabla(s), \quad \omega \in A^1(M), \quad s \in A^0(M, E).$$

Definition 3.2. The composition $K := \nabla \circ \nabla : A^0(M, E) \rightarrow A^2(M, E)$ is called the curvature of the connection $\nabla$.

If $\nabla$ denotes a connection for a vector bundle $E$ of rank $r$ and $E$ is trivial on the open set $U$, i.e., $E|_U \simeq U \times \mathbb{C}^r$ and if $s = (s_1, \ldots, s_r)$ is a frame of $E$ on $U$, then we can write

$$\nabla(s_i) = \sum_{j=1}^r \theta_{ij} \otimes s_j; \quad \theta_{ij} \in A^1(U).$$

The connection matrix with respect to $s$ is $\theta = (\theta_{ij})$. Also, using the curvature definition, we get

$$K(s_i) = \sum_{j=1}^r K_{ij} s_j, \quad \text{where} \quad K_{ij} = d\theta_{ij} - \sum_{k=1}^r \theta_{ik} \wedge \theta_{kj}.$$ 

The curvature matrix with respect to the frame $s$ is $K = (K_{ij})$. Now, to define the Chern class of a vector bundle $E$, we consider $\sigma_i, i = 1, \ldots, r$ the $i$-th elementary symmetric functions in the eigenvalues of the matrix $K$

$$\det(\tau + K) = 1 + \sigma_1(K)t + \sigma_2(K)t^2 + \cdots + \sigma_r(K)t^r.$$ 

Next, we define a $2i$-form of Chern $c_i$ on $U$ by

$$c_i(K) := \sigma_i\left(\frac{\sqrt{-1}}{2\pi}K\right).$$

In general, if $\varphi$ is a symmetric polynomial in $r$ variables of degree $d$, we can write $\varphi = \hat{P}(c_1, \ldots, c_r)$ for some polynomial $\hat{P}$. Then we can define

$$\varphi(K) := \hat{P}(c_1(K), \ldots, c_r(K))$$

which is a closed form on $M$. Therefore, we have a cohomology class of $E$ on $M$, $\varphi(E) := \varphi(K) \in H^{2d}(M; \mathbb{C})$. 
Remark 3.3. Observe that, by a question of rank, if $E$ is a complex vector bundle of rank $r < n$, for an arbitrary connection $\nabla$ for $E$

$$c_p(\nabla) \equiv 0$$

for $r < p \leq n$. Thus for a Chern monomial $\varphi = c_{p_1} \cdots c_{p_k}$, if $r < p_i \leq n$ for some $i$, $\varphi(\nabla) \equiv 0$.

Remark 3.4. If $U$ is an open trivializing the vector bundle $E$ and $(s_1, \ldots, s_r)$ is a frame for $E$ on $U$, we can define a (local) connection for $E$ on $U$ simply doing $\nabla(s_i) = 0$ for all $i$. It is easy to see that the curvature matrix $K$ of this connection is a null matrix, so all Chern class $c_i(E) = 0$, for $i > 0$ on $H^2(U; \mathbb{C})$.

Now let $E_i$, $i = 0, \ldots, q$ be a family of complex vector bundles on a complex manifold $M$, $\xi$ be the virtual bundle $\xi = \sum_{i=0}^q (-1)^i E_i$ and $\varphi$ be a homogeneous symmetric polynomial. By definition \[19\], we have that

$$\varphi(\xi) = \sum_l \varphi_l^{(0)}(E_0) \varphi_l^{(1)}(E_1) \cdots \varphi_l^{(q)}(E_q),$$

where $\varphi_l^{(i)}(E_i)$ is a polynomial in the Chern classes of $E_i$, for each $i$ and $l$.

If $\nabla^{(i)}$ is a connection for $E_i$ consider the family $\nabla^\bullet = (\nabla^{(0)}, \ldots, \nabla^{(0)})$. Then $\varphi(\xi)$ is the cohomology class of the differential form

$$\varphi(\nabla^\bullet) = \sum_l \varphi_l^{(0)}(\nabla^{(0)}) \wedge \varphi_l^{(1)}(\nabla^{(1)}) \wedge \cdots \wedge \varphi_l^{(q)}(\nabla^{(q)}).$$

From above definitions and Remark \[3.4\] we get.

Lemma 3.5. Let $E_i$, $i = 1, \ldots, q$ be a family of complex vector bundles on a complex manifold $M$ and $\xi$ be the virtual bundle $\xi = \sum_{i=0}^q (-1)^i E_i$. Let $U$ be an open trivializing all $E_i$ and $\varphi$ be a homogeneous symmetric polynomial of degree $d > 0$. Then the differential form $\varphi(\xi)$ vanishes on $U$.

Proof. In fact, since each $E_i$ is a trivial bundle on $U$, we can take (see Remark \[8.4\]) $\nabla^{(i)}$ a connection for $E_i$ on $U$ such that $c_j(\nabla^{(i)}) = 0$ for $j > 0$. Writing $\nabla^\bullet = (\nabla^{(0)}, \ldots, \nabla^{(0)})$ we get

$$\varphi(\nabla^\bullet) = \sum_l \varphi_l^{(0)}(\nabla^{(0)}) \varphi_l^{(1)}(\nabla^{(1)}) \cdots \varphi_l^{(q)}(\nabla^{(q)}) = 0$$

since $\varphi_l^{(i)}$ is a polynomial in the Chern classes of $E_i$, for each $i$ and $l$, and the degree of some $\varphi_l^{(i)}$ is greater than zero.

4. Proof of Theorem \[1.1\]

Proof. We will consider $U$ a relatively compact open neighborhood of $Z$ on $M$ disjoint from the other components of $S(F)$. We set $U_0 = U \setminus \{Z\}$ and $U_1 = U$ and consider the open covering $U = \{U_0, U_1\}$ of $U$.

We will use the Chern-Weil theory of characteristic classes, see \[19\] for more details, to compute the Chern class of the normal sheaf $\mathcal{N}_F = \mathcal{N}_{12} \oplus \mathcal{N}_2$ of flag. To do this, we take
resolutions of the normal sheaves $\mathcal{N}_{12}$ and $\mathcal{N}_2$ by real analytic vector bundles $E_i^{12}$ and $E_j^2$ on $U$ see [2].

(4) \[ 0 \to A(U)(E_i^{12}) \to \ldots \to A(U)(E_0^{12}) \to A(U) \otimes \mathcal{N}_{12} \to 0. \]

(5) \[ 0 \to A(U)(E_j^2) \to \ldots \to A(U)(E_0^2) \to A(U) \otimes \mathcal{N}_2 \to 0. \]

These sequences are exact on the sheaf level, but on $U_0$ we have exact sequences of vector bundles, then [3] Definition 4.22 and Lemma 4.17 there exist connections $(12\nabla_0^0, \ldots, 12\nabla_0^{12})$ is compatible with the sequence (4) and $\varphi_1(12\nabla_0^0) = \varphi_1(12\nabla_0)$, where $12\nabla_0^0 = (12\nabla_0^q, \ldots, 12\nabla_0^0)$.

Analogously, there exist connections $1\nabla_0^0$ on $U_0$ for each $E_i^2$ and $2\nabla_0$ for $\mathcal{N}_2^0$ such that the family of connections $(2\nabla_0^q, \ldots, 2\nabla_0^0, 2\nabla_0)$ is compatible with the sequence (5) and $\varphi_2(2\nabla_0^0) = \varphi_2(2\nabla_0)$.

Let $12\nabla_1^0$ be a connection on $U_1$ for each $E_i^{12}$ and set $12\nabla_1^0 = (12\nabla_1^q, \ldots, 12\nabla_1^0)$ (respectively $2\nabla_1^0 = (2\nabla_1^q, \ldots, 2\nabla_1^0)$).

Then the class $\varphi(N_F) = \varphi_1(N_{12}) \sim \varphi_2(N_2)$ in $H^2(d_1 + d_2)(U; \mathbb{C})$ is represented in $A^{2(d_1 + d_2)}(U)$ by the cocycle

\[
\varphi(12\nabla_1^0) = \left( \varphi_1(12\nabla_0^0), \varphi_1(12\nabla_1^0), \varphi_1(12\nabla_0^0, 12\nabla_1^0) \right)
\]

\[
\left. \varphi(2\nabla_0^0), \varphi_2(2\nabla_1^0), \varphi_2(2\nabla_0^0, 2\nabla_1^0) \right)
\]

\[
\left. \varphi_1(12\nabla_0^0) \wedge \varphi_2(2\nabla_0^0), \varphi_1(12\nabla_1^0) \wedge \varphi_2(2\nabla_1^0), \varphi_1(12\nabla_0^0) \wedge \varphi_2(2\nabla_0^0, 2\nabla_1^0) \right)
\]

\[
\varphi_1(12\nabla_0^0, 12\nabla_1^0) \wedge \varphi_2(2\nabla_1^0) \right).
\]

Since the rank of $\mathcal{N}_2^{12}$ is $s_1 - s_2$, the rank of $\mathcal{N}_2^0$ is $s_2$, by hypothesis about the degree of $\varphi_1$ and $\varphi_2$ and the Remark 3.3 we have $\varphi_1(12\nabla_0^0) = 0$ or $\varphi_2(2\nabla_0^0) = 0$. Then

\[
\varphi(12\nabla_1^0) = 0, \varphi_1(12\nabla_1^0) \wedge \varphi_2(2\nabla_1^0), \varphi_1(12\nabla_0^0) \wedge \varphi_2(2\nabla_0^0, 2\nabla_1^0) \]

\[
\varphi_1(12\nabla_0^0, 12\nabla_1^0) \wedge \varphi_2(2\nabla_1^0) \right).
\]

Therefore $\varphi(12\nabla_1^0) \in A^{2(d_1 + d_2)}(U, U_0)$. Denoting $[\varphi(12\nabla_1^0)] = \varphi(Z_{N_F}, F)$ in $H^2(d_1 + d_2)(U, U \setminus Z; \mathbb{C})$ we have the residue $Res_{\varphi_1, \varphi_2}(N_{12}, F; Z) = A(\varphi_Z(Z_{N_F}, F))$ in $H_{2n-2(d_1 + d_2)}(Z; \mathbb{C})$, where $A$ is the Alexander homomorphism, see [11] p. 3028.

\[ \square \]

5. Proof of Theorem 1.2

Proof. Since $p$ is an isolated point of $S(F)$, we can take the open $U$, such that, all the vector bundles, $E_i^{12}, E_j^2$, on the resolutions of $\mathcal{N}_{12}$ and $\mathcal{N}_2$ (see the sequences (4), (5)) are trivial on $U$ and on $U_0 = U \setminus \{p\}$. Therefore by the Lemma 5.5 there exist connections $12\nabla_0^0$ on $U_0$ for each $E_i^{12}$ such that if we denote $12\nabla_0^0$ by $(12\nabla_0^0, \ldots, 12\nabla_0^{12})$

\[
\varphi_1(12\nabla_0^0) = 0.
\]
Analogously, there exist connections $2\nabla^i_1$ on $U_0$ for each $E^i_1$ with the same property. That is, if we denote $2\nabla^i_1$ by $(2\nabla^{(r)}_0, \ldots, 2\nabla^{(d_2)}_0)$. Then

$$\varphi_2(2\nabla^i_1) = 0.$$ Doing the same on $U_1 = U$, there exist connections $12\nabla^i_1$ for each $E^{12}_1$ and $2\nabla^i_1$ for each $E^i_2$ on $U_1$ such that if we denote $12\nabla^r_1$ by $(12\nabla^{(q)}_1, \ldots, 12\nabla^{(r)}_1)$ and $2\nabla^i_1$ by $(2\nabla^{(r)}_0, \ldots, 2\nabla^{(d_2)}_0)$, we have

$$\varphi_1(12\nabla^i_1) = 0, \quad \varphi_2(2\nabla^i_1) = 0.$$ Then the class $\varphi(N_\mathcal{F})$ is represented in $A^{2(d_1+d_2)}(U)$ by the cocycle

$$\varphi(12\nabla^i_1) = \left(\varphi(12\nabla^0_0) \wedge \varphi(2\nabla^i_0), \varphi(12\nabla^i_1) \wedge \varphi(2\nabla^0_1), \varphi(12\nabla^r_1) \wedge \varphi(2\nabla^0_r), 2\nabla^i_1\right)$$

$$+ \varphi(12\nabla^0_0, 12\nabla^i_0) \wedge \varphi(2\nabla^0_1) = (0, 0, 0).$$

Therefore

$$\text{Res}_{\varphi_1, \varphi_2}(\mathcal{F}, N_\mathcal{F}; p) = 0.$$ 

\[\square\]

6. Proof of the Theorem 1.3

In this section we will prove the Theorem 1.3 using the Theorem 1.2.

Proof. In this proof we will use the transversal disc method used by Baum and Bott in [3], Visik in [20] and Corrêa and Lourenço in [9]. By Baum-Bott Theorem for flags (Theorem 1.1), we have

$$\int_M c_1(N_{12})\varphi_2(N_2) = \sum_{Z \in S(F)} \text{Res}_{c_1, \varphi_2}(\mathcal{F}, N_\mathcal{F}; Z),$$

where $\text{Res}_{c_1, \varphi_2}(\mathcal{F}, N_\mathcal{F}; Z)$ denotes the flag residue at component $Z$, $N_\mathcal{F} = N_{12} \oplus N_2$ the normal sheaf and $S(\mathcal{F})$ the singular set of flag $\mathcal{F}$.

In this case we consider $M = \mathbb{P}^3$ and we have $S(\mathcal{F}) = S_0(\mathcal{F}) \cup S_1(\mathcal{F})$, where $S_i(\mathcal{F})$ denotes the components of the singular set of the flag $\mathcal{F}$ of pure dimension $i$, for $i = 0, 1$.

We can rewrite (6) as

$$\int_{\mathbb{P}^3} c_1(N_{12})\varphi_2(N_2) = \sum_{p \in S_0(F)} \text{Res}_{c_1, \varphi_2}(\mathcal{F}, N_\mathcal{F}; p) + \sum_{Z \in S_1(\mathcal{F})} \text{Res}_{c_1, \varphi_2}(\mathcal{F}, N_\mathcal{F}; Z).$$

But, by Theorem 1.2 $\text{Res}_{c_1, \varphi_2}(\mathcal{F}, N_\mathcal{F}; p) = 0$, we have

$$\int_{\mathbb{P}^3} c_1(N_{12})\varphi_2(N_2) = \sum_{Z \in S_1(\mathcal{F})} \text{Res}_{c_1, \varphi_2}(\mathcal{F}, N_\mathcal{F}; Z).$$

Now we will use the commutative diagram [14] Proposition 3.11, p. 55] for foliation $\mathcal{F}_2$

$$\begin{array}{ccc}
H^{2d}(\mathbb{P}^3, \mathbb{P}^3 \setminus Z; \mathbb{C}) & \longrightarrow & H^{2d}(\mathbb{P}^3; \mathbb{C}) \\
\text{Res} & \downarrow & \\
H_{2(3-d)}(Z; \mathbb{C}) & \longrightarrow & H_{2(3-d)}(\mathbb{P}^3; \mathbb{C})
\end{array}$$
where $A$ and $P$ denote, respectively, the Alexander homomorphism (isomorphism if $Z$ is nonsingular) and the Poincaré homomorphism (isomorphism since $P^3$ is nonsingular), see [1, p. 3028], $\iota$ denotes the inclusion map of $Z$ in $P^3$ and $\iota^*$ its induced map in homology group. For go on we use the composition map

$$\alpha = P \circ \iota^* : H_2(S(F_2); \mathbb{C}) \to H^4(P^3; \mathbb{C}).$$

By Theorem 1.2 in [9] and Baum-Bott Theorem for $F_2$ in [3, Theorem 1], we have

$$\sum_{Z \in S_1(F_2)} \alpha(Res_{\varphi_2}(F_2|D;p)[Z]) = \varphi_2(N_2) \text{ in } H^4(P^3; \mathbb{C}),$$

where

$$Res_{\varphi_2}(F_2|D;p) = Res_p \left( \frac{\varphi(Jv|D)}{v_1, v_2} \right),$$

represents the Grothendieck residue of foliation $F_2$ on $D \subset P^3$ at $\{p\}$, with $D$ being a transversal disc to $Z$ such that $D \cap Z = \{p\}$.

Then

$$\sum_{Z \in S_1(F_2)} Res_{\varphi_2}(F_2|D;p) \eta_Z = \varphi_2(N_2)$$

where $\eta_Z = \alpha([Z])$ is the Poincaré dual of $[Z]$.

Utilizing the exact sequence

$$0 \to N_{12} \to N_1 \to N_2 \to 0,$$

we have $c_1(N_{12}) = c_1(N_1) - c_1(N_2)$.

But by the exact sequence

$$0 \to F_1 \to TP^3 \to N_1 \to 0,$$

we have $c(TP^3) = c(F_1)c(N_1)$, and since $F_1 = \mathcal{O}_{P^3}(1 - d_1)$ see [16, p.778], then

$$c_1(N_1) = c_1(TP^3) - c_1(F_1)$$

$$= 4c_1(\mathcal{O}_{P^3}(1)) - (1 - d_1)c_1(\mathcal{O}_{P^3}(1))$$

$$= (3 + d_1)h,$$

where $\mathcal{O}_{P^3}(1)$ is the line bundle associated to a generic hyperplane $H$ on $P^3$ see [16, Definition 2.2.7, p. 69] and $h = c_1(\mathcal{O}_{P^3}(1))$ the hyperplane class, see [14, p. 414]. Combining these Chern classes we have

$$c_1(N_{12}) = (1 + d_1 - d_2)h.$$  

Replacing the equations (8) and (9) in equation (7), we have,
\[
\int_{\mathbb{P}^3} (1 + d_1 - d_2) h \sum_{Z \in S_1(\mathcal{F}_2)} \text{Res}_{\mathcal{F}_2}(\mathcal{F}_2|D; p) \eta_Z = \sum_{Z \in S_1(\mathcal{F})} \text{Res}_{c_1}(\mathcal{F}, N_\mathcal{F}; Z).
\]

We note that \( \int_{\mathbb{P}^3} h \eta_Z = \deg(Z) \). Therefore

\[
(1 + d_1 - d_2) \sum_{Z \in S_1(\mathcal{F}_2)} \text{Res}_{\mathcal{F}_2}(\mathcal{F}_2|D; p) \deg(Z) = \sum_{Z \in S(\mathcal{F})} \text{Res}_{c_1}(\mathcal{F}, N_\mathcal{F}; Z).
\]

\[\square\]

7. Proof of the Theorem 1.4

In this section we prove the Theorem 1.4 and we observe that its consequences in Corollary 1.5 and Corollary 1.6 are immediate.

Proof. We consider \( \pi : \tilde{\mathbb{P}}^3 \to \mathbb{P}^3 \) the blow up of \( \mathbb{P}^3 \) along the curve \( C \) with exceptional divisor \( \mathcal{E} \). We obtain a flag \( \tilde{\mathcal{F}} = (\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2) \) on \( \tilde{\mathbb{P}}^3 \) which has only isolated singularities by hypothesis that \( \tilde{\mathcal{F}}_2 \) is special along \( C \). For this flag we have see \((10)\), Lemma 2.2 (ii) p. 889

\[
\left\{ \begin{array}{l}
\tilde{\mathcal{F}}_1 = \pi^* \mathcal{F}_1 \otimes [\mathcal{E}]^{l_1} \\
\tilde{\mathcal{F}}_2 = \pi^* \mathcal{F}_2 \otimes [\mathcal{E}]^{l_2}.
\end{array} \right.
\]

So by Theorem 1.1 and Theorem 1.2 we have

\[
(11) \quad \int_{\tilde{\mathbb{P}}^3} c_1(N_{12})c_2(N_{2}) = 0.
\]

Now we finish the prove exploring the equation \((11)\). Before we present the necessary data for this. From the expression \((10)\) we get

\[
(12) \quad c_1(\tilde{\mathcal{F}}_1) = (1 - d_1) \pi^* h + l_1 \mathcal{E} \quad \text{and} \quad c_1(\tilde{\mathcal{F}}_2) = (2 - d_2) \pi^* h + 2l_2 \mathcal{E},
\]

where \( h \) is the hyperplane class \( c_1(\mathcal{O}_{\mathbb{P}^3}(1)) \) and by abusing of notation we consider \( c_1([\mathcal{E}]) = \mathcal{E} \).

Now by relation \((12)\) and the short exact sequence on \( \tilde{\mathbb{P}}^3 \),

\[0 \to \tilde{\mathcal{F}}_1 \to \tilde{\mathcal{F}}_2 \to N_{12} \to 0\]

we have \( c_1(N_{12}) = (1 + d_1 - d_2) \pi^* h + (2l_2 - l_1) \mathcal{E} \).

And by the short exact sequence on \( \mathbb{P}^3 \)

\[0 \to \mathcal{F}_2 \to \mathbb{P}^3 \to N_2 \to 0\]

we get

\[
c_2(N_2) = c_2(\mathcal{F}_3) - c_2(\mathcal{F}_2) - c_1(\mathcal{F}_2)c_1(N_2),
\]

where

\[
c_1(N_2) = (2 + d_2) \pi^* h - (2l_2 + l_1) \mathcal{E},
\]

\[
c_2(\mathcal{F}_2) = \pi^* c_2(\mathcal{F}_2) + c_1(\pi^* \mathcal{F}_2)c_1(\mathcal{E}^{l_2}) + l_2^2 \mathcal{E}^2.
\]
From Theorem 3.1 p. 14 in [7] we have
\[ c_2(\mathcal{F}_2) = \left(2 + d_2^2 - \deg(C)\right)h^2 \]
so
\[ c_2(\tilde{\mathcal{F}}_2) = \left(2 + d_2^2 - \deg(C)\right)\pi^*h^2 + l_2(2 - d_2)\pi^*h^\mathcal{C} + l_2^2\mathcal{C}^2. \]

From Porteous Theorem [17, Theorem 2 p. 123],
\[ c_2(\tilde{\mathcal{P}}_3) = 6\pi^*h^2 - \mathcal{C}^2 - \pi^*c_1(TC)\mathcal{C}. \]

With this at hand we have
\[ c_2(\tilde{\mathcal{N}}_2) = \deg(C)\pi^*h^2 + \left(-3l_2d_2 - 2l_2 - d_2 + 2\right)\pi^*h^\mathcal{C} - \pi^*c_1(TC)\mathcal{C} + \left(3l_2^2 + 2l_2 - 1\right)\mathcal{C}^2. \]

So we can calculate the product of Chern classes
\[ c_1(\mathcal{N}_{12})c_2(\mathcal{N}_2) = (1 + d_1 - d_2)\deg(C)\pi^*h^3 + \\
(1 + d_1 - d_2)(-3l_2d_2 - 2l_2 - d_2 + 2)\pi^*h^2\mathcal{C} \\
= -(1 + d_1 - d_2)\pi^*c_1(TC)\mathcal{C}\pi^*h + (1 + d_1 - d_2)(3l_2^2 + 2l_2 - 1)\pi^*h^2\mathcal{C} \\
+ (2l_2 - l_1)\deg(C)\pi^*h^2\mathcal{C} + (2l_2 - l_1)(-3l_2d_2 - 2l_2 - d_2 + 2)\pi^*h^\mathcal{C} \\
- (2l_2 - l_1)\pi^*c_1(TC)\mathcal{C}^2 + (2l_2 - l_1)(3l_2^2 + 2l_2 - 1)\mathcal{C}^3. \]

Now we use the properties of intersection theory see [10].

(1) \[ \int_{\tilde{\mathcal{P}}_3} \pi^*h^3 = 1. \]
(2) \[ \int_{\tilde{\mathcal{P}}_3} \pi^*h^2\mathcal{C} = \int_{\mathcal{C}} \pi^*h^2 = \int_{\mathcal{C}} h^2 = 0. \]
(3) \[ \int_{\tilde{\mathcal{P}}_3} \pi^*h^\mathcal{C} = \int_{\mathcal{C}} \pi^*h^\mathcal{C} = -\int_{\mathcal{C}} h = -\deg(C). \]
(4) \[ \int_{\tilde{\mathcal{P}}_3} \mathcal{C}^3 = \int_{\mathcal{C}} \mathcal{C}^2 = \chi(C) - 4\deg(C). \]
(5) \[ \int_{\tilde{\mathcal{P}}_3} \pi^*h\pi^*c_1(TC)\mathcal{C} = 0. \]
(6) \[ \int_{\tilde{\mathcal{P}}_3} \pi^*c_1(TC)\mathcal{C}^2 = \int_{\mathcal{C}} c_1(TC) = \chi(C). \]

Integrating the Chern class \( c_1(\mathcal{N}_{12})c_2(\mathcal{N}_2) \) on \( \tilde{\mathcal{P}}_3 \) we get
\[
\int_{\tilde{\mathcal{F}}_3} c_1(\mathcal{N}_{12})c_2(\mathcal{N}_2) = (1 + d_1 - d_2) \deg(C) + (1 + d_1 - d_2)(3l_2^2 + 2l_2 - 1)(-\deg(C)) \\
+ (2l_2 - l_1)(-3l_2d_2 - 2l_2 - d_2 + 2)(-\deg(C)) - (2l_2 - l_1)\chi(C) \\
+ (2l_2 - l_1)(3l_2^2 + 2l_2 - 1)(\chi(C) - 4\deg(C)) \\
= \deg(C) \left[(1 + d_1 - d_2)(-l_2(2 + 3l_2) + 2) + \\
(2l_2 - l_1)(-3l_2(2 + 4l_2 - d_2) + 2 + d_2)\right] \\
+ \chi(C)(2l_2 - l_1)\left(l_2(2 + 3l_2) - 2\right) \\
= 0.
\]

Therefore,
\[
\deg(C) \left[(1 + d_1 - d_2)(-l_2(2 + 3l_2) + 2) + (2l_2 - l_1)(-3l_2(2 + 4l_2 - d_2) + 2 + d_2)\right] \\
= \chi(C)(2l_2 - l_1)\left(-l_2(2 + 3l_2) + 2\right). \quad \square
\]

8. Examples

This section is dedicated to show examples to illustrate some results.

**Example 8.1.** Let \( \mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2) \) be a 2-flag on \( \mathbb{P}^3 \) where \( \mathcal{F}_2 \) is the codimension one holomorphic foliation induced by homogeneous 1-form

\[
\omega = -z_0z_3dz_0 - z_1z_3dz_1 - z_2z_3dz_2 + (z_0^2 + z_1^2 + z_2^2)dz_3.
\]

The foliation \( \mathcal{F}_1 \) is induced by homogeneous vector field

\[
X = z_1z_3 \frac{\partial}{\partial z_0} - z_0z_3 \frac{\partial}{\partial z_1} + (z_0^2 + z_1^2 + z_2^2) \frac{\partial}{\partial z_2} + z_2z_3 \frac{\partial}{\partial z_3}.
\]

The singular set of \( \mathcal{F}_2 \) is given by

\[
S(\mathcal{F}_2) = \left\{ C = \{z_3 = z_0^2 + z_1^2 + z_2^2 = 0\}, P = [0 : 0 : 0 : 1] \right\}.
\]

By Theorem 1.3 we have the equality

\[
\sum_{Z \in S(\mathcal{F}_2)} \text{Res}_{c_1c_2}(\mathcal{F}, \mathcal{N}_2 Z) = (1 + d_1 - d_2) \deg(C) \text{Res}_{c_2}((\mathcal{F}_2)_D q),
\]

where \( C \cap D = \{q\} \) with \( D \) a transversal disc to \( C \).

Now we will calculate the sum of residues of flag using the above expression. For this let us consider the chart \( U_0 = \{z_0 = 1\} \), with coordinates \( x = z_1/z_0, y = z_2/z_0 \) and \( z = z_3/z_0 \), so

\[
C_{|U_0} = \{z = 1 + x^2 + y^2 = 0\}.
\]

Let us consider a small transversal disc \( D \subset \{x = 0\} \) such that \( D \cap C = \{(0, \sqrt{-1}, 0)\} = \{q\} \).
The dual vector field of this 1-form is given by

\[ Y_\omega = (1 + y^2) \frac{\partial}{\partial y} + (yz) \frac{\partial}{\partial z}. \]

The Jacobian matrix of the \( Y_\omega \) is

\[ JY_\omega = \begin{bmatrix} 2y & 0 \\ z & y \end{bmatrix}. \]

Therefore, we have the residue

\[ \text{Res}_{c_1}(\mathcal{F}_1|_D; q) = \frac{c_1^2(JY_\omega(q))}{\det(JY_\omega(q))} = \frac{9}{2}. \]

Since \( d_1 = 2, d_2 = 1 \) and \( \deg(C) = 2 \) we have

\[ \sum_{Z \in \mathcal{S}(\mathcal{F})} \text{Res}_{c_1,c_2}(\mathcal{F}, \mathcal{N}_\mathcal{F}; Z) = 18. \]

Example 8.2. We consider now the foliation \( \mathcal{G} \) on \( \mathbb{P}^2 \) induced by 1-form

\[ \eta = \left( (z_0 - z_1)z_1 + z_2(z_0 - z_2) \right) dz_0 + z_0(z_1 - z_0)dz_1 + z_0(z_2 - z_0)dz_2. \]

This foliation has singular set

\[ \mathcal{S}(\mathcal{G}) = \left\{ p_1 = [0 : 1 : \sqrt{-1}], p_2 = [0 : 1 : -\sqrt{-1}], p_3 = [1 : 1 : 1] \right\}. \]

Now let \( \pi \) be the rational map

\[ \pi : \mathbb{P}^3 \rightarrow \mathbb{P}^2, \]

\[ [z_0 : z_1 : z_2 : z_3] \rightarrow [z_0 : z_1 : z_2]. \]

If we consider the pull back of \( \mathcal{G} \) by \( \pi \) we have the holomorphic foliation \( \mathcal{F}_2 = \pi^* \mathcal{G} \) on \( \mathbb{P}^3 \) of degree one \( (d_2 = 1) \) given by 1-form

\[ \omega = \left( (z_0 - z_1)z_1 + z_2(z_0 - z_2) \right) dz_0 + z_0(z_1 - z_0)dz_1 + z_0(z_2 - z_0)dz_2 \]

which its singular set is the union of three lines

\[ \mathcal{S}(\mathcal{F}_2) = L_1 \cup L_2 \cup L_3, \]

where \( L_1 = [0 : z_1 : \sqrt{-1}z_1 : z_3], L_2 = [0 : z_1 : -\sqrt{-1}z_1 : z_3], L_3 = [z_1 : z_1 : z_1 : z_3]. \)

We set the holomorphic one dimensional foliation \( \mathcal{F}_1 \) on \( \mathbb{P}^3 \) of degree two \( (d_1 = 2) \) induced by homogeneous vector field

\[ X = z_0(z_0 - z_1) \frac{\partial}{\partial z_0} + \left( (z_0 - z_1)z_1 + z_2(z_0 - z_2) \right) \frac{\partial}{\partial z_1}. \]

We observe that the foliation \( \mathcal{F}_1 \) is a subfoliation of \( \mathcal{F}_2 \) because \( \omega(X) = 0 \). Furthermore the flag \( \mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2) \) satisfies the hypothesis of Theorem 1.3. Thus we can calculate the sum of residues of the flag.

\[ \sum_{Z \in \mathcal{S}(\mathcal{F})} \text{Res}_{c_1,c_2}(\mathcal{F}, \mathcal{N}_\mathcal{F}; Z) = (1 + d_1 - d_2) \left[ \deg(L_1) \text{Res}_{c_1}(\mathcal{F}_2|_D; t_1) + \right. \]

\[ + \deg(L_2) \text{Res}_{c_1}(\mathcal{F}_2|_D; t_2) + \deg(L_3) \text{Res}_{c_1}(\mathcal{F}_2|_D; t_3) \right]. \]
On chart $U_3 = \{ z_3 \neq 0 \}$ we have coordinates $x = z_0/z_3, y = z_1/z_3$ and $z = z_2/z_3$ with $l_1 = L_1|_{U_3} = (0, y, \sqrt{−1}y), l_2 = L_2|_{U_3} = (0, y, −\sqrt{−1}y)$ and $l_3 = L_3|_{U_3} = (y, y, y)$. Furthermore $\omega$ on $U_3$ is given by

$$\omega|_{U_3} = [(x − y)y + z(x − z)]dx + x(y − x)dy + x(z − x)dz.$$  

Now we choose a small transversal disc at each line $D = \{ z = 1 \}$. The foliation $F_2$ is given on $D$ by 1-form

$$\omega|_{D} = [(x − y)y + (x − 1)]dx + x(y − x)dy$$

and its dual vector field is

$$Y_\omega = x(y − x)\frac{\partial}{\partial x} − [(x − y)y + (x − 1)]\frac{\partial}{\partial y}.$$  

In particular, we have the Jacobian matrix of dual vector field $JY_\omega$

$$JY_\omega = \begin{bmatrix} y−2x & x \\ −y−1 & −x+2y \end{bmatrix}.$$  

Thus

$$\text{Res}_{c_1}^\mathbb{C}(F_2|_D; t_1) = \frac{c_1^2(JY_\omega)(t_1)}{\det(JY_\omega)(t_1)} = \frac{9}{2},$$

$$\text{Res}_{c_1}^\mathbb{C}(F_2|_D; t_2) = \frac{c_1^2(JY_\omega)(t_2)}{\det(JY_\omega)(t_2)} = \frac{9}{2},$$

$$\text{Res}_{c_2}^\mathbb{C}(F_2|_D; t_3) = \frac{c_1^2(JY_\omega)(t_3)}{\det(JY_\omega)(t_3)} = 0,$$

where $t_i$ is the intersect point $D ∩ l_i$.

Therefore

$$\sum_{Z \in S(F)} \text{Res}_{c_1, c_2}(F, N_F; Z) = 2\left(\frac{9}{2} + \frac{9}{2} + 0\right) = 18,$$

since $\deg(l_i) = 1$ for $i = 1, 2, 3$ and $1 + d_1 − d_2 = 2$.

To finish the example we confirm the residue calculation using Theorem 1.1. In this case, one has

$$\int_{\mathbb{P}^3} c_1(N_{\mathbb{P}^2})c_2(N_2) = \int_{\mathbb{P}^3} (1 + d_1 − d_2)h(2 + d_2)^2h^2 = 18.$$  

Acknowledgments. We are grateful to Mauricio Corrêa and Marcio G. Soares for interesting conversations. The authors were partially supported by the FAPEMIG [grant number 38155289/2021].

References

[1] M. Abate, F. Bracci, and F. Tovena, Index Theorems for Holomorphic Maps and Foliations, Indiana University Mathematics Journal, 57(7), 2999-3048.

[2] M. Atiyah, F. Hirzebruch, Analytic cycles on complex manifolds, Topology 1 (1961), 25-45.

[3] P. Baum and R. Bott, Singularities of holomorphic foliations, J. Differential Geom. 7 (1972) 279-342.

[4] P. Baum, Structure of foliation singularities, Adv. Math. 15 (1975) 361-374.

[5] J-P. Brasselet, M. Corrêa, F. Lourenço, Residues for flags of holomorphic foliations, Adv. Math., 320, n.7 (2017), 1158-1184.

[6] F. Bracci and T. Suwa, Perturbation of Baum-Bott residues, Asian Journal of Mathematics. Volume 19 (2015) Number 5. Pages: 871-886.

[7] O. Calvo-Andrade, M. Corrêa and M. Jardim, Codimension One Holomorphic Distributions on the Projective Three-space, International Mathematics Research Notices, Vol. 00, No. 0, pp. 1-64 (2018).
[8] M. Corrêa and M. G. Soares, Inequalities for Characteristic Numbers of Flags of Distributions and Foliations, International Journal of Mathematics Vol. 24, No. 11, 1350093 (2013).
[9] M. Corrêa and F. Lourenço, Determination of Baum-Bott residues for higher dimensional foliations, Asian Journal of Mathematics, (23), 527-538, 2019.
[10] G. N. Costa, Baum-Bott indices for curves of singularities, Bulletin of the Brazilian Mathematical Society, New Series volume 47, pages 883-910 (2016).
[11] G. N. Costa, Holomorphic foliations by curves on $\mathbb{P}^3$ with non-isolated singularities, Annales de la Faculté des sciences de Toulouse: Mathématiques, Serie 6, Volume 15 (2006) no. 2, pp. 297-321.
[12] M. Dia, Sur les résidus de Baum-Bott, Annales de la Faculté des Sciences de Toulouse Vol. XIX, n2, 2010 pp. 363-403.
[13] B.L. Feigin, Characteristic classes of flags of foliations, Functional Analysis and Its Applications. Volume 9, Issue 4, October 1975, Pages 312-317.
[14] P. Griffiths and J. Harris, Principles of algebraic geometry. John Wiley & Sons, New York, 1978.
[15] Huybrechts, D., Complex Geometry An Introduction, Universitext, Springer, 2004.
[16] R. S. Mol, Flags of holomorphic foliations, An. Acad. Bras. Ciênc. vol.83 no.3 Rio de Janeiro Sept. 2011 (775-786).
[17] I.R. Porteous, Blowing up Chern Class, Proc. Cambridge Phil.Soc. 56 p.118-124. (1960).
[18] T. Suwa, Residues of singular holomorphic distributions, European Mathematical. Society Publishing House. Singularities in Geometry and Topology, pp: 207-247 DOI: 10.4171/118-1/12.
[19] T. Suwa, Indices of Vector Fields and Residues of Singular Holomorphic Foliations, Actualités Mathématiques, Hermann, Paris 1998.
[20] M. S. Vishik, Singularities of analytic foliations and characteristic classes, Functional Anal. Appl. 7 (1973) 1-15.

Antonio Marcos Ferreira da Silva, DMA - UFES, Rodovia BR-101, Km 60, Bairro Litorâneo, São Matheus-ES, Brazil, CEP 29932-540
Email address: antonio.m.silva@ufes.br

Fernando Lourenço, DMM - UFLA, Campus Universitário, Lavras MG, Brazil, CEP 37200-000
Email address: fernando.lourenco@ufla.br