The Cauchy Singular Integral Operator on Weighted Variable Lebesgue Spaces

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Abstract. Let \( p : \mathbb{R} \rightarrow (1, \infty) \) be a globally log-Hölder continuous variable exponent and \( w : \mathbb{R} \rightarrow [0, \infty] \) be a weight. We prove that the Cauchy singular integral operator \( S \) is bounded on the weighted variable Lebesgue space \( L^{p(x)}(\mathbb{R}, w) = \{ f : f \in L^{p(x)}(\mathbb{R}) \} \) if and only if the weight \( w \) satisfies
\[
\sup_{-\infty < a < b < \infty} \frac{1}{b-a} \| w \chi_{(a,b)} \|_{p(x)} \| w^{-1} \chi_{(a,b)} \|_{p'(x)} < \infty \quad (1/p(x) + 1/p'(x) = 1).
\]

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1. Introduction

Let \( p : \mathbb{R} \rightarrow [1, \infty] \) be a measurable a.e. finite function. By \( L^{p(x)}(\mathbb{R}) \) we denote the set of all complex-valued functions \( f \) on \( \mathbb{R} \) such that
\[
I_{p(x)}(f/\lambda) := \int_{\mathbb{R}} |f(x)/\lambda|^{p(x)} \, dx < \infty
\]
for some \( \lambda > 0 \). This set becomes a Banach space when equipped with the norm
\[
\| f \|_{p(x)} := \inf \{ \lambda > 0 : I_{p(x)}(f/\lambda) \leq 1 \}.
\]
It is easy to see that if \( p \) is constant, then \( L^{p(x)}(\mathbb{R}) \) is nothing but the standard Lebesgue space \( L^{p}(\mathbb{R}) \). The space \( L^{p(x)}(\mathbb{R}) \) is referred to as a variable Lebesgue space.

A measurable function \( w : \mathbb{R} \rightarrow [0, \infty] \) is referred to as a weight whenever \( 0 < w(x) < \infty \) a.e. on \( \mathbb{R} \). Given a variable exponent \( p : \mathbb{R} \rightarrow [1, \infty] \) and a weight

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$w : \mathbb{R} \to [0, \infty]$, we define the weighted variable exponent space $L^{p(\cdot)}(\mathbb{R}, w)$ as the space of all measurable complex-valued functions $f$ such that $fw \in L^{p(\cdot)}(\mathbb{R})$. The norm on this space is naturally defined by

$$
\|f\|_{p(\cdot), w} := \|fw\|_{p(\cdot)}.
$$

Given $f \in L^1_{\text{loc}}(\mathbb{R})$, the Hardy-Littlewood maximal operator is defined by

$$
Mf(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy
$$

where the supremum is taken over all intervals $Q \subset \mathbb{R}$ containing $x$. The Cauchy singular integral operator $S$ is defined for $f \in L^1_{\text{loc}}(\mathbb{R})$ by

$$
(Sf)(x) := \frac{1}{\pi i} \int_{\mathbb{R}} \frac{f(\tau)}{\tau - x} d\tau \quad (x \in \mathbb{R}),
$$

where the integral is understood in the principal value sense.

Following [4, Section 2] or [5, Section 4.1], one says that $\alpha : \mathbb{R} \to \mathbb{R}$ is locally log-Hölder continuous if there exists $c_1 > 0$ such that

$$
|\alpha(x) - \alpha(y)| \leq \frac{c_1}{\log(e + 1/|x - y|)}
$$

for all $x, y \in \mathbb{R}$. Further, $\alpha$ is said to satisfy the log-Hölder decay condition if there exist $\alpha_\infty \in \mathbb{R}$ and a constant $c_2 > 0$ such that

$$
|\alpha(x) - \alpha_\infty| \leq \frac{c_2}{\log(e + |x|)}
$$

for all $x \in \mathbb{R}$. One says that $\alpha$ is globally log-Hölder continuous on $\mathbb{R}$ if it is locally log-Hölder continuous and satisfies the log-Hölder decay condition. Put

$$
p_- := \text{ess inf}_{x \in \mathbb{R}} p(x), \quad \text{ess sup}_{x \in \mathbb{R}} p(x) := p_+.
$$

As usual, we use the convention $1/\infty := 0$ and denote by $\mathcal{P}^{\log}(\mathbb{R})$ the set of all variable exponents such that $1/p$ is globally log-Hölder continuous. If $p \in \mathcal{P}^{\log}(\mathbb{R})$, then the limit

$$
\frac{1}{p(\infty)} := \lim_{|x| \to \infty} \frac{1}{p(x)}
$$

exists. If $p_+ < \infty$, then $p \in \mathcal{P}^{\log}(\mathbb{R})$ if and only if $p$ is globally log-Hölder continuous.

By [5] Theorem 4.3.8], if $p \in \mathcal{P}^{\log}(\mathbb{R})$ with $p_- > 1$, then the Hardy-Littlewood maximal operator $M$ is bounded on $L^{p(\cdot)}(\mathbb{R})$. Notice, however, that the condition $p \in \mathcal{P}^{\log}(\mathbb{R})$ is not necessary, there are even discontinuous exponents $p$ such that $M$ is bounded on $L^{p(\cdot)}(\mathbb{R})$. Corresponding examples were first constructed by Lerner and they are contained in [5] Section 5.1.

In this paper we will mainly suppose that

$$
1 < p_-, \quad p_+ < \infty. \quad (1.1)
$$
Under these conditions, the space $L^p(\mathbb{R})$ is separable and reflexive, and its Banach space dual $[L^p(\mathbb{R})]^*$ is isomorphic to $L^{p'}(\mathbb{R})$, where

$$1/p(x) + 1/p'(x) = 1 \quad (x \in \mathbb{R})$$

(see [5, Chap. 3]). If, in addition, $w \chi_E \in L^p(\mathbb{R})$ and $\chi_E/w \in L^{p'}(\mathbb{R})$ for any measurable set $E \subset \mathbb{R}$ of finite measure, then $L^p(\mathbb{R},w)$ is a Banach function space and $[L^p(\mathbb{R},w)]^* = L^{p'}(\mathbb{R},w^{-1})$. Here and in what follows, $\chi_E$ denotes the characteristic function of the set $E$.

Probably, one of the simplest weights is the following power weight

$$w(x) := |x - i|^{\lambda_{\infty}} \prod_{j=1}^{m} |x - x_j|^{\lambda_j}, \quad (1.2)$$

where $-\infty < x_1 < \cdots < x_m < +\infty$ and $\lambda_1, \ldots, \lambda_m, \lambda_{\infty} \in \mathbb{R}$. Kokilashvili, Paatashvili, and Samko studied the boundedness of the operators $M$ and $S$ on $L^p(\mathbb{R},w)$ with power weights (1.2). From [11, Theorem A] and [14, Theorem B] one can extract the following result.

**Theorem 1.1.** Let $p \in \mathcal{P}_{\log}(\mathbb{R})$ satisfy (1.1) and $w$ be a power weight (1.2).

(a) (Kokilashvili, Samko). Suppose, in addition, that $p$ is constant outside an interval containing $x_1, \ldots, x_m$. Then the Hardy-Littlewood maximal operator $M$ is bounded on $L^p(\mathbb{R},w)$ if and only if

$$0 < \frac{1}{p(x_j)} + \lambda_j < 1 \quad \text{for } j \in \{1, \ldots, m\}, \quad 0 < \frac{1}{p(\infty)} + \lambda_{\infty} + \sum_{j=1}^{m} \lambda_j < 1. \quad (1.3)$$

(b) (Kokilashvili, Paatashvili, Samko). The Cauchy singular integral operator $S$ is bounded on $L^p(\mathbb{R},w)$ if and only if (1.3) is fulfilled.

Further, the sufficiency portion of this result was extended in [12, 13] to radial oscillating weights of the form $\prod_{j=1}^{m} \omega_j(|x - x_j|)$, where $\omega_j(t)$ are continuous functions for $t > 0$ that may oscillate near zero and whose Matuszewska-Orlicz indices can be different. Notice that the Matuszewska-Orlicz indices of $\omega_j(t) = t^{\lambda_j}$ are both equal to $\lambda_j$.

Very recently, Cruz-Uribe, Diening, and H"ast"o [5, Theorem 1.3] generalized part (a) of Theorem [11] to the case of general weights. To formulate their result, we will introduce the following generalization of the classical Muckenhoupt condition (written in the symmetric form). We say that a weight $w : \mathbb{R} \to [0,\infty]$ belongs to the class $A_{p(\cdot)}(\mathbb{R})$ if

$$\sup_{-\infty < a < b < \infty} \frac{1}{b-a} \|w \chi_{(a,b)}\|_{p(\cdot)}\|w^{-1} \chi_{(a,b)}\|_{p'(\cdot)} < \infty.$$ 

This condition goes back to Berezhnoi [3] (in the more general setting of Banach function spaces), it was studied by the first author [8] (in the case of Banach function spaces defined on Carleson curves) and Kopaliani [15].
Theorem 1.2 (Cruz-Uribe, Diening, Hästö). Let \( p \in \mathcal{P}^{\log}(\mathbb{R}) \) satisfy (1.1) and \( w : \mathbb{R} \to [0, \infty) \) be a weight. The Hardy-Littlewood maximal operator \( M \) is bounded on the weighted variable Lebesgue space \( L^{p(\cdot)}(\mathbb{R}, w) \) if and only if \( w \in A_{p(\cdot)}(\mathbb{R}) \).

The aim of this paper is to generalize part (b) of Theorem 1.1 to the case of general weights. We will prove the following.

Theorem 1.3 (Main result). Let \( p \in \mathcal{P}^{\log}(\mathbb{R}) \) satisfy (1.1) and \( w : \mathbb{R} \to [0, \infty) \) be a weight. The Cauchy singular integral operator \( S \) is bounded on the weighted variable Lebesgue space \( L^{p(\cdot)}(\mathbb{R}, w) \) if and only if \( w \in A_{p(\cdot)}(\mathbb{R}) \).

From this theorem, by using standard techniques, we derive also the following.

Theorem 1.4. Let \( p \in \mathcal{P}^{\log}(\mathbb{R}) \) satisfy (1.1) and \( w \in A_{p(\cdot)}(\mathbb{R}) \). Then \( S^2 = I \) on the space \( L^{p(\cdot)}(\mathbb{R}, w) \) and \( S^* = S \) on the space \( L^{p'(\cdot)}(\mathbb{R}, w^{-1}) \).

The paper is organized as follows. In Section 2 we collect necessary facts on Banach function spaces \( X(\mathbb{R}) \) in the sense of Luxemburg and discuss weighted Banach functions spaces \( X(\mathbb{R}, w) = \{ f : f w \in X(\mathbb{R}) \} \). A special attention is paid to conditions implying that \( X(\mathbb{R}, w) \) is a Banach function space itself, to separability and reflexivity of \( X(\mathbb{R}, w) \), and to density of smooth compactly supported functions in \( X(\mathbb{R}, w) \) and in its dual space \( X'(\mathbb{R}, w^{-1}) \). In Section 3.2 we prepare the proof of a sufficient condition for the boundedness of the operator \( S \) and formulate two key estimates by Lerner [16] and Álvarez and Pérez [4]. On the basis of these results, in Section 3.3 we prove that if \( X(\mathbb{R}) \) is a separable Banach function space and the Hardy-Littlewood maximal function is bounded on the weighted Banach function spaces \( X(\mathbb{R}, w) \) and \( X'(\mathbb{R}, w^{-1}) \), then \( S \) is bounded on \( X(\mathbb{R}, w) \) and \( S^2 = I \). Moreover, if \( X(\mathbb{R}) \) is reflexive, then \( S^* \) coincides with \( S \) on \( X'(\mathbb{R}, w^{-1}) \). In Section 3.4 we prove that if \( S \) is bounded on the weighted Banach function spaces \( X(\mathbb{R}, w) \), then

\[
\sup_{-\infty < a < b < \infty} \frac{1}{b - a} \| w \chi_{(a,b)} \|_{X(\mathbb{R})} \| w^{-1} \chi_{(a,b)} \|_{X'(\mathbb{R})} < \infty
\]

where \( X'(\mathbb{R}) \) is the associate space for \( X(\mathbb{R}) \). Finally, in Section 3.5 we explain that Theorems 1.3 and 1.4 follow from results of Sections 3.3, 3.4, and Theorem 1.2 because \( L^{p(\cdot)}(\mathbb{R}) \) is a Banach function space, which is separable and reflexive whenever \( p \) satisfies (1.1).

2. Weighted Banach function spaces

2.1. Banach function spaces

The set of all Lebesgue measurable complex-valued functions on \( \mathbb{R} \) is denoted by \( \mathcal{M} \). Let \( \mathcal{M}^+ \) be the subset of functions in \( \mathcal{M} \) whose values lie in \([0, \infty]\). The characteristic function of a measurable set \( E \subset \mathbb{R} \) is denoted by \( \chi_E \) and the Lebesgue measure of \( E \) is denoted by \( |E| \).
Proposition 2.1 (2 Chap. 1, Definition 1.1). A mapping \( \rho : M^+ \to [0, \infty] \) is called a \textit{Banach function norm} if, for all functions \( f, g, f_n (n \in N) \) in \( M^+ \), for all constants \( a \geq 0 \), and for all measurable subsets \( E \) of \( \mathbb{R} \), the following properties hold:

\begin{align*}
(A1) & \quad \rho(f) = 0 \Leftrightarrow f = 0 \text{ a.e.}, \quad \rho(a f) = a \rho(f), \quad \rho(f + g) \leq \rho(f) + \rho(g), \\
(A2) & \quad 0 \leq g \leq f \text{ a.e. } \Rightarrow \rho(g) \leq \rho(f) \quad (\text{the lattice property}), \\
(A3) & \quad 0 \leq f_n \uparrow f \text{ a.e. } \Rightarrow \rho(f_n) \uparrow \rho(f) \quad (\text{the Fatou property}), \\
(A4) & \quad |E| < \infty \Rightarrow \rho(\chi_E) < \infty, \\
(A5) & \quad |E| < \infty \Rightarrow \int_E |f(x)| \, dx \leq C_E \rho(f)
\end{align*}

with \( C_E \in (0, \infty) \) which may depend on \( E \) and \( \rho \) but is independent of \( f \).

When functions differing only on a set of measure zero are identified, the set \( X(\mathbb{R}) \) of all functions \( f \in M \) for which \( \rho(|f|) < \infty \) is called a \textit{Banach function space}. For each \( f \in X(\mathbb{R}) \), the norm of \( f \) is defined by

\[ \|f\|_{X(\mathbb{R})} := \rho(|f|). \]

The set \( X(\mathbb{R}) \) under the natural linear space operations and under this norm becomes a Banach space (see [2, Chap. 1, Theorems 1.4 and 1.6]).

If \( \rho \) is a Banach function norm, its associate norm \( \rho' \) is defined on \( M^+ \) by

\[ \rho'(g) := \sup \left\{ \int_\mathbb{R} f(x)g(x) \, dx : f \in M^+, \rho(f) \leq 1 \right\}, \quad g \in M^+. \]

It is a Banach function norm itself [2, Chap. 1, Theorem 2.2]. The Banach function space \( X'(\mathbb{R}) \) determined by the Banach function norm \( \rho' \) is called the \textit{associate space} (Köthe dual) of \( X(\mathbb{R}) \). The associate space \( X'(\mathbb{R}) \) is a subspace of the dual space \( [X(\mathbb{R})]^* \). The construction of the associate space implies the following Hölder inequality for Banach function spaces.

**Lemma 2.2** (2 Chap. 1, Theorem 2.4). Let \( X(\mathbb{R}) \) be a Banach function space and \( X'(\mathbb{R}) \) be its associate space. If \( f \in X(\mathbb{R}) \) and \( g \in X'(\mathbb{R}) \), then \( fg \) is integrable and

\[ \|fg\|_{L^1(\mathbb{R})} \leq \|f\|_{X(\mathbb{R})}\|g\|_{X'(\mathbb{R})}. \]

The next result provides a useful converse to the integrability assertion of Lemma 2.2.

**Lemma 2.3** (2 Chap. 1, Lemma 2.6). Let \( X(\mathbb{R}) \) be a Banach function space. In order that a measurable function \( g \) belong to the associate space \( X'(\mathbb{R}) \), it is necessary and sufficient that \( fg \) be integrable for every \( f \in X(\mathbb{R}) \).

### 2.2. Weighted Banach function spaces

Let \( X(\mathbb{R}) \) be a Banach function space generated by a Banach function norm \( \rho \). We say that \( f \in X_{\text{loc}}(\mathbb{R}) \) if \( f \chi_E \in X(\mathbb{R}) \) for any measurable set \( E \subset \mathbb{R} \) of finite measure. A measurable function \( w : \mathbb{R} \to [0, \infty] \) is referred to as a \textit{weight} if
0 < w(x) < ∞ a.e. on $\mathbb{R}$. Define the mapping $\rho_w : \mathcal{M}^+ \to [0, \infty]$ and the set $X(\mathbb{R}, w)$ by

$$\rho_w(f) := \rho(fw) \quad (f \in \mathcal{M}^+), \quad X(\mathbb{R}, w) := \{ f \in \mathcal{M}^+ : fw \in X(\mathbb{R}) \}.$$  

**Lemma 2.4.** Let $X(\mathbb{R})$ be a Banach function space generated by a Banach function norm $\rho$, let $X'(\mathbb{R})$ be its associate space, and let $w : \mathbb{R} \to [0, \infty]$ be a weight.

(a) The mapping $\rho_w$ satisfies Axioms (A1)-(A3) in Definition 2.1 and $X(\mathbb{R}, w)$ is a linear normed space with respect to the norm

$$\|f\|_{X(\mathbb{R}, w)} := \rho_w(|f|) = \rho(|fw|) = \|fw\|_{X(\mathbb{R})}.$$  

(b) If $w \in X_{loc}(\mathbb{R})$ and $1/w \in X'_{loc}(\mathbb{R})$, then $\rho_w$ is a Banach function norm and $X(\mathbb{R}, w)$ is a Banach function space generated by $\rho_w$.

(c) If $w \in X_{loc}(\mathbb{R})$ and $1/w \in X'_{loc}(\mathbb{R})$, then $X'(\mathbb{R}, w^{-1})$ is the associate space for the Banach function space $X(\mathbb{R}, w)$.

**Proof.** The proof is analogous to that one of [8, Lemma 2.5].

Part (a) follows from Axioms (A1)-(A3) for the Banach function norm $\rho$ and the fact that $0 < w(x) < \infty$ almost everywhere on $\mathbb{R}$.

(b) If $w \in X_{loc}(\mathbb{R})$, then $w\chi_E \in X(\mathbb{R})$ for every measurable set $E \subset \mathbb{R}$ of finite measure. Therefore $\rho_w(\chi_E) = \rho(w\chi_E) < \infty$. Then $\rho_w$ satisfies Axiom (A4).

Since $1/w \in X'_{loc}(\mathbb{R})$, we have $C_E := \rho'(\chi_E/w) < \infty$ for every measurable set $E \subset \mathbb{R}$ of finite measure. On the other hand, by Axiom (A2), for $f \in \mathcal{M}^+$ we have $\rho(fw\chi_E) \leq \rho(fw) = \rho_w(f)$. By Hölder’s inequality for $\rho$ (Lemma 2.2), we obtain

$$\int_E f(x) \, dx = \int_\mathbb{R} f(x)w(x)\chi_E(x) \frac{\chi_E(x)}{w(x)} \, dx \leq \rho(fw\chi_E)\rho'(\chi_E/w) \leq C_E \rho_w(f).$$

Thus $\rho_w$ satisfies Axiom (A5), that is, $X(\mathbb{R}, w)$ is a Banach function space. Part (b) is proved.

(c) For $g \in \mathcal{M}^+$, we have

$$(\rho_w)'(g) = \sup \left\{ \int_\mathbb{R} f(x)g(x) \, dx : f \in \mathcal{M}^+, \rho_w(f) \leq 1 \right\}$$

$$= \sup \left\{ \int_\mathbb{R} (f(x)w(x)) \left( \frac{g(x)}{w(x)} \right) \, dx : f \in \mathcal{M}^+, \rho(fw) \leq 1 \right\}$$

$$= \sup \left\{ \int_\mathbb{R} h(x) \left( \frac{g(x)}{w(x)} \right) \, dx : h \in \mathcal{M}^+, \rho(h) \leq 1 \right\}$$

$$= \rho'(g/w).$$

Hence $(X(\mathbb{R}, w))' = X'(\mathbb{R}, w^{-1})$. □

From Lemma 2.3 and the Lorentz-Luxemburg theorem (see e.g. [2, Chap. 1, Theorem 2.7]) we obtain the following.
Lemma 2.5. Let $X(\mathbb{R})$ be a Banach function space and $w : \mathbb{R} \to [0, \infty]$ be a weight such that $w \in X_{loc}(\mathbb{R})$ and $1/w \in X'_{loc}(\mathbb{R})$. Then

$$
\|f\|_{X(\mathbb{R}, w)} = \sup \left\{ \int_{\mathbb{R}} |f(x)g(x)| \, dx : g \in X'(\mathbb{R}, w^{-1}), \|g\|_{X'(\mathbb{R}, w^{-1})} \leq 1 \right\} \quad (2.1)
$$
for all $f \in X(\mathbb{R}, w)$ and

$$
\|g\|_{X'(\mathbb{R}, w^{-1})} = \sup \left\{ \int_{\mathbb{R}} |f(x)g(x)| \, dx : f \in X(\mathbb{R}, w), \|f\|_{X(\mathbb{R}, w)} \leq 1 \right\} \quad (2.2)
$$
for all $g \in X'(\mathbb{R}, w^{-1})$.

2.3. Reflexivity of weighted Banach function spaces

A function $f$ in a Banach function space $X(\mathbb{R})$ is said to have absolutely continuous norm in $X(\mathbb{R})$ if $\|f\chi_{E_n}\|_X(\mathbb{R}) \to 0$ for every sequence $\{E_n\}_{n=1}^\infty$ of measurable sets on $\mathbb{R}$ satisfying $\chi_{E_n} \to 0$ a.e. on $\mathbb{R}$ as $n \to \infty$. If all functions $f \in X(\mathbb{R})$ have this property, then the space $X(\mathbb{R})$ itself is said to have absolutely continuous norm (see [2, Chap. 1, Section 3]).

Lemma 2.6 ([2, Chap. 1, Lemma 3.4]). Let $X(\mathbb{R})$ be a Banach function space. If $f \in X(\mathbb{R})$ has absolutely continuous norm, then to each $\varepsilon > 0$ there corresponds $\delta > 0$ such that $|E| \leq \delta$ implies $\|f\chi_E\|_X(\mathbb{R}) < \varepsilon$.

Lemma 2.7. Let $X(\mathbb{R})$ be a Banach function space and $w : \mathbb{R} \to [0, \infty]$ be a weight such that $w \in X_{loc}(\mathbb{R})$ and $1/w \in X'_{loc}(\mathbb{R})$. If $X(\mathbb{R})$ has absolutely continuous norm, then $X(\mathbb{R}, w)$ has absolutely continuous norm too.

Proof. The proof is a literal repetition of that one of [8, Proposition 2.6]. By Lemma [2,4(b)], $X(\mathbb{R}, w)$ is a Banach function space. If $f \in X(\mathbb{R}, w)$, then $fw \in X(\mathbb{R})$ has absolutely continuous norm in $X(\mathbb{R})$. Therefore,

$$
\|fw\chi_{E_n}\|_{X(\mathbb{R}, w)} = \|fw\chi_{E_n}\|_{X(\mathbb{R})} \to 0
$$
for every sequence $\{E_n\}_{n=1}^\infty$ of measurable sets on $\mathbb{R}$ satisfying $\chi_{E_n} \to 0$ a.e. on $\mathbb{R}$ as $n \to \infty$. Thus, $f \in X(\mathbb{R}, w)$ has absolutely continuous norm in $X(\mathbb{R}, w)$.

From Lemma [2,4] and [2, Chap. 1, Corollaries 4.3, 4.4] we obtain the following.

Lemma 2.8. Let $X(\mathbb{R})$ be a Banach function space and $w : \mathbb{R} \to [0, \infty]$ be a weight such that $w \in X_{loc}(\mathbb{R})$ and $1/w \in X'_{loc}(\mathbb{R})$.

(a) The Banach space dual $[X(\mathbb{R}, w)]^*$ of the weighted Banach function space $X(\mathbb{R}, w)$ is isometrically isomorphic to the associate space $X'(\mathbb{R}, w^{-1})$ if and only if $X(\mathbb{R}, w)$ has absolutely continuous norm. If this is the case, then the general form of a linear functional on $X(\mathbb{R}, w)$ is given by

$$
G(f) := \int_{\mathbb{R}} f(x)g(x) \, dx \quad \text{for} \quad g \in X'(\mathbb{R}, w^{-1})
$$
and $\|G\|_{[X(\mathbb{R}, w)]^*} = \|g\|_{X'(\mathbb{R}, w^{-1})}$.

(b) The weighted Banach function space $X(\mathbb{R}, w)$ is reflexive if and only if both $X(\mathbb{R}, w)$ and $X'(\mathbb{R}, w^{-1})$ have absolutely continuous norm.
Corollary 2.9. Let $X(\mathbb{R})$ be a Banach function space and $w: \mathbb{R} \to [0, \infty]$ be a weight such that $w \in X_{\text{loc}}(\mathbb{R})$ and $1/w \in X'_{\text{loc}}(\mathbb{R})$. If $X(\mathbb{R})$ is reflexive, then $X(\mathbb{R}, w)$ is reflexive.

Proof. The proof is a literal repetition of that one of [8] Corollary 2.8. If $X(\mathbb{R})$ is reflexive, then, by [2] Chap. 1, Corollary 4.4, both $X(\mathbb{R})$ and $X'(\mathbb{R})$ have absolutely continuous norm. In that case, due to Lemma 2.7, both $X(\mathbb{R}, w)$ and $X'(\mathbb{R}, w^{-1})$ have absolutely continuous norm. By Lemma 2.8(b), $X(\mathbb{R}, w)$ is reflexive. \hfill $\blacksquare$

2.4. Density of smooth compactly supported functions

For a subset $Y$ of $L^\infty(\mathbb{R})$, let $Y_0$ denote the set of all compactly supported functions in $Y$.

Lemma 2.10. Let $X(\mathbb{R})$ be a Banach function space.

(a) $L^\infty_0(\mathbb{R}) \subset X(\mathbb{R})$.

(b) If $X(\mathbb{R})$ has absolutely continuous norm, then $L^\infty_0(\mathbb{R})$, $C_0(\mathbb{R})$, and $C_0^\infty(\mathbb{R})$ are dense in $X(\mathbb{R})$.

Proof. Part (a) follows from the definition of a Banach function space.

Part (b) follows from Lemma 2.7. From [2] Chap. 1, Proposition 3.10 and Theorem 3.11 it follows that $L^\infty_0(\mathbb{R})$ is dense in $X(\mathbb{R})$.

Let us show that each function $u \in L^\infty_0(\mathbb{R})$ can be approximated by a function from $C_0(\mathbb{R})$ in the norm of $X(\mathbb{R})$. We have $\text{supp} \ u \subset Q$ and $|u(x)| \leq a$ for almost all $x \in \mathbb{R}$, where $Q$ is some finite closed segment and $a > 0$. By Axiom (A4), $\chi_Q \in X(\mathbb{R})$ and $\chi_Q$ has absolutely continuous norm by the hypothesis. From Lemma 2.6 it follows that for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|E| < \delta$ implies that $\|\chi_Q 1_E\|_{X(\mathbb{R})} < \varepsilon$. By Luzin’s theorem, for such a $\delta > 0$ there is a continuous function $v$ supported in $Q$ such that $|v(x)| \leq a$ and the measure of the set $\tilde{Q} := \{x \in Q : u(x) \neq v(x)\}$ is less than $\delta$. Then

$$|u(x) - v(x)| \leq 2a\chi_Q(x) \quad (x \in \mathbb{R}).$$

Therefore, by Axiom (A2),

$$\|u - v\|_{X(\mathbb{R})} \leq 2a\|\chi_Q\chi_{\tilde{Q}}\|_{X(\mathbb{R})} < 2a\varepsilon.$$ 

Hence, each function $u \in L^\infty_0(\mathbb{R})$ can be approximated by a function from $C_0(\mathbb{R})$ in the norm of $X(\mathbb{R})$. Thus, $C_0(\mathbb{R})$ is dense in $X(\mathbb{R})$.

Now let us prove that each function $v \in C_0(\mathbb{R})$ can be approximated by a function from $C_0^\infty(\mathbb{R})$ in the norm of $X(\mathbb{R})$. Let $a \in C_0^\infty(\mathbb{R})$ and $\int_\mathbb{R} a(x)dx = 1$.

Consider

$$v_t(x) = \frac{1}{t} \int_\mathbb{R} a\left(\frac{y}{t}\right) v(x - y)dy \quad (t > 0).$$

It is easy to see that $v_t \in C_0^\infty(\mathbb{R})$. Fix an interval $Q$ containing the supports of $v$ and $v_t$. Then for every $\varepsilon > 0$ there is a $t > 0$ such that $|v_t(x) - v(x)| < \varepsilon$ for all $x \in Q$. Hence,

$$\|v_t - v\|_{X(\mathbb{R})} = \|(v_t - v)\chi_Q\|_{X(\mathbb{R})} < \varepsilon \|\chi_Q\|_{X(\mathbb{R})},$$

where $\chi_Q(x)$ is the characteristic function of $Q$.
that is, \( v \in C_0^0(\mathbb{R}) \) can be approximated by a function from \( C_0^\infty(\mathbb{R}) \) in the norm of \( X(\mathbb{R}) \). Thus, \( C_0^\infty(\mathbb{R}) \) is dense in \( X(\mathbb{R}) \). \( \square \)

From [2, Chap. 1, Corollary 5.6] one can extract the following.

**Lemma 2.11.** A Banach function space \( X(\mathbb{R}) \) is separable if and only if it has absolutely continuous norm.

Gathering the results mentioned above, we arrive at the next result.

**Lemma 2.12.** Let \( X(\mathbb{R}) \) be a Banach function space and \( w : \mathbb{R} \to [0, \infty] \) be a weight such that \( w \in X_{loc}^0(\mathbb{R}) \) and \( 1/w \in X_{loc}^0(\mathbb{R}) \).

(a) If \( X(\mathbb{R}) \) is separable, then \( L_0^\infty(\mathbb{R}), C_0(\mathbb{R}), \) and \( C_0^\infty(\mathbb{R}) \) are dense in the weighted Banach function spaces \( X(\mathbb{R}, w) \) and \( X'(\mathbb{R}, w^{-1}) \).

(b) If \( X(\mathbb{R}) \) is reflexive, then \( L_0^\infty(\mathbb{R}), C_0(\mathbb{R}), \) and \( C_0^\infty(\mathbb{R}) \) are dense in the weighted Banach function spaces \( X(\mathbb{R}, w) \) and \( X'(\mathbb{R}, w^{-1}) \).

**Proof.** (a) If \( X(\mathbb{R}) \) is separable, then by Lemma 2.11, \( X(\mathbb{R}) \) has absolutely continuous norm. Therefore \( X(\mathbb{R}, w) \) has absolutely continuous norm too, in view of Lemma 2.7. Hence, from Lemma 2.10(b) we derive that \( L_0^\infty(\mathbb{R}), C_0(\mathbb{R}), \) and \( C_0^\infty(\mathbb{R}) \) are dense in \( X(\mathbb{R}, w) \). Part (a) is proved.

(b) If \( X(\mathbb{R}) \) is reflexive, then by [2, Chap. 1, Corollary 4.4], both \( X(\mathbb{R}) \) and \( X'(\mathbb{R}) \) have absolutely continuous norm. Hence both \( X(\mathbb{R}, w) \) and \( X'(\mathbb{R}, w^{-1}) \) have absolutely continuous norm in view of Lemma 2.7. Thus, from Lemma 2.10(b) we get that \( L_0^\infty(\mathbb{R}), C_0(\mathbb{R}), \) and \( C_0^\infty(\mathbb{R}) \) are dense in \( X(\mathbb{R}, w) \) and in \( X'(\mathbb{R}, w^{-1}) \). \( \square \)

3. Boundedness of the Cauchy singular integral operator on weighted Banach function spaces

3.1. Well-known properties of the Cauchy singular integral operator

One says that a linear operator \( T \) from \( L^1(\mathbb{R}) \) into the space of complex-valued measurable functions on \( \mathbb{R} \) is of weak type \((1, 1)\) if for every \( \alpha > 0 \),

\[
|\{ x \in \mathbb{R} : |(Tf)(x)| > \alpha \}| \leq \frac{C_K}{\alpha} \| f \|_{L^1(\mathbb{R})}
\]

with some absolute constant \( C_K > 0 \).

The following results are proved in many standard texts on Harmonic Analysis, see e.g. [2, Chap. 3, Theorem 4.9(b)] or [6, pp. 51–52].

**Theorem 3.1.** (a) (Kolmogorov). The Cauchy singular integral operator \( S \) is of weak type \((1, 1)\).

(b) (M. Riesz). The Cauchy singular integral operator \( S \) is bounded on \( L^p(\mathbb{R}) \) for every \( p \in (1, \infty) \).
Theorem 3.2. If \( f, g \in L^2(\mathbb{R}) \), then
\[
(S^2 f)(x) = f(x) \quad (x \in \mathbb{R}),
\]
\[
\int_{\mathbb{R}} (Sf)(x)g(x) \, dx = \int_{\mathbb{R}} f(x)(Sg)(x) \, dx.
\]

3.2. Pointwise estimates for sharp maximal functions

For \( \delta > 0 \) and \( f \in L^\delta_{\text{loc}}(\mathbb{R}) \), set
\[
f^\#(x) := \sup \inf_{Q \ni x} \left( \frac{1}{|Q|} \int_Q |f(y) - c|^{\delta} \, dy \right)^{1/\delta}.
\]
The non-increasing rearrangement (see, e.g., [2, Chap. 2, Section 1]) of a measurable function \( f \) on \( \mathbb{R} \) is defined by
\[
f^*(t) := \inf \left\{ \lambda > 0 : |\{x \in \mathbb{R} : |f(x)| > \lambda\}| \leq t \right\} \quad (0 < t < \infty).
\]
For a fixed \( \lambda \in (0, 1) \) and a given measurable function \( f \) on \( \mathbb{R} \), consider the local sharp maximal function \( M^\#_\lambda f \) defined by
\[
M^\#_\lambda f(x) := \sup \inf_{Q \ni x} \left( (f - c) \chi_Q \right)^* (\lambda|Q|).
\]
In all above definitions the suprema are taken over all intervals \( Q \subset \mathbb{R} \) containing \( x \).

The following result was proved by Lerner [16, Theorem 1] for the case of \( \mathbb{R}^n \).

Theorem 3.3 (Lerner). For a function \( g \in L^1_{\text{loc}}(\mathbb{R}) \) and a measurable function \( \varphi \) satisfying
\[
|\{x \in \mathbb{R} : |\varphi(x)| > \alpha\}| < \infty \quad \text{for all} \quad \alpha > 0,
\]
one has
\[
\int_{\mathbb{R}} |\varphi(x)g(x)| \, dx \leq C_L \int_{\mathbb{R}} M^\#_\lambda \varphi(x) Mg(x) \, dx,
\]
where \( C_L > 0 \) and \( \lambda \in (0, 1) \) are some absolute constants.

The sharp maximal functions can be related as follows.

Lemma 3.4 ([9, Proposition 2.3]). If \( \delta > 0, \lambda \in (0, 1), \) and \( f \in L^\delta_{\text{loc}}(\mathbb{R}) \), then
\[
M^\#_\lambda f(x) \leq (1/\lambda)^{1/\delta} f^\#(x) \quad (x \in \mathbb{R}).
\]

The following estimate was proved in [1] Theorem 2.1 for the case of Calderón-Zygmund singular integral operators with standard kernels in the sense of Coifman and Meyer on \( \mathbb{R}^n \). It is well known that the Cauchy kernel is an archetypical example of a standard kernel (see e.g. [6, p. 99]).

Theorem 3.5 (Álvarez-Pérez). If \( 0 < \delta < 1 \), then for every \( f \in C_0^\infty(\mathbb{R}) \),
\[
(S f)^\#(x) \leq C_\delta M f(x) \quad (x \in \mathbb{R})
\]
where \( C_\delta > 0 \) is some constant depending only on \( \delta \).
### 3.3. Sufficient condition

The set of all bounded sublinear operators on a Banach function space $Y(\mathbb{R})$ will be denoted by $\mathcal{B}(Y(\mathbb{R}))$ and its subset of all bounded linear operators will be denoted by $\mathcal{B}(\mathbb{R})$.

**Theorem 3.6.** Let $X(\mathbb{R})$ be a separable Banach function space and $w : \mathbb{R} \to [0, \infty]$ be a weight such that $w \in X_{\text{loc}}(\mathbb{R})$ and $1/w \in X'_{\text{loc}}(\mathbb{R})$. Suppose the Hardy-Littlewood maximal operator $M$ is bounded on $X(\mathbb{R}, w)$ and on $X'(\mathbb{R}, w^{-1})$. Assume that $0 < \delta < 1$ and $T$ is an operator such that

(a) $T$ is of weak type $(1, 1)$;
(b) $T$ is bounded on some $L^p(\mathbb{R})$ with $p \in (1, \infty)$;
(c) for each $f \in C_0^\infty(\mathbb{R})$,

$$
(Tf)^\#(x) \leq C_\delta Mf(x) \quad (x \in \mathbb{R})
$$

where $C_\delta$ is a positive constant depending only on $\delta$.

Then $T \in \mathcal{B}(X(\mathbb{R}, w))$ and

$$
||T||_{\mathcal{B}(X(\mathbb{R}, w))} \leq (1/\lambda)^4 C_L \|M\|_{\mathcal{B}(X(\mathbb{R}, w))} \|M\|_{\mathcal{B}(X'(\mathbb{R}, w^{-1}))} C_\delta,
$$

(3.4)

where $\lambda \in (0, 1)$ and $C_L > 0$ are the constants from Theorem 3.5.

**Proof.** The idea of the proof is borrowed from [27, Theorem 2.7]. By Lemma 2.4, $X(\mathbb{R}, w)$ is a Banach function space whose associate space is $X'(\mathbb{R}, w^{-1})$. Let $f \in C_0^\infty(\mathbb{R})$ and $g \in X'(\mathbb{R}, w^{-1}) \subset L^1_{\text{loc}}(\mathbb{R})$. Taking into account that $T$ is of weak type $(1, 1)$, we see that $Tf$ satisfies (3.3). From Theorem 3.3 we get that there exist constants $\lambda \in (0, 1)$ and $C_L > 0$ independent of $f$ and $g$ such that

$$
\int_\mathbb{R} |(Tf)(x)g(x)| \, dx \leq C_L \int_\mathbb{R} M_\delta^\#(Tf)(x)Mg(x) \, dx.
$$

(3.5)

Since $T$ is bounded on some standard Lebesgue space $L^p(\mathbb{R})$ for $1 < p < \infty$ and $L^s(J) \subset L^r(J)$ whenever $0 < r < s < \infty$ and $J$ is a finite interval, we see that $Tf \in L^1_{\text{loc}}(\mathbb{R})$ for each $\delta \in (0, p]$. From Lemma 3.4 and hypothesis (c) it follows that

$$
M_\delta^\#(Tf)(x) \leq (1/\lambda)^{1/\delta} (Tf)^\#(x) \leq (1/\lambda)^{1/\delta} C_\delta Mf(x) \quad (x \in \mathbb{R})
$$

(3.6)

for some $\delta \in (0, 1)$. Combining (3.5) and (3.6) with Hölder’s inequality (see Lemma 2.2), we obtain

$$
\int_\mathbb{R} |(Tf)(x)g(x)| \, dx \leq C_1 \int_\mathbb{R} Mf(x)Mg(x) \, dx
$$

$$
\leq C_1 \|Mf\|_{X(\mathbb{R}, w)} \|Mg\|_{X'(\mathbb{R}, w^{-1})},
$$

(3.7)

where $C_1 := (1/\lambda)^{1/\delta} C_\delta C_L > 0$ is independent of $f \in C_0^\infty(\mathbb{R})$ and $g \in X'(\mathbb{R}, w^{-1})$.

Taking into account that $M$ is bounded on $X(\mathbb{R}, w)$ and on $X'(\mathbb{R}, w^{-1})$, from (3.7) and we get

$$
\int_\mathbb{R} |(Tf)(x)g(x)| \, dx \leq C_2 \|f\|_{X(\mathbb{R}, w)} \|g\|_{X'(\mathbb{R}, w^{-1})},
$$

(3.7)
where $C_2 := C_1 \|M\|_{\mathcal{B}(X(\mathbb{R}, w))} \|M\|_{\mathcal{B}(X'(\mathbb{R}, w^{-1}))}$. From this inequality and (2.1) we obtain

$$
\|Tf\|_{X(\mathbb{R}, w)} = \sup \left\{ \int_{\mathbb{R}} |(Tf)(x)| g(x) \, dx : g \in X'(\mathbb{R}, w^{-1}), \|g\|_{X'(\mathbb{R}, w^{-1})} \leq 1 \right\}
\leq C_2 \|f\|_{X(\mathbb{R}, w)}
$$

for all $f \in C_0^\infty(\mathbb{R})$. Taking into account that $C_0^\infty(\mathbb{R})$ is dense in $X(\mathbb{R}, w)$ in view of Lemma 2.12(a), from the latter inequality it follows that $T$ is bounded on $X(\mathbb{R}, w)$ and (3.4) holds.  

\[\square\]

**Remark 3.7.** The proof of this result without changes extends to the case of $\mathbb{R}^n$.

**Theorem 3.8.** Let $X(\mathbb{R})$ be a Banach function space and $w : \mathbb{R} \to [0, \infty]$ be a weight such that $w \in X_{\text{loc}}(\mathbb{R})$ and $1/w \in X_{\text{loc}}'(\mathbb{R})$. Suppose the Hardy-Littlewood maximal operator $M$ is bounded on $X(\mathbb{R}, w)$ and on $X'(\mathbb{R}, w^{-1})$.

(a) If the space $X(\mathbb{R})$ is separable, then the Cauchy singular integral operator $S$ is bounded on the space $X(\mathbb{R}, w)$ and $S^2 = I$.

(b) If the space $X(\mathbb{R})$ is reflexive, then the Cauchy singular integral operator $S$ is bounded on the spaces $X(\mathbb{R}, w)$ and $X'(\mathbb{R}, w^{-1})$ and its adjoint $S^*$ coincides with $S$ on the space $X'(\mathbb{R}, w^{-1})$.

**Proof.** From Theorems 3.1 and 3.5 it follows that all hypotheses of Theorem 3.6 are fulfilled. Hence, the operator $S$ is bounded on $X(\mathbb{R}, w)$.

Let now $\varphi \in X(\mathbb{R}, w)$. Then there exists a sequence $f_n \in L_0^\infty(\mathbb{R})$ such that $f_n \rightarrow \varphi$ in $X(\mathbb{R}, w)$ as $n \rightarrow \infty$. From (3.1) we get $S^2 f_n = f_n$ because $L_0^\infty(\mathbb{R}) \subset L^2(\mathbb{R})$. Hence

$$
\|S^2 \varphi - \varphi\|_{X(\mathbb{R}, w)} \leq \|S^2 \varphi - f_n\|_{X(\mathbb{R}, w)} + \|f_n - \varphi\|_{X(\mathbb{R}, w)}
= \|S^2 (\varphi - f_n)\|_{X(\mathbb{R}, w)} + \|\varphi - f_n\|_{X(\mathbb{R}, w)}
\leq (\|S^2 \|_{\mathcal{B}(X(\mathbb{R}, w))} + 1) \|\varphi - f_n\|_{X(\mathbb{R}, w)} \rightarrow 0
$$
as $n \rightarrow \infty$. Thus $S^2 \varphi = \varphi$. Part (a) is proved.

(b) From (3.2) it follows that

$$
\int_{\mathbb{R}} (Sf)(x) \overline{g(x)} \, dx = \int_{\mathbb{R}} f(x) \overline{(Sg)(x)} \, dx.
$$
for all $f, g \in L_0^\infty(\mathbb{R})$. From this equality and Lemmas 2.8 and 2.12(b) it follows that $S$ is a self-adjoint and densely defined operator on $X(\mathbb{R}, w)$ and $X'(\mathbb{R}, w^{-1})$. By the standard argument (see [10] Chap. III, Section 5.5), one can show that $S = S^* \in \mathcal{B}(X'(\mathbb{R}, w^{-1}))$ because $S \in \mathcal{B}(X(\mathbb{R}, w))$ by part (a).  

\[\square\]
3.4. Necessary condition

Let \( X(\mathbb{R}) \) be a Banach function space and \( X'(\mathbb{R}) \) be its associate space. We say that a weight \( w: \mathbb{R} \to [0, \infty) \) belongs to the class \( A_X(\mathbb{R}) \) if

\[
\sup_{-\infty < a < b < \infty} \frac{1}{b - a} \| w \chi_{(a,b)} \|_{X(\mathbb{R})} \| w^{-1} \chi_{(a,b)} \|_{X'(\mathbb{R})} < \infty.
\]

**Theorem 3.9.** Let \( X(\mathbb{R}) \) be a Banach function space and \( w: \mathbb{R} \to [0, \infty) \) be a weight. If the operator \( S \) is bounded on the space \( X(\mathbb{R}, w) \), then

(a) \( w \in X_{\text{loc}}(\mathbb{R}) \) and \( 1/w \in X'_{\text{loc}}(\mathbb{R}) \);

(b) \( X(\mathbb{R}, w) \) is a Banach function space;

(c) \( w \in A_X(\mathbb{R}) \).

**Proof.** (a) The idea of the proof is borrowed from [7, Lemma 3.3]. Let \( E \subset \mathbb{R} \) be a measurable set of finite measure. Then there exist \( a, b \in \mathbb{R} \) such that \( E \subset (a, b) =: J \). It is clear that

\[
w(x) \chi_E(x) \leq w(x) \chi_J(x), \quad \chi_E(x)/w(x) \leq \chi_J(x)/w(x)
\]

for almost all \( x \in \mathbb{R} \). Then by Axiom (A2),

\[
\| w \chi_E \|_{X(\mathbb{R})} \leq \| w \chi_J \|_{X(\mathbb{R})}, \quad \| \chi_E/w \|_{X'(\mathbb{R})} \leq \| \chi_J/w \|_{X'(\mathbb{R})}.
\]

Thus, it is sufficient to prove that \( w \chi_J \in X(\mathbb{R}) \) and \( \chi_J/w \in X'(\mathbb{R}) \).

Obviously, the operator \( (Vf)(x) = \chi_J(x)f(x) \) is bounded on \( X(\mathbb{R}, w) \) and

\[
((SV - VS)f)(x) = \frac{1}{\pi} \int_J f(y) \, dy
\]

for almost all \( x \in \mathbb{R} \). Since the operator \( SV - VS \) is bounded on \( X(\mathbb{R}, w) \), there exists a constant \( C_1 > 0 \) such that

\[
\left\| \frac{1}{\pi} \int_J f(y) \, dy \right\|_{X(\mathbb{R}, w)} \leq C_1 \| f \|_{X(\mathbb{R}, w)} \quad \text{for all} \quad f \in X(\mathbb{R}, w). \tag{3.8}
\]

On the other hand,

\[
\left\| \frac{1}{\pi} \int_J f(y) \, dy \right\|_{X(\mathbb{R}, w)} = \frac{1}{\pi} \left| \int_J f(y) \, dy \right| \| w \chi_J \|_{X(\mathbb{R})}. \tag{3.9}
\]

Since \( w(x) > 0 \) a.e. on \( \mathbb{R} \), we have \( \| w \chi_J \|_{X(\mathbb{R})} > 0 \). Hence, from (3.8) and (3.9) it follows that

\[
\left| \int_J f(y) \, dy \right| \leq \frac{C_1 \pi}{\| w \chi_J \|_{X(\mathbb{R})}} \| f \|_{X(\mathbb{R}, w)}.
\]

Therefore,

\[
\left| \int_{\mathbb{R}} f(y) w(y) \cdot \frac{\chi_J(y)}{w(y)} \, dy \right| \leq \frac{C_1 \pi}{\| w \chi_J \|_{X(\mathbb{R})}} \| fw \|_{X(\mathbb{R})}
\]

for all measurable functions \( f \) such that \( fw \in X(\mathbb{R}) \). By Lemma 2.3, we have \( \chi_J/w \in X'(\mathbb{R}) \).
Let us show that there exists a function \( g_0 \in X(\mathbb{R}) \) such that
\[
C_2 := \frac{1}{\pi} \left| \int_J \frac{g_0(y)}{w(y)} dy \right| > 0. \tag{3.10}
\]
Assume the contrary. Then, taking into account Lemma 2.10(a), we obtain
\[
\int_J \frac{g(y)}{w(y)} dy = 0 \tag{3.11}
\]
for all \( g \) continuous on \( J \). By Axiom (A5), \( (1/w) \chi_J \in L^1(J) \). Without loss of generality, assume that \( |J| = 2\pi \). Let \( \eta : [0, 2\pi] \to J \) be a homeomorphism such that \( |\eta'(x)| = 1 \) for almost all \( x \in [0, 2\pi] \). From (3.11) we get
\[
\int_0^{2\pi} \varphi(x) \frac{\eta(x)}{w(\eta(x))} dx = 0 \quad \text{for all} \quad \varphi \in C[0, 2\pi]. \tag{3.12}
\]
Taking \( \varphi(x) = e^{inx} \) with \( n \in \mathbb{Z} \), we see from (3.12) that all Fourier coefficients of \( 1/(w \circ \eta) \) vanish. This implies that \( 1/w(\eta(x)) = 0 \) for almost all \( x \in [0, 2\pi] \). Consequently, \( w(y) = \infty \) almost everywhere on \( J \). This contradicts the assumption that \( w \) is a weight. Thus, \( C_2 > 0 \).

Clearly, \( f_0 = g_0/w \in X(\Gamma, w) \). Then from (3.8)–(3.10) it follows that
\[
\|w \chi_J\|_{X(\mathbb{R})} \leq \frac{C_1}{C_2} \|f_0\|_{X(\mathbb{R}, w)},
\]
that is, \( w \chi_J \in X(\mathbb{R}) \). Part (a) is proved.

Part (b) follows from part (a) and Lemma 2.4(b).

(c) The idea of the proof is borrowed from [7, Theorem 3.2]. By part(b), \( X(\mathbb{R}, w) \) is a Banach function space.

Let \( Q \) be an arbitrary interval and \( Q_1, Q_2 \) be its two halves. Take a function \( f \geq 0 \) supported in \( Q_1 \). Then for \( \tau \in Q_1 \) and \( x \in Q_2 \) we have \( |\tau - x| \leq |Q| \).

Therefore,
\[
|(Sf)(x)| \leq \frac{1}{\pi} \left| \int_{Q_1} \frac{f(\tau)}{\tau-x} d\tau \right| = \frac{1}{\pi} \int_{Q_1} \frac{f(\tau)}{|\tau-x|} d\tau \geq \frac{1}{\pi |Q_1|} \int_{Q_1} f(\tau) d\tau = \frac{1}{2\pi |Q_1|} \int_{Q_1} f(\tau) d\tau.
\]

Thus,
\[
|(Sf)(x)| \chi_{Q_2}(x) \geq \frac{1}{2\pi |Q_1|} \left( \int_{Q_1} f(\tau) d\tau \right) \chi_{Q_2}(x) \quad (x \in \mathbb{R}).
\]

Then, by Axioms (A1) and (A2),
\[
\|Sf\|_{X(\mathbb{R}, w)} \geq \|(Sf)\chi_{Q_2}\|_{X(\mathbb{R}, w)} \geq \frac{1}{2\pi |Q_1|} \left( \int_{Q_2} f(\tau) d\tau \right) \|\chi_{Q_2}\|_{X(\mathbb{R}, w)}. \tag{3.13}
\]

On the other hand, since \( S \) is bounded on \( X(\mathbb{R}, w) \), we get
\[
\|Sf\|_{X(\mathbb{R}, w)} \leq \|S\|_{B(X(\mathbb{R}, w))} \|f\|_{X(\mathbb{R}, w)} = \|S\|_{B(X(\mathbb{R}, w))} \|f\chi_{Q_1}\|_{X(\mathbb{R}, w)}. \tag{3.14}
\]
Combining (3.13) and (3.14), we arrive at
\[
\frac{1}{|Q_1|} \left( \int_{Q_1} f(\tau) \, d\tau \right) \|w\chi_{Q_2}\|_{X(\mathbb{R})} \leq 2\pi \|S\|_{B(X(\mathbb{R}, w))} \|f\chi_{Q_1}\|_{X(\mathbb{R}, w)}.
\] (3.15)

Taking \( f = \chi_{Q_1} \), from (3.15) we get
\[
\|w\chi_{Q_2}\|_{X(\mathbb{R})} \leq 2\pi \|S\|_{B(X(\mathbb{R}, w))} \|w\chi_{Q_1}\|_{X(\mathbb{R})}.
\] Analogously one can obtain
\[
\|w\chi_{Q_1}\|_{X(\mathbb{R})} \leq 2\pi \|S\|_{B(X(\mathbb{R}, w))} \|w\chi_{Q_2}\|_{X(\mathbb{R})}.
\] (3.16)

From (3.15) and (3.16) it follows that
\[
\frac{1}{|Q_1|} \left( \int_{Q_1} f(\tau) \, d\tau \right) \|w\chi_{Q_1}\|_{X(\mathbb{R})} \leq C \|f\chi_{Q_1}\|_{X(\mathbb{R}, w)},
\] (3.17)

where \( C := (2\pi \|S\|_{B(X(\mathbb{R}, w))})^2 \). Let
\[
Y := \{ g \in X(\mathbb{R}, w) : \|g\|_{X(\mathbb{R}, w)} \leq 1 \}.
\]

If \( g \in Y \), then \( |g|\chi_{Q_1} \geq 0 \) is supported in \( Q_1 \). Then from (3.17) we obtain
\[
\|w\chi_{Q_1}\|_{X(\mathbb{R})} \int_{\mathbb{R}} |g(\tau)|\chi_{Q_1}(\tau) \, d\tau \leq C|Q_1|
\] (3.18)

for all \( g \in Y \). From (2.22) we get
\[
\|w^{-1}\chi_{Q_1}\|_{X(\mathbb{R})} = \|\chi_{Q_1}\|_{X(\mathbb{R}, w^{-1})} = \sup_{g \in Y} \int_{\mathbb{R}} |g(\tau)|\chi_{Q_1}(\tau) \, d\tau.
\] (3.19)

From (3.18) and (3.19) it follows that
\[
\|w\chi_{Q_2}\|_{X(\mathbb{R})} \|w^{-1}\chi_{Q_1}\|_{X(\mathbb{R})} \leq C|Q_1|.
\]
Since \( Q_1 \subset \mathbb{R} \) is an arbitrary interval, we conclude that \( w \in A_X(\mathbb{R}) \). \( \square \)

3.5. The case of weighted variable Lebesgue spaces

We start this subsection with the following well-known fact.

Theorem 3.10 ([5 Theorems 3.2.13 and 3.4.7]). Let \( p : \mathbb{R} \to [1, \infty] \) be a measurable a.e. finite function satisfying (1.1). Then \( L^{p(\cdot)}(\mathbb{R}) \) is a separable and reflexive Banach function space whose associate space is isomorphic to \( L^{p(\cdot)}(\mathbb{R}) \).

Now we are in a position to give a proof of Theorem 1.3.

Proof of Theorem 1.3. Necessity. Theorem 3.10 immediately implies that if \( p \) satisfies (1.1), then \( L^{p(\cdot)}(\mathbb{R}) \) is a Banach function space and
\[
A_{p(\cdot)}(\mathbb{R}) = A_{L^{p(\cdot)}(\mathbb{R})}.
\]

From Theorem 3.9 it follows that if \( S \) is bounded on the space \( L^{p(\cdot)}(\mathbb{R}, w) \), then \( w \in A_{p(\cdot)}(\mathbb{R}) \). The necessity portion is proved.

Sufficiency. From Theorem 3.10 we know that \( L^{p(\cdot)}(\mathbb{R}) \) is a separable and reflexive Banach function space. If \( w \in A_{p(\cdot)}(\mathbb{R}) \), then \( w \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}) \).
and $1/w \in \mathcal{A}_{p'}(\mathbb{R})$. Further, it is easy to see that $p$ is globally log-Hölder continuous if and only if so is $p'$. Hence, by Theorem 1.2, the Hardy-Littlewood maximal function is bounded on $L^{p'}(\mathbb{R}, w)$ and on $L^{p'}(\mathbb{R}, w^{-1})$. Applying Theorem 3.8(a), we see that the operator $S$ is bounded on $L^{p'}(\mathbb{R}, w)$. This finishes the proof of Theorem 1.3.

Theorem 1.4 follows immediately from Theorems 1.2, 3.8, and 3.10.

References

[1] J. Álvarez and C. Pérész, Estimates with $A_{\infty}$ weights for various singular integral operators. Boll. Un. Mat. Ital. A (7) 8 (1994), 123-133.
[2] C. Bennett and R. Sharpley, Interpolation of Operators. Academic Press, New York, 1988.
[3] E. I. Berezhnoi, Two-weighted estimations for the Hardy-Littlewood maximal function in ideal Banach spaces. Proc. Amer. Math. Soc. 127 (1999), 79-87.
[4] D. Cruz-Uribe, L. Diening, and P. Hästö, The maximal operator on weighted variable Lebesgue spaces. Frac. Calc. Appl. Anal. 14 (2011), 361-374.
[5] L. Diening, P. Harjulehto, P. Hästö, and M. Růžička, Lebesgue and Sobolev Spaces with Variable Exponents. Lecture Notes in Mathematics 2017, Springer, Berlin, 2011.
[6] J. Duoandikoetxea, Fourier Analysis. Graduate Studies in Mathematics 29. American Mathematical Society, Providence, RI, 2001.
[7] A. Yu. Karlovich, Singular integral operators with piecewise continuous coefficients in reflexive rearrangement-invariant spaces. Integral Equations and Operator Theory 32 (1998), 436-481.
[8] A. Yu. Karlovich, Fredholmness of singular integral operators with piecewise continuous coefficients on weighted Banach function spaces. J. Integral Equations Appl. 15 (2003), no. 3, 283-320.
[9] A. Yu. Karlovich and A. K. Lerner, Commutators of singular integrals on generalized $L^p$ spaces with variable exponent. Publ. Mat. 49 (2005), 111-125.
[10] T. Kato, Perturbation Theory for Linear Operators. Reprint of the 1980 edition. Springer-Verlag, Berlin, 1995.
[11] V. Kokilashvili, V. Paatashvili, and S. Samko, Boundedness in Lebesgue spaces with variable exponent of the Cauchy singular operator on Carleson curves. In: “Modern operator theory and applications”. Operator Theory: Advances and Applications 170. Birkhäuser Verlag, Basel, 2006, pp. 167-186.
[12] V. Kokilashvili, N. Samko, and S. Samko, The maximal operator in weighted variable spaces $L^{p(\cdot)}$. J. Funct. Spaces Appl. 5 (2007), 299-317.
[13] V. Kokilashvili, N. Samko, and S. Samko, Singular operators in variable spaces $L^{p(\cdot)}(\Omega, \rho)$ with oscillating weights. Math. Nachr. 280 (2007), 1145-1156.
[14] V. Kokilashvili and S. Samko, Boundedness of maximal operators and potential operators on Carleson curves in Lebesgue spaces with variable exponent. Acta Math. Sin. (Engl. Ser.) 24 (2008), 1775-1800.
[15] T. Kopaliani, *Infimal convolution and Muckenhoupt $A_p(\cdot)$ condition in variable $L^p$ spaces*. Arch. Math. (Basel) 89 (2007), 185-192.

[16] A. K. Lerner, *Weighted norm inequalities for the local sharp maximal function*. J. Fourier Anal. Appl. 10 (2004), 465-474.

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