Generators for maximal subgroups of Conway group $\text{Co}_1$

Abstract: The Conway groups are the three sporadic simple groups $\text{Co}_1$, $\text{Co}_2$ and $\text{Co}_3$. There are total of 22 maximal subgroups of $\text{Co}_1$ and generators of 6 maximal subgroups are provided in web Atlas of finite simple groups. The aim of this paper is to give generators of remaining 16 maximal subgroups.

Keywords: Conway group, maximal subgroup, generators, finite group, normalizer

MSC: 20D05, 20B40

1 Introduction

The Conway group $\text{Co}_1$ is one of the 26 sporadic simple groups. The largest of the Conway groups, $\text{Co}_0$, is the group of automorphisms of the Leech lattice $\Gamma$ with respect to addition and inner product. It has order $8,315,553,613,086,720,000$ [1] but it is not a simple group. The simple group $\text{Co}_1$ of order $2^{21}.3^9.5^4.7^2.11.13.23$ is defined as the quotient of $\text{Co}_0$ by its center, which consists of the scalar matrices $\pm 1$ [1]. The local subgroups of $\text{Co}_1$ are found in [2] and the maximal subgroups of $\text{Co}_1$ in [3]. There is also a valuable discussion in [4].

The following theorem is crucial in determining the maximal subgroups of $\text{Co}_1$:

**Theorem 1.1.** [3] If $K$ is a non-Abelian characteristically simple subgroup of $\text{Co}_1$, then $N_{\text{Co}_1}(K)$ is contained either in a local subgroup of $\text{Co}_1$ or in a conjugate of one of six particular groups:

1. $\text{NA}_5 \cong (A_5 \times J_2).2$,
2. $\text{NA}_6 \cong (A_6 \times U_3(3)).2$,
3. $\text{NA}_7 \cong (A_7 \times L_2(7)).2$,
4. $S(2) \cong \text{Co}_2$,
5. $S(3) \cong \text{Co}_3$,
6. $S(2^3) \cong U_6(2).S_3$.

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Note on notation. We use $A \cdot B$ to denote an arbitrary extension of $A$ by $B$, while $A : B$ and $A : B$ denote split and non-split extensions, respectively. The symbol $n$ denotes a cyclic group of that order, while $[n]$ denotes an arbitrary group of order $n$. We follow the ATLAS [5] notation for conjugacy classes. Moreover, we denote $x^y$ by $x^{-1} y x$ and $[x, y]$ by $x^{-1} y^{-1} x y$.

To facilitate the computations in finite simple and almost simple groups, [6], [7] provides the representations and words for generators of most of the maximal subgroups. However, there are still some cases to deal with. A research problem “Words for maximal subgroups in sporadic groups” appears on the web page of R. A. Wilson. We pursue the work initiated by R.A. Wilson of finding words for maximal subgroups of $Co_1$. According to [3] there are 24 conjugacy classes of maximal subgroups, but later on R.A. Wilson pointed out a few errors (in his own paper) in the list of maximal subgroups of $Co_1$ [8] in which he mentions that the two subgroups $3^2.[2.3^6].2A_4$ and $3^2.[2^3.3^4].2A_4$ are not the maximal subgroups of $Co_1$, so the list contains total 22 conjugacy classes of maximal subgroups. There are 22 maximal subgroups of the group $Co_1$. The maximal local subgroups have been determined in [3]. The Atlas of Group Representations [6] contains the words for maximal subgroups of $Co_1$ except 16. The maximal subgroups of $Co_1$ are given below.

1. $*(A_9 \times S_3)$
2. $*(D_{10} \times (A_5 \times A_5)) \cdot 2$
3. $5^{1+12}.GL_2(5)$
4. $3^{1+6} : 2.S_4(3).2$
5. $3^6 : 2.M_{12}$
6. $3^2.U_4(3).D_8$
7. $3^{3+4} : 2.(S_4 \times S_4)$
8. $2^{4+12}.(S_3 \times 3S_6)$
9. $5^3 : (4 \times A_5).2$
10. $7^2 : (3 \times 2.S_6)$
11. $5^2 : 2A_5$
12. $*(A_7 \times L_2(7)) : 2$
13. $*(A_6 \times U_3(3)) : 2$
14. $*(A_4 \times G_2(4)) : 2$
15. $*(A_5 \times J_2) : 2$
16. $2^{2+12}.(A_8 \times S_3)$
17. $U_6(2) : S_3$
18. $2^{1+8}.O_8(2)$
19. $Co_3$
20. $2^{1+1} : M_{24}$
21. $3.Suz : 2$
22. $Co_2$

Next we proceed to find words for 16 maximal subgroups which are marked by asterisk.

## 2 Methods

Most of the maximal subgroups on our list can be generated by two elements. If the group is small enough, a random search will produce the subgroup required. This method was successfully used in [9] but in $Co_1$, the subgroups are too large to use brute force. One more focused way of generating a subgroup is by choosing a pair of conjugacy classes $A$ and $B$ in $G$ such that conjugates of random elements $a \in A$, $b \in B$ have a reasonable high probability of generating a conjugate of the desired subgroup.

In most of the cases we present here, even though the subgroup we wish to construct may be generated by two elements, it may be hard to tell which conjugacy classes they belong to. Even if we know a suitable pair of conjugacy classes it may be that the probability the random elements in these classes generate the desired subgroup is relatively small. In this case, we find some part of the desired subgroup, and work inside
another subgroup, usually an involution centralizer, to find the rest. Once we have found a copy of the desired subgroup, we can get information regarding the generating sets. The maximal subgroups often occur as normalizers of elementary abelian groups. So normalizers, which are crucial of the matter here, were mostly computed by methods given in [10] and [11]. The generators of subgroups, wherever possible, have been obtained from [6].

The subgroups can be identified by determining order, composition series and orbit sizes in several permutation representations. Moreover, comparing our result with the list of maximal subgroups in [5] we find that there is only one possibility of the subgroup.

3 Main Results

We have extensively used the information given in [5], [3] and Atlas of finite group representations [6]. We use GAP [12] for group theoretic calculations. Throughout this paper a and b are the standard generators of Co in permutation representation on 98280 points available at [6].

3.1 Construction of \((A_9 \times S_3)\) inside \(Co_1\)

The required maximal subgroup is the normalizer of an element of class 3D. Here we use power maps to find the representative of class 3D to say that \(a_1\) is the element of said class given by \(a_1 = ((ba)^2 b)^6\). Now the normalizer of \(a_1\) inside \(Co_1\) gives us the required maximal subgroup. The normalizer can be computed by the technique given in [10] and the programs given in [11]. Before computing the normalizer we will give some random elements of \(Co_1\) which will be used later. These words are given below:

\[
\begin{align*}
    b_1 &= (ab)^2 ba, & b_2 &= aba, & b_3 &= ab \\
    b_4 &= (ab)^2, & b_5 &= (ab)^2 a, & b_6 &= (ab)^2 b.
\end{align*}
\]

Consider the group generated by \(a_1\) and \(b\) say \(H_2 = \langle a_1, b \rangle\), then compute the normalizer of \(a_1\) inside \(H_2\). From here we get the partial normalizer of \(a_1\) inside \(Co_1\). Before computing the partial normalizer we will give some random elements of \(H_2\) which will facilitate our computations. These elements are given by:

\[
\begin{align*}
    c_1 &= a_1 b, & c_2 &= a_1 ba_1, & c_3 &= a_1 ba_1 bb, \\
    c_4 &= a_1 ba_1 bba_1, & c_5 &= a_1 ba_1 bba_1 b.
\end{align*}
\]

The words for partial normalizer are given below:

\[
\begin{align*}
    k_1 &= b_2 b_5 b_2 b_5 b_2 b_5 b_2 b_5 b_2 b_5, & k_2 &= b_2 b_5 b_2 b_5 b_2 b_5 b_2 b_5 b_2 b_5, \\
    k_3 &= a_1 c_1 a_1 c_1 a_1 c_1 a_1 c_1, & k_4 &= a_1 c_1 a_1 c_1 a_1 c_1 a_1 c_1, \\
    k_5 &= a_1 c_1 a_1 c_1 a_1 c_1 a_1 c_1, & k_6 &= a_1 c_1 a_1 c_1 a_1 c_1 a_1 c_1.
\end{align*}
\]

Next we will find an involution inside the above partial normalizer. This involution is given by \(d_1 = k_1^3\), then find the centralizer of \(d_1\) inside \(Co_1\) by using the method given by J.Bray [13]. The generators of the centralizer of \(d_1\) inside \(Co_1\) are given by:

\[
\begin{align*}
    d_2 &= a((d_1, a)^2), & d_3 &= [(d_1, b)^2], & d_4 &= ab((d_1, ab)^{10}), \\
    d_5 &= aba((d_1, aba)^7), & d_6 &= [(d_1, ababa)^7], & d_7 &= [(d_1, ababa)^7], \\
    d_8 &= d_2 d_4, & d_9 &= d_2 d_5, & d_{10} &= d_2 d_6, & d_{11} &= d_2 d_7.
\end{align*}
\]

\(H_3 = \langle d_2, d_3, d_4, d_5, d_6, d_7 \rangle\).

The words for the normalizer of \(d_1\) inside the above centralizer \((H_3)\) are given below:

\[
\begin{align*}
    k_7 &= d_2 d_3^5 d_2 d_2^5 d_2 d_2^5 d_2 d_2^5 d_2 d_2^5, & k_8 &= d_3 d_3^5 d_3 d_3^5 d_3 d_3^5 d_3 d_3^5.
\end{align*}
\]
By looking at the [5] we see that these words generate only partial normalizer so we repeat the above process until we find the complete normalizer. Now we give some other elements of $C_{O_1}$ which will be used in further computations.

$$m_1 = (b_1 b_2 b_3 b_2^2 b_0^3 b_0^4)^4, \quad m_2 = (b_1 b_2 b_3 b_2^2 b_5 b_0^5)^6, \quad m_3 = (b_1 b_2 b_3 b_2^2 b_5 b_0^3)^3$$

$$m_4 = (b_1 b_2 b_3 b_5 b_0^5)^6, \quad m_5 = (b_1 b_2 b_3 b_5 b_0^3)^3.$$

Consider the group generated by $a_1$ and $m_5$ say $H_4 = \langle a_1, m_5 \rangle$, then compute the normalizer of $a_1$ inside $H_4$. From here we get the partial normalizer of $a_1$ inside $H_4$. Before computing the partial normalizer we will give some random elements of $H_4$. These elements are given by:

$$n_1 = a_1 m_5, \quad n_2 = a_1 m_5 m_5,$$
$$n_3 = a_1 m_5 m_5 a_1, \quad n_4 = a_1 m_5 m_5 a_1 m_5,$$
$$n_5 = a_1 m_5 m_5 a_1 m_5 a_1, \quad n_6 = a_1 m_5 m_5 a_1 m_5 a_1 m_5,$$
$$n_7 = a_1 m_5 m_5 a_1 m_5 a_1 m_5 m_5.$$

The word for the normalizer of $a_1$ inside $H_4$ is given by:

$$k_{11} = a_1 n_1^a a_1 n_1^a a_1 n_1^a a_1 n_1^a a_1 n_1^a.$$

Now by combining the words for the normalizer of $a_1$ inside $H_2$, $H_3$ and $H_4$ we get the required complete normalizer given by $k_2$, $k_7$ and $k_{11}$. The generators for $(A_9 \times S_3)$ are $k_2$ and $k_7 k_{11}$.

### 3.2 Construction of $(D_{10} \times (A_5 \times A_5)), 2$ inside $C_{O_1}$

The required maximal subgroup is the normalizer of an element of class $5B$. Here we first find an element of class $5B$ and then find the normalizer of that element inside $C_{O_1}$, which gives us the required maximal subgroup. The element of class $5B$ can be calculated by using the power maps. The normalizer can be computed by the technique given in [10] and the programs given in [11] will facilitate us in computing the normalizer. Before computing the normalizer we will give some elements of $C_{O_1}$ which will be used later. These elements are given below:

$$b_1 = ababba, \quad b_2 = aba, \quad b_3 = ab,$$
$$b_4 = abab, \quad b_5 = ababa, \quad b_6 = ababb,$$
$$b_7 = abababb, \quad b_{12} = bababababab.$$

The element of class $5C$ is given by $c = (b_1 b_{12})^2$. Next we will give the strategy of finding the normalizer of $c$ inside $C_{O_1}$. Consider the group generated by $c$ and $a$ say $H_2 = \langle c, a \rangle$, then compute the normalizer of $c$ inside $H_2$. From here we get the partial normalizer of $c$ inside $C_{O_1}$. Before computing the partial normalizer we will give some random elements of $H_2$ which will facilitate us in computations.

$$c_1 = ca, \quad c_2 = cac, \quad c_6 = cacacac,$$
$$c_7 = accaca, \quad c_8 = accacaca, \quad c_9 = accacacac.$$

The words for normalizer of $c$ inside $H_2$ are given by:

$$k_1 = cc_1^1 cc_1^1 cc_1^1 cc_1^1 cc_1^1, \quad k_2 = cc_1^1 cc_1^1 cc_1^1 cc_1^1 cc_1^1,$$
$$k_3 = cc_1^4 cc_1^4 cc_1^4 cc_1^4 cc_1^4, \quad k_4 = ac_3^1 ac_3^3 ac_3^1 ac_3^3 ac_3^3.$$
Here we will give some more elements of $Co_1$. These words are given by:

$$k_5 = ac_9ac_9ac_9c_9ac_9^6.$$ 

The normalizer can be found by using the technique given in [10] i.e. we construct the partial normalizer of $5c$ inside different subgroups of $Co_1$. Then we combine these partial normalizer to get the required normalizer. The computations of these partial normalizers are given below.

Consider the group generated by $5c$ and $c_1$ say $H_1 = < 5c, c_1 >$, then compute the normalizer of $5c$ inside $H_1$. Before computing the partial normalizer we will give some words of $H_1$ which will facilitate us in computations. These words are given by:

$$d_1 = b_{10}e_1(b_{10}^{-1}), \quad d_2 = 5cd_1, \quad d_3 = 5cd_15c,$$

$$d_4 = 5cd_15c5c, \quad d_5 = 5cd_15c5cd_1.$$

Next we use the "TKNormalizertest" given in [11] to compute the words for the partial normalizer of $5c$ inside $H_1$. These words are given by:

$$k_1 = d_1^1d_1^2d_1^3d_1^4d_1^5d_1^6d_1^7d_1^8d_1^9$$

$$k_2 = d_1^2d_1^2d_1^2d_1^2d_1^2d_1^2d_1^2d_1^2d_1^2d_1^2.$$
Again consider the group generated by $5c$ inside $k_{81}$ say $H_2 = \langle 5c, k_{81} \rangle$. Here we compute the partial normalizer of $5c$ inside $H_2$ by giving the similar arguments as mentioned above. We will give some words for $H_2$ which are used in computations. These words are given below:

$$
\begin{align*}
    & k_6 = 5ck_{81}, & d_7 = 5ck_{81}5c, & d_8 = 5ck_{81}5ck_{81}, \\
    & d_9 = 5ck_{81}5ck_{81}k_{81}, & d_{10} = 5ck_{81}5ck_{81}k_{81}5c, & d_{11} = 5ck_{81}5ck_{81}k_{81}5c5c, \\
    & d_{12} = 5ck_{81}5ck_{81}k_{81}5ck_{81}, & d_{13} = 5ck_{81}5ck_{81}k_{81}5ck_{81}k_{81}, \\
    & d_{14} = 5ck_{81}5ck_{81}k_{81}5ck_{81}k_{81}5c, & d_{15} = 5ck_{81}5ck_{81}k_{81}5ck_{81}k_{81}5c, \\
    & d_{16} = d_6d_8, & d_{17} = d_6d_9, & d_{18} = d_6d_{10}, \\
    & d_{19} = d_6d_{11}, & d_{20} = d_6d_{12}, & d_{21} = d_6d_{13}, \\
    & d_{22} = d_6d_{14}, & c_2 = (k_2)^5.
\end{align*}
$$

The words for the partial normalizer are given below:

$$
\begin{align*}
    & k_7 = c_2d_6^3c_2d_6c_2d_6c_2d_6c_2d_6, & k_8 = c_2d_6c_2d_6c_2d_6c_2d_6c_2d_6, \\
    & k_9 = c_2d_6^3c_2d_6c_2d_6c_2d_6c_2d_6.
\end{align*}
$$

Consider the group generated by $k_6, k_7$ and $k_8$ say $H_3 = \langle k_6, k_7, k_8 \rangle$. The words for $H_3$ are given below:

$$
\begin{align*}
    & d_{23} = k_6k_7, & d_{24} = k_6k_8, & d_{25} = k_7k_8, & d_{26} = k_6k_7k_8, \\
    & d_{27} = k_6k_7k_8k_6, & d_{28} = k_6k_7k_8k_7, & d_{29} = k_6k_7k_8k_7k_6, \\
    & d_{30} = k_6k_7k_8k_9k_6, & d_{31} = k_6k_7k_8k_9k_7, & d_{32} = k_6k_7k_8k_9k_8.
\end{align*}
$$

The words for the normalizer of $5c$ inside $H_3$ are given below:

$$
\begin{align*}
    & k_9 = 5ck_6^55ck_6^55ck_6^55ck_6^55ck_6^5, & k_{10} = 5ck_6^55ck_6^55ck_6^55ck_6^55ck_6^55ck_6^5, \\
    & k_{11} = 5ck_6^55ck_6^55ck_6^55ck_6^55ck_6^55ck_6^5, & k_{12} = 5ck_6^55ck_6^55ck_6^55ck_6^55ck_6^55ck_6^5, \\
    & k_{13} = 5cd_{25}^55cd_{25}^55cd_{25}^55cd_{25}^55cd_{25}^5.
\end{align*}
$$

Consider the involution $c_3 = k_2$. Since $c_3$ is an involution so its normalizer can easily be calculated by using the method given by J. Bray [13]. The generators of the normalizer of $c_3$ inside $Co_1$ are given below:

$$
\begin{align*}
    & f_1 = 5c[c_3, 5c]^2, & f_2 = k_{81}5ck_{81}[c_3, 5ck_{81}]^{16}, \\
    & f_3 = [c_3, 5k_{81}5ck_{81}]^{12}, & f_4 = [c_3, 5ck_{81}5ck_{81}]^{6}.
\end{align*}
$$

Now compute the normalizer of $5c$ inside $H_4$. Before computing the normalizer we give some words of $H_4$. These words are given below:

$$
\begin{align*}
    & d_{33} = f_2f_3, & d_{34} = f_2f_3f_2, & d_{40} = f_3f_2f_3f_3f_2f_3.
\end{align*}
$$

The word for the normalizer of $5c$ inside $H_4$ is given below:

$$
\begin{align*}
    & k_{14} = d_{33}d_{41}d_{33}d_{33}d_{40}d_{33}d_{33}d_{33}d_{40}d_{33}d_{33}d_{40}.
\end{align*}
$$

Now combining the above partial normalizers will give us the words for the required maximal subgroup. These words are given by $k_6$, $k_{13}$ and $k_{14}$. 

3.4 Construction of $3^{1+4} : 2.S_6(3).2$ inside $Co_1$

From the information given in Atlas [5] the required maximal subgroup is the normalizer of an element of class 3B. So here we first find an element of class 3B and then find the normalizer of it. We will give some words of $Co_1$. These words are given by:

$$
\begin{align*}
    b_1 &= ababba, \\
    b_2 &= ababa, \\
    b_3 &= ab, \\
    b_4 &= abab, \\
    b_5 &= ababa, \\
    b_6 &= ababb, \\
    b_7 &= ababbab, \\
    c_1 &= b_1b_2b_3b_7b_4b_5^2, \\
    c_2 &= (b_1b_2b_3b_7b_4b_5^2)^3, \\
    c_3 &= (b_1b_2b_3b_7b_4b_5^2)^5, \\
    c_4 &= (b_1b_2b_3b_7b_4b_5^2)^6, \\
    c_5 &= (b_1b_2b_3b_7b_4b_5^2)^{11}, \\
    c_6 &= (b_1b_2b_3b_7b_4b_5^2)^{13}.
\end{align*}
$$

The construction of this group consist of two steps given below.

Step 1

In this step we will find an element of class 3B. This can be done by using the power maps of the above generated elements, then checking whether the centralizer order confirms that the element under consideration belongs to class 3B or not. The element of class 3B is given by "b".

Step 2

In this step we will find the normalizer of $b$ inside $Co_1$. The normalizer can be found by using the technique given in [10] i.e. we construct the partial normalizer of $b$ inside different subgroups of $Co_1$. Then we combine these partial normalizers to get the required normalizer. The computations of these partial normalizers are given below.

Consider the group generated by $b$ and $c_1$ say $H_1 = \langle b, c_1 \rangle$, then compute the normalizer of $b$ inside $H_1$. Before computing the partial normalizer we will give some words of $H_1$ which will facilitates our computations. These words are given below:

$$
\begin{align*}
    e_1 &= bc_1, \\
    e_2 &= bc_1b, \\
    e_3 &= bc_1bc_1.
\end{align*}
$$

Next we use the "TKnormalizertest" [11] to compute the words for the partial normalizer of $b$ inside $H_1$.

$$
\begin{align*}
    k_1 &= be_1be_1be_1^{12}be_1^{14}be_1^{12}, \\
    k_2 &= be_1be_1^6be_1^{10}be_1^{5}be_1^{12}, \\
    k_3 &= be_1be_1^{13}be_1^2be_1^8be_1^4, \\
    k_4 &= be_1be_1^{11}be_1^6be_1^{11}be_1^2, \\
    k_5 &= be_1be_1^6be_1^2be_1^{11}be_1^3, \\
    k_6 &= be_1be_1^6be_1^2be_1^3be_1^2.
\end{align*}
$$

Again consider the group generated by $b$ and $c_4$ say $H_2 = \langle b, c_4 \rangle$. Here we compute the partial normalizer of $b$ inside $H_2$ by giving similar arguments as those mentioned above. We will give some words for $H_2$ which are used in further computations. These words are given by:

$$
\begin{align*}
    e_9 &= bc_4, \\
    e_{10} &= bc_4b, \\
    e_{11} &= bc_4bc_4, \\
    e_{12} &= bc_4bc_4b.
\end{align*}
$$

The words for the partial normalizer are given below:

$$
\begin{align*}
    k_7 &= be_9be_9be_93be_9^{11}be_9^{12}, \\
    k_8 &= be_9be_9^2be_9^2be_9^2be_9^2, \\
    k_9 &= be_9be_9^{13}be_9^2be_9^2be_9^4, \\
    k_{10} &= be_9be_9^{13}be_9^2be_9^4be_9^{16},
\end{align*}
$$

Now combining the above partial normalizers will give the words for the required maximal subgroup. These words are given by $k_5$ and $k_{10}$.

3.5 Construction of $(A_4 \times G_2(4)) : 2$ inside $Co_1$

From the information given in Atlas [5] the required maximal subgroup is the normalizer of $2B^2$ (a four group whose involutions are in class 2B). The construction of this subgroup consist of two steps.
Step 1
In this step we find $2B^2$. First we find an involution of class 2B. This involution is given by $a$, then we find the centralizer inside $Co_1$. This can be done by using the technique given by J. Bray given in [13]. The generators of the centralizer inside $Co_1$ are given by:

$$a_1 = [a, b]^3, \quad a_2 = [a, ba]^3, \quad a_3 = bab[a, bab]^7,$$
$$a_4 = baba[a, baba]^7, \quad a_5 = babab[a, babab]^5, \quad a_6 = abab[a, abab]^7,$$
$$a_7 = ababa[a, ababa]^7.$$

Then we search inside this centralizer for an involution of class 2B to find an elementary abelian group of order 4. This involution is given by $c = a_3^{15}$. Now we have the required $2B^2$ generated by $a$ and $c$ where $c = a_3^{15}$.

Step 2
In this step we will calculate the normalizer of $2B^2$ inside $Co_1$. The normalizer can be found by using the technique given in [10] i.e. we construct the partial normalizer of $2B^2$ inside different subgroups of $Co_1$. Then we combine these partial normalizer to get the required normalizer. We also give some elements of $Co_1$ which are used in computations:

$$b_1 = ab, \quad b_2 = aba, \quad b_3 = abab, \quad b_4 = ababa$$
$$b_5 = abab, \quad b_6 = ababba, \quad b_7 = babba, \quad b_8 = babbab,$$
$$d_{12} = (b_1^2 b_3 b_7 b_5^2 b_3^2)^6, \quad d_{13} = (b_1 b_2 b_3 b_7 b_5^2 b_3^2)^4.$$

Consider the group generated by $a$, $c$ and $d_{12}$ say $H_1 = \langle a, c, d_{12} \rangle$, then compute the normalizer of $2B^2$ inside $H_1$. Before computing the partial normalizer we will some words of $H_1$ which will facilitates our computations. These words are given below:

$$e_1 = ad_{12}, \quad e_2 = cd_{12}, \quad e_3 = cd_{12}a, \quad e_4 = cd_{12}ac.$$

Next we use the "TKnormalizertest" [11] to compute the words for the partial normalizer of $2B^2$ inside $Co_1$. These words are given below:

$$k_1 = ae_1 ae_1 ae_1 ae_1 ae_1^{12}, \quad k_2 = ae_1 ae_1 ae_1 ae_1 ae_1^{13}, \quad k_3 = ae_2 ae_2 ae_2^{11} ae_2^{9} ae_2^{13}$$
$$k_4 = ae_2 ae_2^{9} ae_2^{8} ae_2^{2}, \quad k_5 = ae_2 ae_2^{3} ae_2^{8} ae_2^{3} ae_2.$$

Again consider the group generated by $a$, $c$ and $d_{13}$ say $H_2 = \langle a, c, d_{13} \rangle$. Here we compute the partial normalizer of $2B^2$ inside $H_2$ by giving similar arguments to those mentioned above. We give some words for $H_2$ which are used in further computations:

$$e_{13} = ad_{13}, \quad e_{14} = cd_{13}, \quad e_{15} = ad_{13}c, \quad e_{16} = ad_{13}ca.$$

The words for the partial normalizer are given by:

$$k_9 = ae_{13} ae_{13} ae_{13} ae_{13} ae_{13}^{9}, \quad k_{10} = ae_{13} ae_{13} ae_{13} ae_{13}^{6} ae_{13}^{7}.$$

Now combining the above partial normalizers gives us the words for the required maximal subgroup. These words are given by $k_4$, $k_5$ and $k_{10}$.

### 3.6 Construction of $(2^{2+12})A_8 \times S_3$ inside $Co_1$

From the information given in Atlas [5] the required maximal subgroup is the normalizer of $2A^2$ (a four group whose involutions are in class 2A). The construction of this subgroup consist of two steps.
Step 1
In this step we find $2A^2$. First we find an involution of class 2A. This involution is given by $c = (ab)^{20}$, then we find the centralizer of $c$ inside $Co_1$. This can be done by using the technique in [13]. The generators of the centralizer of $c$ inside $Co_1$ are given below:

$$a_2 = a[1, a]^2, \quad a_3 = b[a_1, b]^2, \quad a_6 = aba[a_1, aba]^2,$$
$$a_5 = bab[1, bab]^2, \quad a_8 = bab[a_1, bab], \quad a_7 = babab[1, babab]^2.$$

Then searching inside this centralizer for an involution of class 2A, which combines with $c$, gives an elementary abelian group of order 4. This involution is given by $d = (a_2a_6)^4$. Now we have the required $2A^2$ generated by $c$ and $d$.

Step 2
In this step we will calculate the normalizer of $2A^2$ inside $Co_1$. The normalizer can be found by using the technique given in [10] i.e. we construct the partial normalizer of $2A^2$ inside different subgroups of $Co_1$. Then we combine these partial normalizers to get the required normalizer. We also give some elements of $Co_1$ which are used in computations:

$$b_1 = ab, \quad b_2 = aba, \quad b_3 = abab, \quad b_4 = ababa,$$
$$b_5 = ababb, \quad b_6 = ababba, \quad b_7 = babba, \quad b_8 = babbab,$$
$$d_1 = (b_1b_2b_3b_4b_5)^3, \quad d_2 = (b_1b_2b_3b_4b_5)^3, \quad d_3 = (b_1b_2b_3b_4b_5)^5,$$
$$d_4 = (b_1b_2b_3b_4b_5)^6, \quad d_5 = (b_1b_2b_3b_4b_5)^5.$$

Consider the group generated by $c$, $d$ and $d_1$. We say $H_1 = \langle c, d, d_1 \rangle$, then compute the normalizer of $2A^2$ inside $H_1$. Before computing the partial normalizer we give some words of $H_1$ which will facilitate our computations:

$$e_1 = cd_1, \quad e_2 = cd_1d, \quad e_3 = cd_1dc, \quad e_4 = cd_1dcd.$$

Next we use the "TKnormalizertest" [11] to compute the words for the partial normalizer of $2B^2$ inside $Co_1$. These words are given by:

$$k_1 = ce_1ce_1ce_1ce_1, \quad k_2 = de_1de_2de_3de_4de_5,$$
$$k_4 = de_2de_3de_4de_5de_6, \quad k_5 = de_2de_3de_4de_5de_6,$$
$$k_7 = de_1de_2de_3de_4de_5, \quad k_8 = de_1de_2de_3de_4de_5.$$

Again consider the group generated by $c$, $d$ and $d_4$. We say $H_2 = \langle c, d, d_4 \rangle$. Here we compute the partial normalizer of $2A^2$ inside $H_2$ by giving similar arguments to those mentioned above. We give some words for $H_2$ which are used in further computations:

$$e_{11} = cd_4, \quad e_{12} = dd_4, \quad e_{13} = dd_4c, \quad e_{14} = dd_4cd.$$

The words for the partial normalizer are given by:

$$k_9 = de_{11}de_{12}de_{13}de_{14}, \quad k_{10} = de_{11}de_{12}de_{13}de_{14},$$
$$k_{11} = de_{11}de_{12}de_{13}de_{14}, \quad k_{12} = de_{11}de_{12}de_{13}de_{14}.$$

Now consider the group generated by $c$, $d$ and $d_{12}$. We say $H_3 = \langle c, d, d_{12} \rangle$. Here we compute the partial normalizer of $2A^2$ inside $H_3$ by giving similar arguments to those mentioned above. We give some words for $H_3$ which are used in further computations:

$$e_{22} = cd_{12}, \quad e_{23} = dd_{12}, \quad e_{24} = dd_{12}c, \quad e_{25} = dd_{12}cd.$$

The words for the partial normalizer are given below:

$$k_{15} = ce_{22}ce_{23}ce_{24}ce_{25}, \quad k_{16} = ce_{23}ce_{24}ce_{25},$$
$$k_{17} = de_{23}de_{24}de_{25}de_{26}.$$

Now combining the above partial normalizers gives the words for the required maximal subgroup. These words are given by $k_6$, $k_{16}$ and $k_{17}$.
3.7 Construction of $3^6 : 2.M_{12}$ inside $Co_1$

From the information given in Atlas [5] the required maximal subgroup is the normalizer of $3^6$ (elementary abelian group of order 729). The construction of this subgroup consist of two steps given below.

Step1

Here we will construct $3^6$. To construct $3^6$ we adopt the following strategy.

i) Find an arbitrary element of order 3. This element is given by $b$.

ii) Find centralizer of $b$ inside $Co_1$. Before calculating the centralizer we give some elements of $Co_1$ as follows:

$$b_1 = ab, \quad b_2 = aba, \quad b_3 = abab, \quad b_4 = ababa,$$

$$b_5 = ababb, \quad b_6 = ababba, \quad b_7 = babba, \quad b_8 = babbab,$$

$$c_1 = (b_1b_2b_3b_7b_4b_5^2), \quad c_2 = (b_1b_2b_3b_7b_4b_5^2)^3,$$

$$c_3 = (b_1b_2b_3b_7b_4b_5^2)^5, \quad c_4 = (b_1b_2b_3b_7b_4b_5^2)^6.$$

The centralizer of $b$ can be found by using the technique given in [10] i.e. we start by constructing the partial centralizer of $3^6$ inside different subgroups of of $Co_1$. Next we combine these partial centralizer to get the required centralizer of $b$ inside $Co_1$. We also give some elements of $Co_1$ which are used in computations. Consider the group generated by $b, c_1$ say $H_1 = < c, c_1 >$. Compute the centralizer of $b$ inside $H_1$. Before computing the partial centralizer we give some random elements of $H_1$ which are used in computations. These elements are given by:

$$e_1 = bc_1, \quad e_2 = bc_1b, \quad e_3 = bc_1b_1c_1, \quad e_5 = bc_1b_1c_1b.$$

The generators of centralizer of $b$ inside $Co_1$ are given below:

$$k_1 = be,b_1be_1^{13}b_1^{11}b_1^{12}, \quad k_2 = be,b_1be_1^{10}b_1^{12}, \quad k_3 = be,b_1be_1^{11}b_1^{11}b_1^{12},$$

$$k_4 = be_1^{13}b_1^{11}b_1^{11}b_1^{2}, \quad k_5 = be_1^{11}b_1^{12}b_1^{11}b_1^{6}, \quad k_6 = be_1^{11}b_1^{12}b_1^{11}b_1^{6}.$$

Again consider the group generated by $b$ and $c_4$ say $H_2 = < b, c_4 >$. Here we compute the partial normalizer of $b$ inside $H_2$. We give some random elements of $H_2$ which are used in further computations:

$$e_9 = bc_4, \quad e_{10} = bc_4b, \quad e_{11} = bc_4bc_4.$$

The generators of centralizer of $b$ inside $H_2$ are given below:

$$k_7 = be_2be_2^3b_1^{11}b_1^{9}, \quad k_8 = be_2be_2^3b_1^{12}, \quad k_9 = be_2be_2^3b_1^{13}b_1^{12}b_1^{9},$$

$$k_{10} = be_2be_2^3b_1^{13}b_1^{12}b_1^{9}.$$

Now combining the above partial normalizers gives us the generators for $3^6$. These generators are given by $k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8, k_9$ and $k_{10}$. Then we can easily find $3^6$ inside the above centralizer. The generators for $3^6$ are given by:

$$f_1 = k_1, \quad f_2 = k_2, \quad f_3 = k_3^2,$$

$$f_4 = k_4^2, \quad f_5 = (k_2k_7)^2, \quad f_6 = (k_2k_{10})^6.$$

Step2

In this step we will find the normalizer of $3^6$ inside $Co_1$ which is the required maximal subgroup. The words for the normalizer of $3^6$ are given below:

$$z_1 = b_1^2b_2^2b_3b_4b_5^3, \quad z_2 = b_1b_2^2b_4b_5, \quad x_1 = f_2k_7,$$

$$z_3 = f_3x_1f_3x_1^2f_3x_1^4x_1^5f_3x_1^6f_3x_1^7, \quad z_4 = f_3x_1^2f_3x_1^2f_3x_1^2f_3x_1^2f_3x_1^3,$$

$$z_5 = (f_3x_1^2)^{3}x_1^2(f_3x_1^2)^2, \quad x_2 = f_6z_2f_5z_6, \quad z_6 = (f_5x_1^2)^5.$$

The words for $3^6 : 2.M_{12}$ are $w_1 = z_3z_4$ and $w_2 = z_5z_6$. 
3.8 Construction of $3^2 \cdot U_4(2) \cdot D_8$ inside $Co_1$

From the information given in Atlas [5] the required maximal subgroup is the normalizer of $3^2$ (elementary abelian group of order 9). Similarly as in the previous cases we will construct this group into two steps given below.

**Step1**
In this step we will find $3^2$. This can be done by taking $3^6$ which we constructed in section 3.7, then searching inside these $3^6$ we can easily find the required $3^2$ given by $f_4$ and $f_5$.

**Step2**
In this step we will find the normalizer of $H_1 = \langle f_4, f_5 \rangle$ inside $Co_1$. The normalizer can be found by using the technique given in [10] i.e., we construct the partial normalizer of $H_1$ inside different subgroups of $Co_1$. Then we combine these partial normalizer to get the required normalizer. The computations of these partial normalizers are given below.

Before computing the partial normalizer we give some words of $Co_1$ which will facilitate our computations. These words are given below:

$$l_1 = b_6 b_7 b_3 b_4 b_5,$$
$$l_2 = b_6 b_7 b_3 b_4 b_5,$$
$$l_3 = b_6 b_7 b_3 b_4 b_5,$$
$$l_4 = b_6 b_7 b_3 b_4 b_5,$$
$$l_5 = b_6 b_7 b_3 b_4 b_5,$$
$$l_6 = b_6 b_7 b_3 b_4 b_5,$$
$$l_7 = b_6 b_7 b_3 b_4 b_5,$$
$$l_8 = b_6 b_7 b_3 b_4 b_5,$$
$$l_9 = b_6 b_7 b_3 b_4 b_5,$$
$$l_{10} = b_6 b_7 b_3 b_4 b_5,$$
$$l_{11} = b_6 b_7 b_3 b_4 b_5,$$
$$l_{12} = b_6 b_7 b_3 b_4 b_5,$$
$$l_{13} = b_6 b_7 b_3 b_4 b_5,$$
$$l_{14} = b_6 b_7 b_3 b_4 b_5,$$
$$l_{15} = b_6 b_7 b_3 b_4 b_5,$$
$$l_{16} = b_6 b_7 b_3 b_4 b_5,$$
$$l_{17} = b_6 b_7 b_3 b_4 b_5,$$
$$l_{18} = b_6 b_7 b_3 b_4 b_5,$$
$$g_1 = f_5 l_1,$$
$$g_2 = f_6 l_1,$$
$$g_3 = f_6 l_5,$$
$$g_4 = f_6 l_5 l_1,$$
$$g_5 = f_6 l_5 f_5 l_6,$$
$$g_6 = f_5 l_5 f_5 l_6.$$

The words for the centralizer are given below:

$$h_1 = [g_{15}, a^2],$$
$$h_2 = b [g_{15}, b],$$
$$h_3 = [g_{15}, ab] a^2,$$
$$h_4 = [g_{15}, aba]^3,$$
$$h_5 = [g_{15}, abab]^2.$$

Now we will find the partial normalizer of $H_1$ inside $Co_1$. The words for the centralizer are given below:

$$g_{16} = h_2 h_3,$$
$$g_{17} = h_2 h_4,$$
$$g_{18} = h_2 h_5,$$
$$g_{19} = h_1 h_4 h_5,$$
$$g_{20} = h_1 h_4 h_5 h_2,$$
$$g_{21} = h_1 h_4 h_5 h_3,$$
$$g_{22} = h_1 h_4 h_5 h_2 h_3 h_4,$$
$$g_{23} = h_1 h_4 h_5 h_2 h_3 h_4 h_5,$$
$$g_{24} = h_1 h_4 h_5 h_2 h_3 h_4 h_5 h_2,$$
$$g_{25} = h_1 h_4 h_5 h_2 h_3 h_4 h_5 h_3,$$
$$g_{26} = h_1 h_4 h_5 h_2 h_3 h_4 h_5 h_4,$$
$$g_{27} = h_1 h_4 h_5 h_2 h_3 h_4 h_5 h_4 h_2,$$
$$g_{28} = h_1 h_2 h_3 h_4 h_5.$$

The words for the partial normalizer of $H_1$ inside $Co_1$ are given by:

$$k_{17} = h_1 g_{16} h_1 g_{16}^2 h_1 g_{16}^2 h_1 g_{16}^2 h_1 g_{16}^2 h_1 g_{16}^2 h_1 g_{16}^2 h_1 g_{16}^8,$$
$$k_{18} = h_1 g_{16} h_1 g_{16}^2 h_1 g_{16}^8 h_1 g_{16}^8 h_1 g_{16}^8,$$
$$k_{19} = h_1 g_{16} h_1 g_{16}^2 h_1 g_{16} h_1 g_{16} h_1 g_{16}^8,$$
$$k_{20} = h_2 g_{27} h_2 g_{27} h_2 g_{27} h_2 g_{27} h_2 g_{27}^3 h_2 g_{27}^3,$$
$$k_{31} = f_5 g_{28}^2 f_5 g_{28}^2 f_5 g_{28}^2 f_5 g_{28}^2 f_5 g_{28}^2.$$

The words for the required maximal subgroup $3^2 \cdot U_4(2) \cdot D_8$ are $k_{11}$ and $k_{19} k_{31}$.
3.9 Construction of $3^{3+4} : 2.(S_4 \times S_4)$ inside $Co_1$

Following [5], we see that the required subgroup is the normalizer of $3^3$. We can easily find $3^3$ from $3^6$ calculated above in 3.7, then the normalizer of it gives us the required subgroup. The generators of $3^3$ are $f_4, f_5$ and $f_6$. Before computing the normalizer we give some elements:

\[
\begin{align*}
    l_1 &= b_4b_3^2b_4b_3, \\
    g_1 &= f_4l_1, \\
    g_7 &= f_4l_1f_5l_1f_6, \\
    g_8 &= f_4f_5f_6l_1f_4l_1, \\
    z_{13} &= f_4g_8f_4g_8^5f_4g_8f_4g_8f_4g_8, \\
    h_1 &= [g_9, a]^3, \\
    h_2 &= b[g_9, b], \\
    h_3 &= ab[g_9, ab]^2, \\
    h_4 &= [g_9, ab]^3, \\
    h_5 &= abab[g_9, abab]^2, \\
    h_{11} &= h_1h_3h_3, \\
    g_{12} &= h_1h_3h_4, \\
    g_{13} &= g_{11}g_{12}.
\end{align*}
\]

The generators for the normalizer of $3^3$ are given below:

\[
\begin{align*}
    z_{11} &= f_6g_3^3(f_6g_3^3)^3f_6g_3^4, \\
    z_{12} &= f_6g_7(f_6g_7)^3f_6g_7^2f_6g_7^8, \\
    z_{14} &= h_1g_9^2(h_1g_9^2)^2h_1g_9^2h_1g_9^2h_1g_9^2h_1g_9^2, \\
    z_{15} &= h_1g_9^2h_1g_9^2h_1g_9^2h_1g_9^2h_1g_9^2h_1g_9^2, \\
    z_{16} &= (z_{14}z_{15})^4(\varepsilon z_{15}^2z_{16})^2z_{16}, \\
    z_{17} &= (z_{14}z_{15})^2z_{14}^2z_{15}^2z_{14}z_{15}^2z_{14}^2.
\end{align*}
\]

The words for $3^{3+4} : 2.(S_4 \times S_4)$ are given by $w_5 = z_{16}z_{17}$ and $w_6 = z_{12}$.

3.10 Construction of $2^{4+12} : (S_3 \times 3S_6)$ inside $Co_1$

From [5], the required subgroup is the normalizer of $2A^4$ (inside $Co_1$) and can be constructed by taking an involution of class $2A$, then searching inside its centralizer. We have already constructed $N_{Co_1}(2A^2)$ in section 3.6, and now we will search $2A^4$ inside $N_{Co_1}(2A^2)$. Now $2A^4 = (k_1, k_3, g_1, g_2)$, where $k_1, k_3$ are same as in section 3.6 and $g_1 = (k_2k_8)^2, g_2 = (k_2k_9)^2$. The words for $k_8$ and $k_9$ are given in section 3.6.

\[
\begin{align*}
    h_1 &= k_1, \\
    h_2 &= k_3, \\
    h_5 &= a[h_1, a]^2, \\
    h_6 &= ba[h_1, ba]^2, \\
    h_8 &= [h_2, ab]^2, \\
    h_9 &= [h_2, ba]^3, \\
    h_{10} &= aba[h_2, ab]^2, \\
    l_1 &= h_8h_9, \\
    l_2 &= h_8h_{10}.
\end{align*}
\]

The generator for the normalizer of $2A^4$ are given below.

\[
\begin{align*}
    k_{10} &= h_5h_7h_5h_7h_5h_7h_5h_7h_5h_7, \\
    k_{11} &= h_5h_7h_5h_7h_5h_7h_5h_7h_5h_7h_5h_7h_7h_7h_7, \\
    k_{12} &= l_1l_2l_1l_2l_1l_2l_1l_2l_1l_2l_1l_2.
\end{align*}
\]

The words for $2^{4+12} : (S_3 \times 3S_6)$ are found to be $k_{11}$ and $k_{13} = k_{12}k_{10}$.

3.11 Construction of $5^3 : (4 \times A_5).2$ inside $Co_1$

From [5], $5^3 : (4 \times A_5).2$ is the normalizer of $5^3$ inside $Co_1$. We give some random elements of $Co_1$ below.

\[
\begin{align*}
    b_1 &= (ab)^2ba, \\
    b_2 &= (ab)^2, \\
    b_3 &= ab, \\
    b_6 &= (ab)^2b, \\
    b_{10} &= (ab)^2(ba)^3b, \\
    b_{13} &= bab(ab)^6, \\
    b_{15} &= bab(ba)^3b, \\
    5c &= (b_1b_3b_{15})^3, \\
    e_1 &= (ab)^20, \\
    e_2 &= (ab)^8, \\
    e_3 &= b_3^6, \\
    c_1 &= (ababa^2b^3)^4, \\
    d_1 &= b_{10}e_1(b_{10}^3), \\
    d_2 &= 5cd_1, \\
    k_2 &= (d_1d_2^3)(d_1d_2^2)^2d_1d_2^2.
\end{align*}
\]
The generators of the Conway group are given by:

\[ k_1 = d_1 d_2 d_1 d_2^{-1}, \quad d_6 = 5 c k_{10}, \quad k_6 = c_2 d_2 c_1 d_2^c d_2 c_2 d_2, \quad k_7 = c_2 d_3 c_2 d_3^c d_2 c_2 d_2^c, \quad k_8 = c_2 d_2 c_1 d_2 c_2 d_2 c_2 d_2, \quad k_9 = 5 c k_5 c k_6 c k_6 c k_3, \quad k_{12} = 5 c k_8 c k_2 c k_2 c k_5 c k_8, \]

Before computations we give some random elements of the centralizer:

\[ g_0 = k_2, \quad g_5 = k_{12} k_{14}, \quad g_2 = (g_0 g_5)^4 g_0 g_5^2, \quad k_{14} = d_{33} d_0 d_{33} d_0 d_{33} d_0 d_{33} d_{33}^7, \]

The generators of \( 5^1 \) are \( h_1 = k_1, h_2 = k_2 \) and \( h_3 = k_3 \).

The generators of the normalizer of \( 5^3 \) are \( h_1, z_1, z_3 \) and \( z_4 \). The words for \( 5^3 : (4 \times A_5) \cdot 2 \) are \( z_5 = h_1 z_1 \) and \( z_6 = z_3 z_4 z_3 \).

### 3.12 Construction of \( 7^2 : (3 \times 2.S_6) \) inside \( Co_1 \)

Following [5], the required subgroup is the normalizer of \( 7^2 \). It is constructed by taking an element of class \( 7B \) and by searching inside its centralizer we find \( 7B^2 \). Before computations we give some random elements of \( Co_1 \):

\[ b_1 = (ab)^2 ba, \quad b_2 = aba, \quad b_3 = ab, \quad b_4 = (ab)^2, \quad b_5 = (ab)^2 a, \quad b_6 = (ab)^2 b, \quad b_7 = (ab)^2 bab, \quad b_8 = (ab)^2 (ba)^2, \quad b_9 = (ab)^2 (ba)^2 b, \quad b_{10} = bab (ba)^3 b^2 a. \]

The elements of \( 7B \) are given by \( a_2 = b_6^{10} \). The generators of \( 7B^2 \) are given below:

\[ f_1 = a_2 e_1 a_2 e_1 a_2 e_3^a a_2 e_7^{a_1}, \quad f_2 = a_2 e_1 a_2 e_3^a a_2 e_8^a a_2 e_1 a_2 e_6^a. \]

Now we find the normalizer of \( 7B^2 \):

\[ k_1 = a_2 e_1 a_2 e_7^a a_2 e_7^a a_2 e_7^a a_2 e_4 a_2 e_7^{10}, \quad k_2 = a_2 e_1 a_2 e_7^a a_2 e_4 a_2 e_1 a_2 e_6^a, \quad g_2 = (b_7 b_8 b_9 b_8 b_7)^8, \quad e_3 = a_2 g_2, \quad e_2 = a_2 g_2 a_2 g_3 a_2, \quad k_3 = (e_2 e_3 e_2 e_3^2)^2 e_3^a e_2 e_3^a. \]

The words for \( 7^2 : (3 \times 2.S_6) \) are \( k_3 \) and \( k_4 = k_1 k_2 \).
3.13 Construction of $5^2 : 2A_5$ inside $Co_1$

The required subgroup is the normalizer of $5C^2[5]$ and it can be constructed by taking an element of order 5 and then search inside its centralizer. We can easily find $5C^2$.

$$b_1 = ababb,$$  
$$b_2 = ababbababa,$$  
$$b_3 = babbabababa,$$  
$$b_4 = babbababababa,$$  
$$c_1 = (ab)^2,$$  
$$c_2 = (ab)^2ba,$$  
$$c_3 = a,$$  
$$c_4 = (ab)^2,$$  
$$d_1 = b_2e_1(b_2^{-1}),$$  
$$d_2 = 5cd_1,$$  
$$d_3 = 5ck_4,$$  
$$d_4 = 5ck_45ck_4,$$  
$$d_5 = k_6k_7.$$

The generators of $5C^2$ are given by $5c$ and $z_1$.

$$g_1 = k_2k_8k_{10},$$  
$$k_11 = (z_1g_1)^5g_1,$$  
$$k_12 = (k_9h_1)^4h_1^4k_9h_1^{-9},$$  
$$j_1 = k_12,$$  
$$j_2 = a[j_1, a]^6,$$  
$$j_3 = b[j_1, b]^7,$$  
$$j_4 = [j_1, ab]^{15},$$  
$$l_1 = j_2j_3j_4,$$  
$$k_13 = j_2l_1^2j_2l_1j_2l_1^2j_2l_1j_2l_1^{16}.$$

The words for $5^2 : 2A_5$ are $k_{11}$ and $k_{13}$.

3.14 Construction of $(A_7 \times L_2(7)) : 2$ inside $Co_1$

The whole strategy for locating $(A_7 \times L_2(7)) : 2$ is given in [3]. First we take an $A_5$ of the type $(2B, 3A, 5A)$ which is in the unique class of $Co_1$ with normalizer $N(A_5) = (A_5 \times J_2).2$ [5], then find $A_4$ inside $A_5$. Then we find an element of class $3A$ which commutes with $A_4$, but not with $A_5$. This $3A$ element with $A_5$ extends $A_5$ to our required $A_7$. Finally we found that $N(A_5) = (A_7 \times L_2(7)) : 2$. We give some random elements of $Co_1$ to facilitate computations.

$$b_1 = (ab)^2ba,$$  
$$b_2 = aba,$$  
$$b_3 = ab,$$  
$$b_4 = (ab)^2,$$  
$$b_5 = (ab)^2a,$$  
$$b_6 = (ab)^2b,$$  
$$b_7 = (ab)^2bab,$$  
$$b_8 = (ab)^2(ba)^2,$$  
$$b_9 = (ab)^2(ba)^2b,$$  
$$b_{10} = (ab)^2(ba)^3,$$  
$$b_{13} = bab(ba)^4,$$  
$$b_{15} = bab^2(ba)^3ba,$$  
$$2b = b_1,$$  
$$3a = b_{15}^3.$$
Here \( f_5 = f_1^2 \) commutes with \( A_4 \) but not with \( A_5 \), so the generators of \( A_7 \) are \( c_1, c_2, c_3 \) and \( f_5 \).

\[
\begin{align*}
  e_1 &= [d_1, ab]^3, & e_2 &= [d_1, ba]^6, & e_3 &= [d_1, bab]^6, \\
  e_6 &= e_2 e_1, & e_7 &= e_1 e_2 e_3, & f_1 &= e_1 e_6 e_1 e_6 e_1 e_6 e_6 e_6^2.
\end{align*}
\]

The words for \( (A_7 \times L_2(7)) : 2 \) are \( k_1 \) and \( k_7 = k_4 k_6 \).

### 3.15 Construction of \( (A_6 \times U_3(3)) : 2 \) inside \( Co_1 \)

The required group is the normalizer of \( A_6 \) which lies in the suzuki chain\([3]\). We easily find \( A_6 \) inside 3.14. The generators of \( A_6 \) are given by \( g_1 = c_1 \) and \( g_2 = (f_5 c_3)^2 c_1 c_2 (c_3 c_2)^2 f_5 c_3 c_1 c_2 c_3 c_2 \). Next we find the normalizer of \( A_6 \).

\[
\begin{align*}
  h_1 &= (b_7 b_8 b_9 b_{10} b_{11} b_{12} b_{13})^7, & e_8 &= g_2 h_1, & k_2 &= g_1^2 g_1 e_8 g_1 e_8 g_1 e_8 g_1 e_8 g_1 e_8, \\
  k_4 &= g_1 e_8 g_1 e_8 g_1 e_8 g_1 e_8 g_1 e_8, & l_6 &= ababa [g_1, ababa]^5, & l_8 &= l_a l_8, \\
  k_7 &= l_{a_5} l_{a_6} l_{a_7} l_{a_8} l_{a_9} l_{a_{10}} l_{a_{11}} l_{a_{12}} l_{a_{13}}^6.
\end{align*}
\]

The words for \( (A_6 \times U_3(3)) : 2 \) are \( k_8 = k_7 k_4 \) and \( k_9 = k_2 k_4 k_7 \).

### 3.16 Construction of \( (A_5 \times J_2) : 2 \) inside \( Co_1 \)

The required group is the normalizer of \( A_5 \) [5] which we already constructed in 3.14. It can also be constructed by taking an involution of class \( 2B \), an element of class \( 3A \) and product of these two elements belongs to class \( 5A \) [5], then \( N(2B, 3A, 5A) = (A_5 \times J_2).2 \).

\[
\begin{align*}
  b_1 &= (ab)^2 a, & b_2 &= bab^2(ab)^3, & b_3 &= bab^2(ab)^3 ba, \\
  3a &= b_1^{13}, & 2b &= a.
\end{align*}
\]

The generators of the required group whose normalizer is to be computed are given below:

\[
\begin{align*}
  c_1 &= (3a)^{b_1^{13}}, & c_2 &= (2b)^{b_1^{13}}, & 3 &= (3a)^{b_1^{13}} (2b)^{b_1^{13}}.
\end{align*}
\]

Before computing the normalizer we give some elements:

\[
\begin{align*}
  d_1 &= a[c_2, a], & d_2 &= [c_2, b]^{12}, & d_3 &= aba[c_2, aba]^7, \\
  d_4 &= d_1 d_3, & k_1 &= d_2 d_4 d_2 d_4 d_2 d_4 d_2 d_4 d_2 d_4, & e_1 &= k_1^{12}, \\
  k_2 &= (d_2 d_4)^3 d_2 d_2 d_4 d_2 d_4 d_2 d_4 d_2 d_4, & e_3 &= [e_1, b]^2, & e_4 &= [e_1, ab]^3, \\
  e_5 &= aba [e_1, aba]^2, & e_6 &= e_4 e_5, & k_3 &= e_3 e_5 e_6 e_6 e_3 e_6 e_6 e_6.
\end{align*}
\]

The words for \( (A_5 \times J_2) : 2 \) are \( k_2 \) and \( k_3 \).
The orders and orbit-shapes of above computed maximal subgroups are mentioned in table given below.

| Group | Order | Orbit – shape |
|-------|-------|---------------|
| $(A_3 \times S_3)$ | 1088640 | $360^2 \cdot 3240^2 \cdot 20160^2 \cdot 25920^2 \cdot 45360^4$ |
| $(D_{10} \times (A_3 \times A_3))$, 2.2 | 144000 | $60^2 \cdot 600 \cdot 720 \cdot 1500 \cdot 3000^4$ |
| | | $12000^4 \cdot 14400^2 \cdot 18000^2 \cdot 24000^2$ |
| $5^{1+2}: GL_2(5)$ | 60000 | $5^1 \cdot 150^1 \cdot 1250^1 \cdot 1500^1 \cdot 1875^1 \cdot 2500^1$ |
| | | $30000^3 \cdot 7500^1 \cdot 15000^1 \cdot 30000^1$ |
| $3^+\times 2: S_5(3).2$ | 25194240 | $27^1 \cdot 3240^2 \cdot 9720^2 \cdot 32805^1 \cdot 52488^1$ |
| $(2^{1+4\times 2}A_3 \times S_3)$ | 1981808640 | $360^2 \cdot 5760^1 \cdot 92160^1$ |
| $3^3 : 2.M_{12}$ | 138568320 | $594 \cdot 17496 \cdot 32076 \cdot 48114^1$ |
| $3^2 \cdot U_3(3).D_{10}$ | 235146840 | $756^1 \cdot 36288^1 \cdot 61236^1$ |
| $3^3 \times 2: (S_5 \times S_3)$ | 25194242 | $108^1 \cdot 1944^1 \cdot 8768^1 \cdot 34992^1 \cdot 52488^1$ |
| $2^{1+1+4} \cdot (S_3 \times 3S_0)$ | 849346560 | $27 \cdot 1440^2 \cdot 23040^2 \cdot 73728^1$ |
| $5^+ : (4 \times A_4).2$ | 60000 | $30^1 \cdot 150^0 \cdot 600^0 \cdot 750^0 \cdot 1500^0 \cdot 3000^0 \cdot 36000^0$ |
| $7^+ : (3 \cdot 2.S_6)$ | 3528 | $84^1 \cdot 588^1 \cdot 882^1 \cdot 1176^1 \cdot 1764^1 \cdot 3528^1$ |
| $5^+ : 4A_4$ | 3000 | $30^1 \cdot 150^0 \cdot 300^0 \cdot 600^0 \cdot 750^0 \cdot 1500^0 \cdot 3000^0 \cdot 3000^0$ |
| $(A_1 \times L_2(7)).2$ | 84672 | $2520^1 \cdot 35280^1 \cdot 60480^1$ |
| $(A_5 \times U(3))$.2 | 435360 | $7560^1 \cdot 90720^1$ |
| $(A_4 \times U(4))$.2 | 6038323200 | $98280^1$ |
| $(A_5 \times L(3))$.2 | 72576000 | $37800^1 \cdot 60480^1$ |

Acknowledgement: The authors would like to thank anonymous referees for their valuable comments.

Competing Interest
The authors do not have any competing interests.

References

[1] Conway J.H., A group of order 8,315,553,613,086,720,000, Bull. London Math. Soc. 1 (1969), 79–88.
[2] Curtis R.T, On subgroups of .0. I. Lattice stabilizers. Journal of Algebra 63 (1980), 413–434.
[3] Wilson R.A., (1983). The maximal subgroups of Conway's group Co1. Journal of Algebra, 85(1), 144–165.
[4] Conway J.H., and Sloane N.J.A., Sphere packings, lattices and groups, Springer 290 (1999), third edition.
[5] Conway J.H, Curtis R.T, Norton S.P, Parker R.A, & Wilson R.A, (1985; reprinted with corrections 2003). ATLAS of Finite Groups, Clarendon Press, Oxford.
[6] Wilson R. A., Nickerson S.J., & Bray J.N., (2004-2014). A World-wide-web Atlas of Group Representations, http://brauer.maths.qmul.ac.uk/Atlas.
[7] Wilson R.A., http://www.maths.qmul.ac.uk/~raw/research.html
[8] Wilson R.A., Maximal subgroups of sporadic groups. arXiv:1701.02095. (2017).
[9] Bray J.H., Ibrahim A.I., Walsh P.G., Wilson R.A., Generating maximal subgroups of sporadic simple groups, Communications in algebra, 29(3), 1325–1337(2001).
[10] Linton S. A, (1995). The art and science of computing in large groups, Computational algebra and number theory (Sydney, 1992), 91–109, Math. Appl, 325, Kluwer Acad. Publ., Dordrecht.
[11] Nickerson S.J., "Maximal Subgroups of the Sporadic Almost-Simple Groups.\ldots", (2003, December). M. Phil Thesis by Simon Jon Nickerson The University of Birmingham.
[12] The GAP Group, (2014). GAP - Groups, Algorithms, and Programming, Version 4.7.5 (http://www.gap-system.org).
[13] Bray J.N., An improved method for generating the centralizer of an involution, Arch. Math. (Basel) 74 (2000), no. 4, 241–245.