Spectral analysis of non-self-adjoint Jacobi operator associated with Jacobian elliptic functions

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1 Complex Jacobi matrices - generalities

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6 Intermezzo II - extremal properties of $|\text{sn}(uK(m) \mid m)|$
Jacobi operators associated with complex semi-infinite Jacobi matrix

To the semi-infinite Jacobi matrix

\[
J = \begin{pmatrix}
    b_1 & a_1 \\
    a_1 & b_2 & a_2 \\
    a_2 & b_3 & a_3 \\
    & & & \ddots & \ddots & \ddots
\end{pmatrix}
\]

where \( b_n \in \mathbb{C} \) and \( a_n \in \mathbb{C} \setminus \{0\} \), we associate two operators \( J_{\text{min}} \) and \( J_{\text{max}} \) acting on \( \ell^2(\mathbb{N}) \).
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- \( J_{\text{min}} \) is the operator closure of \( J_0 \), an operator defined on \( \text{span}\{e_n \mid n \in \mathbb{N}\} \) by
  \[
  J_0 e_n := a_{n-1} e_{n-1} + b_n e_n + a_n e_{n+1}, \quad \forall n \in \mathbb{N}, \ (a_0 := 0).
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\( J_{\max} \) acts as \( J_{\max} x := \mathcal{J} \cdot x \) (formal matrix product) on vectors from

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\text{Dom } J_{\max} = \{ x \in \ell^2(\mathbb{N}) \mid \mathcal{J} \cdot x \in \ell^2(\mathbb{N}) \}.
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- Both operators \( J_{\text{min}} \) and \( J_{\text{max}} \) are closed and densely defined.
Jacobi operators associated with complex semi-infinite Jacobi matrix

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Both operators $J_{\text{min}}$ and $J_{\text{max}}$ are closed and densely defined. They are related as

$$
J_{\text{max}}^* = C J_{\text{min}} C \quad \text{and} \quad J_{\text{min}}^* = C J_{\text{max}} C
$$

where $C$ is the complex conjugation operator, $(Cx)_n = \overline{x_n}$. 

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Complex Jacobi Matrix associated with JEF

June 21-24
Proper case and spectrum of Jacobi operator

- Any closed operator $A$ having $\text{span}\{e_n \mid n \in \mathbb{N}\} \subset \text{Dom}(A)$ and defined by the matrix product satisfies $J_{\text{min}} \subset A \subset J_{\text{max}}$. 

- In general $J_{\text{min}} \neq J_{\text{max}}$. If $J_{\text{min}} = J_{\text{max}}$, the matrix $J$ is called proper and the operator $J := J_{\text{min}} \equiv J_{\text{max}}$ the Jacobi operator associated with $J$. 

- Let $J_{\text{min}} = J_{\text{max}} =: J$. Then $J^* = CJC$. As a consequence, $\sigma_r(J) = \emptyset$. 

- We have the decomposition: $\sigma(J) = \sigma_p(J) \cup \sigma_c(J) = \sigma_p(J) \cup \sigma_{\text{ess}}(J)$, where the essential spectrum has the simple characterization: $\sigma_{\text{ess}}(J) = \{z \in \mathbb{C} \mid \text{Ran}(J - z) \text{ is not closed}\}$. 

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The Jacobi matrix associated with Jacobian elliptic functions

For $\alpha \in \mathbb{C}$, the semi-infinite Jacobi matrix

$$J = \begin{pmatrix}
0 & 1 & 2\alpha \\
1 & 0 & 3 \\
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is proper, and hence it determines a unique densely defined closed operator $J(\alpha)$. 
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is proper, and hence it determines a unique densely defined closed operator $J(\alpha)$.

The aim of this talk is the investigation of spectral properties of $J(\alpha)$ for $\alpha \in \mathbb{C}$. 

We will restrict with $\alpha$ to the unit disk $|\alpha| \leq 1$. The spectral properties of $J(\alpha)$ for $|\alpha| > 1$ are very similar to those for $|\alpha| < 1$. 
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Jacobian elliptic functions

For $0 \leq \alpha \leq 1$, the integral (incomplete elliptic of 1st kind)

$$u = \int_{0}^{\varphi} \frac{d\theta}{\sqrt{1 - \alpha^2 \sin^2 \theta}}$$

measures the arc length of an ellipse.
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Its inverse $\varphi(u) = \text{am}(u, \alpha)$ is known as the amplitude.
Jacobian elliptic functions

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- Its inverse $\varphi(u) = \text{am}(u, \alpha)$ is known as the amplitude.
- The (copolar) triplet of JEF:
  \begin{align*}
  \text{sn}(u, \alpha) &= \sin \text{am}(u, \alpha), \\
  \text{cn}(u, \alpha) &= \cos \text{am}(u, \alpha), \\
  \text{dn}(u, \alpha) &= \sqrt{1 - \alpha^2 \sin^2 \text{am}(u, \alpha)}.
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- Complete elliptic integral of the first kind:
  
  \[ K(\alpha) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \alpha^2 \sin^2 \theta}}. \]
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Complete elliptic integral of the first kind:

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JEFs are meromorphic functions in $u$ (with $\alpha$ fixed) as well as meromorphic functions in $\alpha$ (with $u$ fixed). While $K$ is analytic in the cut-plane $\mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$. 
Jacobian elliptic functions - plotting
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Spectral analysis of $J(\alpha)$ in the self-adjoint case

- We start with the identities

$$\langle e_1, J(\alpha)^{2n+1} e_1 \rangle = 0 \quad \text{and} \quad \langle e_1, J(\alpha)^{2n} e_1 \rangle = C_{2n}(\alpha)$$
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where \( C_{2n} \) are polynomials that can be defined via the generating function formula:

\[
\text{cn}(z, \alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n C_{2n}(\alpha)}{(2n)!} z^{2n}.
\]
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where $C_{2n}$ are polynomials that can be defined via the generating function formula:

$$c_n(z, \alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n C_{2n}(\alpha)}{(2n)!} z^{2n}. $$

- Hence we may write

$$c_n(z, \alpha) = \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} \langle e_1, J(\alpha)^n e_1 \rangle = \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} \int_{\mathbb{R}} x^n d\mu(x) = \int_{\mathbb{R}} e^{ixz} d\mu(x).$$

where we denote $\mu(\cdot) := \langle e_1, E_J(\cdot)e_1 \rangle$. 
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where we denote $\mu(\cdot) := \langle e_1, EJ(\cdot)e_1 \rangle$.

- We get

$$\mathcal{F}[\mu](z) = c_n(z, \alpha).$$
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$$cn(z, \alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n C_{2n}(\alpha)}{(2n)!} z^{2n}.$$ 

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where we denote $\mu(\cdot) := \langle e_1, E_J(\cdot) e_1 \rangle$.

We get

$$F[\mu](z) = cn(z, \alpha).$$

Consequently, by applying the inverse Fourier transform to the function $cn(z; \alpha)$, one may recover the spectral measure $\mu$!
Spectral analysis of $J(\alpha)$ for $\alpha \in (-1, 1)$

For $\alpha \in (-1, 1)$, the evaluation of the inverse Fourier transform yields

$$
\mu(t) = \frac{\pi}{\alpha K} \sum_{n=0}^{\infty} \frac{q^{n+1/2}}{1 + q^{2n+1}} \left[ \delta \left( t - \frac{(2n + 1)\pi}{2K} \right) + \delta \left( t + \frac{(2n + 1)\pi}{2K} \right) \right]
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where the nome $q = q(\alpha)$ ($|q| < 1$).
Spectral analysis - the self-adjoint case

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where the nome \( q = q(\alpha) \) (\(|q| < 1\)).

Hence the measure \( \mu \) is discrete supported by the set

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\text{supp} \mu = \frac{\pi}{2K} (2\mathbb{Z} + 1).
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- Hence the measure $\mu$ is discrete supported by the set

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- This implies that, for $\alpha \in (-1, 1)$, the spectrum of $J(\alpha)$ is discrete and

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- In addition, we can also compute the Weyl $m$-function $m(z; \alpha) := \langle e_1, (J(\alpha) - z)^{-1} e_1 \rangle$, since
  \[
  m(z, \alpha) = iL[\text{cn}(t, \alpha)](-iz), \quad \text{for } \Re z > 0.
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m(z, \alpha) := \langle e_1, (J(\alpha) - z)^{-1} e_1 \rangle,$$

since

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$$

- It results in the formula

$$
m(z, \alpha) = \frac{2\pi z}{\alpha K} \sum_{n=0}^{\infty} \frac{q^{n+1/2}}{1 + q^{2n+1}} \frac{1}{\frac{(2n+1)^2\pi^2}{4K^2} - z^2}.
$$
Spectral analysis of $J(\alpha)$ for $\alpha = \pm 1$

- Recall that

$$\mathcal{F}[\mu](z) = \text{cn}(z, \pm 1) = \frac{1}{\cosh(z)}.$$
Spectral analysis of $J(\alpha)$ for $\alpha = \pm 1$

- Recall that
  \[ \mathcal{F}[\mu](z) = \text{cn}(z, \pm 1) = \frac{1}{\cosh(z)}. \]

- By applying the inverse Fourier transform, one concludes that $\mu$ is absolutely continuous measure supported on $\mathbb{R}$ and its density equals
  \[ \frac{d\mu}{dt} = \frac{1}{2 \cosh(\pi t/2)}, \quad \forall t \in \mathbb{R}. \]
Spectral analysis of $J(\alpha)$ for $\alpha = \pm 1$

- Recall that
  $$\mathcal{F}[\mu](z) = \text{cn}(z, \pm 1) = \frac{1}{\cosh(z)}.$$  

- By applying the inverse Fourier transform, one concludes that $\mu$ is absolutely continuous measure supported on $\mathbb{R}$ and its density equals
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- This implies that the spectrum of $J(\pm 1)$ is purely absolutely continuous and
  $$\sigma(J(\pm 1)) = \sigma_{ac}(J(\pm 1)) = \mathbb{R}.$$
Spectrum of $J(\alpha)$ in the self-adjoint case - animation
Contents

1. Complex Jacobi matrices - generalities
2. The Jacobi matrix associated with Jacobian elliptic functions
3. Intermezzo I - Jacobian elliptic functions
4. Spectral analysis - the self-adjoint case
5. Spectral analysis - the non-self-adjoint case
6. Intermezzo II - extremal properties of $|\text{sn}(uK(m) \mid m)|$
Spectral analysis of $J(\alpha)$ for $|\alpha| < 1$

- For $|\alpha| < 1$, the operator $J(\alpha)$ can be viewed as a perturbation of $J(0)$ with relative bound smaller than 1.
Spectral analysis of $J(\alpha)$ for $|\alpha| < 1$

- For $|\alpha| < 1$, the operator $J(\alpha)$ can be viewed as a perturbation of $J(0)$ with relative bound smaller than 1.
- Consequently, the spectrum of $J(\alpha)$ is discrete if $|\alpha| < 1$. 

In addition, by an analyticity argument it can be shown the formula for the Weyl m-function $m(z,\alpha) = \frac{2\pi z^{\alpha}}{K_\infty} \sum_{n=0}^{\infty} \frac{q^{n+1/2}}{1 + q^{2(n+1)}} \frac{1}{2\pi^2 K^2 - z^2}$. remains true for all $z \in \rho(J(\alpha))$ and $|\alpha| < 1$.

It implies (in the non-self-adjoint case, too!) that $\sigma(J(\alpha)) = \frac{\pi}{2} K(2Z + 1)$. and all the eigenvalues are simple.
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$$m(z, \alpha) = \frac{2\pi z}{\alpha K} \sum_{n=0}^{\infty} \frac{q^{n+1/2}}{1 + q^{2n+1}} \frac{1}{\frac{(2n+1)^2}{4K^2} - z^2}.$$

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$$\sigma(J(\alpha)) = \frac{\pi}{2K} (2N + 1).$$

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**Eigenvectors of $J(\alpha)$ for $|\alpha| < 1$**

**Proposition**

Let $0 < |\alpha| < 1$ and $N \in \mathbb{Z}$, then the vector $v^{(N)}$ given by formulas

$$v_{2k+1}^{(N)} = i(-1)^k \alpha^k \int_0^{2\pi} e^{-i(N+1/2)s} \, \text{cn} \left( \frac{Ks}{\pi}, \alpha \right) \, \text{sn}^{2k} \left( \frac{Ks}{\pi}, \alpha \right) \, ds$$

and

$$v_{2k+2}^{(N)} = (-1)^{k+1} \alpha^k \int_0^{2\pi} e^{-i(N+1/2)s} \, \text{dn} \left( \frac{Ks}{\pi}, \alpha \right) \, \text{sn}^{2k+1} \left( \frac{Ks}{\pi}, \alpha \right) \, ds,$$

for $k \geq 0$, is the eigenvector of $J(\alpha)$ corresponding to the eigenvalue $\frac{\pi}{2K}(2N + 1)$. 
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In addition, the set $\{v^{(N)} \mid N \in \mathbb{Z}\}$ is complete in $\ell^2(\mathbb{N})$. 
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**Interesting open problems:**

1. $\|v^{(N)}\| = ?$ or $\|v^{(N)}\| \sim ?$ for $N \to \pm\infty$.
2. Is $\{v^{(N)} \mid N \in \mathbb{Z}\}$ the Riesz basis of $\ell^2(\mathbb{N})$?
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František Štampach (Stockholm University)
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Spectral analysis - the non-self-adjoint case

Spectrum of $J(\alpha)$ in the non-self-adjoint case - animation
Spectral analysis of $J(\alpha)$ for $|\alpha| = 1$

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If $|\alpha| = 1$, $\alpha \neq \pm 1$, then

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- Then one can verify, indeed, that

$$\lim_{a \to 1^-} \frac{\| (J(\alpha) - z) u(a) \|}{\| u(a) \|} = 0, \quad \text{and} \quad \text{w-\lim}_{a \to 1^-} u(a) = 0.$$
Spectral analysis of \( J(\alpha) \) for \(|\alpha| = 1\) - cont.

Essential for the verification of the “singular property” of the family \( u(a) = a^n u_n \) are two main ingredients:
Spectral analysis of $J(\alpha)$ for $|\alpha| = 1$ - cont.

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   $$\int_0^{2K} e^{-izt} \left\{ \frac{\text{cn}(t, \alpha)}{\text{dn}(t, \alpha)} \right\} \text{sn}^k(t, \alpha) dt, \quad \text{as } k \to \infty.$$
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- The asymptotic formulas (ingredient 2.) can be obtained by applying the saddle point method.
- However, one has to know the location of global maxima of function

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in $(0, 2)$ for $|\alpha| = 1$, $\alpha \neq \pm 1$. 
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in $(0, 2)$ for $|\alpha| = 1, \alpha \neq \pm 1$.

- It can be shown (not trivial!) that the function has unique global maximum at $u = 1$ for every $|\alpha| = 1, \alpha \neq \pm 1$.  

František Štampach (Stockholm University)
Intermezzo II - extremal properties of $|\text{sn}(uK(m) \mid m)|$

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1 Complex Jacobi matrices - generalities

2 The Jacobi matrix associated with Jacobian elliptic functions

3 Intermezzo I - Jacobian elliptic functions

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5 Spectral analysis - the non-self-adjoint case

6 Intermezzo II - extremal properties of $|\text{sn}(uK(m) \mid m)|$
On the extremal properties of $|\text{sn}(uK(m) \mid m)|$

All the necessary properties known when $m \in (0, 1)$
On the extremal properties of $|\text{sn}(uK(m) \mid m)|$

Transf. modulus $m$ from the unit circle to $(0, 1)$
On the extremal properties of $|\text{sn}(uK(m) \mid m)|$

$\Im m > 0$, $m = e^{4i\theta}$ and $s = \text{sn}(uK(\cos^2 \theta) \mid \cos^2 \theta)$, $s_1 = \text{sn}(uK(\sin^2 \theta) \mid \sin^2 \theta)$, etc.

$\text{sn}^2(uK(m) \mid m) = \frac{1}{\sqrt{m}} \frac{c_1^2 + s^2 s_1^2 \cos^2 \theta - cc_1 + iss_1 dd_1}{c_1^2 + s^2 s_1^2 \cos^2 \theta + cc_1 - iss_1 dd_1}$,
On the extremal properties of $|\text{sn}(uK(m) \mid m)|$

$$|\text{sn}(uK(m) \mid m)| \leq 1 \quad \text{for all} \quad m \in \partial \mathbb{D} \quad \text{with the equality only for} \quad m = 1.$$
On the extremal properties of $|\text{sn}(uK(m) | m)|$

Maximum modulus...

$|\text{sn}(uK(m) | m)| \leq 1$ for all $m \in \mathbb{D}$ with the equality only for $m = 1$. 
On the extremal properties of $|\text{sn}(uK(m) \mid m)|$

Another circle $\mathbb{D}_1 = \{ z \mid |z - 1| = 1 \}$
On the extremal properties of $|\text{sn}(uK(m) \mid m)|$

Another circle $D_1 = \{z \mid |z - 1| = 1\}$ and another transformation formula (not displayed) ...
On the extremal properties of $|\text{sn}(uK(m) \mid m)|$

$|\text{sn}(uK(m) \mid m)| < 1$ for all $m \in \partial \mathbb{D}_1$
On the extremal properties of $|\text{sn}(uK(m) \mid m)|$ 

In addition, 

$$\lim_{\epsilon \to 0^+} |\text{sn}(uK(m \pm i\epsilon) \mid m \pm i\epsilon)| < 1$$

for all $m \geq 2$ and the function $m \mapsto \text{sn}(uK(m) \mid m)$ is bounded.
On the extremal properties of $| \text{sn}(uK(m) \mid m)|$

$| \text{sn}(uK(m) \mid m)| \leq 1$ for all $m \notin D_1 \setminus D$ with the equality only for $m = 1$. 
On the extremal properties of $|\text{sn}(uK(m) \mid m)|$

If $0 < u \leq \frac{1}{2}$ the global maximum of $m \mapsto |\text{sn}(uK(m) \mid m)|$

is located at $m = 1$ with the value $= 1$. 
On the extremal properties of $|\text{sn}(uK(m) \mid m)|$

If $\frac{1}{2} < u < 1$, the global maximum of $m \mapsto |\text{sn}(uK(m) \mid m)|$ is located in $(1, 2)$ with the value $> 1$. 
On the extremal properties of $|\text{sn}(uK(m) \mid m)|$ - main theorem

Theorem:
The following statements hold true.
On the extremal properties of $|\text{sn}(uK(m) | m)|$ - main theorem

**Theorem:**

The following statements hold true.

1. For all $u \in (0, 1)$ and $m \notin \{z \in \mathbb{C} \mid |z - 1| < 1 \land |z| > 1\}$, it holds

   $$|\text{sn}(K(m)u | m)| < 1.$$
The following statements hold true.

1. For all $u \in (0, 1)$ and $m \not\in \{ z \in \mathbb{C} \mid |z - 1| < 1 \land |z| > 1 \}$, it holds

   $$|\text{sn}(K(m)u \mid m)| < 1.$$ 

2. For $u \in (0, 1/2]$ the function $m \mapsto |\text{sn}(K(m)u \mid m)|$ has unique global maximum located at $m = 1$ with the value equal to 1, i.e.,

   $$|\text{sn}(K(1)u \mid 1)| = 1 \quad \text{and} \quad |\text{sn}(K(m)u \mid m)| < 1 \quad \text{for all } m \neq 1$$

   (where the value at $m = 1$ is to be understood as the respective limit).
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The following statements hold true.

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   \[
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   \]
   (where the value at \( m = 1 \) is to be understood as the respective limit).

3. For \( u \in (1/2, 1) \), the function \( m \mapsto |\text{sn}(K(m)u | m)| \) has a global maximum located in the interval \((1, 2)\) with the value exceeding 1, i.e.,
   \[
   \max_{m \in \mathbb{C}} |\text{sn}(K(m)u | m)| = |\text{sn}(K(m^*)u | m^*)| > 1 \quad \text{for some} \quad m^* \in (1, 2).
   \]
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2. P. Siegl, F. Š.: Spectral analysis of non-self-adjoint Jacobi operator associated with Jacobian elliptic functions, arXiv:1603.01052.

Thank you!