ON THE DERIVATIVES OF THE LEMPERT FUNCTIONS

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Abstract. We show that if the Kobayashi–Royden metric of a complex manifold is continuous and positive at a given point and any non-zero tangent vector, then the “derivatives” of the higher order Lempert functions exist and equal the respective Kobayashi metrics at the point. It is a generalization of a result by M. Kobayashi for taut manifolds.

1. Introduction and results

Let $\mathbb{D} \subset \mathbb{C}$ be the unit disc. Let $M$ be an $n$-dimensional complex manifold. Recall first the definitions of the Lempert function $k_M$ and the Kobayashi–Royden pseudometric $\kappa_M$ of $M$:

$$k_M^*(z, w) = \inf \{ |\alpha| : \exists f \in \mathcal{O}(\mathbb{D}, M) : f(0) = z, f(\alpha) = w \},$$

$$\kappa_M = \tanh^{-1} k_M^*,$$

$$\kappa_M(z; X) = \inf \{ |\alpha| : \exists f \in \mathcal{O}(\mathbb{D}, M) : f(0) = z, \alpha f^*(d/d\zeta) = X \},$$

where $X$ is a complex tangent vector to $M$ at $z$. Note that such an $f$ always exists (cf. [12]; according to [2], page 49, this was already known by J. Globevnik).

The Kobayashi pseudodistance $k_M$ can be defined as the largest pseudodistance bounded by $\kappa_M$. Note that if $k_M^{(m)}$ denotes the $m$-th Lempert function of $M$, $m \in \mathbb{N}$, that is,

$$k_M^{(m)}(z, w) = \inf \left\{ \sum_{j=1}^{m} k_M(z_{j-1}, z_j) : z_0, \ldots, z_m \in M, z_0 = z, z_m = w \right\},$$

then

$$k_M(z, w) = k_M^{(\infty)} := \inf_m k_M^{(m)}(z, w).$$

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By a result of M.-Y. Pang (see [9]), the Kobayashi–Royden metric is the ”derivative” of the Lempert function for taut domains in \( \mathbb{C}^n \); more precisely, if \( D \subset \mathbb{C}^n \) is a taut domain, then
\[
\kappa_D(z;X) = \lim_{t \to 0} \frac{\kappa_D(z, z + tX)}{t}.
\]

In [6], S. Kobayashi introduces a new invariant pseudometric, called the Kobayashi–Buseman pseudometric \( \hat{\kappa}_M \) of \( M \) is just to set \( \hat{\kappa}_M(z;\cdot) \) to be largest pseudonorm bounded by \( \kappa_M(z;\cdot) \). Recall that
\[
\hat{\kappa}_M(z;X) = \inf\{\sum_{j=1}^m \kappa_M(z;X_j) : m \in \mathbb{N}, \sum_{j=1}^m X_j = X\}.
\]

Thus it is natural to consider the new function \( \kappa_M^{(m)}(z;X) \), \( m \in \mathbb{N} \), namely,
\[
\kappa_M^{(m)}(z;X) = \inf\{\sum_{j=1}^m \kappa_M(z;X_j) : \sum_{j=1}^m X_j = X\}.
\]

We call \( \kappa_M^{(m)} \) the \( m \)-th Kobayashi pseudometric of \( D \). It is clear that \( \kappa_M^{(m)} \geq \kappa_M^{(m+1)} \) and if \( \kappa_M^{(m)}(z;\cdot) = \kappa_M^{(m+1)}(z;\cdot) \) for some \( m \), then \( \kappa_M^{(m)}(z;\cdot) = \kappa_D^{(j)}(z;\cdot) \) for any \( j > m \). It is shown in [8] that \( \kappa_M^{(2n-1)} = \kappa_M^{(\infty)} := \hat{\kappa}_M \), and \( 2n - 1 \) is the optimal number, in general.

We point out that all the introduced objects are upper semicontinuous. Recall that this is true for \( \kappa_M \) (cf. [7]). It remains to check this for \( \hat{\kappa}_M \). We shall use a standard reasoning. Fix \( r \in (0,1) \) and \( z, w \in M \). Let \( f \in \mathcal{O}(\mathbb{D},M) \), \( f(0) = z \) and \( f(\alpha) = w \). Then \( \tilde{f} = (f, \text{id}) : \Delta \to \tilde{M} = M \times \Delta \) is an embedding. Setting \( \tilde{f}_r(\zeta) = \tilde{f}(r\zeta) \), by [10], Lemma 3, we may find a Stein neighborhood \( S \subset M \) of \( \tilde{f}_r(\mathbb{D}) \). Embed \( S \) as a closed complex manifold in some \( \mathbb{C}^N \) and denote by \( \psi \) the respective embedding. Moreover, there is an open neighborhood \( V \subset M \) of \( \psi(S) \) and a holomorphic retraction \( \theta : V \to \psi(S) \). Then, for \( z' \) near \( z \) and \( w' \) near \( w \), we may find, as usual, \( g \in \mathcal{O}(\mathbb{D},V) \) such that \( g(0) = \psi(z',0) \) and \( g(\alpha/r) = \psi(w',\alpha) \). Denote by \( \pi \) the natural projection of \( \tilde{M} \) onto \( M \). Then \( h = \pi \circ \psi^{-1} \circ \theta \circ g \in \mathcal{O}(\mathbb{D},M) \), \( h(0) = z' \) and \( h(\alpha/r) = w' \). So \( r\hat{\kappa}_M^{(\alpha)}(z',w') \leq \alpha \), which implies that
\[
\limsup_{z' \to z, w' \to w} \frac{\hat{\kappa}_M(z',w')}{\alpha} \leq \hat{\kappa}_M(z,w).
\]

To extend Pang’s result on manifolds, we have to define the ”derivatives” of \( \kappa_M^{(m)} \), \( m \in \mathbb{N}^* = \mathbb{N} \cup \{\infty\} \). Let \( (U, \varphi) \) be a holomorphic chart
near $z$. Set

$$D_k^{(m)}(z; X) = \limsup_{t \to 0, w \to z, Y \to \varphi \cdot X} |w|^{-1}(\varphi(w) + tY).$$

Note that this notion does not depend on the chart used in the definition and

$$D_k^{(m)}(z; \lambda X) = |\lambda|D_k^{(m)}(z; X), \quad \lambda \in \mathbb{C}.$$

Replacing $\limsup$ by $\liminf$, we define $D_k^{(m)}$. From M. Kobayashi's paper [5] it follows that, if $M$ is a taut manifold, then

$$\hat{\kappa}_M(z; X) = D_k^{(m)}(z; X) = D_k(z; X),$$

that is, the Kobayashi–Buseman metric is the "derivative" of the Kobayashi distance. The proof there also leads to

$$(*) \quad \kappa_M^{(m)}(z; X) = D_k^{(m)}(z; X) = D_k^{(m)}(z; X), \quad m \in \mathbb{N}^*.$$

We say that a complex manifold $M$ is hyperbolic at $z$ if $k_M(z, w) > 0$ for any $w \neq z$. We point out that the following conditions are equivalent:

(i) $M$ is hyperbolic at $z$;

(ii) $\liminf_{z' \to z, w \in M \setminus U} \hat{\kappa}_M(z', w) > 0$ for any neighborhood $U$ of $z$;

(iii) $\hat{\kappa}_M(z; X) := \liminf_{z' \to z, X' \to X} \kappa_M(z', X') > 0$ for any $X \neq 0$;

The implication (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are almost trivial (cf. [4]) and the implication (iii) $\Rightarrow$ (i) is a consequence of the fact that $k_M$ is the integrated form of $\kappa_M$.

In particular, if $M$ is hyperbolic at $z$, then it is hyperbolic at any $z'$ near $z$.

Since if $M$ is taut, then it is $k$-hyperbolic and $\kappa_M$ is a continuous function, the following theorem is a generalization of $(*).$

**Theorem 1.** Let $M$ be a complex manifold and $z \in M$.

(i) If $M$ is hyperbolic at $z$ and $\kappa_M$ is continuous at $(z, X)$, then

$$\kappa_M(z; X) = D_k(z; X) = D_k^{(m)}(z; X).$$

(ii) If $\kappa_M$ is continuous and positive at $(z, X)$ for any $X \neq 0$, then

$$\kappa_M^{(m)}(z; \cdot) = D_k^{(m)}(z; \cdot) = D_k^{(m)}(z; \cdot), \quad m \in \mathbb{N}^*.$$

The first step in the proof of Theorem 1 is the following

**Proposition 2.** For any complex manifold $M$ one has that

$$\kappa_M^{(m)} \geq D_k^{(m)}, \quad m \in \mathbb{N}^*.$$
Note that when $M$ is a domain, a weaker version of Proposition $2$ can be found in [3], namely, $\hat{k}_M \geq \mathcal{D}k_M$ (the proof is based on the fact that $\mathcal{D}k_M(z; \cdot)$ is a pseudonorm).

2. Examples

The following examples show that the assumption on continuity in Theorem $1$ is essential.

- Let $A$ be a countable dense subset of $\mathbb{C}$, In $[1]$ (see also $[3]$), a pseudoconvex domain $D$ in $\mathbb{C}^2$ is constructed such that:
  
  (a) $(\mathbb{C} \times \{0\}) \cup (A \times \mathbb{C}) \subset D$;
  
  (b) if $z_0 = (0, t) \in D$, $t \neq 0$, then $\kappa_D(z_0; X) \geq C\|X\|$ for some $C = C_t > 0$. (One can be shown that even $\mathcal{D}\hat{k}_D(z_0; X) \geq C\|X\|$).

Then it is easy to see that $\kappa_D(\cdot; e_2) = \mathcal{D}k_D(\cdot; e_2) = k_D^{(3)} = 0$ and $\hat{k}_D(z_0; X) \geq c\|X\|$, where $e_2 = (0, 1)$ and $c > 0$. Thus

$$\hat{k}_D(z_0; X) > \kappa_D(z_0; e_2) = \mathcal{D}k_D^{(3)}(z_0; e_2) = \mathcal{D}k_D^{(5)}(z_0; X), \quad X \neq 0.$$  

This phenomena obviously extends to $\mathbb{C}^n, n \geq 2$ (by considering $D \times \mathbb{D}^{n-2}$). So the inequalities in Proposition 2 are strict in general.

- If $D$ is a pseudoconvex balanced domain with Minkowski function $h_D$, then (cf. [3])

$$h_D = \kappa_D(0; \cdot) = \mathcal{D}\hat{k}_D(0; \cdot).$$

Therefore, $\mathcal{D}\hat{k}_D(0; X) > \mathcal{D}\hat{k}_D(0; X)$ if $\kappa_D(0; \cdot)$ is not continuous at $X$.

On the other hand, if $D$ denotes the convex hull of $D$, then

$$h_D = \hat{k}_D(0; \cdot) = \mathcal{D}k_D(0; \cdot) = \mathcal{D}k_D(0; \cdot) = \hat{k}_D(0; \cdot).$$

- Modifying the first example leads to a pseudoconvex domain $D \subset \mathbb{C}^2$ with

$$L_{\mathcal{D}k_D}(\gamma) > 0 = L_{k_D}(\gamma) = L_{\mathcal{D}\hat{k}_D}(\gamma),$$

where $\gamma : [0, 1] \to \mathbb{C}^2$, $\gamma(t) := (ti/2, 1/2)$, and $L_\gamma(\gamma)$ denotes the respective length.

Indeed, choose a dense sequence $(r_j)$ in $[0, i/2]$. Put

$$u(\lambda) = \sum_{k=1}^{\infty} \frac{1}{k^2} \log \frac{|\lambda-1/k|}{4}, \quad v(\lambda) = \sum_{j=1}^{\infty} \frac{u(\lambda/2 - r_j)}{2j^2}, \quad \lambda \in \mathbb{C},$$

and

$$D = \{ z \in \mathbb{C}^2 : \psi(z) = |z_2| e^{\|z\|^2 + v(z_1)} < 1 \}.$$  

It is easy to see that $v$ is a subharmonic function on $\mathbb{C}$. Hence $D$ is a pseudoconvex domain with $(\mathbb{C} \times \{0\}) \cup (\bigcup_{j,k=1}^{\infty} \{ r_j + 1/k \} \times \mathbb{C}) \subset D$. 

Observe that \( u|_{\mathbb{D}} < -1 \) and so \( D \) contains the unit ball \( \mathbb{B}_2 \). Note also that
\[
k_D(a, b) = 0, \quad a, b \in \gamma([0, 1]).
\]
Set \( \hat{\psi}(z) = \|z\|^2/2 - \log \psi(z) \). Fix \( z^0 \in \mathbb{B}_2 \) with \( \text{Re} z^0_1 \leq 0, \text{Im} z^0_2 \geq 1/e \).
Since \( u(\lambda) \geq u(0) \) for \( \text{Re} \lambda \leq 0 \), we have
\[
\|z^0\|^2/2 < \hat{\psi}(z^0) < 1 - u(0) =: 8C.
\]

Let \( \varphi \in \mathcal{O}(\mathbb{D}, D); \varphi(0) = z^0 \). Following the estimates in the proof of Example 3.5.10 in [3], we see that \( \|\varphi'(0)\| < C \). Hence, \( \kappa_D(z^0, X) \geq C\|X\|, X \in \mathbb{C}^2 \). Since \( k_D \) is the integrated form of \( \kappa_D \), it follows that
\[
k_D(a - te_1) \geq Ct, \quad a \in \gamma([0, 1]), \quad 0 \leq t \leq 1/2 - 1/e, \quad e_1 = (1, 0).
\]
Hence \( \mathcal{D}k_D(a; e_1) \geq C \) and therefore, \( L_{\mathcal{D}k_D}(\gamma) \geq C/2 > 0 \), which completes the proof of this example.

Note that it shows that, with respect to the lengths of curves, \( \mathcal{D}k_D \) behaves different than the ”real” derivative of \( k_D \) (cf. [11] or [4], page 12). Moreover, it implies that, in general, \( \mathcal{D}k_D \neq \tilde{\mathcal{D}}k_D \).

**Questions.** It will be interesting to know examples showing that, in general, \( \kappa_D \neq \mathcal{D}k_D \). It remains also unclear whether \( \mathcal{D}k_D \) is holomorphically contractible (see [3]). Recall that \( \int \mathcal{D}k_D = k_D \); but we do not know if \( \int \tilde{\mathcal{D}}k_D = k_D \).

### 3. Proofs

**Proof of Proposition**

First, we shall consider the case \( m = 1 \). The key is the following

**Theorem 3.** [10] Let \( M \) be an \( n \)-dimensional complex manifold and \( f \in \mathcal{O}(\mathbb{D}, M) \) regular at 0. Let \( r \in (0, 1) \) and \( D_r = r\mathbb{D} \times \mathbb{D}^{n-1} \). Then there exists \( F \in \mathcal{O}(D_r, M) \), which is regular at 0 and \( F|_{r\mathbb{D} \times \{0\}} = f \).

Since \( \kappa_M(z; 0) = \mathcal{D}\tilde{k}_M(z; 0) = 0 \), we may assume that \( X \neq 0 \). Let \( \alpha > 0 \) and \( f \in \mathcal{O}(\mathbb{D}, M) \) be such that \( f(0) = z \) and \( \alpha f_* (d/d\zeta) = X \). Let \( r \in (0, 1) \) and \( F \) as in Theorem 3. Since \( F \) is regular at 0, there exist open neighborhoods \( U = U(z) \subset M \) and \( V = V(0) \subset D_r \) such that \( F|_V : V \rightarrow U \) is biholomorphic. Hence \((U, \varphi) \) with \( \varphi = (F|_V)^{-1} \), is a chart near \( z \). Note that \( \varphi_*(X) = \alpha e_1 \), where \( e_1 = (1, 0, \ldots, 0) \).

If \( w \) and \( Y \) are sufficiently near \( z \) and \( \alpha e_1 \), respectively, then
\[
g(\zeta) := F(\varphi(w) + \zeta Y/\alpha), \quad \zeta \in r^2 \mathbb{D}.
\]

*We may replace Theorem 3 by the approach used in the proof of the upper semicontinuity of \( \tilde{k}_M \).*
belongs to $O(r^2 \mathbb{D}, M)$ with $g(0) = w$ and $g(t) = \varphi^{-1}(\varphi(w) + tY)$, $t < r^2/\alpha$. Therefore, $r^2 k^*_M(w, \varphi^{-1}(\varphi(w) + tY)) \leq \alpha$. Hence $r^2 Dk_M(z; X) \leq \alpha$. Letting $r \to 1$ and $\alpha \to \kappa_M(z; X)$ we get that $Dk_M(z; X) \leq \kappa_M(z; X)$.

Let now $m \in \mathbb{N}$. By definition, $\kappa^{(m)}_M(z; \cdot)$ is the largest function with the following property:

For any $X = \sum_{j=1}^m X_j$ one has that $\kappa^{(m)}_M(z; X) \leq \sum_{j=1}^m \kappa_M(z; X_j)$.

To prove that $\kappa^{(m)}_M \geq Dk^{(m)}_M$ it suffices to check that $Dk^{(m)}_M(z; \cdot)$ has the same property. Following the above notation and choosing $Y_j \to \varphi_* X_j$ with $\sum_{j=1}^m Y_j = Y$, we set $w_0 = w$ and $w_j = \varphi^{-1}(\varphi(w) + t \sum_{k=1}^j Y_j)$. Since

$$k^{(m)}_M(w, w_q) \leq \sum_{j=1}^m \tilde{k}_M(w_{j-1}, w_j),$$

it follows by the case $m = 1$ that

$$Dk^{(m)}_M(z; X) \leq \sum_{j=1}^m Dk_M(z; X_j) \leq \sum_{j=1}^m \kappa_M(z; X_j).$$

Finally, let $m = \infty$ and $n = \dim M$. Since $\tilde{k}_M = \kappa_2^{(2n-1)}$ and $k_M \leq k_2^{(2n-1)}$, we get that $Dk_M \leq \kappa_M$ using the case $m = 2n - 1$.

**Proof of Theorem 1.** We may assume that $X \neq 0$. In virtue of Proposition 2 we have to show that

$$\kappa^{(m)}_M(z; X) \leq Dk^{(m)}_M(z; X).$$

For simplicity we assume that $M$ is a domain in $\mathbb{C}^n$.

(i) Fix a neighborhood $U = U(z) \in M$. Applying the hyperbolicity of $M$ at $z$, there are a neighborhood $V = V(z) \subset U$ and a $\delta \in (0,1)$ such that, if $h \in O(\mathbb{D}, M)$ with $h(0) \in V$, then $h(\delta \mathbb{D}) \subset U$. Hence, by the Cauchy inequalities, $||h^{(k)}(0)|| \leq c/\delta^k$, $k \in \mathbb{N}$.

Now choose sequences $M \ni w_j \to z$, $C_* \ni t_j \to 0$, and $\mathbb{C}^n \ni Y_j \to X$ such that

$$\frac{\tilde{k}_M(w_j, w_j + t_j Y_j)}{|t_j|} \to Dk_M(z; X).$$

There are holomorphic discs $g_j \in O(\mathbb{D}, M)$ and $\beta_j \in (0,1)$ with $g_j(0) = w_j$, $g_j(\beta_j) = w_j + t_j Y_j$, and $\beta_j \leq \tilde{k}_M^*(w_j, w_j + t_j Y_j) + |t_j|/\beta_j$. Note that

$$\tilde{k}_M^*(w_j, w_j + t_j Y_j) \leq c_1 ||t_j Y_j|| \leq c_2 |t_j|.$$

Write

$$w_j + t_j Y_j = g_j(\beta_j) = w_j + g'_j(0) \beta_j + h_j(\beta_j).$$
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Then

$$||h_j(\beta_j)|| \leq c \sum_{k=2}^{\infty} \left( \frac{2}{k} \right)^k \leq c_3 |\beta_j|^2 \leq c_4 |t_j|^2, \quad j \geq j_0.$$ 

Put $\hat{Y}_j = Y_j - h_j(\beta_j)/t_j$. We have that $g_j(0) = w_j$ and $\beta_j g'_j(0)/t_j = \hat{Y}_j \to X$. Therefore,

$$\kappa_M(w_j; \hat{Y}_j) \leq \frac{\beta_j}{|t_j|} \leq \frac{\bar{k}_M^*(z_j, w_j + t_j Y_j)}{|t_j|} + \frac{1}{j}.$$ 

Hence with $j \to \infty$, we get that $\kappa_M(z; X) = \kappa_M(w; Y) \leq D_{\bar{k}}^k M(z; X)$. 

(ii) The proof of the case $m \in \mathbb{N}$ is similar to the next one and we omit it. Now, we shall consider the case $m = \infty$.

Note first that our assumption implies that $M$ is hyperbolic at $z$ and, by the contrary,

$$\forall \varepsilon > 0 \exists \delta > 0 : ||w - z|| < \delta, ||Y - X|| < \delta ||X|| \Rightarrow |\kappa_M(w; Y) - \kappa_M(z; X)| < \varepsilon \kappa_M(z; X).$$

Moreover, the proof of (i) shows that

$$\bar{k}_M(a, b) \geq \kappa_M(a; b - a + o(a, b)), \text{ where } \lim_{a, b \to z} \frac{o(a, b)}{||a - b||} = 0.$$ 

Choose now sequences $M \ni w_j \to z$, $C_* \ni t_j \to 0$, and $\mathbb{C}^n \ni Y_j \to X$ such that

$$\frac{k_M(w_j, w_j + t_j Y_j)}{|t_j|} \to D_{\bar{k}} M(z; X).$$

There are points $w_{j,0} = w_j, \ldots, w_{j,m_j} = w_j + t_j X_j$ in $M$ such that

$$\sum_{k=1}^{m_j} \bar{k}_M(w_j, w_{j,k-1}, w_{j,k}) \leq k_M(w_j, w_j + t_j Y_j) + \frac{1}{j}.$$ 

Set $w_{j,k} = w_j$ for $k > m_j$. Since

$$k_M(w_j, w_{j,l}) \leq \sum_{j=1}^{l} \bar{k}_M(w_j, w_{j,k-1}, w_{j,k}) \leq k_M(w_j, w_j + t_j Y_j) + \frac{1}{j} \leq c_2 |t_j| + \frac{1}{j},$$

then $k_M(w_j, w_{j,l}) \to 0$ uniformly in $l$. Then the hyperbolicity of $M$ at $z$ implies that $w_{j,l} \to z$ uniformly in $l$. Indeed, assuming the contrary and passing to a subsequence, we may suppose that $w_{j,l} \notin U$ for some $U = U(z)$. Then

$$0 = \lim_{j \to \infty} k_M(w_j, w_{j,l}) \geq \lim_{z' \to z, w \in M \setminus U} \bar{k}_M(z', w) > 0,$$

a contradiction.
Fix now $R > 1$. Then (1) implies that
\[ \kappa_M(z; w_{j,k} - w_{j,k-1}) \leq R\kappa_M(w_{j,k}; w_{j,k} - w_{j,k-1} + o(w_{j,k}, w_{j,k-1})), \ j \geq j(R). \]
It follows by this inequality, (2) and (3) that
\[ \sum_{k=1}^{m_j} \kappa_M(z; w_{j,k} - w_{j,k-1}) \leq Rk_M(w_j, w_j + t_j Y_j) + \frac{R}{j}. \]
Since $\hat{\kappa}_M(z; t_j Y_j)$ is bounded by the first sum, we obtain that
\[ \hat{\kappa}_M(z; Y_j) \leq R\frac{k_M(w_j, w_j + t_j Y_j) + 1/j}{|t_j|}. \]
Note that $\hat{\kappa}_M(z; \cdot)$ is a continuous function. Hence with $j \to \infty$ and $R \to 1$, we get that $\hat{\kappa}_M(z; X) \leq Dk_M(z; X).$ \hfill \Box

**Remark.** It follows by the above proofs and a standard diagonal process that $\hat{\kappa}_M(z; \cdot) = D\hat{k}(z; \cdot)$ if $M$ is hyperbolic at $z$.

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