The Internet is a very complex network connecting the large number of computers around the world. The nodes of this network may be interpreted as the routers and links as the cables connecting computers. The network can also be described in the inter-domain level where each domain is represented by a single node and each link is an inter-domain connection. Such a network is well described by a graph consisting of a set of vertices and another set of edges among the vertices. Without assigning any weight with the links between the nodes only the topological structure of the Internet is meaningful. Study of Internet's topological structure may be important for designing efficient routing protocols and modeling Internet traffic.

Internet is one of the large class of real-world networks that exhibit small-world and/or scale-free properties, e.g., social networks, biological networks, electronic communication networks etc. Quantities that characterize a network of $N$ nodes are the diameter $D(N)$ which measures the topological extension of the network, the clustering co-efficient $C$ measures the local correlations among the links of the network and the nodal degree distribution $P(k)$. In a small-world network (SWN) the diameter $D(N)$ of the network scale logarithmically with $N$ where as for a scale-free network (SFN) the degree distribution has a power law tail: $P(k) \propto k^{-\gamma}$. Barabási and Albert (BA) showed that a growing network with preferential link attachment probability is a SFN with $\gamma = 3$.

Waxman first studied probabilistic graph models of Internet where links have weights which are their physical lengths. The link length distribution in such networks decays exponentially: $D(\ell) \sim \exp(-\ell/\ell_0)$. Faloutsos et. al. observed that the out-degree distribution of Internet follows a power law tail. Yook et. al. observed that the distribution of routers of North America is a fractal set and the link length distribution is inversely proportional to the link lengths. It is suggested that in the growing Internet when a new node is becoming member of the network two competing factors control the decision to which node of the already grown Internet the new node will be connected. The factors are the degree $k_i$ of the existing node $i$ and in general the $\alpha$-th power of the length $\ell$ of the link connecting the new node and the node $i$. The preferential attachment probability for the $i$-th node is therefore: $\pi_i \propto k_i \ell^\alpha$.

Recently it has been argued that such a network is scale-free for all values of $\alpha > \alpha_c = 1 - d$ in $d$ dimension and the link length distribution generally follows a power law $D(\ell) \sim \ell^\delta$ where $\delta(\alpha) = \alpha + d - 1$. For $\alpha < \alpha_c$ the degree distribution decays stretched exponentially but $D(\ell)$ still maintains a power law where $\delta$ saturates at $-d-1$. The limit of $\alpha \to -\infty$ is interesting where each node connects only to its nearest earlier node. In a regular network in the form of a linear chain similar studies have been done. An interplay between the preferential attachment and the link length selection within an interaction range for the Euclidean networks is studied in [12].

In this paper we associate a cost function associated with such networks. Each link of the network has the cost equal to its Euclidean length $\ell$ and therefore the cost function of the whole network is the total length of all the links of the network. The question we ask is, how can one construct a small-world scale-free network with minimal cost? To study this we start generating a $N$ node BA SFN on a two-dimensional plane. Links are then interchanged to reduce the cost function keeping the topology i.e. the degree value of each node intact. The optimization of the wiring length of networks on lattices...
points with serial numbers

network with

L ≈ N

cordinates of the set of

(m) embedded in the Euclidean space as follows. Let

ration of the network let a specific set of values of the

1 of rewiring trials.

N ≈ 68 at t=10000 and (d) L ≈ 58 at t=1000000 where t is the number of rewiring trials.

has been studied in [14].

We start with constructing a Barabási-Albert SFN embedded in the Euclidean space as follows. Let

(x₁, x₂, ..., xₜ) and (y₁, y₂, ..., yₜ) be the independent identically and uniformly distributed random variables on the interval [0,1]. To construct one random configuration of the network let a specific set of values of the N pair variables \{(x₁, y₁), (x₂, y₂), ..., (xₜ, yₜ)\} be the co-ordinates of the set of N points on the unit square representing the set of nodes of the network with serial numbers i = 1 to N assigned to them. We use first m + 1 points with serial numbers i = 1 to m + 1 to construct a m + 1-clique by connecting each point with rest of the m points. Then following the serial numbers new points are added to the network one after another and each node is connected to randomly selected m distinct previous nodes. The probability to link the new node with serial number j to a previous node i is linearly proportional to its degree, kᵢ. The network thus constructed up to N nodes is exactly the BA network [8]. At the same time it is a small-world network, i.e. the diameter D(N) of the network measured by the maximal distance between an arbitrary pair of nodes grow as \(\log(N)\). In this paper we restrict ourselves to m = 2.

Let a denote the symmetric adjacency matrix of size N × N for our network such that \(a_{ij} = 1\) if there is a link between the pair of nodes i and j and 0 otherwise. Let \(ℓ_{ij}\) denote the shortest Euclidean distance between the pair of nodes i and j taking into account the periodic boundary condition. Therefore when \(a_{ij} = 1\), \(ℓ_{ij}\) is the length of the connecting wire of the link between i and j. The total cost function \(\mathcal{L}(N)\) is therefore the sum over all link lengths of the network i.e., \(\mathcal{L}(N) = \sum_{i>j} a_{ij} \ell_{ij}\).

For the convenience of discussion we define a degree vector similar to the contact vector generally used in the polymer physics. Our degree vector c describes the topological connectivity of the network and has N elements \(c_i = k_i\), the degree of the i-th node. In the initial BA scale-free network one can associate a notion of time as if nodes are introduced one at each time unit. Therefore the links of the node i introduced at time i are divided into two groups ‘outgoing’ and ‘incoming’. Each node has only \(k_{out} = m\) outgoing links connected to m other nodes which are older than this node and it can be connected to \(k_{in} = k - m\) other nodes which are younger than this node. Consequently the degree vector can be split into two other degree vectors \(c_{out}\) and \(c_{in}\) such that \(c_{out}^i = k_{out}^i\) and \(c_{in}^i = k_{in}^i\) and \(c = c_{out} + c_{in}\).

Next, we perform the optimization dynamics to minimize the total cost function \(\mathcal{L}(N)\). The optimization dynamics conserves the number of links in the network, in addition it not only maintains the same degree vector
c but also $c^\text{out}$ and $c^\text{in}$ separately and thus ensures that the degree distribution of the network remains exactly same as it is before the optimization process starts. We call it as ‘time-ordered’ rewiring. One trial of rewiring in the optimization scheme consists of selecting four nodes $n_1, n_2, n_3$ and $n_4$. The first node $n_1$ is randomly selected from the set of $N$ nodes. $n_2$ is selected randomly from the $k_1$ neighbours of $n_1$. Similarly $n_3(\neq n_1 \neq n_2)$ is selected randomly from $N$ nodes and $n_4(\neq n_1 \neq n_2)$ is again one of the $k_3$ neighbours of $n_3$. The move must maintain the conservation of link numbers as well as degree distribution. We replace the link pair $n_1n_2$ and $n_3n_4$ by another pair of links if either of the following two conditions is satisfied:

i. if $a_{13} = a_{24} = 0$ and $\ell_{12} + \ell_{34} > \ell_{13} + \ell_{24}$ we link $n_1n_3$ and $n_2n_4$.

ii. if $a_{14} = a_{23} = 0$ and $\ell_{12} + \ell_{34} > \ell_{14} + \ell_{23}$ we link $n_1n_4$ and $n_2n_3$.

If both are satisfied we accept one of them with probability $1/2$. If only one is satisfied we accept that (Fig. 1). If none of the two is satisfied we go for a fresh trial. We also study a second type of rewiring process where only the total degree vector $c$ is maintained but not individually $c^\text{out}$ and $c^\text{in}$. Here in the final optimized network a particular node may have all neighbours which are younger than this node. We call this process as the ‘random’ rewiring method. In Fig. 2 we show how an initially complicated network becomes less messy with increasing number of rewiring trials.

Since we accept the move only if the total rewired cost is reduced the trial is similar to the zero temperature Monte Carlo dynamics. The total cost $L(N)$ monotonically decreases with the number of successful trials and the number of un-successful trials between successive accepted moves increases. We typically try around $10(mN)^2$ such trials so that the plot of $L(N)$ with logarithm of the number of trials nearly reaches a plateau.

From each point one can measure $N-1$ distances and if these distances are sorted in an increasing sequence, one has the first neighbour distance, second neighbour distance, ... $(N-1)$-the neighbour distance etc. It is known that the average $n$-th neighbour distance $R_n^N$ varies as $N^{-1/2}$ if $n/N << 1$ and it is of the order of 1 when $n/N \sim 1$ in the limit of $N \to \infty$. There is no other variation like $N^{-x}$ when $x$ is neither 0 nor 1/2 but in between.

In the optimized network the links are not necessarily a fixed $(n)$ neighbour distances but a complex mixture of many neighbour distances. More elaborately it is expected that many of the links of the optimized network are first neighbour distances, less numbers are second neighbour distances, less numbers are third neighbour distances etc. In the optimized network we first calculate the probability density of the link length distribution $D(\ell)$. This distribution on scaling by the average link length $\langle \ell(N) \rangle$ is nearly the same for the time-ordered as well as the random rewiring processes. Contrary to the expectation this distribution has a maximum and it fits very well to a functional form $D(\ell)\langle \ell(N) \rangle \sim x^\alpha e^{-ax^\beta}$ with $x = \ell/\langle \ell(N) \rangle$. The fit on a linear scale gives $\alpha = 1.4, 1.1$ and $\beta = 0.8, 0.9$ approximately for the time-ordered and random rewiring processes respectively. The network has $N_t = 2N - m - 1$ links and therefore $L(N) = N_t\langle \ell(N) \rangle$. We plot in Fig. 3 $\langle \ell(N) \rangle$ with $N/\log N$ and observe excellent straight lines on a double logarithmic scale. Therefore $\langle \ell(N) \rangle \sim (N/\log N)^\mu$ where $\mu = 0.46$ and 0.52 with an error of 0.05 approximately for the time-ordered and random rewiring processes respectively.

The topological size of the network is measured by the diameter of the network. The distance $d_{ij}$ between an arbitrary pair of nodes $i$ and $j$ is the number of links on the shortest path connecting the two nodes. The diameter $d_m$ is the maximal distance on a network. The average diameter $D(N)$ represents the configuration averaged maximal distance $\langle d_m \rangle$. Variations of the average diameter of the optimized network with the network size is shown in Fig. 4. For the time-ordered exchange $D(N) = A + B\log N$ with $A \approx 1.22$ and $B \approx 1.11$ where as for the random exchange $D(N) \sim N^\nu$ with $\nu = 0.31 \pm 0.04$ where the error 0.04 is estimated by the largest difference of the local slopes between successive points and the mean slope. Although the degree distribution remains scale-free in the optimized networks generated by both the time-ordered and in random rewiring procedures, the first network retains some long range connections due to the constraint that both $c^\text{out}$ and $c^\text{in}$ are strictly maintained where as in the second network by random rewirings essentially all the connections are
local, i.e. typically a node has all neighbours within a spatial distance of the order of $\sim N^{-1/2}$.

The local correlation among the links is measured by the clustering co-efficient. The clustering co-efficient $C_i$ of the $i$-th node is measured by the ratio of the number of links $e_i$ within the $k_i$ neighbours of the $i$-th node and the number of links $k_i(k_i-1)/2$ if the $k_i$ nodes have formed an $k_i$-clique i.e., $C_i = 2e_i/k_i(k_i-1)$. The clustering co-efficient of the whole network $C(N)$ is $\langle C \rangle$. Also the average clustering co-efficient for the set of nodes of degree $k$ is defined as $C(k)$. In general both these clustering co-efficients may decrease as power laws: $C(N) \sim N^{-a}$ and $C(k) \sim k^{-b}$. In our case we start from the initial BA network where it is known that $a \approx 3/4$ and $b = 0$.

We also calculate these quantities in the final optimized state. The total clustering coefficient is found to be independent of $N$ and therefore $a=0$ where as unlike a simple power law for $C(k)$ we get a power law with logarithmic correction. In Fig. 5 we plot $C(k)$ with $k/\{\ln(k)\}^{1/2}$ and observe straight lines on a double logarithmic scale implying the variation as:

$$C(k) \sim \{k/\{\ln(k)\}^{1/2}\}^{-b} \quad (1)$$

where $b \approx 0.94$ and 1.1 for the time-ordered and random rewiring processes respectively. We cannot rule out the possibility that $b$ is actually 1 for both the processes. Many networks and models show $b = 1$.

To summarize, we have studied a cost optimized network which has three main features of the real-world networks e.g., it is a small-world network, it is a scale-free network and also it exhibits high clustering properties as well. We studied this network on the two-dimensional Euclidean space which should be relevant in the context of the Internet. While some links in Internet are the cable-less (microwave) links, many connections are made by real physical Ethernet cables. Therefore the question of optimizing cost of the total wiring length of the network arises naturally which is the main point of study in this paper. An optimized geographical embedding algorithm for scale-free networks was recently studied independently [14]. Unlike in [14], our time-ordered optimization produces a statistically non-homogeneous network and preserves a significant number of long-distance connections, permitting the network diameter to still scale as $\log(N)$ as $N \rightarrow \infty$. We also obtain a stretched-exponentially decaying tail of the link length distribution in the optimized network which is unlike the power-law tail observed by Yook et.al. [9] and closer to the Waxman result [8].

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