SEMI-CLASSICAL TRACE ASYMPTOTICS FOR MAGNETIC SCHRÖDINGER OPERATORS WITH ROBIN CONDITION

AYMAN KACHMAR AND MARWA NASRALLAH

Abstract. We compute the sum and number of eigenvalues for a certain class of magnetic Schrödinger operators in a domain with boundary. Functions in the domain of the operator satisfy a (magnetic) Robin condition. The calculations are valid in the semi-classical asymptotic limit and the eigenvalues concerned correspond to eigenstates localized near the boundary of the domain. The formulas we derive display the influence of the boundary and the boundary condition and are valid under a weak regularity assumption of the boundary function. Our approach relies on three main points: reduction to the boundary; construction of boundary coherent states; handling the boundary term as a surface electric potential and controlling the errors by various Lieb-Thirring inequalities.

1. Introduction

Recently, many papers display the influence of the Robin condition on the spectrum of the Laplacian. In planar domains, the papers [13, 25, 30] and references therein contain asymptotics of the principal eigenvalue. The tunneling effect for planar domains with corners is discussed in the paper [15]. In higher dimensions, the low-lying eigenvalues are studied in [31], where the effect of the boundary mean curvature is made precise. Trace semi-classical asymptotics are obtained in [12]. In all the aforementioned papers, there is no magnetic field and the function in the boundary condition is supposed smooth. The new issue addressed in this paper is that we include a magnetic field and we do not assume smoothness of the boundary function in (1.3) below. The discussion in this paper is limited for planar domains. Extensions to higher dimensions does not seem trivial; [29] contains results for the Neumann condition in 3D domains.

Let $\Omega \subset \mathbb{R}^2$ be an open domain with a smooth $C^3$ and compact boundary $\Gamma = \partial \Omega$. We suppose that the boundary $\partial \Omega$ consists of a finite number of connected components. The domain $\Omega$ is allowed to be an interior or exterior domain. By smoothness of the boundary $\partial \Omega$, we can define the unit outward normal vector $\nu$ of $\partial \Omega$.

The magnetic field is defined via a vector field (magnetic potential). Let $A \in C^2(\overline{\Omega}; \mathbb{R}^2)$. The magnetic field is

$$B := \text{curl } A.$$  \hfill (1.1)

Consider a function $\gamma \in L^3(\partial \Omega)$, a number $\alpha \geq 1/2$ and a parameter $h > 0$. The parameter $h$ is called the semi-classical parameter and we shall be concerned with the asymptotic limit of various quantities when the semi-classical parameter tends to 0.

The self-adjoint magnetic Schrödinger operator

$$\mathcal{P}^{\alpha, \gamma}_{h, \Omega} = (-ih\nabla + A)^2,$$ \hfill (1.2)

with a boundary condition of the third type (Robin condition)

$$\nu \cdot (-ih\nabla + A)u + h^{\alpha} \gamma u = 0 \quad \text{on } \partial \Omega,$$ \hfill (1.3)

can be defined by the Friedrich’s Theorem via the closed semi-bounded quadratic form,

$$Q^{\alpha, \gamma}_{h, \Omega}(u) := \|(-ih\nabla + A)u\|^2_{L^2(\Omega)} + h^{1+\alpha} \int_{\partial \Omega} \gamma(x)|u(x)|^2 \,dx.$$ \hfill (1.4)
The assumption $\gamma \in L^3(\partial\Omega)$ ensures that the quadratic form in (1.4) is semi-bounded. Since this does not follow in a straightforward manner, we will recall the main points of the classical proof in the appendix.

As is revealed from (1.3) and (1.4), the role of the parameter $\alpha$ is to control the strength of the boundary condition. Formally, we shall deal with the boundary term in (1.4) as a surface electric potential. This analogy is already observed in [11].

The formula in (1.8) is valid for all $\lambda < b$. This analogy is already observed in [11].

The Neumann problem corresponds to $\gamma \in L^3(\partial\Omega)$.

Concerning the number of eigenvalues, $N(\lambda; h, \gamma, \alpha)$ can be interpreted as the energy of non-interacting fermionic particles in $\Omega$ at chemical potential $\lambda h$. [12]

The Lieb-Thirring inequality will ensure that the sum $E(\lambda; h, \gamma, \alpha)$ is finite for all $\lambda \leq b$. This will be discussed further in Section 3. Concerning the number of eigenvalues, $N(\lambda; h, \gamma, \alpha)$ is finite for all $\lambda < b$. Actually, this energy level is strictly lower than the bottom of the essential spectrum. For exterior domains, we may have that the eigenvalues accumulate near $\lambda h$, i.e. $N(\lambda; h, \gamma, \alpha) = \infty$. In fact, it is proved that this is the case when the magnetic field $B(x)$ is constant, see [3].

The behavior of the two quantities in (1.6) and (1.7) in the semiclassical regime, i.e. when the semiclassical parameter $h$ goes to 0, is studied for the Neumann problem in [10] and [9]. The Neumann problem corresponds to $\gamma$ being identically 0 in (1.3). When the magnetic field $B(x) = b$ is constant, then the results in [10] and [9] assert that, if $h \to 0_+$, then,

$$N(\lambda; h, \gamma, \alpha) = h^{-1/2} c_1(\lambda) + h^{1/2} o(1),$$

$$E(\lambda; h, \gamma, \alpha) = h^{1/2} c_2(\lambda) + h^{1/2} o(1).$$

The formula in (1.8) is valid for all $\lambda < b$ while that in (1.9) is valid for all $\lambda \leq b$. It is pointed in [12] that the formulas in (1.8) and (1.9) are equivalent when $\lambda < b$.

The quantities $c_1(\lambda)$ and $c_2(\lambda)$ are defined by explicit expressions involving spectral quantities for a harmonic oscillator on the semi-axis. In [23], it is derived an analogue of (1.8) valid for a general function $\gamma \in C^\infty(\partial\Omega)$ and constant magnetic fields. The key issue in [23] was the analysis of a modified harmonic oscillator on the semi-axis and a standard approximation of the function $\gamma$ by a constant.

In this paper, we aim to obtain analogues of (1.8) and (1.9) under the relaxed assumptions that the magnetic field is variable and the function $\gamma$ is no more smooth but simply in $L^3(\partial\Omega)$. 
observe that the terms appearing in Theorem 1.1 are well defined.

The approach we follow is by carrying out a reduction to a thin boundary layer. This is easy to do. After localization in the thin boundary layer, we localize in small sub-domains of the boundary layer. In each small sub-domain, the operator is reduced to a one defined with a constant magnetic field and a constant $\gamma$. The reduced operator is defined in the half-plane. The reduction to a constant magnetic field is quiet standard as in [10] and [9]. The non-trivial point is to reduce to a constant $\gamma$ since the smoothness of $\gamma$ is dropped. We do this by dealing with $\gamma$ as being a surface electric potential. With this point of view, we borrow the methods in [28] that allow to approximate a non-smooth electrical potential by a smooth one, and then one passes from the smooth potential to the constant potential in the standard manner. Many errors will arise here. These are controlled by various Lieb-Thirring inequalities, notably the ones in [7, 33, 27] and a remarkable inequality obtained in [9] valid in the torus.

We proceed in the statement of the main result of this paper. We will need some notation regarding a harmonic oscillator in the semi-axis. For $(\gamma, \xi) \in \mathbb{R}^2$, we denote by

$$b[\gamma, \xi] = -\partial_t^2 + (t - \xi)^2 \quad \text{in} \quad L^2(\mathbb{R}_+),$$

the self-adjoint differential operator in $L^2(\mathbb{R}_+)$ associated with the boundary condition $u'(0) = \gamma u(0)$. The increasing sequence of eigenvalues of $b[\gamma, \xi]$ is $\{\mu_j(\gamma, \xi)\}_j$. By Sturm-Liouville theory, these eigenvalues are known to be simple and smooth functions of $\gamma$ and $\xi$. These facts will be recalled precisely in a separate section.

In the following, $(x)_- = \max(-x, 0)$ and $(x)_+ = \max(x, 0)$ denote the negative, respectively positive, part of a number $x \in \mathbb{R}$.

Our main result is

**Theorem 1.1.** Suppose that the magnetic field satisfies,

$$b = \inf_{x \in \Omega} B(x) > 0.$$  

Let $\lambda \leq b$, $\alpha \geq 1/2$ and $\gamma \in L^3(\partial \Omega)$. There holds:

- If $\alpha > 1/2$, then,
  $$\lim_{h \to 0^+} \left( \int_{\partial \Omega} B(x)^{3/2} \left( \mu_1(0, \xi) - \frac{\lambda}{B(x)} \right) - d\xi ds(x) \right),$$

- If $\alpha = 1/2$ and $\gamma \in L^\infty(\partial \Omega)$, then,
  $$\lim_{h \to 0^+} \left( h^{-1/2} E(\lambda; h, \gamma, \alpha) \right)$$

  $$= \frac{1}{2\pi} \sum_{p=1}^{\infty} \int_{\partial \Omega} \int_{\mathbb{R}} B(x)^{3/2} \left( \mu_p \left( B(x)^{-1/2} \gamma(x), \xi \right) - \frac{\lambda}{B(x)} \right) - d\xi ds(x).$$

Here $ds(x)$ denotes integration with respect to arc-length along the boundary $\partial \Omega$, and $E(\lambda; h, \gamma, \alpha)$ is introduced in (1.6).

The results in Theorem 1.1 display the strength of the boundary condition in (1.3). We observe that the influence of the Robin condition is not strong when $\alpha > \frac{1}{3}$, since the leading behavior of $E(\lambda; h, \gamma, \alpha)$ is essentially the same as that for the Neumann condition (i.e. $\gamma = 0$).

The sum

$$\sum_{p=1}^{\infty} \int_{\mathbb{R}} \left( \mu_p (\gamma, \xi) - 1 \right) d\xi$$

is actually a sum of a finite number of terms (for every fixed $\gamma$). The expression in (1.11) is a continuous function of $\gamma$. This will be proved in a separate section of this paper. Thus, we observe that the terms appearing in Theorem 1.1 are well defined.
Due to the implicit nature of the quantity in (1.11), it seems hard to prove that the functional
\[ F(\gamma) = \sum_{p=1}^{\infty} \int_{\partial\Omega} \int_{\mathbb{R}} B(x)^{3/2} \left( \mu_p \left( B(x)^{-1/2} \gamma(x), \xi \right) - \frac{\lambda}{B(x)} \right) d\xi ds(x) \]
is continuous in \( L^1(\partial\Omega) \). If this continuity is true, then the result in Theorem 1.1 continues to hold under the relaxed assumption that \( \alpha = \frac{1}{2} \) and \( \gamma \in L^3(\partial\Omega) \). This will be clear in the proof we provide to Theorem 1.1.

The methods we use do not allow us to obtain versions of Theorem 1.1 valid for \( \alpha < \frac{1}{2} \). In this specific regime, the sign of the function \( \gamma \) will play a significant role, as one can observe the results for the first eigenvalue in [18]. The results in [18] suggest that the localization to the boundary is very strong when \( \alpha < \frac{1}{2} \) and \( \gamma \) is negative. When \( \alpha < \frac{1}{2} \) and \( \gamma > 0 \), then the effect of the boundary is weak, and the situation is closer to the Dirichlet boundary condition, for which the methods in [3] are relevant.

Differentiation of the formulas in Theorem 1.1 with respect to \( \lambda h \) yields a formula for the number of eigenvalues. See [9, 29] for a precise statement of this technique. The formulas for the number of eigenvalues are collected in:

**Corollary 1.2.** Let \( \lambda < b \). Under the assumptions of Theorem 1.1, there holds:

- If \( \alpha > 1/2 \), then
  \[ \lim_{h \to 0} \left( h N(\lambda; h, \gamma, \alpha) \right) = \frac{1}{2\pi} \int_{\{ (x,\xi) \in \partial\Omega \times \mathbb{R} : B(x)\mu_1(0,\xi) < \lambda \}} B(x)^{1/2} d\xi ds(x). \]

- If \( \alpha = 1/2 \), then
  \[ \lim_{h \to 0} \left( h N(\lambda; h, \gamma, \alpha) \right) = \frac{1}{2\pi} \sum_{p=1}^{\infty} \int_{\{ (x,\xi) \in \partial\Omega \times \mathbb{R} : B(x)\mu_p(B(x)^{-1/2} \gamma(x),\xi) < \lambda \}} B(x)^{1/2} d\xi ds(x). \]

Here, \( N(\lambda; h, \gamma, \alpha) \) is the number of eigenvalues below \( \lambda h \), introduced in (1.7).

The proof of Corollary 1.2 is sketched below in Section 7. We mention that a formula for the number of eigenvalues below the energy value \( \lambda = 1 \) is not available yet, even for the case of Neumann boundary condition, i.e. \( \gamma = 0 \). For a matter of illustration, we include the following simple result in the case of Neumann boundary condition and a square domain.

**Theorem 1.3.** Suppose that the domain \( \Omega \) is a square, the magnetic field is constant, \( \text{curl} A = b \), and that \( \gamma = 0 \) in (1.3). As \( h \to 0^+ \), there holds,

\[ \limsup_{h \to 0^+} \left( h N(bh) \right) = \frac{b|\Omega|}{2\pi}. \]

Here,
\[ N(bh) = N(1; h, \gamma = 0, \alpha = 1) \]
is as introduced in (1.7).

In [24], it is proved that the formula for the energy in Theorem 1.1 is still valid when the domain \( \Omega \) is a square and \( \gamma = 0 \). This indicates an interesting observation, namely, the energy
\[ \sum_j (e_j(h) - bh) \]
is localized near the boundary, while the leading order expression of the number of the eigenvalues below \( bh \) is determined by the bulk. The proof we give to Theorems 1.1 and 1.3 suggests that the eigenvalues strictly below \( bh \) are associated with eigenfunctions concentrated near the boundary. A mathematically rigorous explanation of this point is still missing in the literature.
information might be obtained by computing the second correction term in (1.14), expected to be a boundary term. Toward that end, the methods in [3] must prove useful.

If one considers the Dirichlet realization of the operator \( P^D = (-ih\nabla + A)^2 \), then the number \( N(bh) \) is equal to 0. If \( bh \) is an eigenvalue of \( P^D \), then the corresponding ground state can be extended by 0 to all of \( \mathbb{R}^2 \). The min-max principle will yield that this constructed function is an eigenfunction of the Landau Hamiltonian in \( \mathbb{R}^2 \) with constant magnetic field \( bh \). This violates the description of the eigenfunctions of the lowest eigenspace of the Landau Hamiltonian with a constant magnetic field, since this space can not have compactly supported functions. That way we see that the lowest eigenvalue of \( P^D \) is strictly larger than \( bh \).

Remark 1.4. A key ingredient in the proof of Theorem 1.3 is to compare with a model Schrödinger operator with (magnetic) periodic conditions. The advantage of this model operator is that its first eigenvalue is known together with its multiplicity.

Remark 1.5. We list some interesting open problems in connection with Theorem 1.3:

• Inspection of the asymptotics in Theorem 1.3 for general domains.
• Inspecting if the result in Theorem 1.3 is valid with \( \lim \inf \) replacing \( \lim \sup \).
• Inspection of the number \( n(bh) \) of eigenvalues of \( P_{h,b,\Omega} \) in the interval \( (-\infty, bh) \). This question is related to the existence of a non-zero function \( u \) solving the problem:

\[
P_{h,b,\Omega}u = bh u \text{ in } \Omega \text{ and } \nu \cdot (h\nabla - iA_0)u = 0 \text{ on } \partial \Omega.
\]

2. Preliminaries

2.1. Variational principles. In this section, we recall methods used in [28] to establish upper and lower bounds on the energy of eigenvalues.

Lemma 2.1. Let \( \mathcal{H} \) be a semi-bounded self-adjoint operator on \( L^2(\mathbb{R}^3) \) satisfying

\[
\inf \text{Spec}_{\text{ess}}(\mathcal{H}) \geq 0. \tag{2.1}
\]

Let \( \{\nu_j\}_{j=1}^{\infty} \) be the sequence of negative eigenvalues of \( \mathcal{H} \) counting multiplicities. We have,

\[
- \sum_{j=1}^{\infty} (\nu_j)_- = \inf \sum_{j=1}^{N} \langle \psi_j, \mathcal{H}\psi_j \rangle, \tag{2.2}
\]

where the infimum is taken over all \( N \in \mathbb{N} \) and orthonormal families \( \{\psi_1, \psi_2, \cdots, \psi_N\} \subset D(\mathcal{H}) \).

The next lemma states another variational principle. It is used in several papers, e.g. [28].

Lemma 2.2. Let \( \mathcal{H} \) be a self-adjoint semi-bounded operator satisfying the hypothesis (2.1). Suppose in addition that \( (\mathcal{H})_- \) is trace class. For any orthogonal projection \( \gamma \) with range belonging to the domain of \( \mathcal{H} \) and such that \( \mathcal{H}\gamma \) is trace class, we have,

\[
- \sum_{j=1}^{\infty} (\nu_j)_- \leq \text{tr}(\mathcal{H}\gamma). \tag{2.3}
\]

2.2. Existence of discrete spectrum of \( \mathcal{P}^n_{h,\Omega} \). If the domain \( \Omega \) is bounded, it results from the compact embedding of \( D(Q_{h,\Omega}^{n,\gamma}) \) into \( L^2(\Omega) \) that \( \mathcal{P}_h \) has compact resolvent. Hence the spectrum is purely discrete consisting of a sequence of eigenvalues accumulating at infinity.

In the case of exterior domains, the operator \( \mathcal{P}_h \) can have essential spectrum. In particular, we have the inequality

\[
\int_{\Omega} |(-ih\nabla + A)u|^2 dx \geq h \int_{\Omega} B(x)|u|^2 dx, \quad \forall u \in C_0^{\infty}(\Omega). \tag{2.4}
\]

Using then a magnetic version of Persson’s Lemma (see [1, 32]), we get that

\[
\inf \text{Spec}_{\text{ess}} \mathcal{P}_h \geq bh.
\]

This is the reason behind considering the sum of eigenvalues that are below \( bh \).
2.3. Lifting with respect to the dimension. Let \( d \in \mathbb{N} \), and let
\[
A(x) = (a_1(x), a_2(x), \cdots, a_{d+1}(x))^T,
\]
be a magnetic vector potential with real-values entries in \( L^2_{\text{loc}}(\mathbb{R}^{d+1}) \).
We introduce the operator \( H_d(\gamma) \) defined via the quadratic form
\[
h_d(\gamma)[u] = \int_{\mathbb{R}^{d+1}} \left| (-i \nabla + A)u(x) \right|^2 dx - \int_{\mathbb{R}^d} \gamma(x) |u(x)|^2 dx.
\]
Hence and in the sequel \( \mathbb{R}^d_+ = \mathbb{R}^{d-1} \times \mathbb{R}_+ \).
We are going to show the following theorem following a strategy used in [27, Theorem 3.2] to
generalize a Lieb-Thirring type inequality to the case with magnetic field.

**Theorem 2.3.** Let \( d \geq 1 \), \( A \in L^2_{\text{loc}}(\mathbb{R}^{d+1}, \mathbb{R}^{d+1}) \) and \( \gamma \in L^{2\alpha+d}(\mathbb{R}^d) \). Let \( \alpha \geq 1/2 \), then
\[
\text{tr}[H(\gamma)]_-^\alpha \leq 2 L_{\alpha,d}^1 \int_{\mathbb{R}^d} \gamma_+^{2\alpha+d} dx,
\]
where \( L_{\alpha,d}^1 \) is defined by
\[
L_{\alpha,d}^1 = \frac{\Gamma(\alpha+1)}{2^{d+1} \Gamma(1+\alpha+d/2)}.
\]

**Proof.** We shall prove 2.3 by induction over \( d \). Notice that this operator is well-defined for
\( d = 0 \) and \( \gamma \) a non-negative real number. In this case we have \( H_0(\gamma) = (-i \partial_y u + a(y))^2 \) and \( u'(0) = -\gamma u(0) \), and one easily can find that this operator has one negative eigenvalue, namely
\( -\gamma_+^2 \), associated with the eigenfunction \( e^{-i \int_0^\gamma a(x) dx} e^{-\gamma_+ y} \). Hence
\[
\text{tr}_{L^2(\mathbb{R}_+)}[H_0(\gamma)]_-^\alpha = (\gamma_+^2)^\alpha
\]
which is the analogue of 2.3 for \( d = 0 \).

Now fix \( d \geq 1 \) and suppose that the assertion is already proved for all smaller dimensions. We write \( x = (x_1, x') \) when \( x_1 \in \mathbb{R} \) and \( x' \in \mathbb{R}^{d-1} \) and note that
\[
H_d(\gamma) \geq (-i \partial_{x_1} + a_1(x))^2 \otimes 1_{L^2(\mathbb{R}^{d+1}_+)} - [H_{d-1}(\gamma(x_1, \cdot))]_-
\]
We now choose a gauge
\[
\phi(x) = \int_0^{x_1} a_1(\tau, x_2, \cdots, x_{d+1}) d\tau,
\]
and \( \tilde{u}(x) = e^{-i\phi} u(x) \) for all \( u \in \mathcal{D}(H_d(\gamma)) \). Then
\[
\langle H_d(\gamma)u, u \rangle_{L^2(\mathbb{R}^{d+1}_+)} \geq \int_{\mathbb{R}_+^{d+1}} |\partial_{x_1} \tilde{u}|^2 dx - \int_{\mathbb{R}} \langle e^{-i\phi}[H_{d-1}(\gamma(x_1, \cdot))]_--e^{i\phi}\tilde{u}, \tilde{u} \rangle_{L^2(\mathbb{R}_+^{d+1})} dx_1
\]
So by the variational principle
\[
\text{tr}_{L^2(\mathbb{R}^{d+1}_+)}[H_d(\gamma)]_-^\alpha \leq \text{tr}_{L^2(\mathbb{R})} \left[ -\partial_{x_1}^2 \otimes 1_{L^2(\mathbb{R}^d_+)} - e^{-i\phi}[H_{d-1}(\gamma(x_1, \cdot))]_--e^{i\phi} \right]_-^\alpha
\]
and the operator-valued Lieb-Thirring inequality [16, corollary 3.5], it follows that
\[
\text{tr}_{L^2(\mathbb{R})} \left[ -\partial_{x_1}^2 \otimes 1_{L^2(\mathbb{R}^d_+)} - e^{-i\phi}[H_{d-1}(\gamma(x_1, \cdot))]_--e^{i\phi} \right]_-^\alpha \\
\leq 2 L_{\alpha,1}^1 \int_{\mathbb{R}} \text{tr}_{L^2(\mathbb{R}_+^{d+1})} [H_{d-1}(\gamma(x_1, \cdot))]_+^{\alpha+1/2} dx_1,
\]
By induction hypothesis, the right hand side is bounded above by
\[
2 L_{\alpha,1}^1 L_{\alpha+1/2,d}^1 \int_{\mathbb{R}} \gamma_+^{d+2\alpha} dx_1 = 2 L_{\alpha,d}^1 \int_{\mathbb{R}^d} \gamma_+^{d+2\alpha} dx,
\]
which establishes the assertion for dimension \( d \) and completes the proof of Theorem 2.3. \( \square \)
2.4. **Rough energy bound for the cylinder.** In this section, we recall a remarkable inequality for the Schrödinger operator

\[ \mathcal{P}_{h,b,S,T} = (-ih\nabla + bA_0)^2 \text{ in } L^2\left([0,S] \times (0,h^{1/2}T)\right). \tag{2.8} \]

Here \( S, T \) and \( b \) are positive parameters. The magnetic potential \( A_0 \) is

\[ A_0(s,t) = (-t,0). \]

Functions in the domain of the operator \( \mathcal{P}_{h,b,S,T} \) satisfy the periodic conditions

\[ u(0,\cdot) = u(S,\cdot) \text{ on } (0,h^{1/2}T), \]

Neumann condition at \( t = 0 \),

\[ \partial_t u = 0 \text{ on } t = 0, \]

and Dirichlet condition at \( t = h^{1/2}T \).

In this particular case of a bounded domain, the operator has compact resolvent and the spectrum consists of an increasing sequence of eigenvalues \( (e_j)_{j \geq 1} \) tending to \( +\infty \). We define the energy of the sum of the eigenvalues as follows,

\[ E(\lambda,b,S,T) = \sum_j (hb(1 + \lambda) - e_j)_+. \tag{2.9} \]

In [9], the energy in (2.9) is controlled by the product \( ST \). We recall this estimate in the next lemma.

**Lemma 2.4.** There exist positive constants \( T_0 \) and \( \lambda_0 \) such that, for all \( S > 0, b > 0, T \geq \sqrt{b}T_0 \) and \( \lambda \in (0,\lambda_0) \), we have,

\[ E(\lambda,b,S,T) \leq C(1 + \lambda)hb \left( \frac{ST}{\pi h} + 1 \right). \]

2.5. **Boundary coordinates.** The aim of this section is to define a new system of coordinates near the boundary which allows us to approximate the magnetic potential locally near the boundary by a new one corresponding to a constant magnetic field. These coordinates are used in [14]. Let \( \Omega \) be a smooth, simply connected domain in \( \mathbb{R}^2 \). Suppose that the boundary \( \partial\Omega \) is \( C^4 \)-smooth. Let furthermore,

\[ \mathbb{R}/(|\partial\Omega|Z) \ni s \mapsto M(s) \in \partial\Omega \]

be a parametrization of \( \partial\Omega \). The unit tangent vector of \( \partial\Omega \) at the point \( M(s) \) of the boundary is given by

\[ T(s) := M'(s). \]

We define the scalar curvature \( k(s) \) by the following identity

\[ T''(s) = k(s)\nu(s), \]

where \( \nu(s) \) is the unit vector, normal to to the boundary, pointing outward at the point \( M(s) \). We choose the orientation of the parametrization \( M \) to be counterclockwise, so

\[ \det(T(s),\nu(s)) = 1, \quad \forall s \in \mathbb{R}/(|\partial\Omega|Z). \]

For all \( \delta > 0 \), we define

\[ \mathcal{V}_\delta = \{ x \in \partial\Omega : \text{dist}(x,\partial\Omega) < \delta \}. \]

Let \( t_0 > 0 \). The map \( \Phi = \Phi_{t_0} \) is defined as follows:

\[ \Phi : \mathbb{R}/(|\partial\Omega|Z) \times (0,t_0) \ni x = M(s) - tv(s) \in \mathcal{V}_{t_0}. \tag{2.10} \]

By smoothness of the boundary \( \partial\Omega \), we may select \( t_0 \) sufficiently small so that \( \Phi \) is invertible. Thus, for all \( x \in \mathcal{V}_{t_0} \), one can write

\[ x \mapsto \Phi^{-1}(x) := (s(x),t(x)) \in \mathbb{R}/(|\partial\Omega|Z) \times (0,t_0), \tag{2.11} \]
where \( t(x) = \text{dist}(x, \partial \Omega) \) and \( s(x) \in \mathbb{R}/(|\partial \Omega| \mathbb{Z}) \) is associated with the point \( M(s(x)) \in \partial \Omega \) such that \( \text{dist}(x, \partial \Omega) = |x - M(s(x))| \).

The determinant of the Jacobian of the transformation \( \Phi^{-1} \) is
\[
a(s, t) = 1 - tk(s).
\]

For all \( u \in L^2(\mathcal{V}_0) \), we define the function
\[
\tilde{u}(s, t) := u(\Phi(s, t)).
\]

If \( A = (A_1, A_2) \) is a vector field in \( \mathcal{V}_0 \), we define the associated vector potential in the \((s, t)\)-coordinates by
\[
\tilde{A}_1(s, t) = (1 - tk(s))\tilde{A}(\Phi(s, t)) \cdot M'(s),
\]
\[
\tilde{A}_2(s, t) = \tilde{A}(\Phi(s, t)) \cdot \nu(s).
\]

The new magnetic potential \( \tilde{A} \) satisfies,
\[
\left[ \frac{\partial \tilde{A}_2}{\partial s} - \frac{\partial \tilde{A}_1}{\partial t} \right] \, ds \wedge dt = B(\Phi^{-1}(s,t)) \, dx \wedge dy = (1 - tk(s))\tilde{B}(s, t)ds \wedge dt.
\]

For all \( u \in H^1_A(\mathcal{V}_0) \), we have, with \( \tilde{u} = u \circ \Phi \),
\[
\int_{\mathcal{V}_0} |(-i\nabla + A)u|^2 \, dx = \int_0^{[0]} \int_0^t \left[ \left|(-i\partial_s + \tilde{A}_1)\tilde{u}\right|^2 + (1 - tk(s))^{-2}\left|(-i\partial_t + \tilde{A}_2)\tilde{u}\right|^2 \right] (1 - tk(s)) \, ds dt,
\]
and
\[
\int_{\mathcal{V}_0} |u|^2 \, dx = \int_0^{[0]} \int_0^t \left|\tilde{u}(s, t)\right|^2 (1 - tk(s)) \, ds dt.
\]

In the next proposition, it is constructed a gauge transformation such that the magnetic potential in the new coordinates can be approximated up to a small error—by a new one corresponding to a constant magnetic field. The proof is given in [8, Appendix F].

**Proposition 2.5.** Let \( A \in C^2([\Omega, \mathbb{R}^2]) \). There exists a constant \( C > 0 \) such that for all \( S \in (0, |\partial \Omega|), S_0 \in [0, S] \) there exists a gauge function \( \phi \in C^2([0, S] \times [0, t_0]) \) such that \( \tilde{\Omega} := \tilde{A}(s, t) - \nabla(s, t)\phi \), with \( \tilde{A} \) as defined in (2.13), satisfies
\[
\tilde{\Omega}(s, t) = \left( \begin{array}{c} \tilde{A}_1(s, t) \\ \tilde{A}_2(s, t) \end{array} \right) = \left( \begin{array}{c} -B_0t + \beta(s, t) \\ 0 \end{array} \right), \quad (s, t) \in [0, S] \times [0, t_0],
\]
where \( B_0 := \tilde{B}(S_0, 0) \) and for any \( 0 < T \leq t_0 \), we have
\[
\sup_{(s, t) \in [0, S] \times [0, T]} \left| \beta(s, t) \right| \leq C(S^2 + T^2).
\]

We shall frequently make use of the following standard lemma, taken from [10] Lemma 3.5.

**Lemma 2.6.** There exists a constant \( C > 0 \) and for all \( S_1 \in [0, |\partial \Omega|), S_2 \in (S_1, |\partial \Omega|) \), there exists a function \( \phi \in C^2([S_1, S_2] \times [0, t_0]; \mathbb{R}) \) such that, for all
\[
S_0 \in [S_1, S_2], \quad T \in (0, t_0), \quad \varepsilon \in [CT, C t_0],
\]
and for all \( u \in H^1_A(\Omega) \) satisfying
\[
\text{supp} \, \tilde{u} \subset [S_1, S_2] \times [0, T],
\]
one has the following estimate,
\[
\int_{\Omega} |(-i\nabla + A)u|^2 \, dx - \int_{\mathbb{R}^2_+} |(-i\nabla + \tilde{B}A_0)e^{i\phi/h}\tilde{u}|^2 \, ds dt \\
\leq \int_{\mathbb{R}^2_+} \left( \varepsilon \left|(-i\nabla + \tilde{B}A_0)e^{i\phi/h}\tilde{u}\right|^2 + C\varepsilon^{-1}(S^2 + T^2)^2|\tilde{u}|^2 \right) \, ds dt.
\]
Lemma 3.2. The following statements hold true.

(1) For all $\gamma \in \mathbb{R}$, we have, 
$$\Theta_2(\gamma) > \Theta(\gamma).$$

(2) For every $j \in \mathbb{N}$, the function $\xi \mapsto \mu_j(\gamma, \xi)$ is continuous and satisfies 
(a) $\lim_{\xi \to -\infty} \mu_j(\gamma, \xi) = \infty$;
(b) $\lim_{\xi \to -\infty} \mu_j(\gamma, \xi) = 2j + 1$.

3. A family of one-dimensional differential operators

We are concerned in this section with the analysis of a family of ordinary differential operators with Robin boundary condition. For $\xi \in \mathbb{R}$, we consider the operator $h[\gamma, \xi]$ in $L^2(\mathbb{R}^+)$ associated with the operator $-\frac{d^2}{dt^2} + (t - \xi)^2$, i.e.
$$h[\gamma, \xi] := -\frac{d^2}{dt^2} + (t - \xi)^2, \quad D(h[\gamma, \xi]) = \{ u \in B^2(\mathbb{R}^+) : u'(0) = \gamma u(0) \}. \quad (3.1)$$

Here, for a given $k \in \mathbb{N}$, the space $B^k(\mathbb{R}^+)$ is defined as:
$$B^k(\mathbb{R}^+) = \{ u \in L^2(\mathbb{R}^+) : t^p u^{(q)}(t) \in L^2(\mathbb{R}^+), \quad \forall p, q \text{ s.t. } p + q \leq k \}, \quad (3.2)$$
where $u^{(q)}$ denote the distributional derivative of order $q$ of $u$.

The operator $h[\gamma, \xi]$ is associated with the closed quadratic form
$$B^1(\mathbb{R}^+) \ni u \mapsto q[\gamma, \xi] := \int_0^\infty (|u'(t)|^2 + |(t - \xi)u(t)|^2) dt, \quad (3.3)$$
where $B^1(\mathbb{R}^+)$ is defined in (3.2).

It is easy to see that $h[\gamma, \xi]$ has compact resolvent since the embedding $B^1(\mathbb{R}^+)$ into $L^2(\mathbb{R}^+)$ is compact. Hence the spectrum of $h[\gamma, \xi]$ is purely discrete consisting of an increasing sequence of positive eigenvalues $\{ \mu_j(\gamma, \xi) \}_{j=1}^\infty$.

The lowest eigenvalue of $h[\gamma, \xi]$ is defined via the min-max principle by:
$$\mu_1(\gamma, \xi) = \inf_{u \in B^1(\mathbb{R}^+), u \neq 0} \frac{q[\gamma, \xi](u)}{\|u\|_{L^2(\mathbb{R}^+)}^2}.$$ 

It follows from standard Sturm-Liouville theory that all the eigenvalues $\mu_j(\gamma, \xi)$ are simple, and $\mu_1(\gamma, \xi)$ has a positive ground state. Details are given in [5].

We define the functions:
$$\Theta(\gamma) := \inf_{\xi \in \mathbb{R}} \mu_1(\gamma, \xi),$$
and
$$\Theta_j(\gamma) := \inf_{\xi \in \mathbb{R}} \mu_j(\gamma, \xi) \quad (j \geq 2).$$

When $\gamma = 0$, we shall write,
$$h[\xi] := h[0, \xi], \quad \mu_j(\xi) := \mu_j(0, \xi), \quad \forall j \in \mathbb{N} \quad (3.4)$$
$$\Theta_0 := \Theta(0), \quad \xi_0 := \xi(0). \quad (3.5)$$

The result in the next lemma is proved in [10].

Lemma 3.1. For all $\xi \in \mathbb{R}$, we have
$$\mu_2(\xi) > 1.$$ 

Next we collect results proved in [23].

Lemma 3.2. The following statements hold true.

(1) For all $\gamma \in \mathbb{R}$, we have,
$$\Theta_2(\gamma) > \Theta(\gamma).$$

(2) For every $j \in \mathbb{N}$, the function $\xi \mapsto \mu_j(\gamma, \xi)$ is continuous and satisfies 
(a) $\lim_{\xi \to -\infty} \mu_j(\gamma, \xi) = \infty$;
(b) $\lim_{\xi \to -\infty} \mu_j(\gamma, \xi) = 2j + 1$. 

Here, $\mathbb{R}^+_2 = \mathbb{R} \times \mathbb{R}^+$, $S = S_2 - S_1$, $\tilde{B} = \tilde{B}(S_0, 0)$, the function $\tilde{u}$ is associated to $u$ by $0$ on $\mathbb{R}^+_2 \setminus \text{supp } \tilde{u}$. 

Here, $\gamma$, $\xi$ are compact. Hence the spectrum of $h$ is defined in (3.2).

Here, for a given $k \in \mathbb{N}$, the space $B^k(\mathbb{R}^+)$ is defined as:
$$B^k(\mathbb{R}^+) = \{ u \in L^2(\mathbb{R}^+) : t^p u^{(q)}(t) \in L^2(\mathbb{R}^+), \quad \forall p, q \text{ s.t. } p + q \leq k \}. \quad (3.2)$$

Next we define the functions:
$$h[\gamma, \xi] := h[0, \gamma, \xi], \quad \mu_j(\gamma, \xi) := \mu_j(0, \gamma, \xi), \quad \forall j \in \mathbb{N} \quad (3.4)$$
$$\Theta(\gamma) := \Theta(0), \quad \xi(\gamma) := \xi(0). \quad (3.5)$$

The result in the next lemma is proved in [10].

Lemma 3.1. For all $\xi \in \mathbb{R}$, we have
$$\mu_2(\xi) > 1.$$ 

Next we collect results proved in [23].

Lemma 3.2. The following statements hold true.

(1) For all $\gamma \in \mathbb{R}$, we have,
$$\Theta_2(\gamma) > \Theta(\gamma).$$

(2) For every $j \in \mathbb{N}$, the function $\xi \mapsto \mu_j(\gamma, \xi)$ is continuous and satisfies 
(a) $\lim_{\xi \to -\infty} \mu_j(\gamma, \xi) = \infty$;
(b) $\lim_{\xi \to -\infty} \mu_j(\gamma, \xi) = 2j + 1$. 

Next we define the functions:
$$h[\gamma, \xi] := h[0, \gamma, \xi], \quad \mu_j(\gamma, \xi) := \mu_j(0, \gamma, \xi), \quad \forall j \in \mathbb{N} \quad (3.4)$$
$$\Theta(\gamma) := \Theta(0), \quad \xi(\gamma) := \xi(0). \quad (3.5)$$

The result in the next lemma is proved in [10].

Lemma 3.1. For all $\xi \in \mathbb{R}$, we have
$$\mu_2(\xi) > 1.$$ 

Next we collect results proved in [23].
3. Let \( \gamma \in (-\infty, 0) \) and \( j \in \mathbb{N} \). Then \( \Theta_j(\gamma) < 2j + 1 \) and for all \( b_0 \in (\Theta_j(\gamma), 2j + 1) \), the equation \( \mu_j(\gamma, \xi) = b_0 \) has exactly two solutions \( \xi_{j,-}(\gamma, b_0) \) and \( \xi_{j,+}(\gamma, b_0) \). Moreover,
\[ \{ \xi \in \mathbb{R} : \mu_j(\gamma, \xi) < b_0 \} = (\xi_{j,-}(\gamma, b_0), \xi_{j,+}(\gamma, b_0)) . \]

4. Let
\[ U_j = \{(\gamma, b) \in \mathbb{R}^2 : \Theta_j(\gamma) < b < 2j + 1\} . \]

The functions
\[ U_j \ni (\gamma, b) \mapsto \xi_{j,\pm}(\gamma, b) \]

admit continuous extensions
\[ \mathbb{R} \times (-\infty, 2j + 1) \ni \xi_{j,\pm}(\gamma, b) . \]

For later use, we include

Lemma 3.3. Let \( \gamma \in \mathbb{R} \), and let \( u_{j,\gamma}(\cdot; \xi) \) be the normalized eigenfunction associated to the eigenvalue \( \mu_j(\gamma, \xi) \). It holds true that
\[ |u_{j,\gamma}(0; \xi)|^2 \leq C(\mu_j(\gamma, \xi) + (\gamma^2 + 1)) . \]

Proof. Due to the density of \( C_0^\infty(\mathbb{R}_+) \) in \( H^1(\mathbb{R}_+) \), we have for any function \( u \in H^1(\mathbb{R}_+) \),
\[ |u(0)|^2 = -2 \int_{0}^{\infty} u'(\eta)u(\eta)d\eta . \quad (3.6) \]
The inequality of Cauchy-Schwarz gives us that, for any \( \alpha > 0 \),
\[ |u(0)|^2 \leq 2\|u'\|_{L^2(\mathbb{R}_+)}\|u\|_{L^2(\mathbb{R}_+)} \leq \alpha\|u'\|^2_{L^2(\mathbb{R}_+)} + \alpha^{-1}\|u\|^2_{L^2(\mathbb{R}_+)} . \quad (3.7) \]

Assume \( \gamma < 0 \) and choose \( \alpha = -1/(2\gamma) \), it follows that
\[ \gamma|u(0)|^2 \geq -\frac{1}{2}\|u'\|^2 - 2\gamma^2 \|u\|^2 . \quad (3.8) \]

Notice that for \( u := u_{j,\gamma}(\cdot; \xi) \), we have
\[ \|u_{j,\gamma}'\|^2 + \|(t - \xi)u_{j,\gamma}\|^2 + \gamma|u_{j,\gamma}(0; \xi)|^2 = \mu_j(\gamma, \xi) . \]

Using (3.8) with \( u := u_{j,\gamma}(\cdot, \xi) \) and adding \( \|u_{j,\gamma}'\|^2 + \|(t - \xi)u_{j,\gamma}\|^2 \) on both sides, we obtain
\[ \mu_j(\gamma, \xi) \geq \frac{1}{2}\|u_{j,\gamma}'\|^2 - 2\gamma^2 . \quad (3.9) \]

Note also that the inequality in (3.9) is evidently true for \( \gamma \geq 0 \).

We infer from (3.7) that,
\[ |u_{j,\gamma}(0; \xi)|^2 \leq 2\|u_{j,\gamma}'\|^2 + 2 . \]

Now we use the inequality in (3.9) and the assumption that \( u_{j,\gamma} \) is normalized in \( L^2 \) to deduce
\[ |u_{j,\gamma}(0; \xi)|^2 \leq 4\mu_j(\gamma, \xi) + (8\gamma^2 + 2) . \]

□

In the next lemma, using the analysis in [22, Theorem 2.6.2], we establish uniform decay estimates on the eigenfunctions \( u_{j,\gamma} \).

Lemma 3.4. Let \( \epsilon \in (0, 1) \) and \( K > 0 \). There exists a constant \( C_{\epsilon, K} > 0 \) such that, if \( |\xi| \leq K \) and \( \mu_j(\gamma, \xi) \leq 1 \), then,
\[ \left\| e^{\gamma(1/2 - \epsilon)}u_{j,\gamma}(\cdot; \xi) \right\|_{H^1(\mathbb{R}_+; |t - \xi| \geq C_{\epsilon, K})} \leq C_{\epsilon, K}(1 + \gamma_+ + \gamma^2) . \quad (3.10) \]
The estimate in (3.15) is true for all \( \gamma \in (-M, M) \), we have, 
\[
\sum_{j=2}^{\infty} \int_{\mathbb{R}} (\mu_j(\gamma, \xi) - 1)_{-} d\xi = \sum_{j=2}^{\infty} \int_{\mathbb{R}} (\mu_j(\gamma, \xi) - 1)_{-} d\xi \leq C.
\]
We introduce constants $j$.

Consequently, we may find $j > j$.

According to Lemma 3.2, there exists a constant $\ell > 0$ such that

$$\{\xi \in \mathbb{R} : \mu_j(\gamma, \xi) \leq 1\} \subset \{\xi \in \mathbb{R} : \mu_2(\gamma, \xi) \leq 1\},$$

and for all $\gamma \in (-M, M)$, using the monotonicity of $\eta \mapsto \mu_2(\eta, \xi)$, we have,

$$\{\xi \in \mathbb{R} : \mu_j(\gamma, \xi) \leq 1\} \subset \{\xi \in \mathbb{R} : \mu_2(-M, \xi) \leq 1\}.$$

Arguing as in the proof of [23, Lemma 2.5], we get,

$$\lim_{j \to \infty} \mu_j(-M, \xi_j(M)) = \infty. \quad (3.16)$$

We introduce constants $(\xi_j(M))_{j \geq 2} \subset [-\ell, \ell]$ by

$$\mu_j(-M, \xi_j(M)) = \min_{\xi \in [-\ell, \ell]} \mu_j(-M, \xi).$$

Arguing as in the proof of [23] Lemma 2.5, we get,

$$\lim_{j \to \infty} \mu_j(-M, \xi_j(M)) = \infty. \quad (3.16)$$

Consequently, we may find $j_0 \geq 2$ depending solely on $M$ such that

$$\mu_j(-M, \xi_j(M)) > 1, \quad (j > j_0).$$

It follows that, for all $j > j_0$, $\xi \in [-\ell, \ell]$ and $\gamma \in (-M, M)$,

$$\mu_j(\gamma, \xi) \geq \mu_j(-M, \xi) \geq \mu_j(-M, \xi_j(M)) > 1.$$  

The result of Lemma 3.5 now follows upon noticing that, for all $\gamma > -M$ and $\xi \in \mathbb{R}$,

$$\mu_j(\gamma, \xi) > \mu_2(-M, \xi).$$

□

Again, the proof of [23] Lemma 2.5] allows us to obtain:

**Lemma 3.6.** For all $M > 0$, there holds,

$$\lim_{j \to \infty} \left( \inf_{\xi \in \mathbb{R}} \mu_j(-M, \xi) \right) = \infty.$$  

Proof. It has been established in [5] that there exists a sequence $(\xi_j(M))_{j \in \mathbb{N}}$ such that, for all $j$,

$$\inf_{\xi \in \mathbb{R}} \mu_j(-M, \xi) = \mu_j(-M, \xi_j(M)).$$

Let us show that

$$\lim_{j \to \infty} \mu_j(-M, \xi_j(M)) = \infty. \quad (3.17)$$

Suppose that (3.17) were false. Then we can find a constant $M$ and a subsequence $j_n$ such that

$$\inf_{\xi \in \mathbb{R}} \mu_{j_n}(-M, \xi) = \mu_j(-M, \xi_{j_n}(M)) \leq M, \quad \forall j \in \mathbb{N}. \quad (3.18)$$

If $(\xi_{j_n}(M))_{n}$ is unbounded, we may find a subsequence, denoted again by $(\xi_{j_n}(M))_{n}$, such that

$$\lim_{n \to \infty} \xi_{j_n}(M) = \infty.$$

Fix $j_0 \in \mathbb{N}$ and let us observe that for all $j_n \geq j_0$,

$$\mu_{j_n}(-M, \xi_{j_n}(M)) \geq \mu_{j_0}(-M, \xi_{j_n}(M)). \quad (3.19)$$

On account of Lemma 3.2 we know that $\lim_{\xi \to \infty} \mu_{j_0}(-M, \xi) = 2j_0 + 1$. Therefore, passing to the limit $n \to \infty$ in (3.19), we obtain

$$\liminf_{n \to \infty} \mu_{j_n}(-M, \xi_{j_n}(M)) \geq 2j_0 + 1.$$

Letting $j_0 \to \infty$, we conclude,

$$\liminf_{n \to \infty} \mu_{j_n}(-M, \xi_{j_n}(M)) = \infty,$$
Theorem 3.8. There exist constants dominated convergence. □

γ Since the function | ⏐

we have :

τ Let We may find a constant M > 0 and consequently, for all | ⏐

Therefore, we deal with a sum of j ⏐

Hence, we may find We get as in Lemma 2.5 in [23] and Lemma 3.6 :

Proof. Let m > 0. It is sufficient to establish,

\[
\left(\sup_{|\gamma| \leq m} |\mathcal{I}(\gamma + \tau) - \mathcal{I}(\gamma)|\right) \to 0 \text{ as } \tau \to 0.
\] (3.20)

Let \( \tau_1 \in (0, 1) \). By monotonicity, it follows that for all \( \tau \in [-\tau_1, \tau_1] \) and \( j \geq 2 \),

\[
\{ \xi \in \mathbb{R} : \mu_j(\gamma + \tau, \xi) \leq 1 \} \subset \{ \xi \in \mathbb{R} : \mu_j(-m - \tau_1, \xi) \leq 1 \}.
\] (3.21)

We may find a constant \( M > 0 \) depending only on \( m \) such that

\[
\forall \tau \in [-\tau_1, \tau_1], \quad \forall j \geq 2, \quad \{ \xi \in \mathbb{R} : \mu_j(-m - \tau_1, \xi) \leq 1 \} \subset [-M, M].
\] (3.22)

Let \( \xi_j(M) \) be as in the proof of Lemma 3.5, i.e.

\[
\forall \xi \in [-M, M], \quad \forall \tau \in [-\tau_1, \tau_1], \quad \mu_j(\gamma + \tau, \xi) \geq \mu_j(-m - \tau_1, \xi_j(M)).
\]

We get as in Lemma 2.5 in [23] and Lemma 3.6 :\n
\[
\lim_{j \to \infty} \mu_j(-m - \tau_1, \xi_j(M)) = \infty.
\]

Hence, we may find \( j_0 \geq 2 \) depending solely on \( m \) such that, for all \( j \geq j_0 \),

\[
\mu_j(-m - \tau_1, \xi_j(M)) > 1,
\]

and consequently, for all \( |\tau| \leq \tau_1 \), we have,

\[
\sum_{j=2}^{\infty} \int_{-\mathbb{R}} (\mu_j(\gamma + \tau, \xi) - 1)_- d\xi = \sum_{j=2}^{j_0} \int_{-\mathbb{R}} (\mu_j(\gamma, \xi) - 1)_- d\xi.
\]

Therefore, we deal with a sum of \( j_0 \) terms with \( j_0 \) independent from \( \tau \) and \( \gamma \). So given \( k \in \{2, \cdots, j_0\} \) and setting \( I_k(\gamma) = \int_{-\mathbb{R}} (\mu_k(\gamma, \xi) - 1)_- d\xi \), it is sufficient to show that

\[
\lim_{|\tau| \to 0} \left( \sup_{|\gamma| \leq m} |I_k(\gamma + \tau) - I_k(\gamma)| \right) = 0.
\] (3.23)

Since the function \( \gamma \mapsto \mu_k(\gamma, \xi) \) is continuous, the above formula is simply an application of dominated convergence. □

The next theorem is taken from [22] Theorem 2.4.8.

Theorem 3.8. There exist constants \( C > 0 \) and \( \eta > 0 \) such that, for all \( \gamma \in \mathbb{R} \) and \( \xi \in (\eta, +\infty) \), we have :

\[
|\mu_1(\gamma, \xi) - 1| \leq C(1 + |\gamma|)\xi \exp(-\xi^2).
\] (3.24)

Let us introduce the function

\[
\mathcal{J} : \mathbb{R} \ni \gamma \mapsto \sum_{j=1}^{\infty} \int_{\mathbb{R}} (\mu_j(\gamma, \xi) - 1)_- d\xi.
\]
Lemma 3.9. Let \( \{ \gamma_h \}_h \) be a real-sequence such that \( \lim_{h \to 0} \gamma_h = \gamma \in \mathbb{R} \). There holds,
\[
\lim_{h \to 0} J(\gamma_h) = J(\gamma).
\]
Proof. We write,
\[
|J(\gamma_h) - J(\gamma)| \leq \left| \int_{\mathbb{R}} (\mu_1(\gamma_h, \xi) - 1) - d\xi - \int_{\mathbb{R}} (\mu_1(\gamma, \xi) - 1) - d\xi \right| + |I(\gamma_h) - I(\gamma)|. \tag{3.25}
\]
We treat the first term on the right hand side of (3.25) using the inequality (3.23). That way, for every \( \varepsilon \), there exists \( h_0 > 0 \) such that for all \( h \in (0, h_0] \),
\[
\left| \int_{\mathbb{R}} (\mu_1(\gamma_h, \xi) - 1) - d\xi \right| \leq \int_{\mathbb{R}} |\mu_1(\gamma_h, \xi)) - 1| d\xi \leq \int_{\mathbb{R}} g(\xi) d\xi,
\]
with
\[
g(\xi) = C(1 + |\gamma| + \varepsilon)\xi e^{-\xi^2/2} \in L^1(\mathbb{R}).
\]
By continuity of the function \( \gamma \mapsto \mu_1(\gamma, \xi) \) and dominated convergence, it follows that
\[
\int_{\mathbb{R}} (\mu_1(\gamma_h, \xi) - 1) - d\xi \to \int_{\mathbb{R}} (\mu_1(\gamma, \xi) - 1) - d\xi
\]
as \( h \to 0 \). The second term in (3.25) converges to 0 by Lemma 3.7.

4. Eigenprojectors

Recall that \( \mathbb{R}^2_+ = \mathbb{R} \times \mathbb{R}_+ \). Consider \( h, b > 0 \) and the magnetic potential
\[
\mathbb{R}^2_+ \ni (s, t) \mapsto A_0(s, t) = (-t, 0). \tag{4.1}
\]
In this section, we construct projectors on the (generalized) eigenfunctions of the operator
\[
\mathcal{P}^{n, \gamma}_{h, b, \mathbb{R}^2_+} = (-ih\nabla + bA_0)^2 \text{ in } L^2(\mathbb{R}^2_+), \tag{4.2}
\]
whose domain is
\[
\mathcal{D}(\mathcal{P}^{n, \gamma}_{h, b, \mathbb{R}^2_+}) = \{ u \in L^2(\mathbb{R}^2_+) : (-ih\nabla + bA_0)^j \in L^2(\mathbb{R}^2_+), \quad j = 1, 2, \quad \partial_0 u = \gamma u \text{ on } t = 0 \}.
\]
Consider an orthonormal family \( (u_{j, \gamma}(\cdot; \xi))_{j=1}^\infty \) of real-valued eigenfunctions of the operator \( h[\gamma, \xi] \) introduced in (1.10), i.e.
\[
\begin{cases}
-u''_{j, \gamma}(t; \xi) + (t - \xi)^2 u_{j, \gamma}(t; \xi) = \mu_j(\gamma, \xi) u_{j, \gamma}(t; \xi), & \text{in } \mathbb{R}_+, \\
u'_{j, \gamma}(0; \xi) = \gamma u_{j, \gamma}(0; \xi), \\
\int_{\mathbb{R}_+} u_{j, \gamma}(t; \xi)^2 dt = 1.
\end{cases} \tag{4.3}
\]
Let \( u \in \mathcal{D}(\mathcal{P}^{n, \gamma}_{1, 1, \mathbb{R}^2_+}) \). Performing a Fourier transformation with respect to \( s \), we observe the formal relation,
\[
\mathcal{P}^{n, \gamma}_{1, 1, \mathbb{R}^2_+} u = (2\pi)^{-1} F_{\xi \to s}^{-1} \left( -\partial_t^2 + (t - \xi)^2 \right) \mathcal{F}_{s \to \xi} u. \tag{4.4}
\]
By the spectral theorem, we have
\[
h[\gamma, \xi] = \sum_{j=1}^\infty \langle \cdot, u_{j, \gamma}(\cdot; \xi) \rangle_{L^2(\mathbb{R}_+)} u_{j, \gamma}(\cdot; \xi),
\]
and consequently,
\[
\mathcal{P}^{n, \gamma}_{1, 1, \mathbb{R}^2_+} u = (2\pi)^{-1} \sum_{j=1}^\infty \langle \mathcal{F}_{s \to \xi} u, u_{j, \gamma}(\cdot; \xi) \rangle_{L^2(\mathbb{R}_+)} \mathcal{F}_{\xi \to s}^{-1} u_{j, \gamma}(\cdot; \xi).
\]
That way, for every $u \in \mathcal{D}(\mathcal{P}^{\alpha,\gamma}_{1,1,R_+^2})$, we have,
\[
\langle \mathcal{P}^{\alpha,\gamma}_{1,1,R_+^2} u, u \rangle_{L^2(R_+^2)} = (2\pi)^{-1} \sum_{j=1}^{\infty} \left| \langle \mathcal{F}_{s \to \xi} u, u_{j,\gamma} (\cdot; \xi) \rangle_{L^2(\mathbb{R}_+)} \right|^2 d\xi. \tag{4.5}
\]

For every $j \in \mathbb{N}$ and $\xi \in \mathbb{R}$, we introduce the eigenprojector $\Pi_j(\gamma, \xi)$ defined by the corresponding bilinear form,
\[
\langle \Pi_j(\gamma, \xi) u, v \rangle_{L^2(R_+^2)} = \langle \mathcal{F}_{s \to \xi} u, u_{j,\gamma} (\cdot; \xi) \rangle_{L^2(\mathbb{R}_+^2)} \langle \mathcal{F}_{s \to \xi} v, u_{j,\gamma} (\cdot; \xi) \rangle_{L^2(\mathbb{R}_+^2)}.
\]

Through explicit calculations, it is easy to prove:

**Lemma 4.1.** Let $u, v \in L^2(\mathbb{R}_+^2)$. We have
\[
(2\pi)^{-1} \sum_{j=1}^{\infty} \langle \Pi_j(\gamma, \xi) u, v \rangle_{L^2(R_+^2)} d\xi = \langle u, v \rangle_{L^2(\mathbb{R}_+^2)}. \tag{4.6}
\]

If in addition $u \in \mathcal{D}(\mathcal{P}^{\alpha,\gamma}_{1,1,R_+^2})$, then,
\[
\langle u, \mathcal{P}^{\alpha,\gamma}_{1,1,R_+^2} u \rangle_{L^2(R_+^2)} = (2\pi)^{-1} \sum_{j=1}^{\infty} \int_{\mathbb{R}} \mu_j(\gamma, \xi) \langle \Pi_j(\gamma, \xi) u, u \rangle_{L^2(\mathbb{R}_+^2)} d\xi. \tag{4.7}
\]

Let us introduce the unitary operator,
\[
U_{h,b} = L^2(\mathbb{R}_+^2) \ni \varphi \mapsto U_{h,b} \varphi \in L^2(\mathbb{R}_+^2),
\]
such that, for all $x = (x_1, x_2) \in \mathbb{R}_+^2$,
\[
(U_{h,b} \varphi)(x) = \sqrt{b/h} \varphi(\sqrt{b/h} x).
\]

Furthermore, we introduce the family of projectors,
\[
\Pi_j(h, b; \gamma, \xi) = U_{h,b} \Pi_j(\gamma_{h,b}, \xi) U_{h,b}^{-1}, \tag{4.8}
\]
with
\[
\gamma_{h,b} = h^\alpha b^{-1/2} \gamma. \tag{4.9}
\]

It is easy to check that
\[
U_{h,b}^{-1} \mathcal{P}^{\alpha,\gamma}_{h,b,R_+^2} U_{h,b} = \mathcal{P}^{\alpha,\gamma}_{1,1,R_+^2}. \tag{4.10}
\]

That way, we infer from Lemma 4.1.

**Lemma 4.2.** Let $u, v \in L^2(\mathbb{R}_+^2)$. We have
\[
(2\pi)^{-1} \int_{\mathbb{R}} \sum_{j=1}^{\infty} \langle \Pi_j(h, b; \gamma, \xi) u, v \rangle_{L^2(\mathbb{R}_+^2)} d\xi = \langle u, v \rangle_{L^2(\mathbb{R}_+^2)}. \tag{4.11}
\]

If in addition $u \in \mathcal{D}(\mathcal{P}^{\alpha,\gamma}_{h,b,R_+^2})$, then,
\[
\langle u, \mathcal{P}^{\alpha,\gamma}_{h,b,R_+^2} u \rangle = (2\pi)^{-1} \int_{\mathbb{R}} \sum_{j=1}^{\infty} \mu_j(h^\alpha b^{-1/2} \gamma, \xi) \langle \Pi_j(h, b; \gamma, \xi) u, u \rangle_{L^2(\mathbb{R}_+^2)} d\xi. \tag{4.12}
\]

5. Lower bound

In this section, we determine a lower bound of the trace $-E(\lambda; h, \gamma, \alpha)$ consistent with the asymptotics displayed in Theorem 1.1.

Arguing as in [9] Sec. 5.1, it follows from the Lieb-Thirring inequality that the trace $-E(\lambda; h, \gamma, \alpha)$ is finite.
5.1. Decomposition of the energy. Consider a partition of unity of \( \mathbb{R} \),
\[
\chi_1^2 + \chi_2^2 = 1, \quad \text{supp } \chi_1 \subset (\infty, 1), \quad \text{supp } \chi_2 \subset \left[ \frac{1}{2}, \infty \right).
\]  
(5.1)
We set for \( k = 1, 2, x \in \mathbb{R}^2 \),
\[
\zeta_k(x) = \chi_k(t(x)), \quad t(x) = \begin{cases} \text{dist}(x, \partial \Omega) & \text{if } x \in \Omega \\ -\text{dist}(x, \partial \Omega) & \text{otherwise}. \end{cases}
\]
(5.2)
Let \( \delta := \delta(h) \in (0, 1) \) be a small parameter to be chosen later. For \( k = 1, 2 \), we put,
\[
\zeta_{k,h}(x) = \zeta_k \left( \frac{t(x)}{\delta(h)} \right), \quad (x \in \Omega),
\]
(5.3)
where \( \zeta_k \) is introduced in (5.2).

Let \( \{g_j\}_j \) be any orthonormal system in \( \mathcal{D}(P_h^\alpha) \). We aim to prove a uniform lower bound of the following quantity,
\[
\sum_{j=1}^N (Q_{h,\Omega}^{\alpha,\gamma}(g_j) - \lambda_h).
\]
Thanks to the variational principle in Lemma 2.1, this will give us a lower bound of the trace \(-E(\lambda; h, \lambda, \alpha)\).

The IMS localisation formula yields
\[
\sum_{j=1}^N (Q_{h,\Omega}^{\alpha,\gamma}(g_j) - \lambda_h) = \sum_{k=1}^2 \left( Q_{h,\Omega}^{\alpha,\gamma}(\zeta_{k,h} g_j) - \int_{\Omega} (V_h + \lambda h) |\zeta_{k,h} g_j|^2 dx \right), \quad \mathcal{V}_h := \sum_{k=1}^2 |\nabla \zeta_{k,h}|^2.
\]
(5.4)
5.2. The bulk term. We will prove that the bulk term in (5.4) corresponding to \( k = 2 \) is an error term, i.e. of the order \( o(h^{1/2}) \). Thanks to the variational principle in Lemma 2.2, we have,
\[
\sum_{j=1}^N \left( Q_{h,\Omega}^{\alpha,\gamma}(\zeta_{2,h} g_j) - \int_{\Omega} (\mathcal{V}_h + \lambda h) |\zeta_{2,h} g_j|^2 dx \right) \geq \text{Tr} \left( [\tilde{P}_h - (B_h + \mathcal{V}_h)] 1_{(-\infty,0)}(\tilde{P}_h^{\alpha,\gamma} - (B_h + \mathcal{V}_h)) \right),
\]
(5.5)
where \( \tilde{P}_h - (B_h + \mathcal{V}_h) = (-ih\nabla + A)^2 - (B_h + \mathcal{V}_h) \) is the operator acting in \( L^2(\mathbb{R}^2) \). The trace on the right side in (5.5) can be controlled using the Lieb-Thirring inequality. The details are given in [9, Sec. 5.2]. That way, we get,
\[
\sum_{j=1}^N \left( Q_{h,\Omega}^{\alpha,\gamma}(\zeta_{2,h} g_j) - \int_{\Omega} (\mathcal{V}_h + \lambda h) |\zeta_{1,h} g_j|^2 dx \right) \\
\geq -Ch^2 \left( \int_{\mathbb{R}^2} \left\| h^{-1} B \right\|_{L^\infty} \left( -h^{-2} \mathcal{V}_h \right)_{-} + \left( -h^{-2} \mathcal{V}_h \right)_{+}^2 \right) dx \\
\geq -C \left( \frac{h}{\delta(h)} \left( 1 + \frac{h}{\delta(h)^2} \right) \right).
\]
(5.6)
Therefore, we get,
\[
\sum_{j=1}^N (Q_{h,\Omega}^{\alpha,\gamma}(g_j) - \lambda_h) \geq \sum_{j=1}^N \left( Q_{h,\Omega}^{\alpha,\gamma}(\zeta_{1,h} g_j) - \int_{\Omega} (\mathcal{V}_h + \lambda h) |\zeta_{1,h} g_j|^2 \right) - C \left( \frac{h}{\delta(h)} \left( 1 + \frac{h}{\delta(h)^2} \right) \right).
\]
(5.7)
Later on, we shall choose \( \delta(h) \) in a manner that the first term (boundary term) on the right hand side above is the dominant term.
5.3. The boundary term. Here we handle the term corresponding to \( k = 1 \) in (5.3). By assumption, \( \partial \Omega \) has a finite number of connected components. For simplicity of the presentation, we will perform the computations in the case where \( \partial \Omega \) has one connected component. In the general case, we work on each connected component independently and then sum the resulting lower bounds.

Let us introduce a positive, smooth function \( \psi \in L^2(\mathbb{R}) \), supported in \((0, 1)\) with the property that

\[
\int_{\mathbb{R}} \psi^2(s) ds = 1.
\]

Recall the boundary coordinates \((s, t)\) introduced in (2.11). We put

\[
\psi_h(x; \sigma) = \frac{1}{\delta(h)} \psi \left( \frac{s(x) - \sigma}{\delta(h)} \right), \quad (\sigma \in \mathbb{R}).
\]

Using again the IMS decomposition formula, we write,

\[
\sum_{j=1}^{N} \left( Q_{h,j}^{\alpha, \gamma}(\zeta_{1,j} g_{j}) - \int_{\Omega} (\lambda h + \mathcal{V}_h) |\zeta_{1,j} g_{j}|^2 \right)
\]

\[
= \int_{\mathbb{R}} \left( Q_{h,j}^{\alpha, \gamma}(\psi_h(x; \sigma) \zeta_{1,j} g_{j}) - (\lambda h + \mathcal{W}_h) |\psi_h(x; \sigma) \zeta_{1,j} g_{j}|^2 \right) d\sigma, \quad (5.9)
\]

where

\[
\mathcal{W}_h = \mathcal{V}_h + h^2 \int_{\mathbb{R}} |\nabla \psi_h(x, \sigma)|^2 d\sigma. \quad (5.10)
\]

Let us denote by (\( \Phi_{\sigma} \) is the coordinate change (2.11) valid near the boundary)

\[
v_{j,h}(x; \sigma) := \psi_h(x; \sigma) \zeta_{1,h}(x) g_{j}(x), \quad B_{\sigma} = B(\Phi(\sigma, 0)), \quad A_{\sigma}(s, t) = B_{\sigma} A_0(s, t) = (-B_{\sigma} t, 0),
\]

where \( A_0 \) is the magnetic potential introduced in (4.11). From Lemma 2.6 we infer that for all \( \varepsilon \in (0, 1) \),

\[
\int_{\Omega} \left| (-ih \nabla + A) v_{j,h}(x; \sigma) \right|^2 dx 
\]

\[
\geq (1 - \varepsilon) \int_{\mathbb{R}^2_+} \left| (-ih \nabla + A_{\sigma}) \tilde{v}_{j,h,\sigma} \right|^2 ds dt - C \varepsilon^{-1} \delta(h)^4 \int_{\mathbb{R}^2_+} |\tilde{v}_{j,h,\sigma}|^2 ds dt. \quad (5.12)
\]

Here, the function \( \tilde{v}_{j,h,\sigma} \) is defined by the coordinate transformation as follows

\[
\tilde{v}_{j,h,\sigma}(s, t) = e^{i\phi_{\sigma}(s,t)/h} \tilde{v}_{j,h}(\Phi(s,t); \sigma),
\]

where, for a function \( u, \tilde{u} \) is associated to \( u \) by means of (2.12) and \( \phi_{\sigma} \) is the phase factor from Lemma 2.6

Combining the foregoing estimates yields

\[
\int_{\Omega} \left| (-ih \nabla + A) v_{j,h}(x; \sigma) \right|^2 dx - \int_{\Omega} (\lambda h + \mathcal{W}_h) |v_{j,h}(x; \sigma)|^2 dx 
\]

\[
\geq (1 - \varepsilon) \int_{\mathbb{R}^2_+} \left| (-ih \nabla + A_{\sigma}) \tilde{v}_{j,h,\sigma} \right|^2 ds dt - \left( \lambda h (1 + C\delta(h)) + C \varepsilon^{-1} \delta(h)^4 \right) ||\tilde{v}_{j,h,\sigma}||^2_{L^2(\mathbb{R}^2_+)} 
\]

\[
- (1 + C\delta(h)) \int \tilde{W}_h ||\tilde{v}_{j,h,\sigma}||^2 ds dt. \quad (5.13)
\]
Consequently,

\[
\mathcal{Q}^{\alpha,\gamma}_{h,\Omega}(v_{j,h}(x;\sigma)) - \int_{\Omega} (\lambda h + \mathcal{W}_h)|v_{j,h}(x;\sigma)|^2 \, dx \\
\geq (1 - \varepsilon) \int_{\mathbb{R}^2_+} |(-ih\nabla + A_\sigma)\tilde{v}_{j,h,\sigma}|^2 \, dsdt + h^{1+\alpha} \int_{\mathbb{R}} \gamma(s)|\tilde{v}_{j,h,\sigma}(s,0)|^2 \, ds \\
- \left(\lambda h(1 + C\delta(h)) + C\varepsilon^{-1}\delta(h)^4\right) \|\tilde{v}_{j,h,\sigma}\|^2_{L^2(\mathbb{R}^2_+)} - (1 + C\delta(h)) \int \tilde{W}_h|\tilde{v}_{j,h,\sigma}|^2 \, dsdt. \tag{5.14}
\]

The function \(\gamma\) defined on \(\partial\Omega\) can be viewed as a function of the boundary variable \(s \in (0,|\partial\Omega|)\). We extend \(\gamma\) by 0 to a function in \(L^3(\mathbb{R})\).

Hereafter, we distinguish between the easy case when \(\alpha > \frac{1}{2}\) and the harder case when \(\alpha = \frac{1}{2}\).

**The regime** \(\alpha > \frac{1}{2}\). Let \(\eta > 0\). Thanks to \((5.14)\), we have the obvious decomposition,

\[
\sum_{j=1}^{N} \left\{ \mathcal{Q}^{\alpha,\gamma}_{h,\Omega}(v_{j,h}(x;\sigma)) - \int_{\Omega} (\lambda h + \mathcal{W}_h)|v_{j,h}(x;\sigma)|^2 \, dx \right\} \\
\geq \sum_{j=1}^{N} \left[ (1 - \eta)(1 - \varepsilon) \int_{\mathbb{R}^2_+} |(-ih\nabla + A_\sigma)\tilde{v}_{j,h,\sigma}|^2 \, dsdt + h^{1+\alpha} \int_{\mathbb{R}} \tilde{\gamma}_{a,\sigma}|\tilde{v}_{j,h,\sigma}(s,0)|^2 \, ds \\
- \left(\lambda h(1 + C\delta(h)) + C\varepsilon^{-1}\delta(h)^4\right) \|\tilde{v}_{j,h,\sigma}\|^2_{L^2(\mathbb{R}^2_+)} - (1 + C\delta(h)) \int \tilde{W}_h|\tilde{v}_{j,h,\sigma}|^2 \, dsdt \right] + \eta(1 - \varepsilon)R_{h,a,\eta,\sigma}(\tilde{v}_{j,h,\sigma}), \tag{5.15}
\]

where

\[
R_{h,a,\eta,\sigma}(\tilde{v}_{j,h,\sigma}) = \sum_{j=1}^{N} \left[ \int_{\mathbb{R}^2_+} |(-ih\nabla + A_\sigma)\tilde{v}_{j,h,\sigma}|^2 \, dsdt + \eta^{-1}h^{1+\alpha} \int_{\mathbb{R}} \frac{\gamma(s)}{1 - \varepsilon}|\tilde{v}_{j,h,\sigma}(s,0)|^2 \, ds \right]. \tag{5.16}
\]

Furthermore, we define the operator \(\tilde{\Gamma}\) on \(L^2([0,\delta(h)] \times (0,\delta(h)))\),

\[
\tilde{\Gamma}f = \sum_{j=1}^{N} (f,\tilde{v}_{j,h,\sigma})_{L^2([0,\delta(h)] \times (0,\delta(h)))} \tilde{v}_{j,h,\sigma},
\]

which satisfies \(0 \leq \tilde{\Gamma} \leq C\delta(h)^{-1}\) (in the sense of quadratic forms).

Denote by \(\gamma_{h,\eta,\varepsilon} = \frac{h^{\alpha-1/2}B^{1/2}}{\eta(1 - \varepsilon)}\). Thanks to the variational principle in Lemma 2.2 we may write,

\[
R_{h,a,\eta,\sigma}(\tilde{v}_{j,h,\sigma}) = \text{tr} \left[ \mathcal{P}^{\alpha,\gamma/(\eta(1-\varepsilon))}_{h,a,\eta,\sigma} \right] \\
\geq -C\delta(h)^{-1} \text{tr} \left[ \mathcal{P}^{\alpha,\gamma/(\eta(1-\varepsilon))}_{h,a,\sigma,\eta,\sigma} \right] \\
\geq -C\delta(h)^{-1}hB_\sigma \text{tr} \left[ \mathcal{P}^{\alpha,\gamma}_{1,1,1,\sigma,\sigma} \right]. \tag{5.17}
\]

Here the operator \(\mathcal{P}^{\alpha,\gamma}_{h,a,\eta,\sigma}\) has been introduced in \((1.2)\) and identified with the operator \(H_1(-\gamma_{h,\eta,\varepsilon})\) defined in Lemma 2.3. Thus, it follows from Theorem 2.3 (with \(\alpha = 1\)) that

\[
R_{h,a,\eta,\sigma}(\tilde{v}_{j,h,\sigma}) \geq -CB_\sigma^{-1/2}h^\delta h^{-3} \gamma(s)^3 \, ds \\
\geq -CB_\sigma^{-1/2}h^\delta h^{-3} \gamma(s)^3. \tag{5.18}
\]
Integrating (5.18) with respect to \( \sigma \in (-\delta(h), |\partial \Omega|) \), we conclude that,
\[
\eta(1 - \varepsilon) \int R_{\lambda, \alpha, \eta, \sigma}(\tilde{v}_{j,h,s})d\sigma \geq -CB_\sigma^{-1/2}h\delta^{-1}h^3(\alpha - \frac{1}{2})^2(1 - \varepsilon)^{-2}\eta^2\|\gamma\|^3_3 \quad (5.19)
\]

Selecting \( \delta = h^{3/8}, \varepsilon = h^{1/4} \) and \( \eta = h^{1/32} \), we get that the error terms in (5.6) and (5.19) are of the order \( o(h^{1/2}) \). Also, by [9, Proof of (5.26)], we have,
\[
\sum_{j=1}^{N} \{ Q_{h,\Omega}^{\alpha, \gamma}(v_{j,h}(x; \sigma)) - \int_{\Omega} (\lambda h + \mathcal{W}_h)|v_{j,h}(x; \sigma)|^2dx \}
\geq -\frac{h^{1/2}}{2\pi} \int_{\partial \Omega} \int_{\mathbb{R}} B(x)^{3/2}\left(\mu_1(0, \xi) - \frac{\lambda}{B(x)}\right) d\xi ds(x) - h^{1/2}o(1).
\]

Thus, we infer from (5.15), (5.6) and (5.4) that
\[
-E(\lambda; h, \gamma, \alpha) \geq -\frac{h^{1/2}}{2\pi} \int_{\partial \Omega} \int_{\mathbb{R}} B(x)^{3/2}\left(\mu_1(0, \xi) - \frac{\lambda}{B(x)}\right) d\xi ds(x) - h^{1/2}o(1). \quad (5.20)
\]

**The regime** \( \alpha = \frac{1}{2} \). The calculations here are longer compared to the case \( \alpha > \frac{1}{2} \). In the rest of this section, \( \alpha = \frac{1}{2} \). Let \( a > 0 \) and consider
\[
\gamma_a(s) = j_a * \gamma \quad (5.21)
\]
where
\[
j_a(s) = C_s a^{-\frac{1}{2}}j\left(\frac{s}{a}\right), \quad j(s) = e^{-s^2}.
\]

Here \( C_s \) is a normalization constant such that \( \int_{\mathbb{R}} j(s)ds = 1 \). By [26, Theorem 2.16], we know that \( \gamma_a \in C^\infty(\mathbb{R}) \) and, as \( a \to 0 \),
\[
\gamma_a \to \gamma, \quad \text{in} \ L^3(\mathbb{R}).
\]

By smoothness of \( \gamma_a \), we have,
\[
|\gamma_a(s) - \gamma_a(s)| \leq Ca^{-2}|s - \sigma| \leq Ca^{-2}\delta(h), \quad (5.22)
\]
valid on the support of the function \( v_{j,h,s} \).

Also, we have the obvious decomposition,
\[
\int_{\mathbb{R}} \gamma(s)|\tilde{v}_{j,h,s}(s,0)|^2ds = \int_{\mathbb{R}} \gamma_a(s)|\tilde{v}_{j,h,s}(s,0)|^2ds + \int_{\mathbb{R}} (\gamma(s) - \gamma_a(s))|\tilde{v}_{j,h,s}(s,0)|^2ds. \quad (5.23)
\]

Implementing the aforementioned estimates in (5.14), we obtain,
\[
Q_{h,\Omega}^{\alpha, \gamma}(v_{j,h}(x; \sigma)) - \int_{\Omega} (\lambda h + \mathcal{W}_h)|v_{j,h}(x; \sigma)|^2dx
\geq (1 - \varepsilon) \int_{\mathbb{R}^2_{+}} |(-ih\nabla + A_s)|\tilde{v}_{j,h,s}|^2dsdt + h^{3/2} \int_{\mathbb{R}} (\gamma_a(\sigma) - Ca^{-2}\delta(h)) |\tilde{v}_{j,h,s}(s,0)|^2ds
+ h^{3/2} \int_{\mathbb{R}} (\gamma(s) - \gamma_a(s))|\tilde{v}_{j,h,s}(s,0)|^2ds - \left(\lambda h(1 + C\delta(h)) + C\varepsilon^{-1}\delta(h)^4\right) \|	ilde{v}_{j,h,s}\|_{L^2(\mathbb{R}^2_{+})}^2 - (1 + C\delta(h)) \int \tilde{W}_h|\tilde{v}_{j,h,s}|^2dsdt. \quad (5.24)
\]

Let \( \eta > 0 \) and
\[
\gamma_{a, \sigma} = \frac{\gamma_a(\sigma) - Ca^{-2}\delta(h)}{(1 - \varepsilon)(1 - \eta)}. \quad (5.25)
\]
We can rewrite (5.24) in the alternative form,

\[
\begin{align*}
\sum_{j=1}^{N} \left\{ Q_{h,j}^\alpha(v_{j,h}(x;\sigma)) - \int_{\Omega} (\lambda h + W_h)|v_{j,h}(x;\sigma)|^2 dx \right\} \\
\geq (1 - \eta)(1 - \varepsilon) + \sum_{j=1}^{N} \left[ \int_{\mathbb{R}^2_+} \left| (-ih\nabla + A_{\sigma}) \tilde{v}_{j,h,\sigma} \right|^2 dsdt \right. \\
+ h^{3/2} \int_{\mathbb{R}^2_+} \gamma_{\sigma,\sigma} |\tilde{v}_{j,h,\sigma}(s,0)|^2 ds - \lambda h \left\| \tilde{v}_{j,\sigma} \right\|^2_{L^2(\mathbb{R}^2_+)} \\
\left. + \eta(1 - \varepsilon)(1 - \eta_0) R_{h,\eta,\sigma}^{(1)}(\tilde{v}_{j,h,\sigma}) + \eta\eta_0(1 - \varepsilon) R_{h,\eta,\sigma,a}^{(2)}(\tilde{v}_{j,h,\sigma}) \right], \\
\tag{5.26}
\end{align*}
\]

where \( \eta_0 \in (0, 1/2) \),

\[
R_{h,\eta,\sigma}^{(1)}(\tilde{v}_{j,h,\sigma}) = \sum_{j=1}^{N} \left[ \int_{\mathbb{R}^2_+} \left| (-ih\nabla + A_{\sigma}) \tilde{v}_{j,h,\sigma} \right|^2 dsdt - \lambda h \left\| \tilde{v}_{j,\sigma} \right\|^2_{L^2(\mathbb{R}^2_+)} \\
+ \left\{ \frac{2\eta^{-1}\lambda h C\delta(h) + 2C\eta^{-1}\varepsilon^{-1}\delta(h)^4}{(1 - \varepsilon)(1 - \eta_0)} \right\} \left\| \tilde{v}_{j,\sigma} \right\|^2_{L^2(\mathbb{R}^2_+)} \\
- \frac{2\eta^{-1}(1 + C\delta(h))}{(1 - \varepsilon)(1 - \eta_0)} \int_{\mathbb{R}^2_+} \tilde{\mathcal{W}}_h |\tilde{v}_{j,h,\sigma}|^2 ds \right] ,
\]

and

\[
R_{h,\eta,\sigma,a}^{(2)}(\tilde{v}_{j,h,\sigma}) = \sum_{j=1}^{N} \left[ \int_{\mathbb{R}^2_+} \left| (-ih\nabla + A_{\sigma}) \tilde{v}_{j,h,\sigma} \right|^2 dsdt + 2\eta_0^{-1}\eta^{-1}h^{3/2} \int_{\mathbb{R}} \frac{\gamma(s) - \gamma_0(s)}{1 - \varepsilon} |\tilde{v}_{j,h,\sigma}(s,0)|^2 ds \right]. \\
(5.27)
\]

The parameter \( \eta_0 \) will be selected sufficiently small but fixed. Let us define the density matrix

\[
\tilde{\Gamma}f = \sum_{j=1}^{N} \langle f, \tilde{v}_{j,h,\sigma} \rangle_{L^2(\mathbb{R}^2_+)} \tilde{v}_{j,h,\sigma},
\]

which satisfies \( 0 \leq \tilde{\Gamma} \leq C\delta(h)^{-1} \). Denote by \( \gamma_{\text{error}} = 2\eta_0^{-1}\eta^{-1}\frac{\gamma(s) - \gamma_0(s)}{1 - \varepsilon} \). Thanks to the variational principle in Lemma 2.2 and the Lieb-Thirring inequality in (2.3), we may write,

\[
R_{h,\eta,\sigma,a}^{(2)}(\tilde{v}_{j,h,\sigma}) = \text{tr} \left[ \mathcal{P}^{\alpha,\gamma_{\text{error}}} \tilde{\Gamma} \right] \geq -C\delta(h)^{-1} \text{tr} \left[ \mathcal{P}^{\alpha,\gamma_{\text{error}}} \right] \\
- \left| \int_{\mathbb{R}} \gamma(s) - \gamma_0(s) |^2 ds \right| \\
\geq -CB_{\sigma}^{-1/2}h^{3/2}(1 - \varepsilon)^{-3}\eta_0^{-3}\eta^{-3} \left| \int_{\mathbb{R}} \gamma(s) - \gamma_0(s) |^2 ds \right| \tag{5.28}
\]

Let us make the following choice of the parameter \( \delta \) and \( \varepsilon \),

\[
\delta = \eta^{-3/4}h^{1/2}, \quad \varepsilon = h^{1/4}.
\]

Integrating (5.28) with respect to \( \sigma \in (-\delta(h), |\partial\Omega|) \), we conclude that,

\[
\eta_0\eta(1 - \varepsilon) \int R_{h,\eta,\sigma,a}^{(2)}(\tilde{v}_{j,h,\sigma}) d\sigma \geq -CB_{\sigma}^{-1/2}h^{3/2}(1 - \varepsilon)^{-3}\eta_0^{-2}\eta^{-2} \left| \gamma - \gamma_0 \right|^3 \\
= \mathcal{O}(\eta_0^{-5/4}\eta^{-5/4}h^{1/2}) \left| \gamma - \gamma_0 \right|^3. \
(5.30)
\]
We estimate \( R^{(1)}_{h,\eta,\sigma}(\tilde{v}_{j,h,\sigma}) \) using the variational principle in Lemma 2.2 and the rough bound in the cylinder in Lemma 2.3. Indeed, we have

\[
\left\| \frac{2\eta^{-1}(1 + C\delta(h))\tilde{W}_h}{(1-\varepsilon)(1-\eta_0)} - \lambda h \left\{ \left( 1 - \frac{1}{1-\eta_0} \right) - \frac{1}{1-\eta_0} \left( 1 - \frac{1}{1-\varepsilon} \right) \right\} + \frac{2C\eta^{-1}\lambda\delta(h) + C\varepsilon^{-1}\delta(h)^4}{(1-\varepsilon)(1-\eta_0)} \right\| \leq \vartheta B_\sigma h ,
\]

where \( \vartheta = \mathcal{O}(\eta) + \mathcal{O}(\eta_0) + o(1) \). We may select \( \eta \) and \( \eta_0 \) sufficiently small such that \( \vartheta < \lambda_0 \), where \( \lambda_0 \) is the constant in Lemma 2.3. That way, we may apply Lemma 2.3. First, we write by the variational principle,

\[
R^{(1)}_{h,\eta,\sigma}(\tilde{v}_{j,h,\sigma}) \geq \text{Tr} \left[ \left( \mathcal{P}_{h,B_\sigma;\mathbb{R}^2_+}^{\alpha,0} - B_\sigma h(1+\vartheta) \right) \tilde{\Gamma} \right] \geq -C\delta(h)^{-1} \mathcal{E}(\vartheta, B_\sigma, \delta(h), \delta(h)).
\]

Applying Lemma 2.4 and integrating with respect to \( \sigma \in (-\delta(h), |\vartheta\Omega|) \), we arrive at

\[
\eta(1-\varepsilon) \int R^{(1)}_{h,\eta,\sigma}(\tilde{v}_{j,h,\sigma}) d\sigma \geq -C\eta \delta(h) = -C\eta^{1/4} h^{1/2}.
\]

Collecting the estimates in (5.30), (5.32) and (5.26), we get,

\[
\sum_{j=1}^{N} \left\{ \mathcal{Q}_{h,\Omega}^{\gamma_{\sigma}}(v_{j,h}(x;\sigma)) - \int_{\Omega} (\lambda h + W_h)|v_{j,h}(x;\sigma)|^2 dx \right\}
\]

\[
\geq (1-\eta)(1-\varepsilon) \sum_{j=1}^{N} \int_{\mathbb{R}^2_+} |(-ih\nabla + A_\sigma)\tilde{v}_{j,h,\sigma}|^2 ds dt
\]

\[
+ h^{3/2} \int_{\mathbb{R}^2} \gamma_{\alpha,\sigma} |\tilde{v}_{j,h,\sigma}(s,0)|^2 ds - \lambda h \|\tilde{v}_{j,h,\sigma}\|_{L^2(\mathbb{R}^2_+)}^2 - C \left( \eta^{-5/4} \|\gamma - \gamma_{\alpha}\|_3^3 + \eta^{1/4} \right) h^{1/2}. \]

The constant \( C \) in the remainder term depends on the fixed parameter \( \eta_0 \), but independent of the other parameters. Notice that the choice of \( \delta \) and \( \varepsilon \) in (5.29) makes the error in (5.7) of the order \( \mathcal{O}(\sqrt{\eta} h^{1/2}) \). Thus, collecting (5.33), (5.6) and (5.4), we get by the variational principle in (2.1),

\[
-E(\lambda; h, \gamma, \alpha) \geq (1-\eta)(1-\varepsilon) \sum_{j=1}^{N} \int_{\mathbb{R}} \mathcal{Q}_{h,B_\sigma;\mathbb{R}^2_+}^{\alpha,\gamma_{\sigma}}(\tilde{v}_{j,h,\sigma}) - \lambda h \|\tilde{v}_{j,h,\sigma}\|_{L^2(\mathbb{R}^2_+)}^2 \right\} d\sigma
\]

\[
-C \left( \eta^{-5/4} \|\gamma - \gamma_{\alpha}\|_3^3 + \eta^{1/4} \right) h^{1/2}. \]

Here \( \mathcal{Q}_{h,B_\sigma;\mathbb{R}^2_+}^{\alpha,\gamma_{\sigma}} \) is the quadratic form associated to the operator in (4.2).

5.4. The leading order term. Here we continue to handle the case \( \alpha = \frac{1}{2} \). We are going to estimate the leading term in (5.34), i.e.

\[
\sum_{j=1}^{N} \int_{\mathbb{R}} \left\{ \mathcal{Q}_{h,B_\sigma;\mathbb{R}^2_+}^{\alpha,\gamma_{\sigma}}(\tilde{v}_{j,h,\sigma}) - \lambda h \|\tilde{v}_{j,h,\sigma}\|_{L^2(\mathbb{R}^2_+)}^2 \right\} d\sigma.
\]

Here \( \gamma_{\alpha,\sigma} \) is the constant introduced in (5.25). Let

\[
\tilde{\gamma}_{h,\sigma} = h^{\alpha-1/2} B_\sigma^{-1/2} \gamma_{\alpha,\sigma} = B_\sigma^{-1/2} \gamma_{\alpha,\sigma}.
\]

Recall the definition of the eigenprojector \( \Pi_p(h, B_\sigma; \tilde{\gamma}_{h,\sigma}; \xi) \) in (4.8). By Lemma 4.2 we have,

\[
2\pi \sum_{j=1}^{N} \mathcal{Q}_{h,B_\sigma;\mathbb{R}^2_+}^{\alpha,\gamma_{\sigma}}(\tilde{v}_{j,h,\sigma}) = \sum_{j=1}^{N} \sum_{\mu=1}^{\infty} \int_{\mathbb{R}} \mu_p(\tilde{\gamma}_{h,\sigma}, \xi) \left\{ \Pi_p(h, B_\sigma; \tilde{\gamma}_{h,\sigma}, \xi) \tilde{v}_{j,h,\sigma} \right\} d\xi.
\]
Thus,
\[
2\pi \sum_{j=1}^{N} \left\{ Q_{h,B_\sigma,\mathbb{R}^2_{+}}^{a,\gamma_0,\sigma}(\bar{v}_{j,h,\sigma}) - \lambda h \int_{\mathbb{R}^2_{+}} |\bar{v}_{j,h,\sigma}|^2 ds dt \right\} 
\geq -hB_\sigma \sum_{p=1}^{\infty} \int_{\mathbb{R}} \left( \mu_p(\gamma_{h,\sigma}, \xi) - \frac{\lambda}{B_\sigma} \right) - \sum_{j=1}^{N} \langle \Pi_p(h, B_\sigma; \gamma_{h,\sigma}, \xi, \bar{v}_{j,h,\sigma}, \bar{v}_{j,h,\sigma}) \rangle d\xi. \tag{5.36} 
\]

From the definition of \( \Pi_p(h, B_\sigma; \gamma_{h,\sigma}, \xi) \) and the identity \([2.16]\), it follows that
\[
\langle \Pi_p(h, B_\sigma; \gamma_{h,\sigma}, \xi, \bar{v}_{j,h,\sigma}, \bar{v}_{j,h,\sigma}) \rangle_{L^2(\mathbb{R}^2_+)} = \frac{B_\sigma}{\nu} \left| \left( \psi_{j,h,\sigma}, e^{-is\xi(h^{-1}B_\sigma)^{1/2}} u_{p,\gamma_{h,\sigma}}((h^{-1}B_\sigma)^{1/2}t; \xi) \right)_{L^2(\mathbb{R}^2_+)} \right|^2 
\leq (1 + C\delta(h)) \frac{B_\sigma}{\nu} \left\| \left( g_j, \xi_{1,\nu} h(x; \sigma) U^{1}_{\Phi} \right( e^{-is\xi(h^{-1}B_\sigma)^{1/2}} u_{p,\gamma_{h,\sigma}}((h^{-1}B_\sigma)^{1/2}t; \xi) \right) \right\|_{L^2(\Omega)}^2, \tag{5.37} 
\]
where the transformation \( U^{-1}_{\Phi} : \tilde{u} \mapsto u \) is associated with the coordinate transform \( \Phi_{\nu} \) introduced in \([2.11]\). Next, since \( \{g_j\}_{j=1}^{N} \) is an orthonormal system in \( L^2(\Omega) \), we have
\[
\sum_{j=1}^{N} \left\| \xi_{1,\nu} h(x; \sigma) U^{1}_{\Phi} \left( e^{-is\xi(h^{-1}B_\sigma)^{1/2}} u_{p,\gamma_{h,\sigma}}((h^{-1}B_\sigma)^{1/2}t; \xi) \right) \right\|_{L^2(\Omega)}^2 \leq \left\| \xi_{1,\nu} h(x; \sigma) U^{1}_{\Phi} \left( e^{-is\xi(h^{-1}B_\sigma)^{1/2}} u_{p,\gamma_{h,\sigma}}((h^{-1}B_\sigma)^{1/2}t; \xi) \right) \right\|_{L^2(\Omega)}^2. \tag{5.38} 
\]

Putting \([5.37]\) and \([5.38]\) together, we get
\[
0 \leq \sum_{j=1}^{N} \left\| \langle \Pi_p(h, B_\sigma; \gamma_{h,\sigma}, \xi, \bar{v}_{j,h,\sigma}, \bar{v}_{j,h,\sigma}) \rangle_{L^2(\mathbb{R}^2_+)} \right\|_{L^2(\Omega)}^2 
\leq (1 + C\delta(h)) \frac{B_\sigma}{\nu} \left\| \xi_{1,\nu} h(x; \sigma) U^{1}_{\Phi} \left( e^{-is\xi(h^{-1}B_\sigma)^{1/2}} u_{p,\gamma_{h,\sigma}}((h^{-1}B_\sigma)^{1/2}t; \xi) \right) \right\|_{L^2(\Omega)}^2 
\leq (1 + C\delta(h)) \frac{B_\sigma}{\nu} \int_{\mathbb{R}^2_+} \left( 1 - tk(s) \right) |\psi_{j,h}(s; \sigma)|^2 |\xi_{1,h}(t)|^2 |u_{p,\gamma_{h,\sigma}}(h^{-1/2}B_\sigma^{1/2}t; h^{-1/2}B_\sigma^{1/2}, \xi)|^2 ds dt 
\leq (1 + C\delta(h)) \frac{B_\sigma}{\nu} \int_{\mathbb{R}_+} |\psi_{j,h}(s; \sigma)|^2 ds \int_{\mathbb{R}_+} |u_{p,\gamma_{h,\sigma}}(h^{-1/2}B_\sigma^{1/2}t; \xi)|^2 dt 
= (1 + C\delta(h)) h^{-1/2} B_\sigma^{1/2}. \tag{5.39} 
\]

Inserting this into \([5.36]\), we find
\[
2\pi \sum_{j=1}^{N} \left\{ Q_{h,B_\sigma,\mathbb{R}^2_{+}}^{a,\gamma_0,\sigma}(\bar{v}_{j,h,\sigma}) - \lambda h \int_{\mathbb{R}^2_{+}} |\bar{v}_{j,h,\sigma}|^2 ds dt \right\} 
\geq -h^{1/2} B_\sigma^{3/2} \sum_{p=1}^{\infty} \int_{\mathbb{R}} \left( \mu_p(\gamma_{h,\sigma}, \xi) - \frac{\lambda}{B_\sigma} \right) - d\xi - O(h^{1/2}\delta(h)). \tag{5.40} 
\]

Fixing \( a \) and \( \eta \), we have, \( \gamma_{h,\sigma} \to \frac{\gamma_{a}(\sigma)}{1-\eta} \) as \( h \to 0 \). It results from Lemma \([3.3]\) that, if \( h \to 0 \), then,
\[
\sum_{p=1}^{\infty} \int_{\mathbb{R}} \left( \mu_p(\gamma_{h,\sigma}, \xi) - \frac{\lambda}{B_\sigma} \right) - d\xi \to \sum_{p=1}^{\infty} \int_{\mathbb{R}} \left( \mu_p \left( B_\sigma^{-1/2} \frac{\gamma_{a}(\sigma)}{1-\eta}, \xi \right) - \frac{\lambda}{B_\sigma} \right) - d\xi. \tag{5.41} 
\]
Since the function $\gamma_a$ is smooth and bounded (for every fixed $a$), then by dominated convergence,
\[
\int_0^{\partial\Omega} \sum_{p=1}^{\infty} \int_{\mathbb{R}} \left( \mu_p(\tilde{\gamma}_{h,a} ; \xi) - \frac{\lambda}{B_x} \right) - d\xi d\sigma \to \int_0^{\partial\Omega} \sum_{p=1}^{\infty} \int_{\mathbb{R}} \left( \mu_p \left( \frac{B_x^{-1/2} \gamma_a(\sigma)}{1 - \eta}, \xi \right) - \frac{\lambda}{B_x} \right) - d\xi d\sigma.
\]
Inserting this and (5.40) into (5.34), we get,
\[
\liminf_{h \to 0} \left( -2\pi h^{-1/2} E(\lambda; h, \gamma, \alpha) \right) \geq (1 - \eta) \sum_{p=1}^{\infty} \int_{\partial\Omega} \int_{\mathbb{R}} B(x)^{3/2} \left( \mu_p \left( B(x)^{-1/2} \gamma_a(x) \right) \right) - \frac{\lambda}{B(x)} - d\xi ds(x)
\]
\[
- C \left( \eta^{-5/4} ||\gamma - \gamma_a||_2^2 + \eta^{1/4} \right).
\]
Taking successively $\liminf_{a \to 0+}$ then $\liminf_{\eta \to 0+}$, we arrive at,
\[
\liminf_{h \to 0} \left( -2\pi h^{-1/2} E(\lambda; h, \gamma, \alpha) \right) \geq \liminf_{\eta \to 0+} \left\{ \liminf_{a \to 0+} \sum_{p=1}^{\infty} \int_{\partial\Omega} \int_{\mathbb{R}} B(x)^{3/2} \left( \mu_p \left( B(x)^{-1/2} \gamma_a(x) \right) \right) - \frac{\lambda}{B(x)} - d\xi ds(x) \right\}.
\]
If $\gamma \in L^\infty(\partial\Omega)$, then $||\gamma_a||_\infty \leq ||\gamma||_\infty$ and by dominated convergence, the right side in (5.43) is
\[
\sum_{p=1}^{\infty} \int_{\partial\Omega} \int_{\mathbb{R}} B(x)^{3/2} \left( \mu_p \left( B(x)^{-1/2} \gamma(x) \right) \right) - \frac{\lambda}{B(x)} - d\xi ds(x).
\]
Therefore, when $\gamma \in L^\infty(\partial\Omega)$ and $\alpha = \frac{1}{2}$, we have the lower bound,
\[
\liminf_{h \to 0} \left( -2\pi h^{-1/2} E(\lambda; h, \gamma, \alpha) \right) \geq \sum_{p=1}^{\infty} \int_{\partial\Omega} \int_{\mathbb{R}} B(x)^{3/2} \left( \mu_p \left( B(x)^{-1/2} \gamma(x) \right) \right) - \frac{\lambda}{B(x)} - d\xi ds(x).
\]

\section{Upper bound}

Let $\sigma \in [0, |\partial\Omega|]$, $\phi = \phi_\sigma$ be the gauge from Proposition 2.3, $\zeta_{1,h}$ and $\psi_h$ the functions from (5.3) and (5.8) respectively. Let furthermore $\Phi = \Phi_0$ be the coordinate transformation near the boundary given in (2.11), $B_x = B(\Phi(\sigma,0))$ and $\tilde{\gamma}_{h,\sigma}$ the number introduced below in (6.9).

Let $\xi \in \mathbb{R}$. If $\alpha = 1/2$, the function
\[
\tilde{f}_{p,1/2}(s, t; h, \sigma, \xi) := B_x^{1/4} h^{-1/4} e^{-i\xi s} \sqrt{\beta_x/h} u_{p, \tilde{\gamma}_{h,\sigma}}(B_x^{1/2} h^{-1/2} t; \xi) e^{-i\phi_h/h} \psi_h(s; \sigma) \zeta_{1,h}(t),
\]
where $u_{p, \tilde{\gamma}_{h,\sigma}}(\cdot; \xi)$ is the function from (4.3), and if $\alpha > 1/2$, we define
\[
\tilde{f}_{p,\alpha}(s, t; h, \sigma, \xi) := B_x^{1/4} h^{-1/4} e^{-i\xi s} \sqrt{\beta_x/h} u_{p, 0}(B_x^{1/2} h^{-1/2} t; \xi) e^{-i\phi_h/h} \psi_h(s; \sigma) \zeta_{1,h}(t).
\]
Recall the coordinate transformation $\Phi$ valid near a neighborhood of the point $x$ (see Subsection 2.2), and let $x = \Phi^{-1}(y)$. We define $f_p(x; h, \sigma, \xi) := \tilde{f}_p((s, t); h, \sigma, \xi)$ by means of (2.12). Let $K > 0$. If $\alpha = 1/2$, we set,
\[
M_{1/2}(h, \sigma, \xi, p, K) = \mathbf{1}_{((\sigma, \xi, p) \in [0, |\partial\Omega|] \times \mathbb{R} \times \mathbb{N} : \frac{\lambda}{B_x} - \mu_p(\tilde{\gamma}_{h,\sigma} \xi) \geq 0, |\xi| \leq K)},
\]
and if $\alpha > 1/2$,
\[
M_\alpha(h, \sigma, \xi, p, K) = \mathbf{1}_{((\sigma, \xi, p) \in [0, |\partial\Omega|] \times \mathbb{R} \times \mathbb{N} : p=1, \frac{\lambda}{B_x} - \mu_1(0, \xi) \geq 0, |\xi| \leq K)}.
\]
Let \( f \in L^2(\Omega) \). We introduce
\[
(\Gamma f)(x) = (2\pi)^{-1}h^{-1/2} \int B_\sigma^{1/2} \sum_{p=1}^{\infty} M(h, \sigma, \xi, p, K)(f_p(; h, \sigma, \xi), f) f_p(x; h, \sigma, \xi)d\sigma d\xi .
\] (6.3)

In Lemma 6.1 below, we will prove that \( \Gamma \) satisfies the density matrix condition, namely, \( 0 \leq \Gamma \leq 1 + o(1) \). By the variational principle in Lemma 2.2, an upper bound of the sum of eigenvalues of \( \mathcal{P}_{\alpha,\gamma}^{h,\Omega} \) below \( \lambda h \) follows if we can prove an upper bound on
\[
\text{tr}[\mathcal{P}_{h,\Omega}^{\alpha,\gamma} - \lambda h \Gamma] = (2\pi)^{-1}h^{-1/2} \int_{\Omega} B_\sigma^{1/2} M(h, \sigma, \xi, p, K) \left( Q_{h,\Omega}^{\alpha,\gamma}(f_p(x; h, \sigma, \xi)) - \lambda h \| f_p(x; h, \sigma, \xi) \|_2^2 \right) d\sigma d\xi ,
\] (6.4)

where \( Q_{h,\Omega}^{\alpha,\gamma} \) is the quadratic form introduced in (1.4). We will then estimate the quantity in (6.4) in the cases \( \alpha = 1/2 \) and \( \alpha > 1/2 \) independently.

**The regime \( \alpha > 1/2 \).** In this subsection, we suppose that \( \alpha > 1/2 \). We see in (6.2) that the definition of \( M \) involves the first eigenvalue \( \mu_1(\cdot, \cdot) \) only. Consequently, the summation in the definition of \( \Gamma \) is restricted to the first term corresponding to \( p = 1 \). We observe that
\[
Q_{h,\Omega}^{\alpha,\gamma}(f_1(x; h, \sigma, \xi)) = Q_{h,\Omega}^{0,0}(f_1(x; h, \sigma, \xi)) + h^{1+\alpha} \int_{\mathbb{R}} \gamma(s) |f_1((s, 0); h, \sigma, \xi)|^2 ds .
\] (6.5)

Easy computations lead to
\[
\int_{\mathbb{R}} \gamma(s) |f_1((s, 0); h, \sigma, \xi)|^2 ds \leq B_{\sigma}^{1/2} h^{-1/2} |u_{1,0}(0, \xi)|^2 \int_{\mathbb{R}} (\gamma(s))_+ |\psi_h(s; \sigma)|^2 ds .
\]

Inserting this into (6.5), we obtain
\[
Q_{h,\Omega}^{\alpha,\gamma}(f_1(x; h, \sigma, \xi)) \leq Q_{h,\Omega}^{0,0}(f_1(x; h, \sigma, \xi)) + B_{\sigma}^{1/2} h^{\alpha+1/2} |u_{1,0}(0, \xi)|^2 \int_{\mathbb{R}} (\gamma(s))_+ |\psi_h(s; \sigma)|^2 ds .
\] (6.6)

Now, we compute,
\[
\text{tr}[\mathcal{P}_{h,\Omega}^{\alpha,\gamma} - \lambda h \Gamma] = \int_{\Omega} (2\pi)^{-1} B_{\sigma}^{1/2} h^{-1/2} M(h, \sigma, \xi, p = 1, K) \left\{ Q_{h,\Omega}^{\alpha,\gamma}(f_1(x; h, \sigma, \xi)) - \lambda h \| f_1(x; h, \sigma, \xi) \|_2^2 \right\} d\sigma d\xi
\]
\[
\leq \int_{-K}^{K} \int_0^{[\partial \Omega]} (2\pi)^{-1} B_{\sigma}^{1/2} h^{-1/2} \left\{ Q_{h,\Omega}^{0,0}(f_1(x; h, \sigma, \xi)) - \lambda h \| f_1(x; h, \sigma, \xi) \|_2^2 + h^{\alpha+1/2} B_{\sigma}^{1/2} |u_{1,0}(0, \xi)|^2 \int_{\mathbb{R}} (\gamma(s))_+ |\psi_h(s; \sigma)|^2 ds \right\} d\sigma d\xi
\]
\[
\leq \int_{-K}^{K} \int_0^{[\partial \Omega]} (2\pi)^{-1} B_{\sigma}^{1/2} h^{-1/2} \left\{ Q_{h,\Omega}^{0,0}(f_1(x; h, \sigma, \xi)) - \lambda h \| f_1(x; h, \sigma, \xi) \|_2^2 \right\} d\sigma d\xi
\]
\[
+ (2\pi)^{-1} h^\alpha \| B \|_{L^\infty([\partial \Omega])} \int_{-K}^{K} |u_{1,0}(0, \xi)|^2 \int_{\mathbb{R}} \left\{ (\gamma(s))_+ |\psi_h(s; \sigma)|^2 ds \right\} d\sigma d\xi .
\] (6.7)

Using that \( \int_0^{[\partial \Omega]} |\psi_h^2(s; \sigma)| d\sigma = 1 \) and taking into account the regularity of the function \( \xi \mapsto |u_{1,0}(0, \xi)|^2 \), the second term on the right-hand side of (6.7) is estimated from above by
\[
(2\pi)^{-1} h^\alpha 2K \| B \|_{L^\infty([\partial \Omega])} \sup_{\xi \in [-K, K]} |u_{1,0}(0, \xi)|^2 \| \gamma \|_{L^1([\partial \Omega])} ,
\]
which is $o(h^{1/2})$ for fixed $K$. Also, by [9] Proof of (5.37), the first term on the right-hand side of (6.7) is bounded from above by,

$$\frac{h^{1/2}}{2\pi} \int_{\partial \Omega} \int_{-K}^{K} B(x)^{3/2} \left( \mu_1(0, \xi) - \frac{\lambda}{B(x)} \right) d\xi ds(x) - h^{1/2} o(1).$$

Thus, taking the successive limits $\limsup_{h \to 0^+}$ and $\lim_{K \to \infty}$, we obtain,

$$\limsup_{h \to 0} \left( - h^{-1/2} E(\lambda; h, \gamma, 1/2) \right) \leq - \frac{1}{2\pi} \int_{\partial \Omega} \int_{\mathbb{R}} B(x)^{3/2} \left( \mu_1(0, \xi) - \frac{\lambda}{B(x)} \right) d\xi ds(x),$$

which gives the desired upper bound when $\alpha > 1/2$.

The regime $\alpha = 1/2$. In this section, we restrict to the harder case $\alpha = 1/2$. Here, the definition of $M = M_{1/2}$ in (6.1) involves the quantity,

$$\bar{F}_{h, \sigma} = \frac{B_{1}^{-1/2}(\gamma_{a}(\sigma) + C \alpha^{-2} \delta(h))}{1 + \varepsilon}.$$  

(6.9)

In the definition of $\bar{F}_{h, \sigma}$, $\alpha \in (0, 1)$ and $\varepsilon \in (0, 1)$ are fixed parameters, and $\gamma_{a}$ is the function introduced in (5.21). Recall that, as $a \to 0^+$, $\gamma_{a} \to \gamma$ in $L^{3}(\partial \Omega)$.

We start by computing, for all $p \geq 1$,

$$\int_{\Omega} |f_{p}(x; h, \sigma, \xi)|^{2} dx = \int_{0}^{\infty} \int_{0}^{[\partial \Omega]} |\tilde{f}_{p}((s, t); h, \sigma, \xi)|^{2} (1 - tk(s)) ds dt \leq (1 + \delta(h) \|k\|_{\infty}) B_{1}^{1/2} h^{-1/2} \times \int_{0}^{\delta(h)} \int_{0}^{[\partial \Omega]} |u_{p, \bar{F}_{h, \sigma}}(B_{1}^{1/2} h^{-1/2}; t; \xi)|^{2} |\zeta_{1, h}(t)|^{2} |\psi_{h}(s; \sigma)|^{2} ds dt \leq (1 + \delta(h) \|k\|_{\infty}) B_{1}^{1/2} h^{-1/2} \int_{0}^{\delta(h)} |u_{p, \bar{F}_{h, \sigma}}(B_{1}^{1/2} h^{-1/2}; t; \xi)|^{2} dt = (1 + \delta(h) \|k\|_{\infty}),$$

where we have used that the functions $\psi_{h}$ and $u_{p, \bar{F}_{h, \sigma}}$ are normalized in $s$ and $t$ respectively. Again the normalization of $\psi_{h}$ implies that

$$\int_{\mathbb{R}} |\tilde{f}_{p}((s, 0); h, \sigma, \xi)|^{2} ds = B_{1}^{1/2} h^{-1/2} |u_{p, \bar{F}_{h, \sigma}}(0; \xi)|^{2} |\zeta_{1, h}(0)|^{2} \int_{0}^{[\partial \Omega]} |\psi_{h}(s; \sigma)|^{2} ds \leq B_{1}^{1/2} h^{-1/2} |u_{p, \bar{F}_{h, \sigma}}(0; \xi)|^{2} \int_{0}^{[\partial \Omega]} |\psi_{h}(s; \sigma)|^{2} ds = B_{1}^{1/2} h^{-1/2} |u_{p, \bar{F}_{h, \sigma}}(0; \xi)|^{2}.$$  

(6.11)

We also compute

$$\int_{\Omega} |f_{p}(x; h, \sigma, \xi)|^{2} dx = \int \int |\tilde{f}_{p}((s, t); h, \sigma, \xi)|^{2} (1 - tk(s)) ds dt \geq (1 - \delta(h) \|k\|_{\infty}) B_{1}^{1/2} h^{-1/2} \times \int_{0}^{\delta(h)} \int_{0}^{[\partial \Omega]} |u_{p, \bar{F}_{h, \sigma}}(B_{1}^{1/2} h^{-1/2}; t; \xi)|^{2} |\zeta_{1, h}(t)|^{2} |\psi_{h}(s; \sigma)|^{2} ds dt = (1 - \delta(h) \|k\|_{\infty}) B_{1}^{1/2} h^{-1/2} \int_{\mathbb{R}} \int |u_{p, \bar{F}_{h, \sigma}}(B_{1}^{1/2} h^{-1/2}; t; \xi)|^{2} |\zeta_{1, h}(t)|^{2} dt.$$  

(6.12)
Let us write the last integral as
\[
\int_{\mathbb{R}^+} |u_{p, \tilde{\gamma}_{h, \sigma}}(B_{\sigma}^{1/2}h^{-1/2}t; \xi)|^2 |\zeta_{1,h}(t)|^2 dt = \int_{\mathbb{R}^+} |u_{p, \tilde{\gamma}_{h, \sigma}}(B_{\sigma}^{1/2}h^{-1/2}t; \xi)|^2 dt \\
+ \int_{\mathbb{R}^+} |u_{p, \tilde{\gamma}_{h, \sigma}}(B_{\sigma}^{1/2}h^{-1/2}t; \xi)|^2 (|\zeta_{1,h}(t)|^2 - 1) dt \\
= B_{\sigma}^{-1/2}h^{1/2} + \int_{t \geq \delta(h)/2} |u_{p, \tilde{\gamma}_{h, \sigma}}(B_{\sigma}^{1/2}h^{-1/2}t; \xi)|^2 (|\zeta_{1,h}(t)|^2 - 1) dt. \tag{6.13}
\]
Taking into account the support of \(\zeta_{1,h}\), we can write,
\[
\int_{\mathbb{R}^+} |u_{p, \tilde{\gamma}_{h, \sigma}}(B_{\sigma}^{1/2}h^{-1/2}t; \xi)|^2 |\zeta_{1,h}(t)|^2 dt \geq B_{\sigma}^{1/2}h^{-1/2} - \int_{t \geq \delta(h)/2} |u_{p, \tilde{\gamma}_{h, \sigma}}(B_{\sigma}^{1/2}h^{-1/2}t; \xi)|^2 dt \\
= B_{\sigma}^{-1/2}h^{1/2} - \int_{t \geq \delta(h)/2} e^{-\epsilon(B_{\sigma}^{1/2}h^{-1/2}t - \xi)^2/2} e^{(B_{\sigma}^{1/2}h^{-1/2}t - \xi)^2/2} |u_{p, \tilde{\gamma}_{h, \sigma}}(B_{\sigma}^{1/2}h^{-1/2}t; \xi)|^2 dt. \tag{6.14}
\]
In observance of the support of \(M = M_{1/2}\), we see that \(|\xi| \leq K\) and
\[
(B_{\sigma}^{1/2}h^{-1/2}t - \xi)^2 \geq \left(6.14\right) \geq \left(6.17\right) \geq \frac{1}{8}bh^{-1}\delta(h)^2 - 2K^2.
\]
Implementing this into \((6.14)\) and using the exponential decay given in \((6.10)\), we find that
\[
\int_{\mathbb{R}^+} |u_{p, \tilde{\gamma}_{h, \sigma}}(B_{\sigma}^{1/2}h^{-1/2}t; \xi)|^2 |\zeta_{1,h}(t)|^2 dt \\
\geq B_{\sigma}^{1/2}h^{-1/2}\left(1 - C_{\epsilon,k}e^{-\epsilon\left(bh^{-1}\delta(h)^2 - 2K^2\right)/2}(1 + a^{-2}\delta(h) + (a^{-2}\delta(h))^2)\right). \tag{6.15}
\]
In the last step we have used that \(\gamma \in L^\infty\) together with the definition of \(\tilde{\gamma}_{h, \sigma}\) in \((6.11)\).

Inserting this into \((6.14)\), we finally obtain
\[
\frac{1}{2}(f_{p}(x; h, \sigma, \xi))^2 dx \geq \frac{1}{2}(1 - \delta(h)\|k\|_{\infty})(1 - C_{\epsilon,k}e^{-\epsilon\left(bh^{-1}\delta(h)^2 - 2K^2\right)/2}(1 + a^{-2}\delta(h) + (a^{-2}\delta(h))^2)). \tag{6.16}
\]

Next we estimate the quadratic form. By Lemma 2.8, we have for all \(\epsilon > 0\),
\[
Q_{h, \Omega}^{\alpha, \gamma}(f_{p}(x; h, \sigma, \xi)) \\
= \int_{\mathbb{R}^2} |-(ih\nabla + A)f_{p}(x; h, \sigma, \xi)|^2 dx + h^{3/2} \int_{\partial \Omega} \gamma(x)|f_{p}(x; h, \sigma, \xi)|^2 dx \\
\leq (1 + \epsilon) \int_{\mathbb{R}^2} |-(ih\nabla + A_{\alpha})e^{i\xi(\eta/\sqrt{h})}f_{p}(s, t; h, \sigma, \xi)|^2 dsdt \\
+ C e^{-1}\delta(h)^2 \int_{\mathbb{R}^2} \gamma(s)|f_{p}(s, t; h, \sigma, \xi)|^2 dsdt + h^{3/2} \int_{\mathbb{R}^2} \gamma(s)|f_{p}(s, 0; h, \sigma, \xi)|^2 ds. \tag{6.17}
\]
Writing \(\gamma = \gamma_{a} + (\gamma - \gamma_{a})\), it follows that
\[
Q_{h, \Omega}^{\alpha, \gamma}(f_{p}(x; h, \sigma, \xi)) \\
\leq (1 + \epsilon)B_{\sigma}^{1/2}h^{-1/2} \int_{\psi_{h}(s, \sigma)} |-(ih\nabla + A_{\alpha})e^{-i\xi(\eta/\sqrt{h})}u_{p, \tilde{\gamma}_{h, \sigma}}(B_{\sigma}^{1/2}h^{-1/2}t; \xi)|^2 dsdt \\
+ \left(\frac{1}{2}Ch^2\delta(h)^2 - C e^{-1}\delta(h)^4\right) \int_{\mathbb{R}^2} |\tilde{f}_{p}(s, t; h, \sigma, \xi)|^2 dsdt \\
+ h^{3/2} \int_{\partial \Omega} \gamma_{a}(s)|\tilde{f}_{p}(s, 0; h, \sigma, \xi)|^2 ds \\
+ hB_{\sigma}^{1/2}|u_{p, \tilde{\gamma}_{h, \sigma}}(0; \xi)|^2 |\zeta_{1,h}(0)|^2 \int_{\mathbb{R}} (\gamma(s) - \gamma_{a}(s))\psi_{h}(s; \sigma)|^2 ds, \tag{6.18}
\]
where $A_\sigma$ is defined in (5.11). Plugging (6.10) and (6.11) into (6.18), and using (6.22), we find

$$Q_{h,\Omega}^{\alpha,\gamma}(f_p(x; h, \sigma, \xi))$$

where $\sigma > 0$ and $M$ is defined in (5.11). Plugging (6.10) and (6.11) into (6.18), and using (5.22), we find

$$\langle 1 + \epsilon \rangle_{(1 + \epsilon)C h^2 \delta(h)^{-2} + C \varepsilon^{-1} \delta(h)^4}$$

Using Lemma 3.6 and the fact that $\gamma \in L^\infty$, we infer that the number of indices $p$ appearing in the support of $M(h, \sigma, \xi, p, K)$ is finite. More precisely, there exists a constant $p_0 \in \mathbb{N}$ such that, for all $\sigma > 0$, $\alpha \in (0, 1)$ and $K > 0$, the function $M$ in (6.11) vanishes for all $p > p_0$.

Now, we collect (6.19), (6.11), (6.10) and (6.18) to obtain

$$\text{tr}[\mathcal{P}_{h,\Omega}^{\alpha,\gamma} - \lambda h)] \leq (2\pi)^{-1} h^{-1/2} \int \int B_{\sigma}^{1/2} \sum_{p=1}^{\infty} \left\{ Q_{h,\Omega}^{\alpha,\gamma}(f_p(x; h, \sigma, \xi)) - \lambda h \| f_p(x; h, \sigma, \xi) \| ^2 \right\} d\xi d\sigma$$

$$\leq \sum_{p=1}^{p_0} \int \int (2\pi)^{-1} B_{\sigma}^{1/2} h^{-1/2} M(h, \sigma, \xi, p, K) \left\{ (1 + \epsilon) h B_{\sigma} \mu_p(\gamma, h, \sigma, \xi) + \lambda h \right\} d\sigma d\xi$$

We may arrange the terms in (6.20) to obtain,

$$\text{tr}[\mathcal{P}_{h,\Omega}^{\alpha,\gamma} - \lambda h)] \leq -\sum_{p=1}^{p_0} \int_{-K}^{K} \int_{0}^{|\Omega|} (2\pi)^{-1} B_{\sigma}^{1/2} h^{-1/2} \left( \mu_p(\gamma, h, \sigma, \xi) h B_{\sigma} - \lambda h \right) - d\sigma d\xi$$

where

$$R_1 = p_0 \| \partial \Omega \| 2 K \| B \| _{L^\infty(\partial \Omega)} (2\pi)^{-1} h^{-1/2} \times$$

$$\left( \varepsilon + \| k \| _{\infty} \delta(h) + C_{h,K} e^{-T(h)^2/2} (1 + a^{-2} \delta(h) + (a^{-2} \delta(h))^2) \right)$$

$$- \| k \| _{\infty} \delta(h) C_{h,K} e^{-T(h)^2/2} (1 + a^{-2} \delta(h) + (a^{-2} \delta(h))^2),$$

$$R_2 = p_0 \| \partial \Omega \| 2 K \| B \| _{L^\infty(\partial \Omega)} (2\pi)^{-1} h^{-1/2} (1 + \| k \| _{\infty} \delta(h)) \left( 1 + \varepsilon h^2 \delta(h)^{-2} + C \varepsilon^{-1} \delta(h)^4 \right),$$

and

$$R_3 = (2\pi)^{-1} h^{1/2} \| B \| _{L^\infty(\partial \Omega)} \sum_{p=1}^{p_0} \int_{-K}^{K} \int_{0}^{|\Omega|} (\gamma(s) - \gamma_a(s)) + |\psi_h(s; \sigma)|^2 u_{p,\gamma,\sigma}(0; \xi)^2 ds d\sigma d\xi.$$
Choosing $\delta = h^{3/8}$ and $\varepsilon = h^{1/4}$, we see that, for fixed $a$ and $K$,

$$R_1 + R_2 = o(h^{1/2}),$$

and

$$|R_3| \leq (2\pi)^{-1} 2KH^{1/2} \|B\|_{L^\infty(\partial \Omega)} \|\gamma - \gamma_a\|_{L^1(\partial \Omega)} \sum_{p=1}^{p_0} \sup_{\xi \in [-K,K]} |u_{p,\gamma_{h,\sigma}}(0; \xi)|^2.$$

The term $|u_{p,\gamma_{h,\sigma}}(0; \xi)|^2$ is controlled by the estimate in Lemma 3.3. Taking into account the condition of the support of $M = M_{1/2}$ in (6.1), we observe that,

$$|u_{p,\gamma_{h,\sigma}}(0; \xi)|^2 \leq C \left( \mu_p(\gamma_{h,\sigma}; \xi) + \gamma_{h,\sigma}^2 + 1 \right) \leq C(2 + \gamma_{h,\sigma}^2).$$

It follows from the definition of $\gamma_{h,\sigma}$ in (6.9) that, when $h$ and $\sigma$ vary and $K$, $a$ and $p$ remain fixed,

$$\sup_{\xi \in [-K,K]} |u_{p,\gamma_{h,\sigma}}(0; \xi)|^2 \leq C \left( 1 + (a^{-2}\delta(h))^2 \right),$$

thereby giving us that,

$$|R_3| \leq 2CKh^{1/2} \left( 1 + (a^{-2}\delta(h))^2 \right) \|\gamma - \gamma_a\|_{L^1(\partial \Omega)},$$

as long as $a$ and $K$ remain fixed.

Now, we insert (6.25) and (6.26) into (6.27). Thanks to Lemma 6.1 below, we may apply the variational principle in Lemma 2.2. That way, we infer from (6.27),

$$- E(\lambda; h, 1/2) \leq (1 + \|k\|_\infty \delta(h))^{-1} \text{tr}[(\mathcal{P}_{h,\Omega}^a - \lambda h)]$$

$$\leq -(1 + \|k\|_\infty \delta(h))^{-1} \sum_{p=1}^{p_0} \int_0^K \int_{\partial \Omega} (2\pi)^{-1} B_{\sigma}^{3/2} a h^{1/2} \left( \mu_p(\gamma_{h,\sigma}, \xi) - \frac{\lambda}{B_{\sigma}} \right) d\sigma d\xi$$

$$+ o(h^{1/2}) + 2CKh(1 + (a^{-2}\delta(h))^2) \|\gamma - \gamma_a\|_{L^1(\partial \Omega)}.$$

Since $\gamma_{h,\sigma} \to B_{\sigma}^{-1/2} \gamma_a(\sigma)$ as $h \to 0$, and $\gamma_{h,\sigma}$ remains bounded for a fixed $a$, then it results from Lemma 3.9 and dominated convergence (as $h \to 0+$),

$$\sum_{p=1}^{p_0} \int_0^K \int_{\partial \Omega} \left( \mu_p(\gamma_{h,\sigma}, \xi) - \frac{\lambda}{B_{\sigma}} \right) d\sigma d\xi \to \sum_{p=1}^{p_0} \int_0^K \int_{\partial \Omega} \left( \mu_p(B_{\sigma}^{-1/2} \gamma_a(\sigma), \xi) - \frac{\lambda}{B_{\sigma}} \right) d\sigma d\xi.$$

Since the function $\gamma_a$ is smooth and bounded (for every fixed $a$), then by dominated convergence,

$$\int_0^{\partial \Omega} \sum_{p=1}^{p_0} \int_0^K \int_{\partial \Omega} \left( \mu_p(\gamma_{h,\sigma}, \xi) - \frac{\lambda}{B_{\sigma}} \right) d\sigma d\xi \to \int_0^{\partial \Omega} \sum_{p=1}^{p_0} \int_0^K \int_{\partial \Omega} \left( \mu_p(B_{\sigma}^{-1/2} \gamma_a(\sigma), \xi) - \frac{\lambda}{B_{\sigma}} \right) d\sigma d\xi.$$

Taking lim sup$_{h \to 0}$ on both sides in (6.27), it follows that,

$$\limsup_{h \to 0} \left( - h^{1/2} E(\lambda; h, 1/2) \right)$$

$$\leq - \sum_{p=1}^{p_0} \int_0^K \int_{\partial \Omega} (2\pi)^{-1} B(x)^{3/2} \left( \mu_p(B(x)^{-1/2} \gamma_a(x), \xi) - \frac{\lambda}{B(x)} \right) d\xi d\sigma.$$
Since \( \gamma \in L^\infty(\Omega) \), then \( \|\gamma_a\|_\infty \leq \|\gamma\|_\infty \) and by dominated convergence, the right-hand side in (6.29) is
\[
-\frac{1}{2\pi} \sum_{p=1}^{\overline{p}_1} \int_{-\infty}^\infty \int_{\partial \Omega} B(x)^{3/2} \left( \mu_p(B(x)^{-1/2}\gamma(x), \xi) - \frac{\lambda}{B(x)^2} \right) d\xi ds(x).
\]
This finishes the proof of the upper bound in Theorem 1.1.

It remains to verify that the density matrix \( \Gamma \) satisfies the necessary properties to apply the variational principle in Lemma 2.2. That is contained in Lemma 6.1.

**Lemma 6.1.** There exists a constant \( C > 0 \) such that,
\[
\forall f \in L^2(\Omega), \quad 0 \leq \langle \Gamma f, f \rangle_{L^2(\Omega)} \leq (1 + \|k\|_\infty \delta(h)) \|f\|_{L^2(\Omega)}^2, \tag{6.30}
\]
where \( \Gamma \) is as in (6.3).

**Proof.** Let \( f \in L^2(\Omega) \). Due to the support of \( \Gamma \) (in particular \( \zeta_{1,h} \)), we may suppose that \( \text{supp } f \subset \{ x \in \overline{\Omega} : \text{dist}(x, \partial \Omega) \leq \delta(h) \} \). We compute,
\[
\langle f, \Gamma f \rangle_{L^2(\Omega)} = (2\pi)^{-1} h^{-1/2} \sum_{p=1}^{\infty} \int M(h, \sigma, \xi, p, K) B_{\sigma}^{1/2} \left| \int f(s,t) \tilde{f}_p((s,t); h, \sigma, \xi)(1 - tk(s)) ds dt \right|^2 d\sigma d\xi. \tag{6.31}
\]
We estimate from above by replacing \( \int M \times |\cdot|^2 \) by \( \int 1 \times |\cdot|^2 \) in the above expression. Defining
\[
G(s,t) = f(s,t)\psi_h(s; \sigma) \zeta_{1,h}(t)e^{-i\phi_\sigma/h}(1 - tk(s)).
\]
Using Cauchy-Schwarz inequality and the fact that \( u_{p,\gamma_{h,\alpha}}(\cdot, \xi) \) in the case \( \alpha = 1/2 \) (or \( u_{p,0}(\cdot, \xi) \) in the case \( \alpha > 1/2 \)) is an orthonormal basis of \( L^2(\mathbb{R}_+) \) for all \( \xi \), we get,
\[
\sum_{p=1}^{\infty} \left| \int f(s,t) \tilde{f}_p(s,t; h, \sigma, \xi)(1 - tk(s)) ds dt \right|^2 \leq 2\pi \int_{L^0} |(F_{s\rightarrow \xi} G)(B_{\sigma}^{1/2}h^{-1/2}\xi)|^2 dt.
\]
Here, \( F_{s\rightarrow \xi} \) denotes the Fourier transform with respect to the variable \( s \).

Integrating with respect to \( \xi \) and using Plancherel identity, we find that,
\[
2\pi \int_{\mathbb{R}_+} \left| \int_{\mathbb{R}_+} |(F_{s\rightarrow \xi} G)(B_{\sigma}^{1/2}h^{-1/2}\xi)|^2 \right| d\xi ds = 2\pi h^{1/2} B_{\sigma}^{-1/2} \int_{\mathbb{R}_+} |G(s,t)|^2 ds dt.
\]
Consequently,
\[
\langle f, \Gamma f \rangle_{L^2(\Omega)} \leq \int_{L^0} \int_{\mathbb{R}_+} |\tilde{f}(s,t)|^2 (1 - tk(s))^2 \psi_h^2(s; \sigma) \zeta_{1,h}(t) ds dt d\sigma. \tag{6.32}
\]
We do the \( \sigma \)-integration first. The normalization of \( \psi_h \) implies that the result is
\[
\langle f, \Gamma f \rangle_{L^2(\Omega)} \leq \int_{\mathbb{R}_+} |\tilde{f}(s,t)|^2 (1 - tk(s))^2 \zeta_{1,h}^2(t) ds dt \leq (1 + \delta(h) \|k\|_\infty) \int_{\Omega} |f(x)|^2 dx.
\]
This finishes the proof of (6.30). \( \square \)
Replacing $\varepsilon$ \varepsilon>

Let

By combining (7.8) and (7.9), we obtain

$$\limsup_{\varepsilon \rightarrow 0^+} h^{1/2} N(\lambda; h, \gamma, \alpha) \leq \frac{1}{2\pi} \sum_{p=1}^{p_0} \int_{\partial \Omega} f_p(x, \lambda + \varepsilon) - f_p(x, \lambda) \frac{\varepsilon}{\varepsilon} \, d\sigma(x). \quad (7.7)$$

Taking the limit $\varepsilon \rightarrow 0^+$ and using dominated convergence, we deduce that

$$\limsup_{h \rightarrow 0^+} h^{1/2} N(\lambda; h, \gamma, \alpha) \leq \frac{1}{2\pi} \sum_{p=1}^{p_0} \int_{\partial \Omega} \frac{\partial f_p}{\partial \lambda_+}(x, \lambda) d\sigma(x). \quad (7.8)$$

Replacing $\varepsilon$ by $-\varepsilon$ in (7.5) and following the same arguments that led to (7.8), we find

$$\liminf_{h \rightarrow 0^+} h^{1/2} N(\lambda; h, \gamma, \alpha) \geq \frac{1}{2\pi} \sum_{p=1}^{p_0} \int_{\partial \Omega} \frac{\partial f_p}{\partial \lambda_-}(x, \lambda) d\sigma(x). \quad (7.9)$$

By combining (7.8) and (7.9), we obtain

$$\lim_{h \rightarrow 0^+} h^{1/2} N(\lambda; h, \gamma, \alpha) = \frac{1}{2\pi} \sum_{p=1}^{p_0} \int_{\partial \Omega} \frac{\partial f_p}{\partial \lambda_+}(x, \lambda) d\sigma(x). \quad (7.10)$$

7. Proof of Corollary 1.2

We will prove the second assertion in (1.13). The first assertion in (1.12) can be proven similarly. Define

$$f_p(x, \lambda) := \int_{\mathbb{R}} B(x)^{3/2} \left( \mu_p \left( B(x)^{-1/2} \gamma(x), \xi \right) - \frac{\lambda}{B(x)} \right) \, d\xi. \quad (7.1)$$

We start by computing the left- and right- derivatives of the function $\lambda \rightarrow f_p(x, \lambda)$. We thus find

$$\frac{\partial f_p}{\partial \lambda_+}(x, \lambda) = \int_{\{\xi \in \mathbb{R} : B(x)\mu_p(B(x)^{-1/2} \gamma(x), \xi) \leq \lambda\}} B(x)^{1/2} \, d\xi, \quad (7.2)$$

and

$$\frac{\partial f_p}{\partial \lambda_-}(x, \lambda) = \int_{\{\xi \in \mathbb{R} : B(x)\mu_p(B(x)^{-1/2} \gamma(x), \xi) < \lambda\}} B(x)^{1/2} \, d\xi. \quad (7.3)$$

In view of Lemma 3.2, the equation

$$\mu_p(B(x)^{-1/2} \gamma(x), \xi) = \lambda B(x)^{-1},$$

has exactly two solutions

$$\xi_{p, \pm} := \xi_{p, \pm}(\gamma'_x, \lambda'_x); \quad \gamma'_x := B(x)^{-1/2} \gamma(x) \quad \text{and} \quad \lambda'_x := \lambda B(x)^{-1}.$$\n
Since the set $\{\xi : \xi = \xi_{p, \pm}\}$ has measure zero with respect to the $\xi$ integration, it follows that the left- and right- derivatives coincide and we can write

$$\frac{\partial f_p}{\partial \lambda_{\pm}}(x, \lambda) = \int_{\{\xi \in \mathbb{R} : B(x)\mu_p(B(x)^{-1/2} \gamma(x), \xi) \leq \lambda\}} B(x)^{1/2} \, d\xi. \quad (7.4)$$

Let $\varepsilon > 0$. By the variational principle in Lemma 2.2, we have

$$E(\lambda + \varepsilon; h, \gamma, \alpha) - E(\lambda; h, \gamma, \alpha) \geq \varepsilon h N(\lambda; h, \gamma, \alpha). \quad (7.5)$$

On the other hand, it follows from Theorem 1.1 that

$$E(\lambda; h, \gamma, \alpha) = \frac{h^{1/2}}{2\pi} \sum_{p=1}^{\infty} \int_{\partial \Omega} f_p(x, \lambda) d\sigma(x) + h^{1/2} o(1). \quad (7.6)$$

In light of Lemma 3.6, the sum on the right hand side of (7.6) is actually a sum of a finite number of terms. Thus $\sum_{p=1}^{\infty}$ can be replaced by $\sum_{p=1}^{p_0}$. Implementing (7.6) into (7.5), then taking $\limsup_{h \rightarrow 0^+}$, we get

$$\limsup_{h \rightarrow 0^+} h^{1/2} N(\lambda; h, \gamma, \alpha) \leq \frac{1}{2\pi} \sum_{p=1}^{p_0} \int_{\partial \Omega} f_p(x, \lambda + \varepsilon) - f_p(x, \lambda) \frac{\varepsilon}{\varepsilon} \, d\sigma(x). \quad (7.7)$$

Taking the limit $\varepsilon \rightarrow 0^+$ and using dominated convergence, we deduce that

$$\limsup_{h \rightarrow 0^+} h^{1/2} N(\lambda; h, \gamma, \alpha) \leq \frac{1}{2\pi} \sum_{p=1}^{p_0} \int_{\partial \Omega} \frac{\partial f_p}{\partial \lambda_+}(x, \lambda) d\sigma(x). \quad (7.8)$$

Replacing $\varepsilon$ by $-\varepsilon$ in (7.5) and following the same arguments that led to (7.8), we find

$$\liminf_{h \rightarrow 0^+} h^{1/2} N(\lambda; h, \gamma, \alpha) \geq \frac{1}{2\pi} \sum_{p=1}^{p_0} \int_{\partial \Omega} \frac{\partial f_p}{\partial \lambda_-}(x, \lambda) d\sigma(x). \quad (7.9)$$

By combining (7.8) and (7.9), we obtain

$$\lim_{h \rightarrow 0^+} h^{1/2} N(\lambda; h, \gamma, \alpha) = \frac{1}{2\pi} \sum_{p=1}^{p_0} \int_{\partial \Omega} \frac{\partial f_p}{\partial \lambda_+}(x, \lambda) d\sigma(x). \quad (7.10)$$
Now, in light of (7.4), we finally get that
\[
\lim_{h \to 0^+} h^{1/2} N(\lambda; h, \gamma, \alpha) = \frac{1}{2\pi} \sum_{p=1}^{\infty} \iint_{\{(x, \xi) \in \partial \Omega \times \mathbb{R} : B(x) \mu_p(B(x)^{-1/2} \gamma(x) \xi) < \lambda\}} B(x)^{1/2} d\xi ds(x).
\]
(7.11)
This finishes the proof of (1.13).

8. Proof of Theorem 1.3

We will apply a simple scaling argument to pass from the semi-classical to the large area limit. Let \( T \) be a positive number and \( \Omega_T = (0, T) \times (0, T) \). Define the operator,
\[
P_{\Omega_T} = -\left( \nabla - iA_0 \right)^2 \text{ in } L^2(\Omega_T).
\]
Functions in the domain of \( P_{\Omega_T} \) satisfy Neumann condition \( \nu \cdot (\nabla - iA_0)u = 0 \) on the smooth parts of the boundary of \( \Omega_T \). We assume that the vector field \( A_0 \) is given by
\[
A_0(x_1, x_2) = (-x_2, 0), \quad (x_1, x_2) \in \mathbb{R}^2.
\]
(8.1)
The operator \( P_{\Omega_T} \) has compact resolvent and its spectrum consists of an increasing sequence of eigenvalues \( (\mu_j)_{j \geq 1} \) converging to \( \infty \). Note that the terms of the sequence \( (\mu_j) \) are listed counting multiplicities. Given \( \lambda \geq 0 \), the number of eigenvalues below \( 1 + \lambda \) is finite. Denote by
\[
N(\lambda, T) = \text{Card} \{ j : \mu_j \leq 1 + \lambda \}.
\]
(8.2)
By a scaling argument, Theorem 1.3 follows from:

**Theorem 8.1.** There exists a positive number \( \delta \) such that,
\[
\limsup_{T \to \infty} \frac{N(\lambda, T)}{T^2} = \frac{1}{2\pi}, \quad (\lambda \in [0, \delta]).
\]

8.1. Preliminaries.

8.1.1. Variational min-max principle. We shall need the following version of the variational min-max principle.

**Theorem 8.2.** Let \( A \) be a self-adjoint operator in a Hilbert space \( H \). Suppose that \( A \) is semi-bounded (i.e. bounded from below) and has compact resolvent. The terms of the sequence of eigenvalues of \( A \) counting multiplicities are given by,
\[
\mu_n = \inf \left\{ \max_{\phi \in M} \langle A\phi, \phi \rangle_H : M \subset D(A), \dim M = n \right\}.
\]

8.1.2. Rough bound for the operator \( P_{\Omega_T} \). Let \( S \) and \( T \) be positive numbers, and \( \Omega_{S,T} = (0, S) \times (0, T) \). Consider the operator,
\[
P_{\Omega_{S,T}} = -\left( \nabla - iA_0 \right)^2 \text{ in } L^2(\Omega_{S,T}).
\]
a function \( u(x_1, x_2) \) in the domain of \( P_{\Omega_{S,T}} \) satisfies Neumann condition at \( x_2 = 0 \), Dirichlet condition at \( x_2 = T \), and periodic conditions at \( x_1 \in \{0, S\} \). Define,
\[
N(\lambda; S, T) = \text{tr} \left( 1_{(-\infty, (1+\lambda)h]}(P_{h,h,\Omega_{S,T}}) \right).
\]
Along the proof of Lemma 3.1 in [9], a useful rough bound on \( N(\lambda; S, T) \) is given. We recall this bound below.

**Lemma 8.3.** There exist positive constants \( C, T_0 \) and \( \lambda_0 \) such that, for all \( T \geq T_0 \), \( \lambda \in [0, \lambda_0] \) and \( S > 0 \), we have,
\[
N(\lambda; S, T) \leq CST.
\]
(8.3)
8.1.3. The Dirichlet operator in a square. Recall the magnetic potential $A_0$ in (8.1). Consider a positive real number $R$ and the operator $P^D_{\Omega_R} = -\left(\nabla - iA_0\right)^2$ in the square $\Omega_R = (0, R) \times (0, R)$ and with Dirichlet boundary conditions. If $A \in \mathbb{R}$, we define the functions,

$$\nu_0(A) = \frac{1}{2\pi} \text{Card } \{n \in \mathbb{N} : 2n - 1 \leq A\}. \quad (8.4)$$

and

$$N(A, P^D_{\Omega_R}) = \text{tr}(1_{(-\infty, A]}(P^D_{\Omega_R})). \quad (8.5)$$

The next two-sided estimate on the eigenvalue counting function of the operator $P^D_{\Omega_R}$ is proved in [2] Thm. 3.1.

**Lemma 8.4.** There exists a constant $C > 0$ such that, for all $A \in \mathbb{R}$, $R > 0$ and $A \in (0, R/2)$, the following two-sided estimate holds true,

$$(R - A)^2 \nu_0\left(A - \frac{C}{A^2}\right) \leq N(A, P^D_{\Omega_R}) \leq R^2 \nu_0(A).$$

In particular, if $A < 3$, then

$$N(A, P^D_{\Omega_R}) \leq \frac{R^2}{2\pi}. \quad (8.6)$$

8.1.4. The periodic operator. Consider a positive number $R$, the square $\Omega_R = (0, R) \times (0, R)$ and the function space,

$$E_R = \{u \in H^1_{\text{loc}}(\mathbb{R}^2) : u(x_1 + R, x_2) = u(x_1, x_2) \& u(x_1, x_2 + R) = e^{-iRx_1}u(x_1, x_2)\}. \quad (8.7)$$

Recall the magnetic potential $A_0$ in (8.1). If $u \in E_R$, then $|u|$ and $|(\nabla - iA_0)u|$ are periodic with respect to the lattice generated by $\Omega_R$. Consider the self-adjoint operator

$$P^\text{per}_{\Omega_R} = -\left(\nabla - iA_0\right)^2 \text{ in } L^2(\Omega_R),$$

whose domain is that defined by the Friedrichs' extension associated with the quadratic form,

$$E_R \ni f \mapsto \int_{\Omega_R} |(\nabla - iA_0)u|^2 \, dx.$$  

Denote by $(\mu_j)$ the sequence of distinct eigenvalues of the operator $P^\text{per}_{\Omega_R}$. Let us recall the following classical results (see [7, Proposition 2.9]). These results are valid under the assumption that $R^2/(2\pi)$ is a positive integer.

- The first eigenvalue of $P^\text{per}_{\Omega_R}$ is $\mu_1(P^\text{per}_{\Omega_R}) = 1$ and the second eigenvalue $\mu_2(P^\text{per}_{\Omega_R}) \geq 3$.
- The dimension of the eigenspace $\text{Ker}(P^\text{per}_{\Omega_R} - \text{Id})$ is $R^2/(2\pi)$.

As a consequence, we may state the following lemma.

**Lemma 8.5.** Suppose that $R^2 \in 2\pi\mathbb{N}$. If $0 \leq \lambda < 2$ and $N_{\text{per}}(\lambda, R) = \text{tr}(1_{(-\infty, 1+\lambda]}(P_{\Omega_R}))$, then,

$$N_{\text{per}}(\lambda, R) = \frac{R^2}{2\pi}. \quad (8.8)$$

8.1.5. The operator in a sector. Recall the magnetic potential $A_0$ in (8.1). Consider the operator,

$$P_{\Omega_{R, \pi/2}} = -\left(\nabla - iA_0\right)^2 \text{ in } L^2(\Omega_{R, \pi/2}), \quad (8.7)$$

where $\Omega_{R, \pi/2} = \{(r \cos \theta, r \sin \theta) : 0 \leq r < R, \ 0 < \theta < \pi/2\}$. Functions in the domain of $P_{\Omega_{R, \pi/2}}$ satisfy Neumann condition on $\theta = 0$ and $\theta = \pi/2$, and Dirichlet condition on $r = R$.

The operator $P_{\Omega_{R, \pi/2}}$ has compact resolvent and its spectrum consists of an increasing sequence of eigenvalues $(\zeta_j)$ counting multiplicities. We introduce,

$$N_{\text{sec}}(\lambda, R) = \text{Card} \{j : \zeta_j \leq 1 + \lambda\}. \quad (8.8)$$

A useful rough bound on $N_{\text{sec}}(\lambda, R)$ is proved in [23]. We recall this bound in the next lemma.
Lemma 8.6. There exist positive constants \( C, R_0 \) and \( \lambda_1 \) such that, for all \( R \geq R_0 \) and \( \lambda \in [0, \lambda_1] \), we have,

\[
\mathcal{N}_{\text{sec}}(\lambda, R) \leq C(R^2 + 1).
\]

8.2. Proof of Theorem [8.1]. Through this section, the following convention will be used. If \( P \) is a self-adjoint operator and \( \Lambda < \inf \sigma_{\text{ess}}(P) \), denote by

\[
N(\Lambda, P) = \text{tr}(1_{(-\infty, \Lambda]}(P)).
\]

Recall the operator \( P_{\Omega_R} \) and the number \( \mathcal{N}(\lambda, T) = N(1 + \lambda, P_{\Omega_R}) \) introduced in (8.2).

We start by the observation:

Lemma 8.7. Let \( T_n = \sqrt{2\pi n}, n \in \mathbb{N} \). For all \( \lambda \in [0, 2) \), there holds,

\[
\frac{N(1 + \lambda, T_n)}{T_n^2} \geq \frac{1}{2\pi}.
\]

Proof. Recall the operator \( P_{\Omega_R}^{\text{per}} \) introduced in Sec. [8.1.4] together with the number \( N_{\text{per}}(\lambda, R) \) in Lemma [8.3]. Notice that functions in the form domain of \( P_{\Omega_R}^{\text{per}} \) are in \( H^1(\Omega_R) \) and consequently in the form domain of \( P_{\Omega_R} \). The variational min-max principle (Theorem [8.2]) then tells us that the eigenvalues of \( P_{\Omega_R}^{\text{per}} \) are larger than the corresponding ones of \( P_{\Omega_R} \). Consequently (we use \( R = T_n \)),

\[
\mathcal{N}(\lambda, T_n) \geq N_{\text{per}}(\lambda, T_n).
\]

Notice that \( T_n^2 \in 2\pi\mathbb{N} \). Consequently, when \( \lambda \in [0, 2) \), it results from Lemma [8.3] that

\[
N_{\text{per}}(\lambda, T_n) = \frac{T_n^2}{2\pi}.
\]

This proves Lemma [8.4]. \( \square \)

Lemma 8.8. There exist positive constants \( \delta \in (0, 1), T_1 \) and \( C \) such that, for all \( \lambda \in [0, \delta] \) and \( T \geq T_1 \), there holds,

\[
\mathcal{N}(\lambda, T) \leq \frac{T^2}{2\pi} + CT.
\]

Proof. Consider a number \( L \in (0, T) \). We cover the square \( \Omega_T = (0, T) \times (0, T) \) by sets \( U, V_j \) and \( U_j \), \( j \in \{1, 2, 3, 4\} \) defined as follows:

\[
U = \left( \frac{L}{2}, T - \frac{L}{2} \right) \times \left( \frac{L}{2}, T - \frac{L}{2} \right),
\]

\[
U_1 = \left( \frac{L}{2}, T - \frac{L}{2} \right) \times [0, L), \quad U_2 = [0, L) \times \left( \frac{L}{2}, T - \frac{L}{2} \right),
\]

\[
U_3 = \left( \frac{L}{2}, T - \frac{L}{2} \right) \times (T - L, T], \quad U_4 = (T - L, T] \times \left( \frac{L}{2}, T - \frac{L}{2} \right),
\]

\[
V_1 = [0, L) \times [0, L), \quad V_2 = (T - L, T] \times [0, L), \quad V_3 = [0, L) \times (T - L, T], \quad V_4 = (T - L, T] \times (T - L, T).
\]

Let \( P_{V_j} \) and \( P_{U_j} \) be self-adjoint realizations of the operator \( -(\nabla - iA_0)^2 \) in \( L^2(V_j) \) and \( L^2(U_j) \) respectively and defined as follows. For every \( j \) and \( \Omega \in \{V_j, U_j\} \), functions in the domain of \( P_\Omega \) satisfy Neumann condition on the common smooth boundary of \( \Omega \) and \( \Omega_T \) and Dirichlet condition elsewhere.

Notice that the operators \( P_{V_j}, j \in \{1, 2, 3, 4\} \), are unitary equivalent and have the same spectra. Also, it results from the variational min-max principle that the spectrum of \( P_{V_j} \) is below that of the operator \( P_{\Omega_{2L,n/2}} \) introduced in Sec. [8.1.5] thereby obtaining,

\[
\mathcal{N}(\lambda, P_{V_j}) \leq \mathcal{N}_{\text{sec}}(\lambda, 2R).
\]
The operators \( P_{U_j}, j \in \{1, 2, 3, 4\} \), are unitary equivalent also and (recall the operator \( P_{\Omega S,T} \) introduced in Sec. 8.1.2),

\[
\sigma(P_{U_j}) = \sigma(P_{\Omega_T-L,L}) , \quad (j \in \{1, 2, 3, 4\}).
\]

Consider a partition of unity

\[
\sum_{j=1}^{4} \chi_j^2 + \sum_{j=1}^{4} \varphi_j^2 + f^2 = 1 \quad \text{in } \Omega_T,
\]

such that

\[
\sum_{j=1}^{4} (|\nabla \chi_j|^2 + |\nabla \varphi_j|^2) + |\nabla f|^2 \leq \frac{C}{L^2},
\]

\[
\text{supp } \chi_j \subset V_j, \quad \text{supp } f_j \subset U_j, \quad \text{supp } f \subset U,
\]

and \( C \) is a universal constant.

Using the IMS decomposition formula, we may write for any function \( u \) in the form domain of \( P_{\Omega_T} \),

\[
q(u) = q(fu) + \sum_{j=1}^{4} q(\chi_j u) + \sum_{j=1}^{4} q(\varphi_j u) - \int_{\Omega} \left( |\nabla f|^2 + \sum_{j=1}^{4} (|\nabla \chi_j|^2 + |\nabla \varphi_j|^2) \right) |u|^2 \, dx
\]

\[
\geq q(fu) + \sum_{j=1}^{4} q(\chi_j u) + \sum_{j=1}^{4} q(\varphi_j u) - \frac{C}{L^2} \int_{\Omega} |u|^2 \, dx,
\]

where the quadratic form \( q \) is defined by,

\[
q(v) = \int_{\Omega} |(\nabla - iA_0)v|^2 \, dx.
\]

As has been proven in [2], it results from the variational min-max principle (Theorem 8.2):

\[
N(\lambda, P_{\Omega_T}) \leq N(1 + \lambda + \frac{C}{L^2}, P_{\Omega_T-L,L}) + \sum_{j=1}^{4} N(1 + \lambda + \frac{C}{L^2}, P_{V_j}) + \sum_{j=1}^{4} N(1 + \lambda + \frac{C}{L^2}, P_{U_j}). \quad (8.9)
\]

Recall that the operator \( P_{\Omega_T}^{P} \) (with \( R = T - L \)) has been introduced in Sec. 8.1.3. Let \( \lambda_2 = \frac{1}{2} \min(\lambda_0, \lambda_1, 1) \) where \( \lambda_0 \) and \( \lambda_1 \) are as introduced in Lemmas 8.3 and 8.6. Select \( L \) such that,

\[
L \geq T_0 \quad \text{and } \lambda + \frac{C}{L^2} < \min(\lambda_0, \lambda_1, 1), \quad (\lambda \in [0, \lambda_2]),
\]

where \( T_0 \) is as in Lemma 8.3.

Consequently, it follows from Lemma 8.4 that,

\[
N(1 + \lambda + \frac{C}{L^2}, P_{\Omega_T-L,L}^{P}) \leq \frac{(T - L)^2}{2\pi}.
\]

Also, as pointed earlier and using Lemmas 8.3 and 8.6 we get for \( \lambda \in [0, \lambda_2] \) and sufficiently large \( T \),

\[
N(1 + \lambda + \frac{C}{L^2}, P_{V_j}) \leq N_{\text{acc}}(\lambda + \frac{C}{L^2}, 2L) \leq C(4L^2 + 1),
\]

\[
N(1 + \lambda + \frac{C}{L^2}, P_{U_j}) \leq N(\lambda; T - L, L) \leq C(T - L)L.
\]

By substituting the above upper bounds into (8.9), we get the upper bound in Lemma 8.8. □
Proof of Theorem S.1 Let \( \lambda \in [0, \delta] \) with \( \delta \) as in Lemma S.8. In light of Lemma S.7, we get
\[
\limsup_{T \to \infty} \frac{\mathcal{N}(\lambda, T)}{T^2} \geq \frac{1}{2\pi}.
\]
On the other hand, Lemma S.8 tells us that,
\[
\limsup_{T \to \infty} \frac{\mathcal{N}(\lambda, T)}{T^2} \leq \frac{1}{2\pi}.
\]
\( \square \)

Appendix A. The quadratic form in (1.4) is semi-bounded

By density of smooth functions in \( H^1 \) and compactness of the boundary \( \partial \Omega \), the semi-boundedness of (1.4) follows from:

Lemma A.1. Let \( \gamma \in L^3(\partial \Omega) \) and \( x_0 \in \partial \Omega \). There exist constants \( C_0 > 0 \) and \( r_0 > 0 \) such that, for all \( u \in C_0^\infty(B(x_0, r_0) \cap \Omega) \),
\[
\| \nabla u \|^2_{L^2(B(x_0, r_0))} + \int_{B(x_0, r_0) \cap \partial \Omega} \gamma(x)|u(x)|^2 \, ds(x) \geq -C_0\|u\|^2_{L^2(B(x_0, r_0))}.
\]

Proof. Select \( r_0 \) sufficiently small such that the coordinate transformation in (2.11) is defined. Using these coordinates, we may view the function \( u \) as a function in \( C_0^\infty(\mathbb{R}_+^2) \). In the same way, we may view the function \( \gamma \) in \( L^3(\mathbb{R}) \). We have,
\[
\| \nabla u \|^2_{L^2(B(x_0, r_0))} + \int_{B(x_0, r_0) \cap \partial \Omega} \gamma(x)|u(x)|^2 \, ds(x)
\geq C \int_{\mathbb{R}_+} \int_{\mathbb{R}} (|\partial_t u|^2 + |\partial_s u|^2) \, ds \, dt + \int_{\mathbb{R}} \gamma(s)|u(s, 0)|^2 \, ds,
\]
where \( C > 0 \) is a constant.

We have the simple identity,
\[
\gamma(s)|u(s, 0)|^2 = -2 \int_0^\infty \gamma(s) u(s, t) \partial_t u(s, t) \, dt.
\]
By the Cauchy-Schwarz inequality, we get for all \( \epsilon > 0 \),
\[
\gamma(s)|u(s, 0)|^2 \geq -\epsilon \int_0^\infty |\partial_t u(s, t)|^2 \, dt - 4\epsilon^{-1} \int_0^\infty |\gamma(s) u(s, t)|^2 \, dt.
\]
We integrate both sides with respect to \( s \) to obtain,
\[
\int_{\mathbb{R}} |\gamma(s)|u(s, 0)|^2 \, ds \geq -\epsilon \int_{\mathbb{R}_+} \int_{\mathbb{R}} |\partial_t u(s, t)|^2 \, ds \, dt - 4\epsilon^{-1} \int_{\mathbb{R}_+} \int_{\mathbb{R}} |\gamma(s) u(s, t)|^2 \, ds \, dt.
\]
Since \( \gamma \in L^3(\mathbb{R}) \), then the operator
\[
v \mapsto \gamma v
\]
is \( \mathcal{D} \)-compact. Thus, for all \( a \in (0, 1) \), there exists a constant \( b > 0 \) such that,
\[
\int_{\mathbb{R}} |\gamma(s) u(s, t)|^2 \, ds \, dt \leq a \int_{\mathbb{R}} |\partial_s u(s, t)|^2 \, ds \, dt + b \int_{\mathbb{R}} |u(s, t)|^2 \, ds \, dt.
\]
Inserting this into (A.2) and then inserting the resulting inequality into (A.1), we get,
\[
\| \nabla u \|^2_{L^2(B(x_0, r_0))} + \int_{B(x_0, r_0) \cap \partial \Omega} \gamma(x)|u(x)|^2 \, ds(x)
\geq \int_{\mathbb{R}_+} \int_{\mathbb{R}} \left( (C-\epsilon)|\partial_t u|^2 + (C - 4\epsilon^{-1} a)|\partial_s u|^2 - 4\epsilon^{-1} b|u|^2 \right) \, ds \, dt.
\]
We select $\epsilon = C$ and $a$ sufficiently small such that $C - 4\epsilon^{-1}a > 0$. Returning to cartesian coordinates, we get the estimate in Lemma A.1.

REFERENCES

[1] V. Bonnaillie. Analyse mathématique de la supraconductivité dans un domaine à coins: méthodes semi-classiques et numériques. Thèse de doctorat, Université Paris 11 (2003).
[2] Y. Colin de Verdière. L’asymptotique de Weyl pour les bouteilles magnétique. Comm. Math. Phys. 105 (1986), 327–335 (French).
[3] H.D. Cornean, S. Fournais, R.L. Frank, B. Helffer. Sharp trace asymptotics for a class of 2D-magnetic operators. Ann. Inst. Fourier, to appear.
[4] M. Coffeng, A. Kachmar, M. Persson-Sundqvist. Clusters of eigenvalues of the magnetic Laplacian with Robin condition. Preprint.
[5] M. Dauge, B. Helffer. Eigenvalues variation I, Neumann problem for Sturm-Liouville operators. Journal of differential Equations, 104 (1993), no. 2, 243–262.
[6] P.G. deGennes. Superconductivity of Metals and Alloys. Benjamin (1966).
[7] L. Erdős, J.P. Solovej. Semiclassical eigenvalue estimates for the Pauli operator with strong non-homogeneous magnetic fields. II. Leading order asymptotic estimates. Comm. Math. Phys. 188 (1997), 599-656.
[8] S. Fournais, B. Helffer. Spectral methods in surface superconductivity. Progress in Nonlinear Differential Equations and Their Applications, Vol. 77. Birkhäuser Boston (2010).
[9] S. Fournais, A. Kachmar. On the energy of bound states for magnetic Schrödinger operators. J. Lond. Math. Soc. 80 (2009), no. 1, 233–255.
[10] R.L. Frank. On the asymptotic number of edge states for magnetic Schrödinger operators. Pro. London Math. Soc. (3) 95 (2007), no. 1, 1–19.
[11] R.L. Frank, A. Laptev. Spectral inequalities for Schrödinger operators with surface potentials. Spectral theory of differential operators, T. Suslina and D. Yafaev (eds.), 91 - 102, Amer. Math. Soc. Transl. Ser. 2, 225 (2008).
[12] R.L. Frank, L. Geisinger. Semi-classical analysis of the Laplace operator with Robin boundary condition. Bull. Math. Sci. 2 (2012), no. 2, 281–319.
[13] T. Giorgi, R. Smits. Eigenvalue estimates and critical temperature in zero fields for enhanced surface superconductivity. Z. Angew. Math. Phys. 57 (2006), 1–22.
[14] B. Helffer, A. Morame. Magnetic bottles in connection with superconductivity. J. Func. Anal. 181 (2001), no. 2, 604-680.
[15] B. Helffer, K. Pankrashkin. Tunneling between corners for Robin Laplacians. Preprint.
[16] D. Hundertmark, A. Laptev and T. Weidl. New bounds on the Lieb-Thirring constants. Invent. Math., 40 (2000), 693–704.
[17] A. Kachmar, M. Persson. On the essential spectrum of magnetic Schrödinger operators in exterior domains. Arab. J. Math. Sci. 19 (2013), no. 2, 217–222.
[18] A. Kachmar, On the ground state energy for a magnetic Schrödinger operator and the effect of the de Gennes boundary conditions, C. R. Math. Acad. Sci. Paris 332 (2006), 701–706.
[19] A. Kachmar, On the ground state energy for a magnetic Schrödinger operator and the effect of the de Gennes boundary conditions, J. Math. Phys. 47 (7) (2006) 072106, 32 pp.
[20] A. Kachmar, On the stability of normal states for a generalized Ginzburg-Landau model, Asymptot. Anal. 54 (2007), no. 3–4, 145–201.
[21] A. Kachmar, On the perfect superconducting solution for a generalized Ginzburg-Landau equation, Asymptot. Anal. 54 (2007), no. 3–4, 125–164.
[22] A. Kachmar, Problèmes aux limites issues de la supraconductivité, Ph. D. Thesis, University Paris-Sud/Orsay (2007).
[23] A. Kachmar, Weyl asymptotics for magnetic Schrödinger operator and de Gennes’ boundary condition. Reviews in Mathematical Physics Vol 20 (2008), no. 8, 901–932.
[24] A. Kachmar, A. Khochman. Spectral asymptotics for magnetic Schrödinger operators in domains with corners. J. Spectr. Theory, 3 (2013), 553–574.
[25] M. Levitin, L. Parnovski. On the principal eigenvalue of a Robin problem with a large parameter. Math. Nachr. 281 (2008), 272–281.
[26] E. H. Lieb, M. Loss. Analysis. Second edition. Graduate Studies in Mathematics 14, American Mathematical Society, Providence, RI, 2001.
[27] A. Laptev, T. Weidl, Sharp Lieb-Thirring inequalities in high dimensions. Acta Math. 184 (2000), no. 1, 87–111.
[28] E.H. Lieb, J.P. Solovej, J. Yngvason. Asymptotics of heavy atoms in high magnetic fields. II. Semiclassical regions. Comm. Math. Phys. 161 (1994) (1) 77–124.
[29] M. Nasrallah. Energy of surface states for 3D magnetic Schrödinger operators. Preprint.
[30] K. Pankrashkin. On the asymptotics of the principal eigenvalue problem for a Robin problem with a large parameter in a planar domain. Nanosystems: Physics, Chemistry, Mathematics, 2013 4 (4), 474–483.
[31] K. Pankrashkin, N. Popoff. Mean curvature bounds and eigenvalues of Robin Laplacians. Preprint.
[32] A. Persson. Bounds for the discrete part of the spectrum of a semi-bounded Schrödinger operator. Math. Scand. 8 (1960), 143–153.
[33] A. Sobolev. On the Lieb-Thirring estimates for the Pauli operator. Duke J. Math. Vol. 82 (1996), no. 3, 607–635.
[34] F. Truc. Semiclassical asymptotics for magnetic bottles. Asymptot. Anal. 15 (1997), no. 3-4, 385–395.

(A. Kachmar) Lebanese University, Department of Mathematics, Hadath, Lebanon
E-mail address: ayman.kashmar@liu.edu.lb

(M. Nasrallah) Lebanese International University, School of Arts and Sciences, Rayak, Lebanon
E-mail address: marwa.nasrallah@liu.edu.lb