Enhanced Black Hole Horizon Fluctuations

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Abstract

We discuss the possible role of quantum horizon fluctuations on black hole radiance, especially whether they can invalidate Hawking’s analysis based upon transplanckian modes. We are particularly concerned with “enhanced” fluctuations produced by gravitons or matter fields in squeezed vacuum states sent into the black hole after the collapse process. This allows for the possibility of increasing the fluctuations well above the vacuum level. We find that these enhanced fluctuations could significantly alter stimulated emission, but have little effect upon the spontaneous emission. Thus the thermal character of the Hawking radiation is remarkably robust.

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I. INTRODUCTION

Hawking’s discovery [1] of black hole radiance has forged an elegant link between relativity, quantum theory, and thermodynamics. However, some unsolved problems remain, including the information and transplanckian issues. The question of whether information is lost in the black hole evaporation process has been vigorously debated by many authors. (See, for example Ref. [2] and references therein.) The transplanckian issue arises because Hawking’s original derivation used quantum field theory on a fixed background spacetime and requires incoming vacuum modes with frequencies far above the Planck scale. Alternative derivations have been proposed, especially by Unruh [3, 4] and by Jacobson and coworkers [5, 6, 7, 8, 9] which involve cutoffs and a non-linear dispersion relation. The non-linearity can lead to the phenomenon of “mode regeneration”, whereby the modes needed for the outgoing particles are created just before they are needed, rather than being red-shifted transplanckian modes. However, this approach requires a breakdown of local Lorentz invariance and hence new, as yet unobserved, physics.

In this paper, we wish to consider the effects of quantum horizon fluctuations on the Hawking process. This is a topic which has been discussed by several authors from various viewpoints. An early discussion was given by York [10], who used a semiclassical approach to estimate the magnitude of the quantum metric fluctuations near the horizon. Ford and Svaiter [11] used York’s estimate to treat fluctuations of the outgoing rays, and concluded that Hawking’s derivation does not seem to be altered by vacuum fluctuations of linearized quantum gravity. Parentani [12] and Barabes et al. [13] have discussed the possibility that quantum fluctuations could be the source of the non-linearity needed for the mode regeneration picture.

Most of the papers cited in the previous paragraph deal with active fluctuations, the spacetime geometry fluctuations arising from quantization of the dynamical degrees of freedom of gravity itself. Another source of spacetime fluctuations are the quantum fluctuations of matter field stress tensors, which cause passive fluctuations. There has been an extensive discussion of both types of fluctuations in recent years in various contexts, including both black hole spacetimes [10, 11, 12, 13, 14, 15, 16, 17, 18, 19] and more general spacetime geometry fluctuations [20, 21, 22, 23, 24, 25].

In the present paper, we will examine some examples of both active and passive fluctuations. However, our main interest will be in the possibility of enhancing the geometry fluctuations above the vacuum level by use of gravitons or matter fields in squeezed vacuum states. We consider a Schwarzschild black hole formed by gravitational collapse, and then suppose that wavepackets of gravitons or matter fields in squeezed states are sent across the horizon. This will cause greater geometry fluctuations than would occur in the vacuum states of these fields. The key question which we wish to address is whether these fluctuations significantly alter the outgoing modes which will carry the thermal radiation to distant observers. The technique which we employ to study the effects of geometry fluctuations is based upon the geodesic deviation equation. This allows a gauge invariant treatment using the Riemann tensor correlation function.

In Sect. [II] we review Hawking’s derivation of black hole evaporation, and discuss the transplanckian issue. In Sect. [III] we develop some of the formalism of fluctuations of a geodesic separation vector which will be used in subsequent sections. We next turn to the case of active fluctuations. Before considering gravitons, we first investigate a simplified model of “scalar gravitons” in Sect. [IV]. This model reproduced the essential physics of the effects of gravitons, but with reduced complexity. The case of gravitons, quantized linear perturbations of the Schwarzschild geometry, is treated in Sect. [V]. We next turn to passive fluctuation effects in Sect. [VI], where stress tensor fluctuations of a scalar field are treated. We give a unified analysis of the results of all three models in Sect. [VII] and offer our
FIG. 1: The spacetime of a black hole formed by gravitational collapse is illustrated. The shaded region is the interior of the collapsing body, the $r = 0$ line on the left is the worldline of the center of this body, the $r = 0$ line at the top of the diagram is the curvature singularity, and $\mathcal{H}^+$ is the future event horizon. An ingoing light ray with $v < v_0$ from $\mathcal{I}^-$ passes through the body and escapes to $\mathcal{I}^+$ as a $u = \text{constant}$ light ray. Ingoing rays with $v > v_0$ do not escape and eventually reach the singularity.

conclusions in Sect. VIII.

II. DERIVATION OF THE HAWKING EFFECT

In this section, we will briefly review Hawking’s derivation [1] of black hole evaporation. The basic idea is to consider the spacetime of a black hole formed by gravitational collapse, as illustrated in Figure II. Here $v = t + r_*$, and $u = t - r_*$ are respectively the ingoing and outgoing Eddington-Finkelstein coordinates, also referred to as the advanced and retarded times, and $r_* = r + 2M \ln(\frac{r}{2M} - 1)$ is the tortoise coordinate. A quantum field propagating in this spacetime is assumed to be in the in-vacuum state, that is, containing no particles before the collapse. In the case of a massless field, a purely positive frequency mode proportional to $e^{-i\omega v}$ leaves $\mathcal{I}^-$, propagates through the collapsing body, and reaches $\mathcal{I}^+$ after undergoing a large redshift in the region outside of the collapsing matter. At $\mathcal{I}^+$, the mode is now a mixture of positive and negative frequency parts, signalling quantum particle creation. Of special interest are the modes which leave $\mathcal{I}^-$ just before the formation of the horizon, which is the $v = v_0$ ray. These modes give the dominant contribution to the outgoing flux at times long after the black hole has formed. After passing through the collapsing body, they are $u = \text{constant}$ rays, where

$$u = -4M \ln \left( \frac{v_0 - v}{C} \right),$$

where $M$ is the black hole’s mass, and $C$ is a constant. The logarithmic dependence leads to a Planckian spectrum of created particles. It also leads to the “transplanckian issue”,

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the enormous frequency which the dominant modes must have when they leave \( \mathcal{I}^- \). The typical frequency of the radiated particles reaching \( \mathcal{I}^+ \) midway through the evaporation process is of order \( 1/M \), but the typical frequency of these modes at \( \mathcal{I}^- \) is of order

\[
\omega \approx M^{-1} e^{(M/M_{Pl})^2},
\]

where \( M_{Pl} \) is the Planck mass. Another way to state this is to note that the characteristic value of \( u \) for these modes is of order

\[
u_c \approx M \left( \frac{M}{m_p} \right)^2.
\]

A geodesic observer who falls from rest at large distance from the black hole will pass from \( u = u_c \) to the horizon at \( u = \infty \) in a proper time of

\[
d \tau \approx M e^{-u_c/4M} \approx M e^{-M^2/m_p^2}.
\]

which is far smaller than the Planck time. In this sense, the outgoing modes are much less than a Planck length outside the horizon.

If spacetime geometry fluctuations cause these outgoing modes either to be ejected prematurely, or to fall into the singularity, then the outgoing radiation, and possibly the thermal character of the black hole, could be greatly altered. This is the question which we wish to address in this paper.

III. FORMALISM

A. Null Kruskal Coordinates

Most of this work is done using null Kruskal coordinates, for which the line element is

\[
ds^2 = -\frac{32M^3}{r} e^{-r/2M} dU dV + r^2 d\Omega^2
\]

where the coordinates \((U, V)\) are defined by

\[
U = -e^{-u/4M} \quad \text{and} \quad V = e^{v/4M}
\]

and describe surfaces of constant phase, equivalently the path of light rays, in a Schwarzschild space-time. Kruskal coordinates are advantageous because, unlike Schwarzschild coordinates, they do not suffer a coordinate singularity at the horizon.

In null Kruskal coordinates, \( V \) is an affine parameter on \( \mathcal{I}^+ \) and is approximately an affine parameter on outgoing null geodesics very near the horizon, but only on that portion of the geodesic for which \( UV \ll 1 \), or equivalently near the \( r = 2M \) surface. However, outgoing null geodesics spend a long affine time near \( r = 2M \) before finally escaping to infinity, so this should be a good approximation for a large range of \( V \).

Furthermore, on the past horizon of an eternal black hole, \( \mathcal{I}^- \), the Kruskal coordinate \( U \) is an affine parameter for ingoing null geodesics. (See, for example, Eq. 12.5.10 in Ref. [26].) Working in Kruskal coordinates and using \( U \) and \( V \) as affine parameters near the past and future horizons, \( \ell^\mu = (0, 1, 0, 0) \) and \( s^\mu = (1, 0, 0, 0) \) are tangent to, respectively, outgoing and ingoing null geodesics near the horizon.
\[ n^\nu = n^\nu_0 + \delta s n^\mu \]

Perturbing fields fall into the black hole well after the horizon forms.

### B. Geodesic Deviation

The derivation of the Hawking effect outlined in Sect. II requires propagation of a field from \( \mathcal{I}^+ \) backwards along a geodesic, through the collapsing body, and out to \( \mathcal{I}^- \). These tracked geodesics lie very close to the horizon, and are separated from the horizon by some separation vector \( n^\mu \) as in Fig. 2. In this figure, the geodesic (labeled \( \gamma \)) appears to be a straight line at a fixed distance from the horizon so that \( n^\mu \) is constant. This is not quite true; a particle following \( \gamma \) is eventually separated from the horizon by an infinite physical distance. Tracking the evolution of the separation vector from some initial point out to \( \mathcal{I}^+ \) requires integration of the geodesic deviation equation from some initial point out to \( \mathcal{I}^+ \). One finds that the \( U \) component of \( n^\mu \) is constant while the \( V \) component is not. Hawking is actually considering only the fixed \( U \) component as it is only \( n^U \) that is relevant for his derivation of Eq. (1).

Consider \( \mathcal{H}^+ \), parametrized by \( U = 0 \) and affine parameter \( \lambda_1 = V \), and a nearby outgoing null geodesic just outside the horizon parametrized by \( U = n_0 \) and affine parameter \( \lambda_2 \approx V \) for some small \( n_0 \ll 1 \). Let \( n^\mu \) connect points of equal affine parameter on \( \mathcal{H}^+ \) and the geodesic with \( U = n_0 \). Parameterizing the geodesics such that initially \( \lambda_1 = \lambda_2 = V_0 \), the separation vector is initially \( n^\mu_0 = (n_0, 0, 0, 0) \) at some point \( V = V_0 \). In a flat space-time the separation vector would not change and \( n^\mu = n^\mu_0 \) everywhere along the geodesic. By applying the geodesic deviation equation in Kruskal coordinates one can find the subsequent evolution of the separation vector \( n^\mu = n^\mu_0 + \delta s n^\mu \), where \( \delta s n^\mu \) represents the kinematic evolution of \( n^\mu \) due to the classical background. If in addition, there is a perturbation of the background, then \( \tilde{n}^\mu = n^\mu_0 + \delta s n^\mu + \delta p n^\mu \), where \( \delta p n^\mu \) represents the dynamical response of \( \tilde{n}^\mu \) to the perturbation. The bar on \( \tilde{n}^\mu \) is used to differentiate between the background-only space-time separation vector and the background plus perturbation space-time separation vector for this discussion.
1. Background Space-Time

Consider first the unperturbed Schwarzschild space-time of Fig. 2. Let \( \ell^\mu = (0, 1, 0, 0) \) be tangent to the horizon, and let \( n^\mu = n_0^\mu + \delta_0 n^\mu \) denote the separation between the horizon and a nearby outgoing null geodesic with initial separation \( n_0^\mu \) at \( V = V_0 \) such that \( n_0^\mu \) is null with \( n^\mu \ell_\mu = 1 \). It suffices to choose \( n_0^\mu = (n_0, 0, 0, 0) \) with \( n_0 = (g_{UV})^{-1}|_{r=2M} = 1/(8M^2) \). The evolution of the vector \( n^\mu \) characterizes the geodesic deviation of these outgoing rays, and obeys the set of differential equations

\[
\frac{D^2 n^\alpha}{dV^2} = R^\alpha_{\beta\mu\nu} \ell^\beta n^\nu.
\]

Since \( \ell^\mu = (\partial/\partial V^\mu) \), the geodesic deviation is

\[
\frac{D^2 n^\alpha}{dV^2} = \frac{D^2 (\delta_0 n^\alpha)}{dV^2} = R^\alpha_{VV\nu} \ell^V n^\nu.
\]

The only non-zero component of \( R^\alpha_{VV\nu} \) is

\[
R^V_{VVU} = \frac{16M^3}{UVr^3} \left( 1 - \frac{2M}{r} \right),
\]

which near the horizon reduces to

\[
R^V_{VVU} \approx -2e^{-1}.
\]

Also near the horizon, the covariant derivative with respect to \( V \) on the left hand side of Eq. (8) reduces to an ordinary derivative, which may be seen by direct calculation. The second covariant derivative is

\[
\frac{D^2 n^\mu}{dV^2} = \ell^\mu \ell^\sigma n^\sigma,\gamma + \Gamma^\mu_{\delta\lambda} \ell^\delta n^\lambda + 2\Gamma^\mu_{\delta\lambda} \ell^\sigma n^\lambda,\gamma + \Gamma^\mu_{\gamma\beta} \Gamma^\beta_{\delta\lambda} \ell^\delta n^\lambda.
\]

Each term with a Christoffel symbol is of the form \( \Gamma^\alpha_{\nu\mu} \). A straightforward computation of the Christoffel symbols in null Kruskal coordinates reveals that \( \Gamma^\alpha_{V\mu} \rightarrow 0 \) as \( U \rightarrow 0 \) for all \( \alpha \) and \( \mu \). The only non-trivial equation is then

\[
\frac{d^2(\delta_0 n^V)}{dV^2} = -2e^{-1}n_0
\]

which may be integrated from the initial point \( V = V_0 \), resulting in

\[
n^\mu = n_0 \left( 1, -e^{-1}(V - V_0)^2 \right) \quad \text{or} \quad (n^\mu n_\mu)^2 = 4 \left( g_{UV} n_0^2 e^{-1} \right)^2 (V - V_0)^4.
\]

2. Perturbed Space-Time

We wish to let the space-time fluctuate in some way and describe what happens to the outgoing null geodesics. Consider an ensemble of Schwarzschild space-times such that the average apparent horizon is defined by \( r = 2M \) for some chosen value of \( M \). Consider an outgoing geodesic just outside the average apparent horizon. For each space-time in the
ensemble, define the separation vector \( \vec{n}^\mu \) such that in the average space-time \( \vec{n}^\mu = n^\mu \) where \( n^\mu \) is the classical solution for the Schwarzschild space-time of mass \( M \).

Let \( \bar{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} \) be the perturbed space-time, where \( g_{\mu\nu} \) is the unperturbed (Schwarzschild) background space-time and \( h_{\mu\nu} \) is the perturbation. The background metric is used to raise and lower indices. The Riemann tensor is calculated from

\[
\bar{R}^\alpha_{\beta\mu\nu} = \Gamma^\alpha_{\beta\nu,\mu} - \Gamma^\alpha_{\beta\mu,\nu} + \bar{\Gamma}^\alpha_{\beta\mu} \bar{\Gamma}^\beta_{\nu} - \bar{\Gamma}^\alpha_{\nu\sigma} \bar{\Gamma}^\sigma_{\beta\mu},
\]

where

\[
\bar{\Gamma}^\alpha_{\beta\mu} = \frac{1}{2} g^{\alpha\beta} (\bar{g}_{\beta\nu,\mu} + \bar{g}_{\beta\mu,\nu} - \bar{g}_{\mu\nu})
\]

are the connection coefficients in the perturbed space-time. These may be expanded to \( \Gamma^\alpha_{\beta\nu} = \Gamma^\alpha_{\beta\nu} + \delta \Gamma^\alpha_{\beta\nu} \); where \( \Gamma^\alpha_{\beta\nu} \) are the connection coefficients of the background, and \( \delta \Gamma^\alpha_{\beta\nu} \) is due to the perturbation. This yields

\[
\bar{R}^\alpha_{\beta\mu\nu} \approx R^\alpha_{\beta\mu\nu} + \delta R^\alpha_{\beta\mu\nu}
\]

to first order in the metric perturbation, where similarly \( R^\alpha_{\beta\mu\nu} \) and \( \delta R^\alpha_{\beta\mu\nu} \) denote the background and perturbation contributions to the Riemann tensor. Let a semicolon denote covariant differentiation with respect to the background. One may then verify that

\[
\delta R^\alpha_{\beta\mu\nu} = \delta \Gamma^\alpha_{\beta\nu,\mu} - \delta \Gamma^\alpha_{\beta\mu,\nu},
\]

where

\[
\delta \Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} (h_{\beta\mu,\nu} + h_{\beta\nu,\mu} - h_{\mu\nu}) .
\]

Let \( \ell^\mu \) be fixed as tangent to the apparent horizon in the average (or background) space-time. Let \( \vec{n}^\mu = n_0^\mu + \delta_s n^\mu + \delta_p n^\mu \) denote the separation vector, where \( n_0^\mu \) is the same initial separation as before, \( \delta_s n^\mu \) is defined to satisfy the background equation as above, and \( \delta_p n^\mu \) encodes the dynamical response of \( \vec{n}^\mu \) to the perturbation. Letting \( \bar{D} \) be the covariant derivative with respect to the perturbed space-time, the geodesic deviation is

\[
\frac{\bar{D}^2 \vec{n}^\alpha}{dV^2} = \frac{\bar{D}^2 (\delta_s n^\alpha)}{dV^2} + \frac{\bar{D}^2 (\delta_p n^\alpha)}{dV^2} = \bar{R}^\alpha_{\mu\beta\nu} \ell^\mu \ell^\beta \vec{n}^\nu
\]

\[
= (R^\alpha_{\mu\beta\nu} + \delta R^\alpha_{\mu\beta\nu}) \ell^\mu \ell^\beta (n_0^\nu + \delta_s n^\nu + \delta_p n^\nu) .
\]

Consider the left hand side of this equation. The background terms involving \( \Gamma^\alpha_{\nu\mu} \) still vanish near the horizon. There are, however, terms involving \( \delta \Gamma^\alpha_{\nu\mu} \) which do not vanish on the horizon and which enter to first order in the metric perturbation. In particular, the first term on the left hand side is

\[
\frac{\bar{D}^2 (\delta_s n^\mu)}{dV^2} = \ell^\gamma \ell^\sigma (\delta_s n^\mu),_{\sigma\gamma} + 2 \partial \Gamma^\mu_{\delta\lambda,\gamma} \ell^\ell (\delta_s n^\lambda),_\gamma + \delta \Gamma^\mu_{\gamma\beta} \partial \Gamma^\beta_{\delta\lambda} \ell^\ell (\delta_s n^\lambda),_\gamma .
\]

which contributes two terms to first order in the metric perturbation. Similarly for the second term on the left hand side:

\[
\frac{\bar{D}^2 (\delta_p n^\mu)}{dV^2} = \ell^\gamma \ell^\sigma (\delta_p n^\mu),_{\sigma\gamma} + 2 \partial \Gamma^\mu_{\delta\lambda,\gamma} \ell^\ell (\delta_p n^\lambda),_\gamma + \delta \Gamma^\mu_{\gamma\beta} \partial \Gamma^\beta_{\delta\lambda} \ell^\ell (\delta_p n^\lambda),_\gamma .
\]
However, $\delta_p n^\lambda$ is already first order in the metric perturbation, so terms involving both $\delta_p n^\lambda$ and $\delta^{\alpha}_{\mu\nu}$ are second order. Therefore to first order in the metric perturbation, the left hand side of Eq. (19) is

$$\frac{\bar{D}^2 n^\mu}{dV^2} = \frac{d^2(\delta_s n)^\mu}{dV^2} + \frac{d^2(\delta_p n^\mu)}{dV^2} + \delta \Gamma^\mu_{\delta \lambda, \gamma} \ell^\gamma (\delta_s n^\lambda) + 2 \delta \Gamma^\mu_{\delta \lambda} \ell^\gamma (\delta_s n^\lambda), \gamma, \quad (22)$$

and the right hand side of Eq. (19) is

$$R^\alpha_{\mu \beta \nu} \ell^\beta (n_0^\nu + \delta_s n^\nu) + R^\alpha_{\mu \beta \nu} \ell^\beta (\delta_p n^\nu) + \delta R^\alpha_{\mu \beta \nu} \ell^\beta (n_0^\nu + \delta_s n^\nu). \quad (23)$$

The first term is just the result obtained for the background. Since $\delta_s n^\mu$ is by definition the solution to

$$\frac{d^2(\delta_s n)^\mu}{dV^2} = R^\alpha_{\mu \beta \nu} \ell^\beta (n_0^\nu + \delta_s n^\nu), \quad (24)$$

these terms cancel. Furthermore, since $\delta_s n^\mu$ has only a $V$ component and $\delta R^\alpha_{V V V} = 0$ by the antisymmetry of the Riemann tensor on the last two indices, then $\delta R^\alpha_{\mu \beta \nu} \ell^\beta (\delta_s n^\nu) = 0$ and Eq. (19) becomes

$$\frac{d^2(\delta_p n^\nu)}{dV^2} = R^\alpha_{\mu \beta \nu} \ell^\beta (\delta_p n^\nu) + \delta R^\alpha_{\mu \beta \nu} \ell^\beta (n_0^\nu + \delta_s n^\nu) - \delta \Gamma^\alpha_{\delta \lambda, \gamma} \ell^\gamma (\delta_s n^\lambda) - 2 \delta \Gamma^\alpha_{\delta \lambda} \ell^\gamma (\delta_s n^\lambda), \gamma. \quad (25)$$

The solution to this equation involves integrating twice over $V$ from some initial point $V_0$. It has already been found in Eq. (13) that $\delta_s n^\lambda \propto (V - V_0)^2$. We suppose that $\delta_p n^\lambda$ may also be expanded in powers of $(V - V_0)$ and then approach the solution to Eq. (25) iteratively. Suppose $\delta_p n^\lambda = \delta_{p,1} n^\lambda + \delta_{p,2} n^\lambda$ in powers of $(V - V_0)$. Since the constant term is already accounted for in $n^\lambda_0$, then $\delta_{p,1} n^\lambda$ must be $O(V - V_0)$ or smaller. The double integral over the $\delta_{p,1} n^\lambda$ and $\delta_{s,1} n^\lambda$ terms results in terms of higher order in $(V - V_0)$, thus in the first iteration, $\delta_{p,1} n^\lambda$ will be the solution to

$$\frac{d^2(\delta_{p,1} n^\alpha)}{dV^2} = \delta R^\alpha_{\mu \beta \nu} \ell^\beta (n_0^\nu), \quad (26)$$

which turns out to be proportional to $(V - V_0)^2$. The next iteration is the solution to

$$\frac{d^2(\delta_{p,2} n^\alpha)}{dV^2} = R^\alpha_{\mu \beta \nu} \ell^\beta (\delta_{p,1} n^\nu) - \delta \Gamma^\alpha_{\delta \lambda, \gamma} \ell^\gamma (\delta_s n^\lambda) - 2 \delta \Gamma^\alpha_{\delta \lambda} \ell^\gamma (\delta_s n^\lambda), \gamma. \quad (27)$$

Each term on the right hand side is proportional to $(V - V_0)^2$, and the solution is simply a double integral over $V$, giving a result proportional to $(V - V_0)^4$. Consequently, to lowest order in powers of $(V - V_0)$, the solution to $\tilde{n}^\mu$ for the perturbed space-time is

$$\tilde{n}^\alpha = n_0^\alpha + \delta_s n^\alpha + \int_{V_0}^V dW \int_{V_0}^W dV \delta R^\alpha_{\mu \beta \nu} \ell^\beta n_0^\nu. \quad (28)$$

To proceed further requires a model for fluctuations to be specified, and here three different models will be considered in turn:

- A scalar graviton model where the metric is perturbed by a term proportional to the product of a scalar field with the background metric, i.e. $\bar{g}_{\mu\nu} = (1 + \Phi)g_{\mu\nu}$. 


• An ingoing gravitational wave actively perturbs the horizon in a linearized theory of gravity. The perturbation to the Riemann tensor arises from the perturbation to the metric.

• An ingoing scalar field provides a passive perturbation to the horizon. The perturbation to the Riemann tensor is the Ricci tensor contribution that arises from the stress tensor of the scalar field.

In all three cases, the ingoing field is taken to occupy a squeezed quantum state, $|\alpha, \zeta\rangle$. In particular, the expectation value will be evaluated with respect to a multimode squeezed vacuum state, $|0, \zeta\rangle = \sum_{i=0}^{z_1} S(\zeta_i)|0\rangle$, which is further described in Appendix C. The excited modes will be taken to be wavepackets which are sent into the black hole after the collapse, as illustrated by the ingoing arrow in Fig. 2.

It should be noted that the introduction of quantum fluctuations into a black hole spacetime entails a significant conceptual shift. Classical perturbations will shift the location of the horizon, but do not change the fact that there is a precisely defined horizon. Of course, the true event horizon in a classical spacetime, the light ray which barely fails to escape to $\mathcal{I}^+$, can only be known when the complete history of the spacetime is known. Quantum fluctuations introduce an additional ambiguity, whereby the precise event horizon can never be known.

3. Fluctuations

Quantizing the ingoing perturbation field, $\delta_\mu n^\mu$ becomes a quantum operator. To construct the operator $\delta_\mu \hat{n}_\mu$, consider $\delta_\mu \hat{n}_\mu$ as the solution to

$$\frac{d^2 \delta_\mu \hat{n}_\mu}{dV^2} = \delta \hat{R}^\alpha_{\beta\mu\nu} \delta_\mu \hat{f}_\nu. \quad (29)$$

Suppose we characterize fluctuations in the separation vector by the quantity

$$\langle \hat{n}_\mu \delta_\mu \hat{n}_\mu \rangle = \langle \delta_\mu \hat{n}_\mu \delta_\mu \hat{n}_\mu \rangle - \langle \delta_\mu \hat{n}_\mu \rangle \langle \delta_\mu \hat{n}_\mu \rangle. \quad (30)$$

Due to the peculiarities of null Kruskal coordinates, this quantity is not a good comparator for all three models. In particular, it is identically zero for the “scalar graviton” model, of order $(V - V_0)^4$ for the graviton model, and of order $(V - V_0)^2$ for the scalar field model. In order to compare the results of the three models with each other and with the background result, which is of order $(V - V_0)^2$, it is advantageous to consider instead the variance

$$\Delta(\hat{n}_\mu \delta_\mu \hat{n}_\mu)^2 = \langle \langle \hat{n}_\mu (x) \delta_\mu \hat{n}_\mu (x) \rangle \langle \hat{n}_\mu (x') \delta_\mu \hat{n}_\mu (x') \rangle \rangle - \langle \langle \hat{n}_\mu (x) \delta_\mu \hat{n}_\mu (x) \rangle \rangle \langle \langle \hat{n}_\mu (x') \delta_\mu \hat{n}_\mu (x') \rangle \rangle. \quad (31)$$

It is straightforward to show that

$$\Delta(\hat{n}_\mu \delta_\mu \hat{n}_\mu)^2 = 4 (n_0^\mu (x)n_0^\nu (x') + n_0^\nu (x)n_0^\mu (x') + \delta_\mu n^\nu (x) + \delta_\nu n^\mu (x))$$

$$\times [\langle \delta_\mu \hat{n}_\mu (x) \delta_\mu \hat{n}_\mu (x') \rangle - \langle \delta_\mu \hat{n}_\mu (x) \rangle \langle \delta_\mu \hat{n}_\mu (x') \rangle] + 2 (n_0^\mu (x) + \delta_\mu n^\mu (x))$$

$$\times [\langle \delta_\mu \hat{n}_\mu (x) \delta_\mu \hat{n}_\mu (x') \rangle - \langle \delta_\mu \hat{n}_\mu (x) \rangle \langle \delta_\mu \hat{n}_\mu (x') \rangle]$$

$$+ \langle \delta_\mu \hat{n}_\mu (x) \delta_\mu \hat{n}_\mu (x') \rangle^2 - \langle \delta_\mu \hat{n}_\mu (x) \rangle \langle \delta_\mu \hat{n}_\mu (x') \rangle \langle \delta_\mu \hat{n}_\mu (x') \rangle. \quad (32)$$
As will be demonstrated, each $\delta_p \hat{n}_\nu(x) \propto (V - V_0)^2$; additionally, $\delta_s n_\nu(x) \propto (V - V_0)^2$. Therefore, to lowest order in $(V - V_0)$ we have

$$\Delta(n^\mu \bar{n}_\mu) = 4 \left[ \langle (n_0^\mu(x) \delta_p \hat{n}_\mu(x)) (n_0^\nu(x') \delta_p \hat{n}_\nu(x')) \rangle - \langle n_0^\mu(x) \delta_p \hat{n}_\mu(x) \rangle \langle n_0^\nu(x') \delta_p \hat{n}_\nu(x') \rangle \right].$$  \hspace{1cm} (33)

This then is the primary quantity of interest to calculate for the three fluctuation models.

### IV. SCALAR GRAVITON MODEL

Turn now to the scalar graviton model, where the perturbation is simply a scalar field, $\Phi$, (a dilaton) multiplying the background metric, $\bar{g}_{\mu\nu} = (1 + \Phi) g_{\mu\nu}$. Here $g_{\mu\nu}$ is the unperturbed (Schwarzschild) background space-time metric and we may define $\bar{g}^{\mu\nu} = g^{\mu\nu}(1 - \Phi)$ such that to first order in $\Phi$, $\bar{g}^{\mu\nu} g_{\mu\nu} = \delta^{\alpha\beta}$. The scalar field $\Phi$ will be a free quantum scalar field (multiplied by $\ell_P$) and will be defined in the following section. One may object that this model is simply a conformal transformation of the space-time under which the light cone structure remains invariant. However, while it is true that the light cone is invariant under a conformal transformation, the geodesic deviation is affected. This is because the Riemann tensor involves derivatives of the conformal factor, so that the Riemann tensor of the transformed space-time is not equal to a simple conformal transformation of the Riemann tensor. For example, Robertson-Walker space-time is conformally flat but has non-trivial geodesic deviation. This model is useful as a simplified model which reproduces the essential features of the more complicated graviton model of Sect. V.

#### A. Normalized Wave Packets

Let a scalar field propagate from $r_* = \infty$, through the potential barrier of the black hole, to the horizon at $r_* = -\infty$. The wave function must satisfy the wave equation in the Schwarzschild geometry. Following Hawking [1], Fourier decompose solutions of the wave equation with respect to advanced or retarded time, use continuum normalization, and expand in spherical harmonics. Using ingoing Eddington-Finkelstein coordinates, a single ingoing mode is

$$\psi_{\omega \ell m} = \frac{Y_{\ell m}(\theta, \phi)}{r \sqrt{2\pi \omega}} F_\omega(r) e^{-i\omega v}$$  \hspace{1cm} (34)

from which ingoing wave packets may be constructed as

$$\psi_{jn} = \varepsilon_j^{-\frac{1}{4}} \int_{j \varepsilon_j}^{(j+1) \varepsilon_j} e^{-2\pi i n \omega / \varepsilon_j} \psi_{\omega \ell m} d\omega. \hspace{1cm} (35)$$

The integer $j$ controls where in frequency space the wave packet is peaked, while $\varepsilon_j$ controls the width of the wavepacket and has units of frequency. The integer $n$ describes which wave packet is under consideration. This construction allows for wave packets to be sent in at regular intervals of $2\pi / \varepsilon_j$ with various frequencies. Thus $\psi_{jn}$ is the $n^{th}$ wave packet sent in with component frequencies ranging from $j \varepsilon_j$ to $(j + 1) \varepsilon_j$. The function $F_\omega(r)$ is in general a complex function which depends in some complicated way on the geometry of the space-time. For sharply peaked wave packets, $F_\omega(r)$ is of order unity at infinity, and essentially reduces to a transmission coefficient near the horizon. In Appendix [A] it is shown
that these wavepackets are properly normalized. The quantized scalar field is constructed from the wavepackets defined above as

$$\Phi = \sum_{jn} \left( \psi_{jn} \hat{a}_{jn} + \psi_{jn}^* \hat{a}_{jn}^\dagger \right).$$

(36)

In general it suffices to consider a single ingoing wavepacket, thus in what follows the index \(n\) will usually be suppressed and assumed fixed.

**B. Fluctuations**

Since the metric of the full space-time obeys the same symmetries as the background space-time, it is straightforward to calculate the Riemann tensor exactly from Eq. (14). In particular, to first order in \(\Phi\) and its derivatives, one finds the relevant quantity [see Eq. (16)]

$$\delta R^{V \nu \nu} = \Phi_{,\nu}$$

(37)

Applying Eq. (28) one finds, to leading order in \((V - V_0)\),

$$n^\mu_0 (x) \delta_p n_\mu (x) = g_{UV} n^2_0 \int^V_{V_0} dW \int^W_{V_0} dV \delta R^{V \nu \nu} = g_{UV} n^2_0 \int^V_{V_0} dW \int^W_{V_0} dV \Phi_{,\nu}.$$  

(38)

Recall that \(g_{UV} = 8 M^2\) near \(r = 2M\). Clearly, \(\langle \Phi \rangle = 0\) even when evaluating the expectation value with respect to a squeezed vacuum state; therefore the variance, Eq. (33), becomes

$$\Delta (\bar{n}^\mu_0 \bar{n}_\mu) = 4 \langle (n^\mu_0 (x) \delta_p n_\mu (x)) (n^\nu_0 (x') \delta_p n_\nu (x')) \rangle.$$  

(39)

Expanding in terms of the mode functions, one finds

$$\Delta (\bar{n}^\mu_0 \bar{n}_\mu) = 4 (g_{UV} n^2_0)^2 \sum_{j,k} \int^V_{V_0} dW \int^W_{V_0} dV \int^V_{V_0} dW' \int^W_{V_0} dV' \times \left\{ \psi_{j,VU} (x) \psi_{k,V'U'} (x') \langle \zeta, 0 | \hat{a}_j \hat{a}_k | 0, \zeta \rangle + \psi_{j,VU} (x) \psi^*_{k,V'U'} (x') \langle \zeta, 0 | \hat{a}^\dagger_j \hat{a}^\dagger_k | 0, \zeta \rangle + \psi^*_{j,VU} (x) \psi_{k,V'U'} (x') \langle \zeta, 0 | \hat{a}^\dagger_j \hat{a}_k | 0, \zeta \rangle + \psi^*_{j,VU} (x) \psi^*_{k,V'U'} (x') \langle \zeta, 0 | \hat{a}_j \hat{a}^\dagger_k | 0, \zeta \rangle \right\}$$

(40)

where we choose to evaluate the expectation value with respect to a multimode squeezed vacuum state \(|0, \zeta\rangle = \prod_{i = 20} S(\zeta_i) |0\rangle\). Using the results of Appendix C, this becomes

$$\Delta (\bar{n}^\mu_0 \bar{n}_\mu) = 4 (g_{UV} n^2_0)^2 \sum_{j,k} \int^V_{V_0} dW \int^W_{V_0} dV \int^V_{V_0} dW' \int^W_{V_0} dV' \times \delta_{jk} \left\{ \psi_{j,VU} (x) \psi_{k,V'U'} (x') (1 + \Theta_{2021} (j) (cosh \rho_j - 1) (-\Theta_{2021} (k) \sinh \rho_k)) + \psi_{j,VU} (x) \psi^*_{k,V'U'} (x') (1 + \Theta_{2021} (j) (cosh \rho_j - 1) (1 + \Theta_{2021} (k) (cosh \rho_k - 1)) + \psi^*_{j,VU} (x) \psi_{k,V'U'} (x') (-\Theta_{2021} (j) \sinh \rho_j) (-\Theta_{2021} (k) \sinh \rho_k) + \psi^*_{j,VU} (x) \psi^*_{k,V'U'} (x') (-\Theta_{2021} (j) \sinh \rho_j) (1 + \Theta_{2021} (k) (cosh \rho_k - 1)) \right\},$$

(41)
with the integer step function

\[ \Theta_{z_0 z_1}(j) = \begin{cases} 
1, & z_0 \leq j \leq z_1, \\
0, & \text{otherwise}. 
\end{cases} \tag{42} \]

Notice that the factor multiplying \( \psi_{j,VU}(x) \psi^*_k,VU'(x') \) contains a \( \delta_{jk} \) which makes \( \Delta(\bar{n}^\mu \bar{n}_\mu)^2 \) divergent. We therefore take renormalization to correspond to restricting the sum over modes to those occupying an excited squeezed state mode. For the current model and the graviton model, this restriction corresponds to normal ordering. In general, there will also be a vacuum contribution which is being ignored here. This should be a good approximation for highly excited states, that is, the limit of large squeeze parameter, \( \cosh \rho_j \approx \sinh \rho_j \approx e^{\rho_j/2} \). Taking this limit and restricting the sum to \( z_0 \leq j, k \leq z_1 \), the integer step function \( \Theta_{z_0 z_1} = 1 \) and this simplifies to

\[ \Delta(\bar{n}^\mu \bar{n}_\mu)^2 = (g_{UV} n_0^2)^2 \sum_{j=j_0}^{z_1} e^{\rho_j} \int_{V_0}^V dW \int_{V_0}^W dV \left( \psi_{j,VU}(x) - \psi^*_j,VU(x) \right)^2. \tag{43} \]

The \( V \) integral is trivial and we have

\[ \Delta(\bar{n}^\mu \bar{n}_\mu)^2 = (g_{UV} n_0^2)^2 \sum_{j=j_0}^{z_1} e^{\rho_j} \int_{V_0}^V dW \left( \psi_{j,U} \big|_{V=V_0} - \psi^*_j,U \big|_{V=V_0} \right)^2. \tag{44} \]

Consider \( \psi_{j,U}(x) \). Near the horizon, \( F_\omega(r) \) reduces to a transmission coefficient which depends only on \( \omega \) and \( \ell \). One may then find

\[ \psi_{j,U} \approx \int_{\ell j}^{(j+1)\ell j} d\omega \frac{Y_{\ell m} e^{-i\omega \delta_j}}{2M \sqrt{2\pi \omega \varepsilon_j}} F(\omega) V^{1-4iM \omega}, \tag{45} \]

where \( \delta_j = 2\pi n_j / \varepsilon_j \). We recall that \( (1-2\lambda M/r) \approx -UV e^{-1} \), and use the definition of \( V \) to write \( V e^{-i\omega} = V^{1-4iM \omega} \). Next, one finds

\[ \int_{V_0}^V dW \psi_{j,U}(x) \big|_{V=V_0} = \int_{\ell j}^{(j+1)\ell j} d\omega \frac{e^{-1} Y_{\ell m} F(\omega) e^{-i\omega \delta_j}}{2M \sqrt{2\pi \omega \varepsilon_j}} \times \left( (2-4iM \omega)^{-1} (V^{2-4iM \omega} - V_0^{2-4iM \omega}) - V_0^{1-4iM \omega} (V - V_0) \right). \tag{46} \]

Expanding the bracketed terms in powers of \( (V-V_0) \), this becomes

\[ \int_{V_0}^V dW \psi_{j,U}(x) \big|_{V=V_0} = \int_{\ell j}^{(j+1)\ell j} d\omega \frac{e^{-1} Y_{\ell m} F(\omega) (1-4iM \omega) e^{-i\omega (\nu_0 + \delta_j)}}{4M \sqrt{2\pi \omega \varepsilon_j}} e^{-i\omega (\nu_0 + \delta_j)} (V - V_0)^2. \tag{47} \]

Here \( \nu_0 \) is the Eddington-Finkelstein coordinate corresponding to \( V_0 \), i.e. \( V_0 = e^{\nu_0/4M} \). Use Eq. \((47)\) in Eq. \((12)\) and let \( m = 0 \), then \( Y_{\ell m}^0 = Y_{\ell 0} \) and the spherical harmonics factor out. Using this and the results for the classical deviation, Eq. \((13)\), and inserting the appropriate
powers of the Planck length, \( \ell_P \), the fractional fluctuations are found to be

\[
\Delta (\tilde{n}_{\mu})^2 = \sum_{j=0}^{\infty} \int_{\varepsilon_j}^{(j+1)\varepsilon_j} \frac{e^{\rho_j}}{8M\sqrt{2\pi\varepsilon_j}} \frac{Y_{\ell_0}(\ell_P)}{\lambda(0)Y_{\ell_0}(\ell_P)} d\omega \frac{e^{\rho_j}Y_{\ell_0}(\ell_P)}{8M\sqrt{2\pi\varepsilon_j}} \left[ (F(\omega)e^{-i\omega (v_0+\delta_j)} - F^*(\omega)e^{i\omega (v_0+\delta_j)}) - 4i\omega M (F(\omega)e^{-i\omega (v_0+\delta_j)} + F^*(\omega)e^{i\omega (v_0+\delta_j)}) \right]^2.
\]

(48)

Technically, expanding Eq. (46) in powers of \((V-V_0)\) is really an expansion in powers of \(2M\omega(V-V_0)\). In considering the result for the fractional fluctuations, therefore, one must bear in mind that we are considering the limit where \((V-V_0) < 1\) and also \(\omega \leq (2M)^{-1}\). In the limit \(\omega \rightarrow 0\), \(F(\omega) \rightarrow B_\ell (2i\omega M)^\ell + 1\) where \(|B_\ell|\) is of order 1. In this case one finds

\[
\Delta (\tilde{n}_{\mu})^2 = \frac{|B_\ell|^2 Y_{\ell_0}^2}{8} \sum_{j=0}^{\infty} \frac{e^{2\rho_j}i^\ell (2M)^{2\ell}}{\pi\varepsilon_j} \left[ \int_{\varepsilon_j}^{(j+1)\varepsilon_j} d\omega \omega^{\ell+1/2} \cos(\omega (v_0 + \delta_j)) \right]^2.
\]

(49)

Near \(\omega \approx (2M)^{-1}\), the transmission coefficient is of order 1. It follows that if the wavepacket is sharply peaked near \(\omega \approx (2M)^{-1}\)

\[
\Delta (\tilde{n}_{\mu})^2 = \frac{Y_{\ell_0}^2}{16} \sum_{j=0}^{\infty} \frac{e^{2\rho_j}}{M\pi\varepsilon_j} \left[ \int_{\varepsilon_j}^{(j+1)\varepsilon_j} d\omega \sin(\omega (v_0 + \delta_j)) + 2 \cos(\omega (v_0 + \delta_j)) \right]^2.
\]

(50)

Let us leave further analysis until Section VIII and first consider the other models.

V. GRAVITON MODEL

The next model for fluctuations is that of an ingoing gravitational wave occupying a squeezed state. In this case the perturbation field mode functions are constructed from the allowed classical black hole perturbations.

A. Even Parity Classical Perturbations

With Regge and Wheeler paving the way, the subject of classical black hole perturbations was thoroughly studied by Vishveshwara, Eddlestein, Zerilli, Price, Tuckolsky, and others [27, 28, 29, 30, 31, 32, 33, 34]. We begin with the original formulation of metric perturbations by Regge and Wheeler. These come in two varieties, even and odd parity. In this work, we will give an explicit treatment for the even parity case. However, gravitons in odd parity wavepackets can be shown to lead to similar conclusions as we will find here. Purely even parity waves are physically realizable, being generated, for example, by matter falling radially into a black hole [31]. In Schwarzschild coordinates and using the Regge-Wheeler gauge, the even parity metric perturbation is

\[
\Psi_{\mu\nu} = e^{-i\omega t} P_t (\cos \theta) \left( \begin{array}{cccc}
(1 - \frac{2M}{r})H_0(r) & H_1(r) & 0 & 0 \\
H_1(r) & (1 - \frac{2M}{r})^{-1}H_2(r) & 0 & 0 \\
0 & 0 & r^2 K(r) & 0 \\
0 & 0 & 0 & r^2 \sin^2 \theta K(r)
\end{array} \right) \tag{51}
\]
While this equation pertains to a particular choice of gauge, our results are based on the Riemann tensor correlation function and as such are gauge invariant. Zerilli [30, 31] showed that the even parity radial functions $H_0(r), H_1(r), H_2(r),$ and $K(r)$ may be related to a new radial function, $Z(r)$, that obeys a single Schrödinger-type equation

$$\frac{d^2Z(r)}{dr^2} + (\omega^2 - V_{\text{eff}})Z(r) = 0, \quad (52)$$

with an effective potential

$$V_{\text{eff}} = \left( \frac{1 - 2M}{r} \right) \frac{2\lambda^2(\lambda + 1)r^3 + 6\lambda^2Mr^2 + 18\lambda M^2r + 18M^3}{r^3(\lambda r + 3M)^2}, \quad (53)$$

where $\lambda = (\ell - 1)(\ell + 2)/2$.

Equation (51) may be transformed to null Kruskal coordinates, which we indicate with a superscript, $\Psi^{(K)}_{\mu\nu}$. Next, construct the wavepackets

$$\Psi^{(K)}_{(jn)\mu\nu} = \int_{j\varepsilon_j}^{(j+1)\varepsilon_j} d\omega A_{\ell m}(\omega)e^{-i\omega\delta_j}\Psi^{(K)}_{\mu\nu}, \quad (54)$$

where $\delta_j = 2\pi n/\varepsilon_j$ and $A_{\ell m}(\omega)$ is a normalization factor. The normalization of this perturbation mode requires some amount of work, which is relegated to Appendix B. The ingoing linearly quantized graviton perturbation field is then

$$h^{(K)}_{\mu\nu} = \sum_{jn} \left( \Psi^{(K)}_{(jn)\mu\nu}\hat{a}_{jn} + H.C. \right). \quad (55)$$

The interpretation of the integers $j$ and $n$ is exactly the same as discussed in Sect. IV A. Again, it suffices to consider $n$ fixed, so the index $n$ will in general be suppressed. Using the results of Appendix B, the properly normalized even parity wavepacket is written

$$\Psi^{(j)}_{\mu\nu} = \int_{j\varepsilon_j}^{(j+1)\varepsilon_j} d\omega \sqrt{\frac{\pi}{\tilde{L}}} e^{-i\omega\delta_j}\Psi_{\mu\nu}, \quad (56)$$

where

$$\tilde{L} = \frac{1}{2\ell + 1} \left[ 2 + \ell^2(\ell + 1)(\ell + 3) \right] - \frac{(\ell + 1)!}{(\ell - 1)!}. \quad (57)$$

With this normalization, it is understood that the Zerilli radial function takes the asymptotic value $Z(r_\ast \to \infty) = e^{-i\omega r_\ast}$.

### B. Fluctuations

From Eqs. (58) and (55), one finds

$$\delta \tilde{\Gamma}^{\alpha}_{\mu\nu} = \frac{1}{2}\epsilon^{\alpha\beta\gamma} \sum_j \int_{j\varepsilon_j}^{(j+1)\varepsilon_j} d\omega \left[ A_{\ell m}(\omega)e^{-i\omega\delta_j}(\Psi_{\mu\beta\nu} + \Psi_{\beta\mu\nu} - \Psi_{\mu\nu\beta})\hat{a}_j + H.C. \right], \quad (58)$$

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and by extension
\[
\delta \hat{R}_\beta^{\alpha \mu} = \sum_j \int_{j \varepsilon_j}^{(j+1)\varepsilon_j} d\omega \left[ A_{\ell m}(\omega)e^{-i\omega \delta_j} \delta \hat{R}_\beta^{\alpha \mu} \hat{a}_j + H.C. \right].
\] (59)

Here the hat has been reinserted to clarify that the operators \( \delta \Gamma_\mu^{\alpha \nu} \) and \( \delta \hat{R}_\beta^{\alpha \mu} \) may be constructed from the classical quantity corresponding to a classical single mode perturbation. Consequently,
\[
\delta_p \hat{n}^\mu = \sum_j \int_{j \varepsilon_j}^{(j+1)\varepsilon_j} d\omega \left[ A_{\ell m}(\omega)e^{-i\omega \delta_j} \delta_p n^\mu \hat{a}_j + H.C. \right].
\] (60)

The second term of Eq. (33) is zero for squeezed vacuum states. Recognizing that \( A_{\ell m}(\omega) \) is real, the variance for the graviton model is
\[
\Delta(\hat{n}^\mu \hat{n}_\mu) = 4\langle (n_0^\mu(x) \delta_p \hat{n}_\mu(x)) (n_0^\nu(x') \delta_p \hat{n}_\nu(x')) \rangle = \]
\[
4 \sum_j \sum_k \int_{j \varepsilon_j}^{(j+1)\varepsilon_j} d\omega \int_{k \varepsilon_k}^{(k+1)\varepsilon_k} d\omega' A_{\ell m}(\omega) A_{\ell m}(\omega') \]
\[
\left\{ e^{-i(\omega \delta_j + \omega' \delta_k)} (n_0^\mu(x) \delta_p n_\mu(x)) (n_0^\nu(x') \delta_p n_\nu(x')) \langle \hat{a}_j \hat{a}_k \rangle \right. + \]
\[
+ e^{i(\omega \delta_j - \omega' \delta_k)} (n_0^\mu(x) \delta_p n_\mu(x)) (n_0^\nu(x') \delta_p n_\nu(x')) \langle \hat{a}_j^\dagger \hat{a}_k^\dagger \rangle + \]
\[
+ e^{-i(\omega \delta_j - \omega' \delta_k)} (n_0^\mu(x) \delta_p n_\mu(x)) (n_0^\nu(x') \delta_p n_\nu(x')) \langle \hat{a}_j^\dagger \hat{a}_k \rangle \}.
\] (61)

We choose to evaluate the expectation value with respect to a multimode squeezed vacuum state \( |0, \zeta \rangle = \prod_{\ell=0}^{z_1} S(\zeta_\ell) |0 \rangle \). As discussed for the scalar graviton, renormalization amounts to restricting the sum to those states which lie in the range of squeezing. Together with the results of Appendix C and in the limit of large squeeze parameter, \( \rho \), this gives
\[
\Delta(\hat{n}^\mu \hat{n}_\mu) = \sum_{j=0}^{z_1} e^{2\rho_j} \int_{j \varepsilon_j}^{(j+1)\varepsilon_j} d\omega A_{\ell m}(\omega) \left| e^{-i\omega \delta_i} n_0^\mu(x) \delta_p n_\mu(x) - e^{i\omega \delta_j} n_0^\mu(x) \delta_p n_\mu(x) \right|^2.
\] (62)

It remains to calculate \( n_0^\mu \delta_p n_\mu \) due to a single mode classical perturbation. Unfortunately this is difficult to do analytically, but is possible with the use of a computer algebra program. To further simplify the problem we restrict attention to perturbations of purely even parity with \( \ell = 2 \). To calculate \( \delta_p n^\mu \) near the horizon, one must first calculate \( \delta R_\beta^{\alpha \mu} \) near the horizon in null Kruskal coordinates. There are several possible routes toward obtaining this information. The most straightforward might appear to be to first transform the metric perturbation to Kruskal coordinates and then proceed from Eq. (17). The difficulty with this method arises when one tries to take the limit of \( \delta R_\beta^{\alpha \mu} \) near the horizon. In null Kruskal coordinates, the approach to the horizon is along a line of constant \( V \) rather than a line of constant \( t \). The equations, however, still involve \( r \), implicitly defined in terms of \( U \) and \( V \), so taking the limit is not a well defined operation.

It is instead simpler to begin with the metric perturbation in Schwarzschild coordinates and compute \( \delta R_\beta^{\alpha \mu} \) via Eqs. (17) and (18) for a classical single mode ingoing perturbation. Expressing the radial functions \( H(r) \), \( H_1(r) \), and \( K(r) \) in terms of the Zerilli function, \( Z(r) \),
one then uses \( Z(r) \) to first order in \( (r - 2M) \) to expand \( \delta R^\alpha_{\beta \mu \nu} \) in powers of \( (r - 2M) \) near the horizon. It turns out to be necessary to expand the Zerilli function to at least \( O(r - 2M) \) to ensure that the metric perturbation remains finite on the horizon. Since the initial calculation is done in Schwarzschild coordinates, the limit is well defined as \( r \to 2M \) along a line of constant \( t \). Once \( \delta R^\alpha_{\beta \mu \nu} \) is known near the horizon, it may then be transformed to null Kruskal coordinates.

Using this procedure to solve for \( \delta_\mu n^\mu \) as expressed in Eq. (28), one may go on to find

\[
n_0^\mu(x) \delta_\mu n_\mu(x) = -(g_{UV} e^{-1} n_0^2)(V - V_0)^2 \frac{T(\omega)e^{-i\omega v_0}}{324M(i + 4M\omega)} (11800i \\
+ 34321M\omega - 39783i M^2 \omega^2 + 14348M^3 \omega^3) (1 + 3 \cos 2\theta) \tag{63}
\]

Inserting the appropriate powers of the Planck length, the fractional fluctuations are then

\[
\frac{\Delta(\bar{n}^\mu \bar{n}_\mu)}{(n^\mu n_\mu)^2} = \frac{1}{4} \sum_{j=2}^{z_1} e^{2\rho_j} \left| \int_{j\varepsilon_j}^{(j+1)\varepsilon_j} d\omega \frac{\ell_P (1 + 3 \cos 2\theta)}{324M\sqrt{\pi \varepsilon_j L\omega}} \frac{T(\omega)e^{-i\omega(v_0 + \delta_j)}}{(i + 4M\omega)} \right| \times (11800i + 34321M\omega - 39783i M^2 \omega^2 + 14348M^3 \omega^3) - H.C. \right|^2. \tag{64}
\]

Once again, one must consider that the expansion in \( (V - V_0) \) is really an expansion in \( 2M\omega (V - V_0) \). In the limit \( \omega \to 0 \), the transmission coefficient is \( T(\omega) \approx C_\ell (2i\omega M)^{\ell+1} \), where similarly to the scalar case, \( |C_\ell|^2 \approx 1 \). Specializing to \( \ell = 2 \), \( T(\omega) \approx -iC_2 (2M^3 \omega^3 \) and \( \bar{L} = \frac{32}{5} \). Then

\[
\frac{\Delta(\bar{n}^\mu \bar{n}_\mu)}{(n^\mu n_\mu)^2} \approx \frac{5}{2} |C_2|^2 \left( \frac{1475(1 + 3 \cos 2\theta)}{81} \right)^2 \sum_{j=2}^{z_1} e^{2\rho_j} \ell_P^2 (2M)^4 \frac{1}{\pi \varepsilon_j} \times \left[ \int_{j\varepsilon_j}^{(j+1)\varepsilon_j} d\omega \omega^{5/2} \cos \omega (v_0 + \delta_j) \right]^2 \tag{65}
\]

When \( \omega \approx (2M)^{-1} \), the transmission coefficient is of order 1. It follows that if the wavepacket is sharply peaked near \( \omega \approx (2M)^{-1} \), then

\[
\frac{\Delta(\bar{n}^\mu \bar{n}_\mu)}{(n^\mu n_\mu)^2} \approx \frac{(1 + 3 \cos 2\theta)^2}{16} \times \sum_{j=2}^{z_1} \frac{e^{2\rho_j} \ell_P^2}{\pi \varepsilon_j} \frac{1}{M} \left[ \int_{j\varepsilon_j}^{(j+1)\varepsilon_j} d\omega \sin (v_0 + \delta_j) + 54.9 \cos \omega (v_0 + \delta_j) \right]^2. \tag{66}
\]

The physical content of this expression will be explored in Sect. \[\text{VIII}\].

\[\text{VI. PASSIVE FLUCTUATION MODEL}\]

This model differs from the scalar graviton and graviton scenarios in that rather than a quantization of the dynamical degrees of freedom of the gravitational field, the space-time
geometry fluctuations arise passively through fluctuations in the stress tensor of a quantized scalar field. The ingoing scalar field is constructed in the same manner as presented with the scalar graviton model. The quantity of interest is the variance of the squared length of geometry fluctuations arise passively through fluctuations in the stress tensor of a quantized scalar field. The quantity of interest is the variance of the squared length of the separation vector, $\Delta(\bar{n}^\mu\bar{n}_\mu)^2$, given by Eq. \[33\]. Unlike the two previous models, the second term of this equation is not zero for the present model.

The operator nature of $\delta R^\mu_{\alpha
\beta\nu}$ is due to the stress tensor of the scalar field. One can use (see e.g. Ref. \[26\], Eq. 3.2.28)

$$R_{\alpha\beta\mu\nu} = C_{\alpha\beta\mu\nu} + \frac{2}{n-2}(g_{\alpha[\mu} R_{\nu]\beta} - g_{\beta[\mu} R_{\nu]\alpha}) - \frac{2}{(n-1)(n-2)}g_{\alpha[\mu} g_{\nu]\beta} R$$  \hfill (67)

with

$$R_{\mu\nu} = 8\pi \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right), \quad \text{and} \quad R = -8\pi T$$  \hfill (68)

to find

$$\delta R_{\alpha\beta\mu\nu} = 8\pi \left[ g_{\alpha[\mu} T_{\nu]\beta} - g_{\beta[\mu} T_{\nu]\alpha} - \frac{2}{3} g_{\alpha[\mu} g_{\nu]\beta} T \right].$$  \hfill (69)

Here the perturbation of the Weyl tensor vanishes, $\delta C_{\alpha\beta\mu\nu} = 0$, and the number of space-time dimensions is $n = 4$. The stress tensor for a scalar field and its trace are

$$T_{\mu\nu} = \Phi_{(\mu} \Phi_{\nu)} - \frac{1}{2} g_{\mu\nu} g^{\sigma\rho} \Phi_{(\sigma} \Phi_{\rho)}; \quad \text{and} \quad T = -g^{\sigma\rho} \Phi_{(\sigma} \Phi_{\rho)}.$$  \hfill (70)

It follows that

$$\delta R_{\alpha\beta\mu\nu} = 8\pi \left[ g_{\alpha[\mu} T_{\nu]\beta} - g_{\beta[\mu} T_{\nu]\alpha} - \frac{1}{3} g_{\alpha[\mu} g_{\nu]\beta} g^{\sigma\rho} \Phi_{(\sigma} \Phi_{\rho)} \right]$$  \hfill (71)

where it is to be understood that the antisymmetrization proceeds first, i.e.

$$g_{\alpha[\mu} \Phi_{(\nu} \Phi_{\beta)} = \frac{1}{4} \left[ g_{\alpha\mu} (\Phi_{\nu\rho} \Phi_{\beta} + \Phi_{\nu\beta} \Phi_{\rho}) - g_{\alpha\nu} (\Phi_{\rho\mu} \Phi_{\beta} + \Phi_{\rho\beta} \Phi_{\mu}) \right].$$  \hfill (72)

To leading order in $(V - V_0)$,

$$n_0^\mu \delta_p n_\mu = g_{UV} n_0^2 \int_{V_0}^{V} dW \int_{V_0}^{W} dV \delta R^V_{VVU}.$$  \hfill (73)

The variance is therefore proportional to an integral of a component of the Riemann tensor correlation function

$$\Delta(\bar{n}^\mu\bar{n}_\mu)^2 = 4(g_{UV} n_0^2)^2 \int_{V_0}^{V} dW \int_{V_0}^{W} dV \int_{V_0}^{V} dW' \int_{V_0}^{W'} dV' \langle C^V_{VVU} V_{VVU}(x, x') \rangle$$  \hfill (74)

where

$$\langle C^V_{VVU} V_{VVU}(x, x') \rangle = \langle \delta R^V_{VVU}(x) \delta R^V_{VVU}(x') \rangle - \langle \delta R^V_{VVU}(x) \rangle \langle \delta R^V_{VVU}(x') \rangle.$$  \hfill (75)

From Eq. (71), the Riemann tensor component of interest is

$$\delta R^V_{VVU} = \frac{8\pi}{3} \left[ (\Phi_{V'} \Phi_{U'} + \Phi_{U'} \Phi_{V'}) - \frac{1}{4} g_{UV} \left[ g^{\theta\phi} (\Phi_{\theta'} \Phi_{\phi} + \Phi_{\phi} \Phi_{\theta}) + g^{\phi\rho} (\Phi_{\phi'} \Phi_{\rho} + \Phi_{\rho} \Phi_{\phi}) \right] \right].$$  \hfill (76)
For simplicity we restrict to the case $\ell = 0$, so the angular derivatives are zero. Expanding $\Phi$ in terms of its mode functions gives

$$\delta R^{V}_{VVU} = \frac{8\pi}{3} \sum_{j,k} \left[ (\psi_{j,U}\psi_{k,V} + \psi_{j,V}\psi_{k,U}) \hat{a}_j \hat{a}_k + (\psi_{j,U}\pi^*_k + \psi_{j,V}\pi^*_k) \hat{a}_j \hat{a}_k^\dagger \right. \left. + (\psi_{j,U}\psi_{k,V}^* + \psi_{j,V}\psi_{k,U}^*) \hat{a}_j^\dagger \hat{a}_k + (\psi_{j,U}\psi_{k,V}^* + \psi_{j,V}\psi_{k,U}^*) \hat{a}_j \hat{a}_k^\dagger \right].$$  (77)

Simplify the notation by writing

$$\delta R^{V}_{VVU} = \frac{8\pi}{3} \sum_{j,k} \left[ A(x)\hat{a}_j \hat{a}_k + B(x)\hat{a}_j \hat{a}_k^\dagger + B^*(x)\hat{a}_j^\dagger \hat{a}_k + A^*(x)\hat{a}_j^\dagger \hat{a}_k^\dagger \right]$$  (78)

with $A(x)$, $A^*(x)$, $B(x)$, and $B^*(x)$ defined by comparison with Eq. (77).

To find the expectation value with respect to the multimode squeezed state $|0,\zeta\rangle = \prod_{i=0}^{\infty} S(\zeta)|0\rangle$, use the results of Appendix C. We again take renormalization to correspond to restricting the sum over modes to those occupying a squeezed state. In the limit of large squeeze parameter, $\rho$, one finds

$$\langle \zeta, 0 | \delta R^{V}_{VVU}(x) | 0, \zeta \rangle = -\frac{8\pi}{3} \sum_{j,k=0}^{z_1} \frac{e^{\rho_j} e^{\rho_k}}{2^{4}} \delta_{jk} [A(x) + A^*(x) - B(x) - B^*(x)].$$  (79)

and

$$\langle \zeta, 0 | \delta R^{V}_{VVU}(x) \delta R^{V}_{VVU}(x') \rangle = \left( \frac{8\pi}{3} \right)^2 \sum_{j,k=0}^{z_1} \sum_{r,s=0}^{z_1} \frac{e^{\rho_j} e^{\rho_k} e^{\rho_r} e^{\rho_s}}{2^{4}} (\delta_{jr} \delta_{ks} + \delta_{js} \delta_{kr} + \delta_{jk} \delta_{rs}) \times [(A(x) + A^*(x) - B(x) - B^*(x))(A(x') + A^*(x') - B(x') - B^*(x'))].$$  (80)

In general, quartic operator products can be expanded into a sum of a fully normal ordered part, a cross term, and a vacuum part. (See, for example, Ref. [22].) Our procedure of restricting the sum to those modes which lie in the range of squeezing is, in the limit of large squeeze parameter, equivalent to retaining only the fully normal ordered part. To see this, consider the difference between one of the terms above and its normal ordered version. As a concrete example, consider

$$\langle \zeta, 0 | \hat{a}_j \hat{a}_k^\dagger \hat{a}_r \hat{a}_s | 0, \zeta \rangle - \langle \zeta, 0 | \hat{a}_k \hat{a}_r \hat{a}_j \hat{a}_s | 0, \zeta \rangle \delta_{jk} \langle \zeta, 0 | \hat{a}_r \hat{a}_s | 0, \zeta \rangle.$$  (81)

From equations (C15i), (C15m), and (C12a) it is clear that this is proportional to the subdominant term $e^{2\rho}$ (compared to $e^{4\rho}$). Thus, in the limit of large squeeze parameter, restricting the sum over modes to those which lie in the range of squeezing corresponds to taking the fully normal ordered part. Furthermore, the cross and vacuum terms which have been neglected are sub-dominant in this limit.

The $\delta_{jk} \delta_{rs}$ term is the same as $\langle \langle \delta R^{V}_{VVU}(x) : | \langle \delta R^{V}_{VVU}(x') : \rangle \rangle$, which cancels to leave

$$\langle C^{V}_{VVU} V^{V}_{VVU}(x, x') \rangle = \frac{1}{24} \left( \frac{8\pi}{3} \right)^2 \sum_{j,k=0}^{z_1} \sum_{r,s=0}^{z_1} e^{\rho_j} e^{\rho_k} e^{\rho_r} e^{\rho_s} (\delta_{jr} \delta_{ks} + \delta_{js} \delta_{kr}) \times [(A(x) + A^*(x) - B(x) - B^*(x))(A(x') + A^*(x') - B(x') - B^*(x'))].$$  (82)
Using the definitions of $A(x)$ and $B(x)$, noting that the sums extend over the same range, and using the fact that the Kronecker deltas act symmetrically, one may show that

$$
\langle C_{VVU}^V V_{VVU}^V(x, x') \rangle = 4 \left( \frac{8\pi}{3} \right)^2 \sum_{j,k=0}^{z_1} e^{2\rho_j} e^{2\rho_k} \left[ (\text{Im}\psi_j(x))_V (\text{Im}\psi_k(x))_U \right] \times \left[ (\text{Im}\psi_j(x'))_V (\text{Im}\psi_k(x'))_U \right]. \quad (83)
$$

Since $\langle C_{VVU}^V V_{VVU}^V(x, x') \rangle$ is a product of a function of $x$ with a function of $x'$, the integral over $x$ and $x'$ of the product is the product of the integrals such that

$$
\int_{V_0}^V dW \int_{V_0}^W dV \int_{V_0}^W dW' \int_{V_0}^{W'} dV' \langle C_{VVU}^V V_{VVU}^V(x, x') \rangle = 4 \left( \frac{8\pi}{3} \right)^2 \sum_{j,k=0}^{z_1} e^{2\rho_j} e^{2\rho_k} \left| \int_{V_0}^V dW \int_{V_0}^W dV (\text{Im}\psi_j(x))_V (\text{Im}\psi_k(x))_U \right|^2. \quad (84)
$$

Near the horizon, the derivatives of $\psi$ with respect to $U$ and $V$ are straightforwardly calculated. One subsequently finds

$$
(\text{Im}\psi_j)_V (\text{Im}\psi_k)_U = \frac{ie^{-1}}{8\pi^2 M \sqrt{\varepsilon_j \varepsilon_k}} \int_{j\varepsilon_j}^{(j+1)\varepsilon_j} d\omega \int_{k\varepsilon_k}^{(k+1)\varepsilon_k} d\omega' \left( \frac{\omega}{\omega'} \right)^{1/2} \times \left[ F(\omega) F(\omega') e^{-i(\omega\delta_j + \omega'\delta_k)} e^{-iv(\omega + \omega')} - F(\omega) F^*(\omega') e^{-i(\omega\delta_j - \omega'\delta_k)} e^{-iv(\omega - \omega')} + H.C. \right]. \quad (85)
$$

The integration over $V$ of $(\text{Im}\psi_j)_V (\text{Im}\psi_k)_U$ reduces to calculating

$$
\int_{V_0}^V dW \int_{V_0}^W dV e^{\mp iv(\omega + \omega')} = \int_{V_0}^V dW \int_{V_0}^W dV V^{\mp 4iM(\omega + \omega')} \approx \frac{1}{2} e^{\mp iv(\omega + \omega')} (V - V_0)^2, \quad (86)
$$

where the solution has been expanded in powers of $(V - V_0)$. Inserting the appropriate powers of the Planck length, it follows that the fractional fluctuations are characterized by

$$
\frac{\Delta(\bar{n}_\mu \bar{n}_\mu)}{(n^\mu n_\mu)} = \sum_{j,k=0}^{z_1} \left| \frac{e^{\rho_j} e^{\rho_k} \ell_P^2}{3\pi M} \int_{j\varepsilon_j}^{(j+1)\varepsilon_j} d\omega \int_{k\varepsilon_k}^{(k+1)\varepsilon_k} d\omega' \left( F(\omega) e^{-i\omega(\varepsilon_0 + \delta_j)} - F^*(\omega) e^{i\omega(\varepsilon_0 + \delta_j)} \right) \right|^2 \times \left[ F(\omega') e^{-i\omega(\varepsilon_0 + \delta_k)} - F^*(\omega') e^{i\omega(\varepsilon_0 + \delta_k)} \right]^2. \quad (87)
$$

Once again, one must consider that the expansion in $(V - V_0)$ is really an expansion in $2M\omega(V - V_0)$. In the limit $\omega \rightarrow 0$, the transmission coefficient is again $F(\omega) \sim B'_\ell(2i\omega M)^{\ell+1}$. Specializing to $\ell = 0$, $F(\omega) \sim 2iB_0 M\omega$, and the fractional fluctuations are then

$$
\frac{\Delta(\bar{n}_\mu \bar{n}_\mu)}{(n^\mu n_\mu)} = \left( \frac{16}{3} \right)^2 |B_0|^4 \sum_{j,k=0}^{z_1} \frac{e^{2\rho_j} \ell_P^2 M}{\pi \varepsilon_j} \int_{j\varepsilon_j}^{(j+1)\varepsilon_j} d\omega \omega^{3/2} \cos \omega(\varepsilon_0 + \delta_j) \right]^2 \times \frac{e^{2\rho_k} \ell_P^2 M}{\pi \varepsilon_k} \int_{k\varepsilon_k}^{(k+1)\varepsilon_k} d\omega' (\omega')^{1/2} \cos \omega'(\varepsilon_0 + \delta_k). \quad (88)
$$
Near $\omega \approx (2M)^{-1}$, the transmission coefficient is of order 1. It follows that if the wavepacket is sharply peaked near $\omega \approx (2M)^{-1}$, then

$$
\frac{\Delta(\tilde{n}_\mu^\nu \tilde{n}_\mu)}{(n^\mu n_\mu)^2} = \left(\frac{4}{3}\right)^2 \frac{e^{2\rho_j \ell_P^2}}{\pi \varepsilon M} \left[ \int_{\Delta_\omega} (j+1)\varepsilon_j \, d\omega \sin(\omega v_0 + \delta_j) \right]^2 \times \frac{e^{2\rho_k \ell_P^2}}{\pi \varepsilon M} \left[ \int_{\Delta_\omega} (k+1)\varepsilon_k \, d\omega' \sin(\omega' v_0 + \delta_k) \right]^2. 
$$

(89)

VII. DISCUSSION

A. Summary of Results

Consider the results for the three different models – scalar graviton, graviton, and stress tensor induced fluctuations. There are two limits of interest, $\omega \to 0$ and $\omega \approx (2M)^{-1}$.

Case 1: $\omega \ll 1/M$

In the low frequency limit, $\omega \to 0$, we found Eqs. (49), (65), and (88) for the scalar graviton, graviton, and passive fluctuation models, respectively. The solutions to these integrals may be expressed in terms of incomplete gamma functions, but it is not necessary to invoke the machinery of incomplete gamma functions to get an idea of the general behavior of the fluctuations. To simplify the discussion, let us set $\delta_j = 0$. This may be assumed without loss of generality and is equivalent to assuming $n = 0$ in Eq. (35). In this limit we may use the small angle approximation to set $\cos(\omega v_0) \approx 1$. Furthermore, for a sharply peaked wavepacket, we may assume the integrand is approximately constant so that

$$
\int d\omega \, \omega^x \cos(\omega v_0) \approx \omega^x \Delta \omega.
$$

(90)

Ignoring the numerical factors and recognizing that $\varepsilon_j = \Delta \omega$, the general behavior of the fractional fluctuations is

$$
\frac{\Delta(\tilde{n}_\mu^\nu \tilde{n}_\mu)}{(n^\mu n_\mu)^2} \approx \begin{cases} 
\sum_j \frac{1}{\pi} e^{2\rho_j \ell_P^2} (2M\omega)^2 \omega \Delta \omega, & \text{Scalar Graviton,} \\
\sum_j \frac{1}{\pi} e^{2\rho_j \ell_P^2} (2M\omega)^4 \omega \Delta \omega, & \text{Graviton,} \\
\left( \sum_j \frac{1}{\pi} e^{2\rho_j \ell_P^2} (2M\omega) \omega \Delta \omega \right)^2, & \text{Passive.}
\end{cases}
$$

(91)

Case 2: $\omega \approx (2M)^{-1}$

For $\omega \approx (2M)^{-1}$, on the other hand, the fractional fluctuations were given by Eqs. (50), (66), and (89). Since the wavepacket is assumed to be sharply peaked in $\omega$, then $\Delta \omega = \varepsilon_j \ll \omega$ while $j$ is large. Thus we may use the small angle approximation, $\cos(\varepsilon_j v_0) \approx 1$.
and \( \sin(\varepsilon_j v_0) \approx \varepsilon_j v_0 \). Furthermore, as an order of magnitude estimate, we may assume \( \sin(\varepsilon_j v_0) \) is of order 1 so that

\[
\int_{\varepsilon_j}^{(j+1)\varepsilon_j} d\omega \sin(\omega v_0) \approx \varepsilon_j = \Delta \omega, \tag{92}
\]

and similarly

\[
\int_{\varepsilon_j}^{(j+1)\varepsilon_j} d\omega \cos(\omega v_0) \approx \varepsilon_j = \Delta \omega. \tag{93}
\]

The general behavior of the fractional fluctuation in the limit \( \omega \approx (2M)^{-1} \) is now

\[
\frac{\Delta (\bar{n}^\mu \bar{n}_\mu)^2}{(n^\mu n_\mu)^2} \approx \begin{cases} 
\sum_j \frac{e^{2\rho_j} \ell^2 P}{\pi M} \Delta \omega, & \text{Scalar Graviton and Graviton,} \\
\left( \sum_j \frac{e^{2\rho_j} \ell^2 P}{\pi M} \right)^2, & \text{Passive.}
\end{cases} \tag{94}
\]

Consider for a moment the limit of low squeezing, \( \rho \to 1 \). Although our results have been derived for \( \rho \gg 1 \), the order of magnitude behavior should still correspond to what we have derived since \( \sinh \rho \) and \( \cosh \rho \) are of order one for small \( \rho \). In this case the active fluctuations behave as \( \ell^2 P \Delta \omega / M \), which is in agreement with the results of Ford and Svaiter [11] where they find fluctuations in the proper time of an infalling observer become of order one for \( M \sim \ell_p \).

The scalar graviton and graviton models describe the active fluctuations, while the scalar field stress tensor fluctuations are passive fluctuations. Specializing \( \ell = 2 \) in the scalar graviton model, the general behavior is identical to that of the graviton model. In light of Eqs. (48), (64), and (87) one sees that Eqs. (91), and (94) become

\[
\frac{\Delta (\bar{n}^\mu \bar{n}_\mu)^2}{(n^\mu n_\mu)^2} \sim \begin{cases} 
\sum_j \frac{e^{2\rho_j} |F(\omega)|^2 \Delta \omega}{M(M \omega)}, & \text{Active,} \\
\left( \sum_j \frac{e^{2\rho_j} |F(\omega)|^2 \Delta \omega}{M(M \omega)} \right)^2, & \text{Passive.}
\end{cases} \tag{95}
\]

Although we have specialized to \( \ell = 0 \) for the passive fluctuations, it is not difficult to generalize to the more general case of arbitrary \( \ell > 0 \). For \( \ell > 0 \) there are some additional derivatives of \( \Phi \) with respect to the angular variables, but this will not alter the order of magnitude behavior in Eq. (95).

There are some generic features common to both active and passive fluctuations. As \( \omega \to 0 \), the fractional fluctuations are suppressed by some power of \( \omega \). They are further suppressed by a factor of \( \Delta \omega \), which has been assumed small, and some powers of the Planck length. Notice that the passive fluctuations are more heavily suppressed since the passive fluctuations are proportional to \( \ell^2 P \), whereas the active fluctuations are proportional to \( \ell^2 P \). The suppression in the low frequency limit is expected, since the effective potential barrier efficiently prevents low frequency modes from reaching the horizon. However, it seems these suppressions may be overcome by arbitrarily increasing the squeeze parameter.

For \( \omega \approx (2M)^{-1} \) the similarities between the active and passive fluctuations are even more striking. In this case, the fluctuations are again suppressed by the same powers of the Planck length and the width of the wavepacket, but are additionally suppressed by the black hole mass. One would expect this term to become important once the black hole evaporates to the Planck mass. Again, by arbitrarily increasing the squeeze parameter it is possible to overcome the various suppressions.
B. Semiclassical Restriction on Squeezing

Since it appears that the fluctuations may become arbitrarily large by unboundedly increasing $\rho$, we should investigate whether there is an upper bound on $\rho$. After all, a squeezed vacuum state is not necessarily devoid of particle content. From the discussion in Appendix C it follows that for a single mode squeezed vacuum state, $|0, \zeta⟩ = S(\zeta)|0⟩$, the expectation value for the number of particles is

$$\langle \zeta, 0 | \hat{N} | 0, \zeta⟩ = \langle 0 | S^\dagger(\zeta) \hat{a}^\dagger \hat{a} S(\zeta) | 0⟩ = \sinh^2 \rho \rightarrow e^{2\rho}, \rho \gg 1.$$  (96)

This means that increasing the squeeze parameter increases the mean energy density. If the energy density grows too large, then the semiclassical approximation in which backreaction is ignored fails.

Consider the passive fluctuation model. From the semiclassical Einstein equation, these calculations should remain valid as long as

$$\langle \delta R_{\alpha\beta\mu\nu} \rangle \ll R_{\alpha\beta\mu\nu}.$$  (97)

Working in null Kruskal coordinates, a typical component of the background Riemann tensor is

$$R_{VV} = \frac{16M^3}{UVr^3} \left(1 - \frac{2M}{r}\right),$$  (98)

which, since $(1 - \frac{2M}{r}) \approx -UV$ near the horizon, is approximately $-2e^{-1}$. However, $\delta R_{VV}$ is precisely the quantity that was calculated for the passive fluctuation model. Using Eq. (73) it follows that

$$\sum_j e^{2\rho_j} \frac{|F(\omega)|^2 \Delta \omega}{M} \ll 1.$$  (99)

Consider next the scalar graviton model. In this case $\delta R_{\alpha\beta\mu\nu}$ is linear in the field, and therefore $\langle \delta R_{\alpha\beta\mu\nu} \rangle = 0$. One could look for second order perturbations $\delta R_{\alpha\beta\mu\nu}^{(2)} \propto (h_{\mu\nu})^2$ that would be quadratic in the field. Instead, we consider

$$\langle \delta R_{\alpha\beta\mu\nu} \delta R_{\alpha\beta\mu\nu} \rangle \ll R_{\alpha\beta\mu\nu} R_{\alpha\beta\mu\nu}$$  (100)

to be a good indicator of whether the semiclassical treatment is valid for the perturbation. The quantity of interest is

$$\langle \delta R_{VV} \delta R_{VV} \rangle = \langle \Phi_{VV}, \Phi_{VV} \rangle$$  (101)

and the calculations proceed as in Sec. IV. The result for $\langle \Phi_{VV}, \Phi_{VV} \rangle$ is precisely the same as that found for $\Delta(n^\mu n_\mu)^2/(n^\mu n_\mu)^2$ in Eq. (18) and we therefore have

$$\sum_j e^{2\rho_j} \frac{|F(\omega)|^2 \Delta \omega}{M(M\omega)} \ll 1.$$  (102)

For the graviton model, one must again turn to a computer algebra system to calculate $\langle \delta R_{VV} \delta R_{VV} \rangle$. The result agrees with Eq. (102). In the process of seeking to place an upper bound on the results of Eq. (95) via a restriction on the squeeze parameter, $\rho$, a restriction on the results of Eq. (95) themselves has been found. It is therefore sufficient to proceed with this restriction on the general result.
C. Analysis of Results

For the sake of an order of magnitude estimate of the fractional fluctuations, assume the extremal cases where
\[ \sum_j e^{2\rho_j} \ell_p^2 |F(\omega)|^2 \Delta\omega \approx 1 \]
for the active fluctuations, and
\[ \sum_j e^{2\rho_j} \ell_p^2 |F(\omega)|^2 \Delta\omega \sim 1\]
for the passive fluctuations. Using this restriction, the fractional fluctuations become
\[ \frac{\Delta(\bar{n}_\mu \bar{n}_\mu)}{(n_\mu n_\mu)^2} \lesssim \begin{cases} 1, & \text{Active,} \\ 1, & \text{Passive.} \end{cases} \]

At first glance, it appeared that the fractional fluctuations described in Eq. (95) could be made arbitrarily large by increasing the squeeze parameter, a quite surprising result. Since \( \bar{n}_\mu \bar{n}_\mu \) reflects the change in geodesic deviation from the Schwarzschild background, restricting to perturbations that are in some sense small would lead one to expect that \( \bar{n}_\mu \bar{n}_\mu \) should also be small and not deviate very much from the background value, \( n_\mu n_\mu \) (equivalently the expectation value of the fluctuating quantity). Indeed, after imposing restrictions on the amount of allowed squeezing by requiring the induced curvature to be small compared to the background, we find the fluctuations to be no more than of order one. Nonetheless, fractional fluctuations of order unity in \( \Delta(\bar{n}_\mu \bar{n}_\mu)^2/(n_\mu n_\mu)^2 \) have the potential to dramatically alter the outgoing radiation.

D. Implications for Hawking Radiation

Do the space-time fluctuations implied by Eq. (105) have any significant effect on Hawking radiation? Fluctuations of the vector separating the horizon from a nearby outgoing null geodesic have been studied here precisely because of this vector’s importance to the Hawking effect derivation. However, it is the \( U \) component of the separation vector that is crucial to Hawking’s derivation. In fact, \( \delta_p n^U \equiv 0 \) by the symmetry properties of the Riemann tensor. Thus all of the contributions to the quantity \( \Delta(\bar{n}_\mu \bar{n}_\mu)^2/(n_\mu n_\mu)^2 \) come from the \( V \)-component of \( \bar{n}_\mu \).

Fluctuations in \( n^V \) indicate that an outgoing null geodesic may take either a longer or shorter than average affine time in reaching some distance from the black hole. In this sense the results tend to agree with the heuristic picture of Ford and Svaiter [11]. This would correspond to an uncertainty in the knowledge of which wavepacket was under consideration. Recall that in the derivation of the Hawking effect the ingoing wavepackets were controlled by an integer, \( n \), which allowed for successive ingoing wavepackets. A fluctuation in \( n^V \) would mean that the ordering of these successive wavepackets could be disrupted. This would not have any observable effect on the outgoing radiation if the ingoing state is the vacuum. But it is also possible to have stimulated emission in addition to the thermal flux. This would be caused by particles that are initially present during collapse and has been studied by Wald [35]. In this case the stimulated emission only occurs at early times while
the late time behavior remains thermal. Fluctuations in $n^V$ could allow for fluctuations in the arrival time of particles originating from this stimulated flux.

Note that effects near the horizon do not necessarily translate to effects observed at $\mathcal{I}^+$. Indeed, we were restricted to considering fluctuations only along a short segment of the outgoing geodesic. In order to discuss observations made by a distant observer, one would really have to be able to follow the evolution of the separation vector all the way out to the observer. In doing so one would see whether the fluctuations integrate to produce a large effect or not. The presence of the effective potential barrier prevents us from tracking the long term evolution of the fluctuations and so at this point discussing observations of a distant observer is merely speculation. Future research may help resolve this issue. In particular, particle creation near a moving mirror has been shown to be analogous to the Hawking effect for black holes, with an identical thermal spectrum obtained for specific mirror trajectories [36, 37, 38]. Further insight may be obtained into the horizon fluctuations of black holes by considering fluctuations in the trajectory of the mirror. The benefit of considering a moving mirror is that it lacks the complicated potential barrier of a black hole, allowing one to integrate out to an observer.

VIII. SUMMARY AND CONCLUSIONS

Deviation of outgoing null geodesics in a fluctuating Schwarzschild geometry have been considered. The reference geodesic was taken to be that outgoing null geodesic which generates the future horizon in the average (Schwarzschild) space-time. Fluctuations of the space-time were induced both actively and passively. The active fluctuations were first described by a scalar graviton model, consisting of a conformal transformation of the Schwarzschild metric where the conformal factor is a quantized scalar field, and then by a graviton model that is constructed as a linear quantum tensor field where the mode functions are taken to be the classically allowed even parity Schwarzschild perturbations. The passive fluctuations arise from stress tensor fluctuations of a free scalar field. In all models the ingoing perturbation is taken to occupy a multimode squeezed vacuum state. The modes in question are wavepackets being sent into the black hole after the collapse process.

For all models, fractional fluctuations of the quantity $\tilde{n}^\mu \tilde{n}_\mu$ were calculated. The vacuum level fluctuations are very small for large black holes, being of order $(\ell_p/M)^2$ for active fluctuations and of order $(\ell_p/M)^4$ for passive ones. However, the fluctuations could be boosted by increasing the amount of squeezing. An upper bound on the squeeze parameter was imposed by restricting the energy density of the ingoing field to fall within the allowed limits of semiclassical theory ignoring backreaction. This upper bound was then used to estimate an order of magnitude for the fractional fluctuations, $\Delta(\tilde{n}^\mu \tilde{n}_\mu)^2/(n^\mu n_\mu)^2$. It was found that the fractional fluctuations could in principle become of order unity.

Such large fluctuations would seem at first sight to have a dramatic effect upon the outgoing modes which carry the created particles. However, the fluctuations come from the $V$-component of $n^\mu$, not the $U$-component. This implies that the primary effect of these fluctuations is on the time delay of individual wavepackets, which would only be observable in stimulated emission, but not in spontaneous emission arising when the quantum state is the in-vacuum. This suggests that the thermal nature of the Hawking radiation is quite robust and is not altered by the type of enhanced fluctuations we have studied.
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APPENDIX A: SCALAR FIELD NORMALIZATION

The scalar field normalization is found via the Klein-Gordon norm

$$\langle \psi_{jn}, \psi_{j'n'} \rangle = i \int_S d\Sigma_\mu \left( \psi_{jn}^* \overleftrightarrow{\nabla^\mu} \psi_{j'n'} \right) = \delta_{\ell \ell'} \delta_{mm'}.$$  \hfill (A1)

The surface over which the integral is performed is naturally $\mathcal{I}^-$, past null infinity, and the coordinates appropriate in this region are ingoing Eddington-Finkelstein coordinates $(v,r,\theta,\varphi)$. In these coordinates, $\mathcal{I}^-$ is a three-surface isomorphic to $S^2 \times \mathbb{R}$, or a sphere at infinity further parameterized by $v$. The normal to this surface is therefore in the direction of $\partial/\partial r$ and the preceding integral is

$$i \int_{\mathcal{I}^-} d\Sigma_\mu \left( \psi_{jn}^* \overleftrightarrow{\nabla}^\mu \psi_{j'n'} \right) = i \int_{S^2} r^2 d\Omega \int_{-\infty}^{\infty} dv \left( \psi_{jn}^* \overleftrightarrow{\nabla}^v \psi_{j'n'} \right).$$

The field $\psi_{jn}$ is a scalar field, for which $\nabla^\mu = \partial^\mu$. The metric for ingoing Eddington-Finkelstein coordinates is off diagonal, and the $(v,r)$ sector is

$$g_{\mu\nu} = \begin{pmatrix} 1 & -2M \frac{1}{r} \\ 2M \frac{1}{r} & 1 \end{pmatrix} \quad \quad g^{\mu\nu} = \begin{pmatrix} 0 & 1 \\ 1 & (1 - \frac{2M}{r}) \end{pmatrix}.$$  \hfill (A2)

From this one finds

$$\partial^r = g^{rv} \partial_v + g^{rr} \partial_r = \partial_v + \left( 1 - \frac{2M}{r} \right) \partial_r$$  \hfill (A3)

and may proceed to calculate

$$i \int_{S^2} r^2 d\Omega \int_{-\infty}^{\infty} dv \psi_{jn}^* \left[ \overleftrightarrow{\partial_v} + \left( 1 - \frac{2M}{r} \right) \overleftrightarrow{\partial_r} \right] \psi_{j'n'}. \quad \hfill (A4)$$

The calculations are straightforward, but one must bear in mind that while the advanced time is implicitly defined in terms of $r$, when taking the derivative with respect to $v$ or $r$, the other variable is held fixed. Additionally, the $r$ dependence is the same for both $\psi_{jn}^*$ and $\psi_{j'n'}$ so the $r$ derivative terms cancel. Examining the norm in the asymptotically flat region $r_s \rightarrow r \rightarrow \infty$, Eq. (A4) becomes

$$\frac{i}{2\pi \varepsilon_j} \int d\Omega Y_{Lm}^*(\theta, \varphi) Y_{L'm'}(\theta, \varphi) \times \int_{\varepsilon_j}^{(j+1)\varepsilon_j} d\omega \int_{\varepsilon_j}^{(j+1)\varepsilon_j} d\omega' e^{2\pi i (\omega - \omega')/\varepsilon_j} \frac{F^*_\omega(r) F_\omega'(r)}{\sqrt{\omega \omega'}} \int_{-\infty}^{\infty} dv \left( e^{i\omega v} \overleftrightarrow{\partial_v} e^{-i\omega' v} \right). \quad \hfill (A5)$$

From the normalization condition of the spherical harmonics, the integral over the sphere gives a product of delta functions, $\delta_{\ell \ell'} \delta_{mm'}$. The function $F(r)$ was included in the definition
of the ingoing null field to reflect the presence of an effective potential barrier due to the space-time curvature, but \( F(\mathscr{I}^-) = 1 \) where \( r \to \infty \) at \( \mathscr{I}^- \). Lastly, \( e^{i\omega v} \delta_v e^{-i\omega' v} = -i(\omega + \omega')e^{i(\omega - \omega')v} \) so the integration with respect to \( v \) gives a delta function \( \pi \delta(\omega - \omega') \). This reduces the normalization condition at infinity to

\[
\langle \psi_{jn}, \psi_{j'n'} \rangle = (\varepsilon_j)^{-1} \delta_{\ell\ell'} \delta_{mm'} \int_{\varepsilon_j}^{(j+1)\varepsilon_j} d\omega = \delta_{\ell\ell'} \delta_{mm'}.
\]

(A6)

So as given, \( \psi_{jn} \) is properly normalized.

**APPENDIX B: GRAVITON MODE NORMALIZATION**

Some care is required when fixing the normalization of the ingoing graviton wavepackets. We choose to fix the normalization in the asymptotically flat-space region of \( r_* \to \infty \) by setting the vacuum energy of each mode to \( \frac{1}{2} \omega \). One would like to first calculate the effective energy density of the wave, integrate over some volume, and set the result to \( \frac{1}{2} \omega \). The typical approach in calculating the energy density would be to use Eq. 35.70 in Ref. \[39\], which relates the effective energy density of a gravitational wave to derivatives of the perturbation. The problem is that in the Regge-Wheeler gauge, where calculations are most easily performed, \( \Psi_{(j)\mu\nu} \) is not well-behaved at infinity. In fact, \( H_0(r), H_1(r), \) and \( H_2(r) \) all grow linearly with \( r \) as \( r_* \to \infty \). While Eq. 35.70 of Ref. \[39\] is gauge invariant, it implicitly involves some averaging and assumptions about covariant divergences at infinity that do not hold in the Regge-Wheeler gauge. Alternatively, it should be possible to transform the perturbations to the radiation gauge, where they are well-behaved, but the calculations become more difficult in the radiation gauge. Instead, we choose to take a somewhat circuitous route by first calculating the (gauge invariant) Riemann tensor at infinity. We then transform the Riemann tensor to Cartesian coordinates and make use of a special relation between the Riemann tensor and gravitational waves in the Transverse Tracefree (TT) gauge. Calculating the effective stress tensor of a gravity wave in the TT gauge is then straightforward using

\[
T_{\mu\nu}^{\text{eff}} = \frac{1}{32\pi}\left( \Psi_{(j)\mu\nu}^{TT} \Psi_{(j)\mu\nu}^{ab(TT)} \right)^* + \left( \Psi_{(j)\mu\nu}^{TT} \right)^* \Psi_{(j)\mu\nu}^{ab(TT)}.
\]

(B1)

Here the brackets indicate a spatial average over several wavelengths. The components of \( \Psi_{(j)\mu\nu}^{TT} \) have a simple relationship to components of the Riemann tensor, \( \Psi_{(j)\mu\nu}^{TT} = -2R_{0k0l} \). Since \( \Psi_{(j)\mu\nu} \) has a simple sinusoidal time dependence, it follows that \( \Psi_{(j)k\ell0}^{TT} = -i\omega \Psi_{(j)k\ell} \) and so \( \Psi_{(j)k\ell}^{TT} = 2\omega^{-2}R_{0k0l} \). In the asymptotically flat space at infinity,

\[
R_{\alpha\beta\mu\nu} = \frac{1}{2}(\Psi_{(j)\alpha\nu,\beta\mu} - \Psi_{(j)\alpha\mu,\beta\nu} + \Psi_{(j)\beta\mu,\alpha\nu} - \Psi_{(j)\beta\nu,\alpha\mu}).
\]

(B2)

Let \( J_{\nu}^{\mu} \) be the transformation matrix from spherical to Cartesian coordinates. Then in Cartesian coordinates

\[
R_{0\beta\gamma\lambda\sigma} = \delta_0^\beta \delta_0^\gamma R_{\alpha\beta\mu\nu}^\nu = \delta_0^\beta \delta_0^\gamma J_{\nu}^{\mu} J_{\nu}^{\sigma} J_{\nu}^{\lambda} J_{\nu}^{\sigma} R_{\gamma\lambda\sigma} = J_{\beta}^{\lambda} J_{\nu}^{\sigma} R_{0\lambda\sigma}
\]

(B3)

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while the TT metric perturbation is \( \Psi^{TT}_{jkl} = -\omega^{-2}J^\lambda_kJ^\sigma_lR_{0\lambda\sigma} \). Carrying out the above calculations, one finds

\[
T_{00}^{\text{eff}} = \varepsilon^{-1} \left( P_\ell^2 (\cos \theta) \left( 1 + (1 - \ell(\ell + 1))^2 \right) - 2 \cos \theta P_\ell' (\cos \theta) \left( \sin^2 \theta P_\ell'' (\cos \theta) - \cos \theta P_\ell' (\cos \theta) \right) \right)
\times \int d\omega \int d\omega' \frac{CC'\omega\omega'}{16\pi r^2} \cos ((r + t + \delta)(\omega - \omega')).
\] (B4)

Next, for ingoing null waves, the total energy on the surface of a sphere is the same as the flux through the sphere. The flux through the surface of a sphere is therefore

\[
\Phi = \int \Sigma T_{00}^{\text{eff}} r^2 d\Omega = \varepsilon^{-1} 2\pi \int_{-\pi}^{\pi} \sin \theta d\theta \left( P_\ell^2 (\cos \theta) \left( 1 + (1 - \ell(\ell + 1))^2 \right) - 2 \cos \theta P_\ell' (\cos \theta) \left( \sin^2 \theta P_\ell'' (\cos \theta) - \cos \theta P_\ell' (\cos \theta) \right) \right)
\times \int d\omega \int d\omega' \frac{CC'\omega\omega'}{16\pi} \cos ((r + t + \delta)(\omega - \omega')).
\] (B5)

Consider first the angular integral. Let \( x = \cos \theta \). The defining differential equation for Legendre Polynomials is

\[
(1 - x^2) P''_\ell(x) - 2x P'_\ell(x) + \ell(\ell + 1) P_\ell(x) = 0.
\] (B6)

and the associated Legendre functions may be defined by

\[
P^{(m)}_\ell = (-1)^m (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x).
\] (B7)

They satisfy the orthonormality conditions

\[
\int_{-1}^{1} dx P^{(m)}_k(x) P^{(m)}_\ell = \frac{2(\ell + m)!}{(2\ell + 1)(\ell - m)!} \delta_{k\ell}
\] (B8)

and

\[
\int_{-1}^{1} dx \frac{P^{(n)}_\ell(x) P^{(m)}_\ell}{(1 - x^2)} = \begin{cases} 
0, & m \neq n \\
\frac{(\ell+m)!}{m(\ell-m)!}, & m = n \neq 0 \\
\infty, & m = n = 0
\end{cases}.
\] (B9)

Using Eq. (B6) to rewrite \( P''_\ell(x) \) in terms of \( P_\ell(x) \) and \( P'_\ell(x) \), the angular integral in Eq. (B5) becomes

\[
\int_{-1}^{1} dx \left[ P_\ell^2 (x) \left( 1 + (1 - \ell(\ell + 1))^2 \right) - 2x^2 (P'_\ell(x))^2 + 2\ell(\ell + 1)x P'_\ell(x) P_\ell(x) \right].
\] (B10)

The first term satisfies the equal \( \ell \) orthonormality condition, Eq. (B8), so

\[
(1 + (1 - \ell(\ell + 1))) \int_{-1}^{1} dx P_\ell^2(x) = (1 + (1 - \ell(\ell + 1))) \frac{2}{2\ell + 1}.
\] (B11)
The second term is a little more complicated. Begin by integrating by parts
\[ \int_{-1}^{1} dx x^2 (P'_\ell(x))^2 = [x^2 P'_\ell(x)P_\ell(x)]_{-1}^{1} - \int_{-1}^{1} dx [2xP'_\ell(x)P'_\ell(x) + x^2 P_\ell(x)P''_\ell(x)]. \tag{B12} \]
Using Eq. (B6), \( x^2 P''_\ell(x) = P''_\ell(x) - 2xP'_\ell(x) + \ell(\ell+1)P_\ell(x) \) and the preceding is now
\[ [x^2 P'_\ell(x)P_\ell(x)]_{-1}^{1} - \ell(\ell+1) \int_{-1}^{1} dx P'_\ell(x)P_\ell(x) - \int_{-1}^{1} dx P_\ell(x)P''_\ell(x). \tag{B13} \]

The second term here satisfies the orthonormality condition, while integrating the third term by parts and using Eq. (B7) gives
\[ [x^2 P'_\ell(x)P_\ell(x)]_{-1}^{1} - [P'_\ell(x)P_\ell(x)]_{-1}^{1} - \frac{2\ell(\ell+1)}{2\ell+1} + \int_{-1}^{1} dx \frac{P''_\ell(x)P_\ell(x)}{(1-x^2)}, \tag{B14} \]
where the first and second terms cancel. Meanwhile, the last term satisfies Eq. (B9) and finally
\[ \int_{-1}^{1} dx x^2 (P'_\ell(x))^2 = \frac{(\ell+1)!}{(\ell-1)!} - \frac{2\ell(\ell+1)}{2\ell+1}. \tag{B15} \]

Consider now the last term of (B10). An integration by parts gives
\[ \int_{-1}^{1} dx x P'_\ell(x)P_\ell(x) = [xP_\ell(x)P'_\ell(x)]_{-1}^{1} - \int_{-1}^{1} dx x P'_\ell(x)P_\ell(x) - \int_{-1}^{1} dx P'_\ell(x)P_\ell(x). \tag{B16} \]

The second term on the right is the same as the original integral, while the third integral on the right satisfies Eq. (B8). Furthermore, \( P_\ell(1) = 1 \) while \( P_\ell(-1) = (-1)^\ell \) so that \([xP_\ell(x)P'_\ell(x)]_{-1}^{1} = 2\) and therefore
\[ \int_{-1}^{1} dx x P'_\ell(x)P_\ell(x) = \frac{2\ell}{2\ell+1}. \tag{B17} \]

Adding up all the results gives
\[ \int_{-\pi}^{\pi} \sin \theta d\theta \left( P'^2_\ell(\cos \theta) \right) (1 + (1 - \ell(\ell+1))^2)
- 2 \cos \theta P'_\ell(\cos \theta) \left( \sin^2 \theta P''_\ell(\cos \theta) - \cos \theta P'_\ell(\cos \theta) \right)
= \frac{2}{2\ell+1} \left[ 2 + \ell^2(\ell+1)(\ell+3) \right] - \frac{2(\ell+1)!}{(\ell-1)!} \tag{B18} \]

The flux through the surface of the sphere is now
\[ \Phi = \epsilon^{-1} \tilde{L} \int d\omega \int d\omega' \frac{CC'\omega\omega'}{4} \cos \left( (r + t + \delta)(\omega - \omega') \right). \tag{B19} \]
where
\[ \tilde{L} = \frac{1}{2\ell+1} \left[ 2 + \ell^2(\ell+1)(\ell+3) \right] - \frac{(\ell+1)!}{(\ell-1)!}. \tag{B20} \]
The total energy of the wavepacket may now be found by integrating the flux through the surface of the sphere for all time, \( \int dt \Phi \). Writing the cosine function in terms of exponentials, it is clear that
\[
\int dt \cos \left( (r + t + \delta)(\omega - \omega') \right) = 2\pi \delta(\omega - \omega') \cos((r + \delta)(\omega - \omega')).
\]
The delta function takes care of the integration over \( \omega' \), and setting the total energy to \( \omega / 2 \) \((\hbar = 1)\) gives the condition
\[
\varepsilon^{-1} \bar{L} \int d\omega \frac{C^2 \pi \omega^2}{2} = \frac{1}{2} \omega. \tag{B21}
\]
For a wavepacket sharply peaked in frequency, we may approximate
\[
\int_{j\varepsilon}^{(j+1)\varepsilon} \omega^2 d\omega \approx \omega^2 \Delta \omega = \omega^2 \varepsilon, \tag{B22}
\]
then
\[
A_{\ell m}(\omega) = C = \frac{1}{\sqrt{\pi \bar{L} \omega}}. \tag{B23}
\]
This normalization constant is the one contained in the radial function \( Z(r) \), so that when the normalized wavepacket is written
\[
\Psi^{+}_{(j)\mu \nu} = \int_{j\varepsilon}^{(j+1)\varepsilon} \frac{d\omega}{\sqrt{\pi L \varepsilon \omega}} e^{-i\omega \delta \nu} \Psi^{+}_{\mu \nu}, \tag{B24}
\]
it is understood that the radial function takes the value \( Z(r_* \to \infty) = e^{-i\omega r_*} \).

**APPENDIX C: SQUEEZED STATES**

A squeezed state is the natural state for a quantum mechanically created particle occupying an in-vacuum state represented in an out-Fock space. Squeezed quantum states are generated via the unitary displacement and squeeze operators. This Appendix provides a brief summary of the relevant ideas and results for squeezed states, primarily following the notation found in Ref. [40]; see also Refs. [41, 42, 43].

Squeezed states are generated using the unitary displacement and squeeze operators. The displacement operator is
\[
D(\alpha) = e^{\alpha \hat{a} - \alpha^* \hat{a}^\dagger}. \tag{C1}
\]
One can check that \( D(\alpha) \) transforms \( \hat{a} \) and \( \hat{a}^\dagger \) as
\[
D^\dagger(\alpha) \hat{a} D(\alpha) = \hat{a} + \alpha \quad \text{and} \quad D^\dagger(\alpha) \hat{a}^\dagger D(\alpha) = \hat{a}^\dagger + \alpha^*. \tag{C2}
\]
The squeeze operator is
\[
\hat{S}(\zeta) = \exp \left[ \frac{1}{2} \zeta^* \hat{a}^2 - \frac{1}{2} \zeta (\hat{a}^\dagger)^2 \right], \quad \zeta = \rho e^{i\theta}. \tag{C3}
\]
Here the squeezing parameter, \( \zeta \), is an arbitrary complex number. One may show that the squeeze operator transforms \( \hat{a} \) and \( \hat{a}^\dagger \) as
\[
S^\dagger(\zeta) \hat{a} S(\zeta) = \hat{a} \cosh \rho - \hat{a}^\dagger e^{i\theta} \sinh \rho \tag{C4a}
\]
and
\[
S^\dagger(\zeta) \hat{a}^\dagger S(\zeta) = \hat{a}^\dagger \cosh \rho - \hat{a} e^{-i\theta} \sinh \rho. \tag{C4b}
\]
1. Multimode Squeezed State

Suppose the state $|0, \zeta\rangle$ is a multimode squeezed state of the form

$$|0, \zeta\rangle = \prod_{\ell=n}^{m} S(\zeta_{\ell})|0\rangle. \quad (C5)$$

The expectation values under consideration require the calculation of terms such as $\sum_{j} \sum_{k} \langle \zeta, 0 | \hat{a}_{j} \hat{a}_{k} | 0, \zeta \rangle$, $\sum_{j} \sum_{k} \langle \zeta, 0 | \hat{a}_{j} \hat{a}_{k} \hat{a}_{r} \hat{a}_{s} | 0, \zeta \rangle$, etc. Consider for example

$$\langle \zeta, 0 | \hat{a}_{j} \hat{a}_{k} | 0, \zeta \rangle = \langle 0 | \prod_{\ell=n}^{m} S^{\dagger}(\zeta_{\ell}) \hat{a}_{j} \hat{a}_{k} \prod_{\ell=n}^{m} S(\zeta_{\ell}) | 0 \rangle. \quad (C6)$$

This expands as

$$\langle 0 | (S^{\dagger}(\zeta_{m})S^{\dagger}(\zeta_{m-1})...S^{\dagger}(\zeta_{n})\hat{a}_{j}S(\zeta_{n})S(\zeta_{n+1})...S(\zeta_{m}))$$

$$\times (S^{\dagger}(\zeta_{m})S^{\dagger}(\zeta_{m-1})...S^{\dagger}(\zeta_{n})\hat{a}_{k}S(\zeta_{n})S(\zeta_{n+1})...S(\zeta_{m})) | 0 \rangle. \quad (C7)$$

The action of $S(\zeta_{\ell})$ on $\hat{a}_{j}^{\dagger}$ and $\hat{a}_{j}$ is

$$S^{\dagger}(\zeta_{\ell})\hat{a}_{j}S(\zeta_{\ell}) = (\hat{a}_{j}^{\dagger} \cosh \rho_{\ell} - \hat{a}_{j} \sinh \rho_{\ell})\delta_{j\ell} + \hat{a}_{j}^{\dagger}(1 - \delta_{j\ell}) \quad (C8a)$$

and

$$S^{\dagger}(\zeta_{\ell})\hat{a}_{j}S(\zeta_{\ell}) = (\hat{a}_{j} \cosh \rho_{\ell} - \hat{a}_{j}^{\dagger} \sinh \rho_{\ell})\delta_{j\ell} + \hat{a}_{j}(1 - \delta_{j\ell}). \quad (C8b)$$

The action of a range of squeezing is then

$$S^{\dagger}(\zeta_{m})...S^{\dagger}(\zeta_{n})\hat{a}_{j}^{\dagger}S(\zeta_{n})...S(\zeta_{m}) = (\hat{a}_{j}^{\dagger} \cosh \rho_{j} - \hat{a}_{j} \sinh \rho_{j})\Theta_{nm}(j) + \hat{a}_{j}^{\dagger}(1 - \Theta_{nm}(j)), \quad (C9)$$

where the integer step function is defined as

$$\Theta_{nm}(j) = \begin{cases} 1, & n \leq j \leq m, \\ 0, & \text{otherwise}. \end{cases} \quad (C10)$$

It follows that

$$\sum_{jk} \langle \zeta, 0 | \hat{a}_{j} \hat{a}_{k} | 0, \zeta \rangle = \sum_{jk} \langle \zeta, 0 | (\hat{a}_{j}^{\dagger} \cosh \rho_{j} - \hat{a}_{j} \sinh \rho_{j})\Theta_{nm}(j) + \hat{a}_{j}^{\dagger}(1 - \Theta_{nm}(j))$$

$$\times (\hat{a}_{k} \cosh \rho_{k} - \hat{a}_{k}^{\dagger} \sinh \rho_{k})\Theta_{nk}(k) + \hat{a}_{k}(1 - \Theta_{nk}(k)) \rangle | 0, \zeta \rangle$$

$$= \sum_{jk} [(-\Theta_{nm}(j) \sinh \rho_{j} (-\Theta_{nm}(k) \sinh \rho_{k})] \delta_{jk}. \quad (C11)$$

To summarize, one finds

$$\sum_{jk} \langle \zeta, 0 | \hat{a}_{j} \hat{a}_{k} | 0, \zeta \rangle = \sum_{jk} [(1 + \Theta_{nm}(j)(\cosh \rho_{j} - 1)) (-\Theta_{nm} \sinh \rho_{k})] \delta_{jk}, \quad (C12a)$$
\[
\sum_{jk} \langle \zeta, 0 | \hat{a}_j^\dagger \hat{a}_k^\dagger | 0, \zeta \rangle = \sum_{jk} \left[ (-\Theta_{nm}(j) \sinh \rho_j) (1 + \Theta_{nm}(k) \cosh \rho_k) \right] \delta_{jk}, \quad (C12b)
\]
\[
\sum_{jk} \langle \zeta, 0 | \hat{a}_j^\dagger \hat{a}_k^\dagger | 0, \zeta \rangle = \sum_{jk} \left[ (-\Theta_{nm}(j) \sinh \rho_j) (-\Theta_{nm} \sinh \rho_k) \right] \delta_{jk}, \quad (C12c)
\]
and
\[
\sum_{jk} \langle \zeta, 0 | \hat{a}_j^\dagger \hat{a}_k | 0, \zeta \rangle = \sum_{jk} \left[ (1 + \Theta_{nm}(j)(\cosh \rho_j - 1)) (1 + \Theta_{nm}(k) \cosh \rho_k) \right] \delta_{jk}. \quad (C12d)
\]

Note that the result of Eq. (C12d) contains a \(\delta_{jk}\) that makes the sum divergent. Renormalization is therefore taken to correspond to restricting the sum over modes to those occupying an excited squeezed state mode, i.e. \(\sum_{j=n}^m\). For products of two operators this is equivalent to normal ordering; equations (C12a)-(C12d) become

\[
\sum_{jk} \langle \zeta, 0 | \hat{a}_j^\dagger \hat{a}_k | 0, \zeta \rangle = \sum_{jk} - \cosh \rho_j \sinh \rho_k \delta_{jk}, \quad (C13a)
\]
\[
\sum_{jk} \langle \zeta, 0 | \hat{a}_j^\dagger \hat{a}_k^\dagger | 0, \zeta \rangle = \sum_{jk} - \sinh \rho_j \cosh \rho_k \delta_{jk}, \quad (C13b)
\]
\[
\sum_{jk} \langle \zeta, 0 | \hat{a}_j^\dagger \hat{a}_k | 0, \zeta \rangle = \sum_{jk} \sinh \rho_j \sinh \rho_k \delta_{jk}, \quad (C13c)
\]
and
\[
\sum_{jk} \langle \zeta, 0 | \hat{a}_j^\dagger \hat{a}_k^\dagger | 0, \zeta \rangle = \sum_{jk} \cosh \rho_j \cosh \rho_k \delta_{jk}. \quad (C13d)
\]

The study of scalar field stress tensor induced fluctuations further requires the use of the expectation value of four-operator products such as

\[
\sum_{j} \sum_{k} \sum_{r} \sum_{s} \langle \zeta, 0 | \hat{a}_j \hat{a}_k \hat{a}_j^\dagger \hat{a}_s^\dagger | 0, \zeta \rangle =
\sum_{jkrs} \langle 0 | \left( \hat{a}_j \cosh \rho_j - \hat{a}_j^\dagger \sinh \rho_j \right) \Theta_{nm}(j) + \hat{a}_j^\dagger (1 - \Theta_{nm}(j)) \right]
\times \left( \hat{a}_k \cosh \rho_k - \hat{a}_k^\dagger \sinh \rho_k \right) \Theta_{nm}(k) + \hat{a}_k^\dagger (1 - \Theta_{nm}(k)) \right]
\times \left( \hat{a}_r \cosh \rho_r - \hat{a}_r^\dagger \sinh \rho_r \right) \Theta_{nm}(r) + \hat{a}_r^\dagger (1 - \Theta_{nm}(r)) \right]
\times \left( \hat{a}_s^\dagger \cosh \rho_s - \hat{a}_s \sinh \rho_s \right) \Theta_{nm}(s) + \hat{a}_s (1 - \Theta_{nm}(s)) \right] |0\rangle \quad (C14a)
\]
\[
= \sum_{jkrs} \left[ (1 + \Theta_{nm}(j)(\cosh \rho_j - 1))(1 + \Theta_{nm}(k)(\cosh \rho_k - 1)) \right]
\times (1 + \Theta_{nm}(r)(\cosh \rho_r - 1))(1 + \Theta_{nm}(s)(\cosh \rho_s - 1))\delta_{jr}\delta_{ks} + \delta_{ja}\delta_{kr} \right]
\times \left[ (1 + \Theta_{nm}(j)(\cosh \rho_j - 1))(\cosh \rho_k \Theta_{nm}(k)) \right]
\times (1 + \Theta_{nm}(r)(\cosh \rho_r - 1))(\cosh \rho_s \Theta_{nm}(s)) \delta_{jk}\delta_{rs} \right]. \quad (C14b)
\]
In this case there is the term \((\delta_{jr}\delta_{ks} + \delta_{js}\delta_{kr})\) which makes the sum over modes divergent. Again, renormalization corresponds to restricting the sums over modes to those modes which lie in the range of squeezing. As discussed in in Sect. VI, restricting the sum over modes this way for the four-operator products corresponds, in the limit \(\rho_i \gg 1\), to retaining only the fully normal ordered term. The required results are presented here, incorporating the restriction on the sum over modes, but omitting the summation symbol for notational simplification.

\[
\langle \zeta, 0| \hat{a}_j \hat{a}_k \hat{a}_r \hat{a}_s |0, \zeta\rangle = \cosh \rho_j \cosh \rho_k \sinh \rho_r \sinh \rho_s (\delta_{jr}\delta_{ks} + \delta_{js}\delta_{kr}) \\
+ \cosh \rho_j \sinh \rho_k \cosh \rho_r \sinh \rho_s (\delta_{jk}\delta_{rs}), \quad (C15a)
\]

\[
\langle \zeta, 0| \hat{a}_j \hat{a}_k \hat{a}_r \hat{a}_s^\dagger |0, \zeta\rangle = \cosh \rho_j \cosh \rho_k \cosh \rho_r \cosh \rho_s (\delta_{jr}\delta_{ks} + \delta_{js}\delta_{kr}) \\
+ \cosh \rho_j \sinh \rho_k \cosh \rho_r \sinh \rho_s (\delta_{jk}\delta_{rs}), \quad (C15b)
\]

\[
\langle \zeta, 0| \hat{a}_j \hat{a}_k \hat{a}_r \hat{a}_s^\dagger |0, \zeta\rangle = - \cosh \rho_j \cosh \rho_k \sinh \rho_r \cosh \rho_s (\delta_{jr}\delta_{ks} + \delta_{js}\delta_{kr}) \\
- \cosh \rho_j \sinh \rho_k \cosh \rho_r \sinh \rho_s (\delta_{jk}\delta_{rs}), \quad (C15c)
\]

\[
\langle \zeta, 0| \hat{a}_j \hat{a}_k \hat{a}_r \hat{a}_s^\dagger |0, \zeta\rangle = - \cosh \rho_j \cosh \rho_k \cosh \rho_r \sinh \rho_s (\delta_{jr}\delta_{ks} + \delta_{js}\delta_{kr}) \\
- \cosh \rho_j \sinh \rho_k \sinh \rho_r \sinh \rho_s (\delta_{jk}\delta_{rs}), \quad (C15d)
\]

\[
\langle \zeta, 0| \hat{a}_j^\dagger \hat{a}_k^\dagger \hat{a}_r \hat{a}_s |0, \zeta\rangle = \sinh \rho_j \sinh \rho_k \sinh \rho_r \sinh \rho_s (\delta_{jr}\delta_{ks} + \delta_{js}\delta_{kr}) \\
+ \sinh \rho_j \cosh \rho_k \cosh \rho_r \sinh \rho_s (\delta_{jk}\delta_{rs}), \quad (C15e)
\]

\[
\langle \zeta, 0| \hat{a}_j^\dagger \hat{a}_k^\dagger \hat{a}_r \hat{a}_s^\dagger |0, \zeta\rangle = \sinh \rho_j \sinh \rho_k \cosh \rho_r \cosh \rho_s (\delta_{jr}\delta_{ks} + \delta_{js}\delta_{kr}) \\
+ \sinh \rho_j \cosh \rho_k \cosh \rho_r \sinh \rho_s (\delta_{jk}\delta_{rs}), \quad (C15f)
\]

\[
\langle \zeta, 0| \hat{a}_j^\dagger \hat{a}_k^\dagger \hat{a}_r \hat{a}_s^\dagger |0, \zeta\rangle = - \sinh \rho_j \sinh \rho_k \sinh \rho_r \cosh \rho_s (\delta_{jr}\delta_{ks} + \delta_{js}\delta_{kr}) \\
- \sinh \rho_j \cosh \rho_k \cosh \rho_r \sinh \rho_s (\delta_{jk}\delta_{rs}), \quad (C15g)
\]

\[
\langle \zeta, 0| \hat{a}_j^\dagger \hat{a}_k^\dagger \hat{a}_r \hat{a}_s^\dagger |0, \zeta\rangle = - \sinh \rho_j \sinh \rho_k \cosh \rho_r \sinh \rho_s (\delta_{jr}\delta_{ks} + \delta_{js}\delta_{kr}) \\
- \sinh \rho_j \cosh \rho_k \sinh \rho_r \sinh \rho_s (\delta_{jk}\delta_{rs}), \quad (C15h)
\]

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\[ \langle \zeta, 0 | \hat{a}_j \hat{a}^+_k \hat{a}_r \hat{a}^+_s | 0, \zeta \rangle = - \cosh \rho_j \sinh \rho_k \sinh \rho_r \sinh \rho_s (\delta_{jr} \delta_{ks} + \delta_{js} \delta_{kr}) \]
\[ - \cosh \rho_j \cosh \rho_k \cosh \rho_r \sinh \rho_s (\delta_{jk} \delta_{rs}), \quad (C15i) \]

\[ \langle \zeta, 0 | \hat{a}_j \hat{a}^+_k \hat{a}_r \hat{a}^+_s | 0, \zeta \rangle = - \cosh \rho_j \sinh \rho_k \cosh \rho_r \cosh \rho_s (\delta_{jr} \delta_{ks} + \delta_{js} \delta_{kr}) \]
\[ - \cosh \rho_j \cosh \rho_k \sinh \rho_r \cosh \rho_s (\delta_{jk} \delta_{rs}), \quad (C15j) \]

\[ \langle \zeta, 0 | \hat{a}_j \hat{a}^+_k \hat{a}_r \hat{a}^+_s | 0, \zeta \rangle = \cosh \rho_j \sinh \rho_k \cosh \rho_r \cosh \rho_s (\delta_{jr} \delta_{ks} + \delta_{js} \delta_{kr}) \]
\[ + \cosh \rho_j \cosh \rho_k \cosh \rho_r \sinh \rho_s (\delta_{jk} \delta_{rs}), \quad (C15k) \]

\[ \langle \zeta, 0 | \hat{a}_j \hat{a}^+_k \hat{a}_r \hat{a}^+_s | 0, \zeta \rangle = \cosh \rho_j \sinh \rho_k \sinh \rho_r \sinh \rho_s (\delta_{jr} \delta_{ks} + \delta_{js} \delta_{kr}) \]
\[ + \cosh \rho_j \cosh \rho_k \sinh \rho_r \sinh \rho_s (\delta_{jk} \delta_{rs}), \quad (C15l) \]

\[ \langle \zeta, 0 | \hat{a}_j \hat{a}^+_k \hat{a}_r \hat{a}^+_s | 0, \zeta \rangle = - \sinh \rho_j \cosh \rho_k \sinh \rho_r \sinh \rho_s (\delta_{jr} \delta_{ks} + \delta_{js} \delta_{kr}) \]
\[ - \sinh \rho_j \sinh \rho_k \cosh \rho_r \sinh \rho_s (\delta_{jk} \delta_{rs}), \quad (C15m) \]

\[ \langle \zeta, 0 | \hat{a}_j \hat{a}^+_k \hat{a}_r \hat{a}^+_s | 0, \zeta \rangle = - \sinh \rho_j \cosh \rho_k \cosh \rho_r \cosh \rho_s (\delta_{jr} \delta_{ks} + \delta_{js} \delta_{kr}) \]
\[ - \sinh \rho_j \sinh \rho_k \cosh \rho_r \cosh \rho_s (\delta_{jk} \delta_{rs}), \quad (C15n) \]

\[ \langle \zeta, 0 | \hat{a}_j \hat{a}^+_k \hat{a}_r \hat{a}^+_s | 0, \zeta \rangle = \sinh \rho_j \cosh \rho_k \cosh \rho_r \cosh \rho_s (\delta_{jr} \delta_{ks} + \delta_{js} \delta_{kr}) \]
\[ + \sinh \rho_j \sinh \rho_k \cosh \rho_r \cosh \rho_s (\delta_{jk} \delta_{rs}), \quad (C15o) \]

and

\[ \langle \zeta, 0 | \hat{a}_j \hat{a}^+_k \hat{a}_r \hat{a}^+_s | 0, \zeta \rangle = \sinh \rho_j \cosh \rho_k \cosh \rho_r \sinh \rho_s (\delta_{jr} \delta_{ks} + \delta_{js} \delta_{kr}) \]
\[ + \sinh \rho_j k \sinh \rho_k \sinh \rho_r \sinh \rho_s (\delta_{jk} \delta_{rs}). \quad (C15p) \]

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