Three-particle model for the nonlinear response of frictional granular materials

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(Dated: November 8, 2022)

We propose a simple model comprising three particles to study the nonlinear mechanical response of jammed frictional granular materials under oscillatory shear. Owing to the introduction of the simple model, we obtain an exact analytical expression of the complex shear modulus for a system including many monodispersed disks, which satisfies a scaling law in the vicinity of the jamming point. These expressions perfectly reproduce the shear modulus of the many-body system with low strain amplitudes and friction coefficients. Even for disordered many-body systems, the model reproduces results by introducing a single fitting parameter.

Introduction — The rheological property of densely dispersed grains, e.g., granular materials, colloidal suspensions, and emulsions, plays an important role in physics and engineering. This rheological property mainly depends on the packing fraction \( \phi \) of the grains. The materials behave like fluids for \( \phi < \phi_J \) with jamming fraction \( \phi_J \) and exhibit a solid-like elastic response above \( \phi_J \) [1, 2]. In the linear response regime (i.e., for small strains), the shear modulus is characterized by the density of states [3, 4] and satisfies scaling laws [5, 6]. However, the linear response region becomes narrower as \( \phi \) approaches \( \phi_J \) [11, 12], and the nonlinear response becomes relevant [12, 20]. Under quasistatic oscillatory shear, the storage modulus \( G' \) decreases as the strain amplitude \( \gamma_0 \) increases [11, 21, 23]. The loss modulus \( G'' \) remains nonzero in the same region under the quasistatic shear [23, 24].

The theoretical analysis of densely dispersed grains is challenging as a typical many-body problem in non-equilibrium systems. To date, a few theoretical approaches have been proposed for systems related to frictionless particles. The scaling laws for the linear elastic response were derived in terms of the vibrational density of states [3, 4]. The Fourier analysis of particle trajectories helps generate semi-analytical expressions for \( G' \) and \( G'' \) [24]. Unfortunately, these theories cannot apply to frictional particles because of the history-dependent contact force [9, 23].

It is helpful to analyze a simple model with small degrees of freedom to understand the behavior of many-body systems, including densely dispersed grains. This approach has been used in statistical mechanics. The mean-field approximation of the Ising model is a typical example in which the system contains only one Ising spin under the influence of a self-consistently determined mean field [25]. For atomic liquids, a cell model, in which a single atom exists in a cage, was used to derive the equation of state [26, 27]. The coherent potential approximation for disordered solids has been used to understand electronic band structures [26]. The effective medium theory reveals the elastic response of random spring networks [29]. In addition, a simple model consisting of two particles was proposed to reproduce the liquid-solid phase transition [30]. The advantage of such few-body models is that we can obtain exact solutions. The qualitative behavior of the corresponding many-body systems can be determined based on the solutions of the few-body models. Thus, we adopt this approach to determine the nonlinear responses of the frictional dispersed grains.

This study proposes a model consisting of three identical particles to describe the mechanical response of jammed frictional granular materials under oscillatory shear. We demonstrate that the model reproduces the storage and loss moduli of many-particle systems (MPSs) without any fitting parameter if there is no disorder in the particle configuration. This model can be analytically solved for low-strain amplitudes and friction coefficients near the jamming point. We derive a scaling law for the complex shear modulus, which semi-quantitatively agrees with the numerical simulations of the MPS even if disorder exists with the introduction of a fitting parameter.

FIG. 1. (a) Schematic of the ordered MPS. (b) Schematic of the disordered MPS.

Three-Particle Model — We consider two-dimensional granular materials consisting of many frictional particles under oscillatory shear (Fig. 1). Moreover, we introduce a system of three identical particles to simply describe the MPS (Fig. 2). The MPS can contain polydisperse grains, while we assume that the three-particle model (TPM) is a
monodisperse system. In the TPM, the position \( \mathbf{r}_i(t) = (x_i(t), y_i(t)) \) of particle \( i \) with diameter \( d \) at time \( t \) is given by

\[
\mathbf{r}_1(t) = \left( \frac{\sqrt{3} \gamma(t) \ell}{4}, \frac{\sqrt{3} \ell}{4} \right),
\]

\[
\mathbf{r}_2(t) = \left( -\frac{\sqrt{3} \gamma(t) \ell}{4} - \frac{\ell}{2}, \frac{\sqrt{3} \ell}{4} \right),
\]

\[
\mathbf{r}_3(t) = \left( -\frac{\sqrt{3} \gamma(t) \ell}{4} + \frac{\ell}{2}, -\frac{3 \ell}{4} \right),
\]

where \( \ell = d(1 - \epsilon) \) defines the initial distance between particles with compressive strain \( \epsilon \ll 1 \) and \( d \) denotes the diameter of the particle. The compressive strain \( \epsilon \) in the TPM corresponds to \( \phi - \phi_1 \) in the MPS \[31\]. We apply shear strain as

\[
\gamma(\theta) = \gamma_0 \sin \theta
\]

with strain amplitude \( \gamma_0 \), phase \( \theta = \omega t \), and angular frequency \( \omega \). Note that we need at least three particles to realize a stable interlocking state.

We adopt the interaction force \( \mathbf{f}_{ij} \) between particles \( i \) and \( j \) given by

\[
\mathbf{f}_{ij} = \left( f^{(n)}_{ij} \mathbf{n}_{ij} + f^{(t)}_{ij} \mathbf{t}_{ij} \right) H(r_{ij} - d),
\]

where \( f^{(n)}_{ij} \) and \( f^{(t)}_{ij} \) denote the normal and tangential forces between the particles \( i \) and \( j \) \[32\]. The distance between the particles \( i \) and \( j \) is \( r_{ij} = |\mathbf{r}_{ij}| \) with \( \mathbf{r}_{ij} := \mathbf{r}_i - \mathbf{r}_j = (x_{ij}, y_{ij}) \). Here, \( H(x) \) is Heaviside’s step function satisfying \( H(x) = 1 \) for \( x > 0 \) and \( H(x) = 0 \) otherwise. The normal and tangential unit vectors are denoted by \( \mathbf{n}_{ij} := \mathbf{r}_{ij}/r_{ij} = (n_{ij,x}, n_{ij,y}) \) and \( \mathbf{t}_{ij} := (-n_{ij,y}, n_{ij,x}) \), respectively. For simplicity, we do not consider the torque balance and, thus, the rotation of the grains. The effect of the rotation is discussed in Ref. \[31\].

The normal force is assumed to be

\[
f^{(n)}_{ij} = -k_n u^{(n)}_{ij}
\]

with the normal elastic constant \( k_n \) and normal relative displacement \( u^{(n)}_{ij} := r_{ij} - d \). Moreover, the tangential force is assumed to be

\[
f^{(t)}_{ij} = \min \left( f^{(t)}_{ij}, \mu f^{(n)}_{ij} \right) \text{sgn}(\tilde{f}^{(t)}_{ij}),
\]

where \( \tilde{f}^{(t)}_{ij} = -k_t u^{(t)}_{ij} \), \( k_t \) denotes the tangential elastic constant, and \( \mu \) denotes the friction coefficient. Here, \( \min(a, b) \) selects the smaller value between \( a \) and \( b \), \( \text{sgn}(x) = 1 \) for \( x \geq 0 \), and \( \text{sgn}(x) = -1 \) for \( x < 0 \). The tangential displacement \( u^{(t)}_{ij} \) satisfies \( \frac{d}{dt} u^{(t)}_{ij} = v^{(t)}_{ij} \) for \( |\tilde{f}^{(t)}_{ij}| < \mu f^{(n,el)}_{ij} \) with the tangential velocity \( v^{(t)}_{ij} = (\frac{d}{dt} \mathbf{r}_i - \frac{d}{dt} \mathbf{r}_j) \cdot \mathbf{t}_{ij} \), whereas \( u^{(t)}_{ij} \) remains unchanged for \( |\tilde{f}^{(t)}_{ij}| \geq \mu f^{(n,el)}_{ij} \). We refer to the contact with \( |\tilde{f}^{(t)}_{ij}| < \mu f^{(n)}_{ij} \) as the stick contact and the contact with \( |\tilde{f}^{(t)}_{ij}| \geq \mu f^{(n)}_{ij} \) as the slip contact. The tangential displacement, \( u^{(t)}_{ij} \), is initially set to zero.

The (symmetric contact) shear stress is given by

\[
\sigma = \sigma^{(n)} + \sigma^{(t)}
\]

with the normal component of \( \sigma \)

\[
\sigma^{(n)} = -\frac{1}{A} \sum_i \sum_{j>i} x_{ij} y_{ij} f^{(n)}_{ij}
\]

and tangential component of \( \sigma \)

\[
\sigma^{(t)} = -\frac{1}{2A} \sum_i \sum_{j>i} x^2_{ij} - y^2_{ij} f^{(t)}_{ij}.
\]

Here, \( A \) corresponds to the area of the system, and we choose \( A = \sqrt{3} \ell^2 / 2 \) \[31\]. The pressure is given by

\[
P = \frac{1}{2A} \sum_i \sum_{j>i} (x_{ij} f_{ij,x} + y_{ij} f_{ij,y}).
\]

As we are interested in quasistatic processes, we do not consider the kinetic parts of \( \sigma \) and \( P \). After several cycles of oscillatory shear, \( \sigma(\theta) \) becomes periodic. The storage and loss moduli are given by \[33\]

\[
G' = \frac{1}{\pi} \int_0^{2\pi} d\theta \sigma(\theta) \sin \theta / \gamma_0,
\]

\[
G'' = \frac{1}{\pi} \int_0^{2\pi} d\theta \sigma(\theta) \cos \theta / \gamma_0.
\]

**Theoretical analysis**—Assuming \( \gamma_0 \ll \epsilon \ll 1 \), we analytically obtain \( G' \) and \( G'' \) for the TPM. The derivation of the analytical results can be found in \[31\].

First, the normal component of the shear stress is given by \[31\]

\[
\sigma^{(n)}(\theta) = \frac{\sqrt{3} k_n \gamma(\theta)}{4}.
\]

The tangential component of the shear stress is given by

\[
\sigma^{(t)}(\theta) = \frac{\sqrt{3} k_t \gamma(\theta)}{4}
\]

for \( \gamma_0 < \gamma_c(\mu) \) with a critical amplitude

\[
\gamma_c(\mu) = \frac{4 \mu k_n \epsilon}{3 k_t},
\]
which characterizes the transition from stick-to-slip states in the contact between the particles. For \( \gamma_0 \geq \gamma_c(\mu) \), the tangential component of the shear stress is given by

\[
\sigma^{(1)}(\theta) = \begin{cases} 
\frac{\mu k_n\varepsilon}{\sqrt{3}}, & 0 \leq \theta < \frac{\pi}{2} \\
\frac{\mu k_n\varepsilon}{\sqrt{3}} + \sqrt{3}(\gamma(\theta) - \gamma_0), & \frac{\pi}{2} \leq \theta < \frac{3\pi}{2} \\
-\frac{\mu k_n\varepsilon}{\sqrt{3}} + \sqrt{3}(\gamma(\theta) + \gamma_0), & \frac{3\pi}{2} \leq \theta < 2\pi,
\end{cases}
\]

where \( \Theta = \cos^{-1}(1 - 2\gamma_c(\mu)/\gamma_0) \). Regions with \( \frac{\pi}{2} \leq \theta < \frac{3\pi}{2} + \Theta \) and \( \frac{3\pi}{2} \leq \theta < 2\pi + \Theta \) correspond to the stick state, and the other regions correspond to the slip state. Owing to this transition in the contact, the stress–strain curve given by Eqs. (14)–(17) exhibits a hysteresis loop, which is related to the dependence of the shear modulus on \( \gamma_0 \), as explained in Ref. [31].

Substituting Eqs. (8) and (14)–(17) into Eq. (13), we obtain the storage modulus as

\[
G' = \begin{cases} 
\sqrt{3}(k_n + k_t), & \gamma_0 \leq \gamma_c(\mu) \\
\frac{\pi}{3} \left\{ k_n + \frac{k_t}{\Theta - \sin \Theta \Theta \cos \Theta} \right\}, & \gamma_0 > \gamma_c(\mu)
\end{cases}
\]

(18)

As \( \gamma_0 \) increases beyond \( \gamma_c(\mu) \), \( G' \) decreases from a higher value to a lower value. The corresponding behavior has been observed in the MPS in previous studies [9, 23].

Substituting Eqs. (8) and (14)–(17) into Eq. (12), we obtain the loss modulus given by

\[
G'' = \begin{cases} 
0, & \gamma_0 \leq \gamma_c(\mu) \\
\frac{\pi}{3} \left\{ k_n + \frac{k_t}{\Theta - \sin \Theta \Theta \cos \Theta} \right\}, & \gamma_0 > \gamma_c(\mu).
\end{cases}
\]

(19)

The loss modulus \( G'' \) is zero for \( \gamma_0 < \gamma_c(\mu) \), whereas \( G'' \) sharply increases with \( \gamma_0 \) when \( \gamma_0 \) exceeds \( \gamma_c(\mu) \) and decreases to 0 after a peak. The behavior of \( G'' \) for the TPM qualitatively reproduces that of the MPS in previous studies [22].

The pressure \( P_0 \) with \( \gamma = 0 \) is also obtained as [31]

\[
P_0 = \sqrt{3}k_n\varepsilon.
\]

(20)

Equations (16), (18), (19), and (20) satisfy scaling laws for a given \( \epsilon \) as

\[
G' = G'_M(\mu)G'(\frac{k_t\gamma_0}{\mu P_0(\mu)}),
\]

(21)

\[
G'' = G''_M(\mu)G''(\frac{k_t\gamma_0}{\mu P_0(\mu)}),
\]

(22)

where \( G'(x) \) and \( G''(x) \) denote scaling functions. The maximum values of \( G' \) and \( G'' \) are denoted as \( G'_M \) and \( G''_M \), respectively. In the TPM, they are obtained as

\[
G'_M = \sqrt{3}(k_n + k_t)/4, \quad G''_M = \sqrt{3}k_t/(4\pi),
\]

(23)

\[
G'(x) = \begin{cases} 
1, & x \leq x_c, \\
\frac{1}{(1 + \frac{k_t}{\mu k_n}(T(x) - S(x)))} / (1 + \frac{k_t}{3\mu}), & x > x_c,
\end{cases}
\]

(24)

\[
G''(x) = \begin{cases} 
0, & x \leq x_c, \\
1 - \cos^2 T(x), & x > x_c
\end{cases}
\]

(25)

with \( T(x) = \cos^{-1}(1 - 2\epsilon x/\sqrt{\pi}) \), \( S(x) = \sin(2T(x))/2 \), and \( x_c = 4/(3\sqrt{3}) \).

Comparison with the MPS — We demonstrate the relevance of the TPM analysis based on the simulation of a two-dimensional MPS consisting of \( N \) frictional particles. First, we consider a system corresponding to the TPM, where all the particles are identical and initially placed on the triangular lattice with a unit length \( \ell \) (Fig. 1(a)). Next, we consider a bidisperse system where the number of particles with diameter \( d \) is equal to that of particles with diameter \( d/1.4 \), and the particles are randomly placed with packing fraction \( \phi \) (Fig. 1(b)). In both systems, the shear strain given by Eq. (4) is applied for \( N_c \) cycles using the SLLOD equation under the Lees–Edwards boundary condition [34]. In addition to the interactions defined in Eqs. (6) and (7), we adopt the dissipative force characterized by normal and tangential viscous constants \( \eta_n \) and \( \eta_t \). The systems are detailed in Ref. [31]. We measure \( G' \), \( G'' \), and \( P_0 \) in the last cycle using Eq. (11)–(13). The mass densities of the particles are identical. For the ordered MPS shown in Fig. 1(a), we use \( N = 64, k_t/k_n = 1.0, \) and \( \epsilon = 0.001 \), whereas \( N = 1000, k_t/k_n = 0.2, \) and \( \phi = 0.87 \) are used for the disordered MPS shown in Fig. 1(b). In both systems, the other parameters are identical: \( N_c = 20, \eta_n = \eta_t = \sqrt{mk_n}, \) and \( \omega = 0.0001\sqrt{m/k_n} \) with a mass \( m \) of larger particles.

![Fig. 3. Storage modulus \( G' \) against \( \gamma_0 \) with \( k_t/k_n = 1.0 \) and \( \epsilon = 0.001 \) for various values of \( \mu \). The points represent the results of the ordered MPS. The thin solid lines represent the analytical result given by Eq. (15). The vertical dashed lines represent the critical amplitude \( \gamma_c(\mu) \) given by Eq. (18) for \( \mu = 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}, \) and 1 from left to right. As shown in Fig. 3 we plot \( G' \) for the ordered MPS against \( \gamma_0 \) with \( k_t/k_n = 1.0 \) and \( \epsilon = 0.001 \) for various

\[
\gamma_0
\]

10^{-7} 10^{-6} 10^{-5} 10^{-4} 10^{-3} 10^{-2} 10^{-1}

1 0.8 0.6 0.4 0.2 0 0.0 0.2 0.4 0.6 0.8 1

\( G'/\mu \)

\( \gamma_0 \)
values of $\mu$ as points. Moreover, we plot the analytical results obtained using Eq. (18) as thin solid lines. Surprisingly, the results of the TPM agree with those of the MPS for $\gamma_0 < 0.003$ without any fitting parameters. As $\gamma_0$ increases beyond $\gamma_0(\mu)$ shown by the vertical dashed lines, $G'$ for $\mu > 0$ decreases and converges to a constant, which is equal to $G'$ for $\mu = 0$. For larger $\gamma_0$, $G'$ for the MPS decreases again, whereas the theoretical $G'$ based on the TPM is constant. This discrepancy results from the violation of condition $\gamma_0 \ll \epsilon$ for the analytical calculation. If we numerically obtain $G'$ and $G''$ without the assumption $\gamma_0 \ll \epsilon$, the TPM perfectly reproduces the MPS, including the second decrease, as mentioned in Ref. [31].

As shown in Fig. 4, we plot $G''$ for the MPS on the triangular lattice against $\gamma_0$ with $k_t/k_n = 1.0$ and $\epsilon = 0.001$ for various values of $\mu$ as points. Moreover, we plot the analytical results obtained using Eq. (19) as thin solid lines. The analytical result of the TPM agrees perfectly with that of the MPS for $\gamma_0 < 0.003$ without any fitting parameters. As $\gamma_0$ increases beyond $\gamma_0(\mu)$ shown by the vertical dashed lines, $G''$ for $\mu > 0$ increases from 0 and decreases after reaching a peak. The peak position of $G''$ against $\gamma_0$ increases with $\mu$. Thus, we fail to capture the behavior of $G''$ for $\mu = 1$.

Consider the disordered MPS shown in Fig. 1(b). The behaviors of $G'$ and $G''$ in this system are similar to those of the TPM [31]. Therefore, it is expected that the scaling laws in Eqs. (21) and (22) for a given $\epsilon$ in the TPM can be used even in this system with corresponding $\phi$. This expectation is verified by Fig. 5 in which we plot the scaled moduli $G'/G_M'$ and $G''/G_M''$ against the scaled strain $k_t\gamma_0/(\mu P_0)$ for various values of $\mu$ in the disordered MPS. Moreover, we plot the analytical results for the TPM obtained using Eqs. (24) and (25) as solid lines, which qualitatively agree with the MPS results. Here, we chose $k_t/k_n = 1.5$ for the TPM to fit the second plateau to that of the MPS. At present, we do not know the relationship between $\phi$ and the fitting parameter.

**FIG. 5.** (a) Scaled storage modulus $G'/G_M'$ against the scaled strain $k_t\gamma_0/(\mu P_0)$ with $\phi = 0.87$ and $k_t/k_n = 0.2$ for various values of $\mu$ in the disordered MPS. The solid line represents the analytical result of the TPM given by Eq. (24) with $k_t/k_n = 1.5$. (b) Scaled loss modulus $G''/G_M''$ against the scaled strain $k_t\gamma_0/(\mu P_0)$ with $\phi = 0.87$ and $k_t/k_n = 0.2$ for various values of $\mu$ in the disordered MPS. The solid line represents the analytical result of the TPM given by Eq. (25) with $k_t/k_n = 1.5$.

**Conclusions**—We proposed a TPM for the mechanical response of jammed frictional granular materials under oscillatory shear. We analytically obtained $G'$ and $G''$, which led to the derivation of scaling laws given by Eqs. (21) and (22). The results of the TPM agreed with those of the ordered MPS, whereas they are qualitatively similar to those of the MPS for disordered systems. These results indicate that disorder is not essential for the mechanical properties of jammed materials.

Although the values of the plateaus in $G'$ for disordered MPS depended on $\phi - \phi_1$ [3, 9], the corresponding values of the TPM were independent of $\phi - \phi_1$, as expressed in Eq. (18). This is because we ignored the disorder effects, resulting in the $\phi$-dependence of $G'$. To include the disorder effect, we regarded $k_t/k_n$ as a fitting parameter. In previous studies on models with small degrees of freedom, e.g., the coherent potential approximation [23, 28, 29], the corresponding fitting parameters were self-consistently determined. We will discuss the self-consistent determination of the parameter for the TPM in future studies.

**ACKNOWLEDGMENTS**

The authors thank K. Saitoh, D. Ishima, and S. Takada for fruitful discussions. This study was supported by JSPS KAKENHI under Grant Nos. JP19K03670 and JP21H01006.
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Supplemental Material

This supplemental material provides details that are not included in the main text. In Sec. [I] we derive the analytical expressions for σ and P₀ in the three-particle model (TPM). In Sec. [II] we demonstrate the stress–strain curve obtained from the analytical results, which explain the dependence of the shear modulus on γ₀. The shear modulus obtained numerically from the TPM is presented in Sec. [III]. In Sec. [IV] we show the results of the many-particle system (MPS) when the particles are initially placed on a triangular lattice. The effect of particle rotation is described in Sec. [V]. The details of the MPS with disorder are presented in Sec. [VI].

I. ANALYTICAL CALCULATION OF SHEAR STRESS AND PRESSURE

In this section, we briefly explain the derivation of the normal and tangential components of shear stress and pressure for a small value of γ₀. From Eqs. (12)–(14), the relative distance r₁₂(t) is given by

\[ r_{12}(t) = \left( \frac{\sqrt{3} \gamma(t)}{2} + 1, \frac{\sqrt{3} t}{2} \right), \quad \text{(11)} \]

\[ r_{13}(t) = \left( -\frac{\sqrt{3} \gamma(t)}{2} - 1, \frac{\sqrt{3} t}{2} \right), \quad \text{(12)} \]

\[ r_{23}(t) = (0, 0). \quad \text{(13)} \]

Substituting these equations into uⁿᵢⱼ = rᵢⱼ − d, the normal displacements are given by

\[ u_{12}ⁿ(t) = -\epsilon d + \frac{\sqrt{3}}{4} \gamma(t) + O(\gamma₀²), \quad \text{(14)} \]

\[ u_{13}ⁿ(t) = -\epsilon d - \frac{\sqrt{3}}{4} \gamma(t) + O(\gamma₀²), \quad \text{(15)} \]

\[ u_{23}ⁿ(t) = -\epsilon d. \quad \text{(16)} \]

Substituting these equations into Eq. (10), we obtain the normal force as

\[ f_{12}ⁿ = k_n \left( \epsilon d - \frac{\sqrt{3}}{4} \gamma(t) \right), \quad \text{(17)} \]

\[ f_{13}ⁿ = k_n \left( -\epsilon d - \frac{\sqrt{3}}{4} \gamma(t) \right), \quad \text{(18)} \]

\[ f_{23}ⁿ = k_n \epsilon d \quad \text{(19)} \]

up to O(γ₀²).

By differentiating Eqs. (11)–(13) with time t, we obtain the relative velocity as

\[ v_{12}(t) = \left( \frac{\sqrt{3} \dot{\gamma}(t)}{2}, 0 \right), \quad \text{(20)} \]

\[ v_{13}(t) = \left( -\frac{\sqrt{3} \dot{\gamma}(t)}{2}, 0 \right), \quad \text{(21)} \]

\[ v_{23}(t) = (0, 0) \quad \text{(22)} \]

with the strain rate \( \dot{\gamma}(t) = \frac{d}{dt} \gamma(t) \). The tangential unit vector is given by

\[ \mathbf{t}_{12}(t) = \left( \frac{\sqrt{3} \ell}{2}, \frac{\sqrt{3} \gamma(t)}{2} + \frac{1}{2} \ell \right)/|r_{12}|, \quad \text{(23)} \]

\[ \mathbf{t}_{13}(t) = \left( -\frac{\sqrt{3} \ell}{2}, \frac{\sqrt{3} \gamma(t)}{2} - \frac{1}{2} \ell \right)/|r_{13}|, \quad \text{(24)} \]

\[ \mathbf{t}_{23}(t) = (0, -1). \quad \text{(25)} \]

By considering the inner product of \( v_{ij} \) and \( t_{ij} \), the tangential velocity is given by

\[ v_{12}⁽⁽t⁾⁾ = \frac{3}{4} \ell \dot{\gamma}(t) + O(\gamma₀²), \quad \text{(26)} \]

\[ v_{13}⁽⁽t⁾⁾ = -\frac{3}{4} \ell \dot{\gamma}(t) + O(\gamma₀²), \quad \text{(27)} \]

\[ v_{23}⁽⁽t⁾⁾ = 0. \quad \text{(28)} \]

If the transition from the stick state to the slip state does not occur under oscillatory shear, the tangential displacement is obtained by integrating \( v_{ij}⁽⁽t⁾⁾ \) as

\[ u_{12}⁽⁽t⁾⁾ = u_{12}(t) = -\frac{3}{4} \ell \gamma(t) + O(\gamma₀²), \quad \text{(29)} \]

\[ u_{23}⁽⁽t⁾⁾ = 0. \quad \text{(30)} \]

Substituting these equations into \( f_{ij}⁽⁽t⁾⁾ = -k_i u_{ij}⁽⁽t⁾⁾ \) yields

\[ f_{12}⁽⁽t⁾⁾ = f_{13}⁽⁽t⁾⁾ = 3k_1 \gamma(t) \ell / 4, \quad \text{(31)} \]

\[ f_{23}⁽⁽t⁾⁾ = 0 \quad \text{(32)} \]

up to O(γ₀²). The condition that the transition does not occur is satisfied when \( f_{ij}⁽⁽t⁾⁾ < \mu k_i \ell \) for \( \gamma = \gamma₀ \). Using Eqs. (21) and (22) with the assumption \( \gamma₀ \ll \epsilon \), the condition is replaced by \( \gamma₀ < \gamma_c \) given by Eq. (10).

For \( \gamma > \gamma_c \), there exist regions where \( u_{ij}⁽⁽t⁾⁾ \) is unchanged in the slip state as

\[ u_{12}⁽⁽t⁾⁾ = \left\{ \begin{array}{ll}
\frac{\mu k_1 \epsilon d}{k_n}, & 0 \leq \theta(t) < \frac{\pi}{2} \\
\frac{\mu k_1 \epsilon d}{k_n} - \frac{3d(\gamma(t) - \gamma₀)}{4}, & \frac{\pi}{2} \leq \theta(t) < \frac{3\pi}{2} + \Theta \\
\frac{\mu k_1 \epsilon d}{k_n} - \frac{1}{2} \frac{3d(\gamma(t) + \gamma₀)}{4}, & \frac{3\pi}{2} + \Theta \leq \theta(t) < 2\pi,
\end{array} \right. \quad \text{(33)} \]

\[ u_{13}⁽⁽t⁾⁾ = u_{12}(t), \quad \text{(34)} \]

\[ u_{23}⁽⁽t⁾⁾ = 0. \quad \text{(35)} \]
where \( \Theta \) satisfies
\[
-\frac{\mu k_0 \varepsilon_d}{k_t} - 3d^2 \frac{\gamma (\frac{\pi}{2} + \Theta)}{4} = \frac{\mu k_0 \varepsilon_d}{k_t}.
\]
This equation provides \( \Theta = \cos^{-1}(1 - 2\gamma_c/\gamma_0) \). Substituting these equations into \( f^{(1)}_{ij} = -k_t u^{(1)}_{ij} \) yields
\[
f^{(1)}_{12} = \begin{cases} -\mu k_0 \varepsilon_d, & 0 \leq \theta(\theta < \frac{\pi}{2}) \\ -\mu k_0 \varepsilon_d - \frac{3k_0 d(\gamma(\theta) - \gamma_0)}{4}, & \frac{\pi}{2} \leq \theta < \frac{\pi}{2} + \Theta \\ \mu k_0 \varepsilon_d, & \frac{\pi}{2} + \Theta \leq \theta < 2\pi, \\ -\mu k_0 \varepsilon_d - \frac{3k_0 d(\gamma(\theta) + \gamma_0)}{4}, & 2\pi \leq \theta < 2\pi + \Theta \end{cases}
\]
\[
f^{(1)}_{13} = f^{(1)}_{12}, \quad \quad \quad f^{(1)}_{23} = 0.
\]
The normal component of \( \sigma \) in Eq. (9) is given by
\[
\sigma^{(n)} = \sigma_{12}^{(n)} + \sigma_{13}^{(n)}
\]
with
\[
\sigma_{12}^{(n)} = -\frac{1}{A} \frac{x_{12} y_{12}}{r_{12}} f^{(n)}_{12} \quad (S31)
\]
\[
\sigma_{13}^{(n)} = -\frac{1}{A} \frac{x_{13} y_{13}}{r_{13}} f^{(n)}_{12}. \quad (S32)
\]
Substituting Eqs. (S1) and (S2) with Eqs. (S7) and (S8) into Eqs. (S31) and (S32) and using Eq. (S30), we obtain \( \sigma^{(n)} \) as Eq. (14).
The tangential component of \( \sigma \) in Eq. (10) is given by
\[
\sigma^{(t)} = \sigma_{12}^{(t)} + \sigma_{13}^{(t)}
\]
with
\[
\sigma_{12}^{(t)} = -\frac{1}{2A} \frac{x_{12}^2 - y_{12}^2}{r_{12}} f^{(t)}_{12} \quad (S34)
\]
\[
\sigma_{13}^{(t)} = -\frac{1}{2A} \frac{x_{13}^2 - y_{13}^2}{r_{13}} f^{(t)}_{12}. \quad (S35)
\]
Substituting Eqs. (S1) and (S2) with Eqs. (S21) into Eqs. (S33), (S34), and (S35), we obtain \( \sigma^{(t)} \) as Eq. (15) for \( \gamma_0 < \gamma_c \). Using Eqs. (S27) and (S28) instead of Eq. (S21), we obtain \( \sigma^{(t)} \) as Eq. (17) for \( \gamma_0 \geq \gamma_c \).
The pressure, i.e., \( P \), in Eq. (11) is defined as
\[
P = P_{12} + P_{13} + P_{23}
\]
with
\[
P_{ij} = \frac{1}{2A} r_{ij} f^{(n)}_{ij}. \quad (S37)
\]
Substituting Eqs. (S1)- (S3) with Eqs. (S7)- (S9) into Eqs. (S36) and (S37) with \( \gamma = 0 \), we obtain \( P_0 \) as Eq. (20).

II. STRESS–STRAIN CURVE

In this section, we present the stress–strain curve obtained from the analytical results, which explains the dependence of the shear modulus on \( \gamma_0 \). As shown in Fig. S1(a), we plot \( \sigma(\theta) \) against \( \gamma(\theta) \) using Eqs. (4) and (8) with Eqs. (14)-(17) for \( k_t/k_n = 1.0 \) and \( \mu = 0.01 \). For \( \gamma_0 = 0.00001 \), which is lower than \( \gamma_c(\mu) \), \( \sigma \) is proportional to \( \gamma \) with gradient \( \sqrt{3k_0 + k_1}/4 \). For \( \gamma_0 = 0.00005 \) and 0.00003, which are larger than \( \gamma_c(\mu) \), \( \sigma \) exhibits hysteresis loops, including the regions with gradients \( \sqrt{3k_0 + k_1}/4 \) and \( \sqrt{3k_0}/4 \). The region with the lower gradient \( \sqrt{3k_0}/4 \) increases as \( \gamma_0 \) increases. This behavior was observed in the MPS [9],

![FIG. S1. (a) Shear stress \( \sigma \) against \( \gamma \) using Eqs. (4), (8), and (14)-(17) for \( k_t/k_n = 1.0, \epsilon = 0.001, \text{and} \mu = 0.01 \). (b) Scared shear stress \( \sigma/\gamma_0 \) against \( \gamma/\gamma_0 \) using Eqs. (4), (8), and (14)-(17) for \( k_t/k_n = 1.0, \epsilon = 0.001, \text{and} \mu = 0.01 \). The thick black line represents \( \gamma_0 = 0.00001 \). The second black line represents \( \gamma_0 = 0.00005 \). The thin blue line represents \( \gamma_0 = 0.00003 \).](https://example.com/figure_s1.png)
The area $S$ of the curve for $\sigma(\theta)/\gamma_0$ against $\gamma(\theta)/\gamma_0$ is given by

$$S = \int_0^{2\pi} d\theta \frac{1}{\gamma_0} \frac{d\gamma(\theta)}{d\theta} \frac{\sigma(\theta)}{\gamma_0}. \quad (S40)$$

Substituting Eq. (4) into Eq. (S40) with Eq. (12), we obtain

$$S = \int_0^{2\pi} d\theta \sigma(\theta) \cos \theta/\gamma_0 = \pi G'', \quad (S41)$$

which results in $G'' = S/\pi$. As $\gamma_0$ increases, the area $S$ of the scaled stress-strain curve in Fig. S1(b) increases first and decreases later, which explains the $\gamma_0$-dependence of $G''$ provided by Eq. (19).

### III. SHEAR MODULUS NUMERICALLY OBTAINED FROM THE TPM

In this section, we show the behaviors of $G'$ and $G''$ in the TPM without the assumption used to obtain the analytical solution. Here, we numerically obtain $G'$ and $G''$ using Eqs. (12) and (13) based on the left Riemann sum, where the integration of $\Psi(\theta)$, i.e.,

$$\int_0^{2\pi} d\theta \Psi(\theta), \quad (S42)$$

is approximated as

$$\int_0^{2\pi} d\theta \Psi(\theta) \simeq \sum_{n=1}^{M} \Psi(\theta_n) \Delta \theta \quad (S43)$$

with $\Delta \theta = 2\pi/M$ and $\theta_n = (n-1)\Delta \theta$. We use $\epsilon = 0.001$ and $\Delta \theta = 5.0 \times 10^{-5}$ in our simulation.

As shown in Fig. S2, we plot the storage modulus $G'$ numerically obtained from the TPM against $\gamma_0$ with $k_t/k_n = 1.0$ for various values of $\mu$ as points. Moreover, we plot the analytical results derived from Eq. (18) as thin solid lines. The numerical results agree with the analytical results for $\gamma_0 < 0.003$.

Figure S3 shows the loss modulus $G''$ numerically obtained from the TPM against $\gamma_0$ with $k_t/k_n = 1.0$ for various values of $\mu$ as points. We also plot the analytical results given by Eq. (19) as thin solid lines. The numerical results agree with the analytical results for $\gamma_0 < 0.003$.

### IV. ORDERED MPS

This section explains the details of the ordered MPS consisting of monodispersed particles initially placed on a triangular lattice. We consider a two-dimensional assembly of $N$ frictional particles in a periodic box with sizes along the $x$ and $y$ directions $L_x$ and $L_y$, respectively. Here, we initially place $N = 2N_xN_y$ particles of diameter $d$ with integers $N_x$ and $N_y$ at $r_i$ as

$$r_i = \left( n_x \ell - L_x/2, \sqrt{3}n_y \ell - L_y/2 \right) \quad (S44)$$

for $0 \leq i < N_xN_y$ with integers $n_x$, $n_y$, and $i = n_x + N_xn_y$. For $N_xN_y \leq i < 2N_xN_y$, $r_i$ is defined as

$$r_i = \left( (n_x + 1/2) \ell - L_x/2, \sqrt{3}(n_y + 1/2) \ell - L_y/2 \right) \quad (S45)$$

with $i = n_x + N_xn_y + N_xN_y$. The initial configuration is illustrated in Fig. S4. We chose $L_x = N_x \ell$ and $L_y = \sqrt{3}N_y \ell$ with $\ell = d(1 - \epsilon)$.

The position $r_i$ and peculiar momentum $p_i$ of particle $i$ with mass $m_i$ and diameter $d_i$ are driven by the SLLOD equation under the Lees-Edwards boundary condition as
The tangential displacement $u_{ij}^{(t)}$ satisfies $\frac{d}{dt} u_{ij}^{(t)} = v_{ij}^{(t)}$ for $|f_{ij}^{(t)}| < \mu f_{ij}^{(n,el)}$, whereas $u_{ij}^{(t)}$ remains unchanged for $|f_{ij}^{(t)}| \geq \mu f_{ij}^{(n,el)}$. The tangential displacement $u_{ij}^{(t)}$ is set to zero if $i$ and $j$ are detached.

If all the particles are separated, the packing fraction $\phi$ is defined as

$$\phi = \frac{\sum_i \pi d_i^2}{4 L_x L_y},$$  \hspace{1cm} (S54)

Even if contact exists between the particles, we use Eq. (S54) by assuming that the contact length $d_{ij} - r_{ij}$ is sufficiently lower than $d_{ij}$. Using Eq. (S54), $\phi$ is defined as

$$\phi = \frac{\pi}{2\sqrt{3}\epsilon^2},$$  \hspace{1cm} (S55)

The jamming point of this system is

$$\phi_1 = \frac{\pi}{2\sqrt{3}},$$  \hspace{1cm} (S56)

with $\epsilon = 0$. The distance from the jamming point is proportional to $\epsilon$.

$$\phi - \phi_1 \simeq \frac{\pi}{\sqrt{3}\epsilon},$$  \hspace{1cm} (S57)

for $\epsilon \ll 1$.

The shear stress $\sigma$ is defined by Eq. 8 in the main article with the normal component

$$\sigma^{(n)} = -\frac{1}{L_x L_y} \sum_i \sum_{j>i} \frac{x_{ij} y_{ij}}{r_{ij}} f_{ij}^{(n)},$$  \hspace{1cm} (S58)

and tangential component

$$\sigma^{(t)} = -\frac{1}{2L_x L_y} \sum_i \sum_{j>i} \frac{x_{ij}^2 - y_{ij}^2}{r_{ij}} f_{ij}^{(t)},$$  \hspace{1cm} (S59)

The pressure is defined as

$$P = \frac{1}{2L_x L_y} \sum_i \sum_{j>i} (x_{ij} f_{ij,x} + y_{ij} f_{ij,y}).$$  \hspace{1cm} (S60)

We use $N_x = 8$, $N_y = 4$, $N_c = 20$, $k_i = k_n$, and $\eta_n = \eta_t = k_n \sqrt{m/k_n}$, where $m$ denotes the mass of a grain of diameter $d$. This model corresponds to a restitution coefficient $e = 0.043$. We adopt the leapfrog algorithm considering a time step of $\Delta t = 0.05t_0$. We chose $\omega = 1.0 \times 10^{-2} \sqrt{k_n/m}$ as the quasistatic shear deformation because $G'$ and $G''$ are almost independent of $\omega$ for $\omega \leq 1.0 \times 10^{-3} \sqrt{k_n/m}$.

As shown in Figs. 3 and 4, the behaviors of $G'$ and $G''$ of the TPM agree with that of the MPS. We explain the theoretical background of the TPM. The initial configuration is shown in Fig. S4. It contains the unit cell, which is represented by the red rectangle with length $\ell$ and height $\sqrt{3}\ell/2$. It contains interactions between the
three particles, represented by blue lines. Here, we assume that the particles move affinely as
\[ r_i(t) = r_i(0) + \gamma(\theta(t))y_i(0) \mathbf{e}_x. \] (S61)
In this case, the corresponding relative distances between the particles in any unit cell are identical.

In particular, in a unit cell containing particles \( i = i_1, i_2, \) and \( i_3 \) with \( i_1 = N_x N_y, i_2 = 0, \) and \( i_3 = 1, \) the positions of the particles are given by
\[ r_{i_1}(t) = \left( \gamma(\theta(t)) \frac{\sqrt{3}t - L_y}{2} + \frac{\ell - L_x}{2}, \frac{\sqrt{3}t - L_y}{2} \right), \] (S62)
\[ r_{i_2}(t) = \left( -\gamma(\theta(t)) \frac{L_y}{2} - \frac{L_x}{2}, \frac{L_y}{2} \right), \] (S63)
\[ r_{i_3}(t) = \left( -\gamma(\theta(t)) \frac{L_y}{2} + \frac{\ell - L_x}{2}, \frac{L_y}{2} \right). \] (S64)

The relative distances between these particles are identical to those of the TPM, given by Eqs. (11)–(13), which indicates that the TPM provides the interaction forces among the three particles. This system includes 2\( N_x N_y \) unit cells with identical interaction forces. Hence, the normal and tangential components of \( \sigma \) are given by
\[ \sigma^{(n)} = -\frac{2N_x N_y}{L_x L_y} \sum_{i=i_1,i_2,i_3} \left\{ \sum_{j=i_1,i_2} \frac{x_{ij}y_{ij} f^{(n)}_{ij}}{r_{ij}} \right\}, \] (S65)
\[ \sigma^{(t)} = -\frac{N_x N_y}{L_x L_y} \sum_{i=i_1,i_2,i_3} \left\{ \sum_{j=i_1,i_2,i_3} \frac{x_{ij}^2 - y_{ij}^2}{r_{ij}} f^{(t)}_{ij} \right\}. \] (S66)

The pressure is also given by:
\[ P = \frac{N_x N_y}{L_x L_y} \sum_{i=i_1,i_2,i_3} \left\{ \sum_{j=i_1,i_2,i_3} (x_{ij} f_{ij,x} + y_{ij} f_{ij,y}) \right\}. \] (S67)

Using the relation \( L_x L_y/(2N_x N_y) = \sqrt{3}l^2/2 \) corresponding to \( A = \sqrt{3}l^2/2, \sigma^{(n)}, \sigma^{(t)}, \) and \( P \) coincide with Eqs. (8)–(11). Hence, if the assumptions of the affine motion, i.e., Eqs. (S62)–(S64), are satisfied, \( G' \) and \( G'' \) in the ordered MPS coincide with those in the TPM.

V. EFFECT OF PARTICLE ROTATION

In this section, we illustrate the effect of particle rotation, which was not described in Sec. IV. In the model with rotation, the tangential velocity \( v_{ij}^{(t)} \) is given by
\[ v_{ij}^{(t)} = (\mathbf{v}_i - \mathbf{v}_j) \cdot \mathbf{t}_{ij} - (d_i \omega_i + d_j \omega_j)/2 \] (S68)
instead of Eq. (S53), where \( \omega_i \) denotes the angular velocity of particle \( i. \) The time evolution of \( \omega_i \) is given by
\[ I_i \frac{d}{dt} \omega_i = T_i \] (S69)
with the moment of inertia \( I_i = m_i d_i^2/8 \) and torque \( T_i = -\sum_j \mathbf{F}_{ij}^{(t)} \cdot \mathbf{t}_{ij}. \)

As shown in Fig. S5, we plot \( G' \) in the MPS with and without rotation for \( \mu = 0.01 \) and \( \epsilon = 0.001. \) The values of other parameters are the same as those in Sec. IV.

The effect of particle rotation is negligible, except for the region near \( \gamma_c. \)

VI. DISORDERED MPS

In this section, we present the details of the disordered MPS. This model is an extension of the monodisperse model used in Sec. IV including the dispersion of the particles and disordered initial configuration.

The system is bidisperse and includes an equal number of particles with diameters \( d \) and \( d/1.4. \) To simulate the disordered MPS, we randomly place the particles in a rectangular box with an initial packing fraction of \( \phi_1 = 0.75. \) The system is slowly compressed until the packing fraction reaches \( \phi = 0.23. \) In each compression step, the packing fraction is increased by \( \Delta \phi = 1.0 \times 10^{-4} \) with an affine transformation. Thereafter, the particles are relaxed to a mechanical equilibrium state with the kinetic temperature \( T_K = \sum_i p_i^2/(mN) < T_{ih}. \) Here, we chose \( T_{ih} = 1.0 \times 10^{-3} k_B d^2. \) After compression, the oscillatory shear strain given by Eq. (1) in the main text is applied for \( N_c \) cycles. In the last cycle, we measure \( G' \) and \( G'' \) using Eqs. (12) and (13) with Eqs. (8)–(10). The pressure, \( P_0, \) is obtained using Eq. (11) after the last cycle. We use \( \phi = 0.87, N = 1000, N_c = 20, L_y/L_x = 1, \) \( k_z = 0.2 k_B, \) and \( \eta_t = \eta_k = k_B T/m \).

As shown in Fig. S6(a), we plot the shear stress \( \sigma \) against \( \gamma \) in the disordered MPS with \( \mu = 0.0001. \) For
\( \gamma_0 = 0.00001, \sigma \) is almost proportional to \( \gamma_0 \). As \( \gamma_0 \) increases, the stress–strain curve exhibits a loop in which the gradient of the curve is higher near \( \gamma = \pm \gamma_0 \) and lower for \( \gamma \approx 0 \). This loop is similar to that of the TPM (Fig. S1(a)). Figure S6(b) shows the scaled shear stress \( \sigma/\gamma_0 \) against the scaled strain \( \gamma/\gamma_0 \) in the MPS with \( \mu = 0.0001 \). The maximum value \( \tilde{\sigma}_{\text{max}} \) decreases as \( \gamma_0 \) increases. The area \( S \) of the curve is the largest for \( \gamma_0 = 0.00003 \). This behavior is similar to that of the TPM (Fig. S1(b)).