On products of strong Skolem starters

Oleg Ogandzhanyants | Margarita Kondratieva | Nabil Shalaby

Department of Mathematics and Statistics, Memorial University of Newfoundland St. John’s, Newfoundland, Canada

Correspondence
Oleg Ogandzhanyants, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John’s, Newfoundland, Canada A1C 5S7.
Email: oo4706@mun.ca

Abstract
In 1991, Shalaby conjectured that any $\mathbb{Z}_n$, where $n \equiv 1$ or $3 \pmod{8}$, $n \geq 11$, admits a strong Skolem starter. In 2018, the authors fully described and explicitly constructed the infinite “cardioidal” family of strong Skolem starters. No other infinite family of these combinatorial designs was known to date. Statements regarding the products of starters, proven in this paper give a new way of generating strong or skew Skolem starters of composite orders. This approach extends our previous result by generating new infinite families of these starters that are not cardioidal.

KEYWORDS
2-partition, room square, skew starter, Skolem starter, Steiner triple system, strong starter

1 | INTRODUCTION

The objects, we will deal with in this paper, are partitions of a set of even cardinality into pairs. For convenience, we will call these partitions 2-partitions of this set.

A starter in an additive abelian group $G$ of odd order $n$ is a partition of the set $G^*$ of all nonzero elements of $G$ into $q = (n - 1)/2$ pairs $\{(s_i, t_i)\}_{i=1}^q$ such that the elements $s_i - t_i$, $i = 1, ..., q$, comprise $G^*$.

Starters exist in any additive abelian group of odd order $n \geq 3$. For example, the partition $\{x, -x\}|x \in G, x \neq 0\}$ of $G^*$ is a starter in $G$.

In this paper, we will consider only cyclic additive abelian groups, more precisely, groups $\mathbb{Z}_n$ of integers modulo $n$, where $n \geq 3$ is odd.

Definition 1.1. A 2-partition $S = \{(x_i, y_i)\}_{i=1}^q$ of $\mathbb{Z}_n^* = \mathbb{Z}_n \setminus \{0\}, n = 2q + 1, q \geq 1$ is called...
(a) a starter in $\mathbb{Z}_n$, if
\[ \{\pm(x_i - y_i) \pmod{n} | \{x_i, y_i\} \in S, 1 \leq i \leq q\} = \mathbb{Z}_n^*, \tag{1} \]

(b) strong, if
\[ \hat{S} = \{(x_i + y_i) \pmod{n} | \{x_i, y_i\} \in S, 1 \leq i \leq q\} \subset \mathbb{Z}_n^* \quad \text{and} \quad |\hat{S}| = q; \tag{2} \]

(c) skew, if
\[ \{\pm(x_i + y_i) \pmod{n} | \{x_i, y_i\} \in S, 1 \leq i \leq q\} = \mathbb{Z}_n^*; \tag{3} \]

(d) cardioidal \cite{12}, if all its pairs are cardioidal of order $n$, that is, if each pair of the partition
\[ \{x, y\} = \{i, 2i \pmod{n}\} \]
for some $i \in \mathbb{Z}_n^*$;

(e) Skolem, if all its pairs are Skolem of order $n$, that is, if each pair $\{x, y\}$ of the partition is such that
\[ y - x \leq q \pmod{n} \quad \text{is equivalent to} \quad y > x. \tag{5} \]

Here we assume $1 < 2 < \cdots < 2q$ to be the ordering of the nonzero integers modulo $n$.

Remark 1.2. We will refer to 2-partitions of $\mathbb{Z}_n^*$ as 2-partitions of order $n$. A **strong starter** in $\mathbb{Z}_n$ is a 2-partition of order $n$ that possesses properties (1) and (2). A **skew starter** in $\mathbb{Z}_n$ is a 2-partition of order $n$ that possesses properties (1) and (3).

Clearly, if a 2-partition is skew then it is strong. Consequently, any skew starter is strong. Also, it is known that if a partition is cardioidal, then it is Skolem \cite{12}. All the other implications are absent, and a 2-partition may hold any combination of these properties independently from one another. The following examples demonstrate it:

**Example 1.3.**

1. The 2-partition $R = \{\{1, 2\}, \{3, 4\}\}$ of $\mathbb{Z}_5^*$ is strong ($0 \neq 1 + 2 \neq 3 + 4 \neq 0 \pmod{5}$) and cardioidal ($i = 1$ and 4) but not a starter ($2 - 1 = 4 - 3$) and not skew ($\pm 3 = \mp 7 \pmod{5}$).

2. The 2-partition $Q = \{\{1, 3\}, \{2, 5\}, \{4, 6\}, \{7, 8\}\}$ of $\mathbb{Z}_9^*$ is Skolem and skew:
\[ \{\pm(1 + 3) \equiv \pm 4 \pmod{9}, \pm(2 + 5) \equiv \mp 2 \pmod{9}, \pm(4 + 6) \equiv \pm 1 \pmod{9}, \pm(7 + 8) \equiv \mp 3 \pmod{9}\} = \mathbb{Z}_9^*. \] But it is not a starter ($3 - 1 = 6 - 4$), nor is it cardioidal as, for example, the pair $\{1, 3\}$ does not satisfy property (4).

3. The starter $T = \{\{2, 3\}, \{4, 6\}, \{5, 1\}\}$ in $\mathbb{Z}_7$ is strong and skew:
\[ \{\pm(2 + 3) \equiv \pm 5 \pmod{7}, \pm(4 + 6) \equiv \pm 3 \pmod{7}, \pm(5 + 1) \equiv \mp 1 \pmod{7}\} = \mathbb{Z}_7^*, \] but not Skolem, as the pair $\{5, 1\}$ does not satisfy property (5).

4. The starter $S = \{\{1, 2\}, \{10, 12\}, \{3, 6\}, \{4, 8\}, \{11, 16\}, \{9, 15\}, \{7, 14\}, \{5, 13\}\}$ in $\mathbb{Z}_{17}$ is strong as all the pairs yield pairwise different nonzero sums (mod 17) and Skolem.
However, \( S \) is not skew as, for example, the pairs \( \{10, 12\} \) and \( \{4, 8\} \) yield the sums \( 5 \) (mod 17) and 12 (mod 17), respectively, and \( 5 \equiv -12 \) (mod 17). Neither \( S \) is cardioidal as, for example, the pair \( \{5, 13\} \) does not satisfy property (4).

Originally, strong starters were introduced by Mullin and Stanton in 1968 [11] for constructing Room squares and Howell designs. In 1969, Mullin and Nemeth [10] gave a general construction for finding these starters in cyclic groups.

The question of the existence (or nonexistence) of a strong starter in an abelian group is crucial in the theory of Room squares. We refer readers interested in constructions of strong starters to [2, 9, 10] and the references therein.

Strong starters in groups of order 3, 5, and 9 do not exist [3, p. 144]. It is an open question whether there exists a strong starter in every cyclic group of an odd order exceeding 9. In 1981, Dinitz and Stinson [4] found (by a computer search) strong starters in the cyclic group of order \( n \) for all odd \( 7 \leq n \leq 999, n \neq 9 \).

At present, the strongest known general statement on the existence of strong starters is the following [2, p. 625]:

For any \( n > 5 \) coprime to 6, an abelian group of order \( n \) admits a strong starter.

It is known [2, p. 627] that skew starters of order \( n \) do not exist, if \( 3n \).

**Skolem starters**, the objects of our close attention, are defined only in \( \mathbb{Z}_n \).

**Definition 1.4.** Given an odd \( n \geq 3 \), a 2-partition of \( \mathbb{Z}_n^* \) is a Skolem starter in \( \mathbb{Z}_n \) if it possesses properties (1) and (5) from Definition 1.1.

Skolem starters received their name [14] after Skolem sequences. A **Skolem sequence** of order \( q \) is a sequence \( (s_1, ..., s_{2q}) \) of integers from \( D = \{1, ..., q\} \) such that for each \( i \in D \) there is exactly one \( j \in \{1, ..., 2q\} \) such that \( s_j = s_{j+i} = i \). Skolem sequences exist iff \( q \equiv 0 \) or 1 (mod 4) [2].

Clearly, Skolem starters in \( \mathbb{Z}_n \) are in one-to-one correspondence with Skolem sequences of order \( q = (n - 1)/2 \). Therefore, Skolem starters exist in \( \mathbb{Z}_n \) iff \( n \equiv 1 \) or 3 (mod 8).

**Strong Skolem starters** are 2-partitions with properties (1), (2), and (5).

**Theorem 1.5** (Shalaby, 1991 [14, pp. 60–62]). For \( 11 \leq n \leq 57, n \equiv 1 \) or 3 (mod 8), \( \mathbb{Z}_n \) admits a strong Skolem starter.

**Conjecture 1.6** (Shalaby, 1991 [14, p. 62]). Every \( \mathbb{Z}_n \) with \( n \equiv 1 \) or 3 (mod 8) and \( n \geq 11 \), admits a strong Skolem starter.

Up to 2018, there were known only finitely many strong Skolem starters. In 2018, Ogandzhanyants et al. explicitly constructed an infinite family of strong Skolem starters [12], proving the following

**Theorem 1.7.** Let \( n = \prod_{i=1}^m p_i^{k_i} \), where \( p_i > 3, i = 1, ..., m \), are pairwise distinct primes such that \( \text{ord}_{p_i}(2) \equiv 2 \) (mod 4), and \( k_i \in \mathbb{N}, i = 1, ..., m \). Then \( \mathbb{Z}_n \) admits a skew Skolem starter.
In addition, it was shown that all Skolem starters found in [12] are cardioidal starters, that is, they possess property (4), and no strong cardioidal starter lies outside of the family fully described in [12]. The discovery in [12] boosted up the attention of some other researchers towards the proof of Conjecture 1.6, see, for example, [17] and the references therein. They explored alternative approaches to constructing strong Skolem starters, but no infinite family of strong Skolem starters other than strong cardioidal starters has been found.

Theorems 3.10 and 4.2 stated and proved in this paper allow formation of new infinite families of strong (and skew) Skolem starters of composite orders, which are not cardioidal and thus significantly extends the previous result.

Gross [7, p. 170] in 1974 indicated a way to produce a starter for the group $G \oplus H$, the direct sum of two finite abelian groups, given a starter for $H$ and a set of starters for $G$. He showed that under certain conditions, strong starters for $H$ and $G$ give rise to a strong starter for $G \oplus H$. Our constructions of the products given in Definitions 2.4 and 3.24 are inspired by that paper. However, in contrast with Gross who focused on the existence of strong starters and starters with adders in a general setting, our constructions are explicitly defined in cyclic groups $\mathbb{Z}_n$ as we are concerned with skew and strong Skolem starters. In addition, most of our statements have a converse.

Our construction of products resembles the one, given by Turgeon in 1980 [16] for additive sequences of permutations, in the general context of difference sets. Indeed, Skolem starters could be treated as a very special case of difference sets, namely, a very special case of Perfect Difference Families (PDF). PDFs are widely employed in applications to the construction of properly centered permutations, which can be used to construct radar arrays [8]. However, in this paper we avoid over-generalization and adapt the presentation specifically to our needs, as we deal with PDFs containing only blocks of size 2, whereas the mainstream of works over Optical Orthogonal Codes deals with PDFs of much more complex structure [19].

Thus, we first apply the construction to 2-partitions in $\mathbb{Z}_n^*$ without any restrictions imposed on them. Then we endow the 2-partitions with a certain property stated in Definition 1.1, apart from all other properties. The structure of this papers is as follows.

In Section 2, we introduce the notion of a product of a pair of 2-partitions of $\mathbb{Z}_{2p+1}^*$ and $\mathbb{Z}_{2q+1}^*$, respectively.

In Section 3, we focus our attention on starters and other special classes of 2-partitions. Section 3.1 gives some preliminaries. In Section 3.2, we prove several properties of the product of two 2-partitions and give an important intermediate result, Theorem 3.10. In Section 3.3, we generalize the initial definition of the product and compare its properties to the initial one. In Section 3.4, we show explicit ways to apply the products of Skolem starters, thus generating new infinite families of strong and even skew Skolem starters.

Many of our statements about the properties of the product are formulated in direct and converse ways. Most of the direct statements about the product were known before due to Turgeon [16] and Gross [7], whereas the converse statements were out of their concerns.

In addition, Turgeon’s objects of study were not characterized with strongness, while Gross was not interested in Skolemness and skewness of the starters.

One needs the converse statements to eliminate temptations to generate the product of two 2-partitions with required properties imposing the weakest limitations on the ingredients. The complexity of the converse statements is nearly the same as the direct ones. Without claiming novelty of the results, we present our own proofs of the direct statements (see Theorems 2.8, 3.5, 3.6, 3.8, 3.9, 3.18, 3.19, and 3.20) as it makes the proofs of the converse
statements easier to follow. To the best of our knowledge, the converse statements of these theorems have not been known. We also pay close attention to both strongness (skewness) and Skolemness of a starter. Therefore, our paper makes a contribution to the finding of new families of strong Skolem starters.

In Section 4, we conclude with a discussion on several implications of the statements proved in this paper and give the main result of this paper, Theorem 4.2. We also offer a brief discussion on the applications of strong Skolem starters.

2 | THE PRODUCT OF 2-PARTITIONS: CONSTRUCTION

**Lemma 2.1.** Let \( n = 2q + 1, q \geq 1. \) From any 2-partition \( S = \{\{a_i, b_i\}\}_{i=1}^q \) of \( \mathbb{Z}_n^* \), it is possible to make a set of ordered pairs \( \mathcal{S} = \{(x_i, y_i)\}_{i=1}^q \), where either \( x_i = a_i, y_i = b_i \) or \( x_i = b_i, y_i = a_i \), for all \( 1 \leq i \leq q \), such that

\[
\bigcup_{i=1}^q \{\pm x_i\} = \mathbb{Z}_n^*.
\] (6)

**Proof.** Let us denote by \([a, b] \mapsto (x, y)\) the operation of making an ordered pair \((x, y)\) from an unordered pair \([a, b]\) by setting \( x \equiv a, y \equiv b \pmod{n} \), then removing \([a, b]\) from \( S \) and placing \((x, y)\) in \( \mathcal{S} \). Below we describe an explicit construction of \( \mathcal{S} \).

First, for each pair \([a_i, b_i]\), where \( b_i \equiv -a_i \pmod{n} \), if exists, set \([a_i, -a_i] \mapsto (x_i, y_i)\), and place it in the end of the list \( \mathcal{S} \). From all other pairs remaining in \( S \), pick any pair, say, \([a_i, b_i] \in S \), and set \([a_i, b_i] \mapsto (x_i, y_i)\). Then, find a pair containing the element \(-b_1\). Without loss of generality (WLOG), let \( a_k \equiv -b_1 \pmod{n} \). Then set \([-b_1, b_k] \mapsto (x_2, y_2)\).

Then, find a pair which the element \(-b_k\) belongs to. WLOG, let \( a_j \equiv -b_k \pmod{n} \). Then set \([-b_k, b_j] \mapsto (x_3, y_3)\). And so on, until the element \(-a_1\) appears in some pair \([a_m, b_m] \in S\), which will produce \((x_i, y_i)\), where \( y_i \equiv -a_1 \pmod{n} \).

Note that by the construction \( x_i \equiv -y_i \pmod{n} \), \( x_j \equiv -y_2 \pmod{n} \), and so forth. Clearly, such a collection of pairs spans over the subset \( \{x_1, y_1, -y_1, y_2, -y_2, \ldots, -x_i\} \subset \mathbb{Z}_n^* \).

Finally, pick any remaining pair in \( S \), give it an ordering and continue the process until all the pairs are ordered. \( \square \)

**Remark 2.2.** Note that for the set \( \mathcal{S} \) described in Lemma 2.1, we automatically have

\[
\bigcup_{i=1}^q \{\pm y_i\} = \mathbb{Z}_n^*.
\] (7)

**Example 2.3.** For \( q = 6 \) and partition \( \{\{1, 12\}; \{2, 3\}; \{4, 6\}; \{5, 7\}; \{8, 9\}; \{10, 11\}\) we form \( \mathcal{S} \) by making the following three clusters:

\[
\mathcal{S} = \{(2, 3), (-3, -2), (4, 6), (-6, 5), (-5, -4), (1, -1) \pmod{13}\}
\]

with property (6), as required:

\[
\{\pm 2, \mp 3, \pm 4, \mp 6, \mp 5, \pm 1 \pmod{13}\} = \mathbb{Z}_{13}^*.
\]
Let $S = \{(a_i, b_j)\}_{i=1}^q$ be a 2-partition of $\mathbb{Z}_{2q+1}^*$. Denote by $\tilde{S}$ a set of $q$ ordered pairs of $S$ that obey property (6):

$$\tilde{S} = \{(x_i, y_i)\}_{i=1}^q.$$ 

The existence of these sets is ensured by Lemma 2.1. In addition, by $\tilde{S}'$ we denote the set $\{(x_i, -y_i)\}_{i=1}^q$ and by $\tilde{S}$ we denote a set of arbitrary ordered pairs of $S$, $|\tilde{S}| = q$ (the set $\tilde{S}$ does not have to necessarily obey property (6)).

**Definition 2.4.** Given two 2-partitions, $S$ of $\mathbb{Z}_{2q+1}^*$ and $T$ of $\mathbb{Z}_{2p+1}^*$, let us form sets of ordered pairs $\tilde{S}$, and $T, T'$ as specified above. Consider the set $W_{ST} = \{(u_i, v_i)\}_{i=1}^k$ of $k = 2qp + q + p$ pairs of the form

$$(u, v), \quad \text{where} \quad u = (2q + 1)r + x, \quad v = (2q + 1)t + y$$

divided into the following types:

(i) $(2p + 1)q$ pairs: one for each $(r, t) \in T \cup T' \cup \{(0, 0)\}$ and for each $(x, y) \in \tilde{S}$ and

(ii) $p$ pairs: one for each $(r, t) \in T$ and $x = y = 0$.

We will call the set of pairs $W_{ST}$ a product of $S$ and $T$.

Note that the ordering of the pairs of $S$, the first of the two 2-partitions in the product, does not have to obey the property (6), while the ordering of the pairs of $T$ must obey this property. This loosening on the ordering of the pairs of $S$ expands the variety of the products without affecting their properties. The way of ordering the pairs of $\tilde{S}$ is arbitrary, but once we have assigned the ordering, we stick to this ordering while constructing the pairs of the product. Otherwise, we may receive repetitions of the pairs in the product.

**Example 2.5.** Let us construct the set $W_{ST}$ for 2-partitions $S = T = \{(1, 2)\}$ of $\mathbb{Z}_3^*$. Here $q = p = 1$. Take $\tilde{S} = T = \{(1, 2)\}$ and $T' = \{(2, 1)\}$. So we have the pairs of the two types:

(i) $\{3 \times 0 + 1, 3 \times 0 + 2\} = \{1, 2\}, \{3 \times 1 + 1, 3 \times 2 + 2\} = \{4, 8\}, \{3 \times 2 + 1, 3 \times 1 + 2\} = \{7, 5\};$

(ii) $\{3 \times 1 + 0, 3 \times 2 + 0\} = \{3, 6\}.$

The set of these four pairs constitutes $W_{ST}$.

**Remark 2.6.** The pairs of $W_{ST}$ can be formed in various ways, depending on the choices made in the process of constructing $\tilde{S}$ and $T$ from the 2-partitions $S$ and $T$. So, a product of the two 2-partitions is not unique. Nevertheless, the properties proven below hold for $W_{ST}$ regardless of the ordering choices.

We will consider an alternative construction of a product in Sections 3.3 and 3.4. As well, we will outline more possibilities in Section 4.
A product $W_{ST}$ of two 2-partitions, $S$ of $\mathbb{Z}_n^*$ and $T$ of $\mathbb{Z}_m^*$, as we will prove, preserves some properties of the factors. The most general one is given in Theorem 2.8. Before that, we recall the following simple results needed in the proofs.

**Lemma 2.7.** Let $m, n$ be any natural numbers and $a, b, c, d$ be integers.

1. If $a \equiv b \pmod{mn}$ then $a \equiv b \pmod{n}$ and $a \equiv b \pmod{m}$.
2. If $(an + c) \equiv (bn + c) \pmod{mn}$ then $a \equiv b \pmod{m}$.
3. If $(an + c) \equiv (bn + d) \pmod{n}$ then $c \equiv d \pmod{n}$.
4. Let $X^n_c$ be a finite multiset of integers congruent modulo $n$ to a given integer $c$, not necessarily all distinct. If $\left| X^n_c \right| > m$ then there exist $b, d \in X^n_c$ such that $b = d \pmod{mn}$.

**Proof.** The statements are verified by a straightforward check. \qed

**Theorem 2.8.** Let $n, m \geq 3$ be odd integers and $S$ and $T$ be 2-partitions of $\mathbb{Z}_n^*$ and $\mathbb{Z}_m^*$, respectively. Their product $W_{ST}$ (Definition 2.4) is a 2-partition of $\mathbb{Z}_{mn}^*$.

**Proof.** Let $n = 2q + 1$ and $m = 2p + 1$, $p, q \geq 1$. Let us also establish the natural order in $\mathbb{Z}_k^*: 0 < 1 < \cdots < k - 1, k \in \mathbb{N}$. By definition, $W_{ST}$ consists of $2pq + p + q$ pairs, totaling to $4pq + 2p + 2q = mn - 1$ elements, which equals the cardinality of $\mathbb{Z}_{mn}^*$. It remains to show that all these elements of $W_{ST}$ are distinct modulo $mn$. Indeed, all the elements of the pairs of type (ii) are distinct as $T$ is a 2-partition of $\mathbb{Z}_m^*$, and they are multiples of $n$. All the elements of the pairs for $rt = 0$ and $xy \in S$ are distinct and less than $n$ because $S$ is a 2-partition of $\mathbb{Z}_n^*$. All the remaining elements of the pairs of type (i) are greater than $n$ and are not multiples of $n$. Assume for the sake of contradiction the possibility that among them there are a pair $\{u_1, v_1\}$ and a pair $\{u_2, v_2\}$ with a nonempty intersection. Here

$$u_i = r_i n + x_i, \quad v_i = t_i n + y_i, \quad u_2 = r_2 n + x_2, \quad v_2 = t_2 n + y_2, \quad (10)$$

and WLOG, we let $(x_i, y_i) \in \bar{S}, i = 1, 2, (r_i, t_i) \in T, (r_2, t_2) \in T'$.

Let, for example, $u_1 \equiv u_2 \pmod{mn}$, that is, $(r_1 n + x_1) \equiv (r_2 n + x_2) \pmod{mn}$. Then, by Lemma 2.7(1 and 3), $x_1 \equiv x_2 \pmod{n}$. Therefore, by Lemma 2.7(2), $r_1 \equiv r_2 \pmod{m}$. But this is impossible due to property (6). Similar argument will lead to a contradiction if one assumes $v_1 \equiv v_2 \pmod{mn}$ or $u_1 \equiv v_2 \pmod{mn}$ or $u_2 \equiv v_1 \pmod{mn}$.

This proves that all $4pq + 2p + 2q = mn - 1$ elements which appear in the pairs of $W_{ST}$ are distinct. Therefore $W_{ST}$ is a 2-partition of $\mathbb{Z}_{mn}^*$. \qed

# 3 | THE PRODUCT OF SPECIAL CLASSES OF 2-PARTITIONS

## 3.1 | Preliminaries

The 2-partitions of $\mathbb{Z}_n^*$ we mainly concern with are strong and skew Skolem starters. Lemma 2.1 applies to starters in $\mathbb{Z}_{2q+1}$ as they form a 2-partition of $\mathbb{Z}_{2q+1}^*$.
Before we get to the properties of the product of two starters, we present a few additional definitions and lemmas, that will be helpful in the sequel.

**Definition 3.1.** A pair \( \{x, y\} \in S \) is called a canonical pair of order \( k \) if \( \{x, y\} = \{i, -i \mod k\} \) for some \( i \in \mathbb{Z}_k^* \). If all pairs of \( S \) are canonical, then \( S \) is called a canonical starter of order \( k \).

**Definition 3.2.** Two \( 2 \)-partitions \( S \) and \( S' \) in the same group are called conjugate if \( \{x, y\} \in S \) implies \( \{-x, -y\} \in S' \).

Obviously, every \( 2 \)-partition has a conjugate. Note that the \( 2 \)-partition of \( \mathbb{Z}_n^* \) which is a canonical starter is always conjugate to itself. Moreover, a starter is canonical if and only if it is self-conjugate.

The following properties of conjugate \( 2 \)-partitions are rather trivial but very important. Each of them follows immediately from the definitions of their counterparts:

**Lemma 3.3.** If a \( 2 \)-partition is either a starter, or canonical, or strong, or skew, or Skolem, or cardioidal, so is its conjugate.

### 3.2 Properties of the product of two \( 2 \)-partitions

In Example 2.5, the two \( 2 \)-partitions we use are starters in \( \mathbb{Z}_3 \). (We have no choice as the only \( 2 \)-partition of \( \mathbb{Z}_3^* \) is a starter in \( \mathbb{Z}_3 \)). And their product turns out to be a starter in \( \mathbb{Z}_{3,3} = \mathbb{Z}_9 \).

Consider the product of two starters from different groups.

**Example 3.4.** Let us construct the set \( W_{ST} \) for starters \( S = \{\{1, 4\}, \{2, 3\}\} \) in \( \mathbb{Z}_5 \) and \( T = \{1, 2\} \) in \( \mathbb{Z}_3 \). In this case \( n = 5, m = 3 \) and \( q = 2, p = 1 \).

Take \( S = \{(1, 4), (2, 3)\}, T = \{(1, 2)\}, T' = \{(2, 1)\}. \)

Then we have the pairs of the two types:

(i) \( \{5 \times 0 + 1, 5 \times 0 + 4\} = \{1, 4\}, \{5 \times 0 + 2, 5 \times 0 + 3\} = \{2, 3\}, \)

\( \{5 \times 1 + 1, 5 \times 2 + 4\} = \{6, 14\}, \{5 \times 1 + 2, 5 \times 2 + 3\} = \{7, 13\}, \)

\( \{5 \times 2 + 1, 5 \times 1 + 4\} = \{11, 9\}, \{5 \times 2 + 2, 5 \times 1 + 3\} = \{12, 8\}; \)

(ii) \( \{5 \times 1 + 0, 5 \times 2 + 0\} = \{5, 10\}. \)

The set of these seven pairs constitutes \( W_{ST} \). In fact, \( W_{ST} \) is a starter in \( \mathbb{Z}_{3,5} = \mathbb{Z}_{15} \).

This is not coincidental. A product of two starters is a starter. It turns out that the converse is true as well, that is, if \( W_{ST} \) is a starter, then both \( S \) and \( T \) are starters. The following theorem secures this property.

**Theorem 3.5.** Let \( n, m \geq 3 \) be odd integers and \( S \) and \( T \) be \( 2 \)-partitions of \( \mathbb{Z}_n^* \) and \( \mathbb{Z}_m^* \), respectively. Their product \( W_{ST} \) (Definition 2.4) is a starter in \( \mathbb{Z}_{mn} \) if and only if \( S \) is a starter in \( \mathbb{Z}_n \) and \( T \) is a starter in \( \mathbb{Z}_m \).
Proof. (a) Sufficiency.

Let $S$ be a starter in $\mathbb{Z}_n$ and $T$ be a starter in $\mathbb{Z}_m$. To prove that $W_{ST}$ is a starter in $\mathbb{Z}_{mn}$, we need to show that $W_{ST}$ is a partition of $\mathbb{Z}_{mn}^*$ into pairs $\{[u_i, v_i]\}_{i=1}^{(mn-1)/2}$ such that
\[
\{ \pm(u_i - v_i) \ (\text{mod} \ mn) \mid [u_i, v_i] \in W_{ST} \} = \mathbb{Z}_{mn}^* .
\] (11)

Now, let us look at the differences $\pm(u_k - v_k) \ (\text{mod} \ mn)$, $1 \leq k \leq \frac{mn-1}{2}$.

Since $T$ is a starter in $\mathbb{Z}_m$, the pairs of type (ii) make all possible $m - 1$ differences of the form $n\Delta$, where $\Delta \in \mathbb{Z}_n^*$.

Consider two distinct pairs $\{u_1, v_1\}$ and $\{u_2, v_2\}$ of type (i). Suppose, for the sake of contradiction, that $u_1 - v_1 \equiv u_2 - v_2 \ (\text{mod} \ mn)$.

Using notation (10), we have
\[
[(r_1n + x_1) - (t_1n + y_1)] \equiv [(r_2n + x_2) - (t_2n + y_2)] \ (\text{mod} \ mn).
\] (12)

By Lemma 2.7(1), Equation (12) implies
\[
[(r_1n + x_1) - (t_1n + y_1)] \equiv [(r_2n + x_2) - (t_2n + y_2)] \ (\text{mod} \ n).
\]

Then, by Lemma 2.7(3), we obtain
\[
(x_1 - y_1) \equiv (x_2 - y_2) \ (\text{mod} \ n).
\]

Since $S$ is a starter in $\mathbb{Z}_n$, it is possible if and only if $\{x_1, y_1\} = \{x_2, y_2\}$. WLOG, assume that this pair is ordered by $(x_1, y_1) = (x_2, y_2) = (x, y) \in S$. We have
\[
((r_1n + x) - (t_1n + y)) \equiv ((r_2n + x) - (t_2n + y)) \ (\text{mod} \ mn).
\] (13)

By Lemma 2.7(2), Equation (13) implies $(r_1 - t_1) \equiv (r_2 - t_2) \ (\text{mod} \ m)$. Since $T$ is a starter in $\mathbb{Z}_m$, it is possible if and only if $r_1 = r_2$ and $t_1 = t_2$, which contradicts our assumption that $\{u_1, v_1\}$ and $\{u_2, v_2\}$ are distinct pairs.

(b) Necessity.

Suppose that at least one of the 2-partitions $S$ and $T$ is not a starter of the corresponding group. Then to show that $W_{ST}$ is not a starter, it suffices to find at least two pairs of $W_{ST}$ which produce the same differences.

If $T$ is not a starter then it contains at least two pairs $\{r_1, t_1\}, \{r_2, t_2\}$ such that $\{\pm(r_1 - t_1)\} \equiv \{\pm(r_2 - t_2)\} \ (\text{mod} \ m)$. Consequently, two pairs $\{r_1n, t_1n\}, \{r_2n, t_2n\}$ in $W_{ST}$ of type (ii) will yield the same differences modulo $mn$.

If $S$ is not a starter then it contains at least two pairs $\{x_1, y_1\}, \{x_2, y_2\}$ such that $\{\pm(x_1 - y_1)\} \equiv \{\pm(x_2 - y_2)\} \ (\text{mod} \ n)$. Then there are $2m$ pairs in $W_{ST}$ of types (i), which produce differences congruent to $\pm(x_1 - y_1)$ modulo $n$. They are $\{x_1, y_1\}, \{x_2, y_2\}, \{n + x_1, t + y_1\}, \{n + x_2, t + y_2\}, \{-n + x_1, -t + y_1\}, \{-n + x_2, -t + y_2\}, 1 \leq i \leq p$.

Hence, by Lemma 2.7(4), we conclude that there are two pairs among these $2m$ pairs that satisfy $\{\pm(u_1 - v_1)\} \equiv \{\pm(u_2 - v_2)\} \ (\text{mod} \ mn)$. So, (11) is impossible, which means $W_{ST}$ is not a starter in $\mathbb{Z}_{mn}$. This completes the proof of necessity. □
Next statement clarifies the conditions for obtaining a strong 2-partition.

**Theorem 3.6.** Let $n, m \geq 3$ be odd integers and $S$ and $T$ be 2-partitions of $\mathbb{Z}_n^*$ and $\mathbb{Z}_m^*$, respectively. Then their product $W_{ST}$ (Definition 2.4) is a strong 2-partition of $\mathbb{Z}_{mn}^*$ if and only if $S$ is strong and $T$ is skew.

**Proof.** (a) **Sufficiency.**

Let $S$ be strong and $T$ be skew. To show that $W_{ST}$ is strong, we have to show that if $\{u_1, v_1\}$ and $\{u_2, v_2\}$ are two distinct pairs in $W_{ST}$ then $u_1 + v_1 \not\equiv u_2 + v_2 \pmod{mn}$, and for any $\{u, v\} \in W_{ST}$ there holds $u + v \not\equiv 0 \pmod{mn}$.

Suppose, for the sake of contradiction,

$$u_1 + v_1 \equiv u_2 + v_2 \pmod{mn}. \quad (14)$$

Using notation (10), we have

$$(r_1 + t_1)n + x_1 + y_1 \equiv (r_2 + t_2)n + x_2 + y_2 \pmod{mn}.$$  

Consequently, by Lemma 2.7(1), we obtain

$$(r_1 + t_1)n + x_1 + y_1 \equiv (r_2 + t_2)n + x_2 + y_2 \pmod{n}.$$  

Then, by Lemma 2.7(3), $x_1 + y_1 \equiv x_2 + y_2 \equiv C \pmod{n}$. If $C = 0$, we get pairs of type (ii). Otherwise, since $\{x_i, y_i\}, i = 1, 2$, and $S$ is strong, we conclude $\{x_1, y_1\} = \{x_2, y_2\}$. In either case, by Lemma 2.7(2), (14) implies $r_1 + t_1 \equiv r_2 + t_2 \pmod{m}$. Here, the pairs $(r_1, t_1), (r_2, t_2)$ are from either set $T$ or $T'$. By the hypothesis of the theorem, $T$ is skew, which means that all sums of the pairs of $T$ along with $T'$ are different $\pmod{m}$. Thus, there are two options:

1. either $r_1 = r_2$ and $t_1 = t_2$,
2. or $r_1 = t_2$ and $t_1 = r_2$.

Case 1 contradicts our assumption that $\{u_1, v_1\}$ and $\{u_2, v_2\}$ are two distinct pairs in $W_{ST}$.

Case 2 is impossible due to the following reason. Let $n = t_2 = r$ and $t_1 = r_2 = t$. WLOG, assume $(r, t) \in T$ and $(t, r) \in T'$. But $(-r, -t) \in T'$, and $r - t = (-t) - (-r)$, which implies that $t \equiv -r \pmod{m}$. The latter means that $\{-r, r\} \in T$, which contradicts our assumption that $T$ is skew (T is not even strong in that case since $-r + r \equiv 0 \pmod{m}$).

Finally, let $\{u, v\} \in W_{ST}, u = rn + x, v = tn + y$. If $u + v \equiv 0 \pmod{mn}$, then by Lemma 2.7(1 and 3), $x + y \equiv 0 \pmod{n}$, which is impossible since $S$ is strong.

This completes the proof that $W_{ST}$ is a strongly 2-partition of $\mathbb{Z}_{mn}^*$.

(b) **Necessity.**

If $S$ is not strong, then, regardless of the properties of $T$, there are two possible cases:

1. $S$ contains a pair $\{x, y\}$ such that $x + y \equiv 0 \pmod{n}$. Then consider all the pairs of the type (i) and of the form $\{rn + x, tn + y\}$. There are exactly $m$ such pairs. These $m$ pairs along with $(m - 1)/2$ pairs of type (ii) yield $(3m - 1)/2$ sums in $\mathbb{Z}_{mn}$, which are
congruent to 0 modulo \( n \). But these sums cannot be all different and nonzero modulo \( mn \) by Lemma 2.7(4). Thus \( W_{ST} \) is not strong.

2. \( S \) contains two pairs \( \{x_1, y_1\} \) and \( \{x_2, y_2\} \) such that \( x_1 + y_1 \equiv x_2 + y_2 \equiv c \) (mod \( n \)). Then consider all the pairs of the type (i) and of the form \( \{mn + x_i, t_n + y_i\}, i = 1, 2 \). There are \( 2m \) of them, and all of them yield sums in \( \mathbb{Z}_{mn} \), which are congruent to \( c \) modulo \( n \). By Lemma 2.7(4), the sums cannot be all different modulo \( mn \). Thus \( W_{ST} \) is not strong.

If \( T \) is not skew, some of the pairs, say, \( (r_1, t_1) \in T \) and \( (r_2, t_2) \in T' \) yield the same sum (mod \( m \)). Let us take a pair \( (x, y) \in \tilde{S} \). Then pairs \( \{r_1n + x, t_1n + y\} \) and \( \{r_2n + x, t_2n + y\} \) produce the same sum modulo \( mn \). Hence \( W_{ST} \) is not strong. □

Remark 3.7. Not every 2-partition of a composite order is a product of two 2-partitions. For example, there are known [9] strong starters of order \( p^3 \) for some prime \( p > 3 \), but obtaining them by means of a product of two starters of orders 3 and \( p \), respectively, would have contradicted Theorem 3.6.

The following theorem clarifies the question whether or not \( W_{ST} \) is skew.

**Theorem 3.8.** Let \( n, m \geq 3 \) be odd integers and \( S \) and \( T \) be 2-partitions of \( \mathbb{Z}_n^* \) and \( \mathbb{Z}_m^* \), respectively. Then their product \( W_{ST} \) (Definition 2.4) is a skew 2-partition of \( \mathbb{Z}_{mn}^* \) if and only if both \( S \) and \( T \) are skew.

**Proof.** (a) **Sufficiency.**

Let \( S \) and \( T \) be skew and let \( \{u_1, v_1\} \) and \( \{u_2, v_2\} \) be two arbitrary distinct pairs of \( W_{ST} \). By Theorem 3.6, we know that \( W_{ST} \) is strong, that is, \( u_1 + v_1 \not\equiv u_2 + v_2 \) (mod \( mn \)). To show that \( W_{ST} \) is skew, it remains to show that there holds

\[
\begin{align*}
\text{(15)}
\quad u_1 + v_1 & \not\equiv -(u_2 + v_2) \pmod{mn}.
\end{align*}
\]

Suppose, for the sake of contradiction, that (15) is not true, that is, in notation (10),

\[
\begin{align*}
\text{Let } r_1n + x_1 + t_1n + y_1 & \equiv -(r_2n + x_2 + t_2n + y_2) \pmod{mn}.
\end{align*}
\]

By Lemma 2.7(1 and 3), we obtain \( x_1 + y_1 \equiv -(x_2 + y_2) \) (mod \( n \)), which is impossible as \( S \) is skew, unless \( \{\{u_i, v_i\}\}_{i=1}^2 \) are pairs of type (ii). But then \( (r_1n + t_1n) \equiv -(r_2n + t_2n) \) (mod \( mn \)), and hence, by Lemma 2.7(1) \( (r_1 + t_1) \equiv -(r_2 + t_2) \) (mod \( m \)), which is impossible, since \( T \) is skew.

This contradiction implies that for any two distinct pairs in \( W_{ST}, \{u_1, v_1\} \) and \( \{u_2, v_2\} \), there holds (15). Therefore, \( W_{ST} \) is skew.

(b) **Necessity.**

By Theorem 3.6, if \( T \) is not skew, then \( W_{ST} \) is not skew.

Now, suppose, \( S \) is strong but not skew, then \( \tilde{S} \) contains at least two pairs \( (x_1, y_1) \) and \( (x_2, y_2) \) such that \( (x_1 + y_1) \equiv -(x_2 + y_2) \equiv c \) (mod \( n \)). It is clear that \( c \not\equiv 0 \) (mod \( n \)), as \( S \) is strong.

There are \( m \) pairs \( \{mn + x_1, t_n + y_1\} \) of the type (i) in the starter \( W_{ST} \). There are also \( m \) pairs \( \{mn + x_2, t_n + y_2\} \) of the type (i) in the conjugate 2-partition \( W_{ST}' \).
These pairs yield $2m$ sums in $\mathbb{Z}_{mn}$, which are congruent to $c$ modulo $n$. But they could not be all different modulo $mn$ by Lemma 2.7(4). We conclude that there are two pairs among these $2m$ pairs that satisfy $\{\pm(u_1 + v_1)\} = \{\pm(u_2 + v_2)\} \pmod{mn}$. Hence, $W_{ST}$ is not skew. $\square$

Finally, we deal with Skolem 2-partitions.

**Theorem 3.9.** Let $n, m \geq 3$ be odd integers and $S$ and $T$ be 2-partitions of $\mathbb{Z}^*_n$ and $\mathbb{Z}^*_m$, respectively. Then their product $W_{ST}$ (Definition 2.4) is a Skolem 2-partition of $\mathbb{Z}^*_{mn}$ if and only if $S$ and $T$ are both Skolem 2-partitions of $\mathbb{Z}^*_n$ and $\mathbb{Z}^*_m$, respectively.

**Proof.** Let us order $\mathbb{Z}_k : 0 < 1 < \cdots < k - 1, k \in \mathbb{N}$.

(a) **Sufficiency.**

Let $S$ and $T$ be Skolem 2-partitions of orders $n$ and $m$, respectively. To show that $W_{ST}$ is Skolem, we have to show that all its pairs $\{u, v\}$ are Skolem pairs of order $mn$, that is, $u < v$ and $v - u \leq \frac{mn - 1}{2}$.

Let the pair $\{x^*, y^*\} \in S$ make the greatest difference in $S$, that is, $x^* < y^*, y^* - x^* \leq \frac{n - 1}{2}$. As well, consider the pairs in $\{r^*, t^*\} \in T$ and $\{-r^*, -t^*\} \in T'$, which make the greatest difference $t^* - r^* = (-r^*) - (-t^*) \leq \frac{m - 1}{2}$.

Each of the pairs $\{u, v\}$ in the form either $(x_i, y_i)$ or $(r_j n, t_j n)$ is Skolem because for any $n \geq 1$ and $m \geq 1$ we have

$$v - u \leq y^* - x^* \leq \frac{n - 1}{2} < \frac{mn - 1}{2}, \quad v - u \leq (t^* - r^*)n$$

$$\leq \left(\frac{m - 1}{2}\right)n = \frac{mn - n}{2} < \frac{mn - 1}{2}.$$ 

For other pairs of type (i) we consider two cases:

1. $(x^*, y^*) \in \tilde{S}$. Then a pair $\{u, v\} \in W_{ST}$, where $u = r^* n + x^* < v = t^* n + y^*$, makes the greatest possible difference modulo $mn$ among the pairs of $W_{ST}$,

$$v - u = t^* n + y^* - (r^* n + x^*) \leq \frac{m - 1}{2}n - \frac{n - 1}{2} = \frac{mn - 1}{2}.$$ 

2. $(y^*, x^*) \in \tilde{S}$. Then a pair $\{u, v\} \in W_{ST}$, where $u = -r^* n + y^* > v = -t^* n + x^*$, makes the greatest possible difference modulo $mn$ among the pairs of $W_{ST}$,

$$u - v = -r^* n + y^* - (-t^* n + x^*) \leq \frac{mn - 1}{2}.$$ 

All other pairs of type (i) are clearly Skolem as they make no difference greater than $\frac{mn - 1}{2}$.
(b) Necessity.

If \( T \) is not Skolem, then it contains a pair \( \{r, t\} \) which is not Skolem, that is, given \( r < t \), we have \( t - r \geq \frac{m + 1}{2} \). Then the corresponding pair of type (ii), \( \{rn, tn\} \in W_{ST} \), yields a difference greater than \( \frac{mn - 1}{2} \):

\[
 tm - tn \geq \frac{m + 1}{2}n = \frac{mn + n}{2} > \frac{mn - 1}{2}.
\]

If \( S \) is not Skolem, there is \( \{x, y\} \in S \) which is not Skolem, that is, given \( x < y, y - x \geq \frac{n + 1}{2} \). Now, let us take the pair \( \{r, t\} \in T \), such that \( r < t \) and \( t - r \geq \frac{m - 1}{2} \). WLOG, assume \( (x, y) \in \tilde{S} \) and \( (r, t) \in T \). Then the pair \( \{u, v\} \in W_{ST}, u = rn + x < v = tn + y \) makes the difference

\[
 v - u \geq \frac{m - 1}{2}n + \frac{n + 1}{2} = \frac{mn - n + n + 1}{2} = \frac{mn + 1}{2}.
\]

That means \( \{u, v\} \in W_{ST} \) is not a Skolem pair of order \( mn \) and hence, by Definition 1.4, \( W_{ST} \) is not Skolem. \( \square \)

Let us summarize the results of this subsection.

**Theorem 3.10.** Let \( S \) and \( T \) be 2-partitions of \( Z^*_n \) and \( Z^*_m \), respectively.

1. If both \( S \) and \( T \) are Skolem starters and, in addition, \( S \) is strong and \( T \) is skew, then the product \( W_{ST} \) (Definition 2.4) is a strong Skolem starter in \( Z^{\ast n} \). Moreover, if \( S \) and \( T \) are both skew and Skolem in their groups, then \( W_{ST} \) is a skew Skolem starter in \( Z^{\ast n} \).

2. If the product \( W_{ST} \) of 2-partitions \( S \) and \( T \) is a strong but not skew Skolem starter then \( S \) is a strong but not skew Skolem starter and \( T \) is a skew Skolem starter. If the product \( W_{ST} \) of partitions \( S \) and \( T \) is a skew Skolem starter then both \( S \) and \( T \) are skew Skolem starters.

**Proof.** The statement follows from Theorems 3.5, 3.6, 3.9, and 3.8. \( \square \)

**Remark 3.11.** The direct part of Theorem 3.10 can be obtained by combining the idea and constructions of Gross in [7], Turgeon in [16], and Chen et al. [1]. The converse statement cuts off the possibility to generate a strong (skew) Skolem starter by means of “multiplication” of two 2-partitions unless the “factors” possess all the listed properties.

**Example 3.12.** Let us choose the strong (but not skew) Skolem starter \( S \) of order 17 from Example 1.3. Using Definition 2.4, it is possible to generate strong Skolem starters of orders \( 17m \), where \( m \) is one of the orders of the known skew Skolem starters. By Theorem 1.7, there are infinitely many such starters. Paper [12] gives an explicit way of constructing a family of cardioidal starters (4). It was proven in [12] that every cardioidal starter is skew unless its order is divisible by 3. Take, for example, the following cardioidal starter of order 11: \( T = \{\{1, 2\}, \{7, 9\}, \{3, 6\}, \{4, 8\}, \{5, 10\}\} \). Since \( T \) is a skew Skolem starter, \( W_{ST} \) is a strong Skolem starter in \( Z_{17 \cdot 11} = Z_{187} \).
Definition 2.4 can be modified in a variety of ways to achieve diversity of the obtained 2-partitions. For example, we could make an alternative construction of the pairs (9) of $W_{ST}$:

(i*) $2pq$ pairs: one for each $(x, y) \in S \cup \bar{S}$ and for each $(r, t) \in \bar{T}$ and $q$ pairs for $r = t = 0$ and $(x, y) \in S$.

(ii*) $p$ pairs: one for each $(r, t) \in \bar{T}$ and $x = y = 0$.

The proofs of the statements, involving this way of constructing $W_{ST}$, will be analogous.

This modification, while significantly expanding the variety of the obtained starters, does not let us produce a strong starter as a product of two strong starters. For example, we cannot obtain a strong Skolem starter of order $17^2 = 289$ out of the starter $S$ of order 17 used in Example 3.12. To achieve this objective, we have to further modify Definition 2.4.

### 3.3 The product of 2-partitions with a nucleus

In this section, we construct a family of products by introducing the following object.

**Definition 3.13.** The set of ordered pairs $X_m = \{(u_i, v_i), i = 1, m - 1 \in \mathbb{Z}^*_m \times \mathbb{Z}_m^*, m \geq 3, \}$ is called a nucleus of order $m$ if $\bigcup_{i=1}^{m-1} \{u_i\} = \bigcup_{i=1}^{m-1} \{v_i\} = \mathbb{Z}^*_m$.

**Definition 3.14.** A nucleus $X_m$ of order $m$ is called

1. subtractive in $\mathbb{Z}_m$, if $\{(u_i - v_i) \mod m\}_{i=1}^{m-1} = \mathbb{Z}^*_m$;
2. skew in $\mathbb{Z}_m$, if $\{(u_i + v_i) \mod m\}_{i=1}^{m-1} = \mathbb{Z}^*_m$;
3. Skolem in $\mathbb{Z}_m$, if it consists of Skolem pairs of order $m$.

Let $T$ be a 2-partition of $\mathbb{Z}^*_m$. Then $T \cup T'$ is an important example of $X_m$. Let us make a statement on some properties of $T \cup T'$.

**Lemma 3.15.** Let a nucleus $X_m = T \cup T'$, where $T$ is a 2-partition of $\mathbb{Z}^*_m$. Then the following holds:

1. if $T$ is a starter then $X_m$ is subtractive;
2. if $T$ is skew then so is $X_m$;
3. if $T$ is Skolem then so is $X_m$.

**Proof.** The statements follow immediately from Definitions 1.1, 3.13, 3.14, and Lemma 3.3.

Now we define a product with nucleus $X_m$.

**Definition 3.16.** Let $S$ and $T$ be 2-partitions of $\mathbb{Z}^*_n$ and $\mathbb{Z}^*_m$, respectively, $n = 2q + 1, m = 2p + 1, q, p \geq 1$. Let $X_m$ be a nucleus of order $m$. The set $\mathcal{W}_{ST}$ consisting of pairs of the form (9) divided into the following types:
(iX) \( m \) \( q \) pairs: one for each \((x, y) \in \tilde{S}\) and \((r, t) \in \{0, 0\} \cup X_m\);
(iiX) \( p \) pairs: one for each \((r, t) \in \tilde{T}\) and \(x = y = 0\),

is called the \( X \)-generated product of \( S \) and \( T \).

Below we will show that different choices of nucleus \( X_m \) will lead to different properties of \( W_{ST}^X \). It turns out that the use of a proper nucleus allows us to loosen the hypotheses of Theorems 3.6 and 3.10. Consequently, we can extend the family of strong Skolem starters and extend the list of orders \( n \) such that \( Z_n \) admits a strong Skolem starter.

**Theorem 3.17.** The set \( W_{ST}^X \) (Definitions 3.13 and 3.16) is a 2-partition of \( Z^*_{mn} \).

**Proof.** The proof is analogous to that of Theorem 2.8 as the set of ordered pairs \( X_m \) has, by Definition 3.13, the properties of the set \( T \cup T' \) used to prove Theorem 2.8. (Similar proof of this statement was offered in [18].)

**Theorem 3.18.** The set \( W_{ST}^X \) (Definitions 3.13 and 3.16) is a starter in \( Z_{mn} \) if and only if \( X_m \) is subtractive in \( Z_m \), and \( S \) and \( T \) are starters in \( Z_n \) and \( Z_m \), respectively.

**Proof.**

(a) **Sufficiency.**

The proof of sufficiency is analogous to that of Theorem 3.5 as the set of ordered pairs \( X_m \) has, by Lemma 3.15, the properties of the set \( T \cup T' \) used in the part (a) of the proof of Theorem 3.5.

(b) **Necessity.**

Given \( W_{ST}^X \) is a starter, let us assume for the sake of contradiction that \( X_m \) is not subtractive. Then there are two possible cases:

1. there exists a pair \((z, z) \in X_m, z \in Z_m\). Then for every pair \((x, y) \in \tilde{S}\), we have: \([x, y]\) and \([nz + x, nz + y]\) are pairs of \( W_{ST}^X \). But these pairs yield the same difference modulo \( mn \), and hence \( W_{ST}^X \) is not starter;
2. there exist two pairs \((r', t')\) and \((r'', t'')\) in \( X_m \) such that \( r' - t' \equiv r'' - t'' \pmod{m} \).

Then for every pair \((x, y) \in \tilde{S}\), the pairs \([nr' + x, nt' + y]\) and \([nr'' + x, nt'' + y]\) are in \( W_{ST}^X \). Since the two pairs in \( W_{ST}^X \) yield the same differences modulo \( mn \), \( W_{ST}^X \) is not a starter.

Both cases lead to a contradiction. Hence \( X_m \) must be subtractive given \( W_{ST}^X \) is a starter. The assumption that at least one of the two sets, \( S \) and \( T \), is not a starter in its corresponding group, leads to a contradiction, too. The chain of reasonings showing this contradiction is analogous to that of Theorem 3.5.

Thus, Theorems 2.8 and 3.5 are particular cases of Theorems 3.17 and 3.18, respectively, where \( X_m \) coincides with \( T \cup T' \). The direct part of the statement of Theorem 3.5 was proven in [16] as the subtractive and Skolem nucleus can be viewed as a special case of a perfect difference matrix.

**Theorem 3.19.** The set \( W_{ST}^X \) (Definitions 3.13 and 3.16) is a strong (skew) 2-partition of \( Z^*_{mn} \) if and only if \( X_m \) is skew in \( Z_m \), \( S \) is strong (skew) 2-partition of \( Z^*_{n} \) and \( T \) is a strong (skew) 2-partition of \( Z^*_{m} \).
Proof. Here, to obtain a strong 2-partition $W_{ST}^X$, we do not require the second 2-partition $T$ to be skew (unlike in Theorem 3.6), because now the pairs of type $(i_X)$ are formed from the skew nucleus $X_m$ and the strong first 2-partition $S$. The rest of the proof is analogous to those of Theorems 3.6 and 3.8. □

Theorem 3.20. The set $W_{ST}^X$ (Definitions 3.13 and 3.16) is a Skolem 2-partition of $Z_{mn}^*$ if and only if $X_m$ is Skolem in $Z_m$ and $S$ and $T$ are Skolem 2-partitions of $Z_n^*$ and $Z_m^*$, respectively.

Proof. The proof is analogous to that of Theorem 3.9. □

Note that a nucleus $X_m$, which is both skew and subtractive, does not exist for some odd integer orders $m \geq 3$. This can be shown using the notion of a strong permutation of a set of elements in a group.

Remark 3.21. Recall that a permutation $\pi$ is called strong if the maps $i \mapsto (\pi(i) - i)$ and $i \mapsto (\pi(i) + i)$ are permutations, too. In 1973, Wallis and Mullin proved [18] that if $G$ is a group of odd order $n$, $3 \mid n$, and the 3-Sylow subgroup of $G$ is cyclic, then $G$ does not admit a strong permutation. We adjust this statement to our context, as we deal with cyclic groups only, and hence any subgroup is cyclic, too.

Lemma 3.22. A skew and subtractive nucleus $X_m$ of order $m \geq 3$ exists if and only if $3 \nmid m$.

Proof. (a) According to Definitions 3.13 and 3.14, the existence of a skew and subtractive $X_m$ is equivalent to the existence of a strong permutation $i \mapsto \pi(i)$ of the set $\{0, 1, \ldots, m - 1\}$, given $0 \mapsto \pi(0) = 0$:

$$\pi: \nu_i \mapsto u_i, 1 \leq i \leq m - 1.$$ (16)

For the sake of contradiction, assume that $3 \mid m$. Then we can write $m = 3^k t$, $t \geq 1$, $3 \nmid k$.

Assuming that $\pi$ is a strong permutation of the elements of $Z_m$, consider the sums:

$$\sum_{i \in Z_n^*} i^2 \equiv \sum_{i \in Z_n^*} \pi(i)^2 \equiv \sum_{i \in Z_n^*} \pi(i)^2 + \sum_{i \in Z_n^*} i^2 + 2 \sum_{i \in Z_n^*} i\pi(i) \equiv \sum_{i \in Z_n^*} \pi(i)^2 + \sum_{i \in Z_n^*} i^2 - 2 \sum_{i \in Z_n^*} i\pi(i) \pmod{m}. \quad (17)$$

Taking the second line from the first one, we get $\sum_{i \in Z_n^*} i\pi(i) \equiv 0 \pmod{m}$. Hence, $\sum_{i \in Z_n^*} i^2 \equiv 0 \pmod{m}$.

But since $k$, $(2 \cdot 3^k - 1)$, $\frac{3^{k-1}}{2}$ are all coprime to 3, we have

$$\sum_{i=1}^{3^{k-1}/2} i^2 = \frac{3^k(3^k - 1)(2 \cdot 3^k - 1)}{6} = \frac{3^{k-1}k(2 \cdot 3^k - 1)(3^k - 1)}{2} \not\equiv 0 \pmod{m}.$$ 

This is a contradiction. Thus, $3 \nmid m$. 

OGANDZHANYANTS ET AL.
(b) Let \( \gcd(m, 6) = 1 \). Then the map \( \pi : i \mapsto 2i \) of the elements of \( \mathbb{Z}_m \) is clearly a strong permutation of the set \( \{0, 1, \ldots, m - 1\} \). Indeed, the map \( \pi - e : i \mapsto (\pi(i) - i) = i \) is a permutation of this set, so is \( \pi \) since \( m \) is odd, and so is \( \pi + e : i \mapsto (\pi(i) + i) = 3i \) since \( 3 \nmid m \). (The use of the permutation \( \pi : i \mapsto 2i \) for constructing strong starters was offered by Gross in [7].) The permutation \( \pi \) yields the skew and subtractive nucleus, which we denote by \( C_m \)

\[
X_m = C_m = \{(i, 2i \text{ (mod } m)) \}, 1 \leq i \leq m - 1 \]. \hfill (18)

Remark 3.23. A nucleus \( X_m \) of order \( m \) divisible by 3 may be skew, unless we require that it is also subtractive. For example, we can take \( X_9 = Q \cup Q' \) of order 9, where \( Q \) is the 2-partition of \( \mathbb{Z}_9^* \) from Example 1.3. In this case, \( X_9 \) is skew and Skolem, but not subtractive.

Section 3.4 discusses the product with the nucleus \( C_m \) found in the proof of Lemma 3.22.

### 3.4 The cardioidal product

**Definition 3.24.** The set \( C_m \) (18) is called the cardioidal nucleus of order \( m \).

For 2-partitions, \( S \) of \( \mathbb{Z}_m^* \), and \( T \) of \( \mathbb{Z}_n^* \), \( m = 2q + 1, n = 2p + 1, q, p \geq 1 \), the product introduced by Definition 3.16 with nucleus \( X_m = C_m \) is called a cardioidal product of \( S \) and \( T \). It is denoted by \( W_{ST}^c \).

**Lemma 3.25.** If \( T \) is a cardioidal starter then \( W_{ST}^c = W_{ST}^c \).

**Proof.** If \( T \) is a cardioidal starter (refer to Definition 1.1) then \( \bar{T} \cup \bar{T'} = \{(i, 2i \text{ (mod } m)) \}, 1 \leq i \leq m - 1 \) is the cardioidal nucleus of order \( m \). The rest follows from Definitions 2.4 and 3.24. \hfill \Box

Note, that if \( T \) is a cardioidal 2-partition of \( \mathbb{Z}_m^* \), but not a starter, Lemma 3.25 does not work. The following theorem clarifies when the cardioidal nucleus obeys the conditions of each theorem in Section 3.3.

**Theorem 3.26.** The cardioidal nucleus \( C_m \) is subtractive and Skolem for all odd \( m \geq 3 \). The cardioidal nucleus \( C_m \) is skew if and only if \( 3 \nmid m \).

**Proof.** The statement follows from Lemma 3.2 of [12], and Lemma 3.22. \hfill \Box

Then, we have:

**Theorem 3.27.** If \( 3 \nmid m \) and \( S \) in \( \mathbb{Z}_n \) and \( T \) in \( \mathbb{Z}_m \) are strong (skew) Skolem starters, then so is \( W_{ST}^c \).
Proof. The statement follows from Theorems 3.19, 3.20, and 3.26. □

Remarkably, a pair of cardioidal starters does not necessarily produce a cardioidal starter. The following example demonstrates this idea.

**Example 3.28.** Consider the starter \( R = \{(1, 2), (7, 9), (3, 6), (4, 8), (5, 10)\} \) in \( Z_{11} \). It is cardioidal as \( 2 \equiv 1 \cdot 2 \pmod{11}, 7 \equiv 9 \cdot 2 \pmod{11}, 6 \equiv 3 \cdot 2 \pmod{11}, 8 \equiv 4 \cdot 2 \pmod{11}, 10 \equiv 5 \cdot 2 \pmod{11} \).

But the product \( W_{RR} \) of \( R \) with itself contains the pair \( \{7, 9\} \), which is not cardioidal \( \pmod{121} \): \( 9 \equiv 21 \cdot 87 \pmod{121} \) and \( 7 \equiv 21 \cdot 14 \pmod{121} \).

Note that in this case \( W_{RR} = W_{RR}^c \) by Lemma 3.25.

**Lemma 3.29.** Let \( S \) and \( T \) be 2-partitions of \( Z_n^* \) and \( Z_m^* \), respectively. If at least one of these 2-partitions is not cardioidal, then their product \( W_{ST} \) is not cardioidal.

Proof. Let \( T \) be a noncardioidal 2-partition of \( Z_m^* \). Then, by Definition 1.1, there is a pair \( \{x, y\} \in T \) which is not cardioidal of order \( m \). Then by Definition 2.4 the pair \( \{nx, ny\} \in W_{ST} \) is not cardioidal of order \( mn \), which implies that \( W_{ST} \) is not cardioidal.

Now, let \( S \) be a noncardioidal 2-partition of \( Z_n^* \). Then by Definition 1.1, there is a pair \( \{x, y\} \in S \), which is not cardioidal of order \( n \).

Assuming WLOG that \( 0 < x < y < n \), the above means that both \( y \neq 2x \) and \( x \neq 2y - n \). On the other hand, \( \{x, y\} \in W_{ST} \). If \( \{x, y\} \) were cardioidal of order \( mn \), then either \( y = 2x \) (contradiction), or \( x = 2y - mn \), where \( m \geq 3 \), which would imply \( x < 0 \), since \( y < n \). Therefore, \( \{x, y\} \in W_{ST} \) is not cardioidal of order \( mn \), and hence \( W_{ST} \) is not cardioidal.

Let us indicate all possible cases, when a product \( W_{ST} \) of two cardioidal starters, \( S \) and \( T \), is cardioidal.

**Lemma 3.30.** A product \( W_{ST} \) of two cardioidal starters, \( S \) in \( Z_n^* \) and \( T \) in \( Z_m^* \), is cardioidal if and only if \( n = 3 \) and either \( m = 3 \) or \( m > 3 \) and \( \tilde{S} = \{(1, 2)\} \).

Proof.

1. \( n = m = 3 \). The case \( \tilde{S} = \{(1, 2)\} \) is given in Example 2.5. Choosing \( \tilde{S} = \{(2, 1)\} \) does not change \( W_{ST} \).

2. \( n = 3, m > 3 \) and \( \tilde{S} = \{(1, 2)\} \). By Definition 2.4, the pairs of type (ii) are cardioidal as \( T \) is cardioidal. Consider a pair \( \{u, v\} \in W_{ST} \) of type (i). We have \( \{u, v\} = \{in + 1, jn + 2\} \), where \( j \equiv 2i \pmod{m} \). Then \( 2(in + 1) \equiv jn + 2 \pmod{mn} \). That means that \( \{u, v\} \) is cardioidal. Hence, \( W_{ST} \) is a cardioidal starter of order \( mn \).

3. \( n = 3, m > 3 \) and \( \tilde{S} = \{(2, 1)\} \). Then \( \{5, 7\} \in W_{ST} \). But this pair is not cardioidal of order \( mn > 9 \). Hence, \( W_{ST} \) is not cardioidal.

4. \( n > 3 \). Then either \( \{-2, -1\} \pmod{n} \) or \( \{-4, -2\} \pmod{n} \) is a pair in \( S \) as it is cardioidal of order \( n \). Regardless of what kind of starter \( T \) is, either \( \{n - 2, n - 1\} \) or \( \{n - 4, n - 2\} \) appears in \( W_{ST} \). But neither of these pairs is cardioidal of order \( mn > 9 \). Hence, \( W_{ST} \) is not cardioidal. □
To elaborate, if at least one of the factors is not cardioidal, then the product is also not cardioidal by Lemma 3.29, but if both factors are strong (skew) Skolem, then so is the product. If, on the other hand, both factors are cardioidal, and both are also strong (skew), then $mn \geq 1$, so the hypothesis of Lemma 3.30 is not satisfied, and the product of such factors is strong (skew) Skolem but not cardioidal.

Note that all strong Skolem starters referred to in Theorem 1.7 and constructed in [12] are cardioidal. The paper [12] establishes that each strong cardioidal starter is skew, and hence, by Theorem 3.8, the product of strong cardioidal starters is a skew starter. For instance, $W_{RR}$ from Example 3.28 is a skew starter in $Z_{121}$ (as well as $W_{RR'}$ and $W_{RR''}$).

Theorem 1.7 says that $Z_n$ admits a (cardioidal) skew Skolem starter for all $n$ from $C_2 \setminus \{3\}$, that is, from the multiplicative closure of the set $C_2 \setminus \{3\}$, where $C_2 = \{ p \text{ prime} | \text{ord}_p(2) \equiv 2 \pmod{4}\}$. Although in [12], it was proved that skew cardioidal starters do not exist for orders other than indicated in Theorem 1.7, it was also shown that $C_2$ is infinite.

Therefore, Theorem 3.10 implies an explicit construction of an infinite family of skew Skolem starters of all composite orders from $C_2 \setminus \{3\}$. This family is fully new because it consists of starters that are not cardioidal. This family, in its turn, gives rise to further explicit construction of infinitely many strong (and even skew) Skolem starters.

Meanwhile, Theorems 3.10 and 3.27 do not limit us to these composite orders. If we find, by some means, a strong Skolem starter $S$ of an order $n \notin C_2 \setminus \{3\}$, Theorems 3.6 and 3.19 pave an explicit way to construct a strong Skolem starter of any order $nm$, where $m \in C_2 \setminus \{3\}$. The following example illustrates this idea.

**Example 3.31.** Consider a strong Skolem starter $S = \{(25, 26), (20, 22), (21, 24), (8, 12), (18, 23), (10, 16), (7, 14), (1, 9), (2, 11), (3, 13), (4, 15), (5, 17), (6, 19)\}$, found by Shalaby [14], and $T = \{(1, 2), (15, 17), (13, 16), (4, 8), (5, 10), (6, 12), (7, 14), (3, 11), (9, 18)\}$ is skew in $Z_{19}$. Then both $W_{ST}$ and $W_{ST'}$ are strong Skolem starters in $Z_{27 \cdot 19} = Z_{513}$. In general, we can construct a strong Skolem starter in any $Z_{27m}$, $m \in C_2 \setminus \{3\}$. Hence, we receive infinitely many strong Skolem starters of orders divisible by 3.

Moreover, by Theorem 3.27, given a strong Skolem starter $S$ in $Z_n$, where $3 \nmid n$, we can construct a strong Skolem starter of any order $n'm, t \geq 1, m \in C_2 \setminus \{3\}$. If, in addition, $S$ is skew, then we can construct a skew Skolem starter of any order $n'm, t \geq 1, m \in C_2 \setminus \{3\}$.

Finally, we note a possibility of a product with a noncardioidal nucleus, as shown in the following example.

**Example 3.32.** Let $R = W_{PQ}^c$, where $P$ and $Q$ are cardioidal starters of orders $k$ and $l$, respectively, $\{k, l\} \subseteq C_2 \setminus \{3\}$. Then by Theorem 3.27 and Lemma 3.30, $R$ is a skew Skolem starter, but not cardioidal. Consider $X = R \cup R'$. Clearly, $X$ is a skew Skolem nucleus of order $n = kl$, and $X$ is not cardioidal.

Let also $S$ and $T$ be strong Skolem starters of orders $m$ and $n$, respectively. Then by Theorem 3.27, $W_{ST}^X$ is a strong Skolem starter of order $mn$. However, $W_{ST}^X$ is not cardioidal. This can be shown by the reasoning similar to that given in Lemma 3.30, Case 4.
4  |  MAIN RESULT AND APPLICATIONS

4.1  |  Main result

In this paper, we introduced products of two 2-partitions of $\mathbb{Z}_n^*$ and $\mathbb{Z}_m^*$ that give a 2-partition of $\mathbb{Z}_{nm}^*$. These binary multivalued operations are interesting by themselves and deserve further investigations.

The products reveal a remarkable phenomenon: the resulting partition inherits some properties of the initial ones, such as being a starter, or being strong, skew or Skolem. Moreover, in many cases, if the resulting partition has these properties then so do the initial ones. Our results, partly overlapping with findings of Gross [7] and Turgeon [16], extend the results of [7] in the special case of cyclic groups $\mathbb{Z}_n$. Our specialization allowed us to track specific properties of the products of 2-partitions, which were not exposed for more general combinatorial objects discussed in [16].

Remark 4.1. For strong (skew) Skolem starters $S$ and $T$, we consider three specific choices of the nucleus $X$ when forming the pairs of strong (skew) Skolem starter $W_{ST}^X$:

1. $X = T \cup T'$ (in this case $W_{ST}^X = W_{ST}$);
2. $X$ is cardioidal of the same order as $T$;
3. $X = R \cup R'$, where $R$ is a (skew but not cardioidal) Skolem starter of the same order as $T$.

Not every property is passed from 2-partitions to their products. The product $W_{ST}$ of two strong starters, by Theorem 3.6, does not lead to a strong starter unless one of the initial starters is skew. However, the product $W_{ST}^X$ of two strong starters is a strong starter if the nucleus $X_m$ is subtractive and skew, for example, if it is cardioidal $C_m$ (18), where $3 \nmid m$.

On the basis of Lemmas 3.29 and 3.30, we can state that the main objective of the paper has been achieved: we constructed new family of strong Skolem starters that are not cardioidal.

Theorem 4.2. There are infinitely many strong and there are infinitely many skew Skolem starters that are not cardioidal.

Proof: The infinitude of the class of strong (skew) cardioidal starters is proven in [12]. The choices of nucleus outlined in Remark 4.1 in the construction of $W_{ST}^X$ based on two strong (skew) cardioidal starters lead to a new strong (skew) Skolem starter which is not cardioidal, since the smallest order of the factors is 11, which does not satisfy the hypothesis of Lemma 3.30.

By Lemmas 3.29 and 3.30, a product of two strong (skew) Skolem starters is necessarily noncardiological, therefore these products represent a new infinite family of strong (skew) Skolem starters.

Our new results give further support to Shalaby's conjecture stated in 1991.

Observe that the approach of constructing new strong, skew and Skolem starters outlined in this paper, has its limitations because not every starter in a group $\mathbb{Z}_{mn}$, is a product of starters in $\mathbb{Z}_n$ and $\mathbb{Z}_m$, respectively. For example, due to the nonexistence of Skolem starters of orders 5 and 7 and the nonexistence of a strong starter of order 5, the strong Skolem starter of order 35
found in [14] cannot be constructed as a product $W^X_{ST}$ of two starters with a nucleus, as it would have contradicted Theorems 3.20 and 3.19. On the other hand, if we encounter a starter whose structure is that of a product of two “parental” 2-partitions, then, by observing its properties, we can say a lot about the “parents.”

The theorems of this paper establish conditions of existence of starters of certain types and orders and help cut off some unsuccessful directions in their search. Finding a successful way in many cases is yet an open problem and a subject of further investigations.

4.2 Applications of strong Skolem starters

Pursuing our particular interest in constructing (strong, skew) Skolem starters, we generate new Skolem starters out of the known ones, and hence, new Skolem sequences. Skolem sequences were originally used by Skolem in 1957 for the construction of Steiner triple systems [15]. Further applications of Skolem sequences are discussed in [5] and the references therein.

The value of strong Skolem starters of order $2q + 1$ is in their applicability in constructing Room squares and cubes of order $2q + 2$ on the one hand, and Steiner triple systems, STS ($6q + 1$), on the other. Recall that an STS($v$) is a collection of 3-subsets, called blocks, of a $v$-set $S$, such that every two elements of $S$ occur together in exactly one of the blocks. The explicit constructions of Room squares and STS using strong Skolem starters are illustrated in Example 4.3.

Skew starters give rise to special Room squares called skew Room squares, important combinatorial designs.

The STSs, obtained from Skolem starters, are a useful tool for graceful labeling of various triangular cacti ($\Delta$-cacti). (A $\Delta$-cactus is a connected graph whose blocks are all triangles.) Graceful labeling of cacti is widely employed in many areas, such as X-ray crystallography, astronomy, code theory, and communication networks.

Example 4.3. Let $R = \{[s_i, t_i] \}_{i=1}^{5} = \{[1, 2], [7, 9], [3, 6], [4, 8], [5, 10] \}$. It is a strong Skolem starter in $Z_{11} : R = \{1 + 2, 7 + 9, 3 + 6, 4 + 8, 5 + 10 \pmod{11} \} = \{3, 5, 9, 1, 4 \} \subset Z_{11}^*, |R| = 5$. Consequently, $R' = \{9, 10\}, \{2, 4\}, \{5, 8\}, \{3, 7\}, \{1, 6\}$.

- A skew Room square of order 12 on the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \infty \}$ constructed out of $R$. See Table 1.
- The translates of the base blocks $[0, i, t_i + 5], 1 \leq i \leq 5$, obtained from $R$, generate an STS(31):

$$\{0, 1, 7\} \{0, 2, 14\} \{0, 3, 11\} \{0, 4, 13\} \{0, 5, 15\} \pmod{31}.$$  

- The translates of the base blocks $[0, s_i + 5, t_i + 5], 1 \leq i \leq 5$, obtained from $R$, generate another STS(31):

$$\{0, 6, 7\} \{0, 12, 14\} \{0, 8, 11\} \{0, 9, 13\} \{0, 10, 15\} \pmod{31}.$$  

The existence of a strong Skolem starter immediately implies two types of graceful labeling (called $\sigma$- and $\rho$-labeling) of the $\Delta$-cacti as given in Rosa's paper, see [13, p. 90]. Recall that $\sigma$-labeling and $\rho$-labeling of a graph $G(V, E), |E| = n$, are one-to-one
mappings \( \phi : V \rightarrow \{0, 1, 2, \ldots, 2n\} \) such that, respectively, \( |\phi(u) - \phi(v)| : [u, v] \in E \) = \{1, 2, \ldots, n\}, and \( |\phi(u) - \phi(v)| : [u, v] \in E \) = \{\(x_i, x_2, \ldots, x_n\}\}, where either \(x_i = i\) or \(x_i = 2n + 1 - i, 1 \leq i \leq n\). The problem of graceful labeling of \(\Delta\)-cacti is, according to Gallian [6, p. 19], “hopelessly difficult.” The infinite families of strong Skolem starters discovered in [12] and in the current paper bring a tiny dent to the problem.

The following example illustrates the ways of \(\sigma\)-labeling and \(\rho\)-labeling.

**Example 4.4.** Take the strong Skolem starter \(R\) of order \(p = 11\) from Example 4.3 and the two STS(31) constructed from \(R\). Set \(q = (p - 1)/2\) and construct the graph \(B(G)\), called “standard base” for a \(\Delta\)-snake (particular case of a \(\Delta\)-cactus) with \(q\) blocks:

\[
0 - q - 1 - (q - 1) - 2 - (q - 2) - \ldots.
\]

For \(q = 5 : 0 - 5 - 1 - 4 - 2 - 3\). Then translate the base blocks of the two STSs so that the pairs of the arc-ends of \(B(G)\) appear there:

\[
\{2, 3, 9\} \{2, 4, 16\} \{1, 4, 12\} \{1, 5, 14\} \{0, 5, 15\} \pmod{31}
\]

and

\[
\{27, 2, 3\} \{21, 2, 4\} \{24, 1, 4\} \{23, 1, 5\} \{21, 0, 5\} \pmod{31}.
\]

Now we are ready to set up \(\sigma\)-labelings and \(\rho\)-labelings on \(\Delta\)-cacti with 5 blocks as shown in Figures 1–3.

There are 32 different \(\sigma\)-labelings and \(\rho\)-labelings, as wherever two labels are offered, either of them can be chosen independently [10–12, 17].

**TABLE 1** The pairs of \(R\) appear in the first row of the table in the columns according to their sums modulo 11.

|   | 0   | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|---|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0 | ∞   | 0   | 4   | 8   | -   | 1   | 2   | 5   | 10  | 7   | 9   | -   |
| 1 | -   | ∞   | 1   | 5   | 9   | -   | 2   | 3   | 6   | 0   | 8   | 10  | -   |
| 2 | 5   | 8   | -   | ∞   | 2   | 6   | 10  | -   | 3   | 4   | 7   | 1   | 9   | 0   | -   |
| 3 | -   | 6   | 9   | -   | ∞   | 3   | 7   | 0   | -   | 4   | 5   | 8   | 2   | 10  | 1   | -   |
| 4 | -   | -   | 7   | 10  | -   | ∞   | 4   | 8   | 1   | -   | 5   | 6   | 9   | 3   | 0   | 2   |
| 5 | -   | -   | -   | 8   | 0   | -   | ∞   | 5   | 2   | -   | 6   | 7   | 10  | 4   | 1   | 3   |
| 6 | 2   | 4   | -   | -   | -   | 9   | 1   | -   | ∞   | 6   | 10  | 3   | -   | 7   | 8   | 0   | 5   |
| 7 | 1   | 6   | 3   | 5   | -   | -   | -   | 10  | 2   | -   | ∞   | 7   | 0   | 4   | -   | 8   | 9   |
| 8 | 9   | 10  | 2   | 7   | 4   | 6   | -   | -   | -   | 0   | 3   | -   | ∞   | 8   | 1   | 5   | -   |
| 9 | -   | 10  | 0   | 3   | 8   | 5   | 7   | -   | -   | -   | 1   | 4   | -   | ∞   | 9   | 2   | 6   |
| 10| 3   | 7   | -   | 0   | 1   | 4   | 9   | 6   | 8   | -   | -   | -   | 2   | 5   | -   | ∞   | 10  |

*Note:* The pairs of \(R'\) appear in the first column of the table in the rows according to their sums modulo 11.
REFERENCES
1. K. Chen, G. Ge, and L. Zhu, Starters and related codes, J. Statist. Plann. Inference 86 (2000), 379–395.
2. C. J. Colbourn and J. H. Dinitz, Handbook of combinatorial design, 2nd ed., Chapman and Hall/CRC, Boca Raton, FL, 2007.
3. J. H. Dinitz and D. R. Stinson, Contemporary design theory: A collection of surveys, John Wiley & Sons Inc., New York, 1992.
4. J. H. Dinitz and D. R. Stinson, A fast algorithm for finding strong starters, SIAM J. Alg. Disc. Math. 2 (1981), no. 1, 50–56.
5. N. Francetić and E. Mendelsohn, A survey of Skolem-type sequences and Rosa’s use of them, Math. Slovaca. 59 (2009), no. 1, 39–76.
6. J. A. Gallian, A dynamic survey of graph labeling, Electron. J. Combin. (2022), #DS6.
7. K. B. Gross, A multiplication theorem for strong starters, Aeq. Math. 11 (1974), 169–173.
8. G. Ge, Y. Miao, and X. Sun, Perfect difference families, perfect difference matrices, and related combinatorial structures, J. Combin. Des. 18 (2010), 415–449.
9. J. D. Horton, Orthogonal starters in finite abelian groups, Discrete Math. 79 (1989/90), 265–278.

FIGURE 1 A Δ-snake—eight different labelings.

FIGURE 2 A Δ-snake—16 different labelings.

FIGURE 3 A “fish-like” Δ-cactus—eight different labelings.

ORCID
Oleg Ogandzhanyants http://orcid.org/0000-0001-8476-9113
Nabil Shalaby http://orcid.org/0000-0002-7967-5937
10. R. C. Mullin and E. Nemeth, *An existence theorem for room squares*, Canad. Math. Bull. **12** (1969), no. 4, 493–497.
11. R. C. Mullin and R. G. Stanton, *Construction of room squares*, Ann. Math. Statist. **39** (1968), no. 5, 1540–1548.
12. O. Ogandzhanyants, M. Kondratieva, and N. Shalaby, *Strong Skolem starters*, J. Combin. Des. **27** (2018), no. 1, 5–21.
13. A. Rosa, *Cyclic Steiner triple systems and labelings of triangular cacti*, 4th Southeastern Conference on Combinatorics, Graph Theory, and Computing, 1988, pp. 85–95.
14. N. Shalaby, *Skolem sequences: Generalizations and applications*, Ph.D. Thesis, Mc-Master University, Canada, 1991.
15. T. Skolem, *On certain distributions of integers in pairs with given differences*, Math. Scand. **5** (1957), 57–68.
16. J. M. Turgeon, *An upper bound for the length of additive sequences of permutations*, Util. Math. **17** (1980), 189–196.
17. A. Vázquez-Ávila, *On strong Skolem starters*, J. Disc. Math. Sci. Crypt. **25** (2022), 2607–2705.
18. W. D. Wallis and R. C. Mullin, *Recent advances on complementary and room squares* (R. A. Bari, ed.), Proc. 4th Southeastern Conference on Combinatorics, Graph Theory and Computing, 1973, pp. 521–531.
19. D. Wu, M. Cheng, and Z. Chen, *Perfect difference families and related variable-weight optical orthogonal codes*, Aust. J. Combin. **55** (2013), 153–166.

**How to cite this article:** O. Ogandzhanyants, M. Kondratieva, and N. Shalaby, *On products of strong Skolem starters*, J. Combin. Des. (2024), **32**, 464–487.
https://doi.org/10.1002/jcd.21943