The Brody-Hughston Fisher Information Metric

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We study the interrelationships between the Fisher information metric recently introduced, on the basis of maximum entropy considerations, by Brody and Hughston (J. Math. Phys. 41, 2586 [2000]) and the monotone metrics, as explicated by Petz and Sudár (J. Math. Phys. 37, 2662 [1996]). This new metric turns out to be not strictly monotone in nature, and to yield — via its normalized volume element — a prior probability distribution over the Bloch ball of two-level quantum systems that is less noninformative than those obtained from any of the monotone metrics, even the minimal monotone (Bures) metric. We best approximate the additional information contained in the Brody-Hughston prior over that contained in the Bures prior by constructing a certain Bures posterior probability distribution. This is proportional to the product of the Bures prior and a likelihood function based on four pairs of spin measurements oriented along the diagonal axes of an inscribed cube.

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I. INTRODUCTION

In one of their numerous recent contributions [1–4] to the study of the geometry of quantum mechanics, Brody and Hughston (BH) developed a “method for representing probabilistic aspects of quantum systems by means of a density function on the space of pure quantum states,” a maximum entropy argument allowing them “to obtain a natural density function that only reflects the information provided by the density matrix” [1]. (A similarly-motivated study — based, in part, on extensive work of Band and Park [5,6] — can be found in [7].) BH indicated how to associate a Fisher information metric to their family of density functions (i.e., probability distributions), each distribution, of course, corresponding to a density matrix.

In this study, we investigate, in the context of the two-level quantum systems, the interrelationships of the BH (Fisher information) metric to the important, fundamental class of (stochastically) monotone metrics [8–11]. The monotone metrics are quantum extensions of the (classically unique) Fisher information metric. They fulfill the information-theoretic desideratum of being nonincreasing under “coarse-grainings”. Let us note, in particular, that the much studied Bures metric [12–15], in fact, plays the role of the minimal monotone metric. A number of other (non-minimal) monotone metrics have been subjects of detailed investigation, as well [16–18].

First, we find that while the normal/tangential component of the BH metric over the “Bloch ball” of two-level quantum systems appears to be consistent with monotonicity, its radial component is conclusively not (cf. [19]). We are able to verify this failure of monotonicity by finding an example of a pair of density matrices, the BH distance between which increases under a certain completely positive trace-preserving mapping (coarse-graining). Also, as a prior distribution for Bayesian analyses, the normalized volume element ($p_{BH}$) of the BH metric proves to be considerably less noninformative in nature than any of the normalized volume elements of the monotone metrics (even including the Bures/minimal monotone metric). Following the work in [20,21], we approximate the additional information so contained in $p_{BH}$ over that contained in the prior ($p_B$) based on the Bures metric by the construction of a certain Bures posterior distribution ($P_B$). This is the normalized form of the product of $p_B$ and a likelihood function based on four pairs of spin measurements oriented along the diagonals of an inscribed cube. Despite this noninformativity disparity, a certain strong congruence between the BH and Bures metrics emerges. We also study, en passant, a number of other metrics of interest, both monotone and non-monotone in character.
II. MONOTONICITY ANALYSES

BH showed (by integration of a Gaussian distribution) that their generating function for the two-level quantum systems could be written as [1, eq. (18)],

\[ Z(\lambda) = (2\pi)^3 e^{-\lambda_2 - \lambda_1} \frac{e^{-\lambda_2} - e^{-\lambda_1}}{\lambda_1 - \lambda_2}, \]

where \( \lambda_1 \) and \( \lambda_2 \) are the eigenvalues of the corresponding \( 2 \times 2 \) density matrix. The Hessian matrix of \( \ln Z(\lambda) \) [1, eq. (16)] then has the interpretation of being a Fisher information matrix on the parameter space of the (three-dimensional) family of BH probability distributions.

Choosing to parameterize the \( 2 \times 2 \) density matrices (\( \rho \)) by Cartesian coordinates (\( x_1, x_2, x_3 \)) in the Bloch ball,

\[ \rho = \frac{1}{2} \begin{pmatrix} 1 + x_1 & x_2 + ix_3 \\ x_2 - ix_3 & 1 - x_1 \end{pmatrix}, \quad \Sigma^3_{i=1} x_i^2 \leq 1 \]

we have that \( \lambda_1, \lambda_2 = \frac{1 \pm \sqrt{\Sigma^3_{i=1} x_i^2}}{2} \), so that the generating function (1) can be reexpressed as

\[ Z(x_1, x_2, x_3) = 16\pi^3 \frac{\sinh \frac{1}{\lambda} \sqrt{\Sigma^3_{i=1} x_i^2}}{\sqrt{\Sigma^3_{i=1} x_i^2}}. \]

Then, computing the corresponding \( 3 \times 3 \) Fisher information matrix, \( ||\frac{\partial^2 \ln Z(x_1, x_2, x_3)}{\partial x_i \partial x_j}|| \), and converting to spherical coordinates (\( x_1 = r \cos \theta, x_2 = r \sin \theta \cos \phi, x_3 = r \sin \theta \sin \phi \)), we obtain the (diagonal) BH Fisher information metric for the two-level quantum systems,

\[ ds^2_{BH} = \frac{4 - r^2 \text{csch}^2(\frac{\theta}{2})}{4r^2} dr^2 + \frac{r \coth(\frac{\theta}{2}) - 2}{2r^2} d\theta^2, \]

where for the normal (as opposed to radial) component of the metric, \( d\theta^2 = r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \). Now, in spherical coordinates, a monotone metric on the three-dimensional space of \( 2 \times 2 \) density matrices must assume the form [8, eq. (17)],

\[ ds^2_{\text{monotone}} = \frac{1}{1 - r^2} dr^2 + \frac{1}{1 + r} g \left( \frac{1 - r}{1 + r} \right) d\theta^2. \]

(Unfortunately, no explicit demonstration of this proposition seems to be available in the literature.) Here, \( g(t) = 1/f(t) \), where \( f(t) \) is an operator monotone function on the positive real axis such that \( f(t) = tf(t^{-1}) \) for every \( t > 0 \) [8, Thm. 3.1]. (A function \( f \) is called operator monotone if for all pairs of Hermitian operators satisfying \( A \geq B \), that is \( A - B \) is positive semidefinite [all eigenvalues of \( A - B \) being nonnegative], we have \( f(A) \geq f(B) \) [22].)

So, since the radial component in the BH (Fisher information) metric (4) is not identically equal to \( \frac{1}{1 - r^2} \), we can conclude that this BH metric does not strictly fulfill the role of a monotone metric over the two-level quantum systems in the sense of Petz and Sudár [8]. (Presumably, it is also not monotone for the higher-dimensional quantum systems, but we do not at all address this question here.) In Fig. 1 we display these two radial components. (In the vicinity of the fully mixed [classical] state, \( r = 0 \), the two radial components both behave approximately as constants (cf. [23]), that is, 1/12 in the BH case and 1 in the general monotone case.)

![FIG. 1. Radial components of an arbitrary monotone metric (5) — the upper curve — and the BH metric (4)](image-url)
However, if we equate the normal components of the two metrics ((4), (5)), we obtain (setting first $r = (1-t)/(1+t)$ and then taking $f(t) = 1/g(t)$),

$$f(t) = -\left(\frac{(-1+t)^2}{2(1+t) + (-1+t) \coth(\frac{1-t}{2t})}\right).$$

(6)

A plot (Fig. 2) of $f(t)$, along with its first-order series expansion about $t = 0$,

$$f(t) \approx -3 + 4e - e^2 + (7 - 16e + 5e^2)t \approx 6.09929 + 5.70491t = 12(.508274 + .47541t),$$

(7)

reveals $f(t)$ to clearly be a monotonic (remarkably linearly increasing) function on $[0,1]$. (Whether or not $f(t)$ also fulfills the criteria to be an operator monotone function remains to be formally determined, however.)

FIG. 2. Monotone function (6) imputed to the normal component of the BH Fisher information metric (4) on the two-level quantum systems, along with its (slightly smaller) linear approximation (7) for $t \in [0, 1]$.

If we were to divide $f(t)$ by 12 so it met the “Fisher-adjustment” requirement (satisfied by all monotone metrics) of having the value 1 at $t = 1$, it would then have the value .508274 at $t = 0$, remarkably close to the requirement of equaling .5 (met only by the Bures metric) for “Fubini-Study-adjustment” (on the pure states) [26]. (BH performed their integrations over the “quantum phase space” with respect to the Fubini-Study measure [1].) Also, for the Bures metric, $f(t) = .5 + .5t$, which rather closely resembles $508274 + .47541t$, obtained via the series expansion (7).

Consider an arbitrary pair of $2 \times 2$ density matrices, $\rho_1$ and $\rho_2$. Then for any completely positive trace-preserving map $\Phi$ on the space of operators such that $\|\Phi\| < 1$, we must have $d_{mon}(\Phi \rho_1, \Phi \rho_2) \leq d_{mon}(\rho_1, \rho_2)$, where $d_{mon}$ is the distance function corresponding to a monotone metric. We have, in fact, succeeded (through a random search process) in constructing sets of $\Phi, \rho_1, \rho_2$ for which this inequality is violated (but quite rarely) for the BH metric. (We verified our MATHEMATICA program by finding no violations when we replaced the BH metric in the program by the minimal or maximal monotone metrics.) If we take Bloch coordinates, $r = .646675, \theta = 2.51509, \phi = 5.89259$ for $\rho_1$ and add the small differentials $4.17588 \cdot 10^{-6}, -8.44724 \cdot 10^{-6}, 7.82807 \cdot 10^{-6}$ to them, respectively, we generate $\rho_2$. Substituting these values into the formula (4) for $d_{BH}^2$, we obtain a distance of $2.14985 \cdot 10^{-6}$. Now, setting $u = 2.43564, v = .0289153$ in the trigonometric parameterization of $\Phi$ [24, eq. (17)], we obtain images of $\rho_1$ with $r = .546143, \theta = .752553, \phi = .351613$ and of $\rho_2$, $r = .546138, \theta = .752544, \phi = .351621$. (Since the radial distance $r$ has not been preserved by $\Phi$, this mapping is not unitary.) The BH distance between these two images is now $2.15078 \cdot 10^{-6}$, slightly greater than before the application of $\Phi$. If we a priori set $u = 0$, so that $\Phi$ is unital, we found we were more easily able to find such counterexamples to monotonicity. (For the benefit of the reader of [24], we point out that in the right-hand side of eq. (2), the term $w_0$ should occur as a common factor and not just a factor of $I$. Also there, the symbol T was meant — as indicated to this author — to refer only to the corresponding $4 \times 4$ matrix, and not to its $3 \times 3$ submatrix, where just $T$ was meant to be used. A correct presentation is given in [25, sec. III].)

On the other hand, if one lets the (maximum entropy) probability distributions corresponding to $\rho_1, \rho_2$ in the BH framework be $p_{1,2}$, then $d_{\text{Fisher}}(T p_1, T p_2) \leq d_{\text{Fisher}}(p_1, p_2)$, for all $p_1, p_2, T$. We take $T$ to be any (classical) stochastic mapping (Markov morphism), and $d_{\text{Fisher}}$ to be the (classically unique) Fisher information metric, given in our case by $d_{BH}^2$. So, the inequality would be violated for any strictly monotone (in the sense of Petz and Sudár [8]) metric.
III. COMPARATIVE NONINFORMATIVITIES

The volume element of the BH Fisher information metric (4) is

$$\frac{(r \coth\left(\frac{r}{2}\right) - 2)\sqrt{4 - r^2 \csch^2\left(\frac{r}{2}\right)} \sin \theta}{4r}.$$  \hfill (8)

This can be normalized to a probability distribution ("Jeffreys’ prior", in Bayesian terminology) — which we will denote by $p_{BH}$ — over the Bloch ball (unit ball in Euclidean 3-space) by dividing by its integral (.0983103) over the Bloch ball. If we insert this particular integration value into the general formula for the asymptotic minimax/maximin redundancy for universal (classical) data compression [27, eq. (1.4)] (cf. [28]),

$$\frac{d}{2} \ln \frac{n}{2\pi e} + \ln \sqrt{\det I(\psi)} + o(1),$$  \hfill (9)

where $d$ corresponds to the dimensionality of the family of BH probability distributions for which the Fisher information matrix $I(\psi)$ is being computed, we get

$$\frac{3}{2} \ln n - 6.57644 + o(1).$$  \hfill (10)

Here $n$ is the number of observations and we take $d = 3$ and $\sqrt{\det I(\psi)} = .0983103$. For the universal quantum coding of two-level quantum systems [28] (cf. [29–31]), on the other hand, the asymptotic minimax/maximin redundancy (which turns out, quite remarkably, to be intimately associated with the particular monotone metric given by the exponential or identric mean $(1/e)(b^a/a^b)^{1/(b-a)}$ of numbers $a$ and $b$ — the Bures metric itself simply being associated with the arithmetic mean $(a+b)/2$ [8]) is [32],

$$\frac{3}{2} \ln n - 1.77062 + o(1).$$  \hfill (11)

The normalized form of the volume element of the (exponential/identric) monotone metric found in [32] to yield this quantum coding result is

$$p_{GKS} = .226321(1-r)^{\frac{1}{1-r} - (1+r)^{\frac{1}{1+r} - 1} r^2 \sin \theta}.$$  \hfill (12)

(Another monotone metric that plays a distinguished role is the “Bogoliubov-Kubo-Mori” one. It has been shown by Grasselli and Streater [11] that, in finite dimensions, this metric and its constant multiples is the only monotone one for which the (+1) and (-1) affine connections are mutually dual. Also, the connection form [gauge field] pertaining to the generalization of the Berry phase to mixed states proposed by Uhlmann satisfies the source-free Yang-Mills equation $*D*Dw = 0$, where the Hodge star is taken with respect to the Bures metric [14,33].)

The BH Fisher information metric tensor dominates both the Bures (minimal monotone) and the maximal monotone (right logarithmic) metric tensor over the entire Bloch ball, and therefore all the (intermediate) monotone metric tensors. That is, if one subtracts the tensor for any monotone metric from the BH one, the eigenvalues of this difference are always nonnegative.

In [20] it was found that the normalized volume element of the Morozova-Chentsov monotone metric,

$$p_{MC} = 0.00513299(1-r^2)^{-\frac{1}{2}} \log^2\left(\frac{1-r}{1+r}\right) \sin \theta,$$  \hfill (13)

yielded a highly noninformative prior (a desideratum in Bayesian analyses) in the class of monotone metrics. By applying the original test of Clarke [21] (implemented, in part, in [20]), we are now able to reach the conclusion that $p_{MC}$ is also considerably more noninformative than $p_{BH}$.

To develop our argument, let us denote the relative entropy of a probability distribution $p$ with respect to another such distribution $q$ by $D(p||q)$. Then, we compute $D(p_{MC}||p_{BH}) = 1.99971$ and $D(p_{BH}||p_{MC}) = 1.08908$. We regard $p_{MC}$, $p_{BH}$ as prior distributions over the Bloch ball and convert them to posterior distributions $P_{MC}^{(3)}$, $P_{BH}^{(3)}$ by multiplying them by the likelihood, $\Pi_{i=1}^{3} \frac{1-(x_i^2)}{2}$, that three pairs of measurements in the $x_1$, $x_2$, and $x_3$ directions each yield one “up” and one “down” (the same likelihood function principally employed in [20]; cf. [34, eq. (20)]). Normalizing the results, we are able to obtain that $D(P_{BH}^{(3)}||P_{MC}) = 1.43453$ and $D(P_{MC}^{(3)}||P_{BH}) = 1.67748$. Since
by adding information to $p_{MC}$ we render it closer to $p_{BH}$ (that is, $1.67748 < 1.99971$), but not vice versa ($1.43453 > 1.08908$), the indicated conclusion is reached. These inequalities continue to hold (although not so strongly) if instead of imagining three pairs of measurements, we take six, two pairs in each direction, with two “ups” and two “downs” resulting. Then, we have that $D(P_{BH}^{(6)}||p_{MC}) = 1.698$ (which is still less than 1.99971) and $D(P_{MC}^{(6)}||p_{BH}) = 1.74938$. But this phenomenon stemming from the incorporation of additional information into $p_{MC}$ does not continue indefinitely, since $D(P_{MC}^{(9)}||p_{BH}) = 2.02251$, which is now larger than $D(p_{MC}||p_{BH}) = 1.99971$. (We also computed here that $D(\tilde{P}_{MC}||p_{BH}) = 1.698$ (which is still less than 1.99971) and $D(\tilde{P}_{MC}^{(6)}||p_{BH}) = 1.74938$. But this phenomenon stemming from the incorporation of additional information into $p_{MC}$ does not continue indefinitely, since $D(\tilde{P}_{MC}^{(9)}||p_{BH}) = 2.02251$, which is now larger than $D(p_{MC}||p_{BH}) = 1.99971$. (We also computed here that $D(p_{MC}||p_{GKS}) = .386051$, $D(p_{GKS}||p_{MC}) = .329118$ and $D(P_{MC}^{(3)}||p_{GKS}) = .188481$, $D(P_{GKS}^{(3)}||p_{MC}) = .771068$, so $p_{MC}$ is more noninformative than $p_{GKS}$.)

Since the Morozova-Chentsov prior probability distribution was just found to be considerably more noninformative than $p_{BH}$, we investigated whether the least noninformative prior distribution based upon a monotone metric [20], that is, the Bures (minimal monotone) distribution,

$$p_B = \frac{r^2 \sin \theta}{\pi^2 \sqrt{1 - r^2}},$$

was yet itself more noninformative than $p_{BH}$. (Hall [35] had interestingly noted that the Bures metric over the Bloch ball of two-level quantum systems was a specific form of the spatial part of the Robertson-Walker metric, arising in general relativity [36].) This, in fact, turned out to be the case, since $D(p_{BH}||p_B) = .221827$, $D(p_B||p_{BH}) = .342287$, $D(P_{BH}^{(3)}||p_B) = .432781$ and $D(P_B^{(3)}||p_{BH}) = .2343$, which is less than .342287. Also $D(P_{BH}^{(6)}||p_B) = .662496$, $D(P_B^{(6)}||p_{BH}) = .306664$, which is still less than .342287, but $D(P_B^{(9)}||p_{BH}) = .432335$. So, the pattern is similar to that observed for $p_{MC}$, that is, the improvement of noninformativity for the monotone metric prior over $p_{BH}$ breaks down for nine pairs of measurements. So, the posterior distributions based on three pairs of measurements in mutually orthogonal directions — in both the Morozova-Chentsov and Bures case — give the best approximations, in terms of relative entropy, to $p_{BH}$.

In Fig. 3 we show three one-dimensional marginal probability distributions (obtained by integrating out $\theta, \phi$) over the radial coordinate $r$ of $p_{BH}$, $p_B$ and $P_B^{(3)}$. The marginal distribution of $p_{BH}$ is the most linear in nature of the three curves, while that for $p_B$ is the most steeply ascending of the three, so it can be seen — in line with our relative entropy computations — that the marginal for the remaining curve, the posterior $P_B^{(3)}$, is superior to the marginal for $p_B$ in approximating the marginal for the Brody-Hughston prior probability distribution $p_{BH}$.\[5\]
FIG. 3. Marginal (one-dimensional) probability distributions over the radial coordinate $r$ of $p_{BH}$ (the most linear of the three curves), $p_B$ (the most steeply ascending) and the posterior $P^{(3)}_B$, which provides a superior approximation to $p_{BH}$ than does $p_B$.

In Fig. 4 we replace $P^{(3)}_B$ in Fig. 3 with $P^{(3)}_{BH}$, thereby revealing graphically that adding information to $p_{BH}$ makes it less resemble $p_B$, as our computations in terms of relative entropy had indicated.

FIG. 4. Same as Fig. 3, but for the replacement of $P^{(3)}_B$ by $P^{(3)}_{BH}$, which is downward-sloping for $r > .7727551$ and less resembles $p_B$ — in terms of relative entropy — than does $p_{BH}$ itself.

To continue further along these lines, we postulated a likelihood based on four pairs of measurements, using the four diameters of the Bloch sphere obtained from the vertices of an inscribed cube, each pair yielding an “up” and a “down”. Then, adopting our earlier notation, we found that $D(P^{(4/cube)}_B || p_{BH}) = .0774351$, much smaller than any of our previous relative entropy distances. In Fig. 5, we show these two density functions.

FIG. 5. The (relatively flat) BH Fisher information distribution $p_{BH}$ along with our best approximation to it, the posterior distribution $P^{(4/cube)}_B$ formed from the Bures prior $p_B$ and the likelihood that four pairs of spin measurements, each oriented along one of the four diameters formed by an inscribed cube, will each yield an “up” and a “down”.

We then proceeded similarly, but now using six axes of spin measurement, based on an inscribed (regular) icosahedron.
dron. We obtained a larger (that is, inferior) figure of merit, $D(P_B^{(6/icos)}||p_{BH}) = .122255$.

![Marginal probability distribution](image)

**FIG. 6.** The posterior distribution $P_B^{(4/cube)}$ in Fig. 5 is replaced by (the more wiggly) $P_B^{(6/icos)}$.

With ten axes of spin measurement, based on an inscribed (regular) dodecahedron, we obtained a still larger relative entropy, $D(P_B^{(10/dode)}||p_{BH}) = .456816$.

![Marginal probability distribution](image)

**FIG. 7.** The posterior distribution $P_B^{(4/cube)}$ in Fig. 5 is replaced by (the more wiggly) $P_B^{(10/dode)}$.

So, of the Platonic solids [37], we have been able to approximate $p_{BH}$ most closely with the use of measurements based on the cube. (Of course, our initial use above and in [20] of three mutually orthogonal axes of measurement corresponds to the use of an octahedron.) Exploratory efforts of ours employing non-separable measurements [38] for the construction of likelihood functions to use for converting the Bures prior probability distribution $p_B$ to posteriors to well approximate $p_{BH}$, have not so far been at all successful. If we were to replace the radial coordinate of the metric $ds_{BH}^2$ by $\frac{1}{12(1-r^2)}$, in an effort to render it fully monotone (and Fisher-adjusted) in nature, then normalizing the resultant volume element to obtain the modified probability distribution $p_B^\prime$, we find that $D(p_B||p_B^\prime) = 8.36598 \cdot 10^{-6}$ and $D(p_B^\prime||p_B) = 8.37746 \cdot 10^{-6}$, that is both much smaller relative entropies than previously obtained. (Actually, the choice of the scaling constant 12 is irrelevant in this regard, since the volume element just gets renormalized to the same probability distribution in any case.) Also, $D(P_B^{(3)}||p_{BH}) = .138763$ and $D(P_B^{(3)}||p_B) = .143014$, so additional information — as seems plausible — does not diminish these two very small statistics. We were not able — using extended random searches — to find pairs of density matrices, the distance between the members of which increased under stochastic mappings $\Phi$, thus not contradicting the possible monotonicity (we had sought to construct) of $d_{BH}$.

**IV. DISCUSSION**

It would be of interest to find and study the Brody-Hughston Fisher information metrics for higher-dimensional quantum systems (such as the four-dimensional three-level systems examined in [39]) than the three-dimensional two-level ones investigated here and in [1]. However, our efforts along these lines have yet to produce any simple, easily
expressible results.

Another metric over the two-level quantum systems that seems of interest to consider in the general context of this paper is [40, eq. (16)],

$$ds_{BG}^2 = 2 \left( \frac{(1 + r^2)}{(1 - r^2)^2} dr^2 + \frac{1}{(1 - r^2)} dn^2 \right)$$

(15)

This is the Fisher information metric that is obtained by adopting the point of view of Bach [41,42] and of Guiasu [43], among others, that a density matrix can be considered as the covariance matrix of a complex multivariate normal distribution over the points of the corresponding Hilbert space. (As with the approach of Brody and Hughston [1], this too has a maximum-entropy rationale, as a multivariate normal distribution has the maximum entropy of all probability distributions having the same covariance matrix.) If, as before, we equate the normal/tangential component to $1/((1 + r)f[(1 - r)/(1 + r)])$, we obtain $f(t) = t/(1 + t)$, which is obviously a monotonically increasing function for $t > 0$. (In fact, the normal component of $ds_{BG}^2$ is simply twice that for the maximal monotone metric, for which $f(t) = 2t/(1 + t)$.) Clearly, however, since the radial component of (15) is not equal to $1/(1 - r^2)$ (though its behavior [Fig. 8] for $r \in [0, 1]$ is rather similar), this metric is not strictly monotone in nature (in the sense elaborated upon by Petz and Sudár [8]), though one might contend that it was “approximatively” monotone. (At the outset of their paper [26], Sommers and Życzkowski note that the trace metric on density matrices is monotone, but not Riemannian, while the situation is reversed for the Hilbert-Schmidt metric.)

![FIG. 8. Radial components of an arbitrary monotone metric, that is, $1/(1 - r^2)$, over the Bloch ball and the (less-steeply ascending) radial component of the Bach-Guiasu Fisher information metric (15)](image)

Brody and Hughston [1] were interested in finding those probability distributions over the pure states that yielded given density matrices and were of maximum entropy. Another analytical framework in which probability distributions over the pure states arise is in the quantum de Finetti Theorem as applied to density operators with Bose-Einstein symmetry [44]. But there, the probability distributions are unique, yielding exchangeable density operators, so there is no recourse necessary to maximum entropy methods. Let us further note that in their paper [1], “Information content for quantum states,” which has formed the starting point for our analyses here, Brody and Hughston sought the probability distribution over the pure states that was “least informative” (their emphasis), subject to the condition that it is consistent with the prescribed density matrix”. It would seem somewhat paradoxical, then, at least at first glance, that the normalized volume element (which we have denoted by $p_{BH}$) of the BH Fisher information metric for this family of entropy-maximizing (information-minimizing) probability distributions over the pure states, should itself be relatively informative in nature (at least in comparison with the normalized volume elements of the monotone metrics). This phenomenon has been evidenced here by our application of the interesting Bayesian methodology of Clarke [20,21].

So, in conclusion, we have established here that if one adopts the framework of Brody and Hughston, developed in [1], then one must sacrifice the desideratum of the exact monotonicity of metrics — at least in the (quantum) sense of Petz and Sudár [8]. Whether or not this “shortcoming” comprises a “fatal flaw” in the BH scheme (as well as in the complex multivariate normal one of Bach [41,42] and Guiasu [43]) appears to be a matter still open to some discussion.
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