Linear Two-Dimensional MHD of Accretion Disks: Crystalline structure and Nernst coefficient

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Abstract

We analyse the two-dimensional MHD configurations characterising the steady state of the accretion disk on a highly magnetised neutron star. The model we describe has a local character and represents the extension of the crystalline structure outlined in [1], dealing with a local model too, when a specific accretion rate is taken into account. We limit our attention to the linearised MHD formulation of the electromagnetic back-reaction characterising the equilibrium, by fixing the structure of the radial, vertical and azimuthal profiles. Since we deal with toroidal currents only, the consistency of the model is ensured by the presence of a small collisional effect, phenomenologically described by a non-zero constant Nernst coefficient (thermal power of the plasma). Such an effect provides a proper balance of the electron force equation via non-zero temperature gradients, related directly to the radial and vertical velocity components.

We show that the obtained profile has the typical oscillating feature of the crystalline structure, reconciled with the presence of viscosity, associated to the differential rotation of the disk, and with a net accretion rate. In fact, we provide a direct relation between the electromagnetic reaction of the disk and the (no longer zero) increasing of its mass per unit time. The radial accretion component of the velocity results to be few orders of magnitude

1
below the equatorial sound velocity. Its oscillating-like character does not allow a real matter in-fall to the central object (an effect to be searched into non-linear MHD corrections), but it accounts for the out-coming of steady fluxes, favourable to the ring-like morphology of the disk.

**Keywords:** Accretion disks; Plasma physics; Nernst coefficient

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1 Introduction

A long standing problem in astrophysics concerns the mechanism of accretion that compact objects manifest in the presence of lower dense companions. It is well established by observations[2] that such an accretion profile takes place often with the morphology of a thin disk configuration. Since from the very beginning of these studies, increasing interest raised about the details of the angular momentum transport across the disk structure. The solution of this problem appears settled down in the case of a compact object, for which the intrinsic magnetic field does not exert a significant Lorenz force on the elementary charged constituents of the infalling plasma. In this limit, the fluid-dynamics description appears well-grounded and the resulting axisymmetric configurations well understood (see among the first analyses of the problem[3, 4, 5]; for a relativistic analysis of plasma accreting into a black hole, see [6]). In the fluid-dynamics paradigm, the accretion mechanism relies on an angular momentum transfer which is allowed by the shear viscosity properties of the disk material. The differential angular rotation of different disk layers is associated with a non-zero viscosity coefficient, which accounts for the diffusion and turbulence phenomena emerging in the micro-scale structure of the fluid.

However, when the magnetic field of the central object is sufficiently high, the electromagnetic back-reaction of the disk plasma becomes very important. As shown in [1], the Lorenz force introduces a crucial coupling between the radial and the vertical equilibrium, which deeply alters the morphology of the system. In particular, the radial dependence of all the fundamental quantities acquires an oscillating character, modulating the background profile. Furthermore, as outlined in [7], when the disk plasma is characterised by a $\beta$-parameter close to the unity, such an oscillating-like behaviour stands as the dominant effect and the disk is indeed decomposed in a ring profile. However, such a two-MHD approach is pursued neglecting the accretion rate of the disk in the leading order and thus avoiding the problem concerning the azimuthal balance of the force acting on the electrons. In fact, in these two works, toroidal currents and matter fluxes are addressed only.

This question about a consistent electron force balance along the azimuthal direction (the azimuthal electric field is request to vanish by the axial symmetry)
affects also those works where the $z$-dependence of the system is averaged out for a thin disk[8]. Thus, the possibility of presenting a self-consistent MHD description for an accreting disk configuration calls significant attention, especially in view of the implication that the disk morphology can have on the formation of jets from compact astrophysical structures[9]. For a discussion which properly traces the way to face this relevant configuration problem, see [10].

This paper presents a solution to the two-dimensional MHD structure of a thin accretion disk, in the linear regime, when the role of a Nernst coefficient (the so-called plasma thermal power) is taken into account. In particular, we are reconciling the oscillating behaviour outlined by B. Coppi, with the accretion features of the disk. In fact, in our model, radial and vertical velocities are included in the problem, still retaining the poloidal character of the currents living in the plasma. The resulting equilibrium configuration is associated with a radial velocity which, following the poloidal currents, oscillates as a function of the radial coordinates and decays in the vertical direction. Despite its oscillating-like character, this local radial velocity is responsible for a non-zero accretion rate and it describes a mechanism for the appearance of a steady matter flux, able to enforce the ring-profile associated with the crystalline structure. Nonetheless, the oscillating character of this radial velocity prevents a real matter in-fall toward the centre of symmetry. Thus, a real accretion of the central object seems to require, in this scheme, the presence of non-linear MHD features; but the local character of the model does not allow to properly characterise the global accretion profile of the central object, that could imply a connection with the boundary layer physics[11].

The structure of the paper is as follows. In Section 2 we present a description of the main features characterising an accretion disk model. The tools needed to construct the configuration scheme are provided. Section 3 is devoted to fix the local nature of the model, properly defining the addressed approximations. In Section 4, we build up the system of partial differential equations which governs the two-dimensional MHD equilibrium, by assigning the radial, vertical and azimuthal force balance. Section 5 is concerned with the analysis of the linearised equations, based on the assumption that the magnetic field induced into the plasma, by the presence of the toroidal currents is much less in strength than the external field, given by the central object. Section 6 faces the discussion of the electron force balance equation, by introducing a non-zero and constant Nernst coefficient to preserve the presence of radial and vertical non-vanishing components of the matter velocity. Finally we develop some phenomenological considerations on the model, mainly aimed to estimate the strength of the radial velocity and the associated accretion rate. Furthermore, we fix the condition to neglect the temperature gradients into the radial equilibrium. Brief concluding remarks follow in Section 7.
2 Model for an Accretion Disk

Let us fix here the basic statements concerning the steady MHD regime for the specific case of an accretion disk configuration around a compact astrophysical object (a mass over the critical value to have a neutron star within a radius of few kilometres), which is also strongly magnetised (a dipole-like field of about \(10^{12}\) Gauss). Thus, we deal with a typical pulsar, accreting material from a binary companion via a thin disk structure made up of rotating plasma in equilibrium.

To adapt the MHD equations to the disk symmetry, we introduce cylindrical coordinates \(\{r, \phi, z\}\). The gravitational potential of the pulsar takes, in this coordinates, the form

\[
\chi(r, z) = -\frac{GM}{\sqrt{r^2 + z^2}},
\]

where \(G\) denotes the Newton’s constant and \(M\) the mass of the spherical central object, while the self-gravitation of the disk is regarded as a negligible effect.

To describe the magnetic field, we take the potential vector in the form

\[
\vec{A} = \partial_r \Pi \vec{e}_r + \psi \vec{e}_\phi + \partial_z \Pi \vec{e}_z, \tag{2}
\]

where \(\Pi(r, z)\) and \(\psi(r, z^2)\) are arbitrary functions, but only the last one is a physical degree of freedom, since the magnetic field reads

\[
\vec{B} = -\frac{1}{r} \partial_z \psi \vec{e}_r + \frac{1}{r} \psi \vec{e}_z. \tag{3}
\]

The flux function has to be decomposed as \(\psi = \psi_0 + \psi_1\), where \(\psi_0\) accounts for the dipole-like magnetic field of the pulsar (i.e. \(\psi_0 = \mu r^2 (r^2 + z^2)^{-3/2}\), with \(\mu = \text{const.}\)), while \(\psi_1\) is a contribution due to the toroidal currents rising in the disk configuration. Recalling that the axial symmetry prevents any dependence on the azimuthal angle \(\phi\) of all the quantities involved in the problem, the continuity equation, associated to the mass density \(\epsilon\) and to the velocity field \(\vec{v}\), takes the explicit form

\[
\frac{1}{r} \partial_r (r \epsilon \vec{v}_r) + \partial_z (\epsilon \vec{v}_z) = 0, \tag{4}
\]

which admits the solution

\[
\epsilon \vec{v} = -\frac{1}{r} \partial_z \Theta \vec{e}_r + \epsilon \omega(r, z) r \vec{e}_\phi + \frac{1}{r} \partial_r \Theta \vec{e}_z, \tag{5}
\]

\(^1A B_\phi\text{-component of the magnetic field can also be included in the theory by a generic term } K \vec{e}_\phi, \text{ taken, for instance, as a function of } \psi \text{ (i.e. } K = K(\psi))\). For the sake of simplicity and in agreement with Coppi (2005)[1] and Coppi-Rousseau (2006)[7], in our treatment we will fix this component equal to zero.
\( \Theta = \Theta(r, z) \) being a generic function, but for its odd symmetry in the \( z \)-coordinate. This property has to be required in order to ensure that the accretion rate of the disk, as averaged over the vertical direction, be non-vanishing, i.e.

\[
\dot{M}_d = -2\pi r \int_{-z_0}^{z_0} v_r dz = 4\pi \Theta(r, z_0) \equiv 2\pi I \neq 0,
\]

where \( z_0 \) is the half depth of the disk and we recall that \( v_r < 0 \) on the vertical average, to ensure a real accretion of the disk.

The different symmetry of the functions \( \psi \) and \( \Theta \) with respect to the \( z \)-dependence, prevents to fix \textit{a priori} a relation between them. The angular velocity \( \omega(r, z^2) \), describing the differential rotation of the disk, is an even function of \( z \) and, by virtue of the corotation theorem\[12\], can be taken as a function of the flux surfaces, i.e. \( \omega = \omega(\psi) \).

The momentum transfer through the disk structure is characterised by an azimuthal friction between the radial layers, properly described by a viscosity coefficient \( \eta \), responsible for the corresponding non-vanishing stress tensor components. The coefficient \( \eta \) can be decomposed as \( \eta = \eta_0 + \eta_1 \), where the \( \eta_1 \) component is associated with the additional effects arising from the toroidal currents. The necessity to include the electromagnetic back-reaction terms in the viscosity coefficient, is a consequence of the specific coupling such back-reaction establishes. In fact, it was demonstrated\[1\] that the currents induced in the plasma of the disk, provide an intrinsic local coupling between the radial and the vertical equilibrium, absent in the lowest-order approximation.

3 **Local configuration of the thin disk**

Let us now specialise our model to the case of a thin disk, i.e. \( z_0/r \ll 1 \) over the whole configuration. On average, the dominant contribution to the angular velocity of the plasma particles, is then the (equatorial) Keplerian value \( \omega_K \equiv (GM/r^3)^{1/2} \).

For a thin disk, the \( z \)-component of the velocity can be properly identified (see Shakura (1973) \[3\]) with the sound velocity \( v_s \equiv \sqrt{2K_B T/m_i} \) (\( K_B \) denoting the Boltzmann constant, \( T \) the plasma temperature, and \( m_i \) the ion mass), entering the fundamental inequality

\[
\frac{z_0}{r} \sim \frac{v_s}{v_\phi} \ll 1,
\]

which ensures the possibility to neglect the vertical motion with respect to the azimuthal one.

To estimate on average the ratio between the radial velocity and the azimuthal one, we can follow the simple equilibrium condition to fix the asymptotic radial
velocity of a plasma element falling into the disk

\[ \nu_c v_r \sim \omega^2 r = \omega v_\phi \Rightarrow \frac{v_r}{v_\phi} \sim \frac{\omega}{\nu_c}, \quad (8) \]

\(\nu_c\) being a characteristic frequency of particle collision. A rough estimation provides \(\nu_c \sim v_s n^{1/3}\), where \(n\) denotes the number density. Taking \(v_s \sim 10^{-3}c\) and \(n \sim 10^{10} cm^{-3}\), we get \(\nu_c \sim 10^9 Hz\), a value much greater than the typical Keplerian frequency. Thus, we conclude that \(v_\phi \sim \omega_K r\) is, in practice, responsible for the main matter flow within the disk, despite its stationary accretion takes place along the radial direction.

In this limit of approximation, the MHD condition on the balance of the Lorenz force has dominant radial and vertical components, providing the electric field in the form expected by the corotation theorem, i.e.

\[ \vec{E} = -\frac{\vec{v}}{c} \wedge \vec{B} = -\frac{d\Phi}{d\psi} \nabla \psi = -\frac{\omega}{c} \left( \partial_r \psi \vec{e}_r + \partial_z \psi \vec{e}_z \right), \quad (9) \]

\(\Phi\) denoting the electrostatic potential. However, the axial symmetry prevents a dependence of \(\Phi\) on the azimuthal angle and hence, the corresponding \(\phi\)-component of the electric field vanishes identically. This fact requires an additional effect to be included into the problem, able to balance the non-vanishing Lorenz force in the azimuthal direction, due to the velocity components \(v_r\) and \(v_z\). We will address this crucial question in Section 6, in the limit of a linear theory.

In the thin disk approximation, the strength of the gravitational field \(\vec{G}\) reads as

\[ \vec{G} = -\omega_K^2 r \vec{e}_r - \omega_K^2 z \vec{e}_z. \quad (10) \]

We see that a thin disk configuration is justified only in the presence of a sufficiently high rotation to deal with a confining vertical gravitational force (for the linear scheme, here addressed, the Lorenz force can not affect this statement[7]).

According to the analysis in [1], we now develop the disk configuration around a given value of the radial coordinate \(r_0\), limiting our attention to a narrow enough interval to express the local angular velocity field in the form

\[ \omega \simeq \omega_K + \delta \omega \simeq \omega_K + \frac{d\omega}{d\psi_0} \psi_1 \equiv \omega_K + \omega'_0 \psi_1, \quad (11) \]

where \(\psi_1 = \psi_1(r_0, z^2, r - r_0)\). We approximate the dipole surface function, at \(r_0\), as \(\psi_0(r_0) \simeq \frac{z}{r_0}\). We drop the relic \(z\)-dependence in view of the thin nature of the disk.

In what follows, the analysis is performed by addressing the drift ordering approximation that fixes the dominant character of the second spatial derivatives of \(\psi_1\). In particular, the profile of the disk, so outlined, will include viscous features,
as in [3], but making account for the crystalline structure derived in [1]. Indeed, reconciling the accretion feature of the disk with the plasma effects, as described in a bidimensional MHD approach, we will need a significant deviation from the standard mechanism of angular momentum transfer, in the sense that the azimuthal equation holds at the level of electromagnetic back-reaction only.

4 Configuration Equations

In order to fix the profile of the disk around the configuration at \( r_0 \), we have to provide the equations governing the radial, the vertical and the tangential equilibrium in the presence of the toroidal currents. The radial and the vertical configurations are not significantly affected by the viscosity, since its presence mainly concerns the differential rotation of the disk[3]. Therefore, these systems stand in the same form as in [1], while the implications due to a non-zero viscous stress are summarised by the azimuthal equilibrium.

In order to cast the whole system, we split the energy density and the pressure in the form

\[
\epsilon = \bar{\epsilon}(r_0, z^2) + \hat{\epsilon}(r_0, z^2, r - r_0), \quad p = \bar{p}(r_0, z^2) + \hat{p}(r_0, z^2, r - r_0),
\]

where the barred quantities are the contributions existing in absence of the toroidal currents, while the terms denoted with a hat are induced by such an electromagnetic reaction. According to the framework we outlined, the vertical equilibrium is governed by the two relations

\[
D(z^2) \equiv \frac{\bar{\epsilon}}{\epsilon_0(r_0)} = \exp\left(-\frac{z^2}{H_0^2}\right), \quad \epsilon_0(r_0) \equiv \epsilon(r_0, 0), \quad H_0^2 \equiv \frac{2K_BT}{m_\omega^2 K},
\]

\[
\partial_z \hat{\rho} + \omega_K^2 z \hat{\epsilon} + \frac{1}{4\pi r_0^2} \left( \partial_z^2 \psi_1 + \partial_r^2 \psi_1 \right) \partial_z \psi_1 = 0
\]

The radial equation underlying the equilibrium of the rotating layers of the disk, takes the form

\[
\omega \simeq \omega_K + \delta \omega \simeq \omega_0(\psi_0) + \frac{d\omega_0}{d\psi_0} \psi_1,
\]

\[
2\omega_K r_0 (\bar{\epsilon} + \hat{\epsilon}) \frac{d\omega_0}{d\psi_0} \psi_1 - \frac{1}{4\pi r_0^2} \left( \partial_z^2 \psi_1 + \partial_r^2 \psi_1 \right) \partial_r \psi_1 =
\]

\[
= \partial_r \left[ \hat{\rho} + \frac{1}{8\pi r_0^2} (\partial_r \psi_1)^2 \right] + \frac{1}{4\pi r_0^2} \partial_r \psi_1 \partial_z^2 \psi_1
\]

The azimuthal equation exactly reads

\[
e\nu_r \partial_r (\omega r) + e\nu_z \partial_z (\omega r) + e\omega \nu_r = \frac{1}{r^2} \partial_r \left[ \eta r^3 \partial_r \omega \right] + \partial_z \left[ \eta \partial_z (\omega r) \right].
\]
By using equation (5), the tangential equilibrium above easily stands as

$$-\partial_z \Theta \omega + \partial_r \Theta \partial_z \omega - \frac{2}{r} \partial_z \Theta \omega = 3 \eta \partial_r \omega + r \left[ \partial_r \eta \partial_r \omega + \partial_z \eta \partial_z \omega + \eta \left( \partial_r^2 \omega + \partial_z^2 \omega \right) \right].$$

(18)

Accounting for the corotation theorem $\omega = \omega(\psi)$ we can restate the spatial derivatives of $\omega$, in terms of the corresponding ones taken on $\psi$. Recalling the expression $\eta = \eta_0 + \eta_1$, we can now make the reasonable assumption that the viscosity correction $\eta_1$ be written as

$$\eta_1(r_0, \psi_1) \sim \left( \frac{d\eta_1}{d\psi_1} \right)_{\psi_1=0, r=r_0} \psi_1 \equiv \eta'_0 \psi_1,$$

(19)

we recall that for vanishing $\psi_1$, the correction $\eta_1$ vanishes too. In the work [3], a valuable proposal for the expression describing the viscosity coefficient $\eta_0(r_0)$ is provided, i.e.

$$\eta_0(r_0) \equiv \frac{2}{3} \alpha \epsilon_0 \upsilon_{s0} z_0,$$

(20)

where $\upsilon_{s0}$ denotes the sound velocity on the equatorial plane and $\alpha$ is a parameter, whose value must be assigned. Putting together all these statements, the tangential equation rewrites

$$-\partial_z \Theta \left( \partial_{r_0} \psi_0 + \partial_r \psi_1 \right) + \partial_r \Theta \partial_z \psi_1 - \frac{2}{r_0} \partial_z \Theta \frac{\omega_0}{\omega'} = r_0 \left[ \partial_z \psi_1 \eta'_0 \partial_{r_0} \psi_0 + \partial_{r_0} \eta_0 \partial_r \psi_1 + \eta'_0 \left( \partial_r \psi_1 \right)^2 + \eta'_0 \left( \partial_z \psi_1 \right)^2 + \eta_0(r_0) \left( \partial_r^2 \psi_1 + \partial_z^2 \psi_1 \right) \right].$$

(21)

Since we have $\partial_{r_0} \eta_0 \sim \eta_0 / r_0$, the first term on the right-hand-side is negligible in comparison to the term containing second derivatives. Furthermore, the request $\eta_1 \ll \eta_0$ implies that $\eta'_0 \ll \eta_0 / \psi_1$, ensuring that also the quadratic terms in the derivatives of $\psi_1$ are much smaller than the remaining ones.

Therefore the form taken by the azimuthal equation for the steady state of the disk reads as

$$-\partial_z \Theta \left( \partial_{r_0} \psi_0 + \partial_r \psi_1 \right) + \partial_r \Theta \partial_z \psi_1 - \frac{2}{r_0} \partial_z \Theta \frac{\omega_0}{\omega'} = r_0 \left[ \partial_r \psi_1 \eta'_0 \partial_{r_0} \psi_0 + \eta_0(r_0) \left( \partial_r^2 \psi_1 + \partial_z^2 \psi_1 \right) \right].$$

(22)

This equation provides a differential relation between the flux surface $\psi_1$ and the function $\Theta$ characterising the radial and vertical matter fluxes.

5 Configuration in the linear approximation

Let us analyse the compatibility between the radial and the azimuthal equations, in the limit when the induced $z$-field $B_z^1$ is much smaller than the source one $B_{z0}$,
i.e.
\[ \frac{B_1^z}{B_{0z}} \sim k_0 r_0 \frac{\psi_1}{\psi_0} \ll 1, \]  
where the magnetic surface $\psi_1$ admits the explicit dependence
\[ \psi_1 = \psi_1 \left( r_0, k_0(r - r_0), \frac{z^2}{\Delta^2} \right). \]

Here, we set $k_0^2 = 3\omega_K^2/v_{A0}^2$, with $v_{A0}^2 = B_{z0}^2/4\pi\epsilon_0$ being the Alfvén velocity in the plasma. Furthermore, $\Delta$ denotes a narrow interval for the localisation of the $z$-dependence. Under these hypotheses, the radial equation rewrites
\[ \partial_r^2 \psi_1 + \partial_z^2 \psi_1 = -k_0^2 D(z^2) \psi_1. \]

At lowest orders in $z$, we take the following expansion $D(z^2) \simeq 1 - z^2/H_n^2$, $H_n$ being of the same order of magnitude of $H_0$. In this limit, the radial equation admits the solution discussed in [1] which oscillates as a sin function along the radial direction, while it exponentially decays in $z^2$ along the vertical configuration, i.e. $\psi_1$ reads as
\[ \psi_1 = \psi_0^1 \sin [k(r - r_0)] \exp \left\{ -\frac{k_0 z^2}{H_n} \right\}, \]
where we set $k \equiv k_0 \sqrt{1 - \frac{1}{k_0 H_n}}$ and we obtained the identification $\Delta^2 \equiv H_n/k_0$. Such a structure is the linear feature of the crystalline morphology induced in the disk by the toroidal currents.

In the approximation where $k_0 r_0 \gg 1$, we can require that the contribution due to the currents on the viscosity coefficient be sufficiently small to neglect the term in $\eta_0'$ of (22), i.e. by virtue of the inequality
\[ \frac{\eta_0'}{\eta_0} \ll k_0 r_0. \]

Thus, the final form we address for the azimuthal equation is as follows
\[ -\partial_z \Theta \left( \partial_r \psi_0 + \frac{2\omega_0}{r_0 \omega_0} + \partial_r \psi_1 \right) + \partial_r \Theta \partial_z \psi_1 = r_0 \eta_0 (r_0) \left( \partial_r^2 \psi_1 + \partial_z^2 \psi_1 \right). \]

We use this equation to complete the configuration scheme of our thin disk equilibrium. Under the same conditions, fixed above for the linear regime, the azimuthal equation (22) is restated as follows
\[ -\partial_z \Theta \left( \partial_r \psi_0 + \frac{2\omega_0}{r_0 \omega_0} \right) + \partial_r \Theta \partial_z \psi_1 = \eta_0 r_0 \left( \partial_r^2 \psi_1 + \partial_z^2 \psi_1 \right). \]
Let us now assume that the term containing the derivative \( \partial_r \Theta \) be negligible (this is natural because it is multiplied by the small \( r \)-component of the magnetic field) and then divide the equation by \( \partial_{r_0} \psi_0 \). Observing that \( \omega_0' \partial_{r_0} \psi_0 = \partial_{r_0} \omega_0 = -3\omega_0/2 \) (where \( \omega_0 \equiv \omega_K \)), we rewrite (29) as follows

\[
\frac{1}{3} \partial_z \Theta = \frac{\eta_0 r_0}{\partial_{r_0} \psi_0} \left( \partial_r^2 \psi_1 + \partial_z^2 \psi_1 \right). \tag{30}
\]

Comparing the equation above with (25), we arrive to the relation

\[
e_0(r_0)v_r = -\frac{1}{r_0} \partial_z \Theta = 3\eta_0 k_0 Y, \tag{31}
\]

in which we made use of the definition of the dimensionless function \( Y \equiv k_0 \psi_1 / \partial_{r_0} \psi_0 \). Comparing the two equations, we neglected the factor \( D(z^2) \) with respect to the \( z \)-dependence of \( \psi_1 \), because \( H_n \gg \delta \), i.e. \( k_0 H_n \gg 1 \).

Such an accreting profile provides a net increasing of the disk mass, according to the relation

\[
\dot{M}_d = -2\pi r \int_{-\infty}^{\infty} e v_r dz \simeq 6\sqrt{2} \pi^{3/2} \eta_0 (k_0 r_0)^2 \frac{\Delta \psi_0^0}{\mu} r \sin k(r - r_0), \tag{32}
\]

where we made use of the dipole-like expression for \( \psi_0 \) and we extended the integration over the vertical coordinate up to infinity.

It is worth noting that, despite the radial velocity oscillates, the radial matter flux is, on average, an in-falling one. In fact, averaging the expression above between two nodes around \( r_0 \), we get

\[
\langle \dot{M}_d \rangle = 12 \sqrt{2} \pi^{5/2} \eta_0 (k_0 r_0)^2 \frac{\Delta \psi_0^0}{\mu k^2}. \tag{33}
\]

Equation (31) can be easily solved for \( \Theta \) in agreement with the adopted approximation of neglecting the term containing \( \partial_r \Theta \), as follows

\[
\Theta = -3\eta_0 k_0 r_0 \int Y dz. \tag{34}
\]

Hence we get the vertical velocity in the form

\[
v_z = \frac{1}{e_0 r_0} \partial_z \Theta = -\frac{3\eta_0 k_0 k}{e_0} \int Y dz. \tag{35}
\]

We see that, in the linear approximation, the azimuthal equation ensures the existence of a non-zero accretion rate, which is modulated by the oscillating profile of the disk configuration.
6 The Electron Force Balance Equation

Despite we have properly argued that the radial and vertical components of the fluid velocity are significantly smaller than the $\phi$-component of the velocity, nevertheless, the MHD electron force balance, restated here

$$\vec{E} + \vec{v} \wedge \vec{B} = 0,$$

(36)

requires $(\vec{v} \wedge \vec{B})_{\phi} = 0$, because of $E_{\phi} \equiv 0$, by virtue of the axial symmetry. For instance, this feature arises naturally if the corotation theorem is addressed. More simply, this request comes out from the independence of $\phi$ characterising the electrostatic potential. Thus, on the left-hand side of equation (36) an additional term has to appear in order to save the model consistence. In the limit of the linear theory, the most natural term to be included in the analysis, preserving the MHD scenario we addressed, is a weak collisional effect, as phenomenologically described by a non-zero constant Nernst coefficient. However, to introduce this hypothesis we need to give up the idea, pursued above, that the plasma in the disk be perfectly isothermal. Thus, we now introduce temperature gradients[13] in the spirit that they do not affect significantly the three configuration equations developed in the previous sections. We discuss only the implications for the electron force balance when a Nernst coefficient is present and then we fix the restriction on the model parameters to get full consistence.

According to this idea, we split the temperature as $T = T_0 + T_1(r, z^2)$, with $T_0 = \text{const.}$ and $T_1 \ll T_0$. In the presence of a non-zero Nernst coefficient $N$, the equation (36) takes the form

$$\vec{E} + \frac{\vec{v}}{c} \wedge \vec{B} = N \vec{B} \wedge \nabla T.$$ 

(37)

Assuming that the gradients of the temperature are negligible in the radial and vertical component of this equation, its $\phi$-component provides

$$v_r = -N c \partial_r T_1, \quad v_z = -N c \partial_z T_1.$$ 

(38)

We now have to observe that the temperature of the plasma in the static MHD framework, must obey the equation

$$\vec{v} \cdot \nabla T + \frac{2}{3} T \nabla \cdot \vec{v} = 0.$$ 

(39)

Expressing the gradients of the temperature by the relation (38), the equation above takes the form

$$\frac{1}{r} \partial_r (r v_r) + \partial_z v_z = \frac{3}{2N T_0 c} (v_r^2 + v_z^2).$$ 

(40)
It is easy to see that this equation is automatically satisfied in the linear regime, by virtue of the expressions for the radial and vertical velocity (31) and (35) respectively.

The complete consistence of our linear model is then guaranteed by requiring that the radial temperature gradient be negligible in the corresponding configuration equation. Such gradients arise in the radial equation from the spatial variation of the pressure \( p = 2 \epsilon K_B T / m_i \) (from which we get \( \hat{p} = 2 K_B T_0 \hat{\epsilon} / m_i + 2 \epsilon_0 K_B T_1 / m_i \)).

The condition for the negligibility of this term, can be stated, by virtue of equations (31) and (38), in the form

\[
\partial_{r_0} \psi_0 \gg r_0 \sqrt{\frac{6 \eta_0 K_B}{m_i N_c}} ,
\]

(41)

To get this result, we made account for the fact that obtaining equation (25), we divided the original radial equation by the term \( \partial_{r_0} \psi_0 / r_0^2 \). By making use of the expression (20) for \( \eta_0 \), we can rewrite the condition above in terms of the Alfvén and equatorial sound velocities as \( v_A \gg \delta v_{s0} \), with \( \delta \equiv \sqrt{\alpha H_0 v_{s0} / 4 \pi N_c} \) (we approximated everywhere the sound velocity with its value on the equatorial plane and we have taken \( z_0 \equiv H_0 \)). Furthermore we implicitly required that the radial variation of \( \hat{\epsilon} \) be smaller or, at most, equal to the term with the temperature gradient.

Finally, the vertical equation, in the linear approximation reads as

\[
\partial_z \hat{\rho} + \hat{\omega}_K^2 z \hat{\epsilon} = 0 ,
\]

(42)

which easily rewritess as an equation for \( \hat{\epsilon} \), i.e.

\[
\frac{K_B T_0}{m_i} \partial_z \hat{\epsilon} + \frac{\hat{\omega}_K^2}{2} z \hat{\epsilon} + \frac{3 K_B \eta_0 k_0 k}{m_i N_c} \int Y \, dz = 0 .
\]

(43)

Such an equation fixes the following form for the perturbed mass density \( \hat{\epsilon} = \hat{\epsilon}(z^2) \sin k(r - r_0) \), where the \( z \)-dependence comes from the ordinary differential equation

\[
\frac{K_B T_0}{m_i} \frac{d\hat{\epsilon}}{dz} + \frac{\omega_0^2(r_0)}{2} \hat{\epsilon} + \frac{3 K_B \eta_0 k_0 k}{m_i N_c} Y^0 \int \exp \left\{ -\frac{k_0 z^2}{H_n} \right\} \, dz = 0 ,
\]

(44)

having defined \( Y^0 \equiv k_0 \psi_0 / \partial_{r_0} \psi_0 \). The above equation, using the expression (20) for \( \eta_0 \), can be rewritten in the dimensionless form

\[
\xi_1 \frac{dD}{dz} + \bar{z} D + \xi_2 \text{erf} \left( \frac{\bar{z}}{\sqrt{2}} \right) = 0 ,
\]

(45)

\[
\xi_1 = \frac{K_B T_0}{m_i \omega_K^2 \Delta^2} > 1 , \quad \xi_2 = \frac{2 K_B \alpha v_{s0} z_0 k_0 k}{m_i N_c \omega_0^2} Y^0 \sqrt{\pi / 2}
\]
where we defined $\bar{z} \equiv z/\Delta$, $\bar{D} \equiv \bar{\epsilon}/\epsilon_0$ and erf is the error function: $\sqrt{\pi}\text{erf}(z) \equiv 2 \int \exp(-z^2)dz$. This equation admits the formal solution

$$\bar{D} = e^{-\frac{\bar{z}^2}{4\xi_1}} \left( \bar{D}(\bar{z} = 0) - \xi_2 \int_0^{\bar{z}} \frac{e^{\frac{\bar{u}^2}{4\xi_1}} \text{erf}\left(\frac{\bar{u}}{\sqrt{2}}\right)}{\xi_1} du \right)$$

(46)

which is plotted in fig 1 for several values of $\xi_1$ and $\xi_2$.

![Figure 1: Behaviour of $\bar{D}$ for several values of $\xi_1$ and $\xi_2$. The figure outlines an increasing behaviour of the function $\bar{D}$, and a maximum sometimes arises in the range of validity for the linear approximation.](image)

We conclude this analysis, by stressing that, because of (20), the expression (31) provides the following estimation for $v_r$

$$v_r \sim 2\alpha v_{s0} k_0 H_0 \mathcal{O}(Y^0).$$

(47)

We see that the accretion velocity is few orders of magnitude less than the sound velocity on the equatorial plane. In this respect, we observe that the parameter $\alpha$ can be taken in the range $10^{-3} - 1$, while the dimensionless term $k_0 H_0 \sim k_0 H_n \sim \sqrt{\beta}$ can be even much greater than unity and the linear approximation requires $Y^0 \ll 1$.

Despite the picture we fixed above is well-grounded in the linear approximation, it is well-known that the Nernst coefficient is obtained in a kinetic theory via an expansion in the inverse powers of the cyclotronic frequency and it corresponds to a first order approximation scheme. Thus, this effect is, in principle, small and we can expect that it be the proper explanation to the accretion features of a crystalline disk, in the weak field limit only, i.e. we have to require $\beta > 1$, to ensure the model reliability.
7 Concluding Remarks

Our analysis was devoted to a study reconciling the crystalline structure of the disk, as outlined in [1], with the existence of an azimuthal equilibrium configuration. In fact, the impossibility to neglect the radial component of matter velocity, requested by the notion of accretion, leads to include in the equilibrium problem the angular momentum transport across the disk and the unavoidable viscous features, associated to the differential rotation. The problem to get a self-consistent scheme, in which the electron force balance equation properly holds even along the azimuthal direction is then addressed. In fact, the appearance of the radial and vertical component of the matter velocity, together with the vanishing nature of the azimuthal electric field in an axisymmetric configuration, rise the question how to get a consistent equilibrium balancing the Lorenz force along the toroidal symmetry. The proposal we investigate here concerns the role that the Nernst coefficient can have in this perspective. In agreement with the idea that this is a low cyclotronic frequency effect, we treat only the linearised MHD configuration. The presence of such a collisional term allows to link the radial and vertical velocity components to the temperature gradients, as soon as the azimuthal electron force balance equation stands. The linear behaviour is completely self-consistent and the adiabatic equation for the plasma pressure is satisfied as a consequence of the incompressible nature of the linear MHD, out-coming in this axisymmetric scenario. Finally we were able to establish the condition ensuring that the radial temperature gradient be negligible in the corresponding equilibrium, while the linear vertical equilibrium is solved exactly. The main issue of our analysis is that the radial velocity has an oscillating character, with an amplitude few orders of magnitude below the sound velocity. Indeed, we deal with a non-zero accretion rate because the material outside $r_0$ is greater than the one inside (we are estimating the matter density with its value at $r_0$). Such an oscillating profile is a direct consequence of the linear crystalline structure of the model. More than with a real matter in-fall, arising expectably in the non-linear regime, we get a matter flow which explains the formation of the ring configuration within the disk. However our goal opens interesting perspective on the understanding about the existence of a ring-like decomposition in presence of a mechanism of accretion, i.e. significantly non zero radial in-fall of the plasma.

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