AUTOMORPHISMS AND DERIVATIONS OF FINITE-DIMENSIONAL ALGEBRAS

MATEJ BREŠAR

Abstract. Let $A$ be a finite-dimensional algebra over a field $F$ with $\text{char}(F) \neq 2$. We show that a linear map $D : A \to A$ satisfying $xD(x)x \in [A, A]$ for every $x \in A$ is the sum of an inner derivation and a linear map whose image lies in the radical of $A$. Assuming additionally that $A$ is semisimple and $\text{char}(F) \neq 3$, we show that a linear map $T : A \to A$ satisfies $T(x)^3 - x^3 \in [A, A]$ for every $x \in A$ if and only if there exist a Jordan automorphism $J$ of $A$ lying in the multiplication algebra of $A$ and a central element $\alpha$ satisfying $\alpha^3 = 1$ such that $T(x) = \alpha J(x)$ for all $x \in A$. These two results are applied to the study of local derivations and local (Jordan) automorphisms. In particular, the second result is used to prove that every local Jordan automorphism of a finite-dimensional simple algebra $A$ (over a field $F$ with $\text{char}(F) \neq 2, 3$) is a Jordan automorphism.

1. Introduction

The theory of functional identities deals with the description of functions on rings and algebras that satisfy certain identities [5]. In this paper, we consider a more general type of problems where expressions involving functions on an algebra $A$ are, instead of being always 0 as is usually the case with functional identities, contained in a relatively large subset of $A$, namely in $[A, A]$, the linear span of all commutators in $A$. In the special case where $A$ is the matrix algebra $M_n(F)$ this can be equivalently stated as that the trace of these expressions is always zero. Therefore, the relations that we will study are, in some sense, also more general than trace identities (see, e.g., [1]). We will not, however, develop some general theory, but consider only two special cases that merely indicate a possible new approach to “generalized identities” in rings and algebras. The two new type theorems will be shown to have applications to a well-studied research topic, i.e., to the theory of local derivations and local automorphisms.

Let us be more specific. All our results consider a finite-dimensional algebra $A$ over a field $F$ with $\text{char}(F) \neq 2$. In Section 3 we additionally assume that $\text{char}(F) \neq 3$. Section 2 is centered around the condition that a linear map $D : A \to A$ satisfies

\begin{equation}
xD(x)x \in [A, A] \quad \text{for all } x \in A,
\end{equation}

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and Section 3 is centered around the condition that a linear map \( T : A \to A \) satisfies
\[
T(x)^3 - x^3 \in [A, A] \quad \text{for all } x \in A.
\]
It is immediate that inner derivations satisfy (1.1) and inner automorphisms satisfy (1.2).

Our first main result, Theorem 2.3, states that (1.1) implies that \( D \) is the sum of an inner derivation and a linear map having the image in \( \text{rad}(A) \), the radical of \( A \). Although maps with the image in \( \text{rad}(A) \) do not always satisfy (1.1), it is still reasonable that they appear in the conclusion since \( \text{rad}(A) \) is sometimes contained in \([A, A]\).

Condition (1.2) can be studied similarly as condition (1.1), but the results are more involved. Theorem 3.3, which is our second main result, states that if \( A \) is semisimple then condition (1.2) is equivalent to the condition that there exist a central element \( \alpha \) satisfying \( \alpha^3 = 1 \) and a Jordan automorphism \( J \) of \( A \) belonging to the multiplication algebra of \( A \) such that \( T(x) = \alpha J(x) \) for all \( x \in A \). In Corollaries 3.7 and 3.8, we consider the situation where \( A \) is a general, not necessarily semisimple finite-dimensional algebra. However, these results are not as definitive as Theorem 2.3.

The proofs of Theorems 2.3 and 3.3 use the classical theory of finite-dimensional algebras together with some results on Jordan maps. We also provide several examples that justify the assumptions.

As already indicated, these two theorems are applicable to the study of local derivations and local automorphisms. A local derivation of an algebra \( A \) is a linear map \( D : A \to A \) with the property that for each \( x \in A \), there is a derivation \( D_x : A \to A \) such that \( D(x) = D_x(x) \). This notion was introduced in 1990 by Kadison [11] and independently by Larson and Sourour [12] who also introduced local automorphisms. These are defined analogously, i.e., as linear maps \( T : A \to A \) such that for each \( x \in A \), there is an automorphism \( T_x : A \to A \) satisfying \( T(x) = T_x(x) \). The definitions of other types of local maps should now be self-explanatory. The standard question is whether local derivations, local automorphisms, etc. are derivations, automorphisms, etc. Over the last three decades, positive answers were obtained in various algebras \( A \) (occurring not only in algebra but also if not primarily in functional analysis). We refer to a few recent publications [4, 8, 9] which contain some historical remarks and further references.

Theorem 2.3 immediately implies that every local inner derivation of a finite-dimensional semisimple algebra \( A \) (over a field \( F \) with \( \text{char}(F) \neq 2 \)) is an inner derivation (Corollary 2.8). In Example 2.9 we show that a similar conclusion for general derivations does not always hold. The automorphism case is more complex and interesting. As will be explained in Section 3, local Jordan automorphisms are more natural in finite dimensions than local automorphisms. Using Theorem 3.3, we will show that every local Jordan automorphism of a finite-dimensional simple algebra \( A \) over a field \( F \) with \( \text{char}(F) \neq 2, 3 \) is a Jordan automorphism (Theorem 3.11). This is the third main result of this paper.

To the best of our knowledge, our results on local maps are new and cover a basic class of algebras which is quite different from those treated by other authors. Papers on local maps are often based on the existence of some
special elements like idempotents. The class of simple algebras, however, includes division algebras which contain no such elements.

2. Derivations

We start with a simple but important lemma. As usual, we write \([x, y]\) for the commutator \(xy - yx\), and \([A, A]\) for the linear span of all \([x, y]\), \(x, y \in A\).

**Lemma 2.1.** Let \(A\) be a finite-dimensional simple algebra. If \(c \in A\) is such that \(cA \subseteq [A, A]\), then \(c = 0\).

**Proof.** From \(xcy = [x, cy] + cyx \in [A, A]\) we see that the ideal of \(A\) generated by \(c\) is contained in \([A, A]\). However, it is easy to see that \([A, A]\) is a proper subspace of \(A\) (in fact, it has codimension 1 if viewed as a space over the center \(Z\) of \(A\) [Exercise 4.12]). Hence, \(c = 0\).

**Remark 2.2.** We will also need the following technical variation of Lemma 2.1. If \(\text{char}(F) \neq 2\) and \(A\) is as in the lemma, then \(cx^2 \in [A, A]\) for every \(x \in A\) implies \(c = 0\). This follows immediately from \(x = \frac{1}{2}((x+1)^2 - x^2 - 1^2)\) (here we used that a finite-dimensional simple algebra always contains 1). Note that we may replace the condition \(cx^2 \in [A, A]\) by \(xcx \in [A, A]\) since \(xcx = cx^2 + [x, cx]\).

Let \(D\) be a linear map from an algebra \(A\) to itself. A linear map \(\Delta : A \rightarrow A\) is called a Jordan \((D, D)\)-derivation if it satisfies

\[
\Delta(x^2) = D(x)x + xD(x) \quad \text{for all} \ x \in A.
\]

This notion was introduced in [3] (as a special case of more general Jordan \((D, G)\)-derivations) in order to study the classical Jordan derivations (the case where \(\Delta = D\)) on tensor products. Somewhat to the author’s surprise, Jordan \((D, D)\)-derivations naturally occur in the proof of the next theorem, and the result from [3] stating that they satisfy \(\Delta(xy) = D(x)y + xD(y)\) provided that \(A\) is a semiprime algebra (over a field of characteristic not 2) is applicable.

Let us also recall a few standard definitions and facts. The radial of a finite-dimensional algebra \(A\), denoted \(\text{rad}(A)\), is the unique maximal nilpotent ideal of \(A\). An equivalent description is that \(\text{rad}(A)\) is the intersection of all maximal ideals of \(A\). If \(\text{rad}(A) = \{0\}\), then \(A\) is a semisimple algebra, i.e., \(A\) is a direct sum of ideals each of which is a simple algebra. The quotient algebra \(A/\text{rad}(A)\) is always semisimple. Finally, the algebra of all linear maps from \(A\) to \(A\) of the form \(x \mapsto \sum_i a_ixb_i\) for some \(a_i, b_i \in A\) is called the multiplication algebra of \(A\). It will be denoted by \(M(A)\).

We can now state our first main theorem.

**Theorem 2.3.** Let \(A\) be a finite-dimensional algebra over a field \(F\) with \(\text{char}(F) \neq 2\). If a linear map \(D : A \rightarrow A\) satisfies \(xD(x)x \in [A, A]\) for every \(x \in A\), then \(D\) is the sum of an inner derivation of \(A\) and a linear map from \(A\) to \(\text{rad}(A)\).

**Proof.** We write \(x \equiv y\) for \(x - y \in [A, A]\). Our assumption is thus \(xD(x)x \equiv 0\) for all \(x \in A\). As \(\text{char}(F) \neq 2\), replacing \(x\) by \(x \pm y\) implies

\[
yD(x)x + xD(y)x + xD(x)y \equiv 0 \quad \text{for all} \ x, y \in A.
\]
This will be our basic relation in the course of the proof.

Let $M$ be a maximal ideal of $A$. Taking $y \in M$ it follows from (2.1) that $xD(y)x \in M + [A, A]$ for every $x \in A$. Hence, $c = D(y) + M \in A/M$ satisfies $ucu \in [A/M, A/M]$ for every $u \in A/M$. As $A/M$ is simple, Remark 2.2 tells us that $c = 0$, i.e., $D(y) \in M$. We have thus proved that $D(M) \subseteq M$ for every maximal ideal $M$ of $A$.

As $\text{rad}(A)$ is the intersection of all maximal ideals of $A$, it follows that $D(\text{rad}(A)) \subseteq \text{rad}(A)$. We can thus define $\overline{D} : A/\text{rad}(A) \to A/\text{rad}(A)$ by

$$\overline{D}(x + \text{rad}(A)) = D(x) + \text{rad}(A).$$

Note that $\overline{D}$ is a linear map satisfying

$$v \overline{D}(v)v \in [A/\text{rad}(A), A/\text{rad}(A)] \quad \text{for all } v \in A/\text{rad}(A).$$

Assuming that the theorem is true for semisimple algebras, it follows from this relation that $\overline{D}$ is an inner derivation of $A/\text{rad}(A)$, which further implies that $D$ is of the desired form. Therefore, we may assume without loss of generality that $A$ is a semisimple algebra.

Thus, $A = A_1 \oplus \cdots \oplus A_r$ where each $A_i$ is a simple algebra. Since $A_i$ is the intersection of the maximal ideals $A_1 \oplus \cdots \oplus A_{i-1} \oplus A_{i+1} \cdots \oplus A_r$, with $j \neq i$, it is invariant under $D$ by what we proved above. Considering the restriction of $D$ to each $A_i$, we see that there is no loss of generality in assuming that $A$ is a simple algebra.

Let $Z$ denote the center of $A$. Take $z \in Z$. Substituting $zy$ for $y$ in (2.1) we obtain

$$zyD(x)x + xD(zy)x + zxD(x)y \equiv 0 \quad \text{for all } x, y \in A.$$

On the other hand, since $z[A, A] \subseteq [A, A]$, (2.1) shows that

$$zyD(x)x + xD(y)x + zxD(x)y \equiv 0 \quad \text{for all } x, y \in A.$$

Comparing both relations we obtain

$$x(D(zy) - zD(y))x \equiv 0 \quad \text{for all } x, y \in A.$$

Since $A$ is simple, we see from Remark 2.2 that $D(zy) = zD(y)$, i.e., $D$ is $Z$-linear. But then $D$ belongs to the multiplication algebra $M(A)$ [2] Lemma 1.25.

Let $a_i, b_i \in A$ be such that

$$D(x) = \sum_i a_i x b_i \quad \text{for all } x \in A.$$ 

We have

$$xD(y)x = \sum_i x a_i y b_i x = \sum_i b_i x^2 a_i y + \sum_i [x a_i y, b_i x]$$

and so $xD(y)x \equiv \sum_i b_i x^2 a_i y$. Using also $yD(x)x \equiv D(x)xy$ we now see that (2.1) can be rewritten as

$$(D(x)x + \sum_i b_i x^2 a_i + xD(x))y \equiv 0 \quad \text{for all } x, y \in A.$$ 

Therefore, by Lemma 2.1

$$D(x)x + \sum_i b_i x^2 a_i + xD(x) = 0 \quad \text{for all } x \in A.$$
This means that the linear map \( \Delta : A \rightarrow A \) defined by
\[
\Delta(x) = -\sum_i b_i xa_i
\]
is a Jordan \((D, D)\)-derivation. By [3, Theorem 4.3], \( \Delta \) satisfies
\[
(2.2) \quad \Delta(xy) = D(x)y + xD(y) \quad \text{for all } x, y \in A.
\]
Writing first 1 for \( x \) and then 1 for \( y \) we see that \( \beta = D(1) \) lies in \( Z \). From \( \Delta(x) = D(x) + \beta x \) and \((2.2)\) it follows that \( d : A \rightarrow A \) defined by
\[
d(x) = D(x) - \beta x
\]
is a derivation. Moreover, \( d \) is \( Z \)-linear. It is a standard fact that such a derivation is inner (see, e.g., [2, Exercise 4.20]). Thus, there is an \( a \in A \) such that \( D(x) = \beta x + [a, x] \) for all \( x \in A \).

The proof will be complete by showing that \( \beta = 0 \). Since
\[
(2.3) \quad x[a, x]x = [xax, x] \equiv 0
\]
it follows from \( xD(x)x \equiv 0 \) that \( \beta x^3 \equiv 0 \). If \( \beta \) was not 0, every \( x \in A \) would satisfy
\[
(2.4) \quad x^3 \equiv 0.
\]
To show that this is not true, first note that a complete linearization of \((2.3)\) gives
\[
(2.5) \quad x_1 x_2 x_3 + x_1 x_3 x_2 + x_2 x_1 x_3 + x_2 x_3 x_1 + x_3 x_1 x_2 + x_3 x_2 x_1 \equiv 0
\]
for all \( x_1, x_2, x_3 \in A \). Since char\((F) \neq 2 \), we also have
\[
(2.6) \quad x_1^2 x_2 + x_1 x_2 x_1 + x_2 x_1^2 \equiv 0
\]
for all \( x_1, x_2 \in A \). Let \( K \) be the algebraic closure of \( Z \) and let \( A_K = K \otimes A \) be the scalar extension of \( A \) to \( K \). Take \( y = \sum_j k_j \otimes x_j \in A_K \). Observe that \( y^3 \) is a sum of terms of the form
\[
k \otimes x_1^3,
\]
\[
k' \otimes (x_j^2 x_j + x_i x_j x_i + x_j x_i^2),
\]
and
\[
k'' \otimes (x_i x_j x_k + x_i x_k x_j + x_j x_i x_k + x_k x_i x_j + x_k x_j x_i).
\]
Using \((2.4), (2.5), \) and \((2.6)\) it follows that \( y^3 \in [A_K, A_K] \). However, \( A_K \cong M_n(K) \) and so this obviously cannot hold for every \( y \in A_K \) (e.g., for an idempotent of rank 1). This contradiction proves that \( \beta = 0 \). \( \square \)

**Remark 2.4.** The last paragraph of the proof could be shortened if char\((F) \) was different from 3. Indeed, since \( x_1 x_2 x_1 \equiv x_1 x_2 x_1 \equiv x_2 x_1^2 \), in this case it is enough to apply Lemma \((2.1)\) to \((2.6)\). If, however, char\((F) = 3 \), then \((2.6)\) holds for any algebra over \( F \) as \( x_1^2 x_2 + x_1 x_2 x_1 + x_2 x_1^2 \) is then equal to \([x_1, [x_1, x_2]]\).
As noticed in [2,3], every inner derivation $D$ satisfies $xD(x)x \in [A,A]$. Thus, if the algebra $A$ is such that $\text{rad}(A) \subseteq [A,A]$, then Theorem 2.3 turns into an “if and only if” theorem. A simple concrete example is the algebra $A$ of all upper triangular matrices over $F$ (which actually satisfies $\text{rad}(A) = [A,A]$). Another example is of course any semisimple algebra. We record this as a corollary.

**Corollary 2.5.** Let $A$ be a finite-dimensional semisimple algebra over a field $F$ with $\text{char}(F) \neq 2$. The following conditions are equivalent for a linear map $D : A \to A$:

(i) $xD(x)x \in [A,A]$ for every $x \in A$.

(ii) $D$ is an inner derivation.

Inner derivations $D$ also satisfy a simpler condition $xD(x) \in [A,A]$ for all $x \in A$. However, so do many other maps, as the next example shows. There are thus good reasons for considering the condition $xD(x)x \in [A,A]$.

**Example 2.6.** Every map of the form $D(x) = \sum a_i x b_i - b_i x a_i$, satisfies $xD(x) \in [A,A]$ for all $x \in A$. This follows from $x(a_i x b_i - b_i x a_i) = [x a_i, x b_i]$.

The following example shows that the assumption that $\text{char}(F) \neq 2$ is necessary in Corollary 2.5.

**Example 2.7.** Let $F = \mathbb{F}_2$ be the field with 2 elements and let $A = M_2(F)$. Define $D : A \to A$ by

$$D \left( \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \right) = \begin{bmatrix} x_{22} & x_{12} \\ 0 & x_{11} \end{bmatrix}. $$

Using $x y (x + y) = 0$ for all $x, y \in F$ one can check that the trace of the matrix $xD(x)x$ is 0 for every $x \in A$. Therefore, $xD(x)x$ lies in $[A,A]$. However, $D$ is not a derivation.

In the rest of this section we consider local (inner) derivations.

**Corollary 2.8.** Let $A$ be a finite-dimensional semisimple algebra over a field $F$ with $\text{char}(F) \neq 2$. Then every local inner derivation $D : A \to A$ is an inner derivation.

**Proof.** The condition that $D$ is a local inner derivations means that for every $x \in A$, there exists an $a_x \in A$ such that $D(x) = [a_x, x]$. This obviously implies $xD(x)x = [x a_x, x] \in [A,A]$. Therefore, $D$ is an inner derivation by Corollary 2.5.

The next example shows that Corollary 2.8 cannot be extended to general, not necessarily inner derivations. In fact, this fails to hold even when $A$ is a field. Of course, this can occur only if $A$ is an inseparable field extension of $F$ in order to have nontrivial derivations. We remark that our example is similar to the one from [11] which concerns the algebra $\mathbb{C}(X)$ (which, however, is infinite-dimensional over $\mathbb{C}$).

**Example 2.9.** Let $p$ be an odd prime and let $F = \mathbb{F}_p(t)$ be the rational function field over the field with $p$ elements $\mathbb{F}_p$. If $\alpha$ is a root of the (irreducible and inseparable) polynomial $X^p - t$, then $A = F(\alpha)$ has degree $p$ over $F$. For any $f = \sum_{k=0}^{p-1} a_k \alpha^k \in A$, $a_k \in F$, define $f' = \sum_{k=0}^{p-1} k a_k \alpha^{k-1}$.
Observe that $f \mapsto f'$ is an $F$-linear derivation of $A$ whose kernel is $F$. For any $g \in A$, $f \mapsto gf'$ is also an $F$-linear derivation of $A$. This readily implies that every $F$-linear map of $A$ that sends $1$ to $0$ is a local derivation. However, such a map is not necessarily a derivation. For example, if it sends $\alpha$ to $1$ and $\alpha^2$ to $0$, then it certainly is not.

Corollary 2.8 also does not hold without the assumption of finite dimensionality.

**Example 2.10.** There exist (infinite-dimensional) division algebras $D$ in which every nonzero inner derivation is surjective [7]. Every linear map from $D$ to $D$ that vanishes at central elements is then a local inner derivation. However, such a map does not to be an inner derivation.

### 3. Automorphisms, Antiautomorphisms, and Jordan Automorphisms

Let $A$ be an algebra over a field $F$. Recall that a *Jordan automorphism* of $A$ is a bijective linear map $J : A \to A$ satisfying

$$J(xy + yx) = J(x)J(y) + J(y)J(x)$$

for all $x, y \in A$.

If $\text{char}(F) \neq 2$, this condition is equivalent to $J(x^2) = J(x)^2$ for all $x \in A$. Obvious examples of Jordan automorphisms are automorphisms and antiautomorphisms. These obvious examples are also the only examples if $A$ is a simple algebra over a field $F$ with $\text{char}(F) \neq 2$. This is a special case of the classical theorem of Herstein [10] (the assumption from [10] that the characteristic is not $3$ was later removed).

Our first lemma in this section essentially concerns central simple algebras, but for notational consistency we state it in a slightly different form.

**Lemma 3.1.** Let $A$ be a finite-dimensional simple algebra with center $Z$. If $J$ is a $Z$-linear Jordan automorphism of $A$, then $J(x) - x \in [A, A]$ for all $x \in A$.

**Proof.** Let $K$ be the algebraic closure of $Z$ and let $A_K = K \otimes A$ be the scalar extension of $A$ to $K$. Then $J = \text{id}_K \otimes J$ is a Jordan automorphism, and hence an automorphism or an antiautomorphism of $A_K \cong M_n(K)$. If $(u) = u \in [A_K, A_K]$ for every $u \in A_K$, then for $u = 1 \otimes x$ we obtain that $1 \otimes (J(x) - x)$ is equal to an element of the form $\sum k_i [x_i, y_i]$, which implies that $J(x) - x \in [A, A]$. Therefore, it is enough to consider the case where $A$ is the $K$-algebra $M_n(K)$.

If $J$ is an automorphism, then $J$ is inner by the Skolem-Noether Theorem, so there is an invertible $a \in A$ such that

$$J(x) - x = axa^{-1} - x = [a, xa^{-1}] \in [A, A]$$

for every $x \in A$. Assume that $J$ is an antiautomorphism. Then $x \mapsto J(x^t)$, where $x^t$ is the transpose of $x$, is an automorphism of $A$. Hence, $J(x^t) = axa^{-1}$ for all $x \in A$, or written equivalently, $J(x) = ax^t a^{-1}$. Since $ax^t a^{-1}$ and $x$ have the same trace it follows that $J(x) - x \in [A, A]$. \hfill $\square$

We now begin our study of condition $T(x)^3 - x^3 \in [A, A]$. 


Lemma 3.2. Let $A$ be an algebra over a field $F$ with $\text{char}(F) \neq 2, 3$. If a linear map $T : A \to A$ satisfies $T(x)^3 - x^3 \in [A, A]$ for every $x \in A$, then
\begin{equation}
T(x)^2T(y) - x^2y \in [A, A] \quad \text{for all } x, y \in A.
\end{equation}
Moreover, if $A$ is finite-dimensional and $\ker T \cap \text{rad}(A) = \{0\}$, then $T$ is bijective and leaves every maximal ideal of $A$ invariant.

Proof. As in the preceding section, we write $x \equiv y$ for $x - y \in [A, A]$.

Since $\text{char}(F) \neq 2$, replacing $x$ by $x+y$ in $T(x)^3 \equiv x^3$ gives
\begin{align*}
T(x)^2T(y) + T(x)T(y)T(x) + T(y)T(x)^2 \equiv x^2y + xyx + yx^2.
\end{align*}
As $T(x)^2T(y) \equiv T(x)T(y)T(x) \equiv T(y)T(x)^2$, $x^2y \equiv xyx \equiv yx^2$, and $\text{char}(F) \neq 3$, (3.1) follows.

Now assume that $A$ is finite-dimensional and $\ker T \cap \text{rad}(A) = \{0\}$. Take $y \in \ker T$. From (3.1) we see that $x^2y \in [A, A]$ for all $x \in A$. Hence, for any ideal $M$ of $A$ we have $u^2(y + M) \in [A/M, A/M]$ for all $u \in A/M$. Assuming that $M$ is maximal it follows from Remark 2.2 that $y \in M$. This implies that $y \in \text{rad}(A)$ and so $y = 0$ by our assumption. Thus, $T$ is bijective.

Finally, from (3.1) it now follows that $w^2T(y) \in M + [A, A]$ for every ideal $M$ of $A$, $y \in M$, and $w \in A$. As in the preceding paragraph we see that this implies $T(y) \in M$ if $M$ is maximal.

We are ready to prove our second main result.

Theorem 3.3. Let $A$ be a finite-dimensional semisimple algebra over a field $F$ with $\text{char}(F) \neq 2, 3$. The following conditions are equivalent for a linear map $T : A \to A$:

(i) $T(x)^3 \in [A, A]$ for every $x \in A$.

(ii) There exist a Jordan automorphism $J$ of $A$ and an element $\alpha$ from the center $Z$ of $A$ such that $T(x) = \alpha J(x)$ for all $x \in A$. Moreover, $J$ belongs to the multiplication algebra $M(A)$ and $\alpha$ satisfies $\alpha^3 = 1$.

Proof. (i) $\implies$ (ii). Lemma 3.2 tells us that $T$ is bijective and leaves maximal ideals of $A$ invariant. The latter implies that every simple component of $A$ is also invariant under $T$. We may therefore assume without loss of generality that $A$ is a simple algebra.

Take $z$ from the center $Z$ of $A$. Replacing $y$ by $zy$ in (3.1) we obtain
\begin{equation}
T(x)^2T(zy) \equiv zx^2y \equiv T(x)^2zT(y).
\end{equation}
Since $T$ is surjective, we thus have $u^2(T(zy) - zT(y)) \equiv 0$ for every $u \in A$. Hence, $T(zy) = zT(y)$ by Remark 2.2. This implies that $T \in M(A)$ [2]. Lemma 1.25], that is, there exist $a_i, b_i \in A$ such that
\begin{equation}
T(x) = \sum_i a_ixb_i \quad \text{for all } x \in A.
\end{equation}
Hence,
\begin{equation}
T(x)^2T(y) = \sum_i T(x)^2a_iyb_i \equiv \sum_i b_iT(x)^2a_iy,
\end{equation}
which along with (3.1) gives
\begin{equation}
(\sum_i b_iT(x)^2a_i - x^2)y \equiv 0 \quad \text{for all } x, y \in A.
\end{equation}
Therefore, by Lemma 3.1,
\[(3.2) \quad W(T(x)^2) = x^2 \quad \text{for all } x \in A,
\]
where
\[W(x) = \sum b_i x a_i.
\]
Substituting \(x + 1\) for \(x\) in (3.2), it follows that
\[W(T(x)T(1) + T(1)T(x)) = 2x.
\]
This shows that \(W\) is invertible and that \(S = W^{-1}\) satisfies
\[(3.3) \quad 2S(x) = T(x)T(1) + T(1)T(x) \quad \text{for all } x \in A.
\]
By (3.2), \(S(x^2) = T(x)^2\) for all \(x \in A\), and hence
\[(3.4) \quad S(xy + yx) = T(x)T(y) + T(y)T(x) \quad \text{for all } x, y \in A.
\]
As above, we denote by \(K\) the algebraic closure of \(Z\) and by \(A_K = K \otimes A\)
the scalar extension of \(A\) to \(K\). Observe that \(\overline{S} = \text{id}_K \otimes S\) and \(\overline{T} = \text{id}_K \otimes T\)
are \(K\)-linear maps of \(A_K \cong M_n(K)\) satisfying
\[(3.5) \quad \overline{S}(xy + yx) = \overline{T}(x)\overline{T}(y) + \overline{T}(y)\overline{T}(x) \quad \text{for all } x, y \in A_K.
\]
Take an idempotent \(e \in A_K\). From (3.5) we see that \(\overline{S}(e) = \overline{T}(e)^2\) and
\[2\overline{S}(e) = \overline{T}(e)u + u\overline{T}(e) \quad \text{where } u = \overline{T}(1 \otimes 1).
\]
Hence, \(2\overline{T}(e)^2 = \overline{T}(e)u + u\overline{T}(e)\), and so \(\overline{T}(e)u + u\overline{T}(e)\) commutes with \(\overline{T}(e)\). Observe that this can be equivalently stated as that \(\overline{T}(e)^2\) commutes with \(u\). That is, \(\overline{S}(e)\) commutes with \(u\) for every idempotent \(e \in A_K\). The algebra \(A_K \cong M_n(K)\) is linearly spanned by its idempotents (indeed, observe that the matrix unit \(e_{ij}, i \neq j\), is a difference of two idempotents: \(e_{ij} = (e_{ii} + e_{ij}) - e_{ii}\). As \(\overline{S}\) is surjective it follows that \(u\) is a scalar multiple of \(1 \otimes 1\). Since \(u = \overline{T}(1 \otimes 1) = 1 \otimes T(1)\), this shows that \(\alpha = T(1) \in Z\). Of course, \(\alpha \neq 0\). From (3.4) we see that \(\alpha^{-1}S(x) = T(x)\) for all \(x \in A\). Hence, (3.4) implies that \(J(x) = \alpha^{-1}T(x)\) is a Jordan automorphism.

It remains to show that \(\alpha^3 = 1\). By Lemma 3.1 \(J(x)^3 = J(x^3) \equiv x^3\), and
by our assumption, \(T(x^3) \equiv x^3\). Hence, \(J(x)^3 \equiv T(x^3)\). Since \(T(x) = \alpha J(x)\) it follows that \((\alpha^3 - 1)J(x)^3 \equiv 0\). As we saw at the end of the proof of Theorem 2.3 there exist elements in \(A\) whose cube does not lies in \([A, A]\). Since \(J\) is surjective it follows that \(\alpha^3 - 1 = 0\).

(ii) \(\implies\) (i). As \(T\) lies in \(M(A)\), it leaves every ideal of \(A\) invariant. We may therefore assume without loss of generality that \(A\) is simple. Now, \(J \in M(A)\) implies that \(J\) is \(Z\)-linear, and so Lemma 3.1 shows that \(J(x^3) \equiv x^3\). Since
\[J(x^3) = J(x)^3 = \alpha^{-3}T(x)^3 = T(x)^3,
\]
this proves (i). \(\square\)

The author is thankful to Misha Chebotar for suggesting the trick with idempotents after equation (3.5).

The purpose of the following example is to show that the simpler condition \(T(x^2) - x^2 \in [A, A]\) is not characteristic for Jordan automorphisms (compare Example 2.6).
Example 3.4. If $a, b \in A$ are such that $a^2 = ab = ba = b^2 = 0$, then
$T(x) = x + axb - bxa$ satisfies
$$T(x)^2 - x^2 = [xa, xb] + [ax, bx] \in [A, A]$$
and $T(1) = 1$. However, $T$ is not always a Jordan automorphism.

The next two examples show that Theorem 3.3 does not hold if char($F$) is 2 or 3.

Example 3.5. Let $F$ be a field with char($F$) = 2, let $A = M_2(F)$, and let $T : A \to A$ be given by $T(x) = x + \text{tr}(x)1$. It is easy to see that $T$ is neither an automorphism nor an antiautomorphism. However,
$$T(x)^3 - x^3 = \text{tr}(x)(x^2 + \text{tr}(x)x) + \text{tr}(x)^31$$
$$= \text{tr}(x)\text{det}(x)1 + \text{tr}(x)^31$$
$$= (\text{tr}(x)\text{det}(x) + \text{tr}(x)^3)[e_{12}, e_{21}] \in [A, A].$$
Moreover, $T(1) = 1$. Thus, $T$ satisfies condition (i), but does not satisfy condition (ii).

Example 3.6. If $A$ is an algebra over a field $F$ with char($F$) = 3, then for all $x, y \in A$ we have
$$(x + y)^3 - x^3 - y^3 = [x, [x, y]] + [y, [y, x]] \in [A, A]$$
(compare Remark 2.4). Taking any $a \in A$ such that $a^3 \in [A, A]$ (say, $a^3 = 0$) and any linear functional $\varphi$ on $A$, we thus see that the map $T : A \to A$ given by $T(x) = x + \varphi(x)a$ satisfies condition (i). However, $T$ does not necessarily satisfy condition (ii).

The next corollary considers general finite-dimensional algebras.

Corollary 3.7. Let $A$ be a finite-dimensional algebra over a field $F$ with char($F$) $\neq 2, 3$. If a linear map $T : A \to A$ satisfies $T(x)^3 - x^3 \in [A, A]$ for every $x \in A$, then $T(x^4) - T(x)^4 \in \text{rad}(A)$ for every $x \in A$. Moreover, if $A$ is unital and $T(1) = 1$, then $T(x^2) - T(x)^2 \in \text{rad}(A)$ for every $x \in A$.

Proof. Lemma 3.2 implies that $T(\text{rad}(A)) \subseteq \text{rad}(A)$. Therefore, we can define $\overline{T} : A/\text{rad}(A) \to A/\text{rad}(A)$ by
$$\overline{T}(x + \text{rad}(A)) = T(x) + \text{rad}(A).$$
Since
$$\overline{T}(v)^3 - v^3 \in [A/\text{rad}(A), A/\text{rad}(A)] \text{ for all } v \in A/\text{rad}(A)$$
it follows from Theorem 3.3 that there exist a Jordan automorphism $\overline{J}$ of $A/\text{rad}(A)$ and an element $\overline{\alpha}$ from the center of $A/\text{rad}(A)$ such that $\overline{\alpha}^3 = 1$ and $\overline{T}(v) = \overline{\alpha}\overline{J}(v)$ for all $v \in A/\text{rad}(A)$. Accordingly,
$$\overline{T}(v^4) = \overline{\alpha}\overline{J}(v^4) = \overline{\alpha}\overline{J}(v)^4 = \overline{\alpha}^4\overline{J}(v)^4 = \overline{T}(v)^4$$
for all $v \in A/\text{rad}(A)$, which shows that $T(x^4) - T(x)^4 \in \text{rad}(A)$ for all $x \in A$. Finally, if $A$ is unital and $T(1) = 1$, then also $\overline{T}(1) = 1$ and hence $\overline{\alpha} = 1$. Thus, $\overline{T}$ is a Jordan automorphism and so $T(x^2) - T(x)^2 \in \text{rad}(A)$ for every $x \in A$. \qed
Assuming that the field \( F \) is perfect, which makes it possible for us to use the Wedderburn Principal Theorem, we obtain a nicer result that is more similar to Theorem 2.3. We will assume for simplicity that \( A \) is unital and \( T(1) = 1 \).

**Corollary 3.8.** Let \( A \) be a unital finite-dimensional algebra over a perfect field \( F \) with \( \text{char}(F) \neq 2,3 \). If a linear map \( T : A \to A \) satisfies \( T(1) = 1 \) and \( T(x)^3 - x^3 \in [A,A] \) for every \( x \in A \), then \( T \) is the sum of a Jordan endomorphism of \( A \) and a linear map from \( A \) to \( \text{rad}(A) \).

**Proof.** By the Wedderburn Principal Theorem, \( A \) contains a subalgebra \( S \) (isomorphic to \( A/\text{rad}(A) \)) such that \( A \) is the vector space direct sum of \( S \) and \( \text{rad}(A) \). Let \( \pi \) be the projection on \( S \) along \( \text{rad}(A) \). As \( T - \pi T \) has image in \( \text{rad}(A) \) we must only prove that \( \pi T \) is a Jordan endomorphism. Now, Corollary 3.7 tells us that \( \pi(T(x^2) - T(x)^2) = 0 \), and since \( \pi \) is an endomorphism this can be written as \( (\pi T)(x^2) = (\pi T)(x)^2 \).

We continue with an analog of Corollary 2.8.

**Corollary 3.9.** Let \( A \) be a finite-dimensional semisimple algebra over a field \( F \) with \( \text{char}(F) \neq 2,3 \). Then every local inner automorphism \( T : A \to A \) is a Jordan automorphism belonging to \( M(A) \).

**Proof.** Our assumption can be read as that for each \( x \in A \), there is an invertible \( a_x \in A \) such that \( T(x) = a_x x a_x^{-1} \). Hence, \( T(x)^3 - x^3 = [a_x x^3, a_x^{-1}] \in [A,A] \). As \( T(1) = 1 \), Theorem 3.3 gives the desired conclusion. \( \square \)

There are many algebras in which every local automorphism is an automorphism. However, the matrix algebra \( M_n(F) \) is not one of them. Indeed, any matrix \( x \in M_n(F) \) is similar to its transpose \( x^t \), so \( x \mapsto x^t \) is an example of a local inner automorphism which is not an automorphism but an antiautomorphism. This explains why Jordan automorphisms appear in the conclusion of Corollary 2.8. Moreover, it indicates that in just about any reasonable class of finite-dimensional algebras, the question whether local Jordan automorphisms are Jordan automorphisms is more natural than the usual question whether local automorphisms are automorphisms.

Our last theorem gives an answer to the question just raised. In its proof we will use the following elementary lemma. Actually, we will need only its special case where each \( m_i = 3 \). The general form, however, may be useful elsewhere.

**Lemma 3.10.** Let \( F \) be an infinite field, let \( V \) and \( W \) be vector spaces over \( F \), and let \( n \) and \( m_1, \ldots, m_n \) be positive integers. Suppose that \( m_i \)-linear maps \( f_i : V^{m_i} \to W \) are such that for each \( x \in V \), at least one of the elements \( f_1(x, \ldots, x), \ldots, f_n(x, \ldots, x) \) is 0. Then there exists an \( i \in \{1, \ldots, n\} \) such that \( f_i(x, \ldots, x) = 0 \) for every \( x \in V \).

**Proof.** Suppose the lemma is not true. Then for each \( i \in \{1, \ldots, n\} \) there exists an \( x_i \in V \) such that \( f_i(x_i, \ldots, x_i) \neq 0 \). Choose a linear functional \( \tau_i \) on \( W \) such that \( \tau_i(f_i(x_i, \ldots, x_i)) \neq 0 \). For any \( z_1, \ldots, z_n \in F \), define \( p_i(z_1, \ldots, z_n) = \tau_i(f_i(z_1 x_1 + \cdots + z_n x_n, \ldots, z_1 x_1 + \cdots + z_n x_n)) \).
Note that the assumption of the lemma implies that for every \((z_1, \ldots, z_n) \in F^n\), there is an \(i \in \{1, \ldots, n\}\) such that \(p_i(z_1, \ldots, z_n) = 0\). Consequently,
\[
(3.6) \quad p_1(z_1, \ldots, z_n) \cdots p_n(z_1, \ldots, z_n) = 0
\]
for all \((z_1, \ldots, z_n) \in F^n\).

Since \(f_i\) is \(m_i\)-linear, we may consider \(p_i\) as a polynomial in \(z_1, \ldots, z_n\). Its coefficient at \(z_i^{m_i}\) is \(\tau_i(f_i(x_1, \ldots, x_i))\), so \(p_i \neq 0\). The product \(p_1 \cdots p_n\) is therefore a nonzero polynomial too. However, \((3.6)\) shows that this polynomial vanishes at every \((z_1, \ldots, z_n) \in F^n\), which contradicts the assumption that \(F\) is infinite.

Lemma \((3.10)\) will make it possible for us to use Theorem \((3.3)\) in the proof of the following theorem, provided of course that the field \(F\) is infinite. The theorem also holds for finite fields, but for them we will have to use a different method which is more similar to standard methods for tackling local automorphisms.

**Theorem 3.11.** Let \(A\) be a finite-dimensional simple algebra over a field \(F\) with \(\text{char}(F) \neq 2, 3\). Then every local Jordan automorphism \(T : A \to A\) is a Jordan automorphism.

**Proof.** We consider separately two cases.

**Case 1: \(F\) is infinite.** The center \(Z\) of \(A\) is a finite extension of \(F\), so there are only finitely many \(F\)-linear automorphisms of \(Z\). Denote them by \(\sigma_1, \ldots, \sigma_n\). The restriction of every Jordan automorphism of \(A\) to \(Z\) is therefore one of the \(\sigma_i\)'s.

For each \(x \in A\), there is a Jordan automorphism \(T_x\) of \(A\) such that \(T(x) = T_x(x)\). Therefore, \(A\) is the union of its subsets
\[
A_i = \{x \in A \mid T_x|_Z = \sigma_i\}, \quad i = 1, \ldots, n.
\]
For every \(i\) such that \(A_i \neq \emptyset\) choose an \(x_i \in A_i\) and set \(T_i = T_{x_i}\).

Take \(x \in A_i\). Then \(T_i^{-1}T_x\) fixes elements from \(Z\) and is therefore a \(Z\)-linear Jordan automorphism. Hence,
\[
(T_i^{-1}T)(x)^3 - x^3 = (T_i^{-1}T_x)(x)^3 - x^3 = (T_i^{-1}T_x)(x^3) - x^3 \in [A, A]
\]
by Lemma \((5.1)\) This shows that \(A_i\) is a subset of the set
\[
B_i = \{x \in A \mid (T_i^{-1}T)(x)^3 - x^3 \in [A, A]\}.
\]
Therefore, \(A = \bigcup_{i=1}^n B_i\).

Define \(f_i : A^3 \to A/[A, A]\) by
\[
f_i(x, y, z) = (T_i^{-1}T)(x)(T_i^{-1}T)(y)(T_i^{-1}T)(z) - xyz + [A, A].
\]
Note that \(x \in B_i\) if and only if \(f_i(x, x, x) = 0\). The conditions of Lemma \((3.10)\) are therefore satisfied, and so there exists an \(i \in \{1, \ldots, n\}\) such that \(f_i(x, x, x) = 0\) for every \(x \in A\). That is, \((T_i^{-1}T)(x)^3 - x^3 \in [A, A]\) for every \(x \in A\). Theorem \((3.3)\) shows that \(T_i^{-1}T\) is a Jordan automorphism (\(\alpha = 1\) since \((T_i^{-1}T)(1) = 1\)). But then the same holds for \(T = T_i(T_i^{-1}T)\). This completes the proof for this case.

**Case 2: \(F\) is finite.** Without loss of generality we may assume that \(F\) is equal to its prime subfield \(F_p\). The center \(Z\) of \(A\) is then the finite field
Let $\sigma$ denote the Frobenius automorphism of $Z$. It is well known that the only automorphisms of $Z$ are $\sigma^i$, $i = 0, 1, \ldots, n - 1$. Also, it is well known that $Z$ contains a normal basis, i.e., there exists an $a \in Z$ such that the elements $\sigma^i(a) = a^p^i$, $i = 0, 1, \ldots, n - 1$, form a basis of $Z$ over $F$.

The restriction of every Jordan automorphism of $A$ to $Z$ is an automorphism of $Z$. Therefore, for each $z \in Z$ there exists an $i$ such that $T(z) = \sigma^i(z)$. Let $k$ be such that $T(a) = \sigma^k(a)$. Take $j \geq 1$. Then there exist $l, m$ such that

$$T(\sigma^j(a)) = \sigma^l(a)$$

and

$$T(a - \sigma^j(a)) = \sigma^m(a - \sigma^j(a)) = \sigma^m(a) - \sigma^{j+m}(a).$$

On the other hand,

$$T(a - \sigma^j(a)) = T(a) - T(\sigma^j(a)) = \sigma^k(a) - \sigma^l(a).$$

Hence,

$$\sigma^m(a) - \sigma^{j+m}(a) = \sigma^k(a) - \sigma^l(a).$$

Since $\sigma^i(a)$, $i = 0, 1, \ldots, n - 1$, are linearly independent and $\text{char}(F) \neq 2$, it follows that $m = k$ and $j + m \equiv l \pmod{n}$. This means that

$$T(\sigma^j(a)) = \sigma^k(\sigma^l(a)) \quad \text{for every } j \geq 1.$$ 

Consequently, $T|_Z = \sigma^k$. We can extend $\sigma^k$ to an automorphism $S$ of $A$ in the obvious way, that is, $S((z_{ij})) = (\sigma^k(z_{ij}))$ for every matrix $(z_{ij}) \in A$. Note that $S^{-1}T$ is a local Jordan automorphism of $A$ that acts as the identity on $Z$. Therefore, there is no loss of generality in assuming that $T$ itself is the identity on $Z$. Also, we may now assume that $s \geq 2$.

Let $A_0 = M_s(F)$. It is obvious that $T|_{A_0}$, the restriction of $T$ to $A_0$, is an $F$-linear local Jordan homomorphism from $A_0$ to $A$. By [6, Theorem 2.1], $T|_{A_0}$ is the sum of a homomorphism $\Phi$ and an antihomomorphism $\Psi$. Suppose $\Phi \neq 0$. From $\Phi(1) = \Phi(e_{11}) + \cdots + \Phi(e_{ss})$ (where as above $e_{ii}$ are matrix units) we see that $\Phi(1)$ is an idempotent that can be written as a sum of $s$ mutually orthogonal nonzero idempotents in $A$. This implies that $\Phi(1) = 1$. Similarly, $\Psi \neq 0$ yields $\Psi(1) = 1$. However, since $1 = T(1) = \Phi(1) + \Psi(1)$, one of $\Phi$ and $\Psi$ is 0. That is, $T|_{A_0}$ is either a homomorphism or an antihomomorphism. Hence, $R : A \to A$ given by

$$R(\sum z_{ij}e_{ij}) = \sum z_{ij}T(e_{ij}) \quad \text{for all } z_{ij} \in Z$$

is a ($Z$-linear) automorphism or antiautomorphism of $A$. Note that $R^{-1}T$ is a local Jordan automorphism of $A$ that satisfies $(R^{-1}T)(a_0) = a_0$ for every $a_0 \in A_0$. Therefore, without loss of generality we may assume that $T$ itself is the identity on $A_0$.

The proof will be completed by showing that $T$ is the identity on $A$.

We start the proof with a general remark. Observe that a nonzero matrix $a \in A$ has rank one if and only if $aAa \subset ZA$. This shows that every Jordan automorphism of $A$ maps the set of rank one matrices onto itself. The same is then true for every local Jordan automorphism. Thus, $a$ has rank one if and only if $T(a)$ has rank one.
Take and fix an arbitrary rank-one idempotent $e$ belonging to $A_0$. For any $z \in Z$, we have

$$T(ze) = T_{ze}(ze) = T_{ze}(z)T_{ze}(e).$$

Note that $T_{ze}(z) \in Z$ and $T_{ze}(e)$ is a rank-one idempotent. Thus, for every $z \in Z$ there exist a $z' \in Z$ and a rank-one idempotent $e_z \in A$ such that $T(ze) = z'e_z$. Similarly, $T(z(1-e)) = z''f_z$ where $z'' \in Z$ and $f_z$ is an idempotent (of rank $s-1$, but we will not need this). Hence,

$$z = T(z) = T(ze) + T(z(1-e)) = z'e_z + z''f_z.$$

This implies that $z''(e_zf_z - f_z e_z) = 0$. Since $z'' \neq 0$ whenever $z \neq 0$ it follows that $e_z$ and $f_z$ commute for every $z \in Z$. Further, squaring $z = z'e_z + z''f_z$ we obtain

$$z^2 = z'^2e_z + z''^2f_z + 2z'z''f_z e_z = z'(z''f_z) + f_z(z''^2 + 2z'z''e_z),$$

which gives

$$(z - z')z = f_z(z''^2 + 2z'z''e_z - z'z'').$$

Since $f_z$ is a nontrivial idempotent it follows that $z = z'$ or $z = 0$. We have thereby proved that for every $z \in Z$, there is a rank-one idempotent $e_z \in A$ such that $T(ze) = ze_z$. We may assume that $e_0 = e$.

Since $e \in A_0$, $T(e) = e$. For every $z \in Z$ we thus have

$$(z + 1)e_{z+1} = T((z + 1)e) = T(ze) + T(e) = ze_z + e.$$

By squaring we obtain

$$(z + 1)^2e_{z+1} = z^2e_z + z(e_z e + ee_z) + e.$$

On the other hand,

$$(z + 1)^2e_{z+1} = (z + 1)(ze_z + e).$$

Comparing the last two identities we see that $z(e_z - e)^2 = 0$. This shows that $a_z = e_z - e$ has square $0$ for every $z \in Z$. Since $e_z = e + a_z$ is an idempotent it follows that $a_z = ea_z + a_z e$. Consequently, $ea_z e = 0$ and hence

$$a_z = (1-e)a_z e + ea_z(1-e).$$

Suppose $a_z \neq 0$. Then at least one of $(1-e)a_z e$ and $ea_z(1-e)$, let us say the latter one, is nonzero. Take any $u \in (1-e)A_0 e$ such that $(1-e)za_z e \neq u$. Then $ze - u \in Ae$ has rank one, but

$$T(ze - u) = z(e + a_z) - u = ze + ((1-e)za_z e - u) + eza_z(1-e)$$

has rank at least $2$ since it is a sum of an element from $Ze$, a nonzero element from $(1-e)Ae$, and a nonzero element from $eA(1-e)$. This contradiction shows that $a_z = 0$. Thus, $T(ze) = ze$ for every $z \in Z$ and every rank one idempotent $e \in A_0$.

Since $e_{ii}$ and $e_{ii} + e_{ij}$, where $i \neq j$, are rank-one idempotents belonging to $A_0$, it follows that $T(ze_{ii}) = ze_{ii}$ and $T(ze_{ij}) = ze_{ij}$ for all $z \in Z$ and all $1 \leq i, j \leq s$. But then $T(a) = a$ for every $a \in A$.  

\[\square\]
AUTOMORPHISMS AND DERIVATIONS

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Facility of Mathematics and Physics, University of Ljubljana, and Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia

Email address: matej.bresar@fmf.uni-lj.si