NORMAL STRUCTURE OF ISOPTROPIC REDUCTIVE GROUPS OVER RINGS

ANASTASIA STAVROVA AND ALEXEI STEPANOV

Abstract. The paper studies the lattice of subgroups of an isotropic reductive group $G(R)$ over a commutative ring $R$, normalized by the elementary subgroup $E(R)$. We prove the sandwich classification theorem for this lattice under the assumptions that the reductive group scheme $G$ is defined over an arbitrary commutative ring, its isotropic rank is at least 2, and the structure constants are invertible in $R$. The theorem asserts that the lattice splits into a disjoint union of sublattices (sandwiches) $E(R, q) \leq \cdots \leq C(R, q)$ parametrized by the ideals $q$ of $R$, where $E(R, q)$ denotes the relative elementary subgroup and $C(R, q)$ is the inverse image of the center under the natural homomorphism $G(R) \to G(R/I)$. The main ingredients of the proof are the “level computation” by the first author and the universal localization method developed by the second author.

1. INTRODUCTION

Let $G$ be a reductive group scheme over a unital commutative ring $K$ in the sense of [DG70]. The famous result of J. Tits [Tit64] establishes that if $K$ is a field, and $G$ has no normal closed connected smooth $K$-subgroups, then its group of $K$-points $G(K)$ is very close to being simple as an abstract group (except in a few cases where $K = \mathbb{F}_2$ or $\mathbb{F}_3$). Namely, $G(K)$ contains a “large” normal subgroup $G(K)^+$ whose central quotient is simple. If $K$ is a finite field, the corresponding simple group $G(K)^+/C(G(K)^+)$ is a finite simple group of Lie type, and in fact such groups constitute the largest family in the classification of finite simple groups [Wil09].

If $K$ is no longer a field, then any proper ideal $q$ of $K$ determines a normal subgroup $G(K, q)$ of $G(K)$ called the congruence subgroup of level $q$; this subgroup is the kernel of the natural homomorphism $\rho_q : G(K) \to G(K/q)$. Thus $G(K)$ is no longer simple, and its lattice of normal subgroups is at least as rich as the lattice of ideals in $K$. The goal of the present paper is to show that, under the natural assumptions on $G$ and $K$, essentially, all normal subgroups of $G(K)$ are congruence subgroups. In order to state our result in a precise form, we need a few more definitions and notations.

We say that $G$ has isotropic rank $\geq n$, if every semisimple normal $K$-subgroup of $G$ contains an $n$-dimensional split $K$-torus $(\mathbb{G}_{m,K})^n$. If the isotropic rank is $\geq 1$, then $G$ contains a pair of opposite parabolic $K$-subgroups $P^\pm$ [DG70], and one defines the elementary subgroup $E_P(K)$ as the subgroup of $G(K)$ generated by $U_{P^+}(K)$ and $U_{P^-}(K)$, where $U_{P^\pm}$ denotes the unipotent radical of $P^\pm$. If, moreover, the isotropic rank of $G$ is $\geq 2$, the main result of [PS09] implies that $E(K) = E_P(K)$ is independent of the choice of $P^\pm$ and normal in $G(K)$ (see §8 for the details). Under these assumptions, if $G = \text{GL}_n$, then the elementary subgroup is the subgroup generated by the elementary transvections $e + te_{ij}$, $1 \leq i \neq j \leq n$, $t \in K$. And if $K$ is a field, then $E(K) = G(K)^+$ is the above-mentioned group of J. Tits.

1991 Mathematics Subject Classification. 20G35.

Key words and phrases. isotropic reductive groups; parabolic subgroup; elementary subgroup; congruence subgroup; unipotent element; generic element; universal localization; normal structure.

This publication is supported by Russian Science Foundation grant 17-11-01261. The first author is a winner of the contest “Young Russian Mathematics”. 

1
Assume that the isotropic rank of \( G \) is \( \geq 2 \). For any ideal \( q \) of \( K \), let \( U_{P^+}(q) \) be the kernel of the natural homomorphism \( \rho_q \) restricted to \( U_{P^+}(K) \). Denote by \( E_P(K, q) \) the normal closure of \( \langle U_{P^+}(q), U_{P^-}(q) \rangle \) in \( E(K) \). Also, denote by \( C(K, q) \) the full preimage of the center of \( G(K/q) \) under \( \rho_q \).

For any maximal ideal \( m \) of \( K \), denote by \( K/m \) the algebraic closure of the field \( K/m \). The group \( G_{K/m} \) is a reductive algebraic group in the usual sense [Bor91], and thus has a root system \( \Phi \) in the sense of Bourbaki. Note that \( G_{K/m} \) has no closed connected normal smooth \( K \)-subgroups if and only if \( \Phi \) is irreducible. The structure constants of \( \Phi \) are, by definition, the integers \( \pm 1 \), together with \( \pm 2 \), if \( \Phi \) is of type \( B_n, C_n, F_4 \), or \( \pm 2, \pm 3 \), if \( \Phi \) is of type \( G_2 \).

The main result of the present paper is the following theorem.

**Theorem 1.1.** Let \( G \) be a reductive group scheme over a ring \( K \) such that the isotropic rank of its derived group scheme \( [G, G] \) is \( \geq 2 \). Suppose that for any maximal ideal \( m \) of \( K \) the root system of \( G_{K/m} \) is irreducible, and its structure constants are invertible in \( K \). Then

(i) For any ideal \( q \) of \( K \) one has \( E_P(K, q) = [G(K, q), E(K)] \). In particular, \( E_P(K, q) = E(K, q) \) is independent of the choice of a parabolic \( K \)-subgroup \( P \).

(ii) For any subgroup \( H \leq G(K) \) normalized by \( E(K) \), there exists a unique ideal \( q \) in \( K \) such that \( E(K, q) \leq H \leq C(K, q) \).

Since groups of points of reductive group schemes include, in particular, the linear matrix groups \( GL_n(K), SL_n(K), Sp_{2n}(K) \), as well as spinor and special orthogonal groups \( Spin(q)(K), SO(q)(K) \), where \( q \) is a non-degenerate quadratic \( K \)-form, the study of their normal structure has a very long history. Thus, it would take a separate survey paper to describe it in full, and below we only list a few milestone results.

- Simplicity of the groups \( PSL_n(F) \) was proved by Camille Jordan around 1870 (for prime fields) and in early 1900 by Leonard Dickson (for all finite fields).
- A similar result for the \( GL_n \) over a skew-field was obtained by J. Dieudonne in 1943. In [Die48] he established the simplicity of split groups over arbitrary fields. The first uniform proof of their simplicity was given by C. Chevalley [Che55].
- The problem over rings distinct from fields was first approached by J. Brenner [Bre38, Bre44, Bre60] and later by W. Klingenberg [Kli61a, Kli61b, Kli63]. They studied split groups of classical types over quotients of \( \mathbb{Z} \) and general local rings respectively.
- As mentioned above, J. Tits established the simplicity for isotropic reductive groups over fields in [Tit64].
- For the general linear group over an arbitrary ring, the normal structure theorem was proved by H. Bass [Bas64] under the stable range condition. The result was generalized to other quasi-split classical groups by H. Bass himself [Bas73] and A. Bak [Bak69].
- H. Bass, M. Lazard, J.-P. Serre, and J. Milnor [BLS64, BMJP67] elucidated the normal structure of \( SL_n \) and \( Sp_{2n} \) over rings of integers of algebraic number fields, and stated the Congruence subgroup problem. It was solved for all split groups by H. Matsumoto [Mat69].
- The stable range condition for \( GL_n \) was removed by J. Wilson [Wil72] \( (n \geq 4) \) and I. Golubchik [Gol73] \( (n \geq 3) \). Using the ideas of H. Bass and Suslin’s theorem on the normality of the elementary group \( Sus72 \), Z. Borevich and N. Vavilov [BV85] gave a simpler prove of the Wilson–Golubchik theorem.
- In 1974 E. Abe and K. Suzuki proved the normal structure theorem for all Chevalley groups over local rings [AS76].
• In 1976 M.S. Raghunathan established the Congruence subgroup problem for groups of isotropic rank $\geq 2$ over a global field [Rag76]. In 1986 he improved this result, weakening the isotropy conditions [Rag86].

• In 1979 G. Margulis [Mar79] proved his celebrated theorem on lattices in isotropic groups over local fields. His work was based on earlier work of V. P. Platonov [Pla69].

• In 1980 R. Bix established the normal structure theorem for isotropic groups of type $1E_{6,2}$ over local rings [Bix80].

• L. Vaserstein in [Vas81] combined the results of H. Bass, J. Wilson and I. Golubchik proving the standard normal structure of the $GL_n$ under a local stable rank condition.

• After the result of G. Taddei on normality of the elementary group in a Chevalley group L. Vaserstein [Vas86] proved the standard normal structure of Chevalley groups of rank $\geq 2$ over commutative rings provided that the structure constants were invertible. The latter condition was removed by E. Abe in [Abe89].

• In 1988 L. Vaserstein proved the normal structure theorem for isotropic orthogonal groups over commutative rings [Vas88].

• The result of E. Abe has three exceptions where the elementary group is not perfect: types $C_2$ and $G_2$ if the ground ring has a residue field of 2 elements, and type $A_1$. All the exceptions were considered by D. Costa and G. Keller in a series of papers [CK91a, CK91b, CK99]. Of course, for type $A_1$ the ground ring must be low-dimensional with many units, as the description of the normal structure of $SL_2(Z)$ seems to be an unrealistic problem.

• In 2010 Z. Zhang [Zha10] established the normal structure theorem for even hyperbolic unitary groups over a commutative form ring with invertible 2. The latter assumption was removed by Hong You [You12]. A shorter proof was obtained recently by R. Preusser [Pre18].

2. Principal notation

Let $x, y, z$ be elements of an abstract group $G$. Denote by $x^y = y^{-1}xy$ the element conjugate to $x$ by $y$. Sometimes the same element is denoted by $y^{-1}x$. The commutator $x^{-1}y^{-1}xy$ is denoted by $[x, y]$. In the sequel we frequently use the following commutator identity, which can be easily verified by a straightforward calculation. Let $x, y, z$ be elements of an abstract group $G$. Then

$$[x, yz]^{-1} = (x^{-1})^{-1}x^y = [z^{-1}, x] \cdot [x, y].$$

Let $S$ be a subset of $G$. By $\langle S \rangle$ we denote the subgroup spanned by $S$. For subgroups $X$ and $Y$ of $G$ by $X^Y$ we denote the subgroup of $G$ generated by $x^y$ for all $x \in X$ and $y \in Y$. In other words, $X^Y$ is the smallest subgroup containing $X$ and normalized by $Y$. The mutual commutator subgroup $[X, Y]$ is a subgroup of $G$, generated by all the commutators $[x, y]$, $x \in X$, $y \in Y$. The center of an abstract group $G$ is denoted by $C(G)$.

All rings and algebras are assumed to be commutative and to contain a unit. All homomorphisms preserve unit elements. The multiplicative group of a ring $R$ is denoted by $R^\times$. As usual, $\text{Spec} \, R$ denotes the prime spectrum of $R$. For $p \in \text{Spec} \, R$ denote by $R_p = (R \setminus p)^{-1}R$ the localization of $R$ at $p$ and by $k(p)$ the residue field $R_p/pR_p$. The algebraic closure of $k(p)$ is denoted by $\overline{k(p)}$.

Let $s \in R$. The principal localization at the element $s$ (i.e. the localization at the multiplicative subset generated by $s$) is denoted by $R_s$. The localization homomorphism is denoted by $\lambda_p$ or $\lambda_s$ respectively.
Throughout the paper $K$ is a ring, $R$ denotes a $K$-algebra and $G$ stands for a reductive group scheme over $K$ in the sense of [DG70], unless explicitly stated otherwise.

For any ideal $q$ of $R$ we denote by $\rho_q : R \to R/q$ the reduction homomorphism, and, by abuse of notation, the induced homomorphism $G(R) \to G(R/q)$. The principal congruence subgroup $G(R, q)$ is the kernel of $\rho_q : G(R) \to G(R/q)$, whereas the full congruence subgroup $C(R, q)$ is the inverse image of the center $C(G(R/q))$ of $G(R/q)$ under this homomorphism.

3. Elementary subgroup of an isotropic reductive group

Let $P$ be a parabolic subgroup of $G$ in the sense of [DG70]. Since the base Spec $K$ is affine, the group $P$ has a Levi subgroup $L_P$ [DG70, Exp. XXVI Cor. 2.3]. There is a unique parabolic subgroup $P^-$ in $G$ which is opposite to $P$ with respect to $L_P$, that is $P^- \cap P = L_P$, cf. [DG70, Exp. XXVI Th. 4.3.2]. We denote by $U_P$ the unipotent radical of $P$.

Note that if $L_P$ is another Levi subgroup of $P$, then $L_P'$ and $L_P$ are conjugate by an element $u \in U_P(K)$ [DG70, Exp. XXVI Cor. 1.8]. Since our proofs in the present paper do not depend on a particular choice of $L_P$ or $P^-$, we do not pay attention to this choice.

Definition. The elementary subgroup $E_P(R)$ corresponding to $P$ is the subgroup of $G(R)$ generated as an abstract group by $U_P(R)$ and $U_P^-(R)$.

Definition. A parabolic subgroup $P$ in $G$ is called strictly proper, if it intersects properly every normal semisimple subgroup of $G$.

The following theorem is the main result of [PS09].

**Theorem 3.1** ([PS09, Theorem 1]). Assume that for every maximal ideal $m$ of $K$ every normal semisimple subgroup of $G_{K_m}$ contains $(\mathbb{G}_m, K_m)^2$. Then the subgroup $E_P(R)$ of $G(R) = G_R(R)$ is the same for any strictly proper parabolic $R$-subgroup $P$ of $G_R$. In particular, $E_P(R)$ is normal in $G(R)$.

Definition. Under the assumptions of Theorem 3.1 we call $E_P(R)$ the elementary subgroup of $G(R)$ and denote it by $E(R)$.

We also use the following theorems, which are the main results of [KS13] and [LS12]. We denote by $C(G)$ the group scheme center of $G$ in the sense of [DG70], see also discussion around Proposition 6.7 in [Mil12]. The very definition of $C(G)$ implies that $C(G)(R) \subseteq C(G(R))$.

**Theorem 3.2** ([KS13, Theorem 1]). Under the hypothesis of Theorem 3.1 one has

$$C_{G(R)}(E(R)) = C(G)(R) = C(G(R)).$$

**Theorem 3.3** ([LS12, Theorem 1]). Under the hypothesis of Theorem 3.1 assume, moreover, that for any maximal ideal $m$ of $R$, one has $k(m) \neq \mathbb{F}_2$ whenever the root system of $\overline{G_{k(m)}}$ contains an irreducible component of type $B_2 = C_2$ or $G_2$. Then $E(R) = [E(R), E(R)]$.

The following statement will be used twice to get rid of the center.

**Lemma 3.4.** Let $H$ be a subgroup of $G(R)$, normalized by $E(R)$. Under the assumptions of Theorem 3.3 $[H, E(R)] = [H, E(R)] = [H, E(R)] = [H, E(R), E(R)]$. In particular, if $[H, E(R)] \subseteq C(G(R))$, then $H \subseteq C(G(R))$.

**Proof.** By Theorem 3.3 the group $E(R)$ is perfect. Using the Hall–Witt identity we get

$$[E(R), H] = [[E(R), E(R)], H] \subseteq [[H, E(R)], E(R)].$$

The inverse inclusion is obvious. The second assertion follows immediately from the first one and Theorem 3.2. $\square$
4. Root systems corresponding to parabolic subgroups

Let \( S = (\mathbb{G}_{m,K})^N = \text{Spec}(K[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]) \) be a split \( N \)-dimensional torus over \( R \). Recall that the character group \( X^*(S) = \text{Hom}_K(S, \mathbb{G}_{m,K}) \) of \( S \) is canonically isomorphic to \( \mathbb{Z}^N \). If \( S \) acts \( K \)-linearly on a \( K \)-module \( V \), this module has a natural \( \mathbb{Z}^N \)-grading

\[
V = \bigoplus_{\lambda \in X^*(S)} V_\lambda,
\]

where

\[
V_\lambda = \{ v \in V \mid s \cdot v = \lambda(s)v \text{ for any } s \in S(K) \}.
\]

Conversely, any \( \mathbb{Z}^N \)-graded \( K \)-module \( V \) can be provided with an \( S \)-action by the same rule.

Assume that \( S \) acts on \( G \) by \( K \)-group automorphisms. The associated Lie algebra functor \( \text{Lie}(G) \) then acquires a \( \mathbb{Z}^N \)-grading compatible with the Lie algebra structure,

\[
\text{Lie}(G) = \bigoplus_{\lambda \in X^*(S)} \text{Lie}(G)_\lambda.
\]

We will use the following version of [DG70, Exp. XXVI Prop. 6.1].

**Lemma 4.1.** Let \( L = C_G(S) \) be the subscheme of \( G \) fixed by \( S \). Let \( \Psi \subseteq X^*(S) \) be a \( K \)-subsheaf of sets closed under addition of characters. Then there exists a unique smooth connected closed subgroup \( U_\Psi \) of \( G \) normalized by \( L \) and satisfying

\[
(4.1) \quad \text{Lie}(U_\Psi) = \bigoplus_{\lambda \in \Psi} \text{Lie}(G)_\lambda.
\]

Moreover,

1. if \( 0 \in \Psi \), then \( U_\Psi \) contains \( L \);
2. if \( \Psi = \{0\} \), then \( U_\Psi = L \);
3. if \( \Psi = -\Psi \), then \( U_\Psi \) is reductive;
4. if \( \Psi \cup (-\Psi) = X^*(S) \), then \( U_\Psi \) and \( U_{-\Psi} \) are two opposite parabolic subgroups of \( G \) with the common Levi subgroup \( U_{\Psi \cap (-\Psi)} \);
5. If \( 0 \notin \Psi \), then \( U_\Psi \) is unipotent.

**Proof.** The statement immediately follows by faithfully flat descent from the standard facts about the subgroups of split reductive groups proved in [DG70, Exp. XXII]; see the proof of [DG70, Exp. XXVI Prop. 6.1]. \( \square \)

**Definition.** The sheaf of sets

\[
\Phi(S, G) = \{ \lambda \in X^*(S) \setminus \{0\} \mid \text{Lie}(G)_\lambda \neq 0 \}
\]

is called the system of relative roots of \( G \) with respect to \( S \).

**Remark.** Choosing a total ordering on the \( \mathbb{Q} \)-space \( \mathbb{Q} \otimes_{\mathbb{Z}} X^*(S) \cong \mathbb{Q}^n \), one defines the subsets of positive and negative relative roots \( \Phi(S, G)^+ \) and \( \Phi(S, G)^- \), so that \( \Phi(S, G) \) is a disjoint union of \( \Phi(S, G)^+ \), \( \Phi(S, G)^- \), and \( \{0\} \). By Lemma 4.1, the closed subgroups

\[
U_{\Phi(S,G)^+ \cup \{0\}} = P, \quad U_{\Phi(S,G)^- \cup \{0\}} = P^-\n\]

are two opposite parabolic subgroups of \( G \) with the common Levi subgroup \( C_G(S) \). Thus, if a reductive group \( G \) over \( R \) admits a non-trivial action of a split torus, then it has a proper parabolic subgroup. The converse is true Zariski-locally, see Lemma 4.2 below.
Let $P$ be a parabolic subgroup scheme of $G$ over $K$, and let $L$ be a Levi subgroup of $P$. By $[DG70]$ Exp. XXII, Prop. 2.8] the root system $\Phi$ of $G_{k(p)}$, $p \in \text{Spec } K$, is locally constant in the Zariski topology on $\text{Spec } K$. The type of the root system of $L_{k(p)}$ is determined by a Dynkin subdiagram of the Dynkin diagram of $\Phi$, which is also constant Zariski-locally on $\text{Spec } K$ by $[DG70]$ Exp. XXVI, Lemme 1.14 and Prop. 1.15]. In particular, if $\text{Spec } K$ is connected, all these data are constant on $\text{Spec } K$.

**Definition.** Assume that the root system $\Phi$ of $G_{k(p)}$ has the same type for all $p \in \text{Spec } K$. Then we call $\Phi$ the absolute root system of $G$.

**Lemma 4.2** $([Sta16]$ Lemma 3.6$]$). Assume that $K$ is connected. Let $\bar{L}$ be the image of $L$ under the natural homomorphism $G \to G^{\text{ad}} \subseteq \text{Aut}(G)$. Let $D$ be the Dynkin diagram of the absolute root system $\Phi$ of $G$. We identify $D$ with a set of simple roots of $\Phi$ such that $P_{k(p)}$ is a standard positive parabolic subgroup with respect to $D$. Let $J \subseteq D$ be the set of simple roots such that $D \setminus J$ is the subdiagram of $D$ corresponding to $L_{k(p)}$. Then there are a unique maximal split subtorus $S \subseteq C(\bar{L})$ and a subgroup $\Gamma \leq \text{Aut}(D)$ such that $J$ is invariant under $\Gamma$ and for any $p \in \text{Spec } R$ and any split maximal torus $T \subseteq L_{k(p)}$ the kernel of the natural surjection

$$X^*(T) \cong \mathbb{Z}\Phi \xrightarrow{\pi} X^*(S_{k(p)}) \cong \mathbb{Z}\Phi(S,G)$$

is generated by all roots $\alpha \in D \setminus J$, and by all differences $\alpha - \sigma(\alpha)$, $\alpha \in J$, $\sigma \in \Gamma$.

In $[PS09]$, we introduced a system of relative roots $\Phi_P$ with respect to a parabolic subgroup $P$. This system $\Phi_P$ was defined independently over each member $\text{Spec } K = \text{Spec } K_i$ of a suitable finite disjoint Zariski covering

$$\text{Spec } K = \coprod_{i=1}^m \text{Spec } K_i,$$

such that over each $K_i$, $1 \leq i \leq m$, the root system $\Phi$ and the Dynkin diagram $D$ of $G$ is constant. Namely, we considered the formal projection

$$\pi_{J,\Gamma}: \mathbb{Z}\Phi \longrightarrow \mathbb{Z}\Phi/\langle D \setminus J; \alpha - \sigma(\alpha) \mid \alpha \in J, \sigma \in \Gamma \rangle,$$

and set $\Phi_P = \Phi_{J,\Gamma} = \pi_{J,\Gamma}(\Phi) \setminus \{0\}$. The last claim of Lemma $[4.2]$ allows to identify $\Phi_{J,\Gamma}$ and $\Phi(S,G)$ whenever $\text{Spec } K$ is connected.

**Definition.** In the setting of Lemma $[4.2]$ we call $\Phi(S,G) \cong \Phi_{J,\Gamma}$ a system of relative roots with respect to the parabolic subgroup $P$ over $R$ and denote it by $\Phi_P$.

If $K$ is a field or a local ring, and $P$ is a minimal parabolic subgroup of $G$, then $\Phi_P$ is nothing but the relative root system of $G$ with respect to a maximal split subtorus in the sense of $[BT65]$ or, respectively, $[DG70]$ Exp. XXVI §7].

We have also defined in $[PS09]$ irreducible components of systems of relative roots, the subsets of positive and negative relative roots, simple relative roots, and the height of a root. These definitions are immediate analogs of the ones for usual abstract root systems, so we do not reproduce them here.

We will need later the following two lemmas on relative roots.

**Lemma 4.3.** Let $\Phi$ be a root system, and let $\Phi_{J,\Gamma} = \pi(\Phi) \setminus \{0\}$ be a relative root system with the canonical projection $\pi: \mathbb{Z}\Phi \to \mathbb{Z}\Phi_{J,\Gamma}$. Let $\alpha, \beta \in \Phi_{J,\Gamma}$ be two simple relative roots such that $\alpha + \beta \in \Phi_{J,\Gamma}$. Then for any $j \geq 1$ such that $j\beta \in \Phi_{J,\Gamma}$ one has $\alpha + j\beta \in \Phi_{J,\Gamma}$.
Proof. The inclusion $\alpha + \beta \in \Phi_{J,G}$ is equivalent to the existence of two simple roots in $\pi^{-1}(\beta)$ and $\pi^{-1}(\alpha)$ respectively that are connected in the Dynkin diagram by a chain (possibly, empty) of simple roots lying in $\pi^{-1}(0)$. Then for any element $\mu \in \pi^{-1}(j\beta)$ the set $S_\mu \subseteq \Phi$ of simple roots occurring in its decomposition is also connected to a simple root in $\pi^{-1}(\alpha)$ by a chain of simple roots lying in $\pi^{-1}(0)$, which allows to find a root $\nu \in \pi^{-1}(\alpha)$ such that $S_\mu$ and $S_\nu$ are disjoint but adjacent subsets of the Dynkin diagram of $\Phi$, and hence $\mu + \nu \in \Phi$. Then $\pi(\mu + \nu) = \alpha + j\beta \in \Phi_{J,G}$.

Lemma 4.4. Let $\Phi$ be a root system with the scalar product $\langle \cdot, \cdot \rangle$ and a system of simple roots $\Pi$, and let $\Phi_{J,G} = \pi(\Phi) \setminus \{0\}$ be a relative root system with the canonical projection $\pi: \mathbb{Z}\Phi \rightarrow \mathbb{Z}\Phi_{J,G}$. Let $\beta \in \pi(\Pi) \cap \Phi_{J,G}$ be a simple relative root. Then there is a proper parabolic subset $\Sigma$ of $\Phi_{J,G}$ that contains all $\alpha \in \Phi_{J,G}$ such that $\alpha + \beta \notin \Phi_{J,G}$. If $\Phi$ is simply laced, then

$$\Sigma = \{ \alpha \in \Phi_{J,G} \mid (a, \sum_{b \in \pi^{-1}(\beta)} b) \geq 0 \text{ for all } a \in \pi^{-1}(\alpha) \}$$

(4.3)

$$= \{ \alpha \in \Phi_{J,G} \mid (a, \sum_{b \in \pi^{-1}(\beta)} b) \geq 0 \text{ for some } a \in \pi^{-1}(\alpha) \}.$$

Proof. If $\Phi$ is not simply laced, then there is a simply laced root system $\Phi'$ with a system of simple roots $\Pi'$ and a projection $\pi': \mathbb{Z}\Phi' \rightarrow \mathbb{Z}\Phi$ such that $\Phi = \Phi'_{J',G'}$ for some suitable group of automorphisms $\Gamma'$ of the Dynkin diagram of $\Phi'$. Then $\Phi_{J,G} = \Phi'_{\pi'^{-1}(J),G''}$, where $\Gamma''$ is a group of automorphisms of the Dynkin diagram of $\Phi$ generated by $\Gamma'$ and the natural lifting of $\Gamma$. This means that we can assume that $\Phi$ is simply laced from the start.

For any $\alpha \in \Phi_{J,G}$ and any $\mu \in \pi^{-1}(\alpha)$ by [PS09, Lemma 3] the set $\pi^{-1}(\alpha)$ is a union of roots $\mu' \in \Phi$ such that $\sigma(\mu) = \mu_j$ for some $\sigma \in \Gamma$, where $\mu_j$ denotes the shape of $\mu$ with respect to $J$ in the sense of [PS09] or [ABS90]. By [ABS90, Lemma 1] the Weyl group of $\Pi \setminus J$ acts transitively on the set of roots of a fixed shape and length, hence this Weyl group and $\Gamma$ together act transitively on $\pi^{-1}(\alpha)$. Since all elements of these groups are bijections that preserve scalar products and the set $\pi^{-1}(\beta)$, we conclude that $(\mu, \sum_{\nu \in \pi^{-1}(\beta)} \nu)$ is the same for all $\mu \in \pi^{-1}(\alpha)$. Thus, the set $\Sigma$ of (4.3) is well-defined, and it is clear that $\Sigma \cup (-\Sigma) = \Phi_{J,G}$.

Assume that $\alpha_1, \alpha_2 \in \Sigma$ and $\alpha_1 + \alpha_2 \in \Phi_{J,G}$. By [PS09, Lemma 4] for every $\mu \in \pi^{-1}(\alpha_1 + \alpha_2)$ there are $\mu_1 \in \pi^{-1}(\alpha_1)$ and $\mu_2 \in \pi^{-1}(\alpha_2)$ such that $\mu = \mu_1 + \mu_2$. This shows that $\Sigma$ is additively closed. By the same argument, if $\alpha + \beta \notin \Phi_{J,G}$, then $(\mu, \nu) \geq 0$ for all $\mu \in \pi^{-1}(\alpha)$ and $\nu \in \pi^{-1}(\beta)$, and hence $\alpha \in \Sigma$.

It remains to show that $\Sigma$ is a proper subset of $\Phi_{J,G}$. Let $\alpha \in \pi(\Pi) \setminus \{0\}$ be a simple relative root such that $\alpha + \beta \in \Phi_{J,G}$. Since $\alpha - \beta \notin \Phi_{J,G}$, we conclude that $(\mu, \nu) \leq 0$ for all $\mu \in \pi^{-1}(\alpha)$ and $\nu \in \pi^{-1}(\beta)$. On the other hand, since $\alpha + \beta \in \Phi_{J,G}$, there are two simple roots $\mu \in \pi^{-1}(\alpha) \cap J$ and $\nu \in \pi^{-1}(\beta) \cap J$ which are connected on the Dynkin diagram by a chain of roots in $\Pi \setminus J$. Then the scalar product of $\mu$ and $\nu$ (all roots in this chain) is negative. Hence $\alpha \notin \Sigma$. \hfill \Box

5. Relative root subschemes.

For any finitely generated projective $K$-module $V$, we denote by $W(V)$ the natural affine scheme over $K$ associated with $V$, see [DG70, Exp. I, §4.6] or [Mil12, Ch. IV, §1, 1.6]. Any morphism of $K$-schemes $W(V_1) \rightarrow W(V_2)$ is determined by an element $f \in \text{Sym}^d(V_1^\vee) \otimes_K V_2$, where $\text{Sym}^* V_1^\vee$ denotes the symmetric algebra, and $V_1^\vee$ denotes the dual module of $V_1$. If $f \in \text{Sym}^d(V_1^\vee) \otimes_K V_2$, we say that the corresponding morphism is homogeneous of degree $d$. 
By abuse of notation, we also write $f : V_1 \to V_2$ and call it \textit{degree} $d$ \textit{homogeneous polynomial map} from $V_1$ to $V_2$. In this context, one has

$$f(rv) = r^d f(v)$$

for any $v \in V_1$ and $r \in K$.

\textbf{Lemma 5.1.} \cite{Sta16} Lemma 3.9]. \textit{In the setting of Lemma 4.2, for any $\alpha \in \Phi_P = \Phi(S, G)$ there exists a closed $S$-equivariant embedding of $K$-schemes

$$X_\alpha : W(\text{Lie}(G)_\alpha) \to G,$$

satisfying the following condition.

\begin{itemize}
  \item[(*)] Let $R/K$ be any ring extension such that $G_R$ is split with respect to a maximal split $R$-torus $T \subseteq L_R$. Let $e_\delta, \delta \in \Phi$, be a Chevalley basis of $\text{Lie}(G_R)$, adapted to $T$ and $P$, and $x_\delta : \mathbb{G}_a \to G_R, \delta \in \Phi$, be the associated system of 1-parameter root subgroups (e.g. $x_\delta = \exp_\delta$ of \cite{DG70} Exp. XXII, Th. 1.1). Let

$$\pi : \Phi = \Phi(T, G_R) \to \Phi_P \cup \{0\}$$

be the natural projection. Then for any $u = \sum a_\delta e_\delta \in \text{Lie}(G_R)_{\alpha}$ one has

$$X_\alpha(u) = \left( \prod_{\delta \in \pi^{-1}(\alpha)} x_{\delta(a_\delta)} \right) \cdot \prod_{i \geq 2} \left( \prod_{\theta \in \pi^{-1}(i_{\alpha})} p_{\theta} \right),$$

where every $p_{\theta} : \text{Lie}(G_R)_{\alpha} \to R$ is a homogeneous polynomial map of degree $i$, and the products over $\delta$ and $\theta$ are taken in any fixed order.

\textbf{Definition.} Closed embeddings $X_\alpha, \alpha \in \Phi_P$, satisfying the statement of Lemma 5.1 are called \textit{relative root subschemes} of $G$ with respect to the parabolic subgroup $P$.

Relative root subschemes of $G$ with respect to $P$, actually, depend on the choice of a Levi subgroup $L_P$, but their essential properties stay the same, so we usually omit $L_P$ from the notation.

Set $V_\alpha = \text{Lie}(G)_\alpha$ for short. We will use the following properties of relative root subschemes.

\textbf{Lemma 5.2.} \cite{PS09} Theorem 2, Lemma 6, Lemma 9] \textit{Let $X_\alpha, \alpha \in \Phi_P$, be as in Lemma 5.1. Then

(i) There exist degree $i$ homogeneous polynomial maps $q^i_\alpha : V_\alpha \oplus V_\alpha \to V_{i\alpha}, i > 1$, such that for any $K$-algebra $R$ and for any $v, w \in V_\alpha \otimes_K R$ one has

$$X_\alpha(v)X_\alpha(w) = X_\alpha(v + w) \prod_{i \geq 1} X_{i\alpha} \left( q^i_\alpha(v, w) \right).$$

(ii) For any $g \in L(K)$, there exist degree $i$ homogeneous polynomial maps $\varphi^i_{g, \alpha} : V_\alpha \to V_{i\alpha}, i \geq 1$, such that for any $K$-algebra $R$ and for any $v \in V_\alpha \otimes_R R'$ one has

$$gX_\alpha(v)g^{-1} = \prod_{i \geq 1} X_{i\alpha} \left( \varphi^i_{g, \alpha}(v) \right).$$

(iii) (generalized Chevalley commutator formula) For any $\alpha, \beta \in \Phi_P$ such that $m\alpha \neq -k\beta$ for all $m, k \geq 1$, there exist polynomial maps

$$N_{\alpha \beta, ij} : V_\alpha \times V_\beta \to V_{i\alpha + j\beta}, i, j > 0,$$
homogeneous of degree \( i \) in the first variable and of degree \( j \) in the second variable, such that for any \( K \)-algebra \( R \) and for any \( u \in V_\alpha \otimes_K R, v \in V_\beta \otimes_K R \) one has

\[
[X_\alpha(u), X_\beta(v)] = \prod_{i,j>0} X_{i+}\beta}(N_{\alpha\beta,j}(u,v))
\]

(iii) For any subset \( \Psi \subseteq X^*(S) \setminus \{0\} \) that is closed under addition, the morphism

\[
X_{\Psi} : W(\bigoplus_{\alpha \in \Psi} V_\alpha) \to U_{\Psi}, \quad (v_\alpha)_\alpha \mapsto \prod_{\alpha} X_\alpha(v_\alpha),
\]

where the product is taken in any fixed order, is an isomorphism of schemes.

Apart from the above properties of relative root subschemes we will use the following Lemma, which appeared first as \[\text{[PS09, Lemma 10]}\] and in a slightly stronger form in \[\text{[LS12, Lemma 2]}\]. Note that in both cases the original statements erroneously claim that the image \( \Im(N_{\alpha,\beta,1,1}) \) (respectively, the sum of images in (2)) equals \( V_{\alpha+\beta} \), while in reality the respective proofs establish only that it generates \( V_{\alpha+\gamma} \) as an \( R \)-module, and hence as an abelian group. The correct, weaker statement is as follows. One can easily check it is still enough for all the applications in \[\text{[PS09, LS12, Sta14b]}\].

**Lemma 5.3.** Consider \( \alpha, \beta \in \Phi_P \) satisfying \( \alpha + \beta \in \Phi_P \) and \( m\alpha \neq -k\beta \) for any \( m,k \geq 1 \). Denote by \( \Phi^0 \) an irreducible component of \( \Phi \) such that \( \alpha, \beta \in \pi(\Phi^0) \).

(1) In each of the following cases, \( \Im(N_{\alpha,\beta,1,1}) \) generates \( V_{\alpha+\beta} \) as an abelian group:

(a) structure constants of \( \Phi^0 \) are invertible in \( R \) (for example, if \( \Phi^0 \) is simply laced);
(b) \( \alpha \neq \beta \) and \( \alpha - \beta \notin \Phi_P \);
(c) \( \Phi^0 \) is of type \( B_t, C_t, \text{ or } F_4 \), and \( \pi^{-1}(\alpha + \beta) \) consists of short roots;
(d) \( \Phi^0 \) is of type \( B_t, C_t, \text{ or } F_4 \), and there exist long roots \( \alpha \in \pi^{-1}(\alpha), \beta \in \pi^{-1}(\beta) \) such that \( \alpha + \beta \) is a root.

(2) If \( \alpha - \beta \in \Phi_P \) and \( \Phi^0 \neq G_2 \), then \( \Im(N_{\alpha,\beta,1,1}) \), \( \Im(N_{\alpha-\beta,2,1,1}) \), and \( \Im(N_{\alpha-\beta,2,1,2}, -\cdot, v) \) for all \( v \in V_\beta \) together generate \( V_{\alpha+\beta} \) as an abelian group. Here we assume \( \Im(N_{\alpha-\beta,2,1,1}) = 0 \) if \( 2\beta \notin \Phi_P \).

6. Reduction to Extraction of Unipotents

In this section we establish two important reduction statements. Throughout this section, we assume that \( G \) is a reductive group scheme over a connected commutative ring \( K \), and \( R \) is an arbitrary \( K \)-algebra. Let \( P \) be a proper parabolic \( K \)-subgroup of \( G \), and let \( \Phi_P = \Phi_{P,G} \) be the system of relative roots for \( P \), and \( X_\alpha(V_\alpha), \alpha \in \Phi_P \), be the relative root subschemes of \( G \) with respect to \( P \) that exist by Lemmas \[\text{[4.2]} \text{and [5.1]}\].

Recall that for any ideal \( q \) of \( R \) we set \( U_P(q) = U_P(R) \cap G(R, q) \) and denote by \( E_P(R, q) \) the normal closure of \( (U_P(q), U_{P^{-1}}(q)) \) in \( E_P(R) \).

**Lemma 6.1.** Assume that for any maximal ideal \( m \) of \( K \) every semisimple normal subgroup of group \( G_{K_m} \) contains \( (G_{m,K_m})^2 \). Then for any \( K \)-algebra \( R \) the group \( E_P(R, q) \) is independent of the choice of a strictly proper parabolic subgroup \( P \).

**Proof.** Clearly, it is enough to show that for any two strictly proper parabolic subgroups \( P, Q \) one has \( U_P(q) \leq E_Q(R, q) \). Consider the ring of polynomials \( R[x] \). For any \( x \in U_P(xR[x]) \) we have

\[
x \in G(R[x], xR[x]) \cap E_P(R[x]) = G(R[x], xR[x]) \cap E_Q(R[x]) = E_Q(R[x], xR[x])
\]

by the splitting principle for isotropic groups \[\text{[Sta14b, Lemma 4.1]}\]. Now let \( h \in U_P(q) \) be any element. Then, clearly, \( h \) is a product of elements \( X_\alpha(cv) \), where \( \alpha \in \Phi_P, c \in q \) and...
v ∈ V_α. By the previous argument we have X_α(xv) ∈ E_Q(R[x], xR[x]). Specializing x to c, we conclude that X_α(cv) ∈ E_Q(R, q), as required.

\[ \square \]

**Definition.** In the setting of Lemma 6.1 we denote \( E_P(R, q) \) by \( E(R, q) \). We say that the normal structure of the group \( G(R) \) is **standard** if for each subgroup \( H \leq G(R) \) normalized by \( E(R) \), there exists a unique ideal \( q \) of \( R \) such that

\[ E(R, q) \leq H \leq C(R, q). \]

We are going to deduce the existence of standard normal structure from the fact that any \( H \) as above contains an elementary root unipotent. Our starting point is the following theorem established by the first author.

**Theorem 6.2 ([Sta14a, Theorem 2]).** Assume that the structure constants of the absolute root system \( \Phi \) of \( G \) are invertible in \( K \), and for any maximal ideal \( m \) of \( K \) every semisimple normal subgroup of group \( G_{\mathbb{K}_m} \) contains \( (G_{m, \mathbb{K}_m})^2 \). Let \( P \) be a strictly proper parabolic \( K \)-subgroup of \( G \), and let \( E(K) = E_P(K) \) be the elementary subgroup of \( G(K) \). Then for any normal subgroup \( N \leq E(K) \) there exists an ideal \( q = q(N) \) in \( R \) such that \( N \cap X_\alpha(V_\alpha) = X_\alpha(qV_\alpha) \) for any \( \alpha \in \Phi_P \).

Using this theorem, we prove the following statement. The idea of this reduction goes back to [Bas64].

**Proposition 6.3.** In the setting of Theorem 6.2, suppose further that every irreducible component of \( \Phi_P \) contains more than 2 distinct roots, and that for any quotient ring \( K/q \) of \( K \) and any noncentral subgroup \( H \leq G(K/q) \) normalized by \( E(K/q) \) there is \( \alpha \in \Phi_P \) and \( 0 \neq u \in V_\alpha \otimes_K K/q \) such that \( X_\alpha(u) \in H \). Then the normal structure of the group \( G(K) \) is standard.

In order to prove Proposition 6.3 we need the following two statements.

**Lemma 6.4.** Let \( \alpha, \beta \in \Phi_P \) be two relative roots such that \( \alpha + \beta \in \Phi_P \) and \( n\alpha \neq -m\beta \) for all \( n, m \geq 1 \). Assume moreover that \( \alpha - \beta \notin \Phi_P \), or the structure constants of the absolute root system \( \Phi \) of \( G \) are invertible in \( K \). Take \( 0 \neq u \in V_\beta \). Any generating system \( e_1, \ldots, e_n \) of the \( K \)-module \( V_\alpha \) contains an element \( e_i \) such that \( N_{\alpha\beta^{11}}(e_i, u) \neq 0 \).

**Proof.** Consider an affine fpqc-covering \( \prod \text{Spec } R_v \to \text{Spec } K \) that splits \( G \). There is a member \( R_v = R \) of this covering such that the image of \( X_\beta(u) \) under \( G(K) \to G(R) \) is non-trivial. Write

\[ X_\beta(u) = \prod_{\pi(\nu) = \beta} x_\nu(a_\nu) \cdot \prod_{i \geq 2} \prod_{\pi(\nu) = i\beta} x_\nu(c_\nu), \]

where \( \pi : \Phi \to \Phi_P \cup \{0\} \) is the canonical projection of the absolute root system of \( G \) onto the relative one, \( x_\beta \) are root subgroups of the split group \( G_R \), and \( a_\nu \in R \). Since \( X_\beta(u) \neq 0 \), the definition of \( X_\beta \) implies that there exists \( a_\nu \neq 0 \). By [PS09, Lemma 4] there exists a root \( \mu \in \pi^{-1}(\alpha) \) such that \( \mu + \nu \in \Phi \). Let \( v \in V_\lambda \otimes_K R \) be such that \( X_\alpha(v) = x_\mu(1) \prod_{i \geq 2} \prod_{\pi(\eta) = i\alpha} x_\eta(d_\eta) \), for some \( d_\eta \in R \). Then the (usual) Chevalley commutator formula implies that \( [X_\alpha(v), X_\beta(u)] \) contains in its decomposition a factor \( x_{\mu+\nu}(e_{a_\nu}) \), where \( e \) is a structure constant of \( \Phi \).

If \( \alpha - \beta \notin \Phi_P \), then \( e = \pm 1 \), otherwise \( e \) is invertible by assumptions. Hence \( N_{\alpha\beta^{11}}(v, u) \neq 0 \). Since \( N_{\alpha\beta^{11}}(v, u) \) is linear in the first argument, this implies the result.

\[ \square \]

**Lemma 6.5.** Under the hypothesis of Proposition 6.3 one has

\[ [G(K, q), E(K)] = E_P(K, q). \]
Proof. One proves that \([G(K, q), E(K)] \leq E_P(K, q)\) exactly as \([\text{Ste16}, \text{Proposition 5.1}]\), using the splitting principle for isotropic groups \([\text{Sta14b}, \text{Lemma 4.1}]\). To prove the inverse inclusion, by Theorem \(6.2\) it is enough to show that \(X_\tilde{\alpha}(V_\tilde{\alpha} \otimes_K q) \subseteq [G(K, q), E(K)]\), where \(\tilde{\alpha} \in \Phi_P^+\) is a maximal root. Let \(\beta \in \Phi_P^+\) be a simple relative root such that \(\tilde{\alpha} - \beta \in \Phi_P\). Then by Lemma \(5.3\) and the generalized Chevalley commutator formula one has \(X_\tilde{\alpha}(V_\tilde{\alpha} \otimes_K q)\) is generated by all commutators \([X_\beta(u), X_{\tilde{\alpha} - \beta}(v)]\), where \(v \in V_{\tilde{\alpha} - \beta}, u \in V_\beta \otimes_K q\). This finishes the proof. \(\square\)

Proof of Proposition \(6.3\). Let \(N\) be a subgroup of \(G(K)\) normalized by \(E(K)\). If \(N \leq C(G(K))\), there is nothing to prove. Otherwise by our assumption and Theorem \(6.2\) there is an ideal \(q \neq 0\) of \(K\) such that \(N \cap X_\alpha(V_\alpha) = X_\alpha(qV_\alpha)\) for all \(\alpha \in \Phi_P\). Then, clearly, \(E_P(K, q) \leq N\). If \(N\) is not contained in \(C(K, q)\), then \(\rho_q(N)\) is a non-central subgroup of \(G(K/q)\) normalized by \(E(K/q)\), and hence by the same token we have \(N \geq E_P(K/q, q')\) for some ideal \(q' \neq 0\) of \(K/q\). Let \(\tilde{\alpha} \in \Phi_P^+\) be a maximal root, and let \(\beta \in \Phi_P^+\) be a simple relative root such that \(\tilde{\alpha} - \beta \in \Phi_P\). Pick \(0 \neq u \in V_{\tilde{\alpha} - \beta}\) such that \(0 \neq \rho_q(u) \in V_{\tilde{\alpha} - \beta} \otimes_K q'\) and \(X_{\tilde{\alpha} - \beta}(\rho_q(u)) \in \rho_q(N)\). Then there is \(h \in G(K, q)\) such that \(X_{\tilde{\alpha} - \beta}(u)h \in N\). By Lemma \(6.3\) for any generating system \(e_1, \ldots, e_n\) of \(V_\beta\) there is \(1 \leq i \leq n\) such that \(N_{\tilde{\alpha} - \beta, \beta, 1, 1}(\rho_q(u), \rho_q(e_i)) \neq 0\), and hence \(N_{\tilde{\alpha} - \beta, \beta, 1, 1}(u, e_i) \not\in qV_\alpha\). By Lemma \(6.3\) one has

\[X_{\tilde{\alpha} - \beta}(u)h, X_\beta(e) = X_{\tilde{\alpha} - \beta}(u)[h, X_\beta(e)] \cdot X_\tilde{\alpha}(N_{\tilde{\alpha} - \beta, \beta, 1, 1}(u, e_i)) \in E_P(K, q) \cdot X_\tilde{\alpha}(N_{\tilde{\alpha} - \beta, \beta, 1, 1}(u, e_i)).\]

Hence \(X_\tilde{\alpha}(N_{\tilde{\alpha} - \beta, \beta, 1, 1}(u, e_i)) \in N\). However, this contradicts the choice of \(q\). \(\square\)

Thus, we have reduced the proof of normal structure theorem to extraction of unipotents from \(E(K)\)-normalized subgroups. The following lemma shows that one can always extract a unipotent from a parabolic subgroup.

Lemma 6.6. In the setting of Theorem \(6.3\), let \(H\) be a subgroup of \(G(K)\) normalized by the elementary subgroup \(E(K)\). Suppose that \(H \cap P(K)\) is not contained in \(C(G(K))\). Then \(H\) contains a root unipotent element \(X_\alpha(v), \alpha \in \Phi_P, 0 \neq v \in V_\alpha\).

Proof. This is proved exactly as the corresponding statement \([\text{VS08}, \text{Theorem 1}]\) for Chevalley groups. \(\square\)

7. Extraction from the main Gauss cell

As usual, assume that \(G\) is a reductive group scheme over \(K\), and \(P\) is a proper parabolic \(K\)-subgroup of \(G\). Consider the \(K\)-scheme of \(G\)

\[\Omega_P = U_P L_P U_{P-}.\]

This is an open subscheme isomorphic to a \(K\)-scheme to the direct product \(U_P \times L_P \times U_{P-}\), see \([\text{DG70}, \text{Exp. XXVI, Remarque 4.3.6}]\). We call \(\Omega_P\) the main Gauss cell associated with \(P\).

In this section we show how to extract a unipotent from \(\Omega_P(K)\). The following easy lemma will be used several times.

Lemma 7.1. Let \(R\) be a commutative ring, let \(G\) be a reductive group scheme over \(R\), let \(\Omega\) be an open \(R\)-scheme of \(G\) that is finitely presented as an \(R\)-scheme. Let \(g \in G(R)\) be any element. If for all maximal ideals \(m\) of \(R\) one has \(\rho_m(g) \in \Omega(R/m)\), then \(g \in \Omega(R)\).

Proof. First we show that \(\lambda_m(g) \in \Omega(R_m)\) for any fixed maximal ideal \(m\). Since \(\Omega\) is an open subscheme of \(G\), it is a union of principal open \(R\)-subschemas \(G_{f_\alpha} = \text{Spec}(R[G]_{f_\alpha})\) of \(G\) for some \(f_\alpha \in R[G]\). Since \(\rho_m(g) \in \Omega(R/m)\), and \(R/m\) is a field, there is an index \(\alpha\) such
that $\rho_m(g) \in G_{fa}(R/m)$. The latter means that the ring homomorphism $\rho_m(g) : R[G] \to R/m$ satisfies $\rho_m(g)(f_a) \in (R/m)^\times$. Since $R_m/mR_m = R/m$ and $\rho_m(g) = \rho_m(\lambda_m(g))$, we conclude that $\lambda_m(g)(\lambda_m(f_a)) \in (R/m)^\times$. Since $R_m$ is a local ring, this implies that $\lambda_m(g)(\lambda_m(f_a)) \in (R/m)^\times$. Hence $\lambda_m(g) \in G_{fa}(R_m) \subseteq \Omega(R_m)$, as required.

Since $G$ and $\Omega$ are finitely presented $R$-schemes, $\Omega(R_m)$ is the limit of $\Omega(R_s)$, and $G(R_m)$ is the limit of $G(R_s)$, over all $s \in R - m$. Therefore, for any maximal ideal $m$ of $R$ there is $s_m \in R - m$ such that $\lambda_{s_m}(g) \in \Omega(R_{s_m})$. Since all $s_m$ together generate the unit ideal of $R$, the open subschemes $\text{Spec}(R_{s_m})$ constitute an open covering of $\text{Spec}(R)$ for the Zariski topology. Consider the system of points $\lambda_{s_m}(g) \in \Omega(R_{s_m})$. We would like to show that they glue in the Zariski topology to a point in $\Omega(R)$, and this point is $g$. Since the functor of points of $\Omega$ is a sheaf for the Zariski topology on $\text{Spec}(R)$, it is enough to check that for any maximal ideals $m$ and $n$ one has $\lambda_{s_m}(\lambda_{s_n}(g)) = \lambda_{s_n}(\lambda_{s_m}(g))$ inside $\Omega(R_{s_m,s_n})$. This is true, since the same equality holds in $G(R_{s_m,s_n})$, and $\Omega(R_{s_m,s_n})$ injects into $G(R_{s_m,s_n})$. Hence there is $g' \in \Omega(R)$ such that $\lambda_{s_m}(g') = \lambda_{s_m}(g)$ for any maximal ideal $m$ of $R$. It remains to note that $g' = g$, since $\Omega(R)$ injects into $G(R)$.

From now and until the end of this section, assume that $K$ is connected.

**Lemma 7.2.** Let $K$ be a field and $P$ a parabolic subgroup of $G$ over $K$. If $g \in G(K)$ satisfies $[g, U_P(K)] = 1$, then $g \in P(K)$.

**Proof.** Let $Q \leq P$ be a minimal parabolic subgroup of $G$ over $K$, let $L_Q$ be its Levi subgroup contained in $L_P$, and let $Q^-$ be the opposite minimal parabolic subgroup contained in $P^-$. Let $S \leq L_Q$ be the maximal split subtorus of $C(L_Q)$. Bruhat decomposition implies that $g = uwv$, where $u \in U_Q(K)$, $w \in N_G(S)(K)$, $v \in (U_Q)_0(K) = \{x \in U_Q(K) \mid wxw^{-1} \in U_Q(K)\}$, and $u, v, and the class of $w$ in the Weyl group of $\Phi_Q$ are unique. We have $w \in L_P(K)$ if and only if $w$ is a product of elementary reflections $w_{\alpha_i}$ for some simple roots $\alpha_i \in \Phi_Q$ belonging to the root subsystem of $\Phi_Q$ corresponding to $L_P$ [BT65].

Assume first that $w \notin L_P(K)$. Then there is a simple root $\alpha \in \Phi_Q$ not belonging to the root system of $L_P$ such that $w(\alpha) < 0$. Then $x = X_\alpha(\xi), \xi \in V_\alpha$, belongs to $U_P(K)$, and since $[g, x] = 1$, we have $x(uwv) = (uwv)x$. The rightmost factor in the Bruhat decomposition of $x(uwv) = (xu)v$ equals $v$. However, since $\alpha$ is a positive root of minimal height, it is clear that the rightmost factor in the Bruhat decomposition of $(uwv)x$ contains $X_{\alpha}(\eta + \xi)$ in its canonic decomposition, if $v$ contains $X_{\alpha}(\eta)$. Therefore, this rightmost factor is distinct from $v$, a contradiction.

Therefore, $w \in L_P(K)$. Then for any $x \in U_P(K)$ we have $wxw^{-1} \in U_P(K)$, hence by the definition of the Bruhat decomposition $v \in L_P(K \cap U_Q(K)$. This means that $g = uwv \in U_Q(K)L_P(K) = P(K)$. \qed

**Lemma 7.3.** Suppose that the structure constants of the absolute root system of $G$ are invertible in $K$. Let $Q$ be a parabolic $K$-subgroup of $G$, such that $\Phi_Q$ is irreducible and rank$(\Phi_Q) \geq 2$. Let $\beta \in \Phi_Q^+$ be a simple relative root. If $x \in U_{Q^+}(K)$ commutes with $X_\beta(V_\beta)$, then $x = \prod X_{\alpha}(u_{\alpha})$, where each $\alpha$ is a positive (respectively, negative) root of $\Phi_Q$ satisfying $\alpha + \beta \notin \Phi_Q \cup \{0\}$.

**Proof.** First, consider the case where $x \in U_{Q^+}(K)$. We fix a total ordering on $\Phi_Q^+$ compatible with the height. Then there is a unique presentation $x = \prod_{i=1}^{k} X_{\alpha_i}(u_{\alpha_i})$, where all roots are distinct, each of them lies in $\Phi_Q^+$, $0 \neq u_{\alpha_i} \in V_\alpha$, and the product is taken in the fixed order. Let $i_0$ be the smallest index such that $\alpha_i + \beta \in \Phi_Q$. By Lemma 6.3 there is $v \in V_\beta$ such
that $N_{\alpha}\alpha_{i_0,1,1}(v, u_{i_0}) = w \neq 0$. By the generalized Chevalley commutator formula $[X_\beta(v), x]$ contains $X_{\alpha_0+\beta}(w)$ in its reduced decomposition, hence it cannot be equal to 1.

Second, assume that $x = \prod_{i=1}^k X_{\alpha_i}(u_{\alpha_i})$, where each $\alpha_i \in \Phi^-_Q \setminus \mathbb{N} \cdot (-\beta)$, and $0 \neq u_{\alpha_i} \in V_\alpha$. We order these roots $\alpha_i$ according to the following partial order: first we order these roots with respect to the sum of coefficients of all simple relative roots distinct from $\beta$, with the largest (negative) sum of coefficients coming first, and then with respect to the coefficient of $\beta$, with the smallest (negative) coefficient coming first. Assume that there is an index $1 \leq i \leq k$ such that $\alpha_i + \beta \in \Phi_Q$, and let $i_0$ be the smallest such index. By Lemma 6.4 there is $v \in V_\beta$ such that $N_{\beta,\alpha_i,1,1}(v, u_{\alpha_i}) \neq 0$. Then

$$[X_\beta(v), x] = \prod_{i=1}^{i_0-1} X_{\alpha_i}(u_{\alpha_i}) [X_\beta(v), X_{\alpha_0}(u_{\alpha_0})] \prod_{i=1}^{i_0} X_{\alpha_i}(u_{\alpha_i}) [X_\beta(v), \prod_{i=i_0+1}^k X_{\alpha_i}(u_{\alpha_i})].$$

Generalized Chevalley commutator formula then implies that $[X_\beta(v), x]$ contains in its reduced decomposition the non-trivial factor $X_{\beta+\alpha_0}(N_{\beta,\alpha_i,1,1}(v, u_{\alpha_i}))$, and no other factor corresponding to the root $\beta + \alpha_{i_0}$, therefore, $[X_\beta(v), x] \neq 1$, a contradiction.

Finally, let $x \in U_{\Phi^-(K)}$ be an arbitrary element. Write $x = x_1 x_2$, where $x_1$ is a product of negative root elements corresponding to the roots in $\mathbb{N}(-\beta)$, and $x_2$ is a product of other negative root elements. Then $x_2$ belongs to the unipotent radical $U$ of a parabolic subgroup of $G$ corresponding to the parabolic set $\Phi^-_Q \cup (\mathbb{N} \cap \Phi^+_Q)$, and $X_\beta(V_\beta)$ and $x_1$ are contained in the Levi subgroup $L$ of this parabolic subgroup corresponding to the set of roots $\mathbb{Z} \beta \cap \Phi_Q$ (see Lemma 4.11). For any $v \in V_\beta$ one has

$$1 = [X_\beta(v), x_1 x_2] = [X_\beta(v), x_1]^x_1 [X_\beta(v), x_2],$$

where $[X_\beta(v), x_1] \in L(K)$ and $x_1 [X_\beta(v), x_2] \in U(K)$, hence

$$1 = [X_\beta(v), x_1] = [X_\beta(v), x_2].$$

By the previous case we conclude that $x_2$ is a product of root elements corresponding to $\alpha \in \Phi^-_Q$ such that $\alpha + \beta \notin \Phi_Q$; by the definition of $x_2$, we also have $\alpha + \beta \neq 0$, as required. It remains to show that $x_1 = 1$.

Assume that $x_1 \neq 1$. We can write $x_1 = \prod_{i=1}^k X_{-i\beta}(u_{-i\beta})$, $u_{-i\beta} \in V_{-i\beta}$. Let $\alpha \in \Phi^+_Q$ be a simple relative root such that $\alpha + \beta \in \Phi_Q$. Then $\alpha + i\beta \in \Phi_Q$ for any $i \geq 1$ such that $i\beta \in \Phi_Q$ by Lemma 3.1. Let $i \geq 1$ be the smallest natural number such that $u_{-i\beta} \neq 0$. Since $-i\beta - \alpha \in \Phi_Q$, by Lemma 6.4 there is $w \in V_{-\alpha}$ such that $N_{-i\beta,\alpha,1,1}(u_{-i\beta}, w) \neq 0$. Set $y = [x_1, X_{-\alpha}(w)]$. Then generalized Chevalley commutator formula implies that $y \in U(K)$, and $y$ contains $X_{-i\beta,\alpha}(N_{-i\beta,\alpha,1,1}(u_{-i\beta}, w))$ in its reduced decomposition. On the other hand, since $[X_\beta(v), X_{-\alpha}(w)] = 1$ for any $v \in V_\beta$, we conclude that

$$[X_\beta(v), y] = [X_\beta(v), [x_1, X_{-\alpha}(w)]] = 1$$

for any $v \in V_\beta$. By the previous case this implies that $y$ cannot contain $X_{-i\beta,\alpha}(u)$ with non-zero $u$ in its reduced decomposition, a contradiction. This shows that $x_1 = 1$. 

**Lemma 7.4.** Under the hypothesis of Lemma 7.3, let $m \geq 1$ be the maximal positive integer such that $m\beta \in \Phi_Q$. If $x \in U_{(-\beta)}(K) L_Q(K) U_{(-\beta)}(K)$ commutes with $X_\beta(V_\beta)$, then $x \in X_{m\beta}(V_{m\beta}) L_Q(K)$.

**Proof.** Assume that $x = a h b$ for some $a \in U_{(\beta)}(K)$, $h \in L_Q(K)$, and $1 \neq b \in U_{(-\beta)}(K)$. Then $b = \prod_{i=i_0}^m X_{-i\beta}(u_i)$, where $u_i \in V_{-i\beta}$, $u_{i_0} \neq 0$. Let $\alpha \neq \beta \in \Phi_Q$ be a simple relative
root adjacent to \( \beta \). Then \([X_{-\alpha}(V_{-\alpha}), U_\beta(K)] = 1\) by the generalized Chevalley commutator formula, and hence \([X_{-\alpha}(V_{-\alpha}), x^{-1}], X_\beta(V_\beta) = 1\). On the other hand, by Lemma 4.3 one has \(-i_0 \beta - \alpha \in \Phi_Q\), hence by Lemma 6.4 there is \(v \in V_{-\alpha}\) such that \(N_{-i_0 \beta - \alpha, 1}(u_{i_0}, v) \neq 0\). Then

\[
[X_{-\alpha}(v), x^{-1}] = [X_{-\alpha}(v), \left( h \prod_{i=i_0}^m X_{-i \beta}(u_i) \right)^{-1}] \prod_{k=0 \vee k \geq i_0} X_{-j \alpha - k \beta}(w_{j,k}),
\]

where \(w_{1,i_0} \neq 0\). However, this contradicts Lemma 7.3 since \((-\alpha - i_0 \beta) + \beta \in \Phi_Q\).

Therefore, \(b = 1\), and \(x = ah\), where \(a \in U_\beta(K)\) and \(h \in L_Q(K)\). Note that by Lemma 5.3 \(X_\beta(V_\beta)\) generates \(U_\beta\), hence \(x\) commutes with all \(X_\beta(V_\beta)\), \(i \geq 1\). Write \(a = \prod_{i=i_0}^m X_{i \beta}(u_i)\), where \(u_i \in V_{i \beta}, u_{i_0} \neq 0\). For every \(j \geq 1\) and \(v \in V_{j \beta}\) one has

\[
1 = [X_{j \beta}(v), x^{-1}] = [X_{j \beta}(v), h^{-1}]^{h^{-1}}[X_{j \beta}(v), a^{-1}].
\]

By the generalized Chevalley commutator formula we have \(h^{-1}X_{j \beta}(v), a^{-1}] \in U_{\{k \beta \mid k \geq j+1\}}(K)\), and hence \([X_{j \beta}(v), h^{-1}] \in U_{\{k \beta \mid k \geq j+1\}}(K)\). By Lemma 5.2(ii) this implies that \([h, U_\beta(K)] = 1\). Then \([a, X_\beta(V_\beta)] = 1\). By Lemma 7.3 this implies that \(a \in X_{m \beta}(V_{m \beta})\). \(\square\)

**Lemma 7.5.** Suppose that the absolute root system of \(G\) is irreducible and its structure constants are invertible in \(K\). Let \(Q\) be a parabolic \(K\)-subgroup of \(G\) with a Levi subgroup \(L_Q\), such that \(\text{rank}(\Phi_Q) \geq 2\). For any parabolic subgroup \(P \supseteq Q\) and any \(g \in \Omega_P(K) \setminus C(G(K))\) there exists a parabolic subgroup \(M \supseteq L_Q\) of \(G\) such that \([g^{E(K)} \cap M(K), E(K)] \neq 1\).

**Proof.** By Theorem 3.2 \(C(G(K)) = C(G)(K)\) is contained in every parabolic subgroup, hence by Lemma 3.3 it is enough to prove the claim under the assumption that \(G\) is adjoint.

Write \(g = ab\), where \(a \in U_P(K), b \in P^-(K)\). Choose any total order on \(\Phi^+_Q\) compatible with the height of roots. We show that either \(g^{E(K)}\) contains an element \(g' = a'b'\) such that the minimal root element occurring in \(a'\) is strictly larger than the corresponding element of \(a\), or there is a parabolic subgroup \(M\) as in the claim of the lemma. Since the height of roots in \(\Phi_Q\) is bounded, the order induction will eventually lead to \(a' = 1\) and \(g' \in P(K)\).

Assume first that \(P \neq Q\). Let \(L_P > L_Q\) be a Levi subgroup of \(P\). Then there is a simple relative root \(\beta \in \Phi^+_Q\) that does not belongs to the root system of \(L_P\). For any \(v \in V_\beta\) one has

\[
g^{E(K)} \supseteq a^{-1}[X_\beta(v), g] = a^{-1}[X_\beta(v), a] \cdot [X_\beta(v), b].
\]

Here \([X_\beta(v), b] \in P^-(K)\), while \(a^{-1}[X_\beta(v), a] \in U_Q(K)\) by the generalized Chevalley commutator formula (5.3), and all root elements occurring in the decomposition of the latter expression are strictly larger than the minimal root element of \(a\). If \(a^{-1}[X_\beta(v), g]\) is not centralized by \(E(K)\), then we can apply the induction hypothesis.

Assume that \(a^{-1}[X_\beta(v), g]\) is centralized by \(E(K)\). Since \(G\) is adjoint, we have \(a^{-1}[X_\beta(v), g] = 1\). Then \(a^{-1}[X_\beta(v), a] \in U_Q(K) \cap P^-(K) \subseteq L_P(K)\). However, by the choice of \(a\) this implies that \([X_\beta(v), a] = 1\). Then \([X_\beta(v), b] = 1\) as well. Thus, we are in the situation where \([x, a] = [x, b] = 1\) for every \(x \in U_{Q'}(K)\), where

\[
Q' = Q \cap L_P
\]

is a proper parabolic subgroup of \(L_P\). Write \(b = hu\) where \(h \in L_P(K), u \in U_{P^-(K)}\). Then \(1 = [x, hu] = [x, h] \cdot [x, u]\), where \([x, h] \in L_P(K)\) and \([x, u] \in U_{P^-(K)}\), and hence \([x, h] = [x, u] = 1\) as well. For any maximal ideal \(m\) of \(K\) the image of \(h\) in \(L_P(K/m)\)
commutes with $U_Q'(K/m)$, and hence belongs to $Q'(K/m)$ by Lemma 7.2. Since $\Omega_Q'$ is an open subscheme of $L_P$, by Lemma 7.1 we have $h \in \Omega_Q'(K)$. Then
\[
g = ab = ahu \in U_Q(K) \cdot \Omega_Q'(K) \cdot U_{P'}(K) \subseteq \Omega_Q(K).
\]

Thus, we are reduced to the case where $g \in \Omega_Q(K)$, i.e. to the situation $P = Q$. Again, let $\beta \in \Phi_Q$ be a simple relative root. Let $P' = P'^+$ and $P'^-$ be the “comaximal” parabolic subgroups of $G$ with the common Levi subgroup $L_{P'}$ such that $P'^+ = U_{\Phi_Q}^\pm$ and $L_{P'} = U_{\Phi_Q}$ in the sense of Lemma 4.1. Clearly,
\[
g \in \Omega_Q(K) \subseteq \Omega_{P'}(K).
\]

Rerunning, if necessary, the above height induction with respect to the parabolic subgroup $P'$ instead of $P$, we arrive to a new decomposition
\[
g = ahu \in U_Q(K), h \in L_{P'}(K) = U_{\Phi_Q}(K), u \in U_{P'}(K), \text{ and } [a, X_{\beta}(V_{\beta})] = [u, X_{\beta}(V_{\beta})] = 1.
\]

By Lemmas 7.4 and 7.3 this implies that $g$ is a product of root elements of the form $X_{\alpha}(u_a)$ where all $\alpha$ satisfy $\alpha + \beta \not\in \Phi_Q$, and of an element of $L_Q(K)$. Then by Lemma 4.4 there is a proper parabolic set of relative roots $\Sigma \subset \Phi_Q$ such that $g \in U_{\Sigma}(K)$ in the notation of Lemma 4.1. Then we take $M = U_{\Sigma}$.

The following statements follow immediately from Lemmas 7.5 and 6.6.

**Corollary 7.6.** Under assumptions of Lemma 7.5, the subgroup $g^{E(K)}$ contains a nontrivial root unipotent element.

**Corollary 7.7.** Suppose that $G$ is semisimple, $K$ is connected, the absolute root system $\Phi$ of $G$ is irreducible, its structure constants are invertible in $K$, and $G$ contains $(\mathbb{G}_{m,K})^2$. If $H \cap G(K, \text{Rad } K) \not\subseteq C(G(K))$, then $H$ contains a nontrivial root unipotent element.

**Proof.** Since $G$ contains $(\mathbb{G}_{m,K})^2$, there is a parabolic subgroup $Q$ of $G$ such that $\text{rank}(\Phi_Q) \geq 2$. Take $g \in (H \cap G(K, \text{Rad } K)) \setminus C(G(K))$. Then $\rho_m(g) \in \Omega_Q(K/m)$ for any maximal ideal $m$ of $K$. Then by Lemma 7.1 $g \in \Omega_Q(K)$, and the result follows from Corollary 7.6.

8. **Generic element techniques**

Let $G$ be an affine smooth finitely presented group scheme over $K$. Denote by $A = K[G]$ the affine algebra of the scheme $G$. By the definition of an affine scheme, an element $h \in G(R)$ can be identified with a homomorphism $h : A \to R$. We always do this identification, i.e. we always view elements of the group of points $G(R)$ of the scheme $G$ over a $K$-algebra $R$ as homomorphisms from $A$ to $R$. Denote by $g \in G(A)$ the generic element of the scheme $G$, i.e. the identity map $\text{id}_A : A \to A$. An element $h \in G(R)$ induces the homomorphism $G(h) : G(A) \to G(R)$ by the rule $G(h)(a) = h \circ a$ for all $a \in G(A)$. It follows that the image of $g$ under the action of $G(h)$ is equal to $h$.

Recall that for a ring homomorphism $\varphi : R \to R'$ we usually denote the induced group homomorphism $G(\varphi) : G(R) \to G(R')$ again by $\varphi$. This cannot lead to a confusion as one always can determine the meaning of $\varphi$ by the argument type of this homomorphism. In view of this agreement we have $h(g) = h \circ \text{id}_A = h$. If $R'$ is an $R$-algebra, then sometimes we identify elements of $G(R)$ with their canonical images in $G(R')$.

Recall that the fundamental ideal $I$ of $A$ is the kernel of the counit map $\epsilon_K : A \to K$, where $\epsilon_K \in G(K)$ is the identity element of this abstract group. The notation $A$, $I$, and $g$ introduced above is kept till the end of the present section.

The following characterization of the principal congruence subgroup was observed in [Ste16].
Lemma 8.1. An element $h$ of $G(R)$ belongs to the principal congruence subgroup $G(R,q)$ if and only if $h(I) \subseteq q$.

Proof. Denote by $\rho_q : G(R) \to G(R/q)$ the natural homomorphism. It is easy to see that $h \in G(R,q)$ iff the following diagram commutes.

$$
\begin{array}{ccc}
A & \xrightarrow{h} & R \\
\epsilon_K & & \downarrow \rho_q \\
K & \longrightarrow & R/q
\end{array}
$$

And the latter is obviously equivalent to saying that $h(I)$ vanishes modulo $q$, i.e. $h(I) \subseteq q$.

Since $A$ is a finitely presented $K$-algebra, it is a quotient of a polynomial ring in finitely many variables by a finitely generated ideal. Let $K'$ be a $\mathbb{Z}$-subalgebra of $K$, generated by all the coefficients of polynomials that generate this ideal. Then $A \cong A' \otimes_K K'$, were $A'$ is a finitely generated algebra over a Noetherian ring $K'$. Thus, there exists an affine smooth finitely presented group scheme $\tilde{G}$ over a Noetherian ring $K'$ such that $G = \tilde{G}_K$. In the present paper we prove results about the abstract group $G(R)$ for a $K$-algebra $R$. Therefore, we may often assume without loss of generality that $K = K'$, and hence $A$ is a Noetherian ring.

An advantage of the Noetherian property of a ring in this context is the following property [Bak91, Lemma 4.10].

Lemma 8.2. Let $R$ be a Noetherian ring and $s \in R$. There exists $m \in \mathbb{N}$ such that the restriction of the localization homomorphism $\lambda_s : R \to R_s$ to the ideal $s^m A$ is injective.

The next lemma is a version of clearing denominators.

Lemma 8.3. Let $G$ be an affine smooth finitely presented group scheme over ring $K$, $R$ a Noetherian $K$-algebra, $m \in \mathbb{N}$, and $s \in R$. Suppose that the natural map $s^m A \to A_s$ is injective. Given $a \in G(R_s)$ there exists $k \in \mathbb{N}$ such that $[a, \lambda_s(b)] \in \lambda_s(G(R, s^m R))$ for all $b \in G(R, s^k R)$.

Proof. Recall that we assume $A$ to be finitely presented. Therefore, there exists a finite set $J \subseteq A$ that generates $A$ as a $K$-algebra. Since $A = K \oplus I$ as a $K$-module, we may assume that $J \subseteq I$. Obviously, under this condition $J$ spans $I$ as an ideal.

Identifying elements of $G(A)$ and $G(R_s)$ with their canonical images in $G(A \otimes_K R_s)$, consider the commutator $c = [a, \lambda_s(g)] \in G(A \otimes_K R_s, I \otimes_K R_s)$. By Lemma 8.1 the finite set $c(J)$ is contained in $I \otimes_K R_s$. Clearly, there exists $l \in \mathbb{N}$ such that each element of this set can be written as $r/s^l$ for some $r \in I \otimes_K R$. Take $k = l + m$. Take $b \in G(R, s^k R)$ and consider the composition $d = \text{mult} \circ (b \otimes \text{id}) \circ c = [a, \lambda_s(b)]$, where mult : $R \otimes_K R_s \to R_s$ denotes the multiplication homomorphism. By Lemma 8.1 $b(I) \subseteq s^k R$, hence the set $d(J)$ consists of the elements of the form mult $\circ (b \otimes \text{id})(r/s^l) = s^k t/s^l = s^m t$ for some $t \in R$. Since the restriction of the localization homomorphism $\lambda_s$ to $s^m R$ is injective and $J$ generates the $K$-algebra $A$, the homomorphism $d$ factors through $d' : A \to R$, which means that $d = \lambda_s(d')$. Moreover, the natural choice of $d'$ provides that $d'(J) \subseteq s^m R$. Since $J$ generates the fundamental ideal $I$, we have $d'(I) \subseteq s^m R$ and by Lemma 8.1 $d' \in G(R, s^m R)$.

The next lemma is a key step to the proof of the normal structure theorem.

Lemma 8.4. Let $G$ be a reductive group scheme over a Noetherian connected ring $K$, and let $Q$ be a parabolic $K$-subgroup of $G$ such that $\Phi_Q$ is irreducible and $\text{rank}(\Phi_Q) \geq 2$. For any
parabolic subgroup $Q \subseteq P$ there exist elements $c_{ij} \in g^{E(A)} \cap \Omega_P(A)$, $i = 1, \ldots, l$, $j = 1, \ldots, n$, satisfying the following property. Let $S$ be a subscheme defined by the formula

$$S(R) = \{h \in G(R) \mid h(c_{ij}) \in C(G)(R) \text{ for all } i = 1, \ldots, l \text{ and } j = 1, \ldots, n\}$$

for any $K$-algebra $R$. Then the group of points $S(F)$ does not contain $E(F)$ for any $K$-algebra $F$ that is a field.

**Proof.** Since the subscheme $\Omega_P$ is open, hence it is covered by principal open subschemes $\text{Sp}_K A_{\alpha_i}$, $i = 1, \ldots, l$. Let $\alpha, \beta \in \Phi_Q$ be two relative roots such that $-\alpha, -\beta$ are simple roots, $\alpha + \beta \in \Phi_Q$. Let $\{e_1, \ldots, e_n\}$ be a set of generators of the $K$-module $V_\alpha$. Choose $i = 1, \ldots, l$ and $j = 1, \ldots, n$ and put $s = s_i$ and $v = e_j$.

Let $g_s$ be the image of $g$ in $G(A_s)$. In other words, $g_s = \lambda_s$ is the localization homomorphism. It is a tautology that $g_s$ factors through $A_s$, which means that $g_s \in \text{Sp}_K A_s(A_s) \subseteq \Omega_P(A_s)$. Decompose it into a product $g_s = ab$, where $a \in U_P(A_s)$ and $b \in P^-(A_s)$. Since $A$ is Noetherian, by Lemma 8.2 there exists $m \in \mathbb{N}$ such that the restriction of $\lambda_s$ to $s^m A$ is injective. By Lemma 8.3 there exists a positive integer $k \geq m$ such that

$$d_s = X_\alpha(s^k v)^{s-1} \in \lambda_s(U_P(A, s^m A)) \quad \text{and} \quad f_s = d_s^0 = X_\alpha(s^k v)^{b} \in \lambda_s(P^- (A, s^m A)).$$

Let $d = \lambda_s^{-1}(d_s) \in U_P(A, s^m A)$ and $f = \lambda_s^{-1}(f_s) \in P^-(A, s^m A)$ (by definition of $m$ these preimages are unique). Put $c_{ij} = [g, d^{-1}] \in G(A, s^m A)$. Then

$$\lambda_s(c) = [g_s, d_s^{-1}] = d_s^0 d^{-1} = f_s d_s^{-1} \in \lambda_s(\Omega_P(A)).$$

By definition of $m$ we have $c_{ij} = f d^{-1} \in \Omega_P(A)$.

Let $F$ be a field. Put $h = X_\beta(u) \in \Omega_P(F) = \bigcup_{i=1}^l \text{Sp}_K A_{s_i}(F)$ for some $u \in V_\beta \otimes_K F \setminus \{0\}$. Choose $i$ such that $h \in \text{Sp}_K A_{s_i}$ and put $s = s_i$. Then $h$ factors through $A_s$, i.e. $h = \tilde{h} \circ \lambda_s$ for some $\tilde{h} : A_s \to F$. Since $h(g) = h$, we have $h(g_s) = \tilde{h}(a) g(b) = e \cdot X_\beta(u)$. The uniqueness of representation of $h$ as a product of an element from $U_P(F)$ by an element from $P^-(F)$ implies that $h(a) = e$. Thus, we get

$$h(c_{ij}) = [h(g), h(d^{-1})] = [X_\beta(u), X_\alpha(s^k e_j)^{-1}].$$

By Lemma 6.4 there exists $j$ such that $N_{\alpha, \beta}(e_j, u) \neq 0$. By the generalized Chevalley commutator formula $h(c_{ij}) \notin C(G)(F)$. Thus, $h \notin S(F)$, and hence $S(F)$ does not contain $E(F)$, as required. \hfill \Box

9. **Proof of the normal structure theorem**

In this section we prove the main theorem of the present paper.

**Proof of Theorem 1.1.** Let $T \leq G$ be the subgroup isomorphic to $(\mathbb{G}_m, K)^2$. By Lemma 1.1 there are two parabolic subgroups $Q = Q^+ \leq Q^-$ of $G$ with a common Levi subgroup $L_Q = C_G(T)$.

Assume first that $K$ is Noetherian. Then $K$ is a finite product of connected Noetherian rings, so we can assume without loss of generality that $K$ is Noetherian and connected. The relative root system $\Phi_Q$ of Lemma 1.2 is irreducible and $\text{rank}(\Phi_Q) \geq 2$. By Lemma 6.5 we have $E_Q(K, \mathfrak{q}) = [G(K, \mathfrak{q}), E(K)]$. By Lemma 6.1 we have $E_P(K, \mathfrak{q}) = E_Q(K, \mathfrak{q})$ for any other parabolic subgroup $P$. Thus, by Proposition 6.3 it suffices to prove that if $H$ is not inside $C(G(K))$, then it contains a nontrivial relative root unipotent element.

Let $\beta \in \Phi_Q$ be a simple root. Then by Lemma 1.1 $P = P^+ = U_{\beta, \beta}$ is a parabolic subgroup distinct from $Q$, with a Levi subgroup $L_P = U_{\beta, \beta}$. Let $S$ be a subscheme of $G$ satisfying conditions of Lemma 5.4. If there exists $h \in H$ such that $h(c_{ij}) \notin C(G(K))$, then
$h(c_{ij}) \in h^{E(K)} \cap \Omega_P (K)$ is subject to Corollary 7.6 and hence $H$ contains a nontrivial relative root unipotent element.

Otherwise $H$ is contained in the set of $K$-points of the subscheme $S$. Consider a maximal ideal $m$ of $K$. The image $\overline{\mathfrak{f}}$ of the subgroup $H$ under the canonical homomorphism $G(K) \to C(K/m)$ is contained in $S(K/m)$ and is normalized by $E(K/m)$. Tits' simplicity theorem [Tit64] then implies that either $H$ contains $E(K/m)$, or $H$ is contained in $C(G(K/m))$. Since $E(K/m)$ is not contained in $S(K/m)$, $\overline{\mathfrak{f}}$ is contained in $C(G(K/m))$. It follows that the image of the subgroup $[H, E(K)]$ vanishes in $G(K/m)$, i.e. $[H, E(K)] \subseteq G(K, m)$.

Since $m$ is an arbitrary maximal ideal, we have $[H, E(K)] \subseteq G(K, J)$, where $J$ is the Jacobson radical of $K$. On the other hand, by Lemma 3.4 $[H, E(K)]$ is not contained in the center of $G(K)$. Hence, by Corollary 7.7 this subgroup contains a nontrivial relative root unipotent element. This completes the proof of the Noetherian case.

Now let $K$ be arbitrary. For any finite set of elements $\Lambda \subseteq H$ there is a finitely generated subring $\tilde{K}$ of $K$ and a semisimple reductive group scheme $\tilde{G}$ over $\tilde{K}$ with a subgroup $\tilde{T} \cong (G_{m, \tilde{K}})^2$ such that $G = \tilde{G}_{K}$, $T = \tilde{T}_{K}$, and $\Lambda \subseteq \tilde{G}(\tilde{K})$. Clearly, by Lemma 4.1 there is also a parabolic subgroup $\tilde{Q} \subseteq \tilde{G}$ such that $Q = \tilde{Q}_{K}$.

Since for any maximal ideal $\tilde{m}$ of $\tilde{K}$ there is a maximal ideal $m$ of $K$ such that $\tilde{m} \subseteq m$, we conclude that $\tilde{G}(\tilde{K}/\tilde{m})$ also has irreducible root system whose structure constants are invertible in $\tilde{K}$. Thus $\Lambda_{E(K)} \subseteq \tilde{G} (\tilde{K})$ is subject to the Noetherian case of the theorem. Hence there exists an ideal $q(\Lambda) \subseteq \tilde{K}$ such that

$$E(\tilde{K}, q(\Lambda)) \subseteq \Lambda_{E(K)} \subseteq C(\tilde{K}, q(\Lambda)).$$

Clearly, $q(\Lambda)$ is uniquely determined by $\tilde{K}$ and $\Lambda$. Let $q$ be the ideal of $K$ generated by all subsets $q(\Lambda) \subseteq \tilde{K} \subseteq K$. Then, clearly, $H \subseteq C(K, q)$. We show that $E_Q (K, q) \subseteq H$. In order to do that, it is enough to check that $U_{Q^+} ((a))$ is contained in $H$ for any finite $K$-linear combination $a = \sum c_i a_i$ of elements $a_i \in q(\Lambda_i)$. Let $\tilde{K}$ be subring of $K$ corresponding to the finite set $\Lambda = \cup \Lambda_i$, and let $K'$ be the subring of $K$ generated by $\tilde{K}$ and all $c_i$. Then by the Noetherian case of the theorem applied to $G_{K'}$, we conclude that $U_{Q^+} ((a)) = U_{Q^+} ((a))$ are contained in $\Lambda_{E(K')} \subseteq \Lambda_{E(K)} \subseteq H$.

The same argument as above also shows that $E_Q (K, q) = [G(K, q), E(K)]$ and $E_P (K, q) = E_Q (K, q)$ for any other parabolic subgroup $P$ of $\tilde{G}$, since this equality holds for each finitely generated subring $\tilde{K}$ such that $P$ is defined over $\tilde{K}$ and the ideal $\tilde{q} = \tilde{K} \cap q$. □

**References**

[Abe89] E. Abe, *Normal subgroups of Chevalley groups over commutative rings*, Contemp. Math. **83** (1989), 1–17.

[ABS90] H. Azad, M. Barry, and G. Seitz, *On the structure of parabolic subgroups*, Comm. Algebra **18** (1990), no. 2, 551–562.

[AS76] E. Abe and K. Suzuki, *On normal subgroups of Chevalley groups over commutative rings*, Tohoku Math. J. **28** (1976), no. 2, 185–198.

[Bak69] A. Bak, *On modules with quadratic forms*, Lecture Notes in Math. **108** (1969), 55–66.

[Bak91] ——, *Nonabelian K-theory: The nilpotent class of $K_1$ and general stability*, K-Theory **4** (1991), 363–397.

[Bas64] H. Bass, *K-theory and stable algebra*, Publ. Math. Inst. Hautes Études Sci **22** (1964), 5–60.

[Bas73] ——, *Unitary algebraic K-theory*, Lecture Notes in Math. **343** (1973), 57–265.

[Bix80] Robert Bix, *Octonion planes over local rings*, Trans. Amer. Math. Soc. **261** (1980), no. 2, 417–438. MR 580896

[BLS64] H. Bass, M. Lazard, and J.-P. Serre, *Sous-groupes d’indice fini dans $\text{SL}(n, \mathbb{Z})$*, Bull. Amer. Math. Soc. **70** (1964), 385–392. MR 0161913
H. Bass, J. Milnor, and Serre J.-P., *Solution of the congruence subgroup problem for $\text{SL}_n$ ($n \geq 3$) and $\text{Sp}_{2n}$ ($n \geq 2$)*, Publ. Math. Inst. Hautes Études Sci 33 (1967), 59–137.

Armand Borel, *Linear algebraic groups*, second ed., Graduate Texts in Mathematics, vol. 126, Springer-Verlag, New York, 1991. MR 1102012

Joel Brenner, *The linear homogeneous group*, Ann. of Math. (2) 39 (1938), no. 2, 472–493. MR 1503419

Joel Brenner, *The linear homogeneous group. II*, Ann. of Math. (2) 45 (1944), 100–109. MR 0009956

J. L. Brenner, *The linear homogeneous group. III*, Ann. of Math. (2) 71 (1960), 210–223. MR 0110754

A. Borel and J. Tits, *Groupes réductifs*, Publ. Math. Inst. Hautes Études Sci 27 (1965), 55–150.

Z. I. Borevich and N. A. Vavilov, *The distribution of subgroups in the general linear group over a commutative ring*, Proc. Steklov. Inst. Math. 165 (1985), 27–46.

C. Chevalley, *Sur certains groupes simples*, Tohoku Math. J. 7 (1955), 14–66.

D. L. Costa and G. E. Keller, *The $E(2,A)$ sections of $\text{SL}(2,A)$*, Ann. of Math. (2) 134 (1991), no. 1, 159–188.

D. L. Costa and G. E. Keller, *Radix redux: normal subgroups of symplectic groups*, J. Reine Angew. Math. 427 (1991), no. 1, 51–105.

I. Z. Golubchik, *On the general linear group over an associative ring*, Uspekhi Math Nauk 28 (1973), no. 3, 179–180 (Russian).

Wilhelm Klingenberg, *Lineare Gruppen über lokalen Ringen*, Amer. J. Math. 83 (1961), 137–153. MR 0124412

Wilhelm Klingenberg, *Orthogonale Gruppen über lokalen Ringen*, Amer. J. Math. 83 (1961), 281–320. MR 0124414

E. Kulikova and A. Stavrova, *centralizer of the elementary subgroup of an isotropic reductive group*, Vestnik St. Petersburg Univ. 46 (2013), no. 1, 22–28.

A. Luzgarev and A. Stavrova, *Elementary subgroup of an isotropic reductive group is perfect*, St.Petersburg Math. J. 23 (2012), no. 5, 881–890.

G. A. Margulis, *Finiteness of quotient groups of discrete subgroups*, Funktsional. Anal. i Prilozhen. 13 (1979), no. 3, 28–39. MR 545365

H. Matsumoto, *Sur les sous-groupes arithmétiques des groupes semisimples déployés*, Ann. Sci. Éc. Norm. Supér. (4) 2 (1969), no. 1, 1–62.

J. S. Milne, *Basic theory of affine group schemes*, Preprint [http://www.jmilne.org/math/CourseNotes/AGS.pdf](http://www.jmilne.org/math/CourseNotes/AGS.pdf), 2012.

V. Platonov, *The problem of strong approximation and the Kneser-Tits hypothesis for algebraic groups*, Izv. Akad. Nauk SSSR Ser. Mat. 33 (1969), 1211–1219. MR 0258839

Raimund Preusser, *Sandwich classification for $\text{GL}_n(R), \text{O}_{2n}(R)$ and $\text{U}_{2n}(R,\Lambda)$ revisited*, J. Group Theory 21 (2018), no. 1, 21–44. MR 3739342

V. Petrov and A. Stavrova, *Elementary subgroups in isotropic reductive groups*, St.Petersburg Math. J. 20 (2009), no. 4, 625–644.

M. S. Ragunathan, *On the congruence subgroup problem*, Inst. Hautes Études Sci. Publ. Math. (1976), no. 46, 107–161. MR 0507030

MS Ragunathan, *On the congruence subgroup problem, ii*, Inventiones mathematicae 85 (1986), no. 1, 73–117.

A. Stavrova, *On the congruence kernel of isotropic groups over rings*, 2014, Preprint: [http://arxiv.org/abs/1305.0057](http://arxiv.org/abs/1305.0057).

A. K. Stavrova, *Homotopy invariance of non-stable $K_1$-functors*, J. K-Theory 13 (2014), 199–248.

A. Stavrova, *Non-stable $K_1$-functors of multiloop groups*, Canad. J. Math. 68 (2016), 150–178.

A. V. Stepanov, *Structure of Chevalley groups over rings via universal localization*, J. Algebra 450 (2016), 522–548.
NORMAL STRUCTURE OF ISOTROPIC REDUCTIVE GROUPS OVER RINGS

[A. A. Suslin, *On the structure of the special linear group over polynomial rings*, Math. USSR. Izvestija **11** (1977), 221–238.]

[J. Tits, *Algebraic and abstract simple groups*, Ann. of Math. **80** (1964), 313–329.]

[L. N. Vaserstein, *On the normal subgroups of GL_n over a ring*, Lecture Notes in Math. **854** (1981), 456–465.]

[L. N. Vaserstein, *On normal subgroups of Chevalley groups over commutative rings*, Tohoku Math. J. **38** (1986), 219–230.]

[L. N. Vaserstein, *Normal subgroups of orthogonal groups over commutative rings*, Amer. J. Math. **110** (1988), no. 5, 955–973. MR 961501]

[N. A. Vavilov and A. K. Stavrova, *Basic reductions in the description of normal subgroups*, J. Math. Sci. (N. Y.) **151** (2008), no. 3, 2949–2960.]

[J. S. Wilson, *The normal and subnormal structure of general linear groups*, Math. Proc. Cambridge Philos. Soc. **71** (1972), 163–177.]

[Robert A. Wilson, *The finite simple groups*, Graduate Texts in Mathematics **28**, Springer-Verlag, Berlin, New York, 2009.]

[You Hong, *Subgroups of classical groups normalized by relative elementary groups*, J. Pure and Appl. Algebra **216** (2012), 1040–1051.]

[Zhang, Zuhong, *Subnormal structure of non-stable unitary groups over rings*, J. Pure and Appl. Algebra **214** (2010), no. 5, 622–628.]

E-mail address: anastasia.stavrova@gmail.com

E-mail address: stepanov239@gmail.com

St. Petersburg State University