Analytical Writhed Magnetic Flux Rope Model

ANDREAS J. WEISS,1,2 TERESA NIEVES-CHINCHILLA,3,4 CHRISTIAN MÖSTL,1 MARTIN A. REISS,1 TANIA AMERSTORFER,1 AND RACHEL L. BAILEY5

1Space Research Institute, Austrian Academy of Sciences, Schmiedlstraße 6, 8042 Graz, Austria
2Institute of Physics, University of Graz, Universitätsplatz 5, 8010 Graz, Austria
3Heliospheric Physics Laboratory, NASA Goddard Space Flight Center, Greenbelt, MD 20771, USA
4Department of Physics, Catholic University of America, Washington, DC, USA
5Zentralanstalt für Meteorologie und Geodynamik, Hohe Warte 38, 1190 Vienna, Austria

(Received 24 Feb 2022; Revised 11 May 2022; Accepted -)
Submitted to ApJS

ABSTRACT

We present a new analytical approach with the aim to describe writhed magnetic flux rope structures under the constraint of invariant axial magnetic flux. In this paper, we showcase the simplest case of a writhed flux rope with a circular cross-section that is described in terms of the curvature and the torsion of the Frenet-Serret equations. The internal magnetic field configuration for the axial and poloidal field components are described in terms of a radial expansion using a Legendre basis. We further derive equations that allow us to configure our model for any magnetic twist distribution. We show the effects and differences of our proposed model compared to a purely cylindrical or toroidal geometry using a writhed exemplary flux rope structure with a uniformly twisted magnetic field configuration. In order to indirectly compare our model with real in-situ measurements, we generate two synthetic in-situ profiles using virtual spacecraft trajectories, simulating an apex and a flank encounter of an interplanetary coronal mass ejection. This proposed model presents an intermediate steps towards describing more complex flux rope structures with arbitrary cross-section shapes.

1. INTRODUCTION

A magnetic flux rope is a confined magnetic field structure consisting of a flux tube and an axially twisted magnetic field. These structures play a prominent role in heliophysics and in many other astrophysical settings, and are believed to exist at the core of any interplanetary coronal mass ejection (ICME). The in-situ magnetic field measurements of these flux ropes structures within ICMEs were initially named magnetic clouds (Burlaga et al. 1981) before they were associated with coronal mass ejections (Gosling et al. 1991) and successively found to closely follow the signature of a magnetic flux rope (Lepping et al. 1990; Bothmer & Schwenn 1998).

The basic magnetic field structure of a flux rope can be described using cylindrical analytical models such as uniform-twist force-free models (Gold & Hoyle 1960) or a linear force-free configurations (Lundquist 1950; Lepping et al. 1990; Farrugia et al. 1995). From in-situ magnetic field measurements and white light observations using coronagraphs and heliospheric imagers (e.g. Mulligan & Russell 2001; Vandas et al. 2005; Vourlidas et al. 2013; Davies et al. 2021) we know that these cylindrical approximations are highly simplified, and that the geometry of ICMEs can be significantly more complicated due to interaction with the coronal magnetic field (Lugaz et al. 2012; Kay et al. 2015; Möstl et al. 2015) or the solar wind (Riley & Crooker 2004; Liu et al. 2006; Démoülin & Dasso 2009). These general deformations can be very hard to identify in the local in-situ magnetic field measurements. The measurements are also affected by other processes, such as flux rope expansion (Leitner et al. 2007; Gulisano et al. 2012). These problems are additionally exacerbated when only single spacecraft measurements are available.

Recent efforts have focused on constructing models with higher complexity regarding the geometry or the internal magnetic field structure with the aim of better reconstructing the measured in-situ signatures. These studies include purely analytical approaches (Hidalgo et al. 2002; Vandas & Romashets 2017a,b; Nieves-Chinchilla et al. 2018), and also semi-analytical models (Isavnin 2016; Kay & Gopalswamy 2018; Weiss et al. 2021). One of they key components of any recently developed analytical model is axial invariance so that the basic geometry always corresponds to either a
cylinder or torus and only the cross-sections are changed (e.g. Nieves-Chinchilla et al. 2022). This excludes the possibility of modelling any axial deformations which are expected to appear due to interaction of ICME flux ropes with the ambient solar wind (e.g. Rollett et al. 2014; Hinterreiter et al. 2021).

In this paper, we use the mathematical framework developed in Nieves-Chinchilla et al. (2016, 2018) (henceforth referred to as NC16/NC18) and introduce a writhed flux rope model that allows for arbitrary curvature and torsion. We derive our analytical writhed magnetic flux rope model in Section 2. We also derive equations that allow for our model to be configured in terms of an arbitrarily predetermined twist distribution function (see 2.3). An exemplary geometry and related magnetic field solution are shown in Section 3. A discussion of our approach and the results is performed in Section 4. An outline of how our model can be used or extended is given in Section 5.

2. WRITHED FLUX ROPE MODEL

We describe the geometry of our flux rope structure in terms of the flux rope axis that is defined by an arbitrarily parametrized path \( \gamma(s) \). A cylindrical flux rope geometry is thus given by a straight path while we can describe a toroidal geometry using a circular path. A natural set of vectors that can be constructed for any space curve, with non-zero curvature, are the so-called Frenet-Serret vectors that are defined as:

\[
\begin{align*}
\mathbf{t}(s) & = \frac{\partial \gamma(s)}{\| \partial_s \gamma(s) \|}, \\
\mathbf{n}(s) & = \partial_s \mathbf{t}(s)/\| \partial_s \mathbf{t}(s) \|, \\
\mathbf{b}(s) & = \mathbf{t}(s) \times \mathbf{n}(s),
\end{align*}
\]

where we introduce \( v(s) = \|\partial_s \gamma(s)\| \) as a shorthand for the arc-length element. The \( v(s) \) factor is important for calculations but it may disappear in any physical results as any physical quantities must be independent of the used parametrization for \( \gamma(s) \). With the help of these definitions we can set up an appropriate curvilinear coordinate system \((r, s, \varphi)\) for our writhed flux rope geometry:

\[
r(r, s, \varphi) = \gamma(s) - r \sigma \cos(\varphi) \mathbf{n}(s) - r \sigma \sin(\varphi) \mathbf{b}(s),
\]

where \( \sigma \) describes the half-width of the flux rope. As a consequence of our definition used in Eq. (4) the azimuthal coordinate \( \varphi \) is defined so that \( \varphi = 0 \) corresponds to the most “outer” point of the curved flux rope and \( \varphi = \pi \) corresponds to the most “inner” point respectively. The flux rope volume is then defined by the coordinate range \( r \in [0, 1] \), \( s \in [0, s_{\text{max}}] \) and \( \varphi \in [0, 2\pi] \). We will generally omit the \((r, s, \varphi)\) dependence for any quantities or vectors defined within this coordinate system unless required for clarity.

We then attempt to construct the covariant basis vectors of our custom coordinate system which are given as:

\[
\left\{ \mathbf{e}_r, \mathbf{e}_s, \mathbf{e}_\varphi \right\} = \left\{ \frac{\partial \mathbf{r}(r, s, \varphi)}{\partial r}, \frac{\partial \mathbf{r}(r, s, \varphi)}{\partial s}, \frac{\partial \mathbf{r}(r, s, \varphi)}{\partial \varphi} \right\}.
\]

For the evaluation of \( \mathbf{e}_r \) we can make use of the accompanying Frenet-Serret equations that give the derivatives of the Frenet-Serret vectors with respect to the \( s \) coordinate:

\[
\begin{align*}
\frac{\partial_t \mathbf{t}(s)}{s} & = v(s) \kappa(s) \mathbf{m}(s), \\
\frac{\partial_t \mathbf{m}(s)}{s} & = -v(s) \kappa(s) \mathbf{t}(s) + v(s) \tau(s) \mathbf{b}(s), \\
\frac{\partial_t \mathbf{b}(s)}{s} & = -v(s) \tau(s) \mathbf{m}(s),
\end{align*}
\]

where \( \kappa(s) \) is the curvature and \( \tau(s) \) the torsion of the underlying curve. The resulting non-unit coordinate basis vectors take the form:

\[
\begin{align*}
\mathbf{e}_r & = -\sigma \left[ \cos(\varphi) \mathbf{m}(s) + \sin(\varphi) \mathbf{b}(s) \right], \\
\mathbf{e}_s & = \left[ 1 + r \sigma \kappa(s) \cos(\varphi) \right] \mathbf{v}(s) \mathbf{t}(s) + r \sigma \tau(s) v(s) \mathbf{e}_\varphi, \\
\mathbf{e}_\varphi & = r \sigma \left[ \sin(\varphi) \mathbf{m}(s) - \cos(\varphi) \mathbf{b}(s) \right].
\end{align*}
\]

Figure 1 shows an example of a curved flux rope with the Frenet-Serret vectors and the related basis vectors of our curvilinear coordinate system. The basis vectors are not orthogonal due to a very small torsion value in this example. Using the expressions for the basis vectors in Eqs. (9-11) we can subsequently construct the covariant metric tensor.
\( g_{ij} = \epsilon_i \cdot \epsilon_j \). The metric tensor entries and the related scale factors are given by:

\[
\begin{align*}
  g_{rr} &= h_r^2 = \sigma^2, \quad (12) \\
  g_{ss} &= h_s^2 = v^2(s) [1 + r\sigma\kappa(s) \cos(\varphi)]^2, \quad (13) \\
  g_{\varphi\varphi} &= h_{\varphi}^2 = r^2 \sigma^2, \quad (14) \\
  g_{rs} &= g_{sr} = 0, \quad (15) \\
  g_{r\varphi} &= g_{\varphi r} = 0, \quad (16) \\
  g_{s\varphi} &= g_{\varphi s} = r^2 \sigma^2 v(s) \tau(s), \quad (17)
\end{align*}
\]

where we find that only the \( g_{s\varphi} \) off-diagonal component does not vanish. Finally, the metric determinant \( g \) is given by:

\[
g = r^2 \sigma^4 v^2(s) [1 + r\sigma\kappa(s) \cos(\varphi)]^2. \quad (18)
\]

As can be directly seen from the non-zero \( g_{s\varphi} \) entry in the metric tensor, the basis vectors \( \{ \epsilon_r, \epsilon_s, \epsilon_\varphi \} \) will not necessarily be orthonormal if, and only if, the torsion does not vanish. Despite using the orthonormal Frenet-Serret vectors to build the curvilinear coordinate system in Eq (4), the Frenet-Serret equations generate a non-orthogonal coordinate system when constructing the related basis vectors. It is important to note that there are alternative approaches that result in orthogonal coordinate systems for our flux rope geometry. In Eq. (4) one can transform the azimuthal coordinate according to \( \varphi \to \varphi - \int_0^s \tau(s') ds' \) (e.g. Yeh 1986; Prior & Yeates 2016). The drawback of this orthogonalization trick is that it requires the evaluation of the integral over the torsion along the curve. A more sophisticated approach, using parallel transport frames, is described in Bishop (1975). While these alternative approaches lead to more conceptually simpler, and orthogonal, coordinate systems they are computationally more expensive. Properly implemented, the Frenet-Serret approach described in this paper provides the most straightforward framework for describing writhed magnetic flux ropes with circular cross-sections.

The most significant issue with the Frenet-Serret frame is that it is not defined for points on the curve \( \gamma(s) \) where the curvature vanishes as the normal vector \( \mathbf{n}(s) \) is then ill-defined. We will thus only use curves with non-vanishing curvature for our visualizations as we cannot show the full 3D structure of a magnetic flux rope without issues otherwise. Sections on the curve with zero curvature are only problematic for the Frenet-Serret construction. It is still always possible to generate a valid magnetic field as our coordinate system converges towards a cylindrical coordinate system with vanishing curvature and torsion. In this case, the magnetic field only depends on the radial distance from the flux rope axis so that we do not explicitly need to compute the Frenet-Serret vectors and can use a cylindrical coordinate system instead.

As in NC18, due to the non-orthogonal coordinate system, we must distinguish between covariant and contravariant vector quantities and use generalized forms for any vector operations. We will not repeat the required basic definitions as they are explained in detail and used in NC18. We also attempt to use similar conventions in order to reduce any possible confusion. We make use of standard Einstein notation with upper and lower indices indicating contravariant and covariant quantities respectively and raising or lowering of indices via contraction with the metric tensor. Any quantities described in our coordinate system with the non unit basis vectors are denoted with a \( c \) subscript and related to the scaled physical quantities via the appropriate scale factors.

### 2.1. Magnetic Field Solutions

We now attempt to find a magnetic field solution in our custom curvilinear coordinate system. As in NC16/NC18, we implicitly assume that the contravariant radial magnetic field component \( B_r^c \) vanishes. The relevant equations that describe the current density and the magnetic field take the form:

\[
0 = \partial_s \left( \sqrt{\gamma} B_r^c \right) + \partial_\varphi \left( \sqrt{\gamma} B_\varphi^c \right), \quad (19)
\]

\[
\mu_0 j_r^c = \frac{1}{\sqrt{\gamma}} \left[ \partial_s \left( g_{s\varphi} B_r^c + g_{r\varphi} B_\varphi^c \right) - \partial_\varphi \left( g_{ss} B_r^c + g_{s\varphi} B_\varphi^c \right) \right], \quad (20)
\]

\[
\mu_0 j_\varphi^c = -\frac{1}{\sqrt{\gamma}} \partial_s \left( g_{ss} B_r^c + g_{s\varphi} B_\varphi^c \right), \quad (21)
\]

\[
\mu_0 j_r^c = \frac{1}{\sqrt{\gamma}} \partial_\varphi \left( g_{ss} B_r^c + g_{s\varphi} B_\varphi^c \right), \quad (22)
\]

for the contravariant components of the magnetic field \( \mathbf{B}_c \) and the contravariant components of the current density \( \mathbf{j}_c \).

Without loss of generality we assume that there exists a point \( \gamma(s_0) \) so that \( \kappa(s_0) = \tau(s_0) = 0 \) and that all related derivatives also vanish. This point does not necessarily need to exist but serves as a useful point of reference. As explained previously, we cannot construct a Frenet-Serret frame when the curvature vanishes. Since the geometry locally corresponds to a cylinder, we can simply make use of the standard cylindrical coordinate system here instead. We additionally make the assumption that our magnetic field exhibits azimuthal symmetry, at \( \gamma(s_0) \), so that the equations simplify to those given by Eqs. (12-14) in NC18 for a circular cross-section. The solution for \( \gamma(s_0) \) can be written as:

\[
B_r^{c} \big|_{s=s_0} = \frac{1}{v(s)} \left( B_r^{c} \big|_{r=r_0} + \mu_0 \sigma^2 \int_{r_0}^{r} dr' r' j_r^{c} \big|_{s=s_0} \right), \quad (23)
\]

\[
B_\varphi^{c} \big|_{s=s_0} = -\frac{\mu_0 v(s)}{r^2} \int_{r_0}^{r} dr' r' j_\varphi^{c} \big|_{s=s_0}, \quad (24)
\]

where this result is slightly modified, as we need to account for the parameters \( v(s) \) and \( \sigma \). The central magnetic field strength \( B_r^{c} \big|_{r=r_0} \) appears as an integration constant that is independent of the current.
In NC18 the equations for the magnetic field were resolved by describing the current in terms of a radial power series. For our model we will alternatively use a decomposition based on shifted Legendre polynomials, as it can be shown that they have certain beneficial properties for our purposes. We write the axial and poloidal current as:

\[ j^a |_{r=0} = \frac{v(s)}{v(s)} f^a |_{r=0} = \sum_{m=0}^{\infty} \beta_m 1 - r_s \frac{\partial}{\partial s} (r^2 \bar{P}_m(r)) \]
\[ j^\varphi |_{r=0} = r \sigma f^\varphi |_{r=0} = - \sum_{m=1}^{\infty} \alpha_m \partial_r \bar{P}_m(r), \]

where \( \bar{P}_m(r) = P_m(2r - 1) \) are the shifted Legendre polynomials of \( m \)-th order that are defined for \( r \in [0, 1] \). In contrast to NC18 the minimum value for the \( n \)-index now stems from the fact that \( \partial_s P_0(r) = 0 \). Evaluating the integrals in Eqs. (23-24) the magnetic field components then take the form:

\[ B^a |_{r=0} = \frac{1}{v(s)} \left[ B^a \left|_{r=0} - \mu_0 \sigma \sum_{n=1}^{\infty} \alpha_n (\bar{P}_n(r) - \bar{P}_n(0)) \right. \right. \]
\[ + \left. \left. \frac{\mu_0 \sigma}{v(s)} \sum_{n=1}^{\infty} \alpha_n \bar{P}_n(0) \right] \right] \frac{v(s)}{v(s)} \frac{\partial}{\partial s} \bar{P}_n(r), \]
\[ B^\varphi |_{r=0} = -\mu_0 \sum_{m=0}^{\infty} \beta_m \bar{P}_m(r), \]

where we additionally introduce the \( \alpha_0 \) coefficient for \( \bar{P}_0(r) = 1 \) to further simplify the expression and replace the \( B^a |_{r=0} \) parameter.

We continue by making the following ansatz for the general form of the axial magnetic field component:

\[ B^a = A_s (r, s, \varphi) B^a |_{r=0}, \]

where \( A_s \) is an auxiliary function that fully encapsulates the axial and angular dependency of the general expression. We can easily solve for \( A_s \) in the toroidal case with constant curvature \( \kappa(s) = \kappa \), zero torsion \( \tau(s) = 0 \) and no radial current \( j^c = 0 \). For this scenario the Eq. (20) reduces to:

\[ 0 = \frac{1}{\sqrt{\kappa}} \partial_\varphi \left( \frac{\partial s}{\partial \tau} \frac{A_s}{A_s |_{\tau(s)=0}} \right. \]
\[ \left. \left|_{\tau(s)=0} \right) B^a \right] \]

for which we find that \( A_s |_{\kappa(s)=0} = C(r, s) \left[ 1 + r \sigma \kappa(s) \cos(\varphi) \right]^{-2} \). The integration constant \( C(r, s) \) for this particular solution can be found by demanding conservation of the axial flux \( \Phi^a \) for any constant value of \( \kappa(s) \) with the same arrangement of coefficients. Due to the lack of any radial magnetic field component the axial flux must not only be conserved over the entirety of the cross section but also for each radial element \( \partial_r \Phi^a \). We first compute the flux in the cylindrical case \( \partial_r \Phi^a |_{r=0} \) which must be equal to the toroidal expression for \( \partial_r \Phi^a \) from which we can directly infer \( C(r, s) \):

\[ \partial_r \Phi^a |_{r=0} = \int_0^{2\pi} \frac{d\varphi}{\sqrt{\kappa}} B^a |_{r=0} = 2 \pi r \sigma^2 v(s) B^a |_{r=0} \]
\[ = \partial_r \Phi^a = \int_0^{2\pi} \frac{d\varphi}{\sqrt{\kappa}} \left( C(r, s) \right) \frac{2 \pi r \sigma^2 v(s) B^a |_{r=0} \}}{\left[ 1 + r \sigma \kappa(s) \cos(\varphi) \right]^2} \]
\[ = \frac{2 \pi r \sigma^2}{\sqrt{1 - r^2 \sigma^2}} \kappa(s) \]
\[ \implies C(r, s) = \sqrt{1 - r^2 \sigma^2 \kappa^2(s)} \]
\[ \implies A_s |_{\kappa(s)=0} \left|_{\tau(s)=0} = \frac{1}{\sqrt{1 - r^2 \sigma^2 \kappa^2(s)}} \left[ 1 + r \sigma \kappa(s) \cos(\varphi) \right]^2 \right] \]

Given our solution for the axial field in the cylindrical or toroidal case we can now note that \( \sqrt{\kappa} \) only depends on the curvature. As such our previously derived expression for \( A_s \) conserves the axial flux regardless of how the flux rope is curved or twisted. We can thus use the existing toroidal expression for the axial magnetic field for the general case and assume that the poloidal field and the current conform so that the Eqs. (19-22) are resolved. Note that this does not only apply for our given \( A_s \) and one can in fact find a generalized solution with more complexity by modifying the axial field under the constraint of invariant total axial flux.

Applying this assumption to Eq. (19) and further assuming that \( B^a = A_s B^a |_{r=0} \) we can directly reconstruct the poloidal field. Our final combined result for both components is then given as:

\[ B^a = -\mu_0 \sigma \sqrt{1 - r^2 \sigma^2 \kappa^2(s)} \frac{\sum_{n=0}^{\infty} \alpha_n \bar{P}_n(r)}{v(s) \left[ 1 + r \sigma \kappa(s) \cos(\varphi) \right]^2} \]
\[ B^\varphi = -\frac{\mu_0}{1 + r \sigma \kappa(s) \cos(\varphi)} \frac{\sum_{m=0}^{\infty} \beta_m \bar{P}_m(r)}{v(s) \left[ 1 + r \sigma \kappa(s) \cos(\varphi) \right]^2} \]
\[ - \frac{\mu_0 r \sigma^2 \sin(\varphi) \partial_s \kappa(s)}{v(s) \left[ 1 + r \sigma \kappa(s) \cos(\varphi) \right]^2} \frac{\sum_{n=0}^{\infty} \alpha_n \bar{P}_n(r)}{\sqrt{1 - r^2 \sigma^2 \kappa^2(s)}} \]

where we now additionally find a mixture term in the poloidal field that depends on the poloidal current. The dependency of the magnetic field on the torsion \( \tau(s) \) is hidden within the \( \epsilon_s \) basis vector and does not explicitly appear in the expressions themselves. It is important to remember that any “axial” quantity technically includes a poloidal component when the torsion is non-zero due to our non-orthogonal coordinate system. In order to better to understand the structure of this magnetic field solution we can rewrite the solution as:
\[ \mathbf{B} = \mathbf{v}(s) \left[ 1 + r \sigma k(s) \cos(\varphi) \right] \mathbf{B}_c \cdot \mathbf{t}(s) + \mathbf{B}' \cdot \mathbf{\epsilon}_\varphi \] so that:

\[ B^p = -\frac{\mu_0 \sigma r^2 \sin(\varphi) \partial_s k(s)}{v(s) \left[ 1 + r \sigma k(s) \cos(\varphi) \right]^2 \sqrt{1 - r^2 \sigma^2 k^2(s)}} \sum_{n=0}^{\infty} \alpha_n \tilde{P}_n(r) \]

and

\[ -\frac{\mu_0 \sigma \sqrt{1 - r^2 \sigma^2 k^2(s)} \tau(s)}{v(s) \left[ 1 + r \sigma k(s) \cos(\varphi) \right]^2} \sum_{n=0}^{\infty} \alpha_n \tilde{P}_n(r), \]

where we now can see the explicit dependence on the torsion in the last term.

Despite us previously claiming that any terms for \( v(s) \) must drop out in physical quantities as they depend on the choice for parametrization it can be easily seen that this does not appear to be the case in Eq. (34) when additionally accounting for the scale factors. This is due to how derivatives transform under a change of parametrization \( \partial_u/\sqrt{v(u)} \rightarrow \partial_s/v(s) \). As this can be a source of confusion we will from now on assume that we use an arc-length parametrized curve so that \( v(s) = 1 \) and drop the term from all further calculations.

By construction, the current conservation is always fulfilled as long as the current is physical. This may not be the case due to singularities in the current which, due to our chosen description, can only appear at the center of the flux rope structure. An example is the poloidal current \( j_p \) which must vanish at \( r = 0 \). This condition can be shown to be equivalent to:

\[ \sum_{n=1}^{\infty} \alpha_n \left( \partial_r \tilde{P}_n(r) \right) \bigg|_{r=0} = \sum_{n=1}^{\infty} (-1)^{n+1}(n^2 + n) \alpha_n = 0. \]

which sets a constraint on the values for \( \alpha_n \). No such constraint exists for the \( \beta_m \) coefficients as \( j^p \) can take non-zero values at \( r = 0 \).

### 2.1.1. Current Density

By resolving Eqs. (20-22) we could generate the expressions for the current density components in our curved flux rope model. Unfortunately these expressions do not have an easily tractable form and it is very hard to extract general statements on their structure or infer any properties. We instead observe the total amount of current that flows through the flux rope that we denote as \( J_c \):

\[ J_c = \int_0^1 \int_0^{2\pi} dr d\varphi \mathbf{B}_c \cdot \sqrt{\mathbf{g}} J_c^p \]

\[ = \frac{2\pi \sigma^2 \sum_{m=0}^{\infty} \beta_m}{\sqrt{1 - \sigma^2 k^2(s)}} + \frac{2\pi \sigma^2 \tau(s) \sum_{n=0}^{\infty} \alpha_n}{1 - \sigma^2 k^2(s)} \]

where we find that \( J_c \) is now dependent on the curvature and the torsion. In contrast to the magnetic flux, the total axial current is not invariant over the flux rope structure and sections with larger curvature possess a larger axial current. We can similarly compute the net in flowing radial current, denoted \( J_r \), as:

\[ J_r = \int_0^{2\pi} \frac{d\varphi}{\sqrt{g}} \left. f_\varphi \right|_{r=1} = \int_0^{2\pi} d\varphi \sqrt{\mathbf{g}} f_\varphi = -\frac{2\pi \sigma^4 k(s) \partial_s \tau(s)}{1 - \sigma^2 k^2(s)^2} \sum_{n=0}^{\infty} \beta_m \]

\[ - \frac{2\pi \sigma^4 \partial_s \tau(s)}{1 - \sigma^2 k^2(s)^2} \sum_{n=0}^{\infty} \alpha_n. \]

It can now be shown that the following Equation holds:

\[ \partial_s J_r + J_r = 0 \]

which describes a form of current conservation. It shows that there is a non-zero net radial current which feeds the changes in axial current that arise due to changes in the curvature or torsion. Our model thus also sets strong implicit constraints on the external structure of the flux rope which must be able to provide this additional current which could become relatively large depending on the curvature or the torsion.

### 2.1.2. Magnetic Fluxes

We can also derive the magnetic fluxes. We start with the axial magnetic flux which is relatively easy as it is already partially used in the construction of our magnetic field model. We can directly compute:

\[ \Phi^a = \int_0^{1} \int_0^{2\pi} dr d\varphi \mathbf{B}_c \cdot \sqrt{\mathbf{g}} = -\frac{2\pi \mu_0 \sigma^3}{\sum_{m=0}^{\infty} \alpha_m} \int_0^{2\pi} d\varphi \int_0^1 dr r \tilde{P}_n(r) \]

\[ = -\mu_0 \sigma^3 \left( \alpha_0 + \frac{\alpha_1}{3} \right), \]

as the integrals over the Legendre polynomials with index \( n \geq 2 \) vanish. The axial flux is only dependent on two parameters, namely \( \alpha_0 \) and \( \alpha_1 \). Due to our definition of \( \alpha_0 \) there is a hidden interdependency with respect to all other coefficients, the relevance of which depends on how the model is configured.

For the poloidal flux, in a well behaved Frenet-Serret frame, we can show that:

\[ \Phi^p = \int_0^{1} \int_0^{s_{max}} dr ds \mathbf{B}_c \cdot \sqrt{\mathbf{g}} = -\frac{L \mu_0 \sigma^2}{2} \left( \frac{\beta_0}{2} + \frac{\beta_1}{6} \right) \]

\[ - \mu_0 \sigma^4 \sin(\varphi) \left[ \int_0^{1} \int_0^{s_{max}} dr ds \frac{r^2 \partial_s \tau(s)}{1 - \sigma^2 k^2(s)^2} \sum_{n=0}^{\infty} \alpha_n \tilde{P}_n(r) \right] \]

\[ \quad = 0 \quad \text{if } k(0) = k(s_{max}). \]

where \( L \) is the total length of the flux rope and the second integral term vanishes automatically if the flux rope is
closed (for proof see A). Again the total poloidal flux is only determined by the first two coefficients of the axial current. Specifically the poloidal flux also independent of the curvature of the flux rope and scales linearly with the flux rope length $L$.

### 2.1.3. Magnetic Energy

The total magnetic energy stored within the flux rope is given by the following integral:

$$ E = \int_0^1 \int_0^{\tau_{\text{max}}} \int_0^{2\pi} \frac{g_{ij} B_i^c B_j^c}{2 \mu_0} \sqrt{g} \, dr \, d\sigma \, d\phi, \quad (41) $$

which is difficult to analytically calculate as it involves integrals over triple Legendre polynomials. The expression for the magnetic energy density:

$$ g_{ij} B_i^c B_j^c = g_{ss} (B_c^s)^2 + g_{\phi\phi} (B_c^\phi)^2 + g_{nn} B_n^c B_n^c \quad (42) $$

consists of three different terms where the first two always have a positive sign. The third term on the other hand depends on the sign of the torsion $\tau(s)$ and the sign of $B_i^c B_j^c$ which depends on the handedness of the magnetic field.

While more hidden, the magnitude of $(B_c^\phi)^2$ also depends on the handedness of the field due to the second term in the poloidal expression. As such the magnetic energy density is generally split into two separate energy levels whenever there is non-zero torsion or changing curvature. For the general representation we will introduce the following integral:

$$ k E_{nm} = \int_0^1 dr' \rho_n(r') \tilde{P}_m(r') \left[ 1 - r'^2 \sigma^2 k^2(s) \right]^\frac{1}{2}, \quad (43) $$

where higher orders in $k$, $n$ or $m$ will lead to asymptotically smaller values in dependence of the curvature. The total magnetic energy, making use of summation over double indices, can then be written as:

$$ E = \pi \mu_0 \rho_4 \int_0^{\tau_{\text{max}}} ds \left( \frac{E_{nm}^1}{s} \right)^2 \alpha_n \alpha_{n'} + \frac{\sigma^2 \kappa^2(s) \tau^2(s)}{2} \frac{E_{nm}^1}{s} \frac{\sigma^4}{2} \frac{E_{nm}^1}{s} \alpha_n \alpha_{n'} \quad (44) $$

where the last term represents the energy splitting. Note that the terms that depend on the change of the curvature cancel out when integrating over the entire cross-section so that only the axial integral over the torsion produces a difference due to the handedness.

### 2.2. Lorentz Forces

The last property of our model that we will analytically explore are the arising Lorentz forces. The three contravariant components of the Lorentz force can be calculated using the same Equations from Eq. (29) in NC18:

$$ F'_c = g'' r' \sqrt{g} \left( \hat{j}_i B_i^c - \hat{j}_i B_i^c \right) \quad (45) $$

$$ F'_s = \sqrt{g} \left( g''\hat{j}_i B_i^s - g''\hat{j}_i B_i^s \right) \quad (46) $$

$$ F'_n = \sqrt{g} \left( g''\hat{j}_i B_i^n - g''\hat{j}_i B_i^n \right) \quad (47) $$

where now in general all three values will be non-zero as there is a radial current. Nonetheless the typical arising radial currents will be comparatively small and the radial Lorentz force can be expected to be the dominant component. The expressions for these forces will be of similar complexity to those of the current and it is therefore not practical to show their full form. Alternatively we can attempt to compute the net radial force that acts on a slice of our flux rope using:

$$ F_R = < F' > = \int_0^1 \int_0^{2\pi} dr \, d\phi \sqrt{g} e_i F'_c. \quad (48) $$

As the full expression is cumbersome, we expand our result to third order in curvature (first order for derivatives), first order in torsion and only consider the first two coefficients of the Legendre series $(\alpha_0, \beta_0)$. For a torus-like geometry, the net force acting on the cross-section can then be evaluated as:

$$ F_R |_{\tau(s)=0} \approx \frac{\pi}{4} \mu_0 \sigma^4 k(s) \left[ \sigma^2 \left( 2 \alpha_0^2 - 5 \beta_0^2 \right) k^2(s) - 6 \beta_0^2 \right] \cdot n(s), \quad (49) $$

which we find to act fully along the direction defined by the normal vector $n(s)$. The first term $\alpha_0^2$ points inwards and thus acts as the tension force with the other two terms given by $\beta_0$ pointing outwards and thus representing the magnetic hoop force. For the more general form we can investigate any forces acting along the binormal vector $b(s)$ using the same approximations and allowing for changes in curvature:

$$ b(s) \cdot F_R \approx \frac{\pi}{12} \mu_0 \sigma^4 k(s) \left[ \sigma^2 k^2(s) \left[ 19 \beta_0 + 23 \alpha_0 \sigma \tau(s) \right] \right. 
+ \left. 15 \left[ \beta_0 + \alpha_0 \sigma \tau(s) \right] \right). \quad (50) $$

The conclusion of this result is that the flux rope will twist around itself if there is any change in curvature. As this force contains terms of $\alpha_0 \beta_0$ it is also dependent on the handedness of the magnetic field.

### 2.3. Configuration of Coefficients

We will now briefly discuss how we can configure the current coefficients $\alpha_i$ and $\beta_i$ so that the resulting model exhibits certain properties. As our coefficients are defined for a point on the flux rope that is cylindrical it also makes sense to configure the model at our point of reference $s_0$. We start by
defining an arbitrary radial twist function \( Q(r) \) and demand that:

\[
Q(r) = \sum_{l=0}^{\infty} \zeta_l \tilde{P}_l(r) \quad \text{where} \quad \zeta_l = \frac{B^c}{r B^x} = \frac{h_x B_z}{h_z B_x} = \frac{\sum_{m=0}^{\infty} \beta_m \tilde{P}_m(r)}{\sum_{m=0}^{\infty} \alpha_m \tilde{P}_m(r)},
\]

(51)

where \( \zeta_l \) are the coefficients for an expansion, using shifted Legendre polynomials, of our prescribed twist distribution function. Using the following two properties for shifted Legendre polynomials:

\[
\int_0^1 \tilde{P}_m(r) \tilde{P}_n(r) \, dr = \delta_{mn}/(2m+1),
\]

(52)

\[
\int_0^1 \lambda_k \tilde{P}_{m+n-2k}(r) \tilde{P}_m(r) \tilde{P}_n(r) \, dr = \frac{1}{2m+2n-2k+1} \Lambda, \quad \Lambda = \frac{\lambda_k \lambda_{m-k} \lambda_{n-k}}{\lambda_{m+n-k}}, \quad \lambda_k = \frac{(2k)!}{2^{k+1} k!^2},
\]

(53)

(54)

where a derivation for the integral over the triple product is given in Dougall (1953), we can rearrange Eq. (51) and solve for \( \beta_m \) such that:

\[
\beta_m = (2m+1) \int_0^1 dr \tilde{P}_m(r) \left( \sum_{l=0}^{\infty} \zeta_l \tilde{P}_l(r) \right) \left( \sum_{n=0}^{\infty} \alpha_n \tilde{P}_n(r) \right) = (2m+1) \sum_{n=0}^{\infty} \sum_{k=0}^{\min(m,n)} \frac{\alpha_n \lambda_{m+n-2k} \lambda_{m-k}}{2m+2n-2k+1} \lambda_k.
\]

(55)

From this expression we see that we can describe any arbitrary twist distribution for any arbitrary set of coefficients \( \alpha_n \).

We are thus able to halve the degrees of freedom by setting a fixed twist distribution.

In order to determine the \( \alpha_l \) coefficients we require a second constraint for which we can use the radial Lorentz force:

\[
\mu_0^{-1} F^r = -2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \tilde{P}_m(r) \tilde{P}_n(r) \beta_m \beta_n - 2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} r \tilde{P}_m(r) \tilde{P}_n(r) \alpha_m \beta_n + 2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \tilde{P}_m(r) \tilde{P}_n(r) \alpha_n \alpha_m.
\]

(56)

In theory one could repeat the same process as with \( Q(r) \) for \( F^r \) and use a predetermined force distribution. Unfortunately, this turns out to be significantly more complicated and requires the evaluation of integrals over four Legendre polynomials. In practice, when using a limited number of coefficients, it is easier to configure the coefficients so that the flux rope is nearly force free in a cylindrical configuration. This can be achieved by minimizing the function \( \int |F^r|^2 \, dr \).

In the remaining parts of our paper we will exclusively make use of a uniform twist and nearly force-free configuration which can be defined in a more direct way and does not require us to explicitly solve for a vanishing radial Lorentz force profile.

2.4. Model Implementation

In contrast to cylindrical or toroidal flux rope models there is no general straightforward way to transform Cartesian coordinates into our curvilinear coordinates \( (r, s, \varphi) \) and not all transformations are necessarily unique. By default uniqueness is guaranteed within the flux rope volume because \( \sigma \kappa \leq 1 \) so that the flux rope volume does not self-intersect anywhere. If the path \( \gamma(s) \) is given by a purely analytical function, as will be done in Section 3, one can transform the coordinates using standard minimization algorithms.

For numerical implementations, the path \( \gamma(s) \) can be easily implemented as a cubic spline (e.g. Titov et al. 2021). Given a spline \( S(s) \), generated by \( N+1 \) control nodes and \( N \) piecewise cubic polynomials, we can easily construct the Frenet-Serret frame and also compute the curvature and torsion values using:

\[
t(s) = \frac{\partial_s S(s)}{\| \partial_s S(s) \|},
\]

(57)

\[
n(s) = \frac{\partial_s S(s) \times (\partial_s^2 S(s) \times \partial_s S(s))}{\left\| \partial_s S(s) \times (\partial_s^2 S(s) \times \partial_s S(s)) \right\|},
\]

(58)

\[
b(s) = \frac{\partial_s S(s) \times \partial_s^2 S(s)}{\left\| \partial_s S(s) \times \partial_s^2 S(s) \right\|},
\]

(59)

\[
\kappa(s) = \frac{\left\| \partial_s S(s) \right\|^3}{\left\| \partial_s S(s) \times \partial_s^2 S(s) \right\|^2},
\]

(60)

\[
\tau(s) = \frac{\partial_s S(s) \cdot (\partial_s^2 S(s) \times \partial_s^2 S(s))}{\left\| \partial_s S(s) \times \partial_s^2 S(s) \right\|^2},
\]

(61)

where the cubic polynomials guarantee that we are able to take the 3rd order derivative that is necessary to compute the torsion. The problem of transforming Cartesian to curvilinear coordinates is then almost equivalent to the problem of finding the closest point on a cubic spline. This can be achieved using simple iterative algorithms, which should always converge within the flux rope volume (e.g. Wang et al. 2002). The efficiency of these iterative algorithms can be further increased by using the closest control nodes as initial conditions. These algorithms only approximate the \( (r, s) \) coordinates and the \( \varphi \) coordinate must be separately determined in relation to the Frenet-Serret vectors. At this point one can also alternatively use a purely cylindrical geometry to generate the magnetic field if the curvature vanishes and the Frenet-Serret vectors do not exist.

3. MODEL EXAMPLE

We will now show a specific implementation of our model in terms of a purely analytically defined flux rope geometry.
analytical solution for the Gold-Hoyle model (Gold & Hoyle 1960) while the dashed red field line is based on a toroidal approximation which does not take account for effects of changing curvature or torsion. Both field lines start at the same position \((r, s, \phi) = (0.95, 0, 0, 0)\) and the twist is set to \(\zeta_0 = 2\). The position \(s = 0\) corresponds to the bottom and \(s = 1\) to the top.

We use an open parametrized path of the form:

\[
\gamma(s) = \left\{ \left(1 - \frac{s}{2}\right) \sin(3\pi s), \left(1 - \frac{s}{2}\right) \sin(2\pi s), \frac{1 + \sin^2 \left(\frac{s}{2}\right)}{2\pi} \right\},
\]

for the range \(s \in [0, 1]\). The parametrization of this helix-like path was specifically chosen so that the curvature does not completely or nearly vanish at any position. This guarantees that we do not have any issues constructing a Frenet-Serret frame at any point along the path, which would prevent us from visualizing the flux rope geometry. Regarding other model parameters we set \(\sigma = 0.1\) au and \(B_i^0 \bigg|_{s=0} = 15 \text{nT}\) which serve as typical parameters for an ICME at 1 au.

3.1. Uniform Twist Force-Free Configuration

As previously mentioned we will make use of a uniformly twisted magnetic field configuration where we additionally chose the coefficients so that the flux rope is force-free for a cylindrical geometry. From previous calculations it easy to verify that \(\beta_i = \zeta_0 \alpha_i\) for a uniformly twisted flux rope for all indices \(i\). We determine the \(\alpha_i\) coefficients by using the analytical solution for the Gold-Hoyle model (Gold & Hoyle 1960):

\[
\alpha_i = (2i + 1) \int_0^1 dr \left. \frac{B_i^0}{1 + \zeta_0^2 r^2} P_i(r) \right|_{r=0}.
\]

The number of coefficients that are required to deliver sufficient accuracy to approximate the Gold-Hoyle solution depends on the twist parameter \(\zeta_0\) with a higher twist requiring higher order coefficients. Normally only few coefficients are needed for a good approximation of the magnetic field but significantly higher orders are required for accurately depicting the current (and therefore also the Lorentz forces). There also exists an optimal or maximum order beyond which the approximation will begin to diverge that is also further dependent on the twist. For our purposes we will use a dozen coefficients in order to deliver sufficient accuracy for the current and the derived Lorentz forces.

It is at this point that we would also like to highlight why we have chosen our particular approach with the more complicated Legendre polynomials instead of using a radial power series expansion. A Taylor expansion of the Gold-Hoyle solution takes the following form:

\[
\frac{1}{1 + \zeta_0^2 r^2} = 1 - \zeta_0^2 r^2 + \zeta_0^4 r^4 - \zeta_0^6 r^6 + \zeta_0^8 r^8 \ldots
\]

where it is immediately apparent that the power series diverges for any twist values \(\zeta_0 \geq 1\). The models described in NC16/NC18 are thus incapable of describing all uniform twist configurations even with an unlimited number of coefficients. These issues are bypassed by using a description in terms of Legendre polynomials although other, less severe, pitfalls may appear.

3.2. Visualization

Figure 2 shows the exemplary flux rope structure given by Eq. (62) with the parametrized curve \(\gamma(s)\) highlighted in blue. The path initially \((s \approx 0)\) resembles a helix with near constant curvature and torsion and the curvature increases over the axis reaching a maximum near the end \((s \approx 0.8)\). We further plotted two integrated magnetic field lines (red) which show the twisting of the magnetic field within this structure. The first field line, in solid red, shows the magnetic field line of our derived model with a constant twist of \(\zeta_0 = 2\). The second field line, dashed red, shows a magnetic field line of the same twist but as a toroidal approximation. This allows us to visually compare our model solution with classical toroidal solutions and compare the effects on the twisting of the field.
Figure 3. Three different magnetic field $B$, current density $J$, twist $Q$ and $J \times B$ misalignment cross-section contour plots for the flux rope given by Eq. (62) at the positions $s = 0$ (left), $s = 0.75$ (center) and $s = 0.8$ (right). The position $x = -1$ corresponds to $\varphi = \pi$, $x = +1$ to $\varphi = 0$ and $y = \pm 1$ to $\varphi = \pm \frac{\pi}{2}$ respectively. The left column corresponds to a nearly toroidal geometry and the middle/right columns represent strongly twisted geometries with increasing or decreasing curvature. The effect of changing curvature and torsions appears to rotate the magnetic field and current profiles while having strongly asymmetric effects on the local magnetic twist and $J \times B$ misalignment.
lines. From our example here we see that the local twist along
the axis changes significantly and that our writhed solution
exhibits lower twist than the toroidal approximation. As we
will see later this change in twist does not only change over
the axis but also varies across the entire cross-section. As
a consequence this also shows that is significantly harder to
estimate flux rope twist values as they will locally vary from
just deformations of the geometry.

Figure 3 shows 12 panels for the absolute magnetic field
strength B, the absolute current density J, local twist Q and
the J × B misalignment at three varying positions along our
flux rope. The cross-sections are arranged so that x = −1
corresponds to the most inner part of the flux rope (ϕ = π),
x = +1 the most outer part (ϕ = 0) and y = ±1 down and
up respectively. The sign of the y-axis appears inverted be-
cause the binormal vector b(s) generally points towards the
negative 3D Z-axis in our example. The first column (a-d),
for which s = 0, represents the nearly toroidal section of our
flux rope. At this point we see that the structure of our mag-
netic field and the current exhibits a small backwards shift.
The local twist remains nearly constant over the entirety of
the cross-section and it is also mainly force-free with only
small misalignments at the back and front of the structure.
These results are very similar to the results shown in Van-
das & Romashets (2017a) which use a toroidal geometry and
a uniformly twisted magnetic field. The middle (e-h) and
right (i-l) columns represent two strongly twisted sections
of the flux rope with inverted signs for the change of cur-
vature. The magnetic field and current density contour plots
show a stronger shift due to the higher curvature values but
additionally appear to be rotated. The direction of this ro-
tation depends on the sign of the change in curvature, the
sign of the torsion and the handedness of the magnetic field.
The panels (g) & (k) show strongly distorted twist profile
with significant local deviations in the twist number of up
to ±0.5. Despite the appearance of a rough balance for the
size of the higher and lower twist regions the field line in our
example exhibits lower twist. This is due to the fact that a
field line will azimuthally rotate faster when experiencing a
higher twist which leads it to occupy the lower twist region
for longer. Lastly the panels (h) & (l) show two strongly
distorted profiles for the J × B misalignment. In both cases
the flux rope core remains largely force-free but with large
forces arising specifically at the back of structure and weaker
forces at the front. These profiles are also rotated similarly
to the magnetic field and the current with the rotation angle
appearing to be larger.

The Lorentz forces are shown in more detail in Figure 4
where we show the resulting net Lorentz forces, in black, ac-
cording to Eq. (49) which we compute numerically for our
example instead of using the simpler approximations. For
comparison this figure also includes the normal (magenta)
and binormal (orange) Frenet-Serret vectors. For sections of
the flux rope that are nearly toroidal (s ≈ 0) we can see that
the net force mainly consists of the hoop force and also acts
along that direction (inverse of n). For the more twisted sec-
tion of the flux rope we see that the net force does not act
along the same direction as the hoop force, specifically at the
top of our example. We can also see that the net force can
contain a large binormal component and seemingly acts in a
random direction.

3.3. Synthetic Magnetic Field Profiles

While the previously shown global visualizations can fur-
ther the understanding of our flux rope model they cannot be
directly verified or measured. We therefore additionally turn
to synthetic in-situ magnetic field profiles as generated by
virtual spacecraft trajectories in order to generate synthetic
measurements that could be compared to real data.

Figure 5 shows two virtual spacecraft trajectories, from
two different vantage points, and the upper section of our ex-
emplary flux rope structure defined by the range s ∈ [0.5, 1].
The first virtual trajectory, denoted as (a), represents a clas-
sical magnetic flux rope measurement. In the second case,
where the trajectory is denoted as (b), the spacecraft traverses
Figure 5. The upper part of the exemplary flux rope structure defined by $s \in [0.5, 1]$ including two virtual spacecraft trajectories from two different viewing angles. The curvature $\kappa(s)$ in this shown section is highly variable ranging from values of 0.6 up to 5.1. The first trajectory, marked as (a), represents a classical frontal trajectory through of an ICME. The second trajectory, marked as (b), implicates a pass through an ICME flank without an exit to the inside of the flux rope.

through a large portion of the flux rope leading to a significantly longer and highly asymmetric measurement. In both cases the direction of the spacecraft trajectory is given by the arrows. The resulting synthetic in situ magnetic field profiles, for both cases, are shown in Figure 6 and are plotted in terms of an arbitrary length measure along virtual trajectories. The dashed lines within this figure represent local cylindrical approximations which are shown for comparison. Note that in both cases the flux rope size is fixed and that in both our synthetic measurements we do not include any effects due to expansion.

For the first case we see that the profile matches the classical case of a rotating magnetic field profile. The latter part of the synthetic measurement, which corresponds to the compressed inner volume of the curved flux rope, has a stronger absolute magnetic field strength creating a slightly asymmetric profile in terms of magnitude. This asymmetry is not as strong as one would expect from a purely toroidal field as the curvature changes significantly over the trajectory of the virtual spacecraft. The curvature $\kappa(s)$ ranges from a value of 1.9 at the end of the flux rope measurement up to a value of 4.3 at the start. The torsion $\tau(s)$ is fairly constant with values in the range of $-0.7$ at the start down to $-0.9$ at the end.

Our second case is very different showing the typical magnetic field rotation at the start of the measurement then settling into a very static decaying profile as the spacecraft traverses sideways through the flux rope structure. The resulting profile is also significantly different than the result from a local cylindrical approximation (dashed) where the curvature is extremely large. In this case, the curvature $\kappa(s)$ ranges from a value of 5.1 at the start of the flux rope measurement down to a value of 0.8 at the end. The torsion $\tau(s)$ changes by a factor of two with values in the range of $-0.5$ at the start and ranging down to $-1$ at the end. Extreme asymmetries, for which flux rope expansion alone cannot be the cause, can also sometimes be seen in real in-situ data and could therefore be interpreted as flank hits that are similar to our proposed scenario (e.g. Möstl et al. 2010; Owens et al. 2012).

4. DISCUSSION

In this paper we have introduced the basic mathematical concepts that are necessary to describe writhed flux rope structures under the constraint of a circular cross-section. We furthermore derived a specific solution to magnetic field equations for this geometry which requires the introduction of an implicitly defined radial current that imposes further conditions on our flux rope model that we do not investigate in detail. The flux rope still possesses a clear magnetic boundary at its edge but the same cannot be said for the current and this implicitly sets unknown conditions on the external region that surrounds it. As was already remarked in NC18, radial currents can also be introduced into cylindrical or toroidal geometries to break azimuthal symmetry. The exact nature of such a radial current and the resulting physical implications are, at this point in time, not entirely clear to
Figure 6. Two synthetic in-situ profiles generated by the virtual spacecraft trajectories given in Figure 5. The left profile (a) represents a classical frontal fly through and generated the commonly seen rotating magnetic field profile. The right profile (b) describes a flanking pass with an initially fast rotation of the magnetic field and a long largely constant tail. The dashed profile, in both cases, corresponds to a local cylindrical approximation for comparison. The x-axis corresponds to an arbitrary length parameter along the path of the virtual trajectories.

us. Additionally, it may be possible to find more consistent solutions by allowing for a varying cross-section that can be locally adapted to define a boundary through which there is no magnetic flux and no current. A very special case of such a flux rope, for a path $\gamma(s)$ that is confined to a plane and is strictly convex, is given in Yeh (1986).

In the limit of constant curvature, and no torsion, the presented flux rope model is reduced to the classical cylindrical or toroidal flux rope models. It is thus expected that our model can perform at least as well as a toroidal or cylindrical flux rope model when reconstructing real in-situ flux rope measurements. It is very unlikely that it is possible to directly infer the full global structure of a writhed flux rope just from the in-situ magnetic field measurements. Our first example, that is shown in Figure 6, exhibits a profile that is very similar to the result that one would expect of a purely toroidal flux rope despite undergoing a large change in curvature along the virtual spacecraft trajectory.

With the usage of Legendre polynomials and NC16/NC18 there are now at least two approximate approaches for describing the internal magnetic field structures for the type of models that we use in this paper. For uniformly twisted fields the Legendre approach is clearly superior but it may have unknown problems in other scenarios. We also have not attempted to describe the evolution of the coefficients regarding flux rope expansion which will differ depending on the polynomial basis that is used. In the future it may be necessary to more closely analyze these approaches for different scenarios. For both cases, the degrees of freedom that arise when using many coefficients are too high to be properly analyzed in real data. It is thus clear that no matter which approach is used that there must be a simple description of the magnetic field. This can either be a uniform twist number as with the uniformly twisted field, an $\alpha$ parameter for a linear force-free field or another parameter for other type of distributions.

While we have only performed a purely mathematical approach our model is fairly straightforward to implement for numerical applications. The parametrized path $\gamma(s)$ can easily be described using splines, which must be of at least third order, which also allows for easy calculation of the curvature, torsion and the Frenet-Serret vectors (see Section 2.4). Transformation of Cartesian coordinates into our curvilinear coordinate system is fairly straightforward using minimization algorithms under the condition that good initial starting values are used. A basic implementation for some of these procedures was required to generate the synthetic profiles shown in Figure 6 and are contained in the notebooks in the repository linked in the acknowledgements. A spline implementation, in the future, should allow for easy time evolution if the spline control nodes are interpreted as representative particles with mass. This should allow us to build purely analytic simulations and evolve the flux rope geometry according to the arising Lorentz forces or additional external factors such as the solar wind environment or the coronal magnetic field. Using the previous definition for the net radial Lorentz force $F_R$, and using spline approach with representative particles $S(s_i,t)$ of mass $m_i$, we can create basic equations of motion for the underlying spline curve $S(s,t)$:

$$ m_i \frac{d^2 S(s_i,t)}{dt^2} = F_R \bigg|_{s=s_i} + F_{ext} \tag{65} $$

where $F_{ext}$ describes arbitrary external forces. This would allow the curve to evolve over time according to the forces that are shown for the exemplary flux rope structure in Figure 4.

If we want to consider time-dependent changes we also need to consider flux rope expansion or distortions of the cross-section. The results from Figure 3 show highly asymmetric Lorentz force distributions when we include effects due to changing curvature or torsion. It is therefore highly likely that our usage of a circular cross-section is a strong
approximation and that this shape will become additionally distorted over time. The arising Lorentz forces do not necessarily lead to an additional expansion and the flux rope expansion is still expected to be dominated by the pressure gradient at higher distances from the sun. This is specifically the case in our example where the Lorentz forces at the back of the structure point inwards and would thus lead to a distortion of the cross-section that would reduce the total size of the flux rope. In order to test how valid the circular cross-section approximation is, and under which conditions it may be suitable, would require more sophisticated numerical MHD simulations.

5. CONCLUSION

Overall we believe that this paper serves as another useful contribution, building on previous work in NC16/NC18, for extending the framework to describe complex flux rope geometries. Despite using the constraint of a circular cross-section the flexibility that is gained by describing the flux rope in terms of a general space curve allows for an extremely high variety of different configurations to be explored. While our work primarily concerns itself with ICME flux ropes this model can also be applied to flux rope structures significantly closer to the Sun or on the solar surface.

The presented flux rope model, making use of the cubic spline approach, can also be readily implemented in a forward simulation model. An approach similar to the one used in Hinterreiter et al. (2021) could be used to drive the changes in the flux rope geometry due to interactions with the ambient solar wind. Solar wind velocity maps, for the inner heliosphere, can be generated by simulation such as Enlil (Odstrcil 2003), HUXt (Owens et al. 2020) or THUX (Reiss et al. 2020). Assuming that the resulting forward simulations are fast enough, one could attempt to build a fitting pipeline using a Monte-Carlo approach as was done for simpler analytical flux rope models (e.g. Weiss et al. 2021).

Future studies are planned in which we aim to further develop this flux rope model using parallel transport frames (e.g. Bishop 1975). This would allow us to resolve the main issue with the Frenet-Serret frame that occurs for vanishing curvature. Additionally it would allow us to include non-axisymmetric cross-section shapes for writhed flux ropes using the approach described in Nieves-Chinchilla et al. (2022).

ACKNOWLEDGMENTS

A.J.W., R.L.B., M.A.R. and T.A. thank the Austrian Science Fund (FWF): P31521-N27, P31659-N27. T.N-Ch acknowledges the NASA-GSFC Heliophysics Internal Fund (HIF) “Physics-driven modeling of the Interplanetary coronal mass ejections distortions”.

The source code for some of the calculations and the figures are available in terms of Mathematica notebooks at https://github.com/helioforecast/Papers2.

REFERENCES

Bishop, R. L. 1975, The American Mathematical Monthly, 82, 246. http://www.jstor.org/stable/2319846
Bothmer, V., & Schwenn, R. 1998, Annales Geophysicae, 16, 1, doi: 10.1007/s00585-997-0001-x
Burlaga, L., Sittler, E., Mariani, F., & Schwenn, R. 1981, J. Geophys. Res., 86, 6673, doi: 10.1029/JA086iA08p06673
Davies, E. E., Möstl, C., Owens, M. J., et al. 2021, A&A, 656, A2, doi: 10.1051/0004-6361/202040113
Démoulin, P., & Dasso, S. 2009, A&A, 498, 551, doi: 10.1051/0004-6361/200810971
Dougall, J. 1953, Proceedings of the Glasgow Mathematical Association, 1, 121–125, doi: 10.1017/S204061850003590
Farrugia, C. J., Osherovich, V. A., & Burlaga, L. F. 1995, J. Geophys. Res., 100, 12293, doi: 10.1029/95JA00272
Gold, T., & Hoyle, F. 1960, MNRAS, 120, 89, doi: 10.1093/mnras/120.2.89
Gosling, J. T., McComas, D. J., Phillips, J. L., & Bame, S. J. 1991, J. Geophys. Res., 96, 7831, doi: 10.1029/91JA00316
Gulisano, A. M., Démoulin, P., Dasso, S., & Rodriguez, L. 2012, A&A, 543, A107, doi: 10.1051/0004-6361/201118748
Gulisano, A. M., Démoulin, P., Dasso, S., & Rodriguez, L. 2012, A&A, 543, A107, doi: 10.1051/0004-6361/201218901
Hidalgo, M. A., Nieves-Chinchilla, T., & Ced, C. 2002, Geophys. Res. Lett., 29, 1637, doi: 10.1029/2001GL013875
Hinterreiter, J., Amerstorfer, T., Temmer, M., et al. 2021, Space Weather, 19, e02836, doi: 10.1029/2021SW002836
Isavnin, A. 2016, ApJ, 833, 267, doi: 10.3847/1538-4357/833/2/267
Kay, C., & Gopalswamy, N. 2018, Journal of Geophysical Research (Space Physics), 113, 7220, doi: 10.1029/2018JA025780
Kay, C., Opher, M., & Evans, R. M. 2015, ApJ, 805, 168, doi: 10.1088/0004-637X/805/2/168
Leitner, M., Farrugia, C. J., MöStl, C., et al. 2007, Journal of Geophysical Research (Space Physics), 112, A06113, doi: 10.1029/2006JA011940
Lepping, R. P., Jones, J. A., & Burlaga, L. F. 1990, J. Geophys. Res., 95, 11957, doi: 10.1029/JA095iA08p06673
Liu, Y., Richardson, J. D., Bame, S. J., et al. 1991, J. Geophys. Res., 96, 7831, doi: 10.1029/91JA00316
APPENDIX

A. POLOIDAL FLUX INTEGRAL

We want to show that the integral:

$$\mu_0 \sigma^4 \sin(\varphi) \int_0^1 \int_0^{s_{\text{max}}} dr \, ds \, \frac{r^2 \partial_\varphi \kappa(s)}{[1 + r \sigma \kappa(s) \cos(\varphi)] \sqrt{1 - r^2 \sigma^2 \kappa^2(s)}} \sum_{n=1}^{\infty} \alpha_n \tilde{P}_n(r) = 0,$$

(A1)

vanishes. It can already be seen, due to the \(\sin(\varphi)\) dependence, that the term must vanish as the poloidal flux must be invariant with respect to the \(\varphi\) parameter. We start by expanding the fraction using a Taylor series in terms of the curvature \(\kappa(s)\):

$$\frac{1}{[1 + r \sigma \kappa(s) \cos(\varphi)] \sqrt{1 - r^2 \sigma^2 \kappa^2(s)}} = \sum_{k=0}^{\infty} a_k(r, \varphi) k^k(s),$$

(A2)

where \(a_k(r, \varphi)\) are undefined coefficients which are not of importance. Using integration by parts we can then rewrite the integral for every \(k\):

$$\int_0^{s_{\text{max}}} ds \, k^k(s) \partial_\varphi \kappa(s) = [k^k(s) \kappa(s)]_0^{s_{\text{max}}} - k \int_0^{s_{\text{max}}} ds \, k^{k-1}(s) (\partial_\varphi \kappa(s)) \kappa(s).$$

(A3)

Finally, using the condition that \(\kappa(0) = \kappa(s_{\text{max}})\) the first term on the right hand side vanishes and we receive:

$$\int_0^{s_{\text{max}}} ds \, k^k(s) \partial_\varphi \kappa(s) = -k \int_0^{s_{\text{max}}} ds \, k^{k-1}(s) \partial_\varphi \kappa(s)$$

$$\implies \int_0^{s_{\text{max}}} ds \, k^k(s) \partial_\varphi \kappa(s) = 0, \ \forall k, \ k \geq 0,$$

(A4)

with which our claim has been proven.