A precise zeta-function calculation shows that the contribution of the vacuum energy to the observed value of the cosmological constant can possibly have the desired order of magnitude albeit the sign strongly depends on the topology of the universe. The non-renormalizable, infinite contributions which have been recently shown to occur when one physically imposes boundary conditions on quantum fields (Casimir calculations) are considered. It is shown that using a Hadamard regularization in addition to the zeta method, the ordinary, finite results in the literature are exactly recovered.

This report is divided into two parts. In the first one, within a simple cosmological model, the vacuum energy density of a scalar field with a very low (but non-zero) mass is seen to contribute with the right order of magnitude to the observed value of the cosmological constant. In the second, we elaborate on some issues, discussed recently, concerning the physical (and mathematical) meaning of imposing boundary conditions (BC) on quantum fields, which is central to Casimir effect calculations.

1 Vacuum energy in a simple cosmological model

It is not difficult to get the right order of magnitude of the vacuum energy density $\rho_V$, in the range corresponding to astrophysical observations, e.g. $\rho_V \sim 10^{-10} \text{ erg/cm}^3$. For this, one just assumes the existence of a scalar field background, $\phi$, extending through the universe and calculates the contribution to the cosmological constant coming from the Casimir energy density corresponding to this field for some typical boundary conditions. (The ultraviolet contributions are safely set to zero by invoking some mechanism of a fundamental theory.) One further assumes existence of both large and small dimensions (the total number of large spatial coordinates will be always three), some of which (from each class) may be compactified. Indeed, the global topology of the universe plays here an important role. The issue of the possible contribution of the Casimir effect as a source of some sort of cosmic energy, as in the case of the creation of a neutron star, has been considered before, but the emphasis will be here put in obtaining the right order of magnitude for the effect, with a minimum number of assumptions.
Consider a universe with a space-time as: \( \mathbb{R}^{d+1} \times \mathbb{T}^p \times \mathbb{T}^q \), or \( \mathbb{R}^{d+1} \times \mathbb{T}^p \times \mathbb{S}^q \). A (nowadays) free scalar field pervading the universe will satisfy \((-\Box + M^2)\phi = 0\), restricted by the appropriate boundary conditions (e.g., periodic, in the first case considered). Here, \( d \geq 0 \) stands for a possible number of non-compactified dimensions. The vacuum energy density for a \((p, q)\)-toroidal universe (with \(p\) 'large' and \(q\) 'small' dimensions) is

\[
\rho_\phi = \frac{\pi^{-d/2}}{2^{d-1}(d-1)!} \int_0^\infty \frac{dk}{k^{d-1}} \sum_{n_p=1}^\infty \sum_{m_q=1}^\infty \left( \frac{2\pi n_j}{a_j} \right)^2 + \left( \frac{2\pi m_h}{b_h} \right)^2 + M^2 \right]^{1/2}.
\]

We shall use zeta regularization. For the analytic continuation of the zeta function corresponding to (1) we obtain (to simplify, we consider now that all \(a\)'s are equal, and also all \(b\)'s)\(^1\)

\[
\zeta(s) = \frac{2\pi^{s/2+1}}{\alpha^{p-(s+1)/2}b^{p-(s-1)/2}(s/2)} \sum_{n_p=1}^\infty \sum_{m_q=1}^\infty \frac{p}{h} 2^h \sum_{k=1}^h \left( \frac{n_j^2}{m_k^2 + M^2} \right)^{(s-1)/4} K_{(s-1)/2} \left( \frac{2\pi a}{b} \sum_{j=1}^h n_j^2 \right) \left( \sum_{k=1}^h m_k^2 + M^2 \right) \left( \frac{2\pi a}{b} \sum_{j=1}^h n_j^2 \right)^{(s-1)/4},
\]

with \(K_\nu(z)\) the modified Bessel function of the second kind. Having performed already the analytic continuation, this expression is ready for the substitution \(s = -1\), and using also the behaviour of the function \(K_\nu(z)\) for small values of its argument, \(K_\nu(z) \sim \frac{\Gamma(\nu)}{2}(z/2)^{-\nu}, z \to 0\), we obtain, in the case when \(M\) is very small,

\[
\rho_\phi = -\frac{1}{\alpha^{p(b^q+1)}} \left\{ M K_1 \left( \frac{2\pi a}{b} M \right) + \sum_{h=0}^p \left( \frac{p}{h} \right) 2^h \sum_{n_p=1}^\infty \frac{M}{\sum_{j=1}^h n_j^2} \right. \\
\left. \times K_1 \left( \frac{2\pi a}{b} M \sum_{j=1}^h n_j^2 \right) + O \left[ q\sqrt{1 + M^2} K_1 \left( \frac{2\pi a}{b} \sqrt{1 + M^2} \right) \right] \right\}.
\]

We should view all masses appearing here as dimensionless: quotients by the mass-dimensional regularization parameter \(\mu\), as \(M/\mu\) everywhere. This does not affect the small-\(M\) limit, which reads \(M/\mu << b/a\). Replacing in the expression the \(h\) and \(c\) factors, we get

\[
\rho_\phi = -\frac{\hbar c}{2\pi a^{p+1}b^q} \left[ 1 + \sum_{h=0}^p \left( \frac{p}{h} \right) 2^h \alpha \right] + O \left[ qK_1 \left( \frac{2\pi a}{b} \right) \right],
\]

with \(\alpha\) a finite constant (an explicit geometrical sum in the limit \(M \to 0\)).
For the most common variants, the constant $\alpha$ in (4) has been calculated to be of order $10^2$, and the whole factor, in brackets, of the first term in (4) is of order $10^7$. Good coincidence with the observational value for the cosmological constant is obtained for the contribution of a very low mass ($M \leq 1.2 \times 10^{-32}$ eV) scalar field, $\rho_\phi$, for $p = 0$, 1 large compactified dimensions ($a$ the radius of the observable universe) and $q = p + 1$ small compactified dimensions, the small compactification length, $b$, being of the order of 100 to 1000 times the Planck length $l_P$. The best fit is obtained for $p = 1$, $q = 2$, which is a very reasonable result, coinciding with other much more fundamental approaches. Dimensionally speaking, everything is here dictated by the two basic lengths in the problem: its Planck value and the radius of the observable Universe, and the final conclusion appears to be quite robust.

2 Hadamard regularization of the Casimir effect

We luckily chose in the previous section a no-boundary configuration so as to avoid the important problem we will now address: that of imposing BC on a quantum field. To go directly to the heart of the matter, take again the most simple case of a scalar field in one dimension, $\phi(x)$, with a BC of Dirichlet type imposed at a point, e.g. $\phi(0) = 0$. We want to calculate the Casimir energy for this configuration, that is, the difference between the zero point energy corresponding to such field when the BC is enforced, and the zero point energy in the absence of any BC. Both energies are infinite and the regularized difference may still be infinite when the BC point is approached (this is the result in 17) or may be finite (even zero, which is the result given in many standard references on the subject 18, 19).

2.1 Understanding the infinities ‘ab initio’

Let us try to understand this enormous discrepancy. To this end, we propose to go back to the more classical (and mathematical) definition of boundary value problem, as taken e.g. from Courant and Hilbert (see Ref. 21 Chap. V, pp. 275 ff). Unfortunately, there is here no place to go into details. Suffice to say that, within this definition, one imposes smoothness of the solution on the points of the boundary itself (which are not different, in this way, from the rest of the points).\(^\text{a}\) Thus, the boundary is not supposed to divide the space into independent domains (or ‘universes’, so to speak). It can be argued that this sort of mathematical boundary value problem is better suited for the analysis of the quantum vacuum in view that the other

\(^\text{a}\)For a number of physical applications, as reflection of wave packets, BC for perfectly conducting walls, or bag type conditions, it would not be adequate to demand this: the boundary is explicitly excluded from the domain, which is usually broken by it into two or more independent subdomains.
definition cannot solve the dilemma posed by Bob Jaffe and collaborators. Indeed, in the papers
and in Jaffe’s talk at this workshop, it became crystal clear that the ordinary definition is quite useless for describing the Casimir energy density: it leads to misleading results which have nothing to do with the physics of the Casimir effect in QFT.

And here comes the calculation itself. One has to add up all energy modes (trace of $H$). For the mode with energy $\omega$, the field equation is:

$$-\phi''(x) + m^2 \phi(x) = \omega^2 \phi(x).$$

(5)

In the absence of a BC, the solutions to the field equation can be labelled by $k = +\sqrt{\omega^2 - m^2} > 0$, as $\phi_k(x) = Ae^{ikx} + Be^{-ikx}$, with $A, B$ arbitrary complex (for the general complex), or as $\phi_k(x) = a \sin(kx) + b \cos(kx)$, with $a, b$ arbitrary real (for the general real solution). Now, when the mathematical BC of Dirichlet type, $\phi(0) = 0$, is imposed, this does not influence at all the eigenvalues, $k$, which remain exactly the same. However, the number of solutions corresponding to each eigenvalue is reduced by one half to: $\phi_k^{(D)}(x) = A(e^{ikx} - e^{-ikx})$, with $A$ arbitrary complex (complex solution), and $\phi_k^{(D)}(x) = a \sin(kx)$, with $a$ arbitrary real (real solution). In other words, in both cases is the same, a continuous spectrum, but the number of eigenstates corresponding to a given eigenvalue is twice as big in the absence of the BC.

An infinity may originate from the fact that imposing the BC has drastically reduced to one-half the family of eigenfunctions for the spectrum of the operator. And, since this dramatic reduction takes place precisely at the point where the BC is imposed, a physical divergence (infinite energy) may originate there. While this sketchy analysis cannot be taken as a substitute for the actual modelization of Jaffe et al. —where the BC is explicitly enforced through the introduction of an auxiliary, localized field, which probes what happens at the boundary in a much more precise way— it certainly leads one to think that pure mathematical considerations, which include the use of analytic continuation by means of the zeta function, are in no way blind to the infinites of the physical model and do not produce misleading results, when the mathematics are used properly. And it is very remarkable to realize how close the mathematical description of the appearance of an infinite contribution is to the one provided by the more physical realization [see R. Jaffe et al’s contributions to these proceedings].

What I have shown with this little exercise is precisely that, e.g., by defining the 1-dimensional Dirichlet boundary value problem in a classical (or more mathematically minded) fashion, which views the solution of the eigenvalue problem on the whole real line —before and after forcing it to be zero at the origin— we do gain a mathematical understanding of the emergence of a singularity at the boundary, which very closely parallels the physical description of Jaffe et al. This is certainly not the final answer, since the divergence

\[b\]The one not demanding regularity on the boundary.
should be now regularized and renormalized, and the whole argument is based on a maybe non-standard\(^d\) definition of boundary value problem, but it works in this situation (and not only for one dimension \(^{20}\)). The reason why these infinities do not usually show up in the literature on the Casimir effect \(^{18}\) is probably because the other definition of boundary value problem was used generically,\(^d\) and also because textbooks on the subject often focus towards the calculation of the Casimir force (minus the derivative of the energy). Since the infinite terms do not depend on the distance between the plates, they do not contribute to the force (see also \(^{19}\)).

2.2 Splitting the infinities

In the rest of the paper we will focuss on the interpretation of the persistent infinities obtained in the QFT calculations, i.e. those which remain after renormalization \(^{17}\). Surprisingly, by using Hadamard regularization we are going to see that one can split those divergences, in a very natural way, into two terms: a finite and an infinite contribution, each of them having a very precise interpretation. Indeed, the finite part will be recognized as the finite result obtained in the already mentioned usual references on the Casimir effect. The remaining infinite part will be precisely identified as proportional to \(1/\omega\), where \(\omega\) is typically the width of the Gaussian that gives rise to the delta-function (when \(\omega \to 0\)), which enforces the BC in the approach of \(^{17}\). This is undoubtedly a remarkable result.

The naturalness of the splitting of the divergences just mentioned hangs on that of Hadamard’s regularization itself, as a standard and admissible technique. Now, Hadamard regularization is actually a well-established procedure in order to give sense to infinite integrals: in higher-post-Newtonian general relativity \(^{22}\) and also in QFT \(^{23}\). Among mathematicians, it is the standard technique in order to deal with singular differential and integral equations with BCs, both analytically and numerically.

In one dimension, with Dirichlet BC imposed at one \((x = 0)\) and two \((x = \pm a)\) points, respectively, by means of a delta-background of strength \(\lambda\) (see \(^{17}\)), one encounters the two divergent integrals:

\[
E_1(\lambda, m) = \frac{1}{2\pi} \int_m^\infty \frac{dt}{\sqrt{t^2 - m^2}} \left[ t \log \left( 1 + \frac{\lambda}{2t} \right) - \frac{\lambda}{2} \right], \\
E_2(a, \lambda, m) = \frac{1}{2\pi} \int_m^\infty \frac{dt}{\sqrt{t^2 - m^2}} \left\{ t \log \left[ 1 + \frac{\lambda}{t} + \frac{\lambda^2}{4t^2} (1 - e^{-4at}) \right] - \lambda \right\}.
\]

Using Hadamard’s regularization, as described before, we get for the first one

\[
E_1(m) = \frac{\lambda}{4\pi} \left( 1 - \ln \frac{\lambda}{m} \right) \bigg|_{\lambda \to \infty} + \oint,
\]

\(^d\)Better, not-so-useful-in-other-cases.
\(^d\)The one which does not impose regularity of the eigensolution at the boundary.
where the first term is the singular part when the limit $\lambda \to \infty$ is taken, and the second—which is Hadamard’s finite part—yields in this case
\[ \int = \frac{m}{4}. \]
(9)

Such result is coinciding with the classical one (0, for $m = 0$). Note in particular, that the further $\ln m$ divergence as $m \to \infty$ is hidden in the $\lambda$–divergent part, and such behavior does explain why the classical results [13] which are obtained using hard Dirichlet BC—what corresponds as we just prove here to the Hadamard’s regularized part—cannot see it [20].

In the case of a two-point boundary at $x = \pm a$ (separation $2a$), Eq. (7), we get a similar Eq. (8) but now the regularized integral is as follows. For the massless case, we obtain
\[ \int = \frac{-\pi}{48a}, \]
(10)

which is the ‘classical’ regularized result. In the massive case, $m \neq 0$, after additional work the following fast convergent series turns up
\[ \int = -\frac{m}{2\pi} \sum_{k=1}^{\infty} \frac{1}{k} K_1(4akm). \]
(11)

Thus Eq. (8) yields strictly the same result which one already obtains by imposing the Dirichlet BC \textit{ab initio} in the ordinary way [20]. What has now been \textit{gained} is a more clear identification of the singular part, in terms of the strength of the delta potential at the boundary. That is the general conclusion of this analysis, common to all the other cases here considered. (For an alternative approach which also uses Hadamard calculus see the contribution of S. Fulling to these proceedings.)

Correspondingly, for the Casimir force we obtain the finite values
\[ F_2(a) = -\frac{\pi}{96a^2}, \]
(12)
in the massless case, and in the massive one
\[ F_2(a, m) = -\frac{m^2}{\pi} \sum_{k=1}^{\infty} \left[ K_0(4akm) + \frac{1}{4akm} K_1(4akm) \right]. \]
(13)

Those expressions coincide with the ones derived in the above mentioned textbooks on the Casimir effect, and reproduced before by using the zeta function method (for a recent, very simple derivation see [24]).

The two dimensional case turns out to be more singular. After a similar analysis, by using Hadamard regularization we also obtain the usual finite classical results of the literature in this case. For the Casimir energy, we get
\[ E_{\lambda^2}[\tau] = \frac{\lambda^2 a^2}{8} \int_0^\infty dp (ap + 1)^{-\tau} J_0^2(ap) \arctan(p/2m) \Bigg|_{\omega \to 0} \]

\[ = \frac{\lambda^2 a^2}{8} \left\{ \frac{1}{2\omega} + \frac{\gamma + 3 \ln 2}{2\omega} + 4m \left[ \gamma - \frac{2}{\sqrt{\pi}} (1 - \ln(\omega m)) h(4a^2m^2) \right] \right\}, \quad (14) \]

where \( h(z) := {}_2F_3 \left( \frac{1}{2}, \frac{1}{2}; \left| \frac{1}{2}, \frac{3}{2} \right|; z \right) \) and \( \gamma \) is the Euler-Mascheroni constant; in particular, \( h(1) = 1.186711 \), quite a nice value. Here, as previously announced, \( \omega \) is the width of the Gaussian \( \delta \), which is the very physical parameter considered by Candelas.\(^{15}\) The finite part (Hadamard) is

\[ \int_0^\infty dp J_0^2(ap) \arctan(p/2m). \quad (15) \]

Again, this reverts to the results obtained in the literature using Dirichlet BC \textit{ab initio}.

Acknowledgments

I am indebted to the members of the Mathematics Department, MIT, and specially to Dan Freedman, for warm hospitality. I thank the anonymous referee of this contributed paper for critical insights and patient discussions which definitely led to its improvement. The present investigation has been supported by DGICYT (Spain), project BFM2000-0810, and by CI RIT (Generalitat de Catalunya), grants 2002BEAI400019 and 2001SGR-00427.

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