Perfect i.i.d Processes

Pathikrit Basu

20128 White Cloud Circle, USA.

Author’s contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/AJPAS/2022/v18i230443

Open Peer Review History:

This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here: https://www.sdiarticle5.com/review-history/88772

Received: 25 April 2022
Accepted: 29 June 2022
Published: 11 July 2022

Abstract

This note proves a theorem about i.i.d. i.e. independent and indentically distributed processes, when the index space is a measure space. The statement of the problem corresponding to the theorem proved in this paper appears in [1], in which the concept of a sample distribution limit corresponds to the concept of a perfect i.i.d process in this paper.

Theorems proved in this theme, regarding existing and non-existence, have been shown in the economics literature, when the index set is [0, 1], in [2], [3], [4], [5]. The approach taken in this paper is perhaps, surprisingly elementary. We may apply standard measure extension theorems to show existence. These may be found in [6], [7].

Keywords: Index space; probability space; measurable functions; IID process.

1 Model

Suppose \((R, \mathcal{R}, \rho)\) is a probability space that we will call the state space; and \((P, \mathcal{P}, \pi)\) be a probability space called the index space. The following definitions convey the prime theme of the paper.

**Definition 1.1.** A setting is defined as a pair \(<(R, \mathcal{R}, \rho), (P, \mathcal{P}, \pi)\>\) consisting of a state space and an index space.
Definition 1.2. A measure-preserving transformation is any measurable map \( \psi : P \to R \) such that
\[
(\forall B \in \mathcal{R})(\pi(\{ p : \psi(p) \in B \}) = \rho(B)).
\]  

Definition 1.3. A setting \(< (\mathcal{R}, \mathcal{P}, \pi) > \) is said to admit a perfect i.i.d process if there exists a probability space \(< \Omega, \mathcal{F}, \mathbb{P} > \) and measurable functions \( \{ X_p \}_{p \in \mathcal{P}} \) where \( X_p : \Omega \to R \) such that

1. For any finite \( P' \subseteq P \) and collection \( \{ B_p \}_{p \in P'} \subseteq \mathcal{R} \) we have that
\[
\mathbb{P}(\bigcap_{p \in P'} \{ X_p \in B_p \}) = \prod_{p \in P'} \rho(B_p).
\]

2. There exists \( A \in \mathcal{F} \) such that \( \mathbb{P}(A) = 1 \) and
\[
A \subseteq \{ \omega \in \Omega : X_p(\omega) \text{ is measure-preserving in } p \}.
\]

Definition 1.4. An index space \( (P, \mathcal{P}, \pi) \) will be called fine if

1. For every \( p \in P \), \( \{ p \} \in \mathcal{P} \).
2. For every \( p \in P \), \( \pi(\{ p \}) = 0 \).

It follows immediately that any fine index space \( (P, \mathcal{P}, \pi) \) is uncountably infinite. The following is the main theorem of the paper.

Theorem 1.1. Let \(< (\mathcal{R}, \mathcal{P}, \rho), (P, \mathcal{P}, \pi) > \) be a setting. Suppose that the index space \( (P, \mathcal{P}, \pi) \) is fine. Further, suppose that there exists a measure-preserving transformation \( \psi : P \to R \). Then, the setting \(< (\mathcal{R}, \mathcal{P}, \rho), (P, \mathcal{P}, \pi) > \) admits a perfect i.i.d process.

Proof. The proof proceeds in a few steps.

Step 1: We argue that given a measurable-preserving transformation \( \psi : P \to R \); a countable subset \( \hat{P} \subseteq P \); and any function \( \psi' : \hat{P} \to R \), the map \( \psi' : P \to R \) defined as
\[
\psi'(p) = \begin{cases} 
\hat{\psi}(p) & \text{if } p \in \hat{P} \\
\psi(p) & \text{otherwise}
\end{cases}
\]  

is also a measure-preserving transformation. This is true since the probability space \( (P, \mathcal{P}, \pi) \) is assumed to be fine. As \( \mathcal{P} \) includes all singleton sets, it follows that \( \hat{P} \subseteq \mathcal{P} \). Hence, \( \psi' \) is measurable. Further, since singletons have zero probability according to the probability measure \( \pi \), implying that \( \pi(\hat{P}) = 0 \) (due to countable additivity), it follows that \( \psi' \) is also a measure-preserving transformation.

Step 2: We now define the probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). Define \( \Omega \) as
\[
\Omega := \{ \psi : P \to R : \psi \text{ is measure-preserving} \}.
\]

By assumption, we have that \( \Omega \neq \emptyset \). For a finite subset \( \hat{P} \subseteq P \) and collection of sets \( \{ B_p \}_{p \in \hat{P}} \subseteq \mathcal{R} \), define the set
\[
< \hat{P}, \{ B_p \}_{p \in \hat{P}} > := \{ \psi \in \Omega : (\forall p \in \hat{P}) (\psi(p) \in B_p) \}.
\]

The collection of all such sets is defined as
\[
\mathcal{S} := \{ < \hat{P}, \{ B_p \}_{p \in \hat{P}} > : \text{ finite } \hat{P} \subseteq P \text{ and collection } \{ B_p \}_{p \in \hat{P}} \subseteq \mathcal{R} \}.
\]

We show that \( \mathcal{S} \) is a semi-ring (see [8]). Further, we show that the following set function \( \mathbb{P}' \) defines a measure on \( \mathcal{S} \)
\[
\mathbb{P}'(< \hat{P}, \{ B_p \}_{p \in \hat{P}} >) = \prod_{p \in \hat{P}} \rho(B_p).
\]

We first prove that \( \mathcal{S} \) is a semi-ring. This follows simply from the following facts.
1. Suppose that we have a set of the form \( < \hat{P}, \{ B_p \}_{p \in \hat{P}} > \) such that \( B_p = \emptyset \) for some \( p \in \hat{P} \). This immediately implies that \( < \hat{P}, \{ B_p \}_{p \in \hat{P}} > = \emptyset \in \mathcal{S} \).

2. Suppose that \( < \hat{P}, \{ B_p \}_{p \in \hat{P}} >, < \hat{P}^*, \{ B'_p \}_{p \in \hat{P}^*} > \in \mathcal{S} \). Then, it is simple to prove that

\[
< \hat{P}, \{ B_p \}_{p \in \hat{P}} > \cap < \hat{P}^*, \{ B'_p \}_{p \in \hat{P}^*} > = < \hat{P} \cap \hat{P}^*, \{ B_p \cap B'_p \}_{p \in \hat{P} \cap \hat{P}^*} >.
\]

Hence, we have shown that \( < \hat{P}, \{ B_p \}_{p \in \hat{P}} > \cap < \hat{P}^*, \{ B'_p \}_{p \in \hat{P}^*} > \in \mathcal{S} \).

3. Suppose that \( < \hat{P}, \{ B_p \}_{p \in \hat{P}} >, < \hat{P}^*, \{ B'_p \}_{p \in \hat{P}^*} > \in \mathcal{S} \). We wish to show that \( < \hat{P}, \{ B_p \}_{p \in \hat{P}} > \backslash < \hat{P}^*, \{ B'_p \}_{p \in \hat{P}^*} > \) may be represented as a finite union of pairwise disjoint sets in \( \mathcal{S} \). Note that

\[
< \hat{P}, \{ B_p \}_{p \in \hat{P}} > \backslash < \hat{P}^*, \{ B'_p \}_{p \in \hat{P}^*} > =< \hat{P}, \{ B_p \}_{p \in \hat{P}} > \cap (\Omega \backslash < \hat{P}^*, \{ B'_p \}_{p \in \hat{P}^*} >).
\]

Then, it follows that

\[
\Omega \backslash < \hat{P}^*, \{ B'_p \}_{p \in \hat{P}^*} > = \bigcup_{Q \subseteq \hat{P}^* \setminus Q \neq \emptyset} < \hat{P}^*, \{ \Omega \backslash B'_p \}_{p \in Q} \cup \{ B'_p \}_{p \in \hat{P}^* \setminus Q} >.
\]

which is a finite union of disjoint sets in \( \mathcal{S} \). Hence, from 2., we have indeed shown that it is the case \( < \hat{P}, \{ B_p \}_{p \in \hat{P}} > \backslash < \hat{P}^*, \{ B'_p \}_{p \in \hat{P}^*} > \) is a finite union of disjoint sets in \( \mathcal{S} \) as it may be represented as

\[
< \hat{P}, \{ B_p \}_{p \in \hat{P}} > \backslash < \hat{P}^*, \{ B'_p \}_{p \in \hat{P}^*} > = \bigcup_{Q \subseteq \hat{P}^* \setminus Q \neq \emptyset} < \hat{P}, \{ B_p \}_{p \in \hat{P}} > \cap < \hat{P}^*, \{ \Omega \backslash B'_p \}_{p \in Q} \cup \{ B'_p \}_{p \in \hat{P}^* \setminus Q} >.
\]

and we have proved that \( \mathcal{S} \) is a semiring.

We show that \( \mathcal{P}^* \) defines a measure on \( \mathcal{S} \). Suppose, we have a countable collection of pairwise disjoint sets \( \{ < \hat{P}^i, \{ B_p^i \}_{p \in \hat{P}^i} > \}_{i=1}^{\infty} \subseteq \mathcal{S} \) and a set \( < \hat{P}, \{ B_p \}_{p \in \hat{P}} > \in \mathcal{S} \) such that the following holds

\[
< \hat{P}, \{ B_p \}_{p \in \hat{P}} > = \bigcup_{i=1}^{\infty} < \hat{P}^i, \{ B_p^i \}_{p \in \hat{P}^i} >.
\]

We prove it also holds that

\[
\mathcal{P}^* < \hat{P}, \{ B_p \}_{p \in \hat{P}} > = \bigcup_{i=1}^{\infty} \mathcal{P}^* < \hat{P}^i, \{ B_p^i \}_{p \in \hat{P}^i} >.
\]

We prove this as follows. Define the set \( \hat{P}^* = \hat{P} \cup (\cup_{i=1}^{\infty} \hat{P}^i) \). Since \( \hat{P}^* \) is a countable union of finite sets, it is at most countable. We denote the probability space \( (\otimes_{p \in \hat{P}}, R, \otimes_{p \in \hat{P}} R, \otimes_{p \in \hat{P}^*} \rho) \) as the product measure space where \( \otimes_{p \in \hat{P}}, R = \{ \hat{\psi} : \hat{\psi} : \hat{P}^* \to R \} \) is the product space corresponding to \( R \) with index set \( \hat{P}^* \); \( \otimes_{p \in \hat{P}}, R \) is the product \( \sigma \)-field; \( \otimes_{p \in \hat{P}^*} \rho \) is denoted as the associated product measure (see [9]).

Define the map \( T : \mathcal{S} \to \otimes_{p \in \hat{P}^*} R \) as

\[
T(< \hat{P}, \{ B_p \}_{p \in \hat{P}}>) = \{ \hat{\psi} : \hat{\psi} : \hat{P}^* \to R : (\forall p \in \hat{P}' \cap \hat{P}^* (\hat{\psi}(p) \in B_p)) \}.
\]
Hence, it follows that $T(\langle \hat{P}_i, \mathcal{B}_p \rangle_\mathcal{P}) \subseteq \hat{P}_i$. This means there exists a $\hat{P}_i$ such that $\tilde{\psi} \in T(\mathcal{U}_{i=1}^{\infty} \langle \hat{P}_i, \mathcal{B}_p \rangle_\mathcal{P})$. Suppose that we have that $\tilde{\psi} \in \mathcal{U}_{i=1}^{\infty} \langle \hat{P}_i, \mathcal{B}_p \rangle_\mathcal{P}$. Then, there exists a $\hat{P}_i$ such that $\tilde{\psi} \in T(\langle \hat{P}_i, \mathcal{B}_p \rangle_\mathcal{P})$. Hence, $\tilde{\psi} \in \mathcal{U}_{i=1}^{\infty} \langle \hat{P}_i, \mathcal{B}_p \rangle_\mathcal{P}$. By equality 1.3, we have that $\tilde{\psi} \in T(\langle \hat{P}_i, \mathcal{B}_p \rangle_\mathcal{P})$. Hence, $\tilde{\psi} \in \mathcal{U}_{i=1}^{\infty} \langle \hat{P}_i, \mathcal{B}_p \rangle_\mathcal{P}$. By equality 1.3, this implies $\tilde{\psi} \in T(\langle \hat{P}_i, \mathcal{B}_p \rangle_\mathcal{P})$. Hence, $\tilde{\psi} \in \mathcal{U}_{i=1}^{\infty} \langle \hat{P}_i, \mathcal{B}_p \rangle_\mathcal{P}$. By equality 1.3, this implies $\tilde{\psi} \in T(\langle \hat{P}_i, \mathcal{B}_p \rangle_\mathcal{P})$. Hence, $\tilde{\psi} \in \mathcal{U}_{i=1}^{\infty} \langle \hat{P}_i, \mathcal{B}_p \rangle_\mathcal{P}$. By equality 1.3, this implies $\tilde{\psi} \in T(\langle \hat{P}_i, \mathcal{B}_p \rangle_\mathcal{P})$. Hence, $\tilde{\psi} \in \mathcal{U}_{i=1}^{\infty} \langle \hat{P}_i, \mathcal{B}_p \rangle_\mathcal{P}$.

Hence, $\mathcal{P}'$ defines a measure on $\mathcal{S}$.

The probability space $(\Omega, \mathcal{F}, \mathcal{P})$ is then completely defined as $\mathcal{P} := \sigma(\mathcal{S})$ and $\mathcal{P}$ is defined to be the extension of the measure $\mathcal{P}'$ on the defined $\sigma$-field $\mathcal{F}$ by the Caratheodory Extension Theorem.

Step 3: We finish the proof of the theorem. We have defined the appropriate probability space $(\Omega, \mathcal{F}, \mathcal{P})$. For $p \in P$, define $X_p(\psi) := \psi(p)$ and $A := \Omega$.

This completes the proof of the theorem.

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Competing Interests

Author has declared that no competing interests exist.
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