An extension of the Motzkin–Straus theorem to non-uniform hypergraphs and its applications

Yuejian Peng, Hao Peng, Qingsong Tang, Cheng Zhao

College of Mathematics, Hunan University, Changsha 410082, PR China
College of Sciences, Northeastern University, Shenyang, 110819, PR China
Department of Mathematics and Computer Science, Indiana State University, Terre Haute, IN, 47809, USA

A R T I C L E   I N F O
Article history:
Received 18 November 2014
Received in revised form 25 June 2015
Accepted 26 June 2015
Available online 21 July 2015

Keywords:
Lagrangians of hypergraphs
Turán density
Extremal problems

A B S T R A C T
In 1965, Motzkin and Straus established a remarkable connection between the order of a maximum clique and the Lagrangian of a graph and provided a new proof of Turán’s theorem using the connection. The connection of Lagrangians and Turán densities can be also used to prove the fundamental theorem of Erdős–Stone–Simonovits on Turán densities of graphs. Very recently, the study of Turán densities of non-uniform hypergraphs has been motivated by extremal poset problems and suggested by Johnston and Lu. In this paper, we attempt to explore the applications of Lagrangian method in determining Turán densities of non-uniform hypergraphs. We first give a definition of the Lagrangian of a non-uniform hypergraph, then give an extension of the Motzkin–Straus theorem to non-uniform hypergraphs whose edges contain 1 or 2 vertices. Applying it, we give an extension of the Erdős–Stone–Simonovits theorem to non-uniform hypergraphs whose edges contain 1 or 2 vertices. Our approach follows from the approach in Keevash’s paper Keevash (2011).

1. Introduction and main results

Turán problems on uniform hypergraphs have been actively studied. In 1965, Motzkin and Straus provided a new proof of Turán’s theorem based on a remarkable connection between the order of a maximum clique and the Lagrangian of a graph in [9]. In fact, the connection of Lagrangians and Turán densities can be used to give another proof of the fundamental theorem of Erdős–Stone–Simonovits on Turán densities of graphs in [8]. This type of connection aroused interests in the study of Lagrangians and Motzkin–Straus type results of uniform hypergraphs. For example, in [13], Talbot studied properties of Lagrangians of uniform hypergraphs; in [10], Rota Bulò and Pelillo gave an extension of the Motzkin–Straus theorem to uniform hypergraphs. Very recently, the study of Turán densities of non-uniform hypergraphs has been motivated by extremal poset problems (see [4] and [5]). In this paper, we attempt to explore the applications of Lagrangian method in determining Turán densities of non-uniform hypergraphs. We first give a definition of the Lagrangian of a non-uniform hypergraph, then give an extension of the Motzkin–Straus theorem to non-uniform hypergraphs whose edges contain 1 or 2 vertices. Applying it, we give an extension of the Erdős–Stone–Simonovits theorem to non-uniform hypergraphs whose edges contain 1 or 2 vertices.

A hypergraph $H = (V(H), E(H))$ consists of a vertex set $V(H)$ and an edge set $E(H)$, where every edge in $E(H)$ is a subset of $V(H)$. The set $R(H) = \{|F| : F \in E\}$ is called the set of edge types of $H$. We also say that $H$ is a $R(H)$-graph. For example, if $R(H) = \{1, 2\}$, then we say that $H$ is a $\{1, 2\}$-graph. If all edges have the same cardinality $k$, then $H$ is a $k$-uniform hypergraph.

* Corresponding author.
E-mail addresses: ypeng1@hnu.edu.cn (Y. Peng), hpeng@hnu.edu.cn (H. Peng), t_qsong@sina.com.cn (Q. Tang), cheng.zhao@indstate.edu (C. Zhao).
A 2-uniform hypergraph is called a graph. A hypergraph is non-uniform if it has at least two edge types. For any \( k \in R(H) \), the level hypergraph \( H^k \) is the hypergraph consisting of all edges with \( k \) vertices of \( H \). We write \( H^R_n \) for a hypergraph \( H \) on \( n \) vertices with \( R(H) = R \). An edge \( \{i_1, i_2, \ldots, i_k\} \) in a hypergraph is simply written as \( i_1i_2 \cdots i_k \) throughout the paper.

For an integer \( n \), let \([n]\) denote the set \( \{1, 2, \ldots, n\} \). For a set \( V \) and integer \( i \), let \( (V)^i \) be the family of all \( i \)-subsets of \( V \). The complete hypergraph \( K^r_n \) is a hypergraph on \( n \) vertices with edge set \( \bigcup_{i \in R} \binom{[n]}{r} \). For example, \( K^r_n \) is the complete \( r \)-uniform hypergraph on \( n \) vertices. \( K^r_n \) is the non-uniform hypergraph with all possible edges of cardinality at most \( k \). The complete graph on \( n \) vertices \( K^r_n \) is also called a clique. We also let \( [k]^T \) represent the complete \( T \)-graph on vertex set \( [k] \).

Let us briefly review the Turán problem on uniform hypergraphs. For a given \( r \)-uniform hypergraph \( F \) and positive integer \( n \), let \( \text{ex}(n, F) \) be the maximum number of edges an \( r \)-uniform hypergraph on \( n \) vertices can have without containing \( F \) as a subgraph. By a standard averaging argument of Katona, Nemeth, and Simonovits in [7], \( \frac{\text{ex}(n, F)}{\binom{n}{r}} \) decreases as \( n \) increases, therefore \( \lim_{n \to \infty} \frac{\text{ex}(n, F)}{\binom{n}{r}} \) exists. This limit is called the Turán density of \( F \) and denoted by \( \pi(F) \). Turán’s theorem [14] implies that \( \pi(K^{(2)}) = 1 - \frac{1}{\sqrt{1 + \Delta}} \). The fundamental result in extremal graph theory due to Erdős–Stone–Simonovits generalizes Turán’s theorem and it says that for a graph \( F \) with chromatic number \( \chi(F) \geq 3 \), then \( \pi(F) = 1 - \frac{1}{\chi(F) - 1} \). However, we know quite little about Turán density of \( r \)-uniform hypergraphs for \( r \geq 3 \) though some progress has been made.

A useful tool in extremal problems of uniform hypergraphs (graphs) is the Lagrangian of a uniform hypergraph (graph).

**Definition 1.1.** Let \( G \) be an \( r \)-uniform graph with vertex set \([n]\) and edge set \( E(G) \). Let \( S = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 1, x_i \geq 0 \text{ for } i = 1, 2, \ldots, n\} \). For \( \bar{x} = (x_1, x_2, \ldots, x_n) \in S \), define

\[
\lambda(G, \bar{x}) = \sum_{i_1 \leq i_2 \cdots \leq i_r \in E(G)} x_{i_1} x_{i_2} \cdots x_{i_r}.
\]

The Lagrangian of \( G \), denoted by \( \lambda(G) \), is defined as

\[
\lambda(G) = \max_{\bar{x} \in S} \lambda(G, \bar{x}).
\]

Motzkin and Straus in [9] shows that the Lagrangian of a graph is determined by the order of its maximum cliques.

**Theorem 1.1 (Motzkin and Straus [9]).** If \( G \) is a graph in which a largest clique has order \( l \), then \( \lambda(G) = \lambda(K^{(2)}_l) = \lambda([l]^{(2)}) = \frac{1}{2}(1 - \frac{1}{l}) \).

This connection provided another proof of Turán’s theorem. More generally, the connection of Lagrangians and Turán densities can be used to give another proof of the Erdős–Stone–Simonovits result (see Keevash’s survey paper [8]). In 1980s, Sidorenko [11] and Frankl and Füredi [2] developed the method of applying Lagrangians in determining hypergraph Turán densities. More applications of Lagrangians can be found in [3,12] and [8]. Very recently, the study of Turán densities of non-uniform hypergraphs have been motivated by the study of extremal poset problems [4,5]. A generalization of the concept of Turán density to a non-uniform hypergraph was given in [6] by Johnston and Lu.

For a non-uniform hypergraph \( G \) on \( n \) vertices, the Lubell function of \( G \) is defined to be

\[
h_n(G) = \sum_{k \in R(G)} \frac{|E(G^k)|}{\binom{n}{k}}.
\]

Given a family of hypergraphs \( \mathcal{F} \) with common set of edge-types \( R \), the Turán density of \( \mathcal{F} \) is defined to be

\[
\pi(\mathcal{F}) = \lim_{n \to \infty} \max \{h_n(G) : |V(G)| = n, R(G) \subseteq R, \text{ and } G \text{ contains no subgraph in } \mathcal{F}\}.
\]

In [6], it is shown that \( \max \{h_n(G) : |V(G)| = n, R(G) \subseteq R, \text{ and } G \text{ contains no subgraph in } \mathcal{F}\} \) decreases as \( n \) increases, hence the limit exists. For a hypergraph \( F \), \( \pi([F]) \) is simply written as \( \pi(F) \).

**Definition 1.2.** For a hypergraph \( H \) with \( n \) vertices and positive integers \( s_1, s_2, \ldots, s_n \), the blowup of \( H \) is a new hypergraph \( (V, E) \), denoted by \( H(s_1, s_2, \ldots, s_n) \), satisfying

1. \( V = \bigcup_{i=1}^{n} V_i \) is a union of \( n \) pairwise disjoint sets, where \( |V_i| = s_i \);
2. \( E \) is obtained by replacing each edge \( F \in E(H) \) by a complete \( |F| \)-partite \( |F| \)-uniform hypergraph with partition sets \( \bigcup_{i \in F} V_i \).

**Remark 1.2.** For a non-uniform hypergraph \( G \) on \( n \) vertices, the blowup of \( G \) has the following property:

\[
h_{id}(G(t, t, \ldots, t)) \leq h_n(G).
\]

This can be verified easily by a direct calculation.
The Lagrangian of a $k$-uniform graph $G$ is the supremum of the densities of blowups of $G$ multiplying the constant $\frac{1}{k!}$ (see [8]). We define the Lagrangian of a non-uniform hypergraph as follows so that the Lagrangian of a non-uniform hypergraph $H$ is the supremum of the densities of blowups of $H$.

**Definition 1.3.** For a hypergraph $H_n^R$ with $R(H) = R$ and a vector $\bar{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, define

$$
\lambda'(H_n^R, \bar{x}) = \sum_{j \in R} \left( j! \sum_{i_1 \leq i_2 \ldots \leq i_j \in H} x_{i_1} x_{i_2} \ldots x_{i_j} \right).
$$

**Definition 1.4.** Let $S_n = \{\bar{x} = (x_1, x_2, \ldots, x_n) : \sum_{i=1}^n x_i = 1, x_i \geq 0 \text{ for } i = 1, 2, \ldots, n\}$. The Lagrangian of $H_n^R$, denoted by $\lambda'(H_n^R)$, is defined as

$$
\lambda'(H_n^R) = \max\{\lambda'(H_n^R, \bar{x}) : \bar{x} \in S_n\}.
$$

The value $x_i$ is called the weight of the vertex $i$. We call $\bar{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ a legal weighting for $H_n$ if $\bar{x} \in S_n$. A vector $\bar{y} \in S_n$ is called an optimal weighting for $H_n$ if $\lambda'(H_n, \bar{y}) = \lambda'(H_n)$.

**Remark 1.3.** The connection between **Definitions 1.1** and 1.4 is that, if $G$ is a $k$-uniform graph, then

$$
\lambda'(G) = k! \lambda(G).
$$

In this paper, we will prove the following generalization of the Motzkin–Straus result to $\{1, 2\}$-graphs.

**Theorem 1.4.** If $H$ is a $\{1, 2\}$-graph and the order of its maximum complete $\{1, 2\}$-subgraph is $t$ (where $t \geq 2$), then $\lambda'(H) = \lambda'(K_t^{1,2}) = 2 - \frac{1}{t}$.

As an application of **Theorem 1.4**, we will also prove an extension of the Erdős–Stone–Simonovits result to $\{1, 2\}$-graphs as given in the following theorem. This result was proved by Johnston and Lu in [8] using a different approach. Our motivation is to explore the applications of Lagrangian method in the Turán problem. The approach follows from the approach in Keevash’s survey paper [8].

**Theorem 1.5.** If $H$ is a $\{1, 2\}$-graph and $H^2$ is not bipartite, then $\pi(H) = 2 - \frac{1}{\chi(H^2)-1}$.

2. Proofs of the main results

For $\bar{x} = (x_1, x_2, \ldots, x_n)$, let $\sigma(\bar{x}) = \{i \in [n] : x_i > 0\}$. We will impose an additional condition on any optimal weighting $\bar{x} = (x_1, x_2, \ldots, x_n)$ for a hypergraph $H$:

(*) $|\sigma(\bar{x})|$ is minimal, i.e., if $\bar{y}$ is a legal weighting for $H$ satisfying $|\sigma(\bar{y})| < |\sigma(\bar{x})|$, then $\lambda'(G, \bar{y}) < \lambda'(G)$.

2.1. Proof of Theorem 1.4

We need the following two lemmas.

**Lemma 2.1.** Let $\bar{x} = (x_1, x_2, \ldots, x_n)$ be an optimal weighting of a hypergraph $H$, then $\forall i, j \in \sigma(\bar{x})$,

$$
\frac{\partial \lambda'(H, \bar{x})}{\partial x_i} = \frac{\partial \lambda'(H, \bar{x})}{\partial x_j}.
$$

**Proof.** Suppose, for a contradiction, that there exist $i$ and $j$ in $\sigma(\bar{x})$ such that $\frac{\partial \lambda'(H, \bar{x})}{\partial x_i} > \frac{\partial \lambda'(H, \bar{x})}{\partial x_j}$. We define a new legal weighting $\bar{y}$ for $H$ as follows. Let $y_l = x_l$ for $l \neq i, j$, $y_i = x_i + \delta$ and $y_j = x_j - \delta \geq 0$, then

$$
\lambda'(H, \bar{y}) - \lambda'(H, \bar{x}) = \delta \left( \frac{\partial \lambda'(H, \bar{x})}{\partial x_i} - \frac{\partial \lambda'(H, \bar{x})}{\partial x_j} \right) - \delta^2 \frac{\partial^2 \lambda'(H, \bar{x})}{\partial x_i \partial x_j} > 0
$$

for some small enough $\delta$, contradicting to that $\bar{x}$ is an optimal vector. Hence Lemma 2.1 holds. ■

**Lemma 2.2.** Let $\bar{x} = (x_1, x_2, \ldots, x_n)$ be an optimal weighting of a hypergraph $H$ satisfying (*), then $\forall i, j \in \sigma(\bar{x})$ and $i \neq j$, there exists an edge $e \in E(H)$ such that $\{i, j\} \subseteq e$. 
Proof. Suppose, for a contradiction, that there exist \( i \) and \( j \) in \( \sigma(\vec{x}) \) such that \( \{i, j\} \not\subseteq e \) for any \( e \in E(H) \). We define a new weighting \( y \) for \( H \) as follows. Let \( y_l = x_l \) for \( l \neq i, j, y_i = x_i + x_j \) and \( y_j = x_j - x_i = 0 \), then \( y \) is clearly a legal weighting for \( H \), and

\[
\lambda'(H, y) - \lambda'(H, \vec{x}) = x_j \left( \frac{\partial \lambda'(H, \vec{x})}{\partial x_i} - \frac{\partial \lambda'(H, \vec{x})}{\partial x_j} \right) - x_j^2 \frac{\partial^2 \lambda'(H, \vec{x})}{\partial x_i \partial x_j} = 0.
\]

So \( y \) is an optimal vector and \( |\sigma(y)| = |\sigma(\vec{x})| - 1 \), contradicting the minimality of \( |\sigma(\vec{x})| \). Hence Lemma 2.2 holds. ■

Proof of Theorem 1.4. Clearly, \( \lambda'(H) \geq \lambda'(K_t^1) = 2 - \frac{1}{t} \).

Now we proceed to show that \( \lambda'(H) \leq \lambda'(K_t^1) = 2 - \frac{1}{t} \). Let \( \vec{x} = (x_1, x_2, \ldots, x_n) \) be an optimal weighting of \( H \) satisfying (*) with \( k \) positive weights. Without loss of generality, we may assume that \( x_1 \geq x_2 \geq \cdots \geq x_k > x_{k+1} = x_{k+2} = \cdots x_n = 0 \). By Lemma 2.2, \( \forall 1 \leq i < j \leq k, \forall i, j \in H^2 \).

Claim 2.3. \( \forall 1 \leq i \neq j \leq k \), if \( i \in H^1 \) but \( j \not\in H^1 \), then \( x_i - x_j = 0.5 \).

Proof of Claim 2.3. By Lemma 2.1, \( \frac{\partial \lambda'(H, \vec{x})}{\partial x_i} = \frac{\partial \lambda'(H, \vec{x})}{\partial x_j} \). By Lemma 2.2, \( \forall 1 \leq i \neq j \leq k, i, j \in H^2 \), therefore \( 1 + 2(1 - x_i) = 2(1 - x_j) \), i.e. \( x_i - x_j = 0.5 \). ■

Claim 2.4. Either \( i \in H^1 \) for all \( 1 \leq i \leq k \) or \( i \not\in H^1 \) for all \( 1 \leq i \leq k \).

Proof of Claim 2.4. Assume that there are \( 1 \)-sets of \( \{1, 2, \ldots, k\} \) in \( H^1 \), if \( l = k \), then \( i \in H^1 \) for all \( 1 \leq i \leq k \). Now we can assume that \( l+1 < k \). Without loss of generality, assume that \( i \in H^1 \) for \( 1 \leq i \leq l \) and \( j \not\in H^1 \) for \( l+1 \leq j \leq k \). By Claim 2.3, \( x_1 = x_2 = 0.5, \forall 1 \leq i \leq l \) and \( l+1 \leq j \leq k \). Then \( l \leq 1 \). Otherwise, \( x_1 = x_k + 0.5 \) and \( x_2 = x_k + 0.5 \), contradicts to \( \sum_{i=1}^{k} x_i = 1 \) and \( x_i > 0 \) for \( 1 \leq i \leq k \). If \( l = 1 \), then \( x_1 = 0.5 + \frac{0.5}{k}, x_2 = x_3 = \cdots = x_k = \frac{0.5}{k} \) and

\[
\lambda'(H, \vec{x}) = \lambda'(K_k^{1,2}) = 2 - \frac{1}{k} \leq 2 - \frac{1}{l}.
\]

So Claim 2.4 holds. ■

Let us continue the proof of Theorem 1.4.

If \( i \in H^1 \) for all \( 1 \leq i \leq k \), then \( K_k^{1,2} \) is a subgraph of \( H \). Since \( t \) is the order of the maximum complete \( \{1, 2\} \)-graph of \( H \), then \( k \leq t \). We have

\[
\lambda'(H, \vec{x}) = \lambda'(K_k^{1,2}) = 2 - \frac{1}{k} \leq 2 - \frac{1}{t}.
\]

If \( i \not\in H^1 \) for all \( 1 \leq i \leq k \), then

\[
\lambda'(H, \vec{x}) = \lambda'(K_k^{2}) = 1 - \frac{1}{k} \leq 2 - \frac{1}{t}.
\]

2.2. Proof of Theorem 1.5

In [6], Johnston and Lu showed that the Turán density of a blowup of a hypergraph \( H \) is the same as the Turán density of \( G \).

Lemma 2.5 (Theorem 9 in [6]). Let \( \mathcal{F} \) be a finite family of hypergraphs and let \( s \geq 2 \). Then \( \pi(\mathcal{F}(s, s, \ldots, s)) = \pi(\mathcal{F}) \), where \( \mathcal{F}(s, s, \ldots, s) = \{F(s, s, \ldots, s) \mid F \in \mathcal{F}\} \).

Applying this result, one can obtain a result for general hypergraphs similar to Corollary 8 in [1] by Baber and Talbot for uniform hypergraphs.

Corollary 2.6. If any member of a family \( \mathcal{F} \) of hypergraphs is contained in a blow-up of a hypergraph \( G \), then \( \pi(\mathcal{F}) = \pi(\mathcal{F} \cup G) \).
Proof. Note that $\pi(F) \geq \pi(F \cup G)$ is trivial. Let us show another direction. Suppose $G' \in F$ and $G'$ is contained in $G(s, s, \ldots, s)$ for some $s \geq 1$. Lemma 2.5 implies that $\pi(F \cup G(s, s, \ldots, s)) = \pi(F \cup G).$ Hence

$$\pi(F) = \pi(F \cup G') \leq \pi(F \cup G(s, s, \ldots, s)) = \pi(F \cup G).$$

Let $F$ and $G$ be hypergraphs. We say that a function $f : V(F) \to V(G)$ is a homomorphism from $F$ to $G$ if it preserves edges and edge types, i.e., $f(i_1)f(i_2)\cdots f(i_k) \in E(G^k)$ for all $i_1i_2\cdots i_k \in E(F^k)$, where $k \in R(F)$. We say that $G$ is $F$-hom-free if there is no homomorphism from $F$ to $G$.

Remark 2.7. If $G$ is $F$-hom-free, then $G$ is $F$-free.

Proof of Remark 2.7. If $G$ is not $F$-free, then $G$ contains a copy of $F$ as a subgraph. Let $f : V(F) \to V(G)$ be the function defined by $f(v) = v$ for every $v \in V(F)$. Then $f$ is a homomorphism from $F$ to $G$. So $G$ is not $F$-hom-free. ■

Remark 2.8. $G$ is $F$-hom-free if and only if the blowup $G(s, s, \ldots, s)$ is $F$-free for every $s$.

Proof of Remark 2.8. If $G$ is not $F$-hom-free, then there exists a function $f : V(F) \to V(G)$ which is a homomorphism from $F$ to $G$. Let $s = \max\{|f^{-1}(v)| : v \in V(G)\}$. Then $G(s, s, \ldots, s)$ contains $F$ as a subgraph.

Assume that $G(s, s, \ldots, s)$ contains $F$ as a subgraph for some $s$. Then for each $v \in V(F)$, $v$ is contained in a set of some vertices of $G(s, s, \ldots, s)$, which are blowed up by a vertex $w \in V(G)$. Let $f(v) = w$. Then $f$ is a homomorphism from $F$ to $G$. ■

Remark 2.9. If $H$ is a $\{1, 2\}$-graph and $t = \chi(H^2)$, then a complete $\{1, 2\}$-graph $K_n^{(1,2)}$ is $H$-hom-free if and only if $l \leq t - 1$.

Proof of Remark 2.9. Apply Remark 2.8. ■

We can make an analogous definition to the Turán density:

$$\pi_{hom}(F) = \lim_{n \to \infty} \max\{h_n(G) : \left|V(G)\right| = n, G \subseteq K_n^{R(F)}, \text{ and } G \text{ is } F\text{-hom-free}\}.$$ 

Then we have two useful lemmas.

Lemma 2.10. $\pi_{hom}(F) = \pi(F)$.

Proof of Lemma 2.10. If $G$ is $F$-hom-free, then by Remark 2.7, $G$ is $F$-free. So $\pi(F) \geq \pi_{hom}(F)$. Now we show another direction. Note that if hypergraph $H$ is not $F$-hom-free, then $H$ contains a subhypergraph on at most $|F|$ vertices that is not $F$-hom-free. Let $F$ be the finite family of all hypergraphs that are not $F$-hom-free on at most $|F|$ vertices. Note that if $H$ is not $F$-hom-free then $F$ is contained in a blow-up of $H$. Applying Corollary 2.6 repeatedly, we have $\pi(F) = \pi(F \cup F')$. Note that any graph that is $F \cup F'$-free is also $F$-hom-free (since the homomorphic image of $F$ in $H$ would be a copy of some member of $F$). Hence $\pi(F) = \pi(F \cup F') \leq \pi_{hom}(F)$. ■

Lemma 2.11. $\pi(F)$ is the supremum of $\lambda'(G)$ over all $F$-hom-free $G$ with $R(G) \subseteq R(F)$.

Proof of Lemma 2.11. Suppose that $F$ is a hypergraph and $G$ is an $F$-hom-free hypergraph with $n$ vertices and $R(G) \subseteq R(F)$. Let $s = (s_1, s_2, \ldots, s_m)$ be an optimal vector of $\lambda'(G)$. Take any $m$, note that $G(s_1m, s_2m, \ldots, s_mn)$ is an $F$-free hypergraph on $p = \sum_{i=1}^{m}s_i$ vertices with $R(G) \subseteq R(F)$ and

$$h_p(G(ms_1, ms_2, \ldots, ms_m)) = \sum_{j \in R(G)} \sum_{i_1 \leq \ldots \leq i_j \in E(G)} \left(\frac{s_{i_1}s_{i_2}\cdots s_{i_j}}{\binom{im}{j}}\right) \to \lambda'(G, s) \quad \text{as } m \to \infty.$$ 

So $\pi(F) \geq \lambda'(G, s) = \lambda'(G)$.

By a direct calculation as given in [7] and [6], for a hypergraph $G$ with $n$ vertices, the average of $h_n(S)$ over all induced subhypergraphs $S$ of $G$ with $n - 1$ vertices is equal to $h_n(G)$. Therefore, $\max\{h_n(G) : \left|V(G)\right| = n, R(G) \subseteq R(F), \text{ and } G \text{ is } F\text{-hom-free}\}$ decreases as $n$ increase, and converges down to $\pi(F)$. Hence for any fixed $n$, if we take an $F$-hom-free $H$ with $n$ vertices and $R(H) \subseteq R(F)$ such that $h_n(H) = \max\{h_n(G) : \left|V(G)\right| = n, R(G) \subseteq R(F), \text{ and } G \text{ is } F\text{-hom-free}\}$, then $\pi(F) \leq h_n(H)$. Note that

$$\lambda'(H) \geq \lambda'\left(H, \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)\right)$$

$$= \sum_{j \in R(H)} \sum_{i_1 \leq \ldots \leq i_j \in E(H)} \left(\frac{1}{n}\right)^j$$

$$\geq h_n(H) - \epsilon \quad \text{when } n \text{ is large enough.}$$ 

So, $\pi(F) \leq \lambda'(H) + \epsilon$.

Therefore, $\pi(F)$ is the supremum of $\lambda'(G)$ over all $F$-hom-free hypergraphs $G$ with $R(G) \subseteq R(F)$. ■
To continue the proof of Theorem 1.5, we define a dense hypergraph.

Definition 2.1. A hypergraph $G$ is dense if every proper subgraph $G'$ satisfies $\lambda'(G') < \lambda'(G)$.

Remark 2.12. By Theorem 1.1 and Remark 1.3, a graph $G$ is dense if and only if $G$ is $K_t^{[2]}$. By Theorem 1.4, a $\{1,2\}$-graph $G$ with $\chi(G^2) > 2$ is dense if and only if $G$ is $K_t^{(1,2)}$ for some $t > 2$.

Proof of Theorem 1.5. Assume that $H$ is a $\{1,2\}$-graph and $H^2$ is not bipartite. By Lemma 2.11, $\pi(H)$ is the supremum of the Lagrangians of all $H$-hom-free $\{1,2\}$-graphs, all $H$-hom-free graphs and all $H$-hom-free $\{1\}$-graphs. So $\pi(H)$ is the supremum of the Lagrangians of all dense $H$-hom-free $\{1,2\}$-graphs, all dense $H$-hom-free graphs and all $H$-hom-free $\{1\}$-graphs. Let $t = \chi(H^2) \geq 3$. By Remarks 2.9 and 2.12, a dense $H$-hom-free $\{1,2\}$-graph must be $K_t^{(1,2)}$, $2 \leq l \leq t-1$ and a dense $H$-hom-free graph must be $K_l$. Also, note that the Lagrangian of all $\{1\}$-graphs is 1. So,

$$\pi(H) = \max\{\lambda'(K_t^{[1,2]}), \lambda'(K_t), 1\} = \max\left\{2 - \frac{1}{t-1}, 1 - \frac{1}{s}, 1\right\} = 2 - \frac{1}{t-1}. \quad \blacksquare$$

3. Remarks

For hypergraphs other than 2-graphs or $\{1,2\}$-graphs, no result similar to Theorem 1.4 holds. For example, take the $\{1,r\}$-graph $G$ whose edge set is $[l][1] \cup [l-1][r] \cup \{i_1 \cdots i_{t-r-1} : i_1 \cdots i_{t-1} \in [l-2][l-1] \cup \{1 \cdots (r-2)(l-1)(l-1)\}$. Let $\bar{x} = (x_1, \ldots, x_l)$, where $x_1 = x_2 = \cdots = x_{t-2} = \frac{2}{l-1}$ and $x_{t-1} = x_t = \frac{2}{l-1}$. Then $\lambda'(G) \geq \lambda'(G, \bar{x}) > \lambda'(l-1)[r]$. If $H$ is a $\{2,3\}$-hypergraph in which the order of its maximum complete $\{2\}$-subgraph is $s$ and the order of its maximum complete $\{2,3\}$-subgraphs is $t$, when $s \geq f(t, \alpha_1)$, then $\lambda'(s)[2] > \lambda'(t)[2,3]$.

Acknowledgments

We thank both reviewers for reading the manuscript carefully, checking all the details and giving insightful comments to help improve the manuscript. We are thankful to a reviewer for pointing out an error in the proof of Lemma 2.10 and offering a correct proof for it. The proof of Lemma 2.10 in the current version is based on this reviewer’s proof.

The first author was partially supported by National Natural Science Foundation of China (No. 11271116).

References

[1] R. Baber, J. Talbot, New Turán densities for 3-graphs, Electron. J. Combin. 19 (2) (2012) P22.
[2] P. Frankl, Z. Füredi, Extremal problems and the Lagrange function of hypergraphs, Bull. Inst. Math. Acad. Sin. 16 (1988) 305–313.
[3] P. Frankl, Z. Füredi, Extremal problems whose solutions are the blow-ups of the small Witt-designs, J. Combin. Theory Ser. A 52 (1989) 129–147.
[4] J.R. Griggs, G.O.H. Katona, No four subsets forming an N, J. Combin. Theory Ser. A. 115 (2008) 677–685.
[5] J.R. Griggs, L. Lu, On families of subsets with a forbidden subposet, Combin. Probab. Comput. 18 (2009) 731–748.
[6] T. Johnston, L. Lu, Turán problems on non-uniform hypergraphs, Electron. J. Combin. 21 (4) (2014) P4.22.
[7] G. Katona, T. Nemeth, M. Simonovits, On a graph problem of Turán, Mat. Lapok 15 (1964) 228–238.
[8] P. Keevash, Hypergraph Turán Problems, Surveys in Combinatorics, Cambridge University Press, 2011, pp. 83–140.
[9] T.S. Motzkin, E.G. Straus, Maxima for graphs and a new proof of a theorem of Turán, Canad. J. Math. 17 (1965) 533–540.
[10] S. Rota Bulò, M. Pelillo, A generalization of the Motzkin–Straus theorem to Hypergraphs, Optim. Lett. 3 (2009) 287–295.
[11] A.F. Sidorenko, The maximal number of edges in a homogeneous hypergraph containing no prohibited subgraphs, Math. Notes 41 (1987) 247–259. Translated from Mat. Zametki.
[12] A.F. Sidorenko, Solution of a problem of Bollobas on 4-graphs, Mat. Zametki 41 (1987) 433–455.
[13] J. Talbot, Lagrangians of hypergraphs, Combin. Probab. Comput. 11 (2002) 199–216.
[14] P. Turán, On an extremal problem in graph theory, Mat. Fiz. Lapok 48 (1941) 436–452 (in Hungarian).