BOURTOUX ANSATZ FOR THE DEGENERATE THIRD PAINLEVÉ TRANSCENDENTS

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ABSTRACT. For a general solution of the degenerate third Painlevé equation we show the Boutroux ansatz near the point at infinity. It admits an asymptotic representation in terms of the Weierstrass $\wp$-function in cheese-like strips along generic directions. The expression is obtained by using isomonodromy deformation of a linear system governed by the degenerate third Painlevé equation.

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1. Introduction

In the geometrical study of the spaces of initial values for Painlevé equations, Sakai \[25\] classified the third Painlevé equations into three types $P_{III}(D_6)$, $P_{III}(D_7)$ and $P_{III}(D_8)$. For the types $P_{III}(D_7)$ and $P_{III}(D_8)$ Ohyama et al. \[24\] examined basic properties including $\tau$-functions, irreducibility, the spaces of initial values. Equation $P_{III}(D_8)$ is changed into a special case of $P_{III}(D_6)$. Equation $P_{III}(D_7)$ is called the degenerate third Painlevé equation or degenerate $P_{III}$, which may be normalised in the form

\[ v_{\xi\xi} = v^2 \xi - v \xi^2 + \frac{a}{\xi} + \frac{1}{v} \]

($v_\xi = dv/d\xi$) with $a \in \mathbb{C}$. The change of variables

\[ 2\xi = \epsilon b\tau^2, \quad v = \epsilon \tau u \]

takes this equation to the equivalent equation discussed in \[17\], \[18\]

\[ u_{\tau\tau} = \frac{u^2}{u} - \frac{u}{\tau^2} + \frac{1}{\tau}(-8\epsilon u^2 + 2ab) + \frac{b^2}{u} \]

with $\epsilon = \pm 1$, $a \in \mathbb{C}$, $b \in \mathbb{R} \setminus \{0\}$, which governs isomonodromy deformation of linear system \[3.1\]. Using isomonodromy system \[3.1\], Kitaev and Vartanian \[17\], \[18\] obtained asymptotic solutions of \[3.1\] as $\tau \to \pm \infty$, $\pm i\infty$ and $\tau \to \pm 0, \pm i0$, with connection formulas among them. Furthermore, for \[3.1\], a special meromorphic solution is studied by \[16\], \[19\], and truncated solutions are given by \[27\].

As mentioned in \[17\], \[27\], in physical and geometrical applications, degenerate $P_{III}$ appears in contexts independent of $P_{III}(D_6)$, i.e. complete $P_{III}$, and its significant analytic properties are important. Indeed behaviours of solutions of \[3.1\] along real and
imaginary axes [17], [18] are quite different from those for complete $P_{III}$ [12]. For complete $P_{III}$ of the Sine-Gordon type, Novokshenov [22], [23], [5, Chap. 16] provided an asymptotic representation of solutions in terms of the sn-function along generic directions near the point at infinity. It is meaningful to establish the counterpart of this expression for degenerate $P_{III}$.

In this paper we show the Boutroux ansatz [2] for degenerate $P_{III}$, that is, present an elliptic asymptotic representation for a general solution along generic directions near the point at infinity. The main results are described in Section 2. As in Theorems 2.1 and 2.2, degenerate $P_{III}$ admits a general solution written in terms of the Weierstrass $\wp$-function, and so does $P_I$ ([7], [8], [14], [15]). On the other hand for $P_{II}$, $P_{IV}$, $P_{III}(D_6)$ (of Sine-Gordon type) and $P_V$, elliptic asymptotic solutions are given by the sn-function ([5], [9], [10], [11], [15], [20], [21], [22], [23], [20], [28]). This fact reflects the position of degenerate $P_{III}$, i.e. $P_{III}(D_7)$ in the degeneration scheme of the Painlevé equations [24], [25].

For our purpose it is appropriate to treat an equation of the form

$$y'' = \frac{(y')^2}{y} - \frac{y'}{x} - 2y^2 + \frac{3a}{x} + \frac{1}{y}$$

$(y' = dy/dx)$, which comes from (1.1) via the substitution

$$\epsilon \tau u = (x/3)^2 y, \quad \epsilon \tau^2 = 2(x/3)^3.$$

Equation (1.2) with $x = e^{i\phi}t$ governs isomonodromy deformation of the linear system

$$\frac{d\Psi}{d\lambda} = \frac{t}{3} B(\lambda, t) \Psi,$$

$$B(\lambda, t) = -i e^{i\phi} \lambda \sigma_3 + \begin{pmatrix} 0 & -2 i e^{i\phi} y \\ \Gamma_0(t, y, y')/y & 0 \end{pmatrix}$$

$$- (\Gamma_0(t, y, y') + 3(1/2 + i a)t^{-1}) \lambda^{-1} \sigma_3 + 2 e^{i\phi} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \lambda^{-2},$$

where $y$ and $y'$ are arbitrary complex parameters, and

$$\Gamma_0(t, y, y') = \frac{y'}{y} - \frac{i e^{i\phi}}{y} - (1 + 3 ia)t^{-1}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

As shown in Section 3 system (1.4) is a result of transformation of system (3.1) treated in [17], [18]. The isomonodromy deformation of (3.1) is governed by equation (1.1), and solutions of (1.1) are labelled by coordinates on the monodromy manifold for (3.1) defined by Stokes matrices and a connection matrix $G = (g_{ij}) \in SL_2(\mathbb{C})$ for matrix solutions around $\mu = 0$ and $\mu = \infty$. System (1.4) admits the same monodromy manifold as of (3.1), which is described by the same Stokes matrices and $G$ for suitably chosen matrix solutions (cf. Proposition 3.2), so that solutions of (1.1) and (1.2) are labelled by the same monodromy data.
Applying WKB analysis we solve the direct monodromy problem for linear system (1.4) in Section 5 and obtain key relations in Corollary 5.2 containing the monodromy data and certain integrals, which lead to a solution of an inverse problem. Basic necessary materials for this calculation are summarised in Section 4. Asymptotic properties of these integrals are examined in Section 6 by the use of the \( \vartheta \)-function, and from these formulas asymptotic forms in the main theorems are derived in Section 7. Then the justification as a solution of (1.2) is made along the line of Kitaev [13], [15]. The final section is devoted to the Boutroux equations, which determine the modulus contained in the elliptic representation of solutions.

Throughout this paper we use the following symbols:

1. \( \sigma_1, \sigma_2, \sigma_3 \) are the Pauli matrices

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};
\]

2. for complex-valued functions \( f \) and \( g \), we write \( f \ll g \) or \( g \gg f \) if \( f = O(|g|) \), and write \( f \asymp g \) if \( g \ll f \ll g \).

2. Main results

To state our main results we give some explanations on necessary facts.

2.1. Monodromy data. Isomonodromy system (3.1) admits the matrix solutions

\[
Y_k^\infty(\mu) = (I + O(\mu^{-1}))\mu^{-(1/2+ia)\sigma_3}\exp(-i\tau\mu^2\sigma_3)
\]

as \( \mu \to \infty \) through the sector \( \arg\mu + \arg\tau^{1/2} - \pi k/2 < \pi/2 \), and

\[
X_k^0(\mu) = (i/\sqrt{2})\Theta_0^\sigma_3(\sigma_1 + \sigma_3 + O(\mu))\exp(-i\sqrt{\tau\epsilon}b\mu^{-1}\sigma_3)
\]

as \( \mu \to 0 \) through the sector \( \arg\mu - \arg(\tau\epsilon b)^{1/2} - \pi k < \pi \), where \( k \in \mathbb{Z} \) (see Section 3.2). Let the invariant Stokes matrices and a connection matrix be such that \( Y_{j+1}^\infty(\mu) = Y_j^\infty(\mu)S_j^\infty \), \( X_{j+1}^0(\mu) = X_j^0(\mu)S_j^0 \) with \( j \in \mathbb{Z} \) and that \( Y_0^\infty(\mu) = X_0^0(\mu)G \). These are

\[
S_0^\infty = \begin{pmatrix} 1 & 0 \\ s_0^\infty & 1 \end{pmatrix}, \quad S_1^\infty = \begin{pmatrix} 1 & s_1^\infty \\ 0 & 1 \end{pmatrix}, \quad S_0^0 = \begin{pmatrix} 1 & 0 \\ s_0^0 & 1 \end{pmatrix}
\]

with (3.7), and \( G = (g_{ij}) \) with \( g_{11}g_{22} - g_{12}g_{21} = 1 \). The monodromy manifold is given by \( GS_0^\infty S_1^\infty \sigma_3 e^{\pi(i/2-\epsilon)a}\sigma_3 = S_0^0\sigma_1G \), whose generic points admit the coordinates expressed by \( G \) [17]. Solutions \( u(\tau) \) of (1.1) and \( y(x) \) of (1.2) related via (1.3) are labelled by the same monodromy data.

2.2. Elliptic curve and Boutroux equations. For \( A \in \mathbb{C} \) around \( A = 3 \cdot 2^{2/3} \) the polynomial \( 4z^3 - Az^2 + 1 \) has roots \( z_0, z_1 \) close to \( 2^{-1/3} \) and \( z_2 \) close to \( -4^{-2/3} \), and especially, \( z_0 = z_1 = 2^{-1/3}, z_2 = -4^{-2/3} \) when \( A = 3 \cdot 2^{2/3} \). Let \( \Pi_+ \) and \( \Pi_- \) be the copies of \( P^1(\mathbb{C}) \setminus ([\infty, z_2] \cup [z_0, z_1]) \) and set \( \Pi_A = \Pi_+ \cup \Pi_- \) glued along the cuts \( [\infty, z_2] \) and \( [z_0, z_1] \), where \( \text{Re} \, z \to -\infty \) along \( [\infty, z_2] \). Then \( \Pi_A \) is the elliptic curve given by

\[
w(A, z)^2 = 4z^3 - Az^2 + 1,
\]
where the branch of \( \sqrt[4]{z^3 - Az^2 + 1} := 2\sqrt{z - z_0}\sqrt{z - z_1}\sqrt{z - z_2} \) is chosen in such a way that \( \text{Re} \sqrt{z - z_j} \to +\infty \) as \( z \to \infty \) along the positive real axis on the upper plane \( \Pi_+ \). The elliptic curve \( \Pi_A \) does not degenerate as long as \( A \neq 3 \cdot 2^{2/3} e^{2\pi i m/3} \) \((m = 0, \pm 1)\), that is, \( 4z^3 - Az^2 + 1 \) has no double roots, and then we may define \( \Pi_A \) continuously.

As will be shown in Section 8, for any \( \phi \in \mathbb{R} \), there exists \( A_\phi \in \mathbb{C} \) with \( \Pi_{A_\phi} \) such that, for every cycle \( c \) on \( \Pi_{A_\phi} \)

\[
\text{Im} e^{i\phi} \int_c \frac{w(A_\phi, z)}{z^2} \, dz = 0,
\]

and that \( A_\phi \) has the properties (Proposition 8.15):
1. for every \( \phi \), \( A_\phi \) is uniquely determined;
2. \( A_\phi \) is continuous in \( \phi \in \mathbb{R} \), and is smooth in \( \phi \in \mathbb{R} \setminus \{k\pi/3 \mid k \in \mathbb{Z}\} \);
3. \( A_{\phi + 2\pi/3} = e^{\pm 2\pi i/3} A_\phi \); \( A_{\phi + \pi} = A_\phi \); \( A_{-\phi} = \overline{A_\phi} \);
4. \( \Pi_{A_\phi} \) degenerates if and only if \( \phi = k\pi/3 \) with \( k \in \mathbb{Z} \), and then \( A_0 = 3 \cdot 2^{2/3} \); \( A_{\pm \pi/3} = e^{\mp 2\pi i/3} A_0 \); \( A_{\pm \pi} = A_0 \).

In particular, for \( 0 < |\phi| < \pi/3 \) let us consider \( A_\phi \) for specified cycles. For \( A_\phi \) close to \( A_0 = 3 \cdot 2^{2/3} \), by Proposition 8.16 number the roots of \( w(A_\phi, z)^2 \) close to \( 2^{-1/3} \) in such a way that \( \text{Im} z_0 \leq \text{Im} z_1 \) if \( \phi > 0 \) (respectively, \( \text{Im} z_1 \leq \text{Im} z_0 \) if \( \phi < 0 \)) and let the numbering be retained as long as coalescence does not occur. Then for \( 0 < |\phi| < \pi/3 \) we have basic cycles \( a \) and \( b \) on \( \Pi_{A_\phi} \), which are drawn on \( \Pi_+ \) as in Figure 1. For

\[|\phi| < \pi/3\]

the cycles \( a \) and \( b \) may be defined continuously on \( \Pi_{A_\phi} \), and the Boutroux equations are given by

\[
(2.1) \quad \text{Im} e^{i\phi} \int_a \frac{w(A_\phi, z)}{z^2} \, dz = 0, \quad \text{Im} e^{i\phi} \int_b \frac{w(A_\phi, z)}{z^2} \, dz = 0
\]

admitting a unique solution \( A_\phi \). For \( |\phi| < \pi/3 \) the periods of \( \Pi_{A_\phi} \) along \( a \) and \( b \) are defined by

\[
\Omega^a_\phi = \Omega_a = \int_a \frac{dz}{w(A_\phi, z)}, \quad \Omega^b_\phi = \Omega_b = \int_b \frac{dz}{w(A_\phi, z)}
\]

which satisfy \( \text{Im} \Omega_b/\Omega_a > 0 \).
2.3. Main theorems. Let \( y(x) = y(G, x) \) be a solution of (1.2) labelled by the monodromy data \( G = (g_{ij}) \in SL_2(\mathbb{C}) \). Then we have the following, in which \( \varphi(u; g_2, g_3) \) is the Weierstrass \( \varphi \)-function satisfying \( \varphi^2 = 4\varphi^3 - g_2\varphi - g_3 \) ([6], [29]).

**Theorem 2.1.** Suppose that \( 0 < \phi < \pi/3 \) and that \( G = (g_{ij}) \) satisfies \( g_{11}g_{12}g_{22} \neq 0 \). Then

\[
y(x) = \varphi(i(x - x_0^+)) + O(x^{-\delta}) ; g_2(A_\phi), g_3(A_\phi)) + \frac{A_\phi}{12}
\]
as \( x = te^{i\phi} \to \infty \) through the cheese-like strip

\[
S(\phi, t_\infty, \kappa_0, \delta_0) = \{ x = te^{i\phi} | \text{Re} t > t_\infty, \text{Im} t < \kappa_0 \} \cup \{ |x - \sigma| < \delta_0 \}
\]
with

\[
P = \{ \sigma | \varphi(i(\sigma - x_0^+); g_2(A_\phi), g_3(A_\phi)) = \infty \} = \{ x_0^+ - i\Omega_a \mathbb{Z} - i\Omega_b \mathbb{Z} \}.
\]
Here \( \delta \) is some positive number, \( \kappa_0 \) a given positive number, \( \delta_0 \) a given small positive number, \( t_\infty = t_\infty(\kappa_0, \delta_0) \) a sufficiently large number depending on \( (\kappa_0, \delta_0) \); and

\[
g_2(A_\phi) = \frac{A_\phi^2}{12}, \quad g_3(A_\phi) = \frac{A_\phi^3}{216} - 1,
\]

\[
-ix_0^+ = \frac{i}{2\pi} \left( \Omega_a \log \frac{g_{21}}{g_{22}} - \Omega_b (\log (g_{11}g_{22}) - \pi i) \right) - i\Omega_0 \mod \Omega_a \mathbb{Z} + \Omega_b \mathbb{Z}
\]
with

\[
\Omega_0 = \int_{\infty}^{0} \frac{dz}{w(A_\phi, z)},
\]
in which \( 0^+ \) denotes \( 0 \in \Pi_+ \) and the contour \([\infty, 0^+] \subset \Pi_+ \) contains the line from \(-\infty \) to \( z_2 \) along the upper shore of the cut \([\infty, z_2]\).

**Theorem 2.2.** Suppose that \(-\pi/3 < \phi < 0 \) and that \( G = (g_{ij}) \) satisfies \( g_{11}g_{21}g_{22} \neq 0 \). Then \( y(x) \) admits an asymptotic representation of the same form as in Theorem 2.1 with the phase shift

\[
-ix_0^- = \frac{i}{2\pi} \left( \Omega_a \log \frac{g_{21}}{g_{22}} + \Omega_b (\log (g_{11}g_{22}) - \pi i) \right) - i\Omega_0 \mod \Omega_a \mathbb{Z} + \Omega_b \mathbb{Z}.
\]

**Remark 2.1.** From a relation in the proof of Theorem 2.1 we have an expression of \( y'(x) \) for \( 0 < \phi < \pi/3 \) and \(-\pi/3 < \phi < 0 \) of the form

\[
\frac{i y'(x) + 1}{2y(x)^2} = \varphi(i(x - \hat{x}_0^+)) + O(x^{-\delta}) ; g_2(A_\phi), g_3(A_\phi)) + \frac{A_\phi}{12},
\]
respectively, where \( ix_0^\pm = \hat{x}_0^\pm + \Omega_0 \).

The expressions of \( y(x) \) in Theorems 2.1 and 2.2 are determined by \( A_\phi \) and \( x_0 = x_0^+ \) for \( 0 < \phi < \pi/3, \hat{x}_0^+ \) for \(-\pi/3 < \phi < 0 \). Since \( \Omega_{a,b} \) and \( \Omega_0 \) depend on \( A_\phi \), these may be denoted by \( \Omega^\phi_{a,b} \) and \( \Omega^\phi_0 \), respectively. To emphasise this fact, write

\[
y(x) = P(A_\phi, x_0(G, \Omega^\phi_{a,b}, \Omega^\phi_0); x)
\]
for \( 0 < |\phi| < \pi/3 \).
For $\phi$ such that $|\phi - 2m\pi/3| < \pi/3 (m \in \mathbb{Z})$, set $\Omega^\phi_{a,b} = e^{2m\pi i/3} \Omega^{\phi - 2m\pi/3}_{a,b}$. The period, say, $\Omega^\phi_{a}$ may be expressed by the integral on $\Pi_{+}$

$$
\Omega^\phi_{a} = \int_{e^{2m\pi i/3}a}^{d} \frac{dz}{w(A_{g}z)} = \int_{e^{2m\pi i/3}a}^{d} \frac{dz}{w(e^{2m\pi i/3}A_{\phi-2m\pi/3}z)}
$$

Further, for $|\phi - 2m\pi/3| < \pi/3$ set $\Omega^\phi_{0} = e^{2m\pi i/3} \Omega^{\phi - 2m\pi/3}_{0}$. The following provides an analytic continuation of $y(x)$ beyond the sector $|\phi| < \pi/3$.

**Theorem 2.3.** Suppose that $0 < |\phi - 2m\pi/3| < \pi/3$ (respectively, $-\pi/3 < \phi - 2m\pi/3 < 0$) for $m \in \mathbb{Z} \setminus \{0\}$. Then $y(x)$ admits the expression

$$
y(x) = y(G, x) = P(A_{g}, x_{0}(G^{(m)}, \Omega_{a}, \Omega_{b}, \Omega_{0}^{\phi}); x)
$$

as $x = te^{i\phi} \to \infty$ through the cheese-like strip $S(\phi, t_{\infty}, \kappa_{0}, \delta_{0})$, if $g_{11}^{(m)}g_{12}^{(m)}g_{22}^{(m)} \neq 0$ (respectively, $g_{11}^{(m)}g_{21}^{(m)}g_{22}^{(m)} \neq 0$), where

$$
G^{(m)} = \begin{cases}
(S_{0}^{0})^{m}G_{3}^{m}e^{(m\pi/3)(a-i/2)\sigma_{3}} & \text{if } m \geq 1; \\
(S_{1}S_{0}^{0})^{m}G_{3}^{m}e^{(m\pi/3)(i/2-a)\sigma_{3}} & \text{if } m = -n \leq -1.
\end{cases}
$$

**Remark 2.2.** The matrix $G^{(m)}$ has another expression of the form

$$
G^{(m)} = \begin{cases}
G(S_{0}^{0}S_{1}S_{3}^{m}e^{(i/2-a)\sigma_{3}})^{m}S_{3}^{m}e^{(m\pi/3)(a-i/2)\sigma_{3}} & \text{if } m \geq 1; \\
G(S_{3}^{m}e^{(a-i/2)\sigma_{3}}S_{1}S_{0}^{0})^{m}S_{3}^{m}e^{(m\pi/3)(i/2-a)\sigma_{3}} & \text{if } m = -n \leq -1.
\end{cases}
$$

2.4. Examples. For simplicity suppose that $\epsilon = 1$ and $b = 2$ in equation (1.4). Let $G = (g_{ij})$ with $g_{11}g_{22} - g_{12}g_{21} = 1$ be the monodromy data in Kitaev-Vartanian [17], [18], which coincide with ours above. Suppose that $g_{11}g_{12}g_{21}g_{22} \neq 0$. Then [17] Theorem 3.1], [18] Theorems 2.1 and 2.3 with $\epsilon_{1} = \epsilon_{2} = 0$ provide general solutions of (1.4) as in the following examples, in which we write $l(g_{11}g_{22}) = i(2\pi)^{-1}\log(g_{11}g_{22})$.

**Example 2.1.** If $|\text{Re}(l(g_{11}g_{22}))| < 1/6$, equation (1.4) admits a solution of the form

$$
u(\tau) = 2^{-1/3} \tau^{1/3} + 21^{3}3^{-1/4}e^{3\pi i/4}l(g_{11}g_{22})^{1/2}\cosh(\chi(\tau)),
$$

$$
\chi(\tau) = i2^{1/3}3\tau^{2/3} + l(g_{11}g_{22})\log(2^{1/3}3^{2}\tau^{2/3}) + \gamma(g_{11}g_{22}, g_{12}/g_{22}) + o(\tau^{-\delta})
$$

as $\tau \to +\infty$, where $\gamma(g_{11}g_{22}, g_{12}/g_{22})$ is a constant expressed by $(g_{11}g_{22}, g_{12}/g_{22})$, and $\delta$ is some positive number.

**Example 2.2.** For $\text{Re}(l(g_{11}g_{22})) \in (0, 1)$, equation (1.4) admits a solution of the form

$$
u(\tau) = 2^{-1/3} \tau^{1/3} \left(1 - \frac{3}{2\sin^{2}(\chi(\tau))/2} \right) = 2^{-1/3} \tau^{1/3} \frac{\sin(\tilde{\chi}(\tau)/2 - \chi_{0})\sin(\tilde{\chi}(\tau)/2 + \chi_{0})}{\sin^{2}(\tilde{\chi}(\tau)/2)}
$$

with

$$
\chi_{0} = -\pi/2 + (i/2)\log(2 + \sqrt{3}),
$$
\[ \tilde{x}(\tau) = 2^{1/3}3^{3/2} \tau^{2/3} + \lambda(g_{11}g_{22}) \log(2^{1/3}3^{3/2} \tau^{2/3}) + \gamma(g_{ij}) + o(\tau^{-\delta}) \]

as \( \tau \to +\infty \) in a strip \( |\text{Im}\, \tau^{2/3}| < 1 \). Here \( \lambda(g_{11}g_{22}) = (2\pi)^{-1} \log(-g_{11}g_{22}) \) (\( \in \mathbb{R} \)) if \( \text{Re} \, l(g_{11}g_{22}) = 1/2 \), and \(-i(l(g_{11}g_{22}) - 1/2) \) otherwise; and \( \gamma(g_{ij}) \) is a constant expressed by \( (\lambda(g_{11}g_{22}), g_{11}g_{12}, g_{21}g_{22}) \) if \( \text{Re} \, l(g_{11}g_{22}) = 1/2 \), and by \( (l(g_{11}g_{22}), g_{11}g_{12}) \) otherwise.

By the change of variables \( \tau^2 = (x/3)^3 \), \( \tau u = (x/3)^2 y \), these solutions are taken to solutions of \( (1.2) \) on the positive real axis. Proposition 3.2 guarantees the transfer between solutions of \( (1.1) \) and \( (1.2) \) with labels. Observing solutions of \( (1.2) \) on the positive real axis. Proposition 3.2 guarantees the transfer between solutions of \( (1.1) \) and \( (1.2) \) with labels. Observing solutions of \( (1.2) \) on the positive real axis.

Example 2.3. Suppose that \( g_{21} \) or \( g_{12} \) is 0 and that \( g_{11}g_{22} = 1 \). Then \( (1.1) \) admits

\[ u(\tau) = 2^{-1/3} \tau^{1/3} + \left( \frac{s_0 - i e^{-\pi a}}{2 \cdot 3^{1/4} \pi^{1/2}} \exp(\epsilon i (2^{1/3}3^{3/2} \tau^{2/3} + k_\pi/4)) \right)(1 + o(\tau^{-\delta})), \]

as \( \tau \to +\infty \). Here \( s_0 = 2 - \sqrt{3}, \epsilon = -1, k = -1 \) if \( g_{21} = 0 \); and \( s_0 = 2 + \sqrt{3}, \epsilon = 1, k = 3 \) if \( g_{12} = 0 \).

If \( g_{11}g_{22}g_{12} \neq 0 \), \( g_{21} = 0 \) (respectively, \( g_{11}g_{22}g_{21} \neq 0 \), \( g_{12} = 0 \)), Theorem 2.1 for \( 0 < \phi < \pi/3 \) (respectively, 2.2 for \(-\pi/3 < \phi < 0\)) applies to the corresponding solution of \( (1.2) \). In the case, say \( g_{21} = 0 \), this solution is represented by the \( \phi \)-function for \( 0 < \phi < \pi/3 \), and is truncated for \(-\pi < \phi < 0\).

3. ISOMONODROMY DEFORMATION AND MONODROMY DATA

3.1. Isomonodromy deformation. Equation \( (1.1) \) governs isomonodromy deformation of the linear system

\[ \frac{dU}{d\mu} = \mathcal{U}(\mu, \tau) U, \]

\[ \mathcal{U}(\mu, \tau) = -2i \tau \mu \sigma_3 + 2\tau \begin{pmatrix} 0 & 2i\epsilon e^{i\varphi} \\ -e^{i\varphi} & 0 \end{pmatrix} \]

\[ -\frac{1}{\mu} \left( ia + \frac{\tau}{2} (u^\tau / u - i \varphi) \right) \sigma_3 + \frac{1}{\mu^2} \begin{pmatrix} 0 & 2\epsilon e^{i\varphi} (ia - i \tau \varphi / 2) \\ -iue^{-i\varphi} & 0 \end{pmatrix} \]

with \( \varphi = (d/d\tau) \varphi = 2a/\tau + b/u \), that is, the monodromy data remain invariant under small change of \( \tau \) if and only if \( u^\tau = (d/d\tau) u \) holds and \( u(\tau) \) solves \( (1.1) \) \([17]\) Propositions 1.1, 1.2 and 2.1]. Let us change \( (3.1) \) into system \( (1.4) \) associated with \( (1.2) \). After the transformation

\[ U = \begin{pmatrix} e^{i\varphi / 2} & 0 \\ 0 & e^{-i\varphi / 2} \end{pmatrix} \begin{pmatrix} \sqrt{e} \tau^{3/4} & 0 \\ 0 & \tau^{-3/4} / \sqrt{e} \end{pmatrix} \tilde{U}, \quad \mu = \sqrt{2/\kappa \tau^{1/2}}, \]

where \( \kappa \) is a constant.cheon
with \( \kappa \) chosen so that \( \epsilon \kappa b = 2 \). Then (3.1) becomes

\[
\frac{dV}{d\mu} = V(\mu, \xi)V,
\]

\[
V(\mu, \xi) = -4i\xi\tilde{\mu}\sigma_3 + \begin{pmatrix} 0 & 4i \\
-\xi(2\xi q^2/q - 2(1 + i\alpha) - 2i\xi/q) & 0 \end{pmatrix} - \frac{1}{\tilde{\mu}} \left( \frac{q^2}{q} - \frac{1}{2} - \frac{i\xi}{q} \right) \sigma_3 - \frac{i}{\tilde{\mu}^2} \begin{pmatrix} 0 & 1/q \\
q & 0 \end{pmatrix}.
\]

The further change of variables

\[
V = \begin{pmatrix} -i\sqrt{-q} & 0 \\
0 & i\sqrt{-q} \end{pmatrix} \Psi, \quad q = (x/3)^2y, \quad \xi = (x/3)^3, \quad q^2 = y^3 + 2y/x, \quad (x/3)\tilde{\mu} = \lambda/2
\]

with \( x = te^{i\phi} \), i.e. \( t = |x|, \phi = \arg x \) and \( y^t = e^{-i\phi/y^t} \) takes the system above to (1.4):

\[
\frac{d\Psi}{d\lambda} = \frac{t}{3} B(\lambda, t)\Psi,
\]

whose right-hand side is written in the form

(3.2) \[ B(\lambda, t) = b_1\sigma_1 + b_2\sigma_2 + b_3\sigma_3; \]

\[
b_1 = -(i/2)(2e^{i\phi}y + i\Gamma_0(t, y, y^t)y^{-1}) + 2ie^{i\phi}\lambda^{-2},
\]

\[
b_2 = (1/2)(2e^{i\phi}y - i\Gamma_0(t, y, y^t)y^{-1}),
\]

\[
b_3 = -ie^{i\phi}\lambda - (\Gamma_0(t, y, y^t) + 3(1/2 + ia)t^{-1})\lambda^{-1},
\]

\[
\Gamma_0(t, y, y^t) = \frac{y^t}{y} - \frac{ie^{i\phi}}{y} - \frac{1 + 3ia}{t}.
\]

In the linear systems above, \( u, u^t; q, q^2; y, y^t \) are arbitrary complex parameters or functions, and \( 2(\epsilon \kappa)^{-1}q^2 = u^t + u/\tau, q^2 = y^3 + 2y/x \) and \( y^t = e^{-i\phi/y^t} \) are compatible with their derivatives.

**Proposition 3.1.** System (1.4) admits isomonodromy property if and only if \( y^t = (d/dt)y \) holds and \( y = y(e^{i\phi}t) = y(x) \) solves equation (1.2).

3.2. Monodromy data. For each \( j \in \mathbb{Z} \) system (1.4) admits the matrix solutions

(3.3) \[ \hat{Y}^\infty_j(\lambda) = (I + O(\lambda^{-1}))\lambda^{-(1/2 + ia)\sigma_3} \exp(-(i/6)e^{i\phi}tl^{2}\sigma_3) \]

as \( \lambda \to \infty \) through the sector \( |\arg \lambda + \phi/2 + j\pi /2| < \pi /2 \), and

(3.4) \[ \hat{Y}_j^0(\lambda) = (i/\sqrt{2})(\sigma_1 + \sigma_3 + O(\lambda)) \exp(-(2i/3)e^{i\phi}t\lambda^{-1}\sigma_3) \]

as \( \lambda \to 0 \) through the sector \( |\arg \lambda - \phi - j\pi| < \pi \). The Stokes matrices are such that

\[
\hat{Y}^\infty_{j+1}(\lambda) = \hat{Y}^\infty_j(\lambda)\hat{S}_j^\infty, \quad \hat{Y}_j^0(\lambda) = \hat{Y}_j^0(\lambda)\hat{S}_j^0,
\]
and the connection matrix $\hat{G} = (\hat{g}_{ij})$ is defined by

$$\hat{Y}_0^\infty(\lambda) = \hat{Y}_0^0(\lambda)\hat{G}, \quad \hat{g}_{11}\hat{g}_{22} - \hat{g}_{12}\hat{g}_{21} = 1.$$  

The Stokes matrices satisfy

$$\hat{S}_{k+2}^\infty = \sigma_3 e^{-\pi(a-i/2)\sigma_3} \hat{S}_k^\infty e^{\pi(a-i/2)\sigma_3} \sigma_3, \quad \hat{S}_k^0 = \sigma_1 \hat{S}_{k+1}^0 \sigma_1,$$

for $k \in \mathbb{Z}$, and the monodromy manifold is given by

$$\hat{G}\hat{S}_0^\infty \hat{S}_1^\infty \sigma_3 e^{\pi(i/2-a)\sigma_3} = \hat{S}_0^0 \sigma_1 \hat{G}$$

with

$$\hat{S}_0^\infty = \begin{pmatrix} 1 & 0 \\ \hat{s}_0^\infty & 1 \end{pmatrix}, \quad \hat{S}_1^\infty = \begin{pmatrix} 1 & \hat{s}_1^\infty \\ 0 & 1 \end{pmatrix}, \quad \hat{S}_0^0 = \begin{pmatrix} 1 & \hat{s}_0^0 \\ 0 & 1 \end{pmatrix}.$$  

These monodromy data and their relations are obtained by the same argument as in [17, Section 2].

Let $G = (g_{ij})$ be the monodromy data for system (3.1) given in [17, 18]. This connection matrix is defined by

$$Y_0^\infty(\mu) = X_0^0(\mu)G.$$  

Here $Y_0^\infty(\mu)$ and $X_0^0(\mu)$ are matrix solutions of system (3.1) as follows:

$$Y_k^\infty(\mu) = (I + O(\mu^{-1}))\mu^{-(1/2+ia)\sigma_3} \exp(-i\tau \mu^2 \sigma_3)$$

as $\mu \to \infty$ through the sector $|\arg \mu + \arg \tau^{1/2} - \pi k/2| < \pi/2$, and

$$X_k^0(\mu) = (i/\sqrt{2})\Theta_0^\sigma(\sigma_1 + \sigma_3 + O(\mu)) \exp(-i\sqrt{\tau \epsilon b} \mu^{-1} \sigma_3),$$

$$\Theta_0 = (\epsilon \tau)^{1/4} \tau^{-1/4} (-ue^{-i\pi/\tau})^{-1/2}$$

as $\mu \to 0$ through the sector $|\arg \mu - \arg(\tau \epsilon b)^{1/2} - \pi k| < \pi$ [17, Proposition 2.2].

Furthermore Stokes matrices are defined by

$$Y_{j+1}^\infty(\lambda) = Y_j^\infty(\lambda)S_j^\infty, \quad X_{j+1}^0(\lambda) = X_j^0(\lambda)S_j^0,$$

and the monodromy manifold for (3.1) is given by

$$GS_0^\infty S_1^\infty \sigma_3 e^{\pi(i/2-a)\sigma_3} = S_0^0 \sigma_1 G$$

with

$$S_0^\infty = \begin{pmatrix} 1 & 0 \\ \hat{s}_0^\infty & 1 \end{pmatrix}, \quad S_1^\infty = \begin{pmatrix} 1 & \hat{s}_1^\infty \\ 0 & 1 \end{pmatrix}, \quad S_0^0 = \begin{pmatrix} 1 & \hat{s}_0^0 \\ 0 & 1 \end{pmatrix}.$$  

For $k \in \mathbb{Z},$

$$S_{k+2}^\infty = \sigma_3 e^{-\pi(a-i/2)\sigma_3} S_k^\infty e^{\pi(a-i/2)\sigma_3} \sigma_3, \quad S_k^0 = \sigma_1 S_{k+1}^0 \sigma_1.$$  

As shown in [17, Theorems 3.1, 3.2, 3.3] and [18, Theorems 2.1, 2.2, 2.3] solutions of (1.1) are parametrised by the coordinates of the monodromy manifold $g_{11}g_{22}, g_{12}/g_{22}, g_{21}/g_{11}$, provided that (3.1) is an isomonodromy system governed by (1.1). The following relation suggests that we are allowed to use the same monodromy invariants in parametrising our solutions of (1.2) as in [17] and [18] (cf. Examples 2.1, 2.2, 2.3).
Proposition 3.2. Let \((Y_{0,0}^{\infty}\,(\lambda), \hat{Y}_{0,0}^{\infty}\,(\lambda)) = (Y_{0,0}^{(-\lambda)}(\lambda)\Theta_{0,0}^{(-\lambda)}(\lambda), \hat{Y}_{0,0}^{(-\lambda)}(\lambda))\) with \(\Theta_{0,0}^{(-\lambda)}(\lambda) = \Theta_{0,0}(\lambda)^{1/2+ia}\).

\(s_{0}\) be a pair of matrix solutions of (1.4) near \(\lambda = \infty\) and 0, where \(c_{0} = (3/2)^{1/2}e^{\pi/2}x^{1/2-1}\).

Then, for this pair, the corresponding Stokes matrices and connection matrix coincide with \(S_{0,0}^{\infty}, S_{1,0}^{\infty}, S_{0}^{0,0}\) and \(G\) for \((Y_{0,0}^{\infty}(\mu), X_{0,0}^{0,0}(\mu))\) of (3.1).

Proof. Note that (3.1) is changed into (1.4) by the transformation \(U = \Theta_{0,0}^{(-\lambda)}(\lambda)^{1/2}G\). Then

\[
\begin{align*}
&\left(\Theta_{0,0}^{(-\lambda)}Y_{0,0}(c_{0}\lambda), \Theta_{0,0}^{(-\lambda)}X_{0,0}(c_{0}\lambda)\right) = \left(\hat{Y}_{0,0}(\lambda)\Theta_{0,0}^{(-\lambda)}c_{0}^{-1/2+ia}\sigma_{3}, \hat{Y}_{0,0}(\lambda)\right) = \left(Y_{0,0}^{(-\lambda)}(\lambda), \hat{Y}_{0,0}^{(-\lambda)}(\lambda)\right)
\end{align*}
\]

solves (1.4). Insertion of this into \(Y_{0,0}^{\infty}(\mu) = X_{0,0}^{0,0}(\mu)G\) yields \(G = G^{*}\). Let \(S_{0,0}^{\infty}, S_{1,0}^{\infty}\) and \(S_{0}^{0,0}\) be the Stokes matrices for \((Y_{0,0}^{\infty}(\lambda), \hat{Y}_{0,0}^{\infty}(\lambda))\). Then the equation of the monodromy manifold is

\[
G_{0,0}^{\infty}S_{1,0}^{\infty}\sigma_{3}\varepsilon^{(i/2-a)\sigma_{3}} = S_{0}^{0,0}\sigma_{1}G,
\]

which yields the entries of \(S_{0,0}^{\infty}, S_{1,0}^{\infty}\) and \(S_{0}^{0,0}\) in terms of \(g_{ij}\) coinciding with those of \(S_{0,0}^{\infty}, S_{1,0}^{\infty}\) and \(S_{0}^{0,0}\) derived from (3.6) as in [17, p. 1172]. This completes the proof.

Remark 3.1. We have \(G = \Theta_{0,0}^{-\sigma_{3}}c_{0}^{-1/2+ia}\sigma_{3} = \Theta_{0,0}^{-\sigma_{3}}, S_{m}^{\infty} = \Theta_{0,0}^{\sigma_{3}}S_{m}^{\infty}\Theta_{0,0}^{-\sigma_{3}}\) and \(S_{m}^{0,0} = \hat{S}_{m}^{\infty}\).

Equation (3.6) of the monodromy manifold may be extended.

Proposition 3.3. For \(m = 1, 2, 3, \ldots\),

\[
\begin{align*}
&G_{0,0}^{\infty}S_{1,0}^{\infty}\cdots S_{m-2,0}^{\infty}S_{m-1,0}^{\infty}\sigma_{3}^{m}e^{m\pi(i/2-a)\sigma_{3}} = \sigma_{0}^{0,0}\cdots S_{m-1,0}^{m}\sigma_{1}^{m}G,
&G_{m-1,0}^{\infty}S_{2,0}^{\infty}\cdots S_{m-2,0}^{\infty}S_{2,0}^{m}\sigma_{3}^{m}e^{m\pi(a-i/2)\sigma_{3}} = S_{m-1,0}^{\infty}\cdots S_{m-1,0}^{m}\sigma_{1}^{m}G.
\end{align*}
\]

Proof. Recall that \(Y_{k,0}^{\infty}(\mu) = \sigma_{3}Y_{k+1,0}^{\infty}(\mu_{e^{\pi\sigma_{3}}}e^{-\pi(a-i/2)\sigma_{3}})\) and \(X_{k,0}^{0,0}(\mu) = \sigma_{3}X_{k+1,0}^{0,0}(\mu_{e^{\pi\sigma_{3}}})\sigma_{1}^{m}\) [17 (24)]. Then

\[
\begin{align*}
Y_{0,0}^{\infty}(\mu)S_{0,0}^{\infty}S_{1,0}^{\infty}\cdots S_{m-2,0}^{\infty}S_{m-1,0}^{\infty} = Y_{0,0}^{\infty}(\mu) = \sigma_{3}Y_{2,0}^{\infty}(\mu_{e^{-\pi\sigma_{3}}}e^{-\pi(a-i/2)\sigma_{3}})
&= \cdots = \sigma_{3}^{m}Y_{0,0}^{\infty}(\mu_{e^{-m\pi\sigma_{3}}}e^{-m\pi(a-i/2)\sigma_{3}}),
Y_{0,0}^{0,0}(\mu)S_{0,0}^{\infty}\cdots S_{m-1,0}^{\infty} = Y_{0,0}^{0,0}(\mu) = \sigma_{3}Y_{m-1,0}^{\infty}(\mu_{e^{-\pi\sigma_{3}}}e^{-\pi(a-i/2)\sigma_{3}})
&= \cdots = \sigma_{3}^{m}Y_{0,0}^{0,0}(\mu_{e^{-m\pi\sigma_{3}}}e^{-m\pi(a-i/2)\sigma_{3}}).
\end{align*}
\]

Using \(Y_{0,0}^{\infty}(\mu) = Y_{0,0}^{0,0}(\mu)G\) and \(Y_{0,0}^{0,0}(\mu_{e^{-m\pi\sigma_{3}}}) = Y_{0,0}^{0,0}(\mu_{e^{-m\pi\sigma_{3}}})\), we have

\[
\begin{align*}
Y_{0,0}^{0,0}(\mu)GS_{0,0}^{\infty}S_{1,0}^{\infty}\cdots S_{m-2,0}^{\infty}S_{m-1,0}^{\infty} = \sigma_{3}^{m}Y_{0,0}^{0,0}(\mu_{e^{-m\pi\sigma_{3}}}e^{-m\pi(a-i/2)\sigma_{3}})G\sigma_{3}^{m}e^{m\pi(a-i/2)\sigma_{3}}
&= \sigma_{1}^{m}Y_{0,0}^{0,0}(\mu_{e^{-m\pi\sigma_{3}}}e^{-m\pi(a-i/2)\sigma_{3}})G\sigma_{3}^{m}e^{m\pi(a-i/2)\sigma_{3}},
\end{align*}
\]

which implies the first relation.

The formulas above are also written as follows:

Proposition 3.4. For \(m = 1, 2, 3, \ldots\),

\[
G_{0,0}^{\infty}S_{1,0}^{\infty}\cdots S_{m-2,0}^{\infty}S_{m-1,0}^{\infty} = (S_{0,0}^{0,0})^{m}G\sigma_{3}^{m}e^{m\pi(a-i/2)\sigma_{3}},
G_{m-1,0}^{\infty}S_{2,0}^{\infty}\cdots S_{m-2,0}^{\infty}S_{m-2,0}^{m} = (\sigma_{1}^{0,0})^{m}G\sigma_{3}^{m}e^{m\pi(i/2-a)\sigma_{3}}.
\]
Proof. By (3.7) \( S^j_{j-1} \sigma_i^j = \sigma_i S^j_{j-2} \sigma_i^{j-1} = \cdots = \sigma_i^{j+1} S^0 \sigma_i , \) and hence
\[
S^0 \cdots S^m_{m-1} \sigma_i^m G = (S^0 \sigma_i)^m G, \quad S^0_{-1} \cdots S^m_{-m} \sigma_i^m G = (\sigma_i S^0)^m G.
\]
Combining these with Proposition 3.3, we have the desired result.

\( \square \)

4. WKB analysis

4.1. Turning points and Stokes graphs. Let us examine the characteristic roots \( \pm \mu = \pm \mu(t, \lambda) \) of \( B(t, \lambda) \), the turning points, i.e. the roots of \( \mu \), and the Stokes graph, which are used in calculating monodromy data for system (1.4). The characteristic roots are given by

\[
(4.1) \quad \mu^2 = b_i^2 + b_i^2 + b_i^2
\]

\[
= - e^{2i\phi} \lambda^2 + e^{2i\phi} a_\phi \lambda^{-2} - 4e^{2i\phi} \lambda^{-4} + 3ie^{i\phi}(1 + 2ia)\zeta^{-1}
\]

with

\[
(4.2) \quad a_\phi = a_\phi(t) = e^{-2i\phi} \left( \frac{y'}{y} + \frac{1}{2t} \right)^2 + 4y + \frac{1}{y^2} - 3ie^{-i\phi}(1 + 2ia)\zeta^{-1}.
\]

The Stokes graph consists of the Stokes curves and the vertices: each Stokes curve is defined by \( Re \lambda \). The Stokes graph consists of the Stokes curves and the vertices: each Stokes curve is defined by \( Re \lambda \). The vertices are turning points or singular points. Every turning point is simple, and the two-sheeted Riemann surface glued along cuts with ends of turning points or singular points.

First suppose that \( \phi = 0 \). If \( a_0 = a_\phi = 0 = 3 \cdot 2^{2/3} \), then

\[
\mu(\infty, \lambda)^2 |_{\phi = 0} = -\lambda^2 + a_0 \lambda^{-2} - 4\lambda^{-4} = -\lambda^{-4}(\lambda^2 - 2^{1/3})^2(\lambda^2 + 2^{1/3}).
\]

This means that \( \mu(t, \lambda) \) admits six turning points \( \lambda_0, \lambda_1, \lambda_0', \lambda_1', \lambda_2, \lambda_2' \) such that \( \lambda_0 \) and \( \lambda_1 \) coalesce at \( 2^{1/6}, \lambda_0' \) and \( \lambda_1' \) at \( -2^{1/6} \) as \( t \to \infty \), and that \( \lambda_2 \) and \( \lambda_2' \) approach \( \pm 2^{1/3} i \), respectively. The Stokes graph with \( \phi = 0 \) is used in [17, Section 4]. (Note that a solution \( y(x) \) of (1.2) for \( x = t > 0 \) corresponds to \( u(\tau) \) satisfying (1.1) for \( \tau > 0 \) if \( eb > 0 \).) The limit Stokes graph with \( t = \infty \) is as in Figure 2 (c) and \( \mu(\lambda) \) is defined on the two-sheeted Riemann surface \( \mathcal{R}_0 \) glued along, say \([\lambda_2, e^{\pi i/2}] \cup [\lambda_2', e^{-\pi i/2}]\).

The limit Stokes graph for the isomonodromy system (1.4) is considered to reflect the Boutroux equations (2.1). When \( \phi \) increases or decreases, the limit turning points for \( \lambda_0 \) and \( \lambda_1 \) move according to the solution \( A_\phi \) of the Boutroux equations (2.1). By Proposition 8.16 for \( \phi \) close to 0, the double turning point at \( 2^{1/6} \) is resolved into two simple turning points such that \( Im \lambda_0 > 0 > Im \lambda_1 \), \( Re \lambda_0 < 2^{1/6} < Re \lambda_1 \) if \( \phi > 0 \), and that \( Im \lambda_0 < 0 < Im \lambda_1 \), \( Re \lambda_0 < 2^{1/6} < Re \lambda_1 \) if \( \phi < 0 \). As will be shown in Proposition 8.15 for \( 0 < |\phi| < \pi/3 \) the coalescence of turning points does not occur, and then topological properties of the limit Stokes graph remain invariant. Every turning point is simple, and the two-sheeted Riemann surface \( \mathcal{R}_\phi \) of \( \mu(\lambda) \) is glued along the cuts \([\lambda_0, \lambda_1], [\lambda_0', \lambda_1']\) and \([\lambda_2, e^{(\pi - \phi)i/2}] \cup [\lambda_2', e^{-(\pi + \phi)i/2}]\). The Stokes graph lies on the upper sheet of \( \mathcal{R}_\phi \). For \(-\pi/3 < \phi < 0\) and \( 0 < \phi < \pi/3 \), the limit Stokes graphs are as in Figures 2 (b) and (d), in which each limit turning point with \( t = \infty \) is also denoted by \( \lambda_1 \) or \( \lambda_1' \). In
our calculation, for $0 < |\phi| < \pi/3$, we use the Stokes curve from 0 to $\infty$ passing through $\lambda_0$ and $\lambda_1$ appearing as a resolution of the double turning point. For a technical reason, the cut $[\lambda_0, \lambda_1]$ on the upper sheet is made in such a way that the Stokes curve $(\lambda_0, \lambda_1)^\sim$ is located along the lower shore (respectively, the upper shore) of the cut if $0 < \phi < \pi/3$ (respectively, $-\pi/3 < \phi < 0$); and the cut $[0, \lambda_2]$ in such a way that the cut $[\lambda_0, \lambda_1]$ is located on the right-hand side of $[0, \lambda_2]$ (cf. Figures 4, 5).

Let us set

$$\mu(t, \lambda) = i e^{i\phi} \lambda^{-2} \sqrt{4 - a_0 \lambda^2 + \lambda^6 - 3i e^{-i\phi}(1 + 2ia)\lambda^4 t^{-1}}.$$  

Here the branch of the square root is fixed in such a way that

$$-ie^{-i\phi} \lambda^2 \mu(\infty, \lambda) = 2 \sqrt{(1 - \lambda_{0,\infty}^{-2}\lambda^2)(1 - \lambda_{1,\infty}^{-2}\lambda^2)(1 - \lambda_{2,\infty}^{-2}\lambda^2)}$$

$$= 2 \sqrt{1 - \lambda_{0,\infty}^{-2}\lambda^2} \sqrt{1 - \lambda_{1,\infty}^{-2}\lambda^2} \sqrt{1 - \lambda_{2,\infty}^{-2}\lambda^2}$$

with $\lambda_{j,\infty} = \lambda_j(\infty)$, $\lambda_{0,\infty}^{-2}\lambda_{1,\infty}^{-2}\lambda_{2,\infty}^{-2} = -4$, and that $\sqrt{1 - \lambda_{j,\infty}^{-2}\lambda^2} \to 1$ as $\lambda \to 0$ on the upper sheet. Then $\mu(t, \lambda) \to -ie^{i\phi}\lambda + O(1)$ as $\lambda \to \infty$ and $\mu(t, \lambda) \to 2ie^{i\phi}\lambda^{-2} + O(1)$ as $\lambda \to 0$ on the upper sheet.

Figure 2. Limit Stokes graphs for $|\phi| \leq \pi/3$

An unbounded domain $D \subset \mathcal{R}_\phi$ is called a canonical domain if, for each $\lambda \in D$, there exist contours $C_{\pm}(\lambda) \subset D$ terminating in $\lambda$ such that

$$\text{Re} \int_{\lambda_-}^\lambda \mu(\lambda)d\lambda \to -\infty \quad \text{(respectively, Re} \int_{\lambda_+}^\lambda \mu(\lambda)d\lambda \to +\infty \text{)}$$
as $\lambda_- \to \infty$ along $C_-(\lambda)$ (respectively, as $\lambda_+ \to \infty$ along $C_+(\lambda)$) (see [4], [5, p. 242]). The interior of a canonical domain contains exactly one Stokes curve, and its boundary consists of Stokes curves.

4.2. WKB solution. The following WKB solution will be used in our calculus.

**Proposition 4.1.** In the canonical domain whose interior contains a Stokes curve issuing from the turning point $\lambda_0$ or $\lambda_1$, system (1.4) with $B(\lambda, t)$ given by (3.2) admits an asymptotic solution expressed as

$$\Psi_{\text{WKB}}(\lambda) = T(I + O(t^{-\delta})) \exp \left( \int_{\lambda_s}^{\lambda} \Lambda(\tau) d\tau \right), \quad T = \begin{pmatrix} 1 & \frac{b_3 - \mu}{b_1 + ib_2} \\ \frac{b_1 - \mu}{b_1 - ib_2} & 1 \end{pmatrix},$$

outside suitable neighbourhoods of zeros of $b_1 \pm ib_2$ as long as $|\lambda - \lambda_s| \gg t^{-2/3+(2/3)\delta}$ ($i = 0, 1, 2$). Here $\delta$ is an arbitrary number such that $0 < \delta < 1$, $\lambda_s$ is a base point near $\lambda_0$ or $\lambda_1$, and

$$\Lambda(\lambda) = \frac{t}{3} \mu(t, \lambda) \sigma_3 - \text{diag} T^{-1} T_\lambda.$$

**Proof.** This is shown by using $\mu = -i e^{i\phi} \lambda + O(1)$ near $\lambda = \infty$, and $= 2i e^{i\phi} \lambda^{-2} + O(1)$ near $\lambda = 0$ (cf. [5, Theorem 7.2], [26, Proposition 3.8]). \[\square\]

**Remark 4.1.** In the proposition above

$$\text{diag} T^{-1} T_\lambda = \frac{1}{2\mu(\mu + b_3)} (i(b_1 b'_2 - b'_1 b_2) \sigma_3 + (b_3 \mu' - b'_3 \mu) I)$$

$$= \frac{1}{4} \left( 1 - \frac{b_3}{\mu} \right) \frac{\partial}{\partial \lambda} \log \frac{b_1 + ib_2}{b_1 - ib_2} \sigma_3 + \frac{1}{2} \frac{\partial}{\partial \lambda} \log \frac{\mu}{\mu + b_3} t,$$

where $b'_1 = (\partial/\partial \lambda)b_1$.

4.3. Local solution around a turning point. Near turning points the WKB solution above fails in expressing asymptotic behaviour. In the neighbourhood of $\lambda_s$ system (1.4) is reduced to

$$\frac{dW}{d\zeta} = \begin{pmatrix} 0 & 1 \\ \zeta & 0 \end{pmatrix} W,$$

which has the solutions $^T\text{Ai}(\zeta), \text{Ai}_\zeta(\zeta)$, $^T\text{Bi}(\zeta), \text{Bi}_\zeta(\zeta)$ with the Airy function $\text{Ai}(\zeta)$ and $\text{Bi}(\zeta) = e^{-\pi i/6} \text{Ai}(e^{2\pi i/3} \zeta)$ [1], [3]. Then we have the following solution near each simple turning point [5, Theorem 7.3], [26, Proposition 3.9].

**Proposition 4.2.** For each simple turning point $\lambda_i$ ($i = 0, 1, 2$) write $c_k = b_k(\lambda_i)$, $c'_k = (b_k)_i(\lambda_i)$ ($k = 1, 2, 3$), and suppose that $c_k$, $c'_k$ are bounded and $c_1 \pm ic_2 \neq 0$. Let

$$\hat{t} = 2(2\kappa_c)^{-1/3}(c_1 - ic_2)(t/3)^{1/3}$$

with $\kappa_c = c_1 c'_1 + c_2 c'_2 + c_3 c'_3$. Then system (1.4) admits a matrix solution of the form

$$\Phi_i(\lambda) = T_i(I + O(t^{-\delta})) \begin{pmatrix} 1 & 0 \\ 0 & \hat{t}^{-1} \end{pmatrix} W(\zeta), \quad T_i = \begin{pmatrix} 1 & -\frac{c_3}{c_1 + ic_2} \\ \frac{c_3}{c_1 + ic_2} & 1 \end{pmatrix}.$$
in which \( \lambda - \lambda_i = (2\kappa^-)_{1/3}(t/3)^{-2/3}(\zeta + \zeta_0) \) with \(|\zeta_0| \ll t^{-1/3} \), as long as \(|\zeta| \ll t^{(2/3 - \delta')/3} \), that is, \(|\lambda - \lambda_i| \ll t^{-2/3 + (2/3 - \delta')/3} \). Here \( \delta' \) is an arbitrary number such that \( 0 < \delta' < 2/3 \), and \( W(\zeta) \) solves system (1.4) having canonical solutions \( W_\nu(\zeta) (\nu \in \mathbb{Z}) \) such that

\[
W_\nu(\zeta) = \zeta^{-(1/4)\nu} (\sigma_3 + \sigma_1) (I + O(\zeta^{-3/2})) \exp((2/3)\zeta^{3/2}\sigma_3)
\]

as \( \zeta \to \infty \) through the sector \( |\arg \zeta - (2\nu - 1)\pi/3| < 2\pi/3 \), and that \( W_{\nu+1}(\zeta) = W_\nu(\zeta) S_\nu \) with

\[
S_1 = \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix}, \quad S_{\nu+1} = \sigma_1 S_\nu \sigma_1.
\]

**Remark 4.2.** Putting \( \lambda - \lambda_i = (2\kappa^-)_{1/3}(e^{2\pi i/3})^j (t/3)^{-2/3}(\zeta + \zeta_0), \) \( j \in \{0, \pm 1\} \), we have an expression of \( \Phi_i(\lambda) \) with \( \hat{t} = 2(2\kappa^-)^{-1/3}(e^{2\pi i/3})^{2j}(c_1 - ic_2)(t/3)^{1/3} \).

5. Calculation of the connection matrix

We calculate the connection matrix \( \hat{G} = (\hat{g}_{ij}) \) given by (5.3) as a solution of the direct monodromy problem by applying WKB analysis to system (1.4). Suppose that \( a_\phi(t) \) is given by (4.2) with a pair of arbitrary functions \( (y, y') = (y(t), y'(t)) \) not necessarily solving (1.2), and that

\[
a_\phi(t) = A_\phi + \frac{B_\phi(t)}{t}, \quad B_\phi(t) \ll 1
\]

for \( t \in S_\phi(t'_\infty, \kappa_1, \delta_1) \) with given \( \kappa_1 > 0 \), small given \( \delta_1 > 0 \) and sufficiently large \( t'_\infty > 0 \). Here \( A_\phi \) is a solution of the Boutroux equations (2.1), and

\[
S_\phi(t'_\infty, \kappa_1, \delta_1) = \{ t \mid \text{Re} t > t'_\infty, \quad |\text{Im} t| < \kappa_1, \quad |y(t)| + |y'(t)| + |y(t)|^{-1} < \delta_1^{-1} \}.
\]

Let \( 0 < \phi < \pi/3 \). We calculate the analytic continuation of the matrix solution near \( \lambda = \infty \) along the Stokes curve consisting of

\[
c_\infty = (\infty, \lambda_1)^\sim, \quad c_1 = (\lambda_1, \lambda_0)^\sim, \quad c_0 = (\lambda_0, 0)^\sim
\]

starting from \( \infty \) and terminating in 0 on the upper sheet of the Riemann surface \( \mathcal{R}_\phi \) of \( \mu(\infty, \lambda) \) as in Figure 3. Under supposition (5.1) these curves \( c_0, c_1, c_\infty \) lie within the distance \( O(t^{-1}) \) from the limit Stokes graph. Recall that the curve \( c_1 \) is located along the lower shore of the cut \( [\lambda_0, \lambda_1] \).

In the WKB solution, write \( \Lambda(\lambda) \) in the component-wise form \( \Lambda(\lambda) = \Lambda_3(\lambda) + \Lambda_I(\lambda) \) with

\[
\Lambda_3(\lambda) = \frac{t}{3} \mu(t, \lambda) \sigma_3 - \text{diag} T^{-1} T_\lambda |\sigma_3, \quad \Lambda_I(\lambda) = -\text{diag} T^{-1} T_\lambda |I,
\]

in which \( \text{diag} T^{-1} T_\lambda |\sigma_3, \text{diag} T^{-1} T_\lambda |I \in \mathbb{C}I \). In Propositions (1.1 and 4.2) if \( \delta = \delta' = 2/9 - \varepsilon \) with any \( \varepsilon \) such that \( 0 < \varepsilon < 2/9 \), then both propositions are applicable in the annulus

\[
\mathcal{A}_\varepsilon' : \quad t^{-2/3 + (2/3)(2/9 - \varepsilon)} \ll |\lambda - \lambda_i| \ll t^{-2/3 + (2/3)(2/9 + \varepsilon/2)}
\]

(\( \iota = 0, 1 \)). In what follows we set \( \delta = 2/9 - \varepsilon \), and write \( c_k = b_k(\lambda_0), d_k = b_k(\lambda_1) \) (\( k = 1, 2, 3 \)).
(1) Let \( \Psi_\infty(\lambda) \) along \( c_\infty = (\infty, \lambda_1) \) be a WKB solution by Proposition 4.1 and let \( Y_0^{\infty,*}(\lambda) = \dot{Y}_0^{\infty}(\lambda) \Theta_0^{\sigma_3} \) be given by (3.3) and Proposition 3.2. Set \( Y_0^{\infty,*}(\lambda) \Theta_0^{\sigma_3} = \dot{Y}_0^{\infty}(\lambda) = \Psi_\infty(\lambda) \Gamma_\infty \). Note that \( \mu(t, \lambda) = -i e^{i \phi} \lambda - \frac{3}{2} (1 + 2ia) t^{-1} \lambda^{-1} + O(\lambda^{-3}) \) along \( c_\infty \), and \( \mu - b_3 \ll \lambda^{-1} \) as \( \lambda \to \infty \). Then, by Propositions 4.1 and 4.2,
\[
\Gamma_\infty = \Psi_\infty(\lambda)^{-1} \dot{Y}_0^{\infty}(\lambda) = \Psi_\infty(\lambda)^{-1} Y_0^{\infty,*}(\lambda) \Theta_0^{\sigma_3}
\]
\[
= \exp\left(- \int_{\lambda_1}^{\lambda} \Lambda(\tau) d\tau \right) T^{-1} \left( I + O(t^{-\delta} + |\lambda|^{-1}) \right)
\]
\[
\times \exp\left(- \frac{1}{6} \left( i e^{i \phi} t \lambda^2 + 3(1 + 2ia) \log \lambda \sigma_3 \right) \right),
\]
\[
= C_3(\tilde{\lambda}_1) c_I(\tilde{\lambda}_1) (I + O(t^{-\delta}))
\]
\[
\times \exp\left(- \lim_{\lambda \to \infty} \int_{\lambda_1}^{\lambda} \Lambda_3(\tau) d\tau + \frac{1}{6} \left( i e^{i \phi} t \lambda^2 + 3(1 + 2ia) \log \lambda \sigma_3 \right) \right),
\]
in which \( C_3(\tilde{\lambda}_1) = \exp(\int_{\lambda_1}^{\lambda} \Lambda_3(\tau) d\tau) \), \( c_I(\tilde{\lambda}_1) = \exp(- \int_{\lambda_1}^{\lambda} \Lambda_I(\tau) d\tau) \), and \( \tilde{\lambda}_1 \in c_\infty \), \( \tilde{\lambda}_1 - \lambda_1 \sim t^{-1} \).

(2) For \( \Psi_\infty(\lambda) \) and \( \Phi_1^+(\lambda) \) given by Proposition 4.2 in the annulus \( A_1^+ \) around \( \lambda_1 \), set \( \Psi_\infty(\lambda) = \Phi_1^+(\lambda) \Gamma_{1+} \) along \( c_\infty \). Suppose that the curve \((2\kappa_d)^{1/3} (\lambda - \lambda_1) = (t/3)^{-2/3} (\zeta + O(t^{-1/3})) \), \( \kappa_d = d_1 d_1' + d_2 d_2' + d_3 d_3' \) with \( \lambda \in c_\infty \) enters the sector \( |\arg \zeta - 7\pi/3| < 2\pi/3 \) (the other cases are similarly treated by Remark 3.2). Write \( K^{-1} = 2(2\kappa_d)^{-1/3}(d_1 - id_2) \). Then, by Propositions 4.1 and 4.2,
\[
\Gamma_{1+} = \Phi_1^+(\lambda)^{-1} \Psi_\infty(\lambda)
\]
\[
= W(\zeta)^{-1} \left( \begin{array}{ccc} 1 & 0 \\ (t/3)^{-1/3} & K \end{array} \right)^{-1} \left( I + O(t^{-\delta}) \right)
\]
\[
\times \left( \begin{array}{ccc} 1 & -d_3 \\ d_1 - id_2 & 1 \end{array} \right)^{-1}
\]
\[
\times \left( \begin{array}{ccc} 1 & -d_3 \\ \frac{b_3 - \mu}{b_1 + id_2} & 1 \end{array} \right) (I + O(t^{-\delta})) \exp\left( \int_{\lambda_1}^{\lambda} \Lambda(\tau) d\tau \right)
\]
\[
= W(\zeta)^{-1} \left( \begin{array}{ccc} 1 & \frac{d_3}{K(d_1 - id_2)} \\ (t/3)^{1/3} & K(d_1 - id_2) \end{array} \right)(I + O(t^{-\delta})) \exp\left( \int_{\lambda_1}^{\lambda} \Lambda(\tau) d\tau \right)
\]
for $\lambda \in \mathcal{A}_1^i \cap c_\infty$, where $(\mu - b_3)/(b_1 \pm ib_2) = (\mu - d_3)/(d_1 \pm id_2) + O(\eta)$, $\eta = \lambda - \tilde{\lambda_1}$. Since $\mu = (2\kappa d)^{1/2} \eta^{1/2}(1 + O(\eta)) = 2K(d_1 - id_2)(t/3)^{-1/3} + O(1 + O(\eta))$, we have
\[
\Gamma_{1+} = \exp\left(\int_{\tilde{\lambda}_1}^{\lambda} \Lambda(\tau)d\tau \right) \left(1 + O(\tau^{-\delta})\right) \left(1, 0, 0 \right)^T.
\]
By $\Lambda_3(\lambda) = ((2\kappa d)^{1/2} (t/3)^{1/2} + O(\eta))^{1/2} + O(\eta^{-1/2})$ and $\Lambda_I(\lambda) = (-\eta^{-1} + O(\eta^{-1/2}))I$ (cf. Remark 4.1) for $\eta = \lambda - \tilde{\lambda}_1$, $\lambda \in \mathcal{A}_1^i \cap c_\infty$
\[
\Gamma_{1+} = \left(\zeta_1\right)^{1/4} (I + O(t^{-\delta})) C_3(\tilde{\lambda}_1)^{-1} \left(1, 0, 0, \frac{d_1 - id_2}{d_3}\right).
\]
with suitably chosen $\zeta_1 \approx \tilde{\lambda}_1 - \lambda_1$.

(3) Let $\Phi^-_1(\lambda)$ be the solution by Proposition 4.2 near $c_1 = (\lambda_1, \lambda_0)^\sim$, and set $\Phi^+_1(\lambda) = \Phi^-_1(\lambda) \Gamma_{1+}$, where $\Phi^-_1(\lambda)$ is the analytic continuation along an arc in $\mathcal{A}_1^i$ in the clockwise direction. Then by Proposition 4.2
\[
\Gamma_{1+} = \Phi^-_1(\lambda)^{-1} \Phi^+_1(\lambda) = S_2 S_3 = \left(\begin{array}{cc}
1 & 0 \\
0 & 1
\end{array}\right).
\]

(4) For $\Phi^-_1(\lambda)$ and the WKB solution $\Psi^-_1(\lambda)$ along $c_1$, set $\Phi^-_1(\lambda) = \Psi^-_1(\lambda) \Gamma_{1-}$. Then, supposing the curve $(2\kappa d)^{1/3}(\lambda - \tilde{\lambda}_1) = (t/3)^{-2/3}(\zeta + O(t^{-1/3}))$ with $\lambda \in c_1$ to be in the sector $|\arg \zeta - \pi| < 2\pi/3$, we have, for $\tilde{\lambda}_1 \in c_1$, $|\tilde{\lambda}_1 - \lambda_1| \approx t^{-1}$,
\[
\Gamma_{1-} = \Psi^-_1(\lambda)^{-1} \Phi^-_1(\lambda) = \exp\left(-\int_{\tilde{\lambda}_1}^{\lambda} \Lambda(\tau)d\tau\right) (I + O(t^{-\delta})) \left(1, 0, 0 \right)^T.\]

(5) For $\Psi^-_1(\lambda)$ and the WKB solution $\Psi^+_0(\lambda)$ along $c_1$ near $\lambda_0$, set $\Psi^-_1(\lambda) = \Psi^+_0(\lambda) \Gamma_{01}$. Then, for $\tilde{\lambda}_0 \in c_1$, $\lambda_0 - \lambda_0 \approx t^{-1}$,
\[
\Gamma_{01} = \Psi^-_1(\lambda)^{-1} \Psi^+_0(\lambda) = \exp\left(-\int_{\lambda_0}^{\lambda} \Lambda(\tau)d\tau\right) T^{-1} (I + O(t^{-\delta})) T \exp\left(\int_{\lambda_0}^{\lambda} \Lambda(\tau)d\tau\right).
\]
\[ C''_3(\tilde{\lambda}^0)\Gamma_0 = (\tilde{\zeta}_0 + 1)^{1/4}(I + O(t^{-\delta}))C''_3(\tilde{\lambda}^0) \exp \left( \int_{\lambda_0}^\lambda \Lambda_3(\tau)d\tau \right), \]

where \( C''_3(\tilde{\lambda}^0) = \exp \left( \int_{\lambda_0}^{\tilde{\lambda}^0} A_3(\tau)d\tau \right) \), and set \( \Gamma_0 = \Phi_0(\lambda)\Gamma_0 \). Then, by the same argument as in (2) above, we have

\[ \Gamma_{0\pm} = \Phi_0^{\pm}(\lambda)^{-1}\Phi_0^{\pm}(\lambda) = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix}. \]

(7) Let \( \Phi_0(\lambda) \) be the solution by Proposition 4.2 near \( c_0 = (\lambda_0, 0)^\ast \), and set \( \Phi_0^{\pm}(\lambda) = \Phi_0(\lambda)\Gamma_{0\pm} \), where \( \Phi_0^{\pm}(\lambda) \) is the analytic continuation along an arc in \( A^0_\varepsilon \) in the clockwise direction. Then by Proposition 4.2

\[ \Gamma_{0\pm} = \Phi_0^{\pm}(\lambda)^{-1}\Phi_0^{\pm}(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & c_1^{-1} - ic_2 \end{pmatrix} \]

for some \( \tilde{\zeta}_0 < \tilde{\lambda}_0 - \lambda_0 \).

(8) For \( \Phi_0(\lambda) \) and the WKB solution \( \Psi_0(\lambda) \) along \( c_0 \), set \( \Phi_0(\lambda) = \Psi_0(\lambda)\Gamma_{0-} \). By the same argument as in (4), we have

\[ \Gamma_{0-} = \Psi_0(\lambda)^{-1}\Phi_0(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & c_3 \end{pmatrix} \]

with \( \tilde{\Lambda}_3(\lambda)^0 = \exp \left( \int_{\lambda_0}^{\lambda} A_3(\tau)d\tau \right) \) for some \( \tilde{\zeta}_0 < \tilde{\lambda}_0 - \lambda_0 \).

(9) For \( \Psi_0(\lambda) \) and \( \tilde{Y}_0^0(\lambda) \) given by (3.1), set \( \Psi_0(\lambda) = \tilde{Y}_0^0(\lambda)\Gamma_0 \). Then

\[ \Gamma_0 = \tilde{Y}_0^0(\lambda)^{-1}\Psi_0(\lambda) \]

\[ = \exp \left( \frac{2i}{3} e^{i\phi} t \lambda^{-1} \sigma_3 \right) \sqrt{2} \int_{\lambda_0}^\lambda (\sigma_1 + \sigma_3)^{-1}(I + O(t^{-\delta} + |\lambda|))T \exp \left( \int_{\lambda_0}^\lambda \Lambda(\tau)d\tau \right). \]

Note that \( \mu(t, \lambda) = 2ie^{i\phi} t \lambda^{-2} + O(1) \) as \( \lambda \to 0 \) along \( c_0 \). Since

\[ (\sigma_1 + \sigma_3)^{-1} \lim_{\lambda \to 0} T(\lambda) = \frac{1}{2}(\sigma_1 + \sigma_3) \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \sigma_3, \]

we have

\[ \Gamma_0 = \tilde{Y}_0^0(\lambda)^{-1}\tilde{t}(\lambda)^{-1}(\sigma_3 + O(t^{-\delta})) \exp \left( \lim_{\lambda \to 0} \int_{\lambda_0}^\lambda \Lambda(\tau)d\tau + \frac{2i}{3} e^{i\phi} t \lambda^{-1} \sigma_3 \right) \]

with \( \tilde{t}(\lambda)^0 = -\sqrt{2} i \exp(\int_{\lambda_0}^0 A(t)d\tau) \).

Collecting the matrices above, we have the connection matrix

\[ \hat{G} = G^{\theta_0^3}_{\theta_0^3} = \tilde{Y}_0^0(\lambda)^{-1}\tilde{Y}_0^\infty(\lambda)\Theta^{\sigma_3}_{\sigma_3} = \tilde{Y}_0^0(\lambda)^{-1}\tilde{Y}_0^\infty(\lambda) \]

\[ = \Gamma_0\Gamma_{0-}\Gamma_{0+}\Gamma_{01}\Gamma_{1-}\Gamma_{11}\Gamma_\infty \]

\[ = \epsilon_{\pm}(\sigma_3 + O(t^{-\delta})) \exp(J_0\sigma_3) \begin{pmatrix} 1 & 0 \\ 0 & -c_0^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -c_0 \end{pmatrix} \]
\[ \epsilon_+ (I + O(t^{-\delta})) \times \exp(-J_1\sigma_3) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -d_0^{-1} & -d_0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ -i & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -d_0 \end{pmatrix} \exp(-J_\infty\sigma_3) = \epsilon_+ (I + O(t^{-\delta})) \]

if \( 0 < \phi < \pi/3 \), where \( \epsilon_2 = 1 \), \( c_0 = (c_1 - i c_2)/c_3 \), \( d_0 = (d_1 - i d_2)/d_3 \), and

\[ J_0\sigma_3 = \lim_{\lambda \to 0} \left( \int_{\lambda_0}^{\lambda} \Lambda_3(\tau)d\tau + \frac{2i}{3} e^{i\phi t\lambda^{-1}} \right) \sigma_3, \quad J_1\sigma_3 = \int_{\lambda_0}^{\lambda_1} \Lambda_3(\tau)d\tau \quad \text{(along } c_1), \]

\[ J_\infty\sigma_3 = \lim_{\lambda \to \infty} \left( \int_{\lambda_1}^{\lambda} \Lambda_3(\tau)d\tau + \frac{1}{6} (i e^{i\phi t\lambda^2} + 3(1 + 2 i a) \log \lambda) \sigma_3 \right). \]

In the case \(-\pi/3 < \phi < 0\), from the analytic continuation along the Stokes curves as in Figure 4, it follows that

\[ \hat{G} = \epsilon_- i(\sigma_3 + O(t^{-\delta})) \exp(J_0\sigma_3) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -c_0^{-1} & -c_0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ i & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -d_0 \end{pmatrix} \exp(-J_\infty\sigma_3) \]

\[ = \epsilon_- (I + O(t^{-\delta})) \times \begin{pmatrix} -i c_0 d_0^{-1} \exp(J_0 + J_1 - J_\infty) & c_0 \exp(J_1) + d_0 \exp(-J_1) \exp(J_0 + J_\infty) \\ -d_0^{-1} \exp(-J_0 + J_1 - J_\infty) & -i \exp(-J_0 + J_1 + J_\infty) \end{pmatrix} \]

Here \( \epsilon_- = 1 \), and

\[ \hat{J}_1\sigma_3 = \int_{\lambda_0}^{\lambda_1} \Lambda_3(\tau)d\tau \quad \text{(along } \hat{c}_1), \]

in which \( \hat{c}_1 \) is a curve joining \( \lambda_0 \) to \( \lambda_1 \) located along the upper shore of the cut on the upper sheet of \( \mathcal{R}_\phi \). Thus we have the following:

---

**Figure 4.** Stokes curve for \(-\pi/3 < \phi < 0\)
Proposition 5.1. Let \(c_0 = (c_1 - ic_2)/c_3\), \(d_0 = (d_1 - id_2)/d_3\) with \(c_k = b_k(\lambda_0)\), \(d_k = b_k(\lambda_1)\) for \(k = 1, 2, 3\). If \(0 < \phi < \pi/3\), then
\[
\hat{G} = \epsilon_+(I + O(t^{-\delta}))
\times \left( \begin{array}{ccc}
    i \exp(J_0 - J_1 - J_\infty) & -d_0 \exp(J_0 - J_1 - J_\infty) \\
    (c_0^{-1} \exp(-J_1) + d_0^{-1} \exp(J_1)) \exp(-J_0 - J_\infty) & ic_0^{-1} d_0 \exp(-J_0 - J_1 + J_\infty)
\end{array} \right),
\]
and, if \(-\pi/3 < \phi < 0\), then
\[
\hat{G} = \epsilon_-(I + O(t^{-\delta}))
\times \left( \begin{array}{ccc}
    -ic_0 d_0^{-1} \exp(J_0 + \hat{J}_1 - J_\infty) & (c_0 \exp(\hat{J}_1) + d_0 \exp(-\hat{J}_1)) \exp(J_0 + J_\infty) \\
    -d_0^{-1} \exp(-J_0 + \hat{J}_1 - J_\infty) & -i \exp(-J_0 + \hat{J}_1 + J_\infty)
\end{array} \right).
\]

Here \(\epsilon_\pm = 1\), and \(J_0, J_1, \hat{J}_1, J_\infty\) are integrals given by (5.2), (5.3), and (5.4).

From the proposition above with \(\hat{G} = G\Theta_{0,\pm}^\sigma, G = (g_{ij})\) (Remark 3.1), we derive key relations.

Corollary 5.2. If \(0 < \phi < \pi/3\) and \(g_{11} g_{12} g_{22} \neq 0\), then
\[g_{11} g_{22} = -c_0^{-1} d_0 (1 + O(t^{-\delta})) \exp(-2J_1), \quad \frac{g_{12}}{g_{22}} = ic_0 (1 + O(t^{-\delta})) \exp(2J_0).\]

If \(-\pi/3 < \phi < 0\) and \(g_{11} g_{21} g_{22} \neq 0\), then
\[g_{11} g_{22} = -c_0 d_0^{-1} (1 + O(t^{-\delta})) \exp(2\hat{J}_1), \quad \frac{g_{21}}{g_{11}} = -ic_0^{-1} (1 + O(t^{-\delta})) \exp(-2J_0).\]

6. ASYMPTOTIC PROPERTIES OF MONODROMY DATA

6.1. Expressions of \(J_0, J_1\) and \(\hat{J}_1\). To examine asymptotic properties of \(J_0, J_1\) and \(\hat{J}_1\), we make the change of variables \(\lambda^{-2} = z\). Then, by (4.1) and (4.2), \(\mu(t, \lambda)\) becomes
\[
\mu(t, \lambda) d\lambda = \left( \frac{e^{2i\phi}}{z} + e^{2i\phi} a_\phi z - 4e^{2i\phi} z^2 + 3ie^{i\phi} (1 + 2ia) t^{-1} \right)^{1/2} \frac{(-z^{-3/2})^3}{2} \frac{dz}{z^2}.
\]

with \(w(z)^2 = w(a_\phi, z)^2 = 4z^3 - a_\phi z^2 + 1\), for \(z\) such that \(w(z) \gg 1\). The turning points \(\lambda_0, \lambda_1, \lambda_2\) and 0 on \(R_\phi\) are mapped to
\[z_0 = \lambda^{-2}_0, \quad z_1 = \lambda^{-2}_1, \quad z_2 = \lambda^{-2}_2\]
and \(\infty\), respectively, on the elliptic curve \(\Pi_{a_\phi}\) for \(w(a_\phi, z)\) constructed by the same way as in the case of \(\Pi_{\lambda_0}\) in Section 2.2. The branch of \(\mu(t, \lambda)\) is compatible with that of \(w(a_\phi, z)\). Suppose that \(\Pi_{a_\phi}\) is equipped with the cycles \(a\) and \(b\) as in Section 2.2. Then the inverse image of the cycle \(a\) is a closed curve \(a_\lambda\) surrounding the cut \([\lambda_0, \lambda_1]\) anticlockwise (see Figure 5).

Since
\[
\int \frac{w(z)}{z^2} dz = 2 \frac{w(z)}{z} - \frac{dz}{w(z)} + 3 \int \frac{dz}{z^2 w(z)},
\]
we have
\[ \mu(t, \lambda) d\lambda = -ie^{i\phi} w(z) \frac{dz}{z} + \frac{i}{2} e^{i\phi} a_{\phi} \frac{dz}{w(z)} - \frac{3i}{2} e^{i\phi} \frac{dz}{z^2 w(z)} - \frac{3}{4} (1 + 2ia) t^{-1} \frac{dz}{z w(z)} + O(t^{-2}) \]

in which \( w(z)/z = 2z^{1/2} + O(z^{-1/2}) \) as \( z \to \infty \). Hence

\[
\lim_{\lambda \to 0} \left( \int_{c_0}^{c_1} \mu(t, \tau) d\tau + 2ie^{i\phi} \lambda^{-1} \right) = -\frac{i}{4} e^{i\phi} a_{\phi} \int_a^b \frac{dz}{w(z)} + \frac{3}{8} (1 + 2ia) t^{-1} \int_a^b \frac{dz}{z w(z)} + O(t^{-2}),
\]

and

\[
\int_{\lambda_0(e_1)}^{\lambda_1} \mu(t, \tau) d\tau, - \int_{\lambda_0(e_1)}^{\lambda_1} \mu(t, \tau) d\tau = \frac{i}{4} e^{i\phi} a_{\phi} \int_a^b \frac{dz}{w(z)} - \frac{3}{8} (1 + 2ia) t^{-1} \int_a^b \frac{dz}{z w(z)} + O(t^{-2})
\]

in which \( \int_{\lambda_0(e_1)}^{\lambda_1} \) denotes the integral along the contour \( e_1 \). By Remark 4.1

\[
\Lambda_3(t, \lambda) = \left( \frac{t}{3} \mu(t, \lambda) - \text{diag} T^{-1} T_{\lambda} \sigma_3 \right) \sigma_3,
\]
\[
\text{diag } T^{-1}T_\lambda|_{\sigma_3} = \frac{1}{4} \left( 1 - \frac{b_3}{\mu} \right) \frac{\partial}{\partial \lambda} \log \frac{b_1 + ib_2}{b_1 - ib_2}.
\]
To calculate \( J_0, J_1 \) and \( \hat{J}_1 \), it is necessary to know diag \( T^{-1}T_\lambda|_{\sigma_3} \) in addition to (6.1) and (6.2). Note that, by (3.2),

\[
b_1 = 2ie^{i\phi}\lambda^{-2} - iK_+, \quad b_2 = K_-, \quad b_3 = -ie^{i\phi}\lambda - K_0\lambda^{-1}
\]
with \( K_\pm = e^{i\phi}y \pm (i/2)y^{-1}\Gamma_0(t, y, y') \), \( K_0 = \Gamma_0(t, y, y') + 3(1/2 + ia)t^{-1} \), \( \Gamma_0(t, y, y') = y'y^{-1} - ie^{i\phi}y^{-1} - (1 + 3ia)t^{-1} \). Setting \( z_\pm = e^{-i\phi}(K_\pm + K_-)/2 \), i.e.

\[
z_+ = y, \quad z_- = (i/2)e^{-i\phi}y^{-1}\Gamma_0(t, y, y'),
\]
and \( \lambda^2 = z \), we have

\[
b_1 - ib_2 = 2ie^{i\phi}(z - z_+), \quad b_1 + ib_2 = 2ie^{i\phi}(z - z_-).
\]
By (4.1), \( \mu^2 = -e^{2i\phi}\lambda^2w(z)^2 + O(t^{-1}) \), which implies \( \mu = ie^{i\phi}\lambda(w(z) + O(t^{-1}z)) \) on the upper sheet of \( \Pi_{\alpha_0} \), and hence

\[
\frac{b_3}{\mu} = -ie^{-i\phi} \frac{b_3}{\lambda} \left( \frac{1}{w(z)} + O(t^{-1}z^{-2}) \right),
\]
where \( b_3/\lambda = -K_0z - ie^{i\phi} \) satisfies \( (b_3/\lambda)(z_\pm) = -(\mu/\lambda)(z_\pm) = -ie^{i\phi}w(z_\pm) + O(t^{-1}) \), since \( \mu(z_\pm)^2 = (b_1 - ib_2)(b_1 + ib_2)(z_\pm) + b_3(z_\pm)^2 = b_3(z_\pm)^2 \) by (6.5). These facts combined with (6.5) yield

\[
\text{diag } T^{-1}T_\lambda|_{\sigma_3}d\lambda = \frac{1}{4} \left( 1 - \frac{b_3}{\mu} \right) \frac{dl}{dl} \log \frac{b_1 + ib_2}{b_1 - ib_2} d\lambda
\]

\[
= \frac{1}{4} \left( 1 - \frac{b_3}{\mu} \right) \frac{dl}{dz} \log \frac{b_1 + ib_2}{b_1 - ib_2} dz
\]

\[
= \frac{1}{4} \left( 1 + ie^{-i\phi} \frac{b_3}{\lambda} \left( \frac{1}{w(z)} + O(t^{-1}z^{-2}) \right) \right) \left( \frac{1}{z - z_+} - \frac{1}{z - z_-} \right) dz
\]

\[
= -\frac{1}{4} \left( \frac{1}{z - z_+} - \frac{1}{z - z_-} + \frac{w(z_+)}{w(z)} - \frac{w(z_-)}{w(z)} \right) \frac{1}{w(z)} + O(t^{-1}z^{-2}) dz,
\]

which implies

\[
- \lim_{\lambda \to 0} \int_{\lambda_0}^\lambda \text{diag } T^{-1}T_\lambda|_{\sigma_3} d\lambda = -\frac{1}{4} \log \frac{z_0 - z_+}{z_0 - z_-} - \frac{1}{8} \int_a \left( \frac{w(z_+)}{z - z_+} - \frac{w(z_-)}{z - z_-} \right) \frac{dz}{w(z)} + O(t^{-1}).
\]

Here, by (5.1), \( c_0^2 = (c_1 - ic_2)^2/c_3^2 = -(c_1 - ic_2)/(c_1 + ic_2) = -(z_0 - z_+)/(z_0 - z_-) \) and \( \log((z_0 - z_+)/z_0 - z_-)) = \log(-c_0^2) = 2\log(ic_0) \). Similarly,

\[
- \int_{\lambda_0}^{\lambda_1} \text{diag } T^{-1}T_\lambda|_{\sigma_3} d\lambda + \frac{1}{2} \log(c_0d_0^{-1}), \quad \int_{\lambda_0}^{\lambda_1} \text{diag } T^{-1}T_\lambda|_{\sigma_3} d\lambda - \frac{1}{2} \log(c_0d_0^{-1})
\]

\[
= \frac{1}{8} \int_a \left( \frac{w(z_+)}{z - z_+} - \frac{w(z_-)}{z - z_-} \right) \frac{dz}{w(z)} + O(t^{-1}).
\]

Insertion of (6.1), (6.2) and the relations above into (5.2), (5.4) with (6.3) provides the expressions of \( J_0, J_1 \) and \( \hat{J}_1 \). Then by Corollary 5.2 we have
Proposition 6.1. Let
\[ W(z) = \left( \frac{w(z_+) - w(z_-)}{z - z_+ - z - z_-} \right) \frac{1}{w(z)}. \]

1) Suppose that \( g_{11}g_{22} \neq 0, g_{12}/g_{22} \neq 0. \) For \( 0 < \phi < \pi/3, \)
\[ \log \frac{g_{12}}{g_{22}} = \frac{i e^{i \phi} t}{6} \int_{b} \frac{w(z)}{z^2} w(z) - \frac{1}{4} \int_{b} W(z) dz + \frac{1}{4} \int_{b} (1 + 2ia) \int_{b} d\zeta w(z) + O(t^{-\delta}), \]
\[ \log(g_{11}g_{22}) = \frac{i e^{i \phi} t}{6} \int_{a} \frac{w(z)}{z^2} dz - \frac{1}{4} \int_{a} W(z) dz + \frac{1}{4} \int_{a} (1 + 2ia) \int_{a} d\zeta w(z) + \pi i + O(t^{-\delta}). \]

2) Suppose that \( g_{11}g_{22} \neq 0, g_{21}/g_{11} \neq 0. \) For \(-\pi/3 < \phi < 0,\)
\[ \log \frac{g_{21}}{g_{11}} = -\frac{i e^{i \phi} t}{6} \int_{b} \frac{w(z)}{z^2} w(z) + \frac{1}{4} \int_{b} W(z) dz - \frac{1}{4} (1 + 2ia) \int_{b} d\zeta w(z) + O(t^{-\delta}), \]
\[ \log(g_{11}g_{22}) = \frac{i e^{i \phi} t}{6} \int_{a} \frac{w(z)}{z^2} dz - \frac{1}{4} \int_{a} W(z) dz + \frac{1}{4} (1 + 2ia) \int_{a} d\zeta w(z) + \pi i + O(t^{-\delta}). \]

Remark 6.1. In the proposition above
\[ \frac{i e^{i \phi} t}{6} \int_{a, b} \frac{w(z)}{z^2} dz = -\frac{i e^{i \phi} a \phi t}{6} \int_{a, b} \frac{d\zeta}{w(z)} + \frac{i e^{i \phi} t}{2} \int_{a, b} \frac{d\zeta}{z^2 w(z)}. \]

6.2. Expressions by the \( \vartheta \)-function. For \( w(z)^2 = w(a_{\phi}, z)^2 = 4z^3 - a_{\phi}z^2 + 1, \) the differential equation \((dz/du)^2 = w(a_{\phi}, z)^2)\) defines the Weierstrass \( \varphi \)-function
\[ z = \varphi(u; g_2, g_3) + \frac{a_{\phi}}{12}, \quad g_2 = \frac{a_{\phi}^2}{12}, \quad g_3 = -1 + \frac{a_{\phi}^3}{216}. \]

The periods of \( \varphi(u; g_2, g_3) \) are
\[ \omega_a = \int_{a} \frac{d\zeta}{w(a_{\phi}, z)}, \quad \omega_b = \int_{b} \frac{d\zeta}{w(a_{\phi}, z)}, \quad \tau = \frac{\omega_b}{\omega_a}, \quad \text{Im } \tau > 0, \]
where \( a \) and \( b \) are the cycles on the elliptic curve \( \Pi_{a_{\phi}} = \Pi_+ \cup \Pi_- \) for \( w(a_{\phi}, z) \) in Section 6.1 (cf. Figure 5). The \( \vartheta \)-function \( \vartheta(z, \tau) = \vartheta(z) \) is defined by
\[ \vartheta(z, \tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 + 2\pi i n z} \]
and we set
\[ \nu = \frac{1 + \tau}{2} \]
(cf. [6], [29]). For \( z, \bar{z} \in \Pi_{a_{\phi}} = \Pi_+ \cup \Pi_- \), let
\[ F(z, \bar{z}) = \frac{1}{\omega_a} \int_{\bar{z}}^{z} \frac{d\zeta}{w(\zeta)} = \frac{1}{\omega_a} \int_{\bar{z}}^{z} \frac{d\zeta}{w(z)} - \frac{1}{\omega_a} \int_{\infty}^{z} \frac{d\zeta}{w(\zeta)}. \]

For any \( z_0 \in \Pi_{a_{\phi}} \) denote the projections of \( z_0 \) on the respective sheets by \( z_0^+ = (z_0, w(z_0)) = (z_0, w(z_0^+)) \) and \( z_0^- = (z_0, -w(z_0)) = (z_0, -w(z_0^+)) \). If \( z_0 \in \Pi_+ \) (respectively, \( z_0 \in \Pi_- \)), then \( z_0^\pm \in \Pi_+ \) (respectively, \( z_0^\pm \in \Pi_- \)).
Proposition 6.2. For any \( z_0 \in \Pi_{\alpha_\phi} \)

\[
\frac{dz}{(z - z_0)w(z)} = \frac{1}{w(z_0^\pm)} d \log \frac{\vartheta(F(z_0^+, z) + \nu, \tau)}{\vartheta(F(z_0^-, z) + \nu, \tau)} - g_0(z_0) \frac{dz}{w(z)},
\]

\[
g_0(z_0) = \frac{w'(z_0^\pm)}{2w(z_0^\pm)} - \frac{1}{\omega_a} \frac{1}{w(z_0^\pm)} \left( \pi i + \vartheta'(F(z_0^-, z_0^+) + \nu, \tau) \right).
\]

Proof. For \( z_0 = \varphi(u_0) + a_\phi/12 \in \Pi_{\alpha_\phi} \) let \( u_0^\pm \) be such that \( z_0^\pm = \varphi(u_0^\pm) + a_\phi/12 \). Then

\[
\frac{dz}{(z - z_0)w(z)} = \frac{1}{w(z_0^\pm)} \left( \zeta(u - u_0^+) - \zeta(u - u_0^-) + \zeta(u_0^+ - u_0^-) - \frac{1}{2} w'(z_0^+) \right) du
\]

\[
= \frac{1}{w(z_0^\pm)} d \log \frac{\sigma(u - u_0^+)}{\sigma(u - u_0^-)} + \frac{1}{w(z_0^\pm)} \left( \zeta(u_0^+ - u_0^-) - \frac{1}{2} w'(z_0^+) \right) du.
\]

From

\[
d \log \frac{\sigma(u - u_0^+)}{\sigma(u - u_0^-)} = -\frac{2n_a}{\omega_a} (u_0^+ - u_0^-) du + d \log \frac{\vartheta(F(z_0^+, z) + \nu, \tau)}{\vartheta(F(z_0^-, z) + \nu, \tau)},
\]

\[
\zeta(u_0^+ - u_0^-) = \sigma'(u_0^+ - u_0^-) = \frac{2n_a}{\omega_a} (u_0^+ - u_0^-) + \frac{\pi i}{\omega_a} + \frac{1}{\omega_a} \vartheta'(F(z_0^-, z_0^+) + \nu, \tau)
\]

with \( F(z_0^\pm, z) = \omega_a^{-1} \int_{z_0^\pm} \frac{dz}{w(z)} \), the desired formula follows.

Observe that

\[
\log \vartheta(F(z_0^+, z) + \nu, \tau)|_a = 0,
\]

\[
\log \vartheta(F(z_0^+, z) + \nu, \tau)|_b = \log \vartheta(F(z_0^+, z_b) + \tau + \nu, \tau) \vartheta(F(z_0^-, z_b) + \nu, \tau)
\]

\[
= \log \exp(-\pi i (2(F(z_0^+, z_b) + \nu) + \tau))
\]

\[
+ \log \exp(\pi i (2(F(z_0^-, z_b) + \nu) + \tau))
\]

\[
= 2\pi i F(z_0^-, z_0^+)
\]

for \( z_b \in \mathbb{B} \cap (\Pi_+)^\text{cl} \cap (\Pi_-)^\text{cl} \), since \( \vartheta(z \pm \tau, \tau) = e^{-\pi i (\tau \pm 2z)} \vartheta(z, \tau) \), where \( (\Pi_+)^\text{cl} \) denotes the closure of \( \Pi_+ \). Then

\[
\int_a \frac{dz}{(z - z_0)w(z)} = -g_0(z_0) \omega_a,
\]

\[
\int_b \frac{dz}{(z - z_0)w(z)} = \frac{2\pi i}{w(z_0^+)} F(z_0^-, z_0^+) + \tau \int_a \frac{dz}{(z - z_0)w(z)}.
\]

Differentiation of both sides with respect to \( z_0 \) at \( z_0 = 0 \) yields

\[
\int_b \frac{dz}{z^2 w(z)} = \frac{4\pi i}{\omega_a} + \tau \int_a \frac{dz}{z^2 w(z)}.
\]

Using these formulas we have
Proposition 6.3. For \( W(z) \) as in Proposition \([5.1]\) and for \( z_\pm \) by \([6.4]\),

\[
\int_a W(z)dz = -(w(z_+)g_0(z_+) - w(z_-)g_0(z_-))\omega_a
\]

\[
= -\frac{1}{2}(w'(z_+) - w'(z_-))\omega_a
\]

\[
+ \frac{\partial}{\partial \vartheta}(F(z_+, z^\pm_+) + \nu, \tau) - \frac{\partial}{\partial \vartheta}(F(z_-, z^\pm_-) + \nu, \tau),
\]

\[
\left(\int_b -\tau \int_a\right)W(z)dz = 2\pi i(F(z_+, z^\pm_+) - F(z_-, z^\pm_-)),
\]

and

\[
\int_a \frac{dz}{zw(z)} = -g_0(0^+)\omega_a, \quad g_0(0^+) = \frac{1}{\omega_a}\left(\pi i + \frac{\partial}{\partial \vartheta}(F(0^-, 0^+) + \nu, \tau)\right),
\]

\[
\left(\int_b -\tau \int_a\right) \frac{dz}{zw(z)} = -2\pi iF(0^-, 0^+),
\]

\[
\left(\int_b -\tau \int_a\right) \frac{dz}{z^2w(z)} = \frac{4\pi i}{\omega_a}.
\]

Remark 6.2. In the proposition above the first formula is rewritten in the form

\[
\int_a W(z)dz = 2\left(\frac{\partial}{\partial \vartheta}(\frac{1}{2}F(z^+, z^\pm_+) + \nu, \tau) - \frac{\partial}{\partial \vartheta}(\frac{1}{2}F(z^-, z^\pm_-) + \nu, \tau)\right).
\]

The right-hand side is obtained by comparing the poles of \((\partial^\prime/\partial)\left(\frac{1}{2}F(z^+, z^\pm_+) + \nu, \tau\right)\) with those of \(-\frac{1}{2}w'(z^\pm) + (\partial^\prime/\partial)(F(z^-, z^\pm_-) + \nu, \tau)\) on \(\Pi_{a_\vartheta}\), and showing that the difference is a constant (see also \([15, \text{pp. 117-119}]\)).

6.3. Expression of \( B_{\phi}(t) \). Let us write the quantity \( B_{\phi}(t) \) in terms of

\[
\Omega_a = \int_a \frac{dz}{w(A_\phi, z)}, \quad \Omega_b = \int_b \frac{dz}{w(A_\phi, z)},
\]

\[
\epsilon_a = \int_a \frac{w(A_\phi, z)}{z^2}dz, \quad \epsilon_b = \int_b \frac{w(A_\phi, z)}{z^2}dz.
\]

with \(w(A_\phi, z) = \sqrt{4z^3 - A_\phi z^2 + 1}\) and \(a, b\) on \(\Pi_{a_\phi} = \Pi_{a_{\phi}}^* \cup \Pi_{a_{\phi}}^* \lim_{a_\phi(t)\rightarrow A_\phi} \Pi_{a_\phi}\). By \([5.1]\) the cycles \(a\) and \(b\) on \(\Pi_{a_\phi}\) may be regarded as those on \(\Pi_{A_\phi}\), and are independent of \(t\) for sufficiently large \(t\).

Let \(0 < \phi < \pi/3\). By Proposition \([6.3]\) the integral \(\int_a W(z)dz\) is expressed in terms of \(\vartheta(\pm) = (\vartheta^\prime/\vartheta)(\frac{1}{2}F(z^\pm, z^\pm_+) + \nu, \tau)\) (Remark \([6.2]\)) and \(w'(z^\pm)\) and \((\vartheta^\prime/\vartheta)(F(z^\pm, z^\pm_+) + \nu, \tau)\), in which

\[
F(z^\pm, z^\pm_+) = \frac{1}{\omega_a} \int_{z_-}^{z_+} \frac{dz}{w(a_\phi, z)} = \frac{2}{\omega_a} \int_{z_-}^{z_+} \frac{dz}{w(a_\phi, z)}.
\]

Note that \(\int_a W(z)dz\) has no poles or zeros in \(S_{\phi}(t^\prime_{\infty}, \kappa_1, \delta_1)\). Indeed, if, say \(\vartheta_+(\pm)\) or \(\vartheta_-(\pm)\) is \(\infty\) at \(t = t_\pm\), then \(z_+\) or \(z_-\) is \(\infty\), and hence \(t_\pm\) is a pole or a zero of \(y(t)\), or a pole of \(y'(t)\), which is excluded from \(S_{\phi}(t^\prime_{\infty}, \kappa_1, \delta_1)\). Consider \(z_\pm = z_\pm(t)\) (cf. \([6.4]\)) moving on the elliptic curve \(\Pi_{a_\phi}\) crossing \(a\) and \(b\)-cycles, and then \(F(z^\pm, z^\pm_+) = 2p_\pm(t) + 2q_\pm(t)\tau + O(1)\) with \(p_\pm(t), q_\pm(t) \in \mathbb{Z}\). This implies the boundedness of \(\text{Re} (\vartheta^\prime/\vartheta)(\frac{1}{2}F(z^\pm, z^\pm_+) + \nu, \tau)\)
or Re \((\vartheta')/\vartheta\)(F(\(z_{\pm}^\nu, z_{\pm}^\nu\)) + \nu, \tau)\) in \(S_\rho(t'_\infty, \kappa_1, \delta_1)\), and hence the modulus of Re \(\int_a W(z)dz\) is uniformly bounded in \(S_\rho(t'_\infty, \kappa_1, \delta_1)\). Note that, by (5.1),
\[
\frac{1}{z^2}(w(a_\phi, z) - w(A_\phi, z)) = \frac{1}{z^2}(\sqrt{4z^3 - a_\phi z^2 + 1} - \sqrt{4z^3 - A_\phi z^2 + 1}) = -\frac{t^{-1}B_\rho(t)}{2w(A_\phi, z)}(1 + O(t^{-1}B_\rho(t))).
\]
By using this and Proposition 6.3, the second formula in Proposition 6.1 (1) is written in the form
\[
\log(g_{11}g_{22}) = \frac{i e^{i\phi}}{6} \int_a \left( \frac{w(A_\phi, z)}{z^2} - \frac{t^{-1}B_\rho(t)}{2w(A_\phi, z)} \right) dz - \frac{1}{4} \int_a W(z)dz - \frac{1}{4}(1 + 2ia)g_0(0^+)\omega_a + \pi i + O(t^{-\delta}),
\]
which implies
\[
ie^{i\phi} \left( t\mathcal{J}_a - \frac{\Omega_a}{2} B_\rho(t) \right) = \frac{3}{2} \int_a W(z)dz + \frac{3}{2}(1 + 2ia)g_0(0^+)\omega_a + 6\log(g_{11}g_{22}) - 6\pi i + O(t^{-\delta}).
\]
Recall that \(G = \hat{G} \Theta_0^{-\sigma_3} = (g_{ij})\), \(g_{ij} = g_{ij}(t)\) is a solution of the direct monodromy problem. Suppose that
\[
(6.6) \quad |\log(g_{11}g_{22})| \ll 1, \quad |\log(g_{12}/g_{22})| \ll 1 \quad \text{in} \quad S_\rho(t'_\infty, \kappa_1, \delta_1).
\]
By the Bouttroux equations (2.1), \(\text{Im} e^{i\phi} \Omega_a B_\rho(t)\) is bounded as \(e^{i\phi}t \to \infty\) through \(S_\rho(t'_\infty, \kappa_1, \delta_1)\). By using the first formula of Proposition 6.1 (1), we have
\[
ze^{i\phi} \left( t\mathcal{J}_b - \frac{\Omega_b}{2} B_\rho(t) \right) = \frac{3}{2} \int_b W(z)dz + \frac{3}{2}(1 + 2ia)\omega_b + 6\log\frac{g_{12}}{g_{22}} + O(t^{-\delta}),
\]
in which \(\int_b W(z)dz\) admits a similar expression in terms of the \(\vartheta\)-function with \(\hat{\tau} = (-\omega_a)/\omega_b\). This implies the boundedness of \(\text{Im} e^{i\phi} \Omega_b B_\rho(t)\). Then we have \(|B_\rho(t)| \leq C_0\) for some \(C_0 > 0\) in \(S_\rho(t'_\infty, \kappa_1, \delta_1)\). The implied constant of \(B_\rho(t) \ll 1\) in (5.1) may be supposed to be greater than \(2C_0\), which causes no changes in the subsequent equations by choosing \(t'_\infty\) larger if necessary, and hence the boundedness of \(B_\rho(t)\) has been shown independently of (5.1) under (6.6). The case \(-\pi/3 < \phi < 0\) is similarly treated under the supposition
\[
|\log(g_{11}g_{22})| \ll 1, \quad |\log(g_{21}/g_{11})| \ll 1 \quad \text{in} \quad S_\rho(t'_\infty, \kappa_1, \delta_1).
\]
Remark 6.3. The argument above works also under a weaker condition, say \(B_\rho(t) \ll t^{(1-\delta)/2}\). The supposition \(B_\rho(t) \ll 1\) in (5.1) guarantees that each turning point is located within the distance \(O(t^{-1})\) from its limit one, which enables us to use the limit Stokes graph in the WKB analysis.
Proposition 6.4. Suppose that $0 < \phi < \pi/3$ and \eqref{eq:6.6} (respectively, $-\pi/3 < \phi < 0$ and \eqref{eq:5.7}). Then, in $S_\phi(t'_\infty, \kappa_1, \delta_1)$, $B_\phi(t)$ is bounded, and

\[
 ie^{\phi} \left( t J_a - \frac{\Omega_a}{2} B_\phi(t) \right) = \frac{3}{2} \int_a^Z W(z) dz + \frac{3}{2} (1 + 2ia) g_0(0^+) \omega_a + 6 \log(g_{11g_{22}}) - 6 \pi i + O(t^{-\delta}),
\]

\[
= 3 \left( \frac{\partial}{\partial \nu} \left( \frac{1}{2} F(z^+_+, z^+_+) + \nu, \tau \right) - \frac{\partial}{\partial \nu} \left( \frac{1}{2} F(z^+_-, z^+_+) + \nu, \tau \right) \right) + \frac{3}{2} (1 + 2ia) g_0(0^+) \omega_a + 6 \log(g_{11g_{22}}) - 6 \pi i + O(t^{-\delta}),
\]

\[
g_0(0^+) = \frac{1}{\omega_a} \left( \pi i + \frac{\partial}{\partial \nu} (F(0^-, 0^+) + \nu, \tau) \right).
\]

The following fact guarantees the possibility of limitation with respect to $a_\phi$.

Proposition 6.5. Under the same supposition as in Proposition 6.4 we have

\[
\left( \int_{z^+_+}^{z^+_+} - \int_{z^-_+}^{z^-_+} \right) \frac{dz}{w(a_\phi, z)} = \left( \int_{z^+_+}^{z^+_+} - \int_{z^-_+}^{z^-_+} \right) \frac{dz}{w(A_\phi, z)} + O(t^{-1})
\]

uniformly in $z^+_+, z^-_+$ as $te^{i\phi} \to \infty$ through $S_\phi(t'_\infty, \kappa_1, \delta_1)$.

Proof. To show this proposition we note the lemma below, which follows from the relations

\[
\int \frac{w}{z} dz = -\frac{180}{A^2_\phi} \int w dz + \left( \frac{108}{A^2_\phi} - A_\phi \right) \int \frac{dz}{w} - \frac{w - 6 A_\phi w + 72}{A^2_\phi} z w,
\]

\[
\Omega_a J_b - \Omega_b J_a = -\frac{A^2_\phi}{15} \pi i, \quad J_{a, b} = \int_a^b w dz
\]

with $w = w(A_\phi, z)$, the latter equality being obtained by the same way as in the proof of Legendre’s relation \cite{4, 29}.

Lemma 6.6. $\Omega_a J_b - \Omega_b J_a = 12 \pi i$.

From the boundedness of $B_\phi(t)$ it follows that $\omega_{a,b} = \Omega_{a,b} + O(t^{-1})$. By Propositions 6.4 6.3 and Remark 6.1 in the case $0 < \phi < \pi/3$, we have

\[
\log(g_{12}/g_{22}) - \frac{\tau}{\log(g_{11g_{22}})}
\]

\[
= \left( \int_b^a - \tau \right) \frac{ie^{i\phi t}}{6} \cdot \frac{w(a_\phi, z)}{z^2} - \frac{1}{4} W(z) + \frac{1 + 2ia}{4z w(a_\phi, z)} dz - 7 \pi i + O(t^{-\delta}),
\]

\[
= -\frac{2 \pi e^{i\phi t}}{\omega_a} \left( F(z^+_+, z^+_+) - F(z^+_-, z^+_+) \right) + O(1)
\]

\[
= -\frac{2 \pi e^{i\phi t}}{\Omega_a} - \pi i \left( p(t) + \frac{\Omega_b}{\Omega_a} q(t) \right) + O(1) = \tau \ll 1,
\]

with $p(t) = p_+(t) - p_-(t)$, $q(t) = q_+(t) - q_-(t) \in \mathbb{Z}$, since $F(z^+_+, z^+_+) = 2p_+(t) + 2q_+(t) \tau$, $p_+, q_+ \in \mathbb{Z}$. Set $e^{i\phi J_a t/6} + \pi q(t) = X$, $e^{i\phi J_b t/6} - \pi p(t) = Y$, where $X, Y \in \mathbb{R}$ by the
Boutroux equations (2.1). Then, by Lemma 6.6
\[
\Omega_a Y = -2\pi e^{i\phi} t - i(e^{i\phi} t (\Omega_a J_a - \Omega_b J_a) / 6 + \Omega_b X - \Omega_a Y) \\
= -i (\Omega_b X - \Omega_a Y) \ll 1
\]
with \( \text{Im} (\Omega_b / \Omega_a) > 0 \), which implies \( |X|, |Y| \ll 1 \), and hence
\[
\pi p(t) = e^{i\phi} J_a t / 6 + O(1), \quad \pi q(t) = -e^{i\phi} J_a t / 6 + O(1).
\]
Since \( w(a, z)^{-1} - w(A, z)^{-1} = (z^2/2)w(A, z)^{-3}B_{\phi}(t)t^{-1} + O(t^{-2}) \), we have
\[
\left| \left( \int_{z^+} - \int_{z^-} \right) \frac{1}{w(a, z)} - \frac{1}{w(A, z)} \right| dz \ll \left| \left( \int_{z^+} - \int_{z^-} \right) \frac{z^2 B_{\phi}(t)t^{-1}}{w(A, z)^3} \right| + O(t^{-1})
\]
\[
\ll t^{-1} \left| \left( \int_{z^+} - \int_{z^-} \right) \frac{z^2 dz}{w(A, z)^3} \right| + O(t^{-1}) \ll t^{-1} |p(t)ja + q(t)jb| + O(t^{-1})
\]
\[
\ll |J_a j_a - J_a j_b| + O(t^{-1}) = 2(|\partial / \partial A_{\phi})(J_a \Omega_a - J_a \Omega_b)| + O(t^{-1}) \ll t^{-1},
\]
where \( j_{a,b} = \int a,b z^2 w(A, z)^{-3} dz \). This completes the proof of the proposition. \( \square \)

7. Proofs of the main theorems

7.1. Proofs of Theorems 2.1 and 2.2

Suppose that \( 0 < \phi < \pi/3 \). Let \( G = (g_{ij}) \in SL_2(\mathbb{C}) \) be a given matrix with \( g_{11}g_{12}g_{22} \neq 0 \) in the inverse monodromy problem. Then
\[
\log(g_{12}/g_{22}) - \tau \log(g_{11}g_{22}) = \left( \int_b - \tau \int_a \right) \frac{i e^{i\phi} t}{6} \cdot \frac{w(a, z)}{z^2} - \frac{1}{4} W(z) + \frac{1}{4z w(a, z)} dz - \tau \pi i + O(t^{-\delta})
\]
\[
= -\frac{2\pi e^{i\phi} t}{\omega_a} - \frac{\pi i}{2}(F(z^+, z^+_+) - F(z^-, z^-_+)) - \frac{\pi i}{2}(1 + 2ia)F(0^-, 0^+) - \tau \pi i + O(t^{-\delta})
\]
(cf. Proof of Proposition 6.5). By Proposition 6.5 replacing \( a_\phi \) with \( A_\phi \), we have
\[
\log(g_{12}/g_{22}) - \tau \log(g_{11}g_{22}) = -\frac{2\pi e^{i\phi} t}{\Omega_a}
\]
\[
- \frac{\pi i}{2} (F_{A_\phi}(z^+, z^+_+) - F_{A_\phi}(z^-, z^-_+)) - \frac{\pi i}{2}(1 + 2ia)F_{A_\phi}(0^-, 0^+) - \frac{\Omega_b}{\Omega_a} \pi i + O(t^{-\delta})
\]
with \( F_{A_\phi}(z, z) = \frac{1}{\Omega_a} \int z dz / w(A, z) \). Note that
\[
F_{A_\phi}(z^+, z^+_+) - F_{A_\phi}(z^-, z^-_+) = 2(F_{A_\phi}(\infty, z^+_+) - F_{A_\phi}(\infty, z^-_+)), \quad F_{A_\phi}(0^-, 0^+) = 2F_{A_\phi}(\infty, 0^+),
\]
and let \( \varphi(u) = \varphi(w; g_2, g_3) \) with \( g_2 = \frac{1}{12} A_\phi^2, g_3 = \frac{1}{216} A_\phi^3 - 1 \). Let us set
\[
u_+ = \Omega_a F_{A_\phi}(\infty, z^+_+), \quad \nu_- = \Omega_a F_{A_\phi}(\infty, z^-_-), \quad \text{i.e.} \quad z^+_+ = \varphi(\nu_+) + \frac{A_\phi}{12}
\]
to write
\[
u_+ - \nu_- = 2\pi e^{i\phi} t + \frac{i}{\pi} \left( \frac{\Omega_a \log g_{12}}{g_{22}} - \frac{\Omega_b \log g_{11}g_{22}}{g_{22}} \right) - \Omega_b - (1 + 2ia)\Omega_a F_{A_\phi}(\infty, 0^+) + O(t^{-\delta}).
\]
By the addition theorem for the \( \varphi \)-function

\[
\varphi(u_+ + u_-) = -\varphi(u_+) - \varphi(u_-) + \frac{1}{4} \left( \frac{\varphi'(u_+)}{\varphi(u_+)} - \frac{\varphi'(u_-)}{\varphi(u_-)} \right)^2
\]

\[
= -z_+^+ + z_-^- + \frac{A}{6} + \frac{1}{4} \left( \frac{w(z_+^+)}{w(z_-^-)} \right)^2.
\]

By (6.4), \( z_+ = y \) and \( z_- = (i/2)e^{-i\phi}y^{-1}\Gamma_0(t, y, y^t) \) satisfy

\[
z_+^+ + z_-^- = e^{-i\phi}K_+,
\]

\[
w(z_+^+) = ie^{-i\phi}(b_3/\lambda)(z_+^+) = 1 - ie^{-i\phi}\Gamma_0(t, y, y^t)z_+^+ + O(t^{-1}),
\]

and hence

\[
\varphi(u_+ + u_-) = -e^{-i\phi}K_+ + \frac{A}{6} + \frac{1}{4} (ie^{-i\phi}\Gamma_0(t, y, y^t) + O(t^{-1}))^2
\]

\[
= \frac{A}{6} - \frac{1}{4} (4e^{-i\phi}K_+ + e^{-2i\phi}\Gamma_0(t, y, y^t))^2 + O(t^{-1})
\]

\[
= -\frac{A}{12} + O(t^{-1}),
\]

since \( 4e^{-i\phi}K_+ + e^{-2i\phi}\Gamma_0(t, y, y^t)^2 = a_\phi + O(t^{-1}) \). This implies

\[
(7.2) \quad u_+ + u_- = \int_{\infty}^{0^+} \frac{dz}{w(A_\phi, z)} + O(t^{-1}) = \Omega_\phi F_{A_\phi}(\infty, 0^+) + O(t^{-1}).
\]

From (7.1) and (7.2) with \( \Omega_\phi F_{A_\phi}(\infty, 0^+) = \Omega_0 \), it follows that

\[
u = \int_{\infty}^{z_+^+} \frac{dz}{w(A_\phi, z)} + O(t^{-1}) = \Omega_\phi F_{A_\phi}(\infty, 0^+) + O(t^{-1}),
\]

\[
u = \int_{0^+}^{z_-^-} \frac{dz}{w(A_\phi, z)} + O(t^{-1}) = -\frac{i}{2\pi} \left( \Omega_\phi \log \frac{g_{12}}{g_{22}} - \Omega_\phi \log(g_{11}g_{22}) \right) - \frac{\Omega_\phi}{2} - ia\Omega_0 + O(t^{-\delta}),
\]

\[
u = \int_{0^+}^{z_-^-} \frac{dz}{w(A_\phi, z)} + O(t^{-1}) = -\frac{i}{2\pi} \left( \Omega_\phi \log \frac{g_{12}}{g_{22}} - \Omega_\phi \log(g_{11}g_{22}) \right) + \frac{\Omega_\phi}{2} + (1 + ia)\Omega_0 + O(t^{-\delta}),
\]

which leads us to the asymptotic expressions of Theorem 2.1 and Remark 2.1.

**Justification.** The justification of \( y(x) \) as a solution of (1.2) is made along the line in [13] pp. 105–106, pp. 120–121. Let \( \mathcal{G} = (g_{12}/g_{22}, g_{11}g_{22}) \) be a given point such that \( g_{11}g_{12}g_{22} \neq 0 \) on the monodromy manifold for (1.2). In addition to \( y(x) \) obtained above, we have the following expression of \( B_\phi(t) \) from Proposition 6.4.

**Proposition 7.1.** In \( S_\phi(t', \kappa_1, \delta_1) \),

\[
ie^{i\phi} \left( tJ_\alpha - \frac{\Omega_a}{2} B_\phi(t) \right) = \frac{3}{2} \left( \frac{\varphi'(t_0)}{\varphi(t_0)} (\Omega^{-1}_a (x - x_0^+) + \nu, \gamma_0) + \frac{\varphi'(t_0)}{\varphi(t_0)} (\Omega^{-1}_a (i(x - x_0^+) - \Omega_0) + \nu, \gamma_0) \right)
\]

\[
+ \frac{3}{2} (1 + 2ia) g_0(0^+) \Omega_a + 6 \log(g_{11}g_{22}) - 6\pi i + O(t^{-\delta})
\]

with \( x = e^{i\phi} t, \quad \tau_\Omega = \Omega_b / \Omega_a. \)
The equation about $u_+$ and the proposition above provide the leading term expressions
\[ y_{as} = y_{as}(G, t) = \varphi(i(e^{i\phi}t - x_0^+));g_2(A_\phi), g_3(A_\phi) + \frac{A_\phi}{12} \]
and $(B_\phi)_{as}(G, t)$ without $O(t^{-\delta})$, where $x_0^+$ depends on $(g_{12}/g_{22}, g_{11}g_{22})$. Taking (1.2) and (5.1) into account, we set
\[ y_{as}^t = \frac{-y_{as}}{2} t^{-1} + ie^{i\phi} \sqrt{4y_{as}^2 - A_\phi y_{as}^2 + 1 - (3ie^{-i\phi}(1 + 2ia) + (B_\phi)_{as} y_{as}) y_{as} t^{-1}}, \]
where the branch of the square root is chosen in such a way that $y_{as}^t$ is compatible with $(\partial/\partial t)y_{as}$. Then $(y_{as}, y_{as}^t) = (y_{as}(G, t), y_{as}^t(G, t))$ fulfills (5.1) with $B_\phi(t) = (B_\phi)_{as}(G, t)$ in the domain $\tilde{S}(\phi, t_\infty, \kappa_0, \delta_2) = \{ t \mid \text{Re } t > t_\infty, |\text{Im } t| < \kappa_0 \} \cup \{ t \in \mathbb{R} \setminus Z_0 \mid t - e^{-i\phi}\sigma < \delta_2 \}$ with $Z_0 = \{ ix_0^+ + \Omega_\alpha Z + \Omega_\beta Z \} \cup \{ ix_0^+ + \Omega_0 + \Omega_\alpha Z + \Omega_\beta Z \} \cup \{ ix_0^+ + \xi_0 | \varphi(\xi_0) = -A_\phi/12 \}$. Let $G_{as}(t)$ be the monodromy data for system (1.4) containing $(y_{as}, y_{as}^t)$. As a result of the WKB analysis for the direct monodromy problem we have $\|G_{as}(t) - G\| \ll t^{-\delta}$, which holds uniformly in a neighbourhood of $G$. Then the justification scheme of Kitaev [13] applies to our case. Using the maximal modulus principle in each neighbourhood of $\phi = ix_0^+ + \{ \Omega_0 + \Omega_\alpha Z + \Omega_\beta Z \} \cup \{ \xi_0 | \varphi(\xi_0) = -A_\phi/12 \}$, we obtain Theorem 2.1. Theorem 2.2 is proved by the same argument as above.

7.2. Proof of Theorem 2.3. Let (1.4) with $y^t = (d/dt)y$ be an isomonodromy system. Equation (1.2), system (1.4) and the function $a_\phi$ with $y^t = (d/dt)y$ remain invariant under the substitution
\[ \phi = \tilde{\phi} + 2m\pi/3, \quad y = e^{2m\pi i/3} y, \quad x = e^{2m\pi i/3} x, \quad \lambda = e^{2m\pi i/3} \lambda, \quad a_\phi = e^{2m\pi i/3} a_\phi. \]
To show the theorem we use this symmetry (cf. [13]). Let $\phi$ be such that $0 < |\phi - 2m\pi/3| < \pi/3$. Then a new system with respect to $(\lambda, \tilde{y}, \tilde{x}, \tilde{\phi})$ is an isomonodromy system for $0 < |\tilde{\phi}| < \pi/3$. Denote by $G^{(m)}$ a connection matrix as the matrix monodromy data for the system governed by $\tilde{y}(\tilde{x}) = e^{-2m\pi i/3} y(x) = e^{-2m\pi i/3} e^{2m\pi i/3} x$. We would like to know the relation between $G^{(m)}$ and $G$. The matrix solutions of the new system are
\[ \tilde{Y}_{\infty}(\lambda) \sim \tilde{\lambda}^{-(1/2 + ia)} \sigma_3 \exp(-(i/6)e^{i\phi} t \tilde{\lambda}^2 \sigma_3) \]
as $\lambda \to \infty$ through the sector $|\arg \lambda + \tilde{\phi}/2 - j\pi/2| < \pi/2$, and
\[ \tilde{Y}^0_{\infty}(\lambda) \sim (i/\sqrt{2})(\sigma_1 + \sigma_3) \exp(-(2i/3)e^{i\phi} t \tilde{\lambda}^{-1} \sigma_3) \]
as $\lambda \to 0$ through the sector $|\arg \lambda - \phi - j\pi| < \pi$. The connection matrix $G^{(m)}$ is defined by $Y^{\infty, s}_0(\lambda) = \tilde{Y}^{\infty}(\lambda) \Theta_{0, s}^{\sigma_3} = \tilde{Y}^0_{\infty}(\lambda) G^{(m)}$. Note that $\tilde{Y}^{\infty}_0(\tilde{\lambda})$ and $\tilde{Y}^0_{\infty}(\tilde{\lambda})$ are also expressed as
\[ \tilde{Y}^{\infty}_0(\tilde{\lambda}) = \tilde{Y}^{\infty}_0(e^{-2m\pi i/3} \lambda) \sim \lambda^{-(1/2 + ia)} \sigma_1 \exp(-(i/6)e^{i\phi} t \lambda^2 \sigma_3) e^{(2m\pi i/3)(1/2 + ia) \sigma_3} \]
in the sector $|\arg \lambda + \phi/2 - m\pi| < \pi/2$, and that
\[ \tilde{Y}^0_{\infty}(\tilde{\lambda}) = \tilde{Y}^0_{\infty}(e^{-2m\pi i/3} \lambda) \sim (i/\sqrt{2})(\sigma_1 + \sigma_3) \exp(-(2i/3)e^{i\phi} t \lambda^{-1} \sigma_3) \]
in the sector $|\arg \lambda - \phi| < \pi$. Then we have $\tilde{Y}_0^0(\tilde{\lambda}) = \hat{Y}_0^0(\lambda)$ and, if $m \geq 1$,
\[
\tilde{Y}_0^\infty(\tilde{\lambda}) = \tilde{Y}_0^\infty(\lambda)e^{(2m\pi i/3)(1/2+i)\sigma_3} \quad \text{for} \quad \lambda = \frac{e^{(2m\pi i/3)(1/2+i)\sigma_3} \cdot \hat{Y}_0^0(\lambda)}{\tilde{Y}_0^0(\lambda)}
\]
which implies, by Remark 3.1
\[
G^{(m)} = GS_0^\infty \cdot S_1^\infty \cdots S_{2m-2}^\infty S_{2m-1}^\infty e^{(2m\pi i/3)(1/2+i)\sigma_3}.
\]
This combined with Proposition 3.4 yields the expression of $G^{(m)}$ for $m \geq 1$ as in the theorem. Note that
$P(u, A) = \varphi(u; g_2(A), g_3(A)) + \frac{1}{12}A$ solves $(P_u)^2 = 4P_3 - AP^2 + 1$. Then
\[
P_u(u, A) = e^{-2\pi i/3}P_u(e^{2\pi i/3}u, e^{2\pi i/3}A) = e^{2\pi i/3}P_u(e^{-2\pi i/3}u, e^{-2\pi i/3}A).
\]
By Theorems 2.1 and 2.2, $\tilde{y}(\tilde{x}) = -e^{-2\pi i/3}y(x)$ for $0 < |\phi - 2m\pi/3| = |\tilde{\phi}| < \pi/3$ is represented by
\[
e^{-2\pi i/3}y(x) = \tilde{y}(\tilde{x}) = P_u(i(\tilde{x} - x_0(G^{(m)}, \Omega_0^{\tilde{\phi}}, \Omega_0^{\phi})) + O(x^{-\delta}; A_0).
\]
Using the relation above, we have
\[
y(x) = e^{-2\pi i/3}P_u(i(x - x_0(G^{(m)}, \Omega_0^{\tilde{\phi}}, \Omega_0^{\phi})) + O(x^{-\delta}; A_0))
\]
\[
= P_u(i(x - e^{-2\pi i/3}x_0(G^{(m)}, \Omega_0^{\tilde{\phi}}, \Omega_0^{\phi})) + O(x^{-\delta}; A_0))
\]
\[
= P_u(i(x - x_0(G^{(m)}, \Omega_0^{\tilde{\phi}}, \Omega_0^{\phi})) + O(x^{-\delta}; A_0)),
\]
which is denoted by $P(A_0, x_0(G^{(m)}, \Omega_0^{\tilde{\phi}}, \Omega_0^{\phi}); x)$ as in the theorem.

8. Modulus $A_\phi$ and the Boutroux Equations

Recall the elliptic curve $\Pi_A$ for $w(A, z)^2 = 4z^3 - Az^2 + 1$ defined in Section 2.2. For a given $\phi \in \mathbb{R}$ we would like to examine the modulus $A_\phi \in \mathbb{C}$ such that, for every cycle $c \subset \Pi_{A_\phi}$,
\[
\text{Im} e^{i\phi} \int_c \frac{w(A_\phi, z)}{z^2} dz = 0.
\]
First, for $|\phi| \leq \pi/3$, let us consider $A_\phi$ satisfying the Boutroux equations
\[
(BE)_\phi : \quad \text{Im} e^{i\phi} I_a(A_\phi) = 0, \quad \text{Im} e^{i\phi} I_b(A_\phi) = 0,
\]
where $a$, $b$ denote the basic cycles given in Section 2.2 and
\[
I_{a, b}(A) = \int_{a, b} \frac{w(A, z)}{z^2} dz = \int_{a, b} \frac{1}{z^2} \sqrt{4z^3 - Az^2 + 1} dz.
\]
It is easy to see that $w(A, z)^2 = 4z^3 - Az^2 + 1$ has double roots $z_0, z_1$ if and only if
\[
A = a \cdot 2^{2/3}, \quad z_0, z_1 = 2^{-1/3}, \quad z_2 = -4^{-2/3};
\]
\[
A = a \cdot 2^{2/3}e^{\pm 2\pi i/3}, \quad z_0, z_1 = 2^{-1/3}e^{\pm 2\pi i/3}, \quad z_2 = -4^{-2/3}e^{\pm 2\pi i/3}.
\]
Example 8.1. When $\phi = 0$, we have $I_a(3 \cdot 2^{2/3}) = 0$, $I_b(3 \cdot 2^{2/3}) = -2^{4/3}3^{2/3}$. Indeed
\[
I_b(3 \cdot 2^{2/3}) = 2 \int_{-4 - 2^{2/3}}^{4 - 2^{2/3}} \frac{2}{z^2} \left( i \sqrt{2^{1/3} - z} \right)^2 \sqrt{2^{1/3} - z} + 4 - 2^{2/3} dz = -2^{4/3} \int_{-1}^{2} \frac{(2 - t)(3 + 1)}{t^2} dt,
\]
in which the residue of the integrand at $z = 0$ vanishes.
Note that $a$ is a cycle enclosing the cut $[z_0, z_1]$. In accordance with [14, Section 7] we begin with the following:

**Proposition 8.1.** Suppose that $\text{Im} I_n(A) = 0$. Then $A \in \mathbb{R}$.

**Proof.** Set

$$J_n(A) = \int_{-a}^{a} \frac{1}{z^2} v(A, z) \, dz$$

with $v(A, z) = \sqrt{4z^3 + Az^2 - 1} = -iv(A, -z)$. Since $I_n(A) = -iJ_n(A)$, the supposition means $J_n(A) \in i\mathbb{R}$. In this proof, to simplify the description, we write $v(A, z) = v_A(z)$, $v(A, z) = \sqrt{4z^3 + Az^2 - 1}$, and $v(A, z) = (v_A \pm v_A)(z)$. Then

$$0 = J_n(A) + \overline{J_n(A)} = J_n(A) + J_n(A) = J_n(A) - J_n(A) = (A - A) \int_{-a}^{a} \frac{dz}{(v_A + v_A)(z)}.$$

The polynomials $v_A(z)^2$ and $v_A(z)^2$ have the roots $-z_0, -z_1, -z_2$, and $-\overline{z_0}, -\overline{z_1}, -\overline{z_2}$, respectively. The algebraic functions $(v_A \pm v_A)(z)$ may be considered on the two-sheeted Riemann surface glued along the cuts $[-z_0, -z_1], [-\overline{z_0}, -\overline{z_1}], [-z_2, -\overline{z_2}] \cup [-\infty, -\text{Re} z_2]$ (cf. Figure 6). The cycle $-a$ may be supposed to enclose both cuts $[-z_0, -z_1], [-\overline{z_0}, -\overline{z_1}]$, and the cycles as in Figures 6(a.1) and (a.2) may be deformed into contours consisting of horizontal and vertical lines and enclosing the cuts $[-z_2, -\overline{z_2}] \cup [-\infty, -\text{Re} z_2]$ clockwise as in Figures (a*.1) and (a*.2), respectively. Possible extension of this contour is caused by further movement of $-z_0, -z_1$ and $-z_2$, and is given by adding horizontal and vertical lines located in the symmetric position with respect to the real axis. To show $A \in \mathbb{R}$ it is sufficient to verify that, under the supposition $A - A \neq 0$,

$$J = \int_{-a}^{a} \frac{dz}{(v_A + v_A)(z)} \neq 0.$$

Let us compute this integral along the contour $-a$, say as in Figure 6(a*.2) with vertices $\alpha \pm i\beta, \gamma \pm i\beta$ such that $-z_2, -\overline{z_2} = \gamma \pm i\beta$, in which $\alpha \leq \gamma, \beta \geq 0$, and $\alpha$ may be supposed to be $\alpha < 0$. The integral $J$ is decomposed into three parts: $J = J_0 + J_{\text{hor}} + J_{\text{ver}}$ with the real line part

$$J_0 = 2 \int_{-\infty}^{\alpha} \frac{ds}{(v_A + v_A)(s + i\beta)},$$

the horizontal part

$$J_{\text{hor}} = J_{\text{hor}}^+ + J_{\text{hor}}^-,$$

$$J_{\text{hor}}^+ = \int_{\alpha}^{\gamma} \frac{ds}{(v_A + v_A)(s + i\beta)} + \int_{\gamma}^{\alpha} \frac{ds}{(v_A + v_A)(s + i\beta)},$$

$$J_{\text{hor}}^- = \int_{\alpha}^{\gamma} \frac{ds}{(v_A + v_A)(s - i\beta)} + \int_{\gamma}^{\alpha} \frac{ds}{(v_A + v_A)(s - i\beta)},$$

and the vertical part

$$J_{\text{ver}} = J_{\text{ver}}^+ + J_{\text{ver}}^-,$$

$$J_{\text{ver}}^+ = \int_{0}^{\beta} \frac{idt}{(v_A + v_A)(\alpha + it)} + \int_{\beta}^{0} \frac{idt}{(v_A + v_A)(\alpha + it)},$$

$$J_{\text{ver}}^- = \int_{0}^{\beta} \frac{idt}{(v_A + v_A)(\alpha - it)} + \int_{\beta}^{0} \frac{idt}{(v_A + v_A)(\alpha - it)}.$$
Then we have
\[ J_{\text{hor}} = \frac{2}{A - A} \left( \int_{-\infty}^{\alpha} v_A(s + i\beta) \frac{ds}{(s + i\beta)^2} - \int_{\alpha}^{\infty} v_A(s - i\beta) \frac{ds}{(s - i\beta)^2} \right) \in \mathbb{R} \]
and
\[ J_{\text{ver}} = \frac{2i}{A - A} \left( \int_{0}^{\beta} \frac{v_A(\alpha + it)}{(\alpha + it)^2} dt + \int_{0}^{\beta} \frac{v_A(\alpha - it)}{(\alpha - it)^2} dt \right) \in \mathbb{R}, \]
and hence \( J_{\text{hor}} + J_{\text{ver}} \in \mathbb{R} \). Furthermore, observing, for \(-t = x - \alpha, t \geq 0, \alpha < 0,\)
\[ v_A(x) = (-4(t - \alpha)^3 + (\text{Re} A + i\text{Im} A)(t - \alpha)^2 - 1)^{1/2} = i(g(t) - ih(t))^{1/2}, \]
\[ g(t) = 4(t - \alpha)^3 - \text{Re} A \cdot (t - \alpha)^2 + 1, \quad h(t) = \text{Im} A \cdot (t - \alpha)^2, \]
we have
\[ \frac{1}{2} J_0 = \int_{-\infty}^{\infty} \frac{dx}{(v_A + v_A(x))} = -\frac{i}{\sqrt{2}} \int_{0}^{\infty} \frac{dt}{\sqrt{g(t) + \sqrt{g(t)^2 + h(t)^2}}} \in i\mathbb{R} \setminus \{0\}, \]
which implies \( J \neq 0 \) under the supposition \( A - \overline{A} \neq 0 \). In the case where extension by horizontal or vertical lines occurs, the contributions from these parts to \( J \) are integrals analogous to \( J_{\text{hor}} \) or \( J_{\text{ver}} \), and \( J \neq 0 \) are similarly shown. \( \square \)
Let us examine \( I_a(A) \) for \( A \in \mathbb{R} \). It is easy to see that, \( w(A, z)^2 \) has real roots \( z_2 < z_1 < z_0 \) if \( A > 3 \cdot \frac{2}{3} \). Then \( I_a(A) \in i \mathbb{R} \setminus \{0\} \). For \( A = 3 \cdot \frac{2}{3} \) we have \( z_2 < z_1 = z_0 = 2^{-1/3} \), and then \( I_a(3 \cdot \frac{2}{3}) = 0 \).

Suppose that \( A < 3 \cdot \frac{2}{3} \). The roots of \( w(A, z)^2 \) are \( \alpha \pm i \beta \) and \( z_2 \) with \( \alpha, \beta, z_2 \in \mathbb{R} \), and \( A < \) is a cycle enclosing the cut \( [\alpha - i \beta, \alpha + i \beta] \). Then \( I_a(A) \in i \mathbb{R} \), since \( I_a(A) = -I_a(A) \), and the integral

\[
I_a(A) = 2i \int_{-\beta}^{\beta} \frac{w(A, \alpha + it)}{(\alpha + it)^2} dt = 4i \int_0^{\beta} \text{Re} \frac{w(A, \alpha + it)}{(\alpha + it)^2} dt
\]
satisfies, for \( A < 3 \cdot \frac{2}{3} \),

\[
\frac{\partial}{\partial A} \left( \frac{1}{i} I_a(A) \right) = 2 \int_0^\beta \text{Re} w(A, \alpha + it)^{-1} dt = \sqrt{2} \int_0^\beta \frac{\sqrt{g_* + \sqrt{g_*^2 + h_*^2}}}{\sqrt{g_*^2 + h_*^2}} dt > 0,
\]
where

\[
g_* = g_*(t) = \text{Re} w(A, \alpha + it)^2 = 4(\alpha^3 - 3\alpha t^2) - A(\alpha^2 - t^2) + 1,
\]

\[
h_* = h_*(t) = \text{Im} w(A, \alpha + it)^2 = 4(-\beta^3 + 3\alpha^2 t) - 2A\alpha t.
\]

This implies \( I_a(A) \in i \mathbb{R} \setminus \{0\} \) for \( A < 3 \cdot \frac{2}{3} \).

The fact above combined with Proposition [8.1] leads us to the following.

**Proposition 8.2.** If \( \phi = 0 \), then the Boutroux equations (BE)₀ admit a unique solution \( A_0 = 3 \cdot \frac{2}{3} \).

**Corollary 8.3.** For every \( A \in \mathbb{C} \), \( (I_a(A), I_b(A)) \neq (0, 0) \).

**Proof.** If \( I_a(A) = 0 \), then \( A \in \mathbb{R} \) by Proposition [8.1]. By Proposition [8.2] and Example [8.1] \( A = 3 \cdot \frac{2}{3} \) and \( I_b(A) \neq 0 \). □

**Proposition 8.4.** Suppose that, for \( A_\phi \) solving (BE)₀ with \( 0 < |\phi| \leq \pi/3 \), the elliptic curve \( \Pi_{A_\phi} \) degenerates. Then \( \phi = \pi/3 \) or \( -\pi/3 \) and \( A_{\pm \pi/3} = 3 \cdot \frac{2}{3} e^{\pm 2\pi i/3} \).

**Proof.** When \( \Pi_{A_\phi} \) degenerates, \( A_\phi = 3 \cdot \frac{2}{3} e^{2k\pi i/3}, k = 0, \pm 1 \). Suppose that \( A_\phi = 3 \cdot \frac{2}{3} e^{2\pi i/3} \), and that the roots of \( w(A_\phi, z)^2 \) are \( z_0 = z_1 \) and \( z_2 \neq z_0, z_1 \). Then

\[
\mathbb{R} \ni e^{i\phi} \int_{z_0}^{z_2} \frac{1}{z^2} \sqrt{4z^3 - A_\phi z^2 + 1} \, dz = e^{i(\phi - 2\pi/3)} \int_{\zeta_0}^{\zeta_2} \frac{1}{\zeta^2} \sqrt{4\zeta^3 - 3 \cdot 2^{2/3} \zeta^2 + 1} \, d\zeta \neq 0
\]

with \( \zeta_{0,2} = z_{0,2} e^{-2\pi i/3} \in \{2^{-1/3}, -4^{-2/3}\} \), which implies \( \phi = -\pi/3 \). Similarly, if \( A_\phi = 3 \cdot \frac{2}{3} e^{-2\pi i/3} \), the \( \phi = \pi/3 \). □

**Proposition 8.5.** If \( \phi = \pm \pi/3 \), then the Boutroux equations (BE)_{\pm \pi/3} admit a unique solution \( A_{\pm \pi/3} = 3 \cdot \frac{2}{3} e^{\pm 2\pi i/3} \).

**Proof.** For \( \phi = \pi/3 \), (BE)_{\pi/3} are equivalent to

\[
e^{\pi i/3} \int_c \frac{1}{z^2} \sqrt{4z^3 - A_{\pi/3} z^2 + 1} \, dz \in \mathbb{R}
\]
for every cycle $c$ on $\Pi_{A/3}$, which is written as $(\text{BE})_0$ with $\phi = 0$

$$e^{\pi i} \int_{c^{2\pi i/3}} \frac{1}{\zeta} \sqrt{4\zeta^3 - e^{2\pi i/3} A_{2/3} \zeta^2 + 1} \, d\zeta \in \mathbb{R} \quad (z = e^{-2\pi i/3} \zeta).$$

Then by Proposition 8.2, $e^{2\pi i/3} A_{2/3} = 3 \cdot 2^{2/3}$ is a unique solution of $(\text{BE})_{2/3}$. □

The function $h(A) = \mathcal{I}_a(A)/\mathcal{I}_b(A)$ [21, Appendix I] is useful in examining $A_\phi$.

**Proposition 8.6.** Suppose that $A \in \mathbb{C}$.

1. If $A$ solves $(\text{BE})_\phi$ for some $\phi \in \mathbb{R}$ and $\mathcal{I}_b(A) \neq 0$, then $h(A) \in \mathbb{R}$.

2. If $h(A) \in \mathbb{R} \setminus \{0\}$, then, for some $\phi \in \mathbb{R}$, $A$ solves $(\text{BE})_\phi$.

**Proof.** Suppose that $h(A) = \rho \in \mathbb{R}$, and write $\mathcal{I}_a(A) = u + iv$, $\mathcal{I}_b(A) = U + iV$ with $u, v, U, V \in \mathbb{R}$. Then $u = \rho U$, $v = \rho V$, and hence $v/u = V/U = -\tan \phi$ for some $\phi \in [-\pi/2, \pi/2]$. This implies $\text{Im} \, e^{i\phi} \mathcal{I}_a(A) = \text{Im} \, e^{i\phi} \mathcal{I}_b(A) = 0$. □

**Proposition 8.7.** The set $\{ A \in \mathbb{C} \mid A$ solves $(\text{BE})_\phi$ for some $\phi \in \mathbb{R} \}$ is bounded.

**Proof.** The roots of $w(A, z)$ are $z_0, z_1 \sim \pm A^{-1/2}$, $z_2 \sim A/4$ if $A$ is large. Then

$$\int_{z_2}^{z_0} \frac{1}{z^2} w(A, z) \, dz \sim \int_{A/4}^{A^{-1/2}} \frac{1}{z^2} \sqrt{4z^3 - Az^2 + 1} \, dz \sim iA^{1/2} \int_{1}^{A^{-1/2}} \frac{1}{t} \sqrt{1-t} \, dt$$

$$\sim iA^{1/2} (2 + \log(2A^{-3/2})) \sim -\frac{3i}{2} A^{1/2} \log A,$$

and

$$\int_{z_0}^{z_1} \frac{1}{z^2} w(A, z) \, dz \sim \int_{A^{-1/2}}^{A^{-1/2}} \frac{1}{z^2} \sqrt{4z^3 - Az^2 + 1} \, dz \sim A^{1/2} \int_{-1}^{1} \frac{1}{t^2} \sqrt{1-t^2} \, dt \sim \pi A^{1/2}.$$ 

This implies $h(A) \not\in \mathbb{R}$ if $A$ is sufficiently large, which completes the proof. □

The following fact is used in discussing solutions of $(\text{BE})_\phi$.

Let $0 < |\phi| < \pi/3$, and write $\mathcal{I}_a(A) = u(A) + iv(A)$, $\mathcal{I}_b(A) = U(A) + iV(A)$.

Note that $A$ solves $(\text{BE})_\phi$ if and only if

$$\text{Im} \, e^{i\phi} \mathcal{I}_a(A) = u(A) \sin \phi + v(A) \cos \phi = 0, \quad \text{Im} \, e^{i\phi} \mathcal{I}_b(A) = U(A) \sin \phi + V(A) \cos \phi = 0,$$

that is,

$$(8.1) \quad u(A) \tan \phi + v(A) = 0, \quad U(A) \tan \phi + V(A) = 0.$$ 

Then, by the Cauchy-Riemann equations, the Jacobian for (8.1) with $A = x + iy$ is written as

$$\det J(\phi, A) = \det \begin{pmatrix} u_x \tan \phi + v_x & u_y \tan \phi + v_y \\ U_x \tan \phi + V_x & U_y \tan \phi + V_y \end{pmatrix}$$

$$= (1 + \tan^2 \phi)(v_x V_y - v_y V_x)$$

$$= -\frac{1}{4}(1 + \tan^2 \phi)|\Omega_a(A)|^2 \text{Im} \, \frac{\Omega_b(A)}{\Omega_a(A)},$$
where \( \Omega_a(A) \) and \( \Omega_b(A) \) are periods of the elliptic curve \( w(A, z) \). For \( 0 < |\phi| < \pi/3 \), \((d/dt)(8.1)\) with \( t = \tan \phi \) is written as

\[
J(\phi, A) \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} + \begin{pmatrix} u(A) \\ U(A) \end{pmatrix} = 0.
\]

Then we have

\[
(8.3) \quad (x'(t), y'(t)) \neq (0, 0) \quad \text{and} \quad (d/d\phi)A = (x'(t) + i y'(t)) \cos^{-2} \phi \neq 0
\]

for \( 0 < |\phi| < \pi/3 \).

**Proposition 8.8.** Suppose that, for some \( \phi_0 \) such that \( 0 < |\phi_0| < \pi/3 \), \( A_{\phi_0} \) solves \((\text{BE})_{\phi_0}\). Then there exists a trajectory \( T_0 : A = \chi(\phi_0, \phi) \) for \( 0 < |\phi| < \pi/3 \) with the properties:

1. \( \chi(\phi_0, \phi_0) = A_{\phi_0} \);
2. for each \( \phi \), \( A = \chi(\phi_0, \phi) \) solves \((\text{BE})_{\phi} \);
3. \( \chi(\phi_0, \phi) \) is smooth for \( 0 < |\phi| < \pi/3 \).

**Proof.** Since the Jacobian \((8.2)\) satisfies \( \det J(\phi_0, A_{\phi_0}) \in \mathbb{R} \setminus \{0\} \), there exists a local trajectory \( A = \chi_{\text{loc}}(\phi_0, \phi) \) having the properties (1), (2) and (3) above for \( |\phi - \phi_0| < \delta \), where \( \delta \) is sufficiently small. Since \((8.2)\) is in \( \mathbb{R} \setminus \{0\} \) for \( 0 < |\phi| < \pi/3 \), \( \chi_{\text{loc}}(\phi_0, \phi) \) may be extended to the interval \( 0 < |\phi| < \pi/3 \). \( \square \)

**Proposition 8.9.** The trajectory \( T_0 : A = \chi(\phi_0, \phi) \) given above may be extended to \( |\phi| \leq \pi/3 \) such that \( \chi(\phi_0, \phi) \) is continuous in \( \phi \) and that \( \chi(\phi_0, 0) = A_0 = 3 \cdot 2^{2/3}, \chi(\phi_0, \pm \pi/3) = A_{\pm \pi/3} = 3 \cdot 2^{2/3}e^{\mp 2\pi i/3} \).

**Proof.** To show that \( \chi(\phi_0, \phi) \to A_0 \) as \( \phi \to 0 \), suppose to the contrary that there exists a sequence \( \{\phi_\nu\} \) such that \( \phi_\nu \to 0 \) and that \( \chi(\phi_0, \phi_\nu) \) does not converge to \( A_0 \). There exists a subsequence \( \{\phi_\nu(n)\} \) such that \( \chi(\phi_0, \phi_\nu(n)) \to A'_0 \) for some \( A'_0 \neq A_0 \), since, by Proposition 8.7, the trajectory \( T_0 \) for \( 0 < |\phi| < \pi/3 \) is bounded. Then we have \( \text{Im } I_a(A'_0) = \text{Im } I_b(A'_0) = 0 \), which contradicts the uniqueness of a solution of \((\text{BE})_0 \). Hence \( \chi(\phi_0, \phi) \) is extended to \( \phi = 0 \) and is continuous in a neighbourhood of \( \phi = 0 \). By Proposition 8.5, it is possible to extend \( \chi(\phi_0, \phi) \) to \( \phi = \pm \pi/3 \) by the same argument. \( \square \)

**Lemma 8.10.** \( h'(A) = -6\pi i I_b(A)^{-2} \).

**Proof.** From \( I'_{a, b}(A) = -\Omega_{a, b}/2 \) and Lemma 6.6, the conclusion follows. \( \square \)

**Corollary 8.11.** If \( I_b(A) \neq 0, \infty \), then \( h(A) \) is conformal around \( A \).

By Example 8.1, \( h(A) \) is conformal at \( A_0 = 3 \cdot 2^{2/3} \) and \( h(A_0) = 0 \). By Lemma 8.10

\[
h(A) = h'(A_0)(A - A_0) + o(A - A_0) = -\frac{\pi i}{2^{5/3} \cdot 3^2}(A - A_0) + o(A - A_0)
\]

around \( A = A_0 \). By Proposition 8.6, for a sufficiently small \( \varepsilon > 0 \), the inverse image of \( (-\varepsilon, 0) \cup (0, \varepsilon) \) under \( h(A) \) is a trajectory \( T_{0^-} \cup T_{0^+} : A = \chi^+_0(\phi) \) solving \((\text{BE})_\phi \), and is
expressed as
\[(8.4) \quad \chi_0^\pm (\phi) = A_0 + \gamma_0(\phi)i + o(\gamma_0(\phi)),\]
near \(\phi = 0\), where \(\gamma_0(\phi) \in \mathbb{R}\) is continuous in \(\phi\) and \(\gamma_0(0) = 0\).

The fact above implies that there exists a local trajectory solving \((BE)_{\phi}\) near \(\phi = 0\). From this, a trajectory for \(|\phi| \leq \pi/3\) as in Proposition 8.9 may be obtained. Furthermore, if two trajectories \(\chi_1(\phi)\) and \(\chi_2(\phi)\) solving \((BE)_{\phi}\) satisfy \(\chi_1(\phi_0) = \chi_2(\phi_0)\) for some \(\phi_0\) such that \(0 < |\phi_0| < \pi/3\), then \(\chi_1(\phi) = \chi_2(\phi)\). Thus we have the following.

**Proposition 8.12.** There exists a trajectory \(A = A_\phi\) for \(|\phi| \leq \pi/3\) with the properties:

1. For each \(\phi\), \(A_\phi\) is a unique solution of \((BE)_{\phi}\);
2. \(A_\phi\) is smooth in \(\phi\) for \(0 < |\phi| < \pi/3\) and continuous in \(\phi\) for \(|\phi| \leq \pi/3\).

For any cycle \(c\), it is easy to see that
\[
e^{i\phi} \int_{c} \frac{1}{2} w(A_\phi, z) dz = e^{i(\phi + 2\pi/3)} \int_{c} \frac{1}{2} w(e^{-2\pi i/3} A_\phi, \zeta) d\zeta,
\]
\[
e^{i\phi} \int_{c} \frac{1}{2} w(A_\phi, z) dz = -e^{i(\phi + \pi)} \int_{c} \frac{1}{2} w(A_\phi, \zeta) d\zeta,
\]
which yields the following.

**Proposition 8.13.** Set \(A_{\phi + 2\pi/3} = e^{2\pi i/3} A_\phi\) for \(|\phi| \leq \pi/3\). Then for \(|\phi| \leq \pi\), \(A_\phi\) is a unique solution of \((BE)_{\phi}\). Furthermore \(A_{\phi + \pi} = A_\phi\), \(A_{-\phi} = \overline{A_\phi}\).

Let us examine the properties of \(A_\phi\) in more detail. Note that the trajectory \(A = A_\phi = x + iy\) for \(|\phi| < \pi/3\) satisfies \(h(A_\phi) \in \mathbb{R}\). Then, by \((8.3)\),
\[
\frac{d}{dt} h(A_\phi) = (x'(t) + iy'(t))(-6\pi i) I_b(A_\phi)^{-2} \in \mathbb{R} \setminus \{0\}
\]
with \(t = \tan \phi\) for \(0 < |\phi| < \pi/3\). Setting \(I_b(A_\phi)^{-1} = P + iQ\), we have
\[
-\frac{1}{6\pi} \text{Im} \frac{d}{dt} h(A_\phi) = x'(t)(P^2 - Q^2) - 2y'(t)PQ = 0.
\]
If \(x'(t_0) = 0\) for some \(t_0 = \tan(\phi_0) \neq 0, \pm \infty\), then \(PQ = 0\), and hence \(I_b(A_{\phi_0}) \in i\mathbb{R} \setminus \{0\}\) or \(\mathbb{R} \setminus \{0\}\). This is impossible for \(0 < |\phi| < \pi/3\), which implies \(x'(t) \neq 0\) for \(0 < |\phi| < \pi/3\). Since \(A_{\pm \pi/3} = A_0 e^{\pm 2\pi i/3}\), we have \(x'(t) < 0\) for \(0 < \phi < \pi/3\) and \(x'(t) > 0\) for \(-\pi/3 < \phi < 0\). If \(y'(t_0) = 0\) for some \(t_0\) with \(0 < |\phi_0| < \pi/3\), then \(P^2 - Q^2 = 0\), i.e. \(I_b(A_{\phi_0})^{-1} = P(1 + i)\), implying \(\phi_0 = \pm \pi/4\). Note that \(-P < Q < 0\) for \(-\pi/4 < \phi < 0\) and that \(0 < Q < P\) for \(0 < \phi < \pi/4\). It follows that \(y'(t) < 0\) for \(0 < |\phi| < \pi/4\).

**Proposition 8.14.** The trajectory \(A_\phi = x(t) + iy(t)\) with \(t = \tan \phi\) has the properties:

1. \(x'(t) > 0\) for \(-\pi/3 < \phi < 0\), and \(x'(t) < 0\) for \(0 < \phi < \pi/3\);
2. \(y'(t) < 0\) for \(0 < |\phi| < \pi/4\) and \(y'(\tan(\pm \pi/4)) = 0\).

Thus we have
Proposition 8.15. For \( \phi \in \mathbb{R} \) there exists a trajectory \( A = A_\phi \) with the properties:

1. for each \( \phi \), \( A_\phi \) is a unique solution of (BE)_\phi;
2. \( A_{\phi+2\pi/3} = e^{2\pi i/3}A_\phi \), \( A_{\phi+\pi} = A_\phi \), \( A_{-\phi} = \overline{A_\phi} \);
3. \( A_0 = 3 \cdot 2\sqrt{3}, A_{\pm\pi/3} = 3 \cdot 2\sqrt{3}e^{\pi i/3} \);
4. \( A_\phi \) is continuous in \( \phi \in \mathbb{R} \), and smooth in \( \phi \in \mathbb{R} \setminus \{m\pi/3 \mid m \in \mathbb{Z} \} \).

The trajectory of \( A_\phi \) is roughly drawn in Figure 7.

![Figure 7. Trajectory of \( A_\phi \) for \( |\phi| \leq \pi \)](image)

By Proposition 8.14 when \( |\phi| \) is sufficiently small, the location of the turning points may be examined. Small variance of \( A_\phi \) around \( \phi = 0 \) is given by \( A_\phi = A_0 + \delta_\phi \) having the properties: (1) \( \delta_\phi \to 0 \) as \( \phi \to 0 \); (2) Re \( \delta_\phi \leq 0 \); (3) Im \( \delta_\phi \geq 0 \) if \( \phi \leq 0 \) and Im \( \delta_\phi \leq 0 \) if \( \phi \geq 0 \). Then the roots \( z_0, z_1, z_2 = 2^{-1/3} \) and \( z_2 = -4^{-2/3} \) of \( w(A_0, z)^2 \) vary in such a way that

\[
\begin{align*}
z_0 &= 2^{-1/3} + \rho + O(\rho^2), \\
z_1 &= 2^{-1/3} - \rho + O(\rho^2), \\
z_2 &= -4^{-2/3} + O(\rho^2)
\end{align*}
\]

with \( \rho = 2^{-2/3} \cdot 3^{-1/2}\delta_\phi^{1/2} \). Indeed, insertion of \( z_0 = 2^{-1/3} + \rho_+ \), \( z_1 = 2^{-1/3} + \rho_- \), \( z_2 = -4^{-2/3} + \rho_2 \) into \( z_0 + z_1 + z_2 = A_{\phi}/4 \), \( z_1 z_2 + z_2 z_0 + z_0 z_1 = 0 \), \( z_0 z_1 z_2 = -1/4 \) yields

\[
p + \rho_2 = \delta_\phi/4, \quad p + 4\rho_2 + 2^{1/3}q = O(|\rho_2|), \quad p - 2\rho_2 + 2^{1/3}q = O(|\rho_2|) \]

with \( p = \rho_+ + \rho_- \), \( q = \rho_+ \rho_- \), from which the estimates above follow. Thus we have the following.

Proposition 8.16. If \( |\phi| \) is sufficiently small, the turning points \( \lambda_k \) and \( z_k = \lambda_k^{-2} \) \((k = 0, 1, 2)\) are represented as

\[
\begin{align*}
\lambda_0 &= 2^{1/6} - \varepsilon_\phi e^{i\theta_\phi} + O(\varepsilon_\phi^2), \\
\lambda_1 &= 2^{1/6} + \varepsilon_\phi e^{i\theta_\phi} + O(\varepsilon_\phi^2), \\
\lambda_2 &= 2^{2/3} i + O(\varepsilon_\phi^2), \\
z_0 &= 2^{-1/3} + 2^{1/3} \varepsilon_\phi e^{i\theta_\phi} + O(\varepsilon_\phi^2), \\
z_1 &= 2^{-1/3} - 2^{1/3} \varepsilon_\phi e^{i\theta_\phi} + O(\varepsilon_\phi^2), \\
z_2 &= -4^{-2/3} + O(\varepsilon_\phi^2).
\end{align*}
\]

Here \( \varepsilon_\phi \) and \( \theta_\phi \) fulfil

1. \( \varepsilon_\phi > 0 \) and \( \varepsilon_\phi \to 0 \) as \( \phi \to 0 \); and
2. \( \theta_\phi \to \pi/4 \) as \( \phi \to 0 \) with \( \phi < 0 \), and \( \theta_\phi \to -\pi/4 \) as \( \phi \to 0 \) with \( \phi > 0 \).
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