Noncommutative Geometry as a Framework for Unification of all Fundamental Interactions including Gravity. Part I.

Ali H. Chamseddine\textsuperscript{1,3}, Alain Connes\textsuperscript{2,3,4}

\textsuperscript{1}Physics Department, American University of Beirut, Lebanon
\textsuperscript{2}College de France, 3 rue Ulm, F75005, Paris, France
\textsuperscript{3}I.H.E.S. F-91440 Bures-sur-Yvette, France
\textsuperscript{4}Department of Mathematics, Vanderbilt University, Nashville, TN 37240 USA

Abstract

We examine the hypothesis that space-time is a product of a continuous four-dimensional manifold times a finite space. A new tensorial notation is developed to present the various constructs of noncommutative geometry. In particular, this notation is used to determine the spectral data of the standard model. The particle spectrum with all of its symmetries is derived, almost uniquely, under the assumption of irreducibility and of dimension 6 modulo 8 for the finite space. The reduction from the natural symmetry group $SU(2) \times SU(2) \times SU(4)$ to $U(1) \times SU(2) \times SU(3)$ is a consequence of the hypothesis that the two layers of space-time are finite distance apart but is non-dynamical. The square of the Dirac operator, and all geometrical invariants that appear in the calculation of the heat kernel expansion are evaluated. We re-derive the leading order terms in the spectral action. The geometrical action yields unification of all fundamental interactions including gravity at very high energies. We make the following predictions: (i) The number of fermions per family is 16. (ii) The symmetry group is $U(1) \times SU(2) \times SU(3)$. (iii) There are quarks and leptons in the correct representations. (iv) There is a doublet Higgs that breaks the electroweak symmetry to $U(1)$. (v) Top quark mass of 170-175 Gev. (v) There is a right-handed neutrino with a see-saw mechanism. Moreover, the zeroth order spectral action obtained with a cut-off function is consistent with experimental data up to few percent. We discuss a number of open issues. We prepare the ground for computing higher order corrections since the predicted mass of the Higgs field is quite sensitive to the higher order corrections. We speculate on the nature of the noncommutative space at Planckian energies and the possible role of the fundamental group for the problem of generations.
1. Introduction

One of the basic problems facing theoretical physics is to determine the nature of space-time. This is intimately related to the problem of unifying all the fundamental interactions including gravity, and thus is not independent of solving the problem of quantum gravity. In a series of papers we have made important understanding uncovering a first approximation of the hidden structure of space-time. Our assumption is that at energies below the planck scale, space-time can be approximated as a product of a continuous four-dimensional manifold by a finite space. We were able to show in [14] that finite spaces satisfying the axioms of noncommutative geometry are severely restricted, and the corresponding irreducible representations on Hilbert spaces can only have dimensions which are the square of integers, or the double of such a square. The second possibility is the only one allowed when the finite space has dimension 6 modulo 8 (in the sense of K-theory or more pragmatically of the periodicity of Clifford algebras) as imposed by the need to have the total dimension $2 = 4 + 6$ modulo 8 in order to be able to write down the Fermionic part of the action. Together with the restriction of imposing a unitary–symplectic structure and grading on the
finite noncommutative space, this singles out $4^2 = 16$ as the number of physical fermions per generation. Then, in the same way as was shown in [13], this predicts the existence of right-handed neutrinos, and the see-saw mechanism. Our present framework using the classification of finite spaces is stronger and the symmetries of the standard model emerge, rather than assumed and put in by hand. This construction, using the spectral action principle, predicts certain relations between the coupling constants, that can only hold at very high energies of the order of the unification scale. The spectral action principle is the simple statement that the physical action is determined by the spectrum of the Dirac operator $D$. This has now been tested in many interesting models including Superstring theory [6], noncommutative tori [30], Moyal planes [34], 4D-Moyal space [37], manifolds with boundary [12], in the presence of dilatons [10], for supersymmetric models [5] and torsion cases [38]. The additivity of the action forces it to be of the form $\text{Trace } f(D/\Lambda)$. In the approximation where the spectral function $f$ is a cut-off function, the relations given by the spectral action are used as boundary conditions and the couplings are then allowed to run from unification scale to low energy using the renormalization group equations. The equations show, when fitted to the low energy boundary conditions, that the three gauge coupling constants and the Newton constant nearly meet (within few percent) at very high energies, two or three orders from the Planck scale. This might be a coincidence but it can also be an indication that a more fundamental theory exists at unification scale and manifests itself at low scale through integration of the intermediate modes, as in the Wilson understanding of renormalization.

In Part II we shall investigate higher order terms in the perturbative expansion of the spectral function. We shall show that these can give important contributions which effects the low-energy form of the spectral action. A prediction of the Higgs mass is sensitive to these higher order contributions. In this paper we will present our analysis in a transparent setting, geared towards physicists, spelling out the very few assumptions we make, and thus allowing for an exhaustive treatment. This will help us to pave the road for future investigations, and hopefully be of help for students to learn and apply this topic. We shall follow a new and simple tensorial notation to allow physicists to follow our analysis with ease. This will be true for most of the calculations, although we will not re-derive some of the abstract proofs because in this case the tensorial notation is not very practical. Armed with this simplification in our analysis we will evaluate the spectral action rederiving old results in a simple way. The calculation will be extended in Part II, to include higher order terms in a perturbative expansion in function of the inverse of the unification scale. At the end of this Part I, we shall discuss a number of important issues which are:
• The variant of the Einstein-Yang Mills system obtained with the algebra $A_F = M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$, its relation with supersymmetry, and with the unimodularity condition (§9.1).
• The geometric role of $M_4(\mathbb{C})$ (§9.2).
• The possible geometric meaning of several generations (§9.3).
• Unification of couplings (§9.4).
• Mass of the Higgs (§9.5).
• New particles (§9.6).
• Quantum Level (§9.7).

In the appendices we develop the computational tools which will be used in Part II to handle the higher order terms. We also compute an explicit concrete example to check the sign in front of the Yang-Mills interaction.

2. Determining the finite noncommutative space

We first give a very brief summary of the properties of noncommutative spaces. The basic idea is based on physics. The modern way of measuring distances is spectral. The unit of distance is taken as the wavelength of atomic spectra. To adopt this geometrically we have to replace the notion of real variable which one takes as a function $f$ on a set $X$, $f : X \to \mathbb{R}$ to be given now by a self adjoint operator in a Hilbert space as in quantum mechanics. The space $X$ is described by the algebra $\mathcal{A}$ of coordinates which is represented as operators in a fixed Hilbert space $\mathcal{H}$. There is no a priori requirement that this algebra $\mathcal{A}$ is commutative since Hilbert space operators model perfectly the lack of commutativity. In fact if $\mathcal{A}$ is the algebra of functions on a space $X$ and one replaces it by the algebra $\mathcal{B} = M_n(\mathcal{A})$ of matrices of functions, one obtains that the natural gauge invariance group $\mathcal{G}$ of gravity coupled with an $SU(n)$ Yang-Mills theory on $X$, which is the semi-direct product of the group $\text{Map}(X, SU(n))$ by the group of diffeomorphisms $\text{Diff}(X)$,

$$1 \to \text{Map}(X, SU(n)) \to \mathcal{G} \to \text{Diff}(X) \to 1.$$ 

is (locally) nothing else than the group of automorphisms of $\mathcal{B}$

$$1 \to \text{Int}(\mathcal{B}) \to \text{Aut}(\mathcal{B}) \to \text{Out}(\mathcal{B}) \to 1$$

It is rather satisfying that this completely general decomposition of automorphisms of a noncommutative algebra into inner ones (forming the normal subgroup $\text{Int}(\mathcal{B})$) and outer ones (forming the quotient group $\text{Out}(\mathcal{B})$) corresponds in the above simplest example to the decomposition of the gauge symmetries in the internal ones $\text{Map}(X, SU(n))$ and the group $\text{Diff}(X)$ of diffeomorphisms. We have shown in [9] that the study of pure gravity on the “space” associated to the algebra $\mathcal{B}$ yields Einstein gravity on $X$ minimally coupled with Yang-Mills theory for the gauge group $SU(n)$. The Yang-Mills gauge potential appears as the inner part of the metric, in the same way as the group of gauge transformations (for the gauge group $SU(n)$) appears as the group of inner diffeomorphisms.
The meaning of “pure gravity” in the general noncommutative framework comes from Dirac’s solution of the extraction of the square root in Riemann’s formula for the distance between two points

\[ d(a, b) = \inf_{\gamma} \int \sqrt{g_{\mu\nu} dx^\mu dx^\nu} \]

which can be reexpressed, in terms of Dirac’s operator \( D \) on the Hilbert space \( \mathcal{H} \) of spinors, in the form

\[ d(a, b) = \sup \{ |f(a) - f(b)| ; f \in \mathcal{A}, \|[D,f]\| \leq 1 \} \]

This shows that giving the Dirac operator acting in the same Hilbert space \( \mathcal{H} \) as the algebra \( \mathcal{A} \) of coordinates provides an elegant way of giving the geometry of the associated space \( \mathcal{X} \). Moreover this way immediately permits the passage to noncommutative algebras. Thus the geometry of a noncommutative space is determined in terms of the spectral data \((\mathcal{A}, \mathcal{H}, D, J, \gamma)\), where the last two: \( J, \gamma \) should be considered as decorations on the main structure, encoded by the spectral triple \((\mathcal{A}, \mathcal{H}, D)\). In practice a real, even spectral triple is defined by

- \( \mathcal{A} \) an associative algebra with unit 1 and involution \(*\).
- \( \mathcal{H} \) a complex Hilbert space carrying a faithful representation \( \pi \) of the algebra.
- \( D \) a self-adjoint operator on \( \mathcal{H} \) with the resolvent \((D - \lambda)^{-1}, \lambda \notin \mathbb{R}\) of \( D \) compact.
- \( J \) is an anti–unitary operator on \( \mathcal{H} \), a real structure (charge conjugation.)
- \( \gamma \) is a unitary operator on \( \mathcal{H} \), the chirality.

We require the following conditions to hold:

- \( J^2 = \epsilon \), \((\epsilon = 1 \text{ in zero dimensions and } \epsilon = -1 \text{ in 4 dimensions})\).
- \([a, b^\circ] = 0 \text{ for all } a, b \in \mathcal{A}, b^\circ = Jb^*J^{-1} \). This is the zeroth order condition and is needed to define the right action of the algebra on elements of \( \mathcal{H} : \zeta b = b^\circ \zeta \).
- \( DJ = \epsilon' JD, \quad J \gamma = \epsilon'' \gamma J, \quad D \gamma = -\gamma D \) where \( \epsilon, \epsilon', \epsilon'' \in \{-1, 1\} \).

These reality conditions resemble the conditions of existence of Majorana (real) fermions.
- \([[[D, a], b^\circ] = 0 \text{ for all } a, b \in \mathcal{A} \). This is the first order condition.
- \( \gamma^2 = 1 \) and \([\gamma, a] = 0 \text{ for all } a \in \mathcal{A} \). Thus \( \gamma \) is the chirality operator and this gives the decomposition \( \mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R \).

It then follows from the above properties that:

- \( \mathcal{H} \) is endowed with an \( \mathcal{A} \)- bimodule structure \( a \zeta b = ab^\circ \zeta \).
- \( \mathcal{A} \) has a well defined unitary group

\[ \mathcal{U} = \{ u \in \mathcal{A}; \quad uu^* = u^*u = 1 \} \]
The natural adjoint action of $U$ on $H$ is given by $\zeta \rightarrow u\zeta u^* = u\, J\, u^*\zeta \quad \forall \zeta \in H$. Then

$$\langle \zeta, D\zeta \rangle$$

is not invariant under the above transformation but one has:

$$\langle u\, J\, u^* \rangle \, D \, \langle u\, J\, u^* \rangle^* = D + u \, [D, u^*] + \varepsilon' \, J \, (u \, [D, u^*]) \, J^*$$

- The action $\langle \zeta, D_A\zeta \rangle$ is invariant where

$$D_A = D + A + \varepsilon' \, J A J^{-1}, \quad A = \sum_i a^i \, [D, b^i]$$

and $A = A^*$ is self-adjoint. This is similar to the appearance of the interaction term for the photon with the electrons

$$\bar{i} \psi \gamma^\mu \partial_\mu \psi \rightarrow \bar{i} \psi \gamma^\mu (\partial_\mu + ieA_\mu) \psi$$

to maintain invariance under the variations $\psi \rightarrow u\psi = e^{i\alpha(x)}\psi$.

One then extends the familiar geometric notions to this framework:

- The notion of dimension is governed by the growth of eigenvalues of $D$, and may be fractal and involve complex numbers.
- The antilinear isometry $J : \mathcal{H} \rightarrow \mathcal{H}$ gives a real structure of $KO$-dimension $n \in \mathbb{Z}/8$ on a spectral triple $(\mathcal{A}, \mathcal{H}, D)$

$$J^2 = \varepsilon, \quad JD = \varepsilon' DJ, \quad J\gamma = \varepsilon''\gamma J \text{ (even case)}.$$  

The numbers $\varepsilon, \varepsilon', \varepsilon'' \in \{-1, 1\}$ are a function of $n$ mod 8 given by

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----|---|---|---|---|---|---|---|---|
| $\varepsilon$ | 1 | 1 | -1 | -1 | -1 | 1 | 1 |   |
| $\varepsilon'$ | 1 | 1 | -1 | -1 | -1 | 1 | 1 |   |
| $\varepsilon''$ | 1 | 1 | 1 | 1 | -1 | 1 | 1 |   |

Our starting point is the model: space-time is a product of a continuous four-dimensional manifold $M$ times a finite space $F$. And of course we do not assume that the finite space $F$ is commutative. It is described by a spectral data $(A_F, \mathcal{H}_F, D_F, J_F, \gamma_F)$ where all ingredients are finite dimensional.

The algebra $\mathcal{A}$ for the product space is a tensor product. The spectral geometry of $\mathcal{A}$ is given by the product rule

$$\mathcal{A} = C^\infty (M) \otimes A_F$$

$$\mathcal{H} = L^2 (M, S) \otimes \mathcal{H}_F,$$

$$D = D_M \otimes 1 + \gamma_5 \otimes D_F,$$
where $L^2(M,S)$ is the Hilbert space of $L^2$ spinors, and $D_M$ is the Dirac operator of the Levi-Civita spin connection on the four manifold $M$,

$$D_M = \gamma^\mu (\partial_\mu + \omega_\mu).$$

The chirality operator is $\gamma = \gamma_5 \otimes \gamma_F$. The real structure $J$ is $J_M \otimes J_F$ where $J_M$ is charge conjugation.

In order to avoid the fermion doubling problem so that $\zeta, \zeta^c, \zeta^*, \zeta^{c*}$ where $\zeta \in \mathcal{H}$, are not all independent, it was shown in [13] that the finite dimensional space must be taken to be of K-theoretic dimension 6 modulo 8, where in this case $(\epsilon, \epsilon', \epsilon'') = (1, 1, -1)$. This makes the total K-theoretic dimension of the noncommutative space to be 10 and would allow to impose the reality (Majorana) condition and the Weyl condition simultaneously in the Minkowskian continued form, a situation very familiar in ten-dimensional supersymmetry. In the Euclidean version, the use of the $J$ in the fermionic action, would give for the chiral fermions in the path integral, a Pfaffian instead of determinant, and will thus cut the fermionic degrees of freedom by 2. In other words, to have the fermionic sector free of the fermionic doubling problem we must make the choice

$$J_F^2 = 1, \quad J_F D_F = D_F J_F, \quad J_F \gamma_F = -\gamma_F J_F$$

In what follows we will restrict our attention to determination of the finite algebra, and will omit the subscript $F$.

3. Classification of the Finite Space

There are two main constraints on the algebra from the axioms of noncommutative geometry. We first look for involutive algebras $\mathcal{A}$ of operators in $\mathcal{H}$ such that,

$$[a, b^0] = 0, \quad \forall a, b \in \mathcal{A}$$

where for any operator $a$ in $\mathcal{H}$, $a^0 = Ja^*J^{-1}$. This is called the order zero condition. We now look for representations of $\mathcal{A}$ and $J$ in $\mathcal{H}$ which are irreducible. Assume that $e \neq 1$ is a projection in the center $Z(\mathcal{A})$ of $\mathcal{A}$, where $e^2 = e = e^*$, $ea = ae$ $\forall a \in \mathcal{A}$. We then have for the projection $(eJeJ^{-1})^2 = eJeJ^{-1}$,

$$[eJeJ^{-1}, a] = eJeJ^{-1}a - aeJeJ^{-1}$$

$$= eaJeJ^{-1} - aeJeJ^{-1} = 0$$

where we have used the order zero condition $[a, JeJ^{-1}] = 0$. We also have

$$[eJeJ^{-1}, J] = eJeJ^{-1}J - JJJeJ^{-1}e$$

$$= eJe - eJeJ^{-1}e = 0.$$
\(eJ e J^{-1}\) is contained in the range of \(e\). Thus we have

\[ e J e J^{-1} = 0. \]

Similarly if we have two projections \(e_1\) and \(e_2\) in the center \(Z(\mathcal{A})\) of \(\mathcal{A}\), such that \(e_1 e_2 = 0\), then a simple calculation as above shows that the projection

\[(3.4)\]

\[e_1 J e_2 J^{-1} + e_2 J e_1 J^{-1}\]

satisfies

\[(3.5)\]

\[\left[ e_1 J e_2 J^{-1} + e_2 J e_1 J^{-1}, a \right] = 0\]

and thus by irreducibility is equal to 0 or 1. Assume that the center \(Z(\mathcal{A})\) allows for more than two projections then \(\sum_j e_j = 1\) where

\[ e_i^2 = e_i = e_i^*, \quad \forall i, \quad e_i e_j = 0, \quad i \neq j. \]

Thus one gets

\[(3.6)\]

\[1 = \sum_i e_i J \left( \sum_j e_j \right) J^{-1}\]

\[= \sum_{j \neq i} e_i J e_j J^{-1} \quad \text{since} \quad e_i J e_i J^{-1} = 0\]

\[= (e_1 J e_2 J^{-1} + e_2 J e_1 J^{-1}) + (e_1 J e_3 J^{-1} + e_3 J e_1 J^{-1}) + \cdots\]

and therefore only one combination (say \(e_1\) and \(e_2\)) can be equal to 1, the others being zero

\[(3.7)\]

\[e_1 J e_2 J^{-1} + e_2 J e_1 J^{-1} = 1\]

\[(3.8)\]

\[e_i J e_j J^{-1} + e_j J e_i J^{-1} = 0 \quad i \neq 1, 2, \quad \forall j.\]

From this we have that for \(i \notin \{1, 2\}\), \(e_i J e_j J^{-1} = 0\) for all \(j\) and thus

\[(3.9)\]

\[e_i = e_i J \left( \sum_j e_j \right) J^{-1}\]

\[= 0\]

Thus \(e_i = 0\) for \(i \notin \{1, 2\}\), \(e_1 + e_2 = 1\) and one can easily show that

\[(3.10)\]

\[J e_1 J^{-1} = e_2, \quad J e_2 J^{-1} = e_1\]

In general we only assume that the algebra \(\mathcal{A}\) is real and preserved by the involution \(x \mapsto x^*\), but the above argument applies to the complexified extension \(\mathcal{A}_C\). The surprising result is that the classification of irreducible representations of \(\mathcal{A}\) and \(J\) in \(\mathcal{H}\) splits into two cases only. The center for the complexified extension of the algebra can only be \(Z(\mathcal{A}_C) = \mathbb{C}\) for \(e = 1\) or \(Z(\mathcal{A}_C) = \mathbb{C} \oplus \mathbb{C}\) for \(e_1 + e_2 = 1\) with \(J e_1 J^{-1} = e_2\).
Let $\mathcal{H}$ be a Hilbert space of dimension $n$. Then an irreducible solution with $Z(\mathcal{A}_C) = \mathbb{C}$ exists iff $n = k^2$ is a square. It is given by $\mathcal{A}_C = M_k(\mathbb{C})$ acting by left multiplication on itself and antilinear involution $J(x) = x^*$, $\forall x \in M_k(\mathbb{C})$. The dimension of the Hilbert space by left multiplication on itself and antilinear involution $J$ is $k$, because of the reality condition. To have a non-trivial grading on $M_k(\mathbb{C})$ requires $k$ to be a square. It is given by

$$\mathcal{A}_C \otimes \mathcal{A}_C^0 \rightarrow \mathcal{L}(\mathcal{H}), \quad \beta(x \otimes y) = xy^0, \quad \forall x, y \in \mathcal{A}_C,$$

which is injective since $\mathcal{A}_C \otimes \mathcal{A}_C^0 \sim M_k(\mathbb{C})$. Since $\mathcal{A}_C \otimes \mathcal{A}_C^0 \sim M_k^2(\mathbb{C})$ then $n = k^2$ is a square. This determines $\mathcal{A}_C$ and its representations in $(\mathcal{H}, J)$ and allows only for three possibilities for $\mathcal{A}$. These are $\mathcal{A} = M_k(\mathbb{C})$, $M_k(\mathbb{R})$ and $M_a(\mathbb{H})$ for even $k = 2a$, where $\mathbb{H}$ is the field of quaternions. These correspond respectively to the unitary, orthogonal and symplectic case. It can be shown that the case $Z(\mathcal{A}_C) = \mathbb{C}$ is incompatible with the commutation relation $J \gamma = -\gamma J$ and hence with the K-theoretic dimension 6 necessary to impose the reality condition on the spinors to avoid fermion doubling. This implies that the only realistic case to consider is the second possibility.

We thus have to assume that $Z(\mathcal{A}_C) = \mathbb{C} \oplus \mathbb{C}$. Then there exists $k_j \in \mathbb{N}$ such that $\mathcal{A}_C = M_{k_1}(\mathbb{C}) \oplus M_{k_2}(\mathbb{C})$ as an involutive algebra over $\mathbb{C}$. We let $e_j$ be the minimal projections $e_j \in Z(\mathcal{A}_C)$ with $e_j$ corresponding to the component $M_{k_j}(\mathbb{C})$. There is a corresponding decomposition

$$\mathcal{H} = e_1 \mathcal{H} \oplus e_2 \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2, \quad (x_1, x_2)(\xi_1, \xi_2) = (x_1 \xi_1, x_2 \xi_2)$$

One can show (under the natural hypothesis that there is a separating vector in $\mathcal{H}$) that $k_1 = k_2 = k$, the dimension $n$ of the Hilbert space $\mathcal{H}$ is $n = 2k^2$ and that the action of $J$ is given by

$$J(x, y) = (y^*, x^*)$$

We then have six possibilities for the algebra $\mathcal{A}$

$$(3.17) \quad \{ M_k(\mathbb{C}) \text{ or } M_k(\mathbb{R}) \text{ or } M_a(\mathbb{H}) \} \oplus \{ M_k(\mathbb{C}) \text{ or } M_k(\mathbb{R}) \text{ or } M_a(\mathbb{H}) \}.$$

We shall show, at the end of section five, that four of these possibilities can be ruled out immediately and that the choice of

$$\mathcal{A} = M_k(\mathbb{C}) \oplus M_k(\mathbb{C})$$

when $k = 4$ suffers from $U(1)$ anomalies. We thus proceed to make the assumption of imposing an antilinear isometry $I$ such that $I^2 = -1$ on one of the algebras and no condition on the other forcing $\mathcal{A}$ to be

$$\mathcal{A} = M_a(\mathbb{H}) \oplus M_k(\mathbb{C}), \quad k = 2a$$

The dimension of the Hilbert space $n = 2k^2$ gives $k^2$ independent fermions, where $k$ is an even integer, because of the reality condition. To have a non-trivial grading on $M_a(\mathbb{H})$ requires $a$ to be at least 2. Thus the simplest possibility is

$$\mathcal{A} = M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$$

and the grading $\gamma$ reduces $M_2(\mathbb{H})$ to $\mathbb{H} \oplus \mathbb{H}$. This corresponds to a Hilbert space of 16 fermions.
We next examine the order one condition

\[(3.21) \quad [[D, a], b^o] = 0, \quad \forall a, b \in \mathcal{A}\]

First if the Dirac operator commutes with \( Z(A) \)

\[(3.22) \quad [D, Z(A)] = 0 \]

then one can show that the Dirac operator has no non-diagonal elements that connects the two pieces of the algebra \( \mathcal{A} \) and thus \( e_1 D e_2 = 0 \). This will correspond to unbroken color group \( SU(4) \) and with only Dirac masses for the neutrinos. On the other hand if there is a non-trivial mixing such that

\[(3.23) \quad [D, Z(A)] \neq 0 \]

then the non-diagonal operator \( T = e_1 D e_2 : \mathcal{H}_1 \to \mathcal{H}_2 \) must be of rank 1 and thus can only have a singlet non-zero entry forcing elements of the algebra to take the form

\[(3.24) \quad (\lambda, \overline{\lambda}, q) \oplus (\lambda, m), \quad \lambda \in \mathbb{C}, \quad q \in \mathbb{H}, \quad m \in M_3(\mathbb{C}) \]

thus reducing the algebra to

\[(3.25) \quad \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \]

These last steps will be made more transparent in the next section.

### 4. Tensorial notation

To acquaint ourselves with the abstract quantities defined so far, it is useful to use the tensorial notation familiar to physicists. The main advantage of this method is that it can be implemented using computer programs with algebraic manipulations such as Mathematica and Maple. We will restrict to the case where \( Z(\mathcal{A}_\mathbb{C}) = \mathbb{C} \oplus \mathbb{C} \).

An element of the Hilbert space \( \Psi \in \mathcal{H} \) is represented by

\[(4.1) \quad \Psi_M = \begin{pmatrix} \psi_A \\ \psi_A' \end{pmatrix}, \quad \psi_A' = \overline{\psi_A} \]

where \( \psi_A' \) is the conjugate spinor to \( \psi_A \). It is acted on by both the left algebra \( M_2(\mathbb{H}) \) and the right algebra \( M_4(\mathbb{C}) \). Therefore the index \( A \) can take 16 values and is represented by

\[(4.2) \quad A = \alpha I \]

where the index \( \alpha \) is acted on by the quaternionic matrices and the index \( I \) by the \( M_4(\mathbb{C}) \) matrices. Moreover, when grading breaks \( M_2(\mathbb{H}) \) into \( \mathbb{H} \oplus \mathbb{H} \) the index \( \alpha \) is decomposed to \( \alpha = \tilde{a}, a \) where \( \tilde{a} = 1, \tilde{2} \) is acted on by the first quaternionic algebra \( \mathbb{H}_R \) and \( a = 1, 2 \) is acted on by the second quaternionic algebra \( \mathbb{H}_L \). Also when \( M_4(\mathbb{C}) \) breaks into \( \mathbb{C} \oplus M_3(\mathbb{C}) \) the
index \(I\) is decomposed into \(I = 1, i\) where the 1 is acted on by the \(C\) and the \(i\) by \(M_3(\mathbb{C})\). Therefore the various components of the spinor \(\psi_A\) are

\[
\begin{align*}
\psi_{11} &= \nu_R \\
\psi_{21} &= e_R \\
\psi_{a1} &= l_a = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \\
\psi_{1i} &= u_{iR} \\
\psi_{2i} &= d_{iR} \\
\psi_{ai} &= q_{ia} = \begin{pmatrix} u_{iL} \\ d_{iL} \end{pmatrix}
\end{align*}
\]

The Dirac action then take the form

\[
\Psi^* M^N D M \Psi_N
\]

which we can expand to give

\[
\psi^*_A D^B_A \psi_B + \psi^*_A D^B_A' \psi_B' + \psi^*_A D^B_A' \psi_B, + \psi^*_A D^B_A \psi_B'
\]

The Dirac operator can be written in matrix form

\[
D = \begin{pmatrix} D^B_A & D^B_A' \\ D^B_A' & D^B_A \end{pmatrix},
\]

where

\[
A = \alpha I, \quad \alpha = 1, \ldots, 4, \quad I = 1, \ldots, 4
\]

\[
A' = \alpha' I', \quad \alpha' = 1', \ldots, 4', \quad I = 1', \ldots, 4'
\]

Thus \(D^B_A = D^\beta_{\alpha I}\). We start with the algebra

\[
\mathcal{A} = M_4(\mathbb{C}) \oplus M_4(\mathbb{C})
\]

and write

\[
a = \begin{pmatrix} X_{\alpha I} \delta^J_I & 0 \\ 0 & \delta^\alpha_{\alpha'} Y_{I'}^{J'} \end{pmatrix}
\]

For \(J^2 = 1\) we have

\[
J = \begin{pmatrix} 0 & \delta^\beta_{\alpha I} \delta^J_I \\ \delta^\alpha_{\alpha'} \delta^J_I & 0 \end{pmatrix} \times \text{complex conjugation}
\]

In this form

\[
a^0 = J a^* J^{-1} = \begin{pmatrix} \delta^\beta_{\alpha I} Y_{I}^{J} & 0 \\ X_{\alpha' \alpha}^{t \beta} \delta^J_{I'} \end{pmatrix}
\]

where the superscript \(t\) denotes the transpose matrix. This clearly satisfies the commutation relation

\[
[a, b^\alpha] = 0.
\]
The order one condition is
\[ [[D, a], b^a] = 0 \]

Writing
\[ b = \left( \begin{array}{cc} Z^I_\alpha \delta^I_j & 0 \\ 0 & \delta^\beta_\alpha W^I_J^{IJ'} \end{array} \right) \]

then
\[ b^a = \left( \begin{array}{cc} \delta^\beta_\alpha W^I_I^{IJ'} & 0 \\ 0 & Z^\beta_\alpha \delta^{IJ'}_J \end{array} \right) \]

and so \[ [[D, a], b^a] \] is equal to
\[ \left( \begin{array}{cc} [[D, X], W]_A^B & ((DY - XD) Z - W (DY - XD))_A^B' \\ ((DX - YD) W - Z (DX - YD))_A^B' & [[D, Y], Z]_A^B' \end{array} \right) \]

The first two equations can be made explicit by writing:
\[ \left( D^{\gamma K}_\alpha X_i^\beta - X_i^\gamma D^{\beta K}_\alpha \right) \frac{W^I_K}{W^I_I^{IJ'}} - \left( D^{\gamma J}_\alpha X_i^\beta - X_i^\gamma D^{\beta J}_\alpha \right) = 0 \]
\[ \left( D^{\gamma K'}_\alpha Y_{K'}^J - X_i^\gamma D^{\beta K'}_\alpha \right) Z^\beta_\gamma - \left( D^{\gamma J}_\alpha Y_{K'}^J - X_i^\gamma D^{\beta J}_\alpha \right) = 0 \]

Here we have two classes of solutions. First, if all of the \( D^{\beta K'}_\alpha \) are zero, implying that there is no mixing between the fermions and their conjugates. In this case one can easily show that the color group is \( SU(4) \) and not \( SU(3) \) and that there will be no breaking of the left-right symmetry in the leptonic sector. If some of the \( D^{\beta K'}_\alpha \) are non-zero, we have shown that the only solution of the second equation is for \( D^{\beta K'}_\alpha \) to have only one non-zero entry,
\[ D^{\beta K'}_\alpha = \delta_\alpha^1 \delta_{1'}^\beta \delta_{1'}^J \delta_{1'}^K \kappa^{I'J'} \]

where the \( \kappa^{I'J'} \) are matrices in generation space which will be assumed to be \( 3 \times 3 \). We shall discuss the role of families below in \([9.3]\). We thus can write
\[ D^{\beta J}_{\alpha I} = D^{\beta J}_{\alpha I} \delta_1^I \delta_1^J + D^{\beta J}_{\alpha q} \delta_1^I \delta_1^J \]
\[ Y_{I'}^{J'} = \delta_1^{I'} \delta_{1'}^{J'} \]
\[ X_1^I = Y_{I'}^{J'}, \quad X_1^\alpha = 0, \quad \alpha \neq 1 \]

We will be using the notation
\[ \alpha = 1, 2, a \quad \text{where} \quad a = 1, 2 \]

We further impose the condition of symplectic isometry on the first \( M_4(\mathbb{C}) \)
\[ (\sigma_2 \otimes 1) \ (\overline{\sigma}) (\sigma_2 \otimes 1) = a, \quad a \in M_4(\mathbb{C}) \]
reduces it to $M_2(\mathbb{H})$. From the property of commutation of the grading operator

\begin{equation}
   g_\alpha^\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{equation}

\begin{equation}
   [g, a] = 0 \quad a \in M_2(\mathbb{H})
\end{equation}

the algebra $M_2(\mathbb{H})$ reduces to $\mathbb{H} \oplus \mathbb{H}$. This, together with the conditions \[4.27\] and \[4.28\] implies that

\begin{equation}
   X_\alpha^\beta = \delta_\alpha^i \delta^j_\beta X_1^i + \delta_\alpha^p \delta^q_\beta X_2^p
\end{equation}

\begin{equation}
   Y_I^J = \delta_I^I \delta_J^{1'} Y_1^{1'} + \delta_I^p \delta_J^{1'} Y_2^{1'}
\end{equation}

and the algebra $\mathbb{H} \oplus \mathbb{H} \oplus M_4(\mathbb{C})$ reduces to

\begin{equation}
   \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})
\end{equation}

Thus an element of the algebra, to be compatible with the axioms of noncommutative geometry, and the few assumptions we made, must be restricted to the form

\begin{equation}
   a = \begin{pmatrix} X & X \\ q & X \\ m & X \end{pmatrix}, \quad X \in \mathbb{C}, \quad q \in \mathbb{H}, \quad m \in M_3(\mathbb{C}).
\end{equation}

We also note that the property that $DJ = JD$ implies that

\begin{equation}
   D^B_A' = \overline{D^B_A}
\end{equation}

and that $D^\beta K^\gamma_{\alpha I}$ is symmetric matrix, thus $k^{\nu R}$ is symmetric so that $k^{\nu R} = \overline{k^{\nu R}}$. Further restriction is obtained on the form of the Dirac operator $D$ from the property

\begin{equation}
   D \gamma = -\gamma D
\end{equation}

where $\gamma$ is the grading operator. Writing

\begin{equation}
   \gamma = \begin{pmatrix} G^B_A & 0 \\ 0 & -\overline{G^B_A} \end{pmatrix}, \quad G^2 = 1
\end{equation}

we obtain

\begin{equation}
   (GDG)^B_A = -D^B_A
\end{equation}

The grading operator acts only on the first algebra, thus

\begin{equation}
   G^B_A = g_\alpha^\beta \delta_I^J
\end{equation}

which implies that

\begin{equation}
   g_\alpha^\beta D^\gamma_{\alpha \gamma(t)} g_\delta^\beta = -D^\beta_{\alpha(t)}
\end{equation}
thus

\begin{equation}
D^{21}_{\alpha 1} = \begin{pmatrix}
0 & D^{b}_{1} \\
D^{b}_{1} & 0
\end{pmatrix}, \quad D^{b1}_{\alpha 1} = (D^{b1}_{\alpha 1})^* = D^{b}_{a(l)}
\end{equation}

\begin{equation}
D^{2j}_{\alpha i} = \begin{pmatrix}
0 & D^{b}_{a(q)}
D^{b}_{a(q)} & 0
\end{pmatrix}, \quad D^{b}_{a(q)} = (D^{b}_{a(q)})^*
\end{equation}

To summarize, the matrix form for $D^B_A$ is given by

\begin{equation}
\begin{pmatrix}
11 & 21 & a1 & 1i & 2j & ai \\
\nu_R & e_R & l_a & u_{iR} & d_{iR} & q_{iL}
\end{pmatrix}
\begin{pmatrix}
(D)_{1i}^{11} & 0 & (D)_{a1}^{11} & 0 & 0 & 0 \\
0 & (D)_{2i}^{21} & (D)_{a1}^{21} & 0 & 0 & 0 \\
(D)_{ai}^{11} & (D)_{b1}^{11} & (D)_{a1}^{b1} & 0 & 0 & 0 \\
0 & 0 & 0 & (D)_{1j}^{1i} & 0 & (D)_{ai}^{1i} \\
0 & 0 & 0 & 0 & (D)_{2j}^{2i} & (D)_{a1}^{2i} \\
0 & 0 & 0 & (D)_{bj}^{1i} & (D)_{b1}^{2i} & (D)_{a1}^{2i}
\end{pmatrix}
\end{equation}

where the entries above and along the rows and columns denote the corresponding fermion. Finally we require the Dirac operator of the finite space to commute with the element $C \subset \mathbb{C} \otimes \mathbb{H} \otimes M_3(\mathbb{C})$ where

\begin{equation}
C = \begin{pmatrix}
\lambda & \bar{\chi} \\
\bar{\chi} & \lambda
\end{pmatrix}
\end{equation}

This condition will ensure that the photon and not another vector will remain massless. This also reduces $D^{b1}_{\alpha 1}$ to the form

\begin{equation}
D^{b1}_{\alpha 1} = D^{b}_{a(l)} = \begin{pmatrix}
k^{\nu} & 0 \\
0 & k^{*e}
\end{pmatrix}, \quad a = 1, 2, \quad b = 1, 2
\end{equation}

and

\begin{equation}
D^{b}_{a(q)} = \begin{pmatrix}
k^{\nu} & 0 \\
0 & k^{*d}
\end{pmatrix}
\end{equation}

To summarize the finite space Dirac operator is given by

\begin{equation}
(D_F)^{\beta J}_{\alpha I} = \left(\delta^1_0 \delta^2_0 k^{\nu} + \delta^1_0 \delta^2_1 k^{\nu} + \delta^2_0 \delta^2_1 k^{*e} + \delta^2_0 \delta^1_1 k^{d} \right) \delta^1_1 \delta^1_I + \left(\delta^1_0 \delta^1_1 k^{\nu} + \delta^1_0 \delta^2_1 k^{u} \right) \delta^2_1 \delta^2_J
\end{equation}

\begin{equation}
(D_F)^{\beta K'}_{\alpha I} = \delta^1_0 \delta^2_0 \delta^1_1 \delta^1_I \delta^2_J k^{*e} \sigma
\end{equation}
We now form the Dirac operator of the product space of this finite space times a four-dimensional Riemannian manifold

\[ D = D_M \otimes 1 + \gamma_5 \otimes D_F \]

Since \( D_F \) is a \( 32 \times 32 \) matrix tensored with the \( 3 \times 3 \) matrices of generation space, and the \( 4 \times 4 \) Clifford algebra, \( D \) is \( 384 \times 384 \) matrix.

In order for the Dirac action to be invariant under fluctuations of the inner automorphisms of the algebra \( A \), the operator \( D \) must be replaced with the operator

\[ D_A = D + A + JAJ^{-1} \]

where

\[ A = \sum a [D, b] \]

\[ a = \begin{pmatrix} X^\alpha_\beta \delta^I_J & 0 \\ 0 & \delta^\alpha_\beta Y^I_J \end{pmatrix} \]

\[ b = \begin{pmatrix} Z^\alpha_\beta \delta^I_J & 0 \\ 0 & \delta^\alpha_\beta W^I_J \end{pmatrix} \]

To calculate \( A \) we write

\[ A^B_A = \sum a^C_A (D^B_C b^B_D - b^B_C D^B_D) \]

(there are no mixing terms like \( D^B_C b^B_D \) because the matrix \( b \) is block diagonal). Or

\[ A^\beta_J_{\alpha I} = \sum a^\gamma_K_{\alpha I} (D^\delta_L_{\gamma K} b^\beta_J_{L \delta} - b^\beta_J_{\gamma K} D^\delta_L_{\theta L}) \]

Enumerating all possibilities for \( \alpha I \) and \( \beta J \), where \( I = 1, i \) and \( J = 1, j \),

\[ A^\beta_{11}_{\alpha 1} = \sum X^\gamma_{\alpha 1} (D^\delta_{\gamma (l)} Z^\beta_{\delta (l)} - Z^\delta_{\gamma (l)} D^\beta_{\delta (l)}) \]

\[ A^\beta j_{\alpha i} = \delta^j_i \sum X^\gamma_{\alpha i} (D^\delta_{\gamma (q)} Z^\beta_{\delta (q)} - Z^\delta_{\gamma (q)} D^\beta_{\delta (q)}) \]

\[ A^\beta_{11} = A^\beta_{11} = 0 \]

with the mixing terms vanishing. Next we evaluate these, component by component, by taking \( \alpha = 1, 2, a \), and \( \beta = 1, 2, b \):

\[ A^1_{11} = \sum X^1_{11} \left( D^1_{1 (l)} Z^1_{1 (l)} - Z^1_{1 (l)} D^1_{1 (l)} \right) \]

\[ = \sum X^1_{11} \gamma^\mu \partial_\mu Z^1_{1} \equiv -\frac{i}{2} g^1 \gamma^\mu B_\mu \]
\begin{align}
A_{21}^{21} &= \sum X_2^2 \left( D_{2(l)}^2 \frac{Z_2^2}{2} - Z_2^2 \frac{D_{2(l)}}{2} \right) \\
&= \sum X_2^2 \gamma^\mu \partial_\mu Z_2^2 \\
&= \sum X_1^1 \gamma^\mu \partial_\mu Z_1^1 = \frac{i}{2} g_1 \gamma^\mu B_\mu \\
\end{align}

\begin{align}
A_{11}^{11} &= \sum X_1^i \left( D_{1(l)}^i \frac{Z_1^i}{2} - Z_1^i \frac{D_{1(l)}}{2} \right) \\
&= \gamma_5 k^{\kappa \nu} \sum X_1^i \left( Z_1^i - Z_1^i \right) \\
&\equiv \gamma_5 k^{\kappa \nu} H_2
\end{align}

\begin{align}
A_{11}^{21} &= \sum a_{11}^{11} \left( D_{11}^{11} b_{11}^{21} \right) \\
&= \gamma_5 k^{\kappa \nu} \sum X_1^i \left( Z_1^i - Z_1^i \right) \\
&\equiv \gamma_5 k^{\kappa \nu} \left( - H_1 \right)
\end{align}

\begin{align}
A_{21}^{11} &= \sum a_{21}^{21} \left( D_{21}^{21} b_{21}^{11} \right) \\
&= \gamma_5 k^{\kappa \varepsilon} \sum X_2^i \left( Z_2^i - Z_2^i \right) \\
&= \gamma_5 k^{\kappa \varepsilon} H_1
\end{align}

\begin{align}
A_{21}^{21} &= \sum X_2^2 \left( D_{2(l)}^2 \frac{Z_2^2}{2} - Z_2^2 \frac{D_{2(l)}}{2} \right) \\
&= \gamma_5 k^{\kappa \varepsilon} \sum X_2^i \left( Z_2^i - Z_2^i \right) \\
&= \gamma_5 k^{\kappa \varepsilon} H_2
\end{align}

where we have used the relations $X_2^2 = \overline{X}_1^1$ and $Z_2^1 = -\overline{Z}_1^1$ because of the quaternionic property.

Next:

\begin{align}
A_{a1}^{b1} &= \sum a_{a1}^{c1} \left( D_{c1}^{d1} b_{d1}^{b1} - b_{c1}^{d1} D_{d1}^{b1} \right) \\
&= \gamma^\mu \sum X_a^c \left( \partial_\mu Z_c^a \right) \\
&= - \frac{i}{2} g_2 W_\mu^a \left( \sigma^a \right)_{b}
\end{align}

The reason we can write this as an $SU(2)$ gauge field is because it comes from multiplying quaternions:

\begin{align}
q_1 \partial_\mu q_2 &= \left( \begin{array}{c}
\alpha_1 \\
-\beta_1
\end{array} \right) \left( \begin{array}{c}
\beta_1 \\
\alpha_1
\end{array} \right) \\
&= \left( \begin{array}{c}
\alpha_1 \partial_\mu \beta_2 - \beta_1 \partial_\mu \alpha_2 \\
-\beta_1 \partial_\mu \alpha_2 - \alpha_1 \partial_\mu \beta_2
\end{array} \right)
\end{align}
which is of the right form if we note that $A$ is Hermitian. The other components for $A_{\alpha I}^{\beta j}$ give exactly the same results with the replacements $k^\nu \rightarrow k^a$ and $k^e \rightarrow k^d$ and is proportional to $\delta_i^j$. For the $A_{A}^{B'}$ elements we have

$$A_{A}^{B'} = \sum a_{A}^{\alpha' I} \left(D_{C}^{\alpha' I} b_{D'}^{B'} - b_{C}^{D'} D_{D'}^{B'}\right)$$

In terms of components we have

$$A_{\alpha' I}^{\beta' j'} = \sum a_{\alpha' I}^{\gamma' k'} \gamma^\mu \partial_\mu b_{\gamma' l'}^{\beta' j'}$$

$$= \delta_{\alpha'}^{\beta'} \sum Y_{I}^{V_{I'}}^{J} \gamma^\mu \partial_\mu W_{I}^{V_{I'}}$$

$$= \delta_{\alpha'}^{\beta'} \left(-\frac{i}{2} g_{1} \gamma^\mu B_\mu \right)$$

because $Y_1^1 = X_1^1$ and $W_1^1 = Z_1^1$. Next

$$A_{\alpha' I}^{\beta' j'} = \sum a_{\alpha' I}^{\gamma' k'} \left(D_{\gamma' l'}^{\beta' j'} b_{\gamma' k'}^{\beta' j'} - b_{\gamma' k'}^{\beta' j'} D_{\gamma' l'}^{\beta' j'}\right)$$

$$= \delta_{\alpha'}^{\beta'} \sum Y_{I}^{V_{I'}}^{J} \gamma^\mu \partial_\mu W_{I}^{V_{I'}}$$

$$= \delta_{\alpha'}^{\beta'} \gamma^\mu (V_\mu)^{j'}_{I'}$$

We shall require that the field $A$ is unimodular

$$\text{Tr} (A) = 0.$$  

This condition turns out to be equivalent to the cancelation of all chiral anomalies. In this respect, it is important to understand the connection between chiral anomalies and the unimodularity conditions and we refer to [7] and [16]. In fact we shall discuss below in [9.1] the meaning of this unimodularity condition. This condition implies that

$$\sum (A_{I})_{\alpha I}^{\alpha' I'} + (A_{I})_{\alpha' I}^{\alpha I'} = 0.$$  

Thus

$$-\frac{i}{2} g_{1} B_\mu + (V_\mu)^{j'}_{I'} = 0$$

and we can write

$$\left(V_\mu\right)^{j'}_{I'} = \frac{i}{6} g_{1} B_\mu \delta^{j'}_{I'} + \frac{i}{2} g_{3} V_{m}^{m} (\lambda^{m})^{j'}_{I'}$$

where $(\lambda^{m})^{j'}_{I'}$ are the 8 Gell-Mann matrices.

The mixed components are

$$A_{\alpha I}^{\beta' j'} = \sum a_{\alpha I}^{\gamma K} \left(D_{\gamma K}^{J} b_{\gamma L}^{J'} - b_{\gamma K}^{J} D_{\gamma L}^{J'}\right)$$

$$= \sum a_{\alpha I}^{\gamma K} b_{\gamma L}^{J} \left(D_{\gamma K}^{J} b_{\gamma L}^{J'} - b_{\gamma K}^{J} D_{\gamma L}^{J'}\right) D_{\gamma L}^{J'}$$

$$= \delta_{\alpha I}^{\delta_{I}} \delta_{\gamma L}^{\delta_{L}} \sum X_{I}^{1} \left(W_{I}^{V_{I'}} - Z_{I}^{1}\right)$$

$$= 0$$
Thus whatever field would be placed in the mixed component of the Dirac operator would stay unperturbed. Evaluating the matrix $JAJ$ we now have

\begin{align}
(4.77) \quad (JAJ^{-1})_{11}^{11} &= \frac{i}{2} g_1 \gamma^\mu B_\mu \\
(4.78) \quad (JAJ^{-1})_{21}^{21} &= \frac{i}{2} g_1 \gamma^\mu B_\mu \\
(4.79) \quad (JAJ^{-1})_{a1}^{b1} &= \frac{i}{2} g_1 \gamma^\mu B_\mu \delta^b_a \\
(4.80) \quad (JAJ^{-1})_{a\alpha}^{\beta j} &= -\frac{i}{6} g_1 \gamma^\mu B_\mu \delta^j_i - \frac{i}{2} g_3 \gamma^\mu V^m_\mu (\lambda^m)_i^j \delta^\beta_\alpha
\end{align}

Adding $D + A + JAJ^{-1}$ gives the Dirac operator including inner fluctuations. All the components are listed in the appendix A. It is important to note that we have obtained all the correct representations of the fermions, with the correct quantum numbers, including all hypercharges. We stress that the unimodularity condition is essential for obtaining the correct hypercharge assignments.

At this point we can give more details about the cases which were ignored when we restricted our choice of the algebra to the “Symplectic-Unitary”. These are the five possibilities

\begin{align}
(4.81) \quad M_4 (\mathbb{C}) \oplus M_4 (\mathbb{C}) \\
(4.82) \quad M_4 (\mathbb{R}) \oplus M_4 (\mathbb{C}) \\
(4.83) \quad M_2 (\mathbb{H}) \oplus M_2 (\mathbb{H}) \\
(4.84) \quad M_4 (\mathbb{R}) \oplus M_2 (\mathbb{H}) \\
(4.85) \quad M_4 (\mathbb{R}) \oplus M_4 (\mathbb{R})
\end{align}

The last three cases can be discarded immediately. This can be seen as follows. If grading is imposed on $M_4 (\mathbb{R})$ then this will break the algebra into $M_2 (\mathbb{R}) \oplus M_2 (\mathbb{R})$ corresponding to the leptonic group $SO(2) \times SO(2)$ which cannot accommodate the weak symmetry $SU(2)$. If grading is imposed on $M_2 (\mathbb{H})$ this will break the algebra into $\mathbb{H} \oplus \mathbb{H}$ corresponding to the group $SU(2) \times SU(2)$ and in this case the color group $SU(3)$ could not be accommodated. The same reason also hold for the second case with the difference that $M_4 (\mathbb{C})$ will break into $M_2 (\mathbb{C}) \oplus M_2 (\mathbb{C})$. Thus we are left only with the first possibility. This case must be analyzed. When grading is imposed on the first algebra $M_4 (\mathbb{C})$ it will break into $M_2 (\mathbb{C}) \oplus M_2 (\mathbb{C})$. The condition that there is non trivial mixing between fermions and conjugate fermions, would then break the algebra $M_2 (\mathbb{C}) \oplus M_2 (\mathbb{C}) \oplus M_4 (\mathbb{C})$ into

\begin{align}
(4.86) \quad \mathbb{C} \oplus \mathbb{C}^\prime \oplus M_2 (\mathbb{C}) \oplus \mathbb{C} \oplus M_3 (\mathbb{C})
\end{align}

where two of the algebras $\mathbb{C}$ must be identified to satisfy the first order condition. This identification then implies that the first component of the spinor $\psi_A$ will become neutral with respect to all gauge fields. Working
the components of the gauge field $A = \sum a [D, b]$ as was carried out in the $M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$ case shows that we will get two complex Higgs fields instead of one, because in this case the different components will not be related by the quaternionic conditions. If in addition the unimodularity condition is imposed restricting the unitary action of the algebra to be $SU(A)$ the gauge group becomes

$$U(1)^3 \times SU(2) \times SU(3)$$

(4.87)

and there are two additional $U(1)$ gauge fields to those of the standard model. In normal situations it is possible to take one of the $U(1)$ to be the hypercharge and the other $U(1)$ to be the $B-L$, however, the last $U(1)$ if not truncated will be anomalous. However, in this case, because the neutrino will be neutral with respect to the two additional $U(1)$ gauge fields implies that these $U(1)$ are anomalous. Thus this case can only be discarded after analyzing the model, and showing that it is inconsistent at the quantum level. If the unimodularity condition is not imposed, then the gauge group becomes

$$U(1)^4 \times SU(2) \times SU(3)$$

(4.88)

which will also be anomalous. It remains to be seen whether in these cases a Green-Schwarz mechanism can be employed to cancel one of the $U(1)$ anomalies. From this analysis, it should be clear that the only compelling case to consider is when the ”Symplectic-Unitary” symmetry is imposed together with the unimodularity condition.

5. **Spectral Action**

The relevant Dirac operator is $D_A$ which includes both inner and outer automorphisms. The fermionic part of the action is simple and of the Dirac type. Since our considerations are Euclidean, one cannot impose the Majorana condition

$$J \psi = \psi$$

(5.1)

as this could only be done in the Minkowski case. The appropriate action turns out to be given by

$$\langle J \psi, D_A \psi \rangle$$

(5.2)

which is an antisymmetric bilinear form. To show this we have

$$\langle J \zeta', D_A \zeta \rangle = -\langle J \zeta', J^2 D_A \zeta \rangle$$

(5.3)

$$= -\langle JD_A \zeta, \zeta' \rangle = -\langle D_A J \zeta, \zeta' \rangle$$

(5.4)

$$= -\langle J \zeta, D_A \zeta' \rangle$$

(5.5)
where $\zeta, \zeta' \in \mathcal{H}$ are commuting sections, and where we have used $J^2 = -1$, the unitarity of $J$

\begin{equation}
\langle J \zeta', J \zeta \rangle = \langle \zeta, \zeta' \rangle
\end{equation}

and the hermiticity of $D_A$. Because of the anticommutativity of the Grassmann variables $\psi$ the expression $\langle J \psi, D_A \psi \rangle$ is nonzero. Moreover one can impose the chirality condition because

\begin{equation}
\gamma J D = J D \gamma
\end{equation}

The path integral

\begin{equation}
\int \exp \left( -\frac{1}{2} \langle J \psi, D_A \psi \rangle \right) D \psi = \text{Pf} (D_A)
\end{equation}

where the Pfaffian is the square root of the determinant. Thus it is possible to integrate only the chiral fermions $\psi$ and the correct degrees of freedom are obtained because of the appearance of the Pfaffian. All details of the standard model as well as its unification with gravity are achieved by postulating the action

\begin{equation}
\frac{1}{2} \langle J \psi, D_A \psi \rangle + \text{Trace } f(D_A/\Lambda)
\end{equation}

where $\Lambda$ is some scale to be determined, and the trace is taken over all eigenvalues below the scale $\Lambda$. We restrict the function $f$ to be even and positive. It can be shown, using heat kernel methods that this trace can be expressed in terms of the geometrical Seeley deWitt coefficients,

\begin{equation}
\text{Trace } f(D_A/\Lambda) = \sum_{n=0}^{\infty} F_{4-n} A^{4-n} a_n
\end{equation}

where the function $F$ is defined by $F(u) = f(v)$ where $u = v^2$, thus $F(D^2) = f(D)$. We define

\begin{equation}
f_k = \int_0^{\infty} f(v) v^{k-1} dv, \quad k > 0
\end{equation}

then

\begin{equation}
F_4 = \int_0^{\infty} F(u) u du = 2 \int_0^{\infty} f(v) v^3 dv = 2 f_4
\end{equation}

\begin{equation}
F_2 = \int_0^{\infty} F(u) u du = 2 \int_0^{\infty} f(v) vdv = 2 f_2
\end{equation}

\begin{equation}
F_0 = F(0) = f(0) = f_0
\end{equation}

\begin{equation}
F_{-2n} = (-1)^n F^{(n)}(0) = \left[ (-1)^n \left( \frac{d}{2v dv} \right)^n \right] f(0) \quad n \geq 1
\end{equation}

The $a_n$ are the Seeley deWitt coefficients, and fortunately are given by general formulas for any second order elliptic differential operator. These
formulas, derived by Gilkey, can be conveniently used in our case. The first step is to expand $D^2$ into the form
\begin{equation}
D^2 = -(g^{\mu\nu} \partial_{\mu} \partial_{\nu} + A^\mu \partial_{\mu} + B)
\end{equation}
and from this extract the connection $\omega_{\mu}$
\begin{equation}
D^2 = -(g^{\mu\nu} \nabla_{\mu} \nabla_{\nu} + E)
\end{equation}
where
\begin{equation}
\nabla_{\mu} = \partial_{\mu} + \omega_{\mu}.
\end{equation}
This gives
\begin{equation}
\omega_{\mu} = \frac{1}{2} g_{\mu\nu} (A^\nu + \Gamma^\nu)
\end{equation}
\begin{equation}
E = B - g^{\mu\nu} (\partial_{\mu} \omega_{\nu} + \omega_{\mu} \omega_{\nu} - \Gamma^\rho_{\mu\nu} \omega_{\rho})
\end{equation}
\begin{equation}
\Omega_{\mu\nu} = \partial_{\mu} \omega_{\nu} - \partial_{\nu} \omega_{\mu} + [\omega_{\mu}, \omega_{\nu}]
\end{equation}
where $\Gamma^\nu = g^{\rho\sigma} \Gamma^\nu_{\rho\sigma}$ and $\Gamma^\rho_{\mu\nu}$ is the Christoffel connection of the metric $g_{\mu\nu}$.
The first few Seeley-deWitt coefficients $a_n$ for manifolds without boundary, are given by
\begin{equation}
a_0 = \frac{1}{16\pi^2} \int d^4x \sqrt{g} \text{Tr} (1)
\end{equation}
\begin{equation}
a_2 = \frac{1}{16\pi^2} \int d^4x \sqrt{g} \text{Tr} \left( E + \frac{1}{6} R \right)
\end{equation}
\begin{equation}
a_4 = \frac{1}{16\pi^2} \frac{1}{360} \int_M d^4x \sqrt{g} \text{Tr} \left( 12 R_{\mu}^{\quad \mu} + 5 R^2 - 2 R_{\mu\nu} R^{\mu\nu} 
+ 2 R_{\mu\nu} R^{\mu\nu} + 60 E R + 180 E^2 + 60 E_{\mu}^{\quad \mu} + 30 \Omega_{\mu\nu} \Omega^{\mu\nu} \right)
\end{equation}
while the odd ones all vanish for manifolds without boundary
\begin{equation}
a_{2n+1} = 0.
\end{equation}
We will deal with higher order terms such as $a_6$ later. Using these formulas, it is simple and straightforward to compute the spectral action. Having listed all the matrix components of the Dirac operator $D^N_M$ we now proceed to evaluate the matrix $D^2$
\begin{equation}
(D^2)^B_A = D^C_A D^B_C + D^C_A D^B_C,
\end{equation}
\begin{equation}
(D^2)^{B'}_{A'} = D^C_{A'} D^{B'}_C + D^C_{A'} D^{B'}_C,
\end{equation}
\begin{equation}
(D^2)^{B'}_A = D^C_A D^{B'}_C + D^C_A D^{B'}_C,
\end{equation}
\begin{equation}
(D^2)^A_{A'} = D^C_{A'} D^B_C + D^C_{A'} D^B_C.
\end{equation}
We can use the properties
\begin{equation}
D^B_{A'} = D^B_A, \quad D^B_A = D^{B'}_A, \quad D^{B'}_A = D^B_{A'}.
\end{equation}
and thus it will not be necessary to compute all the traces by taking advantage of the fact that some of the traces will be related to each other by complex conjugation. As an example we calculate the first few component of \( D^2 \)

\[(D^2)^{11}_{11} = D_{11}^{11} D_{11}^{11} + D_{11}^{a1} D_{a1}^{11} + k^{\mu\nu} k^{\mu\nu} \sigma^2 \]

\[= \gamma^\mu D_\mu \gamma^\nu D_\nu \otimes 13 + k^{\mu\nu} k^{\mu\nu} H_a \overline{\Pi}^2 + k^{\mu\nu} k^{\mu\nu} \sigma^2 \]

\[(D^2)^{a1}_{11} = D_{11}^{11} D_{11}^{a1} + D_{11}^{b1} D_{b1}^{a1} \]

\[= \gamma^\mu D_\mu \gamma^\nu \epsilon^{ab} H_b + \gamma_5 k^{\mu\nu} \epsilon^{bc} H_c \gamma^\mu \left( \left( D_\mu + \frac{i}{2} g_1 B_\mu \right) \delta^a_b - \frac{i}{2} g_2 W^\alpha_\mu (\sigma^\alpha)_b^{a} \right) \]

\[= \gamma^\mu \gamma_5 k^{\mu\nu} \epsilon^{ab} \nabla_\mu H_b \]

where

\[\nabla_\mu H_a = \left( \left( \partial_\mu - \frac{i}{2} g_1 B_\mu \right) \delta^a_b - \frac{i}{2} g_2 W^\alpha_\mu (\sigma^\alpha)_b^{a} \right) H_b \]

We list all the components of \( D^2 \) in Appendix B. Using the form for \( D^2 \)

\[(D^2) = -(g^{\mu\nu} \partial_\mu \partial_\nu + A^\mu \partial_\mu + B) \]

we can read then \( \omega_\mu \) and \( E \)

\[\omega_\mu = \frac{1}{2} g_{\mu\nu} (A^\nu + \Gamma^\nu) \]

\[E = B - g^{\mu\nu} (\partial_\mu \omega_\nu + \omega_\mu \omega_\nu - \Gamma^\mu_\nu \omega_\rho) \]

The matrix elements \( (\omega_\mu)^N_M \) and \( (E)^N_M \) are listed in Appendix C. It is now possible to summarize the results

\[-\frac{1}{2} \text{Tr} (E) = 4 \left[ 12R + 2a \overline{H}H + c \sigma^2 \right] \]

\[\frac{1}{2} \text{tr} (E^2) = 4 \left[ 5g^2 B^2_{\mu\nu} + 3g^2_2 (W^\alpha_{\mu\nu})^2 + 3g^2_3 (V^m_{\mu\nu})^2 + 3R^2 + aR \overline{H}H \right. \]

\[+ \frac{1}{2} cR \sigma^2 + 2b (\overline{H}H)^2 + 2a |\nabla_\mu H_a|^2 + 4e \overline{H}H \sigma^2 + c (\partial_\mu \sigma)^2 + d \sigma^4 \]

\[- \frac{1}{2} \text{Tr} (\Omega^2_{\mu\nu})^M_N = 4 \left[ -6R^2_{\mu\nu\rho\sigma} - 10g^2_1 B^2_{\mu\nu} - 6g^2_2 (W^\alpha_{\mu\nu})^2 - 6g^2_3 (V^m_{\mu\nu})^2 \right] \]
where
\begin{align*}
\text{(5.40)} \quad a &= \text{tr} \left( k^\nu k^\nu + k^e k^e + 3 \left( k^u k^u + k^d k^d \right) \right) \\
\text{(5.41)} \quad b &= \text{tr} \left( (k^\nu k^\nu)^2 + (k^e k^e)^2 + 3 \left( (k^u k^u)^2 + (k^d k^d)^2 \right) \right) \\
\text{(5.42)} \quad c &= \text{tr} \left( k^{\nu R} k^{\nu R} \right) \\
\text{(5.43)} \quad d &= \text{tr} \left( (k^{\nu R} k^{\nu R})^2 \right) \\
\text{(5.44)} \quad e &= \text{tr} \left( k^{\nu} k^{\nu} k^{\nu R} k^{\nu R} \right)
\end{align*}

The first two Seeley-deWitt coefficients are
\begin{align*}
\text{(5.45)} \quad a_0 &= \frac{1}{16\pi^2} \int d^4x \sqrt{g} \text{Tr} (1) \\
&= \frac{24}{\pi^2} \int d^4x \sqrt{g} \\
\text{(5.46)} \quad a_2 &= \frac{1}{16\pi^2} \int d^4x \sqrt{g} \text{Tr} \left( E + \frac{1}{6} R \right) \\
&= -\frac{2}{\pi^2} \int d^4x \sqrt{g} \left( R + \frac{1}{4} a \bar{H}H + \frac{1}{4} c \sigma^2 \right) \\
\text{(5.47)} \quad a_4 &= \frac{1}{2\pi^2} \int d^4x \sqrt{g} \left[ -\frac{3}{5} C_{\mu\nu\rho\sigma}^2 + \frac{11}{30} R^* R^* + \frac{5}{3} g_1^2 B_{\mu\nu}^2 + g_2^2 (W_{\mu\nu}^\alpha)^2 + g_3^2 (V_{\mu\nu}^m)^2 \\
&\quad + \frac{1}{6} a R \bar{H}H + b (\bar{H}H)^2 \sigma^2 + a |\nabla_{\mu} H_{\alpha}|^2 + 2 e \bar{H}H \sigma^2 \\
&\quad + \frac{1}{2} d \sigma^4 + \frac{11}{12} c R \sigma^2 + \frac{1}{2} c (\partial_{\mu} \sigma)^2 - \frac{2}{5} R_{\mu\nu}^{\mu\nu} - \frac{a}{3} (\bar{H}H)^{\mu\nu} - \frac{c}{6} (\sigma^2)^{\mu\nu} \right]
\end{align*}

Thus the bosonic spectral action to second order is given by
\begin{align*}
\text{(5.48)} \quad S &= F_4 \Lambda^4 a_0 + F_2 \Lambda^2 a_2 + F_0 a_4 + F_{-2} \Lambda^{-2} a_6 + \cdots
\end{align*}

and
\begin{align*}
\text{(5.49)} \quad S_b &= \frac{24}{\pi^2} F_4 \Lambda^4 \int d^4x \sqrt{g} \\
&\quad - \frac{2}{\pi^2} F_2 \Lambda^2 \int d^4x \sqrt{g} \left( R + \frac{1}{2} a \bar{H}H + \frac{1}{4} c \sigma^2 \right) \\
&\quad + \frac{1}{2\pi^2} F_0 \int d^4x \sqrt{g} \left[ \frac{1}{30} (-18 C_{\mu\nu\rho\sigma}^2 + 11 R^* R^*) + \frac{5}{3} g_1^2 B_{\mu\nu}^2 + g_2^2 (W_{\mu\nu}^\alpha)^2 + g_3^2 (V_{\mu\nu}^m)^2 \\
&\quad + \frac{1}{6} a R \bar{H}H + b (\bar{H}H)^2 + a |\nabla_{\mu} H_{\alpha}|^2 + 2 e \bar{H}H \sigma^2 + \frac{1}{2} d \sigma^4 + \frac{11}{12} c R \sigma^2 + \frac{1}{2} c (\partial_{\mu} \sigma)^2 \right] \\
&\quad + F_{-2} \Lambda^{-2} a_6 + \cdots
\end{align*}
It is worth to also summarize the the fermionic action

\[ S_f = \nu_R^* \gamma^\mu D_\mu \nu_R + e_R^* \gamma^\mu (D_\mu + ig_1 B_\mu) e_R + i \eta_L^* \gamma^\mu \left( D_\mu + \frac{i}{2} g_1 B_\mu \right) \delta_a^b \eta_a L + u_R^* \gamma^\mu \left( D_\mu - \frac{2i}{3} g_1 B_\mu \right) \delta_i^j u_i R + d_R^* \gamma^\mu \left( D_\mu + \frac{i}{3} g_1 B_\mu \right) \delta_i^j d_j R + q_i a^* \gamma^\mu \left( D_\mu - \frac{i}{6} g_1 B_\mu \right) \delta_i^b \delta_j^a q_a L + d_i \gamma^5 \nu^* \epsilon^{a b} H_{i b} L + e_R^* \gamma_5 k^* \epsilon^{a b} \frac{\overbar{\gamma}}{\Lambda^2} \eta a L + u_R^* \gamma_5 k^* \epsilon^{a b} \frac{\overbar{\gamma}}{\Lambda^2} \delta_i^j q_j a L + \nu_R^* \gamma_5 k^* \nu^* \epsilon^{a b} \left( \nu_5^* \right) c + h.c. \]

Our strategy is to use the spectral action as an effective action at a fixed scale, of the order of the unification scale, and to impose the additional relations between the independent parameters of the Standard Model coupled to gravity as a boundary condition at that scale. One can then let these parameters run down using the RG equations to their value at ordinary scale. As a first example one has the unification of the three gauge couplings in the form

\[ g_3^2 = g_2^2 = \frac{5}{3} g_1^2 \]

In applying the above strategy we have limited ourselves to the first three terms in the expansion of the spectral action, the reason being that the natural spectral functions \( F(D^2/\Lambda^2) \) used in the spectral action are meant to count the number of eigenvalues of \( D^2 \) which are less than \( \Lambda^2 \). These functions are “cutoff” functions which are completely flat near 0 and thus have all their Taylor coefficients \( F^{(n)}(0) \) vanishing except \( F^{(0)}(0) \). The question of whether we should ignore the higher order terms will be dealt with in Part II.

6. QFT ANALYSIS: ZEROTH ORDER

As we have seen in the previous section, the spectral action gives a well defined form for all interactions. We shall take the Wilsonian point of view where the geometrical action is considered as an effective theory valid at some scale \( \Lambda \), which is related to the low energy action by running all masses and coupling constants as determined by the RG equations. The special relations that exist between the different coupling constants are taken as boundary conditions for the integration of the RG equations. To check the
validity of the model, we examine consequences of these boundary conditions. We shall assume that the spectral action is determined by the cutoff function, so that all higher order terms in the heat kernel expansion are truncated to zero. In this case, the normalization of the kinetic terms imposes a relation between the coupling constants $g_1, g_2, g_3$ and the coefficient $F_0$, of the form

$$\frac{g_3^2 F_0}{2\pi^2} = \frac{1}{4}, \quad g_3^2 = g_2^2 = \frac{5}{3} g_1^2.$$  

This gives that $\sin^2 \theta_W = \frac{3}{8}$ a value also obtained in $SU(5)$ and $SO(10)$ grand unified theories. The three momenta of the function $F_0, F_2$ and $F_4$ can be used to specify the initial conditions on the gauge couplings, the Newton constant and the cosmological constant. The fine structure constant $\alpha_{em}$ is thus given by

$$\alpha_{em} = \sin(\theta_W)^2 \alpha_2, \quad \alpha_i = \frac{g_i^2}{4\pi}$$

Its infrared value is $\sim 1/137.036$ but it is running as a function of the energy and increases to the value $\alpha_{em}(M_Z) = 1/128.09$ already, at the energy $M_Z \sim 91.188$ Gev.

Assuming the “big desert” hypothesis, the running of the three couplings $\alpha_i$ is known. With 1-loop corrections only, it is given by

$$\beta_{g_i} = (4\pi)^{-2} b_i g_i^3,$$  with  $b = (\frac{41}{6}, -\frac{19}{6}, -7)$,

so that

$$\alpha_1^{-1}(\Lambda) = \alpha_1^{-1}(M_Z) - \frac{41}{12\pi} \log \frac{\Lambda}{M_Z},$$

$$\alpha_2^{-1}(\Lambda) = \alpha_2^{-1}(M_Z) + \frac{19}{12\pi} \log \frac{\Lambda}{M_Z},$$

$$\alpha_3^{-1}(\Lambda) = \alpha_3^{-1}(M_Z) + \frac{42}{12\pi} \log \frac{\Lambda}{M_Z},$$

where $M_Z$ is the mass of the $Z^0$ vector boson.

It is known that the predicted unification of the coupling constants does not hold exactly. In fact, if one considers the actual experimental values

$$g_1(M_Z) = 0.3575, \quad g_2(M_Z) = 0.6514, \quad g_3(M_Z) = 1.221,$$

one obtains the values

$$\alpha_1(M_Z) = 0.0101, \quad \alpha_2(M_Z) = 0.0337, \quad \alpha_3(M_Z) = 0.1186.$$  

and one knows that the graphs of the running of the three constants $\alpha_i$ do not meet exactly, hence do not specify a unique unification energy. The discrepancy comes mostly from the running of the $\alpha_1$ coupling as we should expect unification of the gauge couplings with the Newton coupling near the Planck energy.
We first note that the relations between the gauge coupling constants, and the RG equations are carried for the interactions obtained by assuming that the spectral function is a cut-off function, and thus suppressing all higher order terms. In Part II we shall show that if the spectral function $F(D^2)$ deviates by small perturbations from the cut-off function, higher order interactions can lead to small corrections which alter the running of each of the gauge coupling constants. In other words, we shall investigate the possibility that all couplings are unified at $\Lambda$ provided that the function $F$ is chosen appropriately, and higher order corrections from the spectral action are included.

A distinctive feature of the spectral action is that the Higgs coupling is proportional to the gauge couplings. This implies a restriction on its mass. To see this consider the equation

$$\frac{d\lambda}{dt} = \lambda \gamma + \frac{1}{8\pi^2} (12\lambda^2 + B)$$

where

$$\gamma = \frac{1}{16\pi^2} (12y_t^2 - 9g_2^2 - 3g_1^2)$$

$$B = \frac{3}{16} (3g_2^4 + 2g_1^2g_2^2 + g_1^4) - 3y_t^4.$$  

The Higgs mass is then given by

$$m_H^2 = 8\lambda \frac{M^2}{g^2}, \quad m_H = \sqrt{2\lambda} \frac{2M}{g}.$$  

One can solve this equation numerically, provided the boundary condition for $\lambda$ is given. This depends on the value of the gauge coupling at unification, and where the unification scale is taken. If for example the boundary value $\lambda_0 = 0.356$ is taken at $\Lambda = 10^{17}$ Gev, this gives $\lambda(M_Z) \sim 0.241$ and a Higgs mass of the order of 170 Gev which is disfavored by experiment (and was even ruled out for some period). This answer is sensitive to the value of the unification scale, and since we expect that it can have substantial consequences to let the spectral function deviate from the cutoff function, we should include the higher order corrections to the spectral action in our analysis of the Higgs mass. A reliable value for the mass of the Higgs depends on the form of the spectral function, which in turn determines the unification scale.

On the other hand, the mass of the top quark is governed by the top quark Yukawa coupling $k^t$ through the equation

$$m_{top}(t) = \frac{1}{\sqrt{2}} \frac{2M}{g} k^t = \frac{1}{\sqrt{2}} v k^t,$$

where $v = \frac{2M}{g}$ is the vacuum expectation value of the Higgs field. All fermions get their masses by coupling to the Higgs through interactions of
After normalizing the kinetic energy of the Higgs field through the redefinition $H \rightarrow \sqrt{\frac{a}{F_0}} H$, the mass term becomes

\begin{equation}
\frac{\pi}{\sqrt{F_0}} \frac{k}{\sqrt{a}} H \overline{\psi} \psi
\end{equation}

and we notice that $\sum_i \left( \frac{k_i}{\sqrt{a}} \right)^2 = 1$. This gives a relation among the fermions masses and the W- mass

\begin{equation}
\sum_{\text{generations}} m_e^2 + m_{\nu}^2 + 3 m_d^2 + 3 m_u^2 = 8 M_W^2.
\end{equation}

If the value of $g$ at a unification scale of $10^{17}$ Gev is taken to be $\sim 0.517$ and neglecting the $\tau$ neutrino Yukawa coupling, we get

\begin{equation}
k_t = \frac{2}{\sqrt{3}} g \sim 0.597.
\end{equation}

The numerical integration of the differential equation gives a top quark mass of the order of 179 Gev, and the agreement with experiment becomes quite good if one takes into account the Yukawa coupling for neutrinos as explained in details in [13]. This indicates that the top quark mass is less sensitive than the Higgs mass to the unification scale ambiguities. This could be related to the fact that the fermionic action is much simpler than the bosonic one which is only determined by an infinite expansion whose reliability depends on the convergence of the higher order terms.

7. Parity violating terms

It is possible to add to the spectral action terms that will violate parity such as the gravitational term $\epsilon^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}^{a_b} R_{\rho\sigma}^{a_b}$ and the non-abelian $\theta$ term $\epsilon^{\mu\nu\rho\sigma} V^m_{\mu\nu} V^{m}_{\rho\sigma}$. These arise by allowing for the spectral action to include the term

\begin{equation}
\text{Tr} \left( \gamma G \left( \frac{D^2}{\Lambda^2} \right) \right)
\end{equation}

where $G$ is a function not necessarily equal to the function $F$, and

\begin{equation}
\gamma = \gamma_5 \otimes \gamma_F
\end{equation}

is the total grading. In this case it is easy to see that there are no contributions coming from $a_0$ and $a_2$ and the first new term occurs in $a_4$ where there are only two contributions:

\begin{equation}
\frac{1}{16\pi^2} \frac{1}{12} \text{Tr} (\gamma_5 \gamma_F \Omega^2_{\mu\nu}) = \frac{1}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}^{a_b}(24 - 24) = 0
\end{equation}
and

\begin{equation}
\frac{1}{16\pi^2} \frac{1}{2} \text{Tr} (\gamma_5 \gamma_F E^2) = \frac{4}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} \left( \left( 1 - \frac{1}{2} \right)^2 (2) + \left( \frac{2}{3} \right)^2 (3) + \left( \frac{1}{3} \right)^2 (3) - \left( \frac{1}{6} \right)^2 (3) (2) \right) 3g_1^2 B_{\mu\nu} B_{\rho\sigma}
\end{equation}

\begin{equation}
\left( - \left( \frac{1}{2} \right)^2 (2) - \left( \frac{1}{2} \right)^2 (2) (3) \right) 3g_2^2 W_\mu^\alpha W_\rho^\alpha + \left( \left( \frac{1}{2} \right)^2 (2) (1 + 1 - 2) \right) 3V^m_\mu V^m_\rho
\end{equation}

Thus the additional terms to the spectral action, up to orders \( \frac{1}{\Lambda^2} \), are

\begin{equation}
\frac{3G_0}{8\pi^2} \epsilon^{\mu\nu\rho\sigma} \left( 2g_1^2 B_{\mu\nu} B_{\rho\sigma} - 2g_2^2 W_\mu^\alpha W_\rho^\alpha \right)
\end{equation}

where \( G_0 = G(0) \). The \( B_{\mu\nu} B_{\rho\sigma} \) is a surface term, while \( W_\mu^\alpha W_\rho^\alpha \) is topological, and both violate PC invariance. The surprising thing is the vanishing of both the gravitational PC violating term \( \epsilon^{\mu\nu\rho\sigma} R_{\mu\nuab} R_{\rho\sigma}^{ab} \) and the \( \theta \) QCD term \( \epsilon^{\mu\nu\rho\sigma} V^m_\mu V^m_\rho \). In this way the \( \theta \) parameter is naturally zero, and can only be generated by the higher order interactions. The reason behind the vanishing of both terms is that in these two sectors there is a left-right symmetry graded with the matrix \( \gamma_F \) giving an exact cancelation between the left-handed sectors and the right-handed ones. In other words the trace of \( \gamma_F \) vanishes and this implies that the index of the full Dirac operator, using the total grading, vanishes. There is one more condition to solve the strong CP problem which is to have the following condition on the mass matrices of the up quark and down quark

\begin{equation}
\det k^a \det k^d = \text{real}.
\end{equation}

At present, it is not clear what condition must be imposed on the quarks Dirac operator, in order to obtain such relation. If this condition can be imposed naturally, then it will be possible to show that (39)

\begin{equation}
\theta_{QT} + \theta_{QCD} = 0
\end{equation}

at the tree level, and loop corrections can only change this by orders of less than \( 10^{-9} \).

\section{8. Dilaton Interactions}

The scale \( \Lambda \) appears as a free parameter in the spectral action. It is more natural if it can arise as the vev of a dynamical field. We thus introduce the dilaton field \( \phi \) and replace the operator \( D^2 \) in the spectral action by

\begin{equation}
P = e^{-\phi} \mathcal{D}^2 e^{\phi}
\end{equation}
A shift in the dilaton field $\phi \to \phi + \ln \Lambda$ transforms $P \to \frac{1}{\Lambda^2} P$. The interactions of the dilaton can be determined by observing that geometrical constructs $\omega_\mu$ and $E$ that appeared in the heat kernel expansion for $D^2$ are related to $\Omega_\mu$ and $\mathcal{E}$ of $P$ by

\begin{align}
\Omega_\mu &= \omega_\mu - 2\partial_\mu \phi \\
\mathcal{E} &= e^{-2\phi} \left(E + g^{\mu\nu} \nabla^\alpha \nabla^\beta \phi \partial_\mu \phi \partial_\nu \phi + \partial_\mu \phi \partial_\nu \phi \right)
\end{align}

where the covariant derivative $\nabla^\mu$ is with respect to the metric $g_{\mu\nu}$. We have shown that the first four terms in the spectral action are independent of the dilaton field when expressed in the Einstein frame with the metric

\begin{equation}
G_{\mu\nu} = g_{\mu\nu} e^{2\phi}
\end{equation}

and in terms of a rescale Higgs field, except for one term which is the dilaton kinetic energy. The Higgs fields are rescaled according to

\begin{equation}
H' = H e^{-\phi}
\end{equation}

and the fermions according to

\begin{equation}
\psi' = \psi e^{-\frac{3}{2} \phi}
\end{equation}

From the relations between $\mathcal{E}$ and $E$ it should be clear that the full potential of the theory can only get a scaling factor. This factor is absorbed when the rescaled fields are used. In other words, the potential is independent of the dilaton. Thus at the classical level, the vev of the dilaton is undetermined. This situation changes when quantum radiative corrections are taken into account. By taking the corrections to be at the Planck scale, and assuming that there are also non-perturbative effects, one finds that the vev of the dilaton is of order one in Planck units. It is interesting to note that this model is exactly what became to be known as the Randall-Sundrum model, although it was obtained in the noncommutative formulation of the standard model long before that. In this picture the Higgs fields $H$ gets a vev of the order of the Planck scale, however, the physical field $H'$ has its vev suppressed through the dilaton coupling $e^{-\phi}$. Thus if $\langle \phi \rangle \sim 40$ in Planck units, then $e^{-\phi} \sim 10^{-19}$. Thus the problem of explaining the very low mass scale of fermion masses reduces to explaining the origin of a dilaton vev of the order of $10^2$.

9. Conclusions and Outlook

We summarize the main assumptions made in determining the noncommutative space:

1. Space-time is a product of a continuous four-dimensional manifold times a finite space.
2. One of the algebras $M_4(\mathbb{C})$ is subject to symplectic symmetry reducing it to $M_2(\mathbb{H})$.
3. The commutator of the Dirac operator with the center of the algebra is non trivial $[D, Z(A)] \neq 0$. 


(4) The unitary algebra $U(\mathcal{A})$ is restricted to $SU(\mathcal{A})$.

These give rise to the following predictions:

1. The number of fundamental fermions is 16.
2. The algebra of the finite space is $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$.
3. The correct representations of the fermions with respect to $SU(3) \times SU(2) \times U(1)$.
4. Higgs doublet and spontaneous symmetry breaking mechanism. This is highly non-trivial especially that the mass term of the Higgs field comes with the correct negative sign.
5. Mass of the top quark compatible with experiment.
6. See-saw mechanism to give very light left-handed neutrinos.

We give here a brief outline of open directions.

9.1. The variant of the Einstein-Yang Mills system.

Before the reduction to the subgroup $U(1) \times SU(2) \times SU(3)$ (coming from the order one condition and the hypothesis of finite distance between the two copies of the four-dimensional manifold $M$) the model one gets is the product of $M$ with the finite space whose algebra is $A_F = M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$. This model is thus very closely related to the Einstein-Yang Mills system in which one simply replaces the algebra $C^\infty(M)$ of functions by the algebra $C^\infty(M) \otimes M_n(\mathbb{C}) = M_n(C^\infty(M))$.

There are a number of reasons to take seriously the variant of the Einstein-Yang Mills system obtained with the algebra $A_F = M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$.

1. The first one is that, as was shown recently in [5], the usual Einstein-Yang Mills system is closely related to supersymmetry as suggested in [9] and in particular the fermions are in the adjoint representation. While the corresponding $SU(n)$ model are far away from realistic models, the situation changes for the above variant with $A_F = M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$ since the gauge group coming from the even part of the graded algebra is the $SU(2)_L \times SU(2)_R \times SU(4)$ of the Patti-Salam model which is much more realistic already.

2. The second reason is that the $SU(4)$ which appears naturally from inner automorphisms of the algebra $A_F$ is a conceptual explanation for the “unimodularity condition” which is an odd ingredient when taken at the level of the reduction to the subgroup. The point here is that it is only at the level of this algebra $A_F$ that the unimodularity condition does acquire a conceptual meaning instead of being an ad-hoc prescription.

3. The third reason is that the conceptual description of the Hilbert space of Fermions for one generation, i.e. of the irreducible representation of $(A_F, J)$, is as the space of maps $\text{Hom}(E, F) \oplus \text{Hom}(F, E)$ where $E$ is a two dimensional vector space over the quaternions $\mathbb{H}$ and $F$ a 4-dimensional vector space over $\mathbb{C}$. It is hard to miss the
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hint to twistors since the latter involve the relation between the corresponding projective spaces namely \( \mathbb{P}^1 \mathbb{H} \) and \( \mathbb{P}^3 \mathbb{C} \).

While it is natural to try to extend the results of [5] to the above variant by assuming that the Dirac operator of the finite space is 0 as for the Einstein-Yang Mills system, it is also quite desirable to come up with a dynamical mechanism for the reduction to the subgroup \( U(1) \times SU(2) \times SU(3) \) coming from the order one condition and the finite distance between the two copies of the four-dimensional manifold \( M \). At the moment the reduction is imposed by a mathematical requirement which is non-dynamical and the corresponding symmetry breaking should in fact come from additional terms in the action showing a preference for the physically desirable “finite distance” condition.

9.2. Role of \( M_4(\mathbb{C}) \).

Let us first ignore the fact that we have two simple components \( M_2(\mathbb{H}) \oplus M_4(\mathbb{C}) \) and explain briefly in what sense one obtains a simpler presentation by replacing the algebra \( C^\infty(M) \) of functions by the algebra \( C^\infty(M) \otimes M_4(\mathbb{C}) = M_4(C^\infty(M)) \).

We start by the two dimensional case, and give a very simple presentation of the algebra \( C^\infty(S^2) \otimes M_2(\mathbb{C}) = M_2(C^\infty(S^2)) \). The algebra is generated by a symbol \( e \) and the scalar matrices \( m \in M_2(\mathbb{C}) \). Elements of the algebra are sums of words of the form

\[
w = e m_1 e m_2 e \cdots m_k e, \quad m_j \in M_2(\mathbb{C})
\]

One multiplies them according to the following rules. The algebraic rules are the usual ones for \( M_2(\mathbb{C}) \) and one has the additional relations

\[
e = e^* = e^2, \quad \langle e - \frac{1}{2} \rangle = 0.
\]

Here the trace \( X \mapsto \langle X \rangle \) with values in the commutant of \( M_2(\mathbb{C}) \) is

\[
\langle X \rangle = e_{11} X e_{11} + e_{21} X e_{12} + e_{12} X e_{21} + e_{22} X e_{22}
\]

with the standard notation for the 4 matrix units

\[
e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\]

The point then is that one obtains in this way a dense subalgebra of \( C^\infty(S^2) \otimes M_2(\mathbb{C}) = M_2(C^\infty(S^2)) \). This follows since once \( e \) is expressed in matrix form with coefficients in the commutant of \( M_2(\mathbb{C}) \) it takes the form

\[
e = \begin{pmatrix} \frac{1}{2} + t & z \\ z^* & \frac{1}{2} - t \end{pmatrix}
\]

and the equation \( e^2 = e \) implies that \( t, z, z^* \) commute pairwise and fulfill the relation

\[
zz^* + t^2 = \frac{1}{4}
\]
Moreover the orientability condition which fixes the volume form of the metric and guarantees that the metric is non-degenerate takes the simple form

\[(9.6) \quad \left\langle \left( e - \frac{1}{2} \right) [D, e]^2 \right\rangle = \gamma \]

where \(\gamma\) is the chirality operator satisfying

\[(9.7) \quad \gamma^2 = \gamma, \quad \gamma = \gamma^*, \quad \gamma e = e\gamma, \quad D\gamma = -\gamma D\]

In dimension 4 one has a similar description of the commutative solution given by the 4-sphere (with a not necessarily round metric having the prescribed volume form). The algebra \(M_2(\mathbb{C})\) is replaced by \(4 \times 4\) matrices and as above the algebra is generated by \(M_4(\mathbb{C})\) and a projection \(e = e^2 = e^*\) of the form

\[(9.8) \quad e = \begin{pmatrix} \frac{1}{2} + t & 0 & \alpha & \beta \\ 0 & \frac{1}{2} + t & -\beta^* & \alpha^* \\ \alpha^* & -\beta & \frac{1}{2} - t & 0 \\ \beta^* & \alpha & 0 & \frac{1}{2} - t \end{pmatrix} \]

where \(t, \alpha, \alpha^*, \beta\) and \(\beta^*\) all commute and satisfy the relation

\[t^2 + |\alpha|^2 + |\beta|^2 = \frac{1}{4}\]

One can then check that \(A = C(S^4)\). The differential constraints

\[(9.9) \quad \left\langle \left( e - \frac{1}{2} \right) [D, e]^4 \right\rangle = \gamma \]

are then satisfied by any Riemannian structure with a given volume form on \(S^4\).

The really new feature which appears in dimension 4 is that the equations admit non-trivial noncommutative solutions. This fact was discovered in [25] and the problem of classification of solutions has been solved in three dimensions in [22], [23], [24], while the 4-dimensional case is still under investigation.

The simplest noncommutative solution is obtained (25) as a deformation by considering the algebra to be generated by \(M_4(\mathbb{C})\) and \(e\) where

\[(9.10) \quad e = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \]

where each \(q\) is a \(2 \times 2\) matrix of the form

\[(9.11) \quad q = \begin{pmatrix} \alpha & \beta \\ -\lambda \beta & \alpha^* \end{pmatrix} \]
In this case the projection constraints imply that

\[
e = \begin{pmatrix}
\frac{1}{2} + t & 0 & \alpha & \beta \\
0 & \frac{1}{2} + t & -\lambda\beta^* & \alpha^* \\
\alpha^* & -\lambda\beta & \frac{1}{2} - t & 0 \\
\beta^* & \alpha & 0 & \frac{1}{2} - t
\end{pmatrix}
\]

satisfying

\[
\alpha\alpha^* = \alpha^*\alpha, \quad \beta\beta^* = \beta^*\beta, \quad \alpha\beta = \lambda\beta\alpha, \quad \alpha^*\beta = \overline{\lambda}\beta\alpha
\]

giving rise to deformed $S^4$.

Assuming that the unification scale is not far away from the Planck scale, it is natural to modify the basic assumption we made that space-time is a product of a continuous four dimensional manifold times a finite space. This leads us to investigate the postulate that at very high energies, the structure of space-time becomes noncommutative in a nontrivial way, which will change in an intrinsic way the particle spectrum. On the other hand, the encouraging results we obtained about the almost unique prediction of the spectrum of the standard model for the gauge group and particle representations, can be taken as a guide that the true geometry should reproduce at lower energies, the product structure we assumed. The starting point is to look for a noncommutative space whose KO dimension is ten (mod 8) and whose metric dimension as dictated by the growth of eigenvalues of the Dirac operator is four. A good starting point would be to mesh in a smooth manner the four-dimensional manifold with the finite space $M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$. The next step is to define the noncommutative space by marrying the concept of generating a manifold as instantonic solution of a set of equations, and to blend these with the finite space.

9.3. Generations.

In this short section we shall speculate on a possible relation between the fundamental group of space-time and the three generations of fermions. Our starting point is the intimate relation in topology:

Manifold $\leftrightarrow$ Poincaré duality in $KO$-homology

and the coincidence of the basic ingredients of cycles in $KO$-homology, namely spectral triples

\[(A, \mathcal{H}, D), \quad ds = D^{-1}, \quad J, \quad \gamma\]

\[J^2 = \varepsilon, \quad DJ = \varepsilon'JD, \quad J\gamma = \varepsilon''\gamma J, \quad D\gamma = -\gamma D\]

with the ingredients of the quantum theory:

- $\mathcal{H}$: one particle Euclidean Fermions
- $D$: inverse propagator
- $J$: charge conjugation
- $\gamma$: chirality
The point about the fundamental group that we wish to make here is that the above equivalence between “manifolds” $M$ and spaces which fulfill Poincaré duality in $KO$-homology is only fully encoded by the fundamental cycle in $KO$-homology if one also takes into account the fundamental group $\Gamma = \pi_1(M)$. More specifically, in the non-simply connected case when $\pi_1(M)$ is non-trivial, the natural datum is not the Dirac operator $D$ on the manifold $M$ but rather the Dirac operator $\tilde{D}$ on the universal cover $\tilde{M}$ of $M$. Even though these operators look alike locally the action of the group $\Gamma = \pi_1(M)$ on the $L^2$ spinors on the universal cover $\tilde{M}$ breaks this space into “sectors” and there is a (might be superficial) resemblance between this decomposition into sectors and the decomposition of the Hilbert space of Fermions as a sum of Hilbert spaces corresponding to generations. We are fully aware of the subtleties inherent to the mixing of generations from the CKM matrix [26] but there is room in the geometric formalism, with basic examples coming for instance from non-Galois coverings, to investigate the possibility of a geometric origin for the multiplicity of generations.

9.4. Unification of couplings.
The one loop RG equations for the running of the gauge couplings and Newton constant do not meet exactly at one point which is expected to be at the Planck scale. The error, however, is within few percent. Higher order corrections will change the running of all coupling constants and we shall see in Part II that rather surprisingly, if one no longer assumes that the function $f(D/\Lambda)$ is flat at 0 as any cut-off function, then the contributions of the higher order terms alter the simple unification rule involving the $\frac{3}{5}$. This allows one to improve on the above issue under the assumption that the Yukawa coupling of the tau neutrino is of the same order as the Yukawa coupling of the top quark, an hypothesis that already appeared naturally when dealing with the prediction on the top quark mass ([13]).

9.5. Mass of the Higgs.
The mass of the Higgs field in the zeroth order approximation of the spectral action is around 170 Gev. This however, depends on the value of the gauge couplings at the unification scale. Higher order corrections will definitely change this predicted value, but since the prediction comes from the value at unification of the coupling constant of the quartic term in the Higgs potential, the finiteness of this coupling implies the same qualitative results as the hypothesis of the “big desert” and the assumption that the Standard Model is still valid at this very high scale. Of course this hinges on the naturalness problem whose only accepted resolution involves supersymmetry. It is rather striking however that the spectral action naturally contains a quadratic mass term in the Higgs field which has the correct sign and size to allow one to do fine tuning. In any case we consider that the experimental determination of the Higgs mass will give a precious indication.
9.6. **New particles.**

The reduction of the gauge group to $U(1) \times SU(2) \times SU(3)$ was obtained above from the order one condition using the hypothesis that the two layers of space-time corresponding to the two-dimensional center of the algebra $A_F$ of the finite space are at a finite distance apart. This reduces the natural gauge group $SU(2)_L \times SU(2)_R \times SU(4)$ given by the even part of the algebra $A_F = M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$ to the Standard Model gauge group. The justification that we have given, starting in [13], for this reduction, is based on the order one condition for the Dirac operator and is imposed as a mathematical condition. It is desirable to improve this point by finding a dynamical mechanism that effects the same symmetry breaking from $SU(2)_L \times SU(2)_R \times SU(4)$ to $U(1) \times SU(2) \times SU(3)$. Such a mechanism should generate mass terms for the broken part of the gauge sector. This will thus correspond to new particles not present in the Standard Model but well motivated from the above considerations. What is missing at the mathematical level is to understand how the order one condition can be imposed at the dynamical level, and also how the inner fluctuations of the metric behave if one no longer assumes the order one condition.

9.7. **Quantum Level.**

So far we have used the renormalization group in a very straightforward manner starting from the simple idea that the spectral action holds at the unification scale and using the values of the couplings as boundary conditions. The compatibility between the values at low energy (obtained by integration over the fluctuations in the intermediate scales) and observation is a basic test of the general idea but in case this test is passed, one needs to go much further and develop a theory that takes over at higher scales. Since the model we developed contains both gravity and the Standard Model it is clear that this problem is the problem of quantizing gravity. We refer the reader to [57] for interesting suggestions concerning the role of the ghost fields. One challenging problem at this point is to compute the bosonic propagator for the inner fluctuations of the metric using the spectral action and functional derivatives of tracial functions. One may hope that the techniques developed in the context of renormalization of QFT on noncommutative spaces will be useful in the building of the quantum theory of the spectral action. In [27] an analogy was developed between the phase transitions which occur in the number theoretic context and a scenario of spontaneous symmetry breaking involving the full gravitational sector. If substantiated, this could show how geometry would emerge from the computation of the KMS states of an operator theoretic system, closely related to a matrix model with basic variable the Dirac operator $D$. It is worthwhile to note, at this point, that, at the conceptual level, the spectral action is closely related to an entropy since it can be written as the logarithm of a number of states in the second quantized Fermionic Hilbert space.
There is another very interesting mathematical problem which is suggested by the quantum theory. While we have a simple prescription for the inner fluctuations of the metric, the formulas for modifying the “outer” part of the metric are surely more subtle, but we want to point out that

(1) A change of the Weyl factor in the metric is given by a beautifully simple formula for the Dirac operator which extends to the noncommutative case [10], [28].

(2) There is a simple and efficient analogue in noncommutative geometry for the modification of the conformal structure encoded by a Beltrami differential ([18]).

Finally there are interesting developments on cosmology [47], [53] which open a new line of investigations where the KMS condition should play a leading role in the analysis of phase transitions following the model developed in [54] for the case of the electroweak transition.
10. Appendix A: Components of the Dirac Operator

We summarize our results by listing all matrix entries of the full Dirac operator \((D_A)^N_M\), but will omit the index \(A\) of \(D_A\) in what follows:

\[
(D)_{11}^{11} = \gamma^\mu \otimes D_\mu \otimes \mathbf{1}_3, \quad D_\mu = \partial_\mu + \frac{1}{4} \epsilon_{\mu}^{\alpha c d} \chi_{cd}, \quad \mathbf{1}_3 = \text{generations}
\]

\[
(D)_{11}^{01} = \gamma_5 \otimes k_\nu \otimes \epsilon^{ab} H_b \quad k_\nu = 3 \times 3 \text{ neutrino mixing matrix}
\]

\[
(D)_{21}^{21} = \gamma^\mu \otimes (D_\mu + ig_1 B_\mu) \otimes \mathbf{1}_3
\]

\[
(D)_{21}^{01} = \gamma_5 \otimes k_\nu \otimes \mathcal{T}^i
\]

\[
(D)_{11}^{11} = \gamma_5 \otimes k_\nu \otimes \epsilon_{ab} \mathcal{T}^b
\]

\[
(D)_{21}^{21} = \gamma_5 \otimes k_\nu \otimes H_a
\]

\[
(D)_{a1}^{01} = \gamma^\mu \otimes \left( \left( D_\mu + \frac{i}{2} g_1 B_\mu \right) \delta^b_a - \frac{i}{2} g_2 W^{\alpha}_{\mu} (\sigma^\alpha)^b_a \right) \otimes \mathbf{1}_3, \quad \sigma^\alpha = \text{Pauli}
\]

\[
(D)_{11}^{11} = \gamma^\mu \otimes \left( \left( D_\mu - \frac{2i}{3} g_1 B_\mu \right) \delta^j_i - \frac{i}{2} g_3 V^{m}_{\mu} (\lambda^m)^j_i \right) \otimes \mathbf{1}_3, \quad \lambda^i = \text{Gell-Mann}
\]

\[
(D)_{a1}^{01} = \gamma_5 \otimes k_\nu \otimes \epsilon^{ab} H_b \delta^j_i
\]

\[
(D)_{22}^{21} = \gamma^\mu \otimes \left( \left( D_\mu + \frac{i}{3} g_1 B_\mu \right) \delta^j_i \right) \otimes \mathbf{1}_3
\]

\[
(D)_{22}^{01} = \gamma_5 \otimes k_\nu \otimes \mathcal{T}^i \delta^j_i
\]

\[
(D)_{a1}^{11} = \gamma^\mu \otimes \left( \left( D_\mu - \frac{i}{6} g_1 B_\mu \right) \delta^b_a \delta^j_i \right) \otimes \mathbf{1}_3
\]

\[
(D)_{a1}^{11} = \gamma_5 \otimes k_\nu \otimes \epsilon_{ab} \mathcal{T}^b \delta^j_i
\]

\[
(D)_{a1}^{01} = \gamma_5 \otimes k^a \otimes H_a \delta^j_i
\]

\[
(D)_{1}^{11} = \gamma_5 \otimes k_\nu \otimes \sigma \quad \text{generate scale } \mathcal{M}_R \text{ by } \sigma \rightarrow \mathcal{M}_R
\]

\[
(D)_{1}^{11} = \gamma_5 \otimes k_\nu \otimes \sigma
\]

\[
D_{A'}^{B'} = \mathcal{D}_{A'}^{B'}, \quad D_{A'}^{B'} = \mathcal{D}_{A'}^{B'}, \quad D_{A'}^{B'} = \mathcal{D}_{A'}^{B'}
\]
11. Appendix B: Components of the Square of the Dirac Operator

Next we list all the components of the matrix \( (D^2)_M^N \)

\[
(D^2)_{11}^{11} = D_{11}^{01} D_{11}^{11} + D_{11}^{11} D_{01}^{11} + k^{\alpha \beta} k^{\alpha \beta} \sigma^2
\]

\[
= \gamma^\mu D_\mu \gamma^\nu D_\nu \otimes 1_3 + k^{\alpha \beta} k^{\alpha \beta} H_a \overline{\Gamma}^a + k^{\alpha \beta} k^{\alpha \beta} \sigma^2
\]

\[
(D^2)_{11}^{a1} = D_{11}^{11} D_{a1}^{11} + D_{11}^{11} D_{a1}^{11}
\]

\[
= \gamma^\mu D_\mu \gamma^\nu \epsilon^{ab} H_a + \gamma_5 k^{\alpha \beta} \epsilon^{bc} H_c \gamma^\mu \left( \left( D_\mu + \frac{i}{2} g_1 B_\mu \right) \delta^a_b - \frac{i}{2} g_2 W_\mu^a (\sigma^a)^b \right)
\]

where

\[
\nabla_\mu H_a = \left( \frac{\partial_\mu - \frac{i}{2} g_1 B_\mu}{\delta^a_b - \frac{i}{2} g_2 W_\mu^a (\sigma^a)^b} \right) H_b
\]

and we have used the identity \( \epsilon^{ab} (\sigma^a)^d = -(\sigma^a)^c \). Next

\[
(D^2)_{21}^{21} = D_{21}^{21} D_{21}^{21} + D_{21}^{21} D_{a1}^{21}
\]

\[
= \gamma^\mu (D_\mu + ig_1 B_\mu) \gamma^\nu (D_\nu + ig_1 B_\nu) \otimes 1_3 + k^{\alpha \beta} k^{\alpha \beta} \overline{\Gamma}^a
\]

where we have denoted \( \overline{\Gamma} H = \overline{\Gamma}^a H_a = H_a \overline{\Gamma}^a \).

\[
(D^2)_{21}^{a1} = D_{21}^{21} D_{a1}^{21} + D_{a1}^{21} D_{b1}^{21}
\]

\[
= \gamma^\mu (D_\mu + ig_1 B_\mu) \gamma_5 k^{\alpha \beta} \overline{\Gamma}^a + \gamma_5 k^{\alpha \beta} \overline{\Gamma}^a \gamma^\mu \left( \left( D_\mu + \frac{i}{2} g_1 B_\mu \right) \delta^a_b - \frac{i}{2} g_2 W_\mu^a (\sigma^a)^b \right)
\]

\[
\]

\[
(D^2)_{a1}^{b1} = D_{a1}^{a1} D_{b1}^{11} + D_{a1}^{a1} D_{b1}^{11} + D_{a1}^{b1} D_{c1}^{b1}
\]

\[
= k^{\alpha \beta} k^{\alpha \beta} H_a \overline{\Gamma}^a + k^{\alpha \beta} k^{\alpha \beta} \epsilon^{cd} \epsilon^{bc} \overline{\Gamma}^a H_d
\]

\[
+ \gamma^\mu \left( \left( D_\mu + \frac{i}{2} g_1 B_\mu \right) \delta^c_d - \frac{i}{2} g_2 W_\mu^a (\sigma^a)^c \right) \gamma^\nu \left( \left( D_\nu + \frac{i}{2} g_1 B_\nu \right) \delta^b_c - \frac{i}{2} g_2 W_\nu^a (\sigma^a)^b \right) \otimes 1_3
\]
\[(D^2)_{1i}^{lj} = D^{1k}_{1i} D^{lj}_{1k} + D^{ak}_{1i} D^{lj}_{1k} = \]
\[\gamma^\mu \left( \left( D_\mu - \frac{2i}{3} g_1 B_\mu \right) \delta^k_i - \frac{i}{2} g_3 V_\mu^m (\lambda^m)_i^k \right) \gamma^\nu \left( \left( D_\nu - \frac{2i}{3} g_1 B_\nu \right) \delta^k_i - \frac{i}{2} g_3 V_\nu^m (\lambda^m)_i^k \right) \otimes 1_3 + k^{ax} k^b H H \delta^j_i \]

\[(D^2)_{2i}^{aj} = D^{bk}_{2i} D^{aj}_{2k} + D^{ak}_{2i} D^{aj}_{2k} = \gamma_5 k^{xu} H^b \delta^k_i \gamma^\mu \left( \left( D_\mu - \frac{i}{6} g_1 B_\mu \right) \delta^a_b \delta^j_k - \frac{i}{2} g_2 W^\alpha_\mu (\sigma^\alpha)_a^b \delta^j_k - \frac{i}{2} g_3 V_\mu^m (\lambda^m)_k^j \delta^a_b \right) + \gamma^\mu \left( \left( D_\mu - \frac{2i}{3} g_1 B_\mu \right) \delta^k_i - \frac{i}{2} g_3 V_\mu^m (\lambda^m)_i^k \right) \gamma_5 k^{xu} H H \delta^j_i \]

\[(D^2)_{1i}^{aj} = D^{bk}_{1i} D^{aj}_{1k} + D^{1k}_{1i} D^{aj}_{1k} = \gamma_5 k^{xu} \epsilon^{bc} H H \delta^k_i \gamma^\mu \left( \left( D_\mu - \frac{2i}{3} g_1 B_\mu \right) \delta^a_b \delta^j_k - \frac{i}{2} g_2 W^\alpha_\mu (\sigma^\alpha)_a^b \delta^j_k - \frac{i}{2} g_3 V_\mu^m (\lambda^m)_k^j \delta^a_b \right) + \gamma^\mu \left( \left( D_\mu + \frac{i}{3} g_1 B_\mu \right) \delta^k_i - \frac{i}{2} g_3 V_\mu^m (\lambda^m)_i^k \right) \gamma_5 k^{xu} \epsilon^{ab} H H \delta^j_i \]

\[(D^2)_{2i}^{aj} = D^{bk}_{2i} D^{aj}_{2k} + D^{ak}_{2i} D^{aj}_{2k} = \gamma^\mu \left( \left( D_\mu + \frac{i}{3} g_1 B_\mu \right) \delta^k_i - \frac{i}{2} g_3 V_\mu^m (\lambda^m)_i^k \right) \gamma^\nu \left( \left( D_\nu + \frac{i}{3} g_1 B_\nu \right) \delta^j_k - \frac{i}{2} g_3 V_\nu^m (\lambda^m)_k^j \right) + k^{xu} k^a H H \delta^j_i \]

\[(D^2)_{ai}^{lj} = D^{1k}_{ai} D^{lj}_{1k} + D^{ak}_{ai} D^{lj}_{1k} = \gamma_5 k^{xu} \epsilon^{ab} H H \delta^k_i \gamma^\mu \left( \left( D_\mu - \frac{2i}{3} g_1 B_\mu \right) \delta^a_b \delta^j_k - \frac{i}{2} g_2 W^\alpha_\mu (\sigma^\alpha)_a^b \delta^j_k - \frac{i}{2} g_3 V_\mu^m (\lambda^m)_k^j \delta^a_b \right) + \gamma^\mu \left( \left( D_\mu - \frac{2i}{3} g_1 B_\mu \right) \delta^k_i - \frac{i}{2} g_3 V_\mu^m (\lambda^m)_i^k \right) \gamma_5 k^{xu} \epsilon^{ab} H H \delta^j_i \]
\[(D^2)_{a_i}^{2j} = D_{a_i}^{2k} D_{2k}^{2j} + D_{ai}^{ck} D_{ck}^{2j} \]
\[
= \gamma_5 k^d H_a \delta_i^k \gamma^\mu \left( \left( D_\mu + \frac{i}{3} g_1 B_\mu \right) \delta_j^k - \frac{i}{2} g_3 V_\mu^m (\lambda^m)^j_k \right) \\
+ \gamma^\nu \left( \left( D_\nu - \frac{i}{6} g_1 B_\nu \right) \delta_a^c \delta_i^k - \frac{i}{2} g_2 W_\mu^\alpha (\sigma^\alpha)^c_a \delta_i^k - \frac{i}{2} g_3 V_\mu^m (\lambda^m)^i_j \delta_a^c \right) \gamma_5 k^d H_c \delta_i^j \\
= \gamma^\mu \gamma_5 k^d \nabla_\mu H_a \delta_i^j 
\]

Finally

(11.13)

\[(D^2)_{ai}^{bj} = D_{ai}^{1k} D_{1k}^{bj} + D_{ai}^{2k} D_{2k}^{bj} + D_{ai}^{ck} D_{ck}^{bj} \]
\[
= k^u k^v H_a \delta_i^j + k^d k^e \epsilon_{ac} \epsilon_{bd} H_d \delta_i^j \\
+ \left[ \gamma^\mu \left( \left( D_\mu - \frac{i}{6} g_1 B_\mu \right) \delta_a^c \delta_i^k - \frac{i}{2} g_2 W_\mu^\alpha (\sigma^\alpha)^c_a \delta_i^k - \frac{i}{2} g_3 V_\mu^m (\lambda^m)^i_j \delta_a^c \right) \\
\gamma^\nu \left( \left( D_\nu - \frac{i}{6} g_1 B_\nu \right) \delta_b^c \delta_j^k - \frac{i}{2} g_2 W_\nu^\alpha (\sigma^\alpha)^b_c \delta_j^k - \frac{i}{2} g_3 V_\nu^m (\lambda^m)^j_i \delta_b^c \right) \right] 
\]

There are also terms in the off-diagonal part

(11.14)

\[(D^2)_{11}^{a'1'} = D_{11}^{a'1'} D_{11}^{a'1'} \]
\[
= k^{u1} k^{v1} \epsilon_{ab} \nabla_b \sigma 
\]

(11.15)

\[(D^2)_{a'1'}^{11} = D_{a'1'}^{11} D_{11}^{11} \]
\[
= k^{11} k^{uv} \epsilon_{ab} H_b \sigma 
\]

(11.16)

\[(D^2)_{1'1}^{11} = D_{1'1}^{11} D_{11}^{11} \]
\[
= k^{11} k^{uv} \epsilon_{ab} \nabla_b \sigma 
\]

(11.17)

\[(D^2)_{1'1}^{a1} = D_{1'1}^{a1} D_{11}^{a1} \]
\[
= k^{11} k^{uv} \epsilon_{ab} H_b \sigma 
\]
12. Appendix C: Connection $\omega_{\mu}$, Curvature $\Omega_{\mu\nu}$ and Invariant $E$

We list here the entries of the matrices $(\omega_{\mu})^N_M$, $(E)^N_M$ which are defined in terms of the operator $D^2$ and the curvature $(\Omega_{\mu\nu})^N_M$ where

\[
\begin{align*}
(\omega_{\mu})^1_{11} &= \frac{1}{4} \omega_{\mu}^{cd} (e) \gamma_{cd} \otimes 1_3 \\
(\omega_{\mu})^2_{21} &= \left( \frac{1}{4} \omega_{\mu}^{cd} (e) \gamma_{cd} + ig_1 B_{\mu} \right) \otimes 1_3 \\
(\omega_{\mu})^b_{a1} &= \left( \frac{1}{4} \omega_{\mu}^{cd} (e) \gamma_{cd} + \frac{i}{2} g_1 B_{\mu} \right) \delta^b_a - \frac{i}{2} g_2 W_{\mu}^a (\sigma^{\alpha})^b_a \otimes 1_3 \\
(\omega_{\mu})^j_{1i} &= \left( \frac{1}{4} \omega_{\mu}^{cd} (e) \gamma_{cd} - \frac{2i}{3} g_1 B_{\mu} \right) \delta^j_i - \frac{i}{2} g_3 V_{\mu}^m (\lambda^m)^j_i \otimes 1_3 \\
(\omega_{\mu})^2_j_{2i} &= \left( \frac{1}{4} \omega_{\mu}^{cd} (e) \gamma_{cd} + \frac{i}{2} g_1 B_{\mu} \right) \delta^j_i - \frac{i}{2} g_3 V_{\mu}^m (\lambda^m)^j_i \otimes 1_3 \\
(\omega_{\mu})^b_{aj} &= \left( \frac{1}{4} \omega_{\mu}^{cd} (e) \gamma_{cd} - \frac{i}{6} g_1 B_{\mu} \right) \delta^b_a \delta^j_i - \frac{i}{2} g_2 W_{\mu}^a (\sigma^{\alpha})^b_a \delta^j_i - \frac{i}{2} g_3 V_{\mu}^m (\lambda^m)^j_i \delta^b_a \otimes 1_3 \\
(\omega_{\mu})^B'_{A'} &= 0 = (\omega_{\mu})^B_{A'} \text{,} \\
(\omega_{\mu})^B'_{A'} &= (\omega_{\mu})^B_{A} \\
\end{align*}
\]

The components of the curvature $\Omega_{\mu\nu} = \partial_{\mu} \omega_{\nu} - \partial_{\nu} \omega_{\mu} + [\omega_{\mu}, \omega_{\nu}]$ are given by

\[
\begin{align*}
(\Omega_{\mu\nu})^1_{11} &= \frac{1}{4} R_{\mu\nu}^{cd} \gamma_{cd} \otimes 1_3 \\
(\Omega_{\mu\nu})^2_{21} &= \left( \frac{1}{4} R_{\mu\nu}^{cd} \gamma_{cd} + ig_1 B_{\mu\nu} \right) \otimes 1_3 \\
(\Omega_{\mu\nu})^b_{a1} &= \left( \frac{1}{4} R_{\mu\nu}^{cd} \gamma_{cd} + \frac{i}{2} g_1 B_{\mu\nu} \right) \delta^b_a - \frac{i}{2} g_2 W_{\mu\nu}^a (\sigma^{\alpha})^b_a \otimes 1_3 \\
(\Omega_{\mu\nu})^j_{1i} &= \left( \frac{1}{4} R_{\mu\nu}^{cd} \gamma_{cd} - \frac{2i}{3} g_1 B_{\mu\nu} \right) \delta^j_i - \frac{i}{2} g_3 V_{\mu\nu}^m (\lambda^m)^j_i \otimes 1_3 \\
(\Omega_{\mu\nu})^2_j_{2i} &= \left( \frac{1}{4} R_{\mu\nu}^{cd} \gamma_{cd} + \frac{i}{2} g_1 B_{\mu\nu} \right) \delta^j_i - \frac{i}{2} g_3 V_{\mu\nu}^m (\lambda^m)^j_i \otimes 1_3 \\
(\Omega_{\mu\nu})^b_{aj} &= \left( \frac{1}{4} R_{\mu\nu}^{cd} \gamma_{cd} - \frac{i}{6} g_1 B_{\mu\nu} \right) \delta^b_a \delta^j_i - \frac{i}{2} g_2 W_{\mu\nu}^a (\sigma^{\alpha})^b_a \delta^j_i - \frac{i}{2} g_3 V_{\mu\nu}^m (\lambda^m)^j_i \delta^b_a \otimes 1_3 \\
(\Omega_{\mu\nu})^B'_{A'} &= 0 = (\Omega_{\mu\nu})^B_{A'}, \\
(\Omega_{\mu\nu})^B'_{A'} &= (\Omega_{\mu\nu})^B_{A} \\
\end{align*}
\]
Finally

\[-(E)^{11}_{11} = \frac{1}{4} R \otimes 1_3 + \left( k^{a\nu} k^\nu H + k^{a\nu} k^\nu \sigma^2 \right) \]

\[-(E)^{01}_{11} = \gamma^\mu \gamma_5 \otimes k^\nu \otimes \epsilon_{ab} \nabla_\mu H_b \]

\[-(E)^{21}_{21} = \left( \frac{1}{4} R + \frac{1}{2} g_{\mu\nu} (i g_1 B_{\mu\nu}) \right) \otimes 1_3 + \left( k^{a\nu} k^\nu H \right) \]

\[-(E)^{01}_{21} = \gamma^\mu \gamma_5 \otimes k^{a\nu} \otimes \nabla_\mu H^a \]

\[-(E)^{11}_{11} = \left( \gamma^\mu \gamma_5 \otimes k^\nu \otimes \epsilon_{ab} \nabla_\mu H^a \right) \]

\[-(E)^{21}_{11} = \gamma^\mu \gamma_5 \otimes k^{a\nu} \otimes \nabla_\mu H^a \]

\[-(E)^{01}_{21} = \gamma^\mu \gamma_5 \otimes k^{a\nu} \otimes \epsilon_{ab} \nabla_\mu H^a \]

\[-(E)^{11}_{11} = \left( \gamma^\mu \gamma_5 \otimes k^\nu \otimes \epsilon_{ab} \nabla_\mu H^a \right) \]

\[-(E)^{21}_{11} = \gamma^\mu \gamma_5 \otimes k^{a\nu} \otimes \nabla_\mu H^a \]

\[-(E)^{01}_{21} = \gamma^\mu \gamma_5 \otimes k^{a\nu} \otimes \epsilon_{ab} \nabla_\mu H^a \]

\[-(E)^{11}_{11} = \left( \gamma^\mu \gamma_5 \otimes k^\nu \otimes \epsilon_{ab} \nabla_\mu H^a \right) \]

\[-(E)^{21}_{11} = \gamma^\mu \gamma_5 \otimes k^{a\nu} \otimes \nabla_\mu H^a \]

\[-(E)^{01}_{21} = \gamma^\mu \gamma_5 \otimes k^{a\nu} \otimes \epsilon_{ab} \nabla_\mu H^a \]

\[-(E)^{11}_{11} = \left( \gamma^\mu \gamma_5 \otimes k^\nu \otimes \epsilon_{ab} \nabla_\mu H^a \right) \]

\[-(E)^{21}_{11} = \gamma^\mu \gamma_5 \otimes k^{a\nu} \otimes \nabla_\mu H^a \]

\[-(E)^{01}_{21} = \gamma^\mu \gamma_5 \otimes k^{a\nu} \otimes \epsilon_{ab} \nabla_\mu H^a \]
We can easily compute the trace of $E$

\[(12.1) \quad \text{Tr} \ (E) = \text{tr} \left( E_A^A + E_A^A' \right) = \text{tr} \left( E_A^A + \overline{E}_A^A \right) \]

Thus

\[(12.2) \quad -\text{tr} \ (E)_{11}^{11} = \text{tr} \left( \frac{1}{4} R \otimes 1_3 + \left( k^{su} k^v \overline{H} H + k^{su} k^{\nu} k^{\mu} \sigma^2 \right) \right) \]

\[= 4 \left[ \frac{3}{4} R + k^{su} k^v \overline{H} H + k^{nu} k^{\nu} k^{\mu} \sigma^2 \right] \]

\[(12.3) \quad -\text{tr} \ (E)_{21}^{21} = \text{tr} \left( \left( \frac{1}{4} R + \frac{1}{2} \gamma_{\mu\nu} (ig_1 B_{\mu\nu}) \right) 1_3 + \left( k^{se} k^e \overline{H} H \right) \right) \]

\[= 4 \left[ \frac{3}{4} R + k^{se} k^e \overline{H} H \right] \]

\[(12.4) \quad -\text{tr} \ (E)_{a_1}^{a_1} = 4 \left[ \frac{3}{4} R (2) + \left( k^e k^{se} + k^{\nu} k^{\nu} \right) \overline{H} H \right] \]

\[(12.5) \quad -\text{tr} \ (E)_{1_1}^{1_1} = 4 \left[ \frac{3}{4} R (3) + 3k^{su} k^u \overline{H} H \right] \]

\[(12.6) \quad -\text{tr} \ (E)_{1_1}^{1_1} = 4 \left[ \frac{3}{4} R (3) + 3k^{sd} k^d \overline{H} H \right] \]

\[(12.7) \quad -\text{tr} \ (E)_{ai}^{ai} = 4 \left[ \frac{3}{4} R (2) (3) + 3 \left( k^u k^{su} + k^d k^{sd} \right) \overline{H} H \right] \]

Collecting all terms we get

\[(12.8) \quad -\frac{1}{2} \text{Tr} \ (E) = 4 \left[ \frac{3}{4} R (1 + 1 + 2 + 3 + 3 + 6) + 2 \left( k^{su} k^v + k^{se} k^e + 3 \left( k^{su} k^u + k^{sd} k^d \right) \right) \overline{H} H + k^{nu} k^{\nu} k^{\mu} \sigma^2 \right] \]

\[= 4 \left[ 12R + 2a \overline{H} H + c \sigma^2 \right] \]

\[(12.9) \quad \text{Tr} \left( \Omega_{\mu\nu}^{\overline{a}i} \right)_{ai} = \]

\[\text{Tr} \left\{ \left( \left( \frac{1}{4} R_{\gamma\nu}^{\gamma\nu} - \frac{i}{6} g_1 B_{\mu\nu} \right) \delta^b_m \delta^m_n - \frac{i}{2} g_2 W_{\mu\nu}^\alpha (\lambda^\alpha)_{ia} \delta^m_n - \frac{i}{2} g_2 V_{\mu\nu}^m (\lambda^m)_{ij} \delta^b_i \otimes 1_3 \right)^2 \right\} \]

\[= 4 \left[ -\frac{1}{8} R_{\mu\nu}^{\mu\nu} (3) (2) (3) - \frac{1}{36} g_1^2 B_{\mu\nu}^2 (3) (2) (3) - \frac{1}{4} g_2^2 (W_{\mu\nu}^\alpha)^2 (3) (2) (3) - \frac{1}{4} g_3^2 (V_{\mu\nu}^m)^2 (3) (2) (2) \right] \]
Collecting these terms we have

\[
\frac{1}{2} \text{Tr} \left( \Omega_{\mu\nu}^2 \right)_M = 4 \left[ -\frac{3}{8} R^2_{\mu\nu\rho\sigma} (\text{16}) - 3 g_1^2 B_{\mu\nu}^2 \left( \frac{1}{2} + \frac{4}{3} + \frac{1}{3} + \frac{1}{6} \right) - 3 g_2^2 \left( W_{\mu\nu}^a \right)^2 \left( \frac{1}{2} + \frac{3}{2} \right) - 3 g_3^2 \left( V_{\mu\nu}^m \right)^2 \left( \frac{1}{2} + 1 \right) \right]
\]

\[
= 4 \left[ -6 R^2_{\mu\nu\rho\sigma} - 10 g_1^2 B_{\mu\nu}^2 - 6 g_2^2 \left( W_{\mu\nu}^a \right)^2 - 6 g_3^2 \left( V_{\mu\nu}^m \right)^2 \right]
\]

13. Appendix D: Components and Traces of $E^2$ and $\Omega^2$

Next we compute $(E^2)^B_A = E^C_A E^B_C + E^C_A E^B_C$:

\[
(E^2)^{1i}_{1i} = E^{1i}_{1i} E^{1i}_{1i} + E^{a1}_{1i} E^{1i}_{a1} + E^{a^1}_{1i} E^{a1}_{1i} + E^{1i}_{1i} E^{1i}_{1i},
\]

\[
(E^2)^{21}_{21} = E^{a1}_{21} E^{a1}_{21}
\]

\[
(E^2)^{1i}_{2i} = E^{a1}_{21} E^{a1}_{21}
\]

\[
(E^2)^{21}_{21} = E^{a1}_{21} E^{a1}_{21} + E^{a1}_{21} E^{a1}_{21}
\]

\[
(E^2)^{1i}_{1i} = E^{1i}_{1i} E^{1i}_{1i} + E^{b1}_{1i} E^{b1}_{1i} + E^{1i}_{1i} E^{1i}_{1i}
\]

\[
(E^2)^{a1}_{a1} = E^{a1}_{21} E^{a1}_{21} + E^{b1}_{a1} E^{b1}_{a1}
\]

\[
(E^2)^{1i}_{1i} = E^{a1}_{1i} E^{a1}_{1i} + E^{b1}_{1i} E^{b1}_{1i} + E^{1i}_{1i} E^{1i}_{1i}
\]

\[
(E^2)^{21}_{1i} = E^{a1}_{21} E^{a1}_{21} + E^{b1}_{21} E^{b1}_{21}
\]

\[
(E^2)^{a1}_{a1} = E^{a1}_{a1} E^{a1}_{a1} + E^{a1}_{a1} E^{a1}_{a1} + E^{a1}_{a1} E^{a1}_{a1} + E^{a1}_{a1} E^{a1}_{a1}
\]

\[
(E^2)^{1i}_{1i} = E^{1i}_{1i} E^{1i}_{1i} + E^{1i}_{1i} E^{1i}_{1i}
\]

\[
(E^2)^{2j}_{1i} = E^{1k}_{1i} E^{2j}_{1k} + E^{ak}_{1i} E^{2j}_{1k}
\]

\[
(E^2)^{aj}_{1i} = E^{1k}_{1i} E^{aj}_{1k} + E^{ak}_{1i} E^{aj}_{1k}
\]

\[
(E^2)^{2j}_{2i} = E^{2k}_{2i} E^{2j}_{2k} + E^{ak}_{2i} E^{2j}_{2k}
\]

\[
(E^2)^{aj}_{2i} = E^{2k}_{2i} E^{aj}_{2k} + E^{ak}_{2i} E^{aj}_{2k}
\]
\[(E^2)_{ai}^{ij} = E_{ai}^{ik} E_{1k}^{ij} + E_{ai}^{bk} E_{bk}^{ij}\]

\[(E^2)_{ai}^{2j} = E_{ai}^{2k} E_{2k}^{2j} + E_{ai}^{bk} E_{bk}^{2j}\]

\[(E^2)_{ai}^{bj} = E_{ai}^{bk} E_{1k}^{bj} + E_{ai}^{2k} E_{2k}^{bj} + E_{ai}^{ek} E_{ek}^{bj}\]

\[(E^2)_{a'i'}^{11} = E_{a'i'}^{11} E_{11}^{11} + E_{a'i'}^{11} E_{11}^{11} + E_{a'i'}^{11} E_{11}^{11}\]

\[(E^2)_{a'i'}^{b1} = E_{a'i'}^{b1} E_{11}^{b1} + E_{a'i'}^{b1} E_{11}^{b1}\]

\[(E^2)_{a'i'}^{11} = E_{a'i'}^{11} E_{11}^{11} + E_{a'i'}^{11} E_{11}^{11} + E_{a'i'}^{11} E_{11}^{11}\]

\[(E^2)_{a'i'}^{11} = E_{a'i'}^{11} E_{11}^{11} + E_{a'i'}^{11} E_{11}^{11} + E_{a'i'}^{11} E_{11}^{11}\]

\[(E^2)_{a'i'}^{11} = E_{a'i'}^{11} E_{11}^{11} + E_{a'i'}^{11} E_{11}^{11} + E_{a'i'}^{11} E_{11}^{11}\]

We list the various traces for $E^2$

\[(E^2)_{11}^{11} = \text{tr} \left\{ \left( \frac{1}{4} R_{13} + (k^{* \nu} k^{e} \overline{T} H + k^{* \nu} R k^{\nu} \sigma^2) \right)^2 + \gamma^\mu \gamma_5 k^{* \nu} \partial_\mu \sigma \gamma^\nu \gamma_5 k^{* \nu} \partial_\nu \sigma \right\} + \gamma^\mu \gamma_5 k^{* \nu} \epsilon^{ab} \nabla_\mu H_b \gamma^\nu \gamma_5 k^{*} \epsilon_{ac} \nabla_\mu \overline{T} + k^{* \nu} k^{*} \epsilon^{ab} \overline{T} b \epsilon^{ab} k^{* \nu} \epsilon_{ac} H_c \right\}

\[= 4 \left[ \frac{1}{16} R^2 (3) + (k^{* \nu} k^{e} \gamma^\nu \gamma_5 \gamma_5 \otimes k^{e} \nabla_\nu H_a)^2 + \frac{1}{2} R (k^{* \nu} k^{e} \overline{T} H + k^{* \nu} R k^{\nu} \sigma^2) \right] + (k^{* \nu} k^{e} k^{\nu} R)^2 \sigma^2 + k^{* \nu} k^{e} |k^{e} \gamma_5 \gamma_5 \otimes k^{e} \nabla_\nu H_a|^2 + \frac{1}{2} R (k^{* \nu} k^{e} \overline{T} H + k^{* \nu} R k^{\nu} \sigma^2) \right] \]

where we have used $\text{tr}(\gamma^\mu \gamma_5 \gamma^\nu \gamma_5) = 4 g^{\mu \nu}$.

\[(E^2)_{21}^{21} = \text{tr} \left\{ \gamma^\mu \gamma_5 k^{* e} \nabla_\mu \overline{T} ^{i} \gamma^{i} \gamma_5 \otimes k^{e} \nabla_\nu H_a \right\}

\[+ \left( \left( \frac{1}{4} R + \frac{1}{2} \gamma^{\mu \nu} (ig_l B_{\mu \nu}) \right) \right)_{13} + (k^{* e} k^{e} \overline{T} H) \right\}^2 \right\}

\[= 4 \left[ \left( \frac{1}{4} R + \frac{1}{2} \gamma^{\mu \nu} (ig_l B_{\mu \nu}) \right) \right)_{13} (k^{* e} k^{e} \overline{T} H) \right\} + (k^{* e} k^{e} \overline{T} H)^2 + k^{* e} k^{e} |\nabla_\nu H_a|^2 \right] \]
where we have used $\text{tr}(\gamma^{\mu\nu}\gamma_{\kappa\lambda}) = -4 \left( 2\delta^\mu_\delta^\nu_\kappa - 2\delta^\mu_\delta^\nu_\lambda \right)$.

(13.26)

$$
\text{tr} \left( E^2 \right)_{a_1}^{a_1} = \text{tr} \left\{ \left( \frac{R}{4} \delta^i_a + \frac{1}{2} \gamma^{\mu\nu} \left( \frac{i}{2} g_1 B^\mu_{a} \delta^b_i - \frac{i}{2} g_2 W^a_{\mu
u} (\sigma^a)_b \right) \right)^2 + \left( k^e k^{e}\kappa H_a \overline{H}^b + k^{e\nu} k^{e\mu} \epsilon_{a\epsilon} \epsilon^{bd} \overline{H}^b H_d \right) \right\} 
$$

$$
+ \gamma^\mu \gamma_5 k^{e\nu} \epsilon^{ab} \nabla_{\mu} H_b \gamma^\nu \gamma_5 k^{e\nu} \epsilon_{ab} \nabla_{\nu} \overline{H}^b + \gamma^\mu \gamma_5 k^{e\nu} \epsilon^{ab} \nabla_{\mu} \overline{H}^b \gamma^\nu \gamma_5 k^{e\nu} \epsilon_{ab} \nabla_{\nu} H_a \right\} 
$$

$$
= 4 \left\{ \frac{1}{4} (-2) \left( \frac{1}{4} g_1^2 B^{2\mu} (\overline{H})^3 (2) (3) - \frac{1}{4} g_2^2 (W^a_{\mu\nu})^2 (2) (3) \right) + \frac{1}{16} R^2 (2) (3) 

+ \frac{1}{2} R (k^{e\nu} k^{e\nu} + k^{e\nu} k^{e\nu}) \overline{H} H + \left( (k^{e\nu} k^{e\nu})^2 + (k^{e\nu} k^{e\nu})^2 \right) \overline{H} H \right\}^2 
$$

$$
+ (k^{e\nu} k^{e\nu} + k^{e\nu} k^{e\nu}) \left( \nabla_{\mu} H_a \right)^2 + (k^{e\nu} k^{e\nu} + k^{e\nu} k^{e\nu}) \overline{H} H \right\}^2
$$

where the factor (2) = $\delta^a_a$ and $\text{tr}(\sigma^a\sigma^b) = 2\delta^a_b$ and the factor (3) = $\text{tr} \left( 1 \right)$ of the 3 generations. Next

(13.27)

$$
\text{tr} \left( E^2 \right)_{i_1}^{i_1} = \text{tr} \left\{ \left( \frac{R}{4} \delta^i_1 + \frac{1}{2} \gamma^{\mu\nu} \left( \frac{-1}{3} g_1 B^\mu_{i} \delta^j_i - \frac{i}{2} g_2 V^m_{\mu
u} (\lambda^m)_i \right) \right)^2 \right. 
$$

$$
+ \gamma^\mu \gamma_5 k^{e\nu} \epsilon^{ab} \nabla_{\mu} H_b \delta^j_i \gamma^\nu \gamma_5 k^{e\nu} \epsilon_{ab} \nabla_{\nu} \overline{H}^b \delta^i_j \right\} 
$$

$$
= 4 \left\{ \frac{1}{4} (-2) \left( \frac{1}{9} g_1^2 B^{2\mu} (\overline{H})^3 (2) (3) - \frac{1}{4} g_2^2 (V^m_{\mu\nu})^2 (2) (3) \right) + \frac{1}{16} R^2 (2) (3) 

+ (k^{e\nu} k^{e\nu})^2 \overline{H} (2) (3) + \frac{1}{2} R (k^{e\nu} k^{e\nu}) \overline{H} (3) + \left( k^{e\nu} k^{e\nu} \right) \left( \nabla_{\mu} H_a \right)^2 \right\}
$$

where (3) = $\delta^i_i$ and $\text{tr} \left( \lambda^m \lambda^m \right) = 2\delta^m_m$.

(13.28)

$$
\text{tr} \left( E^2 \right)_{2i}^{2i} = \text{tr} \left\{ \left( \frac{R}{4} \delta^i_2 + \frac{1}{2} \gamma^{\mu\nu} \left( \frac{i}{3} g_1 B^\mu_{2} \delta^j_i - \frac{i}{2} g_2 V^m_{\mu
u} (\lambda^m)_i \right) \right)^2 \right. 
$$

$$
+ \gamma^\mu \gamma_5 k^{e\nu} \epsilon^{ab} \nabla_{\mu} \overline{H}_a \delta^j_i \gamma^\nu \gamma_5 k^{e\nu} \epsilon_{ab} \nabla_{\nu} H^a \delta^i_j \right\} 
$$

$$
= 4 \left\{ \frac{1}{4} (-2) \left( \frac{1}{9} g_1^2 B^{2\mu} (\overline{H})^3 (2) (3) - \frac{1}{4} g_2^2 (V^m_{\mu\nu})^2 (2) (3) \right) + \frac{1}{16} R^2 (2) (3) 

+ \left( k^{e\nu} k^{e\nu} \right)^2 \overline{H} (2) (3) + \frac{1}{2} R (k^{e\nu} k^{e\nu}) \overline{H} (3) + \left( k^{e\nu} k^{e\nu} \right) \left( \nabla_{\mu} H_a \right)^2 \right\}
$$
Finally

(13.29)
\[
\begin{align*}
\text{tr } (E^2)_{ai}^{\alpha i} & = \text{tr} \left\{ \left( \frac{R}{4} \delta^a_i \delta^i_j + \frac{1}{2} \gamma^{\mu \nu} \left( -i \frac{g_1}{6} B_{\mu \nu} \delta^a_i \delta^i_j - i \frac{g_2}{2} W_{\mu \nu}^\alpha (\sigma^\alpha)_b \delta^i_j - i \frac{g_3}{2} V_{\mu \nu}^m (\lambda^m)_j \delta^i_b \right) \right) \\
& + k^e k^e H_a \overline{H}_b + k^\nu k^\nu \epsilon_{ac} \epsilon^{bd} \overline{\nabla}_d H_a \right\} \delta^i_j \right)^2 \\
& + \gamma^\mu \gamma^\nu k^a \epsilon_{ab} \nabla_\mu \overline{H}_b \delta^i_j \gamma^\mu \gamma^\nu k^d \nabla_\mu H_a \delta^i_j + \gamma^\nu \gamma^\nu k^d \nabla_\nu \overline{H}_a \delta^i_j \gamma^\nu \gamma^\nu k^d \nabla_\nu \overline{H}_a \delta^i_j \\
& = 4 \left[ \frac{1}{4} \left( -2 \right) \left( -\frac{1}{36} g_1^2 B_{\mu \nu}^2 \right) (3) (2) (3) - \frac{1}{4} g_2^2 (W_{\mu \nu}^\alpha)^2 (3) (2) (3) - \frac{1}{4} g_3^2 (V_{\mu \nu}^m)^2 (3) (2) (3) \right] \\
& + \frac{1}{16} R^2 (3) (2) (3) + \frac{1}{2} R (3) \left[ \left( k^u k^u + k^d k^d \right) (\overline{H} H) + 3 \left( k^u k^u + k^d k^d \right) \right] \nabla_\mu H_a \right]^2 \\
& + 3 \left( \left( k^u k^u \right)^2 + \left( k^d k^d \right)^2 \right) (\overline{H} H)^2 \right]
\end{align*}
\]

Collecting all terms

(13.30)
\[
\begin{align*}
\frac{1}{2} \text{tr } (E^2) &= 4 \left[ g_1^2 B_{\mu \nu}^2 \left( \frac{9}{2} + 2 + \frac{1}{2} + \frac{1}{4} + \frac{3}{4} \right) \right] \\
& + g_2^2 (W_{\mu \nu}^\alpha)^2 \left( \frac{9}{4} + \frac{3}{4} \right) + g_3^2 (V_{\mu \nu}^m)^2 \left( \frac{9}{4} + \frac{3}{4} + \frac{3}{2} \right) \\
& + \frac{1}{16} R^2 (3 + 3 + 9 + 9 + 18 + 6) + \frac{1}{2} R \sigma^2 k^\nu k^\nu + (k^\nu k^\nu) \sigma^2 \\
& + R \overline{H} \left( k^u k^u + k^d k^d + 3 \left( k^u k^u + k^d k^d \right) \right) \\
& + 2 (\overline{H} H)^2 \left( \left( k^u k^u \right)^2 + \left( k^d k^d \right)^2 + 3 \left( \left( k^u k^u \right)^2 + \left( k^d k^d \right)^2 \right) \right) \\
& + 2 \nabla_\mu H_a \left( k^u k^u + k^d k^d + 3 \left( k^u k^u + k^d k^d \right) + 4 k^u k^u k^\nu k^\nu \overline{H} H \sigma^2 \right] \\
& = 4 \left[ 5 g_1^2 B_{\mu \nu}^2 + 3 g_2^2 (W_{\mu \nu}^\alpha)^2 + 3 g_3^2 (V_{\mu \nu}^m)^2 + 3 R^2 + a R \overline{H} H \\
& + \frac{1}{2} c R \sigma^2 + 2 b (\overline{H} H)^2 + 2 a \nabla_\mu H_a \sigma^2 + 4 c \overline{H} H \sigma^2 + c (\partial_\mu \sigma)^2 + d \sigma^4 \right]
\end{align*}
\]
where

\[(13.31)\]
\[a = \text{tr} \left( k^{* \nu} k^{\nu} + (k^{* \nu} k^{\nu})^2 \right) \]
\[(13.32)\]
\[b = \text{tr} \left( (k^{* \nu} k^{\nu})^2 + (k^{* \nu} k^{\nu})^2 \right) \]
\[(13.33)\]
\[c = \text{tr} \left( k^{* \nu} R^{\nu} \right) \]
\[(13.34)\]
\[d = \text{tr} \left( k^{* \nu} k^{\nu} k^{* \nu} R^{\nu} k^{\nu} \right) \]
\[(13.35)\]
\[e = \text{tr} \left( k^{* \nu} k^{\nu} k^{* \nu} R^{\nu} k^{\nu} \right) \]

Next

\[(13.36)\]
\[\text{Tr} \left( \Omega^{2 \mu \nu} \right)_{M}^{A} = 2\text{Tr} \left( \Omega^{2 \mu \nu} \right)_{A}^{A} \]
\[= 2\text{Tr} \left\{ \left( \Omega^{2 \mu \nu} \right)_{11} + \left( \Omega^{2 \mu \nu} \right)_{21} + \left( \Omega^{2 \mu \nu} \right)_{a1} + \left( \Omega^{2 \mu \nu} \right)_{1i} + \left( \Omega^{2 \mu \nu} \right)_{2i} + \left( \Omega^{2 \mu \nu} \right)_{ai} \right\} \]

\[(13.37)\]
\[\text{Tr} \left( \Omega^{2 \mu \nu} \right)_{11}^{i1} = \text{Tr} \left\{ \left( \frac{1}{4} R^{cd \gamma \cd} \otimes 1_3 \right) \right\} \]
\[= 4 \left[ -\frac{1}{8} R^{2 \mu \rho \sigma} (3) \right] \]

\[(13.38)\]
\[\text{Tr} \left( \Omega^{2 \mu \nu} \right)_{21} = \text{Tr} \left\{ \left( \frac{1}{4} R^{cd \gamma \cd} + i g_1 B_{\mu \nu} \right) \otimes 1_3 \right\} \]
\[= 4 \left[ -\frac{1}{8} R^{2 \mu \rho \sigma} (3) - g_1^2 B_{\mu \nu} (3) \right] \]

\[(13.39)\]
\[\text{Tr} \left( \Omega^{2 \mu \nu} \right)_{a1} = \text{Tr} \left\{ \left( \frac{1}{4} R^{cd \gamma \cd} - i \frac{1}{2} g_1 B_{\mu \nu} \right) \delta^{a}_{a} - i \frac{1}{2} g_2 W_{\mu \nu} (\sigma^{a})_{a} \otimes 1_3 \right\} \]
\[= 4 \left[ -\frac{1}{8} R^{2 \mu \rho \sigma} (3) (2) - \frac{1}{4} g_2 B_{\mu \nu} (3) (2) - \frac{1}{4} g_2^2 (W_{\mu \nu})^2 (3) (2) \right] \]

\[(13.40)\]
\[\text{Tr} \left( \Omega^{2 \mu \nu} \right)_{1i}^{i1} = \text{Tr} \left\{ \left( \frac{1}{4} R^{cd \gamma \cd} - \frac{2}{3} i g_1 B_{\mu \nu} \right) \delta^{i}_{i} - i \frac{1}{2} g_2 V_{\mu \nu} (\lambda^{m})_{i} \otimes 1_3 \right\} \]
\[= 4 \left[ -\frac{1}{8} R^{2 \mu \rho \sigma} (3) (3) - \frac{4}{9} g_1^2 B_{\mu \nu} (3) (3) - \frac{1}{4} g_2^2 (V_{\mu \nu})^2 (3) (2) \right] \]
(13.41)
\[
\text{Tr} \left( \Omega_{\mu \nu}^2 \right)_{2i}^{2i} = \text{Tr} \left\{ \left( \left( \frac{1}{4} R_{\mu \nu \rho \sigma} - i g_1 B_{\mu \nu} \right) \delta_{\alpha}^{\rho} \delta_{\beta}^{\sigma} - \frac{i}{2} g_2 W_{\mu \nu}^\alpha (\sigma^\alpha)^a \delta_a^\mu \right) \otimes 1_3 \right\}^2
\]
\[
= 4 \left[ - \frac{1}{8} R_{\mu \nu \rho \sigma}^{2i} (3) (3) - \frac{1}{9} g_1^2 B_{\mu \nu}^{2i} (3) (3) - \frac{1}{4} g_2^2 (V_{\mu \nu}^m)^2 (3) (2) \right]
\]

(13.42)
\[
\text{Tr} \left( \Omega_{\mu \nu}^2 \right)_{ai} = \text{Tr} \left\{ \left( \left( \frac{1}{4} R_{\mu \nu \rho \sigma} - i g_1 B_{\mu \nu} \right) \delta_{\alpha}^{\rho} \delta_{\beta}^{\sigma} - \frac{i}{2} g_2 W_{\mu \nu}^\alpha (\sigma^\alpha)^a \delta_a^\mu \right) \otimes 1_3 \right\}^2
\]
\[
= 4 \left[ - \frac{1}{8} R_{\mu \nu \rho \sigma}^{2i} (3) (3) - \frac{1}{36} g_1^2 B_{\mu \nu}^{2i} (3) (2) - \frac{1}{9} g_2^2 (W_{\alpha \mu \nu}^a)^2 (3) (2) - \frac{1}{4} g_3^2 (V_{\mu \nu}^m)^2 (3) (2) \right]
\]

Collecting these terms we have

(13.43)
\[
\frac{1}{2} \text{Tr} \left( \Omega_{\mu \nu}^2 \right)_{M} = 4 \left[ - \frac{3}{8} R_{\mu \nu \rho \sigma} (16) - 3 g_1^2 B_{\mu \nu}^2 \left( 1 + \frac{1}{2} + \frac{4}{3} + \frac{1}{3} + \frac{1}{6} \right) \right.
\]
\[
- 3 g_2^2 (W_{\alpha \mu \nu}^a)^2 \left( \frac{1}{2} + \frac{3}{2} \right) - 3 g_3^2 (V_{\mu \nu}^m)^2 \left( \frac{1}{2} + \frac{1}{2} + 1 \right) \right]
\]
\[
= 4 \left[ - 6 R_{\mu \nu \rho \sigma} - 10 g_1^2 B_{\mu \nu}^2 - 6 g_2^2 (W_{\mu \nu}^a)^2 - 6 g_3^2 (V_{\mu \nu}^m)^2 \right]
\]

We also have

(13.44)
\[
\frac{1}{6} \text{Tr} \left( E + \frac{1}{5} R \right)_{\mu \nu} = \frac{4}{6} \left[ - 24 R - 4 a H H - 2 c \sigma^2 + \frac{96}{5} R \right]_{\mu \nu}
\]
\[
= - 4 \left[ \frac{4}{5} R + \frac{2}{3} a H H + \frac{1}{3} c \sigma^2 \right]_{\mu \nu}
\]

The first two Seely-de Witt coefficients are, first for \( a_0 \)

(13.45)
\[
a_0 = \frac{1}{16 \pi^2} \int d^4 x \sqrt{g} \text{Tr} (1)
\]
\[
= \frac{1}{16 \pi^2} (4) (32) (3) \int d^4 x \sqrt{g}
\]
\[
= \frac{24}{\pi^2} \int d^4 x \sqrt{g}
\]
then for \( a_2 \):

\[
(13.46) \quad a_2 = \frac{1}{16\pi^2} \int d^4 x \sqrt{g} \text{Tr} \left( E + \frac{1}{6} R \right) \\
= \frac{1}{16\pi^2} \int d^4 x \sqrt{g} \left( (R(-96 + 64) - 16a\Pi H - 8c\sigma^2) \right) \\
= -\frac{2}{\pi^2} \int d^4 x \sqrt{g} \left( R + \frac{1}{4} aHH + \frac{1}{4} c\sigma^2 \right)
\]

With this information we can now compute the Seeley-de Witt coefficient \( a_4 \):

\[
(13.47) \quad a_4 = \frac{1}{16\pi^2} \int d^4 x \sqrt{g} \text{Tr} \left( \frac{1}{360} (5R^2 - 2R_{\mu\nu} + 2R_{\mu\nu\rho\sigma}) + \frac{1}{2} \left( E^2 + \frac{1}{3} RE + \frac{1}{6} \Omega_{\mu\nu}^2 \right) \right)
\]

and where we have omitted the surface terms. Thus

\[
(13.48) \quad \frac{1}{2} \text{Tr} \left( E^2 + \frac{1}{3} RE + \frac{1}{6} \Omega_{\mu\nu}^2 \right) = 4 \left[ 5g_1^2 B_{\mu\nu}^2 + 3g_2^2 (W_{\mu\nu}^\alpha)^2 + 3g_3^2 (V_{\mu\nu}^m)^2 + 3R^2 + aR\Pi H \\
+ \frac{1}{2} cR\sigma^2 + 2b (\Pi H)^2 + 2a |\nabla_{\mu} H_{\alpha}|^2 + 4c\Pi H \sigma^2 + d\sigma^4 + c (\partial_{\mu} \sigma)^2 \right]
\]

Thus

\[
(13.49) \quad a_4 = \frac{1}{2\pi^2} \int d^4 x \sqrt{g} \left[ \frac{1}{30} (5R^2 - 8R_{\mu\nu} - 7R_{\mu\nu\rho\sigma}) + \frac{5}{3} g_1^2 B_{\mu\nu}^2 + g_2^2 (W_{\mu\nu}^\alpha)^2 + g_3^2 (V_{\mu\nu}^m)^2 \\
+ \frac{1}{6} aR\Pi H + b (\Pi H)^2 \sigma^2 + a |\nabla_{\mu} H_{\alpha}|^2 + 2c\Pi H \sigma^2 + \frac{1}{4} d\sigma^4 \right]
\]

Using the identities

\[
(13.50) \quad R_{\mu\nu\rho\sigma} = 2C_{\mu\nu\rho\sigma} + \frac{1}{3} R^2 - R^* R^* \\
(13.51) \quad R_{\mu\nu} = \frac{1}{2} C_{\mu\nu} + \frac{1}{3} R^2 - \frac{1}{2} R^* R^*
\]
where \( R^* R^* = \frac{1}{4} \epsilon_{\alpha\beta\gamma\delta} R_{\mu\nu} \alpha^\beta R_{\rho\sigma} \gamma^\delta \).

(13.52)

\[
\frac{1}{30} (5R^2 - 8R^2_{\mu\nu} - 7R^2_{\mu\nu\rho\sigma}) = R^2 \frac{1}{30} \left( 5 - \frac{8}{3} \frac{7}{3} \right) + \frac{1}{30} C^2_{\mu\nu\rho\sigma} (-4 - 14) + \frac{1}{30} R^* R^* (4 + 7)
\]

\[= -\frac{3}{5} C^2_{\mu\nu\rho\sigma} + \frac{11}{30} R^* R^*
\]

14. Appendix E: a concrete example

We start with a two dimensional example, and on a flat two torus the Dirac operator with coefficient in a trivial bundle \( V \) of dimension 2. We let \( \sigma_j \) be the Pauli matrices acting in \( V \) and use the gauge potential \( W_\mu = \sigma_\mu \). Thus the Dirac is

(14.1)

\[D = \gamma^\mu \otimes (D_\mu + ig\sigma_\mu)\]

We use the notation \( \nabla_\mu = D_\mu + ig\sigma_\mu \) for the covariant derivative. One has \( D_\mu = ip_\mu \) where the \( p_\mu \) are the momenta. The square of \( D \) gives two terms

(14.2)

\[D^2 = 1_S \otimes \Delta_2 - E, \quad \Delta_2 = (p_\mu + g\sigma_\mu)^2, \quad E = -\gamma_1 \gamma_2 \otimes (\nabla_1 \nabla_2 - \nabla_2 \nabla_1)
\]

We begin by computing the eigenvalues of \( \Delta_2 \). It is given by the \( 2 \times 2 \) matrix

(14.3)

\[
\begin{pmatrix}
p_2^2 + p_1^2 + 2g^2 & -2ip_2g + 2p_1g \\
2ip_2g + 2p_1g & p_2^2 + p_1^2 + 2g^2
\end{pmatrix}
\]

whose eigenvalues are

(14.4)

\[\left\{ p_2^2 + p_1^2 + 2g^2 - 2\sqrt{p_2^2g^2 + p_1^2g^2}, p_2^2 + p_1^2 + 2g^2 + 2\sqrt{p_2^2g^2 + p_1^2g^2}\right\}
\]

We compute the asymptotic expansion using the limit of flat space. Thus the trace of \( e^{-t\Delta} \) corresponds to the integral (up to an overall \( 2\pi \))

(14.5)

\[I = \int_0^\infty e^{-t(\rho^2 - 2g\rho + 2g^2)} \rho d\rho + \int_0^\infty e^{-t(\rho^2 + 2g\rho + 2g^2)} \rho d\rho
\]

We take \( g > 0 \) and compute the integrals as follows.

(14.6)

\[I_+ = \int_0^\infty e^{-t(\rho^2 - 2g\rho + 2g^2)} \rho d\rho = e^{-tg^2} \int_{-g}^\infty e^{-tv^2} (v + g) dv
\]

\[= e^{-tg^2} \int_{-g}^\infty e^{-tv^2} v dv + g e^{-tg^2} \int_{-g}^\infty e^{-tv^2} dv
\]

The first integral is the same (since \( e^{-tv^2} v \) is odd) as

(14.7)

\[e^{-tg^2} \int_{-g}^\infty e^{-tv^2} v dv = \frac{e^{-2tg^2}}{2t}
\]

The second integral is expressed using the error function

(14.8)

\[\text{Erf}(u) = \frac{2}{\sqrt{\pi}} \int_0^u e^{-v^2} dv
\]
One has Erf(∞) = 1 and the second integral is

\[ ge^{-tg^2} \int_{-g}^{\infty} e^{-tv^2} dv = ge^{-tg^2} \frac{\sqrt{\pi}}{2\sqrt{t}} (1 + \text{Erf}(g\sqrt{t})) \]

Thus one has

\[ I_1 = \frac{e^{-2tg^2}}{2t} \left( 1 + e^{tg^2} \sqrt{\pi} \sqrt{tg} (1 + \text{Erf}(g\sqrt{t})) \right) \]

Next one has

\[ I_2 = \int_0^{\infty} e^{-x(x^2+2g^2)} x dx = e^{-tg^2} \int_0^{\infty} e^{-v^2} (v - g) dv \]

\[ = e^{-tg^2} \int_0^{\infty} e^{-v^2} v dv - ge^{-tg^2} \int_0^{\infty} e^{-v^2} dv \]

\[ = \frac{e^{-2tg^2}}{2t} - ge^{-tg^2} \frac{\sqrt{\pi}}{2\sqrt{t}} (1 - \text{Erf}(g\sqrt{t})) \]

Thus one has

\[ I_2 = \frac{e^{-2tg^2}}{2t} \left( 1 - e^{tg^2} \sqrt{\pi} \sqrt{tg} (1 - \text{Erf}(g\sqrt{t})) \right) \]

which shows that \( I_2(g) = I_1(-g) \) since Erf is an odd function. One thus gets

\[ I = I_1 + I_2 = \frac{e^{-2tg^2}}{t} + ge^{-tg^2} \frac{\sqrt{\pi}}{\sqrt{t}} \text{Erf}(g\sqrt{t}) \]

One has the Taylor expansion

\[ \frac{\sqrt{\pi}}{2} \text{Erf}(u) = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n+1}}{n!(2n+1)} \]

which gives the expansion

\[ I = \frac{1}{2t} - \frac{2g^4 t}{3} + \frac{8g^6 t^2}{15} - \frac{26g^8 t^3}{105} + \frac{16g^{10} t^4}{189} + O[t^{9/2}] \]

Thus this gives the following formula for the scalar invariants

\[ a_0(x, \Delta_2) = \frac{1}{4\pi} \dim V , \quad a_2(x, \Delta_2) = 0 , \quad a_4(x, \Delta_2) = \frac{1}{4\pi} \dim V \left( -\frac{2g^4}{3} \right) \]

and since \( \dim V = 2 \) and the dimension of spinors is 2 in dimension 2 one gets

\[ a_0(x, 1_S \otimes \Delta_2) = \frac{1}{4\pi} 4 = \frac{1}{\pi} , \quad a_4(x, 1_S \otimes \Delta_2) = \frac{1}{4\pi} 4 \left( -\frac{2g^4}{3} \right) = -\frac{2g^4}{3\pi} \]

We now need to add the contribution coming from \( E \). Note that if one has

\[ \text{Trace}(e^{-t(1_S \otimes \Delta_2)}) \sim a_0 t^{-1} + a_4 t + \ldots \]
and if $E$ is a scalar, one gets
\begin{equation}
\text{Trace}(e^{-t(1S\otimes\Delta^2-E)}) \sim a_0t^{-1} + a_0E + (a_4 + a_0\frac{E^2}{2})t + \ldots
\end{equation}
In our case $E$ is not a scalar, one has
\begin{equation}
E = \gamma_1\gamma_2 \otimes (\nabla_1\nabla_2 - \nabla_2\nabla_1) = g^2\gamma_1\gamma_2 \otimes (2i\sigma_3)
\end{equation}
and $E^2 = 4g^4 \times 1_{S\otimes V}$ is a multiple of the identity operator. The trace of $E$ vanishes and the correction of the $a_4$ is the same as if $E^2 = 4g^4$. Thus the relevant combination is
\begin{equation}
a_4 + a_0\frac{E^2}{2} = a_4 + 2g^4a_0 = a_0\left(2g^4 + \frac{a_4}{a_0}\right)
\end{equation}
Now in our case we have $a_4 = a_0(-\frac{2g^4}{3})$ and thus
\begin{equation}
2g^4 + \frac{a_4}{a_0} = 2g^4 - \frac{2g^4}{3} = \frac{4g^4}{3}
\end{equation}
which gives
\begin{equation}
a_0(x, D^2) = \frac{1}{\pi}, \ a_4(x, D^2) = \frac{1}{\pi^2} \frac{4g^4}{3}
\end{equation}
To obtain a 4-dimensional example we take the product by the flat Dirac in two dimensions, whose expansion gives
\begin{equation}
\text{Trace}(e^{-tD_4^2}) \sim \frac{2}{4\pi t}
\end{equation}
where the 2 comes from the dimension of spinors. Thus for the 4-dimensional Dirac $D_4$ with coefficients in the two dimensional trivial bundle $V$ and connection whose first two components are the $\sigma_\mu$ one gets the heat expansion
\begin{equation}
a_0(x, D_4^2) = \frac{1}{2\pi^2}, \ a_4(x, D_4^2) = \frac{1}{2\pi^2} \frac{4g^4}{3}
\end{equation}
In fact one obtains the full list of the coefficients $a_n$ and the expansion
\begin{equation}
\frac{1}{2\pi^2}\left(\frac{1}{t^2} + \frac{4g^4}{3} + \frac{8g^6t}{15} - \frac{32g^8t^2}{35} + \frac{1088g^{10}t^3}{945} - \frac{9088g^{12}t^4}{10395} + O[t]^9/2\right)
\end{equation}
The above concrete example allows one to check directly that the coefficient of the $\Omega_{\mu\nu}\Omega_{\mu\nu}$ term in $a_4$ is $\frac{1}{12}$ (multiplied by the normalization factor $(4\pi)^{-m/2}$). Indeed, in the example of the two dimensional Laplacian $\Delta_2$, the term in $\frac{1}{4\pi} \dim V(-\frac{2g^4}{3})$ is
\begin{equation}
\frac{2}{4\pi}(-\frac{2g^4}{3}) = \frac{1}{4\pi} \frac{1}{12} \text{Trace}(\Omega_{\mu\nu}\Omega_{\mu\nu})
\end{equation}
since there are two $\Omega_{\mu\nu}\Omega_{\mu\nu}$ each equal to $-4g^2$, as $\nabla_1\nabla_2 - \nabla_2\nabla_1 = -2ig^2\sigma_3$. Thus $\text{Trace}(\Omega_{\mu\nu}\Omega_{\mu\nu}) = -2 \times 2 \times 4g^4 = -16g^4$. With this one can check directly the coefficients of the $a_6$ terms which we have used. The coefficient of the $E^3$ term is $\frac{1}{6}$ (multiplied by the normalization factor $(4\pi)^{-m/2}$) as is
clear by taking $E$ to be a constant. The coefficient of the term $E\Omega^2$ is the same as the coefficient of the $\Omega^2$ term, as is again seen by taking $E$ to be a scalar and multiplying the two series $(t^{-2}a_0 + t^{-1}a_2 + a_4 + ta_6 + ...)(1 + tE + t^2E^2/2 + t^3E^3/6 + ...)$.

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E-mail address: chams@aub.edu.lb
E-mail address: alain@connes.org