Constructing regular graphs with 
smallest defining number

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Abstract

In a given graph $G$, a set $S$ of vertices with an assignment of
colors is a defining set of the vertex coloring of $G$, if there exists
a unique extension of the colors of $S$ to a $\chi(G)$-coloring of the
vertices of $G$. A defining set with minimum cardinality is called
a smallest defining set (of vertex coloring) and its cardinality, the
defining number, is denoted by $d(G, \chi)$. Let $d(n, r, \chi = k)$ be the
smallest defining number of all $r$-regular $k$-chromatic graphs with $n$
vertices. Mahmoodian et. al [7] proved that, for a given $k$ and for
all $n \geq 3k$, if $r \geq 2(k - 1)$ then $d(n, r, \chi = k) = k - 1$. In this paper
we show that for a given $k$ and for all $n < 3k$ and $r \geq 2(k - 1)$,
$d(n, r, \chi = k) = k - 1$.

Keywords: regular graphs, colorings, defining sets, uniquely extendible
colorings.

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1 Introduction

A $k$-coloring of a graph $G$ is an assignment of $k$ different colors to the vertices of $G$ such that no two adjacent vertices receive the same color. The (vertex) chromatic number, $\chi(G)$, of a graph $G$ is the minimum number $k$ for which there exists a $k$-coloring for $G$. A graph $G$ with $\chi(G) = k$ is called a $k$-chromatic graph. In a given graph $G$, a set of vertices $S$ with an assignment of colors is called a defining set of vertex coloring, if there exists a unique extension of the colors of $S$ to a $\chi(G)$-coloring of the vertices of $G$. A defining set with minimum cardinality is called a smallest defining set (of a vertex coloring) and its cardinality is the defining number, denoted by $d(G, \chi)$.

There are some results on defining numbers in [6] (see also [3], and [4]). Here we study the following concept. Let $d(n, r, \chi = k)$ be the smallest value of $d(G, \chi)$ for all $r$-regular $k$-chromatic graphs with $n$ vertices. Note that for any graph $G$, we have $d(G, \chi) \geq \chi(G) - 1$, therefore $d(n, r, \chi = k) \geq k - 1$. By Brooks’ Theorem [2], if $G$ is a connected $r$-regular $k$-chromatic graph which is not a complete graph or an odd cycle, then $k \leq r$. For the case of $r = k$, Mahmoodian and Mendelsohn [5] determined the value of $d(n, k, \chi = k)$ for all $k \leq 5$. Mahmoodian and Soltankhah [8] determined this value for $k = 6$ and $k = 7$. Also in [8], for each $k$, the value of $d(n, k, \chi = k)$ is determined for some congruence classes of $n$. For the case of $k < r$, it is proved in [8] that, for each $n$ and each $r \geq 4$, we have $d(n, r, \chi = 3) = 2$. The following question is raised in [5]:

**Question.** Is it true that for every $k$, there exist $n_0(k)$ and $r_0(k)$, such that for all $n \geq n_0(k)$ and $r \geq r_0(k)$ we have $d(n, r, \chi = k) = k - 1$?

Mahmoodian et. al. [7] proved that the answer to this question is positive and that, for a given $k$ and all $n \geq 3k$, if $r \geq 2(k - 1)$ then $d(n, r, \chi = k) = k - 1$.

We show the above statement for $n < 3k$. In fact we prove that:

**Theorem.** Let $k$ be a positive integer. For each $n < 3k$, if $r \geq 2(k - 1)$ then $d(n, r, \chi = k) = k - 1$.

2 Preliminaries

In this section, we state some known results and definitions which will be used in the sequel.
Definition 1. Let $G$ and $H$ be two graphs, each with a given proper $k$-coloring say $c_G$ and $c_H$, (respectively) with $k$ colors. Then the chromatic join of $G$ and $H$, denoted by $G \vee H$ is a graph where $V(G \vee H) = V(G) \cup V(H)$, and $E(G \vee H) = E(G) \cup E(H)$, together with the set $\{xy \mid x \in V(G), y \in V(H) \text{ such that } c_G(x) \neq c_H(y)\}$.

Theorem A. Let $n$ be a multiple of $k$, say $n = kl$ ($l \geq 2$); then $d(kl, 2(k-1), \chi = k) = k-1$.

To prove this theorem Mahmoodian and Mendelsohn constructed a $2(k-1)$-regular $k$-chromatic graph with $n = kl$ vertices as follows. Let $G_1, G_2, \ldots, G_l$ be vertex disjoint graphs such that $G_1$ and $G_l$ are two copies of $K_k$ and if $l \geq 3$, $G_2, \cdots, G_{l-1}$ are copies of $\overline{K_k}$. Color each $G_i$ with $k$ colors $1, 2, \cdots, k$. Then construct a graph $G$ with $lk$ vertices by taking the union of $G_1 \cup G_2 \cup \ldots \cup G_l$, and by making a chromatic join between $G_i$ and $G_{i+1}$; for $i = 1, 2, \cdots, l-1$. This is the desired graph. We denote such a graph by $G_{l(k)}$ and use this construction in Section 3.

Definition 2. Let $G$ be a $k$-chromatic graph and let $S$ be a defining set for $G$. Then a set $F(S)$ of edges is called nonessential edges, if the chromatic number of $G - F(S)$, the graph obtained from $G$ by removing the edges in $F(S)$, is still $k$, and $S$ is also a defining set for $G - F(S)$.

Remark 1. A necessary condition for the existence of an $r$-regular $k$-chromatic graph is $\frac{r}{k-1} \leq \frac{n}{k}$. For, if $G$ is an $r$-regular $k$-chromatic graph with $n$ vertices, then each chromatic class in $G$ has at most $n - r$ vertices. Therefore $n \leq k(n - r)$. This implies $\frac{n}{k} \leq \frac{r}{k-1}$. Thus, for $r \geq 2(k-1)$ there are not any graph of order $n < 2k$. Hence when $r \geq 2(k-1)$, it is sufficient to investigate $d(n, r, \chi = k)$ only for $n \geq 2k$. Also it is obvious that $n$ and $r$ cannot be both odd.

For the definitions and notations not defined here we refer the reader to texts, such as [9].

3 Main results

In this section in the following four theorems we prove our main result, which was mentioned at the end of Section 1.

Theorem 1. For each $k \geq 3$ and each $r \geq 2(k-1)$, we have $d(3k - 1, r, \chi = k) = k-1$. 


Proof. Let \( n = 3k - 1 \) and \( r = 2(k - 1) + t \). By Remark 1 it is obvious that \( t \leq k - 2 \). First for \( t = 0 \), we construct a \( 2(k - 1) \)-regular \( k \)-chromatic graph \( H \) with \( n \) vertices and \( d(H, \chi) = k - 1 \) as follows. By Theorem A we have \( d(3k, 2(k - 1), \chi = k) = k - 1 \). In graph \( G_{3(l)} \) which was constructed to prove Theorem A, let \( V(G_1) = \{u_1, u_2, ..., u_k\} \), \( V(G_2) = \{v_1, v_2, ..., v_k\} \), and \( V(G_3) = \{w_1, w_2, ..., w_k\} \). Also assume that \( c(u_i) = c(v_i) = c(w_i) = i \), for \( i = 1, 2, ..., k \). Note that the set of vertices adjacent to \( v_k \) is \( N_{G_{3(l)}}(v_k) = \{u_1, ..., u_k\} \cup \{w_1, ..., w_{k-1}\} \). We delete the vertex \( v_k \) and join its neighbors in the following manner: we join \( u_i \) to \( w_i + 1 \) for \( i = 1, 2, ..., k - 2 \) and \( u_{k-1} \) to \( w_1 \). It can be easily seen that the new graph, say \( H \), is \( 2(k - 1) \)-regular \( k \)-chromatic with \( n = 3k - 1 \) vertices with a defining set \( S = \{u_1, u_2, ..., u_{k-1}\} \).

Now for \( 1 \leq t \leq k - 3 \), to construct an \( r \)-regular \( k \)-chromatic graph, we consider the graph \( H \), and we add the edges \( u_iw_{i+j+2} \) (mod \( k \)), for \( i = 1, ..., k \) and \( j = 1, ..., t \), to \( H \). Also, in the case of \( k \) odd, we add the edges of \( t \) mutually disjoint 1-factors of \( K_{k-1} \), and in the case of \( k \) even, the edges of \( \frac{t}{2} \) mutually disjoint 2-factors of \( K_{k-1} \), on vertex set \( \{v_1, ..., v_{k-1}\} \).

Note that if \( t = k - 2 \) then such a graph does not exist. For, if \( G \) is a graph satisfying such conditions then we know that each chromatic class in \( G \) has at most 3 vertices. Since \( n = 3k - 1 \), \( G \) must have \( k - 1 \) chromatic classes of size 3 and one chromatic class of size 2. And each vertex in a chromatic class of size 3 must be adjacent to all other vertices. This implies that the degree of each vertex in the chromatic class of size 2 is \( 3(k - 1) = r + 1 \), which contradicts the \( r \)-regularity of the graph.

Example 1. In Figure 1 we show the graph \( H \) when \( k = 5 \) and \( r = 8 \). The vertices of the defining set are shown by the filled circles.

Figure 1: \( d(H, \chi = 5) = 4 \).
Theorem 2. For each odd number \( k \geq 3 \), and each \( 2k \leq n \leq 3k - 2 \), we have \( d(n, 2(k - 1), \chi = k) = k - 1 \).

Proof. By Theorem A we have \( d(2k, 2(k - 1), \chi = k) = k - 1 \). Let \( n = 2k + s \), where \( s = 1, 2, \ldots, k - 2 \). We construct a \( 2(k - 1) \)-regular \( k \)-chromatic graph \( H_s \) with \( n \) vertices and defining number equals to \( k - 1 \). For this, we consider graph \( G_{2(l)} \) and add \( s \) new vertices to it, delete some suitable edges as follows and join the new vertices to the end vertices of deleted edges. In graph \( G_{2(l)} \), for convenience let \( V(G_1) = \{u_1, \ldots, u_i, \ldots, u_{i - 1}, v_1, \ldots, v_{i - 1}, v_1, \ldots, v_{i - 1}, u_{i - 1}, v_{i - 1}, \ldots, v_1, v_1, \ldots, v_1\} \) and \( V(G_2) = \{v_1, \ldots, v_1, \ldots, v_1, v_1, \ldots, v_1, v_1, \ldots, v_1\} \), where \( i' = i + \frac{k - 1}{2} \), \( i = 1, 2, \ldots, \frac{k - 1}{2} \); and \( c(u_j) = c(v_j) = j \), for \( j = 1, 2, \ldots, k \).

If \( 1 \leq s \leq \frac{k - 1}{2} \), then denote new vertices by \( x_1, \ldots, x_s \). Let \( M_1, M_2, \ldots, M_{\frac{k - 1}{2}} \) be mutually disjoint 1-factors of subgraph \( < u_1, \ldots, u_i, \ldots, u_{i - 1}, v_1, \ldots, v_{i - 1}, v_1, \ldots, v_{i - 1}, u_{i - 1}, v_{i - 1}, \ldots, v_1, v_1, \ldots, v_1 > \) in \( G_{2(l)} \) such that each edge in \( M_i \) has one end in \( \{u_1, u_2, \ldots, u_{\frac{k - 1}{2}}\} \) and the other end in \( \{u_{\frac{k - 1}{2}}, \ldots, u_1, v_1, \ldots, v_1\} \). For each \( 1 \leq i \leq s \) we join \( x_i \) to each of the vertices of \( M_i \), and delete all of the edges of \( M_i \). Also with respect to each \( u_a u_b \in M_i \), we delete the edge \( v_a v_b \) and join \( x_i \) to the vertices \( v_a \) and \( v_b \). Now it can be easily seen that \( deg(x_i) = 2(k - 1) \). Note that the new graph contains a complete subgraph say, \( < u_1, u_2, \ldots, u_{\frac{k - 1}{2}}, v_1, \ldots, v_1, \ldots, v_1 > = K_k \) and a defining set \( S = \{u_1, \ldots, u_{\frac{k - 1}{2}}\} \). Also the colors of vertices of \( G_{2(l)} \) force all new vertices to be colored \( k \).

If \( \frac{k - 1}{2} < s \leq k - 2 \) then we denote the new vertices by \( x_1, x_2, \ldots, x_{\frac{k - 1}{2}}, y_1, y_2, \ldots, y_{s - \frac{k - 1}{2}} \). For \( x_i \) (\( 1 \leq i \leq \frac{k - 1}{2} \)) we proceed as before. For \( y_t \) (\( 1 \leq t \leq s - \frac{k - 1}{2} \)), first we recognize some nonessential edges in \( H_{\frac{k - 1}{2}} \). If for each \( i \), we let \( z_i \) be either \( u_i \) or \( v_i \), and for each \( j \), we let \( w_j \) be either \( u_j \) or \( v_j \), then the following edges form a nonessential set in \( H_{\frac{k - 1}{2}} \):

\[
F = \{v_i v_j \mid 1 \leq i < j \leq \frac{k - 1}{2}\} \cup \{u_i u_{i'} \mid 1 \leq i < i' \leq \frac{k - 1}{2}\} \cup \{x_i u_{i'} \text{ or } x_{i'} v_i \mid 1 \leq i \leq \frac{k - 1}{2}\} \cup \{x_i w_j \mid 2 \leq i \leq \frac{k - 1}{2}, 1 \leq j \leq k - 1\} \cup \{z_i v_k \mid 1 \leq i \leq k - 1\}.
\]

There are two cases to be considered.

Case 1. \( k = 4l + 1 \).

In this case the induced subgraphs \( A = < u_1, u_2, \ldots, u_{\frac{k - 1}{2}} > \) and \( B = < v_1, v_2, v_3, \ldots, v_{k - 1} > \) are complete graphs \( K_{\frac{k - 1}{2}} \). So they are 1-factorable. Let \( F_1, F_2, \ldots, F_{\frac{k - 1}{2}} \) and \( F'_1, F'_2, \ldots, F'_{\frac{k - 1}{2}} \) be 1-factorizations of \( A \) and \( B \), re-
respectively, such that the edge $u_t'v_{(t+1)'} \in F_t$ and $v_tv_{k+1} \in F'_t$. Now for each $t$ ($1 \leq t \leq s - \frac{k-1}{2} \leq \frac{k-3}{2}$) we delete all of the edges of $F_t \setminus \{u_t'v_{(t+1)'}\}$ and $F'_t \setminus \{v_tv_{k+1}\}$. Also we delete the edges $u_tv_{k+1}$ and $u_tv_k$. Finally we delete all the edges $x_1v_t$, $x_2u_{t+1}$, ..., $x_{s-1}u_{t+\frac{k-1}{2}}$ (mod $\frac{k-1}{2}$). We join $y_t$ to the ends of all deleted edges. It can be easily seen that $\deg(y_t) = 2(k-1)$ and the color of $y_t$ is forced to be $k - 1$.

**Case 2.** $k = 4l + 3$.

In this case the induced subgraphs $A = \langle u_t', u_2', ..., u_{(\frac{k+1}{2})'}, u_k \rangle$ and $B = \langle v_1, v_2, ..., v_{(\frac{k+1}{2})}, v_k \rangle$ are complete graphs $K_{\frac{k+1}{2}}$. Thus they are 1-factorable. Let $F_1, F_2, ..., F_{\frac{k-1}{2}}$ and $F'_1, F'_2, ..., F'_{\frac{k-1}{2}}$ be 1-factorizations of $A$ and $B$, respectively, such that $u_t'v_k \in F_t$ and $v_tv_k \in F'_t$, for $1 \leq t \leq \frac{k-1}{2}$. Now for each $t$ ($1 \leq t \leq s - \frac{k-1}{2} \leq \frac{k-3}{2}$) we delete all of the edges of $F_t \setminus \{u_t'v_k\}$ and $F'_t \setminus \{v_tv_k\}$. Also we delete the edge $v_kv_t$. Finally we delete the edges $x_1v_t$, $x_2u_{t+1}$, ..., $x_{s-1}u_{t+\frac{k-1}{2}}$ (mod $\frac{k-1}{2}$). We join $y_t$ to the ends of all deleted edges. It can be easily seen that $\deg(y_t) = 2(k-1)$ and the color of $y_t$ is forced to be $t + \frac{k-1}{2}$.

To illustrate the construction shown in the proof of Theorem 2, we provide the following example.

**Example 2.** Let $k = 7$. For $15 \leq n \leq 19$, we construct a 12-regular 7-chromatic graph of order $n$ with a defining set of size 6. For $n = 14 + s$, $1 \leq s \leq 5$, we add $s$ new vertices to the 12-regular 7-chromatic graph $G_{2(7)}$ of order $14$ and delete some nonessential edges as explained in the proof of Theorem 2.

**Table 1:** New vertices and deleted edges.

| New vertices | $x_1$ | $x_2$ | $x_3$ | $y_1$ | $y_2$ |
|--------------|-------|-------|-------|-------|-------|
| $u_1'u_1'$   | $u_1'u_2'$ | $u_1'u_3'$ | $u_2'u_3'$ | $u_1'u_4'$ | $u_1'u_5'$ |
| $u_2'u_2'$   | $u_2'u_3'$ | $u_2'u_4'$ | $u_2'u_5'$ | $v_1v_1'$ | $v_1v_2'$ |
| $u_3'u_3'$   | $u_3'u_4'$ | $u_3'u_5'$ | $x_1v_1$ | $x_1v_2$ | $x_2u_1$ |
| $v_1v_1'$    | $v_1v_2'$ | $v_1v_3'$ | $x_2u_2$ | $x_2u_3$ | $x_3u_1$ |
| $v_2v_2'$    | $v_2v_3'$ | $v_2v_4'$ | $x_3u_2$ | $x_3u_3$ | $v_4v_1$ |
| $v_3v_3'$    | $v_3v_4'$ | $v_3v_5'$ | $v_7u_1$ | $v_7u_2$ |       |

Table 1 gives all the deleted edges of $G_{2(7)}$ with respect to addition of new vertices. In Figure 2 we show the deleted edges and the added edges.
to construct a 12-regular 7-chromatic graph $H_1$ of order 15 ($s = 1$) with a defining set of size 6. The dotted lines are the deleted edges and the vertices of the defining set are shown by the filled circles.

Figure 2: $d(H_1, \chi = 7) = 6$.

**Theorem 3.** For each even number $k \geq 4$, and each $2k \leq n \leq 3k - 2$, we have $d(n, 2(k - 1), \chi = k) = k - 1$.

**Proof.** By Theorem A we have $d(2k, 2(k - 1), \chi = k) = k - 1$. For $n = 2k + s$, $s = 1, 2, \ldots, k - 2$, we construct a $2(k - 1)$-regular $k$-chromatic graph $H_s$ with $n$ vertices and defining number equal to $k - 1$. To construct $H_s$, we consider graph $G_{2(k)}$ and add $s$ new vertices to it, delete some suitable edges and join the new vertices to the end vertices of the deleted edges as follows. In graph $G_{2(k)}$ for convenience let $V(G_1) = \{u_1, \ldots, u_i, u_{i'}, \ldots, u_{k/2}, u_1', \ldots, u_i', \ldots, u_{k/2}'\}$ and $V(G_2) = \{v_1, \ldots, v_i, \ldots, v_{k/2}, v_1', \ldots, v_i', \ldots, v_{k/2}'\}$, where $i' = i + \frac{k}{2}$, $i = 1, 2, \ldots, \frac{k}{2}$; and $c(u_j) = c(v_j) = j$, for $j = 1, 2, \ldots, k$.

If $1 \leq s \leq \frac{k}{2} - 1$ then we denote the new vertices by $x_1, \ldots, x_s$. Let $M_1, M_2, \ldots, M_{k/2}$ be mutually disjoint 1-factors of the induced subgraph $G_1 = \langle u_1, \ldots, u_i, \ldots, u_{k/2}, u_1', \ldots, u_i', \ldots, u_{k/2}' \rangle$, where, for $i = 1, 2, \ldots, \frac{k}{2}$:

$$M_i = \{u_1u_{i'}, u_2u_{(i+1)'}, \ldots, u_tu_{(i+t-1)'} \} \mod \frac{k}{2}. $$
Also let $M'_1, M'_2, ..., M'_s$ be mutually disjoint 1-factors of the induced subgraph $G_2 = < v_1, ..., v_{s'}, v_1', ..., v_t', ..., v_{(s'+t')'}, >$, where, for $i = 1, 2, ..., \frac{k}{2}$:

$$M'_i = \{v_1v'_i, v_2v_{(i+1)'}, ..., v_{t}v_{(i+t-1)'}, ..., v_{s'}v_{(i+s'-1)'}\} \pmod{\frac{k}{2}}.$$  

Now for each $i$ ($i = 1, 2, ..., s'$) we delete all of the edges of $M_{i+1}\{u_{\frac{s}{2}-i}v_{(\frac{s}{2})'}\}$, and all of the edges of $M'_i\{v_{\frac{s}{2}-i+1}v_{(\frac{s}{2})'}\}$. Finally we delete the edge $u_{\frac{s}{2}-i}v_{\frac{s}{2}-i+1}$. We join $x_i$ to the ends of all deleted edges. Now it can be easily seen that $\deg(x_i) = 2(k-1)$. Note that the new graph contains a complete subgraph say $< u_1, u_2, ..., u_{\frac{s}{2}}, u_{(\frac{s}{2})'}, v_1, ..., v_{(\frac{s}{2}-1)'} > = K_k$ and a defining set $S = \{u_1, ..., u_{k-1}\}$. Also the colors of vertices of $G_{2(k)}$ force the colors of all new vertices to be $k$.

If $\frac{s}{2} \leq s \leq k - 2$ then we denote the new vertices by $x_1, x_2, ..., x_{\frac{s}{2}-1}, y_1, y_2, ..., y_{s-\frac{s}{2}+1}$. For $x_i$ ($1 \leq i \leq \frac{k}{2} - 1$) we treat as before. For $y_t$ ($1 \leq t \leq s - \frac{k}{2} + 1$) first we recognize some nonessential edges in $H_{\frac{s}{2}-1}$. If for each $j$, let $w_j$ be either $u_j$ or $v_j$, then the following edges form a nonessential set in $H_{\frac{s}{2}-1}$:

$$F = \{v_{i}w_{j} \mid 1 \leq i < j \leq \frac{s}{2}, j \neq i+1\} \cup \{u_{i'}u_{j'} \mid 1 \leq i' < j' \leq (\frac{k}{2}) - 1\} \cup \{x_{i}w_{j} \mid 1 \leq i \leq \frac{s}{2} - 1, 1 \leq j \leq k - 1\} \cup \{v_{i}w_{(\frac{s}{2})'} \mid 1 \leq i \leq (\frac{k}{2}) - 1\} \cup M'_1\{u_{\frac{s}{2}}w_{(\frac{s}{2})'}\} \cup M'_s.$$  

There are two cases to be considered.

**Case 1.** $k = 4l$.

In this case the induced subgraphs $A = < u_1, u_2, ..., u_{(\frac{s}{2})'} >$ and $B = < v_1, v_2, ..., v_{s'} >$ are complete graphs $K_{\frac{s}{2}}$. So they are 1-factorable. Let $F_1, F_2, ..., F_{\frac{s}{2}-1}$ and $F'_1, F'_2, ..., F'_{\frac{s}{2}-1}$ be standard 1-factorizations (see [1], page 166) of $A$ and $B$, respectively, such that the edges $u_{i'}u_{(\frac{s}{2})'} \in F_t$ and $v_{i}v_{\frac{s}{2}} \in F'_t$. Now for each $t$ ($1 \leq t \leq s - \frac{k}{2} + 1 \leq \frac{k}{2} - 1$) we delete all of the edges of $F_t\{u_{i'}u_{(\frac{s}{2})'}\}$ and $F'_t$. Also we delete the edge $v_{i}v_{i+1} \pmod{\frac{k}{2} - 1})$. If there exist some edges such as $v_{i}v_{i+1} \in F'_t$, then instead of these edges we delete the edge $v_{i}v_{i+1} \in M'_s$.

Also for an arbitrary index $i$ of such as edges $v_{i}v_{i+1}$ we delete the edge $v_{i}v_{i} \pmod{\frac{k}{2}}$ instead of the edge $v_{i}v_{i+1} \pmod{\frac{k}{2}}$. Finally we delete the edges $x_1u_{t+1}, x_2u_{t+2}, ..., x_{\frac{s}{2}-1}u_{t+\frac{s}{2}-1} \pmod{\frac{k}{2}}$. 


We join \( y_t \) to the ends of all deleted edges. It can be easily seen that \( \deg(y_t) = 2(k - 1) \) and the color of \( y_t \) is forced to be \( t + \frac{k}{2} \), for \( t \neq \frac{k}{2} - 1 \) and the color of \( y_{\frac{k}{2} - 1} \) to be \( \frac{k}{2} - 1 \).

**Case 2.** \( k = 4l + 2 \).

In this case the induced subgraphs \( A = \langle u_{1'}, u_{2'}, ..., u_{\frac{k}{2}'} \rangle, u_1 > \) and \( B = \langle v_1, v_2, ..., v_{\frac{k}{2}}, v_{\frac{k}{2}'} \rangle > \) are complete graphs \( K_{\frac{k}{2} + 1} \). So they are 1-factorable. Let \( F_1, F_2, ..., F_{\frac{k}{2}} \) and \( F'_1, F'_2, ..., F'_{\frac{k}{2}} \) be 1-factorizations of \( A \) and \( B \), respectively, such that \( u_{1}u_{i'} \in F_t \) and \( v_{t}v_{i'} \in F'_t \). Now for each \( t (1 \leq t \leq s - \frac{k}{2} + 1 \leq \frac{k}{2} - 1) \) we delete all of the edges of \( F_1 \setminus \{u_{1}u_{i'}, u_{j'}v_{i'}\} \) and \( F'_t \). Also we delete the edge \( u_{i'}u_{j'} \in M_1 \). If there exist some edges such as \( v_{t}v_{i+1} \in F'_t \) then instead of the edges \( v_{t}v_{i+1} \) we delete the edges \( v_{t}v_{i+1} \in M_{\frac{k}{2}} \). Finally we delete the edges \( x_{1}u_{j+1}, x_{2}u_{j+2}, ..., x_{\frac{k}{2} - 1}u_{j+\frac{k}{2} - 1} \) \((\mod \frac{k}{2})\). We join \( y_t \) to the ends of all deleted edges. It can be easily seen that \( \deg(y_t) = 2(k - 1) \) and the color of \( y_t \) is forced to be \( t + \frac{k}{2} \).

To illustrate the construction shown in the proof of Theorem 3 we provide the following example.

**Example 3.** Let \( k = 8 \). For \( 17 \leq n \leq 22 \), we construct a 14-regular 8-chromatic graph of order \( n \) with a defining set of size 7. For \( n = 16 + s, 1 \leq s \leq 6 \), we add \( s \) new vertices to the 14-regular 8-chromatic graph \( G_{2(8)} \) of order 16 and delete some nonessential edges as explained in the proof of Theorem 3.

**Table 2:** New vertices and deleted edges.

| New vertices | \( x_1 \) | \( x_2 \) | \( x_3 \) | \( y_1 \) | \( y_2 \) | \( y_3 \) |
|--------------|---------|---------|---------|---------|---------|---------|
| Deleted edges | \( u_1u_2 \) | \( u_1u_3 \) | \( u_2u_1 \) | \( u_2u_3 \) | \( u_2u_1 \) | \( u_1u_3 \) |
|              | \( u_2u_3 \) | \( u_3u_1 \) | \( u_3u_2 \) | \( v_1v_4 \) | \( v_2v_4 \) | \( v_3v_4 \) |
|              | \( u_4u_1 \) | \( u_4u_2 \) | \( u_4u_3 \) | \( v_2v_3 \) | \( v_3v_4 \) | \( v_1v_4 \) |
|              | \( v_1v_2 \) | \( v_1v_3 \) | \( v_1v_4 \) | \( v_2v_4 \) | \( v_3v_4 \) | \( v_1v_4 \) |
|              | \( v_2v_3 \) | \( v_2v_4 \) | \( v_3v_4 \) | \( v_2v_4 \) | \( v_3v_4 \) | \( v_1v_4 \) |
|              | \( u_3v_3 \) | \( u_4v_1 \) | \( u_4v_2 \) | \( x_1u_2 \) | \( x_1u_3 \) | \( x_1u_4 \) |
|              | \( u_3v_4 \) | \( u_4v_3 \) | \( u_3v_4 \) | \( x_2u_4 \) | \( x_2u_4 \) | \( x_2u_4 \) |

Table 2 gives all the deleted edges of \( G_{2(8)} \) with respect to addition of new vertices. In Figure 3 we show the deleted edges and the added edges.
to construct a 14-regular 8-chromatic graph $H_1$ of order 17 ($s = 1$) with a defining set of size 7. The dotted lines are the deleted edges and the vertices of the defining set are shown by the filled circles.

Figure 3: $d(H_1, \chi = 8) = 7$.

**Theorem 4.** For each $k \geq 4$, $2k \leq n \leq 3k - 2$, and $r > 2(k - 1)$, we have $d(n, r, \chi = k) = k - 1$.

**Proof.** Let $n = 2k + s$, $0 \leq s \leq k - 2$, and $r = 2(k - 1) + t$. By Remark 1, if there exists an $r$-regular $k$-chromatic graph with $n$ vertices then it is obvious that $t < s$. We construct an $r$-regular $k$-chromatic graph $H$ with $n$ vertices in the following manner.

Consider graph $G_2(k)$, let $V(G_1) = \{u_1, \ldots, u_k\}$ and $V(G_2) = \{v_1, \ldots, v_k\}$, and $c(u_i) = c(v_i) = i$, for $i = 1, 2, \ldots, k$. We add $s$ new vertices say $x_1, \ldots, x_s$ to $G_2(k)$. For each $x_i$ ($1 \leq i \leq s$) we join $x_i$ to each vertex of $V(G_1) \cup V(G_2) \setminus \{u_i, v_i\}$. Also, in the case of $s$ even, we add the edges of $t$ mutually disjoint 1-factors of $K_s$, and in the case of $s$ odd, the edges of $t$ mutually disjoint 2-factors of $K_s$, to $x_1, \ldots, x_s$. The graph obtained in this way, say $H'$, is a $k$-chromatic graph with $n$ vertices and a defining set $S = \{x_2, \ldots, x_s, u_{s+1}, \ldots, v_k\}$ such that $\deg(x_i) = 2(k - 1) + t$ ($1 \leq i \leq s$), $\deg(u_i) = \deg(v_i) = 2(k - 1) + s - 1$ ($1 \leq i \leq s$), and $\deg(u_i) = \deg(v_i) = 2(k - 1) + s + 1$ ($s + 1 \leq i \leq k$). Now we show that by deleting some suitable nonessential edges of $H'$ the desired $r$-regular graph $H$ can be obtained.

In the graph $H'$, for convenience let $A = \{u_1, \ldots, u_{s}\}$, $C = \{u_{s+1}, \ldots, u_s\}$,
$D = \{ u_{s+1}, \ldots, u_{s+\lfloor \frac{k-s}{2} \rfloor} \}$, and $B = \{ u_{s+\lfloor \frac{k-s}{2} \rfloor+1}, \ldots, u_k \}$. Also let $A' = \{ v_1, \ldots, v_{\lfloor \frac{k}{2} \rfloor} \}$, $C' = \{ v_{\lfloor \frac{k}{2} \rfloor+1}, \ldots, v_s \}$, $D' = \{ v_{s+1}, \ldots, v_{s+\lfloor \frac{k-s}{2} \rfloor} \}$, and $B' = \{ v_{s+\lfloor \frac{k-s}{2} \rfloor+1}, \ldots, v_k \}$. Let $i' = i + \lfloor \frac{k-s}{2} \rfloor$ for $s + 1 \leq i \leq s + \lfloor \frac{k-s}{2} \rfloor$.

First we delete a maximal matching of each complete bipartite subgraph with parts $B$ and $D$ of $G_1$ and parts $B'$ and $D'$ of $G_2$. For $k - s$ odd, we assume $u_{k-1}$ and $v_k$ to be vertices unsaturated by the maximal matchings. Then we delete the edge $u_{k-1}v_k$.

Secondly, we delete the edges of $s - t - 1$ mutually disjoint maximal matchings of each complete bipartite subgraph with parts $A \cup B$ and $C \cup D$ of $G_1$ and parts $A' \cup B'$ and $C' \cup D'$ of $G_2$. For $k$ odd, we assume that the following vertices are unsaturated by the maximal matchings: 
\[ \{ u_1, \ldots, u_{\lfloor \frac{k}{2} \rfloor}, u_{(s+1)'}^{s+1}, \ldots, u_{(s+1)'+s-t-2-\lfloor \frac{k}{2} \rfloor} \} \] and 
\[ \{ v_2, \ldots, v_{\lfloor \frac{k}{2} \rfloor}, v_1, v_{(s+1)'+1}, \ldots, v_{(s+1)'+s-t-1-\lfloor \frac{k}{2} \rfloor} \} \]

in the case of $s$ even, or 
\[ \{ u_{\lfloor \frac{k}{2} \rfloor+1}, \ldots, u_{\lfloor \frac{k}{2} \rfloor+s-t-1} \} \]

and 
\[ \{ v_{\lfloor \frac{k}{2} \rfloor+2}, \ldots, v_s, v_{\lfloor \frac{k}{2} \rfloor+1}, v_{s+2}, \ldots, v_{s+s-1-\lfloor \frac{k}{2} \rfloor} \} \]

in the case of $s$ odd. Then we delete the edges 
\[ u_1v_2, u_2v_3, \ldots, u_{\lfloor \frac{k}{2} \rfloor-1}v_{\lfloor \frac{k}{2} \rfloor}, u_{\lfloor \frac{k}{2} \rfloor}v_1, u_{(s+1)'}^{v+(s+1)'}v_{(s+1)'}^{v+(s+1)'} \]

or the edges 
\[ u_{\lfloor \frac{k}{2} \rfloor+1}v_{\lfloor \frac{k}{2} \rfloor+1}, u_{\lfloor \frac{k}{2} \rfloor+s-t-1}v_{\lfloor \frac{k}{2} \rfloor+s-t-1} \]

depending on the parity of $s$, respectively. If $s-t > \lfloor \frac{k}{2} \rfloor$ then in the second step we delete \lfloor $\frac{k}{2} \rfloor - 1$ maximal matchings. Finally we delete the edges of $s - t - \lfloor \frac{k}{2} \rfloor$ mutually disjoint 1-factors $F_j$ $(1 \leq j \leq s - t - \lfloor \frac{k}{2} \rfloor)$ of bipartite subgraph with parts $C \cup D$ and $C' \cup D'$, where

\[ F_j = \{ u_iv_{i+j+1} \mid \lfloor \frac{k}{2} \rfloor + 1 \leq i \leq s + \lfloor \frac{k-s}{2} \rfloor - j - 1 \} \cup \{ u_{i-j}v_{i+j+1} \mid i_0 = s + \lfloor \frac{k-s}{2} \rfloor - j \leq i \leq s + \lfloor \frac{k-s}{2} \rfloor \} \]

In fact if we consider the order $u_{\lfloor \frac{k}{2} \rfloor+1}, \ldots, u_s, u_{s+1}, \ldots, u_{s+\lfloor \frac{k-s}{2} \rfloor}$, and $v_{\lfloor \frac{k}{2} \rfloor+1}, \ldots, v_s, v_{s+1}, \ldots, v_{s+\lfloor \frac{k-s}{2} \rfloor}$, for the vertices in $C \cup D$ and $C' \cup D'$, respectively, then each 1-factor $F_j$ contains the edges in which the $i$th vertex in $C \cup D$ is matched with $(i + j + 1)$th vertex $(C \cup D)$ in $C' \cup D'$. (See Figure 4.)

Also for decreasing the degree of vertex sets $A \cup B$ and $A' \cup B'$, we delete the edges of $s - t - \lfloor \frac{k}{2} \rfloor$ mutually disjoint 1-factors $F'_j$ $(1 \leq j \leq s - t - \lfloor \frac{k}{2} \rfloor)$ of bipartite subgraph with parts $A \cup B$ and $A' \cup B'$ the same as above. Therefore the graph $H$ obtained in this way contains a complete subgraph say $K_k =< A \cup B \cup C' \cup D' >$ and $H$ is an $r$-regular graph.
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References

[1] M. Behzad, G. Chartrand, and L. Lesniak. Graphs and digraphs. Prindle, Boston, 1979.

[2] R.L. Brooks. On coloring the nodes of a network. Mathematical Proceedings of the Cambridge Philosophical Society, 37:194–197, 1941
[3] A.D. Keedwell. Critical sets for latin squares, graphs and block designs: a survey. *Congr. Numer.*, **113**:231–245, 1996. Festschrift for C. St. J. A. Nash-Williams.

[4] E.S. Mahmoodian. Some problems in graph colorings. In S. H. Javadpour and M. Radjabalipour, editors, *Proc. 26th Annual Iranian Math. Conference*, pages 215–218, Kerman, March 1995. Iranian Math. Soc., University of Kerman.

[5] E.S. Mahmoodian and E. Mendelsohn. On defining numbers of vertex coloring of regular graphs. *Discrete Mathematics*, **197/198**:543–554, 1999.

[6] E.S. Mahmoodian, R. Naserasr, and M. Zaker. Defining sets in vertex coloring of graphs and latin rectangles. *Discrete Mathematics*, **167/168**:451–460, 1997.

[7] E.S. Mahmoodian, B. Omoomi, and N. Soltankhah. Smallest defining number of $r$-regular $k$-chromatic graphs: $r \neq k$. *Ars Combinatoria*, **78**: 211–223, 2006.

[8] N. Soltankhah and E.S. Mahmoodian. On defining numbers of $k$-chromatic $k$-regular graphs. *Ars Combinatoria*, **76**: 257-276, 2005.

[9] D.B. West. *Introduction to Graph Theory*. 2nd Edition, Prentice Hall, Upper Saddle River, NJ, 2001.

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