On the violation of Bell’s inequality for all non-product quantum states

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We present an explicit reexamination of Gisin’s 1991 original proof concerning the violation of Bell’s inequality for any pure entangled state of two-particle systems. Given the relevance of Gisin’s work, our analysis is motivated by pedagogical reasons and allows the straightening of a few mathematical points in the original proof that in no way change the physical conclusions reached by Gisin.

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I. INTRODUCTION

Relativity theory, quantum theory, and information theory are fundamental blocks of theoretical physics\cite{1}. The goal of theoretical physics is to describe and, to a certain extent, understand natural phenomena. Unfortunately, complex difficulties arise when one attempts to merge general relativity theory and quantum theory. For example, in classical mechanics it is often said that gravity is a purely geometric theory since the mass does not appear in the usual Newtonian equation of a particle trajectory,

\[
m \frac{d^2 \vec{x}}{dt^2} = -m \nabla_{\vec{x}} \Phi_{\text{gravity}} \iff \frac{d^2 \vec{x}}{dt^2} = \vec{g}.
\]

This is a direct consequence of the equality of the gravitational and inertial masses. In quantum mechanics, the situation is rather different. As a matter of fact, the Schrödinger quantum-mechanical wave equation is given by\cite{2},

\[
\left[ -\frac{\hbar^2}{2m} \nabla_{\vec{x}}^2 + m \Phi_{\text{gravity}} \right] \psi(\vec{x}, t) = i\hbar \frac{\partial}{\partial t} \psi(\vec{x}, t).
\]

The mass \( m \) no longer cancels and, instead, it appears in the combination \( \hbar/m \) (where \( \hbar \equiv h/2\pi \) and \( h \) denotes the Planck constant). Therefore, in an instance where \( \hbar \) appears, \( m \) is also expected to appear\cite{3}. It seems evident that there is the possibility that such difficulties are not simply technical and mathematical, but rather conceptual and fundamental. This viewpoint was recently presented in\cite{4}, where the idea was advanced that quantum causality might shed some light on foundational issues related to the general relativity-quantum mechanics problem.
When describing natural phenomena at the quantum scale, the interaction between the mechanical object under investigation and the observer (or, observing equipment) is not negligible and cannot be predicted \[5–7\]. This fact leads to the impossibility of unambiguously distinguishing between the object and the measuring instruments. This, in turn, is logically incompatible with the classical notion of causality; the possibility of sharply distinguishing between the subject and the object is essential to the ideal of causality. In his attempt to bring consistency in science, Bohr proposed to replace the classical ideal of causality with a more general viewpoint termed complementarity. Roughly speaking, anyone can understand that one cannot bow in front of somebody without showing one’s back to somebody else. This oversimplified statement is behind one of the most revolutionary scientific concepts of the twentieth century, namely Bohr’s complementarity principle \[8\]. This principle is a key feature of quantum physics and represents the dichotomy between the corpuscular (particle) and ondulatory (wave) nature of mechanical objects (matter and light). Within this descriptive framework, particle and wave properties are symbolized by well-defined position and momentum, respectively \[9\].

In 1935, using the complementarity principle, Bohr criticized the Einstein, Podolsky, and Rosen conclusion according to which the quantum mechanical description of physical reality given by wave functions was not complete based on a line of reasoning relying on a thought experiment (gedankenexperiment) \[10\]. In addition to criticizing the Einstein-Podolsky-Rosen argumentation (EPR paradox), Bohr provided a different interpretation of the concept of locality \[5\]. The solution to the EPR paradox is due to John Bell’s 1964 theorem \[11\]. The notion of causality played a key-role in the EPR paradox, Bohr’s complementarity principle, and Bell’s theorem.

In this article, we reexamine Gisin’s 1991 original proof concerning the violation of Bell’s inequality for any pure entangled state of two-particle systems \[12\]. Our investigation is motivated by didactic reasons and permits to straighten a few mathematical points in the original proof that in no way modify the physical content provided by Gisin’s work.

II. BACKGROUND

Einstein, Podolsky, and Rosen considered a composite quantum system consisting of two distant particles, with an entangled wave function \(\psi\) given by \[10\],

\[
\psi = \delta (x_1 - x_2 - L) \delta (p_1 + p_2).
\]  (3)
The following background discussion follows very closely the presentation presented by Asher Peres in Ref. [13]. The quantity $\delta$ denotes a normalizable function with an arbitrarily high and narrow peak. The quantity $L$ is a large distance, much larger than the range of mutual interaction of particles 1 and 2. The physical meaning of the wave function in Eq. (3) is that the two particles have been prepared in such a way that their relative distance is arbitrarily close to $L$, and their total momentum is arbitrarily close to zero. Note that the operators $x_1 - x_2$ and $p_1 + p_2$ commute. In the state $\psi$, one knows nothing about the positions of the individual particles (we only know their distance from each other); and one knows nothing of their individual momenta (one only knows the total momentum). However, if one measures $x_1$, one shall be able to predict with certainty the value of $x_2$, without having in any way disturbed particle 2. At this point, EPR argue that since at the time of measurement the two systems no longer interact, no real change can take place in the second system in consequence of anything that may be done to the first system. Therefore, $x_2$ corresponds to an element of physical reality, as defined by EPR. On the other hand, if one prefers to perform a measurement of $p_1$ rather than $x_1$, one shall then be able to predict with certainty the value of $p_2$, again without having in any way disturbed particle 2. Therefore, by the same argument as above, $p_2$ also corresponds to an element of reality. However, quantum mechanics precludes the simultaneous assignment of precise values to both $x_2$ and $p_2$, since these operators do not commute, and thus EPR are forced to conclude that the quantum mechanical description of physical reality given by wave functions is not complete. However, they prudently leave open the question of whether or not a complete description exists. Reality according to EPR can be described as follows: If, without in any way disturbing a system, one can predict with certainty the value of a physical quantity, then there exists an element of reality corresponding to this physical quantity.

The EPR article was not wrong, but it had been written too early [14]. The EPR argument did not take into account that the observer’s information was localized, like any other physical object. Information is not just an abstract notion. It requires a physical carrier, and the latter is approximately localized.

Let us recall that Einstein’s locality principle asserts that events occurring in a given spacetime region are independent of external parameters that may be controlled, at the same moment, by agents located in distant spacetime regions. In quantum mechanics, one has to accept that a measurement on what seems to be a part of the system is to be considered as a measurement on the whole system. If one persists with keeping Einstein’s locality principle, alternative theories incorporating such a principle lead to a testable inequality (Bell’s inequality, [11]) relation among suitable observables that are not in agreement with the predictions of quantum mechanics. Several violations of Bell’s inequality have been experimentally verified [13]. For this reason, despite the psychological uncomfortable situation,
quantum mechanics has prevailed over alternative theories. *Dura lex sed lex:* this is the experimental verdict. Quantum mechanics predictions are incompatible with Bell’s inequality. There is an experimentally verifiable difference between quantum mechanics and the alternative theories satisfying Einstein’s locality principle. It is somewhat of ironic that Bell’s theorem is the most profound discovery of science because it is not obeyed by experimental facts \[15\]. We remark that Bell’s paper is not about quantum mechanics. Rather, it is a general proof, independent of any specific physical theory, that there is an upper limit to the correlation of distant events, if one solely assumes the validity of Einstein’s principle of local causes. Specifically, Bell showed that in a theory in which parameters are added to determine the results of individual measurements, without changing the statistical predictions, there must be a mechanism whereby the setting of one measuring device can influence the reading of another instrument, however remote. Moreover, the signal involved must propagate instantaneously, so that such a theory could not be Lorenz invariant. In particular, for any nonfactorable quantum state $\psi$, it is possible to find pairs of observables whose correlations violate Bell’s inequality (see Eq. [1]). This means that, for such a state, quantum theory makes statistical predictions which are incompatible with the demand that the outcomes of experiments performed at a given location in space be independent of the arbitrary choice of other experiments that can be performed, simultaneously, at distant locations (this apparently reasonable demand, as stated earlier, is the principle of local causes, also called Einstein locality). Bell’s theorem implies that quantum mechanics is incompatible with the view that physical observables possess pre-existing values independent of the measurement context. A hidden variable theory which would predict individual events must violate the canons of special relativity: there would be no covariant distinction between cause and effect. The EPR paradox is resolved in the way which Einstein would have liked least.

III. REEXAMINATION

Here, we choose to present Bell’s Theorem as presented by Gisin in \[12\] (for a later and stronger derivation by Gisin and Peres, we refer to \[16\]). In what follows, whenever helpful to the discussion, we shall be using the conventional Dirac bra-ket notation where a wave vector $\psi$ will be denoted as $|\psi\rangle$.

Gisin’s Theorem can be stated as follows \[12\]: Let $\psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$. If $\psi$ is entangled (i.e. $\psi$ is not a product), then $\psi$ violates Bell’s inequality. In other words, there are projectors $a, a', b, b'$, such that

$$|P(a, b) - P(a', b')| + P(a', b) + P(a', b') > 2,$$  

(4)
where

\[ P(a, b) \overset{\text{def}}{=} \langle (2a - 1) \otimes (2b - 1) \rangle_\psi . \]  

(5)

The proof proceeds as follows.

• First, from the Schmidt Decomposition Theorem [17], if \( \psi \) is a vector in a tensor product space \( \mathcal{H}_1 \otimes \mathcal{H}_2 \), then there exists an orthonormal basis \( \{ \varphi_i \} \) with \( 1 \leq i \leq n_{\mathcal{H}_1} \) for \( \mathcal{H}_1 \) where \( n_{\mathcal{H}_1} \overset{\text{def}}{=} \text{dim}_\mathbb{C} \mathcal{H}_1 \), and an orthonormal basis \( \{ \theta_j \} \) with \( 1 \leq j \leq n_{\mathcal{H}_2} \) for \( \mathcal{H}_2 \) where \( n_{\mathcal{H}_2} \overset{\text{def}}{=} \text{dim}_\mathbb{C} \mathcal{H}_2 \), and non-negative real numbers \( \{ c_i \} \) so that

\[ \psi = \sum_k c_k \varphi_k \otimes \theta_k . \]  

(6)

The coefficients \( c_k \) are called Schmidt coefficients, and the number of terms in the sum in Eq. (6) will be at most the minimum of \( n_{\mathcal{H}_1} \) and \( n_{\mathcal{H}_2} \). Without loss of generality, in order to have a non-product state of two-particle systems, assume that at least two coefficients are nonzero, \( c_1 \neq 0 \neq c_2 \) while \( c_k = 0 \) for any \( k > 2 \). We point out that, in principle, coefficients \( c_k \) with \( k > 2 \) could also be nonzero. However, since the original Clauser-Horne-Shimony-Holt (CHSH, [18]) version of Bell’s inequality can be violated in any two dimensional Hilbert subspace with nonzero Schmidt coefficient [11, 19], without loss of generality, one can restrict the discussion to the case in which \( c_k = 0 \) for any \( k > 2 \). For the sake of completeness, we point out that the projection of the joint state of \( n \) pairs of particles onto a subspace spanned by states having a common Schmidt coefficients is a key step in the so-called Schmidt projection method, a technique used to study entanglement concentration in any pure state of a bipartite system [20]. Furthermore, for a practical demonstration of the violation of the CHSH version of Bell’s inequality in \( 2D \) subspaces of an higher-dimensional orbital angular momentum Hilbert space, we refer to [21].

• Second, recall that for any separable quantum state, the measure of entanglement is zero. Furthermore, the behavior of entanglement remains unchanged under simple local transformations, i.e. local unitary transformations [22]. A local unitary transformation simply represents a change of basis in which we consider the given entangled state. A change of basis should not change the amount of entanglement that is accessible to us, because at any given time we could just reverse the basis change. Therefore in both bases the entanglement should be the same. Given these remarks, for the sake of convenience and without changing the entanglement behavior of the state \( \psi \), let us apply a local (product) unitary transformation \( U_1 \otimes U_2 \) on \( \psi \) such that,

\[ \psi \rightarrow (U_1 \otimes U_2) \psi = (U_1 \otimes U_2) (c_1 \varphi_1 \otimes \theta_1 + c_2 \varphi_2 \otimes \theta_2) , \]  

(7)
where,

\[(U_1 \otimes U_2) (c_1 \varphi_1 \otimes \theta_1 + c_2 \varphi_2 \otimes \theta_2) = c_1 U_1 \varphi_1 \otimes U_2 \theta_1 + c_2 U_1 \varphi_2 \otimes U_2 \theta_2 = c_1 |+\rangle + c_2 |-\rangle. \tag{8} \]

In the particular case of spin 1/2 systems, we have that \(|+\rangle\) and \(|-\rangle\) in Eq. (8) are given by,

\[|+\rangle \overset{\text{def}}{=} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ and } |-\rangle \overset{\text{def}}{=} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{9} \]

respectively.

• Third, an arbitrary density matrix \(\rho\) for a mixed state qubit can be written as \[\rho \overset{\text{def}}{=} I + \vec{P} \cdot \vec{\sigma}, \tag{10}\]

where \(\vec{P} \in \mathbb{R}^3\) with \(\|\vec{P}\| \leq 1\) is the Bloch vector for the state \(\rho\), \(I\) is the \(2 \times 2\) identity matrix and \(\vec{\sigma}\) is the Pauli matrix vector given by \[\vec{\sigma} = [\sigma_x, \sigma_y, \sigma_z] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{11}\]

A state \(\rho\) is pure if and only if \(\|\vec{P}\| = 1\). A density matrix for a pure state can be described as \(\rho = |\psi\rangle \langle \psi|\) where \(|\psi\rangle \langle \psi|\) is an orthogonal projector since \(\rho^2 = \rho\) and \(\rho^\dagger = \rho\) (the symbol \(\dagger\) denotes the usual Hermitian conjugation operation in quantum mechanics).

Having considered the above mentioned three points, the idea is to check the correctness of the inequality in Eq. (4) for entangled states \(\psi\) in Eq. (6) and for a suitable choice of projectors \(a, a', b, b'\). Consider projectors \(a, a', b, b'\) defined as,

\[a \overset{\text{def}}{=} \frac{1 + \vec{a} \cdot \vec{\sigma}}{2}, \quad a' \overset{\text{def}}{=} \frac{1 + \vec{a}' \cdot \vec{\sigma}}{2}, \quad b \overset{\text{def}}{=} \frac{1 + \vec{b} \cdot \vec{\sigma}}{2}, \quad b' \overset{\text{def}}{=} \frac{1 + \vec{b}' \cdot \vec{\sigma}}{2}, \tag{12}\]

with \(\vec{a}, \vec{a'}, \vec{b}, \vec{b'} \in \mathbb{R}^3\), and

\[\|\vec{a}\| = \|\vec{a}'\| = \|\vec{b}\| = \|\vec{b}'\| = 1, \tag{13}\]

so that \(a^2 = a, a'^2 = a', b^2 = b, b'^2 = b'\). From Eq. (12), \(P(a, b)\) in Eq. (5) becomes

\[P(a, b) \overset{\text{def}}{=} \langle (2a - 1) \otimes (2b - 1) \rangle_\psi = \langle (\vec{a} \cdot \vec{\sigma}) \otimes (\vec{b} \cdot \vec{\sigma}) \rangle_\psi, \tag{14}\]
with $|\psi\rangle \overset{\text{def}}{=} c_1 \left| + + \right> + c_2 \left| - + \right>$ and $c_1, c_2 \in \mathbb{R} \setminus \{0\}$. Let us compute the explicit expression of $P(a, b)$. Using Eq. (11), we obtain

$$
(a \cdot \sigma) \otimes (b \cdot \sigma) = \left( \begin{array}{ccc}
 a_z b_z & a_z (b_x - ib_y) & b_z (a_x - ia_y) \\
 a_z (b_x + ib_y) & -a_z b_z & (a_x - ia_y) (b_x + ib_y) \\
 b_z (a_x + ia_y) & (a_x + ia_y) (b_x - ib_y) & -a_z b_z \\
 (a_x + ia_y) (b_x + ib_y) & -b_z (a_x + ia_y) & -a_z (b_x + ib_y) \\
\end{array} \right) .
$$

Therefore the quantity $P(a, b)$ becomes,

$$
P(a, b) = \langle (a \cdot \sigma) \otimes (b \cdot \sigma) \rangle _\psi = \langle \psi | (a \cdot \sigma) \otimes (b \cdot \sigma) | \psi \rangle
$$

$$
= \left( \begin{array}{ccc}
 0 & c_1 & c_2 \\
 c_1 & c_2 & 0 \\
\end{array} \right) .
$$

$$
= - \left( c_1^2 + c_2^2 \right) a_z b_z + c_1 c_2 (2a_x b_x + 2a_y b_y) ,
$$

that is,

$$
P(a, b) = - \left( c_1^2 + c_2^2 \right) a_z b_z + c_1 c_2 (2a_x b_x + 2a_y b_y) .
$$
Normalization of the wave function demands $\langle \psi | \psi \rangle = 1$. Therefore, we have $c_1^2 + c_2^2 = 1$ and,

$$P(a, b) = +2c_1c_2(a_xb_x + a_yb_y) - a_zb_z.$$  \hfill (18)

This simple mathematical computation leading to Eq. (18) leads to rectify the incorrect sign that appears in [12]. Following Gisin, we assume that convenient expressions for the Bloch vectors are given by,

$$\vec{a} = (a_x, a_y, a_z) = (\sin \alpha, 0, \cos \alpha),$$
$$\vec{b} = (b_x, b_y, b_z) = (\sin \beta, 0, \cos \beta),$$  \hfill (19)

with $\alpha = 0$, $\alpha' = \pm \pi/2$ where the sign is the same as that of $c_1c_2$ (note that due to the previous sign mistake, in [12] it is reported that $\alpha' = \pm \pi/2$ where the sign is the opposite of that of $c_1c_2$). For possible alternative parametrizations of the Bloch vectors, we refer to Appendix A. Let us now consider the expression given by $|P(a, b) - P(a, b')| + P(a', b) + P(a', b')$. Using Eqs. (5), (12), and (19), we obtain

$$|P(a, b) - P(a, b')| = |\cos \beta + \cos \beta'| = |-(\cos \beta - \cos \beta')| = |\cos \beta - \cos \beta'|.$$  \hfill (20)

Furthermore, using the same line of reasoning, $P(a', b) + P(a', b')$ becomes

$$P(a', b) + P(a', b') = (2c_1c_2 \sin \alpha' \sin \beta - \cos \alpha' \cos \beta) + (2c_1c_2 \sin \alpha' \sin \beta' - \cos \alpha' \cos \beta')$$

$$= 2c_1c_2 \sin \alpha' (\sin \beta + \sin \beta') - \cos \alpha' (\cos \beta + \cos \beta'),$$  \hfill (21)

that is,

$$P(a', b) + P(a', b') = 2c_1c_2 \sin \alpha' (\sin \beta + \sin \beta') - \cos \alpha' (\cos \beta + \cos \beta').$$  \hfill (22)

Assuming $c_1c_2 > 0$ and $\alpha' = +\pi/2$, yields

$$P(a', b) + P(a', b') = 2 |c_1c_2| (\sin \beta + \sin \beta').$$  \hfill (23)

Moreover, assuming $c_1c_2 < 0$ and $\alpha' = -\pi/2$, we get

$$P(a', b) + P(a', b') = -2c_1c_2 (\sin \beta + \sin \beta'),$$  \hfill (24)

with $-2c_1c_2 > 0$. Therefore $2 |c_1c_2| = -2c_1c_2 > 0$, and

$$P(a', b) + P(a', b') = 2 |c_1c_2| (\sin \beta + \sin \beta').$$  \hfill (25)
In summary, the expression for \[ |P(a, b) - P(a', b')| + P(a', b) + P(a', b') \] is given by

\[
|P(a, b) - P(a', b')| + P(a', b) + P(a', b') = |\cos \beta - \cos \beta'| + 2|c_1c_2|(\sin \beta + \sin \beta').
\] (26)

At this point, choose \( \beta \) and \( \beta' \) such that \( \sin \beta > 0 \) and \( \sin \beta' > 0 \) so that \( 2|c_1c_2|(\sin \beta + \sin \beta') > 0 \). Take also \( -\cos \beta' = \cos \beta = (1 + 4|c_1c_2|)^{-1/2} \). We finally obtain,

\[
|P(a, b) - P(a', b')| + P(a', b) + P(a', b') \geq 2(1 + 4|c_1c_2|)^{-1/2} > 2.
\] (27)

This concludes the proof.

IV. CONCLUSIONS

In this article, we presented a simple and explicit reexamination of Gisin’s 1991 original proof concerning the violation of Bell’s inequality for any entangled state of two-particle systems. Gisin’s original work on Bell’s inequality was presented in a very synthetic and intuitive manner. After so many years, Gisin’s work continues to be highly regarded also in pure research [27]. For this reason, we have considered it especially helpful for didactic purposes.

For a different alternative pedagogical presentation of Bell’s inequality, we refer to [28, 29]. We remark that we have simply expanded the original proof and in doing so, we have rectified with Eq. (18) an incorrect mathematical sign in the expression of \( P(a, b) \) in [12] that now becomes,

\[
P(a, b) = +2c_1c_2(a_xb_x + a_yb_y) - a_zb_z.
\] (28)

The incorrectness of the sign in \( P(a, b) \) as reported in [12] generates and propagates an incorrect statement. As a consequence of Eq. (28), we have also corrected such a subsequent statement. Namely, the correct statement becomes

\[
\alpha' = \pm \pi/2 \text{ where the sign is the same as that of } c_1c_2.
\] (29)

Notwithstanding these two corrections presented in Eqs. (28) and (29), the physical conclusions provided by Gisin remain unaltered.

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Appendix A: Alternative parametrizations of the Bloch vectors

In general, the Bloch vectors $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^3$ that appear in Eq. (12) can be parametrized in terms of the polar angle $\theta$ with $0 \leq \theta \leq \pi$ and the azimuthal angle $\varphi$ with $0 \leq \varphi < 2\pi$ as follows,

$$\vec{a} = (a_x, a_y, a_z) = (\sin \theta \cos \varphi_1, \sin \theta \sin \varphi_1, \cos \theta_1) ,$$  \hspace{1cm} (A1)

and,

$$\vec{b} = (b_x, b_y, b_z) = (\sin \theta_2 \cos \varphi_2, \sin \theta_2 \sin \varphi_2, \cos \theta_2) ,$$  \hspace{1cm} (A2)

respectively. Using Eqs. (A1) and (A2), the quantity $P(a, b)$ that appears in Eq. (4) becomes

$$P(a, b) = 2c_1c_2 [\sin \theta_1 \sin \theta_2 \cos \varphi_1 \cos \varphi_2 + \sin \theta_1 \sin \theta_2 \sin \varphi_1 \sin \varphi_2 ] - \cos \theta_1 \cos \theta_2 .$$  \hspace{1cm} (A3)

Similar expressions are valid for $P(a, b')$, $P(a', b)$, and $P(a', b')$ in Eq. (4). It would be interesting to provide a thorough numerical investigation for the conditions on the vectors $\vec{a}$ and $\vec{b}$ in Eqs. (A1) and (A2) such that Bell’s inequality is violated. Here, however, we simply point out that Gisin’s original parametrization is not unique and alternative parametrizations may be considered, preserving essentially Gisin’s original line of reasoning. Gisin’s choice was to set $\varphi_1 = \varphi_2 = 0$ in Eqs. (A1) and (A2), respectively, leading to

$$\vec{a} = (\sin \theta_1, 0, \cos \theta_1) , \text{ and } \vec{b} = (\sin \theta_2, 0, \cos \theta_2) .$$  \hspace{1cm} (A4)

Two natural alternatives are the following: i) $\varphi_1 = \varphi_2 = \pi/2$; ii) $\theta_1 = \theta_2 = \pi/2$. In the former and latter cases, we obtain

$$\vec{a} = (0, \sin \theta_1, \cos \theta_1) , \text{ and } \vec{b} = (0, \sin \theta_2, \cos \theta_2) ,$$  \hspace{1cm} (A5)

and,

$$\vec{a} = (\cos \varphi_1, \sin \varphi_1, 0) , \text{ and } \vec{b} = (\cos \varphi_2, \sin \varphi_2, 0) ,$$  \hspace{1cm} (A6)

respectively. Following the line of reasoning presented in this manuscript, it is relatively straightforward to show that Gisin’s argument concerning Bell’s inequality applies essentially in a similar manner to these alternative parametrizations of the Bloch vectors.