A Remark on the Manhattan Distance Matrix of a Rectangular Grid

A. Y. Alfakih

Department of Mathematics and Statistics
University of Windsor
Windsor, Ontario N9B 3P4
Canada.

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Abstract

Consider the Quadratic Assignment Problem (QAP): given two matrices $A$ and $D$, minimize \{ trace ($AXDX^T$) : $X$ is a permutation matrix \}. New lower bounds were obtained recently (Mittelmann and Peng [8]) for the QAP where $D$ is either the Manhattan distance matrix of a rectangular grid or the Hamming distance matrix of a hypercube. In this note, we show that the results in [8, 11] extend to the case where $D$ is a spherical Euclidean distance matrix, which includes the Manhattan distance matrix and the Hamming distance matrix as special cases.

1 Introduction

The Quadratic Assignment Problem (QAP)( see, e.g., [4] and references therein) is the problem:

$$\min_{X \in \Pi} \text{trace}(AXDX^T),$$

where $A$ and $D$ are two given matrices of order $n$ and $\Pi$ is the set of $n \times n$ permutation matrices. New lower bounds were obtained recently (Mittelmann and Peng [8]) for the
QAP where $D$ is either the Manhattan distance matrix of a rectangular grid or the Hamming distance matrix of a hypercube. Mittelmann and Peng exploit the fact that for such matrices $D$ there exists a scalar $\lambda$ such that $\lambda E - D$ is positive semidefinite, where $E$ is the matrix of all 1’s. In this note, we show that the results in [8, 11] extend to the case where $D$ is a spherical Euclidean distance matrix (EDM), which includes the Manhattan distance matrix and the Hamming distance matrix as special cases.

1.1 Notation

The positive semidefiniteness of a symmetric real matrix $A$ is denoted by $A \succeq 0$. The set of symmetric real matrices of order $n$ is denoted by $S_n$. $I_n$ denotes the identity matrix of order $n$ and $e_n$ denotes the vector of all 1’s in $\mathbb{R}^n$. $E_n$ denotes the $n \times n$ matrix of all 1’s, i.e., $E_n = e_n e_n^T$. The subscript $n$ will be omitted if the dimension of $I_n$, $e_n$ and $E_n$ is clear from the context. $A^\dagger$ denotes the Moore-Penrose inverse of $A$. For two matrices $A$ and $B$, $A \otimes B$ denotes the Kronecker product of $A$ and $B$. For a real number $x$, $\lceil x \rceil$ denote the ceiling of $x$. Finally, diag($A$) denotes the vector consisting of the diagonal entries of a matrix $A$.

2 Preliminaries

The following well-known [3] lemmas will be needed in some of the proofs below.

Lemma 2.1 Let $A$ and $B$ be two $n \times n$ real matrices.

1. $(A^T)^\dagger = (A^\dagger)^T := A^\dagger T$.
2. $(AA^T)^\dagger = A^\dagger T A^\dagger$.
3. If $A^T B = 0$ and $B A^T = 0$, then $(A + B)^\dagger = A^\dagger + B^\dagger$.
4. If $A^T B = 0$ and $B A^T = 0$, then rank $(A + B) = \text{rank } A + \text{rank } B$.
5. If $A$ has full column rank then $A^\dagger = (A^T A)^{-1} A^T$.
6. Let $C$ and $Q$ be two real matrices of orders $k \times k$ and $n \times k$ respectively, where $Q^T Q = I_k$. Then $(QCQ^T)^\dagger = QC^\dagger Q^T$.

Lemma 2.2 Let $A$ and $B$ be two $n \times n$ real matrices.

1. $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$.
2. Let $A \succeq 0$ and $B \succeq 0$. Then $(A \otimes B) \succeq 0$. 

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3. \( \text{rank}(A \otimes B) = \text{rank}(A) \text{rank}(B) \).

**Lemma 2.3 (Generalized Schur Complement)** Given the real symmetric partitioned matrix

\[
M = \begin{bmatrix}
A & B \\
B^T & C
\end{bmatrix}.
\]

Then \( M \succeq 0 \) if and only if \( C \succeq 0, \ A - BC^\dagger B^T \succeq 0, \) and the null space of \( C \) is a subset of the null space of \( B \).

### 3 Euclidean Distance Matrices (EDMs)

An \( n \times n \) matrix \( D = (d_{ij}) \) is called a **Euclidean distance matrix (EDM)** if there exist points \( p^1, \ldots, p^n \) in some Euclidean space \( \mathbb{R}^r \) such that

\[
d_{ij} = \|p^i - p^j\|^2 \quad \text{for all } i, j = 1, \ldots, n,
\]

where \( \| \| \) denotes the Euclidean norm. Moreover, the dimension of the affine span of the points \( p^1, \ldots, p^n \) is called the **embedding dimension** of \( D \). Without loss of generality, we make the following assumption.

**Assumption 3.1** The origin coincides with the centroid of the points \( p^1, \ldots, p^n \).

It is well known [12, 14] that a symmetric matrix \( D \) whose diagonal entries are all 0’s is an EDM if and only if \( D \) is negative semidefinite on the subspace

\[
M := \{ x \in \mathbb{R}^n : e^T x = 0 \},
\]

where \( e \) is the vector of all 1’s in \( \mathbb{R}^n \). It easily follows that the orthogonal projection on \( M \) is given by

\[
J := I - ee^T / n = I - E / n. \tag{1}
\]

Let \( S_n \) denote the set of symmetric \( n \times n \) real matrices and let \( S_H \) and \( S_C \) be two subspaces of \( S_n \) such that:

\[
S_H = \{ A \in S_n : \text{diag}(A) = 0 \} \quad \text{and} \quad S_C = \{ A \in S_n : Ae = 0 \}.
\]

Furthermore, let [5] \( \mathcal{T} : S_H \to S_C \) and \( \mathcal{K} : S_C \to S_H \) be the two linear maps defined by

\[
\mathcal{T}(D) := -\frac{1}{2} JDJ, \tag{2}
\]

\[
\mathcal{K}(B) := \text{diag}(B) e^T + e (\text{diag}(B))^T - 2 B. \tag{3}
\]
Then it immediately follows that $T$ and $K$ are mutually inverse between the two subspaces $S_H$ and $S_C$; and that $D$ is an EDM of embedding dimension $r$ if and only if the matrix $T(D)$ is positive semidefinite of rank $r$ [3]. Moreover, it is not difficult to show (see, e.g., [1, 6, 13]) that if $D$ is an EDM of embedding dimension $r$ then rank $(D)$ is equal to either $r + 1$ or $r + 2$.

Note that if $D$ is an EDM of embedding dimension $r$ then $T(D)$ is the Gram matrix of the points $p^1, \ldots, p^n$, i.e., $T(D) = PP^T$, where $P$ is $n \times r$ and $p^iT$ is the $i$th row of $P$. i.e.,

$$P = \begin{bmatrix} p^1T \\ \vdots \\ p^nT \end{bmatrix}.$$  

(4)

$P$ is called a configuration matrix of $D$. Also, note that $P^Te = 0$ which is consistent with our assumption that the origin coincides with the centroid of the points $p^1, \ldots, p^n$.

### 3.1 Spherical EDMs

An EDM $D$ is called spherical if the points $p^1, p^2, \ldots, p^n$ that generate $D$ lie on a hypersphere. Otherwise, $D$ is called non-spherical. The following characterization of spherical EDMs is well known.

**Theorem 3.1** ([10], [6], [13]) Let $D \neq 0$ be a given $n \times n$ EDM with embedding dimension $r$, and let $P$ be a configuration matrix of $D$ such that $P^Te = 0$. Then the following statements are equivalent.

1. $D$ is a spherical EDM.
2. Rank $(D) = r + 1$.
3. The matrix $\lambda ee^T - D$ is positive semidefinite for some scalar $\lambda$.
4. There exists a vector $a$ in $\mathbb{R}^r$ such that:

$$Pa = \frac{1}{2}J\text{diag}(T(D)),$$

(5)

where $J$ is as defined in (1).

Two remarks are in order here. First, spherical EDMs can also be characterized in terms of Gale transform [2] which lies outside the scope of this paper. Second, if $D$ is an $n \times n$ spherical EDM then the points that generate $D$ lie on a hypersphere of center $a$ and radius $\rho = (a^Ta + e^TDe/2n^2)^{1/2}$, where $a$ is given in [5] (see [13]). Furthermore, as the next theorem shows, the minimum value of $\lambda$ such that $\lambda E - D \succeq 0$ is closely related to $\rho$. 


Theorem 3.2 ([10, 13]) Let $D$ be a spherical EDM generated by points that lie on a hypersphere of radius $\rho$. Then $\lambda^* = 2\rho^2$ is the minimum value of $\lambda$ such that $\lambda E - D \succeq 0$.

The following lemma gives an explicit expression for the radius $\rho$ which is more convenient for our purposes.

Lemma 3.1 Let $D$ be a spherical EDM generated by points that lie on a hypersphere of radius $\rho$. Then

$$\rho^2 = \frac{e^T De}{2n^2} + \frac{e^T D(T(D))^\dagger De}{4n^2}. \quad (6)$$

Proof. Let $D$ be a spherical EDM and let $Q = [\begin{smallmatrix} \sqrt{n} \\ V \end{smallmatrix}]$ be an orthogonal matrix, i.e., $J = VV^T$. Then $\lambda E - D \succeq 0$ if and only if $Q^T(\lambda E - D)Q \succeq 0$. But

$$Q^T(\lambda E - D)Q = \begin{bmatrix} \lambda n - e^T De/n & -e^T DV/\sqrt{n} \\ -V^T De/\sqrt{n} & -V^T DV \end{bmatrix}. $$

Clearly $-V^T DV = 2V^T T(D)V \succeq 0$. Thus it follows from Lemma 3.2 that $\lambda E - D \succeq 0$ if and only if $\lambda n - e^T De/n - e^T DV(V^T T(D)V)^\dagger V^T De/2n \geq 0$ and the null space of $V^T DV$ is a subset of the null space $e^T DV$. But it follows from Property 3.2 of Lemma 2.1 that $V(V^T T(D)V)^\dagger V^T = (V^T T(D)V V^T)^\dagger = T(D)^\dagger$ since $V V^T = J$ and since $(T(D))e = 0$. Moreover, since $D$ is a spherical EDM, it follows that the null space of $V^T DV$ is a subset of the null space $e^T DV$ [2]. Thus, $\lambda E - D \succeq 0$ if and only if $\lambda n - e^T De/n - e^T D(T(D))^\dagger De/2n \geq 0$ and the result follows by Theorem 3.2.

Note that $(T(D))^\dagger = (PP^T)^\dagger = P^{\dagger T}P^{\dagger}$. Thus it follows from (6) that $\rho^2 = a^T a + e^T De/2n^2 = e^T De/2n^2 + e^T DP^{\dagger T}P^{\dagger} De/4n^2$. Therefore,

$$a = \frac{1}{2n}P^{\dagger} De = \frac{1}{2n}(P^{T}P)^{-1}P^{T} De, \quad (7)$$

where $P$ is a configuration matrix of $D$.

An interesting class of spherical EDMs is that of regular EDMs. A spherical EDM $D$ is said to be regular (also called EDM of strength 1 [9, 10]) if $D$ is generated by points that lie on a hypersphere centered at the centroid of these points [7], i.e.,

$$na = P^T e = 0,$$

since we assume that the centroid of the points $p^1, \ldots, p^n$ is located at the origin. Therefore, if $D$ is a regular EDM then $\text{diag}(T(D)) = \rho^2 e$. Hence, it follows from (3) that

$$De = 2n\rho^2 e,$$

where $\rho^2 = e^T De/2n^2$. Thus we have the following characterization of regular EDMs.
Theorem 3.3 \([7, 9, 10]\) Let \(D\) be an EDM. Then \(D\) is regular if and only if \(e\) is an eigenvector of \(D\).

4 Main Results

Theorem 4.1 Let \(D_1\) be an \(m \times m\) EDM of embedding dimension \(r_1\), and let \(D_2\) be an \(n \times n\) EDM of embedding dimension \(r_2\). Then \(D = E_m \otimes D_2 + D_1 \otimes E_n\) is an EDM of embedding dimension \(r_1 + r_2\), where \(E_m\) is the \(m \times m\) matrix of all 1’s and \(\otimes\) denotes the Kronecker product.

Proof. \(I_{nm} = I_m \otimes I_n\) and \(E_{nm} = E_m \otimes E_n\). Thus
\[
\mathcal{T}(E_m \otimes D_2) = -\frac{1}{2}(I_m \otimes I_n - \frac{1}{nm}E_m \otimes E_n)(E_m \otimes D_2)(I_m \otimes I_n - \frac{1}{nm}E_m \otimes E_n)
\]
\[
= -\frac{1}{2}E_m \otimes (I_n - \frac{1}{n}E_n)D_2(I_n - \frac{1}{n}E_n)
\]
\[
= E_m \otimes \mathcal{T}(D_2) \succeq 0.
\]
Similarly \(\mathcal{T}(D_1 \otimes E_n) = \mathcal{T}(D_1) \otimes E_n \succeq 0\). Therefore, \(\mathcal{T}(D) = E_m \otimes \mathcal{T}(D_2) + \mathcal{T}(D_1) \otimes E_n \succeq 0\). Hence \(D\) is EDM.

On the other hand, we have from Lemma 2.1 that \(\text{rank } (\mathcal{T}(D)) = \text{rank } (E_m \otimes \mathcal{T}(D_2) + \mathcal{T}(D_1) \otimes E_n) = \text{rank } (E_m \otimes \mathcal{T}(D_2)) + \text{rank } (\mathcal{T}(D_1) \otimes E_n) = \text{rank } (\mathcal{T}(D_2)) + \text{rank } (\mathcal{T}(D_1)) = r_2 + r_1\). Thus the embedding dimension of \(D\) is equal to \(r_1 + r_2\).

\(\square\)

Theorem 4.2 Let \(D_1\) be a spherical EDM of order \(m\) generated by points that lie on a hypersphere of radius \(\rho_1\), and let \(D_2\) be a spherical EDM of order \(n\) generated by points that lie on a hypersphere of radius \(\rho_2\). Then \(D = E_m \otimes D_2 + D_1 \otimes E_n\) is a spherical EDM generated by points that lie on a hypersphere of radius \(\rho = (\rho_1^2 + \rho_2^2)^{1/2}\).

Proof. \(\lambda E_{nm} - D = E_m \otimes (\lambda_2 E_n - D_2) + (\lambda_1 E_m - D_1) \otimes E_n\) where \(\lambda = \lambda_1 + \lambda_2\). Since \(D_1\) and \(D_2\) are spherical EDMs it follows that \(\lambda_2 E_n - D_2 \succeq 0\) for \(\lambda_2 = 2\rho_2^2\), and \(\lambda_1 E_m - D_1 \succeq 0\) for \(\lambda_1 = 2\rho_1^2\). Therefore, for \(\lambda = 2\rho_1^2 + 2\rho_2^2\) we have \(\lambda E_{nm} - D \succeq 0\) and hence \(D\) is a spherical EDM generated by points that lie on a hypersphere of radius \(\rho \leq (\rho_1^2 + \rho_2^2)^{1/2}\). Next we show that \(\rho = (\rho_1^2 + \rho_2^2)^{1/2}\).
By Lemma 3.1 we have
\[
\rho^2 = \frac{e^T De}{2(nm)^2} + \frac{e^T D(T(D))\,\!^\dagger\! De}{4(nm)^2} \\
= \frac{e^T D_2 e_n}{2n^2} + \frac{e^T D_1 e_m}{2m^2} \\
+ \{(me_m^T \otimes e_n^T D_2 + e_m^T D_1 \otimes ne_n^T)(E_m \otimes (T(D_2)) + (T(D_1))\,\!^\dagger\! \otimes \frac{E_n}{m^2})\}... \\
\ldots (me_m \otimes D_2 e_n + D_1 e_m \otimes ne_n)}/4(nm)^2 \\
= \frac{e^T D_2 e_n}{2n^2} + \frac{e^T D_1 e_m}{2m^2} \\
+ \{(me_m^T \otimes e_n^T D_2 + e_m^T D_1 \otimes ne_n^T)(\frac{E_m}{m^2} \otimes (T(D_2)) + (T(D_1))\,\!^\dagger\! \otimes \frac{E_n}{m^2})\}... \\
\ldots (me_m \otimes D_2 e_n + D_1 e_m \otimes ne_n)}/4(nm)^2 \\
= \frac{e^T D_2 e_n}{2n^2} + \frac{e^T D_1 e_m}{2m^2} + \{m^2 e_n^T D_2 (T(D_2))\,\!^\dagger\! D_2 e_n \\
+ n^2 e_m^T D_1 (T(D_1))\,\!^\dagger\! D_1 e_m\}/(4nm)^2 \\
= \rho_2^2 + \rho_1^2
\]

\[\square\]

**Example 4.1** Let \(G_n = (g_{ij})\) be the Manhattan distance matrix of a rectangular grid consisting of 1 row and \(n\) columns. Then
\[g_{ij} = |i - j|\]

Let \(q^1, \ldots, q^n\) be the points in \(\mathbb{R}^{n-1}\) such that the first \(i - 1\) coordinates of \(q^i\) are 1’s and the remaining \(n - i\) coordinates are 0’s. Then it immediately follows that \(q^1, \ldots, q^n\) generate \(G_n\) and \(q^1, \ldots, q^n\) lie on a hypersphere of radius \(\rho = \frac{1}{2}(n - 1)^{1/2}\) and centered at \(b = (1/2, 1/2, \ldots, 1/2)^T\). Thus \(G_n\) is a spherical EDM of embedding dimension \(n - 1\).

Note that in this example, we don’t assume that the origin coincides with the centroid of the points \(q^1, \ldots, q^n\) in order to keep the expressions of the \(q^i\)’s and \(b\) simple.

Now consider a rectangular grid of \(m\) rows and \(n\) columns. Let \(\hat{d}_{ij,kl}\) be the Manhattan distance between the grid point of coordinates \((i, j)\) and the grid point of coordinates \((k, l)\). Then
\[\hat{d}_{ij,kl} = |i - k| + |j - l|\]

Let \(s = i + n(j - 1)\) for \(j = 1, \ldots, m\) and \(i = 1, \ldots, n\); and let \(t = k + n(l - 1)\) for \(l = 1, \ldots, m\) and \(k = 1, \ldots, n\). Then
\[j = \lceil s/n \rceil, i = s - n(\lceil s/n \rceil - 1)\] and \(l = \lceil t/n \rceil, k = t - n(\lceil t/n \rceil - 1)\).
Define the $nm \times nm$ matrix $D = (d_{st})$ such that $d_{st} = \hat{d}_{ij,kl}$. Then it follows \[\] that

$$D = E_m \otimes G_n + G_m \otimes E_n,$$

(8)

where $G_n$ is as defined in Example 4.1. Equation (8) follows since $(E_m \otimes G_n)_{st} = |i - k|$ where $i = s - n([s/n] - 1)$ and $k = t - n([t/n] - 1)$; and since $(G_m \otimes E_n)_{kl} = |j - l|$ where $j = [s/n]$ and $l = [t/n]$.

Thus we have the following two corollaries.

**Corollary 4.1** Let $D$ be the $mn \times mn$ Manhattan distance matrix of a rectangular grid of $m$ rows and $n$ columns. Then $D$ is a spherical EDM of embedding dimension $n + m - 2$. Furthermore, the points that generate $D$ lie on a hypersphere of radius $ho = \frac{1}{2}(n + m - 2)^{1/2}$.

**Proof.** This follows from Theorems 4.1 and 4.2.

**Corollary 4.2** (Mittlemann and Peng [8, Theorem 2.6]) Let $D$ be the $mn \times mn$ Manhattan distance matrix of a rectangular grid of $m$ rows and $n$ columns. Then $(n + m - 2)E_{mn}/2 - D \succeq 0$.

**Example 4.2** Consider the Hamming distance matrix of the $r$-dimensional hypercube $Q_r$ whose vertices are the $2^r$ points in $\mathbb{R}^r$ with coordinates equal to 1 or 0. Let $p^1, \ldots, p^{2^r}$ be the vertices of $Q_r$ and let $D = (d_{ij})$ be the $2^r \times 2^r$ matrix where $d_{ij}$ is the Hamming distance between $p^i$ and $p^j$; i.e., $d_{ij} = \sum_{k=1}^{r} |p^i_k - p^j_k|$. Then $d_{ij}$ is also equal to $\sum_{k=1}^{r} (p^i_k - p^j_k)^2 = \|p^i - p^j\|^2$. Therefore, $D$ is an EDM. Furthermore, the points $p^1, \ldots, p^{2^r}$ lie on a hypersphere, centered at their centroid, of radius $\rho = \frac{1}{2} r^{1/2}$. Thus $D$ is a regular EDM of embedding dimension $r$.

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