Local times for multifractional Brownian motion in higher dimensions: A white noise approach

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Abstract

We present the expansion of the multifractional Brownian (mBm) local time in higher dimensions, in terms of Wick powers of white noises (or multiple Wiener integrals). If a suitable number of kernels is subtracted, they exist in the sense of generalized white noise functionals. Moreover we show the convergence of the regularized truncated local times for mBm in the sense of Hida distributions.
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1 Introduction

Over the last decades fractional Brownian motion (fBm) with Hurst parameter $H$ has become an intensively studied object. This centered Gaussian process $B_H$ with covariance function

$$\mathbb{E}(B_H(t)B_H(s)) = \frac{1}{2} \left( |t|^{2H} + |s|^{2H} - |t-s|^{2H} \right), \quad t, s > 0,$$

was first introduced by Mandelbrot and Van-Ness [MvN68]. Instead of giving an exhaustive overview about fBm we refer to the articles [AN03, Ben03b, Ben03a, DHPD00, EvdH03, HØ03, Nua06, DOS08] and monographs [BHØZ07, Mis08] and the references therein.

Due to its properties such as Hölder continuity of any order less than $H$, long-range dependence and stationary increments, the process is used for modeling problems from telecommunications, finance and engineering. Although there are various problems accessible, the use of fBm involves a restriction to a certain Hölder continuity $H$ of the paths for all times of the process. For many applications this is too restrictive and variable time-dependent Hölder continuities of the paths are needed.

To overcome this Lévy Véhel and Peltier [PLV95] and Benassi et.al. [BJR97] independently introduced multifractional Brownian motion (mBm) $B_h$, where the regularity of the paths is a function of time. The covariance of the centered Gaussian process $B_h$ is given by

$$\mathbb{E}(B_h(t)B_h(s)) = \frac{C \left( \frac{h(t)+h(s)}{2} \right)^2}{C(h(t))C(h(s))} \left[ \frac{1}{2} \left( t^{h(t)+h(s)} + s^{h(t)+h(s)} - |t-s|^{h(t)+h(s)} \right) \right],$$

where $h : [0, T] \rightarrow (1/2, 1)$ is a continuous function and

$$C(x) := \left( \frac{2\pi}{\Gamma(2x+1) \sin(\pi x)} \right)^{1/2},$$

where $\Gamma$ is the Gamma function. Different properties of this process are recently studied, such as Hölder continuity of the paths and Hausdorff dimension [BJR97, BDG08], as well as local times of mBm [ASX11, BDG07, ...].
MWX08] and estimates for the local Hurst parameters [BFG13]. In white noise analysis mBm was treated recently in [LLV14] together with its respective stochastic calculus.

In this article we use a white noise approach to determine the kernels in the Wiener-Itô-Segal chaos decomposition of the (truncated) local time of a $d$-dimensional mBm for $h : [0, T] \rightarrow (1/2, 1)$. As in [DOS08] for fBm, we show the convergence of the regularized local time for mBm to the truncated local time via the convergence of Hida distributions.

2 Gaussian white noise analysis

In this section we review some of the standard concepts and theorems of white noise analysis used throughout this work, and refer to [HKPS93, KLP+96, Kuo96] and references therein for a detailed presentation.

We start with the basic Gel’fand triple

$$S_d \subset L^2_d \subset S_d',$$

where $S_d := S(\mathbb{R}, \mathbb{R}^d), d \in \mathbb{N}$, is the space of vector valued Schwartz test functions, $S_d'$ its topological dual and the central Hilbert space $L^2_d := L^2(\mathbb{R}, \mathbb{R}^d)$ of square integrable vector valued functions, i.e.,

$$|f|_0^2 = \sum_{i=1}^d \int_{\mathbb{R}} f_i^2(x) \, dx, \quad f \in L^2_d.$$

Since $S_d$ is a nuclear space, represented as projective limit of a decreasing chain of Hilbert spaces $(H_p)_{p \in \mathbb{N}}$, see e.g. [RS75] and [GV68], i.e.

$$S_d = \bigcap_{p \in \mathbb{N}} H_p,$$

we have that $S_d$ is a countably Hilbert space in the sense of Gel’fand and Vilenkin [GV68]. We denote the corresponding norm on $H_p$ by $|\cdot|_p$, with the convention $H_0 = L^2_d$. Let $H_{-p}$ be the dual space of $H_p$ and let $\langle \cdot, \cdot \rangle$ denote the dual pairing on $H_{-p} \times H_p$. $H_p$ is continuously embedded into $L^2_d$. By identifying $L^2_d$ with its dual via the Riesz isomorphism, we obtain the chain $H_p \subset L^2_d \subset H_{-p}$. Note that $S_d' = \bigcup_{p \in \mathbb{N}} H_{-p}$, i.e. $S_d'$ is the inductive limit of the increasing chain of Hilbert spaces $(H_{-p})_{p \in \mathbb{N}}$, see e.g. [GV68]. We denote the dual pairing of $S_d'$ and $S_d$ also by $\langle \cdot, \cdot \rangle$. 

3
Let $\mathcal{B}$ be the $\sigma$-algebra generated by cylinder sets on $S'_d$. By Minlos’ theorem there is a unique probability measure $\mu_d$ on $(S'_d, \mathcal{B})$ with characteristic function given by
\[
\hat{e}^{(w, \phi)} d\mu_d(w) = \exp \left( -\frac{1}{2} |\phi|_0^2 \right), \quad \phi \in S_d.
\]
Hence, we have defined the white noise measure space $(S'_d, \mathcal{B}, \mu_d)$. The complex Hilbert space $L^2(\mu_d) := L^2(S'_d, \mathcal{B}, \mu_d)$ is canonically isomorphic to the Fock space of symmetric square integrable functions
\[
L^2(\mu_d) \simeq \left( \bigoplus_{k=0}^{\infty} \text{Sym} L^2(\mathbb{R}^k, k!d^k x) \right)^{\otimes d}
\]
which implies the Wiener-Itô-Segal chaos decomposition for any element $F$ in $L^2(\mu_d)$
\[
F(w) = \sum_{n \in \mathbb{N}^d} \langle :, w^{\otimes n} :, F_n \rangle
\]
with the kernel function $F_n$ in the Fock space. We introduce the following notation for simplicity
\[
n = (n_1, \ldots, n_d) \in \mathbb{N}_0^d, \quad n = n_1 + \ldots + n_d, \quad n! = n_1! \ldots n_d!
\]
and for any $w = (w_1, \ldots, w_d) \in S'_d$
\[
: w^{\otimes n} : = : w_1^{\otimes n_1} : \otimes \ldots \otimes : w_d^{\otimes n_d} :,
\]
where $: w^{\otimes n} :$ denotes the $n$-th Wick tensor power of the element $w \in S'_1$, for its definition see e.g. [HKPS93]. For any $F \in L^2(\mu_d)$, the isomorphism (1) yields
\[
\|F\|_{L^2(\mu_d)}^2 := \sum_{n \in \mathbb{N}^d} n! \|F_n\|_0^2,
\]
where the symbol $| \cdot |_0$ is also preserved for the norms on $L^2(\mathbb{R}, \mathbb{R}^d)^{\otimes n}$, for simplicity. By the standard construction with the space of square-integrable functions w.r.t. $\mu_d$ as central space, we obtain the Gel’fand triple of Hida test functions and Hida distributions.
\[
(S_d) \subset L^2(\mu_d) \subset (S_d)'.
\]
In the following we denote the dual pairing between elements of $(S_d)'$ and $(S_d)$ by $\langle \cdot, \cdot \rangle$. For $F \in L^2(\mu_d)$ and $\varphi \in (S_d)$, with kernel functions $f_n$ and $\varphi_n$, resp. the dual pairing yields

$$\langle \langle F, \varphi \rangle \rangle = \sum_n n! \langle f_n, \varphi_n \rangle$$

This relation extends the chaos expansion to $\Phi \in (S_d)'$ with distribution valued kernels $\Phi_n$ such that

$$\langle \langle \Phi, \varphi \rangle \rangle = \sum_n n! \langle \Phi_n, \varphi_n \rangle,$$

for every generalized test function $\varphi \in (S_d)$ with kernels $\varphi_n$.

Instead of reviewing the detailed construction of these spaces we give a characterization in terms of the $S$-transform.

**Definition 1.** Let $\xi \in S_d$, then $\exp(\langle \cdot, \xi \rangle) := \sum_{k=0}^{\infty} \frac{1}{k!} \langle \cdot \otimes^n \xi \otimes^n \rangle \in (S_d)$ and we define the $S$-transform of $\Phi \in (S_d)'$ by

$$(S\Phi)(\xi) = \langle \langle \Phi, \exp(\langle \cdot, \xi \rangle) \rangle \rangle.$$ 

**Definition 2 (U-functional).** A function $F : S_d \rightarrow \mathbb{C}$ is called a $U$-functional whenever

1. for every $\varphi_1, \varphi_2 \in S_d$ the mapping $\mathbb{R} \ni \lambda \mapsto F(\lambda \varphi_1 + \varphi_2)$ has an entire extension to $z \in \mathbb{C}$,

2. there are constants $K_1, K_2 > 0$ such that

$$|F(z\varphi)| \leq K_1 \exp \left( K_2 |z|^2 \|\varphi\|^2 \right), \quad \forall z \in \mathbb{C}, \varphi \in S_d$$

for some continuous norm $\| \cdot \|$ on $S_d$.

We are now ready to state the aforementioned characterization result.

**Theorem 3** (cf. [KLP+96], [PS91]). The $S$-transform defines a bijection between the space $(S_d)'$ and the space of $U$-functionals. In other words, $\Phi \in (S_d)'$ if and only if $S\Phi : S_d \rightarrow \mathbb{C}$ is a $U$-functional.
Based on Theorem 3 a deeper analysis of the space \((S_d)'\) can be done. The following Corollaries concern the convergence of sequences and the Bochner integration of families of generalized functions in \((S_d)'\) (for more details and proofs see e.g. [HKPS93], [KLP+96], [PS91]).

**Corollary 4.** Let \((\Phi_n)_{n \in \mathbb{N}}\) be a sequence in \((S_d)'\) such that

1. for all \(\varphi \in S_d\), \(((S\Phi_n)(\varphi))_{n \in \mathbb{N}}\) is a Cauchy sequence in \(\mathbb{C}\),
2. there are \(K_1, K_2 > 0\) such that for some continuous norm \(\| \cdot \|\) on \(S_d\) one has
   \[
   |(S\Phi_n)(z\varphi)| \leq K_1 \exp(K_2|z|^2\|\varphi\|^2), \quad \varphi \in S_d, \ n \in \mathbb{N}, \ z \in \mathbb{C}.
   \]

Then \((\Phi_n)_{n \in \mathbb{N}}\) converges strongly in \((S_d)'\) to a unique Hida distribution.

**Corollary 5.** Let \((\Omega, \mathcal{B}, m)\) be a measure space and \(\lambda \mapsto \Phi_\lambda\) be a mapping from \(\Omega\) to \((S_d)'\). We assume that the \(S\)-transform of \(\Phi_\lambda\) fulfills the following two properties:

1. The mapping \(\lambda \mapsto (S\Phi_\lambda)(\varphi)\) is measurable for every \(\varphi \in S_d\),
2. The \(S\Phi_\lambda\) obeys the estimate
   \[
   |(S\Phi_\lambda)(z\varphi)| \leq C_1(\lambda) \exp\left(C_2(\lambda)|z|^2\|\varphi\|^2\right), \quad z \in \mathbb{C}, \varphi \in S_d,
   \]
   for some continuous norm \(\| \cdot \|\) on \(S_d\) and for \(C_1 \in L^1(\Omega, m)\), \(C_2 \in L^\infty(\Omega, m)\).

Then
\[
\int_\Omega \Phi_\lambda \, dm(\lambda) \in (S_d)'
\]
and
\[
S\left(\int_\Omega \Phi_\lambda \, dm(\lambda)\right)(\varphi) = \int_\Omega (S\Phi_\lambda)(\varphi) \, dm(\lambda), \quad \varphi \in S_d.
\]

At the end of this section we introduce the notion of truncated kernels, defined via their Wiener-Itô-Segal chaos decomposition.

**Definition 6.** For \(\Phi \in (S_d)'\) with kernels \(F_n, n \in \mathbb{N}_0\) and \(k \in \mathbb{N}_0\) we define truncated Hida distribution by
\[
\Phi^{(k)} = \sum_{n \in \mathbb{N}_0, n \geq k} \langle \cdot \otimes n \cdot, F_n \rangle.
\]

Obviously one has \(\Phi^{(k)} \in (S_d)'\).
3 Multifractional Brownian motion

Multifractional Brownian motion (mBm) in dimension 1, was introduced by Peltier and Lévy Véhel [PLV95], Benassi et al. [BJR97] as zero mean Gaussian process with covariance given by

\[ R_h(t, s) = \frac{C\left(\frac{h(t)+h(s)}{2}\right)^2}{C(h(t))C(h(s))} \left[ \frac{1}{2}(t^{h(t)+h(s)} + s^{h(t)+h(s)} - |t - s|^{h(t)+h(s)}) \right], \]

where the normalizing constant is defined by

\[ C(x) := \left( \frac{2\pi}{\Gamma(2x + 1)\sin(\pi x)} \right)^{1/2}, \]

where \( \Gamma \) is the Gamma function. Let us fix some notations. By \( \hat{u} \) we denote the Fourier transform of \( u \) and let \( L^1_{\text{loc}}(\mathbb{R}) \) be the set of measurable functions which are locally integrable in \( \mathbb{R} \). Each \( f \in L^1_{\text{loc}}(\mathbb{R}) \) gives rise to an element in \( S'_1 \), denoted by \( T_f \) namely, for any \( \varphi \in S_1 \), \( \langle T_f, \varphi \rangle = \int_{\mathbb{R}} f(x)\varphi(x) \, dx \).

In order to obtain a realization of mBm in the framework of white noise analysis we introduce the following operator, see [LLV14]. For any \( H \in (1/2, 1) \) we define the operator

\[ (\hat{M}_H u)(y) = \frac{\sqrt{2\pi}}{C(H)} |y|^{1/2-H} \hat{u}(y), \]

where \( \hat{u} \) denotes the Fourier transform of the function \( u \). The operator \( M_H \) is well defined in the space

\[ L^2_H(\mathbb{R}) := \{ u \in S'_1 : \hat{u} = T_f; f \in L^1_{\text{loc}}(\mathbb{R}) \text{ and } \|u\|_H < \infty \}, \]

where the norm

\[ \|u\|_H^2 := \frac{1}{C(H)^2} \int_{\mathbb{R}} |x|^{1-2H} |\hat{u}(x)|^2 \, dx \]

is the inner product norm on the Hilbert space \( L^2_H(\mathbb{R}) \) given by

\[ (u, v)_H = \frac{1}{C(H)^2} \int_{\mathbb{R}} |x|^{1-2H} \hat{u}(x)\overline{\hat{v}}(x) \, dx. \]
Another possible representation for the operator $M_H$ is as follows [LLV14, Eq. 2.4]

$$(M_H \varphi)(x) = \gamma(H) \langle \cdot | H^{-3/2}, \varphi(x + \cdot) \rangle =: \gamma(H) \langle \Theta_H, \varphi(x + \cdot) \rangle, \quad \varphi \in S_1(\mathbb{R}),$$

where

$$\gamma(H) := \frac{\sqrt{\Gamma(2H + 1) \sin(\pi H)}}{2 \Gamma(H - 1/2) \cos(\pi(H - 1/2)/2)}.$$

Note that $\Theta_H$ is a generalized function from $S'_1$, i.e. for any $\psi \in S_1$ we have $\langle \Theta_H, \psi \rangle = \langle |y|^{H-3/2}, \psi(y) \rangle$.

The operator $M_H$ establishes an isometry between the Hilbert spaces $L^2_H(\mathbb{R})$ and $L^2(\mathbb{R})$ [LLV14, Prop. 2.10]. Below we review some useful properties of the operator $M_H$.

**Proposition 7.** Let $M_H$ be the operator defined above and $H \in (1/2, 1)$.

1. Then $M_H$ is an isometric isomorphism between the Hilbert spaces $L^2_H(\mathbb{R})$ and $L^2(\mathbb{R})$.

2. For any $f, g \in L^2(\mathbb{R}) \cap L^2_H(\mathbb{R})$ we have

$$\int_{\mathbb{R}} f(x)(M_H g)(x) \, dx = \int_{\mathbb{R}} (M_H f)(x) g(x) \, dx.$$

Moreover, for any $f \in L^1_{\text{loc}}(\mathbb{R}) \cap L^2_H(\mathbb{R})$ and $g \in S_1$, we have

$$\langle f, M_H g \rangle = \langle M_H f, g \rangle_{L^2(\mathbb{R})}.$$

3. There exists a constant $D$ such that for every $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ we have

$$\max_{x \in \mathbb{R}} |(M_H e_k)(x)| \leq D \frac{C(H)}{(k + 1)^{2/3}},$$

where $e_k := M_H^{-1} h_k$ and $h_k$ is the $k$-th Hermite function.

4. There exists $p \in \mathbb{N}$ such that for all $t \in [0, T]$ and $\varphi \in S_1$ we have

$$\left| \int_{\mathbb{R}} \varphi(x)(M_H 1_{[0,t]})(x) \, dx \right| \leq |\gamma(H)| |\Theta_H|_{-p} t \sup_{x \in \mathbb{R}} |\varphi(x + \cdot)|_p,$$

with $\Theta_H$ as in (2).
Proof. The items 1., 2. and 3. are proved in [LLV14, Thm 2.14, 2.15]. Assertion 4. is a direct consequence of 2. under the use of the Cauchy-Schwarz inequality and the representation (2) for the operator $M_H$. \hfill \Box

Next we define the operator $M_H$ for a measurable functional parameter $h : [0, T] \rightarrow (1/2, 1)$. For two indicator functions we define

$$ R_h(t, s) := \langle \mathbb{1}_{[0,t]}, \mathbb{1}_{[0,s]} \rangle_h := \frac{1}{C(h(t))C(h(s))} \int_{\mathbb{R}} |x|^{-2 h(x)} \mathbb{1}_{[0,t]}(x) \overline{\mathbb{1}_{[0,s]}(x)} \, dx, $$

which can be extended by linearity to the pre-Hilbert space of simple functions $(E(\mathbb{R}), (\cdot, \cdot)_h)$.

(A1) From now on we assume that $h : [0, T] \rightarrow (1/2, 1)$ is a continuous function.

Remark 8. For all $h : [0, T] \rightarrow (1/2, 1)$ satisfying (A1), the bilinear form $(\cdot, \cdot)_h$ is an inner product. See [LLV14, Prop. 3.1]. Moreover we define the linear map $M_h : E(\mathbb{R}) \rightarrow L^2(\mathbb{R})$, $\mathbb{1}_{[0,t]} \mapsto M_h \mathbb{1}_{[0,t]} := M_{h(t)} \mathbb{1}_{[0,t]} := M_H \mathbb{1}_{[0,t]}|_{H=h(t)}$.

Definition 9. For $h$ satisfying (A1) we define the $L^2(\mu_1)$ random variable $B_h(t)$ by

$$ B_h(t) = \langle \cdot, M_h \mathbb{1}_{[0,t]} \rangle. $$

1. One can show that the process $(\omega, t) \mapsto B_h(\omega, t)$ is a one-dimensional mBm, see [LLV14].

2. There is a continuous version of the process $B_h(t)$ by Kolmogorov’s theorem and we use the same notation for this continuous version.

Remark 10. The completion of $E(\mathbb{R})$ with respect to $(\cdot, \cdot)_h$ is a Hilbert space denoted by $L^2_h(\mathbb{R})$. Moreover the operator $M_h$ is an isometry between $(E(\mathbb{R}), (\cdot, \cdot)_h)$ and $(L^2(\mathbb{R}), (\cdot, \cdot))$, which can be extended to an isometry between $L^2_h(\mathbb{R})$ and $L^2(\mathbb{R})$.

The next proposition shows certain properties of 1-dimensional mBm.

Proposition 11. The process $B_h(t)$, $t \geq 0$ has the following properties.
1. The characteristic function of $B_h(t)$ is given by
\[
\mathbb{E}(e^{i\lambda B_h(t)}) = \int_{S_1(\mathbb{R})} e^{i\lambda w \cdot M_h 1_{[0,t])}} d\mu_1(w) = \exp \left( -\frac{\lambda^2}{2} |M_{[0,t]}|^2 \right)
\]
\[
= \exp \left( -\frac{\lambda^2}{2} t^{2h(t)} \right).
\]

2. The expectation of $B_h(t)$ is zero.

3. The variance of $B_h(t)$ is given by
\[
\mathbb{E}(B^2_h(t)) = t^{2h(t)}.
\]

4. The covariance of $B_h(t)$ is
\[
R_h(t,s) := \mathbb{E}(B_h(t)B_h(s)) = C \left( \frac{h(t)+h(s)}{2} \right)^2 \left[ \frac{1}{2} (h(t)+h(s) + s^{h(t)+h(s)} - |t-s|^{h(t)+h(s)}) \right].
\]

In Figures 1 and 2 the method of Wood and Chan [CW98] was used to simulate mBm on the interval $[0,1]$ for different Hurst parameter functionals.

Now we are ready to define the $d$-dimensional multifractional Brownian motion.

**Definition 12** ($d$-dimensional mBm). Let $h$ satisfy (A1). A $d$-dimensional mBm with functional parameter $h$ is defined by
\[
B_h(t) = (B_{h,1}(t), \ldots, B_{h,d}(t)), \quad t \geq 0,
\]
where $B_{h,i}(t), i = 1, \ldots, d,$ are $d$ independent 1-dimensional mBms.

Properties of $B_h(t)$:

1. The expectation is zero
\[
\mathbb{E}(B_h(t)) = 0.
\]
Figure 1: Multifractional Brownian motion for parameter functional $h$. Simulated with the method of Wood and Chan with $s = 10000$ discretization points. Here the Hölder continuity of the path is linearly increasing in time.
Figure 2: Multifractional Brownian motion for parameter functional $h$. Simulated with the method of Wood and Chan with $s = 10000$ discretization points. Here the Hölder continuity of the path is the function $h(t) = 0.4 + 0.5\sin(5\pi t)$. 
2. The characteristic function of $B_h(t)$ is given, for any $x \in \mathbb{R}^d$, by

$$
\int_{S_d} e^{i(\lambda, B_h(t))_{\mathbb{R}^d}} d\mu_d(w) = \exp \left( -\frac{1}{2} \sum_{k=1}^d x_k^2 |M_h \mathbb{1}_{[0,t]}|_0^2 \right)
= \exp \left( -\frac{1}{2} t^{2h(t)} |x|_{\mathbb{R}^d}^2 \right).
$$

3. Covariance matrix of $B_h(t)$:

$$\text{cov}(B_h(t)) = (\delta_{ij} t^{2h(t)})_{i,j=1}^d.$$

4 Local time

The time a process spends in a certain point $y \in \mathbb{R}^d$ is called the local time of the process. Formally the local time is given by the time integral over a Dirac delta function of the process. This Donsker’s delta function is a well-defined and studied object in white noise analysis, see e.g. [LLSW93, HKPS93, Oba94, Kuo96]. For mBm recently the concept of local times was introduced and studied, see e.g. [BDG07, MWX08, ASX11] and references therein.

In this section we determine the kernel functions for the local time and the regularized local time of mBm. Moreover we show that for a suitable number of truncated kernel functions the local time of mBm is a Hida distribution and can be obtained as a limit of the regularized local time.

**Proposition 13.** For $t > 0$ the Bochner integral

$$\delta(B_h(t)) := \left( \frac{1}{2\pi} \right)^d \int_{\mathbb{R}^d} e^{i(\lambda, B_h(t))_{\mathbb{R}^d}} d\lambda$$

is a Hida distribution and for any $\varphi \in S_d$ its $S$-transform given by

$$S\delta(B_h(t))(\varphi) = \left( \frac{1}{\sqrt{2\pi} t^{h(t)}} \right)^d \exp \left( -\frac{1}{2t^{2h(t)}} \left| \int_{\mathbb{R}} \varphi(x) (M_h \mathbb{1}_{[0,t]})(x) dx \right|_{\mathbb{R}^d}^2 \right),$$

(4)
Proof. First we compute the $S$-transform of the integrand in (3) for any $\varphi \in S_d$:

$$S e^{i(\lambda, B_h(t))_{\mathbb{R}^d}}(\varphi) = \exp \left( -\frac{1}{2} |\lambda|^2_{\mathbb{R}^d} t^{2h(t)} + i \left( \lambda, \int_{\mathbb{R}} \varphi(x)(M_h \mathbb{1}_{[0,t]})(x) \, dx \right)_{\mathbb{R}^d} \right).$$

(5)

It is clear that the $S$-transform is $\lambda$-measurable for any $\varphi \in S_d$. On the other hand, for any $z \in \mathbb{C}$ and all $\varphi \in S_d$ we obtain

$$|S e^{i(\lambda, B_h(t))_{\mathbb{R}^d}}(z\varphi)| = \left| \exp \left( -\frac{1}{2} |\lambda|^2_{\mathbb{R}^d} t^{2h(t)} + iz \lambda, \int_{\mathbb{R}} \varphi(x)(M_h \mathbb{1}_{[0,t]})(x) \, dx \right)_{\mathbb{R}^d} \right| \right| \leq \exp \left( -\frac{1}{4} |\lambda|^2_{\mathbb{R}^d} t^{2h(t)} \right) \exp \left( -\frac{1}{4} |\lambda|^2_{\mathbb{R}^d} t^{2h(t)} + |z| \left( \lambda, \int_{\mathbb{R}} \varphi(x)(M_h \mathbb{1}_{[0,t]})(x) \, dx \right)_{\mathbb{R}^d} \right) \right| \leq \exp \left( -\frac{1}{4} |\lambda|^2_{\mathbb{R}^d} t^{2h(t)} \right) \exp \left( -\left( \frac{1}{2} |\lambda|^2_{\mathbb{R}^d} t^{h(t)} - \frac{1}{t^{h(t)}} |z| \left( \int_{\mathbb{R}} \varphi(x)(M_h \mathbb{1}_{[0,t]})(x) \, dx \right)_{\mathbb{R}^d} \right)^2 \right) \right| \right| \leq \exp \left( -\frac{1}{4} |\lambda|^2_{\mathbb{R}^d} t^{2h(t)} \right) \exp \left( \frac{1}{t^{2h(t)}} |z|^2 \left( \int_{\mathbb{R}} \varphi(x)(M_h \mathbb{1}_{[0,t]})(x) \, dx \right)_{\mathbb{R}^d} \right) \right| \leq \exp \left( -\frac{1}{4} |\lambda|^2_{\mathbb{R}^d} t^{2h(t)} \right) \exp \left( |z|^2 |\varphi|^2_{\mathbb{R}^d} \right).

Thus we have the following bound

$$|S e^{i(\lambda, B_h(t))_{\mathbb{R}^d}}(z\varphi)| \leq \exp \left( -\frac{1}{4} |\lambda|^2_{\mathbb{R}^d} t^{2h(t)} \right) \exp \left( |z|^2 |\varphi|^2_{\mathbb{R}^d} \right),$$

where, as a function of $\lambda$, the first factor is integrable on $\mathbb{R}^d$ and the second factor is constant. The result (4) follows from (5) and integration with respect to $\lambda$. \hfill \square

In the following theorem we characterize the truncated local time of mBm as a Hida distribution. Therefore we use the notation $\delta^{(N)}$ for the truncated Donsker’s delta function as in 6, i.e. for any $\varphi \in S_d$

$$S \delta^{(N)}(B_h(t))(\varphi) = \left( \frac{1}{\sqrt{2\pi} t^{h(t)}} \right)^d \exp_N \left( -\frac{1}{2 t^{2h(t)}} \left| \int_{\mathbb{R}} \varphi(x)(M_h \mathbb{1}_{[0,t]})(x) \, dx \right|^2_{\mathbb{R}^d} \right),$$

14
with exp\(_N(x) := \sum_{n=N}^{\infty} \frac{1}{n!}x^n \) for \( x \in \mathbb{C} \).

**(A2)** Let \( h \) satisfy (A1) and \( \sup_{t \in [0,T]} h(t) < \frac{1+2N}{2N+d} \), for a fixed \( N \in \mathbb{N}_0 \) and \( d \in \mathbb{N} \).

**Theorem 14.** For \( h \) satisfying (A2) with dimension \( d \in \mathbb{N} \) and \( N \in \mathbb{N}_0 \), the Bochner integral

\[
L_h^{(N)}(T) := \int_0^T \delta^{(N)}(B_h(t)) \, dt
\]

is a Hida distribution.

**Proof.** The proof uses again Corollary 5 with respect to the Lebesgue measure in \([0,T]\). It follows from (4) that

\[
S\delta^{(N)}(B_h(t))(\varphi) = \left( \frac{1}{\sqrt{2\pi t^{h(t)}}} \right)^d \exp_N \left( -\frac{1}{2t^{2h(t)}} \left| \int_{\mathbb{R}} \varphi(x) (M_{h} \mathbb{1}_{[0,t]}(x)) \, dx \right|^2 \right),
\]

which is measurable in \( t \) for every \( \varphi \in S_d \). Using Proposition (7)-4, we obtain the following bound for any \( z \in \mathbb{C} \) and all \( \varphi \in S_d \)

\[
|S\delta^{(N)}(B_h(t))(z\varphi)| \leq \left( \frac{1}{\sqrt{2\pi t^{h(t)}}} \right)^d \exp_N \left( \frac{1}{2t^{2h(t)}} |\gamma(h(t))|^2 |\Theta_{h(t)}|^2 t^2 |z|^2 \left( \sup_{x \in \mathbb{R}} |\varphi(x+\cdot)|_p \right)^2 \right).
\]

\[
\leq \left( \frac{1}{\sqrt{2\pi t^{h(t)}}} \right)^d \exp_N \left( \frac{1}{2} K(h) t^{2h(t)} |z|^2 \|\varphi\|^2 \right),
\]

where \( K(h) \) is independent of \( t \) (note that \( \sup_{t \in [0,T]} |h(t)| = \beta < 1 \)) and we defined the continuous norm on \( S_d \) by

\[
\|\varphi\| := \sup_{x \in \mathbb{R}} |\varphi(x+\cdot)|_p.
\]

The estimation

\[
\exp_N \left( \frac{1}{2} K(h) t^{2-2h(t)} |z|^2 \|\varphi\|^2 \right) \leq t^{2N(1-h(t))} \exp \left( \frac{K(h)}{2} |z|^2 \|\varphi\|^2 \right)
\]

allows us to obtain the bound

\[
|S\delta^{(N)}(B_h(t))(z\varphi)| \leq \left( \frac{1}{\sqrt{2\pi}} \right)^d t^{2N(1-h(t)) - dh(t)} \exp \left( \frac{K(h)}{2} |z|^2 \|\varphi\|^2 \right),
\]
which is integrable in $t \in [0, T)$ due to (A2). The proof follows from the application of Corollary 5.

**Theorem 15.** For $h$ satisfying (A2) for dimension $d$ and $N \in \mathbb{N}_0$, the kernels functions of $L_h^{(N)}(T)$ are given by

$$F_{h,2n}(u_1, \ldots, u_{2n}) = \frac{1}{n!} \left( \frac{1}{\sqrt{2\pi}} \right)^d \left( -\frac{1}{2} \right)^n \int_0^T \frac{1}{t^{2h(t)N+dh(t)}} \prod_{j=1}^{2n} (M_{h(t)} \mathbb{1}_{[0,t]})(u_j) dt \tag{7}$$

for each $n \in \mathbb{N}^d$ such that $n \geq N$. All the other kernels $F_{h,n}$ are zero.

**Proof.** The kernels of $L_h^{(N)}(T)$ are obtained by its $S$-transform. Therefore we use Corollary 5 and integrate (6) over $[0, T]$. For any $\varphi \in S_d$, we have

$$SL_h^{(N)}(T)(\varphi) = \left( \frac{1}{\sqrt{2\pi}} \right)^d \int_0^T \frac{1}{t^{dh(t)}} \sum_{n=N}^\infty \frac{(-1)^n}{2n! 2^{nh(t)}} \prod_{j=1}^d \left( \int_{\mathbb{R}} \varphi_j(x)(M_{h \mathbb{1}_{[0,t]}})(x) dx \right)^{2n_j} dt$$

$$= \left( \frac{1}{\sqrt{2\pi}} \right)^d \int_0^T \sum_{n=N}^\infty \frac{(-1)^n}{t^{2nh(t)+dh(t)}} \prod_{j=1}^d \left( \int_{\mathbb{R}} \varphi_j(x)(M_{h \mathbb{1}_{[0,t]}})(x) dx \right)^{2n_j} dt.$$

Comparing it with the general form of the chaos expansion

$$L_h^{(N)}(T) = \sum_{n \in \mathbb{N}^d} \langle \cdot; w^\otimes n \cdot; F_{h,n} \rangle$$

we obtain $F_{h,n}$ as in (7). This completes the proof.

**Remark 16.** The result of Theorem 14 shows that for $d = 1$ all local times are well-defined, but for $d \geq 2$ they are well-defined after truncation of divergent terms. This motivates the study of a regularized version, namely we discuss

$$L_{h,\varepsilon}(T) := \int_0^T \delta_\varepsilon(B_h(t)) dt, \quad \varepsilon > 0,$$

16
where
\[ \delta_\varepsilon(B_h(t)) := \left( \frac{1}{\sqrt{2\pi \varepsilon}} \right)^d \exp \left( -\frac{1}{2\varepsilon} |B_h(t)|^2_{\mathbb{R}^d} \right). \]

**Theorem 17.** Let \( \varepsilon > 0 \) be given and \( h \) satisfy (A2).

1. The functional \( L_{h,\varepsilon}(T) \) is a Hida distribution with kernels functions given by

\[
F_{h,\varepsilon,2n}(u_1, \ldots, u_{2n}) = \frac{1}{n!} \left( \frac{1}{\sqrt{2\pi}} \right)^d \left( -\frac{1}{2} \right)^n \int_0^T \frac{1}{(\varepsilon + 2h(t))^{n+d/2}} \times \prod_{j=1}^{2n} (M_h \mathbb{1}_{[0,t]})(u_j) \, dt
\]

for each \( n = (n_1, \ldots, n_d) \in \mathbb{N}^d \) and \( F_{h,\varepsilon,n} = 0 \) if at least one of the \( n_j \) is an odd number.

2. For \( \varepsilon \) tends to zero the truncated functional \( L^{(N)}_{h,\varepsilon}(T) \) converges strongly in \( (S_d)' \) to the truncated local time \( L^{(N)}_h(T) \).

**Proof.** 1. First we compute the \( S \)-transform of the integrand of \( L_{h,\varepsilon}(T) \). For any \( \varphi \in S_d \), we obtain

\[
S\delta_\varepsilon(B_h(t))(\varphi) = \left( \frac{1}{\sqrt{2\pi(\varepsilon + t^{2h(t)})}} \right)^d \exp \left( -\frac{1}{2(\varepsilon + t^{2h(t)})} \times \left( \int_{\mathbb{R}} \varphi(x)(M_h \mathbb{1}_{[0,t]})(x) \, dx \right)^2 \right).
\]

which is measurable in \( t \). Thus, for any \( z \in \mathbb{C} \) and \( \varphi \in S_d \), by Proposition 7-4. we arrive at the following bound

\[
|S\delta_\varepsilon(B_h(t))(z\varphi)| \leq \left( \frac{1}{\sqrt{2\pi(\varepsilon + t^{2h(t)})}} \right)^d \exp \left( K(h)|z|^2 \frac{t^2}{2(\varepsilon + t^{2h(t)})} ||\varphi||^2 \right).
\]

On the other hand, \( \frac{t^2}{2(\varepsilon + t^{2h(t)})} \) is bounded in \( [0,T] \) and \( (\varepsilon + t^{2h(t)})^{-d/2} \) is integrable on \( [0,T] \), therefore we may conclude, by Corollary 5, that \( L_{h,\varepsilon}(T) \in \)
\((S_d)'\). In addition, for any \(\varphi \in S_d\), we have

\[
(SL_{h,\varepsilon}(T))(\varphi) = \int_0^T (S\delta_\varepsilon(B_h(t)))(\varphi) \, dt
\]

\[
= \left( \frac{1}{\sqrt{2\pi}} \right)^d \int_0^T \frac{1}{(\varepsilon + t^2 h(t))^{d/2}} \sum_{n=0}^\infty \frac{(-1)^n}{2n(\varepsilon + t^2 h(t))^n} \times \sum_{n_1,\ldots,n_d \in \mathbb{N}} \frac{1}{n_1! \cdots n_d!} \prod_{j=1}^d \left( \int_{\mathbb{R}} \varphi_j(x)(M_h \mathbb{I}_{[0,t]})(x) \, dx \right)^{2n_j} \, dt
\]

\[
= \left( \frac{1}{\sqrt{2\pi}} \right)^d \int_0^T \sum_{n=0}^\infty \left( -\frac{1}{2} \right)^n \frac{1}{(\varepsilon + t^2 h(t))^{n+d/2}} \times \sum_{n_1,\ldots,n_d \in \mathbb{N}} \frac{1}{n_1! \cdots n_d!} \prod_{j=1}^d \left( \int_{\mathbb{R}} \varphi_j(x)(M_h \mathbb{I}_{[0,t]})(x) \, dx \right)^{2n_j} \, dt.
\]

Comparing the latter expression with the kernels \(F_{h,\varepsilon,n}\) from the chaos expansion of \(L_{h,\varepsilon}(T)\)

\[
L_{h,\varepsilon}(T) = \sum_{\mathbf{n} \in \mathbb{N}^d} \langle w^{\otimes \mathbf{n}}, F_{h,\varepsilon,n} \rangle,
\]

we conclude that whenever one of the \(n_j\) in \(\mathbf{n} = (n_1,\ldots,n_d)\) is odd we have \(F_{h,\varepsilon,n} = 0\), otherwise they are given by the expression (8).

2. To check the convergence we shall use Corollary 4 and fact that

\[
(SL^{(N)}_{h,\varepsilon}(T))(\varphi) = \int_0^T (S\delta_\varepsilon(B_h(t)))(\varphi) \, dt.
\]

Thus, for all \(z \in \mathbb{C}\) and all \(\varphi \in S_d\) we estimate \((SL^{(N)}_{h,\varepsilon}(T))(z\varphi)\) by

\[
| (SL^{(N)}_{h,\varepsilon}(T))(z\varphi) | \leq \int_0^T | (S\delta_\varepsilon(B_h(t)))(\varphi) | \, dt
\]

\[
\leq \left( \frac{1}{\sqrt{2\pi\varepsilon}} \right)^d \int_0^T \exp \left( \frac{K(h)}{2\varepsilon} |z|^2 t^2 \|\varphi\|^2 \right) \, dt
\]

\[
\leq \left( \frac{1}{\sqrt{2\pi\varepsilon}} \right)^d \exp \left( \frac{C(h,T)}{2\varepsilon} |z|^2 \|\varphi\|^2 \right),
\]
for a certain constant $C(h, T) > 0$, which shows the uniform boundedness condition. Moreover, using similar calculations as in Theorem 14, for any $t \in [0, T]$, yields

$$\left| (SL_{h, \varepsilon}^{(N)}(T))(z \varphi) \right| \leq \left( \frac{1}{\sqrt{2\pi t^h(t)}} \right)^d \exp_N \left( \frac{K(h)}{2} t^{2-2h(t)} \| \varphi \|^2 \right)$$

$$\leq \left( \frac{1}{\sqrt{2\pi}} \right)^d t^{2N(1-h(t))-dh(t)} \exp \left( \frac{K(h)}{2} \| \varphi \|^2 \right).$$

This upper bound together with the fact that $1/2 < h(t) < 1$, for any $t \in [0, T]$, gives an integrable function on $[0, T]$. Finally, an application of Lebesgue’s dominated convergence theorem implies the other condition in order to apply Corollary 4. This completes the proof.

\[\square\]

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