Learning Stability Certificates from Data

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Abstract

Many existing tools in nonlinear control theory for establishing stability or safety of a dynamical system can be distilled to the construction of a certificate function that guarantees a desired property. However, algorithms for synthesizing certificate functions typically require a closed-form analytical expression of the underlying dynamics, which rules out their use on many modern robotic platforms. To circumvent this issue, we develop algorithms for learning certificate functions only from trajectory data. We establish bounds on the generalization error – the probability that a certificate will not certify a new, unseen trajectory – when learning from trajectories, and we convert such generalization error bounds into global stability guarantees. We demonstrate empirically that certificates for complex dynamics can be efficiently learned, and that the learned certificates can be used for downstream tasks such as adaptive control.

1 Introduction

A fundamental barrier to widespread deployment of reinforcement learning policies on real robots is the lack of formal safety and stability guarantees. While much research has focused on how to train control policies for complex systems, considerably less emphasis has been placed on verifying stability for the resulting closed-loop system. Without any a-priori guarantees, practitioners will be hesitant to deploy learned solutions in the real world regardless of performance in simulation.

Many powerful tools have been developed in nonlinear control theory to address the safety and stability of systems with known dynamics. The most well-known technique is the construction of a Lyapunov function [28, 40] to demonstrate asymptotic stability of a system with respect to an equilibrium point. Similarly, barrier functions [2, 4, 27] are used to show set-invariance, which has been widely used in safety-critical applications to prove that a system does not exit a desired safe set. Contraction analysis [26] provides an alternative view of stability – useful for many problems in nonlinear control and robotics – by considering the convergence of trajectories towards each other rather than to an equilibrium point. The unifying theme among these tools is the

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construction of a certificate function (the Lyapunov/barrier function or contraction metric) that proves a given desirable property for the system of interest. These certificates have strong converse results [8, 12, 17], which implies the existence of a certificate function if the desired property does hold, and can also be used for controller synthesis [2, 21, 41].

The main obstacle for producing certificate functions in modern robotics and reinforcement learning is that existing synthesis and verification tools such as sum-of-squares (SOS) optimization [1] or SMT solvers [11] typically assume the dynamics can be written down analytically in closed form. Furthermore, the functions of interest are often constrained to lie in restrictive classes such as polynomial basis functions of fixed degree. This presents a serious hurdle in modern robotics, where (a) sophisticated physics simulators are widely used to model complex environments and (b) control policies are often represented with complex deep neural networks. Finally, both SOS optimization and formal verification tools are computationally intensive, thus limiting their applicability.

To avoid these limitations, recent approaches have proposed to treat certificate synthesis as a machine learning problem, and train powerful function approximators such as deep neural networks and reproducing kernel Hilbert space (RKHS) predictors on trajectory data collected from a dynamical system [15, 29, 36, 37, 39, 44]. The general strategy is to enforce the desired certificate condition (e.g. the Lie derivative of a function $V$ should be negative) along collected samples. Empirically, this has been shown to be quite effective, and the learned certificate often generalizes well outside of the training data. However, a deeper theoretical understanding of when and why this approach works is missing.

Contributions. Consider Figure 1, where a contraction metric, which certifies pairwise convergence of trajectories, is learned from rollouts of a damped Van der Pol oscillator. Regions of the state space for which the learned metric is not contracting are shown as a function of the number of trajectories $n$. While the size of the violating regions appears to shrink as $n$ increases, Figure 1 raises many questions. How much data does one need to collect so that the violating regions cover at most a prescribed fraction of the relevant state space? Is the learning consistent, i.e. do the regions vanish as $n \to \infty$?

In this paper, we show that learning is indeed consistent. To this end, we compute upper bounds on the volume of the violating regions which tend to zero as $n$ grows. We do this in two steps. First, we formulate a general optimization framework that encompasses learning many existing certificate functions, and use statistical learning theory to prove a fast $\tilde{O}(k/n)$ rate on the generalization error — the probability the learned certificate will not certify a new, unseen trajectory — where $k$ is the effective number of parameters of the function class for the certificate. We then translate bounds on the generalization error into non-probabilistic bounds on the volume of the violating regions. We conclude with experiments, which show that certificates can be efficiently learned from trajectories, and that the learned certificates can perform downstream tasks such as adaptive control against unknown disturbances.
2 Related Work

Prior research generally focuses on learning certificates for a fixed system from trajectories, or on using certificate conditions as regularizers when learning models for control.

**Learning Lyapunov functions from data.** Giesl et al. [13] propose to learn a Lyapunov function from noisy trajectories using a specific reproducing kernel. Their algorithm first fits a dynamics model from data, and then uses interpolation to construct a Lyapunov function from the learned model. The authors prove $L_\infty$ convergence results on the Lie derivative of the constructed Lyapunov function compared to the ground truth, with rates depending on a dense cover of the state space.

Our work circumvents this two-step identification procedure by directly analyzing the generalization error of a Lyapunov function learned by enforcing derivative conditions along the training data. Many other authors have also proposed similar approaches. Kenanian et al. [18] show how to estimate the joint spectral radius of a switched linear system by learning a common quadratic Lyapunov function directly from data. Their analysis heavily exploits properties of linear systems. Richards et al. [36] use a sum-of-squares neural network representation to learn the largest region of attraction of a nonlinear system. Manek and Kolter [29] jointly train a neural network model and Lyapunov function. Neither of these works provide formal guarantees that the learned Lyapunov function will generalize to new trajectories. Both Chang et al. [7] and Ravanbakhsh and Sankaranarayanan [35] propose to use ideas from formal verification to falsify the validity of a learned candidate Lyapunov function. A significant limitation is the requirement of access to the true dynamics.

In many of these works, the Lie derivative constraint that defines a Lyapunov function is relaxed to a soft constraint, so that first-order gradient methods can be used for optimization. We note that our generalization analysis can be modified to handle soft constraints in a straightforward manner.

**Learning barrier functions from data.** Barrier functions are relaxations of Lyapunov functions that demonstrate invariance of a subset of the state space. Recently, many authors have proposed to use and learn barrier functions from data for safety-critical applications. Taylor et al. [44] assume a control barrier function (CBF) is valid for both a nominal and unknown system model, and use the CBF to guide safe learning of the unknown system dynamics. More closely related to our work, Robey et al. [37] learn a CBF for a known nonlinear dynamical system from expert demonstrations, and use Lipschitz arguments to extend the validity of the CBF beyond the training data. Jin et al. [15] propose to jointly learn a Lyapunov, barrier, and a policy function from data. They also prove validity of the learned certificates using Lipschitz arguments.

**Learning contracting vector fields and contraction metrics from data.** The literature on learning contraction metrics [26] from data is more sparse. In an imitation learning context, Sindhwani et al. [38] propose to learn a vector field from demonstrations that satisfies contraction in the identity metric. The authors parameterize the vector field as a vector-valued reproducing kernel. Khadir et al. [20] also learn a vector field from demonstrations by using sum-of-squares to enforce contraction. They argue by smoothness considerations that the learned vector field actually contracts in a tube around the demonstration trajectories. We note that in both these works, the
metric is held fixed and is assumed to be known. Singh et al. [39] jointly learn a model and a control contraction metric from data, and show empirically that using contraction as a regularizer in model learning can lead to better sample efficiency when learning to control. We leave studying the generalization properties of jointly learning a model and a contraction metric to future work.

**Statistical bounds in optimization and control.** Our generalization bounds are similar in spirit to those provided for random convex programs (RCPs) [5, 6]. Random convex programming is concerned with approximating solutions to convex programs with an infinite number of constraints. Such infinitely-constrained problems are approximated by drawing i.i.d. samples from a distribution \( \nu \) over the constraint parameters and enforcing constraints on samples. One can then show that the probability that a new sample from \( \nu \) violates the constraint for the approximate solution scales as \( O(d/n) \) where \( d \) is the number of decision variables. Our results can be viewed as generalizing these bounds beyond convex programs, though our constants are less sharp. In our experiments, we use the RCP bound for numerically computing generalization bounds when the problem is convex.

### 3 Learning Certificates Framework

**3.1 Problem Statement**

We assume the underlying dynamical system is given by a continuous-time autonomous system of the form \( \dot{x} = f(x) \), where \( f \) is continuous, unknown, and the state \( x \in \mathbb{R}^p \) is fully observed. Let \( X \subseteq \mathbb{R}^p \) be a compact set and let \( T \subseteq \mathbb{R}_+ \) be the maximal interval starting at zero for which a unique solution \( \varphi_t(\xi) \) exists for all initial conditions \( \xi \in X \) and \( t \in T \). We assume access to sample trajectories generated from random initial conditions. Specifically, let \( \mathcal{D} \) denote a distribution over \( X \), and let \( \xi_1, \ldots, \xi_n \) be \( n \) i.i.d. samples from \( \mathcal{D} \). We are given access to the \( n \) trajectories \( \{\varphi_t(\xi_i)\}_{i=1,...,n, t \in T} \). For simplicity of exposition, we assume that we can exactly differentiate the trajectories \( \varphi_t(\xi) \) with respect to time. In our experiments, we compute \( \dot{x} \) numerically.

Let \( \mathcal{V} \) be a space of continuously differentiable functions \( V : \mathbb{R}^p \rightarrow \mathbb{R}^q \). Let \( h : \mathbb{R}^p \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^{q 	imes p} \rightarrow \mathbb{R} \) be a fixed and known continuous function. Our goal is to choose a \( V \in \mathcal{V} \) such that

\[
h \left( \varphi_t(\xi), \dot{\varphi}_t(\xi), V(\varphi_t(\xi)), \frac{\partial V}{\partial x}(\varphi_t(\xi)) \right) \leq 0 \quad \forall \xi \in X, t \in T . \tag{3.1}
\]

As we describe below, through suitable choices of the function \( h \), equation (3.1) can be used to enforce various defining conditions for certificates such as Lyapunov functions and contraction metrics. We note that our framework can be modified to allow for higher order derivatives of \( V \).

We study the following optimization problem for searching for a solution to (3.1):

\[
\text{find}_{V \in \mathcal{V}} \quad \text{s.t.} \quad h \left( \varphi_t(\xi_i), \dot{\varphi}_t(\xi_i), V(\varphi_t(\xi_i)), \frac{\partial V}{\partial x}(\varphi_t(\xi_i)) \right) \leq -\gamma , \quad i = 1, \ldots, n, \ t \in T . \tag{3.2}
\]

Here, \( \gamma > 0 \) is a positive margin value which will allow us to generalize the behavior of \( V \) on \( h \) outside of the sampled data. In practice, we often solve (3.2) with a cost term on \( V \) such as its norm. Let \( \hat{V}_n \in \mathcal{V} \) denote a solution to (3.2), assuming one exists. We quantify the generalization of \( \hat{V}_n \) by the probability of violation over trajectories starting from \( \xi \sim \mathcal{D} \):

\[
\text{err}(\hat{V}_n) := \mathbb{P}_{\xi \sim \mathcal{D}} \left\{ \max_{t \in T} h \left( \varphi_t(\xi), \dot{\varphi}_t(\xi), V(\varphi_t(\xi)), \frac{\partial V}{\partial x}(\varphi_t(\xi)) \right) > 0 \right\} . \tag{3.3}
\]
In Section 4, we prove $O(k \cdot \text{polylog}(n)/n)$ decay rates for $\text{err} (\hat{V}_n)$ for various parametric and non-parametric function classes $\mathcal{V}$, where $k$ denotes the effective number of parameters of the class $\mathcal{V}$. In Section 5, we show how $\text{err} (\hat{V}_n) \leq \varepsilon$ bounds translate into global, non-probabilistic results. Before we state our main results, we instantiate our framework for two key certificate functions.

3.1.1 Lyapunov stability analysis

Let zero be an equilibrium point for $\dot{x} = f(x)$. Let $D \subseteq \mathbb{R}^p$ be an open set containing the origin. A Lyapunov function $V : \mathbb{R}^p \rightarrow \mathbb{R}$ is a locally positive definite function such that $V(0) = 0$, $V(x) > 0$ for $x \in D \setminus \{0\}$, and $\langle \nabla V(x), f(x) \rangle < 0$ for $x \in D \setminus \{0\}$. It is well known (see e.g. Slotine and Li [40]) that the existence of such a Lyapunov function $V$ proves the local asymptotic stability of the origin. Our framework can be used to learn a Lyapunov function from stable trajectories by taking $h(x, \dot{x}, V(x), \nabla V(x)) = \langle \nabla V(x), \dot{x} \rangle + \alpha(V(x))$. Here, $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ is a class $\mathcal{K}$ function, i.e. a continuous, strictly increasing function satisfying $\alpha(0) = 0$.

3.1.2 Contraction metrics

A system is said to be contracting in region $D$ with rate $\alpha$ if there exists a uniformly positive definite Riemannian metric $M(x)$ such that $\frac{\partial f}{\partial x}^T M(x) + M(x) \frac{\partial f}{\partial x} + \dot{M}(x) \leq -2\alpha M(x)$ for $x \in D$ [26]. Given knowledge of $\frac{\partial f}{\partial x}$, this condition fits into our framework by taking $h \left( x, \dot{x}, M(x), \frac{\partial f}{\partial x} \right) = \lambda_{\text{max}} \left( \frac{\partial f}{\partial x}^T M(x) + M(x) \frac{\partial f}{\partial x} + \dot{M}(x) + 2\alpha M(x) \right)$.

Without knowledge of $\frac{\partial f}{\partial x}$, it is not immediately clear how to evaluate $h$ from trajectories. Instead, we leverage results from Forni and Sepulchre [10], who reformulate contraction in terms of Lyapunov theory. Consider a candidate differential Lyapunov function $V(x, \delta x) = \delta x^T M(x) \delta x$ for the prolonged system \[
\begin{bmatrix}
\dot{x} \\
\delta \dot{x}
\end{bmatrix} =
\begin{bmatrix}
f(x) \\
\frac{\partial f}{\partial x}(x) \delta x
\end{bmatrix}
\]
defined on the tangent bundle $TD = \bigcup_{x \in D} \{ x \} \times T_x D \simeq D \times \mathbb{R}^p$. The contraction condition is equivalent to:

$$
\langle \nabla_x V(x, \delta x), f(x) \rangle + \langle \nabla_{\delta x} V(x, \delta x), \frac{\partial f}{\partial x}(x) \delta x \rangle \leq -\alpha V(x, \delta x) \quad \forall x \in D, \delta x \in \mathbb{R}^n .
$$ (3.4)

We can enforce (3.4) by directly sampling trajectories on $TD$, by exploiting that the variational dynamics obeyed by $\delta x(t)$ is identical to the local linearization of $f$ around $x(t)$. Specifically, we sample pairs of initial conditions $x_0^{(1)}$ and $x_0^{(2)} = x_0^{(1)} + \delta x_0$ for some small perturbation $\delta x_0$. Numerical differentiation of $x^{(1)}(t)$ and $\delta x(t) = x^{(1)}(t) - x^{(2)}(t)$ provides access to $\dot{x}^{(1)} = f(x^{(1)})$ and $\delta \dot{x}(t) = \frac{\partial f}{\partial x} \delta x(t)$, which then allows us to evaluate (3.4) along system trajectories.

4 Generalization Error Results

We first define the notion of stability we will assume. Recall that $X$ is the containing sample initial conditions, and $T$ is the interval over which our trajectories evolve.

**Assumption 4.1** (Stability in the sense of Lyapunov). We assume there exists a compact set $S \subseteq \mathbb{R}^p$ such that $\varphi_t(\xi) \in S$ for all $\xi \in X$, $t \in T$. Let the constant $B_S := \sup_{x \in \overline{S}} \|x\|$.

Note that contraction implies Assumption 4.1, so that contracting systems are also covered in this setting. Next, we make some regularity assumptions on the function class $\mathcal{V}$.
Assumption 4.2 (Uniform boundedness of $\mathcal{V}$). We assume there exists finite constants $B_V, B_{\nabla V}$ such that $\sup_{V \in \mathcal{V}} \sup_{x \in S} \|V(x)\| \leq B_V$ and $\sup_{V \in \mathcal{V}} \sup_{x \in S} \left\| \frac{\partial V}{\partial x}(x) \right\| \leq B_{\nabla V}$.

Given Assumptions 4.1–4.2, we define $B_h$ (resp. $L_h$) to be an upper bound on $|h(x, f(x), V, \frac{\partial V}{\partial x})|$ (resp. the Lipschitz constant of $(V, \frac{\partial V}{\partial x}) \mapsto h(x, f(x), V, \frac{\partial V}{\partial x})$) over $x \in S$, $|V| \leq B_V$ and $\|\frac{\partial V}{\partial x}\| \leq B_{\nabla V}$. Note that both $B_h$ and $L_h$ are guaranteed to be finite by our assumptions.

We now introduce, with slight abuse of notation, the shorthand $h(\xi, V) := \max_{t \in T} h(\varphi_t(\xi), \hat{\varphi}_t(\xi), V(\varphi_t(\xi)), \frac{\partial}{\partial x}(\varphi_t(\xi)))$. The key insight to our analysis is the simple observation that any feasible solution $V_n$ to (3.2) achieves zero empirical risk on the loss $\hat{R}_n(V) := \frac{1}{n} \sum_{i=1}^{n} 1_{\{h(\xi_i, V) > -\gamma\}}$. In particular, since $\mathbb{P}_{\xi \sim D}(h(\xi, V_n) > 0) = \mathbb{E}_{\xi \sim D} 1_{\{h(\xi, \hat{V}_n) > 0\}}$, we can use results from statistical learning theory which give us fast rates for zero empirical risk minimizers with margin $\gamma$. The following result is adapted from Theorem 5 of Srebro et al. [42].

Lemma 4.1. Fix a $\delta \in (0, 1)$. Assume that Assumption 4.1 and Assumption 4.2 hold. Suppose that the optimization problem (3.2) is feasible and let $\hat{V}_n$ denote a solution. The following statement holds with probability at least $1 - \delta$ over the randomness of $\xi_1, ..., \xi_n$ drawn i.i.d. from $D$:

$$ \mathbb{P}_{\xi \sim D}(h(\xi, \hat{V}_n) > 0) \leq K \left( \frac{\log^3 n}{\gamma^2} R^2_n(V) + \frac{2 \log(4B_h/\gamma)/\delta}{n} \right). $$

Here, $R_n(V) := \sup_{\xi \in X} \mathbb{E}_{\xi \sim \text{Unif}(\{-1, 1\}, \xi^2)} \sup_{V \in \mathcal{V}} \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i h(\xi_i, V)$ is the Rademacher complexity of the function class $\mathcal{V}$ and $K$ is a universal constant.

Lemma 4.1 reduces bounding $\text{err}(\hat{V}_n)$ to bounding the Rademacher complexity $R_n(V)$. Define the norm $\|\cdot\|_\mathcal{V}$ on $\mathcal{V}$ as $\|V\| := \sup_{x \in S} \left\| \frac{\partial V}{\partial x}(x) \right\|$. By Assumptions 4.1–4.2 and Dudley’s entropy inequality [46], we can bound $R_n(V)$ by the estimate $R_n(V) \leq \frac{2M_n}{\sqrt{n}} \int_0^\infty \sqrt{\log N(\varepsilon; \mathcal{V}, \|\cdot\|_\mathcal{V})} d\varepsilon$. Here, $N(\varepsilon; \mathcal{V}, \|\cdot\|_\mathcal{V})$ is the covering number of $\mathcal{V}$ at resolution $\varepsilon$ in the $\|\cdot\|_\mathcal{V}$-norm. We use this strategy to obtain generalization bounds for (3.2) over various representations. For ease of exposition we assume that $q = 1$, i.e. $V : \mathbb{R}^p \mapsto \mathbb{R}$. The extension to $q > 1$ is straightforward.

4.1 Lipschitz parameteric function classes

We consider the following parametric representation:

$$ \mathcal{V} = \{ V_{\theta}(\cdot) = g(x, \theta) : x \in \mathbb{R}^k, \|\theta\| \leq B_{\theta} \}. $$

We assume $g : \mathbb{R}^p \times \mathbb{R}^k \mapsto \mathbb{R}$ is continuously differentiable, which implies that $\mathcal{V}$ satisfies Assumption 4.2. The parameterization (4.1) is very general and encompasses function classes such as neural networks with differentiable activation functions. Furthermore, Dudley’s estimate combined with a volume comparison argument yields $R_n(V)^2 \leq O(k/n)$, which implies the following result.

Theorem 4.2. Under Assumption 4.1, if problem (3.2) over the parametric function class (4.1) is feasible, then any solution $\hat{V}_n$ satisfies with probability at least $1 - \delta$ over $\xi_1, ..., \xi_n$:

$$ \text{err}(\hat{V}_n) \leq O\left( B^2_{\theta} (L_g + L_{\nabla g})^2 L_h^2 \frac{k \log^2 n}{\gamma^2 n} + \frac{\log(B_h/\gamma)/\delta}{n} \right). $$

Here, $L_g := \sup_{x \in S, \|\theta\| \leq B_{\theta}} \| \nabla \theta g(x, \theta) \|$ and $L_{\nabla g} := \sup_{x \in S, \|\theta\| \leq B_{\theta}} \| \frac{\partial^2 g}{\partial \theta \partial x}(x, \theta) \|$.
Often times (4.1) is more structured. For instance, in sum-of-squares (SOS) optimization, we have:

$$V = \left\{ V_Q(x) = m(x)^T Q m(x) : Q \in \mathbb{R}^{d \times d}, Q = Q^T \succeq 0, \|Q\|_F \leq B_Q \right\},$$  \hspace{1cm} (4.3)

where $m : \mathbb{R}^p \mapsto \mathbb{R}^d$ is a monomial feature map. Note that (4.3) is an instance of (4.1) with $k = d(d + 1)/2$. Hence Theorem 4.2 implies a bound of the form $\text{err}(\hat{V}_n) \leq O(d^2/n)$. However, we can actually use the matrix structure of (4.3) to sharpen the bound to $\text{err}(\hat{V}_n) \leq O(d/n)$ by a more careful estimate of $\mathcal{R}_n(V)$ using the dual Sudakov inequality [45].

**Theorem 4.3.** Under Assumption 4.1, if problem (3.2) over the parametric linear function class (4.3) is feasible, then any solution $\hat{V}_n$ satisfies with probability at least $1 - \delta$ over $\xi_1, \ldots, \xi_n$: \hspace{1cm} (4.4)

$$\text{err}(\hat{V}_n) \leq O(1) \left( B_Q^2 (B_m^2 + B_{Dm} B_m)^2 L_h^2 d \log^2 d \log^3 n \frac{\gamma^2}{\gamma^2 n} + \frac{\log(\log(B_h/\gamma)/\delta)}{n} \right).$$

Here, $B_m := \sup_{x \in S} \|m(x)\|$ and $B_{Dm} := \sup_{x \in S} \|\frac{\partial m}{\partial x}(x)\|$.

### 4.2 Reproducing kernel Hilbert space function classes

We now consider the following non-parametric function class:

$$V = \left\{ V_\alpha(\cdot) = \int_{\Theta} \alpha(\theta) \phi(\cdot; \theta) \, d\theta : \|V_\alpha\|_\nu := \sup_{\theta \in \Theta} \left| \frac{\alpha(\theta)}{\nu(\theta)} \right| \leq B_\alpha \right\}.$$  \hspace{1cm} (4.5)

Here, $\phi(\cdot; \theta)$ is a nonlinear function and $\nu$ is a probability distribution over $\Theta$. This function class is a subset of the reproducing kernel Hilbert space (RKHS) defined by the kernel $k(x, y) = \int_{\Theta} \phi(x; \theta) \phi(y; \theta) \nu(\theta) \, d\theta$, and is dense in the RKHS as $B_\alpha \to \infty$ [32]. We further assume that $\phi(x; \theta)$ is of the form $\phi(x; \theta) = \phi(\langle x, w \rangle + b)$ with $\phi$ differentiable and $\theta = (w, b)$. RKHSs of this type often arise naturally. For instance, Bochner’s theorem [31] states that every translation invariant kernel can be expressed in this form.

**Theorem 4.4.** Suppose that $|\phi| \leq 1$, $\phi$ is $L_\phi$-Lipschitz, $\phi$ is differentiable, $\phi'$ is $L_{\phi'}$-Lipschitz, and that $B_\phi := \sup_{\theta \in \Theta} \|\theta\|$ is finite. Under Assumption 4.1, if problem (3.2) over the non-parametric class (4.5) is feasible, then any solution $\hat{V}_n$ satisfies with probability at least $1 - \delta$ over $\xi_1, \ldots, \xi_n$: \hspace{1cm} (4.6)

$$\text{err}(\hat{V}_n) \leq O(1) \left( B_\alpha^2 (1 + B_\theta L_\phi)^2 L_h^2 \frac{\kappa \log^3 n}{\gamma^2 n} + \frac{\log(\log(1 + B_h/\gamma)/\delta)}{n} + 1/n^2 \right),$$

where $\kappa := \frac{B_\phi^2 L_h^2}{\gamma^2} \left( (1 + B_\theta L_\phi)^2 \log n + B_\theta^2 (B_S + 1)^2 (L_\phi + B_\theta L_{\phi'})^2 p \right)$.

### 5 Global Stability Results

In this section, we show how the bounds from Section 4 can be translated into global results for the learned certificate functions. To facilitate our analysis, we assume the dynamics is incrementally stable. Incremental stability is implied by contraction, but is stronger than Lyapunov stability. Before stating the assumption, we say that $\beta(s, t)$ is a class $KL$ function if for every $t$ the map $s \mapsto \beta(s, t)$ is a class $K$ function and for every $s$ the map $t \mapsto \beta(s, t)$ is continuous and non-increasing.
Assumption 5.1 (Incremental stability, c.f. Hanson and Raginsky [14]). There exists a class $\mathcal{KL}$ function $\beta$ such that for all $\xi_1, \xi_2 \in X$, $\|\varphi_t(\xi_1) - \varphi_t(\xi_2)\| \leq \beta(\|\xi_1 - \xi_2\|, t)$ for all $t \in T$.

With Assumption 5.1 in hand, we are ready to state a result regarding learned Lyapunov functions. For what follows, let $\mathbb{B}_2^p(r)$ denote the closed $\ell_2$-ball in $\mathbb{R}^p$ of radius $r$, $\mathbb{S}^{p-1}$ denote the sphere in $\mathbb{R}^p$, and $\mu_{\text{Leb}}(\cdot)$ denote the Lebesgue measure on $\mathbb{R}^p$.

**Theorem 5.1.** Suppose the system satisfies Assumption 5.1, and suppose the set $X$ is full-dimensional and compact. Define the set $S := \cup_{t \in T} \varphi_t(X)$. Let $V : S \mapsto \mathbb{R}$ be a twice-differentiable positive definite function satisfying $V(x) \geq \mu \|x\|^2$ for all $x \in S$. Define the violation set $X_b$ as:

$$X_b := \left\{ \xi \in X : \max_{t \in T} \langle \nabla V(\varphi_t(\xi)), f(\varphi_t(\xi)) \rangle > \lambda V(\varphi_t(\xi)) \right\}. \quad (5.1)$$

Let $\nu$ denote the uniform probability measure on $X$ and suppose that $\nu(X_b) \leq \varepsilon$. Define the function $q(x) := \langle \nabla V(x), f(x) \rangle$, and denote the constants $B_{\nabla q} := \sup_{x \in S} \|\nabla q(x)\|$, $B_{\nabla V} := \sup_{x \in S} \|\nabla V(x)\|$. Let $r(\varepsilon) := \left( \frac{\varepsilon \mu_{\text{Leb}}(X)}{\mu_{\text{Leb}}(\mathbb{B}_2^p(1))} \right)^{1/p}$. Then for all $\eta \in (0, 1)$:

$$\langle \nabla V(x), f(x) \rangle \leq -(1 - \eta)\lambda V(x) \quad \forall x \in S \backslash \mathbb{B}_2^p \left( 0, \sqrt{\frac{\beta(r(\varepsilon), 0)}{\eta \mu}} (B_{\nabla V} + \lambda^{-1} B_{\nabla q}) \right). \quad (5.2)$$

Furthermore, for every $\xi \in X$, let $u_\xi(t)$ denote the solution to the differential equation:

$$\dot{u}_\xi = -\lambda u_\xi + (B_{\nabla q} + \lambda B_{\nabla V}) \beta(r(\varepsilon), t), \quad u_\xi(0) = V(\xi). \quad (5.3)$$

Then for every $\xi \in X$ and $t \in T$, the inequality $V(\varphi_t(\xi)) \leq u_\xi(t)$ holds.

Theorem 5.1 states that the learned Lyapunov function $V$ satisfies the Lie derivative decrease condition on all of $S$ except for a ball of radius $r_b \leq O(\sqrt{\beta(r(\varepsilon), 0)})$ around the origin. Since $r(\varepsilon) \to 0$ as $\varepsilon \to 0$, Theorem 5.1 shows that the quality of our Lyapunov function increases as the measure of the violation set $X_b$ decreases. Furthermore, we can apply the bounds in Section 4 to obtain an upper bound on the radius $r_b$ of the ball as a function of the number of sample trajectories. For example, Theorem 4.2 states that $\nu(X_b) \leq O(k/n)$ if $n$ random samples are drawn uniformly from $X$. For simplicity assume $X = \mathbb{B}_2^p(1)$ and $\beta(r, 0) \leq O(r)$, which implies $r_b \leq O((k/n)^{1/2p})$. Setting $r_b \leq \zeta$ and solving for $n$, we find $n \geq \Omega(k \cdot \zeta^{-2p})$. Such an exponential dependence on the dimension $p$ is unavoidable without assuming more structure.

Equation (5.3) yields bounds of the form $V(\varphi_t(\xi)) \leq V(\xi) e^{-\lambda t} + O(r(\varepsilon) h_\beta(t))$, where $h_\beta$ depends on the specific form of $\beta$. For example if $\beta(s, t) \leq Me^{-\alpha t} s$ for some $\alpha > \lambda$, then $h_\beta(t) = e^{-\lambda t}$. On the other hand, if we have the slower rate $\beta(s, t) \leq Ms/(t + 1)$, then $h_\beta(t) = t^2 e^{-\lambda t}$.

We note that Theorem 5.1 is conceptually similar to the results from Liu et al. [25], but incremental stability assumption dramatically simplifies the proof and enables us to make the constants explicit.

We now state a similar result to Theorem 5.1 for metric learning. Let $\psi_t(\cdot)$ denote the induced flow on the prolonged system $g(x, \delta x) = (f(x), \partial f(x)/\partial x \delta x)$ and $\theta_t(\delta \xi; \xi)$ denote the second element of $\psi_t(\xi, \delta \xi)$. Further let $\zeta_\rho(r)$ be the Haar measure of a spherical cap in $\mathbb{S}^{p-1}$ with arc length $r$. 

8
Theorem 5.2. Fix an $\eta \in (0, 1)$. Suppose that $X \subseteq \mathbb{R}^p$ is full-dimensional and $p \geq 2$. Let $\dot{x} = f(x)$ be contracting in the metric $M_*(x)$ with rate $\lambda > 0$. Assume that $mI \leq M_*(x) \leq LI$. Let $V(x, \delta x) : TS \mapsto \mathbb{R}$ be of the form $V(x, \delta x) = \delta x^TM(x)\delta x$ for some positive definite matrix function $M(x)$ satisfying $M(x) \geq \mu I$. Define the violation set $Z_b$ as:

$$Z_b := \left\{ (\xi, \delta \xi) \in X \times S^{p-1} : \max_{t \in T} \langle \nabla V(\psi_t(\xi, \delta \xi)), g(\psi_t(\xi, \delta \xi)) \rangle > \lambda V(\psi_t(\xi, \delta \xi)) \right\} . \quad (5.4)$$

Suppose that $\nu(Z_b) \leq \varepsilon$, where $\nu$ is the uniform probability measure on $X \times S^{p-1}$. Define $r(\varepsilon) := \sup \left\{ r > 0 : r^p \zeta_p(r) \leq \frac{\nu(\mu \text{Lab}(X))}{\mu \text{Lab}(S_1)} \right\}$, and let the radius $r_b := \sqrt{r(\varepsilon)B_H (B_{\varphi}(L/m)^{3/2}/(\eta \lambda \mu)}$, where $B_{\varphi} := \sup_{x \in S} \left\| \frac{\partial \varphi}{\partial x} \right\|$. Finally, define the sets $X_t(r_b) := \{ \xi \in X : \inf_{\delta \xi \in S^{p-1}} \| \theta_t(\delta \xi; \xi) \| \geq r_b \}$ for $t \in T$. Then the system will be contracting in the metric $M(x)$ at the rate $(1 - \eta)\lambda$ for every $x \in S(r_b) := \cup_{t \in T} \varphi(t)(X_t(r_b))$.

Theorem 5.2 is illustrated in Figure 1, which shows the structure of the violation set. Further details and exploration of the effect of $\eta$ can be found in Section A.5. Finally, in Section D we prove a result similar to Theorem 5.2 for metric learning with known dynamics.

6 Learning Certificates in Practice

We empirically study the generalization behavior of both learning Lyapunov functions and contraction metrics from trajectory data. We consider Lyapunov functions parameterized by $V(x) = x^T(L(x)L(x)^T + I)x$, where $L(x)$ is the (reshaped) value of a fully connected neural network with tanh activations of size $p \times h \times h \times p \cdot (2p)$, where $p$ is the state-dimension of $x$ and $h$ is the hidden width. For metric learning, we study a convex formulation via SOS programming. Each matrix element $M_{ij}(x) = \langle w_{ij}, \phi(x) \rangle$ is given by a polynomial where $w_{ij}$ are the learned weights and $\phi(x)$ is a feature map of monomials in the state vector. In our experiments, we numerically estimate the generalization error of a learned certificate using a test set. We compute an upper confidence bound (UCB) of the estimate using the Chernoff inequality with $\delta = 0.01$, as described by Langford [22]. More experimental details are given in the appendix.

Damped pendulum. We learn a Lyapunov function for the damped pendulum from 1000 training trajectories. Figure 2 (UL) shows the level sets of a typical learned Lyapunov function, where we also numerically rollout a dense set of trajectories starting from $\{ x \in \mathbb{R}^2 : V_0(x) = 10 \}$ to check set invariance. In Figure 2 (UR), we add a disturbance $\langle a, \kappa \phi(t) \rangle$ to the dynamics where $a \in \mathbb{R}^{10}$ is unknown and $\phi(t)$ are random sinusoids. We use an adaptive control law [40] based on the learned Lyapunov function to regulate $x \to 0$ (see the appendix for details). We vary $\kappa \in \{1, 6, 10\}$ to study the robustness of the adaptation. Figure 2 (UR) shows that the learned Lyapunov function is able to provide enough information to robustly regulate the state even as the disturbance $\kappa$ increases by a factor of 10, whereas the system without adaptation is driven far from the origin.

Stable standing for quadrupeds. We learn a discrete-time Lyapunov function for a quadruped robot [19] as it recovers from external forcing. We apply a random impulse force in the $(x, y)$ plane at time $t = 0$ to the Minitaur quadruped environment in PyBullet [9], and use a hand-tuned PD
controller to return the minitaur to a standing position. We train a discrete-time Lyapunov function in order to handle the discontinuities in the trajectories introduced by contact forces.

Figure 2 (LL) shows the result of this experiment. For the Lyapunov curve, the resulting model trained on $n$ trajectories is then validated using a 10000 trajectory test set. The generalization error is the ratio of trajectories which violate the desired decrease condition for any step $k \in \{1, ..., 199\}$. We run 30 trials and plot the 10/50/90-th percentile of the generalization UCB. With $n = 50000$, the median generalization UCB is 1.11%. Since in practice a separate test set may not be available, we also compare to splitting the available training data into an actual training set of size 0.9$n$ and a validation set of size 0.1$n$. The model is trained on the actual training set, and a generalization UCB is calculated from the validation set. We run 30 trials of this setup and plot the 10/50/90-th percentile in the Holdout curve. After $n = 50000$, the median generalization UCB is 1.99%.

6-dimensional gradient system. Gradient flow has recently been explored in the context of Riemannian motion policies for robotics [33, 34], and converges for nonconvex losses with contracting dynamics [47]. We learn a metric for gradient flow on the nonconvex loss $L(x) = \|x\|^2 + \sum_{i \neq j} x_i^2 x_j^2$ for $x \in \mathbb{R}^6$. Figure 2 (LR) shows the generalization error curves for the differential Lyapunov constraints. Because the SOS program for metric learning is convex, we can apply generalization bounds from randomized convex programming (RCP) [5]. We also plot the probability that the learned metric is a true contraction metric with rate .99$\lambda$ on the test set (probability with rate $\lambda$ is low) and a generalization UCB obtained using a validation set. For each curve, we plot the 10/50/90-th percentile of the generalization UCB. As the number of samples increases, the error
probability for differential Lyapunov constraints decreases, and the learned metric becomes a true metric with reduced rate with high probability. With \( n = 4000 \), the median generalization UCBs are 5.85%, 8.02%, and 5.35% for differential Lyapunov on the test set, differential Lyapunov on the validation set, and contraction with rate \(.99\lambda\), respectively.

7 Conclusion

Our work shows that certificate functions can be efficiently learned from data, and raises many interesting questions for future work. Extending the results to handle both noisy state observations and process noise in the dynamics would allow for learning certificates in uncertain environments. Another interesting question is to establish bounds for joint learning of both the unknown dynamics and a certificate, which has shown to be effective in practice [29, 39]. Finally, lower bounds on the learning certificate problem would highlight the amount of conservatism introduced in our results.

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A Experiment Details

A.1 Metric learning algorithm

Here we give pseudocode for the metric learning algorithm used in the main text. Algorithm 1 is written for a parameterization $M_w(x)$ that yields a convex optimization problem, and where uniform positive definiteness may be enforced globally, such as via SOS matrix constraints. It may be readily relaxed to nonconvex parameterizations such as neural networks by using soft constraints and minimizing the loss using a variant of stochastic gradient descent. Uniform positive definiteness can be imposed along trajectories rather than globally.

\begin{algorithm}
\caption{Metric learning}
\label{alg:metric_learning}
\begin{algorithmic}[1]
\STATE \textbf{Hyperparameters:} timestep $\Delta t$, set $X \subseteq \mathbb{R}^p$, horizon length $T$, linear approximation tolerance $\epsilon$, number of samples $n$, lower bound $\mu$, overshoot $L$, contraction rate $\lambda$.
\STATE \# Generate samples.
\WHILE{number of samples less than $n$}
\STATE Draw $x(1) \in X$ from a distribution $D$ on $X$.
\STATE \# Rejection sample to ensure that $x(1) + \delta x \in X$.
\STATE Draw $\delta x \in \mathbb{R}^p$ uniformly from a ball of radius $\epsilon$ around the origin.
\STATE Set $x(2) := x(1) + \delta x$.
\STATE Compute the flows $\varphi_t(x(1))$, $\varphi_t(x(2))$ for $t \in [0, T]$ with timestep $\Delta t$.
\IF{$\|\varphi_t(x(1)) - \varphi_t(x(2))\| \leq L \epsilon \forall t \in [0, T]$}
\STATE Compute numerical time derivatives of $\varphi_t(x(1))$ and $\psi_t(\delta x) := \varphi_t(x(1)) - \varphi_t(x(2))$.
\STATE Add $\varphi_t(x(1))$, $\psi_t(\delta x)$, $\frac{d}{dt} \varphi_t(x(1))$, $\frac{d}{dt} \psi_t(\delta x)$ to the dataset.
\STATE Increment the number of samples.
\ENDIF
\ENDWHILE
\STATE Solve the optimization problem:
\STATE \begin{align*}
& \min_w \frac{1}{2} \|w\|^2 \\
& \text{s.t. } \frac{d}{dt} \langle \psi_t(\delta x_i), M_w(\varphi_t(x_i))\psi_t(\delta x_i) \rangle \leq -\lambda \langle \psi_t(\delta x_i), M_w(\varphi_t(x_i))\psi_t(\delta x_i) \rangle , \ i = 1, \ldots, n, \\
& \quad M_w(x) \succeq \mu I \ \forall x \in \mathbb{R}^p.
\end{align*}
\end{algorithmic}
\end{algorithm}

A.2 Pendulum

The pendulum dynamics are given by $m\ell \ddot{\theta} + b \dot{\theta} + mg\ell \sin \theta = 0$ with $m = 1$, $g = 9.81$, $\ell = 1$, $b = 2$. The state space is $x = (\theta, \dot{\theta})$ and the stable equilibrium is $x = 0$ with $\theta$ wrapped to the interval

\cite{wainwright2019high, wensing2018beyond}
Consider the dynamical system

\[ \dot{x} = f(x) + B(u(t) - Y(x, t)a), \quad (A.2) \]

with \( f \) continuously differentiable and \( Y(x, t) \) locally bounded in \( x \) uniformly in \( t \). Let \( V \) be a twice continuously differentiable positive definite function such that \( \langle \nabla V(x), f(x) \rangle \leq -\rho(x) \) for all \( x \) for some continuously differentiable positive definite function \( \rho(\cdot) \). Let \( M \) be a positive definite matrix. Let \( \hat{a}(t) \) be defined by the differential equation

\[ \dot{\hat{a}} = -M^{-1}Y(x, t)^{\top}B^{\top}\nabla V(x(t)). \quad (A.3) \]

Then the adaptive control law

\[ u(t) = Y(x, t)\hat{a}(t), \quad (A.4) \]

in feedback with (A.2) drives \( x \to 0 \) and \( \dot{x} \to 0 \).

Lemma A.1.

We generate \( n = 1000 \) trajectories initialized at \( x_0 \sim \text{Unif}([-2, 2] \times [-2, 2]) \). Each trajectory is rolled out using the default integrator for \texttt{scipy.integrate.solve_ivp} for \( T = 8 \) seconds with \( dt = 0.02 \), yielding a dataset of size \( 1000 \times 400 \times 2 \). We use \texttt{scipy}'s \texttt{savgol_filter} with \texttt{window_length} = 5 and \texttt{polyorder} = 2 to numerically compute the derivatives \( \dot{x} \). We set the hidden width \( h = 30 \) and minimize the loss \( L(\theta) = \sum_{i=1}^{1000} \sum_{k=1}^{400} \text{ReLU}(\langle \nabla V_\theta(x_i(k)), \dot{x}_i(k) \rangle + \gamma V_\theta(x_i(k))) + \lambda\|\theta\|^2 \), setting \( \gamma = 0.01 \) and \( \lambda = 0.1 \). The loss is minimized for 1000 epochs with Adam using a step size \( 10^{-3} \) and a batch size of 1000.

We repeat this experiment for 30 trials. For each trial we use a test set of size 1000 to compute a UCB on the generalization error. The 10/50/90-th percentile of the UCBs are 0.459%, 0.459%, and 1.163%.

We also uniformly grid the set \([-2, 2] \times [-4, 4] \) with 40000 points and check numerically how often the condition \( \langle \nabla V_\theta(x), \dot{x} \rangle \leq -\gamma V_\theta(x) \) is violated. The 10/50/90-th percentile of these violations over 30 trials are 0.168%, 1.548%, and 3.696%. We note that these numbers are higher than the generalization error because the set \([-2, 2] \times [-4, 4] \) contains points that are outside the flow starting from \([-2, 2] \times [-2, 2] \).

For our adaptive control experiments, the dynamics combined with the added disturbance are

\[ ml^2\ddot{\theta} + b\dot{\theta} + mg\ell \sin \theta + \langle a, \kappa \phi(t) \rangle = u, \]

where \( u \in \mathbb{R} \) is the control input. We sample \( a \in \mathbb{R}^{10} \) from \( a \sim N(0, I) \). We set \( \phi(t) = (\sin(\omega_1 t), \ldots, \sin(\omega_{10} t)) \) where each \( \omega_i \sim \text{Unif}([0, 2\pi]) \) The adaptive control law we use is \( u(t) = \langle \dot{a}(t), \kappa \phi(t) \rangle \) where \( \dot{a}(t) \) evolves according to:

\[ \dot{a}(t) = -\gamma \phi(t)\langle \nabla_x V_\theta(x(t)), e_2 \rangle, \quad \dot{a}(0) = 0, \quad (A.1) \]

where \( \gamma = 15 \), \( V_\theta \) is the learned Lyapunov function, and \( e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^2 \). The idea behind the adaptive control law (A.1) is to rely on the nominal stability of the pendulum dynamics and learn to cancel out the uncertain disturbance. We give a self-contained proof of its correctness.
Proof. Let \(\langle x, y \rangle_M = x^T M y\) and \(\|x\|_M^2 = \langle x, x \rangle_M\). Define the new candidate Lyapunov function
\[
\tilde{V}(t) := V(t) + \frac{1}{2} \|\dot{a} - a\|_M^2 .
\] (A.5)

Let \(\tilde{a} := \dot{a} - a\). Differentiating \(\tilde{V}\) with respect to time:
\[
\dot{\tilde{V}} = \langle \nabla V(x), f(x) + B(u - Y a) \rangle + \langle \tilde{a}, \dot{a} \rangle_M
\]
\[
= \langle \nabla V(x), f(x) \rangle + \langle \nabla V(x), B Y \tilde{a} \rangle + \langle \tilde{a}, \dot{a} \rangle_M
\]
\[
= \langle \nabla V(x), f(x) \rangle + \langle Y^T B^T \nabla V(x), \tilde{a} \rangle + \langle \tilde{a}, \dot{a} \rangle_M
\]
\[
= \langle \nabla V(x), f(x) \rangle + \langle M^{-1} Y^T B^T \nabla V(x), \tilde{a} \rangle_M + \langle \tilde{a}, \dot{a} \rangle_M
\]
\[
\leq -\rho(x) .
\]

Since \(-\rho(x) < 0\) for all \(x \neq 0\), this shows that \(\tilde{V}\) is bounded for all \(t\), which implies both that \(V\) is bounded and that \(\tilde{a}\) is bounded for all \(t\). Since \(V\) is positive definite, \(V\) bounded implies that \(x\) is bounded. Integrating the above inequality shows that
\[
\int_0^\infty \rho(x(\tau)) d\tau \leq \tilde{V}(0) ,
\]
so that \(\rho \in L_1\). Now, \(\dot{\rho} = \langle \nabla \rho, f(x) + B Y \tilde{a} \rangle\). By continuity of \(\nabla \rho, f\), and \(Y\), and by boundedness of \(\tilde{a}\), \(\dot{\rho}\) is bounded. Hence \(\rho\) is uniformly continuous, and by Barbalat’s Lemma (see e.g. Lemma 4.2 of Slotine and Li [40]) \(\rho \to 0\). By positive definiteness of \(\rho\), \(\rho \to 0\) implies that \(x \to 0\) and \(\dot{x} \to 0\).

\[\Box\]

A.3 Minitaur

We collect 50000 random training trajectories and 10000 random test trajectories using the same distribution over the impulse force. For each trajectory, we step the simulator 200 times at \(dt = 0.002\). The state dimension excluding the base orientation is 16.

The PD controller is able to return the joint angles and velocities (excluding the base orientation) to their original standing position up to a small bias of size \(\sim 0.2\) in \(\ell_2\)-norm. Therefore, we train a discrete-time Lyapunov function \(V_\theta\) to satisfy \(V_\theta(e_i(k+1)) \leq \rho V_\theta(e_i(k)) + \gamma\) where \(e_i(k)\) is the error state of the \(i\)-th trajectory at the \(k\)-th step. The specific values we use are \((\rho, \gamma) = (0.945, 0.025)\). The extra slack term \(\gamma\) is necessary for the Lyapunov function to converge to a ball instead of zero.

We use a hidden width of \(h = 40\) and minimize the loss \(L_n(\theta) = \sum_{i=1}^n \sum_{k=1}^{199}\text{ReLU}(V_\theta(e_i(k + 1)) - \rho V_\theta(e_i(k)) - \gamma) + \lambda ||\theta||^2\), setting \(\lambda = 0.01\). The loss is minimized for 1000 epochs with Adam using a step size \(10^{-3}\) with cosine decay\(^1\) and a batch size of 1000.

A.4 6-dimensional gradient system

We parametrize the metric via monomials up to degree two in the state variables. We enforce global positive definiteness \(M(x) \geq \mu I\) via SOS matrix constraints and set \(\mu = 1\). We use a tolerance of \(5 \times 10^{-3}\) for the size of each perturbation \(\delta x\). A pair of trajectories \(\varphi_t(x_1), \varphi_t(x_2)\) with \(x_2 = x_1 + \delta x\)

\[^1\text{See https://www.tensorflow.org/api_docs/python/tf/compat/v1/train/cosine_decay.}\]
is considered to generate a trajectory \( \delta x(t) = \varphi_t(x_2) - \varphi_t(x_1) \) if \( \|\delta x(t)\| < 10^{-2} \) for all \( t \), so that a small overshoot is permitted. Pairs of trajectories not satisfying this requirement are discarded until the desired number of training samples is reached. The size of this overshoot parameter sets the maximum allowed \( L \) where \( M(x) \leq L \) for any metric learned, as we impose \( M(x) \geq I \). In general, the overshoot with respect to the Euclidean norm is given by \( \sqrt{L} \) for \( lI \leq M(x) \leq LI \).

In practice, we require the maximum bound on \( \|\delta x(t)\| \) to be sufficiently small that the dynamics of \( \varphi_t(x_1) - \varphi_t(x_2) \) well-approximates the variational system on the trajectory \( \varphi_t(x_1) \). To search for metrics with larger values of \( L \), we can vary \( \mu \), \( \|\delta x\| \), and the maximum allowed \( \|\delta x(t)\| \) while ensuring \( \|\delta x(t)\| \) remains small throughout its entire trajectory.

Each trajectory is simulated until \( T = 2 \) seconds with a timestep \( dt = 5 \times 10^{-3} \). Because we are interested in convergence of the variational dynamics, we use a small time horizon. This generates a dataset of size \( n \times 400 \times 12 \) where \( n \) is the number of trajectories and 12 is the dimension of the tangent bundle. We subsequently downsample and impose 25 differential Lyapunov constraints along each trajectory. We search for a metric with a rate \( \lambda = 4 \). Initial conditions are drawn uniformly from the ball of radius \( r = 3 \).

The time derivatives \( \dot{x} \) and \( \delta \dot{x} \) are computed numerically by fitting a cubic spline to the corresponding trajectories and analytically differentiating the spline. \( M(x) \) is found by minimizing \( \|w\|^2 \) where \( w \) is a vector containing all parameters. The test set is of size 1000 and each data point in Figure 2 (LR) was computed by averaging over 25 independent draws of the training set. The RCP bound is obtained with a confidence of \( \delta = .01 \).

### A.4.1 Randomized convex programs

Consider the following optimization problem:

\[
\min_{x \in X} \langle c, x \rangle : f(x, \theta) \leq 0 \ \forall \theta \in \Theta . \tag{A.6}
\]

We assume that \( X \subseteq \mathbb{R}^d \) is a convex set and \( x \mapsto f(x, \theta) \) is convex for every \( \theta \in \Theta \). In the case where \( \Theta \) is an infinite (or very large) set, we consider approximations to (A.6) formulated as follows. Let \( \nu \) denote a distribution over \( \Theta \). Let \( \theta_1, \ldots, \theta_n \) be i.i.d. samples from \( \nu \). Let \( \hat{x}_n \) denote a solution to:

\[
\min_{x \in X} \langle c, x \rangle : f(x, \theta_i) \leq 0 , \ i = 1, \ldots, n . \tag{A.7}
\]

**Theorem A.2** (See e.g. Theorem 3.1 of Calafiore [5]). Fix any \( \varepsilon \in [0,1] \). Define \( \beta(\varepsilon) \) as:

\[
\beta(\varepsilon) := \sum_{i=0}^{d-1} \binom{n}{i} \varepsilon^i (1 - \varepsilon)^{n-i} .
\]

With probability at least \( 1 - \beta(\varepsilon) \) over \( \theta_1, \ldots, \theta_n \), we have that:

\[
\mathbb{P}_{\theta \sim \nu}(f(\hat{x}_n, \theta) > 0) \leq \varepsilon .
\]

We use Theorem A.2 as follows. We fix a failure probability \( \delta \in (0,1) \). We then numerically solve for \( \varepsilon \) such that \( \beta(\varepsilon) = \delta \).
To understand the scaling of $\varepsilon$ as a function of $n$ and $\delta$, despite the lack of closed form expression, consider the following. If $n \geq d$ then by a Chernoff bound on the CDF of a Binomial random variable (c.f. Section 5 of Calafiore [5]), we can derive the upper bound

$$\varepsilon \leq c \frac{(d - 1 + \log(1/\delta))}{n},$$

where $c$ is a universal constant.

### A.5 Van der Pol

The study of the Van der Pol (VDP) oscillator was foundational to the development of nonlinear dynamics [43], and its global contraction properties have been analyzed algorithmically via SOS programming [3]. We study the damped VDP to visualize the violation set for the metric condition categorized in Theorem 5.2. The dynamics of the damped Van der Pol are given by

$$\ddot{x} + \alpha (x^2 + k) \dot{x} + \omega^2 x.$$

We set $\alpha = k = \omega = 1$.

We parameterize the metric via monomials up to degree four in the state variables. Similar to the gradient system, we set $M(x) \geq I$ and use a tolerance $\|\delta x\| \leq 5 \times 10^{-3}$, $\|\delta x(t)\| \leq 10^{-2}$ for all $t \in [0, T]$. We use a timestep $dt = 5 \times 10^{-3}$ and simulate until a final time $T = 3$ seconds. This generates a dataset of size $n \times 600 \times 4$ where 4 is the dimension of the tangent bundle and $n$ is the number of training trajectories.

We subsequently downsample and impose 50 differential Lyapunov constraints along each trajectory with a rate $\lambda = 3/4$. Initial conditions are drawn uniformly from a ball of radius $r = 2$.

The same techniques are used for numerical differentiation as for the gradient system.

Theorem 5.2 predicts that the size of the violation set will decrease as the rate tested for the metric condition $\eta \lambda$ with $0 < \eta < 1$ decreases, or as the number of training samples $n$ increases. In both Figure 1 and Figure 3, we use a uniform grid with $9 \times 10^4$ points over $[-2, 2] \times [-2, 2]$ to test the metric condition $\frac{\partial f}{\partial x}(x)^T M(x) + M(x) \frac{\partial f}{\partial x}(x) + M(x) \leq -2\eta \lambda$ for the learned metric and the true dynamics.

In Figure 1, we plot the violation set for fixed $\eta = 1$ as a function of the number of training samples. As discussed in the main text, the size of the violation set decreases, and it is pushed to the boundary of the sampled region as $n$ increases.

In Figure 3, we plot the violation set as a function of $\eta$. As $\eta$ decreases to zero, the size of the violation set decreases and is pushed to the boundary of the sampled region.

### B Proofs for Section 4

Recall that Dudley’s inequality gives us the following estimate on $R_n(V)$:

$$R_n(V) \leq \frac{24L_h}{\sqrt{n}} \int_0^\infty \sqrt{\log N(\varepsilon; V, \|\cdot\|_V)} d\varepsilon.$$

(B.1)
Figure 3: Violation set for contraction metric learning as a function of the tested contraction rate $\eta \lambda$ with $0 < \eta < 1$ for the damped VDP system.

**B.1 Proof of Theorem 4.2**

For every $V_{\theta_1}, V_{\theta_2} \in \mathcal{V}$, $\|V_{\theta_1} - V_{\theta_2}\|_\mathcal{V} \leq (L_g + L_{\nabla g})\|\theta_1 - \theta_2\|$. Furthermore, a standard volume comparison argument tells us that $\log N(\varepsilon; \mathbb{B}_2^k(1), \|\cdot\|) \leq k \log(1 + 2/\varepsilon)$, where $\mathbb{B}_2^k(1)$ is the closed $\ell_2$-ball in $\mathbb{R}^k$ of radius 1. Therefore, by (B.1):

$$R_n(\mathcal{V}) \leq \frac{24B_\theta(L_g + L_{\nabla g})L_h}{\sqrt{n}} \int_0^\infty \sqrt{\log N(\varepsilon; \mathbb{B}_2^k(1), \|\cdot\|)} \, d\varepsilon \leq O(1)B_\theta(L_g + L_{\nabla g})L_h\sqrt{\frac{k}{n}}.$$ 

Theorem 4.2 now follows from Lemma 4.1.

**B.2 Proof of Theorem 4.3**

A simple calculation shows that $\|V_{Q_1} - V_{Q_2}\|_\mathcal{V} \leq (B_m^2 + 2B_{D_m}B_m)\|Q_1 - Q_2\|$. Hence by (B.1),

$$R_n(\mathcal{V}) \leq \frac{24B_Q(B_m^2 + 2B_{D_m}B_m)L_h}{\sqrt{n}} \int_0^\infty \sqrt{\log N(\varepsilon; \mathbb{B}_d^{d \times d}(1), \|\cdot\|)} \, d\varepsilon .$$

Here, $\mathbb{B}_d^{d \times d}(1)$ is the closed ball of $d \times d$ matrices with Frobenius norm bounded by 1, and $\|\cdot\|$ for matrices denotes the operator norm. The metric entropy $\log N(\varepsilon; \mathbb{B}_d^{d \times d}(1), \|\cdot\|)$ can be bounded by the minimum of the standard volume comparison bound and applying the dual Sudakov inequality (see e.g. Theorem 2.2 of Vershynin [45]):

$$\log N(\varepsilon; \mathbb{B}_d^{d \times d}(1), \|\cdot\|) \leq \min\{d^2 \log(1 + 2/\varepsilon), cd/\varepsilon^2\} .$$

Here, $c$ is an absolute constant. By integrating this estimate we arrive at the bound:

$$R_n(\mathcal{V}) \leq O(1)B_Q(B_m^2 + B_{D_m}B_m)L_h\sqrt{\frac{d}{n}} \log d . \quad (B.2)$$

Theorem 4.3 now follows from Lemma 4.1.
B.3 Proof of Theorem 4.4

Define the function classes \( \mathcal{F}(B) \) and \( \hat{\mathcal{F}}(B, \{\theta_k\}) \) as:

\[
\mathcal{F}(B) := \left\{ f(x) = \int_{\Theta} \alpha(\theta) \phi(x; \theta) \, d\theta : \|f\|_\nu := \sup_{\theta \in \Theta} \|\alpha(\theta)\| \leq B \right\},
\]

\[
\hat{\mathcal{F}}(B, \{\theta_k\}_{k=1}^K) := \left\{ f(x) = \sum_{k=1}^{K} c_k \phi(x; \theta_k) : \|c\|_1 \leq B \right\}.
\]

The following is a simple modification of Theorem 3.2 from Rahimi and Recht [32], which also accounts for uniform approximation of the derivatives.

**Lemma B.1.** Fix a \( B > 0 \) and a bounded space \( X \subseteq \mathbb{R}^p \). Let \( B_X := \sup_{x \in X} \|x\| \). Suppose that \( \Theta \subseteq \mathbb{R}^p \times \mathbb{R} \) with \( B_\theta := \sup_{\theta \in \Theta} \|\theta\| \). Furthermore, suppose that \( \phi(x; \theta) = \phi(\langle x, w \rangle + b) \), \( \phi \) is \( L_\phi \)-Lipschitz, and \( \|\phi\| \leq 1 \). Fix a \( f \in \mathcal{F}(B) \) and \( \delta \in (0, 1) \). Let \( \theta_1, \ldots, \theta_K \) be i.i.d. draws from \( \nu \). With probability at least \( 1 - \delta \), there exists a \( \hat{f} \in \hat{\mathcal{F}}(\|f\|_\nu, \{\theta_k\}_{k=1}^K) \) such that:

\[
\sup_{x \in X} |\hat{f}(x) - f(x)| \leq \frac{2\|f\|_\nu}{\sqrt{K}} \left( 1 + \sqrt{\log(1/\delta)} + 2L_\phi \left( B_X \sqrt{\mathbb{E}_\nu \|w\|^2} + \mathbb{E}_\nu \|b\|^2 \right) \right). \tag{B.5}
\]

Furthermore, now assume that \( \phi \) is differentiable and \( \phi' \) is \( L_{\phi'} \)-Lipschitz. Then every \( f \in \mathcal{F}(B) \) is differentiable with

\[
\nabla f(x) = \int_{\Theta} \alpha(\theta) \nabla \phi(x; \theta) \, d\theta. \tag{B.6}
\]

Finally, with probability at least \( 1 - 2\delta \), there exists a \( \hat{f} \in \hat{\mathcal{F}}(\|f\|_\nu, \{\theta_k\}_{k=1}^K) \) such that (B.5) holds and also:

\[
\sup_{x \in X} \|\nabla \hat{f}(x) - \nabla f(x)\| \leq \frac{B_\theta \|f\|_\nu}{\sqrt{K}} \left( L_\phi \sqrt{2\log(1/\delta)} + 4(L_\phi + B_\theta L_{\phi'})(B_X + 1)\sqrt{\nu} \right). \tag{B.7}
\]

**Proof.** Following the proof of Theorem 3.2 of Rahimi and Recht [32], we set \( c_k = \frac{\alpha(\theta_k)}{K \nu(\theta_k)} \), and we define \( v(\theta_1, \ldots, \theta_K) = \|\hat{f} - f\|_\infty \). It is shown in [32] that for all \( \theta_1, \ldots, \theta_K, \theta'_K \in \Theta \):

\[
|v(\theta_1, \ldots, \theta_K, \theta'_K) - v(\theta_1, \ldots, \theta_K)| \leq \frac{2\|f\|_\nu}{K}.
\]

Next, we control the expected value of \( v \):

\[
\mathbb{E}v(\theta_1, \ldots, \theta_K) = \mathbb{E} \sup_{x \in X} |\hat{f}(x) - \mathbb{E} \hat{f}(x)|
\]

\[
\leq 2\mathbb{E} \sup_{x \in X} \left| \sum_{k=1}^{K} \varepsilon_k c_k \phi(\langle x, w_k \rangle + b_k) \right|
\]

\[
= 2\mathbb{E} \sup_{x \in X} \left| \sum_{k=1}^{K} \varepsilon_k c_k (\phi(\langle x, w_k \rangle + b_k) - \phi(0)) + \sum_{k=1}^{K} \varepsilon_k c_k \phi(0) \right|
\]

\[
\leq 2\mathbb{E} \sup_{x \in X} \left| \sum_{k=1}^{K} \varepsilon_k c_k (\phi(\langle x, w_k \rangle + b_k) - \phi(0)) \right| + 2\mathbb{E} \sup_{x \in X} \left| \sum_{k=1}^{K} \varepsilon_k c_k \phi(0) \right|
\]

\[
=: T_1 + T_2.
\]
Above, the first inequality is a standard symmetrization argument where the Rademacher variables \( \{ \varepsilon_k \} \) are introduced, and the second inequality is the triangle inequality. We first bound \( T_1 \). Set \( \psi_k(z) := c_k(\phi(z) - \phi(0)) \). Clearly \( \psi_k(0) = 0 \), and also \( \psi_k \) is \( c_k L_\phi \)-Lipschitz. We bound \( c_k L_\phi \leq \frac{||f||_L}{K} \).

Therefore by the contraction inequality for Rademacher complexities (Theorem 4.12 of Ledoux and Talagrand [23]) followed by Jensen’s inequality:

\[
T_1 \leq \frac{4||f||_L}{K} E_\theta E_\varepsilon \sup_{x \in X} \sum_{k=1}^{K} \varepsilon_k (\langle x, w_k \rangle + b_k) \\
\leq \frac{4||f||_L}{K} LB_X E_\theta E_\varepsilon \left( \sum_{k=1}^{K} \varepsilon_k w_k \right) + \frac{4||f||_L}{K} E_\theta E_\varepsilon \left( \sum_{k=1}^{K} \varepsilon_k b_k \right) \\
\leq \frac{4||f||_L}{K} \sqrt{E \|w_1\|^2 + \sqrt{E} \|b_1\|^2}.
\]

Furthermore we can bound \( T_2 \leq \frac{2||f||_L}{\sqrt{K}} \) by similar arguments. The claim (B.5) now follows by invoking McDiarmid’s inequality.

We note that (B.6) follows from a basic application of the dominated convergence theorem, since we have that:

\[
E_{\theta \sim \nu} \sup_{x \in X} \| \nabla \phi(x; \theta) \| < \infty.
\]

Finally, we focus on the derivative condition (B.7). Let \( g(\theta_1, \ldots, \theta_K) := \sup_{x \in X} \| \nabla \hat{f}(x) - \nabla f(x) \| \).

By symmetrization we have:

\[
E g(\theta_1, \ldots, \theta_K) = E \sup_{x \in X} \| \nabla \hat{f}(x) - E \nabla f(x) \| \\
= E \sup_{x \in X} \sup_{\|q\|=1} \langle q, \nabla \hat{f}(x) \rangle - E \langle q, \nabla f(x) \rangle \\
\leq 2 E_{\theta} E_{\varepsilon} \sup_{x \in X} \sup_{\|q\|=1} \sum_{k=1}^{K} \varepsilon_k c_k \phi'(\langle x, w_k \rangle + b_k) \langle q, w_k \rangle.
\]

We set \( \psi_k(x, q) \) to be:

\[
\psi_k(x, q) := c_k \phi'(\langle x, w_k \rangle + b_k) \langle q, w_k \rangle.
\]

For \((x_1, q_1), (x_2, q_2) \in X \times \mathbb{B}_2^p(1)\) we have:

\[
|\psi_k(x_1, q_1) - \psi_k(x_2, q_2)| = c_k |\phi'(\langle x_1, w_k \rangle + b_k) \langle q_1, w_k \rangle - \phi'(\langle x_2, w_k \rangle + b_k) \langle q_2, w_k \rangle| \\
\leq c_k (L_\phi \|q_1 - q_2\| + \|\phi'(\langle x_1, w_k \rangle + b_k) - \phi'(\langle x_2, w_k \rangle + b_k)\|\langle q_2, w_k \rangle) \\
\leq c_k (B_\theta L_\phi \|q_1 - q_2\| + L_\phi B_\theta^2 \|x_1 - x_2\|) \\
\leq \frac{||f||_L}{K} B_\theta (L_\phi + B_\theta L_\phi') \sqrt{2} \left\| \frac{x_1}{q_1} - \frac{x_2}{q_2} \right\|.
\]

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We can now apply Theorem 3 of Maurer [30] to conclude that:

\[
2\mathbb{E}_\theta \mathbb{E}_e \sup_{x \in X} \sup_{\|q\|=1} \sum_{k=1}^K \varepsilon_k c_k \phi'((x, w_k) + b_k) (q, w_k)
\]

\[
= 2\mathbb{E}_\theta \mathbb{E}_e \sup_{x \in X} \sup_{\|q\|=1} \sum_{k=1}^K \varepsilon_k \psi_k(x, q)
\]

\[
\leq 4 \frac{\|f\|_\nu}{K} B_\theta (L\phi + B_\theta L\phi') \mathbb{E}_e \sup_{x \in X} \sup_{\|q\|=1} \sum_{k=1}^K \left\langle \varepsilon_k, \left[ x \right] \right\rangle
\]

\[
\leq 4 \frac{\|f\|_\nu}{\sqrt{K}} B_\theta (L\phi + B_\theta L\phi')(B_X + 1) \sqrt{p}.
\]

Next, we have for all \(\theta_1, \ldots, \theta_K, \theta'_K \in \Theta:\)

\[
|g(\theta_1, \ldots, \theta_k, \ldots, \theta_K) - g(\theta_1, \ldots, \theta'_k, \ldots, \theta_K)|
\]

\[
\leq \sup_{x \in X} \left\| \frac{\alpha(\theta_k)}{K\nu(\theta_k)} \phi'((x, w_k) + b_k) w_k - \frac{\alpha(\theta'_k)}{K\nu(\theta'_k)} \phi'((x, w'_k) + b'_k) w'_k \right\|
\]

\[
\leq \frac{2B_\theta L\phi \|f\|_\nu}{K}.
\]

The uniform bound on the derivatives (B.7) now follows from another application of McDiarmid’s inequality.

We now turn to the proof of Theorem 4.4. Under the hypothesis of Lemma B.1, we have that for every \(f \in \mathcal{F}(B)\):

\[
\sup_{x \in X} |f(x)| \leq B,
\]

\[
\sup_{x \in X} \|\nabla f(x)\| \leq BB_\theta L\phi.
\]

We also have for any \(\{\theta_k\}_{k=1}^K \subseteq \Theta\) and any \(\hat{f}(x) = \sum_{k=1}^K c_k \phi(x; \theta_k)\) with \(\|c\|_1 \leq B\),

\[
\sup_{x \in X} |\hat{f}(x)| \leq B,
\]

\[
\sup_{x \in X} \|\nabla \hat{f}(x)\| \leq BB_\theta L\phi.
\]

Hence for any \(\{\theta_k\}_{k=1}^K \subseteq \Theta\), the function class \(\mathcal{F}(B_\alpha, \{\theta_k\}_{k=1}^K)\) satisfies Assumption 4.2 with \(B_V = B_\alpha\) and \(B_{\nabla V} \leq B_\alpha B_\theta L\phi\).

Let \(f_n \in \mathcal{F}(B_\alpha)\) denote a feasible solution to (3.2).

At this point, it may be tempting to use the probabilistic method in conjunction with Lemma B.1 to conclude that there exists a set of weights \(\{\bar{\theta}_k\}_{k=1}^K\) such that there exists a \(\hat{f}_n \in \mathcal{F}(B_\alpha, \{\bar{\theta}_k\}_{k=1}^K)\) such that \((f_n, \nabla f_n)\) closely approximates \((\hat{f}_n, \nabla \hat{f}_n)\). This will not work however, since the function class \(\mathcal{F}(B_\alpha, \{\bar{\theta}_k\}_{k=1}^K)\) then becomes a function of the training data \(\xi_1, \ldots, \xi_n\), and hence we would not be able to apply Lemma 4.1 to it.

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To work around this, we need to draw the weights independently of $\xi_1, \ldots, \xi_n$. In particular, we set $K$ such that

$$K = \left\lceil \frac{cB_\alpha^2 L_h^2}{\gamma^2}((1 + B_\theta L_\phi)\sqrt{\log n} + B_\theta (B_S + 1)(L_\phi + B_\theta L_{\phi'})\sqrt{p})^2 \right\rceil,$$

where $c$ is an absolute constant and let $\{\theta_k^*\}_{k=1}^K$ be drawn i.i.d. from $\nu$.

By invoking Lemma B.1 with $K$ as above and $\delta = 1/n^2$, we know there exists an event $\mathcal{E}_{\text{approx}}$ on $\nu^{\otimes K}$ such that on $\mathcal{E}_{\text{approx}}$, there exists a function $\hat{f}_n \in \hat{\mathcal{F}}(B_\alpha, \{\theta_k^*\}_{k=1}^K)$ that satisfies:

$$\sup_{x \in S} |f_n(x) - \hat{f}_n(x)| \leq \gamma/(8\sqrt{2}L_h),$$

$$\sup_{x \in S} \|\nabla f_n(x) - \nabla \hat{f}_n(x)\| \leq \gamma/(8\sqrt{2}L_h).$$

By the definition of $L_h$, these two inequalities imply that

$$\sup_{\xi \in X} |h(\xi; f_n) - h(\xi; \hat{f}_n)| \leq \gamma/4.$$

This means that if $f_n$ is feasible for (3.2) with slack variable $\gamma$, then on $\mathcal{E}_{\text{approx}}$ we have that $\hat{f}_n$ is feasible with slack variable $3\gamma/4$. Specifically:

$$h(\xi_i; \hat{f}_n) \leq -3\gamma/4, \quad i = 1, \ldots, n. \tag{B.8}$$

Observe then that:

$$\mathbb{P}(h(\xi; f_n) > 0) \leq \mathbb{P}(\{h(\xi; f_n) > 0\} \cap \mathcal{E}_{\text{approx}}) + \mathbb{P}(\mathcal{E}_{\text{approx}}^c)$$

$$\leq \mathbb{P}(h(\xi, \hat{f}_n) > -\gamma/4) + 1/n^2$$

$$= \mathbb{P}(h(\xi, \hat{f}_n) + \gamma/4 > 0) + 1/n^2.$$

Here, $\mathbb{P}(\cdot)$ denotes the product measure $\mathcal{D} \otimes \nu^{\otimes n}$ over $(\xi, \{\theta_k^*\}_{k=1}^K)$. Now we define $\bar{h} = h + \gamma/4$. From (B.8),

$$\bar{h}(\xi, \hat{f}_n) = h(\xi, \hat{f}_n) + \gamma/4 \leq -\gamma/2.$$

We can then apply Lemma 4.1 with $\bar{h}$ (with the change $B_h \leftarrow B_h + \gamma/4$ and $\gamma \leftarrow \gamma/2$), to the finite dimensional parametric function class $\hat{\mathcal{F}}(B_\alpha, \{\theta_k^*\}_{k=1}^K)$ (as noted above, this is valid because the elements $\{\theta_k^*\}_{k=1}^K$ are drawn independently from the training data $\xi_1, \ldots, \xi_n$).

The result is that with probability at least $1 - \delta$ over $\xi_1, \ldots, \xi_n$:

$$\mathbb{P}(h(\xi, \hat{f}_n) > -\gamma/4) = \mathbb{P}(\bar{h}(\xi, \hat{f}_n) > 0)$$

$$\leq O(1) \left( \frac{\log^3 n}{\gamma^2} \mathcal{R}_n^2(\hat{\mathcal{F}}(B_\alpha, \{\theta_k^*\}_{k=1}^K)) + \frac{\log(\log(1 + B_h/\gamma)/\delta)}{n} \right).$$

Letting $\hat{f} = \sum_{k=1}^K c_k \phi(x; \theta_k^*)$ and $\hat{g} = \sum_{k=1}^K d_k \phi(x; \theta_k^*)$, we have that $\|\hat{f} - \hat{g}\|_1 \leq (1 + B_\theta L_\phi)\|c - d\|_1$. Hence by (B.1),

$$\mathcal{R}_n(\hat{\mathcal{F}}(B_\alpha, \{\theta_k^*\}_{k=1}^K)) \leq \frac{24B_\alpha(1 + B_\theta L_\phi)L_h}{\sqrt{n}} \int_0^\infty \sqrt{\log N(\varepsilon; \mathbb{H}_1^K(1), \|\cdot\|_1)} d\varepsilon$$

$$\leq O(1)B_\alpha(1 + B_\theta L_\phi)L_h \frac{\sqrt{K}}{n}.$$
Combining the inequalities above:

\[ \mathbb{P}(h(\xi; f_n) > 0) \leq O(1)B_\phi^2(1 + B_\phi L^2)\frac{K\log^3 n}{\gamma^2 n} + O(1)\frac{\log(1 + B_h/\gamma) \delta}{n} + 1/n^2. \]

### C Proofs for Section 5

Before we prove the main results in Section 5, we state and prove a few technical lemmas which we will need. We will let \( \mathbb{B}_d^p(x, r) \) denote the \( \ell_p \) ball in \( \mathbb{R}^d \) centered around \( x \) with radius \( r \), and \( \mu_{\text{Leb}}(\cdot) \) denote the Lebesgue measure on \( \mathbb{R}^d \) (with ambient dimension implicit from context). Let \( X \subseteq \mathbb{R}^d \) be full-dimensional and compact. We denote the uniform measure \( \mu \) on \( X \) to be the measure defined by \( \mu(A) = \frac{\mu_{\text{Leb}}(A)}{\mu_{\text{Leb}}(X)} \) for every measurable \( A \subseteq X \).

**Lemma C.1.** Fix a \( p \in [1, \infty] \). Let \( X \subseteq \mathbb{R}^d \) be a full-dimensional compact set and let \( \mu \) denote its uniform measure. Let \( r_p(\varepsilon) \) be defined as:

\[ r_p(\varepsilon) := \sup_{U \subseteq X : \mu(U) \leq \varepsilon} \sup \{ r > 0 : \exists x \in U : \mathbb{B}_d^p(x, r) \subseteq U \}. \]

Then we have that

\[ r_p(\varepsilon) \leq \left( \frac{\varepsilon \mu_{\text{Leb}}(X)}{\mu_{\text{Leb}}(\mathbb{B}_d^p(0, 1))} \right)^{1/d}. \]

**Proof.** Notice that if \( r > 0 \) satisfies \( \exists x \in U \) such that \( \mathbb{B}_d^p(x, r) \subseteq U \), this implies that \( \mu_{\text{Leb}}(U) \geq \mu_{\text{Leb}}(\mathbb{B}_d^p(x, r)) = r^d \mu_{\text{Leb}}(\mathbb{B}_d^p(0, 1)) \). Hence:

\[ \sup \{ r > 0 : \exists x \in U : \mathbb{B}_d^p(x, r) \subseteq U \} \leq \sup \{ r > 0 : r^d \mu_{\text{Leb}}(\mathbb{B}_d^p(0, 1)) \leq \mu_{\text{Leb}}(U) \} \]

\[ = \left( \frac{\mu_{\text{Leb}}(U)}{\mu_{\text{Leb}}(\mathbb{B}_d^p(0, 1))} \right)^{1/d}. \]

Now if \( \mu(U) \leq \varepsilon \), then \( \mu_{\text{Leb}}(U) = \frac{\mu_{\text{Leb}}(U)}{\mu_{\text{Leb}}(X)} \leq \varepsilon \) and hence \( \mu_{\text{Leb}}(U) \leq \varepsilon \mu_{\text{Leb}}(X) \).

\[ \square \]

**Lemma C.2.** Fix a \( p \in [1, \infty] \). Let \( X \subseteq \mathbb{R}^d \) be a full-dimensional compact set with \( d \geq 2 \) and let \( \mu \) denote its uniform measure. Let \( \nu := \mu \otimes \rho \) denote the product measure on \( X \times \mathbb{S}^{d-1} \) with \( \rho \) the Haar measure. Endow \( \mathbb{R}^d \times \mathbb{S}^{d-1} \) with the metric \( d(x, y) := \max \{ \| x_1 - y_1 \|_p, \rho(x_2, y_2) \} \) where \( \rho \) is the geodesic distance on \( \mathbb{S}^{d-1} \), and let \( B(x, r) \) denote a closed ball of radius \( r \) centered at \( x \) in this metric space. Then, the quantity

\[ r_p(\varepsilon) := \sup_{U \subseteq X \times \mathbb{S}^{d-1} : \nu(U) \leq \varepsilon} \sup \{ r > 0 : \exists x \in U : B(x, r) \subseteq U \} \]

may be upper bounded by the expression

\[ r_p(\varepsilon) \leq \sup \left\{ r > 0 : r^d \zeta_d(r) \leq \frac{\varepsilon \mu_{\text{Leb}}(X)}{\mu_{\text{Leb}}(\mathbb{B}_d^p(0, 1))} \right\}, \]

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with the function \( \zeta_d : \mathbb{R}_+ \mapsto \mathbb{R}_+ \) defined as:

\[
\zeta_d(r) := \begin{cases} 
I \left( \sin^2(r); \frac{d-1}{2}, \frac{1}{2} \right) & \text{if } r \in [0, \pi/2) \\
1 - I \left( \sin^2(\pi - r); \frac{d-1}{2}, \frac{1}{2} \right) & \text{if } r \in [\pi/2, \pi) \\
1 & \text{if } r \geq 1
\end{cases}
\]

where \( I(x; a, b) \) the regularized incomplete beta function.

**Proof.** Let \( B_d(x, r) \) denote the closed ball in \( (\mathbb{S}^{d-1}, \rho) \) centered at \( x \in \mathbb{S}^{d-1} \). Note that \( B(x, r) = B_d(x_1, r) \times B_d(x_2, r) \). Therefore if \( r > 0 \) satisfies \( \exists x \in U \) such that \( B(x, r) \subseteq U \), then

\[
r^d \frac{\mu_{\text{Leb}}(B_d(0, 1))}{\mu_{\text{Leb}}(X)} q(B_d(0, r)) = \frac{\mu_{\text{Leb}}(B_d(x_1, r))}{\mu_{\text{Leb}}(X)} q(B_d(0, r)) = \nu(B(x, r)) \leq \nu(U).
\]

Above, we have used monotonicity of measure and translation invariance of the measure of the respective balls. Now, note that a geodesic ball on \( \mathbb{S}^{d-1} \) is a spherical cap, and hence by Li [24],

\[
q(B_d(0, r)) = \begin{cases} 
I \left( \sin^2(r); \frac{d-1}{2}, \frac{1}{2} \right) & \text{if } r \in [0, \pi/2) \\
1 - I \left( \sin^2(\pi - r); \frac{d-1}{2}, \frac{1}{2} \right) & \text{if } r \in [\pi/2, \pi) \\
1 & \text{if } r \geq 1
\end{cases}
\]

Therefore,

\[
r_p(\varepsilon) \leq \sup \left\{ r > 0 : r^d \zeta_d(r) \leq \frac{\varepsilon \mu_{\text{Leb}}(X)}{\mu_{\text{Leb}}(B_d(0, 1))} \right\}.
\]

which proves the result. \( \square \)

**Lemma C.3.** Let \( \dot{x} = A(t)x \) be a linear time-varying system evolving in \( \mathbb{R}^n \). Let \( \varphi_t(x) \) denote the flow of this system with \( x(0) = x \). Fix a \( t \geq 0 \) and a unit vector \( z \). There exists a positive scalar \( \alpha \) such that \( \alpha z \in \varphi_t(\mathbb{S}^{n-1}) \).

**Proof.** The solution to an LTV system is given by \( x(t) = \Phi(t)x(0) \), where \( \Phi(t) = \exp(\int_0^t A(\tau) \, d\tau) \) and \( \Phi(t) \) is invertible for all \( t \). This means there exists a non-zero \( \xi \) such that \( z = \Phi(t)\xi = \Phi(t)\|\xi\|\xi \). The claim now follows by taking \( \alpha = 1/\|\xi\| \). \( \square \)

**C.1 Proof of Theorem 5.1**

Define \( X_g := X \setminus X_b \). Fix an \( x \in S \). By definition of \( S \), there exists a \( \xi \in X \) and \( t \in T \) such that \( \varphi_t(\xi) = x \). Let \( \xi' \in X_g \) be such that \( \|\xi - \xi'\| \leq r(\varepsilon) \). We know such a \( \xi' \in X_g \) exists by Lemma C.1.

We have the following chain of inequalities:

\[
q(\varphi_t(\xi)) \leq q(\varphi_t(\xi')) + B_{\nabla q}\|\varphi_t(\xi) - \varphi_t(\xi')\|
\]

\[
\leq -\lambda V(\varphi_t(\xi')) + B_{\nabla V}\|\varphi_t(\xi) - \varphi_t(\xi')\|
\]

\[
\leq -\lambda V(\varphi_t(\xi)) + (B_{\nabla q} + \lambda B_{\nabla V})\|\varphi_t(\xi) - \varphi_t(\xi')\|
\]

\[
\leq -\lambda V(\varphi_t(\xi)) + (B_{\nabla q} + \lambda B_{\nabla V})\beta(\|\xi - \xi'\|, t)
\]

\[
\leq -\lambda V(\varphi_t(\xi)) + (B_{\nabla q} + \lambda B_{\nabla V})\beta(r(\varepsilon), t).
\]
This shows that for any \( x \in S \):
\[
q(x) \leq -\lambda V(x) + (B \nabla q + \lambda B \nabla V) \beta(r(\varepsilon), t).
\]

The claim established by (5.3) now follows from the comparison lemma. To establish (5.2), for any \( x \in S \setminus B_{p}^{2}(0, r_b) \),
\[
q(x) \leq -\lambda V(x) + (B \nabla q + \lambda B \nabla V) \beta(r(\varepsilon), t) \\
\leq -((1 - \eta) \lambda + \eta \lambda) V(x) + (B \nabla q + \lambda B \nabla V) \beta(r(\varepsilon), 0) \\
= -(1 - \eta) \lambda V(x) - \eta \lambda V(x) + (B \nabla q + \lambda B \nabla V) \beta(r(\varepsilon), 0) \\
\leq -((1 - \eta) \lambda V(x) - \eta \lambda \mu r_b^2 + (B \nabla q + \lambda B \nabla V) \beta(r(\varepsilon), 0).
\]

The last inequality follows since \( V(x) \geq \mu \|x\|^2 \geq \mu r_b^2 \) for any \( x \in S \setminus B_{p}^{2}(0, r_b) \). The claim (5.2) now follows by setting \( r_b \) such that \( -\eta \lambda \mu r_b^2 + (B \nabla q + \lambda B \nabla V) \beta(r(\varepsilon), 0) \leq 0 \).

### C.2 Proof of Theorem 5.2

We begin with a simple lemma, which shows that if a system evolving on Euclidean space is contracting in the metric \( M(x,t) \), then the corresponding prolongated system on the tangent bundle will be contracting in a block-diagonal metric.

**Lemma C.4.** Let \( \dot{x} = f(x,t) \) be a contracting system with rate \( \gamma \) on \( X \subseteq \mathbb{R}^d \) in the metric \( M(x,t) \). Then the differential dynamics \( \dot{\delta x} = \frac{\partial f}{\partial x}(x,t) \delta x \) is also contracting in the metric \( M(x,t) \) on \( \mathbb{R}^d \). Moreover, the prolongated dynamics defined on the tangent bundle

\[
\dot{x} = f(x,t) \\
\dot{\delta x} = \frac{\partial f}{\partial x}(x,t) \delta x
\]

is contracting on any compact subset of the tangent bundle \( X \times \delta X \subset T X \simeq \mathbb{R}^{2p} \).

**Proof.** Consider the differential dynamics \( \dot{\delta x} = \frac{\partial f}{\partial x}(x,t) \delta x \). This system has Jacobian
\[
\frac{\partial \dot{\delta x}}{\partial \delta x} = \frac{\partial f}{\partial x}(x,t),
\]
where we have noted that \( \delta x \in T_{x(t)}X \simeq \mathbb{R}^p \) is independent of \( x \). This Jacobian induces the second-order variational dynamics
\[
\delta \delta \dot{x} = \frac{\partial f}{\partial x}(x,t) \delta \delta x.
\]

Consideration of the differential Lyapunov function
\[
V = \delta \delta x^T M(x,t) \delta \delta x
\]
shows that \( V \) decreases exponentially by contraction of \( f(x,t) \) in the metric \( M(x,t) \) and hence that the virtual dynamics are contracting. Let \( \Theta(x,t) \) be such that \( \Theta^T \Theta = M \). The metric transformation
\[
\Theta'(x,t) = \begin{pmatrix} \Theta(x,t) & 0 \\ 0 & \epsilon \Theta(x,t) \end{pmatrix}
\]

Furthermore, note that the metric transformation \( \Theta' M \) is exponentially contracting in a metric condition number of \( M \). If the system satisfies Assumption 5.1 with \( \beta \eta \gamma \epsilon \) ensures that \( M \) for \( \epsilon > 0 \) leads to the generalized Jacobian

\[
J'(x, \delta x, t) = 
\begin{pmatrix}
\Theta \frac{\partial f}{\partial x} \Theta^{-1} + \Theta \Theta^{-1} & 0 \\
\Theta \frac{\partial f}{\partial x} \Theta^{-1} + \Theta \Theta^{-1} & \Theta \Theta^{-1}
\end{pmatrix} ,
\]

where \( \left( \frac{\partial^2 f}{\partial x} \delta x \right)_{ij} = \sum_k \frac{\partial^2 f}{\partial x \partial x_k} \delta x_k \). Let \( Q(x, t) = \Theta \frac{\partial f}{\partial x} \Theta^{-1} + \Theta \Theta^{-1} \). Contraction of \( f \) in the metric \( M \) ensures that \( Q(x, t) \leq -\gamma I \), and hence for any vector \( (y^T, z^T)^T \in \mathbb{R}^{2p} \),

\[
(y^T, z^T)^T J'(x, \delta x, t) \left( \begin{array}{c} y \\ z \end{array} \right) = x^T Q(x, t)x + y^T Q(x, t)y + \epsilon y^T \left[ \Theta \frac{\partial^2 f}{\partial x} \delta x \Theta^{-1} \right] x ,
\]

\[
\leq -\gamma \left( 1 - \epsilon \left\| \Theta \right\| \left\| \Theta^{-1} \right\| \left\| \frac{\partial^2 f}{\partial x^2} \right\| \left\| \delta x \right\| \right) \left( \|x\|^2 + \|y\|^2 \right) ,
\]

which shows that the prolonged system is contracting over any compact domain for \( \epsilon \) sufficiently small. In particular, for contraction with rate \( \eta \gamma \) for \( 0 < \eta < 1 \), we may set

\[
\epsilon = \frac{2 (1 - \eta)}{\left\| \Theta \right\| \left\| \Theta^{-1} \right\| \left\| \frac{\partial^2 f}{\partial x^2} \right\| \left( \sup_{x \in \delta x} \| \delta x \| \right)} .
\]

Furthermore, note that the metric transformation \( \Theta' \) corresponds to the block-diagonal metric

\[
M'(x, t) = 
\begin{pmatrix}
M(x, t) & 0 \\
0 & \epsilon^2 M(x, t)
\end{pmatrix}
\]

The proof of Lemma C.4 imposes a metric \( M'(x, t) \) on the second tangent bundle. This construction exploits that the tangent bundle \( T \mathcal{M} \simeq X \times \mathbb{R}^d \) given that \( X \subseteq \mathbb{R}^d \), and hence that the tangent bundle can be described by a single global chart. It is immediate to check that this block-diagonal metric is not invariant under a differentiable change of coordinates between overlapping local parametrizations of a general manifold, and hence the proof does not apply beyond Euclidean space. Canonical metrics on \( T \mathcal{M} \) such as the Sasaki metric or the Cheeger-Gromoll metric may provide a natural generalization of this proof technique to arbitrary differentiable manifolds [16].

We now turn to the proof of Theorem 5.2. Define \( Z_q := (X \times S^{p-1}) \setminus Z_b \). Let \( t \in T \) and \( \xi \in X \) be such that \( \xi \in X_t(r_b) \). Let \( \delta \xi \in S^{p-1} \) be arbitrary. By Lemma C.2, there exists a \( (\xi', \delta \xi') \in Z_q \) such that \( \left\| \left[ \xi - \xi', \delta \xi - \delta \xi' \right] \right\| \leq \sqrt{2} r(\epsilon) \). By Lemma C.4, the prolonged system on the tangent bundle is exponentially contracting in a metric \( M' \), so that there exists an \( \alpha > 0 \) such that the prolonged system satisfies Assumption 5.1 with \( \beta(s, t) = \sqrt{\chi(M')} e^{-\alpha t} \). Here, \( \chi(M') = \frac{\sup_{x \in M' \in \mathcal{M}^c} \lambda_{\text{max}}(M'(x))}{\inf_{x \in M' \in \mathcal{M}^c} \lambda_{\text{min}}(M'(x))} \) is the condition number of \( M' \). We will derive bounds on \( \chi(M') \) and \( \alpha \) later in the proof. Recall that the metric \( M(x) \geq \mu I \), and therefore \( V(x, \delta x) \geq \mu \| \delta x \|^2 \). Then, by the same argument as in the proof of Theorem 5.1, for any fixed \( \eta \in (0, 1) \):

\[
q(\psi_t(\xi, \delta \xi)) \leq -\lambda V(\psi_t(\xi, \delta \xi)) + (B_{\nu} + \lambda B_{\nabla V}) \beta(\sqrt{2} r(\epsilon), 0),
\]

\[
\leq -\lambda (1 - \eta) V(\psi_t(\xi, \delta \xi)) - \eta \lambda \mu \| \theta_t(\delta \xi; \xi) \|^2 + (B_{\nu} + \lambda B_{\nabla V}) \beta(\sqrt{2} r(\epsilon), 0)
\]

\[
\leq -\lambda (1 - \eta) V(\psi_t(\xi, \delta \xi)) - \eta \lambda \mu r^2 + (B_{\nu} + \lambda B_{\nabla V}) \beta(\sqrt{2} r(\epsilon), 0) ,
\]

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where the last inequality follows since $\xi \in X_t(r_b)$.

To find an expression for the condition number and for the contraction rate $\alpha$, consider the block diagonal metric from Lemma C.4, $M'(x, t) = \begin{pmatrix} M_*(x, t) & 0 \\ 0 & \epsilon^2 M_*(x, t) \end{pmatrix}$. Following Lemma C.4, with $\epsilon = \frac{2(1-\zeta)}{\|\Theta(x, t)||\Theta(x, t)^{-1}\| \sup_{\delta x \in \delta X} \|\delta x\|}$ where $\Theta(x, t)^T \Theta(x, t) = M_*(x, t)$, the prolongated system will be contracting with rate $\alpha = \zeta \gamma$ for $0 < \zeta < 1$. Since $mI \preceq M_*(x, t) \preceq LI$ so that $\chi(M_*) = \frac{L_m}{m}$, we immediately have $\epsilon^2 mI \preceq M'(x, t) \preceq LI$. The condition number $\chi(M')$ then simplifies to $\frac{L_m}{m^2} = \frac{L_m}{m} \left( \|\Theta(x, t)||\Theta(x, t)^{-1}\| \sup_{\delta x \in \delta X} \|\delta x\| \right)^2 \frac{1}{4(1-\zeta)^2} = \left( \frac{L_m}{m} \right)^2 B_H^2 \sup_{\delta x \in \delta X} \|\delta x\|^2 \frac{1}{4(1-\zeta)^2}$, where we have used that $M_0 = \Theta^T \Theta$ implies $\|\Theta\| = \sqrt{L}$, $\|\Theta^{-1}\| = \sqrt{\frac{L_m}{m}}$. By contraction of the variational system in the metric $M_*(x, t)$ (see Lemma C.4), and noting that 0 is an equilibrium point of the variational dynamics, $\|\theta_t(\delta x; \xi)\| \leq \frac{L_m}{m}$ for all $t$ if $\delta x \in S^{p-1}$. Hence we may take $\delta X = \mathbb{B}^p_r \left( 0, \sqrt{\frac{L_m}{m}} \right)$, and conclude that $\chi(M') \leq \left( \frac{L_m}{m} \right)^3 B_H^2 \frac{1}{4(1-\zeta)^2} = \chi(M_*)^3 B_H^2 \frac{1}{4(1-\zeta)^2}$.

Now with this expression in hand, we choose $r_b$ such that

$$-\eta \lambda m r_b^2 + (B_{\nabla q} + \lambda B_{\nabla V}) \sqrt{2} r(\epsilon) \chi(M_*)^{3/2} B_H \frac{1}{2(1-\zeta)} \leq 0.$$ 

From this we conclude for every $t \in T$ and $\xi \in X_t(r_b)$, for every $\delta \xi \in S^{p-1}$,

$$q(\psi_t(\xi, \delta \xi)) \leq -\lambda (1-\eta) V(\psi_t(\xi, \delta \xi)).$$

(C.1)

To finish the proof, let $(x, \delta x) \in S(r_b) \times S^{p-1}$ be arbitrary. Let $t \in T$ and $\xi \in X_t(r_b)$ such that $x = \varphi_t(\xi)$. By Lemma C.3, let $\delta \varphi \in S^{p-1}$ be such that there exists an $\alpha \neq 0$ satisfying $\theta_t(\delta \varphi; \xi) = \alpha \delta x$. Observe then that $(x, \alpha \delta x) = (\varphi_t(\xi), \theta_t(\delta \varphi; \xi)) = \psi_t(\xi, \delta \varphi)$ and therefore by (C.1),

$$q(x, \alpha \delta x) = q(\psi_t(\xi, \delta \varphi)) \leq -\lambda (1-\eta) V(\psi_t(\xi, \delta \varphi)) = -\lambda (1-\eta) V(x, \alpha \delta x).$$

By 2-homogeneity of the inequality above, we may divide by $\alpha^2$ on both sides to conclude:

$$q(x, \delta x) \leq -\lambda (1-\eta) V(x, \delta x).$$

Since this inequality holds for arbitrary $x \in S(r_b)$ and $\delta x \in S^{p-1}$,

$$\frac{\partial f^T}{\partial x} M(x) + M(x) \frac{\partial f}{\partial x} + \dot{M}(x) \leq -2(1-\eta) \lambda M(x) \forall x \in S(r_b).$$

### D Known dynamics

In Section 5, we assume access only to trajectories. Here we prove a simple proposition under the assumption that the dynamics is known, so that the defining metric condition for contraction can be sampled directly.

**Proposition D.1.** Let $M(x, t)$ be a uniformly positive definite matrix-valued function satisfying $M(x, t) \succeq LI$. Suppose that $X \subseteq \mathbb{R}^p$ is full-dimensional, and let $\dot{x} = f(x, t)$ denote a dynamical
system evolving on $X$. Let $\phi_t(\cdot)$ denote the corresponding flow, let $\nu$ denote the uniform measure on $X$, and let

$$R(\xi, t) = \frac{\partial f}{\partial x}(\phi_t(\xi), t)^T M(\phi_t(\xi), t) + M(\phi_t(\xi), t) \frac{\partial f}{\partial x}(\phi_t(\xi), t) + \dot{M}(\phi_t(\xi), t),$$

$$X_b = \left\{ \xi \in X : \max_{t \in T} \lambda_{\max} \{ R(\xi, t) \} > 0 \right\}.$$

Suppose that $\nu(X_b) \leq \varepsilon$ for some $\varepsilon \in [0, 1]$. Let $M$, $\nabla M$, and $\frac{\partial f}{\partial x}$ be $L_M$, $L_\nabla M$, and $L_1$-Lipschitz continuous, respectively. Further assume that $\|M\|$, $\|\frac{\partial f}{\partial x}\|$, and $\|\nabla M\|$ are $B_M$, $B_1$, and $B_\nabla M$ uniformly bounded in $x$ and $t$, respectively. Then the system will be globally contracting in the metric $M(x, t)$ with a rate $\lambda/\alpha$ for any $\alpha > 1$ if

$$\varepsilon \leq \left( \frac{2\lambda}{\alpha(2L_M + L_\nabla M B_f + B_\nabla M L_f + 2L_J B_M + 2L_M B_J)} \right)^p \frac{\pi^{p/2}}{\Gamma \left( \frac{p}{2} + 1 \right) \mu(X)}.$$

Proof. Let us partition $X$ into subsets

$$X_g := \left\{ \xi \in X : \max_{t \in T} \lambda_{\max} \{ R(\xi, t) \} \leq 0 \right\},$$

$$X_b := X \setminus X_g,$$

i.e., for any trajectory originating in $X_g$, the metric condition $R(\xi, t)$ remains negative definite along the entire trajectory with rate $\lambda$. By Lemma C.1, for any $\xi \in X_b$, there exists a $\xi' \in X_g$ such that $\|\xi - \xi'\| \leq \left( \frac{\varepsilon \lambda_{\text{lab}}(X)}{\mu_{\text{lab}}(\mathbb{B}_2(0, 1))} \right)^{1/p}$. Let us denote this upper bound by $\eta$. Then $X_g$ forms an $\eta$-net over $X$. Consider evaluating the metric condition for some $\xi_b \in X_b$, and let $\xi_g \in X_g$. Then,

$$\frac{\partial f}{\partial x}(\phi_t(\xi_b), t)^T M(\phi_t(\xi_b), t) + M(\phi_t(\xi_b), t) \frac{\partial f}{\partial x}(\phi_t(\xi_b), t) + \dot{M}(\phi_t(\xi_b), t)
= \frac{\partial f}{\partial x}(\phi_t(\xi_g), t)^T M(\phi_t(\xi_g), t) + M(\phi_t(\xi_g), t) \frac{\partial f}{\partial x}(\phi_t(\xi_g), t) + \dot{M}(\phi_t(\xi_g), t)
+ \frac{\partial f}{\partial x}(\phi_t(\xi_b), t)^T M(\phi_t(\xi_b), t) + M(\phi_t(\xi_b), t) \frac{\partial f}{\partial x}(\phi_t(\xi_b), t) + \dot{M}(\phi_t(\xi_b), t)
- \left( \frac{\partial f}{\partial x}(\phi_t(\xi_g), t)^T M(\phi_t(\xi_g), t) + M(\phi_t(\xi_g), t) \frac{\partial f}{\partial x}(\phi_t(\xi_g), t) + \dot{M}(\phi_t(\xi_g), t) \right)$$

We now control the difference of the terms on the second and third lines of the above equality. To simplify notation, denote $M_g = M(\phi_t(\xi_g), t)$, with analogous shorthands for $\frac{\partial f}{\partial x}$ and for subscript $b$. Then, we have that

$$M_b \frac{\partial f}{\partial x} - M_g \frac{\partial f}{\partial x} = M_b \left( \frac{\partial f}{\partial x} - \frac{\partial f}{\partial x} \right) + (M_b - M_g) \frac{\partial f}{\partial x}
\leq (L_J B_M + L_M B_J) \|\xi_g - \xi_b\| I,$$

with an identical bound for the transpose. Now let $\langle \nabla M, \dot{x} \rangle$ denote the tensor contraction $\langle \nabla M, \dot{x} \rangle_{ij} = \langle \nabla M, \dot{x} \rangle_{ij}$. Then,

$$\dot{M}_b - \dot{M}_g = \langle \nabla M_b, f_b \rangle - \langle \nabla M_g, f_g \rangle
= \langle \nabla M_b - \nabla M_g, f_b \rangle - \langle \nabla M_g, f_b - f_g \rangle
\leq (L_\nabla M B_f + B_\nabla M L_f) \|\xi_g - \xi_b\| I.$$
And clearly $M_g - M_b \leq L_M \|\xi_b - \xi_g\|$. Putting these bounds together, we find that

\[
\begin{align*}
\frac{\partial f}{\partial x}(\varphi_t(\xi_b), t)M(\varphi_t(\xi_b), t) + M(\varphi_t(\xi_b), t)\frac{\partial f}{\partial x}(\varphi_t(\xi_b), t) + \dot{M}(\varphi_t(\xi_b), t) \\
\leq \frac{\partial f}{\partial x}(\varphi_t(\xi_g), t)M(\varphi_t(\xi_g), t) + M(\varphi_t(\xi_g), t)\frac{\partial f}{\partial x}(\varphi_t(\xi_g), t) + \dot{M}(\varphi_t(\xi_g), t) \\
+ (L\nabla M B_f + B\nabla M L_f + 2L_J B_M + 2L_M B_J)\|\xi_g - \xi_b\||I
\end{align*}
\]

\[
\leq -2\lambda M(\varphi_t(\xi_g), t) + (L\nabla M B_f + B\nabla M L_f + 2L_J B_M + 2L_M B_J)\|\xi_g - \xi_b\||I
\]

\[
\leq -2\lambda M(\varphi_t(\xi_b), t) + (2\lambda L_M + L\nabla M B_f + B\nabla M L_f + 2L_J B_M + 2L_M B_J)\|\xi_g - \xi_b\||I
\]

\[
\leq -2\left(\frac{\lambda}{\alpha}\right)M(\varphi_t(\xi_b), t)
\]

\[
+ \left[ (2\lambda L_M + L\nabla M B_f + B\nabla M L_f + 2L_J B_M + 2L_M B_J)\|\xi_g - \xi_b\| - 2\left(\frac{\alpha - 1}{\alpha}\right)\lambda \right] I.
\]

Because $\xi_g$ on the right-hand side is arbitrary, we can can take $\xi_g$ to be the closest point in $X_g$ to $\xi_b$. Then $\|\xi_g - \xi_b\| \leq \eta$. Hence, for contraction at all points in $X$ with a rate $\frac{\lambda}{\alpha}$, we require that

\[
0 \geq (2\lambda L_M + L\nabla M B_f + B\nabla M L_f + 2L_J B_M + 2L_M B_J)\eta - 2\left(\frac{\alpha - 1}{\alpha}\right)\lambda I
\]

\[
2\left(\frac{\alpha - 1}{\alpha}\right)\lambda I \geq \left(\frac{e\mu_{\text{Leb}}(X)}{\mu_{\text{Leb}}(B_p^2(0, 1))}\right)^{1/p} (2\lambda L_M + L\nabla M B_f + B\nabla M L_f + 2L_J B_M + 2L_M B_J)
\]

\[
\varepsilon \leq \frac{2\lambda(\alpha - 1)}{\alpha (2\lambda L_M + L\nabla M B_f + B\nabla M L_f + 2L_J B_M + 2L_M B_J)} \mu_{\text{Leb}}(B_p^2(0, 1)) \mu_{\text{Leb}}(X)
\]