Scaling symmetry, renormalization, and time series modeling

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We present and discuss a stochastic model of financial assets dynamics based on the idea of an inverse renormalization group strategy. With this strategy we construct the multivariate distributions of elementary returns based on the scaling with time of the probability density of their aggregates. In its simplest version the model is the product of an endogenous auto-regressive component and a random rescaling factor embodying exogenous influences. Mathematical properties like increments’ stationarity and ergodicity can be proven. Thanks to the relatively low number of parameters, model calibration can be conveniently based on a method of moments, as exemplified in the case of historical data of the S&P500 index. The calibrated model accounts very well for many stylized facts, like volatility clustering, power law decay of the volatility autocorrelation function, and multiscaling with time of the aggregated return distribution. In agreement with empirical evidence in finance, the dynamics is not invariant under time reversal and, with suitable generalizations, skewness of the return distribution and leverage effects can be included. The analytical tractability of the model opens interesting perspectives for applications, for instance in terms of obtaining closed formulas for derivative pricing. Further important features are: The possibility of making contact, in certain limits, with auto-regressive models widely used in finance; The possibility of partially resolving the endogenous and exogenous components of the volatility, with consistent results when applied to historical series.

I. INTRODUCTION

Time series analysis plays a central role in many disciplines, like physics [1], seismology [2], biology [3], physiology [4], linguistics [5], or economy [6], whenever datasets amount to sequences of measurements or observations. A main goal of such analysis is that of capturing essential regularities of apparently unpredictable signals within a synthetic model, which can be used to get forecasts and a deeper understanding of the mechanisms governing the processes under study. A satisfactory time series modeling for complex systems may become a challenging task, due to the need to account for statistical features of the data connected with the presence of strong correlations. In the last decades, features of this kind have been extensively studied in the context of financial time series, where they strongly stimulated the search for adequate stochastic modeling [7]. The non-Gaussianity of the probability density function (PDF) of aggregated returns of an asset over time intervals in substantial ranges of scales, its anomalous scaling and multiscaling with the interval duration, the long-range dependence of absolute return fluctuations (volatility), the violation of time-reversal symmetry, among other robust statistical features called stylized facts in finance [11], still remain elusive of synthetic and analytically tractable modeling. Besides the standard model of finance based on geometric Brownian motion [7, 12], proposed descriptions include stochastic volatility models (See, e.g., [9] and references therein), multifractal models inspired by turbulence [13–15], multi-timescale models [14, 20], various types of self-similar processes [21–26], multi-agent models [27, 29], and those in the Auto-Regressive Conditional Heteroskedastic (ARCH) and Generalized-ARCH (GARCH) family [8, 30–32].

To be effective, a stochastic model should not only correctly reproduce the statistical features observed in the empirical analysis, but also be easy to calibrate and analytically tractable in order to be useful in applications like derivative pricing and risk evaluation [7, 33]. In this respect, research in stochastic modeling of financial assets is still a challenging topic [15]. Recently, some of us proposed an approach to market dynamics modeling [23, 54] inspired by the renormalization group theory of critical phenomena [43–45]. The background ideas exposed in Refs. [23, 34] already stimulated some contributions along various lines [26, 27, 37, 38]. In particular, in [27] a model with nonstationary increments and lacking a volatility feedback mechanism has been discussed in detail, pointing out its potential interest and missing features. In the present Paper, we make a significant step forward along the lines proposed in [27], by introducing a novel discrete-time stochastic process characterized by both an auto-regressive endogenous component providing a volatility feedback and an exogenous one representing external influences. Many features of the model are under analytical control and a number of basic properties, like...
II. SCALING AS A GUIDING SYMMETRY

Since the pioneering work of Mandelbrot and van Ness on fractional Brownian motion \cite{11}, interest in scaling features has characterized many models of financial and other time series, especially in contributions by members of the physics community. Proposals include the representation of financial processes as truncated Levy flights \cite{21, 22}, or the more sophisticated descriptions through multifractal cascades inspired by turbulence \cite{13, 18}. In the financial literature, a similar focus on scaling properties is harder to find. Indeed, although leptokurtic distributions of aggregated returns are typically obtained in ARCH and similar models by making the conditioned variance of successive elementary increments dependent on the past history \cite{8, 30, 32}, even for more specialized versions of this type of approach, like FIGARCH \cite{42}, a proper description of the correct scaling and multiscaling properties of aggregated increments is still an open issue.

A cornerstone achievement in statistical physics has been the formulation of the renormalization group approach to scaling \cite{43, 45}. In this approach one tries to deduce the scaling properties of a system at criticality by analyzing how coarse-graining operations, which rescale the minimal length at which the system is resolved, change its statistical description in terms of effective interactions or similar parameters. The scaling invariance at criticality then emerges when such changes do not occur (fixed point). In a recent publication \cite{23}, some of the present authors made the proposal that the problem of modeling the stochastic financial process on time scales for which a well defined scaling symmetry holds at least approximately, may be faced by inverting the logic of the standard renormalization group procedure. Given as an input the scaling properties of the aggregated increment PDF over a certain time scale, the idea is to find by fine-graining basic probabilistic rules that apply to elementary increments in order for them to be consistent with the input scaling properties \cite{24, 54}. These rules should describe the system on time scales shorter than that of the aggregation interval, and their knowledge is regarded to be equivalent to that of the effective fixed point interactions in the standard renormalization group approach. Of course, even if properties like the martingale character of the financial process pose strong constraints, there is a degree of arbitrariness in the fine graining operation, and an important task is to show that the proposed fine-graining is plausible at the light of the relevant stylized facts.

This fine-graining, reverse renormalization group strategy for the description of market dynamics has been already exemplified in previous contributions \cite{24, 25, 26}, especially dealing with high frequency processes \cite{35, 37}. Unlike in cases for which a single time series is available, in Refs. \cite{57, 59, 57} we focused on a particular, fixed window of the daily evolution of an asset, and extracted from the available records an ensemble of histories which have been assumed to be independent realizations of the same stochastic process. The manifest time inhomogeneous character of this process and its limited duration in time significantly simplify a modeling approach based on the above fine-graining strategy. Things become more difficult when only one realization of the process one wishes to model is available in the form of a single, long time series. This is the situation we discuss in the present work.

While a precise mathematical definition of our model is postponed to the next Section, in the present one we summarize the basic ideas behind its construction. In particular, we emphasize the inspiration by the renormalization group approach and the basic complementary role played by both endogenous and exogenous mechanisms. Another key aspect concerns the introduction of an auto-regressive dynamical scheme. In our context this endows the endogenous mechanism with sufficiently long memory, guaranteeing at the same time strong mixing,
and hence also the ergodicity of the process \[48\].

Let \( \{X_i\}_{i=1}^\infty \) be a sequence of random variables representing the increments (logarithmic returns in finance) of a discrete-time stochastic process. This process possesses a simple-scaling symmetry if \( X_1 + \cdots + X_1 \) has the same probability law as \( t^H X_1 \) for any \( t, H > 0 \) being the scaling (Hurst) exponent. If this is the case, the property

\[
t^H g_t(t^H x) = g(x)
\]

holds for the PDF \( g_t \) of the aggregated increments \( X_1 + \cdots + X_1 \) where \( g \) is the scaling function (which also coincides with the PDF of \( X_1 \)). One immediate consequence of Eq. (1) is a scaling property for the existing moments of the process:

\[
\mathbb{E}[X_1 + \cdots + X_1]^q = t^{qH} \mathbb{E}[X_1]^q.
\]  
(2)

A normal scaling symmetry is obtained with \( g \) Gaussian and \( H = 1/2 \). Anomalous scaling refers to the fact of \( g \) not being Gaussian and/or \( H \neq 1/2 \). Another kind of anomalous behavior for which Eq. (2) holds with an exponent \( H \) explicitly depending on the moment order \( q \) is called multiscaling and in this case \( H \) is also named generalized Hurst exponent \[16, 17\].

The simple-scaling symmetry can also be expressed in terms of characteristic functions (CF) as

\[
\mathbb{E}[e^{ik(X_1+\cdots+X_1)}] = \mathbb{E}[e^{ik(t^H X_1)}],
\]

or, equivalently, as

\[
\hat{f}_t^X(k_1, \ldots, k_t) = \hat{f}_t^X(t^H k),
\]

(4)

where \( \hat{f}_t^X(k_1, \ldots, k_t) \equiv \mathbb{E}[e^{ik(X_1+\cdots+X_1)}] \) is the joint CF of \( X_1, \ldots, X_t \), i.e. the Fourier transform of the joint PDF \( f_t^X(x_1, \ldots, x_t) \).

Aiming at constructing a model for the increments consistent with Eq. (4) for a given scaling exponent \( H > 0 \) and general scaling function \( g \), we notice that the knowledge of \( f_t^X = g \) combined with Eq. (4) allows us to only fix the CF \( \hat{f}_t^X \) along the diagonal:

\[
\hat{f}_t^X(k_1, \ldots, k_t) = \hat{g}(t^H k) = \int dx e^{i(t^H k)x} g(x).
\]

(5)

The basic inspiration of our approach is thus a quest for the existence of conditions implied by the presence of anomalous scaling which allow us to determine this joint CF also off-diagonal. Namely: “Are there ways of fixing \( \hat{f}_t^X(k_1, \ldots, k_t) \) such that \( \hat{f}_t^X(k_1, \ldots, k_t) = \hat{g}(t^H k) \) with \( g \) non-Gaussian and/or \( H \neq 1/2 \) assumed to be given?”

As a rule, when applying the renormalization group approach, one would be faced with the inverse problem: given a parametric form for \( \hat{f}_t^X \), or for \( f_t^X \), one tries to fix its parameters in such a way that Eq. (4), and thus Eq. (1), with \( H \) and \( g = f_t^X \) to be determined, is satisfied. This amounts to the identification of the fixed point and is generally accomplished by operating a suitable coarse-graining operation on the description of the process. The fixed point is just an instance of the process which is left invariant under such operation. To satisfy our quest, we need to implement a plausible inversion of the coarse-graining operation in which the fixed point scaling is assumed to be known and \( \hat{f}_t^X \) needs to be constructed. This inverse procedure is not unique in general and its plausibility needs to be tested \textit{a posteriori}. We are thus somehow “reverting” the ordinary flux in a renormalization-group approach, as we are trying to realize a \textit{fine-graining} procedure compatible with the existence of an anomalous fixed point scaling.

Given \( H \) and \( g \) as an input, our proposal is to set

\[
\hat{f}_t^X(k_1, \ldots, k_t) = \hat{g}
\]

(6)

where

\[
a_i = \sqrt{i^2 - (i-1)^2 H}
\]

(7)

for any \( i \in \mathbb{N}^+ \), and to find conditions on \( g \) which guarantee that such a \( \hat{f}_t^X \) is a proper CF. If this is the case, \( \hat{f}_t^X \) is the Fourier transform of a PDF and manifestly solves Eq. (4). We thus meet with the problem of characterizing the class of scaling functions \( g \) which make the inverse Fourier transform of our trial \( H \) a non-negative joint PDF. Fortunately, this problem is addressed by Schoenberg’s theorem \[44, 50\], which guarantees that Eq. (6) provides a proper CF for all \( t \) if and only if \( \hat{g} \) is of the form

\[
\hat{g}(k) = \int_0^\infty d\sigma \rho(\sigma) e^{-\sigma^2 k^2/2},
\]

(8)

\( \rho \) being a PDF on the positive real axis. The class of scaling functions suitable for our fine-graining procedure is thus constituted by the Gaussian mixtures

\[
g(x) = \int_0^\infty d\sigma \rho(\sigma) \mathcal{N}_\sigma(x),
\]

(9)

where, here and in the following, \( \mathcal{N}_\sigma \) denotes a Gaussian PDF with mean zero and variance \( \sigma^2 \). Such a class, whose elements are identified by \( \rho \), is rich enough to allow us to account for very general anomalous scaling symmetries. The joint PDF of the variables \( X_i \)’s provided by our inverse strategy and corresponding to \( g \), i.e. to \( \rho \), is then obtained by applying an inverse Fourier transform to Eq. (6) and reads

\[
f_t^X(x_1, \ldots, x_t) = \int_0^\infty d\sigma \rho(\sigma) \prod_{i=1}^t \mathcal{N}_{\sigma}(x_i),
\]

(10)

with the \( a_i \)’s as in Eq. (7).

There are various ways in which Eq. (10) can inspire the construction of a stochastic process suitable for finance. Some possibilities have been tested in Refs. \[24–26, 53, 54\]. In general, the joint PDF of Eq. (10) itself cannot describe a stationary ergodic sequence \( \{X_i\}_{i=1}^\infty \),
but for the problem we address here, i.e. to describe long historical time series, such features are relevant. To recover their statistical features keeping contact with Eq. (10), we here conceive the process of the returns as separated into two components. As shown below, the manifest scaling property of the Gaussian

\[ N_a(x) = \frac{1}{a} N_a \left( \frac{x}{a} \right), \quad (11) \]

which holds for any \( a > 0 \), prompts such a separation. In the financial time series context, one is then naturally led to interpret these components as accounting for endogenous dynamical mechanisms and for the occurrence of exogenous events, respectively.

As far as the former component is concerned, a correlated process \( \{Y_t\}_{t=1}^{\infty} \) with memory order \( M \) is considered. Up to \( t \) equal to \( M \), this process is characterized by the joint PDF of Eq. (10) with \( a_i = 1 \) for all \( i \)'s. This is a sequence of non-Gaussian, dependent random variables and at times up to \( M \) their sum satisfies a form of anomalous scaling with \( H = 1/2 \) and \( g \) given by Eq. (9).

The introduction of a finite \( M \) of course limits the range of time for which this form of scaling is valid. This is not a problem, because empirically we know that anomalous scaling approximately holds within a finite time window. The entire process \( \{Y_t\}_{t=1}^{\infty} \) is then obtained through an auto-regressive scheme of order \( M \). This auto-regressive scheme is such to prevent the dynamics from stabilizing the conditional variance of \( Y_t \)'s, given the past history, to a constant value after an initial transient, thus restoring full ergodicity [24]. At the same time, the conditioning effect of the previous values of the process on the future dynamical evolution is of primary importance in applications like, e.g., those related to derivative pricing or volatility forecasts.

The latter component introduces a multiscale behavior by multiplying each element of the endogenous sequence by the corresponding factor \( a_i \) given in Eq. (7): \( X_t = a_i Y_t \). In principle, these rescaling factors convey a property of the Gaussian \( \sigma(x) \) a constant value after an initial transient, thus restoring full ergodicity [24]. However, a proper randomization of the time argument of the \( a_i \)'s restores the stationarity of the \( X_t \)'s, making them suitable for describing single time series whose statistical properties are thought to be independent of time [10]. This randomization is regarded as mimicking the effects on market evolution of external inputs of information or changing conditions, thus conforming an exogenous character to this second component. The first component, which is responsible for the volatility clustering phenomenon thanks to its long dynamical memory \( M \), is then interpreted as the endogenous part. As we shall review in the Paper, and show in the Supplementary Material [40], relevant properties of the model, like its multiscaling and the power-law decay of non-linear autocorrelations, are determined by the exogenous component. We stress that when combining the endogenous and exogenous processes, together with simple scaling features also the direct link between the Hurst exponent and the exponent \( H \) entering in Eq. (7) is lost. For this reason, in the following we will denote by \( D \), instead of \( H \), the parameter involved in the definition of the exogenous component [See Eq. (19) below].

III. MODEL DEFINITION

On the basis of the background material elaborated in the previous Section, here we precisely define our stochastic process of the increments. Such a process \( \{X_t\}_{t=1}^{\infty} \) is obtained as the product of an endogenous auto-regressive component \( \{Y_t\}_{t=1}^{\infty} \) and an exogenous rescaling factor \( \{a_t\}_{t=1}^{\infty} \), where \( \{Y_t\}_{t=1}^{\infty} \) is a discrete Markovian random time independent of \( \{X_t\}_{t=1}^{\infty} \), and \( \{a_t\}_{t=1}^{\infty} \) is a positive sequence:

\[ X_t \equiv a_t Y_t. \quad (12) \]

The stochastic process \( \{Y_t\}_{t=1}^{\infty} \) is a Markov process taking real values with memory \( M > 0 \). It is defined, through its PDF’s, by the following scheme:

\[ f_t^Y(y_1,\ldots,y_t) \equiv \varphi_t(y_1,\ldots,y_t) \quad (13) \]

if \( t = 1,2,\ldots,M \), and

\[ f_t^Y(y_1,\ldots,y_t) \equiv \varphi_{M+1}(y_{t-M+1},\ldots,y_t), \quad \varphi_M(y_{t-M+1},\ldots,y_{t-1}) \cdot f_{t-1}^Y(y_1,\ldots,y_{t-1}) \quad (14) \]

if \( t > M \). Here, the PDF’s \( \varphi_t \) are given by

\[ \varphi_t(y_1,\ldots,y_t) \equiv \int_0^\infty d\sigma \rho(\sigma) \prod_{n=1}^t N_\sigma(y_n). \quad (15) \]

The process \( \{I_t\}_{t=1}^{\infty} \) is a Markov chain valued in \( \mathbb{N}^+ \) defined by the initial condition

\[ P[I_1 = i] \equiv \nu(1 - \nu)^{i-1} \quad (16) \]

and by the transition probabilities

\[ P[I_{t+1} = i|I_t = j] \equiv \begin{cases} \nu & \text{if } i = 1; \\ 1 - \nu & \text{if } i = j + 1; \\ 0 & \text{otherwise}. \end{cases} \quad (17) \]

In words, we are stating that at time \( t + 1 \) there is a “time-reset” or “restart” \( (I_{t+1} = 1) \) with probability \( \nu > 0 \), whereas with probability \( 1 - \nu \) time flows normally \( (I_{t+1} = I_t + 1) \). For notational simplicity we set \( \pi(i) \equiv P[I_1 = i] \) and we collect the transition probabilities into a stochastic matrix with entries \( W(i,j) \equiv P[I_{t+1} = i|I_t = j] \). We point out that our choice of \( \pi \) corresponds to the invariant distribution of \( W \), with the consequence that \( \{I_t\}_{t=1}^{\infty} \) turns out to be a stationary process:

\[ \sum_{j=1}^\infty W(i,j) \pi(j) = \pi(i). \quad (18) \]
It should be stressed that here we assume that \( \{Y_t\}_{t=1}^{\infty} \) and \( \{I_t\}_{t=1}^{\infty} \) are independent in favor of an initial simplicity. In Section VIII we shall hint at the possibility of making these two processes dependent.

Finally, \( \{a_i\}_{i=1}^{\infty} \) is a positive sequence where, without loss of generality, we can set \( a_1 = 1 \). In analogy with the previous Section, we assume a factor \( a_i \) of the form

\[
a_i = \sqrt{i^{2D} - (i-1)^{2D}}
\]

with \( D > 0 \). The relation between the Hurst exponent and the model parameter \( D \) will be addressed in what follows. For the moment, let us point out that the sequence \( a_i \) is identically equal to 1 if \( D = 1/2 \) while monotonically decays to zero or diverges if \( D < 1/2 \) or \( D > 1/2 \), respectively. For financial applications, the instance \( D < 1/2 \) appears to be the interesting one and, since \( \lim_{i \to \infty} i^{1/2-D} a_i = \sqrt{2D} \), the decay of the rescaling factor is of power-law type. However, in principle other choices for the functional form of \( \{a_i\}_{i=1}^{\infty} \) are possible and could be introduced for further extensions and applications of the model.

The endogenous process \( \{Y_t\}_{t=1}^{\infty} \) recalls the ARCH construction of order \( M \) [20] because the conditional PDF of the current \( Y_t \), given the past history, depends on the previous outcomes only through the sum of the squares of the latest \( M \) ones, as one can easily recognize. As a matter of fact, \( \{Y_t\}_{t=1}^{\infty} \) becomes a genuine ARCH process if the function \( \rho \) is properly chosen, as we shall show in a moment. In general, the basic difference with respect to an ARCH process is that here the whole conditional PDF of \( Y_t \) and not only its variance, changes with time. In spite of this, the process \( \{Y_t\}_{t=1}^{\infty} \) is identified by a small number of parameters independently of the order \( M \). Indeed, besides \( M \), the parameters associated to \( \{Y_t\}_{t=1}^{\infty} \) are only those related to \( \rho \). As we discuss below, satisfactory parametrizations of \( \rho \) for financial time series require just two parameters. This must be contrasted with the fact that in realistic ARCH models the number of parameters can proliferate with the memory, easily becoming of the order of several tens [8]. Such a synthetic result, which we believe to be a most interesting innovative feature of \( \{Y_t\}_{t=1}^{\infty} \), is made possible by the exploitation of the scaling symmetry embodied in Eqs. (13) [15].

A most practical choice for \( \rho \) is one which allows us to explicitly perform the integration over \( \sigma \) in Eq. (14). Indeed, we notice that weighing \( \sigma^2 \) according to an inverse-gamma distribution is the way to reach this goal. Furthermore, in the context of financial modeling, this prescription is in line with the rather common belief that the distribution of the square of the empirical returns can be modeled as an inverse-gamma distribution [23, 52, 54]. This \( \rho \) is identified by two parameters, \( \alpha \) and \( \beta \) governing its form and the scale of fluctuations, respectively, and reads

\[
\rho(\sigma) = \frac{2^{1 - \frac{\beta}{2}}}{\Gamma\left(\frac{\alpha}{2}\right)} \frac{\beta^{\alpha}}{\sigma^{\alpha + 1}} e^{-\frac{\beta}{2\sigma^2}},
\]

where \( \Gamma \) denotes the Euler’s gamma function. Interestingly, making this choice within the model, the endogenous component \( \{Y_t\}_{t=1}^{\infty} \) becomes a true ARCH process of order \( M \) with Student’s t-distributed return residuals, as anticipated above. Indeed, in the Supplementary Material [40] we prove that if \( \rho \) is given by Eq. (20), then we can reformulate our model as \( X_t = a_t Y_t \) with

\[
Y_t = \begin{cases} 
\frac{\beta \cdot Z_1}{\sqrt{\beta^2 + \sum_{n=1}^{\min\{t-1,M\}} Y_{t-n}^2}} \cdot Z_t & \text{if } t = 1; \\
\sqrt{\beta^2 + \sum_{n=1}^{\min\{t-1,M\}} Y_{t-n}^2} \cdot Z_t & \text{if } t > 1,
\end{cases}
\]

and the residual process \( \{Z_t\}_{t=1}^{\infty} \) amounting to a sequence of independent Student’s t-distributed variables:

\[
f_t^Z(z_1, \ldots, z_t) = \prod_{n=1}^{t} \frac{\Gamma\left(\frac{\alpha_n + 1}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{\alpha_n}{2}\right)} \left(1 + z_n^2\right)^{-\frac{\alpha_n + 1}{2}}
\]

with \( \alpha_n \equiv \alpha + \min\{n - 1, M\} \). It is also worth noticing that the Markov-switching character of the volatility, introduced by the exogenous process \( \{a_t\}_{t=1}^{\infty} \), reconciles this particular instance of our model with the SWARCH category proposed by Hamilton and Susmel [55]. The only difference is that we deal with an infinite number of regimes corresponding to the infinite possible values taken by \( I_t \). We stress however that besides \( M \) only two parameters, \( \alpha \) and \( \beta \), are here needed to completely specify \( \{Y_t\}_{t=1}^{\infty} \). This typically applies also to other possible parametrizations of \( \rho \), not related to the inverse-gamma distribution.

The exogenous component entails our model with two further parameters, i.e. \( \nu \) establishing the frequency of occurrence of the “time restarts”, and the exponent \( D \) defining the rescaling factor \( \{a_i\}_{i=1}^{\infty} \). In summary, the model is thus typically identified by 5 parameters, three endogenous and two exogenous. The general fact that both the exogenous and endogenous components are hidden processes, not separately detectable, complicates the effectiveness of a parameter calibration protocol. However, as discussed below, analytical features of the model allow us to identify moment optimization procedures that guarantee, for sufficiently long time-series, proper determination of the input parameters.

In the next Section we clarify in details up to what extent the scaling symmetry is preserved by the process \( \{X_t\}_{t=1}^{\infty} \). For the moment we point out that the contact with the ARCH and Markov-switching models’ literature is particularly interesting. Indeed, thanks to our general results below it sheds some light on how to obtain anomalous scaling properties in auto-regressive models on limited temporal horizons [22].

IV. MODEL PROPERTIES

A number of properties of our model are independent of the choice of the function \( \rho \) and can be analytically
investigated. Here we briefly review these properties referring to the Supplementary Material [40] for detailed derivations.

A. Model joint PDF and stationarity

For any \( t \geq 1 \) the joint PDF of \( X_1, \ldots, X_t \) is given by the formula

\[
f_t^X(x_1, \ldots, x_t) = \sum_{i_1=1}^{\infty} \cdots \sum_{i_n=1}^{\infty} \prod_{n=1}^{t-1} W(i_{n+1}, i_n) \pi(i_1) \cdot \frac{f_t^Y(x_1/a_{i_1}, \ldots, x_t/a_{i_n})}{a_{i_1} \cdots a_{i_n}}.
\] (23)

Since \( f_t^Y \) is defined via mixtures of centered Gaussian variables, Eq. (13), we realize immediately that the conditional expectation of \( S_t \) is distributed as \( \{Y_1\}_{t=1}^{\infty} \) is thus a martingale difference sequence, reflecting the efficient market hypothesis [54, 57]. Moreover, the structure of \( f_t^X \) shows that the observed process cannot retain the Markov property characterizing both endogenous and exogenous components, with the consequence that its future evolution always depends on all past events. This feature reflects the impossibility of directly detecting from the examination of \( \{X_t\} \) the random time \( \{I_t\}_{t=1}^{\infty} \). More importantly, the latter fact makes a maximum-likelihood estimation of the model parameters very difficult because of the too onerous computational work needed. Thus, one is forced to refer to some moment optimization procedure for settling this issue. For this reason, in the next Section we shall propose a simple implementation of a generalized method of moments. A procedure to identify the most probable time restarts by means of the calibrated model, valuable for some applications like, e.g., in option pricing, will be also discussed in Section [VI].

A remarkable feature of the joint PDF \( f_t^X \) is that it does not explicitly depend on the memory range \( M \) at short time scales. Indeed, when \( t \leq M + 1 \) from Eqs. [23] and [13] we have [notice that Eq. (14) gives \( f_M^{M+1} = \varphi_{M+1} \)]

\[
f_t^X(x_1, \ldots, x_t) = \sum_{i_1=1}^{\infty} \cdots \sum_{i_n=1}^{\infty} \prod_{n=1}^{t-1} W(i_{n+1}, i_n) \pi(i_1) \cdot \int_0^{\infty} d\sigma \rho(\sigma) \prod_{n=1}^{t} N_{a_{i_n}}(x_n).
\] (24)

This fact implies that models with different memory orders \( M \) and \( M' > M \), and the same other parameters, cannot be distinguished by looking at their features at times shorter than or equal to \( M + 1 \). Observe also that the Gaussian mixture structure provided by our fine-graining strategy and the random nature of the factor redefining the typical magnitude of the fluctuations is particularly clear in Eq. (24).

Our process is strictly stationary, meaning that \( (X_n, \ldots, X_{n+t-1}) \) is distributed as \( (X_1, \ldots, X_t) \) for any \( n \geq 1 \) and \( t \geq 1 \). This property directly follows from the fact that \( \{Y_t\}_{t=1}^{\infty} \) and \( \{I_t\}_{t=1}^{\infty} \) are both stationary processes and, in particular, tells us that Eqs. [23] and [24] give the PDF of any string of \( t \) consecutive variables extracted from \( \{X_t\}_{t=1}^{\infty} \). We stress that stationarity is a basic assumption in time series analysis, when one is forced to reconstruct the underlying stochastic process on the basis of a single, possibly long, time series.

We also point out that the endogenous component \( \{Y_t\}_{t=1}^{\infty} \) is not only a stationary sequence, but even a reversible one: the law of \( (Y_t, Y_{t-1}, \ldots, Y_1) \) is the same as the law of \( (Y_1, Y_{t-1}, Y_t) \) for any \( t \geq 1 \). In contrast, the observed process \( \{X_t\}_{t=1}^{\infty} \) is not reversible, being such time-reversal symmetry broken by the exogenous component. In Section [VI] we shall better analyze this feature of the model, attempting to quantify the time-reversal asymmetry of \( \{X_t\}_{t=1}^{\infty} \).

The single-variable PDF, which is the same for any \( X_t \) thanks to stationarity, is obtained by setting \( t = 1 \) in Eq. [24] and explicitly reads

\[
f_1^X(x) = \sum_{i=1}^{\infty} \nu(1-\nu)^{i-1} \int_0^{\infty} d\sigma \rho(\sigma) N_{a_i}(x).
\] (25)

The mixture of Gaussian densities with different width can endow this PDF with power law tails, as observed for financial assets [10]. Specifically, when \( \rho(\sigma) \) decays as the power-law \( \sigma^{-\alpha-1} \) for large \( \sigma \), \( f_1^X \) becomes a fat-tailed distribution with the same tail index \( \alpha \). Thus, for example choosing \( \rho \) as in Eq. [20] we have \( \lim_{x \to \infty} |x|^{\alpha+1} f_1^X(x) = c \) with

\[
c \equiv \frac{\beta \Gamma\left(\frac{\alpha+1}{2}\right)}{\sqrt{\pi \Gamma(\frac{\alpha}{2})}} \sum_{i=1}^{\infty} a_i^\alpha \nu(1-\nu)^{i-1} < \infty,
\] (26)

and the above form parameter \( \alpha \) controls the tails of the PDF of the \( X_t \)'s as long as \( \nu \) is finite. It is worth noticing that, even if the above condition on \( \rho \) is necessary for having fat tails in a strict asymptotic sense, there is the possibility of approximately realizing such a feature for returns in empirically accessible ranges by only considering rare enough time restarts. As explained in the Supplementary Material [10], indeed, assuming \( \{a_i\}_{i=1}^{\infty} \) given by Eq. (19) with \( D < 1/2 \), and properly rescaling \( \nu \) in order to avoid \( f_1^X \) to concentrate around zero in the small-\( \nu \) limit, in general the single-variable PDF displays fat tails with index \( 2/(1-2D) \) when the restart probability \( \nu \) approaches zero. Of course, with \( \rho(\sigma) \) behaving as \( \sigma^{-\alpha-1} \) for large \( \sigma \) and \( \alpha < 2/(1-2D) \), the tail index is determined by \( \alpha \) even in the rare-restart limit. In practice, when dealing with small values of \( \nu \) the empirically-accessible power law exponent of \( f_1^X \) depends on all \( \alpha \), \( \nu \), and \( D \). This fact, and the uncertainty affecting the empirical estimate of such exponent [10], lead us to a calibration protocol (See below) which is not based on matching the effective power law tails of \( f_1^X \).
Since we have here stated the stationarity of our model, we also mention that strong mixing properties can be proved under mild assumptions on the function $\rho$ \cite{48}. These mixing properties entail ergodicity, which justifies the comparison between empirical long time averages and theoretical ensemble expectations. They also imply the validity of the central limit theorem, to which we appeal for discussing scaling features of aggregated returns on the long time horizon under, basically, the only hypothesis that the second order moment of the elementary increments is finite. Stating precisely these results and discussing their proof however requires a more rigorous setting \cite{18} which is beyond the scope of the present Paper. The ergodicity has also been numerically verified on the basis of model-based simulations.

### B. Scaling features

Scaling features of $\{X_t\}_{t=1}^\infty$ are at the heart of our approach and two different scaling regimes, corresponding to the empirical evidence found for financial assets \cite{47, 51, 52}, can be identified within the present model: one is an effective multiscaling regime, which is most easily discussed in analytical terms for $t \leq M + 1$, and the other is an asymptotic Gaussian simple-scaling scenario, which prevails for $t \gg M$ as a consequence of the central limit theorem mentioned above \cite{18}. We focus here on the former, which is directly relevant for applications in finance.

The moment time-dependence of the aggregated return $X_1 + \cdots + X_t$ is only ruled by the exogenous component $I_t$. If $t \leq M + 1$, since Eq. (27) enables one to demonstrate \cite{40} that in such a case

$$m_q^X(t) = \frac{\mathbb{E}[|X_1 + \cdots + X_t|^q]}{\mathbb{E}[|X_1|^q]} \approx \left(\frac{\mathbb{E}[a^2_{I_t}]^{\frac{q}{2}}}{\mathbb{E}[a^2_{I_t}]}\right)^{\frac{q}{2}}. \quad (27)$$

Notice that the r.h.s. of Eq. (27) is well defined for any $q$. If the $X_i$’s PDF’s are endowed with fat tails, this is not true for the l.h.s. of the same equation. When $D < 1/2$ and not too small, effective scaling properties of the model follow from the fact that $m_q^X$, although apparently a rather complex function of the time, is well approximated by the power $\nu^q H_q$ for $t \leq M + 1$ \cite{40}. The generalized Hurst-like exponent $H_q$ can be computed using a least squares method over time. Referring for instance to a temporal window extending up to $t = 31$ and adopting a memory order $M \geq 30$, Fig. 1 displays $H_q$ for different pairs of $D$ and $\nu$ values. The exponent $H_q$ stays close to $1/2$ for low orders $q$ up to about $2/(1 - 2D)$, denoting an initial simple-scaling regime. It recovers a dependence on $q$ for larger moment orders, manifesting a multiscaling behavior. A sharp result is found in the limit of small $\nu$, where $H_q = 1/2$ for $q \leq 2/(1 - 2D)$ \cite{40}.

In the perspective of a comparison of our model with data, a remark on the scaling features of empirical financial data is in order (See also \cite{25}). While the simple-
scaling behavior at low q is a stable and robust empirical evidence, multiscaling features occurring at larger q are sensibly dependent on the time series sample, for series of a length comparable with that at our disposal for the S&P500 index. We report this observation in Fig. 2 with respect to the S&P500 daily time series. The empirical exponent \( \bar{H}_q \) is here obtained from the time-average estimation of \( m^X_q \), as computed in the next Section. In turn, from the modeling point of view, the multiscaling region in the moment order axis mostly overlaps the non-existing moment region when fat-tailed distributions are involved.

C. Volatility autocorrelation

In view of financial applications, the volatility autocorrelation of order \( q \) can be introduced as the autocorrelation function of the process \( \{X_t^{[q]}\}_{t=1}^\infty \):

\[
 r^X_q(t) \equiv \frac{E[|X_t^{[q]}|X_t|] - E[|X_t^{[q]}|^2]}{E[|X_t|^{2q}] - E[|X_t|]^2}. \tag{28}
\]

Again, this autocorrelation is easily investigated for \( t \leq M+1 \), where the exogenous component alone determines its decay. Indeed, thanks to the independence of the processes \( \{Y_t\}_{t=1}^\infty \) and \( \{I_t\}_{t=1}^\infty \), \( r^X_q(t) \) can be rewritten as:

\[
v^X_q(t) = u_q + v_q r^\alpha_q(t) \text{ for } 2 \leq t \leq M+1, \text{ while } r^X_q(1) = 1.
\]

Here, \( r^\alpha_q \) is the autocorrelation of \( \{\alpha_t\}_{t=1}^\infty \), and \( u_q, v_q \) are two time-independent coefficients whose explicit expression is provided in the Supplementary Material. We thus see that the time dependence of \( r^X_q \) comes from \( r^\alpha_q \) at short time scales \( t \leq M+1 \). Interestingly, for \( q \) small enough, the smaller the restart probability \( \nu \), the more correlations get persistent: when \( \nu \) approaches zero, we find \( r^\alpha_q(1) = 1 \) for any \( t \) if \( q \leq 1/(1-2D) \). Notice that this last threshold for the moment order \( q \) is now half of that previously discussed for the simple-scaling behavior.

While the initial decay of the volatility autocorrelation \( r^X_q \) is strongly dependent on the parameter setting, in particular through the ratio \( u_q/v_q \), on time scales much larger than \( M \), \( r^X_q \) decays exponentially fast, due to the strong mixing properties of our model. In the Supplementary Material, we show that this is indeed the case focusing on the function \( \rho \) given by Eq. (26) and the instance \( q = 2 \), for which the correlation decay rate can be explicitly computed.

V. MODEL CALIBRATION

An important issue for the application of a model to time series analysis is the implementation of efficient calibration protocols, capable of identifying the model parameters which most effectively reproduce a specific empirical evidence. As anticipated, the inclusion of both an endogenous and an exogenous part in our model complicates the calibration procedure, because the two components cannot be easily resolved along an empirical time series. In order to overcome this difficulty, we devise here a method based on the comparison between empirical and theoretical moments, drawing on the generalized method of moments and taking advantage of the analytical structure of our model.

With the relatively limited amount of daily historical data available for financial assets, the identification of the model parameters is affected by large uncertainties. Since our memory parameter \( M \) establishes the time horizon over which the endogenous dynamical dependence operates, we can choose to fix it on the basis of the time scale associated with the specific application of interest. Given \( M \), we thus optimally exploit the simple analytical structure within the time window \( 1 \leq t \leq M \) for the calibration of the remaining parameters. In order to present the procedure and to test our model on real data, we refer here and in the following to the advantageous function \( \rho \) introduced in Section III by Eq. (20). Once \( M \) is fixed, the further parameters to be estimated are four: the exponent \( D > 0 \), the restart probability \( 0 < \nu \leq 1 \), and the parameters \( \alpha > 0 \) and \( \beta > 0 \) identifying \( \rho \). For simplicity, we collect the first three of them into the vector \( \theta = (D, \nu, \alpha) \) and we denote by \( \Theta \) its feasible range. The parameter \( \beta \) plays a minor role in the model since we only need it to fix the scale of \( X_t \)’s fluctuations.

Given a time series \( \{x_t\}_{t=1}^T \) with empirical mean zero, our calibration protocol is based on the idea of better reproducing, within the model, its scaling and autocorrelation features on times up to \( M \). Thus, in a least square framework, we choose those parameters which minimize the distance between the theoretical \( m^X_q(t) \) and \( r^X_q(t) \), defined by Eqs. (27) and (28) respectively, and the corresponding empirical estimations \( \overline{m}^X_q(t) \) and \( \overline{r}^X_q(t) \) in the window \( 1 \leq t \leq M \). Such empirical estimations are obtained via time averages over the available series. To illustrate the computation, for instance we get \( \overline{m}^X_q(t) \) as \( M_q(t)/M_q(1) \) with

\[
 M_q(t) \equiv \frac{1}{T+1-t} \sum_{n=0}^{T-t} |x_{n+1} + \cdots + x_{n+t}|^q. \tag{29}
\]

We recall that the comparison between empirical time averages and theoretical ensemble expectations is justified by the ergodicity of our process [48].

Being properly normalized, \( m^X_q \) and \( r^X_q \) do not depend on the scale parameter \( \beta \). Denoting by \( \mathbb{Q} \) the set of the moment orders we consider for the calibration purposes, our parameter estimation \( \overline{\theta} = (\overline{D}, \overline{\nu}, \overline{\alpha}) \) results thus to be

\[
 \overline{\theta} = \arg \min_{\theta \in \Theta} \left\{ \sum_{q \in \mathbb{Q}} \sum_{t=1}^M \left[ \frac{m^X_q(\theta; t) - m^X_q(t)}{m^X_q(\theta; t)} \right]^2 + \right. \\
 + \sum_{q \in \mathbb{Q}} \sum_{t=1}^M \left[ \frac{r^X_q(\theta; t) - r^X_q(t)}{r^X_q(\theta; t)} \right]^2 \right\}, \tag{30}
\]

where the dependence of \( m^X_q \) and \( r^X_q \) on \( \theta \) is explicitly indicated. We have directly checked that this calibration
We set S&P500 from January 1st 1950 to December 31st 2010.

Fig. 3a and 4a report the result of this protocol applied to the logarithmic increments of the daily closures of S&P500 from January 1st 1950 to December 31st 2010.

We close the calibration protocol providing a way of estimating the parameter $\beta$. Once $D$, $\nu$, and $\alpha$ have been obtained, we can evaluate $\beta$ by optimizing with respect to $e_q^X \equiv E[X_t^{1/q}]$, where $D$, $\nu$, and $\alpha$ are set equal to $\overline{D}$, $\underline{\nu}$, and $\overline{\alpha}$ respectively. Making explicit the dependence of $e_q^X$ on $\beta$, the relationship $e_q^X (\beta) = e_q^X (1) \beta^q$ is rather evident. If $\underline{\nu}^X$ denotes the empirical counterpart of $e_q^X$, we then get our estimation $\overline{\beta}$ of $\beta$ through the formula

$$\overline{\beta} = \arg \min_{\beta \in (0, \infty)} \left\{ \sum_{t \in \mathcal{Q}} \left[ \frac{e_q^X (\beta) - e_q^X (\beta)}{e_q^X (\beta)} \right]^2 \right\}. \quad (31)$$

Aiming at reducing the computational load of the parameter estimations, in the present Paper we work out calibration with the moment order $q = 1$ only: $\mathcal{Q} = \{1\}$. Fig. 3a and 4a report the result of this protocol applied to the logarithmic increments of the daily closures of S&P500 from January 1st 1950 to December 31st 2010. We set $\overline{x}_t \equiv \ln \overline{x}_t - \ln \overline{x}_{t-1} - \mu$ for $t = 1, \ldots, T$, being $T = 15385$ and $\{\overline{x}_t\}_{t=0}^T$ the considered S&P500 time series. The value of the drift $\mu$ is such that $\sum_{t=1}^T \overline{x}_t = 0$.

In compliance with an application we are developing to derivative pricing, we have chosen $M = 21$ (the operating market days in one month) yielding $(\overline{D}, \underline{\nu}, \overline{\alpha}, \overline{\beta}) = (0.21, 0.030, 4.0, 0.04)$, $M = 42$ (two months) giving $(\overline{D}, \underline{\nu}, \overline{\alpha}, \overline{\beta}) = (0.19, 0.011, 4.5, 0.07)$, and $M = 63$ (three months) for which $(\overline{D}, \underline{\nu}, \overline{\alpha}, \overline{\beta}) = (0.16, 0.004, 5.5, 0.14)$.

Notice how the calibrated model fits the S&P500 scaling features and the volatility autocorrelation well beyond $M$ in the case of two and three months, whereas one month does not seem to be enough to get the correct decay as soon as $t$ is larger than 21.

VI. COMPARISON WITH S&P500 INDEX AND NULL HYPOTHESIS

In order to put into context the performance of our model and to probe the role of the memory $M$, here we consider, as the null hypothesis, a limit version in which $\sigma$ is kept fixed to a constant value $\sigma_0$ [$\rho(\sigma) = \delta(\sigma - \sigma_0)$], which turns out to be the scale parameter. From Eq. (19) we get that this prescription makes the endogenous component a sequence of independent normal variables, preventing the parameter $M$ from playing any role [62]. Even if the null model has no endogenous memory, for the sake of comparison we estimate its parameters (first $D$ and $\nu$, and later $\sigma_0$) by means of the procedure outlined in the previous Section and with the same values of $M$ and $q$ used for the model characterized by the function $\rho$ of Eq. (20), which we name here “the complete model”. Figs. 3b and 4b show the outcome of the calibration protocol, which gives the following results: $(\overline{D}, \underline{\nu}, \overline{\alpha}, \overline{\beta}) = (0.05, 0.0001, 1.01)$ for $M = 21$, $(\overline{D}, \underline{\nu}, \overline{\alpha}, \overline{\beta}) = (0.06, 0.0002, 0.62)$ for $M = 42$, and $(\overline{D}, \underline{\nu}, \overline{\alpha}, \overline{\beta}) = (0.07, 0.0003, 0.45)$ when $M = 63$. The figures indicate that calibration is slightly less successful for the null model than for the complete one.
FIG. 5: Single-variable PDF comparison between S&P500 and calibrated complete model (a), and between S&P500 and the null model (b) in linear scale. Symbols and lines color code is as in the previous plots.

FIG. 6: Single-variable PDF comparison between S&P500 and calibrated complete model (a), and between S&P500 and the null model (b) in log scale. Symbols and lines color code is as in the previous plots.

The unconditional return PDF of the S&P500 is very well reproduced by both the complete and the null calibrated models, both in the central part and in the tails. We realize this fact by an inspection of the linear and log plots of $f_1^X$ in Figs. 5 and 6, respectively. While the function $\rho$ defined by Eq. (20) endows the complete model with fat tails, setting $\sigma = \sigma_0$ prevents the null model from recovering such a feature from a strict mathematical standpoint. However, in Section IV A we mentioned that with a small enough value of the restart probability $\nu$ one recovers an effective fat tails scenario when $D < 1/2$. This circumstance explains why the null model reproduces the empirical fat tails thanks to an estimated value of $\nu$ which is one or two orders of magnitude smaller than the corresponding value for the complete model. The drawback is that a very small restart probability entails very rare but high and strongly time-asymmetric volatility bursts in the typical trajectories of the model, which are not observed in the historical series. Indeed, Fig. 7, showing the comparison of typical simulated realizations of the benchmark model with the S&P500 time series, reports the discrepancy between the S&P500 and the null model paths, where one can immediately identify the time restarts. In contrast, once the endogenous component retains the memory of the previous returns, the combined effect of more frequent restarts and of the volatility clustering phenomenon produces typical trajectories which are pretty similar to the historical S&P500, as shown in Fig. 7 where the above comparison is proposed for the complete model. Notice that restart events become here much harder to identify.

In order to recover the role of the exponent $D$, in Figs. 8a and 8b we also compare the aggregated return scaling features of the calibrated models with those of the S&P500 series. While, as anticipated in Fig. 2, the empirical multiscaling regime is very erratic and dependent on single extreme events, the simple-scaling behavior ($\Pi_q \simeq 1/2$) up to $q \simeq 3$ seems a stable feature of the S&P500. On the other hand, in Section IV B we noticed that our model predicts that the latter extends up to $q = 2/(1 - 2D)$ at low values of $\nu$ when $D < 1/2$, irrespective of the function $\rho$. The complete model provides $2/(1 - 2D) = 3.4$ for the calibration with $M = 21$, $2/(1 - 2D) = 3.2$ for $M = 42$, and $2/(1 - 2D) = 2.9$ if $M = 63$, therefore showing a qualitative agreement with the empirical evidence. The same cannot be said for the null model, which gives $2/(1 - 2D)$ close to 2 for all the three calibrations.

Financial time series are reported to break time-reversal invariance, not only in terms of return-volatility correlation properties (e.g., the leverage effect [7, 39]), but also in terms of volatility-volatility correlations [59]. Although in the form discussed so far our model cannot explain the former (which is an odd-even correlation), it can account for the latter since, as we have already mentioned in Section IV A, the exogenous component breaks the temporal symmetry through the mechanism of time restarts. We thus conclude this Section considering an
FIG. 7: Time series comparison between S&P500 (black) and complete model (a), and between S&P500 (black) and the null model (b). To facilitate the inspection, model time series are shifted by \(-0.3\) \((M = 63)\), \(-0.6\) \((M = 42)\), and \(-0.9\) \((M = 21)\).

Even-even correlation, specifically the historical versus realized volatility correlation \([18, 20, 23, 59]\), and assessing an asymmetry between the past and the future for the calibrated complete model. In Section VIII we shall discuss how to improve the present model in order to also take into account the leverage effect.

Consider two consecutive time windows, named “historical” and “realized”, of width \(t_h \geq 1\) and \(t_r \geq 1\), respectively. The associated “historical volatility” \(S^h_{t_h}\) and “realized volatility” \(S^r_{t_r}\) are defined as the random variables

\[
S^h_{t_h} = \sqrt{\frac{1}{t_h} \sum_{t=1}^{t_h} X^2_t},
\]

\[
S^r_{t_r} = \sqrt{\frac{1}{t_r} \sum_{t=t_h+1}^{t_r} X^2_t},
\]

For a reversible process, the correlation between past and future volatilities \([60]\), namely

\[
\chi(t_h, t_r) = \frac{E[S^h_{t_h} S^r_{t_r}] - E[S^h_{t_h}]E[S^r_{t_r}]}{\sqrt{\text{var}[S^h_{t_h}] \text{var}[S^r_{t_r}]}}.
\]

is a symmetric function of the time horizons \(t_h\) and \(t_r\), as one can easily verify starting from the definition of reversibility given in Section IV A. In contrast, the structure of its empirical estimation \(\chi(t_h, t_r)\) for the S&P500 time series shows some degrees of asymmetry. This is highlighted by the level curves plot in Fig. 9, also named “volatility mug shots” \([18, 19]\). Such an asymmetry is however rather mild and sample dependent, as illustrated by Figs. 9a and 9b where the whole S&P500 sample and the second half only are exploited, respectively. As far as our model is concerned, at variance with what pointed out in Ref. \([25]\) for a different implementation of our ideas, we remark that such a mild time asymmetry is consistently reproduced. For instance, Fig. 9c displays the level curves of \(\chi(t_h, t_r)\) corresponding to the complete model calibrated with \(M = 42\).

In the various comparisons outlined in this Section we have used the average values defined by our calibrated model. In model-generated time series with a length of the order of that of the available S&P500 dataset, we have also inspected the fluctuations around these average values. In general, we have observed fluctuations that are consistent with those associated to the sample-dependence of the S&P500 time series.

VII. ENDOGENOUS AND EXOGENOUS VOLATILITY

An interesting feature for a model of asset evolution is the possibility of distinguishing between endogenous and exogenous contributions to the volatility. Although this should not be regarded as a clear cut distinction, one
may reasonably expect that the first contribution could be ascribed to cooperative influences among the agents, whereas the latter comes, e.g., from news reaching the market. In our model, albeit intimately combined together, the endogenous and exogenous components play their own distinct role in reproducing realistic financial features. In the previous Section we have highlighted that the finite-memory endogenous mechanism is responsible for introducing volatility clustering features which cannot be recovered through a scheme relying only on the exogenous component like that provided by the null hypothesis model. The question then naturally arises about the possibility of identifying these two different contributions. For this reason, we propose here a procedure to localize the time restarts in a given finite realization \( \{ x_t \}_{t=1}^T \) of a process which is assumed to be well represented by our model. Once the restarts are supposed to be known, we can identify the endogenous trajectory \( \{ y_t \}_{t=1}^T \), thus succeeding in distinguishing between the two contributions.

To the purpose of locating restarts, we consider the probability of having a restart at a certain time \( t \) con-
conditioned to the information available in a narrow time window centered in $t$ with half-width $\tau$. Namely,

$$\mathbb{P}[I_t = 1|X_n = \bar{x}_n, |n-t| \leq \tau].$$

The time restarts can thus be tentatively associated to the peaks of this probability, as in Fig. 10 where $\tau = 2$ is used with respect to a model-generated time series. Conditioning the time restarts identification to more time series values by taking a larger value of $\tau$ would in principle provide better results. In practice however computational limitations force us to focus on small values of $\tau$. Despite this restriction, with $\tau = 2$ in Fig. 10 we have been able to identify exactly 60% of the true restarts and about 70% with an uncertainty of two days. Fig. 11 displays the result of the same “time restarts analysis” applied to the S&P500 historical time series. Here, of course, blue circles are absent.

The PDF $k_t$ of this variable is easily obtained when $t \leq M + 1$, due to the fact that $f^{\nu}_s$ reduces to a mixture of factorized Gaussian densities with the same variance. It turns out to be

$$k_t(s) = \int_0^{\infty} d\sigma \rho(\sigma) \frac{2^{1-\frac{\nu}{2}}s^{\nu-1}t^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)\sigma^t} e^{-\frac{s^2}{2\sigma^2}}. \quad (37)$$

In particular, if the function $\rho$ is chosen according to Eq. (20), then $k_t$ is explicitly found as

$$k_t(s) = \frac{2\beta^\alpha t^{\nu-1}t^\frac{\nu}{2}}{B\left(\frac{\alpha}{2}, \frac{\alpha}{2}\right)\left(\beta^2 + s^2t\right)^{\frac{\nu+\alpha}{2}}}, \quad (38)$$

where $B$ is the Euler’s Beta function. On the empirical side, the distribution $k_t$ can be sampled from the estimated endogenous path $\{\overline{y}_t\}_{t=1}^T$. Fig. 12 shows a comparison between theoretical and empirical endogenous volatility distributions, both for model-generated time series and for the S&P500 historical data, for $t = M$ and $M = 63$. Notice that as the model’s time series length $T$ increases, the outcome of the present procedure becomes very close to the theoretical prediction in Eq. (38). Finally, the consistency of the S&P500 histogram with the other ones points out that our procedure for identifying the endogenous component of the volatility could be successfully applied to the real market evolution, having sufficiently long historical time series at disposal.

A distinction between exogenous and endogenous volatility is not a standard practice in finance. However, we think that its consideration could open interesting perspectives in fields like risk evaluation and market regulation.
FIG. 12: Distribution of the endogenous volatility: circles refer to the S&P500 dataset; squares and triangles are related to two time series of \( T \) data generated by the model with the \( M = 63 \) calibration parameter set; the continuous line is the theoretical prediction, Eq. (38).

VIII. IMPROVEMENTS AND FURTHER DEVELOPMENTS

Even if the present version of our model represents a significant advancement in terms of stylized-facts-reproduction-to-analytical-control ratio, some important empirical features like the leverage effect and the skewness of the return distribution are missing. Here we briefly discuss how both these effects can be reproduced by suitable improvements of the model.

The leverage effect refers to the presence in historical time series of a negative odd-even correlation of the kind \( \mathbb{E}[X_1 X_2^2] < 0 \). The model we have presented gives \( \mathbb{E}[X_1 X_2^2] = 0 \) for any \( t \) and, also, a symmetric returns distribution. So far, in favor of an initial simplicity we have kept the endogenous and the exogenous components independent. The introduction of a dependence between these two processes, such as that arising when the latter is assumed somehow affected by the past values of the former (similarly, e.g., to the ideas outlined in Ref. [39]) would make the exogenous rescaling factor not completely exogenous and could produce non-zero sign-volatility correlations like the leverage effect. An appealing and potentially interesting way of doing this within our mathematical construction may simply consist in making time restarts dependent on the sign of the endogenous component. We sketch some arguments about this perspective.

Introducing the process \( \{B_t\}_{t=1}^\infty \) of the signs of \( \{Y_t\}_{t=1}^\infty \), defined as \( B_t = 1 \) if \( Y_t \geq 0 \) and \( B_t = -1 \) if \( Y_t < 0 \), in the Supplementary Material [40] we show that \( \{B_t\}_{t=1}^\infty \) and the sequence \( \{|Y_t|\}_{t=1}^\infty \) of the magnitude of \( Y_t \)’s are mutually independent for the model considered so far. Moreover \( \{B_t\}_{t=1}^\infty \) results in a sequence of i.i.d. binary variables with \( \mathbb{P}[B_1 = 1] = 1/2 \), telling us that we have been tossing a fair coin to decide the sign of returns. These considerations allow one to recast our model as \( X_t = a_t B_t |Y_t| \) with \( \{I_t\}_{t=1}^\infty \), \( \{B_t\}_{t=1}^\infty \), and \( \{Y_t\}_{t=1}^\infty \) independent from each other.

In order to improve the model, we could then think in a different alternative. We could assume that the \( B_t \)’s take value different from \(-1 \) and \(+1 \) as, e.g., in Ref. [17]. Also, keeping \( B_t = \pm 1 \), we could draw \( B_{t+1} \) independently of the past events, but making the restart occurrence \( I_{t+1} = 1 \) dependent on the value of \( B_t \). In such a case, the process \( \{(I_t, B_t)\}_{t=1}^\infty \) would result in a bivariate Markov chain, still independent of \( \{Y_t\}_{t=1}^\infty \). We already know that a simple setting of this kind guarantees the martingale character, the stationarity, and the mixing properties of the returns’ process \( \{X_t\}_{t=1}^\infty \) defined as \( X_t \equiv a_t B_t |Y_t| \). At the same time, a skewness in the return distribution is recovered by properly choosing the values assumed by the \( B_t \)’s. The leverage effect occurs then making negative returns more likely followed by a time restart than positive ones. Work is in progress along these lines.

Coming back to the model discussed in the present Paper, an interesting applicative perspective is the fact that its analytical handiness permits the derivation of closed-formulas for derivative pricing and the associated hedging strategy. As pointed out in [25], in the presence of a Gaussian mixture process for the underlying asset an obvious way of obtaining an arbitrage-free option price is by taking the average Black-Scholes price [7, 33, 61] according to the variance measure of the mixture. In the present approach, such a basic idea must be shaped in order to take into account two basic facts. In first place, the endogenous component implies that an effective variance measure of the Gaussian mixture is conditioned by the previous endogenous values of the process. On the other side, the exogenous process strongly influences the volatility. Thus, an effective way of identifying time restarts according to the scheme discussed in Section VII must be developed. In a related work in progress [38] we have been able to successfully tackle these two aspects and to produce an equivalent martingale measure which allows one to derive European option prices [32] in a closed form and to associate a natural hedging strategy with the underlying asset dynamics.

IX. CONCLUSIONS

Scaling and long range dependence have played a major role in the recent development of stochastic models of financial assets dynamics. This development proceeded parallel to the progressive realization that indeed scal-
ing and multiscaling properties are themselves relevant stylized facts. A key achievement has been the multifractal model of asset returns (MMAR) proposed by Mandelbrot and coworkers. This model introduced important features, like the possibility of multiscale return distributions with finite variance and the long range dependence in the volatility, with uncorrelated returns. This long range dependence had been previously a peculiarity of ARCH or GARCH type models, widely used in empirical finance. The difficulties mainly arising from the strict time reversal invariance of the MMAR has been overcome by subsequent proposals of multi-time-scale models which are somehow intermediate between GARCH processes and descriptions based on multiplicative cascades. However, a limitation of all the approaches mentioned above is due to the scarce analytical tractability and the difficulty in efficiently expressing the conditioning effect of past histories when applying them to VaR estimates or option pricing.

The model we presented here addresses the problem first posed by Bachelier over a century ago and opens some interesting perspectives. From a methodological point of view, due to the roots in renormalization theory, it offers an example where scaling becomes a guiding criterion for the construction of a meaningful dynamics. This direction appears quite natural if we look at the development of complex systems theory in statistical physics. Scaling is normally regarded as a tool for unconditioned forecasting. Thanks to our renormalization group philosophy, here we have shown that scaling can also be exploited in order to obtain conditioned forecasting, which is of major importance in finance. This conditioned forecasting potential is based on the multivariate price return distributions like Eqs. which one can construct on the basis of scaling properties.

The coexistence of exogenous and endogenous effects driving the dynamics of the markets has been recognized since long. Indeed, the variations of the assets’ price and volatility cannot be explained only on the basis of arrival of new information on the market. A remarkable feature of our model is the fact that it embodies a natural and sound distinction between the endogenous and exogenous influences on the volatility. Even if the distinction is model-based, the comparison with the S&P500 dataset has shown consistency with historical data.

In the relatively simple form discussed in this Paper, our model has important requisites for opening the way to useful applications. One of these applications, namely a closed-form formulation for pricing derivative assets, is presently under development. Indeed, in view of the capability to account for a considerable number of stylized facts, our model maintains a high degree of mathematical tractability. This tractability allows to rigorously derive important mathematical properties of the process and to set up successful calibration procedures.

A deep connection of our approach with ARCH models is the fact that we identify an auto-regressive scheme as a natural one on which to base the ergodic and stationary dynamics of our endogenous process. Remarkably, we are naturally led to this choice following our criteria based on scaling and on the quest for ergodicity and stationarity.

Our modeling is not based on a “microscopic”, agent based description, which should be regarded as a most fundamental and advanced stage at which to test the potential of statistical physics methods in finance. However, we believe that our results open in the field novel perspectives thanks to the application of one of the most powerful methods available so far for the study of complexity in physics, the renormalization group approach. This approach provides an original, valuable insight into the statistical texture of return fluctuations, which is a key requisite for successful stochastic modeling.

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[1] H. Kantz, and T. Schreiber, Nonlinear Time Series Analysis, 2nd edn. (Cambridge University Press 2004).
[2] R. Shcherbakov, G. Yakovlev, D.L. Turcotte, and J.B. Rundle, A model for the distribution of aftershock waiting times, Phys. Rev. Lett. 95, 218501 (2005).
[3] D.J. Wilkinson, Stochastic modelling for quantitative description of heterogeneous biological systems, Nature Reviews Genetics 10, 122 (2009).
[4] P.Ch. Ivanov L.A. Nunes Amaral, A.L. Goldberger, S. Havlin, M.G. Rosenblum, Z.R. Struzik, and H.E. Stanley, Multifractality in human heartbeat dynamics Nature 399, 461 (1999).
[5] A.M. Petersen, J. Tenenbaum, S. Havlin, and H.E. Stanley, Statistical Laws Governing Fluctuations in Word Use from Word Birth to Word Death, Scientific Reports 2, 313 (2012).
[6] T. Preis, D.Y. Kenett, H.E. Stanley, D. Helbing, and E. Ben-Jacob, Quantifying the Behavior of Stock Correlations Under Market Stress, Scientific Reports 2, 752 (2012).
[7] J.P. Bouchaud, and M. Potters, Theory of Financial Risk and Derivative Pricing: from Statistical Physics to Risk Management, 2nd edn. (Cambridge University Press, 2003).
[8] R.S. Tsay, Analysis of Financial Time Series (John Wiley & Sons, 2002).
[9] M. Musiela, and M. Rutkowski, Martingale Methods in Financial Modelling, 2nd edn. (Springer Verlag, 2005).
[10] R. Cont, Empirical properties of asset returns: stylized facts and statistical issues, Quant. Fin. 1, 223 (2001).
[11] R. Cont, Long range dependence in financial time series, in E. Lutton, J. Levy Véhel eds. Fractals in Engineering (Springer-Verlag, New York, 2005).
[12] J. Voigt, The Statistical Mechanics of Financial Markets (Springer, Berlin, 2001).
[13] S. Ghasshaie, W. Breymann, J. Peinke, P. Talkner, Y. Dodge, Turbulent cascades in foreign exchange markets, Nature 381, 767 (1996).
[14] B. Mandelbrot, A. Fisher, and L. Calvet, A Multifractal Model of Asset Returns (Cowles Foundation Discussion Papers 1164, Cowles Foundation, Yale University, 1997).
[15] J.C. Vassilicos, A. Demos, and F. Tata, No evidence of chaos but some evidence of multifractals in the foreign exchange and the stock market in A.J. Crilly, R.A. Earnshaw, H. Jones, eds. Applications of Fractals and Chaos (Springer, Berlin, 1993).
[16] E. Bacry, J. Delour, and J.F. Muzy, Modelling financial time series using multifractal random walks, Physica A 299, 84 (2001).
[17] Z. Éisler, and J. Kertész, Multifractal model of asset returns with leverage effect, Physica A 343, 603 (2004).
[18] L. Borland, J.P. Bouchaud, J.F. Muzy, and G.O. Zumbach, The Dynamics of Financial Markets – Mandelbrot’s Multifractal Cascades, and beyond, Science & Finance (CFM) working paper archive 500054, Science & Finance, Capital Fund Management, (2005).
[19] G.O. Zumbach, M.M. Dacorogna, J.L. Olsen, and R.B. Olsen, Measuring shock in financial markets, Int. J. Theor. Appl. Finance 3, 347 (2000).
[20] L. Borland, and J.P. Bouchaud, On a Multi-Timescale Statistical Feedback Model for Volatility Fluctuations, Science & Finance (CFM) working paper archive 500056, Science & Finance, Capital Fund Management, (2005).
[21] R.N. Mantegna, and H.E. Stanley, Scaling behaviour in the dynamics of an economic index, Nature 376, 46 (1995); Nature 383, 587 (1996).
[22] R.N. Mantegna, H.E. Stanley, H. E., An Introduction to Econophysics (Cambridge University Press, Cambridge, UK, 2000).
[23] F. Baldovin, and A.L. Stella, Scaling and efficiency determine the irreversible evolution of a market, Proc. Natl. Natl. Acad. Sci. USA 104, 19741 (2007).
[24] A.L. Stella, and F. Baldovin, Anomalous scaling due to correlations: limit theorems and self-similar processes, J. Stat. Mech. P02018 (2010).
[25] P.P. Peirano, and D. Challet, Baldovin-Stella stochastic volatility process and Wiener process mixtures, Eur. Phys. J. B 85, 276 (2012).
[26] A. Andreoli, F. Caravenna, P. Dai Pra, and G. Posta, Scaling and multiscaling in financial series: a simple model, Adv. in Appl. Probab. 44, 1018 (2012).
[27] T. Lux, and M Marchesi, Scaling and criticality in a stochastic multi-agent model of a financial market, Nature 397, 498 (1999).
[28] B. LeBaron, Short-memory traders and their impact on group learning in financial markets, Proc. Natl. Acad. Sci. USA 99, 7201 (2002).
[29] N. Alii, M. Cristelli, L. Pietronero, and A. Zaccaria, Minimal agent based model for financial markets I and II, Europ. Phys. J. B 67, 385 (2009).
[30] R. Engle, Estimates of the variances of U.S. inflation based upon the ARCH model, Journal of Money, Credit and Banking 15, 286 (1983).
[31] T. Bollerslev, Generalized autoregressive conditional heteroskedasticity, Journal of Econometrics 31, 307 (1986).
[32] T. Bollerslev, R.F. Engle, and D.B. Nelson, ARCH Models, in Handbook of Econometrics, edited by R.F. Engle, D.L. McFadden (Elsevier, 1994), pp. 29593038.
[33] J.C. Hull, Options, Futures and Other Derivatives (Prentice-Hall, 2000).
[34] F. Baldovin, and A.L. Stella, Central limit theorem for anomalous scaling due to correlations, Phys. Rev. E 75, 020101(R) (2007).
[35] F. Baldovin, D. Bovina, F. Camana, and A.L. Stella, Modeling the Non-Markovian, Non-stationary Scaling Dynamics of Financial Markets, in F. Abergel, B.K. Chakrabarti, A. Chakrabarti, and M. Mitra (eds.) Econophysics of order-driven markets (1st edn), (New Economic Windows, Springer 2011) pp. 239–252.
[36] F.Baldovin, F.Camana, M. Caporin, and A. L. Stella, Ensemble properties of high frequency data and intraday trading rules, to be published (2013) [arXiv:1202.2447].
[37] F.Baldovin, F. Camana, M. Caraglio, A.L. Stella, and M. Zamparo Afternoon prediction for high-frequency financial markets’ dynamics in F. Abergel, B.K. Chakrabarti, A. Chakrabarti, A. Ghosh, eds., Econophysics of Systemic Risk and Network Dynamics (New Economic Windows, Springer 2013), pp. 49–58.
[38] F. Baldovin, M. Caporin, M. Caraglio, A.L. Stella, and M. Zamparo, Option pricing with anomalous scaling and stochastic volatility, to be published (2013).
[39] J.-P. Bouchaud, A. Matalcic, and M. Potter, Leverage Effect in Financial Markets: The Retarded Volatility Model Phys. Rev. Lett. 87, 228701 (2001).
[40] M. Zamparo, F. Baldovin, M. Caraglio, and A.L. Stella, Scaling symmetry, renormalization, and time series modeling – Supplementary Material, (2013).
[41] B.B. Mandelbrot, and J.W. Van Ness, Fractional Brownian motions, fractional noises and applications, SIAM Review 10, 422 (1968).
[42] R.T. Baillie, T. Bollerslev, and H.O. Mikkelsen, Fractional integrated generalized autoregressive conditional heteroskedasticity, Journal of Econometrics 74, 3 (1996).
[43] L.P. Kadanoff, Statistical Physics, Statics, Dynamics and Renormalization, (World Scientific, Singapore 2005).
[44] G. Jona-Lasinio, Renormalization group and probability theory, Phys. Rep. 352, 439 (2001).
[45] N. Goldenfeld, Lectures on phase transitions and the renormalization group, (Addison-Wesley, 1993).
[46] U. Frisch, Turbulence: The Legacy of A.N. Kolmogorov (Cambridge University Press, Cambridge, 1995).
[47] T. Di Matteo, Multi-scaling in finance, Quant. Fin. 7, 21 (2007).
[48] M. Zamparo, F. Baldovin, M. Caraglio, and A.L. Stella, in preparation (2013).
[49] I.J. Schoenberg, Metric Spaces and Completely Monotone Functions, Ann. Math. 39, 811 (1938).
[50] D.J. Aldous, Exchaging and related topics, Lecture Notes in Mathematics 1117, 1 (1985).
[51] G. Iori Scaling and Multiscaling in Financial Markets, in Disordered and Complex Systems, ed. P. Sollich et al., AIP Conference Proceedings 553, 297 (2001).
[52] T. Di Matteo, T. Aste, and M.M. Dacorogna, Long-term memories of developed and emerging markets: Using the scaling analysis to characterize their stage of development, J. Bank. & Fin. 29, 827 (2005).
[53] A. Gerig, J. Vicente, and M.A. Fuentes, Model for non-
Gaussian intraday stock returns Phys. Rev. E 80, 065102R (2009).

[54] S. Miccichè, G. Bonanno, F. Lillo, and R.N. Mantegna,
Volatility in financial markets: stochastic models and em-
pirical results, Physica A 314, 756 (2002).

[55] J.D. Hamilton and R. Susmel, Autoregressive Condi-
tional Heteroskedasticity and Changes in Regime, Journal
of Econometrics 64, 307 (1994).

[56] E.F. Fama, Efficient capital markets: review of the-
ory and empirical work, Journal of Finance 25, 383-417
(1970).

[57] F.E. Fama, Efficient capital markets: II, J. Finance 46,
1575 (1991).

[58] A.R. Hall, Generalized Method of Moments (Advanced
Texts in Econometrics) (Oxford University Press, 2005).

[59] G. Zumbach, Time reversal invariance in finance Quant.
Fin. 9, 505 (2009).

[60] P.E. Lynch, and G.O. Zumbach, Market heterogeneities
and the causal structure of volatility Quant. Fin. 3, 320
(2003).

[61] F. Black, and M. Scholes, The Pricing of Options and
Corporate Liabilities, J. Polit. Econ. 81, 637 (1973).

[62] L. Bachelier, Theorie de la speculation, Ann. Sci. Ecole
Norm. Sup. 17, 21 (1900).

[63] This version of the model results to be the discrete-time
counterpart of the one considered in Ref. [26] after one
keeps a constant average volatility.
Scaling symmetry, renormalization, and time series modeling
Supplementary Material

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I. INTRODUCTION

This Supplementary Material provides proofs of the model properties announced in the Main Text. In Section II the joint PDF’s of the process \( \{X_t\}_{t=1}^{\infty} \) are derived from the definition of the endogenous and exogenous components \( \{Y_t\}_{t=1}^{\infty} \) and \( \{I_t\}_{t=1}^{\infty} \), respectively. Section III is devoted to prove the stationarity of \( \{X_t\}_{t=1}^{\infty} \), while in Section IV we show that \( \{Y_t\}_{t=1}^{\infty} \) is a reversible sequence and that the series of its sign is independent of \( \{\delta_y\}_{t=1}^{\infty} \). In Sections V and VI we reconsider the issue of the tail of the single-variable PDF \( f^X \) and the scaling features of our model supplying a more complete study. In Section VII we prove some properties of what we called the “null” and the “complete” models in the Main Text, in particular showing that the endogenous component \( \{Y_t\}_{t=1}^{\infty} \) of the complete model [with the choice in Eq. (20) of the Main Text for the function \( \rho \)] is an ARCH process. Lastly, Section VIII contains a detailed analysis of the autocorrelation \( r^X_q \).

II. PDF’S ASSOCIATED WITH THE MODEL

Here we derive the joint PDF’s of the increments \( \{X_t\}_{t=1}^{\infty} \), reported in Eq. (23) in the Main Text. The derivation exploits the features of the hidden processes \( \{Y_t\}_{t=1}^{\infty} \) and \( \{I_t\}_{t=1}^{\infty} \). To this and further purposes, we shall work with the expectation values of test functions. Unless explicitly stated, we will implicitly assume that such expectation values exist.

Given \( t \geq 1 \) and a test function \( F \) on \( \mathbb{R}^t \), we have

\[
\mathbb{E}[F(X_1, \ldots, X_t)] = \mathbb{E}[F(a_1 Y_1, \ldots, a_t Y_t)] = \sum_{i_1 = 1}^{\infty} \cdots \sum_{i_t = 1}^{\infty} \mathbb{E}[F(a_i Y_1, \ldots, a_i Y_t) \delta_{i_1 i_1} \cdots \delta_{i_t i_t}],
\]

(1)

where as usual \( \delta \) denotes Kronecker’s symbol. On the other hand, recalling that \( \{Y_t\}_{t=1}^{\infty} \) and \( \{I_t\}_{t=1}^{\infty} \) are mutually independent, the following factorization holds:

\[
\mathbb{E}[F(a_i Y_1, \ldots, a_i Y_t) \delta_{i_1 i_1} \cdots \delta_{i_t i_t}] = \mathbb{E}[F(a_i Y_1, \ldots, a_i Y_t) \cdot \mathbb{E}[\delta_{i_1 i_1} \cdots \delta_{i_t i_t}] = \mathbb{E}[F(a_i Y_1, \ldots, a_i Y_t)] \cdot \mathbb{P}[I_1 = i_1, \ldots, I_t = i_t].
\]

(2)

Thus, plugging this in Eq. (1), we obtain the chain of identities

\[
\mathbb{E}[F(X_1, \ldots, X_t)] = \sum_{i_1 = 1}^{\infty} \cdots \sum_{i_t = 1}^{\infty} \mathbb{E}[F(a_i Y_1, \ldots, a_i Y_t)] \cdot \mathbb{P}[I_1 = i_1, \ldots, I_t = i_t]
\]

\[
= \sum_{i_1 = 1}^{\infty} \cdots \sum_{i_t = 1}^{\infty} \mathbb{P}[I_1 = i_1, \ldots, I_t = i_t] \cdot \int_{\mathbb{R}} dy_1 \cdots \int_{\mathbb{R}} dy_t F(a_i y_1, \ldots, a_i y_t) f^Y(y_1, \ldots, y_t)
\]

\[
= \int_{\mathbb{R}} dx_1 \cdots \int_{\mathbb{R}} dx_t F(x_1, \ldots, x_t) \cdot \sum_{i_1 = 1}^{\infty} \cdots \sum_{i_t = 1}^{\infty} \mathbb{P}[I_1 = i_1, \ldots, I_t = i_t] \frac{f^Y(x_1/a_i, \ldots, x_t/a_i)}{a_{i_1} \cdots a_{i_t}}.
\]

(3)
The last equality is the consequence of a simple change of variables in the integrals. This result, combined with the arbitrariness of \( F \), clearly shows that the joint probability density distribution of \((X_1, \ldots, X_t)\) is

\[
\begin{align*}
f_t^X(x_1, \ldots, x_t) & \equiv \sum_{i_1=1}^{\infty} \cdots \sum_{i_t=1}^{\infty} \mathbb{P}[I_1 = i_1, \ldots, I_t = i_t] \frac{f_t^Y(x_1/a_{i_1}, \ldots, x_t/a(i_t))}{a_{i_1} \cdots a_{i_t}} f_t^Y(x_1/a_{i_1}, \ldots, x_t/a_{i_t}) \\
& = \sum_{i_1=1}^{\infty} \cdots \sum_{i_t=1}^{\infty} W(i_t, i_{t-1}) \cdots W(i_2, i_1) \pi(i_1) \frac{f_t^Y(x_3/a_{i_1}, \ldots, x_t/a_{i_t})}{a_{i_1} \cdots a_{i_t}}.
\end{align*}
\]

(4)

As far as the density \( f_t^Y \) is concerned, let us recall that \( f_t^Y = \varphi_t \) for \( t \leq M + 1 \) where

\[ \varphi_t(y_1, \ldots, y_t) = \int_0^\infty d\sigma \rho(\sigma) \prod_{n=1}^t \mathcal{N}_\sigma(y_n). \]

(5)

Moreover, solving Eq. (14) of the Main Text, we explicitly get \( f_t^Y \) also when \( t > M + 1 \) as

\[ f_t^Y(y_1, \ldots, y_t) = \frac{\prod_{n=1}^{t-M} \varphi_{M+1}(y_n, \ldots, y_{n+M})}{\prod_{n=2}^{t-M} \varphi_M(y_n, \ldots, y_{n+M-1})}. \]

(6)

### III. Stationarity

This Section is devoted to prove the strict stationarity of the process \( \{X_t\}_{t=1}^\infty \), for which we must verify that \((X_n, \ldots, X_{n+t-1})\) is distributed as \((X_1, \ldots, X_t)\) for any \( n \geq 1 \) and \( t \geq 1 \). As a matter of fact, \( \{X_t\}_{t=1}^\infty \) inherits this property from the hidden processes \( \{Y_t\}_{t=1}^\infty \) and \( \{I_t\}_{t=1}^\infty \) and so proving the strict stationarity of \( \{Y_t\}_{t=1}^\infty \) and \( \{I_t\}_{t=1}^\infty \) is the main issue. Let us assume for a moment that we know that \((Y_1, \ldots, Y_{n+t-1})\) and \((I_1, \ldots, I_{n+t-1})\) are distributed as \((X_1, \ldots, X_t)\) and \((I_1, \ldots, I_t)\), respectively, for any \( n \geq 1 \) and \( t \geq 1 \). Then, given \( n, t \), and a test function \( F \) on \( \mathbb{R}^t \) and exploiting again the independence between \( \{Y_t\}_{t=1}^\infty \) and \( \{I_t\}_{t=1}^\infty \), we obtain

\[
\mathbb{E}[F(X_n, \ldots, X_{n+t-1})] = \mathbb{E}[F(a_{i_1}Y_1, \ldots, a_{i_{n+t-1}}Y_{n+t-1})] \\
= \sum_{i_1=1}^{\infty} \cdots \sum_{i_t=1}^{\infty} \mathbb{E}[F(a_{i_1}Y_1, \ldots, a_{i_t}Y_{n+t-1})] \cdot \mathbb{P}[I_n = i_1, \ldots, I_{n+t-1} = i_t] \\
= \sum_{i_1=1}^{\infty} \cdots \sum_{i_t=1}^{\infty} \mathbb{E}[F(a_{i_1}Y_1, \ldots, a_{i_t}Y_t)] \cdot \mathbb{P}[I_1 = i_1, \ldots, I_t = i_t] \\
= \mathbb{E}[F(X_1, \ldots, X_t)],
\]

(7)

where the third equality is due to the hypothesis of stationarity of both \( \{Y_t\}_{t=1}^\infty \) and \( \{I_t\}_{t=1}^\infty \). The arbitrariness of \( F \) then tells us that \((X_n, \ldots, X_{n+t-1})\) is distributed as \((X_1, \ldots, X_t)\).

The stationarity of \( \{I_t\}_{t=1}^\infty \) was already discussed in the Main Text and is the consequence of the fact that \( \pi \) is the invariant distribution of \( W \). Thus, now we only have to analyze the process \( \{Y_t\}_{t=1}^\infty \). In order to prove the strict stationarity of this process, let us observe that for any \( n > M + 1 \), isolating the first terms in the products of Eq. (6), we get the identity

\[ f_n^Y(y_1, \ldots, y_n) = \frac{\varphi_{M+1}(y_1, \ldots, y_{M+1})}{\varphi_M(y_2, \ldots, y_{M+1})} f_{n-1}^Y(y_2, \ldots, y_n). \]

(8)

Then, the fact that

\[ \int_{\mathbb{R}} dy_1 \varphi_{M+1}(y_1, \ldots, y_{M+1}) = \varphi_M(y_2, \ldots, y_{M+1}) \]

(9)

leads us to the result

\[ \int_{\mathbb{R}} dy_1 f_n^Y(y_1, \ldots, y_n) = f_{n-1}^Y(y_2, \ldots, y_n), \]

(10)
which is also valid for \( n \leq M + 1 \), where \( f_n^y = \varphi_n \), and hence for any \( n \). This relation allows us to prove that

\[
\mathbb{E}[F(Y_{n+1}, \ldots, Y_{n+t})] = \mathbb{E}[F(Y_n, \ldots, Y_{n+t-1})]
\]  

for any \( n \geq 1 \) and any function \( F \) on \( \mathbb{R}^t \). Indeed

\[
\mathbb{E}[F(Y_{n+1}, \ldots, Y_{n+t})] = \int_{\mathbb{R}} dy_1 \cdots \int_{\mathbb{R}} dy_{n+t} F(y_{n+1}, \ldots, y_{n+t}) f_n^Y(y_1, \ldots, y_{n+t})
\]

\[
= \int_{\mathbb{R}} dy_1 \int_{\mathbb{R}} dy_{n+t} F(y_{n+1}, \ldots, y_{n+t}) f_{n+t-1}^Y(y_1, \ldots, y_{n+t})
\]

\[
= \mathbb{E}[F(Y_n, \ldots, Y_{n+t-1})],
\]

where we have made use of Eq. (10) to obtain the second equality and we have just re–labeled the variables to get the third. The iteration of Eq. (11) then provides

\[
\mathbb{E}[F(Y_n, \ldots, Y_{n+t-1})] = \mathbb{E}[F(Y_1, \ldots, Y_t)],
\]

which states the stationarity of the process \( \{Y_t\}_{t=1}^{\infty} \). \( \Box \)

IV. REVERSIBILITY AND SIGN-MAGNITUDE INDEPENDENCE OF THE ENDOGENOUS COMPONENT

In the Main Text we pointed out that the process \( \{Y_t\}_{t=1}^{\infty} \) is reversible, namely that \( (Y_t, Y_{t-1}, Y_1) \) is distributed as \( (Y_1, \ldots, Y_{t-1}, Y_t) \) for any \( t \geq 1 \). Here we provide the proof verifying that

\[
f_t^Y(y_t, y_{t-1}, \ldots, y_1) = f_t^Y(y_1, \ldots, y_{t-1}, y_t)
\]  

(14)

for any \( t \geq 1 \) and \( (y_1, \ldots, y_{t-1}, y_t) \in \mathbb{R}^t \). This identity descends from the invariance of \( \varphi_t \) with respect to permutations of its arguments and is evident if \( t \leq M + 1 \). At the same time, when \( t > M + 1 \), replacing \( (y_1, \ldots, y_{t-1}, y_t) \) with \( (y_t, y_{t-1}, \ldots, y_1) \) in Eq. (10) and rearranging the indexes, we obtain

\[
f_t^Y(y_t, y_{t-1}, \ldots, y_1) = \prod_{n=1}^{t-M} \varphi_{M+1}(y_{t-n+1}, \ldots, y_{t-n-M+1}) \prod_{n=2}^{t-M} \varphi_M(y_{t-n+1}, \ldots, y_{t-n-M+2})
\]

\[
= \prod_{n=1}^{t-M} \varphi_{M+1}(y_{n+M}, \ldots, y_n) \prod_{n=2}^{t-M} \varphi_M(y_{n+M-1}, \ldots, y_n)
\]

(15)

The exchangeability of the arguments of \( \varphi_{M+1} \) and \( \varphi_M \), again, gives Eq. (14). \( \Box \)

In Section VIII of the Main Text, in order to propose possible extensions of the model, we also mentioned that the sign and the magnitude of \( \{Y_t\}_{t=1}^{\infty} \) constitute two independent processes, the former being a sequence of i.i.d. binary variables taking values in \( \mathbb{Z}_2 = \{-1, +1\} \) with equal probabilities. This fact follows from the symmetry of \( f_t^Y(y_1, \ldots, y_t) \) with respect to any of its arguments. Setting \( B_t = \text{sgn}(Y_t) \) with \( \text{sgn}(y) = 1 \) if \( y \geq 0 \) and \( \text{sgn}(y) = -1 \) if \( y < 0 \), to prove the above two statements we have to check that for any \( t \geq 1 \) and any test functions \( F \) on \( \mathbb{Z}_2^t \) and \( G \) on \( \mathbb{R}^t \) the identity

\[
\mathbb{E}[F(B_1, \ldots, B_t) G(|Y_1|, \ldots, |Y_t|)] = \mathbb{E}[F(B_1, \ldots, B_t)] \cdot \mathbb{E}[G(|Y_1|, \ldots, |Y_t|)]
\]  

(16)

and the relation

\[
\mathbb{E}[F(B_1, \ldots, B_t)] = 2^{-t} \sum_{b_1 \in \mathbb{Z}_2} \cdots \sum_{b_t \in \mathbb{Z}_2} F(b_1, \ldots, b_t)
\]  

(17)
hold.

For a general $F$ on $\mathbb{R}^t$ we have the simple equality

$$\int_{\mathbb{R}} dy_1 \cdots \int_{\mathbb{R}} dy_t \, F(y_1, \ldots, y_t) = \int_0^\infty dy_1 \cdots \int_0^\infty dy_t \sum_{b_1 \in \mathbb{Z}_2} \cdots \sum_{b_t \in \mathbb{Z}_2} F(b_1 y_1, \ldots, b_t y_t).$$  \hspace{1cm} (18)

Thus, we find that

$$E[F(B_1, \ldots, B_t) \, G(|Y_1|, \ldots, |Y_t|)] = \int_{\mathbb{R}} dy_1 \cdots \int_{\mathbb{R}} dy_t \, F(\text{sgn}(y_1), \ldots, \text{sgn}(y_t)) \, G(|y_1|, \ldots, |y_t|) \, f_Y^t(y_1, \ldots, y_t)$$

$$= \int_0^\infty dy_1 \cdots \int_0^\infty dy_t \sum_{b_1 \in \mathbb{Z}_2} \cdots \sum_{b_t \in \mathbb{Z}_2} F(b_1, \ldots, b_t) \, G(y_1, \ldots, y_t) \, f_Y^t(y_1, \ldots, y_t)$$

$$= 2^{-t} \sum_{b_1 \in \mathbb{Z}_2} \cdots \sum_{b_t \in \mathbb{Z}_2} F(b_1, \ldots, b_t) \cdot$$

$$\cdot \left. \right. 2^t \int_0^\infty dy_1 \cdots \int_0^\infty dy_t \, G(y_1, \ldots, y_t) \, f_Y^t(y_1, \ldots, y_t)$$

$$= 2^{-t} \sum_{b_1 \in \mathbb{Z}_2} \cdots \sum_{b_t \in \mathbb{Z}_2} F(b_1, \ldots, b_t) \cdot E[G(|Y_1|, \ldots, |Y_t|)],$$  \hspace{1cm} (19)

where both the second and the last equalities are due to Eq. (18) and the symmetry of $f_Y^t$. Plugging here $G(y_1, \ldots, y_t) = 1$ at first, we get Eq. (17). Eq. (16) is then a consequence of Eqs. (17) and (19). \hfill \Box

V. TAIL BEHAVIOR OF $f_Y^t$

In the Main Text we considered the issue about the tail of $f_Y^t$ very briefly. Here we provide a more extended discussion on this point studying the expectation $E[|X_1|^q]$ for $q > 0$. In addition, we need to consider similar expectations also below in this Supplementary Material.

The independence between $a_i$, and $Y_1$ allows us to write $E[|X_1|^q] = E[a_i^q] E[|Y_1|^q]$. Since Eq. (19) of the Main Text for the sequence $\{a_i\}_{i=1}^\infty$ makes $E[a_i^q]$ always finite, we realize that $E[|X_1|^q]$ is finite if and only if $E[|Y_1|^q]$ is finite. On the other hand, $E[|Y_1|^q]$ is finite if and only if $\int_0^\infty \sigma^q \rho(\sigma) d\sigma$ is so. Indeed, from the definitions of $f_Y^t$ and $\varphi_1$ [Eqs. (13) and (15) of the Main Text, respectively] we get

$$E[|Y_1|^q] = \int_{\mathbb{R}} |x|^q N_1(x) \cdot \int_0^\infty \sigma^q \rho(\sigma) d\sigma$$

$$= \frac{2^q \Gamma \left( q + \frac{1}{2} \right)}{\sqrt{\pi}} \int_0^\infty \sigma^q \rho(\sigma) d\sigma ,$$  \hspace{1cm} (20)

where $\Gamma$ is the Euler’s Gamma function. Thus we discover that $E[|X_1|^q]$ and $E[|Y_1|^q]$, and the tails of $f_Y^t$ as a consequence, are only ruled by the last factor on the r.h.s. of Eq. (20), i.e. by $\rho$. In particular, if the function $\rho$ decays according to a power law as $\sigma^{-\alpha-1}$ for large $\sigma$, then $f_Y^t$ inherits the same feature and displays fat tails with the same tail index $\alpha$.

In the Main Text we also noticed that an effective fat tail scenario can be obtained also by considering suitable small values of the restart probability $\nu$ if $D < 1/2$. In order to shed light on this issue, we need to consider the limit of $f_Y^t$ when $\nu$ goes to zero. Nevertheless, such a limit is meaningless if we do not rescale the function $\rho$ properly with $\nu$ since, if $\nu$ is kept fixed in the limit procedure, then $f_Y^t$ concentrates around zero. The reason is that, when restarts get very rare, the random time $\{I_i\}_{i=1}^\infty$ tends to never go back to 1, reaching very large values in equilibrium conditions. As a consequence, when $D < 1/2$ the rescaling factor $a_i$ tends to vanish and thus the mixture giving $f_Y^t$ becomes dominated by component distributions having a vanishing variance. If we want then to keep the variance of the random variable $X_1$ independent of $\nu$, we must fix a function $\rho'$ which does not depend on $\nu$ and define $\rho$ according to

$$\rho(\sigma) \equiv \sqrt{E[a_i^2]} \rho' \left( \sqrt{E[a_i^2]} \sigma \right).$$  \hspace{1cm} (21)
Indeed, Eq. (20) now provides

\[
\mathbb{E}[|X_1|^q] = \mathbb{E}[a^q(I_1)]\mathbb{E}[|Y_1|^q] = \frac{2^q}{\sqrt{\pi}} \frac{1}{\int_0^\infty \sigma^q \rho'(\sigma) d\sigma},
\]
which in particular entails

\[
\mathbb{E}[X_1^2] = \int_0^\infty \sigma^2 \rho'(\sigma) d\sigma.
\]

In this framework the fluctuations of \( X_1 \) do not shrink in the limit \( \nu \to 0 \) and we obtain

\[
\lim_{\nu \to 0^+} \mathbb{E}[|X_1|^q] = \frac{2^q}{\sqrt{\pi}} \frac{1}{\int_0^\infty \sigma^q \rho'(\sigma) d\sigma}
\]

if \((1/2-D)q < 1\) and \(\lim_{\nu \to 0^+} \mathbb{E}[|X_1|^q] = \infty\) otherwise. We thus get the proof that fat tails with index \(2/(1-2D)\) appear in this limit situation. It is clear that a function \(\rho'\) which endows the model with tails characterized by a tail index \(\alpha < 2/(1-2D)\) hides this effect.

We conclude sketching the computation of this limit. To this purpose, it is convenient to introduce the notation of asymptotic equivalence: given two generic functions \( F \) and \( G \) of \( \nu \), we shall write \( F \sim G \) to say that \( \lim_{\nu \to 0^+} \frac{F(\nu)}{G(\nu)} = 1 \). Then, if \( \{\psi_i\}_{i=1}^\infty \) is a sequence for which there exists \( \gamma > 0 \) and \( l > 0 \) such that \( \lim_{i \to \infty} i^\gamma \psi_i = l \), we have that as \( \nu \) goes to zero

\[
\mathbb{E}[\psi_{i\nu}] = \sum_{i=1}^\infty \psi_i \nu(1-\nu)^{i-1} \sim \begin{cases} 
\frac{l \Gamma(1-\gamma) \nu^\gamma}{\Gamma(1-\gamma)} & \text{if } 0 < \gamma < 1; \\
\nu \ln \nu & \text{if } \gamma = 1; \\
\nu \sum_{i=1}^\infty \psi_i & \text{if } \gamma > 1.
\end{cases}
\]

The last series is convergent. The instance \( \gamma \leq 1 \) is a consequence of the Karamata’s theorem \(1\), whereas the case \( \gamma > 1 \) is due to the Abel’s theorem \(2\). The limit value of Eq. (22) follows then by noticing that \( a_i = \sqrt{i^{2D} - (i-1)^{2D}} \) implies \( \lim_{i \to \infty} i^{1/2-D} a_i = \sqrt{2D} \).

**VI. SCALING FEATURES**

We reconsider here the scaling features of our model at short time scales, providing a deeper insight into the properties outlined in Section IV B of the Main Text.

To begin with, we derive the distribution of the aggregated return \( X_1 + \cdots + X_t \) when \( t \leq M + 1 \). To this aim, let us observe that from Eqs. (1) and (2), thanks to the stability of Gaussian distributions with respect to linear combinations of independent Gaussian variables, we attain

\[
\mathbb{E}[F(X_1 + \cdots + X_t)] = \int_{\mathbb{R}} dx F(x) \mathbb{E} \left[ \int_0^\infty d\sigma \rho(\sigma) N_{\sqrt{a_i^2 + \cdots + a_i^2}}(x) \right]
\]

for any test function \( F \) on \( \mathbb{R} \). This identity clearly shows that, if \( t \leq M + 1 \), the PDF of \( X_1 + \cdots + X_t \) is given as a function of \( x \) by the expression

\[
\mathbb{E} \left[ \int_0^\infty d\sigma \rho(\sigma) N_{\sqrt{a_i^2 + \cdots + a_i^2}}(x) \right].
\]

We notice that this PDF cannot be obtained by simply rescaling \( \mathbb{E}^{X_1} \), except if \( \nu = 1 \) or \( a_i = 1 \) for any \( i \) in which case the normal scaling behavior with exponent 1/2 is recovered. Thus, in general the model accounts for a richer scenario than a perfect time-scale-invariance framework, as we know.

Choosing in Eq. (26) \( F(x) = |x|^q \), with \( q \geq 0 \) such that \( \int_0^\infty \sigma^q \rho(\sigma) d\sigma < \infty \), we get

\[
\mathbb{E}[|X_1 + \cdots + X_t|^q] = \frac{2^q}{\sqrt{\pi}} \frac{1}{\int_0^\infty \sigma^q \rho(\sigma) d\sigma} \mathbb{E}\left[ (a_i^2 + \cdots + a_i^2)^{\frac{q}{2}} \right].
\]

This result allows us to prove Eq. (27) of the Main Text:

\[
m_q^X(t) = \frac{\mathbb{E}[|X_1 + \cdots + X_t|^q]}{\mathbb{E}[|X_1|^q]} = \frac{\mathbb{E}\left[ (a_i^2 + \cdots + a_i^2)^{\frac{q}{2}} \right]}{\mathbb{E}[a_i^2]}.
\]
FIG. 1: Relative deviation of \( m^X_q \) from \( t^{qH_q} \) vs. \( q \).

Notice in passing that the r.h.s. of Eq. (29) is well defined for any real \( q \), even if \( \mathbb{E} |X_1|^q \) diverges.

As we reported in the Main Text, \( m^X_q \) is well approximated for not too small values of \( D \) by the power \( t^{qH_q} \) with a generalized Hurst-like exponent \( H_q \) independent of \( t \), thus allowing the model to exhibit pretty well-defined scaling properties at relatively short time scales. To corroborate this assertion, here we report a study of \( m^X_q \) based on numerical simulations for \( M = 30 \), \( 0.1 \leq D \leq 1/2 \), and \( t \leq M + 1 \). We evaluate the exponent \( H_q \) with the least square method as follow:

\[
qH_q = \arg\min_{H \in \mathbb{R}} \left\{ \frac{1}{M} \sum_{t=2}^{M+1} \left[ \frac{\ln m^X_q(t)}{\ln t} - qH \right]^2 \right\} = \frac{1}{M} \sum_{t=2}^{M+1} \frac{\ln m^X_q(t)}{\ln t}.
\]  

(30)

Then, we measure the distance of \( m^X_q \) from \( t^{qH_q} \) with the relative mean fluctuation

\[
\epsilon_q \equiv \max \left\{ \frac{1}{qH_q} \left[ \frac{1}{M} \sum_{t=2}^{M+1} \left[ \frac{\ln m^X_q(t)}{\ln t} - qH \right] \right]^2 : \nu \in [0,1] \text{ and } D \in [0,1/2] \right\}.
\]  

(31)

Even if not explicitly indicated, it is clear that \( m^X_q \) and \( H_q \) depend on \( \nu \) and \( D \). Fig. 1 shows \( \epsilon_q \) vs. \( q \), for \( q \) in between 0 and 10. The fact that \( \epsilon_2 = 0 \) is not surprising since \( m^X_2(t) = t \) as one can immediately verify recalling the stationarity of \( \{I_i\}_{i \geq 1} \). For \( q \neq 2 \), we find values of \( \epsilon_q \) of few points per cent. This confirms that \( m^X_q \) is close to \( t^{qH_q} \) and motivates the analysis of the exponent \( H_q \). In Fig. 2 we report an explicit comparison between \( m^X_q \) and \( t^{qH_q} \) for \( q = 0.5 \), 1, 3, 4, \( \nu = 0.01 \), and \( D = 0.25 \). The corresponding \( H_q \) vs. \( q \) plot is shown in Fig. 1 of the Main Text.

Before we proceed to further investigate \( H_q \), a remark is in order about small values of \( D \). When \( D \) goes to 0, \( a_i \) vanishes if \( i > 1 \), thus approaching \( \delta_{i1} \). On the other hand, \( \delta_{i1}, \ldots, \delta_{i1} \) are independent identically distributed Bernoulli variables, since in our model there is no correlations between restarts. Then, one can easily verify that in such a limit \( m^X_q(t) \) takes the simple form

\[
\sum_{n=1}^{t} n^{1/2} \binom{t}{n} \nu^{n-1}(1-\nu)^{t-n}.
\]  

(32)

This function is poorly approximated by a power of the time when \( q \) is small and \( \nu \) assumes intermediate values: for instance, with \( q = 0 \) and \( \nu = 1/2 \), it reduces to \( 2(1-2^{-t}) \). This is the reason that leads us to exclude small value of
FIG. 2: Comparison between $m_q^X$ (dots) and $t^{H_q}$ (dashed lines) for $t \leq 31$, $\nu = 0.01$, $D = 0.25$ and some values of $q$. 

$D$ and to focus on $D \geq 0.1$ in the scaling analysis. It is also worth noticing that $\nu$ is typically much smaller than $1/2$ in order to reproduce empirical financial data.

For the scaling exponent $H_q$ defined by Eq. (30), we have $H_q \geq 1/2$ if $q \leq 2$ and $H_q \leq 1/2$ if $q > 2$. Indeed, the function 

$$F(x_1, \ldots, x_t) = \left( x_1^\gamma + \cdots + x_t^\gamma \right)^\alpha$$

is convex when $\alpha \leq 1$ and concave if $\alpha > 1$. Thus, setting $\alpha = q/2$, the Jensen’s inequality and the stationarity of the process $\{I_t\}_{t=1}^\infty$ tell us that if $q \leq 2$ then

$$m_q^X(t) = \frac{\mathbb{E}[F(a_{I_1}^q, \ldots, a_{I_t}^q)]}{\mathbb{E}[a_{I_t}^q]} \geq \frac{F(\mathbb{E}[a_{I_1}^q], \ldots, \mathbb{E}[a_{I_t}^q])}{\mathbb{E}[a_{I_t}^q]} = t^{\frac{q}{2}},$$

while for $q > 2$

$$m_q^X(t) = \frac{\mathbb{E}[F(a_{I_1}^q, \ldots, a_{I_t}^q)]}{\mathbb{E}[a_{I_t}^q]} \leq \frac{F(\mathbb{E}[a_{I_1}^q], \ldots, \mathbb{E}[a_{I_t}^q])}{\mathbb{E}[a_{I_t}^q]} = t^{\frac{q}{2}}.$$  

The bounds on $H_q$ then follow by its definition, Eq. (30). Fig. 3 shows the level curves of $qH_q$ vs. $\nu$ and $D$ in the range $[0, 1]$ and $[0.1, 1/2]$, respectively, and for different values of $q$. The exponent $H_q$ displays large variations for small values of $\nu$ and $D$ when $q > 2$ whereas it is close to $1/2$ in the other cases. Moreover, given $\nu$ we notice that $H_q$ is a decreasing function of $D$ if $q < 2$ and an increasing function if $q > 2$. Disentangling the contribution of $\nu$ and $D$ to the scaling exponent is not easy. However, from the contour lines in Fig. 3 it is possible to appreciate how the variations of one of the two parameters can be compensated by modifications of the other.

We conclude the present analysis by studying analytically the limit of $m_q^X$ when $D \leq 1/2$ and $\nu$ approaches zero. We shall make use of the symbol $\sim$ of asymptotic equivalence introduced in the previous Section and the results in Eq. (25). The computation starts by isolating the trajectories corresponding to at most one restart from the others.
FIG. 3: Scaling exponent $q H_q$ as a function of $\nu$ and $D$ for several $q$. Contour lines are also shown.

in the numerator of $m^X_0$. Then, remembering $a^2_i = i^{2D} - (i - 1)^{2D}$ and assuming $t \geq 2$, we get the first equivalence

$$
\mathbb{E}
\left[
\left(a^2_{I_1} + \cdots + a^2_{I_t}\right)^\frac{q}{2}
\right]
\sim
\sum_{i=1}^\infty
\left[
(i + t - 1)^{2D} - (i - 1)^{2D}
\right]^\frac{q}{2}
\nu(1 - \nu)^{i+t-2}
\left(i + t - 1\right)^{Dq}
\left(i + t - 3\right)^{Dq}.
$$

Noticing now that

$$
\lim_{i \to \infty}
\left[
(i + \tau - 1)^{2D} - (i - 1)^{2D} + (t - \tau + 1)^{2D}
\right]^\frac{q}{2}
= (t - \tau + 1)^{Dq}.
$$

(36)

(37)
and that \((t - \tau + 1)^{Dq}\) does not vanish for \(\tau = 2, \ldots, t\), we see that the second term in Eq. (25) is \(\sim \nu \sum_{\tau=1}^{t-1} \tau^{Dq}\). Thus, Eq. (25) can be recast as

\[
\mathbb{E}\left[(a_{n_1}^\nu + \cdots + a_{n_1}^\nu)^{\frac{q}{2}} \right] \sim \sum_{i=1}^{\infty} [((i + t)^{2D} - i^{2D})^\frac{q}{2} \nu(1 - \nu)^{i+1} + \nu \sum_{\tau=1}^{t} \tau^{Dq}],
\]

which is valid also for \(t = 1\). To obtain this relation we shifted the index \(i\) in the first sum of Eq. (25) and then we moved the first addend to the second term. Finally, since \(\lim_{t \to \infty} i^{(1/2-D)q} [((i + t)^{2D} - i^{2D})^\frac{q}{2} = (2Dt)^\frac{q}{2}\), Eq. (25) allows us to obtain

\[
m_q^X(t) \sim \begin{cases} 
\sum_{i=1}^{\infty} [((i + t)^{2D} - i^{2D})^\frac{q}{2} + \sum_{\tau=1}^{t} \tau^{Dq} & \text{if } (1/2 - D)q \leq 1; \\
\sum_{i=1}^{\infty} [((i + 1)^{2D} - (i+1)^{2D})^\frac{q}{2} + 1 & \text{if } (1/2 - D)q > 1.
\end{cases}
\]

This asymptotic equivalence proves that \(H_q = 1/2\) for \(q \leq 2/(1 - 2D)\) at small values of \(\nu\), as anticipated in the Main Text. In order to intuitively understand the content of such a result, we notice that when the restart probability \(\nu\) goes to zero the random time \(\{I_t\}_{t=1}^{\infty}\) tends to flow without stopping but its starting value becomes affected by very large fluctuations due to its stationarity. At small values of \(q\), the sequence \(\{a_i^\nu\}_{i=1}^{\infty}\) does not decay fast enough to keep under control such fluctuations and only its tail plays a role, providing a normal scaling exponent.

VII. THE “NULL” AND THE “COMPLETE” MODEL

In the Main Text we considered two particular choices for the function \(\rho\): one corresponded to fix \(\sigma\) to a particular value and, being the simplest possible choice, we referred to the “null model” in that case; and the other, given by Eq. (20) of the Main Text, was obtained distributing \(\sigma^2\) according to an inverse-gamma distribution. The model associated to the latter choice for \(\rho\) was named the complete model. Here we give some details about the null model and prove that the endogenous component of the complete model is an ARCH process, as stated in the Main Text.

A. The null model

When \(\sigma\) is fixed to a particular value \(\sigma_0\), the PDF \(\phi_t\) of Eq. (5) factors and, consequently, from Eq. (6) we have that the joint PDF \(f_t^Y\) also factors as

\[
f_t^Y(y_1, \ldots, y_t) = \prod_{n=1}^{t} \mathcal{N}_{\sigma_0}(y_n).
\]

Within this setting, the endogenous component \(\{Y_t\}_{t=1}^{\infty}\) then reduces to a sequence of independent normal variables, which is the simplest possible endogenous process that our model can produce, and the observed compound process becomes a random time change of the Brownian motion. Thus, we are recovering a discrete-time model with random time and constant average volatility. Indeed, without demanding mathematical rigor, if \(\{W_t\}_{t \geq 0}\) is a standard Brownian motion independent of \(\{I_t\}_{t=1}^{\infty}\), then \(X_1 + \cdots + X_t\) is distributed as \(W_{\sigma_0^2[a_{I_1}^2 + \cdots + a_{I_t}^2]}\) for any \(t\) when \(\{Y_t\}_{t=1}^{\infty}\) is a sequence of i.i.d. normal variables with mean zero and variance \(\sigma_0^2\).

As we discussed in the Main Text and in Section VI if \(D < 1/2\) we can have effective fat tails with tail index \(2/(1 - 2D)\) in the distribution of \(X_1\) by considering a small enough value of the restart probability \(\nu\). This is the only possibility to obtain such tails within the present instance of the model. The rescaling of \(\rho\) we considered in Section VI simply consists in taking \(\sigma_0 = \sigma/\sqrt{\mathbb{E}[a_{I_1}^2]}\) here, with \(\sigma\) a parameter independent of \(\nu\).
B. The complete model

The peculiar $\rho$ given in the Main Text by Eq. (20) allows us to explicitly integrate over $\sigma$ in the expression of $\varphi_t$, which reduces to a multivariate Student distribution:

$$\varphi_t(y_1, y_2, \ldots, y_t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{(\pi^n n^{1/2})} \left[1 + \frac{y_1^2 + y_2^2 + \cdots + y_t^2}{\beta^2}\right]^{-\frac{n+1}{2}}. \quad (41)$$

Within this setting, reformulating the endogenous component $\{Y_t\}_{t=1}^\infty$ in terms of stochastic variables, rather than only stating its PDF’s, is interesting and useful. We write such process as

$$Y_t = \begin{cases} \beta \cdot Z_t & \text{if } t = 1; \\ \sqrt{\beta^2 + Y_{2 \max\{1, t-M\}}^2 + \cdots + Y_{t-1}^2} \cdot Z_t & \text{if } t > 1, \end{cases} \quad (42)$$

with a residual sequence $\{Z_t\}_{t=1}^\infty$ obviously defined as

$$Z_t = \begin{cases} Y_t/\beta & \text{if } t = 1; \\ Y_t/\sqrt{\beta^2 + Y_{2 \max\{1, t-M\}}^2 + \cdots + Y_{t-1}^2} & \text{if } t > 1. \end{cases} \quad (43)$$

Here we show that $\{Z_t\}_{t=1}^\infty$ is a sequence of Student’s t-distributed independent variables when $\varphi_t$ is given by Eq. (41). According to the definition of the ARCH process, this fact makes $\{Z_t\}_{t=1}^\infty$ a pure ARCH process with Student’s t-distributed return residuals, as anticipated in the Main Text. To be more precise, we can prove that

$$f_t^Z(z_1, \ldots, z_t) = \prod_{n=1}^t \frac{\Gamma\left(\frac{n+1}{2}\right)}{(\pi^{n/2} n^{1/2})} (1 + z_n^2)^{-\frac{n+1}{2}}, \quad (44)$$

with $\alpha_n = \alpha + \min\{n-1, M\}$. Notice that $Z_t$’s are identically distributed for $t \geq M + 1$, but the stationarity of $\{Y_t\}_{t=1}^\infty$ and the boundary effects at $t = 1$ prevent them to be identically distributed for any $t$. It is also worth mentioning that simulating the process $\{Y_t\}_{t=1}^\infty$ becomes rather simple thanks to the algorithm reported in Ref. [2], which adapts the Box-Muller transform for normally distributed variables to Student’s t-distributed variables.

In order to prove Eq. (41) we study the expectation value $\mathbb{E}[F(Z_1, Z_2, \ldots, Z_t)]$, being $F$ a test function on $\mathbb{R}^t$. By definition, Eq. (45), we have

$$\mathbb{E}[F(Z_1, Z_2, \ldots, Z_t)] = \mathbb{E} \left[ F \left( \frac{Y_1}{\beta}, \frac{Y_2}{\sqrt{\beta^2 + Y_1^2}}, \ldots, \frac{Y_t}{\sqrt{\beta^2 + Y_{2 \max\{1, t-M\}}^2 + \cdots + Y_{t-1}^2}} \right) \right]$$

$$= \int_\mathbb{R} dy_1 \int_\mathbb{R} dy_2 \cdots \int_\mathbb{R} dy_t F \left( \frac{y_1}{\beta_1}, \frac{y_2}{\sqrt{\beta_2^2 + y_1^2}}, \ldots, \frac{y_t}{\sqrt{\beta_t^2 + y_{2 \max\{1, t-M\}}^2 + \cdots + y_{t-1}^2}} \right) \cdot \beta f_t^Y(y_1, y_2, \ldots, y_t). \quad (45)$$

We then perform a change of variables from the old $y_n$’s into the new

$$z_n = \begin{cases} y_1/\beta & \text{if } n = 1; \\ y_n/\sqrt{\beta^2 + y_{2 \max\{1, n-M\}}^2 + \cdots + y_{n-1}^2} & \text{if } 1 < n \leq t. \end{cases} \quad (46)$$

This relation can be inverted to express the $y_n$’s as a function of the $z_n$’s. Since clearly $y_n$ only depends on $z_1, z_2, \ldots, z_n$, the Jacobian matrix of the transformation is triangular and thus its determinant is easily found as

$$\beta \prod_{n=2}^t \sqrt{\beta^2 + y_{2 \max\{1, n-M\}}^2 + \cdots + y_{n-1}^2}. \quad (47)$$

Here and below the $y_n$’s must be thought as functions of the $z_n$’s. Such a change of variables leads us to the identity

$$\mathbb{E}[F(Z_1, Z_2, \ldots, Z_t)] = \int_\mathbb{R} dz_1 \int_\mathbb{R} dz_2 \cdots \int_\mathbb{R} dz_t F(z_1, z_2, \ldots, z_t) \cdot \beta \prod_{n=2}^t \sqrt{\beta^2 + y_{2 \max\{1, n-M\}}^2 + \cdots + y_{n-1}^2} \cdot \beta f_t^Y(z_1, z_2, \ldots, z_t). \quad (48)$$
Now, although lengthy, using Eqs. (6) and (11) at first and Eq. (16) at last, it is straightforward to see that

\[
\beta \prod_{n=2}^{t} \sqrt{\beta^2 + y_{\text{max}}(1,n-M) + \cdots + y_{n-1}} f^Y_i (y_1, y_2, \ldots, y_t) \\
= \frac{\Gamma(\frac{\alpha+n}{2})}{\sqrt{\pi} \Gamma(\frac{\alpha}{2})} \left[ 1 + \frac{y_n^2}{\beta^2} \right] \prod_{n=2}^{t} \frac{\Gamma(\frac{\alpha+n}{2})}{\sqrt{\pi} \Gamma(\frac{\alpha}{2})} \left[ 1 + \beta^2 + y_{\text{max}}(1,n-M) + \cdots + y_{n-1}^2 \right]^{-\frac{\alpha+n}{2}},
\]

where \(\alpha_n = \alpha + \min\{n-1, M\}\). Hence

\[
\mathbb{E}[F(Z_1, Z_2, \ldots, Z_t)] = \int_{\mathbb{R}} dz_1 \int_{\mathbb{R}} dz_2 \cdots \int_{\mathbb{R}} dz_t F(z_1, z_2, \ldots, z_t) \prod_{n=1}^{t} \frac{\Gamma(\frac{\alpha+n}{2})}{\sqrt{\pi} \Gamma(\frac{\alpha}{2})} (1 + z_n^2)^{-\frac{\alpha+n}{2}} \tag{50}
\]

and this eventually confirms that the process \(\{Z_t\}_{t=1}^{\infty}\) is distributed according to Eq. (49). \(\square\)

VIII. AUTOCORRELATION STRUCTURE

Here we discuss in some detail the autocorrelation \(r^X\) of the process \(\{X_i^q\}_{i=1}^{\infty}\) introduced in the Main Text. To begin with, we notice that the independence between \(\{Y_i\}_{i=1}^{\infty}\) and \(\{I_t\}_{t=1}^{\infty}\) allows us to write

\[
r^X(t) = \frac{\mathbb{E}[X_1^q X_1^q] - \mathbb{E}[X_1^q]^2}{\mathbb{E}[X_1^q]^2} = \frac{\mathbb{E}[a_t^q a_t^q] \mathbb{E}[Y_1^q Y_1^q] - \mathbb{E}[a_t^q]^2 \mathbb{E}[Y_1^q]^2}{\mathbb{E}[a_t^q]^2 \mathbb{E}[Y_1^q]^2 - \mathbb{E}[a_t^q]^2 \mathbb{E}[Y_1^q]^2}. \tag{51}
\]

Although expectations involving \(\{a_t\}_{t=1}^{\infty}\) are finite for any \(q\), we restrict on values of \(q\) such that \(\int_0^{\infty} \sigma^2 q \rho(\sigma) d\sigma < \infty\) in order to ensure that also those involving \(\{Y_i\}_{i=1}^{\infty}\) are finite. The analysis of \(r^X\) we propose is based on the preliminary study of the autocorrelations \(r^q\) and \(r_q\) of the processes \(\{a_t\}_{t=1}^{\infty}\) and \(\{Y_i\}_{i=1}^{\infty}\), respectively, which is the subject of the next two paragraphs. Eventually, we bring together the results to go back over \(r^X\). We deal first with \(r^q\).

A. The autocorrelation \(r^q\)

The autocorrelation \(r^q\) can be conveniently manipulated once one knows the probability of \(I_t = j\), given that \(I_1 = i\). When \(j < t\) the event \(I_t = j\) occurs only as a consequence of a restart at the time \(t - j + 1\) and no restarts during the following \(j - 1\) steps, regardless of the value of \(I_1\). Thus, \(\mathbb{P}[I_t = j | I_1 = i] = \nu (1 - \nu)^{j-1}\) if \(j < t\). On the contrary, when \(j \geq t\), the event \(I_t = j\) is only possible if no restart occurs during the whole temporal interval up to time \(t\), since a restart in between 1 and \(t\) would provide a value of \(I_t\) smaller than \(t\). In such circumstances \(I_t = I_1 + t - 1\) and then \(\mathbb{P}[I_t = j | I_1 = i] = (1 - \nu)^{t-1} \delta_j \delta_{i+t-1}\) if \(j \geq t\). Thus,

\[
\mathbb{P}[I_t = j | I_1 = i] = \begin{cases} 
\nu (1 - \nu)^{j-1} & \text{if } j < t; \\
(1 - \nu)^{t-1} \delta_j \delta_{i+t-1} & \text{if } j \geq t 
\end{cases} 
\]

Thus,

\[
\mathbb{P}[I_t = j | I_1 = i] = \mathbb{P}[I_1 = j] + \begin{cases} 
0 & \text{if } j < t; \\
(1 - \nu)^{t-1} \delta_j \delta_{i+t-1} - \nu (1 - \nu)^{j-1} & \text{if } j \geq t. 
\end{cases} \tag{52}
\]

Coming back to \(r^q\) and fixing \(t \geq 2\), we notice that

\[
\mathbb{E}[a_t^q a_t^q] = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_t^q a_t^j \left( \mathbb{P}[I_t = j, I_1 = i] - \mathbb{P}[I_1 = j] \cdot \mathbb{P}[I_1 = i] \right) \\
= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_t^q a_t^j \left( \mathbb{P}[I_t = j | I_1 = i] - \mathbb{P}[I_1 = j] \right) \mathbb{P}[I_1 = i] \\
= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_t^q a_t^j \left( (1 - \nu)^{t-1} \delta_j \delta_{i+t-1} - \nu (1 - \nu)^{j-1} \right) \mathbb{P}[I_1 = i], \tag{53}
\]
where the result stated in Eq. (52) has been used to get the last equality. Then,

\[
\mathbb{E}[a^q_{I_t}a^q_{I_t}] - \mathbb{E}[a^q_{I_t}]^2 = (1 - \nu)^{t-1} \sum_{i=1}^{\infty} a^q_i a^q_{i+t-1} \mathbb{P}[I_t = i] - \mathbb{E}[a^q_{I_t}] \sum_{j=t}^{\infty} a^q_j \nu (1 - \nu)^{j-1}
\]

\[
= (1 - \nu)^{t-1} \mathbb{E}[a^q_{I_t}a^q_{I_{t+1}}] - \mathbb{E}[a^q_{I_t}] \sum_{i=1}^{\infty} a^q_i (1 - \nu)^{i+t-2}
\]

\[
= (1 - \nu)^{t-1} \left( \mathbb{E}[a^q_{I_t}a^q_{I_{t+1}}] - \mathbb{E}[a^q_{I_t}] \mathbb{E}[a^q_{I_{t+1}}] \right),
\]

the second equality being obtained through the substitution \( j = i + t - 1 \) in the second series. In summary, for the autocorrelation \( r^{\alpha_i}_{q} \) we find the more manageable expression:

\[
r^{\alpha_i}_{q}(t) = \frac{\mathbb{E}[a^q_{I_t}a^q_{I_{t+1}}] - \mathbb{E}[a^q_{I_t}]^2}{\mathbb{E}[a^q_{I_t}a^q_{I_{t+1}}] - \mathbb{E}[a^q_{I_t}]^2} \]

\[
= (1 - \nu)^{t-1} \frac{\mathbb{E}[a^q_{I_t}a^q_{I_{t+1}}] - \mathbb{E}[a^q_{I_t}] \mathbb{E}[a^q_{I_{t+1}}]}{\mathbb{E}[a^q_{I_t}a^q_{I_{t+1}}] - \mathbb{E}[a^q_{I_t}]^2},
\]

which is valid also for \( t = 1 \) and allows us to investigate the correlation decay.

We point out that \( r^{\alpha_i}_{q} \) decays approximately according to a power law as \( t \) increases at short time scales if \( D < 1/2 \), whereas in the long time limit an exponential relaxation with rate \(-\ln(1 - \nu)\) dominates:

\[
\lim_{t \to \infty} \frac{1}{t} \ln r^{\alpha_i}_{q}(t) = \ln(1 - \nu).
\]

The former becomes the main trend at short time scales and small \( \nu \). For instance, Fig. 4 reports \( r^{\alpha_i}_{q} \) vs. \( t \) for \( 2 \leq t \leq 31, \nu = 0.01 \) and \( D = 0.25 \) and for four different values of \( q \). Moreover, for \( q \leq 1/(1 - 2D) \), the smaller is \( \nu \), the more the correlations get persistent, as we already mentioned. We can understand this fact by looking at the limit behavior of \( r^{\alpha_i}_{q} \) when \( \nu \) approaches zero. Making use of the symbol of asymptotic equivalence introduced in
We find
\[ r^q_t(t) \sim \begin{cases} 1 & \text{if } (1 - 2D)q \leq 1; \\ \frac{\sum_{i=1}^{\infty} q^i a^q_{i+t-1}}{\sum_{i=1}^{\infty} q^i a^q_i} & \text{if } (1 - 2D)q > 1. \end{cases} \] (57)

This result is an immediate consequence of Eqs. 53 and 25. As for the limit scaling behavior, the large fluctuations of \( I_1 \) affect \( a^q_{I_1} \) when \( q \) is small enough. In addition, they also propagate to \( a^q_{I_1+t-1} \) keeping intact the correlations.

### B. The autocorrelation \( r^2_q \)

The autocorrelation \( r^2_q \) of the process \( \{Y^q_t\}_{t=1}^{\infty} \) is much more difficult to investigate than the previous and in general we must resort to numerical simulations. Nevertheless it has a trivial structure for \( t \leq M + 1 \). Indeed, due to the exchangeability of \( f^q_t = \varphi_t \) if \( t \leq M + 1 \), we have that \( E[|Y^q_t| |Y^q_t|] = E[|Y^q_t| |Y^q_2|] \) when \( 2 \leq t \leq M + 1 \) and thus \( r^2_q \) is independent of the time in such a temporal interval:
\[ r^2_q(t) = \frac{E[|Y^q_t| |Y^q_2|] - E[|Y^q_t|]^2}{E[|Y^q_2|] - E[|Y^q_t|]^2}. \] (58)

To the purpose of analyzing \( r^2_q(t) \) also for \( t > M + 1 \), we need to refer here to a favorable instance, corresponding to the function \( \rho \) given by Eq. (20) of the Main Text with \( \alpha > 4 \) and \( q = 2 \). As we know, such a \( \rho \) makes \( \{Y^q_t\}_{t=1}^{\infty} \) an ARCH process for which the condition \( \alpha > 4 \) guarantees the existence of \( E[|Y^q_t|^2] \). We can then take advantage of the fact that, for ARCH processes, the expectation \( E[F(Y^q_t)Y^2_t] \) can be recursively computed for any \( t \) and any test function \( F \) for which it makes sense. Indeed, recalling Eq. 12, for \( t \geq 2 \) we can write
\[ Y^2_t = (\beta^2 + \sum_{n=1}^{\min(t-1,M)} Y^2_{t-n}) Z^2_t. \] (59)

Thus, noticing that \( Y_n \) is independent of \( Z_t \) if \( n < t \) and bearing in mind that \( Y_1 = \beta Z_1 \), we have the simple chain of equalities
\[
\begin{align*}
E[F(Y^q_1)Y^2_t] - E[F(Y^q_1)] \cdot E[Y^2_t] &= E \left[ F(Y^q_1) \left( \beta^2 + \sum_{n=1}^{\min(t-1,M)} Y^2_{t-n} \right) Z^2_t \right] - E[F(Y^q_1)] \cdot E[Y^2_t] \\
&= E \left[ F(Y^q_1) \left( \beta^2 + \sum_{n=1}^{\min(t-1,M)} Y^2_{t-n} \right) \right] \cdot E[Z^2_t] - E[F(Y^q_1)] \cdot E[Y^2_t] \\
&= E[F(Y^q_1)] \left( \beta^2 E[Z^2_t] - E[Y^2_t] \right) + E[Z^2_t] \cdot \sum_{n=1}^{\min(t-1,M)} E[F(Y^q_1)Y^2_{t-n}] \\
&= \beta^2 E[F(Y^q_1)] \left( E[Z^2_t] - E[Y^2_t] + \min(t - 1, M) E[Z^2_t] E[Y^2_t] \right) + \\
&\quad + E[Z^2_t] \cdot \sum_{n=1}^{\min(t-1,M)} \left( E[F(Y^q_1)Y^2_{t-n}] - E[F(Y^q_1)] \cdot E[Y^2_t] \right) \\
&= \beta^2 E[F(Y^q_1)] \left( E[Z^2_t] - E[Y^2_t] + \min(t - 1, M) E[Z^2_t] E[Z^2_t] \right) + \\
&\quad + E[Z^2_t] \cdot \sum_{n=1}^{\min(t-1,M)} \left( E[F(Y^q_1)Y^2_{t-n}] - E[F(Y^q_1)] \cdot E[Y^2_t] \right). \tag{60}
\end{align*}
\]

On the other hand, for the variables \( Z_t \)’s distributed according to Eq. 14, we have
\[ E[|Z^q_t|] = \frac{\Gamma \left( \frac{q+1}{2} \right) \Gamma \left( \frac{\alpha-4}{2} \right)}{\sqrt{\pi} \Gamma \left( \frac{\alpha}{2} \right)}, \tag{61} \]
FIG. 5: Rate $\lambda$ as a function of $\alpha$ for different values of $M$.

with $\alpha_t = \alpha + \min\{t - 1, M\}$. Thus, to verify that

$$\mathbb{E}[Z_1^2] - \mathbb{E}[Z_t^2] + \min\{t - 1, M\}\mathbb{E}[Z_1^2]\mathbb{E}[Z_t^2] = 0$$

is not difficult once one sets $q = 2$ in Eq. (61). Combining Eq. (60) with Eq. (62), we eventually obtain the result

$$\mathbb{E}[F(Y_1)Y_2^2] - \mathbb{E}[F(Y_1)]\cdot \mathbb{E}[Y_2^2] = \frac{1}{\alpha_t - 2} \sum_{n=1}^{\min\{t - 1, M\}} \left( \mathbb{E}[F(Y_1)Y_{2,n}^2] - \mathbb{E}[F(Y_1)]\cdot \mathbb{E}[Y_{2,n}^2] \right),$$

which establishes a recursive scheme to compute $\mathbb{E}[F(Y_1)Y_2^2] - \mathbb{E}[F(Y_1)]\cdot \mathbb{E}[Y_2^2]$.

Assuming $\alpha > 4$, setting $F(y) = y^2$ in Eq. (63), and dividing by $\mathbb{E}[|Y_1|^4] - \mathbb{E}[|Y_1|^2]^2$, we get a simple tool to evaluate $r_2^Y(t)$ for any $t$ and to study its asymptotic behavior. In particular, as expected we find that if $2 \leq t \leq M + 1$

$$r_2^Y(t) = \frac{1}{\alpha + M - 2} \sum_{n=1}^{M} r_2^Y(t - n).$$

In order to elucidate the asymptotic decay of $r_2^Y$, it is interesting to consider the function

$$F(x) \equiv \frac{1}{\alpha + M - 2} \sum_{n=1}^{M} x^n$$

for positive $x$. This is a strictly decreasing positive continuous function which diverges to infinity when $x \to 0^+$ and goes to zero when $x \to +\infty$. Thus, there exists a unique positive $\lambda$ such that $F(\lambda) = 1$, which is smaller than 1 since $F(1) = M/(M + \alpha - 2) < 1$. The interest of this is that from Eq. (64) we have that $r_2^Y$ decays exponentially fast with rate $-\ln \lambda$. Fig. 5 shows $\lambda$ vs. $\alpha$ for different values of the memory $M$. Not surprisingly, the correlations have a slower decay at higher values of $M$. Also, the decay rate is minimum when $\alpha$ goes towards the lower limit value of 4, namely when the distribution of $Y_1$ displays the most pronounced tails. To show that $\lambda$ really rules the decay of $r_2^Y$, we prove by induction that

$$\frac{\lambda^{t-2}}{\alpha - 1} \leq r_2^Y(t) \leq \frac{\lambda^{t-M-1}}{\alpha - 1}$$

for positive $x$. This is a strictly decreasing positive continuous function which diverges to infinity when $x \to 0^+$ and goes to zero when $x \to +\infty$. Thus, there exists a unique positive $\lambda$ such that $F(\lambda) = 1$, which is smaller than 1 since $F(1) = M/(M + \alpha - 2) < 1$. The interest of this is that from Eq. (64) we have that $r_2^Y$ decays exponentially fast with rate $-\ln \lambda$. Fig. 5 shows $\lambda$ vs. $\alpha$ for different values of the memory $M$. Not surprisingly, the correlations have a slower decay at higher values of $M$. Also, the decay rate is minimum when $\alpha$ goes towards the lower limit value of 4, namely when the distribution of $Y_1$ displays the most pronounced tails. To show that $\lambda$ really rules the decay of $r_2^Y$, we prove by induction that

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$$\frac{\lambda^{t-2}}{\alpha - 1} \leq r_2^Y(t) \leq \frac{\lambda^{t-M-1}}{\alpha - 1}$$
If $t \geq 2$. These bounds then entail that
\[
\lim_{t \to \infty} \frac{1}{t} \ln r_2^Y(t) = \ln \lambda .
\]  
(67)

We focus on the second inequality only, as the first can be treated with the same arguments. Since $\lambda < 1$ we have that $\lambda^{t-M-1}/(\alpha - 1) \geq 1/(\alpha - 1) = r_2^Y(t)$ for $t$ in between 2 and $M + 1$. Fixing then $t > M + 1$ and assuming that the inequality holds up to $t - 1$, we see that
\[
r_2^Y(t) \leq \frac{1}{\alpha + M - 2} \sum_{n=1}^{M} \frac{\lambda^{t-n-M-1}}{\alpha - 1} = \frac{\lambda^{t-M-1}}{\alpha - 1} .
\]  
(68)

C. The autocorrelation $r_q^X$

We now bring together the above results to discuss the behavior of $r_q^X$. Eq. (51) tell us that
\[
r_q^X(t) = u_q + v_q r_q^\alpha(t).
\]  
(69)

If $2 \leq t \leq M + 1$ the coefficients $u_q$ and $v_q$ are independent of time:
\[
u_q = \frac{\mathbb{E}[|Y_1|^q] - \mathbb{E}[|Y_1|^q]^2}{\mathbb{E}[a_{Y_1}^{2q}] \mathbb{E}[|Y_1|^{2q}] - \mathbb{E}[a_{Y_1}^{2q}]^2 \mathbb{E}[|Y_1|^q]^2} \cdot \mathbb{E}[a_{Y_1}^q]^2, \\
u_q = \frac{\mathbb{E}[a_{Y_1}^{2q}] - \mathbb{E}[a_{Y_1}^q]^2}{\mathbb{E}[a_{Y_1}^{2q}] \mathbb{E}[|Y_1|^{2q}] - \mathbb{E}[a_{Y_1}^{2q}]^2 \mathbb{E}[|Y_1|^q]^2} \cdot \mathbb{E}[|Y_1|^q]^2 .
\]  
(70)

(71)

Thus, at short time scales the autocorrelation $r_q^X$ entirely inherits the time dependence of $r_q^\alpha$, as we mentioned in the Main Text. Notice that $u_q = 0$ when the endogenous $\{Y_i\}_{i=1}^\infty$ process reduces to a sequence of independent variables and in such a case Eq. (69) holds for any $t$. More in general, for any $t$ we can rewrite Eq. (51) as
\[
r_q^X(t) = u_q(t) r_q^Y(t) + v_q r_q^\alpha(t)
\]  
(72)

where we have again used the letters $u_q$ and $v_q$ but with a different meaning. Here $u_q$ is the positive function of the time
\[
u_q(t) = \frac{\mathbb{E}[|Y_1|^{2q}] - \mathbb{E}[|Y_1|^q]^2}{\mathbb{E}[a_{Y_1}^{2q}] \mathbb{E}[|Y_1|^{2q}] - \mathbb{E}[a_{Y_1}^{2q}]^2 \mathbb{E}[|Y_1|^q]^2} \cdot \mathbb{E}[a_{Y_1}^q]^2, \\
u_q(t) = \frac{\mathbb{E}[a_{Y_1}^{2q}] - \mathbb{E}[a_{Y_1}^q]^2}{\mathbb{E}[a_{Y_1}^{2q}] \mathbb{E}[|Y_1|^{2q}] - \mathbb{E}[a_{Y_1}^{2q}]^2 \mathbb{E}[|Y_1|^q]^2} \cdot \mathbb{E}[|Y_1|^q]^2 .
\]  
(73)

(74)

Since
\[
\lim_{t \to \infty} u_q(t) = \frac{\mathbb{E}[|Y_1|^{2q}] - \mathbb{E}[|Y_1|^q]^2}{\mathbb{E}[a_{Y_1}^{2q}] \mathbb{E}[|Y_1|^{2q}] - \mathbb{E}[a_{Y_1}^{2q}]^2 \mathbb{E}[|Y_1|^q]^2} \cdot \mathbb{E}[a_{Y_1}^q]^2 > 0 ,
\]  
(75)

when $\rho$ is given by Eq. (20) of the Main Text (for which we know the behavior of $r_q^Y$ in the instance $q = 2$) we find that
\[
\lim_{t \to \infty} \frac{1}{t} \ln r_2^X(t) = \ln \max\{\lambda, 1 - \nu\}.
\]  
(76)

Thus, within this setting the autocorrelation of the observed process decays exponentially fast in the long time limit. The slowest between the relaxation rates of $r_2^Y$ and $r_2^\alpha$ determine the one of $r_q^X$. 
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[1] W. Feller, An Introduction to Probability Theory and its Applications, Vol. II (John Wiley & Sons, New York, 1971).
[2] R.W. Bailey, Mathematics of Computation 62, 779 (1994).
[3] R.S. Tsay, Analysis of Financial Time Series (John Wiley & Sons, 2002).