TROPICAL $\psi$ CLASSES

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Abstract. We introduce a tropical geometric framework that allows us to define $\psi$ classes for moduli spaces of tropical curves of arbitrary genus. We prove correspondence theorems between algebraic and tropical $\psi$ classes for some one-dimensional families of genus-one tropical curves.

1. Introduction

1.1. Results. The main goal of this manuscript is to introduce $\psi$ classes for moduli spaces of tropical curves of arbitrary genus. This is achieved in Definition 6.16: $\psi_i$ is the first Chern class of the $i$-th cotangent line bundle on the stack $\mathcal{M}_{g,n}$ of families of tropical, $n$-marked curves of genus $g$. We postpone by a few paragraphs an overview of the technical work necessary to make such a natural definition, and begin by discussing some features of the resulting theory.

First off, $\psi_i$ is a Chern class on a stack, that is the assignment of a (weak) Chow class on $B$ for any family of tropical curves $\pi: \mathcal{C} \to B$, appropriately compatible with base changes (Definition 6.13). As such, any correspondence statement requires as input a family of curves in algebraic geometry. We provide two non-trivial correspondence theorems for one-dimensional families of tropical curves of genus one. The first arises from a cycle of well-spaced tropical stable maps to $\mathbb{T}\mathbb{P}^2$ passing through eight general points.

**Theorem A** (Proposition 7.3, Proposition 7.4). Let $WS_8(\mathbb{T}\mathbb{P}^2, 3)$ denote the space of well-spaced tropical stable maps of degree three to the tropical projective plane, with eight marked ends. We denote by $B = \cap_{i=1}^8 ev_i^{-1}(P_i) \subset WS_8(\mathbb{T}\mathbb{P}^2, 3)$ the one-dimensional locus of maps where the marked ends are mapped to eight fixed general points in the plane. We define a one-dimensional cycle $\alpha_B$ supported on $B$ and a family of tropical curves $\mathcal{C} \to B$, giving rise to a map $g: B \to \mathcal{M}_{1,17}$. Forgetting the nine non-contracted ends and seven out of the eight marks, one gets eight covering functions $f_i: B \to \mathcal{M}_{1,8}$: For $i = 1, \ldots, 8$, we have

$$\frac{\text{deg}(f_i^* \psi_i \cdot \alpha_B)}{\text{deg}(f_i^* \alpha_B)} = \frac{1}{24}. \tag{1}$$

The second correspondence theorem arises from tropical Hurwitz theory.

**Theorem B** (Proposition 7.7, Proposition 7.8). For $d \geq 2$, denote by $B_d$ (a finite cover of) the space of genus-one tropical admissible covers of degree $d$ of a genus-zero tropical curve with $d + 8$ vertices.

2010 Mathematics Subject Classification. 14T05; 14A20.
with ramification profile \(((d), (d), (2, 1^{d-2}), (2, 1^{d-2}))\). We define a cycle \(\alpha_{B_d}\) supported on \(B_d\) and a family of tropical curves giving rise to a map \(\pi : B_d \to \overline{M}_{1,1}\). We have:

\[
\frac{\deg(\pi^*\psi_1 \cdot \alpha_{B_d})}{\deg(\pi_\ast(\alpha_{B_d}))} = \frac{1}{24}.
\]

Theorems A, B are instances of the tropical counterpart to the algebraic statement

\[
\int_{\overline{M}_{1,1}} \psi_1 = \frac{1}{24}.
\]

As we will discuss more in depth later in the introduction, for \(g > 0\) the moduli spaces \(\overline{M}_{g,n}\) do not carry a universal family, nor a canonical fundamental cycle. Not only can one not make (3) into a tropical statement; in Example 6.17 we construct families of genus-one tropical curves giving rise to maps of equal degree to \(\overline{M}_{1,1}\), but with different evaluations for \(\psi_1\). This is not surprising since the theory we constructed is entirely combinatorial; the best correspondence statement one may expect would compare the tropical and algebraic \(\psi\) classes for any family of tropical curves which arises as the tropicalization of an algebraic family of curves. While we do not prove such a statement in full generality here, Theorems A and B provide some compelling initial evidence.

A key feature of \(\psi\) classes in algebraic geometry is their behavior with respect to pull-back and push-forward via tautological forgetful morphisms. Our next result recovers similar statements for the tropical theory.

**Theorem C** (Theorem 6.19, Theorem 6.23). Let \(g, n \in \mathbb{Z}_{\geq 0}\) such that \(2g - 2 + n > 0\), and let \(\pi_\ast : \overline{M}_{g,n \sqcup \{\ast\}} \to \overline{M}_{g,n}\) denote the morphism forgetting the end marked with \(\ast\). Then:

1. for \(i = 1, \ldots, n\),
   \[
   \psi_i = \pi_\ast^* \psi_i + D_{i,\ast},
   \]
   where \(D_{i,\ast}\) denotes the divisor of curves where the \(i\)- and \(\ast\)-ends are are part of a tripod whose edges are all infinite.

2. \(\pi_\ast(\psi_\ast \cdot [\overline{M}_{g,n \sqcup \{\ast\}}^\text{Mf}]) = (2g - 2 + n) \cdot [\overline{M}_{g,n}^\text{Mf}],\)
   where \([\overline{M}_{g,n}^\text{Mf}]\) denotes the cycle supported with weight one on the cones of tropical curves with all vertices of genus zero.

Finally, our theory agrees with all previously made statements for moduli spaces of tropical curves of genus zero.

**Theorem D** (Theorem 4.32, Corollary 6.22). The tropicalization of the forgetful morphism

\[
\text{Trop}(\overline{M}_{0,n \sqcup \{\ast\}}^\text{alg}) \to \text{Trop}(\overline{M}_{0,n}^\text{alg})
\]

is a family of tropical stable curves inducing an isomorphism between \(\text{Trop}(\overline{M}_{0,n}^\text{alg})\) and \(\overline{M}_{0,n}\).

The class \(\psi_1\) is represented by the cycle on \(\overline{M}_{0,n}\) taking value one on all points corresponding to curves where the \(i\)-marked leg is adjacent to a four-valent vertex, and zero everywhere else.
We now take a few steps back to discuss some of the ingredients used to set up the theory. The first step is to define a notion of families of tropical curves such that the corresponding moduli stack is both reasonably well-behaved and with enough structure to do intersection theory on it.

We define a tropical space $X$ (Definition 2.8) to be a locally polyhedral space endowed with an affine structure $\text{Aff}_X$, a subsheaf of the sheaf of piecewise (affine) linear functions on $X$. The sections of $\text{Aff}_X$ are called affine functions. A tropical curve is then a tropical space of dimension one, with the additional structure of a genus function (Definition 3.2). While at this point there is still a fair amount of freedom in the affine structures allowed for tropical curves, we make the concept restrictive for smooth and nodal curves: germs of affine functions are given by harmonic functions for genus-zero points, and are constant for points of positive genus. We define a family of tropical curves (Definition 3.10) to be a morphism of tropical spaces such that the fibers are nodal tropical curves, with an additional compatibility between the affine structures of base and total space. Formally, this compatibility is stated as requiring the exactness of the sequence of sheaves (16); in simple terms, we are stipulating that any (germ of an) affine function on a fiber should extend to an open thickening of the fiber, and that any (germ of an) affine function that restricts constantly on fibers is the pull-back of an affine function on the base.

The moduli stack $\overline{M}_{g,n}$ (Definition 3.16) associated to families of tropical curves over tropical spaces has some desirable properties.

**Theorem 3.18.** For every pair $g, n \in \mathbb{Z}_{\geq 0}$ of non-negative integers with $2g - 2 + n > 0$, the diagonal of the stack $\overline{M}_{g,n}$ is representable.

Unfortunately, $\overline{M}_{g,n}$ is not a geometric stack: one cannot construct an atlas, i.e. an étale surjective morphism from a tropical space. In fact, one can get pretty close: let $\overline{V}_{g,n}$ denote the extended cone complex parameterizing tropical curves together with a cycle rigidification. One can construct piecewise linear functions on $\overline{V}_{g,n}$ through cross ratios on tropical curves (Definition 4.8): informally, via integration of a one-form along a directed path. Endowing $\overline{V}_{g,n}$ with the affine structure generated by cross ratios, one obtains that the forgetful morphism $\pi_* : \overline{V}_{g,n \cup \{\ast\}} \to \overline{V}_{g,n}$ is a morphism of tropical spaces and even that the exact sequence (16) holds. However, $\pi_*$ is not a family of tropical curves as the fibers do not have enough functions: consider $x \in \overline{V}_{g,n \cup \{\ast\}}$ corresponding to a point in a fiber of $\pi_*$ which is either on a leg adjacent to a vertex of positive genus, or on an edge adjacent to two vertices of positive genus; there are no non-trivial cross ratios involving the leg marked with $\ast$ (which is incident to $x$), and therefore no non-constant functions at $x$. This observation allows one to identify a natural open substack of $\overline{M}_{g,n}$ for which the above rigidification does provide an atlas: we denote by $\overline{M}^\text{M}_{g,n}$ the moduli stack of families of Mumford curves, i.e. tropical curves with no points of positive genus.
Theorem E (Theorem 4.27, Theorem 5.12). The morphism $\gamma^{\mathcal{M}^f}_{g,n} \to \mathcal{M}^f_{g,n}$ defined by the family $\tau_\star: \gamma^{\mathcal{M}^f}_{g,n,\mu_f[\star]} \to \gamma^{\mathcal{M}^f}_{g,n}$ is a covering of $\mathcal{M}^f_{g,n}$. Further, the forgetful morphism $\pi_\star: \mathcal{M}^f_{g,n,\mu_f[\star]} \to \mathcal{M}^f_{g,n}$ represents the universal family.

Theorem E essentially transfers to the Mumford locus of $\mathcal{M}^f_{g,n}$ any statement about moduli spaces of rational tropical curves. In fact, one may canonically construct a family of tropical curves over a locus called $\mathcal{M}^{\mathrm{good}}_{g,n}$, giving rise to a map onto a larger open substack than $\mathcal{M}^f_{g,n}$, as described in Definition 4.22 and Proposition 4.24.

The intuitive meaning of the lack of an atlas for $\mathcal{M}^f_{g,n}$ is that the affine structure on the total space of a family of tropical curves is not determined, even up to automorphisms, by the fibers. This means that one cannot define a fundamental cycle on $\mathcal{M}^f_{g,n}$; in Remark 6.12 this issue is illustrated for $\mathcal{M}^f_{1,1}$, where the obvious candidate for a fundamental cycle is the constant function $1: |\mathcal{M}^f_{1,1}| \to \mathbb{Z}$; the pull-back of 1 to the base of the families $\mathcal{C}_a$ from Example 3.13 is balanced, hence a tropical cycle on $\mathbb{R}$. However, when $a \neq b$ the fiber product $\mathcal{C}_a \times_{\mathcal{M}^f_{1,1}} \mathcal{C}_b$ gives a one dimensional family of tropical curves such that the pull-back of 1 is not balanced. While the absence of a fundamental cycle on $\mathcal{M}^f_{g,n}$ prevents one from globally integrating, one may still do intersection theory on the stack by working with families whose bases admit a fundamental cycle.

1.2. Context, Connections and Considerations. This work is at the confluence of three current areas of mathematical research: tautological classes on moduli spaces of curves, tropical intersection theory, and tropical moduli spaces of curves.

Tautological classes were introduced by Mumford in [Mum83], as a set of Chow classes on the moduli spaces of curves naturally arising from the geometry of the curves parameterized. The notion of $\psi$ classes stems from the observation that cotangent spaces of curves at a marked point naturally organize themselves in families to produce a line bundle on the moduli space of curves, of which $\psi$ is the first Chern class. The intersection theory of $\psi$ classes exhibits a rich combinatorial structure [AC96], culminating in Witten’s conjecture/Kontsevich’s theorem [Wit91, Kon95]: the generating function for intersection numbers of $\psi$ classes satisfies a classical integrable hierarchy. From the perspective of tropical geometry, $\psi$ classes describe the first Chern class of the normal bundles to boundary divisors, and are therefore intimately related to the edge coordinates on tropical moduli spaces of curves.

From its inception, one of the applications of tropical geometry has been to combinatorialize classical enumerative geometric problems. To this end, Mikhalkin [Mik06] sketched the foundations of tropical intersection theory, introducing the notions of tropical cycles and their stable intersection. Allerman and Rau [AR10] gave an alternative definition of intersection products by pulling-back Cartier divisors. Katz [Kat12] showed the agreement of the two constructions by relating both to toric intersection theory of [FS97]. In all these cases, it is essential that the tropical space on which one does intersection theory is embedded in a vector space, which provides a global notion of integral linear functions. Shaw [Sha13] generalizes the intersection product to...
locally matroidal fans, and the second author [Gro18] to weakly embedded cone complexes, arising from tropicalizations of cycles on toroidal embeddings.

It is hard to pinpoint the precise moment in which moduli spaces of tropical rational curves are born. While [Mik06] introduces the name and notation which we currently use, the space of phylogenetic trees studied in [BHV01] and the tropical Grassmannian of [SS04] lay the foundation for the combinatorial analysis of $\mathcal{M}_{0,n}$ as a balanced, rational, polyhedral fan. Work of Kapranov [Kap93] and Tevelev [Tev07] exhibit $\mathcal{M}_{0,n}^{\text{alg}}$ as a tropical compactification, and Gibney-Maclagan [GM10] study its ideal in the Cox ring of the toric variety whose fan is $\mathcal{M}_{0,n}$. François and Hampe [FH13] show that $\mathcal{M}_{0,n}$ represents a moduli functor over smooth tropical spaces. The embedding of $\mathcal{M}_{0,n}^{\text{alg}}$ into a torus induced by the Plücker embedding of $\text{Gr}(2,n)$ is the driving force for all these results. In higher genus there is no such embedding, so tropicalization takes a different perspective: [ACP15] relates the edge lengths of tropical curves to the smoothing parameters of the corresponding nodes of algebraic curves; tropicalization is then a deformation retraction of the analytic moduli space $\mathcal{M}_{g,n}^{\text{an}}$ onto a skeleton induced by the toroidal structure of the boundary and isomorphic, as a cone complex, to the moduli space of tropical curves $\mathcal{M}_{g,n}$. Another perspective on tropicalization uses logarithmic geometry [CCUW17], where it is also shown that $\mathcal{M}_{g,n}$ represents a functor over the category of rational polyhedral cone complexes.

The notion of $\psi$ classes on moduli spaces of rational tropical curves was introduced by Mikhalkin in [Mik07] and later investigated by Kerber and Markwig [KM09], who proved a correspondence theorem stating the equality of intersection numbers of algebraic and tropical $\psi$ classes. Katz [Kat12] later gave a non-computational proof of such equality by combining the connection of tropical and toric intersection theory with the fact that $\mathcal{M}_{0,n}$ gives a tropical compactification. Correspondence theorems for intersection numbers of $\psi$ classes on moduli spaces of higher genus tropical curves appear in Jin’s Phd thesis [Jin19]: higher genus curves produce étale covers of genus zero (orbifold) twisted curves, giving rise to morphisms among the corresponding moduli spaces. The equality of algebraic and tropical intersection numbers is shown using the algebraic degree of the branch morphism as a geometric input datum and doing tropical intersection theory on the moduli spaces of genus zero twisted curves.

In the current work, the construction of $\psi$ classes lies entirely in the tropical world. The main technical issue to overcome is how to make the integral lattices of adjacent cones of the extended cone complex $\mathcal{M}_{g,n}$ communicate with each other. The solution we propose is that families of tropical curves must be endowed with such information, in the form of a sheaf of functions to be considered affine. Germs of functions at points of faces then provide the appropriate transition data among the integral lattices of adjacent cones. Once one knows how to transition affine functions across faces, it is possible to define a notion of balancing, which is the key tool for tropical intersection theory. One may also define sheaves $\text{Aff}_\mathcal{C}(\mathcal{K}_s)$ of affine functions with prescribed order along a linear section, which are torsors over affine functions. This allows to
sidestep the technical issue of making sense of the notions of tangent bundles or relative dualizing sheaves in the category of tropical spaces: the \(i\)-th cotangent line bundle of a family of tropical curves \(\mathcal{C}\) is defined to be \(\text{Aff}_{\mathcal{C}}(-s_i)\), drawing from the algebraic identification of the \(i\)-th cotangent line bundle with the conormal bundle to the \(i\)-th section.

Given the absence of any algebraic input, it is not surprising that one obtains a combinatorial theory which is broader than the algebraic theory. It appears that when a family of tropical curves arises from an algebraic one, then the combinatorial theory agrees with the algebraic one. The computation of the degree of \(\psi\) on \(\mathcal{M}_{1,1}\) we make in Section 7.1 is very much parallel to its classical counterpart [Vak08, Section 3.13]: a pencil of plane cubics has nine base points, and hence it provides a family of genus one curves with nine sections, with total space \(\text{Bl}_{p_1,...,p_9}\mathbb{P}^2\). The \(\psi\) classes on this family are dual to the self-intersections of the exceptional divisors, and hence have degree one. Since the family has twelve rational fibers, the pencil gives a degree-twelve covering of \(\mathcal{M}_{1,1}^{\text{alg}}\). We consider tropical stable maps instead of cubic curves to obtain a covering of \(\mathcal{M}_{1,1}\), as it would be impossible for curves with very large \(j\)-invariant to satisfy the point constraints without contracting any edge. Well-spacedness, which is also a realizability condition [Spe14, RSW17], ensures that the family is pure-dimensional. After that, the proof is parallel to the classical one: the degree of the covering is twelve, as a consequence of the count of rational curves in the family, or from [KM09]. The class \(\psi_i\) is supported on the unique curve where the \(i\)-th leg is incident to a four-valent vertex, and an explicit computation shows that the multiplicity is one.

The intersection theory of \(\psi\) classes in algebraic geometry is controlled by the Virasoro constraints [Vak08]: an infinite sequence of recursive relations which reconstruct all intersection numbers from the initial condition \(\int_{\mathcal{M}_{0,3}^{\text{alg}}} 1 = 1\). The first two relations, known as the string and dilaton equations, follow from the pull-back and push-forward properties of \(\psi\) classes along forgetful morphisms. Theorem C should then be interpreted as the analogous properties (i.e. germs of string and dilaton) holding in the tropical theory.

1.3. Future directions. We consider this work very much as a beginning, rather than a story close to conclusion, and discuss here some of the avenues of investigation that we intend to pursue.

As mentioned earlier, the ultimate goal would be to prove a general correspondence statement, comparing tropical and algebraic \(\psi\) classes for families related via tropicalization.

A first natural step towards such a statement consists in verifying it for a larger class of examples. After some preliminary investigation, we feel confident that the correspondence statement of Section 7.1 will naturally extend to one-dimensional families of tropical curves obtained by resolving pencils of genus one curves on smooth toric surfaces using well-spaced tropical stable maps. We have also analyzed some
one-dimensional families of well-spaced tropical stable maps to tropical $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegrees $(2,3)$ and $(2,4)$, where a correspondence statement depends on establishing a comparison lemma analogous to Theorem 6.19 for the forgetful morphism from the moduli space of maps to the moduli space of curves. A rather general statement which seems reasonable given the current technology is a correspondence theorem for stationary genus one descendant invariants with one $\psi$-insertion.

Tropical stable maps give a natural way to construct families of tropical curves; in positive genus superabundance causes loci of tropical stable maps subject to geometric constraints to not be equidimensional. In genus one, imposing the condition of well-spacedness restores equidimensionality and allows to naturally define intersection cycles of tropical stable maps. One should think of well-spacedness as identifying a virtual fundamental cycle very much analogously to the case of algebraic Gromov-Witten theory. For genus strictly greater than one, such a tool is at present not available.

One may construct families of tropical curves through tropical admissible covers, generalizing the construction in Section 7.2. At present, this seems the most direct avenue to seek correspondence statements for families of arbitrary genus.

Eventually, it would be desirable to gain a complete understanding of the intersection theory of tropical $\psi$ classes, including the description of positive dimensional cycles, and of intersection cycles of multiple $\psi$ classes. While a purely combinatorial analysis seems rather laborious, recent developments in logarithmic geometry [Uli17, Gro18] are offering a conceptual approach to the coordinate-free tropicalization of cycles. While currently the main technical obstacle is that for $g \geq 2$, there is no sufficiently explicit description of a cycle representing the algebraic $\psi$ class, we are hopeful that in the not too distant future it will be possible to answer in the positive the fundamental question: are tropical $\psi$ classes the tropicalization of algebraic $\psi$ classes?

1.4. Structure of the paper. Sections 2 through 6 are devoted to establishing foundations. Section 2 introduces the categories of tropical piecewise linear (TPL) spaces and tropical spaces. Section 3 develops the notion of families of tropical curves and defines the corresponding moduli stack $\mathcal{M}_{g,n}$. This stack has representable diagonal.

In Section 4 we show that the stack $\mathcal{M}_{g,n}$ is not geometric: it does not admit an atlas. We then restrict our attention to an open substack which is geometric: it parameterizes families of Mumford (also called explicit) curves. We show that $\mathcal{M}^{\text{Mf}}_{g,n}$ is essentially as well behaved as spaces of rational tropical curves.

Section 5 establishes that the natural forgetful and section morphisms in the category of TPL-spaces in fact are morphisms in the category of tropical curves. The key technical tool here is to define the notion of the stabilization of a family of tropical curves when forgetting a mark.

In Section 6, the necessary concepts of tropical intersection theory are recalled and adapted to the context necessary to define tropical $\psi$ classes. Basic properties of $\psi$ classes are then investigated.
Finally, Section 7 contains two correspondence statements for tropical $\psi$ classes in genus one. Section 7.1 studies a one-dimensional family of well-spaced tropical stable maps of degree three to the tropical projective plane incident to eight general points. While the section has been written in a rather concise way to highlight the key ideas of the computation, a complete combinatorial description of the family is given in Appendix A. Section 7.2 analyzes a family of one-dimensional spaces of of genus one admissible covers.

1.5. Glossary of Notation. This paper lives almost in its entirety in the tropical world. For this reason, we chose to omit the superscript $\text{trop}$ from our notation; we have instead added the superscript $\text{alg}$ anytime an algebro-geometric moduli space makes an appearance. We acknowledge this is not standard, but maintain it is a sound choice for this work. Here is a list of some of the notation used in the paper.

- $T$: [page 9]; the tropical affine line $\mathbb{R} \cup \{\infty\}$.
- $\mathbb{R}$: [page 9]; the tropical projective line $\mathbb{R} \cup \{\pm \infty\}$.
- $\text{PL}_X$: Definition 2.1; sheaf of piecewise linear functions on $X$.
- $\text{Aff}_X$: Definition 2.8; sheaf of linear functions on $X$.
- $\text{Aff}^\psi(\text{ks})$: Definition 3.23; sheaf of functions with prescribed order along a section.
- $\Omega^1_X$: Definition 2.8; sheaf of one forms on $X$.
- $H^\psi_C$: Definition 3.8; sheaf of fiberwise harmonic functions.
- $\mathcal{M}_{g,n}^{\text{TPL}}$: Definition 3.7; stack of families of TPL curves.
- $\mathcal{M}_{g,n}$: Definition 3.16; stack of families of tropical curves.
- $\mathcal{M}_g$: Definition 4.20; open substack of families of Mumford curves.
- $\mathcal{V}_{g,n}$: Definition 4.3; cone complex of cycle rigidified curves.
- $\mathcal{V}_{g,n}^{\text{good}}$: Definition 4.22; restriction of $\mathcal{V}_{g,n}$ to $\text{good}$ curves.
- $\mathcal{V}_{g,n}^{\text{Mf}}$: Definition 4.22; restriction of $\mathcal{V}_{g,n}$ to Mumford curves.

We conclude by warning the reader of a slight abuse of notation we chose to adopt: it is common to denote by $\mathcal{M}_{g,n}$ the moduli space of families of curves with markings labeled by elements of the finite set $n = \{1, \ldots, n\}$. When we have an additional mark playing a distinguished role we call it $\star$ and denote the indexing set $n \cup \{\star\}$ rather than $[n] \sqcup \{\star\}$.

1.6. Acknowledgements. R.C. is grateful for the support from the Simons collaboration grant 420720. H.M. is grateful for the support of the DFG collaborative research center SFB 1489. We would like to thank Dhruv Ranganathan for answering our questions about well-spacedness, and Martin Ulirsch and Dimitry Zakharov for conversations and comments on the manuscript.

2. Tropical Spaces

This section introduces the category of spaces containing the families of tropical stable curves we consider. What we call tropical spaces are generalizations of (abstract) tropical varieties, previously existing notions of tropical spaces, rational polyhedral
spaces, and affine manifolds with singularities [Mik06, BIMS15, BIMS15, AR10, MZ14, JRS18, GS06, IKMZ19, MRar, Car15, Sha15].

### 2.1. Polyhedral sets, TPL-spaces, and affine structures.

A (rational) polyhedron in $\mathbb{R}^n$ is a finite intersection of half spaces $\{x \in \mathbb{R}^n \mid \langle m, x \rangle \leq a\}$, where $m \in (\mathbb{Z}^n)^\vee$ and $a \in \mathbb{R}$. Let $T = \mathbb{R} \cup \{\infty\}$, topologized to be homeomorphic to a half-line. The space $T^n$ has a natural stratification

$$T^n = \bigcup_{1 \leq i \leq n} T^n_i,$$

where the stratum

$$T^n_i = \{(x_i) \in T^n \mid x_i = \infty \text{ if and only if } i \in I\}$$

can be identified with $\mathbb{R}^{n-|I|}$. A (rational) polyhedron in $T^n$ is the closure of a polyhedron in $T^n_i$ for some subset $I \subseteq \{1, \ldots, n\}$. If $P \subseteq T^n$ is a polyhedron, then the finite faces of the polyhedron $P$ in $T^n$ are the closures of the faces of $P$, and the infinite faces of $P$ are the closures of the nonempty sets of the form $\overline{P} \cap T^n_i$ for some $I \subseteq J \subseteq \{1, \ldots, n\}$. The complement in $P$ of the union of its proper faces (finite or infinite) is called the relative interior of $P$, denoted $\text{relint}(P)$. A polyhedral set in $T^n$ is a finite union of polyhedra.

Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm \infty\}$, topologized to be homeomorphic to a closed interval. A potentially infinite (integral) affine (linear) function on an open subset $U$ of a polyhedral set in $T^n$ is a continuous function $f : U \to \overline{\mathbb{R}}$ such that $f$ is locally a restriction of a function of the form

$$x \mapsto \langle m, x \rangle + a,$$

where $m \in (\mathbb{Z}^n)^\vee$ and $a \in \overline{\mathbb{R}}$. We follow standard conventions to extend operations of addition and multiplication to include $\pm \infty$, with only $\infty - \infty$ being undefined. In particular, if $f(x) = \langle m, x \rangle + a$ and $x \in U \cap T^n_i$, then either $m_i \geq 0$ for all $i \in I$ or $m_i \leq 0$ for all $i \in I$.

A function is (integral) affine (linear) if it is a potentially infinite affine function and has values in $\mathbb{R}$. Being affine is a local condition, so we obtain a sheaf of Abelian groups $\text{Aff}_U$ of affine functions on an open subset $U$ of a polyhedral set in $T^n$.

Let $U \subseteq T^n$ be an open neighborhood of a polyhedral set. A closed subset $A \subseteq U$ of $U$ is locally polyhedral in $U$ if for every $x \in U$ there exists a polyhedral set $P$ in $T^n$ and an open neighborhood $V$ of $x$ in $U$ such that $V \cap A = V \cap P$.

A piecewise (integral, affine) linear function on an open subset $U$ of a polyhedral set in $T^n$ is a continuous function $f : U \to \overline{\mathbb{R}}$ such that for every point $x \in U$ there exist polyhedra $P_1, \ldots, P_k$ in $T^n$ with $U \cap \bigcup_{i=1}^k P_i$, a neighborhood of $x$ in $U$ and $f|_{U \cap P_i}$ a potentially infinite affine function on $U \cap P_i$ for all $1 \leq i \leq k$.

Being piecewise linear is a local condition on $f$, so we obtain a sheaf $\text{PL}_U$ of piecewise linear functions on $U$. Because piecewise linear function can have infinite values, $\text{PL}_U$ is not a sheaf of Abelian groups. However, the subsheaf $\text{PL}_U^{\text{fin}}$ consisting of all piecewise linear functions with finite values is a sheaf of Abelian groups. By definition, the sheaf $\text{Aff}_U$ is a subsheaf of $\text{PL}_U^{\text{fin}}$. 

Definition 2.1. A tropical piecewise linear space, or TPL-space for short, is a pair \((X, \text{PL}_X)\) consisting of a paracompact Hausdorff topological space \(X\) and a sheaf \(\text{PL}_X\) of \(\mathbb{R}\)-valued continuous functions on \(X\), the piecewise (integral, affine) linear functions, such that for every point \(x \in X\) there exists a neighborhood \(U\) of \(x\), an open subset \(V\) of a polyhedral set in \(T^n\) for some \(n \in \mathbb{N}\), and a homeomorphism \(\phi: U \to V\) that induces an isomorphism \(\phi^{-1}\text{PL}_V \to \text{PL}_U\) via precomposition with \(\phi\). The datum \(U \xrightarrow{\phi} V\) is called a chart at \(x\).

We denote by \(\text{PL}^\text{fin}_X\) the sheaf of Abelian groups on \(X\) whose sections are the piecewise linear functions on \(X\) with finite values.

A morphism between two TPL-spaces \(X\) and \(Y\) is a continuous map \(X \to Y\) that pulls back piecewise linear functions to piecewise linear functions.

Example 2.2. If \(\Sigma\) is a cone complex or an extended cone complex \([ACP15, \S2]\), one obtains a sheaf \(\text{PL}_\Sigma\) of \(\mathbb{R}\)-valued functions on \(\Sigma\) by defining \(\Gamma(U, \text{PL}_\Sigma)\) for an open subset \(U \subseteq \Sigma\) as the set of continuous functions \(\phi: U \to \mathbb{R}\) such that the restriction \(\phi|_\sigma \cap U\) is piecewise linear for all \(\sigma \in \Sigma\). Then the pair \((\Sigma, \text{PL}_\Sigma)\) is a TPL-space.

Remark 2.3. Constant functions are piecewise linear, so \(\text{PL}_X\) contains the constant sheaf \(\mathbb{R}_X\) as a subsheaf.

Definition 2.4. A point \(x\) of a TPL-space \(X\) is finite if for every function \(f \in \Gamma(U, \text{PL}_X)\) defined on an open neighborhood \(U\) of \(x\), the equality \(f(x) = \infty\) implies that \(f\) is constant on a neighborhood of \(x\). The irreducible components of a TPL-space \(X\) are the closures in \(X\) of the connected components of the set of finite points of \(X\).

Definition 2.5. A closed subset \(A\) of a TPL-space \(X\) is called locally polyhedral if for every chart \(X \supseteq U \xrightarrow{\phi} V \subseteq T^n\) the image \(\phi(A \cap U)\) is locally polyhedral in \(V\) as defined above.

Definition 2.6. Let \(X\) be a TPL-space. A polyhedron in \(X\) is a locally polyhedral subset \(P\) of \(X\), together with an affine structure \(\text{Aff}_P\) on \(P\) such that there exists a chart \(X \supseteq U \xrightarrow{\phi} V \subseteq T^n\) with \(P \subseteq U\) that maps \(P\) isomorphically (as a tropical space) onto a polyhedron in \(T^n\).

Definition 2.7. Let \(X\) be a TPL-space and let \(x \in X\). A local face structure of \(X\) at \(x\) is a collection \(\Sigma\) of polyhedra in \(X\)

1. for every \(\sigma \in \Sigma\) we have \(x \in \sigma\),
2. if \(\tau\) is a face of \(\sigma \in \Sigma\) such that \(x \in \tau\), then \(\tau \in \Sigma\),
3. for all \(\sigma, \sigma' \in \Sigma\) we have \(\sigma \cap \sigma' \in \Sigma\),
4. the set \(|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma\) is a neighborhood for \(x\), and
5. there exists a chart \(U \xrightarrow{\phi} V \subseteq T^n\) such that \(|\Sigma| \subseteq U\) and such that \(\phi\) maps every \(\sigma \in \Sigma\) isomorphically (as a tropical space) onto a polyhedron in \(T^n\).

Definition 2.8. An affine structure on a TPL-space \(X\) is a subsheaf of Abelian groups \(\text{Aff}_X\) of \(\text{PL}^\text{fin}_X\) containing the constant sheaf \(\mathbb{R}_X\). A TPL-space that is equipped with an
affine structure is called a tropical space. We call the sections of $\text{Aff}_X$ affine (linear) functions on $X$ and the sections of $\Omega^1_X := \text{Aff}_X / \mathbb{R}_X$ tropical one-forms on $X$.

**Example 2.9.** If $P$ is a polyhedral set in $\mathbb{T}^n$, then $P$ together with the inclusion $\text{Aff}_P \to \text{PL}_P$ is a tropical space.

**Example 2.10.** Every TPL-space $X$ has a minimal affine structure given by $\text{Aff}_X = \mathbb{R}_X$ and a maximal affine structure given by $\text{Aff}_X = \text{PL}_X^{\text{fin}}$.

**Example 2.11.** Let $\Sigma$ be a fan in $\mathbb{R}^n$. Then it is a polyhedral set in $\mathbb{R}^n$ and thus has an induced affine structure, making it a tropical space. Let $\Sigma$ denote the tropical toric variety associated to $\Sigma$ [Kaj08, Pay09]. One obtains an affine structure on $\Sigma$ by defining $\Gamma(U, \text{Aff}_\Sigma)$ for an open subset $U \subseteq \Sigma$ as the group of all piecewise linear functions $\phi \in \Gamma(U, \text{PL}_\Sigma)$ that have finite values everywhere and satisfy $\phi|_{U \cap \Sigma} \in \Gamma(U \cap \Sigma, \text{Aff}_\Sigma)$. We can thus consider tropical toric varieties as tropical spaces in a natural way. This construction generalizes to the weakly embedded extended cone complexes of [Gro18]. If $\Sigma$ is the fan of $\mathbb{P}^n$ considered with its natural torus action, we denote the tropical toric variety $\Sigma$ by $\text{TP}_n$, the tropical projective $n$-space. Note that the underlying space of $\text{TP}_1$ is precisely $\mathbb{R}$.

**Remark 2.12.** This definition of affine structures is very general. As seen in Example 2.10, the sheaf of affine functions can be arbitrarily close to the (very large) sheaf of all finite piecewise linear functions. In particular, affine structures do not necessarily admit a combinatorial description. In almost all examples that we consider, including all moduli spaces appearing in this paper, the sheaf of tropical one-forms is constructible in a suitable sense and therefore the affine structure can in fact be described combinatorially on the spaces of interest.

**Definition 2.13.** A morphism between two tropical spaces $X$ and $Y$ is a morphism $f: X \to Y$ of TPL-spaces such that the pull-back $\text{PL}_Y \to f_* \text{PL}_X$ maps $\text{Aff}_Y$ into $f_* \text{Aff}_X$. Sometimes we will call such a morphism a linear morphism from $X$ to $Y$ to stress that it respects the affine (linear) structure.

If $A$ is a locally polyhedral subset of a TPL-space $X$, then the restrictions of the functions in $\text{PL}_X$ to $A$ define a sheaf $\text{PL}_A$ on $A$, and $(A, \text{PL}_A)$ is a TPL-space. If $\text{Aff}_X$ is an affine structure on $X$, then the restrictions of the functions in $\text{Aff}_X$ to $A$ define an affine structure $\text{Aff}_A$ on $A$.

### 2.2. Fiber products of TPL-spaces.

**Definition 2.14.** A morphism $f: X \to Y$ of TPL-spaces is a closed immersion if $f$ maps $X$ homeomorphically onto a closed subset of $Y$, and the morphism $f^{-1} \text{PL}_Y \to \text{PL}_X$ is surjective.

Every locally polyhedral subset $Z$ of a TPL-space $X$ has a unique TPL-structure, the induced TPL-structure, making the inclusion $\iota: Z \to X$ a closed immersion. Namely, one defines $\text{PL}_Z$ as the sheaf consisting of all functions on $Z$ that are locally restrictions of
functions in PL\(_X\). It follows directly from the definitions that any closed immersion \(f: Z \to X\) induces an isomorphism \(\tilde{Z} \to f(Z)\), where \(f(Z)\) is equipped with the induced TPL-structure. Since the underlying maps of closed immersions are injective, they are monomorphisms in the category of TPL-spaces. Moreover, a closed immersion \(f: Z \to X\) satisfies the following universal property: a morphism of TPL-spaces from \(W\) to \(X\) whose image is contained in \(f(Z)\) determines a morphism from \(W\) to \(Z\) making the following diagram commutative:

\[
\begin{array}{ccc}
Z & \xrightarrow{f} & W \\
\downarrow & & \downarrow \\
f(Z) \subseteq X & & \\
\end{array}
\]

**Proposition 2.15.** Let \(p: X \to S\) and \(q: Y \to S\) be morphisms of TPL-spaces. Then the fiber product \(X \times^TPL S Y\) in the category of TPL-spaces exists. Furthermore, the underlying topological space of \(X \times^TPL S Y\) coincides with the fiber product of the underlying topological spaces of \(X\) and \(Y\) over the underlying topological space of \(S\).

**Proof.** Given open covers \([X_i]_{i \in I}\), \([Y_i]_{i \in I}\) and \([S_i]_{i \in I}\) of \(X\), \(Y\), and \(S\), respectively, such that \(p(X_i) \cup q(Y_i) \subseteq S_i\) for all \(i \in I\), the existence of \(X \times^TPL S Y\) is equivalent to the existence of \(X_i \times^TPL S_i Y_i\) for all \(i \in I\). We may thus assume that there exist locally polyhedral sets \(P \subseteq T^m\), \(Q \subseteq T^n\), and \(R \subseteq T^k\) for some \(m, n, k \in \mathbb{N}\) such that \(X\) is an open subset of \(P\), \(Y\) is an open subset of \(Q\), and \(S\) is an open subset of \(R\). After further shrinking \(X\), \(P\), \(Y\), and \(Q\), we may assume that \(p\) and \(q\) extends to morphisms \(p': P \to R\) and \(q': Q \to R\). It then suffices to show that the fiber product \(P \times^TPL R Q\) exists. After replacing \(P\) and \(Q\) by their graphs with respect to \(p'\) and \(q'\), respectively, we may assume that \(p\) is the restriction of the projection \(T^m \to T^k\), and \(q\) is the restriction of the projection \(T^n \to T^k\). It follows immediately that the set-theoretic fiber product \(P \times^set R Q\) is a locally polyhedral subset of \(T^{m+n+k}\). Since \(T^{m+n+k}\) is the fiber product \(T^m \times^TPL T^n \times^TPL T^k\) in the category of TPL-spaces, the locally polyhedral subset \(P \times^set R Q\) of \(T^{m+n+k}\), equipped with the induced TPL-structure, is a fiber product in the category of TPL-spaces.

With the same reductions, the statement about the topology on fiber products on TPL-spaces is reduced to the fact that the topology of \(T^{m+n+k}\) coincides with the topology on the fiber product \(T^{m+k} \times^top T^{n+k}\) taken in the category of topological spaces. \(\square\)

**Remark 2.16.** The PL-functions on a fiber product are in general not the sums of pull-backs of functions from the factors. Indeed, if \(p, q: \mathbb{R}^2 \to \mathbb{R}\) are the two projections, then the morphism

\[
p^{-1}PL\mathbb{R} \oplus q^{-1}PL\mathbb{R} \to PL\mathbb{R}^2
\]

of sheaves is not an epimorphism. For example, the global PL-function

\[
\mathbb{R}^2 \to \mathbb{R}, \ (x, y) \mapsto \min\{x, y\}
\]
is not in the image, and neither are its germs at any point on the diagonal.

**Convention 2.17.** We view the category of TPL-spaces as a site by equipping it with the Grothendieck topology in which the coverings are the jointly surjective collections of local isomorphisms.

**Proposition 2.18.** Let \( f : X \rightarrow Y \) be a morphism of TPL-spaces, and let \( \iota : Z \rightarrow Y \) be a closed immersion. Then the induced morphism \( Z \times_Y^{TPL} X \rightarrow X \) is a closed immersion.

**Proof.** The preimage \( f^{-1}(\iota(Z)) \) is a locally polyhedral subset of \( X \). If we equip it with the induced TPL-structure, then the universal property of closed immersions implies \( f^{-1}(\iota(Z)) = Z \times_Y^{TPL} X \). □

**2.3. Fiber products of tropical spaces.**

**Proposition 2.19.** Let \( p : X \rightarrow S \) and \( q : Y \rightarrow S \) be morphisms of tropical spaces. Then the fiber product \( X \times_S Y \) in the category of tropical spaces exists. Furthermore, the underlying TPL-space of \( X \times_S Y \) coincides with \( X \times_S^{TPL} Y \) and the morphism of sheaves

\[
q'^{-1} \text{Aff}_X \oplus p'^{-1} \text{Aff}_Y \rightarrow \text{Aff}_{X \times_S Y},
\]

where \( p' : X \times_S Y \rightarrow Y \) and \( q' : X \times_S Y \rightarrow X \) are the projection maps, is surjective.

**Proof.** Let \( W = X \times_S^{TPL} Y \) and let \( p' : W \rightarrow Y \) and \( q' : W \rightarrow X \) denote the projections (which are morphisms of TPL-spaces). Define the affine structure \( \text{Aff}_W \) on \( W \) as the image of the morphism of sheaves \( q'^{-1} \text{Aff}_X \oplus p'^{-1} \text{Aff}_Y \rightarrow \text{PL}_W \). Then for every tropical space \( Z \), a map \( f : Z \rightarrow W \) is a morphism of tropical spaces if and only if it is a morphism TPL-spaces and the composites \( p' \circ f \) and \( q' \circ f \) are morphisms of tropical spaces. It follows from this and the universal property of \( W \) as a fiber product in the category of TPL-spaces that \( W \) is also the fiber product of \( X \) and \( Y \) over \( S \) in the category of tropical spaces. By construction, it has all of the asserted properties. □

**Convention 2.20.** We view the category of tropical spaces as a site by equipping it with the Grothendieck topology in which the coverings are the jointly surjective collections of local isomorphisms.

**Definition 2.21.** A morphism \( f : Z \rightarrow X \) of tropical spaces is a **closed immersion** if its underlying morphism of TPL-spaces is a closed immersion and the morphism of sheaves \( f^{-1} \text{Aff}_X \rightarrow \text{Aff}_Z \) is an epimorphism.

Given a locally polyhedral subset \( Z \) of a tropical space \( X \), there is a unique way to make \( Z \) into a tropical space such that the inclusion \( Z \rightarrow X \) is a closed immersion of tropical spaces. Similar as for TPL-spaces, this defines a bijection between isomorphism classes of closed immersions into a given tropical space \( X \) and locally polyhedral subsets of \( X \).

**Proposition 2.22.** Let \( f : X \rightarrow Y \) and \( g : Y' \rightarrow Y \) be a morphisms of tropical spaces. If \( f \) is a closed immersion, then the induced morphism \( X \times_Y Y' \rightarrow Y' \) has the same property.

**Proof.** This follows directly from Proposition 2.18 and Proposition 2.19. □
3. The Stack of Tropical Stable Curves

Abstract tropical curves and their moduli spaces have been studied from various perspectives [MZ08, Mik06, Mik07, GKM09, ACP15, GM10, Tev07, Cap13, Cap18]. Here, we define tropical curves and their families suiting the context of Section 2.

3.1. Tropical curves.

Definition 3.1. A TPL-curve is a purely one-dimensional TPL-space \( \Gamma \), together with a genus function \( \gamma_\Gamma : \Gamma \to \mathbb{Z}_{\geq 0} \) with finite support. The genus of a compact TPL-curve \( \Gamma \) is given by \( h^1(\Gamma) + \sum_{p \in \Gamma} \gamma_\Gamma(p) \).

The underlying space of a TPL-curve \( \Gamma \) is a topological graph. For every point \( p \in \Gamma \) we define the set \( T_p \Gamma \) of direction vectors at \( p \) as the set of germs of immersions \( (\mathbb{R}_{\geq 0}, 0) \to (\Gamma, p) \). One may describe \( T_p \Gamma \) concretely by picking a local face structure \( \Sigma \) at \( p \), and identifying \( T_p \Gamma \) with the set of one-dimensional strata of \( \Sigma \). We call an element of \( T_p \Gamma \) a direction at \( p \) and the number of directions at \( p \) the valence of \( \Gamma \) at \( p \), denoted \( \text{val}(p) \).

Given a function \( m \in \Gamma(\mathcal{U}, \text{PL}_\Gamma^\text{fin}) \) defined on a neighborhood \( \mathcal{U} \) of \( p \) and a direction \( v \in T_p \Gamma \) represented by a germ \( (\mathbb{R}_{\geq 0}, 0) \to (\Gamma, p) \), it makes sense to take the slope of \( m \) with respect to \( v \) by pulling \( m \) back to a neighborhood of zero in \( \mathbb{R}_{\geq 0} \). We denote this slope by \( d_v m \) and call it the slope of \( m \) at \( p \) along \( v \). We say that a function \( m \in \Gamma(\mathcal{U}, \text{PL}_\Gamma^\text{fin}) \) on an open subset \( \mathcal{U} \) of a TPL-curve \( \Gamma \) is harmonic if at every \( p \in \mathcal{U} \) the sum of all slopes at \( p \) is zero, that is if

\[
\sum_{v \in T_p \Gamma} d_v m = 0. \tag{7}
\]

Since harmonicity is a local condition, harmonic functions on \( \Gamma \) give a subsheaf of Abelian groups \( H_\Gamma \) of \( \text{PL}_\Gamma^\text{fin} \).

Definition 3.2. A tropical curve is a TPL-curve equipped with an affine structure. The genus of a tropical curve is the genus of the underlying TPL-curve. A point \( p \) of a tropical curve \( \Gamma \) is said to be smooth if one of the following conditions hold:

1. \( \gamma_\Gamma(p) > 0 \), \( p \) is finite, and \( \text{Aff}_{\Gamma,p} = \mathbb{R} \);
2. \( \gamma_\Gamma(p) = 0 \), \( p \) is finite, and \( \text{Aff}_{\Gamma,p} = H_{\Gamma,p} \);
3. \( \gamma_\Gamma(p) = 0 \), \( p \) is infinite with \( \text{val}(p) = 1 \), and \( \text{Aff}_{\Gamma,p} = \mathbb{R} \) (= \( H_{\Gamma,p} \)).

We say \( \Gamma \) is smooth if all of its points are smooth. A point \( p \in \Gamma \) is a node at infinity if \( \gamma_\Gamma(p) = 0 \), \( \text{val}(p) = 2 \), and \( \text{PL}_\Gamma^\text{fin} = \mathbb{R} \). We say that \( \Gamma \) is nodal if every point in \( \Gamma \) is either smooth or a node at infinity.

On a nodal tropical curve, the affine structure is uniquely determined by the underlying TPL-curve. We say that a TPL-curve is smooth (nodal) if it is the underlying TPL-curve of a smooth (nodal) tropical curve.

Remark 3.3. For \( d \geq 2 \), define \( C_{r,d} \) as the union in \( \mathbb{R}^{d-1} \) of the open rays \( \{ t e_i | 0 \leq t < r \} \) for \( 0 \leq i \leq d - 1 \), where \( e_i \) is the \( i \)-th standard basis vector for \( i > 0 \), and \( e_0 = -\sum_{i=1}^{d-1} e_i \); then, a point \( p \) with \( \gamma_\Gamma(p) = 0 \) of a tropical curve \( \Gamma \) is smooth if and only if \( p \) has a
neighborhood isomorphic to either $C_{r,d}$ for some $r \in \mathbb{R}_{>0}$ and $d \geq 2$, or to $(s,\infty]$ for some $s \in \mathbb{R}$. A point of $\Gamma$ is a node at infinity if an only if it is not smooth and has a neighborhood isomorphic to $(r,\infty] \cup (\infty, r, \infty] \subseteq T^2$ for some $r \in \mathbb{R}$.

**Definition 3.4.** Let $I$ be a finite set. An $I$-marked semi-stable tropical (resp. TPL) curve is a tuple $(\Gamma, (s_i)_{i \in I})$ such that $\Gamma$ is a compact connected nodal tropical (resp. TPL) curve, and $i \mapsto s_i$ is a bijection from $I$ to the set of infinite smooth points of $\Gamma$. An $I$-marked semi-stable tropical (resp. TPL) curve is called stable if every irreducible component of $\Gamma$ of genus zero contains at least three special points (marked points or nodes at infinity), and every irreducible component of $\Gamma$ of genus one contains at least one special point.

For the purposes of this paper a graph $G$ is a tuple

$$G = (V(G), F(G), L(G) \subseteq E_\infty(G) \subseteq E(G), v_G, e_G, \gamma_G),$$

where $V(G)$, $F(G)$, $E(G)$, are finite sets, the sets of vertices, flags, and edges, respectively; $v_G : F(G) \to V(G)$ is a map assigning to every flag its adjacent vertex, $e_G : F(G) \to E(G)$ is a surjective map with fibers of cardinality at most two, and $\gamma_G : V(G) \to \mathbb{N}$ is a map assigning to every vertex its genus. The set $L(G)$ of legs of $G$ is the subset of $E(G)$ consisting of edges whose fiber under $e_G$ has cardinality one. The subset $E_\infty(G)$ that comes with the datum $G$ is the set of unbounded edges of $G$, and we always require that $L(G) \subseteq E_\infty(G)$. We denote the set of bounded edges of $G$ by $E_b(G) = E(G) \setminus E_\infty(G)$. For every vertex $v$ we denote by $T_v G = v_G^{-1}(v)$ its set of adjacent flags and by $\operatorname{val}(v) = \# T_v G$ the valence of $v$. A graph is stable if it is connected and for every vertex $v \in V(G)$ one has

$$\operatorname{val}(v) + 2\gamma_G(v) \geq 3. \tag{8}$$

The genus $g(G)$ of $G$ is given by

$$g(G) = \# E(G) - \# L(G) - \# V(G) + 1 + \sum_{v \in V(G)} \gamma_G(v). \tag{9}$$

For any finite set $I$, an $I$-marked graph is a graph $G$ together with a bijection $I \to L(G)$. In the geometric realization of a graph, we assume the legs to be compact segments, but we do not consider the lone endpoint to be a vertex.

**Definition 3.5.** Let $\Gamma$ be an $I$-marked stable TPL-curve. The combinatorial type of $\Gamma$ is the unique stable graph $G$ whose geometric realization is homeomorphic to the underlying topological graph of $\Gamma$ in such a way that the unbounded edges of $G$ are precisely those edges that contain an infinite point of $\Gamma$. In particular, the set of vertices of $G$ consists of the points of $\Gamma$ which have strictly positive genus or valence strictly greater than two.

Given a stable graph $G$ and a bounded edge $e \in E_b(G)$, one can specialize $e$ in two ways. Namely, one can stretch $e$, which adds $e$ to $E_\infty(G)$, or one can contract $e$, which is achieved by identifying the endpoints of $e$ if it is not a loop, and by deleting $e$
and increasing the genus of its adjacent vertex by one if it is a loop. Any graph obtained from $G$ by a sequence of edge specializations is called a specialization of $G$. A morphism between two stable graphs is given by a sequence of graph specializations and graph isomorphisms.

3.2. Families of tropical curves. The starting point for defining families of tropical curves is the cone over an $I$-marked stable graph $G$, as introduced in [CCUW17, Definition 4.1]. In our case, we choose the base to equal the extended cone $\sigma_G = R_{\leq 0}^{E_b(G)}$. We recall the construction in this context.

For every edge $e \in E_b(G)$, denote by $l_e$ the coordinate on $\sigma_G$ corresponding to $e$. For every $v \in V(G)$, define $\sigma_{G,v} = \sigma_G$. For every bounded edge $e \in E_b(G)$, define $\sigma_{G,e}$ as the extended cone of $\sigma_G = \sigma_{G,e} = \{(x,y) \in E^{E_b(G)} \times R_{\geq 0} | y \leq l_e(x)\}$.

If $v_1$ and $v_2$ are the vertices of $e$, then the two faces of $\sigma_{G,e}$ given by the extended cones of the faces $\{ (x,0) \in \sigma_{G,e} \}$ and $\{ (x,y) \in \sigma_{G,e} | y = l_e(x) \}$ of $\sigma_{G,e}$ can be naturally identified with $\sigma_{G,v_1}$ and $\sigma_{G,v_2}$, respectively. For every leg $l \in L(G)$ define $\sigma_{G,l} = \sigma_G \times R_{\geq 0}$.

The cone $\sigma_{G,(v)} \cong \sigma_G \times \{0\}$ is naturally identified with a face of $\sigma_{G,l}$. Finally, for every $e \in E_{\infty}(G) \setminus L(G)$, we define $\sigma_{G,e} = \sigma_G \times (R_{\geq 0} \times \{0\} \cup \{\infty\})$. If $v_1$ and $v_2$ are the vertices of $e$, then the two subsets $\sigma_G \times \{0\}$ and $\sigma_G \times \{\infty\}$ of $\sigma_{G,e}$ can be naturally identified with $\sigma_{G,v_1}$ and $\sigma_{G,v_2}$.

Denote by $\pi_G : \mathcal{C}_G \to \sigma_G$ the TPL-space over $\sigma_G$ obtained by gluing the cones $\sigma_{G,v}, \sigma_{G,e}, \sigma_{G,l}$ according to how edges and vertices are glued in $G$. Define a function $\gamma_{\mathcal{C}_G} : \mathcal{C}_G \to \mathbb{N}$ by

$$\gamma_{\mathcal{C}_G} (x) = \sum_{v \in V(G)} \gamma_G (v) \cdot 1_{\sigma_{G,v}} (x) + \dim \left( \lim_{x \to \infty} H_1 \left( \cup \cap \pi_{G^{-1}}(\mathcal{C}_G); \mathbb{Q} \right) \right),$$

where $1_{\sigma_{G,v}}$ denotes the indicator function associated to $\sigma_{G,v}$ and the direct limit is taken over all neighborhoods $U$ of $x$.

By construction, the fiber of $\pi_G : \mathcal{C}_G \to \sigma_G$ over a point $x \in \text{relint}(\sigma_G)$, together with the restriction of $\gamma_{\mathcal{C}_G}$, is a stable TPL-curve whose combinatorial type can be identified with $G$ in such a way that the length of the edge of $\pi_G^{-1}(x)$ corresponding to $e \in E_b(G)$ is given by $l_e(x)$.

Any morphism $G \to G'$ of stable graphs induces a morphism $\overline{\sigma}_G \to \overline{\sigma}_G'$ of extended cones that identifies $\overline{\sigma}_G$ with a (potentially infinite) face of $\overline{\sigma}_G'$. By construction, the TPL-space $\mathcal{C}_G$ can be naturally identified with the TPL-space $\mathcal{C}_{G'} \times \overline{\sigma}_{G'} \to \overline{\sigma}_G$ in a way that respects the projection to $\sigma_G$.

**Definition 3.6.** Let $B$ be a TPL-space and $n$ a non-negative integer. A family of $n$-marked genus-$g$ stable TPL-curves over $B$ is a tuple $(\pi : \mathcal{C} \to B, \{s_i\}_{i \in [n]} ; \gamma_{\mathcal{C}})$ consisting of a morphism $\pi : \mathcal{C} \to B$, a section $s_i$ of $\pi$ for every $i \in [n]$, and a genus function $\gamma_{\mathcal{C}} : \mathcal{C} \to \mathbb{N}$ such that
(1) for every \( b \in B \), the fiber \( C_b := \{ b \} \times_B \mathcal{C} \), together with the restrictions of the sections and the genus function, is an \( n \)-marked genus-\( g \) stable TPL-curve.

(2) for every \( b \in B \) there exists a local face structure \( \Sigma \) of \( B \) at \( b \) such that for every \( \sigma \in \Sigma \) there exists an \( n \)-marked genus-\( g \) stable graph \( G_{\sigma} \), a linear morphism \( f_\sigma : \sigma \to \pi_{G_\sigma} \) with
\[
\begin{align*}
  f_\sigma(\text{relint}(\sigma)) \subseteq \text{relint}(\sigma_{G_\sigma}),
\end{align*}
\]
and an isomorphism
\[
\begin{align*}
  \chi_\sigma : C \times_B \mathcal{C} \xrightarrow{\cong} C_{G_\sigma} \times_B \mathcal{C}_\sigma
\end{align*}
\]
respecting the induced genus functions and the induced sections.

The datum \((\Sigma, (G_\sigma, f_\sigma, \chi_\sigma)_{\sigma \in \Sigma})\) is called a \textbf{TPL-trivialization} of the family \( \mathcal{C} \to B \) at \( b \).

It follows from the definition that if \( \pi : \mathcal{C} \to B \) is a family of \( n \)-marked genus-\( g \) stable TPL-curves and \( B' \to B \) is a morphism of TPL-spaces, then \( \mathcal{C} \times_B B' \to B' \), with the induced sections and genus function, is a family of \( n \)-marked genus-\( g \) stable TPL-curves again. Therefore, the following definition makes sense:

\textbf{Definition 3.7.} We denote by \( \mathcal{M}_{g,n}^{\text{TPL}} \) the category fibered in groupoids over the category of TPL-spaces with
\[
\mathcal{M}_{g,n}^{\text{TPL}}(B) = \{\text{families of } n \text{-marked genus-} g \text{-stable TPL-curves over } B\}
\]
for any TPL-space \( B \). Considering the category of TPL-spaces as a site, as outlined in Convention 2.17, the fibered category \( \mathcal{M}_{g,n}^{\text{TPL}} \) is a stack.

Since the affine functions on smooth tropical curves are all harmonic, the following definition and lemma will be of great importance to us:

\textbf{Definition 3.8.} Let \( \pi : \mathcal{C} \to B \) be a family of stable TPL-curves. The \textbf{sheaf of fiberwise harmonic functions}, denoted by \( H_{\mathcal{C}} \), is the subsheaf of \( \mathcal{L}_{\mathcal{C}}^{\text{fin}} \) whose sections, when restricted to any fiber, are harmonic.

\textbf{Lemma 3.9.} Let \( \pi : \mathcal{C} \to B \) be a family of stable TPL-curves, let \( x \in C \) be a point with \( \gamma_\mathcal{C}(x) = 0 \), and let \( b = \pi(x) \). Then the sequence
\[
\begin{align*}
  0 \to \mathcal{L}_{\mathcal{C},b}^{\text{fin}} \to H_{\mathcal{C},x} \to H_{\mathcal{C},x}/R \to 0
\end{align*}
\]
is exact.

\textbf{Proof.} It is clear that (15) is a complex and that it is exact on the left. To see exactness in the middle, given a function \( \phi \) on an open neighborhood \( W \) of \( x \) such that \( \phi|_{W_b} \) is constant, we show that \( \phi \) must be constant along fibers for all points \( c \) in a neighborhood of \( b \). Using a TPL-trivialization at \( b \), one may construct for every \( d \in T_x C_b \) a local section \( s_d \) of \( \pi \) such that that \( s_d(b) \) is a point arbitrarily close to \( b \) in the direction given by \( d \), and \( s_d \) does not intersect the locus of vertices of the fibers of \( \mathcal{C} \). We denote by \( C \) a connected neighborhood of \( b \) contained in the common locus of definition of the sections \( s_d \).
Let \( U \) be the connected component of \( (W \cap \pi^{-1}(C)) \setminus \bigcup_{d \in T_x C_b} s_d(B) \) containing \( x \). After potentially shrinking \( C \), the fiber \( \overline{U}_c \) of a point \( c \in C \) is a tree whose leaves are precisely the points \( s_d(c) \), \( d \in T_x C_b \). Therefore, the restriction \( \phi|_{U_c} \) is uniquely determined by the slopes of \( \phi|_{W_c} \) at the points \( s_d(c) \), \( d \in T_x C_b \), and by its value at \( s_{d_0}(c) \) for some fixed \( d_0 \in T_x C_b \). For any \( d \in T_x C_b \), the slope of \( \phi|_{W_c} \) at \( s_d(c) \) is the same for all \( c \in C \) since \( C \) is connected and slopes take values in a discrete set. As the slope of \( \phi|_{W_b} \) is zero everywhere, it follows that \( \phi \) is constant on all fibers in \( C \), and therefore that \( \phi = \pi^*(s^*_d \phi) \) on \( U \). This proves the exactness in the middle.

With all notation as above, we observe that given a harmonic function \( \xi \) on \( W_b \), we can define a fiberwise harmonic function \( \tilde{\xi} \) on \( \overline{U} \) by defining it as a fiber \( \prod_{c} \) as the unique harmonic function with \( \tilde{\xi}(s_{d_0}(c)) = \xi(s_{d_0}(b)) \) and slope at \( s_d(c) \) equal to the slope of \( \xi \) at \( s_d(b) \) for all \( d \in T_x C_b \). By construction, one then has \( \tilde{\xi}|_{\overline{U}_b} = \tilde{\xi}|_{\overline{U}_b} \), which proves exactness on the right.

**Definition 3.10.** Let \( B \) be a tropical space. A family of \( n \)-marked genus-\( g \) stable tropical curves over \( B \) is a family \( \pi: \mathcal{C} \to B \) of \( n \)-marked genus-\( g \) stable TPL-curves, together with an affine structure on \( \mathcal{C} \) making \( \pi \) linear, and such that for every \( b \in B \) the fiber \( \mathcal{C}_b = \{ b \} \times_B \mathcal{C} \) is a stable tropical curve; further we require that the sequence

\[
0 \to \Omega^1_{B,b} \to \Omega^1_{\mathcal{C}|B \times B} \to \Omega^1_{\mathcal{C}_b} \to 0 \tag{16}
\]

is an exact sequence of sheaves on \( \mathcal{C}_b \).

**Remark 3.11.** The exactness of the sequence displayed in (16) is equivalent to asking that every affine function on \( \mathcal{C} \) whose restriction to the fiber over \( b \) is constant is locally a pull-back of an affine function defined on a neighborhood of \( b \) in \( B \).

**Remark 3.12.** As a consequence of Lemma 3.9, every family of stable TPL-curves \( \pi: \mathcal{C} \to B \) can be equipped with affine structures on base and total space to yield a family of stable tropical curves. More precisely, the lemma proves that if \( \tilde{H}_{\mathcal{C}} \) denotes the affine structure on \( \mathcal{C} \) with stalks

\[
\tilde{H}_{\mathcal{C},x} = \begin{cases} 
H_{\mathcal{C},x} & \text{if } \gamma_{\mathcal{C}}(x) = 0 \\
\text{PL}_{B,\pi(x)}^{\text{fin}} & \text{if } \gamma_{\mathcal{C}}(x) > 0
\end{cases}
\]

then \( (\mathcal{C}, \tilde{H}_{\mathcal{C}}) \to (B, \text{PL}_{B}^{\text{fin}}) \) is a family of stable tropical curves. Families of stable tropical curves obtained in this way are not particularly useful from the point of view of tropical intersection theory; there are "too many" affine functions on the base to have positive-dimensional tropical cycles on \( (B, \text{PL}_{B}^{\text{fin}}) \) (see (84) for the definition of balancing used to define tropical cycles on tropical spaces).

**Example 3.13.** For any integer \( a \), we describe a family \( \mathcal{C}^a \) of elliptic curves over \( \mathbb{T}^1 = \mathbb{R} \); refer to Figure 1 to keep up with the notation. As a cone complex, this is the union of four extended plane orthants \([0, \infty) \times [0, \infty] \), denoted \( L_+, \tilde{T}_+ \). We denote by \((u_\pm, v_\pm) \) (integral) coordinate functions for \( L_\pm \), and by \((x_\pm, y_\pm) \) coordinate functions for \( \tilde{T}_\pm \); we denote by \( x \) a coordinate on the base \( \mathbb{T}^1 \).
Figure 1. A neighborhood of the fiber $C_0$ of the family $C^a$ in the case $a = 1$. Using the affine structure, the finite part of $T_+ \cup T_-$, indicated in gray, can be embedded into the plane. This embedding depends on $a$, as illustrated by the integral lattice depicted in a neighborhood of $v$.

The finite one dimensional faces of $L_-$ are identified, and glued with the horizontal axis of $T_-$. (and similarly for the cones labeled by $+$. The finite vertical axes of $T_-$ and $T_+$ are identified. We define a genus function $\gamma_{C^a} : C^a \to \mathbb{Z}$ to have value one at the point $v$ of $C^a$ obtained gluing the four $(0,0)$-vertices of the extended cones, and zero elsewhere.

The map $\pi : C^a \to \mathbb{TP}^1 = [-\infty, \infty]$ defined by

\[
\begin{align*}
\pi|_{T_-}(x_-, y_-) &= -x_- , \\
\pi|_{L_-}(u_-, v_-) &= -u_- - v_- , \\
\pi|_{T_+}(x_+, y_+) &= x_+ , \\
\pi|_{L_+}(u_+, v_+) &= u_+ + v_+ ,
\end{align*}
\]

(17)

has tropical elliptic curves for fibers.

To make the extended cone complex $C^a$ into a tropical space, we endow it with a sheaf of affine linear functions. Let $\widetilde{\text{Aff}}_{C^a}$ denote the subsheaf of $\text{PL}_{fin}^{a}$ consisting of all piecewise linear function whose restriction to any fiber is harmonic and whose pull-backs to $T_\pm$ and $L_\pm$ are affine, where we consider $[0, \infty] \times [0, \infty]$ with its standard affine structure. One quickly checks that with this affine structure, the projection $\pi$ is linear, all fibers of $\pi$ are stable tropical curves, and the sequence (16) is exact away from the central fiber $C_0^a$. The sequence is not exact on the central fiber because the horizontal slopes of a function in $\widetilde{\text{Aff}}_{C^a}$ defined in a neighborhood of a point $x \in C_0^a$
in the positive (on $T_+$) and negative (on $T_-$) directions are independent. To fix this, one needs to replace $\mathcal{A}(\mathcal{F}_{\mathcal{C}})$ by a subsheaf $\mathcal{A}(\mathcal{F}_{\mathcal{C}})$ in which the transition from $T_+$ to $T_-$ is specified:
\[ x_+ = -x_- \quad y_+ = y_- - ax_-, \] (18)
for some $a \in \mathbb{Z}$; different values of $a$ yield different affine structures on the same underlying TPL-space, that may be interpreted as different local embeddings of $T_- \cup T_+$ into $T^2$: the value $a$ measures the difference of the slopes of the images of the horizontal axes, as shown in Figure 1.

**Lemma 3.14.** Let $\pi: \mathcal{C} \to B$ be a family of 1-marked genus-$g$ stable tropical curves and let $f: B' \to B$ be a morphism of tropical spaces. If $\mathcal{C}' = B' \times_B \mathcal{C}$ and $\pi': \mathcal{C}' \to B'$ is the projection, then $\pi'$, together with the induced sections and genus function, is a family of 1-marked genus-$g$ stable tropical curves.

**Proof.** It suffices to prove for every $b \in B'$ the exactness of the sequence displayed in (16). Let $x \in \mathcal{C}'_b$ and let $\phi$ be an affine function defined in a neighborhood of $x$ in $\mathcal{C}'$ such that $\phi$ is constant on a neighborhood of $x$ in $\mathcal{C}'_b$. Let $f': \mathcal{C}' \to \mathcal{C}$ denote the projection. By Proposition 2.19, we may assume that $\phi = f'^*\phi$ for an affine function $\phi$ defined in a neighborhood of $f'(x)$ in $\mathcal{C}$. By assumption, $\phi$ is constant on a neighborhood of $f'(x)$ in $\mathcal{C}'_b$. Since $\pi: \mathcal{C} \to B$ is a family of tropical curves, it follows that $\phi = \pi^*\chi$ for some affine function $\chi$ defined on a neighborhood of $f(b)$ in $B$. It follows that $\phi = \pi'^*\chi$ in a neighborhood of $x$, finishing the proof. \[ \square \]

**Example 3.15.** Let $\mathbb{Z} \times \mathbb{Z}$ act on $\mathbb{R}^2$ via $(a, b) \cdot (x, y) = (a + x, (-1)^a(b + y))$ and let $\mathcal{C} = \mathbb{R}^2/\mathbb{Z} \times \mathbb{Z}$ be the quotient (homeomorphic to a Klein bottle), equipped with the induced affine structure. Let $B = \mathbb{R}/\mathbb{Z}$, also equipped with the induced affine structure, let $\pi: \mathcal{C} \to B$ denote the morphism with $\pi([x, y]) = [x, 0]$, and let $s: B \to C$ denote the section of $\pi$ with $s([x, 0]) = (x, 0)$. Attaching to $C$ a copy of $\mathbb{R}_{\geq 0} \times B$ along $s(B)$ we obtain an isotrivial family $\mathcal{C} \to B$ of one-marked genus one stable TPL-curves over $B$. Define an affine structure on $\mathcal{C}$ by declaring a piecewise linear function $\phi$ on $\mathcal{C}$ to be affine if it is harmonic on all fibers, $\phi|_{\mathcal{C}(\mathbb{R})} = \phi$ on $C$, and $\phi|_{\mathbb{R}_{\geq 0} \times B}$ is affine on $\mathbb{R}_{\geq 0} \times B$. One checks that with this affine structure on the total space, $\mathcal{C} \to B$ is an isotrivial family of one-marked genus one stable tropical curves. Because $\mathcal{C}$ is not homeomorphic to a trivial $S^1$-bundle over $B$, the family $\mathcal{C} \to B$ is not isomorphic to a trivial family, neither as a family of TPL-curves, nor as a family of tropical curves. However, the pull-back of $\mathcal{C} \to B$ along a connected double cover $B' \to B$ is isomorphic to a trivial family.

With Lemma 3.14 established, we can make the following definition:

**Definition 3.16.** We denote by $\mathcal{M}_{g,n}$ the category fibered in groupoids over the category of tropical spaces with $\mathcal{M}_{g,n}(B) = \{\text{families of } n\text{-marked genus-}g \text{ stable tropical curves over } B\}$ for any tropical space $B$. Considering the category of tropical spaces as a site, as outlined in Convention 2.20, the fibered category $\mathcal{M}_{g,n}$ is a stack.
3.3. Representability of the diagonal. Recall that for any category \( \mathcal{M} \) fibered in groupoids over a base category \( \mathcal{S} \) with finite products, the following three conditions are equivalent:

1. The diagonal \( \Delta_\mathcal{M}: \mathcal{M} \to \mathcal{M} \times \mathcal{S} \mathcal{M} \) is representable.
2. For any two objects \( U \) and \( V \) of \( \mathcal{S} \) and morphisms \( U \to \mathcal{M} \) and \( V \to \mathcal{M} \), the fiber product \( U \times_{\mathcal{M}} V \) is representable by an object in \( \mathcal{S} \).
3. For any object \( U \) of \( \mathcal{S} \) and any two elements \( \xi_1 \) and \( \xi_2 \) of \( \mathcal{M}(U) \), the presheaf \( \text{Isom}_U(\xi_1, \xi_2) \) on \( (\mathcal{S}/U) \) that assigns to \( T \to U \) the set of isomorphisms in \( \mathcal{M}(T) \) from \( T^\ast \xi_1 \) to \( T^\ast \xi_2 \) is representable.

**Proposition 3.17.** For every pair \( g, n \in \mathbb{Z}_{\geq 0} \) of nonnegative integers with \( 2g - 2 + n > 0 \), the stack \( \mathcal{M}_{g,n}^{\text{TPL}} \) over the category of TPL-spaces has representable diagonal.

**Proof.** We need to show that given two families \( \mathcal{C}_1 \to B \) and \( \mathcal{C}_2 \to B \) of \( n \)-marked genus-\( g \) stable TPL-curves over a base \( B \), the functor \( \text{Isom}_B(\mathcal{C}_1, \mathcal{C}_2) \) is represented by a TPL-space over \( B \). Let \( b \in B \), and for \( i \in \{1, 2\} \) let \( (\Sigma_i, (G_{\sigma,i}, f_{\sigma,i}, \chi_{\sigma,i}))_{\sigma \in \Sigma_i} \) be a TPL-trivialization of \( \mathcal{C}_i \to B \) at \( b \). After taking a common refinement of the local face structures \( \Sigma_1 \) and \( \Sigma_2 \) at \( b \), we may assume that \( \Sigma_1 = \Sigma_2 \) and we denote this local face structure by \( \Sigma \). Since TPL-spaces can be glued along open subsets, we may assume that \( B = |\Sigma| \).

Consider the set \( \Delta \) consisting of all pairs \( (\sigma, \phi) \), where \( \sigma \in \Sigma \) and \( \phi: G_{\sigma,1} \to G_{\sigma,2} \) is an isomorphism of marked graphs. If \( \tau \) is a face of \( \sigma \), then \( G_{\tau,1} \) and \( G_{\tau,2} \) are obtained from \( G_{\sigma,1} \) and \( G_{\sigma,2} \), respectively, via edge specializations. An isomorphism \( \phi: G_{\sigma,1} \to G_{\sigma,2} \) induces an isomorphism \( G_{\tau,1} \to G_{\tau,2} \) if and only if \( \phi \) maps contracted (resp. stretched) edges to contracted (resp. stretched) edges. We define a partial order \( \preceq \) on \( \Delta \) by stipulating that \( (\tau, \phi) \preceq (\sigma, \phi) \) if and only if \( \tau \) is a face of \( \sigma \) and \( \phi \) is induced by \( \phi \). For every \( \delta = (\tau, \phi) \in \Delta \), define \( \phi_\delta := \tau \). Let \( T \) be the TPL-space obtained by gluing the cones \( \sigma_\delta \) for every \( \delta \in \Delta \) according to the partial order on \( \Delta \), that is we define \( T \) as

\[
T = \lim_{\delta \in \Delta} \sigma_\delta .
\]  
(19)

For every element \( \delta = (\tau, \phi) \) of \( \Delta \), the isomorphism \( \phi \) defines an isomorphism \( \phi_G: \varpi_{G_{\tau,1}} \to \varpi_{G_{\tau,2}} \). The set

\[
P_\delta = \{ x \in \tau \mid \phi_G(f_{\tau,1}(x)) = f_{\tau,2}(x) \}
\]  
(20)

is a polyhedron in \( \tau = \sigma_\delta \). Let

\[
I = \bigcup_{\delta \in \Delta} P_\delta \subseteq T ,
\]  
(21)

and let \( g: I \to B \) denote the natural morphism of TPL-spaces induced by the projection \( T \to |\Sigma| \). By construction, the isomorphisms \( \chi_{\sigma,i} \) induce an isomorphism \( \chi: g^\ast \mathcal{C}_1 \to g^\ast \mathcal{C}_2 \) of TPL-spaces such that for \( \delta = (\tau, \phi) \in \Delta \) and \( x \in \text{relint}(\delta) \) the combinatorial type of the restriction of \( \chi \) to the fiber over \( x \) is given by \( \phi \).

To finish the proof, we need to show that \( (I, g, B, \chi) \) represents \( \text{Isom}_B(\mathcal{C}_1, \mathcal{C}_2) \). Let \( B' \to B \) be a morphism of TPL-spaces, and let \( \xi: h^\ast \mathcal{C}_1 \to h^\ast \mathcal{C}_2 \) be an isomorphism of
families of stable TPL-curves. Let \( b' \in B' \) and let \( \Theta \) be a local face structure at \( b' \) such that \( h \) maps the polyhedra of \( \Theta \) linearly into polyhedra of \( \Sigma \). For \( \emptyset \in \Theta \), let \( \sigma \in \Sigma \) be the minimal polyhedron in \( \Sigma \) containing \( h(\emptyset) \). Then for every \( x \in \text{relint}(\emptyset) \), the fiber \( (h^*e_1)_x \) has combinatorial type \( G_{\sigma,i} \). Furthermore, the isomorphism \( (h^*e_1)_x \to (h^*e_2)_x \) induced by \( \zeta \) is induced by the same isomorphism \( \zeta_\emptyset : G_{\sigma,1} \to G_{\sigma,2} \) of stable graphs for all \( x \in \text{relint}(\emptyset) \). It follows that the lift \( k_\emptyset : \emptyset \to \text{relint}(\emptyset) \) of \( \emptyset \) is the unique lift of \( \emptyset \) to \( I \) via which \( \chi \) induces the restriction \( \zeta_\emptyset : (h^*e_1)_\emptyset \to (h^*e_2)_\emptyset \) of \( \zeta \) to \( \emptyset \). By the uniqueness of \( k_\emptyset \), the morphisms \( k_\emptyset \) glue to a unique lift \( k : B' \to I \) of \( h \) via which \( \chi \) induces \( \zeta \).

**Theorem 3.18.** For every pair \( g,n \in \mathbb{Z}_{\geq 0} \) of non-negative integers with \( 2g - 2 + n > 0 \), the diagonal of the stack \( \mathcal{M}_{g,n} \) is representable.

**Proof.** We need to show that given two families \( \mathcal{E}_1 \to B \) and \( \mathcal{E}_2 \to B \) of \( n \)-marked genus-\( g \) stable tropical curves over a base \( B \), the functor \( \text{Isom}_B(\mathcal{E}_1, \mathcal{E}_2) \) is represented by a tropical space over \( B \). Since families of stable tropical curves are families of stable TPL-curves, we can apply Proposition 3.17 and obtain a morphism of TPL-spaces \( J \to B \) and a universal isomorphism \( \chi : g^*\mathcal{E}_1 \to g^*\mathcal{E}_2 \). We equip \( J \) with the minimal affine structure such that \( g \) becomes linear, that is we let \( \text{Aff}_J \) consist of all functions that are locally pull-backs of affine functions on \( B \). Denote \( g^*\mathcal{E}_i \) by \( \mathcal{E}'_i \), where the pull-back is taken in the category of tropical spaces, and let \( \pi'_i : \mathcal{E}'_i \to J \) be the projections. Let \( y \in \mathcal{E}'_1 \), and let \( l' \) be an affine function on \( \mathcal{E}'_2 \) defined on a neighborhood of \( \chi(y) \). Since the restriction \( (\mathcal{E}'_1)_{\pi'_1(y)} \to (\mathcal{E}'_2)_{\pi'_2(y)} \) of \( \chi \) to the fiber over \( \pi'_1(y) \) is linear, there exists an affine function \( l \) on a neighborhood of \( y \) in \( \mathcal{E}'_1 \) such that the germs of the restrictions of \( l \) and \( \chi^*l' \) to \( (\mathcal{E}'_1)_{\pi'_1(y)} \) at \( y \) coincide. Since both \( l \) and \( \chi^*l' \) are harmonic on all fibers, by Lemma 3.9 there exists a piecewise linear function \( m \) on a neighborhood of \( \pi'_1(y) \) such that the germ of \( \chi^*l' - l \) coincides with \( \pi'_1*m \) on a neighborhood of \( y \). Let \( \text{Aff}_I \) be the sub-sheaf of \( \text{PL}_I^\text{fin} \) generated by \( \text{Aff}_I \) and all piecewise linear functions \( m \) obtained this way, let \( I \) be the tropical space obtained by replacing \( \text{Aff}_I \) by \( \text{Aff}_I \), and let \( f : I \to B \) be the linear morphism induced by \( g \). Then by construction, \( \chi \) induces an isomorphism \( \varphi : f^*\mathcal{E}_1 \to f^*\mathcal{E}_2 \) of families of stable tropical curves. It follows directly from the universal property of \( J \) and \( \chi \) and the construction of \( \text{Aff}_I \) that \( (I, \varphi) \) represents \( \text{Isom}_B(\mathcal{E}_1, \mathcal{E}_2) \).

**Example 3.19.** Let \( a, b \in \mathbb{Z} \). Example 3.13 yields two families \( \mathcal{E}_a \) and \( \mathcal{E}_b \) over \( \text{TP}^1 \) of one-marked genus-one stable tropical curves. Let \( I = \text{Isom}_{\text{TP}^1}(\mathcal{E}_a, \mathcal{E}_b) \), \( f : I \to \text{TP}^1 \) the structure morphism, and \( \chi : f^*\mathcal{E}_a \to f^*\mathcal{E}_b \) the universal isomorphism. For every \( x \in \text{TP}^1 \setminus \{0\} \), there are two isomorphisms \( \mathcal{E}_a^x \to \mathcal{E}_b^x \), whereas there exists one isomorphism \( \mathcal{E}_a^0 \to \mathcal{E}_b^0 \). According to the proof of Proposition 3.17, the underlying TPL-space of \( I \) is given by two copies of \( \text{TP}^1 \) glued together at their respective origins. Let \( 0' \) denote the unique preimage under \( f \) of \( 0 \). From the description of the affine structures on \( \mathcal{E}_a \) and \( \mathcal{E}_b \) in Example 3.13 and from the description of the affine structure on \( I \) in the proof of Theorem 3.18 one concludes that the restriction \( I \setminus \{0'\} \to \text{TP}^1 \) of \( f \) is a
local isomorphism. To determine the affine structure at $0'$ denote by $\phi_c$, for $c \in (a, b)$, the function on a neighborhood in $C^c$ of the finite genus-zero points of $C^c_0$ whose restriction to $T_x$ is given by $y_c$, notation as in Example 3.13. Denoting by $f^*\phi_c$ the pull-back of $\phi_c$ to $f^*C^c$, the two functions $f^*\phi_a$ and $\chi^*f^*\phi_b$ are both defined on a neighborhood in $f^*C^c$ of the finite genus-zero points of the central fiber $(f^*C^c)_0'$, and their restrictions to the central fiber have the same slope. Therefore, the difference $f^*\phi_a - \chi^*f^*\phi_b$ determines an affine function $m$ on a neighborhood of $0'$ in $I$. The horizontal slope of $\phi_c$ are $0$ over $\mathbb{R}_{\geq 0}$ and $-c$ over $\mathbb{R}_{\leq 0}$. It follows that $m$ has slope $0$ on the two rays of $I$ mapping to $\mathbb{R}_{\geq 0}$ and slope $b - a$ on the two rays of $I$ mapping to $\mathbb{R}_{\leq 0}$. The pair of functions $(x, m)$ defines a map $g : I \to S \subset TP \times TP$, with

$$S = \mathbb{R}_{\geq 0} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \cup \mathbb{R}_{\geq 0} \left( \begin{array}{c} -1 \\ b - a \end{array} \right)$$

such that $f$ is given by the composite of $g$ with the projection to the first coordinate, $g$ is a double cover away from $0'$, and the affine functions on $I$ are the pull-backs via $g$ of the affine functions on $S$.

3.4. Affine functions on families of curves. In order to describe the affine structures of the moduli spaces $\mathcal{M}_{g,n}$, we need to have good control over the affine structures on the total spaces of families of stable tropical curves. By definition, affine functions on the total space of a family are fiberwise harmonic. In what follows, we investigate under which conditions a fiberwise harmonic function is affine.

**Lemma 3.20.** Let $\pi : C \to B$ be a family of stable tropical curves, and let $x \in C$ be a point that is contained in an open edge of $C_{\pi(x)}$. Then there exists an $\epsilon > 0$ and an open neighborhood $U$ of $\pi(x)$ such that $x$ has a neighborhood in $C$ which is isomorphic to $U \times (0, \epsilon)$ over $B$.

**Proof.** Using a TPL-trivialization, we find a neighborhood $W$ of $\pi(x)$ and a $\delta > 0$ such that $x$ has an open neighborhood $V$ in $C$ allowing an isomorphism $f : V \to W \times (0, \delta)$ of TPL-spaces over $B$. Equipping $W \times (0, \delta)$ with the affine structure induced by $f$, call it Aff', the map $f$ becomes an isomorphism of tropical spaces over $B$. Since $\pi : C \to B$ is a family of tropical curves, there exists an integral affine function $\phi$ at $x$ that has slope one on $0(0, \delta) \equiv (\pi(x)) \times (0, \delta)$. After potentially shrinking $V$, $W$ and $\delta$, we can assume that $\phi$ is defined all of $W \times (0, \delta)$. Because $\phi$ is harmonic on all fibers, it follows that $\phi$ has slope one on all fibers $\{w\} \times (0, \delta)$ for $w \in W$. Therefore, $\phi$ induces a linear map

$$W \times (0, \delta) \xrightarrow{id_W \times \phi} W \times \mathbb{R}$$

that is an isomorphism of TPL-spaces onto an open subset $V'$ of $W \times \mathbb{R}$. Since the affine structure Aff' is generated by $\phi$ and the pull-backs of affine function from $W$, it follows that $id_W \times \phi$ is an isomorphism of tropical spaces onto $V'$. The proof is concluded by observing that $(\pi(x), \phi(f(x)))$ has a neighborhood in $V'$ isomorphic to $U \times (0, \epsilon)$ for some $0 < \epsilon < \delta$ and some neighborhood $U$ of $\pi(x)$ in $W$. □
Lemma 3.21. Let $\pi: \mathcal{C} \to B$ be a family of stable tropical curves, let $U \subseteq \mathcal{C}$ be an open subset such that $\gamma|_U = 0$, let $\phi \in \Gamma(U, H_{\mathcal{C}})$, and let $b \in B$. Then the set
\[ V = \{ x \in \mathcal{C}_b \cap U \mid \phi_x \in \text{Aff}_{\mathcal{C},b} \} \] (24)
is open and closed in $\mathcal{C}_b \cap U$.

Proof. Refer to Figure 2 throughout this proof. It is clear that $V$ is open in $\mathcal{C}_b \cap U$. To show that it is closed, let $x \in V \cap U$. By Definition 3.2, since $\phi$ restricts to a harmonic function on $\mathcal{C}_b$, it defines a one-form on $\mathcal{C}_b$; by the short exact sequence of cotangent sheaves (16), such a form admits a lift, that is there exists $m \in \text{Aff}_{\mathcal{C},b}$ such that the restrictions of $m$ and $\phi$ to $\mathcal{C}_b$ agree in a neighborhood of $x$ in $\mathcal{C}_b$. By Lemma 3.9, there exists $\phi_b \in \text{PL}_{\mathcal{C},b}$ such that $\phi_x = m_x + (\pi^* \phi)_x$. We need to show that $\phi_b \in \text{Aff}_{B,b}$. Since $x \in V$, there exists a point $y \in V$ that is contained in the interior of an edge of $\mathcal{C}_b$ and such that $m$ is defined at $y$ and $\phi_y - m_y = (\pi^* \phi)_y$. By Lemma 3.20, there exists a linear section $s: W \to U$ of $\pi$ in a neighborhood $W$ of $b$ such that $s(b) = y$. We see that
\[ \phi_b = s^*(\pi^* \phi)_y = s^* \phi_y - s^* m_y \in \text{Aff}_{B,b}. \] (25)

Lemma 3.22. Let $\pi: \mathcal{C} \to B$ be a family of stable tropical curves and let $s: B \to \mathcal{C}$ be a linear section of $\pi$ such that $s(c)$ is a smooth point of $\mathcal{C}_c$ for all $c \in B$. Moreover, let $b \in B$ such that $\gamma(s(b)) = 0$, and let $\phi \in H_{\mathcal{C},s(b)}$. Then $\phi \in \text{Aff}_{\mathcal{C},s(b)}$ if and only if $s^* \phi \in \text{Aff}_{B,b}$.

Proof. The only if part is clear, so assume that $s^* \phi \in \text{Aff}_{B,b}$. By the exact sequence for cotangent sheaves (16), there exists $m \in \text{Aff}_{\mathcal{C},s(b)}$ such that the restrictions $\phi|_{\mathcal{C}_b}$ and...
m|_{\mathcal{E}_b}$ coincide. By Lemma 3.9, there exists a piecewise linear function $\varphi \in \text{PL}_{B,b}$ such that $\phi - m = \pi^* \varphi$. Since
\[
\varphi = s^* \pi^* \varphi = s^* \phi - s^* m \in \text{Aff}_{B,b}
\] it follows that $\phi \in \text{Aff}_{s, s(b)}$.

**Definition 3.23.** Let $\pi: C \to B$ be a family of stable tropical curves and let $s: B \to C$ be a linear section of $\pi$ such that $s(b)$ is a smooth genus-zero point of $\mathcal{C}_b$ for all $b \in B$. For an integer $k \in \mathbb{Z}$, we define the **sheaf of functions with prescribed order $k$ along $s$**, denoted $\text{Aff}_s^k(ks)$: its sections over an open subset $U \subseteq C$ are functions $m \in \Gamma(U, \text{PL}_C)$, possibly with value $\pm \infty$ on $s(B)$, that are affine away from the support of $s$, that is
\[
m |_{U \setminus s(B)} \in \Gamma(U \setminus s(B), \text{Aff}_s^k),
\]
and such that for all $b \in s^{-1} U$ we have
\[
\sum_{v \in I_b(s, \mathcal{E}_b)} d_v(m|_{\mathcal{E}_b}) = k.
\]

Since the constant sheaf $\mathbb{R}$ acts on $\text{Aff}_s^k(ks)$ additively, we can take the quotient $\Omega^1_C(ks) := \text{Aff}_s^k(ks)/\mathbb{R}$, the **sheaf of tropical one-forms with prescribed order $k$ along $s$**.

If $r \in \Gamma(U, \text{Aff}_s^k(ks))$ and $m \in \Gamma(U, \text{Aff}_s^k)$, then $m + r \in \Gamma(U, \text{Aff}_s^k(ks))$. This endows $\text{Aff}_s^k(ks)$ with an action by $\text{Aff}_s^k$. The following two propositions shows that $\text{Aff}_s^k(ks)$ is in fact an $\text{Aff}_s^k$-torsor.

**Proposition 3.24.** Let $\pi: C \to B$ be a family of stable tropical curves, let $s: B \to C$ be a linear section of $\pi$ such that $s(b)$ is a smooth genus-zero point of $\mathcal{C}_b$ for all $b \in B$, and let $k \in \mathbb{Z}$. Moreover, let $U \subseteq C$ be an open subset and let $l, l' \in \Gamma(U, \text{Aff}_s^k(ks))$. Then there exists a unique $m \in \Gamma(U, \text{Aff}_s^k)$ such that $l' = l + m$.

**Proof.** We refer to Figure 3 to illustrate the proof. Let
\[
N = \{b \in B \mid s(b) \text{ is infinite in } \mathcal{C}_b\}.
\]
The set $s(N)$ closed in $U$ and $V := U \setminus s(N)$ is a dense open subset of $U$. The restrictions of $l$ and $l'$ to $V$ have finite values everywhere, so we can take their difference $m := l|_V - l'|_V$. By definition of $\text{Aff}_s^k(ks)$ and Lemma 3.21, $m$ is affine. It suffices to prove that $m$ extends to a function in $\Gamma(U, \text{Aff}_s^k)$. Since such an extension is unique by the denseness of $V$, we can do this locally at a point in $x \in s(N)$, that is we can shrink $U$ to arbitrary small neighborhoods of $x$. In particular, we may assume that $U$ has connected fibers. Let $b = \pi(x)$, and let $y \in \mathcal{C}_b \cap V$ be a point in the interior of the leg that contains $x$. By Lemma 3.20, there exists a neighborhood $W$ of $b$ and a section $t: W \to V$ with $t(b) = y$. After potentially shrinking $U$ and $W$ we may assume that $\pi(U) = W$ and that there exists a section $u: W \to U$ with $u(b) = x$ and such that $u(b')$ is an infinite point of $\mathcal{E}_{b'}$ on the same leg as $t(b')$ for all $b' \in W$. Let $b' \in W$. Since $l|_{\mathcal{E}_{b'}}$ and $l'|_{\mathcal{E}_{b'}}$ have the same slope in the unique fiber direction leaving from $u(b')$,
the function $m|_{C_b'}$ has slope zero in that direction. It follows that $m|_{C_b'}$ is constant on the edge segment between $t(b')$ and $u(b')$. Since this is true for every $b' \in W$, we see that $m = ((t \circ \pi)^* m)|_V$ on the connected component of $x$ in $U \setminus t(W)$.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{In the situation depicted here, we have $N = \{b\}$. The affine function $m = l - l'$ is constant along all fibers in the gray region, and therefore extends across the section $u$ at infinity.}
\end{figure}

Proposition 3.25. Let $\pi: \mathcal{C} \to B$ be a family of stable tropical curves, let $s: B \to \mathcal{C}$ be a linear section of $\pi$ such that $s(c)$ is a smooth point of $\mathcal{C}_c$ for all $c \in B$, and let $k \in \mathbb{Z}$. Then for every $b \in B$ and $x \in \mathcal{C}_b$, the map

$$\text{Aff}_{\mathcal{C}}(k s)_x \to \text{Aff}_{\mathcal{C}_b}(k \cdot s(b))_x$$

induced by the restriction of functions is surjective.

Proof. If $x \neq s(b)$ the assertion follows from the short exact sequence of the cotangent sheaves (16), so we can assume that $x = s(b)$. Let $\phi \in \text{Aff}_{\mathcal{C}_b}(k s(b))_x$. First assume that $x$ is an infinite point in $\mathcal{C}_b$ and refer to Figure 4.

Let $u: B \to \mathcal{C}$ be the section of $\pi$ such that $u(b) = x$ and such that $u(c)$ is an infinite point in $\mathcal{C}_c$ for all $c \in B$. After potentially shrinking $B$, Lemma 3.20 guarantees the existence of a section $t: B \to \mathcal{C}$ such that for all $c \in B$, the point $t(c)$ is contained in the interior of the edge of $\mathcal{C}_c$ adjacent to $u(c)$; further we can request that $t(c) \neq s(c)$ for all $c \in B$. Let $U$ be the connected component of $\mathcal{C} \setminus t(B)$ containing $x$. Let $\varphi$ be the function on a small neighborhood of $U$ that has constant value $\phi(t(b))$ on $t(B)$, value $\phi(x)$ on $u(B)$, and whose slope on $U \cap \mathcal{C}_c$ in the fiber direction is $k$ for all $c \in B$ (use a TPL-trivialization to see that $\varphi$ is piecewise linear). By Lemma 3.21 and Lemma 3.22, $\varphi$ is affine on $U \setminus u(B)$. Let $V$ be the connected component of $U \setminus s(B)$ that contains the
Figure 4. Extending the function $\phi$ to a neighborhood of $C_b$: it is extended constantly along the section $t$, by a function $\varphi$ with the appropriate vertical slope between $t$ and $s$, and constantly above $s$.

edge segment between $s(b)$ and $t(b)$ in $C_b$. Then the function

$$U \to \mathbb{R}, \ y \mapsto \begin{cases} \varphi(y), & y \in V \\ (s \circ \pi)^* \varphi, & y \notin V \end{cases}$$

defines an element in $\text{Aff}_{C_b}(s(b))_x$ that lifts $\phi$.

Now assume that $x$ is not an infinite point in $C_b$, and refer to Figure 5. Since every element in $\text{Aff}_{C_b}(s(b))_x$ is a linear combination of elements in $\text{Aff}_{C_b}(s(b))_x$, we may assume that $\phi \in \text{Aff}_{C_b}(s(b))_x$. In fact, we may assume that there exists a direction $v \in T_x C_b$ such that $d_v \phi = 1$ and $d_w \phi = 0$ for all $w \in T_x C_b \setminus \{v\}$. Let $w \in T_x C_b$. By Lemma 3.20, there exists, after potentially shrinking $B$, a linear section $t_w : B \to C$ of $\pi$ such that $t_w(b)$ is on the edge adjacent to $x$ in the direction specified by $w$, and such that all values of $t_w$ are in the interior of edges of their fibers. We may assume that all these sections are disjoint from $s$, and there exists a neighborhood $U$ of $x$ with simply connected fibers such that $\gamma_{C_b}|U = 0$ and such that for every $c \in B$, every non-compact edge of $U \cap C_c$ contains one of the points $t_w(c)$. For every $c \in B$ and $w \in T_x C_b$ with $w \neq v$, choose a one-form on $U \cap C_c$ whose support contains the path $[t_w(c), t_v(c)]$ and has slope one in the direction from $t_w(c)$ to $t_v(c)$. There exists a unique affine function $\chi_w, c$ on $U \cap C_c$ that lifts this one-form and satisfies $\chi_w(c(t_v(c)) = 0$. Let

$$\chi_w : U \to \mathbb{R}, \ y \mapsto \chi_w, \pi(y)(y).$$
Figure 5. The function $\varphi$ extending $\phi$ may be described informally as follows: it is constantly equal to zero along the section $t_v$; along a given fiber $s$, it has vertical slope one until either it hits the section $s$, or it hits a vertex; in the latter case, it continues with slope one towards the section $s$, and it is extended constantly along other directions. According to (33), $\varphi$ is defined at the points $y_i$ as follows:

$$
\begin{align*}
\varphi(y_1) &= \chi_{w_1}(y_1) = \chi_{w_2}(y_1) = \chi_{w_3}(y_1), \\
\varphi(y_2) &= \chi_{w_1}(s(c)), \\
\varphi(y_3) &= \chi_{w_2}(y_3) = \chi_{w_3}(y_3), \\
\varphi(y_4) &= \varphi(y_5) = \varphi(y_6) = \chi_{w_2}(s(c)) = \chi_{w_3}(s(c)).
\end{align*}
$$

By construction, $\chi_w$ is fiberwise harmonic and satisfies $t^*_w \chi_w = 0$. Therefore, $\chi_w$ is affine by Lemma 3.21 and Lemma 3.22. Define $\hat{U}$ to be the connected component of $U \setminus \bigcup_{w \in T_x \mathcal{C}_b} t_w(B)$ containing $x$, and construct a function $\varphi: \hat{U} \to \mathbb{R}$ as follows:

$$
\varphi(y) = \begin{cases} 
\chi_w(y), & \text{if } s(c) \in [t_w(c), t_v(c)] \\
& \text{and } y \in [s(c), t_v(c)]. \\
\chi_w(s(c)), & \text{else, with } w \text{ such that } y \in [t_w(c), t_v(c)].
\end{cases} \quad (33)
$$

Using a TPL-trivialization one sees that $\varphi$ is piecewise integral linear, and hence

$$
\varphi|_{\hat{U}(B)} \in \Gamma(\hat{U} \setminus s(B), H_{\mathcal{C}}). 
$$

By construction, we have $t^*_w \varphi = 0$ and

$$
t^*_w \varphi = s^* \chi_w \quad (35)
$$
for $w \in T_x \mathcal{C}_b$ with $w \neq v$. Therefore, we have

$$
\varphi|_{\hat{U}(B)} \in \Gamma(\hat{U} \setminus s(B), \text{Aff}_{\mathcal{C}}) \quad (36)
$$
by Lemma 3.21 and Lemma 3.22 and hence
\[ \varphi \in \Gamma(\tilde{U}, \text{Aff}_{\mathcal{E}}(s)) . \]  
(37)
Since \( \varphi \) restricts to \( \phi \) on \( \mathcal{C}_b \), this finishes the proof. \( \square \)

4. IN PURSUIT OF AN ATLAS

In this section we attempt to construct an atlas of \( \mathcal{M}_{g,n} \), that is a local isomorphism \( B \to \mathcal{M}_{g,n} \), where \( B \) is a tropical space. First we eliminate non-trivial automorphisms by considering the space \( \mathcal{V}_{g,n} \) of cycle-rigidified curves and then consider the universal TPL-curve over \( \mathcal{V}_{g,n} \). This has similarities to the construction of tropical Teichmüller space in [Uli20]. The major challenge is to equip the rigidification and its universal family with affine structures to obtain a family of stable tropical curves. This essentially amounts to recovering the affine structure of \( \mathcal{M}_{g,n} \) from its functorial description. Both the fact that the isotropy groups at curves with higher-genus vertices are too small and the fact that the affine structure on the total space of a family is not determined by the underlying TPL-family (see Example 3.13) make it impossible to obtain an atlas for all of \( \mathcal{M}_{g,n} \), but we can obtain a natural surjection onto the locus of good curves and a local isomorphism onto the locus of Mumford curves.

4.1. The rigidification \( \mathcal{V}_{g,n} \) of \( \mathcal{M}_{g,n} \).

Definition 4.1. Let \( G \) be an \( n \)-marked stable graph of genus \( g \). An oriented cycle basis for \( G \) is a collection \( c_i^G, \ldots, c_g^G \in H_1(G;\mathbb{Z}) \) that generates \( H_1(G;\mathbb{Z}) \) and such that each \( c_i^G \) is either zero, or corresponds to a primitive cycle in \( G \). We call a stable graph equipped with an oriented cycle basis a cycle-rigidified stable graph. A morphism between two cycle-rigidified stable graphs \( G \) and \( G' \) of the same genus \( g \) is a morphism \( f : G \to G' \) of graphs such that \( f_*c_i^G = c_i^{G'} \) for all \( 1 \leq i \leq g \).

If \( G \) is a cycle-rigidified stable graph of genus \( g \) and \( G' \) is obtained as a specialization of \( G \), then the cycles \( c_i^G, \ldots, c_g^G \) induce and oriented cycle basis \( c_i^{G'}, \ldots, c_g^{G'} \) for \( G' \).

Lemma 4.2. Let \( G \) be a cycle-rigidified stable graph, and let \( f : G \to G' \) be an automorphism of \( G \). Then \( f \) is the identity.

For a proof of Lemma 4.2, see, for example, [Zim96, Lemma 1].

Definition 4.3. For every \( g, n \in \mathbb{Z}_{\geq 0} \) with \( n + 2g - 2 > 0 \) we define a topological space \( \mathcal{V}_{g,n} \) by gluing for each cycle-rigidified \( n \)-marked stable graph \( G = (G, (c_i^G)_1 \leq i \leq g) \) the extended cone \( \overline{\sigma}_G := \overline{\sigma}_G \) along the face maps \( \overline{\sigma}_G \to \overline{\sigma}_G \) associated to morphisms \( G \to G' \). For an open subset \( V \subseteq \mathcal{V}_{g,n} \) we define \( \Gamma(V, \text{PL}_{\mathcal{V}_{g,n}}) \) as the set of continuous functions from \( V \) to \( \overline{\mathbb{R}} \) such that for all \( G \), the pull-back to \( \overline{\sigma}_G \) is a piecewise linear function.
The space \((\overline{T}_{g,n}, \text{PL}_{\overline{T}_{g,n}})\) is a TPL-space and it follows from Lemma 4.2 that the morphism \(\text{relint}(\sigma_G) \to \overline{T}_{g,n}\) are injective for all cycle-rigidified 1-marked stable graphs of genus \(g\). We now define an affine structure on \(\overline{T}_{g,n}\). The idea is to pretend we had an affine structure on the universal family over \(\overline{T}_{g,n}\) and then check which piecewise linear functions on \(\overline{T}_{g,n}\) are forced to be affine by the exactness of the sequence (16). Informally, we integrate relative one-forms along paths in the fibers of the total space to obtain these functions. To formalize this, we introduce the notions of thickened paths, one-forms along thickened paths, and cross ratios.

**Definition 4.4.** Let \(G\) be a stable graph, and denote by \(G^f\) the subset of genus zero vertices or points of bounded edges. A **thickened path** \(\gamma^\pm = (\gamma, f_s, f_e)\) is a path \(\gamma: [0,1] \to G^f\) together with the choice of an initial flag \(f_s\) incident to \(\gamma(0)\) and a terminal flag \(f_e\) incident to \(\gamma(1)\).

While not strictly necessary, we include the following simplifying assumptions in the definition:
- \(\gamma(0) = v_s, \gamma(1) = v_e\) are vertices;
- if \(\gamma\) is not constant, \(f_s\) is distinct from the flag determined by \(\gamma'(0)\) and \(f_e\) is distinct from \(\gamma'(1)\);
- if \(\gamma\) is constant, \(f_s \neq f_e\).

**Remark 4.5.** Note that in the definition of thickened paths, \(\gamma\) may be constant: in this case a thickened path consists only of a genus-zero vertex \(v\) and two flags \(f_s\) and \(f_e\) with \(v_G(f_s) = v_G(f_e) = v\).

**Definition 4.6.** Let \(G\) be a stable graph, \(\Gamma\) a tropical curve of combinatorial type \(G\) and \(\gamma^\pm\) be a thickened path on \(G\). A **one-form along** \(\gamma^\pm\) is an element \(w \in H^0([0,1], \gamma^{-1}\Omega^1_{\Gamma})\) such that any local one-form \(\omega\) representing \(w\) near \(\gamma(0)\) (resp. \(\gamma(1)\)) has slope zero along \(f_s\) (resp. \(f_e\)).

Denote by \(v_0 = v_s, v_1, \ldots, v_n = v_e\) the ordered multiset of vertices traversed by \(\gamma\). Explicitly, a one-form along \(\gamma^\pm\) consists of the combinatorial datum of a collection of maps

\[
\begin{align*}
w_i &: T_{v_i} G \to \mathbb{Z} \\
\end{align*}
\]

such for \(0 \leq i \leq n\) such that

1. \(w_0(f_s) = w_n(f_e) = 0\),

2. for every \(0 \leq i \leq n\) we have

\[
\sum_{f \in T_{v_i} G} w_i(f) = 0 ,
\]

3. if \(1 \leq i \leq n\) and \(f^+_i, f^-_i\) denote the two flags belonging to the edge \(e_i\) traversed by \(\gamma\) between \(v_i-1\) and \(v_i\), then

\[
w_{i-1}(f^+_i) = -w_i(f^-_i) .
\]
We may interpret condition (3) as saying that $w$ assigns an integer $w(e_i)$ to any directed edge when traversed by $\gamma$.

**Definition 4.7.** A **cross ratio datum** on a stable graph $G$ is a pair $(\gamma^\pm, w)$ consisting of a thickened path $\gamma^\pm$ on $G$ and a one-form $w$ along $\gamma^\pm$.

If a stable graph $G$ is obtained as a specialization of a stable graph $\tilde{G}$, then any thickened path $\gamma^\pm$ on $G$, which by definition is contained in the set of genus-zero points of $G$, can be lifted uniquely to a thickened path $\tilde{\gamma}^\pm$ of a trail on $\tilde{G}$: insert into $\gamma$ the edges of $\tilde{G}$ that are contracted to vertices that $\gamma$ passes through. Any one-form $w$ along $\gamma^\pm$ can be uniquely lifted to a one-form $\tilde{w}$ along $\tilde{\gamma}^\pm$, the values of $\tilde{w}$ on contracted flags being determined by part (2) in Definition 4.6. Therefore, any cross ratio datum on $G$ lifts uniquely to a cross ratio datum on $\tilde{G}$.

**Definition 4.8.** Let $G$ be an $n$-marked cycle-rigidified stable graph of genus $g$, and let $c = (\gamma^\pm, w)$ be a cross ratio datum on $G$. We define a piecewise-linear function $\xi_c$, the **cross ratio associated to** $c$, on

$$V(G) = \bigcup_{\tilde{G}} \mathrm{relint}(\tilde{G})$$

where the union is taken over all cycle-rigidified stable graphs $\tilde{G}$ specializing to $G$. Namely, given a stable graph $\tilde{G}$ specializing to $G$, let $\tilde{c} = (\tilde{\gamma}^\pm, \tilde{w})$ denote the unique lift of $c$ to $\tilde{G}$. We then define the restriction of $\xi_c$ to $\mathrm{relint}(\sigma_{\tilde{G}})$ by

$$\xi_c|_{\mathrm{relint}(\sigma_{\tilde{G}})} = \sum_{1 \leq i \leq m} \tilde{w}(e_i)|_{e_i},$$

where $e_1, \ldots, e_m$ denote the ordered multiset of directed edges traversed by $\tilde{\gamma}$.

**Definition 4.9.** We define $\mathbb{Aff}_{g,n}$ as the subsheaf of $\mathbb{PL}_{g,n}$ generated by the cross ratios $\xi_c$, where $c$ is a cross ratio datum on some $n$-marked cycle-rigidified stable graph of genus $g$.

To ease the use of cross ratio data, we introduce a generating set for cross ratio data.

**Definition 4.10.** A cross ratio datum $c = (\gamma^\pm, w)$ on a stable graph $G$ is said to be **primitive** if it is of one of the following two types:

- **(Edge)** the path $\gamma$ parameterizes a bounded edge $e$ connecting two genus-zero vertices $v_0$ and $v_1$. The one-form $w$ has $w(e) = 1$, and there exist precisely one flag in $T_{v_0}G \setminus \{f_s, \gamma'(0)\}$ and one flag in $T_{v_1}G \setminus \{f_e, \gamma'(1)\}$ where $w$ has non-zero slope. By the balancing condition the slopes along these two special flags are $\pm 1$; we label the flags $f_{\pm 1}$ so that $w(f_{-1}) = -1$ and $w(f_1) = 1$. We denote this cross ratio datum by $e_{(f_s, f_e), (f_{-1}, f_1)}$ and the associated cross ratio by $\xi_{e_{(f_s, f_e), (f_{-1}, f_1)}}$.

- **(Vertex)** The image of $\gamma$ consists of a single genus-zero vertex $v$. There exist two flags $f_{-1}, f_1 \in T_vG \setminus \{f_s, f_e\}$ on which $w$ is nonzero, and we have $w(f_{-1}) = -1$ and
\[ \Gamma \]

**Figure 6.** A primitive cross ratio datum on a tropical elliptic curve.

\[ w(f_1) = 1. \] We denote this cross ratio datum by \( c((f_s, f_e), (f_{-1}, f_1)) \) associated cross ratio by \( \xi_{c((f_s, f_e), (f_{-1}, f_1))} \).

**Example 4.11.** Let \( \Gamma \) denote a cycle-rigidified tropical elliptic curve with no points of genus one, as depicted in Figure 6. As an example of a non-trivial primitive cross ratio of edge type, one may take \( \gamma^\pm \) to be the path parameterizing the loop in the direction of the cycle-rigidification, with \( f_s = f_e \) being the tail of \( \Gamma \). The one-form \( w \) may be chosen to be the (pull-back via \( \gamma^\pm \) of the) global one-form \( \omega \) with slope zero on the tail of \( \Gamma \) and one along the loop. This cross ratio datum produces the function on \((0, \infty) \subset V_{1,1}\) that assigns to each tropical elliptic curve the length of its loop. There are several other choices for primitive cross ratio data, but up to sign they all produce the length of the loop as a function.

**Remark 4.12.** Given a cycle-rigidified stable graph \( \tilde{G} \) specializing to \( G \), a primitive cross ratio datum \( c \) on \( G \) determines two oriented paths on \( \tilde{G} \): \( \gamma \) connecting \( v_{\tilde{G}}(f_s) \) to \( v_{\tilde{G}}(f_e) \), and \( \gamma_w \) connecting \( v_{\tilde{G}}(f_{-1}) \) to \( v_{\tilde{G}}(f_1) \). For a point \( x \) in \( \text{relint}(\sigma_{\tilde{G}}) \), the value \( \xi_{c}(x) \) equals the signed length of the intersection \( \tilde{\gamma} \cap \gamma_w \), where the sign is positive if the two paths are oriented the same way and negative otherwise, see Figure 7. Thus, a primitive cross ratio datum is intimately related to the notions of tropical cross ratios in the literature, which can be viewed as tropicalizations of cross ratios in algebraic geometry [Mik07, GM08, Tyo17, Gol18].

**Proposition 4.13.** The sheaf \( \text{Aff}_{\tau_{g,n}} \) is generated by the cross ratios \( \xi_c \) associated to primitive cross ratio data \( c \).

**Proof.** Let \( G \) be a cycle-rigidified stable graph and let \( c = (\gamma^\pm, w) \) be a cross ratio datum on \( G \). We show that \( \xi_c \) is a linear combination of primitive cross ratios. Let \( v_0, \ldots, v_m \) be the ordered multiset of vertices traversed by \( \gamma \). We proceed by induction on \( m \). First suppose \( m = 0 \). The one-form \( w \) along \( \gamma^\pm \) is uniquely determined by the map \( w_0 : T_{v_0}G \setminus \{f_s, f_e\} \to \mathbb{Z} \). We may think of \( w_0 \) as an element of the sublattice \( \mathbb{H} \subset \mathbb{Z}^{\text{val}_{v_0}-2} \) of integral vectors whose coordinates add to zero. It is well known that \( w_0 \) may be written as an integral linear combination of vectors with exactly two non-zero entries, equal to 1 and -1. Each such vector corresponds to a primitive cross ratio of vertex type, concluding the proof of the base case.
Now suppose $m > 0$. Denote by $e_{m-1}, e_m$ the last two oriented edges traversed by $\gamma$, by $f_{m-1}, f_m^+$ the flags tangent to $\gamma$ and incident to $v_{m-1}$, and by $h$ a flag incident to $v_{m-1}$ which is not traversed by $\gamma$. Consider the integral affine function

$$\xi := \xi_{e} - \sum_{f \in T_{v_{m-1}} G \setminus \{e\}} w_m(f) \cdot \xi_{(e_{m-1}, f, e)}.$$ (44)

One may verify that the affine function $\xi$ can be written as a cross ratio $\xi = \xi_{(\tilde{\gamma}^\pm, \tilde{w})}$, where $\tilde{\gamma}^\pm$ is the thickened path in $G$ obtained by truncating $\gamma$ at $v_{m-1}$ and choosing $f_e = f_{m-1}^+$, and $\tilde{w}$ is the one-form given by $\tilde{w}_i = w_i$ for $i < m-1$ and

$$\tilde{w}_{m-1}(f) = \begin{cases} 0, & f = f_{m-1}^+, \\ w_m(h) + w_m(f_{m-1}^+), & f = h, \\ w_m(f), & \text{else.} \end{cases}$$ (45)

The statement is immediate for any point in $\text{relint}(\sigma_G)$; it is a combinatorial exercise to show that the functions $\xi = \xi_{(\tilde{\gamma}^\pm, \tilde{w})}$ agree on the relative interior of cones $\sigma_{\tilde{G}}$, where $\tilde{G}$ is a graph specializing to $G$. Using the induction hypothesis finishes the proof.

\[\square\]

**Lemma 4.14.** Let $G$ be a stable graph and let $v \in V(G)$ be a genus-zero vertex, and let $f_1, f_2, f_3, f_4 \in T_v G$ be distinct flags.

1. We have

$$\xi_{((f_1, f_2), (f_3, f_4))} = \xi_{((f_3, f_4), (f_1, f_2))}.$$ (1)

2. We have

$$\xi_{((f_1, f_2), (f_3, f_4))} = -\xi_{((f_2, f_1), (f_3, f_4))}.$$ (2)

3. We have

$$\xi_{((f_1, f_3), (f_2, f_4))} = \xi_{((f_1, f_2), (f_3, f_4))} + \xi_{((f_1, f_4), (f_2, f_3))}.$$ (3)
Lemma 4.15. Let \( G \) be a stable graph, let \( e = \{f, f'\} \in E_b(G) \) be a bounded oriented edge connecting two (not necessarily distinct) genus-zero vertices \( v_0 \) and \( v_1 \); Let \( f_s, f_{-1} \in T_{v_0} G \setminus \{f\} \) be two distinct flags, and let \( f_e, f_1 \in T_{v_1} G \setminus \{f'\} \) be two distinct flags.

1. We have
\[
\xi_{((f_s, f_e), (f_{-1}, f_1))} = \xi_{(\{-e, e\}, (f_1, f_{-1}))}.
\]
2. We have
\[
\xi_{(e, (f_s, f_e), (f_{-1}, f_1))} = \xi_{(e, (f_s, f_e), (f_{-1}, f_1))}.
\]
3. If \( h \in T_{v_1} G \setminus \{f_e, f_{-1}, f'\} \), then
\[
\xi_{(e, (f_s, f_e), (f_{-1}, f_1))} = \xi_{(e, (f_s, h), (f_{-1}, f_1))} + \xi_{(f_s, f_e), (f_{-1}, f_1)}.
\]

Proof. These statements and their proofs are analogous to those in Lemma 4.14. \( \square \)

The next technical lemma describes cross ratios on cones of cycle-rigidified graphs with some edges of infinite length.

Lemma 4.16. Let \( G \) be a cycle-rigidified 1-marked genus-\( g \) stable graph, let \( G_b \) the stable graph obtained by making all edges that are not legs bounded, and let \( G_{\infty} \) denote the stable...
A graph obtained by contracting the edges in $G_b$ that correspond to the bounded edges of $G$. Using the notation $V(G)$ as in Definition 4.8, the restriction
\[ \Gamma(V(G), \text{Aff}_{\mathcal{F}_{g,n}}) \to \Gamma(V(G_b), \text{Aff}_{\mathcal{F}_{g,n}}) \] (46)
identifies $\Gamma(V(G), \text{Aff}_{\mathcal{F}_{g,n}})$ with the group of all integral affine functions on $V(G_b)$ that are constant on $x + \text{relint}(\sigma_{G_b})$ for some (and hence any) $x \in \text{relint}(\sigma_{G_b})$.

**Proof.** Since $V(G)$ is contained in the closure of $V(G_b)$, the restriction of functions from $V(G)$ to $V(G_b)$ is injective. By definition, the thickened paths appearing in cross ratio data on $G$ do not include any infinite edges of $G$, which implies that restrictions of integral affine functions on $V(G)$ to $V(G_b)$ are constant on $x + \text{relint}(\sigma_{G_b})$ for any $x \in \text{relint}(\sigma_{G,b})$. Conversely, let $\xi$ be an integral affine function on $V(G_b)$ that is constant on $x + \text{relint}(\sigma_{G_b})$ for any $x \in \text{relint}(\sigma_{G_b})$. We may write $\xi = \xi_b + \xi_\infty$, where $\xi_b$ is a linear combination of primitive cross ratios on $G_b$ that do not involve any of the infinite edges of $G$, and $\xi_\infty$ is a linear combination of primitive cross ratios of edge-type on $G_b$ that involve only edges in $\mathcal{E}_{\infty}(G)$. By part (3) of Lemma 4.15, two edge-type primitive cross ratios that use the same edge are proportional modulo primitive cross ratios of vertex-type. Hence, we may assume that
\[ \xi_\infty = \sum_{e \in \mathcal{E}_{\infty}(G)} \xi_e, \] (47)
where $\xi_e$ is a primitive edge-type cross ratio that uses the edge $e$. Noting that the $\xi_e$'s are linearly independent on $x + \text{relint}(\sigma_{G_b})$ for any $x \in \text{relint}(\sigma_{G_b})$ implies that $\xi_\infty = 0$. If $c$ is a primitive cross ratio datum on $G_b$ that does not involve any infinite edge of $G$, then $c$ induces a primitive cross ratio datum $c'$ on $G$ and $\xi_c = \xi_c'(V(G_b))$. Since $\xi = \xi_b$ is a linear combination of such primitive cross ratios, it follows that $\xi$ is the restriction of an affine function on $V(G)$. \qed

### 4.2. Tautological maps as morphisms of tropical spaces.

In [CCUW17, Section 4], tautological morphisms of moduli spaces of tropical curves are defined as morphisms of cone stacks. We show they are in fact morphisms of stacks on tropical spaces.

Set theoretically, each point $u \in \mathcal{F}_{g,n,\bullet}(\ast)$ corresponds to a cycle-rigidified $(\mathcal{C}_i \sqcup \{\ast\})$-marked tropical stable curve $(C_i(x_i)_{i \in \mathcal{I}(\ast)})$, of genus $g$. Omitting the marking $s_*$ and stabilizing yields a cycle-rigidified $n$-marked tropical stable curve of genus $g$, which corresponds to a point in $\mathcal{F}_{g,n}$. This map is called the forgetful morphism, and we denote it by $\pi_*$. For each $i \in \mathcal{I}$ the map $\pi_*$ has a section: if $(C_i(x_i)_{i \in \mathcal{I}})$ is an $I$-marked tropical stable curve of genus $g$, then we can attach a tripod (with three infinite edges) to the point $x_i$, move the marking $x_i$ to one of the two newly created ends, and mark the other end by $\ast$. The new compact edge formed has infinite length. We denote this section by $s_i$.

**Proposition 4.17.** Let $G$ be a cycle-rigidified stable graph. If $\gamma: \mathcal{F}_{g,n,\bullet}(\ast) \to \mathbb{Z}$ is the function that assigns to every point in $\text{relint}(\sigma_G)$ the genus of the vertex of $G$ that is adjacent to
the leg marked by $\star$, then $(\pi_*: \mathcal{F}_{g,n,\mbox{\scriptsize{(\star)}}} \to \mathcal{F}_{g,n})$ is a family of $n$-marked stable TPL-curves of genus $g$.

Moreover, if $G$ is a cycle-rigidified $n$-marked genus-$g$ stable graph and $b \in \text{relint}(\sigma_G)$, then the combinatorial type of the fiber $(\mathcal{F}_{g,n,\mbox{\scriptsize{(\star)}}})_b$ can be naturally identified with the underlying graph $\hat{G}_b$ and the edge of $(\mathcal{F}_{g,n,\mbox{\scriptsize{(\star)}}})_b$ corresponding to $e \in E_b(G)$ has length $l_e(b)$.

**Proof.** This proposition follows from [CCUW17, Proposition 4.5] by observing that by construction $\pi_*$ pulls-back piecewise linear functions to piecewise-linear functions.  

\[ \square \]

**Proposition 4.18.** The maps $\pi_*$ and $s_i$ are morphism of tropical spaces. Furthermore, for every $x \in \mathcal{F}_{g,n,\mbox{\scriptsize{\{\star\}}}}$, the sequence

\[ 0 \to \Omega_{\mathcal{F}_{g,n,\mbox{\scriptsize{\{\star\}}}}}^{\pi_*} \to \Omega_{\mathcal{F}_{g,n,\mbox{\scriptsize{\{\star\}}}}}^{\pi_*} \to \Omega_{\mathcal{F}_{g,n,\mbox{\scriptsize{\{\star\}}}}}^{\pi_*} \to 0 \tag{48} \]

is exact.

**Proof.** Let $G$ be a cycle-rigidified $n$-marked genus-$g$ stable graph, and let $c$ be a cross ratio datum. For any cycle-rigidified $(n \cup \{\star\})$-marked genus-$g$ stable graph $G$ whose stabilization after forgetting the $\star$-marked leg is $G_\star$, $c$ defines a cross ratio datum $c'$ on $G$ that does not involve the $\star$-marked leg. It follows immediately from the definitions that $(\pi_*^* \xi_c)_\pi = (\xi_{c'})_\pi$ at any point of $\text{relint}(\sigma_G)$. It follows that $\pi_*$ is linear. It also follows that for any cross ratio datum $c'$ on $H$ which does not involve the $\star$-marked leg, the cross ratio $\xi_{c'}$ agrees with the pull-back of a cross ratio defined by a cross ratio datum $c$ on $G$.

To see that $s_i$ is linear, let $y \in \mathcal{F}_{g,n}$, and let $G$ be the cycle-rigidified $(n \cup \{\star\})$-marked genus-$g$ stable graph such that $s_i(y) \in \text{relint}(\sigma_G)$. By construction of $s_i$, the $\star$-marked leg in $G$ is adjacent to a three-valent genus-zero vertex, all of whose adjacent edges are unbounded. It follows that there is no cross ratio datum on $G$ that involves the $\star$-marked leg. Therefore, every integral affine linear function $\xi$, at $s_i(y)$ is locally the pull-back via $\pi_*$ of an integral-linear $\xi'$ function at $y$. Since $s_i$ is a section of $\pi_*$, we then have $s_i^* \xi = \xi'$. We conclude that $s_i$ is linear.

To prove the exactness of the sequence, let $x \in \mathcal{F}_{g,n,\mbox{\scriptsize{\{\star\}}}}$. Since the integral affine functions on the fiber over $\pi_*(x)$ are precisely the restrictions of integral affine functions on $\mathcal{F}_{g,n,\mbox{\scriptsize{\{\star\}}}}$, the sequence is exact on the right. The exactness on the left follows immediately from the fact that images of neighborhoods of $x$ under $\pi_*$ are neighborhoods of $\pi_*(x)$. It remains to show exactness in the middle. Let $G$ be the cycle-rigidified $(n \cup \{\star\})$-marked genus-$g$ stable graph such that $x \in \text{relint}(\sigma_G)$ and let $\xi \in (\text{Aff}_{\mathcal{F}_{g,n,\mbox{\scriptsize{\{\star\}}}}})_x$ be a function whose restriction to the fiber $(\mathcal{F}_{g,n,\mbox{\scriptsize{\{\star\}}}})_x$ is constant in a neighborhood of $x$. Denote by $f_* \in L(G)$ the $\star$-marked leg in $G$ and let $V := v_G(f_*)$. If $y_G(V) > 0$ or $V$ is a three-valent vertex whose adjacent edges are all unbounded, there is no cross ratio datum on $G$ involving $f_*$. In that case, all integral affine functions at $x$ are pull-backs under $\pi_*$ of integral affine functions at $\pi_*(x)$ and we are done. We may thus assume $V$ is a genus-zero vertex $v$ that is either at least four-valent, or three-valent and adjacent to at least one bounded edge whose vertices have genus zero.
First assume the latter case. Let \( \{f_1, f_2\} = T_v G \setminus f_* \). Without loss of generality we may assume that \( e_G(f_1) \) is a bounded edge whose vertices have genus zero. Let \( v_1 \) be the second vertex of \( e_G(f_1) \) and let \( f_1,e, f'_1,e \in T_{v_1} G \) be distinct flags such that \( e_G(f_1,e) \neq e_G(f_1) \neq e_G(f'_1,e) \). If \( e_G(f_2) \) is either unbounded or adjacent to a vertex of positive genus, by Lemma 4.15, we have
\[
\xi \equiv \lambda \cdot \xi_{e_G(f_1), (f_1,e), (f_2,f'_1,e)}
\]
for some \( \lambda \in \mathbb{Z} \), where the congruence holds modulo cross ratios pulled back from \( F_{g,n} \). The slope of the restriction of the cross ratio \( \xi_{e_G(f_1), (f_1,e), (f_2,f'_1,e)} \) to the fiber \( (F_{g,n\setminus \{s\}}, \tau_s)_{\pi_* (x)} \) in the direction corresponding to \( f_1 \) is \(-1\), so \( \xi \), being constant on the fiber implies \( \lambda = 0 \). It follows that \( \xi \) equals the pull-back of an integral affine function from \( F_{g,n} \). If \( e_G(f_2) \) is bounded and its vertices have genus zero, define \( v_2, f_{2,e}, \) and \( f'_{2,e} \) similarly as for \( f_1 \). Then again by Lemma 4.15, we have
\[
\xi \equiv \lambda_1 \cdot \xi_{e_G(f_1), (f_1,e), (f_2,f'_1,e)} + \lambda_2 \cdot \xi_{e_G(f_2), (f_2,e), (f_1,f'_2,e)}
\]
for some \( \lambda_1, \lambda_2 \in \mathbb{Z} \), where again the congruence holds modulo cross ratios pulled back from \( F_{g,n} \). One may observe that \( \xi \) being constant on the fiber implies \( \lambda_1 = \lambda_2 \). Let \( G \) be the cycle-rigidified \( 1 \)-marked genus-\( g \) stable graph obtained by removing \( f_* \) and \( v \) from \( G \) and replacing the edges \( e_G(f_1) \) and \( e_G(f_2) \) by a single edge \( e \) connecting \( v_1 \) and \( v_2 \). Then \( \tau_s(x) \in \text{relint}(\sigma_G) \). Then one sees that
\[
\tau^*_s(\xi_{e_G(f_1), (f_1,e), (f_2,f'_1,e)}) = \xi_{e_G(f_1), (f_1,e), (f_2,f'_1,e)} + \xi_{e_G(f_2), (f_2,e), (f_1,f'_2,e)}.
\]
It follows that \( \xi \) equals the pull-back of an integral affine function from \( F_{g,n} \). This concludes the analysis of \( v \) being a three-valent vertex.

Now assume that \( v \) is a genus-zero vertex of valence at least four. Observe that if \( \xi \) is a primitive cross ratio of edge type containing \( f_* \), by Lemma 4.15 (3), it may be replaced by a cross ratio of edge type not containing \( f_* \) plus a cross ratio of vertex type. We may therefore assume that we start with \( \xi \), a linear combination of cross ratios of vertex type containing \( f_* \).

We manipulate \( \xi \) using the relations provided by Lemma 4.14. Fix three distinct flags \( f_1, f_2, f_3 \in T_v G \setminus \{f_*\} \). Through repeated applications of part (4) we may guarantee that all cross ratios in \( \xi \) involve the flags \( f_*, f_1, f_2 \). We may use parts (1), (2), (3) to express \( \xi \) as a linear combinations of the cross ratios \( \xi_{((f_*, f_1), (f_2, h))} \) and \( \xi_{((f_*, f_3), (f_1, f_2))} \), for any \( h \in T_v G \setminus \{f_* , f_1, f_2\} \). With one more application of (4) we observe:
\[
\xi_{((f_*, h), (f_1, f_2))} = \xi_{((f_*, f_3), (f_1, f_2))} + \xi_{((f_3, h), (f_1, f_2))},
\]
hence modulo cross ratios not involving \( f_* \), we may assume that \( h = f_3 \) in the second family of cross ratios.

In conclusion, we may write
\[
\xi \equiv \mu \cdot \xi_{((f_*, f_3), (f_1, f_2))} + \sum_{h \in T_v G \setminus \{f_1, f_2\}} \lambda_h \cdot \xi_{((f_*, f_1), (f_2, h))}
\]
for appropriate choices of $\mu, \lambda_h \in \mathbb{Z}$, i.e. $\xi$ is a linear combination of $\text{val}(v) - 2$ cross ratios. Notice that $\xi$, being constant on the fiber of $\pi_*(x)$ imposes $\text{val}(v) - 2$ independent linear homogeneous conditions on the coefficients $\mu, \lambda_v$ hence $\xi \equiv 0$ modulo cross ratios not involving $f_*$, which is equivalent to $\xi$, being-pulled back from an integral affine function from $\mathcal{T}_{g,n}$.

**Proposition 4.19.** Let $\pi: \mathcal{C} \to B$ be a family of $n$-marked genus-$g$ stable tropical curves, and let $f: B \to \mathcal{T}_{g,n}$ and $f_*: \mathcal{C} \to \mathcal{T}_{g,n}(\ast)$ be morphisms of TPL-spaces such that the diagram

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{f_*} & \mathcal{T}_{g,n}(\ast) \\
\downarrow \pi & & \downarrow \pi_* \\
B & \xrightarrow{f} & \mathcal{T}_{g,n}
\end{array}
$$

is a Cartesian diagram of TPL-spaces exhibiting the family of $n$-marked stable TPL-curve $\pi: \mathcal{C} \to B$ as the pull-back of $\pi_*: \mathcal{T}_{g,n}(\ast) \to \mathcal{T}_{g,n}$. Then $f$ and $f_*$ are linear.

**Proof.** Since the affine structure on $\mathcal{T}_{g,n}$ is generated by cross ratios, to show that $f$ is linear we need to show that cross ratios induce linear functions on $B$. Let $b \in B$, and let $G$ be the cycle-rigidified, $n$-marked, genus-$g$ stable graph such that $f(b) \in \text{relint}(\sigma_G)$. Since $f_*$ induces an isomorphism $\mathcal{C}_b \cong (\mathcal{T}_{g,n}(\ast))_{f(b)}$, the combinatorial type of $\mathcal{C}_b$ is naturally identified with $G$ and the edge-length of the edge in $\mathcal{C}_b$ corresponding to $e \in E_b(G)$ is given by $l_e(f(b))$. Let $c = (\gamma^+, \omega)$ be a cross ratio datum on $G$, and let us first assume that the image of $\gamma$ is a simple curve. Refer to Figure 9 for the notation in this proof. Let $V_\gamma$ denote a contractible neighborhood of the image of $\gamma$ and $\omega \in \Omega^1_{\mathcal{C}_b}$ such that $\omega = \gamma^* \omega$. By the exactness of (16), $\omega$ may be lifted to a one-form $\tilde{\omega} \in \Omega^1_{\mathcal{C}}(\tilde{V}_\gamma)$, with $\tilde{V}_\gamma$ an open set in $\mathcal{C}$ restricting to $V_\gamma$ on $\mathcal{C}_b$. Further, we may choose $\tilde{\Phi} \in \text{Aff}_{\mathcal{C}}(\tilde{V}_\gamma)$ lifting $\tilde{\omega}$.

By Lemma 3.20, there exists a neighborhood $V$ of $b$ and linear sections $\epsilon_s, \epsilon_\mathcal{C}: V \to \mathcal{C}$ of $\pi$, whose images are contained in $f_*$ and $f_*$. By construction, we have

$$
(f_* \xi_c)_b = \left( \epsilon_\mathcal{C} \tilde{\Phi} - \epsilon_s \tilde{\Phi} \right)_b.
$$

The right side of this equation is contained in $\text{Aff}_{B,b}$. We conclude that $f$ is linear at $b$, and since $b$ was arbitrary that $f$ is linear. If $\gamma$ is not a simple path, one may apply a standard local lifting argument to a refinement of $\gamma$ and conduct essentially the same proof.

To show that $f_*$ is linear let $x \in \mathcal{C}$ be a point in $\mathcal{C}_b$. The position of $x$ in $\mathcal{C}_b$, together with the identification of the combinatorial type of $\mathcal{C}_b$ with $G$ induced by $g$ uniquely determines the $(n \cup \{\ast\})$-marked graph $G_\ast$ with $f_*(x) \in \text{relint}(\sigma_{G_\ast})$. Let $c = (\gamma^+, \omega)$ be a cross ratio datum on $G_\ast$. By Proposition 4.13 we may assume that $c$ is primitive. If $c$ does not involve the $\ast$-marked leg then $c$ defines a cross ratio datum $c'$ on $G$ and $\xi_c = \pi_* \xi_{c'}$. Then $f_* \xi_c = (f \circ \pi)$ is integral affine at $x$ by the linearity of $\pi$. 

and \(f\). We may thus assume that \(c\) involves the \(\ast\)-marked leg. By Lemma 4.14 and Lemma 4.15 we may further assume that the thickened path \(\gamma^\pm\) does not involve the \(\ast\)-marked leg, but only the one-form does. Let \(\Gamma\) be the stable tropical curve obtained by attaching to \(\mathscr{C}_b\) an additional leg at \(x\) and marking it by \(\ast\). Then \(w\) is an element of \(\Gamma([0,1],\gamma^{-1}\Omega^1_{\gamma})\). Because the inclusion \(\mathscr{C}_b \to \Gamma\) is not linear, \(w\) cannot be pulled back to an element in \(\Gamma([0,1],\gamma^{-1}\Omega^1_{\gamma}\mathscr{C}_b)\). However, it can be pulled back to an element \(\omega \in \Gamma([0,1],\gamma^{-1}\Omega^1_{\mathscr{C}_b}(kx))\) for an appropriate choice of \(k \in \mathbb{Z}\) (since \(c\) is primitive, \(k = \pm 1\)). Let \(\mathscr{D} = \mathscr{C} \times_B \mathscr{C}\) and let \(\Delta: \mathscr{C} \to \mathscr{D}\) be the diagonal map. Then the projection \(\mathscr{D} \to \mathscr{C}\), together with the pull-backs of the sections and the genus function, is an \(n\)-marked genus-\(g\) stable tropical curve. The fiber \(\mathscr{D}_x\) can be canonically identified with \(\mathscr{C}_b\), and \(x\) maps to \(\Delta(x)\) in this identification. Therefore, \(\gamma\) can be viewed as a path in \(\mathscr{D}_x\), and \(\omega\) can be viewed as an element in \(\Gamma([0,1],\gamma^{-1}\Omega^1_{\mathscr{D}_x}(k\Delta(x)))\). By Proposition 3.24 and Proposition 3.25, \(\omega\) lifts to an element in \(\Gamma([0,1],\gamma^{-1}\Omega^1_{\mathscr{D}}(k\Delta))\). Using Proposition 3.24, one can now proceed similarly as in the proof of the linearity of \(f\) above.

\[\square\]

**Definition 4.20.** We say that a stable tropical curve is a **Mumford-curve** if the genus-function of its combinatorial type is identically zero. We denote by \(\mathcal{M}^{\text{Mf}}_{g,n}\) the full subcategory of \(\mathcal{M}_{g,n}\) consisting of all families of Mumford curves.
Proposition 4.21. For every pair $g,n \in \mathbb{Z}_{>0}$ with $2g-2+n > 0$, the stack $\mathcal{M}_g^{M}$ is an open substack of $\overline{\mathcal{M}}_{g,n}$, that is the inclusion $\mathcal{M}^{M}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$ is represented by an open immersion.

Proof. We need to show that for every family $\mathcal{C} \rightarrow B$ of stable tropical curves the locus

$$U := \{ b \in B \mid \mathcal{C}_b \text{ is a Mumford curve} \}$$

is open in $B$. Let $b \in U$, and let $(\Sigma, (G_{\sigma}, f_{\sigma}, \chi_{\sigma})_{\sigma \in \Sigma})$ be a TPL-trivialization of $\mathcal{C} \rightarrow B$ at $b$. Let $\tau \in \Sigma$ be the inclusion-minimal polyhedron in $\Sigma$ containing $b$. By assumption, the function $\gamma_{G_{\sigma}}$ is identically zero. For every $\sigma \in \Sigma$, the polyhedron $\tau$ is a face of $\sigma$, and therefore the combinatorial type $G_{\tau}$ is a contraction of $G_{\sigma}$. It follows immediately that $\gamma_{G_{\sigma}}$ is identically zero as well. It follows that $U$ contains the interior of $|\Sigma|$. \hfill \Box

Definition 4.22. We denote by $\mathcal{V}_{g,n}^{\text{good}} \subseteq \overline{\mathcal{V}}_{g,n}$ the union of the relative interiors of the cones $\sigma_G$ corresponding to cycle-rigidified stable graphs $G$ where every bounded edge is adjacent to at least one genus-zero vertex, and every vertex of an unbounded edge has genus zero. Moreover, we denote by $\mathcal{V}^{M}_{g,n} \subseteq \mathcal{V}_{g,n}^{\text{good}}$ the union of the relative interiors of the cones $\sigma_G$ corresponding to cycle-rigidified stable graphs $G$ with $\gamma_{G_{\sigma}}$ identically zero.

Lemma 4.23. The subset $\mathcal{V}^{M}_{g,n}$ is open in $\overline{\mathcal{V}}_{g,n}$. Furthermore, we have $\pi_{*}^{-1}\mathcal{V}^{M}_{g,n} = \mathcal{V}^{M}_{g,n,\{\dagger\}}$.

Proof. The openness follows directly from Proposition 4.21 and Proposition 4.17. The second statement follows from the fact that forgetting a marking and stabilizing never increases the genus function. \hfill \Box

Proposition 4.24. All integral linear functions on $\overline{\mathcal{V}}_{g,n,\{\dagger\}}$ are harmonic on the fibers of $\pi_{*} : \overline{\mathcal{V}}_{g,n,\{\dagger\}} \rightarrow \overline{\mathcal{V}}_{g,n}$. Moreover, the fibers of $\pi_{*}$ over $\mathcal{V}_{g,n}^{\text{good}}$ are stable tropical curves and $\pi_{*}^{-1}\mathcal{V}_{g,n}^{\text{good}} \rightarrow \mathcal{V}_{g,n}^{\text{good}}$ is a family of $n$-marked genus-$g$ stable tropical curves. In particular, $\mathcal{V}^{M}_{g,n,\{\dagger\}} \rightarrow \mathcal{V}^{M}_{g,n}$ is a family of $n$-marked genus-$g$ stable tropical curves.

Proof. Let $G$ be a cycle-rigidified $(n \sqcup \{\dagger\})$-marked genus-$g$ stable graph and let $c$ be a cross ratio datum on $G$. We need to show that $\xi_{c}$ is harmonic on all fibers. By Proposition 4.13 we may assume that $c$ is primitive. If $c$ does not involve the $\dagger$-marked flag $f_{\dagger}$, then $\xi_{c}$ coincides with the pull-back of a cross ratio from $\overline{\mathcal{V}}_{g,n}$ and thus is constant on all fibers. We may thus assume that $c$ involves $f_{\dagger}$. By Lemma 4.14 and Lemma 4.15, we may assume that either $c = c([f_{\dagger}, f_{\pm 1}, (f_{\pm 1}, f_{1})])$ or $c = c([e, f_{\dagger}, (f_{\pm 1}, f_{1})])$. In the first case, the restriction of $\xi_{c}$ to a fiber near a point in relint$(\sigma_{G})$ has slope 1 in the direction corresponding to $f_{-1}$, slope $\pm 1$ in the direction of corresponding to $f_{\pm 1}$, and slope 0 in all other directions. It follows that the restrictions of $\xi_{c}$ to the fibers of $\pi_{*}$ are harmonic. Moreover, by taking appropriate linear combinations of primitive vertex-type cross ratios, we see that if $v$ is a genus-zero vertex of valence at least three in a fiber of $\pi_{*}$, then all harmonic functions in a neighborhood of $v$ in $\pi_{*}^{-1}(\pi_{*}(v))$ are restrictions of integral affine functions on $\overline{\mathcal{V}}_{g,n,\{\dagger\}}$.\hfill \Box
In the second case, the restriction of $ξ_c$ to a fiber near a point in $\text{relint}(σ_G)$ has slope $-1$ in the direction corresponding to $e$, slope $1$ in the direction corresponding to $f_{-1}$, and slope $0$ in all other directions. It follows that the restriction of $ξ_c$ to the fibers of $π_*$ is harmonic as well. Moreover, we see that if $v$ is a point in the interior of an edge in a fiber of $π_*$ that is adjacent to at least one finite genus-zero vertex, then all harmonic functions in a neighborhood of $v$ in $π_{-1}^{-1}(π_*(v))$ are restrictions of integral affine functions on $\mathcal{F}_{g,n,\Sigma_b}^{\pi}$. 

If $b ∈ \mathcal{F}_{g,n,\Sigma_b}^{\pi}$, then all points of the fiber $\left(\mathcal{Y}_{g,n,\Sigma_b}^{\pi}\right)_b$ are either nodes, markings, genus-zero vertices of valence at least three, or points in the interior of an edge that is adjacent to at least one finite genus-zero vertex. By what we have seen, the restrictions of integral affine functions on $\mathcal{F}_{g,n,\Sigma_b}^{\pi}$ to $\left(\mathcal{Y}_{g,n,\Sigma_b}^{\pi}\right)_b$ are precisely the harmonic functions. It follows that $\left(\mathcal{Y}_{g,n,\Sigma_b}^{\pi}\right)_b$ is a stable tropical curve. Together with Proposition 4.18, it follows that the $π_{-1}^{-1}\mathcal{Y}_{g,n}^{\pi} → \mathcal{Y}_{g,n}^{\pi}$ is a family of $n$-marked genus-$g$ stable tropical curves.

The “in particular”-statement is a consequence of Lemma 4.23 and the fact that $\mathcal{Y}_{g,n}^{\pi}$ is contained in $\mathcal{Y}_{g,n}^{\pi}$.

**Lemma 4.25.** Let $π: C → B$ and $π': C' → B$ be two families of $n$-marked genus-$g$ pre-stable TPL-curves, and let $f: C → C'$ be a bijective morphism of families of $n$-marked pre-stable TPL-curves such that the induced morphisms $C_b → C'_b$ are isomorphisms for every $b ∈ B$. Then $f$ is an isomorphism of TPL-curves. Moreover, if both $π$ and $π'$ are families of tropical curves and $f$ is linear, then $f$ is an isomorphism of tropical curves.

**Proof.** Let $b ∈ B$. Let $(Σ, (G_σ, f_σ, X_σ)_{σ ∈ Σ})$ and $(Σ', (G'_σ, f'_σ, X'_σ)_{σ ∈ Σ})$ be TPL-trivializations of $π$ and $π'$, respectively. After taking a common refinement of $Σ$ and $Σ'$, we may assume that $Σ = Σ'$. Since $f$ is an isomorphism on fibers it induces for every $σ ∈ Σ$ an isomorphism $G_σ \cong G'_τ$ such that the resulting morphism

$$C ×_{B}^{TPL} σ \xrightarrow{X_σ} G_σ ×_{π_σ}^{TPL} σ \cong G'_τ ×_{π'_τ}^{TPL} σ \xrightarrow{X'_τ} C' ×_{B}^{TPL} σ \quad (56)$$

coincides with the restriction of $f$ to $C ×_{B}^{TPL} σ$. Therefore, the restriction of $f$ to $π^{-1}σ$ is an isomorphism onto its image for all $σ ∈ Σ$, which implies that the restriction of $f$ to $π^{-1}|Σ|$ is an isomorphism of TPL-spaces. Since being an isomorphism is local, we conclude that $f$ is an isomorphism of TPL-spaces.

Now assume that both $π$ and $π'$ are families of tropical curves and that $f$ is linear. It suffices to show that the pull-back $Ω^{1}_{C'} → Ω^{1}_{C}$ is an isomorphism. So let $p ∈ C'$, let $p' = f(p)$, and let $b = π(b)$. Then by the definition of families of semi-stable curves, there exists a commutative diagram

$$
\begin{array}{cccccc}
0 & → & Ω^{1}_{b,b} & → & Ω^{1}_{C',p'} & → & Ω^{1}_{C',p'} \\
\downarrow{id} & & \downarrow{id} & & \downarrow{id} & & \downarrow{id} \\
0 & → & Ω^{1}_{b,b} & → & Ω^{1}_{C,p} & → & Ω^{1}_{C,p} \\
\end{array}
$$

(57)
with exact rows. The leftmost vertical arrow is the identity, and hence an isomorphism, and the rightmost vertical arrow is an isomorphism because the morphism \( C_b \to C'_b \) of the fibers induced by \( f \) is an isomorphism by assumption. Therefore, the vertical arrow in the middle is an isomorphism as well, finishing the proof. \( \square \)

**Proposition 4.26.** Let \( \pi: \mathcal{C} \to B \) be a family of n-marked genus-\( g \) Mumford TPL-curves, let \( b \in B \), and assume we are given a Cartesian diagram of TPL-spaces

\[
\begin{array}{ccc}
C_b & \xrightarrow{f} & \mathcal{V}^{\text{Mf}}_{g,n,\{\star\}} \\
\downarrow & & \downarrow \pi_* \\
\{b\} & \xrightarrow{f_*} & \mathcal{V}^{\text{Mf}}_{g,n}
\end{array}
\]

that is compatible with the sections.

Then there exists a neighborhood \( U \) of \( b \) in \( B \) and morphisms

\[
\begin{align*}
  f & : U \to \mathcal{V}^{\text{Mf}}_{g,n} \\
  f_* & : C_U \to \mathcal{V}^{\text{Mf}}_{g,n,\{\star\}}
\end{align*}
\]

that are compatible with the sections, such that the diagram

\[
\begin{array}{ccc}
C_b & \xrightarrow{f} & \mathcal{V}^{\text{Mf}}_{g,n,\{\star\}} \\
\downarrow & & \downarrow \pi_* \\
\{b\} & \xrightarrow{f_*} & \mathcal{V}^{\text{Mf}}_{g,n}
\end{array}
\]

is commutative and both squares are Cartesian in the category of TPL-spaces. Moreover, if \( \mathcal{C} \to B \) is a family of tropical curves, then both \( f \) and \( f_* \) are linear and the squares are Cartesian in the category of tropical spaces.

**Proof.** Let \( (\Sigma, (G_\sigma, f_\sigma, \chi_\sigma)_{\sigma \in \Sigma}) \) be a TPL-trivialization of \( \pi \) at \( b \), and let \( \tau \in \Sigma \) be the unique stratum with \( b \in \text{relint}(\tau) \). Let \( x \) be the image of \( b \) in \( \mathcal{V}^{\text{Mf}}_{g,n} \), and let \( H \) be the unique cycle-rigidified n-marked genus-\( g \) stable graph with \( x \in \text{relint}(\sigma_H) \). The given isomorphism \( C_b \cong (\mathcal{V}^{\text{Mf}}_{g,n,\{\star\}})_b \) induces an isomorphism \( H \cong G_\tau \) of stable graphs, and thus induces a cycle rigidification on \( G_\tau \). For every face \( \sigma \in \Sigma \), the stable graph \( G_\sigma \) specializes to \( G_\tau \). Since \( \mathcal{C} \to B \) is a family of Mumford curves, the cycle rigidification on \( G_\tau \) lifts uniquely to a cycle-rigidification on \( G_\sigma \). With these cycle rigidifications, each pair of morphisms \((f_\sigma, \chi_\sigma)\) defines a Cartesian square

\[
\begin{array}{ccc}
C_\sigma & \xrightarrow{f_\sigma} & \mathcal{V}^{\text{Mf}}_{g,n,\{\star\}} \\
\downarrow & & \downarrow \pi_* \\
\sigma & \xrightarrow{f_\sigma} & \mathcal{V}^{\text{Mf}}_{g,n}
\end{array}
\]
of TPL-spaces that is compatible with the sections. By construction, the morphisms $f_{\sigma}$ and $f_{\tau}$ glue to morphisms $f: |\Sigma| \to \mathcal{Y}^{\text{MF}}_{g,n}$ and $f_*: \mathcal{C}_|\Sigma| \to \mathcal{Y}^{\text{MF}}_{g,n|\Sigma|}$, so we can take $U = |\Sigma|$.

Now assume that $\mathcal{C} \to \mathcal{B}$ is a family of tropical curves. By Proposition 4.19, both $f$ and $f_*$ are linear. By Lemma 4.25, it follows that the given squares are Cartesian in the category of tropical spaces.

**Theorem 4.27.** The morphism $\mathcal{Y}^{\text{MF}}_{g,n} \to \mathcal{M}^{\text{MF}}_{g,n}$ defined by the family $\pi_*: \mathcal{Y}^{\text{MF}}_{g,n|\Sigma|} \to \mathcal{Y}^{\text{MF}}_{g,n}$ is a covering of $\mathcal{M}^{\text{MF}}_{g,n}$, that is for any morphism $T \to \mathcal{M}^{\text{MF}}_{g,n}$ whose domain is a tropical space, the projection $T \times \mathcal{M}^{\text{MF}}_{g,n} \to T$ is a local isomorphism whose underlying morphism of topological spaces is a covering.

**Proof.** Let $T \to \mathcal{M}^{\text{MF}}_{g,n}$ be a morphism from a tropical space $T$, represented by a family $\mathcal{C} \to T$ of $n$-marked genus-$g$ stable tropical curves. Then the fiber product $F := T \times \mathcal{M}^{\text{MF}}_{g,n}$ $\mathcal{Y}^{\text{MF}}_{g,n}$ represents the functor that assigns to any tropical $T$-space $S \to T$ the set of pairs $(\phi, \chi)$ consisting of a morphism $\phi: S \to \mathcal{Y}^{\text{MF}}_{g,n}$ and an isomorphism $\chi: \phi^*\mathcal{C} \to \phi^*\mathcal{Y}^{\text{MF}}_{g,n|\Sigma|}$. First we show that $F \to T$ is finite and that the cardinality of the fibers is locally constant on $T$. Let $t \in T$, and let $G$ be the combinatorial type of $\mathcal{C}_t$. For every isomorphism type $G'$ of cycle-rigidified graph whose underlying stable graph is isomorphic to $G$ there is a point $x$ in the relative interior of $\Sigma_{G'}$ such that $\mathcal{C}_x$ is isomorphic to $(\mathcal{Y}^{\text{MF}}_{g,n|\Sigma|})_x$. If $C$ denotes the set of cycle-rigidifications of $G$, then there are precisely $|C/\text{Aut}(G)|$ choices for $G'$. By Lemma 4.2, that is $|C/|\text{Aut}(G)||$ many choices. Now for each such choice of $G'$, the group Aut$(G)$ acts naturally on $\Sigma_{G'}$, and it acts transitively on the set of all $x \in \text{relint}(\Sigma_{G'})$ such that $\mathcal{C}_x$ is isomorphic to $(\mathcal{Y}^{\text{MF}}_{g,n|\Sigma|})_x$. Finally, for every such $x$ there are $\text{Aut}(G)_x$ many isomorphisms $\mathcal{C}_x \to (\mathcal{Y}^{\text{MF}}_{g,n|\Sigma|})_x$. Applying the Orbit-Stabilizer Theorem yields that $t$ has precisely

$$|C/|\text{Aut}(G)|| \cdot |\text{Aut}(G)| = |C|$$

inverse images in $F$. We conclude that the number of inverse images of a point $t \in T$ is the number of cycle-rigidifications of the combinatorial type of $\mathcal{C}_t$. To see the that this number is locally constant on $T$, let $(\Sigma, (G_{\sigma}, f_{\sigma}, \chi_{\sigma})_{\sigma \in \Sigma})$ be a TPL-trivialization of $\mathcal{C} \to T$ at a point $t \in T$. Let $\tau \in \Sigma$ be the unique minimal polyhedron in $\Sigma$. Then $t \in \text{relint}(\tau)$, and in particular $G_{\tau}$ is the combinatorial type of $\mathcal{C}_t$. For all $\sigma \in \Sigma$, the combinatorial type $G_{\tau}$ is a specialization of $G_{\sigma}$, so cycle-rigidifications on $G_{\tau}$ lift uniquely to cycle-rigidification of $G_{\sigma}$. Since for every $t' \in \text{relint}(\sigma)$, the curve $\mathcal{C}_{t'}$ is of combinatorial type $G_{\sigma}$, this shows that the fibers of $F \to T$ have the same cardinality on $|\Sigma|$. Together with Proposition 4.26, this finishes the proof.

Since in genus zero every curve is a Mumford curve, we immediately obtain the following corollary.

**Corollary 4.28.** For $n \geq 3$, the stack $\overline{\mathcal{M}}_{g,n}$ is represented by $\overline{\mathcal{Y}}_{0,n} = \mathcal{Y}^{\text{MF}}_{0,n}$.

**Example 4.29.** We have already noted earlier that there does not exist a local isomorphism onto the full moduli space $\overline{\mathcal{M}}_{g,n}$ for any $g > 0$: in fact, the issue is topological;
because the isotropy group of a tropical curve with vertices of genus greater one is too small compared to the isotropy groups of Mumford curves that are close to it in the moduli space, there does not exist a local homeomorphism onto \( \mathcal{M}_g,n \). This beckons the question whether there is a well-behaved class of morphisms of tropical spaces besides local isomorphisms in which one could look for an atlas for \( \mathcal{M}_g,n \). We do not treat this question in detail, but instead provide a computation that makes this seem like a fruitless endeavor. Namely, we discuss the surjections \( f_a : TP^1 \to \mathcal{M}_{1,1} \) associated to the families \( \mathcal{C}^a \to TP^1 \) from Example 3.13. To answer the question if these are “well-behaved”, we compute the fiber products \( F_{a,b} = TP^1 \times_{f_a,\mathcal{M}_{1,1},f_b} TP^1 \) for all \( a, b \in \mathbb{Z} \). If \( p_1, p_2 : TP^1 \times TP^1 \to TP^1 \) denote the projections to the first and second factor, the fiber product \( F_{a,b} \) is given by \( \text{Isom}_{TP^1 \times TP^1}(p_1^* \mathcal{C}^a, p_2^* \mathcal{C}^b) \). This is computed following the proof of Theorem 3.18. Analogously to Example 3.19, one sees that \( F_{a,b} \) is the tropical space obtained by doubling the rays of the tropical subspace

\[
S = \overline{\mathbb{R}}_{\geq 0} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cup \overline{\mathbb{R}}_{\geq 0} \begin{pmatrix} 1 \\ -1 \\ b \end{pmatrix} \cup \overline{\mathbb{R}}_{\geq 0} \begin{pmatrix} -1 \\ -1 \\ b - a \end{pmatrix} \cup \overline{\mathbb{R}}_{\geq 0} \begin{pmatrix} -1 \\ 1 \\ -a \end{pmatrix}
\]

(59)
of \((TP^1)^3\). More precisely, the fiber product \( F_{a,b} \) is obtained by gluing two copies of \( S \) at their respective origins \((0,0,0)\); the affine functions are generated by pull-backs of affine functions on \( S \) on the glued space. The two projections from \( F_{a,b} \) to \( TP^1 \) are given by the composite of the natural map \( F_{a,b} \to S \) and the projections from \( S \) onto the first and second coordinate of \((TP^1)^3\). One first notes that, as expected, neither of the projections is a local isomorphism, or even a local homeomorphism, over the origin \( 0 \in TP^1 \) due to the lack of nontrivial automorphisms of \( \mathcal{C}^a_0 \) and \( \mathcal{C}^b_0 \). Moreover, neither \( F_{a,b} \) nor \( S \) is smooth over the origin even though \( TP^1 \) is; if \( a \neq b \) there even is a non-harmonic affine function on \( F_{a,b} \). In particular, tropical cycles on either of the \( TP^1 \) factors can in general not be pulled back to \( F_{a,b} \) (see Section 6 for the definition of tropical cycles on tropical spaces).

4.3. The moduli space \( \mathcal{M}_{0,n} \) is a tropical toric variety. It is known [SS04, GKM09] that the underlying set of \( \mathcal{M}_{0,n} \) can be mapped onto a fan in affine space using the distances between the markings. Let us quickly recall the details. One embeds \( \mathbb{R}^n \) into \( \mathbb{R}^{n \choose 2} \) (where \( {n \choose 2} \) counts the two-element subsets of \( \{n\} \)) by mapping a tuple \( (x_i) \) to the tuple \( (x_i + x_j)_{\{i,j\}} \). We can then map a moduli point \( [\Gamma] \in \mathcal{M}_{0,n} \) to the point \( d(\Gamma) = (d_{\Gamma}(i,j))_{2 \choose \{i,j\}} \) in the quotient \( \mathbb{R}^{n \choose 2} / \mathbb{R}^n \), where \( d_{\Gamma}(i,j) \) denotes the distance between the vertices opposite to the markings \( i \) and \( j \), respectively. The image of all curves of a given combinatorial type is an open cone, and the closures of these cones define a rational, polyhedral fan in \( \mathbb{R}^{n \choose 2} / \mathbb{R}^n \), which we denote by \( \Sigma_n \).

Remark 4.30. The factor \( \frac{1}{2} \) in the definition of \( d(\Gamma) \) does not appear in [GKM09] or [SS04], but has been shown in [Gro16] to appear when tropicalizing the Plücker-coordinates on the algebraic moduli space \( \mathcal{M}_{0,n}^{alg} \) of \( n \)-marked genus-zero stable curves.
Lemma 4.31. Let $C$ be the subgroup of the integral linear functions on $\mathbb{R}^{(2)} / \mathbb{R}^n$ generated by the functions

$$x_{ij} + x_{kl} - x_{ik} - x_{jl}$$

corresponding to four distinct elements $i, j, k, l \in [n]$. Then the functions in $C$ are precisely the integral linear functions on $\mathbb{R}^{(2)} / \mathbb{R}^n$.

Proof. First of all, note that the functions $x_{ij} + x_{kl} - x_{ik} - x_{jl}$ are integral linear functions on $\mathbb{R}^{(2)}$ that vanish on $\mathbb{R}^n$. Therefore, they define integral linear functions on the quotient space.

Assume there exists a nonzero integral linear function $f$ on $\mathbb{R}^{(2)} / \mathbb{R}^n$ that is not contained in $C$. It can be represented by an integral linear function $F$ on $\mathbb{R}^{(2)}$ that vanishes on $\mathbb{R}^n$. We can uniquely express $F$ as

$$F = \sum_{e \in \mathbb{R}^{(2)}} a_e x_e ,$$

where the coefficients $a_e \in \mathbb{Z}$. Order $[(2)]$ lexicographically. Let $S_F = \{ e \in [(2)] | a_e \neq 0 \}$ and

$$s = \max \{ \min \{ e \in S_F | F \not\in C \} \} ;$$

in simple terms, any function supported on elements strictly greater than $s$ must belong to $C$. It is an elementary linear algebra exercise to verify that if a function’s support is a subset of the four largest coordinate functions, then it is identically zero and hence it belongs to $C$. We may therefore assume that $s = \{ m < k | \} < (n - 3 < n)$. Choose $F \not\in C$ such that $\min(S_F) = s$; since $F$ must vanish on $\mathbb{R}^n$, $k < n$. Choose one element $i \in \{ n - 3, n - 2, n - 1 \} \backslash \{ m, k \}$.

Define $F'$ as

$$F' = F - a_{mk} \cdot (x_{mk} + x_{in} - x_{mn} - x_{ik}) .$$

Then the coefficient of $x_{mk}$ is zero in $F'$ and for all coordinates $x_e$ that appear in $F'$ with nonzero coefficient, we have $e > \{ m < k \}$ . As the integral linear function $f'$ on $\mathbb{R}^{(2)} / \mathbb{R}^n$ induced by $F'$ is still not contained in $C$, this is a contradiction. \hfill $\Box$

Theorem 4.32. The map $d: \mathcal{M}_{0,n} \to \mathbb{R}^{(2)} / \mathbb{R}^n$ is a closed embedding, which extends to an isomorphism $\overline{\mathcal{M}}_{0,n} \to \overline{\Sigma}_n$.

Proof. Since we defined the TPL-space $\mathcal{M}_{0,n}$ as an extended cone complex in the same way as in [SS04, GKM09], it follows that $d$ extends to an isomorphism $\overline{\mathcal{M}}_{0,n} \to \overline{\Sigma}_n$ of TPL-spaces. To show that $d$ is linear, it suffices to show that $d^* [x_{ij} + x_{kl} - x_{ik} - x_{jl}]$ is integral affine on $\mathcal{M}_{0,n}$ for all distinct $i, j, k, l \in [n]$ by Lemma 4.31. For any moduli point $[\Gamma] \in \mathcal{M}_{0,n}$, the value of $x_{ij} + x_{kl} - x_{ik} - x_{jl}$ on $d(\Gamma)$ only depends on the subtree of $\Gamma$ spanned by the leaves marked by $i$, $j$, $k$, and $l$. By checking this on the three combinatorial types for this subtree, one sees that

$$d^*(x_{ij} + x_{kl} - x_{ik} - x_{jl}) = \xi_{i((f_i,t),(f_i,t))} ,$$
where for \( m \in \{ i, j, k, l \} \) we denote by \( f_m \) the leg marked by \( m \) in the \( n \)-marked tree \( T_0 \) consisting of one vertex with \( n \) legs attached. To show that \( d \) is a closed embedding we need to show that at every moduli point \( p \in \mathcal{M}_{0,n} \), the stalk \( \text{Aff} \mathcal{M}_{0,n} \) is generated by pull-backs of integral affine functions via \( d \). It follows from \((64)\) that this is true if \( p = [T_0] \); it then suffices to show that the stalk \( \text{Aff} \mathcal{M}_{0,n} \) is generated by the cross ratios on \( T_0 \). Let \( G \) be the unique \( n \)-marked genus-zero graph with \( p \in \text{relint}(\sigma_G) \), and let \( \xi \) be a primitive cross ratio on \( G \). Then either \( \xi = \xi_{(e,((f_1,f_2),(f_3,f_4)))} \) or \( \xi = \xi_{((f_1,f_2),(f_3,f_4)))} \). For \( 1 \leq i \leq 4 \) choose a leg \( g_i \) of \( G \) in the connected component of \( G \setminus f_i \) that does not contain any other of the \( f_i \)’s (informally, make a simple path in the graph starting in the direction pointed by \( f_i \) until you reach a leg). The marked legs \( g_i \) can also be viewed as legs on \( T_0 \) and one sees that \( \xi = \xi_{((g_1,g_2),(g_3,g_4)))} \), where the latter is defined by a cross ratio datum on \( T_0 \).

To show that \( d \) induces an isomorphism \( \overline{\mathcal{M}}_{0,n} \to \Sigma_n \) we need to compare the affine structures at the infinity points. Let \( x \) be an infinite point of \( \Sigma_n \), and let \( \sigma \in \Sigma_n \) such that \( x \in \mathcal{G}, \) and let \( \tau \) be the face of \( \sigma \) corresponding to the infinite face of \( \mathcal{G} \) that contains \( x \) in its relative interior. By definition, the integral affine functions at \( x \) are precisely the functions in a neighborhood of \( \text{relint}(\sigma) \) that are constant on \( y + \text{relint}(\tau) \) for any \( y \in \text{relint}(\sigma) \). By Lemma \( 4.16 \), the analogous statement holds on \( \mathcal{M}_{0,n} \). As we have already shown that \( d \) defines an isomorphism between \( \mathcal{M}_{0,n} \) and \( \Sigma_n \), this completes the proof. □

5. Tautological morphisms

The universal family over \( \mathcal{M}^M_{g,n} \) is constructed using the forgetful morphism (of tropical spaces) described in Section 4.2. In this section we show the forgetful morphism is a morphism of stacks; to do so, one needs to define the stabilization of a family of stable curves after forgetting a section.

**Definition 5.1.** Let \( \pi : \mathcal{D} \to B \) be a family of \( (n \sqcup \{\ast\}) \)-marked genus-\( g \) stable tropical curves. When one forgets the \( \ast \)-section, the fibers of the family are not \( n \)-marked stable curves because they have an unmarked smooth infinite point. A **stabilization** of \( \pi \) consists of a family \( e : \mathcal{C} \to B \) of \( n \)-marked genus-\( g \) stable tropical curves, together with a morphism \( \phi : \mathcal{D} \to \mathcal{C} \) over \( B \) that respect the genus functions and the \( i \)-marked section for all \( i \in I \), and such that

1. For every \( b \in B \), the induced morphism \( \mathcal{D}_b \to \mathcal{C}_b \) is the stabilization of the smooth tropical curve \( \mathcal{D}_b \) after forgetting the \( \ast \)-marking.
2. \( \mathcal{C} \) has the quotient topology with respect to \( \phi \).
3. For every open subset \( U \subseteq \mathcal{C} \), we have
   \[
   \Gamma(U, \text{PL}_\mathcal{C}) = \{ \phi : U \to \mathbb{R} \mid \phi \circ \phi \in \Gamma(\phi^{-1}U, \text{PL}_\mathcal{D}) \}.
   \] (65)
4. The pull-back morphism \( \text{Aff}_\mathcal{C} \to \phi_* \text{Aff}_\mathcal{D} \) is an isomorphism.

See Figure 10 for an illustration of Definition 5.1. The following Lemma shows that if a stabilization exists, then it is unique up to isomorphism.
Figure 10. The stabilization of a family of curves. The picture shows a neighborhood $U$ of a point in the image under $\phi$ of the $\ast$-marked leg; its inverse image contains infinite $\ast$-marked legs, which implies that any function in $\phi_* \text{Aff}_D(U)$ must have zero slope along the $\ast$-leg.

**Lemma 5.2.** Let $\phi: D \to C$ be the stabilization of an $(n \sqcup \{\ast\})$-marked family of stable tropical curves $D \to B$ after forgetting the $\ast$-section, let $C' \to B$ be an $n$-marked family of stable tropical curves, and let $\phi': D \to C'$ be a morphism over $B$ that exhibits $C'_b$ as the stabilization of $D_b$ for all $b \in B$. Then there exists a unique morphism $\varphi: C \to C'$ such that $\varphi \circ \phi = \phi'$, and this morphism $\varphi$ is an isomorphism of families of tropical curves.

**Proof.** The uniqueness of $\varphi$ immediately follows from the fact that $\phi$ is surjective. To show the existence of $\varphi$, we note that for all $b \in B$, the morphisms

$$\phi|_{D_b}: D_b \to C_b$$

and

$$\phi'|_{D_b}: D_b \to C'_b$$

are both stabilizations of $D_b$. This implies the existence of a bijection $\varphi: C \to C'$ such that $\varphi \circ \phi = \phi'$. This map is continuous because $C$ has the quotient topology with respect to $\phi$, and it is a morphism of tropical spaces because functions on $C$ are piecewise integral linear or integral affine if and only if their pull-backs under $\phi$ are. By Lemma 4.25, it follows that $\varphi$ is an isomorphism of families of tropical curves. $\square$
Lemma 5.3. Let $\pi: \mathcal{D} \to B$ be a family of $(n \sqcup \{\ast\})$-marked stable tropical curves. Then there exists a stabilization of $\pi$ when forgetting the $\ast$-section.

Proof. Because stabilizations are unique if they exists by Lemma 5.2, we can prove the existence locally around a point $b \in B$, that is we may assume that $B = [\Sigma]$ for some TPL-trivialization $(\Sigma, (G_\sigma, f_\sigma, \chi_\sigma)_{\sigma \in \Sigma})$ at $b$. For each $\sigma \in \Sigma$, the stable graph $G_\sigma$ is $(n \sqcup \{\ast\})$-marked. Forgetting the $\ast$-marked leg and stabilizing one obtains an $n$-marked stable graph $G'_\sigma$ and $f_\sigma$ induces a map $f'_\sigma: \sigma \to \Sigma_{G'_\sigma}$. Gluing the families of TPL-curves $\sigma \times \Sigma_{G'_\sigma}$ yields an $n$-marked TPL-curve $\mathcal{C} \to B$ and a morphism $\phi: \mathcal{D} \to \mathcal{C}$ of TPL-spaces such that the induced morphisms $\mathcal{D}_b \to \mathcal{C}_b$ are stabilizations for all $b \in B$. Since $\mathcal{C}_{G'_\sigma}$ has the quotient topology with respect to the natural morphism $\mathcal{C}_{G_\sigma} \to \mathcal{C}_{G'_\sigma}$ and the piecewise linear functions on $\mathcal{C}_{G'_\sigma}$ are precisely those functions which pull-back to piecewise linear functions on $\mathcal{C}_{G_\sigma}$, the same is true for $\mathcal{C}$, that is $\mathcal{C}$ has the quotient topology with respect to $\phi$ and the piecewise linear functions on $\mathcal{C}$ are precisely those functions whose pull-back under $\phi$ is piecewise linear on $\mathcal{D}$. We define an affine structure $\text{Aff}_\mathcal{C}$ on $\mathcal{C}$ by declaring its sections on an open subset $U \subseteq \mathcal{C}$ to be

$$\Gamma(U, \text{Aff}_\mathcal{C}) = \{m \in \Gamma(U, \text{PL}_{\mathcal{C}}) \mid \phi^*m \in \Gamma(\mathcal{C}, \text{Aff}_\mathcal{D})\}$$  \hspace{1cm} (66)

Let $m' \in \Gamma(\mathcal{C}, \text{Aff}_\mathcal{D})$. Since $\phi$ stabilizes $\mathcal{D}$ fiberwise, the function $m'$ is constant on the fibers of $\phi$. Therefore, $m' = \phi^*m$ for a unique function $m$ on $U$, and by definition of $\text{Aff}_\mathcal{C}$ we have $m \in \Gamma(U, \text{Aff}_\mathcal{C})$. It follows that the pull-back defines an isomorphism

$$\text{Aff}_\mathcal{C} \to \phi_* \text{Aff}_\mathcal{D}.$$ \hspace{1cm} (67)

It is left to show that the fibers of $\mathcal{C}$ are stable tropical curves and that the sequences of cotangent sheaves on the fibers of $\mathcal{C}$ are exact. Let $b \in B$. Since

$$0 \to \text{Aff}_{B,b} \to \text{Aff}_{\mathcal{D}_b} \to \Omega^1_{\mathcal{D}_b} \to 0$$  \hspace{1cm} (68)

is an exact sequence of sheaves on $\mathcal{D}_b$, it follows that the lower row in the commutative diagram

$$
\begin{array}{ccc}
0 & \to & \text{Aff}_{B,b} \\
\downarrow & & \downarrow \\
0 & \to & \phi_* \text{Aff}_{B,b}
\end{array}

\begin{array}{ccc}
\to & \text{Aff}_{\mathcal{D}_b} |_{\mathcal{C}_b} & \to & \Omega^1_{\mathcal{C}_b} & \to & 0 \\
\downarrow & & & & & \\
\to & \phi_* (\text{Aff}_{\mathcal{D}_b} |_{\mathcal{C}_b}) & \to & \phi_* \Omega^1_{\mathcal{D}_b} & \to & 0
\end{array}
$$

\hspace{1cm} (69)

is left-exact. Note that since $\phi$ is proper, one has

$$\phi_* (\text{Aff}_{\mathcal{D}_b} |_{\mathcal{C}_b}) \cong (\phi_* \text{Aff}_{\mathcal{D}_b}) |_{\mathcal{C}_b}$$ \hspace{1cm} (70)

so that the diagram makes sense. The fibers of $\phi = \phi |_{\mathcal{D}_b}$ are contractible and $\phi_b$ is proper, so we have $R^1\phi_{b*} \text{Aff}_{B,b} = 0$ and the lower row in the commutative diagram is exact. For the same reason, the vertical morphism to the left is an isomorphism. It follows from the fact that $\phi_* \text{Aff}_{\mathcal{D}_b}$ is an isomorphism, that the vertical morphism in the middle is an isomorphism as well. Together with the fact that the upper
row of the diagram is exact to the right, this implies that the upper row is exact and the vertical morphism to the right is an isomorphism as well. It remains to show that $\operatorname{Aff}_{\mathcal{C}^b}$ is the sheaf of harmonic functions on $\mathcal{C}^b$. Since $R^1\phi_b^*R = 0$ and $\phi_b^*R = R$, the fact that $\Omega_{\mathcal{C}^b}^1 \to \phi_b^*\Omega_{\mathcal{D}^b}^1$ is an isomorphism implies that $\operatorname{Aff}_{\mathcal{C}^b} \to \phi_b^*\operatorname{Aff}_{\mathcal{D}^b}$ is an isomorphism. But because $\phi_b$ is the stabilization of $\mathcal{D}^b$, one has

$$\phi_b^*\operatorname{Aff}_{\mathcal{D}^b} = H_{\mathcal{C}^b}. \quad (71)$$

\[\square\]

**Proposition 5.4.** Choosing for every family of $(n \sqcup \{\ast\})$-marked genus-$g$ stable tropical curves $\mathcal{C} \to B$ a stabilization of $\mathcal{C} \to B$ after forgetting the $\ast$-section defines a morphism $\mathcal{M}_{g,n,\sqcup\{\ast\}} \to \mathcal{M}_{g,n}$.

**Proof.** We need to show that if $\mathcal{D}' \to B'$ is a family of $(n \sqcup \{\ast\})$-marked tropical curves that is the pull-back of a family $(n \sqcup \{\ast\})$-marked tropical curves $\mathcal{D} \to B$ under a morphism $f: B' \to B$, then the pull-back under $f$ of stabilization $\mathcal{C} \to B$ of $\mathcal{D} \to B$ after forgetting the $\ast$-section is the stabilization of $\mathcal{D}' \to B'$ after forgetting the $\ast$-section. Let $\phi: \mathcal{D} \to \mathcal{C}$ be the morphism exhibiting $\mathcal{C}$ as the stabilization of $\mathcal{D}$. Then the composite

$$\mathcal{D}' \to \mathcal{D} \to \mathcal{C} \quad (72)$$

induces a morphism $g: \mathcal{D}' \to f^*\mathcal{C}$. Since $\mathcal{D}'$ and $\mathcal{D}$ have the same fibers and $\mathcal{C}$ is obtained by stabilizing $\mathcal{D}$, the morphism $g$ is a fiber-wise stabilization of $\mathcal{D}'$. By Lemma 5.2, it follows that $f^*\mathcal{C}$ is the stabilization of $\mathcal{D}'$. \[\square\]

**Definition 5.5.** We call the morphism from Proposition 5.4 the **forgetful morphism** associated to the $\ast$-marking, and we also denote it by

$$\pi_{\ast}: \mathcal{M}_{g,n,\sqcup\{\ast\}} \to \mathcal{M}_{g,n}. \quad (73)$$

The following result shows that the forgetful morphisms $\pi_{\ast}: \mathcal{Y}_{g,n,\sqcup\{\ast\}} \to \mathcal{Y}_{g,n}$ used in §4.2 are compatible with the forgetful morphisms $\pi_{\ast}: \mathcal{M}_{g,n,\sqcup\{\ast\}} \to \mathcal{M}_{g,n}$.

**Proposition 5.6.** Let $g, n$ be two non-negative integers such that $n + 2g - 2 > 0$. Then the morphism

$$\mathcal{Y}_{g,n,\sqcup\{\ast, \diamond\}} \to \mathcal{Y}_{g,n,\sqcup\{\ast\}} \times \mathcal{Y}_{g,n} \quad (74)$$

is the stabilization of the family of $(n \sqcup \{\ast\})$-marked genus-$g$ stable tropical curves

$$\mathcal{Y}_{g,n,\sqcup\{\ast, \diamond\}} \to \mathcal{Y}_{g,n,\sqcup\{\ast\}} \quad (75)$$

when forgetting the $\ast$-section.
Proof. Consider the diagram

\[
\begin{array}{c}
\mathcal{M}_{g,n,[t,\infty]} \times \mathcal{M}_{g,n,[0,\infty]} \xrightarrow{\pi_0 \times \pi_*} \mathcal{M}_{g,n,[0,\infty]} \\
\downarrow \pi_0 \quad \downarrow \pi_0 \quad \downarrow \pi_0 \\
\mathcal{M}_{g,n,[t,\infty]} \quad \mathcal{M}_{g,n,[0,\infty]} \quad \mathcal{M}_{g,n,[0,\infty]}
\end{array}
\]

(76)

According to the definition of $\pi_0$ in Section 4.2, the morphism $\pi_0 \times \pi_*$ stabilizes the family $\pi$ fiber-wise. By 5.2 and 5.3, it follows that it is the stabilization. \[ \square \]

Lemma 5.7. Let $\pi: \mathcal{C} \to \mathcal{B}$ be a family of $n$-marked genus-$g$ tropical stable curves and let $x \in \mathcal{C}_b$ be a vertex with $\gamma_{\mathcal{C}_b}(x) = 0$ and $\text{val}(x) \geq 3$. Then, denoting $J = \mathcal{T}_x(\mathcal{C}_b)$, there exist open neighborhoods $U$ and $V$ of $b = \pi(x)$ and $x$, respectively, and linear morphisms $f: U \to \mathcal{M}_{0,J}$ and $f_*: V \to \mathcal{M}_{0,J}$ such that the diagram

\[
\begin{array}{c}
V \xrightarrow{f_*} \mathcal{M}_{0,J} \\
\downarrow \pi \quad \downarrow \pi_* \\
U \xrightarrow{f} \mathcal{M}_{0,J}
\end{array}
\]

is commutative and the induced morphism $V \to U \times_{\mathcal{M}_{0,J}} \mathcal{M}_{0,J}$ is an open immersion.

Proof. For each $v \in T(x(\mathcal{C}_b))$, let $x_v$ be a point in the interior of the edge in $\mathcal{C}_b$ that is adjacent to $x$ in the direction of $v$. By Lemma 3.20, there exists an open neighborhood $U$ of $b$, neighborhoods $W_v$ of $x_v$ for all $v \in T(x(\mathcal{C}_b))$, an $\epsilon > 0$, and isomorphisms $W_v \cong (0, \epsilon) \times U$ over $b$. Let $V'$ the connected component of

\[
\mathcal{C}_b \setminus \bigcup_{v \in T_x(\mathcal{C}_b)} W_v
\]

(77)

containing $x$, and let

\[
V = V' \cup \bigcup_{v \in T_x(\mathcal{C}_b)} W_v.
\]

(78)

Gluing for each $v \in T_x(\mathcal{C}_b)$ a copy of $(0, \infty) \times U$ to $V$ via the identification

\[
W_v \cong (0, \epsilon) \times U,
\]

(79)

we obtain a family $\mathcal{C} \to U$ of $J$-marked genus-zero stable tropical curves over $U$. By Corollary 4.28, there exist a Cartesian diagram

\[
\begin{array}{c}
\mathcal{C} \xrightarrow{\pi_*} \mathcal{M}_{0,J} \\
\downarrow \pi \quad \downarrow \pi_* \\
U \xrightarrow{f} \mathcal{M}_{0,J}
\end{array}
\]

(80)

Noting that by construction, $V$ is an open subset of $\mathcal{C}$ finishes the proof. \[ \square \]
Remark 5.8. Lemma 5.7 may be generalized to apply to points \( x \) in the interior of an edge of a fiber (where \( \text{val}(x) = 2 \)); one must attach products of \( U \) with a tripod, rather than just copies of \( (0, \infty) \), to the sets \( W_{\nu} \).

Lemma 5.9. Let \( \pi : \mathcal{C} \to B \) be a family of \( n \)-marked genus-\( g \) tropical stable curves and let \( t : B \to \mathcal{C} \) be a linear section such that \( \gamma_{t}(t(b)) = 0 \) for all \( b \in B \). Then there exists a family of \((n \sqcup \{\ast\})\)-marked genus-\( g \) tropical stable curves \( \mathcal{D} \to B \) and a morphism \( \phi : \mathcal{D} \to \mathcal{C} \) over \( B \) that exhibits \( \mathcal{C} \to B \) as the stabilization of \( \mathcal{D} \to B \) after forgetting the \( \ast \)-section \( s_{\ast} : B \to \mathcal{D} \) and such that \( \phi \circ s_{\ast} = t \). Moreover, \( \phi : \mathcal{D} \to \mathcal{C} \) is unique up to unique isomorphism, that is if \( \phi' : \mathcal{D}' \to \mathcal{C} \) is a second such family, then there exists a unique isomorphism \( \varphi : \mathcal{D} \to \mathcal{D}' \) with \( \phi' \circ \varphi = \phi \).

Proof. Since any morphism \( \phi : \mathcal{D} \to \mathcal{C} \) in question is an isomorphism over \( \mathcal{C} \setminus t(B) \), the construction of \( \phi \) is local around \( t(B) \). Using Lemma 5.7, we may thus assume that \( \mathcal{C} \to B \) is a family tropical curves of genus zero. For the remainder of the proof, refer to the following diagram:

\[
\begin{array}{c}
\mathcal{D} \\
\downarrow f_{\ast} \\
\mathcal{C} \\
\downarrow \pi_{\ast} \\
B \\
\end{array}
\quad \xymatrix{ 
\pi_{\ast} \times \pi_{\ast} \\
\ar^{\pi_{\ast}}[u] \\
\ar_{\pi_{\ast}}[d] \\
\mathcal{M}_{0, n_{\ast}} \\
\ar_{\pi_{\ast}}[u] \\
\ar^{\pi_{\ast}}[d] \\
\mathcal{M}_{0, n} \\
\ar_{\pi_{\ast}}[u] \\
\ar^{\pi_{\ast}}[d] \\
\mathcal{M}_{0, n}\} \\
\end{array}
\]

(81)

By Corollary 4.28, there exist morphisms \( f : B \to \mathcal{M}_{0, n} \) and \( f_{\ast} : \mathcal{C} \to \mathcal{M}_{0, n, \ast} \) identifying \( \mathcal{C} \) with \( f_{\ast}(\mathcal{M}_{0, n, \ast}) \). Since \( t \) is a section of \( \pi_{\ast} \), we obtain an induced identification of \( \mathcal{C} \) with \( (f_{\ast} \circ t)^{\ast}(\mathcal{M}_{0, n, \ast} \times_{\mathcal{M}_{0, n}} \mathcal{M}_{0, n, 0}) \). It follows from Proposition 5.6 that the morphism

\[
\mathcal{D} := (f_{\ast} \circ t)^{\ast}(\mathcal{M}_{0, n, \ast}) \to (f_{\ast} \circ t)^{\ast}(\mathcal{M}_{0, n, \ast} \times_{\mathcal{M}_{0, n}} \mathcal{M}_{0, n, 0}) \cong \mathcal{C}
\]

is a stabilization such that the \( \ast \)-section of \( \mathcal{D} \) is mapped to \( t \) in \( \mathcal{C} \). This shows existence. Uniqueness follows from the fact that a family of genus-zero stable tropical curves is uniquely determined by its fibers. \( \square \)

We may now use the construction from Lemma 5.9 to define sections \( s_{i} \) as morphisms of moduli spaces.

Proposition 5.10. The morphism \( s_{i} : \mathcal{M}_{g, n} \to \mathcal{M}_{g, n, \ast} \) is defined as follows. To a family of \( n \)-marked genus-\( g \) stable tropical curves \( \mathcal{C} \to B, \{s_{i}\}_{i \in I}, \gamma_{\mathcal{C}} \), assign a family of \((n \sqcup \{\ast\})\)-marked genus-\( g \) curves \( \mathcal{D} \to B \) such that:

- there is a morphism \( \phi : \mathcal{D} \to \mathcal{C} \) over \( B \) exhibiting \( \mathcal{C} \) as the stabilization of \( \mathcal{D} \) after forgetting the \( \ast \)-section;
- \( \phi \circ s_{\ast} = s_{i} \) after forgetting the \( \ast \)-section.
Proof. By Lemma 5.9, the morphism \( \phi : \mathcal{D} \to \mathcal{C} \) exists and is unique up to unique isomorphism. To show that this defines a morphism of stacks one needs to show that the choice of \( \mathcal{D} \) is compatible with fiber products. For this, one proceeds similarly as in the proof of Proposition 5.4. \( \square \)

Definition 5.11. We call the morphism \( s_i : M_{g,n} \to M_{g,n} \sqcup \{ \star \} \) from Proposition 5.10 the \( i \)-th section.

Theorem 5.12. The forgetful morphism \( \pi_* : M_{g,n}^{Mf} \to M_{g,n}^{Mf} \) represents the universal family. More precisely, if \( \mathcal{B} \to \mathcal{M}_{g,n} \) is defined via a family of \( n \)-marked genus-\( g \) tropical Mumford curves \( \pi : \mathcal{C} \to \mathcal{B} \), then there is a morphism \( \mathcal{C} \to \mathcal{M}_{g,n}^{Mf} \sqcup \{ \star \} \) such that the resulting square

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\pi} & M_{g,n}^{Mf} \\
\downarrow{\pi} & & \downarrow{\pi_*} \\
\mathcal{B} & \xrightarrow{s_\star} & M_{g,n}^{Mf} \\
\end{array}
\]

is Cartesian. Moreover, the pull-back of the \( i \)-th section from \( M_{g,n}^{Mf} \) to \( \mathcal{B} \) coincides with the \( i \)-th section on \( \mathcal{B} \).

Proof. For any tropical space \( T \to \mathcal{B} \) over \( \mathcal{B} \), the groupoid of morphisms from \( T \) to \( \mathcal{B} \times M_{g,n}^{Mf} \sqcup M_{g,n}^{Mf} \) over \( \mathcal{B} \) has as objects the pairs \((\mathcal{D}, \phi)\) consisting of a family \((n \sqcup \{ \star \})\)-marked genus-\( g \) tropical Mumford curves and a morphism \( \phi : \mathcal{D} \to \mathcal{C} \) of \( n \)-marked families of tropical curves that exhibits \( \mathcal{C} = \mathcal{f}_g \mathcal{C} \) as the stabilization of \( \mathcal{D} \) after forgetting the \( \star \)-section. By Lemma 5.9, between any two objects in this groupoid there is at most one isomorphism. Therefore, we can take isomorphism classes of these pairs \((\mathcal{D}, \phi)\) and represent \( \mathcal{B} \times M_{g,n}^{Mf} \sqcup M_{g,n}^{Mf} \) by a functor. Again by Lemma 5.9, the equivalence class of one of the pairs \((\mathcal{D}, \phi)\) only depends on the composite \( \phi \circ s_\star \) of the \( \star \)-section and \( \phi \). Lemma 5.9 also states that every linear section \( T \to \mathcal{C} \) appears as \( \phi \circ s_\star \) for some pair \((\mathcal{D}, \phi)\). Therefore, the fiber product \( \mathcal{B} \times M_{g,n}^{Mf} \sqcup M_{g,n}^{Mf} \) is the functor assigning to \( T/\mathcal{B} \) the sections of \( \mathcal{C} \). This is naturally equivalent to the functor represented by the family \( \pi : \mathcal{C} \to \mathcal{B} \). The statement about the sections follows immediately. \( \square \)

6. Tropical intersection theory on \( \mathcal{M}_{g,n} \)

The main goal of this section is to define \( \psi \)-classes on \( \mathcal{M}_{g,n} \). To achieve this goal, we need to develop some elements of intersection theory for tropical spaces and stacks over the category of tropical spaces. For a computational approach to tropical intersection theory, in particular of moduli spaces for curves of genus zero, see the POLYMAKE [GJ00] extension a-tint [Ham14].

6.1. Cycles and divisors on tropical spaces and stacks. Much of the tropical intersection theory developed for tropical varieties in \( \mathbb{R}^n \) [Mik06, AR10, GKM09, Rau16, Flo14,
and more generally for weakly embedded cone complexes [Gro18], tropical manifolds [Sha13, FR13], and rational polyhedral spaces [JRS18, GS19], carries over to tropical spaces. We explain the adaptions that have to be made.

6.1.1. Tropical cycles. Let \( X \) be a tropical space. We say that a function \( f: X \to \mathbb{Z} \) is **constructible**, if at every \( x \in X \) there exists a local face structure \( \Sigma \) such that \( f|_{\sigma} \) is constant for every \( \sigma \in \Sigma \). For a constructible function \( f: X \to \mathbb{Z} \), we denote by

\[
|f| = \{ x \in X \mid f(x) \neq 0 \} \tag{83}
\]

its support. Let \( W_k(X) \) denote the quotient of the Abelian group of constructible functions \( X \to \mathbb{Z} \) whose support is at most \( k \)-dimensional by the subgroup of all constructible functions \( X \to \mathbb{Z} \) whose support is at most \( (k-1) \)-dimensional. Given an element \( \alpha \in W_k(X) \) represented by a function \( f: X \to \mathbb{Z} \), the closure of the set of all points \( x \in |f| \) where the local dimension of \( |f| \) is \( k \) only depends on \( \alpha \). We call it the **support** of \( \alpha \) and denote it by \( |\alpha| \).

Let \( \phi: X \to Y \) be a morphism of TPL-spaces, where \( X \) is purely \( n \)-dimensional and \( \dim(Y) = m \). Let \( x \in X \). Then there exist local face structures \( \Sigma \) and \( \Delta \) of \( X \) and \( Y \) at \( x \) and \( \phi(x) \), respectively, such that \( \phi(\sigma) \in \Delta \) and the restriction \( \phi|_{\sigma}: \sigma \to \phi(\sigma) \) is linear for all \( \sigma \in \Sigma \). For every point \( z \) in the relative interior of some \( n \)-dimensional \( \sigma \in \Sigma \) we define the **multiplicity** \( \text{mult}_z(\phi) \) of \( \phi \) at \( z \) as follows: if \( \dim(\phi(\sigma)) < n \), then \( \text{mult}_z(\phi) = 0 \), otherwise \( \phi|_{\sigma} \) can be expressed in suitable coordinates as \( \mathbb{R}^n \to \mathbb{R}^m, x \mapsto Ax + b \) for appropriate choices of a matrix \( A \in \mathbb{Z}^{m \times n} \) and a vector \( b \in \mathbb{R}^m \), and we set \( \text{mult}_z(\phi) \) equal to the index of \( A(Z^n) \) in \( Z^m \). Note that this can be expressed as the gcd of all basic minors of \( A \). The domain of \( z \mapsto \text{mult}_z(\phi) \) depends on the choices of \( \Sigma \) and \( \Delta \), but its values do not. Therefore, we obtain a well-defined element \( \text{mult}(\phi) \in W_n(X) \).

Still assuming that \( X \) is purely \( n \)-dimensional, we say that \( |\alpha| \in W_n(X) \) is a **tropical \( n \)-cycle** if it satisfies the following **balancing condition**: for every open set \( U \subseteq X \), affine function \( \phi \in \Gamma(U, \text{Aff}_X) \), local face structure \( \Sigma \) on \( U \) such that \( \alpha \) and \( \text{mult}(\phi) \) have constant value on the relative interiors of any \( \sigma \in \Sigma \), and for every \( (n-1) \)-dimensional polyhedron \( \tau \in \Sigma \) with \( \phi|_\tau = 0 \) we have

\[
\sum_{\sigma : \tau \subseteq \sigma} \alpha \cdot \text{sign}(\phi|_{\sigma}) \cdot \text{mult}(\phi) = 0 \tag{84}
\]

where the sum is taken over all \( n \)-dimensional \( \sigma \in \Sigma \) containing \( \tau \). Since \( \phi|_\tau = 0 \), the sign \( \text{sign}(\phi|_{\sigma}) \) of \( \phi \) on \( \sigma \) is well-defined for all \( \sigma \in \Sigma \) containing \( \tau \). We denote the subgroup of \( W_n(X) \) consisting of all tropical \( n \)-cycles by \( Z_n(X) \).

Now assume that \( X \) is an arbitrary tropical space, and let \( k \) be a nonnegative integer. The group \( Z_k(X) \) of **tropical \( k \)-cycles** is the subgroup consisting of all elements of \( W_k(X) \) represented by a function \( \alpha: X \to \mathbb{Z} \) such that there exists a purely \( k \)-dimensional locally polyhedral subset \( Y \subseteq X \) such that \( \alpha|_{X \setminus Y} = 0 \) and \( \alpha|_Y \) represents an element in \( Z_k(Y) \).

**Example 6.1.** If \( X \) is a smooth connected tropical curve, then the tropical one-cycles on \( X \) are represented by constant functions \( X \to \mathbb{Z} \).
Example 6.2. If $X$ can be embedded in $\mathbb{R}^d$ for some $d$, then our definition of balancing is equivalent to the definition in [AR10]. In particular, the constant function with value one defines a tropical cycle $[\mathcal{M}_{0,n}]$ on $\mathcal{M}_{0,n}$ by Theorem 4.32 and [GKM09, Theorem 3.7]. The proof in [GKM09] generalizes easily to higher genus in that the constant function with value one defines a tropical cycle $[\mathcal{M}_{g,n}]$ on $\mathcal{M}_{g,n}$.

Let $f: X \to Y$ be a proper morphism of tropical spaces, and let $\alpha \in W_k(X)$. Let $f_\alpha: |\alpha| \to f(|\alpha|)$ denote the morphism induced by $f$. Then we define the push-forward $f_* \alpha$ by $f_* \alpha = 0$ if $\dim(f(|\alpha|)) < k$, and by

$$f_* \alpha: f(|\alpha|) \to \mathbb{Z}, \; y \mapsto \sum_{x \in f_\alpha^{-1}(y)} \text{mult}_x(f_\alpha) \cdot \alpha(x)$$

otherwise. Note that $f_* \alpha$ is a well-defined element of $W_k(Y)$ when extended by zero to all of $Y$. If $\alpha \in Z_k(X)$, then a straightforward adaption of the proof of [GKM09, Proposition 2.25] to the setting of tropical spaces shows that $f_* \alpha \in Z_k(Y)$.

6.1.2. Tropical Cartier divisors.

**Definition 6.3.** Let $X$ be a tropical space. A piecewise linear function $\phi \in \Gamma(X, \text{PL}_X)$ is a rational function if at every $x \in X$ there exists a local face structure $\Sigma$ such that for every $\sigma \in \Sigma$ there are an open neighborhood $U$ of $\sigma$ and $m \in \Gamma(U, \text{Aff}_X)$ with $\phi|_\sigma = m|_\sigma$.

We denote by $\text{Rat}_X$ the subsheaf of $\text{PL}_X$ whose sections over an open subset $U \subseteq X$ are rational functions on $U$.

By definition, every integral affine function is rational.

**Definition 6.4.** Let $X$ be a tropical space. The sheaf of tropical Cartier divisors on $X$, denoted by $\text{Div}_X$, is defined by

$$\text{Div}_X := \text{Rat}_X / \text{Aff}_X.$$  

The group $\text{Div}(X)$ of tropical Cartier divisors on $X$ is defined by

$$\text{Div}(X) := \Gamma(X, \text{Div}_X).$$

For any $\phi \in \Gamma(X, \text{Rat}_X)$ we denote its image in $\text{Div}(X)$ by $\text{div}(\phi)$.

Let $X$ be a purely $n$-dimensional tropical space, let $\alpha \in Z_n(X)$ be a tropical $n$-cycle on $X$, and let $\phi \in \Gamma(X, \text{Rat}_X)$ be a rational function on $X$. By definition, there exists at any $x \in X$ a local face structure $\Sigma$ such that for every $\sigma \in \Sigma$ there exists a neighborhood $U_\sigma$ of $\sigma$ and an integral affine function $m_\sigma \in \Gamma(U, \text{Aff}_X)$ such that $\phi|_\sigma = m_\sigma|_\sigma$, and such that $\alpha$ is constant on the relative interior of every $\sigma \in \Sigma$. Let $\tau \in \Sigma$ be an $(n-1)$-dimensional polyhedron. Since $\alpha$ is a tropical cycle, the sum

$$d_\tau = \sum_{\sigma: \tau \leq \sigma} \text{sign}(m_\sigma - m_\tau) \cdot \text{mult}(m_\sigma - m_\tau) \cdot \alpha$$

is independent of the choice of $m_\tau$. Define a function $|\Sigma| \to \mathbb{Z}$ whose value is $d_\tau$ on the relative interior of an $(n-1)$-dimensional polyhedron $\tau \in \Sigma$, and zero everywhere else.
Since this construction is local, it defines an element in $W_{n-1}(X)$ which we denote by $\text{div}(\phi) \cdot \alpha$. If $\phi \in \Gamma(X, \text{Aff}_X)$, then by construction, one has $\text{div}(\phi) \cdot \alpha = 0$. So because the construction of $\text{div}(\phi) \cdot \alpha$ is local and bilinear, it induces a bilinear pairing

$$\text{Div}(X) \times \mathbb{Z}_n(X) \to W_{n-1}(X).$$

(89)

We denote the image of a pair $(D, \alpha)$ under this pairing by $D \cdot \alpha$. Exactly as in [AR10, Proposition 3.7 (a)], one proves that $D \cdot \alpha \in \mathbb{Z}_{n-1}(X)$.

The above construction generalizes to any tropical space $X$, $D \in \text{Div}(X)$ and $\alpha \in Z_k(X)$ for any non-negative integer $k$, by setting

$$D \cdot \alpha := D \mid_{|\alpha|} \cdot \alpha.$$

(90)

For any $k \geq 0$ one thus obtains a bilinear pairing

$$\text{Div}(X) \times Z_k(X) \to Z_{k-1}(X), \ (D, \alpha) \mapsto D \cdot \alpha.$$  

(91)

**Remark 6.5.** The constant function with value one on $X = \mathbb{T}$ defines a cycle $[X] \in Z_1(X)$. One would expect that for every zero-cycle $\alpha \in Z_0(X)$ there exists a divisor $D \in \text{Div}(X)$ with $D \cdot [X] = \alpha$. However, with our definitions this is true only if $|\alpha|$ does not contain the point $\infty$. This can be fixed by modifying the sheaf $\text{Rat}_X$ in a way that allows functions with value $\pm \infty$ so that there would be a rational function on $X$ approaching $\infty$ with nonzero slope. As doing so would introduce several technical difficulties without increasing the generality of our statements, we decided to work with the simpler version of $\text{Rat}_X$ (and hence $\text{Div}(X)$) introduced above.

**Definition 6.6.** Given a tropical space $X$, we define the weak Chow group $\mathbb{A}_s(X)$ of $X$ as the quotient of $Z_0(X)$ by its subgroup generated by all elements of the form $\text{div}(\phi) \cdot \alpha$, where $\alpha \in Z_k(X)$ for some non-negative integer $k$, and $\phi \in \Gamma(|\alpha|, \text{Rat}_{|\alpha|})$. We say that two tropical cycles $\alpha$ and $\beta$ on $X$ are weakly rational equivalent if they have the same image in $\mathbb{A}_s(X)$.

Exactly as in [AR10, Proposition 3.7 (b)], one proves that for any two divisors $D, E \in \text{Div}(X)$ and $\alpha \in Z_0(X)$ one has

$$D \cdot (E \cdot \alpha) = E \cdot (D \cdot \alpha).$$

(92)

It follows from this that every divisor $D \in \text{Div}(X)$ defines a morphism

$$\mathbb{A}_s(X) \to \mathbb{A}_s(X), \ [\alpha] \mapsto [D \cdot \alpha].$$

(93)

If $X$ is a compact tropical space, then every tropical zero-cycle $\alpha$ on $X$ can only have finite support. Therefore, one can integrate zero-cycles and obtain an integer

$$\int_X \alpha := \sum_{x \in X} \alpha(x),$$

(94)

the degree of $\alpha$. Since the sum over all points of the sum of all outgoing slopes at a point of a rational function on a compact tropical curve is zero, the degree induces a
map

\[ \int_X : A_0(X) \to \mathbb{Z} \]  \hspace{1cm} (95)

on the Chow group of a compact tropical curve \( X \).

6.1.3. Tropical line bundles. If \( X \) is a tropical space, then a tropical line bundle may be described in the following equivalent ways:

1. a tropical space \( \pi : \mathcal{L} \to X \) over \( X \) such that \( X \) can be covered by open subsets \( U \) such that \( \mathcal{L}_U = \pi^{-1}U \cong U \times \mathbb{T} \) over \( U \).

2. an \( \text{Aff}_X \)-torsor, that is a sheaf \( \mathcal{A} \) of sets with an \( \text{Aff}_X \)-action such that \( \mathcal{A}_x \) is an \( \text{Aff}_{X,x} \)-torsor for all \( x \in X \).

3. a cocycle \( \{ \xi_{\alpha,\alpha'} \} \) determining an element of \( H^1(X, \text{Aff}_X) \).

If \( \mathcal{L} \) is a tropical space as in (1), the linear sections \( U \to \mathcal{L}_U \) whose values are never infinite form an \( \text{Aff}_X \)-torsor, showing that (1) implies (2). Given an \( \text{Aff}_X \)-torsor \( \mathcal{A} \), differences of local sections for the torsor give the elements of a cocycle, yielding (2) implies (3). Finally, a cocycle provides patching data to construct a tropical space \( \mathcal{L} \), showing that (3) implies (1).

**Example 6.7.** Given a family of tropical curves \( \pi : \mathcal{C} \to B \) together with an affine linear section \( s : B \to \mathcal{C} \) supported on genus zero-points, the sheaf \( \text{Aff}_\mathcal{C}(xa) \) from Definition 3.23 defines a tropical line bundle on \( \mathcal{C} \).

**Example 6.8** (Line bundles on \( TP^1 \)). For \( a \in \mathbb{Z} \), we describe a line bundle that we denote \( \mathcal{O}_{TP^1}(a) \) in the three different ways introduced in this section:

1. Let \( \mathcal{L} = \mathcal{L}_{-\infty} \cup \mathcal{L}_\infty \), where \( \mathcal{L}_{-\infty} = (-\infty, \infty) \times \mathbb{T} \), with affine integral coordinates \( (x_-, y_-) \) and \( \mathcal{L}_\infty = (-\infty, \infty) \times \mathbb{T} \), with affine integral coordinates \( (x_+, y_+) \). The two charts are glued together via the transition functions:

\[ x_+ = x_- \quad y_+ = y_- + ax_- \]  \hspace{1cm} (96)

Projection to the first factor provides a map to \( TP^1 \), and by construction \( \mathcal{L} \) is covered by two open sets \( U \) with \( \mathcal{L}_U \cong U \times \mathbb{T} \). We wish to consider the constant section \( s_0 : (-\infty, \infty) \to \mathcal{L}_{-\infty} \) given by \( y_- = 0 \), and observe that in the \( + \)-coordinates the local equation for \( s_0 \) is \( y_+ = ax_+ \).

2. Consider the sheaf \( \text{Aff}_{TP^1}(a\infty) \): its sections in a (small) neighborhood of \( -\infty \) are constant, whereas its sections in a neighborhood of \( +\infty \) have slope \(-a\). Given a section of \( \text{Aff}_{TP^1}(a\infty) \), one may obtain a section of \( \mathcal{L} \) by adding to it the distinguished section \( s_0 \).

3. Given two open sets \([-\infty, \infty) \) and \((-\infty, \infty] \) covering \( TP^1 \), we get the cocycle

\[ \xi_{-\infty,-\infty} = ax. \]  \hspace{1cm} (97)

**Definition 6.9.** Let \( X \) be a tropical space. The **Picard group** \( \text{Pic}(X) \) of \( X \) is given by

\[ \text{Pic}(X) = H^1(X, \text{Aff}_X). \]  \hspace{1cm} (98)
A rational section $s$ of a tropical line bundle $\mathcal{L} \to X$ is a continuous section $s : X \to \mathcal{L}$ such that whenever $U \subseteq X$ is an open subset and $\mathcal{L}_U \cong T \times U$ is an isomorphism, the composite
\[ U \xrightarrow{s|_U} \mathcal{L}_U \to T \times U \to T \] is contained in $\Gamma(U, \text{Rat}_X)$.

Given a tropical Cartier divisor $D \in \text{Div}(X)$, one defines a tropical line bundle $\mathcal{L}(D)$ by taking the image of $D$ under the connecting morphism $\text{Div}(X) \to \text{Pic}(X)$ associated to the short exact sequence
\[ 0 \to \text{Aff}_X \to \text{Rat}_X \to \text{Div}_X \to 0. \] (100)

The bundle $\mathcal{L}(D)$ comes with a canonical rational section, and conversely every rational section of a tropical line bundle $\mathcal{L}$ defines a Cartier divisor. Just as in algebraic geometry, one obtains a bijection between $\text{Div}(X)$ and isomorphism classes of Pairs $(\mathcal{L}, s)$ consisting of a tropical line bundle $\mathcal{L}$ on $X$ and a rational section $s$ of $\mathcal{L}$. Differently from algebraic geometry, tropical line bundles need not have any rational section, as shown in Example 6.10.

Example 6.10. Let $X = \mathbb{R}$ be the tropical space with the exotic affine structure generated by the restriction of the identity to $(-1,1)$: it satisfies $\text{Aff}_x = \mathbb{Z} \oplus \mathbb{R}$ for $-1 < x < 1$, and $\text{Aff}_{\mathcal{L}} = \mathbb{R}$ otherwise. Let $U_1 = (-\infty,1)$ and $U_{-1} = (-1,\infty)$ be an open cover of $X$, and let $m \in \text{Aff}_x(-1,1) \setminus \mathbb{R}$ be a cocycle defining a line bundle $\mathcal{L}$ on $X$. Since the only sections of $\text{Aff}_x(U_{\pm 1})$ are constants, we deduce that $\mathcal{L}$ does not admit any rational section.

We denote by $\text{RPic}(X)$ the subgroup of $\text{Pic}(X)$ consisting of all line bundles that admit a rational section. In other words, $\text{RPic}(X)$ is the image of $\text{Div}(X)$ in $\text{Pic}(X)$. For every $\mathcal{L} \in \text{RPic}(S)$ there exists a divisor $D \in \text{Div}(X)$ such that $\mathcal{L} = \mathcal{L}(D)$. This divisor induces a map
\[ \Lambda_* \mathcal{L} \to \Lambda_* \alpha, \quad \alpha \mapsto D \cdot \alpha, \] (101)
which does not depend on the choice of $D$. This map is called the first Chern class of $\mathcal{L}$, denoted by $c_1(\mathcal{L})$. The image of $\alpha \in \Lambda_* \mathcal{L}$ under $c_1(\mathcal{L})$ is denoted by $c_1(\mathcal{L}) \cdot \alpha$.

Remark 6.11. If every point of a tropical space $X$ has a neighborhood that allows a closed embedding into $T^n$ for some positive integer $r$, then the sheaf $\text{Rat}_X$ agrees with the sheaf $\mathcal{M}$ from [JRS18]. It is shown in Lemma 4.5 in loc. cit. that $H^1(X, \mathcal{M}) = 0$, which implies that $\text{RPic}(X) = \text{Pic}(X)$ in this case.

6.1.4. Line bundles and their Chern classes on stacks over tropical spaces. We define divisors and line bundles on a stack $\mathcal{M}$ over the site of tropical spaces with representable diagonal. Because we want our definitions to apply to $\mathcal{M}_{g,n}$, we do not assume that $\mathcal{M}$ has an atlas. The lack of an atlas makes it impossible to define a group of tropical cycles on $\mathcal{M}$ and we settle with considering tropical cycles on tropical spaces $T$ equipped with a morphism $T \to \mathcal{M}$. 
Remark 6.12. One might be tempted to define tropical cycles on the stack \( \overline{\mathcal{M}}_{g,n} \) as balanced functions on its underlying set \( \overline{\mathcal{M}}_{g,n} = \mathcal{M}_{g,n}(pt) \) of isomorphism classes of \( n \)-marked genus-\( g \) stable tropical curves. The underlying set \( \overline{\mathcal{M}}_{g,n} \) does have a natural TPL-structure, obtained for example by viewing it as a quotient of the underlying set \( \mathcal{M}_{g,n} \), but since there is no natural surjection from a tropical space onto \( \overline{\mathcal{M}}_{g,n} \), this TPL-space \( \overline{\mathcal{M}}_{g,n} \) does not have a natural affine structure. In particular, there is no notion of balancing on \( \overline{\mathcal{M}}_{g,n} \), and thus no notion of tropical cycles. In the case \( g = n = 1 \) this is illustrated by Example 4.29: there is a surjection \( f_a : \text{TP}^1 \to \mathcal{M}_{1,1} \) for every \( a \in \mathbb{Z} \) via which one could induce an affine structure on \( \overline{\mathcal{M}}_{1,1} \); however, the fiber products \( \text{TP}^1 \times_{f_a, \mathcal{M}_{1,1}, f_b} \text{TP}^1 \) are pathological: for example, the constant weights do not define a one-cycle on them if \( a \neq b \). This shows that these affine structure are incompatible.

It is possible to define tropical line bundles and their Chern classes on \( \mathcal{M} \). A tropical line bundle on \( \mathcal{M} \) is given by specifying for every morphism \( T \to \mathcal{M} \) a line bundle \( f^* \mathcal{L} \) in a way that is compatible with pull-backs. Since line bundles can be tensored in a way compatible with pull-backs, all tropical line bundles on \( \mathcal{M} \) form a group, the tropical Picard group \( \text{Pic}(\mathcal{M}) \) of \( \mathcal{M} \). By construction, one can pull-back tropical line bundles along any morphism from a tropical space to \( \mathcal{M} \). The first Chern class of \( \mathcal{L} \) on the stack may be defined using the first Chern class of \( f^* \mathcal{L} \) for all suitable pull-backs.

Definition 6.13. Given a tropical line bundle \( \mathcal{L} \) on \( \mathcal{M} \), a morphism \( T \to \mathcal{M} \) such that \( f^* \mathcal{L} \in \text{RPic}(T) \), and a cycle \( \alpha \in \mathbb{Z}^*_+(T) \), we define the cycle

\[
c_1(\mathcal{L}) \cdot \alpha := c_1(f^* \mathcal{L}) \cdot \alpha \tag{102}
\]

We call the collections of the morphisms

\[
\mathbb{A}_*(T) \to \mathbb{A}_*(T), \quad \alpha \mapsto c_1(\mathcal{L}) \cdot \alpha \tag{103}
\]

for all maps \( T \to \mathcal{M} \) such that \( f^* \mathcal{L} \in \text{RPic}(T) \) the first Chern class of \( \mathcal{L} \) on \( \mathcal{M} \).

6.2. Boundary divisors and \( \psi \)-classes. We begin by defining tropical line bundles that are closely related to the cone subcomplexes at infinity that are usually thought of as tropical boundary divisors ([ACP15]).

Definition 6.14. Let \( g,n \in \mathbb{Z}_{\geq 0} \) with \( n + 2g - 2 > 1 \), and let \( i,j \in I \) with \( i \neq j \). We define the boundary line bundle \( \mathcal{L}(D_{ij}) \in \text{Pic}(\overline{\mathcal{M}}_{g,n}) \) by defining

\[
f^* \mathcal{L}(D_{ij}) = s_i^* \phi^*(\text{Aff}_{C}(\phi \circ s_j)) \tag{104}
\]

for a morphism \( f : B \to \overline{\mathcal{M}}_{g,n} \) corresponding to a family of tropical stable curves \( \mathcal{D} \to B \) with sections \( (s_k)_{k \in I} \), where \( \phi : \mathcal{D} \to C \) is the stabilization of \( \mathcal{D} \) after forgetting the \( i \)-th mark.

Remark 6.15. As observed in Remark 6.5, our definition of tropical Cartier divisors does not allow them to be supported on strata at infinity. Therefore, given a family
of stable tropical curves $\mathcal{C} \to B$ and a section corresponding to a marked end $s: B \to \mathcal{C}$, there is no tropical Cartier divisor on $\mathcal{C}$ that is supported on $s(B)$. As already explained in Remark 6.5, one can enlarge the sheaf $\text{Rat}_X$ to remedy this, which would allow us to define boundary divisors $D_{i,j}$ whose associated line bundles $\mathcal{L}(D_{i,j})$ agree with the boundary line bundles from Definition 6.14. But because this involves several technicalities while not increasing the generality of our results, we refrain from doing so.

While the definition of $\mathcal{L}(D_{i,j})$ is asymmetrical in $i$ and $j$, one has $\mathcal{L}(D_{i,j}) = \mathcal{L}(D_{j,i})$. We omit the proof of this fact as it is a technical exercise relying on the intuitive fact that for families of tropical curves where the $i$-th and $j$-th leg of every fiber are attached at the same vertex, there exists an isomorphism of tropical spaces exchanging the two legs. We have all ingredients to define the tropical $\psi$-classes.

**Definition 6.16.** Let $g,n \in \mathbb{Z}_{\geq 0}$ with $n + 2g - 2 > 0$, and let $i \in [n]$. We define the $i$-th cotangent line bundle $L_i \in \text{Pic}(\overline{\mathcal{M}}_{g,n})$ by defining $f^*L_i = s_i^*(\text{Aff}_X(-s_i))$ for a morphism $f: B \to \overline{\mathcal{M}}_{g,n}$ corresponding to a family of $n$-marked genus-$g$ stable tropical curves $\mathcal{C} \to B$ with $i$-th section $s_i$. The $i$-th psi-class $\psi_i$ is defined by

$$\psi_i = c_1(L_i).$$

**Example 6.17.** We compute the class $f^*_a \psi_1$ for the family of elliptic curves described in Example 3.13, corresponding to a map $f_a: TP^1 \to \overline{\mathcal{M}}_{1,1}$.

Consider the section $s_1: TP^1 \to \mathcal{C}^a$, defined by $s_1(x) = (x, \infty)$.

Denote by $T = T_- \cup T_+ \subset \mathcal{C}^a$, and construct two open sets covering $s_1$:

$$T_{-\infty} = T \setminus \mathcal{C}^a_{\infty} \quad T_{\infty} = T \setminus \mathcal{C}^a_{-\infty}.$$  \hfill (105)

For these two open sets, we consider sections of the sheaf $\text{Aff}_X(-s_1)$:

$$y_- = y_+ - ax_+ \in \text{Aff}_X(-s_1)(T_{-\infty}) \quad y_- - ax_- = y_+ \in \text{Aff}_X(-s_1)(T_{\infty}).$$  \hfill (106)

Observe that neither $y_-$ nor $y_+$ restrict to the germ of an affine function for any point $x \in s_1$, making it impossible to pull them back to $TP^1$ via $s_1$. However, we may consider the cocycle $\xi \in \text{Aff}_X(T_{-\infty} \cap T_{\infty})$:

$$\xi = y_+ - y_- = ax_+ = -ax_-.$$  \hfill (107)

Since $\xi$ is constant along vertical fibers, its pull-back via $s_1$ determines a cocycle for $f^*_a L_1$.

$$f^*_a L_1 = (s_1^*(T_{-\infty}) \cap s_1^*(T_{-\infty}), s_1^*(\xi)) = ((-\infty, \infty), ax).$$  \hfill (108)

As seen in Example 6.8, (108) shows that $f^*_a L_1 \cong \partial_{TP^1}(a)$, which implies that the degree of $f^*_a \psi_1 = c_1(f^*_a L_1) = a$.

This example illustrates that the degree of the class $f^*_a \psi_1$ does not depend only on the set theoretical map from the base to the moduli space, but it detects in an essential way the affine structure on the family of curves.

The following lemma will be needed in the proof of the pull-back formula for the tropical $\psi$-classes.
**Lemma 6.18.** Let $\mathcal{C} \to B$ be a family of $n$-marked stable tropical curves and let $i \in [n]$. Assume that for every $b \in B$ the vertex $v(b)$ of $\mathcal{C}_b$ that is adjacent to the $i$-marked leg is three-valent. Then the map $v: B \to \mathcal{C}$ is linear.

**Proof.** Let $b \in B$, and let $J = T_{v(b)} \mathcal{C}_b$. By Lemma 5.7, there is an open neighborhood $V$ of $v(b)$ in $\mathcal{C}$, an open neighborhood $U$ of $b$ in $B$, and linear morphisms $f: U \to \overline{\mathcal{M}}_{0,1}$ and $f_*: V \to \overline{\mathcal{M}}_{0,1,J}$ such that the diagram

$$
\begin{array}{ccc}
V & \xrightarrow{f_*} & \overline{\mathcal{M}}_{0,1,J} \\
\downarrow & & \downarrow \pi_* \\
U & \xrightarrow{f} & \overline{\mathcal{M}}_{0,1}
\end{array}
$$

is commutative and the induced morphism $V \to U \times_{\overline{\mathcal{M}}_{0,1}} \overline{\mathcal{M}}_{0,1,J}$ is an open immersion. Since $\#J = 3$, this means that $\mathcal{C} \to B$ is isomorphic to an open subset of a constant family in a neighborhood of $v(b)$. It follows immediately that $v$ is linear. \hfill $\Box$

**Theorem 6.19.** Let $g, n \in \mathbb{Z}_{\geq 0}$ with $n + 2g - 2 > 0$, and let $i \in [n]$. Then

$$L_i = \pi_*^i L_i \otimes \mathcal{L}(D_{i,*}),$$

where $\pi_*$ denotes the forgetful morphism from Definition 5.5.

**Proof.** Let $\mathcal{D} \to B$ be a family of $(n \cup \{\ast\})$-marked genus-$g$ stable tropical curves and $\phi: \mathcal{D} \to \mathcal{C}$ its stabilization after forgetting the $\ast$-section. Let $Y$ denote the open subset of $B$ consisting of all $b \in B$ such that the $i$-marked and $\ast$-marked legs of $\mathcal{D}_b$ are adjacent to the same three-valent vertex $v(b)$, and let $D$ denote the closed subset of $Y$ consisting of all $b \in Y$ such that all three edges adjacent to $v(b)$ are infinite. Alternatively, one may characterize the points $b \in D$ as those whose fiber’s $i$-th marked leg is contracted by $\phi$. Let $\{U_{\delta}\}_{\delta \in \Delta}$ be an open cover of $D$, and $\{U_{\lambda}\}_{\lambda \in \Lambda}$ an open cover of $B \setminus Y$ such that:

- for any $\lambda \in \Lambda$, $\lambda \in \Lambda$, one has $U_\delta \cap U_\lambda = \emptyset$;
- for any $\lambda \in \Lambda$, there is a linear section $t_\lambda: U_\lambda \to \mathcal{D}_{U_\lambda}$ whose image is contained in the $i$-th marked leg of $\mathcal{D}_b$ for all $b \in U_\lambda$;
- for any $\delta \in \Delta$, there is a linear section $s_\delta: U_\delta \to \mathcal{C}_{U_\delta}$ whose image is contained in the $i$-th marked leg of $\mathcal{C}_b$ for all $b \in U_\delta$.

All this data is organized in the following diagram:

$$
\begin{array}{ccc}
\bigcup_{\lambda \in \Lambda} \mathcal{D}_{U_\lambda} & \xrightarrow{\phi} & \mathcal{C} & \xleftarrow{\mathcal{D}} \mathcal{C}_{U_\delta} \\
\bigcup_{\lambda \in \Lambda} \mathcal{D}_{U_\lambda} & \xleftarrow{B \setminus Y} & Y & \xleftarrow{D} \bigcup_{\delta \in \Delta} U_\delta \\
\bigcup_{\lambda \in \Lambda} \mathcal{D}_{U_\lambda} & \xrightarrow{\bigcup_{\lambda \in \Lambda} t_\lambda} & B & \xrightarrow{\bigcup_{\delta \in \Delta} s_\delta} \bigcup_{\delta \in \Delta} U_\delta
\end{array}
$$

(110)
Let \( \Lambda = \Delta \cup \{v\} \cup \Lambda \); denote \( \mathbb{U}_v = Y \setminus D \) and \( t_v : \mathbb{U}_v \to \mathcal{D}_{\mathbb{U}_v} \) to be the linear section \( b \mapsto v(b) \). Then \( \{ \mathbb{U}_\alpha \}_{\alpha \in \Lambda} \) is an open cover of \( B \), which we will use to describe the cocycles for the line bundles we are interested in comparing.

To describe the cocycle for \( L_i \) we need to construct local sections of \( s_i^*(\text{Aff}_B(-s_i)) \) for all open sets in the open cover. For \( \alpha \in \Delta \cup \{v\} \), let \( \varphi_\alpha \) be the fiberwise harmonic function defined in a fiberwise connected neighborhood of \( t_\alpha(\mathbb{U}_\alpha) \cup s_i(\mathbb{U}_\alpha) \) in \( \mathcal{D}_{\mathbb{U}_\alpha} \) that has constant value zero on \( t_\alpha(\mathbb{U}_\alpha) \) and slope one in the direction of \( s_i(\mathbb{U}_\alpha) \). For \( \alpha \in \Delta \), define and denote \( \varphi_\alpha = \varphi_v \), the fiberwise harmonic function equal to zero at \( v(b) \) and with slope one towards \( s_i(\mathbb{U}_\alpha) \). Note that the local section \( t_v \) is linear by Lemma 6.18, hence by Lemma 3.21 and Lemma 3.22 we see that \( \varphi_\alpha \) defines a section of \( \text{Aff}_B(-s_i) \) for all \( \alpha \in \Lambda \). The line bundle \( L_i \) is represented by the cocycle \( (\xi_{\alpha, \alpha'}) \in \check{H}^1(\{ \mathbb{U}_\alpha \}_{\alpha \in \Lambda}, \text{Aff}_B) \), where

\[
\xi_{\alpha, \alpha'} = \begin{cases} 
  s_i^*(\varphi_v - \varphi_v) = 0, & \alpha, \alpha' \in \Delta \\
  s_i^*(\varphi_v - \varphi_v) = 0, & \alpha \in \Delta, \alpha' = v \\
  s_i^*(\varphi_\alpha - \varphi_v), & \alpha \in \Lambda, \alpha' = v \\
  s_i^*(\varphi_\alpha - \varphi_\alpha'), & \alpha, \alpha' \in \Lambda .
\end{cases}
\tag{111}
\]

To describe the cocycles for \( \pi^* L_i \) and \( \mathcal{L}(D_{i,s}) \), we need to work with linear (local) sections of the stabilization \( \mathscr{G} \). For \( \delta \in \Delta \), we are given sections \( r_\delta \) by construction. For \( \alpha \in \Lambda \cup \{v\} \), define \( r_\alpha = \phi \circ t_\alpha ; \) for \( \alpha \in \Lambda \), let \( \chi_\alpha \) be the fiberwise harmonic function defined in a fiberwise connected neighborhood of \( r_\alpha(\mathbb{U}_\alpha) \cup \delta(\mathbb{U}_\alpha) \) in \( \mathcal{G}_{\mathbb{U}_\alpha} \) that has constant value zero on \( r_\alpha(\mathbb{U}_\alpha) \) and slope one in the direction of \( \delta(\mathbb{U}_\alpha) \). By Lemma 3.21 and Lemma 3.22, each \( \chi_\alpha \) defines a section of \( \text{Aff}_B(-\delta) \). For \( \alpha \in \Lambda \cup \{v\} \), note that \( \phi^*(\chi_\alpha) \) is constantly zero along the end marked by \( * \).

The line bundle \( \pi^* L_i \) is represented by the cocycle

\[
((\phi \circ s_i)^*(\chi_\alpha - \chi_\alpha')),
\tag{112}
\]

and \( \mathcal{L}(D_{i,s}) \) is represented by the cocycle \( (\zeta_{\alpha, \alpha'}) \), where

\[
\zeta_{\alpha, \alpha'} = \begin{cases} 
  (\phi \circ s_\alpha)^*(\chi_\alpha' - \chi_\alpha), & \alpha, \alpha' \in \Delta \\
  (\phi \circ s_\alpha)^*(-\chi_\alpha), & \alpha \in \Delta, \alpha' = v \\
  0, & \alpha \in \Lambda, \alpha' = v \\
  0, & \alpha, \alpha' \in \Lambda .
\end{cases}
\tag{113}
\]

We implicitly assume here that \( \chi_\delta \) is defined on \( (\phi \circ s_\alpha)(\mathbb{U}_\delta) \) for all \( \delta \in \Delta \). This can always be achieved after potentially shrinking the open sets \( \mathbb{U}_\delta \).

Since \( \chi_\alpha \) and \( \chi_\alpha' \) always have the same slope on the i-marked leg, we have

\[
(\phi \circ s_\alpha)^*(\chi_\alpha' - \chi_\alpha) = (\phi \circ s_\alpha)^*(\chi_\alpha' - \chi_\alpha)
\tag{114}
\]

for all \( \alpha, \alpha' \in \Delta \cup \{v\} \). Moreover, for \( \alpha \in \Lambda \cup \{v\} \), in a neighborhood of \( s_i(\mathbb{U}_\alpha) \), we have

\[
\phi^* \chi_\alpha = \varphi_\alpha .
\tag{115}
\]
Combining (111), (112), (113), (114), (115) with the fact that \((\phi \circ s_{i})^{*} \chi_{v} = 0\), one sees that for every \((\alpha, \alpha') \in A \times A,\)

\[
(\phi \circ s_{i})^{*}(X_{\alpha} - X_{\alpha'}) + \xi_{\alpha, \alpha'} = \xi_{\alpha, \alpha'},
\]

which implies

\[
\pi_{i}^{*}L_{i} \otimes \mathcal{L}(D_{i, s}) \cong L_{i}.
\]

In case the vertex \(v\) adjacent to the \(i\)-marked leg has always genus zero, one can find a Cartier divisor \(D\) with \(\mathcal{L}(D) \cong L_{i}\) such that \(D\) is supported at the points \(b\) where \(v(b)\) has valence greater than three.

**Proposition 6.20.** Let \(\mathcal{C} : \mathcal{G} \to B\) be a family of \(n\)-marked genus-\(g\) stable tropical curves corresponding to a map \(f : B \to \mathcal{M}_{g,n}\), let \(i \in \mathcal{I}[n]_{i}\), and assume that for every \(b \in B\) the vertex \(v(b)\) of \(\mathcal{C}_{b}\) that is adjacent to the \(i\)-marked leg has genus zero. For every \(b \in B\), let \(J_{b} = T_{v(b)} \mathcal{C}_{b}\), where we identify \(i\) with the element of \(J_{b}\) corresponding to the \(i\)-marked leg, let \(j_{b}, k_{b} \in J_{b} \setminus \{i\}\) be distinct, and let \(V_{b}\) be an open neighborhood of \(\mathcal{C}_{b}\) that is isomorphic to an open subset of a family of \(J_{b}\)-marked genus-zero stable tropical curves \(\mathcal{C}_{b} \to U_{b}\) over \(U_{b} = \pi(V_{b})\), which exists by Lemma 5.7. For every \(x \in U_{b}\), the intersection in \(\mathcal{C}_{b}\) of the paths from \(v(x)\) to the \(j_{b}\)-th and \(k_{b}\)-th marked point, respectively, is contained in \(V_{b}\) and we denote by \(\chi_{b}(x)\) the length of that path. Then the datum \(((U_{b}, \chi_{b}))_{b \in B}\) defines a Cartier divisor \(D \in \text{Div}(B)\) such that \(\mathcal{L}(D) \cong f^{*}L_{i}\).

**Proof.** For every \(b \in B\), the identification of \(V_{b}\) with an open subset of the family of genus-zero curves \(\mathcal{C}_{b} \to U_{b}\) induces a morphism \(f_{b} : V_{b} \to \mathcal{M}_{0,j_{b},(i,k_{b})}\). By definition of \(\chi_{b}\), we have

\[
\chi_{b} = (f_{b} \circ v)^{*} \xi_{(s_{j_{b}}, (i,k_{b})}.
\]

It follows from this that \(\chi_{b} \in \Gamma(U_{b}, \text{Rat}_{\mathcal{C}})\). Moreover, \(f_{b}^{*} \xi_{(s_{j_{b}}, (i,k_{b})}\) has outgoing slope one from \(v(b)\) in the direction determined by \(i\). By Lemma 3.21 and Lemma 3.22, \(f_{b}^{*} \xi_{(s_{j_{b}}, (i,k_{b})}\) can be extended to a section \(\xi_{b}\) of \(\text{Aff}_{(s_{j})/\{s_{i}\}}\) over open set including \(V_{b}\) and the whole \(i\)-marked leg of \(\mathcal{C}_{b}\) for every \(x \in U_{b}\). Given two points \(b, b' \in B\), the functions \(\xi_{b}\) and \(\xi_{b'}\) have the same slope on the \(i\)-marked leg, and hence their difference is constant on that leg of \(\mathcal{C}_{b}\) for all \(x \in U_{b} \cap U_{b'}\). It follows that

\[
\delta_{i}^{*}(\xi_{b} - \xi_{b'}) = v^{*}(\xi_{b} - \xi_{b'}) = \chi_{b} - \chi_{b'},
\]

from which we conclude that \(((U_{b}, \chi_{b}))_{b \in B}\) defines a divisor \(D\) on \(B\) for which \(\mathcal{L}(D) \cong f^{*}L_{i}\).

**Corollary 6.21.** Let \(\mathcal{C} : \mathcal{G} \to B\) be a family of \(n\)-marked tropical stable curves corresponding to a morphism \(f : B \to \mathcal{M}_{g,n}\), and let \(i \in \mathcal{I}[n]_{i}\) such that the vertex \(v(b)\) adjacent to the \(i\)-marked leg of \(\mathcal{C}_{b}\) has genus zero for all \(b \in B\). Moreover, assume that \(v(b)\) is three-valent for all \(b\) in a dense open subset of \(B\). Then there exists a divisor \(D \in \text{Div}(B)\) with \(\mathcal{L}(D) \cong f^{*}L_{i}\) and whose support is contained in the set of all \(b \in B\) such that \(v(b)\) is at least four-valent.
**Corollary 6.22.** Let \( \overline{\mathcal{M}}_{0,n} \) denote the top-dimensional cycle on \( \mathcal{M}_{0,n} \) represented by the constant function with value one. Then \( \psi_i \cdot [\overline{\mathcal{M}}_{0,n}] \) is represented by the function on \( \mathcal{M}_{0,n} \) that is one on all points corresponding to curves where the \( i \)-marked leg is adjacent to a four-valent vertex, and zero everywhere else.

**Proof.** Let \( \{(U_b, X_b)\}_{b \in B} \) be as in the statement of Proposition 6.20, and let \( D \) be the Cartier divisor represented by that data. As we have seen in the proof of Corollary 6.21, the support of \( D \) is precisely those points of \( \mathcal{M}_{0,n} \) corresponding to curves where the \( i \)-marked leg is adjacent to a vertex of valence at least four. Let \( G \) be a \( n \)-marked genus-zero graph such that all edges except legs of \( G \) are bounded and all vertices of \( G \) are three-valent, except the vertex \( v \) adjacent to the \( i \)-marked leg, which is four-valent. Let \( b \in \text{relint}(\sigma_G) \) be a point corresponding to a curve \( \Gamma \). Then \( \chi_b \) vanishes on \( U_b \cap \sigma_G \). Let \( j_b, k_b \in T_b G \setminus \{i\} \) be distinct, as in the statement of Proposition 6.20, and let \( l_b \) be the fourth element of \( T_b G \). The cone \( \sigma_G \) is adjacent to precisely three top-dimensional cones of \( \overline{\mathcal{M}}_{0,n} \), corresponding to three-valent stable graphs \( G_{j_b}, G_{k_b} \) and \( G_{l_b} \), where the subscript denotes the tangent direction which remains adjacent to the \( i \)-marked leg. By construction, the rational function \( \chi_b \) has multiplicity one on \( U_b \cap \sigma_{G_{j_b}} \) and it is identically zero on \( U_b \cap \sigma_{G_{k_b}} \) and on \( U_b \cap \sigma_{G_{l_b}} \). It follows that \( \psi_i \cdot [\mathcal{M}_{0,n}] \) has weight one at \( b \).

**Theorem 6.23.** Let \( \pi: \mathcal{C} \to B \) be a family of \( n \)-marked genus-\( g \) tropical Mumford curves corresponding to a map \( f: B \to \mathcal{M}^\text{trop}_{g,n} \) and let \( f_*: \mathcal{C} \to \text{relint} \mathcal{M}_{0,n} \) be the pull-back of \( f \) under the forgetful morphism. Furthermore, assume that \( B \) is purely \( n \)-dimensional and let \( \alpha \in Z_n(B) \). Then \( \alpha \circ \pi \) is a tropical \((n+1)\)-cycle on \( \mathcal{C} \), and \( \psi_* \cdot (\alpha \circ \pi) \) is represented by the tropical \( n \)-cycle \( \mathcal{C} \to \mathbb{Z} \) that has value zero on all genus zero, one-valent vertices of fibers, and \( (\text{val}(x) - 2) \cdot \alpha(\text{val}(x)) \) at all other points \( x \in \mathcal{C} \).

**Proof.** By Proposition 5.7, the total space \( \mathcal{C} \) is locally isomorphic to the product of \( B \) and a stable tropical curve away from a codimension-two locus on \( \mathcal{C} \). The composition \( \alpha \circ \pi \) is a tropical cycle because of this, the facts that \( \alpha \) is a tropical cycle on \( B \) and that constant integer weights define tropical cycles on smooth tropical curves.

To compute \( \psi_* \cdot (\alpha \circ \pi) \), let \( \{(U_X, X_X)\}_{X \in \mathcal{C}} \) be as in the statement of Proposition 6.20 (applied to the \((n \sqcup \{\ast\})\)-marked family over \( \mathcal{C} \)), and let \( D \) be the Cartier divisor represented by that data. As in the proof of Corollary 6.21, the support of \( D \) is the set of all \( x \in \mathcal{C} \) having valence at least three in its fiber. Let \( x \) be a generic point in the support of \( D \), and let \( v \) be the vertex adjacent to the \( \ast \)-marked leg in the fiber over \( x \) in the \((n \sqcup \{\ast\})\)-marked family \( \mathcal{C}_x \to \mathcal{C} \). The rational function \( \chi_x \) depends on a choice of two
distinct tangent vectors in $T_v(C_{\pi(x)})$ that do not point in the direction of the $\ast$-marked leg. This corresponds to the choice of two distinct tangent vectors $j, k \in T_x(C_{\pi(x)})$. With this choice, the function $\chi_x$ has slope one in the direction corresponding to any $t \in T_x(C_{\pi(x)})$, and slope zero in the directions corresponding to $j$ and $k$. It follows that the multiplicity of $D \cdot (\alpha \circ \pi)$ at $x$ is given by $(\text{val}(x) - 2) \alpha(\pi(x))$.

\[ \square \]

Remark 6.24. Theorem 6.23 can be interpreted as the germ of a tropical dilaton equation. Although we have not mentioned this in §6, the push-forward of tropical cycles respects rational equivalence. Together with the fact the sum $\sum_v (\text{val}(v) - 2)$ over all finite vertices $v$ of an $n$-marked genus-$g$ stable graph equals $2g - 2 + n$, we conclude that

$$\pi_* (\psi_* \cdot (\alpha \circ \pi)) = (2g - 2 + n) \cdot \alpha$$

in the situation of the theorem. In the case where the base is $\mathcal{M}_{g,n}^{\text{MF}}$, we obtain

$$\pi_* (\psi_* \cdot [\mathcal{M}_{g,n,\{\ast\}}^{\text{MF}}]) = (2g - 2 + n) \cdot [\mathcal{M}_{g,n}^{\text{MF}}].$$

7. Correspondence theorems in genus 1

In this section, we evaluate $\psi$ classes on some one-dimensional cycles of genus one tropical curves. The first example is a Gromov-Witten cycle, i.e. a family of tropical stable maps to the plane subject to point conditions [Tyo12, GKM09, Ran17, RSW19]. The second family of examples consists of one-dimensional spaces of admissible covers of the projective line [Cap14, CMR16]. In both cases, the theory of tropical $\psi$-classes provides a correspondence with its algebro-geometric counterpart.

7.1. Families of tropical curves from well-spaced stable maps. Given a degree $d$ and a non-negative integer $n$, we consider the space of all tropical stable maps into $\mathbb{P}^2$ of degree $d$ with $n$ marked points as described in [Ran17]. We slightly modify the construction of loc. cit. by adding cycle-rigidifications to the source curves and taking the extended cone complex to obtain a compact TPL-space $\text{TSM}_{1,n}(\mathbb{P}^2, d)$. As some combinatorial types of these tropical stable maps are super-abundant, the dimension of $\text{TSM}_{1,n}(\mathbb{P}^2)$ is larger than the expected dimension. We consider the closed subspace $\text{WS}_n(\mathbb{P}^2, d) \subset \text{TSM}_{1,n}(\mathbb{P}^2)$ consisting of all tropical stable maps that are well-spaced in the sense of [RSW19, Definition 4.4.4] (following well-spacedness as a realizability condition in [Spe05, Spe14]). This space is pure-dimensional of dimension

$$\dim(\text{WS}_n(\mathbb{P}^2, d)) = 3d + n.$$  

Evaluating stable maps at the marked points yields the evaluation maps

$$\text{ev}_i : \text{WS}_n(\mathbb{P}^2, d) \to \mathbb{P}^2$$  

for each $1 \leq i \leq n$; forgetting the map and stabilizing the source curve defines a map

$$\text{src} : \text{WS}_n(\mathbb{P}^2, d) \to \overline{\mathcal{M}}_{1,n+3d}.$$  

We endow $\text{WS}_n(\mathbb{P}^2, d)$ with the smallest affine structure such that src and $\text{ev}_i$ for all $1 \leq i \leq n$ become morphisms of tropical spaces.
Choose eight general points \( p_1, \ldots, p_8 \) in \( \mathbb{R}^2 \). We may assume that the points are horizontally stretched: they lie in a thin neighborhood of a line of very small negative slope, their numbering and horizontal spacing increases from left to right. Because the points are in general position and \( \text{WS}_8(\mathbb{T} \mathbb{P}^2, 3) \) is 17-dimensional, the set

\[
B = \bigcap_{i=1}^{8} \text{ev}_i^{-1}\{p_i\}
\]

is a one-dimensional subset of \( \text{WS}_8(\mathbb{T} \mathbb{P}^2, 3) \). The corresponding family of tropical stable maps is described explicitly in Appendix A. Since \( B \) is defined as a generic intersection of the sets \( \text{ev}_i^{-1}\{p_i\} \), we may define a tropical one-cycle on \( B \) via tropical intersection theory.

**Lemma 7.1.** There exists an open neighborhood \( U \) of \( B \) in \( \text{WS}_8(\mathbb{T} \mathbb{P}^2, 3) \) such that the function \( \alpha_U \) assigning to a point \( b \in B \) corresponding to a tropical stable map \( \Gamma_b \to \mathbb{T} \mathbb{P}^2 \) the value zero, if \( \Gamma \) has a loop attached to a trivalent vertex or the midpoint of an edge of multiplicity one, and the number of automorphisms of \( \Gamma_b \), else, defines a tropical 17-cycle on \( U \).

**Proof.** We define \( U \) to be the union of relative interiors of cones in \( \text{WS}_8(\mathbb{T} \mathbb{P}^2, 3) \) (considered as an extended cone complex) that have nonempty intersection with \( B \). Because the eight points used to define \( B \) are chosen generically, \( U \) is open and only contains the relative interiors of 16- and 17-dimensional cones. We have to check balancing around every 16-dimensional facet. If the facet is contained in the locus of Mumford curves, this follows from [Tor14, Theorem 3.3.8]. If, on the other hand, a facet \( \tau \) is contained in the locus of curves that have a genus one vertex, then it parametrizes maps whose source curve has a two-valent genus one vertex that is mapped to the midpoint of an edge of multiplicity two. The integer affine functions defined in a neighborhood of \( \tau \) are the linear combinations of the edge lengths \( l_e \) for all edges \( e \) contained in the set \( E_0 \) of edges not adjacent to the genus one vertex and pull-backs of integer affine functions on \( \mathbb{T} \mathbb{P}^2 \) via the evaluation maps. There are two 17-dimensional faces adjacent to \( \tau \). In one of them, call it \( \sigma_1 \), the genus one vertex is resolved by inserting a loop that is contracted by the map. In the other one, call it \( \sigma_2 \), the genus one vertex is resolved by inserting two parallel edges on which the map has multiplicity one. In either case, every combination of values of evaluation maps and edge lengths \( l_e \) for \( e \in E_0 \) on \( \sigma_i \) is also attained on \( \tau \). More precisely, we have

\[
\left( \prod_{e \in E_0} l_e \times \prod_{i=1}^{8} \text{ev}_i \right)(\sigma_k) = \left( \prod_{e \in E_0} l_e \times \prod_{i=1}^{8} \text{ev}_i \right)(\tau)
\]

for all \( k \in \{1, 2\} \). It follows that every integral affine function that vanishes on \( \tau \) also vanishes on a small neighborhood of \( \tau \). Therefore, the balancing condition is vacuously true around \( \tau \). \( \square \)

For the following definition, note that for every \( p \in \mathbb{R}^2 \), the tropical line \( L_p \) with vertex at \( p \) defines a tropical Cartier divisor on \( \mathbb{T} \mathbb{P}^2 \) and that \( L_p^2 = |p| \).
Definition 7.2. We define the cycle $\alpha_B$ on $B$ as

$$
\alpha_B = \left( \prod_{i=1}^{n} (ev_i^* L_{p_i})^2 \right) \cdot \alpha_U ,
$$

where $U$ is a sufficiently small neighborhood of $B$ in $WS_8(\mathbb{T}P^2, 3)$ and $\alpha_U$ is defined as in Lemma 7.1.

No leg of the source curve of any tropical stable map corresponding to some $b \in B$ is adjacent to a genus one vertex, hence the set $\text{src}(B)$ is contained in $\mathcal{V}_{1,17}^{\text{good}}$. Therefore, there exists a natural family of $17$-marked genus one stable tropical curves $C_1 \to B$ by Proposition 4.24. For $1 \leq i \leq 8$, let $C_i \to B$ be the family of one-marked genus one stable tropical curves obtained from $C_1 \to B$ by forgetting all marks but the $i$-th one.

Proposition 7.3. For $i = 1, \ldots, 8$, let $f_i : B \to \overline{\mathcal{M}}_{1,\{i\}}$ be the morphism corresponding to the family $C_i \to B$. Then we have

$$
\int_B f_i^* \psi_i \cdot \alpha_B = 432 .
$$

Proof. First we observe that the $i$-th marked point never sits on a component that is contracted by the stabilization $C \to C_i$. So by Theorem 6.19, it suffices to prove that

$$
\int_B g^* \psi_i \cdot \alpha_B = 432 ,
$$

where $g : B \to \overline{\mathcal{M}}_{1,17}$ is the morphism induced by the 17-marked family $C \to B$.

The $i$-marked leg is attached to a genus-zero vertex in every fiber of $C$, allowing us to use Proposition 6.20 to compute the degree of $\psi_i$. Using the representative for $\psi_i$ produced by the proposition, we see that there is a contribution for every curve in the family contained in the subset $\Psi$ of $B$ consisting of curves where the $i$-marked leg is adjacent to a four-valent vertex. As one sees combinatorially (see Figure 17), all points in $B$ for which this happens have the same image curve $\Gamma'$, that is they correspond to the same well-spaced map to $\mathbb{T}P^2$ up to orientation of the cycle and marking of the legs on which the map is not constant.

There are two cases to consider. In the first case, for $i \in \{3, 4, 6, 7, 8\}$, the image curve $\Gamma'$ does not have any edges of multiplicity two. There are $2 \cdot (3!)^3 = 432$ different curves in $B$ with given image curve $\Gamma'$, so $\Psi$ has 432 elements. For each $b = [\Gamma \to \mathbb{T}P^2] \in \Psi$, the $i$-th marked point maps to a three-valent vertex in $\Gamma'$. Moreover, the star of $\Gamma'$ at that vertex is a tropical line in suitable coordinates. Therefore, locally around $b$, the space $B$ is isomorphic to the space of $(i)$-marked plane tropical lines passing through a given point in the plane (see Figure 11). By [GKM09, Proposition 4.7], this space is isomorphic to $\overline{\mathcal{M}}_{0,4}$. The weights of $\alpha_B$ are computed as explained in Appendix A; computations analogous to the one illustrated in Figure 19 show the weights are equal to one locally around $b$. By Corollary 6.22 it follows that the contribution of $b$ to the degree of $\psi_i$ on $B$ is thus equal to 432.
In the second case, occurring when \( i \in \{1,2,5\} \), the image curve \( \Gamma' \) has an edge of multiplicity two. If \( i = 5 \), then the \( i \)-th leg is adjacent to the loop of \( \Gamma \), whereas when \( i \in \{1,2\} \) it is adjacent to two legs on which the map is non-constant. In either case, \( \Psi \) consists of 216 points. In any map \( b = [\Gamma \to \mathbb{TP}^2] \in \Psi \), the \( i \)-th leg is adjacent to a four-valent vertex \( v \) of \( \Gamma \) that maps to the interior of the multiplicity-two edge in \( \Gamma' \). Let \( f_1, f_2, f_3 \), the three remaining directions of \( \Gamma \) at \( v \) such that the map has dilation factor two in direction \( f_1 \). Each of these flags corresponds to the three directions \( e_1, e_2, e_3 \) on the base at \( b \), as shown in Figure 12. Computing the weights as in Appendix A, Figure 19, one sees the weight \( \alpha_B \) is two in the direction of \( e_1 \) and one in the other two directions. Using Lemma 5.7, the contribution of \( \psi_i \) at \( b \) is uniquely determined by the map \( \phi: V \to \mathcal{M}_{0,4} \) defined in a neighborhood \( V \) of \( b \) by restricting one’s attention to a neighborhood of \( v \) in the total space of the family. In each direction \( e_i \) one obtains a different combinatorial type of four-marked curves, so \( \phi \) maps each of the directions \( e_i \) to different directions at the cone-point of \( \mathcal{M}_{0,4} \). Furthermore, \( \phi \) has slope one in the direction of \( e_1 \). Therefore, \( \phi^* (\alpha_B | V) \) has weight two everywhere. By the tropical projection formula and Corollary 6.22, we conclude that the contribution at \( b \) to \( \psi_i \) is two. The total degree of \( \psi_i \) on \( B \) is thus equal to \( 2 \cdot 216 = 432 \). \( \square \)

**Proposition 7.4.** Let \( g: B \to \mathcal{M}_{1,1} \) be the composition \( B \hookrightarrow \text{WS}_B(\mathbb{TP}^2,3) \xrightarrow{\text{src}} \mathcal{M}_{1,1} \hookrightarrow \mathcal{M}_{1,1} \). Then we have

\[
g_\ast \alpha_B = 12 \cdot 432 \cdot [\mathcal{M}_{1,1}] \tag{129}\]

**Proof.** Since every point in \( \mathcal{M}_{1,1} \) except the initial point has a neighborhood isomorphic to an open subset of \( \mathcal{T} \), every tropical one-cycle on \( \mathcal{M}_{1,1} \) is a multiple of \( [\mathcal{M}_{1,1}] \). Therefore, we have to prove that \( g \) has degree \( 12 \cdot 432 \), where the degree is defined with respect to \( \alpha_B \). By definition of \( \alpha_B \), the degree of \( g \) is the same as the degree with
Figure 12. The flag \( f_i \) corresponds to the direction \( e_i \in T_bB \) where the \( i \)-marked leg moves away from the vertex in the direction given by \( f_i \).

respect to \( \alpha_U \) of the map

\[
U \xrightarrow{g \times \prod_{i=1}^g \ev_i} \mathcal{T}_{1,1} \times \prod_{i=1}^g \TP^2 ,
\]

where \( U \) is the neighborhood of \( B \) from Proposition 7.7. The weights of \( \alpha_U \) are defined exactly as in [KM09, Definition 3.5], except that pairs of strata appearing in part (c) of that definition are replaced by one stratum, parameterizing curves with a loop contracting at the mid-point of an edge of weight two, with twice the multiplicity. The degree of \( g \) is \( 12 \cdot 432 \) from [KM09, Theorem 6.3] and the fact that there are 12 tropical rational curves passing through eight general points in \( \TP^2 \) when counted with multiplicity [BM09]. The factor of 432 does not appear in [KM09]: it arises from orienting the cycle and marking all unbounded edges. \( \square \)

We can interpret Proposition 7.3 and Proposition 7.4 combined as the tropical version of the equality

\[
\psi_1 = \frac{1}{24} [pt]
\]

that holds on the algebraic moduli space of genus one stable curves \( \mathcal{M}_{1,1} \). The lack of a fundamental class for \( \mathcal{M}_{1,1} \) does not allow to integrate the class of a point. However, we may view \( \mathcal{T}_{1,1} \) as a two-to-one cover of \( \mathcal{M}_{1,1} \). Then the map \( f: B \to \mathcal{M}_{1,1} \) defined by the family \( \mathcal{C}_1 \to B \) has degree \( 24 \cdot 432 \). We thus have obtained that the pull-back of \( \psi_1 \) via a map of degree \( 24 \cdot 432 \) is a zero-dimensional cycle of degree 432.

7.2. Families of tropical curves from admissible covers. We turn our attention to families coming from tropicalizing admissible covers. Adapting the notions from
Figure 13. Classes of combinatorial types of degree-$d$ tropical admissible covers of a tropical genus-zero curve with ramification profile $((d), (d), (2,1^{d-2}), (2,1^{d-2}))$

[Cap14, CMR16] to allow nodes at infinity, we say that a tropical admissible cover is a harmonic map $f : \Gamma_1 \to \Gamma_2$ of stable curves such that $f$ does not contract any edges, preimages of marked points are marked points, preimages of nodes at infinity are nodes at infinity, and the local Riemann-Hurwitz condition

$$2g_x - 2 + \text{val}(x) = d_x(2g_{f(x)} - 2 + \text{val}(f(x)))$$

(132)

is satisfied for any point $x \in \Gamma_1$. Note that $f$ might not be a linear morphism of tropical curves because the affine structure of $\Gamma_1$ at a point $x \in \Gamma_1$ of genus greater zero is trivial, whereas the affine structure of $\Gamma_2$ at $f(x)$ need not be trivial. However, $f$ is linear away from the locus of points of higher genus since there harmonicity is equivalent to linearity. Modifying the construction from [CMR16, §3.2.2] of a cone complex of tropical admissible covers by adding cycle-rigidifications to the source curves and taking the extended cone complex, one obtains compact TPL-spaces of cycle-oriented tropical admissible covers.

We focus on a specific class of one dimensional examples. For $d \geq 2$, denote by $B_d$ the compact TPL-space of tropical admissible covers of degree-$d$ of a genus-zero tropical curve with ramification profile $((d), (d), (2,1^{d-2}), (2,1^{d-2}))$. By the Riemann-Hurwitz formula the genus of the source curve is one.

The combinatorial types of covers in $B_d$ can be divided into four classes, depicted in Figure 13, where one has to add markings and cycle-rigidifications to obtain a combinatorial type. There are $(d-2)!$ combinatorial types of class I, $\binom{d-2}{2}(d-4)!$ types of class II, $2(d-1)$ combinatorial types of class III, and $(d-2)^2(d-3)!$ types of class IV. So $B_d$ is isomorphic, as a TPL-space, to the space obtained by gluing together
\[(d - 2)! + (d - 2)^2 (d - 4)! + 2(d - 1) + (d - 2)^2 \cdot (d - 3)!\] copies of \([0, \infty]\) at their respective element 0.

By construction, there are natural source and branch maps

\[
\text{src}: B_d \rightarrow \mathcal{P}_{1,2d}, \quad [\Gamma_1 \rightarrow \Gamma_2] \mapsto [\Gamma_1], \quad \text{and}
\]
\[
\text{br}: B_d \rightarrow \mathcal{M}_{0,4}, \quad [\Gamma_1 \rightarrow \Gamma_2] \mapsto [\Gamma_2].
\]

Define \(\text{Aff}_{B_d}\) to be the smallest affine structure on \(B_d\) that makes src and br linear.

Let \(\mathcal{C}_1 = B_d \times_{\mathcal{P}_{1,2d}} \mathcal{P}_{1,2d,\{\ast\}}, \) and let \(\mathcal{C}_2 = B_d \times_{\mathcal{M}_{0,4}} \mathcal{M}_{0,4,\{\ast\}}.\) Then the projection \(\pi_1: \mathcal{C}_1 \rightarrow B_d\) is a family of 2d-marked genus-one stable TPL-curves and the projection \(\pi_2: \mathcal{C}_2 \rightarrow B_d\) is a family of four-marked genus-zero stable tropical curves. By construction of \(B_d\), there is a morphism \(\mathcal{F}: \mathcal{C}_1 \rightarrow \mathcal{C}_2\) of TPL-spaces over \(B_d\) such that the induced morphism \(F_b: ([\mathcal{C}_1]_b \mapsto ([\mathcal{C}_2]_b)\) identifies \(b\) with the modular point \([(\mathcal{C}_1]_b \mapsto ([\mathcal{C}_2]_b)\).

Unfortunately, \(\mathcal{C}_1 \rightarrow B_d\) is not a family of tropical curves because \(\mathcal{P}_{1,2d,\{\ast\}} \rightarrow \mathcal{P}_{1,2d}\) is not a family of tropical curves. The problem is not the exactness of the sequence of cotangent sheaves, but the lack of enough affine functions on \(\mathcal{C}_1\) to make the fibers stable tropical curves. The idea to fix this is to define the pull-back of affine functions via the map \(\mathcal{F}\) to be affine functions on \(\mathcal{C}_1\). On a leg of a fiber of \(\mathcal{C}_1 \rightarrow B_d\) where the multiplicity of \(\mathcal{F}\) is \(d\), one then obtains a function with slope \(d\). One would like to divide this function by \(d\) to obtain a fiberwise harmonic function on \(\mathcal{C}_1\) with vertical slope one, but if the horizontal slopes of that function are not all divisible by \(d\), which they usually are not, this does not define a piecewise integral linear function. To fix this, one takes a degree-lcm(2, \(d\)) cover \(\widetilde{B}_d\) of \(B_d\) whose underlying topological space is \(B_d\), whose sheaf of piecewise linear functions is given by \(\text{PL}_{\widetilde{B}_d} = \frac{1}{\text{lcm}(2, d)} \text{PL}_{B_d} \) and whose sheaf of affine functions \(\text{Aff}_{\widetilde{B}_d}\) is given by the saturation of \(\text{Aff}_{B_d}\) in \(\text{PL}_{\widetilde{B}_d}\).

In other words, a piecewise integral linear function \(\phi\) on \(\widetilde{B}_d\) is a section in \(\text{Aff}_{\widetilde{B}_d}\) if and only if there exists a positive integer \(n\), such that \(n \phi\) is a section of \(\text{Aff}_{B_d}\). Let \(\mathcal{E}_1 = \widetilde{B}_d \times_{\mathcal{P}_{1,2d}} \mathcal{C}_1, \) let \(\mathcal{E}_2 = \widetilde{B}_d \times_{B_d} \mathcal{C}_2, \) and let \(\widetilde{\mathcal{F}}: \mathcal{E}_1 \rightarrow \mathcal{E}_2\) be the morphism of TPL-spaces induced by \(\mathcal{F}.\) Finally, \(\mathcal{E}_1\) is made into a tropical space by equipping it with the smallest affine structure that is saturated in \(\text{PL}_{\mathcal{E}_1}\), makes the projection \(\mathcal{E}_1 \rightarrow \mathcal{C}_1\) linear everywhere, and makes \(\widetilde{\mathcal{F}}\) linear at all points of genus zero.

**Lemma 7.5.** \(\mathcal{E}_1 \rightarrow \widetilde{B}_d\) is a family of tropical stable curves.

**Proof.** It easily follows from the fact that the cotangent sequence (16) is exact on \(\mathcal{C}_1\) and \(\mathcal{C}_2\) by Proposition 4.18, together with the fact that \(\text{Aff}_{\widetilde{B}_d}\) is saturated, that the cotangent sequence is also exact on \(\mathcal{E}_1\) as well. Since all functions in \(\text{Aff}_{\mathcal{E}_1}\) are fiberwise harmonic by construction, it remains to be shown that the fibers are tropical stable curves, that is that all harmonic functions on the fibers are affine. By the proof of Proposition 4.24, this is clear everywhere except at points \(x \in \mathcal{E}_2\) that are contained in a leg in its fiber that is adjacent to a genus-one vertex. So assume that \(x\) is contained in such a leg. Let \(m \in \{1, 2, d\}\) denote the multiplicity of \(\widetilde{\mathcal{F}}\) at \(x\), and let \(\phi\) be an integral affine function
in a neighborhood of \( \tilde{F}(x) \) with vertical slope one. Then by construction \( \frac{1}{m} \tilde{F}^*(\phi) \) is an integral affine function in a neighborhood of \( x \) of slope one (for this we needed to replace \( B_d \) by \( \tilde{B}_d \)). This finishes the proof. \( \square \)

We will not refer to \( B_d \) again and work exclusively with \( \tilde{B}_d \) in the rest of this section. To ease the notation, we replace \( B_d \) with \( \tilde{B}_d \), \( C_i \) with \( \tilde{C}_i \), and \( \text{br} \) (src) with the composite of \( \text{br} \) (src) and the natural map \( \tilde{B}_d \to B_d \).

We define a weight on \( B_d \) following [BBM11, Definition 2.6]: the value at a point \( f: \Gamma_1 \to \Gamma_2 \) is the automorphism-weighted product of multiplicities on the edges and the local Hurwitz numbers at the vertices of the corresponding source curve:

**Definition 7.6.** The weight \( \alpha_{B_d} \) on \( B_d \) depends on the class of the combinatorial type of the covers parameterized, as labeled in Figure 13. Specifically:

\[
\alpha_{B_d}(b) = \begin{cases} 
(d-1)! & \text{if } F_b \text{ is of class I}, \\
\frac{1}{2}(d-4)! \cdot (d-1)(d-2)(d-3) & \text{if } F_b \text{ is of class II}, \\
gcd(a, d-a) [(d-2)!]^2 & \text{if } F_b \text{ is of class III}, \\
\frac{1}{3}(d-3)! \cdot (d-1)(d-2) & \text{if } F_b \text{ is of class IV}.
\end{cases}
\]  

(133)

We assume that the first marked point of the family \( C_i \to B_d \) is one of the two points of ramification order \( d \). Let \( \tilde{\tau}: B_d \to \overline{\mathcal{T}}_{1,1} \) the the composition of the source map src with the forgetful morphism forgetting all but the first marked point.

**Proposition 7.7.**

a) The weight \( \alpha_{B_d} \) defines a tropical one-cycle on \( B_d \).

b) We have

\[
\text{br}_* \alpha_{B_d} = \frac{\text{lcm}(2,d) [(d-2)!]^2 (d-1)d(d+1)}{6} \overline{\mathcal{M}}_{0,4} \cdot \overline{\mathcal{M}}_{1,1}.
\]  

(134)

c) We have

\[
\tilde{\tau}_* \alpha_{B_d} = 2\text{lcm}(2,d) [(d-2)!]^2 (d-1)(d+1) \overline{\mathcal{T}}_{1,1}.
\]  

(135)

**Proof.** For a) we need to check that the weights are balanced at the vertex of \( B_d \). For germs of affine functions obtained by pull-back via \( \text{br} \), balancing is equivalent to the fact that \( \text{br} \) has a well-defined degree, hence this follows from b). There are non-trivial germs of affine functions on the vertex of \( \overline{\mathcal{T}}_{1,2d} \), hence balancing holds vacuously for functions pulled-back via src.

Part b) is [BBM11, Theorem 2.11], which in particular shows that the degree of the branch morphism is well-defined.

For c), we compute the contribution to \( \tilde{\tau}_* \alpha_{B_d} \) by rays of \( B_d \) of combinatorial types belonging to each of the three classes identified in Figure 13. For a ray parameterizing curves of class I, \( \tilde{\tau} \) is an integral map of slope \( 2\text{lcm}(2,d) \). There are \( (d-2)! \) such rays, and to each of them \( \alpha_{B_d} \) assigns weight \( (d-1)! \).

Rays of class II and IV are contracted to the vertex of \( \overline{\mathcal{T}}_{1,1} \), and hence do not contribute.
The function $\tilde{\pi}$ restricted to a ray parameterizing curves of class III has slope $\text{lcm}(2,d)\text{lcm}(a,d-a)\left(\frac{1}{a}+\frac{1}{d-a}\right)$. The weight on the ray is $\gcd(a,d-a) [(d-2)!]^2$, hence the contribution of the ray is independent of $a$, equal to $\text{lcm}(2,d)[(d-2)!]^2$. There are $2(d-1)$ rays of class III. Putting all contributions together we see that the degree of $\tilde{\pi}$ equals

\[
(\text{lcm}(2,d))((d-2)! \cdot 2\text{lcm}(2,d) + 2(d-1) \cdot \text{lcm}(2,d) d)[(d-2)!]^2 = \\
= 2\text{lcm}(2,d)[(d-2)!]^2(d-1)(d+1) \quad (136)
\]

\[ \square \]

**Proposition 7.8.** Let $\pi : B_d \to \overline{\mathcal{M}}_{1,1}$ the the composition of the source map src with the forgetful morphism forgetting all but the first marked point. We have

\[
\int_{B_d} \pi^* \psi_1 \cdot \alpha_{B_d} = \frac{\text{lcm}(2,d)[(d-2)!]^2(d-1)(d+1)}{6}. \quad (137)
\]

**Proof.** The map $F : \mathcal{C}_1 \to \mathcal{C}_2$ has degree $d$ on the one-marked leg in every fiber, so if $F$ maps the one-marked leg in $\mathcal{C}_1$ to the $\circ$-marked leg in $\mathcal{C}_2$, then we have

\[
d \cdot \pi^* \psi_1 = \text{br}^* \psi_\circ. \quad (138)
\]

Using the tropical projection formula [AR10, Proposition 7.7] and Proposition 7.7 b), we thus obtain

\[
\int_{B_d} \pi^* \psi_1 \cdot \alpha_{B_d} = \frac{1}{d} \int_{B_d} \text{br}^* \psi_\circ \cdot \alpha_{B_d} = \frac{1}{d} \int_{\overline{\mathcal{M}}_{0,4}} \text{br}_* (\text{br}^* \psi_\circ \cdot \alpha_{B_d}) = \frac{1}{d} \int_{\overline{\mathcal{M}}_{0,4}} \psi_\circ \cdot \text{br}_* \alpha_{B_d} = \\
= \frac{[\text{lcm}(2,d)(d-2)]^2(d-1)(d+1)}{6} \int_{\overline{\mathcal{M}}_{0,4}} \psi_\circ \cdot \text{br}_* \alpha_{B_d} = \frac{\text{lcm}(2,d)[(d-2)!]^2(d-1)(d+1)}{6},
\]

where the last equality follows from Corollary 6.22. \[ \square \]

Proposition 7.8 together with Proposition 7.7 c) show that the ratio of the evaluation of $\pi^* \psi_1 \cdot \alpha_{B_d}$ with the degree of src is $1/12$. Together with the fact that stack $\overline{\mathcal{M}}_{1,1}$ has a generic $B_{1,2}$ stabilizer, we have that for the admissible covers families studied $\psi$ behaves as the pull-back of a class of degree $1/24$.

**Appendix A. Cycle of degree 3 tropical stable maps**

In this Appendix we explicitly describe the one dimensional cycle $B = \bigcap_{i=1}^8 \text{ev}_i^{-1} \{ p_i \} \subset \text{WS}_8(\mathbb{T}^2,3)$ studied in Section 7.1. To be precise, Figure 14 depicts a quotient of the cycle $B$ by the action of the group $S_3^3$ permuting the labels of the infinite ends. Equivalently, one may think of parameterizing tropical stable maps where the infinite ends are unlabeled. The vertical map to $\mathcal{M}_{1,1}^{\text{trop}}$ is given by the tropical $j$-invariant.

The different combinatorial types of curves appearing, together with the corresponding lattice paths and floor diagrams [Mik05, FM10], are depicted in Figure 15 (spanning multiple pages).
Figure 14. Genus one, degree three maps through 8 general points in the plane. The part of B which is drawn in black coincides with a tropical pencil of cubics through 8 points. In gray are the tropical stable maps with un underlying constant image curve.

The vertices of B contain eighth rational nodal curves, as depicted in Figure 16. Among the vertices of B are when each of the marked points sits at a four-valent vertex (or equivalently the mark is incident to a three-valent vertex of the image cubic). These are the points of B where the various ψ classes are supported. These are depicted in Figure 17. The remaining vertices in B are collected in Figure 18.
The combinatorial types of tropical stable maps associated to each segment of the cycle $B$.

The weights of the cycle $\alpha_B$ are computed using the techniques of [KM09, Remark 4.8], [GKM09, Corollary 2.27]. Given an edge $e$ of $B$, corresponding to a given topological type of maps, choose a tree containing all eight markings of the source curve; $x, y$ and the edge lengths $l_1, \ldots, l_{15}$ of its compact edges may be used as integral linear coordinates for the cone of $\text{WS}_8(\mathbb{TP}^2, 3)$ containing $e$. Then $\alpha_B$ assigns to $e$ the gcd of the $16 \times 16$ minors of the matrix representing the linear map $\prod_{i=1}^8 \text{ev}_i$. In Figure 19, we show as an example the case of $e$ corresponding to the edge labeled 5 in Figure
Figure 17. Vertices of $B$ corresponding to where one of the marks sits at a four-valent vertex.

Figure 18. The remaining vertices of $B$.

14. For the choice of tree of $\Gamma$ and labeling of its compact edges depicted in the figure, the resulting matrix is explicitly computed. Deleting the fifth column, we shaded a block triangular decomposition of the remaining matrix that is easily seen to have determinant one. It follows that the weight of the edge labeled 5 in $B$ is equal to one.
Figure 19. In the upper part of the Figure, we choose a tree containing all eight markings for a curve $\Gamma$ in the topological type of maps denoted 5 in Figure 14. The matrix below represents the function $\prod_{i=1}^{8} \text{ev}_i$ in the coordinates $x, y, l_1, \ldots, l_{15}$ for the cone parameterizing maps of this topological type.

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