Existence, uniqueness and asymptotic behavior of time periodic solutions for extended Fisher-Kolmogorov equations with delays

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Abstract

In this paper, we investigate the global existence, uniqueness and asymptotic stability of time $\omega$-periodic classical solution for a class of extended Fisher-Kolmogorov equations with delays and general nonlinear term. We establish a general framework to find time $\omega$-periodic solutions for nonlinear extended Fisher-Kolmogorov equations with delays and general nonlinear function, which will provide an effective way to deal with such kinds of problems. The discussion is based on the theory of compact and analytic operator semigroups and maximal regularization method.

Keywords: Extended Fisher-Kolmogorov equation with delays; Time periodic solution; Global existence and uniqueness; Classical solution; Asymptotic behavior

Mathematics Subject Classification (2010): 34K13; 47J35

1 Introduction

The extended Fisher-Kolmogorov (EFK) equation

$$\frac{\partial u}{\partial t} + \gamma \frac{\partial^4 u}{\partial x^4} - \frac{\partial^2 u}{\partial x^2} - u + u^3 = 0, \quad \gamma > 0$$

(1.1)
was proposed in 1987 by Coullet, Elphick, and Repaux [1] and in 1988 by Dee and van Saarloos [2] as a generalization of the classical Fisher-Kolmogorov equation which was firstly propounded by Fisher and Kolmogorov in 1937. The extended Fisher-Kolmogorov equation are a family of models arising in population dynamics problems, cancer modelling, chemical kinetics, the description of propagating crystallization/polymerization fronts, geochemistry and many other fields. These equations do not admit a Lagrangian density depending on the field $u$ and thus the variational formulation for the effective particle parameters cannot be written in the usual way. Therefore, substantial attention has been focused on the steady-state equation

$$-\gamma u'''' + u'' + u - u^3 = 0, \quad \gamma > 0 \quad (1.2)$$

corresponding to EFK equation (1.1), see [3], [4], [5], [6] and references therein for more comments and citations.

Recently, Danumjaya and Paniuse employ the Galerkin finite element approximation method and orthogonal cubic spline collocation method studied the existence, uniqueness and regularity of EFK equation (1.1) in [7] and [8]. In 2011, by using a Crank-Nicolson type finite difference scheme and the method of Lyapunov functional, Khiari and Omrani [9] studied the existence of approximate solutions for the following extended Fisher-Kolmogorov equation in two space dimension with Dirichlet boundary conditions

$$\begin{cases}
  u_t + \gamma \Delta^2 u - \Delta u - u + u^3 = 0, & \text{in } (0, T] \times \Omega, \\
  u = 0, \quad \Delta u = 0, & (t, x, y) \in (0, T] \times \partial \Omega, \\
  u(0, x, y) = u_0(x, y), & \text{in } \Omega,
\end{cases} \quad (1.3)$$

where $\Omega$ is a bounded domain in $\mathbb{R}^2$ with boundary $\partial \Omega$, $\gamma > 0$ is a constant.

On the other hand, evolution equations with delays have attracted increasing attention in recent years and the existence or attractivity of periodic solutions for evolution equations with delays have been considered by several authors, see [10-18] and references listed therein for more comments and citations. Most of these results are established by applying semigroup theory [11,13-17], corresponding fixed point theorems [12-14,16], coincidence degree theory [15] and so on. Recently, Liu and Li [12] obtain the existence of periodic solutions for a class of parabolic evolution equations with delay by utilizing Schaefer type theorem, which extend the corresponding results of Burton and Zhang [10]. Latter, in 2008, by using the method of constructing some suitable Lyapunov functionals and establishing the prior bound for all possible periodic solutions, Zhu, Liu and Li [19] investigated the existence, uniqueness and global attractivity of time periodic solutions for the following one-dimensional parabolic evolution equation with delays
\[
\frac{\partial}{\partial t} u(t, x) = \frac{\partial^2}{\partial x^2} u(t, x) + a u(t, x) + g(t, x) \\
+ f(u(t - \tau_1, x), \ldots, u(t - \tau_n, x)), \quad (t, x) \in \mathbb{R} \times (0, 1),
\]

which is usually used to model some process of biology, where \(a \in \mathbb{R}, f : \mathbb{R}^n \to \mathbb{R}\) is locally Lipschitz continuous, \(g : \mathbb{R} \times [0, 1] \to \mathbb{R}\) is Hölder continuous and \(g(t, x)\) is \(\omega\)-periodic in \(t, \tau_1, \tau_2, \ldots, \tau_n\) are positive constants. In addition, the dynamical characteristics (including stable, unstable, attract, oscillatory and chaotic behavior) of differential equations have become a subject of intense research activities. For the details of this field, we refer the reader to the monographs of Burton [18], Hale [15] and the papers of Caicedo, Cuevasa, Mophoub and N’Guérékata [14], Chen and Guo [20], Li and Wang [21] and Wang, Liu and Liu [22]. As far as we know, no work has been done for the asymptotic behavior of time periodic solutions for the extended Fisher-Kolmogorov equations. This is an interesting and important problem that needs to be solved. Also, it is one of motivations of this paper.

To our best knowledge, up until now the time periodic solutions for extended Fisher-Kolmogorov equations with delays have not been considered in the literature. Motivated by the above consideration, in this paper, we are concerned with the existence, uniqueness and asymptotic behavior of time \(\omega\)-periodical classical solutions for the following extended Fisher-Kolmogorov (EFK) equations with delays and general nonlinear term of the form

\[
\begin{aligned}
\frac{\partial}{\partial t} u(t, x) + \gamma \frac{\partial^4}{\partial x^4} u(t, x) - \frac{\partial^2}{\partial x^2} u(t, x) - u(t, x) = & \ g(t, x) \\
+ f(u(t - \tau_1, x), \ldots, u(t - \tau_n, x)), \quad \text{in} \ \mathbb{R} \times (0, 1),
\end{aligned}
\]

where \(\gamma > 0\) is a constant, \(f : \mathbb{R}^n \to \mathbb{R}\) is a nonlinear continuous function, \(g : \mathbb{R} \times [0, 1] \to \mathbb{R}\) is continuous and \(g(t, x)\) is \(\omega\)-periodic in \(t, \tau_1, \tau_2, \ldots, \tau_n\) are positive constants.

In [19], the authors required that \(n \leq 3\), which means that there are at most three delays in nonlinear term and is a strong restriction. In this paper, we will completely delete this condition. By defining a positive definite selfadjoint operator \(A\), which generates a compact semigroup \(T(t)\) \((t \geq 0)\) in Hilbert space \(H\), we can transfer the extended Fisher-Kolmogorov equations with delays (1.5) into the abstract form for a class of nonlinear evolution equation in the frame of Hilbert space \(H\), and then applying corresponding fixed point theorems, the theory of compact operator
semigroups and nonlinear analysis theory to discuss the existence and uniqueness of ω-periodic mild solutions for abstract nonlinear evolution equation. Further, by applying the maximal regularity of linear evolution equations with positive definite operator combined with the regularization method via the theory of analytic semigroups, we proved the existence and uniqueness of time ω-periodic classical solution for extended Fisher-Kolmogorov equations with delays (1.5). In addition, based on the uniqueness of time ω-periodic classical solution, we obtained the global asymptotic stability of time ω-periodic classical solution for extended Fisher-Kolmogorov equations with delays (1.5) by using the exponentially stability of analytic semigroup $T(t)$ ($t \geq 0$) and an integral inequality of Bellman type with delays.

The main results of this paper are as follows:

**Theorem 1.1.** Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz continuous, $g : \mathbb{R} \times [0,1] \to \mathbb{R}$ is Hölder continuous and $g(t,x)$ is ω-periodic in $t$. If the following conditions

(H1) There exist positive constants $\beta_1, \beta_2, \cdots, \beta_n$ and $K$ such that

$$|f(\xi_1, \cdots, \xi_n) + g(t,x)| \leq \sum_{k=1}^{n} \beta_k |\xi_k| + K \quad \text{for } x \in [0,1], \ (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n;$$

(H2) $\sum_{k=1}^{n} \beta_k < \gamma \pi^4 + \pi^2 - 1$,

hold, then EFK equation (1.5) has at least one time ω-periodic classical solution $u \in C^{1,2}(\mathbb{R} \times [0,1])$.

If we strengthen condition (H1), then we have the following uniqueness result of time ω-periodic classical solution for EFK equation (1.5).

**Theorem 1.2.** Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz continuous, $g : \mathbb{R} \times [0,1] \to \mathbb{R}$ is Hölder continuous and $g(t,x)$ is ω-periodic in $t$. If the following condition

(H3) There exist positive constants $\beta_1, \beta_2, \cdots, \beta_n$ such that

$$|f(\xi_1, \cdots, \xi_n) - f(\eta_1, \cdots, \eta_n)| \leq \sum_{k=1}^{n} \beta_k |\xi_k - \eta_k| \quad \text{for } (\xi_1, \cdots, \xi_n), (\eta_1, \cdots, \eta_n) \in \mathbb{R}^n,$$

and condition (H2) hold, then EFK equation (1.5) exists a unique time ω-periodic classical solution $u \in C^{1,2}(\mathbb{R} \times [0,1])$.

If we strengthen condition (H2), then we can obtain the global asymptotic stability of time ω-periodic classical solution for EFK equation (1.5).

**Theorem 1.3.** Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz continuous and $g : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ is Hölder continuous. If condition (H3) and the following condition
(H2) $\sum_{k=1}^{n} \beta_k e^{(\gamma \pi^4 + \pi^2 - 1) \tau_k} < \gamma \pi^4 + \pi^2 - 1$,

hold, then EFK equation (1.5) exists a unique time $\omega$-periodic classical solution $\bar{u} \in C^{1, 2}(\mathbb{R} \times [0, 1])$ and it is globally asymptotically stable.

The rest of this paper is organized as follows: In the following section we first introduce some notations and preliminaries which are used throughout this paper. Especially, the extended Fisher-Kolmogorov equations with delays (1.5) is transformed into an abstract nonlinear evolution equation in Hilbert space $H$. In section 3 we prove the global existence and uniqueness of time $\omega$-periodic classical solutions for extended Fisher-Kolmogorov equations with delays (1.5) (Theorems 1.1 and 1.2). In the last paragraph, we prove the global asymptotic stability of time $\omega$-periodic classical solution for extended Fisher-Kolmogorov equations with delays (1.5) (Theorem 1.3).

2 Preliminaries

Let $H = L^2([0, 1], \mathbb{R})$ be a real Hilbert space with the $L^2$-norm $\| \cdot \|_2$ defined by

$$\|u\|_2 = \left( \int_0^1 |u(x)|^2 dx \right)^{\frac{1}{2}}, \quad \forall u \in L^2([0, 1], \mathbb{R})$$

and inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle u, v \rangle = \int_0^1 u(x)v(x) dx, \quad \forall u, v \in L^2([0, 1], \mathbb{R}).$$

We define an operator $A$ in Hilbert space $H$ by

$$Au = \gamma \frac{\partial^4 u}{\partial x^4} - \frac{\partial^2 u}{\partial x^2} - u,$$

$$D(A) = \{W^{4,2}[0, 1] \mid u(0) = u(1) = u''(0) = u''(1) = 0\}. \quad (2.1)$$

From (2.1) it is easy to know that $D(A)$ is densely defined in $H$.

Let $u(t) = u(t, \cdot)$, $f(u(t - \tau_1), \cdots, u(t - \tau_n)) = f(u(t - \tau_1, \cdot), \cdots, u(t - \tau_n, \cdot))$, $g(t) = g(t, \cdot)$. Then the extended Fisher-Kolmogorov (EFK) equation with delays (1.5) can be transformed into the abstract form of delay evolution equation

$$u'(t) + Au(t) = f(u(t - \tau_1), \cdots, u(t - \tau_n)) + g(t), \quad t \in \mathbb{R}, \quad (2.2)$$
in Hilbert space $H = L^2([0, 1], \mathbb{R})$.

**Lemma 2.1.** The operator $A : D(A) \subset H \to H$ defined by (2.1) is a symmetric operator.

**Proof.** For any $u, v \in D(A)$, using integration by parts one gets that

$$\langle Au, v \rangle = \gamma \int_0^1 \frac{\partial^4 u(x)}{\partial x^4} v(x) dx - \int_0^1 \frac{\partial^2 u(x)}{\partial x^2} \frac{\partial^2 v(x)}{\partial x^2} dx - \int_0^1 u(x)v(x) dx$$

(2.3)

(2.3) means that the operator $A$ is a symmetric operator. This completes the proof of Lemma 2.1.

**Lemma 2.2.** The operator $A : D(A) \subset H \to H$ defined by (2.1) is a positive definite operator.

**Proof.** For any $u \in D(A)$, by (2.1) and Poincare inequality, we get that

$$\langle Au, u \rangle = \gamma \int_0^1 \frac{\partial^4 u(x)}{\partial x^4} u(x) dx - \int_0^1 \frac{\partial^2 u(x)}{\partial x^2} \frac{\partial^2 u(x)}{\partial x^2} dx - \int_0^1 u(x)^2 dx$$

and $\langle Au, u \rangle = 0$ if and only if $u = 0$. Therefore, $A$ is a positive definite operator. This completes the proof of Lemma 2.2.

**Lemma 2.3.** $\mathcal{R}(A) = H$.

**Proof.** We only need to prove that for any $\phi \in H$ there exist $u \in D(A)$ such that $Au = \phi$. This fact is equivalent to resolve the following linear boundary value problem
of forth-order ordinary differential equation
\[
\begin{cases}
\gamma \frac{d^4u(x)}{dx^4} - \frac{d^2u(x)}{dx^2} - u(x) = \phi(x), \quad x \in [0, 1], \\
u(0) = u(1) = u_{xx}(0) = u_{xx}(1) = 0,
\end{cases}
\]
(2.4)
namely
\[Au = \gamma \left( - \frac{d^2}{dx^2} + \mu_1 \right) \left( - \frac{d^2}{dx^2} + \mu_2 \right) u(x) = \phi(x),\]
(2.5)
where
\[\mu_1 = \frac{1 + \sqrt{1 + 4\gamma^2}}{2\gamma}, \quad \mu_2 = \frac{1 - \sqrt{1 + 4\gamma^2}}{2\gamma}.\]
(2.6)
Since \(\gamma > 0\), it is obvious that \(\mu_1 > 0\). From (2.6) one gets that
\[\mu_2 + \pi^2 = \frac{1 - \sqrt{1 + 4\gamma^2}}{2\gamma} + \pi^2 = \frac{1 - \sqrt{1 + 4\gamma^2} + 2\gamma\pi^2}{2\gamma} > \frac{1 + 4\gamma - \sqrt{1 + 4\gamma^2}}{2\gamma} > 0,
\]
which means that \(-\pi^2 < \mu_2 < 0\). By [23] we know that the solution of linear boundary value problem (2.4) can be expressed by
\[u(x) = \gamma \int_0^1 \int_0^1 G_1(x, y) G_2(y, z) \phi(z) dz dy, \quad (2.7)
\]
where \(G_1(x, y) (i = 1, 2)\) is the Green’s function of the second order linear boundary value problem
\[
\begin{cases}
-u''(x) + \mu_i u(x) = 0, \quad x \in [0, 1], \\
u(0) = u(1) = 0,
\end{cases}\]
(2.8)
and \(G_i(x, y)\) can be expressed by
\[
G_1(x, y) = \begin{cases}
\sinh \frac{\sqrt{\mu_1} x \sinh \sqrt{\mu_1} (1-y)}{\sqrt{\mu_1} \sinh \sqrt{\mu_1}}, & 0 \leq x \leq y \leq 1, \\
\sinh \frac{\sqrt{\mu_1} y \sinh \sqrt{\mu_1} (1-x)}{\sqrt{\mu_1} \sinh \sqrt{\mu_1}}, & 0 \leq y \leq x \leq 1,
\end{cases}
\]
and
\[
G_2(x, y) = \begin{cases}
\sin \frac{\sqrt{\mu_1} x \sin \sqrt{\mu_1} (1-y)}{\sqrt{\mu_1} \sin \sqrt{\mu_1}}, & 0 \leq x \leq y \leq 1, \\
\sin \frac{\sqrt{\mu_1} y \sin \sqrt{\mu_1} (1-x)}{\sqrt{\mu_1} \sin \sqrt{\mu_1}}, & 0 \leq y \leq x \leq 1.
\end{cases}
\]
(2.5) and (2.7) mean that for any \(\phi \in H\) there exist \(u \in D(A)\) such that \(Au = \phi\). Therefore, \(R(A) = H\). This completes the proof of Lemma 2.3. \(\square\)

Therefore, from Lemmas 2.1, 2.2 and 2.3, we know that the operator \(A : D(A) \subset H \to H\) defined by (2.1) is a positive definite selfadjoint operator and the first eigenvalue of the operator \(A\) is \(\lambda_1 = \gamma \pi^4 + \pi^2 - 1\). Furthermore, by Lemma 2.3 and [23]
one can easily to prove that the operator \( A : D(A) \subset H \to H \) defined by (2.1) has compact resolvent. Hence, it is well known from [17, 24] that the operator \( A \) defined by (2.1) is a sectorial operator, and therefore \(-A\) generates an analytic and compact semigroup \( T(t) (t \geq 0) \) in \( H \), which is exponentially stable and satisfies
\[
\|T(t)\|_2 \leq e^{-(\gamma \pi^4+\pi^2-1)t}, \quad t \geq 0.
\] (2.9)

Next, we give some concepts and conclusions on the fractional powers of \( A \). For \( \alpha > 0 \), \( A^{-\alpha} \) is defined by
\[
A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty s^{\alpha-1}T(s)ds,
\] (2.10)
where \( \Gamma(\cdot) \) is the Gamma function. \( A^{-\alpha} \in B(H) \) is injective, and \( A^\alpha \) can be defined by \( A^\alpha = (A^{-\alpha})^{-1} \) with the domain \( D(A^\alpha) = A^{-\alpha}(H) \), where \( B(H) \) denote by the Banach space of all linear bounded operators from \( H \) to \( H \) endowed with the topology defined by operator norm. For \( \alpha = 0 \), let \( A^0 = I \). We endow an inner product
\[
\langle \cdot, \cdot \rangle_\alpha = \langle A^\alpha \cdot, A^\alpha \cdot \rangle \text{ to } D(A^\alpha).
\]
Then \( D(A^\alpha) \) is a Hilbert space. We denote by \( H_\alpha \) the Hilbert space \( D(A^\alpha), \langle \cdot, \cdot \rangle_\alpha \). Especially, \( H_0 = H \) and \( H_1 = D(A) \). For \( 0 \leq \alpha < \beta \), \( H_\beta \) is densely embedded into \( H_\alpha \) and the embedding \( H_\beta \hookrightarrow H_\alpha \) is compact. For the details, we refer to [11] and [17].

From [24, Chapter 4, Corollary 2.5], we know that for any \( u_0 \in D(A) \), if the linear function \( \varphi \) is continuously differentiable on \( \mathbb{R}_+ \), then the initial value problem of linear evolution equation (LIVP)
\[
\begin{cases}
  u'(t) + Au(t) = \varphi(t), & t \in \mathbb{R}_+,
  \\
  u(0) = u_0
\end{cases}
\] (2.11)
exists a unique classical solution \( u \in C^1((0, +\infty), H) \cap C((0, +\infty), H_1) \cap C(\mathbb{R}_+, H) \) expressed by
\[
u(t) = T(t)u_0 + \int_0^t T(t-s)\varphi(s)ds.
\] (2.12)
If \( u_0 \in H \) and \( \varphi \in L^1(\mathbb{R}_+, H) \), the function \( u \) given by (2.12) belongs to \( C(\mathbb{R}_+, H) \), which is known as a mild solution of the LIVP (2.11). If a mild solution \( u \) of the LIVP (2.11) belongs to \( W^{1,1}(\mathbb{R}_+, H) \cap L^1(\mathbb{R}_+, H_1) \) and satisfies the equation for a.e. \( t \in \mathbb{R}_+ \), we call it a strong solution. By [24, Chapter 4, Corollary 2.10], we know that for any \( u_0 \in D(A) \), if the linear function \( \varphi \) is differentiable on \( \mathbb{R}_+ \), then LIVP (2.11) exists a unique strong solution.

The following regularity result will be used in the proof of our main results.
Lemma 2.4 ([25, Chapter II, Theorem 3.3]). Assume that $V$ and $H$ are two Hilbert space, $V \subset H$, $V$ denses in $H$, the injection is continuous and compact, $A : D(A) \subset H \to V$ is a positive definite self-adjoint operator in $H$. Then for any $u_0 \in V$ and $\varphi \in L^2(\mathbb{R}_+, V)$, the mild solution of the LIVP (2.11) has the regularity

$$u \in W^{1,2}(\mathbb{R}_+, H) \cap L^2(\mathbb{R}_+, H_1) \cap C(\mathbb{R}_+, V).$$

Denote by

$$C_\omega(\mathbb{R}, H) = \{u \mid u : \mathbb{R} \to H \text{ is continuous and } u(t+\omega) = u(t) \text{ for every } t \in \mathbb{R}\}.$$ 

Then it is easy to verify that $C_\omega(\mathbb{R}, H)$ is a Banach space endowed with the norm

$$\|u\|_C = \max_{t \in [0,\omega]} \|u(t)\|_2, \ \forall u \in C_\omega(\mathbb{R}, H).$$

Lemma 2.5. For every $\varphi \in C_\omega(\mathbb{R}, H)$, the linear evolution equation

$$u'(t) + Au(t) = \varphi(t), \quad t \in \mathbb{R} \tag{2.13}$$

has a unique $\omega$-periodic mild solution $u \in C_\omega(\mathbb{R}, H)$ which is given by

$$u(t) = T(t)(I - T(\omega))^{-1}\int_0^\omega T(\omega - s)\varphi(s)ds + \int_0^t T(t-s)\varphi(s)ds, \quad t \in \mathbb{R}. \tag{2.14}$$

Proof. By the above discussion, we know that the evolution equation (2.11) exists a unique mild solution $u$ given by (2.12) and

$$u(\omega) = T(\omega)u_0 + \int_0^\omega T(\omega - s)\varphi(s)ds. \tag{2.15}$$

From (2.9) one gets that $\|T(\omega)\|_2 \leq e^{-((\gamma+\pi^2-1)\omega)} < 1$. Therefore, operator spectrum theorem we know that $I - T(\omega)$ has a bounded inverse operator $(I - T(\omega))^{-1}$. Hence, there exists a unique initial value

$$u_0 = (I - T(\omega))^{-1}\int_0^\omega T(\omega - s)\varphi(s)ds \tag{2.16}$$

such that the unique mild solution $u$ of LIVP (2.11) expressed by (2.12) satisfies the periodic boundary condition $u(0) = u_0 = u(\omega)$. Therefore, from (2.12), (2.15), (2.16) and the fact that $\varphi(t) = \varphi(t+\omega)$ for $t \in \mathbb{R}$, we get that for every $t \in \mathbb{R}_+$

$$u(t+\omega) = T(t+\omega)u_0 + \int_0^\omega T(t+\omega - s)\varphi(s)ds + \int_0^t T(t+\omega - s)\varphi(s)ds$$

$$= T(t)[T(\omega)u_0 + \int_0^\omega T(\omega - s)\varphi(s)ds] + \int_0^t T(t-s)\varphi(s)ds$$

$$= T(t)u(\omega) + \int_0^t T(t-s)\varphi(s)ds$$

$$= u(t).$$
Therefore, the $\omega$-periodic extension of $u$ on $\mathbb{R}$ is a unique $\omega$-periodic mild solution of linear evolution equation (2.13). Combining (2.12) and (2.16), we get that the mild solution $u$ of linear evolution equation (2.13) satisfies (2.14).

Conversely, we can verify directly that the function $u \in C_{\omega}(\mathbb{R}, H)$ given by (2.14) is a mild solution of linear evolution equation (2.13). This completes the proof of Lemma 2.5. □

In what follows, we recall the Bellman type inequality with delays (see [13, Lemma 4.1]), which will be used in the proof of our main results.

**Lemma 2.6.** Denote $r = \max\{\tau_1, \tau_2, \ldots, \tau_n\}$. Let $\psi \in C([-r, \infty), \mathbb{R}_+)$. If there exist positive constants $b_1, b_2, \ldots, b_n$ such that $\psi$ satisfy the integral inequality

$$
\psi(t) \leq \psi(0) + \sum_{k=1}^{n} b_k \int_{0}^{t} \psi(s - \tau_k) ds, \quad t \geq 0.
$$

Then for every $t \geq 0$,

$$
\psi(t) \leq \|\psi\|_{C[-r,0]} e^{(\sum_{k=1}^{n} b_k)t},
$$

where $\|\psi\|_{C[-r,0]} = \max_{t \in [-r,0]} |\psi(t)|$.

### 3 Existence and uniqueness of periodic solutions

In this section, we will prove the global existence and uniqueness of time $\omega$-periodic classical solutions to the extended Fisher-Kolmogorov equations with delays and general nonlinear term (1.5), i.e., Theorems 1.1 and 1.2.

**Proof of Theorem 1.1.** By the discussions in Section 2, we know that EFK equation (1.5) can be transformed into the abstract delay evolution equation (2.2) in Hilbert space $H = L^2([0, 1], \mathbb{R})$. In what follows, we prove the existence of time $\omega$-periodic mild solutions for abstract delay evolution equation (2.2). Consider the operator $\mathcal{F}$ on $C_{\omega}(\mathbb{R}, H)$ defined by

$$
(\mathcal{F}u)(t) = T(t)(I - T(\omega))^{-1} \int_{0}^{\omega} T(\omega - s)[f(u(s - \tau_1), \ldots, u(s - \tau_n)) + g(s)] ds
$$

$$
+ \int_{0}^{t} T(t - s)[f(u(s - \tau_1), \ldots, u(s - \tau_n)) + g(s)] ds, \quad t \in \mathbb{R}.
$$

(3.1)
By the assumptions that \( f : \mathbb{R}^n \to \mathbb{R} \) is locally Lipschitz continuous, \( g : [0, 1] \times \mathbb{R} \to \mathbb{R} \) is Hölder continuous and \( g(x, t) \) is \( \omega \)-periodic in \( t \) combined with Lemma 2.5 one can easily see that the operator \( \mathcal{F} \) maps \( C_\omega(\mathbb{R}, \mathcal{H}) \) to \( C_\omega(\mathbb{R}, \mathcal{H}) \) is continuous and the time \( \omega \)-periodic mild solutions of abstract delay evolution equation (2.2) is equivalent to the fixed point of operator \( \mathcal{F} \) defined by (3.1).

Denote

\[
\Omega_R = \{ u \in C_\omega(\mathbb{R}, \mathcal{H}) : \| u \|_C \leq R \},
\]

then \( \Omega_R \) is a closed ball in \( C_\omega(\mathbb{R}, \mathcal{H}) \) with center \( \theta \) and radius \( R \). By the condition (H1) we know that for any \( u \in C_\omega(\mathbb{R}, \mathcal{H}) \)

\[
\| f(u(t - \tau_1), \ldots, u(t - \tau_n)) + g(t) \|_2 \leq \sum_{k=1}^{n} \beta_k \| u(t - \tau_k) \|_2 + K, \quad t \in \mathbb{R}. \tag{3.2}
\]

Furthermore, from the fact that \( \| T(\omega) \|_2 \leq e^{-((\gamma \pi^4 + \pi^2 - 1)\omega) < 1 \) combined with Neumann expression, \( (I - T(\omega))^{-1} \) can be expressed by

\[
(I - T(\omega))^{-1} = \sum_{n=0}^{\infty} T^n(\omega).
\]

Therefore, by the above equality and (2.9) one gets that

\[
\| (I - T(\omega))^{-1} \|_2 = \| \sum_{n=0}^{\infty} T^n(\omega) \|_2 \leq \sum_{n=0}^{\infty} e^{-(\gamma \pi^4 + \pi^2 - 1)n\omega} = \frac{1}{1 - e^{-(\gamma \pi^4 + \pi^2 - 1)\omega}}. \tag{3.3}
\]

Next, we prove that there exists a constant \( R \) big enough such that the operator \( \mathcal{F} \) maps \( \Omega_R \) to \( \Omega_R \). In fact, choosing

\[
R \geq \frac{K}{\gamma \pi^4 + \pi^2 - 1 - \sum_{k=1}^{n} \beta_k}. \tag{3.4}
\]

For any \( u \in \Omega_R \) and \( t \in \mathbb{R} \), by (2.9), (3.1)-(3.4) and the condition (H2), we have

\[
\| (\mathcal{F}u)(t) \|_2 \leq \frac{e^{-(\gamma \pi^4 + \pi^2 - 1)t}}{1 - e^{-(\gamma \pi^4 + \pi^2 - 1)\omega}} \int_{0}^{\omega} e^{-(\gamma \pi^4 + \pi^2 - 1)(s - \tau)} \left[ \sum_{k=1}^{n} \beta_k \| u(s - \tau_k) \|_2 + K \right] ds
\]

\[
+ \int_{0}^{t} e^{-(\gamma \pi^4 + \pi^2 - 1)(t - s)} \left[ \sum_{k=1}^{n} \beta_k \| u(s - \tau_k) \|_2 + K \right] ds
\]

\[
\leq \left[ \frac{e^{-(\gamma \pi^4 + \pi^2 - 1)t}}{\gamma \pi^4 + \pi^2 - 1} + \frac{1 - e^{-(\gamma \pi^4 + \pi^2 - 1)t}}{\gamma \pi^4 + \pi^2 - 1} \right] \cdot \left[ \sum_{k=1}^{n} \beta_k \| u \|_C + K \right]
\]

\[
\leq \frac{1}{\gamma \pi^4 + \pi^2 - 1} \left( R \sum_{k=1}^{n} \beta_k + K \right)
\]

\[
\leq R.
\]
Therefore, 
\[
\|\mathcal{F}u\|_C = \max_{t \in [0, \omega]} \|(\mathcal{F}u)(t)\|_2 \leq R,
\]
which means that \( \mathcal{F}u \in \Omega_R \). Therefore, we proved that \( \mathcal{F} : \Omega_R \to \Omega_R \) is a continuous operator.

Next, we demonstrate that \( \mathcal{F} : \Omega_R \to \Omega_R \) is a compact operator. To prove this, we first show that \( \{(\mathcal{F}u)(t) : u \in \Omega_R\} \) is relatively compact in \( H \) for every \( t \in \mathbb{R} \). From the periodicity of the operator \( (\mathcal{F}u)(t) \) for \( t \in \mathbb{R} \) and \( u \in \Omega_R \), we only need to prove that \( \{(\mathcal{F}u)(t) : u \in \Omega_R\} \) is relatively compact in \( H \) for \( 0 \leq t \leq \omega \). It is easy to see that for every \( u \in \Omega_R \),

\[
(\mathcal{F}u)(0) = \left( I - T(\omega) \right)^{-1} \int_0^\omega T(\omega - s)[f(u(s - \tau_1), \ldots, u(s - \tau_n)) + g(s)]ds. \tag{3.5}
\]

For any \( 0 < \varepsilon < \omega \) and \( u \in \Omega_R \), we define the operator \( \mathcal{F}^\varepsilon \) by

\[
(\mathcal{F}^\varepsilon u)(0) = \left( I - T(\omega) \right)^{-1} \int_0^{\omega - \varepsilon} T(\omega - s)[f(u(s - \tau_1), \ldots, u(s - \tau_n)) + g(s)]ds = T(\varepsilon) \left( I - T(\omega) \right)^{-1} \int_0^{\omega - \varepsilon} T(\omega - s - \varepsilon) (f(u(s - \tau_1), \ldots, u(s - \tau_n)) + g(s)]ds. \tag{3.6}
\]

Since \( T(t) \) is compact for every \( t > 0 \), the set \( \{(\mathcal{F}^\varepsilon u)(0) : u \in \Omega_R\} \) is relatively compact in \( H \) for every \( \varepsilon \in (0, \omega) \). Moreover, for every \( u \in \Omega_R \), by (3.2), (3.3), (3.5) and (3.6), we get that

\[
\|(\mathcal{F}u)(0) - (\mathcal{F}^\varepsilon u)(0)\|_2 \leq \frac{1}{1 - e^{-((\pi^4 + \pi^2 - 1)\omega - \varepsilon)}[\sum_{k=1}^n \beta_k \|u(s - \tau_k)\|_2 + K]} ds \leq \frac{1}{(\gamma \pi^4 + \pi^2 - 1)(1 - e^{-(\pi^4 + \pi^2 - 1)\omega})} (R \sum_{k=1}^n \beta_k + K) \to 0 \text{ as } \varepsilon \to 0.
\]

Therefore, we have proved that there exists relatively compact set \( \{(\mathcal{F}^\varepsilon u)(0) : u \in \Omega_R\} \) arbitrarily close to the set \( \{(\mathcal{F}u)(0) : u \in \Omega_R\} \), this means that the set \( \{(\mathcal{F}u)(0) : u \in \Omega_R\} \) is relatively compact in \( H \). Let \( 0 < t \leq \omega \) be given, \( 0 < \varepsilon < t \) and \( u \in \Omega_R \), we
define the operator $\mathcal{F}^\epsilon u$ by

$$(\mathcal{F}^\epsilon u)(t) = T(t)(\mathcal{F}u)(0) + \int_0^{t-\epsilon} T(t-s)[f(u(s-\tau_1), \ldots, u(s-\tau_n)) + g(s)]ds$$

$$(\mathcal{F}^\epsilon u)(t) = T(t)(\mathcal{F}u)(0) + T(\epsilon) \int_0^{t-\epsilon} T(t-s-\epsilon)[f(u(s-\tau_1), \ldots, u(s-\tau_n)) + g(s)]ds. \quad (3.7)$$

By compactness of the operator $T(t)$ for $t > 0$ combined with the fact that the set $\{(\mathcal{F}u)(0) : u \in \Omega_R\}$ is relatively compact in $H$, the set $\{(\mathcal{F}^\epsilon u)(t) : u \in \Omega_R\}$ is relatively compact in $H$ for every $\epsilon \in (0, t)$ and $0 < t \leq \omega$. Furthermore, for every $u \in \Omega_R$, by (3.1), (3.2) and (3.7), we get that

$$\| (\mathcal{F}u)(t) - (\mathcal{F}^\epsilon u)(t) \|_2 = \left\| \int_{t-\epsilon}^{t} T(t-s)[f(u(s-\tau_1), \ldots, u(s-\tau_n)) + g(s)]ds \right\|_2$$

$$\leq \int_{t-\epsilon}^{t} e^{-(\gamma\pi^4 + \gamma^2 - 1)(t-s)} \left[ \sum_{k=1}^{n} \beta_k \| u(s-\tau_k) \|_2 + K \right] ds$$

$$\leq \frac{1 - e^{-(\gamma\pi^4 + \gamma^2 - 1)\epsilon}}{\gamma\pi^4 + \pi^2 - 1} \left( R \sum_{k=1}^{n} \beta_k + K \right)$$

$$\rightarrow 0 \text{ as } \epsilon \rightarrow 0.$$

Hence, we have proved that there exists relatively compact set $\{(\mathcal{F}^\epsilon u)(t) : u \in \Omega_R\}$ arbitrarily close to the set $\{(\mathcal{F}u)(t) : u \in \Omega_R\}$ in $H$ for $0 < t \leq \omega$. Therefore, the set $\{(\mathcal{F}u)(t) : u \in \Omega_R\}$ is also relatively compact in $H$ for $0 < t \leq \omega$, which combined with the fact that the set $\{(\mathcal{F}u)(0) : u \in \Omega_R\}$ is relatively compact in $H$ we get the relatively compactness of the set $\{(\mathcal{F}u)(t) : u \in \Omega_R\}$ in $H$ for $0 \leq t \leq \omega$.

In the following, we prove that $\mathcal{F} : \Omega_R \rightarrow \Omega_R$ is an equicontinuous operator. For any $u \in \Omega_R$ and $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$, we get form (2.9), (3.1) and (3.2) that

$$\| (\mathcal{F}u)(t_2) - (\mathcal{F}u)(t_1) \|_2 \leq \left\| (T(t_2) - T(t_1)) \left( I - T(\omega) \right)^{-1} \int_{0}^{\omega} T(\omega - s) [f(u(s-\tau_1), \ldots, u(s-\tau_n)) + g(s)] ds \right\|_2$$

$$+ \left( R \sum_{k=1}^{n} \beta_k + K \right) \int_{t_1}^{t_2} e^{-(\gamma\pi^4 + \gamma^2 - 1)(t_2-s)} ds$$
\[
\begin{align*}
\sum_{k=1}^n \beta_k + K \int_0^{t_1} \|T(t_2 - s) - T(t_1 - s)\|ds \\
\quad + \left( R \sum_{k=1}^n \beta_k + K \right) \int_0^{t_1} \|T(t_2) - T(t_1)\|ds \\
= I_1 + I_2 + I_3,
\end{align*}
\]

where

\[
\begin{align*}
I_1 &= \left\| (T(t_2) - T(t_1)) \left( I - T(\omega) \right)^{-1} \int_0^\omega T(\omega - s) [f(u(s - \tau_1), \ldots, u(s - \tau_n)) + g(s)] ds \right\|_2, \\
I_2 &= \left( R \sum_{k=1}^n \beta_k + K \right) \int_0^{t_2} e^{-(\gamma \pi^4 + \pi^2 - 1)(t_2 - s)} ds, \\
I_3 &= \left( R \sum_{k=1}^n \beta_k + K \right) \int_0^{t_1} \|T(t_2 - s) - T(t_1 - s)\| ds.
\end{align*}
\]

Therefore, we only need to check \( I_i \) tend to 0 independently of \( u \in \Omega_R \) when \( t_2 - t_1 \to 0 \) for \( i = 1, 2, 3 \). For \( I_1 \), by the definition of \( I_1 \), (2.9), (3.2) and (3.3), we get that

\[
\begin{align*}
\left\| (I - T(\omega))^{-1} \int_0^\omega T(\omega - s) [f(u(s - \tau_1), \ldots, u(s - \tau_n)) + g(s)] ds \right\|_2 \\
&\leq \frac{1}{1 - e^{-(\gamma \pi^4 + \pi^2 - 1)\omega}} \int_0^\omega e^{-(\gamma \pi^4 + \pi^2 - 1)(\omega - s)} \left[ \sum_{k=1}^n \beta_k \|u(s - \tau_k)\|_2 + K \right] ds \\
&\leq \frac{R \sum_{k=1}^n \beta_k + K}{\gamma \pi^4 + \pi^2 - 1}.
\end{align*}
\]

The above inequality combined with the strongly continuity of the semigroup \( T(t) \) \( (t \geq 0) \) and the definition of \( I_1 \), we can easily to get that \( I_1 \to 0 \) as \( t_2 - t_1 \to 0 \). For \( I_2 \), we can get by direct calculus that

\[
I_2 \leq \frac{R \sum_{k=1}^n \beta_k + K}{\gamma \pi^4 + \pi^2 - 1} \left[ 1 - e^{-(\gamma \pi^4 + \pi^2 - 1)(t_2 - t_1)} \right]
\]

\[
\to 0 \quad \text{as} \quad t_2 - t_1 \to 0.
\]

For \( I_3 \), by the definition of \( I_3 \), the property of Lebesgue integral and the norm continuity of \( T(t) \) for \( t > 0 \), we get that

\[
I_3 \leq \left( R \sum_{k=1}^n \beta_k + K \right) \int_0^{t_1} \|T(t_2 - t_1 + s) - T(s)\| ds
\]

\[
\to 0 \quad \text{as} \quad t_2 - t_1 \to 0.
\]

\[14\]
As a result, \( \| (Fu)(t_2) - (Fu)(t_1) \|_2 \) tends to zero independently of \( u \in \Omega_R \) as \( t_2 - t_1 \to 0 \), which means that the operator \( F : \Omega_R \to \Omega_R \) is an equicontinuous operator. Therefore, \( \{ Fu : u \in \Omega_R \} \) is relatively compact by Arzela-Ascoli theorem. Hence, the continuity of operator \( F \) and relatively compactness of the set \( \{ Fu : u \in \Omega_R \} \) imply that \( F : \Omega_R \to \Omega_R \) is a completely continuous operator. It follows from Schauder’s fixed point theorem that \( F \) has at least one fixed point \( u \in \Omega_R \), which is just a time \( \omega \)-periodic mild solution of abstract delay evolution equation (2.2).

In what follows, we prove the regularity for the time \( \omega \)-periodic mild solution \( u \) of abstract delay evolution equation (2.2). Since \( u \) is the mild solution of linear evolution equation (2.13) for \( \varphi(\cdot) = f(u(\cdot - \tau_1), \cdots, u(\cdot - \tau_n)) + g(\cdot) \in L^2(\mathbb{R}, \mathbb{H}) \), by the maximal regularity of linear evolution equations with positive definite operator in Hilbert spaces (see for details Lemma 2.4), when \( u_0 \in V := H_{1/2} \), the mild solution of LIVP (2.11) has the regularity
\[
 u \in W^{1,2}(\mathbb{R}, H) \cap L^2(\mathbb{R}, H_1) \cap C(\mathbb{R}, H_{1/2})
\]
and it is a strong solution. We noticed that \( u(t) \) is the mild solution of LIVP (2.11) for
\[
 u_0 = (I - T(\omega))^{-1} \int_0^\omega T(\omega - s)\varphi(s)ds.
\]

By the representation (2.12) of mild solution, \( u(t) = T(t)u_0 + v(t) \), where \( v(t) = \int_0^t T(t - s)\varphi(s)ds \). Since the function \( v(t) \) is a mild solution of LIVP (2.11) with the null initial value \( u(0) = \theta \), \( v \) has the regularity (3.8). By the analytic property of the semigroup \( T(t) \), \( T(\omega)u_0 \in D(A) \subset H_{1/2} \). Hence,
\[
 u_0 = u(\omega) = T(\omega)u_0 + v(\omega) \in H_{1/2}.
\]

Using the regularity (3.8) again, we obtain that \( u \in W^{1,2}(\mathbb{R}, H) \cap L^2(\mathbb{R}, D(A)) \) and it is a time \( \omega \)-periodic strong solution of linear evolution equation (2.13), which means that the fixed point \( u \) of the operator \( F \) defined by (3.1) belongs to \( W^{1,2}(\mathbb{R}, H) \cap L^2(\mathbb{R}, D(A)) \) is the time \( \omega \)-periodic strong solution of the abstract delay evolution equation (2.2). Furthermore, By the usual regularization method via the theory of analytic semigroups of linear operators which used in [26, Lemma 4.2] combined with the fact that \( f : \mathbb{R}^n \to \mathbb{R} \) is locally Lipschitz continuous and \( g : \mathbb{R} \times [0, 1] \to \mathbb{R} \) is Hölder continuous, we can prove that \( u \in C^{1,2}(\mathbb{R} \times [0, 1]) \) is a time \( \omega \)-periodic classical solution of EFK equation (1.5). This completes the proof of Theorem 1.1. □

**Proof of Theorem 1.2.** By the proof of Theorem 1.1 we know that EFK equation (1.5) can be transformed into the abstract delay evolution equation (2.2) in Hilbert space \( H = L^2([0, 1], \mathbb{R}) \) and the time \( \omega \)-periodic mild solutions of abstract delay evolution equation (2.2) is equivalent to the fixed point of operator \( F \) defined by (3.1),
which maps $C_\omega(\mathbb{R}, H)$ to $C_\omega(\mathbb{R}, H)$. By the condition (H3) we know that for every $t \in \mathbb{R}$ and $u, v \in C_\omega(\mathbb{R}, H)$

$$\|f(u(t - \tau_1), \cdots, u(t - \tau_n)) - f(v(t - \tau_1), \cdots, v(t - \tau_n))\|_2 \leq \sum_{k=1}^n \beta_k \|u(t - \tau_k) - v(t - \tau_k)\|_2. \quad (3.9)$$

Therefore, for any $u, v \in C_\omega(\mathbb{R}, H)$, by (2.9), (3.1), (3.3), (3.9) and the condition (H2), we get that

$$\|F u(t) - (F v)(t)\|_2 \leq \frac{e^{-(\gamma_1^4 + \pi^2 - 1)t}}{1 - e^{-(\gamma_1^4 + \pi^2 - 1)}} \int_0^\omega e^{-(\gamma_1^4 + \pi^2 - 1)(\omega - s)} \left(\sum_{k=1}^n \beta_k \|u(s - \tau_k) - v(s - \tau_k)\|_2\right) ds$$

$$+ \int_0^t e^{-(\gamma_1^4 + \pi^2 - 1)(t - s)} \left(\sum_{k=1}^n \beta_k \|u(s - \tau_k) - v(s - \tau_k)\|_2\right) ds \leq \left[\frac{e^{-(\gamma_1^4 + \pi^2 - 1)t}}{\gamma_1^4 + \pi^2 - 1} + \frac{1 - e^{-(\gamma_1^4 + \pi^2 - 1)t}}{\gamma_1^4 + \pi^2 - 1}\right] \cdot \left(\sum_{k=1}^n \beta_k \|u - v\|_C\right)$$

$$= \frac{\sum_{k=1}^n \beta_k}{\gamma_1^4 + \pi^2 - 1} \|u - v\|_C$$

$$< \|u - v\|_C,$$

which means that,

$$\|F u - F v\|_C = \max_{t \in [0, \omega]} \|(F u)(t) - (F v)(t)\|_2 < \|u - v\|_C.$$

Hence, $F : C_\omega(\mathbb{R}, H) \to C_\omega(\mathbb{R}, H)$ is a contraction operator, and therefore $F$ has a unique fixed point $u \in C_\omega(\mathbb{R}, H)$, which is in turn the unique time $\omega$-periodic mild solution of the abstract delay evolution equation (2.2). By using a completely similar method with which used in the proof of Theorem 1.1 combined with the fact that $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz continuous and $g : \mathbb{R} \times [0, 1] \to \mathbb{R}$ is Hölder continuous, we can prove that $u \in C^{1,2}(\mathbb{R} \times [0, 1])$ is the unique time $\omega$-periodic classical solution of EFK equation (1.5). This completes the proof of Theorem 1.2. \hfill \Box
Global asymptotic stability of periodic solutions

In this section, we will prove the global asymptotic stability of time $\omega$-periodic classical solution for EFK equation (1.5), i.e., Theorem 1.3. For this purpose, we firstly discuss the existence of classical solutions for the initial value problem of extended Fisher-Kolmogorov equations with delays

$$
\begin{aligned}
\frac{\partial}{\partial t} u(t, x) + \gamma \frac{\partial^4}{\partial x^4} u(t, x) - \frac{\partial^2}{\partial x^2} u(t, x) - u(t, x) &= g(t, x) + f(u(t - \tau_1, x), \ldots, u(t - \tau_n, x)), \text{ in } \mathbb{R}_+ \times (0, 1), \\
\end{aligned}
$$

with

$$
\begin{aligned}
u(t, 0) = u(t, 1) = 0, \quad u_{xx}(t, 0) = u_{xx}(t, 1) = 0, \quad t \in \mathbb{R}_+, \\
u(t, x) = \kappa(t, x), \quad t \in [-r, 0], \quad x \in (0, 1),
\end{aligned}
$$

where $\gamma > 0$ is a constant, $f : \mathbb{R}^n \to \mathbb{R}$ is a nonlinear continuous function, $g : \mathbb{R}_+ \times [0, 1] \to \mathbb{R}$ is continuous, $\tau_1, \tau_2, \ldots, \tau_n$ are positive constants, $r = \max\{\tau_1, \tau_2, \ldots, \tau_n\}$, $\kappa \in C([-r, 0] \times (0, 1), \mathbb{R})$.

Let $\kappa(t) = \kappa(t, \cdot)$ for $t \in [-r, 0]$. Then from the discussion in Section 2 we know that the initial value problem of extended Fisher-Kolmogorov equations with delays (4.1) can be transformed into the abstract form of initial value problem to delay evolution equation

$$
\begin{aligned}
\begin{cases}
u' + Au(t) = f(u(t - \tau_1), \ldots, u(t - \tau_n)) + g(t), & t \in \mathbb{R}_+, \\
u(t) = \kappa(t), & t \in [-r, 0]
\end{cases}
\end{aligned}
$$

in Hilbert space $H = L^2((0, 1), \mathbb{R})$. A function $u \in C([-r, \infty), H)$ is said to be a mild solution of initial value problem (4.2) if $u(t)$ satisfies

$$
u(t) = T(t)u(0) + \int_0^t T(t-s)[f(u(s-\tau_1), \ldots, u(s-\tau_n)) + g(s)]ds \quad \text{for} \quad t \in \mathbb{R}_+, \quad (4.3)
$$

and the initial condition

$$
u(t) = \kappa(t) \quad \text{for} \quad t \in [-r, 0]. \quad (4.4)
$$

**Theorem 4.1.** Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz continuous and $g : \mathbb{R}_+ \times [0, 1] \to \mathbb{R}$ is Hölder continuous. If the condition (H3) is satisfied, the initial value problem of extended Fisher-Kolmogorov equations with delays (4.1) exists a unique classical solution $u \in C^{1,2}(\mathbb{R} \times [0, 1])$.

**Proof.** By the above discussion, we know that the initial value problem of extended Fisher-Kolmogorov equations with delays (4.1) can be transformed into the abstract

...
form of initial value problem to delay evolution equation (4.2) in Hilbert space $H = L^2([0, 1], \mathbb{R})$. Define the operator $Q$ on $C([-r, \infty), H)$ as follows

$$
(Qu)(t) = \begin{cases}
T(t)u(0) + \int_0^t T(t-s)[f(u(s-\tau_1), \ldots, u(s-\tau_n)) + g(s)]ds, & t \in \mathbb{R}_+,
\kappa(t), & t \in [-r, 0]
\end{cases}
$$

(4.5)

Then by the assumptions that $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is locally Lipschitz continuous, $g : \mathbb{R}_+ \times [0, 1] \rightarrow \mathbb{R}$ is Hölder continuous and $\kappa \in C([-r, 0] \times (0, 1), \mathbb{R})$ one can easily see that $Q$ maps $C([-r, \infty), H)$ to $C([-r, \infty), H)$ and the mild solutions of initial value problem for delay evolution equation (4.2) is equivalent to the fixed point of operator $Q$ defined by (4.5).

For any $u, v \in C([-r, \infty), H)$, by (2.9), (3.9) and (4.5), we get that

$$
\|(Qu)(t) - (Qu)(t)\|_2 \leq \int_0^t e^{-(\gamma \pi^4 + \pi^2 - 1)(t-s)} \left( \sum_{k=1}^n \beta_k \|u(s-\tau_k) - v(s-\tau_k)\|_2 \right)ds
$$

$$
\leq \frac{\sum_{k=1}^n \beta_k}{\gamma \pi^4 + \pi^2 - 1} \|u - v\|_C < \|u - v\|_C,
$$

which means that,

$$
\|Q u - Q v\|_C = \sup_{t \in [-r, \infty)} \|(Q u)(t) - (Qu)(t)\|_2 < \|u - v\|_C.
$$

Therefore, $Q : C([-r, \infty), H) \rightarrow C([-r, \infty), H)$ is a contraction operator, and therefore $Q$ has a unique fixed point $u \in C([-r, \infty), H)$, which in turn is the unique mild solution of the initial value problem to delay evolution equation (4.2). By using a completely similar method with which used in the proof of Theorem 1.1 combined with the fact that $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is locally Lipschitz continuous and $g : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$ is Hölder continuous, we can prove that $u \in C^{1,2}(\mathbb{R}_+ \times [0, 1])$ is the unique classical solution for the initial value problem of extended Fisher-Kolmogorov equations with delays (4.1). This completes the proof of Theorem 4.1.

**Proof of Theorem 1.3.** One can easily see that $(H2)' \Rightarrow (H2)$. Therefore, By Theorem 1.2 we know that EFK equation (1.5) exists a unique time $\omega$-periodic classical solution $\varphi \in C^{1,2}(\mathbb{R} \times [0, 1])$. Furthermore, Theorem 4.1 means that the initial value problem of extended Fisher-Kolmogorov equations with delays (4.1) exist a unique classical solution $u_\kappa \in C^{1,2}(\mathbb{R}_+ \times [0, 1])$. 


By (2.9), (3.1), (3.5), (3.9) and (4.5), we get that
\[
\|\overline{u}(t) - u_\kappa(t)\|_2 \\
\leq e^{-(\gamma^4 + \pi^2 - 1)t} \|\overline{u}(0) - u_\kappa(0)\|_2 \\
+ \int_0^t e^{-(\gamma^4 + \pi^2 - 1)(t-s)} \left( \sum_{k=1}^n \beta_k \|\overline{u}(s - \tau_k) - u_\kappa(s - \tau_k)\|_2 \right) ds \\
= e^{-(\gamma^4 + \pi^2 - 1)t} \|\overline{u}(0) - u_\kappa(0)\|_2 \\
+ e^{-(\gamma^4 + \pi^2 - 1)t} \sum_{k=1}^n \beta_k e^{(\gamma^4 + \pi^2 - 1)\tau_k} \int_0^t e^{(\gamma^4 + \pi^2 - 1)(s-\tau_k)} \|\overline{u}(s - \tau_k) - u_\kappa(s - \tau_k)\|_2 ds,
\]
from which one gets that
\[
e^{(\gamma^4 + \pi^2 - 1)t} \|\overline{u}(t) - u_\kappa(t)\|_2 \\
\leq \|\overline{u}(0) - u_\kappa(0)\|_2 \\
+ \sum_{k=1}^n \beta_k e^{(\gamma^4 + \pi^2 - 1)\tau_k} \int_0^t e^{(\gamma^4 + \pi^2 - 1)(s-\tau_k)} \|\overline{u}(s - \tau_k) - u_\kappa(s - \tau_k)\|_2 ds
\]
\hspace{1cm} (4.6)

Letting
\[
\psi(t) = e^{(\gamma^4 + \pi^2 - 1)t} \|\overline{u}(t) - u_\kappa(t)\|_2, \quad [-r, \infty).
\]
Then from (4.6) we get that
\[
\psi(t) \leq \psi(0) + \sum_{k=1}^n \beta_k e^{(\gamma^4 + \pi^2 - 1)\tau_k} \int_0^t \psi(s - \tau_k) ds, \quad t \geq 0.
\]
\hspace{1cm} (4.7)

Therefore, by (4.7) and Lemma 2.6, we know that for every \( t \geq 0, \)
\[
e^{(\gamma^4 + \pi^2 - 1)t} \|\overline{u}(t) - u_\kappa(t)\|_2 = \psi(t) \leq \max_{t \in [-r,0]} e^{(\gamma^4 + \pi^2 - 1)t} \|\overline{u}(t) - \kappa(t)\|_2 \\
\cdot e^{[\sum_{k=1}^n \beta_k e^{(\gamma^4 + \pi^2 - 1)\tau_k}] t},
\]
from which one gets that
\[
\|\overline{u}(t) - u_\kappa(t)\|_2 \leq \max_{t \in [-r,0]} e^{(\gamma^4 + \pi^2 - 1)t} \|\overline{u}(t) - \kappa(t)\|_2 e^{[\sum_{k=1}^n \beta_k e^{(\gamma^4 + \pi^2 - 1)\tau_k} - (\gamma^4 + \pi^2 - 1)] t}.
\]
\hspace{1cm} (4.8)

By the condition (H2)', we get that
\[
\sum_{k=1}^n \beta_k e^{(\gamma^4 + \pi^2 - 1)\tau_k} - (\gamma^4 + \pi^2 - 1) < 0.
\]
\hspace{1cm} (4.9)
Hence, from (4.8) and (4.9) we know that

\[ \left( \int_0^1 |\bar{u}(t, x) - u_\kappa(t, x)|^2 dx \right)^{\frac{1}{2}} = \| u(t) - u_\kappa(t) \|_2 \to 0 \quad \text{as} \quad t \to +\infty. \]

Therefore, the \( \omega \)-periodic classical solution \( u \) of EFK equation (1.5) is globally asymptotically stable and it exponentially attracts every classical solution for the initial value problem of extended Fisher-Kolmogorov equations with delays. This completes the proof of Theorem 1.3.

\[ \square \]

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