Non-rational divisors over non-Gorenstein terminal singularities

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Abstract

Let $(X, o)$ be a germ of a 3-dimensional terminal singularity of index $m \geq 2$. If $(X, o)$ has type $cAx/4$, $cD/3 - 3$, $cD/2 - 2$, or $cE/2$, then assume that the standard equation of $X$ in $\mathbb{C}^4/\mathbb{Z}_m$ is non-degenerate with respect to its Newton diagram. Let $\pi: Y \to X$ be a resolution. We show that there are not more than 2 non-rational divisors $E_i$, $i = 1, 2$, on $Y$ such that $\pi(E_i) = o$ and discrepancy $a(E_i, X) \leq 1$. When such divisors exist, we describe them as exceptional divisors of certain blowups of $(X, o)$ and study their birational type.

1 Introduction

In this paper, we continue the study of resolutions of terminal singularities started in [9] and [10].

Let $(X, o)$ be a germ of a 3-dimensional terminal singularity defined over the field $\mathbb{C}$ of complex numbers. Consider a resolution $\pi: Y \to X$ and let $E \subset Y$ be a prime divisor such that $\pi(E) = o$ and discrepancy $a(E, X) \leq 1$. Note that if $\pi$ is a divisorial resolution, then $E$ does exist (see [3], [5]). On the other hand, the number of such divisors is finite (here we identify two divisors over $X$ if they give the same discrete valuations of the field $k(X)$).

What can be said about the birational type of the algebraic surface $E$? It is known that $E$ is birationally ruled ([8], Corollary 2.14). Moreover, if the singularity $(X, o)$ is of type $cA/m$, $m \geq 1$, then the surface $E$ is rational ([7], Proposition 2.4). When the singularity $(X, o)$ is of type $cD$, the surface $E$ is either rational or birationally isomorphic to $\mathbb{P}^1 \times C$, where $C$ is a (hyper)elliptic curve. If this non-rational divisor $E$ exists, it is unique ([9]). When $(X, o)$ is a general singularity of type $cE$, the non-rational divisor with

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low discrepancy is again unique and birational to the surface \( \mathbb{P}^1 \times C \), but the curve \( C \) can be non-hyperelliptic ([10]).

In this paper, we study the case when \((X, o)\) is a general non-Gorenstein (i.e., the canonical divisor \( K_X \) is not a Cartier divisor) 3-dimensional terminal singularity. By “general” we mean the following. Any non-Gorenstein terminal singularity is analitically isomorhic to one of singularities of Theorem 2.1 which we call standard. The singularity \((X, o)\) is general if its standard equation in \( \mathbb{C}^4/\mathbb{Z}_m \) is non-degenerate with respect to its Newton diagram.

**Theorem.** Let \( \pi : Y \to X \) be a resolution of 3-dimensional non-Gorenstein terminal singularity \((X, o)\). If \((X, o)\) is of type \(cAx/4, cD/3-3, cD/2-2\), or \(cE/2\), then additionally assume that the standard defining equation of \( X \) is non-degenerate with respect to its Newton diagram. Then there are not more than 2 non-rational divisors \( E_i, i = 1, 2 \), such that \( \pi(E_i) = o \) and discrepancy \( a(E_i, X) \leq 1 \).

In all the cases when the non-rational divisors exist, we describe them as exceptional divisors of certain blowups of the singularity \((X, o)\) and study their birational type.

In section 2 we recall the analylic classification of 3-dimensional non-Gorenstein terminal singularities and state some lemmas useful for working with discrepancies and resolutions. In section 3 we prove our Theorem by case-by-case analysis of all types of non-Gorenstein terminal singularities. We do not consider the case of \(cA/m\)-singularities because it was completely studied by Yu. G. Prokhorov in [7].

## 2 Preliminaries

Let the cyclic group \( \mathbb{Z}_m \) act on the space \( \mathbb{C}^n \) as follows: \( x_i \to \varepsilon^{a_i r} x_i, i = 1, \ldots, n \), where \( x_i \) are the coordinates in \( \mathbb{C}^n \), \( \varepsilon \) is a primitive \( m \)-th root of unity, \( a_i \in \mathbb{Z} \), and \( r \in \mathbb{Z}_m \) is a residue modulo \( m \). We shall denote the quotient space \( \mathbb{C}^n/\mathbb{Z}_m \) by \( \mathbb{C}^n/\mathbb{Z}_m(a_1, a_2, \ldots, a_n) \) or by \( \frac{1}{m}(a_1, a_2, \ldots, a_n) \).

The classification of 3-dimensional non-Gorenstein terminal singularities was obtained by Danilov, Mori, Kollár, and Shephard-Barron.

**Theorem 2.1.** (6) Let \( X \) be a germ of a 3-dimensional terminal singularity of index \( \geq 2 \). Then there is an embedding of \( X \) to \( \mathbb{C}^4/\mathbb{Z}_m \) such that one of the following holds:

\( \text{(cA/m)} \) \( X \simeq \{ xy + f(z, u) = 0 \} \subset \frac{1}{m}(\alpha, -\alpha, 1, 0) \) where \( \alpha \) is an integer prime to \( m \) and \( f(z, u) \in \mathbb{C}\{z, u\} \) is a \( \mathbb{Z}_m \)-invariant.
(cAx/4) \( X \simeq \{ x^2 + y^2 + f(z, u) = 0 \} \subset \frac{1}{2}(1, 3, 1, 2) \) where \( f(z, u) \in \mathbb{C}\{z, u\} \) is a \( \mathbb{Z}_4 \)-semi-invariant and \( u \notin f(z, u) \) (the coefficient of the monomial \( u \) in the power series \( f \) is zero).

(cAx/2) \( X \simeq \{ x^2 + y^2 + f(z, u) = 0 \} \subset \frac{1}{2}(0, 1, 1, 1) \) where \( f(z, u) \in (z, u)^4\mathbb{C}\{z, u\} \) is a \( \mathbb{Z}_2 \)-invariant.

(cD/3) \( X \simeq \{ \varphi(x, y, z, u) = 0 \} \subset \frac{1}{3}(1, 2, 2, 0) \) where \( \varphi \) has one of the following forms:

\[ (cD/3-1) \quad \varphi = u^2 + x^3 + yz(y + z), \]
\[ (cD/3-2) \quad \varphi = u^2 + x^3 + yz^2 + xy^4\lambda(y^3) + y^6\mu(y^3) \text{ where } \lambda(y^3), \mu(y^3) \in \mathbb{C}\{y^3\} \text{ and } 4\lambda^3 + 27\mu^2 \neq 0, \]
\[ (cD/3-3) \quad \varphi = u^2 + x^3 + y^3 + xz^3\alpha(z^3) + xz^4\beta(z^3) + yz^5\gamma(z^3) + z^6\delta(z^3) \text{ where } \alpha(z^3), \beta(z^3), \gamma(z^3), \delta(z^3) \in \mathbb{C}\{z^3\}. \]

(cD/2) \( X \simeq \{ \varphi(x, y, z, u) = 0 \} \subset \frac{1}{4}(1, 1, 0, 1) \) where \( \varphi \) has one of the following forms:

\[ (cD/2-1) \quad \varphi = u^2 + xyz + x^2a + y^2b + z^c \text{ where } a, b \geq 2, c \geq 3, \]
\[ (cD/2-2) \quad \varphi = u^2 + y^2z + \lambda yx^{2a+1} + g(x, z) \text{ where } \lambda \in \mathbb{C}, a \geq 1, \]
\[ g(x, z) \in (x^4, x^2z^2, z^3)\mathbb{C}\{x, z\}. \]

(cE/2) \( X \simeq \{ u^2 + x^3 + g(y, z)x + h(y, z) = 0 \} \subset \frac{1}{2}(0, 1, 1, 1) \) where \( g(y, z) \in (y, z)^4\mathbb{C}\{y, z\}, h(y, z) \in (y, z)^4\mathbb{C}\{y, z\} \setminus (y, z)^5\mathbb{C}\{y, z\}. \)

The index of \( X \) is equal to the order of the cyclic group \( \mathbb{Z}_m \).

**Theorem 2.2.** ([12]) Let \( X \) be one of the hyperquotient singularities

\[ \{ \varphi(x, y, z, u) = 0 \} \subset \mathbb{C}^4/\mathbb{Z}_m \]

listed in Theorem 2.1. Assume that \( \varphi(x, y, z, u) = 0 \) defines an isolated singularity at 0 and the action of \( \mathbb{Z}_m \) is free on \( X \) outside 0. Then \( X \) is terminal.

Let \( f = f(x_1, x_2, \ldots, x_n) \) be a convergent power series such that \( f(0) = 0 \) and (\( \{ f = 0 \}, 0 \) \( \subset \mathbb{C}^n, 0 \)) is an isolated singularity. We denote by \( \Gamma(f) \) the Newton diagram of the series \( f \). If \( f \) is non-degenerate with respect to its Newton diagram (in the sequel, we say simply that \( f \) is non-degenerate), then there is a Varchenko-Hovanskiǐ embedded toric resolution of the singularity (\( \{ f = 0 \}, 0 \) ) (see [13]). Moreover, if the group \( \mathbb{Z}_m \) acts on \( \mathbb{C}^n \) and \( f \) is its
semi-invariant, then we can repeat the construction from [11] and obtain an embedded toric resolution of the quotient singularity

\[(X, o) = (\{ f = 0 \}, 0)/\mathbb{Z}_m \subset \mathbb{C}^n / \mathbb{Z}_m.\]

Here all necessary toric varieties and morphisms are built with respect to the lattice \(N'\) dual to the lattice \(M'\) of monomials invariant under the action of \(\mathbb{Z}_m, M' \subset \mathbb{Z}^n\). This easy observation was pointed out to us by S. A. Kudryavtsev.

Recall that the embedded toric resolution \(\pi: Y \to X\) of the singularity \((X, o)\) is determined by a certain subdivision of the non-negative octant \(\mathbb{R}^n_{\geq 0}\). If \(\Sigma\) is the corresponding fan, then let \(\widetilde{C}^n = \mathbb{X}(\Sigma, N')\) be the toric variety built from \(\Sigma\) and let \(\widetilde{\pi}: \widetilde{C}^n \to \mathbb{C}^n / \mathbb{Z}_m\) be the natural birational morphism. Then \(\pi\) is the restriction of the morphism \(\widetilde{\pi}\) to the proper transform \(Y\) of the singularity \(X\).

Exceptional divisors of the morphism \(\widetilde{\pi}\) are in one-to-one correspondence with 1-dimensional cones of the fan \(\Sigma\). Take a 1-dimensional cone \(\tau\), its exceptional divisor \(E_\tau \subset \widetilde{C}^n\), and let \(E_\tau|_Y = \sum m_j E_j\). Further, let \(w = (w_1, \ldots, w_n)\) be the primitive vector of the lattice \(N''\) along the cone \(\tau\). The diagram \(\Gamma(f)\) lies in the space \((\mathbb{R}^n)^*\) dual to \(\mathbb{R}^n = \mathbb{R} \otimes \mathbb{Z}^n\); we denote the corresponding pairing by \(\langle \cdot, \cdot \rangle\). Now we want to calculate discrepancy \(a(E_j, X)\).

**Lemma 2.3.** \(a(E_j, X) = m_j(w_1 + \cdots + w_n - 1 - w(f))\), where \(w(f) = \min\{\langle w, v \rangle \mid v \in \Gamma(f)\}\).

**Proof.** Arguing as in [11], §10, we find an affine neighborhood \(U \simeq \mathbb{C}^n\) of the generic point \(E_\tau\) in \(\mathbb{C}^n\) with coordinates \(y_1, \ldots, y_n\) such that the equation \(y_1 = 0\) defines \(E_\tau \cap U\) and the morphism \(\widetilde{\pi}|_U: U \to \mathbb{C}^n / \mathbb{Z}_m\) is given by the formulae:

\[
x_1 = y_1^{a_1^1} y_2^{a_1^2} \cdots y_n^{a_1^n},
\]

\[
\ldots.
\]

\[
x_n = y_1^{a_n^1} y_2^{a_n^2} \cdots y_n^{a_n^n}
\]

for some \(a^i = (a_1^i, \ldots, a_n^i) \in N' \cap \mathbb{R}^n_{\geq 0}\). To prove the lemma, it remains only to lift the differential form \(dx_1 \wedge \cdots \wedge dx_n\) to \(U\) and to apply the adjunction formula. \(\square\)

**Corollary 2.4.** If \(a(E_j, X) \leq 1\), then \(w_1 + \cdots + w_n - 1 - w(f) \leq 1\).

Note that if the vectors \(w, e_1, e_2, \ldots, e_n\), where \(e_i = (0, \ldots, 1, \ldots, 0)\), generate the lattice \(N'\), then the exceptional divisors \(E_j\) are birationally
isomorphic to the divisors $E_{w,j}$ respectively, \( \sum E_{w,j} = E_w|_{X_w} \), where $X_w$ is the proper transform of $X$ under the weighted blowup

\[ \nu_w : \mathbb{C}_w^n \to \mathbb{C}^n/\mathbb{Z}_m. \]

This follows from the fact that for any two subdivisions of the non-negative octant $\mathbb{R}_{\geq 0}^n$ there is a common sub subdivision. The exceptional divisor $E_w$ of $\nu_w$ is isomorphic to the weighted projective space $\mathbb{P}(w_1, \ldots, w_n)$. The divisor $\sum E_{w,j}$ is defined in $\mathbb{P}(w_1, \ldots, w_n)$ by the equation

\[ f_{\rho(w)}(x_1, \ldots, x_n) = 0, \]

where $f_{\rho}$ corresponds to the face

\[ \rho(w) = \{ v \in \Gamma(f) \mid \langle w, v \rangle = w(f) \}, \]

\[ f_{\rho(w)} = \sum_{(m_1, \ldots, m_n) \in \rho(w)} a_{m_1 \ldots m_n} x_1^{m_1} \ldots x_n^{m_n} \]

if $f = \sum_{(m_1, \ldots, m_n) \in \mathbb{Z}_{\geq 0}} a_{m_1 \ldots m_n} x_1^{m_1} \ldots x_n^{m_n}$.

Now the letters $x_1, \ldots, x_n$ denote the quasihomogeneous coordinates in the space $\mathbb{P}(w_1, \ldots, w_n)$. We often use this abuse of notation in the sequel; the weight meaning of the letters is clear from the context.

Now suppose that the vectors $w, e_1, \ldots, e_n$ generate some sublattice $N'' \subset N'$. Consider the subdivision of the octant $\mathbb{R}_{\geq 0}^n$ by the vector $w$, i.e., the fan $\Sigma_w$ consisting of the cones $\sigma_i = \langle e_1, \ldots, w, \ldots, e_n \rangle$ and all their faces. So obtained morphism

\[ \mu_w : \widetilde{\mathbb{C}}_w^n \to \mathbb{C}^n/\mathbb{Z}_m \]

is not a weighted blowup. We shall call it a pseudo blowup with the weight $w$. It is easily proved that its exceptional divisor $\widetilde{E}_w \simeq \mathbb{P}(w_1, \ldots, w_n)/G$, where $G = N'/N''$ is a cyclic group, and the equation of $\sum \widetilde{E}_{w,j}$ is the same as above.

Let $(X, o)$ be one of the terminal singularities listed in Theorem 2.1 let $\nu_w$ be its weighted blowup or pseudo blowup, and let $E_w$ be the exceptional divisor of the morphism $\nu_w : X_w \to X$. Denote by $E'$ the surface in $\mathbb{P}(w_1, \ldots, w_n)$ covering $E_w$ (if $\nu_w$ is a weighted blowup, then $E' = E_w$).

**Lemma 2.5.** Suppose that the surface $E'$ is irreducible and has only rational singularities. Then the surface $E_w$ is rational.
Proof. We can consider $E'$ as a divisor over some terminal $cDV$-point. Take a resolution $\pi: \tilde{E}' \to E'$ of singularities of the surface $E'$. According to [8], Corollary 2.14, $E'$ is birationally ruled. Thus $P_2(\tilde{E}') = h^0(2K_{\tilde{E}'}) = 0$. On the other hand, $E'$ is a hypersurface in the space $\mathbb{P}(w_1, \ldots, w_n)$, hence $h^1(\mathcal{O}_{E'}) = 0$. Since $E'$ has only rational singularities, we have $h^1(\mathcal{O}_{\tilde{E}'}) = h^1(\mathcal{O}_{E'}) = 0$. Therefore $\tilde{E}'$ is rational by Castelnuovo’s criterion and thus $E_w$ is rational too.

If the blowup $\nu$ has a non-rational exceptional divisor, we shall sometimes say that the blowup $\nu$ is non-rational.

3 Proof of Theorem

3.1 Terminal singularities of type $cAx/4$

Consider the singularity $(X, o)$ of type $cAx/4$, i. e.,

$$X \simeq \{ \varphi = x^2 + y^2 + f(z, u) = 0 \} \subset \mathbb{C}^4(1, 3, 1, 2),$$

where $f(z, u) \in \mathbb{C}\{z, u\}$ is a $\mathbb{Z}_4$-semi-invariant and $u \notin f(z, u)$. We assume that the defining series $\varphi$ is non-degenerate. Then the singularity $(X, o)$ has an embedded toric resolution $\pi: Y \to X$. Divisors with center at $o$ and discrepancy $a \leq 1$ belong to any divisorial resolution of $X$, so if there is a non-rational divisor $E$ over $(X, o)$, center($E$) = $o$, $a(E, X) \leq 1$, then $E$ belongs to the resolution $\pi$. We saw in section 2 that $E$ is birationally isomorphic to the exceptional divisor (or to its irreducible component) of some weighted blowup or pseudo blowup $\nu_w$. Thus we can suppose that $E$ is given in $\mathbb{P}(w_1, w_2, w_3, w_4)$ (or in $\mathbb{P}(w_1, w_2, w_3, w_4)/G$) by the part $\varphi_w$ of the series $\varphi$. It is clear that if $E$ is non-rational, then the polynomial $\varphi_w$ contains at least one of the monomials $x^2$ or $y^2$. But if it contains both of them, i. e., $\varphi_w = x^2 + y^2 + f_w(z, u)$, then in the affine chart $u \neq 0$ the surface $E_w$ is defined as

$$\{ x^2 + y^2 + f_w(z, 1) = 0 \} \subset \mathbb{C}^4/G_1,$$

and in the chart $z \neq 0$ as

$$\{ x^2 + y^2 + f_w(1, u) = 0 \} \subset \mathbb{C}^4/G_2,$$

where $G_1, G_2$ are finite cyclic subgroups of $\text{GL}_\mathbb{C}(3)$. Now it is obvious that $E_w$ has only rational singularities, thus by Lemma 2.5 $E_w$ is rational. So, we can assume that $\varphi_w = x^2 + f_w(z, u)$ or $\varphi_w = y^2 + f_w(z, u)$. 


Since \( \varphi \) is a \( \mathbb{Z}_4(1,3,1,2) \)-semi-invariant and the singularity \((X,o)\) is isolated, the series \( f \) contains the monomial \( u^{2n+1} \). If \( n \) is the minimal number with the property \( u^{2n+1} \in f \), we say that the singularity \((X,o)\) is of type \( cA_{2n+1}x/4 \). Now let us find all blowups and pseudo blowups \( \nu \) of the singularity \((X,o)\) of type \( cA_{2n+1}x/4 \) such that the exceptional divisor of \( \nu \) can be non-rational and has discrepancy \( \alpha \leq 1 \).

We have to find all primitive vectors \( w \in \mathbb{Z}^4 + \frac{1}{4}(1,3,1,2)\mathbb{Z} \) such that either (i) \( w_1 + w_2 + w_3 + w_4 - 1 - 2w_1 \leq 1, \ w_2 > w_1, \ (2n + 1)w_4 \geq 2w_1 \), or (ii) \( w_1 + w_2 + w_3 + w_4 - 1 - 2w_2 \leq 1, \ w_1 > w_2, \ (2n + 1)w_4 \geq 2w_2 \).

**Proposition 3.1.1.** Let the primitive vector \( w \in \mathbb{Z}^4 + \frac{1}{4}(1,3,1,2)\mathbb{Z} \) satisfy conditions (i) or (ii). Then \( w \) is one of the following:
1) \( \frac{1}{4}(4k + 1, 4k + 3, 1, 2), \ \kappa \leq n/2, \ \kappa \in \mathbb{Z}_{\geq 0} \);
2) \( \frac{1}{4}(4k + 3, 4k + 5, 3, 2), \ \kappa \leq (n - 1)/2, \ \kappa \in \mathbb{Z}_{\geq 0} \);
3) \( \frac{1}{4}(4k + 5, 4k + 3, 1, 2), \ \kappa \leq (n - 1)/2, \ \kappa \in \mathbb{Z}_{\geq 0} \);
4) \( \frac{1}{4}(4k + 3, 4k + 1, 3, 2), \ \kappa \leq n/2, \ \kappa \in \mathbb{Z}_{\geq 0} \).

**Proof.** It is an easy arithmetic calculation. For example, assume (i). Since \( w_2 > w_1 \), the inequality for discrepancy has the form \( w_3 + w_4 < 2 \). Taking into account that \( w_3 \in \frac{1}{4}\mathbb{Z}, \ w_4 \in \frac{1}{2}\mathbb{Z} \), and \( 2w_3 \equiv w_4 \mod \mathbb{Z} \), we get the following possibilities:

\[
\begin{align*}
&w_4 = 1/2, \ w_3 = 1/4, \ 3/4, \ 5/4; \\
&w_4 = 1, \ w_3 = 1/2; \\
&w_4 = 3/2, \ w_3 = 1/4.
\end{align*}
\]

Assume \( (w_3, w_4) = (1/4, 1/2) \). Then we have \( w_2 - w_1 + 3/4 - 1 \leq 1 \), i.e., \( w_2 - w_1 \leq 5/4 \). On the other hand, \( w_1 \leq (2n + 1)w_4 = n/2 + 1/4 \). If we combine these inequalities with \( w_1 \equiv w_3 \mod \mathbb{Z} \), we obtain \( w_1 = \frac{1}{4}(4k + 1), \ w_2 = \frac{1}{4}(4k + 3), \ \kappa \leq n/2, \ i.e., \ case \ 1 \).

Let us also consider the possibility \( (w_3, w_4) = (5/4, 1/2) \). It follows that \( w_2 - w_1 \leq 1/4 \). But this is impossible because the difference \( w_2 - w_1 \) is always multiple of \( 1/2 \).

Other cases can be done in a similar way. In the sequel, we omit such calculations. \( \square \)

Note that vectors 1)–4) give weighted blowups but not pseudo blowups.

The exceptional divisor \( E_1 \) of the blowup \( \nu_1 = \frac{1}{4}(4k + 1, 4k + 3, 1, 2) \) (weighted blowup with the weight \( \frac{1}{4}(4k + 1, 4k + 3, 1, 2) \)) is defined by the equation

\[
\{x^2 + f_{2k+\frac{1}{2}} (z,u) = 0\} \subset \mathbb{P}(4k + 1, 4k + 3, 1, 2).
\]

If \( E_1 \) is non-rational, it is irreducible and reduced. Then discrepancy

\[
a(E_1, X) = (1/4)(4k + 1 + 4k + 3 + 1 + 2) - 1 - 2k - 1/2 = 1/4.
\]
It is obvious that $E_1$ is a cone over the hyperelliptic curve $C = \{x^2 + f_{2k+\frac{1}{2}}(z, u) = 0\} \subset \mathbb{P}(4k + 1, 1, 2)$. Genus of a curve in a weighted projective plane can be found by methods from [1]. Genus of the curve $C$ is $g(C) \leq 2k$.

The exceptional divisor $E_2$ of the blowup $\nu_2 = \frac{1}{4}(4k + 3, 4k + 5, 3, 2)$ is defined by the equation
\[
\{x^2 + f_{2k+4}(z, u) = 0\} \subset \mathbb{P}(4k + 3, 4k + 5, 3, 2).
\]
If it is irreducible and reduced, its discrepancy $a(E_2) = 3/4$. The divisor $E_2$ is a cone over a hyperelliptic curve of genus
\[
g \leq \begin{cases} 2m - 1, & k = 3m, \\ 2m + 1, & k = 3m + 1, \\ 2m + 2, & k = 3m + 2. \end{cases}
\]

The exceptional divisor $E_3$ of the blowup $\nu_3 = \frac{1}{4}(4k + 5, 4k + 3, 1, 2)$ is defined by the equation
\[
\{y^2 + f_{2k+4}(z, u) = 0\} \subset \mathbb{P}(4k + 5, 4k + 3, 1, 2).
\]
If $E_3$ is irreducible and reduced, discrepancy $a(E_3) = 1/4$. The surface $E_3$ is a cone over a hyperelliptic curve of genus $g \leq 2k + 1$.

The exceptional divisor $E_4$ of the blowup $\nu_4 = \frac{1}{4}(4k + 3, 4k + 1, 3, 2)$ is defined by the equation
\[
\{y^2 + f_{2k+4}(z, u) = 0\} \subset \mathbb{P}(4k + 3, 4k + 1, 3, 2).
\]
If $E_4$ is irreducible and reduced, discrepancy $a(E_4) = 3/4$. Divisor $E_4$ is a cone over a hyperelliptic curve of genus
\[
g \leq \begin{cases} 2m, & k = 3m, \\ 2m + 1, & k = 3m + 1 \text{ or } k = 3m + 2. \end{cases}
\]

It is clear that the blowups $\nu_1$ and $\nu_3$, $\nu_2$ and $\nu_4$ can not simultaneously be non-rational. Indeed, assume for example that $\nu_1$ is non-rational. It follows that for the weights $w(z) = 1$, $w(u) = 2$ the function $f$ has the weight $w(f) = 8k_1 + 2$. But if $\nu_3$ is also non-rational, then $w(f) = 8k_2 + 6$, contradiction. Other pairs of blowups can be non-rational.

**Example 3.1.2.** Consider the singularity
\[
\{x^2 + y^2 + z^{18} + z^6 u^6 + u^{15} = 0\} \subset \frac{1}{4}(1, 3, 1, 2)
\]
of type $cA_{14}x/4$. Make the blowups $\nu_1 = \frac{1}{4}(9,11,1,2)$ and $\nu_2 = \frac{1}{4}(15,17,3,2)$. The exceptional divisor

$$E_1: \{x^2 + z^{18} + z^6u^6 = 0\} \subset \mathbb{P}(9,11,1,2)$$

of the first one is a cone over a singular curve of genus $g = 2$. The exceptional divisor

$$E_2: \{x^2 + z^6u^6 + u^{15} = 0\} \subset \mathbb{P}(15,17,3,2)$$

of the second blowup is a cone over a singular curve of genus $g = 1$.

We see that there is not more than 2 non-rational divisors with discrepancy $a \leq 1$ over a non-degenerate singularity of type $cAx/4$.

### 3.2 Terminal singularities of type $cAx/2$

Consider the singularity $(X,o)$ of type $cAx/2$, i.e.,

$$X \simeq \{x^2 + y^2 + f(z,u) = 0\} \subset \frac{1}{2}(0,1,1,1), \quad (3.2.1)$$

where $f(z,u) \in (z,u)^4\mathbb{C}\{z,u\}$ is a $\mathbb{Z}_2$-semi-invariant. Here our proof does not depend on the fact whether given singularity is non-degenerate. Following [2], §8, assume that if the weights of variables are $w(z) = w(u) = 1/2$, then the weight $w(f)$ of the series $f$ equals $k$. If $k$ is even, we make the weighted blowup $\nu_0 = \frac{1}{2}(k,k+1,1,1)$. If $k$ is odd, we make the blowup $\nu_1 = \frac{1}{2}(k+1,k,1,1)$. We shall only consider $\nu_0$, the other case can be done in a similar way.

We have $\nu_0: \widehat{\mathbb{C}}^4 \to \mathbb{C}^4/\mathbb{Z}_2(0,1,1,1)$ and the variety $\widehat{\mathbb{C}}^4$ is covered by 4 affine charts. In the first one $U_1 \simeq \frac{1}{k}(1,-1,-1,-1)$, the proper transform $\widetilde{X}$ of the singularity $X$ is given by the equation

$$1 + xy^2 + f_k(z,u) + x(\ldots) = 0.$$ 

It is clear that in $U_1$ the variety $\widetilde{X}$ is non-singular.

In the second chart $U_2 \simeq \frac{1}{k+1}(1,1,-1,-1),$ 

$$\widetilde{X} \cap U_2: x^2 + y + f_k(z,u) + y(\ldots) = 0.$$ 

Here $\widetilde{X}$ is non-Gorenstein at the origin only, where it has a cyclic terminal quotient singularity of type $\frac{1}{k+1}(1,-1,-1)$. The third and the fourth charts are isomorphic to $\mathbb{C}^4$. In the third one the variety $\widetilde{X} \cap U_3$ is defined by the equation

$$x^2 + y^2z + f_k(1,u) + z(\ldots) = 0.$$
Since \((X, o)\) is an isolated singularity, singularities of \(\tilde{X} \cap U_3\) lie only on the exceptional divisor \(\{ z = 0 \}\). It is obvious that all of them are isolated \(cDV\)-points. Similarly, in the fourth chart the variety \(\tilde{X}\) has only isolated \(cDV\)-points.

Let \(E\) be the exceptional divisor of the blowup \(\nu_0\) of the singularity \((X, o)\). We have

\[
E \simeq \{ x^2 + f_k(z, u) = 0 \} \subset \mathbb{P}(k, k + 1, 1, 1).
\]

If it is non-rational, it is irreducible and reduced, discrepancy \(a(E, X) = (1/2)(k + k + 1 + 1 + 1) - 1 - k = 1/2\), and the surface \(E\) is a cone over a hyperelliptic curve of genus \(g \leq k - 1\).

Take an arbitrary resolution \(\pi: Y \to \tilde{X} \to X\). All non-rational divisors with discrepancy \(a \leq 1\) appear in \(\pi\). But \(cDV\)-singularities of the variety \(\tilde{X}\) produce only divisors with discrepancies \(a(E_i, \tilde{X}) \geq 1\), so that \(a(E_i, X) > 1\). Any resolution of the cyclic quotient singularity from the second chart of \(\tilde{X}\) contains with discrepancies \(\leq 1\) only rational divisors. Thus \(E\) is the unique non-rational divisor with \(a \leq 1\) over the singularity \((X, o)\). We have proved the following

**Proposition 3.2.1.** Any resolution of the singularity \((X, o)\) of type \(cAx/2\) contains not more than 1 non-rational divisor \(E\) with discrepancy \(a(E, X) \leq 1\) and center \(X(E) = o\). Let \((X, o)\) be defined by the equation \((3.2.1)\). Then the non-rational divisor \(E\) can be realized as the exceptional divisor of the weighted blowup \(\nu_0 = \frac{1}{2}(k, k + 1, 1, 1)\) (if \(k\) is even), or as the exceptional divisor of the weighted blowup \(\nu_1 = \frac{1}{2}(k + 1, k, 1, 1)\) (if \(k\) is odd). In both cases \(E\) is a cone over a hyperelliptic curve of genus \(g \leq k - 1\).

**Example 3.2.2.** Consider the singularity

\[
\{ x^2 + y^2 + z^6 + u^6 = 0 \} \subset \frac{1}{2}(0, 1, 1, 1)
\]

and its weighted blowup \(\frac{1}{2}(4, 3, 1, 1)\). Its exceptional divisor

\[
E \simeq \{ y^2 + z^6 + u^6 = 0 \} \subset \mathbb{P}(4, 3, 1, 1)
\]

is a cone over a curve of genus 2.

### 3.3 Terminal singularities of type \(cD/3\)

#### 3.3.1 \(cD/3 - 1\)

Consider the singularity \((X, o)\) of type \(cD/3 - 1\), i. e.,

\[
X \simeq \{ u^2 + x^3 + yz(y + z) = 0 \} \subset \frac{1}{3}(1, 2, 2, 0).
\]
This singularity can be resolved by an explicit calculation. There are no non-rational divisors with discrepancy $a \leq 1$ over $(X, o)$.

### 3.3.2 $cD/3 - 2$

Consider the singularity $(X, o)$ of type $cD/3 - 2$, i.e.,

$$X \simeq \{ u^2 + x^3 + yz^2 + xy^4\lambda(y^3) + y^6\mu(y^3) = 0 \} \subset \mathbb{P}(1, 2, 2, 0),$$

where $\lambda(y^3), \mu(y^3) \in \mathbb{C}\{y^3\}$ and $4\lambda^3 + 27\mu^2 \neq 0$. Note that the last condition guarantees that the singularity $(X, o)$ is non-degenerate. However, we shall not use this fact. We shall proceed as in case $cAx/2$ in section 3.2.

Consider the weighted blowup $\nu = \frac{1}{3}(2, 1, 4, 3)$ (see [2], §9) of the given singularity. It can be easily verified that in the first, in the second, and in the fourth charts the blown up variety $\tilde{X}$ is non-singular. In the third chart $U_3 \simeq \frac{1}{4}(2, 3, 3, 1)$

$$\tilde{X}_3 = \tilde{X} \cap U_3 \simeq \{ u^2 + x^3 + yz + \lambda_0xy^4 + \mu_0y^6 + z(\ldots) = 0 \}.$$  

At the origin the variety $\tilde{X}_3$ has a singularity analytically isomorphic to

$$\{ u^2 + y^2 + z^2 + x^3 = 0 \} \subset \frac{1}{4}(2, 3, 3, 1).$$

It is obvious that it has type $cAx/4$ and is non-degenerate. We described all blowups of non-degenerate $cAx/4$-singularities which can have non-rational exceptional divisors with small discrepancies in section 3.1. But in this case all of them are rational. It follows that only the blowup $\nu$ of $(X, o)$ can have a non-rational exceptional divisor. It has the form

$$E = \{ u^2 + x^3 + \lambda_0xy^4 + \mu_0y^6 = 0 \} \subset \mathbb{P}(2, 1, 4, 3).$$

This is a cone over a curve of genus $g \leq 1$. Discrepancy $a(E, X)$ is equal to $(1/3)(2 + 1 + 4 + 3) - 1 - 2 = 1/3$. We have proved

**Proposition 3.3.1.** There is not more than $1$ non-rational divisor $E$ with discrepancy $a \leq 1$ over the singularity $(X, o)$ or type $cD/3 - 2$. If $X$ is defined by equation (3.3.1), then the non-rational divisor $E$ is birational to the exceptional divisor of the blowup $\frac{1}{3}(2, 1, 4, 3)$. It is a cone over a curve of genus $1$. 

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3.3.3 \( cD/3 - 3 \)

Consider the singularity \((X, o)\) of type \(cD/3 - 3\), i.e.,

\[
X \simeq \{ \varphi = u^2 + x^3 + y^3 + xyz^3 \alpha(z^3) + xz^4 \beta(z^3) + yz^5 \gamma(z^3) + z^6 \delta(z^3) \},
\]

where \(\alpha(z^3), \beta(z^3), \gamma(z^3), \delta(z^3) \in \mathbb{C}\{z^3\}\). Here we additionally assume that the defining series \(\varphi\) is non-degenerate. If \(E\) is a non-rational divisor with \(a(E, X) \leq 1\) and \(\text{cent}_X(E) = o\), then, as in section 3.1, we can consider \(E\) as an exceptional divisor of some weighted blowup or pseudo blowup. Let \(w\) be weight of this blowup. The Newton diagram \(\Gamma(f)\) is spanned by the monomials \(u^2, x^3, y^3, xyz^{3b_1}, xz^{4+3b_2}, yz^{5+3b_3}, z^{6+3b_4}\), where \(b_i \in \mathbb{Z}_{\geq 0}\). Thus if \(E\) is non-rational, its equation \(\varphi_w\) contains the monomials \(u^2\) and \(x^3\), \(u^2\) and \(y^3\), or \(x^2\) and \(y^3\). Using the condition \(a(E) \leq 1\), we come to the following:

- Either (i) \(w_1 + w_2 + w_3 + w_4 - 1 - 2w_4 \leq 1, 2w_4 = 3w_1, 3w_2 \geq 2w_4\);
- or (ii) \(w_1 + w_2 + w_3 + w_4 - 1 - 2w_4 \leq 1, 2w_4 < 3w_1, 3w_2 = 2w_4\);
- or (iii) \(w_1 + w_2 + w_3 + w_4 - 1 - 2w_1 \leq 1, w_1 = w_2, 2w_4 > w_1\).

**Proposition 3.3.2.** Let the primitive vector \(w \in \mathbb{Z}^4 + \frac{1}{3}(1, 2, 2, 0)\mathbb{Z}\) satisfy one of conditions (i), (ii), or (iii). Then \(w\) is one of the following:

1) \(\frac{1}{3}(5, 4, 1, 6)\);
2) \(\frac{1}{3}(2, 4, 1, 3)\);
3) \(\frac{1}{3}(4, 5, 2, 6)\);
4) \((2, 2, 1, 3)\).

**Proof.** It is an easy arithmetic calculation. \(\square\)

Note that weight 4) corresponds to a pseudo blowup, other weights correspond to weighted blowups.

The exceptional divisor \(E_1\) of the blowup \(\nu_1 = \frac{1}{3}(5, 4, 1, 6)\) is defined in \(\mathbb{P}(5, 4, 1, 6)\) by the equation

\[
u_2^2 + y^3 + \gamma_1 y^8 + \delta_2 z^{12} = 0.
\]

(Recall that we assume that \(E_1\) is non-rational. It follows that \(\alpha_0 = \beta_0 = \beta_1 = \gamma_0 = \delta_0 = \delta_1 = 0\). Discrepancy \(a(E) = 1/3\). The divisor \(E\) is a cone over a curve of genus 1.

The exceptional divisor \(E_2\) of the blowup \(\nu_2 = \frac{1}{3}(2, 4, 1, 3)\) is given in \(\mathbb{P}(2, 4, 1, 3)\) by the equation

\[
u_2^2 + x^3 + \beta_0 x z^4 + \delta_0 z^6 = 0.
\]

Discrepancy \(a(E_2) = 1/3\) and \(E_2\) is again a cone over a curve of genus 1.

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The exceptional divisor $E_3$ of the blowup $\nu_3 = \frac{1}{3}(4, 5, 2, 6)$ is defined in $\mathbb{P}(4, 5, 2, 6)$ by the equation
\[ u^2 + x^3 + \beta_0 x z^4 + \delta_0 z^6 = 0. \]
It follows that $E_3 \simeq \{u^2 + x^3 + \beta_0 x z^4 + \delta_0 z^6 = 0\} \subset \mathbb{P}(2, 5, 1, 3)$. It is again a cone over a curve of genus 1. Discrepancy $a(E_3) = 2/3$.

The exceptional divisor $E_4$ of the blowup $\nu_4 = (2, 2, 1, 3)$ is defined as
\[ E_4 \simeq \{u^2 + x^3 + y^3 + \beta_0 x z^4 + \delta_0 z^6 = 0\} \subset \mathbb{P}(2, 2, 1, 3)/G, \]
where $G$ is a cyclic group. But the surface $\{u^2 + x^3 + y^3 + \beta_0 x z^4 + \delta_0 z^6 = 0\} \subset \mathbb{P}(2, 2, 1, 3)$ has only rational singularities. According to Lemma 2.5, the surface $E_4$ is rational.

It is clear that the blowups $\nu_2$ and $\nu_3$, $\nu_1$ and $\nu_3$ can not simultaneously be non-rational. But if one of the blowups $\nu_2$, $\nu_3$ is non-rational, then the other is too.

Example 3.3.3. Consider the singularity $\{u^2 + x^3 + y^3 + z^6 = 0\} \subset \mathbb{P}(1, 2, 2, 0)$ of type $cD/3 - 3$ and its blowups $\nu_2$ and $\nu_3$. Their exceptional divisors
\[ E_2 = \{u^2 + x^3 + z^6 = 0\} \subset \mathbb{P}(2, 4, 1, 3) \]
and $E_3 = \{u^2 + x^3 + z^6 = 0\} \subset \mathbb{P}(2, 5, 1, 3)$ are cones over elliptic curves. It is interesting that they are given by the same equations. But the blowups $\nu_2$ and $\nu_3$ are not isomorphic since their discrepancies are different: $a(E_2) = 1/3$, $a(E_3) = 2/3$.

Thus there are not more than 2 non-rational divisors with discrepancy $a \leq 1$ over a singularity of type $cD/3 - 3$.

3.4 Terminal singularities of type $cD/2$

3.4.1 $cD/2 - 1$

Consider the singularity $(X, o)$ of type $cD/2 - 1$, i. e.,
\[ X \simeq \{\varphi = u^2 + x y z + x^{2a} + y^{2b} + z^c = 0\} \subset \frac{1}{2}(1, 1, 0, 1), \]
where $a, b \geq 2$, $c \geq 3$. This singularity is non-degenerate. Thus all divisors with discrepancy $a \leq 1$ correspond to faces of the Newton diagram $\Gamma(\varphi)$. But it is easy to show that all faces produce rational divisors, hence there are no non-rational divisors with discrepancy $a \leq 1$ over a singularity of type $cD/2 - 1$. 

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3.4.2 $cD/2 - 2$

Consider the singularity $(X, o)$ of type $cD/2 - 2$, i.e.,

$$X \simeq \{ \varphi = u^2 + g^2z + \lambda yx^{2a+1} + g(x, z) = 0 \} \subset \frac{1}{2}(1, 1, 0, 1).$$

where $\lambda \in \mathbb{C}$, $a \geq 1$, $g(x, z) \in (x^4, x^2z^2, z^3)\mathbb{C}\{x, z\}$. Here we assume that the series $\varphi$ is non-degenerate. Since the singularity $(X, o)$ is isolated and the function $g$ is $\mathbb{Z}_2$-invariant, we see that $g$ contains a monomial of the form $z^{n-1}$. If $n$ is the minimal integer with this property, then we say that $(X, o)$ is of type $cD_n/2 - 2$.

Divisors with discrepancy $a \leq 1$ over $(X, o)$ correspond to faces of the Newton diagram $\Gamma(\varphi)$. In the same way as in sections 3.4.1 and 3.3.3, we come to the following problem: find all primitive vectors $w \in \mathbb{Z}^4 + \frac{1}{2}(1, 1, 0, 1)\mathbb{Z}$ such that either (i) $w_1 + w_2 + w_3 + w_4 - 1 - 2w_4 \leq 1$, $2w_2 + w_3 \geq 2w_4$, $(n-1)w_3 \geq 2w_4$; or (ii) $w_1 + w_2 + w_3 + w_4 - 1 - 2w_2 - w_3 \leq 1$, $2w_4 > 2w_2 + w_3$, $(n-1)w_3 \geq 2w_4$.

The answer is given by the following

**Proposition 3.4.1.** Let the primitive vector $w \in \mathbb{Z}^4 + \frac{1}{2}(1, 1, 0, 1)\mathbb{Z}$ satisfy conditions (i) or (ii). Then $w$ is one of the following:

1) $\frac{1}{2}(1, m, 2, m)$, $m = 2k - 1$, $m \leq n - 1$;
2) $\frac{1}{2}(1, m - 2, 4, m)$, $m = 2k - 1$, $m \leq 2(n - 1)$;
3) $\frac{1}{2}(1, m - 1, 2, m + 1)$, $m = 2k$, $m \leq n - 1$;
4) $(1, k, 2, k)$, $k \leq (n - 1)/2$;
5) $(1, k - 1, 2, k)$, $k \leq n - 1$;
6) $(1, k - 1, 1, k)$, $k \leq n/2$.

**Proof.** It is an easy arithmetic calculation. \(\square\)

Blowups $\nu_1 = \frac{1}{2}(1, m, 2, m)$, $\nu_2 = \frac{1}{2}(1, m - 2, 4, m)$, $\nu_3 = \frac{1}{2}(1, m - 1, 2, m + 1)$ are weighted, and $\nu_4 = (1, k, 2, k)$, $\nu_5 = (1, k - 1, 2, k)$, and $\nu_6 = (1, k - 1, 1, k)$ are pseudo blowups. Actually, only the blowups $\nu_1$, $\nu_3$ (with discrepancy $a = 1/2$), $\nu_4$, and $\nu_6$ (with discrepancy $a = 1$) can be non-rational.

**Example 3.4.2.** Consider the singularity

$$\{ u^2 + y^2z + z^{2k} + x^{2k} = 0 \} \subset \frac{1}{2}(1, 1, 0, 1)$$

of type $cD_{2k+1}/2$ and its pseudo blowup $\nu_4 = (1, k, 1, k)$. Assume that the number $k$ is even. Then the affine chart $U_1 = X(\sigma_1, N')$ of the blown up variety $\tilde{\mathbb{C}}^4/(1, k, 1, k)$ (for the notation see section 2) is isomorphic to

$$\mathbb{C}^4/\mathbb{Z}_2(1, 1 - k, -1, 1 - k) = \frac{1}{2}(1, 1, 1, 1).$$

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\[ \bar{X} \cap U_1 = \{ y_4^2 + y_1 y_2 y_3 + y_3^{2k} + 1 = 0 \} \subset \frac{1}{2} (1, 1, 1, 1), \]
the exceptional divisor \((y_1 = 0)\)
\[ E \simeq \{ y_4^2 + y_3^{2k} + 1 = 0 \} \subset \frac{1}{2} (1, 1, 1) \]
is a cone over the curve \(\{ y_4^2 + y_3^{2k} + 1 = 0 \} \subset \frac{1}{2} (1, 1, 1)\). It is a hyperelliptic curve of genus \(k/2\).

For all the blowups \(\nu_1, \nu_3, \nu_4, \nu_6\), the exceptional divisor is a cone over a hyperelliptic curve of genus \(g \leq k - 1\) for \(\nu_1\); \(g \leq k\) for \(\nu_3\); \(g \leq k/2\) for even \(k\) and \(g \leq (k-1)/2\) for odd \(k\) for \(\nu_4\); \(g \leq (k-1)/2\) for odd \(k\) and \(g \leq (k-2)/2\) for even \(k\) for \(\nu_6\). In the latter case the exceptional divisor \(E_6\) splits onto 2 components, one of them is rational.

The pairs of blowups \(\nu_1\) and \(\nu_3\), \(\nu_4\) and \(\nu_6\) can not simultaneously be non-rational; the others can.

**Example 3.4.3.** Consider the singularity
\[ \{ u^2 + y^2 z + z^{12} + z^6 x^6 + x^{18} = 0 \} \subset \frac{1}{2} (1, 1, 0, 1) \]
of type \(cD_{13}/2 - 2\), its weighted blowup \(\nu_1 = \frac{1}{2} (1, 9, 2, 9)\), and its pseudo blowup \(\nu_4 = (1, 6, 1, 6)\).

The exceptional divisors are \(E_1\) and \(E_4\).
\[ E_1 = \{ u^2 + z^6 x^6 + x^{18} = 0 \} \subset \mathbb{P}(1, 9, 2, 9) \]
is a cone over a singular curve of genus 2;
\[ E_4 = \{ u^2 + z^{12} + z^6 x^6 = 0 \} \subset \mathbb{P}(1, 6, 1, 6)/\mathbb{Z}_2 \]
is a cone over a singular curve of genus 1.

So, there are not more than 2 non-rational divisors with discrepancy \(a \leq 1\) over a non-degenerate singularity of type \(cD/2 - 2\).

### 3.5 Terminal singularities of type \(cE/2\)
Consider the singularity \((X, o)\) of type \(cE/2\), i. e.,
\[ X \simeq \{ \varphi = u^2 + x^3 + g(y, z)x + h(y, z) = 0 \} \subset \frac{1}{2} (0, 1, 1, 1), \]
where \(g(y, z) \in (y, z)^4 \mathbb{C}\{y, z\}, h(y, z) \in (y, z)^4 \mathbb{C}\{y, z\} \setminus (y, z)^5 \mathbb{C}\{y, z\}\). We assume that the series \(\varphi\) is non-degenerate. In addition, by permutation of
coordinates $y$ and $z$ if necessary, we can suppose that $y^4$, $y^3z$, or $y^2z^2 \in h(y,z)$. The argument here is similar to that of sections 3.1, 3.3.3, and 3.4.2, so we just formulate the final results.

For $cE/2$-singularities, the non-rational divisors again can be represented as exceptional divisors of certain weighted blowups and pseudo blowups. All these divisors are cones. In the following proposition we list all possible non-rational blowups, discrepancies of their exceptional divisors, and genuses of the corresponding curves.

**Proposition 3.5.1.** (cf. [2], §10) Let $E$ be a non-rational divisor over the singularity $(X, o)$ such that $\text{center}_X(E) = o$ and $a(E, X) \leq 1$. Then $E$ is birational to the exceptional divisor of one of the following blowups.

1) $\nu_1 = \frac{1}{2}(2, 3, 1, 3), a = 1/2, g = 1$;
2) $\nu_2 = \frac{1}{2}(2, 1, 3, 3), a = 1/2, g = 1$;
3) $\nu_3 = \frac{1}{2}(4, 3, 1, 5), a = 1/2, g = 1$;
4) $\nu_4 = \frac{1}{2}(4, 3, 1, 7), a = 1/2, g \leq 3$;
5) $\nu_5 = \frac{1}{2}(6, 5, 1, 9), a = 1/2, g = 1$;
6) $\nu_6 = (2, 2, 1, 3), a = 1, g = 1$;
7) $\nu_7 = (3, 2, 1, 4), a = 1, g = 1$.

Note that curve for the blowup $\nu_4$ is not necessarily hyperelliptic.

**Example 3.5.2.**

\[ \{u^2 + x^3 + y^3 + z^{12} = 0\} \subset \frac{1}{2}(0, 1, 1, 1). \]

The exceptional divisor of the weighted blowup $\nu_4$ is given by the equation

\[ \{x^3 + y^4 + z^{12} = 0\} \subset \mathbb{P}(4, 3, 1, 7). \]

It is a cone over a non-hyperelliptic curve of genus 3.

Only the following pairs of blowups can simultaneously be non-rational: $\nu_1$ and $\nu_2$, $\nu_1$ and $\nu_6$.

**Example 3.5.3.**

\[ \{u^2 + x^3 + y^3 z^2 + y^6 + z^6 = 0\} \subset \frac{1}{2}(0, 1, 1, 1). \]

The exceptional divisor of the blowup $\nu_1$ is

\[ \{u^2 + x^3 + z^6 = 0\} \subset \mathbb{P}(2, 3, 1, 3); \]

the exceptional divisor of the blowup $\nu_2$ is

\[ \{u^2 + x^3 + y^6 = 0\} \subset \mathbb{P}(2, 1, 3, 3). \]

Both of them are cones over elliptic curves.

So, there are not more than 2 non-rational divisors with discrepancy $a \leq 1$ over a non-degenerate singularity of type $cE/2$. 

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