SEIBERG–WITTEN VANISHING THEOREM FOR S\(^1\)–MANIFOLDS WITH FIXED POINTS

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Abstract. In this paper we show that the Seiberg–Witten invariant is zero for all smooth 4–manifolds with \(b_+ > 1\) which admit circle actions that have at least one fixed point. Furthermore, we show that all symplectic 4–manifolds which admit circle actions with fixed points are rational or ruled, and thus admit a symplectic circle action.

1. Introduction

This paper addresses two problems in 4–manifold theory. The first concerns the computation of 4–dimensional diffeomorphism invariants. Ever since the introduction of Donaldson invariants in the early 1980’s, efforts to calculate diffeomorphism invariants centered upon large classes of smooth manifolds that have some additional structure. One such class of manifolds thought to have promise was 4–manifolds with effective circle actions, but the extra structure given by such manifolds turned out to be insufficient for calculating Donaldson invariants.

With the introduction of Seiberg–Witten invariants in 1994, old problems were revisited with new hope. S. Donaldson showed how to calculate the Seiberg–Witten invariants in the simplest case where the 4–manifold was a product of a 3–manifold and a circle \([7]\). In 1997 T. Mrowka, P. Ozvath, and B. Yu \([16]\), and simultaneously L. Nicolaescu \([17]\), studied the 3–dimensional Seiberg–Witten equations of Seifert–fibered spaces. In 2001, Seiberg–Witten invariants for manifolds with free and fixed point free circle actions were calculated in \([4]\), \([5]\). In this paper we finish this line of research by calculating the Seiberg–Witten invariants for \(S^1\)–manifolds with fixed points. We prove:

Theorem 1. If \(X\) is a smooth, closed, oriented 4–manifold with \(b_+ > 1\) and admits a circle action which has at least one fixed point, then the Seiberg–Witten invariant vanishes for all Spin\(^c\) structures.

When the action on \(X^4\) is free, the quotient of the \(S^1\)–action is a smooth, closed 3–manifold \(Y\). In this case \(X\) can be thought of as a unit circle bundle of a complex line bundle over \(Y\). Hence \(X\) with the given circle action is specified by the quotient \(Y\) and the Euler class \(\chi \in H^2(Y; \mathbb{Z})\) of the line bundle.

A fixed point free circle action will have nontrivial isotropy groups, forcing the orbit space to be an orbifold rather than a manifold. As in the free case, a manifold with a fixed point free \(S^1\)–action can still be considered a unit circle bundle, but it is furthermore a principal \(S^1\)–bundle of an orbifold line bundle over a 3–dimensional orbifold. In this setup, \(H^2(Y; \mathbb{Z})\) is replaced by the group \(\text{Pic}^f(Y)\) which records local data around the singular set (see \([\text{6}]\)). The manifold \(X\) with given \(S^1\)–action is then determined by the orbifold \(Y\) and the Euler class of the orbifold line bundle which is now an element of \(\text{Pic}^f(Y)\).

Manifolds which admit circle actions with fixed points come with more complicated local data, yet this extra structure gives more control when computing the Seiberg–Witten invariants. The
intuition behind the calculation is to find an essential sphere of nonnegative square; the existence of such spheres force the Seiberg–Witten invariants to vanish.

Theorem 1 combined with the formula derived in [5] gives the means for calculating the Seiberg–Witten invariants of any $S^1$–manifold $X$ with $b_+ > 1$. See [12] for an introduction to Seiberg–Witten theory or [18] for a more detailed analysis.

**Theorem 2** (General Formula). Let $X$ be a smooth, closed, oriented 4–manifold with $b_+ > 1$ and a smooth circle action.

1. If the action has fixed points, then $SW_X(\xi) = 0$ for any Spin$^c$ structure $\xi$.
2. If $X$ has a fixed point free $S^1$–action, let $Y^3$ be the orbifold quotient space and suppose that $\chi \in \text{Pic}^d(Y)$ is the orbifold Euler class of the circle action. If $\xi$ is a Spin$^c$ structure over $X$ with $SW_X^3(\xi) \neq 0$, then $\xi = \pi^*(\xi_0)$ for some Spin$^c$ structure on $Y$ and

$$SW_X^4(\xi) = \sum_{\xi' \equiv \xi_0 \mod \chi} SW_Y^3(\xi'),$$

where $\xi' - \xi_0$ is a well–defined element of $\text{Pic}^d(Y)$. See [5] with respect to the $b_+ = 1$ fixed point free case.

A theorem of Taubes [21] says that the Spin$^c$ structure associated with the first Chern class of a symplectic 4–manifold must have Seiberg–Witten invariant $\pm 1$. Symplectic 4–manifolds always have $b_+ > 0$ because the wedge product of symplectic form with itself is the volume form. Putting these facts together with Theorem 1 implies the following corollary.

**Corollary 3.** A symplectic 4–manifold which admits a circle action with fixed points must have $b_+ = 1$.

This application shows the usefulness of the Seiberg–Witten invariants even when they vanish. It also provides a nice introduction to the second problem: classifying symplectic 4–manifolds which admit circle actions with fixed points.

Such manifolds have been extensively studied. If the circle action preserves the symplectic form in the sense that the generating vector field of the action $T$ satisfies $L_T \omega = d\iota_T \omega = 0$ then the action is called symplectic. If in addition, $\iota_T \omega$ is exact, then the action is called Hamiltonian because there is an $f \in C^\infty(X)$ such that $\iota_T \omega = df$. D. McDuff showed for 4–manifolds that the existence of a fixed point implied that a symplectic circle action must also be a Hamiltonian action [15]. The same is true in any dimension if the action is semifree with only isolated fixed points [20].

In 1990 M. Audin classified 4–manifolds with symplectic circle actions having fixed points using McDuff’s result and symplectic reduction techniques [2] (or [1]). But it remained unknown whether every symplectic 4–manifold with a circle action having fixed points also admitted a symplectic circle action. Corollary 3 attacks this classification problem from the opposite direction by restricting which $S^1$–manifolds can have symplectic structures. It turns out that the restrictive theory can be made to meet with Audin’s theory and thus gives a complete classification.

Theorem 1 can be profitably restated as saying that every manifold with $b_+ > 0$ which admits a circle action with fixed points has an essential sphere of nonnegative square. A theorem of T.J. Li [13] implies that symplectic 4–manifolds with this condition must be rational or ruled. Thus:

**Theorem 4.** Every symplectic 4–manifold which admits a circle action with at least one fixed point is $\mathbb{CP}^2$, an $S^2$–bundle over a surface, or $\mathbb{CP}^2$ blowups of $\mathbb{CP}^2$ or an $S^2$–bundle over a surface.

While the circle action in Theorem 4 is not necessarily symplectic, the manifolds listed above are exactly the ones which Audin classified (c.f. [2] or [3]). Putting the two ideas together gives:
Corollary 5. Every symplectic 4–manifold which admits a circle action with at least one fixed point also admits a symplectic circle action.

This corollary leads us to an interesting question posed by C. Taubes [22]: if $Y^3 \times S^1$ is symplectic, does $Y$ fiber over the circle? Partial positive results have been posted in [6], [8]. One can ask a much more general question for any $S^1$–manifold (c.f. [4]):

Conjecture 6. Every symplectic 4–manifold which admits a circle action also admits (possibly a different) symplectic form and circle action which are symplectic with respect to each other.

If the conjecture holds it effectively answers Taubes’ question. Corollary 5 proves the conjecture when the action has fixed points. While the full proof of this conjecture seems out of reach at the moment, Corollary 5 does lend support to the hypothesis that symplectic 4–manifolds with $S^1$–actions are very special.

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2. Proofs

The theorem below proves both Theorem 1 and Theorem 4:

Theorem 7. Suppose $X$ is a smooth, closed, oriented, 4–manifold with $b_+(X)>0$ which admits a smooth $S^1$–action with at least one fixed point. Then $X$ contains an essential embedded sphere of nonnegative self–intersection.

If a 4–manifold $X$ has an essential embedded sphere of nonnegative self–intersection and $b_+(X)>1$, then the Seiberg–Witten invariant vanishes (c.f. [11]), proving Theorem 1. Likewise, the existence of such a sphere in a symplectic 4–manifold implies that the manifold is rational or ruled [13].

The proof of Theorem 7 relies heavily upon R. Fintushel’s foundational work on 4–manifolds with $S^1$–actions [9], [10]. Every such 4–manifold $\pi : X \to Y$ has a quotient 3–manifold $Y$ together with the following data (altogether called a legally weighted 3–manifold $Y$):

1. A finite collection of weighted arcs and circles in $\text{int} Y$.
2. A finite set of isolated fixed points in $\text{int} Y$ disjoint from the sets of (1).
3. A class (the “Euler class”) $\chi \in H_1(Y,S)$ where $S$ is the union of $\partial Y$, points of (2), and arcs of (1).

The endpoints of weighted arcs are fixed points and there may be a finite set of fixed points in the interior of a weighted arc or circle. To each component of an arc or circle that runs between two fixed points (without passing through another fixed point) one assigns a weight which records the local isotropy data of the circle action. If a weighted circle does not contain any fixed points, it is called simply–weighted; otherwise it is multiply–weighted.

To each component of (1) and (2) there is an associated index which is simply the Euler number of the principal $S^1$–bundle over the boundary of a tubular neighborhood of the component. Similarly, an index can be given to each boundary component (see Section 3.3 of [9]). The sum of the indices of all of the components is equal to 0 [13, Page 380].

We also need Fintushel’s classification result for simply connected 4–manifolds with circle actions [9], [10].

Theorem 8 (Fintushel). Let $S^1$ act smoothly on a simply connected 4–manifold $X$, and suppose the quotient space $Y \simeq S^3$ is not a counterexample to the 3–dimensional Poincaré conjecture. Then $X$ is a connected sum of copies of $S^4$, $\mathbb{C}P^2$, $\mathbb{C}P^2$, and $S^2 \times S^2$. 
We can already use this result to prove Theorem 1 when $X$ is simply connected. In that case, $X$ is the equivariant connect sum of two $b_+ > 0$ pieces where one of the quotient spaces is $S^3$. Thus there is a $\mathbb{CP}^2$ or $S^2 \times S^2$ summand in that piece with an essential embedded sphere of nonnegative square, as required. We are now ready to generalize this idea.

**Proof.** Let $X$ be a smooth, closed 4–manifold with a smooth $S^1$–action whose quotient is a legally weighted 3–manifold $Y$. First we show how to reduce to the case where there are no multiply–weighted circles in $Y$.

Suppose $Y$ contains a multiply–weighted circle $C$ with 3 or more fixed points. Note that $C$ could represent a nontrivial class in $H_1(Y; \mathbb{Z})$ or be embedded in $Y$ nontrivially as in Figure 1.

![Figure 1. Example of a multiply–weighted circle](image1)

In this situation $X$ can be decomposed into an equivariant connect sum of two 4–manifolds $X = X_0 \# N_1$, both with circle actions. The weighted orbit space of $X_0$ is the same as before except the weighted circle $C$ has exactly two fixed points; the weighted orbit space of $N_1$ is $S^3$ with a trivially embedded multiply–weighted circle with the original weights (Figure 2). The equivariant connect sum is performed by removing the preimage of a $D^3$ neighborhood of the fixed point between the weights $(5, 2)$ and $(12, 5)$ from both $X_0$ and $N_1$ and then gluing equivariantly along the $S^3$ boundary.

![Figure 2. Decomposing $X$ into $X_0 \# N_1$](image2)

If we repeat this for all multiply–weighted circles in $X$ with 3 or more fixed points, we can decompose $X$ into

$$X = X_0 \# N_1 \# N_2 \cdots \# N_k$$

and we can further assume each $N_i$ decomposes into connect sums of $\mathbb{CP}^2$, $\mathbb{CP}^2$, $S^2 \times S^2$, or $S^4$ by Theorem 8. If any of the $N_i$’s have a $\mathbb{CP}^2$ or $S^2 \times S^2$ factor, then $X$ has an essential
embedded sphere of nonnegative square. So we may assume that $X$ decomposes into $X_0\#k\mathbb{C}P^2$
where $b_+(X_0) > 0$ and $k \geq 0$. If there is an essential sphere of nonnegative square in $X_0$, then
such a sphere survives in any of its blowups. Hence it is enough to find a sphere of nonnegative
square in $X_0$.

We can get rid of the multiply–weighted circle in $X_0$ using P. Pao’s “replacement trick” [19].
He noticed that the same 4–manifold often admits many different $S^1$–actions and showed how to
replace a complicated circle action with a simpler one.

**Lemma 9** (Pao). Let $X$ be a 4–manifold with an $S^1$–action whose weighted orbit space $Y$
contains a weighted circle $C$ with exactly two fixed points. Then $X$ admits a different $S^1$–action
whose weighted orbit spaces is either $Y$ with $C$ replaced with a pair of isolated fixed points or
$Y \setminus \text{int } D^3$ with $C$ removed from the collection of weighted circles.

Thus we can work with $X_0$, rename it $X$, and assume it has only boundary components,
weighted arcs, isolated fixed points, or simply–weighted circles. If there are two or more boundary
components in the quotient, the preimage of an arc that runs from one boundary component to
another boundary component is an essential sphere of square zero; so we may reduce to the case
where $Y$ has only one boundary component. This case can be eliminated using a short argument
in the lemma below.

Thus we only need to consider $S^1$–manifolds $X$ with weighted arcs, isolated fixed points, and
simply–weighted circles. Isotope the arcs and fixed points into a smooth ball $D^3 \subset Y$ and enclose
them by a sphere $\partial D^3 \subset Y$. Since the sum of the indices of the components is 0, the Euler class
of the preimage of the sphere in $X$ is zero, i.e., we can realize $X$ as the fiber sum of two manifolds
$X_1$ and $N$ by

$$X = (X_1 \setminus (D^3 \times S^1)) \cup_{S^2 \times S^1} (N \setminus (D^3 \times S^1))$$

where the orbit space of $N$ contains all of the weighted arcs and isolated fixed points. The
quotient of $X_1$ contains no fixed points but it could still have complicated topology. $N$ is a
simply connected 4–manifold with an $S^1$ action and quotient $S^3$, so it is diffeomorphic to a
connect sum of $S^4$’s, $\mathbb{C}P^2$’s, $\overline{\mathbb{C}P^2}$’s, and $S^2 \times S^2$’s by Theorem [3]. We can in fact build $N$ by
starting with a circle action on $S^3$ with two fixed points and equivariantly connect summing
the other factors. This may be yet a different $S^1$–action, but the resulting manifold is still
diffeomorphic to $N$. Since $N$ is simply connected, any two embedded circles are isotopic; hence
the fiber sum of $X_1$ with $N$ along $S^2 \times S^1$ using the new $S^1$–action will still be diffeomorphic to
$X$. Once again we can eliminate all connect sum factors of $N$ except for the $S^1$ we started with.

Thus we have reduced the problem to finding an essential sphere of nonnegative square in

$$X = (X_1 \setminus (D^3 \times S^1)) \cup_{S^2 \times S^1} (S^4 \setminus (D^3 \times S^1))$$

where the quotient of $S^4$ has 2 fixed points. Note that $X$ is a 4–manifold with an $S^1$–action,
$b_+ > 0$, and 2 isolated fixed points $F \subset Y$. Because the sum of the indices is zero, one of the
fixed points comes with a $+1$ index and the other comes with a $-1$ index. In this situation,
$b_+(X) > 0$ forces $b_1(X) > 0$ by the formula \( \chi(F) = \chi(X) = 2 - 2b_1(X) + b_2(X) \) derived from
the Smith–Gysin sequence. By Corollary 10.3 of [10] we have that $b_1(Y) = b_1(X) > 0$. Hence
$H_1(Y; \mathbb{Z})$ in the long exact sequence of the pair $(Y, F)$ is nonzero:

$$0 \to H_1(Y; \mathbb{Z}) \to H_1(Y, F; \mathbb{Z}) \xrightarrow{\partial} H_0(F; \mathbb{Z}) \to H_0(Y; \mathbb{Z}) \to 0.$$ 

Since $\partial \chi = (1, -1) \in H_0(F; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$ there exists a 1–cycle in $H_1(X, F; \mathbb{Z})$ represented by a
closed loop $\gamma$ which is not a multiple of the Euler class $\chi$ of the action. The preimage $\pi^{-1}(\gamma)$ is
an essential torus of self–intersection 0. Isotope $\gamma$ to the fixed point set such that $\gamma = \gamma_1 + \gamma_2$
where $\gamma_1$’s are two arcs running from one fixed point to the other. The preimage of both of these
arcs is a sphere of self-intersection 0. Since the preimage of $\gamma$ is essential, one of these spheres must be essential as Theorem 9 demands.

The following lemma is a slight generalization of Proposition 4 in [9].

**Lemma 10.** Let $X$ be a smooth, closed oriented $b_+ > 1$ 4–manifold with a smooth circle action whose orbit space $Y$ has weighted circles and arcs, isolated fixed points, and one boundary component. Then there exist an essential sphere of nonnegative square in $X$.

**Proof.** As before, we eliminate cases which have spheres of nonnegative square by using the local description of equivariant plumbing given in Section 4 of [9], and by using techniques used in the proof above. Thus we can assume that the quotient space of $X$ contains only simply-weighted circles, $n$ isolated fixed points $\{x_1, x_2, \ldots, x_n\}$ each with a $+1$ index, and one boundary component with index $-n$. Denote the fixed point set by $F$.

A linearly independent subset of $H_2(X; \mathbb{Z})$ can be constructed as follows (c.f. Section 8 of [9]). Let $i = 1$ to $n$, let $\gamma_i$ be an arc that runs from $x_i$ to a point on $\partial Y$ such that all arcs are mutually disjoint. The preimage $\pi^{-1}(\gamma_i)$ of each of these arcs is an essential sphere $S_i$ which represents a 2–cycle in $H_2(X; \mathbb{Z})$. These linearly independent classes have an intersection matrix with respect to each other given by

$$S_i \cdot S_j = \begin{cases} -1 & i = j \\ 0 & \text{otherwise.} \end{cases}$$

Because the intersection form of $X$ is not negative definite, we must have $b_2(X) > n$. Let $g$ be the genus of $\partial Y$. Using the fact $\chi(X) = \chi(F)$, we get

$$b_1(X) = \frac{1}{2}(b_2(X) - n + 2g).$$

In particular, $b_1(Y) > g$ by Corollary 10.3 of [9] again. The long exact sequence for the pair $(Y, F)$ implies that there is a 1–cycle in $H_1(Y, F; \mathbb{Z})$ which is represented by a closed loop $\gamma$ which is not a multiple of the Euler class $\chi$. The preimage $\pi^{-1}(\gamma)$ is an essential torus of self–intersection zero. The loop $\gamma$ is homologous to an arc which starts and ends on $\partial Y$ but is otherwise disjoint from $F$; and the preimage of the arc is a sphere which is homologous to the torus. This is an essential sphere of nonnegative square, proving Lemma 10. □

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