Exceptional Coupling Constants for the Coulomb-Dirac Operator with Anomalous Magnetic Moment

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Dedicated to E.B. Davies on the occasion of his 60th birthday

Abstract. It was recently shown that the point spectrum of the separated Coulomb-Dirac operator \( H_0(k) \) is the limit of the point spectrum of the Dirac operator with anomalous magnetic moment \( H_a(k) \) as the anomaly parameter tends to 0; this spectral stability holds for all Coulomb coupling constants \( c \) for which \( H_0(k) \) has a distinguished self-adjoint extension if the angular momentum quantum number \( k \) is negative, but for positive \( k \) there are certain exceptional values for \( c \). Here we obtain an explicit formula for these exceptional values. In particular, it implies spectral stability for the three-dimensional Coulomb-Dirac operator if \( |c| < 1 \), covering all physically relevant cases.

1. Introduction

By separation of variables in spherical polar coordinates, the Dirac operator of relativistic quantum mechanics with a Coulomb potential,

\[
H_0 := -i\alpha \cdot \nabla + \beta + \frac{c}{|x|},
\]

where \( c < 0 \) and \( \alpha_1, \alpha_2, \alpha_3 \) and \( \beta = \alpha_0 \) are symmetric \( 4 \times 4 \) matrices satisfying the anticommutation relations

\[
\alpha_i \alpha_j + \alpha_j \alpha_i = \delta_{ij} \quad (i, j \in \{0, 1, 2, 3\}),
\]

is unitarily equivalent to a direct sum of one-dimensional Dirac operators on the half-line,

\[
H_0(k) = -i\sigma_2 \frac{d}{dr} + \sigma_3 + \frac{k}{r} \sigma_1 + \frac{c}{r} \quad (r \in (0, \infty)),
\]

where \( \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), \( \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \) and \( \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) are the Pauli matrices and \( k \in \mathbb{Z} \setminus \{0\} \) is the angular momentum quantum number [13, Appendix to Section 1]. For an electron orbiting a nucleus of charge number \( Z \in \mathbb{N} \), the Coulomb coupling constant is \( c = -Z\alpha \), with the Sommerfeld fine structure constant \( \alpha \approx 1/137 \).
\( H_0(k) \) is essentially self-adjoint on its minimal domain if and only if \( c^2 \leq k^2 - 1/4 \) [13, Theorem 6.9]. For \( c^2 \in (k^2 - 1/4, k^2) \), there still exists a distinguished self-adjoint extension, characterised by the requirement that the wave-functions in its domain have a finite potential (or kinetic) energy [13, Theorem 6.10].

Pauli suggested a modification of the Dirac operator which takes into account the anomalous magnetic moment of the electron (for the historical background cf. [7]). With suitably normalised constants, the operator with a potential \( V \) for an electron of magnetic moment \((1 + a)\mu_B\), where \( \mu_B \) is the Bohr magneton, takes the form

\[
\mathcal{H}_a = -i\alpha \cdot \nabla + \beta + V - i\alpha \cdot \nabla V.
\]

In the case of the Coulomb potential, the corresponding half-line operators after variable separation will be

\[
H_a(k) = -i\sigma_2 \frac{d}{dr} + \sigma_3 + \left( \frac{k}{r} + \frac{a}{r^2} \right) \sigma_1 + \frac{c}{r} \quad (r \in (0, \infty)).
\]

The mathematical investigation of the properties of this operator was initiated by Behncke [2, 3, 4]. \( H_a(k) \) is essentially self-adjoint on its minimal domain for all values \( c < 0 \) of the coupling constant (cf. [1, 5]). The essential spectrum of \( H_a(k) \), as that of \( H_0(k) \), is \((-\infty, -1] \cup [1, \infty)\).

Let us now focus on the case \( c^2 < k^2 \), so that a distinguished self-adjoint realisation of \( H_0(k) \) exists. We always assume \( a < 0 \) in the following; the case of positive \( a \) can be reduced to this (with a sign change of \( k \)) by a suitable unitary transformation. Although only integer values of \( k \) are relevant for the three-dimensional operator, we admit general non-zero real values of \( k \). As is well known, \( H_0(k) \) has infinitely many eigenvalues in the spectral gap \((-1, 1)\), which accumulate at 1. One would expect that the eigenvalues of \( H_a(k) \) will be perturbations of those of \( H_0(k) \), such that each eigenvalue of \( H_0(k) \) will be the limit of exactly one eigenvalue branch of \( H_a(k) \) in the limit \( a \to 0 \) (spectral stability, cf. [8, Chapter VIII §1.4]). This expectation is partly corroborated by the strong resolvent convergence of \( H_a(k) \) to \( H_0(k) \), at least for \( c^2 \leq k^2 - 1/4 \), which implies that the spectrum cannot suddenly expand in the limit. Nevertheless, due to the strong singularity of the \( a/r^2 \) term at the origin, this limit problem is highly non-trivial and does not subject itself to the standard perturbation techniques. Behncke [4, Theorem 2] was able to prove spectral stability for \( c^2 < k^2 - (3/2)^2 \) if \( k < 0 \), and for \( c^2 < k - 5/2 \) if \( k > 0 \).

The surprising asymmetry with respect to the sign of \( k \) is not an artefact of Behncke’s method, but inherent in the problem, as observed in the recent treatment [7], where spectral stability was proved, based on an asymptotic analysis of the (non-linear) Prüfer and Riccati equations equivalent to the Dirac system

\[
(H_a(k) - \lambda)u = 0,
\]

for all \( c \in (k, 0) \) if \( k < 0 \), and for all \( c \in (-k, 0) \setminus \{ c_0, c_1, \ldots \} \) if \( k > 0 \). Here \( c_j \in (-k, 0) \), with \( c_j > c_{j+1} \), are certain exceptional values of the coupling constant at which a
transition in the behaviour of the eigenfunctions of $H_a(k)$ occurs: for $c \in (c_n, c_{n-1})$, the eigenfunctions of $H_a(k)$ show $n$ additional rapid oscillations very close to the origin, compared to the corresponding eigenfunctions of $H_0(k)$. It remains a fairly subtle open question whether or not spectral stability holds if $c$ is equal to one of the exceptional values.

The present note is devoted to a study of the number and positions of the exceptional values of $c$ for a given $k > 0$.

2. Exceptional values

The exceptional values for $c$ appear in a stability analysis of the differential equation [7, Eq.(1)]

$$\varrho \vartheta'(\varrho) = c + (k - \frac{1}{\varrho}) \sin 2\vartheta \quad (\varrho > 0).$$

(2)

This equation arises from the Prüfer transformation (cf. Appendix A) of the Dirac system (1) after omitting the lowest-order term at 0, $\sigma_3 - \lambda$, and rescaling $r = |a| \varrho$ to absorb the parameter $a$.

In the limit $\varrho \to \infty$, the right-hand side of (2) — which is $\pi$-periodic in $\vartheta$ — has asymptotic zeros $\vartheta_{\pm}(c, k)$ satisfying $0 < \vartheta_-(c, k) < \pi/4 < \vartheta_+(c, k) < \pi/2$,

$$\sin 2\vartheta_{\pm}(c, k) = -\frac{c}{k}, \quad \tan \vartheta_{\pm}(c, k) = \frac{k \pm \sqrt{k^2 - c^2}}{-c}.$$

An asymptotic study of the direction field of (2) for $\varrho \to 0$ and $\varrho \to \infty$ yields the following result, which shows that this differential equation has (up to addition of a constant multiple of $\pi$) a unique unstable solution at 0 and at $\infty$ (see [7, Lemma 2.1, 2.2]).

**Lemma 1.** Let $k > 0$ and $c \in (-k, 0)$. Then the following statements hold.

a) There is a unique solution $\vartheta_0(\cdot, c)$ of (2) such that $\lim_{\varrho \to 0} \vartheta_0(\varrho, c) = \pi$.

All other solutions either differ from $\vartheta_0$ by a constant integer multiple of $\pi$, or else satisfy $\lim_{\varrho \to 0} \vartheta(\varrho, c) = \frac{\pi}{2} \text{ mod } \pi$.

For fixed $\hat{\varrho} > 0$, $\vartheta_0(\hat{\varrho}, \cdot)$ is continuous non-decreasing.

b) There is a unique solution $\vartheta_\infty(\cdot, c)$ of (2) such that $\lim_{\varrho \to \infty} \vartheta_\infty(\varrho, c) = \vartheta_-(c, k)$.

All other solutions either differ from $\vartheta_\infty$ by a constant integer multiple of $\pi$, or else satisfy $\lim_{\varrho \to \infty} \vartheta(\varrho, c) = \vartheta_+(c, k) \text{ mod } \pi$.

For fixed $\hat{\varrho} > 0$, $\vartheta_\infty(\hat{\varrho}, \cdot)$ is continuous and strictly decreasing.

The solution $\vartheta_0(\cdot, c)$ asymptotically corresponds to the Prüfer angle of the $L^2(0, \infty)$ solution of (1).

The **exceptional values** are defined as those values of $c$ for which $\vartheta_0(\cdot, c)$ and $\vartheta_\infty(\cdot, c)$ match up mod $\pi$, so that the unstable solution of (2) at 0 is unstable at
infinity as well. More precisely, in view of the monotonicity properties of \( \vartheta_0 \) and \( \vartheta_\infty \) with respect to \( c \), we have
\[
\lim_{\varrho \to \infty} \vartheta_0(\varrho, c_m) = \vartheta_-(c_m, k) - m\pi \quad (m = 0, 1, 2, \ldots).
\]

By the transformation \( \varrho = e^t \), \( \vartheta(\varrho) = \varphi(\log \varrho) \), the equation (2) is equivalent to
\[
\varphi'(t) = c + (k - e^{-t}) \sin 2\varphi(t) \quad (t \in \mathbb{R}),
\]
the differential equation for the Prüfer angle \( \varphi \) (cf. Appendix A) of a \( \mathbb{R}^2 \)-valued solution \( u \) of the Dirac system
\[
\begin{aligned}
(-i\sigma_2 \frac{d}{dt} + (k - e^{-t})\sigma_1 + c) u(t) &= 0,
\end{aligned}
\]
which can be rewritten using the definition of the Pauli matrices as
\[
\begin{aligned}
u_1' &= (e^{-t} - k)u_1 - cu_2, \\
u_2' &= cu_1 + (k - e^{-t})u_2.
\end{aligned}
\]

In (4), \( c \) takes on the role of a spectral parameter, while the coefficient of \( \sigma_1 \) can be interpreted as a constant mass term with an exponentially decaying perturbation. In the sense of the analogue of Kneser’s Theorem for Dirac systems [9], this perturbation is subcritical, and indeed the method developed in [9] can be used to show that, for each \( k > 0 \), there are only finitely many exceptional values for \( c \). Moreover, an asymptotic analysis of (4) along the lines of [10; see also 12] reveals that the number of exceptional values is asymptotic to \( k \) in the limit \( k \to \infty \) with fixed ratio \( c/k \).

This was verified in a computational investigation of (4) based on a piecewise-constant approximation of the exponential function, following the approach of [11]. The numerical findings suggested that the number of exceptional values in \((-k, 0)\) is always equal to the unique integer in \([k - 1, k)\); more precisely, whenever \( k \) reaches an integer value, an additional exceptional value, initially equal to \(-k\), appears and moves into the interval \((-k, 0)\) as \( k \) increases.

This regularity raised the suspicion that (4) has an underlying solvable structure, and indeed, on closer scrutiny it turns out that this equation can be analysed by means of a variant of the factorisation method, resulting not only in a proof of the above conjecture, but even in an explicit formula for the exceptional values, which makes any asymptotic and computational analysis of (4) superfluous.

**Theorem 1.** For given \( k > 0 \), the exceptional values for \( H_\alpha(k) \) are given by
\[
c\alpha_{n-1}(k) = -\sqrt{2kn - n^2} \quad (n \in \mathbb{N}, n < k).
\]

Consequently there are exactly \( N \) exceptional values in \((-k, 0)\) if \( k \in (N, N + 1] \), \( N \in \mathbb{N}_0 \).
In view of Theorems 1.1 and 1.2 of [7], this implies in particular that the eigenvalues of $H_a(k)$ converge to those of $H_0(k)$ as $a \to 0$ if $c^2 < 2k - 1$, confirming Behncke’s linear bound (although with a different constant). However, his conjecture [4, p. 2558] that spectral stability may hold for all $c^2 < k^2 - 5/2$ turns out to be very questionable.

Since the three-dimensional Coulomb-Dirac Hamiltonian $H_a$ is the direct sum of $H_a(k)$ for $k \in \mathbb{Z} \setminus \{0\}$, we can thus infer spectral stability for this operator in all cases in which $H_0$ has a distinguished self-adjoint realisation.

**Corollary 1.** Let $H_0$ be the Dirac operator and $H_a$ the Dirac operator with anomalous magnetic moment with a Coulomb potential with coupling constant $c \in (-1, 0)$. Then the eigenvalues of $H_0$ are the limits of the eigenvalues of $H_a$ as $a \to 0$.

**3. Proof of Theorem 1**

The following proof of Theorem 1 will be based on squaring the Dirac-type operator in (4), which is equivalent to deriving second-order differential equations for $u_1$ and $u_2$. In a general situation, this is usually not a good idea except to obtain a quick and rough heuristic result, since it involves derivatives of the coefficients and consequently requires unnecessarily strong regularity. In the present instance, however, we are dealing with a specific Dirac system with analytic coefficients, and as it turns out, the resulting second-order equation system decouples and can be solved by the factorisation method in the relevant cases (the second-order equation (5) is closely related to the radial Schrödinger equation with a Morse potential [6, Section 5.2]). Here the factorisation is incidentally provided by the original Dirac system; see the discussion in Appendix B. The key observation is contained in the following result.

**Theorem 2.** Let $\kappa \geq 0$. Let $v_1(t, \kappa) := e^{-e^{-t} - \kappa t}$, and define $v_j$ recursively for $j \in \mathbb{N} + 1$ by

$$v_{j+1}(t, \kappa) := (\kappa + j - e^{-t})v_j(t) - v_j'(t) \quad (t \in \mathbb{R}).$$

Then $v_j(\cdot, \kappa)$ is a nontrivial solution of

$$v'' = (e^{-2t} - (2\kappa + 2j - 1)e^{-t} + \kappa^2) v,$$

and it has the asymptotic properties

$$\lim_{t \to \infty} v_j(t, \kappa) = 0, \quad \lim_{t \to \infty} \frac{v_j'(t, \kappa)}{v_j(t, \kappa)} = -\kappa, \quad \text{and} \quad \lim_{t \to \infty} \frac{v_j'(t, \kappa)}{v_j(t, \kappa)} e^t = 1 \quad (j \in \mathbb{N}).$$

The proof can be done by induction with respect to $j$. The fact that $v_j$ is a solution of (5) is readily verified by differentiating twice. The asymptotic properties are obvious for

$$\frac{v_j'(t, \kappa)}{v_1(t, \kappa)} = e^{-t} - \kappa.$$
Now assume that \( j \in \mathbb{N} \) is such that the assertion is true. If \( v_{j+1} \) were trivial, this would imply that (up to multiplication by a constant) \( v_j(t, \kappa) = e^{(\kappa+j)t} t \to \infty \) (\( t \to \infty \)), contradicting the first limit property of \( v_j \). The asymptotic properties of \( v_{j+1} \) are easily checked using the identity

\[
\frac{v_{j+1}'(t, \kappa)}{v_{j+1}(t, \kappa)} = \frac{-e^{-2t} + 2(\kappa + j)e^{-t} - \kappa^2}{\kappa + j - e^{-t} - \frac{v_j'(t, \kappa)}{v_j(t, \kappa)}}.
\]

**Corollary 2.** Let \( \kappa \geq 0, j \in \mathbb{N} \) and \( v_j \) be as in Theorem 1. Then

\[
\lim_{t \to \infty} \frac{v_j(t)}{v_{j+1}(t)} = \frac{1}{2\kappa + j}, \quad \lim_{t \to -\infty} \frac{v_j(t)}{v_{j+1}(t)} = 0.
\]

This is a direct consequence of Theorem 1 in view of

\[
\frac{v_j(t, \kappa)}{v_{j+1}(t, \kappa)} = \frac{1}{\kappa + j - e^{-t} - \frac{v_j'(t, \kappa)}{v_j(t, \kappa)}}.
\]

Furthermore, the following conclusion can be verified by a straightforward calculation.

**Corollary 3.** Let \( v_j \) be defined as in Theorem 1, and let \( j \in \mathbb{N}, k \geq j \). Then

\[
 u(t) = \left( \frac{1}{\sqrt{2kj - j^2}} v_{j+1}(t, k-j) \right) \quad (t \in \mathbb{R})
\]

is a nontrivial solution of the Dirac system (4) with \( c = -\sqrt{2kj - j^2} \). The Prüfer angle \( \varphi \) of \( u \) satisfies

\[
\lim_{t \to \infty} \tan \varphi(t) = \frac{j}{\sqrt{2kj - j^2}}, \quad \lim_{t \to -\infty} \tan \varphi(t) = 0.
\]

Since, with \( c \) as in Corollary 3,

\[
\tan \theta_+(c) = \frac{k - \sqrt{k^2 - c^2}}{-c} = \frac{k - \sqrt{k^2 - 2kj + j^2}}{\sqrt{2kj - j^2}} = \frac{j}{\sqrt{2kj - j^2}},
\]

where we have used \( j \leq k \), the Prüfer angle \( \varphi \) in Corollary 3 corresponds via \( \vartheta(g) = \varphi(\log g) \) to a solution of (3) which coincides (up to addition of integer multiples of \( \pi \)) with both \( \vartheta_0(\cdot, c) \) and \( \vartheta_\infty(\cdot, c) \); hence such values of \( c \) are exceptional values.
We now conclude the proof of Theorem 1 by showing, based on the continuity and monotonicity properties of $\vartheta_0$ and $\vartheta_\infty$, that all exceptional values are obtained in this way. Corollary 3 specifies the asymptotic behaviour of the relevant angle functions mod $\pi$; a study of the zeros of $v_j$ reveals their global behaviour as shown in Lemma 2 below. Theorem 1 then follows in view of

$$\vartheta(\varrho, -\sqrt{2kj - j^2}) = \varphi_j(\log \varrho, k - j) \quad (\varrho > 0, k > 0, j \in \mathbb{N}, j < k),$$

where (with $\arctan 0 = 0$)

$$\varphi_j(\cdot, \kappa) := \arctan \left( \frac{v_j(\cdot, \kappa)}{v_{j+1}(\cdot, \kappa)} \sqrt{2\kappa j + j^2} \right) + \pi.$$

**Lemma 2.** Let $\kappa \geq 0$. Then $v_j(\cdot, \kappa)$ has exactly $j - 1$ zeros. The associated angle function $\varphi_j$ satisfies

$$\lim_{t \to -\infty} \varphi_j(t, \kappa) = \pi$$

and

$$\lim_{t \to \infty} \varphi_j(t, \kappa) = \vartheta_-( -\sqrt{2\kappa j + j^2}, \kappa + j ) - (j - 1)\pi \quad (j \in \mathbb{N}).$$

**Proof.** A look at the direction field of (3) with $k = \kappa + j$ shows that $\varphi_j(\cdot, \kappa)$, being the unstable solution at both $\pm \infty$, is monotone decreasing. $v_1(\cdot, \kappa)$ has no zeros, so $\varphi_1(\cdot, \kappa)$ has no zeros mod $\pi$, and the assertion follows for $j = 1$.

To conclude the proof by induction, assume now that $j \in \mathbb{N}$ is such that the assertion is true. Then $\varphi_j(\cdot, \kappa) - \pi/2$ has exactly $j$ zeros mod $\pi$, so $v_{j+1}(\cdot, \kappa)$ has exactly $j$ zeros. Since the $v_{j+1}(\cdot, \kappa)$ is a non-trivial solution of a second-order equation, $v'_{j+1}(\cdot, \kappa)$ does not vanish at zeros of $v_{j+1}(\cdot, \kappa)$. Consequently, $v_{j+1}(\cdot, \kappa)$ and $v_{j+2}(\cdot, \kappa)$ have no common zeros. Hence $\varphi_{j+1}(\cdot, \kappa)$ has exactly $j$ zeros mod $\pi$, and therefore must converge to $\vartheta_-( -\sqrt{2\kappa j + j^2}, \kappa + j ) - (j - 1)\pi$ as $t \to \infty$.

**Appendix**

A. The Prüfer Transformation

Consider a general Dirac system

$$(-i\sigma_2 \frac{d}{dx} + m(x)\sigma_3 + l(x)\sigma_1 + q(x))u(x) = 0 \quad (x \in I) \quad (6)$$

on an interval $I \subset \mathbb{R}$ with locally integrable, real-valued coefficients $m, l$ and $q$ (we have absorbed the spectral parameter in the latter). If $u$ is a solution of (6), then so is its (component-wise) complex conjugate, so it is sufficient to study $\mathbb{R}^2$-valued solutions.
Since (6) is linear, a non-trivial solution will never take the value \( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \), so \( u(x) \) traces out an absolutely continuous curve in the punctured plane as \( x \) varies.

Introducing polar coordinates in this plane by writing

\[
u(x) = |u(x)| \begin{pmatrix} \cos \vartheta(x) \\ \sin \vartheta(x) \end{pmatrix},
\]

where the absolutely continuous function \( \vartheta \) (the Prüfer angle of \( u \)) is determined uniquely up to addition of an integer multiple of \( 2\pi \) — or indeed \( \pi \), since \( -u \) is again a solution of (6) —, a straightforward calculation yields the differential equation for \( \vartheta \) (Prüfer equation)

\[
\vartheta' = m \cos 2\vartheta + l \sin 2\vartheta + q.
\]

This non-linear equation is equivalent to (6), because one can recover \( u \) from \( \vartheta \) by noticing that (choosing \( x_0 \in I \))

\[
|u(x)| = |u(x_0)| \exp \int_{x_0}^x (m \sin 2\vartheta - l \cos 2\vartheta) \quad (x \in I).
\]

B. Dirac systems and the factorisation method

Known to students of quantum mechanics as a trick to treat the Schrödinger equation for the harmonic oscillator, the factorisation method is in fact a tool of much wider scope for studying second-order differential equations (see the extensive discussion in [6]). It is based on the close link between the eigenvalue equations

\[
A^*Au = \lambda u \quad \text{and} \quad AA^*v = \lambda v,
\]

where \( A \) is typically a first-order differential operator: a solution \( u \) of the first equation gives rise to a solution \( v = Au \) of the second.

Here we remark on how this method can serve to solve eigenvalue equations for certain one-dimensional Dirac operators, which in turn represent a factorisation of the Sturm-Liouville operators arising as their formal squares.

Starting from a Dirac system (6), or equivalently

\[
\begin{align*}
u_1' &= -lu_1 + (m - q)u_2, \\
u_2' &= (m + q)u_1 + lu_2,
\end{align*}
\]

with absolutely continuous coefficients \( l, m, q \), we find that \( u_1 \) and \( u_2 \) satisfy the second-order equations

\[
\begin{align*}
u_1'' &= (m^2 - q^2 + l^2 - l')u_1 + (m' - q')u_2, \\
u_2'' &= (m^2 - q^2 + l^2 + l')u_2 + (m' + q')u_1.
\end{align*}
\]
This ordinary differential equation system decouples if \( m \) and \( q \) are constant. If we assume this in the following and set \( c := m - q, \ d := m + q \) and \( a := cd \), we are dealing with the Dirac system

\[
\begin{align*}
  u'_1 &= -lu_1 + cu_2 \\
  u'_2 &= lu_1 + du_2
\end{align*}
\]

and corresponding second-order equations

\[
\begin{align*}
  u''_1 &= (a + l^2 - l')u_1, \quad (7) \\
  u''_2 &= (a + l^2 + l')u_2. \quad (8)
\end{align*}
\]

These two Sturm-Liouville equations differ only in the sign of \( l' \). Hence we can construct a chain of interlocking equation systems, along with special solutions, in the following way.

**Theorem 3.** Let \((a_n)_{n \in \mathbb{N}}\) be a sequence of real numbers with \( a_1 = 0 \), and let \((l_n)_{n \in \mathbb{N}}\) be a sequence of real-valued absolutely continuous functions on \( I \) such that

\[
l^2_{n+1} + l'_{n+1} + a_{n+1} = l^2_n - l'_n + a_n \quad (n \in \mathbb{N}).
\]

Let \( v_1(x) := \exp \int^x l_1 \ (x \in I) \), and define recursively for \( n \in \mathbb{N} + 1 \)

\[
v_n := l_nv_{n-1} - v'_{n-1}.
\]

Then \( v_n \) is a solution of \((7)\) with \( a = a_n, \ l = l_n \), and of \((8)\) with \( a = a_{n+1}, \ l = l_{n+1} \). Furthermore, for \( n \in \mathbb{N} + 1 \) and arbitrary \( d_n \in \mathbb{R} \setminus \{0\} \),

\[
u := \begin{pmatrix}
  -\frac{1}{d_n}v_n \\
  v_{n-1}
\end{pmatrix}
\]

is a solution of the Dirac system

\[
\begin{align*}
  u'_1 &= -lu_1 + (a_n/d_n)u_2 \\
  u'_2 &= du_1 + lu_2. \quad (9)
\end{align*}
\]

**Proof.** Because of \( a_1 = 0 \), the first equation of the Dirac system \((9)\) with \( n = 1 \) decouples from the second and can easily be solved to obtain the formula for \( v_1 \). As the first component of a solution of this Dirac system, \( v_1 \) satisfies equation \((7)\) with \( l = l_1, \ a = a_1 \). In view of the assumptions on the coefficients \( l_n \), this equation is the same as \((8)\) with \( l = 2, \ a = a_2 \), the Sturm-Liouville equation satisfied by the second component of a solution of the Dirac system for \( n = 2 \). The corresponding first component of this solution can easily be calculated from the second equation of the Dirac system. It will
depend on \(d_2\), but note that this dependence can be eliminated by normalisation as far as obtaining a solution \(v_2\) of the Sturm-Liouville equation (7) with \(l = l_2, a = a_2\) is concerned.

This process is then iterated to find the solutions for \(n \in \{3, 4, \ldots\}\).

Examples.
1. The classical application of the factorisation method to the one-dimensional Schrödinger equation for a quantum-mechanical harmonic oscillator,

\[-v''(x) + x^2v(x) = \lambda v(x),\]

is captured in the above scheme if we choose \(l_n(x) := x\ (n \in \mathbb{N})\). The hypothesis of Theorem 3 is then satisfied with \(a_n := -2(n - 1)\ (n \in \mathbb{N})\), and we obtain the well-known solutions

\[v_n(x) = \left(-\frac{d}{dx} + x\right)^{n-1}e^{-x^2/2}\]

with eigenvalues \(\lambda_n := 2n - 1\ (n \in \mathbb{N})\).

2. The case of equation (5) is treated by choosing \(l_n(x) := \kappa + (n - 1) - e^{-x}\ (n \in \mathbb{N})\); we immediately find \(v_1(x) = e^{-e^{-x}+\kappa x}\) and the recursion formula of Theorem 2.

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